DISCRETIZATION IN SPACE AND TIME OF SUBDIFFUSION EQUATIONS WITH MEMORY

by

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DECLARATION

It is certified that the work contained in this thesis entitled **“ Discretization in space and time of subdiffusion equations with memory”** has done by me, under the supervision of **Dr. Rajen Kumar Sinha**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is certified that the work contained in this thesis entitled **“ Discretization in space and time of subdiffusion equations with memory”** by **Shantiram Mahata (156123015)**, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

November, 2021

Dr. Rajen Kumar Sinha
Professor
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To my

Parents and family members


भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
Indian Institute of Technology Guwahati

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With regards,

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Abstract

The main objective of this thesis is to investigate the numerical analysis of finite element method for the subdiffusion equations with memory. Both smooth and singular kernels are considered covering smooth and nonsmooth initial data.

Prior to the development of numerical schemes, we first study the existence, uniqueness, and stability results of the continuous solution with respect to various Sobolev regularity assumptions on the problem data, i.e., the initial data and the source function. For both smooth and singular kernel cases, we show that the solution corresponding to the homogeneous problem is infinitely differentiable with respect to time variable when the initial data is simply an element of $L^2(\Omega)$. In each case, we study the semidiscrete as well as the fully discrete schemes and carry out the convergence analysis for both smooth and nonsmooth initial data. Error bounds are expressed directly in terms of the problem data. We discretize the spatial variable based on Galerkin finite element method (GFEM) by using piecewise linear functions. The temporal discretization is performed by convolution quadrature with the generating function given by the backward Euler (BE) and the second-order backward difference (SBD) schemes and the L1 scheme. For the smooth kernel case, we derive optimal order error bounds for smooth initial data and almost optimal order for nonsmooth data in both $L^2(\Omega)$- and $H^1(\Omega)$-norms for the spatially discrete scheme. For a particular choice of the kernel operator in the memory term, we have achieved optimal error estimates for nonsmooth initial data. In addition, we study the convergence analysis with nonsmooth data by energy arguments which enables us to recover optimal order error bounds even for a more general kernel in the memory term. For the temporal discretization, we demonstrate error bounds of first-order and second-order accuracy in time for the BE and SBD schemes, respectively. In the singular kernel case, we prove optimal error bounds for the GFEM to the spatial discretization and an error estimate of order $O(k)$ in time for the L1 scheme. In all cases, we validate our theoretical convergence rates of the approximate solutions by extensive numerical experiments.

Contents

NOMENCLATURE

Ω	Bounded convex polygonal or polyhedral domain
$\partial\Omega$	Boundary of Ω
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{C}$	Set of all complex numbers
IBVP	Initial-boundary value problem
SE	Subdiffusion equations
SEM	Subdiffusion equations with memory
PIDE	Parabolic integro-differential equations
FEM	Finite element method
GFEM	Galerkin finite element method
BE	Backward Euler
SBD	Second-order backward difference
CR	Convergence rate
A	Negative Laplacian operator $-\Delta$
$\{\lambda_j, \phi_j\}$	Dirichlet eigenpairs of the operator A over Ω, $j \geq 1$
A_h	Discrete analogue of A
$\{\lambda_j^h, \phi_j^h\}$	Dirichlet eigenpairs of the operator A_h over Ω, $1 \leq j \leq N$
$B(t,s)$	General second-order partial differential operator of the form $B(t,s) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(b_{ij}(x;t,s)\frac{\partial}{\partial x_i}\Big) + \sum_{j=1}^{d} b_j(x;t,s)\frac{\partial}{\partial x_j} + b_0(x;t,s)$ with smooth coefficients $b_{ij}(x;t,s)$, $b_j(x;t,s)$, and $b_0(x;t,s)$
$B_t(t,s)$	An operator obtained from $B(t,s)$ by differentiating the coefficients with respect to t
$B_s(t,s)$	An operator obtained from $B(t,s)$ by differentiating the coefficients with respect to s
$B(t,s;\cdot,\cdot)$	Bilinear form associated to the operator $B(t,s)$

$B_h(t,s)$	Discrete analogue of the operator $B(t,s)$
$B_{ht}(t,s)$	Discrete analogue of the operator $B_t(t,s)$
u	Exact solution
u_0	Initial data
f	Source function
T	Fixed positive real number
h	Mesh size
$\mathcal{T}_h$	Quasi-uniform triangulation of the domain Ω
S_h	Finite dimensional vector space of piecewise continuous and linear functions over $\mathcal{T}_h$
P_h	L^2-projection operator
R_h	Ritz projection operator
V_h	Ritz-Volterra projection operator
u_h	Spatial finite element approximation
k	Time step size
U_h^n	Fully discrete approximate solution
$L^p(\Omega)$	p-integrable functions over the measurable set Ω supplied with the norm $\\|f\\|_{L^p(\Omega)} = \big(\int_\Omega \lvert f(x)\rvert^p\,dx\big)^{1/p}$, $1 \leq p < \infty$
$L^\infty(\Omega)$	Space of all bounded functions supplied with the norm $\\|f\\|_{L^\infty(\Omega)} = \inf\\{C : \lvert f(x)\rvert \leq C$ almost everywhere on $\Omega\\}$
$(\cdot,\cdot)$	Standard $L^2(\Omega)$-inner product
$H^m(\Omega)$	Standard Sobolev spaces $W^{m,2}(\Omega)$, $m = 1, 2$
$H_0^1(\Omega)$	Collection of all functions in $H^1(\Omega)$ that are zero on $\partial\Omega$ in the sense of trace
$\dot{H}^r(\Omega)$	Subspace of $L^2(\Omega)$ supplied with the norm $\\|v\\|_r = \big(\sum_{j=1}^{\infty} \lambda_j^r (v,\phi_j)^2\big)^{1/2}$, where r is a nonnegative real number
$L^p(0,T;X)$	The standard Bochner space with X is a Banach space supplied with the norm $\\|f\\|_{L^p(0,T;X)} = \big(\int_0^T \\|f(t)\\|_X^p\,dt\big)^{1/p}$, $1 \leq p < \infty$

$L^\infty(0,T;X)$	The standard Bochner space with X is a Banach space supplied with the norm $\|f\|_{L^\infty(0,T;X)}=$ $\inf\{C : \|f(t)\|_X \leq C$ almost everywhere on $(0,T)\}$
$C([0,T];L^2(\Omega))$	The standard Bochner space supplied with the norm $\|f\|_{C([0,T];L^2(\Omega))} = \max\limits_{t\in[0,T]} \|f(t)\|_{L^2(\Omega)}$
$C^\beta([0,T];L^2(\Omega))$	The standard Bochner space supplied with the norm $\|f\|_{C^\beta([0,T];L^2(\Omega))} = \|f\|_{C([0,T];L^2(\Omega))} + \sup\limits_{0\leq s<t\leq T} \frac{\|f(t)-f(s)\|_{L^2(\Omega)}}{\|t-s\|^\beta}$
$C^1([0,T];L^2(\Omega))$	The standard Bochner space supplied with the norm $\|f\|_{C^1([0,T];L^2(\Omega))} = \max\limits_{t\in[0,T]} \|f(t)\|_{L^2(\Omega)} + \max\limits_{t\in[0,T]} \|\frac{\partial f(t)}{\partial t}\|_{L^2(\Omega)}$
C	Positive generic constant (with or without subscripts)

List of Figures

List of Tables

1

Introduction

In this chapter, we introduce fractional diffusion with memory and state some existing well-posedness results and numerical methods available for such class of problems. For future purposes to this thesis, we introduce some notations and discuss some preliminary results. To start with, we first introduce standard diffusion, standard diffusion with memory, and then fractional diffusion. In the end, we present the summary of the thesis with our contributions to the literature.

Standard diffusion. It is one of the most important equations in partial differential equations and mathematical physics. It describes diffusion phenomena in nature such as heat conduction, chemical concentration etc.. Mathematically, the initial-boundary value problem (IBVP) of the standard diffusion reads

$$\begin{aligned} \partial_t u(x,t) + Au(x,t) &= f(x,t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^d (d \ge 1)$ with boundary $\partial\Omega$, T is a positive fixed value, and $u = u(x,t)$ is the unknown which describes the temperature or the concentration at the point x at time t. Further, $\partial_t = \frac{\partial}{\partial t}$ denotes partial derivative with respect to time, $-A = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian, and the source $f = f(x,t)$ and initial data $u_0(x)$ are given functions. It is well-known that there is a close relationship between the diffusion model (1.1) and the Brownian motion. In model (1.1), the displacement of the particles follows linear power law in time, i.e.,

$$\langle (\Delta X)^2 \rangle \sim t,$$

where $\langle (\Delta X)^2 \rangle$ stands for the mean-squared displacement of the particle. This process is known as Gaussian process.

Standard diffusion with memory. Although the model (1.1) describes successfully the evolution process of the density of some physical quantities in homogeneous and isotropic media, there are some physical phenomena in nature where it is not. For example, while describing the situations like heat conduction in materials with memory, the theory of nuclear reactors, the theory of viscoelasticity, and the compression in porous media at a given time, it does not include the effect of the previous history. This fact invites an integral term (the memory term) in the system and we obtain an integro-differential equation. A particular model for the standard diffusion with memory, also known as the parabolic integro-differential equation (PIDE) is of the form

$$\begin{aligned} &\partial_t u(x,t) + Au(x,t) = f(x,t) + \int_0^t B(t,s)u(x,s)\,ds \quad \text{in} \quad \Omega, \quad 0 < t \le T, \\ &u(x,t) = 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ &u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{1.2}$$

where the memory kernel $B(t,s)$ is a second-order partial differential operator of the form

$$B(t,s) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(b_{ij}(x;t,s)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x;t,s)\frac{\partial}{\partial x_i} + b_0(x;t,s), \tag{1.3}$$

with smooth coefficients $b_{ij}(x;t,s)$, $b_i(x;t,s)$, and $b_0(x;t,s)$ for our purposes. We refer the interested readers to Yanik and Fairweather [73], Chen and Shih [8], and the references listed therein for further details.

Fractional diffusion. We now introduce the fractional diffusion. In the last two decades, several experimental results demonstrate that there are various interesting situations where the mean-squared displacement of the diffusing particles follows a fractional order power law in time rather than the Gaussian process. Mathematically, we write

$$\langle (\Delta X)^2 \rangle \sim t^\alpha, \quad \alpha > 0.$$

Such diffusion processes are known as anomalous diffusion. When α lies in $(0,1)$, the diffusion processes are called subdiffusion and for $1 < \alpha < 2$, they are called superdiffusion. The prototype of the IBVP for the subdiffusion equations (SE) is of the form

$$\begin{aligned} &\partial_t^\alpha u(x,t) + Au(x,t) = f(x,t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad 0 < \alpha < 1, \\ &u(x,t) = 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ &u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{1.4}$$

where $\partial_t^\alpha u(x,t)$ denotes the Caputo time fractional derivative [55, Chapter 2] of order α defined by

$$\partial_t^\alpha u(x,t) = \int_0^t \kappa_{1-\alpha}(t-s)u'(x,s)\,ds, \quad \kappa_\alpha(t) := t^{(\alpha-1)}/\Gamma(\alpha),\ u'(x,s) = \partial_s u(x,s).$$

In the limiting case, i.e., when $\alpha \to 1^-$, the term $\partial_t^\alpha u(x,t)$ in (1.4) becomes the usual integer order partial derivative $\partial_t u(x,t)$ and we have standard diffusion model (1.1). Over the last two decades subdiffusion describes with huge success in many interesting phenomena of various fields of science and engineering. These applications include, for example, diffusion phenomenon in medium with fractal geometry [53], underground environmental problem [19], highly heterogeneous aquifer [2], for a detailed list, see Metzler et al. [49]. We refer our readers to Henry et al. [21] for an introduction to fractional diffusion and [77] for the numerical methods for fractional diffusion problems.

Fractional diffusion with memory. In this part, we introduce the subdiffusion equations with memory (SEM). The kernel function considered here in the memory term will be of the form (1.3) or of the form

$$B(t,s) = -\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}A =: -\kappa_\beta(t-s)A, \ \ s<t,\ 0<\beta<1,$$

i.e., a weakly singular kernel. In the former case, the IBVP for SEM reads

$$\begin{aligned} \partial_t^\alpha u(x,t) + Au(x,t) &= f(x,t) + \int_0^t B(t,s)u(x,s)\,ds \quad \text{in} \quad \Omega,\ \ 0<t\le T,\\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega, \quad 0<t\le T,\\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{1.5}$$

and in the latter case

$$\begin{aligned} \partial_t^\alpha u(x,t) + Au(x,t) + \int_0^t \kappa_\beta(t-s)Au(x,s)\,ds &= f(x,t) \quad \text{in} \quad \Omega,\ \ 0<t\le T,\\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega, \quad 0<t\le T,\\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{1.6}$$

where $\alpha \in (0,1)$. The motivations in the study of SEM problems (1.5)-(1.6) and nonlinear variants of them are concerned with a wide range of applications in nature. For example, heat conduction in materials with memory, in the theory of viscoelasticity, the theory of nuclear reactors, and compression in porous media, see [4, 5, 7, 8, 12, 22, 31, 32, 33, 56, 58, 62] for further details.

1.1 Existing Well-posedness Results and Numerical Schemes for SEM

The study of both theory and numerics of SEM has become of paramount importance due to their applications in various fields of science and engineering. In the theory direction, Krasnoschok et al. [32] have investigated the solvability of the following linear SEM

$$\partial_t^\alpha u(x,t) - L_1 u(x,t) - \int_0^t K_1(t-s) L_2 u(x,s)\, ds = f(x,t) \quad \text{in} \quad \Omega, \quad 0 < t \leq T,$$

with the initial condition $u(x,0) = u_0(x)$ and the boundary condition of the following type:

$$u(x,t) = v(x,t) \quad \text{on } \partial\Omega \times [0,T],$$

or

$$M_1 u(x,t) + \int_0^t K_2(t-s) M_2 u(x,s)\, ds = v_1(x,t) \quad \text{on } \partial\Omega \times [0,T].$$

In the above, for $i = 1, 2$, L_i are second-order linear elliptic operators with time-dependent coefficients, M_i are first-order operators, the memory kernels K_i are summable on $(0,T)$, and f, u_0, v, v_1 are given functions. They have studied the well-posedness and regularity of the solution in the class of smooth functions. The same authors have proved the well-posedness of the semilinear SEM problems in one and multi-dimensional spatial domain Ω, respectively, in [31] and [33]. Very recently, Abadias and Alvarez [1] have studied the Cauchy problem for semilinear SEM in a Banach space setting and investigated the existence and uniqueness of local and global mild solutions and blow-up of the mild solutions. In addition, they have also proved the existence and uniqueness of strong solutions for the linear problem. In [56], Ponce has established a subordination principle for SEM in a Banach space. Here, the author has first derived a subordination formula for the solution and then studied the existence of mild solution. Ponce also studied in [57], a discrete SEM. Based on the backward operator and the fractional resolvent families, he has expressed the solution to the discrete problem as a variation of parameter formula.

On the other hand, in the numerics direction, many authors have developed several methods for numerical solutions of the following SEM and their nonlinear variants

$$D_t^{\alpha}u(x,t)+\mu Au(x,t)+\int_0^t \kappa_\beta(t-s)Au(x,s)\,ds=f(x,t),\;\; 0<\alpha,\beta<1,\; t,\mu>0, \tag{1.7}$$

where the time fractional differential operator D_t^{α} is defined either in the Riemann-Liouville sense or Caputo sense. For finite difference schemes, including alternating direction implicit (ADI) schemes, we refer to [50, 59, 60, 74, 76] and the references quoted therein. In [50], the author has considered the problem (1.7) with a particular $\beta(=1/2)$ and has developed a high-order unconditionally stable and convergent finite difference scheme. In [59], Qiao et al. have discussed a new numerical approximation to the problem (1.7) with multi-term time-fractional derivative. The authors have used high-order orthogonal spline collocation method for the spatial discretization and the L1 approximation along with ADI method in time to establish an optimal order error estimates. The same authors have provided a fast and efficient numerical scheme by employing finite difference approach for spatial discretization and an ADI method combined with convolution quadrature and the L1 scheme to derive the error estimates. In [74], Zaeri et al. have proposed an approximate solution by applying a second-order time difference approximation followed by fractional integral operator to the homogeneous problem ($f=0$ in (1.7)) with a specific choice of $\beta=1/2$. They have considered the stability of the method in the sense of von Neumann analysis. In [76], Zhou and Xu have obtained an optimal order error estimate for an ADI difference scheme, where they have used the L1 scheme and the convolution quadrature to discretize the Caputo derivative and the integral term, respectively. More recently, Wang et al. [70] have established a fully discrete scheme for a nonlinear variant of (1.7) based on the weak Galerkin finite element method in the spatial direction and the L1 scheme in the temporal direction. The authors have investigated the stability and the convergence of the scheme.

At all of the above numerical studies, the stability estimates of the solution are not discussed. In addition, the convergence analysis is carried out under the assumption that the continuous solution u is sufficiently smooth with respect to both space and time variables, including the time at $t=0$, which is not realistic even for smooth initial data, see Theorem 6.2.2. Further, the well-posedness results of SEM (1.5) with respect to different Sobolev regularity assumptions on the problem data, i.e., the initial data and the source function are yet to be proved and the numerical analysis of such problems by means of finite element method is to be explored. This thesis attempts to provide answers to some of the above issues.

The model problems (1.5)-(1.6) at one hand can be interpreted as a perturbation of SE perturbed by the Volterra integral term, and on the other hand it can be viewed as a more general PIDE involving the Caputo fractional derivative in time. It is, therefore, natural to see how the well-posedness results of [63] and [75] and the error analysis [77] and [75] of the SE and PIDE can be applied to SEM. We wish to emphasize that due to the presence of the Volterra integral term and the Caputo fractional derivative, such extensions are not straightforward. Although, literature for SE and PIDE is very vast, the above mentioned issues are not addressed for SEM and this thesis fills up some of the gaps. Before discussing our contribution to the literature and organization of the thesis, we now introduce some notations and recall some preliminaries which will be used throughout the thesis.

1.2 Notations and Preliminaries

This section introduces some notations and preliminary results which will be useful in this thesis. For a non-negative real number r, introduce the Hilbert space

$$\dot{H}^r(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^r (v, \phi_j)^2 < \infty \right\}$$

supplied with the norm

$$\|v\|_r = \left(\sum_{j=1}^{\infty} \lambda_j^r (v, \phi_j)^2 \right)^{1/2},$$

where $(\cdot,\cdot)$ represents the usual $L^2(\Omega)$-inner product, $\{\lambda_j\}_{j=1}^{\infty}$ is an increasing sequence of positive eigenvalues of A with $\lambda_j \to \infty$ when $j \to \infty$, and $\{\phi_j\}_{j=1}^{\infty}$ is the corresponding eigenfunctions which form an orthonormal basis of $L^2(\Omega)$. Thus, for example, the space $\dot{H}^0(\Omega)$ $(= L^2(\Omega))$ is equipped with the norm $\|v\|_0 = \|v\| = (v,v)^{\frac{1}{2}}$, $\dot{H}^1(\Omega)$ is equipped with the norm $\|v\|_1$, and $\dot{H}^2(\Omega)$ has the norm $\|v\|_2$. The norms $\|v\|_1$ and $\|v\|_2$, respectively, are equivalent to the norms on the standard Sobolev spaces $H^1(\Omega)$ and $H^2(\Omega)$ when $v = 0$ on $\partial\Omega$ [68, Lemma 3.1]. Throughout the thesis C, with or without subscript, denotes a positive generic constant (gets some constant value after some finite number of steps) and it is free from the initial data u_0, the mesh size h, the time step size k, and the solution u, but may depend on α, β, and the time T.

Next, we recall the definition and some properties of the Mittag-Leffler function. The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(\cdot)$, where $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $\beta \in \mathbb{C}$, is defined by (cf. [30, p. 42])

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha + \beta)}, \quad z \in \mathbb{C}, \tag{1.8}$$

plays a vital role in studying fractional differential equations. Note that, with $\alpha = \beta = 1$, $E_{1,1}(z) = e^z$. Therefore, $E_{\alpha,\beta}(z)$ is a generalization of the exponential function e^z. Further, the family of functions $E_{\alpha,\beta}(z)$ are entire functions in z of order $[\mathrm{Re}(\alpha)]^{-1}$. In particular, the functions $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha}(z)$ are proved to be convenient for our analysis.

The following lemma provides a bound for $E_{\alpha,\beta}(z)$. For a proof, we refer to [30, 55].

Lemma 1.2.1. *Let $\alpha \in (0,2)$ and β be an arbitrary real number. Assume that r_1 be a real number with $\frac{\alpha\pi}{2} < r_1 < \min\{\pi, \alpha\pi\}$. Then there exists a constant $C = C(\alpha,\beta,r_1) > 0$ such that, for $z \in \mathbb{C}$ with $r_1 \leq |\arg z| \leq \pi$,*

$$|E_{\alpha,\beta}(z)| \leq \begin{cases} \dfrac{C}{1+|z|^2}, & (\beta-\alpha) \in \mathbb{Z}^- \cup \{0\}, \\ \dfrac{C}{1+|z|}, & \textit{otherwise} . \end{cases}$$

We also need the following space-time valued function spaces:

$$C(L^2(\Omega)) = C([0,T];L^2(\Omega)) = \Big\{ f : [0,T] \to L^2(\Omega) \mid f \text{ is continuous} \Big\}$$

supplied with the norm

$$\|f\|_{C(L^2(\Omega))} = \sup_{0\leq t\leq T} \|f(t)\| .$$

Similarly, we define

$$C^1(L^2(\Omega)) = \Big\{ f : [0,T] \to L^2(\Omega) \mid f' \text{ is continuous} \Big\}$$

equipped with the norm

$$\|f\|_{C^1(L^2(\Omega))} = \|f\|_{C(L^2(\Omega))} + \|f'\|_{C(L^2(\Omega))} ,$$

where $f' = \frac{\partial f}{\partial t}$. For $\beta \in (0,1)$, we now introduce the space

$$C^\beta(L^2(\Omega)) = \Big\{ f : [0,T] \to L^2(\Omega) \text{ is continuous such that } \sup_{0\leq s<t\leq T} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} < \infty \Big\}$$

supplied with the norm

$$\|f\|_{C^\beta(L^2(\Omega))} = \|f\|_{C(L^2(\Omega))} + \sup_{0\leq s<t\leq T} \frac{\|f(t)-f(s)\|}{|t-s|^\beta}.$$

In an analogous way, we can define the functions spaces $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T; H_0^1(\Omega))$, where the space $L^\infty(0,T)$ is defined by

$$L^\infty(0,T) = \{ f : (0,T) \to \mathbb{R} \text{ such that } |f(t)| \text{ is bounded almost everywhere on } (0,T) \}$$

supplied with the norm

$$\|f\|_{L^\infty(0,T)} = \inf\{C : |f(t)| \leq C \text{ almost everywhere on } (0,T)\}.$$

In the stability and regularity estimates of SEM (1.5), we shall make use of the following important inequality. For a proof, see [8, p.23].

Lemma 1.2.2 (Gronwall's lemma). *Let $G_0(t)$ and $G(t)$ be two non-negative integrable functions over $[0,t]$, where $t \in (0,T]$. Further, let a non-negative integrable function $V(t)$ be related by*

$$V(t) \leq G_0(t) + \int_0^t G(s)V(s)\,ds.$$

Then

$$V(t) \leq G_0(t) + \int_0^t G_0(s)G(s)\exp\Big(\int_s^t G(\tau)\,d\tau\Big)\,ds.$$

Throughout our analysis, we shall make the following assumption:

($\mathcal{A}$) The operator $B(t,s)$ is dominated by operator A, i.e., for $\mathcal{D}(A)$ and $\mathcal{D}(B(t,s))$, the dense domains of A and $B(t,s)$, respectively, the following two conditions are satisfied:

$$\mathcal{D}(A) \subset \mathcal{D}(B(t,s)) = \mathcal{D}(B^*(t,s)) \subset L^2(\Omega), \quad 0 \leq s \leq t \leq T,$$

and there is a positive constant C such that for all $\psi \in \mathcal{D}(A)$,

$$\|B(t,s)\psi\| + \|B^*(t,s)\psi\| \leq C\,\|A\psi\|\,, \tag{1.9}$$

where $B^*(t,s)$ is the adjoint operator of $B(t,s)$. In addition to $B(t,s)$, we assume the time derivative $B_t(t,s)$ of $B(t,s)$ is also dominated by A.

As an example, we have for the operator $A = -\Delta$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\mathcal{D}(B(t,s)) = H^2(\Omega)$ for the general operator $B(t,s)$ as in (1.3). For simplicity, in the rest part of the thesis we will write $u(x,t)$ as $u(t)$ and there will not be any confusion.

1.3 Contributions and Outline of the Thesis

The present dissertation first investigates the well-posedness of the SEM for different Sobolev regularity assumptions on the problem data, i.e., the initial function u_0 and the source function f. Then, we derive error estimates for both semidiscrete as well as fully discrete schemes which cover both smooth and nonsmooth initial data, including $u_0 \in L^2(\Omega)$. The semidiscrete error analysis includes Galerkin finite element method (GFEM) and computes the errors in both $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$ norms. The fully discrete error analysis is based on the convolution quadrature with the generating function given by the backward Euler (BE), the second-order backward difference (SBD) schemes and the L1 scheme, and calculates the error in the $L^\infty(0,T;L^2(\Omega))$ norm. We validate our theoretical results in each case by extensive numerical illustrations. The summary of this thesis is described below.

In Chapter 2, we prove the existence and uniqueness of the solution to problem (1.5) and establish a priori bounds for the solution under various regularity assumptions on the initial data and the source function. The main tools used for the well-posedness results are the eigenfunction expansion, theory of Volterra integral equation, and Banach fixed point theorem. Our study includes the initial data in the spaces $\dot{H}^2(\Omega)$, $H_0^1(\Omega)$, and $L^2(\Omega)$ while the source function belongs to the class of Hölder continuous and bounded functions. It is shown that the solution of the corresponding homogeneous problem ($f = 0$ in (1.5)) is infinitely differentiable with respect to time t when the initial function is an element of $L^2(\Omega)$ (see Theorem 2.3.1). That is, it is shown that for $u_0 \in L^2(\Omega)$ and $i = 0, 1$, we have

$$\|A^i D_t^n u(t)\| \le C t^{-(i\alpha+n)} \|u_0\|, \quad t > 0, \quad n \in \mathbb{N} \cup \{0\}.$$

Further, for the analysis of numerical schemes it is also crucial to establish stability estimates for the Caputo fractional derivative $\partial_t^\alpha u(t)$ and the usual partial derivative $u'(t) = \frac{\partial u(t)}{\partial t}$ of the continuous solution $u(t)$. In this regard, we derive some general stability results for the solution of the homogeneous and nonhomogeneous problems, see respectively, Theorems 2.4.1, 2.4.2, and 2.4.3: For $t > 0$,

$$\|\partial_t^\alpha u(t)\|_p \le C t^{-\alpha(1+\frac{p-q}{2})} \|u_0\|_q, \quad 0 \le p \le q \le 2,$$

$$\|\partial_t^\alpha u(t)\|_p \le C\big(t^{-\alpha p/2}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big), \quad p = 1, 2,$$

and for $p = 1, 2$,

$$\|u'(t)\|_p \le C\big(t^{-1+(2-p)\alpha/2}[\|u_0\|_2 + t^{(p-1)\alpha/2}\|f(0)\|_{(p-1)}] + t^{\alpha/2}\|f\|_{C^1(\dot{H}^{(p-1)}(\Omega))}\big).$$

Chapter 3 discusses the Galerkin finite element approximation for problem (1.5) and derive error estimates. Both smooth and nonsmooth initial data cases are considered and analyzed. In the error analysis, with $\tilde{u}$ as a suitable projection of u, we split the error $u - u_h$ (u being the exact solution and u_h be its semidiscrete approximation) in a canonical way as for parabolic problems

$$u(t) - u_h(t) = (u(t) - \tilde{u}(t)) + (\tilde{u}(t) - u_h(t)) =: \rho(t) + \theta(t)$$

and then proceed to estimate ρ and θ in a suitable Sobolev norm. The estimation of θ relies on the bound of $\partial_t^\alpha \rho(t)$, the Caputo fractional time derivative of ρ, which is crucial to our error analysis. With the choice of $\tilde{u} = R_h u$, the Ritz projector of u (cf. (3.11)), and observing the fact that the operators ∂_t^α and R_h commute, i.e., $\partial_t^\alpha R_h = R_h \partial_t^\alpha$, we have the following optimal order error estimates (see Theorem 3.2.1):

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \leq Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big), \;\; 0 \leq t \leq T,$$

where h is the mesh size. We would like to emphasize the fact that the finite element error analysis for the standard PIDE involves an elegant use of the Ritz-Volterra projection [6, 36] to simplify the complication arising from the Volterra integral term. An attempt has been made in this chapter to investigate the error analysis with $\tilde{u}$ as $V_h u$, the Ritz-Volterra projector of u (cf. (3.14)). Since the Caputo fractional derivative operator ∂_t^α and the operator V_h, in general, do not commute, i.e., $\partial_t^\alpha V_h \neq V_h \partial_t^\alpha$, the bounds for $\|\theta\|$ and $\|\nabla\theta\|$ are not straightforward from the approximation property of V_h. Therefore, we first establish bounds for $\partial_t^\alpha \rho(t)$ (see Lemma 3.2.2) which lead to the following error estimates (cf. Theorem 3.2.2):

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \leq Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big), \;\; 0 \leq t \leq T.$$

In the case of nonsmooth initial data (i.e., $u_0 \in L^2(\Omega)$), we use the L^2-projection to derive almost optimal order error estimates for the semidiscrete scheme (cf. Theorem 3.2.3):

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \leq Ch^2 |\ln h| t^{-\alpha}\,\|u_0\|\,, \;\; t > 0. \tag{1.10}$$

In the last section of this chapter, we remove the log-factor $|\ln h|$ in the estimate (1.10) and obtain optimal order error bounds with nonsmooth initial data for a particular choice of the kernel operator ,i.e., $B(t,s) = -A$. The analysis is based on an operator trick introduced by Fujita and Suzuki [15]. The nonsmooth data error estimates read (cf. Theorem 3.3.1):

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \leq Ch^2 t^{-\alpha}\,\|u_0\|\,, \;\; t > 0.$$

In the end of this chapter, we perform extensive numerical tests to validate the theoretical results.

In Chapter 4, we investigate the a priori error analysis of a semidiscrete GFEM for problem (1.5) by energy arguments. Since the energy method is the most elementary technique in the a priori analysis, therefore, it is natural to study the a priori error analysis for SEM (1.5) by energy arguments. The convergence analysis covers both smooth and nonsmooth initial data. Since the continuous solution u of (1.5) has singularity at $t = 0$ even for smooth initial data, we use t^j, $j = 1, 2$, type of weights along with time integral operator in the error analysis. The use of energy arguments enable us to prove optimal order error estimates in both $L^2(\Omega)$- and $H^1(\Omega)$-norms with respect to both convergence order of the approximate solution and regularity of the initial data. The present approach gets rid of the log-factor $|\ln h|$ from the nonsmooth data error estimate (cf. (1.10)) even for general second-order operator $B(t, s)$ as in (1.3). In addition, we also establish a quasi-optimal pointwise in time error bound in the maximum norm for smooth initial data. We provide ample numerical experiments to confirm our theoretical findings.

The fifth chapter of this thesis concerns the error analysis for the fully discrete finite element approximation of problem (1.5) with $B(t, s) = -A$. The convergence analysis is based on the convolution quadrature generated by the BE and the SBD schemes. The main advantage of using the convolution quadrature is that it allows the discretization of the time derivative and the integral term in (1.5) simultaneously. We derive error estimates of first and second order accuracy in time for both BE and SBD schemes, respectively. For the nonhomogeneous problem with smooth initial data u_0, the BE error estimate reads (see (5.10) of Theorem 5.2.1): For t_n positive,

$$\|u(t_n) - U_h^n\| \leq Ch^2\big(\|u_0\|_2 + t_n^{-2}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big) + Ckt_n^{\alpha-1}\big(\|u_0\|_2 + \|f(0)\| + t_n\,\|f'\|_{C(L^2(\Omega))}\big),$$

where $\widehat{f}$ is the Laplace transform of f and the number $\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}$ is given by (3.64). In the case of homogeneous problem with nonsmooth initial data, we prove (see (5.11) of Theorem 5.2.1):

$$\|u(t_n) - U_h^n\| \leq C(h^2 t_n^{-\alpha} + kt_n^{-1})\,\|u_0\|\,, \quad t_n > 0. \tag{1.11}$$

The estimate (1.11) has the following features similar to the case of SE [27]:

- Robust with respect to data regularity, i.e., for $u_0 \in L^2(\Omega)$ and t_n fixed, the scheme is second-order accurate in space and first-order in time.
- When t_n is too close to zero, the estimate deteriorates. The term t_n^{-1} in this case reveals the singularity behavior.

Further, we demonstrate an improved second-order accurate in time for the SBD scheme. Numerical illustrations are presented to support our theoretical results.

Chapter 6 is devoted to the systematic study to the weakly singular case where the memory kernel is given by $B(t, s) = -(t - s)^{\beta-1}/\Gamma(\beta)A =: -\kappa_\beta(t - s)A, \quad 0 < \beta <$

1. We first establish the Sobolev regularity of the solution to (1.6) and then develop the semisdiscrete GFEM with piecewise linear functions and fully discrete L1 scheme. Our analysis uses the Laplace transform to represent the solution as a contour integral along a suitably chosen contour in the complex plane. Then, we establish the following smoothing properties of the solution to problem (1.6) with respect to both nonsmooth and smooth initial data (cf. Theorem 6.2.2): For $t > 0$, $j \in \{0, 1\}$, and $m \in \mathbb{N}$, we have

   (a) for $u_0 \in L^2(\Omega)$,

$$\|A^j u^{(m)}(t)\| \le C\big(t^{-(m+j\alpha)}\,\|u_0\| + t^{-(m+1)-(j-1)\alpha}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big),$$

   (b) for $u_0 \in \dot{H}^2(\Omega)$,

$$\|A^j u^{(m)}(t)\| \le C\big(t^{-m+(1-j)\alpha}\,\|u_0\|_2 + t^{-(m+1)-(j-1)\alpha}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big).$$

Next, we propose the spatially discrete GFEM for problem (1.6) and derive optimal order error bounds for both homogeneous and nonhomogeneous problems in $L^2(\Omega)$- and $H^1(\Omega)$-norms. For the homogeneous problem, the semidiscrete error estimates (cf. Theorem 6.3.1) read: For $t > 0$,

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \le Ch^2 t^{-\alpha(2-q)/2}\,\|u_0\|_q\,, \quad q = 0, 2. \tag{1.12}$$

In addition, the following error estimates hold for the nonhomogeneous problem with zero initial data (cf. Theorem 6.3.2): For $t > 0$,

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \le Ch^2 t^{-1}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\,.$$

Next, we concentrate on the time discretization by the L1 scheme to the problem (1.6). The L1 scheme is the simplest and widely used numerical method for discretizing the Caputo fractional derivative in time. For the homogeneous problem, we obtain the following error bounds (see Theorem 6.4.1): For $n \ge 1$,

$$\|u_h(t_n) - U_h^n\| \le Ckt_n^{-1+q\alpha/2}\,\|u_0\|_q\,, \quad q = 0, 2,$$

which combine with estimate (1.12) produces the fully discrete error estimate: For $n \ge 1$,

$$\|u(t_n) - U_h^n\| \le C\big(h^2 t_n^{-\alpha(2-q)/2} + kt_n^{-1+q\alpha/2}\big)\,\|u_0\|_q\,, \quad q = 0, 2. \tag{1.13}$$

Further, we also derive the fully discrete error estimate for the nonhomogeneous problem with zero initial data (see Theorem 6.4.4) which is stated as: For $n \ge 1$,

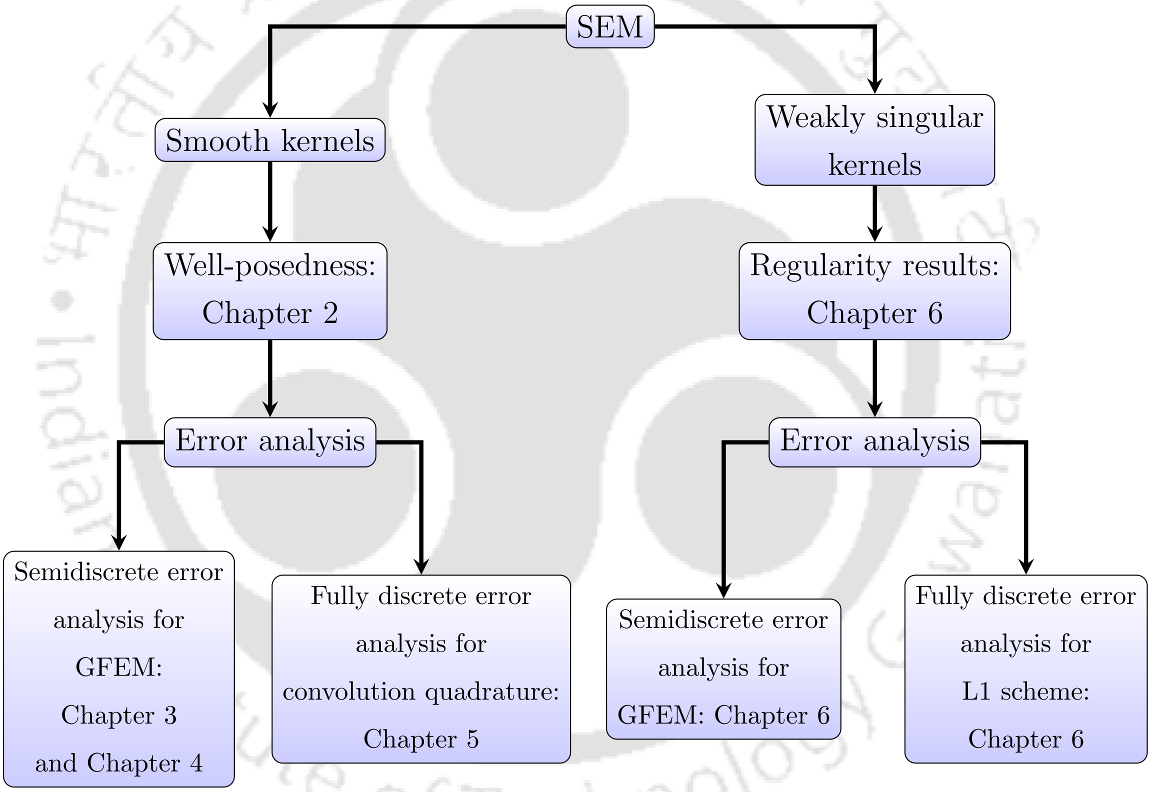
$$\|u(t_n) - U_h^n\| \le C\Big(h^2 t_n^{-1}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} + k\big(t_n^{\alpha-1}\,\|f(0)\| + \int_0^{t_n} (t_n - s)^{\alpha-1}\,\|f'(s)\|\, ds\big)\Big).$$

We perform extensive numerical tests to support theoretical analysis.

Finally, Chapter 7 presents the summary of the results highlighting the contributions of the thesis to the literature and discusses possible extensions for future work.

All the numerical computations in this thesis are carried out by using the software FreeFem++ [20].

The pictorial representation of the our study to SEM problems (1.5)-(1.6) is given below.



2

Well-posedness and Regularity Results

In this chapter we prove the existence and uniqueness of the solution to problem (1.5) and establish a priori bounds for the solution under various regularity assumptions on the initial data and the source function. The main tools used for the well-posedness results are the eigenfunction expansion, theory of Volterra integral equation, and Banach fixed point theorem. Our study includes the initial data in the spaces $\dot{H}^2(\Omega)$, $H_0^1(\Omega)$, and $L^2(\Omega)$ while the source function belongs to the class of Hölder continuous and bounded functions. It is shown that the solution of the corresponding homogeneous problem ($f = 0$ in (1.5)) is infinitely differentiable with respect to time t when the initial function is an element of $L^2(\Omega)$. For the analysis of numerical schemes it is also crucial to have stability estimates for $\partial_t^\alpha u(t)$ and $\partial_t u(t)$ of the continuous solution $u(t)$. In this regard, we prove some more general stability results for the solution of the homogeneous and nonhomogeneous problems. In the end, we present some examples to illustrate our theory.

2.1 Introduction

We now recall the problem (1.5), i.e.,

$$\begin{aligned} \partial_t^\alpha u(t) + Au(t) &= f(t) + \tilde{B}u(t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad 0 < \alpha < 1, \\ u(t) &= 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ u(0) &= u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{2.1}$$

where $\tilde{B}u(t) = \int_0^t B(t,s)u(s)\,ds$, Ω is a bounded convex polygonal domain in $\mathbb{R}^d$, $u(t)$ is the unknown function of x and t, $\partial_t^\alpha u(t)$ is the α-th order Caputo fractional derivative of $u(t)$ with respect to t and it is defined by

$$\partial_t^\alpha u(t) = \int_0^t \kappa_{1-\alpha}(t-s)u'(s)\,ds, \quad \kappa_\alpha(t) := t^{(\alpha-1)}/\Gamma(\alpha),\ u'(s) = \partial_s u(s),$$

where $\Gamma(\cdot)$ denotes the standard Gamma function. Here,

$$-A = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

is the Laplacian, $f\colon \Omega \times (0,T] \to \mathbb{R}$ and $u_0\colon \Omega \to \mathbb{R}$ are given functions, and $T > 0$ is a constant. Further, the operator $B(t,s)$ under the integral sign is a second-order partial differential operator of the form

$$B(t,s) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(b_{ij}(x;t,s)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(x;t,s)\frac{\partial}{\partial x_i} + b_0(x;t,s)I,$$

where the coefficients $b_{ij}(x;t,s)$, $b_i(x;t,s)$, and $b_0(x;t,s)$ are assumed to be sufficiently smooth. Note that when $\alpha \to 1^-$, the operator ∂_t^α in (2.1) becomes the usual integer order partial derivative ∂_t and we have the standard PIDE.

The present study is motivated by the work of Sakamoto and Yamamoto [63]. In [63], the authors have established the well-posedness results to SE (i.e., $B(t,s) = 0$ in (2.1)) with respect to various Sobolev regularity assumptions on the initial data and the source function. Thus, it is natural to see how the well-posedness results of [63] of the SE can be generalized to SEM. We wish to emphasize that due to the presence of the Volterra integral term, such an extension is not straightforward.

The outline of this chapter is as follows. Section 2.2 is devoted to the study of existence and uniqueness of the solution and proves a priori bounds for the solution under various regularity assumptions on the problem data. Section 2.3 discusses the infinite differentiability of the solution with respect to time to the corresponding homogeneous problem. In Section 2.4, we establish some more general stability results for the solutions of the homogeneous and nonhomogeneous problems which are crucial for the development of the numerical scheme and provide some examples to illustrate Theorems 2.2.1 and 2.2.2.

2.2 Existence, Uniqueness and Stability Results

This section concerns existence, uniqueness and stability of the solution of problem (2.1). The stability estimates are proved under various regularity assumptions on the initial data u_0 and the source function f. First, we prove the existence of solution of (2.1) with zero initial condition, i.e., $u_0 = 0$. Subsequently, this result will be used to establish the uniqueness and the stability results.

Lemma 2.2.1. *There exists a solution to the problem*

$$
\begin{aligned}
&\partial_t^\alpha u(t) + Au(t) = f(t) + \tilde{B}u(t) \quad in \quad \Omega, \quad 0 < t \leq T, \quad 0 < \alpha < 1, \\
&u(t) = 0 \quad on \quad \partial\Omega, \quad 0 < t \leq T, \\
&u(0) = 0 \quad in \quad \Omega,
\end{aligned} \tag{2.2}
$$

where $\tilde{B}u(t) = \int_0^t B(t,s)u(s)\,ds$.

Proof. Let $\phi_j(x)$ be the eigenfunction of A corresponding to the eigenvalue λ_j on Ω. Multiply Eq. (2.2) by $\phi_j(x)$ and integrate with respect to x. Then use Green's formula with $\phi_j = 0$ on $\partial\Omega$ to have

$$
\begin{aligned}
&\partial_t^\alpha u_j(t) + \lambda_j u_j(t) = f_j(t) + G_j(t), \quad t > 0, \\
&u_j(0) = 0.
\end{aligned} \tag{2.3}
$$

Here $u_j(t) = (u(t), \phi_j(x))$, $f_j(t) = (f(t), \phi_j)$ and $G_j(t) = (\tilde{B}u(t), \phi_j)$. Arguing as in [30, 55, 63], the solution of (2.3) has the representation

$$
u(x,t) = \sum_{j=1}^\infty \left\{ \int_0^t (f_j(\tau) + G_j(\tau))(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha)\,d\tau \right\} \phi_j(x), \tag{2.4}
$$

where $E_{\alpha,\alpha}(\cdot)$ is given by (1.8) and this proves the lemma. □

In the following, we present a proof of an important result which will be useful in the subsequent analysis. To start with, we introduce the operators $E(t)$ and $\widehat{E}(t)$ be defined by

$$
E(t)u_0 = \sum_{j=1}^\infty E_{\alpha,1}(-\lambda_j t^\alpha)(u_0, \phi_j)\phi_j(x), \tag{2.5}
$$

$$
\widehat{E}(t)\psi = \sum_{j=1}^\infty t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha)(\psi, \phi_j)\phi_j(x), \quad t > 0, \ \psi \in L^2(\Omega), \tag{2.6}
$$

where $E_{\alpha,1}(\cdot)$ and $E_{\alpha,\alpha}(\cdot)$ respectively, are one and two parameters Mittag-Leffler functions. It follows from the boundedness of $E_{\alpha,1}(-\lambda_j t^\alpha)$ (cf. Lemma 1.2.1) that

$$
\|E(t)u_0\| \leq C\|u_0\|.
$$

Lemma 2.2.2. *Let the operator* $R\colon C(H^2(\Omega) \cap H_0^1(\Omega)) \to L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ *be defined by*

$$
Rf(t) = \int_0^t \widehat{E}(t-\tau)\tilde{B}f(\tau)\,d\tau,
$$

where $\tilde{B}f(t) = \int_0^t B(t,\tau)f(\tau)\,d\tau$ *and* $\widehat{E}(t)$ *is defined by (2.6). Further, assume that (1.9) holds for the operators* B *and* B_t*. Then there exists a positive constant* C *such that for* $0 \le t \le T$*,*

$$\|ARf(t)\| \le C\int_0^t \|Af(\tau)\|\,d\tau.$$

Proof. Let $g \in C^1(L^2(\Omega))$ with $g(0) = 0$. By the change of order of integration we write

$$\begin{aligned}\tilde{\widehat{E}}(t)g(t) &:= \int_0^t \widehat{E}(t-\tau)g(\tau)\,d\tau \\ &= \int_0^t \widehat{E}(t-\tau)\int_0^\tau g_s(s)\,ds\,d\tau = \int_0^t\int_s^t \widehat{E}(t-\tau)\,d\tau g_s(s)\,ds.\end{aligned} \tag{2.7}$$

An easy calculation shows that $\int_s^t \widehat{E}(t-\tau)\,d\tau = A^{-1}(I - E(t-s))$ (see Eq. (2.21)). Since $E(t)$ is bounded, (2.7) implies

$$\|A\tilde{\widehat{E}}(t)g(t)\| \le C\int_0^t \|g_\tau\|\,d\tau.$$

And hence,

$$\|ARf(t)\| = \|A\tilde{\widehat{E}}(t)\tilde{B}f(t)\| \le C\int_0^t \|Af(\tau)\|\,d\tau,$$

where we have used the fact that the operators B and B_t are dominated by A. ☐

The following theorem presents the existence of unique solution of (2.1) and the stability properties of the solution for smooth initial data.

Theorem 2.2.1. *Let* $u_0 \in \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ *and* $f \in C^\beta(L^2(\Omega))$*,* $\beta \in (0,1)$*. Then problem (2.1) has a unique solution* u *in* $C(\mathcal{D}(A))$ *such that* $\partial_t^\alpha u \in C(L^2(\Omega))$*. Furthermore, there exists a positive constant* C *such that*

$$\|u(t)\|_2 + \|\partial_t^\alpha u(t)\| \le C\big(\|u_0\|_2 + \|f\|_{C^\beta(L^2(\Omega))}\big), \quad 0 \le t \le T.$$

Proof. We first decompose problem (2.1) in two parts: (I) homogeneous SE with nonzero initial data ($f = 0, B(t,s) = 0, u_0 \ne 0$); (II) nonhomogeneous SEM with zero initial data ($f \ne 0, B(t,s) \ne 0, u_0 = 0$). For (I), the solution $u_1(x,t)$ is given by Sakamoto and Yamamoto [63], where

$$u_1(t) = E(t)u_0 = \sum_{j=1}^\infty E_{\alpha,1}(-\lambda_j t^\alpha)(u_0,\phi_j)\phi_j(x),$$

with

$$\|u_1\|_{C(H^2(\Omega))} + \|\partial_t^\alpha u_1\|_{C(L^2(\Omega))} \le C\|u_0\|_2\,,$$

and the solution of (II) is given by (2.4). Thus, combining the above solutions we have the formal representation of the solution of (2.1) as

$$u(t) = E(t)u_0 + \int_0^t \widehat{E}(t-\tau)\{f(\tau) + \tilde{B}u(\tau)\}\,d\tau. \tag{2.8}$$

Now rewriting the above equation as

$$\begin{aligned} u(t) &= \big(E(t)u_0 + \int_0^t \widehat{E}(t-\tau)f(\tau)\,d\tau\big) + \int_0^t \widehat{E}(t-\tau)\tilde{B}u(\tau)\,d\tau \\ &= Q(x,t) + Ru(t), \end{aligned} \tag{2.9}$$

where $Ru(t) = \int_0^t \widehat{E}(t-\tau)\tilde{B}u(\tau)\,d\tau$ and $Q(x,t)$ is the solution of the problem

$$\begin{aligned} &\partial_t^\alpha Q(t) + AQ(t) = f(t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad 0 < \alpha < 1, \\ &Q(t) = 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ &Q(0) = u_0 \quad \text{in} \quad \Omega. \end{aligned}$$

It now follows from [63, Theorem 2.4] that $Q \in C(\mathcal{D}(A))$ with $\partial_t^\alpha Q \in C(L^2(\Omega))$, and for $0 \le t \le T$,

$$\|AQ(t)\| + \|\partial_t^\alpha Q(t)\| \le C\big(\|u_0\|_2 + \|f\|_{C^\beta(L^2(\Omega))}\big). \tag{2.10}$$

The uniqueness of the solution of (2.1) is a consequence of Lemma 2.2.2 and the standard argument for the existence of unique solution for Volterra integral equation. Using (2.10) and Gronwall's lemma, we obtain

$$\begin{aligned} \|u(t)\|_2 &\le \|ARu(t)\| + \|AQ(t)\| \\ &\le C\int_0^t \|Au(s)\|\,ds + C\big(\|u_0\|_2 + \|f\|_{C^\beta(L^2(\Omega))}\big) \\ &\le C\big(\|u_0\|_2 + \|f\|_{C^\beta(L^2(\Omega))}\big). \end{aligned}$$

Thus, $\tilde{B}u + f$ belongs to $C^\beta(L^2(\Omega))$. Therefore, $u \in C(\mathcal{D}(A))$ with $\partial_t^\alpha u \in C(L^2(\Omega))$ is the unique solution of (2.1), a SE problem with the source $\tilde{B}u + f$. Furthermore, it follows from (2.1) and (1.9) that

$$\|\partial_t^\alpha u(t)\| \le \|Au(t)\| + \|\tilde{B}u(t)\| + \|f(t)\| \le C\big(\|u_0\|_2 + \|f\|_{C^\beta(L^2(\Omega))}\big).$$

□

As an immediate consequence of Theorem 2.2.1, for the homogeneous SEM ($f = 0$) with $u_0 \in \mathcal{D}(A)$, we have

$$\|u(t)\|_2 \le C\,\|u_0\|_2 \quad \text{and} \qquad \|\partial_t^\alpha u(t)\| \le C\,\|u_0\|_2\,, \quad 0 \le t \le T. \tag{2.11}$$

Next, we discuss the existence of a unique solution of (2.1) for comparatively less smoothness assumption on data in comparison to Theorem 2.2.1. In the following theorem, we consider the initial data $u_0 \in H_0^1(\Omega)$ and the source function $f \in L^\infty(0,T;L^2(\Omega))$. In view of [63, Theorems 2.1 and 2.2], the solution Q of the SE (defined as in (2.9)) belongs to $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ such that $\partial_t^\alpha u \in L^2(0,T;L^2(\Omega))$ when $u_0 \in H_0^1(\Omega)$ and $f \in L^\infty(0,T;L^2(\Omega))$. Moreover, the following a priori estimate holds:

$$\|Q\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha Q\|_{L^2(0,T;L^2(\Omega))} \le C\big(\,\|u_0\|_1 + \|f\|_{L^2(0,T;L^2(\Omega))}\big). \tag{2.12}$$

Under the same regularity assumptions on the initial function u_0 and the source term f, we have the following theorem for SEM (2.1).

Theorem 2.2.2. *Assume that $f \in L^\infty(0,T;L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$. Then there is a unique solution $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ to (2.1) with $\partial_t^\alpha u \in L^2(0,T;L^2(\Omega))$. Moreover, for some constant $C>0$, the following estimate holds:*

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))} \le C\big(\,\|u_0\|_1 + \|f\|_{L^2(0,T;L^2(\Omega))}\big). \tag{2.13}$$

Proof. To prove the theorem we first note that the estimate of Lemma 2.2.2 is still valid even if we choose the domain of the operator R as $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $g \in H^1([0,T];L^2(\Omega))$ with $g(0) = 0$. Then the uniqueness of the solution follows from the contraction principle of a suitable map. With the definition of R as in (2.9), we have

$$R^2u(t) = \int_0^t \widehat{E}(t-\tau)\tilde{B}Ru(\tau)\,d\tau,$$

and by Lemma 2.2.2 followed by the change of order of integration

$$\|AR^2u(t)\| \le C\int_0^t \|ARu(\tau)\|\,d\tau \le C\int_0^t (t-s)\,\|Au(s)\|\,ds.$$

Repeated applications of the above lead to

$$\|AR^ku(t)\| \le \frac{C}{\Gamma(k)}\int_0^t (t-s)^{k-1}\,\|Au(s)\|\,ds \le \frac{CT^{k-1}}{\Gamma(k)}\int_0^t \|Au(s)\|\,ds, \tag{2.14}$$

where in the last step we have used the fact $(t-s)\le T$. Now, set $a_k = \frac{CT^{k-1}}{\Gamma(k)}$. Squaring (2.14) and then integrating over $(0,T)$, it follows that

$$\int_0^T \|AR^ku(t)\|^2\,dt \le (a_kT)^2\int_0^T \|Au(s)\|^2\,ds,$$

i.e.,

$$\|AR^k u\|_{L^2(0,T;L^2(\Omega))} \le a_k T \|Au\|_{L^2(0,T;L^2(\Omega))}. \tag{2.15}$$

Since $\Gamma(k)$ goes to ∞ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} T a_k = T \lim_{k\to\infty} \frac{CT^{k-1}}{\Gamma(k)} = 0. \tag{2.16}$$

Therefore, there exists a natural number k_0 such that for all $k \ge k_0$, $|Ta_k| < 1$. Next, set $\hat{R}u = Q + Ru$ and let $v_1, v_2 \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$. Then, an easy computation shows that

$$\hat{R}^k v_1 - \hat{R}^k v_2 = R^k v_1 - R^k v_2 = R^k(v_1 - v_2), \quad k \in \mathbb{N}. \tag{2.17}$$

Then by virtue of (2.15)-(2.17), $\hat{R}^k : L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \to L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ is a contraction map. Consequently, the map $\hat{R}^k$ has a unique fixed point $\hat{u}$ in $L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, i.e., $\hat{R}^k\hat{u} = \hat{u}$. Also, note that $\hat{R}^k(\hat{R}\hat{u}) = \hat{R}\hat{u}$ and therefore $\hat{R}\hat{u} = \hat{u} = Q + R\hat{u}$ by the uniqueness of $\hat{u}$. Hence there exists a unique solution to the representation $u = Q + Ru$, and by (2.12), we obtain

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C\|Q\|_{L^2(0,T;H^2(\Omega))} \le C\big(\|u_0\|_1 + \|f\|_{L^2(0,T;L^2(\Omega))}\big).$$

From (2.1), since $B(t,s)$ is dominated by A, we have

$$\|\partial_t^\alpha u\|_{L^2(0,T;L^2(\Omega))} \le C\big(\|u_0\|_1 + \|f\|_{L^2(0,T;L^2(\Omega))}\big).$$

This proves the theorem. □

Remark 2.2.1. *The idea of the proof of Theorem 2.2.2 is taken from* Gorenflo et al. *[16, Theorem 4.2], where the authors have studied the well-posedness of a more general SE problem with $u_0 = 0$ and $f \in L^2(0,T; L^2(\Omega))$.*

Now we study the stability properties of the homogeneous SEM with nonsmooth initial data.

Theorem 2.2.3. *Assume that $u_0 \in L^2(\Omega)$ and $f = 0$ in (2.1). Then the homogeneous problem has a unique solution u in $C(L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ with $\partial_t^\alpha u \in C((0,T]; L^2(\Omega))$. Furthermore, there exists a positive constant C such that*

$$\|u(t)\| \le C\|u_0\|, \quad 0 \le t \le T,$$

and

$$\|u(t)\|_2 + \|\partial_t^\alpha u(t)\| \le Ct^{-\alpha}\|u_0\|, \quad 0 < t \le T.$$

To prove the above theorem, we need the following two lemmas.

Lemma 2.2.3. *Consider the integral* $\tilde{B}E(t)u_0 = \int_0^t B(t,\tau)E(\tau)u_0\, d\tau$, *where* $E(t)u_0$ *is given by (2.5). Assume that the operator* A *dominates* B. *Then for* $0 \le t \le T$,

$$\|\tilde{B}E(t)u_0\| \le C\|u_0\|\,.$$

Proof. We have

$$E(t)u_0 = \sum_{j=1}^{\infty}(u_0,\phi_j)E_{\alpha,1}(-\lambda_j t^\alpha)\phi_j(x) \;\text{ and }\; AE(t)u_0 = \sum_{j=1}^{\infty}\lambda_j(u_0,\phi_j)E_{\alpha,1}(-\lambda_j t^\alpha)\phi_j(x).$$

Then we have

$$\|AE(t)u_0\| \le Ct^{-\alpha}\|u_0\|\,, \quad t>0. \tag{2.18}$$

Since B is dominated by A, we get

$$\|\tilde{B}E(t)u_0\| \le \int_0^t \|B(t,\tau)E(\tau)u_0\|\, d\tau \le C\int_0^t \|AE(\tau)u_0\|\, d\tau \le C\|u_0\|\,, \quad 0\le t\le T,$$

and this completes the proof. □

Lemma 2.2.4. *With the operator* R *as in (2.9), consider the integral*

$$RE(t)u_0 = \int_0^t \widehat{E}(t-\tau)\int_0^\tau B(\tau,s)E(s)u_0\, ds\, d\tau.$$

Let the assumption (1.9) holds for the operators B *and* B_t. *Then for* $0\le t\le T$, *we have*

$$\|ARE(t)u_0\| \le C\|u_0\|\,.$$

Proof. From (2.6), an easy calculation shows that

$$\|\widehat{E}(t)\chi\| \le Ct^{\alpha-1}\,\|\chi\|\,, \quad t>0,$$

which combine with Lemma 2.2.3 yields

$$\begin{aligned}\|RE(t)u_0\| &\le \int_0^t \|\widehat{E}(t-\tau)\int_0^\tau B(\tau,s)E(s)u_0\, ds\|\, d\tau\\ &\le C\int_0^t (t-\tau)^{\alpha-1}\,\|\int_0^\tau B(\tau,s)E(s)u_0\, ds\|\, d\tau \le C\|u_0\|\,.\end{aligned}$$

Thus, $RE(t)u_0 \in C(L^2(\Omega))$. Next, we need to estimate $\big\|ARE(t)u_0\big\|$. Before that we need a relation between the operators $E(t)$ and $\widehat{E}(t)$. We know that the Mittag-Leffler functions $E_{\alpha,1}(\cdot)$ and $E_{\alpha,\alpha}(\cdot)$ are related by (see Sakamoto and Yamamoto [63, p.432])

$$-\frac{1}{\lambda_j}\frac{d}{dt}E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha) = (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha) \tag{2.19}$$

and

$$\frac{1}{\lambda_j}\frac{d}{d\tau}E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha) = (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha). \tag{2.20}$$

Using the relations (2.19) and (2.20), we get

$$\begin{aligned}
\widehat{E}(t-\tau)\chi &= \sum_{j=1}^{\infty}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha)(\chi,\phi_j)\phi_j(x)\\
&= \sum_{j=1}^{\infty} -\frac{1}{\lambda_j}\frac{d}{dt}E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha)(\chi,\phi_j)\phi_j(x)\\
&= \sum_{j=1}^{\infty} \frac{1}{\lambda_j}\frac{d}{d\tau}E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha)(\chi,\phi_j)\phi_j(x).
\end{aligned}$$

Hence

$$\begin{aligned}
A\widehat{E}(t-\tau)\chi &= \sum_{j=1}^{\infty}\frac{d}{d\tau}E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha)(\chi,\phi_j)\phi_j(x)\\
&= \frac{d}{d\tau}E(t-\tau)\chi. \qquad (2.21)
\end{aligned}$$

We apply integration by parts to obtain

$$\begin{aligned}
ARE(t)u_0 &= \int_0^t A\widehat{E}(t-\tau)\int_0^\tau B(\tau,s)E(s)u_0\,ds\,d\tau\\
&= \int_0^t \frac{d}{d\tau}E(t-\tau)\int_0^\tau B(\tau,s)E(s)u_0\,ds\,d\tau\\
&= \tilde{B}E(t)u_0 - \int_0^t E(t-\tau)\int_0^\tau B_\tau(\tau,s)E(s)u_0\,ds\,d\tau\\
&\quad - \int_0^t E(t-\tau)B(\tau,\tau)E(\tau)u_0\,d\tau.
\end{aligned}$$

Using Lemma 2.2.3, the boundedness of $E(t)$ and the fact B_t, B are dominated by A, we conclude that

$$\|ARE(t)u_0\| \le C\|u_0\|.$$

Thus, $RE(t)u_0 \in C(\mathcal{D}(A))$. This completes the proof. □

Now we are ready to prove the theorem.

Proof of Theorem 2.2.3. Let $v(t) = u(t) - E(t)u_0$. Then $v(0) = 0$, and

$$\partial_t^\alpha v(t) + Av(t) = \tilde{B}v(t) + \tilde{B}E(t)u_0.$$

The equality that gives solutions

$$\begin{aligned} v(t) &= \int_0^t \widehat{E}(t-\tau)(\tilde{B}v(\tau) + \tilde{B}E(\tau)u_0)\, d\tau \\ &= Rv(t) + RE(t)u_0. \end{aligned} \tag{2.22}$$

We now show that (2.22) is a well-posed Volterra type equation in the Banach space $C(\mathcal{D}(A))$. Using Lemmas 2.2.2 and 2.2.4, we conclude that Eq. (2.22) has a unique solution in $C(\mathcal{D}(A))$, and

$$\|Av(t)\| \leq C \int_0^t \|Av(\tau)\|\, d\tau + C\,\|u_0\| \leq C\,\|u_0\|.$$

Also, $\big\|v(t)\big\| \leq C\|u_0\|$, for $0 \leq t \leq T$. Therefore, using (2.18) we find that for $t > 0$,

$$\|Au(t)\| \leq \|Av(t)\| + \|AE(t)u_0\| \leq Ct^{-\alpha}\,\|u_0\|, \tag{2.23}$$

and

$$\|u(t)\| \leq \|v(t)\| + \|E(t)u_0\| \leq C\,\|u_0\|. \tag{2.24}$$

Remark 2.2.2. *It is known that [63, Corollary 2.6], for the homogeneous SE with the initial condition* $u_0 \in L^2(\Omega)$, *the solution* $u \in C([0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H_0^1(\Omega))$ *. However, in the case of SEM the solution* $u \in C(L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$*. Further, the inequality (2.23) reveals the smoothing property of homogeneous SEM with respect to space for positive time when* $u_0 \in L^2(\Omega)$*.*

2.3 Infinite Differentiability of the Solution

It is known that the solution of the homogeneous SE is infinitely differentiable with respect to time t when $u_0 \in L^2(\Omega)$ for $t > 0$ (cf. [63, Corollary. 2.6]). We would like to see whether this property is valid in the case of SEM. In fact, we shall demonstrate that this infinite differentiabilty with respect to time holds true when $u_0 \in L^2(\Omega)$. For this, we now introduce some operators and a sequence of lemmas involving these operators.

Let us define operator $M\colon C(L^2(\Omega)) \to C(L^2(\Omega))$ by

$$\begin{aligned} (Mf)(t) = \int_0^t \widehat{E}(t-\tau)f(\tau)\, d\tau &= \int_0^{t/2} \widehat{E}(t-\tau)f(\tau)\, d\tau + \int_{t/2}^t \widehat{E}(t-\tau)f(\tau)\, d\tau \\ &:= M_1f(t) + M_2f(t), \quad 0 \leq t \leq T, \end{aligned}$$

where $f \in C(L^2(\Omega))$. Also, let $f^{(i)}(t) = D_t^i f(t)$ be the i-th derivative of f with respect to time t.

Lemma 2.3.1. *For $i \in \{0,1\}$ and $n \in \mathbb{N}\cup\{0\}$, we have $A^iD_t^nM_2f \in C((0,T];L^2(\Omega))$ and the following estimates hold true for $0 < t \le T$,*

$$\|A^iD_t^nM_2f(t)\| \le C\sum_{j=0}^{n-1+i} t^{(1-i)\alpha-n+j}\,\|f^{(j)}(t/2)\| + C\int_{t/2}^{t}(t-\tau)^{(\alpha-1)+i(1-\alpha)}\,\|f^{(n+i)}(\tau)\|\,d\tau.$$

Proof. We first prove this assertion for $i = 0$. Note that,

$$\|M_2f(t)\| \le C\int_{t/2}^{t}(t-\tau)^{(\alpha-1)}\,\|f(\tau)\|\,d\tau.$$

Now, since $D_t\widehat{E}(t-\tau)\chi = -D_\tau\widehat{E}(t-\tau)\chi$, integration by parts gives

$$\begin{aligned}
D_tM_2f(t) &= \int_{t/2}^{t} D_t\widehat{E}(t-\tau)f(\tau)\,d\tau - \frac{1}{2}\widehat{E}(t/2)f(t/2)\\
&= -\frac{1}{2}\widehat{E}(t/2)f(t/2) - \int_{t/2}^{t} D_\tau\widehat{E}(t-\tau)f(\tau)\,d\tau\\
&= -\frac{1}{2}\widehat{E}(t/2)f(t/2) + \widehat{E}(t/2)f(t/2) + \int_{t/2}^{t} \widehat{E}(t-\tau)f^{(1)}(\tau)\,d\tau\\
&= \frac{1}{2}\widehat{E}(t/2)f(t/2) + M_2f^{(1)}(t). \qquad (2.25)
\end{aligned}$$

Operating D_t^{n-1} in (2.25), we obtain

$$\begin{aligned}
D_t^nM_2f(t) &= \frac{1}{2}\sum_{m=1}^{n} D_t^{m-1}(\widehat{E}(t/2)f^{n-m}(t/2)) + M_2f^n(t)\\
&= \sum_{m=1}^{n}\left(\frac{1}{2}\right)^m\sum_{j=0}^{m-1}\binom{m-1}{j}(D_t^j\widehat{E})(t/2)f^{(n-j-1)}(t/2) + M_2f^n(t). \qquad (2.26)
\end{aligned}$$

We know that for $t > 0$, $\|\widehat{E}(t)\chi\| \le Ct^{\alpha-1}\|\chi\|$. By Lemma 3.2 of [63]

$$\begin{aligned}
(D_t\widehat{E})(t)\chi &= \sum_{j=1}^{\infty}\frac{d}{dt}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_jt^\alpha))(\chi,\phi_j)\phi_j(x)\\
&= \sum_{j=1}^{\infty} t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_jt^\alpha)(\chi,\phi_j)\phi_j(x). \qquad (2.27)
\end{aligned}$$

Therefore,

$$\|(D_t\widehat{E})(t)\chi\| \le Ct^{\alpha-2}\|\chi\|, \quad t > 0.$$

Similarly, we have for $t > 0$,

$$(D_t^n\widehat{E})(t)\chi = \sum_{j=1}^{\infty} t^{\alpha-(n+1)}E_{\alpha,\alpha-n}(-\lambda_jt^\alpha)(\chi,\phi_j)\phi_j(x) \qquad (2.28)$$

and

$$\left\| (D_t^n \widehat{E})(t)\chi \right\| \le C t^{\alpha-(n+1)} \left\| \chi \right\|. \tag{2.29}$$

With the help of (2.29), (2.26) provides

$$\begin{aligned}
\left\| D_t^n M_2 f(t) \right\| &\le C \sum_{m=1}^{n} \sum_{j=0}^{m-1} t^{\alpha-(j+1)} \left\| f^{(n-1-j)}(t/2) \right\| + C \int_{t/2}^{t} (t-\tau)^{\alpha-1} \left\| f^{(n)}(\tau) \right\| d\tau \\
&\le C \sum_{j=0}^{n-1} t^{\alpha-n+j} \left\| f^{(j)}(t/2) \right\| + C \int_{t/2}^{t} (t-\tau)^{\alpha-1} \left\| f^{(n)}(\tau) \right\| d\tau, \qquad (2.30)
\end{aligned}$$

which proves the lemma for $i = 0$. Next, for $i = 1$, operating A in (2.28), we have for $t > 0$

$$A(D_t^n \widehat{E})(t)\chi = \sum_{j=1}^{\infty} \lambda_j t^{\alpha-(n+1)} E_{\alpha,\alpha-n}(-\lambda_j t^{\alpha})(\chi, \phi_j)\phi_j(x) \tag{2.31}$$

and hence,

$$\left\| A(D_t^n \widehat{E})(t)\chi \right\| \le C t^{-(n+1)} \left\| \chi \right\|. \tag{2.32}$$

Now operating A in (2.26), we get

$$AD_t^n M_2 f(t) = \sum_{m=1}^{n} \left(\frac{1}{2}\right)^m \sum_{j=0}^{m-1} \binom{m-1}{j} A(D_t^j \widehat{E})(t/2) f^{(n-j-1)}(t/2) + AM_2 f^n(t).$$

Then by (2.32),

$$\begin{aligned}
\left\| AD_t^n M_2 f(t) \right\| &\le C \sum_{m=1}^{n} \sum_{j=0}^{m-1} t^{-(j+1)} \left\| f^{(n-1-j)}(t/2) \right\| + \left\| AM_2 f^n(t) \right\| \\
&\le C \sum_{j=0}^{n-1} t^{-n+j} \left\| f^{(j)}(t/2) \right\| + \left\| AM_2 f^n(t) \right\|. \qquad (2.33)
\end{aligned}$$

In order to bound the second term on the right-hand side of (2.33), we use (2.21) and then integrate by parts to have

$$\begin{aligned}
AM_2 f(t) = \int_{t/2}^{t} A\widehat{E}(t-\tau) f(\tau)\, d\tau &= \int_{t/2}^{t} (D_\tau E(t-\tau)) f(\tau)\, d\tau \\
&= f(t) - E(t/2) f(t/2) - \int_{t/2}^{t} E(t-\tau) f'(\tau)\, d\tau \\
&= (I - E(t/2)) f(t/2) - \int_{t/2}^{t} E(t-\tau) f'(\tau)\, d\tau + \int_{t/2}^{t} f'(\tau)\, d\tau.
\end{aligned}$$

The boundedness of the operator $E(t)$ implies

$$\|AM_2 f^{(n)}(t)\| \le C\,\|f^{(n)}(t/2)\| + C\int_{t/2}^{t} \|f^{(n+1)}(\tau)\|\,d\tau. \tag{2.34}$$

Using (2.34) in (2.33), we have

$$\|AD_t^n M_2 f(t)\| \le C\sum_{j=0}^{n} t^{-n+j}\,\|f^{(j)}(t/2)\| + C\int_{t/2}^{t} \|f^{(n+1)}(\tau)\|\,d\tau,$$

and this yields the desired bound for $i=1$. □

In the following lemma, we derive a similar bound for the operator M_1.

Lemma 2.3.2. *For $i \in \{0,1\}$ and $n \in \mathbb{N}\cup\{0\}$, we have $A^i D_t^n M_1 f \in C((0,T];L^2(\Omega))$ and the following estimates*

$$\|A^i D_t^n M_1 f(t)\| \le C\sum_{j=1}^{n-1} t^{(1-i)\alpha-n+j}\,\|f^{(j)}(t/2)\| + Ct^{(1-i)\alpha-n}\max_{\tau\le t}\|f(\tau)\|\,,\ \ 0<t\le T,$$

hold.

Proof. Analogous to Lemma 2.3.1, we first prove for the case $i=0$. We have

$$\|M_1 f(t)\| \le C\int_0^{t/2} (t-\tau)^{\alpha-1}\,\|f(\tau)\|\,d\tau \le Ct^{\alpha}\max_{\tau\le t}\|f(\tau)\|\,.$$

Differentiating $M_1 f(t)$ with respect to t, we get

$$D_t M_1 f(t) = \frac{1}{2}\widehat{E}(t/2)f(t/2) + \int_0^{t/2} D_t\widehat{E}(t-\tau)f(\tau)\,d\tau. \tag{2.35}$$

Operating D_t^{n-1} in both sides of (2.35), it follows that

$$\begin{aligned} D_t^n M_1 f(t) &= \frac{1}{2}\sum_{m=1}^{n} D_t^{m-1}\Big((D_t^{n-m}\widehat{E})(t/2)f^{n-m}(t/2)\Big)\\ &\quad + \int_0^{t/2} (D_t^n\widehat{E}(t-\tau))f(\tau)\,d\tau\\ &= \sum_{m=1}^{n}\left(\frac{1}{2}\right)^m\sum_{j=0}^{m-1}\binom{m-1}{j}(D_t^{n-j-1}\widehat{E})(t/2)f^{(j)}(t/2)\\ &\quad + \int_0^{t/2} (D_t^n\widehat{E}(t-\tau))f(\tau)\,d\tau. \end{aligned} \tag{2.36}$$

Now using (2.29), we obtain

$$\begin{aligned}\left\|D_t^n M_1 f(t)\right\| &\leq C \sum_{m=1}^{n} \sum_{j=0}^{m-1} t^{\alpha-n+j} \left\|f^j(t/2)\right\| + C \int_0^{t/2} (t-\tau)^{\alpha-(1+n)} \left\|f(\tau)\right\| \, d\tau \\ &\leq C \sum_{j=1}^{n-1} t^{\alpha-n+j} \left\|f^{(j)}(t/2)\right\| + C t^{\alpha-n} \max_{\tau \leq t} \left\|f(\tau)\right\|,\end{aligned}$$

which proves the lemma for $i = 0$. Next, operating A in both sides of (2.36) and using (2.32), we have for $i = 1$

$$\left\|A D_t^n M_1 f(t)\right\| \leq C \sum_{j=1}^{n-1} t^{-n+j} \left\|f^{(j)}(t/2)\right\| + C t^{-n} \max_{\tau \leq t} \left\|f(\tau)\right\|,$$

and this completes the proof. □

The following lemma plays an important role in proving the infinite differentiability with respect to time of the solution of the homogeneous problem.

Lemma 2.3.3. *Let* u_1 *be the solution of (2.1) with* $f = 0$*,* $B(t,s) = 0$*, and* $u_0 \in L^2(\Omega)$*. Then for each natural number* n*,* $A D_t^n u_1 \in C((0,T]; L^2(\Omega))$*. Further, the following estimate*

$$\left\|A D_t^n u_1(t)\right\| \leq C t^{-(n+\alpha)} \left\|u_0\right\|, \quad t > 0,$$

holds.

Proof. Since $u_1(t) = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j t^\alpha)(u_0,\phi_j)\phi_j(x)$, it follows from [63, Lemma 3.2] that for $n \in \mathbb{N}$,

$$D_t^n u_1(t) = \sum_{j=1}^{\infty} -\lambda_j t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda_j t^\alpha)(u_0,\phi_j)\phi_j(x). \tag{2.37}$$

Then

$$A D_t^n u_1(t) = \sum_{j=1}^{\infty} -\lambda_j^2 t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda_j t^\alpha)(u_0,\phi_j)\phi_j(x).$$

Now, the boundedness of the two-parameter Mittag-Leffler function $E_{\alpha,\alpha}(\cdot,\cdot)$ (cf. Lemma 1.2.1) yields

$$\begin{aligned}\left\|A D_t^n u_1(t)\right\|^2 &= \sum_{j=1}^{\infty} \Big\{ -\lambda_j^2 t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda_j t^\alpha)(u_0,\phi_j) \Big\}^2 \\ &\leq C \sum_{j=1}^{\infty} \Big\{ -\lambda_j^2 t^{\alpha-n} \Big(\frac{C}{1+(\lambda_j t^\alpha)^2} \Big)(u_0,\phi_j) \Big\}^2 \\ &\leq C t^{-2(n+\alpha)} \sum_{j=1}^{\infty} (u_0,\phi_j)^2 \leq C t^{-2(n+\alpha)} \left\|u_0\right\|^2,\end{aligned}$$

which proves the desired bound. ☐

We are now ready to state the theorem regarding infinite differentiability of the solution of the homogeneous equation ($f = 0$ in (2.1)) with respect to time. Below, we use the notations $B_t^m(t,\tau) = D_t^m B(t,\tau)$ and $\widetilde{B_t^m} f(t) = \int_0^t B_t^m(t,\tau) f(\tau)\, d\tau$.

Theorem 2.3.1. *Assume that $u_0 \in L^2(\Omega)$, $f \equiv 0$ in (2.1), and let u be the corresponding solution. Then we have for $i \in \{0,1\}$ and $n \in \mathbb{N} \cup \{0\}$, $A^i D_t^n u \in C((0,T]; L^2(\Omega))$. Further, assume that (1.9) holds for the operators $B_t^{(n)}(t,\tau)$ and $B_t^{(n+1)}(t,\tau)$. Then, for $t > 0$*

$$\|A^i D_t^n u(t)\| \le Ct^{-(\alpha i+n)} \|u_0\|.$$

Proof. The proof is based on the induction hypothesis. Analogous to the proof of Theorem 2.2.3, let $v(t) = u(t) - E(t)u_0$. Then

$$v(t) = M_1 \tilde{B} u(t) + M_2 \tilde{B} u(t). \tag{2.38}$$

Also, we know from Theorem 2.2.3 that, for $t > 0$

$$\|A^i u(t)\| \le Ct^{-\alpha i} \|u_0\|, \quad i = 0, 1. \tag{2.39}$$

Let us assume that for $i = 0, 1$, and $1 \le m < n$,

$$\|A^i D_t^m u(t)\| \le Ct^{-(i\alpha+m)} \|u_0\|, \quad t > 0. \tag{2.40}$$

We now show that (2.40) is valid for $m = n$ as well. Applying Lemma 2.3.7 [75, p.25], induction hypothesis (2.40) and (2.39) with $i = 1$, we have for $m \le n$

$$\|D_t^m \tilde{B} u(t)\| \le C \sum_{j=0}^{m-1} \|A D_t^j u(t)\| + \|\widetilde{B_t^m} u(t)\| \le Ct^{-(\alpha+m-1)} \|u_0\|. \tag{2.41}$$

Now, by Lemma 2.3.2 with $i = 0$ and (2.41), we find that

$$\begin{aligned} \|D_t^n M_1 \tilde{B} u(t)\| &\le C \sum_{j=1}^{n-1} t^{(\alpha-n+j)} \|D_t^{(j)} \tilde{B} u(t/2)\| + Ct^{\alpha-n} \max_{\tau \le t} \|\tilde{B} u(\tau)\| \\ &\le C \sum_{j=1}^{n-1} t^{(\alpha-n+j)-\alpha-j+1} \|u_0\| + Ct^{\alpha-n} t^{1-\alpha} \|u_0\| \le Ct^{-n+1} \|u_0\|. \end{aligned} \tag{2.42}$$

Similarly, Lemma 2.3.1 with $i = 0$ and (2.41) leads to

(2.43)

$$\begin{aligned} \|D_t^n M_2 \tilde{B} u(t)\| &\le C \sum_{j=1}^{n-1} t^{(\alpha-n+j)} \|D_t^{(j)} \tilde{B} u(t/2)\| + C \max_{t/2 \le \tau \le t} \|D_t^n \tilde{B} u(\tau)\| \int_{t/2}^t (t-\tau)^{\alpha-1}\, d\tau \\ &\le C \sum_{j=1}^{n-1} t^{\alpha-n+j-\alpha-j+1} \|u_0\| + Ct^{-(\alpha+n-1)} \|u_0\| t^{\alpha} \le Ct^{-n+1} \|u_0\|. \end{aligned}$$

With an aid of (2.42) and (2.43), we obtain

$$\|D_t^n v(t)\| \leq Ct^{-n+1}\|u_0\|. \tag{2.44}$$

Again, using (2.37) we have for $u_0 \in L^2(\Omega)$

$$\|D_t^n u_1(t)\| \leq Ct^{-n}\|u_0\|. \tag{2.45}$$

The combination of (2.44) and (2.45) completes the proof for $i = 0$, as

$$\|D_t^n u(t)\| \leq \|D_t^n u_1(t)\| + \|D_t^n v(t)\| \leq Ct^{-n}\|u_0\| + Ct^{-n+1}\|u_0\| \leq Ct^{-n}\|u_0\|.$$

Next, we prove (2.40) for $i = 1$. Putting $i = 1$ in Lemma 2.3.2 and with the help of (2.41), we obtain

$$\begin{aligned}\|AD_t^n M_1\tilde{B}u(t)\| &\leq Ct^{-n-\alpha+1}\|u_0\| + Ct^{-n}\max_{\tau\leq t}\|\tilde{B}u(\tau)\| \\ &\leq Ct^{-n-\alpha+1}\|u_0\| + Ct^{-n-\alpha+1}\|u_0\| \leq Ct^{-(n+\alpha-1)}\|u_0\|.\end{aligned} \tag{2.46}$$

Analogously, using Lemma 2.3.1 and (2.41), we find that

$$\|AD_t^n M_2\tilde{B}u(t)\| \leq Ct^{-n-\alpha+1}\|u_0\| + C\int_{t/2}^{t}\|D_\tau^{(n+1)}\tilde{B}u(\tau)\|\,d\tau. \tag{2.47}$$

Now, for the time being assume that $AD_t^n v(t) \in C((0,T];L^2(\Omega))$. Arguing as (2.41) and by Lemma 2.3.3 we obtain

$$\begin{aligned}\|D_\tau^{(n+1)}\tilde{B}u(\tau)\| &\leq C\tau^{-(\alpha+n-1)}\|u_0\| + C\|AD_\tau^n u(\tau)\| \\ &\leq C\tau^{-(\alpha+n-1)}\|u_0\| + C\|AD_\tau^n u_1(\tau)\| + C\|AD_\tau^n v(\tau)\| \\ &\leq C\tau^{-(\alpha+n)}\|u_0\| + C\|AD_\tau^n v(\tau)\|.\end{aligned} \tag{2.48}$$

Application of (2.48) in (2.47) yields

$$\begin{aligned}\|AD_t^n M_2\tilde{B}u(t)\| &\leq Ct^{-(\alpha+n-1)}\|u_0\| + Ct^{-(\alpha+n-1)}\|u_0\| + C\int_{t/2}^{t}\|AD_\tau^n v(\tau)\|\,d\tau \\ &\leq Ct^{-(\alpha+n-1)}\|u_0\| + C\int_{t/2}^{t}\|AD_\tau^n v(\tau)\|\,d\tau.\end{aligned} \tag{2.49}$$

Therefore, by (2.46) and (2.49), we have

$$\|AD_t^n v(t)\| \leq Ct^{-(\alpha+n-1)}\|u_0\| + C\int_{t/2}^{t}\|AD_\tau^n v(\tau)\|\,d\tau.$$

Hence,

$$t^{(\alpha+n-1)}\left\|AD_t^n v(t)\right\| \leq C\left\|u_0\right\| + C\int_0^t \tau^{(\alpha+n-1)}\left\|AD_\tau^n v(\tau)\right\| d\tau.$$

Invite Gronwall's lemma to obtain

$$t^{(\alpha+n-1)}\left\|AD_t^n v(t)\right\| \leq C\left\|u_0\right\|. \tag{2.50}$$

Now, in view of Lemma 2.3.3 and (2.50), we obtain for $i=1$,

$$\left\|AD_t^n u(t)\right\| \leq Ct^{-(\alpha+n)}\left\|u_0\right\| + Ct^{-(\alpha+n-1)}\left\|u_0\right\| \leq Ct^{-(\alpha+n)}\left\|u_0\right\|.$$

Thus, the relation (2.40) holds for $m = n$. ☐

Remark 2.3.1. *Theorem 2.3.1 shows the regularity property in time of the homogeneous SEM with the initial data $u_0 \in L^2(\Omega)$. Observe that when $\alpha \to 1^-$, the presented result is similar to the standard PIDE [75, Theorem 2.3.4].*

2.4 General Stability Results

In this section, we establish some more general stability estimates of the Caputo fractional derivative and the usual partial derivative of the solution of the homogeneous and nonhomogeneous problems. We also present examples of SEM to illustrate Theorems 2.2.1 and 2.2.2.

The following stability results hold.

Theorem 2.4.1. *Let u satisfies (2.1) with $f=0$. Then for $t>0$,*

$$\left\|u(t)\right\|_p \leq Ct^{-\alpha\left(\frac{p-q}{2}\right)}\left\|u_0\right\|_q, \quad 0 \leq q \leq p \leq 2, \tag{2.51}$$

and

$$\left\|\partial_t^\alpha u(t)\right\|_p \leq Ct^{-\alpha\left(1+\frac{p-q}{2}\right)}\left\|u_0\right\|_q, \quad 0 \leq p \leq q \leq 2. \tag{2.52}$$

Proof. The proofs are based on the interpolation of estimates corresponding to the homogeneous problem. First, we shall prove (2.51). For $u_0 \in L^2(\Omega)$, Theorem 2.2.3 yields

$$\left\|u(t)\right\| \leq C\left\|u_0\right\|, \tag{2.53}$$

and

$$\left\|u(t)\right\|_2 \leq Ct^{-\alpha}\left\|u_0\right\|. \tag{2.54}$$

Also, we have from (2.11)

$$\|u(t)\|_2 \leq C\,\|u_0\|_2\,. \tag{2.55}$$

For $0 \leq p \leq 2, q = 0$, the estimate (2.51) follows from the interpolation of estimates (2.53) and (2.54). Again, interpolation of (2.54) and (2.55) gives (2.51), for $p = 2, 0 \leq q \leq 2$. Finally, we invite interpolation of (2.53) and (2.55) to obtain (2.51) for $0 \leq q = p \leq 2$.

Next, to estimate $\|\partial_t^\alpha u(t)\|_p$ for different real number p, $0 \leq p \leq 2$, we first show that $\|\partial_t^\alpha u(t)\|_p$ makes sense for $p = 2$. The estimates of $\|\partial_t^\alpha u(t)\|_p$ for $p = 0$ are contained in Theorems 2.2.1 and 2.2.3 when the data u_0 belongs to $\dot{H}^2(\Omega)$ and $L^2(\Omega)$, respectively. Then, we apply interpolation of estimates to obtain (2.52).

Putting $f = 0$ in (2.8) and taking the Caputo fractional derivative of order α, we have

$$\begin{aligned} \partial_t^\alpha u(t) &= \sum_{j=1}^{\infty} -\lambda_j(u_0,\phi_j)E_{\alpha,1}(-\lambda_j t^\alpha)\phi_j(x) + \tilde{B}u(t) \\ &\quad - \sum_{j=1}^{\infty}\left(\int_0^t \lambda_j(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha)(\tilde{B}u(\tau),\phi_j)\,d\tau\right)\phi_j(x), \end{aligned} \tag{2.56}$$

where $\partial_t^\alpha E(t)u_0 = \sum_{j=1}^{\infty} -\lambda_j(u_0,\phi_j)E_{\alpha,1}(-\lambda_j t^\alpha)\phi_j(x)$. Setting

$$F_j = \int_0^t \lambda_j(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-\tau)^\alpha)(\tilde{B}u(\tau),\phi_j)\,d\tau,$$

using (2.20) and integrating by parts, we first note that

$$\begin{aligned} F_j &= \int_0^t \frac{d}{d\tau}E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha)(\tilde{B}u(\tau),\phi_j)\,d\tau \\ &= (\tilde{B}u(t),\phi_j) - \int_0^t E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha)(\tilde{B}_\tau u(\tau),\phi_j)\,d\tau \\ &\qquad - \int_0^t E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha)(B(\tau,\tau),\phi_j)\,d\tau. \end{aligned}$$

The last term of the right-hand side of (2.56) can be expressed as

$$\sum_{j=1}^{\infty} F_j\phi_j(x) = \tilde{B}u(t) - \sum_{j=1}^{\infty}\left(\int_0^t E_{\alpha,1}(-\lambda_j(t-\tau)^\alpha)(\tilde{B}_\tau u(\tau) + B(\tau,\tau)u(\tau),\phi_j)\,d\tau\right)\phi_j(x).$$

From (2.56), we can write

(2.57)

$$\partial_t^\alpha u(t) = \sum_{j=1}^{\infty} -\lambda_j (u_0, \phi_j) E_{\alpha,1}(-\lambda_j t^\alpha)\phi_j(x) + \int_0^t E(t-\tau)(\tilde{B}_\tau u(\tau) + B(\tau,\tau)u(\tau))\, d\tau$$
$$:= S_1 + S_2.$$

The quantity $\|AS_1\|$ is meaningful only when the data $u_0 \in \dot{H}^2(\Omega)$. Following Jin et al. [25, Theorem 2.2, $\ell = 1$], we have $\|AS_1\| \le Ct^{-\alpha}\|u_0\|_2$. Now we show that $\|AS_2\|$ makes sense for $u_0 \in \dot{H}^2(\Omega)$. Since B_t and B are dominated by A (cf. (1.9)), we obtain using (2.11)

$$\begin{aligned}\|AS_2\| &\le \int_0^t \|AE(t-\tau)(\tilde{B}_\tau u(\tau) + B(\tau,\tau)u(\tau))\|\, d\tau\\ &\le C\int_0^t (t-\tau)^{-\alpha}\Big(\|\tilde{B}_\tau u(\tau)\| + \|B(\tau,\tau)u(\tau)\|\Big)\, d\tau\\ &\le C\|u_0\|_2 \le Ct^{-\alpha}\|u_0\|_2 .\end{aligned}$$

It follows from (2.57) that

$$\|A\partial_t^\alpha u(t)\| \le Ct^{-\alpha}\|u_0\|_2 . \tag{2.58}$$

Further, the $L^2(\Omega)$-estimates of $\partial_t^\alpha u(t)$ for nonsmooth and smooth data u_0, are respectively (see Theorems 2.2.3 and 2.2.1),

$$\|\partial_t^\alpha u(t)\| \le Ct^{-\alpha}\|u_0\|, \quad t > 0, \tag{2.59}$$

and

$$\|\partial_t^\alpha u(t)\| \le C\|u_0\|_2 . \tag{2.60}$$

Interpolation of the estimates (2.59) and (2.60) gives (2.52) for $p = 0$ and $0 \le q \le 2$. The estimates for $0 \le p = q \le 2$ follow from (2.59) and (2.58). Finally, again interpolation of the estimates (2.60) and (2.58) yields (2.52), for $0 < p < q \le 2$. This completes the proof. □

Remark 2.4.1. *(i) By Theorem 2.4.1, for the homogeneous problem with* $u_0 \in H_0^1(\Omega)$*, we get* $\|u(t)\|_2 + \|\partial_t^\alpha u(t)\| \le Ct^{-\alpha/2}\|u_0\|_1$, $t > 0$*, which is an immediate consequence of the interpolation of estimates for* $q = 0$ *and* $q = 2$.

(ii) The estimate (2.52) invites the restriction $p \le q$*. Such restriction do present for SE (see [25, Theorem 2.2]). This shows that like SE, SEM also has a narrow smoothing property.*

In the following, we prove some stability results for the solution of the nonhomogeneous problem. The first theorem presents stability estimates for the Caputo derivative while the second one describes those for the usual partial derivative of the continuous solution. These results will be useful in the semidiscrete error analysis.

Theorem 2.4.2. *Assume that* $u_0 \in \dot{H}^2(\Omega)$ *and* $f \in C^1(L^2(\Omega))$. *With the assumption* $(\mathcal{A})$, *we have*

$$\|u(t)\|_2 \le C\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big), \quad 0 \le t \le T, \tag{2.61}$$

and

$$\|\partial_t^\alpha u(t)\|_p \le C\big(t^{-\alpha p/2}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big), \quad p = 1, 2, \ t > 0. \tag{2.62}$$

Proof. The proof of (2.61) is given by Theorem 2.2.1 and the proof of (2.62) follows from Theorem 2.4.1. □

Remark 2.4.2. *We remark that the a priori bound (2.61) remains valid for Hölder continuous source functions* f. *However, the differentiability of* f *is needed only to estimate (2.62). The singularity* $t^{-\alpha p/2}$, $p = 1, 2$, *on* $f(0)$ *in (2.62) can be removed if we consider* $f(0)$ *is an element of the space* $\dot{H}^1(\Omega)$ *for* $p = 1$ *and* $f(0)$ *in* $\dot{H}^2(\Omega)$ *for* $p = 2$.

Theorem 2.4.3. *Assume that* $u_0 \in \dot{H}^2(\Omega)$ *and* $f \in C^1(\dot{H}^1(\Omega))$. *With the assumption* $(\mathcal{A})$ *and* $p = 1, 2$, *we have for positive time* t,

$$\|u'(t)\|_p \le C\big(t^{-1+(2-p)\alpha/2}[\|u_0\|_2 + t^{(p-1)\alpha/2}\|f(0)\|_{(p-1)}] + t^{\alpha/2}\|f\|_{C^1(\dot{H}^{(p-1)}(\Omega))}\big).$$

Proof. To prove the assertion, we decompose the problem (2.1) in two parts as

$$\begin{aligned} \partial_t^\alpha v(t) + Av(t) &= \tilde{B}v(t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad 0 < \alpha < 1, \\ v(t) &= 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ v(0) &= u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{2.63}$$

where $\tilde{B}v(t) = \int_0^t B(t,s)v(s)\,ds$ and

$$\begin{aligned} \partial_t^\alpha w(t) + Aw(t) &= f(t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad 0 < \alpha < 1, \\ w(t) &= 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ w(0) &= 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{2.64}$$

Following Theorem 2.3.1, for smooth initial data we can obtain $\|v'(t)\| \leq Ct^{-1+\alpha}\|u_0\|_2$ and $\|v'(t)\|_2 \leq Ct^{-1}\|u_0\|_2$ for $t>0$. By interpolation we get

$$\|v'(t)\|_1 \leq Ct^{-1+\alpha/2}\|u_0\|_2\,, \quad t>0. \tag{2.65}$$

Now, the solution for problem (2.64) (cf. [24]) is given by

$$w(t) = \int_0^t \widehat{E}(t-s)f(s)\,ds = \int_0^t \widehat{E}(s)f(t-s)\,ds, \tag{2.66}$$

where the operator $\widehat{E}(t)$ is defined by

$$\widehat{E}(t)\chi = \sum_{j=1}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha)(\chi,\phi_j)\phi_j(x), \quad t>0,\ \chi\in L^2(\Omega),$$

with $E_{\alpha,\alpha}(-\lambda_j t^\alpha) = \sum_{j=0}^{\infty}\frac{(-\lambda_j t^\alpha)^j}{\Gamma(j\alpha+\alpha)}$. Differentiating (2.66) with respect to time to have

$$w'(t) = \int_0^t \widehat{E}(s)f'(t-s)\,ds + \widehat{E}(t)f(0) = \int_0^t \widehat{E}(t-s)f'(s)\,ds + \widehat{E}(t)f(0).$$

Invoking Lemma 2.2 [24, p. 565] with $p=1, q=0$, we obtain for $t>0$

$$\begin{aligned}\|w'(t)\|_1 &\leq \int_0^t \|\widehat{E}(t-s)f'(s)\|_1\,ds + \|\widehat{E}(t)f(0)\|_1\\ &\leq \int_0^t (t-s)^{-1+\alpha/2}\|f'(s)\|\,ds + Ct^{-1+\alpha/2}\|f(0)\|\\ &\leq Ct^{\alpha/2}\|f\|_{C^1(L^2)} + Ct^{-1+\alpha/2}\|f(0)\|\,.\end{aligned} \tag{2.67}$$

In view of (2.65) and (2.67), we get for $t>0$

$$\|u'(t)\|_1 \leq \|v'(t)\|_1 + \|w'(t)\|_1 \leq Ct^{-1+\alpha/2}\big(\|u_0\|_2 + \|f(0)\|\big) + Ct^{\alpha/2}\|f\|_{C^1(L^2)}\,. \tag{2.68}$$

Similarly, putting $p=2, q=1$, in Lemma 2.2 [24, p. 565], we obtain

$$\|w'(t)\|_2 \leq Ct^{\alpha/2}\|f\|_{C^1(\dot{H}^1)} + Ct^{-1+\alpha/2}\|f(0)\|_1\,, \quad t>0.$$

Thus, for positive time t,

$$\|u'(t)\|_2 \leq \|v'(t)\|_2 + \|w'(t)\|_2 \leq Ct^{-1}\big(\|u_0\|_2 + t^{\alpha/2}\|f(0)\|_1\big) + Ct^{\alpha/2}\|f\|_{C^1(\dot{H}^1)}\,. \tag{2.69}$$

The estimates (2.68)-(2.69) complete the proof of the assertion. □

In the following, as applications to Theorems 2.2.1 and 2.2.2, we provide two examples of SEM in one and two space dimensions.

Example 2.1. *Let* $\Omega = (0,1)$ *and* $T > 0$ *be a fixed value. Consider the SEM (cf. [75])*

$$\partial_t^\alpha u(t) - \frac{\partial^2 u(t)}{\partial x^2} = f(t) + \int_0^t e^{(t-s)} u(s)\, ds \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad 0 < \alpha < 1,$$
$$u(0,t) = u(1,t) = 0, \quad 0 < t \le T,$$
$$u(x,0) = \sin(\pi x) \quad \text{in} \quad \Omega.$$

Here $A = -\frac{\partial^2}{\partial x^2}$ *and* $B(t,s) = e^{(t-s)} I$*. In addition,* $B(t,s)$ *is dominated by* A*. Choose the source function* f *as*

$$f(x,t) = \frac{e^t \gamma(1-\alpha, t)}{\Gamma(1-\alpha)} \sin(\pi x) + \pi^2 e^t \sin(\pi x) - t e^t \sin(\pi x),$$

where $\gamma(1-\alpha,t)$ *is an incomplete gamma function [23, 44]. Then* f *is a Hölder continuous function. The exact solution* u *is given by* $u(x,t) = e^t \sin(\pi x)$ *and Theorems 2.2.1 and 2.2.2 hold.*

Example 2.2. *Let* $\Omega = (0,1)^2$*. For* $\alpha \in (0,1)$ *and* $T < \infty$*, consider the following IBVP in* $u(x,y,t)$*:*

$$\begin{aligned} &\partial_t^\alpha u(t) - \Delta u(t) = f(t) + \int_0^t e^{(t-s)} \Delta u(s)\, ds \quad \text{in} \quad \Omega, \quad 0 < t \le T, \\ &u(x,0,t) = u(x,1,t) = 0 \quad \text{for} \quad 0 \le x \le 1, \quad 0 < t \le T, \\ &u(0,y,t) = u(1,y,t) = 0 \quad \text{for} \quad 0 \le y \le 1, \quad 0 < t \le T, \\ &u(x,y,0) = x(1-x)y(1-y) \quad \text{in} \quad \Omega, \end{aligned} \tag{2.70}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$*. Here we see that* $A = -\Delta$ *and* $B(t,s) = e^{(t-s)}\Delta$*. Consider the source function* f *as*

$$\begin{aligned} f(x,y,t) =& \frac{e^t \gamma(1-\alpha,t)}{\Gamma(1-\alpha)} x(1-x)y(1-y) \\ &+ 2e^t(x(1-x) + y(1-y)) + 2te^t(x(1-x) + y(1-y)). \end{aligned}$$

Then, it is not so hard to verify that the function $u(x,y,t) = e^t x(1-x)y(1-y)$ *satisfies problem (2.70) and the estimates of Theorems 2.2.1 and 2.2.2 hold for the solution* u*.*

At the end, we remind our readers that our analysis of (2.1) is still valid when the operator A is a more general elliptic operator of the form:

$$A = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(c_{ij}(x) \frac{\partial}{\partial x_i}\Big) + c_0(x) I.$$

Here, the coefficients $c_{ij} = c_{ji}$ and $c_0 \ge 0$ are smooth functions of the space variable only and c_{ij} form a uniformly positive definite matrix on $\overline{\Omega}$.

2.5 Concluding Remarks

In this chapter, we have studied the well-posedness results of SEM (2.1) with a smooth kernel with respect to different Sobolev regularity assumptions on the initial data and the source function. The proofs use the idea of the eigenfunction expansion, theory of Volterra integral equation, and Banach fixed point theorem. We have demonstrated that the solution corresponding to the homogeneous problem is infinitely differentiable with respect to time when the initial data is an element of $L^2(\Omega)$. In addition, we have also established some more general stability results for the solutions of both homogeneous and nonhomogeneous problems which are crucial for the development of the numerical schemes. Finally, some examples are provided to illustrate our theory.

3

Semidiscrete Error Analysis

This chapter deals with the semidiscrete error analysis to problem (1.5). The GFEM is considered and analyzed by using piecewise continuous and linear finite elements. The convergence analysis is carried out for both smooth and nonsmooth data cases. We prove optimal order error bounds in both $L^2(\Omega)$- and $H^1(\Omega)$-norms for smooth initial data by using both Ritz and Ritz-Volterra projections. Moreover, almost optimal order nonsmooth initial data error estimates are proved using the L^2-projection. Finally, for a special choice of the kernel operator $B(t,s)$, namely, $B(t,s) = -A$, we demonstrate that the almost optimal error bounds can be converted to optimal order.

3.1 Introduction

To start with, we recall the following IBVP for the SEM of the form

$$\begin{aligned} &\partial_t^\alpha u(t) + Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad 0 < \alpha < 1, \\ &u(t) = 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\ &u(0) = u_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{3.1}$$

where the symbols are defined as in the previous chapters.

Here, we study the spatially discrete finite element approximation to problem (3.1) and derive the error estimates for both smooth and nonsmooth initial data. The properties of the operator $\mathbb{E}_h$ (see (3.7)) and the inverse inequality play an important role in the error analysis. While both Ritz and Ritz-Volterra projections are employed to estimate the errors for smooth initial data, nonsmooth data error estimates rely only on the L^2-projection. In the absence of the memory term, i.e., when $B(t,s) = 0$ in (3.1), Jin et al. [24, 25] studied semidiscrete GFEM using piecewise linear finite elements and established the convergence analysis. In this chapter, an effort has been made to generalize the convergence analysis of the SE [24, 25] to problem (3.1). For smooth data error estimates, in addition to the Ritz projection we have used a novel Ritz-Volterra projection [6, 36] which greatly simplifies our error analysis. We conclude this section by introducing an operator $\mathbb{B}(t,s): H_0^1(\Omega) \to L^2(\Omega)$ defined by

$$(\mathbb{B}(t,s)v, w) = B(t,s;v,w), \quad \forall v, w \in H_0^1(\Omega),$$

where $B(t,s;\cdot,\cdot)$ is the bilinear form associated to the operator $B(t,s)$, i.e.,

$$B(t,s;v,w) = \int_\Omega \left(\sum_{i,j=1}^{d} b_{ij}(x;t,s) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + \sum_{i=1}^{d} b_i(x;t,s) \frac{\partial v}{\partial x_i} w + b_0(x;t,s) v w \right) dx.$$

Note that, the operator $\mathbb{B}(t,s)$ is well-defined on $H_0^1(\Omega)$ and will be used in the error analysis.

This chapter is organized as follows. In Section 3.2, we develop the semidiscrete GFEM and derive error bounds for both smooth and nonsmooth initial data. Finally, in Section 3.3, for a special choice of $B(t,s)$, namely, $B(t,s) = -A$, we recover the optimal order error bounds in the $L^2(\Omega)$- and $H^1(\Omega)$-norms for the nonsmooth data. Numerical tests in Section 3.4 confirm the theoretical rates of convergence.

3.2 Standard Galerkin Finite Element Method

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain Ω. Consider the finite element space $S_h \equiv S_h(\Omega)$ given by

$$S_h = \{\chi \in H_0^1(\Omega) \colon \chi \text{ is a linear polynomial on each element of } \mathcal{T}_h\}.$$

The semidiscrete GFEM for (3.1) is to seek a function $u_h : [0,T] \to S_h$ such that

$$\begin{aligned} &(\partial_t^\alpha u_h(t), \chi) + (\nabla u_h(t), \nabla \chi) = (f(t), \chi) + \int_0^t B(t,s; u_h(s), \chi)\, ds, \quad \forall \chi \in S_h, \\ &u_h(0) = u_{0h} \in S_h, \end{aligned} \tag{3.2}$$

where

$$B(t,s;u_h(s),\chi) = \int_\Omega \left(\sum_{i,j=1}^{d} b_{ij}(x;t,s) \frac{\partial u_h}{\partial x_i} \frac{\partial \chi}{\partial x_j} + \sum_{i=1}^{d} b_i(x;t,s) \frac{\partial u_h}{\partial x_i} \chi + b_0(x;t,s) u_h \chi \right) dx.$$

Define the discrete Laplacian $A_h \colon S_h \to S_h$ by

$$(A_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \forall \psi, \chi \in S_h. \tag{3.3}$$

Let $B_h(t,s) : S_h \to S_h$, the discrete version of the operator $B(t,s)$, be defined by

$$(B_h(t,s)\psi, \chi) = B(t,s;\psi,\chi),$$

for $0 \le s \le t \le T$ and $\psi, \chi \in S_h$. Then semidiscrete problem (3.2) can be written as

$$\begin{aligned} &\partial_t^\alpha u_h(t) + A_h u_h(t) = f_h(t) + \tilde{B}_h u_h(t), \quad t > 0, \\ &u_h(0) = u_{0h}, \end{aligned} \tag{3.4}$$

where $\tilde{B}_h u_h(t) = \int_0^t B_h(t,s)u_h(s)\,ds$ and $f_h = P_h f$ is the L^2-projection of f. The solution $u_h(x,t)$ of (3.4) has the representation given by

$$u_h(t) = E_h(t)u_{0h} + \int_0^t \mathbb{E}_h(t-\tau)(f_h(\tau) + \tilde{B}_h u_h(\tau))\,d\tau, \tag{3.5}$$

where the operators $E_h(t)$ and $\mathbb{E}_h(t)$ are the discrete analogues of $E(t)$ and $\widehat{E}(t)$ (see Chapter 2), respectively. For $\chi \in S_h$, these discrete operators are defined by

$$E_h(t)\chi = \sum_{j=1}^N E_{\alpha,1}(-\lambda_j^h t^\alpha)(\chi, \phi_j^h)\phi_j^h(x), \tag{3.6}$$

and

$$\mathbb{E}_h(t)\chi = \sum_{j=1}^N t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j^h t^\alpha)(\chi, \phi_j^h)\phi_j^h(x), \quad t > 0, \tag{3.7}$$

where $\{\lambda_j^h, \phi_j^h\}$, $j = 1, \ldots, N$ are the Dirichlet eigenpairs of A_h on Ω.

Now, for all $\chi \in S_h$ and for any $p \in (-\infty, \infty)$, we define

$$|||\chi|||_p^2 = \sum_{j=1}^N (\lambda_j^h)^p (\chi, \phi_j^h)^2. \tag{3.8}$$

It can be shown that $|||\cdot|||_p$ is a norm on S_h. For $\chi \in S_h$, we have $|||\chi|||_0 = \|\chi\|$ and $|||\chi|||_1 = \|\chi\|_1$.

In the following, we present the stability estimates of the operator $\mathbb{E}_h(t)$. For a proof, see [25, p.452].

Lemma 3.2.1. *Let $\mathbb{E}_h(t)$ be defined by (3.7) and let χ be any element of S_h. Then for any $t > 0$,*

$$|||\mathbb{E}_h(t)\chi|||_p \le \begin{cases} Ct^{-1+\alpha(1+\frac{q-p}{2})}|||\chi|||_q, & \text{if} \quad p-2 \le q \le p, \\ Ct^{-1+\alpha}|||\chi|||_q, & \text{if} \quad p < q. \end{cases} \tag{3.9}$$

Further for $\chi \in S_h$, the following inverse estimate holds:

$$\|\nabla \chi\| \leq Ch^{-1}\|\chi\|\,. \tag{3.10}$$

We now recall some well-known projection operators and their approximation properties. Let the L^2-projection $P_h\colon L^2(\Omega) \to S_h$ and the Ritz projection $R_h\colon H_0^1(\Omega) \to S_h$ be defined by

$$(P_h v - v, \chi) = 0, \ \ \forall \chi \in S_h, \, v \in L^2(\Omega),$$

and

$$(\nabla(R_h v - v), \nabla \chi) = 0, \ \ \forall \chi \in S_h, \, v \in H_0^1(\Omega), \tag{3.11}$$

respectively. It is known that P_h and R_h satisfy the following approximation properties [13, 68]: For $v \in \dot{H}^p(\Omega)$, $p = 1, 2$,

$$\|P_h v - v\| + h\|\nabla(P_h v - v)\| \leq Ch^p \|v\|_p\,, \tag{3.12}$$

$$\|R_h v - v\| + h\|\nabla(R_h v - v)\| \leq Ch^p \|v\|_p\,. \tag{3.13}$$

We also need the Ritz-Volterra projection $V_h\colon C(H_0^1(\Omega)) \to C(S_h)$ be defined by

$$(\nabla(V_h v - v)(t), \nabla \chi) = \int_0^t B(t, s; (V_h v - v)(s), \chi)\, ds, \ \ \forall \chi \in S_h, \tag{3.14}$$

with the following approximation property [6, 36]:

$$\|V_h v(t) - v(t)\| + h\,\|\nabla(V_h v(t) - v(t))\| \leq Ch^2\Big(\,\|v(t)\|_2 + \int_0^t \|v(s)\|_2\, ds\Big), \tag{3.15}$$

for $v(t) \in \dot{H}^2(\Omega)$.

3.2.1 Smooth Data Error Estimates

In this section, we focus our attention to derive error estimates for smooth initial data, i.e., $u_0 \in \dot{H}^2(\Omega)$.

Error estimates using the Ritz projection. Using the Ritz projection as the comparison function, we split the error $u(t) - u_h(t)$ as usual way like for parabolic problems

$$u(t) - u_h(t) = \rho(t) + \theta(t), \ \ \rho = u - R_h u, \ \theta = R_h u - u_h. \tag{3.16}$$

From the approximation property (3.13) and the stability estimate (2.61), we have the bounds for $\rho(t)$ as

$$\|\rho(t)\| + h\,\|\nabla \rho(t)\| \leq Ch^2\,\|u(t)\|_2 \leq Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big). \tag{3.17}$$

Now it remains to estimate $\|\theta(t)\|$ and $\|\nabla\theta(t)\|$. It is easy to verify that θ satisfies the following error equation

$$\begin{aligned}\partial_t^\alpha\theta(t)+A_h\theta(t)&=-P_h\partial_t^\alpha\rho(t)+P_h\int_0^t\mathbb{B}(t,s)\rho(s)\,ds+\int_0^t B_h(t,s)\theta(s)\,ds\\&=-P_h\partial_t^\alpha\rho(t)+P_h\tilde{\mathbb{B}}\rho(t)+\tilde{B}_h\theta(t),\end{aligned}\tag{3.18}$$

where $(\tilde{\mathbb{B}}\rho(t),\chi)=(\int_0^t\mathbb{B}(t,s)\rho(s)\,ds,\chi)=\int_0^t B(t,s;\rho(s),\chi)\,ds$, for $\chi\in S_h$. With $u_{0h}=u_h(0)=R_hu_0$, we have $\theta(0)=0$. By Duhamel's principle (3.5), we write

$$\begin{aligned}\theta(t)=&-\int_0^t\mathbb{E}_h(t-\tau)(P_h\partial_\tau^\alpha\rho(\tau))\,d\tau+\int_0^t\mathbb{E}_h(t-\tau)P_h\tilde{\mathbb{B}}\rho(\tau)\,d\tau\\&+\int_0^t\mathbb{E}_h(t-\tau)\tilde{B}_h\theta(\tau)\,d\tau=:I_1+I_2+I_3.\end{aligned}\tag{3.19}$$

We first estimate the $L^2(\Omega)$-norm of $\theta(t)$ and then its gradient. Applying (3.9) with $p=q=0$, the L^2-stability of P_h, (3.13), and the estimate (2.62) with $p=2$, we have for the term I_1

$$\begin{aligned}\|I_1\|&\le C\int_0^t(t-\tau)^{\alpha-1}\left\|\partial_\tau^\alpha\rho(\tau)\right\|\,d\tau\\&\le Ch^2\int_0^t(t-\tau)^{\alpha-1}\left\|\partial_\tau^\alpha u(\tau)\right\|_2\,d\tau\\&\le Ch^2\int_0^t(t-\tau)^{\alpha-1}\big(\tau^{-\alpha}[\|u_0\|_2+\|f(0)\|]+\|f\|_{C^1(L^2(\Omega))}\big)\,d\tau\\&\le Ch^2\big(\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\big).\end{aligned}\tag{3.20}$$

Now we proceed to estimate the term I_2. Following [69], we define the discrete solution operator $T_h:L^2\to S_h$ of the elliptic problem by

$$(\nabla T_hf,\nabla\chi)=(f,\chi),\quad\forall\chi\in S_h.$$

Here the operator T_h is self-adjoint, positive definite on S_h and positive semidefinite on $L^2(\Omega)$. Therefore $T_h:S_h\to S_h$ is invertible and it is bounded on S_h. By Bounded Inverse Theorem $T_h^{-1}:S_h\to S_h$ is bounded. Thus, for each $\chi\in S_h$, there exists a unique $\psi\in S_h$ such that $T_h\psi=\chi$ imply $\psi=T_h^{-1}\chi$, and

$$\|\psi\|\le\|T_h^{-1}\|\,\|\chi\|\,.$$

With $\chi=T_h\psi$, we note that

$$\Big(\int_0^t\mathbb{B}(t,s)\rho(s)\,ds,\chi\Big)=\Big(\int_0^t\mathbb{B}(t,s)\rho(s)\,ds,T_h\psi\Big)=\int_0^t B(t,s;\rho(s),T_h\psi)\,ds.\tag{3.21}$$

Invoking [69, Lemma 2.1], (3.13) and (2.61), we find that

$$\begin{aligned} B(t,s;\rho(s),T_h\psi) &\leq C(h\,\|\nabla\rho(s)\| + \|\rho(s)\|)\,\|\psi\| \\ &\leq C(Ch^2\,\|u(s)\|_2 + Ch^2\,\|u(s)\|_2)\,\|T_h^{-1}\|\,\|\chi\| \\ &\leq Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\,\big)\|\chi\|\,. \end{aligned} \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$\|\int_0^t \mathbb{B}(t,s)\rho(s)\,ds\|_{0,h} := \sup_{\substack{\chi\in S_h\\ \chi\neq 0}} \frac{(\int_0^t \mathbb{B}(t,s)\rho(s)\,ds,\chi)}{\|\chi\|} \leq Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\,\big), \tag{3.23}$$

where $\|\cdot\|_{0,h}$ denotes the discrete $L^2(\Omega)$-norm. With $p = 0$, $q = 0$ in the stability estimate (3.9) of the operator $\mathbb{E}_h(t)$ and (3.23), we get

$$\begin{aligned} \|I_2\| &\leq C\int_0^\tau (t-\tau)^{\alpha-1}\|\int_0^t \mathbb{B}(\tau,s)\rho(s)ds\|_{0,h}\,d\tau \\ &\leq C\int_0^t (t-\tau)^{\alpha-1}Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\,\big)\,d\tau \\ &\leq Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\,\big). \end{aligned} \tag{3.24}$$

To estimate I_3, we proceed as in the case of I_2 and apply the inverse estimate and the change of order of integration to obtain

$$\begin{aligned} \|I_3\| &\leq C\int_0^t (t-\tau)^{\alpha-1}\|\int_0^\tau B_h(\tau,s)\theta(s)\,ds\|_{0,h}\,d\tau \\ &\leq C\int_0^t (t-\tau)^{\alpha-1}\left(\int_0^\tau \Big\{\,\|\theta(s)\| + h\,\|\nabla\theta(s)\|\,\Big\}\,ds\right)d\tau \\ &\leq C\int_0^t (t-\tau)^{\alpha-1}\left(\int_0^\tau \|\theta(s)\|\;ds\right)d\tau \\ &\leq C\int_0^t \left(\int_s^t (t-\tau)^{\alpha-1}\,\|\theta(s)\|\;d\tau\right)ds \\ &= C\int_0^t (t-s)^{\alpha}\,\|\theta(s)\|\;ds. \end{aligned} \tag{3.25}$$

Using (3.20), (3.24), (3.25) and applying Gronwall's lemma, we obtain

$$\|\theta(t)\| \leq Ch^2\big(\,\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\,\big). \tag{3.26}$$

Our next task is to bound $\left\|\nabla\theta(t)\right\|$. Putting $p=1, q=0$ in (3.9), the L^2-stability of P_h, (3.13), and (2.62), we have

$$\begin{aligned}
\|\nabla I_1\| &\le C\int_0^t (t-\tau)^{\alpha/2-1}\,\|\partial_\tau^\alpha \rho(\tau)\|\,d\tau\\
&\le Ch\int_0^t (t-\tau)^{\alpha/2-1}\,\|\partial_\tau^\alpha u(\tau)\|_1\,d\tau \qquad (3.27)\\
&\le Ch[\|u_0\|_2+\|f(0)\|]\int_0^t (t-\tau)^{\alpha/2-1}\tau^{-\alpha/2}\,d\tau\\
&\quad + Ch\,\|f\|_{C^1(L^2(\Omega))}\int_0^t (t-\tau)^{\alpha/2-1}\,d\tau\\
&\le Ch\,\|u_0\|_2\,\mathcal{B}(\alpha/2,-\alpha/2+1)+Ch\,\|f\|_{C^1(L^2(\Omega))}\\
&\le Ch\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\,\big), \qquad (3.28)
\end{aligned}$$

where $\mathcal{B}(\cdot,\cdot)$ is the Beta function. Again, with $p=1, q=0$ in (3.9) and (3.23) yield

$$\begin{aligned}
\|\nabla I_2\| &\le C\int_0^t (t-\tau)^{-1+\alpha/2}\Big\|\int_0^\tau \mathbb{B}(\tau,s)\rho(s)\,ds\Big\|_{0,h}\,d\tau\\
&\le Ch^2\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\,\big)\int_0^t (t-\tau)^{-1+\alpha/2}\,d\tau\\
&\le Ch^2\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\,\big). \qquad (3.29)
\end{aligned}$$

Arguing as I_3, we estimate $\|\nabla I_3\|$ as

$$\begin{aligned}
\|\nabla I_3\| &\le C\int_0^t (t-\tau)^{-1+\alpha/2}\Big\|\int_0^\tau B_h(\tau,s)\theta(s)\,ds\Big\|_{0,h}\,d\tau\\
&\le C\int_0^t (t-s)^{\alpha/2}\,\|\theta(s)\|\,ds \le Ch^2\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\,\big), \qquad (3.30)
\end{aligned}$$

where we have used (3.26). Altogether (3.28)-(3.30) now lead to

$$\|\nabla\theta(t)\| \le Ch\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\,\big).$$

Combining the estimates for ρ and θ we have the following theorem for smooth data error estimate.

Theorem 3.2.1. *Let u and u_h be the solutions of (3.1) and (3.2), respectively. With $u_0\in \dot H^2(\Omega)$, $f\in C^1(L^2(\Omega))$ and $u_{0h}=R_hu_0$, $f_h=P_hf$, we have for $0\le t\le T$,*

$$\|u(t)-u_h(t)\|+h\,\|\nabla(u(t)-u_h(t))\| \le Ch^2\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\,\big).$$

Remark 3.2.1. *(i) One can improve the estimate of* $\|\nabla\theta(t)\|$ *to* $O(h^2)$ *with an additional factor* $t^{-\alpha/2}$ *by using the estimate*

$$\|\partial_\tau^\alpha \rho(\tau)\| \leq Ch^2 \|\partial_\tau^\alpha u(\tau)\|_2 \leq Ch^2\big(\tau^{-\alpha}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big)$$

in (3.27) together with the estimates (3.29)-(3.30).

(ii) When the initial data $u_0 \in \dot{H}^1(\Omega)$*, the Ritz projection* $R_h u_0$ *is well-defined. Arguing as in Theorem 3.2.1, it is easy to show that for the homogeneous problem (Eq. (2.1) with* $f = 0$*), by Theorem 2.4.1,*

$$\|\nabla(u(t) - u_h(t))\| \leq Cht^{-\alpha/2}\|u_0\|_1, \quad t > 0.$$

Error estimates using the Ritz-Volterra projection. In this part we provide an alternate approach to derive error estimates for smooth initial data, where the Ritz-Volterra projection is used as an intermediate solution to split the error as

$$u(t) - u_h(t) = (u(t) - V_h u(t)) + (V_h u(t) - u_h(t)) = \hat{\rho}(t) + \hat{\theta}(t). \tag{3.31}$$

Using the approximation property of the Ritz-Volterra projection (3.15) and the stability estimate (2.61), the bounds of $\hat{\rho}(t)$ are obtained as follows:

$$\|\hat{\rho}(t)\| + h\|\nabla\hat{\rho}(t)\| \leq Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big). \tag{3.32}$$

It remains to bound $\hat{\theta}(t)$ in the $L^2(\Omega)$ and gradient norms which relies on a bound for the term $\|\partial_t^\alpha\hat{\rho}(t)\|$. Unlike the Ritz projection, in this case $\partial_t^\alpha V_h \neq V_h\partial_t^\alpha$, in general. As a result, the bounds for θ in both $L^2(\Omega)$- and $H^1(\Omega)$-norms are not straightforward from the approximation properties of the Ritz-Volterra projection V_h. The following lemma provides a bound for $\|\partial_t^\alpha\hat{\rho}(t)\|$.

Lemma 3.2.2. *Let* $\hat{\rho}(t)$ *be defined as in (3.31). Then*

$$\|\partial_t^\alpha\hat{\rho}(t)\| + h\|\nabla\partial_t^\alpha\hat{\rho}(t)\| \leq Ch^2\big(t^{-\alpha}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big), \quad t > 0.$$

Proof. We first prove the gradient norm estimate, and then $L^2(\Omega)$-norm estimate follows by duality argument. Taking α-th order Caputo fractional time derivative to (3.14), we have

(3.33)
$$
\begin{aligned}
(\nabla\partial_t^\alpha\hat{\rho}(t),\nabla\chi) &= \sum_{i,j=1}^{d}\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} b_{ij}(x;\tau,\tau)\frac{\partial\hat{\rho}(\tau)}{\partial x_i}\,d\tau,\,\frac{\partial\chi}{\partial x_j}\Big)\\
&+\sum_{i,j=1}^{d}\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\Big(\int_0^\tau \frac{\partial b_{ij}}{\partial\tau}(x;\tau,s)\frac{\partial\hat{\rho}(s)}{\partial x_i}\,ds\Big)\,d\tau,\,\frac{\partial\chi}{\partial x_j}\Big)\\
&+\sum_{i=1}^{d}\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} b_i(x;\tau,\tau)\frac{\partial\hat{\rho}(\tau)}{\partial x_i}\,d\tau,\chi\Big)\\
&+\sum_{i=1}^{d}\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\Big(\int_0^\tau \frac{\partial b_i}{\partial\tau}(x;\tau,s)\frac{\partial\hat{\rho}(s)}{\partial x_i}\,ds\Big)\,d\tau,\,\chi\Big)\\
&+\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} b_0(x;\tau,\tau)\hat{\rho}(\tau)\,d\tau,\chi\Big)\\
&+\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\Big(\int_0^\tau \frac{\partial b_0}{\partial\tau}(x;\tau,s)\hat{\rho}(s)\,ds\Big)\,d\tau,\,\chi\Big)\\
&=: J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}
$$

It is easy to check by the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\|J_1\| &\le Ch\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\big)\,\|\nabla\chi\|\,,\\
\|J_2\| &\le Ch\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\big)\,\|\nabla\chi\|\,.
\end{aligned}
$$

Since $\chi\in S_h$, by the Cauchy-Schwarz inequality and the Poincaré inequality we have

$$\|J_3\|+\|J_4\|\le Ch\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\big)\,\|\nabla\chi\|\,,$$

and

$$\|J_5\|+\|J_6\|\le Ch^2\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\big)\,\|\nabla\chi\|\,.$$

Hence, by (3.33), for any $\chi\in S_h$ we have

(3.34) $$(\nabla\partial_t^\alpha\hat{\rho}(t),\nabla\chi)\le Ch\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\big)\,\|\nabla\chi\|\,.$$

We now split the term $\partial_t^\alpha\hat{\rho}(t)$ as

$$\partial_t^\alpha\hat{\rho}(t)=(\partial_t^\alpha u(t)-\partial_t^\alpha R_h u(t))+(\partial_t^\alpha R_h u(t)-\partial_t^\alpha V_h u(t))=:\hat{\rho}_1+\hat{\theta}_1.$$

By (3.34) and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\|\nabla\partial_t^\alpha\hat{\rho}(t)\|^2 &= (\nabla\partial_t^\alpha\hat{\rho}(t),\nabla\partial_t^\alpha\hat{\rho}(t))\\
&= (\nabla\partial_t^\alpha\hat{\rho}(t),\nabla\hat{\theta}_1(t))+(\nabla\partial_t^\alpha\hat{\rho}(t),\nabla\hat{\rho}_1(t))\\
&\le Ch\big(\,\|u_0\|_2+\|f\|_{C^1(L^2(\Omega))}\big)\,\|\nabla\hat{\theta}_1\|+\|\nabla\partial_t^\alpha\hat{\rho}(t)\|\,\|\nabla\hat{\rho}_1(t)\|\,. \qquad (3.35)
\end{aligned}
$$

Note that,

$$\|\nabla\hat{\rho}_1(t)\| = \|\nabla(\partial_t^\alpha u(t) - R_h\partial_t^\alpha u(t))\| \leq Ch\big(t^{-\alpha}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big).$$

Since $\|\nabla\hat{\theta}_1(t)\| \leq \|\nabla\hat{\rho}_1(t)\| + \|\nabla\partial_t^\alpha\hat{\rho}(t)\|$, an application of Young's inequality to (3.35) yields

$$\begin{aligned}\|\nabla\partial_t^\alpha\hat{\rho}(t)\|^2 &\leq Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big)^2 + Ch^2\big(t^{-\alpha}[\|u_0\|_2 + \|f(0)\|]\\ &\quad + \|f\|_{C^1(L^2(\Omega))}\big)^2 + Ch^2\big(t^{-\alpha}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big)^2\\ &\leq Ch^2\big(t^{-\alpha}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big)^2,\end{aligned}$$

from which the desired estimate for $\|\nabla\partial_t^\alpha\hat{\rho}(t)\|$ follows immediately. Next, to estimate $\partial_t^\alpha\hat{\rho}(t)$ in the $L^2(\Omega)$-norm, we proceed by duality argument. For $\varphi \in L^2(\Omega)$, let $g \in \dot{H}^2(\Omega)$ be the solution of the auxiliary problem

$$\begin{aligned} Ag &= \varphi \quad \text{in } \Omega,\\ g &= 0 \quad \text{on } \partial\Omega,\end{aligned}$$

satisfying the regularity estimate $\|g\|_2 \leq C\,\|\varphi\|$. For $g_h \in S_h$, we have

$$\begin{aligned}(\partial_t^\alpha\hat{\rho}(t), \varphi) &= (\nabla\partial_t^\alpha\hat{\rho}(t), \nabla g)\\ &= (\nabla\partial_t^\alpha\hat{\rho}(t), \nabla(g - g_h) + (\nabla\partial_t^\alpha\hat{\rho}(t), \nabla g_h)\\ &=: L_1 + L_2.\end{aligned} \tag{3.36}$$

With $g_h = R_h g$, the Ritz projector of g, we apply the Cauchy-Schwarz inequality and the gradient norm estimate of $\partial_t^\alpha\hat{\rho}(t)$ to obtain

$$\|L_1\| \leq Ch^2\big(t^{-\alpha}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big)\,\|\varphi\|\,. \tag{3.37}$$

In view of (3.33), the term L_2 can be expressed as

$$L_2 = J_1 + J_2 + J_3 + J_4 + J_5 + J_6,$$

where

$$
\begin{aligned}
J_1 &= \sum_{i,j=1}^{d} \Big(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} b_{ij}(x;\tau,\tau) \frac{\partial \rho(\tau)}{\partial x_i}\, d\tau, \frac{\partial g_h}{\partial x_j}\Big) \\
&= \sum_{i,j=1}^{d} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \Big(b_{ij}(x;\tau,\tau)\frac{\partial \rho(\tau)}{\partial x_i}, \frac{\partial (g_h - g)}{\partial x_j}\Big)\, d\tau \\
&\quad - \sum_{i,j=1}^{d} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \Big(\rho(\tau), \frac{\partial}{\partial x_i}\big(b_{ij}(x;\tau,\tau)\frac{\partial g}{\partial x_j}\big)\Big)\, d\tau \\
&=: J_1^1 + J_1^2.
\end{aligned}
\tag{3.38}
$$

Since $g_h = R_h g$, an easy calculation shows that

$$
\begin{aligned}
\|J_1^1\| &\le Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big)\|\varphi\|, \\
\|J_1^2\| &\le Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big)\|\varphi\|.
\end{aligned}
$$

Using (3.38), it now follows that

$$
\|J_1\| \le Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big)\|\varphi\|.
$$

Analogous to J_1, we can show that

$$
\|J_2\| \le Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big)\|\varphi\|.
$$

Similarly, we split J_i as $J_i = J_i^1 + J_i^2$ for $i = 3, 4, 5, 6$. Using the facts

$$
\|g_h - g\| \le Ch^2\|g\|_2 \le Ch^2\|\varphi\|, \quad \|\nabla g\| \le C\|g\|_2, \text{ and } \|g\| \le C\|g\|_2,
$$

we have for $i = 3, 4, 5, 6$,

$$
\|J_i\| \le \|J_i^1\| + \|J_i^2\| \le Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big)\|\varphi\|.
$$

Thus,

$$
\|L_2\| \le \sum_{i=1}^{6}\|J_i\| \le Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big)\|\varphi\|. \tag{3.39}
$$

Utilizing (3.37), (3.39), and (3.36), it now leads to

$$
\|\partial_t^\alpha \hat{\rho}(t)\| \le Ch^2\big(t^{-\alpha}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big), \tag{3.40}
$$

and this completes the proof. □

We are now ready to bound $\hat{\theta}(t)$ and its gradient in the $L^2(\Omega)$ norms. It is easy to verify that $\hat{\theta}$ satisfies the relation

$$\partial_t^\alpha \hat{\theta}(t) + A_h \hat{\theta}(t) = -P_h \partial_t^\alpha \hat{\rho}(t) + \tilde{B}_h \hat{\theta}(t). \tag{3.41}$$

With $u_{0h} = V_h u_0$, we have $\hat{\theta}(0) = 0$. Then the solution $\hat{\theta}(t)$ for the above equation is represented by

$$\begin{aligned}\hat{\theta}(t) &= \int_0^t \mathbb{E}_h(t-\tau)(-P_h \partial_\tau^\alpha \hat{\rho}(\tau))\,d\tau + \int_0^t \mathbb{E}_h(t-\tau)\tilde{B}_h \hat{\theta}(\tau)\,d\tau \\ &=: I_4 + I_5.\end{aligned}$$

Using (3.9) for $p = 0 = q$, the L^2-stability of P_h, and Lemma 3.2.2 we obtain

$$\begin{aligned}\|I_4\| &\le C \int_0^t (t-\tau)^{-1+\alpha} \|\partial_\tau^\alpha \hat{\rho}(\tau)\|\,d\tau \\ &\le Ch^2[\|u_0\|_2 + \|f(0)\|] \int_0^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,d\tau \\ &\quad + Ch^2 \|f\|_{C^1(L^2(\Omega))} \int_0^t (t-\tau)^{-1+\alpha}\,d\tau \\ &\le Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big).\end{aligned} \tag{3.42}$$

Similarly, taking $p = 1, q = 0$ in (3.9), using the L^2-stability of P_h and Lemma 3.2.2, it follows that

$$\begin{aligned}\|\nabla I_4\| &\le C \int_0^t (t-\tau)^{\alpha/2-1} \|\partial_\tau^\alpha \hat{\rho}(\tau)\|\,d\tau \\ &\le Ch^2[\|u_0\|_2 + \|f(0)\|] \int_0^t (t-\tau)^{\alpha/2-1}\tau^{-\alpha}\,d\tau \\ &\quad + Ch^2 \|f\|_{C^1(L^2(\Omega))} \int_0^t (t-\tau)^{-1+\alpha/2}\,d\tau \\ &\le Ch^2\big(t^{-\alpha/2}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big).\end{aligned} \tag{3.43}$$

The bounds for I_5 and its gradient in the $L^2(\Omega)$-norms are analogous to those of the Ritz projection and hence, they are obtained as

$$\|I_5\| \le C \int_0^t (t-s)^\alpha \|\hat{\theta}(s)\|\,ds \tag{3.44}$$

and

$$\|\nabla I_5\| \le C \int_0^t (t-s)^{\alpha/2} \|\hat{\theta}(s)\|\,ds. \tag{3.45}$$

To bound $\hat{\theta}(t)$ in the $L^2(\Omega)$-norm, we take the help of (3.42), (3.44) and Gronwall's lemma to obtain

$$\|\hat{\theta}(t)\| \leq Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big), \tag{3.46}$$

and for the gradient norm estimate, we use inverse estimate (3.10) and (3.46) to obtain

$$\|\nabla\hat{\theta}(t)\| \leq Ch^{-1}\,\|\hat{\theta}(t)\| \leq Ch\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big). \tag{3.47}$$

Altogether these estimates lead to the following theorem.

Theorem 3.2.2. *Let u be the solution of (3.1) with $f \in C^1(L^2(\Omega))$ and $u_0 \in \dot{H}^2(\Omega)$. Further, let u_h be the solution of the discrete problem (3.2) with $u_{0h} = V_h u_0$ and $f_h = P_h f$. Then there exists a positive constant C such that*

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \leq Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(L^2(\Omega))}\big), \quad 0 \leq t \leq T.$$

Remark 3.2.2. (*i*) *The main advantage of using the Ritz-Volterra projection as a comparison function is that the term $\int_0^t \mathbb{B}(t,s)\hat{\rho}(s)\,ds$ does not appear in the error equation of $\hat{\theta}(t)$ (cf. (3.41)) in contrast to that of the Ritz projection (cf. (3.18)), which greatly simplifies the analysis.*

(*ii*) *The gradient norm error estimate for $\hat{\theta}$ in (3.47) can be improved to $O(h^2)$ with an additional factor $t^{-\alpha/2}$ by utilizing (3.43), (3.45), (3.46), and we obtain*

$$\|\nabla\hat{\theta}(t)\| \leq Ch^2\big(t^{-\alpha/2}[\|u_0\|_2 + \|f(0)\|] + \|f\|_{C^1(L^2(\Omega))}\big).$$

3.2.2 Nonsmooth Data Error Estimates

This section explores the error analysis for the homogeneous SEM with nonsmooth initial data, i.e., when $f = 0$ and $u_0 \in L^2(\Omega)$. Instead of the Ritz projection, we use this time the orthogonal L^2-projection P_h. As before, we decompose the error $u(t) - u_h(t)$ in the following way:

$$u(t) - u_h(t) = (u(t) - P_h u(t)) + (P_h u(t) - u_h(t)) = \tilde{\rho}(t) + \tilde{\theta}(t)$$

With the help of (3.12) and (2.51), $\tilde{\rho}(t)$ is estimated as

$$\|\tilde{\rho}(t)\| + h\,\|\nabla\tilde{\rho}(t)\| \leq Ch^2\,\|u(t)\|_2 \leq Ch^2 t^{-\alpha}\,\|u_0\|, \quad t > 0. \tag{3.48}$$

It now remains to find a bound for the expression $\|\tilde{\theta}(t)\| + h\,\|\nabla\tilde{\theta}(t)\|$. Using the relation $A_h R_h = P_h A$, an easy calculation shows that for $t > 0$,

$$\partial_t^\alpha \tilde{\theta}(t) + A_h\tilde{\theta}(t) = -A_h(R_h u - P_h u)(t) + P_h\tilde{\mathbb{B}}\tilde{\rho}(t) + \tilde{B}_h\tilde{\theta}(t).$$

Choose $u_{0h} = P_h u(0)$ so that $\tilde{\theta}(0) = 0$. By Duhamel's principle

$$\begin{aligned}\tilde{\theta}(t) = &-\int_0^t \mathbb{E}_h(t-\tau)(A_h(R_h u(\tau) - P_h u(\tau)))\,d\tau + \int_0^t \mathbb{E}_h(t-\tau)P_h\tilde{\mathbb{B}}\tilde{\rho}(\tau)\,d\tau \\ &+ \int_0^t \mathbb{E}_h(t-\tau)\tilde{B}_h\tilde{\theta}(\tau)\,d\tau =: I_6 + I_7 + I_8.\end{aligned} \tag{3.49}$$

Following Jin et al. [25, Section 3.3], the bounds for the term I_6 in both $L^2(\Omega)$ and gradient norms are given by

$$\|I_6\| + h\,\|\nabla I_6\| \le Ch^2|\ln h|t^{-\alpha}\,\|u_0\|\,, \quad t > 0. \tag{3.50}$$

Next, we are going to estimate the terms I_7 and I_8 separately. Since the discrete $L^2(\Omega)$-norm of $\int_0^t \mathbb{B}(t,s)\tilde{\rho}(s)\,ds$ is bounded by

$$\|\int_0^t \mathbb{B}(t,s)\tilde{\rho}(s)\,ds\|_{0,h} \le Ch^2 t^{-\alpha}\,\|u_0\|\,, \tag{3.51}$$

with $p = 0, q = 0$ in (3.9) and (3.51), we have for the term I_7

$$\begin{aligned}\|I_7\| &\le C\int_0^t (t-\tau)^{-1+\alpha}\|\int_0^\tau \mathbb{B}(\tau,s)\tilde{\rho}(s)\,ds\|_{0,h}\,d\tau \\ &\le Ch^2\,\|u_0\|\int_0^t (t-\tau)^{-1+\alpha}\tau^{-\alpha}\,d\tau \le Ch^2\,\|u_0\|\,.\end{aligned} \tag{3.52}$$

We apply (3.9) with $p = 1, q = 0$ and (3.51) to obtain

$$\|\nabla I_7\| \le Ch^2\,\|u_0\|\int_0^t (t-\tau)^{-1+\alpha/2}\tau^{-\alpha}\,d\tau \le Ch^2 t^{-\alpha/2}\,\|u_0\|\,. \tag{3.53}$$

Arguing as in the case of smooth data error estimate, it is easy to show that

$$\|I_8\| \le C\int_0^t (t-s)^{\alpha}\,\|\tilde{\theta}(s)\|\,ds, \tag{3.54}$$

and

$$\|\nabla I_8\| \le C\int_0^t (t-s)^{\alpha/2}\,\|\tilde{\theta}(s)\|\,ds. \tag{3.55}$$

Now, using (3.50), (3.52), (3.54), and Gronwall's lemma we conclude

$$\|\tilde{\theta}(t)\| \le Ch^2|\ln h|t^{-\alpha}\,\|u_0\|\,, \quad t > 0. \tag{3.56}$$

Similarly, with the help of (3.50), (3.53), (3.55), and (3.56) we find that

$$\begin{aligned}\|\nabla\tilde{\theta}(t)\| &\le Ch|\ln h|t^{-\alpha}\,\|u_0\| + Ch^2 t^{-\alpha/2}\,\|u_0\| + C\int_0^t (t-s)^{\alpha/2}\,\|\tilde{\theta}(s)\|\,ds \\ &\le Ch|\ln h|t^{-\alpha}\,\|u_0\| + Ch^2|\ln h|\,\|u_0\|\int_0^t t^{\alpha/2}s^{-\alpha}\,ds \\ &\le Ch|\ln h|t^{-\alpha}\,\|u_0\|\,, \quad t > 0.\end{aligned} \tag{3.57}$$

Combining the above estimates for $\tilde{\theta}(t)$ and $\tilde{\rho}(t)$, we obtain the following almost optimal order error estimate for the homogeneous problem with nonsmooth data.

Theorem 3.2.3. *Let u and u_h be the solutions of (3.1) and (3.2), respectively with $f = 0$. Assume that $u_{0h} = P_h u_0$. Then, we have*

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \leq Ch^2 |\ln h| t^{-\alpha}\,\|u_0\|\,, \quad t > 0.$$

3.3 Optimal Error Estimates for Nonsmooth Data

In this section, we derive optimal error bounds for the semidiscrete Galerkin approximation of (3.1) for a particular choice of $B(t,s)$, namely, $B(t,s) = -A$, in both $L^2(\Omega)$- and $H^1(\Omega)$-norms. We use the Laplace transform method to represent the solution of this problem as a contour integral along a suitably chosen contour Γ in the complex plane. In doing so, we can get rid of the log-factor $|\ln h|$ in Theorem 3.2.3 by applying an operator trick introduced by Fujita and Suzuki [15] and which was later adapted in [5, 27]. To begin with, we introduce the sector

$$S_\vartheta = \{z \in \mathbb{C} : |\arg z| < \vartheta\}, \quad 0 < \vartheta \leq \pi,$$

and for $\epsilon > 0$, the contour (see Fig. 3.1)

$$\begin{aligned}\Gamma_{\epsilon,\vartheta} &= \{z \in \mathbb{C} : |z| = \epsilon, |\arg z| \leq \vartheta\} \cup \{z \in \mathbb{C} : z = re^{\mp i\vartheta}, r \geq \epsilon\} \\ &=: \Gamma_1 \cup \Gamma_2.\end{aligned} \tag{3.58}$$

Moreover, we shall need the following well-known estimate for the resolvent of $-A = \Delta$. For $0 < \vartheta < \pi/2$,

$$\|(zI + A)^{-1}\| \leq C|z|^{-1}, \quad z \in S_{\pi-\vartheta}. \tag{3.59}$$

Now, the Laplace transform to (3.1) with $B(t,s) = -A$ produces

$$\Big(z^\alpha I + (1 + \frac{1}{z})A\Big)\widehat{u}(z) = z^{\alpha-1}u_0 + \widehat{f}(z),$$

where $\widehat{u}(z) = \mathcal{L}\{u(t)\}$ and $\widehat{f}(z) = \mathcal{L}\{f(t)\}$ denote the Laplace transforms of $u(t)$ and $f(t)$, respectively. Therefore,

$$\widehat{u}(z) = \frac{l(z)}{z}(l(z)I + A)^{-1}u_0 + \frac{l(z)}{z^\alpha}(l(z)I + A)^{-1}\widehat{f}(z),$$

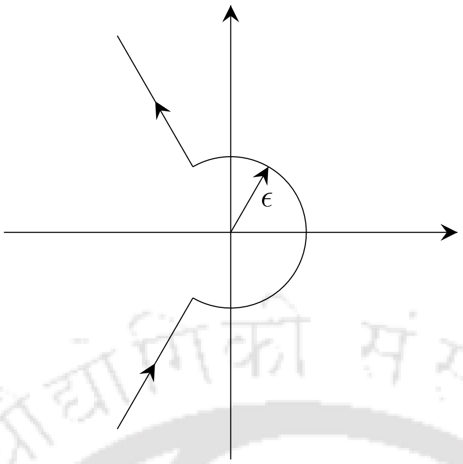


Figure 3.1: Contour $\Gamma_{\epsilon,\vartheta}$

where

$$l(z) = \frac{z^{\alpha+1}}{z+1}. \tag{3.60}$$

Hence, by inverting $\widehat{u}(z)$, the contour integral representation of the solution is

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}\Big[\frac{l(z)}{z}(l(z)I+A)^{-1}u_0 + \frac{l(z)}{z^{\alpha}}(l(z)I+A)^{-1}\widehat{f}(z)\Big]\,dz, \tag{3.61}$$

where $\pi/2 < |\vartheta| < \tilde{\vartheta}$ with $\tilde{\vartheta} = \frac{\pi}{1+\alpha}$. Clearly, $\tilde{\vartheta} \in (\pi/2, \pi)$ and hence the sector $S_{\tilde{\vartheta}}$ contains the right half-plane along with to some extent of the left half-plane of the complex plane. For further development towards the error analysis, it is important to obtain a bound of the function l. The following lemma provides a bound of $l(z)$.

Lemma 3.3.1. *Let the function l be as in (3.60), and let $z \in S_{\tilde{\vartheta}}$ with $\tilde{\vartheta} = \frac{\pi}{1+\alpha}$. Then*

$$l(z) \in S_{\pi}.$$

Furthermore,

$$|l(z)| \le C\min\{|z|^{\alpha}, |z|^{\alpha+1}\}, \quad \textit{for } z \in S_{\tilde{\vartheta}}.$$

Proof. We borrow the idea of [5]. Let $z \in S_{\tilde{\vartheta}}$ with $z = re^{i\vartheta}$. Then $|\vartheta| < \tilde{\vartheta}$. Now,

$$l(z) = \frac{z^{\alpha+1}}{z+1} = \frac{r^{\alpha+1}e^{i(\alpha+1)\vartheta}}{re^{i\vartheta}+1} = \frac{r^{\alpha+1}e^{i(\alpha+1)\vartheta} + r^{\alpha+2}e^{i\alpha\vartheta}}{1+2r\cos\vartheta + r^2} =: l_1(z) + l_2(z).$$

Using the fact $0 < \alpha < 1$, it is obvious that $l_1(z) + l_2(z) \in S_\pi$. Next, we show the estimate of l. Since $|\vartheta| < \tilde{\vartheta}$,

$$|z+1|^2 = (1 + r\cos\vartheta)^2 + r^2\sin^2\vartheta > (1 + r\cos\tilde{\vartheta})^2 + r^2\sin^2\tilde{\vartheta} > r^2\sin^2\tilde{\vartheta},$$

and hence,

$$|l(z)| = \frac{|z|^{\alpha+1}}{|z+1|} < \frac{|z|^{\alpha+1}}{|z|\sin\tilde{\vartheta}} = \frac{|z|^{\alpha}}{\sin\tilde{\vartheta}}.$$

Again, note that

$$|z+1|^2 = 1 + 2r\cos\vartheta + r^2 > 1 + 2r\cos\tilde{\vartheta} + r^2.$$

Since $\cos\tilde{\vartheta}$ is a constant, it is easy to see that the minimum value of the expression $1 + 2r\cos\tilde{\vartheta} + r^2$ is $\sin^2\tilde{\vartheta}$. Therefore,

$$|l(z)| = \frac{|z|^{\alpha+1}}{|z+1|} < \frac{|z|^{\alpha+1}}{\sin\tilde{\vartheta}},$$

which completes the proof of the lemma. $\square$

Now, Lemma 3.3.1 together with the resolvent estimate (3.59) yields

$$\|(l(z)I + A)^{-1}\| \le C|l(z)|^{-1}, \quad z \in S_{\tilde{\vartheta}}. \tag{3.62}$$

Next, following the approach of Fujita and Suzuki [15] and Bazhlekova et al. [5], we remove the $|\ln h|$ factor in Theorem 3.2.3. For this purpose, the following lemma is crucial for the rest of our work.

Lemma 3.3.2. *Assume that* $z \in S_{\tilde{\vartheta}}$ *and* $\Psi \in H_0^1(\Omega)$. *Then*

$$|l(z)|\,\|\Psi\|^2 + \|\nabla\Psi\|^2 \le C\big|\, l(z)\,\|\Psi\|^2 + \|\nabla\Psi\|^2\,\big|.$$

Proof. Let $z \in S_{\tilde{\vartheta}}$. Then by Lemma 3.3.1, $l(z) \in S_\pi$. We apply [15, Lemma 7.1] to complete the proof. $\square$

Set $v = (l(z)I + A)^{-1}u_0$ and its discrete analogue by $v_h = (l(z)I + A_h)^{-1}P_h u_0$, where $A_h = -\Delta_h$. Also, set $W = (l(z)I + A)^{-1}\widehat{f}(z)$ and $W_h = (l(z)I + A_h)^{-1}P_h\widehat{f}(z)$, $z \in S_{\tilde{\vartheta}}$. Proceeding as in [5, Lemma 3.4], the difference between v and v_h, W and W_h in the $L^2(\Omega)$ and gradient norms are estimated in the following lemmas. The details are thus omitted.

Lemma 3.3.3. *Assume that $z \in S_{\tilde{\vartheta}}$ and the initial data $u_0 \in L^2(\Omega)$. Then*

$$\|v - v_h\| + h\,\|\nabla(v - v_h)\| \leq Ch^2\,\|u_0\|\,.$$

Lemma 3.3.4. *For $z \in S_{\tilde{\vartheta}}$, the following estimate holds:*

$$\|W - W_h\| + h\,\|\nabla(W - W_h)\| \leq Ch^2\,\|\widehat{f}(z)\|\,.$$

Analogous to the continuous case, the discrete solution $u_h(t)$ is given by

$$u_h(t) = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}\Big[\frac{l(z)}{z}(l(z)I + A_h)^{-1}u_h(0) + \frac{l(z)}{z^\alpha}(l(z)I + A_h)^{-1}P_h\widehat{f}(z)\Big]\,dz, \tag{3.63}$$

where $u_h(0)$ is an approximation of u_0 and $\pi/2 < |\vartheta| < \tilde{\vartheta}$.

In the following theorem, we prove the error estimates which are independent of the factor $|\ln h|$.

Theorem 3.3.1. *Assume that $u_0 \in L^2(\Omega)$ and $f = 0$. Let u and u_h be the corresponding solutions given by (3.61) and (3.63), respectively. Then*

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \leq Ch^2 t^{-\alpha}\,\|u_0\|\,,\quad t > 0.$$

Proof. By virtue of (3.61) and (3.63), for $\widehat{f} = 0$, we have

$$u(t) - u_h(t) = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}\frac{l(z)}{z}(v - v_h)\,dz,$$

where $v = (l(z)I + A)^{-1}u_0$ and $v_h = (l(z)I + A_h)^{-1}P_h u_0$. With $\epsilon = \frac{1}{t}$, an application of Lemma 3.3.3 and Lemma 3.3.1 yield

$$\begin{aligned}
\|u(t) - u_h(t)\| &\leq Ch^2\,\|u_0\|\Big(\int_{\Gamma_1}|e^{zt}\frac{l(z)}{z}|\,|dz| + \int_{\Gamma_2}|e^{zt}\frac{l(z)}{z}|\,|dz|\Big)\\
&\leq Ch^2\,\|u_0\|\Big(\int_{-\vartheta}^{\vartheta} e^{\cos\sigma}(\frac{1}{t})^\alpha\,d\sigma + \int_{1/t}^{\infty} r^{\alpha-1}e^{rt\cos\vartheta}\,dr\Big)\\
&\leq Ch^2 t^{-\alpha}\,\|u_0\|\,.
\end{aligned}$$

Similarly, we can show that $\|\nabla(u(t) - u_h(t))\| \leq Cht^{-\alpha}\,\|u_0\|$. The above two estimates constitute the proof of the theorem. □

In the following, we provide an intermediate error bounds for the nonhomogeneous problem where the bounds are expressed in terms of $\widehat{f}$ rather than f.

Let $X := \Gamma_{\epsilon,\vartheta} + \mathbb{R}_+$ be the right-hand side part of the contour $\Gamma_{\epsilon,\vartheta}$ (see Fig. 3.1) in the complex plane. We assume that the Laplace transform $\widehat{f}(z)$ of $f(t)$ is analytic and bounded on $\overline{X}$. For $\tilde{\vartheta} \in (\pi/2, \pi)$ and $X \subset S_{\tilde{\vartheta}}$, we define

$$\|\widehat{f}\|_X := \sup_{z\in X} \|\widehat{f}(z)\| .$$

Then, by the maximum principle, we conclude that

$$\|\widehat{f}\|_X = \|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} . \tag{3.64}$$

Theorem 3.3.2. *With* $u_0 \in \dot{H}^2(\Omega)$ *and* $u_{0h} = R_h u_0$, *the Ritz projector of* u_0, *let* u *and* u_h *be represented by (3.61) and (3.63), respectively. Then for* $t > 0$, *we have*

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \le Ch^2\big(\,\|u_0\|_2 + t^{-2}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\,\big),$$

where the contour $\Gamma_{\epsilon,\vartheta}$ *is given by (3.58).*

Proof. In view of (3.61) and (3.63), the difference $u(t) - u_h(t)$ is given by

$$u(t) - u_h(t) =: T_1 + T_2,$$

where

$$T_1 = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}\frac{l(z)}{z}(v - v_h)\,dz, \quad \text{and} \quad T_2 = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}\frac{l(z)}{z^\alpha}(W - W_h)\,dz,$$

where $v = (l(z)I + A)^{-1}u_0$, $v_h = (l(z)I + A_h)^{-1}R_h u_0$ and $W = (l(z)I + A)^{-1}\widehat{f}(z)$, $W_h = (l(z)I + A_h)^{-1}P_h\widehat{f}(z)$. To bound the term T_1, we use the following identities:

$$\frac{l(z)}{z}(l(z)I + A)^{-1} = \frac{1}{z}(I - (l(z)I + A)^{-1}A),$$

and

$$A_h R_h = P_h A.$$

Following the arguments of Theorem 3.3.1, it is easy to show that

$$\|T_1\| + h\,\|\nabla T_1\| \le Ch^2\,\|u_0\|_2\,, \quad \|T_2\| + h\,\|\nabla T_2\| \le Ch^2 t^{-2}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} .$$

The result follows from the estimates of T_1 and T_2. □

Remark 3.3.1. *The order of singularity in Theorem 3.3.2 can be reduced from* t^{-2} *to* t^{-1} *by choosing* $|l(z)| \le C|z|^\alpha$ *in Lemma 3.3.1. Then, one obtains* $\|T_2\| + h\,\|\nabla T_2\| \le Ch^2 t^{-1}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}$.

3.4 Numerical Illustrations

In this section, we verify numerically our theoretical rates of convergence of the finite element solution for the semidiscrete scheme. Numerical experiments presented below include both smooth and nonsmooth initial data cases. Let $t_j = jk$, $j = 0, 1, \dots, N$ be a uniform partition of the time interval $[0, T]$ with time step size k, where T is the time of interest. We take the time step size k small during the spatial discretization to negligible the temporal discretization error. For our numerical experiments, we take $\Omega = (0, 1)^2$ and consider the following two cases:

(a) Smooth initial data: $u_0(x, y) = \sin(\pi x)\sin(\pi y)$ and the source function $f(x, y, t) = 1$.

(b) Nonsmooth initial data $u_0(x, y) = \chi_{(1/2,1)\times(0,1)}$ and $f(x, y, t) = 0$, where χ_S denotes the characteristic function on the set S.

Table 3.1: The $L^2(\Omega)$ and $H^1(\Omega)$-errors and CRs for the case (a) at $T = 0.01$ with $k = 5 \times 10^{-4}$ for various mesh size h and for different α

$h\diagdown\alpha = 1/10$	$L^2(\Omega)$-norm	$H^1(\Omega)$-norm	$L^2(\Omega)$-CR	$H^1(\Omega)$-CR
$1/20$	8.27206e-03	3.47017e-01	–	–
$1/40$	2.09455e-03	1.88804e-01	1.98161	0.87811
$1/80$	5.25322e-04	9.96484e-02	1.99536	0.92196
$1/160$	1.31436e-04	5.23127e-02	1.99884	0.92968
$\alpha = 5/10$	7.26200e-03	3.83680e-01	–	–
	1.82977e-03	2.03989e-01	1.98870	0.91141
	4.58347e-04	1.06323e-01	1.99715	0.94003
	1.14644e-04	5.54443e-02	1.99929	0.93935
$\alpha = 9/10$	5.83103e-03	3.08148e-01	–	–
	1.46916e-03	1.61058e-01	1.98876	0.93604
	3.68020e-04	8.26353e-02	1.99713	0.96275
	9.20512e-05	4.28593e-02	1.99928	0.94715

To verify the spatial convergence rates (CRs) of the semidiscrete solution, we fix time step size $k = 5 \times 10^{-4}$, and perform numerical experiments for the case (a) with mesh size $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}$ and for the case (b) with $h = \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}$. Table 3.1 shows the CRs for the errors in $L^2(\Omega)$ and $H^1(\Omega)$-norms for different values of the fractional order derivative α at time $T = 0.01$. From Table 3.1, we observe that the numerical CRs agree with the theoretical CRs of Theorems 3.2.1, 3.2.2, and 3.3.2. Table 3.2 contains the results for the case (b). Optimal CRs for the errors in both $L^2(\Omega)$ and $H^1(\Omega)$-norms at the time of interest $T = 0.01$ and $T = 0.001$ with $\alpha = 0.5$ are demonstrated which validate the theoretical CRs of Theorems 3.2.3 and 3.3.1.

Table 3.2: The $L^2(\Omega)$ and $H^1(\Omega)$-errors and CRs for the case (b) for $\alpha = 0.5$ at $T = 0.01$ and $T = 0.001$ with $k = 5 \times 10^{-4}$ for various mesh size h

$h\diagdown T = 0.01$	$L^2(\Omega)$-norm	$H^1(\Omega)$-norm	$L^2(\Omega)$-CR	$H^1(\Omega)$-CR
1/40	7.49812e-03	3.24271e-01	–	–
1/80	2.31520e-03	1.78410e-01	1.69539	0.86200
1/160	6.44194e-04	9.15200e-02	1.84557	0.96304
1/320	1.68894e-04	4.57219e-02	1.93138	1.00120
$T = 0.001$	1.72270e-03	6.77893e-02	–	–
	4.84994e-04	3.60570e-02	1.82863	0.91077
	1.28071e-04	1.81201e-02	1.92102	0.99269
	3.26725e-05	9.00333e-03	1.97080	1.00906

3.5 Concluding Remarks

In this chapter, we have studied the convergence analysis based on the GFEM by using piecewise linear functions of the SEM problem in a bounded convex polygonal domain. The error analysis is carried out for both smooth and nonsmooth initial data cases. For smooth initial data, we have derived optimal error bounds in both $L^2(\Omega)$- and $H^1(\Omega)$-norms by employing both Ritz and Ritz-Volterra projections. In the case of nonsmooth initial data, relying only on the L^2-projection, almost optimal error bounds are established for the homogeneous problem. Further, we recover optimal order error bounds in both $L^2(\Omega)$- and $H^1(\Omega)$-norms for nonsmooth initial data for a particular choice of the kernel $B(t,s)$, namely, $B(t,s) = -A$. Numerical tests confirm the theoretical rates of convergence.

4

Error Analysis by Energy Arguments

In this chapter we investigate the a priori error analysis of a semidiscrete GFEM for problem (1.5) by energy arguments. Since the energy method is the most elementary technique in the a priori analysis, therefore, it is natural to study the a priori error analysis for SEM (1.5) by energy arguments. The convergence analysis covers both smooth and nonsmooth initial data. Since the solution u of (1.5) has singularity at $t = 0$ even for smooth initial data, we use t^j, $j = 1, 2$, type of weights along with time integral operator in the error analysis. The use of energy arguments enable us to prove optimal order error estimates in both $L^2(\Omega)$- and $H^1(\Omega)$-norms with respect to both convergence order of the approximate solution and regularity of the initial data. The present approach gets rid of the log-factor $|\ln h|$ from the nonsmooth data error analysis (cf. Theorem 3.2.3) even for a general second-order operator $B(t,s)$ as in (1.3). In addition, we also establish a quasi-optimal pointwise in time error bound in the maximum norm for smooth initial data.

4.1 Introduction

To begin with, we first recall the IBVP for the SEM, i.e.,

$$
\begin{aligned}
&\partial_t^\alpha u(t) + Au(t) = \int_0^t B(t,s)u(s)\,ds + f(t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \quad \alpha \in (0,1),\\
&u(t) = 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \qquad\qquad (4.1)\\
&u(0) = u_0(x) \quad \text{in} \quad \Omega,
\end{aligned}
$$

where the notations have the same meanings as in the previous chapters.

This chapter investigates convergence analysis of the semidiscrete GFEM using energy arguments. Optimal order error bounds for both smooth and nonsmooth initial data are derived. We also prove a quasi-optimal pointwise in time error bound in the maximum norm for smooth initial data. A novel Ritz-Volterra projection is used to derive optimal error estimates. For the SE, i.e., when $B(t,s) = 0$ in (4.1), Karaa et al. [29] and Mustapha [51] have studied the energy arguments and derived optimal error bounds in both $L^2(\Omega)$- and $H^1(\Omega)$-norms covering both smooth and nonsmooth initial data cases. Further, when $\alpha = 1$ in (4.1), i.e., for the PIDE, Goswami et al. [17] have derived optimal $L^2(\Omega)$-norm error estimate based on energy arguments in conjunction with repeated use of time integration operator for the semidiscrete GFEM with nonsmooth initial data. However, developing energy arguments for (4.1), i.e., in the presence of both integral term and Caputo fractional derivative, is not straightforward. This chapter attempts to extend the analysis of [29] to SEM (4.1). To the best of authors' knowledge, this work is reported for first time in the literature.

The plan of this chapter is as follows. In Section 4.2, we introduce some notations and preliminaries which are useful in the convergence analysis. The error analysis is carried out in Section 4.3. Section 4.4 contains numerical experiments to support the theoretical rates of convergence.

4.2 Notations and Preliminary Results

Let Riemann-Liouville fractional integral operator $\mathcal{I}^\alpha$ of order $\alpha \in (0,\infty)$ be defined by

$$\mathcal{I}^\alpha g(t) = \int_0^t k_\alpha(t-\tau)g(\tau)\,d\tau, \quad k_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

The properties of the above operator play a vital role in the convergence analysis. In the subsequent parts of this chapter we use $\mathcal{I}^{1-\alpha}u'(t)$ for the Caputo fractional derivative $\partial_t^\alpha u(t)$ in (4.1). For the integral operators $\mathcal{I}^\alpha$ and $\mathcal{I}^\beta$, the following semigroup property

$$\mathcal{I}^\alpha \mathcal{I}^\beta = \mathcal{I}^{\alpha+\beta}, \quad \alpha, \beta \in (0,\infty)$$

holds. Next, we recall some interesting properties of the operator $\mathcal{I}^\alpha$, $\alpha \in (0,1)$. Let $g : [0,T] \to L^2(\Omega)$ be a piecewise time continuous function. Then, by [52, Lemma 3.1(ii)], the following positivity property holds for $\mathcal{I}^\alpha$:

$$\int_0^T (\mathcal{I}^\alpha g, g)\,dt \geq C_\alpha \int_0^T \left\|\mathcal{I}^{\alpha/2} g\right\|^2 dt \geq 0, \quad \alpha \in (0,1),\ C_\alpha = \cos(\alpha\pi/2). \tag{4.2}$$

Set $g' = \partial_t g$. For piecewise time continuous function $g' : [0,T] \to L^2(\Omega)$, we have the following important functional inequality between g and g', follows from [34, Lemma 2.1] and the positivity property (4.2) of $\mathcal{I}^{1-\alpha}$:

$$\|g(t) - g(0)\|^2 \leq Ct^\alpha \int_0^t (\mathcal{I}^{1-\alpha} g', g')\,ds, \quad t > 0. \tag{4.3}$$

In addition to the above, we require the following continuity property of $\mathcal{I}^\alpha$ [52, Lemma 3.1(iii)]: For g, $\phi \in L^2(0,T;L^2(\Omega))$ and any positive ϵ,

$$\int_0^t (\mathcal{I}^{1-\alpha} g, \phi)\, ds \le \epsilon \int_0^t (\mathcal{I}^{1-\alpha} g, g)\, ds + \frac{1}{4\epsilon(1-\alpha)^2} \int_0^t (\mathcal{I}^{1-\alpha}\phi, \phi)\, ds. \tag{4.4}$$

For notational convenience, we set $g_1(t) = tg(t)$ and $g_2(t) = t^2 g(t)$. The following three identities will be used in our subsequent analysis. For a proof, one may refer to [29, 51]:

$$t\mathcal{I}^\alpha g(t) = \mathcal{I}^\alpha g_1(t) + \alpha \mathcal{I}^{\alpha+1} g(t), \tag{4.5}$$

$$t\mathcal{I}^\alpha g'(t) = \mathcal{I}^\alpha (g_1)'(t) + (\alpha - 1)\mathcal{I}^\alpha g(t) - tk_\alpha(t) g(0), \tag{4.6}$$

$$t^2\mathcal{I}^\alpha g'(t) = \mathcal{I}^\alpha (g_2)'(t) + 2(\alpha - 1)\mathcal{I}^\alpha g_1(t) + \alpha(\alpha-1)\mathcal{I}^{\alpha+1} g(t) - t^2 k_\alpha(t) g(0). \tag{4.7}$$

We now state some relations among the integer order integral operators. We introduce the symbols

$$\begin{aligned}
\hat{g}(t) &= \mathcal{I}g(t) = \int_0^t g(s)\, ds\\
\hat{\hat{g}}(t) &= \mathcal{I}^2 g(t) = \int_0^t \int_0^s g(\tau)\, d\tau\, ds\\
\hat{\hat{\hat{g}}}(t) &= \mathcal{I}^3 g(t) = \int_0^t \int_0^s \int_0^\tau g(\mu)\, d\mu\, d\tau\, ds.
\end{aligned}$$

Then, we have the following identity

$$t\hat{g}(t) = \hat{g}_1(t) + \hat{\hat{g}}(t). \tag{4.8}$$

Observe that when $\alpha \to 1^-$, (4.5) coincides with (4.8). Further, with the help of (4.8), it is easy to obtain

$$t^2\hat{g}(t) = \hat{g}_2(t) + 2\hat{\hat{g}}_1(t) + 2\hat{\hat{\hat{g}}}(t). \tag{4.9}$$

For u and v in $H_0^1(\Omega)$, it is easy to obtain

$$|B(t,s;u,v)| \le C \left\|\nabla u\right\| \left\|\nabla v\right\|, \quad 0 \le s \le t \le T, \tag{4.10}$$

where $B(t,s;\cdot,\cdot)$ is the bilinear form associated to the operator $B(t,s)$. For convenience, we recall from Chapter 3 the following semidiscrete problem and approximation properties of the Ritz-Volterra projection. The semidiscrete GFEM reads: Find $u_h(t) \in S_h$ such that for $0 < t \le T$,

$$\begin{aligned}
(\partial_t^\alpha u_h(t), \chi) + (\nabla u_h(t), \nabla \chi) &= \int_0^t B(t,s;u_h(s),\chi)\, ds + (f(t),\chi) \quad \forall \chi \in S_h,\\
u_h(0) &= u_{0h} \in S_h.
\end{aligned} \tag{4.11}$$

Here u_{0h} is an approximation of u_0 in S_h and in the convergence analysis, we take $u_{0h} = P_h u_0$, where $P_h : L^2(\Omega) \to S_h$ is the standard L^2-projection defined by

$$(P_h v, \chi) = (v, \chi) \quad \text{for all } \chi \in S_h.$$

We also need the Ritz-Volterra projection $V_h \colon C(H_0^1(\Omega)) \to C(S_h)$ defined by

$$(\nabla(V_h v - v)(t), \nabla \chi) = \int_0^t B(t, s; (V_h v - v)(s), \chi)\, ds, \ \ \forall \chi \in S_h,$$

with the following approximation properties [6, 36, 54]: For $p = 1, 2$

$$\|V_h v(t) - v(t)\| + h\, \|\nabla(V_h v(t) - v(t))\| \le Ch^p \Big(\|v(t)\|_p + \int_0^t \|v(s)\|_p \, ds \Big), \tag{4.12}$$

and

$$\|(V_h v(t) - v(t))'\| + h\, \|\nabla(V_h v(t) - v(t))'\| \le Ch^p \Big(\|v(t)\|_p + \|v'(t)\|_p + \int_0^t \|v(s)\|_p \, ds \Big). \tag{4.13}$$

In addition to the above, for the maximum norm estimate, we need the following discrete Sobolev inequality

$$\|\chi\|_{L^\infty(\Omega)} \le C |\ln h|^{1/2}\, \|\nabla \chi\| \quad \text{for all} \quad \chi \in S_h. \tag{4.14}$$

4.3 Convergence Analysis

In this section we estimate the error between the semidiscrete solution and the continuous solution in both $L^2(\Omega)$- and $H^1(\Omega)$-norms. The convergence properties are proved allowing both smooth and nonsmooth initial data u_0. We split the error $u_h(t) - u(t)$ in a canonical way as follows

$$u_h(t) - u(t) = (u_h(t) - V_h u(t)) - (u(t) - V_h u(t)) = \vartheta(t) - \rho(t).$$

Using (4.12) with $p = 2$ and (2.61), the bounds of $\rho(t)$ for smooth initial data are

$$\|\rho(t)\| + h\, \|\nabla \rho(t)\| \le Ch^2 \big(\|u_0\|_2 + \|f\|_{C^1(L^2)} \big). \tag{4.15}$$

Similarly, for nonsmooth data u_0, (4.12) with $p = 2$ and (2.51) with $p = 2, q = 0$ lead to

$$\|\rho(t)\| + h\, \|\nabla \rho(t)\| \le Ch^2 t^{-\alpha}\, \|u_0\|, \ \ t > 0. \tag{4.16}$$

Thus, we are left with the job to bound $\|\vartheta\|$ and $\|\nabla\vartheta\|$ for both smooth and nonsmooth initial data. It is easy to verify that $\vartheta = u_h - V_h u \in S_h$ satisfies the following relation for $0 < t \le T$,

$$(\mathcal{I}^{1-\alpha}\vartheta', \chi) + (\nabla\vartheta, \nabla\chi) = -(\mathcal{I}^{1-\alpha}\rho', \chi) + \int_0^t B(t, s; \vartheta(s), \chi)\, ds, \quad \forall \chi \in S_h, \tag{4.17}$$

where we have used $\mathcal{I}^{1-\alpha}\vartheta' = \partial_t^\alpha \vartheta$ and $\mathcal{I}^{1-\alpha}\rho' = \partial_t^\alpha \rho$. For the convenience, we set

$$\vartheta_i(t) = t^i \vartheta(t), \quad i = 1, 2,$$

and state some lemmas involving $\vartheta(t)$, $\vartheta_1(t)$, and $\vartheta_2(t)$. Below, we first prove a lemma involving ϑ which will be used later.

Lemma 4.3.1. *Let ϑ satisfy (4.17). Then*

$$\int_0^t (\mathcal{I}^{1-\alpha}\hat{\vartheta}, \hat{\vartheta})\, ds + \|\nabla\hat{\hat{\vartheta}}(t)\|^2 \le C \int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho}, \hat{\rho})\, ds, \quad 0 \le t \le T.$$

Proof. To prove the assertion we will start from error Eq. (4.17). Integrate it with respect to time, utilize the identity $\mathcal{I}^{2-\alpha}g'(t) = \mathcal{I}^{1-\alpha}g(t) - k_{2-\alpha}(t)g(0)$, and recall the definition of the L^2-projection to arrive at

$$\begin{aligned}(\mathcal{I}^{1-\alpha}\vartheta, \chi) + (\mathcal{I}\nabla\vartheta, \nabla\chi) &= (\mathcal{I}^{1-\alpha}\rho - k_{2-\alpha}(t)(\rho(0) - \vartheta(0), \chi) \\ &\quad + \int_0^t \int_0^s B(s, \tau; \vartheta(\tau), \chi)\, d\tau\, ds \\ &= (\mathcal{I}^{1-\alpha}\rho, \chi) + \int_0^t \int_0^s B(s, \tau; \vartheta(\tau), \chi)\, d\tau\, ds.\end{aligned}$$

Again integrate this with respect to time and then choose $\chi = \hat{\vartheta}(t)$ to get

$$\begin{aligned}(\mathcal{I}^{1-\alpha}\hat{\vartheta}, \hat{\vartheta}(t)) + (\nabla\hat{\hat{\vartheta}}, \nabla\hat{\vartheta}(t)) &= (\mathcal{I}^{1-\alpha}\hat{\rho}, \hat{\vartheta}(t)) + \int_0^t \int_0^s \int_0^\tau B(\tau, \mu; \vartheta(\mu), \hat{\vartheta}(t))\, d\mu\, d\tau\, ds \\ &=: (\mathcal{I}^{1-\alpha}\hat{\rho}, \hat{\vartheta}(t)) + T_1.\end{aligned} \tag{4.18}$$

Now, we concentrate on the term T_1. Observe that

$$\int_0^\tau B(\tau, \mu; \vartheta(\mu), \hat{\vartheta}(t))\, d\mu = B(\tau, \tau; \hat{\vartheta}(\tau), \hat{\vartheta}(t)) - \int_0^\tau B_\mu(\tau, \mu; \hat{\vartheta}(\mu), \hat{\vartheta}(t))\, d\mu.$$

Integrating the above twice, we obtain

$$
\begin{aligned}
T_1 = {} & \int_0^t B(s,s;\hat{\hat{\vartheta}}(s),\hat{\vartheta}(t))\,ds - 2\int_0^t\int_0^s B_\tau(\tau,\tau;\hat{\hat{\vartheta}}(\tau),\hat{\vartheta}(t))\,d\tau\,ds \\
& + \int_0^t\int_0^s\int_0^\tau B_{\tau\mu}(\tau,\mu;\hat{\hat{\vartheta}}(\mu),\hat{\vartheta}(t))\,d\mu\,d\tau\,ds \\
= {} & -B(t,t;\hat{\hat{\vartheta}}(t),\hat{\hat{\vartheta}}(t)) - \frac{d}{dt}\Big(\int_0^t B(s,s;\hat{\hat{\vartheta}}(s),\hat{\hat{\vartheta}}(t))\,ds\Big) + 2\int_0^t B_s(s,s;\hat{\hat{\vartheta}}(s),\hat{\hat{\vartheta}}(t))\,ds \\
& - 2\frac{\partial}{\partial t}\Big(\int_0^t\int_0^s B_\tau(\tau,\tau;\hat{\hat{\vartheta}}(\tau),\hat{\hat{\vartheta}}(t))\,d\tau\,ds\Big) - \int_0^t\int_0^s B_{s\tau}(\tau,\tau;\hat{\hat{\vartheta}}(\tau),\hat{\hat{\vartheta}}(t))\,d\tau\,ds \\
& + \frac{\partial}{\partial t}\Big(\int_0^t\int_0^s\int_0^\tau B_{\tau\mu}(\tau,\mu;\hat{\hat{\vartheta}}(\mu),\hat{\hat{\vartheta}}(t))\,d\mu\,d\tau\,ds\Big).
\end{aligned}
$$

Insert this expression for T_1 in (4.18), integrate over $(0,t)$, use the continuity property (4.4) of the operator $\mathcal{I}^{1-\alpha}$ (with $\epsilon = 1/2$), invoke the bound (4.10), and finally apply Young's inequality to yield

$$
\begin{aligned}
\int_0^t (\mathcal{I}^{1-\alpha}\hat{\vartheta},\hat{\vartheta})\,ds + \int_0^t (\nabla\hat{\hat{\vartheta}},\nabla\hat{\vartheta})\,ds \le \frac{1}{2}\int_0^t (\mathcal{I}^{1-\alpha}\hat{\vartheta},\hat{\vartheta})\,ds + C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds \\
+ C_\epsilon \|\nabla\hat{\hat{\vartheta}}(t)\|^2 + C\int_0^t \|\nabla\hat{\hat{\vartheta}}(s)\|^2\,ds.
\end{aligned}
$$

Note that,

$$
(\nabla\hat{\hat{\vartheta}},\nabla\hat{\vartheta}) = \frac{1}{2}\frac{d}{dt}(\nabla\hat{\hat{\vartheta}},\nabla\hat{\hat{\vartheta}}).
$$

Choose C_ϵ such that $(1 - C_\epsilon) > 0$. An application of Gronwall's lemma leads to

$$
\int_0^t (\mathcal{I}^{1-\alpha}\hat{\vartheta},\hat{\vartheta})\,ds + \|\nabla\hat{\hat{\vartheta}}(t)\|^2 \le C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds,
$$

which completes the proof. □

Lemma 4.3.2. *Let $\vartheta_1 = t\vartheta$, $\rho_1 = t\rho$, and $\hat{\rho}(t) = \int_0^t \rho(s)\,ds$. Further, let ϑ satisfy the equality (4.17). Then for $0 \le t \le T$, we have*

$$
\int_0^t (\mathcal{I}^{1-\alpha}\vartheta_1,\vartheta_1)\,ds + \|\nabla\hat{\vartheta}_1(t)\|^2 \le C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds + C\int_0^t (\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,ds.
$$

Proof. Multiply both sides of (4.17) by t, use the identity (4.6), and the definition of the L^2-projection to obtain

$$
\begin{aligned}
(\mathcal{I}^{1-\alpha}\vartheta_1',\chi) + (\nabla\vartheta_1,\nabla\chi) = {} & \alpha(\mathcal{I}^{1-\alpha}\vartheta,\chi) + (\mathcal{I}^{1-\alpha}\rho_1' - \alpha\mathcal{I}^{1-\alpha}\rho,\chi) \\
& + t\int_0^t B(t,s;\vartheta(s),\chi)\,ds, \quad \chi \in S_h,
\end{aligned}
$$

where $(\rho(0)-\vartheta(0),\chi)=0$. Now, integrate the above equality in time and put $\chi=\vartheta_1(t)$ to reach

(4.19)

$$\begin{aligned}(\mathcal{I}^{1-\alpha}\vartheta_1,\vartheta_1(t))+(\nabla\hat{\vartheta}_1,\nabla\vartheta_1(t)) &= \alpha(\mathcal{I}^{1-\alpha}\hat{\vartheta},\vartheta_1(t))+(\mathcal{I}^{1-\alpha}\rho_1-\alpha\mathcal{I}^{1-\alpha}\hat{\rho},\vartheta_1(t))\\ &\quad+\int_0^t s\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_1(t))\,d\tau\,ds\\ &=: \alpha(\mathcal{I}^{1-\alpha}\hat{\vartheta},\vartheta_1(t))+(\mathcal{I}^{1-\alpha}\rho_1-\alpha\mathcal{I}^{1-\alpha}\hat{\rho},\vartheta_1(t))+T_2.\end{aligned}$$

We now proceed to bound the term T_2. Using the identity (4.8) we get

$$\begin{aligned}s\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_1(t))\,d\tau &= \int_0^s B(s,\tau;\vartheta_1(\tau),\vartheta_1(t))\,d\tau\\ &\quad+\int_0^s\int_0^\tau B(\tau,\mu;\vartheta(\mu),\vartheta_1(t))\,d\mu\,d\tau.\end{aligned}$$

Following the T_1 term calculation as in Lemma 4.3.1 we obtain

$$\begin{aligned}T_2 &= \int_0^t\int_0^s B(s,\tau;\vartheta_1(\tau),\vartheta_1(t))\,d\tau\,ds+\int_0^t\int_0^s\int_0^\tau B(s,\mu;\vartheta(\mu),\vartheta_1(t))\,d\mu\,d\tau\,ds\\ &= \int_0^t B(s,s;\hat{\vartheta}_1(s),\vartheta_1(t))\,ds-\int_0^t\int_0^s B_\tau(s,\tau;\hat{\vartheta}_1(\tau),\vartheta_1(t))\,d\tau\,ds\\ &\quad+\int_0^t B(s,s;\hat{\hat{\vartheta}}(s),\vartheta_1(t))\,ds-2\int_0^t\int_0^s B_\tau(s,\tau;\hat{\hat{\vartheta}}(\tau),\vartheta_1(t))\,d\tau\,ds\\ &\quad+\int_0^t\int_0^s\int_0^\tau B_{\mu\mu}(s,\mu;\hat{\hat{\vartheta}}(\mu),\vartheta_1(t))\,d\mu\,d\tau\,ds\\ &= -B(t,t;\hat{\vartheta}_1(t),\hat{\vartheta}_1(t))+\frac{\partial}{\partial t}\int_0^t B(s,s;\hat{\vartheta}_1(s),\hat{\vartheta}_1(t))\,ds+\int_0^t B_\tau(t,\tau;\hat{\vartheta}_1(\tau),\hat{\vartheta}_1(t))\,d\tau\\ &\quad-\frac{\partial}{\partial t}\Big(\int_0^t\int_0^s B_\tau(s,\tau;\hat{\vartheta}_1(\tau),\hat{\vartheta}_1(t))\,d\tau\,ds\Big)-B(t,t;\hat{\hat{\vartheta}}(t),\hat{\vartheta}_1(t))\\ &\quad+\frac{\partial}{\partial t}\int_0^t B(s,s;\hat{\hat{\vartheta}}(s),\hat{\vartheta}_1(t))\,ds+2\int_0^t B_\tau(t,\tau;\hat{\hat{\vartheta}}(\tau),\hat{\vartheta}_1(t))\,d\tau\\ &\quad-2\frac{\partial}{\partial t}\Big(\int_0^t\int_0^s B_\tau(s,\tau;\hat{\hat{\vartheta}}(\tau),\hat{\vartheta}_1(t))\,d\tau\,ds\Big)-\int_0^t\int_0^\tau B_{\mu\mu}(t,\mu;\hat{\hat{\vartheta}}(\mu),\hat{\vartheta}_1(t))\,d\mu\,d\tau\\ &\quad+\frac{\partial}{\partial t}\Big(\int_0^t\int_0^s\int_0^\tau B_{\mu\mu}(s,\mu;\hat{\hat{\vartheta}}(\mu),\hat{\vartheta}_1(t))\,d\mu\,d\tau\,ds\Big).\end{aligned}$$

Insert this expression for T_2 in (4.19), integrate over $(0,t)$, use the continuity property of the operator $\mathcal{I}^{1-\alpha}$ with a suitable ϵ, invite the bound (4.10), and then apply Young's inequality to obtain

$$\begin{aligned}\int_0^t (\mathcal{I}^{1-\alpha}\vartheta_1,\vartheta_1)\,ds+\int_0^t(\nabla\hat{\vartheta}_1,\nabla\vartheta_1)\,ds &\le \frac{1}{2}\int_0^t(\mathcal{I}^{1-\alpha}\vartheta_1,\vartheta_1)\,ds + C\int_0^t(\mathcal{I}^{1-\alpha}\hat{\vartheta},\hat{\vartheta})\,ds\\ &\quad + C_\epsilon\left\|\nabla\hat{\vartheta}_1(t)\right\|^2 + C\int_0^t\left\|\nabla\hat{\vartheta}_1(s)\right\|^2 ds + C\int_0^t\left\|\nabla\hat{\hat{\vartheta}}(s)\right\|^2 ds.\end{aligned}$$

Using the fact

$$(\nabla\hat{\vartheta}_1,\nabla\vartheta_1)=\frac{1}{2}\frac{d}{dt}(\nabla\hat{\vartheta}_1,\nabla\hat{\vartheta}_1),$$

we choose the number C_ϵ such that $(1-C_\epsilon)>0$. Finally, an application of Lemma 4.3.1 and Gronwall's lemma completes the rest of the proof. ☐

Lemma 4.3.3. *Let ϑ satisfy the identity (4.17). With $\vartheta_i = t^i\vartheta$, $\rho_i = t^i\rho$ for $i=1,2$, and $\hat{\rho}(t)=\int_0^t\rho(s)\,ds$, we have for $t>0$,*

$$\|\vartheta(t)\|^2 + t^\alpha\|\nabla\vartheta(t)\|^2 \le Ct^{\alpha-4}\Big(\int_0^t\Big[(\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})+(\mathcal{I}^{1-\alpha}\rho_1,\rho_1)+(\mathcal{I}^{1-\alpha}\lambda,\lambda)\Big]\,ds\Big),$$

where $\lambda(t)=\rho_2'(t)-2\alpha\rho_1(t)-\alpha(1-\alpha)\hat{\rho}(t)$.

Proof. Multiply both sides of (4.17) by t^2, use the identity (4.7), and the definition of the L^2-projection to reach

$$\begin{aligned}(\mathcal{I}^{1-\alpha}\vartheta_2',\chi)+(\nabla\vartheta_2,\nabla\chi) &= 2\alpha(\mathcal{I}^{1-\alpha}\vartheta_1,\chi)+\alpha(1-\alpha)(\mathcal{I}^{1-\alpha}\hat{\vartheta},\chi)+(\mathcal{I}^{1-\alpha}\lambda,\chi)\\ &\quad + t^2\int_0^t B(t,s;\vartheta(s),\chi)\,ds, \quad \text{for each } \chi\in S_h,\end{aligned}$$

where $\lambda(t)=\rho_2'(t)-2\alpha\rho_1(t)-\alpha(1-\alpha)\hat{\rho}(t)$ and $(\rho(0)-\vartheta(0),\chi)=0$. Since the above identity holds true for each $\chi\in S_h$, choose $\chi=\vartheta_2'(t)$, integrate from 0 to t in time, and use the continuity property of the operator $\mathcal{I}^{1-\alpha}$ with a suitable ϵ to obtain

$$\begin{aligned}&\int_0^t(\mathcal{I}^{1-\alpha}\vartheta_2',\vartheta_2')\,ds+\int_0^t(\nabla\vartheta_2,\nabla\vartheta_2')\,ds \le \frac{1}{2}\int_0^t(\mathcal{I}^{1-\alpha}\vartheta_2',\vartheta_2')\,ds + C\int_0^t(\mathcal{I}^{1-\alpha}\vartheta_1,\vartheta_1)\,ds\\ &+C\int_0^t(\mathcal{I}^{1-\alpha}\hat{\vartheta},\hat{\vartheta})\,ds + C\int_0^t(\mathcal{I}^{1-\alpha}\lambda,\lambda)\,ds + \int_0^t s^2\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_2'(s))\,d\tau\,ds.\end{aligned}$$

Using the fact

$$2\int_0^t(\nabla\vartheta_2,\nabla\vartheta_2')\,ds = \|\nabla\vartheta_2(t)\|^2,$$

the inequality (4.3), and Lemmas 4.3.1-4.3.2 in the above inequality to have

(4.20)

$$
\begin{aligned}
t^{-\alpha}\left\|\vartheta_2(t)\right\|^2+\left\|\nabla\vartheta_2(t)\right\|^2 \le C\int_0^t(\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds + C\int_0^t(\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,ds \\
+ C\int_0^t(\mathcal{I}^{1-\alpha}\lambda,\lambda)\,ds + \int_0^t s^2\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_2'(s))\,d\tau\,ds.
\end{aligned}
$$

Now it is suffices to obtain a bound for the last term in (4.20). Let

$$
T_3 := \int_0^t s^2\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_2'(s))\,d\tau\,ds.
$$

Integrating by parts we find that

$$
\begin{aligned}
T_3 = \int_0^t s^2\frac{\partial}{\partial s}\Big(\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_2(s))\,d\tau\Big)\,ds - \int_0^t s^2\int_0^s B_s(s,\tau;\vartheta(\tau),\vartheta_2(s))\,d\tau\,ds \\
- \int_0^t s^2 B(s,s;\vartheta(s),\vartheta_2(s))\,ds.
\end{aligned}
$$

Again integrating the first term by parts we obtain

(4.21)

$$
\begin{aligned}
T_3 &= t^2\int_0^t B(t,s;\vartheta(s),\vartheta_2(t))\,ds - 2t\int_0^t\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_2(s))\,d\tau\,ds \\
&\quad + 2\int_0^t\int_0^s\int_0^\tau B(\tau,\mu;\vartheta(\mu),\vartheta_2(\tau))\,d\mu\,d\tau\,ds - \int_0^t s^2\int_0^s B_s(s,\tau;\vartheta(\tau),\vartheta_2(s))\,d\tau\,ds \\
&\quad - \int_0^t s^2 B(s,s;\vartheta(s),\vartheta_2(s))\,ds \\
&=: T_3^1 + T_3^2 + T_3^3 + T_3^4 + T_3^5.
\end{aligned}
$$

Utilizing the identity (4.9) we get

$$
\begin{aligned}
T_3^1 = \int_0^t B(t,s;\vartheta_2(s),\vartheta_2(t))\,ds + 2\int_0^t\int_0^s B(t,\tau;\vartheta_1(\tau),\vartheta_2(t))\,d\tau\,ds \\
+ 2\int_0^t\int_0^s\int_0^\tau B(t,\mu;\vartheta(\mu),\vartheta_2(t))\,d\mu\,d\tau\,ds.
\end{aligned}
$$

Repeating the steps of the term T_1 as in Lemma 4.3.1 we write

$$\begin{aligned}
T_3^1 &= \int_0^t B(t,s;\vartheta_2(s),\vartheta_2(t))\,ds + 2\int_0^t B(t,s;\hat{\vartheta}_1(s),\vartheta_2(t))\,ds \\
&\quad - 2\int_0^t\int_0^s B(t,\tau;\hat{\vartheta}_1(\tau),\vartheta_2(t))\,d\tau\,ds + 2\int_0^t\int_0^s B(t,\tau;\hat{\vartheta}(\tau),\vartheta_2(t))\,d\tau\,ds \\
&\quad - 2\int_0^t\int_0^s\int_0^\tau B_\mu(t,\mu;\hat{\vartheta}(\mu),\vartheta_2(t))\,d\mu\,d\tau\,ds \\
&= \int_0^t B(t,s;\vartheta_2(s),\vartheta_2(t))\,ds + 2\int_0^t B(t,s;\hat{\vartheta}_1(s),\vartheta_2(t))\,ds \\
&\quad - 2\int_0^t\int_0^s B(t,\tau;\hat{\vartheta}_1(\tau),\vartheta_2(t))\,d\tau\,ds + 2\int_0^t B(t,s;\hat{\hat{\vartheta}}(s),\vartheta_2(t))\,ds \\
&\quad - 4\int_0^t\int_0^s B_\tau(t,\tau;\hat{\hat{\vartheta}}(\tau),\vartheta_2(t))\,d\tau\,ds + 2\int_0^t\int_0^s\int_0^\tau B_{\mu\mu}(t,\mu;\hat{\hat{\vartheta}}(\mu),\vartheta_2(t))\,d\mu\,d\tau\,ds.
\end{aligned}$$

Now the estimate (4.10), Young's inequality, and Lemmas 4.3.1-4.3.2 lead to

$$\begin{aligned}
|T_3^1| &\le C_\epsilon \|\nabla\vartheta_2(t)\|^2 + C\int_0^t \|\nabla\hat{\vartheta}_1(s)\|^2\,ds + C\int_0^t \|\nabla\hat{\hat{\vartheta}}(s)\|^2\,ds \\
&\le C_\epsilon \|\nabla\vartheta_2(t)\|^2 + C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds + C\int_0^t (\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,ds.
\end{aligned}$$

Use the identity (4.8) to have

$$T_3^2 = -2\int_0^t s\int_0^s B(s,\tau;\vartheta(\tau),\vartheta_2(s))\,d\tau\,ds - 2\int_0^t\int_0^s\int_0^\tau B(\tau,\mu;\vartheta(\mu),\vartheta_2(\tau))\,d\mu\,d\tau\,ds.$$

Similar calculations as of T_2 term in Lemma 4.3.2 and Lemmas 4.3.1-4.3.2 yield

$$\begin{aligned}
|T_3^2| &\le C\int_0^t \|\nabla\hat{\vartheta}_1(s)\|^2\,ds + C\int_0^t \|\nabla\hat{\hat{\vartheta}}(s)\|^2\,ds + C\int_0^t \|\nabla\vartheta_2(s)\|^2\,ds \\
&\le C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds + C\int_0^t (\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,ds + C\int_0^t \|\nabla\vartheta_2(s)\|^2\,ds.
\end{aligned}$$

The computations of bounds for T_3^3 and T_3^4 terms are analogous to T_3^1. Invoking Lemmas 4.3.1-4.3.2 it follows that

$$|T_3^3| \le C_\epsilon \|\nabla\vartheta_2(t)\|^2 + C\int_0^t \|\nabla\hat{\hat{\vartheta}}(s)\|^2\,ds \le C_\epsilon \|\nabla\vartheta_2(t)\|^2 + C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds,$$

and

$$|T_3^4| \le C_\epsilon \|\nabla\vartheta_2(t)\|^2 + C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho},\hat{\rho})\,ds + C\int_0^t (\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,ds.$$

Finally, using the bound (4.10) we obtain

$$|T_3^5| \leq C\int_0^t \|\nabla\vartheta_2(s)\|^2 \, ds.$$

Now we combine all the bounds for T_3^i, $i = 1, \dots, 5$ to have

$$|T_3| \leq C_\epsilon \|\nabla\vartheta_2(t)\|^2 + C\int_0^t \|\nabla\vartheta_2(s)\|^2 \, ds + C\int_0^t (\mathcal{I}^{1-\alpha}\hat{\rho}, \hat{\rho})\, ds + C\int_0^t (\mathcal{I}^{1-\alpha}\rho_1, \rho_1)\, ds.$$

Insert the bound of T_3 in (4.20) and choose C_ϵ such that $(1 - C_\epsilon) > 0$. Invite Gronwall's lemma to complete (as $\vartheta_2(t) = t^2\vartheta(t)$) the proof. □

Now, we are ready to bound the finite element term ϑ in both $L^2(\Omega)$- and $H^1(\Omega)$-norms for both smooth and nonsmooth initial data. In all cases, the bounds follow from Lemma 4.3.3. By the Cauchy-Schwarz inequality we get

(4.22)

$$\begin{aligned} \|\vartheta(t)\|^2 + t^\alpha \|\nabla\vartheta(t)\|^2 &\leq Ct^{\alpha-4}\int_0^t \Big(\|\mathcal{I}^{1-\alpha}\hat{\rho}\| \, \|\hat{\rho}\| + \|\mathcal{I}^{1-\alpha}\rho_1\| \, \|\rho_1\| + \|\mathcal{I}^{1-\alpha}\lambda\| \, \|\lambda\| \Big)\, ds \\ &\leq Ct^{\alpha-4}\int_0^t \Big(\|\mathcal{I}^{1-\alpha}\hat{\rho}\| + \|\mathcal{I}^{1-\alpha}\rho_1\| + \|\mathcal{I}^{1-\alpha}\rho_2'\| \Big)\Big(\|\hat{\rho}\| + \|\rho_1\| + \|\rho_2'\| \Big)\, ds \\ &=: T_4. \end{aligned}$$

For smooth initial data, using the approximation property (4.13) and Theorem 2.4.3 with $p = 2$ we find

$$T_4 \leq Ct^{\alpha-4}h^4\big(\|u_0\|_2 + \|f\|_{C^1(\dot{H}^1)} \big)^2 \int_0^t s^{(2-\alpha)+1}\, ds \leq Ch^4\big(\|u_0\|_2 + \|f\|_{C^1(\dot{H}^1)} \big)^2,$$

which implies

$$\|\vartheta(t)\| + t^{\alpha/2}\|\nabla\vartheta(t)\| \leq Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(\dot{H}^1)} \big). \tag{4.23}$$

Next, for nonsmooth initial data, by (4.13) with $p = 2$ and Theorem 2.3.1 we obtain

$$T_4 \leq Ct^{\alpha-4}h^4 \|u_0\|^2 \int_0^t s^{(2-2\alpha)+(1-\alpha)}\, ds \leq Ch^4t^{-2\alpha}\|u_0\|^2,$$

which yields

$$\|\vartheta(t)\| + t^{\alpha/2}\|\nabla\vartheta(t)\| \leq Ch^2t^{-\alpha}\|u_0\|, \quad t > 0. \tag{4.24}$$

Again, for nonsmooth data, if we apply the estimate (4.13) with $p = 1$, then we bound the term T_4 as

$$T_4 \leq Ct^{\alpha-4} h^2 \|u_0\|^2 \int_0^t s^{(2-3\alpha/2)+(1-\alpha/2)}\, ds \leq Ch^2 t^{-\alpha} \|u_0\|^2 ,$$

which leads to

$$\|\nabla\vartheta(t)\| \leq Cht^{-\alpha} \|u_0\| . \tag{4.25}$$

With the above preparation, we are now ready to establish the error between the continuous solution u to problem (4.1) and its semidiscrete approximation u_h for both smooth and nonsmooth initial data.

Theorem 4.3.1. *Assume that* $u_0 \in \dot{H}^2(\Omega)$, $f \in C^1(\dot{H}^1)$ *and* $u_{0h} = P_h u_0$, $f_h = P_h f$. *Let* u *be the solution of continuous problem (4.1), and let* u_h *be that of spatially semidiscrete problem (4.11). Then,*

$$\|u(t) - u_h(t)\| \leq Ch^2\big(\|u_0\|_2 + \|f\|_{C^1(\dot{H}^1)} \big), \quad 0 \leq t \leq T,$$

$$\|\nabla(u(t) - u_h(t))\| \leq Ch \max\{1, ht^{-\alpha/2}\}\big(\|u_0\|_2 + \|f\|_{C^1(\dot{H}^1)} \big), \quad 0 < t \leq T.$$

Proof. The proof immediately follows from the triangle inequality and estimates (4.15) and (4.23). □

Remark 4.3.1. *Since the triangulation is quasi-uniform, as a result of the inverse estimate* $\|\nabla\chi\| \leq Ch^{-1} \|\chi\|$, *we obtain* $\|\nabla(u(t) - u_h(t))\| \leq Ch(\|u_0\|_2 + \|f\|_{C^1(\dot{H}^1)})$.

Theorem 4.3.2. *Assume that* $u_0 \in L^2(\Omega)$ *and* $u_{0h} = P_h u_0$. *Let* u *and* u_h *be, respectively, the solutions of continuous problem (4.1) and spatially semidiscrete problem (4.11) with* $f = 0$. *Then,*

$$\|u(t) - u_h(t)\| + h \|\nabla(u(t) - u_h(t))\| \leq Ch^2 t^{-\alpha} \|u_0\| , \quad 0 < t \leq T.$$

Proof. Using the decomposition $u_h - u = \vartheta - \rho$, the proof follows from the triangle inequality and estimates (4.16), (4.24), and (4.25). □

Remark 4.3.2. *Theorems 4.3.1-4.3.2 extend the results of SE [29, 51] to SEM. Further, it is noteworthy that when* $\alpha \to 1^-$, *the* $L^2(\Omega)$*-error estimate in Theorem 4.3.2 is similar to that of [17, Theorem 1.1].*

Maximum norm error estimate. In this section we derive a quasi-optimal pointwise in time $L^\infty(\Omega)$-error bound for the semidiscrete approximation (4.11) for smooth initial data. We assume that $\Omega \subset \mathbb{R}^2$. Recall that

$$u_h(t) - u(t) = (u_h(t) - V_h u(t)) - (u(t) - V_h u(t)) = \tilde{\vartheta}(t) - \tilde{\rho}(t).$$

Hence

$$\|u(t)-u_h(t)\|_{L^\infty(\Omega)} \le \|\tilde{\vartheta}(t)\|_{L^\infty(\Omega)} + \|\tilde{\rho}(t)\|_{L^\infty(\Omega)}\,.$$

In view of discrete Sobolev inequality (4.14) and gradient estimate in (4.23) we obtain for positive time

$$(4.26)\qquad \|\tilde{\vartheta}(t)\|_{L^\infty(\Omega)} \le C|\ln h|^{1/2}\,\|\nabla\tilde{\vartheta}(t)\| \le Ch^2|\ln h|^{1/2}t^{-\alpha/2}\Big(\|u_0\|_2 + \|f\|_{C^1(\dot{H}^1)}\Big).$$

Thus, it suffices to bound $\|\tilde{\rho}(t)\|_{L^\infty(\Omega)}$. We use the results of [36] and interpolation theory to arrive

$$\|\tilde{\rho}(t)\|_{L^\infty(\Omega)} \le C_\epsilon h^{2-\epsilon}\Big(\|u(t)\|_{2-\epsilon,\infty} + \int_0^t \|u(s)\|_{2-\epsilon,\infty}\,ds\Big),\quad \epsilon>0,$$

where $\|\cdot\|_{p,\infty}$ denotes the norm of the standard Sobolev space $W^{p,\infty}(\Omega)$. Using the Sobolev embedding theorem $\|u(t)\|_{2-\epsilon,\infty} \le C_\epsilon\,\|u(t)\|_{H^3(\Omega)}$, we bound $\|\tilde{\rho}(t)\|_{L^\infty(\Omega)}$ as

$$(4.27)\qquad \|\tilde{\rho}(t)\|_{L^\infty(\Omega)} \le C_\epsilon h^{2-\epsilon}\Big(\|u(t)\|_{H^3(\Omega)} + \int_0^t \|u(s)\|_{H^3(\Omega)}\,ds\Big).$$

We now bound $\|u(t)\|_{H^3(\Omega)}$. From the elliptic regularity estimate, Eq. (4.1), and (2.62) we get

$$\begin{aligned}\|u(t)\|_{H^3(\Omega)} &\le C\,\|Au(t)\|_{H^1(\Omega)} \le C\Big(\|\partial_t^\alpha u(t)\|_1 + \|f(t)\|_1 + \int_0^t \|B(t,s)u(s)\|_{H^1(\Omega)}\,ds\Big)\\ &\le C\Big(t^{-\alpha/2}(\|u_0\|_2 + \|f(0)\|) + \|f\|_{C^1(L^2)}\Big) + \|f(t)\|_1 + C\int_0^t \|u(s)\|_{H^3(\Omega)}\,ds\\ &\le Ct^{-\alpha/2}\,\|u_0\|_2 + C\,\|f\|_{C^1(H^1)} + C\int_0^t \|u(s)\|_{H^3(\Omega)}\,ds.\end{aligned}$$

Note that, the singularity in the above on $f(0)$ can be removed (see Remark 2.4.2), as $f(0)\in \dot{H}^1(\Omega)$. Invoking Gronwall's lemma we obtain

$$\|u(t)\|_{H^3(\Omega)} \le C\big(t^{-\alpha/2}\,\|u_0\|_2 + \|f\|_{C^1(H^1)}\big),\quad t>0.$$

Inserting this bound in (4.27) we reach

$$(4.28)\qquad \|\tilde{\rho}(t)\|_{L^\infty(\Omega)} \le C_\epsilon h^{2-\epsilon}\big(t^{-\alpha/2}\,\|u_0\|_2 + \|f\|_{C^1(H^1)}\big),\quad t>0.$$

With an aid of the bounds of $\tilde{\vartheta}(t)$ and $\tilde{\rho}(t)$, we have the following theorem for the error estimate in the $L^\infty(\Omega)$-norm.

Theorem 4.3.3. *Let $\Omega \subset \mathbb{R}^2$. Assume that $u_0 \in \dot{H}^2(\Omega)$, $f \in C^1(\dot{H}^1)$ and $u_{0h} = P_h u_0$, $f_h = P_h f$. Let u be the solution of continuous problem (4.1), and let u_h be that of spatially semidiscrete problem (4.11). Then, for $0 < t \leq T$*

$$\|u(t) - u_h(t)\|_{L^\infty(\Omega)} \leq C_\epsilon h^{2-\epsilon} |\ln h|^{1/2} t^{-\alpha/2} \big(\|u_0\|_2 + \|f\|_{C^1(H^1)} \big).$$

Proof. The proof follows from the triangle inequality and estimates (4.26) and (4.28). □

Remark 4.3.3. *For nonsmooth initial data, i.e., $u_0 \in L^2(\Omega)$, the $L^\infty(\Omega)$-norm error estimate with the Ritz-Volterra projection as a comparison function is not possible as we need $H^3(\Omega)$ regularity of the solution to bound $\tilde{\rho} = u - V_h u$, whereas the solution cannot be smoother than $H^2(\Omega)$ for positive time, see Remark 2.2.2 of Chapter 2.*

4.4 Numerical Illustrations

In this part we perform some numerical experiments to validate the convergence results established in Theorems 4.3.1-4.3.3. In our computation, we consider $\Omega = (0,1)^2$ and carry out the numerical experiments with three test problems. Let $t_j = jk$, $j = 0, 1, \ldots, N$ be a uniform partition of the time interval $[0, T]$ with time step size k, where T is the time of interest. We take the time step size k small during the spatial discretization to negligible the temporal discretization error. Below, Examples 4.1 and 4.2, respectively, verify numerically the convergence rate (CR) for $L^2(\Omega)$-, $H^1(\Omega)$-, and $L^\infty(\Omega)$-norms, and Example 4.3 validates the same for both $L^2(\Omega)$- and $H^1(\Omega)$-norms.

Example 4.1. *Consider the following problem with the memory kernel as the identity operator, i.e., $B(t,s) = I$:*

$$\begin{aligned}
\partial_t^\alpha u(t) - \Delta u(t) &= \int_0^t u(s)\, ds + f(t) \quad \text{in} \quad \Omega, \quad t \in (0,T], \quad \alpha \in (0,1),\\
u(x,0,t) &= 0 = u(x,1,t) \quad \text{for} \quad x \in [0,1], \quad t \in (0,T],\\
u(0,y,t) &= 0 = u(1,y,t) \quad \text{for} \quad y \in [0,1], \quad t \in (0,T],\\
u(x,y,0) &= \sin(\pi x)\sin(\pi y) \quad \text{in} \quad \Omega,
\end{aligned}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. With the source function $f(x,y,t) = (\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\pi^2(t^2+1) - (t + \frac{t^3}{3}))\sin(\pi x)\sin(\pi y)$, the exact solution is given by $u(x,y,t) = (t^2+1)\sin(\pi x)\sin(\pi y)$.

To compute the CRs of the discrete solution, we employ the backward Euler convolution quadrature scheme to discretize both fractional derivative and Volterra integral term. We compute numerical solutions with mesh size $h = 1/10, 1/20, 1/40, 1/80$ at the time of interest $T = 0.1$ with $\alpha = 0.55$ and a constant time step size $k = 0.0005$. We demonstrate the CRs for $L^2(\Omega)$-, $H^1(\Omega)$-, and $L^\infty(\Omega)$-norms. Table 4.1 shows the CRs in $L^2(\Omega)$-, $H^1(\Omega)$-, and $L^\infty(\Omega)$-norms which validate the theoretical CRs of Theorems 4.3.1 and 4.3.3.

Table 4.1: The $L^2(\Omega)$-, $H^1(\Omega)$-, and $L^\infty(\Omega)$-errors and CRs for Example 4.1 at $T = 0.1$ with $\alpha = 0.55$ and $k = 0.0005$

$h \diagdown \alpha = 0.55$	L^2-norm	H^1-norm	L^∞-norm	L^2-CR	H^1-CR	L^∞-CR
1/10	8.88149e-03	3.46124e-01	2.22819e-02	–	–	–
1/20	2.26615e-03	1.75417e-01	5.61473e-03	1.97056	0.98049	1.98859
1/40	5.61455e-04	8.80081e-02	1.38677e-03	2.01300	0.99508	2.01749
1/80	1.32071e-04	4.40417e-02	3.25957e-04	2.08786	0.99876	2.08898

Example 4.2. *Consider the following IBVP with $B(t,s) = ts\Delta$ in the unknown u:*

$$\begin{aligned} \partial_t^\alpha u(t) - \Delta u(t) &= \int_0^t ts\Delta u(s)\,ds + f(t) \quad \text{in} \quad \Omega, \quad t \in (0,T], \quad \alpha \in (0,1), \\ u(x,0,t) &= 0 = u(x,1,t) \quad \text{for} \quad x \in [0,1], \quad t \in (0,T], \\ u(0,y,t) &= 0 = u(1,y,t) \quad \text{for} \quad y \in [0,1], \quad t \in (0,T], \\ u(x,y,0) &= \sin(\pi x)\sin(\pi y) \quad \text{in} \quad \Omega. \end{aligned}$$

Choose the source function $f(x,y,t) = (\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\pi^2(t^2+1) + 2\pi^2(\frac{t^3}{2} + \frac{t^5}{4}))\sin(\pi x)\sin(\pi y)$ *such that the exact solution is given by* $u(x,y,t) = (t^2+1)\sin(\pi x)\sin(\pi y)$.

To compute the CRs of the semidiscrete solution, we apply the backward Euler convolution quadrature scheme to discretize the fractional derivative and the (left) rectangular rule for the integral term. We perform numerical experiments with mesh size $h = 1/5, 1/10, 1/20, 1/40$ with a constant $k = 0.0005$ at $T = 0.01$ and for the fractional order $\alpha = 0.35, 0.55, 0.75$. The empirical CRs in Table 4.2 for $L^2(\Omega)$-, $H^1(\Omega)$-, and $L^\infty(\Omega)$-norms are consistent with the theoretical CRs of Theorems 4.3.1 and 4.3.3.

Table 4.2: The $L^2(\Omega)$-, $H^1(\Omega)$-, and $L^\infty(\Omega)$-errors and CRs for Example 4.2 at $T = 0.01$ with $\alpha = 0.35, 0.55, 0.75$ and $k = 0.0005$

$h\diagdown\alpha = 0.35$	L^2-norm	H^1-norm	L^∞-norm	L^2-CR	H^1-CR	L^∞-CR
1/5	2.95865e-02	6.50904e-01	7.28892e-02	–	–	–
1/10	8.14368e-03	3.42789e-01	2.04331e-02	1.86119	0.92512	1.83480
1/20	2.10394e-03	1.73704e-01	5.21296e-03	1.95259	0.98069	1.97074
1/40	5.50182e-04	8.71456e-02	1.35893e-03	1.93511	0.99513	1.93963
$\alpha = 0.55$	2.49862e-02	6.52727e-01	6.18854e-02	–	–	–
	6.78462e-03	3.43038e-01	1.70277e-02	1.88079	0.92811	1.86171
	1.74272e-03	1.73735e-01	4.31827e-03	1.96093	0.98147	1.97936
	4.51392e-04	8.71492e-02	1.11495e-03	1.94889	0.99533	1.95348
$\alpha = 0.75$	1.69396e-02	6.58551e-01	4.26484e-02	–	–	–
	4.49217e-03	3.43845e-01	1.12878e-02	1.91492	0.93753	1.91773
	1.14382e-03	1.73839e-01	2.83516e-03	1.97356	0.98400	1.99326
	2.92733e-04	8.71624e-02	7.23110e-04	1.96620	0.99597	1.97114

Example 4.3. *Consider the following homogeneous IBVP with* $B(t,s) = e^{-\pi^2(t-s)}\Delta$ *in the unknown* u*:*

$$\begin{aligned}
\partial_t^\alpha u(t) - \Delta u(t) &= \int_0^t e^{-\pi^2(t-s)}\Delta u(s)\,ds \quad \mathit{in} \quad \Omega, \;\; t \in (0,T], \;\; \alpha \in (0,1),\\
u(x,0,t) &= 0 = u(x,1,t) \quad \mathit{for} \quad x \in [0,1], \quad t \in (0,T],\\
u(0,y,t) &= 0 = u(1,y,t) \quad \mathit{for} \quad y \in [0,1], \quad t \in (0,T],\\
u(x,y,0) &= u_0(x,y) \quad \mathit{in} \quad \Omega,
\end{aligned}$$

where the initial condition $u_0(x,y)$ *is given by*

$$u_0(x,y) = \chi_{(1/2,1)\times(0,1)}.$$

Here the exact solution is unknown.

In order to compute errors for both $L^2(\Omega)$- and $H^1(\Omega)$-norms and the related CRs, we calculate the discrete solution (discretizing the fractional derivative and the integral term as in Example 4.1) with mesh size $h = 1/20, 1/40, 1/80, 1/160, 1/320$ with a fixed time step size $k = 0.00002$. Since the nonsmooth initial data error estimate reflects a singularity at $t = 0$ (see Theorem 4.3.2), we particularly interested in the behavior of the errors when T is close to zero. For this purpose, we choose $T = 0.001$ in our computation. The empirical CRs of the errors in both $L^2(\Omega)$- and $H^1(\Omega)$-norms are listed in Table 4.3 which validate the theoretical CRs of Theorem 4.3.2.

Table 4.3: The $L^2(\Omega)$- and $H^1(\Omega)$-errors and CRs for Example 4.3 at $T = 0.001$ with $\alpha = 0.35, 0.55, 0.75$ and $k = 0.00002$

$h \diagdown \alpha = 0.35$	L^2-norm	H^1-norm	L^2-CR	H^1-CR
1/40	1.19900e-03	4.48736e-02	–	–
1/80	3.16727e-04	2.33723e-02	1.92052	0.94106
1/160	8.10442e-05	1.16394e-02	1.96646	1.00578
1/320	2.03453e-05	5.77652e-03	1.99401	1.01075
$\alpha = 0.55$	2.52952e-03	9.68415e-02	–	–
	6.85072e-04	5.08080e-02	1.88453	0.93057
	1.77321e-04	2.53730e-02	1.94989	1.00176
	4.47661e-05	1.25941e-02	1.98588	1.01055
$\alpha = 0.75$	5.65607e-03	2.16012e-01	–	–
	1.57299e-03	1.14010e-01	1.84629	0.92194
	4.08456e-04	5.67961e-02	1.94526	1.00530
	1.03102e-04	2.81395e-02	1.98610	1.01320

4.5 Concluding Remarks

In this chapter, we studied the a priori error analysis of a semidiscrete GFEM by energy arguments applied to the SEM in a bounded convex polygonal domain. The present analysis deals with both smooth and nonsmooth data cases. Energy arguments in conjunction with repeated application of the time integral operators are used to derive optimal order error bounds in both $L^2(\Omega)$- and $H^1(\Omega)$-norms. The derived results are optimal with respect to the regularity of the initial function as well as the convergence order of the approximate solution for both $L^2(\Omega)$- and $H^1(\Omega)$-norms. Further, we have established a quasi-optimal pointwise in time error bound in the $L^\infty(\Omega)$-norm for smooth initial data. Extensive numerical illustrations confirm our theoretical convergence results.

5

Fully Discrete Schemes

This chapter studies the error estimates for the fully discrete finite element approximations for problem (3.1) with $B(t,s) = -A$. We consider two fully discrete schemes based on the convolution quadrature, namely, the backward Euler (BE) and the second-order backward difference (SBD). The convergence analysis is analyzed for both smooth and nonsmooth data cases and optimal error bounds in time are derived in the $L^\infty(L^2)$-norm for both schemes.

5.1 Introduction

We first recall the SEM with $B(t,s) = -A$, i.e.,

$$\begin{aligned} &\partial_t^\alpha u(t) + Au(t) + \int_0^t Au(s)\,ds = f(t) \quad \text{in} \quad \Omega, \quad 0 < t \leq T, \quad 0 < \alpha < 1, \\ &u(t) = 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \leq T, \\ &u(0) = u_0(x) \quad \text{in} \quad \Omega. \end{aligned} \tag{5.1}$$

We now briefly discuss the convolution quadrature. This numerical method was first introduced by Lubich in [38, 39] and subsequently used by many authors in [5, 11, 27, 28, 40, 41, 61, 64]. The main advantage of using the convolution quadrature is that it allows the discretization of the derivative and the integral term in (5.1) simultaneously. However, this is not the case for the standard temporal discretization where the derivative and the integral terms are discretized separately. This procedure requires a higher temporal regularity of the solution u in the error analysis which problem (5.1) does not possess even for smooth initial data (see Theorems 2.4.1, 2.4.2, and 2.4.3). The convolution quadrature framework provides excellent stability properties and higher order accuracy in time. It also permits omission of the term corresponding to $t = 0$ during discretization keeping the convergence rate unchanged.

Our study to (5.1) is motivated by the works of Bazhlekova et al. [5] and Jin et al. [27]. In [5], the authors have studied the Rayleigh-Stokes problem for a generalized second-grade fluid. Both semidiscrete and fully discrete schemes are discussed and optimal order error bounds are established covering both smooth and nonsmooth initial data cases. The semidiscrete scheme is based on the piecewise linear GFEM in space and the fully discrete scheme is based on the convolution quadrature in time with the generating functions given by the BE and SBD methods. In [27], the authors have considered and analyzed the fully discrete BE and SBD schemes for the SE and derived error estimates of first-order and second-order accuracy in time, respectively, for both smooth and nonsmooth initial data. In this chapter, we have made an attempt to extend these results to SEM.

Next, we describe the convolution quadrature framework. Let K be a complex-valued or operator valued analytic function in the sector $S_{\pi-\vartheta} = \{z \in \mathbb{C} : |\arg\, z| \leq \vartheta\}, 0 < \vartheta < \pi/2$ and satisfies

$$\|K(z)\| \leq M_0 |z|^{-\gamma}, \tag{5.2}$$

for some real numbers M_0, γ and for all $z \in S_{\pi-\vartheta}$. Then there exists a distribution $\kappa(t)$ on the real line $\mathbb{R}$ such that $K(z) = \mathcal{L}\{\kappa(t)\}$, the Laplace transform of $\kappa(t)$. Here the distribution $\kappa(t)$ for positive time is given by the following inversion formula

$$\kappa(t) = \frac{1}{2\pi i} \int_\Gamma K(z) e^{zt}\, dz,$$

where Γ is a contour in the sector $S_{\pi-\vartheta}$, parallel to its boundary and with imaginary part increasing. Further, $\kappa(t)$ is zero on the negative real axis and it has singular support empty or concentrated at the origin. Let us define $K(D_t)$ with $D_t = \frac{d}{dt}$ as the operator of (distributional) convolution with the kernel κ: $K(D_t)f := \kappa * f = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau,\ t > 0$, for a suitable function $f(t)$. Let K_1 and K_2 be two operators of type K generated by the kernels κ_1 and κ_2, respectively. Then we have the following relation using the convolution rule

$$K_1(D_t)K_2(D_t)f(t) = (K_1K_2)(D_t)f(t). \tag{5.3}$$

Now consider a partition of $[0, T]$ with equidistant points $\{t_n\}_{n=0}^N$, where $t_n = nk, n = 0, \ldots, N$ with $k = T/N$. Let U_h^n be the approximation of $u(t_n)$. The convolution quadrature of the continuous convolution $K(D_t)f(t)$ is denoted by $K(\bar{D}_k)f(t)$ and is defined by (cf. [11, 39])

$$K(\bar{D}_k)f(t) = \sum_{0 \leq jk \leq t} \omega_j f(t - jk), \quad t > 0, \tag{5.4}$$

where the quadrature weights $\omega_j, j = 0, 1, \ldots$ are determined by $\sum_{j=0}^{\infty} \omega_j \xi^j = K(\frac{\delta(\xi)}{k})$, δ is the quotient of the generating polynomials of a stable and consistent linear multistep method. For the BE method $\delta(\xi) = (1-\xi)$ and for the SBD method $\delta(\xi) = \frac{3}{2} - 2\xi + \frac{\xi^2}{2}$. Further, similar to (5.3) the convolution quadrature satisfies the following relations:

$$K_1(\bar{D}_k)K_2(\bar{D}_k)f(t) = (K_1K_2)(\bar{D}_k)f(t), \quad \text{and} \quad K_1(\bar{D}_k)(\kappa_2 * f) = (K_1(\bar{D}_k)\kappa_2) * f. \tag{5.5}$$

We now proceed to estimate the errors in the fully discrete approximations. This is accomplished by reformulating the problem by transforming the Caputo fractional derivative to the Riemann-Liouville, and then apply the convolution quadrature. It is known fact that the Caputo and the Riemann-Liouville fractional time derivatives (cf. [30, p.91] or [27, p.A150]) are related as follows: For $\tilde{\alpha} \in (m-1, m), m \in \mathbb{N}$,

$$\partial_t^{\tilde{\alpha}} f(t) = D_t^{\tilde{\alpha}} \Big(f(t) - \sum_{j=0}^{m-1} \frac{f^j(0)}{j!} t^j \Big), \tag{5.6}$$

where $D_t^{\tilde{\alpha}} f(t)$ is the Riemann-Liouville fractional time derivative (left-sided) of order $\tilde{\alpha}$ [55], defined by

$$D_t^{\tilde{\alpha}} f(t) = \frac{1}{\Gamma(m-\tilde{\alpha})} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{1+\tilde{\alpha}-m}} \, d\tau. \tag{5.7}$$

In our analysis, we consider $m = 1$ in (5.6) and hence, $\partial_t^{\tilde{\alpha}} f(t) = D_t^{\tilde{\alpha}}(f(t) - f(0))$. Using this identity and $B_h(t,s) = -A_h$ we can rewrite the semidiscrete approximation (3.4) as

$$u_h(t) = \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} D_t^{\alpha} u_{0h} + \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} f_h(t), \quad t > 0. \tag{5.8}$$

As usual, the smooth data error analysis is carried out for the nonhomogeneous problem whereas the homogeneous problem is considered for nonsmooth data error estimate.

The outline of this chapter is as follows. We study the BE and SBD schemes, respectively, in Sections 5.2 and 5.3. In Section 5.4, we present some numerical illustrations to validate our theoretical findings.

5.2 Analysis for the BE Method

This section is devoted to error analysis for the fully discrete BE approximation to (5.8). By the associativity of the convolution quadrature (5.5), we can express (5.8) as: Given $U_h^0 = u_{0h}$, compute U_h^n for $n = 1, \ldots, N$, such that

$$U_h^n = \Big(\bar{D}_k^{\alpha} + (I + \bar{D}_k^{-1})A_h\Big)^{-1} \bar{D}_k^{\alpha} u_{0h} + \Big(\bar{D}_k^{\alpha} + (I + \bar{D}_k^{-1})A_h\Big)^{-1} f_h(t_n). \tag{5.9}$$

We need the following lemma (cf. [39, Theorem 5.2]).

Lemma 5.2.1. *Let $K(z)$ be analytic in S_ϑ, $\pi/2 < |\vartheta| \leq \tilde{\vartheta} = \frac{\pi}{1+\alpha}$ and satisfy (5.2). Then with $f(t) = Ct^{\mu-1}$, the following inequality holds by the convolution quadrature for the BE method:*

$$\|K(D_t)f(t) - K(\bar{D}_k)f(t)\| \leq \begin{cases} Ct^{\gamma-1}k^{\mu}, & 0 < \mu \leq 1, \\ Ct^{\gamma+\mu-2}k, & 1 \leq \mu. \end{cases}$$

We are now in a position to state the error estimate of the BE method for both smooth and nonsmooth initial data.

Theorem 5.2.1. *Let u be the solution of (5.1), and let U_h^n be its fully discrete approximation given by (5.9). Then for $u_0 \in \dot{H}^2(\Omega)$, $f' \in C(L^2(\Omega))$, $u_{0h} = R_h u_0$ and $f_h = P_h f$, we have at the nth time level t_n, $n \geq 1$,*

$$\|u(t_n) - U_h^n\| \leq Ch^2\big(\|u_0\|_2 + t_n^{-2}\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big) + Ckt_n^{\alpha-1}\big(\|u_0\|_2 + \|f(0)\| + t_n\|f'\|_{C(L^2(\Omega))}\big), \tag{5.10}$$

where the number $\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}$ is given by (3.64). Moreover, for the homogeneous problem with $u_0 \in L^2(\Omega)$ and $u_{0h} = P_h u_0$, we have

$$\|u(t_n) - U_h^n\| \leq C(h^2 t_n^{-\alpha} + kt_n^{-1})\|u_0\|, \quad t_n > 0. \tag{5.11}$$

Proof. For the proof of both (5.10) and (5.11), we split the error $u(t_n) - U_h^n$ as

$$u(t_n) - U_h^n = (u(t_n) - u_h(t_n)) + (u_h(t_n) - U_h^n).$$

We first prove (5.10). The first term $u(t_n) - u_h(t_n)$ is bounded as desired by Theorem 3.3.2. To bound the second term $u_h(t_n) - U_h^n$, we need the following identity

$$l(z)(l(z)I + A_h)^{-1} = I - (l(z)I + A_h)^{-1}A_h,$$

where $l(z) = \frac{z^{\alpha+1}}{z+1}$. We set $F_1(z) = (l(z)I + A_h)^{-1}$ and $\widetilde{F}_1(z) = \frac{l(z)}{z^\alpha}(l(z)I + A_h)^{-1}$, $z \in S_\vartheta$. Using (3.62) and Lemma 3.3.1, we find that

$$\begin{aligned} \|F_1(z)\| &\leq \frac{C}{|l(z)|} = C\frac{|1+z|}{|z|^{\alpha+1}} \leq C\{|z|^{-(\alpha+1)} + |z|^{-\alpha}\}, \\ \|\widetilde{F}_1(z)\| &= \frac{|l(z)|}{|z|^\alpha}\,\|(l(z)I + A_h)^{-1}\| \leq C|z|^{-\alpha}. \end{aligned}$$

Again, using (5.8), (5.9), and the relation $f_h(t) = f_h(0) + (1 * f_h')(t)$, the difference

$u_h(t_n) - U_h^n$ can be expressed as

$$
\begin{aligned}
u_h(t_n) - U_h^n &= -\big(F_1(D_t) - F_1(\bar{D}_k)\big)A_h R_h u_0 + \big(\widetilde{F}_1(D_t) - \widetilde{F}_1(\bar{D}_k)\big)(f_h(0) + (1 * f_h')(t_n)) \\
&= -\big(F_1(D_t) - F_1(\bar{D}_k)\big)A_h R_h u_0 + \big(\widetilde{F}_1(D_t) - \widetilde{F}_1(\bar{D}_k)\big) f_h(0) \\
&\qquad + \big(\big(\widetilde{F}_1(D_t) - \widetilde{F}_1(\bar{D}_k)\big)1\big) * f_h'(t_n) \\
&=: T_3 + T_4 + T_5,
\end{aligned}
$$

where in the second step we have used (5.5). Invoking Lemma 5.2.1 with $\mu = 1$ and $\gamma = \alpha$, we obtain

$$
\begin{aligned}
&\|T_3\| \le Ckt_n^{\alpha-1} \|u_0\|_2, \quad \|T_4\| \le Ckt_n^{\alpha-1} \|f_h(0)\| \le Ckt_n^{\alpha-1} \|f(0)\|, \\
&\|T_5\| \le \int_0^{t_n} \|\big(\big(\widetilde{F}_1(D_t) - \widetilde{F}_1(\bar{D}_k)\big)1\big)(t_n - s) f_h'(s)\| \, ds \\
&\qquad \le Ck \int_0^{t_n} (t_n - s)^{\alpha-1} \|f_h'(s)\| \, ds \le Ckt_n^{\alpha} \|f'\|_{C(L^2(\Omega))},
\end{aligned}
$$

where we have used the relation $A_h R_h = P_h A$ and the L^2-stability of P_h. Thus, for smooth initial data we obtain

$$
\|u_h(t_n) - U_h^n\| \le Ckt_n^{\alpha-1}\big(\|u_0\|_2 + \|f(0)\| + t_n \|f'\|_{C(L^2(\Omega))} \big). \tag{5.12}
$$

Now, the triangle inequality, Theorem 3.3.2 together with (5.12) provide the following estimate: For $t_n > 0$,

$$
\begin{aligned}
\|u(t_n) - U_h^n\| &\le Ch^2\big(\|u_0\|_2 + t_n^{-2} \|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} \big) \\
&\qquad + Ckt_n^{\alpha-1}\big(\|u_0\|_2 + \|f(0)\| + t_n \|f'\|_{C(L^2(\Omega))} \big),
\end{aligned} \tag{5.13}
$$

which completes the proof of (5.10).

We next turn to nonsmooth data error estimate (5.11). In view of Theorem 3.3.1, it is enough to bound $\|u_h(t_n) - U_h^n\|$. For this purpose, set

$$
\begin{aligned}
F_2(z) &= z^{\alpha}\Big(z^{\alpha} I + (1 + \frac{1}{z})A_h\Big)^{-1} \\
&= l(z)(l(z)I + A_h)^{-1}, \quad z \in S_{\vartheta}, \ \pi/2 < |\vartheta| \le \tilde{\vartheta}.
\end{aligned}
$$

With an aid of (3.62), we obtain

$$
\|F_2(z)\| = |l(z)| \, \|(l(z)I + A_h)^{-1}\| \le C.
$$

Subtracting (5.9) from (5.8), we have

$$u_h(t_n) - U_h^n = \big(F_2(D_t) - F_2(\bar{D}_k)\big)u_{0h},$$

where $u_{0h} = P_h u_0$. Now, by Lemma 5.2.1 with $\mu = 1$ and $\gamma = 0$ we obtain

$$\|u_h(t_n) - U_h^n\| \le Ckt_n^{-1}\,\|u_0\|\,. \tag{5.14}$$

Use of the triangle inequality, Theorem 3.3.1, and estimate (5.14) yield

$$\|u(t_n) - U_h^n\| \le C(h^2 t_n^{-\alpha} + kt_n^{-1})\,\|u_0\|\,, \;\; t_n > 0, \tag{5.15}$$

and this completes the proof of the theorem. □

Remark 5.2.1. (*i*) *For the homogeneous problem ($f = 0$ in (5.1)) with $u_0 \in \dot{H}^q(\Omega)$, $0 < q < 2$, interpolation of estimates (5.15) and (5.13) yield the error estimate*

$$\|u(t_n) - U_h^n\| \le C(h^2 t_n^{-\alpha\frac{(2-q)}{2}} + kt_n^{\frac{\alpha q}{2}-1})\,\|u_0\|_q\,, \;\; t_n > 0.$$

(*ii*) *Observe that even if for the smooth initial data u_0, estimate (5.12) produces a singularity of order $t_n^{\alpha-1}$ as $t \to 0^+$. When $\alpha \to 1^-$, this bound similar to the standard PIDE [3, Theorem 4.1].*

5.3 Analysis for the SBD Method

In the previous section we have seen that the scheme (5.4) yields first-order accuracy in time provided $f(0) \neq 0$. This section considers a modification of the scheme (5.4) so-called the SBD method which leads to the second-order accuracy in time.

Following [11, 27], we introduce the notation $D_t^{-p}, p > 0$ for the Riemann-Liouville fractional integral of order p which is defined by

$$D_t^{-p}u(t) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}u(\tau)\,d\tau.$$

An easy calculation yields the following identity

$$\Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} = D_t^{-\alpha} - \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} D_t^{-\alpha}(I + D_t^{-1})A_h. \tag{5.16}$$

Using (5.16) and the identity $f_h(t) = f_h(0) + tf_h'(0) + t * f_h''$, Eq. (5.8) can be expressed as

(5.17)

$$\begin{aligned}
u_h(t) = u_{0h} &- \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} D_t^{-\alpha}(I + D_t^{-1})A_h D_t^{\alpha} u_{0h} \\
&+ \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} f_h(0) + \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} (tf_h'(0) + t * f_h'') \\
= u_{0h} &- \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} D_t^{-\alpha}(I + D_t^{-1}) D_t^{\alpha} D_t D_t^{-1}(A_h u_{0h}) \\
&+ \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} D_t(D_t^{-1} f_h(0)) \\
&+ \Big(D_t^{\alpha} + (I + D_t^{-1})A_h\Big)^{-1} (tf_h'(0) + t * f_h'').
\end{aligned}$$

Taking the convolution quadrature for the SBD scheme, we have for $n = 1, 2, \ldots, N$,

(5.18)

$$\begin{aligned}
U_h^n = u_{0h} &- \Big(\bar{D}_k^{\alpha} + (I + \bar{D}_k^{-1})A_h\Big)^{-1} \bar{D}_k^{-\alpha}(I + \bar{D}_k^{-1}) \bar{D}_k^{\alpha} \bar{D}_k D_t^{-1}(A_h u_{0h}) \\
&+ \Big(\bar{D}_k^{\alpha} + (I + \bar{D}_k^{-1})A_h\Big)^{-1} \bar{D}_k(tf_h(0)) + \Big(\bar{D}_k^{\alpha} + (I + \bar{D}_k^{-1})A_h\Big)^{-1} (tf_h'(0) + t * f_h''),
\end{aligned}$$

with $U_h^0 = u_{0h}$. The presence of the operator D_t^{-1} in (5.18) leads to the second-order accuracy. Analogous to Lemma 5.2.1 for the BE method, we have the following lemma regarding the SBD scheme for the convolution quadrature (cf. [39, Theorem 5.2, p.144]).

Lemma 5.3.1. *Let $K(z)$ be analytic in $S_{\vartheta}, \pi/2 < |\vartheta| \leq \tilde{\vartheta} = \frac{\pi}{1+\alpha}$ and satisfy (5.2). Then with $f(t) = Ct^{\mu-1}$, the following inequality is satisfied by the convolution quadrature for the SBD method:*

$$\|K(D_t)f(t) - K(\bar{D}_k)f(t)\| \leq \begin{cases} Ct^{\gamma-1}k^{\mu}, & 0 < \mu \leq 2, \\ Ct^{\gamma+\mu-3}k^2, & 2 \leq \mu. \end{cases}$$

The error estimates for the fully discrete SBD method for both smooth and nonsmooth data are presented in the following theorem.

Theorem 5.3.1. *Let u be the solution of (5.1), and let U_h^n be the solution of the fully discrete scheme given by (5.18). Then for $u_0 \in \dot{H}^2(\Omega)$, $f'' \in C(L^2(\Omega))$, $u_{0h} = R_h u_0$ and $f_h = P_h f$, we have for $t_n > 0$,*

$$\begin{aligned}
\|u(t_n) - U_h^n\| &\leq Ch^2\big(\,\|u_0\|_2 + t_n^{-2}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big) \\
&\quad + Ck^2 t_n^{\alpha-2}\big(\,\|u_0\|_2 + \|f(0)\| + t_n\,\|f'(0)\| + t_n^2\,\|f''\|_{C(L^2(\Omega))}\big),
\end{aligned} \tag{5.19}$$

where the number $\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}$ *is given by (3.64). Furthermore, for the homogeneous problem with* $u_0 \in L^2(\Omega)$ *and* $u_{0h} = P_h u_0$*, we have*

$$\|u(t_n) - U_h^n\| \leq C(h^2 t_n^{-\alpha} + k^2 t_n^{-2}) \|u_0\| , \quad t_n > 0. \tag{5.20}$$

Proof. As before, we split the error $u(t_n) - U_h^n$ at t_n as the sum of $u(t_n) - u_h(t_n)$ and $u_h(t_n) - U_h^n$. For smooth initial data u_0, a bound of the first term $u(t_n) - u_h(t_n)$ is given by Theorem 3.3.2. Thus, it remains to bound the term $u_h(t_n) - U_h^n$. Subtracting (5.18) from (5.17) and utilizing (5.5), we get

$$\begin{aligned} u_h(t_n) - U_h^n &= (F_3(\bar{D}_k) - F_3(D_t))D_t^{-1} A_h u_{0h}(t_n) + (\widetilde{F}_3(D_t) - \widetilde{F}_3(\bar{D}_k))(t f_h(0)) \\ &\quad + (\widetilde{F}_4(D_t) - \widetilde{F}_4(\bar{D}_k))(t f_h'(0)) + ((\widetilde{F}_4(D_t) - \widetilde{F}_4(\bar{D}_k))t) * f_h'' \\ &=: T_6 + T_7 + T_8 + T_9, \end{aligned}$$

where for $z \in S_\vartheta$, $\pi/2 < |\vartheta| \leq \tilde{\vartheta}$,

$$F_3(z) = z(l(z)I + A_h)^{-1}, \ \widetilde{F}_3(z) = z^{1-\alpha} l(z)(l(z)I + A_h)^{-1}, \ \widetilde{F}_4(z) = \frac{l(z)}{z^\alpha}(l(z)I + A_h)^{-1}.$$

As before, using (3.62) and (3.60) we obtain

$$\begin{aligned} &\|F_3(z)\| \leq C|z|/|l(z)| \leq C|z|\frac{(1+|z|)}{|z|^{\alpha+1}} = C(|z|^{-\alpha} + |z|^{1-\alpha}), \\ &\|\widetilde{F}_3(z)\| \leq C|z|^{1-\alpha}, \ \|\widetilde{F}_4(z)\| \leq C|z|^{-\alpha}. \end{aligned}$$

We now invoke Lemma 5.3.1 with $\gamma = \alpha - 1$ and $\mu = 2$ to have

$$\|T_6\| \leq Ck^2 t_n^{\alpha-2} \|u_0\|_2 , \ \|T_7\| \leq Ck^2 t_n^{\alpha-2} \|f_h(0)\| \leq Ck^2 t_n^{\alpha-2} \|f(0)\| , \tag{5.21}$$

where we have used the fact $A_h R_h = P_h A$ and the L^2-stability of P_h. Further, invoke Lemma 5.3.1 with $\gamma = \alpha$ and $\mu = 2$ to obtain

$$\begin{aligned} &\|T_8\| \leq Ck^2 t_n^{\alpha-1} \|f_h'(0)\| \leq Ck^2 t_n^{\alpha-1} \|f'(0)\| , \\ &\|T_9\| \leq \int_0^{t_n} \|((\widetilde{F}_4(D_t) - \widetilde{F}_4(\bar{D}_k))t)(t_n - s) f_h''(s)\| \, ds \\ &\qquad \leq Ck^2 \int_0^{t_n} (t_n - s)^{\alpha-1} \|f_h''(s)\| \, ds \leq Ck^2 t_n^\alpha \|f''\|_{C(L^2(\Omega))} , \end{aligned} \tag{5.22}$$

where we have used the L^2-stability of P_h. Using (5.21)-(5.22), we get

$$\begin{aligned} \|u_h(t_n) - U_h^n\| &\leq \|T_6\| + \|T_7\| + \|T_8\| + \|T_9\| \\ &\leq Ck^2 t_n^{\alpha-2} \big(\|u_0\|_2 + \|f(0)\| + t_n \|f'(0)\| + t_n^2 \|f''\|_{C(L^2(\Omega))} \big). \end{aligned} \tag{5.23}$$

Now, Theorem 3.3.2 together with (5.23) yields

$$
\begin{aligned}
\|u(t_n) - U_h^n\| &\le Ch^2\big(\,\|u_0\|_2 + t_n^{-2}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\,\big) \\
&\quad + Ck^2 t_n^{\alpha-2}\big(\,\|u_0\|_2 + \|f(0)\| + t_n\,\|f'(0)\| + t_n^2\,\|f''\|_{C(L^2(\Omega))}\,\big),
\end{aligned} \tag{5.24}
$$

which proves the first assertion (5.19).

We next attempt to prove (5.20). In this case, a bound for the first term $u(t_n)-u_h(t_n)$ is given by Theorem 3.3.1. Further, in view of (5.17) and (5.18), we have

$$u_h(t_n) - U_h^n = (F_4(\bar{D}_k) - F_4(D_t))D_t^{-1}u_{0h}(t_n),$$

where for $z \in S_\vartheta$, $\pi/2 < |\vartheta| \le \tilde{\vartheta}$,

$$
\begin{aligned}
F_4(z) &= \Big(z^\alpha I + (1+\frac{1}{z})A_h\Big)^{-1}\frac{1}{z^\alpha}(1+\frac{1}{z})z^\alpha z A_h \\
&= z(l(z)I + A_h)^{-1}A_h \\
&= z - zl(z)(l(z)I + A_h)^{-1}.
\end{aligned}
$$

By (3.62), $\|F_4(z)\| \le C|z|$. Now, inviting Lemma 5.3.1 with $\gamma = -1$ and $\mu = 2$ we conclude

$$\|u_h(t_n) - U_h^n\| \le Ck^2 t_n^{-2}\,\|u_{0h}\| \le Ck^2 t_n^{-2}\,\|u_0\|\,. \tag{5.25}$$

Finally, use of Theorem 3.3.1 and estimate (5.25) will lead to

$$\|u(t_n) - U_h^n\| \le C(h^2 t_n^{-\alpha} + k^2 t_n^{-2})\,\|u_0\|\,, \tag{5.26}$$

and this completes the proof of the theorem. □

Remark 5.3.1. *For the homogeneous problem ($f = 0$ in (5.1)), as a consequence of interpolation to error estimates (5.26) and (5.24), we have for $u_0 \in \dot{H}^q(\Omega)$, $0 < q < 2$,*

$$\|u(t_n) - U_h^n\| \le C(h^2 t_n^{\alpha\frac{(q-2)}{2}} + k^2 t_n^{-2+\frac{q\alpha}{2}})\,\|u_0\|_q\,,\ \ t_n > 0.$$

5.4 Numerical Illustrations

In this section, we verify numerically our theoretical results obtained in the fully discrete schemes. Numerical experiments presented below include both smooth and nonsmooth initial data cases. For our computation, the time discretization is done for both BE and SBD schemes over an equidistant grid points $t_j = jk$, $j = 0, 1, 2, \ldots$, where k being the time step size. We take a small spatial mesh size h during the temporal discretization so that the spatial discretization error is negligible. For our numerical experiments, we take $\Omega = (0,1)\times(0,1)$ and consider the following two cases:

(a) Nonhomogeneous equation with smooth initial data

$$u_0(x,y) = \sin(\pi x)\sin(\pi y), \quad \text{and} \quad f = 1.$$

(b) Homogeneous equation with nonsmooth initial data

$$f = 0, \quad \text{and} \quad u_0(x,y) = \begin{cases} 0, & \text{for } 0 < x \leq 1/2, 0 < y < 1, \\ 1, & \text{for } 1/2 < x < 1, 0 < y < 1. \end{cases}$$

We now perform numerical tests for the BE and SBD methods for three different values of α at $T = 0.1$ (for the case (a)) and $T = 0.01$ (for the case (b)) with a fixed mesh size $h = \frac{1}{100}$. For this purpose, we choose the time step size $k = \frac{T}{N}$, where N is the number of subintervals. For the BE method, Tables 5.1 and 5.2, respectively, show the convergence rates (CRs) for the smooth data (case (a)) and the nonsmooth data (case (b)), respectively. Our numerical experiments reveal $O(k)$ accuracy in time as expected from the theoretical results of Theorem 5.2.1 (see Tables 5.1 and 5.2). Numerical results for the SBD method are presented in Tables 5.3 and 5.4. For both the cases, we obtain $O(k^2)$ convergence which confirm our theoretical findings in Theorem 5.3.1.

Table 5.1: The $L^2(\Omega)$-errors and CRs for the BE method for the case (a) for $\alpha = 0.25, 0.5, 0.75$ at $T = 0.1$ with $h = 10^{-2}$

$N\diagdown\alpha$	$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR
2	2.34406e-01	–	2.08059e-01	–	1.57457e-01	–
4	1.28886e-01	0.86291	1.14104e-01	0.86665	8.83437e-02	0.83376
8	5.29829e-02	1.28249	4.77695e-02	1.25618	4.82901e-02	0.87139
16	2.65424e-02	0.99722	2.42926e-02	0.97557	2.33384e-02	1.04902

5.5 Concluding Remarks

In this chapter, we have considered and analyzed two fully discrete schemes based on the convolution quadrature with the generating functions given by the BE and the SBD schemes for SEM. The presented convergence analysis includes both smooth and nonsmooth initial data cases. Optimal error bounds in time for both schemes are established,

Table 5.2: The $L^2(\Omega)$-errors and CRs for the BE method for the case (b) for $\alpha = 0.25, 0.5, 0.75$ at $T = 0.01$ with $h = 10^{-2}$

$N\diagdown\alpha$	$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR
2	4.77180e-01	–	4.09861e-01	–	3.69959e-01	–
4	2.34956e-01	1.02214	1.77235e-01	1.20947	1.85927e-01	0.99263
8	1.27223e-01	0.88502	1.05974e-01	0.74195	1.00724e-01	0.88432
16	5.50487e-02	1.20858	4.83068e-02	1.13341	3.92947e-02	1.35801

Table 5.3: The $L^2(\Omega)$-errors and CRs for the SBD method for the case (a) for $\alpha = 0.25, 0.5, 0.75$ at $T = 0.1$ with $h = 10^{-2}$

$N\diagdown\alpha$	$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR
2	1.22933e-01	–	9.04593e-02	–	6.22231e-02	–
4	2.85011e-02	2.10878	2.39659e-02	1.91628	1.60547e-02	1.95446
8	7.64753e-03	1.89795	5.70285e-03	2.07123	3.92891e-03	2.03079
16	1.89314e-03	2.01421	1.39786e-03	2.02847	9.79432e-04	2.00411

Table 5.4: The $L^2(\Omega)$-errors and CRs for the SBD method for the case (b) for $\alpha = 0.25, 0.5, 0.75$ at $T = 0.01$ with $h = 10^{-2}$

$N\diagdown\alpha$	$\alpha = 0.25$		$\alpha = 0.50$		$\alpha = 0.75$	
	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR	$L^2(\Omega)$-norm	L^2-CR
2	7.47078e-01	–	7.20421e-01	–	7.06709e-01	–
4	1.57776e-01	2.24338	1.95315e-01	1.88304	2.01472e-01	1.81054
8	3.65519e-02	2.10986	5.15920e-02	1.92058	5.47703e-02	1.87911
16	9.38070e-03	1.96218	1.30698e-02	1.98091	1.38370e-02	1.98486

namely, for the BE scheme first-order accuracy in time and the second-order accuracy in time for the SBD scheme. Finally, we have provided some numerical illustrations to validate the theoretical findings.

6

Cases with Weakly Singular Kernels

This chapter is devoted to the study of SEM in which the memory kernel is weakly singular. By means of the Laplace transform, we first represent the solution as a contour integral along a suitably chosen contour in the complex plane. We establish the Sobolev regularity results of the solution for the nonhomogeneous problem with respect to both smooth and nonsmooth initial data u_0, including $u_0 \in L^2(\Omega)$. The spatially semidiscrete Galerkin scheme is then developed by using piecewise linear and continuous finite elements. We derive optimal order error bounds with respect to the various regularity assumptions on the initial data for both homogeneous and nonhomogeneous problems. Moreover, the L1 scheme for the temporal discretization is analyzed and error estimates of order $O(k)$ for both homogeneous and nonhomogeneous problems are proved.

6.1 Introduction

We now recall the following IBVP for SEM (1.6) in the unknown $u(t) = u(x,t)$:

$$
\begin{aligned}
\partial_t^\alpha u(t) + Au(t) + \int_0^t \kappa_\beta(t-s)Au(s)\,ds &= f(t) \quad \text{in} \quad \Omega, \quad 0 < t \le T, \\
u(t) &= 0 \quad \text{on} \quad \partial\Omega, \quad 0 < t \le T, \\
u(0) &= u_0(x) \quad \text{in} \quad \Omega,
\end{aligned}
\tag{6.1}
$$

where $0 < \alpha, \beta < 1$. Here $\partial_t^\alpha u(t)$ denotes the Caputo fractional derivative defined by

$$
\partial_t^\alpha u(t) = \int_0^t \kappa_{1-\alpha}(t-s)u'(s)\,ds, \quad \kappa_\alpha(t) := t^{(\alpha-1)}/\Gamma(\alpha), \ u'(s) = \partial_s u(s),
$$

where $\Gamma(z) = \int_0^\infty e^{-w}w^{z-1}\,dw$, $\mathrm{Re}(z) > 0$ and $A = -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ with given source function $f(t) = f(x,t)$ and the initial data u_0.

The present study is motivated by the interesting works of Lubich et al. [41], McLean et al. [47], McLean and Mustapha [46], and Jin et al. [26]. In [41], the authors have studied the semidiscrete as well as the fully discrete approximations of an integro-differential equation with a weakly singular kernel. Using the piecewise linear functions in space and the BE and SBD schemes in time, the authors have derived optimal order error bounds for the semidiscrete and fully discrete schemes for both homogeneous and nonhomogeneous problems for nonsmooth problem data. McLean et al. [47] have considered and analyzed both semidiscrete and fully discrete schemes of a PIDE with a convolution type memory term in a Banach space setting. By using the Laplace transform, the authors have first represented the solution along a suitable contour in the complex plane and established some stability and regularity results of the solution. They have applied a quadrature formula for the time discretization and FEM in space variable to obtain fully discrete method. In [46], the authors have discussed the discontinuous Galerkin method with respect to time for SE and derived optimal error estimates for both smooth and nonsmooth initial data cases. The authors of [26] have analyzed the L1 scheme in the absence of memory term (SE) and derived error estimates of order $O(k)$ for both smooth and nonsmooth initial data. The same authors have extended the convergence theory of the SE to space-time fractional diffusion problems. The L1 scheme is the simplest and widely used numerical method for discretizing the Caputo fractional derivative in time. For a survey of convergence results with the L1 scheme to discretize the Caputo fractional derivative in the time-fractional problems, we refer the readers to [66].

Our work differs from the above studies in the simultaneous presence of both fractional derivative and Riemann-Liouville fractional integral in problem (6.1). The analysis presented demands that the fractional derivative order α and the fractional integral order β should not be arbitrary− their sum $(\alpha + \beta)$ should always be bigger than or equal to 1 (see Remark 6.2.1).

The structure of the chapter is as follows. In Section 6.2, we establish the Sobolev regularity results of the solution. In Section 6.3, we analyze GFEM and derive optimal order error bounds for both homogeneous and nonhomogeneous problem in both $L^2(\Omega)$ and $H^1(\Omega)$-norms. Next, the error analysis for the fully discrete L1 scheme is carried out in Section 6.4. In the last section, we perform numerical tests to validate the theoretical analysis.

6.2 Regularity Results of the Solution

This section establishes the Sobolev regularity results for the solution of (6.1) with respect to both smooth and nonsmooth initial data. For convenience, we recall some notations from Chapter 3. We introduce the sector

$$S_\vartheta = \{z \in \mathbb{C} : |\arg z| < \vartheta\}, \;\; 0 < \vartheta \le \pi,$$

and for $\epsilon > 0$, the contour

$$\begin{aligned}\Gamma_{\epsilon,\vartheta} =&\{z \in \mathbb{C} : |z| = \epsilon, |\arg z| \le \vartheta\} \cup \{z \in \mathbb{C} : z = re^{-i\vartheta}, r \ge \epsilon\} \\ &\cup \{z \in \mathbb{C} : z = re^{i\vartheta}, r \ge \epsilon\}.\end{aligned} \tag{6.2}$$

Further, it is well-known that the resolvent of $-A = \Delta$ satisfies the following estimate: For $\vartheta \in (0, \pi/2)$,

$$\|(zI + A)^{-1}\| \le C|z|^{-1}, \;\; z \in S_{\pi-\vartheta}. \tag{6.3}$$

Now, the Laplace transform to (6.1) produces

$$\Big(z^\alpha I + (1 + \frac{1}{z^\beta})A\Big)\widehat{u}(z) = z^{\alpha-1}u_0 + \widehat{f}(z),$$

where $\widehat{u}(z) = \mathcal{L}\{u(t)\}$ and $\widehat{f}(z) = \mathcal{L}\{f(t)\}$ denote the Laplace transforms of $u(t)$ and $f(t)$, respectively. Therefore,

$$\widehat{u}(z) = (l(z)I + A)^{-1}\Big(\frac{l(z)}{z}u_0 + \frac{l(z)}{z^\alpha}\widehat{f}(z)\Big),$$

where

$$l(z) = \frac{z^{(\alpha+\beta)}}{z^\beta + 1}. \tag{6.4}$$

Hence, by inverting $\widehat{u}(z)$, the contour integral representation of the solution is

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}(l(z)I + A)^{-1}\Big(\frac{l(z)}{z}u_0 + \frac{l(z)}{z^\alpha}\widehat{f}(z)\Big)\,dz, \tag{6.5}$$

where $\pi/2 < |\vartheta| < \tilde{\vartheta} = \frac{\pi}{(\alpha+\beta)}$ with $(\alpha + \beta) \ge 1$. Clearly, $\tilde{\vartheta} \in (\pi/2, \pi]$ and hence the sector $S_{\tilde{\vartheta}}$ contains the right half-plane along with to some extent of the left half-plane of the complex plane.

Remark 6.2.1. *Note that we cannot choose the numbers α and β arbitrarily in the analysis but with the restriction $(\alpha+\beta) \ge 1$. For example, if $\alpha = 1/2$, then $\beta \in [1/2, 1)$. If $(\alpha + \beta) < 1$, we cannot apply the resolvent estimate (6.6) (see below) as in this case $l(z)$ will not be an element of the sector $S_{\pi-\vartheta}$, $\vartheta \in (0, \pi/2)$.*

In order to proceed further to this work, it is important to obtain a bound of the function l. In the following lemma, we establish a bound of $l(z)$. The proof goes analogous to Lemma 3.3.1 of Chapter 3 and hence the details are thus omitted.

Lemma 6.2.1. *Let the function l be as in (6.4), and let $z \in S_{\tilde{\vartheta}}$ with $\tilde{\vartheta} = \frac{\pi}{(\alpha+\beta)}$ with $(\alpha+\beta) \geq 1$. Then $l(z) \in S_{\pi}$. Moreover,*

$$|l(z)| \leq C \min\{|z|^{\alpha}, |z|^{\alpha+\beta}\}, \quad \textit{for all } z \in S_{\tilde{\vartheta}}.$$

Now, Lemma 6.2.1 and the resolvent estimate (6.3) yield

$$\|(l(z)I + A)^{-1}\| \leq C|l(z)|^{-1}, \;\; z \in S_{\tilde{\vartheta}}. \tag{6.6}$$

Below, we first establish a stability estimate of the solution (6.5).

Theorem 6.2.1. *For the solution $u(t)$ of (6.5), we have*

$$\|u(t)\| \leq C\big(\, \|u_0\| + t^{\alpha-1}\, \|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} \big), \;\; t > 0.$$

Proof. Let $\epsilon = \frac{1}{t}$. In view of (6.5), the resolvent estimate (6.6), and Lemma 6.2.1 we find that

$$\begin{aligned}
\|u(t)\| &\leq C \int_{\Gamma_{\epsilon,\vartheta}} e^{Re(z)t}\, \|(l(z)I + A)^{-1}\| \Big(\frac{|l(z)|}{|z|}\, \|u_0\| + \frac{|l(z)|}{|z|^{\alpha}}\, \|\widehat{f}(z)\| \Big)\, |dz| \\
&\leq C\Big(\int_{-\vartheta}^{\vartheta} e^{\cos\psi}\, d\psi + t \int_{\frac{1}{t}}^{\infty} e^{rt\cos\vartheta}\, dr \Big)\, \|u_0\| \\
&\quad + C\Big(\int_{-\vartheta}^{\vartheta} t^{\alpha-1} e^{\cos\psi}\, d\psi + t^{\alpha} \int_{\frac{1}{t}}^{\infty} e^{rt\cos\vartheta}\, dr \Big)\, \|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} \leq C\big(\, \|u_0\| + t^{\alpha-1}\, \|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} \big),
\end{aligned}$$

and this completes the proof. □

We now establish a more conventional stability estimate for solution (6.5). By the convolution rule, we can write the solution $u(t)$ as

$$u(t) = H(t)u_0 + \int_0^t F(t-s)f(s)\, ds, \tag{6.7}$$

where the operators $H(t)$ and $F(t)$ are defined by

$$\begin{aligned}
H(t) &= \frac{1}{2\pi i} \int_{\Gamma_{\epsilon,\vartheta}} e^{zt}(l(z)I + A)^{-1} \frac{l(z)}{z}\, dz, \\
F(t) &= \frac{1}{2\pi i} \int_{\Gamma_{\epsilon,\vartheta}} e^{zt}(l(z)I + A)^{-1} \frac{l(z)}{z^{\alpha}}\, dz.
\end{aligned} \tag{6.8}$$

Following the proof of Theorem 6.2.1, we can arrive at

$$\|H(t)\| \le C \quad \text{and} \quad \|F(t)\| \le Ct^{\alpha-1}, \tag{6.9}$$

which together with (6.7) give a familiar stability estimate

$$\|u(t)\| \le C\Big(\|u_0\| + \int_0^t (t-s)^{\alpha-1}\,\|f(s)\|\,ds\Big).$$

Below, we now present smoothness properties of solution (6.5) of nonhomogeneous problem (6.1). Let $u^{(m)}(t) := \frac{\partial^m u(t)}{\partial t^m}$ be the m-th order partial derivative of $u(t)$ with respect to time.

Theorem 6.2.2. *Let $j \in \{0,1\}$, $m \in \mathbb{N}$, and $u_0 \in L^2(\Omega)$. Then, for the solution $u(t)$ of (6.5), the following a priori estimates hold:*

$$\|A^j u^{(m)}(t)\| \le C\big(t^{-(m+j\alpha)}\,\|u_0\| + t^{-(m+1)-(j-1)\alpha}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big), \quad t > 0. \tag{6.10}$$

Moreover, for $u_0 \in \dot{H}^2(\Omega)$, we have

$$\|A^j u^{(m)}(t)\| \le C\big(t^{-m+(1-j)\alpha}\,\|u_0\|_2 + t^{-(m+1)-(j-1)\alpha}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big), \quad t > 0. \tag{6.11}$$

Remark 6.2.2. (*i*) *When $\alpha \to 1^-$, Theorem 6.2.1 and Theorem 6.2.2, respectively, produce similar stability estimates as in Theorem 2.1 and Theorem 2.2 of [47].*
(*ii*) *Such smoothing properties were discussed in Chapter 2 for smooth kernel but with a more general second-order partial differential operator under the integral sign. We wish to emphasize the fact that similar to the standard PIDE, the smoothing properties with respect to the spatial variable is limited to only one application of A.*
(*iii*) *For the solution of the homogeneous problem, we have the following stability estimates by the interpolation of (6.10) and (6.11): For $t > 0$*

$$\|u^{(m)}(t)\|_p \le Ct^{-(m+\frac{(p-q)\alpha}{2})}\,\|u_0\|_q, \quad 0 \le p, q \le 2.$$

Now, we present the proof of Theorem 6.2.2.

Proof of Theorem 6.2.2. For both estimates (6.10) and (6.11), we choose $\epsilon = \frac{1}{t}$. We first prove the estimate (6.10). From (6.5) it is easy to see that

$$A^j u^{(m)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^m e^{zt} A^j (l(z)I + A)^{-1}\Big(\frac{l(z)}{z}u_0 + \frac{l(z)}{z^\alpha}\widehat{f}(z)\Big)\,dz. \tag{6.12}$$

We first establish the estimate for $j=0$ and $m\in\mathbb{N}$. By virtue of the resolvent estimate (6.6) we obtain

$$\begin{aligned}
\|u^{(m)}(t)\| &\le C\int_{\Gamma_{\epsilon,\vartheta}} |z|^m e^{Re(z)t}\,\|(l(z)I+A)^{-1}\|\Big(\frac{|l(z)|}{|z|}\,\|u_0\|+\frac{|l(z)|}{|z|^\alpha}\,\|\widehat{f}(z)\|\Big)\,|dz| \\
&\le C\,\|u_0\|\,\Big(\int_{-\vartheta}^{\vartheta} t^{-m}e^{\cos\psi}\,d\psi+\int_{\frac{1}{t}}^{\infty} r^{m-1}e^{rt\cos\vartheta}\,dr\Big) \\
&+C\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\,\Big(\int_{-\vartheta}^{\vartheta} t^{-(m+1)+\alpha}e^{\cos\psi}\,d\psi+\int_{\frac{1}{t}}^{\infty} r^{m-\alpha}e^{rt\cos\vartheta}\,dr\Big) \\
&\le C\big(t^{-m}\,\|u_0\|+t^{-(m+1)+\alpha}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big).
\end{aligned}$$

Now, to prove for $j=1$ and $m\in\mathbb{N}$, we use the identity $A(l(z)I+A)^{-1}=(I-l(z)(l(z)I+A)^{-1})$ and (6.6) to have

$$\|A(l(z)I+A)^{-1}\|\le C.$$

Application of this bound and Lemma 6.2.1 with $|l(z)|\le C|z|^\alpha$ to (6.12) lead to

$$\begin{aligned}
\|Au^{(m)}(t)\| &\le C\int_{\Gamma_{\epsilon,\vartheta}} |z|^m e^{Re(z)t}\Big(\frac{|l(z)|}{|z|}\,\|u_0\|+\frac{|l(z)|}{|z|^\alpha}\,\|\widehat{f}(z)\|\Big)\,|dz| \\
&\le C\,\|u_0\|\,\Big(\int_{-\vartheta}^{\vartheta} t^{-(m+\alpha)}e^{\cos\psi}\,d\psi+\int_{\frac{1}{t}}^{\infty} r^{m+\alpha-1}e^{rt\cos\vartheta}\,dr\Big) \\
&+C\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\,\Big(\int_{-\vartheta}^{\vartheta} t^{-(m+1)}e^{\cos\psi}\,d\psi+\int_{\frac{1}{t}}^{\infty} r^{m+1-1}e^{rt\cos\vartheta}\,dr\Big) \\
&\le C\big(t^{-(m+\alpha)}\,\|u_0\|+t^{-(m+1)}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big),
\end{aligned}$$

which completes the proof for (6.10). Next, we turn our attention to prove assertion (6.11). We observe from (6.12) that, for $j=1$ and $m\in\mathbb{N}$, the estimate is similar to the estimate (6.10) with $j=0$ where u_0 is replaced by Au_0, hence we obtain

$$\|Au^{(m)}(t)\|\le C\big(t^{-m}\,\|Au_0\|+t^{-(m+1)}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big),\quad t>0.$$

Now, we prove the estimate for $j=0$ and $m\in\mathbb{N}$. Using the identity

$$A^{-1}(l(z)I+A)^{-1}=\frac{1}{l(z)}(A^{-1}-(l(z)I+A)^{-1})$$

and (6.12), we write

$$
\begin{aligned}
u^{(m)}(t) &= \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^m e^{zt} A^{-1}(l(z)I+A)^{-1}\frac{l(z)}{z}Au_0\,dz \\
&\quad + \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^m e^{zt}(l(z)I+A)^{-1}\frac{l(z)}{z^\alpha}\widehat{f}(z)\,dz \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^{(m-1)}e^{zt}u_0\,dz - \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^{(m-1)}e^{zt}(l(z)I+A)^{-1}Au_0\,dz \\
&\quad + \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^m e^{zt}(l(z)I+A)^{-1}\frac{l(z)}{z^\alpha}\widehat{f}(z)\,dz \\
&= -\frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^{(m-1)}e^{zt}(l(z)I+A)^{-1}Au_0\,dz \\
&\quad + \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} z^m e^{zt}(l(z)I+A)^{-1}\frac{l(z)}{z^\alpha}\widehat{f}(z)\,dz, \\
&=: I_1 + I_2,
\end{aligned}
$$

where we have used the fact that $\frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}z^{(m-1)}\,dz = 0$ for $m\in\mathbb{N}$. The bound for the term I_2 is same as before and thus it suffices to bound I_1. By (6.6) and from the definition of $l(z)$ as in (6.4) we get

$$\|(l(z)I+A)^{-1}\| \leq C/|l(z)| \leq C\frac{(1+|z|^\beta)}{|z|^{\alpha+\beta}} \leq C(|z|^{-(\alpha+\beta)}+|z|^{-\alpha}).$$

With an aid of the above bound, we proceed as previous calculations to conclude

$$\|I_1\| \leq C\int_{\Gamma_{\epsilon,\vartheta}} |z|^{(m-1)}e^{Re(z)t}(|z|^{-(\alpha+\beta)}+|z|^{-\alpha})\,\|Au_0\|\,|dz| \leq Ct^{-m+\alpha}\,\|Au_0\|\,.$$

Now combine the estimates of I_1 and I_2 to arrive at

$$\|u^{(m)}(t)\| \leq \|I_1\|+\|I_2\| \leq C\big(t^{-m+\alpha}\,\|Au_0\| + t^{-(m+1)+\alpha}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\big),\quad t>0,$$

and this completes the proof. $\square$

6.3 Error Analysis for the Semidiscrete GFEM

In this section, we consider the spatial semidiscrete GFEM for problem (6.1) and derive optimal order error bounds for both homogeneous and nonhomogeneous problems. Let $\{\mathcal{T}_h\}_{0<h<1}$ be a family of quasi-uniform triangulations of the domain $\overline{\Omega}$, where h is the largest diameter among all finite elements. On $\mathcal{T}_h$, construct the finite element function space $S_h = S_h(\Omega) \subset H_0^1(\Omega)$ of continuous, piecewise linear functions:

$$S_h = \{\chi \in H_0^1(\Omega)\colon \chi|_\tau \text{ is a linear polynomial over } \tau \in \mathcal{T}_h\}.$$

We now state the semidiscrete Galerkin finite element approximation to (6.1) as follows: Find $u_h(t) \in S_h$ such that for $0 < t \leq T$,

$$\begin{aligned} (\partial_t^\alpha u_h(t), \chi) + (A_h u_h(t), \chi) + \int_0^t \kappa_\beta(t-s)(A_h u_h(s), \chi)\, ds = (f(t), \chi) \ \ \forall \chi \in S_h, \\ u_h(0) = u_{0h} \in S_h. \end{aligned} \tag{6.13}$$

Here u_{0h} is an approximation of u_0 in S_h and the operator $A_h : S_h \to S_h$ is the discrete version of the operator A defined by

$$(A_h \psi, \chi) = (\nabla \psi, \nabla \chi) \ \text{ for all } \ \psi, \chi \in S_h.$$

With $f_h = P_h f$, the L^2-projection of f on S_h, we can rewrite semidiscrete scheme (6.13) as

$$\begin{aligned} \partial_t^\alpha u_h(t) + A_h u_h(t) + \int_0^t \kappa_\beta(t-s) A_h u_h(s)\, ds = f_h(t) \ \ \forall \chi \in S_h, \\ u_h(0) = u_{0h} \in S_h. \end{aligned} \tag{6.14}$$

Analogous to the continuous case, the discrete solution $u_h(t)$ is given by

$$u_h(t) = \frac{1}{2\pi i} \int_{\Gamma_{\epsilon,\vartheta}} e^{zt} (l(z)I + A_h)^{-1} \Big(\frac{l(z)}{z} u_{0h} + \frac{l(z)}{z^\alpha} \widehat{f}_h(z) \Big)\, dz, \tag{6.15}$$

where $\pi/2 < |\vartheta| < \tilde{\vartheta}$. On S_h, we now introduce the following discrete norm

$$|||\chi|||^2_{\dot{H}_h^p(\Omega)} = \sum_{j=1}^{N} (\lambda_j^h)^p (\chi, \phi_j^h)^2, \ \ \chi \in S_h, \ \ p \in \mathbb{R},$$

where $\{\lambda_j^h\}_{j=1}^N$ are the eigenvalues with the corresponding eigenfunctions $\{\phi_j^h\}_{j=1}^N$ of the discrete operator A_h on Ω. With respect to this discrete norm, it is easy to establish similar a priori estimates for semidiscrete solution (6.15) as in Theorem 6.2.2. Thus, the details are omitted.

In the error analysis, we take $u_{0h} = P_h u_0$, where $P_h : L^2(\Omega) \to S_h$ is the standard L^2-projection defined by

$$(P_h v, \chi) = (v, \chi) \quad \text{for all } \chi \in S_h.$$

We also use the Ritz projection $R_h \colon H_0^1(\Omega) \to S_h$ defined by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi) \ \ \forall \chi \in S_h, \tag{6.16}$$

with the following approximation properties [68]:

$$\|R_h v - v\| + h\,\|\nabla(R_h v - v)\| \le Ch^p \|v\|_p\,, \quad v \in \dot{H}^p(\Omega),\ p = 1, 2. \tag{6.17}$$

Now we are ready to state optimal error bounds for semidiscrete scheme (6.14) for both homogeneous and nonhomogeneous problems. The first theorem is about the homogeneous problem covering both smooth and nonsmooth initial data cases while the second one related to the nonhomogeneous problem with zero initial data.

Theorem 6.3.1. *Let u be the solution of continuous problem (6.1) with $f = 0$, and let u_h be its finite element approximation given by (6.15). Then, for $u_0 \in L^2(\Omega)$ and $u_{0h} = P_h u_0$, we have*

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \le Ch^2 t^{-\alpha}\,\|u_0\|\,, \quad t > 0. \tag{6.18}$$

Moreover, if $u_0 \in \dot{H}^2(\Omega)$ and $u_{0h} = R_h u_0$, then

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \le Ch^2\,\|u_0\|_2\,, \quad t \ge 0. \tag{6.19}$$

Proof. The proofs of the above assertions are similar to Theorems 3.3.1-3.3.2 of Chapter 3 and hence we refrain from giving the details. ☐

We now state the semidiscrete error estimate of the nonhomogeneous problem with zero initial data.

Theorem 6.3.2. *Let u be the solution of continuous problem (6.1) satisfying the zero initial condition, and let u_h be its finite element approximation. Then, for $t > 0$,*

$$\|u(t) - u_h(t)\| + h\,\|\nabla(u(t) - u_h(t))\| \le Ch^2 t^{-1}\,\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}\,,$$

where the number $\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}$ is given by (3.64).

Proof. The proof of the theorem is analogous to Theorem 3.3.2 of Chapter 3 with the choice $|l(z)| \le C|z|^\alpha$ in Lemma 6.2.1. ☐

Remark 6.3.1. *By linearity, Theorems 6.3.1 and 6.3.2 establish the semidiscrete error estimate for (6.1).*

Next, for the purpose of the fully discrete analysis, we set $v_h(t) = u_h(t) - u_{0h}$. Then $v_h(0) = 0$. Further, v_h satisfies the following identity

(6.20)

$$\partial_t^\alpha v_h(t) + A_h v_h(t) + \int_0^t \kappa_\beta(t-s) A_h v_h(s)\,ds = -A_h u_{0h} - \int_0^t \kappa_\beta(t-s) A_h u_{0h}\,ds + f_h(t),$$

where $f_h(t) = P_h f(t)$. Now, the Laplace transform to (6.20) yields

$$(l(z)I + A_h)\widehat{v}_h(z) = -\frac{1}{z}A_h u_{0h} + \frac{z^\beta}{z^\beta+1}\widehat{f}_h(z) = -\frac{1}{z}A_h u_{0h} + \frac{l(z)}{z^\alpha}\widehat{f}_h(z).$$

By inverting $\widehat{v}_h(z)$, we obtain the following contour integral representation for $v_h(t)$:

$$v_h(t) = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}} e^{zt}(l(z)I + A_h)^{-1}\Big(-\frac{1}{z}A_h u_{0h} + \frac{l(z)}{z^\alpha}\widehat{f}_h(z)\Big)\,dz. \tag{6.21}$$

6.4 Error Analysis for the L1 Scheme

This section considers the fully discrete L1 scheme for problem (6.1) and derives the related error estimates. Consider a uniform partition of the interval $[0, T]$ with grid points $\{t_n\}_{n=0}^N$, where $t_n = nk$, $n = 0, \dots, N$ and $k = T/N$, the time step size. The L1 approximation at t_n of the Caputo fractional derivative $\partial_t^\alpha u(t)$ is given by (cf. [37, 67])

$$\begin{aligned}
\partial_t^\alpha u(t_n) &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_n-s)^{-\alpha}\frac{\partial u(s)}{\partial s}\,ds\\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}\frac{u(t_{j+1})-u(t_j)}{k}\int_{t_j}^{t_{j+1}}(t_n-s)^{-\alpha}\,ds\\
&= \sum_{j=0}^{n-1} d_j\frac{u(t_{n-j})-u(t_{n-j-1})}{k^\alpha}\\
&= k^{-\alpha}\Big(d_0u(t_n) - d_{n-1}u(t_0) + \sum_{i=1}^{n-1}(d_j-d_{j-1})u(t_{n-j})\Big) =: Q_n(u),
\end{aligned} \tag{6.22}$$

where the weights d_j are given by

$$d_j = \frac{(j+1)^{1-\alpha}-j^{1-\alpha}}{\Gamma(2-\alpha)}, \quad j = 0, \dots, N-1.$$

For the integral term in (6.1), we apply a quadrature formula of the form

$$q_n(\psi) = \sum_{j=0}^{n}\kappa_{\beta,n-j}\psi^j \approx \int_0^{t_n}\kappa_\beta(t_n-s)\psi(s)\,ds, \tag{6.23}$$

where $\psi^j = \psi(t_j)$. In our case, $\kappa_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ and the quadrature weights $\kappa_{\beta,j}$ are determined by the following generating function

$$\sum_{j=0}^{\infty}\kappa_{\beta,j}\xi^j = K(\delta(\xi)/k),$$

where $K(z) = \mathcal{L}\{\kappa_\beta(t)\} = \frac{1}{z^\beta}$ and δ is the quotient of the generating polynomials of a stable and consistent linear multistep method. Here, we consider the BE method and hence $\delta(\xi) = (1-\xi)$. Therefore, we get

$$\kappa_{\beta,j} = k^\beta(-1)^j\binom{-\beta}{j} = k^\beta\beta_j, \quad \beta_j = (-1)^j\binom{-\beta}{j} \text{ for all } j.$$

Following [41], we omit the term for $j=0$, i.e., ψ^0 in (6.23) and rewrite it as $q_n(\psi) = \sum_{j=1}^n \kappa_{\beta,n-j}\psi^j$. Let U_h^n be the fully discrete approximation of $u(t_n)$. The fully discrete L1 scheme thus reads: Seek $U_h^n \in S_h$ for $n = 1, 2, \ldots, N$ such that

$$Q_n(U_h) + A_hU_h^n + q_n(A_hU_h) = f_h^n, \quad \text{with } U_h^0 = u_{0h}, \quad f_h^n = P_hf(t_n).$$

Expanding this we get

(6.24)

$$[d_0I + k^\alpha(k^\beta\beta_0+1)A_h]U_h^n = d_{n-1}U_h^0 + \sum_{j=1}^{n-1}[(d_{n-j-1}-d_{n-j}) - k^{\alpha+\beta}\beta_{n-j}A_h]U_h^j + k^\alpha f_h^n,$$

with $U_h^0 = u_{0h}$ and $f_h^n = P_hf(t_n)$. We first decompose the error $u(t_n) - U_h^n$ as follows

$$u(t_n) - U_h^n = (u(t_n) - u_h(t_n)) + (u_h(t_n) - U_h^n).$$

For both homogeneous and nonhomogeneous problems, error bounds for the term $u(t_n) - u_h(t_n)$ are contained in Theorems 6.3.1 and 6.3.2. So, our next task is to bound the term $u_h(t_n) - U_h^n$.

To analyze the fully discrete scheme (6.24), we first derive a discrete version of the solution (6.21). Like before, in this case we set $W_h^n = U_h^n - U_h^0$, so that $W_h^0 = 0$. Then the fully discrete approximation W_h^n satisfies for $n = 1, 2, \ldots, N$,

$$Q_n(W_h) + A_hW_h^n + q_n(A_hW_h) = -A_hu_{0h} - q_n(1)A_hu_{0h} + f_h^n, \quad \text{with } W_h^0 = 0. \tag{6.25}$$

Now, for a sequence $(\omega_j)_{j=0}^\infty$, define $\tilde{\omega}(\xi) := \sum_{j=0}^\infty \omega_j\xi^j$ to represent a generating function or a discrete Laplace transform. Multiply both sides of (6.25) by ξ^n and sum from 1 to ∞ to obtain

(6.26)

$$\sum_{n=1}^\infty Q_n(W_h)\xi^n + A_h\tilde{W}_h(\xi) + \sum_{n=1}^\infty q_n(A_hW_h)\xi^n = -\frac{\xi}{1-\xi}A_hu_{0h} - \sum_{n=1}^\infty q_n(1)\xi^nA_hu_{0h} + \frac{\xi}{1-\xi}f_h(0) + \tilde{G}_h(\xi),$$

where $G_h(t) = f_h(t) - f_h(0)$. Following [26], we have

$$\sum_{n=1}^{\infty} Q_n(W_h)\xi^n = \frac{k^{-\alpha}(1-\xi)^2 \mathrm{Li}_{\alpha-1}(\xi)}{\xi\Gamma(2-\alpha)} \tilde{W}_h(\xi),$$

where $\mathrm{Li}_{\alpha-1}$ denotes the polylogarithm function (cf. [14, 35]) defined by

$$\mathrm{Li}_{\alpha-1}(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^{\alpha-1}}, \quad |z| < 1.$$

Further, using $W_h^0 = 0$ and the convolution property of the discrete Laplace transforms, we obtain

$$\sum_{n=1}^{\infty} q_n(A_h W_h)\xi^n = \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n} \kappa_{n-j} A_h W_h^j\Big)\xi^n = \tilde{\kappa}(\xi) A_h \tilde{W}_h(\xi) = \frac{k^\beta}{(1-\xi)^\beta} A_h \tilde{W}_h(\xi),$$

where $\tilde{\kappa}(\xi) = \sum_{j=0}^{\infty} \kappa_j \xi^j = \frac{k^\beta}{(1-\xi)^\beta}$. Similarly, with $1_k = (0,1,1,\ldots)$, for which $\tilde{1}_k(\xi) = \frac{\xi}{1-\xi}$, it follows that

$$\sum_{n=1}^{\infty} q_n(1)\xi^n A_h u_{0h} = \frac{k^\beta \xi}{(1-\xi)^{\beta+1}} A_h u_{0h}.$$

Inserting all these in (6.26) and simplifying the resulting expression we arrive at

$$\begin{aligned}\tilde{W}_h(\xi) =& \Big(\frac{(1-\xi)^2}{\xi k^\alpha \Gamma(2-\alpha)} \frac{(1-\xi)^\beta}{(1-\xi)^\beta + k^\beta} \mathrm{Li}_{\alpha-1}(\xi) I + A_h\Big)^{-1} \\ & \Big(-\frac{\xi}{(1-\xi)} A_h u_{0h} + \frac{\xi}{(1-\xi)} \frac{(1-\xi)^\beta}{(1-\xi)^\beta + k^\beta} f_h(0) + \frac{(1-\xi)^\beta}{(1-\xi)^\beta + k^\beta} \tilde{G}_h(\xi)\Big).\end{aligned}$$

It is easy to see that $\tilde{W}_h(\xi)$ is analytic at $\xi = 0$ and hence the generating function $\tilde{W}_h(\xi) = \sum_{n=0}^{\infty} W_h^n \xi^n$ has a positive radius of convergence. Using Cauchy's integral formula, for small ρ we obtain

$$\begin{aligned}W_h^n = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{1}{\xi^{n+1}} & \Big(\frac{(1-\xi)^2}{\xi k^\alpha \Gamma(2-\alpha)} \frac{(1-\xi)^\beta}{(1-\xi)^\beta + k^\beta} \mathrm{Li}_{\alpha-1}(\xi) I + A_h\Big)^{-1} \\ & \Big(-\frac{\xi}{(1-\xi)} A_h u_{0h} + \frac{\xi}{(1-\xi)} \frac{(1-\xi)^\beta}{(1-\xi)^\beta + k^\beta} f_h(0) + \frac{(1-\xi)^\beta}{(1-\xi)^\beta + k^\beta} \tilde{G}_h(\xi)\Big)\, d\xi.\end{aligned}$$

A simple change of variable ξ to z by the transformation $\xi = e^{-zk}$ leads to

$$\begin{aligned}W_h^n =& \frac{1}{2\pi i} \int_{\Gamma_1} k e^{z t_n} \big(M(z) I + A_h\big)^{-1} \\ & \Big(-\frac{e^{-zk}}{(1-e^{-zk})} A_h u_{0h} + \frac{e^{-zk}}{(1-e^{-zk})} \gamma(z) f_h(0) + \gamma(z) \tilde{G}_h(e^{-zk})\Big)\, dz,\end{aligned}$$

where $\gamma(z) = \mu(z)/(1+\mu(z))$ with $\mu(z) = (1-e^{-zk})^\beta/k^\beta$,

$$M(z) = \frac{(1-e^{-zk})^2}{e^{-zk}k^\alpha\Gamma(2-\alpha)}\gamma(z)\mathrm{Li}_{\alpha-1}(e^{-zk}),$$

and Γ_1 is the contour $\{z = -k^{-1}\ln\rho + iy : |y| \le \pi/k\}$ oriented counterclockwise. Finally, deform the contour Γ_1 to the contour $\Gamma_k = \{z \in \Gamma_{\epsilon,\vartheta} : |\mathrm{Im}(z)| \le \pi/k\}$ and apply the Cauchy theorem (see Fig. 6.1) to reach

$$\begin{aligned} W_h^n =& \frac{1}{2\pi i}\int_{\Gamma_k} ke^{zt_n}\big(M(z)I + A_h\big)^{-1} \\ & \Big(-\frac{e^{-zk}}{(1-e^{-zk})}A_h u_{0h} + \frac{e^{-zk}}{(1-e^{-zk})}\gamma(z)f_h(0) + \gamma(z)\tilde{G}_h(e^{-zk})\Big)\,dz. \end{aligned} \tag{6.27}$$

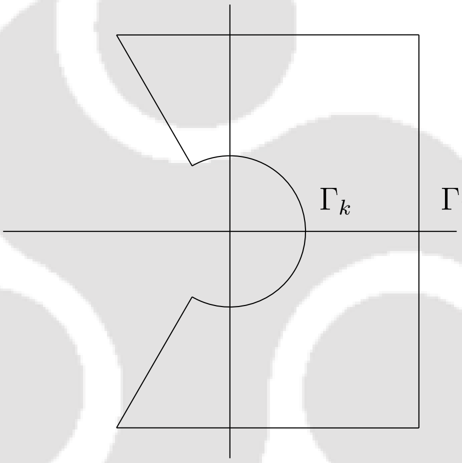


Figure 6.1: The integration contour

6.4.1 Auxiliary Results

In this section we shall derive fully discrete error estimates for the scheme (6.27). In view of (6.21) and (6.27) we get

(6.28)
$$\begin{aligned} v_h(t_n) - W_h^n =& \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}\setminus\Gamma_k} e^{zt_n}K_1(z)u_{0h}\,dz + \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}(K_1(z) - e^{-zk}K_3(z))u_{0h}\,dz \\ & + \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}\setminus\Gamma_k} e^{zt_n}K_2(z)\widehat{f}_h(z)\,dz \\ & + \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}(K_2(z)\widehat{f}_h(z) - K_4(z)\tilde{G}_h(e^{-zk}) - K_5(z)f_h(0))\,dz. \end{aligned}$$

Here the kernels K_i, $i = 1, \dots, 5$ are given by

$$\begin{aligned} K_1(z) &= -\frac{1}{z}(l(z)I + A_h)^{-1}A_h =: \frac{1}{z}M_1(z),\\ K_2(z) &= \frac{l(z)}{z^\alpha}(l(z)I + A_h)^{-1} =: \frac{l(z)}{z^\alpha}M_2(z),\\ K_3(z) &= -\frac{k}{(1-e^{-zk})}\Big(\frac{(1-e^{-zk})}{k^\alpha}\varphi(zk)I + A_h\Big)^{-1}A_h =: \frac{k}{(1-e^{-zk})}M_3(z),\\ K_4(z) &= k\gamma(z)\Big(\frac{(1-e^{-zk})}{k^\alpha}\varphi(zk)I + A_h\Big)^{-1} =: k\gamma(z)M_4(z),\\ K_5(z) &= \frac{ke^{-zk}}{(1-e^{-zk})}\gamma(z)\Big(\frac{(1-e^{-zk})}{k^\alpha}\varphi(zk)I + A_h\Big)^{-1} =: \frac{ke^{-zk}}{(1-e^{-zk})}\gamma(z)M_4(z), \end{aligned} \tag{6.29}$$

where the function φ is defined by

$$\varphi(zk) = \frac{(e^{zk}-1)}{\Gamma(2-\alpha)}\gamma(z)\mathrm{Li}_{\alpha-1}(e^{-zk}).$$

We first discuss the error analysis for the homogeneous problem and then treat the nonhomogeneous problem separately. To compute the error $v_h(t_n) - W_h^n$ in (6.28), we now proceed to bound the functions $K_1(z)$ and $(K_1(z) - e^{-zk}K_3(z))$. Using the identity $(l(z)I + A_h)^{-1}A_h = I - l(z)(l(z)I + A_h)^{-1}$ and the resolvent estimate (6.6) we obtain

$$\|K_1(z)\| = \|-\frac{1}{z}(l(z)I + A_h)^{-1}A_h\| = \|-\frac{1}{z}(I - l(z)(l(z)I + A_h)^{-1})\| \le C|z|^{-1}. \tag{6.30}$$

It now remains to estimate $\|(K_1(z) - e^{-zk}K_3(z))\|$. For this purpose, we shall prove below a series of lemmas. Let us recall the following estimates from [26, Lemma 3.1] of the function $g(z) := (1 - e^{-zk})/k$ on the contour Γ_k; there exist two positive constants C_1 and C_2 such that

$$C_1|z| \le |g(z)| \le C_2|z| \quad \text{and} \quad |g(z) - z| \le Ck|z|^2, \tag{6.31}$$

where $C_1 = \min\{(e^{-\cos\vartheta} - 1), 3 - e\}$, $\vartheta \in (\pi/2, \pi)$.

Lemma 6.4.1. *For $\mu(z) = g(z)^\beta = (1 - e^{-zk})^\beta/k^\beta$, there exist two positive constants C_1 and C_2 such that*

$$C_1^\beta|z|^\beta \le |\mu(z)| \le C_2^\beta|z|^\beta \quad \textit{and} \quad |\mu(z) - z^\beta| \le Ck|z|^{1+\beta} \quad \textit{for all} \;\; z \in \Gamma_k.$$

Proof. We apply the first statement of (6.31) to obtain the first assertion. Next, following [28, Lemma 3.2], we can conclude

$$|\mu(z) - z^\beta| \le Ck|z|^{1+\beta} \quad \text{for all} \;\; z \in \Gamma_k.$$

□

It is easy to see that the bounds of the function $|g(z)|$ in (6.31) remain true if we extend the contour Γ_k to some extent to right in the complex plane. Precisely, now onwards we consider for some small positive constant C_0,

$$|z| \geq \frac{(1+C_0)}{C_1}, \tag{6.32}$$

for which Lemma 6.4.1 also remains valid. This technicality enables us to obtain upper as well as lower bounds of the function $\gamma(z) = \mu(z)/(1+\mu(z))$, $z \in \Gamma_k$ in the next lemma. In the error analysis, the bounds of the function $|\gamma(z)|$ play a crucial role.

Lemma 6.4.2. *Let condition (6.32) be satisfied. Then, for* $\gamma(z) = \mu(z)/(1+\mu(z))$, *there exist two positive constants* C_3 *and* C_4 *such that*

$$C_3 \leq |\gamma(z)| \leq C_4 \quad \textit{for all} \quad z \in \Gamma_k.$$

Proof. From Lemma 6.4.1, an easy calculation shows

$$|\gamma(z)| \geq \frac{C_1|z|^\beta}{1+C_2|z|^\beta},$$

which combine with (6.32) completes the lower bound of $|\gamma(z)|$ on Γ_k. Next, we prove the upper bound. In view of Lemma 6.4.1 and (6.32), we have for all $z \in \Gamma_k$,

$$|\mu(z)| \geq C_1^\beta |z|^\beta \geq (1+C_0)^\beta \quad \Longrightarrow \quad |(\mu(z)| - 1 \geq (1+C_0)^\beta - 1 = C.$$

Thus,

$$|1+\mu(z)| \geq ||\mu(z)| - 1| \geq C \quad \text{for } z \in \Gamma_k. \tag{6.33}$$

Now apply (6.33) to get

$$|\gamma(z)| = \frac{|\mu(z)|}{|1+\mu(z)|} = |1 - \frac{1}{1+\mu(z)}| \leq 1 + \frac{1}{C} \leq C,$$

which completes the proof of the lemma. ☐

Remark 6.4.1. *From (6.32), it follows that* $|z^\beta + 1| \geq ||z^\beta| - 1| \geq C$ *and hence we get* $1/|1+z^\beta| \leq C$. *Further, applying this bound we get* $|z|^\beta/|1+z^\beta| = |1 - 1/(1+z^\beta)| \leq C$. *An alternative way to bound the function* $|z|^\beta/|1+z^\beta|$ *on* Γ_k *using Lemma 6.2.1 with* $|l(z)| \leq C|z|^\alpha$ *is*

$$\frac{|z|^\beta}{|1+z^\beta|} = \frac{|l(z)|}{|z^\alpha|} \leq C, \quad \textit{for all} \quad z \in S_{\tilde{\vartheta}}.$$

Since the contour Γ_k *lies in the sector* $S_{\tilde{\vartheta}}$, *the inequality follows.*

Remark 6.4.2. *The restriction (6.32) requires only to obtain a lower bound for the function $|1+\mu(z)|$ on Γ_k. If we can find a lower bound of $|1+\mu(z)|$ on Γ_k by some other means with no restriction on $|z|$, then we take $\epsilon = 1/t_n$ in the error estimates (see Theorem 6.4.1 below) as in the case of the subdiffusion problems [26].*

Let $\lambda(z) = k^\beta e^{-zk}/(k^\beta + (1-e^{-zk})^\beta) g(z)^{(1-\beta)}$. The following bounds are proved to be convenient in the error analysis.

Lemma 6.4.3. *With $\lambda(z)$ as above, we have*

$$|\lambda(z)| \le C\big(k + \frac{1}{|z|}\big) \quad \textit{and} \quad |\lambda(z) - \frac{l(z)}{z^{\alpha+1}}| \le Ck \quad \textit{for all } z \in \Gamma_k.$$

Proof. In view of the estimate (6.31), Lemma 6.4.1, and the bound (6.33) we obtain

$$\begin{aligned}
|\lambda(z)| &= |\frac{k^\beta e^{-zk}}{(k^\beta + (1-e^{-zk})^\beta) g(z)^{1-\beta}}| = |\frac{(e^{-zk}-1)+1}{(1+\mu(z))g(z)^{1-\beta}}| \\
&\le \frac{1}{|g(z)^{1-\beta}|}\Big(\frac{|(e^{-zk}-1)|}{|1+\mu(z)|} + \frac{1}{|1+\mu(z)|}\Big) \le \frac{C}{|z|^{1-\beta}}\Big(\frac{k|z|^{1-\beta}|z|^\beta}{|1+\mu(z)|} + \frac{1}{C_1|z|^\beta}\frac{C_1|z|^\beta}{|1+\mu(z)|}\Big) \\
&\le \Big(Ck\frac{|\mu(z)|}{|1+\mu(z)|} + \frac{1}{C_1|z|}\frac{|\mu(z)|}{|1+\mu(z)|}\Big) \le \Big(Ck|1 - \frac{1}{1+\mu(z)}| + \frac{1}{C_1|z|}|1 - \frac{1}{1+\mu(z)}|\Big) \\
&\le Ck + \frac{1}{C_1|z|},
\end{aligned}$$

which completes the proof of the first assertion. We now proceed to prove the second estimate. Inviting estimate (6.31), Lemma 6.4.1, and Remark 6.4.1 we arrive at

$$\begin{aligned}
|\frac{l(z)}{z^{\alpha+1}} - \lambda(z)| &= |\frac{1}{z^{1-\beta}(1+z^\beta)} - \frac{e^{-zk}}{g(z)^{1-\beta}(1+\mu(z))}| = |\frac{1}{z^{1-\beta}(1+z^\beta)} - \frac{(e^{-zk}-1)+1}{g(z)^{1-\beta}(1+\mu(z))}| \\
&\le \frac{|e^{-zk}-1|}{|g(z)^{1-\beta}(1+\mu(z))|} + \frac{|(g(z)^{1-\beta} - z^{1-\beta}) + (g(z)^{1-\beta}\mu(z) - z)|}{|z^{1-\beta}(1+z^\beta)g(z)^{1-\beta}(1+\mu(z))|} \\
&\le \frac{Ck|z|}{|g(z)^{1-\beta}(1+\mu(z))|} + \frac{|(g(z)^{1-\beta} - z^{1-\beta})| + |(g(z)^{1-\beta}\mu(z) - z)|}{|z^{1-\beta}(1+z^\beta)g(z)^{1-\beta}(1+\mu(z))|} \\
&\le \frac{Ck|z|^{1-\beta}|z|^\beta}{|g(z)^{1-\beta}(1+\mu(z))|} + \frac{Ck|z|^{2-\beta} + Ck|z|^2}{|z^{1-\beta}(1+z^\beta)g(z)^{1-\beta}(1+\mu(z))|} \\
&\le Ck + \Big(Ck\frac{|z|^\beta}{|1+z^\beta|}\frac{1}{|1+\mu(z)|} + Ck\frac{|z|^\beta}{|1+z^\beta|}\frac{|z|^\beta}{|1+\mu(z)|}\Big) \le Ck,
\end{aligned}$$

and this completes the rest of the proof. □

For subsequent use in our error analysis, we now calculate the error between the functions $\frac{(1-e^{-zk})}{k^\alpha}\varphi(zk)$ and $l(z)$.

Lemma 6.4.4. *For* $\varphi(zk) = \frac{(e^{zk}-1)(1-e^{-zk})^\beta}{\Gamma(2-\alpha)((1-e^{-zk})^\beta+k^\beta)} Li_{\alpha-1}(e^{-zk})$ *and* $\vartheta \in (\pi/2, 5\pi/6)$, *the following estimate*

$$|\frac{(1-e^{-zk})}{k^\alpha}\varphi(zk) - l(z)| \le C\big(k^{2-\alpha}|z|^2 + k|z|^{1+\alpha-\beta}\big) \quad \textit{for } z \in \Gamma_k$$

holds.

Proof. Observe that $\varphi(zk) = \frac{(e^{zk}-1)(1-e^{-zk})^\beta}{\Gamma(2-\alpha)((1-e^{-zk})^\beta+k^\beta)}\mathrm{Li}_{\alpha-1}(e^{-zk}) = \frac{(e^{zk}-1)}{\Gamma(2-\alpha)}\mathrm{Li}_{\alpha-1}(e^{-zk})\gamma(z)$, where $\gamma(z) = \mu(z)/1+\mu(z)$ with $\mu(z) = (1-e^{-zk})^\beta/k^\beta$. Thus,

$$\begin{aligned}\frac{(1-e^{-zk})}{k^\alpha}\varphi(zk) - l(z) =& \Big(\frac{(1-e^{-zk})}{k^\alpha}\frac{(e^{zk}-1)}{\Gamma(2-\alpha)}\mathrm{Li}_{\alpha-1}(e^{-zk}) - z^\alpha\Big)\gamma(z)\\ &+ \Big(z^\alpha\gamma(z) - l(z)\Big) =: I_3 + I_4.\end{aligned}$$

Using [26, Lemma 3.5] and Lemma 6.4.2 we obtain

$$|I_3| \le Ck^{2-\alpha}|z|^2 \quad \text{for } z \in \Gamma_k.$$

Further,

$$I_4 = z^\alpha\gamma(z) - l(z) = \frac{z^\alpha\mu(z)}{1+\mu(z)} - \frac{z^{\alpha+\beta}}{1+z^\beta} = \frac{z^\alpha(\mu(z)-z^\beta)}{(1+\mu(z))(1+z^\beta)}$$

Applications of Lemma 6.4.1, Lemma 6.4.2, and Remark 6.4.1 yield

$$\begin{aligned}|I_4| = \frac{|z^\alpha(\mu(z)-z^\beta)|}{|(1+\mu(z))(1+z^\beta)|} &\le \frac{Ck|z|^{\alpha+\beta+1}}{|(1+\mu(z))(1+z^\beta)|} \le Ck|z|^{1+\alpha-\beta}\frac{|z|^\beta}{|(1+\mu(z))|}\frac{|z|^\beta}{|1+z^\beta|}\\ &\le Ck|z|^{1+\alpha-\beta}\frac{|\mu(z)|}{|(1+\mu(z))|}\frac{|z|^\beta}{|1+z^\beta|} \le Ck|z|^{1+\alpha-\beta}.\end{aligned}$$

Combine the bounds of I_3 and I_4 to complete the proof. □

From now onwards we take $\vartheta \in (\pi/2, \pi)$. Now, let $\chi(z) = \frac{(1-e^{-zk})}{k^\alpha}\varphi(zk)$. Then, it has the following sector-preserving property: For some $\vartheta_0 \in (\pi/2, \pi)$, $\chi(z) \in S_{\vartheta_0}$ for all $z \in S_\vartheta$. This property plays a crucial role in the convergence analysis.

Lemma 6.4.5. *For* $\chi(z) = \frac{(1-e^{-zk})}{k^\alpha}\varphi(zk)$ *with* $\varphi(zk) = \frac{(e^{zk}-1)(1-e^{-zk})^\beta}{\Gamma(2-\alpha)((1-e^{-zk})^\beta+k^\beta)}Li_{\alpha-1}(e^{-zk})$, *there exists* $\vartheta_0 \in (\pi/2, \pi)$ *such that*

$$\chi(z) \in S_{\vartheta_0} \quad \textit{for all} \quad z \in S_\vartheta \textit{ with } \vartheta \textit{ close to } \pi/2.$$

Proof. To prove the assertion, we rewrite the function $\chi(z)$ as

$$\chi(z) = \frac{(1-e^{-zk})^\beta}{(1-e^{-zk})^\beta + k^\beta}\chi_1(z) = \gamma(z)\chi_1(z), \quad \text{where}$$
$$\chi_1(z) = \frac{(1-e^{-zk})(e^{zk}-1)}{k^\alpha\Gamma(2-\alpha)}\mathrm{Li}_{\alpha-1}(e^{-zk}).$$

Now, when ϑ is close to $\pi/2$, by the virtue of [26, Remark 3.8] we get $\chi_1(z) \in S_{\vartheta_1}$, where the sector S_{ϑ_1} is contained in the sector $S_{3\pi/4-\delta}$ for any positive δ. We choose $\delta = \pi/4 - \delta_1$ for some small positive δ_1 so that $\chi_1(z)$ lies in the sector $S_{\pi/2+\delta_1}$. Thus, we write $\chi_1(z) = R_1 e^{i\vartheta_1}$ with $\vartheta_1 \in (\pi/2, \pi/2+\delta_1)$. Next, we find the representation for the function $\gamma(z)$. When ϑ is close to $\pi/2$, by [18, Lemma 3.4, Eqs (3.13)-(3.14)] we can write

$$\frac{(1-e^{-zk})^\beta}{k^\beta} \in S_{\beta\vartheta_2} \ \text{ with } \beta\vartheta_2 \le \pi/2 - \delta_1.$$

Using this fact we obtain

$$\gamma(z) = \frac{R_2 e^{i\beta\vartheta_2}}{1+R_2 e^{i\beta\vartheta_2}} = \frac{R_2 e^{i\beta\vartheta_2}(1+R_2 e^{-i\beta\vartheta_2})}{R_2^*}, \quad \text{where } \ R_2^* = |(1+R_2 e^{i\beta\vartheta_2})|^2.$$

With the help of the functions $\gamma(z)$ and $\chi_1(z)$ we arrive at

$$\chi(z) = \frac{R_2 e^{i\beta\vartheta_2}(1+R_2 e^{-i\beta\vartheta_2})}{R_2^*} R_1 e^{i\vartheta_1} = R_3 e^{i(\beta\vartheta_2+\vartheta_1)} + R_4 e^{i\vartheta_1},$$

where $R_3 = (R_1R_2)/R_2^*$ and $R_4 = (R_1R_2^2)/R_2^*$. Since $(\beta\vartheta_2+\vartheta_1) \le (\pi/2-\delta_1)+(\pi/2+\delta_1) = \pi$, we conclude by the parallelogram law $\chi(z) \in S_{\vartheta_0}$, for some $\vartheta_0 \in (\pi/2, \pi)$.

□

Now, we are ready to calculate the error for the time discretization. For the homogeneous problem, in order to derive error bounds for both smooth and nonsmooth data we need to bound the term $\|K_1(z) - e^{-zk}K_3(z)\|$ as in (6.28). In view of Lemma 6.4.1, the boundedness of K_1 in (6.30), and the uniform boundedness of the function $|e^{-zk}|$ on Γ_k we obtain

$$\begin{aligned} \|K_1(z) - e^{-zk}K_3(z)\| &\le C|(1-e^{-zk})|\,\|K_1(z)\| + |e^{-zk}|\,\|K_1(z)-K_3(z)\| \\ &\le Ck + C\,\|K_1(z)-K_3(z)\|\,. \end{aligned} \tag{6.34}$$

In view of (6.34), it suffices to bound the term $\|K_1(z) - K_3(z)\|$. By the triangle inequality, estimate (6.31), and the bound $\|M_1(z)\| \le C$ it follows that

$$\begin{aligned} \|K_1(z) - K_3(z)\| &\le |g(z)|^{-1}\,\|M_1(z) - M_3(z)\| + |z^{-1} - g(z)^{-1}|\,\|M_1(z)\| \\ &\le C|z|^{-1}\,\|M_1(z) - M_3(z)\| + Ck. \end{aligned} \tag{6.35}$$

Thus, we are left with finding a bound for $\|M_1(z) - M_3(z)\|$. The following lemma provides a bound for $\|M_1(z) - M_3(z)\|$.

Lemma 6.4.6. *Let the functions $M_1(z)$ and $M_3(z)$ be defined as in (6.29). Then for $\epsilon < \pi/2k$ and ϑ close to $\pi/2$, we have*

$$\|M_1(z) - M_3(z)\| \le C\big(k^{2-\alpha}|z|^{2-\alpha} + k|z|^{1-\beta}\big) \quad \text{for all } \ z \in \Gamma_k.$$

Proof. With $\chi(z) = \frac{(1-e^{-zk})}{k^\alpha}\varphi(zk)$, we can write from (6.29)

$$M_1(z) = -I + l(z)(l(z)I + A_h)^{-1} \ \text{ and } \ M_3(z) = -I + \chi(z)(\chi(z)I + A_h)^{-1}.$$

Now, utilizing [72, Lemma 2.6 (Eq. 24)] and Lemma 6.4.2, we obtain

$$|\chi(z)| = |\frac{(1-e^{-zk})}{k^\alpha}\frac{(e^{zk}-1)}{\Gamma(2-\alpha)}\mathrm{Li}_{\alpha-1}(e^{-zk})||\gamma(z)| \ge C|z|^\alpha. \tag{6.36}$$

Applications of Lemma 6.4.4 and the above inequality yield

$$\begin{aligned}
&\|M_1(z) - M_3(z)\| \\
&= \|l(z)((l(z)I + A_h)^{-1} - (\chi(z)I + A_h)^{-1}) + (l(z) - \chi(z))(\chi(z)I + A_h)^{-1}\| \\
&\le |l(z)|\,\|(l(z)I + A_h)^{-1} - (\chi(z)I + A_h)^{-1}\| + |l(z) - \chi(z)|\,\|(\chi(z)I + A_h)^{-1}\| \\
&\le C|l(z) - \chi(z)|\,\|(\chi(z)I + A_h)^{-1}\| + |l(z) - \chi(z)|\,\|(\chi(z)I + A_h)^{-1}\| \\
&\le C|l(z) - \chi(z)||\chi(z)|^{-1} \le C\big(k^{2-\alpha}|z|^{2-\alpha} + k|z|^{1-\beta}\big),
\end{aligned}$$

which completes the proof of the assertion. ☐

Applying Lemma 6.4.6 and (6.35) in (6.34), we can obtain for $z \in \Gamma_k$,

$$\|K_1(z) - e^{-zk}K_3(z)\| \le C|z|^{-1}\,\|M_1(z) - M_3(z)\| + Ck \le C\big(k + k|z|^{-\beta}\big). \tag{6.37}$$

6.4.2 Error Estimates for the Homogeneous Problem

In this part we calculate the error for the homogeneous problem, i.e., $f = 0$ in (6.1). The following theorem states the error estimates for the homogeneous problem for both smooth and nonsmooth initial data.

Theorem 6.4.1. *Assume that $f = 0$. Let u_h be the solution represented by (6.15), and let U_h^n be the solution of fully discrete problem (6.24). Then, for $u_0 \in L^2(\Omega)$ and $u_{0h} = P_h u_0$,*

$$\|u_h(t_n) - U_h^n\| \le Ckt_n^{-1}\,\|u_0\|\,, \quad n \ge 1. \tag{6.38}$$

Furthermore, for $u_0 \in \dot{H}^2(\Omega)$ *and* $u_{0h} = R_h u_0$*, we obtain*

$$\|u_h(t_n) - U_h^n\| \le Ckt_n^{\alpha-1}\|u_0\|_2, \quad n \ge 1. \tag{6.39}$$

Proof. To prove (6.38) and (6.39), we choose $\epsilon = (1 + C_0)/C_1 \ge 1$ as in (6.32). We first prove the nonsmooth data estimate (6.38). From (6.28), we have the following representation of the error $v_h(t_n) - W_h^n$,

$$\begin{aligned} v_h(t_n) - W_h^n &= \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}\setminus\Gamma_k} e^{zt_n} K_1(z) u_{0h}\, dz + \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}(K_1(z) - e^{-zk}K_3(z))u_{0h}\, dz \\ &=: I_5 + I_6. \end{aligned}$$

Using the fact $\|K_1(z)\| \le C|z|^{-1}$ (see (6.30)) and the L^2-stability of P_h we obtain

$$\|I_5\| \le C\|u_{0h}\| \int_{\pi/k\sin\vartheta}^{\infty} e^{rt_n\cos\vartheta} r^{-1}\, dr \le Ckt_n^{-1}\|u_0\|.$$

Now, the L^2-stability of P_h and estimate (6.37) yield

$$\begin{aligned} \|I_6\| &\le C\|u_{0h}\| \int_{\Gamma_k} |e^{zt_n}|\, \|K_1(z) - e^{-zk}K_3(z)\|\, |dz| \\ &\le C\|u_0\| \int_{\Gamma_k} e^{t_n Re(z)}\big(k + k|z|^{-\beta}\big)\, |dz| =: \|I_6^1\| + \|I_6^2\|, \end{aligned}$$

where

$$\begin{aligned} \|I_6^1\| &= C\|u_0\| \int_{\Gamma_k} e^{t_n Re(z)} k\, |dz| \le Ck\|u_0\| \left(\int_{-\vartheta}^{\vartheta} e^{t_n\epsilon\cos\psi}\epsilon\, d\psi + \int_{\epsilon}^{\pi/k\sin\vartheta} e^{rt_n\cos\theta}\, dr\right) \\ &\le Ck\|u_0\|\left(\epsilon e^{\epsilon T} + t_n^{-1}\right) \le Ckt_n^{-1}\|u_0\|. \end{aligned}$$

On the other-hand

$$\begin{aligned} \|I_6^2\| &= Ck^{\alpha}\|u_0\| \int_{\Gamma_k} e^{t_n Re(z)}|z|^{-\beta}\, |dz| \\ &\le Ck\|u_0\|\left(\int_{-\vartheta}^{\vartheta} \epsilon^{-\beta} e^{t_n\epsilon\cos\psi}\epsilon\, d\psi + \int_{\epsilon}^{\pi/k\sin\vartheta} e^{rt_n\cos\theta} r^{-\beta}\, dr\right) \\ &\le Ck\|u_0\|\left(\int_{-\theta}^{\theta} \epsilon^{1-\beta} e^{t_n\epsilon\cos\psi}\epsilon\, d\psi + \int_{0}^{\infty} e^{-r(t_nC)} r^{(1-\beta)-1}\, dr\right) \\ &\le Ck\|u_0\|\left(\epsilon^{1-\beta}e^{\epsilon T} + t_n^{\beta-1}\right) \le Ckt_n^{\beta-1}\|u_0\|, \end{aligned} \tag{6.40}$$

and this implies

$$\|I_6\| \le \|I_6^1\| + \|I_6^2\| \le Ckt_n^{-1}\|u_0\| + Ckt_n^{\beta-1}\|u_0\| \le Ckt_n^{-1}\|u_0\|.$$

Since $v_h(t_n) - W_h^n = u_h(t_n) - U_h^n$, the proof follows by the triangle inequality and appealing the bounds of I_5 and I_6. Next, we turn our attention to smooth data error estimate (6.39). In this case, $u_0 \in \dot{H}^2(\Omega)$ and $u_{0h} = R_h u_0$. Let

$$\begin{aligned} K_1^s(z) &= -z^{-1}(l(z)I + A_h)^{-1} =: z^{-1}M_1^s(z), \\ K_3^s(z) &= -\frac{k}{(1-e^{-zk})}(\chi(z)I + A_h)^{-1} =: \frac{k}{(1-e^{-zk})}M_3^s(z). \end{aligned}$$

Following the proof of Lemma 6.4.6 we obtain for all $z \in \Gamma_k$,

$$\begin{aligned} \|M_1^s(z) - M_3^s(z)\| &= \|(l(z)I + A_h)^{-1} - (\chi(z)I + A_h)^{-1}\| \\ &\le C\big(k^{2-\alpha}|z|^{2-2\alpha} + k^{2-\alpha}|z|^{2-2\alpha-\beta} + k|z|^{1-\alpha-\beta} + k|z|^{1-\alpha-2\beta}\big), \end{aligned} \tag{6.41}$$

where we have used $1/|l(z)| \le (1+|z|^\beta)/|z|^{\alpha+\beta} \le \big(|z|^{-\alpha} + |z|^{-\alpha-\beta}\big)$. Use of estimates (6.31) and (6.41) leads to

$$\begin{aligned} \|K_1^s(z) - K_3^s(z)\| &\le |z^{-1} - g(z)^{-1}|\,\|M_1^s(z)\| + |g(z)|^{-1}\,\|M_1^s(z) - M_3^s(z)\| \\ &\le \frac{|g(z) - z|}{|z||g(z)|}\,\|M_1^s(z)\| + C|z|^{-1}\,\|M_1^s(z) - M_3^s(z)\| \\ &\le C\big(k|z|^{-\alpha} + k|z|^{-\alpha-\beta} + k|z|^{-\alpha-2\beta}\big). \end{aligned} \tag{6.42}$$

Now from (6.28), for the homogeneous problem we write

$$\begin{aligned} v_h(t_n) - W_h^n = \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}\setminus\Gamma_k} e^{zt_n} K_1^s(z) A_h u_{0h}\, dz \\ + \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}(K_1^s(z) - e^{-zk}K_3^s(z))A_h u_{0h}\, dz =: I_7 + I_8. \end{aligned}$$

Our next task is to bound the kernels $K_1^s(z)$ and $(K_1^s(z) - e^{-zk}K_3^s(z))$. An easy calculation shows that $\|K_1^s(z)\| \le C|z|^{-1}/|l(z)| \le C\big(|z|^{-(1+\alpha)} + |z|^{-(1+\alpha+\beta)}\big)$. Using this bound, (6.31), and (6.42) we obtain

$$\begin{aligned} \|K_1^s(z) - e^{-zk}K_3^s(z)\| &\le |(1-e^{-zk})|\,\|K_1^s(z)\| + |e^{-zk}|\,\|K_1^s(z) - K_3^s(z)\| \\ &\le C\big(k|z|^{-\alpha} + k|z|^{-\alpha-\beta} + k^\alpha|z|^{-\alpha-2\beta}\big). \end{aligned} \tag{6.43}$$

Utilizing the bound of $K_1^s(z)$, $A_h R_h = P_h A$, and the L^2-stability of P_h, we get

$$\begin{aligned} \|I_7\| &\le C\,\|A_h u_{0h}\|\left(\int_{\pi/k\sin\vartheta}^{\infty} e^{rt_n\cos\vartheta} r^{-(1+\alpha)}\, dr + \int_{\pi/k\sin\vartheta}^{\infty} e^{rt_n\cos\vartheta} r^{-(1+\alpha+\beta)}\, dr\right) \\ &\le Ckt_n^{\alpha-1}\,\|A_h R_h u_0\| + Ck^{1+\beta}t_n^{\alpha-1}\,\|A_h R_h u_0\| \le Ckt_n^{\alpha-1}\,\|u_0\|_2\,. \end{aligned}$$

It follows from (6.43) and similar calculations as in (6.40) that

$$\begin{aligned}\|I_8\| &\leq C\,\|A_h u_{0h}\| \int_{\Gamma_k} |e^{zt_n}|\,\|K_1^s(z) - e^{-zk}K_3^s(z)\|\,|dz| \\ &\leq C\,\|A_h R_h u_0\|\,kt_n^{\alpha-1} \leq Ckt_n^{\alpha-1}\,\|u_0\|_2\,,\end{aligned}$$

where we have used the relation $A_h R_h = P_h A$ and the L^2-stability of P_h. Since $v_h(t_n) - W_h^n = u_h(t_n) - U_h^n$, the proof of (6.39) follows from the triangle inequality and the bounds of I_7 and I_8. □

Remark 6.4.3. *Theorem 6.4.1 produces an error estimate of order $O(k)$ for both smooth and nonsmooth initial data. This is in contrast to the result obtained in [71, Theorem 3.1], where the author has established a suboptimal error estimate with a general second-order memory kernel and a particular β, namely, $\beta = 1/2$. Moreover, our smooth data error bound (6.39) is similar to the result proved in [9, Theorem 3] for the standard PIDE ($\alpha = 1$).*

6.4.3 Error Estimates for the Nonhomogeneous Problem

In this part we calculate the error for the nonhomogeneous problem with zero initial data, i.e., $u_0 = 0$ in (6.1). Recall Eq. (6.28):

$$\begin{aligned}v_h(t_n) - W_h^n =& \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}\setminus\Gamma_k} e^{zt_n} K_2(z)\widehat{f}_h(z)\,dz \\ &+ \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}(K_2(z)\widehat{f}_h(z) - K_4(z)\tilde{G}_h(e^{-zk}) - K_5(z)f_h(0))\,dz,\end{aligned} \tag{6.44}$$

where the kernels K_2, K_4, and $K_5(z)$ are given by (6.29). Since, $f_h(t) = f_h(0) + (1 * f_h'(t))$, we first carry out the error analysis when $f_h(t) = f_h(0)$, i.e., $f_h(t)$ is time-independent and then $f_h(t) = 1 * f_h'(t)$. By linearity, these estimates will serve the error etimate for the nonhomogeneous problem. In the first case, i.e., for $f_h(t) = f_h(0)$, we obtain the following error estimate.

Lemma 6.4.7. *Let u_h be the solution represented by (6.15) with $u_0 = 0$ and U_h^n be the solution of fully discrete problem (6.24) with $U_h^0 = 0$. With $f_h = P_h f(0)$, the following error estimate*

$$\|u_h(t_n) - U_h^n\| \leq Ckt_n^{\alpha-1}\,\|f(0)\|\,, \quad n \geq 1$$

holds.

Proof. For this choice $f_h(t) = f_h(0)$, the representation of $v_h(t_n) - W_n^n$ in (6.44) takes the form

$$\begin{aligned} v_h(t_n) - W_h^n &= \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}\setminus\Gamma_k} e^{zt_n}\frac{l(z)}{z^{\alpha+1}}(l(z)I + A_h)^{-1} f_h(0)\,dz \\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}\big[\frac{l(z)}{z^{\alpha+1}}(l(z)I + A_h)^{-1} - \lambda(z)(\chi(z)I + A_h)^{-1}\big] f_h(0)\,dz \\ &=: I_9 + I_{10}, \end{aligned}$$

where $\lambda(z) = \frac{k^\beta e^{-zk}}{(k^\beta + (1-e^{-zk})^\beta) g(z)^{(1-\beta)}}$. Thus, it suffices to bound the terms I_9 and I_{10}. Similar to I_5 bound we obtain

$$\|I_9\| \leq Ckt_n^{\alpha-1}\,\|f(0)\|\,.$$

To bound I_{10}, let $\eta(z) = \frac{l(z)}{z^{\alpha+1}}(l(z)I + A_h)^{-1} - \lambda(z)(\chi(z)I + A_h)^{-1}$, $z \in \Gamma_k$. Our next aim is to bound $\eta(z)$. By the triangle inequality we write

$$\begin{aligned} \|\eta(z)\| &= \|l(z)/z^{\alpha+1}(l(z)I + A_h)^{-1} - \lambda(z)(\chi(zk)I + A_h)^{-1}\| \\ &= \|\lambda(z)(M_2(z) - M_4(z)) + (l(z)/z^{\alpha+1} - \lambda(z))M_2(z)\| \qquad (6.45) \\ &\leq |\lambda(z)|\,\|M_2(z) - M_4(z)\| + |l(z)/z^{\alpha+1} - \lambda(z)|\,\|M_2(z)\|\,, \end{aligned}$$

where the functions $M_2(z)$ and $M_4(z)$ are defined in (6.29). Now, analogous to (6.41) we find for $z \in \Gamma_k$ that

$$\|M_2(z) - M_4(z)\| \leq C\big(k^{2-\alpha}|z|^{2-2\alpha} + k^{2-\alpha}|z|^{2-2\alpha-\beta} + k|z|^{1-\alpha-\beta} + k|z|^{1-\alpha-2\beta}\big).$$

By the resolvent estimate (6.6), we get

$$\|M_2(z)\| \leq C/|l(z)| \leq C\big(|z|^{-\alpha} + |z|^{-\alpha-\beta}\big), \quad z \in \Gamma_k.$$

Using the above two bounds and Lemma 6.4.3 in (6.45) it now leads to

$$\|\eta(z)\| \leq C\big(k|z|^{-\alpha} + k|z|^{-\alpha-\beta} + k|z|^{-\alpha-2\beta}\big),$$

where we have used $k|z| \leq \pi/\sin\vartheta \leq C$. Utilizing this bound, we proceed as in (6.40) to obtain

$$\|I_{10}\| \leq C\int_{\Gamma_k} |e^{zt_n}|\,\|\eta(z)\|\,\|f_h(0)\|\,|dz| \leq Ckt_n^{\alpha-1}\,\|f(0)\|\,.$$

Altogether the bounds of I_9 and I_{10} yield

$$\|u_h(t_n) - U_h^n\| \leq C\big(kt_n^{\alpha-1}\,\|f(0)\| + kt_n^{\alpha-1}\,\|f(0)\|\,\big) \leq Ckt_n^{\alpha-1}\,\|f(0)\|\,.$$

□

The following lemma derives error estimate when the source term f_h is of the form $f_h = 1 * f_h'$ with $f_h(0) = 0$.

Lemma 6.4.8. *Let u_h be the solution represented by (6.15) with $u_0 = 0$, and let U_h^n be the solution of fully discrete problem (6.24) satisfying $U_h^0 = 0$. Then, with $f_h = P_h f$, $f(0) = 0$, and $f' = \frac{\partial f}{\partial t}$, we have*

$$\|u_h(t_n) - U_h^n\| \le Ck \int_0^{t_n} (t_n - s)^{\alpha-1} \|f'(s)\| \, ds.$$

Proof. Since $f_h(0) = 0$, we write $f_h(t) = f_h(0) + (1 * f_h'(t)) = 1 * f_h'(t)$. The error analysis for this situation proceeds as follows. For $u_0 = 0$, we have by (6.7)

$$u_h(t_n) = \int_0^{t_n} F(t_n - s) f_h(s) \, ds = (F * (1 * f_h'))(t_n) = ((F * 1) * f_h')(t_n), \tag{6.46}$$

where the operator $F(t)$ is given by

$$F(t) = \frac{1}{2\pi i} \int_{\Gamma_{\epsilon,\vartheta}} e^{zt} K_2(z) \, dz = \frac{1}{2\pi i} \int_{\Gamma_{\epsilon,\vartheta}} e^{zt} \frac{l(z)}{z^\alpha} (l(z)I + A_h)^{-1} \, dz.$$

Now, using the generating function $\tilde{f}_h(\xi) = \sum_{j=1}^{\infty} f_h(t_j)\xi^j$ and $\tilde{U}_h(\xi) := \widetilde{F}(\chi(\xi))\tilde{f}_h(\xi) = \frac{(1-\xi)^\beta}{(1-\xi)^\beta + k^\beta} (\chi(\xi)I + A_h)^{-1} \tilde{f}_h(\xi)$, we write the fully discrete solution U_h^n as

$$U_h^n = \sum_{j=0}^{n} F_k^{n-j} f_h(t_j) \quad \text{with} \quad \widetilde{F}(\chi(\xi)) = \sum_{j=0}^{\infty} F_k^j \xi^j.$$

An easy calculation to the above yields

$$F_k^n = \frac{k}{2\pi i} \int_{\Gamma_k} e^{zt_n} \frac{(1 - e^{-zk})^\beta}{(1 - e^{-zk})^\beta + k^\beta} (\chi(z)I + A_h)^{-1} \, dz.$$

In addition to the above, using the facts $|(1 - e^{-zk})^\beta / ((1 - e^{-zk})^\beta + k^\beta)| \le C$ and Lemma 6.4.4, we get

$$\|F_k^n\| \le Ckt_n^{\alpha-1}. \tag{6.47}$$

Let σ be a number in $(0, k)$. Also, let $\delta_{t_n - \sigma}$ be the Dirac delta function at $t_{n-\sigma}$. With the function $F_{k,\sigma}(t) = \sum_{n=0}^{\infty} F_k^n \delta_{t_n - \sigma}(t)$, we rewrite the fully discrete solution U_h^n as

$$U_h^n = \lim_{\sigma \to 0} (F_{k,\sigma} * f_h)(t_n) = \lim_{\sigma \to 0} (F_{k,\sigma} * (1 * f_h'))(t_n) = \Big(\lim_{\sigma \to 0} (F_{k,\sigma} * 1) * f_h' \Big)(t_n). \tag{6.48}$$

By the discrete Laplace transform, the discrete convolution rule, and Cauchy's integral formula we deduce that

$$\begin{aligned}\lim_{\sigma\to 0}(F_{k,\sigma}*1)(t_n) &= \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}\frac{ke^{-zk}(1-e^{-zk})^{\beta}}{((1-e^{-zk})^{\beta}+k^{\beta})(1-e^{-zk})}(\chi(z)I+A_h)^{-1}\,dz\\ &= \frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}\lambda(z)(\chi(z)I+A_h)^{-1}\,dz.\end{aligned}$$

From this relation and the definition of $F(t)$ it follows that

$$\begin{aligned}\lim_{\sigma\to 0}\big((F-F_{k,\sigma})*1\big)(t_n) &= \frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}}(e^{zt_n}-1)\frac{l(z)}{z^{\alpha+1}}(l(z)I+A_h)^{-1}\,dz\\ &\quad-\frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}\lambda(z)(\chi(z)I+A_h)^{-1}\,dz\\ &= \Big(\frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}\setminus\Gamma_k} e^{zt_n}\frac{l(z)}{z^{\alpha+1}}(l(z)I+A_h)^{-1}\,dz\\ &\quad+\frac{1}{2\pi i}\int_{\Gamma_k} e^{zt_n}\big[\frac{l(z)}{z^{\alpha+1}}(l(z)I+A_h)^{-1}-\lambda(z)(\chi(z)I+A_h)^{-1}\big]\,dz\Big)\\ &\quad-\frac{1}{2\pi i}\int_{\Gamma_{\epsilon,\vartheta}}\frac{l(z)}{z^{\alpha+1}}(l(z)I+A_h)^{-1}\,dz\\ &=: I_{11}+I_{12},\end{aligned}$$

Following Lemma 6.4.7, we conclude that

$$\|I_{11}\|\le Ckt_n^{\alpha-1}.$$

Next, by resolvent estimate (6.6) and choosing $\epsilon = 1/k^{\frac{1}{\alpha}}$ we obtain

$$\begin{aligned}\|I_{12}\| &\le C\int_{\Gamma_{\epsilon,\vartheta}}|z|^{-\alpha-1}\,|dz|\\ &= C\Big(\int_{\frac{\pi}{k^{\frac{1}{\alpha}}\sin\vartheta}}^{\infty} r^{-\alpha-1}\,dr + k^{\frac{1}{\alpha}}\int_{\frac{1}{k^{\frac{1}{\alpha}}}}^{\frac{\pi}{k^{\frac{1}{\alpha}}\sin\vartheta}} r^{-\alpha}\,dr + k^{\frac{1}{\alpha}}\int_{-\vartheta}^{\vartheta} k^{(1-\frac{1}{\alpha})}\,d\psi\Big)\le Ck.\end{aligned}$$

Thus,

$$\|\lim_{\sigma\to 0}\big((F-F_{k,\sigma})*1\big)(t_n)\|\le\|I_{11}\|+\|I_{12}\|\le Ckt_n^{\alpha-1}. \tag{6.49}$$

In order to bound the error, we need estimate (6.49) to valid for all $t\in(0,T)$. For this purpose, we use the Taylor expansion of $\big((F-F_{k,\sigma})*1\big)(t)$ at $t=t_n$ to get

$$\big((F-F_{k,\sigma})*1\big)(t)=\big((F-F_{k,\sigma})*1\big)(t_n)-\int_t^{t_n}(F-F_{k,\sigma})(s)\,ds.$$

Therefore,

$$\big(\lim_{\sigma\to 0}(F - F_{k,\sigma}) * 1\big)(t) = \big(\lim_{\sigma\to 0}(F - F_{k,\sigma}) * 1\big)(t_n) + \lim_{\sigma\to 0}\int_t^{t_n} F_{k,\sigma}(s)\,ds - \int_t^{t_n} F(s)\,ds$$
$$=: I_{13} + I_{14} + I_{15}.$$

The term I_{13} is bounded by (6.49). Now, using (6.47) we obtain

$$\|I_{14}\| = \|F_k^n\| \le Ckt_n^{\alpha-1}.$$

Finally, by (6.9)

$$\|I_{15}\| \le \int_t^{t_n} \|F(s)\|\,ds \le C\int_t^{t_n} s^{\alpha-1}\,ds \le Ckt^{\alpha-1}.$$

Altogether the bounds of I_{13}, I_{14}, and I_{15} lead to

$$\|\big(\lim_{\sigma\to 0}(F - F_{k,\sigma}) * 1\big)(t)\| \le Ckt^{\alpha-1}. \tag{6.50}$$

With the above preparation, we now proceed to derive the error bound. In view of the representations (6.46) and (6.48), we write

$$u_h(t_n) - U_h^n = \Big(\lim_{\sigma\to 0}\big((F - F_{k,\sigma}) * 1\big) * f_h'\Big)(t_n).$$

Application of (6.50) and the L^2-stability of P_h lead to

$$\begin{aligned}\|u_h(t_n) - U_h^n\| &\le \int_0^{t_n} \|\lim_{\sigma\to 0}\big((F - F_{k,\sigma}) * 1\big)(t_n - s)\|\,\|f_h'(s)\|\,ds\\ &\le Ck\int_0^{t_n} (t_n - s)^{\alpha-1}\,\|f'(s)\|\,ds,\end{aligned}$$

which completes the proof of the assertion. □

Now, Lemmas 6.4.7 and 6.4.8 constitute the error estimate for the time discretization for the nonhomogeneous problem with zero initial data.

Theorem 6.4.2. *Let u_h be the solution represented by (6.15) with $u_0 = 0$, and let U_h^n be the solution of fully discrete problem (6.24) satisfying $U_h^0 = 0$. With $f_h = P_h f$ and $f' = \frac{\partial f}{\partial t}$, the following error estimate*

$$\|u_h(t_n) - U_h^n\| \le Ck\Big(t_n^{\alpha-1}\,\|f(0)\| + \int_0^{t_n} (t_n - s)^{\alpha-1}\,\|f'(s)\|\,ds\Big)$$

holds.

We are now ready to state the fully discrete error estimates for scheme (6.24) for both homogeneous and nonhomogeneous problems.

Theorem 6.4.3. *Assume that $f = 0$ in (6.1) and let u be the corresponding solution given by (6.5). Further, let U_h^n be the fully discrete solution given by (6.24) with $f_h = 0$. Then, for $u_0 \in L^2(\Omega)$ with $u_{0h} = P_h u_0$, we have*

$$\|u(t_n) - U_h^n\| \le C\big(h^2 t_n^{-\alpha} + k t_n^{-1}\big)\, \|u_0\|\,, \quad n \ge 1. \tag{6.51}$$

Moreover, if we choose $u_0 \in \dot{H}^2(\Omega)$ with $u_{0h} = R_h u_0$, then

$$\|u(t_n) - U_h^n\| \le C\big(h^2 + k t_n^{\alpha-1}\big)\, \|u_0\|_2\,, \quad n \ge 1. \tag{6.52}$$

Proof. We split the error $u(t_n) - U_h^n$ in a canonical way as follows

$$u(t_n) - U_h^n = (u(t_n) - u_h(t_n)) - (U_h^n - u_h(t_n)).$$

Now, estimate (6.51) follows from the triangle inequality, (6.18), and (6.38). The second assertion (6.52) is obtained by using the triangle inequality, estimates (6.19) and (6.39). □

Theorem 6.4.4. *Assume that $u_0 = 0$ in (6.1) and let u be the corresponding solution given by (6.5). Further, let U_h^n be the fully discrete solution given by (6.24) with $U_h^0 = 0$ and $f_h = P_h f$. Then, with $f' = \frac{\partial f}{\partial t}$, we have for $n \ge 1$,*

$$\|u(t_n) - U_h^n\| \le C\Big(h^2 t_n^{-1}\, \|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}} + k\big(t_n^{\alpha-1}\, \|f(0)\| + \int_0^{t_n} (t_n - s)^{\alpha-1}\, \|f'(s)\|\, ds\big)\Big),$$

where the number $\|\widehat{f}\|_{\Gamma_{\epsilon,\vartheta}}$ is given by (3.64).

Proof. Since $u(t_n) - U_h^n = (u(t_n) - u_h(t_n)) - (U_h^n - u_h(t_n))$, the desired result follows from the triangle inequality and Theorems 6.3.2 and 6.4.2. □

6.5 Numerical Illustrations

In this section, we perform numerical experiments to validate the convergence results established in Sections 3 and 4. For numerical simulation, we consider $\Omega = (0,1)^2$ and triangulate it first by dividing it into small equal squares and then connecting the diagonal of each square. Next, in the temporal direction, let $t_j = jk$, $j = 0, 1, \dots, N$ be a uniform partition of the time interval $[0, T]$ with time step size $k = T/N$, where T is the time of interest. We take the time step size k (or the mesh size h, respectively) small during the spatial discretization (or the temporal discretization) to negligible the temporal discretization (or the spatial discretization) error. To compute the convergence rates (CRs) of the discrete solution, we employ the L1 scheme to discretize the fractional derivative and the BE convolution quadrature scheme for the Volterra integral term. Below, we carry out the numerical tests on the following three set of data:

(a) Smooth initial data: $u_0(x, y) = \sin(2\pi x)\sin(2\pi y)$ and the source function $f(x, y, t) = 0$.

(b) Nonsmooth initial data: $u_0(x, y) = \chi_{(1/2,1)\times(0,1)}$ and $f(x, y, t) = 0$, where χ_S denotes the characteristic function on the set S.

(c) Zero initial data: $u_0(x, y) = 0$ and $f(x, y, t) = 1$.

We first report CRs for the spatial discretization and then the same for the temporal discretization for the cases (a), (b), and (c).

For the semidiscrete scheme with the case (a), we take the mesh size $h \in \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}\}$ with a fixed time step size $k = 10^{-4}$ and the time of interest $T = 0.1$. The computed errors and related CRs are listed in Table 6.1. From Table 6.1, we see that the empirical CRs are consistent with the theoretical CRs of the estimate (6.19) of Theorem 6.3.1. For the case (b), we perform the numerical tests for $h \in \{\frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}\}$ with a fixed time step size $k = 10^{-5}$. Since the nonsmooth initial data error estimate reflects the singularity at $t = 0$ (see the estimate (6.18)), we particularly interested in the numerical experiments while T is near zero. Thus, with $T = 0.001$, numerical results listed in Table 6.2 show an $O(h^2)$ CR for the $L^2(\Omega)$-norm and an $O(h)$ CR for the $H^1(\Omega)$-norm, which are consistent with the theoretical CRs of the estimate (6.18). For the case (c), we fix $k = 10^{-5}$ and $h \in \{\frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}, \frac{1}{160}\}$. The empirical CRs are listed in Table 6.3 which are compatible with the theoretical results of Theorem 6.3.2.

Next, we now report the CRs of the discrete solutions for the time discretization. We compute the errors and the related CRs with a fixed mesh size $h = 10^{-2}$ at $T = 0.1$ with different values of α, β, and the time step size $k = \{1/10, 1/20, 1/40, 1/80, 1/160\}$ for the cases (a), (b), and (c). Table 6.4 shows the convergence results for the case (a) for various values of α and β which are consistent with the theoretical CR of Theorem 6.4.1 (see (6.39)). The empirical CRs for the cases (b) and (c), respectively, are provided in Tables 6.5 and 6.6 which agree with the theoretical CRs of Theorem 6.4.1 (see (6.38)) and Theorem 6.4.2.

Table 6.1: The $L^2(\Omega)$- and $H^1(\Omega)$-errors and CRs for the case (a) at $T = 0.1$ with $\alpha = 0.55, 0.7, 0.85$, $\beta = 0.5$ and $k = 10^{-4}$

$h \diagdown \alpha = 0.55$	$L^2(\Omega)$-norm	$H^1(\Omega)$-norm	$L^2(\Omega)$-CR	$H^1(\Omega)$-CR
1/10	1.91298e-03	3.25344e-02	–	–
1/20	5.13069e-04	1.73565e-02	1.89860	0.90649
1/40	1.24948e-04	7.83976e-03	2.03782	1.14659
1/80	2.58016e-05	4.00651e-03	2.27580	0.96846
$\alpha = 0.7$	1.91478e-03	3.16654e-02	–	–
	5.15012e-04	1.68597e-02	1.89450	0.90932
	1.25501e-04	7.61218e-03	2.03691	1.14720
	2.58853e-05	3.88944e-03	2.27749	0.96874
$\alpha = 0.85$	1.58259e-03	2.38837e-02	–	–
	4.29018e-04	1.26321e-02	1.88318	0.91892
	1.04728e-04	5.69472e-03	2.03438	1.14940
	2.15234e-05	2.90775e-03	2.28267	0.96972

6.6 Concluding Remarks

In this chapter, we have studied systematically the SEM problem with a weakly singular kernel in a bounded convex polygonal domain. The Sobolev regularity results for the solution to the nonhomogeneous problem are established. The semidiscrete GFEM is developed by using piecewise linear functions, and optimal order error bounds are derived for both smooth and nonsmooth initial data cases. In addition, the L1 scheme for the temporal discretization is analyzed, and error estimates are demonstrated for both homogeneous and nonhomogeneous problems. In the end, we have shown some numerical examples to support our theoretical convergence rates of the approximate solutions.

Table 6.2: The $L^2(\Omega)$- and $H^1(\Omega)$-errors and CRs for the case (b) at $T = 0.001$ with $\beta = 0.4$, $\alpha = 0.65, 0.75, 0.85$ and $\beta = 0.8$, $\alpha = 0.25, 0.45$ and $k = 10^{-5}$

$h\diagdown(\alpha,\beta) = (0.65, 0.4)$	$L^2(\Omega)$-norm	$H^1(\Omega)$-norm	$L^2(\Omega)$-CR	$H^1(\Omega)$-CR
1/20	5.03082e-03	1.67357e-01	–	–
1/40	1.34471e-03	8.48954e-02	1.90350	0.97917
1/80	3.30587e-04	3.97480e-02	2.02419	1.09481
1/160	6.83889e-05	1.88514e-02	2.27320	1.07621
$(\alpha,\beta) = (0.75, 0.4)$	7.56082e-03	2.54268e-01	–	–
	2.04288e-03	1.28264e-01	1.88794	0.98723
	5.02653e-04	5.99218e-02	2.02297	1.09796
	1.03892e-04	2.83494e-02	2.27448	1.07977
$(\alpha,\beta) = (0.85, 0.4)$	1.14046e-02	3.90995e-01	–	–
	3.12474e-03	1.95023e-01	1.86781	1.00350
	7.69996e-04	9.07585e-02	2.02081	1.10354
	1.58860e-04	4.28499e-02	2.27709	1.08274
$(\alpha,\beta) = (0.25, 0.8)$	1.04293e-03	3.19134e-02	–	–
	2.68402e-04	1.63742e-02	1.95818	0.96273
	6.48996e-05	7.71133e-03	2.04812	1.08637
	1.33296e-05	3.71386e-03	2.28357	1.05406
$(\alpha,\beta) = (0.45, 0.8)$	2.29698e-03	7.55259e-02	–	–
	6.01139e-04	3.85926e-02	1.93397	0.96864
	1.46876e-04	1.81341e-02	2.03309	1.08962
	3.03333e-05	8.67674e-03	2.27563	1.06348

Table 6.3: The $L^2(\Omega)$- and $H^1(\Omega)$-errors and CRs for the case (c) at $T = 0.001$ with $\alpha = 0.4, 0.6, 0.8$, $\beta = 0.7$ and $k = 10^{-5}$

$h \diagdown \alpha = 0.4$	$L^2(\Omega)$-norm	$H^1(\Omega)$-norm	$L^2(\Omega)$-CR	$H^1(\Omega)$-CR
1/10	6.87249e-04	2.45035e-02	–	–
1/20	1.73573e-04	1.31078e-02	1.98529	0.90255
1/40	4.21269e-05	5.95246e-03	2.04273	1.13887
1/80	9.25867e-06	2.92131e-03	2.18587	1.02687
$\alpha = 0.6$	4.35642e-04	1.6812e-02	–	–
	1.12377e-04	9.07483e-03	1.95480	0.88955
	2.75457e-05	4.15377e-03	2.02845	1.12745
	6.13600e-06	1.97698e-03	2.16646	1.07113
$\alpha = 0.8$	2.81522e-04	1.11887e-02	–	–
	7.61341e-05	6.23212e-03	1.88663	0.84425
	1.89479e-05	2.89203e-03	2.00650	1.10764
	4.25725e-06	1.33948e-03	2.15405	1.11041

Table 6.4: The $L^2(\Omega)$-errors and CRs for the case (a) at $T = 0.1$ for different values of α and β with $h = 10^{-2}$

N	(α, β)	$L^2(\Omega)$-norm	CR	(α, β)	$L^2(\Omega)$-norm	CR
10	$(0.4, 0.7)$	2.81925e-03	–	$(0.5, 0.8)$	2.61072e-03	–
20		1.40439e-03	1.00537		1.30813e-03	0.99694
40		6.99180e-04	1.00621		6.53997e-04	1.00015
80		3.48320e-04	1.00525		3.26774e-04	1.00099
	$(0.8, 0.5)$	2.89131e-03	–	$(0.9, 0.4)$	2.87150e-03	–
		1.42304e-03	1.02275		1.40469e-03	1.03155
		6.99825e-04	1.02391		6.83546e-04	1.03914
		3.45034e-04	1.02026		3.33751e-04	1.03426

Table 6.5: The $L^2(\Omega)$-errors and CRs for the case (b) at $T = 0.1$ for different values of α and β with $h = 10^{-2}$

N	(α, β)	$L^2(\Omega)$-norm	CR	(α, β)	$L^2(\Omega)$-norm	CR
10	$(0.4, 0.7)$	1.10984e-03	–	$(0.5, 0.8)$	1.65363e-03	–
20		5.27448e-04	1.07325		7.67213e-04	1.10794
40		2.56676e-04	1.03908		3.67264e-04	1.06281
80		1.26397e-04	1.02198		1.78739e-04	1.03896
	$(0.8, 0.5)$	6.14450e-03	–	$(0.9, 0.4)$	7.86486e-03	–
		2.90823e-03	1.07915		3.93304e-03	0.99977
		1.38139e-03	1.07402		1.94045e-03	1.01925
		6.59978e-04	1.06563		9.52214e-04	1.02703

Table 6.6: The $L^2(\Omega)$-errors and CRs for the case (c) at $T = 0.1$ for different values of α and β with $h = 10^{-2}$

N	(α, β)	$L^2(\Omega)$-norm	CR	(α, β)	$L^2(\Omega)$-norm	CR
10	$(0.4, 0.7)$	1.62634e-05	–	$(0.5, 0.8)$	1.92113e-05	–
20		9.05574e-06	0.844727		7.86503e-06	1.28843
40		4.74119e-06	0.933582		3.50824e-06	1.16471
80		2.42118e-06	0.969539		1.64207e-06	1.09523
	$(0.8, 0.5)$	1.09577e-04	–	$(0.9, 0.4)$	2.34236e-04	–
		4.10268e-05	1.41731		1.02514e-04	1.19214
		1.65956e-05	1.30577		4.61151e-05	1.15251
		7.08484e-06	1.22800		2.12135e-05	1.12026

7

Conclusions and Future Scope

This chapter summarizes the results highlighting the contributions made in this thesis and presents some research problems for future study.

7.1 Summary of the Results

The study of both theory and numerics of SEM has become of paramount interest due to its wide range of applications in science and engineering. There are not many literature available till date on the study to SEM. A few of them investigate the well-posedness results in the class of smooth functions. On the other hand, the existing convergence analysis in the numerical studies are carried out under the assumption that the exact solution is sufficiently regular with respect to both space and time variables, including the time at $t = 0$, which is not realistic even for smooth initial data. In addition to the above, the well-posedness results of SEM (1.5) with respect to different Sobolev regularity assumptions on the problem data, i.e., the initial data and the source function are not proved and the finite element error analysis is not studied. Here, we fill up some of these gaps. Our study includes both smooth and weakly singular kernels in the memory term and covers both smooth and nonsmooth initial data cases.

In Chapter 2, we study the well-posedness results for smooth kernel case. We prove existence, uniqueness, and stability results for the solution under several Sobolev regularity assumptions on the initial and the source functions. The main tools used for the well-posedness results are the eigenfunction expansion, theory of Volterra integral equation, and Banach fixed point theorem. For nonsmooth initial data u_0, i.e., when u_0 simply is an element of $L^2(\Omega)$, we demonstrate that the solution corresponding to the homogeneous problem is infinitely differentiable with respect to time (Theorem 2.3.1). Further, for the development of the GFEM, it is essential to establish some stability estimates for the Caputo fractional derivative and the usual partial derivative of the continuous solution $u(t)$. In this regard, we prove such stability results for the solution and expressed directly in terms of the problem data (Theorems 2.4.2 and 2.4.3). This chapter concludes with some examples to illustrate our theory.

In Chapter 3, we have considered and analyzed the standard GFEM by using piecewise linear and continuous finite elements. We have derived a priori error bounds for both smooth and nonsmooth initial data cases in $L^2(\Omega)$- and $H^1(\Omega)$-norms. An use of suitable projection (Ritz, Ritz-Volterra, and L^2-projections) as an intermediate solution we split the error as $u(t) - u_h(t) = (u(t) - \tilde{u}(t)) + (\tilde{u}(t) - u_h(t)) =: \rho(t) + \theta(t)$ and then proceed to estimate $\rho(t)$ and $\theta(t)$ in a suitable Sobolev norms. The estimation of θ relies on the bound of $\partial_t^\alpha \rho(t)$, the Caputo fractional time derivative of ρ, which is crucial to our error analysis. With the choice of $\tilde{u} = R_h u$, the Ritz projector of u, and observing the fact that the operators ∂_t^α and R_h commute, i.e., $\partial_t^\alpha R_h = R_h \partial_t^\alpha$, we derive optimal error bounds for smooth initial data (Theorem 3.2.1). Further, a novel use of the Ritz-Volterra projection greatly simplifies the error analysis for the semidiscrete case. It is important to noteworthy that unlike the Ritz projection, the Ritz-Volterra projection does not posses the commutative property, i.e., $\partial_t^\alpha V_h \neq V_h \partial_t^\alpha$. Therefore, it is not a straightforward exercise to bound $\|\theta\|$ and $\|\nabla\theta\|$. The bounds for $\partial_t^\alpha \rho(t)$ in both $L^2(\Omega)$- and $H^1(\Omega)$-norms (Lemma 3.2.2) are the key technical tools to bound $\|\theta\|$ and $\|\nabla\theta\|$. These bounds together with the approximation property of the Ritz-Volterra projection lead to optimal error estimates for the smooth initial data (Theorem 3.2.2). The nonsmooth data (i.e., $u_0 \in L^2(\Omega)$) error estimate relies on the L^2-projection and almost optimal order error estimates are obtained (Theorem 3.2.3). However, for a special choice of the kernel $B(t,s)$, namely, $B(t,s) = -A$, we are able to recover the optimal order error bounds in both $L^2(\Omega)$- and $H^1(\Omega)$-norms for nonsmooth initial data (Theorem 3.3.1). Numerical experiments are provided to illustrate the theoretical results.

Chapter 4 deals with the a priori error analysis of a semidiscrete GFEM by energy arguments for SEM (1.5). Since the energy method is the most elementary technique in the a priori analysis, therefore, it is natural to study the convergence analysis by energy arguments. The error analysis covers both smooth and nonsmooth initial data. Since the solution u has singularity at $t = 0$ even for smooth initial data, we use t^j, $j = 0, 1$, type of weights along with time integral operator in the error analysis. We simply use energy arguments to prove optimal order error estimates in both $L^2(\Omega)$- and $H^1(\Omega)$-norms with respect to both convergence order of the approximate solution and regularity of the initial data (Theorems 4.3.1-4.3.2). The present approach removes the log-factor $|\ln h|$ from the nonsmooth data error analysis (Theorem 3.2.3) even for general second-order operator $B(t,s)$ as in (1.3). In addition, we also establish a quasi-optimal pointwise in time error bound in the maximum norm for smooth initial data (Theorem 4.3.3). We provide ample numerical experiments to confirm our theoretical findings.

Based on the convolution quadrature generated by the BE and SBD schemes, we discuss the a priori error analysis for the fully discrete schemes in Chapter 5. Optimal order error bounds in time in $L^\infty(L^2(\Omega))$-norm are demonstrated for both schemes covering both smooth and nonsmooth initial data (see respectively, Theorems 5.2.1 and 5.3.1). We validate the theoretical findings by extensive numerical tests.

Chapter 6 is devoted to the systematic study to the weakly singular case. By means of the Laplace transform, we first represent the solution as a contour integral along a suitably chosen contour in the complex plane. Then, we establish the Sobolev regularity results of the solution for the nonhomogeneous problem with respect to both smooth and nonsmooth initial data u_0, including $u_0 \in L^2(\Omega)$ (Theorem 6.2.2). The spatially semidiscrete Galerkin scheme is then developed by using piecewise linear and continuous finite elements. We derive optimal order error bounds with respect to the various regularity assumptions on the initial data for both homogeneous and nonhomogeneous problems (Theorems 6.3.1-6.3.2). Moreover, the L1 scheme for the temporal discretization is considered and analyzed. We prove error estimates of order $O(k^\alpha), 0<\alpha<1$ for both homogeneous and nonhomogeneous problems (Theorems 6.4.1-(6.4.2)). In the end, extensive numerical illustrations verify the theoretical convergence rates of the approximate solutions.

7.2 Future Scope

This section proposes some interesting research problems to be taken up in future.

- **Analysis of the fractional evolution equations with a positive-type memory term.** Consider the model problem of the form

$$\begin{aligned}
&\partial_t^\alpha u(t) + \int_0^t \kappa_\beta(t-s)Au(s)\,ds = f(t) \quad \text{in} \quad \Omega, \quad 0<t\le T, \quad 0<\alpha<1,\\
&u(t) = 0 \quad \text{on} \quad \partial\Omega, \quad 0<t\le T, \\
&u(0) = u_0(x) \quad \text{in} \quad \Omega.
\end{aligned} \tag{7.1}$$

Here $\partial_t^\alpha u(t)$ denotes the Caputo fractional derivative defined by

$$\partial_t^\alpha u(t) = \int_0^t \kappa_{1-\alpha}(t-s)u'(s)\,ds, \quad \kappa_\alpha(t) := t^{(\alpha-1)}/\Gamma(\alpha),\ u'(s) = \partial_s u(s),$$

where $\Gamma(z) = \int_0^\infty e^{-w} w^{z-1}\, dw$, $\mathrm{Re}(z) > 0$ and $A = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ with given source function $f(t) = f(x,t)$ and the initial data u_0. Further, $\kappa_\beta(t)$ is a real-valued positive definite kernel. If $(\alpha+\beta) \geq 1$, then model (7.1) describes phenomena in viscoelasticity and heat conduction in materials with memory, see MacCamy [42, 43], Jin Choi and MacCamy [10], and for $(\alpha+\beta) < 1$, it represents fractional diffusion (cf. Schneider and Wyss [65]).

Numerical treatments by FEM to problem (7.1) have been analyzed by McLean and Thomee [48], Lubich et al. [41], and McLean and Mustapha [45] for the case $\alpha = 1$. We strongly believe that the analysis of Chapters 5 and 6 would be useful to study this problem. It would be interesting to investigate both theory and numerics of (7.1) in future.

- **Analysis of the L1 scheme for weakly singular case on a nonuniform time mesh.** We have considered and analyzed the L1 scheme on a uniform time-stepping for SEM problem (1.6) in Chapter 6 and derived an $O(k)$ error estimate for both smooth and nonsmooth initial data. Since the solution to problem (1.6) exhibits a singular behavior near $t = 0$ even for smooth initial data, it is also an excellent open question to rigorously establish error analysis on a nonuniform time step size for both smooth and nonsmooth data cases. This problem would be considered in future.

- **Study of the Crank-Nicolson scheme to the weakly singular case:** One of the exciting and highly challenging works is to study the higher time discretization scheme namely, Crank-Nicolson scheme for both smooth and nonsmooth initial data for weakly singular model (1.6).

- **Analysis of the superdiffusion equations with memory:** When the fractional derivative order α in model problems (1.5)-(1.6) lies in $(1,2)$, we call these models superdiffusion equations with memory. Both theory and numerical analysis for such models are interesting open questions for research.

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