STUDY OF NON-LOCAL ELLIPTIC PROBLEMS INVOLVING VARIABLE ORDER AND VARIABLE EXPONENTS

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by

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To My Family

Certificate

This is to certify that the thesis entitled **"Study of non-local elliptic problems involving variable order and variable exponents"** submitted by **Ms. Reshmi Biswas** to the Indian Institute of Technology Guwahati, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Guwahati *Reshmi Biswas*

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Abstract

The main objective of the thesis is to introduce the fractional Sobolev spaces with variable-order $s(\cdot,\cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, (0,1))$ and variable exponent $p(\cdot,\cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, (1,\infty))$ and to study some non-local elliptic problems involving the variable-order fractional $p(\cdot)$-Laplacian, defined as

$$(-\Delta)_{p(\cdot)}^{s(\cdot)} u(x) := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{| u(x) - u(y) |^{p(x,y)-2} (u(x) - u(y))}{| x - y |^{N+s(x,y)p(x,y)}} dy, \quad x \in \mathbb{R}^N,$$

up to a normalized constant. We also study here some doubly non-local problems involving the operator $(-\Delta)_{p(\cdot)}^{s(\cdot)}$ and Choquard type of non-local nonlinearity defined as

$$\left(\int_{\mathbb{R}^N} \frac{F(y, u(y))}{|x - y|^{\mu(x,y)}} dy \right) f(x, u(x)),$$

where $\mu(\cdot,\cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, (0, N))$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and $F(x,t)$ is the anti-derivative of f. Furthermore, we explore non-local elliptic system involving fractional $p(\cdot)$-Laplacian $(-\Delta)_{p(\cdot)}^{s}$.

In the first chapter, we discuss the motivation and the main objectives of the thesis. We also recall some definitions and results from the literature which are used in the subsequent chapters of the thesis. Then we briefly describe the salient features of our main problems and the significance of our works.

In the second chapter, we introduce the variable order fractional Sobolev spaces with

variable exponents and explore basic properties of these spaces. Here we also establish continuous and compact embedding from these spaces to some appropriate variable exponent Lebesgue spaces. Such spaces describe the solution spaces for the non-local elliptic problems involving the operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$, which we study in the subsequent chapters.

Our third chapter of the thesis deals the existence of weak solutions of a class of elliptic equations involving the variable-order fractional $p(\cdot)$-Laplacian and generalized Choquard type (also called Hartree type) nonlinearity in a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary, given below:

$$(\mathscr{P}_1) \quad \begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) &= \left(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x,u(x)), \quad x \in \Omega, \\ u(x) &= 0, \qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

with some appropriate assumptions on f. For studying the problem $(\mathscr{P}_1)$, first we establish a Hardy-Littlewood-Sobolev type inequality result suitable for the variable order fractional Sobolev spaces with variable exponents and then apply it to the problem $(\mathscr{P}_1)$. In addition, we also discuss the combined effect of concave and convex nonlinearities on the multiplicity of solutions of the following Choquard problem:

$$(\mathscr{P}'_1) \quad \begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) &= \lambda |u(x)|^{\alpha(x)-2} u(x) + \left(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x,u(x)), \ x \in \Omega, \\ u &= 0, \qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda > 0$ is a real parameter, f and F are as in problem $(\mathscr{P}_1)$, and the variable exponent $\alpha(\cdot) : \Omega \to \mathbb{R}$ is a continuous function satisfying some appropriate assumption. In the forth chapter of the thesis, we study the following Kirchhoff-Choquard problem involving the variable-order fractional $p(\cdot)$-Laplacian and generalized nonlinearity:

$$(\mathscr{P}_2) \quad \begin{cases} m\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} dxdy + \int_\Omega V(x) \frac{|u(x)|^{\overline{p}(x)}}{\overline{p}(x)} dx \right) \\ \qquad \left[(-\Delta)^{s(\cdot)}_{p(\cdot)} u + V(x)|u|^{\overline{p}(x)-2} u \right] = \left(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x,u), \quad x \in \Omega, \\ u = 0, \qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $m(\cdot) : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ and $V(\cdot) : \mathbb{R} \to \mathbb{R}^+_0$ are two continuous functions. Here the reaction term f does not satisfy the well known Ambrosetti-Rabinowitz type condition. First we show the existence of non-trivial weak solution of the problem $(\mathscr{P}_2)$ using the mountain pass theorem with Cerami condition. Then we ensure the existence of the ground state solution of the problem $(\mathscr{P}_2)$ using Nehari manifold and fibering map analysis. In addition, we study existence of infinitely many solutions with unbounded critical energy and with negative critical energy of the problem $(\mathscr{P}_2)$ using fountain theorem with Cerami condition and dual fountain theorem with Cerami* condition, respectively.

The fifth chapter of the thesis is dedicated to the study of the regularity results for a class of doubly non-local equations involving fractional p-Laplacian and Choquard type nonlinearity:

$$(\mathscr{P}_3) \qquad \begin{cases} (-\Delta)_p^s u &= \left(\displaystyle\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u), \qquad x \in \Omega, \\ \qquad u &= 0, \qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $1 < p < \infty$ and $0 < s < 1$ such that $sp < N$, $0 < \mu < \min\{N, 2sp\}$, and the nonlinearity $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ is having at most critical growth in the sense of Hardy-Littlewood-Sobolev inequality. Next, for $p \geq 2$, we discuss the Sobolev versus Hölder minimizers of the energy functional associated to this problem, and hence establish the existence of the local minimizer of the associated energy functional.

Finally, the last chapter of the thesis deals with the multiplicity result of the weak solutions of the following elliptic system involving the fractional $p(\cdot)$-Laplacian and concave-convex nonlinearity:

$$(\mathscr{P}_4) \qquad \begin{cases} (-\Delta)_{p(\cdot)}^s u &= \lambda\, a(x)|u|^{r(x)-2}u + \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}, \;\; x \in \Omega, \\ (-\Delta)_{p(\cdot)}^s v &= \zeta\, b(x)|v|^{r(x)-2}v + \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}, \;\; x \in \Omega, \\ \quad u = v &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda, \zeta > 0$ are the parameters, $s \in (0,1)$. Here $r(\cdot), \alpha(\cdot), \beta(\cdot) \in C(\overline{\Omega}, (1,\infty))$ are the variable exponents and $a(\cdot), b(\cdot), c(\cdot) : \overline{\Omega} \to [0,\infty)$ are the non-negative weight functions satisfying some suitable integrability assumptions. We establish the existence of two

non-trivial and non-negative weak solutions to the problem $(\mathscr{P}_4)$ using Nehari manifold approach and analysis of the associated fibering map.

Contents

List of Symbols

Symbol	**Meaning**
$\mathbb{R}^+$	Set of all positive real numbers
$\mathbb{R}_0^+$	Set of all non-negative real numbers
$meas(A)$	Lebesgue measure of the set A
$\overline{A}$	Closure of the set A
$A \Supset U$	The closure of U is a compact subset of A
$o_n(1)$	$\lim\limits_{n\to+\infty} o_n(1) = 0$
$f(x) = o(g(x))$ as $x \to x_0$	$\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 0$
$\overline{f}(x)$	$f(x,x)$, for x in $\mathbb{R}^N$
$f^+(x)$	$\max\{f(x),0\}$
p'	Conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$
$\rightharpoonup$	Weak convergence
$\hookrightarrow$	Continuous embedding
$\hookrightarrow\hookrightarrow$	Compact embedding
$B_r(x)$	Ball of radius r centered at x
C_0^∞	Set of infinitely differentiable functions with compact support
$C(A,B)$	Set of all continuous functions defined from A to B
$L^p(\Omega)$	Lebesgue space
$L^{p(\cdot)}(\Omega)$	Variable exponent Lebesgue space

Symbol	**Meaning**
$W^{s,p}(\Omega)$	Fractional Sobolev space with order s and exponent p
$W_0^{s,p}(\Omega)$	Closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\mathbb{R}^N)$
$W_0^{-s,p'}(\Omega)$	Topological dual of $W_0^{s,p}(\Omega)$
$W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)$	Fractional Sobolev space with variable order $s(x,y)$ and variable exponents $\beta(x)$, $p(x,y)$
X^*	Topological dual space of any norm linear space X
$\|u\|_{L^p(\Omega)}$	Norm of u in $L^p(\Omega)$
$\|u\|_{L^{p(\cdot)}(\Omega)}$	Norm of u in $L^{p(\cdot)}(\Omega)$
$\|u\|_{s,p}$	Norm of u in $W_0^{s,p}(\Omega)$
$\|u\|_{-s,p'}$	Norm of u in $W_0^{-s,p'}(\Omega)$
$\|u\|_W$	Norm of u in $W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)$
$\langle\cdot,\cdot\rangle_X$	Duality pair between any norm linear space X and its dual X^*
Δ	Laplacian
Δ_p	p-Laplacian
$\Delta_{p(x)}$	$p(x)$-Laplacian
$(-\Delta)^s$	Fractional Laplacian
$(-\Delta)^{s(\cdot)}$	Variable-order fractional Laplacian
$(-\Delta)_p^s$	Fractional p-Laplacian
$(-\Delta)_{p(\cdot)}^{s(\cdot)}$	Variable-order fractional $p(\cdot)$-Laplacian

1

Introduction

1.1 Objective of the thesis

The main objective of this thesis is to introduce the variable order non-local Sobolev spaces with variable exponents for studying some non-local elliptic problems involving the associated variable-order fractional $p(\cdot)$-Laplacian, as well as, to discuss the qualitative properties of the weak solutions to the doubly non-local problems involving non-local nonlinearity.

1.1.1 Motivation of the thesis: Real world application and mathematical significance

In recent years, problems involving non-local operators have gained a lot of attentions due to their occurrence in real-world applications, such as, the thin obstacle problem, optimization, finance, phase transitions and also in pure mathematical research, such as, minimal surfaces, conservation laws (see for e.g., [22] and the references therein for more details), etc. The non-local fractional p-Laplacian (see [30]) is defined as

$$(-\Delta)_p^s u(x) := 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+sp}} dy, \quad x \in \mathbb{R}^N, \qquad (1.1.1)$$

up to a normalized constant.

Consider the following prototype of non-local equation:

$$(-\Delta)_p^s u = f(x,u), \; x \in \Omega, \quad u = 0, \; x \in \mathbb{R}^N \setminus \Omega, \qquad (1.1.2)$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. The celebrated work of Nezza et al. [30] provides the necessary functional set up to study these non-local problems using variational method. For $p = 2$, we refer to [15] and references therein for more details on problems involving semi-linear fractional Laplace operator. In continuation to this, the nonlinear problems involving fractional p-Laplacian are extensively studied by many researchers including Squassina, Palatucci, Mosconi, Brasco, Parini, Sreenadh, Rădulescu et al. (see [16, 17, 40, 48, 71, 76, 77]), where the authors studied various aspects, viz., existence, multiplicity, and regularity of the solutions of the quasi-linear non-local problem of type (1.1.2) involving fractional p-Laplacian.

One salient feature of the problem involving (1.1.1) is the non-locality, in the sense that the value of $(-\Delta)_p^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on Ω, but actually on the entire space $\mathbb{R}^N$, which makes the equation of type (1.1.2) with the operator (1.1.1) to be no longer a point-wise identity. Therefore, the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$ (which is different from the classical case of the local p-Laplacian) and not simply on $\partial\Omega$. Hence, it is often called non-local problem. This makes the study of such a problem interesting.

On the other hand, in the last decades, problems involving variable exponents have been center of attraction. The variable exponent $p(x)$-Laplacian (see [32]) is defined as

$$(-\Delta)_{p(x)}u := (|\nabla u|^{p(x)-2}\nabla u), \quad x \in \mathbb{R}^N. \tag{1.1.3}$$

Investigation of structure properties, as well as, operator theoretic aspects of variable exponent spaces interest many researchers for not only having a rich and interesting mathematical structure but also due to their major appearances in various mathematical models in the study of some non-homogeneous materials (such as electrorheological fluids), image restoration (see for e.g., [1, 27]), etc.

The notion of variable exponent Lebesgue spaces started by Orlicz in [83]. Then the theory of modular function and modular spaces was developed, which provides the required framework for discussing various types of function spaces, viz., classical (weighted) Lebesgue spaces, Orlicz spaces, and variable exponent Lebesgue spaces. The properties of these modular spaces were extensively investigated by many authors like Nakano [81]-[82], Hudzik, Kaminska, and Musielak [80]. For further development on the study of such spaces, associated variable exponent Laplacian, and problems involving it, we refer to [32, 37, 38, 90, 93] and references there in.

Hence, the idea of replacing fractional p-Laplacian, as defined in (1.1.1), by its variable version sparks to expect some better mathematical modeling. Also, the presence of the variable exponents makes the mathematical structure more complex. In this aspect, Kaufmann et al. [57] first introduced fractional $p(\cdot)$-Laplacian $(-\Delta)^s_{p(\cdot)}$, defined as

$$(-\Delta)^s_{p(\cdot)}u(x) := \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(0)} \frac{|\, u(x)-u(y)\,|^{p(x,y)-2}\,(u(x)-u(y))}{|\, x-y\,|^{N+sp(x,y)}}dy, \quad x \in \mathbb{R}^N, \tag{1.1.4}$$

up to a normalized constant. In [57], the authors also defined the fractional Sobolev spaces with variable exponents associated to (1.1.4). Then Rădulescu et al. [10], Bahrouni [9], Ho and Kim [50] studied the extensive properties of these spaces and associated problems involving fractional $p(\cdot)$-Laplacian.

Now a days a great attention has been devoted to the study of variable order derivatives and corresponding variable order fractional spaces. The study of the problems involving variable-order fractional Laplacian $(-\Delta)^{s(\cdot)}$ associated with variable order fractional Sobolev spaces, defined as

$$(-\Delta)^{s(\cdot)}u(x) := \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(0)} \frac{u(x)-u(y)}{\mid x-y\mid^{N+2s(x,y)}}dy, \quad x\in\mathbb{R}^N, \tag{1.1.5}$$

up to some normalized constant is attributed to Zhang et al. [114]. The fractional variable order derivatives were proposed by Lorenzo and Hartley [68], which have wide applications in the study of nonlinear diffusion processes reacting to temperature changes, see also [69] for more details. Especially, in [94] the authors considered some Gaussian processes defined by elliptic pseudodifferential equations, in which the inner product of a fractional Sobolev space of variable order defines the covariance function of these random processes. For application of variable order fractional model in efficient modeling of variable memory property and hereditary property of complex systems, we refer to [103]. Interested readers may also see [58, 59] and references therein for some more studies on variable order derivatives in different areas.

1.2 Preliminaries

In this section, first we briefly discuss some basic properties of the variable exponent Lebesgue spaces and fractional Sobolev spaces, which have been used as tools to study our results. First we fix some notations as follows: For any real valued function Φ, defined on any set $\mathcal{D}$, we set

$$\Phi^- := \inf_{\mathcal{D}}\Phi(x) \text{ and } \Phi^+ := \sup_{\mathcal{D}}\Phi(x). \tag{1.2.6}$$

We also define the function space

$$C_+(\mathcal{D}) := \{\Phi \in C(\mathcal{D}, \mathbb{R}) : 1 < \Phi^- \leq \Phi^+ < \infty\}.$$

1.2.1 Variable exponent Lebesgue spaces

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. For $\Theta \in C_+(\Omega)$, the variable exponent Lebesgue space $L^{\Theta(\cdot)}(\Omega)$ is defined as

$$L^{\Theta(\cdot)}(\Omega) := \Big\{u : \Omega \to \mathbb{R} \text{ is measurable} : \int_\Omega |u(x)|^{\Theta(x)}\, dx < \infty\Big\},$$

which is a separable, reflexive, and uniformly convex Banach space (see [32, 38, 93]) with respect to the following Luxemburg norm

$$\|u\|_{L^{\Theta(\cdot)}(\Omega)} := \inf \Big\{\eta > 0 : \int_\Omega \Big|\frac{u(x)}{\eta}\Big|^{\Theta(x)} dx \leq 1\Big\}.$$

From [38], we have that $C_0^\infty(\Omega)$ is dense in the space $(L^{\Theta(\cdot)}(\Omega), \|\cdot\|_{L^{\Theta(\cdot)}(\Omega)})$ with respect to the norm $\|\cdot\|_{L^{\Theta(\cdot)}(\Omega)}$. Define the modular function $\rho_\Theta : L^{\Theta(\cdot)}(\Omega) \to \mathbb{R}$ as

$$\rho_\Theta(u) := \int_\Omega |u|^{\Theta(x)}\, dx, \text{ for all } u \in L^{\Theta(\cdot)}(\Omega).$$

Lemma 1.2.1. ([38])*Let $u \in L^{\Theta(\cdot)}(\Omega) \setminus \{0\}$, then the following properties hold:*

(i) *For $\eta > 0$, $\eta = \|u\|_{L^{\Theta(\cdot)}(\Omega)}$ if and only if $\rho(\frac{u}{\eta}) = 1$.*

(ii) *$\rho(u) > 1$ $(= 1;\ < 1)$ if and only if $\|u\|_{L^{\Theta(\cdot)}(\Omega)} > 1$ $(= 1;\ < 1)$, respectively.*

(iii) *If $\|u\|_{L^{\Theta(\cdot)}(\Omega)} > 1$, then $\|u\|^{\Theta^-}_{L^{\Theta(\cdot)}(\Omega)} \leq \rho(u) \leq \|u\|^{\Theta^+}_{L^{\Theta(\cdot)}(\Omega)}$.*

(iv) *If $\|u\|_{L^{\Theta(\cdot)}(\Omega)} < 1$, then $\|u\|^{\Theta^+}_{L^{\Theta(\cdot)}(\Omega)} \leq \rho(u) \leq \|u\|^{\Theta^-}_{L^{\Theta(\cdot)}(\Omega)}$.*

Lemma 1.2.2. *Let $u, u_n \in L^{\Theta(\cdot)}(\Omega)$, $n = 1, 2, 3, \cdots$. Then the following statements are equivalent:*

(i) $\lim_{n\to+\infty} \| u_n - u \|_{L^{\Theta(\cdot)}(\Omega)} = 0.$

(ii) $\lim_{n\to+\infty} \rho_\Theta(u_n - u) = 0.$

(iii) *u_n converges to u in Ω in measure and $\lim_{n\to+\infty} \rho_\Theta(u_n) = \rho_\Theta(u)$.*

Let Θ' be conjugate function of Θ, that is, $\frac{1}{\Theta(x)} + \frac{1}{\Theta'(x)} = 1$.

Lemma 1.2.3. *(Hölder inequality)* ([38]) *For any $u \in L^{\Theta(\cdot)}(\Omega)$ and $v \in L^{\Theta'(\cdot)}(\Omega)$, we have*

$$\Big|\int_\Omega uv\,dx\Big| \le 2\|u\|_{L^{\Theta(\cdot)}(\Omega)}\|v\|_{L^{\Theta'(\cdot)}(\Omega)}.$$

Lemma 1.2.4. ([32, Corollary 3.3.4]) *Let $\alpha(\cdot), \beta(\cdot) \in C_+(\overline{\Omega})$ such that $\alpha(x) \le \beta(x)$, for all $x \in \overline{\Omega}$. Then we have*

$$\|u\|_{L^{\alpha(\cdot)}(\Omega)} \le 2\big[1 + meas(\Omega)\big]\|u\|_{L^{\beta(\cdot)}(\Omega)}, \quad \textit{for all}\ \ u \in L^{\alpha(\cdot)}(\Omega) \cap L^{\beta(\cdot)}(\Omega).$$

We recall the next lemma from [47].

Lemma 1.2.5. *Let $\vartheta_1(x) \in L^\infty(\Omega)$ such that $\vartheta_1 \ge 0$, $\vartheta_1 \not\equiv 0$. Let $\vartheta_2 : \Omega \to \mathbb{R}$ be a measurable function such that $\vartheta_1(x)\vartheta_2(x) \ge 1$ a.e. in Ω. Then for every $u \in L^{\vartheta_1(x)\vartheta_2(x)}(\Omega)$,*

$$\|\ |u|^{\vartheta_1(\cdot)}\ \|_{L^{\vartheta_2(x)}(\Omega)} \le \|\ u\ \|^{\vartheta_1^-}_{L^{\vartheta_1(x)\vartheta_2(x)}(\Omega)} + \|\ u\ \|^{\vartheta_1^+}_{L^{\vartheta_1(x)\vartheta_2(x)}(\Omega)}.$$

1.2.2 Fractional Sobolev spaces

Here we collect some known results about the fractional Sobolev spaces (see [15, 30]). Let $\Omega \subseteq \mathbb{R}^N$ be any open set. Then for $0 < s < 1$ and $1 < p < \infty$, constants, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \Big\{u \in L^p(\Omega) : \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy < \infty\Big\}$$

endowed with norm

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \Big(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\Big)^{1/p}.$$

Next, we recall the following embedding result for the space $W^{s,p}(\Omega)$:

Proposition 1.2.1. ([30]) *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary. Also, let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < N$. Then $W^{s,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ continuously, for $1 \le \gamma \le p_s^* := \frac{Np}{N-sp}$. Moreover, this embedding is compact for $\gamma < p_s^*$.*

Here p_s^* denotes the Sobolev type critical exponent in the fractional framework. The space $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)})$ is separable, reflexive, and uniformly convex Banach space (see [30]).

Proposition 1.2.2. ([30]) *Let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < N$. Then there exists a positive constant $C = C(N,p,s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$, we have*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \leq C \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}.$$

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Then we define the following space:

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

equipped with the norm

$$\|u\|_{s,p} := \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}},$$

where $Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$. When $p = 2$, we denote the norm in the above as $\|\cdot\|_s$. The set $C_0^\infty(\Omega)$ is dense in $W_0^{s,p}(\Omega)$ with respect to the norm $\|\cdot\|_{s,p}$ (see [15, 30]).

Proposition 1.2.3. ([30]) *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and let $\gamma \in [1, p_s^*]$. Then for $u \in W_0^{s,p}(\Omega)$, there exists a positive constant $C = C(N,p,s,\gamma)$ such that*

$$\|u\|_{s,p} \leq C \|u\|_{L^\gamma(\Omega)}.$$

Moreover, this embedding is compact for each $\gamma \in [1, p_s^*)$.

The best constant S_s for this embedding is given below:

$$S_s = \inf_{u \in W_0^{s,p}(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy}{\left(\int_\Omega |u|^{p_s^*} \, dx \right)^{p/p_s^*}}. \tag{1.2.7}$$

The space $(W_0^{s,p}(\Omega), \|\cdot\|_{s,p})$ is separable, reflexive, and uniformly convex Banach space (see [30]). The dual of the space $W_0^{s,p}(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$ with the norm $\|\cdot\|_{-s,p'}$, where $p' = \frac{p}{p-1}$ is the conjugate to p.

The next lemma states the monotonicity property of the fractional p-Laplacian for $p \geq 2$.

Lemma 1.2.6. ([54, Lemma 2.3]) *Let $p \geq 2$. There exists $C = C(p) > 0$ such that, for all $u, v \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ and all $q \geq 1$*

$$\left\| (u-v)^{\frac{p+q-1}{p}} \right\|_{s,p}^p \leq C\, q^{p-1} \langle (-\Delta)_s^p u - (-\Delta)_s^p v, (u-v)^q \rangle_{W_0^{s,p}(\Omega)}.$$

The strong maximum principle for fractional p-Laplacian is given as follows:

Lemma 1.2.7. ([76, Lemma 2.3]) *(Strong Maximum Principle) Let $u \in W_0^{s,p}(\Omega)$ satisfy*

$$\left. \begin{aligned} (-\Delta)_p^s u &\geq 0 \quad \text{weakly in } \Omega, \\ u &\geq 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned} \right\} \tag{1.2.8}$$

Then u has a lower semi-continuous representative in Ω, which is either identically 0 or positive.

Next we recall some weighted Hölder spaces. Let the distance function $d : \overline{\Omega} \to \mathbb{R}_+$ be defined by

$$d(x) := \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega), \ x \in \overline{\Omega}. \tag{1.2.9}$$

The weighted Hölder type spaces are defined as follows:

$$C_d^0(\overline{\Omega}) := \left\{ u \in C^0(\overline{\Omega}) : u/d^s \text{ admits a continuous extension to } \overline{\Omega} \right\},$$
$$C_d^{0,\alpha}(\overline{\Omega}) := \left\{ u \in C^0(\overline{\Omega}) : u/d^s \text{ admits a } \alpha \text{-Hölder continuous extension to } \overline{\Omega} \right\}$$

equipped with the norms

$$\|u\|_{C_d^0(\overline{\Omega})} := \|u/d^s\|_{L^\infty(\Omega)},$$
$$\|u\|_{C_d^{0,\alpha}(\overline{\Omega})} := \|u\|_{C_d^0(\overline{\Omega})} + \sup_{x,y \in \overline{\Omega},\, x \neq y} \frac{|u(x)/d^s(x) - u(y)/d^s(y)|}{|x-y|^\alpha},$$

respectively. The embedding $C_d^{0,\alpha}(\overline{\Omega}) \hookrightarrow\hookrightarrow C_d^0(\overline{\Omega})$ is compact, for all $\alpha \in (0,1)$.

We have the following result from [19]:

Theorem 1.2.1. *(Brezis-Lieb) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\{u_n\} \subset L^p(\Omega)$, $1 \leq p < \infty$ be such that $\|u_n\|_{L^p} \leq C$ for some $C > 0$ and $u_n \to u$ a.e. in Ω. Then*

$$\lim_{n\to+\infty}[\|u_n - u\|^p_{L^p(\Omega)} - \|u_n\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] = 0.$$

Let us now recall the well known Simon's inequalities (see [99]). For all $\zeta, \xi \in \mathbb{R}^N$, we have the following:

$$\left.\begin{aligned} |\zeta-\xi|^p &\leq \tfrac{1}{p-1}\Big[\Big(|\zeta|^{p-2}\zeta - |\xi|^{p-2}\xi\Big).(\zeta-\xi)\Big]^{\frac{p}{2}}\left(|\zeta|^p+|\xi|^p\right)^{\frac{2-p}{2}}, 1<p<2,\\ |\zeta-\xi|^p &\leq 2^p\Big(|\zeta|^{p-2}\zeta - |\xi|^{p-2}\xi\Big).(\zeta-\xi)\ , \qquad p\geq 2. \end{aligned}\right\} \tag{1.2.10}$$

We also recall the following standard inequalities which follows from [17, Lemma A.2] for $g(t) = t^+$. For all $a \geq b$ and $1 < p < \infty$, there exists some constant $C_p > 0$ such that

$$|a^+ - b^+|^p \leq (a-b)^{p-1}(a^+ - b^+) \text{ and } (a-b)^{p-1} \leq C_p(a^{p-1} - b^{p-1}). \tag{1.2.11}$$

Another important inequality which is used repeatedly in this thesis is given below.

$$\left.\begin{aligned} (x_1+x_2)^p &\leq x_1^p + x_2^p,\ 0<p<1,\ x_1,x_2\geq 0,\\ (x_1+x_2)^p &\leq 2^{p-1}(x_1^p + x_2^p),\ p>1,\ x_1,x_2\geq 0. \end{aligned}\right\} \tag{1.2.12}$$

1.3 A brief outline of the thesis

In this section, we give a brief discussion regarding our works, as well as, relevance and novelty of our findings. Now we assume that the variable order $s(\cdot,\cdot)$ and the variable exponent $p(\cdot,\cdot)$ satisfy the following hypotheses throughout this thesis:

(S_1) $s : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous and symmetric function, that is, $s(x,y) = s(y,x)$, for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $0 < s^- \leq s^+ < 1$.

(P_1) $p \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ is continuous and symmetric function, that is, $p(x,y) = p(y,x)$, for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

We also set $\overline{p}(x) := p(x,x)$ and $\overline{s}(x) := s(x,x)$, for all $x \in \mathbb{R}^N$.

1.3.1 Variable order fractional Sobolev spaces with variable exponents

Considering complex mathematical structure of variable order derivative, as well as, its significant amount of applications in various fields, it is interesting to explore the rich mathematical structure of the fractional $p(\cdot)$-Laplacian by imposing variable growth on its order s, and expect more accurate mathematical modeling. In this aspect, to the best of our knowledge, this is the first work on the problems driven by the variable-order fractional $p(\cdot)$-Laplacian. Motivated by the work of Kaufmann et al. [57], for any measurable function $u \in \mathbb{R}^N \to \mathbb{R}$, we define the variable-order fractional $p(\cdot)$-Laplacian $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ as

$$(-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) := \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{| u(x) - u(y) |^{p(x,y)-2} (u(x) - u(y))}{| x - y |^{N+s(x,y)p(x,y)}} dy, \quad x \in \mathbb{R}^N, \tag{1.3.13}$$

up to a normalized constant. Note that, such operator is non-homogeneous in nature.

In the second chapter, we define the variable order fractional Sobolev spaces with variable exponents equipped with some appropriate norm to study the problem involving the operator of type (1.3.13). Moreover, we define the associated modular function and establish its interplay with the norm. We also prove some important embedding results from these spaces to the appropriate variable exponent Lebesgue spaces. Then we discuss the important properties, viz., completeness, reflexivity, separability, density, etc., of these normed linear spaces.

1.3.2 Non-local problems involving generalized Choquard type nonlinearity

In the third chapter, we study the following non-local Choquard problem involving variable order and variable exponents:

$$(\mathscr{P}_1) \qquad \begin{cases} (-\Delta)_{p(\cdot)}^{s(\cdot)} u(x) &= \left(\displaystyle\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x,u(x)), \quad x \in \Omega, \\ u(x) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1) respectively, with $s^+p^+ < N$, and $\mu(\cdot,\cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, (0,N))$. Also, $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ with the anti derivative $F(x,t)$ given by

$$F(x,t) := \int_0^t f(x,\tau) d\tau.$$

The main feature of the problem $(\mathscr{P}_1)$ is the presence of both the non-local operator and the non-local nonlinearity of Choquard type together, due to which the problem $(\mathscr{P}_2)$ remains no longer a point wise identity. Hence, it is categorized as a doubly non-local problem. The Choquard type equation $(\mathscr{P}_1)$ is motivated by the work of Pekar [84], studying the following nonlinear Schrödinger-Newton equation:

$$-\Delta u + V(x)u = (\mathcal{K}_\mu * u^2)u + \lambda f(x,u), \tag{1.3.14}$$

where $\mathcal{K}_\mu$ denotes the Riesz potential, defined as

$$\mathcal{K}_\mu = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}}.$$

Now we recall the following result from [63], which is very much crucial for handling the non-local Choquard type (also called Hartree type) of nonlinearity in the constant exponent set up:

Proposition 1.3.1. *(Hardy-Littlewood-Sobolev inequality) Let $q, t > 1$ and $0 < \mu < N$ with $1/q + \mu/N + 1/t = 2$, $g \in L^q(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C(q, t, N, \mu)$, independent of g, h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\mu} dxdy \leq C(q, t, N, \mu)\|g\|_{L^q(\mathbb{R}^N)}\|h\|_{L^t(\mathbb{R}^N)}. \tag{1.3.15}$$

Remark 1.3.1. *Note that, by the Hardy-Littlewood-Sobolev inequality, the following integration*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^r|u(y)|^r}{|x-y|} dxdy$$

is finite whenever $|u|^r \in L^q(\mathbb{R}^N)$, for some $q > 1$, satisfying

$$\frac{2}{q} + \frac{\mu}{N} = 2.$$

Hence, by the fractional Sobolev embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{qr}(\mathbb{R}^N)$, for $u \in W^{s,p}(\mathbb{R}^N)$, the last integration is finite provided $qr \in [p, p_s^*]$. Hence, r has to satisfy

$$\tilde{p}_{s,\mu}^* := \frac{p(2N-\mu)}{2N} \leq r \leq \frac{p(2N-\mu)}{2(N-sp)} := p_{s,\mu}^* \tag{1.3.16}$$

Here $\tilde{p}_{s,\mu}^*$ is called the lower critical exponent and $p_{s,\mu}^*$ is said to be the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. Observe that $p_{\mu,s}^* = \frac{p_s^*}{\frac{2N}{2N-\mu}} < p_s^*$. From now on wards, by critical Choquard exponent, we mean the exponent $p_{s,\mu}^*$.

This Hartree type of nonlinearity describes the self gravitational collapse of a quantum mechanical wave function (see [85]) and also plays a key role in the Bose–Einstein condensation (see [28]). For $V(x) = 1, \lambda = 0$, the equations of type (1.3.14) were extensively studied in [62, 65, 73, 74].

In the fractional Laplacian set up, Wu [110] investigated existence and stability of solutions for the equation

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\mu * |u|^r)|u|^{r-2}u + \lambda f(x, u) \ \text{ in } \mathbb{R}^N, \tag{1.3.17}$$

where $r = 2,\ \lambda = 0$ and $\mu \in (N-2s, N)$. In the critical case, that is, $r = 2^*_{\mu,s} := (2N - 2\mu/2)/(N-2s)$, Mukherjee and Sreenadh [78] obtained existence and multiplicity results for solutions of (1.3.17) in a smooth bounded domain, for $w = 0$ and $f(x,u) = u$. Recently, Gao et al. [45] discussed the existence of ground state solution of Pohozaev type and existence of infinitely many solutions for the following problem with $V \in C^1(\mathbb{R}^N, [0,\infty))$ and F satisfying general Berestycki–Lions type assumptions:

$$(-\Delta)^s u + V(x)u = (\mathcal{K}_\mu * F(u))F'(u) \ \text{ in } \mathbb{R}^N. \tag{1.3.18}$$

For more results on Choquard problem in local and non-local set up involving constant exponents, we refer to [73–75, 78, 79, 106, 107] and references therein.

Concerning the Choquard type nonlinearity in the variable exponents frame-work, we first recall the introductory works of Samko (see [95, 96]) regarding the convolution operators of the form $K(f) := \int_{\mathbb{R}^N} k(x-y)f(y)dy$ in the variable exponents setting. Though, in [95], the author investigated Young's theorem for variable exponents set up to have appropriate estimations of the potential type operators with the kernel $\frac{1}{|x-y|^{N-\alpha(x)}}$, but these estimates are not applicable in the study of the doubly non-local problems like $(\mathscr{P}_1)$. Recently in [4], Alves et al. introduced a new kernel given by the formula

$$A(x,y) := \frac{1}{|x-y|^{\mu(x,y)}}, (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\mu(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous two-point function with some appropriate assumptions. Then using the properties of this kernel the authors established Hardy-Littlewood-Sobolev type inequality (see [4, Proposition 2.4]) for variable exponents.

Motivated by all the above works, in our work, we first establish a Hardy-Littlewood-Sobolev type result following the approach as in [4] for the functions in fractional Sobolev spaces with variable order and variable exponents as defined in the Chapter 2. Then using this result, we study the problem $(\mathscr{P}_1)$. Moreover, we further discuss the combined effect of concave and convex nonlinearities on the multiplicity of solutions of the following

Choquard problem:

$$(\mathscr{P}_1') \quad \begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) & = \lambda |u(x)|^{\alpha(x)-2}u(x) + \left(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x,u(x)), \, x \in \Omega, \\ u & = 0, \qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3.19}$$

where $\lambda > 0$ is a real parameter, the exponent $\mu(\cdot,\cdot)$ and the nonlinearities f and F are as in problem $(\mathscr{P}_1)$, and the variable exponent $\alpha(\cdot) : \Omega \to \mathbb{R}$ is continuous and satisfies some appropriate assumption. The doubly non-local feature of the problems $(\mathscr{P}_1)$ and $(\mathscr{P}_1')$ due to the presence of non-local operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ and the Choquard term involving variable exponents makes the problem mathematically rich and further challenging. To the best of our knowledge, this is the first work addressing the variable order non-local Choquard problem with variable exponents. In this work, we use the standard variational tools as given below:

Definition 1.3.1. *(Palais-Smale condition) Let X be a Banach space and X^* be its topological dual. Suppose that $\Phi \in C^1(X, \mathbb{R})$. We say that Φ satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ (the $(PS)_c$-condition for short) if any sequence $\{u_n\} \subseteq X$ such that $\Phi(u_n) \to c$ and*

$$\Phi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to +\infty$$

admits a strongly convergent subsequence. If this condition holds at every level $c \in \mathbb{R}$, then we say that Φ satisfies the Palais-Smale condition (the PS-condition for short).

Theorem 1.3.1. ([108]) *(Mountain pass theorem with Palais-Smale condition) Let X be a real Banach space and X^* be its topological dual. Suppose that $\Phi \in C^1(X, \mathbb{R})$ satisfies the condition*

$$\max\{\Phi(0), \Phi(u_0)\} \le i < j \le \inf_{\|u\|_X = \varrho_0} \Phi(u),$$

for some $i < j$, $\varrho_0 > 0$ and $u_0 \in X$ with $\|u_0\|_X = \varrho_0$. Let $c \ge j$ be characterized by $c = \inf_{\nu \in \Gamma} \max_{0 \le t \le 1} \Phi(\nu(t))$, where $\Gamma = \{\nu \in C([0,1], X), \nu(0) = 0, \nu(1) = u_0\}$ is the set of continuous paths joining 0 and u_0. Then there exists a Palais-Smale sequence $\{u_n\} \subset X$ such that $\Phi(u_n) \to c \ge j$ and $\|\Phi'(u_n)\|_{X^*} \to 0$ as $n \to +\infty$.

Theorem 1.3.2. ([35]) *(Ekeland's variational principle) Let (X, d_X) be a complete metric* *space and* $\Phi : X \to \mathbb{R}\cup\{+\infty\}$ *lower semi-continuous function that is bounded below. Then for every* $\epsilon > 0$, *there exists* $v_\epsilon \in X$ *such that*

$$\inf_X \Phi \leq \Phi(v_\epsilon) \leq \inf_X \Phi + \epsilon$$

and for all $w \neq v_\epsilon$,

$$\Phi(v_\epsilon) < \Phi(w) + \epsilon d_X(v_\epsilon, w).$$

1.3.3 Kirchhoff-Choquard equations without Ambrosetti - Rabinowitz type condition

The fourth chapter of the thesis deals with the study of the following Kirchhoff-Choquard type problem:

$$(\mathscr{P}_2)\quad \begin{cases} m\left(\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy+\int_\Omega V(x)\frac{|u(x)|^{\overline{p}(x)}}{\overline{p}(x)}dx\right) \\ \qquad\left[(-\Delta)^{s(\cdot)}_{p(\cdot)}u+V(x)|u|^{\overline{p}(x)-2}u\right]=\left(\displaystyle\int_\Omega\frac{F(y,u(y))}{|x-y|^{\mu(x,y)}}dy\right)f(x,u), \quad x\in\Omega, \\ \qquad u=0, \qquad x\in\mathbb{R}^N\setminus\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1) respectively, with $s^+p^+ < N$, $m(\cdot) : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, $V(\cdot) : \Omega \to \mathbb{R}^+_0$, $\mu(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to (0, N)$ are continuous functions. The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous function, and F is the primitive of f.

The Kirchhoff type problems arise in various models of physical and biological systems and hence, have received more attentions in recent years. Precisely, Kirchhoff established a model given by the following equation:

$$\rho\frac{\partial^2 u}{\partial^2 t}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial t}\right|^2dx\right)\frac{\partial^2 u}{\partial^2 t}=0,$$

which extends the classical D'Alembert wave equation by taking into account the effects of the changes in the length of the strings during the vibrations, where the constants

ρ, p_0, h, E, L represent physical parameters of the string. A typical prototype of Kirchhoff function $m(\cdot)$ in $(\mathscr{P}_2)$ is given as $m(t) = a + bt^{\theta-1},\ a \geq 0, b > 0$. Here $a = 0$ represents the degenerate Kirchhoff equation and $a > 0$ represents non-degenerate Kirchhoff equation. In the case of the degenerate Kirchhoff problems, the transverse oscillations of a stretched string, with non-local flexural rigidity, depends continuously on the Sobolev deflection norm of u via $m(\cdot)$. The fact $m(0) = 0$ means that the base tension of the string is zero, a very realistic model from a physical point of view. For the first time, in [39], Fiscella and Valdinoci studied a fractional stationary Kirchhoff model in a bounded domain $\Omega \subset \mathbb{R}^N$ as follows:

$$m(\|u\|_s^2)(-\Delta)^s u = f \quad \text{in } \Omega, \quad \text{and } u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{1.3.20}$$

where $m(t) = m_0 + 2bt$ with $m_0 > 0, b > 0$ and f is the external force field. In the equation (1.3.20) the authors considered the non-local aspect of the tension arising from non-local measurements of the fractional length of the string. For some recent works regarding degenerate and non-degenerate Kirchhoff type problems, we refer to [8, 14, 86–88, 112, 113] and the references therein for more details.

Subsequently, using the Nehari manifold and the concentration compactness principle in [70], Lü studied the Kirchhoff-Choquard type equation

$$\Big(-a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)\Delta u + V_\lambda(x)u = (\mathcal{K}_\mu * |u|^q)|u|^{q-2}u \ \text{ in } \mathbb{R}^3, \tag{1.3.21}$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}_0^+$, $q \in (2, 6-\mu)$, $V_\lambda(x) = 1 + \lambda g(x)$, $\lambda > 0$ and g is a continuous potential function. Recently, Pucci et al. [89] extensively studied the existence and asymptotic behavior of entire solutions of the following superlinear Kirchhoff-Schrödinger-Choquard equation involving fractional p-Laplacian:

$$m(\|u\|^p_{W^{s,p}(\Omega)})[(-\Delta)^s_p u + V(x)|u|^{p-2}u] = \lambda f(x,u) + (\mathcal{K}_\mu * |u|^{p^*_{\mu,s}})|u|^{p^*_{\mu,s}-2}u \quad \text{in } \mathbb{R}^N, \tag{1.3.22}$$

where $m(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is degenerate type Kirchhoff function, $V(\cdot) : \mathbb{R}^N \to \mathbb{R}^+$ is a scalar potential, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with superlinear growth. Further, Liang and Rădulescu [61] studied the existence of infinitely many solutions of (1.3.22) using symmetric mountain pass lemma under some appropriate assumptions on f.

Inspired by all the above works, in problem $(\mathscr{P}_2)$, we consider the study of non-local Kirchhoff-Choquard type problems with variable order and variable exponents. We mention that in our work, we explore both the degenerate and non-degenerate cases for the problem $(\mathscr{P}_2)$.

Another salient feature of the problem $(\mathscr{P}_2)$ is that the nonlinearity f does not satisfy Ambrosetti-Rabinowitz type condition (see [7]). In [7] the Ambrosetti and Rabinowitz studied the following Laplace equation in a smooth bounded domain Ω in $\mathbb{R}^N$:

$$(-\Delta)u = g \ \ in\ \Omega, \ \ u = 0 \ \ on\ \ \partial\Omega.$$

Here the authors assumed the following condition on the nonlinearity $g(x,t)$ which is later on known as Ambrosetti-Rabinowitz type condition ($(A.R.)$ in short).
$(A.R.)$ There exists $\omega > 2,\ t_0 > 0$ such that

$$0 < \omega G(x,t) \leq t g(x,t), \ \text{for all } |t| > t_0, \ x \in \Omega,$$

where G is the primitive of g. Note that the Ambrosetti-Rabinowitz type condition ensures the boundedness of the Palais-Smale sequence of the functional and plays a pivotal role in proving the compactness result. Observe that the function, $g(x,t) = t\log(1+|t|)$ does not satisfy $(A.R.)$ condition. Therefore, relaxing the condition of type $(A.R.)$ not only includes a larger class of nonlinearities but also calls for delicate analysis to establish the compactness results and hence, interests many studies (see for e.g., [2, 60, 66, 104] and references therein).

Therefore, to discuss the existence of nontrivial weak solution and ground state solution of problem $(\mathscr{P}_2)$, we use variant of mountain pass theorem with Cerami condition and Nehari manifold technique, respectively. For that, we recall the following:

Definition 1.3.2. *(Cerami condition) Let X be a Banach space and X^* be its topological dual. Suppose that $\Phi \in C^1(X,\mathbb{R})$. We say that Φ satisfies the Cerami condition at the level $c \in \mathbb{R}$ (the $(C)_c$-condition for short) if any sequence $\{u_n\} \subseteq X$ such that $\Phi(u_n) \to c$ and*

$$(1 + \|u_n\|_X)\Phi^{'}(u_n) \to 0 \ \text{in}\ X^* \ \text{as}\ n \to +\infty$$

admits a strongly convergent subsequence. If this condition holds at every level $c \in \mathbb{R}$, then we say that Φ satisfies the Cerami condition (the C-condition, in short).

Now we state the following version of mountain pass theorem which uses Cerami condition:

Theorem 1.3.3. *(Mountain pass theorem with Cerami condition) Let X be a real Banach space and X^* be its topological dual. Suppose that $\Phi \in C^1(X,\mathbb{R})$ satisfies the condition*

$$\max\{\Phi(0), \Phi(u_0)\} \leq i < j \leq \inf_{\|u\|_X = \varrho_0} \Phi(u),$$

for some $i < j$, $\varrho_0 > 0$ and $u_0 \in X$ with $\|u_0\|_X = \varrho_0$. Let $c \geq j$ be characterized by

$$c = \inf_{\nu \in \Gamma} \max_{0 \leq t \leq 1} \Phi(\nu(t)),$$

where $\Gamma = \{\nu \in C([0,1], X), \nu(0) = 0, \nu(1) = u_0\}$ is the set of continuous paths joining 0 and u_0. Then there exists a Cerami sequence $\{u_n\} \subset X$ such that

$$\Phi(u_n) \to c \geq j \quad \text{and}\ (1 + \|u_n\|_X)\|\Phi'(u_n)\|_{X^*} \to 0 \quad \text{as}\ n \to +\infty.$$

Furthermore, for the odd nonlinearity f, we investigate the existence results of infinitely many solutions using fountain theorem with Cerami condition and dual fountain theorem with Cerami* condition, as stated below. We first recall the following lemma from [36]:

Lemma 1.3.1. *Let X be a reflexive and separable Banach space. Then there exist sequences $\{e_n\} \subset X$ and $\{f_n^*\} \subset X^*$ such that*

$$X = \overline{span\{e_n : n = 1, 2, 3..\}}, \quad X^* = \overline{span\{f_n^* : n = 1, 2, 3..\}},$$

and

$$\langle f_i^*, e_j\rangle_X = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Let us denote

$$X_n = span\{e_n\}, \quad Y_k = \bigoplus_{n=1}^{k} X_n, \quad \text{and } Z_k = \overline{\bigoplus_{n=k}^{\infty} X_n}. \tag{1.3.23}$$

Now we state the following version of fountain theorem from [2], which is motivated by the fountain theorem of Bartsch [12, Theorem 2.5]; (see also [108, Theorem 3.6]).

Theorem 1.3.4. *(Fountain theorem with Cerami condition) Let X be a real Banach space and X^* be its topological dual. Assume that $\Phi \in C^1(X, \mathbb{R})$ satisfies the Cerami condition $(C)_c$ and $\Phi(-u) = \Phi(u)$. If for each sufficiently large $k \in \mathbb{N}$, there exists $\varrho_k > \delta_k > 0$ such that*

$(\mathscr{B}_1)$ $b_k := \inf\{\Phi(u) : u \in Z_k, \ \|u\|_X = \delta_k\} \to +\infty$ *as* $k \to +\infty$,

$(\mathscr{B}_2)$ $a_k := \max\{\Phi(u) : u \in Y_k, \ \|u\|_X = \varrho_k\} \leq 0$,

then Φ has a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \to +\infty$ as $k \to +\infty$.

Next, inspired by Bartsch-Willem dual fountain theorem (see [108] Theorem 3.18), we state a variant of dual fountain theorem with Cerami* condition for studying problem $(\mathscr{P}_2)$.

Definition 1.3.3. *(Cerami* condition) Let X be a Banach space and X^* be its topological dual. Suppose that $\Phi \in C^1(X, \mathbb{R})$. We say that Φ satisfies the Cerami*condition (the $(C)_c^*$ condition in short) (with respect to (Y_n) in (1.3.23)) if any sequence $\{u_n\}$ in X with $u_n \in Y_n$ such that*

$$\Phi(u_n) \to c \quad \text{and} \quad \|\Phi'_{|Y_n}(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0 \quad \text{as} \quad n \to +\infty,$$

contains a convergent subsequence.

Theorem 1.3.5. *(Dual fountain theorem with Cerami* condition) Let X be a real Banach space and X^* be its topological dual. Assume that $\Phi \in C^1(X, \mathbb{R})$ such that $\Phi(-u) = \Phi(u)$. If for each $k \geq k_0$, there exist $\varrho_k > \delta_k > 0$ such that*

($\mathscr{A}_1$) $a_k = \inf\{\Phi(u) : u \in Z_k,\ \|u\|_X = \varrho_k\} \geq 0$,

($\mathscr{A}_2$) $b_k = \sup\{\Phi(u) : u \in Y_k,\ \|u\|_X = \delta_k\} < 0$,

($\mathscr{A}_3$) $d_k = \inf\{\Phi(u) : u \in Z_k,\ \|u\|_X \leq \varrho_k\} \to 0$ *as* $k \to +\infty$,

($\mathscr{A}_4$) Φ *satisfies the* $(C)_c^*$ *condition for every* $c \in [d_{k_0}, 0[$,

then Φ *has a sequence of negative critical values converging to* 0.

Remark 1.3.2. *Here we would like to mention that in [108], assuming that* Φ *satisfies* $(PS)_c^*$ *condition the dual fountain theorem is proved using deformation theorem under the Palais-Smale condition which is also true under Cerami condition. Therefore, we see that the dual fountain theorem is true under* $(C)_c^*$ *condition.*

It is to be noted that, due to the presence of variable order and variable exponents, problem ($\mathscr{P}_2$) possess non-homogeneous nature. This fact together with doubly non-local feature of the problem ($\mathscr{P}_2$) induces further mathematical complexities for adapting classical methods of nonlinear analysis. According to best of our knowledge, the equation of type ($\mathscr{P}_2$) involving the operator $(-\Delta)_{p(\cdot)}^{s(\cdot)}$ is studied for the first time in this work and also, the same results hold in the case of variable-order fractional Laplacian, as well as, fractional and local $p(\cdot)$-Laplacian, which are also new in the literature. The main novelty of this work lies in relaxing the Ambrosetti-Rabinowitz type assumption on the nonlinearity f and to the best of our knowledge, the conditions we impose on f are attempted for the first time for the Choquard type nonlinearity even in the case of fractional, as well as, local p-Laplacian. It is also worth mentioning that, as far as we are aware, there are only few results on Choquard type equations involving fractional p-Laplacian (or local p-Laplacian) without Ambrosetti-Rabinowitz type condition in the literature, which mostly deal with Palais-Smale sequence (see for e.g., [56, 73]). But in our work, we establish the compactness results via Cerami sequence with the assumptions (f_1)-(f_4), which require delicate analysis.

1.3.4 Regularity results for doubly non-local problems

In the fifth chapter, we discuss the regularity of weak solutions of the following p-fractional Choquard equation :

$$(\mathscr{P}_3)\quad \begin{cases} (-\Delta)_p^s u &= \left(\displaystyle\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u), \quad x \in \Omega, \\ \\ u &= 0, \qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $1 < p < \infty$ and $0 < s < 1$ such that $sp < N$, $0 < \mu < \min\{N, 2sp\}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with at most critical growth condition (in the sense of Hardy-Littlewood-Sobolev inequality (1.3.15)). Here $F(x,t) = \int_0^t f(x,\tau)d\tau$ is the primitive of f.

The regularity of weak solutions has been one of the most interesting topics since years and the literature available on the regularity of weak solutions for both local and non-local problems is quite vast. For the regularity results of the local elliptic problems, we refer to [31, 64, 105]. A systematic study on the regularity results of the non-local elliptic problems started with the pioneering work of Caffarelli and Silvestre in [23]. Consider the following non-local problem:

$$(-\Delta)_p^s u = g \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \tag{1.3.24}$$

When $p = 2$, in [23], Caffarelli and Silvestre established the interior $C^{1+\alpha}$ regularity for viscosity solutions to (1.3.24). The authors also proved interior $C^{2s+\alpha}$ regularity for the convex equation (see [24]). For the regularity of weak solutions to free boundary problem involving the fractional Laplacian ($p = 2$), we refer to [98]. Concerning the boundary regularity for the solution of (1.3.24), for $p = 2$ and $g \in L^\infty(\Omega)$, we refer to the work of Ros-Oton and Serra in [91]. Here the authors used a barrier function and the interior regularity results for the fractional Laplacian to show that any weak solution u of (1.3.24) belongs to $C^s(\mathbb{R}^N)$ and $\frac{u}{d^s}|_\Omega \in C^\alpha$, up to the boundary $\partial\Omega$, for some $\alpha \in (0,1)$. In [92], the authors discussed the high integrability of these weak solution by using the regularity of Riesz potential established in [101]. The regularity results for the non-local quasi-linear problem is explored by Squassina et al. in [52], where the authors studied the global Hölder regularity for the weak solutions to (1.3.24), for $p > 1$ and $g \in L^\infty(\Omega)$. Also, regarding the fine boundary regularity results for the problems of type (1.3.24), for the degenerate case ($p \geq 2$), we cite [53]. Here the authors exhibited a weighted Hölder regularity up to the boundary, that is, $\frac{u}{d^s}|_\Omega \in C^{\alpha,}$ up to the boundary $\partial\Omega$, for some $\alpha \in (0,1)$. We would like to mention that the fine boundary regularity for the singular case ($1 < p < 2$) is still an open problem.

Concerning the regularity of the Choquard equations, we refer to [43], in which Gao and Yang studied the Dirichlet problem involving local Laplacian and the critical Choquard type nonlinearity (in view of (1.3.15)). Moroz and Schaftingen [73] established the $W^{2,q}_{\text{loc}}(\mathbb{R}^N)$-regularity ($q > 1$) of the weak solutions to the following Choquard problems involving local Laplacian :

$$-\Delta u + u = (\mathcal{K}_\mu * F(u))f(u) \ \text{ in } \ \mathbb{R}^N, \tag{1.3.25}$$

where

$$|tf(t)| < C(|t|^{\frac{N+\mu}{N}} + |t|^{\frac{N+\mu}{N-2}}), \quad \text{for some constant } \ C > 0.$$

Although an extensive research is done on the existence of solutions for the doubly non-local problems, there are very few results present in the literature regarding the regularity of weak solutions to such problems. By generalizing the idea of [72], in [34], for the fractional Laplacian framework, the authors established the regularity results for solutions of the following Choquard equation :

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\mu * |u|^r)|u|^{r-2}u, \ \ u \in H^s(\mathbb{R}^N),$$

where $\omega > 0$, $N \geq 3$, $\mu \in (0, N)$, $s \in (0,1)$, and $\tilde{2}^*_{s,\mu} < r < 2^*_{s,\mu}$. In [102], the authors studied the $L^\infty(\mathbb{R}^N)$ bound of the non-negative ground state solution to some Kirchhoff-Choquard equation driven by the fractional Laplacian with critical Choquard term (in view of (1.3.15)). Very recently, Giacomoni et al. [46] studied the regularity result for the following generalized doubly non-local problem in a smooth bounded domain Ω in $\mathbb{R}^N$:

$$\left.(-\Delta)^s u = g(x,u) + \left(\int_\Omega \frac{F(u)(y)}{|x-y|^\mu} dy\right) f(u)(x) \text{ in } \Omega, \qquad u = 0 \ \text{ in } \mathbb{R}^N \setminus \Omega, \right\} \tag{1.3.26}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists a constant $C > 0$,

$$|tf(t)| \leq C(|t|^{\frac{2N-\mu}{N}} + |t|^{\frac{2N-\mu}{N-2s}})$$

and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying Sobolev type critical (or singular) growth assumption.

We mention that the techniques used in [46, 73] cannot be implemented straightforward to problem $(\mathscr{P}_3)$ due to lack of Hilbert nature of the solution space associated to the problem. The regularity result for the quasilinear Choquard equations involving the local (or fractional) p-Laplacian are very few. For instance, consider the following equation studied in [13]:

$$(-\Delta)_p^s u + \omega u = \left(\frac{1}{|u|^{\mu}} * F(u)\right) f(u) \text{ in } \mathbb{R}^N, \tag{1.3.27}$$

where $\omega > 0$ is a real number and f has sub-critical growth in terms of (1.3.15). For the case $s = 1$, we cite [5], in which the authors studied (1.3.27) in the local p-Laplacian set up. In both the aforementioned works, the authors proved local Hölder regularity of the weak solutions of (1.3.27) with some restrictive conditions, viz., $\mu < sp$ and $\mu < p$, respectively.

Inspired by all these works, by using a unified boot-strap technique for $1 < p < \infty$, first we investigate *a priori* bound for the weak solutions to the problem $(\mathscr{P}_3)$ which covers a large class of nonlinearities (up to the critical level in the sense of Hardy-Littlewood-Sobolev inequality (1.3.15)). After achieving $L^{\infty}(\Omega)$ estimate on the weak solution to problem $(\mathscr{P}_3)$, we use the result by Squassina et al. [52] along with Hardy-Littlewood-Sobolev inequality (1.3.15), to infer the Hölder regularity result. To the best of our knowledge, the $L^{\infty}(\Omega)$ bound on the weak solutions to the doubly non-local problem of type $(\mathscr{P}_3)$ involving critical Choquard type nonlinearity is established for the first time in this work.

Next, we discuss the Sobolev versus Hölder minimizers for the energy functional associated to the problem $(\mathscr{P}_3)$. We show that local minimizers of the energy functional associated to $(\mathscr{P}_3)$ with respect to $C_d^0(\overline{\Omega})$-topology are also local minimizers of the same energy functional with respect to $W_0^{s,p}(\Omega)$-topology. In variational problems, this result plays an important role in establishing the multiplicity of solutions. In the local framework, Brezis and Nirenberg [20] were the first to study this type of result, where the authors showed that the local minima of the associated energy functional in C^1 topology and in H^1, topology coincides. In the fractional framework ($p = 2$), the analogous result is proved in [51]. In [54], this result is further generalized for the fractional p-Laplacian set up for $p \geq 2$. For non-local nonlinearity, Gao and Yang [44] studied such result for the following Brezis-Nirenberg type Critical Choquard problem involving local Laplacian under some appropriate assumptions on f:

$$-\Delta u = \lambda f(u) + \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u(x)|^{2^*_\mu - 2} u \quad \text{in} \ \Omega, \quad u = 0 \ \text{in} \quad \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain having smooth boundary, $\lambda > 0$, $0 < \mu < N$ and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical Choquard exponent in view of (1.3.15). In the case of doubly non-local equation, Giacomoni et al. [46] investigated H^s versus C^0- weighted minimizers of the functional associated to (1.3.26).

But, to the best of our knowledge, there is no such work regarding Sobolev versus Hölder minimizers for the problems involving the fractional p-Laplacian and critical (or sub-critical) Choquard type nonlinearity. Also, the tools used in [44, 46] to prove this result can not be adapted for the general case of $1 < p < \infty$. Therefore, we establish this result for the problem $(\mathscr{P}_3)$ considering the degenerate case ($p \geq 2$).

Finally, we show that if problem $(\mathscr{P}_3)$ has a weak sub-solution and a weak supersolution, then it attains a solution in between the sub-super solutions pair, which also appears as a local minimizer of the associated energy functional to the problem $(\mathscr{P}_3)$ in $W_0^{s,p}(\Omega)$ topology.

For doubly non-local problems like $(\mathscr{P}_3)$, the main difficulty arises due to the non-Hilbert nature of the solution space and the presence of the nonlinear operator $(-\Delta)^p_s$, as well as, the non-local nonlinearity of Choquard type. Hence, most of the results and techniques that were used in establishing the similar kind of regularity results in the fractional Laplacian or in the local Laplacian set up (for instance, see [43, 46, 73]) are not applicable to our problem $(\mathscr{P}_3)$. Therefore, we need to carry out some extra delicate analysis in our proofs to overcome the stated difficulties. In [100], the regularity of solutions of critical Choquard equations involving the p-Laplacian is posed as an open problem and in this work, we come up with the answer to it. In this regard, we would like to remark that the regularity results we establish for the problem $(\mathscr{P}_3)$ are also valid in the local p-Laplacian framework, which are also new to the literature.

1.3.5 Non-local elliptic system involving fractional $p(\cdot)$- Laplacian

The last chapter of the thesis encompasses the study of the following non-local elliptic system involving variable exponents:

$$(\mathscr{P}_4) \quad \begin{cases} (-\Delta)^s_{p(\cdot)} u &= \lambda\, a(x)|u|^{r(x)-2}u + \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}, \quad x \in \Omega, \\ (-\Delta)^s_{p(\cdot)} v &= \zeta\, b(x)|v|^{r(x)-2}v + \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}, \quad x \in \Omega, \\ u = v &= 0, \quad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with Lipschitz boundary, $\lambda, \zeta > 0$ are the parameters, $s \in (0,1)$, $p(\cdot,\cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, (1,\infty))$ with $sp^+ < N$. Here $r(\cdot), \alpha(\cdot), \beta(\cdot) \in C(\overline{\Omega}, (1,\infty))$ are the variable exponents and $a(\cdot), b(\cdot), c(\cdot) : \overline{\Omega} \to [0,\infty)$ are the non-negative weight functions with some appropriate integrability assumption.

Using the Nehari manifold and the fibering map, in the case of local p-Laplacian, Brown and Wu [111] obtained multiple solutions of an elliptic system with sign changing weight functions and concave-convex nonlinearities. In the non-local set up, Sreenadh and Goyal [49] studied the same for the single fractional p-Laplacian equation . Also, we cite [26] where the authors studied the fractional p-Laplacian system involving concave-convex nonlinearities via Nehari manifold and fibering map. In [42], Pucci et al. modified the definition of Nehari manifold and fibering map for the fractional (p,q)-Laplacian system and studied the corresponding Dirichlet problem. Recently Alves et al. [3] used this Nehari manifold method to prove the multiplicity of solutions for local $p(\cdot)$-Laplacian problems in the whole of $\mathbb{R}^N$.

Motivated by the above works, we address the multiplicity of the solutions of the non-local elliptic system with variable exponents involving concave and convex nonlinearities using the analysis of the fibering map and Nehari manifold. We also show that the solutions are non-negative. We note that the Nehari manifold approach through the fibering map analysis for the functional involving variable exponents is interesting due to the non-homogeneity that arises from the variable exponents. It is also worth mentioning that due to the presence of the variable exponents, most of the estimates do not hold immediately, unlike in the constant exponent set-up. Hence, in our present work, we need to carry out some extra careful analysis to overcome this issue. To the best of our knowledge, this is the first work dealing with existence of non-negative solutions of a fractional $p(\cdot)$-Laplacian system involving concave and convex nonlinearities, using fibering map approach.

Note: Throughout this thesis, $C,\ C_i,\ K_i,\ i \in \mathbb{N}$, are considered to be generic positive constants, which may vary from line to line and chapter to chapter.

Organization of the Thesis

We present the entire work of this thesis in six chapters as described below.

- Chapter 1: Introduction
- Chapter 2: Variable order fractional Sobolev spaces with variable exponents
- Chapter 3: Non-local problems involving generalized Choquard type nonlinearity
- Chapter 4: Kirchhoff-Choquard equations without Ambrosetti-Rabinowitz type condition
- Chapter 5: Regularity results for doubly non-local problems
- Chapter 6: Non-local elliptic system involving fractional $p(\cdot)$-Laplacian

2

Variable order fractional Sobolev spaces with variable exponents

2.1 The spaces

In this chapter, we introduce variable order fractional Sobolev spaces with variable exponents and the associated norms. We further establish the embedding results and fundamental functional properties of these spaces which are used in the subsequent chapters. Assume that $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1), respectively. Let $\beta(\cdot) \in C_+(\overline{\Omega})$ be any variable exponent. Then, recalling the definition of the Lebesgue spaces with variable exponents in [32, 38] and the definition of the fractional Sobolev spaces with variable exponents in [57], for any open subset Ω in $\mathbb{R}^N$, we introduce the variable order fractional

Sobolev space with variable exponents as follows:

$$W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)$$
$$:= \left\{ u \in L^{\beta(\cdot)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}} dxdy < \infty, \text{ for some } \eta > 0 \right\}.$$

For simplicity in notation, we denote $W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)$ as W, in short. We set

$$[u]_\Omega^{s(x,y),p(x,y)} := \inf \left\{ \eta > 0 : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}} dxdy < 1 \right\},$$

which defines a semi-norm on W. The norm on W is defined as

$$|u|_W := \|u\|_{L^{\beta(\cdot)}(\Omega)} + [u]_\Omega^{s(x,y),p(x,y)}.$$

On W, we also define the following norm:

$$\|u\|_W := \inf \left\{ \eta > 0 : \rho_W \left(\frac{u}{\eta} \right) < 1 \right\},$$

where

$$\rho_W(u) := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} dxdy + \int_\Omega |u|^{\beta(x)}\, dx$$

defines a convex modular on W. It is not difficult to show that $\|\cdot\|_W$ and $|\cdot|_W$ are equivalent norms on W with the relation

$$\frac{1}{2}\|u\|_W \le |u|_W \le 2\|u\|_W, \quad \text{for all } u \in W. \tag{2.1.1}$$

Remark 2.1.1. *If $A_1 \subseteq A_2$ are two bounded open sets in $\mathbb{R}^N$, then it is easy to check that*

$$[u]_{A_1}^{s(x,y)p(x,y)} \le [u]_{A_2}^{s(x,y)p(x,y)}.$$

The next two results establish the relation between the norm and the modular functions on W, as defined above. The proofs follow adapting the similar arguments as in [38].

Proposition 2.1.1. *For $u \in W \setminus \{0\}$, we have the following:*

(i) *For* $\eta > 0$, $\eta = \|u\|_W$ *if and only if* $\rho_E(\frac{u}{\eta}) = 1$.

(ii) $\rho_W(u) > 1$ $(= 1;\ < 1)$ *if and only if* $\|u\|_W > 1$ $(= 1;\ < 1)$, *respectively.*

(iii) *If* $\|u\|_W \geq 1$, *then* $\|u\|_W^{p^-} \leq \rho_W(u) \leq \|u\|_W^{p^+}$.

(iv) *If* $\|u\|_W < 1$, *then* $\|u\|_W^{p^+} \leq \rho_W(u) \leq \|u\|_W^{p^-}$.

As a consequence of the above proposition, we can derive the following result:

Proposition 2.1.2. *Let* $u, u_n \in W, n \in \mathbb{N}$. *Then the following statements are equivalent:*

(i) $\lim_{n\to+\infty} \|u_n - u\|_W = 0$.

(ii) $\lim_{n\to+\infty} \rho_W(u_n - u) = 0$.

2.1.1 Embedding results

Here first we prove the Sobolev type embedding theorem for $W = W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)$, when the set $\Omega(\subset \mathbb{R}^N)$ is open and bounded. In the proof, we follow the approach as in [57], in which the authors studied this result for $s(\cdot,\cdot) = s$, constant. Also, we would like to mention that we improve the result by including the case $\beta(x) = \overline{p}(x)$, unlike in [57], where the authors considered $\beta(x) > \overline{p}(x)$.

Theorem 2.1.1. *Let* Ω *be an open and bounded set in* $\mathbb{R}^N$, $N \geq 2$, *with Lipschitz boundary. Assume that* $s(\cdot,\cdot)$ *and* $p(\cdot,\cdot)$ *satisfy* (S_1) *and* (P_1), *respectively, with* $s^+p^+ < N$ *and* $\beta(\cdot) \in C_+(\overline{\Omega})$ *such that* $\overline{p}(x) \leq \beta(x)$, *for all* $x \in \overline{\Omega}$. *Then for any* $\gamma(\cdot) \in C_+(\overline{\Omega})$ *with* $1 < \gamma(x) < p_s^*(x) := \frac{N\overline{p}(x)}{N-\overline{s}(x)\overline{p}(x)}$, *for all* $x \in \overline{\Omega}$, *there exits a constant* $C = C(N, s, p, \beta, \gamma, \Omega) > 0$ *such that, for every* $u \in W$,

$$\|u\|_{L^{\gamma(\cdot)}(\Omega)} \leq K\|u\|_W.$$

Moreover, this embedding is compact.

Proof. Since $\overline{p}(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, s are continuous on the compact set $\overline{\Omega}$, it follows that

$$\inf_{x\in\overline{\Omega}} \left\{ \frac{N\overline{p}(x)}{N - \overline{s}(x)\overline{p}(x)} - \gamma(x) \right\} = k_0 > 0. \tag{2.1.2}$$

Using (2.1.2) and continuity of the functions $p(\cdot,\cdot), \beta(\cdot), \gamma(\cdot)$, and $s(\cdot,\cdot)$, we get a finite family of disjoint open balls $\{B_i'\}_{i=1}^k$ with radius $\epsilon = \epsilon(p, \beta, \gamma, s, k_0)$ satisfying $\overline{\Omega} \subseteq \cup_{i=1}^k B_i'$

such that

$$\frac{Np(z,y)}{N - s(z,y)p(z,y)} - \gamma(x) = \frac{k_0}{2} > 0, \tag{2.1.3}$$

for all $(z,y) \in B_i \times B_i$ and for $x \in B_i$, $i = 1, 2, \ldots, k$, where $B_i = \Omega \cap B_i'$, for each $i = 1, 2, \ldots, k$. Set

$$s_i := \inf_{(z,y)\in B_i \times B_i} s(z,y). \tag{2.1.4}$$

Again by using continuity of $p(\cdot,\cdot), \beta(\cdot), \gamma(\cdot)$, and $s(\cdot,\cdot)$, we can choose $\delta = \delta(k_0)$, with $0 < \delta < p^- - 1$, $t_i \in (0, s_i)$, and $\epsilon > 0$ such that (2.1.3)-(2.1.4) give us

$$p^*_{t_i} := \frac{Np_i}{N - t_i p_i} \geq \frac{k_0}{3} + \gamma(x) \tag{2.1.5}$$

and

$$\beta(x) \geq \overline{p}(x) > p_i, \tag{2.1.6}$$

for all $x \in B_i$, $i = 1, 2, \ldots, k$, where

$$p_i := \inf_{(z,y)\in B_i \times B_i} (p(z,y) - \delta). \tag{2.1.7}$$

Indeed, since by (2.1.7), $p_i = \inf_{(z,y)\in B_i \times B_i} p(z,y) - \delta(k_0) < \overline{p}(x) \leq \beta(x)$, for each $x \in B_i$, we have (2.1.6). Using the continuous embedding $W^{t_i,p_i}(B_i) \hookrightarrow L^{p^*_{t_i}}(B_i)$ (see Proposition 1.2.1), we get a constant $C = C(N, p_i, t_i, \epsilon, B_i) > 0$ such that

$$\| u \|_{L^{p^*_{t_i}}(B_i)} \leq C\left(\|u\|_{L^{p_i}(B_i)} + \left(\int_{B_i}\int_{B_i} \frac{|u(x) - u(y)|^{p_i}}{|x-y|^{N+t_i p_i}} dxdy \right)^{\frac{1}{p_i}} \right). \tag{2.1.8}$$

Since $|u(x)| = \sum_{i=1}^{k} |u(x)|\chi_{B_i}$, we have

$$\|u\|_{L^{\gamma(\cdot)}(\Omega)} \leq \sum_{i=1}^{k} \|u\|_{L^{\gamma(\cdot)}(B_i)}. \tag{2.1.9}$$

Now in the rest of the proof, we consider $K_i,\ i \in \mathbb{N}$, to be generic positive constants which may vary from line to line but depends only on $p, s, \beta, \gamma, \epsilon, \Omega$ and B_1', where B_1' denotes the open ball centered at the origin with radius $\epsilon > 0$.

From (2.1.5), we get that $\gamma(x) < p_{t_i}^*$, for all $x \in B_i$, $i = 1, ..., k$. Hence we can choose $a_i(\cdot) \in C_+(\Omega)$ such that $\frac{1}{\gamma(x)} = \frac{1}{p_{t_i}^*} + \frac{1}{a_i(x)}$. By applying Lemma 1.2.3, we obtain

$$\|u\|_{L^{\gamma(\cdot)}(B_i)} \leq K_2 \|u\|_{L^{p_{t_i}^*}(B_i)} \|1\|_{L^{a_i(\cdot)}(B_i)} \leq K_3 \|u\|_{L^{p_{t_i}^*}(B_i)}. \tag{2.1.10}$$

Hence, from (2.1.9) and (2.1.10), we deduce

$$\|u\|_{L^{\gamma(\cdot)}(\Omega)} \leq K_4 \sum_{i=1}^{k} \|u\|_{L^{p_{t_i}^*}(B_i)}. \tag{2.1.11}$$

Now using (2.1.6) and arguing in a similar way as above, we obtain

$$\sum_{i=1}^{k} \|u\|_{L^{p_i}(B_i)} \leq K_5 \|u\|_{L^{\beta(\cdot)}(\Omega)}. \tag{2.1.12}$$

Next, for each $i = 1, ..., k$, we can choose $b_i(\cdot, \cdot) \in C_+(B_i \times B_i)$ such that

$$\frac{1}{p_i} = \frac{1}{p(x,y)} + \frac{1}{b_i(x,y)}. \tag{2.1.13}$$

We define a measure $d\tilde{\mu}$ on $B_i \times B_i$ as

$$d\tilde{\mu}(x,y) = \frac{dxdy}{|x-y|^{N+(t_i - s(x,y))p_i}}. \tag{2.1.14}$$

Also, we define the function $U : B_i \times B_i \to \mathbb{R}$ as $U(x,y) = \frac{|u(x)-u(y)|}{|x-y|^{s(x,y)}}$, $x \neq y$. Using Hölder's inequality combining with (2.1.13) and (2.1.14), it follows that there exist some constants $K_6, K_7 > 0$ such that

$$\left(\int_{B_i} \int_{B_i} \frac{|u(x) - u(y)|^{p_i}}{|x-y|^{N+t_i p_i}} dxdy \right)^{\frac{1}{p_i}} = \left(\int_{B_i} \int_{B_i} \left(\frac{|u(x) - u(y)|}{|x-y|^{s(x,y)}} \right)^{p_i} \frac{dxdy}{|x-y|^{N+(t_i - s(x,y))p_i}} \right)^{\frac{1}{p_i}}$$

$$= \left(\int_{B_i} \int_{B_i} (U(x,y))^{p_i} d\tilde{\mu}(x,y) \right)^{\frac{1}{p_i}}$$
$$\leq K_6 \, \|U\|_{L^{p(\cdot,\cdot)}(\tilde{\mu},\ B_i \times B_i)} \, \|1\|_{L^{b_i(\cdot,\cdot)}(\tilde{\mu},\ B_i \times B_i)}$$
$$\leq K_7 \, \|U\|_{L^{p(\cdot,\cdot)}(\tilde{\mu},\ B_i \times B_i)}. \tag{2.1.15}$$

Now let $\lambda' > 0$ be such that

$$\int_{B_i} \int_{B_i} \frac{|u(x) - u(y)|^{p(x,y)}}{(\lambda')^{p(x,y)} |x-y|^{N + s(x,y)p(x,y)}} dxdy < 1. \tag{2.1.16}$$

Choose

$$d_i := \sup \left\{ 1, \sup_{(x,y) \in B_i \times B_i} |x-y|^{s(x,y) - t_i} \right\} \quad \text{and} \quad \overline{\lambda_i} = \lambda' d_i. \tag{2.1.17}$$

Combining (2.1.16) and (2.1.17), we deduce

$$\int_{B_i} \int_{B_i} \left(\frac{U(x,y)}{\overline{\lambda_i}} \right)^{p(x,y)} d\tilde{\mu}(x,y)$$
$$= \int_{B_i} \int_{B_i} \left(\frac{|u(x) - u(y)|}{\overline{\lambda_i} |x-y|^{s(x,y)}} \right)^{p(x,y)} \frac{dxdy}{|x-y|^{N + (t_i - s(x,y))p_i}}$$
$$= \int_{B_i} \int_{B_i} \frac{|x-y|^{(s(x,y) - t_i)p_i}}{d_i^{\,p(x,y)}} \frac{|u(x) - u(y)|^{p(x,y)}}{(\lambda')^{p(x,y)} |x-y|^{N + s(x,y)p(x,y)}} dxdy$$
$$\leq \int_{B_i} \int_{B_i} \frac{|u(x) - u(y)|^{p(x,y)}}{(\lambda')^{p(x,y)} |x-y|^{N + s(x,y)p(x,y)}} dxdy < 1.$$

Thus from the last relation, we obtain $\|U\|_{L^{p(\cdot,\cdot)}(\tilde{\mu},\ B_i \times B_i)} \leq \overline{\lambda_i} = \lambda' d_i$, which together with Remark 2.1.1 implies that

$$\|U\|_{L^{p(\cdot,\cdot)}(\tilde{\mu},\ B_i \times B_i)} \leq K_8 [u]_{B_i}^{s(x,y), p(x,y)} \leq K_8 [u]_{\Omega}^{s(x,y), p(x,y)}, \tag{2.1.18}$$

where $K_8 = \max\limits_{i \in \{1,2,\ldots,k\}} \{d_i\} > 1$. Taking into account (2.1.15) and (2.1.18), we get

$$\left(\int_{B_i} \int_{B_i} \frac{|u(x) - u(y)|^{p_i}}{|x-y|^{N + t_i p_i}} dxdy \right)^{\frac{1}{p_i}} \leq K_8 [u]_{\Omega}^{s(x,y), p(x,y)}. \tag{2.1.19}$$

From (2.1.19), we infer that

$$\sum_{i=1}^{m}\left(\int_{B_i}\int_{B_i}\frac{|u(x)-u(y)|^{p_i}}{|x-y|^{N+t_ip_i}}dxdy\right)^{\frac{1}{p_i}} \leq K_9[u]_{\Omega}^{s(x,y),p(x,y)}. \tag{2.1.20}$$

Thus, using (2.1.8), (2.1.11), (2.1.12) and (2.1.20), we deduce

$$\begin{aligned}\|u\|_{L^{\gamma(\cdot)}(\Omega)} &\leq K_4\sum_{i=1}^{k}\|u\|_{L^{p_i^*}(B_i)}\\ &\leq K_{10}\sum_{i=1}^{m}\left(\|u\|_{L^{p_i}(B_i)}+\left(\int_{B_i\times B_i}\frac{|u(x)-u(y)|^{p_i}}{|x-y|^{N+t_ip_i}}dxdy\right)^{\frac{1}{p_i}}\right)\\ &\leq K_{11}\left(\|u\|_{L^{\beta(\cdot)}(\Omega)}+[u]_{\Omega}^{s(x,y),p(x,y)}\right)=C\|u\|_W,\end{aligned} \tag{2.1.21}$$

where the constant $C := C(N, s, p, \beta, \gamma, \Omega) > 0$. This proves that the space W is continuously embedded in $L^{\gamma(\cdot)}(\Omega)$.

Now for showing the compactness of the embedding (2.1.21), let us consider $u_n \rightharpoonup u$ weakly in W as $n \to +\infty$. Since by (2.1.5) and (2.1.19), $W \hookrightarrow W^{t_i,p_i}(\Omega_i)$, we have $u_n \rightharpoonup u$ weakly in $W^{t_i,p_i}(\Omega_i)$ as $n \to +\infty$. Now using continuous embedding $W^{t_i,p_i}(B_i) \hookrightarrow L^{p_{t_i}^*}(B_i)$ (see Proposition 1.2.1), it follows that up to a sub-sequence, still denoted by $\{u_n\}$, $u_n \to u$ strongly in $L^{p_{t_i}^*}(B_i)$ which eventually, together with (2.1.21) yields that $u_n \to u$ strongly in $L^{\gamma(\cdot)}(\Omega)$ as $n \to +\infty$. Hence, $W \hookrightarrow\hookrightarrow L^{\gamma(\cdot)}(\Omega)$. This completes the proof of the theorem. □

Next, we have the Sobolev type embedding theorem for $W = W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$. The proof follows using the similar arguments as in [50], in which the authors considered the case $s(x, y) = s$, constant and $\beta(x) = \overline{p}(x)$. But here we refine the result by taking $\beta(x) \geq \overline{p}(x)$ and $s(\cdot, \cdot)$ to be of variable growth.

Theorem 2.1.2. *Let $\Omega = \mathbb{R}^N$ and $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ be uniformly continuous functions satisfying* (S$_1$) *and* (P$_1$)*, respectively, with $s^+p^+ < N$. Assume that $\gamma(\cdot) \in C_+(\mathbb{R}^N)$ is uniformly continuous function such that $\gamma(x) \geq \overline{p}(x)$, for all $x \in \mathbb{R}^N$ and* $\inf_{x\in\mathbb{R}^N}(p_s^*(x) - \gamma(x)) >$

0. *Then there exists a positive constant* $C := C(s, p, N, \gamma, \beta)$ *such that, for every* $u \in W$,

$$\|u\|_W \leq C\|u\|_{L^{\gamma(\cdot)}(\mathbb{R}^N)}. \tag{2.1.22}$$

Moreover, if $\gamma(\cdot) \in C_+(\mathbb{R}^N)$ *satisfies* $\gamma(x) < p_s^*(x)$, *for all* $x \in \mathbb{R}^N$, *we have the compact embedding*

$$W \hookrightarrow\hookrightarrow L_{loc}^{\gamma(\cdot)}(\mathbb{R}^N). \tag{2.1.23}$$

Proof. It is sufficient to prove the theorem for the case $\inf_{x\in\mathbb{R}^N}(\gamma(x) - \overline{p}(x)) > 0$. Now we can express $\mathbb{R}^N$ as $\mathbb{R}^N = \cup_{i=1}^{\infty} Q_i$, where Q_i is the cube parallel to the co-ordinate axes of side length $\epsilon > 0$ for $i = 1, 2, 3, \cdots$. Set

$$p_i^- := \inf_{(x,y)\in Q_i\times Q_i} p(x, y),\ \ s_i^- := \inf_{(x,y)\in Q_i\times Q_i} s(x, y),\ \ \gamma_i^- := \inf_{x\in Q_i} \gamma(x), \gamma_i^+ := \sup_{x\in Q_i} \gamma(x).$$

Using the uniform continuity of $p(\cdot,\cdot)$ and $\gamma(\cdot)$, we can choose $\epsilon > 0$ sufficiently small and $t_i \in (0, s_i^-)$ such that, on every Q_i, the following holds:

$$\beta(x) \geq p_i^- \leq \gamma_i^- \leq \gamma_i^+ \leq (p_{t_i}^-)^* := \frac{Np_i^-}{N - t_i p_i^-}, \quad \text{for all } \ x \in Q_i. \tag{2.1.24}$$

Let $u \in W \setminus \{0\}$. Set $v := \frac{u}{\|u\|_W}$. Then using Proposition 2.1.1, we obtain

$$\int_{\mathbb{R}^N} |v|^{\beta(x)} dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} dxdy = 1. \tag{2.1.25}$$

The last relation implies that

$$\int_{Q_i} |v|^{\beta(x)} dx + \int_{Q_i}\int_{Q_i} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} dxdy \leq 1, \qquad \text{for all } i \in \mathbb{N}. \tag{2.1.26}$$

Here in the rest of the proof, we consider C to be a generic positive constant which may vary from line to line but depends only on $p, s, \beta, \gamma, \epsilon$ and Q_1, where Q_1 denotes the cube centered at the origin with sides of length $\epsilon > 0$ and parallel to the coordinate axes. By

Lemma 1.2.4, we get

$$\|v\|_{L^{\gamma(\cdot)}(Q_i)} \le C\|v\|_{L^{\gamma_i^+}(Q_i)} \tag{2.1.27}$$

Again using Lemma 1.2.4 and (2.1.24), we have

$$\|v\|_{L^{p_i^-}(Q_i)} \le C\|v\|_{L^{\beta(\cdot)}(Q_i)}. \tag{2.1.28}$$

Set $W_i := W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(Q_i)$. Then, arguing in a similar way as obtained in [50, (3.22)] and using (2.1.28), we deduce

$$\|v\|_{W^{t_i,p_i^-}(Q_i)} \le C\|v\|_{W_i}. \tag{2.1.29}$$

Arguing as in [30, Proof Theorem 5.4] with a translation, we can find an extension $\widetilde{v} \in W^{t_i,p_i^-}(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N$ such that $\widetilde{v} = v$ on Q_i, and the following holds:

$$\|\widetilde{v}\|_{W^{t_i,p_i^-}(\mathbb{R}^N)} \le C\|v\|_{W^{t_i,p_i^-}(Q_i)}. \tag{2.1.30}$$

Now from Proposition 1.2.2, it follows that

$$\|\widetilde{v}\|^{p_i^-}_{L^{(p_{t_i}^-)^*}(\mathbb{R}^N)} \le C\|\widetilde{v}\|_{W^{t_i,p_i^-}(\mathbb{R}^N)},$$

which implies that

$$\|v\|_{L^{\gamma_i^+}(Q_i)} \le C\|\widetilde{v}\|_{W^{t_i,p_i^-}(\mathbb{R}^N)}. \tag{2.1.31}$$

By taking into account (2.1.27)-(2.1.31), there exists a positive constant C such that

$$\|v\|_{L^{\gamma(\cdot)}(Q_i)} \le C\|v\|_{W_i}, \quad \text{for all } i \in \mathbb{N}. \tag{2.1.32}$$

Now for further estimating (2.1.32), we require to consider the following two cases:

Case $\|v\|_{L^{\gamma(\cdot)}(Q_i)} \ge 1$**:** Then using Lemma 1.2.1, Proposition 2.1.1, (2.1.26), and (2.1.32), we get

$$\int_{Q_i} |v|^{\gamma(x)}dx \le \|v\|^{\gamma_i^+}_{L^{\gamma(\cdot)}(Q_i)} \le C^{\gamma_i^+}\|v\|^{\gamma_i^+}_{W_i}$$

$$\leq C^{\gamma_i^+}\left(\int_{Q_i}|v|^{\beta(x)}dx+\int_{Q_i}\int_{Q_i}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dxdy\right)^{\frac{\gamma_i^+}{p_i^+}}$$
$$\leq C^{\gamma_i^+}\left(\int_{Q_i}|v|^{\beta(x)}dx+\int_{Q_i}\int_{Q_i}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dxdy\right).$$

Case $\|v\|_{L^{\gamma(\cdot)}(Q_i)}<1$: Then again combining Lemma 1.2.1, Proposition 2.1.1, (2.1.26), and (2.1.32), we obtain

$$\int_{Q_i}|v|^{\gamma(x)}dx\leq\|v\|^{\gamma_i^-}_{L^{\gamma(\cdot)}(Q_i)}\leq C^{\gamma_i^-}\|v\|_{W_i}$$
$$\leq C^{\gamma_i^-}\left(\int_{Q_i}|v|^{\beta(x)}dx+\int_{Q_i}\int_{Q_i}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dxdy\right)^{\frac{\gamma_i^-}{p_i^+}}$$
$$\leq C^{\gamma_i^-}\left(\int_{Q_i}|v|^{\beta(x)}dx+\int_{Q_i}\int_{Q_i}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dxdy\right).$$

Therefore, in both the cases, for any $i\in\mathbb{N}$, we have

$$\int_{Q_i}|v|^{\gamma(x)}dx\leq\left(C^{\gamma^-}+C^{\gamma^+}\right)\left(\int_{Q_i}|v|^{\beta(x)}dx+\int_{Q_i}\int_{Q_i}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dxdy\right).$$

Summing up the last inequality over all $i\in\mathbb{N}$ and using (2.1.25), we obtain

$$\int_{\mathbb{R}^N}|v|^{\gamma(x)}dx\leq C^{\gamma^-}+C^{\gamma^+}.$$

Thus, by closed graph theorem we infer that $W\hookrightarrow L^{\gamma(\cdot)}(\mathbb{R}^N)$ and hence, we get (2.1.22).

Next, we prove the compactness of this embedding. Let B be any ball in $\mathbb{R}^N$. Let $u_n\rightharpoonup 0$ weakly in $W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$ and hence, $u_n\rightharpoonup 0$ weakly in $W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(B)$ as $n\to+\infty$. Now applying Theorem 2.1.1, we get $u_n\to 0$ in $L^{\gamma(\cdot)}(B)$. This completes the proof of the theorem. □

Remark 2.1.2. *Note that the exponent $p_s^*(x):=\frac{N\overline{p}(x)}{N-\overline{s}(x)\overline{p}(x)}$ is termed as the Sobolev type critical exponent with variable order and variable exponent in the fractional framework.*

2.1.2 Functional properties of the spaces

In this sub-section, we discuss the completeness, uniform convexity, separability, and density properties of the space W.

Lemma 2.1.1. *$(W, \|\cdot\|_W)$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in W. For any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that if $n, k \geq N_\epsilon$

$$\|u_n - u_k\|_W \leq \epsilon. \tag{2.1.33}$$

Now using Theorem 2.1.1, (2.1.33), and the fact that $(L^{\beta(\cdot)}(\Omega), \|\cdot\|_{L^{\beta(\cdot)}(\Omega)})$ is a Banach space, we get that there exists $u \in L^{\beta(\cdot)}(\Omega)$ such that $u_n \to u$ in $L^{\beta(\cdot)}(\Omega)$ strongly as $n \to +\infty$. So, there exists a subsequence $\{u_{n_j}\}$ such that $u_{n_j}(x) \to u(x)$ a.e. $x \in \Omega$. By Fatou's lemma, (1.2.12), and (2.1.33) with $\epsilon = 1$, we obtain

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\, dxdy \\
&\leq \liminf_{n\to+\infty} \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\, dxdy \\
&\leq \liminf_{n\to+\infty} \int_\Omega \int_\Omega \frac{|u_n(x) - u_{N_1}(x) - u_{N_1}(y) + u_{N_1}(x) + u_{N_1}(y) - u_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\, dxdy \\
&\leq 2^{p^+} \liminf_{n\to+\infty} \Big[\int_\Omega \int_\Omega \frac{(|u_n(x) - u_{N_1}(x)) - (u_n(y) - u_{N_1}(y))|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\, dxdy \\
&\qquad + \int_\Omega \int_\Omega \frac{|u_{N_1}(x) - u_{N_1}(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\, dxdy \Big] \\
&\leq 2^{p^+} \Big[1 + \int_\Omega \int_\Omega \frac{|u_{N_1}(x) - u_{N_1}(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\, dxdy \Big] < \infty.
\end{aligned} \tag{2.1.34}$$

Therefore, $u \in W$. Now again (2.1.33) together with Fatou's lemma implies that, for all $n, n_j \geq N_\epsilon$,

$$\rho_W(u_n - u) \leq \liminf_{j\to\infty} \rho_W(u_n - u_{n_j}) \leq \epsilon. \tag{2.1.35}$$

Finally, from Proposition 2.1.2, we infer that $u_n \to u$ in W as $n \to +\infty$. Hence, $(W, \|\cdot\|_W)$

is a Banach space. □

Lemma 2.1.2. *The space $(W, \|\cdot\|_W)$ is uniformly convex, reflexive, and separable.*

Proof. For proving uniform convexity of W, we define the map $T : W \to L^{\beta(\cdot)}(\Omega) \times L^{p(\cdot,\cdot)}(\Omega\times\Omega)$ as

$$T(u) = \left(u(x), \ \frac{|u(x - u(y)|}{|x-y|^{s(x,y)+\frac{N}{p(x,y)}}} \right).$$

The norm on $L^{\beta(\cdot)}(\Omega) \times L^{p(\cdot,\cdot)}(\Omega\times\Omega)$ is given as

$$\|u\| = \|u\|_{L^{\beta(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot,\cdot)}(\Omega\times\Omega)}.$$

Clearly T is an isometry. Hence, $T(W)$ is uniformly convex being a closed subspace of the uniformly convex Banach space $L^{\beta(\cdot)}(\Omega) \times L^{p(\cdot,\cdot)}(\Omega\times\Omega)$ (see [18, Proposition 3.20]) and consequently, W is uniformly convex and hence, it is reflexive. Arguing similarly, we get that W is separable (see [18, Proposition 3.25]). □

Next, we have the following density result. For proving this result, we adapt the approach as in [10], in which the authors studied the same result for $s(\cdot,\cdot) = s$, constant. For completeness, we give the proof here.

Lemma 2.1.3. *If $\Omega = \mathbb{R}^N$, the space $C_0^\infty(\mathbb{R}^N)$ is a dense subspace of W.*

Proof. Let $u \in W$. Take $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \geq 0$ with $supp(\varphi) \subset \tilde{A}$ and

$$\int_{\tilde{A}} \varphi(x)dx = 1.$$

We set u_ϵ as the mollifier of u, that is,

$$u_\epsilon := \int_{\mathbb{R}^N} \varphi_\epsilon(x-y)u(y)dy, \quad x \in \mathbb{R}^N,$$

where $\varphi_\epsilon(x) = \epsilon^{-N}\varphi\left(\frac{x}{\epsilon}\right)$. Since $u \in L^{\beta(\cdot)}(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{\beta(\cdot)}(\mathbb{R}^N)$, we get

$$\|u_\epsilon - u\|_{L^{\beta(\cdot)}(\mathbb{R}^N)} \to 0 \quad \text{as } \epsilon \to 0. \tag{2.1.36}$$

We claim that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\epsilon(x)-u(x)-u_\epsilon(y)+u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\to 0 \quad \text{as } \epsilon\to 0. \tag{2.1.37}$$

First using Lemma 1.2.3 together with Tonelli's theorem and Fubini's theorem, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\epsilon(x)-u(x)-u_\epsilon(y)+u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[\int_{\mathbb{R}^N}(u(x-w)-u(y-w))\varphi_\epsilon(w)dw-u(x)+u(y)\right]^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[\int_{\tilde{A}}(u(x-\epsilon w)-u(y-\epsilon w)-u(x)+u(y))\varphi(w)dw\right]^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\
&\leq (meas(\tilde{A}))^{p^-+p^+-1}\\
&\quad\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[\int_{\tilde{A}}|u(x-\epsilon w)-u(y-\epsilon w)-u(x)+u(y)|^{p(x,y)}\varphi^{p(x,y)}(w)dw\right]}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\
&\leq (meas(\tilde{A}))^{p^-+p^+-1}\\
&\quad\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\tilde{A}}\frac{|u(x-\epsilon w)-u(y-\epsilon w)-u(x)+u(y)|^{p(x,y)}\varphi^{p(x,y)}(w)}{|x-y|^{N+s(x,y)p(x,y)}}dxdydw\\
&\leq (meas(\tilde{A}))^{p^-+p^+-1}\\
&\quad\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\tilde{A}}\frac{|u(x-\epsilon w)-u(y-\epsilon w)-u(x)+u(y)|^{p(x,y)}(\varphi^{p^-}(w)+\varphi^{p^+}(w))}{|x-y|^{N+s(x,y)p(x,y)}}dxdydw.
\end{aligned} \tag{2.1.38}$$

Next, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x-\epsilon w)-u(y-\epsilon zw)-u(x)+u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\to 0 \quad \text{as } \epsilon\to 0. \tag{2.1.39}$$

Indeed, we fix $w\in\tilde{A}$ and set $z:=(w,w)\in\mathbb{R}^N\times\mathbb{R}^N$. Define the function $v:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ as

$$v(x,y)=\left(\frac{(u(x)-u(y))}{|x-y|^{N+s(x,y)p(x,y)}}\right)^{\frac{1}{p(x,y)}}, \quad \text{for all } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N.$$

Clearly $v\in L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)$. Again using the fact that $C_0^\infty(\mathbb{R}^N\times\mathbb{R}^N)$ is dense in $L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)$, we get that for a given $\epsilon'>0$, there exists $h\in C_0^\infty(\mathbb{R}^N\times\mathbb{R}^N)$ such that $\|v-h\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)}<\frac{\epsilon'}{3}$, which implies

$$\begin{aligned}\|v(\cdot-\epsilon z)-v\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)} &\leq \|v(\cdot-\epsilon z)-h(\cdot-\epsilon z)\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)}\\ &\quad+\|h(\cdot-\epsilon z)-h\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)}+\|v-h\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)}\\ &\leq \frac{\epsilon'}{3}+\frac{\epsilon'}{3}+\frac{\epsilon'}{3}=\epsilon',\end{aligned}$$

for ϵ' sufficiently small. Therefore we get (2.1.39). Furthermore, for a.e. $w\in\tilde{A}$ and for any $\epsilon>0$, using (1.2.12), we have

$$\begin{aligned}&(\varphi^{p^-}(w)+\varphi^{p^+}(w))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x-\epsilon w)-u(y-\epsilon w)-u(x)+u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\ &\leq 2^{p^+-1}(\varphi^{p^-}(w)+\varphi^{p^+}(w))\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x-\epsilon w)-u(y-\epsilon w)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\ &\qquad\qquad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\Big)\\ &\leq 2^{p^+}(\varphi^{p^-}(w)+\varphi^{p^+}(w))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\in L^\infty(\tilde{A}).\end{aligned}\tag{2.1.40}$$

Therefore, combining (2.1.39) and (2.1.40) and using Lebesgue dominated convergence theorem, we obtain

$$\int_{\tilde{A}}\varphi^{p(x,y)}(w)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x-\epsilon w)-u(y-\epsilon w)-u(x)+u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdydw\to 0\ \text{ as }\epsilon\to 0.$$

Hence, (2.1.37) holds true. Finally, we conclude the proof of the lemma from (2.1.36) and (2.1.37). □

2.1.3 Functional spaces for non-local Dirichlet BVP

To study non-local problems involving the operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ with Dirichlet boundary data via variational methods, we define another new fractional type variable order Sobolev spaces with variable exponents. One can refer to [15] and references therein for this type of spaces in the fractional p-Laplacian framework. Let Ω be an open bounded subset of $\mathbb{R}^N$. Assume that (S_1), (P_1) hold true and the variable exponent $\beta(\cdot)\in C_+(\overline{\Omega})$ such that

$\beta(x) \geq \overline{p}(x)$, for all $x \in \overline{\Omega}$. Set $Q := (\mathbb{R}^N \times \mathbb{R}^N) \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$ and define

$$X^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega) := \Big\{u : \mathbb{R}^N \to \mathbb{R} : u_{|\Omega} \in L^{\beta(\cdot)}(\Omega),$$
$$\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}}dxdy < \infty, \text{ for some } \eta > 0\Big\}.$$

The space $X^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)$ is a normed linear space equipped with the following norm:

$$\|u\|_{X^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)} := \|u\|_{L^{\beta(\cdot)}(\Omega)} + \inf\Big\{\eta > 0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\eta^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}}dxdy < 1\Big\}.$$

Next, for any open and bounded set $\Omega \subset \mathbb{R}^N$, we define the following subspace of $W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$:

$$X_0^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega) := \{u \in W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N) \ : \ u = 0 \ a.e. \ in \ \Omega^c\}.$$

From now on wards, we denote $X_0^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\Omega)$ by X_0, in short. By Lemma 2.1.3, it follows that $C_0^\infty(\Omega)$ is dense in X_0. It can be verified that the following defines a norm on X_0 :

$$\|u\|_{X_0} := \inf\Big\{\eta > 0 : \rho_{X_0}\left(\frac{u}{\eta}\right) < 1\Big\},$$

where $\rho_{X_0} : X_0 \to \mathbb{R}$ is the modular function, defined as

$$\rho_{X_0}(u) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy, \quad \text{for } \ u \in X_0. \tag{2.1.41}$$

The interplay between the norm in X_0 and the modular function ρ_{X_0} can be studied in the following lemmas:

Lemma 2.1.4. *Let $u \in X_0$ and ρ_{X_0} be defined as in* (2.1.41)*. Then we have the following results:*

(*i*) $\|u\|_{X_0} < 1(=1;>1) \iff \rho_{X_0}(u) < 1(=1;>1)$.

(*ii*) *If* $\|u\|_{X_0} > 1$, *then* $\|u\|_{X_0}^{p^-} \leq \rho_{X_0}(u) \leq \|u\|_{X_0}^{p^+}$.

(*iii*) *If* $\|u\|_{X_0} < 1$, *then* $\|u\|_{X_0}^{p^+} \leq \rho_{X_0}(u) \leq \|u\|_{X_0}^{p^-}$.

Lemma 2.1.5. *Let $u, u_n \in X_0$, $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\lim_{n\to+\infty} \|u_n - u\|_{X_0} = 0$.

(ii) $\lim_{n\to+\infty} \rho_{X_0}(u_n - u) = 0$.

The proofs of Lemma 2.1.4 and Lemma 2.1.5 follow in the same line as the proofs of Theorem 3.1 and Theorem 3.2, respectively, in [38].

Now we study the following Sobolev type embedding theorem for the space X_0. The proof of this theorem is motivated by [9], where the author studied the result, for $s(\cdot,\cdot) = s$, constant.

Theorem 2.1.3. *Let Ω be an open and bounded set in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary. Assume that $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1), respectively, with $s^+p^+ < N$ and $\beta(\cdot) \in C_+(\overline{\Omega})$ such that $\overline{p}(x) \leq \beta(x) < p_s^*(x)$, for all $x \in \overline{\Omega}$. Then for any $\gamma \in C_+(\overline{\Omega})$ with $1 < \gamma(x) < p_s^*(x)$, for all $x \in \overline{\Omega}$, there exits a constant $C = C(N, s, p, \gamma, \beta, \Omega) > 0$ such that, for every $u \in X_0$,*

$$\|u\|_{L^{\gamma(\cdot)}(\Omega)} \leq C\|u\|_{X_0}.$$

Moreover, this embedding is compact.

Proof. First we note that, using Tietze extension theorem, we can extend $\beta(\cdot)$ and $\gamma(\cdot)$ on $\mathbb{R}^N$ continuously, such that $\overline{p}(x) \leq \beta(x) < p_s^*(x)$ and $1 < \gamma(x) < p_s^*(x)$, for all $x \in \mathbb{R}^N$. Next, we claim that there exists a constant $C' > 0$ such that

$$\|u\|_{L^{\beta(\cdot)}(\Omega)} \leq \frac{1}{C'}\|u\|_{X_0}, \text{ for all } u \in X_0. \tag{2.1.42}$$

This is equivalent to proving that, for $A := \{u \in X_0 : \|u\|_{L^{\beta(\cdot)}(\Omega)} = 1\}$, $\inf_{u\in A} \|u\|_{X_0}$ is achieved. Let $\{u_n\} \subset A$ be a minimizing sequence, that is, $\|u_n\|_{X_0} \downarrow \inf_{u\in A} \|u\|_{X_0} := C'$ as $n \to +\infty$. This implies that $\{u_n\}$ is bounded in X_0 and $L^{\beta(\cdot)}(\Omega)$ and hence, in W. Therefore, up to a sub-sequence, still denoted by $\{u_n\}$, $u_n \rightharpoonup u_0$ in W as $n \to +\infty$. Now from Theorem 2.1.1, it follows that $u_n \to u_0$ strongly in $L^{\beta(\cdot)}(\Omega)$ as $n \to +\infty$. We extend u_0 to $\mathbb{R}^N$ by setting $u_0(x) = 0$ on $x \in \mathbb{R}^N \setminus \Omega$. This implies $u_n(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^N$ as

$n \to +\infty$. Hence by using Fatou's Lemma, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy,$$

which implies that $\|u_0\|_{X_0} \leq \liminf_{n\to+\infty}\|u_n\|_{X_0} = C'$ and thus, $u_0 \in X_0$. Also, since $\|u_0\|_{L^{\beta(\cdot)}(\Omega)} = 1$, we get that $u_0 \in A$. Therefore, $\|u_0\|_{X_0} = C'$. This proves (2.1.42). and hence, from (2.1.42), it follows that

$$\|u\|_W = \|u\|_{L^{\beta(\cdot)}(\Omega)} + [u]_{\Omega}^{s(x,y),p(x,y)} \leq \|u\|_{L^{\beta(\cdot)}(\Omega)} + \|u\|_{X_0} \leq (1+\frac{1}{C'})\|u\|_{X_0}.$$

This implies that X_0 is continuously embedded in W. As Theorem 2.1.1 gives that W is continuously embedded in $L^{\gamma(\cdot)}(\Omega)$, it follows that there exists a constant $C(N,s,p,\gamma,\beta,\Omega)$ > 0 such that

$$\|u\|_{L^{\gamma(\cdot)}(\Omega)} \leq C(N,s,p,\gamma,\beta,\Omega)\|u\|_{X_0}. \tag{2.1.43}$$

Next, we prove that the embedding given in (2.1.43) is compact. Let $\{v_n\}$ be a bounded sequence in X_0. This implies that $\{v_n\}$ is bounded in W. Hence using Theorem 2.1.1, we infer that there exists $v_0 \in L^{\gamma(\cdot)}(\Omega)$ such that, up to a subsequence, still denoted by $\{v_n\}$, $v_n \to v_0$ strongly in $L^{\gamma(\cdot)}(\Omega)$ as $n \to +\infty$. This completes the theorem. □

Using Theorem 2.1.3 together with the fact that X_0 is a closed subspace of the separable reflexive Banach space $W = W^{s(\cdot,\cdot),\beta(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_W$, we have the following proposition:

Proposition 2.1.3. $(X_0, \|\cdot\|_{X_0})$ *is a uniformly convex, reflexive, and separable Banach space.*

Remark 2.1.3. *From now on-wards, in the subsequent chapters, we consider* $\beta(x) = \overline{p}(x)$ *and for brevity, we still denote the space* $X_0^{s(\cdot),\overline{p}(\cdot),p(\cdot,\cdot)}(\Omega)$ *by* X_0.

2.2 Conclusion

In this chapter, first we have defined a new type of variable order fractional Sobolev spaces with variable exponents and then we have established the basic functional properties such as, completeness, separability, reflexivity, density, etc., for these spaces. The main highlight of this chapter is proving the continuous and compact embedding of the variable order fractional Sobolev spaces with variable exponents into appropriate variable exponent Lebesgue spaces, for $s^+p^+ < N$. Here we would like to mention that we have improved the embedding results as proved in [50, 57] by imposing variable growth in the order s and by considering $\beta(x) \geq \overline{p}(x)$, for all $x \in \mathbb{R}^N$. $\square$

3

Non-local problems involving generalized Choquard type nonlinearity

In this chapter, we study the existence of solutions of the non-local problem involving the operator $-(\Delta)^{s(\cdot)}_{p(\cdot)}$ with Choquard type nonlinearity. First we establish a Hardy-Littlewood-Sobolev type inequality appropriate for the functions belonging to variable order fractional Sobolev spaces with variable exponents. Then using this inequality, we study the existence and multiplicity of solutions for some generalized Choquard equations involving variable-order fractional $p(\cdot)$-Laplacian.

3.1 Hardy-Littlewood-Sobolev inequality type result in $W^{s(\cdot,\cdot),\bar{p}(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$ framework

For studying Choquard type problem with non-local operator involving variable order and variable exponents, we require the theorem stated below. We consider the variable exponent $\mu(\cdot,\cdot)$ satisfying the following assumption:

(μ_1) $\mu : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is uniformly continuous and symmetric function, that is, $\mu(x,y) = \mu(y,x)$, for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, with $0 < \mu^- \le \mu^+ < N$.

Theorem 3.1.1. *Let $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ be uniformly continuous functions on $\mathbb{R}^N \times \mathbb{R}^N$ satisfying (S_1) and (P_1), respectively, with $s^+p^+ < N$ and let $\mu(\cdot,\cdot)$ satisfy (μ_1). Assume that $q \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ such that*

$$\frac{2}{q(x,y)} + \frac{\mu(x,y)}{N} = 2, \quad \textit{for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Set $\mathcal{M} := \left\{ r(\cdot) \in C_+(\mathbb{R}^N) : p(x,x) \le r(x)q^- \le r(x)q^+ < p_s^*(x), \text{ for all } x \in \mathbb{R}^N \right\}$. Then for $u \in W^{s(\cdot,\cdot),\bar{p}(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$ and $r(\cdot) \in \mathcal{M}$, we have $|u|^{r(\cdot)} \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{r(x)}|u(y)|^{r(y)}}{|x-y|^{\mu(x,y)}}dxdy \le C\left(\||u|^{r(\cdot)}\|^2_{L^{q^+}(\mathbb{R}^N)} + \||u|^{r(\cdot)}\|^2_{L^{q^-}(\mathbb{R}^N)} \right) \tag{3.1.1}$$

where $C > 0$ is a constant, independent of u.

To prove the above theorem, first we establish the following Hardy-Littlewood-Sobolev inequality type result in the variable exponents framework for two-point functions $p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ defined in $\mathbb{R}^N \times \mathbb{R}^N$. In [4], Alves et al. established the Hardy-Littlewood-Sobolev type result for variable exponents taking $p(\cdot), q(\cdot)$ as one-point functions defined in $\mathbb{R}^N$.

Proposition 3.1.1. *Let $p(\cdot,\cdot)$, $q(\cdot,\cdot)$ be any two uniformly continuous functions satisfying (P_1) and let $\mu(\cdot,\cdot)$ satisfy (μ_1) such that the following relation holds:*

$$\frac{1}{p(x,y)} + \frac{1}{q(x,y)} + \frac{\mu(x,y)}{N} = 2. \tag{3.1.2}$$

Assume that there exist two sequences $\{(x_n, y_n)\}$ and $\{(x'_n, y'_n)\}$ in $\mathbb{R}^N \times \mathbb{R}^N$ such that $\lim_{n\to+\infty} p(x_n, y_n) = p^+$, $\lim_{n\to+\infty} q(x_n, y_n) = q^+$ and $\lim_{n\to+\infty} p(x'_n, y'_n) = p^-$, $\lim_{n\to+\infty} q(x'_n, y'_n) = q^-$. Then for $h \in L^{p^+}(\mathbb{R}^N) \cap L^{p^-}(\mathbb{R}^N)$ and $g \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$, we have

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h(x)g(y)}{|x-y|^{\mu(x,y)}} dxdy \right| \leq K \left(\|h\|_{L^{p^+}(\mathbb{R}^N)} \|g\|_{L^{q^+}(\mathbb{R}^N)} + \|h\|_{L^{p^-}(\mathbb{R}^N)} \|g\|_{L^{q^-}(\mathbb{R}^N)} \right),$$

where $C := C(p, q, \mu, N) > 0$ is a constant, independent of h and g.

Proof. Using (3.1.2), first we obtain the following upper bound on $\mu(\cdot,\cdot)$. We have

$$\mu^+ = \sup_{\mathbb{R}^N \times \mathbb{R}^N} \mu(x,y) = 2N \lim_{n\to\infty} \left(1 - \frac{1}{2p(x,y)} - \frac{1}{2q(x,y)}\right) \leq 2N \left(1 - \frac{1}{2p^+} - \frac{1}{2q^+}\right).$$

Also, from the continuity assumption on $\mu(\cdot,\cdot)$, $p(\cdot,\cdot)$, and $q(\cdot,\cdot)$, we get

$$\begin{aligned} \lim_{n\to+\infty} \mu(x_n, y_n) &= 2N \lim_{n\to+\infty} \left(1 - \frac{1}{2p(x_n,y_n)} - \frac{1}{2q(x_n,y_n)}\right) \\ &= 2N \left(1 - \frac{1}{2p^+} - \frac{1}{2q^+}\right). \end{aligned}$$

This implies that $\mu^+ = 2N\left(1 - \frac{1}{2p^+} - \frac{1}{2q^+}\right)$. Similarly, we have $\mu^- = 2N\left(1 - \frac{1}{2p^-} - \frac{1}{2q^-}\right)$. Now, using the Hardy-Littlewood-Sobolev inequality for constant exponent (see (1.3.15)), we have the following estimate.

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{h(x)g(y)}{|x-y|^{\mu(x,y)}} dxdy \right| &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h(x)g(y)|}{|x-y|^{\mu(x,y)}} dxdy \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h(x)g(y)|}{|x-y|^{\mu^+}} dxdy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h(x)g(y)|}{|x-y|^{\mu^-}} dxdy \\ &\leq C \left(\|h\|_{L^{p^+}(\mathbb{R}^N)} \|g\|_{L^{q^+}(\mathbb{R}^N)} + \|h\|_{L^{p^-}(\mathbb{R}^N)} \|g\|_{L^{q^-}(\mathbb{R}^N)} \right), \end{aligned}$$

This completes the proof of the proposition. □

Proof of Theorem 3.1.1: Using Theorem 2.1.2, it is easy to check that for $u \in W^{s(\cdot,\cdot),\overline{p}(\cdot),p(\cdot,\cdot)}(\mathbb{R}^N)$, we have $|u|^{r(\cdot)} \in L^{q^+}(\mathbb{R}^N) \cap L^{q^-}(\mathbb{R}^N)$. Now the proof follows by taking $h(x) = g(x) = |u|^{r(x)}$ and $q(x,y) = p(x,y)$ in Proposition 3.1.1. □

Remark 3.1.1. *From Theorem 3.1.1, we can define the variable order and variable exponent Hardy-Littlewood-Sobolev critical exponent as*

$$p^*_{s,\mu}(x) := \frac{p^*_s(x)}{\overline{q}(x)} = \frac{\overline{p}(x)}{2}\left(\frac{2N-\overline{\mu}(x)}{N-\overline{s}(x)\overline{p}(x)}\right),$$

where $\overline{q}(x) = q(x,x)$ *and* $\overline{\mu}(x) = \mu(x,x)$, *for all* $x \in \mathbb{R}^N$.

3.2 Existence of solution of non-local problem with purely Choquard type nonlinearity

In this section, we study the existence of the solution of the following variable order non-local Choquard equation with variable exponents:

$$\left.\begin{aligned}(-\Delta)^{s(\cdot)}_{p(\cdot)}u(x) &= \left(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}}dy\right) f(x,u(x)), \quad x\in\Omega,\\ u &= 0, \qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{R}^N\setminus\Omega,\end{aligned}\right\} \tag{3.2.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $s(\cdot,\cdot)$, $p(\cdot,\cdot)$, and $\mu(\cdot,\cdot)$ satisfy (S_1), (P_1), and (μ_1), respectively. The nonlinearity $f \in C(\Omega\times\Omega,\mathbb{R})$ and $F(x,t) = \int_0^t f(x,\tau)d\tau$ a.e. $x \in \Omega$ with the following assumptions:

(F_1) There exists a constant $M_0 > 0$ and a function $r(\cdot) \in C_+(\mathbb{R}^N)\cap\mathcal{M}$ with $r^- > p^+$ such that

$$|f(x,t)| \leq M_0(|t|^{r(x)-1}), \text{ for all } (x,t)\in\Omega\times\mathbb{R}.$$

(F_2) There exist a constant $\nu > p^+$ such that $0 < \nu F(x,t) \leq 2tf(x,t)$, for all $t \in \mathbb{R}\setminus\{0\}$ and for a.e. $x \in \Omega$.

Definition 3.2.1. *The energy functional* $\mathscr{J} : X_0 \to \mathbb{R}$ *associated to* (3.2.3) *is defined as*

$$\mathscr{J}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\mu(x,y)}}dxdy. \tag{3.2.4}$$

Note that the first term in the definition of $\mathscr{J}$ is C^1. Set

$$\Psi := \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u(x)F(y,u(y))}{|x-y|^{\mu(x,y)}}, \quad u \in X_0.$$

Now under the assumption (F_1), we study the differentiability of the functional Ψ. The proof follows using the similar argument as in [4, Section 3]. For completeness, we give the proof here.

Lemma 3.2.1. *The functional Ψ is well defined and belongs to $C^1(X_0,\mathbb{R})$ with*

$$\langle \Psi'(u), v\rangle_{X_0} = \int_\Omega\int_\Omega \frac{F(y,u(y)f(x,u(x))v(x))}{|x-y|^{\mu(x,y)}} dxdy,$$

for all $u, v \in X_0$, where $\langle\cdot,\cdot\rangle_{X_0}$ denotes the dual pairing between X_0 and its topological dual X_0^*.

Proof. By (F_1) and Proposition 3.1.1, Ψ is well defined. Now we divide the proof into three parts:

Step 1: First we show the existence of the Fréchet derivative. Let $u, v \in X_0$ and $t \in [-1,1]$. Then we have

$$\begin{aligned} &\frac{\Psi(u+tv)-\Psi(u)}{t} \\ &= \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u(x)+tv(x))F(y,u(y)+tv(y)) - F(x,u(x))F(y,u(y))}{t|x-y|^{\mu(x,y)}} dxdy. \end{aligned} \tag{3.2.5}$$

Set

$$\begin{aligned} \mathcal{I}_* :=& \frac{F(x,u(x)+tv(x))(F(y,u(y)+tv(y)) - F(y,u(y))}{t} \\ &+ \frac{F(y,u(y))(F(x,u(x)+tv(x)) - F(x,u(x)))}{t}. \end{aligned}$$

By the mean value theorem, there exist $\theta(x,t), \xi(y,t) \in [0,1]$ such that

$$F(y,u(y)+tv(y)) - F(y,u(y)) = f(y,u(y)+\xi(y,t)tv(y))v(y)t$$

and

$$F(x,u(x)+tv(x))-F(x,u(x))=f(x,u(x)+\theta(x,t)tv(x))v(x)t.$$

From (3.2.5), we have

$$\left|\frac{\Psi(u+tv)-\Psi(u)}{t}-\int_\Omega\int_\Omega\frac{F(x,u(x))f(y,u(y))v(y)}{|x-y|^{\mu(x,y)}}dxdy\right|\leq|A_1^t|+|A_2^t|,$$

where

$$A_1^t:=\int_\Omega\int_\Omega\frac{F(x,u(x)+tv(x))f(y,u(y)+\xi(y,t)tv(y))v(y)-F(x,u(x)f(y,u(y))v(y))}{2|x-y|^{\mu(x,y)}}dxdy$$

and

$$A_2^t:=\int_\Omega\int_\Omega\frac{F(y,u(y))f(x,u(x)+\theta(x,t)tv(x))v(x)-F(x,u(x))f(y,u(y))v(y)}{2|x-y|^{\mu(x,y)}}dxdy.$$

Using (μ_1) and Fubini's Theorem, we deduce

$$\begin{aligned}\int_\Omega\int_\Omega\frac{F(x,u(x))f(y,u(y))v(y)}{|x-y|^{\mu(x,y)}}dxdy&=\int_\Omega\int_\Omega\frac{F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\mu(y,x)}}dydx\\&=\int_\Omega\int_\Omega\frac{F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\mu(y,x)}}dxdy\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\mu(x,y)}}dxdy.\end{aligned}$$

Therefore, A_2^t can be rewritten as

$$A_2^t:=\frac{1}{2}\int_\Omega\int_\Omega\frac{F(y,u(y))f(x,u(x)+\theta(x,t)tv(x))v(x)-F(y,u(y))f(x,u(x))v(x)}{|x-y|^{\mu(x,y)}}dxdy.$$

Then using Proposition 3.1.1, we obtain

$$\begin{aligned}|A_2^t|\leq\,&C\|F(\cdot,u)\|_{L^{q^+}(\Omega)}\|f(\cdot,u+\theta(\cdot,t)v)v-f(\cdot,u)v\|_{L^{q^+}(\Omega)}\\&+C\|F(\cdot,u)\|_{L^{q^-}(\Omega)}\|f(\cdot,u+\theta(\cdot,t)v)v-f(\cdot,u)v\|_{L^{q^-}(\Omega)},\end{aligned}$$

where the constant $C>0$ is independent of u,v,θ. Since $\theta(x,t)\in[0,1]$ and $t\in[-1,1]$, by

(F_1), we get

$$|f(x,u(x)+\theta(t,x)tv(x))v(x)-f(x,u(x))v(x)|^{q^+} \\ \leq C(|u(x)|^{q^+(r(x)-1)}|v(x)|^{q^+}+|v(x)|^{q^+r(x)}), \quad (3.2.6)$$

where $C>0$ is some constant that does not depend on u, v, θ. Now Theorem 2.1.3 ensure that the right-hand side of the inequality (3.2.6) is integrable. Thus, the Lebesgue dominated convergence theorem implies that

$$\|f(\cdot,u+\theta(\cdot,t)v)v-f(\cdot,u)v\|_{L^{q^+}(\Omega)}\to 0 \quad \text{as} \quad t\to 0.$$

Similarly, we get

$$\|f(\cdot,u+\theta(\cdot,t)tv)v-f(\cdot,u)v\|_{L^{q^-}(\mathbb{R}^N)}\to 0 \quad \text{as} \quad t\to 0.$$

The last two limits yield that $A_2^t\to 0$ as $t\to 0$. Now we have the following estimation on A_1^t:

$$|A_1^t|\leq \frac{1}{2}\int_\Omega\int_\Omega \frac{|F(x,u(x))||f(y,u(y)+\xi(y,t)tv(y))v(y)-f(y,v(y))v(y)|}{|x-y|^{\mu(x,y)}}dxdy \\ +\frac{1}{2}\int_\Omega\int_\Omega \frac{|f(y,u(y)+\xi(y,t)tv(y))v(y)||F(x,u(x)+tv(x))-F(x,u(x))|}{|x-y|^{\mu(x,y)}}dxdy.$$

Arguing as above, we deduce

$$\int_\Omega\int_\Omega \frac{|F(x,u(x))||f(y,u(y)+\xi(y,t)tv(y))v(y)-f(y,v(y))v(y)|}{|x-y|^{\mu(x,y)}}dxdy\to 0$$

as $t\to 0$. Again, by the Lebesgue dominated convergence theorem, we have that

$$\|F(\cdot,u+tv)-F(\cdot,u)\|_{L^{q^+}(\Omega)}\to 0 \text{ as } t\to 0 \quad (3.2.7)$$

and

$$\|F(\cdot,u+tv)-F(\cdot,u)\|_{L^{q^-}(\Omega)}\to 0 \text{ as } t\to 0. \quad (3.2.8)$$

As in (3.2.6), $\|f(\cdot, u+\xi(\cdot,t)tv)v\|_{L^{q^+}(\Omega)}$ and $\|f(\cdot, u+\xi(\cdot,t)tv)v\|_{L^{q^-}(\Omega)}$ are uniformly bounded by a constant which is independent of $t \in [-1,1]$. Thus, Proposition 3.1.1, (3.2.7), and (3.2.8) give us

$$\int_\Omega\int_\Omega \frac{|f(y,u(y)+\xi(y,t)tv(y))v(y)||F(x,u(x)+tv(x))-F(x,u(x))|}{|x-y|^{\mu(x,y)}}dxdy \to 0$$

as $t\to 0$, and so, $A_1^t \to 0$ as $t \to 0$. Therefore, from the above discussion, we get

$$\lim_{t\to 0}\frac{\Psi(u+tv)-\Psi(u)}{t} = \int_\Omega\int_\Omega \frac{F(x,u(x)f(y,u(y))v(y))}{|x-y|^{\mu(x,y)}}dxdy.$$

Hence, the last relation shows the existence of the Fréchet derivative $\frac{\partial\Psi(u)}{\partial v}$.

Step 2: Next, we prove that $\frac{\partial\Psi(u)}{\partial(\cdot)} \in X_0^*$, for all $u\in X_0$.

Clearly $\frac{\partial\Psi(u)}{\partial v}$ is linear at v for each fixed u. Next, we aim to show that

$$\left|\frac{\partial\Psi(u)}{\partial v}\right| \le C_u\|v\|, \quad \text{for all} \quad v\in X_0, \tag{3.2.9}$$

where $C_u>0$ is a constant, independent of $v\in X_0$. From (F_1) and Proposition 3.1.1, we obtain

$$\begin{aligned}\left|\int_\Omega\int_\Omega \frac{F(x,u(x))f(y,u(y))v(y)}{|x-y|^{\mu(x,y)}}dxdy\right| &\le C\|F(\cdot,u)\|_{L^{q^+}(\Omega)}\|f(\cdot,u)v\|_{L^{q^+}(\Omega)} \\ &\quad + C\|F(\cdot,u)\|_{L^{q^-}(\Omega)}\|f(\cdot,u)v\|_{L^{q^-}(\Omega)},\end{aligned} \tag{3.2.10}$$

where $C>0$ is a constant that does not depend on u,v. If $\|v\|_{X_0}\le 1$, then using Lemma 1.2.1, Lemma 1.2.3, Lemma 1.2.5, (F_1), and Theorem 2.1.3, we deduce

$$\begin{aligned}\|f(\cdot,u)v\|_{L^{q^-}(\Omega)} &= \int_\Omega |f(y,u(y))v(y)|^{q^+}dy \\ &\le \frac{1}{2}\||u|^{q^+(r(\cdot)-1)}\|_{L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)}\||v|^{q^+}\|_{L^{r(\cdot)}(\Omega} \\ &\le \frac{1}{2}\left(\|u\|^{q^+(r^+-1)}_{L^{q^+r(\cdot)}(\Omega)} + \|u\|^{q^+(r^--1)}_{L^{q^+r(\cdot)}(\Omega)}\right)\|v\|^{q^+}_{L^{q^+r(\cdot)}(\Omega)} \\ &\le K_1\left(\|u\|^{q^+(r^+-1)}_{X_0} + \|u\|^{q^+(r^--1)}_{X_0}\right)\|v\|^{q^+}_{X_0} \\ &\le C_{u_1},\end{aligned} \tag{3.2.11}$$

where $C_{u_1} := K_1 \left(\|u\|_{X_0}^{q^+(r^+-1)} + \|u\|_{X_0}^{q^+(r^--1)} \right)$ and K_1 is a positive constant that does not depend on u and v. Similarly,

$$\|f(\cdot, u)v\|_{L^{q^-}(\Omega)} \leq C_{u_2}, \text{ for all } v \in X_0 \text{ with } \|v\|_{X_0} \leq 1, \tag{3.2.12}$$

where $C_{u_2} := K_2 \left(\|u\|_{X_0}^{q^+(r^+-1)} + \|u\|_{X_0}^{q^+(r^--1)} \right)$ and K_2 is a positive constant that does not depend on u and v. Now combining (3.2.10), (3.2.11), and (3.2.12), we get (3.2.9). Therefore, for each $u \in X_0$, $\frac{\partial \Psi(u)}{\partial(\cdot)} \in X_0^*$, that is, the map $v \longmapsto \frac{\partial \Psi(u)}{\partial v}$ is a continuous linear functional.

Step 3: Finally, we claim that if for any sequence $\{u_n\}$ in X_0, $u_n \to u$ strongly in X_0, then

$$\sup_{\|v\|_{X_0} \leq 1} \left| \frac{\partial \Psi(u_n)}{\partial v} - \frac{\partial \Psi(u)}{\partial v} \right| \to 0 \quad \text{as } n \to +\infty. \tag{3.2.13}$$

Consider $v \in X_0$ with $\|v\|_{X_0} \leq 1$ and note that

$$\begin{aligned} \left| \frac{\partial \Psi(u_n)}{\partial v} - \frac{\partial \Psi(u)}{\partial v} \right| \leq & \int_\Omega \int_\Omega \frac{|F(x, u_n(x)) - F(x, u(x))||f(y, u_n(y))v(y)|}{|x-y|^{\mu(x,y)}} dxdy \\ & + \int_\Omega \int_\Omega \frac{|F(x, u(x))||f(y, u_n(y))v(y) - f(y, u(y))v(y)|}{|x-y|^{\mu(x,y)}} dxdy \\ & := A_f^n + A_F^n. \end{aligned}$$

By Proposition 3.1.1, we get

$$\begin{aligned} A_f^n \leq & C\|F(\cdot, u_n) - F(\cdot, u))\|_{L^{q^+}(\Omega)} \|f(\cdot, u_n)v\|_{L^{q^+}(\Omega)} \\ & + C\|F(\cdot, u_n) - F(\cdot, u))\|_{L^{q^-}(\Omega)} \|f(\cdot, u_n)v\|_{L^{q^-}(\Omega)}, \end{aligned}$$

where $C > 0$ is a constant that does not depend on u, v, θ. Using the continuity of the function F along with (F_1), Theorem 2.1.3, and Lebesgue dominated converges theorem, we obtain

$$\|F(\cdot, u_n) - F(\cdot, u)\|_{L^{q^+}(\Omega)}, \|F(\cdot, u_n) - F(\cdot, u)\|_{L^{q^-}(\Omega)} \to 0 \quad \text{as } n \to +\infty.$$

Since $\{u_n\}$ is bounded in X_0, arguing similarly as obtained in (3.2.11) and (3.2.12), we get that the sequences $\{\|f(\cdot,u_n)v\|_{L^{q^+}(\Omega)}\}$ and $\{\|f(\cdot,u_n)v\|_{L^{q^-}(\Omega)}\}$ are bounded. Therefore, using the last two limits, we have

$$\sup_{\substack{v\in X_0\\ \|v\|_{X_0}\leq 1}} \|F(\cdot,u_n)-F(\cdot,u))\|_{L^{q^+}(\Omega)}\|f(\cdot,u_n)v\|_{L^{q^+}(\Omega)} \to 0 \tag{3.2.14}$$

and

$$\sup_{\substack{v\in X_0\\ \|v\|_{X_0}\leq 1}} \|F(\cdot,u_n)-F(\cdot,u))\|_{L^{q^-}(\Omega)}\|f(\cdot,u_n)v\|_{L^{q^-}(\Omega)} \to 0 \tag{3.2.15}$$

as $n\to+\infty$. Thus,

$$A_n^f \to 0 \quad \text{as } n\to+\infty. \tag{3.2.16}$$

Now we estimate A_F^n. Note that by Proposition 3.1.1, we get

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|F(x,u(x))||(f(y,u_n(y))-f(y,u(y)))v(y)|}{|x-y|^{\mu(x,y)}}dxdy\\
&\leq C\|F(\cdot,u)\|_{L^{q^+}(\Omega)}\|(f(\cdot,u_n)-f(\cdot,u))v\|_{L^{q^+}(\Omega)}\\
&\quad + C\|F(\cdot,u)\|_{L^{q^-}(\Omega)}\|(f(\cdot,u_n)-f(\cdot,u))v\|_{L^{q^-}(\Omega)}\\
&\leq C\left(\|(f(\cdot,u_n)-f(\cdot,u))v\|_{L^{q^+}(\Omega)}+\|(f(\cdot,u_n)-f(\cdot,u))v\|_{L^{q^-}(\Omega)}\right)
\end{aligned} \tag{3.2.17}$$

where $C>0$ is some constant, independent of v,u_n,n. Now for $v\in X_0$ with $\|v\|_{X_0}\leq 1$, using (F_1) together with Lemma 1.2.3, Lemma 1.2.5, and Theorem 2.1.3, we deduce

$$\begin{aligned}
&\int_\Omega |(f(y,u_n(y))-f(x,u(x)))v(y)|^{q^+}dy\\
&\leq C\left(\||u_n|^{q^+(r(\cdot)-1)}\|_{L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)}\||v|^{q^+}\|_{L^{r(\cdot)}(\Omega)}+\||u|^{q^+(r(\cdot)-1)}\|_{L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)}\||v|^{q^+}\|_{L^{r(\cdot)}(\Omega)}\right)\\
&\leq C\left(\|u_n\|^{q^+(r^+-1)}_{L^{q^+r(\cdot)}(\Omega)}+\|u_n\|^{q^+(r^--1)}_{L^{q^+r(\cdot)}(\Omega)}+\|u\|^{q^+(r^+-1)}_{L^{q^+r(\cdot)}(\Omega)}+\|u\|^{q^+(r^--1)}_{L^{q^+r(\cdot)}(\Omega)}\right)\|v\|^{q^+}_{L^{q^+r(\cdot)}(\Omega)}\\
&\leq \tilde{C}_0,
\end{aligned} \tag{3.2.18}$$

where the constants $C,\tilde{C}_0>0$ do not depend on v,u_n,n. Thus, using the continuity of f and Lebesgue dominated convergence theorem, together with (3.2.18), we have

$$\sup_{\substack{v\in X_0\\ \|v\|_{X_0}\leq 1}} \|(f(\cdot,u_n)-f(\cdot,u))v\|_{L^{q^+}(\Omega)} \to 0 \text{ as } n\to+\infty. \tag{3.2.19}$$

Likewise

$$\sup_{\substack{v\in X_0\\ \|v\|_{X_0}\leq 1}} \|(f(\cdot,u_n)-f(\cdot,u))v\|_{L^{q^-}(\Omega)} \to 0 \text{ as } n\to+\infty. \tag{3.2.20}$$

Taking into account (3.2.17), (3.2.19) and (3.2.20), it follows that

$$\sup_{\substack{v\in X_0\\ \|v\|_{X_0}\leq 1}} \int_\Omega\int_\Omega \frac{|F(x,u(x))||(f(y,u_n(y))-f(y,u(y)))v(y)|}{|x-y|^{\mu(x,y)}}dxdy \to 0 \tag{3.2.21}$$

as $n\to+\infty$. Thus, using (3.2.14), (3.2.15), and (3.2.21), we get (3.2.13). Hence, the proof of the lemma is complete. □

Thus, using Lemma 3.2.1, we have that $\mathscr{J}$ is well-defined and C^1 on X_0 with the derivative $\mathscr{J}': X_0 \to X_0^*$, given as

$$\begin{aligned}\langle \mathscr{J}'(u),w\rangle_{X_0} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\ &\quad -\int_\Omega\int_\Omega \frac{F(y,u(y))f(x,u(x))w(x)}{|x-y|^{\mu(x,y)}}dxdy, \quad \text{for } u,w\in X_0. \end{aligned}\tag{3.2.22}$$

We define the weak solution of problem (3.2.3) in the functional space X_0 (defined in Chapter 2) as follows:

Definition 3.2.2. *A function $u\in X_0$ is called a weak solution of* (3.2.3)*, if for every $w\in X_0$, we have*

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\ &= \int_\Omega\int_\Omega \frac{F(x,u(x))f(y,u(y))w(y)}{|x-y|^{\mu(x,y)}}dxdy.\end{aligned}$$

Now by the standard critical point theory, the weak solutions of (3.2.3) are characterized by the critical points of $\mathscr{J}$. Now we have the following existence result for the non-trivial solution of (3.2.3):

Theorem 3.2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $\mu(\cdot,\cdot)$, $s(\cdot,\cdot)$, $p(\cdot,\cdot)$ satisfy (μ_1), (S_1), (P_1), respectively, with $s^+p^+ < N$ and $q(\cdot,\cdot)$ be as in Theorem 3.1.1. Also, assume that f satisfies (F_1) and (F_2). Then (3.2.3) admits a non-trivial weak solution.*

First we show that $\mathscr{J}$ admits the mountain pass geometry. Precisely, we have the following lemma:

Lemma 3.2.2. *(Mountain pass geometry for $\mathscr{J}$) Let the assumptions in Theorem 3.2.1 hold. Then we have the following assertions:*

(i) There exists $\delta > 0$ such that $\mathscr{J}(u) \geq R > 0$, for all $u \in X_0$ with $\|u\|_{X_0} = \delta$.

(ii) There exists $\phi \in X_0$ with $\|\phi\|_{X_0} > \delta$ such that $\mathscr{J}(\phi) < 0$.

Proof. (i) Using (F_1), Lemma 2.1.4 and Theorem 2.1.3, we note that for $u \in X_0$, $F(x, u(x)) \in L^{q^+}(\Omega) \cap L^{q^-}(\Omega)$. Indeed, from (F_1), we have $F(x, 0) = 0$ and thus, using Lemma 1.2.5 and Theorem 2.1.3, we get

$$\begin{aligned}\|F(\cdot, u(\cdot))\|_{L^{q^+}(\mathbb{R}^N)} = \|F(\cdot, u(\cdot))\|_{L^{q^+}(\Omega)} &\leq C_1 \left(\int_\Omega |u(x)|^{r(x)q^+} dx\right)^{1/q^+}\\ &\leq C_1 \left(\|u\|^{r^+}_{L^{r(\cdot)q^+}(\Omega)} + \|u\|^{r^-}_{L^{r(\cdot)q^+}(\Omega)}\right)\\ &\leq C_2 \left(\|u\|^{r^+}_{X_0} + \|u\|^{r^-}_{X_0}\right), \qquad (3.2.23)\end{aligned}$$

where the constant C_1, $C_2 > 0$ are independent of u. Similarly for $u \in X_0$, we have that $F(x, u(x)) \in L^{q^-}(\Omega)$ and

$$\|F(\cdot, u(\cdot))\|_{L^{q^-}(\mathbb{R}^N)} = \|F(\cdot, u(\cdot))\|_{L^{q^-}(\Omega)} \leq C_3 \left(\|u\|^{r^+}_{X_0} + \|u\|^{r^-}_{X_0}\right), \qquad (3.2.24)$$

for some constant $C_3 > 0$, independent of u. Hence, from Theorem 3.1.1, (3.2.23), and (3.2.24), we infer that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x, u(x))F(y, u(y))}{|x-y|^{\mu(x,y)}} dxdy\right| &\leq C\left(\|F(\cdot, u(\cdot))\|^2_{L^{q^+}(\mathbb{R}^N)} + \|F(\cdot, u(\cdot))\|^2_{L^{q^-}(\mathbb{R}^N)}\right)\\ &\leq C_4 \left(\|u\|^{2r^+}_{X_0} + \|u\|^{2r^-}_{X_0}\right), \qquad (3.2.25)\end{aligned}$$

where the constants $C, C_4 > 0$ are independent of u. Using Lemma 2.1.4 and (3.2.25), for $\|u\|_{X_0} < 1$, we obtain

$$\begin{aligned}
\mathscr{J}(u) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \frac{1}{2}\int_{\Omega}\int_{\Omega}\frac{F(x,u(x))F(y,u(y))}{|x-y|^{\mu(x,y)}}dxdy\\
&\geq \frac{1}{p^+}\|u\|_{X_0}^{p^+} - C_5\left(\|u\|_{X_0}^{2r^+} + \|u\|_{X_0}^{2r^-}\right)\\
&\geq \frac{1}{p^+}\|u\|_{X_0}^{p^+} - C_5\|u\|_{X_0}^{2r^-},
\end{aligned}$$

where the constant $C_5 > 0$ is independent of $u \in X_0$. Now noting that $r^- > p^+$, we can choose $\delta > 0$ sufficiently small such that $\mathscr{J}(u) \geq R > 0$, for all $u \in X_0$ with $\|u\|_{X_0} = \delta$.
(ii) Recalling [97, Lemma 4] and using (F_2), it follows that there exist two constants l_1, $l_2 > 0$ such that

$$F(x,t) \geq l_1|t|^{\nu/2}, \tag{3.2.26}$$

for all $x \in \Omega$ and $|t| \geq l_2$. Now for $\xi \in X_0$ with $\xi > 0$ and $t > 0$ sufficiently large, using Lemma 2.1.4 and (3.2.26), we deduce

$$\begin{aligned}
\mathscr{J}(t\xi) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|t\xi(x)-t\xi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \frac{1}{2}\int_{\Omega}\int_{\Omega}\frac{F(x,t\xi(x))F(y,t\xi(y))}{|x-y|^{\mu(x,y)}}dxdy\\
&\leq \frac{t^{p^+}}{p^-}\|\xi\|_{X_0}^{p^+} - \frac{l_1^2 t^{\nu}}{2}\int_{\Omega}\int_{\Omega}\frac{|\xi(x)|^{\nu/2}|\xi(y)|^{\nu/2}}{|x-y|^{\mu(x,y)}}dxdy.
\end{aligned}$$

Since $p^+ < \nu$ in (F_2), it follows that $J(t\xi) \to -\infty$ as $t \to +\infty$. This guarantees the existence of $\phi \in X_0$ with $\|\phi\|_{X_0} > \delta$ such that $J(\phi) < 0$. □

Now we show that the energy functional $\mathscr{J}$ satisfies Palais-Smale condition.

Lemma 3.2.3. *Let the assumptions in Theorem 3.2.1 hold. Then for any $c \in \mathbb{R}$, the functional $\mathscr{J}$ satisfies the Palais-Smale (in short $(PS)_c$) condition.*

Proof. Let $\{u_n\} \subset X_0$ be a $(PS)_c$ sequence for the functional $\mathscr{J}$, that is, $\mathscr{J}(u_n) \to c$ and $\|\mathscr{J}'(u_n)\|_{X_0^*} \to 0$ as $n \to +\infty$. Note that, $\{u_n\}$ is bounded in X_0. Indeed, if $\{u_n\}$ is unbounded in X_0, there exists some constant $d_0 > 0$ such that, for sufficiently large $n \in \mathbb{N}$, using Theorem 2.1.3, Lemma 2.1.4, together with (F_2), we deduce

$$\begin{aligned}
& d_0 + d_0\|u_n\|_{X_0} \\
& \geq \mathscr{J}(u_n) - \frac{1}{\nu}\langle \mathscr{J}'(u_n), u_n\rangle_{X_0} \\
& = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u_n(x))F(y,u_n(y))}{|x-y|^{\mu(x,y)}}dxdy \\
& \quad - \frac{1}{\nu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy + \frac{1}{\nu}\int_\Omega\int_\Omega \frac{F(y,u_n(y))f(x,u_n(x))u_n(x)}{|x-y|^{\mu(x,y)}}dxdy \\
& > \left(\frac{1}{p^+} - \frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy \\
& \quad + \int_\Omega\int_\Omega \frac{F(y,u_n(y))\left[\frac{1}{\nu}f(x,u_n(x))u_n(x) - \frac{1}{2}F(x,u_n(x))\right]}{|x-y|^{\mu(x,y)}}dxdy \\
& > \left(\frac{1}{p^+} - \frac{1}{\nu}\right)\|u_n\|_{X_0}^{p^-}.
\end{aligned}$$

Since $1 < p^- \leq p^+ < \nu$ in (F_2), from the above expression, we get a contradiction and hence, the sequence $\{u_n\}$ is bounded in X_0. Since X_0 is a reflexive Banach space (Proposition 2.1.3), it follows that there exists $u_0 \in X_0$ such that up to a sub-sequence (still denoted by u_n), $u_n \rightharpoonup u_0$ weakly in X_0 and $u_n(x) \to u_0(x)$ point-wise a.e. $x \in \mathbb{R}^N$ as $n \to +\infty$. We claim that $u_n \to u_0$ strongly in X_0 as $n \to +\infty$. We define $I : X_0 \to X_0^*$ as

$$\langle I(u), w\rangle_{X_0} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy, \tag{3.2.27}$$

where $u, w \in X_0$. Since $\|\mathscr{J}'(u_n)\|_{X_0^*} \to 0$ as $n \to \infty$, taking $w = u_n - u_0$, in (3.2.22) and using (3.2.27), we get

$$\begin{aligned}
o_n(1) &= \left\langle \mathscr{J}'(u_n), (u_n - u_0)\right\rangle_{X_0} \\
&= \langle I(u_n), (u_n-u_0)\rangle_{X_0} - \int_\Omega\int_\Omega \frac{F(x,u_n(x)f(y,u_n(y))(u_n-u_0)(y)}{|x-y|^{\mu(x,y)}}dxdy. \qquad (3.2.28)
\end{aligned}$$

Now using Lemma 1.2.5, (1.2.12), Theorem 2.1.3 and Proposition 3.1.1, we estimate the second term in the right-hand side of (3.2.28) as follows: First we note that

$$\begin{aligned}
&\left|\int_\Omega\int_\Omega \frac{F(y,u_n(y))f(x,u_n(x))(u_n(x)-u_0(x))}{|x-y|^{\mu(x,y)}}dxdy\right| \\
&\leq C\|F(\cdot,u_n(\cdot))\|_{L^{q^+}(\Omega)}\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|_{L^{q^+}(\Omega)} \\
&\quad + C\|F(\cdot,u_n(\cdot))\|_{L^{q^-}(\Omega)}\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|_{L^{q^-}(\Omega)} \\
&\leq C_6\left(\|u_n\|^{r^+}_{L^{r(\cdot)q^+}(\Omega)}+\|u_n\|^{r^-}_{L^{r(\cdot)q^+}(\Omega)}\right)\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|_{L^{q^+}(\Omega)} \\
&\quad + C_6\left(\|u_0\|^{r^+}_{L^{r(\cdot)q^-}(\Omega)}+\|u_0\|^{r^-}_{L^{r(\cdot)q^-}(\Omega)}\right)\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|_{L^{q^-}(\Omega)} \\
&\leq C_7\left(\|u_n\|^{r^+}_{X_0}+\|u_n\|^{r^-}_{X_0}\right) \\
&\qquad\left(\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|_{L^{q^+}(\Omega)}+\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|_{L^{q^-}(\Omega)}\right), \quad (3.2.29)
\end{aligned}$$

where the constants $C, C_6, C_7 > 0$ are independent of n, u_n. Combining (F_1), together with Lemma 1.2.3 and Lemma 1.2.5 and the fact that $u_n \to u_0$ strongly in $L^{q^- r(\cdot)}(\Omega)$ as $n \to +\infty$, we have

$$\begin{aligned}
&\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|^{q^+}_{L^{q^+}(\Omega)} \\
&= \int_\Omega |f(x,u_n(x))(u_n(x)-u_0(x))|^{q^+}dx \\
&\leq M_0^{q^+}\int_\Omega |u_n(x)|^{(r(x)-1)q^+}|\,|(u_n(x)-u_0(x))|^{q^+}dx \\
&\leq C_8\|u_n^{(r(\cdot)-1)q^+}\|_{L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)}\|(u_n-u_0)^{q^+}\|_{L^{r(\cdot)}(\Omega)} \\
&\leq C_9\left(\|u_n\|^{(r^+-1)q^+}_{L^{r(\cdot)q^+}(\Omega)}+\|u_n\|^{(r^--1)q^+}_{L^{r(\cdot)q^+}(\Omega)}\right)\|(u_n-u_0)\|^{q^+}_{L^{q^+r(\cdot)}(\Omega)} \\
&\leq C_{10}\left(\|u_n\|^{(r^+-1)q^+}_{X_0}+\|u_n\|^{(r^--1)q^+}_{X_0}\right)\|(u_n-u_0)\|^{q^+}_{L^{q^+r(\cdot)}(\Omega)} \\
&\leq C_{11}\|(u_n-u_0)\|^{q^+}_{L^{q^+r(\cdot)}(\Omega)} = o_n(1). \quad (3.2.30)
\end{aligned}$$

Here C_8, C_9, C_{10} and C_{11} are non-negative constants which do not depend on n, u_n. Again arguing similarly as above, we obtain

$$\|f(\cdot,u_n(\cdot))(u_n(\cdot)-u_0(\cdot))\|_{L^{q^-}(\Omega)} = o_n(1). \quad (3.2.31)$$

Thus, combining (3.2.28)-(3.2.31), we deduce $\lim_{n\to+\infty}\langle I(u_n),(u_n-u_0)\rangle_{X_0}=0$. Also, one can check that for any $v\in X_0$, $\langle I(u_0),v\rangle_{X_0}$ defines a linear functional. Hence, using the fact that $u_n \rightharpoonup u_0$ weakly in X_0, we get $\lim_{n\to+\infty}\langle I(u_0),(u_n-u_0)\rangle_{X_0}=0$. From the last two limits, it follows that

$$\big\langle (I(u_n)-I(u_0)),(u_n-u_0)\big\rangle_{X_0}=o_n(1). \tag{3.2.32}$$

We denote $v_n:=u_n-u_0$ and define $\Omega_1:=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : 1<p(x,y)<2\}$, $\Omega_2:=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : p(x,y)\geq 2\}$. Then we have the following estimate:

$$\begin{aligned}\rho_{X_0}(v_n)&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\&=\int_{\Omega_1}\frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy+\int_{\Omega_2}\frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy.\end{aligned} \tag{3.2.33}$$

Let us set the following:

$$\begin{aligned}g_n^{(1)}(x,y)&:=\Bigg[\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(v_n(x)-v_n(y))}{|x-y|^{N+s(x,y)p(x,y)}}\\&\qquad-\frac{|u_0(x)-u_0(y)|^{p(x,y)-2}(u_0(x)-u_0(y))(v_n(x)-v_n(y))}{|x-y|^{N+s(x,y)p(x,y)}}\Bigg];\\g_n^{(2)}(x,y)&:=\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}};\ g_n^{(3)}(x,y):=\frac{|u_0(x)-u_0(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}.\end{aligned}$$

For all $n\in\mathbb{N}$, using (1.2.10), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}g_n^{(1)}(x,y)dxdy\geq 0,$$

which together with (3.2.32) implies that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}g_n^{(1)}(x,y)dxdy=0. \tag{3.2.34}$$

Now for $(x,y)\in\Omega_1$, taking into account Lemma 1.2.3, Lemma 1.2.5, Simon's inequality (1.2.10), (1.2.12), Lemma 2.1.4, and (3.2.34), we deduce

$$\begin{aligned}
&\int_{\Omega_1} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy \\
&\leq \frac{1}{(p^- -1)} \int_{\Omega_1} \left(g_n^{(1)}(x,y)\right)^{\frac{p(x,y)}{2}} \left(g_n^{(2)}(x,y)+g_n^{(3)}(x,y)\right)^{\frac{2-p(x,y)}{2}} dxdy \\
&\leq C_{12} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left[\left((g_n^{(1)})^{\frac{p(x,y)}{2}} \cdot (g_n^{(2)})^{\frac{2-p(x,y)}{2}}\right) + \left((g_n^{(1)})^{\frac{p(x,y)}{2}} \cdot (g_n^{(3)})^{\frac{2-p(x,y)}{2}}\right)\right] dxdy \\
&\leq C_{12}\Big[\|(g_n^{(1)})^{\frac{p(\cdot,\cdot)}{2}}\|_{L^{\frac{2}{p(\cdot,\cdot)}}(\mathbb{R}^N\times\mathbb{R}^N)} \\
&\quad \times \left(\|(g_n^{(2)})^{\frac{2-p(\cdot,\cdot)}{2}}\|_{L^{\frac{2}{2-p(\cdot,\cdot)}}(\mathbb{R}^N\times\mathbb{R}^N)} + \|(g_n^{(3)})^{\frac{2-p(\cdot,\cdot)}{2}}\|_{L^{\frac{2}{2-p(\cdot,\cdot)}}(\mathbb{R}^N\times\mathbb{R}^N)}\right)\Big] \\
&\leq C_{12}\Big[\left(\|g_n^{(1)}\|^{\frac{p^+}{2}}_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} + \|g_n^{(1)}\|^{\frac{p^-}{2}}_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}\right) \\
&\quad \times \left(\|g_n^{(2)}\|^{\frac{2-p^+}{2}}_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} + \|g_n^{(2)}\|^{\frac{2-p^-}{2}}_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} + \|g_n^{(3)}\|^{\frac{2-p^+}{2}}_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} + \|g_n^{(3)}\|^{\frac{2-p^-}{2}}_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}\right)\Big] \\
&= o_n(1), \qquad (3.2.35)
\end{aligned}$$

where $C_{12} > 0$ is a constant, independent of n, u_n. Next, for $(x,y) \in \Omega_2$, taking into account Lemma 1.2.3, Lemma 1.2.5, (1.2.10), (1.2.12), Lemma 2.1.4, and (3.2.32), we get

$$\int_{\Omega_2} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy \leq 2^{p^+}\langle (I(u_n)-I(u_0)),(u_n-u_0)\rangle_{X_0} = o_n(1). \qquad (3.2.36)$$

Thus, from (3.2.33), (3.2.35), and (3.2.36), we have $\lim_{n\to+\infty} \rho_{X_0}(v_n) = 0$. This implies that $\lim_{n\to+\infty} \|v_n\|_{X_0} = \lim_{n\to+\infty} \|u_n-u_0\|_{X_0} = 0$, thanks to Lemma 2.1.5. Hence, $u_n \to u_0$ strongly in X_0 as $n \to +\infty$. This completes the lemma. □

Now we give the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1: From Lemma 3.2.2 and Lemma 3.2.3, it follows that $\mathscr{J}$ satisfies the mountain pass geometry and Palais-Smale condition. Therefore, by using mountain pass theorem (Theorem 1.3.1), we infer that there exists $u_0 \in X_0$, a critical point of $\mathscr{J}$, with

$$\mathscr{J}(u_0) = \bar{c} > 0. \qquad (3.2.37)$$

Since $\mathscr{J}(0) = 0$, thanks to (F_1), we get that $u_0 (\not\equiv 0)$ is a non-trivial weak solution of (3.2.3). □

3.3 Existence of multiple solutions of non-local Choquard problem with nonlinearity of concave-convex type

Motivated by the pioneer work of Cerami et al. [6] on problems involving concave and convex nonlinearities in the case of local operator and [11] in the case of non-local operator, we study the existence of multiple solutions for variable order non-local Choquard problem with variable exponents involving concave-convex nonlinearities. Consider the equation

$$\left.\begin{aligned} (-\Delta)^{s(\cdot)}_{p(\cdot)}u(x) &= \lambda|u(x)|^{\alpha(x)-2}u(x) + \left(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}}dy\right) f(x,u(x)),\ x\in\Omega,\\ u &= 0, \qquad\qquad\qquad\qquad x\in\mathbb{R}^N\setminus\Omega, \end{aligned}\right\} \tag{3.3.38}$$

where $\lambda > 0$ is a real parameter, the functions $s(\cdot,\cdot)$, $p(\cdot,\cdot)$, $\mu(\cdot,\cdot)$, and the nonlinearities f and F are as in Theorem 3.2.1, and the variable exponent $\alpha(\cdot) \in C_+(\overline{\Omega})$ with some appropriate assumption.

Definition 3.3.1. *A function $u \in X_0$ is called a weak solution of* (3.3.38)*, if for every $w \in X_0$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy \\ &= \lambda\int_\Omega |u(x)|^{\alpha(x)-2}u(x)w(x)dx + \int_\Omega\int_\Omega \frac{F(y,u(y))f(x,u(x))w(x)}{|x-y|^{\mu(x,y)}}dxdy. \end{aligned}$$

Definition 3.3.2. *The energy functional $J_\lambda : X_0 \to \mathbb{R}$ associated to* (3.3.38) *is defined as*

$$\begin{aligned} J_\lambda(u) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \lambda\int_\Omega \frac{|u|^{\alpha(x)}}{\alpha(x)}dx \\ &- \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\mu(x,y)}}dxdy. \end{aligned}$$

Note that in view of Lemma 3.2.1, $J_\lambda \in C^1(X_0,\mathbb{R})$ and the weak solutions of (3.3.38) are characterized by the critical points of J_λ. Now we state and prove the existence of two weak solutions of (3.3.38).

Theorem 3.3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $\mu(\cdot,\cdot)$, $s(\cdot,\cdot)$, $p(\cdot,\cdot)$ satisfy (μ_1), (S_1), (P_1), respectively, with $s^+p^+ < N$ and $q(\cdot,\cdot)$ be as in Theorem 3.1.1. Also, assume that the variable exponent $\alpha(\cdot) \in C_+(\overline{\Omega})$ such that $\alpha^+ < p^-$ and f satisfies $(F_1)-(F_2)$. Then there exists $\Lambda_* > 0$ such that, for all $\lambda \in (0, \Lambda_*)$, (3.3.38) admits at least two distinct non-trivial weak solutions.*

For proving Theorem 3.3.1, we require the following two results:

Lemma 3.3.1. *(Mountain pass geometry for J_λ) Let the assumptions in Theorem 3.3.1 hold. Then we have the following assertions:*

(i) There exists $\Lambda_* > 0$ such that for every $\lambda \in (0, \Lambda_*)$, there exist $R_\lambda > 0$ and $0 < \delta_\lambda << 1$ such that $J_\lambda(u) \geq R_\lambda > 0$, for all $u \in X_0$ with $\|u\|_{X_0} = \delta_\lambda$.

(ii) There exists $\Phi \in X_0$ with $\|\Phi\|_{X_0} > \delta_\lambda$ such that $J_\lambda(\Phi) < 0$.

(iii) There exists $\psi \in X_0$, $\psi > 0$ such that $J_\lambda(t\psi) < 0$, for all $t \to 0^+$.

Proof. (i) Using Lemma 1.2.5, Theorem 2.1.3, Theorem 3.1.1, (3.2.25), together with (F_1) and (F_2), for $\|u\|_{X_0} < 1$, we have

$$\begin{aligned}
J_\lambda(u) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \lambda\int_\Omega \frac{|u|^{\alpha(x)}}{\alpha(x)}dx \\
&\quad - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\mu(x,y)}}dxdy \\
&\geq \frac{1}{p^+}\|u\|_{X_0}^{p^+} - \frac{\lambda}{\alpha^-}\left(\|u\|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^-} + \|u\|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^+}\right) - C_{13}\left(\|u\|_{X_0}^{2r^-} + \|u\|_{X_0}^{2r^+}\right) \\
&\geq \frac{1}{p^+}\|u\|_{X_0}^{p^+} - \frac{\lambda C_{14}}{\alpha^-}\|u\|_{X_0}^{\alpha^-} - C_{13}\|u\|_{X_0}^{2r^-} \\
&\geq \left(\frac{1}{p^+} - \frac{\lambda C_{14}}{\alpha^-}\|u\|_{X_0}^{\alpha^- - p^+} - C_{13}\|u\|_{X_0}^{2r^- - p^+}\right)\|u\|_{X_0}^{p^+}, \qquad (3.3.39)
\end{aligned}$$

where the constants $C_{13}, C_{14} > 0$ are independent of u. Now for each $\lambda > 0$, we define the function, $T_\lambda : (0, +\infty) \to \mathbb{R}$ as

$$T_\lambda(t) = C_{14}\frac{\lambda}{\alpha^-}t^{\alpha^- - p^+} + C_{13}t^{2r^- - p^+}.$$

Since we have $1 < \alpha^- < p^+ < r^-$, it follows that $\lim_{t\to 0} T_\lambda(t) = \lim_{t\to\infty} T_\lambda(t) = +\infty$.

Thus, we can find infimum of T_λ. Equating

$$T'_\lambda(t) = \frac{\alpha^- - p^+}{\alpha^-}\lambda C_{14} + C_{13}(2r^- - p^+)t^{2r^- - \alpha^-} = 0,$$

we note that $t_0 := \left(\lambda \frac{p^+ - \alpha^-}{(2r^- - p^+)\alpha^-} \cdot \frac{C_{14}}{C_{13}}\right)^{1/(2r^- - \alpha^-)}$ is the critical point of $T_\lambda(t)$. Clearly $t_0 > 0$. Also, it can be checked that $T''_\lambda(t_0) > 0$ and hence, the infimum of $T_\lambda(t)$ is achieved at t_0. Now observe that

$$\begin{aligned}
&T_\lambda(t_0) \\
&= \lambda \frac{C_{14}}{\alpha^-}\left(\lambda \frac{p^+ - \alpha^-}{(2r^- - p^+)\alpha^-} \cdot \frac{C_{14}}{C_{13}}\right)^{\frac{\alpha^- - p^+}{2r^- - \alpha^-}} + C_{13}\left(\lambda \frac{p^+ - \alpha^-}{(2r^- - p^+)\alpha^-} \cdot \frac{C_{14}}{C_{13}}\right)^{\frac{2r^- - p^+}{r^- - \alpha^-}} \\
&= \lambda^{\frac{2r^- - p^+}{r^- - \alpha^-}} \cdot C_{15} \to 0 \text{ as } \lambda \to 0^+,
\end{aligned} \tag{3.3.40}$$

where $C_{15} > 0$, is a constant that is independent of u. Therefore, we infer from (3.3.39) that there exists $\Lambda_* > 0$ such that for any $\lambda \in (0, \Lambda_*)$, we can choose $R_\lambda > 0$ and $0 < \delta_\lambda << 1$ such that

$$J_\lambda(u) \geq R_\lambda > 0, \text{ for all } u \in X_0 \text{ with } \|u\|_{X_0} = \delta_\lambda. \tag{3.3.41}$$

(ii) For $\xi \in X_0, \xi > 0$, we obtain

$$\begin{aligned}
J_\lambda(t\xi) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|t\xi(x) - t\xi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \lambda\int_\Omega \frac{|t\xi|^{\alpha(x)}}{\alpha(x)}dx \\
&- \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x, t\xi(x))F(y, t\xi(y))}{|x-y|^{\mu(x,y)}}dxdy \\
\leq &\frac{t^{p^+}}{p^-}\|\xi\|_{X_0}^{p^+} - \frac{l_1^2 t^\nu}{2}\int_\Omega\int_\Omega \frac{|\xi(x)|^{\nu/2}|\xi(y)|^{\nu/2}}{|x-y|^{\mu(x,y)}}dxdy \to -\infty \text{ as } t \to +\infty.
\end{aligned}$$

Thus, there exists $t_* > 0$ with $\Phi = t_*\xi \in X_0$ such that

$$\|\Phi\|_{X_0} > \delta_\lambda \text{ and } J_\lambda(\Phi) < 0. \tag{3.3.42}$$

(iii) For $\psi \in X_0$, $\psi > 0$, and for $t > 0$ sufficiently small, we have $\|t\psi\|_{X_0} < 1$. Then from

(F_2) and Lemma 2.1.4, we obtain

$$\begin{aligned}
J_\lambda(t\psi) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|t\psi(x)-t\psi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \lambda\int_\Omega \frac{|t\psi|^{\alpha(x)}}{\alpha(x)}dx \\
&\quad - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,t\psi)F(y,t\psi)}{|x-y|^{\mu(x,y)}}dxdy \\
&\le \frac{t^{p^-}}{p^-}\|\psi\|_{X_0}^{p^-} - \frac{\lambda t^{\alpha^+}}{\alpha^+}\int_\Omega |\psi(x)|^{\alpha(x)}dx.
\end{aligned}$$

Since $\alpha^+ < p^-$, it follows that $J_\lambda(t\psi) < 0$ as $t \to 0^+$.

□

Lemma 3.3.2. *Let the assumptions in Theorem 3.3.1 hold. Then for any $c \in \mathbb{R}$, the functional J_λ satisfies the Palais-Smale condition.*

Proof. Let $\{w_n\} \subset X_0$ be a $(PS)_c$ sequence for the functional J_λ. So, $J_\lambda(w_n) \to c$ and $\|J'_\lambda(w_n)\|_{X_0^*} \to 0$ as $n \to +\infty$. We claim that $\{w_n\}$ is bounded in X_0. Indeed, if not, $\|w_n\|_{X_0} \to +\infty$ as $n \to +\infty$. Therefore, there exists some constant $C_{16} > 0$ such that for large $n \in \mathbb{N}$, using (F_2), Lemma 1.2.5, Theorem 2.1.3, and Lemma 2.1.4, we obtain

$$\begin{aligned}
&C_{16} + C_{16}\|w_n\|_{X_0} \\
&\ge J_\lambda(w_n) - \frac{1}{\nu}\langle J'_\lambda(w_n), w_n\rangle_{X_0} \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)-w_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy - \lambda\int_\Omega \frac{|u|^{\alpha(x)}}{\alpha(x)}dx \\
&\qquad - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,w_n(x))F(y,w_n(y))}{|x-y|^{\mu(x,y)}}dxdy \\
&\quad - \frac{1}{\nu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)-w_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy + \frac{\lambda}{\nu}\int_\Omega |u|^{\alpha(x)}dx \\
&\qquad + \frac{1}{\nu}\int_\Omega\int_\Omega \frac{F(y,w_n(y))f(x,w_n(x))w_n(x)}{|x-y|^{\mu(x,y)}}dxdy \\
&> \left(\frac{1}{p^+}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)-w_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy - \lambda\left(1-\frac{1}{\nu}\right)\left(\|u\|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^+} + \|u\|_{L^{\alpha(\cdot)}(\Omega)}^{\alpha^-}\right) \\
&\quad + \int_\Omega\int_\Omega \frac{F(y,w_n(y))\left[\frac{1}{\nu}f(x,w_n(x))w_n(x) - \frac{1}{2}F(x,w_n(x))\right]}{|x-y|^{\mu(x,y)}}dxdy \\
&> \left(\frac{1}{p^+}-\frac{1}{\nu}\right)\|w_n\|_{X_0}^{p^-} - \lambda C_{17}\left(1-\frac{1}{\nu}\right)\|u\|_{X_0}^{\alpha^+},
\end{aligned} \tag{3.3.43}$$

where C_{17} is a positive constant, independent of u. Since $1 < \alpha^+ < p^- \leq p^+ < \nu$ in (F_2) and we have assumed $\|w_n\|_{X_0} \to +\infty$, from (3.3.43), we get a contradiction. Hence, the sequence $\{w_n\}$ is bounded in X_0. Since X_0 is reflexive, there exists $w_0 \in X_0$ such that $w_n \rightharpoonup w_0$ weakly as $n \to +\infty$. Since $\|J'_\lambda(w_n)\|_{X_0^*} \to 0$ as $n \to +\infty$, using (3.2.27), we have

$$\begin{aligned} o_n(1) &= \big\langle J'_\lambda(w_n), (w_n - w_0)\big\rangle_{X_0} \\ &= \langle I(w_n), (w_n - w_0)\rangle_{X_0} - \lambda \int_\Omega |w_n(x)|^{\alpha(x)-2} w_n(x)(w_n - w_0)(x)dx \\ &\quad - \int_\Omega \int_\Omega \frac{F(x, w_n(x)f(y, w_n(y))(w_n - w_0)(y)}{|x-y|^{\mu(x,y)}} dxdy. \end{aligned} \tag{3.3.44}$$

Now we estimate the second term in the right hand side of (3.3.44). Taking into account Lemma 1.2.3, Lemma 1.2.5, and Theorem 2.1.3, we deduce

$$\begin{aligned} \left| \int_\Omega |w_n(x)|^{\alpha(x)-2} w_n(x)(w_n - w_0)(x)dx \right| &\leq \| |w_n|^{\alpha(\cdot)-1} \|_{L^{\frac{\alpha}{\alpha-1}(\cdot)}(\Omega)} \|w_n - w_0\|_{L^{\alpha(\cdot)}(\Omega)} \\ &\leq \left(\|w_n\|^{\alpha^+-1}_{L^{\alpha(\cdot)}(\Omega)} + \|w_n\|^{\alpha^--1}_{L^{\alpha(\cdot)}(\Omega)} \right) \|w_n - w_0\|_{L^{\alpha(\cdot)}(\Omega)} \\ &= o_n(1). \end{aligned} \tag{3.3.45}$$

Thus, combining (3.2.28)-(3.2.31), (3.3.44), and (3.3.45), we obtain (3.2.32). Hence, as in the proof of Lemma 3.2.3 , it follows that $w_n \to w_0$ strongly in X_0 as $n \to +\infty$. □

Proof of Theorem 3.3.1. Note that by (F_1), $J_\lambda(0) = 0$. From Lemma 3.3.1$(i), (ii)$, it follows that there exists $\Lambda_* > 0$ such that for $\lambda \in (0, \Lambda_*)$, J_λ admits a mountain pass geometry. Also, by Lemma 3.3.2, the functional J_λ satisfies the Palais-Smale condition $(PS)_c$ for any $c \in \mathbb{R}$. Hence, by applying mountain pass theorem (Theorem 1.3.1), we infer that for $\lambda \in (0, \Lambda_*)$, there exists a non-trivial weak solution, $u_{1\lambda}$ (say) of (3.3.38) with $J_\lambda(u_{1\lambda}) > 0$.

Next, we prove the existence of the second weak solution of (3.3.38). From Lemma 3.3.1(iii), it yields that

$$\inf_{u \in \overline{B}_{\delta_\lambda}(0)} J_\lambda(u) = \underline{c} < 0, \tag{3.3.46}$$

where $\overline{B}_{\delta_\lambda}(0) = \{u \in X_0 : \|u\|_{X_0} \leq \delta_\lambda\}$. Now by applying Ekeland's variational principle (Theorem 1.3.2), for given any $\epsilon > 0$, there exists $\tilde{w}_\epsilon \in \overline{B}_{\delta_\lambda}(0)$ such that

$$J_\lambda(\tilde{w}_\epsilon) < \inf_{u \in \overline{B}_{\delta_\lambda}(0)} J_\lambda(u) + \epsilon \tag{3.3.47}$$

and

$$J_\lambda(\tilde{w}_\epsilon) < J_\lambda(u) + \epsilon \|u - \tilde{w}_\epsilon\|_{X_0}, \text{ for all } u \in B_{\delta_\lambda}(0),\ u \neq \tilde{w}_\epsilon. \tag{3.3.48}$$

Using Lemma 3.3.1(i) and (3.3.46), we choose $\varrho_\lambda > 0$ such that

$$0 < \varrho_\lambda < \inf_{u \in \partial B_{\delta_\lambda}(0)} J_\lambda(u) - \inf_{u \in \overline{B}_{\delta_\lambda}(0)} J_\lambda(u). \tag{3.3.49}$$

Putting together (3.3.47) and (3.3.49), we obtain $J_\lambda(\tilde{w}_\epsilon) < \inf\limits_{u \in \partial B_{\delta_\lambda}(0)} J_\lambda(u)$, which implies $\tilde{w}_\epsilon \in B_{\delta_\lambda}(0)$. By taking $u = \tilde{w}_\epsilon + tv$ in (3.3.48) with $t > 0$ and $v \in B_{\delta_\lambda}(0) \setminus \{0\}$, we deduce

$$J_\lambda(\tilde{w}_\epsilon) - J_\lambda(\tilde{w}_\epsilon + tv) \leq \delta_\lambda t \|v\|_{X_0}.$$

Thus,

$$\lim_{t \to 0} \frac{J_\lambda(\tilde{w}_\epsilon) - J_\lambda(\tilde{w}_\epsilon + tv)}{t} \leq \delta_\lambda \|v\|_{X_0},$$

that is, for all $v \in B_{\delta_\lambda}(0)$, we have

$$\langle -J'_\lambda(\tilde{w}_\epsilon), v \rangle_{X_0} \leq \delta_\lambda \|v\|_{X_0}. \tag{3.3.50}$$

Replacing v by $-v$ in (3.3.50), we get

$$(J'_\lambda(\tilde{w}_\epsilon), v) \leq \delta_\lambda \|v\|_{X_0}. \tag{3.3.51}$$

Taking into account (3.3.50) and (3.3.51), we obtain

$$\|J'_\lambda(\tilde{w}_\epsilon)\|_{X_0^*} \leq \delta_\lambda. \tag{3.3.52}$$

From (3.3.52), it follows that there exists a sequence $\{\tilde{w}_n\} \subset B_{\delta_\lambda}(0)$ such that $J_\lambda(\tilde{w}_n) \to \underline{c}$ and $J'_\lambda(\tilde{w}_n) \to 0$ in X_0^* as $n \to +\infty$. Therefore, from Lemma 3.3.2 and (3.3.46), we infer that there exists $u_{2\lambda} \in \overline{B}_{\delta_\lambda}(0) \subset X_0$ such that $\tilde{w}_n \to u_{2\lambda}$ strongly in X_0 as $n \to +\infty$ with

$$J_\lambda(u_{2\lambda}) = \underline{c} < 0. \tag{3.3.53}$$

Thus, we get that $u_{2\lambda}$ is a non-trivial weak solution of (3.3.38). Now from (3.2.37) and (3.3.53), we have $J_\lambda(u_{1\lambda}) > 0 > J_\lambda(u_{2\lambda})$, and hence $u_{1\lambda} \neq u_{2\lambda}$. This completes the proof of the theorem. □

3.4 Conclusion

In this chapter, we have discussed the existence of non-trivial solution of a class of doubly non-local problems involving variable-order fractional $p(\cdot)$-Laplacian and Choquard type of nonlinearity. Here the crucial step of proving the C^1-smoothness of the associated energy functional has been achieved by establishing a Hardy-Littlewood-Sobolev type inequality appropriate for the variable order fractional spaces with variable exponents. We have further studied the combined effect of the concave type perturbation term and the Choquard type nonlinearity with sub-critical growth on the multiplicity of solutions.

The multiplicity result for the problem with concave type perturbation term and critical Choquard type nonlinearity (in view of Remark 3.1.1) involving variable order and variable exponents is an open problem in this direction. Here we would like to mention that the attainment of the best constant for Hardy-Littlewood-Sobolev inequality for both local and non-local framework, for $p \neq 2$, is yet to be explored. Also, it will be interesting to study (3.2.3) involving singular type perturbation term, together with the Choquard type of nonlinearity. □

4

Kirchhoff-Choquard equations without Ambrosetti-Rabinowitz type condition

The purpose of this chapter is to study the following Kirchhoff-Choquard type problem:

$$\left.\begin{aligned} m\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy+\int_\Omega V(x)\frac{|u(x)|^{\overline{p}(x)}}{\overline{p}(x)}dx\Big)&\\ \Big[(-\Delta)^{s(\cdot)}_{p(\cdot)}u+V(x)|u|^{\overline{p}(x)-2}u\Big]=\Big(\int_\Omega\frac{F(y,u(y))}{|x-y|^{\mu(x,y)}}dy\Big)f(x,u),\quad & x\in\Omega,\\ u=0,\quad & x\in\mathbb{R}^N\setminus\Omega,\end{aligned}\right\} \tag{4.0.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s(\cdot,\cdot)$, $p(\cdot,\cdot)$, and $\mu(\cdot,\cdot)$ satisfy (S_1), (P_1), and (μ_1), respectively, with $s^+p^+<N$. The function $V(\cdot)$ satisfies the

following:

(V_1) $V(\cdot) \in C(\mathbb{R}, \mathbb{R})$ such that $V(x) \geq 0$, for all $x \in \Omega$.

Next, the assumption on the Kirchhoff function $m(\cdot)$ in (4.0.1) is given as follows:

(M_1) $m(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined as $m(t) = a + bt^{\theta-1}$, $a \geq 0, b > 0$ such that $\theta \in [1, 2\frac{p_s^{*-}}{p^+ q^+})$, where $p^+ \geq 2$, $p_s^{*-} := \frac{Np^-}{N - s^- p^-}$, and $q(\cdot,\cdot) \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ verifies

$$\frac{2}{q(x,y)} + \frac{\mu(x,y)}{N} = 2, \quad \text{for all } x, y \in \mathbb{R}^N. \tag{4.0.2}$$

Let $M(t) := \int_0^t m(\tau)d\tau$ denote the primitive of $m(t)$. We have the following remarks from the assumption (M_1):

Remark 4.0.1. *When $a = 0$, we have the following observations:*

(i) For any $\tau > 0$, there exists $m_0 := m_0(\tau) > 0$ such that $m(t) \geq m_0$, whenever $t \geq \tau$.

(ii) $\theta M(t) - m(t)t$ is non-decreasing in $t > 0$ and $\theta M(t) - m(t)t = 0$, for all $t \geq 0$.

(iii) $M(t) = M(1)t^\theta$, where $M(1) = \frac{b}{\theta}$.

Remark 4.0.2. *When $a > 0$, the following results hold:*

(i) $m(t) = a + bt^{\theta-1}, a > 0$ and $m(t) \geq \inf_{t \geq 0} m(t) = a > 0$.

(ii) $\theta M(t) - m(t)t$ is non-decreasing in $t > 0$ and $\theta M(t) - m(t)t \geq 0$, for all $t \geq 0$.

(iii) For each $t \geq 0$, we have

$$\left.\begin{aligned} M(t) &\geq M(1)t^\theta, && \textit{for all } t \in [0,1], \\ &\leq M(1)t^\theta, && \textit{for all } t \geq 1, \\ &\leq M(1)(1 + t^\theta), && \textit{for all } t \geq 0, \end{aligned}\right\} \tag{4.0.3}$$

where $M(1) = a + \frac{b}{\theta}$.

The hypotheses that we consider on the nonlinearity f in (4.0.1) are as follows:

(f_1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ such that $|f(x,t)| \leq \mathcal{C}\left(1 + |t|^{r(x)-1}\right)$, where $\mathcal{C} > 0$ is a constant, $r(\cdot) \in C_+(\mathbb{R}^N)$ satisfies $1 < r(x)q^- \leq r(x)q^+ < p_s^*(x)$, for all $x \in \mathbb{R}^N$, $r^- > \frac{\theta p^+}{2}$ and $q(\cdot,\cdot) \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ verifies (4.0.2).

(f_2) $f(x,t) = o\left(|t|^{\frac{\theta p^+}{2} - 2}t\right)$ as $|t| \to 0$, uniformly in $x \in \Omega$.

(f_3) $\lim\limits_{|t|\to+\infty} \dfrac{F(x,t)}{|t|^{\frac{\theta p^+}{2}}} = +\infty$ uniformly in $x \in \Omega$, where $F(x,t) := \int_0^t f(x,s)ds$ is the primitive of f.

(f_4) There exists $\vartheta > 1$ such that $\vartheta \mathcal{F}(x,t) \geq \mathcal{F}(x,\tau t)$, for $(x,t) \in \Omega \times \mathbb{R}$ and $\tau \in [0,1]$, where $\mathcal{F}(x,t) = 2tf(x,t) - \theta p^+ F(x,t)$.

The condition (f_4) is originally due to Jeanjean [55] in the case $p(\cdot,\cdot) = 2$, and is used in [67] for p-Laplacian equations in bounded domain. Note that the assumptions (f_1)-(f_4) allow us to consider the nonlinearities which do not satisfy the following Ambrosetti-Rabinowitz type condition ((AR) condition, in short):

(AR) There exists $\omega > \theta p^+$ such that

$$0 < \omega F(x,t) \leq 2tf(x,t), t \neq 0, \text{ for all } x \in \Omega.$$

An example of a function which does not satisfy (AR) but satisfies (f_1)-(f_4) is

$$f(x,t) = t|t|^{\frac{\theta p^+}{2}-2} \log(1 + |t|).$$

Next, we make following remark about f :

Remark 4.0.3. *Since $f(x,0) = 0 = F(x,0)$, thanks to (f_2), from (f_4) we get*

$$\mathcal{F}(x,t) = 2tf(x,t) - \theta p^+ F(x,t) \geq 0, \quad \textit{for all} \quad (x,t) \in \Omega \times \mathbb{R}. \tag{4.0.4}$$

The following remark is studied in [66] for local p-Laplacian.

Remark 4.0.4. $F(x,t) \geq 0$, *for all* $(x,t) \in \Omega \times \mathbb{R}$.

Proof. For $t > 0$, using (4.0.4), we have

$$\frac{d}{dt} \frac{F(x,t)}{t^{\frac{\theta p^+}{2}}} = \frac{t^{\frac{\theta p^+}{2}} f(x,t) - \frac{\theta p^+}{2} t^{\frac{\theta p^+}{2}-1} F(x,t)}{t^{\theta p^+}} = \frac{\mathcal{F}(x,t)}{2\, t^{\frac{\theta p^+}{2}+1}} \geq 0.$$

Furthermore, from (f_2), we can easily deduce that $\lim\limits_{t\to 0^+} \dfrac{F(x,t)}{t^{\frac{\theta p^+}{2}}} = 0$. Using the above two facts, it follows that $F(x,t) \geq 0$, for all $(x,t) \in \Omega \times \mathbb{R}$, $t \geq 0$. Similarly, for $t < 0$, proceeding as above, we get $\lim_{t\to 0^-}\dfrac{F(x,t)}{(-t)^{\frac{\theta p^+}{2}}}=0$, and therefore, $F(x,t)\geq 0$, for all $(x,t)\in\Omega\times\mathbb{R},\ t\leq 0$. □

In view of the above remark, we have the following assertion:

Remark 4.0.5. *From Remark 4.0.3 and Remark 4.0.4, we obtain $f(x,t)\geq 0$, for all $(x,t)\in\Omega\times\mathbb{R},\ t\geq 0$ and $f(x,t)\leq 0$, for all $(x,t)\in\Omega\times\mathbb{R},\ t\leq 0$. Therefore, for all $x\in\Omega$, we have that $F(x,t)$ is non-decreasing in $t\geq 0$ and is non-increasing in $t\leq 0$.*

4.1 Functional settings

In this section, we give the suitable functional setting to study (4.0.1). We assume that (V_1) holds and define the following space:

$$E=\Big\{u\in X_0:\int_\Omega\frac{V(x)|u|^{\overline{p}(x)}}{\eta^{\overline{p}(x)}}dx<+\infty,\ \text{for some }\eta>0\Big\}.$$

We note that E is a normed linear space equipped with the norm

$$\|u\|_E:=\inf\Big\{\eta>0:\rho_E\left(\frac{u}{\eta}\right)<1\Big\},$$

where

$$\rho_E(u):=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}\,dxdy+\int_\Omega V(x)\,|u|^{\overline{p}(x)}\,dx$$

defines a convex modular on E. We can also define another norm on E as follows:

$$|u|_E:=[u]_V+\|u\|_{X_0},$$

where

$$[u]_V=\inf\Big\{\eta>0:\int_\Omega V(x)\frac{|u(x)|^{\overline{p}(x)}}{\eta^{\overline{p}(x)}}dx<1\Big\}.$$

One can easily verify that $\|\cdot\|_E$ and $|\cdot|_E$ are equivalent norms on E with the relation

$$\frac{1}{2}\|u\|_E\leq|u|_E\leq 2\|u\|_E,\quad \text{for all } u\in E. \tag{4.1.5}$$

Next, we study the interaction between $\|\cdot\|_E$ and $\rho_E(\cdot)$ in the next two results. The proofs of these results follows using the similar arguments as in [38].

Proposition 4.1.1. *For $u \in E \setminus \{0\}$, we have the following:*

(i) *For $\eta > 0$, $\eta = \|u\|_E$ if and only if $\rho_E(\frac{u}{\eta}) = 1$.*

(ii) *$\rho_E(u) > 1$ $(= 1;\ < 1)$ if and only if $\|u\|_E > 1$ $(= 1;\ < 1)$, respectively.*

(iii) *If $\|u\|_E \geq 1$, then $\|u\|_E^{p^-} \leq \rho_E(u) \leq \|u\|_E^{p^+}$.*

(iv) *If $\|u\|_E < 1$, then $\|u\|_E^{p^+} \leq \rho_E(u) \leq \|u\|_E^{p^-}$.*

As a consequence of the above proposition, we have the following result:

Proposition 4.1.2. *Let $u, u_n \in E, n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\lim\limits_{n\to+\infty} \|u_n - u\|_E = 0$.

(ii) $\lim\limits_{n\to+\infty} \rho_E(u_n - u) = 0$.

In the following lemma, we derive the separability and reflexivity of the space E.

Lemma 4.1.1. *$(E, \|\cdot\|_E)$ is a reflexive, separable Banach space.*

Proof. First we show that $(E, \|\cdot\|_E)$ is a Banach space. For this, let $\{u_n\}$ be any Cauchy sequence in E. Therefore, for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that, if $n, k \geq N_\epsilon$,

$$\|u_n - u_k\|_E \leq \epsilon. \tag{4.1.6}$$

Since $\|u\|_E \geq \|u\|_{X_0}$ and $(X_0, \|\cdot\|_{X_0})$ is a Banach space, there exists $u \in X_0$ such that $u_n \to u$ in X_0 strongly as $n \to +\infty$. So, there exists a sub-sequence $\{u_{n_j}\}$ such that $u_{n_j}(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Now using Fatou's lemma, (1.2.12), and (4.1.6) with $\epsilon = 1$, we have

$$\begin{aligned}
\int_\Omega V(x)|u(x)|^{\overline{p}(x)}dx &\leq \liminf_{n\to\infty} \int_\Omega V(x)|u_n(x)|^{\overline{p}(x)}dx \\
&\leq \liminf_{n\to\infty} \int_\Omega V(x)|u_n(x) - u_{N_1}(x) - u_{N_1}(x)|^{\overline{p}(x)}dx \\
&\leq 2^{p^+} \liminf_{n\to\infty} \left[\int_\Omega V(x)|u_n(x) - u_{N_1}(x)|^{\overline{p}(x)} + \int_\Omega V(x)|u_{N_1}(x)|^{\overline{p}(x)}dx\right] \\
&\leq 2^{p^+} \left[1 + \int_\Omega V(x)|u_{N_1}(x)|^{\overline{p}(x)}dx\right] < \infty.
\end{aligned}$$

Therefore, $u \in E$. Now again by Fatou's lemma and (4.1.6), we get

$$\rho_E(u_n - u) \leq \liminf_{j \to +\infty} \rho_E(u_n - u_{n_j}) \leq \epsilon, \text{ for all } n, n_j \geq N_\epsilon. \tag{4.1.7}$$

Thus, from Proposition 4.1.2, we infer that $u_n \to u$ in E as $n \to +\infty$. Hence, $(E, \|\cdot\|_E)$ is a Banach space.

For proving reflexivity of E, we define the map $T : E \to L^{\overline{p}(\cdot)}(\Omega) \times L^{p(\cdot,\cdot)}(\mathbb{R}^N \times \mathbb{R}^N)$ as

$$T(u) = \left(V^{1/\overline{p}(x)} u, \ \frac{|u(x - u(y)|}{|x - y|^{s(x,y) + \frac{N}{p(x,y)}}} \right).$$

The norm on the product space $L^{\overline{p}(\cdot)}(\Omega) \times L^{p(\cdot,\cdot)}(\mathbb{R}^N \times \mathbb{R}^N)$ is given by

$$\|u\| = \|u\|_{L^{\overline{p}(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^N \times \mathbb{R}^N)}.$$

Since $T(E)$ is a closed subspace of the reflexive Banach space $L^{\overline{p}(\cdot)}(\Omega) \times L^{p(\cdot,\cdot)}(\mathbb{R}^N \times \mathbb{R}^N)$, by [18, Proposition 3.20], we have that $T(E)$ is reflexive and consequently, E is reflexive. Arguing similarly, we get that E is separable (see [18, Proposition 3.25]). □

Using Theorem 2.1.3 and the fact $\|u\|_E \geq \|u\|_{X_0}$, we have the following Sobolev type embedding theorem for the space E:

Theorem 4.1.1. *Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 2$, with Lipschitz boundary. Let $V(\cdot)$, $s(\cdot,\cdot)$, and $p(\cdot,\cdot)$ satisfy (V_1), (S_1), and (P_1), respectively, such that $s^+p^+ < N$. Then for any $\gamma(\cdot) \in C_+(\overline{\Omega})$ with $1 < \gamma(x) < p_s^*(x)$, for all $x \in \overline{\Omega}$, there exits a constant $C = C(N, s, p, \gamma, \Omega) > 0$ such that, for every $u \in E$, $\|u\|_{L^{\gamma(\cdot)}(\Omega)} \leq C\|u\|_E$. Moreover, this embedding is compact.*

Now we give the weak formulation for (4.0.1). For $u \in E$, we set

- $\sigma(u) := \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+s(x,y)p(x,y)}} dxdy + \int_\Omega V(x) \frac{|u(x)|^{\overline{p}(x)}}{\overline{p}(x)} dx.$
- $\Psi(u) := \displaystyle\frac{1}{2} \int_\Omega \int_\Omega \frac{F(x, u(x))F(y, u(y))}{|x - y|^{\mu(x,y)}} dxdy.$
- $\tilde{I}(u) := \displaystyle\int_\Omega \int_\Omega \left(\frac{F(y, u(y))}{|x - y|^{\mu(x,y)}} dy \right) f(x, u(x))u(x)dx.$

Definition 4.1.1. *The energy functional $\mathcal{J} : E \to \mathbb{R}$ associated to* (4.0.1) *is defined as*

$$\mathcal{J}(u) = M(\sigma(u)) - \Psi(u).$$

Lemma 4.1.2. *The functional $\mathcal{J}$ as defined in the Definition 4.1.1 is of class C^1 and for all $u, w \in E$,*

$$\langle J'(u), w\rangle_E = m(\sigma(u))\Big[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy + \int_\Omega V(x)|u(x)|^{\overline{p}(x)-2}u(x)w(x)dx\Big] - \int_\Omega\int_\Omega \frac{F(y,u(y))f(x,u(x))w(x)}{|x-y|^{\mu(x,y)}}dxdy.$$

Proof. Clearly, $\mathcal{J}$ is well defined. Also, it is easy to see that $M(\sigma(\cdot))$ is Gateaux-differentiable in E and the derivative function at $u \in E$ is given as

$$m(\sigma(u))\Big[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy + \int_\Omega V|u|^{\overline{p}(x)-2}uwdx\Big],$$

for all $w \in E$. Let $\{u_n\}$ be any sequence in E such that $u_n \to u$ strongly in E as $n \to +\infty$. Thus, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Let p' and $\overline{p}'$ denote the conjugate of p and $\overline{p}$, respectively. Then the sequences $\left\{\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))}{|x-y|^{(N+s(x,y)p(x,y))/p'(x,y)}}\right\}$ and $\left\{[V(x)]^{1/\overline{p}'(x)}|u_n(x)|^{\overline{p}(x)-2}u_n(x)\right\}$ are bounded in $L^{p'(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)$ and in $L^{\overline{p}'(\cdot)}(\Omega)$, respectively, and as $n \to +\infty$,

$$\mathcal{U}_n(x,y) := \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))}{|x-y|^{(N+s(x,y)p(x,y))/p'(x,y)}} \to \mathcal{U}(x,y) := \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{(N+s(x,y)p(x,y))/p'(x,y)}}, \quad \text{for a.e. } x,y \in \mathbb{R}^N,$$

and

$$V(x)^{1/\overline{p}'(x)}|u_n(x)|^{\overline{p}(x)-2}u_n(x) \to V(x)^{1/\overline{p}'(x)}|u(x)|^{\overline{p}(x)-2}u(x), \text{ for a.e. } x \in \Omega.$$

Thus, by [109, Proposition 5.4.7], letting $n \to +\infty$, we have $\mathcal{U}_n \rightharpoonup \mathcal{U}$ weakly in $L^{p'(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)$ and $V(\cdot)^{1/\overline{p}'(\cdot)}|u_n(\cdot)|^{\overline{p}(\cdot)-2}u_n(\cdot) \rightharpoonup V(\cdot)^{1/\overline{p}'(\cdot)}|u(\cdot)|^{\overline{p}(\cdot)-2}u(\cdot)$ weakly in

$L^{\overline{p}'(\cdot)}(\Omega)$. Hence, for any $w \in E$, by Theorem 4.1.1 and the definition of weak convergence, we have the following:

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy$$
$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy, \tag{4.1.8}$$

and

$$\lim_{n\to+\infty}\int_{\Omega}V(x)|u_n(x)|^{\overline{p}(x)-2}u_n(x)w(x)dx=\int_{\Omega}V(x)|u(x)|^{\overline{p}(x)-2}u(x)w(x)dx. \tag{4.1.9}$$

Next, using (f_1), Theorem 4.1.1 and arguing similarly as in Lemma 3.2.1, we get that Ψ is of class C^1 such that the Gateaux derivative of Ψ is given as

$$\langle\Psi'(u),w\rangle_E=\int_{\Omega}\int_{\Omega}\frac{F(y,u)}{|x-y|^{\mu(x,y)}}f(x,u)w(x)dxdy,$$

for all $w \in E$, where $\langle\cdot,\cdot\rangle_E$ defines the dual pairing between E and E^*. Moreover, by letting $n \to +\infty$, we get

$$\int_{\Omega}\int_{\Omega}\frac{F(y,u_n)}{|x-y|^{\mu(x,y)}}f(x,u_n)w(x)dxdy\to\int_{\Omega}\int_{\Omega}\frac{F(y,u)}{|x-y|^{\mu(x,y)}}f(x,u)w(x)dxdy. \tag{4.1.10}$$

Finally, combining (4.1.8)-(4.1.10), we obtain

$$\|\mathcal{J}'(u_n)-\mathcal{J}'(u)\|_{E^*}=\sup_{w\in E,\|w\|_E=1}|\langle\mathcal{J}'(u_n)-\mathcal{J}'(u),w\rangle_E|\to 0 \text{ as } n\to\infty.$$

This completes the proof. □

Definition 4.1.2. *A function $u \in E$ is said to be weak solution of* (4.0.1), *if for all $w \in E$*

$$m(\sigma(u))\Big[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy$$
$$+\int_{\Omega}V(x)|u(x)|^{\overline{p}(x)-2}u(x)w(x)dx\Big]=\int_{\Omega}\int_{\Omega}\frac{F(y,u(y))f(x,u(x))w(x)}{|x-y|^{\mu(x,y)}}dx.$$

The weak solutions of (4.0.1) are characterized as the critical point of the associated energy functional $\mathcal{J}$.

4.2 Existence of solution via mountain pass theorem with Cerami condition

In this section, we state and prove the existence of non-trivial weak solution of (4.0.1) using mountain pass theorem with Cerami condition.

Theorem 4.2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1), respectively, such that $s^+p^+ < N$. Assume that $(\mu_1), (V_1)$, and (M_1) hold. Also, let f satisfy (f_1)-(f_4). Then* (4.0.1) *admits a non-trivial weak solution.*

To give the proof of Theorem 4.2.1, we first show that the functional $\mathcal{J}$ achieves the mountain pass geometry and then we prove that $\mathcal{J}$ satisfies the Cerami condition.

4.2.1 Mountain pass geometry

Lemma 4.2.1. *Let the assumptions in Theorem 4.2.1 hold. Then there exist some positive constants R and δ such that $\mathcal{J}(u) > R$, for all $u \in E$ with $\|u\|_E = \delta$.*

Proof. First we estimate $\Psi(u)$ appearing in the expression of $\mathcal{J}$ (see Definition 4.1.1). Note that, using (f_1) and Theorem 4.1.1, one can easily check that $F(\cdot,u) \in L^{q^-}(\Omega) \cap L^{q^+}(\Omega)$. Hence, by Proposition 3.1.1, we get

$$\begin{aligned}\Psi(u) &= \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u(x))F(y,u(y))}{|x-y|^{\mu(x,y)}}dxdy \\ &\leq C\Big[\|F(\cdot,u)\|^2_{L^{q^+}(\Omega)} + \|F(\cdot,u)\|^2_{L^{q^-}(\Omega)}\Big],\end{aligned} \tag{4.2.11}$$

where the constant $C > 0$ does not depend on u. From (f_1), (f_2) and the definition of $F(x,t)$, we deduce that, for any $\epsilon > 0$, there exist some constant $C(\epsilon) > 0$ such that

$$|F(x,t)| \leq \epsilon|t|^{\frac{\theta p^+}{2}} + C(\epsilon)|t|^{r(x)}, \quad \text{for all } t \in \mathbb{R} \text{ and for a.e. } x \in \Omega. \tag{4.2.12}$$

Using Lemma 1.2.5, (1.2.12), and (4.2.12), we have

$$\begin{aligned}\|F(\cdot,u)\|_{L^{q^+}(\Omega)} &\leq \Big[\int_\Omega \Big(\epsilon|u(x)|^{\frac{\theta p^+}{2}} + C(\epsilon)|u(x)|^{r(x)}\Big)^{q^+} dx\Big]^{1/q^+} \\ &\leq 2\Big[\epsilon\Big(\int_\Omega |u(x)|^{\frac{\theta p^+}{2}q^+} dx\Big)^{1/q^+} + C(\epsilon)\| \, |u|^{r(\cdot)} \, \|_{L^{q^+}(\Omega)}\Big] \\ &\leq 2\Big[\epsilon\|u\|^{\frac{\theta p^+}{2}}_{L^{\frac{\theta p^+}{2}q^+}(\Omega)} + C(\epsilon)\Big\{\|u\|^{r^+}_{L^{r(\cdot)q^+}(\Omega)} + \|u\|^{r^-}_{L^{r(\cdot)q^+}(\Omega)}\Big\}\Big]. \end{aligned} \tag{4.2.13}$$

Similarly, we deduce

$$\|F(\cdot,u)\|_{L^{q^-}(\Omega)} \leq 2\Big[\epsilon\|u\|^{\frac{\theta p^+}{2}}_{L^{\frac{\theta p^+}{2}q^-}(\Omega)} + C(\epsilon)\Big\{\|u\|^{r^+}_{L^{r(\cdot)q^-}(\Omega)} + \|u\|^{r^-}_{L^{r(\cdot)q^-}(\Omega)}\Big\}\Big] \tag{4.2.14}$$

Plugging (4.2.13) and (4.2.14) into (4.2.11), we derive

$$\begin{aligned}\Psi(u) \leq C_1\Big[&\Big\{\epsilon^2\|u\|^{\theta p^+}_{L^{\frac{\theta p^+}{2}q^+}(\Omega)} + C(\epsilon)^2\Big(\|u\|^{2r^+}_{L^{r(\cdot)q^+}(\Omega)} + \|u\|^{2r^-}_{L^{r(\cdot)q^+}(\Omega)}\Big)\Big\} \\ &+ \Big\{\epsilon^2\|u\|^{\theta p^+}_{L^{\frac{\theta p^+}{2}q^-}(\Omega)} + C(\epsilon)^2\Big(\|u\|^{2r^+}_{L^{r(\cdot)q^-}(\Omega)} + \|u\|^{2r^-}_{L^{r(\cdot)q^-}(\Omega)}\Big)\Big\}\Big], \end{aligned} \tag{4.2.15}$$

where $C_1 > 0$ is a constant, independent of u. Now by applying Theorem 4.1.1, from (4.2.15) we obtain

$$\Psi(u) \leq C_2\left[\epsilon^2\|u\|^{\theta p^+}_E + C(\epsilon)^2\left\{\|u\|^{2r^+}_E + \|u\|^{2r^-}_E\right\}\right], \tag{4.2.16}$$

where the constant $C_2 > 0$ does not depend on u. Let $u \in E$, $\|u\|_E < 1$. Therefore, using (4.2.16), Remark 4.0.1(iii), for $a = 0$ in (M_1) (or Remark 4.0.2(iii), for $a > 0$ in (M_1)), and Proposition 4.1.1, we get

$$\begin{aligned}\mathcal{J}(u) &\geq M(1)\{\sigma(u)\}^\theta - C_2\left[\epsilon^2\|u\|^{\theta p^+}_E + C(\epsilon)^2\left\{\|u\|^{2r^+}_E + \|u\|^{2r^-}_E\right\}\right] \\ &\geq \frac{M(1)}{(p^+)^\theta}\{\rho_E(u)\}^\theta - C_2\epsilon^2\|u\|^{\theta p^+}_E - 2C_2C(\epsilon)^2\|u\|^{2r^-}_E \\ &\geq \frac{M(1)}{(p^+)^\theta}\|u\|^{\theta p^+}_E - C_2\epsilon^2\|u\|^{\theta p^+}_E - 2C_2C(\epsilon)^2\|u\|^{2r^-}_E. \end{aligned} \tag{4.2.17}$$

By taking $0 < \epsilon < \big[M(1)/(2C_2(p^+)\theta)\big]^{1/2}$ in (4.2.17), since $\|u\|_E < 1$ and $\frac{\theta p^+}{2} < r^-$, we can choose $0 < \delta < 1$ sufficiently small so that (4.2.17) implies that there exists some $R > 0$ such that $\mathcal{J}(u) > R > 0$, for $\|u\|_E = \delta$. □

Lemma 4.2.2. *Let the assumptions in Theorem 4.2.1 hold. Then there exists $e \in X_0$ with $\|e\|_{X_0} > \delta$ such that $\mathcal{J}(e) < 0$, where δ is as given in Lemma 4.2.1.*

Proof. Choose $u \in E, u > 0$ such that $\|u\|_E = 1$ and $\int_\Omega \int_\Omega \frac{|u(x)|^{\frac{\theta p^+}{2}} |u(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}} dxdy > 0$. Now for $t > 1$ large, using Remark 4.0.1(iii), for $a = 0$ in (M_1) (or Remark 4.0.2(iii), for $a > 0$ in (M_1)), and Proposition 4.1.1, we get

$$\begin{aligned} \mathcal{J}(tu) &\leq \frac{M(1)}{(p^-)^\theta} (\rho_E(tu))^\theta - \Psi(tu) \\ &\leq \frac{M(1)}{(p^-)^\theta} t^{\theta p^+} (\rho_E(u))^\theta - \Psi(tu) \\ &= \frac{M(1)}{(p^-)^\theta} t^{\theta p^+} - \Psi(tu). \end{aligned} \tag{4.2.18}$$

It follows from (f_3) that for any $l > 0$, there exists a real number $C_l > 0$ such that

$$F(x, tu(x)) > l|tu(x)|^{\frac{\theta p^+}{2}},$$

whenever $|tu(x)| > C_l$, for a.e. $x \in \Omega$. Therefore, using the above inequality in (4.2.18), we deduce that

$$\mathcal{J}(tu) \leq \frac{M(1)}{(p^-)^\theta} t^{\theta p^+} - \frac{l^2}{2} t^{\theta p^+} \Big(\int_\Omega \int_\Omega \frac{|u(x)|^{\frac{\theta p^+}{2}} |u(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}} dxdy \Big). \tag{4.2.19}$$

After taking $0 < l < \big[4 \frac{M(1)}{(p^-)^\theta} \big(\int_\Omega \int_\Omega \frac{|u(x)|^{\frac{\theta p^+}{2}} |u(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}} dxdy \big)^{-1} \big]^{1/2}$ in (4.2.19), we can choose $t_* > 0$ large enough so that $|t_* u(x)| > C_l$, for a.e. $x \in \Omega$ with $\|t_* u\|_E > \delta$ such that $\mathcal{J}(t_* u) < 0$. Thus, by fixing $e = t_* u$, the result follows. □

4.2.2 Cerami compactness condition

Proposition 4.2.1. *Let the assumptions in Theorem 4.2.1 hold. Then the functional $\mathcal{J}$ satisfies the Cerami condition $(C)_c$ for any $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset E$ be a Cerami sequence for $\mathcal{J}$ at level $c \in \mathbb{R}$. Then by Definition 1.3.2,

$$\mathcal{J}(u_n) \to c \qquad \text{and} \qquad (1 + \|u_n\|_E)\|\mathcal{J}'(u_n)\|_{E^*} \to 0 \text{ as } n \to +\infty, \tag{4.2.20}$$

which implies that

$$\langle \mathcal{J}'(u_n), u_n\rangle_E \to 0 \quad \text{as } n \to +\infty, \tag{4.2.21}$$

where $\langle \cdot, \cdot \rangle_E$ denotes the duality pairing between E and its dual E^*.

First we discuss the degenerate case, i.e., $a = 0$. We divide the proof into two parts.

(I) $\inf\limits_{n\in\mathbb{N}} \|u_n\|_E = d_* > 0$**:** First we prove that the sequence $\{u_n\}$ is bounded in E. Indeed, arguing by contradiction, we assume that $\{u_n\}$ is unbounded in E, that is,

$$\|u_n\|_E \to +\infty \quad \text{as} \quad n \to +\infty. \tag{4.2.22}$$

Without loss of generality, we assume $\|u_n\|_E > 1$. Set $w_n := \frac{u_n}{\|u_n\|_E}$. Then $w_n \in E$ with $\|w_n\|_E = 1$ and since E is reflexive, $w_n \rightharpoonup w$ weakly in E and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$ for some $w \in E$. Now by applying Theorem 4.1.1, for any $\gamma(\cdot) \in C_+(\Omega)$ with $\gamma(x) < p_s^*(x)$, we have

$$w_n \to w \text{ strongly in } L^{\gamma(\cdot)}(\Omega) \text{ as } n \to +\infty. \tag{4.2.23}$$

Let $\Omega_0 := \{x \in \Omega : w(x) \neq 0\}$. Thus, we get

$$|u_n(x)| \to +\infty \ \text{ a.e. } x \in \Omega_0 \ \text{ as } n \to +\infty. \tag{4.2.24}$$

When $x \in \Omega_0$, we have $|w_n(x)| > 0$ for large $n \in \mathbb{N}$. Therefore, using this fact together with (f_3) and (4.2.24), for each $x \in \Omega_0$ and sufficiently large $n \in \mathbb{N}$, we get

$$\lim_{|u_n(x)|\to+\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{\frac{\theta p^+}{2}}} |w_n(x)|^{\frac{\theta p^+}{2}} = +\infty. \tag{4.2.25}$$

Using Remark 4.0.4, (4.2.25) and Fatou's lemma, we derive

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{\Omega}\frac{F(y,u_n(y))|w_n(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}|u_n(y)|^{\frac{\theta p^+}{2}}}dy\\
&\geq \liminf_{n\to+\infty}\int_{\Omega_0}\frac{F(y,u_n(y))|w_n(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}|u_n(y)|^{\frac{\theta p^+}{2}}}dy\\
&\geq \int_{\Omega_0}\liminf_{n\to+\infty}\frac{F(y,u_n(y))|w_n(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}|u_n(y)|^{\frac{\theta p^+}{2}}}dy\\
&= +\infty. \qquad (4.2.26)
\end{aligned}$$

Combining (4.2.25) and (4.2.26), for each $x \in \Omega_0$, we obtain

$$\left(\int_{\Omega}\frac{F(y,u_n(y))|w_n(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}|u_n(y)|^{\frac{\theta p^+}{2}}}dy\right)\frac{F(x,u_n(x))}{|u_n(x)|^{\frac{\theta p^+}{2}}}|w_n(x)|^{\frac{\theta p^+}{2}} \to +\infty \text{ as } n\to+\infty,$$

that is,

$$\frac{1}{\|u_n\|_E^{\theta p^+}}\Big(\int_{\Omega}\frac{F(y,u_n(y))}{|x-y|^{\mu(x,y)}}dy\Big)F(x,u_n(x)) \to +\infty \text{ as } n\to+\infty. \qquad (4.2.27)$$

We claim that $meas(\Omega_0) = 0$. Indeed, if not, then using the definitions of $\rho_E(\cdot)$ and $\sigma(\cdot)$, (4.2.20), (4.2.22), (4.2.27) and Remark 4.0.1(iii), and Remark 4.0.4 along with Fatou's lemma, we get

$$\begin{aligned}
\frac{1}{(p^-)^\theta} &\geq \liminf_{n\to\infty}\frac{1}{(p^-)^\theta}\frac{[\rho_E(u_n)]^\theta}{\|u_n\|_E^{\theta p^+}}\\
&\geq \liminf_{n\to+\infty}\frac{[\sigma(u_n)]^\theta}{\|u_n\|_E^{\theta p^+}}\\
&= \liminf_{n\to+\infty}\frac{1}{\|u_n\|_E^{\theta p^+}}\frac{[J(u_n)+\Psi(u_n)]}{M(1)}\\
&\geq \liminf_{n\to+\infty}\frac{\Psi(u_n)}{M(1)\|u_n\|_E^{\theta p^+}} - 1\\
&\geq \frac{1}{2M(1)}\liminf_{n\to+\infty}\int_{\Omega_0}\frac{1}{\|u_n\|_E^{\theta p^+}}\left(\int_{\Omega}\frac{F(y,u_n(y))}{|x-y|^{\mu(x,y)}}dy\right)F(x,u_n(x))dx - 1
\end{aligned}$$

$$\geq \frac{1}{2M(1)} \int_{\Omega_0} \liminf_{n\to\infty} \frac{1}{\|u_n\|_E^{\theta p^+}} \left(\int_\Omega \frac{F(y,u_n(y))}{|x-y|^{\mu(x,y)}} dy \right) F(x,u_n(x))dx - 1$$
$$= +\infty,$$

which gives contradiction and hence, $meas(\Omega_0) = 0$. Therefore,

$$w(x) = 0 \ \text{ a.e. } x \in \Omega. \tag{4.2.28}$$

In the rest of the proof we consider $C > 0$ to be a generic positive constant, independent of n, u_n, w_n, which may vary from line to line.
Given any real number $\kappa > 1$, by (f_1), it follows that there exists some real number $C > 0$ such that $F(x, \kappa t) \leq C\left(|\kappa t| + |\kappa t|^{r(x)}\right)$, for any $x \in \Omega$ and for all $t \in \mathbb{R}$, which together with Theorem 4.1.1 yields that $|F(x,\kappa w_n(x)|^{q^+} \leq \overline{h}(x)$, $|F(x,\kappa w_n(x)|^{q^-} \leq \underline{h}(x)$ a.e. $x \in \Omega$, for some $\overline{h}$, $\underline{h} \in L^1(\Omega)$. Note that, from (4.2.28), we have $w_n \to 0$ strongly in $L^{\gamma(\cdot)}(\Omega)$, for all $1 < \gamma(x) < p_s^*(x)$, and $w_n(x) \to 0$ a.e. in Ω. Hence, using the continuity of F, we deduce that $\lim_{n\to+\infty} F(x,\kappa w_n(x)) = F(x,0) = 0$ a.e. $x \in \Omega$. Therefore, by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty} \|F(\cdot,\kappa w_n(\cdot))\|_{L^{q^+}(\Omega)} = 0 \text{ and } \lim_{n\to+\infty} \|F(\cdot,\kappa w_n(\cdot))\|_{L^{q^-}(\Omega)} = 0. \tag{4.2.29}$$

Using Proposition 3.1.1 and (4.2.29) and letting $n \to +\infty$, we get

$$0 \leq \Psi(\kappa w_n) \leq C\left[\|F(\cdot,\kappa w_n(\cdot))\|^2_{L^{q^+}(\Omega)} + \|F(\cdot,\kappa w_n(\cdot))\|^2_{L^{q^-}(\Omega)}\right] \to 0. \tag{4.2.30}$$

Since $\mathcal{J}(tu_n)$ is continuous in $t \in [0,1]$, for each $n \in \mathbb{N}$, there exists $t_n \in [0,1]$ such that

$$\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(tu_n). \tag{4.2.31}$$

We claim that

$$\mathcal{J}(t_n u_n) \to +\infty \text{ as } n \to +\infty. \tag{4.2.32}$$

For any real number $C_0 > 1$, choose $\kappa = [C_0/(\min\{1, \frac{m_0}{\theta p^+}\})]^{1/p^-}$. Using (4.2.22), we have $\frac{\kappa}{\|u_n\|_E} \in (0,1)$, for $n \in \mathbb{N}$, sufficiently large. Thus, by (M_1), Remark 4.0.1, (4.2.30), (4.2.31), and Proposition 4.1.1, we get

$$\begin{aligned}\mathcal{J}(t_n u_n) \geq \mathcal{J}\left(\frac{\kappa}{\|u_n\|_E}u_n\right) = \mathcal{J}(\kappa w_n) &= M(\sigma(\kappa w_n)) - \Psi(\kappa w_n)\\ &= \frac{1}{\theta}m(\sigma(\kappa w_n))\sigma(\kappa w_n) + o_n(1)\\ &\geq \frac{m_0}{\theta p^+}\rho_E(\kappa w_n) + o_n(1)\\ &\geq \frac{m_0}{\theta p^+}(\kappa)^{p^-} + o_n(1) \geq C_0 + o_n(1).\end{aligned}$$

This proves the claim in (4.2.32). Since $\mathcal{J}(0) = 0$ and $\mathcal{J}(u_n) \to c$ as $n \to +\infty$, we have

$$t_n \in (0,1) \ \text{ and } \langle \mathcal{J}'(t_n u_n), t_n u_n\rangle_E = t_n \frac{d}{dt}\Big|_{t=t_n}\mathcal{J}(t u_n) = 0. \tag{4.2.33}$$

Combining Remark 4.0.1, Remark 4.0.4 and Remark 4.0.5 with (f_4), (4.2.20), (4.2.21), and (4.2.33), we obtain

$$\begin{aligned}&\frac{1}{\vartheta}\mathcal{J}(t_n u_n) + o_n(1)\\ &= \frac{1}{\vartheta}\Big[\mathcal{J}(t_n u_n) - \frac{1}{\theta p^+}\langle \mathcal{J}'(t_n u_n), t_n u_n\rangle_E\Big]\\ &= \frac{1}{\vartheta}\Big[M(\sigma(t_n u_n)) - \frac{1}{\theta p^+}m(\sigma(t_n u_n))\rho_E(t_n u_n)\Big] + \frac{1}{2\theta p^+}\int_\Omega\Big(\int_\Omega \frac{F(y, t_n u_n)}{|x-y|^{\mu(x,y)}}dy\Big)\frac{\mathcal{F}(x, t_n u_n)}{\vartheta}dx\\ &\leq \Big[M(\sigma(t_n u_n)) - \frac{1}{\theta p^+}m(\sigma(t_n u_n))\rho_E(t_n u_n)\Big] + \frac{1}{2\theta p^+}\int_\Omega\int_\Omega \frac{F(y, u_n(y))}{|x-y|^{\mu(x,y)}}\mathcal{F}(x, u_n(x))dxdy\\ &= \Big[M(1)\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|t_n u_n(x) - t_n u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\Big)^{\theta-1}\\ &\qquad \times \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\frac{1}{p(x,y)} - \frac{1}{p^+}\Big)\frac{|t_n u_n(x) - t_n u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\Big] - \Psi(u_n) + \frac{1}{\theta p^+}\tilde{I}(u_n)\\ &\leq \Big[M(1)\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p(x,y)}\frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\Big)^{\theta-1}\\ &\qquad \times \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{p(x,y)} - \frac{1}{p^+}\right)\frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\Big] - \Psi(u_n) + \frac{1}{\theta p^+}\tilde{I}(u_n)\\ &= \mathcal{J}(u_n) - \frac{1}{\theta p^+}\langle \mathcal{J}'(u_n), u_n\rangle_E\\ &= c + o_n(1),\end{aligned}$$

which contradicts (4.2.32). Hence, the sequence $\{u_n\}$ is bounded in E. Therefore, from Theorem 4.1.1, up to a sub-sequence (still denoted by u_n), we have $u_n \rightharpoonup u$ weakly in E, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and $u_n \to u$ strongly in $L^{\gamma(\cdot)}(\Omega)$, for all $\gamma \in C_+(\overline{\Omega})$ with $1 < \gamma(x) < p_s^*(x)$. Now to prove that $\{u_n\}$ converges strongly to u in E as $n \to +\infty$, we define the following functional. Let $\phi \in E$ be fixed and let $\mathcal{B}_\phi$ denote the linear functional on E defined by

$$\begin{aligned}\mathcal{B}_\phi(v) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^{p(x,y)-2}(\phi(x)-\phi(y))(v(x)-v(y))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\\ &+ \int_\Omega V(x)|\phi(x)|^{\overline{p}(x)-2}\phi(x)v(x)\,dx, \text{ for all } v \in E.\end{aligned}$$

For $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, let us denote

$$\Upsilon(x,y) := \frac{|\phi(x)-\phi(y)|}{|x-y|^{\frac{N}{p(x,y)}+s(x,y)}} \quad \text{and} \quad U(x,y) := \frac{|v(x)-v(y)|}{|x-y|^{\frac{N}{p(x,y)}+s(x,y)}}.$$

Then from Lemma 1.2.3 and Lemma 1.2.5, it follows that

$$\begin{aligned}|\mathcal{B}_\phi(v)| &\le C\Big[\big\||\Upsilon|^{p(\cdot,\cdot)-1}\big\|_{L^{p'(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)}\big\|U\big\|_{L^{p(\cdot,\cdot)}(\mathbb{R}^N\times\mathbb{R}^N)}\\ &\quad + \big\||V^{\frac{1}{\overline{p}(\cdot)}}\phi|^{\overline{p}(\cdot)-1}\big\|_{L^{\overline{p}'(\cdot)}(\Omega)}\big\||V|^{\frac{1}{\overline{p}(\cdot)}}v\big\|_{L^{\overline{p}(\cdot)}(\Omega)}\Big]\\ &\le C\left[\left(\|\phi\|_{X_0}^{p^--1}+\|\phi\|_{X_0}^{p^+-1}\right)\|v\|_{X_0}+\left([\phi]_V^{p^--1}+[\phi]_V^{p^+-1}\right)[v]_V\right]\\ &\le C\left[\|\phi\|_E^{p^--1}+\|\phi\|_E^{p^+-1}\right]\|v\|_E.\end{aligned}$$

Thus, for each $\phi \in E$ the linear functional $\mathcal{B}_\phi$ is continuous on E. Note that for $v_n := u_n - u$, we have $v_n \rightharpoonup 0$ weakly in E and hence, by Theorem 4.1.1,

$$v_n \to 0 \quad \text{strongly in } L^{\gamma(\cdot)}(\Omega),\ 1 < \gamma(x) < p_s^*(x),\ \text{for all } x \in \mathbb{R}^N, \tag{4.2.34}$$

as $n \to +\infty$. This implies that

$$\lim_{n\to+\infty}\mathcal{B}_u(v_n) = 0. \tag{4.2.35}$$

Since $\{m(\sigma(u_n)) - m(\sigma(u))\}$ is a bounded sequence in $\mathbb{R}$, from (4.2.35), we get

$$\lim_{n\to+\infty}[m(\sigma(u_n)) - m(\sigma(u))]\mathcal{B}_u(v_n) = 0. \tag{4.2.36}$$

Now using the boundedness of $\{u_n\}$ in E, (f_1) and Theorem 4.1.1, we obtain $f(\cdot, u_n(\cdot))v_n(\cdot) \in L^{q^+}(\Omega)$, for each $n \in \mathbb{N}$. Furthermore, by Lemma 1.2.3, (1.2.12), Theorem 4.1.1, and (4.2.34), we have

$$\begin{aligned}
&\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^+}(\Omega)} \\
&\leq C\Big[\Big(\int_\Omega |v_n(x)|^{q^+}dx\Big)^{\frac{1}{q^+}} + \Big(\int_\Omega |u_n(x)|^{(r(x)-1)q^+}\ |v_n(x)|^{q^+}dx\Big)^{\frac{1}{q^+}}\Big] \\
&\leq C\Big[\|v_n\|_{L^{q^+}(\Omega)} + \|\ |u_n|^{(r(\cdot)-1)q^+}\|^{\frac{1}{q^+}}_{L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)}\ \|v_n\|_{L^{r(\cdot)q^+}(\Omega)}\Big] \\
&\leq C\Big[\|v_n\|_{L^{q^+}(\Omega)} + \Big(\|u_n\|^{(r^+-1)}_{L^{r(\cdot)q^+}(\Omega)} + \|u_n\|^{(r^--1)}_{L^{r(\cdot)q^+}(\Omega)}\Big)\ \|v_n\|_{L^{r(\cdot)q^+}(\Omega)}\Big] \\
&\leq C\Big[\|v_n\|_{L^{q^+}(\Omega)} + \Big(\|u_n\|^{(r^+-1)}_{E} + \|u_n\|^{(r^--1)}_{E}\Big)\|v_n\|_{L^{r(\cdot)q^+}(\Omega)}\Big] = o_n(1).
\end{aligned} \tag{4.2.37}$$

Similarly, we can deduce that $f(\cdot, u_n(\cdot))v_n(\cdot) \in L^{q^-}(\Omega)$ and

$$\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^-}(\Omega)} = o_n(1). \tag{4.2.38}$$

Therefore, using (f_1), (1.2.12), Proposition 3.1.1, Theorem 4.1.1 with (4.2.37) and (4.2.38), we get

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{F(y, u_n(y))f(x, u_n(x))v_n(x)}{|x-y|^{\mu(x,y)}}dxdy \\
&\leq C\Big[\Big(\int_\Omega |F(x, u_n(x))|^{q^+}dx\Big)^{1/q^+}\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^+}(\Omega)} \\
&\qquad + \Big(\int_\Omega |F(x, u_n(x))|^{q^-}dx\Big)^{1/q^-}\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^-}(\Omega)}\Big] \\
&\leq C\Big(\|u_n\|_{L^{q^+}(\Omega)} + \|\ |u_n|^{r(\cdot)}\ \|_{L^{q^+}(\Omega)}\Big)\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^+}(\Omega)} \\
&\qquad + C\Big(\|u_n\|_{L^{q^-}(\Omega)} + \|\ |u_n|^{r(\cdot)}\ \|_{L^{q^-}(\Omega)}\Big)\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^-}(\Omega)} \\
&\leq C\Big[\Big\{\|u_n\|_{L^{q^+}(\Omega)} + \Big(\|u_n\|^{r^+}_{L^{r(\cdot)q^+}(\Omega)} + \|u_n\|^{r^-}_{L^{r(\cdot)q^+}(\Omega)}\Big)\Big\}\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^+}(\Omega)} \\
&\qquad + \Big\{\|u_n\|_{L^{q^-}(\Omega)} + \Big(\|u_n\|^{r^+}_{L^{r(\cdot)q^-}(\Omega)} + \|u_n\|^{r^-}_{L^{r(\cdot)q^-}(\Omega)}\Big)\Big\}\|f(\cdot, u_n(\cdot))v_n(\cdot)\|_{L^{q^-}(\Omega)}\Big]
\end{aligned}$$

$$\leq C\left[\|u_n\|_E + \left\{\|u_n\|_E^{r^+} + \|u_n\|_E^{r^-}\right\}\right]$$
$$\times\left[\|f(\cdot,u_n(\cdot))v_n(\cdot)\|_{L^{q^+}(\Omega)} + \|f(\cdot,u_n(\cdot))v_n(\cdot)\|_{L^{q^-}(\Omega)}\right] = o_n(1). \tag{4.2.39}$$

Now again from (f_1), it follows that $f(\cdot,u(\cdot))v_n(\cdot) \in L^{q^+}(\Omega)\cap L^{q^-}(\Omega)$ and hence, by arguing similarly as above, we obtain

$$\int_\Omega\int_\Omega \frac{F(y,u(y))f(x,u(x))v_n(x)}{|x-y|^{\mu(x,y)}}dxdy$$
$$\leq C\Big[\left\{\|u\|_E + \left(\|u\|_E^{r^+} + \|u\|_E^{r^-}\right)\right\}$$
$$\times\left\{\|f(\cdot,u(\cdot))v_n(\cdot)\|_{L^{q^+}(\Omega)} + \|f(\cdot,u(\cdot))v_n(\cdot)\|_{L^{q^-}(\Omega)}\right\}\Big] = o_n(1). \tag{4.2.40}$$

Since $\{u_n\}$ is bounded, combining (4.2.34), (4.2.36) ,(4.2.39), and (4.2.40), we get

$$\begin{aligned}
&o_n(1)\\
&= \langle \mathcal{J}'(u_n) - \mathcal{J}'(u), v_n\rangle_E\\
&= m(\sigma(u_n))\mathcal{B}_{u_n}(v_n) - m(\sigma(u_n))\mathcal{B}_u(v_n) + [m(\sigma(u_n)) - m(\sigma(u))]\,\mathcal{B}_u(v_n)\\
&\quad + \int_\Omega\int_\Omega \frac{F(y,u_n(y))f(x,u_n(x))v_n(x)}{|x-y|^{\mu(x,y)}}dxdy - \int_\Omega\int_\Omega \frac{F(y,u(y))f(x,u(x))v_n(x)}{|x-y|^{\mu(x,y)}}dxdy\\
&= m(\sigma(u_n))[\mathcal{B}_{u_n}(v_n) - \mathcal{B}_u(v_n)] + o_n(1),
\end{aligned}$$

that is,

$$\lim_{n\to+\infty}\left[m(\sigma(u_n))\left(\mathcal{B}_{u_n}(v_n) - \mathcal{B}_u(v_n)\right)\right] = 0. \tag{4.2.41}$$

Using Remark 4.0.1 (i), we have in particular

$$\lim_{n\to+\infty}[\mathcal{B}_{u_n}(v_n) - \mathcal{B}_u(v_n)] = 0. \tag{4.2.42}$$

Note that by Proposition 4.1.2, $v_n \to 0$ strongly in E is equivalent to $\rho_E(v_n) \to 0$ as

$n \to +\infty$. First we derive the following estimates required to obtain $\lim_{n\to\infty} \rho_E(v_n)$. Set

$$g_n(x) := V(x)\Big(|u_n|^{\overline{p}(x)-2}u_n - |u|^{\overline{p}(x)-2}u\Big)v_n(x).$$

For all $n \in \mathbb{N}$, using (1.2.10), we have $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_n(x)dx \geq 0$, which together with (4.2.42) yields that

$$\lim_{n\to+\infty}\int_{\Omega} g_n(x)dx = 0. \tag{4.2.43}$$

Next, to handle the variable exponents $p(\cdot,\cdot)$ and $\overline{p}(\cdot)$ we divide the domains $\mathbb{R}^N \times \mathbb{R}^N$ and Ω into four subdomains as follows:

$\Delta_1 := \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : 1 < p(x,y) < 2\}$; $\quad \Delta_2 := \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : p(x,y) \geq 2\}$;

$\widetilde{\Delta}_1 := \{x \in \Omega : 1 < \overline{p}(x) < 2\}$; $\quad \widetilde{\Delta}_2 := \{x \in \Omega : \overline{p}(x) \geq 2\}$.

(i) Case $(x,y) \in \Delta_1$: Since $\{u_n\}$ is bounded in E, arguing similarly as for obtaining (3.2.35), we have

$$\widehat{I}_1 := \int_{\Delta_1} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy = o_n(1). \tag{4.2.44}$$

(ii) Case $(x,y) \in \Delta_2$: In this case also, using the arguments similar to (3.2.36), we obtain

$$\widehat{I}_2 := \int_{\Delta_2} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy = o_n(1). \tag{4.2.45}$$

(iii) Case $x \in \widetilde{\Delta}_1$: Since $\{u_n\}$ is bounded, Lemma 1.2.3, Lemma 1.2.5, (1.2.10), (1.2.12), and (4.2.43) imply

$$\begin{aligned}\widehat{I}_3 &:= \int_{\widetilde{\Delta}_1} V(x)|v_n(x)|^{\overline{p}(x)}dx \\ &\leq \frac{1}{(p^- - 1)}\int_{\widetilde{\Delta}_1} (g_n(x))^{\overline{p}(x)/2} \times \Big\{V(x)(|u_n(x)|^{\overline{p}(x)} + |u(x)|^{\overline{p}(x)})\Big\}^{\frac{2-\overline{p}(x)}{2}} dx\end{aligned}$$

$$
\begin{aligned}
&\leq C\Big[\int_\Omega (g_n(x))^{\frac{\overline{p}(x)}{2}}(V(x)|u_n|^{\overline{p}(x)})^{\frac{2-\overline{p}(x)}{2}}dx+\int_\Omega (g_n(x))^{\frac{\overline{p}(x)}{2}}(V(x)|u|^{\overline{p}(x)})^{\frac{2-\overline{p}(x)}{2}}dx\Big]\\
&\leq C\|(g_n)^{\frac{\overline{p}(\cdot)}{2}}\|_{L^{\frac{2}{\overline{p}(\cdot)}}(\Omega)}\Big[\|(V|u_n|^{\overline{p}(\cdot)})^{\frac{2-\overline{p}(\cdot)}{2}}\|_{L^{\frac{2}{2-\overline{p}(\cdot)}}(\Omega)}+\|(V|u|^{\overline{p}(\cdot)})^{\frac{2-\overline{p}(\cdot)}{2}}\|_{L^{\frac{2}{2-\overline{p}(\cdot)}}(\Omega)}\Big]\\
&\leq C\Big[\|g_n\|_{L^1(\Omega)}^{\frac{p^-}{2}}+\|g_n\|_{L^1(\Omega)}^{\frac{p^+}{2}}\Big]\\
&\quad\times\Big[\|V|u_n|^{\overline{p}(\cdot)}\|_{L^1(\Omega)}^{\frac{2-p^-}{2}}+\|V|u_n|^{\overline{p}(\cdot)}\|_{L^1(\Omega)}^{\frac{2-p^+}{2}}+\|V|u|^{\overline{p}(\cdot)}\|_{L^1(\Omega)}^{\frac{2-p^-}{2}}+\|V|u|^{\overline{p}(\cdot)}\|_{L^1(\Omega)}^{\frac{2-p^+}{2}}\Big]\\
&=o_n(1). \qquad (4.2.46)
\end{aligned}
$$

(iv) Case $x\in\widetilde{\Delta}_2$: Using Lemma 1.2.3, (1.2.10), (1.2.12), and (4.2.43), we deduce

$$
\widehat{I}_4:=\int_{\widetilde{\Delta}_2}V(x)|v_n(x)|^{\overline{p}(x)}dx\leq 2^{p^+}\int_\Omega g_n^{(4)}(x)dx=o_n(1). \qquad (4.2.47)
$$

Now taking into account (4.2.42) and (4.2.44)-(4.2.47), we get

$$
\begin{aligned}
\rho_E(v_n)&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy+\int_\Omega V(x)|v_n(x)|^{\overline{p}(x)}dx\\
&=\widehat{I}_1+\widehat{I}_2+\widehat{I}_3+\widehat{I}_4=o_n(1).
\end{aligned}
$$

Hence $v_n\to 0$ strongly in E, thanks to Proposition 4.1.2, which implies $u_n\to u$ strongly in E as $n\to+\infty$.

(II) $\inf_{n\in\mathbb{N}}\|u_n\|_E=0$**:** If 0 is an isolated point for the sequence $\{\|u_n\|_E\}$ then there exists a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$
\inf_{n\in\mathbb{N}}\|u_{n_k}\|_E=d_*>0
$$

and therefore, we can proceed as before. Otherwise, 0 is an accumulation point of the sequence $\{\|u_{n_k}\|_E\}$. Hence, there exists a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}\to 0$ in E strongly. This completes the lemma in the degenerate case, i.e. $a=0$.

Next, we consider the non-degenerate case, i.e., $a>0$. Hence, the above proof reduces to Case **(I)**. Then using Remark 4.0.2 and (4.0.3) in place of Remark 4.0.1 in Case **(I)** and arguing similarly, the result follows. □

Now we give the proof of the Theorem 4.2.1.

Proof of Theorem 4.2.1: Since $\mathcal{J}$ satisfies Lemma 4.2.1 and Lemma 4.2.2, by Theorem 1.3.3, there exists a Cerami sequence $\{u_n\}$ for $\mathcal{J}$ in E such that

$$\mathcal{J}(u_n) \to c_* \qquad \text{and} \qquad (1 + \|u_n\|_E)\|\mathcal{J}'(u_n)\|_{E^*} \to 0 \text{ as } n \to \infty,$$

where $c_* > 0$ is the mountain pass level defined by

$$c_* := \inf_{\nu \in \Gamma} \sup_{t \in [0,1]} \mathcal{J}(\nu(t)).$$

Now Proposition 4.2.1 implies that $u_n \to u_*$ strongly in E, for some $u_* \in E$, and hence, $\mathcal{J}'(u_*) = 0$. This yields that u_* is a critical point of $\mathcal{J}$ and therefore, a weak solution to 4.0.1. Also, $\mathcal{J}(u_*) = c_* > 0$ and since $\mathcal{J}(0) = 0$, we conclude that $u_* \not\equiv 0$. □

4.3 Existence of ground state solution via Nehari manifold and fibering map

In this section, we study the existence of ground state solution of (4.0.1) using Nehari manifold and fibering map approach. Here we consider the assumption:

$(f_4)'$ $\frac{f(x,t)}{|t|^{\frac{\theta p^+}{2}-2}t}$ is increasing in $t > 0$ and decreasing in $t < 0$, for all $x \in \Omega$.

Note that the conditions $(f_2), (f_3), (f_4)'$ are weaker than (AR). We say $u \in E$ is a ground state solution of (4.0.1), if

$$\mathcal{J}(u) = \inf\{\mathcal{J}(v) : v \in E \setminus \{0\} \text{ is a weak solution of (4.0.1)}\}.$$

Theorem 4.3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1), respectively, such that $s^+p^+ < N$. Assume that $(\mu_1), (V_1)$, and (M_1) hold. Also, let f satisfy (f_1)-(f_3) and $(f_4)'$. Then (4.0.1) admits a non-trivial ground state solution.*

Proof. First we note that the condition (f_4) is the consequence of $(f_4)'$. Indeed, for $t_2 \geq$

$t_1 > 0$, $(f_4)'$ implies that

$$\begin{aligned}
&\mathcal{F}(x,t_2) - \mathcal{F}(x,t_1)\\
&= \theta p^+ \left[\frac{2}{\theta p^+} \left(f(x,t_2)t_2 - f(x,t_1)t_1 \right) - \left(F(x,t_2) - F(x,t_1) \right) \right]\\
&= \theta p^+ \Big[\int_0^{t_2} \frac{f(x,t_2)}{t_2^{\frac{\theta p^+}{2}-1}} \tau^{\frac{\theta p^+}{2}-1} d\tau - \int_0^{t_1} \frac{f(x,t_1)}{t_1^{\frac{\theta p^+}{2}-1}} \tau^{\frac{\theta p^+}{2}-1} d\tau - \int_{t_1}^{t_2} \frac{f(x,\tau)}{\tau^{\frac{\theta p^+}{2}-1}} \tau^{\frac{\theta p^+}{2}-1} d\tau \Big]\\
&= \theta p^+ \Big[\int_{t_1}^{t_2} \Big(\frac{f(x,t_2)}{t_2^{\frac{\theta p^+}{2}-1}} - \frac{f(x,\tau)}{\tau^{\frac{\theta p^+}{2}-1}} \Big) \tau^{\frac{\theta p^+}{2}-1} d\tau + \int_0^{t_1} \Big(\frac{f(x,t_2)}{t_2^{\frac{\theta p^+}{2}-1}} - \frac{f(x,t_1)}{t_1^{\frac{\theta p^+}{2}-1}} \Big) \tau^{\frac{\theta p^+}{2}-1} d\tau \Big]\\
&\geq 0.
\end{aligned}$$

Similarly, for $0 > t_1 \geq t_2$, we can deduce $\mathcal{F}(x,t_2) - \mathcal{F}(x,t_1) \geq 0$, that is, $\mathcal{F}(\cdot,t)$ is increasing in $t \geq 0$ and decreasing in $t \leq 0$. Hence, (f_4) follows. Therefore, there exists a weak solution $v_* \neq 0$ of (4.0.1), thanks to Theorem 4.2.1, with $\mathcal{J}'(v_*) = 0$ and $\mathcal{J}(v_*) = b_*$, where b_* is given as

$$b_* := \inf_{\nu \in \Gamma} \max_{0 \leq t \leq 1} \mathcal{J}(\nu(t)),$$

where $\Gamma = \{u \in C([0,1],E) : \nu(0) = 0, J(\nu(1)) < 0\}$. We claim that v_* is a ground state solution of (4.0.1). The Nehari manifold associated with the functional $\mathcal{J}$ is defined as

$$\mathcal{N} := \{u \in E \setminus \{0\} : \langle \mathcal{J}'(u), u \rangle_E = 0\}.$$

Since v_* is a critical point of $\mathcal{J}$, we have $v_* \in \mathcal{N}$. Let $\alpha_* = \inf_{u \in \mathcal{N}} \mathcal{J}(u)$. Hence, $\alpha_* \leq b_*$. Therefore, it is left to show that $b_* \leq \alpha_*$. For $u \in \mathcal{N}$, define the fibering map $\mathcal{H} : [0,+\infty) \to \mathbb{R}$ by $\mathcal{H}(t) := \mathcal{J}(tu)$. Then $\mathcal{H}$ is differentiable with respect to t and

$$\begin{aligned}
\mathcal{H}'(t) &= \langle \mathcal{J}'(tu), u \rangle_E\\
&= m(\sigma(tu)) \Big[\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} t^{p(x,y)-1} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} dxdy\\
&\quad + \int_\Omega t^{\overline{p}(x)-1} V(x) |u|^{\overline{p}(x)} dx \Big] - \int_\Omega \int_\Omega \frac{F(y,tu) f(x,tu) u(x)}{|x-y|^{\mu(x,y)}} dxdy. \qquad (4.3.48)
\end{aligned}$$

In addition, we have $\langle \mathcal{J}'(u), u\rangle_E = 0$, that is,

$$m(\sigma(u))\rho_E(u) = I(u) := \int_\Omega \int_\Omega \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu(x,y)}} dxdy. \tag{4.3.49}$$

Now we discuss the degenerate and non-degenerate cases separately as follows:

Case I ($a = 0$): From (M_1), Remark 4.0.1, (4.3.48) and (4.3.49), for $t > 1$, it follows that

$$\begin{aligned}
&\mathcal{H}'(t)\\
&= m(\sigma(tu))\Big[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} t^{p(x,y)-1}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}}dxdy + \int_\Omega t^{\overline{p}(x)-1}V|u|^{\overline{p}(x)}dx\Big]\\
&\quad - \int_\Omega\int_\Omega \frac{F(y,tu)f(x,tu)u(x)}{|x-y|^{\mu(x,y)}}dxdy - t^{\theta p^+-1}m(\sigma(u))\rho_E(u) + t^{\theta p^+-1}I(u)\\
&\le b[\sigma(tu)]^{\theta-1}t^{p^+-1}\rho_E(u) - t^{\theta p^+-1}b[\sigma(u)]^{\theta-1}\rho_E(u) + \mathcal{G}(t)\\
&\le bt^{p^+(\theta-1)}[\sigma(u)]^{\theta-1}t^{p^+-1}\rho_E(u) - t^{\theta p^+-1}b[\sigma(u)]^{\theta-1}\rho_E(u) + \mathcal{G}(t) = \mathcal{G}(t),
\end{aligned} \tag{4.3.50}$$

where

$$\mathcal{G}(t) = t^{\theta p^+-1}\int_\Omega\int_\Omega \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu(x,y)}}dxdy - \int_\Omega\int_\Omega \frac{F(y,tu)f(x,tu)u(x)}{|x-y|^{\mu(x,y)}}dxdy.$$

Since $\mathcal{F}(x,\tau) = 2\tau f(x,\tau) - \theta p^+ F(x,\tau) \ge 0$, for all $x \in \mathbb{R}^N, \tau \in \mathbb{R}$, it follows that

$$\begin{aligned}
\frac{d}{dt}\frac{F(x,tu)}{t^{\frac{\theta p^+}{2}}} &= \frac{t^{\frac{\theta p^+}{2}}f(x,tu)u(x) - \frac{\theta p^+}{2}t^{\frac{\theta p^+}{2}-1}F(x,tu)}{t^{\theta p^+}}\\
&= \frac{f(x,tu)tu(x) - \frac{\theta p^+}{2}F(x,tu)}{t^{\frac{\theta p^+}{2}-1}} \ge 0.
\end{aligned}$$

Thus, $\frac{F(x,tu)}{t^{\frac{\theta p^+}{2}}}$ is increasing function in $t > 0$, for all $u \in E$. Now for $t > 1$, using $(f_4)'$ and Remark 4.0.4, we deduce

$$\begin{aligned}
&\mathcal{G}(t)\\
&= t^{\theta p^+-1}\Big[\int_\Omega\int_\Omega \frac{F(y,u)f(x,u)|u|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}|u|^{\frac{\theta p^+}{2}-2}u}dxdy - \int_\Omega\int_\Omega \frac{F(y,tu)f(x,tu)|u|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}t^{\frac{\theta p^+}{2}}|tu|^{\frac{\theta p^+}{2}-2}tu}dxdy\Big]
\end{aligned}$$

$$\leq t^{\theta p^+-1}\Big[\int_\Omega\int_\Omega \frac{F(y,u)f(x,u)|u|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}|u|^{\frac{\theta p^+}{2}-2}u}dxdy - \int_\Omega\int_\Omega \frac{F(y,tu)f(x,u)|u|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)}t^{\frac{\theta p^+}{2}}|u|^{\frac{\theta p^+}{2}-2}\,u}dxdy\Big]$$

$$= t^{\theta p^+-1}\Big[\int_\Omega\Big(\int_\Omega \frac{F(y,u)-\frac{F(y,tu)}{t^{\frac{\theta p^+}{2}}}}{|x-y|^{\mu(x,y)}}dy\Big)f(x,u)u\,dx\Big] \leq 0. \tag{4.3.51}$$

Combining (4.3.50) and (4.3.51), we get $\mathcal{H}'(t) \leq 0$, for $t > 1$. Arguing similarly as above, we can deduce that $\mathcal{H}'(t) \geq 0$, for $t \leq 1$. Therefore, 1 is the maximum point of $\mathcal{H}$, that is, $\mathcal{J}(u) = \max_{t\geq 0} J(tu)$. Next, we define the map $\nu : [0,1] \to E$ as $\nu(t) = (t_0u)t$, where $t_0 > 1$ satisfies $\mathcal{J}(t_0u) < 0$. This map is well-defined due to Lemma 4.2.2. So, $\nu \in \Gamma$. Hence,

$$b_* \leq \max_{0\leq t\leq 1}\mathcal{J}(\nu(t)) \leq \max_{0\leq t\leq 1}\mathcal{J}(tu) = J(u).$$

Since $u \in \mathcal{N}$ is arbitrary, we get $b_* \leq \alpha_*$. Therefore,

$$\inf_{u\in\mathcal{N}} J(u) = \alpha_* = b_* = J(v_*).$$

Case II ($a > 0$): By replacing Remark 4.0.1 with Remark 4.0.2 in (4.3.50) and arguing in a similar way as in the Case I, we conclude the proof. □

4.4 Existence of infinitely many solutions

In this section, we consider the nonlinearity f of odd nature and exploit this property to establish the existence of infinitely many solutions of (4.0.1) with unbounded energy and negative energy.

4.4.1 Solutions with unbounded critical energy

Here we establish the existence of a sequence of solutions to (4.0.1) with unbounded energy (in the limiting sense) using the fountain theorem with Cerami condition (Theorem 1.3.4).

Theorem 4.4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1), respectively, such that $s^+p^+ < N$. Assume that $(\mu_1), (V_1)$,* *and* (M_1) *hold. Also, let* f *satisfy* (f_1)-(f_4) *with* $f(x,-t) = -f(x,t)$*. Then* (4.0.1) *has a sequence of non-trivial weak solutions* $\{v_n\}$, $n \in \mathbb{N}$ *such that* $\mathcal{J}(v_n) \to +\infty$ *as* $n \to +\infty$*.*

Proof. For the reflexive, separable Banach space E, define Y_k and Z_k appropriately, following (1.3.23). The functional $\mathcal{J}$ satisfies Cerami condition $(C)_c$, for all $c \in \mathbb{R}$, thanks to Proposition 4.2.1, and $\mathcal{J}$ is even. Now we verify the conditions $(\mathscr{B}_1)$-$(\mathscr{B}_2)$ in Theorem 1.3.4.

Verification of $(\mathscr{B}_1)$: For $k \in \mathbb{N}$ large enough, let us denote

$$\alpha_k = \sup_{u \in Z_k,\ \|u\|_E = 1} \|u\|_{L^{\gamma(\cdot)}(\Omega)}, \tag{4.4.52}$$

where $\gamma(\cdot) \in C_+(\overline{\Omega})$ such that $1 < \gamma(x) < p_s^*(x)$, for all $x \in \overline{\Omega}$. Then we have

$$\lim_{k \to +\infty} \alpha_k = 0. \tag{4.4.53}$$

If not, supposing to the contrary, there exist $\epsilon_0 > 0, k_0 \geq 0$ and a sequence $\{u_k\}$ in Z_k such that

$$\|u_k\|_E = 1 \text{ and } \ \|u_k\|_{L^{\gamma(\cdot)}(\Omega)} \geq \epsilon_0,$$

for all $k \geq k_0$. Since $\{u_k\}$ is bounded in E, there exists $\tilde{u}_0 \in E$ such that up to a subsequence, still denoted by $\{u_k\}$, we have $u_k \rightharpoonup \tilde{u}_0$ weakly in E as $k \to +\infty$. Now by Lemma 1.3.1, we get

$$\langle f_j^*, \tilde{u}_0 \rangle_E = \lim_{k \to +\infty} \langle f_j^*, u_k \rangle_E = 0, \quad \text{for } j = 1, 2, 3, \cdots.$$

Thus, we have $\tilde{u}_0 = 0$. Furthermore, using Theorem 4.1.1, we obtain

$$\epsilon_0 \leq \lim_{k \to +\infty} \|u_k\|_{L^{\gamma(\cdot)}(\Omega)} = \|\tilde{u}_0\|_{L^{\gamma(\cdot)}(\Omega)} = 0,$$

which is a contradiction to the fact $\epsilon_0 > 0$. Hence, (4.4.53) holds true. Let $u \in Z_k$ with $\|u\|_E > 1$. By using Remark 4.0.1 (iii), for $a = 0$ in (M_1) (or Remark 4.0.2 (iii), for $a > 0$

in (M_1)), and Proposition 4.1.1, we have

$$\mathcal{J}(u) \geq \frac{M(1)}{(p^+)^\theta}\{\rho(u)\}^\theta - \Psi(u) \geq \frac{M(1)}{(p^+)^\theta}\|u\|_E^{\theta p^-} - \Psi(u). \tag{4.4.54}$$

Now from (4.4.53), we infer that $\alpha_k < 1$, for large $k \in \mathbb{N}$. Therefore, using (f_1), Lemma 1.2.5, (1.2.12), Proposition 3.1.1, and (4.4.52), for sufficiently large $k \in \mathbb{N}$, we get

$$\begin{aligned}\Psi(u) \leq C\Big[&\Big\{\|u\|^2_{L^{q^-}(\Omega)} + \Big(\|u\|^{2r^-}_{L^{r(\cdot)q^-}(\Omega)} + \|u\|^{2r^+}_{L^{r(\cdot)q^-}(\Omega)}\Big)\Big\} \\ &+ \Big\{\|u\|^2_{L^{q^+}(\Omega)} + \Big(\|u\|^{2r^-}_{L^{r(\cdot)q^+}(\Omega)} + \|u\|^{2r^+}_{L^{r(\cdot)q^+}(\Omega)}\Big)\Big\}\Big] \\ \leq\ & 2C\Big\{\|u\|_E^2\alpha_k^2 + \Big(\|u\|_E^{2r^-}\alpha_k^{2r^-} + \|u\|_E^{2r^+}\alpha_k^{2r^+}\Big)\Big\} \\ \leq\ & \widehat{C}\alpha_k\|u\|_E^{2r^+},\end{aligned} \tag{4.4.55}$$

where $\widehat{C}$ is a positive constant, independent of k, u. Thus, (4.4.54) and (4.4.55) give us

$$\mathcal{J}(u) \geq \frac{M(1)}{(p^+)^\theta}\|u\|_E^{\theta p^-} - \widehat{C}\alpha_k\|u\|_E^{2r^+}. \tag{4.4.56}$$

Consider the real function $G : \mathbb{R} \to \mathbb{R}$,

$$G(t) = \frac{M(1)}{(p^+)^\theta}t^{\theta p^-} - \widehat{C}\alpha_k t^{2r^+}.$$

Then from the elementary calculus, it follows that G attains its maximum at

$$\delta_k = \left(\frac{M(1)\theta p^-}{2r^+(p^+)^\theta\widehat{C}\alpha_k}\right)^{\frac{1}{(2r^+-\theta p^-)}}$$

and the maximum value of G is given by

$$\begin{aligned}G(\delta_k) &= \frac{M(1)}{(p^+)^\theta}\left(\frac{M(1)\theta p^-}{2r^+(p^+)^\theta\widehat{C}\alpha_k}\right)^{\frac{\theta p^-}{(2r^+-\theta p^-)}} - \widehat{C}\alpha_k\left(\frac{M(1)\theta p^-}{2r^+(p^+)^\theta\widehat{C}\alpha_k}\right)^{\frac{2r^+}{(2r^+-\theta p^-)}} \\ &= \left(\frac{M(1)}{(p^+)^\theta}\right)^{\frac{2r^+}{(2r^+-\theta p^-)}}\left(\frac{1}{\widehat{C}\alpha_k}\right)^{\frac{\theta p^-}{(2r^+-\theta p^-)}}\left(\frac{\theta p^-}{2r^+}\right)^{\frac{\theta p^-}{(2r^+-\theta p^-)}}\left(1 - \frac{\theta p^-}{2r^+}\right).\end{aligned}$$

Since $\theta p^- < 2r^+$ and $\alpha_k \to 0$ as $k \to +\infty$, we have

$$G(\delta_k) \to +\infty \text{ as } k \to +\infty. \tag{4.4.57}$$

Again, using (4.4.53), we get $\delta_k \to +\infty$ as $k \to +\infty$. Thus, for $u \in Z_k$ with $\|u\|_E = \delta_k$, combining (4.4.56) and (4.4.57), it readily follows that

$$b_k = \inf_{u \in Z_k, \|u\|_E = \delta_k} \mathcal{J}(u) \to +\infty \text{ as } k \to +\infty.$$

Verification of $(\mathscr{B}_2)$: Due to the presence of Choquard type nonlinearity, here we use an indirect argument. Suppose the assertion $(\mathscr{B}_2)$ of Theorem 1.3.4 does not hold true for some given $k \in \mathbb{N}$. Then there exists a sequence $\{u_n\} \subset Y_k$ such that

$$\|u_n\|_E \to +\infty, \qquad J(u_n) \geq 0. \tag{4.4.58}$$

Let us take $w_n := \frac{u_n}{\|u_n\|_E}$, then $w_n \in E$ and $\|w_n\|_E = 1$. Since Y_k is of finite dimension, there exists $w \in Y_k \setminus \{0\}$ such that up to a sub-sequence, still denoted by $\{w_n\}$, $w_n \to w$ strongly in Y_k and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$ as $n \to +\infty$. If $w(x) \neq 0$ then $|u_n(x)| \to +\infty$ as $n \to +\infty$. Similar to as obtained in (4.2.27), for each $x \in \Omega$, it follows that

$$\left(\int_\Omega \frac{F(y, u_n(y)) |w_n(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)} |u_n(y)|^{\frac{\theta p^+}{2}}} dy \right) \frac{F(x, u_n(x))}{|u_n(x)|^{\frac{\theta p^+}{2}}} |w_n(x)|^{\frac{\theta p^+}{2}} \to +\infty. \tag{4.4.59}$$

Thus, using (4.4.58), (4.4.59) with Remark 4.0.4 and applying Fatou's lemma, we get

$$\begin{aligned} \frac{\Psi(u_n)}{\|u_n\|_E^{\theta p^+}} &= \frac{1}{2} \int_\Omega \left(\int_\Omega \frac{F(y, u_n) |w_n(y)|^{\frac{\theta p^+}{2}}}{|x-y|^{\mu(x,y)} |u_n(y)|^{\frac{\theta p^+}{2}}} dy \right) \frac{F(x, u_n)}{|u_n(x)|^{\frac{\theta p^+}{2}}} |w_n(x)|^{\frac{\theta p^+}{2}} dx \\ &\to +\infty \text{ as } n \to +\infty. \end{aligned} \tag{4.4.60}$$

Since, for large $n \in \mathbb{N}$, $\|u_n\|_E > 1$, using Remark 4.0.1 (iii), for $a = 0$ in (M_1) (or Remark

4.0.2 (iii), for $a > 0$ in (M_1)), Proposition 4.1.1, and (4.4.60), we deduce that

$$\begin{aligned}\mathcal{J}(u_n) &\leq \frac{M(1)}{(p^-)^\theta}\|u_n\|^{\theta p^+} - \Psi(u_n)\\ &= \left(\frac{M(1)}{(p^-)^\theta} - \frac{1}{\|u_n\|_E^{\theta p^+}}\Psi(u_n)\right)\|u_n\|_E^{\theta p^+} \to -\infty,\end{aligned}$$

as $n \to +\infty$. This contradicts (4.4.58). Thus, for sufficiently large $k \in \mathbb{N}$, we can have $\varrho_k > \delta_k > 0$ such that, for $u \in Y_k$ with $\|u\|_E = \varrho_k$, the assertion $(\mathscr{B}_2)$ follows.
Hence, by Theorem 1.3.4, there exists a sequence of solutions $\{v_n\}$, $n \in \mathbb{N}$, of (4.0.1) such that $\mathcal{J}(v_n) \to +\infty$ as $n \to +\infty$.

□

4.4.2 Solutions with negative critical energy

For the reflexive, separable Banach space E, define Y_k and Z_k by following the argument as in (1.3.23). Now we establish the existence of a sequence of solutions to (4.0.1) with negative energy using the dual fountain theorem with Cerami* condition (Theorem 1.3.5):

Theorem 4.4.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $s(\cdot,\cdot)$ and $p(\cdot,\cdot)$ satisfy (S_1) and (P_1), respectively, such that $s^+p^+ < N$. Assume that $(\mu_1), (V_1)$, and (M_1) hold. Also, let f satisfy (f_1)-(f_4) with $f(x,-t) = -f(x,t)$. Then (4.0.1) has a sequence of non-trivial weak solutions $\{w_n\}$, $n \in \mathbb{N}$ such that $\mathcal{J}(w_n) \leq 0$ and $\mathcal{J}(w_n) \to 0$ as $n \to +\infty$.*

First we prove the following lemma:

Lemma 4.4.1. *Suppose that the hypotheses in Theorem 4.4.2 hold, then $\mathcal{J}$ satisfies the $(C)_c^*$ condition.*

Proof. Since E is reflexive separable Banach space, using Lemma 1.3.1 we define Y_k and Z_k as in (1.3.23). Let $c \in \mathbb{R}$ and $\{u_k\}$ be any sequence in E such that $u_k \in Y_k$, for all $k \in \mathbb{N}$, $\mathcal{J}(u_k) \to c$ and $\left\|\mathcal{J}'_{|Y_k}(u_k)\right\|_{E^*}(1 + \|u_k\|_E) \to 0$ as $k \to +\infty$. Therefore, we have

$$c = \mathcal{J}(u_k) + o_k(1) \text{ and } \langle \mathcal{J}'(u_k), u_k\rangle_E = o_k(1).$$

Analogously to the proof of Proposition 4.2.1, we can show that $\{u_k\}$ is bounded in E. Hence, there exists a sub-sequence, still denoted by $\{u_k\}$, and $u \in E$ such that $u_k \rightharpoonup u$ weakly in E as $k \to +\infty$. On the other hand, Lemma 1.3.1 implies $E = \overline{\cup_k Y_k} = \overline{span\{e_k : k \geqslant 1\}}$ and thus, we can choose $v_k \in Y_k$ such that $v_k \to u$ strongly in E as $k \to +\infty$. Therefore, using the facts $\mathcal{J}'_{|Y_k}(u_k) \to 0$ and $u_k - v_k \rightharpoonup 0$ in Y_k, (see [18, Proposition 3.5]), we achieve

$$\lim_{k\to+\infty} \langle \mathcal{J}'(u_k), u_k - u\rangle_E = \lim_{k\to+\infty} \langle \mathcal{J}'(u_k), u_k - v_k\rangle_E + \lim_{k\to+\infty} \langle \mathcal{J}'(u_k), v_k - u\rangle_E = 0.$$

Again recalling the proof of Proposition 4.2.1, we can deduce $u_k \to u$ strongly in E as $k \to +\infty$. Then, we conclude that $\mathcal{J}$ satisfies the $(C)^*_c$ condition. Thus, we obtain that $\mathcal{J}'(u_k) \to \mathcal{J}'(u)$ as $k \to +\infty$. Let us prove $\mathcal{J}'(u) = 0$. Indeed, taking $\omega_j \in Y_j$, for $k \geq j$, we have

$$\begin{aligned}\langle \mathcal{J}'(u), \omega_j\rangle_E &= \lim_{k\to+\infty} [\langle \mathcal{J}'(u) - \mathcal{J}'(u_k), \omega_j\rangle_E + \langle \mathcal{J}'(u_k), \omega_j\rangle_E] \\ &= \lim_{k\to+\infty} [\langle \mathcal{J}'(u) - \mathcal{J}'(u_k), \omega_j\rangle_E + \langle \mathcal{J}'_{|Y_k}(u_k), \omega_j\rangle_E] = 0.\end{aligned}$$

Therefore, $\mathcal{J}'(u) = 0$ in E^* and hence, $\mathcal{J}$ satisfies the $(C)^*_c$ condition, for every $c \in \mathbb{R}$. □

Proof of Theorem 4.4.2: Note that the fact that $\mathcal{J}$ is even and Lemma 4.4.1 ensures that $\mathcal{J}$ satisfies Cerami* condition $(C)^*_c$, for all $c \in \mathbb{R}$. So, to prove Theorem 4.4.2, it is enough to verify the conditions $(\mathscr{A}_1)$-$(\mathscr{A}_3)$ of Theorem 1.3.5.

Verification in $(\mathscr{A}_1)$*:* For all $u \in Z_k$ with $\|u\|_E < 1$, arguing in a similar fashion as obtained in (4.4.55), we can derive

$$\Psi(u) \leq C_3\alpha_k\|u\|_E, \tag{4.4.61}$$

which together with Remark 4.0.1 (iii), for $a = 0$ in (M_1) (or Remark 4.0.2 (iii), for $a > 0$ in (M_1)), and Proposition 4.1.1 imply

$$\mathcal{J}(u) \geq \frac{M(1)}{(p^+)^\theta}\|u\|_E^{\theta p^+} - C_3\alpha_k\|u\|_E, \tag{4.4.62}$$

where $C_3 > 0$ is a constant, independent on k, u. Let us choose $\varrho_k = [(p^+)^\theta C_3\alpha_k/M(1)]^{1/(\theta p^+-1)}$. Since $\theta p^+ > 1$, (4.4.52) yields that

$$\varrho_k \to 0 \quad \text{as } k \to +\infty. \tag{4.4.63}$$

Thus, for $u \in Z_k$ with $\|u\|_E = \varrho_k$ and for sufficiently large $k \in \mathbb{N}$, from (4.4.62), we get $\mathcal{J}(u) \geq 0$.

Verification of $(\mathscr{A}_2)$*:* Suppose the assertion $(\mathscr{A}_2)$ of Theorem 1.3.5 does not hold true for some given $k \in \mathbb{N}$. Then there exists a sequence $\{u_n\} \subset Y_k$ such that

$$\|u_n\|_E \to +\infty, \qquad \mathcal{J}(u_n) \geq 0. \tag{4.4.64}$$

Now arguing similar to *Verification of* $(\mathscr{B}_2)$ in the proof of Theorem 4.4.1, we obtain (4.4.59) and (4.4.60) which together with Remark 4.0.1(iii), for $a = 0$ in (M_1)(or Remark 4.0.2(iii), for $a > 0$ in (M_1)), and Proposition 4.1.1 imply that as $n \to +\infty$

$$\begin{aligned}\mathcal{J}(u_n) &\leq \frac{M(1)}{(p^-)^\theta}\|u_n\|^{\theta p^+} - \Psi(u_n)\\ &= \left(\frac{M(1)}{(p^-)^\theta} - \frac{1}{\|u_n\|_E^{\theta p^+}}\Psi(u_n)\right)\|u_n\|_E^{\theta p^+} \to -\infty.\end{aligned}$$

Hence, we get a contradiction to (4.4.64). Thus, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, we have $1 > \varrho_k > \delta_k > 0$ so that for $u \in Y_k$ with $\|u\|_E = \delta_k$, the assertion $(\mathscr{A}_2)$ follows.

Verification of $(\mathscr{A}_3)$*:* Since $Y_k \cap Z_k \neq \emptyset$, we get $d_k \leq b_k < 0$. Now for $u \in Z_k$, $\|u\|_E \leq \varrho_k$, using (4.4.61), we have

$$\mathcal{J}(u) \geq -C_3\alpha_k\|u\|_E \geq -C_3\alpha_k\varrho_k.$$

Therefore, using (4.4.52) and (4.4.63), we obtain

$$d_k \geq -C_3\alpha_k\varrho_k \to 0 \quad \text{as } k \to +\infty.$$

Since $d_k < 0$, we finally conclude that $\lim_{k\to+\infty} d_k = 0$. This completes the proof of the theorem. □

4.5 Conclusion

In this chapter, we have studied a class of doubly non-local Kirchhoff type Choquard equations involving variable-order fractional $p(\cdot)$-Laplacian in a bounded domain with Lipschitz boundary. The salient feature of the problem is relaxing the Ambrosetti-Rabinowitz (AR) type condition on the reaction term and hence covering a more general class of non-linearities. We mention that the (AR)-condition on the reaction term guarantees the boundedness of the Palais-Smale sequence for the associated functional which is not possible for the nonlinearities without (AR)-condition. Therefore, here we have worked with the Cerami sequences and established the compactness result which provides us a weak solution in the form of the weak limit of the Cerami sequence. Next, we have obtained the ground state solution of the problem using the Nehari manifold approach and the analysis of the associated fibering map. We have also shown the existence of a sequence of solutions with unbounded energy and a sequence of solutions with negative energy converging to zero by additionally taking the odd nature on the nonlinearity $f(x,t)$ with respect to the variable t and considering the fountain theorem with Cerami condition and the dual fountain theorem with Cerami* condition, respectively. Another important feature of the problem is that here we have covered both the degenerate and the non-degenerate cases of the Kirchhoff function.

The research in this chapter has risen up some open questions to be further investigated. Here we have assumed the continuity and non-negativity of the potential function $V(x)$. It will be interesting to explore the results obtained in this work under the weaker integrability and the sign changing assumptions on $V(x)$ in the whole of $\mathbb{R}^N$. In this case, one of the main challenges would be the application of the associated fibering map due to the non-homogeneous feature of the problem, together with sign changing potential $V(\cdot)$. Also, establishing compactness result (Cerami type) for such problem calls for the attention. □

5

Regularity results for doubly non-local problems

In this chapter, we study the regularity of weak solutions of the following class of p-fractional Choquard equations :

$$\left.\begin{aligned} (-\Delta)_p^s u &= \left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u), \quad && x \in \Omega, \\ u &= 0, && x \in \mathbb{R}^N \setminus \Omega, \end{aligned}\right\} \tag{5.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $1 < p < \infty$ and $0 < s < 1$ such that $sp < N$, $0 < \mu < \min\{N, 2sp\}$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with at most critical growth condition (in the sense of Hardy-Littlewood-Sobolev inequality (1.3.15)). Here $F(x,t) = \int_0^t f(x,\tau)d\tau$ is the primitive of f. We also study $W^{s,p}(\Omega)$ versus $C_d^0(\Omega)$ local minimizers result for (5.0.1).

Definition 5.0.1. *A function $u \in W_0^{s,p}(\Omega)$ is said to be weak solution of* (5.0.1)*, if for all $w \in W_0^{s,p}(\Omega)$*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(w(x)-w(y))}{|x-y|^{N+sp}}dxdy = \int_\Omega\int_\Omega \frac{F(y,u)f(x,u)}{|x-y|^\mu}w(x)\ dxdy. \tag{5.0.2}$$

Definition 5.0.2. *The energy functional $J : W_0^{s,p}(\Omega) \to \mathbb{R}$ associated to* (5.0.1) *is given by*

$$J(u) = \frac{1}{p}\|u\|_{s,p} - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(y,u)F(x,u)}{|x-y|^\mu}dxdy. \tag{5.0.3}$$

Motivated by the inequality (1.3.15), we assume the following hypothesis on f:

(**H**) There exists some constant K_0 such that, for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$,

$$|f(x,t)| \leq K_0\left(1+|t|^{r-1}\right)$$

with $1 < r \leq p^*_{\mu,s}$, where $p^*_{\mu,s} := \frac{(pN-p\mu/2)}{(N-ps)}$ denotes the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality (see Remark 1.3.1). Note that (1.3.15) ensures that (5.0.3) is well defined.

5.1 L^∞ bound and Hölder smoothness of weak solution

In this section we derive *a priori*-L^∞ bound and establish the Hölder regularity of weak solutions to (5.0.1).

Theorem 5.1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary and $1 < p < \infty$, $s \in (0,1)$ such that $sp < N$. Assume that* (**H**) *holds. Then for any weak solution*

$u \in W_0^{s,p}(\Omega)$ of (5.0.1), there exists $\alpha \in (0,s]$ depending upon s,p,N,Ω such that $u \in L^\infty(\mathbb{R}^N) \cap C^{0,\alpha}(\mathbb{R}^N)$.

In order to prove Theorem 5.1.1, we first recall the following inequalities:

Lemma 5.1.1. ([16, Lemma C.1]) *Let $1 < p < \infty$ and $\beta \geq 1$. For every $a,b,t \geq 0$, it holds that*

$$|a-b|^{p-2}(a-b)(a_t^\beta - b_t^\beta) \geq \frac{\beta p^p}{(\beta+p-1)^p}\left|a_t^{\frac{\beta+p-1}{p}} - b_t^{\frac{\beta+p-1}{p}}\right|^p,$$

where $a_t = \min\{a,t\}$ and $b_t = \min\{b,t\}$.

Lemma 5.1.2. ([17, Lemma A.1]) *Let $1 < p < \infty$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a differentiable convex function. Then*

$$\begin{aligned}|a-b|^{p-2}(a-b)\left[c\ |\phi'(a)|^{p-2}\phi'(a) - t\ |\phi'(b)|^{p-2}\phi'(b)\right]\\ \geq |\phi(a)-\phi(b)|^{p-2}(\phi(a)-\phi(b))(c-t),\end{aligned}$$

for every $a,b \in \mathbb{R}$ and every $c,t \geq 0$.

We also recall [53, Theorem 1.1], which deals with the regularity of the solution for the following problem:

$$\left.\begin{aligned}(-\Delta)_p^s u &= g \quad x \in \Omega,\\ u &= 0, \quad x \in \mathbb{R}^N \setminus \Omega.\end{aligned}\right\} \tag{5.1.4}$$

Proposition 5.1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary and $2 \leq p < \infty$, $s \in (0,1)$ such that $sp < N$. Assume that $g \in L^\infty(\Omega)$. Then there exist constants C and α, both positive and depending upon s,p,N,Ω such that, any weak solution $u \in W_0^{s,p}(\Omega)$ of (5.1.4) satisfies*

$$\|u\|_{C_d^{0,\alpha}(\overline{\Omega})} \leq C\|g\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}.$$

In the next result, we derive a-priori L^∞ bound on the weak solution of (5.0.1).

Lemma 5.1.3. *Let the assumptions in Theorem 5.1.1 hold. Then any weak solution $u \in W_0^{s,p}(\Omega)$ of (5.0.1) belongs to $L^\infty(\Omega)$. Moreover, there exist two positive constants C_**

and C^* depending upon s, p, μ, N, Ω such that

$$\|u\|_{L^\infty(\Omega)} \leq (C_*)^{\frac{1}{p^*_{\mu,s} - p}} (C^*)^{\frac{p-1}{\sqrt{p}(\sqrt{p^*_{\mu,s}} - \sqrt{p})}} \|u\|_{L^{p^*_s}(\Omega)}.$$

Proof. First from the given assumption on μ, we have $p^*_{\mu,s} > p$. Now for every $0 < \epsilon << 1$, we define the smooth convex Lipschitz function

$$h_\epsilon(t) = (\epsilon^2 + t^2)^{\frac{1}{2}}$$

and take the test function $\phi = \psi |h'_\epsilon(u)|^{p-2} h'_\epsilon(u)$ in (5.0.2), where $\psi \in C_c^\infty(\Omega), \psi > 0$. In addition, by choosing $a = u(x)$, $b = u(y)$, $c = \psi(x)$, and $t = \psi(y)$ in Lemma 5.1.2, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|h_\epsilon(u(x)) - h_\epsilon(u(y))|^{p-2}(h_\epsilon(u(x)) - h_\epsilon(u(y)))(\psi(x) - \psi(y))}{|x-y|^{N+sp}} dxdy \\ \leq \int_\Omega \int_\Omega \frac{|F(y,u)|\, |f(x,u)|}{|x-y|^\mu} |h'_\epsilon(u(x))|^{p-1} \psi(x)\, dxdy. \tag{5.1.5}$$

Since $h_\epsilon(t)$ converges to $h(t) = |t|$ as $\epsilon \to 0^+$ and $|h'_\epsilon(t)| \leq 1$, using Fatou's lemma in (5.1.5), we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big||u(x)| - |u(y)|\Big|^{p-2}\Big(|u(x)| - |u(y)|\Big)\Big(\psi(x) - \psi(y)\Big)}{|x-y|^{N+sp}} dxdy \\ \leq \int_\Omega \int_\Omega \frac{|F(y,u)|\, |f(x,u)|}{|x-y|^\mu} \psi(x)\, dxdy. \tag{5.1.6}$$

Since $C_c^\infty(\Omega)$ is dense in $W_0^{s,p}(\Omega)$, (5.1.6) holds true, for $0 \leq \psi \in W_0^{s,p}(\Omega)$. Next, for $l \in \mathbb{N}$, we define

$$u_l = \min\{l, |u(x)|\}.$$

Clearly $u_l \in W_0^{s,p}(\Omega)$. For $k \geq 1$, let us set

$$\beta := kp - p + 1.$$

So, $\beta > 1$. Choosing $\psi = u_l^\beta$ in (5.1.6) and using Lemma 5.1.1, we obtain

$$\frac{\beta p^p}{(\beta+p-1)^p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(u_l(x))^{\frac{\beta+p-1}{p}}-(u_l(y))^{\frac{\beta+p-1}{p}}\right|^p}{|x-y|^{N+sp}}dxdy \leq \int_\Omega\int_\Omega\frac{|F(y,u)|\,|f(x,u)|}{|x-y|^\mu}(u_l(x))^\beta\,dxdy. \tag{5.1.7}$$

By observing that

$$\frac{1}{\beta}\left(\frac{\beta+p-1}{p}\right)^p \leq \left(\frac{\beta+p-1}{p}\right)^{p-1}, \quad \text{for large } \beta$$

and using the the relation $k = \frac{\beta+p-1}{p}$ along with continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$ (see Proposition 1.2.3), from (5.1.7), we get

$$\left\|u_l^k\right\|^p_{L^{p_s^*}(\Omega)} \leq \frac{(k)^{p-1}}{S_s^p}\int_\Omega\int_\Omega\frac{|F(y,u)|\,|f(x,u)|}{|x-y|^\mu}(u_l(x))^\beta\,dxdy, \tag{5.1.8}$$

where the best Sobolev constant S_s is as given in (1.2.7). Now we will estimate the right-hand side in (5.1.8). Using the inequality (1.2.12), (**H**), Proposition 1.3.1 with $q = t = \frac{2N}{2N-\mu}$, and the fact $u_l \leq |u|$, we deduce

$$\begin{aligned}
&\int_\Omega\int_\Omega\frac{|F(y,u)|\,|f(x,u)|}{|x-y|^\mu}(u_l(x))^\beta\,dxdy\\
&\leq C(N,p,\mu)\|F(\cdot,u(\cdot))\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\left(\int_\Omega\left(|f(x,u)|\,|u_l(x)|^\beta\right)^{\frac{2N}{2N-\mu}}\right)^{\frac{2N-\mu}{2N}}\\
&\leq C_1\left(\|u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}+\left\||u|^{p^*_{\mu,s}}\right\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\right)\\
&\quad\times\left(\int_\Omega|u_l|^{\frac{2N\beta}{2N-\mu}}dx+\int_\Omega\left(|u|^{p^*_{\mu,s}-2}|u\,u_l^\beta|\right)^{\frac{2N}{2N-\mu}}dx\right)^{\frac{2N-\mu}{2N}}\\
&= C_1\left(\|u\|_{L^{\frac{p_s^*}{p^*_{\mu,s}}}(\Omega)}+\|u\|^{p^*_{\mu,s}}_{L^{p_s^*}(\Omega)}\right)\left[\int_{\Omega\cap\{|u|<\Lambda\}}|u_l|^{\beta\frac{p_s^*}{p^*_{\mu,s}}}dx+\int_{\Omega\cap\{|u|\geq\Lambda\}}|u_l|^{\beta\frac{p_s^*}{p^*_{\mu,s}}}dx\right.\\
&\quad\left.+\int_{\Omega\cap\{|u|<\Lambda\}}\left(|u|^{p^*_{\mu,s}-2}|u\,u_l^\beta|\right)^{\frac{p_s^*}{p^*_{\mu,s}}}dx+\int_{\Omega\cap\{|u|\geq\Lambda\}}\left(|u|^{p^*_{\mu,s}-2}|u\,u_l^\beta|\right)^{\frac{p_s^*}{p^*_{\mu,s}}}dx\right]^{\frac{p^*_{\mu,s}}{p_s^*}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \tilde{C}\Bigg[\left(\int_{\Omega\cap\{|u|<\Lambda\}} |u|^{\beta\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}} + \left(\int_{\Omega\cap\{|u|\geq\Lambda\}} \left(|u|^{p_{\mu,s}^*+\beta-1}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}} \\
&\quad + \left(\int_{\Omega\cap\{|u|<\Lambda\}} \left(|u|^{p_{\mu,s}^*+\beta-1}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}} + \left(\int_{\Omega\cap\{|u|\geq\Lambda\}} \left(|u|^{p_{\mu,s}^*+\beta-1}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}}\Bigg] \\
&\leq \tilde{C}\Bigg[\Lambda^{1-p}\left(\int_{\Omega\cap\{|u|<\Lambda\}} \left(|u|^{p+\beta-1}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}} \\
&\qquad\qquad + \Lambda^{p_{\mu,s}^*-p}\left(\int_{\Omega\cap\{|u|<\Lambda\}} \left(|u|^{p+\beta-1}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}} \\
&\qquad\qquad\quad + 2\left(\int_{\Omega\cap\{|u|\geq\Lambda\}} \left(|u|^{p_{\mu,s}^*-p}\,|u|^{p+\beta-1}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}}\Bigg] \\
&\leq \tilde{C}\Big[(\Lambda^{1-p}+\Lambda^{p_{\mu,s}^*-p})\,\|u\|^{kp}_{L^{\frac{kpp_s^*}{p_{\mu,s}^*}}(\Omega)} + 2\left(\int_{\Omega\cap\{|u|\geq\Lambda\}} \left(|u|^{p_{\mu,s}^*-p}\,|u|^{kp}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}}\Big],
\end{aligned} \tag{5.1.9}
$$

where $\Lambda > 1$ will be chosen later, $C_1 > 0$ is a constant, and $\tilde{C} = C\left(\|u\|_{L^{\frac{p_s^*}{p_{\mu,s}^*}}(\Omega)} + \|u\|^{p_{\mu,s}^*}_{L^{p_s^*}(\Omega)}\right)$. By plugging (5.1.9) into (5.1.8) and applying Fatou's lemma, we get

$$
\begin{aligned}
\|u\|^{kp}_{L^{kp_s^*}(\Omega)} \leq \tilde{C}\frac{k^{p-1}}{(S_s)^p}\Bigg[&(\Lambda^{1-p}+\Lambda^{p_{\mu,s}^*-p})\,\|u\|^{kp}_{L^{\frac{kpp_s^*}{p_{\mu,s}^*}}(\Omega)} \\
&+ 2\left(\int_{\Omega\cap\{|u|\geq\Lambda\}} \left(|u|^{p_{\mu,s}^*-p}\,|u|^{kp}\right)^{\frac{p_s^*}{p_{\mu,s}^*}}\,dx\right)^{\frac{p_{\mu,s}^*}{p_s^*}}\Bigg].
\end{aligned} \tag{5.1.10}
$$

Now we estimate the second term on the right-hand side in (5.1.10). For this, using Hölder inequality for constant exponents, we obtain

$$\begin{aligned}
&\left(\int_{\Omega\cap\{|u|\geq\Lambda\}}\left(|u|^{p^*_{\mu,s}-p}|u|^{kp}\right)^{\frac{p^*_s}{p^*_{\mu,s}}}dx\right)^{\frac{p^*_{\mu,s}}{p^*_s}}\\
&\leq C_2\left(\int_{\Omega\cap\{|u|\geq\Lambda\}}\left(|u|^{(p^*_{\mu,s}-p)\frac{p^*_s}{p^*_{\mu,s}}}\right)^{\frac{p^*_{\mu,s}}{p^*_{\mu,s}-p}}dx\right)^{\frac{p^*_{\mu,s}}{p^*_s}\cdot\frac{p^*_{\mu,s}-p}{p^*_{\mu,s}}}\\
&\quad\times\left(\int_{\Omega\cap\{|u|\geq\Lambda\}}\left(|u|^{kp\cdot\frac{p^*_s}{p^*_{\mu,s}}}\right)^{\frac{p^*_{\mu,s}}{p}}dx\right)^{\frac{p^*_{\mu,s}}{p^*_s}\cdot\frac{p}{p^*_{\mu,s}}}\\
&\leq C_2\left(\int_{\Omega\cap\{|u|\geq\Lambda\}}|u|^{p^*_s}dx\right)^{\frac{p^*_{\mu,s}-p}{p^*_s}}\left(\int_\Omega|u|^{kp^*_s}dx\right)^{\frac{p}{p^*_s}}\\
&=C(\Lambda)\|u\|^{kp}_{L^{kp^*_s}(\Omega)},
\end{aligned}\tag{5.1.11}$$

where $C_2>0$ is a constant and $C(\Lambda)=C_2\left(\int_{\Omega\cap\{|u|\geq\Lambda\}}|u|^{p^*_s}dx\right)^{\frac{p^*_{\mu,s}-p}{p^*_s}}$. Combining (5.1.10) and (5.1.11), we have

$$\|u\|^{kp}_{L^{kp^*_s}(\Omega)}\leq\tilde{C}\frac{k^{p-1}}{(S_s)^p}\left[(\Lambda^{1-p}+\Lambda^{p^*_{\mu,s}-p})\|u\|_{L^{\frac{kpp^*_s}{p^*_{\mu,s}}}(\Omega)}+2C(\Lambda)\|u\|^{kp}_{L^{kp^*_s}(\Omega)}\right].\tag{5.1.12}$$

Now by applying Lebesgue dominated convergence theorem in (5.1.11), we can choose $\Lambda>1$ large enough such that $C(\Lambda)<\dfrac{(S_s)^p}{4\tilde{C}(k)^{p-1}}$ and hence, from (5.1.12), it follows that

$$\|u\|_{L^{kp^*_s}(\Omega)}\leq\left(C_*^{\frac{1}{k}}\right)^{\frac{1}{p}}(k^{\frac{1}{k}})^{\frac{p-1}{p}}\|u\|_{L^{\frac{kpp^*_s}{p^*_{\mu,s}}}(\Omega)},\tag{5.1.13}$$

where $C_*=\dfrac{2\tilde{C}\ (\Lambda^{1-p}+\Lambda^{p^*_{\mu,s}-p})}{(S_s)^p}>1$. Next, we start bootstrap argument on (5.1.13). Choose $k=k_1:=\frac{p^*_{\mu,s}}{p}>1$ as the first iteration. Thus, (5.1.13) yields that

$$\|u\|_{L^{k_1p^*_s}(\Omega)}\leq\left(C_*^{\frac{1}{k_1}}\right)^{\frac{1}{p}}(k_1^{\frac{1}{k_1}})^{\frac{p-1}{p}}\|u\|_{L^{p^*_s}(\Omega)}.\tag{5.1.14}$$

Again by taking $k = k_2 := k_1 \frac{p^*_{\mu,s}}{p}$ as the second iteration in (5.1.13) and then inserting (5.1.14) in (5.1.13), we get

$$\begin{aligned} \|u\|_{L^{k_2 p^*_s}(\Omega)} &\leq \left(C_*^{\frac{1}{k_2}}\right)^{\frac{1}{p}} \left[(k_2)^{\frac{1}{k_2}}\right]^{\frac{p-1}{p}} \|u\|_{L^{k_1 p^*_s}(\Omega)} \\ &\leq \left(C_*^{\frac{1}{k_1}+\frac{1}{k_2}}\right)^{\frac{1}{p}} \left[(k_1)^{\frac{1}{k_1}}.(k_2)^{\frac{1}{k_2}}\right]^{\frac{p-1}{p}} \|u\|_{L^{p^*_s}(\Omega)}. \end{aligned} \tag{5.1.15}$$

In this fashion, taking $k = k_n := k_{n-1} \frac{p^*_{\mu,s}}{p}$ as the n^{th} iteration and iterating for n times, we obtain

$$\begin{aligned} \|u\|_{L^{k_n p^*_s}(\Omega)} &\leq \left(C_*^{\frac{1}{k_n}}\right)^{\frac{1}{p}} \left[(k_n)^{\frac{1}{k_n}}\right]^{\frac{p-1}{p}} \|u\|_{L^{k_{n-1} p^*_s}(\Omega)} \\ &\leq \left(C_*^{\dfrac{1}{k_1}+\dfrac{1}{k_2}\cdots+\dfrac{1}{k_n}}\right)^{\frac{1}{p}} \left[(k_1)^{\dfrac{1}{k_1}}.(k_2)^{\dfrac{1}{k_2}}\cdots(k_n)^{\dfrac{1}{k_n}}\right]^{\frac{p-1}{p}} \|u\|_{L^{p^*_s}(\Omega)} \\ &= \left(C_*^{\sum_{j=1}^{n}\dfrac{1}{k_j}}\right)^{\frac{1}{p}} \left(\prod_{j=1}^{n}\left(k_j^{\sqrt{1/k_j}}\right)^{\sqrt{1/k_j}}\right)^{\frac{p-1}{p}} \|u\|_{L^{p^*_s}(\Omega)}, \end{aligned} \tag{5.1.16}$$

where $k_j = \left(\frac{p^*_{\mu,s}}{p}\right)^j$. Since $\frac{p^*_{\mu,s}}{p} > 1$, we have $k_j^{\sqrt{1/k_j}} > 1$, for all $j \in \mathbb{N}$ and

$$\lim_{j \to +\infty} k_j^{\sqrt{1/k_j}} = 1.$$

Hence, it follows that there exists a constant $C^* > 1$, independent of $j, n \in \mathbb{N}$ such that $k_j^{\sqrt{1/k_j}} < C^*$ and thus, (5.1.16) gives

$$\|u\|_{L^{k_n p^*_s}(\Omega)} \leq \left(C_*^{\sum_{j=1}^{n}\dfrac{1}{k_j}}\right)^{\frac{1}{p}} \left(C^{*\sum_{j=1}^{n}\sqrt{1/k_j}}\right)^{\frac{p-1}{p}} \|u\|_{L^{p^*_s}(\Omega)}. \tag{5.1.17}$$

Observe that

$$\sum_{j=1}^{\infty}\frac{1}{k_j}=\sum_{j=1}^{n}\left(\frac{p}{p^*_{\mu,s}}\right)^j=\frac{p/p^*_{\mu,s}}{1-p/p^*_{\mu,s}}=\frac{p}{p^*_{\mu,s}-p}$$

and

$$\sum_{j=1}^{\infty}\frac{1}{\sqrt{k_j}}=\sum_{j=1}^{n}\left(\sqrt{\frac{p}{p^*_{\mu,s}}}\right)^j=\frac{\sqrt{p}}{\sqrt{p^*_{\mu,s}}-\sqrt{p}},$$

from (5.1.17), we get that

$$\|u\|_{L^{\nu_n}(\Omega)}\leq (C_*)^{\frac{1}{p^*_{\mu,s}-p}}(C^*)^{\frac{p-1}{\sqrt{p}(\sqrt{p^*_{\mu,s}}-\sqrt{p})}}\|u\|_{L^{p^*_s}(\Omega)}, \tag{5.1.18}$$

where $\nu_n:=k_np^*_s$. Note that, $\nu_n\to+\infty$ as $n\to+\infty$. Therefore, we claim that

$$\|u\|_{L^\infty(\Omega)}=\|u\|_{L^\infty(\Omega)}\leq (C_*)^{\frac{1}{p^*_{\mu,s}-p}}(C^*)^{\frac{p-1}{\sqrt{p}(\sqrt{p^*_{\mu,s}}-\sqrt{p})}}\|u\|_{L^{p^*_s}(\Omega)}. \tag{5.1.19}$$

Indeed, if not, let us assume $\|u\|_{L^\infty(\Omega)}>C_3\|u\|_{L^{p^*_s}(\Omega)}$, where

$$C_3=(C_*)^{\frac{1}{p^*_{\mu,s}-p}}(C^*)^{\frac{p-1}{\sqrt{p}(\sqrt{p^*_{\mu,s}}-\sqrt{p})}}.$$

Then there exists $C_4>0$ and a subset $\mathcal{S}$ of Ω with $meas(\mathcal{S})>0$ such that

$$u(x)>C_3\|u\|_{L^{p^*_s}(\Omega)}+C_4,\quad \text{for } x\in\mathcal{S}.$$

The above implies

$$\begin{aligned}\liminf_{\nu_n\to+\infty}\left(\int_\Omega|u(x)|^{\nu_n}dx\right)^{\frac{1}{\nu_n}}&\geq\liminf_{\nu_n\to+\infty}\left(\int_{\mathcal{S}}|u(x)|^{\nu_n}dx\right)^{\frac{1}{\nu_n}}\\&\geq\liminf_{\nu_n\to+\infty}\left(C_3\|u\|_{L^{p^*_s}(\Omega)}+C_4\right)(meas(\mathcal{S}))^{\frac{1}{\nu_n}}\\&=\liminf_{\nu_n\to+\infty}\left(C_3\|u\|_{L^{p^*_s}(\Omega)}+C_4\right),\end{aligned}$$

a contradiction to (5.1.18). Therefore, (5.1.19) holds and hence, $u\in L^\infty(\Omega)$. □

Proof of Theorem 5.1.1. Now for proving Hölder regularity, we first claim that

$$\left(\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right) f(x,u) \in L^\infty(\Omega). \tag{5.1.20}$$

Indeed, by Lemma 5.1.3, we get $u \in L^\infty(\Omega)$ and thus, by (**H**), we have $f(\cdot, u(\cdot))$, $F(\cdot, u(\cdot)) \in L^\infty(\Omega)$, which imply that

$$\begin{aligned}\left|\int_\Omega \frac{F(y,u)}{|x-y|^\mu}dy\right| &\leq \|F(\cdot,u(\cdot))\|_{L^\infty(\Omega)}\left[\int_{\Omega\cap\{|x-y|<1\}} \frac{dy}{|x-y|^\mu} + \int_{\Omega\cap\{|x-y|\geq 1\}} \frac{dy}{|x-y|^\mu}\right]\\ &\leq \|F(\cdot,u(\cdot))\|_{L^\infty(\Omega)}\left[\int_{\Omega\cap\{\overline{r}\leq 1\}} \overline{r}^{N-1-\mu}d\overline{r} + |\Omega|\right] < \infty,\end{aligned}$$

and since $0<\mu<N$, (5.1.20) holds. Now by applying Proposition 5.1.1, we finally can conclude that there exists some $\alpha \in [0,s)$ such that $u \in C^{0,\alpha}(\overline{\Omega})$. Hence, the proof is complete. □

5.2 Sobolev versus Hölder minimizers

In this section, using the L^∞ regularity and Hölder regularity of the weak solutions to (5.0.1), as obtained in Theorem 5.1.1, we study the following result about the Sobolev versus Hölder minimizers for the energy functional J:

Theorem 5.2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary and $2 \leq p < \infty$, $s \in (0,1)$ such that $sp < N$. Suppose that* (**H**) *holds. Then for any $w_0 \in W_0^{s,p}(\Omega)$, the following assertions are equivalent:*

- (i) *There exists $\varrho > 0$ such that $J(w_0 + w) \geq J(w_0)$, for all $w \in W_0^{s,p}(\Omega) \cap C_d^0(\overline{\Omega})$ with $\|w\|_{C_d^0(\overline{\Omega})} \leq \varrho$.*
- (ii) *There exists $\delta > 0$ such that $J(w_0+w) \geq J(w_0)$, for all $w \in W_0^{s,p}(\Omega)$ with $\|w\|_{s,p} \leq \delta$.*

Proof. We divide the proof into two cases depending on r in (**H**) .

(a) *Critical Case: $r = p^*_{\mu,s}$ in* (**H**)*:*

First, we show that (i) implies (ii). From (i), it follows that $\langle J'(w_0), \phi\rangle_{W_0^{s,p}(\Omega)} \geq 0$, for all $\phi \in W_0^{s,p}(\Omega) \cap C_d^0(\overline{\Omega})$, where $\langle\cdot,\cdot\rangle_{W_0^{s,p}(\Omega)}$ denotes the dual pairing between $W_0^{s,p}(\Omega)$ and

$W^{-s,p'}(\Omega)$. Since $W_0^{s,p}(\Omega) \cap C_d^0(\overline{\Omega})$ is a dense subspace of $W_0^{s,p}(\Omega)$, we have

$$\langle J'(w_0), \phi \rangle_{W_0^{s,p}(\Omega)} = 0, \quad \text{for all } \phi \in W_0^{s,p}(\Omega).$$

Therefore, by Theorem 5.1.1, we infer that $w_0 \in C_d^0(\overline{\Omega}) \cap L^\infty(\Omega)$. Here we argue by contradiction. Suppose (ii) does not hold. Then there exists a sequence, say $\{\tilde{w}_n\}$ in $W_0^{s,p}(\Omega)$ such that $\tilde{w}_n \to w_0$ strongly in $W_0^{s,p}(\Omega)$ as $n \to +\infty$ and $J(\tilde{w}_n) < J(w_0)$, for all $n \in \mathbb{N}$. Now we introduce a suitable truncation to the nonlinearity f to handle its critical growth (in the sense of Hardy-Littlewood-Sobolev inequality (1.3.15)). For each $j \in \mathbb{N}$, we define $f_j : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$f_j(x,t) = \max\{f(x,-j), \min\{f(x,j), f(x,t)\}\}$$

We define the corresponding truncated energy functional $J_j : W_0^{s,p}(\Omega) \to \mathbb{R}$ as

$$J_j(u) = \frac{\|u\|_{s,p}^p}{p} - \int_\Omega \int_\Omega \frac{F_j(x,u)F_j(y,u)}{|x-y|^\mu} dxdy,$$

where $F_j(x,t) = \int_0^t f_j(x,\tau)\, d\tau$. One can check that $J_j \in C^1(W_0^{s,p}(\Omega), \mathbb{R})$. Note that, by (**H**), we get that

$$|f_j(x,t)| \le \tilde{C}_j := K_0\,(1 + j^{p^*_{\mu,s}-1}), \quad |F_j(x,t)| \le K_j(1+|t|)$$

are of sub-critical growth (in the sense of Hardy-Littlewood-Sobolev inequality (1.3.15)), where $\tilde{C}_j, K_j,\ j \in \mathbb{N}$, are positive constants. Now by applying Lebesgue dominated convergence theorem, for all $u \in W_0^{s,p}(\Omega)$, we have

$$\lim_{j\to+\infty} F_j(x,u) = \lim_{j\to+\infty} \int_0^u f_j(x,t)\, dt = F(x,u). \tag{5.2.21}$$

For fixed $n \in \mathbb{N}$ and $0 < \xi_n < J(w_0) - J(\tilde{w}_n)$, using (5.2.21), we can find $j_n > C(\|w_0\|_{L^\infty(\Omega)}) + 1 > 0$ such that

$$\Big| \int_\Omega \int_\Omega \frac{F_{j_n}(x,\tilde{w}_n)F_{j_n}(y,\tilde{w}_n)}{|x-y|^\mu}\, dxdy - \int_\Omega \int_\Omega \frac{F(x,\tilde{w}_n)F(y,\tilde{w}_n)}{|x-y|^\mu}\, dxdy \Big| < \xi_n. \tag{5.2.22}$$

For all $n \in \mathbb{N}$, let us set

$$\sigma_n := \|\tilde{w}_n - w_0\|_{L^{p_s^*}(\Omega)}, \qquad B_{\sigma_n} := \{u \in W_0^{s,p}(\Omega) : \|u - w_0\|_{L^{p_s^*}(\Omega)} \leq \sigma_n\}.$$

Using the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$ (see Proposition 1.2.3), we have $\sigma_n \to 0$ as $n \to +\infty$. Now B_{σ_n} is a closed convex subset of $W_0^{s,p}(\Omega)$ and hence, weakly closed subset of $W_0^{s,p}(\Omega)$. Therefore, J_{j_n} is sequentially weakly lower semi-continuous and coercive on B_{σ_n}. Thus, for any $n \in \mathbb{N}$, there exists $w_n \in B_{\sigma_n}$ such that

$$J_{j_n}(w_n) = \inf_{u \in B_{\sigma_n}} J_{j_n}(u). \tag{5.2.23}$$

In view of (5.2.22) and by the choice of ξ_n and j_n, we get

$$J_{j_n}(w_n) \leq J_{j_n}(\tilde{w}_n) \leq J(\tilde{w}_n) + \xi_n < J(w_0) = J_{j_n}(w_0). \tag{5.2.24}$$

We claim that there exists $m_n \geq 0$ such that

$$(-\Delta)_p^s\, w_n + m_n(w_n - w_0)^{p_s^*-1} = \left(\int_\Omega \frac{F_{j_n}(y,w_n)}{|x-y|^\mu} dy \right) f_{j_n}(x,w_n). \tag{5.2.25}$$

Since $w_n \in B_{\sigma_n}$, while proving our claim we encounter two possible cases as following:
Case $\|w_n - w_0\|_{L^{p_s^*}(\Omega)} < \sigma_n$**:** Then (5.2.23) yields that w_n is a local minimizer of J_{j_n} in $W_0^{s,p}(\Omega)$ and hence, $J'_{j_n}(w_n) = 0$. Thus, (5.2.25) holds with $m_n = 0$.

Case $\|w_n - w_0\|_{L^{p_s^*}(\Omega)} = \sigma_n$**:** We define the functional $\mathcal{I} : W_0^{s,p}(\Omega) \to \mathbb{R}$ as

$$\mathcal{I}(u) := \frac{\|u - w_0\|_{L^{p_s^*}(\Omega)}^{p_s^*}}{p_s^*}.$$

One can check that $\mathcal{I} \in C^1(W_0^{s,p}(\Omega), \mathbb{R})$. Next, we consider the following C^1-manifold in $W_0^{s,p}(\Omega)$:

$$\mathscr{M}_n := \Big\{ u \in W_0^{s,p}(\Omega) : \mathcal{I}(u) := \frac{\sigma_n^{p_s^*}}{p_s^*} \Big\}.$$

Now (5.2.23) yields that w_n is a global minimizer of J_{j_n} on $\mathscr{M}_n$. Therefore, by the method of Lagrange's multipliers, there exists $m_n \in \mathbb{R}$ such that in $W^{-s,p'}(\Omega)$,

$$J'_{j_n}(w_n) + m_n \mathcal{I}'(w_n) = 0,$$

the PDE form of which is (5.2.25). Furthermore, using (5.2.23) again, we can derive

$$m_n = -\frac{\langle J'_{j_n}(w_n), w_0 - w_n \rangle_{W_0^{s,p}(\Omega)}}{\langle \mathcal{I}'(w_n), w_0 - w_n \rangle_{W_0^{s,p}(\Omega)}} \geq 0$$

such that, possibly $m_n \to +\infty$ as $n \to +\infty$. Hence, our claim is proved. As per the construction, $w_n \to w_0$ strongly in $L^{p_s^*}(\Omega)$ as $n \to +\infty$. Moreover, applying Lemma 5.1.3 for (5.2.25), we have that $w_n \in L^\infty(\Omega)$, for all $n \in \mathbb{N}$. Next, we show

$$w_n \to w_0 \ \text{ strongly in } L^\infty(\Omega) \text{ as } n \to +\infty. \tag{5.2.26}$$

Subtracting (5.0.1) from (5.2.25), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} &(-\Delta)_p^s w_n - (-\Delta)_p^s w_0 + m_n (w_n - w_0)^{p_s^*-1} \\ &\quad = \int_\Omega \left(\frac{F_{j_n}(y, w_n)}{|x-y|^\mu} dy \right) f_{j_n}(x, w_n) - \int_\Omega \left(\frac{F(y, w_0)}{|x-y|^\mu} dy \right) f(x, w_0). \end{aligned} \tag{5.2.27}$$

We set $v_n := w_n - w_0 \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$. For $\beta := kp - p + 1,\ k \geq 1$, taking $v_n^\beta \in W_0^{s,p}(\Omega)$ as a test function in the weak formulation of (5.2.27), we get

$$\begin{aligned} &\langle (-\Delta)_p^s w_n - (-\Delta)_p^s w_0, v_n^\beta \rangle_{W_0^{s,p}(\Omega)} + m_n \int_\Omega |v_n|^{p_s^* + \beta - 1}\, dx \\ &\quad = \int_\Omega \int_\Omega \frac{F_{j_n}(y, w_n) f_{j_n}(x, w_n) - F(y, w_0) f(x, w_0)}{|x-y|^\mu} (v_n(x))^\beta \, dx dy. \end{aligned} \tag{5.2.28}$$

In the rest of the proof, we consider $C > 0$ to be positive constant which may vary from line to line but is independent of n, j, k, β, v_n, w_n.
Using Proposition 1.2.3 and Lemma 1.2.6, from the left-hand side of (5.2.28), we deduce that

$$\left[\int_\Omega |v_n|^{kp_s^*} \, dx \right]^{\frac{p}{p_s^*}} \leq C \left\| v_n^{\frac{p+\beta-1}{p}} \right\|_{s,p}^p \leq C \, \beta^{p-1} \langle (-\Delta)_p^s w_n - (-\Delta)_p^s w_0, v_n^\beta \rangle_{W_0^{s,p}(\Omega)}. \quad (5.2.29)$$

Now we estimate the right-hand side in (5.2.28). For that, first observe that by the construction, for $(x,t) \in \Omega \times \mathbb{R}$, we have $|F_{j_n}(x,t)| \leq |F(x,t)|$ and $|f_{j_n}(x,t)| \leq |f(x,t)|$. Therefore, using (**H**) and the fact $w_0 \in L^\infty(\Omega)$ together with (1.2.12), we have

$$\begin{aligned}
&|F_{j_n}(y, w_n) f_{j_n}(x, w_n) - F(y, w_0) f(x, w_0)| \\
&\leq |F(y, w_n)| \, |f(x, w_n)| + |F(y, w_0)| \, |f(x, w_0)| \\
&\leq C \left[1 + \left(1 + |(v_n + w_0)(y)|^{p^*_{\mu,s}} \right) \left(1 + |(v_n + w_0)(x)|^{p^*_{\mu,s}-1} \right) \right] \\
&\leq 2^{2p^*_{\mu,s}} C \Big[1 + \Big\{ \left(1 + |v_n(y)|^{p^*_{\mu,s}} + |w_0(y)|^{p^*_{\mu,s}} \right) \\
&\qquad\qquad \times \left(1 + |v_n(x)|^{p^*_{\mu,s}-1} + |w_0(x)|^{p^*_{\mu,s}-1} \right) \Big\} \Big] \\
&\leq \tilde{K} \left[1 + \left(1 + |v_n(y)|^{p^*_{\mu,s}} \right) \left(1 + |v_n(x)|^{p^*_{\mu,s}-1} \right) \right] \qquad (5.2.30)
\end{aligned}$$

for some constant $\tilde{K} > 0$ (independent of j, n). Let us denote $\widehat{g}(x,t) := 1 + |t|^{p^*_{\mu,s}-1}$ and $\widehat{G}(x,t) := 1 + |t|^{p^*_{\mu,s}}$. Therefore, using (5.2.28)-(5.2.30) along with $m_n \geq 0$ and Proposition 1.3.1, for all $n \in \mathbb{N}$ and $k, \beta \geq 1$, we obtain

$$\begin{aligned}
&\|v_n\|_{L^{kp_s^*}(\Omega)}^{kp} \\
&\leq C \, \beta^{p-1} \Bigg[\left(\int_\Omega |v_n|^{\beta \frac{p_s^*}{p^*_{\mu,s}}} \, dx \right)^{\frac{p^*_{\mu,s}}{p_s^*}} \\
&\quad + \Bigg\{ \left(\int_\Omega |\widehat{g}(x, v_n) \, |v_n|^\beta|^{\frac{p_s^*}{p^*_{\mu,s}}} \, dx \right)^{\frac{p^*_{\mu,s}}{p_s^*}} \left(\int_\Omega |\widehat{G}(x, v_n)|^{\frac{p_s^*}{p^*_{\mu,s}}} \, dx \right)^{\frac{p^*_{\mu,s}}{p_s^*}} \Bigg\} \Bigg]. \qquad (5.2.31)
\end{aligned}$$

Next, using similar the arguments as obtained in the proof of Lemma 5.1.3, we deduce that

$$\|v_n\|_{L^\infty(\Omega)} \leq C\|v_n\|_{L^{p_s^*}(\Omega)},$$

which yields that $v_n \to 0$ strongly in $L^\infty(\Omega)$ being $v_n \to 0$ strongly in $L^{p_s^*}(\Omega)$ as $n \to +\infty$. Thus, (5.2.26) follows. Now for sufficiently large $n \in \mathbb{N}$, (5.2.25) can be rewritten as

$$(-\Delta)_p^s\, w_n = \left(\int_\Omega \frac{F(y,w_n)}{|x-y|^\mu}dy\right) f(x,w_n) - m_n(w_n - w_0)^{p_s^*-1} \text{ in } W^{-s,p'}(\Omega). \tag{5.2.32}$$

From (5.2.26), we infer that the sequence $\{w_n\}$ is bounded in $L^\infty(\Omega)$ and hence by (**H**), it follows that the sequence $\left\{\left(\int_\Omega \frac{F(y,w_n)}{|x-y|^\mu}dy\right) f(\cdot,w_n)\right\}$ is also uniformly bounded. Therefore, again taking v_n^β as test function in the weak formulation of (5.2.27) and applying Lemma 1.2.6, for all sufficiently large $n \in \mathbb{N}$, we achieve

$$\begin{aligned} m_n \int_\Omega |v_n|^{p_s^*+\beta-1}\,dx &\leq C\int_\Omega |v_n|^\beta\,dx \\ &\leq C\Big[\int_\Omega |v_n|^{p_s^*+\beta-1}\,dx\Big]^{\frac{\beta}{p_s^*+\beta-1}} (meas(\Omega))^{\frac{p_s^*-1}{p_s^*+\beta-1}}, \end{aligned}$$

The above implies

$$m_n\,\|v_n\|^{p_s^*-1}_{L^{p_s^*+\beta-1}(\Omega)} \leq C(meas(\Omega))^{\frac{p_s^*-1}{p_s^*+\beta-1}}.$$

Letting $k \to +\infty$, we have $\beta \to +\infty$ and hence, from the last relation, we deduce that

$$m_n\,\|v_n\|^{p_s^*-1}_{L^\infty(\Omega)} \leq C,$$

that is, $\{m_n\,(w_n - w_0)^{p_s^*-1}\}$ is a bounded sequence in $L^\infty(\Omega)$. Hence, from (5.2.32) and Lemma 5.1.1 we infer that $\{w_n\}$ is bounded in $C_d^\alpha(\overline{\Omega})$. By the compact embedding $C_d^\alpha(\overline{\Omega}) \hookrightarrow\hookrightarrow C_d^0(\bar{\Omega})$, passing to a sub-sequence, still denoted by $\{w_n\}$, we have $w_n \to w_0$ strongly in $C_d^0(\overline{\Omega})$ as $n \to +\infty$. So, for all $n \in \mathbb{N}$ large enough, we infer that $\|w_n - w_0\|_{C_d^0} \leq \varrho$. Also, since $\{w_n\}$ is bounded in $L^\infty(\Omega)$, we get that $J_{j_n}(w_n) = J(w_n)$ for sufficiently large $n \in \mathbb{N}$. Hence, from (5.2.24), it follows that $J(w_n) < J(w_0)$. But this contradicts (i). Therefore, (i) implies (ii).

Next, we show (ii) implies (i). By (ii), we have $\langle J'(w_0), v\rangle_{W_0^{s,p}(\Omega)} = 0$, for all $v \in W_0^{s,p}(\Omega)$.

Therefore, Lemma 5.1.3 and Proposition 5.1.1 imply that $w_0 \in C_d^0(\overline{\Omega})$. Supposing the contrary, let there exist a sequence $\{\tilde{u}_n\}$ in $W_0^{s,p}(\Omega) \cap C_d^0(\overline{\Omega})$ such that $\tilde{u}_n \to w_0$ in $C_d^0(\overline{\Omega})$ as $n \to +\infty$ and $J(\tilde{u}_n) < J(w_0)$, for all $n \in \mathbb{N}$. Thus, we have $\tilde{u}_n \to w_0$ strongly in $L^\infty(\Omega)$ as $n \to +\infty$. Hence, by the continuity and (**H**), the sequence $\{F(\cdot, \tilde{u}_n)\}$ is bounded in $L^\infty(\Omega)$ and

$$F(\cdot, \tilde{u}_n) \to F(\cdot, w_0) \text{ strongly in } L^\infty(\Omega) \text{ as } n \to +\infty. \tag{5.2.33}$$

Observe that

$$\begin{aligned} I_0 :&= \int_\Omega \int_\Omega \frac{F(x, \tilde{u}_n)F(y, \tilde{u}_n)}{|x-y|^\mu} dxdy - \int_\Omega \int_\Omega \frac{F(x, w_0)F(y, w_0)}{|x-y|^\mu} dxdy \\ &= \underline{I}_1 + \underline{I}_2, \end{aligned} \tag{5.2.34}$$

where

$$\begin{aligned} \underline{I}_1 &:= \int_\Omega \int_\Omega \frac{F(x, \tilde{u}_n)\left[F(y, \tilde{u}_n) - F(y, w_0)\right]}{|x-y|^\mu} dxdy; \\ \underline{I}_2 &:= \int_\Omega \int_\Omega \frac{\left[F(x, \tilde{u}_n) - F(x, w_0)\right]F(y, w_0)}{|x-y|^\mu} dxdy. \end{aligned}$$

By Proposition 1.3.1 and (5.2.33), we observe that

$$\begin{aligned} |\underline{I}_1| &\leq C\|F(\cdot, \tilde{u}_n)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \|F(\cdot, \tilde{u}_n) - F(\cdot, w_0)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \\ &\to 0 \quad \text{ as } n \to +\infty. \end{aligned} \tag{5.2.35}$$

Arguing similarly, as $n \to +\infty$ we obtain $|\underline{I}_2| \to 0$, which together with (5.2.34) and (5.2.35) implies that $I_0 \to 0$, that is,

$$\int_\Omega \int_\Omega \frac{F(x, \tilde{u}_n)F(y, \tilde{u}_n)}{|x-y|^\mu} dxdy \to \int_\Omega \int_\Omega \frac{F(x, w_0)F(y, w_0)}{|x-y|^\mu} dxdy \text{ as } n \to +\infty. \tag{5.2.36}$$

Using (5.2.36), we obtain

$$\begin{aligned}\limsup_{n\to+\infty}\frac{\|\tilde{u}_n\|_{s,p}^p}{p} &= \limsup_{n\to+\infty}\left[J(\tilde{u}_n)+\frac{1}{2}\int_\Omega\int_\Omega\frac{F(x,\tilde{u}_n)F(y,\tilde{u}_n)}{|x-y|^\mu}dxdy\right]\\ &\leq J(w_0)+\frac{1}{2}\int_\Omega\int_\Omega\frac{F(x,w_0)F(y,w_0)}{|x-y|^\mu}dxdy\\ &=\frac{\|w_0\|_{s,p}^p}{p},\end{aligned}\tag{5.2.37}$$

that is, $\{\tilde{u}_n\}$ is bounded in $W_0^{s,p}(\Omega)$. Hence, passing to a sub-sequence, still denoted by $\{\tilde{u}_n\}$, we have $\tilde{u}_n \rightharpoonup w_0$ weakly in $W_0^{s,p}(\Omega)$ as $n\to+\infty$. Therefore, using the lower semi-continuity property of the norm, we get

$$\liminf_{n\to+\infty}\|\tilde{u}_n\|_{s,p}\geq\|w_0\|_{s,p},\tag{5.2.38}$$

which together with the fact that $W_0^{s,p}(\Omega)$ is uniformly convex, gives us $\|\tilde{u}_n\|_{s,p}\to\|w_0\|_{s,p}$ as $n\to+\infty$. Consequently, by Theorem 1.2.1, $\|\tilde{u}_n-w_0\|_{s,p}\to 0$ as $n\to+\infty$. So for $n\in\mathbb{N}$, sufficiently large enough, we have $\|\tilde{u}_n-w_0\|_{s,p}\leq\delta$ with $J(\tilde{u}_n)<J(w_0)$, which contradicts (ii). Therefore, (i) holds.

(b) *Sub-critical Case*: $r<p^*_{\mu,s}$ in $(\mathbf{H})$:

In this case, the proof follows using the similar arguments as in the *Critical Case*, discussed above, by taking $f_j(x,t)=f(x,t)$ for each $j\in\mathbb{N}$ and using the compact embedding $W_0^{s,p}(\Omega)\hookrightarrow\hookrightarrow L^\gamma(\Omega),\ \ 1<\gamma<p_s^*$ (see Proposition 1.2.3). This completes the proof of the theorem. □

5.3 An application

Finally, we discuss the following theorem as an application of Theorem 5.2.1. First we consider the following definition:

Definition 5.3.1. *Let* $u\in W_0^{s,p}(\Omega)$. *Then*

(i) *u is a super-solution of* (5.0.1), *if* $u\geq 0$ *in* $\mathbb{R}^N\setminus\Omega$ *and for all* $v\in W_0^{s,p}(\Omega)$ *with*

$v \geq 0$ a.e. in Ω, we have

$$\langle (-\Delta)_s^p u, v \rangle_{W_0^{s,p}(\Omega)} \geq \int_\Omega \int_\Omega \frac{F(y,u) f(x,u)}{|x-y|^\mu} v(x);$$

(*ii*) *u is a sub-solution of* (5.0.1), *if $u \leq 0$ in $\mathbb{R}^N \setminus \Omega$ and for all $v \in W_0^{s,p}(\Omega)$ with $v \geq 0$ a.e. in Ω, we have*

$$\langle (-\Delta)_s^p u, v \rangle_{W_0^{s,p}(\Omega)} \leq \int_\Omega \int_\Omega \frac{F(y,u) f(x,u)}{|x-y|^\mu} v(x).$$

Theorem 5.3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary and $2 < p < \infty$, $s \in (0,1)$ with $sp < N$. Let* (**H**) *hold and $f(x,\cdot)$ be non decreasing function in $\mathbb{R}$, for all $x \in \Omega$. Suppose $\underline{v}, \overline{v} \in W_0^{s,p}(\Omega)$ are a weak sub-solution and a weak super-solution, respectively, to* (5.0.1), *which are not solutions. Then there exists a solution $v_0 \in W_0^{s,p}(\Omega)$ to* (5.0.1) *such that $\underline{v} \leq v_0 \leq \overline{v}$ a.e in Ω and v_0 is a local minimizer of J in $W_0^{s,p}(\Omega)$.*

Proof. First, we define the following truncated function $\widehat{f}(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$\widehat{f}(x,t) =: \begin{cases} f(x, \underline{v}(x)) & \text{if } t \leq \underline{v}(x), \\ f(x,t) & \text{if } \underline{v}(x) < t < \overline{v}(x), \\ f(x, \overline{v}(x)) & \text{if } t \geq \overline{v}(x). \end{cases}$$

Clearly by (**H**), $\widehat{f}$ is continuous and there exist constants $C_5, C_6 > 0$ such that

$$\left.\begin{aligned} |\widehat{f}(x,t)| &\leq C_5(1 + |\underline{v}|^{r-1} + |\overline{v}|^{r-1}), \\ |\widehat{F}(x,t)| &= \left| \int_0^t \widehat{f}(x,\tau) d\tau \right| \leq C_6(1 + |\underline{v}|^{r-1} + |\overline{v}|^{r-1})|t|, \end{aligned}\right\} \tag{5.3.39}$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$. We define the operator $T : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ as

$$\langle T(u), v \rangle_{W_0^{s,p}(\Omega)} = -\int_\Omega \int_\Omega \frac{\widehat{F}(y,u) \widehat{f}(x,u) v(x)}{|x-y|^\mu} \, dxdy, \quad \text{for all } u, v \in W_0^{s,p}(\Omega).$$

In view of Proposition 1.3.1 and (5.3.39), T is well posed. We show that T is strongly continuous (see [25, Definition 2.95 (*iv*)]). Indeed, let $\{u_n\}$ be a sequence in $W_0^{s,p}(\Omega)$ such that $u_n \rightharpoonup u_*$ weakly in $W_0^{s,p}(\Omega)$ as $n \to +\infty$. Now using Proposition 1.2.3 passing to a sub-sequence, still denoted by $\{u_n\}$, we have $u_n \to u_*$ strongly in $L^\gamma(\Omega)$, $u_n(x) \to u_*(x)$ as $n \to +\infty$ and hence, there exists some $\tilde{h} \in L^{\frac{2N}{2N-\mu}}(\Omega)$ with $|u_n(x)| \le \tilde{h}(x)$, for a.e. $x \in \Omega$. Moreover, for a.e. $x \in \Omega$, we have $\widehat{f}(x, u_n(x)) \to \widehat{f}(x, u_*(x))$ and $\widehat{F}(x, u_n(x)) \to \widehat{F}(x, u_*(x))$ as $n \to +\infty$ and the sequence $\{\widehat{F}(\cdot, u_n)\}$ is bounded in $L^{\frac{2N}{2N-\mu}}(\Omega)$. Thus, $\widehat{F}(\cdot, u_n) \rightharpoonup \widehat{F}(\cdot, u_*)$ weakly in $L^{\frac{2N}{2N-\mu}}(\Omega)$ as $n \to +\infty$. Consider the linear continuous map $\Sigma : L^{\frac{2N}{2N-\mu}}(\Omega) \to \mathbb{R}$, defined as

$$\Sigma(w) = \int_\Omega \int_\Omega \frac{w(y)\widehat{f}(x, u_*)v(x)}{|x-y|^\mu} dxdy, \quad \text{for } v \in L^{\frac{2N}{2N-\mu}}(\Omega).$$

Then by letting $n \to +\infty$, we get $\Sigma(\widehat{F}(y, u_n)) \to \Sigma(\widehat{F}(y, u_*))$ in $\mathbb{R}$, that is,

$$\int_\Omega \int_\Omega \frac{\widehat{F}(y, u_n)\widehat{f}(x, u_*)v(x)}{|x-y|^\mu} dxdy \to \int_\Omega \int_\Omega \frac{\widehat{F}(y, u_*)\widehat{f}(x, u_*)v(x)}{|x-y|^\mu} dxdy. \tag{5.3.40}$$

Therefore, for any $v \in W_0^{s,p}(\Omega)$, using Proposition 1.2.3, Proposition 1.3.1, (5.3.39), (5.3.40) and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
|\langle T(u_n) - T(u_*), v\rangle_{W_0^{s,p}(\Omega)}| &\le \left| \int_\Omega \int_\Omega \frac{\widehat{F}(y, u_n)\left[(\widehat{f}(x, u_n) - \widehat{f}(x, u_*))v\right]}{|x-y|^\mu} dxdy \right| \\
&\quad + \left| \int_\Omega \int_\Omega \frac{\left[\widehat{F}(y, u_n) - \widehat{F}(y, u_*)\right]\widehat{f}(x, u_*)v}{|x-y|^\mu} dxdy \right| \\
&\le \|\widehat{F}(\cdot, u_n)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \|(\widehat{f}(\cdot, u_n) - \widehat{f}(\cdot, u_*))v\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \\
&\quad + \left| \int_\Omega \int_\Omega \frac{\left[\widehat{F}(y, u_n) - \widehat{F}(y, u_*)\right]\widehat{f}(x, u_*)v}{|x-y|^\mu} dxdy \right| \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{5.3.41}$$

This implies $T(u_n) \to T(u_*)$ in $W^{-s,p'}(\Omega)$ as $n \to +\infty$ and thus, T is strongly continuous. Hence, [25, Lemma 2.98 (ii)] yields that T is pseudomonotone.

Next, using Lemma 1.2.6 and arguing as in [41, Lemma 3.2] we get that $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is pseudomonotone. Therefore, $(-\Delta)_p^s + T$ is a pseudomonotone operator.

On the other hand, again using the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$, $1 < \gamma < p_s^*$

(Proposition 1.2.3), Proposition 1.3.1, (5.3.39), we deduce that

$$\begin{aligned}
\|T(u)\|_{-s,p'} &= \sup_{\|v\|_{s,p}\le 1} \int_\Omega\int_\Omega \frac{\widehat{F}(y,u)\widehat{f}(x,u)v(x)}{|x-y|^\mu}\,dxdy\\
&\le C_7\|\widehat{f}(\cdot,u)v\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\|\widehat{F}(\cdot,u)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\\
&\le C_8\left(1+\|\underline{v}\|^{r-1}_{L^{r\frac{2N}{2N-\mu}}(\Omega)}+\|\overline{v}\|^{r-1}_{L^{r\frac{2N}{2N-\mu}}(\Omega)}\right)^2,
\end{aligned}$$

where C_7, C_8 are positive constants and do not depend on u, v. Again arguing as in [41, Lemma 3.2], it follows that $(-\Delta)^s_p + T$ is bounded.

Finally, we show that $(-\Delta)^s_p + T$ is coercive. Indeed, using Proposition 1.2.3, Proposition 1.3.1, and (5.3.39), for all $u \in W^{s,p}_0(\Omega)\setminus\{0\}$, we have

$$\begin{aligned}
&\frac{\langle(-\Delta)^s_p(u)+T(u),u\rangle_{W^{s,p}_0(\Omega)}}{\|u\|_{s,p}}\\
&= \|u\|^{p-1}_{s,p} - \frac{1}{\|u\|_{s,p}}\int_\Omega\int_\Omega\frac{\widehat{F}(y,u)\widehat{f}(x,u)u(x)}{|x-y|^\mu}\,dxdy\\
&\ge \|u\|^{p-1}_{s,p} - \frac{C_9}{\|u\|_{s,p}}\Big(\|u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}+\|\underline{v}\|^{r-1}_{L^{r\frac{2N}{2N-\mu}}(\Omega)}\|u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\\
&\qquad\qquad + \|\overline{v}\|^{r-1}_{L^{r\frac{2N}{2N-\mu}}(\Omega)}\|u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\Big)^2\\
&\ge \|u\|^{p-1}_{s,p} - C_{10}\|u\|^2_{s,p} \to +\infty \qquad \text{as } \|u\|_{s,p}\to+\infty,
\end{aligned}$$

where $C_9, C_{10} > 0$ are the constants, independent of u. Applying [25, Theorem 2.99], we get that there exists a solution $v_0 \in W^{s,p}_0(\Omega)$ to the following equation:

$$(-\Delta)^s_p(u) + T(u) = 0 \text{ in } W^{-s,p'}(\Omega), \tag{5.3.42}$$

that is,

$$\widehat{J}(v_0) = \min_{u\in W^{s,p}_0(\Omega)} \widehat{J}(u),$$

where $\widehat{J} \in C^1(W^{s,p}_0(\Omega),\mathbb{R})$ is the energy functional associated to (5.3.42) and is defined as

$$\widehat{J}(u) = \frac{\|u\|^p_{s,p}}{p} - \int_\Omega\int_\Omega \frac{\widehat{F}(x,u)\widehat{F}(y,u)}{|x-y|^\mu}dxdy.$$

Now we claim that

$$\underline{v} \leq v_0 \leq \overline{v} \quad \text{in } \Omega. \tag{5.3.43}$$

Observe that, (5.3.43) holds in $\mathbb{R}^N \setminus \Omega$. Taking $(v_0 - \overline{v})^+ \in W_0^{s,p}(\Omega)$ as test function in the weak formulation of (5.3.42) and using the fact that $\overline{v}$ is a super-solution of (5.0.1), we deduce

$$\begin{aligned}
\langle (-\Delta)_p^s u, (v_0 - \overline{v})^+ \rangle_{W_0^{s,p}(\Omega)} &= \int_\Omega \int_\Omega \frac{\widehat{F}(y, v_0)}{|x-y|^\mu} \widehat{f}(x, v_0)(v_0 - \overline{v})^+(x)\, dxdy \\
&= \int_\Omega \int_\Omega \frac{F(y, \overline{v})}{|x-y|^\mu} f(x, \overline{v})(v_0 - \overline{v})^+(x)\, dxdy \\
&\leq \langle (-\Delta)_p^s \overline{v}, (v_0 - \overline{v})^+ \rangle_{W_0^{s,p}(\Omega)}.
\end{aligned}$$

Therefore, we get

$$\langle (-\Delta)_p^s v_0 - (-\Delta)_p^s \overline{v}, (v_0 - \overline{v})^+ \rangle_{W_0^{s,p}(\Omega)} \leq 0. \tag{5.3.44}$$

Using (1.2.11) along with (5.3.44), we obtain

$$\begin{aligned}
&\|(v_0 - \overline{v})^+\|_{s,p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(v_0(x) - \overline{v}(x))^+ - (v_0(y) - \overline{v}(y))^+|^p}{|x-y|^{N+sp}}\, dxdy \\
&\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[(v_0 - \overline{v})(x) - (v_0 - \overline{v})(y)]^{p-1}[(v_0 - \overline{v})^+(x) - (v_0 - \overline{v})^+(y)]}{|x-y|^{N+sp}}\, dxdy \\
&\leq C_p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_0(x) - v_0(y))^{p-1}[(v_0 - \overline{v})^+(x) - (v_0 - \overline{v})^+(y)]}{|x-y|^{N+sp}}\, dxdy \\
&\quad - C_p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\overline{v}(x) - \overline{v}(y))^{p-1}[(v_0 - \overline{v})^+(x) - (v_0 - \overline{v})^+(y)]}{|x-y|^{N+sp}}\, dxdy \\
&= C_p \langle (-\Delta)_p^s v_0 - (-\Delta)_p^s \overline{v}, (v_0 - \overline{v})^+ \rangle_{W_0^{s,p}(\Omega)} \\
&\leq 0,
\end{aligned}$$

and thus, we have $(v_0 - \overline{v})^+ = 0$ which implies that $v_0 \leq \overline{v}$ in Ω. Similarly, using $(v_0 - \underline{v})^- \in W_0^{s,p}(\Omega)$ as a test function in the weak formulation of (5.3.42), we can prove that $v_0 \geq \underline{v}$ and hence, (5.3.43) holds true.

Next, we show that v_0 is a local minimizer of J in $W_0^{s,p}(\Omega)$. By exploiting the monotonicity and the definition of $\widehat{f}$ along with (5.3.43), we obtain that

$$(-\Delta)_p^s(\overline{v} - v_0) \geq \left(\int_\Omega \frac{F(y,\overline{v})}{|x-y|^\mu}\,dy\right) f(x,\overline{v}) - \left(\int_\Omega \frac{\widehat{F}(y,v_0)}{|x-y|^\mu}\,dy\right) \widehat{f}(x,v_0)$$
$$\geq 0$$

weakly in Ω. Also, by definition, $\overline{v} - v_0 \geq 0$ in $\mathbb{R}^N \setminus \Omega$. In view of the fact that $\overline{v}$ is not a solution to (5.0.1), we have $v_0 \neq \overline{v}$. Therefore, by the strong maximum principle for fractional p-Laplacian (Lemma 1.2.7), it follows that $\overline{v} - v_0 > 0$ in Ω. Similarly $v_0 - \underline{v} > 0$ in Ω. Thus, it follows that $v_0 \in W_0^{s,p}(\Omega)$ is a weak solution to (5.0.1). Again, using Lemma 1.2.7 and Hopf's lemma for fractional p-Laplacian (see [29, Theorem 1.5]), we can have $\overline{v} - v_0 \geq Rd^s$ in Ω, for some $R > 0$, where d is the distance function, defined as in (1.2.9). Likewise, $v_0 - \underline{v} \geq Rd^s$ in Ω, for some $R > 0$. Also, from Proposition 5.1.1 and Lemma 5.1.3, we get that $v_0 \in C_d^{0,\alpha}(\overline{\Omega})$. Let us denote

$$\bar{B}^d_{R/2}(v_0) := \{u \in W_0^{s,p}(\Omega) : \|u - v_0\|_{0,d} \leq R/2\}.$$

For each $w \in \bar{B}^d_{R/2}(v_0)$, we have

$$\frac{\overline{v} - w}{d^s} = \frac{\overline{v} - v_0}{d^s} + \frac{v_0 - w}{d^s} \geq R - \frac{R}{2} = \frac{R}{2} \quad \text{in } \overline{\Omega}.$$

The last relation implies that $\overline{v} - w > 0$ in Ω. On a similar note, we have $w - \underline{v} > 0$ in Ω. Therefore, in $W_0^{s,p}(\Omega) \cap \bar{B}^d_{R/2}(v_0)$, $\widehat{J}$ agrees with J and thus, v_0 emerges as a local minimizer of J in $W_0^{s,p}(\Omega) \cap \bar{B}^d_{R/2}(v_0)$. Finally, Theorem 5.2.1 implies that v_0 is a local minimizer of J in $W_0^{s,p}(\Omega)$. Hence, the proof of the theorem is complete. □

5.4 Conclusion

In this chapter, we have established a global uniform bound and global Hölder bound on the weak solutions of a class of doubly non-local equations involving fractional p-Laplacian

($1 < p < \infty$) with the Choquard type nonlinearity up to critical growth assumption in the sense of Hardy-Littlewood-Sobolev inequality. Then for $p \geq 2$, we have proved the analogous 'Sobolev versus Hölder minimizers' type result. Finally, we have applied these two results to show the existence of a weak solution of (5.0.1) which is also a local minimizer of the associated energy functional. Here it is worthy to note that the non-Hilbert nature of the associated solution space have restricted us applying the many important tools and results as applied for the case $p = 2$. Also, to the best of our knowledge, this is the first work dealing with the regularity result for p-fractional critical Choquard equation.

The fine boundary regularity for the non-local problem involving fractional p-Laplacian is still an open question for $1 < p < 2$. Provided the answer for this is affirmative, Theorem 5.2.1 could be proved using the similar approach for the singular case ($1 < p < 2$) using [54, Lemma 2.4] instead of Lemma 1.2.6. Also, studying the regularity of the solutions of (5.0.1) by considering the critical perturbation of the form $|u|^{p_s^*-2}u$ in the right-hand side will be an interesting problem. Here we remark that the approaches, used in [43, 46, 73] for studying these type of problem, for the case $p = 2$, will not be applicable for any general $1 < p < \infty$ and hence, sought for developing new techniques to address such problems.

Next, considering the variable order and variable exponents in (5.0.1) opens new problem to be explored. $\square$

6

Non-local elliptic system involving fractional $p(\cdot)$-Laplacian

In this chapter, we consider the following non-local elliptic system with variable exponents:

$$\left.\begin{aligned} (-\Delta)^s_{p(\cdot)}u &= \lambda\ a(x)|u|^{r(x)-2}u + \tfrac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}, \quad x\in\Omega, \\ (-\Delta)^s_{p(\cdot)}v &= \zeta\ b(x)|v|^{r(x)-2}v + \tfrac{\alpha(x)}{\alpha(x)+\beta(x)}c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}, \quad x\in\Omega, \\ u=v &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x\in\mathbb{R}^N\setminus\Omega, \end{aligned}\right\} \tag{6.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $\lambda, \zeta > 0$ are the real parameters, $s\in(0,1)$, $p(\cdot,\cdot)$ satisfies (P_1) with $sp^+ < N$. We assume that the variable exponents $r(\cdot), \alpha(\cdot), \beta(\cdot)$ and the weight functions $a(\cdot), b(\cdot), c(\cdot)$ satisfy the following hypotheses:

(A_1) The variable exponents $r(\cdot), \alpha(\cdot), \beta(\cdot) \in C_+(\overline{\Omega})$ verify the following:

$$1 < r^- \leq r^+ < p^- \leq p^+ < \alpha^- + \beta^- \leq \alpha^+ + \beta^+ < (p_s^*)^-,$$

where $(p_s^*)^- := \frac{Np^-}{N-sp^-}$.

(A_2) It holds that

$$\frac{p^-}{\alpha^+ + \beta^+} < \left(\frac{p^- - r^+}{\alpha^+ + \beta^+ - r^+}\right)\left(\frac{\alpha^- + \beta^- - r^-}{p^+ - r^-}\right).$$

Observe that, when all the exponents are constants, the above inequality is equivalent to the condition $0 < p < \alpha + \beta$.

(A_3) The non-negative weight functions $a(\cdot), b(\cdot) \in L^{r_*(\cdot)}(\Omega)$, where

$$r_*(x) = \frac{\alpha(x) + \beta(x)}{\alpha(x) + \beta(x) - r(x)}.$$

(A_4) The non-negative weight function $c \in L^\infty(\Omega)$.

For brevity, we still denote $X_0 := X_0^{s,\overline{p}(\cdot),p(\cdot,\cdot)}(\Omega)$ by taking $\beta(x) = \overline{p}(x)$ and $s(\cdot,\cdot) = s$, constant (see Chapter 2). We define $\mathscr{E} := X_0 \times X_0$ as the solution space corresponding to (6.0.1), equipped with the norm $\|(u,v)\| = \max\{\|u\|_{X_0}, \|v\|_{X_0}\}$. Clearly $(\mathscr{E}, \|(\cdot,\cdot)\|)$ is a reflexive, separable Banach space. Now we define the weak solution of (6.0.1) in the functional space $\mathscr{E}$ as follows:

Definition 6.0.1. *We say that $(u,v) \in \mathscr{E}$ is a weak solution of* (6.0.1) *if for all $(\phi,\psi) \in \mathscr{E}$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp(x,y)}}dxdy \\
&= \int_\Omega \Big(\lambda a(x)|u|^{r(x)-2}u\phi + \zeta b(x)|v|^{r(x)-2}v\psi\Big)dx \\
&\quad + \int_\Omega \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}\phi dx \\
&\quad + \int_\Omega \frac{\beta(x)}{\alpha(x)+\beta(x)} c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}\psi dx \quad .
\end{aligned} \tag{6.0.2}$$

The energy functional $J_{\lambda,\zeta} : \mathscr{E} \to \mathbb{R}$ associated to (6.0.1) is defined as

$$\begin{aligned} J_{\lambda,\zeta}(u,v) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \\ &- \int_\Omega \frac{1}{r(x)} \Big(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\Big) dx \\ &- \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)}|v|^{\beta(x)} dx. \end{aligned} \tag{6.0.3}$$

By a direct computation, it can be checked that $J_{\lambda,\zeta} \in C^1(\mathscr{E},\mathbb{R})$ and for any $(\phi,\psi) \in \mathscr{E}$, we have

$$\begin{aligned} \langle J'_{\lambda,\zeta}(u,v),(\phi,\psi)\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp(x,y)}} dxdy \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp(x,y)}} dxdy \\ &- \int_\Omega \Big(\lambda a(x)|u|^{r(x)-2}u\phi + \zeta b(x)|v|^{r(x)-2}v\psi\Big) dx \\ &- \int_\Omega \frac{\alpha(x)}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)}\phi dx \\ &- \int_\Omega \frac{\beta(x)}{\alpha(x)+\beta(x)} c(x)|v|^{\alpha(x)-2}v|u|^{\beta(x)}\psi dx. \end{aligned} \tag{6.0.4}$$

Therefore, the weak solutions of (6.0.1) are the critical points of the functional $J_{\lambda,\zeta}$. Note that $J_{\lambda,\zeta}$ is not bounded below on $\mathscr{E}$. The main result in this chapter is stated as follows:

Theorem 6.0.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $s \in (0,1)$, and $p(\cdot,\cdot)$ satisfy (P_1) with $sp^+ < N$. Assume that the hypotheses $(A_1)-(A_4)$ hold. Then there exists a positive constant $\Lambda_0 = \Lambda_0(N,s,p,r,\alpha,\beta,a,b,c,\Omega)$ such that, for any pair of positive parameters (λ,ζ) with $\lambda+\zeta < \Lambda_0$, (6.0.1) admits at least two non-trivial, non-negative weak solutions.*

6.1 Nehari manifold and fibering map analysis

The nehari manifold for the functional $J_{\lambda,\zeta}$ is a subset of $\mathscr{E}$, defined as

$$\mathscr{N}_{\lambda,\zeta} := \{(u,v) \in \mathscr{E} \setminus \{(0,0)\} \ : \ \langle J'_{\lambda,\zeta}(u,v),(u,v)\rangle_{\mathscr{E}} = 0\}.$$

Now by (6.0.4), $(u,v) \in \mathscr{N}_{\lambda,\zeta}$ if and only if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\ &-\int_\Omega \left(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\right)dx - \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx = 0. \end{aligned} \tag{6.1.5}$$

The Nehari manifold is closely associated with the behavior of the fibering maps $\varphi_{u,v} : \mathbb{R}^+ \to \mathbb{R}$, defined as $\varphi_{u,v}(t) = J_{\lambda,\zeta}(tu,tv)$, where $(u,v) \in \mathscr{E}$. These maps were first given by Drabek and Pohozaev in [33] and are discussed in details in [21] and [49]. For $(u,v) \in \mathscr{E}$, we have the following:

$$\begin{aligned}\varphi_{u,v}(t) &= J_{\lambda,\zeta}(tu,tv)\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{t^{p(x,y)}}{p(x,y)}\left(\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right)dxdy\\ &\quad - \int_\Omega \frac{t^{r(x)}}{r(x)}\left(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\right)dx - \int_\Omega \frac{t^{\alpha(x)+\beta(x)}}{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.\end{aligned} \tag{6.1.6}$$

$$\begin{aligned}\varphi'_{u,v}(t) &= \langle J'_{\lambda,\zeta}(tu,tv),(u,v)\rangle_{\mathscr{E}}\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} t^{p(x,y)-1}\left(\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right)dxdy\\ &\quad - \int_\Omega t^{r(x)-1}\left(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\right)dx - \int_\Omega t^{\alpha(x)+\beta(x)-1}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.\end{aligned} \tag{6.1.7}$$

$$\begin{aligned}\varphi''_{u,v}(t) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (p(x,y)-1)t^{p(x,y)-2}\left(\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right)dxdy\\ &- \int_\Omega (r(x)-1)t^{r(x)-2}\left(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\right)dx\\ &- \int_\Omega (\alpha(x)+\beta(x)-1)t^{\alpha(x)+\beta(x)-2}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.\end{aligned} \tag{6.1.8}$$

Hence, for any $(u,v) \in \mathscr{N}_{\lambda,\zeta}$, from (6.1.5), (6.1.7), and (6.1.8), we deduce

$$\begin{aligned}\varphi''_{u,v}(1) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} p(x,y)\left(\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right\}dxdy\\ &- \int_\Omega r(x)\Big(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\Big)dx\\ &- \int_\Omega (\alpha(x)+\beta(x))c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx. \qquad (6.1.9)\end{aligned}$$

Now using the fact that $\varphi'_{u,v}(t) = \langle J'_{\lambda,\zeta}(tu,tv),(u,v)\rangle_{\mathscr{E}}$, we can see that $(tu,tv) \in \mathscr{N}_{\lambda,\zeta}$ if and only if $\varphi'_{u,v}(t) = 0$. In particular, $(u,v) \in \mathscr{N}_{\lambda,\zeta}$ if and only if $\varphi'_{u,v}(1) = 0$. Thus, it is natural to split $\mathscr{N}_{\lambda,\zeta}$ into three parts corresponding to the points of local minima, local maxima and inflection of the function $\varphi_{u,v}$ as following:

$$\mathscr{N}^+_{\lambda,\zeta} := \{(u,v) \in \mathscr{N}_{\lambda,\zeta} \ : \ \varphi''_{u,v}(1) > 0\} = \{(tu,tv) \in \mathscr{E}\backslash\{(0,0)\} \ : \ \varphi'_{u,v}(t) = 0,\ \varphi''_{u,v}(1) > 0\};$$

$$\mathscr{N}^-_{\lambda,\zeta} := \{(u,v) \in \mathscr{N}_{\lambda,\zeta} \ : \ \varphi''_{u,v}(1) < 0\} = \{(tu,tv) \in \mathscr{E}\backslash\{(0,0)\} \ : \ \varphi'_{u,v}(t) = 0,\ \varphi''_{u,v}(1) < 0\};$$

$$\mathscr{N}^0_{\lambda,\zeta} := \{(u,v) \in \mathscr{N}_{\lambda,\zeta} \ : \ \varphi''_{u,v}(1) = 0\} = \{(tu,tv) \in \mathscr{E}\backslash\{(0,0)\} \ : \ \varphi'_{u,v}(t) = 0,\ \varphi''_{u,v}(1) = 0\}.$$

6.1.1 Functional settings

For a given pair of functions $(u,v) \in \mathscr{E}$, we set

$$P(u,v) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right\}dxdy,$$

$$Q(u,v) := \int_\Omega \Big(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\Big)dx$$

and

$$R(u,v) := \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

In the next lemma, we obtain some estimations on P, Q and R.

Lemma 6.1.1. *Let $(u,v) \in \mathscr{E}$. Then we have the following:*

$$(i)\quad \begin{cases} \|(u,v)\|^{p^+}, & \|(u,v)\|<1 \\ \|(u,v)\|^{p^-}, & \|(u,v)\|>1 \end{cases} \leq P(u,v) \leq \begin{cases} 2\|(u,v)\|^{p^-}, & \|(u,v)\|<1 \\ 2\|(u,v)\|^{p^+}, & \|(u,v)\|>1. \end{cases}$$

(ii) *There exists a constant* $\overline{C}_1 = \overline{C}_1(N,s,p,r,\alpha,\beta,a,b,\Omega) > 0$ *such that*

$$Q(u,v) \leq \overline{C}_1(\lambda+\zeta)\max\{\|(u,v)\|^{r^-},\ \|(u,v)\|^{r^+}\}.$$

(iii) *There exists a constant* $\overline{C}_2 = \overline{C}_2(N,s,p,\alpha,\beta,c,\Omega) > 1$ *such that*

$$R(u,v) \leq \overline{C}_2 \max\{\|(u,v)\|^{r^-},\ \|(u,v)\|^{r^+}\}.$$

Proof. (i) Clearly $P(u,v) = \rho_{X_0}(u) + \rho_{X_0}(v)$. Hence, we have

$$\max\{\rho_{X_0}(u), \rho_{X_0}(v)\} \leq P(u,v) \leq 2\max\{\rho_{X_0}(u), \rho_{X_0}(v)\}. \tag{6.1.10}$$

For $\|(u,v)\| > 1$, we have two cases.

Case I. $\|u\|_{X_0} > 1$ and $\|v\|_{X_0} > 1$: Then from Lemma 2.1.4 , we obtain

$$\|u\|_{X_0}^{p^-} < \rho_{X_0}(u) < \|u\|_{X_0}^{p^+} \quad \text{and} \quad \|v\|_{X_0}^{p^-} < \rho_{X_0}(v) < \|v\|_{X_0}^{p^+}. \tag{6.1.11}$$

Thus, from (6.1.10) and (6.1.11), we get

$$P(u,v) \leq 2\max\{\|u\|_{X_0}^{p^+},\ \|v\|_{X_0}^{p^+}\} = 2\|(u,v)\|^{p^+},$$

$$P(u,v) \geq \max\{\|u\|_{X_0}^{p^-},\ \|v\|_{X_0}^{p^-}\} = \|(u,v)\|^{p^-}.$$

Case II. Without loss of generality, let $\|v\|_{X_0} < 1 < \|u\|_{X_0}$: Then $\|(u,v)\| = \|u\|_{X_0}$. Now Lemma 2.1.4 implies that

$$\|u\|_{X_0}^{p^-} < \rho_{X_0}(u) < \|u\|_{X_0}^{p^+} \ \textit{and}\ \ \|v\|_{X_0}^{p^+} < \rho_{X_0}(v) < \|v\|_{X_0}^{p^-}. \tag{6.1.12}$$

Combining (6.1.10) and (6.1.12), we deduce

$$P(u,v) \leq 2\max\{\|u\|_{X_0}^{p^+}, \|v\|_{X_0}^{p^+}\} = 2\|(u,v)\|^{p^+} \text{ and}$$

$$P(u,v) \geq \max\{\|u\|_{X_0}^{p^-}, \|v\|_{X_0}^{p^-}\} = \|(u,v)\|^{p^-}.$$

Next, for $\|(u,v)\| < 1$, we have $\|u\|_{X_0} < 1$ and $\|v\|_{X_0} < 1$. By Lemma 2.1.4, we obtain

$$\|u\|_{X_0}^{p^+} < \rho_{X_0}(u) < \|u\|_{X_0}^{p^-} \ \ and \ \ \|v\|_{X_0}^{p^+} < \rho_{X_0}(v) < \|v\|_{X_0}^{p^-}. \tag{6.1.13}$$

Hence, from (6.1.10) and (6.1.13), it follows that $P(u,v) \leq 2\max\{\|u\|_{X_0}^{p^-}, \|v\|_{X_0}^{p^-}\} = 2\|(u,v)\|^{p^-}$ and $P(u,v) \geq \max\{\|u\|_{X_0}^{p^+}, \|v\|_{X_0}^{p^+}\} = \|(u,v)\|^{p^+}$. Thus, we get (i).

(ii) Using Lemma 1.2.3, Lemma 1.2.5, and Theorem 2.1.3, we obtain

$$\begin{aligned}
Q(u,v) &= \int_\Omega \Big(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\Big)dx \\
&\leq 2\lambda\|a\|_{L^{r_*(\cdot)}(\Omega)}\||u|^{q(\cdot)}\|_{L^{\frac{\alpha+\beta}{q}(\cdot)}(\Omega)} + 2\zeta\|b\|_{L^{r_*(\cdot)}(\Omega)}\||v|^{q(\cdot)}\|_{L^{\frac{\alpha+\beta}{q}(\cdot)}(\Omega)} \\
&\leq 2\lambda\|a\|_{L^{r_*(\cdot)}(\Omega)}\Big(\|u\|^{r^-}_{L^{(\alpha+\beta)(\cdot)}(\Omega)} + \|u\|^{r^+}_{L^{(\alpha+\beta)(\cdot)}(\Omega)}\Big) \\
&\quad + 2\zeta\|b\|_{L^{r_*(\cdot)}(\Omega)}\Big(\|v\|^{r^-}_{L^{(\alpha+\beta)(\cdot)}(\Omega)} + \|v\|^{r^+}_{L^{(\alpha+\beta)(\cdot)}(\Omega)}\Big) \\
&\leq \tilde{K}_1\Big[\lambda\Big(\|u\|_{X_0}^{r^-} + \|u\|_{X_0}^{r^+}\Big) + \zeta\Big(\|v\|_{X_0}^{r^-} + \|v\|_{X_0}^{r^+}\Big)\Big] \\
&\leq \overline{C}_1(\lambda+\zeta)\max\Big\{\|u\|_{X_0}^{r^-}, \|u\|_{X_0}^{r^+}, \ \|v\|_{X_0}^{r^-}, \ \|v\|_{X_0}^{r^+}\Big\} \\
&= \overline{C}_1(\lambda+\zeta)\max\Big\{\ \max\Big\{\|u\|_{X_0}^{r^-}, \ \|v\|_{X_0}^{r^-}\Big\}, \ \ \max\Big\{\|u\|_{X_0}^{r^+}, \ \|v\|_{X_0}^{r^+}\Big\}\Big\} \\
&= \overline{C}_1(\lambda+\zeta)\max\Big\{\|(u,v)\|^{r^-}, \ \|(u,v)\|^{r^+}\Big\},
\end{aligned}$$

where $\tilde{K}_1 = 2\Big(\|a\|_{L^{r_*(\cdot)}(\Omega)} + \|b\|_{L^{r_*(\cdot)}(\Omega)}\Big)\max\Big\{\big(C(N,s,p,\alpha,\beta,\Omega)\big)^{r^-}, \big(C(N,s,p,\alpha,\beta,\Omega)\big)^{r^+}\Big\}$ and $\overline{C}_1 = 4\tilde{K}_1$.

(iii) Using Young's inequality, Lemma 1.2.5 and Theorem 2.1.3, we deduce

$$\begin{aligned}
R(u,v) &= \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\leq \|c\|_{L^\infty(\Omega)}\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\leq \|c\|_{L^\infty(\Omega)}\int_\Omega \Big(\frac{\alpha(x)}{\alpha(x)+\beta(x)}|u|^{\alpha(x)+\beta(x)} + \frac{\beta(x)}{\alpha(x)+\beta(x)}|v|^{\alpha(x)+\beta(x)}\Big)dx \\
&\leq \|c\|_{L^\infty(\Omega)}\Big[\Big(\|u\|^{\alpha^+ + \beta^+}_{L^{(\alpha+\beta)(\cdot)}(\Omega)} + \|u\|^{\alpha^- + \beta^-}_{L^{(\alpha+\beta)(\cdot)}(\Omega)}\Big) \\
&\qquad\qquad + \Big(\|v\|^{\alpha^+ + \beta^+}_{L^{(\alpha+\beta)(\cdot)}(\Omega)} + \|v\|^{\alpha^- + \beta^-}_{L^{(\alpha+\beta)(\cdot)}(\Omega)}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \tilde{K}_2\Big[\Big(\|u\|_{X_0}^{\alpha^++\beta^+}+\|u\|_{X_0}^{\alpha^-+\beta^-}\Big)+\Big(\|v\|_{X_0}^{\alpha^++\beta^+}+\|v\|_{X_0}^{\alpha^-+\beta^-}\Big)\Big]\\
&\leq \overline{C}_2 \max\Big\{\|u\|_{X_0}^{\alpha^-+\beta^-},\ \|u\|_{X_0}^{\alpha^++\beta^+},\ \|v\|_{X_0}^{\alpha^-+\beta^-},\ \|v\|_{X_0}^{\alpha^++\beta^+}\Big\}\\
&= \overline{C}_2 \max\Big\{\max\Big\{\|u\|_{X_0}^{\alpha^-+\beta^-}, \|v\|_{X_0}^{\alpha^-+\beta^-}\Big\},\ \max\Big\{\|u\|_{X_0}^{\alpha^++\beta^+},\ \|v\|_{X_0}^{\alpha^++\beta^+}\Big\}\Big\}\\
&= \overline{C}_2 \max\Big\{\|(u,v)\|^{\alpha^-+\beta^-},\ \|(u,v)\|^{\alpha^++\beta^+}\Big\},
\end{aligned}
$$

where $\tilde{K}_2 = \|c\|_{L^\infty(\Omega)} \max\big\{\big(C(N,s,p,\alpha,\beta,\Omega)\big)^{\alpha^-+\beta^-}, \big(C(N,s,p,\alpha,\beta,\Omega)\big)^{\alpha^++\beta^+}\big\}$ and $\overline{C}_2 = 4\tilde{K}_2+1$. □

6.2 Some technical results

Lemma 6.2.1. *Let $\{u_n\},\{v_n\}$ be any two bounded sequences in X_0 and $c(\cdot),\alpha(\cdot),\beta(\cdot)$ be as in Theorem 6.0.1. Then*

$$\lim_{n\to\infty}\int_\Omega c(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx = \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

Proof. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in X_0 and X_0 is reflexive, up to subsequences, still denoted by $\{u_n\}$ and $\{v_n\}$, $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in X_0, respectively, as $n\to+\infty$. First we claim that

$$
\begin{aligned}
&\lim_{n\to+\infty}\int_\Omega |u_n-u|^{\alpha(x)}|v_n-v|^{\beta(x)}dx\\
&= \lim_{n\to\infty}\int_\Omega |u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx - \int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx
\end{aligned}
\tag{6.2.14}
$$

For $t\in(0,1)$, we note that

$$
\begin{aligned}
&\int_\Omega\int_0^1 \alpha(x)|u_n-tu|^{\alpha(x)-2}(u_n-tu)u|v_n|^{\beta(x)}dxdt\\
&\quad - \int_\Omega\int_0^1 \beta(x)|u_n-u|^{\alpha(x)}|v_n-tv|^{\beta(x)-2}v(v_n-tv)dxdt\\
&= \int_\Omega |u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx - \int_\Omega |u_n-u|^{\alpha(x)}|v_n-v|^{\beta(x)}dx.
\end{aligned}
\tag{6.2.15}
$$

Set

$$\left.\begin{aligned} f_n(x,t): \quad &= |u_n - tu|^{\alpha(x)-2}(u_n - tu)|v_n|^{\beta(x)} \\ g_n(x,t): \quad &= |u_n - u|^{\alpha(x)}|v_n - tv|^{\beta(x)-2}(v_n - tv). \end{aligned}\right\}$$

Now from the given assumptions, we have

$$\left.\begin{aligned} &f_n(x,t) \to (1-t)^{\alpha(x)-1}|u|^{\alpha(x)-2}u|v|^{\beta(x)} \text{ a.e. in } \mathbb{R}^N \times (0,1) \text{ as } n \to +\infty, \\ &g_n(x,t) \to 0 \text{ a.e. in } \mathbb{R}^N \times (0,1) \text{ as } n \to +\infty. \end{aligned}\right\} \tag{6.2.16}$$

Next, using Lemma 1.2.3 and Theorem 2.1.3, we obtain

$$\begin{aligned} &\int_\Omega \int_0^1 |f_n|^{\frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)-1}} dxdt \\ &\leq \| |u_n - \cdot u|^{\{(\alpha-1)(\frac{\alpha+\beta}{\alpha+\beta-1})\}(\cdot)} \|_{L^{\frac{\alpha(\cdot)+\beta(\cdot)-1}{\alpha(\cdot)-1}}(\Omega\times(0,1))} \| |v_n|^{\beta(\cdot)} \|_{L^{\frac{\alpha(\cdot)+\beta(\cdot)-1}{\beta(\cdot)}}(\Omega\times(0,1))} \\ &< C_1, \end{aligned} \tag{6.2.17}$$

and

$$\begin{aligned} &\int_\Omega \int_0^1 |g_n|^{\frac{\alpha(x)+\beta(x)}{\alpha(x)+\beta(x)-1}} dxdt \\ &\leq \| |u_n - u|^{\{\alpha(\frac{\alpha+\beta}{\alpha+\beta-1})\}(\cdot)} \|_{L^{\frac{\alpha(\cdot)+\beta(\cdot)-1}{\alpha(\cdot)}}(\Omega\times(0,1))} \| |v_n - \cdot v|^{(\beta-1)\frac{\alpha+\beta}{\alpha+\beta-1}(\cdot)} \|_{L^{\frac{\alpha(\cdot)+\beta(\cdot)-1}{\beta(\cdot)-1}}(\Omega\times(0,1))} \\ &< C_2, \end{aligned} \tag{6.2.18}$$

where C_1, C_2 are two positive constants independent of n, u_n, v_n. Hence, the sequences $\{f_n\}$ and $\{g_n\}$ are uniformly bounded in $L^{\frac{\alpha(\cdot)+\beta(\cdot)}{\alpha(\cdot)+\beta(\cdot)-1}}(\Omega \times (0,1))$ and thus we have, up to sub-sequences, as $n \to \infty$

$$\left.\begin{aligned} &f_n \rightharpoonup (1-t)^{\alpha(x)-1}|u|^{\alpha(x)-2}u|v|^{\beta(x)} \text{ weakly in } L^{\frac{\alpha(\cdot)+\beta(\cdot)}{\alpha(\cdot)+\beta(\cdot)-1}}(\Omega \times (0,1)) \text{ and} \\ &g_n \rightharpoonup 0 \text{ weakly in } L^{\frac{\alpha(\cdot)+\beta(\cdot)}{\alpha(\cdot)+\beta(\cdot)-1}}(\Omega \times (0,1)). \end{aligned}\right\} \tag{6.2.19}$$

Using (6.2.19), we deduce

$$\lim_{n\to+\infty}\int_\Omega\int_0^1 \alpha(x)f_n u\,dxdt = \lim_{n\to+\infty}\int_\Omega\int_0^1 \alpha(x)fu\,dxdt = \lim_{n\to+\infty}\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx, \tag{6.2.20}$$

and

$$\lim_{n\to+\infty}\int_\Omega\int_0^1 \beta(x)g_n v\,dxdt = 0. \tag{6.2.21}$$

Thus, plugging (6.2.20) and (6.2.21) into (6.2.15), we obtain (6.2.14). Note that, from Lemma 1.2.2 and Theorem 2.1.3, we get

$$\int_\Omega |u_n-u|^{\alpha(x)+\beta(x)}dx \to 0 \text{ and } \int_\Omega |v_n-v|^{\alpha(x)+\beta(x)}dx \to 0 \text{ as } n\to+\infty. \tag{6.2.22}$$

Now using (6.2.22) and Young's inequality, we have

$$\begin{aligned}
&\int_\Omega |u_n-u|^{\alpha(x)}|v_n-v|^{\beta(x)}dx\\
&\leq \int_\Omega \left(\frac{\alpha(x)}{\alpha(x)+\beta(x)}|u_n-u|^{\alpha(x)+\beta(x)} + \frac{\alpha(x)}{\alpha(x)+\beta(x)}|v_n-v|^{\alpha(x)+\beta(x)}\right\}dx\\
&\leq \frac{\alpha^+}{\alpha^-+\beta^-}\int_\Omega |u_n-u|^{\alpha(x)+\beta(x)}dx + \frac{\beta^+}{\alpha^-+\beta^-}\int_\Omega |v_n-v|^{\alpha(x)+\beta(x)}dx\\
&\to 0 \quad \text{as } n\to+\infty.
\end{aligned} \tag{6.2.23}$$

Thus, inserting (6.2.23) into (6.2.14), we obtain

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx = \int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx. \tag{6.2.24}$$

Now

$$\begin{aligned}
&\left|\int_\Omega c(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx - \int_\Omega c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx\right|\\
&\leq \|c\|_{L^\infty(\Omega)}\int_\Omega \left||u_n|^{\alpha(x)}|v_n|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\right|dx.
\end{aligned} \tag{6.2.25}$$

Define

$$w_n := |u_n|^{\alpha(x)}|v_n|^{\beta(x)} + |u|^{\alpha(x)}|v|^{\beta(x)} - \left||u_n|^{\alpha(x)}|v_n|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\right| \geq 0.$$

Since $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ as $n \to +\infty$, we have

$$w_n(x) \to 2|u(x)|^{\alpha(x)}|v(x)|^{\beta(x)} \text{ a.e. in } \mathbb{R}^N \text{ as } n \to +\infty.$$

Thus, by Fatou's Lemma, we get

$$\liminf_{n\to+\infty} \int_\Omega w_n(x)dx \geq 2\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx. \tag{6.2.26}$$

Again from (6.2.24), we find

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_\Omega w_n(x)dx \\
&\leq \lim_{n\to\infty} \int_\Omega |u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx + \lim_{n\to\infty} \int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx \\
&\quad - \limsup_{n\to+\infty} \int_\Omega \left||u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx - |u|^{\alpha(x)}|v|^{\beta(x)}\right| dx \\
&= 2\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}dx - \limsup_{n\to+\infty} \int_\Omega \left||u_n|^{\alpha(x)}|v_n|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\right| dx
\end{aligned} \tag{6.2.27}$$

Combining (6.2.26) and (6.2.27), we have

$$\limsup_{n\to+\infty} \int_\Omega \left||u_n|^{\alpha(x)}|v_n|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\right| dx \leq 0,$$

that is, $\lim_{n\to\infty} \int_\Omega \left||u_n|^{\alpha(x)}|v_n|^{\beta(x)} - |u|^{\alpha(x)}|v|^{\beta(x)}\right| dx = 0$. Thus, combining this together with (6.2.25), we get our final result. □

Lemma 6.2.2. *Let $\{u_n\}, \{v_n\}$ be any two bounded sequences in X_0 and $c(\cdot), \alpha(\cdot), \beta(\cdot)$ be as in Theorem 6.0.1. Then*

$$\lim_{n\to\infty} \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx = \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

The proof of the above lemma follows similarly as in Lemma 6.2.1 and using the fact that $\alpha(\cdot), \beta(\cdot) \in C_+(\overline{\Omega})$. □

In the following lemma, we characterize the critical points of $J_{\lambda,\zeta}$ as the local minimizers of $J_{\lambda,\zeta}$ on $\mathscr{N}^+_{\lambda,\zeta}$ (or $\mathscr{N}^-_{\lambda,\zeta}$).

Lemma 6.2.3. *Let $(u^*, v^*) \in \mathscr{N}^+_{\lambda,\zeta}$ (or $\mathscr{N}^-_{\lambda,\zeta}$) be a local minimizer for $J_{\lambda,\zeta}$ on $\mathscr{N}^+_{\lambda,\zeta}$ (or $\mathscr{N}^-_{\lambda,\zeta}$). Then (u^*, v^*) is a critical point of $J_{\lambda,\zeta}$.*

Proof. First assume that $(u^*, v^*) \in \mathscr{N}^+_{\lambda,\zeta}$ is a local minimizer for $J_{\lambda,\zeta}$ on $\mathscr{N}^+_{\lambda,\zeta}$. Let $I_{\lambda,\zeta}(u,v) = \langle J'_{\lambda,\zeta}(u,v), (u,v)\rangle_{\mathscr{E}}$. Note that for $(u,v) \in \mathscr{E} \setminus \{0\}$ with $I_{\lambda,\zeta}(u,v) = 0$, we have $\varphi''_{u,v}(1) > 0$ if and only if $\langle I'_{\lambda,\zeta}(u,v), (u,v)\rangle_{\mathscr{E}} > 0$. Since (u^*, v^*) is a local minimizer for $J_{\lambda,\zeta}$ on $\mathscr{N}^+_{\lambda,\zeta}$, using Lagrange's multiplier theorem, we get a real number τ such that

$$J'_{\lambda,\zeta}(u^*, v^*) = \tau I'_{\lambda,\zeta}(u^*, v^*).$$

Therefore,

$$0 = \langle J'_{\lambda,\zeta}(u^*, v^*), (u^*, v^*)\rangle_{\mathscr{E}} = \tau \langle I'_{\lambda,\zeta}(u^*, v^*), (u^*, v^*)\rangle_{\mathscr{E}} = \tau \phi''_{(u^*,v^*)}(1).$$

Since $(u^*, v^*) \in \mathscr{N}^+_{\lambda,\zeta}$, we have $\phi''_{(u^*,v^*)}(1) > 0$ and hence, $\tau = 0$. This completes the proof. Similarly we can prove the result when $(u^*, v^*) \in \mathscr{N}^-_{\lambda,\zeta}$ is a local minimizer for $J_{\lambda,\zeta}$ on $\mathscr{N}^-_{\lambda,\zeta}$. □

Next, we show that the set of points of inflection of the function $\varphi_{u,v}$ is empty for certain values of the parameters λ and ζ.

Lemma 6.2.4. *There exists $\delta > 0$, given by*

$$\delta = \frac{1}{\overline{C}_1}\left(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - r^-}\right)\left(\frac{p^- - r^+}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{p^+ - r^-}{\alpha^- + \beta^- - p^+}}$$

such that, for any pair of $(\lambda, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $\lambda + \zeta < \delta$, we have $\mathscr{N}^0_{\lambda,\zeta} = \emptyset$, where the positive constants $\overline{C}_1, \overline{C}_2$ are given as in Lemma 6.1.1.

Proof. We prove this lemma by contradiction. Let us assume that there exist $\lambda, \zeta > 0$ with $\lambda + \zeta < \delta$ such that $\mathscr{N}^0_{\lambda,\zeta} \neq \emptyset$. Hence, there is $(u, v) \in \mathscr{N}^0_{\lambda,\zeta}$.
Now, if $\|(u, v)\| < 1$, then using (6.1.5), (6.1.9) and Lemma 6.1.1 $(i), (ii)$, we obtain

$$\begin{aligned} 0 = \varphi''_{(u,v)}(1) &\leq p^+ P(u, v) - r^- Q(u, v) - (\alpha^- + \beta^-) R(u, v) \\ &= (p^+ - (\alpha^- + \beta^-)) P(u, v) + (\alpha^- + \beta^- - r^-) Q(u, v) \\ &\leq (p^+ - (\alpha^- + \beta^-)) \|(u, v)\|^{p^+} + (\alpha^- + \beta^- - r^-) \overline{C}_1 (\lambda + \zeta) \|(u, v)\|^{r^-}. \end{aligned}$$

This implies

$$\|(u, v)\|^{p^+ - r^-} \leq \frac{(\alpha^- + \beta^- - r^-)}{(\alpha^- + \beta^- - p^+)} \overline{C}_1 (\lambda + \zeta). \tag{6.2.28}$$

Again using (6.1.5), (6.1.9), and Lemma 6.1.1 $(i), (iii)$, we deduce

$$\begin{aligned} 0 = \varphi''_{(u,v)}(1) &\geq p^- P(u, v) - r^+ Q(u, v) - (\alpha^+ + \beta^+) R(u, v) \\ &= (p^- - r^+) P(u, v) - (\alpha^+ + \beta^+ - r^+) R(u, v) \\ &\geq (p^- - r^+) \|(u, v)\|^{p^+} - (\alpha^+ + \beta^+ - r^+) \overline{C}_2 \, \|(u, v)\|^{\alpha^- + \beta^-}. \end{aligned}$$

This yields that

$$1 \geq \|(u, v)\|^{\alpha^- + \beta^- - p^+} \geq \frac{(p^- - r^+)}{\overline{C}_2 (\alpha^+ + \beta^+ - r^+)}. \tag{6.2.29}$$

Putting together (6.2.28) and (6.2.29), we get

$$\lambda + \zeta \geq \frac{1}{\overline{C}_1} \left(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - r^-} \right) \left(\frac{p^- - r^+}{\overline{C}_2 (\alpha^+ + \beta^+ - r^+)} \right)^{\frac{p^+ - r^-}{\alpha^- + \beta^- - p^+}},$$

which is a contradiction.
Next, if $\|(u, v)\| > 1$, combining (6.1.5), (6.1.9), and Lemma 6.1.1 $(i), (ii)$, we find

$$0 = \varphi''_{u,v}(1) \leq (p^+ - (\alpha^- + \beta^-)) \|(u, v)\|^{p^-} + (\alpha^- + \beta^- - r^-) \overline{C}_1 (\lambda + \zeta) \|(u, v)\|^{r^+}.$$

This implies

$$\|(u,v)\|^{p^- - r^+} \le \frac{(\alpha^- + \beta^- - r^-)}{(\alpha^- + \beta^- - p^+)}\overline{C}_1(\lambda+\zeta). \tag{6.2.30}$$

On the other hand, by taking into account (6.1.5), (6.1.9) and Lemma 6.1.1 $(i), (iii)$, it follows that

$$0 = \varphi''_{u,v}(1) \ge (p^- - r^+)\|(u,v)\|^{p^-} - (\alpha^+ + \beta^+ - r^+)\overline{C}_2\, \|(u,v)\|^{\alpha^+ + \beta^+},$$

that is,

$$\|(u,v)\|^{\alpha^+ + \beta^+ - p^-} \ge \frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}. \tag{6.2.31}$$

Thus, combining (6.2.30) and (6.2.31), we obtain

$$\lambda + \zeta \ge \frac{1}{\overline{C}_1}\left(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - r^-}\right)\left(\frac{p^- - r^+}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{p^- - r^+}{\alpha^+ + \beta^+ - p^-}}. \tag{6.2.32}$$

Since $0 < \left(\frac{p^- - r^+}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right) < 1$ and $\frac{p^- - r^+}{\alpha^+ + \beta^+ - p^-} < \frac{p^+ - r^-}{\alpha^- + \beta^- - p^+}$, from (6.2.32) we infer that

$$\lambda + \zeta \ge \frac{1}{\overline{C}_1}\left(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - r^-}\right)\left(\frac{p^- - r^+}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{p^+ - r^-}{\alpha^- + \beta^- - p^+}},$$

which is a contradiction. Hence, the lemma is proved. □

In the next result, we discuss the behavior of the functional $J_{\lambda,\zeta}$ on $\mathcal{N}_{\lambda,\zeta}$.

Lemma 6.2.5. *Let δ be as in Lemma 6.2.4. Then for $\lambda + \zeta < \delta$, $J_{\lambda,\zeta}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\zeta}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\zeta}$. Then for $\|(u,v)\| > 1$, from (6.0.3) and (6.1.5) and Lemma 6.1.1 (ii), we deduce

$$J_{\lambda,\zeta}(u,v) \ge \frac{1}{p^+}P(u,v) - \frac{1}{r^-}Q(u,v) - \frac{1}{\alpha^- + \beta^-}R(u,v)$$

$$= \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) P(u,v) - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big) Q(u,v)$$
$$\geq \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) \|(u,v)\|^{p^-} - \overline{C}_1(\lambda+\zeta)\Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big) \|(u,v)\|^{r^+}. \tag{6.2.33}$$

Since from (A_1), we have $1 < r^- \leq r^+ < p^- \leq p^+ < \alpha^- + \beta^-$, (6.2.33) yields that $J_{\lambda,\zeta}(u,v) \to +\infty$ as $\|(u,v)\| \to +\infty$. Therefore, $J_{\lambda,\zeta}$ is coercive and bounded below on $\mathscr{N}_{\lambda,\zeta}$. □

Lemma 6.2.6. *We have the following results:*

(i) *If $(u,v) \in \mathscr{N}^+_{\lambda,\zeta}$, then $Q(u,v) > 0$.*

(ii) *If $(u,v) \in \mathscr{N}^-_{\lambda,\zeta}$, then $R(u,v) > 0$.*

Proof. *(i)* Since $(u,v) \in \mathscr{N}^+_{\lambda,\zeta}$, we have $\phi''_{(u,v)}(1) > 0$. Thus, using (6.1.5) and (6.1.9), we obtain

$$0 < \varphi''_{(u,v)}(1) \leq p^+ P(u,v) - r^- Q(u,v) - (\alpha^- + \beta^-) R(u,v)$$
$$= \{p^+ - (\alpha^- + \beta^-)\} P(u,v) + (\alpha^- + \beta^- - r^-) Q(u,v).$$

This implies that $Q(u,v) \geq \frac{(\alpha^- + \beta^- - p^+)}{(\alpha^- + \beta^- - r^-)} P(u,v) > 0$.

(ii) Since $(u,v) \in \mathscr{N}^-_{\lambda,\zeta}$, $\phi''_{(u,v)}(1) < 0$. Thus, taking into account (6.1.5) and (6.1.9), we get

$$0 > \varphi''_{(u,v)}(1) \geq p^- P(u,v) - r^+ Q(u,v) - (\alpha^+ + \beta^+) R(u,v)$$
$$= (p^- - r^+) P(u,v) - (\alpha^+ + \beta^+ - r^+) R(u,v),$$

that is, $R(u,v) \geq \frac{(\alpha^+ + \beta^+ - p^-)}{(\alpha^+ + \beta^+ - r^+)} P(u,v) > 0$.

□

Remark 6.2.1. *From Lemma 6.2.4 and Lemma 6.2.5, we conclude that, for any pair of parameters $(\lambda,\zeta) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $\lambda + \zeta < \delta$, $\mathscr{N}_{\lambda,\zeta} = \mathscr{N}^-_{\lambda,\zeta} \cup \mathscr{N}^+_{\lambda,\zeta}$ and $J_{\lambda,\zeta}$ is coercive and* *bounded below on* $\mathcal{N}^-_{\lambda,\zeta}$ *and* $\mathcal{N}^+_{\lambda,\zeta}$. *Therefore, we can define*

$$\theta_{\lambda,\zeta} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\zeta}} J_{\lambda,\zeta}(u,v);\ \theta^+_{\lambda,\zeta} = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\zeta}} J_{\lambda,\zeta}(u,v);\ \theta^-_{\lambda,\zeta} = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\zeta}} J_{\lambda,\zeta}(u,v).$$

The following two lemmas give the signs of $\theta^+_{\lambda,\zeta}$ and $\theta^-_{\lambda,\zeta}$, respectively.

Lemma 6.2.7. *Let* δ *be as in Lemma 6.2.4. Then for* $\lambda+\zeta<\delta$, $\theta_{\lambda,\zeta}\le\theta^+_{\lambda,\zeta}<0$.

Proof. Let $(u,v)\in\mathcal{N}^+_{\lambda,\zeta}$. Then $\varphi''_{u,v}(1)>0$. Now combining (6.1.5) and (6.1.9), we obtain

$$\begin{aligned}0<\varphi''_{u,v}(1)&<p^+P(u,v)-r^-Q(u,v)-(\alpha^-+\beta^-)R(u,v)\\&=(p^+-r^-)P(u,v)-(\alpha^-+\beta^--r^-)R(u,v),\end{aligned}$$

that is,

$$R(u,v)<\frac{(p^+-r^-)}{(\alpha^-+\beta^--r^-)}P(u,v). \tag{6.2.34}$$

Using (6.0.3), (6.1.5), and (6.2.34), we deduce

$$\begin{aligned}J_{\lambda,\zeta}(u,v)&\le\frac{1}{p^-}P(u,v)-\frac{1}{r^+}Q(u,v)-\frac{1}{\alpha^++\beta^+}R(u,v)\\&=\Big(\frac{1}{p^-}-\frac{1}{r^+}\Big)P(u,v)+\Big(\frac{1}{r^+}-\frac{1}{\alpha^++\beta^+}\Big)R(u,v)\\&\le\left[\Big(\frac{1}{p^-}-\frac{1}{r^+}\Big)+\Big(\frac{1}{r^+}-\frac{1}{\alpha^++\beta^+}\Big)\frac{(p^+-r^-)}{(\alpha^-+\beta^--r^-)}\right]P(u,v)\\&=\left[\frac{(r^+-p^-)(\alpha^++\beta^+)+p^-(\alpha^++\beta^+-r^+)\frac{(p^+-r^-)}{(\alpha^-+\beta^--r^-)}}{p^-r^+(\alpha^++\beta^+)}\right]P(u,v).\end{aligned} \tag{6.2.35}$$

From (A_2), we have $(r^+-p^-)(\alpha^++\beta^+)+p^-(\alpha^++\beta^+-r^+)\frac{(p^+-r^-)}{(\alpha^-+\beta^--r^-)}<0$. Hence, (6.2.35) implies that $J_{\lambda,\zeta}(u,v)<0$. Therefore, from the definition of $\theta_{\lambda,\zeta}$ and $\theta^+_{\lambda,\zeta}$, it follows that $\theta_{\lambda,\zeta}\le\theta^+_{\lambda,\zeta}<0$. □

Lemma 6.2.8. *Let* δ *be as in Lemma 6.2.4. Then for* $\lambda+\zeta<\Big(\frac{r^-}{p^+}\Big)\delta$, *there exists a positive constant* $\underline{K}$ *depending on* $N,s,p,r,\alpha,\beta,a,b,\lambda,\zeta,\Omega$ *such that* $\theta^-_{\lambda,\zeta}>\underline{K}$.

Proof. Let $(u,v)\in\mathcal{N}^-_{\lambda,\zeta}$. Then $\varphi''_{u,v}(1)<0$. Therefore, from (6.2.29) and (6.2.30), we obtain

$$\|(u,v)\| \geq \begin{cases} \left(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{1/(\alpha^- + \beta^- - p^+)}, & \|(u,v)\| < 1; \\ \left(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{1/(\alpha^+ + \beta^+ - p^-)}, & \|(u,v)\| > 1. \end{cases} \tag{6.2.36}$$

Now for $\|(u,v)\| < 1$, plugging (6.1.5) into (6.0.3) and using Lemma 6.1.1 $(i),(ii)$ and (6.2.36), we deduce

$$\begin{aligned} & J_{\lambda,\zeta}(u,v) \\ & \geq \frac{1}{p^+}P(u,v) - \frac{1}{r^-}Q(u,v) - \frac{1}{\alpha^- + \beta^-}R(u,v) \\ & = \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)P(u,v) - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)Q(u,v) \\ & \geq \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{p^+} - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta)\|(u,v)\|^{r^-} \\ & = \|(u,v)\|^{r^-}\Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{p^+ - r^-} - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta)\Big] \\ & \geq \left(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{r^-}{(\alpha^- + \beta^- - p^+)}}\Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big) \\ & \qquad \left(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{(p^+ - r^-)}{(\alpha^- + \beta^- - p^+)}} - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta)\Big] := d_1. \end{aligned} \tag{6.2.37}$$

If

$$\lambda + \zeta < \Big(\frac{r^-}{p^+}\Big)\delta = \Big(\frac{r^-}{p^+}\Big)\frac{1}{\overline{C}_1}\Big(\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - r^-}\Big)\left(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{(p^+ - r^-)}{(\alpha^- + \beta^- - p^+)}},$$

then

$$\lambda + \zeta < \frac{\alpha^- + \beta^- - p^+}{p^+(\alpha^- + \beta^-)}\left(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{(p^+ - r^-)}{(\alpha^- + \beta^- - p^+)}}\frac{(\alpha^- + \beta^-)r^-}{\alpha^- + \beta^- - r^-}.\frac{1}{\overline{C}_1},$$

that is,

$$\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\left(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\right)^{\frac{(p^+ - r^-)}{(\alpha^- + \beta^- - p^+)}} - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta) > 0,$$

and thus, from (6.2.37), we get that $d_1 > 0$.

Similarly for $\|(u,v)\| > 1$, again plugging (6.1.5) in (6.0.3) and using Lemma 6.1.1 $(i), (ii)$ and (6.2.36), we obtain

$$\begin{aligned}
&J_{\lambda,\zeta}(u,v)\\
&\geq \frac{1}{p^+}P(u,v) - \frac{1}{r^-}Q(u,v) - \frac{1}{\alpha^- + \beta^-}R(u,v)\\
&= \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)P(u,v) - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)Q(u,v)\\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{p^-} - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta)\|(u,v)\|^{r^+}\\
&= \|(u,v)\|^{r^+}\Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\|(u,v)\|^{p^- - r^+} - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta)\Big]\\
&\geq \Big(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\Big)^{\frac{r^+}{(\alpha^+ + \beta^+ - p^-)}}\Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\Big(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\Big)^{\frac{(p^- - r^+)}{(\alpha^+ + \beta^+ - p^-)}}\\
&\qquad - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta)\Big]. \qquad (6.2.38)
\end{aligned}$$

Combining the facts that $\frac{(p^- - r^+)}{(\alpha^+ + \beta^+ - p^-)} < \frac{(p^+ - r^-)}{(\alpha^- + \beta^- - p^+)}$ and $\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)} < 1$, and taking into account (6.2.37) and (6.2.38), we deduce

$$\begin{aligned}
J_{\lambda,\zeta}(u,v) &\geq \Big(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\Big)^{\frac{r^+}{(\alpha^- + \beta^- - p^+)}}\Big[\Big(\frac{1}{p^+} - \frac{1}{\alpha^- + \beta^-}\Big)\\
&\qquad \Big(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\Big)^{\frac{(p^+ - r^-)}{(\alpha^- + \beta^- - p^+)}} - \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)\overline{C}_1(\lambda+\zeta)\Big]\\
&\geq \Big(\frac{(p^- - r^+)}{\overline{C}_2(\alpha^+ + \beta^+ - r^+)}\Big)^{\frac{(r^+ - r^-)}{(\alpha^- + \beta^- - p^+)}} d_1 := d_2 > 0.
\end{aligned}$$

Finally by choosing $\underline{K} = \min\{d_1, d_2\} > 0$, the lemma holds. □

Next lemma describes the nature of the fibering map $\varphi_{u,v}$. We refer to [111] and [26] for similar result in the case of local p-Laplacian and non-local p-Laplacian, respectively, and [3, 37] for variable exponent Laplacian.

Lemma 6.2.9. *For $(u,v) \in \mathscr{E} \setminus \{(0,0)\}$, there exists $\delta' > 0$ such that, for all $\lambda + \zeta < \delta'$, we have the following:*

(*i*) *If* $Q(u,v)=0$, *then there exists a unique* $t^- := t^-(u,v)$ *such that* $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\zeta}^-$ *and* $J_{\lambda,\zeta}(t^-u, t^-v) = \sup_{t\geq 0} J_{\lambda,\zeta}(tu, tv)$.

(*ii*) *If* $Q(u,v)>0$, *then there exist* $t^*>0$ *and unique positive numbers* $t^+ := t^+(u,v) < t^- = t^-(u,v)$ *such that* $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\zeta}^-$, $(t^+u, t^+v) \in \mathcal{N}_{\lambda,\zeta}^+$ *and*

$$J_{\lambda,\zeta}(t^+u, t^+v) = \inf_{0\leq t\leq t^*} J_{\lambda,\zeta}(tu, tv);\ J_{\lambda,\zeta}(t^-u, t^-v) = \sup_{t\geq 0} J_{\lambda,\zeta}(tu, tv).$$

Proof. (*i*) Using the given assumption, for $0<t<1$ sufficiently small, we obtain

$$\varphi_{u,v}(t) > \frac{t^{p^+}}{p^+}P(u,v) - \frac{t^{\alpha^+ + \beta^+}}{\alpha^+ + \beta^+}R(u,v) > 0$$

and for $t>1$ sufficiently large, we get

$$\varphi_{u,v}(t) < \frac{t^{p^+}}{p^-}P(u,v) - \frac{t^{\alpha^+ + \beta^+}}{\alpha^- + \beta^-}R(u,v) < 0.$$

Hence, $\varphi_{u,v}$ achieves its maximum at some point $t^-(u,v)$ on $[0,\infty)$. Thus, $\varphi'_{u,v}(t^-) = \langle J'_{\lambda,\zeta}(t^-u, t^-v), (u,v)\rangle_{\mathscr{E}} = 0$. Set $(t^-u, t^-v) := (\tilde{u}, \tilde{v})$. Then $\langle J'_{\lambda,\zeta}(\tilde{u},\tilde{v}), (\tilde{u},\tilde{v})\rangle_{\mathscr{E}} = 0$, which implies $(\tilde{u},\tilde{v}) \in \mathcal{N}_{\lambda,\zeta}$. Thus, from (6.1.5), we get

$$P(\tilde{u},\tilde{v}) = R(\tilde{u},\tilde{v}). \tag{6.2.39}$$

Now we define the function $\Theta_{\tilde{u},\tilde{v}} : [0,\infty) \to \mathbb{R}$ as $\Theta_{\tilde{u},\tilde{v}}(t) = J_{\lambda,\zeta}(t\tilde{u}, t\tilde{v})$. We know that $\Theta_{\tilde{u},\tilde{v}}(1) = J_{\lambda,\zeta}(\tilde{u},\tilde{v}) = \max_{t\in[0,\infty)} \Theta_{\tilde{u},\tilde{v}}(t)$ and $\Theta'_{\tilde{u},\tilde{v}}(1) = \langle J'_{\lambda,\zeta}(\tilde{u},\tilde{v}), (\tilde{u},\tilde{v})\rangle_{\mathscr{E}} = 0$. For $t>1$, by (6.2.39), we deduce

$$\begin{aligned}\Theta'_{\tilde{u},\tilde{v}}(t) &= \langle J'_{\lambda,\zeta}(t\tilde{u}, t\tilde{v}), (\tilde{u},\tilde{v})\rangle_{\mathscr{E}} \\ &\leq t^{p^+-1}P(\tilde{u},\tilde{v}) - t^{\alpha^- + \beta^- - 1}R(\tilde{u},\tilde{v}) < 0,\end{aligned}$$

and on the other hand, for $t\in(0,1)$, again using (6.2.39), we obtain

$$\Theta'_{\tilde{u},\tilde{v}}(t) = \langle J'_{\lambda,\zeta}(t\tilde{u}, t\tilde{v}), (\tilde{u},\tilde{v})\rangle_{\mathscr{E}} \geq t^{p^- -1}P(\tilde{u},\tilde{v}) - t^{\alpha^+ + \beta^+ - 1}R(\tilde{u},\tilde{v}) > 0.$$

This shows that the point t^- is unique. Hence, the result follows.

(ii) To prove this , first we set

$$f_1(t) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} t^{p(x,y)}\left(\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right)dxdy;$$
$$f_2(t) := \int_\Omega t^{r(x)}\left(\lambda a(x)|u|^{r(x)} + \zeta b(x)|v|^{r(x)}\right)dx;$$
$$f_3(t) := \int_\Omega t^{\alpha(x)+\beta(x)}c(x)|u|^{\alpha(x)}|v|^{\beta(x)}dx.$$

Then f_i's are continuous and strictly increasing functions with $f_i(0) = 0$, for $i = 1, 2, 3$. Also, we observe the following:

(I) $\displaystyle\lim_{t\to 0^+}\frac{f_3(t)}{f_1(t)} = 0.$

(II) $\displaystyle\lim_{t\to+\infty} f_2(t) = +\infty.$

(III) $\displaystyle\lim_{t\to+\infty}\frac{(f_1-f_3)(t)}{f_2(t)} = 0.$

(IV) $f_1 - f_3$ has unique point of maximum, say t_{max} and $(f_1 - f_3)(t) \to -\infty$ as $t \to +\infty$.

(V) There exists $\tilde{t} \in (0, t_{max})$ such that $\dfrac{f_1 - f_3}{f_2}$ is strictly increasing on $(0, \tilde{t})$.

From (I), we note that $(f_1 - f_3)(t) > 0$, for $t \to 0^+$ sufficiently small. Hence, by (V) and intermediate value theorem, for each choice of the pair $(\lambda, \zeta) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $f_2(\tilde{t}) < (f_1 - f_3)(\tilde{t})$, there exists a unique $t^+ = t^+(\lambda, \zeta) \in (0, \tilde{t})$ such that

$$\frac{(f_1 - f_3)(t^+)}{f_2(t^+)} = 1. \tag{6.2.40}$$

Since $\frac{(f_1-f_3)}{f_2}$ is strictly monotone increasing in $(t^+, \tilde{t})$, from (6.2.40), we get

$$1 = \frac{(f_1 - f_3)(t^+)}{f_2(t^+)} < \frac{(f_1 - f_3)(t)}{f_2(t)}, \quad \text{for all } t \in (t^+, \tilde{t}),$$

that is,

$$f_2(t) < (f_1 - f_3)(t), \quad \text{for all} \quad t \in (t^+, \tilde{t}). \tag{6.2.41}$$

Now we can fix $(\lambda^*, \zeta^*) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that, for all $\lambda \in (0, \lambda^*)$, $\zeta \in (0, \zeta^*)$, taking into account (6.2.41), we have

$$f_2(t) < (f_1 - f_3)(t), \text{ for all } t \in (t^+, t_{\max}). \tag{6.2.42}$$

Since $f_1 - f_3$ is strictly decreasing in $(t_{\max}, \infty)$ and f_2 is monotonically increasing in $(0, \infty)$, by (II) and (6.2.42), there exists a unique positive real number $t^- > t_{max}$ such that

$$f_2(t^-) = (f_1 - f_3)(t^-), \quad \text{for all} \quad (\lambda, \zeta) \in (0, \lambda^*) \times (0, \zeta^*). \tag{6.2.43}$$

Hence, combining (6.2.40) and (6.2.43), we infer that the function $\varphi'_{u,v}(t) = (f_1 - f_2 - f_3)(t)$ has exactly two nontrivial zeroes, $t^+ < t^-$, that is, t^+ and t^- are critical points of $\varphi_{u,v}(t)$. For $\delta' := \lambda^* + \zeta^*$, we can choose $\lambda^*, \zeta^* > 0$ sufficiently small such that $\delta' < \delta$, where δ is given as in Lemma 6.2.4. Since $\varphi_{u,v}(0) = 0$ and $\varphi_{u,v}(t) < 0$, for $t \to 0^+$ sufficiently small, we get that $\varphi'_{u,v}(t) < 0$, for all $t \in (0, t^+)$ and $\varphi'_{u,v}(t) > 0$, for all $t \in (t^+, t_{max})$ and $\varphi'_{u,v}(t^+) = 0$. Again, by the Lemma 6.2.4, we have $\mathcal{N}^0_{\lambda,\zeta} = \emptyset$ and thus, we infer that $\varphi_{u,v}$ attains a local minimum at t^+ and consequently $\varphi''_{u,v}(t^+) > 0$. Hence, $(t^+u, t^+v) \in \mathcal{N}^+$.

Similarly, since $\varphi'_{u,v}(t) > 0$, for all $t \in [t_{max}, t^-)$, $\varphi'_{u,v}(t) < 0$, for all $t > t^-$, and $\varphi'_{u,v}(t^-) = 0$, using the fact that $\mathcal{N}^0_{\lambda,\zeta} = \emptyset$ from Lemma 6.2.4, it follows that t^- is the point of global maximum for $\varphi_{u,v}$ and consequently $\varphi''_{u,v}(t^-) < 0$. Hence, $(t^-u, t^-v) \in \mathcal{N}^-$. Now by appealing Lemma 6.2.7 and Lemma 6.2.8, we obtain $\varphi_{u,v}(t^+) < 0$ and $\varphi_{u,v}(t^-) > 0$. Also, by the above discussions, $\varphi_{u,v}$ is strictly increasing on $[t^+, t^-]$ and strictly decreasing in $t > t^-$ with $\varphi_{u,v}(t) \to -\infty$ as $t \to +\infty$. Thus, there exists a unique $t^* \in (t^+, t^-)$ such that $\varphi_{u,v}(t^*) = 0$. Therefore,

$$J_{\lambda,\zeta}(t^+u, t^+v) = \varphi_{u,v}(t^+) = \inf_{0 \le t \le t^*} \phi_{u,v}(t) = \inf_{0 \le t \le t^*} J_{\lambda,\zeta}(tu, tv);$$
$$J_{\lambda,\zeta}(t^-u, t^-v) = \varphi_{u,v}(t^-) = \sup_{t \ge 0} \phi_{u,v}(t) = \sup_{t \ge 0} J_{\lambda,\zeta}(tu, tv).$$

This completes the lemma. □

6.3 Existence of non-negative multiple solutions

In this section, we prove the existence of at least two distinct non-trivial and non-negative weak solutions of (6.0.1). The next two propositions ensure the existence of minimizers for the functional $J_{\lambda,\zeta}$ in $\mathscr{N}^{+}_{\lambda,\zeta}$ and $\mathscr{N}^{-}_{\lambda,\zeta}$, respectively, which serve as the weak solutions of (6.0.1). We set $\Lambda_0 := \min\left\{\left(\frac{r^-}{p^+}\right)\delta, \delta'\right\}$, where δ and δ' are given as in Lemma 6.2.4 and Lemma 6.2.9, respectively.

Proposition 6.3.1. *For $\lambda + \zeta < \Lambda_0$, the functional $J_{\lambda,\zeta}$ has a minimizer (u_0, v_0) in $\mathscr{N}^{+}_{\lambda,\zeta}$, which satisfies the following assertions:*

(*i*) $J_{\lambda,\zeta}(u_0, v_0) = \theta^{+}_{\lambda,\zeta} < 0$;

(*ii*) (u_0, v_0) *is a solution of* (6.0.1)

Proof. (*i*) Since $J_{\lambda,\zeta}$ is bounded below on $\mathscr{N}_{\lambda,\zeta}$ and hence on $\mathscr{N}^{+}_{\lambda,\zeta}$, there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathscr{N}^{+}_{\lambda,\zeta}$ such that

$$\lim_{n\to+\infty} J_{\lambda,\zeta}(u_n, v_n) = \inf_{(u,v)\in\mathscr{N}^{+}_{\lambda,\zeta}} J_{\lambda,\zeta}(u, v).$$

By Lemma 6.2.5, we have $J_{\lambda,\zeta}$ is coercive on $\mathscr{N}^{+}_{\lambda,\zeta}$, which yields that the sequence $\{(u_n, v_n)\}$ is bounded on $\mathscr{E}$. Therefore, there exists $(u_0, v_0) \in \mathscr{E}$ such that, passing to a sub-sequence, still denoted by $\{(u_n, v_n)\}$,

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0 \quad \text{in} \quad X_0 \quad \text{as } n \to +\infty$$

and hence, using Sobolev type embedding (see Theorem 2.1.3), we have

$$u_n \to u_0, \quad v_n \to v_0 \quad \text{strongly in } L^{r(\cdot)}(\Omega) \text{ and } L^{(\alpha+\beta)(\cdot)}(\Omega),$$

$$u_n(x) \to u_0(x) \text{ and } v_n(x) \to v_0(x) \text{ a.e. in } \Omega$$

as $n \to +\infty$. Now by applying Lemma 1.2.2 and Lebesgue dominated convergence theorem, one can check that

$$\lim_{n\to+\infty}\int_\Omega a(x)|u_n|^{r(x)}dx = \int_\Omega a(x)|u_0|^{r(x)}dx,\ \lim_{n\to+\infty}\int_\Omega b(x)|v_n|^{r(x)}dx = \int_\Omega b(x)|v_0|^{r(x)}dx, \tag{6.3.44}$$

and

$$\lim_{n\to+\infty}\int_\Omega a(x)\frac{|u_n|^{r(x)}}{r(x)}dx = \int_\Omega a(x)\frac{|u_0|^{r(x)}}{r(x)}dx,\ \lim_{n\to+\infty}\int_\Omega b(x)\frac{|v_n|^{r(x)}}{r(x)}dx = \int_\Omega b(x)\frac{|v_0|^{r(x)}}{r(x)}dx. \tag{6.3.45}$$

Also, by Lemma 6.2.1 and Lemma 6.2.2, we have

$$\lim_{n\to+\infty} R(u_n,v_n) = R(u_0,v_0),$$
$$\text{and}\ \lim_{n\to+\infty}\int_\Omega \frac{c(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}}{\alpha(x)+\beta(x)}dx = \int_\Omega \frac{c(x)|u_0|^{\alpha(x)}|v_0|^{\beta(x)}}{\alpha(x)+\beta(x)}dx, \tag{6.3.46}$$

respectively. We claim that $(u_0,v_0)\not\equiv(0,0)$. Note that $Q(u_0,v_0)>0$. Indeed, if not, then from (6.3.44), we have

$$Q(u_n,v_n)\to Q(u_0,v_0)=0\ as\ n\to\infty. \tag{6.3.47}$$

Since $(u_n,v_n)\in\mathscr{N}^+_{\lambda,\zeta}$, using (6.0.3) and (6.1.5), we get

$$J_{\lambda,\zeta}(u_n,v_n)\geq\Big(\frac{1}{p^+}-\frac{1}{\alpha^-+\beta^-}\Big)P(u_n,v_n)-\Big(\frac{1}{r^-}-\frac{1}{\alpha^-+\beta^-}\Big)Q(u_n,v_n).$$

Now letting $n\to+\infty$ in the both side of the last expression and using (6.3.47), we obtain

$$\lim_{n\to\infty}J_{\lambda,\zeta}(u_n,v_n)\geq 0. \tag{6.3.48}$$

But Lemma 6.2.7 gives that $\lim_{n\to\infty}J_{\lambda,\zeta}(u_n,v_n)=\inf_{(u,v)\in\mathscr{N}^+_{\lambda,\zeta}}J_{\lambda,\zeta}(u,v)<0$, which contradicts (6.3.48). Thus, the claim is proved and we get that $(u_0,v_0)\in\mathscr{E}\setminus\{(0,0)\}$. Next, we claim that

$$u_n\to u_0 \text{ and } v_n\to v_0 \text{ strongly in } X_0 \text{ as } n\to+\infty. \tag{6.3.49}$$

If not, then $u_n \not\to u_0$ or $v_n \not\to v_0$ in X_0 as $n \to +\infty$. Therefore, using uniform convexity of X_0 (see Remark 2.1.3), it follows that

$$\begin{aligned}
&\text{either} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u_0(x)-u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\qquad\qquad < \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\text{or} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\qquad\qquad < \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy. \qquad (6.3.50)
\end{aligned}$$

Thus, combining (6.0.3), (6.3.45), (6.3.46) and (6.3.50), we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} J_{\lambda,\zeta}(u_n,v_n) \\
&= \liminf_{n\to+\infty} \Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \left(\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right)dxdyk \\
&\quad - \int_\Omega \frac{1}{r(x)}\left(\lambda a(x)|u_n|^{r(x)} + \zeta b(x)|v_n|^{r(x)}\right)dx - \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx.\Bigg] \\
&\geq \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad + \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad - \lim_{n\to+\infty} \int_\Omega \frac{1}{r(x)}\left(\lambda a(x)|u_n|^{r(x)} + \zeta b(x)|v_n|^{r(x)}\right)dx \\
&\quad - \lim_{n\to+\infty} \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx \\
&> \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u_0(x)-u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|v_0(x)-v_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad - \int_\Omega \frac{1}{r(x)}\left(\lambda a(x)|u_0|^{r(x)} + \zeta b(x)|v_0|^{r(x)}\right)dx \\
&\quad - \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|u_0|^{\alpha(x)}|v_0|^{\beta(x)}dx = J_{\lambda,\zeta}(u_0,v_0) \qquad (6.3.51)
\end{aligned}$$

By Lemma 6.2.9 (ii), for $(u_0,v_0) \in \mathscr{E} \setminus \{(0,0)\}$, there exists a positive real number $t_0^+ := t_0^+(u_0,v_0)$ such that $(t_0^+u_0, t_0^+v_0) \in \mathscr{N}_{\lambda,\zeta}^+$. Again, considering the assumption $u_n \not\to u_0$ or $v_n \not\to v_0$ in X_0 as $n \to +\infty$, we have

$$\rho_{X_0}(t_0^+u_0) < \liminf_{n\to+\infty} \rho_{X_0}(t_0^+u_n) \text{ or } \rho_{X_0}(t_0^+v_0) < \liminf_{n\to+\infty} \rho_{X_0}(t_0^+v_n). \qquad (6.3.52)$$

Furthermore, using Lemma 1.2.2 and Lebesgue dominated convergence theorem, we get

$$Q(t_0^+u_0, t_0^+v_0) = \lim_{n\to+\infty} Q(t_0^+u_n, t_0^+v_n) \tag{6.3.53}$$

and by Lemma 6.2.1, we obtain

$$R(t_0^+u_0, t_0^+v_0) = \lim_{n\to+\infty} R(t_0^+u_n, t_0^+v_n). \tag{6.3.54}$$

Taking into account (6.1.7), (6.3.52), (6.3.53), and (6.3.54), we deduce

$$\begin{aligned}
&\lim_{n\to+\infty} \varphi'_{u_n,v_n}(t_0^+)\\
&= \liminf_{n\to+\infty} \Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (t_0^+)^{p(x,y)-1}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&\qquad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (t_0^+)^{p(x,y)-1}\frac{|v_n(x)-v_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&\qquad - \int_\Omega (t_0^+)^{r(x)-1}\Big(\lambda a(x)|u_n|^{r(x)} + \zeta b(x)|v_n|^{r(x)}\Big)dx\\
&\qquad - \int_\Omega (t_0^+)^{\alpha(x)+\beta(x)-1}c(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}dx\Bigg]\\
&\geq \frac{1}{t_0^+}\Bigg[\liminf_{n\to+\infty}\rho_{X_0}(t_0^+u_n) + \liminf_{n\to+\infty}\rho_{X_0}(t_0^+v_n) - \lim_{n\to\infty} Q(t_0^+u_n, t_0^+v_n)\\
&\qquad - \lim_{n\to+\infty} R(t_0^+u_n, t_0^+v_n)\Bigg]\\
&> \frac{1}{t_0^+}\Big[\rho_{X_0}(t_0^+u_0) + \rho_{X_0}(t_0^+v_n) - Q(t_0^+u_0, t_0^+v_0) - R(t_0^+u_0, t_0^+v_0)\Big]\\
&= \varphi'_{u_0,v_0}(t_0^+) = 0.
\end{aligned} \tag{6.3.55}$$

Thus, for sufficiently $n \in \mathbb{N}$, $\varphi'_{u_n,v_n}(t_0^+) > 0$. Since $(u_n, v_n) \in \mathscr{N}^+_{\lambda,\zeta}$, for all $n \in \mathbb{N}$, we have $\varphi'_{u_n,v_n}(1) = 0$ and $\varphi''_{u_n,v_n}(1) > 0$. Then using Lemma 6.2.9 (ii), we get $\varphi'_{u_n,v_n}(t) < 0$ for all $t \in (0,1)$ and therefore, from (6.3.55), we must have $t_0^+ > 1$. Since $(t_0^+u_0, t_0^+v_0) \in \mathscr{N}^+_{\lambda,\zeta}$, again by Lemma 6.2.9 (ii), we obtain that $\varphi_{u_0,v_0}(t)$ is monotone decreasing on $(0, t_0^+)$. Hence using (6.3.51), we achieve

$$J_{\lambda,\zeta}(t_0^+u_0, t_0^+v_0) \leq J_{\lambda,\zeta}(u_0, v_0) < \lim_{n\to+\infty} J_{\lambda,\zeta}(u_n, v_n) = \inf_{(u,v)\in\mathscr{N}^+_{\lambda,\zeta}} J_{\lambda,\zeta}(u, v).$$

This is a contradiction to the fact that $(t_0^+u_0, t_0^+v_0) \in \mathcal{N}_{\lambda,\zeta}^+$. Hence, the claim (6.3.49). So, $(u_n, v_n) \to (u_0, v_0)$ strongly in $\mathscr{E}$ as $n \to +\infty$ and thus, $(u_0, v_0) \in \mathcal{N}_{\lambda,\zeta}$. Since by Lemma 6.2.4, $\mathcal{N}_{\lambda,\zeta}^0 = \emptyset$ and from Lemma 6.2.7, we have $J_{\lambda,\zeta}(u_0, v_0) = \lim_{n\to+\infty} J_{\lambda,\zeta}(u_n, v_n) < 0$, we infer that $(u_0, v_0) \in \mathcal{N}_{\lambda,\zeta}^+$.

(ii) Using Lemma 6.2.3, we conclude that (u_0, v_0) is a solution of (6.0.1). □

Proposition 6.3.2. *If $\lambda + \zeta < \delta_0$, then $J_{\lambda,\zeta}$ has a minimizer (w_0, z_0) in $\mathcal{N}_{\lambda,\zeta}^-$ such that the following assertions hold:*

- (i) $J_{\lambda,\zeta}(w_0, z_0) = \theta_{\lambda,\zeta}^- > 0$.
- (ii) *(w_0, z_0) is a non-semi trivial solution of* (6.0.1).

Proof. (i) By Lemma 6.2.5, $J_{\lambda,\zeta}$ is bounded below on $\mathcal{N}_{\lambda,\zeta}^-$. Hence, there exists a minimizing sequence $\{(w_n, z_n)\} \subset \mathcal{N}_{\lambda,\zeta}^-$ such that

$$\lim_{n\to+\infty} J_{\lambda,\zeta}(w_n, z_n) = \inf_{(u,v)\in\mathcal{N}_{\lambda,\zeta}^-} J_{\lambda,\zeta}(u, v).$$

Again from Lemma 6.2.5, we have $J_{\lambda,\zeta}$ is coercive, which implies that the sequence $\{(w_n, z_n)\}$ is bounded on $\mathscr{E}$ and thus, there exists $(w_0, z_0) \in \mathscr{E}$ such that up to a subsequence, $(w_n, z_n) \rightharpoonup (w_0, z_0)$ weakly and by Sobolev type embedding result (see Theorem 2.1.3), we get

$$w_n \to w_0,\ z_n \to z_0 \ \text{ strongly in } L^{r(\cdot)}(\Omega) \text{ and } L^{(\alpha+\beta)(\cdot)}(\Omega),$$

$$w_n(x) \to w_0(x) \text{ and } z_n(x) \to z_0(x) \text{ a.e. in } \Omega$$

as $n \to +\infty$. Using Lemma 1.2.2 and Dominated convergence theorem, we derive

$$\lim_{n\to+\infty} \int_\Omega a(x)|w_n|^{r(x)}dx = \int_\Omega a(x)|w_0|^{r(x)}dx\ ;\ \lim_{n\to+\infty} \int_\Omega b(x)|z_n|^{r(x)}dx = \int_\Omega b(x)|z_0|^{r(x)}dx. \tag{6.3.56}$$

Also, Lemma 6.2.1 gives that

$$R(w_n, z_n) \to R(w_0, z_0) \text{ as } n \to +\infty. \tag{6.3.57}$$

Next, we have $(w_0, z_0) \not\equiv (0,0)$. Indeed, if $(w_0, z_0) = (0,0)$, from (6.3.57), we obtain

$$R(w_n, z_n) \to R(w_0, z_0) = 0 \text{ as } n \to +\infty. \tag{6.3.58}$$

Since $(w_n, z_n) \in \mathscr{N}^-_{\lambda,\zeta}$, using (6.0.3), (6.1.5) and Lemma 6.2.7, we deduce

$$\begin{aligned} 0 < \underline{K} &< J_{\lambda,\zeta}(w_n, z_n) \\ &\leq \Big(\frac{1}{p^+} - \frac{1}{r^-}\Big)P(w_n, z_n) + \Big(\frac{1}{r^-} - \frac{1}{\alpha^- + \beta^-}\Big)R(w_n, z_n) + o_m(1). \end{aligned}$$

Now letting $n \to +\infty$, from the last expression and (6.3.58), we get

$$0 < \underline{K} < \lim_{n\to+\infty} J_{\lambda,\zeta}(w_n, z_n) \leq 0,$$

which is a contradiction. Thus, $(w_0, z_0) \in \mathscr{E} \setminus \{(0,0)\}$. If $Q(w_0, z_0) = 0$, then we use Lemma 6.2.9 (i) and if $Q(w_0, z_0) > 0$, then we use Lemma 6.2.9 (ii). In both the cases, there exists a positive real number $t_0^- := t_0^-(w_0, z_0)$ such that $(t_0^- w_0, t_0^- z_0) \in \mathscr{N}^-_{\lambda,\zeta}$. Next, we claim that

$$w_n \to w_0 \text{ strongly in } X_0 \text{ and } z_n \to z_0 \text{ strongly in } X_0 \text{ as } n \to +\infty. \tag{6.3.59}$$

Suppose the claim does not hold true. Then $t_0^- w_n \not\to t_0^- w_0$ or $t_0^- z_n \not\to t_0^- z_0$ in X_0 as $n \to +\infty$. This implies that

$$\text{either } \rho_{X_0}(t_0^- w_0) < \liminf_{n\to+\infty} \rho_{X_0}(t_0^- w_n) \text{ or } \rho_{X_0}(t_0^- z_0) < \liminf_{n\to+\infty} \rho_{X_0}(t_0^- z_n). \tag{6.3.60}$$

Now using uniform convexity of X_0 (see Remark 2.1.3), we have the following:

$$\begin{aligned} \text{either} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- w_0(x) - t_0^- w_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \\ < \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- w_n(x) - t_0^- w_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \end{aligned}$$

$$\text{or} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- z_0(x) - t_0^- z_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy$$
$$< \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- z_n(x) - t_0^- z_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy. \tag{6.3.61}$$

Note that, using Lemma 1.2.2 and Lebesgue dominated converges theorem, we deduce

$$\lim_{n\to+\infty} \int_\Omega \frac{1}{r(x)}\Big(\lambda a(x)|t_0^- w_n|^{r(x)} + \zeta b(x)|t_0^- z_n|^{r(x)}\Big)dx$$
$$= \int_\Omega \frac{1}{r(x)}\Big(\lambda a(x)|t_0^- w_0|^{r(x)} + \zeta b(x)|t_0^- z_0|^{r(x)}\Big)dx. \tag{6.3.62}$$

Also, by Lemma 6.2.2, we have

$$\lim_{n\to+\infty} \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|t_0^- w_n|^{\alpha(x)}|t_0^- z_n|^{\beta(x)}dx$$
$$= \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|t_0^- w_0|^{\alpha(x)}|t_0^- z_0|^{\beta(x)}dx. \tag{6.3.63}$$

Thus, combining (6.0.3), (6.3.61), (6.3.62), and (6.3.63), we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} J_{\lambda,\zeta}(t_0^- w_n, t_0^- z_n)\\
&= \liminf_{n\to+\infty}\Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)}\left(\frac{|t_0^- w_n(x) - t_0^- w_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} + \frac{|t_0^- z_n(x) - t_0^- z_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right)dxdy\\
&\quad - \int_\Omega \frac{1}{r(x)}\Big(\lambda a(x)|t_0^- w_n|^{r(x)} + \zeta b(x)|t_0^- z_n|^{r(x)}\Big)dx\\
&\quad - \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|t_0^- w_n|^{\alpha(x)}|t_0^- z_n|^{\beta(x)}dx\Bigg]\\
&\geq \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- w_n(x) - t_0^- w_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&\quad + \liminf_{n\to+\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|t_0^- z_n(x) - t_0^- z_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&\quad - \lim_{n\to+\infty} \int_\Omega \frac{1}{r(x)}\Big(\lambda a(x)|t_0^- w_n|^{r(x)} + \zeta b(x)|t_0^- z_n|^{r(x)}\Big)dx\\
&\quad - \lim_{n\to+\infty} \int_\Omega \frac{1}{\alpha(x)+\beta(x)} c(x)|t_0^- w_n|^{\alpha(x)}|t_0^- z_n|^{\beta(x)}dx
\end{aligned}$$

$$
\begin{aligned}
&> \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{p(x,y)}\left(\frac{|t_0^- w_0(x)-t_0^- w_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}+\frac{|t_0^- z_0(x)-t_0^- z_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\right)dxdy\\
&\quad -\int_\Omega \frac{1}{r(x)}\Big(\lambda a(x)|t_0^- w_0|^{r(x)}+\zeta b(x)|t_0^- z_0|^{r(x)}\Big)dx\\
&\quad -\int_\Omega \frac{1}{\alpha(x)+\beta(x)}c(x)|t_0^- w_0|^{\alpha(x)}|t_0^- z_0|^{\beta(x)}dx\\
&= J_{\lambda,\zeta}(t_0^- w_0, t_0^- z_0).
\end{aligned} \tag{6.3.64}
$$

Again, using the facts that $w_n \to w_0$ and $z_n \to z_0$ in $L^{r(\cdot)}(\Omega)$, we get

$$
\lim_{n\to+\infty} Q(t_0^- w_n, t_0^- z_n) = Q(t_0^- w_0, t_0^- z_0) \tag{6.3.65}
$$

and Lemma 6.2.1 gives us

$$
\lim_{n\to+\infty} R(t_0^- w_n, t_0^- z_n) = R(t_0^- w_0, t_0^- z_0). \tag{6.3.66}
$$

Considering (6.1.7), (6.3.60), (6.3.65), and (6.3.66), we obtain

$$
\begin{aligned}
&\lim_{n\to+\infty}\varphi'_{w_n,z_n}(t_0^-)\\
&= \liminf_{n\to+\infty}\Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(t_0^-)^{p(x,y)-1}\frac{|w_n(x)-w_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&\qquad +\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(t_0^-)^{p(x,y)-1}\frac{|z_n(x)-z_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\
&\qquad -\int_\Omega (t_0^-)^{r(x)-1}\Big(\lambda a(x)|w_n|^{r(x)}+\zeta b(x)|z_n|^{r(x)}\Big)dx\\
&\qquad -\int_\Omega (t_0^-)^{\alpha(x)+\beta(x)-1}c(x)|w_n|^{\alpha(x)}|z_n|^{\beta(x)}dx\Bigg]\\
&\geq \frac{1}{t_0^-}\Bigg[\liminf_{n\to+\infty}\rho_{X_0}(t_0^- w_n)+\liminf_{n\to+\infty}\rho_{X_0}(t_0^- z_n)-\lim_{n\to\infty}Q(t_0^- w_n,t_0^- z_n)\\
&\qquad -\lim_{n\to+\infty}R(t_0^- w_n,t_0^- z_n)\Bigg]\\
&> \frac{1}{t_0^-}\Big[\rho_{X_0}(t_0^- w_0)+\rho_{X_0}(t_0^- z_n)-Q(t_0^- w_0,t_0^- z_0)-R(t_0^- w_0,t_0^- z_0)\Big]\\
&= \varphi'_{u_0,v_0}(t_0^-)=0.
\end{aligned} \tag{6.3.67}
$$

For sufficiently large $n \in \mathbb{N}$, $\varphi'_{w_n,z_n}(t_0^-) > 0$. Since $(w_n, z_n) \in \mathscr{N}^-_{\lambda,\zeta}$, for all $n \in \mathbb{N}$, we have $\varphi'_{w_n,z_n}(1) = 0$ and $\varphi''_{w_n,z_n}(1) < 0$, for all $n \in \mathbb{N}$. Now using the Lemma 6.2.9, we get $\varphi'_{w_n,z_n}(t) < 0$, for all $t > 1$. Then from (6.3.67), we must have $t_0^- < 1$. Since $(t_0^- w_0, t_0^- z_0) \in \mathscr{N}^-_{\lambda,\zeta}$, again using Lemma 6.2.9, we obtain that 1 is the global maximum point for $\varphi_{w_n,z_n}(t)$. Therefore, from (6.3.64), it follows that

$$J_{\lambda,\zeta}(t_0^- w_0, t_0^- z_0) < \lim_{n\to+\infty} J_{\lambda,\zeta}(t_0^- w_n, t_0^- z_n) \leq \lim_{n\to+\infty} J_{\lambda,\zeta}(w_n, z_n) = \inf_{(u,v)\in\mathscr{N}^-_{\lambda,\zeta}} J_{\lambda,\zeta}(u,v).$$

This is a contradiction to the fact that $(t_0^- w_0, t_0^- z_0) \in \mathscr{N}^-_{\lambda,\zeta}$. Hence, the claim (6.3.59) and $(w_n, z_n) \to (w_0, z_0)$ strongly in $\mathscr{E}$ as $n \to +\infty$ and $(w_0, z_0) \in \mathscr{N}$. Now using the fact that $\mathscr{N}^0_{\lambda,\zeta} = \emptyset$ from Lemma 6.2.4, and noticing that $J_{\lambda,\zeta}(w_0, z_0) = \inf_{(u,v)\in\mathscr{N}^-_{\lambda,\zeta}} J_{\lambda,\zeta}(u,v) > 0$, we conclude that $(w_0, z_0) \in \mathscr{N}^-_{\lambda,\zeta}$.

(ii) Using Lemma 6.2.3, we infer that (w_0, z_0) is a solution of (6.0.1). Now we show that (w_0, z_0) is not semi-trivial, that is, (w_0, z_0) is not of the form $(W_0, 0)$ (or $(0, z_0)$). The proof follows adapting the similar approach as in [26]. If $(w_0, 0)$ (or $(0, z_0)$) is a semi-trivial solution of (6.0.1), then from (6.0.2) we get

$$\rho_{X_0}(w_0) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_0(x) - w_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy = \lambda \int_\Omega a(x)|u|^{r(x)} dx.$$

Therefore,

$$\begin{aligned} J_{\lambda,\zeta}(w_0, 0) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|w_0(x) - w_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy - \lambda \int_\Omega \frac{1}{r(x)} a(x)|w_0|^{r(x)} dx \\ &\leq \frac{1}{p^-}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_0(x) - w_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy - \frac{\lambda}{r^+}\int_\Omega a(x)|w_0|^{r(x)} dx \\ &= \left(\frac{1}{p^-} - \frac{1}{r^+}\right)\rho_{X_0}(w_0) < 0, \end{aligned}$$

since by Lemma 6.2.8, $J_{\lambda,\zeta}(w_0, z_0) > 0$, it follows that (w_0, z_0) is not semi-trivial. □

Proof of Theorem 6.0.1. Let Λ_0 be as in Section 6.3. Let (u_0, v_0) be obtained as in Proposition 6.3.1. Now using Lemma 6.2.6 and the fact that $(u_0, v_0) \in \mathscr{N}^+_{\lambda,\zeta}$, for

$(|u_0|,|v_0|) \in \mathscr{E} \setminus \{(0,0)\}$, we have $Q(|u_0|,|v_0|) = Q(u_0,v_0) > 0$, and thus by Lemma 6.2.9 (ii), there exists $t_1 > 0$ such that $(t_1|u_0|, t_1|v_0|) \in \mathscr{N}^+_{\lambda,\zeta}$. This implies

$$0 = \varphi'_{|u_0|,|v_0|}(t_1) \leq \varphi'_{u_0,v_0}(t_1). \tag{6.3.68}$$

Combining (6.3.68) with the facts that $(u_0,v_0) \in \mathscr{N}^+_{\lambda,\zeta}$, $\varphi'_{u_0,v_0}(1) = 0$, and again using Lemma 6.2.9 (ii), we get $t_1 \geq 1$. This yields that

$$J_{\lambda,\zeta}(t_1|u_0|, t_1|v_0|) \leq J_{\lambda,\zeta}(|u_0|,|v_0|) \leq J_{\lambda,\zeta}(u_0,v_0) = \inf_{(u,v)\in\mathscr{N}^+_{\lambda,\zeta}} J_{\lambda,\zeta}(u,v).$$

Therefore, there exists a non-negative minimizer for $J_{\lambda,\zeta}$ on $\mathscr{N}^+_{\lambda,\zeta}$, which is a solution of (6.0.1) by Lemma 6.2.3. Next, we assert that there exists a non-negative minimizer for $J_{\lambda,\zeta}(w,z)$ on $\mathscr{N}^-_{\lambda,\zeta}$. Indeed, for $(|w_0|,|z_0|) \in \mathscr{E} \setminus \{(0,0)\}$, by Lemma 6.2.9, there exists $t_2 > 0$ such that $(t_2|w_0|, t_2|z_0|) \in \mathscr{N}^-_{\lambda,\zeta}$, where (w_0,z_0) is as in Proposition 6.3.2. Since $(w_0,z_0) \in \mathscr{N}^-_{\lambda,\zeta}$, again by Lemma 6.2.9, we get

$$J_{\lambda,\zeta}(t_2|w_0|, t_2|z_0|) \leq J_{\lambda,\zeta}(t_2w_0, t_2z_0) \leq J_{\lambda,\zeta}(w_0,z_0) = \inf_{(u,v)\in\mathscr{N}^-_{\lambda,\zeta}} J_{\lambda,\zeta}(u,v).$$

Thus, we get a non-negative minimizer for $J_{\lambda,\zeta}$ on $\mathscr{N}^-_{\lambda,\zeta}$, which is a solution of (6.0.1), thanks to Lemma 6.2.3. Hence for all $0 < \lambda + \zeta < \Lambda_0$, (6.0.1) admits two non-trivial and non-negative solutions in $\mathscr{N}^+_{\lambda,\zeta}$ and in $\mathscr{N}^-_{\lambda,\zeta}$, respectively. Since $\mathscr{N}^+_{\lambda,\zeta} \cap \mathscr{N}^-_{\lambda,\zeta} = \emptyset$, these solutions are distinct. This completes the proof. □

6.4 Conclusion

In this chapter, we have studied the existence of non-negative multiple solutions for a non-local elliptic system involving fractional $p(\cdot)$-Laplacian with the weighted concave and convex nonlinearities, when the pair of the parameters (λ,ζ) is lying in a suitable subset of $\mathbb{R}^+ \times \mathbb{R}^+$. We have also shown that one of the solutions is not semi-trivial, that is, not of the form $(u,0)$ or $(0,u)$. Here we have used the technique of constrained minimization over the Nehari manifold and the analysis of the associated fibering map.

We would like to mention that the analysis of the fibering map involved very delicate analysis due to the non-homogeneity arisen from variable exponents, unlike the constant exponent case as in [26, 48, 111].

Next, we remark that it will be challenging to consider the sign changing weight functions on the right hand-side of (6.0.1) due to the non-homogeneity coming from variable exponents. Also, the doubly non-local system of type (6.0.1) with concave (or singular) type nonlinearity, together with Choquard type nonlinearity will be an interesting problem to be explored. □

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Publications

<u>List of Publications from Thesis work</u>

1. Reshmi Biswas and Sweta Tiwari. Variable order nonlocal Choquard problem with variable exponents. *Complex Var. Elliptic Equ.* 1–23, 2020.
   DOI: 10.1080/17476933.2020.1751136.
2. Reshmi Biswas and Sweta Tiwati. Nehari manifold approach for fractional $p(\cdot)$-Laplacian system involving concave-convex nonlinearities. *Electron. J. Differential Equations*, 2020(98):1–29,2020 .
3. Reshmi Biswas and Sweta Tiwati. On a class of Kirchhoff-Choquard equations involving variable-order fractional $p(\cdot)$-Laplacian and without Ambrosetti-Rabinowitz type condition. Accepted in *Topol. Methods Nonlinear Anal.*, 2020.
4. Reshmi Biswas and Sweta Tiwari. Regularity results for Choquard equations involving fractional p-Laplacian. (Under review).
   DOI: https://arxiv.org/abs/2008.07398.

RESHMI BISWAS

EDUCATION

- **Ph.D in Mathematics**, Department of Mathematics, IIT Guwahati, December, 2015 to current.
- **M.Sc. in Mathematics**, Department of Mathematics, University of Hyderabad, Hyderabad, July, 2014, with first class.
- **B. Sc.** (Mathematics Hons.), St. Xavier's College, Kolkata, July, 2012, with first class.

AWARDS/SCHOLARSHIP

- **Senior Research Fellowship**: Given by Ministry of Human Resource and Development, India, January, 2018 - till present.
- **Junior Research Fellowship**: Given by Ministry of Human Resource and Development, India, January, 2016 - December, 2017.
- **Joint CSIR-UGC National Level Test**: Secured All India Rank 78 in the in Mathematical Science, India, June, 2015.
- **West Bengal Board Scholarship**: Given to the top 400 students in 12^{th} standard board examination under WBHSE, West Bengal, India, 2008.

ARTICLES ACCEPTED/PUBLISHED/PREPRINT

- Reshmi Biswas and Sweta Tiwari, *Variable order nonlocal Choquard problem with variable exponents*, Complex Var. Elliptic Equ., (2020), 1–23. DOI: 10.1080/17476933.2020.1751136.
- Reshmi Biswas and Sweta Tiwati, *Nehari manifold approach for fractional $p(\cdot)$- Laplacian system involving concave-convex nonlinearities*, Electron. J. Differential Equations, 2020 (2020), no. 98, 1–29 .
- Reshmi Biswas and Sweta Tiwati, *On a class of Kirchhoff-Choquard equations involving variable-order fractional $p(\cdot)$-Laplacian and without Ambrosetti-Rabinowitz type condition*, accepted in Topol. Methods Nonlinear Anal., 2020.
- Reshmi Biswas and Sweta Tiwari, *Regularity results for Choquard equations involving*

fractional p-Laplacian.

DOI: https://arxiv.org/abs/2008.07398.

ACADEMIC VISIT AND PROJECT

- Research visit to the Dept. of Mathematics, Indian Institute of Technology Delhi, Delhi, India, December 18 - 31, 2019.
- A Short Course in **Differential Topology**, Institute's training program in the Stat-Math Unit, Indian Statistical Institute, Kolkata, May 6 - June 20, 2012.

  Project Guide: Prof. Mahuya Datta.

WORKSHOP AND CONFERENCE

- Presented a talk at **International Conference on Advances in Differential Equations and Numerical Analysis (ADENA 2020)**, Indian Institute of Technology Guwahati, Guwahati, India, October, 2020.

  Title of the talk: *"Variable order Nonlocal Choquard problem with variable exponents"*.
- Attended **Advanced Instructional School: Linear Partial Differential Equations**, TIFR Centre for Applicable Mathematics, Bangalore, India, June 19 - July 8, 2017.
- Attended **Annual Conference of Indian Women and Mathematics**, School of Mathematics and Statistics, University of Hyderabad, India, June 29 - July 1, 2016.

TEACHING ASSISTANT-SHIP

- MA102: *Linear Algebra and Ordinary Differential Equations* (offered for 1^{st} year B.Tech. students), IIT Guwahati, India, January - March, 2018 - 2020.
- MA201: *Complex Analysis and Partial Differential Equations* (offered for 2^{nd} year B.Tech. students), IIT Guwahati, India. July - November, 2017 - 2019.
- MA101: *Single variable calculus and Multi variable calculus* (offered for 1^{st} year B.Tech. students), IIT Guwahati, India. November, 2020 - current.