A POSTERIORI ERROR ESTIMATES FOR FINITE ELEMENT DISCRETIZATIONS OF PARABOLIC OPTIMAL CONTROL PROBLEMS

by

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A POSTERIORI ERROR ESTIMATES FOR FINITE ELEMENT DISCRETIZATIONS OF PARABOLIC OPTIMAL CONTROL PROBLEMS

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by

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Dedicated to my beloved parents

Ram Vishal Namdev (Father)

and

Sumiran Devi (Mother)

I am greatly indebted to them for their encouragement and support.

– Gratitude makes sense of our past, brings peace for today,

and creates a vision for tomorrow

⋆ ⋆ ⋆

DECLARATION

I hereby declare that the information presented in this thesis entitled "**A Posteriori Error Estimates for Finite Element Discretizations of Parabolic Optimal Control Problems**", has done by my own account of research, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, under the supervision of **Dr. Rajen Kumar Sinha,** Professor, for the award of the degree of Doctor of Philosophy and this work has not been previously submitted by for a degree or diploma at this institute or any tertiary educational institution.

September, 2021

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CERTIFICATE

It is certified that the work contained in this thesis entitled "**A Posteriori Error Estimates for Finite Element Discretizations of Parabolic Optimal Control Problems**", submitted by **Ram Manohar**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy, has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

September, 2021

Dr. Rajen Kumar Sinha
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– *"Research is like a roller-coaster ride"*,

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September, 2021

With regards
Ram Manohar

Abstract

The main objective of this thesis is to derive *a posteriori* error estimates for finite element discretizations of optimal control problems governed by parabolic partial differential equations. Both distributed and boundary parabolic optimal control problems are considered and analyzed.

We first study $L^\infty(0,T;L^2(\Omega))-$*a posteriori* error analysis for parabolic optimal control problem (POCP) with distributed control. To discretize the state and co-state variables we use the piecewise linear and continuous finite elements, while the piecewise constant functions are used to discretize the control variable. The backward Euler scheme is applied for the time discretization. An elliptic reconstruction technique in conjunction with energy argument is used to derive *a posteriori* error estimates for the state and costate variables in the $L^\infty(0,T;L^2(\Omega))$-norm. The first-order necessary optimality condition is used to derive the error estimate for the control variable in the $L^\infty(0,T;L^2(\Omega))$-norm. The second problem considers the POCP with distributed control and discusses *a posteriori* error analysis for both semi-discrete and fully discrete finite element method. The variational discretization is used to approximate the state and co-state variables with the piecewise linear and continuous functions, while the control variable is computed by using the implicit relation between the control and co-state variables. The temporal discretization is based on the backward Euler method. The key feature of this approach is not to discretize the control variable but to implicitly utilize the optimality conditions for the discretization of the control variable. We use the elliptic reconstruction technique in conjunction with heat kernel estimates for linear parabolic problem to derive *a posteriori* error estimates for the state, co-state and control variables in the $L^\infty(0,T;\ L^\infty(\Omega))$-norm. Use of elliptic reconstruction technique greatly simplifies the analysis by allowing us to take the advantage of existing elliptic maximum norm error estimates and the heat kernel estimate. Our next problem focuses on finite element approximations of the POCP with controls acting on a lower dimensional manifold. The manifold considered here is either a point, or a curve or a surface which is lying completely in the space domain. In addition, the manifold is assumed to be either time independent or evolved with the time. The state and co-state variables are approximated by the piecewise linear and continuous finite elements whereas the piecewise constant functions are employed to approximate the control variable. Moreover, the discrete-in-time scheme is based on the backward Euler method. We derive *a posteriori* error estimates for the various dimensions of the manifold.

We next turn our attention to study local *a posteriori* error estimates for the space-time finite element approximations of parabolic boundary control problem (PBCP). In many engineering applications, it is often useful to study the behavior of the state and co-state variables in a small neighborhood of the boundary. Therefore, *a posteriori* error estimators in some suitable local norms have become more useful, and the derivation of these estimates is not straightforward. Therefore, an attempt has been made in this thesis to derive local *a posteriori* error estimates for the PBCP. The space-time discretization is accomplished by using the piecewise linear and continuous finite element approximations for the state and co-state variables while the piecewise constant function spaces employed for the control variable. We use the backward Euler method to approximate the time derivative. We derive three different reliable local *a posteriori* error estimates for Neumann boundary control problems with the observations of the boundary state, the distributed state and the final state. Our derived estimators are of local character in the sense that the leading terms of the estimators depend on the small neighborhood of the boundary. These new local *a posteriori* error bounds can be used to study the behaviour of the state and co-state variables around the boundary and provide the necessary feedback in terms of the error indicators for the adaptive mesh refinements in the finite element method. Our last problem is devoted to study finite element approximation for nonlinear PBCP. The error analysis is carried out by using the piecewise linear and continuous finite elements for the approximation of the state and co-state variables whereas the approximation of the control variable is done with the piecewise constant functions. The backward Euler method is used to approximate the time derivative. The reliable type local *a posteriori* error bounds for the state, co-state and control variables in the $L^2(0,T;L^2(\Gamma))$-norm are derived, while the *a posteriori* error estimates for the control variable is established by assuming the second-order optimality condition.

Computational results are presented to illustrate the performance of the derived estimators.

Contents

NOMENCLATURE

POCP	Parabolic Optimal Control Problem
PBCP	Parabolic Boundary Control Problem
LDMF	Lower Dimensional Manifold
$\mathbb{R}^d$	d-dimensional Euclidean space
Ω	Bounded convex domain
T	Final time
Ω_T	$\Omega \times (0, T]$
$\Gamma = \partial\Omega$	Boundary of Ω
Γ_T	$\partial\Omega \times [0, T]$
$\mathcal{A}$	Second order linear elliptic partial differential operator of the form $-\nabla \cdot (A\nabla) + a_0$
$\mathcal{A}^*$	Adjoint of the operator $\mathcal{A}$
$A = \{a_{ij}(x)\}$	Coefficient matrix corresponding to the operator $\mathcal{A}$
$a(\cdot, \cdot)$	Bilinear form corresponding to the elliptic operator $\mathcal{A}$
α_0	Continuity constant for $a(\cdot, \cdot)$
α_1	Coercivity constant for $a(\cdot, \cdot)$
α	A regularization parameter
y	Exact state
p	Exact co-state
u	Exact control
y_0	Initial state
y_{ds}	desired state
u_a	Lower bound of the control variable for POCP/PBCP
u_b	Upper bound of the control variable for POCP/PBCP
$\mathcal{J}(\cdot, \cdot)$	Cost functional for POCP/PBCP
$j(\cdot)$	Reduced cost functional for POCP/PBCP

γ_s	Positive constant related to second-order optimality condition
$\gamma(t)$	Lower dimensional manifold
$L^p(\Omega),\ 1 \leq p \leq \infty$	Standard Lebesgue space of order p over Ω
$\|\cdot\|_{L^p(\Omega)}$	Norm on space $L^p(\Omega)$
$\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$	Norm on space $L^2(\Omega)$
$\|\cdot\|_{\mathscr{B}}$	Norm on space $\mathscr{B} := L^2(\Gamma)$
$(\cdot,\cdot)$	Standard L^2-inner product on Ω
$(\cdot,\cdot)_{\Omega_T}$	L^2-inner product on Ω_T
$(\cdot,\cdot)_{\mathscr{B}}$	L^2-inner product on Γ
$W^{k,p}(\Omega)$	Standard Sobolev space of order (k,p) over Ω
$\|\cdot\|_{W^{k,p}(\Omega)} = \|\cdot\|_{k,p}$	Norm on space $W^{k,p}(\Omega)$
$H^k(\Omega)$	Hilbert space $W^{k,2}(\Omega)$
$H_0^1(\Omega)$	Space of functions in $H^1(\Omega)$ that vanish on the boundary of Ω in the sense of trace
$\mathcal{C}^k(\overline{\Omega})$	Space of functions with continuous derivatives up to and including order k in $\overline{\Omega}$
$\mathcal{C}_0^k(\Omega)$	Space of all $C^k(\Omega)$ functions with compact support in Ω
$\mathcal{C}_0^\infty(\Omega)$	Space of all infinitely differentiable functions with compact support in Ω
$\text{supp}(\phi)$	Support of ϕ
$\mathscr{T}_h$	Shape regular, conforming triangulations of the domain Ω
$\mathscr{T}_h^n$	Shape regular, conforming triangulations of domain Ω at time level $t = t_n$
$\mathscr{T}_{\mathscr{B}}$	Triangulations of the boundary
$\mathscr{T}_{\mathscr{B}}^n$	Triangulations of the boundary at time level $t = t_n$
$\text{diam}(K)$	Longest side of the element K

h_K	diam(K)
h	$\max_{K\in\mathscr{T}_h} h_K$
$\mathcal{E}_h$	Collection of all edges (faces) E of interelement K of $\mathscr{T}_h$
$\mathcal{A}_h^n$	Discrete elliptic operator
$\mathcal{E}_n/\mathcal{E}_h^n$	Set of internal edges (faces) of $K \in \mathscr{T}_h^n$
$\mathcal{M}_h^n$	Collection of all triangles $K \in \mathscr{T}_h^n$ such that K lying on the manifold $\gamma(t)$
$\left[\frac{\partial v}{\partial n_E}\right]$	Jump of ∇v across across an element edge or face E
Σ_h	Union of all internal sides $\bigcup_{E\in\mathcal{E}_h} E$
Σ_n	Union of all internal sides $\bigcup_{E\in\mathcal{E}_n} E$
I_n	n-th sub-interval of $[0,T]$
k_n	Time step size
$\mathcal{L}_h^n$	L^2-projection operator
$\mathcal{L}_n$	L^2-projection operator onto the piecewise constant space $\mathbb{P}_0(I_n)$ for each time interval I_n
Π_n	Clément-type interpolation operator
Π_h/Π_h^n	Standard Lagrange interpolation operator
$\hat{\Pi}_h$	Average interpolation operator
$\mathbb{P}_m$	Space of polynomials with *degree* less than or equals to m
V_h^0/V_h	Finite element space for the state and co-state variables
U_{ad}	Set of admissible controls
$U_{ad,h}$	Finite element space of U_{ad}
V_h^n	Finite element space for the state and co-state variables at time level $t = t_n$
$U_{ad,h}^n$	Finite element space for the control at time $t = t_n$
$\beta_{1,n}$	Space error estimator for the state variable

$\beta_{2,n}$	Temporal error estimator for the state variable
$\delta_{1,n}$	Space error estimator for the co-state variable
$\delta_{2,n}$	Temporal error estimator for the co-state variable
$\beta_{3,n},\ \beta_{4,n},\ \delta_{3,n},\ \delta_{4,n}$	Data approximation error estimators
$\mathfrak{R}_{p,-j}$	Elementwise error indicator for $K \in \mathscr{T}_h$ with $1 \leq p \leq \infty$ and $j \geq 0$
$\mathscr{R}_{p,-j}$	Global error estimator corresponding to $\mathfrak{R}_{p,-j}$
$\hat{\mathfrak{R}}_{p,-j}$	Elementwise error indicator for $\hat{K} \in \mathscr{T}_{h_1} \wedge \mathscr{T}_{h_2}$ with $1 \leq p \leq \infty$ and $j \geq 0$
$\hat{\mathscr{R}}_{p,-j}$	Global error estimator corresponding to $\hat{\mathfrak{R}}_{p,-j}$
$\mathscr{E}_{3,1}$	Initial data estimator
$\mathscr{E}_{3,2}$	Spatial error estimator of the state variable for POCP with distributed control
$\mathscr{E}_{3,3}$	Temporal error estimator of the state variable for POCP with distributed control
$\mathscr{E}_{3,4}$	Spatial error estimator of the co-state variable for POCP with distributed control
$\mathscr{E}_{3,5}$	Temporal error estimator of the co-state variable for POCP with distributed control
$\mathscr{E}_{3,6}$	Error estimator of the control variable for POCP with distributed control
$\mathscr{E}_{3,7}$	L^{∞}- type error estimator of the state variable for POCP with distributed control
$\mathscr{E}_{3,8}$	L^{∞}- type error estimator of the co-state variable for POCP with distributed control
$\mathscr{E}^n_{4,i},\ \tilde{\mathscr{E}}^n_{4,i},\ \widehat{\mathscr{E}}^n_{4,i},\ i = 1,\, 2$	Control error estimators on the set of triangulations $\mathcal{M}^n_h$ for POCP with control acting on LDMF

$\mathscr{E}^n_{4,3}$, $\widehat{\mathscr{E}}^n_{4,3}$	Spatial error estimator of the co-state variable for POCP with control acting on LDMF
$\mathscr{E}^n_{4,4}$, $\widehat{\mathscr{E}}^n_{4,4}$	Temporal error estimator of the co-state variable for POCP with control acting on LDMF
$\mathscr{E}^n_{4,5}$, $\widehat{\mathscr{E}}^n_{4,6}$	Spatial error estimator of the state variable for POCP with control acting on LDMF
$\widehat{\mathscr{E}}^n_{4,5}$, $\mathscr{E}^n_{4,6}$	Space-time error estimator of the state variable for POCP with control acting on LDMF
$\mathscr{E}^n_{4,0}$, $\mathscr{E}^n_{4,7}$, $\widehat{\mathscr{E}}^n_{4,7}$, $\widehat{\mathscr{E}}^n_{4,8}$, $\tilde{\mathscr{E}}^n_{4,0}$, $\widehat{\mathscr{E}}^n_{4,0}$, $\widehat{\mathscr{E}}^n_{4,8}$	Data approximation errors of the state variable for POCP with control acting on lower dimensional manifold
$\mathscr{E}_{5,1}$, $\hat{\mathscr{E}}_{5,1}$, $\tilde{\mathscr{E}}_{5,1}$	Control error estimators for PBCP
$\mathscr{E}_{5,2}$, $\hat{\mathscr{E}}_{5,2}$, $\tilde{\mathscr{E}}_{5,2}$	Approximation error estimator of the co-state variable for the PBCP
$\mathscr{E}_{5,3}$, $\hat{\mathscr{E}}_{5,3}$, $\tilde{\mathscr{E}}_{5,3}$	Temporal error estimator of the co-state variable for the PBCP
$\mathscr{E}_{5,4}$, $\hat{\mathscr{E}}_{5,4}$, $\tilde{\mathscr{E}}_{5,4}$	Approximation error estimator of the co-state variable for the PBCP
$\mathscr{E}_{5,5}$, $\tilde{\mathscr{E}}_{5,5}$	Approximation error estimator of the state variable for the PBCP
$\mathscr{E}_{5,6}$, $\tilde{\mathscr{E}}_{5,6}$	Temporal error estimator of the state variable for the PBCP
$\hat{\mathscr{E}}_{5,5}$	Temporal error estimator of the state variable for the PBCP
$\hat{\mathscr{E}}_{5,6}$, $\mathscr{E}_{5,7}$, $\tilde{\mathscr{E}}_{5,7}$,	Approximation error estimator of the state variable for the PBCP

$\hat{\mathscr{E}}_{5,7}$, $\mathscr{E}_{5,8}$,$\tilde{\mathscr{E}}_{5,8}$, $\hat{\mathscr{E}}_{5,8}$, $\mathscr{E}_{5,9}$, $\tilde{\mathscr{E}}_{5,9}$, $\tilde{\mathscr{E}}_{5,10}$	Data approximation error estimators of the state variable for the PBCP
$\mathscr{E}_{6,1}$	Control error error estimators for nonlinear PBCP
$\mathscr{E}_{6,2}$	Approximation error estimator of the co-state variable for the nonlinear PBCP
$\mathscr{E}_{6,3}$	Temporal error estimator of the co-state variable for the nonlinear PBCP
$\mathscr{E}_{6,4}$	Approximation error estimator of the co-state variable for the nonlinear PBCP
$\mathscr{E}_{6,6}$	Temporal error estimator of the state variable for the nonlinear PBCP
$\mathscr{E}_{6,7}$	Approximation error estimator of the state variable for the nonlinear PBCP
$\mathscr{E}_{6,8}$	Approximation error estimator of the state variable for the nonlinear PBCP
$\mathscr{E}_{6,5}$, $\mathscr{E}_{6,9}$, $\mathscr{E}_{6,10}$	Data approximation error estimator of the state variable for the nonlinear PBCP

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1

INTRODUCTION

This chapter is introductory. At the outset we describe the model equations to be investigated in the thesis. More precisely, we introduce the optimal control problems governed by parabolic partial differential equations with distributed controls as well as boundary controls. Some useful notations and the preliminary materials to be used throughout the thesis are presented. It also contains a brief survey of the relevant literature and motivation for the present study. The last section of this chapter describes the structure of the thesis.

1.1 Description of the problems

In this section, we describe the parabolic optimal control problems (POCPs) with distributed and boundary controls. In addition, we also discuss a special case of the distributed optimal control problem where the controls are acting on a lower dimensional manifold. The model equations considered in this thesis are described below.

Model Equation I. Consider the following POCPs with distributed control:

$$\min_{u\in U_{ad}} \mathcal{J}(u,y) = \min_{u\in U_{ad}} \{\mathcal{G}(y) + \mathcal{H}(u)\} \tag{1.1}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f + Bu & \text{in } \Omega_T, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_T, \end{cases} \tag{1.2}$$

and the control constraints

$$u_a \leq u(x,t) \leq u_b \quad \text{a.e.} \quad \text{in } \Omega_T, \tag{1.3}$$

where Ω is a convex bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\Gamma := \partial\Omega$, $\Omega_T = \Omega \times (0, T]$ and $\Gamma_T = \partial\Omega \times (0, T]$ with $T < \infty$. The second order Laplacian operator Δ is defined as $\Delta := \displaystyle\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$. Here, $y = y(x, t)$ represents the state variable and $u = u(x, t)$ is the control variable. The set of admissible controls is defined by

$$U_{ad} = \big\{ u \in X : \;\; u_a \leq u \leq u_b \;\; a.e. \;\text{ in }\; \Omega_T \big\}$$

with u_a, $u_b \in \mathbb{R}$ fulfill $u_a < u_b$. Further, U_{ad} is a nonempty closed convex set in $X = L^2(0, T; L^2(\Omega))$. We shall make the following assumptions on the functionals $\mathcal{G}$, $\mathcal{H}$ and the operator B:

A1. The functionals $\mathcal{G}$ and $\mathcal{H}$ are continuously differentiable on the state space (V) and the control space (X), respectively. In addition, $\mathcal{G}$ and $\mathcal{H}$ are strictly convex on V and X, respectively with $\mathcal{H}(u) \to +\infty$ as $\|u\|_X \to +\infty$. Further, $\mathcal{G}$ is bounded below and $\mathcal{G}'$ is locally Lipschitz continuous.

A2. B is a continuous and linear operator from X to X.

The functionals $\mathcal{G}$ and $\mathcal{H}$ are of the form

$$\mathcal{G}(y) := \frac{1}{2} \int_0^T \|y - y_{ds}\|^2_{L^2(\Omega)} dt, \;\; \text{and} \;\; \mathcal{H}(u) := \frac{\alpha}{2} \int_0^T \|u\|^2_{L^2(\Omega)} dt,$$

where the constant $\alpha > 0$ which can be viewed as a measure of the energy costs needed to implement the control u. Mathematically, this term is known as a regularization parameter. The control problem (1.1)–(1.3) arises in many engineering and physical processes such as heat conduction [61], fluid mechanics [89], and material sciences [83]. The functional $\mathcal{J}$ in (1.1) is to be minimized over the subject to the state equations which satisfies the control constraints. Further, the functional $\mathcal{J}$ in (1.1) is referred as the *cost functional* or the *performance measure*. An optimal control problem reads: Find an input control variable u such that the cost functional $\mathcal{J}$ is minimized, and the pair (u, y) is subjected to satisfy the state equation and the constraints inequality. Note that the second term of the functional $\mathcal{J}(u, y)$, i.e., $\mathcal{H}(u) := \frac{\alpha}{2} \int_0^T \|u\|^2_{L^2(\Omega)}\, dt$ is proportional to the consumed energy. Thus, the functional $\mathcal{J}$ to be minimized is a compromise between the energy consumption and finding the control variable (u) such that the state variable (y) is close to the desired state (y_{ds}).

Model Equation II. Let Ω be an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\Gamma := \partial\Omega$. We now consider the following control problem with controls acting on lower dimensional manifold:

$$\min_{u \in U_{ad}} \mathcal{J}(u, y) = \min_{u \in U_{ad}} \big\{ \mathcal{G}(y) + \mathcal{H}(u) \big\} \tag{1.4}$$

subject to state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = u(x,t)\delta_{\gamma(t)}(x) & \text{in } \Omega_T, \\ y(\cdot,0) = y_0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_T, \end{cases} \tag{1.5}$$

where $\Omega_T = \Omega \times (0,T]$ and $\Gamma_T = \partial\Omega \times (0,T]$ with $T < \infty$. In the above, the initial state $y_0 \in H_0^1(\Omega)$, the desired state $y_{ds} \in L^2(0,T;L^2(\Omega))$, $y = y(x,t)$ represents the state variable and $u = u(x,t)$ refers the control variable. The set of admissible controls U_{ad} is defined by

$$U_{ad} := \big\{u \in X : u_a \le u(x,t) \le u_b \ \text{ a.e. on } \gamma(t) \ \text{ a.a. } t \in [0,T]\big\}, \tag{1.6}$$

where u_a, u_b are real constants with $u_a < u_b$ or constant vectors according to the dimension of the manifold $\gamma(t)$. Here, $\gamma(t) \subset \Omega$ is an $r-$dimensional manifold in Ω with $0 \le r \le d-1$ $(d = 2, 3)$ for all $t \in [0,T]$. If $r = 0$, $\gamma(t)$ represents a single point or a finite number of points for each $t \in [0,T]$. When $r = 1$, $\gamma(t)$ is a $\mathcal{C}^2$-curve for each $t \in [0,T]$. For $r = 2$, $\gamma(t)$ refers to a $\mathcal{C}^2$-surface for each $t \in [0,T]$. The control space X may be considered as $L^2(0,T;\mathbb{R}^m)$ if $r = 0$, and $X = L^2(0,T;L^2(\gamma(t)))$ for $r \ge 1$. For the existence and uniqueness of the control problem (1.4) – (1.6), we refer to [33].

The control problems of the form (1.4) – (1.6) with Dirac measure are used in the design and management of wastewater treatment systems, the disposal of sea outfalls discharging polluting effluent from a sewerage systems [67]. There are also applications in inverse problems [3, 52]: For example, in environmental data simulation one seeks to determine the magnitude of pollutant emissions at some fixed locations from measurements. Another application in this context concerns control of pollution sources in industrial areas, ensuring required air and/or water quality in populated areas [104].

Model Equation III. In this problem, we consider the boundary control problem governed by parabolic partial differential equations (PDEs) of the form:

$$\min_{u\in U_{ad}} \mathcal{J}(u,y) = \min_{u\in U_{ad}} \big\{\mathcal{G}(y) + \mathcal{H}(u)\big\} \tag{1.7}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + \mathcal{A}y = f & \text{in } \Omega_T, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ (A\nabla y)\cdot \mathbf{n}_E = Bu + \mathrm{x}_b & \text{on } \Gamma_T, \end{cases} \tag{1.8}$$

and the control constraints

$$u_a \leq u(x,t) \leq u_b \quad \text{a.e.} \quad \text{in } \Gamma_T, \tag{1.9}$$

where Ω is a convex bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\Gamma := \partial\Omega$, $\Omega_T = \Omega \times (0,T]$ and $\Gamma_T = \partial\Omega \times (0,T]$ with $T < \infty$. The operator $\mathcal{A}$ is a second order elliptic partial differential operator of the form

$$\mathcal{A}y := -\nabla \cdot (A(x)\nabla y) + a_0(x)y \tag{1.10}$$

with $A(\cdot) = (a_{i,j}(x))_{d\times d} \in (W^{1,\infty}(\Omega))^{d\times d}$ and $a_0(x) \in L^\infty(\Omega)$. Further, we assume that the coefficient matrix $A(\cdot)$ is a symmetric and positive definite such that there exists a positive constant c_0 satisfying $a_0(x) \geq c_0$, $\forall x \in \Omega$ and the following ellipticity condition

$$\sum_{i,j=1}^{d} a_{i,j}(x)\mathrm{x}_i\mathrm{x}_j \geq c_0\|\mathrm{x}\|^2_{\mathbb{R}^d}, \qquad \forall \mathrm{x} \in \mathbb{R}^d, \;\; x \in \Omega, \tag{1.11}$$

where $\|\cdot\|_{\mathbb{R}^d}$ denotes the standard Euclidean norm on d-dimensional Euclidean space $\mathbb{R}^d$. Note that the control variable u appears in the Neumann boundary condition, and $\mathbf{n}_E$ is the unit outward normal to the boundary. The set of admissible controls is defined by

$$U_{ad} = \left\{u \in X : \; u_a \leq u \leq u_b \;\; a.e. \;\; \text{in } \Gamma_T\right\}$$

with $u_a, u_b \in \mathbb{R}$ fulfill $u_a < u_b$. Further, U_{ad} is a nonempty closed convex set in $X = L^2(0,T;\mathscr{B})$ with $\mathscr{B} = L^2(\Gamma)$. We assume that

$$f \in L^2(0,T;L^2(\Omega)), \;\; y_0 \in H^1(\Omega), \;\; \mathbf{x}_b \in \mathscr{B},$$

and the operator B is a linear and continuous operator from X to X. Moreover, the functionals $\mathcal{G}$ and $\mathcal{H}$ satisfy the following assumptions:

B1. The functionals $\mathcal{G}$ and $\mathcal{H}$ are continuously differentiable on the observation space $\mathcal{O}_{sp}$ (the space $\mathcal{O}_{sp}$ may be $L^2(0,T;L^2(\Omega))$ or $L^2(0,T;L^2(\Gamma))$ or $L^2(\Omega)$) and the control space X.

B2. The functionals $\mathcal{G}$ and $\mathcal{H}$, respectively are strictly convex on the spaces $\mathcal{O}_{sp}$ and X with $\mathcal{H}(u) \to +\infty$ as $\|u\|_X \to +\infty$. Further, $\mathcal{G}$ is bounded below, and $\mathcal{G}'$ satisfies the locally Lipschitz continuity condition.

We shall investigate local *a posteriori* error estimates for the problem (1.7) – (1.9) with three different choices of observation spaces. This leads to the following three cases.

To begin with, we first discuss the case for the observation space of the boundary $\mathcal{O}_{sp_1} = L^2(0,T;\mathscr{B})$.

Case I (*Observation space* $\mathcal{O}_{sp} = \mathcal{O}_{sp_1}$). In this case, we consider the control problem (1.7)-(1.9) with observation of the boundary state. Define

$$\mathcal{G}(y) := \int_0^T \int_\Gamma g(y)ds\,dt \;\text{ and }\; \mathcal{H}(u) := \int_0^T \int_\Gamma h(u)ds\,dt,$$

where $g(\cdot)$ and $h(\cdot)$ are strictly convex and continuously differentiable functions so that our conditions on $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ are fulfilled. Note that, both functionals $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ are defined on the boundary. For example, functionals $\mathcal{G}$ and $\mathcal{H}$ are

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|^2_{\mathscr{B}}dt, \;\text{ and }\; \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2_{\mathscr{B}}dt,$$

where $\alpha > 0$ is a fixed constant.

Next, we consider the problem $(1.7)-(1.9)$ with observation of the distributed state, i.e., the observation space $\mathcal{O}_{sp_2} = L^2(0,T;L^2(\Omega))$.

Case II (*Observation space* $\mathcal{O}_{sp} = \mathcal{O}_{sp_2}$). This case considers the parabolic boundary control problem (PBCP) $(1.7)-(1.9)$ with observation of the distributed state. Again, let $g(\cdot)$ and $h(\cdot)$ be strictly convex and continuously differentiable functions on $\mathcal{O}_{sp_2}$ and X, respectively. Let

$$\mathcal{G}(y) := \int_0^T \int_\Omega g(y)dx\,dt \quad\text{and}\quad \mathcal{H}(u) := \int_0^T \int_\Gamma h(u)ds\,dt$$

be such that our assumptions on $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ are fulfilled. For example, we may think of the functionals $\mathcal{G}$ and $\mathcal{H}$ as:

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|^2_{L^2(\Omega)}dt \quad\text{and}\quad \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2_{\mathscr{B}}dt.$$

The last case considers PBCP $(1.7)-(1.9)$ with observation of the final state $y(x,T)$. That is, the observation space is $\mathcal{O}_{sp_3} = L^2(\Omega)$.

Case III (*Observation space* $\mathcal{O}_{sp} = \mathcal{O}_{sp_3}$). This is a very vital and factual case. In this case, the co-state equation for the control problem $(1.7)-(1.9)$ with observation of the final state turns out to be different from the previous ones. Let us define

$$\mathcal{G}(y) := \int_\Omega g(y(x,T))dx \;\text{ and }\; \mathcal{H}(u) := \int_0^T \int_\Gamma h(u)ds\,dt,$$

where again $g(\cdot)$ and $h(\cdot)$ are strictly convex and continuously differentiable functions. For example, $\mathcal{G}$ and $\mathcal{H}$ are given by

$$\mathcal{G}(y) := \frac{1}{2}\|y(x,T) - y_{ds}\|^2_{L^2(\Omega)} \;\text{ and }\; \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2_{\mathscr{B}}dt,$$

respectively.

Note that $g(\cdot)$ and $h(\cdot)$ satisfies the following convexity assumptions:

$$\begin{aligned} (g'(v_1) - g'(v_2), v_1 - v_2)_{\mathcal{O}_{sp}} &\geq 0 \qquad \forall v_1,\ v_2 \in \mathcal{O}_{sp}, & (1.12)\\ (h'(w_1) - h'(w_2), w_1 - w_2)_{\mathscr{B}} &\geq C\|w_1 - w_2\|^2_{\mathscr{B}} \qquad \forall w_1,\ w_2 \in \mathscr{B}. & (1.13) \end{aligned}$$

The convexity condition (1.12) is satisfied in all three cases with $\mathcal{O}_{sp} = \mathcal{O}_{sp_1}$ for the **Case I**, $\mathcal{O}_{sp} = \mathcal{O}_{sp_2}$ for the **Case II** and $\mathcal{O}_{sp} = \mathcal{O}_{sp_3}$ for the **Case III**, respectively.

The boundary control problem of the form (1.7)–(1.9) are generally considered to be physically more realistic because the control is applied only through the boundary. From physical point of view, the actuators and sensors are often applied through the boundary in the thermal process; food processing and packaging systems; industrial processes; heating/cooling process. In the thermal process, boundary controls are known as actuate heat flux. Thermal flow sensors are used in the detection of boundary layer separation (flow separation) and reattachment on airfoils [6, 90].

Next, we consider the boundary control problem governed by nonlinear PDEs, where $\mathcal{G}(y)$ and $\mathcal{H}(u)$ are defined as

$$\mathcal{G}(y) = \frac{1}{2}\int_0^T \|y - y_{ds}\|^2_{\mathscr{B}}\, dt, \quad \text{and} \quad \mathcal{H}(u) = \frac{\alpha}{2}\int_0^T \|u\|_{\mathscr{B}}\, dt,$$

respectively, where $\mathscr{B} = L^2(\Gamma)$.

Model Equation IV. In this problem, we consider the following nonlinear PBCP:

$$\min_{u \in U_{ad}} \mathcal{J}(u, y) = \min_{u \in U_{ad}} \frac{1}{2}\int_0^T \left\{\|y - y_{ds}\|^2_{\mathscr{B}} + \alpha\|u\|^2_{\mathscr{B}}\right\} dt \tag{1.14}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + \mathcal{A}y + \varphi(y) = f \ \text{ in } \ \Omega_T, \\ y(x, 0) = y_0(x) \ \text{ in } \ \Omega, \\ (A\nabla y) \cdot \mathbf{n}_E = Bu + \mathrm{x}_b \quad \text{on } \ \Gamma_T, \end{cases} \tag{1.15}$$

and the control constraints

$$u_a \leq u(x, t) \leq u_b \quad \text{a.e. in } \ \Gamma_T, \tag{1.16}$$

where Ω is a convex bounded domain in $\mathbb{R}^d$ $(d \leq 3)$ with Lipschitz boundary $\Gamma := \partial\Omega$. Further, we set $\Omega_T = \Omega \times (0, T]$ and $\Gamma_T = \partial\Omega \times (0, T]$ with fixed $T < \infty$. The operator $\mathcal{A}$ is given by (1.10) and $\mathbf{n}_E$ is the unit outward normal to the boundary. Here, the unknown functions $y = y(x,t)$ and $u = u(x,t)$, respectively, represent the state and control variables. We shall make the following assumptions on y_{ds} (the desired state), f (the source function), y_0 (the initial data) and $\mathtt{x}_b$ (the boundary function):

$$y_0 \in C(\overline{\Omega}), \;\; y_{ds} \in L^\infty(0,T;L^\infty(\Gamma)), \;\; f \in L^\infty(0,T;L^\infty(\Omega)), \;\; \mathtt{x}_b \in L^\infty(\Gamma).$$

Let B be a linear and continuous operator from $L^\infty(0,T;L^\infty(\Gamma))$ to $L^\infty(0,T;L^\infty(\Gamma))$. The nonlinear function φ satisfies the following conditions:

C1. Let $\varphi : \Omega_T \times \mathbb{R} \to \mathbb{R}$ be a measurable function with respect to (x,t) and twice continuously differentiable with respect to $y \in \mathbb{R}$.

C2. Moreover, φ is a monotone non-decreasing with respect to $y \in \mathbb{R}$ and for all $(x,t) \in \Omega_T$, i.e., $\varphi_y(x,t,y) \geq 0, \;\forall (x,t) \in \Omega_T$.

C3. There exists a positive constant C_M such that

$$|\varphi(x,t,\cdot)| + |\varphi_y(x,t,\cdot)| + |\varphi_{yy}(x,t,\cdot)| \leq C_M \quad \text{for all} \quad (x,t) \in \Omega_T,$$

where $\varphi_y(x,t,\cdot)$ and $\varphi_{yy}(x,t,\cdot)$ denote the first and second order partial derivatives of φ with respect to y, respectively.

C4. For all $L > 0$ and all $y_1, \; y_2 \in \mathbb{R}$ with $|y_1 - y_2| < L$, there exists a positive constant C_L such that

$$|\varphi_{yy}(x,t,y_1) - \varphi_{yy}(x,t,y_2)| \leq C_L |y_1 - y_2| \quad \text{for all} \quad (x,t) \in \Omega_T.$$

Note that, the control variable u presents in the Neumann boundary conditions. The set of admissible controls is defined by

$$U_{ad} = \big\{ u \in X : \;\; u_a \leq u(x,t) \leq u_b \quad a.e. \quad (x,t) \in \Gamma_T \big\}$$

with $u_a, \, u_b \in \mathbb{R}$ fulfil $u_a < u_b$. Furthermore, U_{ad} is a nonempty closed convex set in $L^\infty(0,T;L^\infty(\Gamma))$.

Many applications of nonlinear parabolic boundary control problems arise in stefan-like problems, thermo-chemical flow phenomenon, solidification/ dissolution of crystals from a solution of their own liquid. Some other applications of nonlinear parabolic control problems include heat transfer in large reservoirs; heat transfer in porous media and frost propagation. For a more detailed discussion on mathematical formulation for this kind of problem, we refer to [27, 72].

1.2 Notations and preliminaries

This section introduces some standard notations and preliminary materials, which are important in the set-up of the thesis. All the functions considered in the thesis are real-valued. Let Ω be a convex bounded domain in $\mathbb{R}^d$. We denote the boundary of Ω by $\Gamma := \partial\Omega$. Let $\mu = (\mu_1, \ldots, \mu_d)$ be a non negative integer with d-components and the order of μ is denoted by $|\mu| = \mu_1 + \ldots + \mu_d$. Further, let $x = (x_1, x_2, \ldots, x_d) \in \Omega$ and $dx = dx_1, \ldots, dx_d$. Then, the μ-th derivative of a function ψ is denoted by $D^\mu\psi$ with

$$D^\mu\psi := \frac{\partial^{|\mu|}\psi}{\partial x_1^{\mu_1} \ldots \partial x_d^{\mu_d}}.$$

We now recall some standard function spaces. Let $L^p(\Omega)$, $1 \leq p < \infty$, be the linear space of equivalence classes of measurable functions ψ in Ω such that the measure of $\int_\Omega |\psi|^p \, dx$ exists and finite. Moreover, the associated norm on $L^p(\Omega)$ with $1 \leq p < \infty$, is

$$\|\psi\|_{L^p(\Omega)} := \Big(\int_\Omega |\psi|^p \, dx \Big)^{\frac{1}{p}}.$$

For $p = \infty$, $L^\infty(\Omega)$ is a Banach space of all Lebesgue measurable and essentially bounded functions, equipped with the norm

$$\|\psi\|_{L^\infty(\Omega)} := ess \sup_{x\in\Omega} |\psi| < \infty.$$

Again, for $p = 2$, $L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(\psi_1, \psi_2) = \int_\Omega \psi_1 \, \psi_2 \, dx, \quad \forall \psi_1, \, \psi_2 \in L^2(\Omega).$$

We denote the support of a function ψ by $supp(\psi)$ which is a closure of all points x with $\psi(x) \neq 0$, i.e.,

$$supp(\psi) \;:=\; \overline{\{x : \;\; \psi(x) \neq 0\}}.$$

Let $\mathcal{C}^k(\overline{\Omega})$ be the collection of all functions with continuous derivatives upto and including order k in $\overline{\Omega}$, where k is a non-negative integer. Further, let $\mathcal{C}_0^k(\Omega)$ be the space of all $\mathcal{C}^k(\Omega)$ functions with compact support in Ω. Moreover, the space $\mathcal{C}_0^\infty(\Omega)$ consists of all infinitely differentiable functions with compact support in Ω. For a non-negative integer k, let $W^{k,p}(\Omega)$ $(1 \leq p < \infty)$ be the collection of all equivalence class of functions in $L^p(\Omega)$ such that the distributional derivatives upto order k are also belongs to $L^p(\Omega)$, i.e.,

$$W^{k,p}(\Omega) := \big\{\psi \in L^p(\Omega) : \;\; D^\mu\psi \in L^p(\Omega) \;\text{ for }\; 0 \leq |\mu| \leq k\big\}.$$

The space $W^{k,p}(\Omega)$ is known as the Sobolev space of order (k,p) on Ω and is endowed with the norm

$$\|\psi\|_{k,p} := \Big(\int_\Omega \sum_{0\le|\mu|\le k} |D^\mu \psi|^p \, dx\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

For $p=\infty$, we define the norm on $W^{k,\infty}(\Omega)$ by

$$\|\psi\|_{k,\infty} := \max_{0\le|\mu|\le k} \|D^\mu\psi\|_{L^\infty(\Omega)}.$$

Further, we denote the semi-norm on $W^{k,p}(\Omega)$ by $|\cdot|_{W^{k,p}(\Omega)}$ which is defined as

$$|\psi|_{W^{k,p}(\Omega)} := \sum_{|\mu|=k} \|D^\mu\psi\|_{L^p(\Omega)}.$$

When $p=2$, we denote $W^{k,p}(\Omega)=H^k(\Omega)$. Moreover, if $k=0$ and $p=2$ then we have $W^{0,2}(\Omega)=H^0(\Omega)=L^2(\Omega)$. Let $H^k(\Omega)$ be the Hilbert space with inner product

$$(\psi_1,\psi_2)_k := \sum_{0\le|\mu|\le k} \int_\Omega D^\mu\psi_1 D^\mu\psi_2 \, dx, \qquad \psi_1,\psi_2\in H^k(\Omega).$$

Observe that the Sobolev space $H^k(\Omega)$ and $H_0^k(\Omega)$, respectively, are defined to be the closure of $\mathcal{C}^\infty(\Omega)$ and $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_k$ and semi-norm $|\cdot|_k$. Further, we shall denote the space $H^{-1}(\Omega)$ as the dual of $H_0^1(\Omega)$ with the norm

$$\|\psi_1\|_{H^{-1}(\Omega)} := \sup_{\psi_2\in H_0^1(\Omega),\,\psi_2\neq 0} \frac{(\psi_1,\psi_2)}{\|\psi_2\|_{H^1(\Omega)}}.$$

For $1\le p\le\infty$, we now define the standard Bochner spaces $L^p(0,T;\mathcal{B})$, where $\mathcal{B}$ is a real Banach space equipped with the norm $\|\cdot\|_{\mathcal{B}}$, consisting of all measurable functions $\psi:[0,T]\to\mathcal{B}$ for which

$$\begin{aligned}
\|\psi\|_{L^p(0,T;\mathcal{B})} &:= \Big(\int_0^T \|\psi(t)\|_{\mathcal{B}}^p \, dt\Big)^{\frac{1}{p}} < \infty \quad && \text{for } \ 1\le p<\infty,\\
\|\psi\|_{L^\infty(0,T;\mathcal{B})} &:= \ ess \sup_{t\in(0,T)} \|\psi(t)\|_{\mathcal{B}} < \infty && \text{for } \ p=\infty.
\end{aligned}$$

In the succeeding chapters, we use the following spaces. For a given Banach space $\mathcal{B}$, the space $H^1(0,T;\mathcal{B})$ consisting of all the measurable functions $\psi:(0,T)\to\mathcal{B}$, such that

$$\|\psi\|_{H^1(0,T;\mathcal{B})} := \Big(\int_0^T \Big\{\|\psi(t)\|_{\mathcal{B}}^2 + \Big\|\frac{\partial\psi}{\partial t}(t)\Big\|_{\mathcal{B}}^2\Big\}\, dt\Big)^{\frac{1}{2}} < \infty.$$

Furthermore, let $\mathcal{C}(0,T;\mathcal{B})$ be the space of continuous functions $\psi : [0,T] \to \mathcal{B}$ and the norm on $\mathcal{C}(0,T;\mathcal{B})$ is defined as $\|\psi\|_{\mathcal{C}(0,T;\mathcal{B})} := \max_{t\in[0,T]} \|\psi(t)\|_{\mathcal{B}} < \infty$. For a more detailed discussion on Sobolev spaces, one may refer to Adams and Fournier [1], and Grisvard [36].

Time to time we need the following inequalities for our analysis. For a proof, see Hardy *et al.* [37].

Young's inequality. *For all non-negative real numbers* a, b, *and* $1 < p < \infty$, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *the following inequality*

$$\mathsf{ab} \leq \frac{\mathsf{a}^p}{p} + \frac{\mathsf{b}^q}{q}$$

holds.

Young's inequality with ϵ-form. *For all non-negative real numbers* a, b *and every* $\epsilon > 0$, *the following inequality*

$$\mathsf{ab} \leq \frac{\mathsf{a}^2}{2\epsilon} + \frac{\epsilon\,\mathsf{b}^2}{2}$$

holds.

Discrete version of Hölder's inequality. *Let* a_i, b_i, $i = 1,\dots,d$ *be the positive real numbers. Then*

$$\sum_{i=1}^{d} |\mathsf{a}_i \mathsf{b}_i| \leq \Big(\sum_{i=1}^{d} |\mathsf{a}_i|^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{d} |\mathsf{b}_i|^q\Big)^{\frac{1}{q}},$$

where $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *In particular, for* $p = 2$ *and* $q = 2$, *the above inequality is known as the Cauchy-Schwarz inequality in* $\mathbb{R}^d$.

Integral form of the Hölder's inequality. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For real valued functions $\psi_1 \in L^p(\Omega)$ and $\psi_2 \in L^q(\Omega)$, we have

$$\int_\Omega |\psi_1\, \psi_2|\, dx \leq \|\psi_1\|_{L^p(\Omega)} \|\psi_2\|_{L^q(\Omega)}.$$

For $p = q = 2$, the above inequality is known as the Cauchy-Schwarz inequality.

The following lemma is proved to be convenient for later use.

Lemma 1.2.1 ([48])**.** *Given* $\hat{\mathsf{a}} = (\mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_d) \in \mathbb{R}^{d+1}$, $\hat{\mathsf{b}} = (\mathsf{b}_0, \mathsf{b}_1, \ldots, \mathsf{b}_d) \in \mathbb{R}^{d+1}$ *and* $\hat{\mathsf{c}} \in \mathbb{R}$ *such that*

$$|\hat{\mathsf{a}}|^2 \leq \hat{\mathsf{c}}^2 + \hat{\mathsf{a}} \cdot \hat{\mathsf{b}},$$

then we have the following

$$|\hat{\mathsf{a}}| \leq |\hat{\mathsf{b}}| + |\hat{\mathsf{c}}|,$$

where $|\cdot|$ *and* $\hat{\mathsf{a}} \cdot \hat{\mathsf{b}}$ *denote the standard Euclidean vector norm and inner product on* $\mathbb{R}^{d+1}$*, respectively.*

In this thesis, we shall use the Poincaré inequality which is stated below.

Lemma 1.2.2 (Poincaré inequality)**.** *Assume that the* $\Omega \subset \mathbb{R}^d$ *is a convex bounded domain. Then there exists a constant* $C > 0$ *such that*

$$\|\psi\|_{L^2(\Omega)} \leq C\|\nabla\psi\|_{L^2(\Omega)}, \qquad for\ every\ \psi \in H_0^1(\Omega),$$

where the constant C *depends on the domain* Ω*.*

The above inequality implies that, for any $\psi \in H_0^1(\Omega)$, the L^2-norm is bounded by the semi H^1-norm.

We now borrow some useful definitions from F. Tröltzsch [96].

Definition 1.2.1 (Directional derivative)**.** *Assume that* $\tilde{X}$ *and* $\tilde{Y}$ *are two Banach spaces. Let* $\tilde{X}_0$ *be a non-empty open subset of* $\tilde{X}$ *and* $\mathbf{f}$ *be a given map from* $\tilde{X}_0$ *to* $\tilde{Y}$*. If the limit*

$$\delta\mathbf{f}(\mathbf{x}, \mathbf{v}) := \lim_{h \to 0} \frac{\mathbf{f}(\mathbf{x} + h\mathbf{v}) - \mathbf{f}(\mathbf{x})}{h} \qquad for \quad \mathbf{x} \in \tilde{X}_0 \quad and \quad \mathbf{v} \in \tilde{X},$$

exists in $\tilde{Y}$*, then it is called the directional derivative of* $\mathbf{f}$ *at* $\mathbf{x}$ *in the direction of* $\mathbf{v}$*. If this limit exists for all* $\mathbf{v} \in \tilde{X}_0$*, then the mapping* $\mathbf{v} \mapsto \delta\mathbf{f}(\mathbf{x}, \mathbf{v})$ *is termed the first variation of* $\mathbf{f}$ *at* $\mathbf{x}$*.*

Definition 1.2.2 (Gâteaux derivative)**.** *Suppose that the first variation at* $\mathbf{x} \in \tilde{X}_0$ *exists, and suppose there exists a continuous linear operator* $T : \tilde{X}_0 \to \tilde{Y}$ *such that*

$$\delta\mathbf{f}(\mathbf{x}, \mathbf{v}) = T\mathbf{v} \qquad \forall \mathbf{v} \in \tilde{X}_0.$$

Then $\mathbf{f}$ *is said to be Gâteaux differentiable at* $\mathbf{x} \in \tilde{X}_0$*, and* T *is referred as the Gâteaux derivative of* f *at* $\mathbf{x}$*. We write* $T = \mathbf{f}'(\mathbf{x})$*.*

Definition 1.2.3 (Fréchet derivative)**.** *Assume that $\tilde{X}$ and $\tilde{Y}$ are two Banach spaces, and $\tilde{X}_0$ is a non-empty open subset of $\tilde{X}$. A mapping $\mathbf{f} : \tilde{X}_0 \to \tilde{Y}$ is said to be Fréchet differentiable at $\mathbf{x} \in \tilde{X}_0$ if there exists an operator $T \in \mathcal{L}(\tilde{X}, \tilde{Y})$ and a mapping $\mathscr{R}(\mathbf{x}, \cdot) : \tilde{X}_0 \to \tilde{Y}$ with the following properties: for all $\mathbf{v} \in \tilde{X}$ such that $\mathbf{x} + \mathbf{v} \in \tilde{X}_0$, we have*

$$\mathbf{f}(\mathbf{x} + \mathbf{v}) = \mathbf{f}(\mathbf{x}) + T\mathbf{v} + \mathcal{R}(\mathbf{x}, \mathbf{v}),$$

where the remainder $\mathcal{R}$ satisfies the condition

$$\frac{\|\mathcal{R}(\mathbf{x}, \mathbf{v})\|_{\tilde{Y}}}{\|\mathbf{v}\|_{\tilde{X}}} \to 0 \quad as \quad \|\mathbf{v}\|_{\tilde{X}} \to 0.$$

Then operator T is called the Fréchet derivative of $\mathbf{f}$ at $\mathbf{x}$, and we write $T = \mathbf{f}'(\mathbf{x})$. If T is Fréchet differentiable at every point of $\mathbf{x} \in \tilde{X}_0$, then T is said to be Fréchet differentiable in $\tilde{X}_0$.

Definition 1.2.4 (Convex functional)**.** *Let $\tilde{X}_0$ be a non-empty convex subset of $\mathbb{R}^d$ $(d =$* 2, 3*). A functional $\mathcal{F}$ from $\tilde{X}_0$ to $\mathbb{R}$ is said to be convex if it satisfies the following property: For all $\nu \in [0, 1]$ and $\mathbf{x}_1, \mathbf{x}_2 \in \tilde{X}_0$ such that*

$$\mathcal{F}(\nu \mathbf{x}_1 + (1 - \nu)x_2) \leq \nu \mathcal{F}(\mathbf{x}_1) + (1 - \nu)\mathcal{F}(\mathbf{x}_2).$$

A functional $\mathcal{F}$ is said to be strictly convex, if it satisfies the above condition with strict inequality whenever $\mathbf{x}_1 \neq \mathbf{x}_2$, $\nu \in (0, 1)$.

Before introducing the auxiliary problems, we first collect some function spaces which will be useful in our subsequent analysis. For simplicity, we set

$$\begin{aligned} \mathcal{W}_1(0,T) &:= L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega)), \\ \mathcal{W}_2(0,T) &:= L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)). \end{aligned}$$

We observe that $\mathcal{W}_1(0,T) \hookrightarrow \mathcal{C}(0,T; H_0^1(\Omega))$ and $\mathcal{W}_2(0,T) \hookrightarrow \mathcal{C}(0,T; L^2(\Omega))$ (cf., [55]).

We now introduce some auxiliary problems. For any $f \in L^2(0,T; L^2(\Omega))$, let Ψ, $\Phi \in \mathcal{W}_1(0,T)$ be solutions of the following forward and backward in time standard parabolic problems

$$\begin{cases} \Psi_t + \mathcal{A}\Psi = f & \text{in} \quad \Omega_T, \\ \Psi(\cdot, 0) = 0 & \text{in} \quad \Omega, \\ \Psi = 0 & \text{on} \quad \Gamma_T, \end{cases} \tag{1.17}$$

and

$$\begin{cases} -\Phi_t + \mathcal{A}^*\Phi = f & \text{in} \quad \Omega_T, \\ \Phi(\cdot, T) = 0 & \text{in} \quad \Omega, \\ \Phi = 0 & \text{on} \quad \Gamma_T, \end{cases} \tag{1.18}$$

respectively.

The solutions Ψ and Φ satisfy the following regularity results [55].

Lemma 1.2.3. *For a given function $f \in L^2(0,T;L^2(\Omega))$, let $\Psi,\ \Phi \in \mathcal{W}_1(0,T)$ be the solutions of the problems* (1.17) *and* (1.18)*, respectively. Then there exists a positive generic constant C_R such that*

$$\begin{aligned} \|\Psi_t\|_{L^2(0,T;L^2(\Omega))} + \|\Psi\|_{L^2(0,T;H^2(\Omega))} &\leq C_R\|f\|_{L^2(0,T;L^2(\Omega))}, \\ \|\Phi_t\|_{L^2(0,T;L^2(\Omega))} + \|\Phi\|_{L^2(0,T;H^2(\Omega))} &\leq C_R\|f\|_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

and

$$\|\Psi(\cdot,T)\|_{H^1(\Omega)} \leq C_R\|f\|_{L^2(0,T;L^2(\Omega))}, \quad \textit{and} \quad \|\Phi(\cdot,0)\|_{H^1(\Omega)} \leq C_R\|f\|_{L^2(0,T;L^2(\Omega))}.$$

Throughout the thesis, we assume that Ω is convex polygonal bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\Gamma := \partial\Omega$, where the dimension d will be specified in each chapter.

1.3 Background and motivation

This section provides a brief survey of the relevant literature pertaining to the POCPs with distributed and boundary controls. It also elucidates the motivation for the present study.

The goal of any optimal control problem is to switch the input control variable u in such a way that the output state variable $y(u)$ optimizes the cost functional $\mathcal{J}$. If the input control variable u is distributed all over the domain, it is known as the distributed control. When the control is prescribed on the boundary, it is known as the boundary control. The output state variable y is the solution of the given state equation. One can make some assumptions and restrictions on the control or state variables due to physical and technical limitations. Theory for the optimal control problems governed by PDEs are extensively discussed by J. L. Lions and F. Tröltzsch in [53, 96]. It is known that the finite element method is a widely used numerical technique for the approximations of optimal control problems. In the following section, we provide a brief survey of the relevant literature concerning finite element method for POCPs.

Finite element method for POCPs. The numerical analysis of control problems is very challenging due to the presence of control constraints which leads to stronger restriction on the regularity of the optimal solution. We refer to [50] for a discussion on the regularity of the solution of POCPs with control constraints. The numerical treatment of control problems by means of finite element method has gained popularity among the researchers [39, 40, 84, 96]. The advantage of finite element method over other numerical techniques lies in its capability to handle complex geometry in a systematic way and it has a rigorous mathematical foundation. The error analysis of finite element method is grouped into two categories: *A priori* error analysis and *a posteriori* error analysis. In the *a priori* error analysis, one predicts the error estimates of the form

$$\|y - Y_h\|_{\mathcal{X}} \leq C(y, data)\, h^r, \tag{1.19}$$

where $\|\cdot\|_{\mathcal{X}}$ is a specified norm on $\mathcal{X}$, y is the exact solution and Y_h be its finite element approximation. The constant $C(y, data)$ in (1.19) depends on the exact solution y and the given data. In the above, h denotes the mesh parameter and r refers to the order of convergence of the finite element method. In general, *a priori* error bound (1.19) is not practically usable as the constant $C(y, data)$ depends on the exact solution y, which is unknown for most of the problems. The estimate (1.19) only provides asymptotic order of convergence as $h \to 0$, and it fails to quantify the actual error. This brings to a new approximation technique of error estimation, which characterizes the accuracy of approximate solutions and is known as *a posteriori* error estimation technique. An *a posteriori* error estimate predicts a bound of the form

$$\|y - Y_h\|_{\mathcal{X}} \leq \eta(Y_h, h_K^r, data), \tag{1.20}$$

where the estimator $\eta(Y_h, h_K^r, data)$ is a computable quantity which depends on finite element solution Y_h, given data and the mesh parameter h_K. The estimator reduces with optimal order as the mesh parameter h_K decreases. The *a posteriori* error estimator provides an important feedback for the design of adaptive algorithm. The use of adaptive method based on *a posteriori* error estimation is well accepted in the context of finite element discretizations of partial differential equations [24, 25] and [97]. The adaptive finite element method (AFEM) can save substantial computational work and ensures a higher density of nodes in a specific area of the given domain where the solution is more difficult to approximate. *A posteriori* error estimates provide crucial information to the adaptive algorithm whether further refinement of the meshes is needed or not to achieve the desired accuracy. This is accomplished via *a posteriori* error indicators. We now give a brief account of the relevant literature concerning *a priori* and *a posteriori* error analysis of POCPs.

A priori error estimates for POCPs. There are wide range of articles available in the literature regarding *a priori* error estimates for POCPs. The time-dependent optimal control problems play an important role in many applications in engineering and sciences. The numerical treatment of these problems has been an active research topic in recent years. The pioneering work of [18, 30, 43, 68, 77, 85] and [100] consider numerical approximation of POCPs by finite element method. In [30], Gong and Hinze have presented *a priori* error estimates for the POCP with control and state constraints. They have used piecewise linear and continuous finite elements to discretize the state variable, while the time discretization is based on the backward Euler method. The authors of [70] have employed the space-time finite element discretizations of POCPs with pointwise inequality constraints on the control variable to derive *a priori* error estimates. The conforming finite element space is used to approximate the state variable, whereas the discontinuous Galerkin method is used for the time discretization. Subsequently, Meidner and Vexler have applied Petrov-Galerkin Crank-Nicolson scheme for the discretization of POCPs in [71]. However, these articles are concentrating on distributed control problems. There are many real-life applications for the optimal control problems where the control is acting locally on the finitely many points of the region. For instance, the problems arise in the environmental sciences [67, 104] such as air pollution and wastewater treatment problems. This type of controls are known as pointwise control. Chryssoverghi [14] has analyzed the convergence properties for the state and control variables of the POCPs with pointwise controls. For the semilinear POCPs with pointwise controls, we refer to [22]. The authors of [31] and [51] have derived the *a priori* error estimates for finite element approximations to POCPs with pointwise control. Recently, *a priori* error estimates for the sparse POCPs which utilizes a formulation with control variable in measure spaces can be found in [11]. The authors of [94] have considered the control problem with mixed boundaries and derived the existence and uniqueness of the optimal controls. Further, they have also proved the first-order optimality conditions in terms of the adjoint state and the related convergence results.

A posteriori error estimates for POCPs. We now provide some relevant literature in the context of *a posteriori* error analysis for POCPs. *A posteriori* error analysis for POCP has been extensively studied by various researches in [49, 59, 91, 92, 93] and [101]. Most of the *a posteriori* error estimates can classify into the residual and recovery types. In the residual type error estimates, various residual quantities such as element and jump residuals are enclosed in the error. In the recovery type error estimates, a gradient recovery (post-processing) operator is applied to the finite element solution and then it is compared with the gradient of the exact solution to assess the error. Further, *a posteriori* error estimates can also be derived by using the hierarchic bases or equilibrated residuals, see [2]. The residual type *a posteriori* error estimates of finite element methods for POCPs are discussed in [59] and [101]. Later, Tang and Chen [92] have studied a recovery type *a posteriori* error estimates for fully discrete finite element approximation to POCPs. Subsequently, Sun *et al.* have established both lower and upper bounds for the errors for POCP in [91]. In [93], Tang and Hua have derived a reliable type *a posteriori* error bounds in the $L^\infty(0,T;L^2(\Omega))$-norm using elliptic reconstruction for the semi-discrete finite element approximations to POCPs. The authors of [28] have established the *a posteriori* error estimates for the POCPs using the method of lumped masses for the approximation of convex optimal control problems. Moreover, they have discussed a reliable type *a posteriori* error estimates for both the state and control variables in the $L^\infty(0,T;L^2(\Omega))$-norm. Recently, Hou *et al.* [42] have discussed the semilinear POCP using the variational discretization and derived the residual type *a posteriori* error estimates. For a functional type *a posteriori* error estimates for POCPs, we refer to [49], where the authors have derived *a posteriori* upper bounds for state and co-state errors as well as for the cost functional.

Finite element method for PBCPs. The boundary control problems governed by parabolic PDEs arise in modeling of processes of the thermal conductivity, diffusion and filtering, see [9, 23, 54, 76]. In [65], Mackenroth has derived necessary optimality conditions for convex parabolic boundary control problems with control constraints and pointwise state constraints. Moreover, the author has shown the existence of an optimal solution to the dual problem. The authors of [29] have considered nonlinear parabolic boundary control problems with control constraints and the state constraints, and derived the sufficient second-order optimality conditions. Later in [44], Kowalewski and Krakowaik have analyzed necessary and sufficient optimality conditions for the Neumann boundary control problem governed by retarded parabolic equations. Recently, Gong *et al.* [32] have studied boundary control problems governed by parabolic PDEs on convex polygonal domains. The authors have used both piecewise linear, continuous finite element approximation and variational discretization for the approximation of the control problem. They have derived *a priori* error bounds for the state, co-state and control variables in the $L^2(0,T;L^2(\Gamma))$-norm. Some studies on *a posteriori* analysis for boundary control problems are discussed in [34, 56, 57, 60, 63]. Liu and Yan [56] have derived *a posteriori* error estimates for the model boundary control problems on polygonal domain. They have derived reliable type of *a posteriori* error estimates for convex boundary control problems in [57]. Subsequently in [60], the same authors have discussed both upper and lower *a posteriori* error estimates for the finite element approximations of elliptic boundary control problems with two different observations: The observation of the boundary state and the observation of the distributed state. They have obtained *a posteriori* error estimates for the co-state and the control variables in the $L^2(\Gamma)$-norm for both the observations, while error bound for the state variable is proved in the $L^2(\Gamma)$-norm or $L^2(\Omega)$-norm according to the observation spaces. In the context of PBCP, Gong and Yan [34] have studied the boundary control problems governed by parabolic differential equations and derived *a posterirori* error bounds in three different observations namely, observation of the boundary state, observation of the distributed state, and observation of the final state. Later on, Lu *et al.* [63] have discussed reliable type *a posteriori* error analysis for the parabolic Nuemann boundary control problems. Moreover, the authors have derived the upper bounds for the state and co-state variables in the $L^2(0,T;H^1(\Omega))$-norm while the error bound for the control variable in the $L^2(0,T;L^2(\Gamma))$-norm. The author of [62] has considered nonlinear quadratic PBCP and obtained reliable type *a posteriori* error estimates for the state and co-state variables in the $L^2(0,T;H^1(\Omega))$-norm, while the control error in the $L^2(0,T;L^2(\Gamma))$-norm.

Our contributions. In order to put the results of this thesis into a proper prospective, we present here the related literature that motivate us for the present study. For parabolic problem without controls, Picasso and Verfürth [82, 98] have used energy method to derive *a posteriori* error estimates in the $L^\infty(0,T;L^2(\Omega))$-norm. These estimates provide suboptimal rates of convergence in the $L^\infty(0,T;L^2(\Omega))$-norm. Since energy method is the most elementary technique for estimating the error in the *a priori* error analysis, it is therefore a natural to raise a question that whether one can recover optimal *a posteriori* error estimates in the norm $L^\infty(0,T;L^2(\Omega))$. Subsequently, Makridakis and Nochetto in [66] have successfully addressed this issue by introducing a novel elliptic reconstruction operator $\mathcal{R}_h : V_h \longrightarrow H_0^1(\Omega)$, where V_h is the finite element space. This operator is an *a posteriori* dual analogue of Wheeler's elliptic projection operator [99] introduced in the context of *a priori* error analysis to restore the optimality in the norm $L^\infty(0,T;L^2(\Omega))$. The idea behind the introduction of elliptic reconstruction operator is to extend the traditional energy method in *a priori* error analysis to *a posteriori* error analysis to obtain optimal order estimates in the $L^\infty(0,T;L^2(\Omega))$-norm. To restore the optimality, the usual strategy is to split the total error $e := y - Y_h$ into two parts $e := (y - \mathcal{R}_hY_h) + (\mathcal{R}_hY_h - Y_h)$, where

- the term $y - \mathcal{R}_hY_h$ refers to as the parabolic error and satisfies the original partial differential equation with a right hand side quantity which can be bounded *a posteriori* in an optimal way,

- the difference $\mathcal{R}_h Y_h - Y_h$ refers to as the elliptic reconstruction error which can be controlled by using some well established results for the elliptic problem.

Subsequently, Lakkis and Makridakis [48] have studied the fully discrete backward Euler approximation via reconstruction approach and obtained optimal order estimates in the $L^\infty(0, T; L^2(\Omega))$-norm. Our first work in this thesis is to extend the idea of [48] to study the *a posteriori* error analysis for the optimal control problems. We consider the fully-discrete finite element approximations to the optimal control problems of the form $(1.1) - (1.3)$ with distributed control. To discretize the state and co-state variables, we use the piecewise linear and continuous finite element while the piecewise constant functions are used to approximate the control variable. The backward Euler scheme is applied for the time discretization. Optimal order *a posterirori* error estimates for the state, co-state, and control variables are established in the $L^\infty(0, T; L^2(\Omega))$-norm using an elliptic reconstruction technique in conjunction with energy arguments. Numerical experiments are carried out to illustrate the performance of the derived estimators.

Our next goal is to study *a posteriori* error analysis for the POCP (1.1)-(1.3) with distributed control in the $L^\infty(0, T; L^\infty(\Omega))$-norm. In many situations, it is natural to ensure the good pointwise approximation of the exact solution. For instance, the pointwise error control is a natural goal while computing the free boundaries. Recently, for purely parabolic problem, Demlow *et al.* [21] have proved *a posteriori* error estimates in the $L^\infty(0, T; L^\infty(\Omega))$-norm for both semi-discrete and fully-discrete finite element approximations. The elliptic reconstruction technique is essential to their error analysis for both semi-discrete and fully discrete case. In our second problem, an attempt has been made to extend the analysis of [21] to POCP of the form $(1.1) - (1.3)$. Both the semi-discrete and fully-discrete finite element approximations of the control problem $(1.1) - (1.3)$ have been analyzed. The variational discretization is used to approximate the state and co-state variables with the piecewise linear and continuous functions, while the control variable is computed by using the implicit relation between the control and co-state variables. The temporal discretization is based on the backward Euler method. The key feature of this approach is not to discretize the control variable but to implicitly utilize the optimality conditions for the discretization of the control variable. We have derived a reliable type *a posteriori* error estimates for the state, co-state, and control variables in the $L^\infty(0, T; L^\infty(\Omega))$-norm. The results presented in this work are rigorously proven and do not appear to be reported elsewhere in the literature. Numerical tests are conducted to study the effectiveness of the derived estimators.

Our next problem focuses on POCP with controls acting on lower dimensional manifold. In this work, we address different *a posteriori* error estimates for the control problem (1.4) − (1.6) for various dimension of the manifold. There are many real life applications for the optimal control problem where the state variable exhibits less regularity due to the pointwise controls. Some applications can be found in the inverse problems, e.g., in the environmental model such as air pollution and wastewater treatment problems. Most of the achievements are made by assuming that the control is acting on the entire domain (cf. [69, 70]), or on a subdomain $\Omega_1 \subset \Omega$ (cf. [58, 59]), or only on the boundary of the domain Ω (see, [12, 32]). Essentially, the support of the controls requires to be very small compared to the actual size of the domain Ω if we are restricted by the cost of controls. This fact motivates us to consider such kind of control problems where the controls are located in the lower dimensional manifold. Few of the relevant work can be found in [12, 32, 39] for the Dirichlet and Neumann boundary control problems with controls acting on the boundary of the domain. Castro and Zuazua [13] have considered a heat equation with the Dirichlet boundary conditions in a bounded domain with a control acting on a lower dimensional time-dependent manifold. They have provided several controlablity results for a large class of lower dimensional moving controls and proved the convergence results for the rapidly oscillating controllers. Further, Nguyen and Raymond have considered the semilinear parabolic and convection-diffusion equations in [79] and [78], respectively. The authors of [79] have proved the existence of solutions for the control problem and computed the derivative of the functional with respect to deformations of the manifold. Whereas in [78], the authors have considered an optimal control problem for convection-diffusion equation with a pointwise control or a control localized on a smooth manifold and derived the regularity results. Subsequently, Leykekhman and Vexler have considered POCPs with a pointwise (Dirac type) control in a convex polygonal domain $\Omega \subset \mathbb{R}^2$ (cf., [51]) and obtained the order of convergence almost $\mathcal{O}(h^2+k)$ in the $L^2(0,T;L^1(\Omega))$-norm but not in the $L^2(0,T;L^2(\Omega))$-norm. Casas *et al.* have considered an elliptic optimal control problem in [10], where the control measure is approximated by a linear combination of Dirac measures. Moreover, they have proved the convergence of the discretized problems to the continuous one. The same authors have discussed POCP in [11], and they have derived the convergence results for POCP. Some recent works on *a priori* error analysis of elliptic optimal control problems with measure data in space can be found in [46] and for POCP with measure data in time, we refer to [88]. Further, the authors of [87] have considered a POCP with measure data in space as well as measure data in time and derived *a posteriori* error estimates. Moreover, Gong *et al.* have studied the finite element approximations to the POCP with pointwise control in [31], where they have considered the control acting only on the finitely many points (which are time-independent). The authors of [33] have extensively discussed the finite element approximations to elliptic control problems with controls acting on the lower dimensional manifold. Thereafter, Gong and Yan [35] have generalized the *a priori* error analysis results for the optimal control problem governed by parabolic partial differential equations where the manifold evolves with time. Motivated by the work of [35], we have made an effort to investigate the *a posteriori* error analysis for the control problem $(1.4)-(1.6)$ with controls acting on the lower dimensional manifolds. We employ the piecewise linear and continuous finite elements for the state and co-state variables whereas the piecewise constant functions are used to approximate the control variable. We have proved different types of *a posteriori* error estimates for the state variable in the $L^2(0,T;L^2(\Omega))$-norm while error for the control variable in the $L^2(0,T;\mathbb{R}^m)$-norm or $L^2(0,T;L^2(\gamma(t)))$-norm according to dimension of the manifolds. To the best of authors' knowledge this work is reported for the first time in the literature.

Our next attention of the thesis work is to investigate local *a posteriori* error analysis for the optimal control problems $(1.7)-(1.9)$. Liu and Yan [56] have investigated *a posteriori* error estimates for both the state and control variables for model boundary control problems governed by elliptic PDEs on a polygonal domain. The authors have proved reliable type of *a posteriori* error estimates for convex boundary control problems using global Sobolev norms in [57]. Later on, the authors of [41] have considered boundary control problems governed by elliptic equations, and derived related *a posteriori* error estimates. The discretization of the control problem is done by using the continuous, piecewise linear finite elements for the state and the co-state variables, and elementwise constant approximations of the control variable. They have derived the residual-type *a posteriori* error estimates for the global discretization errors in the state, co-state and control variables. In [34], Gong and Yan have considered the boundary control problems governed by parabolic differential equations and proved three different types *a posteriori* error estimates in the $L^2(0,T;H^1(\Omega))$-norm with observations of the distributed state, the boundary state, and the final state. In many engineering applications, it is often useful to study the behavior of the state and co-state variables in a small neighborhood of the boundary. Therefore, *a posteriori* error estimators in some suitable local norms have become more useful, and the derivation of these estimates is not straightforward. Subsequently, in [60], they have derived local *a posteriori* error estimates for finite element approximation of elliptic boundary control problems using a duality argument. Further, it is known that *a posteriori* error estimates provide important feedback for adaptive mesh refinement procedures. These observations motivate us to make an effort to derive local *a posteriori* error estimates for the control problem $(1.7)-(1.9)$. The space-time discretization is accomplished by using the piecewise linear and continuous finite elements to approximate the state and co-state variables while the piecewise constant function spaces are used for the control variable. We employ the backward Euler method for approximating the time derivative. We derive a reliable type local *a posteriori* error estimates for fully discrete finite element approximation of boundary control problem in three different observations: observations of the boundary state, the distributed state, and the final state. The key ingredients for deriving a reliable type local *a posteriori* error estimates for the aforementioned observations include cutoff function, duality technique, and the trace result to prove the *a posteriori* error estimates. Further, the derived *a posteriori* estimators are local in character because the leading terms of the estimators depend on the small neighborhood of the boundary.

Our last problem concerns local *a posteriori* error estimates for the finite element approximation of nonlinear parabolic boundary control problems $(1.14)-(1.16)$. In many engineering applications, it is often useful to study the behavior of the state and co-state variables in a small neighborhood of the boundary. In the boundary control problem $(1.14)-(1.16)$, the control and the objective functionals are defined on the boundary Γ. The main difficulty associated with this problem is that the cost functional $\mathcal{J}$ is not differentiable in $L^2(\Gamma)$ and hence it fails to satisfy the first-order optimality condition. This difficulty is overcome by observing the fact that $\mathcal{J}$ is differentiable in $L^\infty(\Gamma)$, which is necessary for the existence of the control variable. The error analysis is carried out by using the piecewise linear and continuous finite elements for the approximation of the state and co-state variables whereas the approximation of the control variable is done with the piecewise constant functions. The backward Euler method is used to approximate the time derivative. The reliable type local *a posteriori* error bounds for the state, co-state and control variables in the $L^2(0,T;L^2(\Gamma))$-norm are derived, while the *a posteriori* error estimates for the control variable is established by assuming the second-order optimality condition. Numerical results are provided to support the theoretical findings.

1.4 Organization of the thesis

This thesis consists of seven chapters and is organized as follows.

Chapter 1 introduces the model equations. It contains some basic notations and preliminary materials to be used in the thesis. A brief survey of the relevant literature and motivation for the present study are also presented.

Chapter 2 considers the optimal control problems governed by parabolic equations $(1.1)-(1.3)$ with distributed control. We derive *a posteriori* error estimates for the state, co-state, and control variables in the $L^\infty(0,T;L^2(\Omega))$-norm using the elliptic reconstruction technique and energy argument. Numerical results are presented to illustrate the performance of the derived estimators.

In Chapter 3, we address the control of maximum norm errors for the control problem $(1.1)-(1.3)$. We use the variational discretization for the finite element approximation for the control problem $(1.1)-(1.3)$. The discretization of the state and the co-state variables are done by using the piecewise linear and continuous finite elements while the control variable is evaluated by using the implicit relation between the control and co-state variables. The temporal discretization is based on the backward Euler method. We derive *a posteriori* upper bound for the state, co-state and the control variables in the $L^\infty(0,T;L^\infty(\Omega))$-norm. Essential to our error analysis is the elliptic reconstruction technique introduced by [66] which greatly simplifies the error analysis. The use of elliptic reconstruction technique allow us to take advantage of the existing elliptic maximum norm error estimators and the heat kernel estimate. Numerical tests are conducted to study the effectiveness of our estimators.

Chapter 4 concerns *a posteriori* error analysis for the state, co-state, and control variables to the fully discrete finite element approximations of the control problem $(1.4)-(1.6)$ with controls acting on a lower-dimensional manifold. We use the piecewise linear and continuous elements to discretize the state and co-state variables, while the piecewise constant function spaces are used to discretize the control variable. The implicit backward Euler scheme is employed to discretize the time-derivative. We derive *a posteriori* error estimates for the state variable in the $L^2(0,T;L^2(\Omega))$-norm while error for the control variable is established in the $L^2(0,T;\mathbb{R}^m)$-norm or $L^2(0,T;L^2(\gamma(t)))$-norm according to the dimension of the manifold $\gamma(t)$. Numerical assessments of the error estimators are provided.

In Chapter 5, we study the finite element approximations of the boundary control problem $(1.7)-(1.9)$. This chapter analyzes three different types of local *a posteriori* error estimates for the Neumann boundary control problems with the observations of the boundary state, the distributed state, and the final state. More precisely, *a posteriori* error bounds for the state, co-state and control variables in the $L^2(0,T;L^2(\Gamma))$-norm for the observation of the boundary state as well as for the final state. In addition, for the observation of the distributed state, *a posteriori* error bound for the state variable is derived in the $L^2(0,T;L^2(\Omega))$-norm. Moreover, error bounds for the co-state and control variables are proved in the $L^2(0,T;L^2(\Gamma))$-norm. Our derived estimators are of local character in the sense that the leading terms of the estimators depend on the small neighborhood of the boundary. These new local *a posteriori* error bounds can be used to study the behavior of the state and co-state variables near the boundary, and provide the necessary feedback in terms of the error indicators for the adaptive mesh refinements in the finite element method. Computational results are presented to validate the theoretical analysis.

In Chapter 6, we devote our attention to study *a posteriori* error analysis for the finite element approximation of nonlinear PBCP (1.14) – (1.16). The local *a posteriori* error bounds for the state, co-state, and control variables are established in the $L^2(0,T;L^2(\Gamma))$-norm. Numerical results are provided to support the theoretical findings.

Finally, Chapter 7 discusses the critical evaluation of the results highlighting the contributions made by this thesis and scope of future investigations.

For clarity of the presentation we have repeatedly recall the set of equations (1.1) – (1.3) or (1.4) – (1.6) or (1.7) – (1.9) or (1.14) – (1.16) and define the cost function $\mathcal{J}$ at the beginning of the subsequent chapters.

2

$L^\infty(L^2)$–A POSTERIORI ERROR ESTIMATES FOR POCP

In this chapter, we derive space-time *a posteriori* error estimates of finite element method for POCP (1.1) – (1.3) in a bounded convex polygonal domain. Here we consider the functionals $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ as

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|^2 \, dt \quad \text{and} \quad \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2 \, dt.$$

For simplicity, we assume that the regularization parameter $\alpha = 1$. To discretize the control problem (1.1) – (1.3), we use piecewise linear and continuous finite elements for the approximations of the state and costate variables, while the piecewise constant functions are employed for the control variable. The temporal discretization is based on the backward Euler implicit scheme. An elliptic reconstruction technique in conjunction with energy argument is used to derive an optimal order *a posteriori* error estimates for the state, costate and control variables in the $L^\infty(0,T;L^2(\Omega))$-norm. Numerical results validate the theoretical analysis.

2.1 Introduction

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^d$ $(d \geq 1)$ with Lipschitz boundary $\Gamma := \partial\Omega$. Set $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \partial\Omega \times (0,T]$ with $T < \infty$. We consider the following POCP:

$$\min_{u \in U_{ad}} \mathcal{J}(y,u) := \frac{1}{2}\int_0^T \left\{ \|y - y_{ds}\|^2 + \|u\|^2 \right\} dt \tag{2.1}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f + u & \text{in } \Omega_T, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_T, \end{cases} \tag{2.2}$$

and the control constraints

$$u_a \le u(x,t) \le u_b \quad \text{a.e. in } \Omega_T, \tag{2.3}$$

where the initial state $y_0 \in H_0^1(\Omega)$ and the source function $f \in L^2(0,T;L^2(\Omega))$. Here $y = y(x,t)$ denotes the state variable, $u = u(x,t)$ is the control variable. We denote the partial derivative y with respect to time t (i.e., $\frac{\partial y}{\partial t}$) by y_t in the subsequent analysis. The set of admissible controls is defined by

$$U_{ad} = \left\{ u \in L^2(0,T;L^2(\Omega)) : \ u_a \le u \le u_b \ \ a.e. \ \text{in } \Omega_T \right\}$$

with $u_a, u_b \in \mathbb{R}$ fulfill $u_a < u_b$. Further, we shall take the space for the state variable by $V = L^\infty(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$. We now define the bilinear form $a(\cdot,\cdot)$ on $H_0^1(\Omega)$ by

$$a(v,w) \ = \ \int_\Omega \nabla v \cdot \nabla w \, dx \quad \forall v, w \in H_0^1(\Omega),$$

where $H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : \ v = 0 \text{ on } \partial\Omega \right\}$. It follows that the bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $H_0^1(\Omega)$, i.e., $\exists \ \alpha_0, \ \alpha_1 > 0$ such that

$$|a(v,w)| \ \le \ \alpha_0 \|v\|_1 \|w\|_1, \qquad \forall v, w \in H_0^1(\Omega),$$

and

$$a(v,v) \ \ge \ \alpha_1 \|v\|_1^2, \qquad \forall v \in H_0^1(\Omega).$$

The weak formulation of (2.1) – (2.3) is to seek a pair $(y,u) \in V \times U_{ad}$ such that

$$\min_{u \in U_{ad}} \frac{1}{2} \int_0^T \left\{ \|y - y_{ds}\|^2 + \|u\|^2 \right\} dt \tag{2.4}$$

subject to

$$\begin{cases} (y_t, v) + a(y,v) = (f+u, v) & \forall v \in H_0^1(\Omega), \\ y(\cdot,0) = y_0(x) & x \in \Omega. \end{cases} \tag{2.5}$$

Using standard energy techniques we have the following *a priori* bounds for the state variable y (cf., [55]).

Proposition 2.1.1. *For every* $u \in L^2(0,T;L^2(\Omega))$*, the state equation (2.5) admits a unique solution* $y \in V$*, and the following a priori estimate*

$$\|y\|_{L^2(0,T;H^2(\Omega))} + \|y_t\|_{L^2(0,T;L^2(\Omega))} \leq C\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T,L^2(\Omega))} + \|u\|_{L^2(0,T;L^2(\Omega))}\right)$$

holds.

It is well known [53] that the convex optimal control problem $(2.4) - (2.5)$ has a unique solution (y,u) if and only if there exists co-state variable p such that (y,p,u) satisfies the following optimality conditions for $t \in [0,T]$:

$$\begin{aligned} (y_t,v) + a(y,v) &= (f+u,v) \quad \forall v \in H_0^1(\Omega), && (2.6)\\ y(x,0) &= y_0(x) \quad x \in \Omega, && (2.7)\\ -(p_t,v) + a(p,v) &= (y - y_{ds},v) \quad \forall v \in H_0^1(\Omega), && (2.8)\\ p(x,T) &= 0 \quad x \in \Omega, && (2.9)\\ (u+p,w-u) &\geq 0 \quad \forall w \in U_{ad}. && (2.10) \end{aligned}$$

We introduce the reduced cost functional

$$\begin{aligned} j : L^2(0,T;L^2(\Omega)) &\to \mathbb{R} \quad \text{as}\\ u &\mapsto j(u) := \mathcal{J}(u,y(u)), \end{aligned}$$

where $y(u)$ is the solution of (2.5). Hence the optimal control problem $(2.4) - (2.5)$ can be equivalently reformulated as

$$\min_{u \in U_{ad}} j(u).$$

This chapter studies *a posteriori* error analysis of fully discrete finite element approximation to POCP. Some relevant work in this direction can be found in [59, 91, 92, 93]. The authors of [59] have considered the finite element approximation for the distributed control problem governed by linear parabolic equations and derived a residual type *a posteriori* upper bounds for the state and co-state variables in the $\mathcal{C}(0,T;L^2(\Omega))$-norm, and the control error in the $L^\infty(0,T;L^2(\Omega))$-norm. Later, Tang and Chen [92] have discussed a recovery type *a posteriori* error estimates using the superconvergence properties of finite element solutions for fully discrete finite element approximation to POCP. Subsequently, Sun *et al.* [91] have analyzed the control problem by means of multi-meshes finite element approximation in the backward Euler scheme. They have proved *a posteriori* error bounds for both the state and the co-state variables in the $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$ norms, respectively, and the error for the control variable in the

$L^2(0,T;L^2(\Omega))$-norm. The authors of [93] have considered finite element approximations of the control problem governed by parabolic equation with integral control constraints and derived a reliable type *a posteriori* error bounds in the $L^\infty(0,T;L^2(\Omega))$-norm using elliptic reconstruction for the semi-discrete problem. In this chapter, our emphasis is on the fully-discrete control problem with distributed control allowing mesh modification in time which is natural in adaptive schemes for time-dependent problems. Our main technique in deriving the error estimates is a legitimate adaptation to the fully discrete analogue of the elliptic reconstruction technique introduced by Lakkis and Makridakis in [48]. We study residual-based energy estimates for space-time POCP (2.1) – (2.3). A main characteristic of this approach, in contrasts with other direct techniques in the literature is that we can virtually use any available *a posteriori* estimates for elliptic problem to control the main part of the spatial error. We derive upper bounds of the errors for the state, co-state and control variables in the $L^\infty(0,T;L^2(\Omega))$-norm.

The layout of this chapter is as follows. In Section 2.2, we discuss finite element approximation of POCP (2.1) – (2.3). We derive *a posteriori* error estimates in Section 2.3. Section 2.4 is devoted to the numerical experiments to illustrate the performance of the derived estimators. Finally, a concluding remark is presented in the last section.

2.2 Discrete optimal control problem

This section considers the space-time finite element discretizations of optimal control problem (2.4) – (2.5) and formulate the discrete optimal control problem.

Let $\mathscr{T}_h$ be a family of regular triangulations of $\overline{\Omega}$ such that $\overline{\Omega} = \cup_{K\in\mathscr{T}_h}\overline{K}$. For each element $K \in \mathscr{T}_h$, we denote parameter $\rho(K)$ as the diameter of the element K and the mesh size by $h = \max_K \rho(K)$. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of the interval $[0,T]$, with $I_n = (t_{n-1}, t_n]$ and $k_n := t_n - t_{n-1}$. We now make the following assumptions on the triangulation $\mathscr{T}_h^n := \{K\}\,(0 \le n \le N)$ of $\overline{\Omega}$ at time level t_n (cf. [48]):

D1. If $K_1, K_2 \in \mathscr{T}_h^n$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$, or $K_1 \cap K_2$ share a common edge, or a common vertex.

D2. Two simplicial decompositions $\mathscr{T}_h^{n-1}$ and $\mathscr{T}_h^n$ of $\overline{\Omega}$ are said to be compatible if they are derived from the same macro triangulation $\mathscr{T}_h = \mathscr{T}_h^0$ by an admissible refinement procedure which preserves the shape regularity (cf. [8]) and assures that for any elements $K \in \mathscr{T}_h^{n-1}$ and $K' \in \mathscr{T}_h^n$, either $K \cap K' = \varnothing$, $K \subset K'$, or $K' \subset K$. There is a natural partial ordering on a set of compatible triangulations namely, $\mathscr{T}_h^{n-1} \le \mathscr{T}_h^n$ if $\mathscr{T}_h^n$ is a refinement of $\mathscr{T}_h^{n-1}$. Then, for a given pair of successive compatible triangulations $\mathscr{T}_h^{n-1}$ and $\mathscr{T}_h^n$, we define naturally the finest common coarsening $\hat{\mathscr{T}}_h^n := \mathscr{T}_h^n \wedge \mathscr{T}_h^{n-1}$ with local mesh sizes are given by $\hat{h}_n := \max\{h_{n-1}, h_n\}$. These conditions allow us to bound the elliptic errors which lie in two adjacent finite element spaces, see [48].

We shall also need the following notations for future use. For $0 \leq n \leq N$, let $\mathcal{E}_n := \{E\}$ be the set of all edges of the triangles $K \in \mathscr{T}_h^n$ which do not lie on Γ, and $\Sigma_n := \cup_{E \in \mathcal{E}_n} E$. Furthermore, we will also use the sets $\hat{\Sigma}_n := \Sigma_n \cap \Sigma_{n-1}$ and $\check{\Sigma}_n := \Sigma_n \cup \Sigma_{n-1}$. For each $n = 0, \ldots, N$, we consider the finite element spaces V_h^n and U_{ad}^n corresponding to the triangulation $\mathscr{T}_h^n$ as follows:

$$\begin{aligned} V_h^n &:= \{\chi \in H_0^1(\Omega) : \ \chi|_K \in \mathbb{P}_1(K) \qquad \forall K \in \mathscr{T}_h^n\}, \\ U_{ad}^n &:= \{\chi \in U_{ad} : \ \chi|_K = constant \qquad \forall K \in \mathscr{T}_h^n\}, \end{aligned}$$

where $\mathbb{P}_1(K)$ is the space of polynomials of degree less than or equal to 1 on K. We shall use the following notations for our subsequent analysis:

$$\phi^n := \phi(\cdot, t_n), \quad \bar{\partial}\phi^n := \frac{1}{k_n}(\phi^n - \phi^{n-1}), \quad \text{and} \quad \mathcal{L}_h^n(\bar{\partial}\phi^n) := \frac{1}{k_n}(\phi^n - \mathcal{L}_h^n\phi^{n-1}),$$

where $\mathcal{L}_h^n$ is the L^2-projection from $L^2(\Omega)$ to V_h^n.

Representation of the bilinear form: For a function $v_h \in V_h^n$ $(0 \leq n \leq N)$, the bilinear form $a(v_h, w)$ can be represented as

$$a(v_h, w) = -\sum_{K \in \mathscr{T}_h^n} (\Delta v_h, w)_K + \sum_{E \in \mathcal{E}_n} (J_1[v_h], w)_E \qquad \forall w \in H_0^1(\Omega).$$

Here, $J_1[v_h]$ denotes the spatial jump of the field ∇v_h across an element side $E \in \mathcal{E}_n$ and is defined as

$$J_1[v_h]|_E(x) := [[\nabla v_h]]_E(x) := \lim_{\epsilon \to 0} \Big(\nabla v_h(x + \epsilon \mathbf{n}_E) - \nabla v_h(x - \epsilon \mathbf{n}_E)\Big) \cdot \mathbf{n}_E,$$

where $\mathbf{n}_E$ is a unit normal vector to E at the point x.

For $v_h \in V_h^n$, let $\mathcal{A}_{el}\, v_h$ be the regular part of the distribution Δv_h, which is defined as a piecewise continuous function such that

$$(\mathcal{A}_{el} v_h, w) := -\sum_{K \in \mathscr{T}_h^n} (\Delta v_h, w)_K \qquad \forall w \in H_0^1(\Omega).$$

Thus, we can represent our bilinear form $a(\cdot, \cdot)$ as

$$a(v_h, w) = (\mathcal{A}_{el} v_h, w) + (J_1[v_h], w)_{\Sigma_n} \qquad \forall w \in H_0^1(\Omega). \tag{2.11}$$

Definition 2.2.1. (Discrete elliptic operator). *The discrete elliptic operator associated with the bilinear form* $a(\cdot,\cdot)$ *and the finite element space* V_h^n *is the operator* $\mathcal{A}_h^n : H_0^1(\Omega) \to V_h^n$ *such that for* $v \in H_0^1(\Omega)$ *and* $0 \le n \le N$,

$$(\mathcal{A}_h^n v, w_h) = a(v, w_h) \qquad \forall w_h \in V_h^n. \tag{2.12}$$

The fully discrete finite element approximation of (2.4) – (2.5) is defined as follows: For $n \ge 1$, find $(y_h^n, u_h^n) \in V_h^n \times U_{ad}^n$ such that

$$\min_{u_h^n \in U_{ad}^n} \sum_{n=1}^{N} \frac{1}{2} \int_{I_n} \left\{ \|y_h^n - y_{ds}^n\|_{L^2(\Omega)}^2 + \|u_h^n\|_{L^2(\Omega)}^2 \right\} dt \tag{2.13}$$

subject to

$$\begin{cases} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = (f^n + u_h^n, v_h) & \forall v_h \in V_h^n, \\ y_h^0 = y_{h,0}, \end{cases} \tag{2.14}$$

where $y_{h,0}$ is the suitable approximation of y_0 in V_h^0.

The optimal control problem (2.13) – (2.14) admits a unique solution (y_h^n, u_h^n) if and only if there exists a co-state variable $p_h^{n-1} \in V_h^n$ such that the following optimality conditions are satisfied

$$\begin{aligned} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) &= (f^n + u_h^n, v_h) \quad \forall v_h \in V_h^n, & (2.15)\\ y_h^0 &= y_{h,0}, & (2.16)\\ -(\bar{\partial} p_h^n, v_h) + a(p_h^{n-1}, v_h) &= (y_h^n - y_{ds}^n, v_h) \quad \forall v_h \in V_h^n, & (2.17)\\ p_h^N &= 0, & (2.18)\\ (u_h^n + p_h^{n-1}, w_h^n - u_h^n) &\ge 0 \quad \forall w_h^n \in U_{ad}^n. & (2.19) \end{aligned}$$

Given a sequence of discrete values $\{y_h^n\}$, $n = 0, 1, \ldots, N$, we associate a continuous function of time defined by the continuous piecewise linear interpolant $Y_h(t)$, $t \in I_n$ as

$$Y_h(t) := \frac{(t_n - t)}{k_n} y_h^{n-1} + \frac{(t - t_{n-1})}{k_n} y_h^n.$$

Similarly, we define $P_h(t)$, $t \in I_n$ from the set of values $\{p_h^n\}$, $n = 0, 1, \ldots, N$ as

$$P_h(t) := \frac{(t_n - t)}{k_n} p_h^{n-1} + \frac{(t - t_{n-1})}{k_n} p_h^n,$$

and

$$U_h(t)|_{t \in I_n} := u_h^n.$$

Then the optimality conditions (2.15) − (2.19) can be rewritten as

$$\begin{aligned}
(\bar\partial Y_h^n, v_h) + a(Y_h^n, v_h) &= (f^n + U_h, v_h) \quad \forall v_h \in V_h^n, && (2.20)\\
Y_h^0 &= y_{h,0}, && (2.21)\\
-(\bar\partial P_h^n, v_h) + a(P_h^{n-1}, v_h) &= (Y_h^n - y_{ds}^n, v_h) \quad \forall v_h \in V_h^n, && (2.22)\\
P_h^N &= 0, && (2.23)\\
(U_h + P_h^{n-1}, w_h^n - U_h) &\geq 0 \quad \forall w_h^n \in U_{ad}^n. && (2.24)
\end{aligned}$$

Analogous to the continuous case, we reformulate the discrete optimal control problem (2.13) − (2.14) as

$$\min_{U_h \in U_{ad}^n} j_h^n(U_h) := J(U_h, Y_h(U_h)).$$

2.3 A posteriori error analysis

In this section, we first derive some intermediate error estimates for the state and co-state variables in the $L^\infty(0,T;L^2(\Omega))$-norm which will be used to prove our main results. This is accomplished by introducing elliptic reconstructions for the state and co-state variables as intermediate objects in the error analysis. For this, we introduce some intermediate variables.

For $\hat u \in U_{ad}$, let the pair $(y(\hat u), p(\hat u)) \in V \times V$ be the solutions of the following equations:

$$\begin{aligned}
(y_t(\hat u), v) + a(y(\hat u), v) &= (f + \hat u, v) \quad \forall v \in H_0^1(\Omega), && (2.25)\\
y(\hat u)(\cdot, 0) &= y_0(x) \quad x \in \Omega, && (2.26)\\
-(p_t(\hat u, v) + a(p(\hat u), v) &= (y(\hat u) - y_{ds}, v) \quad \forall v \in H_0^1(\Omega), && (2.27)\\
p(\hat u)(\cdot, T) &= 0 \quad x \in \Omega. && (2.28)
\end{aligned}$$

For the purpose of error analysis, we shall define the errors for the state and co-state variables as

$$e_y := Y_h - y(U_h), \text{ and } e_p := P_h - p(U_h),$$

respectively. From (2.20), (2.22) and (2.25), (2.27) with $\hat u = U_h$, we derive the following error equations for $t \in I_n$, $n = 1, \dots N$ and $v \in H_0^1(\Omega)$:

$$\begin{aligned}
(e_{y,t}, v) + a(e_y, v) &= r_y^n(v) + (k_n^{-1}(\mathcal{L}_h^n y_h^{n-1} - y_h^{n-1}), v) + a(Y_h - y_h^n, v)\\
&\quad + (f^n - f, v), && (2.29)\\
-(e_{p,t}, v) + a(e_p, v) &= r_p^n(v) - (k_n^{-1}(\mathcal{L}_h^n p_h^{n-1} - p_h^{n-1}), v) + a(P_h - p_h^{n-1}, v)\\
&\quad + (Y_h^n - y(U_h), v) - (y_{ds}^n - y_{ds}, v), && (2.30)
\end{aligned}$$

where

$$\begin{aligned} r_y^n(v) &:= (\mathcal{L}_h^n(\bar{\partial} y_h^n), v) + a(y_h^n, v) - (f^n + U_h, v), \\ r_p^n(v) &:= (\mathcal{L}_h^n(\bar{\partial} p_h^n), v) + a(p_h^{n-1}, v) - (y_h^n - y_{ds}^n, v). \end{aligned}$$

We now define the *elliptic reconstructions* at $t = t_n$, $n \in [1 : N]$ as follows: For given y_h^n, p_h^{n-1}, seek $\tilde{y}_h^n$, $\tilde{p}_h^{n-1} \in H_0^1(\Omega)$ satisfying

$$a(\tilde{y}_h^n - y_h^n, v) = -r_y^n(v) \quad \forall v \in H_0^1(\Omega), \tag{2.31}$$

$$a(\tilde{p}_h^{n-1} - p_h^{n-1}, v) = -r_p^n(v) + (\tilde{y}_h^n - y_h^n, v) \quad \forall v \in H_0^1(\Omega). \tag{2.32}$$

Since $r_y^n(v_h) = 0$ and $r_p^n(v_h) = 0$, $\forall v_h \in V_h^n$, we observe that y_h^n denotes the elliptic projection of $\tilde{y}_h^n$ at time level $t = t_n$. Using a sequence of discrete values $\{\tilde{y}_h^n\}$ for $n = 0, 1, \ldots, N$, we set a continuous function of time defined by piecewise linear interpolant $\tilde{y}(t)$, $t \in [0, T]$ as

$$\tilde{y}(t) := \frac{(t_n - t)}{k_n} \tilde{y}_h^{n-1} + \frac{(t - t_{n-1})}{k_n} \tilde{y}_h^n, \quad t_{n-1} \le t \le t_n, \ \ n = 1, \ldots, N.$$

Similarly, we define $\tilde{p}(t)$, $t \in [0, T]$, from the set of values $\{\tilde{p}_h^n\}$, $n = 1, \ldots, N$ as

$$\tilde{p}(t) := \frac{(t_n - t)}{k_n} \tilde{p}_h^{n-1} + \frac{(t - t_{n-1})}{k_n} \tilde{p}_h^n, \quad t_{n-1} \le t \le t_n, \ \ n = 1, \ldots, N.$$

We note that functions $\tilde{y}$ and $\tilde{p}$ satisfy, for each $t \in [0, T]$, the following equations:

$$\begin{aligned} a(\tilde{y} - Y_h, v) &= -r_y(v) \quad \forall v \in H_0^1(\Omega), \\ a(\tilde{p} - P_h, v) &= -r_p(v) + (\tilde{y} - Y_h, v) \quad \forall v \in H_0^1(\Omega), \end{aligned}$$

where r_y and r_p are piecewise linear interpolant of $\{r_y^n\}_{n=1}^N$ and $\{r_p^n\}_{n=1}^N$, respectively. Using elliptic reconstruction, we split the errors as follows:

$$\begin{aligned} e_y &= (\tilde{y} - y(U_h)) - (\tilde{y} - Y_h) =: \ \xi_y - \eta_y, \\ e_p &= (\tilde{p} - p(U_h)) - (\tilde{p} - P_h) =: \ \xi_p - \eta_p. \end{aligned}$$

Using (2.31) – (2.32) in (2.29) – (2.30), we find that

$$\begin{aligned} (\xi_{y,t}, v) + a(\xi_y, v) &= (\eta_{y,t}, v) + a(\tilde{y} - \tilde{y}_h^n, v) + (k_n^{-1}(\mathcal{L}_h^n - I) y_h^{n-1}, v) \\ &\quad + (f^n - f, v) \quad \forall v \in H_0^1(\Omega), \end{aligned} \tag{2.33}$$

$$\begin{aligned} -(\xi_{p,t}, v) + a(\xi_p, v) &= -(\eta_{p,t}, v) + a(\tilde{p} - \tilde{p}_h^{n-1}, v) - (k_n^{-1}(\mathcal{L}_h^n - I) p_h^{n-1}, v) \\ &\quad - (y_{ds}^n - y_{ds}, v) + (\tilde{y}_h^n - y(U_h), v) \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{2.34}$$

Note that

$$\tilde{y}-\tilde{y}_h^n := -\frac{(t_n-t)}{k_n}(\tilde{y}_h^n-\tilde{y}_h^{n-1}) \quad \text{and} \quad \tilde{p}-\tilde{p}_h^{n-1} := \frac{(t-t_{n-1})}{k_n}(\tilde{p}_h^n-\tilde{p}_h^{n-1}).$$

Moreover, the equations (2.33) − (2.34) can be written as

$$\begin{aligned}(\xi_{y,t},v)+a(\xi_y,v) &= (\eta_{y,t},v)-\frac{(t_n-t)}{k_n}a(\tilde{y}_h^n-\tilde{y}_h^{n-1},v)+(k_n^{-1}(\mathcal{L}_h^n-I)y_h^{n-1},v)\\ &\quad +(f^n-f,v) \qquad \forall v\in H_0^1(\Omega), \qquad (2.35)\\ -(\xi_{p,t},v)+a(\xi_p,v) &= -(\eta_{p,t},v)+\frac{(t-t_{n-1})}{k_n}a(\tilde{p}_h^n-\tilde{p}_h^{n-1},v)-(k_n^{-1}(\mathcal{L}_h^n-I)p_h^{n-1},v)\\ &\quad -(y_{ds}^n-y_{ds},v)+(\tilde{y}_h^n-y(U_h),v) \qquad \forall v\in H_0^1(\Omega). \qquad (2.36)\end{aligned}$$

We need the following two propositions for future use (cf. [86]). The first one is about the interpolation error estimate for the Clément-type interpolation operator. The second proposition is related to the approximation property which reflects the change of behaviour for the finest common coarsening of $\mathscr{T}_h^n$ and $\mathscr{T}_h^{n-1}$.

Proposition 2.3.1. *Let* $\Pi_n : H_0^1(\Omega)\to V_h^n$ *be the Clément-type interpolation operator. Then, for sufficiently smooth* ψ *and finite element polynomial space of degree* l*, there exist constants* $\bar{C}_{1,j}$ *and* $\bar{C}_{2,j}$ *depending only upon the shape-regularity of the family of triangulations such that for* $j\le l+1$

$$\|h_n^{-j}(\psi-\Pi_n\psi)\| \;\le\; \bar{C}_{1,j}|\psi|_j, \qquad (2.37)$$

and

$$\|h_n^{1/2-j}(\psi-\Pi_n\psi)\|_{\Sigma_n} \;\le\; \bar{C}_{2,j}|\psi|_j. \qquad (2.38)$$

Proposition 2.3.2. *Let* $\hat{\Pi}_n : X\to V_h^n\cap V_h^{n-1}$ *be the Clément-type interpolation operator with respect to the finest common coarsening of* $\mathscr{T}_h^n$ *and* $\mathscr{T}_h^{n-1}$*, i.e.,* $\hat{\mathscr{T}} := \mathscr{T}_h^n\wedge\mathscr{T}_h^{n-1}$ *corresponding to the finite element space* $V_h^n\cap V_h^{n-1}$ *with mesh size* $\hat{h}_n := \max\{h_n,h_{n-1}\}$*. Then the following inequality holds:*

$$\|\hat{h}_n^{1/2-j}(\psi-\hat{\Pi}_n\psi)\|_{\Sigma_n\backslash\hat{\Sigma}_n} \;\le\; \bar{C}_{3,j}|\psi|_j,$$

where the constant $\bar{C}_{3,j}$ *depends on the shape regularity of the family of triangulations and on the number of steps required to move from* $\mathscr{T}_h^{n-1}$ *to* $\mathscr{T}_h^n$*. Further, the approximation properties (2.37) and (2.38) hold true in the finite element space* $V_h^n\cap V_h^{n-1}$ *with* $\hat{h}_n$ *replacing* h_n*.*

In the following lemma, we provide the bounds for the elliptic reconstruction errors associated with the state and co-state variables.

Lemma 2.3.1 (Elliptic reconstruction errors)**.** *Let* $(\tilde{y}_h^n, \tilde{p}_h^{n-1}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfy* $(2.31) - (2.32)$*. Then,* $0 \leq n \leq N$*, the following estimate holds:*

$$\|\tilde{y}_h^n - y_h^n\| \;\leq\; \bar{C}_{1,2} h_n^2 \|(\mathcal{A}_h^n - \mathcal{A}_{el}) y_h^n\| + \bar{C}_{2,2} h_n^{3/2} \|J_1[y_h^n]\|_{\Sigma_n}. \tag{2.39}$$

For $n \in [1:N]$*, we have*

$$\|\tilde{p}_h^{n-1} - p_h^{n-1}\| \;\leq\; \bar{C}_{1,2} h_n^2 \|(\mathcal{A}_h^n - \mathcal{A}_{el}) p_h^{n-1}\| + \bar{C}_{2,2} h_n^{3/2} \|J_1[p_h^{n-1}]\|_{\Sigma_n} + \|\tilde{y}_h^n - y_h^n\|, \tag{2.40}$$

where the constants $\bar{C}_{1,2}$ *and* $\bar{C}_{2,2}$ *are defined in Proposition 2.3.1.*

Proof. The proof of the lemma will proceed by the duality argument. Let $w : [0,T] \to H_0^1(\Omega)$ be the solution of the elliptic problem in the weak form as

$$a(v, w(t)) \;=\; (\tilde{y}_h^n - y_h^n, v) \qquad \forall v \in H_0^1(\Omega), \;\; a.e. \;\; t \in [0,T], \tag{2.41}$$

Further, the solution w of (2.41) satisfies the following regularity result

$$\|w\|_2 \;\leq\; C_R \|\tilde{y}_h^n - y_h^n\|, \tag{2.42}$$

where C_R denotes to the regularity constant. Setting $v = \tilde{y}_h^n - y_h^n \in V_h^n$ in (2.41) and use of (2.31), (2.11) and (2.12) yields

$$\begin{aligned}
\|\tilde{y}_h^n - y_h^n\|^2 &= a(\tilde{y}_h^n - y_h^n, w) \\
&= a(\tilde{y}_h^n - y_h^n, w - \Pi_n w) \\
&= ((\mathcal{A}_h^n - \mathcal{A}_{el}) y_h^n, w - \Pi_n w) - (J_1[y_h^n], w - \Pi_n w)_{\Sigma_n}.
\end{aligned}$$

An application of the Cauchy-Schwarz inequality, Proposition 2.3.1 and (2.42) implies

$$\|\tilde{y}_h^n - y_h^n\| \;\leq\; \bar{C}_{1,2} h_h^2 \|(\mathcal{A}_h^n - \mathcal{A}_{el}) y_h^n\| + \bar{C}_{2,2} h_n^{\frac{3}{2}} \|J_1[y_h^n]\|,$$

which proves (2.39).

Next, we shall prove the inequality (2.40). Setting $v = \tilde{p}_h^{n-1} - p_h^{n-1} \in V_h^n$ in

$$a(v, w) \;=\; (\tilde{p}_h^{n-1} - p_h^{n-1}, v),$$

we obtain

$$\begin{aligned}
\|\tilde{p}_h^{n-1} - p_h^{n-1}\|^2 &= a(\tilde{p}_h^{n-1} - p_h^{n-1}, w), \\
&= a(\tilde{p}_h^{n-1} - p_h^{n-1}, w - \Pi_n w) + a(\tilde{p}_h^{n-1} - p_h^{n-1}, \Pi_n w), \\
&= (\mathcal{A}_h^n p_h^{n-1} - \mathcal{A}_{el} p_h^{n-1}, w - \Pi_n w) + (\tilde{y}_h^n - y_h^n, \Pi_n w).
\end{aligned}$$

An application of Proposition 2.3.1 and the Cauchy-Schwarz inequality with $w = \tilde{p}_h^{n-1} - p_h^{n-1}$ yields

$$\|\tilde{p}_h^{n-1} - p_h^{n-1}\| \;\leq\; \bar{C}_{1,2} h_n^2 \|(\mathcal{A}_h^n - \mathcal{A}_{el}) p_h^{n-1}\| + \bar{C}_{2,2} h_n^{3/2} \|J_1[p_h^{n-1}]\|_{\Sigma_n} + \|\tilde{y}_h^n - y_h^n\|,$$

and this completes the proof. □

In the following lemma, we derive the bound for ξ_y.

Lemma 2.3.2 (Parabolic error for the state variable)**.** *Let ξ_y satisfy (2.35). Then the following estimate holds, $1 \leq m \leq N$:*

$$\Big(\max_{t\in[0,t_m]} \|\xi_y(t)\|^2 + 2\alpha_1 \int_0^{t_m} \|\xi_y\|_1^2\, dt\Big)^{\frac{1}{2}} \;\leq\; \|\xi_y(0)\| + 4\Big(\mathscr{E}_{1,m}^2 + \mathscr{E}_{2,m}^2\Big)^{\frac{1}{2}},$$

where

$$\mathscr{E}_{1,m} \;:=\; \sum_{n=1}^{m} k_n\big(\beta_{1,n} + \beta_{2,n} + \beta_{4,n}\big),$$

and

$$\mathscr{E}_{2,m} \;:=\; \sum_{n=1}^{m} \beta_{3,n}^2\, k_n.$$

Here, for $n \in [1:N]$, $\beta_{1,n}$ is the space error estimator and is defined by

$$\beta_{1,n} \;:=\; \Big(\bar{C}_{1,2}\|\hat{h}_n^2 \bar{\partial} R_y^n\| + \bar{C}_{2,2}\|\hat{h}_n^{3/2}\bar{\partial} J_y^n\|_{\hat{\Sigma}_n} + \bar{C}_{3,2}\|\hat{h}_n^{3/2}\bar{\partial} J_y^n\|_{\check{\Sigma}_n\setminus\hat{\Sigma}_n}\Big),$$

$\beta_{2,n}$ is the temporal error estimator defined by

$$\beta_{2,n} \;:=\; \|k_n \bar{\partial} \tilde{r}_y^n\|,$$

and the data approximation error estimators are defined by

$$\beta_{3,n} \;:=\; \bar{C}_{3,1}\|h_n k_n^{-1}(\mathcal{L}_h^n - I) y_h^{n-1}\|,$$

$$\beta_{4,n} \;:=\; \frac{1}{k_n}\int_{t_{n-1}}^{t_n} \|f^n - f\|\, dt.$$

The constants $\bar{C}_{i,2}|_{i=1,2,3}$ and $\bar{C}_{3,1}$ are positive and independent of the discretization parameters.

Proof. By setting $v = \xi_y$ in (2.35) and using the coercive property of the bilinear form, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\xi_y\|^2 + \alpha_1\|\xi_y\|_1^2 \;\leq\; (\eta_{y,t}, \xi_y) - \frac{(t_n - t)}{k_n} a(\tilde{y}_h^n - \tilde{y}_h^{n-1}, \xi_y) + (k_n^{-1}(L_h^n - I) y_h^{n-1}, \xi_y) + (f^n - f, \xi_y). \tag{2.43}$$

Integrating (2.43) with respect to time from 0 to t_m, $m \in [1:N]$, and using the fact $\left|\frac{(t_n-t)}{k_n}\right| \leq 1$ in I_n, it follows that

$$\begin{aligned}
&\frac{1}{2}\|\xi_y(t_m)\|^2 - \frac{1}{2}\|\xi_y(0)\|^2 + \alpha_1\int_0^{t_m} \|\xi_y\|_1^2\,dt \\
&\leq \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} \big\{|(\eta_{y,t},\xi_y)| + |a(\tilde{y}_h^n - \tilde{y}_h^{n-1},\xi_y)| + |(k_n^{-1}(\mathcal{L}_h^n - I)y_h^{n-1},\xi_y)| \\
&\quad + |(f^n - f,\xi_y)|\big\}\,dt \;=:\; \sum_{n=1}^{m}\left(\zeta_n^1 + \zeta_n^2 + \zeta_n^3 + \zeta_n^4\right) = \zeta_m.
\end{aligned} \tag{2.44}$$

Since $\xi_y(t)$ is continuous in $[0,t_m]$, there exists $t_{m^*} \in [0,t_m]$ for which

$$\max_{t\in[0,t_m]} \|\xi_y(t)\| \;=\; \|\xi_y(t_{m^*})\| \;=:\; \|\xi_y^{m^*}\|. \tag{2.45}$$

Integrate (2.43) with respect to time from 0 to t_{m^*} to have

$$\begin{aligned}
&\frac{1}{2}\|\xi_y(t_{m^*})\|^2 - \frac{1}{2}\|\xi_y(0)\|^2 + \alpha_1\int_0^{t_{m^*}} \|\xi_y\|_1^2\,dt \\
&\leq \int_0^{t_{m^*}} \big\{|(\eta_{y,t},\xi_y)| + |a(\tilde{y}_h^n - \tilde{y}_h^{n-1},\xi_y)| \\
&\quad + |(k_n^{-1}(\mathcal{L}_h^n - I)y_h^{n-1},\xi_y)| + |(f^n - f,\xi_y)|\big\}\,dt \\
&\leq \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} \big\{|(\eta_{y,t},\xi_y)| + |a(\tilde{y}_h^n - \tilde{y}_h^{n-1},\xi_y)| \\
&\quad + |(k_n^{-1}(\mathcal{L}_h^n - I)y_h^{n-1},\xi_y)| + |(f^n - f,\xi_y)|\big\}\,dt \\
&=: \sum_{n=1}^{m}\left(\zeta_n^1 + \zeta_n^2 + \zeta_n^3 + \zeta_n^4\right) \;=:\; \zeta_m.
\end{aligned}$$

In view of (2.45), we have

$$\frac{1}{2}\|\xi_y^{m^*}\|^2 - \frac{1}{2}\|\xi_y(0)\|^2 + \alpha_1\int_0^{t_{m^*}} \|\xi_y\|_1^2\,dt \;\leq\; \zeta_m. \tag{2.46}$$

Combining (2.44) and (2.46), it now leads to

$$\frac{1}{2}\|\xi_y^{m^*}\|^2 + \alpha_1\int_0^{t_m} \|\xi_y\|_1^2\,dt \;\leq\; \|\xi_y(0)\|^2 + 2\,\zeta_m. \tag{2.47}$$

Now, we estimate ζ_n^i, $i = 1,\ldots,4$. To estimate the term ζ_n^1, which measures the space error and mesh change, we will exploit the orthogonality property of the elliptic reconstruction definition (2.31). Observe that for each $n \in [1:N]$, we have

$$\zeta_n^1 \;=\; \int_{t_{n-1}}^{t_n} |(\eta_{y,t},\xi_y)|\,dt \;=\; k_n^{-1}\int_{t_{n-1}}^{t_n} |(\tilde{y}_h^n - \tilde{y}_h^{n-1} - y_h^n + y_h^{n-1},\xi_y)|\,dt.$$

Since $\tilde{y}_h^n - y_h^n$ is orthogonal to V_h^n with respect to $a(\cdot,\cdot)$, the first term inside the brackets is orthogonal to $V_h^n \cap V_h^{n-1}$.

Let $\psi : [0,T] \to H_0^1(\Omega)$ be such that

$$a(\chi, \psi(t)) \;=\; (\xi_y(t), \chi) \qquad \forall \chi \in H_0^1(\Omega),\ t \in [0,T].$$

Using the interpolation operator $\hat{\Pi}_n$ defined in Proposition 2.3.2, we write

$$\begin{aligned}(\tilde{y}_h^n - \tilde{y}_h^{n-1} - y_h^n + y_h^{n-1}, \xi_y) \;&=\; a(\tilde{y}_h^n - \tilde{y}_h^{n-1} - y_h^n + y_h^{n-1}, \psi(t)) \\ &=\; a(\tilde{y}_h^n - \tilde{y}_h^{n-1} - y_h^n + y_h^{n-1}, \psi(t) - \hat{\Pi}_n\psi(t)).\end{aligned}$$

With an aid of (2.31) and Green's formula, we have

$$\begin{aligned}(\tilde{y}_h^n - \tilde{y}_h^{n-1} - y_h^n + y_h^{n-1}, \xi_y) \;&=\; (R_y^n - R_y^{n-1}, \psi(t) - \hat{\Pi}_n\psi(t)) \\ &\quad +(J_y^n - J_y^{n-1}, \psi(t) - \hat{\Pi}_n\psi(t))_{\check{\Sigma}_n}, \qquad (2.48)\end{aligned}$$

where

$$R_y^n \;:=\; \mathcal{L}_h^n(\bar{\partial} y_h^n) - div(\nabla y_h^n) - f^n - U_h \;\text{ and }\; J_y^n = J_1[y_h^n].$$

Equation (2.48) can be expressed as

$$(\tilde{y}_h^n - \tilde{y}_h^{n-1} - y_h^n + y_h^{n-1}, \xi_y) \;=\; k_n(\bar{\partial} R_y^n, \psi(t) - \hat{\Pi}_n\psi(t)) + k_n(\bar{\partial} J_y^n, \psi(t) - \hat{\Pi}_n\psi(t))_{\check{\Sigma}_n}.$$

By Proposition 2.3.2, it follows that

$$\zeta_n^1 \;\le\; \Big(\int_{t_{n-1}}^{t_n} |\psi(t)|_2\, dt\Big) \times \Big(\bar{C}_{1,2}\|\hat{h}_n^2 \bar{\partial} R_y^n\| + \bar{C}_{2,2}\|\hat{h}_n^{3/2}\bar{\partial} J_y^n\|_{\hat{\Sigma}_n} + \bar{C}_{3,2}\|\hat{h}_n^{3/2}\bar{\partial} J_y^n\|_{\check{\Sigma}_n \setminus \hat{\Sigma}_n}\Big).$$

Use of elliptic regularity result, $|\psi|_2 \le \|\xi_y\|$, yields

$$\zeta_n^1 \;\le\; \max_{t \in I_n} \|\xi_y(t)\| \Big(\bar{C}_{1,2}\|\hat{h}_n^2 \bar{\partial} R_y^n\| + \bar{C}_{2,2}\|\hat{h}_n^{3/2}\bar{\partial} J_y^n\|_{\hat{\Sigma}_n} + \bar{C}_{3,2}\|\hat{h}_n^{3/2}\bar{\partial} J_y^n\|_{\check{\Sigma}_n \setminus \hat{\Sigma}_n}\Big) k_n,$$

and hence,

$$\sum_{n=1}^{m} \zeta_n^1 \;\le\; \|\xi_y^{m^*}\| \sum_{n=1}^{m} k_n\, \beta_{1,n}.$$

In order to estimate ζ_n^2, which accounts for the time discretization error, we use the definition of elliptic reconstruction (2.31) to obtain

$$\zeta_n^2 \;=\; \int_{t_{n-1}}^{t_n} |a(\tilde{y}_h^n - \tilde{y}_h^{n-1}, \xi_y)|\, dt. \qquad (2.49)$$

From (2.31), we have

$$a(\tilde{y}_h^n, v) \;=\; -\tilde{r}_y^n(v), \tag{2.50}$$

where $\tilde{r}_y^n(v) = (\mathcal{L}_h^n(\bar{\partial} y_h^n), v) - (f^n + U_h, v)$. Altogether (2.50) with (2.49) yields

$$\zeta_n^2 \;=\; \int_{t_{n-1}}^{t_n} |(k_n \bar{\partial} \tilde{r}_y^n, \xi_y)|\, dt \;\leq\; \int_{t_{n-1}}^{t_n} \|k_n \bar{\partial} \tilde{r}_y^n\| \|\xi_y\|\, dt,$$

and hence,

$$\sum_{n=1}^{m} \zeta_n^2 \;\leq\; \|\xi_y^{m^*}\| \sum_{n=1}^{m} k_n\, \beta_{2,n}.$$

We now bound the term ζ_n^3. Since $V_h^n \subset ker(\mathcal{L}_h^n - I)$, we have

$$\begin{aligned}
\zeta_n^3 \;&=\; \int_{t_{n-1}}^{t_n} |(k_n^{-1}(\mathcal{L}_h^n - I) y_h^{n-1}, \xi_y)|\, dt \\
&=\; \int_{t_{n-1}}^{t_n} |(k_n^{-1}(\mathcal{L}_h^n - I) y_h^{n-1}, \xi_y - \Pi_n \xi_y)|\, dt \\
&=\; \int_{t_{n-1}}^{t_n} |(h_n k_n^{-1}(\mathcal{L}_h^n - I) y_h^{n-1}, h_n^{-1}(\xi_y - \Pi_n \xi_y))|\, dt \\
&\leq\; \int_{t_{n-1}}^{t_n} \|h_n k_n^{-1}(\mathcal{L}_h^n - I) y_h^{n-1}\| \|h_n^{-1}(\xi_y - \Pi_n \xi_y)\|\, dt \\
&\leq\; C_{1,1} k_n^{1/2} \|h_n k_n^{-1}(\mathcal{L}_h^n - I) y_h^{n-1}\| \Big(\int_{t_{n-1}}^{t_n} \|\xi_y\|_1^2\, dt \Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus, we have

$$\sum_{n=1}^{m} \zeta_n^3 \;\leq\; \sum_{n=1}^{m} \beta_{3,n} k_n^{1/2} \Big(\int_{t_{n-1}}^{t_n} \|\xi_y\|_1^2\, dt \Big)^{\frac{1}{2}}.$$

To estimate ζ_n^4, we first note that

$$\zeta_n^4 \;\leq\; \Big(\max_{t \in I_n} \|\xi_y(t)\| \Big) \int_{t_{n-1}}^{t_n} \|f^n - f\|\, dt,$$

and hence,

$$\sum_{n=1}^{m} \zeta_n^4 \;\leq\; \|\xi_y^{m^*}\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \|f^n - f\|\, dt.$$

Combining the estimates of ζ_n^i, $i = 1, \ldots, 4$, together with (2.47), we obtain

$$\begin{aligned}
\frac{1}{2}\|\xi_y^{m^*}\|^2 + \alpha_1 \int_0^{t_m} \|\xi_y\|_1^2\, dt \;&\leq\; \frac{1}{2}\|\xi_y(0)\|^2 + 2\|\xi_y^{m^*}\| \sum_{n=1}^{m} k_n \big(\beta_{1,n} + \beta_{2,n} + \beta_{4,n}\big) \\
&\quad + 2 \sum_{n=1}^{m} \beta_{3,n} k_n^{1/2} \Big(\int_{t_{n-1}}^{t_n} \|\xi_y\|_1^2\, dt \Big)^{1/2}.
\end{aligned}$$

Now, we use a standard inequality due to [48]. For $\hat{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$, $\hat{b} = (b_0, b_1, \ldots, b_n) \in \mathbb{R}^{n+1}$ and $\hat{c} \in \mathbb{R}$, if

$$|\hat{a}|^2 \;\le\; \hat{c}^2 + \hat{a}\cdot\hat{b},$$

then, we have

$$|\hat{a}| \;\le\; |\hat{c}| + |\hat{b}|.$$

Finally, for $1 \le n \le m$, we take

$$\begin{aligned} a_0 \;&:=\; \frac{1}{\sqrt{2}}\|\xi_y^{m^*}\|, \quad a_n \;:=\; \left(\int_{t_{n-1}}^{t_n} \|\xi_y\|_1^2\,dt\right)^{1/2}, \quad \hat{c} \;:=\; \frac{1}{\sqrt{2}}\|\xi_y(0)\|,\\ b_0 \;&:=\; 2\sqrt{2}\sum_{n=1}^{m} k_n\left(\beta_{1,n}+\beta_{2,n}+\beta_{4,n}\right), \qquad b_n \;:=\; 2\beta_{3,n}\,k_n^{1/2}, \end{aligned}$$

to complete the rest of the proof of the lemma. □

The following lemma provides a bound for ξ_p.

Lemma 2.3.3 (Parabolic error for the co-state variable)**.** *Let ξ_p satisfy (2.36). Then the following estimate holds, $0 \le m \le N$:*

$$\left(\max_{t\in[t_m,T]} \|\xi_p(t)\|^2 + 2\alpha_1\int_{t_m}^{T}\|\xi_p\|_1^2\,dt\right)^{1/2} \le 4\left(\mathscr{E}_{3,m}^2 + \mathscr{E}_{4,m}^2\right)^{\frac{1}{2}} + \int_{t_m}^{T}\|\xi_y\|\,dt,$$

where

$$\mathscr{E}_{3,m} \;:=\; \sum_{n=m+1}^{N} k_n\left(\delta_{1,n}+\delta_{2,n}+\delta_{4,n}\right),$$

and

$$\mathscr{E}_{4,m} \;:=\; \sum_{n=m+1}^{N} \delta_{3,n}^2\,k_n.$$

In the above, for $n \in [0:N]$, $\delta_{1,n}$ represents the space error and is defined by

$$\delta_{1,n} \;:=\; \left(\bar{C}_{1,2}\|\hat{h}_n^2\bar{\partial}R_p^{n+1}\| + \bar{C}_{2,2}\|\hat{h}_n^{3/2}\bar{\partial}J_p^n\|_{\hat{\Sigma}_n} + \bar{C}_{3,2}\|\hat{h}_n^{3/2}\bar{\partial}J_p^n\|_{\check{\Sigma}_n\setminus\hat{\Sigma}_n}\right) + \|\bar{\partial}\eta_y^{n+1}\|,$$

and the temporal error is defined as

$$\delta_{2,n} \;:=\; \|k_n(\bar{\partial}\tilde{r}_p^n + \bar{\partial}\eta_y^{n+1})\|,$$

and the data approximation error estimators are defined by

$$\begin{aligned} \delta_{3,n} \;&:=\; \bar{C}_{3,1}\|h_n k_n^{-1}(\mathcal{L}_h^n - I)p_h^{n-1}\|,\\ \delta_{4,n} \;&:=\; \frac{1}{k_n}\int_{t_{n-1}}^{t_n}\|y_{ds}^n - y_{ds}\|\,dt. \end{aligned}$$

The constants $\bar{C}_{i,2}|_{i=1,2,3}$ and $\bar{C}_{3,1}$ are positive and independent of the discretization parameters.

Proof. Choose $v = \xi_p$ in (2.36) and use the coercive property of the bilinear form to obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|\xi_p\|^2 + \alpha_1\|\xi_p\|_1^2 \\
&\leq -(\eta_{p,t},\xi_p) + \frac{(t-t_{n-1})}{k_n}a(\tilde{p}_h^n - \tilde{p}_h^{n-1},\xi_p) - (k_n^{-1}(\mathcal{L}_h^n - I)p_h^{n-1},\xi_p) \\
&\quad - (y_{ds}^n - y_{ds},\xi_p) + (\tilde{y}_h^n - y(U_h),\xi_p).
\end{aligned} \tag{2.51}$$

Integrating (2.51) with respect to time from t_m to T, we find that

$$\begin{aligned}
&\frac{1}{2}\|\xi_p(t_m)\|^2 + \alpha_1\int_{t_m}^T \|\xi_p\|_1^2\,dt \\
&\leq \frac{1}{2}\|\xi_p(T)\|^2 + \int_{t_m}^T \Big\{ - (\eta_{p,t},\xi_p) + \frac{(t-t_{n-1})}{k_n}a(\tilde{p}_h^n - \tilde{p}_h^{n-1},\xi_p) \\
&\quad - (k_n^{-1}(\mathcal{L}_h^n - I)p_h^{n-1},\xi_p) - (y_{ds}^n - y_{ds},\xi_p) + (\tilde{y}_h^n - y(U_h),\xi_p)\Big\}\,dt \\
&\leq \sum_{n=m+1}^N \int_{t_{n-1}}^{t_n} \Big\{|(\eta_{p,t},\xi_p)| + |a(\tilde{p}_h^n - \tilde{p}_h^{n-1}),\xi_p)| \\
&\quad + |(k_n^{-1}(\mathcal{L}_h^n - I)p_h^{n-1},\xi_p)| + |(y_{ds}^n - y_{ds},\xi_p)| + |(\tilde{y}_h^n - y(U_h),\xi_p)|\Big\}\,dt \\
&=: \sum_{n=m+1}^N \Big\{\vartheta_n^1 + \vartheta_n^2 + \vartheta_n^3 + \vartheta_n^4 + \vartheta_n^5\Big\} =: \Theta_m.
\end{aligned} \tag{2.52}$$

Since ξ_p is a continuous in $[t_m,T]$, there exists $t_{m^*} \in [t_m,T]$ such that

$$\max_{t\in[t_m,T]} \|\xi_p\| \;=\; \|\xi_p(t_{m^*})\| \;=:\; \|\xi_p^{m^*}\|.$$

As Lemma 2.3.2, we deduce that

$$\frac{1}{2}\|\xi_p^{m^*}\|^2 + \alpha_1\int_{t_{m^*}}^T \|\xi_p\|_1^2\,dt \;\leq\; \Theta_m. \tag{2.53}$$

Consequently, from (2.52) and (2.53), it follows that

$$\frac{1}{2}\|\xi_p^{m^*}\|^2 + \alpha_1\int_{t_{m^*}}^T \|\xi_p\|_1^2\,dt \;\leq\; 2\Theta_m. \tag{2.54}$$

Now we estimate each of the summands $\vartheta_n^i|_{i=1,\ldots,5}$ appearing on the right hand side of (2.54). To estimate the term ϑ_n^1, we write

$$\vartheta_n^1 \;=\; \int_{t_{n-1}}^{t_n} |(\eta_{p,t},\xi_p)|\,dt \;=\; k_n^{-1}\int_{t_{n-1}}^{t_n} |(\tilde{p}_h^n - \tilde{p}_h^{n-1} - p_h^n + p_h^{n-1},\xi_p)|\,dt.$$

Let $\Psi : [0,T] \to H_0^1(\Omega)$ such that $a(\chi, \Psi(t)) = (\xi_p, \chi),\ \forall \chi \in H_0^1(\Omega)$. A similar argument as in the case of the state variable and use of interpolation operator $\hat{\Pi}_n$ defined as in Proposition 2.3.2 leads to

$$\begin{aligned}(\tilde{p}_h^n - \tilde{p}_h^{n-1} - p_h^n + p_h^{n-1}, \xi_p) &= a(\tilde{p}_h^n - \tilde{p}_h^{n-1} - p_h^n + p_h^{n-1}, \Psi) \\ &= a(\tilde{p}_h^n - \tilde{p}_h^{n-1} - p_h^n + p_h^{n-1}, \Psi - \hat{\Pi}_n \Psi) \\ &\quad + a(\tilde{p}_h^n - \tilde{p}_h^{n-1} - p_h^n + p_h^{n-1}, \hat{\Pi}_n \Psi).\end{aligned}$$

By the definition of elliptic reconstruction (2.32) and use of integration by parts formula leads to

$$\begin{aligned}(\tilde{p}_h^n - \tilde{p}_h^{n-1} - p_h^n + p_h^{n-1}, \xi_p) &= -k_n(\bar{\partial} R_p^{n+1}, \Psi - \hat{\Pi}_n \Psi) - k_n(\bar{\partial} J_p^n, \Psi - \hat{\Pi}_n \Psi)_{\check{\Sigma}_n} \\ &\quad + k_n(\bar{\partial}\eta_y^{n+1}, \Psi),\end{aligned}$$

where

$$\begin{aligned}R_p^n &:= -\mathcal{L}_h^n(\bar{\partial} p_h^n) - div(\nabla p_h^{n-1}) - (y_h^n - y_{ds}^n), \\ J_p^n &:= J_1[p_h^n].\end{aligned}$$

An application of Proposition 2.3.2 yields

$$\begin{aligned}\vartheta_n^1 \leq &\left(\int_{t_{n-1}}^{t_n} |\Psi|_2 \, dt\right) \times \left(\bar{C}_{1,2}\|\hat{h}_n^2 \bar{\partial} R_p^{n+1}\| + \bar{C}_{2,2}\|\hat{h}_n^{3/2}\bar{\partial} J_p^n\|_{\hat{\Sigma}_n} + \bar{C}_{3,2}\|\hat{h}_n^{3/2}\bar{\partial} J_p^n\|_{\check{\Sigma}_n \setminus \hat{\Sigma}_n}\right) \\ &+ k_n\|\bar{\partial}\eta_y^{n+1}\|\|\Psi\|.\end{aligned}$$

Using elliptic regularity and summing $n = m+1$ to N, we obtain

$$\sum_{n=m+1}^{N} \vartheta_n^1 \leq \|\xi_p^{m^*}\| \sum_{n=m+1}^{N} k_n \, \delta_{1,n}.$$

To estimate ϑ_n^2, use (2.32) together with the definition of r_p^n to have

$$\sum_{n=m+1}^{N} \vartheta_n^2 \leq \|\xi_p^{m^*}\| \sum_{n=m+1}^{N} k_n \, \|k_n(\bar{\partial}\tilde{r}_p^n + \bar{\partial}\eta_y^{n+1})\| = \|\xi_p^{m^*}\| \sum_{n=m+1}^{N} k_n \, \delta_{2,n},$$

where

$$\tilde{r}_p^n := -\mathcal{L}_h^n(\bar{\partial} p_h^n) - (y_h^n - y_{ds}^n).$$

Now, we estimate ϑ_n^3 and ϑ_n^4. Arguing as in Lemma 2.3.2, we arrive at

$$\sum_{n=m+1}^{N} \vartheta_n^3 \leq \sum_{n=m+1}^{N} \delta_{3,n} k_n^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\xi_p\|_1^2 \, dt\right)^{\frac{1}{2}}.$$

Note that

$$\vartheta_n^4 \;\le\; \Big(\max_{t\in I_n}\|\xi_p(t)\|\Big)\int_{t_{n-1}}^{t_n}\|y_{ds}^n - y_{ds}\|\,dt,$$

and hence

$$\sum_{n=m+1}^{N}\vartheta_n^4 \;\le\; \|\xi_p^{m^*}\|\Big(\sum_{n=m+1}^{N}\int_{t_{n-1}}^{t_n}\|y_{ds}^n - y_{ds}\|\,dt\Big).$$

Finally, for ϑ_n^5, we find that

$$\sum_{n=m+1}^{N}\vartheta_n^5 \;\le\; \|\xi_p^{m^*}\|\Big(\sum_{n=m+1}^{N}\int_{t_{n-1}}^{t_n}\|\xi_y\|\,dt\Big).$$

Proceed as in Lemma 2.3.2 to complete the rest of the proof. □

We are now in a position to present the main intermediate results for the state and co-state variables which will be used to derive the main result.

Theorem 2.3.1 (Intermediate error estimates)**.** *Let* (Y_h, P_h, U_h) *and* $(y(U_h), p(U_h))$, *respectively, be the solutions of* (2.20) – (2.24) *and* (2.25) – (2.28) *with* $\hat{u} = U_h$. *Then, for each* $1 \le m \le N$, *the following a posteriori error estimate holds:*

$$\max_{[0,t_m]}\|Y_h(t) - y(U_h)(t)\| \;\le\; \|\tilde{y}_h^0 - y_0\| + 2\max_{n\in[0:m]}\mathscr{F}_{y,n} + 4\Big(\mathscr{E}_{1,n}^2 + \mathscr{E}_{2,n}^2\Big)^{1/2}, \tag{2.55}$$

and

$$\max_{[t_m,T]}\|P_h(t) - p(U_h)(t)\| \;\le\; 2\max_{n\in[m:N]}\mathscr{F}_{p,n} + 4\Big(\mathscr{E}_{3,n}^2 + \mathscr{E}_{4,n}^2\Big)^{1/2} + \int_{t_m}^{T}\|\xi_y\|\,dt, \tag{2.56}$$

with

$$\begin{aligned}\mathscr{F}_{y,n} &:= \bar{C}_{1,2}h_n^2\|(\mathcal{A}_h^n - A_{el})y_h^n\| + \bar{C}_{2,2}h_n^{3/2}\|J_1[y_h^n]\|_{\Sigma_n},\\ \mathscr{F}_{p,n} &:= \bar{C}_{1,2}h_n^2\|(\mathcal{A}_h^n - A_{el})p_h^n\| + \bar{C}_{2,2}h_n^{3/2}\|J_1[p_h^{n-1}]\|_{\Sigma_n} + \mathscr{F}_{y,n},\end{aligned}$$

and $\mathscr{E}_{i,n}|_{i=1,2}$, $\mathscr{E}_{i,n}|_{i=3,4}$ *are defined in Lemma 2.3.2 and Lemma 2.3.3, respectively.*

Proof. By the triangle inequality, we have

$$\begin{aligned}\|e_y(t)\| &\le \|\xi_y(t)\| + \|\eta_y(t)\|, \quad t\in I_n\\ &\le \max_{t\in[0,t_m]}\|\xi_y(t)\| + \max_{t\in[0,t_m]}\|\eta_y(t)\|.\end{aligned} \tag{2.57}$$

For $t \in I_n$, we note that

$$\begin{aligned}\|\eta_y(t)\| &= \Big\|\frac{(t_n - t)}{k_n}\eta_y^{n-1} + \frac{(t - t_{n-1})}{k_n}\eta_y^n\Big\|\\ &\le \|\eta_y^{n-1}\| + \|\eta_y^n\| \le 2\max_{n\in[1,m]}\big\{\|\eta_y^{n-1}\|,\ \|\eta_y^n\|\big\}.\end{aligned}$$

Again, for $t \in [0, t_m]$, use of Lemma 2.3.1 yields

$$\|\eta_y(t)\| \;\leq\; 2 \max_{n\in[0,m]} \|\eta_y^n\| \leq 2 \max_{n\in[0,m]} \mathscr{F}_{y,n}. \tag{2.58}$$

Altogether (2.57), (2.58) and Lemma 2.3.2 yield the first inequality (2.55). The second inequality (2.56) is proved in a similar way by using Lemma 2.3.3. This completes the rest of the proof. □

Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (2.6) − (2.10) and (2.20) − (2.24), respectively. In order to derive *a posteriori* error bounds for the state, co-state and control variables, we decompose the errors as follows:

$$\begin{aligned} Y_h - y &= (Y_h - y(U_h)) + (y(U_h) - y) =: e_y - \tilde{e}_y, \\ P_h - p &= (P_h - p(U_h)) + (p(U_h) - p) =: e_p - \tilde{e}_p. \end{aligned}$$

From (2.6), (2.8) and (2.25), (2.27), we derive the following error equations:

$$(\tilde{e}_{y,t}, v) + a(\tilde{e}_y, v) \;=\; (u - U_h, v) \qquad \forall v \in H_0^1(\Omega), \tag{2.59}$$

$$-(\tilde{e}_{p,t}, v) + a(\tilde{e}_p, v) \;=\; (\tilde{e}_y, v) \qquad \forall v \in H_0^1(\Omega). \tag{2.60}$$

We now provide the bounds for $\tilde{e}_y$ and $\tilde{e}_p$.

Lemma 2.3.4. *Let (y, p, u) and $(y(U_h), p(U_h))$, respectively, be the solutions of* (2.6) − (2.10) *and* (2.25) − (2.28) *with $\hat{u} = U_h$. Then, for $m \in [1 : N]$, we have*

$$\|\tilde{e}_y(t_m)\|^2 + \int_0^{t_m} \|\tilde{e}_y\|_1^2 \, dt \;\leq\; C_{2,1} \int_0^{t_m} \|u - U_h\|_{L^2(\Omega)}^2 \, dt, \tag{2.61}$$

and

$$\|\tilde{e}_p(t_m)\|^2 + \int_{t_m}^{T} \|\tilde{e}_p\|_1^2 \, dt \;\leq\; C_{2,2} \int_{t_m}^{T} \|u - U_h\|_{L^2(\Omega)}^2 \, dt, \tag{2.62}$$

where $C_{2,1}$ and $C_{2,2}$ are the positive constants depend on the Poincaré inequality constant C and the coercivity constant α_1.

Proof. Set $v = \tilde{e}_y$ in (2.59) and use the coercive property of the bilinear form to obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{e}_y\|^2 + \alpha_1 \|\tilde{e}_y\|_1^2 \;\leq\; |(u - U_h, \tilde{e}_y)|.$$

Apply the Cauchy-Schwarz and the Young's inequality to have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{e}_y\|^2 + \alpha_1 \|\tilde{e}_y\|_1^2 \;\leq\; \frac{1}{2}\|u - U_h\|^2 + \frac{1}{2}\|\tilde{e}_y\|^2.$$

Integrating the above with respect to time from 0 to t_m, and use the fact that $\tilde{e}_y|_{t=0} = 0$ to have

$$\|\tilde{e}_y(t_m)\|^2 + 2\alpha_1 \int_0^{t_m} \|\tilde{e}_y\|_1^2 \, dt \;\leq\; \int_0^{t_m} \|u - U_h\|^2 \, dt + \int_0^{t_m} \|\tilde{e}_y\|^2 \, dt.$$

An application of the Poincaré inequality yields the first inequality (2.61).

To prove (2.62), choose $v = \tilde{e}_p$ in (2.60). Then, using the coercive property of bilinear form, the Cauchy-Schwarz inequality and the Young's inequality to obtain

$$-\frac{1}{2}\frac{d}{dt}\|\tilde{e}_p\|^2 + \alpha_1\|\tilde{e}_p\|_1^2 \;\leq\; \frac{1}{2}\|\tilde{e}_y\|^2 + \frac{1}{2}\|\tilde{e}_p\|^2. \tag{2.63}$$

Integrating (2.63) from t_m to T and using the fact that $\tilde{e}_p|_{t=T} = 0$, it follows that

$$\|\tilde{e}_p(t_m)\|^2 + 2\alpha_1 \int_{t_m}^{T} \|\tilde{e}_p\|_1^2 \, dt \;\leq\; \int_{t_m}^{T} \|\tilde{e}_y\|^2 \, dt + \int_{t_m}^{T} \|\tilde{e}_p\|^2 \, dt.$$

An application of the Poincaré inequality together with (2.61) completes the rest of the proof. □

The following lemma presents the *a posteriori* error estimate for the control variable in the $L^2(0,T;L^2(\Omega))$-norm.

Lemma 2.3.5 (Control error estimate). *Let* (y,p,u) *and* (Y_h,P_h,U_h) *be the solutions of* (2.6) − (2.10) *and* (2.20) − (2.24)*, respectively. Assume that* $U_{ad}^n \subset U_{ad}$, $(U_h + P_h^{n-1})|_K \in H^1(K)$ *and* $w_h \in U_{ad}^n$*, and there exists a positive constant* $C_{2,3}$ *such that*

$$\left| \int_0^T (U_h + P_h^{n-1}, w_h - u)\, dt \right| \;\leq\; C_{2,3} \int_0^T \sum_{K\in\mathscr{T}_h^n} h_K |U_h + P_h^{n-1}|_{H^1(K)} \|u - U_h\|_{L^2(K)} \, dt. \tag{2.64}$$

Then, we have

$$\|u - U_h\|_{L^2(0,T;L^2(\Omega))}^2 \leq C_{2,4} \Big(\int_0^T \sum_{K\in\mathscr{T}_h^n} h_K^2 |U_h + P_h^{n-1}|_{H^1(K)}^2 \, dt + \|P_h^{n-1} - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \Big),$$

where $C_{2,4} = \frac{3}{2}\max\{C_{2,3}^2, 1\}$*, and* $(y(U_h), p(U_h))$ *is the solution of* (2.25) − (2.28) *with* $\hat{u} = U_h$*.*

Proof. From (2.10) with $w = U_h$, we have

$$(u, u - U_h) \;\leq\; -(p, u - U_h),$$

using the above inequality, we find that

$$\begin{aligned}
\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T (u-U_h,u-U_h)\,dt \\
&\leq \int_0^T \{-(p,u-U_h)-(U_h,u-U_h)\}\,dt \\
&= -\int_0^T (P_h^{n-1}+U_h,u-w_h)\,dt - \int_0^T (U_h+P_h^{n-1},w_h-U_h)\,dt \\
&\quad + \int_0^T (P_h^{n-1}-p(U_h),u-U_h)\,dt + \int_0^T (p(U_h)-p,u-U_h)\,dt.
\end{aligned}$$

With an aid of (2.24), we get

$$\begin{aligned}
\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))} &\leq \int_0^T (U_h+P_h^{n-1},w_h-u)\,dt + \int_0^T (P_h^{n-1}-p(U_h),u-U_h)\,dt \\
&\quad + \int_0^T (p(U_h)-p,u-U_h)\,dt \\
&=: E_1+E_2+E_3. \qquad (2.65)
\end{aligned}$$

In view of the assumption (2.64), it follows that

$$\begin{aligned}
|E_1| &= \left|\int_0^T (U_h+P_h^{n-1},w_h-u)\,dt\right| \\
&\leq C_{2,3}\int_0^T \sum_{K\in\mathscr{T}_h^n} h_K|U_h+P_h^{n-1}|_{H^1(K)}\|u-U_h\|_{L^2(K)}\,dt \\
&\leq \frac{3C_{2,3}^2}{4}\Big(\int_0^T \sum_{K\in\mathscr{T}_h^n} h_K^2|U_h+P_h^{n-1}|^2_{H^1(K)}\,dt\Big) + \frac{1}{4}\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}. \qquad (2.66)
\end{aligned}$$

The term E_2 is bounded as

$$\begin{aligned}
|E_2| &= \left|\int_0^T (P_h^{n-1}-p(U_h),u-U_h)\,dt\right| \\
&\leq \frac{3}{4}\|P_h^{n-1}-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{4}\|u-U_h\|^2_{L^2(0,T;L^2(\Omega))}. \qquad (2.67)
\end{aligned}$$

It now remains to estimate the term E_3. For this, we first note that

$$y(x,0)-y(U_h)(x,0) = 0 \quad \text{and} \quad p(x,T)-p(U_h)(x,T) = 0.$$

Utilizing (2.6) and (2.25), we write E_3 as

$$\begin{aligned}
E_3 &= \int_0^T (u-U_h,p(U_h)-p)\,dt \\
&= \int_0^T \Big\{-(y-y(U_h),p_t(U_h)-p_t)+a(y-y(U_h),p(U_h)-p)\Big\}\,dt,
\end{aligned}$$

which combine with (2.8) and (2.27) yields

$$E_3 \;=\; -\int_0^T \|y-y(U_h)\|^2\,dt \;\leq\; 0. \tag{2.68}$$

Altogether (2.65) – (2.68) yields the desired estimate. This completes the proof. □

Remark 2.3.1. *The assumption (2.64) refers to the auxiliary finite element solution for the control variable, which help us in simplifying the proof of Lemma 2.3.5 for the control variable. It is easy to verify this condition for many applications (cf., [57]).*

By collecting Theorem 2.3.1 and Lemmas 2.3.4 – 2.3.5, the main results of this chapter are presented in the following theorem.

Theorem 2.3.2 ($L^\infty(L^2)-$error estimates)**.** *Let (y,p,u) and (Y_h,P_h,U_h) be the solutions of* (2.6) – (2.10) *and* (2.20) – (2.24)*, respectively. Then, for each $m\in[1:N]$, the following a posteriori error estimates*

$$\begin{aligned}\max_{[0,t_m]}\|Y_h(t)-y(t)\| \;&\leq\; \|\tilde{y}_h^0-y_{h,0}\|+2\max_{n\in[0:m]}\mathscr{F}_{y,n}+4\Big(\mathscr{E}_{1,n}^2+\mathscr{E}_{2,n}^2\Big)^{1/2}\\ &\quad+\Big(\int_0^{t_m}\|u-U_h\|^2\,dt\Big)^{\frac12},\\ \max_{[t_m,T]}\|P_h(t)-p(t)\| \;&\leq\; 2\max_{n\in[1:m]}\mathscr{F}_{p,n}+4\Big(\mathscr{E}_{3,n}^2+\mathscr{E}_{4,n}^2\Big)^{1/2}+\int_{t_m}^T\|\xi_y\|\,dt\\ &\quad+\Big(\int_{t_m}^T\|u-U_h\|^2\,dt\Big)^{\frac12},\end{aligned}$$

and

$$\begin{aligned}\max_{[0,t_m]}\|u(t)-U_h(t)\| \;&\leq\; C_{2,4}\Big(\int_0^T\sum_{K\in\mathscr{T}_h^n}h_K^2|U_h+P_h^{n-1}|_{H^1(K)}^2dt\Big)^{1/2}\\ &\quad+2\max_{n\in[1:m]}\mathscr{F}_{p,n}+4\Big(\mathscr{E}_{3,n}^2+\mathscr{E}_{4,n}^2\Big)^{1/2}+\int_{t_m}^T\|\xi_y\|\,dt\end{aligned}$$

hold, where

$$\begin{aligned}\mathscr{F}_{y,n} \;&:=\; \bar{C}_{1,2}h_n^2\|(\mathcal{A}_h^n-A_{el})y_h^n\|+\bar{C}_{2,2}h_n^{3/2}\|J_1[y_h^n]\|_{\Sigma_n},\\ \mathscr{F}_{p,n} \;&:=\; \bar{C}_{1,2}h_n^2\|(\mathcal{A}_h^n-A_{el})p_h^n\|+\bar{C}_{2,2}h_n^{3/2}\|J_1[p_h^{n-1}]\|_{\Sigma_n}+\mathscr{F}_{y,n},\end{aligned}$$

and $\mathscr{E}_{i,n}$, $i=1,2$, and $\mathscr{E}_{i,n}$, $i=3,4$, are defined in Lemma 2.3.2 and 2.3.3, respectively.

Proof. From (2.10) and (2.3), we obtain

$$u(x,t) \;=\; \Pi_{[u_a,u_b]}\big(-p(x,t)\big), \tag{2.69}$$

where $\Pi_{[u_a,u_b]}(u(x,t)) := \max(u_a, \min(u_b, -p(x,t))$ (cf. [96]). Similarly, we have

$$U_h = \Pi_{[u_a,u_b]}\big(-P_h^{n-1}\big). \tag{2.70}$$

Using (2.69) and (2.70), we obtain

$$\|u - U_h\|_{L^\infty(0,T;L^2(\Omega))} \le \|p - P_h^{n-1}\|_{L^\infty(0,T;L^2(\Omega))}.$$

Inviting Theorem 2.3.1 together with Lemmas 2.3.4−2.3.5 leads to the desired estimates. This completes the proof of the theorem. ☐

2.4 Numerical assessments

This section presents some numerical experiments to illustrate the performance of the derived *a posteriori* error estimators. We use this loop

$$SOLVE \to ESTIMATE \to MARK \to REFINE$$

to achieve a refinement from the initializing triangulation. For simplicity, all constants involved in the estimators are taken to be 1. All computations are carried out using the software FreeFem++ [38].

Note that, for the solution of the minimization problem, we use the projection gradient method. In Algorithm 2.1, we first decrease the time step size to keep the time error estimator below the tolerance ϵ_{time} in Step 2 while keeping the space mesh unchanged. In Step 3, the refinement procedure is carried out until the time and space error estimators satisfy the desired tolerances. In the last step, if the time error estimator is very less than the prescribe time tolerance ϵ_{time} then we increase the time step size by multiplying a factor σ_2. For marking and refinement of the elements $K \in \mathscr{T}_h^n$, we follow the strategy of Morin, Nochetto and Siebert, see [75]. For both the test example problems, we choose tolerances for time and space as $\epsilon_{time} = .01$ and $\epsilon_{space} = .001$, respectively.

We now consider the optimal control problem (2.1) − (2.3). Then u, f and y_{ds} are easily evaluated by

$$u = \max\big\{u_a, \min\{u_b, -p\}\big\}, \tag{2.71}$$

$$f = \frac{\partial y}{\partial t} - \Delta y - u, \tag{2.72}$$

$$y_{ds} = \frac{\partial p}{\partial t} + \Delta p + y. \tag{2.73}$$

Numerical results are based on the *a posteriori* error indicators developed in Section 2.3. We implement the space-time adaptive Algorithm 2.1 to illustrate the efficiency of

Algorithm 2.1 (Space-time adaptive algorithm)

```
Given parameters $\sigma_1 \in (0,1)$, $\sigma_2 > 1$, $\theta_1 \in (0,1)$, $\theta_2 \in (0,\theta_1)$, and the space tolerance $\epsilon_{space}$
and time tolerance $\epsilon_{time}$. Suppose that $(y_h^{n-1}, p_h^n, u_h^{n-1})$ is evaluated on mesh $\mathscr{T}_h^{n-1}$ at time
$t_{n-1}$ and time step size $k_{n-1}$.
Step 1. Set $\mathscr{T}_h^n := \mathscr{T}_h^{n-1}$, $k_n := k_{n-1}$, $t_n := t_{n-1} + k_n$.
            Evaluate the discrete problem (2.15) – (2.19) for $(y_h^n, p_h^{n-1}, u_h^n)$ on $\mathscr{T}_h^n$.
            Compute error estimators $\beta_{j,n}$, and $\delta_{j,n}$, $j = 1, \ldots, 4$ on $\mathscr{T}_h^n$.
Step 2. While $\sum_{j=2}^4 (\beta_{j,n} + \delta_{j,n}) > \theta_1 \cdot \epsilon_{time}$, do
            $k_n := \sigma_1 k_{n-1}$, $t_n := t_{n-1} + k_n$.
            Evaluate the discrete problem (2.15) – (2.19) for $(y_h^n, p_h^{n-1}, u_h^n)$ on $\mathscr{T}_h^n$.
            Compute error estimators $\beta_{j,n}$, $\delta_{j,n}$, and $j = 1, \ldots, 4$ on $\mathscr{T}_h^n$.
        End while.
Step 3. While $\beta_{1,n} + \delta_{1,n} > \epsilon_{space}$, do
            Refine mesh $\mathscr{T}_h^n$ generate a new modify mesh $\mathscr{T}_{h_n}^n$.
            Evaluate the discrete problem (2.15) – (2.19) for $(y_h^n, p_h^{n-1}, u_h^n)$ on $\mathscr{T}_{h_n}^n$.
            Compute error estimators $\beta_{j,n}$, $\delta_{j,n}$, and $j = 1, \ldots, 4$ on $\mathscr{T}_{h_n}^n$.
            While $\sum_{j=2}^4 (\beta_{j,n} + \delta_{j,n}) > \theta_1 \cdot \epsilon_{time}$, do
                $k_n' := \sigma_1 k_{n-1}$, $t_n := t_{n-1} + k_n'$.
                Evaluate the discrete problem (2.15) – (2.19) for $(y_h^n, p_h^{n-1}, u_h^n)$ on $\mathscr{T}_{h_n}^{n,k_n'}$.
                Compute error estimators $\beta_{j,n}$, $\delta_{j,n}$, and $j = 1, \ldots, 4$ on $\mathscr{T}_{h_n}^{n,k_n'}$.
            End while.
        End while.
Step 4. If $\sum_{j=2}^4 (\beta_{j,n} + \delta_{j,n}) \leq \theta_2 \cdot \epsilon_{time}$, do
            Set $k_n' := \sigma_2 k_{n-1}$, $t_n := t_{n-1} + k_n'$.
        End if.
```

the *a posteriori* error indicator. The adaptive meshes generated via the error estimators $\beta_{1,n}, \beta_{2,n}, \beta_{3,n}, \beta_{4,n},\ \delta_{1,n}, \delta_{2,n}, \delta_{3,n}$ and $\delta_{4,n}$ enable us to save convincing computational work in comparison to the uniform meshes. In each example, we present some mesh information at the final time $T = 1.0$ for the state, co-state and the control variables and their errors in the $L^\infty(0,T;L^2(\Omega))$-norm.

In the first example we have considered the problem where the solution is continuous whereas in the second example the solution is discontinuous.

Example 2.1. *This example considers the continuous solution of problem* (2.1) – (2.3) *with the following data:*

$$y = 0.1 \times \big(1 - \exp\{-10000(t-0.5)^2\}\big) \times \exp\Big\{ - [(x_1 - t + 0.5)^2 + (x_2 - t + 0.5)^2]/0.04\Big\},$$

$p = (t-1) \times y,\ u_a = -0.0125$ *and* $u_b = 0.0025$. *The control* u, *the source function* f *and the desired state* y_{ds} *are determined by* (2.71) – (2.73).

Numerical results on uniform and adaptive meshes at the final time $T = 1.0$ are listed in Table 2.1. From Figure 2.5, we observe that the time-step size drops in an interval around $t = 0.5$ and remains constant away from this interval due to the term $\big(1 - \exp\{-10000(t-0.5)^2\}\big)$. We can easily see that the term $(1 - \exp\{-10000(t-0.5)^2\})$ falls exponentially around the time interval $t = 0.5$. Note that the term $\big(1 - \exp\{-10000(t-0.5)^2\}\big)$ changes exponentially from 1 to 0 and 0 to 1 in the neighborhood

Table 2.1: *Numerical results on uniform and adaptive mesh* (Step-(II)).

		uniform mesh	adaptive mesh
Mesh Info.	h_{min}	0.0100	0.00613
	h_{max}	0.0100	0.75025
	♯ elements	23556	4708
	♯ nodes	11979	2375
$L^\infty(L^2)$-Error	$y - Y_h$	1.2002e-2	1.1953e-2
	$p - P_h$	2.1053e-2	2.0760e-2
	$u - U_h$	2.1398e-2	2.1230e-2

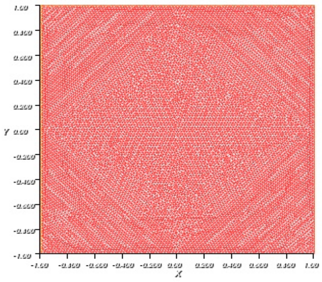


Figure 2.1: *Uniform mesh*

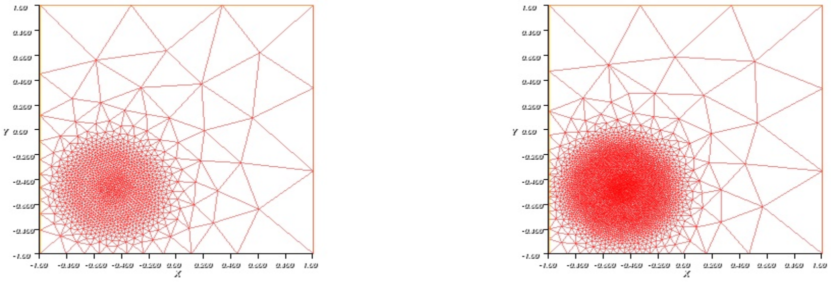


Figure 2.2: *Adaptive mesh* Step (I) *(left) and* Step (II) *(right) at time* $T = 0.1$

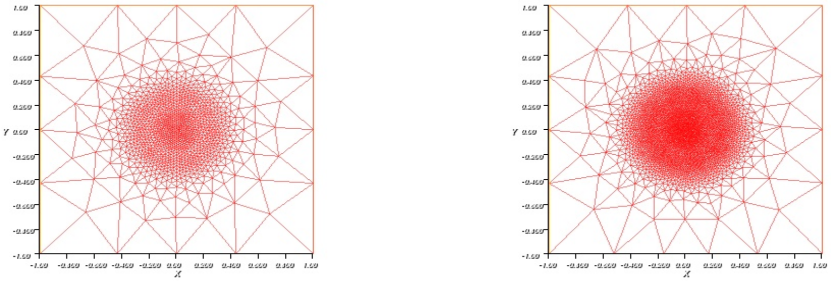


Figure 2.3: *Adaptive mesh* Step (I) *(left) and* Step (II) *(right) at time* $T = 0.5$

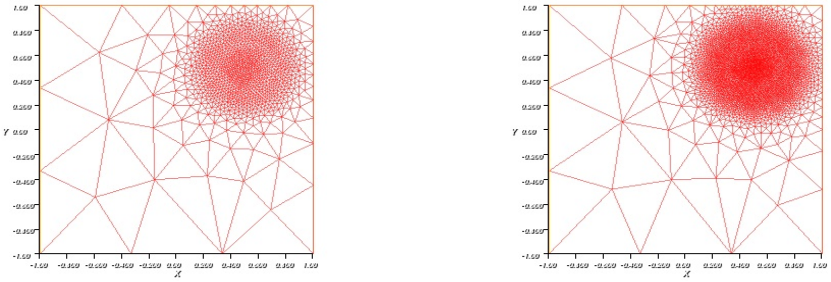


Figure 2.4: *Adaptive mesh* Step (I) *(left) and* Step (II) *(right) at time* $T = 1$

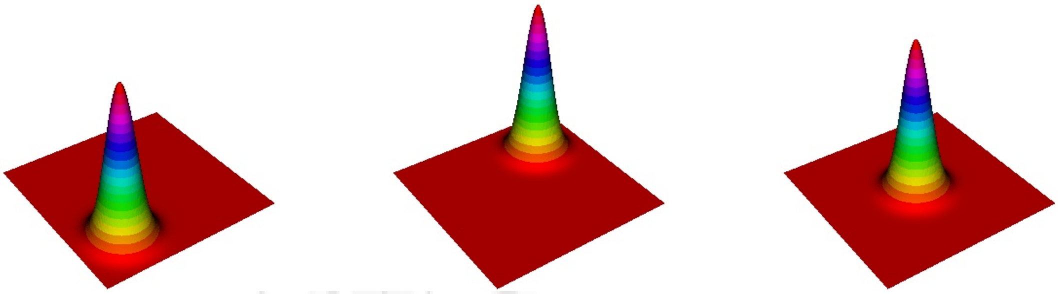


Figure 2.5: *Approximate state solution surfaces on adaptive mesh* (Step-(II)) *at* $T = 0.1$, $T = 0.5$ *and* $T = 1.0$, *respectively.*

of $t = 0.5$. Further, we observe that the mesh moves at constant speed away from the neighborhood of $t = 0.5$ but the shape of the solution remains unchanged. Figures 2.1 – 2.4 show that the mesh adapt very well by our derived error estimators. We plot the approximate solutions on the adaptive meshes (Step-(II)) at the time level $T = 0.1$, $T = 0.5$ and $T = 1.0$, respectively.

Example 2.2. *In this example, we consider the discontinuous solution to problem* (2.1)–(2.3) *with the following data:*

$$T = 1, \quad \Omega = [0,1] \times [0,1],$$

$$y = \begin{cases} t\sin(2\pi x_1)\sin(2\pi x_2) & x_1 + x_2 \leq 1, \\ 2t\sin(2\pi x_1)\sin(2\pi x_2) & x_1 + x_2 > 1, \end{cases}$$

$$p = \begin{cases} (t-1)\sin(2\pi x_1)\sin(2\pi x_2) & x_1 + x_2 \leq 1, \\ 2(t-1)\sin(2\pi x_1)\sin(2\pi x_2) & x_1 + x_2 > 1 \end{cases}$$

with $u_a = -0.25$ *and* $u_b = 0.25$.

We approximate the time derivative by the backward Euler method. We choose the time step-size $\Delta t \approx 6.6 \times 10^{-3}$, $N = T/\Delta t(= 150) \in \mathbb{Z}^+$, $t_n = n\Delta t$, $n = 1, 2, ...N$ in $[0, 1]$. Table 2.2 shows that the number of nodes required for adaptive meshes is much less in comparison to the uniformly meshes in our numerical experiment. Figure 2.7 depicts the approximate controls on the uniform, adaptive meshes at Step (I) and Step (II). From Figure 2.6, it is clear that the mesh adapts very well in the neighborhood of the discontinuous line $x_1 + x_2 = 1$. The higher density of the nodes are distributed

Table 2.2: *Numerical results on uniform and adaptive mesh*

		uniform mesh	adaptive mesh
Mesh Info.	h_{min}	0.0066	0.00041
	h_{max}	0.0066	0.30601
	♯ elements	37796	25615
	♯ nodes	19350	14354
$L^\infty(L^2)$-Error	$y - Y_h$	2.7410e-2	1.6924e-2
	$p - P_h$	2.8644e-2	1.9767e-2
	$u - U_h$	1.4049e-2	1.1989e-2

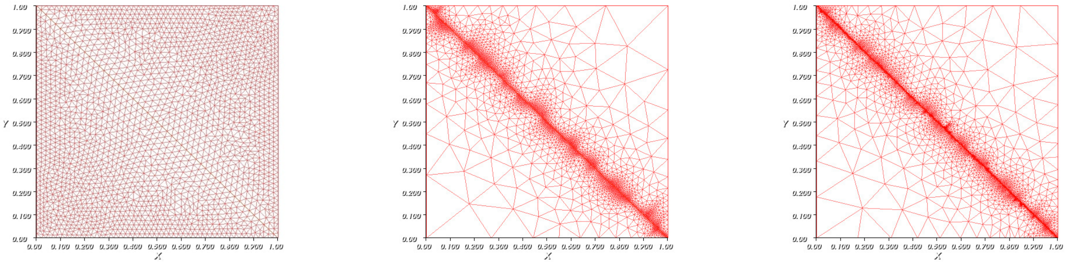


Figure 2.6: *Uniform mesh and adaptive meshes (*Step (I) *and* Step (II)*) generated via indicators at initial nodes* $N = 150$.

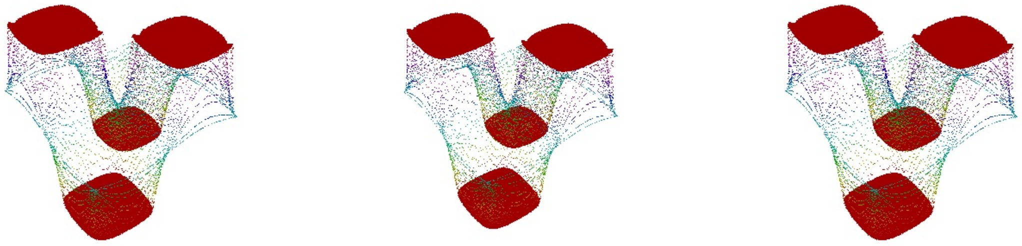


Figure 2.7: *Profiles of approximated controls on uniform mesh and adaptive meshes (Step (I) and Step (II)).*

along the line $x_1 + x_2 = 1$. It is clear from Table 2.2 that the adaptive mesh generated via the error indicators yields promising numerical results.

2.5 Concluding remarks

In this chapter, we studied the fully discrete finite element approximation for linear POCP (2.1) – (2.3) using elliptic reconstruction technique. We derived $L^\infty(0, T; L^2(\Omega))$ *a posteriori* upper bounds for errors in the state, co-state and control variables. Our main technique in deriving the error estimates is an appropriate adaptation of the fully discrete elliptic reconstruction technique introduced by Lakkis and Makridakis in [48]. The main feature of the proposed technique, in contrast to the direct approach in the literature, is that we can essentially use any available *a posteriori* error estimates for elliptic equation to control the main part of the spatial error. The present work extends the analysis of [48] from standard parabolic problem to POCP.

3

$L^\infty(L^\infty)$−A POSTERIORI ERROR ESTIMATES FOR POCP

This chapter devotes for the space-time *a posteriori* error estimates of finite element method for linear POCP (1.1) − (1.3) in a bounded polyhedral domain. For clarity, we define the functionals $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ as

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|^2 \, dt \;\text{ and }\; \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2 \, dt.$$

with the regularization parameter $\alpha = 1$. The variational discretization is used to approximate the control problem (1.1) − (1.3). The error analysis is carried out by using the piecewise linear and continuous finite elements for the approximation of the state and co-state variables, while the control variable is computed using the implicit relation between the control and co-state variables. The temporal discretization is based on the backward Euler method. The key feature of this approach is not to discretize the control variable but to implicitly utilize the optimality conditions for the discretization of the control variable. We use the elliptic reconstruction technique introduced by Makridakis and Nochetto [SIAM J. Numer. Anal., 41(2003), pp. 1585-1594] in conjunction with heat kernel estimates for linear parabolic problem to derive *a posteriori* error estimates for the state, co-state and control variables in the $L^\infty(0, T; L^\infty(\Omega))$-norm. Use of elliptic reconstruction technique greatly simplifies the analysis by allowing us to take the advantage of existing elliptic maximum norm error estimates and the heat kernel estimate. Numerical experiments are conducted to illustrate the performance of the derived estimators.

3.1 Introduction

Let Ω be a convex bounded polyhedral domain in $\mathbb{R}^d\,(d = 2, 3)$ with Lipschitz boundary $\Gamma =: \partial\Omega$. Set $\Omega_T = \Omega \times (0, T]$, $\Gamma_T = \partial\Omega \times (0, T]$ with $T < \infty$. We consider the following POCP:

$$\min_{u\in U_{ad}} \mathcal{J}(u, y) := \frac{1}{2}\int_0^T \left\{\|y - y_{ds}\|^2 + \|u\|^2\right\} dt \tag{3.1}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f + u & \text{in } \Omega_T, \\ y(x, 0) = y_0(x) & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma_T, \end{cases} \tag{3.2}$$

and the control constraints

$$u_a \leq u(x, t) \leq u_b \quad \text{a.e. in } \Omega_T, \tag{3.3}$$

where the initial state $y_0 \in L^\infty(\Omega)$, the desired state $y_{ds} \in L^\infty(0, T; L^\infty(\Omega))$ and the source function $f \in L^\infty(0, T; L^\infty(\Omega))$. Here $y = y(x, t)$ and $u = u(x, t)$ denote the state and the control variables, respectively. The set of admissible controls is defined by

$$U_{ad} = \left\{u \in L^\infty(0, T; L^\infty(\Omega)) : \; u_a \leq u \leq u_b \;\; a.e. \text{ in } \Omega_T\right\}$$

with $u_a, u_b \in \mathbb{R}$ fulfill $u_a < u_b$. Moreover, we shall denote the state space $V = L^\infty(0, T; L^\infty(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$. Observe that $V \subset \mathcal{C}(0, T; L^\infty(\Omega))$. The bilinear form $a(\cdot, \cdot)$ on $H_0^1(\Omega)$ is defined by

$$a(v, w) \;=\; \int_\Omega \nabla v \cdot \nabla w \, dx \quad \forall v, w \in H_0^1(\Omega),$$

where $H_0^1(\Omega) = \left\{v \in H^1(\Omega) : \; v = 0 \text{ on } \partial\Omega\right\}$. We assume that the bilinear form $a(\cdot, \cdot)$ satisfies the continuity and the coercivity properties, i.e., $\exists\ \alpha_0,\ \alpha_1 > 0$ such that

$$|a(v, w)| \;\leq\; \alpha_0\|v\|_1\|w\|_1, \quad \forall v, w \in H_0^1(\Omega),$$

and

$$a(v, v) \;\geq\; \alpha_1\|v\|_1^2, \quad \forall v \in H_0^1(\Omega).$$

The weak form of POCP (3.1) – (3.3) is defined as follows: Find a pair $(y, u) \in V \times U_{ad}$ such that

$$\min_{u \in U_{ad}} \frac{1}{2} \int_0^T \left\{ \|y - y_{ds}\|^2 + \|u\|^2 \right\} dt \tag{3.4}$$

subject to

$$\begin{cases} \left(\dfrac{\partial y}{\partial t}, v\right) + a(y, v) = (f + u, v) \qquad \forall v \in H_0^1(\Omega), \\ y(\cdot, 0) = y_0(x) \qquad x \in \Omega. \end{cases} \tag{3.5}$$

It is well known that the convex optimal control problem (3.4) – (3.5) has a unique solution (y, u) if and only if there exists a co-state variable p such that the triplet (y, p, u) satisfies the following optimality conditions for $t \in [0, T]$ (cf. [53]):

$$\begin{aligned} \left(\frac{\partial y}{\partial t}, v\right) + a(y, v) &= (f + u, v) \qquad \forall v \in H_0^1(\Omega), && (3.6)\\ y(x, 0) &= y_0(x) \qquad x \in \Omega, && (3.7)\\ -\left(\frac{\partial p}{\partial t}, v\right) + a(p, v) &= (y - y_{ds}, v) \qquad \forall v \in H_0^1(\Omega), && (3.8)\\ p(x, T) &= 0 \qquad x \in \Omega, && (3.9)\\ (u + p, w - u) &\geq 0 \qquad \forall w \in U_{ad}. && (3.10) \end{aligned}$$

Let $\Pi_{[u_a, u_b]}$ be a pointwise projection on the admissible set U_{ad}, and is defined as

$$\Pi_{[u_a, u_b]}(\chi(x, t)) := \min\{u_b, \max\{u_a, \chi(x, t)\}\}.$$

Following [73], it is easy to express the equivalent form of (3.10) as

$$u(x, t) = \Pi_{[u_a, u_b]}(-p(x, t)). \tag{3.11}$$

Introducing the reduced cost functional

$$\begin{aligned} j : L^\infty(0, T; L^\infty(\Omega)) &\to \mathbb{R} \\ u \mapsto j(u) &:= \mathcal{J}(u, y(u)), \end{aligned}$$

where $y(u)$ is the solution of (3.5), the optimal control problem (3.4) – (3.5) can be equivalently reformulated as

$$\min_{u \in U_{ad}} j(u).$$

We now collect some lemma for the pointwise *a posteriori* error estimates of elliptic problems and heat kernel estimate for the parabolic problem.

Elliptic a posteriori estimates. For $\psi \in L^\infty(\Omega)$, let $\Phi \in H_0^1(\Omega)$ be the solution of

$$\begin{aligned} -\Delta\Phi &= \psi \quad \text{in} \quad \Omega, \\ \Phi &= 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a convex polyhedral domain. Let $\mathscr{T}_h$ be a regular triangulation of Ω such that $\overline{\Omega} = \cup_{K\in\mathscr{T}_h}\bar{K}$, and if $K_1, K_2 \in \mathscr{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$, or $K_1 \cap K_2$ share a common edge, or a common vertex. Associated with $\mathscr{T}_h$ is a finite dimensional subspace V_h of $\mathcal{C}(\overline{\Omega})$, such that $v_h|_K$ is the polynomial of degree less than or equal to 1, for all $v_h \in V_h$. Now we set $V_h^0 = V_h \cap H_0^1(\Omega)$. Let $\Phi_h \in V_h^0$ be the finite element approximation to Φ such that

$$\int_\Omega \nabla\Phi_h \nabla v_h \, dx = \int_\Omega \psi \, v_h \, dx, \qquad \forall \, v_h \in V_h^0.$$

For $K_1, K_2 \in \mathscr{T}_h$, let E be the edge or face of the element such that $E = K_1 \cap K_2$. We now define the jump residual across an element edge E as

$$[[\nabla\Phi_h]]_E(x) := \lim_{\epsilon\to 0} \big(\nabla\Phi_h(x + \epsilon\mathbf{n}_E) - \nabla\Phi_h(x - \epsilon\mathbf{n}_E)\big) \cdot \mathbf{n}_E,$$

where $\mathbf{n}_E$ is a unit normal vector to E at the point x. Let h_K be the diameter of the element K. For $1 \le p \le \infty$ and $j \ge 0$, we define the elementwise error indicator as

$$\Re_{p,-j}(K) = h_K^{2+j}\|\psi + \Delta\Phi_h\|_{L^p(K)} + h_K^{j+1+\frac{1}{p}}\|[[\nabla\Phi_h]]\|_{L^p(\partial K)},$$

and the global estimator as

$$\mathscr{R}_{p,-j}(\Phi_h, \psi) = \begin{cases} \Big[\displaystyle\sum_{K\in\mathscr{T}_h} (\Re_{p,-j}(K))^p\Big]^{1/p} & 1 \le p < \infty, \\ \displaystyle\max_{K\in\mathscr{T}_h} \Re_{\infty,-j}(K) & p = \infty. \end{cases} \tag{3.12}$$

We state an elliptic pointwise error estimate from [81].

Lemma 3.1.1. *Let Ω be a convex bounded polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$, and $\bar{h} = \min_{K\in\mathscr{T}_h} h_K$. Then the following a posteriori error estimate*

$$\|\Phi - \Phi_h\|_{L^\infty(\Omega)} \;\le\; C_{3,1}\,(\ln \bar{h})^2\,\mathscr{R}_{\infty,0}(\Phi_h, \psi),$$

holds, where the constant $C_{3,1}$ depends on the domain Ω.

To bound some of our fully discrete *a posteriori* error estimates of the form $\Phi_1 - \Phi_2 - (\Phi_{h_1} - \Phi_{h_2})$, where Φ_{h_1} and Φ_{h_2} are related to different finite element spaces defined on meshes at adjacent time steps, we recall the following results from [21].

Let $V_{h_1}^0$ and $V_{h_2}^0$ be the finite element spaces associated on different meshes $\mathscr{T}_{h_1}$ and $\mathscr{T}_{h_2}$. Let $\Phi_{h_1} \in V_{h_1}^0$ and $\Phi_{h_2} \in V_{h_2}^0$ be the finite element approximations of Φ_1 and Φ_2, respectively and satisfy

$$-\Delta\Phi_1 = \psi_1 \quad \text{in} \quad \Omega \quad \text{and} \quad \Phi_1 = 0 \quad \text{on} \quad \Gamma,$$

and

$$-\Delta\Phi_2 = \psi_2 \quad \text{in} \quad \Omega \quad \text{and} \quad \Phi_2 = 0 \quad \text{on} \quad \Gamma.$$

For $1 \leq p \leq \infty$ and $j \geq 0$, we define the elementwise error indicator for $\hat{K} \in \mathscr{T}_{h_1} \wedge \mathscr{T}_{h_2}$ by

$$\hat{\Re}_{p,-j}(\hat{K}) = \hat{h}_{\hat{K}}^{2+j} \|\psi_1 - \psi_2 + \Delta(\Phi_{h_1} - \Phi_{h_2})\|_{L^p(\hat{K})} + \hat{h}_{\hat{K}}^{j+1+\frac{1}{p}} \|[[\nabla(\Phi_{h_1} - \Phi_{h_2})]]\|_{L^p(\Sigma_{\hat{K}})},$$

where $\Sigma_{\hat{K}} = (\Sigma_1 \cup \Sigma_2) \cap \hat{K}$ (Σ_1 and Σ_2 be the collection of all edges of elements $\mathscr{T}_{h_1}$ and $\mathscr{T}_{h_2}$, *respectively*) and the global estimator is defined by

$$\hat{\mathscr{R}}_{p,-j}(\Phi_{h_1} - \Phi_{h_2}, \psi_1 - \psi_2; \mathscr{T}_{h_1}, \mathscr{T}_{h_2}) = \begin{cases} \left[\displaystyle\sum_{\hat{K} \in \mathscr{T}_{h_1} \wedge \mathscr{T}_{h_2}} (\hat{\Re}_{p,-j}(\hat{K}))^p \right]^{1/p} & 1 \leq p < \infty, \\ \displaystyle\max_{\hat{K} \in \mathscr{T}_{h_1} \wedge \mathscr{T}_{h_2}} \hat{\Re}_{p,-j}(\hat{K}) & p = \infty. \end{cases}$$

Lemma 3.1.2. *Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a convex bounded polyhedral domain, and let $\mathscr{T}_{h_1}$ and $\mathscr{T}_{h_2}$ be compatible triangulations with $\hat{h} = \min_{x \in \Omega} \min\{h_1(x), h_2(x)\}$. Then we have*

$$\|\Phi_1 - \Phi_2 - (\Phi_{h_1} - \Phi_{h_2})\|_{L^\infty(\Omega)} \leq C_{3,2} (\ln \hat{h})^2 \hat{\mathscr{R}}_{\infty,0}(\Phi_{h_1} - \Phi_{h_2}, \psi_1 - \psi_2; \mathscr{T}_{h_1}, \mathscr{T}_{h_2}),$$

where $C_{3,2}$ depends on the number of refinement steps used to pass from $\mathscr{T}_{h_1}$ to $\mathscr{T}_{h_2}$.

As our analysis depends heavily on the properties of the Green's function for the heat equation, we invoke the necessary results in the following two lemmas. The proof of first lemma can be found in [64], and for the second lemma, we refer to ([5, 21]).

Lemma 3.1.3. *With $\mathrm{F} \in L^2(0,T; L^2(\Omega))$, let $\Psi \in \mathcal{W}_1$ be the solution of*

$$\Psi_t - \Delta\Psi = \mathrm{F} \quad \text{in } \Omega_T, \tag{3.13}$$

$$\Psi(x,0) = \Psi_0 \quad \text{in } \Omega, \tag{3.14}$$

$$\Psi = 0 \quad \text{on } \Gamma_T. \tag{3.15}$$

Moreover, we have the following a priori estimate

$$\|\Psi\|_{L^2(0,T;H^2(\Omega))} \ \leq \ C_R \left(\|\mathrm{F}\|_{L^2(0,T;L^2(\Omega))} + \|\Psi_0\|_{L^2(\Omega)}\right),$$

where C_R *is the regularity constant.*

Lemma 3.1.4. *Let* $\Omega \subset \mathbb{R}^d$ $(d = 2,\ 3)$ *be a convex bounded polyhedral domain. Then there exists a Green's function* $\mathfrak{F}(x,t;w,s)$ *for the problem* $(3.13)-(3.15)$*, i.e., there exists a kernel* $\mathfrak{F}$*, for* $(x,t) \in \Omega \times (0,T]$*, the solution* $\Psi(x,t)$ *for* $(3.13)-(3.15)$ *is given by*

$$\Psi(x,t) \ = \ \int_\Omega \mathfrak{F}(x,t;w,0)\Psi_0(w)\,dw + \int_0^t\int_\Omega \mathfrak{F}(x,t;w,s)\mathrm{F}(w,s)\,dw\,ds. \tag{3.16}$$

Moreover, $s < t$*,* $\mathfrak{F}$ *satisfies the bound*

$$\|\mathfrak{F}(x,t;\cdot,s)\|_{L^1(\Omega)} \ \leq \ 1. \tag{3.17}$$

Our goal in this work is to study pointwise *a posteriori* error estimates for the control problem $(3.1)-(3.3)$. Many applications only require knowledge of exact solution on some subset of the given domain $\Omega \times (0,T]$. For example, one may consider a thermal evolution problem in which one desires to monitor the temperature evolution at a single point (i.e., $x^* \times (0,T]$ for a given $x^* \in \Omega$) or to calculate the temperature distribution accurately only at the final time T. The pointwise error control is also a natural goal when computing free boundaries. Some relevant literature on maximum norm error for controlling pointwise errors for elliptic and parabolic problems are contained in [17, 19, 20, 80, 81] and [7, 21, 26], respectively. We acknowledge the work of Demlow *et al.* [21], where the authors have used known *a posteriori* error estimates for elliptic problems and the heat kernel estimate to derive $L^\infty(0,T;L^\infty(\Omega))$-norm error estimates for the purely parabolic problem. This chapter extends the work of [21] from purely parabolic problem to POCP with distributed controls. The variational discretization is used to approximate the control problem $(3.1)-(3.3)$. Both semidiscrete and fully discrete control problems are considered and analyzed. The state and co-state variables are approximated by using the piecewise linear and continuous functions, while the control variable is computed by using implicit relation between the control and co-state variables. We derive *a posteriori* error estimates for the state, co-state and control variables in the $L^\infty(0,T;\ L^\infty(\Omega))$-norm for both the semidiscrete and fully discrete variational discretization approximations. Our error analysis rely on the known elliptic reconstruction error estimates and the heat kernel estimate for linear parabolic problem.

The chapter is organized as follows. In Section 3.2, we discuss variational discretization approximation for the control problem (3.1) – (3.3) and derive *a posteriori* error estimates for the semidiscrete problem. Section 3.3 is devoted to the fully discrete approximations of the control problem (3.1) – (3.3) and related *a posteriori* error estimates are established. Numerical results are provided to illustrate the performance of the derived estimators in Section 3.4. Some concluding remarks are presented in the last section.

3.2 Analysis for semidiscrete control problem

This section is devoted to the spatially discrete optimization problem and *a posteriori* upper bounds for the state, co-state and control variables are derived in the $L^\infty(0,T;L^\infty(\Omega))$-norm.

The semidiscrete variational discretization approximations of (3.4) – (3.5) is to seek a pair $(y_h,u_h)\in\mathcal{C}(0,T;V_h^0)\times U_{ad}$ such that

$$\min_{u_h\in U_{ad}}\frac{1}{2}\int_0^T\left\{\|y_h-y_{ds}\|^2+\|u_h\|^2\right\}dt \tag{3.18}$$

subject to

$$\begin{cases}\left(\dfrac{\partial y_h}{\partial t},v_h\right)+a(y_h,v_h) \;=\; (f+u_h,v_h) \quad \forall v_h\in V_h^0,\\ y_h(\cdot,0) \;=\; y_{h,0}(x) \quad x\in\Omega,\end{cases} \tag{3.19}$$

where $y_{h,0}\in V_h^0$ is a suitable approximation or projection of y_0.

It follows from [53] that the convex optimal control problem (3.18) – (3.19) has a unique solution (y_h,u_h) if and only if there exists a co-state variable $p_h\in C(0,T;V_h^0)$ such that the triplet (y_h,p_h,u_h) satisfies the following optimality conditions for $t\in[0,T]$:

$$\left(\frac{\partial y_h}{\partial t},v_h\right)+a(y_h,v_h) = (f+u_h,v_h) \quad \forall v_h\in V_h^0, \tag{3.20}$$

$$y_h(\cdot,0) = y_{h,0}, \tag{3.21}$$

$$-\left(\frac{\partial p_h}{\partial t},v_h\right)+a(p_h,v_h) = (y_h-y_{ds},v_h) \quad \forall v_h\in V_h^0, \tag{3.22}$$

$$p_h(\cdot,T) = 0 \tag{3.23}$$

$$(u_h+p_h,w_h-u_h) \geq 0 \quad \forall w_h\in U_{ad}. \tag{3.24}$$

Similar to the continuous case, we can express (3.24) equivalently to

$$u_h(x,t) = \Pi_{[u_a,u_b]}\big(-p_h(x,t)\big). \tag{3.25}$$

Equation (3.25) reveals that the control variable u_h is the projection of a finite element function (approximate co-state variable) onto the admissible space U_{ad}.

Definition 3.2.1 (Discrete elliptic operator)**.** *The discrete elliptic operator associated with the bilinear form* $a(\cdot,\cdot)$ *and the finite element space* V_h^0 *is the operator* $-\mathcal{A}_h : H_0^1(\Omega) \to V_h^0 + \mathcal{L}_h f$ *such that for* $w \in H_0^1(\Omega)$ *and* $t \in (0,T]$,

$$(-\mathcal{A}_h w, v_h) \;=\; a(w, v_h), \qquad \forall v_h \in V_h^0,$$

where $\mathcal{L}_h$ *be the* L^2*-projection onto the finite element space* V_h.

Therefore, we have the following pointwise form of (3.20) and (3.22)

$$\begin{aligned} -\mathcal{A}_h y_h &= \mathcal{L}_h f + u_h - \frac{\partial y_h}{\partial t}, \\ -\mathcal{A}_h p_h &= y_h - \mathcal{L}_h y_{ds} + \frac{\partial p_h}{\partial t}, \end{aligned}$$

respectively.

To begin with, we first establish some intermediate error estimates for the state and co-state variables in the $L^\infty(0,T;L^\infty(\Omega))$-norm which will enable us to prove the main results of this section. This is accomplished by introducing elliptic reconstructions for the state and co-state variables. For this, we now introduce some auxiliary problems. For $\hat{u} \in U_{ad}$, let the pair $(y(\hat{u}), p(\hat{u})) \in V \times V$ be the solutions of the following equations:

$$\begin{aligned} \Big(\frac{\partial y(\hat{u})}{\partial t}, v\Big) + a(y(\hat{u}), v) &= (f + \hat{u}, v) \qquad \forall v \in H_0^1(\Omega), && (3.26)\\ y(\hat{u})(\cdot, 0) &= y_0(x) \qquad x \in \Omega, && (3.27)\\ -\Big(\frac{\partial p(\hat{u})}{\partial t}, v\Big) + a(p(\hat{u}), v) &= (y(\hat{u}) - y_{ds}, v) \qquad \forall v \in H_0^1(\Omega), && (3.28)\\ p(\hat{u})(\cdot, T) &= 0 \qquad x \in \Omega. && (3.29) \end{aligned}$$

Define the errors for the state and co-state variables as follows:

$$\hat{e}_y := y_h - y(u_h) \;\text{ and }\; \hat{e}_p := p_h - p(u_h). \tag{3.30}$$

From (3.20), (3.22), (3.26) and (3.28) with $\hat{u} = u_h$, we obtain the following error equations for $v \in H_0^1(\Omega)$:

$$\begin{aligned} \Big(\frac{\partial \hat{e}_y}{\partial t}, v\Big) + a(\hat{e}_y, v) &= -(\mathscr{G}_y, v) + (\nabla y_h, \nabla v), && (3.31)\\ -\Big(\frac{\partial \hat{e}_p}{\partial t}, v\Big) + a(\hat{e}_p, v) &= -(\mathscr{G}_p, v) + (\nabla p_h, \nabla v) + (\hat{e}_y, v), && (3.32) \end{aligned}$$

where $\mathscr{G}_y = f + u_h - \dfrac{\partial y_h}{\partial t}$ and $\mathscr{G}_p = y_h - y_{ds} + \dfrac{\partial p_h}{\partial t}$.

For $t \in (0,T]$, we now define the elliptic reconstructions for the state and co-state variables as follows: For given $y_h,\ p_h$, seek $\tilde{y},\ \tilde{p} \in H_0^1(\Omega)$ such that

$$a(\tilde{y}, v) \;=\; (\mathscr{G}_y, v) \qquad \forall v \in H_0^1(\Omega), \tag{3.33}$$

$$a(\tilde{p}, v) \;=\; (\mathscr{G}_p, v) + (\tilde{y} - y_h, v) \qquad \forall v \in H_0^1(\Omega). \tag{3.34}$$

With the help of elliptic reconstructions $\tilde{y}$ and $\tilde{p}$ as an intermediate object, we split the errors:

$$\begin{aligned}\hat{e}_y &= (\tilde{y} - y(u_h)) - (\tilde{y} - y_h) =: \hat{\xi}_y - \hat{\eta}_y,\\ \hat{e}_p &= (\tilde{p} - p(u_h)) - (\tilde{p} - p_h) =: \hat{\xi}_p - \hat{\eta}_p.\end{aligned}$$

Using (3.31) – (3.34), for all $v \in H_0^1(\Omega)$, we obtain

$$\Big(\frac{\partial \hat{\xi}_y}{\partial t}, v\Big) + a(\hat{\xi}_y, v) \;=\; \Big(\frac{\partial \hat{\eta}_y}{\partial t}, v\Big), \tag{3.35}$$

$$-\Big(\frac{\partial \hat{\xi}_p}{\partial t}, v\Big) + a(\hat{\xi}_p, v) \;=\; -\Big(\frac{\partial \hat{\eta}_p}{\partial t}, v\Big) + (\hat{\xi}_y, v). \tag{3.36}$$

As a consequence of elliptic error estimate in Lemma 3.1.1, we obtain the following bounds for the elliptic reconstruction errors.

Lemma 3.2.1 (Elliptic reconstruction errors). *Let* $(\tilde{y}, \tilde{p}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfy* (3.33) – (3.34) *and let Lemma 3.1.1 be valid. Then, for each* $t \in [0,T]$, *the following estimates hold:*

$$\|\hat{\eta}_y(t)\|_{L^\infty(\Omega)} \;\le\; C_{3,3}\,(\ln \bar{h})^2 \mathscr{R}_{\infty,0}(y_h(t), \mathscr{G}_y(t)),$$

and

$$\|\hat{\eta}_p(t)\|_{L^\infty(\Omega)} \;\le\; C_{3,4}\,(\ln \bar{h})^2 \mathscr{R}_{\infty,0}(p_h(t), \mathscr{G}_p(t)) + \|\hat{\eta}_y(t)\|_{L^\infty(\Omega)},$$

where the constants $C_{3,3}$ *and* $C_{3,4}$ *depend on the constant* $C_{3,1}$.

We next turn our attention to derive the bounds for $\hat{\xi}_y$ and $\hat{\xi}_p$.

Lemma 3.2.2 (Parabolic errors for the state and co-state variables). *Let* $\hat{\xi}_y$ *and* $\hat{\xi}_p$ *satisfy (3.35) and (3.36), respectively. Then, for any* $t \in [0,T]$, *the following estimates hold:*

$$\|\hat{\xi}_y(t)\|_{L^\infty(\Omega)} \;\le\; \|\hat{\xi}_y(0)\|_{L^\infty(\Omega)} + C_{3,5}\,(\ln \bar{h})^2 \|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t}, \frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,T]},$$

and

$$\|\hat{\xi}_p(t)\|_{L^\infty(\Omega)} \;\le\; C_{3,6}\,(\ln \bar{h})^2 \|\mathscr{R}_{\infty,0}(\frac{\partial p_h}{\partial t}, \frac{\partial \mathscr{G}_p}{\partial t})\|_{L^1[0,T]} + \|\hat{\xi}_y(t)\|_{L^\infty(\Omega)},$$

where $\mathscr{R}_{\infty,0}$ *is the* L^∞*-type residual estimator defined in (3.12). The constants* $C_{3,5}$ *and* $C_{3,6}$ *are positive and depend on the domain* Ω.

Proof. We know that $\hat{\xi}_y$ satisfies (3.35). For any $(x,t) \in \Omega \times (0,T]$, use of (3.16) leads to

$$\hat{\xi}_y(x,t) \;=\; \int_\Omega \mathfrak{F}(x,t;w,0)\,\hat{\xi}_y(w,0)dw + \int_0^t \int_\Omega \mathfrak{F}(x,t;w,s)\,\frac{\partial \hat{\eta}_y}{\partial t}(w,s)dw\,ds.$$

An application of the Hölder's inequality yields

$$\|\hat{\xi}_y(t)\|_{L^\infty(\Omega)} \;\leq\; \|\mathfrak{F}(x,t;w,0)\|_{L^1(\Omega)}\|\hat{\xi}_y(0)\|_{L^\infty(\Omega)} + \|\mathfrak{F}(x,t;w,s)\|_{L^1(\Omega)}\|\frac{\partial \hat{\eta}_y}{\partial t}\|_{L^1(0,t;L^\infty(\Omega))}.$$

With an aid of (3.17), we have

$$\|\hat{\xi}_y(t)\|_{L^\infty(\Omega)} \;\leq\; \|\hat{\xi}_y(0)\|_{L^\infty(\Omega)} + \|\frac{\partial \hat{\eta}_y}{\partial t}\|_{L^1(0,t;L^\infty(\Omega))},$$

which combine with Lemma 3.1.1 to obtain

$$\|\hat{\xi}_y(t)\|_{L^\infty(\Omega)} \;\leq\; \|\hat{\xi}_y(0)\|_{L^\infty(\Omega)} + C_{3,5}(\ln \bar{h})^2\|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t}, \frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,t]},$$

where the constant $C_{3,5}$ depends on Ω, and this proves the first inequality. The proof of the second inequality can be treated in a similar manner using the fact that $\hat{\xi}_p(T)=0$. This completes the proof of the lemma. □

Let (y,p,u) and (y_h,p_h,u_h) be the solutions of (3.6) – (3.10) and (3.20) – (3.24), respectively. In order to derive *a posteriori* error bounds for the state and the co-state variables, we decompose the errors as follows:

$$y - y_h = (y - y(u_h)) + (y(u_h) - y_h) := \hat{r}_y - \hat{e}_y,$$

and

$$p - p_h = (p - p(u_h)) + (p(u_h) - p_h) := \hat{r}_p - \hat{e}_p,$$

where $\hat{r}_y = y - y(u_h)$, $\hat{r}_p = p - p(u_h)$ and $\hat{e}_y$, $\hat{e}_p$ are defined in (3.30). With the help of (3.6), (3.8), (3.26) and (3.28), we derive the following error equations for each $t \in (0,T]$:

$$\Big(\frac{\partial \hat{r}_y}{\partial t}, v\Big) + a(\hat{r}_y, v) \;=\; (u - u_h, v) \qquad \forall v \in H_0^1(\Omega), \tag{3.37}$$

and

$$-\Big(\frac{\partial \hat{r}_p}{\partial t}, v\Big) + a(\hat{r}_p, v) \;=\; (\hat{r}_y, v) \qquad \forall v \in H_0^1(\Omega). \tag{3.38}$$

In the following lemma, we derive the bounds for $\hat{r}_y$ and $\hat{r}_p$.

Lemma 3.2.3. *Let* (y, p, u) *be the solution of* (3.6) – (3.10)*, and let* $(y(u_h), p(u_h))$ *be the solution of* (3.26) – (3.29) *with* $\hat{u} = u_h$*. Then the following estimates hold:*

$$\|\hat{r}_y\|_{L^\infty(0,T;L^\infty(\Omega))} \ \leq\ C_{3,7}\,\|u - u_h\|_{L^\infty(0,T;L^2(\Omega))},$$

and

$$\|\hat{r}_p\|_{L^\infty(0,T;L^\infty(\Omega))} \ \leq\ C_{3,8}\,\|u - u_h\|_{L^\infty(0,T;L^2(\Omega))},$$

where the constants $C_{3,7}$ *and* $C_{3,8}$ *depend on the regularity constant* C_R*.*

Proof. Note that, for any $t \in [0, T]$

$$\|\hat{r}_y(t)\|_{L^\infty(\Omega)} \ \leq\ \|\hat{r}_y(t)\|_{\mathcal{C}(\overline{\Omega})}.$$

Using the embedding result $H^2(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ and Lemma 3.1.3 we obtain

$$\|\hat{r}_y\|_{L^\infty(0,T;L^\infty(\Omega))} \ \leq\ \|\hat{r}_y\|_{L^\infty(0,T;H^2(\Omega))} \ \leq\ C_{3,7}\,\|u - u_h\|_{L^\infty(0,T;L^2(\Omega))},$$

where we have used the fact $\hat{r}_y(0) = 0$, and this proves the first inequality.

Similarly, the second inequality can easily be proved for $\hat{r}_p$ by using the fact $\hat{r}_p(T) = 0$. This completes the rest of the proof. □

The following lemma presents the *a posteriori* error estimate for the control variable in the $L^\infty(0, T; L^2(\Omega))$-norm.

Lemma 3.2.4. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (3.6)–(3.10) *and* (3.20)–(3.24)*, respectively. Assume that* $(u_h + p_h)|_K \in H^1(K)$ *and there exists a positive constant* $C_{3,9}$*, and* $w_h \in U_{ad}$ *such that*

$$\big|(u_h + p_h, w_h - u)\big| \ \leq\ C_{3,9} \sum_{K\in\mathscr{T}_h} h_K |u_h + p_h|_{H^1(K)} \|u - u_h\|_{L^2(K)}. \tag{3.39}$$

Then, we have

$$\|u - u_h\|_{L^\infty(0,T;L^2(\Omega))} \ \leq\ C_{3,10}\Bigg[\Big(\max_{t\in[0,T]} \sum_{K\in\mathscr{T}_h} h_K^2 |u_h + p_h|^2_{H^1(K)}\Big)^{1/2} + \|p_h - p(u_h)\|_{L^\infty(0,T;L^2(\Omega))}\Bigg],$$

where $C_{3,10} = \sqrt{\frac{3}{2}}\,\max\{1, C_{3,9}\}$*, and* $(y(\hat{u}), p(\hat{u}))$ *is solution of the system* (3.26)–(3.29) *with* $\hat{u} = u_h$*.*

Proof. Note that

$$\begin{aligned}\|u-u_h\|^2_{L^2(\Omega)} &= (u-u_h, u-u_h)\\ &= (u, u-u_h)-(u_h, u-u_h).\end{aligned}$$

An application of (3.10) and a simple calculation with an aid of (3.24) yields

$$\begin{aligned}\|u-u_h\|^2_{L^2(\Omega)} &\leq -(p, u-u_h)-(u_h, u-u_h)\\ &= -(u_h+p_h, u-w_h)-(u_h+p_h, w_h-u_h)\\ &\quad +(p_h-p(u_h), u-u_h)+(p(u_h)-p, u-u_h)\\ &\leq (p_h-p(u_h), u-u_h)+(p(u_h)-p, u-u_h)\\ &\quad +(u_h+p_h, w_h-u)\\ &=: E_1+E_2+E_3. \end{aligned} \tag{3.40}$$

To bound E_1, we use the Cauchy-Schwarz inequality and the Young's inequality to have

$$\begin{aligned}E_1 &\leq \|p_h-p(u_h)\|_{L^2(\Omega)}\|u-u_h\|_{L^2(\Omega)}\\ &\leq \frac{3}{4}\|p_h-p(u_h)\|^2_{L^2(\Omega)}+\frac{1}{4}\|u-u_h\|^2_{L^2(\Omega)}. \end{aligned}\tag{3.41}$$

Setting $v=p(u_h)-p$ in (3.37), and integrate the resulting equation from 0 to T. Then, an integration by parts formula with $\hat{r}_y(0)=\hat{r}_p(T)=0$ leads to

$$\int_0^T\left(\hat{r}_y, \frac{\partial\hat{r}_p}{\partial t}\right)dt-\int_0^T a(\hat{r}_y,\hat{r}_p)\,dt = \int_0^T(u-u_h, p(u_h)-p)\,dt. \tag{3.42}$$

Again, choose $v=y(u_h)-y$ in (3.38) and integrate with respect to time from 0 to T to obtain

$$\int_0^T\left(\frac{\partial\hat{r}_p}{\partial t}, \hat{r}_y\right)dt-\int_0^T a(\hat{r}_p,\hat{r}_y)\,dt = \int_0^T(y-y(u_h), y(u_h)-y)\,dt. \tag{3.43}$$

Use of (3.42) and (3.43) leads to

$$\begin{aligned}E_2 &= (u-u_h, p(u_h)-p)\\ &= (y-y(u_h), y(u_h)-y)\\ &= -\|y-y(u_h)\|^2_{L^2(\Omega)} \leq 0. \end{aligned}\tag{3.44}$$

Finally to bound of E_3, we use (3.39) and the Young's inequality to have

$$\begin{aligned}E_3 &\leq C_{3,9}\sum_{K\in\mathscr{T}_h} h_K|u_h+p_h|_{H^1(K)}\|u-u_h\|_{L^2(K)}\\ &\leq \frac{3}{4}C^2_{3,9}\sum_{K\in\mathscr{T}_h} h^2_K|u_h+p_h|^2_{H^1(K)}+\frac{1}{4}\|u-u_h\|^2_{L^2(\Omega)}. \end{aligned}\tag{3.45}$$

Altogether (3.40), (3.41), (3.44) and (3.45) yields

$$\|u-u_h\|^2_{L^2(\Omega)} \le \frac{3}{2}\max\{1, C^2_{3,9}\}\Big[\sum_{K\in\mathscr{T}_h} h_K^2|u_h+p_h|^2_{H^1(K)} + \|p_h - p(u_h)\|^2_{L^2(\Omega)}\Big].$$

Taking both side maximum over the time domain $[0,T]$, we achieve the desired estimates. This completes the proof. □

By collecting Lemmas 3.2.1 – 3.2.4, we finally derive the main results for the state and co-state variables in the $L^\infty(0,T;L^\infty(\Omega))$-norm.

Theorem 3.2.1 ($L^\infty(L^\infty)-$error estimates for the state and co-state variables)**.** *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of* (3.6) – (3.10) *and* (3.20) – (3.24)*, respectively. Let* $f \in L^\infty(0,T;L^\infty(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega))$*. Then the following a posteriori error estimates hold for each* $t\in(0,T]$*:*

$$\begin{aligned}\|u-u_h\|_{L^\infty(0,T;L^2(\Omega))} \;\le\; & C_{3,11}\Big[\Big(\max_{t\in[0,T]}\sum_{K\in\mathscr{T}_h} h_K^2|u_h+p_h|^2_{H^1(K)}\Big)^{1/2} + \|\hat{\xi}_y(0)\|_{L^\infty(\Omega)}\\ & + \|\hat{\eta}_y(t)\|_{L^\infty(\Omega)} + (\ln\bar{h})^2\Big\{\|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t},\frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,T]}\\ & + \mathscr{R}_{\infty,0}(p_h(t),\mathscr{G}_p(t)) + \|\mathscr{R}_{\infty,0}(\frac{\partial p_h}{\partial t},\frac{\partial \mathscr{G}_p}{\partial t})\|_{L^1[0,T]}\Big\}\Big],\end{aligned}$$

where $C_{3,11}$ *depends on the domain* Ω *and the constant* $C_{3,10}$ *as defined in Lemma 3.2.4,*

$$\begin{aligned}\|y-y_h\|_{L^\infty(0,T;L^\infty(\Omega))} \;\le\; & \|y_0-y_{h,0}\|_{L^\infty(\Omega)} + C_{3,12}\,(\ln\bar{h})^2\Big[\mathscr{R}_{\infty,0}(y_h(0),\mathscr{G}_y(0))\\ & + \mathscr{R}_{\infty,0}(y_h(t),\mathscr{G}_y(t)) + \|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t},\frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,T]}\Big]\\ & + C_{3,7}\,\|u-u_h\|_{L^\infty(0,T;L^2(\Omega))},\end{aligned}$$

$$\begin{aligned}\|p-p_h\|_{L^\infty(0,T;L^\infty(\Omega))} \;\le\; & \|y_0-y_{h,0}\|_{L^\infty(\Omega)} + C_{3,13}\,(\ln\bar{h})^2\Big[\mathscr{R}_{\infty,0}(p_h(t),\mathscr{G}_p(t))\\ & + \|\mathscr{R}_{\infty,0}(\frac{\partial p_h}{\partial t},\frac{\partial \mathscr{G}_p}{\partial t})\|_{L^1[0,T]} + \mathscr{R}_{\infty,0}(y_h(0),\mathscr{G}_y(0))\\ & + \mathscr{R}_{\infty,0}(y_h(t),\mathscr{G}_y(t)) + \|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t},\frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,t]}\Big]\\ & + C_{3,8}\,\|u-u_h\|_{L^\infty(0,T;L^2(\Omega))},\end{aligned}$$

where the constants $C_{3,12}$ *and* $C_{3,13}$ *depend on the domain* Ω*, and the constants* $C_{3,7}$*,* $C_{3,8}$ *are defined in Lemma 3.2.3.*

Proof. The first inequality follows from Lemmas 3.2.1, 3.2.2 and 3.2.4. To prove the second inequality, we decompose the error in the state variable as

$$y-y_h \;=\; (y-y(u_h)) + (y(u_h)-\tilde{y}) + (\tilde{y}-y_h) = \hat{r}_y - (\hat{\xi}_y - \hat{\eta}_y).$$

For any $t \in (0,T]$, we have

$$\|(y-y_h)(t)\|_{L^\infty(\Omega)} \;\leq\; \|\hat{r}_y(t)\|_{L^\infty(\Omega)} + \|\hat{\xi}_y(t)\|_{L^\infty(\Omega)} + \|\hat{\eta}_y(t)\|_{L^\infty(\Omega)}.$$

An application of Lemma 3.2.1 yields

$$\|\hat{\eta}_y(t)\|_{L^\infty(\Omega)} \;\leq\; C_{3,3}\,(\ln \bar{h})^2 \mathscr{R}_{\infty,0}(y_h(t), \mathscr{G}_y(t)).$$

By using Lemma 3.2.2, it now follows that

$$\begin{aligned}\|\hat{\xi}_y(t)\|_{L^\infty(\Omega)} \;&\leq\; \|y_0 - y_{h,0}\|_{L^\infty(\Omega)} + C_{3,3}(\ln \bar{h})^2 \mathscr{R}_{\infty,0}(y_h(0), \mathscr{G}_y(0)) \\ &\quad + C_{3,5}(\ln \bar{h})^2 \|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t}, \frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,t]}.\end{aligned}$$

Altogether these estimates and Lemma 3.2.3 leads to the desired result, where $C_{3,12} = \max\{C_{3,3}, C_{3,5}\}$.

Similarly for the co-state variable, we use the triangle inequality to write

$$\|(p-p_h)(t)\|_{L^\infty(\Omega)} \;\leq\; \|\hat{r}_p(t)\|_{L^\infty(\Omega)} + \|\hat{\xi}_p(t)\|_{L^\infty(\Omega)} + \|\hat{\eta}_p(t)\|_{L^\infty(\Omega)}.$$

Again, by using Lemmas 3.2.1 – 3.2.3 and a similar argument as above, we conclude that

$$\begin{aligned}\|(p-p_h)(t)\|_{L^\infty(\Omega)} \;&\leq\; \|y_0 - y_{h,0}\|_{L^\infty(\Omega)} + (\ln \bar{h})^2 \Big[C_{3,3}\, \mathscr{R}_{\infty,0}(y_h(0), \mathscr{G}_y(0)) \\ &\quad + C_{3,3}\, \mathscr{R}_{\infty,0}(y_h(t), \mathscr{G}_y(t)) + C_{3,4}\, \mathscr{R}_{\infty,0}(p_h(t), \mathscr{G}_p(t)) \\ &\quad + C_{3,5}\|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t}, \frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,t]} + C_{3,6}\|\mathscr{R}_{\infty,0}(\frac{\partial p_h}{\partial t}, \frac{\partial \mathscr{G}_p}{\partial t})\|_{L^1[0,t]}\Big] \\ &\quad + C_{3,8}\,\|u - u_h\|_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

Setting $C_{3,13} = \max\{C_{3,3}, C_{3,4}, C_{3,5}, C_{3,6}\}$, we complete the rest of the proof. □

Theorem 3.2.2 ($L^\infty(L^\infty)-$error estimate for the control variable)**.** *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (3.6) – (3.10) *and* (3.20) – (3.24)*, respectively. Assume that all the conditions in Theorem 3.2.1 are valid. Then, for each* $t \in (0, T]$*, there exists a positive constant* $C_{3,14}$ *such that the following error estimate*

$$\begin{aligned}\|u-u_h\|_{L^\infty(0,T;L^\infty(\Omega))} \;&\leq\; C_{3,14}\Big\{\|y_0 - y_{h,0}\|_{L^\infty(\Omega)} + (\ln \bar{h})^2 \Big[\mathscr{R}_{\infty,0}(y_{h,0}, G_y(0)) \\ &\quad + \mathscr{R}_{\infty,0}(y_h(t), G_y(t)) + \mathscr{R}_{\infty,0}(p_h(t), \mathscr{G}_p(t)) \\ &\quad + \|\mathscr{R}_{\infty,0}(\frac{\partial y_h}{\partial t}, \frac{\partial \mathscr{G}_y}{\partial t})\|_{L^1[0,T]} + \|\mathscr{R}_{\infty,0}(\frac{\partial p_h}{\partial t}, \frac{\partial \mathscr{G}_p}{\partial t})\|_{L^1[0,T]}\Big] \\ &\quad + \Big(\max_{t\in[0,T]} \sum_{K\in\mathscr{T}_h} h_K^2 |u_h + p_h|^2_{H^1(K)}\Big)^{1/2}\Big\},\end{aligned}$$

holds, where the constant $C_{3,14}$ *depends on the domain* Ω*, the regularity constant* C_R *and the constant* $C_{3,11}$ *as defined in Theorem 3.2.1.*

Proof. From (3.11) and (3.25) we obtain

$$\begin{aligned}\|u-u_h\|_{L^\infty(0,T;L^\infty(\Omega))} &= \|\Pi_{[u_a,u_b]}(-p)-\Pi_{[u_a,u_b]}(-p_h)\|_{L^\infty(0,T;L^\infty(\Omega))}\\ &\le \|p_h-p\|_{L^\infty(0,T;L^\infty(\Omega))},\end{aligned}$$

where we have used the Lipschitz continuity of $\Pi_{[u_a,u_b]}$ with Lipschitz constant 1. An application of Lemma 3.2.1 and Theorem 3.2.1 completes the rest of the proof. □

3.3 Analysis for fully discrete control problem

This section concerns the fully-discrete variational discretization approximations of POCP (3.18) – (3.19). Let $0=t_0<t_1<\ldots<t_N=T$, be a partition of $[0,T]$ with $I_n=(t_{n-1},t_n]$ and $k_n:=t_n-t_{n-1}$. Let $\mathscr{T}_h^n:=\{K\}\,(0\le n\le N)$ be the triangulation of $\bar{\Omega}$ at the time level t_n. We assume that $\mathscr{T}_h^n$ satisfies the conditions **D1** and **D2** of Section 2.2 of Chapter 2. Now we introduce the following notation for the fully discrete error analysis. For $0\le n\le N$, let $\mathcal{E}_n:=\{E\}$ be the set of all edges of the triangles $K\in\mathscr{T}_h^n$ which do not lie on $\partial\Omega$, and $\Sigma_n:=\cup_{E\in\mathcal{E}_n}E$. Furthermore, we will also use the sets $\hat{\Sigma}_n:=\Sigma_n\cap\Sigma_{n-1}$ and $\check{\Sigma}_n:=\Sigma_n\cup\Sigma_{n-1}$. For each $n=0,\ldots,N$, we consider the finite element spaces V^n corresponding to the triangulation $\mathscr{T}_h^n$ as follows:

$$V^n := \{\chi\in\mathcal{C}(\overline{\Omega}):\ \chi|_K\in\mathbb{P}_1(K),\quad \forall K\in\mathscr{T}_h^n\},$$

where $\mathbb{P}_1(K)$ is the space of polynomials of degree less than or equal to 1 on K. Set $V_0^n=V^n\cap H_0^1(\Omega)$. For the purpose of fully discrete approximation, we need the following notation

$$\phi^n := \phi(\cdot,t_n),\ \ \bar{\partial}\phi^n := \frac{1}{k_n}(\phi^n-\phi^{n-1})\ \ \text{and}\ \ \mathcal{L}_h^n(\bar{\partial}\phi^n):=\frac{1}{k_n}(\phi^n-\mathcal{L}_h^n\phi^{n-1}),$$

where $\mathcal{L}_h^n$ is the L^2-projection from $L^2(\Omega)$ to V^n.

Representation of the bilinear form: For a function $v_h\in V_0^n$ $(0\le n\le N)$, the bilinear form $a(v_h,w)$ can be represented as

$$a(v_h,w)=\sum_{K\in\mathscr{T}_h^n}<-div(\nabla v_h),w>_K+\sum_{E\in\mathcal{E}_n}<J_1[v_h],w>_E,\qquad \forall w\in H_0^1(\Omega),$$

where $J_1[v_h]$ denotes the spatial jump of the field ∇v_h across an element side $E\in\mathcal{E}_n$ and is defined as

$$J_1[v_h]|_E(x):=\lim_{\epsilon\to 0}\big(\nabla v_h(x+\epsilon\mathbf{n}_E)-\nabla v_h(x-\epsilon\mathbf{n}_E)\big)\cdot\mathbf{n}_E,$$

where $\mathbf{n}_E$ is a unit normal vector to E at the point x.

Let $\mathcal{L}_h^n$ and $\mathcal{L}_{h,0}^n$ be the L^2-projections onto V^n and V_0^n such that

$$(\mathcal{L}_h^n \phi, \psi_n) = (\phi, \psi_n) \quad \forall\, \psi_n \in V^n, \quad \text{and} \quad (\mathcal{L}_{h,0}^n \phi, \psi_n) = (\phi, \psi_n) \quad \forall\, \psi_n \in V_0^n.$$

Discrete elliptic operator: The discrete elliptic operator associated with the bilinear form $a(\cdot,\cdot)$ and the finite element space V_0^n is the operator $\mathcal{A}_h^n : H_0^1(\Omega) \to V_0^n + \mathcal{L}_h^n f^n$ such that for $v \in H_0^1(\Omega)$ and $0 \leq n \leq N$,

$$(-\mathcal{A}_h^n v, w_h) = a(v, w_h), \quad \forall w_h \in V_0^n.$$

The fully discrete variational discretization approximations of the problem (3.18) – (3.19) is defined as follows: Find $(y_h^n, u_h^n) \in V_0^n \times U_{ad}$, for $n \in [1:N]$, such that

$$\min_{u_h^n \in U_{ad}} \frac{1}{2} \sum_{n=1}^{N} \int_{I_n} \left\{ \|y_h^n - y_{ds}^n\|_{L^2(\Omega)}^2 + \|u_h^n\|_{L^2(\Omega)}^2 \right\} dt \tag{3.46}$$

subject to

$$\begin{cases} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = (f^n + u_h^n, v_h) \quad \forall v_h \in V_0^n, \\ y_h^0 = y_{h,0}, \end{cases} \tag{3.47}$$

where $y_{h,0}$ is the suitable approximation or projection of y_0 in V_0^0.

The optimal control problem (3.46) – (3.47) admits a unique solution (y_h^n, u_h^n) if and only if there exists a co-state $p_h^{n-1} \in V_0^n$ such that the following optimality conditions are satisfied: For each $n \in [1:N]$,

$$\begin{aligned} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) &= (f^n + u_h^n, v_h) \quad \forall v_h \in V_0^n, && (3.48)\\ y_h^0 &= y_{h,0}, && (3.49)\\ -(\bar{\partial} p_h^n, v_h) + a(p_h^{n-1}, v_h) &= (y_h^n - y_{ds}^n, v_h) \quad \forall v_h \in V_0^n, && (3.50)\\ p_h^N &= 0, && (3.51)\\ (u_h^n + p_h^{n-1}, w_h^n - u_h^n) &\geq 0 \quad \forall w_h^n \in U_{ad}. && (3.52) \end{aligned}$$

Given a sequence of discrete values $\{y_h^n\}$, $n = 0, 1, \ldots, N$, we associate a continuous function of time defined by the continuous piecewise linear interpolant $Y_h(t)$, $t \in I_n$ as

$$Y_h(t) := \frac{(t_n - t)}{k_n} y_h^{n-1} + \frac{(t - t_{n-1})}{k_n} y_h^n.$$

Similarly, we define $P_h(t)$, $t \in I_n$, from the set of values $\{p_h^n\}$, $n = 0, 1, \ldots, N$ as

$$P_h(t) := \frac{(t_n - t)}{k_n} p_h^{n-1} + \frac{(t - t_{n-1})}{k_n} p_h^n,$$

and

$$U_h(t)|_{t\in I_n} := u_h^n.$$

Finally, we define $Y_{h,t}^n = \frac{\partial Y_h}{\partial t}|_{I_n}$ and $P_{h,t}^n = \frac{\partial P_h}{\partial t}|_{I_n}$. Further, we note that the values of $Y_h(t)$ and $P_h(t)$ at the nodal point $t = t_n$, $n = 1, 2, \ldots, N$ are coincided with y_h^n and p_h^n, respectively.

The weak-form of fully discrete schemes (3.48) and (3.50) can be easily transformed into the pointwise form as

$$\begin{aligned} \frac{y_h^n - \mathcal{L}_{h,0}^n y_h^{n-1}}{k_n} - \mathcal{A}_h^n y_h^n &= \mathcal{L}_h^n f^n + u_h^n, \\ -\frac{p_h^n - \mathcal{L}_{h,0}^n p_h^{n-1}}{k_n} - \mathcal{A}_h^n p_h^{n-1} &= y_h^n - \mathcal{L}_h^n y_{ds}^n. \end{aligned}$$

This implies

$$Y_{h,t}^n - \mathcal{A}_h^n y_h^n = \mathcal{L}_h^n f^n + u_h^n + \frac{\mathcal{L}_{h,0}^n y_h^{n-1} - y_h^{n-1}}{k_n}, \qquad n \geq 1, \tag{3.53}$$

$$-P_{h,t}^n - \mathcal{A}_h^n p_h^{n-1} = y_h^n - \mathcal{L}_h^n y_{ds}^n - \frac{\mathcal{L}_{h,0}^n p_h^{n-1} - p_h^{n-1}}{k_n}, \qquad n \geq 1. \tag{3.54}$$

Then the optimality conditions (3.48) – (3.52) can be stated as follows:

$$(\bar{\partial} Y_h^n, v_h) + a(Y_h^n, v_h) = (f^n + U_h, v_h) \quad \forall v_h \in V_0^n, \tag{3.55}$$

$$Y_h^0 = y_{h,0}, \tag{3.56}$$

$$-(\bar{\partial} P_h^n, v_h) + a(P_h^{n-1}, v_h) = (Y_h^n - y_{ds}^n, v_h) \quad \forall v_h \in V_0^n, \tag{3.57}$$

$$P_h^N = 0, \tag{3.58}$$

$$(U_h + P_h^{n-1}, w_h^n - U_h) \geq 0 \quad \forall w_h^n \in U_{ad}. \tag{3.59}$$

Analogous to the continuous case, we reformulate the discrete optimal control problem (3.46) – (3.47) as

$$\min_{U_h \in U_{ad}} j_h^n(U_h) := J(U_h, Y_h(U_h)).$$

As in the case of semidiscrete error analysis, we first derive some intermediate error estimates for the state and co-state variables in the $L^\infty(0,T; L^\infty(\Omega))$-norm. Here, the fully discrete analogue of elliptic reconstructions for the state and co-state variables are treated as intermediate objects in the error analysis.

For the purpose of error analysis, we shall define the errors for the state and co-state variables as follows:

$$e_y := Y_h - y(U_h) \quad \text{and} \quad e_p := P_h - p(U_h).$$

From (3.26), (3.28), (3.55) and (3.57) with $\hat{u} = U_h$, we have the following error equations for $v \in H_0^1(\Omega)$:

$$\begin{aligned} \left(\frac{\partial e_y}{\partial t}, v\right) + a(e_y, v) &= -\omega_y^n(v) + a(Y_h - y_h^n, v) + (f^n - f, v), \qquad (3.60)\\ -\left(\frac{\partial e_p}{\partial t}, v\right) + a(e_p, v) &= \omega_p^n(v) + a(P_h - p_h^{n-1}, v) + (y_h^n - y(U_h), v)\\ &\quad +(y_{ds} - y_{ds}^n, v), \qquad (3.61) \end{aligned}$$

where

$$\begin{aligned} \omega_y^n(v) &:= (f^n - \mathcal{L}_h^n f^n, v) + \left(\frac{y_h^{n-1} - \mathcal{L}_{h,0}^n y_h^{n-1}}{k_n}, v\right),\\ \omega_p^n(v) &:= (y_{ds}^n - \mathcal{L}_h^n y_{ds}^n, v) + \left(\frac{p_h^{n-1} - \mathcal{L}_{h,0}^n p_h^{n-1}}{k_n}, v\right). \end{aligned}$$

We now define the elliptic reconstructions at $t = t_n$, $n \in [1:N]$ as follows: For given y_h^n, p_h^{n-1}, seek $\tilde{y}_h^n$, $\tilde{p}_h^{n-1} \in H_0^1(\Omega)$ satisfying

$$a(\tilde{y}_h^n, v) = (\mathscr{G}_y^n, v) \quad \forall v \in H_0^1(\Omega), \qquad (3.62)$$

and

$$a(\tilde{p}_h^{n-1}, v) = (\mathscr{G}_p^n, v) + (\tilde{y}_h^n - y_h^n, v) \quad \forall v \in H_0^1(\Omega), \qquad (3.63)$$

where

$$\mathscr{G}_y^n = \begin{cases} -\mathcal{A}_h^0 y_h^0 + f^0 - \mathcal{L}_h^0 f^0 & n = 0,\\ f^n + u_h^n - Y_{h,t}^n & n \geq 1, \end{cases}$$

and

$$\mathscr{G}_p^n = \begin{cases} -\mathcal{A}_h^0 p_h^0 + y_{ds}^0 - \mathcal{L}_h^0 y_{ds}^0 & n = 0,\\ y_h^n - \mathcal{L}_h^n y_{ds}^n + P_{h,t}^n & n \geq 1. \end{cases}$$

Using a sequence of discrete values $\{\tilde{y}_h^n\}$ for $n = 0, 1, \ldots, N$, we set a continuous function of time defined by piecewise linear interpolant $\tilde{y}(t)$ as

$$\tilde{y}(t) := \frac{(t_n - t)}{k_n}\tilde{y}_h^{n-1} + \frac{(t - t_{n-1})}{k_n}\tilde{y}_h^n \qquad t_{n-1} \leq t \leq t_n, \quad n = 1, \ldots, N.$$

Similarly, we define $\tilde{p}(t)$ from the set of values $\{\tilde{p}_h^n\}$, $n = 1, \ldots, N$ as

$$\tilde{p}(t) := \frac{(t_n - t)}{k_n}\tilde{p}_h^{n-1} + \frac{(t - t_{n-1})}{k_n}\tilde{p}_h^n \qquad t_{n-1} \leq t \leq t_n, \quad n = 1, \ldots, N.$$

We note that functions $\tilde{y}$ and $\tilde{p}$ satisfy, for each $t \in [0,T]$, the following equations:

$$\begin{aligned}
a(\tilde{y} - Y_h, v) &= \omega_y(v) \qquad \forall v \in H_0^1(\Omega),\\
a(\tilde{p} - P_h, v) &= -\,\omega_p(v) + (\tilde{y} - Y_h, v) \qquad \forall v \in H_0^1(\Omega).
\end{aligned}$$

From (3.53) and (3.54), we obtain

$$\begin{aligned}
\mathscr{G}_y^n &= f^n + u_h^n - Y_{h,t}^n = -\mathcal{A}_h^n y_h^n + f^n - \mathcal{L}_h^n f^n - \frac{\mathcal{L}_{h,0}^n y_h^{n-1} - y_h^{n-1}}{k_n}, \quad n \geq 1,\\
\mathscr{G}_p^n &= y_h^n - y_{ds}^n + P_{h,t}^n = -\mathcal{A}_h^n p_h^{n-1} + \mathcal{L}_h^n y_{ds}^n - y_{ds}^n + \frac{\mathcal{L}_{h,0}^n p_h^{n-1} - p_h^{n-1}}{k_n}, \quad n \geq 1.
\end{aligned}$$

Using elliptic reconstruction, we decompose the errors as

$$e_y = (\tilde{y} - y(U_h)) - (\tilde{y} - Y_h) \;=:\; \xi_y - \eta_y, \;\text{ and }\; e_p \;=\; (\tilde{p} - p(U_h)) - (\tilde{p} - P_h) \;=:\; \xi_p - \eta_p.$$

Note that

$$\tilde{y} - \tilde{y}_h^n \;:=\; -\,(1 - l(t))(\tilde{y}_h^n - \tilde{y}_h^{n-1}) \quad \text{and} \quad \tilde{p} - \tilde{p}_h^{n-1} \;:=\; l(t)(\tilde{p}_h^n - \tilde{p}_h^{n-1}),$$

where $l(t) = \frac{t - t_{n-1}}{k_n}$. Using (3.62) – (3.63) in (3.60) – (3.61), for all $v \in H_0^1(\Omega)$, we obtain

$$\begin{aligned}
\left(\frac{\partial \xi_y}{\partial t}, v\right) + a(\xi_y, v) &= \left(\frac{\partial \eta_y}{\partial t}, v\right) + (f^n - f, v) + (1 - l(t))(\mathscr{G}_y^{n-1} - \mathscr{G}_y^n, v), \quad (3.64)\\
-\left(\frac{\partial \xi_p}{\partial t}, v\right) + a(\xi_p, v) &= -\left(\frac{\partial \eta_p}{\partial t}, v\right) + (y_{ds} - y_{ds}^n, v) + (\tilde{y}_h^n - y(U_h), v)\\
&\quad + l(t)(\mathscr{G}_p^n - \mathscr{G}_p^{n+1}, v). \qquad (3.65)
\end{aligned}$$

The fully discrete analogue of Lemma 3.2.1 is stated in the following lemma.

Lemma 3.3.1 (Elliptic reconstruction errors)**.** *Let* $(\tilde{y}_h^n, \tilde{p}_h^{n-1}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *satisfy* (3.62) – (3.63). *Then,* $0 \leq n \leq N$, *we have*

$$\|\tilde{y}_h^n - y_h^n\|_{L^\infty(\Omega)} \;\leq\; C_{3,15}\,(\ln \hat{h}_n)^2 \mathscr{R}_{\infty,0}(y_h^n, \mathscr{G}_y^n).$$

Moreover, for $n \in [1:N]$, *we have*

$$\|\tilde{p}_h^{n-1} - p_h^{n-1}\|_{L^\infty(\Omega)} \;\leq\; C_{3,16}\,(\ln \hat{h}_n)^2 \mathscr{R}_{\infty,0}(p_h^{n-1}, \mathscr{G}_p^n) + \|\tilde{y}_h^n - y_h^n\|_{L^\infty(\Omega)},$$

where the constants $C_{3,15}$ *and* $C_{3,16}$ *depend on the domain* Ω.

In the following lemma, we derive the bounds for ξ_y and ξ_p.

Lemma 3.3.2 (Parabolic error estimates for the state and co-state variables)**.** *Let ξ_y and ξ_p satisfy (3.64) and (3.65), respectively. Then, for any $1 \le m \le N$ with $\hat{h}_m = \min\limits_{1\le n\le m} \min\limits_{K\in\mathscr{T}_h^n} h_K$, the following estimates hold:*

$$\begin{aligned}\|\xi_y(t_m)\|_{L^\infty(\Omega)} \;&\le\; \|\xi_y(0)\|_{L^\infty(\Omega)} + \sum_{n=1}^{m}\int_{I_n} \|f^n - f\|_{L^\infty(\Omega)}\, ds + \frac{k_n}{2}\|\mathscr{G}_y^{n-1} - \mathscr{G}_y^n\|_{L^\infty(\Omega)} \\ &\quad + C_{3,17}\,(\ln \hat{h}_m)^2 \sum_{n=1}^{m} k_n \hat{\mathscr{R}}_{\infty,0}\Big(\frac{y_h^n - y_h^{n-1}}{k_n}, \mathscr{G}_y^n - \mathscr{G}_y^{n-1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\Big),\end{aligned} \tag{3.66}$$

$$\begin{aligned}\|\xi_p(t_m)\|_{L^\infty(\Omega)} \;&\le\; \|\xi_y(t_m)\|_{L^\infty(\Omega)} + \sum_{n=m+1}^{N}\int_{I_n} \|y_{ds} - y_{ds}^n\|_{L^\infty(\Omega)} ds + \frac{k_n}{2}\|\mathscr{G}_p^{n} - \mathscr{G}_p^{n+1}\|_{L^\infty(\Omega)} \\ &\quad + C_{3,18}\,(\ln \hat{h}_m)^2 \sum_{n=m+1}^{N} k_n \hat{\mathscr{R}}_{\infty,0}\Big(\frac{p_h^{n-1} - p_h^{n}}{k_n}, \mathscr{G}_p^n - \mathscr{G}_p^{n+1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\Big).\end{aligned} \tag{3.67}$$

In the above, the constants $C_{3,17}$ and $C_{3,18}$ are positive and depend on the constant $C_{3,2}$.

Proof. Note that ξ_y satisfies (3.64). For any $t_m \in [0,T]$ and a fixed $x_m \in \Omega$, an application of (3.16) leads to

$$\begin{aligned}|\xi_y(x_m,t_m)| \;&\le\; \int_\Omega |\mathfrak{F}(x_m,t_m;w,0)\,\xi_y(w,0)|dw \\ &\quad + \int_0^{t_m}\int_\Omega |\mathfrak{F}(x_m,t_m;w,s)\,\frac{\partial \eta_y}{\partial t}(w,s)|\,dw\,ds \\ &\quad + \sum_{n=1}^{m}\int_{I_n}\int_\Omega |\mathfrak{F}(x_m,t_m;w,s)\,(f^n - f)|\,dw\,ds \\ &\quad + \sum_{n=1}^{m}\int_{I_n}\int_\Omega |\mathfrak{F}(x_m,t_m;w,s)\,(1 - l(s))(\mathscr{G}_y^{n-1} - \mathscr{G}_y^n)|\,dw\,ds,\end{aligned}$$

using the Hölder's inequality and (3.17) with $|\xi_y(x_m,t_m)| = \|\xi_y(t_m)\|_{L^\infty(\Omega)}$ (since x_m is fixed), we obtain

$$\begin{aligned}\|\xi_y(t_m)\|_{L^\infty(\Omega)} \;&\le\; \|\xi_y(0)\|_{L^\infty(\Omega)} + \|\frac{\partial \eta_y}{\partial t}\|_{L^1([0,t_m];L^\infty(\Omega))} + \sum_{n=1}^{m}\int_{I_n}\|f^n - f\|_{L^\infty(\Omega))}\,ds \\ &\quad + \frac{k_n}{2}\|\mathscr{G}_y^{n-1} - \mathscr{G}_y^n\|_{L^\infty(\Omega)}.\end{aligned}$$

Use of Lemma 3.1.2 leads to

$$\begin{aligned}\|\xi_y(t_m)\|_{L^\infty(\Omega)} \;\leq\;& \|\xi_y(0)\|_{L^\infty(\Omega)} + \sum_{n=1}^{m}\int_{I_n}\|f^n - f\|_{L^\infty(\Omega)}\,ds + \frac{k_n}{2}\|\mathscr{G}_y^{n-1} - \mathscr{G}_y^{n}\|_{L^\infty(\Omega)}\\ &+C_{3,17}\,(\ln \hat{h}_m)^2\sum_{n=1}^{m} k_n\hat{\mathscr{R}}_{\infty,0}\big(\frac{y_h^n - y_h^{n-1}}{k_n}, \mathscr{G}_y^n - \mathscr{G}_y^{n-1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\big),\end{aligned}$$

and this completes the proof of (3.66).

To prove (3.67), we first note that ξ_p satisfies (3.65). For any $t_m \in [0, T]$ and fixed $x_m \in \Omega$, a similar argument as before leads to

$$\begin{aligned}|\xi_p(x_m, t_m)| \;\leq\;& \int_\Omega |\mathfrak{F}(x_m, t_m; w, T)\,\xi_p(w, T)|dw\\ &+\int_{t_m}^{T}\int_\Omega |\mathfrak{F}(x_m, t_m; w, s)\,\frac{\partial \eta_p}{\partial t}(w, s)|dw\,ds\\ &+\sum_{n=m+1}^{N}\int_{I_n}\int_\Omega |\mathfrak{F}(x_m, t_m; w, s)\,(y_{ds} - y_{ds}^n)dw|\,ds\\ &+\sum_{n=m+1}^{N}\int_{I_n}\int_\Omega |\mathfrak{F}(x_m, t_m; w, s)\,(\tilde{y}_h^n - y(U_h))|dw\,ds\\ &+\sum_{n=m+1}^{N}\int_{I_n}\int_\Omega |\mathfrak{F}(x_m, t_m; w, s)\,l(s)(\mathscr{G}_p^n - \mathscr{G}_p^{n+1})|\,dw\,ds.\end{aligned}$$

An application of the Hölder's inequality and (3.17) with $|\xi_p(x_m, t_m)| = \|\xi_p(t_m)\|_{L^\infty(\Omega)}$ yields

$$\begin{aligned}\|\xi_p(t_m)\|_{L^\infty(\Omega)} \;\leq\;& \|\xi_p(T)\|_{L^\infty(\Omega)} + \|\frac{\partial \eta_p}{\partial t}\|_{L^1([t_m,T];L^\infty(\Omega))} + \sum_{n=m+1}^{N}\int_{I_n}\|y_{ds} - y_{ds}^n\|_{L^\infty(\Omega)}ds\\ &+\sum_{n=m+1}^{N}\int_{I_n}\|\tilde{y}_h^n - y(U_h)\|_{L^\infty(\Omega)}ds + \frac{k_n}{2}\|\mathscr{G}_p^n - \mathscr{G}_p^{n+1}\|_{L^\infty(\Omega)}.\end{aligned}$$

Utilization of Lemma 3.1.2 and $\xi_p(T) = 0$ imply

$$\begin{aligned}\|\xi_p(t_m)\|_{L^\infty(\Omega)} \leq\;& C_{3,18}\,(\ln \hat{h}_m)^2\sum_{n=1}^{m} k_n\hat{\mathscr{R}}_{\infty,0}\big(\frac{p_h^{n-1} - p_h^n}{k_n}, \mathscr{G}_p^n - \mathscr{G}_p^{n+1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\big)\\ &+\sum_{n=m+1}^{N}\int_{I_n}\|y_{ds} - y_{ds}^n\|_{L^\infty(\Omega)}ds + \frac{k_n}{2}\|\mathscr{G}_p^n - \mathscr{G}_p^{n+1}\|_{L^\infty(\Omega)} + \|\xi_y(t_m)\|_{L^\infty(\Omega)},\end{aligned}$$

which completes the rest of the proof. □

Let (y,p,u) and (Y_h,P_h,U_h) be the solutions of (3.6) – (3.10) and (3.55) – (3.59), respectively. In order to derive *a posteriori* error bounds for the state and co-state variables, we decompose the errors as follows:

$$\begin{aligned} y-Y_h &= (y-y(U_h))+(y(U_h)-Y_h) =: r_y-e_y, \\ p-P_h &= (p-p(U_h))+(p(U_h)-P_h) =: r_p-e_p. \end{aligned}$$

From (3.6), (3.8), (3.26) and (3.28) with $\hat{u}=U_h$, we derive the following error equations:

$$\Big(\frac{\partial r_y}{\partial t},v\Big)+a(r_y,v) = (u-U_h,v) \quad \forall v\in H_0^1(\Omega), \tag{3.68}$$

$$-\Big(\frac{\partial r_p}{\partial t},v\Big)+a(r_p,v) = (r_y,v) \quad \forall v\in H_0^1(\Omega). \tag{3.69}$$

The following lemma provides the bounds for r_y and r_p.

Lemma 3.3.3. *Let* (y,p,u) *be the solution of the problem* (3.6)−(3.10)*, and let* $(y(\hat{u}),p(\hat{u}))$ *be the solution of the problem* (3.26) – (3.29) *with* $\hat{u}=U_h$*. Then, for any* $1\le m\le N$*, we have*

$$\|r_y(t_m)\|_{L^\infty(\Omega)} \le C_{3,19}\,\|u-U_h\|_{L^\infty(0,T;L^2(\Omega))}, \tag{3.70}$$

and

$$\|r_p(t_m)\|_{L^\infty(\Omega)} \le C_{3,20}\,\|u-U_h\|_{L^\infty(0,T;L^2(\Omega))}, \tag{3.71}$$

where the constants $C_{3,19}$ *and* $C_{3,20}$ *depend on the regularity constant* C_R*.*

Proof. Following the lines of argument of Lemma 3.2.3, the proof of inequalities (3.70) and (3.71) can easily be obtained. The details are thus omitted. □

In the following lemma, we derive the *a posteriori* error estimate for the control variable in the $L^2(0,T;L^2(\Omega))$-norm.

Lemma 3.3.4. *Let* (y,p,u) *and* (Y_h,P_h,U_h) *be the solutions of* (3.6)−(3.10) *and* (3.55)−(3.59)*, respectively. Assume that* $(U_h+P_h^{n-1})|_K\in H^1(K)$ *and* $w_h\in U_{ad}$*, and there exists a positive constant* $C_{3,21}$ *such that*

$$\big|(U_h+P_h^{n-1},w_h-u)\big| \le C_{3,21}\sum_{K\in\mathscr{T}_h^n} h_K|U_h+P_h^{n-1}|_{H^1(K)}\|u-U_h\|_{L^2(K)}.$$

Then, we have

$$\begin{aligned}\|u-U_h\|_{L^\infty(0,T;L^2(\Omega))} \le C_{3,22}\Bigg[&\Big(\max_{n\in[1,N]}\sum_{K\in\mathscr{T}_h^n} h_K^2|U_h+P_h^{n-1}|^2_{H^1(K)}\Big)^{1/2} \\ &+\|P_h^{n-1}-p(U_h)\|_{L^\infty(0,T;L^\infty(\Omega))}\Bigg],\end{aligned}$$

where $C_{3,22} = \sqrt{\frac{3}{2}}\max\{1, C_{3,21}\}$*, and* $(y(\hat{u}), p(\hat{u}))$ *is defined by the system* $(3.26)-(3.29)$ *with* $\hat{u} = U_h$*.*

Proof. From (3.10) with $w = U_h$, we have

$$(u, u - U_h) \;\le\; -(p, u - U_h).$$

Using the above inequality, it follows that

$$\begin{aligned}
\|u - U_h\|^2_{L^2(\Omega)} &= (u - U_h, u - U_h)\\
&\le -(p, u - U_h) - (U_h, u - U_h)\\
&= -(P_h^{n-1} + U_h, u - w_h) - (U_h + P_h^{n-1}, w_h - U_h)\\
&\quad +(P_h^{n-1} - p(U_h), u - U_h) + (p(U_h) - p, u - U_h).
\end{aligned}$$

Inviting (3.59) we obtain

$$\begin{aligned}
\|u - U_h\|^2_{L^2(\Omega)} &\le (U_h + P_h^{n-1}, w_h - u) + (P_h^{n-1} - p(U_h), u - U_h)\\
&\quad +(p(U_h) - p, u - U_h)\\
&=: \hat{E}_1 + \hat{E}_2 + \hat{E}_3.
\end{aligned}$$

Following the idea of Lemma 3.2.4, it is easy to bound the term $\hat{E}_i$, $i = 1, 2, 3$. So, we omit the details. This completes the proof. $\square$

By collecting Lemmas $3.3.1 - 3.3.4$, we finally derive the main results of this section.

Theorem 3.3.1 ($L^\infty(L^\infty)-$error estimates for the state and co-state variables)**.** *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* $(3.6)-(3.10)$ *and* $(3.55)-(3.59)$*, respectively. Then there exists positive constants* $C_{3,24}$*,* $C_{3,25}$ *(depend on* Ω*), for each* $t \in (0, T]$*, and any* $1 \le m \le N$ *with* $\hat{h}_m = \min_{1\le n\le m}\min_{K\in\mathscr{T}_h^n} h_K$*, the following estimates*

$$\begin{aligned}
\|u - U_h\|_{L^\infty(0,T;L^2(\Omega))} \;\le\; C_{3,23}\Bigg[\Bigg(\max_{n\in[1,N]}\sum_{K\in\mathscr{T}_h^n} h_K^2|U_h + P_h^{n-1}|^2_{H^1(K)}\Bigg)^{1/2}\\
+\|\xi_p\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\eta_p\|_{L^\infty(0,T;L^\infty(\Omega))}\Bigg],
\end{aligned} \tag{3.72}$$

where the constant $C_{3,23}$ *depends on the domain* Ω *and the constant* $C_{3,22}$ *as defined in*

Lemma 3.3.4,

$$\begin{aligned}\|y - Y_h\|_{L^\infty(0,T;L^\infty(\Omega))} \;\le\;& \|y_0 - y_{h,0}\|_{L^\infty(\Omega)} + C_{3,24}\,(\ln \hat{h}_m)^2 \\ & \times\Big[\mathscr{R}_{\infty,0}(y_{h,0},\mathscr{G}_y^0) + \mathscr{R}_{\infty,0}(y_h^m,\mathscr{G}_y^m) \\ & + \sum_{n=1}^{m} k_n \hat{\mathscr{R}}_{\infty,0}\big(\frac{y_h^n - y_h^{n-1}}{k_n}, \mathscr{G}_y^n - \mathscr{G}_y^{n-1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\big)\Big] \\ & + \sum_{n=1}^{m} \int_{I_n} \|f^n - f\|_{L^\infty(\Omega)}\, ds + \frac{k_n}{2}\|\mathscr{G}_y^{n-1} - \mathscr{G}_y^n\|_{L^\infty(\Omega)} \\ & + C_{3,19}\,\|u - U_h\|_{L^\infty(0,T;L^2(\Omega))}, \end{aligned} \tag{3.73}$$

and

$$\begin{aligned}\|p - P_h\|_{L^\infty(0,T;L^\infty(\Omega))} \;\le\;& \|y_0 - y_{h,0}\|_{L^\infty(\Omega)} + C_{3,25}\,(\ln \hat{h}_m)^2 \Big[\mathscr{R}_{\infty,0}(y_h^m,\mathscr{G}_y^m) \\ & + \mathscr{R}_{\infty,0}(y_{h,0},\mathscr{G}_y^0) + \mathscr{R}_{\infty,0}(p_h^m,\mathscr{G}_p^m) \\ & + \sum_{n=1}^{m} k_n \hat{\mathscr{R}}_{\infty,0}\big(\frac{y_h^n - y_h^{n-1}}{k_n}, \mathscr{G}_y^n - \mathscr{G}_y^{n-1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\big) \\ & + \sum_{n=m+1}^{N} k_n \hat{\mathscr{R}}_{\infty,0}\big(\frac{p_h^{n-1} - p_h^{n}}{k_n}, \mathscr{G}_p^n - \mathscr{G}_p^{n+1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\big)\Big] \\ & + \sum_{n=m+1}^{N} \int_{I_n} \|y_{ds} - y_{ds}^n\|_{L^\infty(\Omega)} ds + \frac{k_n}{2}\|\mathscr{G}_p^{n} - \mathscr{G}_p^{n+1}\|_{L^\infty(\Omega)} \\ & + \sum_{n=1}^{m} \int_{I_n} \|f^n - f\|_{L^\infty(\Omega)}\, ds + \frac{k_n}{2}\|\mathscr{G}_y^{n-1} - \mathscr{G}_y^n\|_{L^\infty(\Omega)} \\ & + C_{3,20}\,\|u - U_h\|_{L^\infty(0,T;L^2(\Omega))} \end{aligned} \tag{3.74}$$

hold, where the constants $C_{3,19}$ *and* $C_{3,20}$ *are defined in Lemma 3.3.3.*

Proof. The first inequality (3.72) follows from Lemma 3.3.4. Next, to prove error estimate for the state variable, we write

$$y - Y_h \;=\; (y - y(U_h)) + (y(U_h) - \tilde{y}) + (\tilde{y} - Y_h) = r_y - (\xi_y - \eta_y).$$

For a fix $x_m \in \Omega$ and $t_m \in (0, T]$, we have

$$\|(y - Y_h)(t_m)\|_{L^\infty(\Omega)} \;\le\; \|r_y(t_m)\|_{L^\infty(\Omega)} + \|\xi_y(t_m)\|_{L^\infty(\Omega)} + \|\eta_y(t_m)\|_{L^\infty(\Omega)}.$$

Using Lemma 3.3.1, the last term of the right hand side is bounded as

$$\|\eta_y(t_m)\|_{L^\infty(\Omega)} \;\le\; C_{3,15}\,(\ln \hat{h}_m)^2 \mathscr{R}_{\infty,0}(y_h^m, \mathscr{G}_y^m).$$

By Lemma 3.3.2, we have

$$\begin{aligned}\|\xi_y(t_m)\|_{L^\infty(\Omega)} &\leq \|y_0 - y_{h,0}\|_{L^\infty(\Omega)} + C_{3,15}\,(\ln \hat{h}_m)^2 \mathscr{R}_{\infty,0}(y_{h,0}, \mathscr{G}_y^0)\\ &\quad + C_{3,17}\,(\ln \hat{h}_m)^2 \sum_{n=1}^{m} k_n \hat{\mathscr{R}}_{\infty,0}\big(\frac{y_h^n - y_h^{n-1}}{k_n}, \mathscr{G}_y^n - \mathscr{G}_y^{n-1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\big)\\ &\quad + \sum_{n=1}^{m} \int_{I_n} \|f^n - f\|_{L^\infty(\Omega)}\, ds + \frac{k_n}{2} \|\mathscr{G}_y^{n-1} - \mathscr{G}_y^n\|_{L^\infty(\Omega)}.\end{aligned}$$

An application of Lemma 3.3.3 yields

$$\|r_y(t_m)\|_{L^\infty(\Omega)} \leq C_{3,19}\, \|u - U_h\|_{L^\infty(0,T;L^2(\Omega))}.$$

Combining the above estimates and setting $C_{3,24} = \max\{C_{3,15}, C_{3,17}\}$, we accomplish (3.73).

Next, we estimate the error for the co-state variable. By the triangle inequality, for any $t_m \in (0, T]$, we have

$$\|(p - P_h)(t_m)\|_{L^\infty(\Omega)} \leq \|r_p(t_m)\|_{L^\infty(\Omega)} + \|\xi_p(t_m)\|_{L^\infty(\Omega)} + \|\eta_p(t_m)\|_{L^\infty(\Omega)}.$$

We apply Lemmas 3.3.1 – 3.3.3 to arrive at

$$\begin{aligned}\|(p - P_h)(t_m)\|_{L^\infty(\Omega)} &\leq (\ln \hat{h}_m)^2 \Big[C_{3,15}\, \mathscr{R}_{\infty,0}(y_h^m, \mathscr{G}_y^m) + C_{3,16}\, \mathscr{R}_{\infty,0}(p_h^m, \mathscr{G}_p^m)\\ &\quad + C_{3,18} \sum_{n=m+1}^{N} k_n \hat{\mathscr{R}}_{\infty,0}\big(\frac{p_h^{n-1} - p_h^n}{k_n}, \mathscr{G}_p^n - \mathscr{G}_p^{n+1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n\big)\Big]\\ &\quad + \sum_{n=m+1}^{N} \int_{I_n} \|y_{ds} - y_{ds}^n\|_{L^\infty(\Omega)} ds + \frac{k_n}{2} \|\mathscr{G}_p^n - \mathscr{G}_p^{n+1}\|_{L^\infty(\Omega)}\\ &\quad + \|\xi_y\|_{L^\infty(0,T;L^\infty(\Omega))} + C_{3,20}\, \|u - U_h\|_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

Substituting the bound of ξ_y and setting $C_{3,25} = \max\{C_{3,15}, C_{3,16}, C_{3,17}, C_{3,18}\}$, we complete the rest of the proof. □

Theorem 3.3.2 ($L^\infty(L^\infty)-$error estimate for the control variable). *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* (3.6) – (3.10) *and* (3.55) – (3.59), *respectively. Assume that all the conditions in Theorem 3.3.1 are valid. For each* $t \in (0, T]$, *there exists a* *positive constant $C_{3,26}$ such that the following error estimate*

$$
\begin{aligned}
\|u-U_h\|_{L^\infty(0,T;L^\infty(\Omega))} \leq\ & C_{3,26}\Big[\|y_0-y_{h,0}\|_{L^\infty(\Omega)}+(\ln\hat{h}_m)^2\Big\{\mathscr{R}_{\infty,0}(y_{h,0},\mathscr{G}_y^0) \\
&+\mathscr{R}_{\infty,0}(y_h^m,\mathscr{G}_y^m)+\mathscr{R}_{\infty,0}(p_h^m,\mathscr{G}_p^m) \\
&+\sum_{n=1}^{m}k_n\hat{\mathscr{R}}_{\infty,0}\big(\frac{y_h^n-y_h^{n-1}}{k_n},\mathscr{G}_y^n-\mathscr{G}_y^{n-1};\mathscr{T}_h^{n-1},\mathscr{T}_h^n\big) \\
&+\sum_{n=m+1}^{N}k_n\hat{\mathscr{R}}_{\infty,0}\big(\frac{p_h^{n-1}-p_h^n}{k_n},\mathscr{G}_p^n-\mathscr{G}_p^{n+1};\mathscr{T}_h^{n-1},\mathscr{T}_h^n\big)\Big\} \\
&+\sum_{n=1}^{m}\int_{I_n}\|f^n-f\|_{L^\infty(\Omega)}\,ds+\frac{k_n}{2}\|\mathscr{G}_y^{n-1}-\mathscr{G}_y^n\|_{L^\infty(\Omega)} \\
&+\sum_{n=m+1}^{N}\int_{I_n}\|y_{ds}-y_{ds}^n\|_{L^\infty(\Omega)}ds+\frac{k_n}{2}\|\mathscr{G}_p^n-\mathscr{G}_p^{n+1}\|_{L^\infty(\Omega)} \\
&+\Big(\max_{n\in[1,N]}\sum_{K\in\mathscr{T}_h^n}h_K^2|U_h+P_h^{n-1}|_{H^1(K)}^2\,dt\Big)^{1/2}\Big]
\end{aligned}
$$

holds, where the constant $C_{3,26}$ depends on the domain Ω, the regularity constant C_R, and the constant $C_{3,23}$ as defined in Theorem 3.3.1.

Proof. Use of pointwise projection of u and U_h leads to

$$
\begin{aligned}
\|u-U_h\|_{L^\infty(0,T;L^\infty(\Omega))} &= \|\Pi_{[u_a,u_b]}(-p)-\Pi_{[u_a,u_b]}(-P_h^{n-1})\|_{L^\infty(0,T;L^\infty(\Omega))} \\
&\leq \|p-P_h^{n-1}\|_{L^\infty(0,T;L^\infty(\Omega))}.
\end{aligned}
$$

In the above, we have used the Lipschitz continuity of $\Pi_{[u_a,u_b]}$ with Lipschitz constant 1. Inviting Theorem 3.3.1, we complete the rest of the proof. □

3.4 Numerical assessments

This section performs a numerical experiment to illustrate the theoretical results of the previous section. For the purpose of adaptive refinement, we need the following error estimators:

- Initial data estimator $\mathscr{E}_{3,1} := \|y_0-y_{h,0}\|_{L^\infty(\Omega)}$,
- spatial estimator for the state $\mathscr{E}_{3,2} := (\ln\hat{h}_m)^2\mathscr{R}_{\infty,0}(y_h^m,\mathscr{G}_y^m)$,
- temporal error estimator for the state

$$
\mathscr{E}_{3,3} := \sum_{n=1}^{m}\int_{I_n}\|f^n-f\|_{L^\infty(\Omega)}\,ds+\frac{k_n}{2}\|\mathscr{G}_y^{n-1}-\mathscr{G}_y^n\|_{L^\infty(\Omega)}
$$

,

- spatial estimator for the co-state $\mathscr{E}_{3,4} := (\ln \hat{h}_m)^2 \mathscr{R}_{\infty,0}(p_h^m, \mathscr{G}_p^m)$,

- temporal error estimator for the co-state

$$\mathscr{E}_{3,5} := \sum_{n=m+1}^{N} \int_{I_n} \|y_{ds} - y_{ds}^n\|_{L^\infty(\Omega)} ds + \frac{k_n}{2} \|\mathscr{G}_p^n - \mathscr{G}_p^{n+1}\|_{L^\infty(\Omega)}$$

- a control error estimator $\mathscr{E}_{3,6} := \Big(\max_{n \in [1:N]} \sum_{K \in \mathscr{T}_h} h_K^2 |U_h + P_h^{n-1}|_{H^1(K)}^2 \, dt \Big)^{1/2}$, and

- $L^\infty-$type residual error estimators

$$\mathscr{E}_{3,7} := (\ln \hat{h}_m)^2 \sum_{n=1}^{m} k_n \hat{\mathscr{R}}_{\infty,0}\big(\frac{y_h^n - y_h^{n-1}}{k_n}, \mathscr{G}_y^n - \mathscr{G}_y^{n-1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n \big),$$

$$\mathscr{E}_{3,8} := (\ln \hat{h}_m)^2 \sum_{n=m+1}^{N} k_n \hat{\mathscr{R}}_{\infty,0}\big(\frac{p_h^{n-1} - p_h^n}{k_n}, \mathscr{G}_p^n - \mathscr{G}_p^{n+1}; \mathscr{T}_h^{n-1}, \mathscr{T}_h^n \big).$$

The numerical simulation is carried out with the help of the software *FreeFem++* [38] and all the constants involved in the estimators are taken to be 1. We use the following loop

$$SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE$$

to achieve a refinement from the initializing triangulation. The role of Step 2 in Algorithm 3.1 is to reduce the time step size to keep the time error estimator below the tolerance ϵ_{time} while keeping the space mesh unchanged. In Step 3, the refinement procedure is carried out until the time and space error estimators satisfy the desired tolerances. In the last step, if the time error estimator is much less than the prescribe time tolerance ϵ_{time} then we increase the time step size by multiplying a factor δ_2. For marking and refinement of the elements $K \in \mathscr{T}_h^n$, we follow the strategy of Morin, Nochetto and Siebert, see [75]. For both the test example problems, we choose tolerances for time and space as $\epsilon_{time} = \epsilon_{space} = 0.0001$.

Example 3.1. *We consider the spatial domain* $\Omega = [0,1] \times [0,1]$ *and the time interval* $[0,T] = [0,1]$. *We shall use the following data for the optimal control problem* (3.1) $-$

Algorithm 3.1 (Space-time adaptive algorithm)

Given space and time tolerances ϵ_{space}, ϵ_{time} and the parameters $\delta_1 \in (0,1)$, $\delta_2 > 1$, $\lambda_1 \in (0,1)$, $\lambda_2 \in (0,\lambda_1)$. Suppose that $(y_h^{n-1}, p_h^n, u_h^{n-1})$ is computed on the mesh $\mathscr{T}_h^{n-1}$ at time level t_{n-1} with time step size k_{n-1} by using the variational discretization algorithm (see, [92]).

```
Step 1. Set $\mathscr{T}_h^n := \mathscr{T}_h^{n-1}$, $k_n := k_{n-1}$, $t_n := t_{n-1} + k_n$.
            Compute $(y_h^n, p_h^{n-1}, u_h^n)$ from the problem (3.48) – (3.52) on $\mathscr{T}_h^n$.
            Compute the estimators $\mathscr{E}_{3,j}$, $j = 1, \ldots, 8$ on $\mathscr{T}_h^n$.
Step 2. While $\left(\sum_{j\in\{3,5\}} \mathscr{E}_{3,j}\right) > \lambda_1 \cdot \epsilon_{time}$, do
            $k_n := \delta_1 k_{n-1}$, $t_n := t_{n-1} + k_n$.
            Compute $(y_h^n, p_h^{n-1}, u_h^n)$ from the discrete problem (3.48) – (3.52) on $\mathscr{T}_h^n$.
            Compute the estimators $\mathscr{E}_{3,j}$, $j = 1, \ldots, 8$ on $\mathscr{T}_h^n$.
        End while
Step 3. While $\left(\sum_{j\in\{1,2,4,6,7,8\}} \mathscr{E}_{3,j}\right) > \epsilon_{space}$, do
            Refine mesh $\mathscr{T}_h^n$ generate a modified mesh (say) $\mathscr{T}_{h_n}^n$.
            Compute $(y_h^n, p_h^{n-1}, u_h^n)$ from the problem (3.48) – (3.52) on $\mathscr{T}_{h_n}^n$.
            Compute the estimators $\mathscr{E}_{3,j}$, $j = 1, \ldots, 8$ on $\mathscr{T}_{h_n}^n$.
          While $\left(\sum_{j\in\{3,5\}} \mathscr{E}_{3,j}\right) > \lambda_1 \cdot \epsilon_{time}$, do
              $k_n' := \delta_1 k_{n-1}$, $t_n := t_{n-1} + k_n'$.
              Compute $(y_h^n, p_h^{n-1}, u_h^n)$ from the problem (3.48) – (3.52) on $\mathscr{T}_{h_n}^{n,k_n'}$.
              Compute the estimators $\mathscr{E}_{3,j}$ $j = 1, \ldots, 8$ on $\mathscr{T}_{h_n}^{n,k_n'}$.
          End while
        End while
Step 4. If $\left(\sum_{j\in\{3,5\}} \mathscr{E}_{3,j}\right) \leq \lambda_2 \cdot \epsilon_{time}$, do
          Set $k_n' := \delta_2 k_{n-1}$, $t_n := t_{n-1} + k_n'$
        End if
```

(3.3)*:*

$$y(x,t) = \begin{cases} t\sin(2\pi x_1)\sin(2\pi x_2) & x_1+x_2 \le 1, \\ 2t\sin(2\pi x_1)\sin(2\pi x_2) & x_1+x_2 > 1, \end{cases}$$

$$p(x,t) = \begin{cases} (t-1)\sin(2\pi x_1)\sin(2\pi x_2) & x_1+x_2 \le 1, \\ 2(t-1)\sin(2\pi x_1)\sin(2\pi x_2) & x_1+x_2 > 1, \end{cases}$$

with $u_a = -0.125$, *and* $u_b = +0.125$.

Note that functions f, y_{ds} and u are easily determined from the control problem (3.1)-(3.3) as

$$f = \frac{\partial y}{\partial t} - \Delta y - u, \tag{3.75}$$

$$y_{ds} = \frac{\partial p}{\partial t} + \Delta p + y, \tag{3.76}$$

$$u = \min\{u_b, \max\{u_a, -p\}\}. \tag{3.77}$$

We partition the time interval $[0,1]$ with the step-size $k_n = \Delta t \approx 5.56 \times 10^{-3}$ such that $t_n = n\Delta t$, $n = 1, 2, ...N$ with the initial mesh $N = T/\Delta t (= 180)$. In the variational discretization, we use piecewise linear and continuous functions for approximations of the state (y) and co-state (p) variables whereas the control variable (u) is computed by using implicit relation between u and p. The time derivative is approximated by the backward Euler method. The variational discretization algorithm is used to solve the fully discrete optimal control problem $(3.46)-(3.47)$. The adaptive meshes are generated via the error estimators $\mathscr{E}_{3,j}$, $j = 1, 2, \ldots, 8$. We present some computational results by setting tolerances 0.0001 and the time step size $k_n = \Delta t = 5.56 \times 10^{-3}$. In Figures 3.2 and 3.3, the plots of approximate solutions of y and u are depicted on uniform mesh, adaptive mesh step-(I) and adaptive mesh step-(II), respectively, at final time $T = 1.0$. Table 3.1 presents mesh information and errors for the state, co-state and control variables in the $L^\infty(0,T;L^\infty(\Omega))$-norm. This table also reveals that the number of nodes required for adaptive mesh is much less in comparison to the uniform mesh. It is clear from Figure 3.1 that the mesh adapts very well in the neighbourhood of the line $x_1 + x_2 = 1$ where the solution is discontinuous. The higher density of the node points are distributed along the line $x_1 + x_2 = 1$ enable us to save convincing computational work in comparison to uniform mesh.

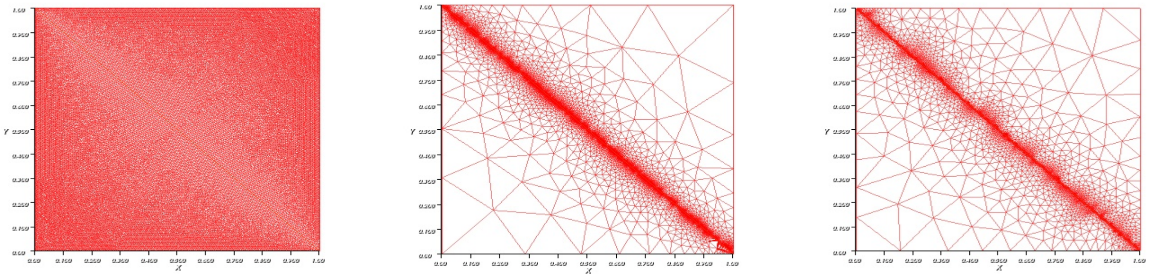


Figure 3.1: *Uniform mesh, adaptive mesh step-(I) and adaptive mesh step-(II)*

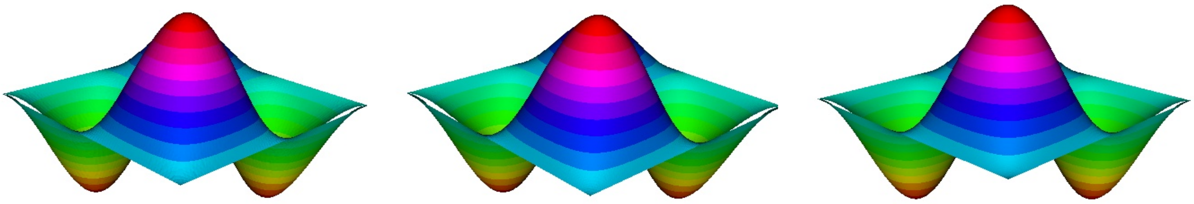


Figure 3.2: *Plots of discrete solution on uniform mesh, adaptive mesh step-(I) and adaptive mesh step-(II), respectively.*

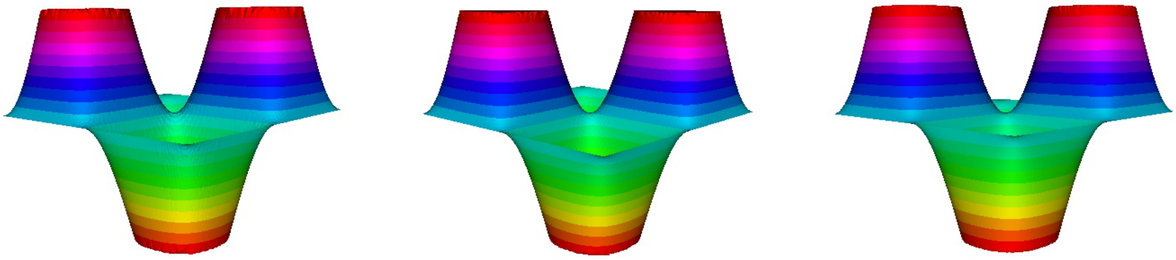


Figure 3.3: *Approximate controls corresponding to uniform mesh, adaptive mesh step-(I) and adaptive mesh step-(II), respectively.*

Table 3.1: *Comparison of data on uniform mesh, adaptive mesh step-(I) and adaptive mesh step-(II) for the initial mesh* $N = 180$.

		On uniform mesh	On adaptive mesh-(I)	On adaptive mesh-(II)
Mesh Information	h_{min}	0.00666	3.5126e-04	3.0408e-04
	h_{max}	0.00666	3.0645e-01	1.9218e-01
	# nodes	17715	8879	7047
	# elements	31835	15929	12549
$L^\infty(L^\infty)$-Error	$y - Y_h$	2.0341e-02	2.1958e-02	2.8827e-03
	$p - P_h$	1.6870e-02	3.3837e-03	1.2073e-03
	$u - U_h$	3.0998e-03	3.0750e-03	2.1170e-04

3.5 Concluding remarks

In this chapter, we have derived maximum-norm *a posteriori* error estimates for variational discretization approximations of POCP (3.1) − (3.3). The variational discretization is used to discretize the control problem. The state and co-state variables are approximated by using the piecewise linear and continuous functions, and the control variable is computed using the implicit relation between control and co-state variables (see, (3.11)). The salient feature of the technical tools include elliptic reconstruction technique and heat kernel estimate. Interestingly, the constants involved in Theorems 3.3.1 and 3.3.2 are independent of time but may depend on the domain Ω. In fact, some of the constants are stemming from the use of elliptic *a posteriori* error estimates. The proposed method does not require the discretization of the admissible control set but to implicitly utilize the optimality conditions for the discretization of the control variable. Our theoretical analysis is supported by numerical experiments which reveals that the adaptive scheme is able to save the substantial computational work.

4

A POSTERIORI ERROR ESTIMATES FOR POCP WITH CONTROLS ACTING ON LOWER DIMENSIONAL MANIFOLDS

This chapter concerns space-time *a posteriori* error analysis to the finite element approximation of POCP (1.4) – (1.6) where the control is acting on a lower dimensional manifold. The manifold considered in this chapter may be a point, a curve or a surface which is lying completely in the space domain. Further, the manifold is assumed to be either time independent or evolved with the time. The space discretization consists of piecewise linear and continuous finite elements for the state and co-state variables, and the piecewise constant functions are employed to approximate the control variable. Moreover, the time derivative is approximated by using the backward Euler scheme. We derive *a posteriori* error estimates for the various dimensions of the manifold. Numerical experiments exhibits the effectiveness of the derived error estimators.

4.1 Introduction

This section considers POCP (1.4) – (1.6) with the functionals

$$\mathcal{G}(y) := \frac{1}{2}\|y - y_{ds}\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{and} \quad \mathcal{H}(u) := \frac{\alpha}{2}\|u\|^2_X,$$

where α is a regularization parameter. The model control problem is stated as follows:

$$\min_{u \in U_{ad}} \mathcal{J}(y,u) := \frac{1}{2}\Big\{\|y - y_{ds}\|^2_{L^2(0,T;L^2(\Omega))} + \alpha\|u\|^2_X\Big\} \tag{4.1}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = u(x,t)\delta_{\gamma(t)}(x) \quad \text{in } \ \Omega_T, \\ y(\cdot,0) = y_0 \quad \text{in } \ \Omega, \\ y = 0 \quad \text{on } \ \Gamma_T, \end{cases} \tag{4.2}$$

where Ω is an open bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz boundary $\Gamma := \partial\Omega$ and $I = [0,T]$ (a fixed real $T < \infty$). Moreover, we denote $\Omega_T = \Omega \times I$ and $\Gamma_T = \partial\Omega \times I$. In the above, the initial state $y_0 \in H_0^1(\Omega)$ and the desired state $y_{ds} \in L^2(0,T;L^2(\Omega))$. Later on, we will specify the control space X. Further, we denote the state variable by $y = y(x,t)$, control variable $u = u(x,t)$ and denote $\frac{\partial y}{\partial t} = y_t$. The set of admissible controls U_{ad} is defined by

$$U_{ad} := \big\{ u \in X : \ \ u_a \leq u(x,t) \leq u_b \quad \text{a.e.} \quad \text{on} \ \ \gamma(t) \quad \text{a.a.} \quad t \in I \big\}, \tag{4.3}$$

where u_a, u_b are real constants with $u_a < u_b$ or constant vectors according to manifold's dimension, and $\delta_{\gamma(t)}$ is a Dirac measure on $\gamma(t)$. Assume that, for all $t \in [0,T]$, the lower dimensional manifold $\gamma(t)$ is precisely contained in Ω. Note that the lower dimensional manifold represents a point, or a curve if $d \geq 2$, or a surface if $d = 3$ which can be time independent or evolves in the time horizon. For the well-posedness of the problem (4.1) – (4.3), we refer to [13, 35].

We now assume that the control variable $u(x,t) \in L^2(0,T;\mathbb{R}^m)$ or $u(x,t) \in L^2(0,T;L^2(\gamma(t)))$ with norm

$$\int_0^T \|u(t)\|^2_{\mathbb{R}^m}\, dt \ < \ \infty \qquad \text{if} \ \ r = 0,$$

or

$$\int_0^T \int_{\gamma(t)} |u(x,t)|^2\, dx\, dt \ < \ \infty \qquad \text{if} \ \ r \neq 0,$$

respectively, where the norm $\|\cdot\|_{\mathbb{R}^m}$ denotes the standard Euclidean norm on m-dimensional Euclidean space. Henceforth, we denote the control space as $X := L^2(0,T;\mathcal{U})$ with $\mathcal{U} = \mathbb{R}^m$ if $r = 0$ (or $\mathcal{U} := L^2(\gamma(t))$ if $r \geq 1$), and the inner product $\langle \cdot,\cdot \rangle_X$ is a duality pairing between X and $(X)'$. Here $(X)'$ denotes the dual space of X.

We now define the inner products on $L^2(\Omega)$ and $L^2(0,T;L^2(\Omega))$ as

$$(\psi,\phi) \ = \ \int_\Omega \psi\,\phi\, dx, \qquad \forall \psi,\ \phi \in L^2(\Omega),$$

and

$$(\psi,\phi)_{\Omega_T} \;=\; \int_0^T\int_\Omega \psi\,\phi\,dxdt = \int_{\Omega_T}\psi\,\phi\,dxdt, \qquad \forall\psi,\,\phi\in L^2(0,T;L^2(\Omega)),$$

respectively. The bilinear forms $a(\cdot,\cdot)$ and $a(\cdot,\cdot)_{\Omega_T}$ are defined as

$$a(v,w) \;=\; \int_\Omega \nabla v\cdot\nabla w\,dx, \qquad \forall v,w\in H_0^1(\Omega),$$

and

$$a(v,w)_{\Omega_T} \;=\; \int_{\Omega_T} \nabla v\cdot\nabla w\,dx\,dt, \qquad \forall v,w\in L^2(0,T;H_0^1(\Omega)),$$

respectively. Then there exists α_0, $\alpha_1>0$ such that $a(\cdot,\cdot)$ the bilinear form satisfies

$$|a(v,w)| \;\le\; \alpha_0\|v\|_1\|w\|_1, \qquad \forall v,w\in H_0^1(\Omega),$$

and

$$a(v,v) \;\ge\; \alpha_1\|v\|_1^2, \qquad \forall v\in H_0^1(\Omega),$$

respectively, i.e., $a(\cdot,\cdot)$ is bounded and coercive.

Next, we discuss the well-posed property for the state equation (4.2) (cf., [35]). In order to state the existence and uniqueness result for the problem (4.2), we define following notation:

$$H_0=\begin{cases} H^{-1}(\Omega) & \text{if } d-r=1,\\ L^2(\Omega) & \text{if } d-r>1; \end{cases} \qquad \text{and} \qquad H_2=\begin{cases} H_0^1(\Omega) & \text{if } d-r=1,\\ H^2(\Omega)\cap H_0^1(\Omega) & \text{if } d-r>1. \end{cases}$$

For $\Phi\in L^2(0,T;H_2)$, we define

$$\langle u(x,t)\delta_{\gamma(t)},\Phi\rangle_I=\begin{cases} \displaystyle\sum_{j=1}^m\int_0^T u_j(t)\Phi(\gamma_j(t),t)\,dt & \text{if} \quad r=0,\\ \displaystyle\int_0^T\int_{\gamma(t)} u(x,t)\Phi(x,t)\,dxdt & \text{if} \quad r\ge 1. \end{cases}$$

Note that

$$\Big|\sum_{j=1}^m\int_0^T u_j(t)\Phi(\gamma_j(t),t)\,dt\Big| \;\le\; \|u\|_{L^2(0,T;\mathbb{R}^m)}\|\Phi\|_{L^2(0,T;L^\infty(\Omega))} \qquad \text{if } r=0,$$

and

$$\Big|\int_0^T\int_{\gamma(t)} u(x,t)\Phi(x,t)\,dxdt\Big| \;\le\; \|u\|_{L^2(0,T;L^2(\gamma(t)))}\|\Phi\|_{L^2(0,T;L^2(\gamma(t)))} \qquad \text{if} \;\; r\ge 1.$$

From the trace theorem and the standard embedding result $H^2(\Omega)\cap H_0^1(\Omega)\hookrightarrow \mathcal{C}(\overline{\Omega})$ we know that

$$\|\Phi|_{\gamma(t)}(\cdot,t)\|_{L^2(\gamma(t))} \;\le\; C_{4,1}\|\Phi(\cdot,t)\|_{H_0^1(\Omega)} \quad \text{for } d=2,\,3 \;\text{ and }\; d-r=1,$$

and

$$\|\Phi\|_{L^2(0,T;L^\infty(\Omega))} \;\le\; C_{4,2}\|\Phi\|_{L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))},$$

respectively, where $C_{4,1}$ and $C_{4,2}$ depend on the manifold $\gamma(t)$. Hence,

$$\|\Phi\|_{L^2(0,T;L^2(\gamma(t)))} \le \begin{cases} C_{4,1}\|\Phi\|_{L^2(0,T;H_0^1(\Omega))} & \text{if } d=2,3;\;\; d-r=1,\\ C_{4,3}\|\Phi\|_{L^2(0,T;L^\infty(\Omega))} \le C_{4,4}\|\Phi\|_{L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)} & \text{if } d=3;\; r=1,\end{cases}$$

with $C_{4,4}=C_{4,2}C_{4,3}$. Thus, we find that

$$\langle u(x,t)\delta_{\gamma(t)},\Phi\rangle_I \le \begin{cases} C_{4,5}\|u\|_{L^2(0,T;\mathbb{R}^m)}\|\Phi\|_{L^2(0,T;H_2)} & \text{if } r=0,\\ C_{4,6}\|u\|_{L^2(0,T;L^2(\gamma(t)))}\|\Phi\|_{L^2(0,T;H_2)} & \text{if } r\ge 1,\end{cases} \tag{4.4}$$

where $C_{4,5}$ and $C_{4,6}$ depend on the manifold $\gamma(t)$. Since $\langle\cdot,\cdot\rangle_I$ denotes the duality pairing between $L^2(0,T;H_2)$ and $L^2(0,T;(H_2)')$, $u(x,t)\delta_{\gamma(t)}\in L^2(0,T;(H_2)')$, where $(H_2)'$ is the dual space of H_2.

The following lemma concerns the existence of a solution for the problem (4.2). The weak solution of (4.2) can be defined by employing the transposition technique (cf. [53] and [55]).

Lemma 4.1.1. *Let $f\in L^2(0,T;H_0)$, and let Φ be the solution of (1.18). Then the control problem (4.2) has a weak solution $y\in L^2(0,T;(H_0)')$ with $u\in U_{ad}$, if*

$$(y,f)_{\Omega_T} \;=\; (y_0,\Phi(\cdot,0)) + \langle u(x,t)\delta_{\gamma(t)},\Phi\rangle_I, \tag{4.5}$$

for $\Phi\in L^2(0,T;H_2)$, where $(H_0)'$ denotes the dual of H_0.

Invoking the well-known Lax-Milgram theorem, we can show that the control problem (4.2) admits a unique solution $y\in L^2(0,T;(H_0)')$ in the sense of (4.5). Following theorem presents the regularity of the solution for the problem (4.2) in various dimension of the manifold $\gamma(t)$. For a proof, we refer to [35].

Theorem 4.1.1. *Assume that $y_0\in H_0^1(\Omega)$ and $f\in L^2(0,T;H_0)$. Let $y\in L^2(0,T;L^2(\Omega))$ be the solution of the state equation (4.2) in the sense of (4.5), i.e.,*

$$(y,f)_{\Omega_T} \;=\; \langle u\delta_{\gamma(t)},\Phi\rangle_I + (y_0,\Phi(\cdot,0)).$$

Then, we have the following regularity results:

$$
\begin{aligned}
&y \in L^2(0,T;W_0^{1,s}(\Omega)) \cap H^1(0,T;W^{-1,s}(\Omega)), && s \in \Big(1, \frac{d}{d-1}\Big) \ \ \textit{when} \ \ r=0, \ d=2,\,3;\\
&y \in L^2(0,T;W_0^{1,\sigma}(\Omega)) \cap H^1(0,T;W^{-1,\sigma}(\Omega)), && \sigma \in (1,2) \ \ \textit{when} \ \ r=1, \ d=3;\\
&y \in L^2(0,T;H^{\frac{3-\epsilon}{2}}(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;H^{\frac{-1+\epsilon}{2}}(\Omega)), && \forall\, \epsilon > 0 \ \ \textit{when} \ \ r \geq 1, \ d-r=1.
\end{aligned}
$$

The weak formulation of the optimal control problem (4.1) – (4.3) is to seek a pair $(y,u) \in L^2(0,T;L^2(\Omega)) \times U_{ad}$ such that

$$
\min_{u \in U_{ad}} \frac{1}{2}\{\|y-y_{ds}\|^2_{L^2(0,T;L^2(\Omega))} + \alpha\|u\|^2_X\} \tag{4.6}
$$

subject to

$$
-(y,v_t)_{\Omega_T} - (y,\Delta v)_{\Omega_T} \;=\; \langle u\delta_{\gamma(t)}, v\rangle_I + (y_0, v(\cdot,0)) \qquad \forall v \in \mathcal{W}_1(0,T). \tag{4.7}
$$

It is well known (cf. [53]) that the control problem (4.6) – (4.7) has a unique optimal pair (y,u) iff there exists a co-state variable $p \in L^2(0,T;H_0^1(\Omega))$ such that the following optimality conditions are satisfied:

$$
\begin{aligned}
-(y,v_t)_{\Omega_T} - (y,\Delta v)_{\Omega_T} &= \langle u\delta_{\gamma(t)}, v\rangle_I + (y_0, v(\cdot,0)) \qquad \forall v \in \mathcal{W}_1(0,T), && (4.8)\\
-(p_t,v)_{\Omega_T} + a(p,v)_{\Omega_T} &= (y-y_{ds}, v)_{\Omega_T} \qquad \forall v \in \mathcal{W}_2(0,T), && (4.9)\\
p(\cdot,T) &= 0, && (4.10)\\
(\alpha u + p|_{\gamma}, w-u)_X &\geq 0 \qquad \forall w \in U_{ad}, && (4.11)
\end{aligned}
$$

where $p|_{\gamma}$ stands for the restriction of p on the manifold $\gamma(t)$. From (4.11), the control variable u can be expressed in terms of the co-state variable p as follows: For $t \in [0,T]$,

$$
u(t) = \begin{cases} \mathscr{P}_{U_{ad}}\Big(-\frac{1}{\alpha}p(\gamma_j(t))(t)\Big)_{j=1}^{m} & \text{for} \quad r=0,\\ \mathscr{P}_{U_{ad}}\Big(-\frac{1}{\alpha}p(x,t)|_{\gamma}\Big) & \text{for} \quad r \geq 1, \end{cases}
$$

where $\mathscr{P}_{U_{ad}}$ is the orthogonal projection onto U_{ad}.

The following theorem collects the regularity results of the solutions (y,p,u) to the control problem (4.8) – (4.11).

Theorem 4.1.2. *Let* (y,p,u) *be the solution of the optimization problem* (4.8) – (4.11)*. Assume that* $y_0 \in H_0^1(\Omega)$*, then we have the following regularity results:*

Case I (if $r = 0,\, d = 2,\, 3$):

$$\begin{cases} y \in L^2(I; W_0^{1,s}(\Omega)) \cap H^1(I; W^{-1,s}(\Omega)) & s \in (1, \frac{d}{d-1}), \\ p \in L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega)) & u \in L^2(I; \mathbb{R}^m). \end{cases}$$

Case II (if $r = 1,\ d = 3$):

$$\begin{cases} y \in L^2(I; W_0^{1,\sigma}(\Omega)) \cap H^1(I; W^{-1,\sigma}(\Omega)) & \sigma \in (1, 2), \\ p \in L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega)) & u \in L^2(I; H^1(\gamma(t)). \end{cases}$$

Case III (if $r \geq 1,\ d - r = 1$):

$$\begin{cases} y \in L^2(I; H^{\frac{3-\epsilon}{2}}(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; H^{\frac{-1+\epsilon}{2}}(\Omega)) & \textit{for any } \epsilon > 0, \\ p \in L^2(I; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(I; L^2(\Omega)) & u \in L^2(I; H^1(\gamma(t)). \end{cases}$$

In this direction, for time-independent control problem, Gong *et al.* [33] have extensively discussed the finite element approximations to elliptic control problems with controls acting on the lower dimensional manifold. They have derived *a priori error* estimates for the state variable in the $L^2(\Omega)$-norm while the error for the control variable in the $\mathcal{U}$-norm with $\mathcal{U} = L^2(\gamma(t))$ or $\mathcal{U} = \mathbb{R}^m$ according to the dimension of the manifold. Gong *et al.* [31] have studied *a priori* error analysis for the control problem governed by parabolic equations, where the control acts only on finitely many points which are time-independent. Thereafter, Gong and Yan [35] have generalized the *a priori* error analysis results for the optimal control problem governed by parabolic partial differential equations where the manifold evolves with time. In all of the above studies, *a priori* error estimates are derived and the support of the controls requires to be very small compared to the actual size of the domain Ω if we are restricted by the cost of controls. To the best of author's knowledge, the *a posteriori* error analysis are yet to be explored. This chapter attempts to study *a posteriori* error analysis of the fully discrete finite element approximations to the state, co-state and control variables for the control problems (4.1) – (4.3) with controls acting on a lower dimensional manifold. We derive *a posteriori* error estimates for the state variable in the $L^2(0, T; L^2(\Omega))$-norm while error for the control variable in the $L^2(0, T; \mathbb{R}^m)$ or $L^2(0, T; L^2(\gamma(t)))$-norm according to dimension of the manifold.

The layout of the chapter is as follows. Section 4.2 is devoted to the fully discrete finite element approximation of the model problem (4.1) – (4.3). *A posteriori* error estimates for the state, co-state and control variables are derived in Section 4.3. We perform numerical tests to illustrate the performance of the derived estimators in Section 4.4. In the last section, we present some concluding remarks.

4.2 Discrete optimal control problem

This section is devoted to the finite element approximation of the optimal control problem (4.1) – (4.3). To discretize the space variable, we employ the piecewise continuous and linear finite elements whereas the piecewise constant functions are used to discretize the control variable. The backward Euler scheme is applied to approximate temporal derivative.

Let $\mathscr{T}_h$ be a shape regular triangulations of $\overline{\Omega}$ such that $\overline{\Omega} = \bigcup_{K\in\mathscr{T}_h} \bar{K}$. Let $\mathcal{E}_h$ be the collection of all interelement edges or faces in the interior of the mesh. The jump of $\nabla \mathrm{v}$ across an element edge or face $E \in \mathcal{E}_h$ is measured as

$$\left[\frac{\partial \mathrm{v}}{\partial n_E}\right] = (\nabla \mathrm{v})_{K_1}\cdot n_{K_1} + (\nabla \mathrm{v})_{K_2}\cdot n_{K_2} \tag{4.12}$$

with $E = \bar{K}_1 \cap \bar{K}_2$.

Let $\tilde{V}_h$ be a finite dimensional subspace of $\mathcal{C}(\overline{\Omega})$ associated with $\mathscr{T}_h$, such that $\forall \chi \in \tilde{V}_h$, $\chi|_K \in P_1(K)$, i.e., $\chi|_K$ are piecewise linear functions with respect to $K \in \mathscr{T}_h$. Set $V_h = \tilde{V}_h \cap H_0^1(\Omega)$. We approximate the control variable with piecewise constant functions and the corresponding finite dimensional space $U_{ad,h}$ be given by

$$U_{ad,h} = \big\{u \in U_{ad} : \quad u|_K \in \mathbb{P}_0, \quad \forall\, K \in \mathscr{T}_h\big\},$$

where $\mathbb{P}_0$ denotes the space of piecewise constant functions. Clearly, $U_{ad,h} \subset U_{ad}$.

For the purpose of fully discrete case, we consider time points $0 = t_0 < \ldots < t_{N-1} < t_N = T$ in the time domain $I = [0, T]$ with time subintervals $I_n = (t_{n-1}, t_n]$ of size $k_n = t_n - t_{n-1}$. Clearly, $I = \cup_{n=1}^{N} \bar{I}_n$. Let $\mathscr{T}_h^n = \{K\}(0 \le n \le N)$ be the triangulation of $\overline{\Omega}$ at the time level t_n, and let $\mathcal{M}_h^n$ be the collection of all triangles $K \in \mathscr{T}_h^n$ such that K lying on the manifold $\gamma(t)$. Further, let V_h^n, $n = 1, 2, \ldots, N$, be the finite dimensional spaces (similar to V_h) with respect to $\mathscr{T}_h^n$, and let

$$U_{ad,h}^n = \big\{u \in U_{ad} : \ u|_K \in \mathbb{P}_0, \ \ \forall\, K \in \mathscr{T}_h^n\big\}$$

be a finite dimensional control space corresponding to the triangulation $\mathscr{T}_h^n$. We define $\mathcal{E}_h^n$, similar to $\mathcal{E}_h$, the union of the interelement edges or faces E of $K \in \mathscr{T}_h^n$. For a sequence of function $\{\phi^n\}$ in $L^2(\Omega)$, we set $\bar{\partial}\phi^n := \frac{1}{k_n}(\phi^n - \phi^{n-1}), \quad n \ge 1$.

Then the fully discrete finite element approximation of the optimal control problem (4.6) – (4.7) is to seek a pair $(y_h^n, u_h^n) \in V_h^n \times U_{ad,h}^n$, for $n = 1, 2, \ldots, N$, such that

$$\min_{u_h^n \in U_{ad,h}^n} \frac{1}{2} \sum_{n=1}^{N} \int_{I_n} \{ \|y_h^n - y_{ds}^n\|_{L^2(\Omega)}^2 + \alpha \|u_h^n\|_{\mathcal{U}}^2 \} \, dt \tag{4.13}$$

subject to

$$\begin{cases} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) \;=\; \langle u_h^n \delta_{\gamma(t)}, v_h \rangle_{I_n} \qquad \forall v_h \in V_h^n, \\ y_h^0 = y_{h,0}, \end{cases} \tag{4.14}$$

where $y_{h,0}$ is a suitable projection or approximation of y_0.

The control problem (4.13) – (4.14) admits a unique pair (y_h^n, u_h^n) for each $n = 1, \ldots, N$ iff there exists a co-state $p_h^{n-1} \in V_h^n$ such that $(y_h^n, p_h^{n-1}, u_h^n)$ satisfies the following optimality conditions, for each $n \in [1 : N]$:

$$\begin{aligned} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) &= \langle u_h^n \delta_{\gamma(t)}, v_h \rangle_{I_n} \qquad \forall v_h \in V_h^n, && (4.15)\\ y_h^0 &= y_{h,0}, && (4.16)\\ -(\bar{\partial} p_h^n, v_h) + a(p_h^{n-1}, v_h) &= (y_h^n - y_{ds}^n, v_h) \qquad \forall v_h \in V_h^n, && (4.17)\\ p_h^N &= 0, && (4.18)\\ (\alpha u_h^n + p_h^{n-1}|_{\gamma(t|_{I_n})}, w_h^n - u_h^n)_{\mathcal{U}} &\geq 0 \qquad \forall w_h^n \in U_{ad,h}^n. && (4.19) \end{aligned}$$

Given a sequence of discrete values $\{p_h^n\}$, $n = 0, 1, \ldots, N$, we associate a continuous function of time defined by the continuous piecewise linear interpolant $P_h(t)$ as

$$P_h(t)|_{t \in I_n} := \frac{(t_n - t)}{k_n} p_h^{n-1} + \frac{(t - t_{n-1})}{k_n} p_h^n.$$

For $n = 1, 2, \ldots, N$, we define $Y_h(t)$ and $U_h(t)$ as $Y_h(t)|_{t \in I_n} := y_h^n$ and $U_h(t)|_{t \in I_n} := u_h^n$, respectively.

Then the optimality conditions (4.15) – (4.19) can be stated as follows:

$$\begin{aligned} (\bar{\partial} Y_h^n, v_h) + a(Y_h^n, v_h) &= \langle U_h \delta_{\gamma(t)}, v_h \rangle_{I_n} \qquad \forall v_h \in V_h^n, && (4.20)\\ Y_h^0 &= y_{h,0}, && (4.21)\\ -(\bar{\partial} P_h^n, v_h) + a(P_h^{n-1}, v_h) &= (Y_h^n - y_{ds}^n, v_h) \qquad \forall v_h \in V_h^n, && (4.22)\\ P_h^N &= 0, && (4.23)\\ (\alpha U_h + P_h^{n-1}|_{\gamma(t|_{I_n})}, w_h^n - U_h)_{\mathcal{U}} &\geq 0 \qquad \forall w_h^n \in U_{ad,h}^n. && (4.24) \end{aligned}$$

From (4.24), for $t \in I_n$, $n = 1, \ldots, N$, the control variable $U_h(t)$ can be expressed as

$$U_h(t) = \begin{cases} \mathscr{P}_{U_{ad}}\Big(-\frac{1}{\alpha}(P_h^{n-1}(\gamma_j(t|_{I_n})))\Big)_{j=1}^{m} & \text{for} \quad r = 0, \\ \mathscr{P}_{U_{ad}}\Big(-\frac{1}{\alpha}P_h^{n-1}|_{\gamma(t|_{I_n})}\Big) & \text{for} \quad r \geq 1. \end{cases}$$

Analogous to the continuous case, we reformulate the fully discrete optimal control problem (4.13) – (4.14) as:

$$\min_{u_h^n \in U_{ad,h}^n} J_h^n(u_h^n), \quad \text{for} \ \ n = 1, \ldots, N, \quad \text{where} \ \ J_h^n := \mathcal{J}(u_h^n, y_h^n(u_h^n)).$$

4.3 A posteriori error analysis

In this section, we derive *a posteriori* error estimates for the state, co-state and the control variables when $\gamma(t)$ is a r-dimensional manifold with $r = 0,\ 1,\ 2$. The following two lemmas are proved to be convenient. The first lemma is about the interpolation approximation properties and the second one refers to the trace inequality.

Lemma 4.3.1 ([16]). *Let* $\Pi_h : \mathcal{C}(\overline{\Omega}) \to V_h^0$ *be the standard Lagrange interpolation operator. Then for* $m = 0,\ 1,$ *we have*

$$\|v - \Pi_h v\|_{m,K} \ \leq \ C_{I,m} h_K^{2-m} |v|_{2,K} \qquad \forall v \in H^2(K).$$

For all $v \in W^{k+1,p}(K)$

$$\|v - \Pi_h v\|_{m,q,K} \ \leq \ C_{I,2}(meas(K))^{(1/q-1/p)} h_K^{k+1-m} |v|_{k+1,p,K},$$

and

$$\|v - \Pi_h v\|_{\infty,K} \ \leq \ C_{I,\infty} h_K^{2-d/2} \|v\|_{2,K} \qquad \forall v \in H^2(K).$$

Lemma 4.3.2 ([45]). *For all* $v \in W^{1,p}(\Omega)$, $1 \leq p < \infty$,

$$\|v\|_{p,E} \ \leq \ C_{I,E}\Big(h_K^{-1/p} \|v\|_{p,K} + h_K^{1-1/p} |v|_{1,p,K}\Big).$$

In order to derive *a posteriori* error estimates for the optimal control problem (4.1) – (4.3), we first define the auxiliary problem: To find a pair $(y(\hat{u}_h), p(\hat{u}_h)) \in L^2(0,T; L^2(\Omega)) \times L^2(0,T; H_0^1(\Omega))$ such that

$$-(y(\hat{u}_h), v_t)_{\Omega_T} - (y(\hat{u}_h), \Delta v)_{\Omega_T} \ = \ \langle \hat{u}_h \delta_{\gamma(t)}, v\rangle_I + (y_0, v(\cdot, 0)) \quad \forall v \in \mathcal{W}_1(0,T), \quad (4.25)$$

$$-(p_t(\hat{u}_h), v)_{\Omega_T} + a(p(\hat{u}_h), v)_{\Omega_T} \ = \ (y(\hat{u}_h) - y_{ds}, v)_{\Omega_T} \quad \forall v \in \mathcal{W}_2(0,T), \quad (4.26)$$

$$p(\hat{u}_h)(x, T) \ = \ 0 \quad \forall x \in \Omega. \quad (4.27)$$

Before developing the error analysis, we collect some stability results for the state variable. Since y and $y(\hat{u}_h)$ satisfy equation (4.5), the following stability results hold for the state variable.

Lemma 4.3.3. *Let (y, p, u) and $(y(\hat{u}_h), p(\hat{u}_h))$ with $\hat{u}_h = U_h$ be the solutions of the problems (4.8) – (4.11) and (4.25) – (4.27), respectively. Then there exists positive constants $C_{4,i}$, $i = 7,\ 8,\ 9$ such that the following estimates hold:*

(i) For $r = 0$ and $d = 2$ or $d = 3$, we have

$$\|y - y(U_h)\|_{L^2(I;W_0^{1,s}(\Omega))} \le C_{4,7}\, \|u - U_h\|_{L^2(I;\mathbb{R}^m)}. \tag{4.28}$$

(ii) For $r = 1$ and $d = 3$, we have

$$\|y - y(U_h)\|_{L^2(I;W_0^{1,\sigma}(\Omega))} + \|y_t - y_t(U_h)\|_{L^2(I;W^{-1,\sigma}(\Omega(t)))} \le C_{4,8}\, \|u - U_h\|_{L^2(I;L^2(\gamma(t)))}. \tag{4.29}$$

(iii) For $d - r = 1$, $r = 1$ or 2, we have

$$\|y - y(U_h)\|_{L^2(I;H^{\frac{3-\epsilon}{2}}(\Omega)\cap H_0^1(\Omega))} + \|y_t - y_t(U_h)\|_{L^2(I;H^{\frac{-1+\epsilon}{2}}(\Omega))} \le C_{4,9}\, \|u - U_h\|_{L^2(I;L^2(\gamma(t)))}. \tag{4.30}$$

The constants $C_{4,7}$, $C_{4,8}$ and $C_{4,9}$ denote the stability constant.

Let $\mathcal{L}_n$ be the L^2-projection operator onto the piecewise constant space $P_0(I_n)$ for each time interval I_n. Define

$$\mathcal{L}_n\Phi \;:=\; \frac{1}{k_n}\int_{I_n} \Phi\, dt.$$

Further, we define Φ_I such that $\Phi_I|_{I_n} = \Pi_h^n(\mathcal{L}_n\Phi) \in V_h^n$ on each time interval I_n, where Π_h^n is the standard Lagrange interpolation operator onto V_h^n. Note that $\Phi \in L^2(0,T;H^2(\Omega)) \hookrightarrow L^2(0,T;\mathcal{C}(\overline{\Omega}))$, thus Lagrange interpolation is well defined and the following approximation properties

$$\|\Phi - \mathcal{L}_n\Phi\|_{L^\infty(I_n;L^2(\Omega))} \;\le\; C_{I,3}\, k_n^{1/2}\, \|\Phi_t\|_{L^2(0,T;L^2(\Omega))}, \tag{4.31}$$

and

$$\|\Phi - \Pi_h^n\Phi\|_{L^2(0,T;L^2(\Omega))} \;\le\; C_{I,0}\, h_K^2\, \|\Phi\|_{L^2(0,T;H^2(\Omega))} \tag{4.32}$$

hold. We now discuss the *a posteriori* error analysis for the optimal control problem (4.1) – (4.3). We first discuss the case $d - r = 1$ with $r = 1$ or 2.

Theorem 4.3.1. *Let $\gamma(t)$ be an r-dimensional manifold which is completely lying inside Ω with $d-r=1$, and let $X = L^2(0,T;L^2(\gamma(t)))$. Assume that $(y,p,u) \in L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;H_0^1(\Omega)) \times U_{ad}$ and $(Y_h,P_h,U_h) \in V_h^n \times V_h^n \times U_{ad,h}^n$, respectively, be the solutions of the continuous problem (4.8) − (4.11) and (4.20) − (4.24). Further, we assume that $(P_h^{n-1}+U_h)|_K \in H^1(K)$ and there exists positive constants $C_{4,10}$, $C_{4,11}$ and $w_h \in U_{ad,h}^n$ such that*

$$\Big|(U_h+P_h^{n-1}|_{\gamma(t)}, w_h-u)_X\Big| \le C_{4,10}\int_0^T \sum_{K\in\mathcal{M}_h^n} h_K|U_h+P_h^{n-1}|_{H^1(K)}\|u-U_h\|_{L^2(K)}dt. \quad (4.33)$$

Then, the following estimate holds:

$$\|u-U_h\|^2_{L^2(0,T;L^2(\gamma(t)))} + \|y-Y_h\|^2_{L^2(0,T;L^2(\Omega))} \le C_{4,11}\Big(\mathscr{E}_{4,0} + \sum_{n=1}^{N} k_n\Big(\sum_{i=1}^{7}\mathscr{E}_{4,i}^n\Big)\Big)$$

with

$$\begin{aligned}
\mathscr{E}_{4,1}^n &:= \frac{1}{k_n}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n} h_K^2|P_h^{n-1}+U_h|^2_{H^1(K)}\Big\}dt,\\
\mathscr{E}_{4,2}^n &:= \frac{1}{k_n}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n} h_K^2\|U_h\|^2_{L^2(K)}\Big\}dt,\\
\mathscr{E}_{4,3}^n &:= \sum_{K\in\mathscr{T}_h^n} h_K^2\|-k_n^{-1}(P_h^n-P_h^{n-1})-\Delta P_h^{n-1}-Y_h^n+y_{ds}^n\|^2_{L^2(K)}\\
&\quad + \sum_{E\in\mathcal{E}_h^n} h_E\Big\|\Big[\frac{\partial P_h^{n-1}}{\partial n}\Big]\Big\|^2_{L^2(E)},\\
\mathscr{E}_{4,4}^n &:= \|P_h^n-P_h^{n-1}\|^2_{H^1(\Omega)},\\
\mathscr{E}_{4,5}^n &:= \sum_{K\in\mathscr{T}_h^n} h_K^4\|-\Delta Y_h^n\|^2_{L^2(K)} + \sum_{E\in\mathcal{E}_h^n} h_E^3\Big\|\Big[\frac{\partial Y_h^n}{\partial n_E}\Big]\Big\|^2_{L^2(E)},\\
\mathscr{E}_{4,6}^n &:= \|Y_h^n-Y_h^{n-1}\|^2_{L^2(\Omega)} + \sum_{K\in\mathscr{T}_h^n} h_K^4\|k_n^{-1}(Y_h^n-Y_h^{n-1})\|^2_{L^2(K)},\\
\mathscr{E}_{4,7}^n &:= \frac{1}{k_n}\int_{I_n}\|y_{ds}-y_{ds}^n\|^2_{L^2(\Omega)}dt, \quad \text{and} \quad \mathscr{E}_{4,0} := \|y_0-Y_h^0\|^2_{L^2(\Omega)},
\end{aligned}$$

where the quantity $\Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big]$ is defined in (4.12).

Proof. First, we discuss error estimate for the control variable for the case $d-r=1$ with $r \ge 1$. Without loss of generality, we take $\alpha = 1$. From (4.11) with $w = U_h$, we have

$$(u, u-U_h)_X \le -(p|_{\gamma(t)}, u-U_h)_X.$$

Using the above, we find that

$$\begin{aligned}\|u-U_h\|_X^2 &= (u-U_h,u-U_h)_X\\ &\le -(p|_{\gamma(t)},u-U_h)_X-(U_h,u-U_h)_X\\ &= -(U_h+P_h^{n-1}|_{\gamma(t)},u-w_h)_X-(U_h+P_h^{n-1}|_{\gamma(t)},w_h-U_h)_X\\ &\quad+((P_h^{n-1}-p(U_h))|_{\gamma(t)},u-U_h)_X+((p(U_h)-p)|_{\gamma(t)},u-U_h)_X.\end{aligned}$$

With the help of (4.24), it now yields

$$\begin{aligned}\|u-U_h\|_X^2 &\le -(U_h+P_h^{n-1}|_{\gamma(t)},u-w_h)_X+((P_h-p(U_h))|_{\gamma(t)},u-U_h)_X\\ &\quad+((p(U_h)-p)|_{\gamma(t)},u-U_h)_X\\ &=: I_1+I_2+I_3. \end{aligned}\tag{4.34}$$

Now, we estimate I_1, I_2 and I_3, separately. Using (4.33) we estimate the first term I_1 as

$$\begin{aligned}|I_1| &= \left|(U_h+P_h^{n-1}|_{\gamma(t)},u-w_h)_X\right|\\ &\le C_{4,10}\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}h_K|P_h^{n-1}+U_h|_{H^1(K)}\|u-U_h\|_{L^2(K)}\Big\}\,dt\\ &\le \frac{3}{4}C_{4,10}^2\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}h_K^2|P_h^{n-1}+U_h|_{H^1(K)}^2\Big\}\,dt+\frac{1}{4}\|u-U_h\|_X^2\\ &\le \frac{3}{4}C_{4,10}^2\sum_{n=1}^{N}k_n\mathscr{E}_{4,1}^n+\frac{1}{4}\|u-U_h\|_X^2. \end{aligned}\tag{4.35}$$

Next, we estimate I_2 as follows:

$$\begin{aligned}|I_2| &= \Big|\int_0^T\int_{\gamma(t)}(P_h^{n-1}-p(U_h))(u-U_h)\,dxdt\Big|\\ &\le \|P_h^{n-1}-p(U_h)\|_{L^2(0,T;L^2(\gamma(t)))}\|u-U_h\|_X\\ &\le \frac{3}{4}\|P_h^{n-1}-p(U_h)\|_{L^2(0,T;L^2(\gamma(t)))}^2+\frac{1}{4}\|u-U_h\|_X^2. \end{aligned}\tag{4.36}$$

For I_3, we note that $y(x,0)-y(U_h)(x,0)=0$ and $p(x,T)-p(U_h)(x,T)=0$. Using

(4.8) and (4.25) with $\hat{u}_h = U_h$ we obtain

$$\begin{aligned} I_3 &= \int_0^T \int_{\gamma(t)} (u - U_h)(p(U_h) - p)\, dxdt \\ &= \int_0^T \{-(y - y(U_h), p_t(U_h) - p_t) + a(y - y(U_h), p(U_h) - p)\}\, dt, \\ &= -\int_0^T \|y - y(U_h)\|^2_{L^2(\Omega)}\, dt \le 0. \end{aligned} \tag{4.37}$$

Combining (4.35) – (4.37) with (4.34) yields

$$\|u - U_h\|^2_X \;\le\; C_{4,12}\Big(\sum_{n=1}^{N} k_n \mathscr{E}^n_{4,1} + \|P_h^{n-1} - p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))}\Big), \tag{4.38}$$

where $C_{4,12} = \frac{3}{2}\max\{C^2_{4,10}, 1\}$.

Note that the estimation of the error in control variable depends on the co-state error $P_h^{n-1} - p(U_h)$ in the $L^2(0,T;L^2(\gamma(t)))$-norm. To estimate this error we introduce the following auxiliary problem. Let ψ be the solution of

$$\begin{cases} \psi_t - \Delta\psi \;=\; (P_h - p(U_h))\delta_{\gamma(t)}(x) \quad \text{in} \quad \Omega_T, \\ \psi(\cdot,0) \;=\; 0 \quad \text{in} \quad \Omega, \\ \psi \;=\; 0 \quad \text{on} \quad \Gamma_T. \end{cases} \tag{4.39}$$

It follows from Theorem 4.1.1 that $\psi \in L^2(0,T;H^{\frac{3-\epsilon}{2}}(\Omega)\cap H^1_0(\Omega))\cap H^1(0,T;H^{-\frac{1+\epsilon}{2}}(\Omega))$, for any $\epsilon > 0$, and there exists a positive constant $C_{4,13}$ such that

$$\|\psi\|_{L^2(0,T;H^{\frac{3-\epsilon}{2}}(\Omega)\cap H^1_0(\Omega))} + \|\psi_t\|_{L^2(0,T;H^{-\frac{1+\epsilon}{2}}(\Omega))} \;\le\; C_{4,13}\,\|P_h - p(U_h)\|_{L^2(0,T;L^2(\gamma(t)))}. \tag{4.40}$$

Use of duality argument leads to

$$\begin{aligned} \|P_h - p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))} &= \int_0^T (\psi_t - \Delta\psi, P_h - p(U_h))\, dt \\ &= \int_0^T \big\{ -(P_{h,t} - p_t(U_h), \psi) + a(P_h - p(U_h), \psi)\big\}\, dt \\ &= \int_0^T \big\{ -(P_{h,t} - p_t(U_h), \psi - \psi_I) \\ &\quad + a(P_h^{n-1} - p(U_h), \psi - \psi_I)\big\}\, dt - \int_0^T \big\{(P_{h,t} - p_t(U_h), \psi_I) \\ &\quad - a(P_h^{n-1} - p(U_h), \psi_I)\big\}\, dt + \int_0^T a(P_h - P_h^{n-1}, \psi)\, dt, \end{aligned}$$

and hence,

$$\begin{aligned}
\|P_h - p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))} &= \Big\{ \int_0^T (-P_{h,t} - \Delta P_h^{n-1} - (Y_h^n - y_{ds}^n), \psi - \psi_I)\, dt \\
&\quad + \int_0^T \sum_{E\in\mathcal{E}_h^n} \int_E \Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big] (\psi - \psi_I)\, ds\, dt \Big\} \\
&\quad + \int_0^T (Y_h^n - y(U_h), \psi)\, dt + \int_0^T (y_{ds} - y_{ds}^n, \psi)\, dt \\
&\quad + \int_0^T a(P_h - P_h^{n-1}, \psi)\, dt \\
&=: I_4 + I_5 + I_6 + I_7. \qquad (4.41)
\end{aligned}$$

Now, we estimate $I_j,\ j = 4,\ 5,\ 6,\ 7$. Inviting Lemma 4.3.1 and shape regularity of $\mathscr{T}_h^n$, we obtain

$$\begin{aligned}
|I_4| = & \Big| \sum_{n=1}^N \int_{I_n} \Big\{ (-k_n^{-1}(P_h^n - P_h^{n-1}) - \Delta P_h^{n-1} - Y_h^n + y_{ds}^n, \psi - \psi_I) \\
& + \sum_{E\in\mathcal{E}_h} \int_E \Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big] (\psi - \psi_I)\, dl \Big\} dt \Big| \\
\leq & \, C_{I,2} \max\{1, C_{I,E}\} \Big\{ \sum_{n=1}^N \int_{I_n} \Big(\sum_{K\in\mathscr{T}_h^n} h_K^2 \| - k_n^{-1}(P_h^n - P_h^{n-1}) - \Delta P_h^{n-1} \\
& - Y_h^n + y_{ds}^n \|^2_{L^2(K)} + \sum_{E\in\mathcal{E}_h^n} h_E \Big\| \Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big] \Big\|^2_{L^2(E)} \Big) dt \Big\}^{1/2} \|\psi\|_{L^2(0,T;H^1(\Omega))}. \qquad (4.42)
\end{aligned}$$

With an aid of (4.40), we get

$$\begin{aligned}
|I_4| \leq & \; c(\epsilon) C_{I,2}^2 C_{4,13}^2 \max\{1, C_{I,E}^2\} \sum_{n=1}^N k_n \Big\{ \sum_{K\in\mathscr{T}_h^n} h_K^2 \| - k_n^{-1}(P_h^n - P_h^{n-1}) \\
& - \Delta P_h^{n-1} - Y_h^n + y_{ds}^n \|^2_{L^2(K)} + \sum_{E\in\mathcal{E}_h^n} h_E \Big\| \Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big] \Big\|^2_{L^2(E)} \Big\} \\
& + \epsilon \|P_h - p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))}. \qquad (4.43)
\end{aligned}$$

We apply the Cauchy-Schwartz inequality and (4.40) to bound the terms I_5 and I_6 as

$$\begin{aligned}
|I_5| \leq & \; \|Y_h^n - y(U_h)\|_{L^2(0,T;L^2(\Omega))} \|\psi\|_{L^2(0,T;L^2(\Omega))} \\
\leq & \; c(\epsilon) C_{4,13}^2 \|Y_h^n - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))} + \epsilon \|P_h - p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))}, \qquad (4.44)
\end{aligned}$$

and

$$\begin{aligned}|I_6| \leq\ & \Big(\sum_{n=1}^{N} k_n \Big\{\frac{1}{k_n}\int_{I_n} \|y_{ds}-y_{ds}^n\|^2_{L^2(\Omega)}dt\Big\}\Big)^{1/2}\|\psi\|_{L^2(0,T;L^2(\Omega))}\\ \leq\ & c(\epsilon)C_{4,13}^2\sum_{n=1}^{N} k_n\Big\{\frac{1}{k_n}\int_{I_n}\|y_{ds}-y_{ds}^n\|^2_{L^2(\Omega)}dt\Big\} + \epsilon\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t))}. \quad (4.45)\end{aligned}$$

Finally,

$$\begin{aligned}|I_7| \leq\ & \|P_h-P_h^{n-1}\|_{L^2(0,T;H^1(\Omega))}\|\psi\|_{L^2(0,T;H^1(\Omega))}\\ \leq\ & c(\epsilon)C_{4,13}^2\sum_{n=1}^{N} k_n\|P_h^n-P_h^{n-1}\|^2_{H^1(\Omega)}+\epsilon\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))}. \quad (4.46)\end{aligned}$$

Combining (4.42) – (4.46) with (4.41) and set $C_{4,14} = 2c(\epsilon)C_{4,13}^2 \max\{C_{I,2}^2, C_{I,2}^2C_{I,E}^2, 1\}$ with $\epsilon = 1/8$, we find that

$$\begin{aligned}\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))} \leq\ & C_{4,14}\sum_{n=1}^{N} k_n\big(\mathscr{E}_{4,3}^n+\mathscr{E}_{4,4}^n+\mathscr{E}_{4,7}^n\big)\\ & + C_{4,13}^2\|Y_h^n-y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}. \quad (4.47)\end{aligned}$$

The estimation of $Y_h^n - y(U_h)$ in the $L^2(0,T;L^2(\Omega))$- norm rely on the error of $Y_h - y(U_h)$ in the $L^2(0,T;L^2(\Omega))$-norm. For this, let Φ be a solution of the problem (1.18) with $f \in L^2(0,T;L^2(\Omega))$. We apply duality argument and use the fact $\Phi = 0$ on $\partial\Omega$, $\Phi^N = \Phi(\cdot,T) = 0$, to have

$$\begin{aligned}\int_0^T (Y_h-y(U_h),f)\,dt =\ & \int_0^T \{(Y_h-y(U_h),-\Phi_t)-(Y_h-y(U_h),\Delta\Phi)\}\,dt\\ =\ & \int_0^T\{(y(U_h),\Phi_t)-(\nabla y(U_h),\nabla\Phi)\}\,dt\\ & -\sum_{n=1}^{N}\int_{I_n}\{(Y_h^n,\Phi_t)-a(Y_h^n,\Phi)\}\,dt\\ =\ & -\langle U_h\delta_{\gamma(t)},\Phi\rangle_I-(y_0-Y_h^0,\Phi(\cdot,0))\\ & +\sum_{n=1}^{N}\int_{I_n}\Big\{k_n^{-1}(Y_h^n-Y_h^{n-1},\Phi^{n-1})+a(Y_h^n,\Phi)\Big\}\,dt. \quad (4.48)\end{aligned}$$

From (4.20), we note that

$$\sum_{n=1}^{N}\Big\{k_n^{-1}(Y_h^n-Y_h^{n-1},\Phi_I)+a(Y_h^n,\Phi_I)\Big\} = \sum_{n=1}^{N}\langle U_h\delta_{\gamma(t)},\Phi_I\rangle_{I_n}. \quad (4.49)$$

Use of (4.49) in (4.48) yields

$$\begin{aligned}\int_0^T (Y_h - y(U_h), f)\,dt = & \sum_{n=1}^{N} \int_{I_n} k_n^{-1}(Y_h^n - Y_h^{n-1}, \Phi^{n-1} - \Phi_I)\,dt \\ & + \sum_{n=1}^{N} \int_{I_n} a(Y_h^n, \Phi - \Phi_I) - (y_0 - Y_h^0, \Phi(\cdot,0)) \\ & - \langle U_h \delta_{\gamma(t)}, \Phi - \Phi_I \rangle_I \\ & =: \; I_8 + I_9 + I_{10} + I_{11}. \end{aligned} \tag{4.50}$$

Now we estimate the terms I_j, $j = 8, 9, 10, 11$. An application of integration by parts formula and Lemma 4.3.1 leads to the estimation of I_8 as

$$\begin{aligned} I_8 = & \sum_{n=1}^{N} \int_{I_n} k_n^{-1}(Y_h^n - Y_h^{n-1}, \Phi^{n-1} - \mathcal{L}_n\Phi + \mathcal{L}_n(\Phi - \Pi_h^n\Phi))\,dt \\ = & \sum_{n=1}^{N} \int_{I_n} \Big\{ \int_\Omega k_n^{-1}(Y_h^n - Y_h^{n-1})(\Phi^{n-1} - \mathcal{L}_n\Phi)\,dx \\ & + \int_\Omega k_n^{-1}(Y_h^n - Y_h^{n-1})\mathcal{L}_n(\Phi - \Pi_h^n\Phi)\,dx \Big\}\,dt. \end{aligned}$$

Using (4.31), (4.32) and Lemma 1.2.3 with $f = Y_h - y(U_h)$, we obtain

$$\begin{aligned} |I_8| \le & \max\{C_{I,0}, C_{I,3}\} \Big[\sum_{n=1}^{N} k_n \Big(\|Y_h^n - Y_h^{n-1}\|^2_{L^2(\Omega)} + \sum_{K\in\mathscr{T}_h^n} h_K^4 \|k_n^{-1}(Y_h^n - Y_h^{n-1})\|^2_{L^2(K)} \Big) \Big]^{\frac{1}{2}} \\ & \times \Big(\|\Phi_t\|_{L^2(0,T;L^2(\Omega))} + \|\Phi\|_{L^2(0,T;H^2(\Omega))} \Big) \\ & \le C_R \max\{C_{I,0}, C_{I,3}\} \Big[\sum_{n=1}^{N} k_n \Big(\|Y_h^n - Y_h^{n-1}\|^2_{L^2(\Omega)} + \sum_{K\in\mathscr{T}_h^n} h_K^4 \|k_n^{-1}(Y_h^n - Y_h^{n-1})\|^2_{L^2(K)} \Big) \Big]^{\frac{1}{2}} \\ & \times \|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

and hence,

$$\begin{aligned} |I_8| \le \; & c(\epsilon) C_R^2 \max\{C_{I,0}^2, C_{I,3}^2\} \sum_{n=1}^{N} k_n \Big(\|Y_h^n - Y_h^{n-1}\|^2_{L^2(\Omega)} \\ & + \sum_{K\in\mathscr{T}_h^n} h_K^4 \|k_n^{-1}(Y_h^n - Y_h^{n-1})\|^2_{L^2(K)} \Big) + \epsilon \|Y_h - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{4.51}$$

For the term I_9, the definition of L^2-projection operator $\mathcal{L}_n$ and Lemma 1.2.3 yields

$$\begin{aligned}
|I_9| = {} & \Big|\sum_{n=1}^{N}\int_{I_n} a(Y_h^n, \Phi - \Phi_I)\,dt\Big| = \Big|\sum_{n=1}^{N}\int_{I_n} a(Y_h^n, \Phi - \Pi_h^n\Phi)\,dt\Big| \\
\leq {} & \sum_{n=1}^{N}\int_{I_n}\Big(\sum_{K\in\mathscr{T}_h^n}\|-\Delta Y_h^n\|_{L^2(K)}\|\Phi - \Pi_h^n\Phi\|_{L^2(K)} \\
& + \sum_{E\in\mathscr{E}_h^n}\Big\|\Big[\frac{\partial Y_h^n}{\partial n_E}\Big]\Big\|_{L^2(E)}\|\Phi - \Pi_h^n\Phi\|_{L^2(E)}\Big)\,dt \\
\leq {} & C_{I,0}\max\{1, C_{I,E}\}\Big[\sum_{n=1}^{N}k_n\Big(\sum_{K\in\mathscr{T}_h^n}h_K^4\|-\Delta Y_h^n\|_{L^2(K)}^2 \\
& + \sum_{E\in\mathscr{E}_h^n}h_E^3\Big\|\Big[\frac{\partial Y_h^n}{\partial n_E}\Big]\Big\|_{L^2(E)}^2\Big)\Big]^{1/2}\times\|\Phi\|_{L^2(0,T;H^2(\Omega))} \\
\leq {} & c(\epsilon)C_R^2C_{I,0}^2\max\{1, C_{I,E}^2\}\Big[\sum_{n=1}^{N}k_n\Big(\sum_{K\in\mathscr{T}_h^n}h_K^4\|-\Delta Y_h^n\|_{L^2(K)}^2 \\
& + \sum_{E\in\mathscr{E}_h^n}h_E^3\Big\|\Big[\frac{\partial Y_h^n}{\partial n_E}\Big]\Big\|_{L^2(E)}^2\Big)\Big] + \epsilon\|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2. \qquad (4.52)
\end{aligned}$$

Again using Lemma 1.2.3 with $f = Y_h - y(U_h)$ we estimate the term I_{10} as

$$\begin{aligned}
|I_{10}| & = |(y_0 - Y_h^0, \Phi(\cdot, 0))| \leq \|y_0 - Y_h^0\|_{L^2(\Omega)}\|\Phi(\cdot, 0)\|_{L^2(\Omega)} \\
& \leq c(\epsilon)C_R^2\|y_0 - Y_h^0\|_{L^2(\Omega)}^2 + \epsilon\|Y_h - y(U_h)\|_{L^2(\Omega)}^2. \qquad (4.53)
\end{aligned}$$

From the structure of I_{11} and Lemma 4.3.1 with the fact $d - r = 1$ and $r \geq 1$, we obtain

$$\begin{aligned}
|I_{11}| = {} & \Big|-\sum_{n=1}^{N}\int_{I_n}\int_{\gamma(t)} U_h(\Phi - \Phi_I)\,dxdt\Big| \leq \int_0^T\Big\{\sum_{K\in\mathcal{M}_h^n}\|U_h\|_{L^2(\gamma(t))}\|\Phi - \Phi_I\|_{L^2(\gamma(t))}\Big\}\,dt \\
\leq {} & \sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}\|U_h\|_{L^2(K)}\|\Phi - \Phi_I\|_{H^1(K)}\Big\}\,dt \\
\leq {} & C_{I,1}\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}\|U_h\|_{L^2(K)}h_K\|\Phi\|_{H^2(K)}\Big\}\,dt \\
\leq {} & C_{I,1}\Big[\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}h_K^2\|U_h\|_{L^2(K)}^2\Big\}\,dt\Big]^{1/2}\times\|\Phi\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned}$$

Invoking Lemma 1.2.3 with $f = Y_h - y(U_h)$, we find that

$$|I_{11}| \leq c(\epsilon)C_R^2C_{I,1}^2\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}h_K^2\|U_h\|_{L^2(K)}^2\Big\}\,dt + \epsilon\|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2. \qquad (4.54)$$

Combining (4.51) – (4.54) with (4.50) and setting $C_{4,15} = 2c(\epsilon)C_R^2 \max\{C_{I,0}^2, C_{I,0}^2 C_{I,E}^2, C_{I,1}^2, 1\}$ with $\epsilon = 1/8$, we obtain

$$\|Y_h - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))} \leq C_{4,15}\left\{\mathscr{E}_{4,0} + \sum_{n=1}^{N} k_n(\mathscr{E}^n_{4,2} + \mathscr{E}^n_{4,5} + \mathscr{E}^n_{4,6})\right\}. \tag{4.55}$$

It follows from the triangle inequality and (4.30) that

$$\begin{aligned}\|y - Y_h\|^2_{L^2(0,T;L^2(\Omega))} &\leq \|y - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))} + \|Y_h - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\\ &\leq C_{4,9}\,\|u - U_h\|^2_{L^2(0,T;L^2(\gamma(t)))} + \|Y_h - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned}\tag{4.56}$$

Combining (4.38), (4.47), (4.55) and (4.56) with $C_{4,11} = \max\{C_{4,12}(C_{4,9}+1), C_{4,12}C_{4,14}(C_{4,9}+1), (C_{4,12}C_{4,13}^2(C_{4,9}+1)+1)C_{4,15}\}$, we complete the rest of the proof. □

The following theorem provides the error estimates for the case $d - r > 1$ with $r = 1$ and $d = 3$. Here we use the control space as $X = L^2(0,T;L^2(\gamma(t)))$.

Theorem 4.3.2. *Let $\gamma(t)$ be an r-dimensional manifold which is completely lying inside Ω with $d - r > 1$ when $r = 1$ and $d = 3$. Further, let $(y,p,u) \in L^2(0,T;W_0^{1,s}(\Omega)) \times L^2(0,T;H_0^1(\Omega)) \times U_{ad}$ and $(Y_h, P_h, U_h) \in V_h^n \times V_h^n \times U^n_{ad,h}$ be the solutions of (4.8) – (4.11) and (4.20) – (4.24), respectively. Assume that $(P_h^{n-1} + U_h)|_K \in W^{1,\sigma'}(K)$ and there exists positive constants $C_{4,16}$, $C_{4,17}$ and $w_h \in U^n_{ad,h}$ such that*

$$\left|(U_h + P_h^{n-1}|_{\gamma(t)}, w_h - u)_X\right| \leq C_{4,16}\int_0^T \sum_{K\in\mathcal{M}_h^n} h_K^{\frac{3}{\sigma}-\frac{1}{2}}|P_h^{n-1} + U_h|_{W^{1,\sigma'}(K)}\|u - U_h\|_{L^2(K)}dt. \tag{4.57}$$

Then, the following estimate holds:

$$\|u - U_h\|^2_{L^2(0,T;L^2(\gamma(t)))} + \|y - Y_h\|^2_{L^2(0,T;L^2(\Omega))} \leq C_{4,17}\Big(\tilde{\mathscr{E}}_{4,0} + \sum_{n=1}^{N} k_n\Big(\sum_{i=1}^{7}\tilde{\mathscr{E}}^n_{4,i}\Big)\Big), \tag{4.58}$$

where

$$\begin{aligned}\tilde{\mathscr{E}}^n_{4,1} &:= \frac{1}{k_n}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n} h_K^{\frac{6}{\sigma}-1}|P_h^{n-1} + U_h|^2_{W^{1,\sigma'}(K)}\Big\}\,dt,\\ \tilde{\mathscr{E}}^n_{4,2} &:= \frac{1}{k_n}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n} h_K^{5-\frac{6}{\sigma}}\|U_h\|^2_{L^2(K)}\Big\}\,dt,\end{aligned}$$

with $\tilde{\mathscr{E}}_{4,0} = \mathscr{E}_{4,0}$ and $\tilde{\mathscr{E}}^n_{4,i} = \mathscr{E}^n_{4,i}$, $i = 3, 4, 5, 6, 7$. Here σ' is a conjugate exponent of $\sigma \in (\frac{3}{2}, 2)$.

Proof. Here we discuss the case for $r = 1$ and $d = 3$. Arguing as Theorem 4.3.1, we arrive at (4.34). We now estimate $\tilde{I}_1 (= I_1)$, $\tilde{I}_2 (= I_2)$ and $\tilde{I}_3 (= I_3)$, separately. For $\tilde{I}_1$, we note that

$$\begin{aligned}
|\tilde{I}_1| &\leq \left|(U_h + P_h^{n-1}|_{\gamma(t)}, u - w_h)_X\right| \\
&\leq C_{4,16} \sum_{n=1}^{N} \int_{I_n} \Big\{ \sum_{K \in \mathcal{M}_h^n} h_K^{\frac{3}{\sigma}-\frac{1}{2}} |P_h^{n-1} + U_h|_{W^{1,\sigma'}(K)} \|u - U_h\|_{L^2(K)} \Big\}\, dt \\
&\leq \frac{3}{4} C_{4,16}^2 \sum_{n=1}^{N} \int_{I_n} \Big\{ \sum_{K \in \mathcal{M}_h^n} h_K^{\frac{6}{\sigma}-1} |P_h^{n-1} + U_h|^2_{W^{1,\sigma'}(K)} \Big\}\, dt + \frac{1}{4} \|u - U_h\|^2_{L^2(0,T;L^2(\gamma(t)))} \\
&\leq \frac{3}{4} C_{4,16}^2 \sum_{n=1}^{N} k_n \tilde{\mathscr{E}}_{4,1}^{n} + \frac{1}{4} \|u - U_h\|^2_{L^2(0,T;L^2(\gamma(t)))}. \qquad (4.59)
\end{aligned}$$

We apply the Cauchy-Schwartz inequality with $X = L^2(0,T; L^2(\gamma(t))$ to estimate I_2,

$$\begin{aligned}
|\tilde{I}_2| &= \left|(P_h^{n-1} - p(U_h)|_{\gamma(t)}, u - U_h)_X\right| \\
&\leq C \|P_h^{n-1} - p(U_h)\|_{L^2(0,T;L^2(\gamma(t)))} \|u - U_h\|_{L^2(0,T;L^2(\gamma(t)))} \\
&\leq \frac{3}{4} \|P_h^{n-1} - p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))} + \frac{1}{4} \|u - U_h\|^2_{L^2(0,T;L^2(\gamma(t)))}. \qquad (4.60)
\end{aligned}$$

The estimation of $\tilde{I}_3$ is similar to (4.37). Altogether (4.34), (4.37), (4.59), (4.60) and setting $C_{4,18} = \frac{3}{2} \max\{C_{4,16}^2, 1\}$ we conclude that

$$\|u - U_h\|^2_{L^2(0,T;L^2(\gamma(t))} \leq C_{4,18} \Big(\sum_{n=1}^{N} k_n \tilde{\mathscr{E}}_{4,1}^{n} + \|P_h^{n-1} - p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))} \Big). \qquad (4.61)$$

To estimate the second term in the above we need the following auxiliary problem. Let ψ be the solution of

$$\begin{cases} \psi_t - \Delta \psi = (P_h - p(U_h)) \delta_{\gamma(t)}(x) \ \text{ in } \Omega_T, \\ \psi(\cdot, 0) = 0 \quad \text{in } \Omega, \\ \psi = 0 \quad \text{on } \Gamma_T. \end{cases} \qquad (4.62)$$

It follows from Theorem 4.1.1 that $\psi \in L^2(0,T; W_0^{1,\sigma}(\Omega)) \cap H^1(0,T; W^{-1,\sigma}(\Omega))$ with $\sigma \in (1,2)$ when $r = 1$, $d = 3$. Further, there exists a constant $C_{4,19} > 0$ such that ψ satisfies the following regularity result:

$$\|\psi\|_{L^2(0,T;W_0^{1,\sigma}(\Omega))} + \|\psi_t\|_{L^2(0,T;W^{-1,\sigma}(\Omega))} \leq C_{4,19} \|P_h - p(U_h)\|_{L^2(0,T;L^2(\gamma(t)))}. \qquad (4.63)$$

Following the arguments similar to the previous case (see the proof of (4.47) in Theorem 4.3.1, for $d-r=1,\ r\geq 1$) and using the regularity result (4.63) it is easy to obtain

$$\begin{aligned}\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\gamma(t)))} &\leq C_{4,20}\sum_{n=1}^{N}k_n\big(\tilde{\mathscr{E}}^n_{4,3}+\tilde{\mathscr{E}}^n_{4,4}+\tilde{\mathscr{E}}^n_{4,7}\big)\\ &\quad +C^2_{4,19}\,\|Y^n_h-y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}. \qquad (4.64)\end{aligned}$$

Next, we estimate the intermediate state error $Y_h-y(U_h)$ in the $L^2(0,T;L^2(\Omega))$-norm. We proceed as in Theorem 4.3.1 to directly reach at (4.50) with $\tilde{I}_8(=I_8)$, $\tilde{I}_9(=I_9)$, $\tilde{I}_{10}(=I_{10})$ and $\tilde{I}_{11}(=I_{11})$. Since the estimation of $\tilde{I}_8$, $\tilde{I}_9$ and $\tilde{I}_{10}$ are similar to Theorem 4.3.1, it is enough to bound the term $\tilde{I}_{11}$. An application of the trace theorem and the approximation result $\|\Phi-\Phi_I\|_{W^{1,\sigma}(\Omega)}\leq C_{4,21}\,h_K^{5/2-3/\sigma}\|\Phi\|_{H^2(\Omega)}$, $\sigma\in(\frac{3}{2},2)$ yields the bound for $\tilde{I}_{11}$ as

$$\begin{aligned}|\tilde{I}_{11}| = \Big|-\langle U_h\delta_{\gamma(t)},\Phi-\Phi_I\rangle_I\Big| &\leq \Big|\sum_{n=1}^{N}\int_{I_n}\int_{\gamma(t)}U_h(\Phi-\Phi_I)\,dx\,dt\Big|\\ \leq \sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}&\|U_h\|_{L^2(K)}\|\Phi-\Phi_I\|_{L^2(K)}\Big\}\,dt\\ \leq \sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}&\|U_h\|_{L^2(K)}\|\Phi-\Phi_I\|_{W_0^{1,\sigma}(K)}\Big\}\,dt\\ \leq C_{4,21}\Big\{\sum_{n=1}^{N}\int_{I_n}\sum_{K\in\mathcal{M}_h^n}&h_K^{5-\frac{6}{\sigma}}\|U_h\|^2_{L^2(K)}\,dt\Big\}^{1/2}\|\Phi\|_{L^2(0,T;H^2(\Omega))}\\ \leq c(\epsilon)C^2_{4,21}\sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathcal{M}_h^n}&h_K^{5-\frac{6}{\sigma}}\|U_h\|^2_{L^2(K)}\Big\}\,dt+\epsilon\|\Phi\|^2_{L^2(0,T;H^2(\Omega))}. \qquad (4.65)\end{aligned}$$

Substituting (4.51), (4.52), (4.53), (4.65) in (4.50), we use Lemma 1.2.3 with $f=Y_h-y(U_h)$ and set $\epsilon=1/8$ to have

$$\|Y_h-y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\leq C_{4,22}\Big\{\tilde{\mathscr{E}}_{4,0}+\sum_{n=1}^{N}k_n\big(\tilde{\mathscr{E}}^n_{4,2}+\tilde{\mathscr{E}}^n_{4,5}+\tilde{\mathscr{E}}^n_{4,6}\big)\Big\}. \qquad (4.66)$$

Since

$$\|y-Y_h\|^2_{L^2(0,T;L^2(\Omega))}\leq \|y-y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}+\|Y_h-y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}, \qquad (4.67)$$

utilizing (4.29), (4.61), (4.64), (4.66) and (4.67), the desired result is obtained. This completes the proof. □

In the next theorem, we consider the case $d-r>1$ with $r=0$, $d=2$ or 3 and the control space $X=L^2(0,T;\mathbb{R}^m)$.

Theorem 4.3.3. *Let $\gamma(t)$ be an r-dimensional manifold which is completely lying inside Ω with $d-r>1$ when $r=0$ and $d=2$ or 3. Further, let $(y,p,u)\in L^2(0,T;W_0^{1,s}(\Omega))\times L^2(0,T;H_0^1(\Omega))\times U_{ad}$ and $(Y_h,P_h,U_h)\in V_h^n\times V_h^n\times U_{ad,h}^n$ be the solutions of $(4.8)-(4.11)$ and $(4.20)-(4.24)$, respectively. Assume that $(P_h^{n-1}+U_h)|_K\in L^\infty(K)$ and there exists positive constants $C_{4,23}$ and $w_h\in U_{ad,h}^n$ such that*

$$\Big|(U_h+P_h^{n-1}|_{\gamma(t)},w_h-u)_X\Big| \;\le\; C_{4,23}\int_0^T\sum_{K\in\mathcal{M}_h^n}\rho^{-1}_{(d,h_K)}\|P_h^{n-1}+U_h\|_{L^\infty(K)}\|u-U_h\|_{\mathbb{R}^m}dt. \tag{4.68}$$

Further, with $y_{ds}\in L^2(0,T;L^\infty(\Omega))$, there exists a positive constant $C_{4,24}$ such that the following estimates hold:

$$\begin{aligned}
&\|u-U_h\|^2_{L^2(0,T;\mathbb{R}^m)}+\|y-Y_h\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad\le\; C_{4,24}\Big(\widehat{\mathscr{E}}_{4,0}+\sum_{n=1}^{N}k_n\big(\widehat{\mathscr{E}}^n_{4,1}+\widehat{\mathscr{E}}^n_{4,8}\big)+\sum_{n=1}^{N}k_nh_K^{-d}\big(\widehat{\mathscr{E}}^n_{4,3}+\widehat{\mathscr{E}}^n_{4,4}+\widehat{\mathscr{E}}^n_{4,7}\big)\\
&\quad\;+\;\sum_{n=1}^{N}k_n\big(1+h_K^{-d}\big)\big(\widehat{\mathscr{E}}^n_{4,2}+\widehat{\mathscr{E}}^n_{4,5}+\widehat{\mathscr{E}}^n_{4,6}\big)\Big),
\end{aligned}\tag{4.69}$$

where

$$\widehat{\mathscr{E}}^n_{4,1} \;=\; \frac{1}{k_n}\int_{I_n}\Big(\sum_{K\in\mathcal{M}_h^n}\rho^{-2}_{(d,h_K)}\|P_h^{n-1}+U_h\|^2_{L^\infty(K)}\Big)\,dt,$$

with

$$\rho_{(d,h_K)} \;=\; \begin{cases}\sqrt{|\log h_K|} & d=2,\\ h_K^{-1/2} & d=3,\end{cases}$$

$$\begin{aligned}
\widehat{\mathscr{E}}^n_{4,2} &= \frac{1}{k_n}\int_{I_n}h_K^{4-d}\|U_h\|^2_{\mathbb{R}^m}dt,\\
\widehat{\mathscr{E}}^n_{4,3} &= \sum_{K\in\mathscr{T}_h^n}h_K^4\|-k_n^{-1}(P_h^n-P_h^{n-1})-\Delta P_h^{n-1}-Y_h^n+y_{ds}^n\|_{L^2(K)}\\
&\quad+\sum_{E\in\mathcal{E}_h}h_E^3\Big\|\Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big]\Big\|_{L^2(E)},\\
\widehat{\mathscr{E}}^n_{4,4} &= \|P_h^n-P_h^{n-1}\|^2_{H^1(\Omega)},\\
\widehat{\mathscr{E}}^n_{4,5} &= \|Y_h^n-Y_h^{n-1}\|^2_{L^2(\Omega)}+\sum_{K\in\mathscr{T}_h^n}h_K^4\|k_n^{-1}(Y_h^n-Y_h^{n-1})\|^2_{L^2(K)},
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathscr{E}}^n_{4,6} &= \sum_{K\in\mathscr{T}^n_h} h^4_K\|-\Delta Y^n_h\|^2_{L^2(K)} + \sum_{E\in\mathcal{E}^n_h} h^3_E\Big\|\Big[\frac{\partial Y^n_h}{\partial n_E}\Big]\Big\|^2_{L^2(E)},\\
\widehat{\mathscr{E}}^n_{4,7} &= \frac{1}{k_n}\int_{I_n}\|y_{ds}-y^n_{ds}\|^2_{L^2(\Omega)}dt,\\
\widehat{\mathscr{E}}^n_{4,8} &= \frac{1}{k_n}\int_{I_n} h^{4-d}_K\|y_{ds}\|^2_{L^\infty(\Omega)}dt,\\
\widehat{\mathscr{E}}_{4,0} &= \big(h^{4-d}_K + h^4_K\big)\|y_0\|^2_{H^1(\Omega)},
\end{aligned}$$

Proof. We proceed as in Theorem 4.3.1 to obtain (4.34) with $\widehat{I}_1(=I_1)$, $\widehat{I}_2(=I_2)$, and $\widehat{I}_3(=I_3)$. To estimate $\widehat{I}_1$, use (4.68) to have

$$\begin{aligned}
|\widehat{I}_1| &\le \Big|(U_h+P^{n-1}_h|_{\gamma(t)}, u-w_h)_X\Big|\\
&\le C_{4,23}\sum_{n=1}^N\int_{I_n}\sum_{K\in\mathcal{M}^n_h}\rho^{-1}_{(d,h_K)}\|P^{n-1}_h+U_h\|_{L^\infty(K)}\|u-U_h\|_{\mathbb{R}^m}dt\\
&\le \frac{3}{4}C^2_{4,23}\sum_{n=1}^N\int_{I_n}\Big(\sum_{K\in\mathcal{M}^n_h}\rho^{-2}_{(d,h_K)}\|P^{n-1}_h+U_h\|^2_{L^\infty(K)}\Big)\,dt+\frac{1}{4}\|u-U_h\|^2_X\\
&\le \frac{3}{4}C^2_{4,23}\sum_{n=1}^N k_n\widehat{\mathscr{E}}^n_{4,1}+\frac{1}{4}\|u-U_h\|^2_X.
\end{aligned}$$

Moreover, it is clear that

$$\begin{aligned}
|\widehat{I}_2| &= |((P_h-p(U_h))|_{\gamma(t)}, u-U_h)_X|\\
&\le \frac{3}{4}\|P_h-p(U_h)\|^2_{L^2(0,T;L^\infty(\Omega))}+\frac{1}{4}\|u-U_h\|^2_X.
\end{aligned}$$

Now, it remains to estimate $\widehat{I}_3$. Analogous to (4.37), and using the fact $y(x,0)-y(U_h)(x,0)=0=p(x,T)-p(U_h)(x,T)$ we find that

$$\widehat{I}_3 \le 0.$$

Substituting the bounds of $\widehat{I}_1$, $\widehat{I}_2$, and $\widehat{I}_3$ in (4.34) we obtain

$$\|u-U_h\|^2_{L^2(0,T;\mathbb{R}^m)} \le C_{4,25}\Big(\sum_{n=1}^N k_n\widehat{\mathscr{E}}^n_{4,1}+\|P_h-p(U_h)\|^2_{L^2(0,T;L^\infty(\Omega))}\Big), \tag{4.70}$$

where $C_{4,25}=\frac{3}{2}\max\{C^2_{4,23},1\}$.

Note that we have found the estimate of $\|u-U_h\|_{L^2(0,T;\mathbb{R}^m)}$ in terms of $\widehat{\mathscr{E}}^n_{u,1}$ and $\|P_h-p(U_h)\|_{L^2(0,T;L^\infty(\Omega))}$. By the triangle inequality, we have

$$\|P_h-p(U_h)\|^2_{L^2(0,T;L^\infty(\Omega))} \le \|p(U_h)-\Pi^n_h p(U_h)\|^2_{L^2(0,T;L^\infty(\Omega))}+\|P_h-\Pi^n_h p(U_h)\|_{L^2(0,T;L^\infty(\Omega))}.$$

Using Lemma 4.3.1 and inverse estimate $\|v_h\|_{L^\infty(\Omega)} \leq C_{4,26}\, h_K^{-d/2}\|v_h\|_{L^2(\Omega)},\ \forall v_h \in V_h^n$ and the stability results from [51] to obtain

$$\begin{aligned}
&\|P_h - p(U_h)\|^2_{L^2(0,T;L^\infty(\Omega))}\\
&\quad\leq\ C_{I,\infty}h_K^{4-d}\|p(U_h)\|^2_{L^2(0,T;H^2(\Omega))} + C_{4,26}h_K^{-d}\|P_h - \Pi_h^n p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad\leq\ C_{I,\infty}h_K^{4-d}\|p(U_h)\|^2_{L^2(0,T;H^2(\Omega))} + C_{4,26}h_K^{-d}\big(\|P_h - p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\\
&\qquad + \|p(U_h) - \Pi_h^n p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\big)\\
&\quad\leq\ C_{4,27}h_K^{4-d}\big(\|U_h\|^2_{L^2(0,T;\mathbb{R}^m)} + \|y_{ds}\|^2_{L^2(0,T;L^\infty(\Omega))}\big)\\
&\qquad + C_{4,26}h_K^{-d}\|P_h - p(U_h)\|^2_{L^2(0,T;L^2(\Omega))},
\end{aligned} \tag{4.71}$$

where $C_{4,27} = C_{I,\infty}(1+C_{4,26})$. Next, we estimate the error $P_h - p(U_h)$ in the $L^2(0,T;L^2(\Omega))$-norm. Use of duality technique and setting $f = P_h - p(U_h)$ it follows that

$$\begin{aligned}
&\int_0^T \|P_h - p(U_h)\|^2_{L^2(\Omega)}dt\\
&= \int_0^T (P_h - p(U_h), \Psi_t - \Delta\Psi)dt = -\int_0^T (P_{h,t} - p_t(U_h), \Psi)dt + \int_0^T a(P_h - p(U_h), \Psi)dt\\
&= -\int_0^T (P_{h,t} - p_t(U_h), \Psi - \Psi_I)dt + \int_0^T a(P_h^{n-1} - p(U_h), \Psi - \Psi_I)dt\\
&\quad -\int_0^T (P_{h,t} - p_t(U_h), \Psi_I)dt + \int_0^T a(P_h^{n-1} - p(U_h), \Psi_I)dt + \int_0^T a(P_h - P_h^{n-1}, \Psi)dt\\
&= \int_0^T (-P_{h,t} - Y_h^n + y_{ds}^n, \Psi - \Psi_I)dt + \int_0^T a(P_h^{n-1}, \Psi - \Psi_I)dt + \int_0^T (Y_h^n - y(U_h), \Psi)dt\\
&\quad + \int_0^T (y_{ds} - y_{ds}^n, \Psi)dt + \int_0^T a(P_h - P_h^{n-1}, \Psi)dt.
\end{aligned} \tag{4.72}$$

Thus we have

$$\begin{aligned}
\int_0^T \|P_h - p(U_h)\|^2_{L^2(\Omega)}dt =\ & \int_0^T \Big\{(-P_{h,t} - \Delta P_h^{n-1} - Y_h^n + y_{ds}^n, \Psi - \Psi_I)\\
& + \sum_{E\in\mathcal{E}_h}\int_E \Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big](\Psi - \Psi_I)\, dl\Big\}dt + \int_0^T (Y_h^n - y(U_h), \Psi)dt\\
& + \int_0^T (y_{ds} - y_{ds}^n, \Psi)dt + \int_0^T a(P_h - P_h^{n-1}, \Psi)dt\\
=:\ & I_{12} + I_{13} + I_{14} + I_{15}.
\end{aligned} \tag{4.73}$$

We now estimate each terms of the equation (4.73), separately. Application of the Cauchy-Schwarz inequality, Lemmas 4.3.1 − 4.3.2 and Lemma 1.2.3 with $f = P_h - p(U_h)$

yield the estimate of I_{12} as,

$$\begin{aligned}I_{12} &\leq \sum_{n=1}^{N}\int_{I_n}\Big\{\sum_{K\in\mathscr{T}_h^n}\|-k_n^{-1}(P^n-P_h^{n-1})-\Delta P_h^{n-1}-Y_h^n+y_{ds}^n\|_{L^2(K)}\|\Psi-\Psi_I\|_{L^2(K)}\\&\quad+\sum_{E\in\mathcal{E}_h}\Big\|\Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big]\Big\|_{L^2(E)}\|\Psi-\Psi_I\|_{L^2(E)}\Big\}dt\\&\leq\sum_{n=1}^{N}\int_{I_n}\Big\{C_{I,0}\sum_{K\in\mathscr{T}_h^n}h_K^2\|-k_n^{-1}(P^n-P_h^{n-1})-\Delta P_h^{n-1}-Y_h^n+y_{ds}^n\|_{L^2(K)}\|\Psi\|_{H^2(K)}\\&\quad+C_{I,0}C_{I,E}\sum_{E\in\mathcal{E}_h}h_E^{3/2}\Big\|\Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big]\Big\|_{L^2(E)}\|\Psi\|_{H^2(K)}\Big\}dt\\&\leq C(\epsilon)C_R^2C_{I,0}^2\max\{1,C_{I,E}^2\}\sum_{n=1}^{N}k_n\Big\{\sum_{K\in\mathscr{T}_h^n}h_K^4\|-k_n^{-1}(P^n-P_h^{n-1})-\Delta P_h^{n-1}-Y_h^n\\&\quad+y_{ds}^n\|_{L^2(K)}+\sum_{E\in\mathcal{E}_h}h_E^3\Big\|\Big[\frac{\partial P_h^{n-1}}{\partial n_E}\Big]\Big\|_{L^2(E)}\Big\}dt+\epsilon\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}\tag{4.74}$$

To estimate the term I_{13}, we apply the Cauchy-Schwarz inequality to have

$$\begin{aligned}I_{13} &\leq \int_0^T\|Y_h^n-y(U_h)\|_{L^2(\Omega)}\|\Psi\|_{L^2(\Omega)}dt\\&\leq C(\epsilon)C_R^2\int_0^T\|Y_h^n-y(U_h)\|^2_{L^2(\Omega)}dt+\epsilon\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}\tag{4.75}$$

Arguing as I_{13}, we estimate I_{14} as

$$I_{14} \leq C(\epsilon)C_R^2\sum_{n=1}^{N}\int_{I_n}\|y_{ds}-y_{ds}^n\|^2_{L^2(\Omega)}dt+\epsilon\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}.\tag{4.76}$$

Finally, using the boundedness property of the bilinear form and $|\frac{t-t_{n-1}}{k_n}|<1,\ \forall t_n\in I_n,\ n\in[1:N]$, we estimate the last term as

$$\begin{aligned}I_{15} &\leq \alpha_0\int_0^T\|P_h-P_h^{n-1}\|_{H^1(\Omega)}\|\Psi\|_{H^1(\Omega)}dt\\&\leq C(\epsilon)C_R^2\alpha_0^2\sum_{n=1}^{N}k_n\|P_h^n-P_h^{n-1}\|^2_{H^1(\Omega)}dt+\epsilon\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}\tag{4.77}$$

Combining (4.74) – (4.77) with (4.73) and setting with $C_{4,28}=2C(\epsilon)C_R^2\max\big\{C_{I,0},C_{I,0}C_{I,E},\alpha_0^2,1\big\}$ with $\epsilon=1/8$, we obtain

$$\begin{aligned}\|P_h-p(U_h)\|^2_{L^2(0,T;L^2(\Omega))} &\leq C_{4,28}\Big(\sum_{n=1}^{N}k_n\{\widehat{\mathscr{E}}_{4,3}^n+\widehat{\mathscr{E}}_{4,4}^n+\widehat{\mathscr{E}}_{4,7}^n\}\\&\quad+\|Y_h^n-y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}\Big).\end{aligned}\tag{4.78}$$

Combine (4.71) and (4.78) with $C_{4,29} = C_{4,26}C_{4,28}$ to have

$$\begin{aligned}\|P_h - p(U_h)\|^2_{L^2(0,T;L^\infty(\Omega))} \le\ & C_{4,27}h_K^{4-d}\big(\|U_h\|^2_{L^2(0,T;\mathbb{R}^m)} + \|y_{ds}\|^2_{L^2(0,T;L^\infty(\Omega))}\big)\\ & + C_{4,29}\sum_{n=1}^{N} k_n h_K^{-d}\{\widehat{\mathscr{E}}^n_{4,3} + \widehat{\mathscr{E}}^n_{4,4} + \widehat{\mathscr{E}}^n_{4,7}\}\\ & + C_{4,29}h_K^{-d}\|Y_h^n - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{4.79}$$

Next we estimate the error $Y_h^n - y(U_h)$ in the norm $L^2(0,T;L^2(\Omega))$. For this, we invoke the duality argument to have

$$\int_0^T (Y_h - y(U_h), f)dt = \int_0^T (Y_h - y(U_h), -\Phi_t - \Delta\Phi)dt.$$

Note that $\Phi = 0$ on Γ_T and $\Phi(\cdot,T) = \Phi^N = 0$. From (4.25) with $\hat{u}_h = U_h$ and integrating by parts, it follows that

$$\begin{aligned}\int_0^T (Y_h - y(U_h), f)dt =\ & \int_0^T \{(y(U_h),\Phi_t) + (\nabla y(U_h), \nabla\Phi)\}dt\\ & - \sum_{n=1}^{N}\int_{I_n}\{(Y_h^n,\Phi_t)dt - a(Y_h^n,\Phi)\}dt\\ =\ & -\langle U_h\delta_{\gamma(t)},\Phi\rangle_I - (y_0 - Y_h^0, \Phi(\cdot,0))\\ & + \sum_{n=1}^{N}\int_{I_n}\{k_n^{-1}(Y_h^n - Y_h^{n-1}, \Phi^{n-1}) + a(Y_h^n,\Phi)\}dt.\end{aligned}$$

We proceed as in (4.50) to arrive at

$$\begin{aligned}\int_0^T (Y_h - y(U_h), f)\,dt =\ & \sum_{n=1}^{N}\int_{I_n} k_n^{-1}(Y_h^n - Y_h^{n-1}, \Phi^{n-1} - \Phi_I)\,dt + \sum_{n=1}^{N}\int_{I_n} a(Y_h^n, \Phi - \Phi_I)\\ & - (y_0 - Y_h^0, \Phi(\cdot,0)) - \langle U_h\delta_{\gamma(t)}, \Phi - \Phi_I\rangle_I\\ =:\ & I_{16} + I_{17} + I_{18} + I_{19}.\end{aligned} \tag{4.80}$$

It remains to estimate I_j, $j = 16, 17, 18, 19$. Following the arguments of I_8 and I_9, and using Lemma 1.2.3 with $f = Y_h - y(U_h)$ we estimate the terms I_{16} and I_{17} as

$$\begin{aligned}I_{16} \le\ & C(\epsilon)C_R^2\max\{C_{I,0}^2, C_{I,3}^2\}\sum_{n=1}^{N} k_n\Big(\|Y_h^n - Y_h^{n-1}\|^2_{L^2(\Omega)}\\ & + \sum_{K\in\mathscr{T}_h^n} h_K^4\|k_n^{-1}(Y_h^n - Y_h^{n-1})\|^2_{L^2(K)}\Big) + \epsilon\|Y_h - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))},\end{aligned} \tag{4.81}$$

and

$$\begin{aligned} I_{17} &\le C(\epsilon)C_R^2 C_{I,0}\max\{1, C_{I,E}^2\}\Big[\sum_{n=1}^{N} k_n\Big(\sum_{K\in\mathscr{T}_h^n} h_K^4\| -\Delta Y_h^n\|_{L^2(K)}^2 \\ &\quad + \sum_{E\in\mathcal{E}_h^n} h_E^3\Big\|\Big[\frac{\partial Y_h^n}{\partial n_E}\Big]\Big\|_{L^2(E)}^2\Big)\Big] + \epsilon\|Y_h - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{4.82}$$

Let $\mathcal{L}_h^0$ be the L^2-projection operator form $L^2(\Omega)$ to V_h^0 such that $(\mathcal{L}_h^0 v, w) = (v, w)$, $\forall w \in V_h^0$. Then the following approximation property holds:

$$\|v - \mathcal{L}_h^0 v\|_{H^{-1}(\Omega)} + h_K\|v - \mathcal{L}_h^0 v\|_{L^2(\Omega)} \le C_{4,30}\, h_K^2\|v\|_{H^1(\Omega)}. \tag{4.83}$$

By setting $Y_h^0 = \mathcal{L}_h^0 y_0$ and using (4.83), we estimate the term I_{18} as

$$\begin{aligned} I_{18} &\le \|y_0 - Y_h^0\|_{H^{-1}(\Omega)}\|\Phi(\cdot,0)\|_{H^1(\Omega)} \\ &\le C(\epsilon)C_R^2 C_{4,30}^2 h_K^4\|y_0\|^2_{H^1(\Omega)} + \epsilon\|Y_h - y(U_h)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.84}$$

Finally, for the last term, we use Lemma 4.3.1 to have

$$\begin{aligned} I_{19} &= \Big| -\sum_{n=1}^{N}\int_{I_n}\sum_{j=1}^{m}(U_h)_j(t)(\Phi - \Phi_I)(\gamma_j(t))\,dxdt\Big| \\ &\le \Big(\sum_{n=1}^{N}\int_{I_n}\|U_h\|^2_{\mathbb{R}^m}dt\Big)^{1/2}\|\Phi - \Phi_I\|_{L^2(0,T;L^\infty(\Omega))} \\ &\le C_{I,\infty}h_K^{2-d/2}\Big(\sum_{n=1}^{N}\int_{I_n}\|U_h\|^2_{\mathbb{R}^m}dt\Big)^{1/2}\|\Phi\|_{L^2(0,T;H^2(\Omega))} \\ &\le C(\epsilon)C_R^2 C_{I,\infty}^2 h_K^{4-d}\Big(\sum_{n=1}^{N}\int_{I_n}\|U_h\|^2_{\mathbb{R}^m}dt\Big) + \epsilon\|Y_h - y(U_h)\|^2_{L^2(0,T;H^2(\Omega))}. \end{aligned} \tag{4.85}$$

Combining (4.81) – (4.85) with (4.80), and by setting $C_{4,31} = 2C(\epsilon)C_R^2\max\{C_{I,0}^2, C_{I,3}^2, C_{I,0}^2C_{I,E}^2, C_{I,\infty}^2, C_{4,30}^2\}$ with $\epsilon = 1/8$, we arrive at

$$\|Y_h - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))} \le C_{4,31}\Big(h_K^4\,\|y_0\|^2_{H^1(\Omega)} + \sum_{n=1}^{N}k_n\big(\widehat{\mathscr{E}}_{4,2}^n + \widehat{\mathscr{E}}_{4,5}^n + \widehat{\mathscr{E}}_{4,6}^n\big)\Big). \tag{4.86}$$

By the triangle inequality and (4.28), we have

$$\begin{aligned} &\|u - U_h\|^2_{L^2(0,T;\mathbb{R}^m)} + \|y - Y_h\|^2_{L^2(0,T;L^2(\Omega))} \\ &\le \|u - U_h\|^2_{L^2(0,T;\mathbb{R}^m)} + \|y - y(U_h)\|^2_{L^2(0,T;L^2(\Omega))} + \|y(U_h) - Y_h\|^2_{L^2(0,T;L^2(\Omega))} \\ &\le C_{3,32}\,\|u - U_h\|^2_{L^2(0,T;\mathbb{R}^m)} + \|y(U_h) - Y_h\|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

with $C_{4,32} = 1 + C_{4,7}$. Using (4.70), (4.79) and (4.86) in the above equation, and adjustment of the constants yields

$$\begin{aligned}
&\|u - U_h\|^2_{L^2(0,T;\mathbb{R}^m)} + \|y - Y_h\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad\leq\; C_{4,24} \Big\{ (h_K^{4-d} + h_K^4)\|y_0\|^2_{H^1(\Omega)} + \sum_{n=1}^{N} k_n \big(\widehat{\mathscr{E}}^n_{4,1} + \widehat{\mathscr{E}}^n_{y_{4,8}}\big) + \sum_{n=1}^{N} k_n h_K^{-d} \big(\widehat{\mathscr{E}}^n_{4,3} + \widehat{\mathscr{E}}^n_{4,4} + \widehat{\mathscr{E}}^n_{4,7}\big) \\
&\qquad + \sum_{n=1}^{N} k_n (1 + h_K^{-d}) \big(\widehat{\mathscr{E}}^n_{4,2} + \widehat{\mathscr{E}}^n_{4,5} + \widehat{\mathscr{E}}^n_{4,6}\big) \Big\},
\end{aligned}$$

where $C_{4,24} = C_{4,25}\, C_{4,32} \max\{1, C_{4,28}, C_{4,29}, C_{4,28}C_{4,31}\}$. This completes the rest of the proof. □

4.4 Numerical assessments

In this section, we perform numerical experiments to illustrate our theoretical results of the previous section. The simulation is carried out with the help of the software FreeFem++ [38] and all the constants involved in the estimators are taken to be 1. We have implemented the following steps

$$SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE$$

to acquire the refinement from the initializing triangulation. The construction of time-step size and the mesh refinement algorithm is based on the error equi-distribution strategy. We now define some notations for the purpose of adaptive algorithm. Set

$$\begin{aligned}
\mathscr{E}^n_{space} &:= \mathscr{E}^n_{4,1} + \mathscr{E}^n_{4,2} + \mathscr{E}^n_{4,3} + \mathscr{E}^n_{4,5} + \mathscr{E}^n_{4,7}, \\
\mathscr{E}^n_{time} &:= \mathscr{E}^n_{4,4} + \mathscr{E}^n_{4,6}, \\
\mathscr{E}_{initial} &:= \mathscr{E}_{4,0}
\end{aligned}$$

for Theorem 4.3.1. Similarly, we set $\mathscr{E}^n_{space} := \tilde{\mathscr{E}}^n_{4,1} + \tilde{\mathscr{E}}^n_{4,2} + \tilde{\mathscr{E}}^n_{4,3} + \tilde{\mathscr{E}}^n_{4,5} + \tilde{\mathscr{E}}^n_{4,7}$, $\mathscr{E}^n_{time} := \tilde{\mathscr{E}}^n_{4,4} + \tilde{\mathscr{E}}^n_{4,6}$, $\mathscr{E}_{initial} := \tilde{\mathscr{E}}_{y_0}$ for Theorem 4.3.2, and $\mathscr{E}^n_{space} := \widehat{\mathscr{E}}^n_{4,1} + \widehat{\mathscr{E}}^n_{4,2} + \widehat{\mathscr{E}}^n_{4,3} + \widehat{\mathscr{E}}^n_{4,6} + \widehat{\mathscr{E}}^n_{4,7} + \widehat{\mathscr{E}}^n_{4,8}$, $\mathscr{E}^n_{time} := \widehat{\mathscr{E}}^n_{4,4} + \widehat{\mathscr{E}}^n_{4,5}$, $\mathscr{E}_{initial} := \widehat{\mathscr{E}}_{y_0}$ for Theorem 4.3.3. The space-time adaptive algorithm is presented below.

The role of Step 2 in Algorithm 4.1 is to reduce the time step size to keep the time error estimator below the tolerance ϵ_{time} while keeping the space mesh unchanged. In Step 3, the refinement procedure is carried out until the time and space error estimators satisfy the desired tolerances. In the last step, if the time error estimator is much less

Algorithm 4.1 (Space-time adaptive algorithm)

Given tolerances ϵ_{tpace}, ϵ_{time} and the parameters $\delta_1 \in (0,1)$, $\delta_2 > 1$, $\lambda_1 > 0$, $\lambda_2 \in (0, \lambda_1)$. Suppose that $(y_h^{n-1}, p_h^n, u_h^{n-1})$ is computed on the mesh $\mathscr{T}_h^{n-1}$ at the time level t_{n-1} with time step size k_{n-1}.

```
Step 1.  Set  $\mathscr{T}_h^n := \mathscr{T}_h^{n-1}$,  $k_n := k_{n-1}$,  $t_n := t_{n-1} + k_n$.
             Compute $(y_h^n, p_h^{n-1}, u_h^n)$ by solving the discrete problem on $\mathscr{T}_h^n$.
             Compute all the estimators on $\mathscr{T}_h^n$.
Step 2.  While  $\mathscr{E}_{time}^n > \frac{\epsilon_{time}}{T}$,  do
             $k_n := \delta_1 k_{n-1}$,  $t_n := t_{n-1} + k_n$.
             Compute $(y_h^n, p_h^{n-1}, u_h^n)$ by solving the discrete problem on $\mathscr{T}_h^n$.
             Compute all the estimators on $\mathscr{T}_h^n$.
         End while
Step 3.  While  $\mathscr{E}_{space}^n > \frac{\epsilon_{space}}{T}$,  do
             Refine mesh $\mathscr{T}_h^n$ and generate the new modified mesh (say) $\mathscr{T}_{h_n}^n$.
             Compute $(y_h^n, p_h^{n-1}, u_h^n)$ by solving the discrete problem on $\mathscr{T}_{h_n}^n$.
             Compute all the estimators on $\mathscr{T}_{h_n}^n$.
           While  $\mathscr{E}_{time}^n > \frac{\epsilon_{time}}{T}$,  do
               $k_n^{'} := \delta_1 k_{n-1}$,  $t_n := t_{n-1} + k_n^{'}$.
               Compute $(y_h^n, p_h^{n-1}, u_h^n)$ by solving the discrete problem on $\mathscr{T}_{h_n}^{n,k_n^{'}}$.
               Compute all the estimators on $\mathscr{T}_{h_n}^{n,k_n^{'}}$.
           End while
         End while
Step 4.  If  $\mathscr{E}_{time}^n \leq \lambda_2 \frac{\epsilon_{time}}{T}$,  do
             Set $k_n^{'} := \delta_2 k_{n-1}$,  $t_n := t_{n-1} + k_n^{'}$.
         End if
```

than the prescribe time tolerance ϵ_{time} then we increase the time step size by multiplying a factor δ_2. We use MNS strategy [75] for marking the elements $K \in \mathscr{T}_h^n$ in the refinement process.

We have considered two test examples from [35] for numerical verification of our estimators. The first example considers the optimal control problem with lower dimensional manifold as a moving point $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ whereas the second one with moving interval. For the space discretization, we use the piecewise linear and continuous finite elements for the state and co-state variables, while piecewise constant functions are employed for the control variable. Further, we approximate the time derivative using the backward Euler method. We partition the time interval $[0,1]$ and set time step-size $k_n = \mathcal{O}(h_K^2)$ to achieve the optimal order *a posteriori* error estimates. For both the test examples, we choose time and space tolerances as $\epsilon_{time} = \epsilon_{space} = 10^{-4}$. Adaptive meshes generated via Algorithm 4.1 show the effectiveness of our derived estimators.

Example 4.1. *Consider the optimal control problem* $(4.1)-(4.3)$ *with* $\Omega_T = B(0,1) \times [0,1]$, *where* $B(0,1)$ *denotes the circle with radius* 1 *and center at the origin,* $\gamma(t) = (\gamma_1(t), \gamma_2(t)) = (0.5\cos(2\pi t), 0.5\sin(2\pi t))$ *is a moving point (*$d = 2$ *and* $r = 0$*) in* Ω *and* $\alpha = 1$. *The datum for the problem are taken as follow.*

$$\begin{aligned} y &= \frac{(e^t - e)}{2\pi}\log|x-\gamma(t)|, \\ p &= (e^t - e)\sin(2\pi|x|^2)\cos\left(\frac{\pi}{2}|x-\gamma(t)|^2\right), \\ u &= \mathscr{P}_{U_{ad}}(e^t - e), \end{aligned}$$

with corresponding y_{ds}. *Here the control space is*

$$U_{ad} := \left\{u \in L^2[0,T] \ : \ u_a \leq u(t) \leq u_b, \ \ a.e. \ \ t \in [0,T]\right\},$$

with $u_a = -0.005$ *and* $u_b = 0.0125$.

Table 4.1 shows the mesh information with error estimates for the state and control variable in the $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;\mathbb{R}^m)$ norms, respectively, on the uniform mesh, adaptive mesh-(I) and adaptive mesh-(II). While Figure 4.1 contains the uniform mesh, Figures 4.2 $(i)-(iv)$ depict the adaptive meshes at the different time levels $t = 0.3333,\ 0.5632,\ 0.7219,$ and 0.9889. From Figures 4.2 $(i)-(iv)$, it is observed that mesh adapts very well near the singularity and moves with the point $\gamma(t)$ at different time levels. We plot the approximate state solution at different time levels $t = 0.3333,\ 0.5632,\ 0.7219$ and 0.9889 for the corresponding adaptive meshes (Step-(II)) in Figure 4.3 $(i)-(iv)$.

Table 4.1: *Error estimates for the state and co-state variables in the* $L^2(0,T;L^2(\Omega))$*-norm whereas the error for the control variable in the* $L^2(0,T;\mathbb{R}^m)$*-norm, on the uniform mesh, adaptive mesh Step-(I) and adaptive mesh Step-(II), respectively.*

Time levels	Meshes	# Elements	# Nodes	$y-Y_h$	$p-P_h$	$u-U_h$
$t=0.3333$	Uniform	43200	21851	1.7791e-02	5.8776e-02	4.4918e-02
	Adapt-(I)	14878	7488	1.7881e-03	9.3737e-03	3.0029e-03
	Adapt-(II)	6821	3446	8.5483e-04	4.6054e-03	1.4060e-04
$t=0.5632$	Uniform	43200	21851	1.2054e-02	1.7889e-02	1.3409e-02
	Adapt-(I)	15395	7743	1.2209e-03	9.2882e-03	8.5660e-03
	Adapt-(II)	7059	3563	4.9037e-04	5.0179e-04	8.0788e-04
$t=0.7219$	Uniform	43200	21851	1.7644e-02	3.8195e-02	1.3336e-02
	Adapt-(I)	15130	7613	3.0903e-03	3.1066e-03	1.1632e-02
	Adapt-(II)	7025	3547	7.5919e-04	1.0676e-03	1.1193e-03
$t=0.9889$	Uniform	43200	21851	2.9303e-02	8.0552e-02	1.3219e-02
	Adapt-(I)	15116	7606	5.5206e-03	1.3617e-02	1.4529e-03
	Adapt-(II)	6873	3471	1.9250e-04	1.4022e-03	1.2674e-03

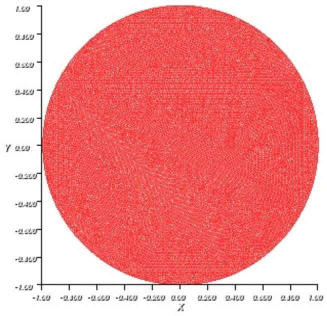
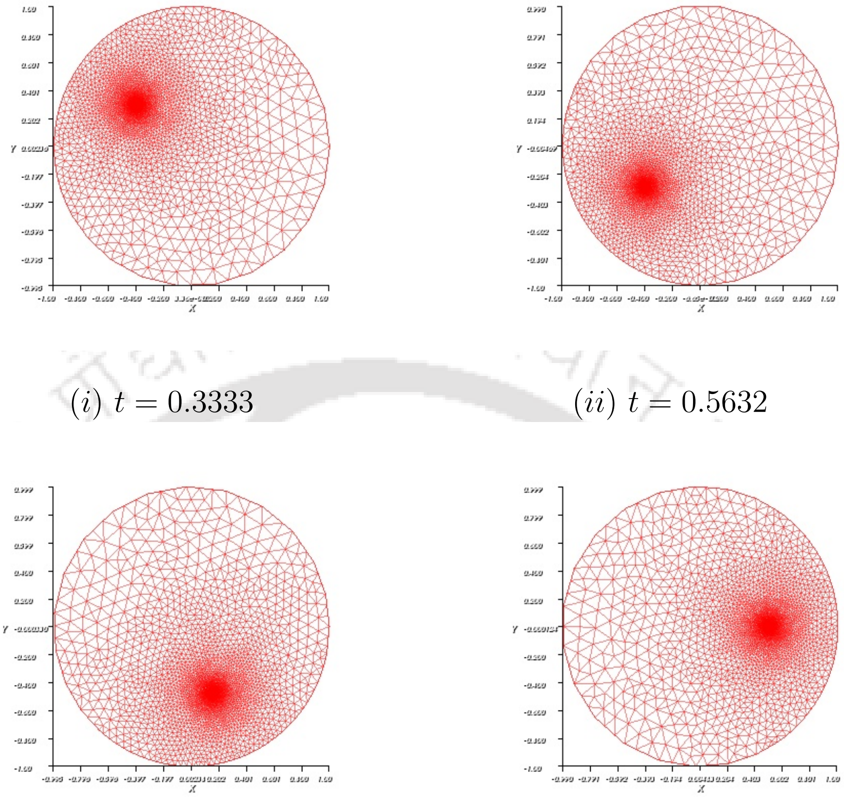


Figure 4.1: *Uniform mesh*



(i) $t = 0.3333$ (ii) $t = 0.5632$

(iii) $t = 0.7219$ (iv) 0.9889

Figure 4.2: *Adaptive meshes* $(i)-(iv)$ *(Step-(II)) at the different time levels* $t = 0.3333,\ 0.5632,\ 0.7219,\ 0.9889$ *for Example 4.1.*

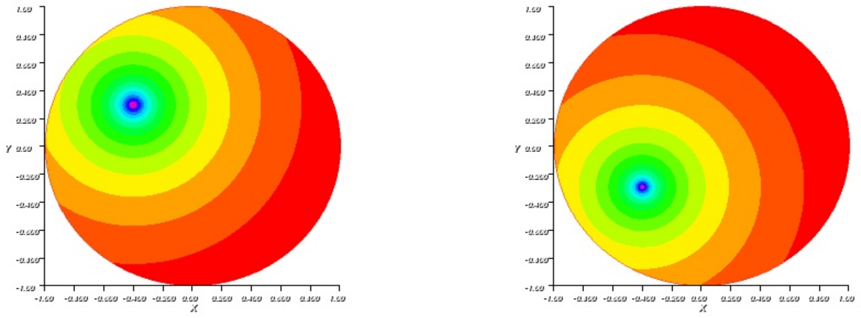
In the following example we consider the optimal control problem (4.1) – (4.3) with manifold as a moving interval.

Example 4.2. *We consider the optimal control problem* (4.1)–(4.3) *with spatial domain* $\Omega = (-1, 1) \times (-1, 1)$ *and* $\gamma(t) = \{2t - 0.5\} \times [-0.25, 0.25]$ *be a moving interval* ($d = 2$ *and* $r = 1$) *strictly contained in the domain. Here, we set the following data*

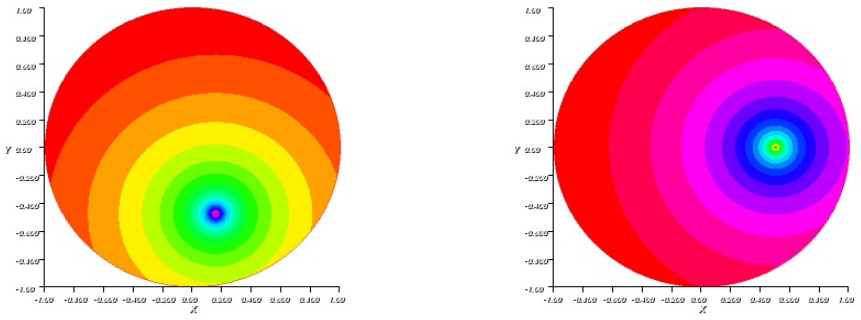
$$\begin{aligned}&T = 0.5, \quad \alpha = 0.01, \quad u_a = -0.025, \quad u_b = 0.025,\\ &y_{ds} = \pi \sin(\pi t) \sin\big(\frac{\pi x_1}{2}\big) \sin\big(\frac{\pi x_2}{2}\big), \quad (x, t) \in \Omega \times [0, T],\end{aligned}$$

with $x = (x_1, x_2)$.

Similar to Example 4.1, here we present the mesh information and errors. The state and the co-state errors are computed in the $L^2(0, T; L^2(\Omega))$-norm whereas the control



$(i)\ t = 0.3333$ $(ii)\ t = 0.5632$



$(iii)\ t = 0.7219$ $(iv)\ t = 0.9889$

Figure 4.3: *Profiles of the approximate state solutions* $(i) - (iv)$ *on the corresponding adaptive meshes (Figure 4.2 (i)-(iv)) at the different time levels* $t = 0.3333,\ 0.5632,\ 0.7219,\ 0.9889$ *for Example 4.1.*

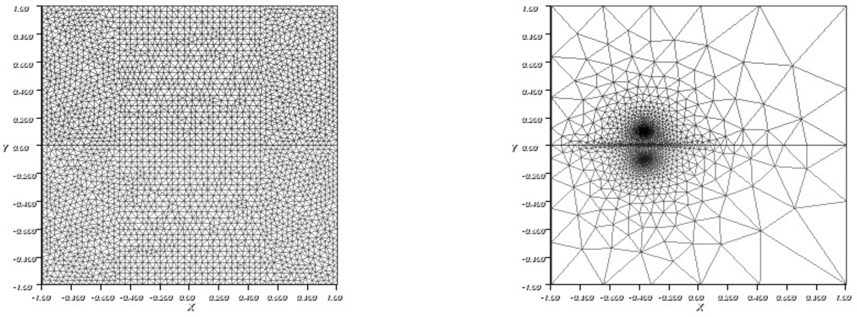
error in the $L^2(0,T;L^2(\gamma(t)))$-norm at the different time levels $t = 0.0625,\ 0.1875,\ 0.325$ and 0.450. From Table 4.2, we observe that the number of nodes in adaptive meshes at Step-(II) are very less in comparison to the uniform mesh. Figure 4.4 (i) shows the uniform mesh and Figures 4.4 $(ii)-(vi)$ show the adaptive meshes at the different time levels $t = 0.0625,\ 0.1875,\ 0.325,\ 0.450$ and 0.500, respectively. We observe from Figures 4.4 $(ii)-(vi)$ that the mesh is adopted along with the moving interval. The approximate solutions of the state variable on the adaptive meshes (Step-(II)) at the different time levels $t = 0.0625,\ 0.1875,\ 0.325,\ 0.450$ and 0.500 are respectively shown in Figures 4.5 $(i)-(v)$.

Table 4.2: *Error estimates for the state and co-state variables in the* $L^2(0,T;L^2(\Omega))$*-norm, and the control error in the* $L^2(0,T;L^2(\gamma(t)))$*-norm, on the uniform mesh, adaptive mesh Step-(I) and adaptive mesh Step-(II).*

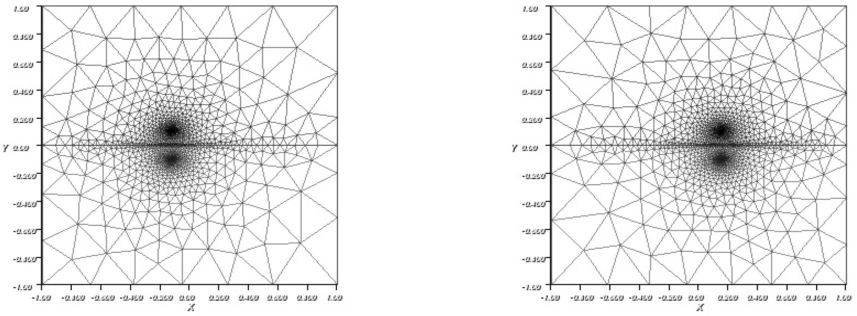
Time levels	Meshes	# Elements	# Nodes	$y - Y_h$	$p - P_h$	$u - U_h$
$t = 0.0265$	Uniform	43526	22014	3.8642e-03	3.2087e-03	1.3517e-02
	Adapt-(I)	17894	8796	6.3108e-04	4.7880e-04	1.3435e-03
	Adapt-(II)	11358	5716	5.9650e-04	1.3449e-04	3.9063e-04
$t = 0.1875$	Uniform	43526	22014	4.9392e-03	1.9437e-03	6.3116e-02
	Adapt-(I)	17920	8996	1.4438e-03	2.6609e-04	6.5696e-03
	Adapt-(II)	9368	4710	7.0320e-04	2.5278e-04	6.2938e-04
$t = 0.325$	Uniform	43526	22014	4.3590e-03	5.1341e-03	6.2876e-02
	Adapt-(I)	17910	9000	8.0753e-04	6.6223e-04	2.2923e-03
	Adapt-(II)	9199	4631	7.6867e-04	6.1575e-04	1.3397e-04
$t = 0.450$	Uniform	43526	22014	5.1942e-03	1.4087e-03	6.3117e-02
	Adapt-(I)	17902	8993	9.2268e-04	2.0554e-04	4.2844e-02
	Adapt-(II)	9204	4636	8.7283e-04	1.9590e-04	6.5842e-03

4.5 Concluding remarks

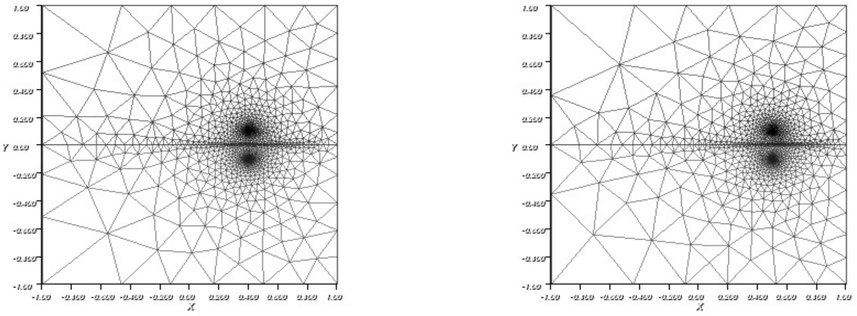
In this chapter, we have studied *a posteriori* error estimates for the finite element approximations to POCP (4.1) – (4.3) with control acting on a lower dimensional manifold. We derive *a posteriori* error estimates for various type of manifolds. The solution of state equation for this kind of problems exhibits a low regularity due to the involvement of Dirac measure on $\gamma(t)$ in the source term. The development of the adaptive finite element method provides necessary feedback to such kind of problems for sake of accuracy enhancement. Our numerical results shows the effectiveness of the derived estimators.



(i) Uniform mesh (ii) $t = 0.0625$

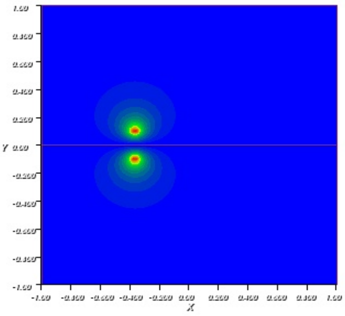


(iii) $t = 0.1875$ (iv) $t = 0.325$

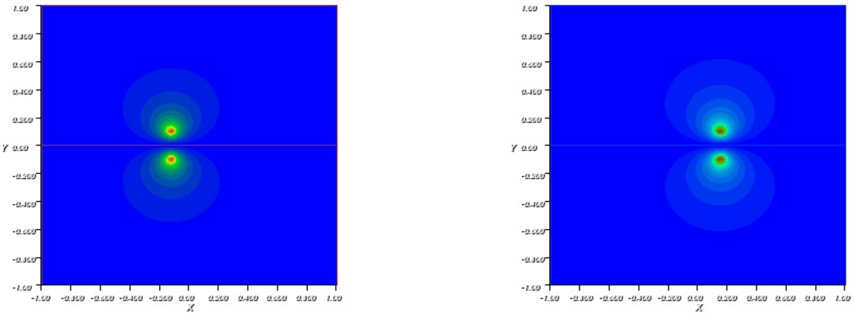


(v) $t = 0.450$ (vi) $t = 0.500$

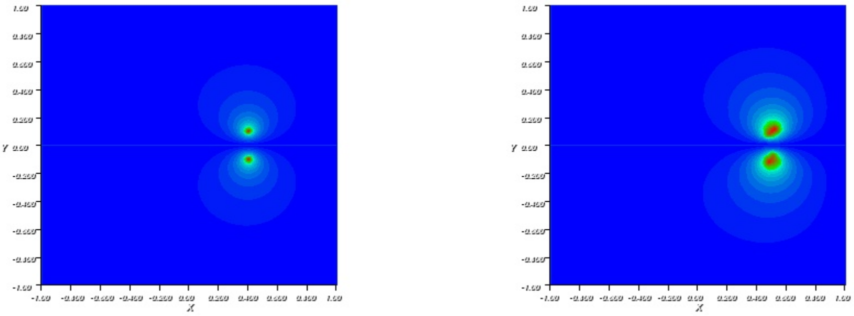
Figure 4.4: *Uniform mesh* (i) *and Adaptive meshes* $(ii) - (vi)$ *(at Step-(II)) at different time levels* $t = 0.0625,\ 0.1875,\ 0.325,\ 0.450, 0.500,$ *for Example 4.2.*



(i) $t = 0.0625$



(ii) $t = 0.1875$ (iii) $t = 0.325$



(iv) $t = 0.450$ (v) $t = 0.500$

Figure 4.5: *Profiles of the approximate solutions* $(i) - (v)$ *corresponding to the adaptive meshes (Figure 4.4* $(ii) - (vi)$*) at different time levels* $t = 0.0625,\ 0.1875,\ 0.325,\ 0.450,$ *and* 0.500*, respectively, for Example 4.2.*

5

Local A Posteriori Error Estimates for PBCP

In this chapter, we derive space-time local *a posteriori* error estimates of finite element approximations to Neumann boundary control problems (1.7) – (1.9) governed by parabolic partial differential equations in a convex bounded domain $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ with Lipschitz boundary. The piecewise linear and continuous finite elements are used for the approximation of the state and co-state variables, while the approximation of the control variable is done by piecewise constant functions. The discrete-in-time scheme is based on the backward Euler implicit scheme. We derive reliable type local *a posteriori* error bounds for the state, co-state and the control variables with three different observations: Observation of the boundary state, the distributed state and the final state. The *a posteriori* error analysis for the state and the co-state variables uses cutoff function, approximation properties and duality argument, while the first-order necessary optimality condition plays an important role for the development of the *a posteriori* error estimate for the control variable. The derived estimators are of local character in the sense that the leading terms of the estimators are depending on the small neighborhood of the boundary and the resulting mesh refinement is much concentrating around the boundary.

5.1 Introduction

This chapter deals with local *a posteriori* error estimates for the following PBCP:

$$\min_{u \in U_{ad}} \mathcal{J}(u, y) = \min_{u \in U_{ad}} \left\{ \mathcal{G}(y) + \mathcal{H}(u) \right\} \tag{5.1}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + \mathcal{A}y = f & \text{in } \ \Omega_T, \\ (A\nabla y)\cdot n = Bu + \mathbf{x}_b & \text{on } \ \Gamma_T, \\ y(x,0) = y_0(x) & \text{in } \ \Omega, \end{cases} \tag{5.2}$$

and the pointwise control constraints are given by

$$u_a \leq u(x,t) \leq u_b \quad \text{a.e.} \quad \text{in } \Gamma_T, \tag{5.3}$$

where $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \partial\Omega \times (0,T]$ with $T < \infty$, and Ω is a convex bounded domain in $\mathbb{R}^d$ $(d \leq 3)$ with Lipschitz boundary $\Gamma := \partial\Omega$. We now assume that

$$f \in L^2(0,T;L^2(\Omega)), \quad y_0 \in H^1(\Omega), \quad \mathbf{x}_b \in \mathscr{B}, \quad a_0 \in L^\infty(\Omega),$$

and the operator B is a linear and continuous operator from X to X. The coefficient matrix $A(\cdot) = (a_{ij}(\cdot))_{d\times d} \in (W^{1,\infty}(\Omega))^{d\times d}$ is a symmetric and positive definite matrix such that there exists a constant $c_0 > 0$ satisfying $a_0(x) \geq c_0$, $\forall x \in \Omega$ and the ellipticity condition

$$\mathbf{x}^t A\mathbf{x} \geq c_0 \|\mathbf{x}\|^2_{\mathbb{R}^d}, \quad \forall \mathbf{x} \in \mathbb{R}^d,$$

where $\mathbf{x}^t$ is the transpose of $\mathbf{x}$ and $\|\cdot\|_{\mathbb{R}^d}$ denotes the standard Euclidean norm on d-dimensional Euclidean space. Here, $y = y(x,t)$ is the state variable and the control variable u appears in the Neumann boundary condition. Let U_{ad} denotes the admissible control set which is given by

$$U_{ad} = \big\{ u \in X : \ u_a \leq u \leq u_b \ \ a.e. \ \text{ in } \Gamma_T \big\}$$

with $u_a < u_b$, where $u_a, u_b \in \mathbb{R}$. Further, U_{ad} is a nonempty closed convex set in $X = L^2(0,T;\mathscr{B})$ with $\mathscr{B} = L^2(\Gamma)$. We define the inner product on $L^2(\Gamma)$ by

$$(v,w)_{\mathscr{B}} \ = \ \int_\Gamma v\, w\, ds, \quad \forall v,\, w \in L^2(\Gamma).$$

Moreover, we shall consider the state space $\mathcal{W} = L^2(0,T;H^2(\Omega)\cap V)\cap H^1(0,T;V^*)$, where $V := H^1(\Omega) = \big\{ v : v \in L^2(\Omega) \ \text{ and } \ \nabla v \in (L^2(\Omega))^d \big\}$. In addition, we shall make the following assumptions on the functionals $\mathcal{G}$ and $\mathcal{H}$:

B1. The functionals $\mathcal{G}$ and $\mathcal{H}$ are continuously differentiable on the observation space $\mathcal{O}_{sp}$ (the space $\mathcal{O}_{sp}$ may be $L^2(0,T;\mathscr{B})$ or $L^2(0,T;L^2(\Omega))$ or $L^2(\Omega)$) and the control space X.

B2. The functionals $\mathcal{G}$ and $\mathcal{H}$, respectively are strictly convex on $\mathcal{O}_{sp}$ and X with $\mathcal{H}(u) \to +\infty$ as $\|u\|_X \to +\infty$. Additionally, $\mathcal{G}$ is bounded below, and $\mathcal{G}'$ satisfies the locally Lipschitz continuity condition.

We now define the bilinear form $a(\cdot,\cdot)$ on $H^1(\Omega)$ by

$$a(v,w) \ = \ \int_\Omega \{(A\nabla v)\cdot\nabla w + a_0 vw\}\,dx, \qquad \forall v,w \in H^1(\Omega),$$

where $H^1(\Omega) = \{v : v \in L^2(\Omega) \ \text{ and } \ \nabla v \in L^2(\Omega)\}$. We assume that the bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $H^1(\Omega)$, i.e., $\exists\ \alpha_0,\ \alpha_1 > 0$ such that

$$|a(v,w)| \ \le \ \alpha_0\|v\|_1\|w\|_1, \qquad \forall v,w \in H^1(\Omega),$$

and

$$a(v,v) \ \ge \ \alpha_1\|v\|_1^2, \qquad \forall v \in H^1(\Omega).$$

The weak form of PBCP $(5.1)-(5.3)$ is defined as follows: Find a pair $(y,u) \in \mathcal{W}\times U_{ad}$ such that

$$\min_{u\in U_{ad}} \mathcal{J}(u,y) = \min_{u\in U_{ad}} \{\mathcal{G}(y) + \mathcal{H}(u)\} \tag{5.4}$$

subject to the state equation

$$\begin{cases} \left(\dfrac{\partial y}{\partial t}, v\right) + a(y,v) = (f,v) + (Bu + \mathbf{x}_b, v)_{\mathscr{B}} & \forall v \in V, \\ y(\cdot,0) = y_0(x) & x \in \Omega. \end{cases} \tag{5.5}$$

We shall investigate local *a posteriori* error estimates for the above control problem with three different choices of observation spaces ($\mathcal{O}_{sp}$) and optimality conditions, which are described as follows.

Case I (Observation space $\mathcal{O}_{sp} = \mathcal{O}_{sp_1}$). This case considers the control problem $(5.1)-(5.3)$ with observation of the boundary state, i.e., the observation space is $\mathcal{O}_{sp_1} = L^2(0,T;\mathscr{B})$. Let

$$\mathcal{G}(y) := \int_0^T \int_\Gamma g(y)ds\,dt \ \text{ and } \ \mathcal{H}(u) := \int_0^T \int_\Gamma h(u)ds\,dt,$$

where $g(\cdot)$ and $h(\cdot)$ are strictly convex and continuously differentiable functions on $\mathcal{O}_{sp_1}$ and X, respectively, so that our conditions on $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ are fulfilled. Note that, both functionals $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ are defined on the boundary. For example, functionals $\mathcal{G}$ and $\mathcal{H}$ are

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|_{\mathscr{B}}^2 dt \ \text{ and } \ \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|_{\mathscr{B}}^2 dt.$$

It is known that the convex boundary control problem $(5.4)-(5.5)$ has a unique solution (y,u) iff there exists a co-state variable $p \in \mathcal{W}$ such that the triplet (y,p,u) satisfies the following optimality conditions for $t \in [0,T]$ (cf. [53]):

$$\begin{aligned}\Big(\frac{\partial y}{\partial t}, v\Big) + a(y,v) &= (f,v) + (Bu + \mathbf{x}_b, v)_{\mathscr{B}} \quad \forall v \in V, && (5.6)\\ y(x,0) &= y_0(x) \quad x \in \Omega, && (5.7)\\ -\Big(\frac{\partial p}{\partial t}, v\Big) + a(p,v) &= (g'(y), v)_{\mathscr{B}} \quad \forall v \in V, && (5.8)\\ p(x,T) &= 0 \quad x \in \Omega, && (5.9)\\ (h'(u) + B^* p, w-u)_{\mathscr{B}} &\geq 0 \quad \forall w \in U_{ad}, && (5.10)\end{aligned}$$

where B^* is the adjoint operator of B.

Next, we consider control problem $(5.1)-(5.3)$ with observation of the distributed state, i.e., the observation space is $\mathcal{O}_{sp_2} = L^2(0,T;L^2(\Omega))$.

Case II (Observation space $\mathcal{O}_{sp} = \mathcal{O}_{sp_2}$). In this case, we consider the control problem $(5.1)-(5.3)$ with observation of the distributed state. Assume that the functions $g(\cdot)$ and $h(\cdot)$ are strictly convex and continuously differentiable on $\mathcal{O}_{sp_2}$ and X, respectively. Define

$$\mathcal{G}(y) := \int_0^T \int_\Omega g(y)dx\,dt \;\text{ and }\; \mathcal{H}(u) := \int_0^T \int_\Gamma h(u)ds\,dt$$

be such that our assumptions on $\mathcal{G}(\cdot)$ and $\mathcal{H}(\cdot)$ are fulfilled. For example, we may think of the functionals $\mathcal{G}$ and $\mathcal{H}$ as:

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|^2_{L^2(\Omega)}dt \;\text{ and }\; \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2_{\mathscr{B}}dt.$$

The optimality conditions in this case read: For $t \in [0,T]$, the triplet (y,p,u) satisfies

$$\begin{aligned}\Big(\frac{\partial y}{\partial t}, v\Big) + a(y,v) &= (f,v) + (Bu + \mathbf{x}_b, v)_{\mathscr{B}} \quad \forall v \in V, && (5.11)\\ y(x,0) &= y_0(x) \quad x \in \Omega, && (5.12)\\ -\Big(\frac{\partial p}{\partial t}, v\Big) + a(p,v) &= (g'(y), v) \quad \forall v \in V, && (5.13)\\ p(x,T) &= 0 \quad x \in \Omega, && (5.14)\\ (h'(u) + B^* p, w-u)_{\mathscr{B}} &\geq 0 \quad \forall w \in U_{ad}, && (5.15)\end{aligned}$$

where B^* is as defined before.

The only difference between the above two cases can be observed in their corresponding co-state equations (5.8) and (5.13). More precisely, Neumann boundary condition for the co-state equation in (5.8) is $g'(y)$ compared to 0 in (5.13), and the source function is changing from 0 to $g'(y)$.

The last case deals with control problem (5.1) – (5.3) with observation of the final state $y(x,T)$. That is, the observation space is $\mathcal{O}_{sp_3} = L^2(\Omega)$.

Case III (Observation space $\mathcal{O}_{sp} = \mathcal{O}_{sp_3}$). This is a very vital and factual case. In this case, the co-state equation for the control problem (5.1) – (5.3) with observation of the final state turns out to be different from the previous ones. Let us define

$$\mathcal{G}(y) := \int_\Omega g(y(x,T))dx \text{ and } \mathcal{H}(u) := \int_0^T \int_\Gamma h(u)ds\,dt,$$

where again $g(\cdot)$ and $h(\cdot)$ are strictly convex and continuously differentiable functions. For example, $\mathcal{G}$ and $\mathcal{H}$ are given by

$$\mathcal{G}(y) := \frac{1}{2}\|y(x,T) - y_{ds}\|^2_{L^2(\Omega)} \text{ and } \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2_{\mathscr{B}}dt,$$

respectively. By Lagrangian approach, one may directly reach to the adjoint problem

$$\begin{aligned} -\frac{\partial p}{\partial t} - \nabla\cdot(A\nabla p) + a_0 p &= 0 \quad \text{in} \quad \Omega_T, \\ (A\nabla p)\cdot n &= 0 \quad \text{on} \quad \Gamma_T, \\ p(\cdot,T) &= g'(y(\cdot,T)) \quad \text{in} \quad \Omega. \end{aligned}$$

The optimality conditions for this case are as follows :

$$\begin{aligned} \left(\frac{\partial y}{\partial t}, v\right) + a(y,v) &= (f,v) + (Bu + \mathbf{x}_b, v)_{\mathscr{B}} \quad \forall v \in V, && (5.16)\\ y(x,0) &= y_0(x) \quad x \in \Omega, && (5.17)\\ -\left(\frac{\partial p}{\partial t}, v\right) + a(p,v) &= 0 \quad \forall v \in V, && (5.18)\\ p(x,T) &= g'(y(x,T)) \quad x \in \Omega, && (5.19)\\ (h'(u) + B^* p, w - u)_{\mathscr{B}} &\geq 0 \quad \forall w \in U_{ad}, && (5.20) \end{aligned}$$

where B^* has the same meaning as before. Note that $p(x,T) = 0$ in previous two cases but in the third case $p(x,T) = g'(y)(T)$. In this case, we are required to pay more attention to derive the *a posteriori* error estimate for the co-state variable.

In the above three cases, the convexity assumptions on $g(\cdot)$ and $h(\cdot)$ imply that

$$\begin{aligned} (g'(v_1) - g'(v_2), v_1 - v_2)_{\mathcal{O}_{sp}} &\geq 0 \qquad \forall v_1,\ v_2 \in \mathcal{O}_{sp}, && (5.21)\\ (h'(w_1) - h'(w_2), w_1 - w_2)_{\mathscr{B}} &\geq C_{5,1}\|w_1 - w_2\|^2_{\mathscr{B}} \qquad \forall w_1,\ w_2 \in \mathscr{B}, && (5.22) \end{aligned}$$

where $C_{5,1} > 0$ denotes the convexity constant.

Some previous work in this direction can be found in [34, 56, 57, 60]. Liu and Yan [56] have derived *a posteriori* error estimates for the model boundary control problems on polygonal domain. They have derived reliable type of *a posteriori* error estimates for convex boundary control problems in [57]. Subsequently in [60], the same authors have discussed both upper and lower *a posteriori* error estimates for the finite element approximations of elliptic boundary control problems with two different observations: The observation of the boundary state and the observation of the distributed state. They have obtained *a posteriori* error estimates for the co-state and the control variables in the $L^2(\Gamma)$-norm for both the observations, while error bound for the state variable is proved in the $L^2(\Gamma)$-norm or $L^2(\Omega)$-norm according to the observation spaces. In the context of PBCP, Gong and Yan [34] have considered the boundary control problems governed by parabolic differential equations and derived *a posterirori* error bounds in three different observations namely, observation of the boundary state, observation of the distributed state, and observation of the final state. Using the energy argument, Clémet type interpolation approximation properties and the trace result, they have established *a posteriori* error estimates for the state and the co-state variables in the $L^2(0,T;H^1(\Omega))$-norm whereas error for the control variable in the $L^2(0,T;\mathscr{B})$-norm. These estimates are not local in character. In many engineering applications, it is often useful to study the behaviour of the state and co-state variables in a small neighborhood of the boundary. Therefore, *a posteriori* error estimators in some suitable local norms have become more useful and the derivation of these estimates are not straightforward. In this chapter, we intend to investigate local *a posteriori* error analysis for the finite element approximation to PBCP (5.1) $-$ (5.3) with three different observations. We derive local *a posteriori* upper bounds for the state, co-state and the control variables in the $L^2(0,T;\mathscr{B})$-norm for the observation of the boundary state as well as for the final state. Moreover, the *a posteriori* error estimate to the state variable for the observation of the distributed state is proved in the $L^2(0,T;L^2(\Omega))$-norm, whereas the error bounds for the co-state and control variables in the $L^2(0,T;\mathscr{B})$-norm. However, the global influences can be seen by the higher order terms in the derived estimators. These estimators can serve as indicators in the adaptive finite element method for the mesh refinements. Our numerical results confirm the reliability of the derived estimators.

The outline of this chapter is as follows. Section 5.2 contains finite element approximation to the control problem (5.1) $-$ (5.3) and some basic results for our subsequent use. We derive local *a posteriori* error estimates for the fully discrete finite element approximation of boundary control problem (5.1)$-$(5.3) with observations of the boundary state, the distributed state and the final state in Section 5.3. Numerical results presented in Section 5.4. The last section contains some concluding remarks.

5.2 Discrete optimal control problems

This section describes the finite element approximation of convex boundary control problem (5.4) – (5.5) and recall some interpolation approximation properties.

Let Ω_h be a polygonal approximation to Ω with a boundary Γ_h. Let $\mathscr{T}_h$ be a partitioning of Ω_h into disjoint regular d-simplices K, so that $\overline{\Omega}_h = \cup_{K\in\mathscr{T}_h}\bar{K}$, and if $K_1,\ K_2 \in \mathscr{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ share a common vertex or a common edge. Note that, when $\Omega \neq \Omega_h$, it is bit complicated to set up and analyze the problem (cf., [57]). For simplicity, we investigate the problem with $\Omega = \Omega_h$. Associated with $\mathscr{T}_h$, we consider a finite dimensional subspace V_h of $\mathcal{C}(\overline{\Omega}_h)$ consisting piecewise linear and continuous polynomials, i.e.,

$$V_h = \big\{v(t) \in \mathcal{C}(\overline{\Omega}_h) :\ v|_K \in \mathbb{P}_1(K),\ \ \forall K \in \mathscr{T}_h,\ t \in (0,T]\big\},$$

where $\mathbb{P}_1$ is the space of polynomials of degree less than or equal to 1 on $K \in \mathscr{T}_h$.

Let us consider $\mathscr{T}_{\mathscr{B}}$ be the partition of Γ_h into disjoint regular $d-1$ simplices τ such that $\Gamma_h = \cup_{\tau\in\mathscr{T}_{\mathscr{B}}}\bar{\tau}$, and if $\tau_1,\ \tau_2 \in \mathscr{T}_{\mathscr{B}}$ and $\tau_1 \neq \tau_2$, then either $\bar{\tau}_1$ and $\bar{\tau}_2$ share a whole edge, or a face, or one common vertex, or they are disjoint. Further, let $\mathscr{B}_h$ be the finite dimensional subspace of $L^2(\Gamma_h)$ consisting piecewise constant polynomials associated with $\mathscr{T}_{\mathscr{B}}$, that is,

$$\mathscr{B}_h = \big\{w(t) \in L^2(\Gamma_h) :\ w(t)|_\tau = constant,\ \ \forall \tau \in \mathscr{T}_{\mathscr{B}},\ t \in (0,T]\big\}.$$

Now we set $U_{ad,h} = X_h \cap U_{ad}$ with $X_h = L^2(0,T;\mathscr{B}_h)$. Let h_K and h_τ denote the maximum diameters of the elements $K \in \mathscr{T}_h$ and $\tau \in \mathscr{T}_{\mathscr{B}}$, respectively.

First, we shall discuss the finite element approximation for the boundary control problem with observation of the boundary state, i.e., for **Case I**.

The semidiscrete finite element approximation of (5.4) – (5.5) is to seek a pair $(y_h, u_h) \in L^2(0,T;V_h) \times U_{ad,h}$ such that

$$\min_{u_h\in U_{ad,h}} \mathcal{J}(u_h, y_h) = \min_{u_h\in U_{ad,h}} \{\mathcal{G}(y_h) + \mathcal{H}(u_h)\} \tag{5.23}$$

subject to

$$\begin{cases} \Big(\dfrac{\partial y_h}{\partial t}, v_h\Big) + a(y_h, v_h) = (f, v_h) + (Bu_h + \mathbf{x}_b, v_h)_{\mathscr{B}} & \forall v_h \in V_h, \\ y_h(\cdot,0) = y_{h,0}(x) \quad\ x \in \Omega, \end{cases} \tag{5.24}$$

where $y_{h,0} \in V_h$ is a suitable approximation of y_0.

It is well known [53] that the convex optimal control problem (5.23) – (5.24) admits a unique solution $(y_h, u_h) \in L^2(0,T;V_h) \times U_{ad,h}$ iff there is a co-state variable $p_h \in L^2(0,T;V_h)$ such that the triplet $(y_h, p_h, u_h) \in L^2(0,T;V_h) \times L^2(0,T;V_h) \times U_{ad,h}$ satisfies the following optimality conditions, for $t \in [0,T]$:

$$\begin{aligned}
\Big(\frac{\partial y_h}{\partial t}, v_h\Big) + a(y_h, v_h) &= (f, v_h) + (Bu_h + \mathbf{x}_b, v_h)_{\mathscr{B}} \quad \forall v_h \in V_h,\\
y_h(\cdot, 0) &= y_{h,0},\\
-\Big(\frac{\partial p_h}{\partial t}, v_h\Big) + a(p_h, v_h) &= (g'(y_h), v_h)_{\mathscr{B}} \quad \forall v_h \in V_h,\\
p_h(\cdot, T) &= 0,\\
(h'(u_h) + B^* p_h, w_h - u_h)_{\mathscr{B}} &\geq 0 \quad \forall w_h \in U_{ad,h}.
\end{aligned}$$

We now consider the space-time finite element discretization of boundary control problem (5.23) – (5.24). The temporal discretization is based on the backward Euler scheme.

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of $[0,T]$ with $I_n = (t_{n-1}, t_n]$ and $k_n := t_n - t_{n-1}$. Let $\mathscr{T}_h^n := \{K\}\,(0 \leq n \leq N)$ be the triangulation of $\overline{\Omega}_h$ at the time level t_n. For each $n = 0, \ldots, N$, we construct the finite element space V_h^n corresponding to the triangulation $\mathscr{T}_h^n$ as

$$V_h^n := \big\{v \in \mathcal{C}(\overline{\Omega}_h) :\ v|_K \in \mathbb{P}_1(K),\ \ \forall K \in \mathscr{T}_h^n\big\}.$$

Let $\mathscr{T}_{\mathscr{B}}^n$ be a partition of Γ_h at the time level $t = t_n$. For $n = 0, \ldots, N$, the finite element space $U_{ad,h}^n$ associated with the mesh $\mathscr{T}_{\mathscr{B}}^n$ is given by $U_{ad,h}^n = \mathscr{B}_h^n \cap U_{ad}$, where

$$\mathscr{B}_h^n = \big\{w \in L^2(\Gamma_h) :\ w|_\tau = constant,\ \ \forall \tau \in \mathscr{T}_{\mathscr{B}}^n\big\}.$$

Let h_{K_n} and h_{τ_n} denote the maximum diameters of the elements $K_n \in \mathscr{T}_h^n$ and $\tau_n \in \mathscr{T}_{\mathscr{B}}^n$, respectively. Now we define mesh size functions $h_K(\cdot)$ and $h_\tau(\cdot)$ associated with mesh functions $K(\cdot)$ and $\tau(\cdot)$, respectively, such that

$$h_K(t)|_{t\in I_n} = h_{K_n},\quad h_\tau(t)|_{t\in I_n} = h_{\tau_n},\quad K(t)|_{t\in I_n} = K_n \ \text{ and }\ \tau(t)|_{t\in I_n} = \tau_n.$$

For simplicity, frequently we use $h_K(t) = h_K,\ \ h_\tau(t) = h_\tau,\ \ K(t) = K$ and $\tau(t) = \tau$. For the purpose of fully discrete approximation, we use the following notation

$$\psi^n := \psi(\cdot, t_n) \ \text{ and }\ \bar{\partial}\psi^n := \frac{1}{k_n}(\psi^n - \psi^{n-1}).$$

The space-time finite element discretization of (5.23) – (5.24) is defined as follows: To seek a pair $(y_h^n, u_h^n) \in V_h^n \times U_{ad,h}^n$, for $1 \leq n \leq N$, such that

$$\min_{u_h^n \in U_{ad,h}^n} \sum_{n=1}^{N} \int_{I_n} \int_\Gamma \{g(y_h^n) + h(u_h^n)\}\, ds\, dt \tag{5.25}$$

subject to

$$\begin{cases} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) = (\bar{f}^n, v_h) + (Bu_h^n + \mathbf{x}_b, v_h)_{\mathscr{B}} \qquad \forall v_h \in V_h^n, \\ y_h^0 = y_{h,0}, \end{cases} \tag{5.26}$$

where $\bar{f}^n = \dfrac{1}{k_n} \displaystyle\int_{t_{n-1}}^{t_n} f(t)\, dt$ and $y_{h,0}$ is a suitable approximation of y_0 in V_h^0.

The optimal control problem (5.25) – (5.26) has a unique solution (y_h^n, u_h^n) iff there is a co-state variable $p_h^{n-1} \in V_h^n$ such that the triplet $(y_h^n, p_h^{n-1} u_h^n)$ satisfies the following optimality conditions, for $1 \leq n \leq N$:

$$\begin{aligned} (\bar{\partial} y_h^n, v_h) + a(y_h^n, v_h) &= (\bar{f}^n, v_h) + (Bu_h^n + \mathbf{x}_b, v_h)_{\mathscr{B}} \qquad \forall v_h \in V_h^n, && (5.27) \\ y_h^0 &= y_{h,0}, && (5.28) \\ -(\bar{\partial} p_h^n, v_h) + a(p_h^{n-1}, v_h) &= (g'(y_h^n), v_h)_{\mathscr{B}} \qquad \forall v_h \in V_h^n, && (5.29) \\ p_h^N &= 0, && (5.30) \\ (h'(u_h^n) + B^* p_h^{n-1}, w_h^n - u_h^n)_{\mathscr{B}} &\geq 0 \qquad \forall w_h^n \in U_{ad,h}^n. && (5.31) \end{aligned}$$

For $t \in I_n, \quad 1 \leq n \leq N$, we define

$$\begin{aligned} Y_h(t) &:= \frac{(t_n - t)}{k_n} y_h^{n-1} + \frac{(t - t_{n-1})}{k_n} y_h^n, \\ P_h(t) &:= \frac{(t_n - t)}{k_n} p_h^{n-1} + \frac{(t - t_{n-1})}{k_n} p_h^n, \\ U_h(t) &:= u_h^n. \end{aligned}$$

Here $Y_h(t)$, $P_h(t)$ represent the continuous piecewise linear interpolant functions in time, and $U_h(t)$ denotes the piecewise constant function in time. For any function $\omega \in \mathcal{C}(0, T; L^2(\Omega))$, we set

$$\hat{\omega}(x,t)|_{(t_{n-1},t_n]} = \omega(x, t_n) \quad \text{and} \quad \tilde{\omega}(x,t)|_{(t_{n-1},t_n]} = \omega(x, t_{n-1}).$$

Then the optimality conditions (5.27)-(5.31) can be stated as follows:

$$\begin{aligned} \Big(\frac{\partial Y_h}{\partial t}, v_h\Big) + a(\hat{Y}_h, v_h) &= (\bar{f}^n, v_h) + (BU_h + \mathbf{x}_b, v_h)_{\mathscr{B}} \qquad \forall v_h \in V_h^n, && (5.32) \\ Y_h^0 &= y_{h,0}, && (5.33) \\ -\Big(\frac{\partial P_h}{\partial t}, v_h\Big) + a(\tilde{P}_h, v_h) &= (g'(\hat{Y}_h), v_h)_{\mathscr{B}} \qquad \forall v_h \in V_h^n, && (5.34) \\ P_h^N &= 0, && (5.35) \\ (h'(U_h) + B^* \tilde{P}_h, w_h^n - U_h)_{\mathscr{B}} &\geq 0 \qquad \forall w_h^n \in U_{ad,h}^n. && (5.36) \end{aligned}$$

For the purpose of deriving local *a posteriori* error estimates we consider the set

$$\Omega_l = \big\{ x \in \Omega \,|\; \mathrm{dist}(x, \Gamma) < l \big\} \subset \Omega,$$

where $l > 0$ is a constant, and independent of mesh parameters h and k. For each time level $t_n \in (0, T]$, we first introduce the subset $\mathscr{T}^n_{\Omega_l}$ of $\mathscr{T}^n_h$ as follows:

$$\mathscr{T}^n_{\Omega_l} = \big\{ K \in \mathscr{T}^n_h \,|\quad \bar{K} \cap \overline{\Omega}_l \neq \emptyset \big\}.$$

For the purpose of understanding of the triangulations $\mathscr{T}^n_h$ and $\mathscr{T}^n_{\Omega_l}$, a typical discretization of $\Omega \subset \mathbb{R}^2$ is shown in Figures 5.1 (a) and (b).

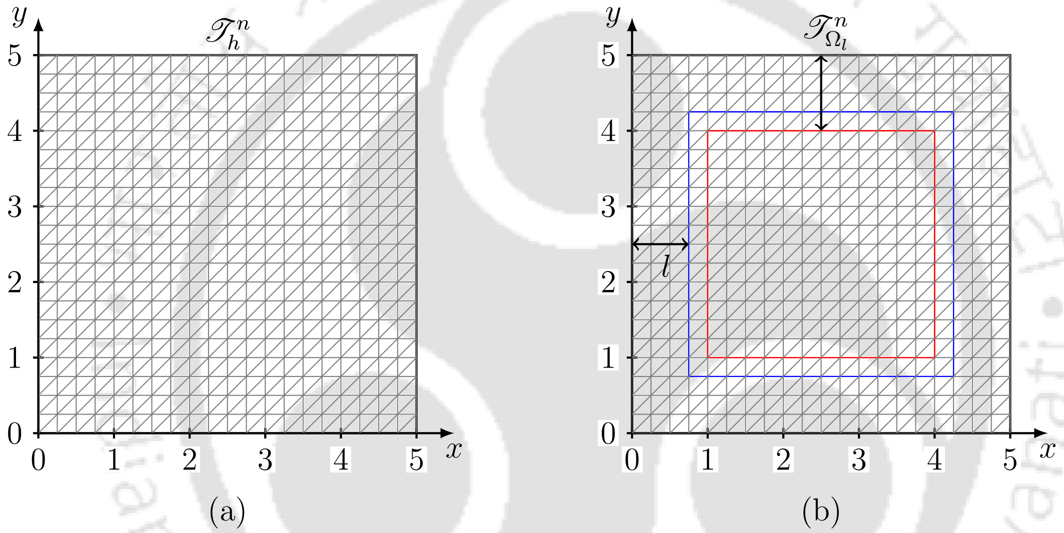


Figure 5.1: (a) $\mathscr{T}^n_h$ corresponding to Ω; (b) $\mathscr{T}^n_{\Omega_l}$ corresponding to Ω_l

We now recall some important interpolation approximation properties and trace result in the following lemmas. These lemmas will be useful in deriving the residual type *a posteriori* error estimates.

Lemma 5.2.1 ([15]). *For $m = 0$ or 1, $q > \frac{d}{2}$ and $\psi \in W^{2,q}(\Omega)$,*

$$|\psi - \Pi_h \psi|_{W^{m,q}(\Omega_h)} \;\leq\; \tilde{C}_{I,1}\, h^{2-m} |\psi|_{W^{2,q}(\Omega_h)},$$

where Π_h is the standard Lagrange interpolation operator.

Lemma 5.2.2. *For $m = 0$ or 1, $1 \leq q \leq \infty$ and $\forall\, \psi \in W^{1,q}(\Omega_h)$,*

$$|\psi - \hat{\Pi}_h \psi|_{W^{m,q}(K)} \;\leq\; \sum_{\bar{K}' \cap \bar{K} \neq \emptyset} \tilde{C}_{I,2}\, h_K^{1-m} |\psi|_{W^{1,q}(K')}, \tag{5.37}$$

where $\hat{\Pi}_h$ is the average interpolation operator as defined in [86].

For $t \in I_n$, let $\phi \in \mathcal{C}_0^\infty(\Omega_l)$ be a cut-off function with $\phi = 1$ on Γ. Set $\bar{\psi} = \phi^2\psi$ and $\bar{\psi}^I = \hat{\Pi}_h\bar{\psi}$, it follows from Lemma 5.2.2 that

$$\sum_K h_K^{-2}\|\bar{\psi} - \bar{\psi}^I\|^2_{L^2(K)} + |\bar{\psi} - \bar{\psi}^I|^2_{H^1(\Omega)} \;\le\; \tilde{C}^2_{I,4}\|\bar{\psi}\|^2_{H^1(\Omega)} \;\le\; \tilde{C}^2_{I,5}\|\phi\psi\|^2_{H^1(\Omega)}, \qquad (5.38)$$

where the constant $\tilde{C}_{I,4}$ depends on the constant $\tilde{C}_{I,2}$.

Lemma 5.2.3 ([45])**.** *For all* $\psi \in W^{1,q}(\Omega)$ *and* $1 \le q < \infty$*,*

$$\|\psi\|_{W^{0,q}(\partial K)} \;\le\; \tilde{C}_{I,3}\Big(h_K^{\frac{-1}{q}}\|\psi\|_{W^{0,q}(K)} + h_K^{1-\frac{1}{q}}\|\psi\|_{W^{1,q}(K)}\Big).$$

5.3 A posteriori error analysis

In this section, we shall derive local *a posteriori* error estimates for the control problem (5.1) – (5.3) in three different observations (the boundary state, the distributed state and the final state).

5.3.1 Control problem with observation space $\mathcal{O}_{sp} = \mathcal{O}_{sp_1}$

We first consider the control problem (5.1) – (5.3) with observation of the boundary state, i.e., observation space $\mathcal{O}_{sp_1}$ and derive reliable type local *a posteriori* error bounds for the **Case I**. To derive the main result we first need to establish some intermediate *a posteriori* error estimates. For this purpose, we introduce the following auxiliary problems.

For any control variable $u^* \in U_{ad}$, we define that the pair $(y(u^*), p(u^*)) \in \mathcal{W} \times \mathcal{W}$ satisfies the following system:

$$\begin{aligned}
\Big(\frac{\partial y(u^*)}{\partial t}, v\Big) + a(y(u^*), v) &= (f, v) + (Bu^* + \mathbf{x}_b, v)_{\mathscr{B}} \qquad \forall v \in V, && (5.39)\\
y(u^*)(\cdot, 0) &= y_0(x) \qquad x \in \Omega, && (5.40)\\
-\Big(\frac{\partial p(u^*)}{\partial t}, v\Big) + a(p(u^*), v) &= (g'(y(u^*)), v)_{\mathscr{B}} \qquad \forall v \in V, && (5.41)\\
p(u^*)(\cdot, T) &= 0 \qquad x \in \Omega. && (5.42)
\end{aligned}$$

In the following lemma, we present the error estimate for the control variable in the $L^2(0, T; \mathscr{B})$-norm.

Lemma 5.3.1. *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* (5.6) – (5.10) *and* (5.32) – (5.36)*, respectively. Assume that* $U^n_{ad,h} \subset U_{ad}$*,* $(h'(U_h) + B^*\tilde{P}_h)|_\tau \in H^1(\tau)$ *and there* *exists a positive constant* $C_{5,2}$ *and* $w_h \in U^n_{ad,h}$ *such that*

$$\Big|\int_0^T (h'(U_h)+B^*\tilde{P}_h, w_h-u)_{\mathscr{B}}\,dt\Big| \;\le\; C_{5,2}\int_0^T \sum_{\tau\in\mathscr{T}^n_{\mathscr{B}}} h_\tau |h'(U_h)+B^*\tilde{P}_h|_{H^1(\tau)}\|u-U_h\|_{L^2(\tau)}\,dt. \tag{5.43}$$

Then, we have

$$\|u-U_h\|^2_{L^2(0,T;\mathscr{B})} \;\le\; C_{5,3}\left(\mathscr{E}^2_{5,1} + \|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;\mathscr{B})}\right),$$

with

$$\mathscr{E}^2_{5,1} \;=\; \int_0^T \sum_{\tau\in\mathscr{T}^n_{\mathscr{B}}} h^2_\tau |h'(U_h)+B^*\tilde{P}_h|^2_{H^1(\tau)}\,dt.$$

In the above, $C_{5,3} = \dfrac{3}{2(2C_{5,1}-1)}\max\left\{C^2_{5,2}, 1\right\}$ *and the pair* $(y(U_h), p(U_h))$ *is a solution of the problem* $(5.39)-(5.42)$ *with* $u^* = U_h$.

Proof. From (5.10) with $w = U_h$, we have

$$(h'(u), u-U_h)_{\mathscr{B}} \;\le\; -(B^*p, u-U_h)_{\mathscr{B}}.$$

An application of convexity condition (5.22) yields

$$\begin{aligned}
C_{5,1}\int_0^T \|u-U_h\|^2_{\mathscr{B}}dt \;&\le\; \int_0^T (h'(u)-h'(U_h), u-U_h)_{\mathscr{B}}dt \\
&\le\; \int_0^T \Big\{-(B^*p, u-U_h)_{\mathscr{B}} - (h'(U_h), u-U_h)_{\mathscr{B}}\Big\}\,dt \\
&=\; -\int_0^T (B^*\tilde{P}_h + h'(U_h), u-w_h)_{\mathscr{B}}\,dt \\
&\quad -\int_0^T (h'(U_h) + B^*\tilde{P}_h, w_h-U_h)_{\mathscr{B}}\,dt \\
&\quad +\int_0^T (B^*\tilde{P}_h - B^*p(U_h), u-U_h)_{\mathscr{B}}\,dt \\
&\quad +\int_0^T (B^*(p(U_h)-p), u-U_h)_{\mathscr{B}}\,dt,
\end{aligned}$$

using (5.36), it now follows that

$$\begin{aligned}
C_{5,1}\int_0^T \|u-U_h\|^2_{\mathscr{B}}dt \;&\le\; \int_0^T (B^*\tilde{P}_h + h'(U_h), w_h-u)_{\mathscr{B}}\,dt \\
&\quad +\int_0^T (B^*(\tilde{P}_h - p(U_h)), u-U_h)_{\mathscr{B}}\,dt \\
&\quad +\int_0^T (B^*(p(U_h)-p), u-U_h)_{\mathscr{B}}\,dt \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{5.44}$$

In view of (5.43), an application of the Young's inequality leads to the bound of J_1,

$$\begin{aligned} J_1 &\leq C_{5,2}\int_0^T \sum_{\tau\in\mathscr{T}_{\mathscr{B}}^n} h_\tau |h'(U_h)+B^*\tilde{P}_h|_{H^1(\tau)}\|u-U_h\|_{L^2(\tau)}\,dt \\ &\leq \frac{3}{4}C_{5,2}^2\int_0^T \sum_{\tau\in\mathscr{T}_{\mathscr{B}}^n} h_\tau^2 |h'(U_h)+B^*\tilde{P}_h|_{H^1(\tau)}^2 + \frac{1}{4}\int_0^T \|u-U_h\|_{\mathscr{B}}^2 dt. \qquad (5.45)\end{aligned}$$

To bound J_2, we use the Cauchy-Schwarz inequality and the Young's inequality to have

$$\begin{aligned} J_2 &\leq \int_0^T \|B^*(\tilde{P}_h - p(U_h))\|_{\mathscr{B}}\|u-U_h\|_{\mathscr{B}}\,dt \\ &\leq \frac{3}{4}\int_0^T \|\tilde{P}_h - p(U_h)\|_{\mathscr{B}}^2 dt + \frac{1}{4}\int_0^T \|u-U_h\|_{\mathscr{B}}^2 dt. \qquad (5.46)\end{aligned}$$

From (5.6) and (5.39) with $u^* = U_h$ and $v = p(U_h) - p$, we obtain after integrating with respect to time from 0 to T,

$$\begin{aligned} \int_0^T \Big(\frac{\partial}{\partial t}(y-y(U_h)), p(U_h)-p\Big)\,dt + \int_0^T a(y-y(U_h), p(U_h)-p)dt \\ = \int_0^T (B(u-U_h), p(U_h)-p)_{\mathscr{B}}dt = \int_0^T (u-U_h, B^*(p(U_h)-p))_{\mathscr{B}}dt. \qquad (5.47)\end{aligned}$$

Set $v = y(U_h) - y$ in (5.8) and $u^* = U_h$ in (5.41). An integration from 0 to T yields

$$\begin{aligned} -\int_0^T \Big(\frac{\partial}{\partial t}(p-p(U_h)), y(U_h)-y\Big)\,dt + \int_0^T a(p-p(U_h), y(U_h)-y)\,dt \\ = \int_0^T (g'(y)-g'(y(U_h)), y(U_h)-y)_{\mathscr{B}}\,dt.\end{aligned}$$

Use integration by parts formula and the fact of $(p-p(U_h))(T) = 0 = (y-y(U_h))(0)$ to have

$$\begin{aligned} \int_0^T \Big(p-p(U_h), \frac{\partial}{\partial t}(y(U_h)-y)\Big)dt + \int_0^T a(p-p(U_h), y(U_h)-y)\,dt \\ = \int_0^T (g'(y)-g'(y(U_h)), y(U_h)-y)_{\mathscr{B}}\,dt. \qquad (5.48)\end{aligned}$$

From (5.47) and (5.48), we have

$$J_3 = \int_0^T (B^*(p(U_h)-p), u-U_h)_{\mathscr{B}}\,dt = -\int_0^T (g'(y)-g'(y(U_h)), y-y(U_h))_{\mathscr{B}}\,dt,$$

use of convexity condition (5.21) implies

$$J_3 \leq 0. \qquad (5.49)$$

Putting (5.45), (5.46), (5.49) and (5.44) together, we choose $C_{5,1} > 0$ in such a way that $2C_{5,1} - 1 > 0$, and this leads to the desire result. ☐

Before developing the next results, we first define the A-normal derivative jumps quantities over an edge or face (E) as:

$$[A\nabla\hat{Y}_h \cdot n_E] = (A\nabla\hat{Y}_h|_{K_1^E} - A\nabla\hat{Y}_h|_{K_2^E}) \cdot n_E, \tag{5.50}$$

and

$$[A\nabla\tilde{P}_h \cdot n_E] = (A\nabla\tilde{P}_h|_{K_1^E} - A\nabla\tilde{P}_h|_{K_2^E}) \cdot n_E, \tag{5.51}$$

where n_E is the outward unit normal vector to K_1^E on E with $E = \bar{K}_1^E \cap \bar{K}_2^E$.

Lemma 5.3.1 shows that the error in the control variable depends on the intermediate co-state error in the $L^2(0,T;\mathscr{B})$-norm. We now estimate this co-state error in the following lemma.

Lemma 5.3.2. *Let* (Y_h, P_h, U_h) *and* $(y(u^*), p(u^*))$ *be the solutions of* (5.32) – (5.36) *and* (5.39) – (5.42) *with* $u^* = U_h$*, respectively. Let the assumptions of Lemma 5.3.1 be fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_1}$*. Then there exists a positive constant* $C_{5,4}$ *such that*

$$\int_0^T \|p(U_h) - P_h\|^2_{\mathscr{B}} dt \;\le\; C_{5,4}\Big(\sum_{i=2}^{4} \mathscr{E}^2_{5,i} + \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}} dt \Big),$$

with

$$\begin{aligned}
\mathscr{E}^2_{5,2} &= \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \int_K h_K^2\Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h\Big)^2 dx\,dt \\
&\quad + \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset,\,E\subset\mathscr{T}^n_{\Omega_l}} \int_E h_E([A\nabla\tilde{P}_h\cdot n_E])^2 ds\,dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E(A\nabla\tilde{P}_h\cdot n_E - g'(\hat{Y}_h))^2 ds\,dt, \\
\mathscr{E}^2_{5,3} &= \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + \int_0^T \sum_{K\in\mathscr{T}^n_h} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt, \\
\mathscr{E}^2_{5,4} &= \int_0^T \sum_{K\in\mathscr{T}^n_h} \int_K h_K^4\Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h\Big)^2 dx\,dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E h_E^3([A\nabla\tilde{P}_h\cdot n_E])^2 ds\,dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E^3(A\nabla\tilde{P}_h\cdot n_E - g'(\hat{Y}_h))^2 ds\,dt,
\end{aligned}$$

where h_E *and* h_K *are the diameters of* E *and* K*, respectively, and* $[A\nabla\tilde{P}_h\cdot n_E]$ *is given by (5.51). The constant* $C_{5,4}$ *depends on* α_0*,* α_1*, the Lipschitz-constant* C_L*, the bound* C_M *of the cut-off function, the regularity constant* C_R *and the constants* $\tilde{C}_{I,j}$*,* $j = 1,\ldots,5$ *defined in Lemmas* 5.2.1 − 5.2.3 *and the inequality* (5.38)*.*

Proof. From (5.34), $\forall v \in V$ and $v_h \in V_h^n$, $t \in I_n$, we have

$$\begin{aligned}
-\Big(\frac{\partial P_h}{\partial t}, v\Big) + a(\tilde{P}_h, v) &= -\Big(\frac{\partial P_h}{\partial t}, v - v_h\Big) + a(\tilde{P}_h, v - v_h) + (g'(\hat{Y}_h), v_h)_{\mathscr{B}} \\
&= -\Big(\frac{\partial P_h}{\partial t}, v - v_h\Big) + a(\tilde{P}_h, v - v_h) + (g'(\hat{Y}_h), v)_{\mathscr{B}} \\
&\quad - (g'(\hat{Y}_h), v - v_h)_{\mathscr{B}}.
\end{aligned} \tag{5.52}$$

Subtracting (5.52) from (5.41) with $u^* = U_h$ we obtain

$$\begin{aligned}
-\Big(\frac{\partial}{\partial t}(p(U_h) - P_h), v\Big) + a(p(U_h) - P_h, v) = {}& \Big(\frac{\partial P_h}{\partial t}, v - v_h\Big) - a(\tilde{P}_h, v - v_h) \\
&+ (g'(\hat{Y}_h), v - v_h)_{\mathscr{B}} + a(\tilde{P}_h - P_h, v) \\
&+ (g'(y(U_h)) - g'(\hat{Y}_h), v)_{\mathscr{B}}.
\end{aligned} \tag{5.53}$$

Note that, $e_p = p(U_h) - P_h$ and the defined cut-off function ϕ which is independent of time imply the following identity

$$\int_\Omega A\nabla(e_p)\cdot\nabla(\phi^2 e_p)dx = \int_\Omega A\nabla(\phi e_p)\cdot\nabla(\phi e_p)dx - \int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\, dx.$$

A simple calculation now leads to

$$a(e_p, \phi^2 e_p) = a(\phi e_p, \phi e_p) - \int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\, dx. \tag{5.54}$$

Setting $v = \phi^2 e_p := w_p$ and $v_h = \hat{\Pi}_h(\phi^2 e_p) := w_p^I$ in (5.53), where w_p^I is the interpolation of w_p, we arrive at

$$\begin{aligned}
-\Big(\frac{\partial e_p}{\partial t}, \phi^2 e_p\Big) + a(e_p, \phi^2 e_p) = {}& \Big(\frac{\partial P_h}{\partial t}, w_p - w_p^I\Big) - a(\tilde{P}_h, w_p - w_p^I) + a(\tilde{P}_h - P_h, w_p) \\
&+ (g'(\hat{Y}_h), w_p - w_p^I)_{\mathscr{B}} + (g'(y(U_h)) - g'(\hat{Y}_h), w_p)_{\mathscr{B}}.
\end{aligned}$$

Using (5.54) and then integrating from t_{n-1} to t_n with respect to time t, we have

$$\begin{aligned}
&\frac{1}{2}\big(\|\phi e_p(t_{n-1})\|^2_{L^2(\Omega)} - \|\phi e_p(t_n)\|^2_{L^2(\Omega)}\big) + \int_{t_{n-1}}^{t_n} a(\phi e_p, \phi e_p)dt \\
&= \int_{t_{n-1}}^{t_n} \Big\{\Big(\frac{\partial P_h}{\partial t}, w_p - w_p^I\Big) + a(\tilde{P}_h, w_p^I - w_p) + (g'(\hat{Y}_h), w_p - w_p^I)_{\mathscr{B}}\Big\}dt \\
&\quad + \int_{t_{n-1}}^{t_n} a(\tilde{P}_h - P_h, w_p)dt + \int_{t_{n-1}}^{t_n} (g'(y(U_h)) - g'(\hat{Y}_h), w_p)_{\mathscr{B}}dt \\
&\quad + \int_{t_{n-1}}^{t_n}\int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi \, dx\, dt.
\end{aligned}$$

Summing up from 1 to N, we find that

$$\begin{aligned}
&\frac{1}{2}\big(\|\phi e_p(0)\|^2_{L^2(\Omega)} - \|\phi e_p(T)\|^2_{L^2(\Omega)}\big) + \int_0^T a(\phi e_p, \phi e_p)dt \\
&= \int_0^T \Big\{\Big(\frac{\partial P_h}{\partial t}, w_p - w_p^I\Big) + a(\tilde{P}_h, w_p^I - w_p) + (g'(\hat{Y}_h), w_p - w_p^I)_{\mathscr{B}}\Big\}dt \\
&\quad + \int_0^T a(\tilde{P}_h - P_h, w_p)dt + \int_0^T (g'(y(U_h)) - g'(\hat{Y}_h), w_p)_{\mathscr{B}}dt \\
&\quad + \int_0^T\int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi \, dx\, dt \quad =: I_1 + I_2 + I_3 + I_4. \qquad (5.55)
\end{aligned}$$

We now proceed to estimate I_j, $j = 1, 2, 3, 4$. For I_1, first we note that

$$\begin{aligned}
I_1 &= \int_0^T \Big\{\Big(\frac{\partial P_h}{\partial t}, w_p - w_p^I\Big) + a(\tilde{P}_h, w_p^I - w_p) + (g'(\hat{Y}_h), w_p - w_p^I)_{\mathscr{B}}\Big\}dt, \\
&= \int_0^T \sum_K \int_K \Big\{\Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h\Big)(w_p - w_p^I)\Big\}dx\, dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset}\int_E [A\nabla\tilde{P}_h\cdot n_E](w_p - w_p^I)ds\, dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma}\int_E (A\nabla\tilde{P}_h\cdot n_E - g'(\hat{Y}_h))(w_p - w_p^I)ds\, dt.
\end{aligned}$$

Note that $\phi = 0$ on $\Omega\setminus\Omega_l$. Thus, an application of Lemma 5.2.3, inequality (5.38) and the Young's inequality imply

$$\begin{aligned}
I_1 \leq\ & c(\delta)C^2_{5,5}\Big\{\int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}}\int_K h_K^2\Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h\Big)^2 dx\, dt \\
&+ \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset, E\subset\mathscr{T}^n_{\Omega_l}}\int_E h_E([A\nabla\tilde{P}_h\cdot n_E])^2 dsdt \\
&+ \int_0^T \sum_{E\subset\Gamma}\int_E h_E(A\nabla\tilde{P}_h\cdot n_E - g'(\hat{Y}_h))^2 ds\, dt\Big\} + \delta\int_0^T \|\phi e_p\|^2_{H^1(\Omega)}dt,
\end{aligned}$$

where $C_{5,5}^2 = \tilde{C}_{I,5}^2 \max\{\tilde{C}_{I,3}^2, 1\}$. By the continuity of the bilinear form and the Young's inequality, we have for I_2,

$$\begin{aligned} I_2 &= \int_0^T a(\tilde{P}_h - P_h, w_p)dt, \\ &\leq c(\delta)\alpha_0^2 C_{5,6}^2 \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)}dt + \delta\int_0^T \|\phi e_p\|^2_{H^1(K)}dt, \end{aligned}$$

where $C_{5,6}^2 = \tilde{C}_{I,5}^2/\tilde{C}_{I,4}^2$. For I_3, we use the Lipschitz continuity of $g'(\cdot)$ and the Young's inequality to have

$$\begin{aligned} I_3 &= \int_0^T (g'(y(U_h)) - g'(\hat{Y}_h), w_p)_{\mathscr{B}}dt \\ &\leq c(\delta)C_{5,6}^2 C_L^2 \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}}dt + \delta\int_0^T \|\phi e_p\|^2_{H^1(\Omega)}dt, \end{aligned}$$

where $C_L > 0$ is the Lipschitz constant. Now we estimate our last term I_4. We have

$$I_4 = \int_0^T\int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\, dx\, dt \leq \|\phi\|^2_{1,\infty,\Omega}\int_0^T \|e_p\|^2_{L^2(\Omega)}dt.$$

Since ϕ is a predefined cut-off function, there exists a positive constant C_M such that $\|\phi\|_{1,\infty,\Omega} \leq C_M$. Thus, we bound the term I_4 as

$$I_4 \leq C_M^2 \int_0^T \|e_p\|^2_{L^2(\Omega)}dt.$$

Combining these bounds together with (5.55), $e_p(T) = 0$ and the coercive property of the bilinear, we obtain

$$\begin{aligned} &\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \alpha_1\int_0^T \|\phi e_p\|^2_{H^1(\Omega)}dt \\ &\leq c(\delta)\Big(C_{5,5}^2\mathscr{E}_{5,2}^2 + \alpha_0^2 C_{5,6}^2\int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)}dt + C_{5,6}^2 C_L^2\int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}}dt\Big) \\ &\quad + C_M^2\int_0^T \|e_p\|^2_{L^2(\Omega)}dt + 3\delta\int_0^T \|\phi e_p\|^2_{H^1(\Omega)}dt. \end{aligned}$$

An application of the trace theorem with cut-off function ϕ implies

$$\|e_p\|_{L^2(0,T;\mathscr{B})} = \|\phi e_p\|_{L^2(0,T;\mathscr{B})} \leq \|\phi e_p\|_{L^2(0,T;H^1(\Omega))}. \qquad (5.56)$$

Setting $C_{5,7} = \max\left\{c(\delta)C_{5,5}^2, c(\delta)\alpha_0^2 C_{5,6}^2, c(\delta)C_{5,6}^2 C_L^2, C_M^2\right\}$ with $\delta = \alpha_1/6$, and we use the standard kick-back argument and (5.56) to obtain

$$\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2}\int_0^T \|e_p\|^2_{\mathscr{B}}dt \;\le\; C_{5,7}\Big(\mathscr{E}^2_{5,2} + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)}dt$$
$$+ \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}} + \int_0^T \|e_p\|^2_{L^2(\Omega)}dt\Big). \quad (5.57)$$

We now turn to estimate the term e_p in the $L^2(0,T;L^2(\Omega))$-norm. By using duality argument, we have

$$\begin{aligned}
\int_0^T \|e_p\|^2_{L^2(\Omega)}dt &= \int_0^T \Big\{\Big(e_p, \frac{\partial\chi}{\partial t}\Big) + a(e_p,\chi)\Big\}dt \\
&= -\int_0^T \Big(\frac{\partial e_p}{\partial t},\chi\Big)dt + \int_0^T a(e_p,\chi)dt \\
&= -\int_0^T \Big\{-\Big(\frac{\partial P_h}{\partial t},\chi\Big) + a(\tilde{P}_h,\chi)\Big\}dt \\
&\quad + \int_0^T a(\tilde{P}_h - P_h,\chi)dt + \int_0^T (g'(y(U_h)),\chi)_{\mathscr{B}}dt.
\end{aligned} \quad (5.58)$$

Utilization of (5.52) in the above equation leads to

$$\begin{aligned}
\int_0^T \|e_p\|^2_{L^2(\Omega)}dt &= \int_0^T \Big\{\Big(\frac{\partial P_h}{\partial t},\chi-\chi^I\Big) - a(\tilde{P}_h,\chi-\chi^I) + (g'(\hat{Y}_h),\chi-\chi^I)_{\mathscr{B}}\Big\}dt \\
&\quad + \int_0^T a(\tilde{P}_h - P_h,\chi)dt + \int_0^T (g'(y(U_h)) - g'(\hat{Y}_h),\chi)_{\mathscr{B}}dt, \\
&= \int_0^T \Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h, \chi-\chi^I\Big)dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset}\int_E [A\nabla\tilde{P}_h\cdot n_E](\chi^I-\chi)ds\,dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma}\int_E (A\nabla\tilde{P}_h\cdot n_E - g'(\hat{Y}_h))(\chi^I-\chi)ds\,dt \\
&\quad + \int_0^T a(\tilde{P}_h - P_h,\chi)dt + \int_0^T (g'(y(U_h)) - g'(\hat{Y}_h),\chi)_{\mathscr{B}}dt,
\end{aligned}$$

where χ^I is the Lagrange interpolation of χ as defined in Lemma 5.2.1. We apply Lemmas 5.2.1 and 5.2.3, Lipschitz continuity of g', the continuity property of the bilinear form and the Young's inequality to obtain

$$
\begin{aligned}
\int_0^T \|e_p\|^2_{L^2(\Omega)} dt \le\ & c(\delta)\, C^2_{5,8}\Big(\int_0^T \sum_{K\in\mathscr{T}_h^n} \int_K h_K^4 \Big(\frac{\partial P_h}{\partial t} + div(A\nabla \tilde{P}_h) - a_0 \tilde{P}_h \Big)^2 dx\, dt \\
& + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E h_E^3 ([A\nabla \tilde{P}_h \cdot n_E])^2 ds\, dt \\
& + \int_0^T \sum_{E\subset\Gamma} \int_E h_E^3 (A\nabla \tilde{P}_h \cdot n_E - g'(\hat{Y}_h))^2 ds\, dt \Big) \\
& + c(\delta)\alpha_0^2 \int_0^T \sum_{K\in\mathscr{T}_h^n} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + c(\delta) C_L^2 \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}} dt \\
& + \delta \int_0^T \|\chi\|^2_{\mathscr{B}} dt + \delta \int_0^T \|\chi\|^2_{H^2(\Omega)} dt,
\end{aligned}
$$

where $C^2_{5,8} = \tilde{C}^2_{I,1} \max\{\tilde{C}^2_{I,3}, 1\}$. We first use trace theorem, Lemma 1.2.3 with $\Psi = \chi$ and $f = p(U_h) - P_h$. Then, choose $\delta > 0$ such that $1 - 2\delta C_R^2 > 0$ and use the standard kick-back argument we obtain

$$
\int_0^T \|e_p\|^2_{L^2(\Omega)} dt \le C_{5,9}\Big(\mathscr{E}^2_{5,4} + \int_0^T \sum_{K\in\mathscr{T}_h^n} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}} dt \Big), \quad (5.59)
$$

with $C_{5,9} = \frac{c(\delta)}{1-2\delta C_R^2} \max\{C^2_{5,8}, \alpha_0^2, C_L^2\}$. Combining (5.57) with (5.59) and setting $C_{5,4} = C_{5,7}(1 + C_{5,9})$, we complete rest of the proof. □

It is evident from Lemma 5.3.2 that the estimation of $p(U_h) - P_h$ depends on the error $y(U_h) - \hat{Y}_h$. Since the bound of $y(U_h) - \hat{Y}_h$ is controlled by estimating $y(U_h) - Y_h$ in the $L^2(0, T; \mathscr{B})$-norm, we estimate the error $y(U_h) - Y_h$ in the following lemma.

Lemma 5.3.3. *Let* (Y_h, P_h, U_h) *and* $(y(u^*), p(u^*))$ *be the solutions of* (5.32) – (5.36) *and* (5.39) – (5.42) *with* $u^* = U_h$*, respectively. Let the assumptions of Lemma 5.3.1 be fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_1}$*. Then there exists a positive constant* $C_{5,10}$ *such that*

$$
\int_0^T \|y(U_h) - Y_h\|^2_{\mathscr{B}} dt \ \le\ C_{5,10}\Big(\sum_{i=5}^{9} \mathscr{E}^2_{5,i} \Big),
$$

with

$$
\begin{aligned}
\mathscr{E}_{5,5}^2 &= \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \int_K h_K^2 \Big(\bar{f}^n - \frac{\partial Y_h}{\partial t} + div(A\nabla\hat{Y}_h) - a_0 Y_h^n\Big)^2 dx\, dt \\
&\quad + \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset, E\subset\mathscr{T}^n_{\Omega_l}} \int_E h_E([A\nabla\hat{Y}_h\cdot n_E])^2 ds\, dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E(A\nabla\hat{Y}_h\cdot n_E - BU_h - \mathbf{x}_b)^2 ds\, dt, \\
\mathscr{E}_{5,6}^2 &= \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + \int_0^T \sum_{K\in\mathscr{T}^n_h} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt, \\
\mathscr{E}_{5,7}^2 &= \int_0^T \sum_{K\in\mathscr{T}^n_h} \int_K h_K^4 \Big(\bar{f}^n - \frac{\partial Y_h}{\partial t} + div(A\nabla\hat{Y}_h) - a_0 \hat{Y}_h\Big)^2 dx\, dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E h_E^3([A\nabla\hat{Y}_h\cdot n_E])^2 ds\, dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E^3(A\nabla\hat{Y}_h\cdot n_E - BU_h - \mathbf{x}_b)^2 ds\, dt, \\
\mathscr{E}_{5,8}^2 &= \int_0^T \|f - \bar{f}^n\|^2_{L^2(\Omega)} dt, \\
\mathscr{E}_{5,9}^2 &= \frac{1}{2}\|y_0 - y_{h,0}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where h_E and h_K be the diameters of E and K, respectively, and $[A\nabla\hat{Y}_h\cdot n_E]$ is defined as in (5.50). The constant $C_{5,10}$ depends on α_0, α_1, the bound C_M of the cut-off function, the regularity constant C_R and the constants $\tilde{C}_{I,j}$, $j = 1,\dots,5$ defined in Lemmas 5.2.1-5.2.3 and the inequality (5.38).

Proof. For all $v \in V$ and $v_h \in V_h^n$, $t \in I_n$, we have from (5.32)

$$
\begin{aligned}
\Big(\frac{\partial Y_h}{\partial t}, v\Big) + a(\hat{Y}_h, v) &= \Big(\frac{\partial Y_h}{\partial t}, v - v_h\Big) + a(\hat{Y}_h, v - v_h) + (BU_h + \mathbf{x}_b, v_h)_{\mathscr{B}} + (\bar{f}^n, v_h) \\
&= -\Big(\bar{f}^n - \frac{\partial Y_h}{\partial t}, v - v_h\Big) + a(\hat{Y}_h, v - v_h) + (BU_h + \mathbf{x}_b, v_h)_{\mathscr{B}} \\
&\quad + (\bar{f}^n, v). \qquad (5.60)
\end{aligned}
$$

Using (5.60) and (5.39) with $u^* = U_h$ we obtain

$$\begin{aligned}\Big(\frac{\partial}{\partial t}(y(U_h - Y_h)), v\Big) + a(y(U_h) - Y_h, v) &= \Big(\bar{f}^n - \frac{\partial Y_h}{\partial t}, v - v_h\Big) - a(\hat{Y}_h, v - v_h)\\ &\quad + (BU_h + \mathbf{x}_b, v - v_h)_{\mathscr{B}} + a(\hat{Y}_h - Y_h, v)\\ &\quad + (f - \bar{f}^n, v). \qquad (5.61)\end{aligned}$$

With $e_y = y(U_h) - Y_h$, a cut-off function ϕ defined as earlier and using the identity

$$\int_\Omega A\nabla(e_y)\cdot\nabla(\phi^2 e_y)dx = \int_\Omega A\nabla(\phi e_y)\cdot\nabla(\phi e_y)dx - \int_\Omega (e_y)^2 A\nabla\phi\cdot\nabla\phi dx,$$

we obtain

$$a(e_y, \phi^2 e_y) = a(\phi e_y, \phi e_y) - \int_\Omega (e_y)^2 A\nabla\phi\cdot\nabla\phi dx. \qquad (5.62)$$

Define $v = \phi^2 e_y := w_y$ and $v_h = \hat{\Pi}_h(\phi^2 e_y) := w_y^I$. Thus, from (5.61), it follows that

$$\begin{aligned}\Big(\frac{\partial e_y}{\partial t}, \phi^2 e_y\Big) + a(e_y, \phi^2 e_y) &= \Big(\bar{f}^n - \frac{\partial Y_h}{\partial t}, w_y - w_y^I\Big) + a(\hat{Y}_h, w_y^I - w_y)\\ &\quad + (BU_h + \mathbf{x}_b, w_y - w_y^I)_{\mathscr{B}} + a(\hat{Y}_h - Y_h, w_y) + (f - \bar{f}^n, w_y),\end{aligned}$$

with an aid of (5.62) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi e_y\|^2_{L^2(\Omega)} + a(\phi e_y, \phi e_y) &= \Big(\bar{f}^n - \frac{\partial Y_h}{\partial t}, w_y - w_y^I\Big) + a(\hat{Y}_h, w_y^I - w_y)\\ &\quad + (BU_h + \mathbf{x}_b, w_y - w_y^I)_{\mathscr{B}} + a(\hat{Y}_h - Y_h, w_y)\\ &\quad + (f - \bar{f}^n, w_y) + \int_\Omega (e_y)^2 A\nabla\phi\cdot\nabla\phi\, dx.\end{aligned}$$

Integrating the above from t_{n-1} to t_n with respect to time t to have

$$\begin{aligned}&\frac{1}{2}\big(\|\phi e_y(t_n)\|^2_{L^2(\Omega)} - \|\phi e_y(t_{n-1})\|^2_{L^2(\Omega)}\big) + \int_{t_{n-1}}^{t_n} a(\phi e_y, \phi e_y)dt\\ &= \int_{t_{n-1}}^{t_n}\Big\{\Big(\bar{f}^n - \frac{\partial Y_h}{\partial t}, w_y - w_y^I\Big) + a(\hat{Y}_h, w_y^I - w_y) + (BU_h + \mathbf{x}_b, w_y - w_y^I)_{\mathscr{B}}\Big\}dt\\ &\quad + \int_{t_{n-1}}^{t_n} a(\hat{Y}_h - Y_h, w_y)dt + \int_{t_{n-1}}^{t_n}(f - \bar{f}^n, w_y)dt + \int_{t_{n-1}}^{t_n}\int_\Omega (e_y)^2 A\nabla\phi\cdot\nabla\phi\, dx\, dt,\end{aligned}$$

summing up from 1 to N, we obtain

$$
\begin{aligned}
&\frac{1}{2}\big(\|\phi e_y(T)\|^2_{L^2(\Omega)} - \|\phi e_y(0)\|^2_{L^2(\Omega)}\big) + \int_0^T a(\phi e_y, \phi e_y)dt \\
&= \int_0^T \Big\{\Big(\bar{f}^n - \frac{\partial Y_h}{\partial t}, w_y - w_y^I\Big) + a(\hat{Y}_h, w_y^I - w_y) + (BU_h + \mathbf{x}_b, w_y - w_y^I)_{\mathscr{B}}\Big\}dt \\
&\quad + \int_0^T a(\hat{Y}_h - Y_h, w_y)dt + \int_0^T (f - \bar{f}^n, w_y)dt + \int_0^T \int_\Omega (e_y)^2 A\nabla\phi \cdot \nabla\phi \, dx\, dt \\
&=: I_5 + I_6 + I_7 + I_8. \qquad (5.63)
\end{aligned}
$$

We proceed to estimate the terms I_j, $j = 5, 6, 7, 8$. First, the term I_5 may be written as

$$
\begin{aligned}
I_5 \;=\; & \int_0^T \sum_K \int_K \Big\{(\bar{f}^n - \frac{\partial Y_h}{\partial t} + div(A\nabla\hat{Y}_h) - a_0\hat{Y}_h)(w_y - w_y^I)\Big\}dx\, dt \\
& + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E [A\nabla\hat{Y}_h \cdot n_E](w_y - w_y^I)ds\, dt \\
& + \int_0^T \sum_{E\subset\Gamma} \int_E (A\nabla\hat{Y}_h \cdot n_E - BU_h - \mathbf{x}_b)(w_y - w_y^I)ds\, dt.
\end{aligned}
$$

Since $\phi = 0$ on $\Omega \setminus \Omega_l$, use of Lemma 5.2.3, the inequality (5.38) and the Young's inequality yield

$$
I_5 \;\leq\; c(\delta)\, C_{5,5}^2\, \mathscr{E}_{5,5}^2 + \delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)}dt.
$$

For I_6, the continuity of the bilinear form and the Young's inequality imply

$$
\begin{aligned}
I_6 \;&=\; \int_0^T a(\hat{Y}_h - Y_h, w_y)dt \\
&\leq\; c(\delta)\, \alpha_0^2\, C_{5,6}^2 \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)}dt + \delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)}dt.
\end{aligned}
$$

To estimate I_7, we use (5.38), the Cauchy-Schwarz inequality and the Young's inequality to obtain

$$
\begin{aligned}
I_7 \;&\leq\; \int_0^T \|f - \bar{f}^n\|_{L^2(\Omega)}\|w_y\|_{L^2(\Omega)}dt \\
&\leq\; c(\delta)\, C_{5,6}^2 \int_0^T \|f - \bar{f}^n\|^2_{L^2(\Omega)}dt + \delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)}dt.
\end{aligned}
$$

Arguing as I_4, we finally estimate I_8 as

$$
I_8 \;\leq\; C_M^2 \int_0^T \|e_y\|^2_{L^2(\Omega)}dt.
$$

Combining the above estimates with (5.63), and using the coercive property of the bilinear form we conclude that

$$\begin{aligned}
&\frac{1}{2}\|\phi e_y(T)\|^2_{L^2(\Omega)} + \alpha_1 \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt \\
&\leq \; c(\delta)\, C^2_{5,5}\, \mathscr{E}^2_{5,5} + c(\delta)\, \alpha_0^2\, C^2_{5,6} \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt \\
&\quad + c(\delta)\, C^2_{5,6} \int_0^T \|f - \bar{f}^n\|^2_{L^2(\Omega)} dt + \frac{C^2_M}{2}\|y_0 - y_{h,0}\|^2_{L^2(\Omega)} \\
&\quad + C^2_M \int_0^T \|e_y\|^2_{L^2(\Omega)} dt + 3\delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt.
\end{aligned}$$

Setting $C_{5,11} = \max\{c(\delta)C^2_{5,5}, c(\delta)\alpha_0^2 C^2_{5,6}, c(\delta)C^2_{5,6}, C^2_M\}$ with $\delta = \alpha_1/6$ and use the kick-back argument by adjusting the constants to arrive at

$$\begin{aligned}
&\frac{1}{2}\|\phi e_y(T)\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2} \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt \\
&\leq C_{5,11}\Big(\mathscr{E}^2_{5,5} + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + \int_0^T \|f - \bar{f}^n\|^2_{L^2(\Omega)} dt \\
&\quad + \frac{1}{2}\|y_0 - y_{h,0}\|^2 + \int_0^T \|e_y\|^2_{L^2(\Omega)} dt\Big).
\end{aligned}$$

An application of the trace inequality with cut-off function ϕ implies

$$\|e_y\|_{L^2(0,T;\mathscr{B})} \;=\; \|\phi e_y\|_{L^2(0,T;\mathscr{B})} \;\leq\; \|\phi e_y\|_{L^2(0,T;H^1(\Omega))}, \tag{5.64}$$

using (5.64) in the above leads to

$$\begin{aligned}
&\frac{1}{2}\|\phi e_y(T)\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2} \int_0^T \|e_y\|^2_{\mathscr{B}} dt \\
&\leq C_{5,11}\Big(\mathscr{E}^2_{5,5} + \mathscr{E}^2_{5,8} + \mathscr{E}^2_{5,9} + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + \int_0^T \|e_y\|^2_{L^2(\Omega)} dt\Big).
\end{aligned} \tag{5.65}$$

To estimate the error e_y in the norm $L^2(0,T;L^2(\Omega))$, we employ duality trick and integration by parts formula to have

$$\begin{aligned}
\int_0^T \|e_y\|^2_{L^2(\Omega)} dt &= \int_0^T \Big(e_y, -\frac{\partial\psi}{\partial t}\Big) + a(e_y,\psi)\, dt \\
&= \int_0^T \Big\{\Big(\frac{\partial e_y}{\partial t},\psi\Big) + a(e_y,\psi)\Big\}\, dt + (y_0 - y_{h,0}, \psi(0)) \\
&= \int_0^T (f,\psi) dt + \int_0^T (BU_h + \mathbf{x}_b, \psi)_{\mathscr{B}}\, dt - \int_0^T \Big\{\Big(\frac{\partial Y_h}{\partial t},\psi\Big) + a(\hat{Y}_h,\psi)\Big\} dt \\
&\quad + \int_0^T a(\hat{Y}_h - Y_h, \psi) dt + (y_0 - y_{h,0}, \psi(0)).
\end{aligned}$$

For any $w \in V$, $w_h \in V_h^n$, a similar argument as (5.52) yields

$$\begin{aligned}\Big(\frac{\partial Y_h}{\partial t}, w\Big) + a(\hat{Y}_h, w) \;&= -\Big(\bar{f}^n - \frac{\partial Y_h}{\partial t}, w - w_h\Big) + a(\hat{Y}_h, w - w_h) + (\bar{f}^n, w) \\ &\quad + (BU_h + \mathbf{x}_b, w_h)_{\mathscr{B}}. \end{aligned} \tag{5.66}$$

Use of (5.66) in the above to have

$$\begin{aligned}\int_0^T \|e_y\|^2_{L^2(\Omega)}dt \;&=\; \int_0^T \Big\{(\bar{f}^n - \frac{\partial Y_h}{\partial t}, \psi - \psi^I) - a(\hat{Y}_h, \psi - \psi^I)\Big\}dt \\ &\quad + \int_0^T (BU_h + \mathbf{x}_b, \psi - \psi^I)_{\mathscr{B}}dt + \int_0^T a(\hat{Y}_h - Y_h, \psi)dt \\ &\quad + \int_0^T (f - \bar{f}^n, \psi)dt + (y_0 - y_{h,0}, \psi(0)).\end{aligned}$$

Which may be rewritten as

$$\begin{aligned}\int_0^T \|e_y\|^2_{L^2(\Omega)}dt \;&=\; \int_0^T \Big(\bar{f}^n - \frac{\partial Y_h}{\partial t} + div(A\nabla\hat{Y}_h) - a_0\hat{Y}_h, \psi - \psi^I\Big)dt \\ &\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E [A\nabla\hat{Y}_h\cdot n_E](\psi^I - \psi)ds\,dt \\ &\quad + \int_0^T \sum_{E\subset\Gamma} \int_E (A\nabla\hat{Y}_h\cdot n_E - BU_h - \mathbf{x}_b)(\psi^I - \psi)ds\,dt \\ &\quad + \int_0^T a(\hat{Y}_h - Y_h, \psi)dt + \int_0^T (f - \bar{f}^n, \psi)dt + (y_0 - y_{h,0}, \psi(0)),\end{aligned}$$

where ψ^I is the interpolation of ψ, as defined in Lemma 5.2.1. An application of Lemmas 5.2.1 and 5.2.3, the continuity property of bilinear form and the Young's inequality leads to

$$\begin{aligned}\int_0^T \|e_y\|^2_{L^2(\Omega)}dt \le c(\delta)\Big(C^2_{5,8}\mathscr{E}^2_{5,7} + \alpha_0^2\int_0^T \sum_{K\in\mathscr{T}_h^n} \|\hat{Y}_h - Y_h\|^2_{H^1(K)}dt + \int_0^T \|f - \bar{f}^n\|^2_{L^2(\Omega)}dt\Big) \\ + \frac{1}{2}\|y_0 - y_{h,0}\|^2_{L^2(\Omega)}dt + \frac{1}{2}\|\psi(0)\|^2_{L^2(\Omega)} + \delta\int_0^T \|\psi\|^2_{H^2(\Omega)}dt.\end{aligned}$$

We apply Lemma 1.2.3 with $\Phi = \psi$, $f = y(U_h) - Y_h$. Then setting $\delta > 0$ such that $1 - 2\delta C_R^2 > 0$ with $C_{5,12} = \frac{2}{1-2\delta C_R^2}\max\{c(\delta)C^2_{5,8}, c(\delta)\alpha_0^2, 1\}$, and employing the standard kick-back argument to obtain

$$\int_0^T \|e_y\|^2_{L^2(\Omega)}dt \;\le\; C_{5,12}\Big(\mathscr{E}^2_{5,7}+\mathscr{E}^2_{5,8}+\mathscr{E}^2_{5,9}+\int_0^T \sum_{K\in\mathscr{T}_h^n} \|\hat{Y}_h - Y_h\|^2_{H^1(K)}dt\Big). \tag{5.67}$$

Now we substitute (5.67) in (5.65), and set $C_{5,10} = C_{5,11}(1 + C_{5,12})$ to obtain the desire result. □

Now we are in the position to develop the main result of this section by collecting Lemmas 5.3.1 – 5.3.3.

Theorem 5.3.1. *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* (5.6) – (5.10) *and* (5.32) – (5.36)*, respectively. Let the assumptions of Lemma 5.3.1 be fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_1}$*. Then there exists a positive constant* $C_{5,13}$ *such that*

$$\|u - U_h\|^2_{L^2(0,T;\mathscr{B})} + \|y - Y_h\|^2_{L^2(0,T;\mathscr{B})} + \|p - P_h\|^2_{L^2(0,T;\mathscr{B})} \ \le\ C_{5,13} \Big(\sum_{i=1}^{9} \mathscr{E}^2_{5,i} \Big),$$

where $\mathscr{E}^2_{5,i}$*,* $i = 1, \ldots, 9$ *are defined as in Lemmas* 5.3.1 – 5.3.3*. The constant* $C_{5,13}$ *depends on* α_0*,* α_1*, the Lipschitz-constant* C_L*, the bound* C_M *of the cut-off function, the stability constant* C_R*, the constant* $C_{5,3}$ *defined in Lemma 5.3.1, and the constants* $\tilde{C}_{I,j}$*,* $j = 1, \ldots, 5$ *defined in Lemmas* 5.2.1 – 5.2.3 *and the inequality* (5.38)*.*

Proof. It follows from the triangle inequality that

$$\begin{aligned} \|y - Y_h\|^2_{L^2(0,T;\mathscr{B})} &\le \|y - y(U_h)\|^2_{L^2(0,T;\mathscr{B})} + \|y(U_h) - Y_h\|^2_{L^2(0,T;\mathscr{B})} \\ &\le C^2_{5,14}\|u - U_h\|^2_{L^2(0,T;\mathscr{B})} + \|y(U_h) - Y_h\|^2_{L^2(0,T;\mathscr{B})}, \end{aligned} \tag{5.68}$$

and

$$\begin{aligned} \|p - P_h\|^2_{L^2(0,T;\mathscr{B})} &\le \|p - p(U_h)\|^2_{L^2(0,T;\mathscr{B})} + \|p(U_h) - P_h\|^2_{L^2(0,T;\mathscr{B})} \\ &\le C^2_L\|y - y(U_h)\|^2_{L^2(0,T;\mathscr{B})} + \|p(U_h) - P_h\|^2_{L^2(0,T;\mathscr{B})} \\ &\le C^2_{5,14}C^2_L\|u - U_h\|^2_{L^2(0,T;\mathscr{B})} + \|p(U_h) - P_h\|^2_{L^2(0,T;\mathscr{B})}, \end{aligned} \tag{5.69}$$

where $C_{5,14}$ is a positive constant. We know that

$$\|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;\mathscr{B})} \ \le\ \|y(U_h) - Y_h\|^2_{L^2(0,T;\mathscr{B})} + \|\hat{Y}_h - Y_h\|^2_{L^2(0,T;H^1(\Omega))}. \tag{5.70}$$

Similarly,

$$\|p(U_h) - \tilde{P}_h\|^2_{L^2(0,T;\mathscr{B})} \ \le\ \|p(U_h) - P_h\|^2_{L^2(0,T;\mathscr{B})} + \|\tilde{P}_h - P_h\|^2_{L^2(0,T;H^1(\Omega))}. \tag{5.71}$$

Combine (5.68) – (5.71) with Lemmas 5.3.1 – 5.3.3 to complete the rest of the proof. □

Remark 5.3.1. (*i*) *The a posteriori error estimate in Theorem 5.3.1 consists of three parts. The estimators* $\mathscr{E}^2_{5,i}$*,* $i \in \{2, 4, 5, 7\}$ *are contributed from the approximation error of the state and co-state equations, whereas* $\mathscr{E}^2_{5,1}$ *reflects the approximation error for the* *control variable which is derived from the variational inequality. Further,* $\mathscr{E}^2_{5,3}$ *and* $\mathscr{E}^2_{5,6}$ *denote the temporal error estimators for the state and the co-state variables, respectively. Moreover, the data approximation errors are given by* $\mathscr{E}^2_{5,8}$ *and* $\mathscr{E}^2_{5,9}$*.*

(ii) For the sake of comparison, we refer to [34]. The derived estimators in Theorem 5.3.1 are different from the estimators of [34, Theorem 3.7]. Our estimators are of local character, i.e., the leading terms of $\mathscr{E}^2_{5,i}$*,* $i \in \{1,2,3,5,6\}$ *depend only on the data in a smaller neighborhood of the boundary. But, the global influences are reflected by the higher order terms* $\mathscr{E}^2_{5,4}$ *and* $\mathscr{E}^2_{5,7}$ *which can not be ignored. By making* Ω_l *smaller (i.e., reducing the distance* l*), the resulting mesh refinement could be more concentrated around the boundary.*

5.3.2 Control problem with observation space $\mathcal{O}_{sp} = \mathcal{O}_{sp_2}$

This section analyses the control problem (5.1) – (5.3) with observation space $\mathcal{O}_{sp_2}$. We derive a local *a posteriori* error bounds for the **Case II**. Similar to Section 5.3.1, use of space-time finite element discretization for the boundary control problem (5.1) – (5.3) leads to the following optimality conditions for $1 \leq n \leq N$:

$$\begin{aligned}
\Big(\frac{\partial Y_h}{\partial t}, v_h\Big) + a(\hat{Y}_h, v_h) &= (\bar{f}^n, v_h) + (BU_h + \mathbf{x}_b, v_h)_{\mathscr{B}} \quad \forall v_h \in V_h^n, && (5.72)\\
Y_h^0 &= y_{h,0}, && (5.73)\\
-\Big(\frac{\partial P_h}{\partial t}, v_h\Big) + a(\tilde{P}_h, v_h) &= (g'(\hat{Y}_h), v_h) \quad \forall v_h \in V_h^n, && (5.74)\\
P_h^N &= 0, && (5.75)\\
(h'(U_h) + B^*\tilde{P}_h, w_h^n - U_h)_{\mathscr{B}} &\geq 0 \quad \forall w_h^n \in U^n_{ad,h}. && (5.76)
\end{aligned}$$

To begin with, we first establish some intermediate error estimates for the state and co-state variables which will enable us to prove the main results. For this purpose, we now introduce some intermediate problems. Let $(y(u^*), p(u^*)) \in \mathcal{W} \times \mathcal{W}$ with $u^* \in U_{ad}$ be the solutions of the following systems:

$$\begin{aligned}
\Big(\frac{\partial y(u^*)}{\partial t}, v\Big) + a(y(u^*), v) &= (f, v) + (Bu^* + \mathbf{x}_b, v)_{\mathscr{B}} \quad \forall v \in V, && (5.77)\\
y(u^*)(\cdot, 0) &= y_0(x) \quad x \in \Omega, && (5.78)\\
-\Big(\frac{\partial p(u^*)}{\partial t}, v\Big) + a(p(u^*), v) &= (g'(y(u^*)), v) \quad \forall v \in V, && (5.79)\\
p(u^*)(\cdot, T) &= 0 \quad x \in \Omega. && (5.80)
\end{aligned}$$

Analogous to Lemma 5.3.1 the proof of the following lemma is easily obtained, and hence we omit the details.

Lemma 5.3.4. *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* $(5.11) - (5.15)$ *and* $(5.72) - (5.76)$, *respectively. Assume that* $U^n_{ad,h} \subset U_{ad}$, $(h'(U_h) + B^*\tilde{P}_h)|_\tau \in H^1(\tau)$ *and there exists a positive constant* $C_{5,15}$ *and* $w_h \in U^n_{ad,h}$ *such that*

$$\left|\int_0^T (h'(U_h)+B^*\tilde{P}_h, w_h-u)_{\mathscr{B}}\,dt\right| \le C_{5,15}\int_0^T \sum_{\tau\in\mathscr{T}^n_{\mathscr{B}}} h_\tau |h'(U_h)+B^*\tilde{P}_h|_{H^1(\tau)}\|u-U_h\|_{L^2(\tau)}\,dt.$$

Then, we have

$$\|u - U_h\|^2_{L^2(0,T;\mathscr{B})} \le C_{5,16}\Big(\hat{\mathscr{E}}^2_{5,1} + \|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;\mathscr{B})}\Big),$$

where $C_{5,16} = \frac{3}{2(2C_{5,1}-1)}\max\{C^2_{5,15}, 1\}$, *and* $\hat{\mathscr{E}}^2_{5,1} = \mathscr{E}^2_{5,1}$ *is defined in Lemma 5.3.1 and the pair* $(y(U_h), p(U_h))$ *is a solution of the problem* $(5.77) - (5.80)$ *with* $u^* = U_h$.

Lemma 5.3.5. *Let* (Y_h, P_h, U_h) *and* $(y(u^*), p(u^*))$ *be the solutions of* $(5.72) - (5.76)$ *and* $(5.77) - (5.80)$ *with* $u^* = U_h$, *respectively. Let the assumptions of Lemma 5.3.4 be fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_2}$. *Then there exists a positive constant* $C_{5,17}$ *such that*

$$\int_0^T \|p(U_h) - P_h\|^2_{\mathscr{B}}dt \le C_{5,17}\left(\sum_{i=2}^4 \hat{\mathscr{E}}^2_{5,i} + \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{L^2(\Omega)}dt\right),$$

with

$$\begin{aligned}
\hat{\mathscr{E}}^2_{5,2} &= \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \int_K h^2_K\Big(g'(\hat{Y}_h) + \frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h\Big)^2 dx\,dt\\
&\quad + \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset, E\subset\mathscr{T}^n_{\Omega_l}} \int_E h_E([A\nabla\tilde{P}_h\cdot n_E])^2 ds\,dt\\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E(A\nabla\tilde{P}_h\cdot n_E)^2 ds\,dt,\\
\hat{\mathscr{E}}^2_{5,4} &= \int_0^T \sum_{K\in\mathscr{T}^n_h} \int_K h^4_K\Big(g'(\hat{Y}_h) + \frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h\Big)^2 dx\,dt\\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E h^3_E([A\nabla\tilde{P}_h\cdot n_E])^2 ds\,dt\\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h^3_E(A\nabla\tilde{P}_h\cdot n_E)^2 ds\,dt,
\end{aligned}$$

and $\hat{\mathscr{E}}^2_{5,3} = \mathscr{E}^2_{5,3}$, *where* $\mathscr{E}^2_{5,3}$ *is defined in Lemma 5.3.2. Here,* h_E *and* h_K *be the diameters of* E *and* K, *respectively, and* $[A\nabla\tilde{P}_h\cdot n_E]$ *is defined in (5.51).*

In the next lemma, we provide an intermediate *a posteriori* error estimate for the state variable. The proof follows from Lemma 5.3.3.

Lemma 5.3.6. *Let* (Y_h, P_h, U_h) *and* $(y(u^*), p(u^*))$*, respectively be the solutions of* (5.72)−(5.76) *and* (5.77) − (5.80) *with* $u^* = U_h$*. Let the assumptions of Lemma 5.3.4 are fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_2}$*. Then there exists a constant* $C_{5,18}$ *such that*

$$\int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt \; \leq \; C_{5,18} \Big(\sum_{i=5}^{8} \hat{\mathscr{E}}^2_{5,i} \Big),$$

with

$$\hat{\mathscr{E}}^2_{5,5} := \int_0^T \sum_{K \in \mathscr{T}_h^n} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt,$$

where $\hat{\mathscr{E}}^2_{5,i} = \mathscr{E}^2_{5,i+1}$*,* $i = 6, 7, 8$*, are defined in Lemma 5.3.3.*

By collecting Lemmas 5.3.4 − 5.3.6, we now present our main result for this section.

Theorem 5.3.2. *Let* (y, p, u) *and* (Y_h, P_h, U_h) *be the solutions of* (5.11) − (5.15) *and* (5.72) − (5.76)*, respectively. Let the assumptions of Lemma 5.3.4 are fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_2}$*. Then there exists a positive constant* $C_{5,19}$ *such that*

$$\|u - U_h\|^2_{L^2(0,T;\mathscr{B})} + \|y - Y_h\|^2_{L^2(0,T;L^2(\Omega))} + \|p - P_h\|^2_{L^2(0,T;\mathscr{B})} \; \leq \; C_{5,19} \Big(\sum_{i=1}^{8} \hat{\mathscr{E}}^2_{5,i} \Big),$$

where $\hat{\mathscr{E}}^2_{5,i}$*,* $i = 1, \ldots, 8$*, are defined as in Lemmas* 5.3.4 − 5.3.6*, respectively. The constant* $C_{5,19}$ *depends on* α_0*,* α_1*, the convexity constant* $C_{5,1}$*, the Lipschitz-constant* C_L*, the bound* C_M *of the cut-off function, the regularity constant* C_R*, and the constants* $\tilde{C}_{I,j}$*,* $j = 1, \ldots, 5$ *defined in Lemmas* 5.2.1 − 5.2.3 *and the inequality* (5.38)*.*

5.3.3 Control problem with observation space $\mathcal{O}_{sp} = \mathcal{O}_{sp_3}$

This section concerns our last case i.e., **Case III** which is very important. We shall derive *a posteriori* error estimates for the control problem (5.16)−(5.20) with observation of the final state (i.e. observation space $\mathcal{O}_{sp_3}$). Here the objective functional $\mathcal{J}(\cdot, \cdot)$ consists of terms involving the functional $\mathcal{G}(\cdot)$ at the final time T and the functional $\mathcal{H}(\cdot)$ on the boundary.

In this case, for $1 \leq n \leq N$, the optimality conditions read as:

$$\begin{aligned}
\Big(\frac{\partial Y_h}{\partial t}, v_h\Big) + a(\hat{Y}_h, v_h) &= (\bar{f}^n, v_h) + (BU_h + \mathbf{x}_b, v_h)_{\mathscr{B}} \quad \forall v_h \in V_h^n, && (5.81)\\
Y_h^0 &= y_{h,0}, && (5.82)\\
-\Big(\frac{\partial P_h}{\partial t}, v_h\Big) + a(\tilde{P}_h, v_h) &= 0 \quad \forall v_h \in V_h^n, && (5.83)\\
P_h^N &= g'(Y_h(\cdot, T))_h, && (5.84)\\
(h'(U_h) + B^* \tilde{P}_h, w_h^n - U_h)_{\mathscr{B}} &\geq 0 \quad \forall w_h^n \in U_{ad,h}^n. && (5.85)
\end{aligned}$$

Note that, $g'(Y_h(\cdot, T))_h \in V_h^N$ is an approximation of $g'(Y_h(\cdot, T))$. We now derive some intermediate *a posteriori* error estimates for which we need to introduce the following auxiliary problem. Let $(y(u^*), p(u^*)) \in \mathcal{W} \times \mathcal{W}$ with $u^* \in U_{ad}$ satisfies the following equations:

$$\begin{aligned}
\Big(\frac{\partial y(u^*)}{\partial t}, v\Big) + a(y(u^*), v) &= (f, v) + (Bu^* + \mathbf{x}_b, v)_{\mathscr{B}} \quad \forall v \in V, && (5.86)\\
y(u^*)(\cdot, 0) &= y_0(x) \quad x \in \Omega, && (5.87)\\
-\Big(\frac{\partial p(u^*)}{\partial t}, v\Big) + a(p(u^*), v) &= 0 \quad \forall v \in V, && (5.88)\\
p(u^*)(\cdot, T) &= g'(y(u^*)(\cdot, T)) \quad x \in \Omega. && (5.89)
\end{aligned}$$

Observe that we have non zero conditions for the co-state variable at final time $t = T$ which will play crucial role in our analysis. In the following lemma, we now estimate error for the control variable in the $L^2(0, T; \mathscr{B})$-norm.

Lemma 5.3.7. *Let (y, p, u) and (Y_h, P_h, U_h) be the solutions of (5.16) – (5.20) and (5.81) – (5.85), respectively. Assume that $U_{ad,h}^n \subset U_{ad}$, $(h'(U_h) + B^* \tilde{P}_h)|_\tau \in H^1(\tau)$ and there exists a positive constant $C_{5,20}$ and $w_h \in U_{ad,h}^n$ such that*

$$\Big| \int_0^T (h'(U_h) + B^* \tilde{P}_h, w_h - u)_{\mathscr{B}} \, dt \Big| \leq C_{5,20} \int_0^T \sum_{\tau \in \mathscr{T}_{\mathscr{B}}^n} h_\tau |h'(U_h) + B^* \tilde{P}_h|_{H^1(\tau)} \|u - U_h\|_{L^2(\tau)} \, dt.$$

Then, we have

$$\|u - U_h\|_{L^2(0,T;\mathscr{B})}^2 \leq C_{5,21} \Big(\tilde{\mathscr{E}}_{5,1}^2 + \|\tilde{P}_h - p(U_h)\|_{L^2(0,T;\mathscr{B})}^2 \Big),$$

with $C_{5,21} = \dfrac{3}{2(2C_{5,1} - 1)} \max\big\{C_{5,20}^2, 1\big\}$. *Here* $\tilde{\mathscr{E}}_{5,1}^2 = \mathscr{E}_{5,1}^2$ *is defined in Lemma 5.3.1, and $(y(U_h), p(U_h))$ is solution of the problem (5.86) – (5.89) with $u^* = U_h$.*

Proof. Arguing as in the proof of Lemma 5.3.1, we reach at (5.44) with the last term J_3 is replaced by

$$J_4 := \int_0^T (B^*(p(U_h) - p), u - U_h)_{\mathscr{B}} \, dt.$$

Since the estimation of the first two terms of (5.44) is treated in a similar manner as in Lemma 5.3.1, it is enough to estimate the term J_4. For this, we subtract (5.88) from (5.18) with $u^* = U_h$ and $v = y(U_h) - y$, then integrate the resulting equation from 0 to T to have

$$-\int_0^T \Big(\frac{\partial}{\partial t}(p - p(U_h)), y(U_h) - y\Big)\, dt + \int_0^T a(p - p(U_h), y(U_h) - y)\, dt = 0.$$

Use integration by parts formula with $(y(u) - y(U_h))(0) = 0$ and $(p(u) - P(U_h))(T) = (g'(y) - g'(y(U_h)))(T)$ to obtain

$$\int_0^T \Big(p - p(U_h), \frac{\partial}{\partial t}(y(U_h) - y)\Big) dt + \int_0^T a(p - p(U_h), y(U_h) - y)\, dt$$
$$= \Big((g'(y) - g'(y(U_h)))(T), (y(U_h) - y)(T)\Big). \quad (5.90)$$

From (5.47) and (5.90), use of (5.21) leads to

$$J_4 = -\big(g'(y)(T) - g'(y(U_h))(T), y(T) - y(U_h)(T)\big) \leq 0. \quad (5.91)$$

Altogether (5.45), (5.46), (5.91) with (5.44) completes the rest of the proof. □

Next, we bound the error $P_h - p(U_h)$ in the $L^2(0, T; \mathscr{B})$-norm for the boundary control problem (5.1) – (5.3) with observation of the final state $y(x, T)$.

Lemma 5.3.8. *Let* (Y_h, P_h, U_h) *and* $(y(u^*), p(u^*))$ *be the solutions of* (5.81) – (5.85) *and* (5.86) – (5.89) *with* $u^* = U_h$*, respectively. Let the assumption of Lemma 5.3.7 are fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_3}$*. Then there exists a positive constant* $C_{5,22}$ *such that*

$$\int_0^T \|p(U_h) - P_h\|^2_{\mathscr{B}} dt \ \leq \ C_{5,22} \Big(\sum_{i=2}^{4} \tilde{\mathscr{E}}^2_{5,i} + \tilde{\mathscr{E}}^2_{5,10} + \|y(U_h)(T) - Y_h(T)\|^2_{L^2(\Omega)} \Big),$$

with

$$\begin{aligned}
\tilde{\mathscr{E}}^2_{5,2} \ &= \ \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \int_K h_K^2 \Big(\frac{\partial P_h}{\partial t} + div(A\nabla \tilde{P}_h) - a_0 \tilde{P}_h\Big)^2 dx\, dt \\
&+ \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset, E\subset\mathscr{T}^n_{\Omega_l}} \int_E h_E([A\nabla \tilde{P}_h \cdot n_E])^2 ds\, dt \\
&+ \int_0^T \sum_{E\subset\Gamma} \int_E h_E(A\nabla \tilde{P}_h \cdot n_E)^2 ds\, dt,
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathscr{E}}^2_{5,4} &= \int_0^T \sum_{K\in\mathscr{T}_h^n} \int_K h_K^4 \Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h\Big)^2 dx\,dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E h_E^3([A\nabla\tilde{P}_h\cdot n_E])^2 ds\,dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E^3(A\nabla\tilde{P}_h\cdot n_E)^2 ds\,dt, \\
\tilde{\mathscr{E}}^2_{5,10} &= \frac{1}{2}\|g'(Y_h(T)) - g'(Y_h(T))_h\|^2_{L^2(\Omega)},
\end{aligned}$$

and $\tilde{\mathscr{E}}^2_{5,3} = \mathscr{E}^2_{5,3}$*, where* $\mathscr{E}^2_{5,3}$ *is defined in Lemma 5.3.2 and the jump* $[A\nabla\tilde{P}_h\cdot n_E]$ *is given by (5.51). The constant* $C_{5,22}$ *depends on* α_0*,* α_1*, the Lipschitz-constant* C_L*, the bound* C_M *of the cut-off function, the regularity constant* C_R *and the constants* $\tilde{C}_{I,j}$*,* $j = 1,\dots,5$ *defined in Lemmas* 5.2.1 – 5.2.3 *and the inequality* (5.38)*.*

Proof. From (5.83), for all $v\in V$ and $v_h\in V_h^n$, $t\in I_n$, we have

$$-\Big(\frac{\partial P_h}{\partial t}, v\Big) + a(\tilde{P}_h, v) = -\Big(\frac{\partial P_h}{\partial t}, v - v_h\Big) + a(\tilde{P}_h, v - v_h). \tag{5.92}$$

Subtracting (5.92) from (5.88) with $u^* = U_h$, we obtain

$$\begin{aligned}
-\Big(\frac{\partial}{\partial t}(p(U_h) - P_h), v\Big) + a(p(U_h) - P_h, v) &= \Big(\frac{\partial P_h}{\partial t}, v - v_h\Big) - a(\tilde{P}_h, v - v_h) \\
&\quad + a(\tilde{P}_h - P_h, v).
\end{aligned} \tag{5.93}$$

Set $v = \phi^2 e_p := w_p$ and $v_h = \hat{\Pi}_h(\phi^2 e_p) := w_p^I$ in (5.93), then use of (5.54) leads to

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\|\phi e_p\|^2_{L^2(\Omega)} + a(\phi e_p, \phi e_p) &= \Big(\frac{\partial P_h}{\partial t}, w_p - w_p^I\Big) - a(\tilde{P}_h, w_p - w_p^I) + a(\tilde{P}_h - P_h, w_p) \\
&\quad + \int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\,dx.
\end{aligned}$$

Integrating with respect to t from t_n to t_{n-1}, we obtain

$$\begin{aligned}
&\frac{1}{2}\big(\|\phi e_p(t_{n-1})\|^2_{L^2(\Omega)} - \|\phi e_p(t_n)\|^2_{L^2(\Omega)}\big) + \int_{t_{n-1}}^{t_n} a(\phi e_p, \phi e_p)dt \\
&= \int_{t_{n-1}}^{t_n} \Big\{\Big(\frac{\partial P_h}{\partial t}, w_p - w_p^I\Big) + a(\tilde{P}_h, w_p^I - w_p)\Big\}dt + \int_{t_{n-1}}^{t_n} a(\tilde{P}_h - P_h, w_p)dt \\
&\quad + \int_{t_{n-1}}^{t_n}\int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\,dx\,dt.
\end{aligned} \tag{5.94}$$

Summing up from 1 to N, we have

$$\begin{aligned}
&\frac{1}{2}\big(\|\phi e_p(0)\|^2_{L^2(\Omega)} - \|\phi e_p(T)\|^2_{L^2(\Omega)}\big) + \int_0^T a(\phi e_p, \phi e_p)dt \\
&= \int_0^T \Big\{\Big(\frac{\partial P_h}{\partial t}, w_p - w_p^I\Big) + a(\tilde{P}_h, w_p^I - w_p)\Big\}dt + \int_0^T a(\tilde{P}_h - P_h, w_p)dt \\
&\quad + \int_0^T \int_\Omega (e_p)^2 A\nabla\phi \cdot \nabla\phi \, dx \, dt =: (III)_1 + (III)_2 + (III)_3.
\end{aligned} \tag{5.95}$$

We now proceed to estimate $(III)_j$, $j = 1, 2, 3$, separately. To bound $(III)_1$, we argue as the term I_1 of Lemma 5.3.2 and we use inequality (5.38), the Cauchy Schwarz inequality and the Young's inequality to obtain

$$(III)_1 \;\leq\; c(\delta)\, C^2_{5,5}\, \tilde{\mathscr{E}}^2_{5,2} + \delta \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt.$$

By the continuity of the bilinear form and the Young's inequality, we have for the term $(III)_2$,

$$(III)_2 \;\leq\; c(\delta)\alpha_0^2 C^2_{5,6} \int_0^T \sum_{K \in \mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + \delta \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt.$$

Finally, we estimate $(III)_3$ as

$$(III)_3 \;\leq\; \|\phi\|^2_{1,\infty,\Omega} \int_0^T \|e_p\|^2_{L^2(\Omega)} dt \leq C_M^2 \int_0^T \|e_p\|^2_{L^2(\Omega)} dt.$$

Substituting all the above estimates in (5.95) and using the coercivity of the bilinear form, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \alpha_1 \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt \\
&\leq c(\delta) C^2_{5,5} \tilde{\mathscr{E}}^2_{5,2} + c(\delta)\alpha_0^2 C^2_{5,6} \int_0^T \sum_{K \in \mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + C_M^2 \int_0^T \|e_p\|^2_{L^2(\Omega)} dt \\
&\quad + \frac{C_M^2}{2} \|p(U_h)(T) - P_h(T)\|^2_{L^2(\Omega)} + 2\delta \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt.
\end{aligned}$$

An application of the trace property (5.56) and the standard kick-back argument with $\delta = \alpha_1/4$ to obtain

$$\begin{aligned}
&\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2} \int_0^T \|e_p\|^2_{\mathscr{B}} dt \\
&\leq c(\delta) C^2_{5,5} \tilde{\mathscr{E}}^2_{5,2} + c(\delta)\alpha_0^2 C^2_{5,6} \int_0^T \sum_{K \in \mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + C_M^2 \int_0^T \|e_p\|^2_{L^2(\Omega)} dt \\
&\quad + \frac{C_M^2}{2} \|g'(y(U_h))(T) - g'(Y_h(T))\|^2_{L^2(\Omega)} + \frac{C_M^2}{2} \|g'(Y_h(T)) - g'(Y_h(T))_h\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using the Lipschitz continuity of g' and set $C_{5,23} = \max\left\{c(\delta)C_{5,5}^2, c(\delta)\alpha_0^2 C_{5,6}^2, C_M^2 C_L^2, C_M^2\right\}$ with $\delta = \alpha_1/4$, then it follows that

$$\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2}\int_0^T \|e_p\|^2_{\mathscr{B}} dt \leq C_{5,23}\Big(\tilde{\mathscr{E}}^2_{5,2} + \tilde{\mathscr{E}}^2_{5,10} + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + \frac{1}{2}\|(y(U_h) - Y_h)(T)\|^2_{L^2(\Omega)} + \int_0^T \|e_p\|^2_{L^2(\Omega)} dt\Big). \tag{5.96}$$

Next, our requirement is to estimate the last term of (5.96) in the $L^2(0,T;L^2(\Omega))$-norm. Note that, the condition for the co-state variable at final time T is non zero. Therefore, duality technique as (5.58) leads to

$$\begin{aligned}\int_0^T \|e_p\|^2_{L^2(\Omega)} dt &= \big(g'(y(U_h))(T) - g'(Y_h(T))_h, \chi(T)\big) \\ &\quad - \int_0^T \Big\{ -\Big(\frac{\partial P_h}{\partial t}, \chi\Big) + a(\tilde{P}_h, \chi)\Big\} dt + \int_0^T a(\tilde{P}_h - P_h, \chi) dt.\end{aligned}$$

Use (5.92) to obtain

$$\begin{aligned}\int_0^T \|e_p\|^2_{L^2(\Omega)} dt &= \int_0^T \Big(\frac{\partial P_h}{\partial t} + div(A\nabla \tilde{P}_h) - a_0\tilde{P}_h, \chi - \chi^I\Big) dt \\ &\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E [A\nabla\tilde{P}_h \cdot n_E](\chi^I - \chi) ds\, dt \\ &\quad + \int_0^T \sum_{E\subset\Gamma} \int_E (A\nabla\tilde{P}_h \cdot n_E)(\chi^I - \chi) ds\, dt \\ &\quad + \int_0^T a(\tilde{P}_h - P_h, \chi) dt + \big(g'(y(U_h))(T) - g'(Y_h(T))_h, \chi(T)\big).\end{aligned}$$

Again, we treat the last term $\big(g'(y(U_h))(T) - g'(Y_h(T))_h, \chi(T)\big)$ as before, and use Lemmas 5.2.1 and 5.2.3, the Lipschitz continuity of g', the continuity property of the bilinear form and the Young's inequality to obtain

$$\begin{aligned}\int_0^T \|e_p\|^2_{L^2(\Omega)} dt \ &\leq\ c(\delta)\Big(C^2_{5,8}\tilde{\mathscr{E}}^2_{5,4} + \tilde{\mathscr{E}}^2_{5,10} + \alpha_0^2 \int_0^T \sum_{K\in\mathscr{T}^n_h} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt \\ &\quad + C_L^2\|y(U_h)(T) - Y_h(T)\|^2_{L^2(\Omega)}\Big) + \delta\|\chi(T)\|^2_{H^1(\Omega)} + \delta\int_0^T \|\chi\|^2_{H^2(\Omega)} dt.\end{aligned}$$

We now apply Lemma 1.2.3 with $\Psi = \chi$ and $f = p(U_h) - P_h$. Then, choosing δ appropriately such that $1 - 2\delta C_R^2 > 0$ and by setting $C_{5,24} = \frac{c(\delta)}{1-2\delta C_R^2}\max\left\{C^2_{5,8}, \alpha_0^2, C_L^2, 1\right\}$

we conclude that

$$\int_0^T \|e_p\|^2_{L^2(\Omega)}dt \leq \; C_{5,24}\Big(\tilde{\mathscr{E}}^2_{5,4}+\tilde{\mathscr{E}}^2_{5,10}+\int_0^T\sum_{K\in\mathscr{T}_h^n}\|\tilde{P}_h-P_h\|^2_{H^1(K)}dt + \frac{1}{2}\|y(U_h)(T)-Y_h(T)\|^2_{L^2(\Omega)}\Big). \tag{5.97}$$

Combining (5.96) with (5.97) and setting $C_{5,22} = C_{5,23}(1+C_{5,24})$ the desired result follows. □

Finally, collecting Lemmas 5.3.3, 5.3.7 and 5.3.8, we present the main theorem of this section.

Theorem 5.3.3. *Let* (y,p,u) *and* (Y_h,P_h,U_h) *be the solutions of* (5.16) – (5.20) *and* (5.81) – (5.85), *respectively. Let the assumption of Lemma 5.3.7 are fulfilled, and let* $g'(\cdot)$ *be a locally Lipschitz continuous function in a neighborhood of* y *with observation space* $\mathcal{O}_{sp_3}$. *Then there exists a positive constant* $C_{5,25}$ *such that*

$$\|u-U_h\|^2_{L^2(0,T;\mathscr{B})}+\|y-Y_h\|^2_{L^2(0,T;\mathscr{B})}+\|p-P_h\|^2_{L^2(0,T;\mathscr{B})} \;\leq\; C_{5,25}\Big(\sum_{i=1}^{10}\tilde{\mathscr{E}}^2_{5,i}\Big),$$

with $\tilde{\mathscr{E}}^2_{5,1}$, $\tilde{\mathscr{E}}^2_{5,i}$, $i=2,3,4,10$ *and* $\tilde{\mathscr{E}}^2_{5,i} = \mathscr{E}^2_{5,i}$, $j=5,\ldots,9$ *are defined as in Lemmas 5.3.7, 5.3.8 and 5.3.3, respectively. The constant* $C_{5,25}$ *depends on* α_0, α_1, *the Lipschitz-constant* C_L, *the bound* C_M *of the cut-off function, the stability constant* C_R, *the constant* $C_{5,21}$ *defined in Lemma 5.3.7 and the constants* $\tilde{C}_{I,j}$, $j=1,\ldots,5$ *defined in Lemmas* 5.2.1 – 5.2.3 *and the inequality* (5.38).

Proof. Set $u^* = U_h$ and $v = y-y(U_h)$ in (5.86) and (5.16). Then, subtracting one from other and using integration by parts formula with an aid of the Cauchy-Schwartz inequality leads to

$$\|y(T)-y(U_h)(T)\|^2_{L^2(\Omega)}+\alpha_1\|y-y(U_h)\|^2_{L^2(0,T;H^1(\Omega))} \;\leq\; C^2_{5,26}\|u-U_h\|^2_{L^2(0,T;\mathscr{B})},$$

apply trace inequality to have

$$\|y(T)-y(U_h)(T)\|^2_{L^2(\Omega)}+\alpha_1\|y-y(U_h)\|^2_{L^2(0,T;\mathscr{B})} \;\leq\; C^2_{5,26}\|u-U_h\|^2_{L^2(0,T;\mathscr{B})}. \tag{5.98}$$

Similarly, using (5.18) and (5.88) with $u^* = U_h$ and the Lipschitz continuity of g' imply

$$\|p-p(U_h)\|^2_{L^2(0,T;H^1(\Omega))} \;\leq\; \frac{C^2_{5,26}C^2_L}{2\alpha_1}\|y(T)-y(U_h)(T)\|^2_{L^2(\Omega)}.$$

By the trace inequality and (5.98), we obtain

$$\|p-p(U_h)\|^2_{L^2(0,T;\mathscr{B})} \;\leq\; C_{5,27}\,\|u-U_h\|^2_{L^2(0,T;\mathscr{B})}, \tag{5.99}$$

where $C_{5,27} = \frac{C_{5,26}^2 C_L^2}{2\alpha_1}$, and $C_{5,26}$ is a positive constant. Note that

$$\|y - Y_h\|^2_{L^2(0,T;\mathscr{B})} \leq \|y - y(U_h)\|^2_{L^2(0,T;\mathscr{B})} + \|y(U_h) - Y_h\|^2_{L^2(0,T;\mathscr{B})}, \tag{5.100}$$

and

$$\|p - P_h\|^2_{L^2(0,T;\mathscr{B})} \leq \|p - p(U_h)\|^2_{L^2(0,T;\mathscr{B})} + \|p(U_h) - P_h\|^2_{L^2(0,T;\mathscr{B})}. \tag{5.101}$$

Also, by the triangle inequality

$$\|p(U_h) - \tilde{P}_h\|^2_{L^2(0,T;\mathscr{B})} \leq \|p(U_h) - P_h\|^2_{L^2(0,T;\mathscr{B})} + \|\tilde{P}_h - P_h\|^2_{L^2(0,T;H^1(\Omega))}. \tag{5.102}$$

Altogether (5.98) – (5.102) with Lemmas 5.3.3, 5.3.7 and 5.3.8 complete the proof of the theorem. □

Remark 5.3.2. *We observe that the leading terms of the derived estimators $\tilde{\mathscr{E}}^2_{5,i}$, $i \in \{1, 2, 3, 5, 6\}$ are of local character because they depend only on the data in a smaller neighborhood of the boundary, which differ from the estimators derived in [34, Theorem 3.10]. But, the global influences are reflected by the higher order terms $\tilde{\mathscr{E}}^2_{5,i}$, $i = 4, 7$, which cannot be ignored.*

5.4 Numerical assessments

This section reports numerical assessments of the estimators derived in Sections 5.3.1 – 5.3.3. We solve our optimization problem by inviting the projection gradient algorithm. All the constants involved in the estimators are chosen to be one and for the computations we use the software *FreeFem++* [38]. In the first example, we consider the **Case I**, i.e., the observation of the boundary state ($\mathcal{O}_{sp} = \mathcal{O}_{sp_1}$).

Example 5.1. *We consider the time interval $I = [0, T]$ with $T = 1$ and the domain $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ with $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_1 = [0, 1] \times \{0\}$, $\Gamma_2 = \{1\} \times [0, 1]$, $\Gamma_3 = [0, 1] \times \{1\}$, and $\Gamma_4 = \{0\} \times [0, 1]$. Let*

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|^2_{\mathscr{B}} dt \quad \textit{and} \quad \mathcal{H}(u) := \frac{\alpha}{2}\int_0^T \|u\|^2_{\mathscr{B}} dt,$$

with $y_{ds} \in L^2(0, T; \mathscr{B})$ is given by

$$y_{ds} = \begin{cases} \sin(2\pi x_1)\exp(-2t) & x \in \Gamma_1 \cup \Gamma_3, \quad t > 0, \\ 0 & x \in \Gamma_2 \cup \Gamma_4, \quad t > 0. \end{cases}$$

We consider the problem (5.4) − (5.5) *with observation on the boundary state, that is,*

$$\min_{u\in U_{ad}} \mathcal{J}(u,y) = \min_{u\in U_{ad}} \frac{1}{2}\int_0^T \{\|y-y_{ds}\|^2_{\mathscr{B}} + \alpha\|u\|^2_{\mathscr{B}}\}dt$$

subject to

$$\begin{cases} \left(\dfrac{\partial y}{\partial t}, v\right) + a(y,v) = (f,v) + (Bu + \mathbf{x}_b, v)_{\mathscr{B}} & \forall v \in V, \\ y(\cdot,0) = y_0(x) \quad x\in\Omega. \end{cases}$$

For our computation, we take $\alpha = 1$, $Bu = u$, *for any* $u \in U_{ad}$, $f = 0$, $y_0 = \sin(2\pi x_1)\sin(2\pi x_2)$ *and*

$$\mathbf{x}_b = \begin{cases} 1 & x \in (\frac{1}{2},\frac{3}{4}] \times \{0\}, \\ -1 & x \in (\frac{3}{4}, 1] \times \{0\}, \\ 0 & \textit{otherwise.} \end{cases}$$

We take $u_a = -0.1 \times |t\sin(6\pi x_1)|$ *and* $u_b = 2 + t\cos(11\pi x_1)$.

The adaptive meshes generated via the estimators $\mathscr{E}_{5,i}$, $i = 1,\dots,9$ reduce much computational work in comparison to the uniform meshes. Results in Table 5.1 display the mesh information together with the state, co-state and control errors in the $L^2(0,T;\mathscr{B})$-norm at the final time $T = 1$. We have chosen the equal space and time tolerances, i.e., $\epsilon_{space} = \epsilon_{time} = \epsilon$ (say). Let $\mathscr{E}^L_{5,i}$, $i = 1,2,3,5,6$ denote the leading terms of the estimators $\mathscr{E}_{5,i}$, $i = 1,2,3,5,6$, respectively. Tables 5.2 − 5.4 provide information of the leading terms on the triangulations $\mathscr{T}^n_{\Omega_l}$ with different values of l (i.e., $l = 0.3$, 0.2, 0.1). Note that the number of elements (# Elements) presented in Tables 5.2 − 5.4 is the sum of the interior elements and the boundary (Bdry) elements as shown in Figure 5.1 (b). We observe that the number of elements and nodes on adaptive meshes are very less with comparison to the uniform meshes. The values of the leading terms $\mathscr{E}^L_{5,i}$, $i = 1,2,3,5,6$ are decreasing when the size of l is decreasing.

In the next example, we perform numerical experiment for the **Case II**, i.e., the observation of the distributed state ($\mathcal{O}_{sp} = \mathcal{O}_{sp_2}$).

Example 5.2. *In this example, we consider all the datum same as in Example 5.1 except* $\mathcal{G}(y)$. *We assume that*

$$\mathcal{G}(y) := \frac{1}{2}\int_0^T \|y - y_{ds}\|^2_{L^2(\Omega)}dt,$$

Table 5.1: *Comparison of the results on uniform mesh* $\mathscr{T}_h^n$ *and adaptive mesh with* $l = 0.1$ *and tolerance* $\epsilon = 10^{-4}$.

			Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}_h^n$	$y-$	Elements	23764	2824
		Nodes	12083	1466
		Bdry Elements	400	106
	$p-$	Elements	23764	806
		Nodes	12083	441
		Bdry Elements	400	74
	$u-$	Elements	23764	4163
		Nodes	12083	2162
		Bdry Elements	400	294
Errors		$\|y - Y_h\|_{L^2(0,T;\mathscr{B})}$	6.811×10^{-2}	5.076×10^{-4}
		$\|p - P_h\|_{L^2(0,T;\mathscr{B})}$	5.759×10^{-2}	1.365×10^{-3}
		$\|u - U_h\|_{L^2(0,T;\mathscr{B})}$	1.040×10^{-2}	1.567×10^{-3}

Table 5.2: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}_{\Omega_l}^n$ *with* $l = 0.3$.

	$l = 0.3$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}_{\Omega_l}^n$	$y-$ Elements	18414	2510
	$y-$ Nodes	9407	1308
	$p-$ Elements	18414	729
	$p-$ Nodes	9407	401
	$u-$ Elements	18414	3313
	$u-$ Nodes	9407	1732
Values of the leading terms on $\mathscr{T}_{\Omega_l}^n$	$\mathscr{E}_{5,1}^L$	9.889×10^{-3}	2.063×10^{-4}
	$\mathscr{E}_{5,2}^L$	1.629×10^{-2}	8.697×10^{-5}
	$\mathscr{E}_{5,3}^L$	4.897×10^{-3}	2.753×10^{-4}
	$\mathscr{E}_{5,5}^L$	1.662×10^{-2}	6.871×10^{-4}
	$\mathscr{E}_{5,6}^L$	1.414×10^{-3}	1.952×10^{-4}

Table 5.3: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.2$.

	$l = 0.2$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	13912	1898
	$y-$ Nodes	7156	1002
	$p-$ Elements	13912	603
	$p-$ Nodes	7156	338
	$u-$ Elements	13912	2834
	$u-$ Nodes	7156	1496
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\mathscr{E}^L_{5,1}$	9.017×10^{-3}	4.661×10^{-5}
	$\mathscr{E}^L_{5,2}$	1.877×10^{-3}	7.618×10^{-6}
	$\mathscr{E}^L_{5,3}$	4.602×10^{-3}	2.535×10^{-5}
	$\mathscr{E}^L_{5,5}$	1.093×10^{-2}	5.755×10^{-5}
	$\mathscr{E}^L_{5,6}$	1.209×10^{-3}	2.320×10^{-5}

Table 5.4: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.1$.

	$l = 0.1$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	8058	924
	$y-$ Nodes	4229	515
	$p-$ Elements	8058	392
	$p-$ Nodes	4229	234
	$u-$ Elements	8058	2378
	$u-$ Nodes	4229	1268
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\mathscr{E}^L_{5,1}$	2.433×10^{-3}	3.452×10^{-6}
	$\mathscr{E}^L_{5,2}$	1.718×10^{-3}	9.265×10^{-7}
	$\mathscr{E}^L_{5,3}$	3.888×10^{-3}	5.284×10^{-6}
	$\mathscr{E}^L_{5,5}$	1.611×10^{-3}	2.926×10^{-6}
	$\mathscr{E}^L_{5,6}$	1.140×10^{-3}	1.066×10^{-6}

with $y_{ds} \in L^2(0,T;L^2(\Omega))$ *which is defined as*

$$y_{ds}(x,t) = \begin{cases} t\sin(2\pi x_1)\sin(2\pi x_2) & x_1 + x_2 \leq 1, \quad t > 0, \\ 2t\sin(2\pi x_1)\sin(2\pi x_2) & x_1 + x_2 > 1, \quad t > 0. \end{cases}$$

Further, we consider $f = 0$, $y_0 = 0$, $Bu = u$, u_a *and* u_b *as defined in the previous example.*

Similar to Example 5.1, we present some mesh information and errors on the uniform mesh and the adaptive mesh (Step-II) with $l = 0.1$ in Table 5.5. Note that, the co-state and control errors are presented in the $L^2(0,T;\mathscr{B})$-norm while the state error is computed in the $L^2(0,T;L^2(\Omega))$-norm. Let $\hat{\mathscr{E}}^L_{5,i}$, $i = 1, 2, 3$ be the leading terms of the estimators $\hat{\mathscr{E}}_{5,i}$, $i = 1, 2, 3$, respectively. From Table 5.6 – 5.8, we observe that the values of the leading terms are decreasing when the distance l is decreased. Tables 5.6 – 5.8 show the behaviour of the leading terms.

Table 5.5: *Comparison of the results on uniform mesh* $\mathscr{T}_h^n$ *and adaptive mesh with* $l = 0.1$ *and tolerance* $\epsilon = 10^{-4}$.

			Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}_h^n$	$y-$	Elements	23764	4440
		Nodes	12083	2324
		Bdry Elements	400	276
	$p-$	Elements	23764	1548
		Nodes	12083	847
		Bdry Elements	400	173
	$u-$	Elements	23764	5686
		Nodes	12083	2914
		Bdry Elements	400	326
Errors		$\|y - Y_h\|_{L^2(0,T;L^2(\Omega))}$	4.856×10^{-3}	7.932×10^{-4}
		$\|p - P_h\|_{L^2(0,T;\mathscr{B})}$	2.139×10^{-3}	8.288×10^{-4}
		$\|u - U_h\|_{L^2(0,T;\mathscr{B})}$	1.442×10^{-2}	1.646×10^{-3}

Table 5.6: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.3$.

	$l = 0.3$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	18976	2743
	$y-$ Nodes	9768	1421
	$p-$ Elements	18976	1069
	$p-$ Nodes	9768	588
	$u-$ Elements	18976	3676
	$u-$ Nodes	9768	1958
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\hat{\mathscr{E}}^L_{5,1}$	1.884×10^{-2}	5.700×10^{-3}
	$\hat{\mathscr{E}}^L_{5,2}$	7.037×10^{-2}	3.359×10^{-5}
	$\hat{\mathscr{E}}^L_{5,3}$	9.522×10^{-3}	2.561×10^{-5}

Table 5.7: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.2$.

	$l = 0.2$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	14192	2556
	$y-$ Nodes	7096	1278
	$p-$ Elements	14192	924
	$p-$ Nodes	7096	441
	$u-$ Elements	14192	2183
	$u-$ Nodes	7096	1054
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\hat{\mathscr{E}}^L_{5,1}$	1.096×10^{-2}	9.759×10^{-4}
	$\hat{\mathscr{E}}^L_{5,2}$	4.603×10^{-2}	1.371×10^{-6}
	$\hat{\mathscr{E}}^L_{5,3}$	6.313×10^{-3}	8.951×10^{-6}

Table 5.8: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.1$.

	$l = 0.1$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	9180	2106
	$y-$ Nodes	4990	1216
	$p-$ Elements	9180	612
	$p-$ Nodes	4990	285
	$u-$ Elements	9180	836
	$u-$ Nodes	4990	439
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\hat{\mathscr{E}}^L_{5,1}$	8.092×10^{-3}	2.100×10^{-5}
	$\hat{\mathscr{E}}^L_{5,2}$	2.638×10^{-3}	2.708×10^{-8}
	$\hat{\mathscr{E}}^L_{5,3}$	1.456×10^{-4}	5.890×10^{-7}

For validating the **Case III**, we consider the following example with observation of the final state ($\mathcal{O}_{sp} = \mathcal{O}_{sp_3}$).

Example 5.3. *This example considers the same data as given in Example 5.1 except* $\mathcal{G}(y)$. *Here, we take*

$$\mathcal{G}(y) := \frac{1}{2}\|y(x,T) - y_{ds}(x)\|^2_{L^2(\Omega)},$$

with $y(x,T) = 0.7$, $T = 1$, *and* $y_{ds}(x) \in L^2(\Omega)$ *is defined as*

$$y_{ds}(x) \;=\; \frac{1}{4}\big(1 - x_1^2\big)\big(1 - x_2^2\big), \qquad \forall (x_1, x_2) \in \Omega.$$

Similar to the previous cases, the mesh information and the errors in the $L^2(0,T;\mathscr{B})$-norm are presented in Table 5.9. Let the leading terms of the estimators $\tilde{\mathscr{E}}_{5,i}$, $i = 1, 2, 3, 5, 6$ be denoted as $\tilde{\mathscr{E}}^L_{5,i}$, $i = 1, 2, 3, 5, 6$, respectively. Tables 5.10 – 5.12 show the values of the leading terms which reveals the effectivity of the estimators.

5.5 Concluding remarks

In this chapter, we have derived local *a posteriori* error estimates of the finite element approximation to the PBCP (5.1) – (5.3) in three different observations namely, the boundary state, the distributed state and the final state. For the numerical discretization we have used the standard piecewise linear and continuous finite elements

Table 5.9: *Comparison of the results on uniform mesh* $\mathscr{T}_h^n$ *and adaptive mesh with* $l = 0.1$ *and tolerance* $\epsilon = 10^{-5}$.

			Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}_h^n$	$y-$	Elements	23764	2580
		Nodes	12083	1357
		Bdry Elements	400	132
	$p-$	Elements	23764	681
		Nodes	12083	377
		Bdry Elements	400	71
	$u-$	Elements	23764	3568
		Nodes	12083	1863
		Bdry Elements	400	156
Errors		$\Vert y - Y_h\Vert_{L^2(0,T;\mathscr{B})}$	1.243×10^{-2}	4.676×10^{-4}
		$\Vert p - P_h\Vert_{L^2(0,T;\mathscr{B})}$	1.005×10^{-3}	6.174×10^{-5}
		$\Vert u - U_h\Vert_{L^2(0,T;\mathscr{B})}$	5.561×10^{-3}	7.785×10^{-5}

Table 5.10: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}_{\Omega_l}^n$ *with* $l = 0.3$.

	$l = 0.3$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}_{\Omega_l}^n$	$y-$ Elements	17438	1236
	$y-$ Nodes	8719	778
	$p-$ Elements	17438	642
	$p-$ Nodes	8719	322
	$u-$ Elements	17438	2808
	$u-$ Nodes	8719	1624
Values of the leading terms on $\mathscr{T}_{\Omega_l}^n$	$\tilde{\mathscr{E}}_{5,1}^L$	9.191×10^{-2}	2.080×10^{-4}
	$\tilde{\mathscr{E}}_{5,2}^L$	3.020×10^{-2}	5.098×10^{-5}
	$\tilde{\mathscr{E}}_{5,3}^L$	1.114×10^{-3}	5.000×10^{-5}
	$\tilde{\mathscr{E}}_{5,5}^L$	3.394×10^{-3}	3.871×10^{-6}
	$\tilde{\mathscr{E}}_{5,6}^L$	4.256×10^{-2}	1.661×10^{-4}

Table 5.11: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.2$.

	$l = 0.2$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	11198	710
	$y-$ Nodes	5599	337
	$p-$ Elements	11198	492
	$p-$ Nodes	8719	246
	$u-$ Elements	11198	2258
	$u-$ Nodes	5599	1180
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\tilde{\mathscr{E}}^L_{5,1}$	1.568×10^{-2}	1.823×10^{-5}
	$\tilde{\mathscr{E}}^L_{5,2}$	2.746×10^{-2}	1.083×10^{-5}
	$\tilde{\mathscr{E}}^L_{5,3}$	1.058×10^{-3}	4.267×10^{-6}
	$\tilde{\mathscr{E}}^L_{5,5}$	3.146×10^{-3}	5.012×10^{-7}
	$\tilde{\mathscr{E}}^L_{5,6}$	2.984×10^{-2}	9.902×10^{-5}

Table 5.12: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.1$.

	$l = 0.1$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	7358	590
	$y-$ Nodes	3598	326
	$p-$ Elements	7358	324
	$p-$ Nodes	3598	167
	$u-$ Elements	7358	1612
	$u-$ Nodes	3598	970
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\tilde{\mathscr{E}}^L_{5,1}$	8.832×10^{-3}	2.065×10^{-7}
	$\tilde{\mathscr{E}}^L_{5,2}$	2.652×10^{-3}	5.461×10^{-7}
	$\tilde{\mathscr{E}}^L_{5,3}$	1.139×10^{-4}	5.246×10^{-8}
	$\tilde{\mathscr{E}}^L_{5,5}$	4.421×10^{-4}	6.769×10^{-9}
	$\tilde{\mathscr{E}}^L_{5,6}$	3.677×10^{-4}	1.406×10^{-6}

for the approximations of the state and co-state variables. Whereas, the control variable is approximated by employing the piecewise constant functions. The discrete-in-time scheme is based on the implicit backward Euler method. Our derived estimators (see Theorems 5.3.1 – 5.3.3) are local in character in the sense that the leading terms of the estimators depend on the small neighborhood of the boundary. These estimators are different from the estimators derived in [34] (see Remarks 5.3.1 and 5.3.2), and will be useful for the adaptive mesh refinements in the finite element method.

6

LOCAL A POSTERIORI ERROR ESTIMATES FOR NONLINEAR PBCP

In this chapter, we derive a reliable type local *a posteriori* error estimates for Neumann boundary control problems (1.14) − (1.16) governed by nonlinear parabolic equations. To discretize the boundary control problems, we use piecewise linear and continuous finite elements for the approximations of the state and the co-state variables whereas piecewise constant functions are employed for the control variable. The backward Euler method is used to approximate the time derivative. We derive local *a posteriori* error estimates in the $L^2(0,T;\mathscr{B})$-norm. The key to our error analysis includes a technique from [102] and [103], the cut-off function and interpolation approximation properties. The derived estimators are of local character in the sense that the leading terms of the estimators are depending on the small neighborhood of the boundary and the resulting mesh refinement is much concentrating around the boundary. However, the global influences can be seen by the higher order terms in the derived estimators. These derived estimators can be implemented as indicators in the adaptive finite element method for the mesh refinements.

6.1 Introduction

Let $\Omega_T = \Omega \times (0,T]$ and $\Gamma_T = \partial\Omega \times (0,T]$, where Ω is a convex bounded domain in $\mathbb{R}^d$ $(d \leq 3)$ with Lipschitz boundary $\Gamma = \partial\Omega$ and T is a fixed real number $(T < \infty)$. We consider the following nonlinear PBCP:

$$\min_{u\in U_{ad}} \mathcal{J}(u,y) = \min_{u\in U_{ad}} \frac{1}{2}\int_0^T \left\{\|y - y_{ds}\|^2_{\mathscr{B}} + \alpha\|u\|^2_{\mathscr{B}}\right\} dt \tag{6.1}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + \mathcal{A}y + \varphi(y) = f \ \text{ in } \ \Omega_T, \\ y(x,0) = y_0(x) \ \text{ in } \ \Omega, \\ (A\nabla y)\cdot n = Bu + \mathtt{x}_b \quad \text{on } \ \Gamma_T, \end{cases} \tag{6.2}$$

and the control constraints

$$u_a \leq u(x,t) \leq u_b \quad \text{a.e.} \quad \text{in } \ \Gamma_T, \tag{6.3}$$

where $\mathscr{B} = L^2(\Gamma)$. Here $y = y(x,t)$ and $u = u(x,t)$ denote the state and the control variables, respectively. Further, we assume that

$$y_{ds} \in L^\infty(0,T;L^\infty(\Gamma)), \ \ f \in L^\infty(0,T;L^\infty(\Omega)), \ \ y_0 \in C(\overline{\Omega}), \ \ \mathtt{x}_b \in L^\infty(\Gamma), \ a_0 \in L^\infty(\Omega).$$

The coefficient matrix $A = (a_{ij}(\cdot))_{d\times d} \in (W^{1,\infty}(\Omega))^{d\times d}$ is assumed to be symmetric and positive definite matrix, i.e., there is a positive constant $c_0 > 0$ satisfying $a_0(x) \geq c_0$, $\forall x \in \Omega$ and the following ellipticity condition

$$\mathtt{x}^t A \mathtt{x} \geq c_0 \|\mathtt{x}\|^2_{\mathbb{R}^d} \quad \forall \mathtt{x} \in \ \mathbb{R}^d,$$

where $\mathtt{x}^t$ is the transpose of $\mathtt{x}$. Moreover, the nonlinear function φ satisfies the following conditions:

C1. Let $\varphi : \Omega_T \times \mathbb{R} \to \mathbb{R}$ be a measurable function with respect to (x,t) and twice continuously differentiable with respect to $y \in \mathbb{R}$.

C2. Moreover, φ is a monotone non-decreasing with respect to $y \in \mathbb{R}$ and for all $(x,t) \in \Omega_T$, i.e., $\varphi_y(x,t,y) \geq 0, \ \ \forall (x,t) \in \Omega_T$.

C3. There exists a positive constant $\tilde{C}_M$ such that

$$|\varphi(x,t,\cdot)| + |\varphi_y(x,t,\cdot)| + |\varphi_{yy}(x,t,\cdot)| \leq \tilde{C}_M, \quad \text{for all} \quad (x,t) \in \Omega_T,$$

where $\varphi_y(x,t,\cdot)$ and $\varphi_{yy}(x,t,\cdot)$ denote the first and second order partial derivatives of φ with respect to y, respectively.

C4. For all y_1, $y_2 \in \mathbb{R}$ and $L > 0$ with $|y_1 - y_2| < L$, there exists a positive constant $\tilde{C}_L$ such that

$$|\varphi_{yy}(x,t,y_1) - \varphi_{yy}(x,t,y_2)| \leq \tilde{C}_L |y_1 - y_2|, \quad \text{for all} \quad (x,t) \in \Omega_T.$$

Let B be a linear and continuous operator from $L^\infty(0,T;L^\infty(\Gamma))$ to $L^\infty(0,T;L^\infty(\Gamma))$. Note that the control variable u presents in the Neumann boundary condition. The set of admissible controls is defined by

$$U_{ad} = \Big\{ u \in L^\infty(0,T;L^\infty(\Gamma)) \ : \ u_a \leq u(x,t) \leq u_b \quad a.e. \quad (x,t) \in \Gamma_T \Big\}$$

with u_a, $u_b \in \mathbb{R}$ fulfil $u_a < u_b$. Furthermore, U_{ad} is a nonempty closed convex set in $L^\infty(0,T;L^\infty(\Gamma))$. Further, we shall use the state space $W = \mathcal{W}(0,T) \cap \mathcal{C}(\overline{\Omega}_T)$ with $\mathcal{W}(0,T) := \big\{ v \in L^2(0,T;V) \text{ such that } \frac{\partial v}{\partial t} \in L^2(0,T;V^*) \big\}$, where V^* is the dual space of V with $V := H^1(\Omega) = \{ v : v \in L^2(\Omega) \text{ and } \nabla v \in (L^2(\Omega))^d \}$. Now we define the bilinear form $a(\cdot,\cdot)$ on $H^1(\Omega)$ as follows

$$a(v,w) \ = \ \int_\Omega \{ (A\nabla v) \cdot \nabla w + a_0 v\, w \}\, dx, \qquad \forall v, w \in V.$$

We assume that the bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $H^1(\Omega)$, i.e., $\exists$ α_0, $\alpha_1 > 0$ such that

$$|a(v,w)| \ \leq \ \alpha_0 \|v\|_V \|w\|_V, \qquad \forall v, w \in V,$$

and

$$a(v,v) \ \geq \ \alpha_1 \|v\|_V^2, \qquad \forall v \in V.$$

The weak formulation of the problem (6.1) – (6.3) is defined as follows: To find a pair $(y,u) \in W \times U_{ad}$ such that

$$\min_{u \in U_{ad}} \frac{1}{2} \int_0^T \Big\{ \|y - y_{ds}\|_{\mathscr{B}}^2 + \alpha \|u\|_{\mathscr{B}}^2 \Big\}\, dt \tag{6.4}$$

subject to

$$\begin{cases} \Big(\dfrac{\partial y}{\partial t}, v \Big) + a(y,v) + (\varphi(y), v) = (f,v) + (Bu + \mathtt{x}_b, v)_{\mathscr{B}} \qquad \forall v \in V, \\ y(\cdot,0) = y_0(x) \qquad x \in \Omega. \end{cases} \tag{6.5}$$

To ensure the existence of the control variable u, we introduce the reduced cost functional $j : L^\infty(0,T;L^\infty(\Gamma)) \to \mathbb{R}$ by

$$j(u) := \mathcal{J}(u, y(u)),$$

where $y := y(u)$ is the solution of (6.5). Hence, we can reformulate our optimal control problem (6.4) – (6.5) as

$$\min_{u \in U_{ad}} j(u). \tag{6.6}$$

Note that, the first order necessary optimality condition is characterize as follows:

$$j'(u)(u^* - u) \ \geq \ 0, \qquad \forall u^* \in U_{ad}, \tag{6.7}$$

where the derivative $j'(u)(u^* - u)$ is given by

$$j'(u)(u^* - u) \ = \ \int_0^T (\alpha u + B^* p, u^* - u)_{\mathscr{B}}\, dt,$$

where B^* is the adjoint operator of B and the co-state variable $p := p(u)$ is a solution of the problem

$$\begin{cases} -\Big(\dfrac{\partial p}{\partial t}, v\Big) + a(p, v) + (\varphi'(y)p, v) = (y - y_{ds}, v)_{\mathscr{B}} \qquad \forall v \in V, \\ p(x, T) = 0 \qquad x \in \Omega. \end{cases} \tag{6.8}$$

Further, we assume the second-order optimality condition on j, i.e., for all $u \in U_{ad}$ and $w \in X$, where $X = L^2(0, T; \mathscr{B})$, there exists a radius $r(\epsilon) > 0$ $(0 < \epsilon < \gamma_s)$ corresponding to γ_s (a positive constant) such that

$$j''(u)(w, w) \geq \gamma_s \|w\|_X^2 \quad \text{with} \quad \|u - u^*\|_{L^\infty(0,T;L^\infty(\Gamma))} \leq r(\epsilon). \tag{6.9}$$

That is, $j''(u)(\cdot, \cdot)$ satisfies the positive definiteness condition.

The existence of a unique weak solution for the state variable is now stated in the following theorem. For a proof, we refer to [96].

Theorem 6.1.1. *Assume that φ satisfies the conditions **C1**− **C4**, and let $u \in U_{ad}$, $y_0 \in C(\overline{\Omega})$, $y_{ds} \in L^\infty(0, T; L^\infty(\Omega))$ and $f \in L^\infty(0, T; L^\infty(\Omega))$. Then, the state equation* (6.2) *has a unique weak solution $y \in W$ such that*

$$-\Big(y, \frac{\partial v}{\partial t}\Big) + a(y, v) + (\varphi(y), v) = (y_0, v) + (f, v) + (Bu + \mathrm{x}_b, v)_{\mathscr{B}} \qquad \forall v \in V,$$

with $v(\cdot, T) = 0$.

We now recall the following result from [96] concerning the existence of the control variable.

Theorem 6.1.2. *Assume that φ satisfies the assumptions **C1** − **C4**. Assume that the pair (y^*, u^*) complies with both the first order necessary optimality condition* (6.7) *and the constraints* (6.3) *of the boundary control problem. Further, assume that there is a constant $\gamma_s > 0$ such that the positive definite condition* (6.9) *is fulfilled. Then there exists constants $\epsilon > 0$ and $\sigma > 0$ such that for every $u \in U_{ad}$ with $\|u - u^*\|_{L^\infty(0,T;L^\infty(\Gamma))} \leq r(\epsilon)$ and associated state $y \in W$, the quadratic growth condition holds:*

$$J(u, y) \geq \mathcal{J}(u^*, y^*) + \sigma \|u - u^*\|^2_{L^2(0,T;L^2(\Omega))}.$$

In particular, $u^* \in U_{ad}$ is the local optimal in the sense of $L^\infty(0, T; L^\infty(\Gamma))$.

As a consequences of above Theorems 6.1.1 − 6.1.2, the following theorem states that the optimization problem (6.6) has a solution.

Theorem 6.1.3. *There exists an optimal pair* $(y^*, u^*) \in W \times U_{ad}$ *that satisfies the optimization problem* (6.6)*.*

Earlier, Liu and Yan [60] have derived local *a posteriori* error estimates for finite element approximation of elliptic boundary control problems using the duality argument. For linear parabolic boundary control problem, Gong and Yan [34] have derived *a posteriori* error estimates with three different observations, i.e., observation the boundary state, the distributed state and the final state. They have obtained $L^2(0, T; H^1(\Omega))$-errors for the state and co-state variables whereas the error for the control variable in the $L^2(0, T; L^2(\Gamma))$-norm with the above observations. Later, the author of [62] has considered nonlinear PBCP and derived reliable type *a posteriori* error estimates for the state and co-state errors in the $L^2(0, T; H^1(\Omega))$-norm, while the control error in the $L^2(0, T; L^2(\Gamma))$-norm. Since in many applications it is often useful to study the behaviour of the state and co-state variables in a small neighborhood of the boundary, an effort has been made in this chapter to study local *a posteriori* error estimates for the nonlinear PBCP (6.1) – (6.3) in a bounded convex domain.

The rest of the chapter is organized as follows. Section 6.2 contains finite element approximation of the boundary control problem (6.1) – (6.3) and some basic prerequisite materials for future use. The local *a posteriori* error estimates for fully discrete nonlinear PBCP (6.1) – (6.3) is carried out in Section 6.3. In Section 6.4, we present the numerical experiment which illustrates the performance of the derived estimators. Finally, a concluding remark is presented in the last section.

6.2 Discrete optimal control problem and stability results

In this section, we describe the finite element approximation for the optimal control problem (6.4) – (6.5). Further, we collect some stability and interpolation results. Here we allow only d-simplex conforming Lagrange elements.

Let Ω_h be a convex polygonal approximation to Ω with a boundary Γ_h. Let $\mathscr{T}_h$ be a partitioning of Ω_h into disjoint regular d-simplices K, so that $\bar{\Omega}_h = \cup_{K \in \mathscr{T}_h} \bar{K}$, and if K_1, $K_2 \in \mathscr{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$, or $K_1 \cap K_2$ share a common edge, or a common vertex. Note that, when $\Omega \neq \Omega_h$, then it is bit complicated to set up and analyze the problem (cf. [57]). For simplicity, we investigate the problem with

$\Omega = \Omega_h$. Let V_h be the finite dimensional subspace of $\mathcal{C}(\overline{\Omega}_h)$ consisting piecewise linear and continuous polynomials associated with $\mathscr{T}_h$, i.e.,

$$V_h \;=\; \big\{v(t) \in C(\overline{\Omega}_h) : v|_K \in \mathbb{P}_1(K),\;\; \forall K \in \mathscr{T}_h,\; t \in (0,T]\big\}.$$

Let $\mathscr{T}_{\mathscr{B}}$ be the partition of Γ_h into disjoint regular $(d-1)$ simplices τ such that $\Gamma_h = \cup_{\tau\in\mathscr{T}_{\mathscr{B}}}\bar{\tau}$, and if $\tau_1, \tau_2 \in \mathscr{T}_{\mathscr{B}}$ and $\tau_1 \neq \tau_2$, then either $\bar{\tau}_1$ and $\bar{\tau}_2$ share a whole edge, or a face, or one common vertex, or they are disjoint. Further, let U_h be the finite dimensional subspace of $L^\infty(\Gamma_h)$ consisting piecewise constant polynomials associated with $\mathscr{T}_{\mathscr{B}}$ and is defined as

$$\mathscr{B}_h = \big\{w(t) \in L^\infty(\Gamma_h) : w|_\tau = constant,\;\; \forall \tau \in \mathscr{T}_{\mathscr{B}},\;\; t \in (0,T]\big\}.$$

Now we set $U_{ad,h} = L^\infty(0,T;\mathscr{B}_h) \cap U_{ad}$. Let h_K and h_τ be the maximum diameter of the element $K \in \mathscr{T}_h$ and $\tau \in \mathscr{T}_{\mathscr{B}}$, respectively.

We now consider the fully discrete finite element approximation for nonlinear boundary control problem (6.4)–(6.5). The temporal discretization is followed by using the backward Euler scheme. Let $0 = t_0 < t_1 < \ldots < t_N = T$, be a partition of the time interval $I = [0,T]$ with $I_n = (t_{n-1}, t_n]$ and $k_n := t_n - t_{n-1}$. Let $\mathscr{T}_h^n := \{K\}$ $(0 \le n \le N)$ be the triangulation of $\overline{\Omega}$ at the time level t_n. For each $n = 0,\ldots,N$, we construct the finite element space V_h^n corresponding to the triangulation $\mathscr{T}_h^n$ as follows:

$$V_h^n \;=\; \big\{v \in C(\overline{\Omega}_h) :\; v|_K \in \mathbb{P}_1(K), \quad \forall K \in \mathscr{T}_h^n\big\},$$

where $\mathbb{P}_1(K)$ is the space of polynomials of degree less than or equal to 1 on K.

Let $\mathscr{T}_{\mathscr{B}}^n$ be the partition of Γ at the time level $t = t_n$. For $n = 0,\ldots,N$, we construct the finite element space for the control variable as $U_{ad,h}^n$ associated with the mesh $\mathscr{T}_{\mathscr{B}}^n$ and is defined as $U_{ad,h}^n = \mathscr{B}_h^n \cap U_{ad}$, where

$$\mathscr{B}_h^n \;=\; \big\{w \in L^\infty(\Gamma_h) :\; w|_\tau = constant, \qquad \forall \tau \in \mathscr{T}_{\mathscr{B}}^n\big\}.$$

Now we define the mesh size functions $h_K(\cdot)$ and $h_\tau(\cdot)$ with reference of the mesh functions $K(\cdot)$ and $\tau(\cdot)$, respectively, such that

$$h_K(t)|_{t\in I_n} = h_{K_n},\;\; h_\tau(t)|_{t\in I_n} = h_{\tau_n},\;\; K(t)|_{t\in I_n} = K_n \;\text{ and }\; \tau(t)|_{t\in I_n} = \tau_n,$$

where h_{K_n} and h_{τ_n} are the maximum diameters of the elements $K_n \in \mathscr{T}_h^n$ and $\tau_n \in \mathscr{T}_{\mathscr{B}}^n$, respectively. For simplicity, we will use $h_K(t) = h_K$, $\;h_\tau(t) = h_\tau$, $\;K(t) =$

K and $\tau(t) = \tau$. For the purpose of fully discrete approximation, we use the following notation:

$$\phi^n := \phi(\cdot, t_n) \text{ and } \bar{\partial}\phi^n := \frac{1}{k_n}(\phi^n - \phi^{n-1}).$$

The fully discrete finite element approximation of (6.4)–(6.5) is defined as follows: Find $(y_h^n, u_h^n) \in V_h^n \times U_{ad,h}^n$, for $n = 1, \ldots, N$, such that

$$\min_{u_h^n \in U_{ad,h}^n} \sum_{n=1}^{N} \frac{k_n}{2} \int_{I_n} \{\|y_h^n - y_{ds}^n\|_{\mathscr{B}}^2 + \alpha\|u_h^n\|_{\mathscr{B}}^2\}\, dt \tag{6.10}$$

subject to the state equation

$$\begin{cases} (\bar{\partial}y_h^n, v_h) + a(y_h^n, v_h) + (\varphi(y_h^n), v_h) = (f^n, v_h) + (Bu_h^n + \mathtt{x}_b, v_h)_{\mathscr{B}} & \forall v_h \in V_h^n, \\ y_h^0 = y_{h,0}, \end{cases} \tag{6.11}$$

where $y_{h,0}$ is the suitable approximation of y_0 in V_h^0.

The optimal control problem (6.10) – (6.11) has a solution $(y_h^n, u_h^n) \in V_h^n \times U_{ad,h}^n$ if and only if there exists a co-state variable $p_h^{n-1} \in V_h^n$ such that the following optimality conditions are satisfied, for all $v_h \in V_h^n$, $w_h^n \in U_{ad,h}^n$, $n = 1, 2, \ldots, N$:

$$(\bar{\partial}y_h^n, v_h) + a(y_h^n, v_h) + (\varphi(y_h^n), v_h) = (f^n, v_h) + (Bu_h^n + \mathtt{x}_b, v_h)_{\mathscr{B}}, \tag{6.12}$$

$$y_h^0 = y_{h,0}, \tag{6.13}$$

$$-(\bar{\partial}p_h^n, v_h) + a(p_h^{n-1}, v_h) + (\varphi'(y_h^n)p_h^{n-1}, v_h) = (y_h^n - y_{ds}^n, v_h)_{\mathscr{B}}, \tag{6.14}$$

$$p_h^N = 0, \tag{6.15}$$

$$(\alpha u_h^n + B^* p_h^{n-1}, w_h^n - u_h^n)_{\mathscr{B}} \geq 0. \tag{6.16}$$

Given a sequence of discrete values $\{y_h^n\}$, $n = 0, 1, \ldots, N$, we associate a continuous function of time defined by the continuous piecewise linear interpolant $Y_h(t)$, $t \in I_n$ as

$$Y_h(t) := \frac{(t_n - t)}{k_n} y_h^{n-1} + \frac{(t - t_{n-1})}{k_n} y_h^n.$$

Similarly, we define $P_h(t)$, $t \in I_n$, from the set of values $\{p_h^n\}$, $n = 0, 1, \ldots, N$ as

$$P_h(t) := \frac{(t_n - t)}{k_n} p_h^{n-1} + \frac{(t - t_{n-1})}{k_n} p_h^n,$$

and

$$U_h(t)|_{I_n} := u_h^n.$$

For any function $\omega \in \mathcal{C}(0,T;L^2(\Omega))$, we set

$$\hat{\omega}(x,t)|_{(t_{n-1},t_n]} = \omega(x,t_n) \quad \text{and} \quad \tilde{\omega}(x,t)|_{(t_{n-1},t_n]} = \omega(x,t_{n-1}).$$

Then, the optimality conditions (6.12) – (6.16) can be stated as follows, for all $v_h \in V_h^n$, $w_h^n \in U_{ad,h}^n$, $n = 1,\ldots,N$:

$$\Big(\frac{\partial Y_h}{\partial t}, v_h\Big) + a(\hat{Y}_h, v_h) + (\varphi(\hat{Y}_h), v_h) = (f^n, v_h) + (BU_h + \mathtt{x}_b, v_h)_{\mathscr{B}}, \tag{6.17}$$

$$Y_h^0 = y_{h,0}, \tag{6.18}$$

$$-\Big(\frac{\partial P_h}{\partial t}, v_h\Big) + a(\tilde{P}_h, v_h) + (\varphi'(\hat{Y}_h)\tilde{P}_h, v_h) = (\hat{Y}_h - y_{ds}^n, v_h)_{\mathscr{B}}, \tag{6.19}$$

$$P_h^N = 0, \tag{6.20}$$

$$(\alpha U_h + B^*\tilde{P}_h, w_h^n - U_h)_{\mathscr{B}} \geq 0. \tag{6.21}$$

Analogous to the continuous case, we now reformulate the optimal control problem (6.10) – (6.11) as

$$\min_{U_h \in U_{ad,h}^n} j_h^n(U_h) := \mathcal{J}(U_h, Y_h(U_h)). \tag{6.22}$$

We now introduce the following auxiliary problems: Given $F \in L^2(0,T;L^2(\Omega))$, let χ and ψ be the solutions of the two dual auxiliary problems

$$\begin{cases} \dfrac{\partial \chi}{\partial t} + \mathcal{A}\chi + \varphi'(y(U_h))\chi = F, & \text{in} \quad \Omega_T, \ \ t > 0, \\ \chi(x,0) = 0, & \text{in} \quad \Omega, \\ \chi(\cdot,\cdot) = 0, & \text{on} \quad \Gamma_T, \end{cases} \tag{6.23}$$

and

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} + \mathcal{A}\psi + \bar{\varphi}\psi = F, & \text{in} \quad \Omega_T, \quad t < T, \\ \psi(x,T) = 0, & \text{in} \quad \Omega, \\ \psi(\cdot,\cdot) = 0, & \text{on} \quad \Gamma_T, \end{cases} \tag{6.24}$$

respectively, where

$$\bar{\varphi} = \begin{cases} \dfrac{\varphi(y(U_h)) - \varphi(Y_h)}{y(U_h) - Y_h}, & y(U_h) \neq Y_h, \\ \varphi'(Y_h), & y(U_h) = Y_h. \end{cases}$$

Then we have the following stability results (cf., [47]).

Lemma 6.2.1. *Assume that Ω is a convex domain. Let χ and ψ be the solutions of the problems (6.23) and (6.24), respectively. Then the following estimates hold for $\Phi = \chi$ or $\Phi = \psi$,*

$$\begin{aligned}
\|\Phi\|_{L^\infty(0,T;L^2(\Omega))} &\leq \|F\|_{L^2(0,T;L^2(\Omega))},\\
\left\|\frac{\partial \Phi}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} &\leq \|F\|_{L^2(0,T;L^2(\Omega))},\\
\|\nabla\Phi\|_{L^2(0,T;L^2(\Omega))} &\leq \|F\|_{L^2(0,T;L^2(\Omega))},\\
\left\|\frac{\partial^2 \Phi}{\partial x_i \partial x_j}\right\|_{L^2(0,T;L^2(\Omega))} &\leq \|F\|_{L^2(0,T;L^2(\Omega))}, \quad 1 \leq i,\ j \leq n.
\end{aligned}$$

Let S_h be the finite element subspace of $L^2(\Omega)$ and is defined as $S_h = \{w \in L^2(\Omega) : w|_{\tilde{K}} \in L^2(\tilde{K}), K \in \mathscr{T}_h\}$ (cf., [74]). Then, we have

$$w(\xi) - w(\eta) = -\bar{w}'(\xi)(\eta - \xi) = -\bar{w}'(\eta)(\eta - \xi) + \bar{w}''(\xi)(\eta - \xi)^2, \tag{6.25}$$

with

$$\begin{aligned}
\bar{w}'(\xi) &= \int_0^1 w'(\xi + \lambda(\eta - \xi))dt,\\
\bar{w}''(\xi) &= \int_0^1 (1-\lambda) w''(\eta + \lambda(\xi - \eta))dt,
\end{aligned}$$

where functions $\bar{w}'(\cdot)$ and $\bar{w}''(\cdot)$ are bounded in $\overline{\Omega}$.

It is very useful to have local *a posteriori* error estimate bounded by some local norms which is not straight forward to derive. To accomplish this, we consider the set

$$\Omega_l = \big\{x \in \Omega \,|\; dist(x, \Gamma) < l\big\} \subset \Omega,$$

where $l > 0$ is a constant independent of h. For each time level $t_n \in (0, T]$, we first characterize a subset $\mathscr{T}^n_{\Omega_l}$ of $\mathscr{T}^n_h$ as follows:

$$\mathscr{T}^n_{\Omega_l} = \big\{K \in \mathscr{T}^n_h \,|\quad \bar{K} \cap \bar{\Omega}_l \neq \emptyset\big\},$$

where $l > 0$ is a constant and independent of the mesh parameters h and k.

6.3 A posteriori error analysis

This section is devoted to the *a posteriori* error analysis for the optimal control problem (6.1)−(6.3). To begin with, we first establish some intermediate error estimates for the state and the co-state variables which will enable us to prove the main results of this section. For this, we now introduce some auxiliary problems. For $u^* \in U_{ad}$, let the pair $(y(u^*), p(u^*)) \in W \times W$ be the solutions of the following equations:

$$\begin{aligned}
\Big(\frac{\partial y(u^*)}{\partial t}, v\Big) + a(y(u^*), v) + (\varphi(y(u^*)), v) &= (f, v) + (Bu^* + \mathtt{x}_b, v)_{\mathscr{B}} \quad \forall v \in V, && (6.26)\\
y(u^*)(\cdot, 0) &= y_0(x) \quad x \in \Omega, && (6.27)\\
-\Big(\frac{\partial p(u^*)}{\partial t}, v\Big) + a(p(u^*), v) + (\varphi'(y(u^*))p(u^*), v) &= (y(u^*) - y_{ds}, v)_{\mathscr{B}} \quad \forall v \in V, && (6.28)\\
p(u^*)(\cdot, T) &= 0 \quad x \in \Omega. && (6.29)
\end{aligned}$$

In the following lemma, we derive local *a posteriori* error estimate for the control variable in the $L^2(0,T;\mathscr{B})$-norm .

Lemma 6.3.1. *Let u and U_h be the solutions of (6.6) and (6.22), respectively. Assume that $U^n_{ad,h} \subset U_{ad}$, $(\alpha U_h + B^*\tilde{P}_h)|_\tau \in H^1(\tau)$ and there exist a positive constant $C_{6,1}$ and $w_h \in U^n_{ad,h}$ such that*

$$\Big|\int_0^T (\alpha U_h + B^*\tilde{P}_h, w_h - u)_{\mathscr{B}}\, dt\Big| \le C_{6,1}\int_0^T \sum_{\tau\in\mathscr{T}^n_{\mathscr{B}}} h_\tau |\alpha U_h + B^*\tilde{P}_h|_{H^1(\tau)} \|u - U_h\|_{L^2(\tau)}\, dt. \tag{6.30}$$

Further, we assume that the first-order necessary condition (6.7) *and the second-order sufficient condition* (6.9) *hold. Then, we have*

$$\|u - U_h\|^2_{L^2(0,T;\mathscr{B})} \le C_{6,2}\left(\mathscr{E}^2_{6,1} + \|\tilde{P}_h - p(U_h)\|^2_{L^2(0,T;\mathscr{B})}\right),$$

with

$$\mathscr{E}^2_{6,1} = \int_0^T \sum_{\tau\in\mathscr{T}^n_{\mathscr{B}}} h^2_\tau |\alpha U_h + B^*\tilde{P}_h|^2_{H^1(\tau)}\, dt,$$

where $C_{6,2} = \frac{3}{2(2\gamma_s - 1)}\max\{1, C^2_{6,1}\}$ and $(y(u^*), p(u^*))$ is the solution of the problem (6.26) – (6.29) *with $u^* = U_h$.*

Proof. From (6.9) with $\zeta = \lambda u + (1-\lambda)U_h$, $\lambda \in [0,1]$, we note that

$$\begin{aligned}
\gamma_s \int_0^T \|u - U_h\|^2_{\mathscr{B}} dt &\le j''(\zeta)(u - U_h, u - U_h)\\
&= j'(u)(u - U_h) - j'(U_h)(u - U_h).
\end{aligned}$$

Applying the first-order optimality condition (6.7) for the first term and the definition of the derivative $j'(\cdot)$ for the second term, we obtain

$$\begin{aligned}\gamma_s \int_0^T \|u-U_h\|^2_{\mathscr{B}} dt \;&\leq\; -\int_0^T (\alpha U_h + B^* p(U_h), u-U_h)_{\mathscr{B}}\, dt \\ &=\; -\int_0^T (B^*\tilde{P}_h + \alpha U_h, u-w_h)_{\mathscr{B}}\, dt \\ &\quad\; -\int_0^T (\alpha U_h + B^*\tilde{P}_h, w_h-U_h)_{\mathscr{B}}\, dt \\ &\quad\; +\int_0^T (B^*\tilde{P}_h - B^* p(U_h), u-U_h)_{\mathscr{B}}\, dt.\end{aligned}$$

Again, the condition (6.7) implies $-\int_0^T (\alpha U_h + B^*\tilde{P}_h, w_h - U_h)_{\mathscr{B}}\, dt \leq 0$, and hence

$$\begin{aligned}\gamma_s \int_0^T \|u-U_h\|^2_{\mathscr{B}} dt \;&\leq\; -\int_0^T (B^*\tilde{P}_h + \alpha U_h, u-w_h)_{\mathscr{B}}\, dt \\ &\quad\; +\int_0^T (B^*\tilde{P}_h - B^* p(U_h), u-U_h)_{\mathscr{B}}\, dt. \\ &:=\; E_1 + E_2. \qquad (6.31)\end{aligned}$$

To estimate the term E_1, we use assumption (6.30) and the Young's inequality to have

$$\begin{aligned}|E_1| \;&\leq\; C_{6,1} \int_0^T \sum_{\tau\in\mathscr{T}^n_{\mathscr{B}}} h_\tau |\alpha U_h + B^*\tilde{P}_h|_{H^1(\tau)} \|u-U_h\|_{L^2(\tau)}\, dt \\ &\leq\; \frac{3}{4} C^2_{6,1} \int_0^T \sum_{\tau\in\mathscr{T}^n_{\mathscr{B}}} h^2_\tau |\alpha U_h + B^*\tilde{P}_h|^2_{H^1(\tau)} + \frac{1}{4}\int_0^T \|u-U_h\|^2_{\mathscr{B}} dt. \qquad (6.32)\end{aligned}$$

To bound the term E_2, use of the Cauchy-Schwarz inequality and the Young's inequality yields

$$\begin{aligned}E_2 \;&\leq\; \int_0^T \|B^*(\tilde{P}_h - p(U_h))\|_{\mathscr{B}} \|u-U_h\|_{\mathscr{B}}\, dt \\ &\leq\; \frac{3}{4}\int_0^T \|\tilde{P}_h - p(U_h)\|^2_{\mathscr{B}} dt + \frac{1}{4}\int_0^T \|u-U_h\|^2_{\mathscr{B}} dt. \qquad (6.33)\end{aligned}$$

Combining (6.32), (6.33) with (6.31), the desired estimate follows. This completes the rest of the proof. □

Before deriving the intermediate *a posteriori* error estimates for the state and co-state variables, we first define the jumps quantities $[A\nabla\hat{Y}_h \cdot n_E]$ and $[A\nabla\tilde{P}_h \cdot n_E]$ over the edge (or face) E as:

$$[A\nabla\hat{Y}_h \cdot n_E] = (A\nabla\hat{Y}_h|_{K_1^E} - A\nabla\hat{Y}_h|_{K_2^E}) \cdot n_E, \qquad (6.34)$$

and

$$[A\nabla\tilde{P}_h \cdot n_E] = (A\nabla\tilde{P}_h|_{K_1^E} - A\nabla\tilde{P}_h|_{K_2^E}) \cdot n_E, \tag{6.35}$$

where n_E is the unit outward normal vector to K_1^E on E with $E = \bar{K}_1^E \cap \bar{K}_2^E$.

In Lemma 6.3.1, the error in the control variable relies on the estimate of $p(U_h) - \tilde{P}_h$ in the $L^2(0,T;\mathscr{B})$-norm. Since the bound of error $p(U_h) - \tilde{P}_h$ is the sum of the error $p(U_h) - P_h$ and the temporal error $P_h - \tilde{P}_h$, the error $p(U_h) - \tilde{P}_h$ is controlled by estimating the error $p(U_h) - P_h$ in the $L^2(0,T;\mathscr{B})$-norm.

The following lemma represents the intermediate error estimates for the co-state variable.

Lemma 6.3.2. *Let* (Y_h, P_h, U_h) *and* $(y(u^*), p(u^*))$ *with* $u^* = U_h$ *be the solutions of the problems* (6.17)–(6.21) *and* (6.26)–(6.29)*, respectively. Assume that all the assumptions of Lemma 6.3.1 are fulfilled, then there exists a positive constant* $C_{6,3}$ *such that the following estimate holds:*

$$\|p(U_h) - P_h\|^2_{L^2(0,T;\mathscr{B})} \;\le\; C_{6,3}\left(\sum_{i=2}^{6}\mathscr{E}^2_{6,i} + \|y(U_h) - Y_h\|^2_{L^2(0,T;L^2(\Omega))} + \|y(U_h) - \hat{Y}_h\|^2_{L^2(0,T;\mathscr{B})}\right)$$

with

$$\begin{aligned}
\mathscr{E}^2_{6,2} \;&=\; \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \int_K h_K^2\Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h - \varphi'(\hat{Y}_h)\tilde{P}_h\Big)^2 dx\,dt \\
&\quad + \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset,\; E\in\mathscr{T}^n_{\Omega_l}} \int_E h_E([A\nabla\tilde{P}_h \cdot n_E])^2 ds\,dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E(A\nabla\tilde{P}_h \cdot n_E - \hat{Y}_h + y^n_{ds})^2 ds\,dt, \\
\mathscr{E}^2_{6,3} \;&=\; \int_0^T \Big\{\sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} + \sum_{K\in\mathscr{T}^n_h} \|\tilde{P}_h - P_h\|^2_{H^1(K)}\Big\} dt, \\
\mathscr{E}^2_{6,4} \;&=\; \int_0^T \sum_{K\in\mathscr{T}^n_h} \int_K h_K^4\Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h - \varphi'(\hat{Y}_h)\tilde{P}_h\Big)^2 dx\,dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E h_E^3([A\nabla\tilde{P}_h \cdot n_E])^2 ds\,dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E^3(A\nabla\tilde{P}_h \cdot n_E - \hat{Y}_h + y^n_{ds})^2 ds\,dt, \\
\mathscr{E}^2_{6,5} \;&=\; \int_0^T \|y^n_{ds} - y_{ds}\|^2_{\mathscr{B}} dt,
\end{aligned}$$

$$\mathscr{E}_{6,6}^2 = \int_0^T \Big\{ \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} + \sum_{K\in\mathscr{T}^n_h} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} \Big\} dt,$$

where h_K *and* h_E *be the maximum diameter of* K *and* E*, respectively, and the jump* $[A\nabla \tilde{P}_h \cdot n_E]$ *is given by* (6.35).

Proof. From (6.19), $\forall v \in V$, $t \in (0,T]$, we obtain

$$\begin{aligned}
&-\Big(\frac{\partial P_h}{\partial t}, v\Big) + a(\tilde{P}_h, v) + (\varphi'(\hat{Y}_h)\tilde{P}_h, v) \\
&= -\Big(\frac{\partial P_h}{\partial t}, v - v_h\Big) + a(\tilde{P}_h, v - v_h) + (\varphi'(\hat{Y}_h)\tilde{P}_h, v - v_h) + (\hat{Y}_h - y^n_{ds}, v_h)_{\mathscr{B}} \\
&= -\Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, v - v_h\Big) + a(\tilde{P}_h, v - v_h) - (\hat{Y}_h - y^n_{ds}, v - v_h)_{\mathscr{B}} \\
&\quad + (\hat{Y}_h - y^n_{ds}, v)_{\mathscr{B}}.
\end{aligned} \tag{6.36}$$

Subtracting (6.36) from (6.28) with $u^* = U_h$, it follows that

$$\begin{aligned}
&-\Big(\frac{\partial}{\partial t}(p(U_h) - P_h), v\Big) + a(p(U_h) - \tilde{P}_h, v) + (\varphi'(y(U_h))p(U_h) - \varphi'(\hat{Y}_h)\tilde{P}_h, v) \\
&= \Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, v - v_h\Big) - a(\tilde{P}_h, v - v_h) + (\hat{Y}_h - y^n_{ds}, v - v_h)_{\mathscr{B}} \\
&\quad + (y(U_h) - \hat{Y}_h, v)_{\mathscr{B}} + (y^n_{ds} - y_{ds}, v)_{\mathscr{B}},
\end{aligned}$$

and hence,

$$\begin{aligned}
&-\Big(\frac{\partial}{\partial t}(p(U_h) - P_h), v\Big) + a(p(U_h) - P_h, v) = \Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, v - v_h\Big) - a(\tilde{P}_h, v - v_h) \\
&+ (\hat{Y}_h - y^n_{ds}, v - v_h)_{\mathscr{B}} + (y^n_{ds} - y_{ds}, v)_{\mathscr{B}} + a(\tilde{P}_h - P_h, v) + (y(U_h) - \hat{Y}_h, v)_{\mathscr{B}} \\
&- (\varphi'(y(U_h))p(U_h) - \varphi'(\hat{Y}_h)\tilde{P}_h, v).
\end{aligned} \tag{6.37}$$

Setting $e_p = p(U_h) - P_h$ and using the cut-off function ϕ defined in Chapter 5 (which is independent of time t) defined earlier, we compute the following identity

$$\int_\Omega A\nabla(e_p)\cdot\nabla(\phi^2 e_p)dx \;=\; \int_\Omega A\nabla(\phi e_p)\cdot\nabla(\phi e_p)dx - \int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\, dx.$$

That is,

$$a(e_p, \phi^2 e_p) = a(\phi e_p, \phi e_p) - \int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\, dx. \tag{6.38}$$

With $v = \phi^2 e_p := w_p$ and $v_h = \hat{\Pi}_h \phi^2 e_p := w^I_p$, equation (6.37) implies

$$\begin{aligned}
-\Big(\frac{\partial e_p}{\partial t}, \phi^2 e_p\Big) + a(e_p, \phi^2 e_p) \;&=\; \Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p - w^I_p\Big) - a(\tilde{P}_h, w_p - w^I_p) \\
&+ (\hat{Y}_h - y^n_{ds}, w_p - w^I_p)_{\mathscr{B}} + a(\tilde{P}_h - P_h, w_p) + (y(U_h) - \hat{Y}_h, w_p)_{\mathscr{B}} + (y^n_{ds} - y_{ds}, w_p)_{\mathscr{B}} \\
&- \big(\varphi'(y(U_h))p(U_h) - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p\big).
\end{aligned}$$

Using the identity (6.38), we obtain

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|\phi e_p\|^2 + a(\phi e_p, \phi e_p) = \Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p - w_p^I\Big) - a(\tilde{P}_h, w_p - w_p^I)\\
&\quad + (\hat{Y}_h - y_{ds}^n, w_p - w_p^I)_{\mathscr{B}} + a(\tilde{P}_h - P_h, w_p) + (y(U_h) - \hat{Y}_h, w_p)_{\mathscr{B}} + (y_{ds}^n - y_{ds}, w_p)_{\mathscr{B}}\\
&\quad - \big(\varphi'(y(U_h))p(U_h) - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p\big) + \int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\,dx.
\end{aligned}
$$

Integrating from t_{n-1} to t_n with respect to time t to have

$$
\begin{aligned}
&\frac{1}{2}\big(\|\phi e_p(t_{n-1})\|^2 - \|\phi e_p(t_n)\|^2\big) + \int_{t_{n-1}}^{t_n} a(\phi e_p, \phi e_p)dt\\
&= \int_{t_{n-1}}^{t_n} \Big\{\Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p - w_p^I\Big) - a(\tilde{P}_h, w_p - w_p^I) + (\hat{Y}_h - y_{ds}^n, w_p - w_p^I)_{\mathscr{B}}\Big\}dt\\
&\quad + \int_{t_{n-1}}^{t_n} a(\tilde{P}_h - P_h, w_p)dt - \int_{t_{n-1}}^{t_n} \big(\varphi'(y(U_h))p(U_h) - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p\big)dt\\
&\quad + \int_{t_{n-1}}^{t_n} (y(U_h) - \hat{Y}_h, w_p)_{\mathscr{B}}dt + \int_{t_{n-1}}^{t_n}\int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\,dxdt + \int_{t_{n-1}}^{t_n} (y_{ds}^n - y_{ds}, w_p)_{\mathscr{B}}dt.
\end{aligned}
$$

Taking summation from 1 to N, we find that

$$
\begin{aligned}
&\frac{1}{2}\big(\|\phi e_p(0)\|^2 - \|\phi e_p(T)\|^2\big) + \int_0^T a(\phi e_p, \phi e_p)dt\\
&= \int_0^T \Big\{\Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p - w_p^I\Big) - a(\tilde{P}_h, w_p - w_p^I) + (\hat{Y}_h - y_{ds}^n, w_p - w_p^I)_{\mathscr{B}}\Big\}dt\\
&\quad + \int_0^T a(\tilde{P}_h - P_h, w_p)dt - \int_0^T \big(\varphi'(y(U_h))p(U_h) - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p\big)dt\\
&\quad + \int_0^T (y(U_h) - \hat{Y}_h, w_p)_{\mathscr{B}}\,dt + \int_0^T\int_\Omega (e_p)^2 A\nabla\phi\cdot\nabla\phi\,dx\,dt + \int_0^T (y_{ds}^n - y_{ds}, w_p)_{\mathscr{B}}dt\\
&=:\; I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
\tag{6.39}
$$

We now estimate the terms I_j, $j = 1, 2, \ldots, 6$. For I_1, we find that

$$
\begin{aligned}
I_1 &= \int_0^T \Big\{\Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, w_p - w_p^I\Big) - a(\tilde{P}_h, w_p - w_p^I) + (\hat{Y}_h - y_{ds}^n, w_p - w_p^I)_{\mathscr{B}}\Big\}dt\\
&= \int_0^T \sum_{K\in\mathscr{T}_h^n}\int_K \Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h - \varphi'(\hat{Y}_h)\tilde{P}_h\Big)(w_p - w_p^I)dx\,dt\\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset}\int_E [A\nabla\tilde{P}_h\cdot n_E](w_p - w_p^I)ds\,dt\\
&\quad + \int_0^T \sum_{E\subset\Gamma}\int_E (A\nabla\tilde{P}_h\cdot n_E - \hat{Y}_h + y_{ds}^n)(w_p - w_p^I)ds\,dt.
\end{aligned}
$$

Note that $\phi = 0$ on $\Omega \setminus \Omega_l$. Thus, use of Lemma 5.2.3, the inequality (5.38) and the Young's inequality leads to

$$\begin{aligned} I_1 \;\leq\; & c(\delta) C_{6,4}^2 \Big\{ \int_0^T \sum_{K \in \mathscr{T}_{\Omega_l}^n} \int_K h_K^2 \Big(\frac{\partial P_h}{\partial t} + div(A \nabla \tilde{P}_h) - a_0 \tilde{P}_h - \varphi'(\hat{Y}_h) \tilde{P}_h \Big)^2 dx\, dt \\ & + \int_0^T \sum_{E \cap \partial \mathscr{T}_{\Omega_l}^n = \emptyset,\; E \in \mathscr{T}_{\Omega_l}^n} \int_E h_E ([A \nabla \tilde{P}_h \cdot n_E])^2 ds\, dt \\ & + \int_0^T \sum_{E \subset \Gamma} \int_E h_E (A \nabla \tilde{P}_h \cdot n_E - \hat{Y}_h + y_{ds}^n)^2 ds\, dt \Big\} + \delta \int_0^T \|\phi e_p\|_{H^1(\Omega)}^2 dt, \end{aligned}$$

where $C_{6,4}^2 = \tilde{C}_{I,5}^2 \max\{\tilde{C}_{I,3}^2, 1\}$. Using the continuity of the bilinear form and the Young's inequality, we have for I_2,

$$\begin{aligned} I_2 \;=\; & \int_0^T a(\tilde{P}_h - P_h, w_p) dt, \\ \leq\; & c(\delta) \alpha_0^2 C_{6,5}^2 \int_0^T \sum_{K \in \mathscr{T}_{\Omega_l}^n} \|\tilde{P}_h - P_h\|_{H^1(K)}^2\, dt + \delta \int_0^T \|\phi e_p\|_{H^1(\Omega)}^2 dt \end{aligned}$$

with $C_{6,5}^2 = \tilde{C}_{I,5}^2 / \tilde{C}_{I,4}^2$. Next, utilizing (6.25), we estimate I_3 as

$$\begin{aligned} |I_3| \;\leq\; & \int_0^T \big| \big(\varphi'(y(U_h))(p(U_h) - P_h), w_p \big) \big| dt + \int_0^T \big| \big((\varphi'(y(U_h)) - \varphi'(Y_h)) P_h, w_p \big) \big| dt \\ & + \int_0^T \big| \big((\varphi'(Y_h) - \varphi'(\hat{Y}_h)) P_h, w_p \big) \big| dt + \int_0^T \big| \big(\varphi'(\hat{Y}_h)(P_h - \tilde{P}_h), w_p \big) \big| dt \\ \leq\; & \int_0^T \big| \big(\varphi'(y(U_h))(p(U_h) - P_h), w_p \big) \big| dt + \int_0^T \big| \big(\bar{\varphi}''(Y_h)(y(U_h) - Y_h) P_h, w_p \big) \big| dt \\ & + \int_0^T \big| \big((\varphi'(Y_h) - \varphi'(\hat{Y}_h)) P_h, w_p \big) \big| dt + \int_0^T \big| \big(\varphi'(\hat{Y}_h)(P_h - \tilde{P}_h), w_p \big) \big| dt, \end{aligned}$$

and hence

$$\begin{aligned} |I_3| \;\leq\; & c(\delta)\, C_{6,5}^2\, C_{6,6}^2 \Big\{ \int_0^T \|p(U_h) - P_h\|_{L^2(\Omega)}^2 dt + \int_0^T \|y(U_h) - Y_h\|_{L^2(\Omega)}^2 dt \\ & + \int_0^T \sum_{K \in \mathscr{T}_{\Omega_l}^n} \|\hat{Y}_h - Y_h\|_{H^1(K)}^2 dt + \int_0^T \sum_{K \in \mathscr{T}_{\Omega_l}^n} \|\tilde{P}_h - P_h\|_{H^1(K)}^2 dt \Big\} \\ & + \delta \int_0^T \|\phi e_p\|_{H^1(\Omega)}^2 dt, \end{aligned} \tag{6.40}$$

where $C_{6,6}^2 = \max\{\tilde{C}_M^2, \tilde{C}_L^2\}$. To estimate I_4, we apply the Cauchy inequality and the Young's inequality to obtain

$$I_4 \;\leq\; c(\delta)\, C_{6,5}^2 \int_0^T \|y(U_h) - \hat{Y}_h\|_{\mathscr{B}}^2 dt + \delta \int_0^T \|\phi e_p\|_{H^1(\Omega)}^2 dt.$$

The term I_5 may be estimated as

$$I_5 \;=\; \int_0^T \int_\Omega (e_p)^2 A\nabla\phi\nabla\phi \, dx\, dt \;\le\; \|\phi\|^2_{1,\infty,\Omega} \int_0^T \|e_p\|^2_{L^2(\Omega)} dt.$$

Since ϕ is a cut-off function, there exists a positive constant C_M such that $\|\phi\|_{1,\infty,\Omega} \le C_M$, thus we obtain

$$I_5 \;\le\; C_M^2 \int_0^T \|e_p\|^2_{L^2(\Omega)} dt.$$

Finally, we apply the Cauchy inequality and the Young's inequality to estimate the last term I_6 as

$$I_6 \;\le\; c(\delta) C_{6,5}^2 \int_0^T \|y_{ds}^n - y_{ds}\|^2_{\mathscr{B}} dt + \delta \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt.$$

Combining the above estimates with (6.39), and using the coercive of bilinear form and $e_p(T) = 0$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \alpha_1 \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt \\
&\le C_{6,7} \Big\{ \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \int_K h_K^2 \Big(\frac{\partial P_h}{\partial t} + div(A\nabla \tilde{P}_h) - a_0 \tilde{P}_h - \varphi'(\hat{Y}_h)\tilde{P}_h \Big)^2 dx\, dt \\
&\quad + \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset,\, E\subset\mathscr{T}^n_{\Omega_l}} \int_E h_E([A\nabla\tilde{P}_h \cdot n_E])^2 ds\, dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E(A\nabla\tilde{P}_h \cdot n_E - \hat{Y}_h + y^n_{ds})^2 ds\, dt + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt \\
&\quad + \int_0^T \|p(U_h) - P_h\|^2_{L^2(\Omega)} dt + \int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt + \int_0^T \|y^n_{ds} - y_{ds}\|^2_{\mathscr{B}} dt \\
&\quad + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}} dt \Big\} + 5\delta \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt,
\end{aligned}$$

where $C_{6,7} = c(\delta)\max\{C_{6,4}^2, C_{6,5}^2, \alpha_0^2 C_{6,5}^2, C_{6,5}^2 C_{6,6}^2, C_M^2\}$. Setting $\delta = \alpha_1/10$ and using standard kick-back argument to obtain

$$\begin{aligned}
&\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2} \int_0^T \|\phi e_p\|^2_{H^1(\Omega)} dt \\
&\le C_{6,7} \Big\{ \mathscr{E}^2_{6,2} + \mathscr{E}^2_{6,5} + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt \\
&\quad + \int_0^T \|p(U_h) - P_h\|^2_{L^2(\Omega)} dt + \int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt + \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}} dt \Big\}.
\end{aligned} \tag{6.41}$$

An application of the trace inequality with cut-off function ϕ implies

$$\|e_p\|_{L^2(0,T;\mathscr{B})} = \|\phi e_p\|_{L^2(0,T;\mathscr{B})} \leq \|\phi e_p\|_{L^2(0,T;H^1(\Omega))},$$

and utilizing the above inequality in (6.41) we obtain

$$\begin{aligned}
&\frac{1}{2}\|\phi e_p(0)\|^2_{L^2(\Omega)} + \frac{\alpha_1}{2}\int_0^T \|e_p\|^2_{\mathscr{B}} dt \\
&\leq C_{6,7}\Big\{\mathscr{E}^2_{6,2} + \mathscr{E}^2_{6,5} + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\tilde{P}_h - P_h\|^2_{H^1(K)} dt + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt \\
&\quad + \int_0^T \|p(U_h) - P_h\|^2_{L^2(\Omega)} dt + \int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt + \int_0^T \|y(U_h) - \hat{Y}_h\|^2_{\mathscr{B}} dt\Big\}.
\end{aligned} \tag{6.42}$$

Perusing equation (6.42), we need to estimate the term $p(U_h) - P_h$ in the $L^2(0,T;L^2(\Omega))$-norm. For this, we shall use the duality trick. Thus, we have

$$\begin{aligned}
&\int_0^T \|p(U_h) - P_h\|^2_{L^2(\Omega)} dt = \int_0^T (p(U_h) - P_h, F) dt \\
&= \int_0^T \Big\{\Big(p(U_h) - P_h, \frac{\partial \chi}{\partial t}\Big) + a(p(U_h) - P_h, \chi) + (\varphi'(y(U_h))(p(U_h) - P_h), \chi)\Big\} dt \\
&= -\int_0^T \Big(\frac{\partial}{\partial t}(p(U_h) - P_h), \chi\Big) dt + \int_0^T a(p(U_h) - P_h, \chi) dt \\
&\quad + \int_0^T \big(\varphi'(y(U_h))p(U_h) - \varphi'(\hat{Y}_h)\tilde{P}_h, \chi\big) dt + \int_0^T \big(\varphi'(\hat{Y}_h)\tilde{P}_h - \varphi'(y(U_h))P_h, \chi\big) dt, \\
&= -\int_0^T \Big\{-\Big(\frac{\partial P_h}{\partial t}, \chi\Big) + a(\tilde{P}_h, \chi) + \big(\varphi'(\hat{Y}_h)\tilde{P}_h, \chi\big)\Big\} dt + \int_0^T (y(U_h) - y_{ds}, \chi)_{\mathscr{B}} dt \\
&\quad + \int_0^T a(\tilde{P}_h - P_h, \chi) dt + \int_0^T \big(\varphi'(\hat{Y}_h)\tilde{P}_h - \varphi'(y(U_h))P_h, \chi\big) dt,
\end{aligned}$$

use of (6.25) and (6.36) lead to

$$\begin{aligned}
&\int_0^T \|p(U_h) - P_h\|^2_{L^2(\Omega)} dt \\
&= \int_0^T \Big\{\Big(\frac{\partial P_h}{\partial t} - \varphi'(\hat{Y}_h)\tilde{P}_h, \chi - \chi^I\Big) - a(\tilde{P}_h, \chi - \chi^I) + (\hat{Y}_h - y^n_{ds}, \chi - \chi^I)_{\mathscr{B}}\Big\} dt \\
&\quad + \int_0^T a(\tilde{P}_h - P_h, \chi) dt + \int_0^T (y(U_h) - \hat{Y}_h, \chi)_{\mathscr{B}} dt + \int_0^T \big(\varphi'(\hat{Y}_h)(\tilde{P}_h - P_h), \chi\big) dt \\
&\quad + \int_0^T \big((\varphi'(\hat{Y}_h) - \varphi'(Y_h))P_h, \chi\big) dt + \int_0^T \big((\varphi'(Y_h) - \varphi'(y(U_h)))P_h, \chi\big) dt \\
&\quad + \int_0^T (y^n_{ds} - y_{ds}, \chi)_{\mathscr{B}} dt,
\end{aligned}$$

and hence,

$$
\begin{aligned}
\int_0^T \|p(U_h)-P_h\|^2_{L^2(\Omega)}dt \;&= \int_0^T \Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h - \varphi'(\hat{Y}_h)\tilde{P}_h\Big)(\chi-\chi^I)dt \\
&+ \int_0^T \sum_{E\cap\Gamma=\emptyset}\int_E [A\nabla\tilde{P}_h\cdot n_E](\chi^I-\chi)ds\,dt \\
&+ \int_0^T \sum_{E\subset\Gamma}\int_E (A\nabla\tilde{P}_h\cdot n_E - \hat{Y}_h + y^n_{ds})(\chi^I-\chi)ds\,dt + \int_0^T a(\tilde{P}_h-P_h,\chi)dt \\
&+ \int_0^T (\varphi'(\hat{Y}_h)(\tilde{P}_h-P_h),\chi)dt + \int_0^T ((\varphi'(\hat{Y}_h)-\varphi'(Y_h))P_h,\chi)dt \\
&+ \int_0^T (\varphi''(Y_h)(Y_h-y(U_h))P_h,\chi)dt + \int_0^T (y(U_h)-\hat{Y}_h,\chi)_{\mathscr{B}}dt + \int_0^T (y^n_{ds}-y_{ds},\chi)_{\mathscr{B}}dt,
\end{aligned}
$$

where χ^I is the interpolation of χ. Application of Lemmas 5.2.1 and 5.2.3, continuity property of the bilinear form, the Young's inequality and arguing as (6.40) to obtain

$$
\begin{aligned}
&\int_0^T \|p(U_h)-P_h\|^2_{L^2(\Omega)}dt \\
&\le c(\delta)C_{6,8}\Big\{\int_0^T \sum_{K\in\mathscr{T}_h^n}\int_K h_K^4\Big(\frac{\partial P_h}{\partial t} + div(A\nabla\tilde{P}_h) - a_0\tilde{P}_h - \varphi'(\hat{Y}_h)\tilde{P}_h\Big)^2 dx\,dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset}\int_E h_E^3([A\nabla\tilde{P}_h\cdot n_E])^2 ds\,dt + \int_0^T \sum_{E\subset\Gamma}\int_E h_E^3(A\nabla\tilde{P}_h\cdot n_E - \hat{Y}_h + y_{ds})^2 ds\,dt\Big\} \\
&\quad + c(\delta)(\alpha_0^2+\tilde{C}_M^2)\int_0^T \sum_{K\in\mathscr{T}_h^n}\|\tilde{P}_h-P_h\|^2_{H^1(K)}dt + c(\delta)\tilde{C}_M^2\int_0^T \|y(U_h)-Y_h\|^2_{L^2(\Omega)}dt \\
&\quad + c(\delta)\tilde{C}_L^2\int_0^T \sum_{K\in\mathscr{T}_h^n}\|\hat{Y}_h-Y_h\|^2_{H^1(K)}dt + c(\delta)\int_0^T \|y(U_h)-\hat{Y}_h\|^2_{\mathscr{B}}dt \\
&\quad + c(\delta)\int_0^T \|y^n_{ds}-y_{ds}\|^2_{\mathscr{B}}dt + \delta\int_0^T \|\chi\|^2_{\mathscr{B}}dt + \delta\int_0^T \|\chi\|^2_{H^2(\Omega)}dt,
\end{aligned}
$$

where $C_{6,8} = \tilde{C}^2_{I,1}\max\{\tilde{C}^2_{I,3},1\}$. For the last two terms on the right hand side, use the trace result and the stability result of Lemma 6.2.1 with $\Phi=\chi$, $F=p(U_h)-P_h$. Then, choose $\delta = 1/4$ and set $C_{6,9} = 2c(\delta)\max\{C_{6,8},\alpha_0^2+\tilde{C}_M^2,\tilde{C}_M^2,\tilde{C}_L^2,1\}$ to obtain

$$
\begin{aligned}
\int_0^T \|p(U_h)-P_h\|^2_{L^2(\Omega)}dt \;&\le\; C_{6,9}\Big\{\mathscr{E}^2_{6,4} + \mathscr{E}^2_{6,5} + \int_0^T \sum_{K\in\mathscr{T}_h^n}\|\tilde{P}_h-P_h\|^2_{H^1(K)}\,dt \\
&+ \int_0^T \sum_{K\in\mathscr{T}_h^n}\|\hat{Y}_h-Y_h\|^2_{H^1(K)}\,dt + \int_0^T \|y(U_h)-Y_h\|^2_{L^2(\Omega)}\,dt + \int_0^T \|y(U_h)-\hat{Y}_h\|^2_{\mathscr{B}}\,dt\Big\}.
\end{aligned}
\tag{6.43}
$$

Combining above with (6.42) with (6.43) and adjusting the constants we complete the rest of the proof. □

It is evident from the above lemma that our intermediate error estimate of $p(U_h)-P_h$ is depending on the error $y(U_h)-Y_h$ in the norms $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;\mathscr{B})$. In the following lemma, we present an intermediate error estimate for the state variable.

Lemma 6.3.3. *Let* (Y_h,P_h,U_h) *and* $(y(u^*),p(u^*))$*, respectively, be the solutions of* (6.17)–(6.21) *and* (6.26) – (6.29) *with* $u^*=U_h$*. Assume that all the assumptions of Lemma 6.3.1 are fulfilled. Then there exists a positive constant* $C_{6,10}$ *such that the following estimate holds:*

$$\|y(U_h)-Y_h\|^2_{L^2(0,T;\mathscr{B})} \;\leq\; C_{6,10}\Big(\sum_{i=6}^{10}\mathscr{E}^2_{6,i}\Big),$$

with

$$\begin{aligned}
\mathscr{E}^2_{6,6} &= \int_0^T\Big\{\sum_{K\in\mathscr{T}^n_{\Omega_l}}\|\hat{Y}_h-Y_h\|^2_{H^1(K)}+\sum_{K\in\mathscr{T}^n_h}\|\hat{Y}_h-Y_h\|^2_{H^1(K)}\Big\}dt,\\
\mathscr{E}^2_{6,7} &= \int_0^T\sum_{K\in\mathscr{T}^n_{\Omega_l}}\int_K h_K^2\Big(f^n-\frac{\partial Y_h}{\partial t}+div(A\nabla\hat{Y}_h)-a_0\hat{Y}_h-\varphi(\hat{Y}_h)\Big)^2dx\,dt\\
&\quad+\int_0^T\sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset,\ E\in\mathscr{T}^n_{\Omega_l}}\int_E h_E([A\nabla\hat{Y}_h\cdot n_E])^2ds\,dt\\
&\quad+\int_0^T\sum_{E\subset\Gamma}\int_E h_E(A\nabla\hat{Y}_h\cdot n_E-BU_h-\mathrm{x}_b)^2ds\,dt,\\
\mathscr{E}^2_{6,8} &= \int_0^T\sum_{K\in\mathscr{T}^n_h}\int_K h_K^4\Big(f^n-\frac{\partial Y_h}{\partial t}+div(A\nabla\hat{Y}_h)-a_0\hat{Y}_h-\varphi(\hat{Y}_h)\Big)^2dx\,dt\\
&\quad+\int_0^T\sum_{E\cap\Gamma=\emptyset}\int_E h_E^3([A\nabla\hat{Y}_h\cdot n_E])^2ds\,dt\\
&\quad+\int_0^T\sum_{E\subset\Gamma}\int_E h_E^3(A\nabla\hat{Y}_h\cdot n_E-BU_h-\mathrm{x}_b)^2ds\,dt,\\
\mathscr{E}^2_{6,9} &= \int_0^T\|f-f^n\|^2_{L^2(\Omega)}dt,\\
\mathscr{E}^2_{6,10} &= \frac{1}{2}\|y_0-y_{h,0}\|^2_{L^2(\Omega)},
\end{aligned}$$

where h_K *and* h_E *be the maximum diameters of* K *and* E*, respectively, and the jump* $[A\nabla\hat{Y}_h\cdot n_E]$ *is defined as in* (6.34).

Proof. For all $v \in V$ and $t \in (0, T]$, we have from (6.17)

$$\begin{aligned}
&\Big(\frac{\partial Y_h}{\partial t}, v\Big) + a(Y_h^n, v) + (\varphi(\hat{Y}_h), v) \\
&= \Big(\frac{\partial Y_h}{\partial t}, v - v_h\Big) + a(Y_h^n, v - v_h) + (\varphi(\hat{Y}_h), v - v_h) + (BU_h + \mathtt{x}_b, v_h)_{\mathscr{B}} + (f^n, v_h) \\
&= -\Big(f^n - \frac{\partial Y_h}{\partial t} - \varphi(\hat{Y}_h), v - v_h\Big) + a(Y_h^n, v - v_h) \\
&\quad + (BU_h + \mathtt{x}_b, v_h)_{\mathscr{B}} + (f^n, v). \qquad (6.44)
\end{aligned}$$

With the help of (6.26) and (6.44) with $u^* = U_h$, we get

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t}(y(U_h) - Y_h), v\Big) + a(y(U_h) - Y_h, v) \\
&= \Big(f^n - \frac{\partial Y_h}{\partial t} - \varphi(\hat{Y}_h), v - v_h\Big) - a(Y_h^n, v - v_h) + (BU_h + \mathtt{x}_b, v - v_h)_{\mathscr{B}} \\
&\quad + a(\hat{Y}_h - Y_h, v) + (f - f^n, v) - \big(\varphi(y(U_h)) - \varphi(\hat{Y}_h), v\big).
\end{aligned}$$

Setting $v = \phi^2 e_y := w_y$ and $v_h = \hat{\Pi}_h(\phi^2 e_y) := w_y^I$ with $e_y = y(U_h) - Y_h$, where w_y^I is interpolation of w_y, we obtain

$$\begin{aligned}
&\Big(\frac{\partial e_y}{\partial t}, \phi^2 e_y\Big) + a(e_y, \phi^2 e_y) = \Big(f^n - \frac{\partial Y_h}{\partial t} - \varphi(\hat{Y}_h), w_y - w_y^I\Big) - a(\hat{Y}_h, w_y^I - w_y) \\
&\quad + (BU_h + \mathtt{x}_b, w_y - w_y^I)_{\mathscr{B}} + a(\hat{Y}_h - Y_h, w_y) + (f - f^n, w_y) \\
&\quad - \big(\varphi(y(U_h)) - \varphi(\hat{Y}_h), w_y\big). \qquad (6.45)
\end{aligned}$$

Using the definition of cut-off function ϕ, we have the following identity:

$$\int_\Omega A\nabla(e_y) \cdot \nabla(\phi^2 e_y)dx = \int_\Omega A\nabla(\phi e_y) \cdot \nabla(\phi e_y)dx - \int_\Omega (e_y)^2 A\nabla\phi \cdot \nabla\phi dx,$$

i.e.,
$$(e_y, \phi^2 e_y) = a(\phi e_y, \phi e_y) - \int_\Omega (e_y)^2 A\nabla\phi \cdot \nabla\phi dx.$$

Applying the above identity in (6.45) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi e_y\|^2 + a(\phi e_y, \phi e_y) = \Big(f^n - \frac{\partial Y_h}{\partial t} - \varphi(\hat{Y}_h), w_y - w_y^I\Big) - a(\hat{Y}_h, w_y^I - w_y) \\
&\quad + (BU_h + \mathtt{x}_b, w_y - w_y^I)_{\mathscr{B}} + a(\hat{Y}_h - Y_h, w_y) + (f - f^n, w_y) \\
&\quad - \big(\varphi(y(U_h)) - \varphi(\hat{Y}_h), w_y\big) + \int_\Omega (e_y)^2 A\nabla\phi \cdot \nabla\phi \, dx.
\end{aligned}$$

Integrate the above from t_{n-1} to t_n with respect to time to have

$$
\begin{aligned}
&\frac{1}{2}\big(\|\phi e_y(t_n)\|^2 - \|\phi e_y(t_{n-1})\|^2\big) + \int_{t_{n-1}}^{t_n} a(\phi e_y, \phi e_y)dt \\
&= \int_{t_{n-1}}^{t_n} \Big\{\Big(f^n - \frac{\partial Y_h}{\partial t} - \varphi(\hat{Y}_h), w_y - w_y^I\Big) - a(\hat{Y}_h, w_y^I - w_y) \\
&\quad + (BU_h + \mathrm{x}_b, w_y - w_y^I)_{\mathscr{B}}\Big\}dt + \int_{t_{n-1}}^{t_n} a(\hat{Y}_h - Y_h, w_y)dt + \int_{t_{n-1}}^{t_n} (f - f^n, w_y)dt \\
&\quad - \int_{t_{n-1}}^{t_n} \big(\varphi(y(U_h)) - \varphi(\hat{Y}_h), w_y\big)dt + \int_{t_{n-1}}^{t_n}\int_\Omega (e_y)^2 A\nabla\phi\cdot\nabla\phi\,dx\,dt.
\end{aligned}
$$

Summing up from 1 to N, we get

$$
\begin{aligned}
&\frac{1}{2}\big(\|\phi e_y(T)\|^2 - \|\phi e_y(0)\|^2\big) + \int_0^T a(\phi e_y, \phi e_y)dt \\
&= \int_0^T \Big\{\Big(f^n - \frac{\partial Y_h}{\partial t} - \varphi(\hat{Y}_h), w_y - w_y^I\Big) - a(\hat{Y}_h, w_y^I - w_y) + (BU_h + \mathrm{x}_b, w_y - w_y^I)_{\mathscr{B}}\Big\}dt \\
&\quad + \int_0^T a(\hat{Y}_h - Y_h, w_y)dt + \int_0^T (f - f^n, w_y)dt - \int_0^T \big(\varphi(y(U_h)) - \varphi(\hat{Y}_h), w_y\big)dt \\
&\quad + \int_0^T\int_\Omega (e_y)^2 A\nabla\phi\cdot\nabla\phi\,dx\,dt \;\; =: I_7 + I_8 + I_9 + I_{10} + I_{11}.
\end{aligned}
\tag{6.46}
$$

We need to bound the terms $I_j,\ j = 7, \ldots, 11$, separately. First, we rewrite I_7 as

$$
\begin{aligned}
I_7 &= \int_0^T \Big\{\Big(f^n - \frac{\partial Y_h}{\partial t} - \varphi(\hat{Y}_h), w_y - w_y^I\Big) - a(\hat{Y}_h, w_y^I - w_y) + (BU_h + \mathrm{x}_b, w_y - w_y^I)_{\mathscr{B}}\Big\}dt \\
&= \int_0^T \sum_{K\in\mathscr{T}_h^n}\int_K \Big(f^n - \frac{\partial Y_h}{\partial t} + div(A\nabla\hat{Y}_h) - a_0\hat{Y}_h - \varphi(\hat{Y}_h)\Big)(w_y - w_y^I)\,dx\,dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset}\int_E [A\nabla\hat{Y}_h\cdot n_E](w_y - w_y^I)ds\,dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma}\int_E (A\nabla\hat{Y}_h\cdot n_E - BU_h - \mathrm{x}_b)(w_y - w_y^I)ds\,dt.
\end{aligned}
$$

Since $\phi = 0$ on $\Omega\setminus\Omega_l$, applying Lemma 5.2.3, the inequality (5.38) and the Young's inequality, we arrive at

$$
\begin{aligned}
I_7 \;\le\; c(\delta)\,C_{6,4}^2\Big\{&\int_0^T \sum_{K\in\mathscr{T}_{\Omega_l}^n}\int_K h_K^2\Big(f^n - \frac{\partial Y_h}{\partial t} + div(A\nabla\hat{Y}_h) - a_0\hat{Y}_h - \varphi(\hat{Y}_h)\Big)^2 dx\,dt \\
&+ \int_0^T \sum_{E\cap\partial\mathscr{T}_{\Omega_l}^n=\emptyset,\ E\in\mathscr{T}_{\Omega_l}^n}\int_E h_E([A\nabla\hat{Y}_h\cdot n_E])^2 ds\,dt \\
&+ \int_0^T \sum_{E\subset\Gamma}\int_E h_E(A\nabla\hat{Y}_h\cdot n_E - BU_h - \mathrm{x}_b)^2 ds\,dt\Big\} + \delta\int_0^T \|\phi e_y\|_{H^1(\Omega)}^2 dt.
\end{aligned}
$$

Next, for I_8, the continuity of the bilinear form and the Young's inequality imply

$$I_8 \;\leq\; c(\delta)\, C_{6,5}^2\, \alpha_0^2 \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + \delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt.$$

An application of the Cauchy-Schwarz inequality for I_9 leads to

$$\begin{aligned} I_9 &= \int_0^T (f - f^n, w_y) dt \leq \int_0^T \|f - f^n\|_{L^2(\Omega)} \|w_y\|_{L^2(\Omega)} dt \\ &\leq \int_0^T \|f - f^n\|_{L^2(\Omega)} \|\phi e_y\|_{H^1(\Omega)} dt. \end{aligned}$$

With the help of the Young's inequality we obtain

$$I_9 \;\leq\; c(\delta) C_{6,5}^2 \int_0^T \|f - f^n\|^2_{L^2(\Omega)} dt + \delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt.$$

To estimate I_{10}, we use (6.25) to have

$$\begin{aligned} I_{10} &= \int_0^T \big(\varphi(\hat{Y}_h) - \varphi(Y_h)), w_y\big) dt + \int_0^T \big(\varphi(Y_h) - \varphi(y(U_h)), w_y\big) dt \\ &= \int_0^T \big(\varphi'(\hat{Y}_h)(\hat{Y}_h - Y_h), w_y\big) dt + \int_0^T \big(\bar{\varphi}'(Y_h)(Y_h - y(U_h)), w_y\big) dt, \end{aligned}$$

an application of the Chaucy-Schwarz inequality and the Young's inequality leads to

$$\begin{aligned} I_{10} \;\leq\; & c(\delta)\, C_{6,5}^2\, \tilde{C_M}^2 \Big\{ \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + \int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt \Big\} \\ & + \delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt. \end{aligned} \tag{6.47}$$

Finally, arguing as I_5 we estimate the last term I_{11}

$$I_{11} \;\leq\; C_M^2 \int_0^T \|e_y\|^2_{L^2(\Omega)} dt.$$

Combining bounds for $I_j,\ j = 7, \ldots,\ 11$, with (6.46) and using the coercivity of the bilinear form we conclude that

$$
\begin{aligned}
&\frac{1}{2}\|\phi e_y(T)\|^2 + \alpha_1 \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt \\
&\le\; C_{6,11}\Big\{ \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \int_K h_K^2 \Big(f^n - \frac{\partial Y_h}{\partial t} + div(A\nabla \hat{Y}_h) - a_0\hat{Y}_h - \varphi(\hat{Y}_h)\Big)^2 dx\, dt \\
&\quad + \int_0^T \sum_{E\cap\partial\mathscr{T}^n_{\Omega_l}=\emptyset,\; E\in\mathscr{T}^n_{\Omega_l}} \int_E h_E([A\nabla\hat{Y}_h\cdot n_E])^2 ds\, dt \\
&\quad + \int_0^T \sum_{E\subset\Gamma} \int_E h_E(A\nabla\hat{Y}_h\cdot n_E - BU_h - \mathrm{x}_b)^2 ds\, dt \\
&\quad + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + \int_0^T \|f - f^n\|^2_{L^2(\Omega)} dt + \frac{1}{2}\|y_0 - y_{h,0}\|^2 \\
&\quad + \int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt \Big\} + 4\delta \int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt
\end{aligned}
$$

with $C_{6,11} = \max\{c(\delta)C^2_{6,4}, c(\delta)C^2_{6,5}\alpha_0^2, c(\delta)C^2_{6,5}, c(\delta)C^2_{6,5}\tilde{C}^2_M, C^2_M\}$. Choose $\delta = \alpha_1/8$ and then use the standard kick-back argument to have

$$
\begin{aligned}
\frac{1}{2}\|\phi e_y(T)\|^2 + \frac{\alpha_1}{2}\int_0^T \|\phi e_y\|^2_{H^1(\Omega)} dt \;&\le\; C_{6,11}\Big\{\mathscr{E}^2_{6,7} + \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt \\
&+ \int_0^T \|f - f^n\|^2_{L^2(\Omega)} dt + \frac{1}{2}\|y_0 - y_{h,0}\|^2 + \int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt\Big\}. \quad (6.48)
\end{aligned}
$$

An application of the trace inequality with the cut-off function ϕ implies

$$
\|e_y\|_{L^2(0,T;\mathscr{B})} \;=\; \|\phi e_y\|_{L^2(0,T;\mathscr{B})} \;\le\; \|\phi e_y\|_{L^2(0,T;H^1(\Omega))},
$$

which combine with (6.48) yields

$$
\begin{aligned}
\frac{1}{2}\|\phi e_y(T)\|^2 + \frac{\alpha_1}{2}\int_0^T \|e_y\|^2_{\mathscr{B}} dt \le C_{6,11}\Big\{\mathscr{E}^2_{6,7} &+ \int_0^T \sum_{K\in\mathscr{T}^n_{\Omega_l}} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt \\
+ \int_0^T \|f - f^n\|^2_{L^2(\Omega)} dt + \frac{1}{2}\|y_0 - y_{h,0}\|^2 &+ \int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt\Big\}. \quad (6.49)
\end{aligned}
$$

Next, our requirement is to evaluate the term $(y(U_h) - Y_h)$ in the $L^2(0,T;L^2(\Omega))$-norm.

A standard duality argument leads to

$$\begin{aligned}
&\int_0^T \|y(U_h)-Y_h\|^2_{L^2(\Omega)}dt\\
&=\int_0^T \Big\{\Big(y(U_h)-Y_h,-\frac{\partial\psi}{\partial t}\Big)+a(y(U_h)-Y_h,\psi)+(\bar{\varphi}\,(y(U_h)-Y_h),\psi)\Big\}dt\\
&=\int_0^T \Big\{(\frac{\partial}{\partial t}(y(U_h)-Y_h),\psi)+a(y(U_h)-Y_h,\psi)\\
&\quad+\big(\varphi(y(U_h))-\varphi(Y_h),\psi\big)\Big\}dt+((y(U_h)-Y_h)(0),\psi(0))\\
&=-\int_0^T \Big\{\Big(\frac{\partial Y_h}{\partial t},\psi\Big)+a(Y_h^n,\psi)+(\varphi(\hat{Y}_h),\psi)\Big\}dt+\int_0^T (f,\psi)dt\\
&\quad+\int_0^T (BU_h+\mathtt{x}_b,\psi)_{\mathscr{B}}\,dt+\int_0^T a(Y_h^n-Y_h,\psi)dt\\
&\quad+\int_0^T \big(\varphi(\hat{Y}_h)-\varphi(Y_h),\psi\big)dt+(y_0-y_{h,0},\psi(0)),
\end{aligned}$$

which together with (6.44) and (6.25) imply

$$\begin{aligned}
&\int_0^T \|y(U_h)-Y_h\|^2_{L^2(\Omega)}dt\\
&=\int_0^T \Big\{\Big(f^n-\frac{\partial Y_h}{\partial t}-\varphi(\hat{Y}_h),\psi-\psi^I\Big)-a(\hat{Y}_h,\psi-\psi^I)\\
&\quad+(BU_h+\mathtt{x}_b,\psi-\psi^I)_{\mathscr{B}}\Big\}dt+\int_0^T a(Y_h^n-Y_h,\psi)dt+\int_0^T (f-f^n,\psi)dt\\
&\quad+\int_0^T \big(\varphi'(\hat{Y}_h)(\hat{Y}_h-Y_h),\psi\big)dt+(y_0-y_{h,0},\psi(0)),\\
&=\int_0^T \Big(f^n-\frac{\partial Y_h}{\partial t}+div(A\nabla\hat{Y}_h)-a_0\hat{Y}_h-\varphi(\hat{Y}_h)\Big)(\psi-\psi^I)dt\\
&\quad+\int_0^T \sum_{E\cap\Gamma=\emptyset}\int_E [A\nabla\hat{Y}_h\cdot n_E](\psi^I-\psi)ds\,dt\\
&\quad+\int_0^T \sum_{E\subset\Gamma}\int_E (A\nabla\hat{Y}_h\cdot n_E-BU_h-\mathtt{x}_b)(\psi^I-\psi)ds\,dt\\
&\quad+\int_0^T a(Y_h^n-Y_h,\psi)dt+\int_0^T (f-f^n,\psi)dt\\
&\quad+\int_0^T \big(\varphi'(\hat{Y}_h)(\hat{Y}_h-Y_h),\psi\big)dt+(y_0-y_{h,0},\psi(0)),
\end{aligned}$$

where ψ^I is the interpolation of ψ. By continuity of the bilinear form, the Young's inequality, Lemmas 5.2.1 and 5.2.3, and arguing as (6.47), we find that

$$\begin{aligned}
&\int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt \\
&\leq c(\delta) C^2_{6,8} \Big\{ \int_0^T \sum_{K\in\mathscr{T}_h^n} \int_K h_K^4 \Big(f^n - \frac{\partial Y_h}{\partial t} + div(A\nabla \hat{Y}_h) - a_0 \hat{Y}_h - \varphi(\hat{Y}_h) \Big)^2 dx\, dt \\
&\quad + \int_0^T \sum_{E\cap\Gamma=\emptyset} \int_E h_E^3 ([A\nabla\hat{Y}_h \cdot n_E])^2 ds\, dt + \int_0^T \sum_{E\subset\Gamma} \int_E h_E^3 (A\nabla \hat{Y}_h \cdot n_E - BU_h - \mathrm{x}_b)^2 ds\, dt \Big\} \\
&\quad + c(\delta)(\alpha_0^2 + \tilde{C}_M^2) \int_0^T \sum_{K\in\mathscr{T}_h^n} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt + c(\delta) \int_0^T \|f - f^n\|^2_{L^2(\Omega)} dt \\
&\quad + \frac{1}{2} \|y_0 - y_{h,0}\|^2_{L^2(\Omega)} dt + \frac{1}{2} \|\psi(0)\|^2_{L^2(\Omega)} dt + \delta \int_0^T \|\psi\|^2_{H^2(\Omega)}.
\end{aligned}$$

Invoking stability results from Lemma 6.2.1 with $\Phi = \psi$, $F = y(U_h) - Y_h$ and setting $C_{6,12} = 4\max\{c(\delta)C^2_{6,8}, c(\delta)(\alpha_0^2 + \tilde{C}_M^2), c(\delta), 1\}$ with $\delta = 1/4$, the standard kick-back argument leads to

$$\begin{aligned}
\int_0^T \|y(U_h) - Y_h\|^2_{L^2(\Omega)} dt \;\leq\; & C_{6,12} \Big\{ \mathscr{E}^2_{6,8} + \int_0^T \sum_{K\in\mathscr{T}_h^n} \|\hat{Y}_h - Y_h\|^2_{H^1(K)} dt \\
& + \int_0^T \|f - f^n\|^2_{L^2(\Omega)} dt + \frac{1}{2} \|y_0 - y_{h,0}\|^2_{L^2(\Omega)} dt \Big\}. \qquad (6.50)
\end{aligned}$$

Putting (6.49) and (6.50) together, and then adjusting the constants the desire result follows. This completes the proof. ☐

We are now in a position to establish the main result by collecting the Lemmas 6.3.1 – 6.3.3.

Theorem 6.3.1. *Let* (y, p, u) *be the solutions of* (6.5) – (6.8), *and let* (Y_h, P_h, U_h) *be the solution of* (6.17) – (6.21). *Further, we assume that Lemma 6.3.1 holds. Then there exists a positive constant* $C_{6,13}$ *such that*

$$\|u - U_h\|^2_{L^2(0,T;\mathscr{B})} + \|y - Y_h\|^2_{L^2(0,T;\mathscr{B})} + \|p - P_h\|^2_{L^2(0,T;\mathscr{B})} \;\leq\; C_{6,13} \Big(\sum_{i=1}^{10} \mathscr{E}^2_{6,i} \Big),$$

where $\mathscr{E}^2_{6,1}$, $\mathscr{E}^2_{6,i}$, $i = 2, \ldots, 6$ *and* $\mathscr{E}^2_{6,i}$, $i = 7, \ldots, 10$ *are defined in Lemmas 6.3.1, 6.3.2 and 6.3.3, respectively.*

Proof. An use of stability results in [95] leads to

$$\|y - y(U_h)\|^2_{L^2(0,T;\mathscr{B})} \;\leq\; \|y - y(U_h)\|^2_{L^2(0,T;H^1(\Omega))} \;\leq\; C^2_{6,14} \|u - U_h\|^2_{L^2(0,T;\mathscr{B})},$$

and

$$\begin{aligned}\|p-p(U_h)\|^2_{L^2(0,T;\mathscr{B})} &\leq \|p-p(U_h)\|^2_{L^2(0,T;H^1(\Omega))}\\ &\leq \frac{C^2_{6,15}}{\alpha_1}\|y-y(U_h)\|^2_{L^2(0,T;\mathscr{B})} \leq C^2_{6,17}\,\|u-U_h\|^2_{L^2(0,T;\mathscr{B})},\end{aligned}$$

where $C^2_{6,17}=\frac{C^2_{6,14}C^2_{6,15}}{\alpha_1}$, and $C_{6,14}$, $C_{6,15}$ are positive constants. Thus, we have

$$\begin{aligned}\|y-Y_h\|^2_{L^2(0,T;\mathscr{B})} &\leq \|y-y(U_h)\|^2_{L^2(0,T;\mathscr{B})}+\|y(U_h)-Y_h\|^2_{L^2(0,T;\mathscr{B})}\\ &\leq C^2_{6,14}\,\|u-U_h\|^2_{L^2(0,T;\mathscr{B})}+\|y(U_h)-Y_h\|^2_{L^2(0,T;\mathscr{B})}, \qquad (6.51)\end{aligned}$$

and

$$\begin{aligned}\|p-P_h\|^2_{L^2(0,T;\mathscr{B})} &\leq \|p-p(U_h)\|^2_{L^2(0,T;\mathscr{B})}+\|p(U_h)-P_h\|^2_{L^2(0,T;\mathscr{B})}\\ &\leq C^2_{6,17}\|u-U_h\|^2_{L^2(0,T;\mathscr{B})}+\|p(U_h)-P_h\|^2_{L^2(0,T;\mathscr{B})}. \qquad (6.52)\end{aligned}$$

Further,

$$\|p(U_h)-\tilde{P}_h\|^2_{L^2(0,T;\mathscr{B})} \leq \|p(U_h)-P_h\|^2_{L^2(0,T;\mathscr{B})}+\|\tilde{P}_h-P_h\|^2_{L^2(0,T;H^1(\Omega))}. \qquad (6.53)$$

Altogether (6.51) – (6.53) and Lemmas 6.3.1 – 6.3.3 complete the rest of the proof. □

Remark 6.3.1. *For the sake of comparison, we refer to [62]. The derived estimators in Theorem 6.3.1 are different from the estimators of [62, Theorem 3.4]. Our estimators are of local character, that is, the leading terms of* $\mathscr{E}^2_{6,i}$, $i=1,2,3,6,7$ *depend only on the data in a smaller neighborhood of the boundary. But, the global influences are reflected by the higher order terms* $\mathscr{E}^2_{6,4}$ *and* $\mathscr{E}^2_{6,8}$ *which can not be dislodged. By making* Ω_l *smaller (i.e., reducing the distance* l*), the resulting mesh refinement could be more concentrated around the boundary.*

The a posteriori error estimate in Theorem 6.3.1 consists of three parts. The estimators $\mathscr{E}^2_{6,i}$, $i=2,4,7,8$ *are contributed from the approximation error of the state and co-state equations, whereas* $\mathscr{E}^2_{6,1}$ *reflects the approximation error for the control variable which is derived from the variational inequality. Further,* $\mathscr{E}^2_{6,3}$ *and* $\mathscr{E}^2_{6,6}$ *denote the temporal error estimators for the state and the co-state variables, respectively. Moreover, the data approximation errors are given by* $\mathscr{E}^2_{6,5}$, $\mathscr{E}^2_{6,9}$ *and* $\mathscr{E}^2_{6,10}$. *The positive constants* $C_{6,3}$, $C_{6,10}$ *and* $C_{6,13}$ *depend on the Lipschitz-constants* $\tilde{C}_L$, *distance* l, *and the bound* $\tilde{C}_M$ *of the nonlinear function and it's derivatives, the bound* C_M *of the cut-off function and the constants* $\tilde{C}_{I,j}$, $j,\ldots,5$ *defined in Chapter 5.*

6.4 Numerical assessments

This section carry out numerical experiment to implement the *a posteriori* error estimators derived in the previous section. All computation is carried out using the software *FreeFem++* [38]. Here, all constants appearing in the estimators are taken to be one.

Example 6.1. *Here we consider the optimal control problem* (6.1) − (6.3) *with the following data:*

$$\Omega = [0,1]\times[0,1], \quad T = 1.0 \quad \textit{and} \quad \alpha = 1,$$

$$y_{ds} = \exp(2x_1 t)\sin(2\pi x_1) \quad \textit{on} \quad \Gamma_T,$$

$$f = 0, \quad \mathrm{x}_b = 0 \quad \textit{and} \quad y(x,0) = 0,$$

$$A = B = I \quad u_a = -0.50 \quad \textit{and} \quad u_b = 0.50,$$

with $\varphi(y) = y^3$.

For our computation, we have used the piecewise linear and continuous finite element approximation for the state and co-state variables and piecewise constant functions are used for the control variable. The time derivative is approximated by the backward Euler scheme. This experiment shows the reliability of the *a posteriori* error indicators. The adaptive meshes generated via the error estimators $\mathscr{E}_{6,i}$, $i = 1, \ldots, 10$ enable us to save convincing computational work in comparison with the uniform mesh. Table 6.1 contains the mesh information of $\mathscr{T}_h^n$, error in the $L^2(0,T;\mathscr{B})$-norm for the state, co-state and the control variables at the final time $T = 1.0$. Note that, the estimators are reducing with the increase of degrees of freedom. Compared to the uniform mesh, the adaptive meshes require much less number of nodes which shows the effectiveness of our estimators. Let $\mathscr{E}_{6,i}^L$, $i = 1, 2, 3, 6, 7$ denote the leading terms of the estimators $\mathscr{E}_{6,i}$, $i = 1, 2, 3, 6, 7$, respectively. Further, we measure the behaviour of the leading terms $\mathscr{E}_{6,i}^L$, $i = 1, 2, 3, 6, 7$, respectively, on the set $\mathscr{T}_{\Omega_l}^n$ with $l = 0.3, 0.2, 0.1$ which is prescribed in Tables 6.2 − 6.4. Results of the Table shows that the values of the leading terms are decreasing when the distance l is reducing. That is, numerical results of Tables 6.2 − 6.4 show the effectiveness of the leading terms of the estimators.

6.5 Concluding remarks

In this chapter, we have derived local *a posteriori* error estimates for the finite element approximation to the boundary control problem (6.1) − (6.3). The main difficulty

Table 6.1: *Comparison of the results on uniform mesh* $\mathscr{T}_h^n$ *and adaptive mesh with* $l = 0.1$ *and tolerance* $\epsilon = 10^{-5}$.

			Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}_h^n$	$y-$	Elements	31226	2654
		Nodes	15844	1403
		Bdry Elements	460	150
	$p-$	Elements	31226	4638
		Nodes	15844	2423
		Bdry Elements	460	206
	$u-$	Elements	31226	8963
		Nodes	15844	4620
		Bdry Elements	460	275
Errors		$\|y - Y_h\|_{L^2(0,T;\mathscr{B})}$	1.517×10^{-3}	9.183×10^{-5}
		$\|p - P_h\|_{L^2(0,T;\mathscr{B})}$	4.559×10^{-3}	7.522×10^{-5}
		$\|u - U_h\|_{L^2(0,T;\mathscr{B})}$	2.796×10^{-2}	5.797×10^{-3}

Table 6.2: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}_{\Omega_l}^n$ *with* $l = 0.3$.

	$l = 0.3$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}_{\Omega_l}^n$	$y-$ Elements	21876	2569
	$y-$ Nodes	11238	1384
	$p-$ Elements	21876	4066
	$p-$ Nodes	11238	2165
	$u-$ Elements	21876	6098
	$u-$ Nodes	11238	3160
Values of the leading terms on $\mathscr{T}_{\Omega_l}^n$	$\mathscr{E}_{6,1}^L$	1.581×10^{-2}	8.579×10^{-3}
	$\mathscr{E}_{6,2}^L$	2.837×10^{-3}	1.844×10^{-4}
	$\mathscr{E}_{6,3}^L$	2.398×10^{-3}	4.669×10^{-4}
	$\mathscr{E}_{6,6}^L$	1.314×10^{-4}	2.999×10^{-5}
	$\mathscr{E}_{6,7}^L$	3.745×10^{-4}	5.841×10^{-6}

Table 6.3: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.2$.

	$l = 0.2$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	14656	2014
	$y-$ Nodes	7658	1125
	$p-$ Elements	14656	3671
	$p-$ Nodes	7658	1988
	$u-$ Elements	14656	4110
	$u-$ Nodes	7658	2214
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\mathscr{E}^L_{6,1}$	5.210×10^{-3}	6.517×10^{-4}
	$\mathscr{E}^L_{6,2}$	9.925×10^{-4}	3.235×10^{-7}
	$\mathscr{E}^L_{6,3}$	1.645×10^{-4}	6.176×10^{-7}
	$\mathscr{E}^L_{6,6}$	1.121×10^{-4}	5.888×10^{-6}
	$\mathscr{E}^L_{6,7}$	2.380×10^{-4}	3.133×10^{-7}

Table 6.4: *Mesh information and values of leading terms of the estimators on* $\mathscr{T}^n_{\Omega_l}$ *with* $l = 0.1$.

	$l = 0.1$	Uniform mesh	Adaptive mesh
Mesh Information of $\mathscr{T}^n_{\Omega_l}$	$y-$ Elements	9948	1132
	$y-$ Nodes	5414	625
	$p-$ Elements	9948	1341
	$p-$ Nodes	5414	807
	$u-$ Elements	9948	3080
	$u-$ Nodes	5414	1739
Values of the leading terms on $\mathscr{T}^n_{\Omega_l}$	$\mathscr{E}^L_{6,1}$	1.132×10^{-4}	7.177×10^{-5}
	$\mathscr{E}^L_{6,2}$	6.600×10^{-5}	6.809×10^{-9}
	$\mathscr{E}^L_{6,3}$	1.995×10^{-5}	5.942×10^{-7}
	$\mathscr{E}^L_{6,6}$	1.051×10^{-5}	3.369×10^{-8}
	$\mathscr{E}^L_{6,7}$	8.371×10^{-6}	2.472×10^{-8}

associated with this problem is that the cost functional $\mathcal{J}$ is not differentiable in $\mathscr{B}$ and hence it fails to satisfy the first-order optimality condition. This difficulty is overcome by observing the fact that $\mathcal{J}$ is differentiable in $L^\infty(\Gamma)$, which is necessary for the existence of control variable. Further, assuming the second-order optimality condition, we have derived the *a posteriori* error estimates for the control variable. Our derived estimators (see, Theorems 6.3.1) are of local in character, i.e., the leading terms of the estimators depend only on the small neighborhood of the boundary. They also serve as important ingredients for adaptive algorithm. We have noticed that the refinement meshes are effectively reduced for the state and co-state variables.

7

SUMMARY AND FUTURE EXTENSIONS

In this chapter, we summarize the results highlighting the contributions made by this thesis and the techniques involved in deriving these *a posteriori* error bounds. It also contains some information on the scope for the possible extensions and future works.

7.1 Summary of the results

This thesis studied *a posteriori* error analysis of finite element approximations for POCPs with distributed and boundary controls. The main emphasis is on deriving reliable type *a posteriori* error estimates in suitable Sobolev norm. The *a posteriori* error bounds for the state, co-state and the control variables are derived for POCPs and PBCPs. Moreover, the theoretical analysis is supported by the computational results.

In Chapter 2, we have considered the finite element approximations of POCP (2.1) – (2.3) with distributed control. The spatial discretization uses the piecewise linear and continuous finite elements for approximations of the state and co-state variables, and the piecewise constant functions are employed for the control variable. The time discretization is based on the backward Euler method. We have established *a posteriori* error estimates for the state, co-state and control variables in the $L^\infty(0,T;L^2(\Omega))$-norm (Theorem 2.3.2) by using elliptic reconstruction technique and energy arguments. The basic idea is to split the main errors for the state and co-state variables into two parts namely, elliptic reconstruction error and the parabolic error. While Lemma 2.3.1 contains elliptic reconstruction error estimates, the parabolic errors for the state and co-state variables are established in Lemma 2.3.2 and Lemma 2.3.3, respectively. These lemmas are then utilized to prove intermediate *a posteriori* error bounds for the state and co-state variables (Theorem 2.3.1), which are crucial for obtaining the main results. With the help of first-order optimality condition we prove error for the control variable (Lemma 2.3.5). The key technical tools used in this chapter include elliptic reconstruction, approximation properties, first-order optimality condition and energy arguments. Numerical results are provided to illustrate the theoretical analysis.

Chapter 3 is devoted to $L^\infty(0,T;L^\infty(\Omega))-$type *a posteriori* error analysis for finite element approximations of POCP $(3.1)-(3.3)$ with distributed control. In this chapter, we have considered both semidiscrete and fully discrete finite element approximations of the control problem. The variational discretization is used to approximate the state and co-state variables with the piecewise linear and continuous functions, while the control variable is computed by using the implicit relation between the control and co-state variables. The key feature of this approach is not to discretize the control variable but to implicitly utilize the optimality conditions for the discretization of the control variable. The time derivative is approximated by the backward Euler method. We have used the elliptic reconstruction technique introduced earlier by Demlow *et al.* [21] in conjunction with heat kernel estimate (3.17) as main technical ingredients. We derive reliable type *a posteriori* error estimates for the state, co-state and control variables in the $L^\infty(0,T;L^\infty(\Omega))$-norm for both the semidiscrete approximations (Theorems $3.2.1-3.2.2$) and fully discrete approximations (Theorems $3.3.1-3.3.2$). Numerical experiments show the effectiveness of the derived estimators.

Chapter 4 deals with *a posteriori* error analysis of fully discrete finite element approximation to POCP $(4.1)-(4.3)$ with controls acting on lower dimensional manifolds. The manifold is assumed to be a point, a curve or a surface which is lying completely in the space domain, and it is either time independent or evolved with the time horizon. The space discretization consists of piecewise linear and continuous finite elements for the state and co-state variables, while the piecewise constant functions are employed to approximate the control variable. The discrete-in-time scheme is based on the backward Euler implicit scheme. We have derived different types of *a posteriori* error estimates for the state variable in the $L^2(0,T;L^2(\Omega))$-norm (Theorems $4.3.1-4.3.3$) whereas errors for the control variable are demonstrated in the $L^2(0,T;\mathbb{R}^m)$-norm or $L^2(0,T;L^2(\gamma(t)))$-norm according to the dimension of the manifolds (Theorems $4.3.1-4.3.3$). The essential technical tools involve interpolation approximation properties, trace inequality, inverse estimate, duality argument and the first-order optimality condition. Numerical results validate the theoretical analysis.

Chapter 5 discussed local *a posteriori* error analysis for finite element approximations of PBCP $(5.1)-(5.3)$ in three different observations: Observations of the boundary state, the distributed state and the final state. The piecewise linear and continuous finite elements are used for the approximation of the state and co-state variables, while the piecewise constant function spaces are employed for the control variable. Moreover, we have used the backward Euler method to approximate the time derivative. The development of intermediate *a posteriori* error estimates for the state and the co-state variables rely on the use of cutoff function, the inequality (5.38), approximation properties and duality argument. The *a posteriori* error bound for the control variable is established by using first-order optimality condition along with the convexity condition (5.22) (Lemmas 5.3.1, 5.3.4 and 5.3.7). More precisely, *a posteriori* error bounds for the state, co-state and control variables are derived in the $L^2(0,T;\mathscr{B})$-norm for the observation of the boundary state as well as for the final state (see, Theorem 5.3.1 and Theorem 5.3.3). In addition, for the observation of the distributed state, *a posteriori* error bound for the state variable is established in the $L^2(0,T;L^2(\Omega))$-norm, whereas error for the co-state and control variables in the $L^2(0,T;\mathscr{B})$-norm (Theorem 5.3.2). Further, the derived *a posteriori estimators* are of local in character means that the leading terms of the estimators depend on the small neighborhood of the boundary. These new local *a posteriori* error bounds can be used to study the behavior of the state and co-state variables near the boundary. The derived error indicators can be used as a guide to show how the refinement might be accomplished most efficiently in the adaptive algorithm. Computational results are presented to validate the theoretical analysis.

Finally, Chapter 6 considered a nonlinear PBCP (6.1) – (6.3) and derived reliable type local *a posteriori* error estimates for the fully discrete finite element approximations. For the error analysis, we have employed the standard space-time discretization as in Chapter 5. Using the second-order optimality condition, we have obtained the control error in the $L^2(0,T;\mathscr{B})$-norm. Moreover, the cutoff function, duality technique, and trace result are the key ingredients of the error analysis. The *a posteriori* error estimates for the state, co-state and the control variables are derived in the $L^2(0,T;\mathscr{B})$-norm (Theorem 6.3.1). The numerical results are provided to support the theoretical analysis.

For the computational aspects, we have used the software *FreeFem++* [38] which illustrate the theoretical analysis of the Chapters $2-6$. Further, the projection gradient algorithm is used for solving the optimization problems. Numerical experiments reveal the effectiveness of derived estimators by saving considerable computational efforts.

7.2 Extensions and remarks

This section makes some informal observations of the possible extensions of our results to new research directions. Many applications of control problems are naturally arising in engineering and sciences. The convergence analysis of these problems is one of the greatest challenges in computational mathematics today and has attracted a lot of attention over the past years. To the best of the author's knowledge, the results presented in this thesis are reported for the first time in the context of POCPs and PBCPs. Here, we shall briefly outline some interesting problems to be persuaded in the future.

1. *A posteriori error analysis of POCP in a non-convex domain.* Consider the following optimal control problem:

$$\min_{u\in U_{ad}} \mathcal{J}(u,y) := \frac{1}{2}\int_0^T \{\|y-y_{ds}\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega)}\}dt \tag{7.1}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f + u \quad \text{in } \Omega_T, \\ y(\cdot,0) = y_0(x) \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \Gamma_T, \end{cases} \tag{7.2}$$

and the control constraints

$$u_a \le u(x,t) \le u_b \ \ a.e. \ \text{in } \Omega_T, \tag{7.3}$$

where $u_a, u_b \in \mathbb{R}$ fulfills $u_a < u_b$, and the regularization parameter $\alpha > 0$ is a fixed constant. Further, $y = y(x,t)$, $y_{ds} = y_{ds}(x,t)$ and $u = u(x,t)$ denote the state, desired state and control variables, respectively. Assume that Ω is a non-convex polygonal domain in $\mathbb{R}^2$. For non-convex domain, the solutions of the state and co-state variables possess low regularity, i.e., y, $p \in L^2(0,T;H^{1+s}(\Omega))$, $s \in (1/2,1)$ (cf. [4]). It is therefore interesting to investigate *a posterirori* error analysis for the problem (7.1) – (7.3). We strongly believe the techniques used in Chapters 2 and 3 will be useful to study these problems.

2. The *a posteriori* error analysis developed in Chapters 5 and 6 can be extended to deal with boundary control problems on a non-convex domains.

3. *Optimal control problems governed by time-fractional diffusion equation*: Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2,3)$ with sufficiently smooth boundary $\Gamma := \partial\Omega$. Further, let $\Omega_T = \Omega \times (0,T]$ and $\Gamma_T = \partial\Omega \times (0,T]$. We consider the following time-fractional optimal control problem:

$$\min_{u\in U_{ad}} \mathcal{J}(u,y) := \frac{1}{2}\int_0^T \left\{\|y-y_{ds}\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega)}\right\}dt \tag{7.4}$$

subject to

$$\begin{cases} {}_0D_t^\sigma y - \Delta y = f + u \quad \text{in } \Omega_T, \\ y(\cdot,0) = y_0(x) \ \text{ in } \quad \Omega, \\ y = 0 \quad \text{on} \quad \Gamma_T, \end{cases} \tag{7.5}$$

and

$$U_{ad} = \left\{ \phi \in L^1(0,T;L^1(\Omega)) : \quad \int_0^T \int_\Omega \phi \, dx \, dt \leq \epsilon \right\}, \tag{7.6}$$

where $0 < \sigma < 1$ and a real number $\epsilon > 0$. Further, the left and right Riemann-Liouville fractional derivatives are defined as follows:

$$ {}_0D_t^\sigma w = \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{w(\tau)}{(t-\tau)^\sigma} d\tau, $$

and

$$ {}_tD_T^\sigma w = \frac{-1}{\Gamma(1-\alpha)} \int_t^T \frac{w(\tau)}{(\tau-t)^\sigma} d\tau. $$

The *a posteriori* error analysis for the control problem (7.4) – (7.6) is not straightforward due to the presence of time-fractional derivative. It would be challenging to investigate such problems in future.

4. *A posteriori error analysis for optimal control problems governed by hyperbolic PDEs*: Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded convex domain with boundary $\Gamma := \partial\Omega$, and set $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \partial\Omega \times (0,T]$. We consider the following control problem

$$\min_{u \in U_{ad}} \mathcal{J}(u,y) := \frac{1}{2} \int_0^T \left\{ \|y - y_{ds}\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{L^2(\Omega)} \right\} dt \tag{7.7}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = f + u \quad \text{in } \Omega_T, \\ y(\cdot,0) = y_0(x) \quad \text{in } \Omega, \\ \dfrac{\partial y}{\partial t}(\cdot,0) = y_1(x) \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \Gamma_T, \end{cases} \tag{7.8}$$

and the control constraints

$$u_a \leq u(x,t) \leq u_b \quad a.e. \quad \text{in} \quad \Omega_T, \tag{7.9}$$

where $u_a,\ u_b \in \mathbb{R}$ fulfills $u_a < u_b$. Assume that the given function $y_0,\ y_1 \in H^1(\Omega)$ and $f,\ y_{ds} \in L^2(0,T;L^2(\Omega))$. The constant $\alpha > 0$ stands for the regularization parameter. The set of admissible controls is given by

$$U_{ad} = \big\{u \in L^2(0,T;L^2(\Omega)) : \quad u_a \leq u \leq u_b \ \ a.e. \ \ (x,t) \in \Omega_T\big\}.$$

The analysis developed in Chapters 2 and 3 can be extended to treat the hyperbolic optimal control problem (7.7) – (7.9). We wish to investigate both theory and numerics of *a posteriori* error analysis for the above optimization problem in future.

5. *A posteriori error analysis for Dirichlet boundary control problems governed by parabolic partial differential equations:* We consider the following problem on a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz-continuous boundary $\Gamma := \partial\Omega$:

$$\min_{u \in U_{ad}} \mathcal{J}(u,y) := \frac{1}{2}\int_0^T \{\|y - y_{ds}\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{\mathscr{B}}\}dt \tag{7.10}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = f & \text{in} \ \ \Omega_T, \\ y(\cdot,0) = y_0(x) & \text{in} \ \ \Omega, \\ y = u & \text{on} \ \ \Gamma_T, \end{cases} \tag{7.11}$$

and the control constraints

$$u_a \leq u(x,t) \leq u_b \quad a.e. \quad \text{in} \ \Gamma_T, \tag{7.12}$$

where $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \partial\Omega \times (0,T]$, $\mathscr{B} = L^2(\Gamma)$ and $u_a, u_b \in \mathbb{R}$ fulfills $u_a < u_b$. Assume that the desired state $y_{ds} \in L^2(0,T;L^2(\Omega))$ and the source function $f \in L^2(0,T;L^2(\Omega))$. Further, $y = y(x,t)$ and $u = u(x,t)$ denote the state and control variables, respectively. The analysis of Chapters 5 and 6 can be applied to study the *a posteriori* error analysis for the finite element approximations to the Dirichlet boundary control problem (7.10) – (7.12), which is an interesting future project.

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List of Research Papers

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