Nearly Invariant Subspaces with Finite Defect in Vector Valued Hardy Spaces and its Applications

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May, 2023

Nearly Invariant Subspaces with Finite Defect in Vector Valued Hardy Spaces and its Applications

A thesis submitted
in partial fulfilment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

by

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to the

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May, 2023

Declaration

I hereby declare that the work contained in the thesis entitled **"Nearly Invariant Subspaces with Finite Defect in Vector Valued Hardy Spaces and its Applications"** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, under the supervision of **Dr. Arup Chattopadhyay**, Indian Institute of Technology Guwahati, for the award of **Doctor of Philosophy** and this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in the thesis titled **"Nearly Invariant Subspaces with Finite Defect in Vector Valued Hardy Spaces and its Applications"** by **Soma Das (186123016)**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of **Doctor of Philosophy** has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

Guwahati
May, 2023

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Dedicated

To

My

Parents:
Jyotsna Das and Tapan Kumar Das

Acknowledgement

I would like to express my sincere gratitude to my thesis supervisor Professor Arup Chattopadhyay, whose constant guidance and motivation have greatly inspired my research work. I am really grateful to him for sharing some of his expertise with me by carefully correcting my mathematical errors. I am blessed to have him as my Ph.D. supervisor and many thanks to him for keeping faith in me. Also, I would like to thank all his family members for providing me with a nice family-like environment throughout my Ph.D. life at IITG.

I would especially like to thank Professor Tirthanakar Bhattacharyya for giving his valuable time through many fruitful discussions. It is my privilege to get his kind support for my research work.

I am extremely grateful to Professor Jaydeb Sarkar for introducing the research area concerning "nearly S^*-invariant subspaces"to us.

My sincere thanks to Professor Bata Krishna Das for his valuable comments and many helpful suggestions.

I would like to thank Professor Haripada Sau (Hari Da) for always being so helpful and inspiring us in doing good research from our initial stage.

I am also thankful to Dr. Srijan Sarkar (Srijan Da), for being an amazing research collaborator and also an elder brother to me. I have learned various new things from him through our many helpful discussions.

I am truly grateful to all my IITG Doctoral committee members Prof. M. Guru. Prem. Prasad (Chairperson), Prof. Jitendriya Swain, and Prof. Anjan K. Chakrabarty for all their helpful suggestions and valuable comments regarding my research work. My special thanks to Prof. Subhamay Saha and Prof. Pratyoosh Kumar for always being so nice to us and providing a friendly atmosphere in the department at IITG.

I would like to declare my gratitude to the Department of Mathematics, Indian Institute of Technology Guwahati for their financial support, which help me to complete my research work presented in this thesis. I am, especially thankful to all the office staff of our department for all their technical support and official help.

I always feel blessed to have some wonderful teachers as well as some kind people whose contributions are endless. I would like to gratefully acknowledge Prof. Srimanta Sinharoy and

Prof. Ranen Bhattacharyya for instilling my love for mathematics and always believing in me. I am grateful to all my teachers in the Department of Mathematics, at Serampore College especially Prof. Madhabendu Majumdar, Prof. Supriya Pal, Prof. Manas Kumar Ghosh, Prof. Ranjan Das, and Prof Ranita Roy for teaching me how to do flawless mathematics and giving inspiration for continuing with mathematics.

I am overwhelmed and grateful to my dear friends Priya, Suparna, and Debarati for being so supportive and always being there. Also thanks to my IITG friends Riya, and Menaxi and my two special brother cum friends Rupchand and Saikat for all their necessary help and for providing a happy time to rest my mind outside of my research.

My heartiest thanks to Chandan, for being a true friend, a well-wisher, a benefactor, and also an honest person towards our work in this PhD-journey. I am truly grateful to him for being there during my difficult times and always motivating me to dedicate myself to research.

Above all, I want to convey my deepest thanks to my Father Tapan Kumar Das and my mother Jyotsna Das, my sister and her husband (Sangita Das Kar and Ayan Kar), my brother Tushar Das, and my Masi Sandhya Mandal for their enormous support and encouragement every time. Especially I would like to thank and give major credit to my mother for always being my mental strength and my the strongest supporter without whom I could never start my journey of education. Lastly, I would like to thank our youngest family member of affection, Ayantika Kar (Gublai) for being my stress reliever and giving me happy moments.

Soma Das

Abstract

In this dissertation, we characterize nearly invariant subspaces of finite defect for the backward shift operator acting on the vector valued Hardy space. Using this characterization we completely describe the almost invariant subspaces for the shift and its adjoint acting on the vector valued Hardy space. Moreover, as an application, we also identify the kernel of perturbed Toeplitz operator in terms of backward shift-invariant subspaces in various important cases using our characterization in connection with nearly invariant subspaces of finite defect for the backward shift operator acting on the vector valued Hardy space.

Going further, in this report, we briefly describe nearly T^{-1} invariant subspaces with finite defect for a shift operator T of finite multiplicity acting on a separable Hilbert space $\mathcal{H}$ in terms of backward shift invariant subspaces of finite defect in vector valued Hardy spaces. We also provide the representation of nearly T_B^{-1} invariant subspaces with finite defect in a scale of Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in [-1, 1]$ for a finite Blaschke product B.

Finally, this dissertation deals with the study of *Schmidt* subspaces in vector valued Hardy spaces. More precisely, *Schmidt* subspaces for a bounded Hankel operator are in correspondence with weighted model spaces, and they are closely related to nearly S^*-invariant subspaces. In this direction, we prove that these subspaces in vector valued Hardy spaces are nearly S^*-invariant with finite defect in general. Furthermore, we also describe the structure of such subspaces using our characterization of nearly invariant subspaces of finite defect in vector valued Hardy space providing a short proof compared to scalar valued case. At the end,

we calculate the precise action of the associated Hankel operator on some particular *Schmidt* subspaces.

Introduction

In 1988, Hitt [34] first introduces the notion of nearly invariant subspaces under the backward shift operator acting on the scalar valued Hardy space to classify the simply shift-invariant subspaces of the Hardy space of an annulus. Later Sarason [49] further investigated these spaces and modified Hitt's algorithm for scalar valued Hardy space to study the kernels of Toeplitz operators. In 2010, Chalendar-Chevrot-Partington (C-C-P) [9] gives a complete characterization of nearly invariant subspaces under the backward shift operator acting on the vector valued Hardy space. Recently in 2020, Chalendar-Gallardo-Partington (C-G-P) [11] introduce the notion of nearly invariant subspace of finite defect for the backward shift operator acting on the scalar valued Hardy space as a generalization of nearly invariant subspaces and provides a complete characterization of these spaces in terms of backward shift invariant subspaces. In this dissertation, we completely characterize nearly invariant subspaces of finite defect under the backward shift operator acting on the vector valued Hardy space providing a vectorial generalization of C-G-P algorithm. Furthermore, using the characterization, we completely describe the almost invariant subspaces for the shift and its adjoint acting on the vector valued Hardy space. Chapter 2 is devoted for such characterization results.

The kernel of a Toeplitz operator is nearly invariant under the backward shift operator acting on the scalar valued Hardy space. In this context, Liang and Partington [38] recently provide a connection between kernels of finite-rank perturbations of Toeplitz operators and nearly invariant subspaces with finite defect under the backward shift operator acting on the scalar valued Hardy space for several important cases by applying the recent theorem of Chalendar–Gallardo–Partington (C-G-P)[11]. In Chapter 3, we study the kernels of finite-rank perturbations of Toeplitz operators and its connection with nearly invariant subspaces with finite defect under the backward shift operator acting on the vector valued Hardy space. Moreover, we explicitly identify the kernel of perturbed Toeplitz operator in terms of backward shift-invariant subspaces by applying the characterization result obtained in Chapter 2 in various cases as mentioned by Liang and Partington in [38].

In 2021, Liang and Partington introduce the notion of nearly T^{-1} invariant subspaces in general Hilbert space setting [39] and provide a representation of such spaces for the shift operator T with finite multiplicity acting on a separable Hilbert space $\mathcal{H}$ in terms of backward shift invariant subspaces on the vector valued Hardy spaces [9]. They also give a description of the nearly T_B^{-1} invariant subspaces for the operator T_B of multiplication by finite Blaschke B in a scale of Dirichlet-type spaces [39]. In Chapter 4, we introduce the notion of nearly T^{-1} invariant subspaces with finite defect for an left invertible operator T acting on $\mathcal{H}$ as a generalization of nearly T^{-1} invariant subspaces. Moreover, we characterize nearly T^{-1} invariant subspaces with finite defect, where T is a shift operator having finite multiplicity, in terms of backward shift invariant subspaces in vector-valued Hardy spaces by using the characterization result obtained in Chapter 2. Furthermore, we also provide a concrete representation of the nearly T_B^{-1} invariant subspaces with finite defect in a scale of Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in [-1, 1]$ for any finite Blaschke product B.

Finally in Chapter 5, we extend the study of *Schmidt* subspaces associated with a scalar valued Hankel operator to a matrix-valued Hankel operator. In scalar case, Gérard and Pushnitski study these spaces in [30]. Later in [31], they established an excellent connection between these *Schmidt* subspaces with nearly S^*-invariant subspaces and using Hitt's [34] characterization of nearly S^*- invariant subspaces, authors gave an alternative proof (indeed, a short proof) of the main result of [30] concerning the characterization of such *Schmidt* subspaces of scalar valued Hankel operator. In this Chapter, we prove that any non-trivial *Schmidt* subspaces of a matrix-valued Hankel operator are nearly S^*-invariant with finite defect in general. As a consequence, we obtain again a short proof of the characterization results concerning the structure of *Schmidt* subspaces in scalar valued Hardy space, in an alternative way compared to [30, 31]. At the end, we describe the action of a specific class of Hankel operators on their

Schmidt subspaces.

Papers

The thesis is based on four papers

[1] Chapter 2 is based on [14] (jointly with Arup Chattopadhyay and Chandan Pradhan, and published in Integral Equations Operator Theory).

[2] Chapter 3 is based on [15] (jointly with Arup Chattopadhyay and Chandan Pradhan, and published in Advances in Operator Theory).

[3] Chapter 4 is based on [12] (jointly with Arup Chattopadhyay, and published in Proc. Indian Acad. Sci. Math. Sci.).

[4] Chapter 5 is based on [13] (jointly with Arup Chattopadhyay and Chandan Pradhan, and available in arXiv).

Abbreviation and Notation

$\mathbb{N}$	The set of all natural numbers.
$\mathbb{Z}$	The set of all integers.
$\mathbb{Z}_+$	The set of all non-negative integers.
$\mathbb{R}$	The set of all real numbers.
$\mathbb{C}$	The set of all complex numbers.
$\mathbb{C}^m$	The m dimensional standard complex Hilbert space.
$\mathcal{H}$	Separable infinite dimensional complex Hilbert space.
$\mathcal{L}(\mathcal{H})$	The set of all bounded linear operators on $\mathcal{H}$.
$\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m)$	The set of all bounded linear operators from $\mathbb{C}^r$ to $\mathbb{C}^m$.
$\ell^2(\mathbb{N})$	The space of all complex sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \lvert x_n\rvert^2 < \infty$.
$\ell^2(\mathbb{Z}_+)$	The space of all complex sequences $\{x_n\}_{n\in\mathbb{Z}_+}$ such that $\sum_{n=0}^{\infty} \lvert x_n\rvert^2 < \infty$.
$\ell^2(\mathbb{Z})$	The space of all two sided sequences $\{x_n\}_{n\in\mathbb{Z}}$ such that $\sum_{n=-\infty}^{\infty} \lvert x_n\rvert^2 < \infty$.
$\mathbb{D}$	The open unit disk in $\mathbb{C}$.
$\mathbb{T}$	The unit circle in $\mathbb{C}$.
$\mathbf{m}$	The normalized Lebesgue measure (arc-length measure) on $\mathbb{T}$.

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CHAPTER 1

Preliminaries

In this chapter, we will revisit some useful definitions and preliminary results in Hardy space theory. Throughout this thesis $\mathcal{H}$ will denote a separable infinite dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. We denote the unit circle by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and the open unit disk by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

1.1 L^p spaces

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a finite dimensional normed linear space. The $\mathcal{X}$-valued $L^p(\mathbb{T}, \mathcal{X})$- spaces are defined to be

$$L^p(\mathbb{T}, \mathcal{X}) := \{f : \mathbb{T} \to \mathcal{X} \text{ measurable} \mid \|f\|_p^p := \int_{\mathbb{T}} \|f\|_{\mathcal{X}}^p \, d\mathbf{m} < \infty\}, \tag{1.1}$$

where $d\mathbf{m}$ is the normalized arc-length measure on $\mathbb{T}$ (see, e.g., [35, Definition 1.2.15]). For an integer k and an $L^1(\mathbb{T}, \mathcal{X})$-function f, the k-th order Fourier coefficient $\hat{f}(k)$ of f is given by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ik\theta} \, d\theta,$$

where the above integration is interpreted by the Bochner integral sense. For more on the theory of Bochner integration, we refer to [35, Chapter 1]. Note that the space of all essentially bounded measurable functions on $\mathbb{T}$ is denoted by $L^\infty(\mathbb{T}, \mathcal{X})$. In short, we will denote by $L^p(\mathbb{T}) \equiv L^p(\mathbb{T}, \mathbb{C})$ for $1 \leq p \leq \infty$.

1.2 Hardy spaces

The Hardy space is a Banach space of all analytic functions on the disk $\mathbb{D}$ and then view them in another light as subspaces of functions defined on the unit circle $\mathbb{T}$ which we equip with normalized Lebesgue measure $\mathbf{m}$.

Definition 1.2.1. *For $1 \leq p < \infty$, the Hardy space H^p is defined as the space of all analytic functions f on $\mathbb{D}$ for which the norm*

$$||f|| = \sup_{0 \leq r < 1} \left(\int_0^{2\pi} |f(re^{i\theta}|^p d\theta \right)^{1/p} \tag{1.2}$$

is finite. The space H^∞ consists of all bounded analytic functions with the norm

$$||f||_\infty = \sup_{|z|<1} |f(z)|. \tag{1.3}$$

It is easy to observe that for $p \leq q$, $H^q \subseteq H^p$. Therefore we have $H^1 \supseteq H^2 \supseteq \ldots \supseteq H^\infty$.

Theorem 1.2.2. *For any function $f \in H^p$ with $1 \leq p \leq \infty$, the radial limit*

$$\tilde{f}(e^{i\theta}) = \lim_{r \to 1-} (re^{i\theta})$$

exists almost everywhere on $\mathbb{T}$, and $\tilde{f} \in L^p(\mathbb{T})$ with $||f||_{H^p} = ||\tilde{f}||_{L^p}$.

Therefore we can think H^p as a subspace of $L^p(\mathbb{T})$, identifying f with $\tilde{f}$ and we denote these closed subspace by $H^p(\mathbb{T})$. In other words, for $f \in H^p$ we have $\tilde{f} \in H^p(\mathbb{T})$ for $1 \leq p \leq \infty$. For more detail see [23, Theorem 4.1].

For $p = 2$, $H^2 \equiv H^2_{\mathbb{C}}(\mathbb{D})$ is a Hilbert space known as the Hardy Hilbert space which can also be defined as follows :

$$H^2_{\mathbb{C}}(\mathbb{D}) := \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty, a_n \in \mathbb{C}\}. \tag{1.4}$$

Here the inner product, $\langle f, g \rangle := \sum_{n=0}^{\infty} a_n \overline{b_n}$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two H^2 functions. H^2 has nice properties because of it's Hilbert space structure. From now onwards , we will deal with H^2 and by the Hardy space we will mean H^2 unless otherwise stated. Again, we can think H^2 as a closed subspace $H^2(\mathbb{T})$ of $L^2(\mathbb{T})$ which consists of functions whose negative Fourier coefficients are zero. Let P_{H^2} denote the orthogonal projection of $L^2(\mathbb{T})$ onto H^2 :

$$P_{H^2}: \quad \sum_{n=-\infty}^{\infty} a_n z^n \longrightarrow \sum_{n=0}^{\infty} a_n z^n.$$

Regarding H^2 as a closed subspace of $L^2(\mathbb{T})$, the space $L^2(\mathbb{T})$ can be decomposed in the following way $L^2(\mathbb{T}) = H^2 \oplus \overline{H}_0^2$, where $\overline{H}_0^2 = \{f \in L^2(\mathbb{T}) : \overline{f} \in H^2 \textit{ and } f(0) = 0\}$.

The $\mathbb{C}^m$- valued Hardy space [45] over the unit disc $\mathbb{D}$ is denoted by $H^2_{\mathbb{C}^m}(\mathbb{D})$ and defined by

$$H^2_{\mathbb{C}^m}(\mathbb{D}) := \{f(z) := \sum_{n=0}^{\infty} A_n z^n : \sum_{n=0}^{\infty} ||A_n||^2 < \infty, A_n \in \mathbb{C}^m\}. \tag{1.5}$$

We can also view the above Hilbert space as the direct sum of m-copies of $H^2_{\mathbb{C}}(\mathbb{D})$ or sometimes it is useful to see the above space as a tensor product of two Hilbert spaces $H^2_{\mathbb{C}}(\mathbb{D})$ and $\mathbb{C}^m$, that is,

$$H^2_{\mathbb{C}^m}(\mathbb{D}) \equiv \underbrace{H^2_{\mathbb{C}}(\mathbb{D}) \oplus \cdots \oplus H^2_{\mathbb{C}}(\mathbb{D})}_{m} \equiv H^2_{\mathbb{C}}(\mathbb{D}) \otimes \mathbb{C}^m.$$

Therefore if $F, G \in H^2_{\mathbb{C}^m}(\mathbb{D})$, then $F = (f_1, f_2, \ldots, f_m)$ and $G = (g_1, g_2, \ldots, g_m)$ so the inner product is defined as follows:

$$\langle F, G\rangle = \sum_{i=1}^{m} \langle f_i, g_i\rangle \text{ where each } f_i, g_i \in H^2_{\mathbb{C}}(\mathbb{D}).$$

The Banach space of all $\mathcal{L}(\mathbb{C}^r, \mathbb{C}^m)$-valued bounded analytic functions on $\mathbb{D}$ is denoted by $H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$. The elements of $H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ are called the *multipliers*.

In more general setting we can also define the Hardy spaces as follows:

Let $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ be a complex separable Hilbert space, then the $\mathcal{E}$-valued Hardy space over the unit disk $\mathbb{D}$ is denoted by $H^2_{\mathcal{E}}(\mathbb{D})$ and defined by

$$H^2_{\mathcal{E}}(\mathbb{D}) := \Big\{F(z) = \sum_{n\geq 0} h_n z^n : \|F\|^2 = \sum_{n\geq 0} \|h_n\|^2_{\mathcal{E}} < \infty, \ h_n \in \mathcal{E}, z \in \mathbb{D}\Big\}. \tag{1.6}$$

1.3 The Unilateral shift and factorization of functions

Now we will discuss about an important operator, that is, the unilateral shift, the study of the invariant subspaces of this operator in the Hardy space yields a nice factorization of functions in H^2.

1.3.1 The Shift operators

Definition 1.3.1. *On $\ell^2(\mathbb{Z}_+)$ we define the unilateral shift U by*

$$U(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots) \quad for \quad (a_0, a_1, a_2, \ldots) \in l^2(\mathbb{Z}_+).$$

Theorem 1.3.2. *(i) The unilateral shift is an isometry.*
(ii) The adjiont U^* of the unilateral shift has the following form :

$$U^*(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots).$$

There are also bilateral shifts, defined on the space of all two-sided square-summable sequences.

Definition 1.3.3. *The bilateral shift operator W on $\ell^2(\mathbb{Z})$ is defined by*

$$W(\ldots, a_{-2}, a_{-1}, \boldsymbol{a_0}, a_1, a_2, \ldots) = (\ldots, a_{-3}, a_{-2}, \boldsymbol{a_{-1}}, a_0, a_1, \ldots).$$

Theorem 1.3.4. *(i) The bilateral shift is a unitary operator.*
(ii) The adjiont W^* of the bilateral shift known as backward bilateral shift and is given by :

$$W^*(\ldots, a_{-2}, a_{-1}, \boldsymbol{a_0}, a_1, a_2, \ldots) = (\ldots, a_{-1}, a_0, \boldsymbol{a_1}, a_2, a_3, \ldots).$$

Definition 1.3.5. *The operator M_z ("multiplication by z") on H^2 is defined by*

$$(M_z f)(z) = zf(z).$$

So, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$(M_z f)(z) = \sum_{n=0}^{\infty} a_n z^{n+1}.$$

Therefore M_z acts like the unilateral shift. Usually, the shift operator on H^2 is denoted by S.

Theorem 1.3.6. *The operator M_z on H^2 is unitarily equivalent to the unilateral shift.*

Definition 1.3.7. *The operators $M_{e^{i\theta}}$ and $M_{e^{-i\theta}}$ are defined on $L^2(\mathbb{T})$ by*

$$(M_{e^{i\theta}} f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta}) \quad and \quad (M_{e^{-i\theta}} f)(e^{i\theta}) = e^{-i\theta} f(e^{i\theta}).$$

Theorem 1.3.8. *The operator $M_{e^{i\theta}}$ on $L^2(\mathbb{T})$ is unitarily equivalent to the bilateral shift W on $\ell^2(\mathbb{Z})$ and the operator $M_{e^{-i\theta}}$ is unitarily equivalent to W^*.*

The following theorem is important in the sequel.

Theorem 1.3.9. *The operator $M_{e^{i\theta}}$ leaves the subspace $H^2(\mathbb{T})$ of $L^2(\mathbb{T})$ invariant and the restriction of $M_{e^{i\theta}}$ to $H^2(\mathbb{T})$ is the unilateral shift on $H^2(\mathbb{T})$. On $\ell^2(\mathbb{Z})$ the operator W leaves the subspace $\ell^2(\mathbb{Z}_+)$, consisting of those sequences whose coordinates in negative positions are 0, invariant, and the restriction of W to $\ell^2(\mathbb{Z}_+)$ is the unilateral shift on $\ell^2(\mathbb{Z}_+)$.*

1.3.2 Invariant and Reducing subspaces

There are some obvious invariant subspaces of the unilateral shift. As for example the subspace in $\ell^2(\mathbb{Z}_+)$ consisting of those sequences whose first n coordinates are zero is invariant under U and the corresponding invariant subspace for M_z in H^2 is the subspace of H^2 consisting of the functions whose first n derivatives (including the 0^{th} derivative) vanish at the origin.
The unilateral shift has many invariant subspaces which are very difficult to describe in $\ell^2(\mathbb{Z}_+)$ but all invariant subspaces of the unilateral shift can be nicely described as subspaces of H^2.

Theorem 1.3.10. *The only reducing subspaces of the unilateral shift are $\{0\}$ and the entire space.*

We know that the commutant of a bounded linear operator T is the set of all bounded linear operators that commute with T. For any $\phi \in L^\infty$, the multiplication operator M_ϕ is defined by $M_\phi(f) = \phi f$ for every $f \in L^2(\mathbb{T})$.

Theorem 1.3.11. *The commutant of W (regarded as an operator on $L^2(\mathbb{T})$) is $\{M_\phi : \phi \in L^\infty(\mathbb{T})\}$.*

Now we can explicitly describe the reducing subspaces of the bilateral shift.

Corollary 1.3.1. *The reducing subspaces of the bilateral shift on $L^2(\mathbb{T})$ are the subspaces $\mathcal{M}_E = \{f \in L^2(\mathbb{T}) : f(e^{i\theta}) = 0 \text{ a.e. on } E\}$ for measurable subsets $E \subset \mathbb{T}$.*

A description of the nonreducing invariant subspaces of the bilateral shift can be given as follows:

Theorem 1.3.12. *The subspaces of $L^2(\mathbb{T})$ that are invariant but not reducing for the bilateral shift are of the form $\mathcal{M} = \phi H^2(\mathbb{T})$, where ϕ is a function in $L^\infty(\mathbb{T})$ such that $|\phi(e^{i\theta})| = 1$ a.e. and this ϕ is unique up to constant factor of unit modulus.*

1.3.3 Inner and Outer functions

The unilateral shift is a restriction of the bilateral shift to an invariant subspace, invariant subspaces of the unilateral shift are determined by Theorem 1.3.12 which are the invariant subspaces of the bilateral shift that are contained in $H^2(\mathbb{T})$. In this case, the functions generating the invariant subspaces are certain analytic functions whose structures are important.

Definition 1.3.13. *A function $\phi \in H^\infty$ satisfying $|\widetilde{\phi}(e^{i\theta})| = 1$ a.e. is an inner function.*

The definition of inner functions requires that the functions to be in H^∞ , but some time it is useful to know that a function in H^2 whose boundary value is of modulus 1 a.e. then it is also an inner function.
Now we are going to state the famous **Beurling's Theorem** [6] which completely describes the shift invariant subspaces of H^2.

Theorem 1.3.14. *Every invariant subspace of the unilateral shift other than $\{0\}$ has the form ϕH^2, where ϕ is an inner function. Conversely any subspace of the form ϕH^2 is invariant subspace of H^2 for the shift operator.*

Therefore, the invariant subspaces for S^* are given by,

$$K_\theta := (\theta H^2)^\perp, \text{where } \theta \text{ is an inner function.}$$

Now in vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$, each $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ induces a bounded linear transformation $T_\Theta \in \mathcal{L}\big(H^2_{\mathbb{C}^r}(\mathbb{D}), H^2_{\mathbb{C}^m}(\mathbb{D})\big)$ defined by

$$T_\Theta F(z) = \Theta(z)F(z). \quad (F \in H^2_{\mathbb{C}^r}(\mathbb{D})) \tag{1.7}$$

The operator T_Θ for $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ is known as the multiplication operator having symbol Θ. As discussed eariler the elements of $H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ are known as the *multipliers* and are determined by

$$\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D}) \textit{ if and only if } ST_\Theta = T_\Theta S, \tag{1.8}$$

where the shift S on the left hand side and the right hand side act on $H^2_{\mathbb{C}^m}(\mathbb{D})$ and $H^2_{\mathbb{C}^r}(\mathbb{D})$ respectively.

Definition 1.3.15. *A multiplier* $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ *is said to be* inner *if* M_Θ *is an isometry, or equivalently,* $\Theta(z) \in \mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$ *is an isometry almost everywhere with respect to the Lebesgue measure on* $\mathbb{T}$ *(unit circle).*

Inner multipliers are among the most important tools for classifying invariant subspaces of the vector valued Hardy spaces. For instance:

Theorem 1.3.16. ***(Beurling-Lax-Halmos [51])*** *A non-zero closed subspace* $\mathcal{M} \subseteq H^2_{\mathbb{C}^m}(\mathbb{D})$ *is shift invariant if and only if there exists an inner multiplier* $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ *such that*

$$\mathcal{M} = \Theta H^2_{\mathbb{C}^r}(\mathbb{D}),$$

for some r $(1 \leq r \leq m)$*.*

Consequently, the space $\mathcal{M}^\perp$ of $H^2_{\mathbb{C}^m}(\mathbb{D})$ is invariant under S^* (the backward shift), and it can be represented as $\mathcal{K}_\Theta := \mathcal{M}^\perp = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^r}(\mathbb{D})$ which also known as model spaces ([23, 24, 42, 40]).

Now we will introduce outer functions in $H^2_{\mathbb{C}}(\mathbb{D})(\equiv H^2)$.

Definition 1.3.17. *The function* $f \in H^2$ *is an outer function if* f *is a cyclic vector for the unilateral shift. That is,* f *is an outer function if*

$$\bigvee_{k=0}^{\infty} \{S^k f\} = H^2.$$

An important property of an outer function is that it has no zeros in $\mathbb{D}$. A beautiful result that follows from **Beurling's theorem** is the following factorization of functions in H^2.

Theorem 1.3.18. *If* g *is a function in* H^2 *that is not identically zero, then* $g = \phi f$*, where* ϕ *is an inner function and* f *is an outer function. This factorization is unique up to constant factors.*

Definition 1.3.19. *For* $g \in H^2$*, if* $g = \phi f$ *with* ϕ *inner and* f *outer, we call* ϕ *the inner part of* g *and* f *the outer part of* g*.*

Therefore the zeros of an H^2 function are precisely the zeros of its inner part. Similarly, the inner outer factorization of a H^2-function also holds in vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$.

1.3.4 Blaschke products

Blaschke products are special type of inner functions in $H^2_{\mathbb{C}}(\mathbb{D})$. For some particular type of invariant spaces of the shift operator in H^2 the corresponding inner function can be explicitly formulated. From this, the concept of a Blaschke product came.

Definition 1.3.20. *Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of non zero complex numbers in $\mathbb{D}$ and assume that $\sum_{k=1}^{\infty}(1-|z_k|) < \infty$. Let s be a nonnegative integer. Then the Blaschke product with zeros $\{z_k\}$ and a zero of multiplicity s at 0 is defined by*

$$B(z) = z^s \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k} z}.$$

Note that s could be zero and there could be only a finite number of z_k's.

Theorem 1.3.21. *Every Blaschke product*

$$B(z) = z^s \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k} z},$$

where s is a nonnegative integer and $\{z_k\}$ is a sequence of nonzero numbers in $\mathbb{D}$ satisfynig $\sum_{k=1}^{\infty}(1-|z_k|) < \infty$, converges for every $z \in D$. Moreover, B is an inner function whose nonzero zeros are precisely the $\{z_k\}$, counting multiplicity, and a zero of multiplicity s at 0.

Corollary 1.3.2. *Suppose the inner function ϕ has a zero of multiplicity s at 0 and has nonzero zeros at the point $z_1, z_2, \cdots$ in $\mathbb{D}$(repeated according to multiplicity). Let*

$$B(z) = z^s \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k} z}$$

be the Blaschke product formed from those zeros. Then ϕ can be written as a product $\phi = BS$, where S is an inner function that has no zero in $\mathbb{D}$.

Now there are two types of Blaschke products; if

$$B(z) = z^s \prod_{k=1}^{n} \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k} z}, \tag{1.9}$$

where s is finite, then this Blaschke product B is known as finite Blaschke product and if

$$B(z) = z^s \prod_{k=1}^{\infty} \frac{\overline{z_k}}{|z_k|} \frac{z_k - z}{1 - \overline{z_k} z}, \tag{1.10}$$

then B is called infinite Blaschke product.

It is sometimes useful to know which invariant subspaces of the shift operator in H^2 have finite codimension.

Theorem 1.3.22. *[40, Theorem 2.6.8] If ϕ is an inner function, then ϕH^2 has finite codimension if and only if ϕ is a constant multiple of a Blaschke product with a finite number of factors.*

1.4 Toeplitz operators

The most studied and best-known operators on the Hardy Hilbert space are the Toeplitz operators. The forward and backward shift operators are simple examples of Toeplitz operators; more generally, the Toeplitz operators are those operators whose matrices with respect to the standard basis of H^2 have constant diagonals.

Definition 1.4.1. *For each ϕ in $L^\infty(\mathbb{T})$, the Toeplitz operator with symbol ϕ is the operator T_ϕ defined by*

$$T_\phi(f) = P\phi(f)$$

for each $f \in H^2(\mathbb{T})$, where $P = P_{H^2}$ is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

The following theorem determine the Toeplitz operators and for detailed proof see [40, Corollary 3.2.7.].

Theorem 1.4.2. *The operator T is a Toeplitz operator if and only if $S^*TS = T$, where S is the unilateral shift operator.*

The most tractable Toeplitz operators are the analytic ones.

Definition 1.4.3. *A Toeplitz operator T_ϕ is an analytic Toeplitz operator if $\phi \in H^\infty(\mathbb{T})$. The Toeplitz operator T_ϕ is called an coanalytic Toeplitz operator if T_ϕ^* is analytic.*

Now we will state some important results concerning Toeplitz operators.

Theorem 1.4.4. *[40, Theorem 3.2.5.] The commutant of the shift operator acting on $H^2(\mathbb{T})$ is $\{T_\phi : \phi \in H^\infty(\mathbb{T})\}$.*

Theorem 1.4.5. *[40, Theorem 3.2.11.] For ψ and ϕ in $L^\infty(\mathbb{T})$, $T_\psi T_\phi$ is a Toeplitz operator if and only if either T_ψ is coanalytic or T_ϕ is analytic. In both of those cases, $T_\psi T_\phi = T_{\psi\phi}$.*

Theorem 1.4.6. *[40, Theorem 3.2.15.] A Toeplitz operator is self-adjoint if and only if its symbol is real-valued almost everywhere.*

Theorem 1.4.7. *The only compact Toeplitz operator is 0.*

Theorem 1.4.8. ***(Coburn Alternative)*** *If ϕ is a function in $L^\infty(\mathbb{T})$ other than 0, then at least one of T_ϕ and T_ϕ^* is injective.*

In vector valued case, let $P_m : L^2(\mathbb{T}, \mathbb{C}^m) \to H^2_{C^m}(\mathbb{D})$ be the orthogonal projection of $L^2(\mathbb{T}, \mathbb{C}^m)$ onto $H^2_{C^m}(\mathbb{D})$, and defined by

$$\sum_{n=-\infty}^{\infty} A_n e^{int} \mapsto \sum_{n=0}^{\infty} A_n e^{int}.$$

Therefore $P_m(F) = (Pf_1, Pf_2, \ldots, Pf_m)$, where P is the Riesz projection on $H^2_{\mathbb{C}}(\mathbb{D})$ [23] and $F = (f_1, f_2, \ldots, f_m) \in L^2(\mathbb{T}, \mathbb{C}^m)$.

Definition 1.4.9. *For any $\Phi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$, the Toeplitz operator $T_\Phi : H^2_{\mathbb{C}^m}(\mathbb{D}) \to H^2_{\mathbb{C}^m}(\mathbb{D})$ is defined by*

$$T_\Phi(F) = P_m(\Phi F)$$

for any $F \in H^2_{\mathbb{C}^m}(\mathbb{D})$.

Since $H^2_{\mathbb{C}^m}(\mathbb{D})$ can be written as the direct sum of m-copies of $H^2_{\mathbb{C}}(\mathbb{D})$, then we have the following matrix-representation of T_Φ :

$$T_\Phi = \begin{bmatrix} T_{\phi_{11}} & T_{\phi_{12}} & \cdots & T_{\phi_{1m}} \\ T_{\phi_{21}} & T_{\phi_{22}} & \cdots & T_{\phi_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\phi_{m1}} & T_{\phi_{m2}} & \cdots & T_{\phi_{mm}} \end{bmatrix}_{m\times m}, \tag{1.11}$$

where

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1m} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1} & \phi_{m2} & \cdots & \phi_{mm} \end{bmatrix}_{m\times m} \tag{1.12}$$

is an element of $L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ and each $T_{\phi_{ij}}$ is a Toeplitz operator in $H^2_{\mathbb{C}}(\mathbb{D})$ [18]. Furthermore, it is well known that $T_\Phi^* = T_{\Phi^*}$, where $\Phi^* = \left(\overline{\phi}_{ji}\right)_{m\times m}$ [18].

1.5 Nearly invariant subspace for the Backward shift S^* in scalar valued Hardy space

We will describe certain subspaces of H^2, modulo one dimension, are invariant under the backward shift operator. These subspaces are called nearly S^*-invariant subspaces. The nearly S^*-invariant subspaces are linked to study the kernel of Toeplitz operators.

Definition 1.5.1. *A closed subspace $\mathcal{M} \subset H^2_{\mathbb{C}}(\mathbb{D})$ is said to be nearly invariant for S^* (or, nearly S^*-invariant) if whenever $f \in \mathcal{M}$ and $f(0) = 0$, then $S^* f \in \mathcal{M}$.*

Lemma 1.5.2. *Let $\mathcal{M}$ be a nontrivial nearly S^*-invariant subspace of H^2. Then dim$(\mathcal{M} \ominus (\mathcal{M} \cap H^2_0)) = 1$, where $H^2_0 = \{f \in H^2 : f(0) = 0\}$.*

Example 1.5.3. *The kernel of a Toeplitz operator is nearly S^*-invariant subspace.*

Proof. Consider a Toeplitz operator T_ϕ, for $\phi \in L^\infty(\mathbb{T})$. Let $f \in KerT_\phi$ with $f(0) = 0$. So, $f = Sg$, for some $g \in H^2_{\mathbb{C}}(\mathbb{D})$. Now by Theorem 1.4.4 we have,

$$T_\phi(S^* f) = T_\phi(g) = S^* T_\phi S(g) = S^* T_\phi(f) = 0$$

which imples that $S^* f \in KerT_\phi$ and hence $KerT_\phi$ is a nearly S^*-invariant subspace. □

In the scalar case, nearly S^*-invariant subspaces of $H^2(\mathbb{D})$ were introduced by Hitt [34] as a tool for classifying the simply S-invariant subspaces of $H^2(A)$, where $H^2(A)$ denotes the Hardy space on the annulus $A = \{z \in \mathbb{C} : r_0 < |z| < 1\}$, r_0 is a positive real number less than unity, and studied further by Sarason [49], who used them to study the kernels of Toeplitz operators. The follwing theorem mainly describes nearly S^*-invariant subspace of H^2 in scalar valued case as described in [34, 49] and also one can see it in [24].

Theorem 1.5.4. *Let $\mathcal{M}$ be a nontrivial closed subspace of H^2 . The following are equivalent.*
(i) $\mathcal{M}$ is a nearly S^*-invariant subspace of H^2.
(ii) There exists a unique function g of unit norm in $\mathcal{M}$ such that $g(0) > 0$ and a unique S^* invariant subspace $\mathcal{M}'$ of H^2 on which T_g acts isometrically such that $\mathcal{M} = T_g\mathcal{M}'$, where T_g is the multiplication operator on $\mathcal{M}'$ which sends $m' \in \mathcal{M}'$ to $gm' \in \mathcal{M}$.

1.6 Almost invariant space

Let X be an infinite dimensional separable complex Banach space, and let $T \in \mathcal{L}(X)$ be a bounded linear operator on X. We say that a closed subspace $\mathcal{M}$ of X is invariant under T if $T(X) \subset X$.

Definition 1.6.1. *A closed subspace $\mathcal{M} \subset X$ is said to be almost invariant for $T \in \mathcal{L}(X)$ if there exists a finite dimensional subspace $\mathcal{F}$ of X such that*

$$T(\mathcal{M}) \subset \mathcal{M} + \mathcal{F}.$$

Definition 1.6.2. *A subspace $\mathcal{M}$ of a Banach space X is said to be a half-space if it has infinite dimension and as well as infinite codimension.*

The study of almost invariant half-spaces of operators T acting on complex Banach space is a highly interesting subject of the recent research area. For more on almost invariant half-spaces see [3, 47, 50, 53].

1.7 Nearly invariant subspaces for Backward shift on vector valued Hardy spaces

The study of the shift operator S (multiplication by the independent variable) on certain Hardy spaces consisting of vector valued analytic functions on the unit disc $\mathbb{D}$, is the main aim of this section. The characterization of nearly invariant subspaces for Backward shift on vector valued Hardy spaces was initiated by Chalendar-Chevrot-Partington (C-C-P) [9] as a vectorial generalization of Hitt [34] and Sarason's [49] work. Let us now discuss some useful definitions:

Definition 1.7.1. *A bounded linear operator T on $\mathcal{H}$ belongs to the class $C_{\cdot 0}$ if for all $x \in \mathcal{H}$*

$$\lim_{n\to\infty} ||T^{*n}x|| = 0.$$

Let $v \otimes w$ dentes the operator defined by $(v \otimes w)(x) = \langle x, w\rangle v$, for each $x \in \mathcal{H}$ and $u, v \in \mathcal{H}$.

1.7.1 Nearly invariant subspaces in $H^2_{\mathbb{C}^m}(\mathbb{D})$

For $F \in H^2_{\mathbb{C}^m}(\mathbb{D})$, the shift operator S is defined by

$$S(F)(z) := zF(z),$$

and the adjoint of the shift operator S is given by

$$S^*(F)(z) := \frac{F(z) - F(0)}{z}.$$

Definition 1.7.2. *A subspace $\mathcal{F}$ of $H^2_{\mathbb{C}^m}(\mathbb{D})$ is said to be nearly S^* -invariant if $\mathcal{F}$ is closed, and if every element $F \in \mathcal{F}$ with $F(0) = 0$ satisfies $S^*F \in \mathcal{F}$.*

A preliminary question concerning a nearly S^*-invariant subspace is whether there exists $F \in \mathcal{F}$ such that $F(0) \neq 0$. Indeed, if $\mathcal{F} \subset zH^2_{\mathbb{C}^m}(\mathbb{D})$ then by definition $\mathcal{F}$ is nearly S^*-invariant if and only if $\mathcal{F}$ is S^* invariant.

Lemma 1.7.3. *Let $\mathcal{F}$ be a nearly S^*-invariant subspace with $\mathcal{F} \subset zH^2_{\mathbb{C}^m}(\mathbb{D})$. Then $\mathcal{F} = \{0\}$.*

This lemma has an important consequence for characterizing nearly S^*-invariant subspaces in vector valued case.

Corollary 1.7.1. *If $\mathcal{F}$ is a nearly S^* invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ and $\mathcal{F} \neq \{0\}$, then $1 \leq dim(\mathcal{F} \ominus (\mathcal{F} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) \leq m$.*

Now a full description of nearly S^*-subspaces will be given by linking them with S^*-invariant subspaces. The following theorem is due to Chalendar-Chevrot-Partington (C-C-P) [9] which is an adaptation to the vectorial case of Hitt's algorithm [34] and the operatorial version of Sarason [49].

Theorem 1.7.4. *Let $\mathcal{F}$ be a nearly S^*-invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ and let $(W_1, W_2, \cdots, W_r)$ be an orthonormal basis of $\mathcal{W} := \mathcal{F} \ominus (\mathcal{F} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$.*
Let F_0 be the $m \times r$ matrix whose columns are $W_1, W_2, \cdots, W_r$. Then there exists an isometric mapping

$$J : \mathcal{F} \longleftrightarrow \mathcal{F}' \quad \text{given by} \quad F_0 G \mapsto G,$$

where $\mathcal{F}' := \{G \in H^2_{\mathbb{C}^r}(\mathbb{D}) : \exists F \in \mathcal{F}, F = F_0 G\}$. Moreover $\mathcal{F}'$ is S^* invariant.

Corollary 1.7.2. *Let $\mathcal{F}$ be a nearly S^*-invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ and let $(W_1, W_2, \cdots, W_r)$ be an orthonormal basis of $\mathcal{W} := \mathcal{F} \ominus (\mathcal{F} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$.*
Let F_0 be the $m \times r$ matrix whose columns are $W_1, W_2, \cdots, W_r$. Then there exists an inner function $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^r))$, which is unique up to unitary equivalance and vanishes at zero, such that

$$\mathcal{F} = F_0(H^2_{\mathbb{C}^r}(\mathbb{D}) \ominus \Phi H^2_{r'}(\mathbb{D})).$$

Remark 1.7.5. *It should be noted that nearly S^*-invariant subspaces are linked to the problem of injectivity of Toeplitz operators. Indeed, suppose that Ψ is a function in $L^\infty(\mathbb{T}, \mathbb{C}^{m\times m})$ not identically equals to zero, such that the Toeplitz operator T_Ψ on $H^2_{\mathbb{C}^m}(\mathbb{D})$ has a nontrivial kernel. Then $KerT_\Psi$ is a nontrivial nearly S^*-invariant subspace. Therefore, according to our previous result, it equals $T_{F_0}\mathcal{M}'$, where $\mathcal{M}'$ is a S^*-invariant subspace, containing the constant functions, and on which F_0 acts isometrically. In the scalar case, Sarason first noticed this link, providing an alternative proof of Hayashi's result.*

1.8 Nearly invariant subspaces of finite defect in scalar valued Hardy space

In order to describe the almost-invariant subspaces for S, the definition of nearly invariant subspaces with defect m for S^* has been first introduced by Chalendar-Gallardo-Partington [11] (C-G-P) as a generalization of nearly invariant subspaces. They gave a complete characterization of nearly S^*-invariant subspaces with finite defect in the scalar valued Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$ in [11].

Definition 1.8.1. *A closed subspace $M \subset H^2_{\mathbb{C}}(\mathbb{D})$ is said to be nearly S^*-invariant with defect m if and only if there is an m-dimensional subspace F (which may be taken to be orthogonal to M) such that if $f \in M,\ f(0) = 0$ then $S^*f \in M \oplus F$. We say that M is S^* almost-invariant with defect m if and only if $S^*M \subset M \oplus F$, where dim $F = m$.*

The first observation provides a link between nearly invariant subspaces for S^* and almost-invariant spaces for S. For detailed proof of the following results one is refer to see [11].

Proposition 1.8.2. *[11, Proposition 1.2] Every nearly invariant subspace $M = gK_\theta$ for S^* is* *almost-invariant for S with defect 1. Moreover, if θ is not rational, it is an almost-invariant half-space with defect 1.*

The next result will state that the orthocomplement of certain nearly invariant subspaces for S^* are also almost-invariant for S of defect 1. Before that, the following lemma is needed.

Lemma 1.8.3. *[11, Lemma 1.3] Let ψ and θ be non-constant inner functions. Then $(\psi K_\theta)^\perp = \theta\psi H^2 \oplus K_\psi$.*

Now using Lemma 1.8.3, we have the following result.

Proposition 1.8.4. *[11, Proposition 1.4] Let ψ and θ be non-constant inner functions. Then $(\psi K_\theta)^\perp$ is an almost-invariant space of defect 1. Moreover, if ψ is not rational (finite Blaschke product); or if ψ is rational but θ is not a rational inner function, then $(\psi K_\theta)^\perp$ is an almost-invariant half-space of defect 1.*

Clearly S^* almost-invariance implies nearly S^* -invariance (with the same defect). Next we will describe the generalization of Hitt's algorithm to obtain a representation of nearly S^* -invariant subspaces with defect m (finite), (done by **C-G-P**) as follows:

Theorem 1.8.5. *[11, Theorem 2.2] Let M be a closed subspace that is nearly S^*-invariant with defect m. Then:*

(i) in the case where there are functions in M that do not vanish at 0,

$$M = \{f : f(z) = k_0(z)f_0(z) + z\sum_{j=1}^{m} k_j(z)e_j(z) : (k_0, k_1, \dots, k_m) \in K\},$$

where f_0 is the normalized reproducing kernel for M at 0, $\{e_1, \dots, e_m\}$ is any orthonormal basis for F , and K is a closed $S^* \oplus \cdots \oplus S^*$ invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^{m+1}}(\mathbb{D})$, and $||f||^2 = \sum_{j=0}^{m} ||k_j||^2$.
(ii) In the case where all functions in M vanish at 0,

$$M = \{f : f(z) = z\sum_{j=1}^{m} k_j(z)e_j(z) : (k_1, k_2, \dots, k_m)\}$$

with the same notation as in (i), except that K is now a closed $S^* \oplus \cdots \oplus S^*$ invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$, and $||f||^2 = \sum_{j=1}^{m} ||k_j||^2$.
Conversely, if a closed subspace $M \subset H^2$ has a representation as in (i) or (ii), then it is a nearly S^* -invariant subspace of defect m.

It should also be noted that $K^\perp$ can be described using the Beurling-Lax-Halmos theorem.

Corollary 1.8.1. *[11, Corollary 2.4] A closed subspace M is an almost-invariant subspace for S^* with defect m if and only if it satisfies the conditions of Theorem 1.8.5, together with the extra condition that $S^* f_0 \in M \oplus F$ in case (i), while case (ii) is unchanged.*

Remark 1.8.6. *[11, Remark 2.5] Note also that $S^* M \subset M \oplus F$ is equivalent to the condition that*

$$S((M \oplus F)^\perp) \subset M^\perp = (M \oplus F)^\perp \oplus G,$$

where $G = F \ominus M$ and dim G = dim F ; this gives an expression for S almost-invariant subspaces.

CHAPTER 2

Almost invariant subspaces of the shift operator on vector valued Hardy spaces

2.1 Introduction

In 1988, Hitt [34] first introduces the notion of nearly invariant subspaces under the backward shift operator acting on the scalar valued Hardy space which he used as a tool for classifying the simply shift-invariant subspaces of the Hardy space of an annulus. In his paper, he rather called it as "weakly invariant subspace under the backward shift ". Later Sarason [49] further investigated these spaces and modified Hitt's algorithm for scalar valued Hardy space to study the kernels of Toeplitz operators. In 2010, Chalendar-Chevrot-Partington (C-C-P) [9] gives a complete characterization of nearly invariant subspaces under the backward shift operator acting on the vector valued Hardy space, providing a vectorial generalization of a result of Hitt. Recently Chalendar-Gallardo-Partington (C-G-P) [11] introduce the notion of nearly invariant subspace of finite defect for the backward shift operator acting on the scalar valued Hardy space as a generalization of nearly invariant subspaces and provides a complete characterization of these spaces in terms of backward shift invariant subspaces. Using this characterization they also described the almost-invariant subspaces for the shift and its adjoint acting on the scalar valued Hardy space. In this connection, we should mention that the relation between nearly invariant subspaces under the backward shift and the kernel of Toeplitz operators has been discussed in [16].

In this chapter, we further study nearly invariant subspaces of finite defect under the backward shift operator acting on the vector valued Hardy space and provides a vectorial generalization of C-G-P algorithm. As a consequences, we completely characterize nearly invariant subspaces of finite defect under the backward shift in terms of backward shift invariant subspaces. Furthermore, using the characterization of nearly invariant subspace under the backward shift we completely describe the almost invariant subspaces for the shift and its adjoint acting on the vector valued Hardy space. Moreover, at the end we also provide a connection between the orthocomplement of a nearly invariant subspaces of finite defect under the backward shift and the shift invariant subspaces on the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$ (see Definition (1.5) of chapter 1).

Now we revisit the Defintion 1.6.1 mentioned in Chapter 1 of almost invariant subspaces regarding the shift operator in vector valued Hardy space.

Definition 2.1.1. *A closed subspace $\mathcal{M}$ of $H^2_{\mathbb{C}^m}(\mathbb{D})$ is said to be* almost-invariant *for S if there exists a finite dimensional subspace $\mathcal{F}$ of $H^2_{\mathbb{C}^m}(\mathbb{D})$ such that*

$$S(\mathcal{M}) \subseteq \mathcal{M} + \mathcal{F}.$$

The space $\mathcal{F}$ is called the defect space and the smallest possible dimension of $\mathcal{F}$ is called defect of the space $\mathcal{M}$. Moreover, a space $\mathcal{M}$ is called a *half-space* (see Definition 1.6.2) if $\mathcal{M}$ has infinite dimension and infinite co-dimension. The study of almost-invariant half-spaces of any bonded linear operators T acting on complex Banach spaces was initiated in 2009 due to Androulakis, Popov, Tcaciuc and Troitsky [3] and later it further studied by Popov, Tcaciuc, Sirotkin and Wallis [47, 50, 53] to investigate the structure of almost-invariant half-spaces in more general setting. In this connection, it is easy to observe that every subspace which is not a half-space is clearly almost-invariant under any operator. A well-known result due to Beurling [6] states that if $\mathcal{M}$ is a S-invariant subspace of $H^2_{\mathbb{C}}(\mathbb{D})$, then $\mathcal{M}$ can be represented as

$$\mathcal{M} = \theta H^2_{\mathbb{C}}(\mathbb{D}),$$

(also mentioned in Chapter 1 as Theorem 1.3.14) where $\theta \in H^\infty_{\mathbb{C}}(\mathbb{D})$ is an inner function (that is, θ is a bounded holomorphic function on $\mathbb{D}$ and $|\theta| = 1$ a.e. on $\mathbb{T}$). In this regard, it is not difficult to conclude that the shift invariant subspace $\mathcal{M}$ of $H^2_{\mathbb{C}}(\mathbb{D})$ is a half-space if and only if $\mathcal{M} = \theta H^2_{\mathbb{C}}(\mathbb{D})$ with θ is not rational (that means, θ is not a product of finitely many Blaschke factor) [40].

The purpose of this chapter is to characterize almost invariant subspaces for the shift and its adjoint acting on the vector valued Hardy space in terms of invariant subspaces for the adjoint S^* with finite defect. To achieve our goal we give a connection between nearly invariant subspaces with finite defect and invariant subspaces for S^* in the vector valued Hardy space (see. Theorem 2.3.5).

The chapter is organized as follows: In Section 2.2 we give a connection between nearly invariant subspaces for S^* and almost-invariant subspaces for S in vector valued Hardy spaces. In other words, we generalize some results of [11, Section 2] in the vector valued setting. Section 2.3 deals with the main result of this chapter. At the end, in Section 2.4, we obtain a connection between the orthocomplement of a class of nearly invariant subspaces of finite defect under the backward shift and the shift invariant subspaces on the vector valued Hardy space.

2.2 Preliminary results

The space $H^2_{\mathbb{C}^m}(\mathbb{D})$ as defined in (1.5) of Chapter 1, can also be defined as the collection of all $\mathbb{C}^m$-valued analytic functions F on $\mathbb{D}$ such that

$$\|F\| = \Big[\sup_{0\leq r<1} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 \, d\theta \Big]^{\frac{1}{2}} < \infty.$$

Moreover, the nontangential boundary limit (or radial limit)

$$F(e^{i\theta}) := \lim_{r\to 1-} F(re^{i\theta})$$

exists almost everywhere on $\mathbb{T}$ (for more details see [42], I.3.11). Therefore $H^2_{\mathbb{C}^m}(\mathbb{D})$ can be embedded isomertically as a closed subspace of $L^2(\mathbb{T},\mathbb{C}^m)$ by identifying $H^2_{\mathbb{C}^m}(\mathbb{D})$ through the nontangential boundary limits of $H^2_{\mathbb{C}^m}(\mathbb{D})$ functions. Furthermore, $L^2(\mathbb{T},\mathbb{C}^m)$ can be decomposed in the following way

$$L^2(\mathbb{T},\mathbb{C}^m) = H^2_{\mathbb{C}^m}(\mathbb{D}) \oplus \overline{H^2_{0\,\mathbb{C}^m}},$$

where $\overline{H^2_{0\,\mathbb{C}^m}} = \{F \in L^2(\mathbb{T},\mathbb{C}^m) : \overline{F} \in H^2_{\mathbb{C}^m}(\mathbb{D}) \textit{ and } F(0) = 0\}$.

We already noticed that nearly S^*-invariant subspaces of $H^2_{\mathbb{C}}(\mathbb{D})$ (see Definition 1.5.1) were introduced and characterized by Hitt [34] and Sarason [49]. For vector valued nearly S^* invariant subspaces in $H^2_{\mathbb{C}^m}(\mathbb{D})$ one is refer to see Definition 1.7.2 and the vectorial generalization of Hitt's and Sarason's result was due to Chalendar-Chevrot-Partington (C-C-P) [9] which says the following: Every non trivial nearly S^*-invariant subspace $\mathcal{M}$ of $H^2_{\mathbb{C}^m}(\mathbb{D})$ has the form $\mathcal{M} = F_0\mathcal{K}$, where F_0 is the $m \times r$ $(1 \leq r \leq m)$ matrix whose columns are $\{W_1, W_2, ..., W_r\}$ which forms an orthonormal basis of $\mathcal{W} := \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$ and $\mathcal{K}$ is a S^*- invariant subspace of $H^2_{\mathbb{C}^r}(\mathbb{D})$. Therefore by Beurling-Lax-Halmos theorem (see Theorem 1.3.16), there exists an inner multiplier $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^{r'},\mathbb{C}^r)}(\mathbb{D})$ for some r' $(\leq r)$ such that

$$\mathcal{K} = \mathcal{K}_\Theta := H^2_{\mathbb{C}^r}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^{r'}}(\mathbb{D})$$

with an extra property that $\Theta(0) = 0$. The operator

$$T_{F_0} : H^2_{\mathbb{C}^r}(\mathbb{D}) \to H^2_{\mathbb{C}^m}(\mathbb{D})$$

$$G \longmapsto P(F_0G),$$

where P is the Fourier projection of the $L^1(\mathbb{T}, \mathbb{C}^m)$ function F_0G, as in the scalar case it is an isometry from $\mathcal{K}_\Theta$ *onto* $\mathcal{M}$.

As discussed earlier, throughout this section we will provide vectorial generalization of similar results given in ([11] , Section 2). Now we are in a position to prove our first result which produces a connection between nearly invariant subspaces for S^* and almost-invariant subspaces for S in the vector valued case.

Proposition 2.2.1. *Let F_0 and $\mathcal{K}_\Theta$ be as above. Then the nearly S^*- invariant subspace $\mathcal{M} = F_0\mathcal{K}_\Theta$ is an almost invariant for S with defect r'. In particular, if the inner multiplier $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r)}(\mathbb{D})$ is of the form:*

$$\Theta = \begin{pmatrix} \theta_1 & 0 & 0 & . & . & . & 0 \\ 0 & \theta_2 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \theta_r \end{pmatrix}_{r\times r}, \tag{2.1}$$

where $\{\theta_1, \theta_2, \ldots, \theta_r\}$ is a collection of inner functions of $H^2_{\mathbb{C}}(\mathbb{D})$ with at least one θ_i (say) is not rational, then $\mathcal{M} = F_0\mathcal{K}_\Theta$ is an almost-invariant half-space with defect r.

Proof. Since $\Theta \in H^{\infty}_{\mathcal{L}(\mathbb{C}^{r\prime},\mathbb{C}^{r})}(\mathbb{D})$ is an inner multiplier, then the map

$$T_{\Theta} : H^2_{\mathbb{C}^{r\prime}}(\mathbb{D}) \to H^2_{\mathbb{C}^{r}}(\mathbb{D})$$

is an isometry. Let $\{e_i\}_{i=1}^{r\prime}$ be an orthonormal basis of $\mathbb{C}^{r\prime}$. Now consider $\widetilde{\Theta_i} = \Theta e_i \in H^2_{\mathbb{C}^r}(\mathbb{D})$, for $i = 1, 2, ..., r\prime$. Note that T_{Θ} is an isometry implies $\{\widetilde{\Theta_i}\}_{i=1}^{r\prime}$ is linearly independent in $H^2_{\mathbb{C}^r}(\mathbb{D})$. Moreover

$$(\mathcal{K}_{\Theta} + \ span\{\widetilde{\Theta_i}\}_{i=1}^{r\prime})^{\perp} = \Theta H^2_{\mathbb{C}^{r\prime}}(\mathbb{D}) \cap (span\{\widetilde{\Theta_i}\}_{i=1}^{r\prime})^{\perp} = z\Theta H^2_{\mathbb{C}^{r\prime}}(\mathbb{D}).$$

On the other hand for $G \in H^2_{\mathbb{C}^{r\prime}}(\mathbb{D})$ and $F \in \mathcal{K}_{\Theta}$ we have $\langle z\Theta G, zF\rangle = 0$ and hence $z\Theta H^2_{\mathbb{C}^{r\prime}}(\mathbb{D}) \subseteq (z\mathcal{K}_{\Theta})^{\perp}$. Thus $S\mathcal{K}_{\Theta} \subseteq \mathcal{K}_{\Theta} + span\{\widetilde{\Theta_i}\}_{i=1}^{r\prime}$. Since T_{F_0} is an isometry from $\mathcal{K}_{\Theta}$ onto $\mathcal{M}$ we have $S\mathcal{M} \subseteq \mathcal{M} + span\{F_0\widetilde{\Theta_i}\}_{i=1}^{r\prime}$. This proves that $\mathcal{M}$ is almost invariant under S with defect r'.

For the second part we assume that θ_j is not rational for some $j \in \{1, \dots, r\}$ which immediately implies that $\theta_j H^2_{\mathbb{C}}(\mathbb{D})$ is a half space. Note that since Θ is of the form (2.1), then $\Theta H^2_{\mathbb{C}^r}(\mathbb{D})$ is again a half space which concludes that $\mathcal{M}$ is also a half space. This concludes the proof. □

The following two lemmas are very useful to conclude that the orthocomplement of some nearly invariant subspaces for S^* are also almost invariant for S of some finite defect in $H^2_{\mathbb{C}^m}(\mathbb{D})$.

Lemma 2.2.2. *Let $\Psi \in H^{\infty}_{\mathcal{L}(\mathbb{C}^m)}(\mathbb{D})$ be an inner multiplier of the form*

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 & . & . & . & 0 \\ 0 & \psi_2 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \psi_m \end{pmatrix}_{m\times m.}, \tag{2.2}$$

where $\{\psi_1, \psi_2, \dots, \psi_m\}$ is a collection of inner functions of $H^2_{\mathbb{C}}(\mathbb{D})$. Then $(\Psi\mathcal{K}_{\Theta})^{\perp} = \Psi\Theta H^2_{\mathbb{C}^r}(\mathbb{D}) \oplus \mathcal{K}_{\Psi}$ for any inner multiplier $\Theta \in H^{\infty}_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ with $r \leq m$.

Proof. Let $F \in H^2_{\mathbb{C}^m}(\mathbb{D})$. Then for all $K \in \mathcal{K}_{\Theta}$ we have

$$\langle F, \Psi K\rangle_{H^2_{\mathbb{C}^m}(\mathbb{D})} = \langle F, \Psi K\rangle_{L^2(\mathbb{T},\mathbb{C}^m)} = \langle T^*_{\Psi}F, K\rangle_{L^2(\mathbb{T},\mathbb{C}^m)}.$$

Therefore $T^*_{\Psi}F \in \Theta H^2_{\mathbb{C}^r}(\mathbb{D}) \oplus \overline{H^2_0}$ *if and only if* $F \in \Psi\Theta H^2_{\mathbb{C}^r}(\mathbb{D}) \oplus \mathcal{K}_{\Psi}$, where $\mathcal{K}_{\Psi} = (\Psi H^2_{\mathbb{C}^m}(\mathbb{D}))^{\perp} = H^2_{\mathbb{C}^m}(\mathbb{D}) \cap \Psi\overline{H^2_0}$. This completes the proof. □

Lemma 2.2.3. *Let $\Psi \in H^\infty_{\mathcal{L}(\mathbb{C}^m)}(\mathbb{D})$ be as in the statement of Lemma 2.2.2 with an extra assumption that $\psi_i(0) \neq 0$ for each $i \in \{1, 2, \dots, m\}$. Then $\Psi\mathcal{K}_\Theta$ is nearly S^* invariant for any inner multiplier $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ with $r \leq m$.*

Proof. Let $F \in \Psi\mathcal{K}_\Theta$ be such that $F(0) = 0$. Then

$$F = \Psi K, \text{ where } K \in \mathcal{K}_\Theta \text{ and } F(0) = \Psi(0)K(0) = 0.$$

Since each $\psi_i(0) \neq 0$ for each $i \in \{1, 2, \dots, m\}$, then from the above we conclude that $K(0) = 0$. Thus

$$S^*F(z) = \frac{F(z) - F(0)}{z} = \frac{\psi(z)K(z)}{z} = \Psi(z)S^*K(z), \forall z \in D.$$

Since $\mathcal{K}_\Theta$ is S^* invariant, then $S^*F \in \Psi\mathcal{K}_\Theta$. This completes the proof. □

Combining Lemma 2.2.2 and Lemma 2.2.3 we have the following result.

Proposition 2.2.4. *Let $\Psi \in H^\infty_{\mathcal{L}(\mathbb{C}^m)}(\mathbb{D})$ be as in the statement of Lemma 2.2.2 and let $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ for $r \leq m$. Then $(\Psi\mathcal{K}_\Theta)^\perp$ is an almost invariant subspace for S in $H^2_{\mathbb{C}^m}(\mathbb{D})$ with defect m. In particular if*

$$\Theta = \begin{pmatrix} \theta_1 & 0 & 0 & . & . & . & 0 \\ 0 & \theta_2 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \theta_m \end{pmatrix}_{m\times m,} , \tag{2.3}$$

where $\{\theta_1, \theta_2, \dots, \theta_m\}$ is a collection of inner functions of $H^2_{\mathbb{C}}(\mathbb{D})$ with at least one θ_i (say) is not rational, then $(\Psi\mathcal{K}_\Theta)^\perp$ is an almost invariant half space of defect m.

Proof. By repeating the similar kind of argument as in the proof of Proposition 2.2.1 we conclude that

$$S\mathcal{K}_\Psi \subset \mathcal{K}_\Psi + span\{\widetilde{\Psi_i}\}_{i=1}^m,$$

where $\widetilde{\Psi_i} = \Psi e_i$ and $\{e_1, e_2, \dots, e_m\}$ is an orthonormal basis of $\mathbb{C}^m$. On the other hand by Lemma 2.2.2 we have $(\Psi\mathcal{K}_\Theta)^\perp = \Psi\Theta H^2_{\mathbb{C}^m}(\mathbb{D}) \oplus \mathcal{K}_\Psi$. Thus by combining this two results we have the following :

$$S(\Psi\mathcal{K}_\Theta)^\perp \subset (\Psi\mathcal{K}_\Theta)^\perp + span\{\widetilde{\Psi_i}\}_{i=1}^m.$$

Since by hypothesis Θ is of the form (2.3), then the dimensions of both $(\Psi\mathcal{K}_\Theta)^\perp$ and $\Psi\mathcal{K}_\Theta$ are infinite and hence it is a half space. This completes the proof. □

In Proposition 2.2.1, we have seen that every nearly invariant subspace for S^* is an almost invariant subspace for S. The next proposition says that the converse of this result is not true that is, there exists an almost invariant half space for S which is not nearly S^*- invariant.

Proposition 2.2.5. *Let $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^m)}(\mathbb{D})$ be as in (2.3) with an extra assumption that $\Theta(0) = 0$. Then $(\Theta\mathcal{K}_\Theta)^\perp$ is an almost invariant half space for S of defect m but not nearly invariant for S^*.*

Proof. Since Θ is of the form (2.3), then by Proposition 2.2.4 we conclude that $(\Theta\mathcal{K}_\Theta)^\perp$ is an almost invariant for S. Note that Θ is an inner multiplier of the form (2.3) with at least one θ_j (say) is not rational. Now $\Theta(0) = 0$ implies that $\theta_i(0) = 0$ for all $i \in \{1, 2, \dots, m\}$. Let $F = \Theta^2 e_j$, where $\{e_1, e_2, \dots, e_m\}$ is an orthonormal basis of $\mathbb{C}^m$. Then $F \in H^2_{\mathbb{C}^m}(\mathbb{D})$ and $F(0) = 0$. Note that for any $K \in \mathcal{K}_\Theta$, $\langle F, \Theta K\rangle = \langle \Theta^2 e_j, \Theta K\rangle = 0$ and hence $F \in (\Theta\mathcal{K}_\Theta)^\perp$. On the contrary, let us assume that $(\Theta\mathcal{K}_\Theta)^\perp$ is nearly invariant for S^*. Therefore $S^*F \in (\Theta\mathcal{K}_\Theta)^\perp = \Theta^2 H^2_{\mathbb{C}^m}(\mathbb{D}) \oplus \mathcal{K}_\Theta$ (by Lemma 2.2.2) and hence

$$S^*F(z) = \frac{F(z)}{z} = \Theta^2(z)H(z) + K(z), \tag{2.4}$$

for some $H \in H^2_{\mathbb{C}^m}(\mathbb{D})$ and $K \in \mathcal{K}_\Theta$. On the other hand $\Theta(0) = 0$ implies that there exists an another inner multiplier $\Theta_1 \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D},)$ such that $\Theta(z) = z\Theta_1(z)$, $\forall z \in \mathbb{D}$ and hence

$$S^*F(z) = \frac{F(z)}{z} = \frac{\Theta^2(z)e_j}{z} = \Theta(z)\Theta_1(z)e_j \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D}). \tag{2.5}$$

Combining (2.4) and (2.5) we conclude that $K \in \mathcal{K}_\Theta \cap \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ and therefore $K(z) = 0$. This implies that $H(z) = \frac{1}{z} \otimes e_j \in H^2_{\mathbb{C}^m}(\mathbb{D})$ which is not the case. This completes the proof.

□

2.3 Classification of almost invariant subspaces

The main aim of this section is to describe completely the almost invariant subspaces for the shift and its adjoint acting on the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$. At first we begin with the definition of nearly invariant subspace for S^* with finite defect on the vector valued Hardy space. For scalar valued case, it has been introduced in [11] (see Definition 1.8.1).

Definition 2.3.1. *A closed subspace $\mathcal{M} \subset H^2_{\mathbb{C}^m}(\mathbb{D})$ is said to be nearly S^*-invariant with defect p if and only if there is an p-dimensional subspace $\mathcal{F} \subset H^2_{\mathbb{C}^m}(\mathbb{D})$ (which is orthogonal to $\mathcal{M}$) such that if $F \in M, F(0) = 0$ then $S^*F \in M \oplus \mathcal{F}$. This p dimensional subspace $\mathcal{F}$ is known as the defect space corresponding to the nearly S^*-invariant subspace $\mathcal{M}$ with finite defect p. We say that M is S^* almost invariant with defect p if and only if $S^*\mathcal{M} \subset \mathcal{M} \oplus \mathcal{F}$; where dim $\mathcal{F} = p$.*

Characterization of the subspaces in a vector valued Hardy space that are nearly S^* invariant was due to Chalendar-Chevrot-Partington [9] which provides a vectorial generalization of a result of Hitt [34]. Recently Chalendar-Gallardo-Partington (C-G-P) gives a complete characterization of nearly S^* invariant subspaces with finite defect in the scalar valued Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$ [11]. Here our principle aim is to provide a complete characterization of nearly S^* invariant subspaces with finite defect in the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$. Before going to the main result of this section we first start with the following two lemmas. Note that the first lemma already proved in [9] but for reader's convenience we are providing a proof herewith.

Lemma 2.3.2. *Let $\mathcal{M}$ be a closed subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ such that all functions in $\mathcal{M}$ do not vanish at 0. Then*

$$1 \leq \dim\left(\mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))\right) \leq m.$$

Proof. Note that by hypothesis the space $\mathcal{W} := \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$ is non-trivial. Let $\dim \mathcal{W} = r$. For $i \in \{1, 2, ..., m\}$, let $F_i = P_{\mathcal{M}}(k_0 \otimes e_i)$, where $P_{\mathcal{M}}$ is the orthogonal projection of $H^2_{\mathbb{C}^m}(\mathbb{D})$ onto $\mathcal{M}$ and k_0 is the reproducing kernel at 0 in $H^2_{\mathbb{C}}(\mathbb{D})$. Then $F_i \in \mathcal{M}$ and $\langle F_i, G\rangle = 0$ for all $G \in \mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D})$ which implies that $G(0) = 0$. This shows that $\{F_i\}_{i=1}^m$ generates the space $\mathcal{W}$ and hence $\dim \mathcal{W} \leq m$. This completes the proof. □

Recall the Definition 1.7.1 concerning $C_{\cdot 0}$ contraction and we have the following result.

Lemma 2.3.3. *([5, Lemma 3.3]) Suppose $T \in C_{\cdot 0}$ and $\dim\ \mathcal{D}_T\big(= \overline{Ran}(I - T^*T)^{\frac{1}{2}}\big) < \infty$. Let $\mathcal{F} \subset \mathcal{H}$ be a closed subspace of finite codimension. Then $TP_{\mathcal{F}} \in C_{\cdot 0}$, where $P_{\mathcal{F}}$ denotes the orthogonal projection onto $\mathcal{F}$.*

Now we are in a position to state and prove the main theorem in this section.

Theorem 2.3.4. *Let $\mathcal{M}$ be a closed subspace that is nearly S^*-invariant with defect 1 in $H^2_{\mathbb{C}^m}(\mathbb{D})$ and let $E_1 \in H^2_{\mathbb{C}^m}(\mathbb{D})$ be such that $\mathcal{F} = \langle E_1\rangle$ (subspace spanned by the vector E_1)*

is the defect space with $\|E_1\| = 1$. Let $\{W_1, W_2, \ldots, W_r\}$ be an orthonormal basis of $\mathcal{W} := \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$, and let F_0 be the $m \times r$ matrix whose columns are $W_1, W_2, \ldots, W_r$. Then

(i) in the case where there are functions in $\mathcal{M}$ that do not vanish at 0,

$$\mathcal{M} = \Big\{F \in H^2_{\mathbb{C}^m}(\mathbb{D}) : F = F_0K_0 + zk_1E_1 \quad \text{and} \quad (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\}, \tag{2.6}$$

where $\mathcal{K}$ is a closed $S^* \oplus \cdots \oplus S^*$-invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^{r+1}}(\mathbb{D})$, and $\|F\|^2 = \|K_0\|^2 + \|k_1\|^2$.

(ii) In the case where all functions in $\mathcal{M}$ vanish at 0.

$$\mathcal{M} = \{F : F(z) = zk_1(z)E_1(z) : k_1 \in \mathcal{K}\},$$

where $\mathcal{K}$ is now a closed S^*- invariant subspace of the Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$ and $\|F\|^2 = \|k_1\|^2$.

Conversely, if a closed subspace $\mathcal{M}$ of the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$ has a representation like (i) or (ii) as above, then it is a nearly S^*-invariant subspace of defect 1.

Proof. (*i*) By hypothesis $\mathcal{M} \nsubseteq zH^2_{\mathbb{C}^m}(\mathbb{D})$. Let $P_{\mathcal{W}}$ denote the orthogonal projection of $\mathcal{M}$ onto $\mathcal{W}$. If $F \in \mathcal{M}$, then $P_{\mathcal{W}}(F)$ can be written as

$$P_{\mathcal{W}}(F)(z) = a_{0,1}W_1(z) + \cdots + a_{0,r}W_r(z),$$

and hence for each $z \in \mathbb{D}$, $F(z) = P_{\mathcal{W}}(F)(z) + F_1(z)$, where $F_1 \in \mathcal{M} \cap \mathcal{W}^{\perp}$. Since $\{W_1, W_2, \ldots, W_r\}$ forms an orthonormal basis of $\mathcal{W}$, then we have the following norm identity:

$$\begin{aligned}\|F\|^2 &= |a_{0,1}|^2 + |a_{0,2}|^2 + \cdots + |a_{0,r}|^2 + \|F_1\|^2 \\ &= \|A_0\|^2 + \|F_1\|^2, \text{ where } A_0 = (a_{0,1}, a_{0,2}, \ldots, a_{0,r})^T.\end{aligned}$$

Note that $F_1 \in \mathcal{M} \cap \mathcal{W}^{\perp}$ and hence $F_1(0) = 0$. On the other hand since $\mathcal{M}$ is a nearly S^*-invariant subspace with defect 1, then $S^*F_1 = G_1 + \beta_1 E_1$, where $G_1 \in \mathcal{M}$ and $\beta_1 \in \mathbb{C}$. Therefore $F_1 = S(G_1 + \beta_1 E_1)$ because $F_1(0) = 0$ and $SS^*F_1 = F_1$. Thus for $z \in \mathbb{D}$ we have

$$F(z) = F_0(z)A_0 + zG_1(z) + \beta_1 zE_1(z) \quad \text{and} \quad \|F\|^2 = \|A_0\|^2 + \|G_1\|^2 + |\beta_1|^2. \tag{2.7}$$

Now we repeat the above process starting with G_1. Then $G_1 = F_0A_1 + F_2$ with $F_2 \in \mathcal{M}$ and $F_2(0) = 0$. Similarly by using the properties of $\mathcal{M}$ we conclude that $S^*F_2 = G_2 + \beta_2 E_1$ for some $G_2 \in \mathcal{M}$ and $\beta_2 \in \mathbb{C}$ which again implies that $F_2 = zG_2 + \beta_2 z E_1$. Therefore in the second iteration we have

$$F(z) = F_0(z)(A_0 + A_1 z) + z^2 G_2(z) + (\beta_1 z + \beta_2 z^2)E_1(z) \qquad (z \in \mathbb{D})$$

and $\|F\|^2 = \|A_0\|^2 + \|A_1\|^2 + \|G_2\|^2 + |\beta_1|^2 + |\beta_2|^2$. If we continue the above process at the k-th iteration we obtain,

$$F(z) = F_0(z)(A_0 + A_1 z + \cdots + A_{k-1}z^{k-1}) + z^k G_k(z) + (\beta_1 z + \cdots + \beta_k z^k)E_1(z) \tag{2.8}$$

and

$$\|F\|^2 = \sum_{j=0}^{k-1} \|A_j\|^2 + \|G_k\|^2 + \sum_{j=1}^{k} |\beta_j|^2. \tag{2.9}$$

Moreover from the above iterations we also note that $G_k = P_1 S^* P_2(G_{k-1})$, where P_1 and P_2 are the orthogonal projections with kernel $\langle E_1 \rangle$ and $\mathcal{W}$ respectively. Since $S \in C_{\cdot 0}$, $\dim \mathcal{D}_S < \infty$ and P_1 is an orthogonal projection with finite dimensional kernel, then by applying Lemma 2.3.3 we conclude that $SP_1 \in C_{\cdot 0}$. Furthermore, since $\dim \mathcal{D}_{SP_1} < \infty$ and P_2 is an orthogonal projection with finite dimensional kernel, then by applying Lemma 2.3.3 once again we conclude that $SP_1P_2 \in C_{\cdot 0}$. Again from the above iterations we note that

$$G_k = (P_1 S^* P_2)^k (F) = P_1 S^* (P_2 P_1 S^*)^{k-1} P_2(F) = P_1 S^* (S P_1 P_2)^{*k-1} P_2(F)$$

and hence $\|G_k\| \leq \|(SP_1P_2)^{*k-1}P_2(F)\| \to 0$, as $k \to \infty$. Consequently from the above equations (2.8) and (2.9) we can write

$$F(z) = F_0(z)K_0(z) + zk_1(z)E_1(z) \quad \text{and} \quad \|F\|^2 = \|K_0\|^2 + \|k_1\|^2, \tag{2.10}$$

where

$$K_0(z) = \sum_{j=0}^{\infty} A_j z^j, \; k_1(z) = \sum_{j=1}^{\infty} \beta_j z^{j-1}$$

and the sums converge in $H^2_{\mathbb{C}^r}(\mathbb{D})$ and $H^2_{\mathbb{C}}(\mathbb{D})$ norm respectively. Thus finally we say that if $F \in \mathcal{M}$ then

$$F = F_0 K_0 + z k_1 E_1,$$

where $(K_0, k_1) \in H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})$. Recall that $H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})$ can be identified with $H^2_{\mathbb{C}^{r+1}}(\mathbb{D})$. Define the subspace $\mathcal{K}$ of $H^2_{\mathbb{C}^{r+1}}(\mathbb{D})$ as follows:

$$\mathcal{K} = \Big\{ (K_0, k_1) \in H^2_{\mathbb{C}^{r+1}}(\mathbb{D}) : \exists \; F \in \mathcal{M} \text{ such that}$$

$$F = F_0K_0 + zk_1E_1\Big\}.$$

Then by using (2.10) we conclude that $\mathcal{K}$ is a closed subspace of $H^2_{\mathbb{C}^{r+1}}(\mathbb{D})$. Now it remains to show that $\mathcal{K}$ is $S^* \oplus \cdots \oplus S^*$-invariant in $H^2_{\mathbb{C}^{r+1}}(\mathbb{D})$. Indeed, let $(K_0, k_1) \in \mathcal{K}$. Then there exists F in $\mathcal{M}$ such that $F = F_0K_0 + zk_1E_1$. On the other hand

$$\begin{aligned} F &= F_0K_0 + zk_1E_1 = F_0A_0 + F_0(K_0 - K_0(0)) + zk_1E_1 \\ &= F_0A_0 + \{F_0(K_0 - K_0(0)) + z(k_1 - k_1(0))E_1 + zk_1(0)E_1\} \\ &= F_0A_0 + \underbrace{z(F_0S^*K_0 + zS^*k_1E_1)} + z\beta_1E_1, \end{aligned}$$

which along with equation (2.7) implies $F_0S^*K_0 + zS^*k_1E_1 = G_1 \in \mathcal{M}$, which proves that $\mathcal{K}$ is $\underbrace{S^* \oplus \cdots \oplus S^*}_{r+1}$-invariant.

Conversely, let $\mathcal{M}$ be a closed subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ which has a representation like (2.6). Let $F \in \mathcal{M}$ be such that $F(0) = 0$. Then there exists (K_0, k_1) in $\mathcal{K}$ such that $F = F_0K_0 + zk_1E_1$. Now it is easy to observe that $\{W_i(0)\}_{i=1}^r$ is linearly independent which follows from the fact that $\mathcal{W}$ is a linear space and all functions in $\mathcal{W}$ do not vanish at 0. On the other hand $F(0) = 0$ and the fact $\{W_i(0)\}_{i=1}^r$ is linearly independent together implies $K_0(0) = 0$. Thus

$$\begin{aligned} S^*F(z) &= \frac{F(z) - F(0)}{z} = \frac{F_0(z)K_0(z) + zk_1(z)E_1(z) - F_0(0)K_0(0)}{z} \\ &= F_0(z)S^*K_0(z) + (SS^*k_1(z))E_1(z) + k_1(0)E_1(z), \end{aligned}$$

and hence $S^*F = F_0S^*K_0 + zS^*k_1E_1 + k_1(0)E_1$. On the other hand since $\mathcal{K}$ is $S^* \oplus \cdots \oplus S^*$-invariant in $H^2_{\mathbb{C}^{r+1}}(\mathbb{D})$, then we have $(S^*K_0, S^*k_1) \in \mathcal{K}$ and hence $F_0S^*K_0 + zS^*k_1E_1 \in \mathcal{M}$. This shows that $S^*F \in \mathcal{M} \oplus \mathcal{F}$ and consequently $\mathcal{M}$ is nearly S^*- invariant with defect one.

(ii) If $\mathcal{M} \subseteq zH^2_{\mathbb{C}^m}(\mathbb{D})$, then $\mathcal{W} = \{0\}$. Therefore by applying the similar kind of algorithm as in (i), F can be written as $F(z) = zk_1(z)E_1(z)$ for some $H^2_{\mathbb{C}}(\mathbb{D})$ fucntion k_1. In other words $\mathcal{M}$ has the following representation.

$$\mathcal{M} = \{F : F(z) = zk_1(z)E_1(z) : k_1 \in \mathcal{K}\},$$

where $\mathcal{K}$ is a closed S^*- invariant subspace of the Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$ and $\|F\|^2 = \|k_1\|^2$. This completes the proof. □

In general for finite defect p the analogous calculations produce the following result.

Theorem 2.3.5. *Let $\mathcal{M}$ be a closed subspace that is nearly S^*-invariant with defect p in $H^2_{\mathbb{C}^m}(\mathbb{D})$ and let $\{E_1, E_2, ...E_p\}$ be any orthonormal basis for the p-dimensional defect space $\mathcal{F}$. Let $\{W_1, W_2, \dots, W_r\}$ be an orthonormal basis of $\mathcal{W} := \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$, and let F_0 be the $m \times r$ matrix whose columns are $W_1, W_2, \dots, W_r$. Then*

(i) in the case where there are functions in $\mathcal{M}$ that do not vanish at 0,

$$\mathcal{M} = \Big\{F : F(z) = F_0(z)K_0(z) + \sum_{j=1}^{p} zk_j(z)E_j(z) : (K_0, k_1, \dots, k_p) \in \mathcal{K}\Big\}, \tag{2.11}$$

where $\mathcal{K} \subset H^2_{\mathbb{C}^r}(\mathbb{D}) \times \underbrace{H^2_{\mathbb{C}}(\mathbb{D}) \times \cdots \times H^2_{\mathbb{C}}(\mathbb{D})}_{p}$ is a closed $S^* \oplus \cdots \oplus S^*$- invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$ and

$$\|F\|^2 = \|K_0\|^2 + \sum_{j=1}^{p} \|k_j\|^2.$$

(ii) In the case where all the functions in $\mathcal{M}$ vanish at 0,

$$\mathcal{M} = \Big\{F : F(z) = \sum_{j=1}^{p} zk_j(z)E_j(z) : (k_1, \dots, k_p) \in \mathcal{K}\Big\}, \tag{2.12}$$

with the same notion as in (i) except that $\mathcal{K}$ is now a closed $S^* \oplus \cdots \oplus S^*$- invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^p}(\mathbb{D})$ and

$$\|F\|^2 = \sum_{j=1}^{p} \|k_j\|^2.$$

Conversely, if a closed subspace $\mathcal{M}$ of the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$ has a representation like (i) or (ii) as above, then it is a nearly S^*-invariant subspace of defect p.

It is worth mentioning that, the above Theorem 2.3.5 is also obtained independently by R. O'Loughlin (see [44, Theorem 3.4]). Below we provide few differences between the proof given by us and R. O'Loughlin.

1. In the description of functions of a nearly S^* invariant subspace of defect 1 (just before the [44, Theorem 3.4]), O'Loughlin use [9, Proposition 2.1] and similar kind of argument given in [9] to show $||G_{n+1}|| \to 0$ in the convergence part.
   In our case, we use Lemma 2.3.3 given by Benhida-Timotin and directly prove such type of convergence in Theorem 2.3.4, providing the description of a nearly invariant subspace of defect 1.

2. We have used the fact $\{W_i(0)\}_{i=1}^r$ are linearly independent directly to prove that $\mathcal{K}$ is S^* invariant in the converse part of Theorem 2.3.4, whereas O'Loughlin proved that K is S^* invariant in a different way while using the fact $\{W_i(0)\}_{i=1}^r$ are linearly independent.

Corollary 2.3.1. *A closed subspace $\mathcal{M} \subset H^2_{\mathbb{C}^m}(\mathbb{D})$ is an almost invariant subspace for S^* with defect p if and only if it satisfies the conditions of the above Theorem 2.3.5 together with an extra condition that $S^*W_i \in \mathcal{M} \oplus \mathcal{F}$ for all $i = 1, 2, \ldots, r$ in case (i), while case (ii) is unchanged.*

Proof. If $\mathcal{M}$ is an almost invariant subspace for S^* with defect p, then it is nearly S^*-invariant subspace with defect p. Thus it satisfies the conditions of the above Theorem 2.3.5 and since $W_1, W_2, \ldots, W_r \in \mathcal{M}$, then from the hypothesis it follows that $S^*W_i \in \mathcal{M} \oplus \mathcal{F}$ for all $i = 1, 2, \ldots, r$. Conversely, assume that $\mathcal{M}$ satisfies the conditions of Theorem 2.3.5 together with $S^*W_i \in \mathcal{M} \oplus \mathcal{F}$ for all $i = 1, 2, \ldots, r$. Thus for any $F \in \mathcal{M}$ we have

$$F = F_0K_0 + \sum_{j=1}^{p} zk_jE_j = F_0K_0(0) + K,$$

where

$$K(z) = F_0(z)(K_0(z) - K_0(0)) + \sum_{j=1}^{p} zk_j(z)E_j(z).$$

Observe that $K(0) = 0$ and $F_0K_0(0) \in \mathcal{M}$ implies $K \in \mathcal{M}$. Since $\mathcal{M}$ is nearly S^* invariant with defect p, then $S^*K \in \mathcal{M} \oplus \mathcal{F}$ which along with the fact $S^*W_i \in \mathcal{M} \oplus \mathcal{F}$ for all $i = 1, \ldots, r$ implies $S^*F \in \mathcal{M} \oplus \mathcal{F}$ and hence $\mathcal{M}$ is an almost invariant with defect p. This completes the proof. □

Remark 2.3.6. *It is easy to observe that $S^*\mathcal{M} \subset \mathcal{M} \oplus \mathcal{F}$ is equivalent to the condition that $S(\mathcal{M} \oplus \mathcal{F})^\perp \subset (\mathcal{M} \oplus \mathcal{F})^\perp \oplus \mathcal{F}$. Therefore almost invariant subspaces for S can be characterized by S^*- invariant subspaces.*

2.4 Characterization of nearly invariant subspaces in terms of shift invariant subspaces

In this section, our main aim is to give a connection between nearly S^*-invariant subspaces and S-invariant subspaces using our main result in the previous section. In other words, we try to characterize $\mathcal{M}^\perp$ in terms of shift invariant subspaces, where $\mathcal{M}$ is a nearly S^*- invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ with finite defect p. Note that in general it is difficult to deal with the case when the defect has an orthonormal basis consist of arbitrary functions of $H^2_{\mathbb{C}^m}(\mathbb{D})$. Therefore we restrict ourselves in the special case when the orthonormal basis for the defect space are bounded analytic functions that is, $\mathcal{F} = span\{E_1, E_2, \dots, E_p\}$, where $E_i \in H^\infty_{\mathcal{L}(\mathbb{C},\mathbb{C}^m)}(\mathbb{D})$ for $i = 1, \dots, p$. Let $F_0 \in H^2_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$, and $T_{F_0} : H^2_{\mathbb{C}^r}(\mathbb{D}) \to H^2_{\mathbb{C}^m}(\mathbb{D})$ be an operator defined by $T_{F_0}(G) = P(F_0G)$, where P is the Fourier projection of the $L^1(\mathbb{T}, \mathbb{C}^m)$ function F_0G.

First we consider the case when $\mathcal{M}$ is a nearly S^*- invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ which contain functions that do not vanish at 0 with an extra assumption that $\mathcal{W} := \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$ has an orthonormal basis consist of $H^\infty_{\mathbb{C}^m}(\mathbb{D})$-functions, that is, $\{W_1, \dots, W_r\} \subseteq H^\infty_{\mathbb{C}^m}(\mathbb{D})$. Therefore $\mathcal{M}$ has the form (2.11) by Theorem 2.3.5 (i). Now consider $G \in \mathcal{M}^\perp$, then $\langle G, F\rangle = 0, \forall F \in \mathcal{M}$. But for any $F \in \mathcal{M}$ we have

$$F = F_0K_0 + \sum_{j=1}^{p} Sk_jE_j,$$

where $(K_0, k_1, \dots, k_p) \in \mathcal{K}$ and $\mathcal{K}$ is a S^*- invariant subspace of $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$. Therefore

$$\begin{aligned}\langle G, F\rangle = \langle G, F_0K_0\rangle_2 + \sum_{j=1}^{p}\langle G, Sk_jE_j\rangle_2 &= \langle G, T_{F_0}K_0\rangle_2 + \sum_{j=1}^{p}\langle S^*G, k_jE_j\rangle_2\\ &= \langle T^*_{F_0}G, K_0\rangle_{H^2_{\mathbb{C}^r}(\mathbb{D})} + \sum_{j=1}^{p}\langle T^*_{E_j}S^*G, k_j\rangle_{H^2_{\mathbb{C}}(\mathbb{D})},\end{aligned}$$

which ensures

$$G \in \mathcal{M}^\perp \textit{ if and only if } T^*_{F_0}G \oplus T^*_{E_1}S^*G \oplus T^*_{E_2}S^*G \oplus \cdots \oplus T^*_{E_p}S^*G \in \mathcal{K}^\perp.$$

Thus

$$\mathcal{M}^\perp = \Big\{G \in H^2_{\mathbb{C}^m}(\mathbb{D}) : T^*_{F_0}G \oplus T^*_{E_1}S^*G \oplus T^*_{E_2}S^*G \oplus \cdots \oplus T^*_{E_p}S^*G \in \mathcal{K}^\perp\Big\}, \tag{2.13}$$

where $\mathcal{K}^\perp$ is a S- invariant subspace of $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$. Conversely, if $\mathcal{M}$ is a closed subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ such that $\mathcal{M}^\perp$ is of the form (2.13), then $\mathcal{M}$ is a nearly S^*- invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ with defect p. Similarly we can also obtain the expression of $\mathcal{M}^\perp$ in the case when $\mathcal{M} \subset zH^2_{\mathbb{C}^m}(\mathbb{D})$. We can then formulate our main result in this section as follows.

Theorem 2.4.1. *Let $\mathcal{M}$ be a closed subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ which is nearly S^*- invariant with defect p with an extra condition that the orthonormal basis for both the defect space and the* *space $\mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$ are consist of bounded analytic functions. Then,*

(i) in the case where there are functions in $\mathcal{M}$ that do not vanish at 0,

$$\mathcal{M}^{\perp} = \left\{G \in H^2_{\mathbb{C}^m}(\mathbb{D}) : T^*_{F_0}G \oplus T^*_{E_1}S^*G \oplus T^*_{E_2}S^*G \oplus \cdots \oplus T^*_{E_p}S^*G \in \mathcal{N}\right\},$$

where $\{W_i\}_{i=1}^r$ is an orthonormal basis of $\mathcal{W} := \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$, F_0 is the $m \times r$ matrix whose columns are $W_1, W_2, \ldots, W_r$, the defect space $\mathcal{F}$ has an orthonormal basis

$$\{E_1, E_2, \ldots, E_p\} \subseteq H^{\infty}_{\mathcal{L}(\mathbb{C},\mathbb{C}^m)}(\mathbb{D}),$$

and $\mathcal{N} \subseteq H^2_{\mathbb{C}^r}(\mathbb{D}) \times \underbrace{H^2_{\mathbb{C}}(\mathbb{D}) \times \cdots \times H^2_{\mathbb{C}}(\mathbb{D})}_{p}$ is a closed $S \oplus \cdots \oplus S$- invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$.

(ii) In the case where all functions in $\mathcal{M}$ vanish at 0,

$$\mathcal{M}^{\perp} = \left\{G \in H^2_{\mathbb{C}^m}(\mathbb{D}) : T^*_{E_1}S^*G \oplus T^*_{E_2}S^*G \oplus \cdots \oplus T^*_{E_p}S^*G \in \mathcal{N}\right\},$$

where $\{E_1, E_2, ...E_p\} \subseteq H^{\infty}_{\mathcal{L}(\mathbb{C},\mathbb{C}^m)}(\mathbb{D})$ is an orthonormal basis for the p dimensional defect space $\mathcal{F}$, and $\mathcal{N} \subseteq \underbrace{H^2_{\mathbb{C}}(\mathbb{D}) \times \cdots \times H^2_{\mathbb{C}}(\mathbb{D})}_{p}$ is a closed $S \oplus \cdots \oplus S$- invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^p}(\mathbb{D})$. Conversely, if a closed subspace $\mathcal{M} \subset H^2_{\mathbb{C}^m}(\mathbb{D})$ is such that $\mathcal{M}^{\perp}$ has a representation as in (i) or (ii), then $\mathcal{M}$ is a nearly S^*- invariant subspace of defect p.

Finally we end the section with the following remark.

Remark 2.4.2. *In this section we describe nearly invariant subspaces under the backward shift with finite defect with an extra assumption that bounded analytic functions form an orthonormal basis of the defect space. But we do expect that the version of Theorem 2.4.1 still hold without this assumption.*

CHAPTER 3

Kernels of perturbed Toeplitz operators in vector valued Hardy spaces

3.1 Introduction

In Chapter 2, we characterizes nearly invariant subspace of finite defect (see Definition 2.3.1) for the backward shift operator acting on the vector valued Hardy space, providing a vectorial generalization of a result of Chalendar-Gallardo-Partington (C-G-P). A similar type of characterization result is also obtained independently by R. O'Loughlin in [44]. Furthermore, in this context, Liang and Partington [38] recently provide a connection between kernels of finite-rank perturbations of Toeplitz operators and nearly invariant subspaces with finite defect (see Definition 1.8.1 in Chapter 1) under the backward shift operator acting on the scalar valued Hardy space. In other words, they give an affirmative answer to the following question, which is closely related to the invariant subspace problem:

Given a Toeplitz operator T acting on the scalar valued Hardy space, is the kernel of a finite-rank perturbation of T is nearly backward shift invariant with finite defect? (3.1)

Moreover, they also identify the kernel of perturbed Toeplitz operator in terms of backward shift-invariant subspaces in several important cases by applying a recent theorem by Chalendar–Gallardo–Partington (C-G-P).

The purpose of this chapter is to study the kernels of finite-rank perturbations of Toeplitz operators and its connection with nearly invariant subspaces with finite defect under the backward shift operator acting on the vector valued Hardy space. In other words, we give an affirmative answer to the above question (3.1) in the context of vector valued Hardy space, providing a vectorial generalization of a result of Liang and Partington. Furthermore, we also identify the kernel of perturbed Toeplitz operator in terms of backward shift-invariant subspaces by applying Theorem 2.3.5 of Chapter 2 in connection with nearly invariant subspaces of finite defect for the backward shift operator acting on the vector valued Hardy space in several important cases as mentioned by Liang and Partington in [38]. For more information on this direction of research, we refer the reader to ([33, 16, 53]) and the references therein.

In Chapter 2, we have shown a connection between nearly S^*-invariant subspaces and S^* invariant subspaces in vector valued Hardy space. It can be easily seen that the kernel of a Toeplitz operator is nearly S^*-invariant (see Definition 1.7.2) in vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$ (1.5). Next we consider a finite rank perturbation (say rank n) of a Toeplitz operator $T_\Phi : H^2_{\mathbb{C}^m}(\mathbb{D}) \to H^2_{\mathbb{C}^m}(\mathbb{D})$, (see Definition 1.4.9 and for properties of T_Φ one is refer to see Section 1.4 in Chapter 1) denoted by T_n and defined by

$$T_n(F) = T_\Phi(F) + \sum_{i=1}^{n} \langle F, G_i \rangle H_i \ , \quad \forall F \in H^2_{\mathbb{C}^m}(\mathbb{D}),$$

where $\{G_i\}_{i=1}^n$ and $\{H_i\}_{i=1}^n$ are orthonormal sets in $H^2_{\mathbb{C}^m}(\mathbb{D})$. Therefore it is natural to ask whether the kernel of T_n is nearly S^*-invariant subspace with a finite defect or not. In this chapter, we provide an affirmative answer to this question in several important cases mentioned by Liang and Partington in [38]. In other words, we solve the above-mentioned problem (3.1) in various important cases in vector valued Hardy spaces, providing a vectorial generalization of a result of Liang and Partington [38]. For simplicity, we first discuss the problem for rank two perturbation, that is for T_2, and then we state our main theorem for rank n perturbation of T_Φ, that is for T_n.

The rest of the chapter is organized as follows: In Section 3.2, we study the kernel of T_n whenever $\Phi = 0$ almost everywhere on the circle and provide some applications of our earlier Theorem 2.3.5 of Chapter 2. Section 3.3, 3.4 and 3.5 deal with the study of the kernel of T_n in several important cases as mentioned by Liang and Partington [38] whenever Φ is non zero almost everywhere on the circle along with few applications of our earlier Theorem 2.3.5 of Chapter 2.

3.2 Kernel of finite rank perturbation of Toeplitz operator having symbol zero almost everywhere on the circle

In this section, we study the kernel of $T_n = T_\Phi + \sum_{i=1}^{n} \langle ., G_i \rangle H_i$, where $\Phi = 0$ almost everywhere on $\mathbb{T}$. As we have discussed earlier first we study the kernel of T_2 and later we will state the main theorem corresponding to T_n. Note that if $\Phi = 0$ almost everywhere on $\mathbb{T}$, then the kernel of T_2 is given by

$$\operatorname{Ker} T_2 = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \bigvee \{G_1, G_2\}.$$

It is easy to check that the kernel of T_2 is nearly S^*-invariant with defect 2 because if we consider any element $F \in \operatorname{Ker} T_2$ with $F(0) = 0$, then

$$S^*(F) \in \operatorname{Ker} T_2 \cup \left(\bigvee \{G_1, G_2\} \right) = H^2_{\mathbb{C}^m}(\mathbb{D}),$$

and the defect space is $\mathcal{F} = \bigvee \left\{ G_1, G_2 \right\}$. Now for the general case we have the following result:

Theorem 3.2.1. *Suppose $\Phi = 0$ almost everywhere on $\mathbb{T}$. Then the subspace $\operatorname{Ker} T_n$ is nearly S^*-invariant with defect n and the defect space is $\mathcal{F} = \bigvee \{G_1, G_2, ..., G_n\}$.*

Next, we provide a nice application of Theorem 2.3.5 obtained by us in Chapter 2 as well as obtained independently by Ryan O'Loughlin (see Theorem 3.4, [44]) to understand the kernel of perturbed Toeplitz operator in a better way in terms of backward shift-invariant subspaces.

For simplicity, we will deal with rank one perturbation of Toeplitz operator and let it be denoted by T , that is $T = T_\Phi + \langle ., G \rangle H$ with $\|G\|_2 = 1$ *and* $S^*H \neq 0$. Now as an application of the above Theorem 2.3.5 of Chapter 2, our aim is to represent the kernel of the operator T in some special cases. It should also be observed that, like the scalar case we can find $\mathcal{K}$ (see Theorem 2.3.5 given in Chapter 2) as the S^*-invariant subspace in $H^2_{\mathbb{C}^m}(\mathbb{D})$ such as

$$F_0(z) S^{*n} K_0(z) + z \sum_{j=1}^{p} S^{*n} k_j(z) E_j(z) \in \mathcal{M} \text{ or } z \sum_{j=1}^{p} S^{*n} k_j(z) E_j(z) \in \mathcal{M}$$

for all $n \in \mathbb{N} \cup \{0\}$ and $(K_0, k_1, \ldots, k_p) \in \mathcal{K}$. In the case $\Phi = 0$, we have $\mathcal{M} = \operatorname{Ker} T = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \langle G \rangle$ which is nearly S^*-invariant with defect space $\mathcal{F} = \langle G \rangle$ by Theorem 3.2.1. Assume furthermore that the function $G \in H^\infty_{\mathcal{L}(\mathbb{C},\mathbb{C}^m)}(\mathbb{D}) = H^\infty_{\mathbb{C}^m}(\mathbb{D})$. Now consider $F_i = P_{\mathcal{M}}(k_0 \otimes e_i)$, where $P_{\mathcal{M}}$ denotes the orthogonal projection from $H^2_{\mathbb{C}^m}(\mathbb{D})$ onto $\mathcal{M}$, and k_0 is the reproducing kernel at 0 and $\{e_i : 1 \leq i \leq m\}$ is a standard orthonormal basis of $\mathbb{C}^m$, generating the subspace $\mathcal{W}$ in Theorem 2.3.5 of Chapter 2. Without loss of generality we assume that $\{F_1, F_2, \dots F_r\}$ (*where* $r \leq m$) is a basis of $\mathcal{W}$. By using Gram-Schmidt orthonormalization we find an orthonormal basis of $\mathcal{W}$ as follows: $W_1 = C_{11}F_1,\ W_2 = C_{21}F_1 + C_{22}F_2,\ \dots,\ W_r = C_{r1}F_1 + C_{r2}F_2 + \cdots + C_{rr}F_r$, where the constant C_{ij} can be determined via the process of orthonormalization. Now if we consider $G = (g_1, g_2, \dots, g_m) \in H^\infty_{\mathbb{C}^m}(\mathbb{D})$ with $\|G\|_2 = 1$ and $\mathcal{M} = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \langle G\rangle$, then for any $i \in \{1, 2, \dots, m\}$ we have

$$F_i = P_{\mathcal{M}}(k_0 \otimes e_i) = (-\overline{g_i(0)}g_1, -\overline{g_i(0)}g_2, \dots, 1 - \overline{g_i(0)}g_i, \dots, -\overline{g_i(0)}g_m).$$

Therefore by Theorem 2.3.5 of Chapter 2, the $m\times r$ matrix F_0 whose columns are $\{W_1, W_2, \dots, W_r\}$ has the following representation

$$F_0 = \begin{bmatrix} C_{11}(1-\overline{g_1(0)}g_1) & C_{21}(1-\overline{g_1(0)}g_1)+C_{22}(-\overline{g_2(0)}g_1) & \cdots & C_{r1}(1-\overline{g_1(0)}g_1)+\cdots+C_{rr}(-\overline{g_r(0)}g_1) \\ C_{11}(-\overline{g_1(0)}g_2) & C_{21}(-\overline{g_1(0)}g_2)+C_{22}(1-\overline{g_2(0)}g_2) & \cdots & C_{r1}(-\overline{g_1(0)}g_2)+\cdots+C_{rr}(-\overline{g_r(0)}g_2) \\ \vdots & \vdots & \ddots & \vdots \\ C_{11}(-\overline{g_1(0)}g_m) & C_{21}(-\overline{g_1(0)}g_m)+C_{22}(-\overline{g_2(0)}g_m) & \cdots & C_{r1}(-\overline{g_1(0)}g_m)+\cdots+C_{rr}(-\overline{g_r(0)}g_m) \end{bmatrix}_{m\times r}. \quad (3.2)$$

Therefore, according to Theorem 2.3.5 of Chapter 2, $\mathcal{M}$ has the following representation :

(i) In the case when $\mathcal{W} \neq \{0\}$, $\mathcal{M} = \Big\{F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subseteq H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\}$. Suppose $|G|^2 = |g_1|^2 + |g_2|^2 + \ldots + |g_m|^2$ and if we consider

$$G_0 = \begin{bmatrix} \overline{C}_{11}P(g_1 - g_1(0)|G|^2) \\ \overline{C}_{21}P(g_1 - g_1(0)|G|^2) + \overline{C}_{22}P(g_2 - g_2(0)|G|^2) \\ \vdots \\ \overline{C}_{r1}P(g_1 - g_1(0)|G|^2) + \cdots + \overline{C}_{rr}P(g_r - g_r(0)|G|^2) \end{bmatrix} \in H^2_{\mathbb{C}^r}(\mathbb{D}) \quad (3.3)$$

and $g = P(\overline{z}|G|^2) \in H^2_{\mathbb{C}}(\mathbb{D})$ (here, P is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ mentioned in Definition 1.4.1), then the $S^* \oplus S^*$-invariant subspace corresponding to $\mathcal{M}$ is :

$$\mathcal{K} = \Big\{(K_0, k_1) \in H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D}) : \langle K_0, z^nG_0\rangle_{H^2_{\mathbb{C}^r}(\mathbb{D})} + \langle k_1, z^ng\rangle_{H^2_{\mathbb{C}}(\mathbb{D})} = 0 \ for\ n \in \mathbb{N} \cup \{0\}\Big\}.$$

(ii) In case $\mathcal{W} = \{0\}$, $\mathcal{M} = \Big\{F : F(z) = zk_1(z)G(z) : k_1 \in \mathcal{K}\Big\}$ with S^* invariant subspace is $\mathcal{K} = \Big\{k_1 \in H^2_{\mathbb{C}}(\mathbb{D}) : \langle k_1, z^ng\rangle_{H^2_{\mathbb{C}}(\mathbb{D})} = 0 \ for\ n \in \mathbb{N} \cup \{0\}\Big\}$.

Remark 3.2.2. *If* $G = (g_1, g_2, \dots, g_m)$ *is an arbitrary element of* $H^2_{\mathbb{C}^m}(\mathbb{D})$*, then* $\mathcal{M} = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \langle G \rangle$ *will be of the form :*

1. *In the case when* $\mathcal{W} \neq \{0\}$,

$$\mathcal{M} = \Big\{F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subseteq H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\},$$

where the $S^* \oplus S^*$*-invariant subspace corresponding to* $\mathcal{M}$ *is*

$$\mathcal{K} = \big\{(K_0, k_1) \in H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D}) : \langle F_0 S^{*n} K_0 + zS^{*n} k_1 G, G\rangle = 0 \ for \ n \in \mathbb{N} \cup \{0\}\big\}.$$

2. *In case* $\mathcal{W} = \{0\}$, $\mathcal{M} = \Big\{F : F(z) = zk_1(z)G(z) : k_1 \in \mathcal{K}\Big\}$*, where the* S^* *invariant subspace corresponding to* $\mathcal{M}$ *is* $\mathcal{K} = \Big\{k_1 \in H^2_{\mathbb{C}}(\mathbb{D}) : \langle zS^{*n} k_1 G, G\rangle = 0 \ for \ n \in \mathbb{N} \cup \{0\}\Big\}$.

Next, we give some concrete examples without any detail which are based on the examples given [38] (see Section 3) with minor modifications through which we calculate the space $\mathcal{K}$ explicitly. To proceed further, we need the following useful result in the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$ setting.

Proposition 3.2.3. *Let* $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}^m)}(\mathbb{D}))\backslash\{0\}$ *be of the form*

$$\Phi = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_m \end{bmatrix}_{m\times m,} \tag{3.4}$$

and let Θ *be the inner part of* Φ*. Then* $\operatorname{Ker} T_{\Phi^*} = \mathcal{K}_\Theta = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$*. In particular if,*

$$\Psi = \begin{bmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_m \end{bmatrix}_{m\times m,} \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D}) \tag{3.5}$$

with each ψ_i *is an outer function in* $H^2_{\mathbb{C}}(\mathbb{D})$*, then* T_{Ψ^*} *is an injective Toeplitz operator.*

Proof. Let Ψ be of the above form with each ψ_i is an outer function and consider the element $F = (f_1, f_2, \dots, f_m) \in \operatorname{Ker} T_{\Psi^*}$. Then for all $i \in \{1, 2, \dots, m\}$, $P(\overline{\psi}_i f_i) = 0$ which implies that

$\overline{\psi}_i f_i \in \overline{H}_0^2$. Since each ψ_i is outer, then $f_i \in \overline{H}_0^2$ and hence each $f_i = 0$. Therefore $F = 0$ and hence T_{Ψ^*} is injective. Now using the canonical factorization of each $\phi_i = \psi_i\theta_i$, where ψ_i is outer and θ_i is inner, we have the following decomposition of Φ:

$$\Phi = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_m \end{bmatrix} = \begin{bmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_m \end{bmatrix} \times \begin{bmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_m \end{bmatrix} = \Psi \cdot \Theta \text{ (say)}. \tag{3.6}$$

Therefore Θ is an inner multiplier, and Ψ is an outer function in the sense of V. I. Smirnov [36]. We call Θ as the inner part of Φ and Ψ as the outer part of Φ. Note that

$$\operatorname{Ker} T_{\Phi^*} = \left\{F \in H^2_{\mathbb{C}^m}(\mathbb{D}) : T_{\Psi^*}T_{\Theta^*}(F) = 0\right\} = \left\{F \in H^2_{\mathbb{C}^m}(\mathbb{D}) : T_{\Theta^*}(F) = 0\right\} = \mathcal{K}_\Theta,$$

where at the second equality we have used the fact that T_{Ψ^*} is injective. This completes the proof. □

Example 3.2.4. *(i) Let* $G = (1, 0, \ldots, 0) = k_0 \otimes e_1$, *then*

$$\mathcal{M} = span\{e_2, e_3, \ldots, e_m\} \oplus zH^2_{\mathbb{C}^m}(\mathbb{D}).$$

In this case, $\mathcal{W} = span\{e_2, e_3, \ldots, e_m\}$ *and consider*

$$F_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{m\times(m-1)}. \tag{3.7}$$

Therefore

$$\mathcal{M} = \left\{F : F(z) = F_0(z)K_0(z) + zk_1(z) \otimes e_1 : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}^{m-1}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\right\},$$

where $\mathcal{K} = H^2_{\mathbb{C}^m}(\mathbb{D})$ *is a trivial* S^**-invariant subspace.*

(ii) Let $G = \frac{1}{\sqrt{m}}(\theta_1, \theta_2, \ldots, \theta_m)$ *with each* θ_i *a non constant inner function in* $H^2_{\mathbb{C}}(\mathbb{D})$ *and each* $\theta_i(0) = 0$. *If*

$$\Theta = \begin{bmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_m \end{bmatrix}_{m\times m}, \tag{3.8}$$

then

$$\mathcal{M} = K_\Theta \oplus z\Theta H^2_{\mathbb{C}^m}(\mathbb{D}) \oplus \Theta(\mathbb{C}^m \ominus \langle(1,1,\dots,1)\rangle).$$

In this case,

$$F_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{m\times m}, \tag{3.9}$$

$$G_0 = \begin{bmatrix} \frac{1}{\sqrt{m}}\theta_1 \\ \frac{1}{\sqrt{m}}\theta_2 \\ \vdots \\ \frac{1}{\sqrt{m}}\theta_m \end{bmatrix} \in H^2_{\mathbb{C}^m}(\mathbb{D}), \; g = 0. \tag{3.10}$$

Therefore by using the case(i), $\mathcal{M}$ has the following representation:

$$\mathcal{M} = \Big\{F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0,k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}^m}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\}$$

with the S^*-invariant subspace $\mathcal{K} = \operatorname{Ker} T_{\xi^*} \times H^2_{\mathbb{C}}(\mathbb{D})$, where $\xi \in H^\infty_{\mathcal{L}(\mathbb{C},\mathbb{C}^m)}(\mathbb{D})$ is given by

$$\xi = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{bmatrix}_{m\times 1}. \tag{3.11}$$

Using the fact (1.8) one can easily check that $\mathcal{K}$ is S^*-invariant subspace of $H^2_{\mathbb{C}^{m+1}}(\mathbb{D})$.

(iii) If we consider $G(z) = (\frac{1+z^k}{\sqrt{2}}, 0, \dots, 0)$ for $k \geq 1$, then

$$\mathcal{M} = span\Big\{1 - z^k, z, z^2, \dots z^{k-1}, z^{k+1}, \dots\Big\} \oplus \underbrace{H^2_{\mathbb{C}}(\mathbb{D}) \oplus H^2_{\mathbb{C}}(\mathbb{D}) \oplus \dots \oplus H^2_{\mathbb{C}}(\mathbb{D})}_{m-1}.$$

In this case,

$$F_0 = \begin{bmatrix} \frac{1-z^k}{\sqrt{2}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{m\times m}, G_0 = \begin{bmatrix} \frac{z^k}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in H^2_{\mathbb{C}^m}(\mathbb{D}), \quad and \quad g(z) = \frac{z^{k-1}}{2} \in H^2_{\mathbb{C}}(\mathbb{D}).$$

Thus by using case(i),

$$\mathcal{M} = \left\{F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}^m}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\right\}$$

with the S^* invariant subspace

$$\mathcal{K} = \left\{(K_0, k_1) \in H^2_{\mathbb{C}^m}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D}) : (S^*)^{k-1}k_1(z) = -\langle (S^*)^k K_0, k_z \otimes e_1\rangle, K_0 \in H^2_{\mathbb{C}^m}(\mathbb{D})\right\}.$$

(iv) *Now consider $G(z) = \sqrt{\frac{1-|\alpha|^2}{m}}(k_\alpha(z), k_\alpha(z), \dots, k_\alpha(z))$, where k_α is a reproducing kernel at $\alpha \in \mathbb{D}\backslash\{0\}$ in $H^2_{\mathbb{C}}(\mathbb{D})$. Then $\mathcal{M} = \left\{F = (f_1, f_2, \dots, f_m) \in H^2_{\mathbb{C}^m}(\mathbb{D}) : \sum_{i=1}^m f_i(\alpha) = 0\right\}$. Since $P_{\mathcal{M}}(k_0 \otimes e_1)$ is non zero, then $\mathcal{W}$ is non trivial and hence F_0 has the same form as in (3.2) with each $g_i(z) = k_\alpha(z)$. In this case $G_0 = 0$, $g(z) = m\overline{\alpha}k_\alpha(z)$ which is an outer function. Moreover by using case(i), $\mathcal{M} = \left\{F : F(z) = F_0(z)K_0(z) + zk_1(z)G(z) : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\right\}$ with the S^*-invariant subspace $\mathcal{K} = H^2_{\mathbb{C}^r}(\mathbb{D}) \times \{0\}$.*

For the case $n = m$ we have the following interesting example in this context.

Example 3.2.5. *If we consider $T_m = T_\Phi + \sum_{i=1}^m \langle ., G_i\rangle H_i$ with $\Phi = 0$ almost everywhere on $\mathbb{T}$. Then the defect space $\mathcal{F} = span\{G_1, G_2, \dots, G_m\}$. Consider $G_1 = \theta_1 \otimes e_1, G_2 = \theta_2 \otimes e_2, \dots, G_m = \theta_m \otimes e_m$, where each θ_i is a non constant inner function in $H^2_{\mathbb{C}}(\mathbb{D})$. Let*

$$\Theta = \begin{bmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_m \end{bmatrix}_{m\times m}. \tag{3.12}$$

Therefore $\mathcal{M} = \operatorname{Ker} T_m = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \mathcal{F} = K_\Theta \oplus z\Theta H^2_{\mathbb{C}^m}(\mathbb{D})$. Thus, $\mathcal{W} \neq \{0\}$ and consider

$$F_0 = \begin{bmatrix} \frac{1-\overline{\theta_1(0)}\theta_1}{1-|\theta_1(0)|^2} & 0 & \cdots & 0 \\ 0 & \frac{1-\overline{\theta_2(0)}\theta_2}{1-|\theta_2(0)|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1-\overline{\theta_m(0)}\theta_m}{1-|\theta_m(0)|^2} \end{bmatrix}_{m\times m}.$$

Therefore by using case (i) of Theorem 2.3.5 of Chapter 2 we have

$$\mathcal{M} = \left\{F : F(z) = F_0(z)K_0(z) + \sum_{i=1}^m zk_i(z)G_i(z) : (K_0, k_1, k_2, \dots, k_m) \in \mathcal{K} \subseteq H^2_{\mathbb{C}^m}(\mathbb{D}) \times H^2_{\mathbb{C}^m}(\mathbb{D})\right\},$$

where the $S^* \oplus S^* \cdots \oplus S^*$-invariant subspace $\mathcal{K}$ corresponding to $\mathcal{M}$ is $\mathcal{K} = K_\zeta \times H^2_{\mathbb{C}^m}(\mathbb{D})$ and ζ is the inner part of the $H^\infty_{\mathcal{L}(\mathbb{C}^m)}(\mathbb{D})$ function

$$\xi = \begin{bmatrix} \theta_1 - \theta_1(0) & 0 & \cdots & 0 \\ 0 & \theta_2 - \theta_2(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_m - \theta_m(0) \end{bmatrix}_{m\times m}.$$

One can check that $\mathcal{K}$ is $S^* \oplus S^* \cdots \oplus S^*$-invariant subspace of $H^2_{\mathbb{C}^{2m}}(\mathbb{D})$ by using Proposition 3.2.3.

Next we consider the case when $\Phi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ is non zero almost everywhere on $\mathbb{T}$ and with this assumption we consider three important subcases in the next three sections. Note that if $\Phi = [\phi_{ij}]_{m\times m} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$, then $\Phi^*(z) = [\overline{\phi_{ij}(z)}]^t$. To proceed further we need the following useful results :

Theorem 3.2.6. *For $\Phi, \Psi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$; if either $\Psi^* \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D})$ or $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D})$, then $T_\Psi T_\Phi$ is a Toeplitz operator; in both cases $T_\Psi T_\Phi = T_{\Psi\Phi}$.*

The proof of the above Theorem 3.2.6 follows similarly as in the scalar valued case and hence left it to the reader. On the other hand, it is important to observe that the converse of the above Theorem 3.2.6 is not true in general. For example, consider

$$\Psi = \begin{bmatrix} e^{i\theta} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times m,} , \text{ and } \Phi = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{-i\theta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times m}. \tag{3.13}$$

Then it is easy to observe that $\Psi \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D})$ and $\Phi^* \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D})$ but $T_\Psi T_\Phi = 0$ is a Toeplitz operator. Now we denote

$$\mathbf{Z} = \begin{bmatrix} z & 0 & \cdots & 0 \\ 0 & z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z \end{bmatrix}_{m\times m,} \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D}) \quad \text{and hence} \quad T^*_{\mathbf{Z}} = T_{\mathbf{Z}^*}. \tag{3.14}$$

Thus

$$T_{\mathbf{Z}^*} T_\Phi = T_{\mathbf{Z}^*\mathbf{\Phi}} = T_{\Phi\mathbf{Z}^*}\ ,\ \forall\ \Phi \in L^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)).$$

Next we provide an equivalent condition on the element F to be in the kernel of $T_2 = T_\Phi + \sum_{i=1}^{2}\langle ., G_i\rangle H_i$. To do that suppose $F \in \operatorname{Ker} T_2$ with $F(0) = 0$. Then the following equivalent conditions hold.

$$T_2(F) = 0 \Leftrightarrow \; T_\Phi(F) + \sum_{i=1}^{2}\langle F, G_i\rangle H_i = 0 \tag{3.15}$$

$$\Leftrightarrow \; P_m\Big(\Phi F + \sum_{i=1}^{2}\langle F, G_i\rangle H_i\Big) = 0 \Leftrightarrow \; \Phi F + \sum_{i=1}^{2}\langle F, G_i\rangle H_i \in \overline{H}^2_0.$$

Applying $T^*_{\mathbf{Z}}$ on both sides of (3.15) and using Theorem 3.2.6, we have the following equivalent conditions.

$$T_{\Phi\mathbf{Z}^*}(F) + \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i = 0 \Leftrightarrow \; P_m\Big(\Phi\frac{F}{z} + \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i\Big) = 0$$

$$\Leftrightarrow \; T_\Phi\big(\frac{F}{z}\big) + \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i = 0 \Leftrightarrow \; \Phi\frac{F}{z} + \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i \in \overline{H}^2_0. \tag{3.16}$$

Now if we recapitulate our problem, actually we have to show the kernel of T_2 is nearly S^*-invariant with finite defect and to do so we have to find a vector V in some appropriate finite dimensional subspace $\mathcal{F}$ such that

$$S^*F + V \in \operatorname{Ker} T_2 \quad \text{with} \quad F \in \operatorname{Ker} T_2,\; F(0) = 0,$$

which is equivalent to the following equations

$$T_2(S^*F + V) = 0 \Leftrightarrow \; T_\Phi(\frac{F}{z} + V) + \sum_{i=1}^{2}\langle \frac{F}{z} + V, G_i\rangle H_i = 0 \tag{3.17}$$

$$\Leftrightarrow \; P_m\Big(\Phi(\frac{F}{z} + V) + \sum_{i=1}^{2}\langle \frac{F}{z} + V, G_i\rangle H_i\Big) = 0$$

$$\Leftrightarrow \; P_m\Big(\Phi V - \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i + \sum_{i=1}^{2}\langle \frac{F}{z} + V, G_i\rangle H_i\Big) = 0, (\text{using (3.16)}) \tag{3.18}$$

$$\Leftrightarrow \; \Phi V - \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i + \sum_{i=1}^{2}\langle \frac{F}{z} + V, G_i\rangle H_i \in \overline{H}^2_0. \tag{3.19}$$

In the following three sections, we are going to show that kernel of T_2 is nearly S^*-invariant with a finite defect in various important cases mentioned by Liang and Partington in [38]. Furthermore, we also calculate the defect space $\mathcal{F}$ explicitly in those mentioned cases.

3.3 Kernel of finite rank perturbation of Toeplitz operator having symbol an inner multiplier

In this section, we consider $\Phi = \Theta$ is an inner function. Our main aim in this section is to show that the kernel of $T_n = T_\Theta + \sum_{i=1}^{n} \langle ., G_i \rangle H_i$ is nearly S^*-invariant with finite defect and to calculate the defect space explicitly. To avoid complicacies in the calculations, we restrict our self in the case $n = 2$ and finally we state our result in general setting. For this purpose let us consider $F \in \operatorname{Ker} T_2$ with $F(0) = 0$. Then the equation (3.15) can be rewritten as,

$$\Theta F + \sum_{i=1}^{2} \langle F, G_i \rangle H_i = 0.$$

In this context, the equation (3.16) becomes

$$\Theta \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i = 0. \tag{3.20}$$

Next by acting T_Θ^* on both sides of (3.20) we have

$$T_{\Theta^*} \Big(\sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \Big) = \Theta^* \Big(\sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \Big) = -\frac{F}{z} \in H^2_{\mathbb{C}^m}(\mathbb{D}).$$

Therefore, (3.17) becomes $\Theta \left(\frac{F}{z} + V \right) + \sum_{i=1}^{2} \left\langle \frac{F}{z} + V, G_i \right\rangle H_i = 0$. Using (3.20), the above equation is equivalent to

$$\Theta V - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \sum_{i=1}^{2} \langle \frac{F}{z} + V, G_i \rangle H_i = 0. \tag{3.21}$$

Let $V \in H^2_{\mathbb{C}^m}(\mathbb{D})$ be such that

$$\Theta V = \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \in H^2_{\mathbb{C}^m}(\mathbb{D}).$$

The above yields that

$$V = \sum_{i=1}^{2} \langle F, G_i \rangle T_{\Theta^*}(S^* H_i) = \sum_{i=1}^{2} \langle F, G_i \rangle S^* (T_{\Theta^*}(H_i)), \tag{3.22}$$

because $T_{\Theta^*} S^* = T_{\Theta^*} T_{\overline{\mathbf{z}}} = T_{\overline{\mathbf{z}}} T_{\Theta^*} = S^* T_{\Theta^*}$ and therefore by using (3.20) we conclude that V satisfies the above equation (3.21), that is

$$\sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \sum_{i=1}^{2} \langle \frac{F}{z} + V, G_i \rangle H_i$$

$$= \sum_{i=1}^{2} \Big\langle \Theta(\frac{F}{z} + V), \Theta G_i \Big\rangle H_i$$
$$= \sum_{i=1}^{2} \Big\langle \Theta\frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i, \Theta G_i \Big\rangle H_i = 0.$$

Using the above construction of V in (3.22) we define the defect space

$$\mathcal{F} = \bigvee_{i=1}^{2} \Big\{ T_{\Theta^*}(S^* H_i) \Big\} = \bigvee_{i=1}^{2} \Big\{ S^*(T_{\Theta^*} H_i) \Big\},$$

with dimension at most 2. Hence $\operatorname{Ker} T_2$ is nearly S^*-invariant with defect at most 2. Therefore by repeating the above calculations again we have the following theorem regarding the kernel of T_n.

Theorem 3.3.1. *If $\Phi = \Theta$ is an inner multiplier, then the subspace $\operatorname{Ker} T_n$ is nearly S^*-invariant subspace with defect at most n and the defect space is*

$$\mathcal{F} = \bigvee_{i=1}^{n} \Big\{ T_{\Theta^*}(S^* H_i) \Big\} = \bigvee_{i=1}^{n} \Big\{ S^*(T_{\Theta^*} H_i) \Big\}.$$

The following example is similar to example given in [38] (see Section 2) for scalar valued case.

Example 3.3.2. *If* $\Phi(z) = \begin{bmatrix} z^p & 0 & \cdots & 0 \\ 0 & z^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^p \end{bmatrix}_{m \times m,}$ *, $p \in \mathbb{N}$ and $n = 1$.. Then $\operatorname{Ker} T_1$ is nearly S^*-invariant subspace with defect 1 and the defect space is $\mathcal{F} = \big\langle (S^*)^{p+1} H_1 \big\rangle$.*

Now coming back to the application part of Theorem 2.3.5 of Chapter 2, as discussed earlier we deal with rank one perturbation of Toeplitz operator. Thus $T = T_\Theta + \langle ., G \rangle H$, with $\|G\|_2 = 1$ and $S^* H \neq 0$. Therefore $\mathcal{M} = \operatorname{Ker} T \subseteq \langle \Theta^* H \rangle$ *and* $\mathcal{F} = \langle S^*(T_{\Theta^*} H) \rangle = \langle S^*(\Theta^* H) \rangle$. Next we consider $F = \mu \Theta^* H \in \mathcal{M}$ satisfying $T(F) = 0$ which is equivalent to

$$\mu(1 + \langle \Theta^* H, G \rangle) = 0.$$

Therefore we have the following two cases to consider:

Case 1. If $1 + \langle \Theta^* H, G \rangle \neq 0$, then $\mathcal{M} = \{0\}$ is a trivial S^*-invariant subspace.

Case 2. If $1 + \langle \Theta^* H, G \rangle = 0$, then it yields $\mathcal{M} = \langle \Theta^* H \rangle$. Let $\{A_k\}_{k=0}^{\infty}$ be the coefficients of the Taylor series expansion of Θ^*H (since $\Theta^*H \in H^2_{\mathbb{C}^m}(\mathbb{D})$) and we calculate the subspace $\mathcal{K}$ in two sub-cases.

(i) If $A_0 \neq 0$, then there exists at least one function in $\mathcal{M}$ that do not vanish at 0. Now $\mathcal{W} = \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$ has dimension 1 (since dim $\mathcal{M} = 1$). Therefore exactly one $F_i \neq 0$ for some i $(1 \leq i \leq m)$ which generates $\mathcal{W}$. On the other hand

$$F_i = P_{\mathcal{M}}(k_0 \otimes e_i) = \frac{\langle k_0 \otimes e_i, \Theta^*H\rangle \Theta^*H}{\|\Theta^*H\|_2^2},$$

and hence $F_0 = \Big[C_{11}F_i\Big]_{m\times 1}$. Thus

$$\begin{aligned}
\mathcal{M} &= \langle \Theta^*H\rangle \\
&= \Big\{F : F = F_0K_0 + zk_1\frac{S^*(\Theta^*H)}{\|S^*(\Theta^*H)\|} : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\} \\
&= \Big\{F : F = K_0C_{11}\frac{\langle k_0 \otimes e_i, \Theta^*H\rangle \Theta^*H}{\|\Theta^*H\|_2^2} + \frac{k_1(\Theta^*H - A_0)}{\|S^*(\Theta^*H)\|} : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\}.
\end{aligned}$$

Note that the structure of $\mathcal{M}$ forces to conclude that $K_0 \in \mathbb{C}$ and $k_1(\Theta^*H - A_0) = \xi\Theta^*H$ with $\xi \in H^2_{\mathbb{C}}(\mathbb{D})$ which will be valid if and only if $K_0 \in \mathbb{C}$ and $k_1 = 0$. Therefore the required nearly S^*-invariant subspace of finite defect is

$$\mathcal{M} = \Big\{F : F = K_0C_{11}\frac{\langle k_0 \otimes e_i, \Theta^*H\rangle}{\|\Theta^*H\|_2^2}\Theta^*H : (K_0, 0) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\},$$

and the corresponding $S^* \oplus S^*$-invariant subspace is $\mathcal{K} = \mathbb{C} \times \{0\}$ of $H^2_{\mathbb{C}^2}(\mathbb{D})$.

(ii) If $A_0 = 0$, then $\mathcal{W} = \{0\}$ and hence $F_0 = 0$. In that case the nearly S^*-invariant subspace of finite defect is

$$\mathcal{M} = \langle \Theta^*H\rangle = \Big\{F : F = k_1\frac{\Theta^*H}{\|S^*(\Theta^*H)\|} : k_1 \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D})\Big\}$$

with the associated S^*-invariant subspace $\mathcal{K} = \mathbb{C}$ of $H^2_{\mathbb{C}}(\mathbb{D})$.

3.4 Kernel of finite rank perturbation of Toeplitz operator with symbol having factorization in $\mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m))$

In this section, we deal with very special type of $\Phi \in L^\infty(T, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ such that $\Phi = F_1^*F_2$, with $F_j \in \mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ for $j = 1, 2$. Here $\mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ denotes the set of all invertible elements in $H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D})$. For more details in the literature we refer to [4] and the references cited therein. Note that the commutativity property makes a major difference while dealing with the vector valued case in comparison to scalar valued case. Indeed, in our setting we do not have the commutativity between F_1 and F_2 whereas in the scalar valued case they do so (see Section 2.3). Therefore it is essential to give little more details of the analysis in our setting for better understanding of the reader. Note that for any vector $F \in \operatorname{Ker} T_2$ with $F(0) = 0$, (3.15) can be rewritten as

$$T_{F_1^*}T_{F_2}(F) + \sum_{i=1}^{2}\langle F, G_i\rangle H_i = 0. \tag{3.23}$$

Since $F(0) = 0$ and $T_{\bar{\mathbf{z}}}T_{F_1^*} = T_{F_1^*}T_{\bar{\mathbf{z}}}$, then by using these fact along with the action of $T_{\mathbf{Z}^*}$ on both sides of (3.23) we get

$$T_{F_1^*}(F_2\frac{F}{z}) + \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i = 0. \tag{3.24}$$

Now by applying $T_{F_2^{-1}}T_{F_1^{*-1}}$ on both sides of (3.24) we have

$$\frac{F}{z} + \sum_{i=1}^{2}\langle F, G_i\rangle T_{F_2^{-1}}T_{F_1^{*-1}}(S^*H_i) = 0. \tag{3.25}$$

Thus by using (3.24) our desired result (3.17) is equivalent to

$$T_{F_1^*}T_{F_2}(V) - \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i + \sum_{i=1}^{2}\langle \frac{F}{z} + V, G_i\rangle H_i = 0. \tag{3.26}$$

Now we choose V in such a way so that

$$T_{F_1^*}(F_2V) = \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i \Leftrightarrow \quad V = \sum_{i=1}^{2}\langle F, G_i\rangle T_{F_2^{-1}}T_{F_1^{*-1}}(S^*H_i).$$

Next if we consider the above V, then by using (3.25) the left hand side of (3.26) becomes

$$\sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i - \sum_{i=1}^{2}\langle F, G_i\rangle S^*H_i + \sum_{i=1}^{2}\left\langle \frac{F}{z} + \sum_{i=1}^{2}\langle F, G_i\rangle T_{F_2^{-1}}T_{F_1^{*-1}}(S^*H_i), G_i\right\rangle H_i = 0.$$

Moreover from the choice of V it is clear that the defect space should be $\mathcal{F} = \bigvee_{i=1}^{2}\left\{F_2^{-1}S^*(T_{F_1^{*-1}}H_i)\right\}$ with dimension at most 2. Therefore in general setting we have the following theorem regarding the kernel of T_n:

Theorem 3.4.1. *Let $\Phi = F_1^* F_2$ with $F_1, F_2 \in \mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m))$. Then the subspace* $\operatorname{Ker} T_n$ *is nearly S^*-invariant with defect at most n and the defect space is*

$$\mathcal{F} = \bigvee_{i=1}^{n} \left\{ F_2^{-1} T_{F_1^{*-1}}(S^* H_i) \right\} = \bigvee_{i=1}^{n} \left\{ F_2^{-1} S^*(T_{F_1^{*-1}} H_i) \right\}.$$

Remark 3.4.2.

1. *If we suppose $\Phi^* \in \mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m))$, then the kernel of T_n is nearly S^*-invariant with defect at most n and the defect space is $\mathcal{F} = \bigvee_{i=1}^{n} \left\{ S^*(T_{\Phi^{-1}} H_i) \right\}$.*
2. *If we consider $\Phi \in \mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m))$, then the kernel of T_n is also nearly S^*-invariant with defect at most n and the defect space will be $\mathcal{F} = \bigvee_{i=1}^{n} \left\{ T_{\Phi^{-1}} S^*(H_i) \right\}$.*

For simplicity if we assume $n = 1$ and $\Phi \in \mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m))$, then we have the following corollary.

Corollary 3.4.1. *For $n = 1$ and $\Phi \in \mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m))$, the subspace* $\operatorname{Ker} T_1$ *(i.e.,* $\operatorname{Ker} T$*) is nearly S^*-invariant with defect at most 1 and the defect space is $\mathcal{F} = \langle \frac{S^* H}{\Phi} \rangle$.*

We now discuss about the application part of Theorem 2.3.5 of Chapter 2. Since in this section we consider that $\Phi = F_1^* F_2$, where $F_1, F_2 \in \mathcal{G}H^\infty(\mathcal{L}(\mathbb{C}^m))$ almost everywhere on $\mathbb{T}$, then $\mathcal{M} = \operatorname{Ker} T \subseteq \langle F_2^{-1}(T_{F_1^{*-1}} H) \rangle$ and the defect space is

$$\mathcal{F} = \langle F_2^{-1} T_{F_1^{*-1}}(S^* H) \rangle = \langle F_2^{-1} S^*(T_{F_1^{*-1}} H) \rangle.$$

Let us consider any vector $F = \lambda F_2^{-1}(T_{F_1^{*-1}} H) \in \mathcal{M}$ satisfying $T(F) = 0$. Then it is equivalent to the following

$$\lambda(1 + \langle F_2^{-1}(T_{F_1^{*-1}} H), G \rangle) = 0. \tag{3.27}$$

Therefore according to (3.27) we have the following two cases.

Case I: If $1 + \langle F_2^{-1}(T_{F_1^{*-1}} H), G \rangle \neq 0$, then it yields that $\mathcal{M} = \{0\}$ which is a trivial S^*-invariant subspace.

Case II: If $1 + \langle F_2^{-1}(T_{F_1^{*-1}} H), G \rangle = 0$, then $\mathcal{M} = \langle F_2^{-1}(T_{F_1^{*-1}} H) \rangle$.

Let $\{A_k\}_{k=0}^\infty$ be the Taylor coefficients of $T_{F_1^{*-1}} H$ and let $\{\Phi_k\}_{k=0}^\infty$ be the Taylor coefficients of F_2^{-1}. Therefore it follows that

$$\left[F_2^{-1}(T_{F_1^{*-1}} H) \right](z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \Phi_k A_{n-k} \right) z^n, \text{ and } z S^*(T_{F_1^{*-1}} H)(z) = (T_{F_1^{*-1}} H)(z) - A_0.$$

Depending upon the context, we again divide our analysis into two sub-cases to calculate $\mathcal{K}$.

Sub-case I: If $F_2^{-1}(T_{F_1^{*-1}}H)(0) \neq 0$, then $\mathcal{W}$ has dimension exactly 1 and therefore only one F_i generates $\mathcal{W}$. Thus $F_i = P_{\mathcal{M}}(k_0 \otimes e_i) = \dfrac{\langle k_0 \otimes e_i, F_2^{-1}(T_{F_1^{*-1}}H)\rangle F_2^{-1}(T_{F_1^{*-1}}H)}{\|F_2^{-1}(T_{F_1^{*-1}}H\|^2}$ and hence

$$F_0 = [C_{11}F_2^{-1}(T_{F_1^{*-1}}H)]_{m\times 1}, \text{ and } E(z) = \frac{F_2^{-1}S^*(T_{F_1^{*-1}}H)}{\|F_2^{-1}S^*(T_{F_1^{*-1}}H)\|^2}.$$

Therefore from the first case of Theorem 2.3.5 of Chapter 2 it follows that

$$\mathcal{M} = \langle F_2^{-1}(T_{F_1^{*-1}}H)\rangle = \Big\{F : F = F_0K_0 + zk_1E_1 : (K_0, k_1) \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D}) \times H^2_{\mathbb{C}}(\mathbb{D})\Big\}.$$

Following the above identities we have $K_0 \in \mathbb{C}$, and $k_1\dfrac{F_2^{-1}(T_{F_1^{*-1}}H - A_0)}{\|F_2^{-1}S^*(T_{F_1^{*-1}}H)\|^2} = \xi F_2^{-1}(T_{F_1^{*-1}}H)$ with $\xi \in \mathbb{C}$, which will be true if and only if $K_0 \in \mathbb{C}$ and $k_1 = 0$. Thus the subspace $\mathcal{M}$ has the following representation

$$\mathcal{M} = \Big\{F : F = C_{11}K_0\frac{F_2^{-1}(T_{F_1^{*-1}}H)}{\|F_2^{-1}(T_{F_1^{*-1}}H)\|^2} : (K_0, 0) \in \mathcal{K}\Big\}$$

with $\mathcal{K} = \mathbb{C} \times \{0\}$ is a $S^* \oplus S^*$-invariant subspace of $H^2(\mathbb{D}, \mathbb{C}^2)$.

Sub-case II: If $F_2^{-1}(T_{F_1^{*-1}}H)(0) = \Phi_0A_0 = 0$, then $\mathcal{W} = \{0\}$ and hence from the case(ii) of Theorem 2.3.5 of Chapter 2 we have

$$\mathcal{M} = \langle F_2^{-1}(T_{F_1^{*-1}}H)\rangle = \Big\{F : F = k_1\frac{F_2^{-1}(T_{F_1^{*-1}}H - A_0)}{\|F_2^{-1}S^*(T_{F_1^{*-1}}H)\|} : k_1 \in \mathcal{K} \subset H^2_{\mathbb{C}}(\mathbb{D})\Big\}$$

which will be true if and only if $A_0 = 0$ and $k_1 \in \mathbb{C}$. Thus we have the subspace

$$\mathcal{M} = \Big\{F : F = k_1\frac{F_2^{-1}(T_{F_1^{*-1}}H)}{\|F_2^{-1}S^*(T_{F_1^{*-1}}H)\|} : k_1 \in \mathcal{K}\Big\}$$

with $\mathcal{K} = \mathbb{C}$ an S^* invariant subspace of $H^2_{\mathbb{C}}(\mathbb{D})$.

3.5 Kernel of finite rank perturbation of Toeplitz operator having symbol adjoint of an inner multiplier

In this section, we consider $\Phi(z) = \Theta^*(z)$, where Θ is a non constant inner multiplier. It is well known that the kernel of the Toeplitz operator T_{Θ^*} is the model space $\mathcal{K}_\Theta$. In other words $\operatorname{Ker} T_{\Theta^*} = \mathcal{K}_\Theta = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$. Compare to scalar valued case our analysis mainly differ in two places. Firstly, in Subsection 3.5.1 to show that the subspace $\mathcal{K}$ is backward shift invariant and secondly, in Subsection 3.5.2 conditions (3.44) and (3.45) are not exactly similar with conditions (3.8) and (3.9) given in [38] (see Section 3) to get the explicit expression of $\mathcal{K}$. Perhaps it is better for the reader if we provide little details of the analysis in this section. To proceed further, like in the previous section first we find an equivalent condition (3.16) for any vector $F \in \operatorname{Ker} T_2$ with $F(0) = 0$. Note that

$$\Theta^* \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \in \overline{H}_0^2 \Leftrightarrow \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i \in \Theta \overline{H}_0^2. \tag{3.28}$$

Therefore in this case the equations (3.18) and (3.19) changed into

$$P_m \left(\Theta^* V - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \sum_{i=1}^{2} \langle \frac{F}{z} + V, G_i \rangle H_i \right) = 0. \tag{3.29}$$

$$\Leftrightarrow \quad \Theta^* V - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \sum_{i=1}^{2} \langle \frac{F}{z} + V, G_i \rangle H_i \in \overline{H}_0^2.$$

Next, we consider three cases for the construction of the defect space $\mathcal{F}$ in this case.

Case 1: Suppose $G_1 \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ and $G_2 \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$. Due to condition (3.28), it follows that

$$\left\langle \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i, G_1 \right\rangle = 0, \tag{3.30}$$

and

$$\left\langle \frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i, G_2 \right\rangle = 0. \tag{3.31}$$

Now we consider $V = \sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j$. Then substituting this V and using the above two identities (3.30) and (3.31), the left hand side of (3.29) becomes

$$P_m \left(\Theta^* \left(\sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i \right) - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \sum_{i=1}^{2} \left\langle \frac{F}{z} + \sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j, G_i \right\rangle H_i \right) = 0.$$

Thus we define the defect space as

$$\mathcal{F} = \bigvee_{i=1}^{2} \{\Theta S^* H_i\}, \tag{3.32}$$

and hence $\operatorname{Ker} T_2$ is nearly S^*-invariant with defect at most 2.

Case 2: In this case we consider $G_1 \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ and $G_2 \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ (on a similar note, one can assume that $G_1 \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ and $G_2 \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$, then the corresponding analysis will be of similar kind). First we note that (3.30) holds in this case. Now if the identity (3.31) also hold in this case, then the defect space will be exactly same as in Case 1. Therefore we assume that the identity (3.31) does not hold in this case, then the vector G_2 can be decomposed in the following way

$$G_2 = U + W \quad \text{with} \quad U \neq 0 \in \mathcal{K}_\Theta = \operatorname{Ker} T_{\Theta^*} \quad \text{and} \quad W \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D}).$$

Thus

$$T_{\Theta^*}(U) = P_m(\Theta^* U) = 0. \tag{3.33}$$

Now we choose $V = \sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j + \xi_F U$, where ξ_F is a constant which satisfies the following identity

$$\xi_F \|U\|^2 = -\left\langle \frac{F}{z} + \sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j, G_2 \right\rangle \neq 0. \tag{3.34}$$

If we substitute V in the left hand side of (3.29), and then using (3.33) we have

$$\begin{aligned}
&P_m\Bigg(\Theta^* (\sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j + \xi_F U) - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i \\
&\qquad\qquad + \sum_{i=1}^{2} \left\langle \frac{F}{z} + (\sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j + \xi_F U), G_i \right\rangle H_i \Bigg) \\
=&P_m\left(\xi_F \Theta^* U + \sum_{i=1}^{2} \left\langle \frac{F}{z} + (\sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j + \xi_F U), G_i \right\rangle H_i \right) \\
&\sum_{i=1}^{2} \left\langle \frac{F}{z} + (\sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j + \xi_F U), G_i \right\rangle H_i \\
=& \left\langle \frac{F}{z} + (\sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j + \xi_F U), G_2 \right\rangle H_2 \quad (\text{using (3.30) and } \langle U, G_1 \rangle = 0) \\
=& \left(\left\langle \frac{F}{z} + \sum_{j=1}^{2} \langle F, G_j \rangle \Theta S^* H_j, G_2 \right\rangle + \xi_F \|U\|^2 \right) H_2 = 0. \qquad (\text{using (3.34)})
\end{aligned}$$

Therefore by using the construction of V we define the defect space as

$$\mathcal{F} = \bigvee \{\Theta S^* H_1, \Theta S^* H_2, P_{\mathcal{K}_\Theta} G_2\}, \tag{3.35}$$

where $P_{\mathcal{K}_\Theta} : L^2(\mathbb{T}) \to \mathcal{K}_\Theta$ is an orthogonal projection onto $\mathcal{K}_\Theta$. Therefore $\operatorname{Ker} T_2$ is nearly S^*-invariant with defect at most 3. The defect space is either given by (3.35) or given by (3.32) depending upon the condition (3.31).

Case 3: In this case, we consider that none of G_1, G_2 is in $\Theta H^2_{\mathbb{C}^m}(\mathbb{D})$, that is $G_1 \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ and $G_2 \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$. Therefore we decompose the vectors G_1, G_2 in the following way

$$G_1 = X_1 + \Theta Y_1 \quad \text{and} \quad G_2 = X_2 + \Theta Y_2,$$

where $X_i(\neq 0) \in \mathcal{K}_\Theta$ and $Y_i \in H^2_{\mathbb{C}^m}(\mathbb{D})$ for $i = 1, 2$. From the condition (3.28) we have the following

$$\frac{F}{z} + \sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i = \Theta F^0, \text{ for some } F^0 \in \overline{H^2_0}. \tag{3.36}$$

In this context, we will find such a V so that the equivalent condition (3.17) holds that is,

$$\Theta^*(\frac{F}{z} + V) + \sum_{i=1}^{2} \langle \frac{F}{z} + V, G_i \rangle H_i \in \overline{H^2_0} \tag{3.37}$$

which by (3.36) is equivalent to

$$F^0 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V + \sum_{i=1}^{2} \langle F^0 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V, \Theta^* G_i \rangle H_i \in \overline{H^2_0}. \tag{3.38}$$

Now we can decompose F^0 as $F^0 = F^0_1 + F^0_2$, where

$$F^0_1 \in \mathcal{N} := \bigvee \{\Theta^* X_1, \Theta^* X_2\} \subseteq \overline{H^2_0} \text{ and } F^0_2 \in \overline{H^2_0} \ominus \mathcal{N}.$$

Then it implies that

$$\langle F^0_2, \Theta^* X_i \rangle = 0 \text{ and } \langle F^0_2, Y_i \rangle = 0 \text{ for } i = 1, 2.$$

Hence, using the above facts, (3.38) is equivalent to

$$F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V + \sum_{i=1}^{2} \left\langle F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V, \Theta^* G_i \right\rangle H_i \in \overline{H^2_0}$$
$$\Leftrightarrow \quad F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V + \sum_{i=1}^{2} \left\langle F^0_1 - \sum_{i=1}^{2} \langle F, G_i \rangle S^* H_i + \Theta^* V, \Theta^* X_i + Y_i \right\rangle H_i \in \overline{H^2_0}.$$

Let us choose $V = \sum_{i=1}^{2} \langle F, G_i \rangle \Theta S^* H_i - \Theta F^0_1$. Thus our chosen vector V fulfill all the requirements and from the construction of V we can define the defect space as follows

$$\mathcal{F} = \bigvee \left\{ \Theta S^* H_1, \Theta S^* H_2, P_{\mathcal{K}_\Theta} G_1, P_{\mathcal{K}_\Theta} G_2 \right\}$$

with dimension at most 4. Consequently, the kernel of T_2 is nearly S^*-invariant with defect at most 4. Therefore by repeating the above argument once again we have the following theorem in general context regarding the kernel of T_n.

Theorem 3.5.1. *Suppose $\Phi(z) = \Theta^*(z)$, where $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D})$ is an inner multiplier, then the following statements hold.*

(i) If $G_j \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D}), j \in \{1, 2, \dots, n\}$, then the subspace $\operatorname{Ker} T_n$ is nearly S^*-invariant with defect at most n and the defect space is $\mathcal{F} = \bigvee \Big\{\Theta S^* H_j : j = 1, 2, \dots n\Big\}$.

(ii) If $G_j \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ for $j \in \Lambda_l \subset \{1, 2, \dots, n\}$, then the kernel of T_n is nearly S^*-invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ with defect at most $n + l$ and the defect space is

$$\mathcal{F} = \bigvee \Big\{\Theta S^* H_i, P_{K_\Theta} G_j : i = 1, 2, \dots, n \text{ and } j \in \Lambda_l\Big\}.$$

Like other sections we now discuss the application part of Theorem 2.3.5 of Chapter 2 in this context. For the operator T, the equation (3.15) is equivalent to

$$T_{\Theta^*} F + \langle F, G\rangle H = 0 \Leftrightarrow \Theta^* F + \langle F, G\rangle H \in \overline{H^2_0} \Leftrightarrow F + \langle F, G\rangle \Theta H \in \Theta \overline{H^2_0}.$$

Observing the above equivalent criteria we say that the kernel of the operator T satisfies $\mathcal{M} = \operatorname{Ker} T \subset (H^2 \cap \Theta \overline{H^2_0}) \oplus \langle \Theta H\rangle = K_\Theta \oplus \langle \Theta H\rangle$. Now consider the vector $F \in \mathcal{M} = \operatorname{Ker} T$ which is of the form $F = F_\zeta + \mu \Theta H$, where $F_\zeta \in K_\Theta$ and $\mu \in \mathbb{C}$. Then the above equivalent condition reduces to

$$\mu(1 + \langle \Theta H, G\rangle) = -\langle F_\zeta, G\rangle. \tag{3.39}$$

Therefore from the first part of this section we conclude the defect space of the nearly S^*-invariant subspace $\operatorname{Ker} T$ and which is as follows:

$$\mathcal{F} = \begin{cases} \langle \Theta S^* H\rangle, & \text{for } G \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D}), \\ \bigvee \{\Theta S^* H, P_{K_\Theta} G\}, & \text{for } G \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D}). \end{cases}$$

Accordingly, we analyze the kernel of T in two different subsections due to the above two different representation of defect spaces. Before we proceed, we would like to mention the following: We know that the subspace $\mathcal{W} = \mathcal{M} \ominus (\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$ (if non trivial) is generated by the vectors $\{F_1, F_2, \dots, F_m\}$, where $F_i = P_{\mathcal{M}}(k_0 \otimes e_i)$. If $dim \mathcal{W} = r (1 \leq r \leq m)$ and $\{W_1, W_2, \dots, W_r\}$ is an orthonormal basis for $\mathcal{W}$, then F_0 is the $m \times r$ matrix whose columns are

$W_1, W_2, \ldots, W_r$. Using Gram-Schmidt orthonormalization we can form an orthonormal basis for $\mathcal{W}$ from the generating set $\{F_1, F_2, \ldots, F_m\}$ and the resulting vectors are nothing but the linear combination of $\{F_1, F_2, \ldots, F_m\}$. Therefore it is sufficient to prove that everything holds good for the generating set of vectors. For simplicity of calculations we do the whole analysis for single F_i, that means from now onward we assume that $dim\mathcal{W} = 1$ and $F_0 = [\alpha_i F_i]_{m\times 1}$ in each cases and subsections.

3.5.1 $G \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$

In this subsection, we assume that $G \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$. Then the corresponding defect space for the nearly S^*-invariant subspace $\mathcal{M} = \operatorname{Ker} T$ is $\mathcal{F} = \langle \Theta S^* H\rangle$. Therefore the equation (3.39) reduces to

$$\mu(1 + \langle \Theta H, G\rangle) = 0.$$

Now considering the defect space as $\mathcal{F} = \langle \Theta S^* H\rangle$ we have the following two cases.

Case 1. If $1 + \langle \Theta H, G\rangle = 0$, then $\mathcal{M} = K_\Theta + \langle \Theta H\rangle$. Moreover,

$$\begin{aligned} F_i &= P_M(k_0 \otimes e_i) \\ &= P_{K_\Theta}(k_0 \otimes e_i) + P_{\langle \Theta H\rangle}(k_0 \otimes e_i) \\ &= k_0 \otimes e_i - \sum_{j=1}^{m} \langle k_0 \otimes e_i, \Theta e_j\rangle \Theta e_j + \frac{\langle k_0 \otimes e_i, \Theta H\rangle \Theta H}{\|\Theta H\|^2} \\ &= k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H\rangle \Theta H}{\|\Theta H\|^2}. \end{aligned}$$

Therefore the case (i) of Theorem 2.3.5 of Chapter 2 implies that the nearly invariant subspace $\mathcal{M}$ for S^* with the finite defect can be written in the following form.

$$\begin{aligned} &\mathcal{M} = K_\Theta + \langle \Theta H\rangle \\ &= \left\{ F : F = F_0 K_0 + z k_1 \frac{\Theta S^* H}{\|S^* H\|} : (K_0, k_1) \in \mathcal{K} \right\} \end{aligned}$$

$$= \left\{ F : F = \alpha_i (k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\|\Theta H\|^2}) K_0 + k_1 \frac{\Theta(H - H(0))}{\|S^* H\|} : (K_0, k_1) \in \mathcal{K} \right\}$$

$$\supseteq \left\{ F : F = \alpha_i \Big(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} \Big) K_0 + k_1 \frac{\Theta H(0)}{\|S^* H\|} \right.$$
$$\left. + \Big(\alpha_i \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{\|\Theta H\|^2} K_0 + \frac{k_1}{\|S^* H\|} \Big) \Theta H : (K_0, k_1) \in \mathcal{K} \right\},$$

where $\mathcal{K} = \{(K_0, k_1) \in H^2_{\mathbb{C}^2}(\mathbb{D}) : K_0$ and k_1 satisfies the following (3.40) $\}$

$$\begin{cases} \alpha_i \Big(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} \Big) K_0 + k_1 \frac{\Theta H(0)}{\|S^* H\|} \in K_\Theta \\ \alpha_i \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{\|\Theta H\|^2} K_0 + \frac{k_1}{\|S^* H\|} \in \mathbb{C}. \end{cases} \tag{3.40}$$

Our next aim is to show the subspace $\mathcal{K}$ of $H^2_{\mathbb{C}^2}(\mathbb{D})$ is a $S^* \oplus S^*$-invariant subspace which is not exactly same as in the scalar valued case. First we observe that if we replace K_0, k_1 by $S^* K_0, S^* k_1$, then the second condition of (3.40) is trivially true. Thus we only have to check that the first condition also true if we replace K_0, k_1 by $S^* K_0, S^* k_1$. Since $\mathcal{K}_\Theta$ is an S^*-invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$, then

$$H_\Theta : \;= S^* \left(\alpha_i \Big(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} \Big) K_0 + k_1 \frac{\Theta H(0)}{\|S^* H\|} \right) \in \mathcal{K}_\Theta.$$

On the other hand

$$\alpha_i\Big(k_0\otimes e_i-\Theta\begin{pmatrix}\overline{\theta_{i1}(0)}\\ \overline{\theta_{i2}(0)}\\ \vdots\\ \overline{\theta_{im}(0)}\end{pmatrix}\Big)\;S^*K_0+S^*k_1\frac{\Theta H(0)}{\|S^*H\|}$$

$$= H_\Theta+S^*\left(\alpha_i\Theta\begin{pmatrix}\overline{\theta_{i1}(0)}\\ \overline{\theta_{i2}(0)}\\ \vdots\\ \overline{\theta_{im}(0)}\end{pmatrix}K_0-k_1\frac{\Theta H(0)}{\|S^*H\|}\right)-\alpha_i\Theta\begin{pmatrix}\overline{\theta_{i1}(0)}\\ \overline{\theta_{i2}(0)}\\ \vdots\\ \overline{\theta_{im}(0)}\end{pmatrix}S^*K_0+S^*k_1\frac{\Theta H(0)}{\|S^*H\|}$$

$$= H_\Theta+\alpha_i\frac{\Theta(z)-\Theta(0)}{z}\begin{pmatrix}\overline{\theta_{i1}(0)}\\ \overline{\theta_{i2}(0)}\\ \vdots\\ \overline{\theta_{im}(0)}\end{pmatrix}K_0(0)-\frac{k_1(0)}{\|S^*H\|}\frac{\Theta(z)-\Theta(0)}{z}H(0)$$

$$= H_\Theta+\alpha_iS^*\Theta\begin{pmatrix}\overline{\theta_{i1}(0)}\\ \overline{\theta_{i2}(0)}\\ \vdots\\ \overline{\theta_{im}(0)}\end{pmatrix}K_0(0)-\frac{k_1(0)}{\|S^*H\|}S^*\Theta(H(0))\in\mathcal{K}_\Theta.$$

Since $H_\Theta\in\mathcal{K}_\Theta$, $S^*\Theta\begin{pmatrix}\overline{\theta_{i1}(0)}\\ \overline{\theta_{i2}(0)}\\ \vdots\\ \overline{\theta_{im}(0)}\end{pmatrix}\in\mathcal{K}_\Theta$ and $S^*\Theta(H(0))\in\mathcal{K}_\Theta$, then from the above equation we conclude that $\mathcal{K}$ is an $S^*\oplus S^*$-invariant subspace of $H^2_{\mathbb{C}^2}(\mathbb{D})$.

Case 2. If $1+\langle\Theta H,G\rangle\neq 0$, then $\mathcal{M}=\mathcal{K}_\Theta$. Therefore

$$F_i=P_{\mathcal{M}}(k_0\otimes e_i)=P_{\mathcal{K}_\Theta}(k_0\otimes e_i)=k_0\otimes e_i-\sum_{j=1}^{m}\langle k_0\otimes e_i,\Theta e_j\rangle\Theta e_j=k_0\otimes e_i-\Theta\begin{pmatrix}\overline{\theta_{i1}(0)}\\ \overline{\theta_{i2}(0)}\\ \vdots\\ \overline{\theta_{im}(0)}\end{pmatrix}.$$

Using the case(i) of Theorem 2.3.5 in Chapter 2 we have

$$\mathcal{M}=\mathcal{K}_\Theta$$

$$= \left\{F : F = F_0K_0 + zk_1\frac{\Theta S^*H}{\|S^*H\|} : (K_0, k_1) \in \mathcal{K}\right\}$$

$$= \left\{F : F = \alpha_i(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix})K_0 + k_1\frac{\Theta(H - H(0))}{\|S^*H\|} : (K_0, k_1) \in \mathcal{K}\right\}$$

$$\supseteq \left\{F : F = \alpha_i\Big(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix}\Big) K_0 : (K_0, 0) \in \mathcal{K}\right\}.$$

Therefore the corresponding $S^* \oplus S^*$-invariant subspace is

$$\mathcal{K} = \left\{(K_0, 0) \in H^2_{\mathbb{C}^2}(\mathbb{D}) : K_0 \text{ satisfies the following (3.41)}\right\}$$

such that

$$\alpha_i\Big(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix}\Big) K_0 \in \mathcal{K}_\Theta,$$

which is equivalent to the fact that

$$K_0 \in \bigcap_{j=1}^{m} \operatorname{Ker} T_{\overline{\theta_{ij} - \theta_{ij}(0)}} = \bigcap_{j=1}^{m} K_{\zeta_{ij}} \text{ where } \zeta_{ij} \text{ is an inner factor of } \theta_{ij} - \theta_{ij}(0). \tag{3.41}$$

Furthermore, it is easy to check that $\mathcal{K}$ is an $S^* \oplus S^*$-invariant subspace of $H^2_{\mathbb{C}^2}(\mathbb{D})$ using the scalar version of Proposition 3.2.3.

3.5.2 $G \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$

In this subsection, we consider the case that $G \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$, then the kernel of T is a nearly S^*-invariant subspace with defect at most 2 and the defect space is $\mathcal{F} = \bigvee\{\Theta S^*H, P_{K_\Theta}G\}$. Since $G \notin \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$, then we find a nonzero $G_\zeta \in K_\Theta$ and $G_\Theta \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ such that $G = G_\zeta + G_\Theta$. Therefore the identity (3.39) reduces to

$$\mu(1 + \langle \Theta H, G_\Theta \rangle) = -\langle F_\zeta, G_\zeta \rangle. \tag{3.42}$$

As we have already mentioned in the beginning of Section 3.5 that the obtained conditions (3.44) and (3.45) in our setting are not exactly similar to (3.8) and (3.9) in the scalar case [38] (see Section 3). In other words, we can not break the inner product mentioned in conditions (3.44) and (3.45) into smaller pieces because we do not know whether the individual term lies in respective $L^2(\mathbb{T})$-spaces or not. To proceed further, we need the following remark concerning the projection $P_{\mathcal{M}}(k_0 \otimes e_i)$.

Remark 3.5.2. *If $\mathcal{M} = \operatorname{Ker} T \subset \mathcal{N} := K_\Theta \oplus \langle \Theta H \rangle$ satisfies $\mathcal{N} = \mathcal{M} \oplus \langle R \rangle$, where $R = U + \rho\Theta H$ with $U \in K_\Theta$ and $\rho \in \mathbb{C}$. Then*

$$P_{\mathcal{M}}(k_0 \otimes e_i) = k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\|\Theta H\|^2} - \frac{\langle k_0 \otimes e_i, U + \rho\Theta H \rangle}{\|U + \rho\Theta H\|^2}(U + \rho\Theta H). \quad (3.43)$$

Now we need to analyze two cases according to whether $v_\Theta := 1 + \langle \Theta H, G_\Theta \rangle$ is zero or not along with the defect space $\mathcal{F} = \bigvee\{\Theta S^* H, P_{K_\Theta} G\}$.

Case 1. If $v_\Theta = 0$, then (3.42) holds if and only if $F_\zeta \in \langle P_{K_\Theta} G \rangle^\perp = \langle G_\zeta \rangle^\perp$. Thus $\mathcal{M} = \mathcal{N} \ominus \langle G_\zeta \rangle$ and therefore substituting $U = G_\zeta$ and $\rho = 0$ in (3.43) we have

$$P_{\mathcal{M}}(k_0 \otimes e_i) = k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\|\Theta H\|^2} - \frac{\langle k_0 \otimes e_i, G_\zeta \rangle}{\|G_\zeta\|^2} G_\zeta.$$

Therefore $F_0 = [\alpha_i P_{\mathcal{M}}(k_0 \otimes e_i)]_{m \times 1}$ and hence by using the case (i) of Theorem 2.3.5 in Chapter 2, we have the following representation of $\mathcal{M}$:

$$\mathcal{M} = \left\{ F : F = \alpha_i \left(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\|\Theta H\|^2} - \frac{\langle k_0 \otimes e_i, G_\zeta \rangle}{\|G_\zeta\|^2} G_\zeta \right) K_0 \right.$$
$$\left. + z k_1 \frac{\Theta S^* H}{\|S^* H\|} + z k_2 \frac{G_\zeta}{\|G_\zeta\|} : (K_0, k_1, k_2) \in \mathcal{K} \right\}$$

$$\supseteq \left\{ F : F = \alpha_i \left(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} \right) K_0 - \frac{k_1}{\|S^*H\|} \Theta H(0) + \left(\frac{k_1}{\|S^*H\|} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{\|\Theta H\|^2} K_0 \right) \Theta H \right.$$
$$\left. - \left(\frac{\langle k_0 \otimes e_i, G_\zeta \rangle \alpha_i K_0}{\|G_\zeta\|^2} - \frac{z k_2}{\|G_\zeta\|} \right) G_\zeta : (K_0, k_1, k_2) \in \mathcal{K} \right\},$$

and the corresponding $S^* \oplus S^* \oplus S^*$-invariant subspace is

$$\mathcal{K} = \left\{ (K_0, k_1, k_2) \in H^2_{\mathbb{C}^3}(\mathbb{D}) : K_0, k_1, k_2 \text{ satisfies the following (3.44)} \right\},$$

where

$$\begin{cases} \alpha_i \left(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} \right) K_0 - \frac{k_1}{\|S^*H\|} \Theta H(0) \in \mathcal{K}_\Theta \\ \frac{k_1}{\|S^*H\|} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{\|\Theta H\|^2} K_0 \in \mathbb{C} \\ \langle K_0, z^n \overline{\alpha_i} G_i \rangle - \left\langle \left(\frac{\langle k_0 \otimes e_i, G_\zeta \rangle \alpha_i S^{*n} K_0}{\|G_\zeta\|^2} - \frac{z S^{*n} k_2}{\|G_\zeta\|} \right) G_\zeta, G_\zeta \right\rangle = 0, \ for \ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.44}$$

and $G_\zeta = (G_1, G_2, \dots, G_m)$. In a similar fashion like (3.40) we prove that the first two conditions of (3.44) also hold for S^*K_0, S^*k_1 . Moreover, the last condition also holds for S^*K_0 and S^*k_2 trivially. Thus $\mathcal{K}$ is an $S^* \oplus S^* \oplus S^*$ invariant subspace of $H^2_{\mathbb{C}^3}(\mathbb{D})$.

Case 2. Next we consider $v_\Theta \neq 0$, then the equation (3.42) gives $\mu = -v_\Theta^{-1} \langle F_\zeta, G_\zeta \rangle$ and thus we have

$$\mathcal{M} = \operatorname{Ker} T = \{F : F = K - v_\Theta^{-1} \langle K, G_\zeta \rangle \Theta H, K \in K_\Theta\}.$$

Therefore by simple calculations we conclude that $\mathcal{N} = \mathcal{M} \oplus \langle G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2} \Theta H \rangle$. Thus by letting $U = G_\zeta$ and $\rho = \frac{\overline{v}_\Theta}{\|H\|^2}$ in (3.43), we get

$$P_{\mathcal{M}}(k_0 \otimes e_i) = k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\|\Theta H\|^2} - \frac{\langle k_0 \otimes e_i, G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H \rangle}{\|G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H\|^2}(G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H)$$

$$= k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\|H\|^2} - \omega_\Theta(G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H),$$

where

$$\omega_\Theta = \frac{\langle k_0 \otimes e_i, G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H \rangle}{\|G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H\|^2}.$$

Therefore from the Case *(i) of Theorem 2.3.5 in Chapter 2 we have the following representation of $\mathcal{M}$:*

$$\mathcal{M} = \left(K_\Theta \oplus \langle \Theta H \rangle\right) \ominus \left\langle G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H \right\rangle$$

$$= \left\{ F : F = \alpha_i \left(k_0 \otimes e_i - \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + \frac{\langle k_0 \otimes e_i, \Theta H \rangle \Theta H}{\|H\|^2} - \omega_\Theta(G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H) \right) K_0 \right.$$

$$\left. + k_1 \frac{\Theta(H - H(0))}{\|S^* H\|} + k_2 \frac{zG_\zeta}{\|G_\zeta\|} : (K_0, k_1, k_2) \in \mathcal{K} \right\}$$

$$\supseteq \left\{ F : F = \alpha_i (k_0 \otimes e_i) K_0 - \left(\alpha_i K_0 \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + k_1 \frac{\Theta H(0)}{\|S^* H\|} \right) \right.$$

$$+ \left(\alpha_i \frac{\langle k_0 \otimes e_i, \Theta H \rangle}{\|H\|^2} K_0 + \frac{k_1}{\|S^* H\|} - \frac{zk_2}{\|G_\zeta\|} \frac{\overline{v}_\Theta}{\|H\|^2} \right) \Theta H$$

$$\left. + \left(-\alpha_i K_0 \omega_\Theta + \frac{zk_2}{\|G_\zeta\|} \right) (G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2}\Theta H) : (K_0, k_1, k_2) \in \mathcal{K} \right\},$$

and the corresponding $S^* \oplus S^* \oplus S^*$-invariant subspace is

$$\mathcal{K} = \big\{(K_0, k_1, k_2) : K_0, k_1, k_2 \text{ satisfies the following } (3.45)\big\},$$

where

$$\begin{cases} \alpha_i (k_0 \otimes e_i) K_0 - \left(\alpha_i K_0 \Theta \begin{pmatrix} \overline{\theta_{i1}(0)} \\ \overline{\theta_{i2}(0)} \\ \vdots \\ \overline{\theta_{im}(0)} \end{pmatrix} + k_1 \dfrac{\Theta H(0)}{\|S^* H\|} \right) \in \mathcal{K}_\Theta \\ \alpha_i \dfrac{\langle k_0 \otimes e_i, \Theta H \rangle}{\|H\|^2} K_0 + \dfrac{k_1}{\|S^* H\|} - \dfrac{z k_2}{\|G_\zeta\|} \dfrac{\overline{v}_\Theta}{\|H\|^2} \in \mathbb{C} \\ \langle K_0, z^n \overline{\alpha_i} G_i \rangle + L(n) + \left\langle \left(-\alpha_i S^{*n} K_0 \omega_\Theta + \dfrac{z S^{*n} k_2}{\|G_\zeta\|} \right) (G_\zeta + \dfrac{\overline{v}_\Theta}{\|H\|^2} \Theta H), G_\zeta + \dfrac{\overline{v}_\Theta}{\|H\|^2} \Theta H \right\rangle = 0, \\ \hfill for\ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.45}$$

$G_\zeta = (G_1, G_2, \ldots, G_m)$, *and*

$$L(n) = \begin{cases} \dfrac{(S^{*n-1} k_2)(0)}{\|G_\zeta\|} \cdot \dfrac{|v_\Theta|^2}{\|H\|^2} & \text{if } n \in \mathbb{N} \\ \left(\dfrac{\alpha_i \langle k_0 \otimes e_i, \Theta H \rangle}{\|H\|^2} K_0 + \dfrac{k_1}{\|S^* H\|} - \dfrac{z k_2 \overline{v_\Theta}}{\|G_\zeta\| \|H\|^2} \right) v_\Theta & \text{if } n = 0. \end{cases}$$

By repeating the similar explanations given for (3.40), one can check that the conditions of (3.45) also hold for $S^* K_0$, $S^* k_1$, $S^* k_2$ and hence we conclude that, $\mathcal{K}$ is an $S^* \oplus S^* \oplus S^*$-invariant subspace of $H^2_{\mathbb{C}^3}(\mathbb{D})$. Now we give an example motivated from [38] with minor modifications to understand the case $\Phi = \Theta^*$.

Example 3.5.3. *Suppose*

$$\Theta(z) = \begin{bmatrix} z^s & 0 & \cdots & 0 \\ 0 & z^s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^s \end{bmatrix}_{m \times m} \in H^\infty_{\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)}(\mathbb{D}), \text{ where } s \geq 1, \tag{3.46}$$

$G = G_\zeta + G_\Theta$ with $G_\zeta = (z^{s-1}, 1, \ldots, 1)$, $G_\Theta \in \Theta H^2_{\mathbb{C}^m}(\mathbb{D}) = z^s H^2_{\mathbb{C}}(\mathbb{D}) \oplus \cdots \oplus z^s H^2_{\mathbb{C}}(\mathbb{D})$ and $H = \dfrac{1}{\sqrt{2}}(1 - z, 0, \ldots, 0)$. Then it follows that, $\mathcal{M}$ is a nearly S^*-invariant subspace with defect

space

$$\mathcal{F} = \bigvee \{z^s S^* H, G_\zeta\}.$$

Case 1. *If* $v_\Theta = 1 + \langle \Theta H, G_\Theta \rangle = 0$*, then it follows that*

$$\begin{aligned}\mathcal{M} &= \bigvee \Big\{ 1 \otimes e_1, \{z \otimes e_i\}_{i=1}^m, \{z^2 \otimes e_i\}_{i=1}^m, \dots, \{z^{s-2} \otimes e_i\}_{i=1}^m, \{z^{s-1} \otimes e_j\}_{j=2}^m \Big\} \oplus \langle z^s H \rangle \\ &= \Big\{ F : F = K_0 \otimes e_1 - k_1 \frac{z^s H(0)}{\|S^* H\|} + k_1 \frac{z^s H}{\|S^* H\|} + z k_2 \frac{G_\zeta}{\|G_\zeta\|} : (K_0, k_1, k_2) \in \mathcal{K} \Big\}\end{aligned}$$

with an $S^* \oplus S^* \oplus S^*$*-invariant subspace* $\mathcal{K} = \Big\{ (K_0, k_1, k_2) : K_0, k_1$ *satisfies the following* (3.47) *and* $k_2 \in H^2_{\mathbb{C}}(\mathbb{D}) \Big\}$ *such that*

$$K_0 \otimes e_1 - k_1 \frac{z^s H(0)}{\|S^* H\|} \in \mathcal{K}_\Theta, \quad \frac{k_1}{\|S^* H\|} \in \mathbb{C}, \text{ and } \quad \langle K_0, z^{n+s-1} \rangle = 0 \text{ for } n \in \mathbb{N} \cup \{0\}. \tag{3.47}$$

Case 2. *If* $v_\Theta = 1 + \langle \Theta H, G_\Theta \rangle \neq 0$*, then it follows that*

$$\begin{aligned}\mathcal{M} &= \bigvee \Big\{ 1 \otimes e_1, \{z \otimes e_i\}_{i=1}^m, \{z^2 \otimes e_i\}_{i=1}^m, \dots, \{z^{s-2} \otimes e_i\}_{i=1}^m, \{z^{s-1} \otimes e_j\}_{j=1}^m \Big\} \oplus \langle z^s H \rangle \\ &\qquad \ominus \langle G_\zeta + \frac{\overline{v}_\Theta}{\|H\|^2} z^s H \rangle \\ &= \Big\{ F : F = K_0 \otimes e_1 - k_1 \frac{z^s H(0)}{\|S^* H\|} + \left(\frac{k_1}{\|S^* H\|} - \frac{z k_2}{\|G_\zeta\|} \overline{v}_\Theta \right) z^s H \\ &\qquad + \frac{z k_2}{\|G_\zeta\|} (G_\zeta + \overline{v}_\Theta z^s H) : (K_0, k_1, k_2) \in \mathcal{K} \Big\}\end{aligned}$$

with an $S^* \oplus S^* \oplus S^*$*-invariant subspace* $\mathcal{K} = \Big\{ (K_0, k_1, k_2) : K_0, k_1, k_2$ *satiesfies the following* (3.48)$\Big\}$ *such that*

$$\begin{cases} K_0 \otimes e_1 - k_1 \dfrac{z^s H(0)}{\|S^* H\|} \in \mathcal{K}_\Theta, \quad \dfrac{k_1}{\|S^* H\|} - \dfrac{z k_2}{\|G_\zeta\|} \overline{v}_\Theta \in \mathbb{C}, \text{ and} \\ \left\langle K_0, z^n \left(z^{s-1} + \dfrac{\overline{v}_\Theta z^s}{\sqrt{2}} (1 - z) \right) \right\rangle + \left\langle k_1, \dfrac{\overline{v}_\Theta}{\sqrt{2}} z^n \right\rangle + \left\langle k_2, \dfrac{z^n \overline{v}_\Theta}{\sqrt{2}} (1 - z) \right\rangle = 0 \text{ for } n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{3.48}$$

CHAPTER 4

Study of nearly invariant subspaces with finite defect in Hilbert spaces

4.1 Introduction

The structure of the invariant subspaces of an operator T plays an important role to study the action of T on the full space in a better way. To that aim, the study of (almost) invariant subspaces (see Definition 1.6.1) were initiated and a suitable investigation of these brings the concept such as near invariance. As already discussed in Chapter 2 and Chapter 3 the study of nearly invariant subspaces for the backward shift in the scalar valued Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$ were introduced by Hayashi [33], Hitt[34], and then Sarason [49] in the context of kernels of Toeplitz operators. Going further, Chalendar-Chevrot-Partington (C-C-P) [9] gives a complete characterization of nearly invariant subspaces under the backward shift operator acting on the vector valued Hardy space, providing a vectorial generalization of a result of Hitt. In 2004, Erard investigated the nearly invariant subspaces related to multiplication operators in Hilbert spaces of analytic functions in [21]. The concept of nearly invariant subspaces of finite defect for the backward shift in the scalar valued Hardy space was introduced by Chalendar- Gallardo-Partington (C-G-P) in [11] and provides a complete characterization of these spaces in terms of backward shift invariant subspaces. In Chapter 2, we have characterized nearly invariant subspace of finite defect for the backward shift operator acting on the vector valued Hardy space *and provide a vectorial generalization of C-G-P algorithm. In this connection, we also mention that similar type of connection also obtained independently by R. O'Loughlin in [44]. Recently, Liang and Partington introduce the notion of nearly T^{-1} invariant subspaces in general Hilbert space setting [39] and provide a representation of nearly T^{-1} invariant subspaces for the shift operator T with finite multiplicity acting on a separable infinite dimensional Hilbert space $\mathcal{H}$ in terms of backward shift invariant subspaces on the vector valued Hardy spaces as an application of the result given in [9, Corollary 4.5.]. Moreover, they also give a description of the nearly T_B^{-1} invariant subspaces for the operator T_B (see Definition 1.7) of multiplication by B in a scale of Dirichlet-type spaces [39], where B is any finite Blashcke product (see Definition 1.3.20 and 1.9).*

Motivated by the work of Liang and Partington in [39], we also introduce the notion of nearly T^{-1} invariant subspaces with finite defect (see Definition 2.1) for an left invertible operator T acting on a separable infinite dimensional Hilbert space as a generalization of nearly T^{-1} invariant subspaces. The purpose of this chapter is to study nearly T^{-1} invariant subspaces with finite defect for a shift operator T with finite multiplicity acting on a separable Hilbert space. In other words, we provide a characterization of nearly T^{-1} invariant subspaces with finite defect in terms of backward shift invariant subspaces in vector valued Hardy spaces by using Theorem 2.3.5 of Chapter 2. Moreover, we also give a concrete representation of the nearly T_B^{-1} invariant subspaces with finite defect in a scale of Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in [-1, 1]$ corresponding to any finite Blashcke product B by using the similar ideas mentioned in [21, 39] with an appropriate modification, providing a generalization of results of Liang and Partington in a sense that they already proved the result for defect zero setting (see Section 3, [39]). There are also many other contributions related with this topic and the interested reader can also refer to [7, 20] and the references cited therein. In order to state the precise contribution of this chapter, we need to recapitulate some useful notations and definitions mentioned in Chapter 1.

We introduce a special family of Hilbert spaces of analytic functions. Let α be any real number. Then the Dirichlet-type spaces are denoted by $\mathcal{D}_\alpha \equiv \mathcal{D}_\alpha(\mathbb{D})$ and defined by

$$\mathcal{D}_\alpha \equiv \mathcal{D}_\alpha(\mathbb{D}) := \Big\{ f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 < \infty \Big\}.$$

Then each $\mathcal{D}_\alpha$ is a Hilbert space with respect to the norm

$$\|f\|_\alpha := \left(\sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 \right)^{\frac{1}{2}}.$$

Note that the particular instances of α yield well-known Hilbert spaces of analytic functions on $\mathbb{D}$. More precisely, when $\alpha = 0$ we get the Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$, for $\alpha = -1$ we have the classical Bergman space $\mathcal{A}^2$, and $\alpha = 1$ corresponds to the Dirichlet space $\mathcal{D}$. Since $\|f\|_\gamma < \|f\|_\beta$ for $\gamma < \beta$, then the continuous inclusion $\mathcal{D}_\beta \subset \mathcal{D}_\gamma$ holds for any $\gamma < \beta$. For more information about Dirichlet-type spaces we refer to [7] and the references cited therein. Recall that an analytic function u is said to be an multiplier of $\mathcal{D}_\alpha$ if for any $f \in \mathcal{D}_\alpha$, $uf \in \mathcal{D}_\alpha$ that is, the analytic Toeplitz operator $T_u : f \to uf$ is defined everywhere on $\mathcal{D}_\alpha$ (hence bounded by closed graph theorem). Furthermore, one can easily check that any finite Blaschke product B is a multiplier for each $\mathcal{D}_\alpha$ spaces. Note that a finite Blaschke product (also see (1.9)) is given by

$$B(z) = e^{i\theta} \prod_{k=1}^{N} \frac{z - z_k}{1 - \overline{z_k} z}, \quad (z \in \mathbb{D})$$

where $\alpha_i \in \mathbb{D}$ and the degree of B is just the number of zeros $\{z_1, \dots, z_N\}$, counted with multiplicity. Moreover, finite Blashcke products play an important role in mathematics. We refer [52] and [26] for more on multipliers of $\mathcal{D}_\alpha$ and the qualitative study of finite Blaschke product respectively. The famous Wold Decomposition Theorem [17] implies that for any Blaschke product B, each element $f \in H^2_{\mathbb{C}}(\mathbb{D})$ has the following decomposition:

$$f(z) = \sum_{n=0}^{\infty} B^n(z) h_n(z),$$

where h_n belongs to the model space $\mathcal{K}_B = H^2_{\mathbb{C}}(\mathbb{D}) \ominus BH^2_{\mathbb{C}}(\mathbb{D})$. An analogous theorem for Dirichlet-type spaces $\mathcal{D}_\alpha(\mathbb{D})$ is the following:

Theorem 4.1.1. *[25, Theorem 3.1][10, Theorem 2.1]*

Suppose $\alpha \in [-1, 1]$ and B is a finite Blaschke product. Then $f \in \mathcal{D}_\alpha(\mathbb{D})$ if and only if $f = \sum_{n=0}^{\infty} B^n h_n$ (convergence in $\mathcal{D}_\alpha(\mathbb{D})$ norm) with $h_n \in \mathcal{K}_B = H^2_{\mathbb{C}}(\mathbb{D}) \ominus BH^2_{\mathbb{C}}(\mathbb{D})$ and

$$\sum_{n=0}^{\infty} (n+1)^\alpha \|h_n\|^2_{H^2} < \infty. \tag{4.1}$$

Moreover, since B is a finite Blaschke product, then $\mathcal{K}_B$ is finite dimensional and hence we can consider other (equivalent) norms here, such as $\|h\|_{\mathcal{D}_\alpha}$.

The nearly invariant subspaces related to the multiplication operator M_u in the Hilbert space of analytic functions has been studied by C. Erard in [21]. In fact Erard gave the definition of "nearly invariant under division by u ", which is same as "nearly M_u^{-1} invariant", a special *case of the notion of nearly T^{-1} invariant subspaces for any left invertible operator $T \in \mathcal{L}(\mathcal{H})$ recently introduced by Liang and Partington in [39] and the definition is the following:*

Definition 4.1.2. *[39, Definition 1.2] Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be left invertible. Then a closed subspace $\mathcal{M} \subset \mathcal{H}$ is said to be nearly T^{-1} invariant if for every $g \in \mathcal{H}$ such that $Tg \in \mathcal{M}$ then it holds that $g \in \mathcal{M}$.*

It is well known that the shift operator acting on a separable Hilbert space is a generalization of the unilateral shift S and the operator T_B on $H^2_{\mathbb{C}^m}(\mathbb{D})$. Recall that, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a shift operator if it is an isometry and T^* converges strongly to zero that is, $\|T^{*n}h\| \to 0$ as $n \to \infty$ for all $h \in \mathcal{H}$ [48]. Equivalently, an isometry $T \in \mathcal{L}(\mathcal{H})$ is a shift operator if and only if T is pure that is, $\cap_{n=0}^{\infty} T^n\mathcal{H} = \{0\}$. Therefore it is easy to observe that shift operator is an isometry and left invertible. Moreover, the multiplicity of a shift operator $T \in \mathcal{L}(\mathcal{H})$ is defined to be the dimension of $KerT^* = \mathcal{H} \ominus T\mathcal{H}$. As we have discussed earlier, Liang and Partington have characterized nearly T^{-1} invariant subspaces for a shift operator $T \in \mathcal{L}(\mathcal{H})$ with finite multiplicity and furthermore they also studied the nearly T_B^{-1} invariant subspaces corresponding to a finite Blaschke product B in a scale of Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in [-1, 1]$ in [39]. The main aim of this chapter is to first introduce the notion of nearly T^{-1} invariant subspaces with finite defect for a shift operator $T \in \mathcal{L}(\mathcal{H})$ with finite multiplicity and then characterize those subspaces in terms of backward shift invariant subspaces in vector valued Hardy spaces. Furthermore, we also study the nearly T_B^{-1} invariant subspaces in a scale of Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in [-1, 1]$ corresponding to a finite Blaschke product B and provide a concrete representation of it by generalizing some results of C. Erard [21] and using the concept of the equivalent norm introduced by Liang and Partington (see Section 3, [39]) in our context.

The rest of the chapter is organized as follows: In Section 4.2, we introduce the notion of nearly T^{-1} invariant subspaces with finite defect for an left invertible operator $T \in \mathcal{L}(\mathcal{H})$ and give a chracterization of nearly T^{-1} invariant subspaces with finite defect for the shift operator $T \in \mathcal{L}(\mathcal{H})$ with finite multiplicity. In Section 4.3, we deal with the study of nearly T_B^{-1} invariant subspaces with finite defect corresponding to a finite Blaschke product B in a scale of Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in [-1, 1]$.

4.2 Characterization of nearly invariant subspaces with finite defect for the shift operator

In this section, we study nearly T^{-1} invariant subspaces with finite defect for a shift operator $T \in \mathcal{L}(\mathcal{H})$ having finite multiplicity. Now we introduce the notion of nearly T^{-1} invariant subspaces with finite defect for any left invertible operator $T \in \mathcal{L}(\mathcal{H})$ as a generalization of nearly T^{-1} invariant subspaces.

Definition 4.2.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be left invertible. Then a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be nearly T^{-1} invariant with finite defect p if there exists a p dimensional subspace $\mathcal{F}$ (which may be taken to be orthogonal to $\mathcal{M}$) such that for any $f \in \mathcal{H}$ with $Tf \in \mathcal{M}$, then it holds that $f \in \mathcal{M} \oplus \mathcal{F}$.*

The following lemma which is almost similar to Lemma 2.2 ([39]) gives a connection of nearly invariant subspaces with same defect between similar operators.

Lemma 4.2.2. *Let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ be two left invertible operators such that they are similar by some invertible operator $V : \mathcal{H}_1 \to \mathcal{H}_2$, so that $T_2 = VT_1V^{-1}$. Let $\mathcal{M}$ be a nearly T_1^{-1} invariant subspace with defect p in $\mathcal{H}_1$, then $V(\mathcal{M})$ is also a nearly T_2^{-1} invariant subspace with the same defect p in $\mathcal{H}_2$.*

Proof. Suppose $g \in \mathcal{H}_2$ such that $T_2 g \in V\mathcal{M}$, then we want to show $g \in V\mathcal{M} \oplus V\mathcal{F}$, where $\mathcal{F}$ is the p dimensional defect space for $\mathcal{M}$ in $\mathcal{H}_1$. Since $T_2 g = VT_1V^{-1}g \in V\mathcal{M}$, then it implies that $T_1V^{-1}g \in V\mathcal{M}$. Moreover, since $\mathcal{M}$ is nearly T_1^{-1} invariant with defect space $\mathcal{F}$, then we must have $V^{-1}g \in \mathcal{M} \oplus \mathcal{F}$. Thus $g \in V(\mathcal{M} \oplus \mathcal{F}) = V\mathcal{M} \oplus V\mathcal{F}$, proving that $V(\mathcal{M})$ is a nearly T_2^{-1} invariant subspace with defect p in $\mathcal{H}_2$. □

Now onwards we always assume $T \in \mathcal{L}(\mathcal{H})$ is a shift operator with multiplicity m throughout this section. Let $\{e_1, e_2, \dots, e_m\}$ be an orthonormal basis of $\mathcal{K} = \mathcal{H} \ominus T\mathcal{H}$ and let $\delta_j^m = (0, 0, \dots, 1, \dots, 0)$ with 1 in the jth place be an orthonormal basis of $\mathcal{K}_z = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus zH^2_{\mathbb{C}^m}(\mathbb{D})$ for $j = 1, 2, \dots, m$. By considering the following two orthogonal decompositions

$$\mathcal{H} = \bigoplus_{i=0}^{\infty} T^i\mathcal{K} \text{ and } H^2_{\mathbb{C}^m}(\mathbb{D}) = \bigoplus_{i=0}^{\infty} z^i\mathcal{K}_z,$$

we have an unitary mapping $U : \mathcal{H} \to H^2_{\mathbb{C}^m}(\mathbb{D})$ *defined by*

$$U(T^i e_j) = z^i \delta_j^m. \tag{4.2}$$

Therefore the following diagram (4.3) *corresponding to the shift operator* $T : \mathcal{H} \to \mathcal{H}$ *with multiplicity* m *and the unilateral shift* $S : H^2_{\mathbb{C}^m}(\mathbb{D}) \to H^2_{\mathbb{C}^m}(\mathbb{D})$ *is commutative.*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad T \quad} & \mathcal{H} \\ U \Big\downarrow & & \Big\downarrow U \\ H^2_{\mathbb{C}^m}(\mathbb{D}) & \xrightarrow{\quad S \quad} & H^2_{\mathbb{C}^m}(\mathbb{D}) \end{array} . \tag{4.3}$$

Therefore from the above commutative diagram (4.3) *we get*

$$S^n U = UT^n, \forall n \in \mathbb{N} \cup \{0\}. \tag{4.4}$$

Now onwards we denote by $P_{\mathcal{M}}$ *as the orthogonal projection of* $\mathcal{H}$ *onto a closed subspace* $\mathcal{M}$ *of* $\mathcal{H}$*. The following lemma gives an upper bound concerning the dimension of the subspace* $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})$*:*

Lemma 4.2.3. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be a shift operator with multiplicity* m *and let* $\mathcal{M}$ *be a non trivial closed subspace of* $\mathcal{H}$ *such that* $\mathcal{M} \nsubseteq T\mathcal{H}$ *(that means* $\mathcal{M}$ *is not properly contained in* $T\mathcal{H}$*). Then*

$$1 \leq r := dim(\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})) \leq m. \tag{4.5}$$

Proof. Since T is a shift operator with multiplicity m, then $dim(\mathcal{H} \ominus T\mathcal{H}) = m$. Moreover, since $\mathcal{M} \nsubseteq T\mathcal{H}$, then $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H}) \neq \{0\}$. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $\mathcal{H} \ominus T\mathcal{H}$. Our claim is that $\{P_{\mathcal{M}} e_1, \ldots, P_{\mathcal{M}} e_m\}$ generates $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})$. Indeed, for any $g \in \mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})$ with $\langle g, P_{\mathcal{M}} e_i \rangle = 0$ for all $i \in \{1, \ldots, m\}$ implies $g = 0$ and hence $1 \leq r := dim(\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})) \leq m$. □

Next by using condition (4.4) *and the above Lemma 4.2.3 we have the following result.*

Lemma 4.2.4. *Let* $\mathcal{M}$ *be a non trivial nearly* T^{-1} *invariant subspace with finite defect* p *and let* $G_0 = [g_1, g_2, \ldots, g_r]^t$ *be an* $r \times 1$ *matrix with* $\{g_1, g_2, \ldots, g_r\}$ *is an orthonormal basis of* $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})$ *(note that the superscript* t *denotes the transpose of a matrix). Then* $F_0 = [Ug_1, Ug_2, \ldots, Ug_r]^t$ *be an* $r \times m$ *matrix with* $\{Ug_1, Ug_2, \ldots, Ug_r\}$ *is an orthonormal basis for* $U\mathcal{M} \ominus (U\mathcal{M} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$*.*

Proof. The proof is straightforward and we leave it to the reader. □

Going further, we need the following useful lemma similar to Liang and Partington in [39].

Lemma 4.2.5. *Suppose $T:\mathcal{H}\to\mathcal{H}$ is a shift operator, and let U be as in (4.4). Let $g_1, g_2, \dots, g_n \in \mathcal{H}$ and $h_1, h_2, \dots, h_n \in H^2_{\mathbb{C}}(\mathbb{D})$ be such that*

$$(Ug_1)h_1 + (Ug_2)h_2 + \cdots + (Ug_n)h_n \in H^2_{\mathbb{C}^m}(\mathbb{D}),$$

and suppose there exist sequences of polynomials $\left\{\{p_l^i\}_l : 1 \le i \le n,\ l \in \mathbb{N}\right\}$ with $p_l^i \to h_i$ in $H^2_{\mathbb{C}}(\mathbb{D})$ as $l\to\infty$ for $1 \le i \le n$ so that $(Ug_1)p_l^1 + (Ug_2)p_l^2 + \cdots + (Ug_n)p_l^n \to (Ug_1)h_1 + (Ug_2)h_2 + \cdots + (Ug_n)h_n$ in $H^2_{\mathbb{C}^m}(\mathbb{D})$ as $l \to \infty$. Then

$$U^*[(Ug_1)h_1 + (Ug_2)h_2 + \cdots + (Ug_n)h_n] = h(T)g, \tag{4.6}$$

where

$$h(T)g = \lim_{l\to\infty}[p_l^1(T)g_1 + p_l^2(T)g_2 + \cdots + p_l^n(T)g_n],$$

and $h(T) = [h_1(T), h_2(T), \dots, h_n(T)]$, $g = [g_1, g_2, \dots, g_n]^t$.

Now we are in a position to state and prove our main result in this section which provides an isometric relation between nearly T^{-1} invariant subspaces with defect p and the backward shift invariant subspaces of $H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) = H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}^p}(\mathbb{D})$.

Theorem 4.2.6. *Suppose T is a shift operator with multiplicity m and $\mathcal{M} \subset \mathcal{H}$ is a non trivial nearly T^{-1} invariant subspace with defect p and let $\mathcal{F}$ be the corresponding p dimensional defect space. Let $F_1 = [f_1, f_2, \dots, f_p]^t$ be a $p\times 1$ matrix containing an orthonormal basis $\{f_1, f_2, \dots, f_p\}$ of $\mathcal{F}$. Then*

(i) in the case when $\mathcal{M} \not\subseteq T\mathcal{H}$, there exists a non negative integer $r' \le r+p$ and an inner multiplier $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}^{r'},\mathbb{C}^{r+p}))}(\mathbb{D})$, unique upto an unitary equivalence such that

$$\mathcal{M} = \left\{ f \in \mathcal{H} : f = K_0(T)G_0 + TK_1(T)F_1 : (K_0, K_1) \in H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{r'}}(\mathbb{D}) \right\}, \tag{4.7}$$

where $G_0 = [g_1, g_2, \dots, g_r]^t$ is an $r\times 1$ matrix with $\{g_1, g_2, \dots, g_r\}$ is an orthonormal basis of $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H})$ and also there exists an isometry

$$Q : \mathcal{M} \to H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \quad \text{defined by} \quad Q(f) = (K_0, K_1).$$

(ii) In the case, when $\mathcal{M} \subseteq T\mathcal{H}$, there exists a non negative integer $p' \leq p$ and an inner multiplier $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^{p'},\mathbb{C}^p))}(\mathbb{D})$ which is unique upto unitary constant such that

$$\mathcal{M} = \Big\{f \in \mathcal{H} : f = TK_1(T)F_1 : K_1 \in H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^{p'}}(\mathbb{D})\Big\}, \tag{4.8}$$

and also there exists an isometry

$$R : \mathcal{M} \to H^2_{\mathbb{C}^p}(\mathbb{D}) \quad \text{defined by} \quad R(f) = K_1.$$

Proof. From Lemma 4.2.2 and using (4.2), we say $U\mathcal{M}$ is a nearly S^*-invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ with defect p and the correosponding defect space is $U\mathcal{F} \subseteq H^2_{\mathbb{C}^m}(\mathbb{D})$. Therefore by applying Theorem 2.3.5 (case (i)) of Chapter 2 (see also Theorem 3.4, [44]) corresponding to nearly S^*-invariant subspace with finite defect in vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$, we have

$$U\mathcal{M} = \left\{F \in H^2_{\mathbb{C}^m}(\mathbb{D}) : F(z) = F_0(z)^t K_0(z) + \sum_{j=1}^{p} zk_j(z)Uf_j(z) : (K_0, k_1, \dots, k_p) \in \mathcal{K}\right\},$$

where $\mathcal{K} \subset H^2_{\mathbb{C}^r}(\mathbb{D}) \times \underbrace{H^2_{\mathbb{C}}(\mathbb{D}) \times \cdots \times H^2_{\mathbb{C}}(\mathbb{D})}_{p}$ is a closed $S^* \oplus \cdots \oplus S^*$- invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$,

$$\|F\|^2 = \|K_0\|^2 + \sum_{j=1}^{p} \|k_j\|^2, \tag{4.9}$$

and F_0 given in Lemma 4.2.4. Therefore by the Beurling-Lax-Halmos theorem on $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$, there exists a non negative integer $r' \leq r+p$ and an inner multiplier $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}^{r'},\mathbb{C}^{r+p})}(\mathbb{D})$ unique upto unitary equivalance such that $\mathcal{K} = H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{r'}}(\mathbb{D})$. Thus if we consider $f \in \mathcal{M}$, then there exists $(K_0, k_1, k_2, \dots, k_p) \in \mathcal{K}$ such that

$$Uf = [Ug_1, Ug_2, \dots, Ug_r]K_0 + \sum_{j=1}^{p} Sk_jUf_j$$

and

$$\|f\|^2 = \|Uf\|^2 = \|K_0\|^2 + \sum_{j=1}^{p} \|k_j\|^2. \tag{4.10}$$

Let $K_0 = (k_1^0, k_2^0, \dots, k_r^0) \in H^2_{\mathbb{C}^r}(\mathbb{D})$, then

$$Uf = [Ug_1, Ug_2, \dots, Ug_r]K_0 + \sum_{j=1}^{p} Sk_jUf_j = \sum_{i=1}^{r} (Ug_i)k_i^0 + \sum_{j=1}^{p} S(Uf_j)k_j$$

and therefore by using Lemma 4.2.5 we get

$$U^*(Uf) = U^*\Big[\sum_{i=1}^{r}(Ug_i)k_i^0 + \sum_{j=1}^{p} S(Uf_j)k_j\Big] = U^*\Big[\sum_{i=1}^{r}(Ug_i)k_i^0 + \sum_{j=1}^{p} U(Tf_j)k_j\Big]$$
$$= K_0(T)G_0 + TK_1(T)F_1,$$

and hence

$$f = K_0(T)G_0 + TK_1(T)F_1,$$

where $K_1 = (k_1, k_2, \dots, k_p) \in H^2_{\mathbb{C}^p}(\mathbb{D})$. Therefore

$$\mathcal{M} = \Big\{ f \in \mathcal{H} : f = K_0(T)G_0 + TK_1(T)F_1 : (K_0, K_1) \in H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{r'}}(\mathbb{D}) \Big\}.$$

Moreover, the relation (4.9) gives the existence of an isometry $V : U\mathcal{M} \to H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{r'}}(\mathbb{D})$. Now if we define $Q = VU$, then $Q : \mathcal{M} \to H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{r'}}(\mathbb{D})$ is an isometry and the isometric relation is given by (4.10) . This completes the proof of (i).

For case (ii), we assume $\mathcal{M} \subset T\mathcal{H}$ and hence $\mathcal{M} \ominus (\mathcal{M} \cap T\mathcal{H}) = \{0\}$. Therefore again by applying Theorem 2.3.5 [case (ii)] of Chapter 2 we have

$$U\mathcal{M} = \left\{ F \in H^2_{\mathbb{C}^m}(\mathbb{D}) : F(z) = \sum_{j=1}^{p} z k_j(z) U f_j(z) : (k_1, \dots, k_p) \in \mathcal{K} \right\},$$

where $\mathcal{K} \subset \underbrace{H^2_{\mathbb{C}}(\mathbb{D}) \times \cdots \times H^2_{\mathbb{C}}(\mathbb{D})}_{p}$ is a closed $S^* \oplus \cdots \oplus S^*$- invariant subspace of the vector valued Hardy space $H^2_{\mathbb{C}^p}(\mathbb{D})$ and

$$\|F\|^2 = \sum_{j=1}^{p} \|k_j\|^2. \tag{4.11}$$

Similarly as in case (i), there exists a non negative integer $p' \leq p$ and an inner multiplier $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^{p'}, \mathbb{C}^p)}(\mathbb{D})$ unique upto an unitary equivalance such that $\mathcal{K} = H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{p'}}(\mathbb{D})$. Moreover, if $K_1 = (k_1, k_2, \dots, k_p) \in H^2_{\mathbb{C}^p}(\mathbb{D})$, then

$$\mathcal{M} = \Big\{ f \in \mathcal{H} : f = TK_1(T)F_1 : K_1 \in H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^{p'}}(\mathbb{D}) \Big\}.$$

Furthermore, the equation (4.11) gives an existence of an isometry $W : U\mathcal{M} \to H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{p'}}(\mathbb{D})$ and therefore if we define $R = WU$, then $R : \mathcal{M} \to H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^{p'}}(\mathbb{D})$ is an isometry. This completes the proof of (ii). □

Motivated from the Corollary 2.6 in [39] we have the following corollary which characterize the nearly T_B^{-1} invariant subspace with finite defect p in $H^2_{\mathbb{C}}(\mathbb{D})$ as a consequence of the above

Theorem 4.2.6. Note that for any finite Blaschke B with degree m, the operator $T_B : H^2_{\mathbb{C}}(\mathbb{D}) \to H^2_{\mathbb{C}}(\mathbb{D})$ is a shift operator with multiplicity m.

Corollary 4.2.1. *Let $\mathcal{M} \subset H^2_{\mathbb{C}}(\mathbb{D})$ be a non trivial nearly T_B^{-1} invariant subspace with defect p, where B is a finite Blaschke of degree m having atleast one zero in $\mathbb{D} \setminus \{0\}$. Let $G_0 = [g_1, g_2, \dots, g_r]^t$ be an $r \times 1$ matrix with $\{g_1, g_2, \dots, g_r\}$ is an orthonormal basis of $\mathcal{M} \ominus (\mathcal{M} \cap T_B\mathcal{H})$ and let $F_1 = [f_1, f_2, \dots, f_p]^t$ be a $p \times 1$ matrix containing an orthonormal basis $\{f_1, f_2, \dots, f_p\}$ of the defct space $\mathcal{F}$. Then there exists a non negative integer $r' \leq r+p$ and an inner multiplier $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^{r+p})}(\mathbb{D})$, unique upto unitary equivalence such that*

$$\mathcal{M} = \Big\{ f \in \mathcal{H} : f = K_0(T_B)G_0 + T_B K_1(T_B)F_1 : (K_0, K_1) \in H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{r'}}(\mathbb{D}) \Big\}. \tag{4.12}$$

The following example gives a better understanding of the above corollary which is same as Example 2.7 in [39] with a small variation.

Example 4.2.7. *Let us define $B_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ for any $a \in \mathbb{D} \setminus \{0\}$. Now consider the subspace*

$$\mathcal{M} = B_a(z) \cdot \Big\{ \bigvee \{1, z^2, z^6, z^8, z^{10}, \dots\} \oplus \bigvee \{z, z^3, z^5, \cdots, z^{2m+1}\} \Big\}$$

for some $m \in \mathbb{N} \cup \{0\}$. Then $\mathcal{M}$ is a nearly $T^*_{z^2}$-invariant subspace of $H^2_{\mathbb{C}}(\mathbb{D})$ with defect 1. It is easy to observe that $\dim(\mathcal{M} \ominus (\mathcal{M} \cap T_{z^2} H^2_{\mathbb{C}}(\mathbb{D}))) = 2$, $G_0 = B_a(z) \cdot [1, z]^t$ and the defect space is $\mathcal{F} = \langle z^4 \phi_a(z) \rangle$ with $F_1 = [z^4 \phi_a(z)]$. Therefore for any $f \in \mathcal{M}$, we have

$$f(z) = \left[\sum_{k=0}^{\infty} a_{k1} z^{2k}, \sum_{k=0}^{\infty} a_{k2} z^{2k} \right] G_0(z) + T_{z^2} \left[\sum_{k=0}^{\infty} b_k z^{2k} \right] F_1,$$

where the constants a_{k1}, a_{k2} and b_k satisfy the following:

$$\begin{cases} a_{k1} \in \mathbb{C} \text{ for } k \in \{0, 1\} \text{ and } a_{k1} = 0 \text{ for } k \geq 2, \\ a_{k2} \in \mathbb{C} \text{ for } k \in \{0, 1, \cdots, m\} \text{ and } a_{k2} = 0 \text{ for } k \geq m+1, \\ b_k \in \mathbb{C} \text{ for } k \geq 0. \end{cases}$$

Moreover, the equation (4.12) along with above discussions conclude

$$\mathcal{M} = \Big\{ f \in \mathcal{H} : f = K_0(T_{z^2})G_0 + T_{z^2} K_1(T_{z^2})F_1 : (K_0, K_1) \in H^2_{\mathbb{C}^{2+1}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}}(\mathbb{D}) \Big\},$$

where $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}, \mathbb{C}^3)}(\mathbb{D})$ is an inner multiplier such that $\Phi(z) = (z^2, z^{m+1}, 0) \in \mathbb{C}^3$.

4.3 Description of nearly T_B^{-1} invariant subspaces with defect for finite Blaschke B in $\mathcal{D}_\alpha$ Spaces

In this section, we discuss about nearly T_B^{-1} invariant subspaces with finite defect corresponding to any finite Blaschke product B in a scale of $\mathcal{D}_\alpha$ spaces for $\alpha \in [-1, 1]$ by combining the ideas of Erard [21] and Liang and Partington [39] with appropriate changes. Recall that any finite Blaschke product B is a multiplier of each $\mathcal{D}_\alpha$, that is, the multiplication operator $T_B : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is defined everywhere and bounded. Moreover, the operator T_B is bounded below but not an isometry. We refer to the reader concerning the work of Lance and Stessin [37] in connection with the study of multiplication invariant subspaces of Hardy spaces. In [21], C. Erard studied the nearly invariant subspaces corresponding to lower bounded multiplication operator M_u on the Hilbert space of analytic functions $\mathcal{H}$ and there are four conditions concerning the pairs $(\mathcal{H}, u)$ which are as follows:

(i) $\mathcal{H}$ *is a Hilbert space and a linear subspace of* $\mathcal{O}(\mathcal{W}) := \{f : \mathcal{W} \to \mathbb{C} |\ f \text{ is analytic}\}$, *where* $\mathcal{W}$ *is an open subset of* $\mathbb{C}^d$ $(d \in \mathbb{N})$,

(ii) $u \in \mathcal{O}(\mathcal{W})$ *satisfies* $uh \in \mathcal{H}$ *for all* $h \in \mathcal{H}$,

(iii) for all $w \in \mathcal{W}$ *the evaluation* $\mathcal{H} \to \mathbb{C}$, $h \to h(w)$ *is continuous,*

(iv) there exists $c > 0$ *such that for all* $h \in \mathcal{H}$ $c\|h\|_{\mathcal{H}} \le \|uh\|_{\mathcal{H}}$.

Corresponding to the above pair $(\mathcal{H}, u)$, the lower bound *of the multiplication operator M_u relative to the norm $\|.\|_{\mathcal{H}}$ is defined by*

$$\gamma_{\mathcal{H},M_u} = sup\{c > 0 : \forall h \in \mathcal{H}, c\|h\|_{\mathcal{H}} \le \|uh\|_{\mathcal{H}}\} \in (0, \infty). \tag{4.13}$$

For simplicity we denote $\gamma_{\mathcal{H},M_u}$ by γ. In particular, for the pair $(\mathcal{H}, u(z) = z)$, Erard gives a connection between nearly backward shift invariant subspaces in $\mathcal{H}$ and a backward shift invariant subspaces in $H^2_{\mathbb{C}}(\mathbb{D})$ (see Theorem 5.1 in [21]). Note that the operator $T_B : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is more general than $M_z : H^2_{\mathbb{C}}(\mathbb{D}) \to H^2_{\mathbb{C}}(\mathbb{D})$ and the characterizations for nearly T_B^{-1} invariant subspaces in $\mathcal{D}_\alpha$ for $\alpha \in [-1, 1]$ corresponding to the finite Blaschke product B is due to Liang and Partington (see Theorem 3.4 and Theorem 3.7 in [39]) by applying some results of Erard

[21]. Here our main aim is to characterize nearly T_B^{-1} invariant subspaces with finite defect in $\mathcal{D}_\alpha$ for $\alpha \in [-1,1]$ corresponding to the finite Blaschke product B. To achieve our goal we need to first extend two important results (namely Approximation Lemma and Factorization Theorem) due to Erard [21]. Before we proceed, note that if $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator that is bounded from below, then T has closed range and T^*T is invertible. Now using the ideas in [21] with suitable modification we have the following lemma which provides a generalization of Lemma 2.1. in [21].

Lemma 4.3.1 (Approximation Lemma). *Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a bounded operator such that for all $h \in \mathcal{H}$, $\|h\|_{\mathcal{H}} \leq \|Th\|_{\mathcal{H}}$. Suppose $\mathcal{M}$ is a nearly T^{-1} invariant subspace of $\mathcal{H}$ with defect p (i.e. the dimension of the defect space $\mathcal{F}$ is p). We set $R = (T^*T)^{-1}T^*P_{\mathcal{M}\cap T\mathcal{H}}$, $Q = P_{\mathcal{M}\ominus(\mathcal{M}\cap T\mathcal{H})}$, $S = P_{\mathcal{F}}$. Then $\|R\| \leq 1$, and for all $h \in \mathcal{M}$ and $m \in \mathbb{N}$, we have*

$$h = \sum_{k=0}^{m} T^kQR^kh + T^{m+1}R^{m+1} + T\sum_{k=1}^{m} T^{k-1}SR^kh \tag{4.14}$$

and

$$\|h\|^2_{\mathcal{H}} \geq \sum_{k=0}^{\infty}\|QR^kh\|^2_{\mathcal{H}} + \sum_{k=1}^{\infty}\|SR^kh\|^2_{\mathcal{H}}. \tag{4.15}$$

Proof. Consider $h \in \mathcal{H}$ and write $P_{\mathcal{M}\cap T\mathcal{H}}(h) = Th_0$. Then we have

$$TRh = T(T^*T)^{-1}T^*Th_0 = Th_0 = P_{\mathcal{M}\cap T\mathcal{H}}(h). \tag{4.16}$$

Thus for any $h \in \mathcal{H}$, we have $\|Rh\| \leq \|TRh\| = \|P_{\mathcal{M}\cap T\mathcal{H}}(h)\| \leq \|h\|$ and hence $\|R\| \leq 1$. Suppose $h \in \mathcal{M}$ and therefore by using (4.16) we conclude that $TRh \in \mathcal{M}$. Since $\mathcal{M}$ is a nearly T^{-1} invariant subspace with defect p, then we have

$$Rh \in \mathcal{M} \oplus \mathcal{F}. \tag{4.17}$$

Moreover, by using (4.16) and since T is bounded below we have for any $h \in \mathcal{M}$,

$$h = Qh + TRh \tag{4.18}$$

and

$$\|h\|^2 \geq \|Qh\|^2 + \|TRh\|^2 \geq \|Qh\|^2 + \|Rh\|^2. \tag{4.19}$$

Since $Rh \in \mathcal{M} \oplus \mathcal{F}$ (by (4.17)), then we have

$$Rh = P_{\mathcal{M}}Rh + SRh$$

which implies that $Rh - SRh \in \mathcal{M}$. Note that since (4.18) is true for any $h \in \mathcal{M}$, therefore if we replace h by $Rh - SRh$ in (4.18) we get

$$Rh = QRh + TR^2h + SRh. \tag{4.20}$$

Now it is easy to observe that $R(\mathcal{M} \oplus \mathcal{F}) \subset \mathcal{M} \oplus \mathcal{F}$ and hence $R^m h \in \mathcal{M} \oplus \mathcal{F}$, $\forall m \in \mathbb{N}$. Therefore by induction from (4.20) we get for any $m \in \mathbb{N}$,

$$R^m h = QR^m h + TR^{m+1}h + SR^m h \tag{4.21}$$

and since T is bounded below we have

$$\|R^m h\|^2 \geq \|QR^m h\|^2 + \|R^{m+1}h\|^2 + \|SR^m h\|^2. \tag{4.22}$$

Finally by combining (4.18) and (4.21) we have

$$h = \sum_{k=0}^{m} T^k QR^k h + T^{m+1}R^{m+1}h + T\sum_{k=1}^{m} T^{k-1}SR^k h, \quad m \in \mathbb{N}$$

and moreover equations (4.19) and (4.22) yield that

$$\|h\|^2_{\mathcal{H}} \geq \sum_{k=0}^{\infty} \|QR^k h\|^2_{\mathcal{H}} + \sum_{k=1}^{\infty} \|SR^k h\|^2_{\mathcal{H}}.$$

This completes the proof. □

Remark 4.3.2. *Under the same assumtion as in Lemma 4.3.1, let $\mathcal{M}$ be a nearly T^{-1} invariant subspace of $\mathcal{H}$ with defect p such that $\mathcal{M} \subseteq T\mathcal{H}$ and let $\mathcal{F}$ be the corresponding p dimensional defect space having an orthonormal basis $\{e_j\}_{j=1}^{p}$. Then for any $h \in \mathcal{M}$ and $m \in \mathbb{N}$ we have*

$$h = T^{m+1}R^{m+1} + T\sum_{k=1}^{m} T^{k-1}SR^k h \quad \text{and} \quad \|h\|^2_{\mathcal{H}} \geq \sum_{k=1}^{\infty} \|SR^k h\|^2_{\mathcal{H}}. \tag{4.23}$$

Next we denote $D(0,a) := \{z \in \mathbb{C} : |z| < a\}$. As an application of the above Approximation Lemma and mimicking the ideas given in [21] with appropriate changes we have the following theorem which gives a generalization of Theorem 3.2 in [21].

Theorem 4.3.3 (Factorization Theorem). *Assume that the pair $(\mathcal{H}, u)$ satisfies the four conditions (i)-(iv) given in the beginning of Section 4.3. Let $\mathcal{M}$ be a nearly M_u^{-1} invariant subspace of $\mathcal{H}$ with defect p and let $\mathcal{F}$ be the corresponding defect space. Let $\{g_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{M} \ominus (\mathcal{M} \cap M_u\mathcal{H})$ and let $\{e_j\}_{j=1}^p$ be an orthonormal basis of $\mathcal{F}$. Moreover, we also assume that*

$$\bigcap_{n\in\mathbb{N}} u^n|_{u^{-1}(D(0,\gamma))}\mathcal{H}|_{u^{-1}(D(0,\gamma))} = \{0\}, \tag{4.24}$$

where $\mathcal{H}|_{u^{-1}(D(0,\gamma))}$ consists of the restrictions to $u^{-1}(D(0,\gamma))$ of the functions of $\mathcal{H}$. Then (i) in the case when $\mathcal{M} \nsubseteq M_u\mathcal{H}$, for all $h \in \mathcal{M}$, there exist $(q_i)_{i\in I}$ and $(h_j)_{j=1}^p$ in $\mathcal{O}(u^{-1}(D(0,\gamma)))$ such that

$$h = \sum_{i\in I} g_i q_i + \gamma^{-1} M_u \sum_{j=1}^{p} e_j h_j$$

on $u^{-1}(D(0,\gamma))$ for all $i \in I$ and $j \in \{1,\ldots,p\}$, and also there exist $(c_{ki})_{k\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}}$ and $(b_{kj})_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with

$$q_i = \sum_{k=0}^{\infty} c_{ki} \left(\frac{u}{\gamma}\right)^k, \quad h_j = \sum_{k=1}^{\infty} b_{kj} \left(\frac{u}{\gamma}\right)^{k-1} \tag{4.25}$$

and

$$\sum_{i\in I}\sum_{k=0}^{\infty} |c_{ki}|^2 + \sum_{j=1}^{p}\sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|_{\mathcal{H}}^2. \tag{4.26}$$

(ii) In the case when $\mathcal{M} \subseteq M_u\mathcal{H}$, then for all $h \in \mathcal{M}$ there exists $(h_j)_{j=1}^p$ in $\mathcal{O}(u^{-1}(D(0,\gamma)))$ such that

$$h = \gamma^{-1} M_u \sum_{j=1}^{p} e_j h_j \quad \text{on } u^{-1}(D(0,\gamma))$$

for all $j \in \{1,2,\ldots,p\}$ and also there exists $(b_{kj})_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that

$$h_j = \sum_{k=1}^{\infty} b_{kj} \left(\frac{u}{\gamma}\right)^{k-1} \quad \text{and} \quad \sum_{j=1}^{p}\sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2.$$

Proof. (i) First we consider $T = \gamma^{-1}M_u$. Then T satisfies the hypothesis of Lemma 4.3.1. Now we define R, Q, S as in Lemma 4.3.1 and let $h \in \mathcal{M}$. Then we define a family of sequences $\{(c_{ki})_{k\in\mathbb{N}_0}\}_{i\in I}$, $\{(b_{kj})_{k\in\mathbb{N}}\}_{j=1}^p$ of complex numbers by the following equations

$$QR^k h = \sum_{i\in I} c_{ki} g_i, \quad k \in \mathbb{N}_0 \quad \text{and} \quad SR^k h = \sum_{j=1}^{p} b_{kj} e_j \quad k \in \mathbb{N}.$$

Therefore by using (4.14) and (4.15) we get

$$\begin{aligned} h &= \sum_{k=0}^{m} T^k Q R^k h + T^{m+1} R^{m+1} h + \sum_{k=1}^{m} T^k S R^k h \\ &= \sum_{k=0}^{m} \sum_{i \in I} c_{ki} T^k g_i + T^{m+1} R^{m+1} h + T \sum_{k=1}^{m} \sum_{j=1}^{p} b_{kj} T^{k-1} e_j, \end{aligned}$$

and hence

$$h = \sum_{k=0}^{m} \sum_{i \in I} c_{ki} \left(\frac{u}{\gamma}\right)^k g_i + \left(\frac{u}{\gamma}\right)^{m+1} R^{m+1} h + \gamma^{-1} u \sum_{k=1}^{m} \sum_{j=1}^{p} b_{kj} \left(\frac{u}{\gamma}\right)^{k-1} e_j, \tag{4.27}$$

and

$$\sum_{i \in I} \sum_{k=0}^{\infty} |c_{ki}|^2 + \sum_{j=1}^{p} \sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2, \tag{4.28}$$

so that for all $i \in I$ and $j \in \{1, 2, \ldots, p\}$,

$$\sum_{k=0}^{\infty} |c_{ki}|^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |b_{kj}|^2 < \infty.$$

Therefore it follows that for all $i \in I$, the series $\sum\limits_{k=0}^{\infty} c_{ki} \left(\frac{u}{\gamma}\right)^k$ converges uniformly on compact subsets of $u^{-1}(D(0,\gamma))$, so that its sum, which we denote by q_i, belongs to $\mathcal{O}(u^{-1}(D(0,\gamma)))$. Similarly the series $\sum\limits_{k=1}^{\infty} b_{kj} \left(\frac{u}{\gamma}\right)^{k-1}$ also converges uniformly on compact subsets of $u^{-1}(D(0,\gamma))$ and hence the sum of the series denoted by h_j also belongs to $\mathcal{O}(u^{-1}(D(0,\gamma)))$. Let $w \in u^{-1}(D(0,\gamma))$, then by using Cauchy-Schwarz inequality and (4.28) we obtain

$$\begin{aligned} \sum_{i \in I} |(g_i q_i)(w)| &\leq \left(\sum_{i \in I} |g_i(w)|^2\right)^{\frac{1}{2}} \left(\sum_{i \in I} |q_i(w)|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i \in I} |\langle g_i, k_w \rangle|^2\right)^{\frac{1}{2}} \left(\sum_{i \in I} (\sum_{k=0}^{\infty} |c_{ki}|^2)(\sum_{k=0}^{\infty} \frac{|u(w)|^{2k}}{\gamma^{2k}})\right)^{\frac{1}{2}} \leq \|Qk_w\|_{\mathcal{H}} \|h\|_{\mathcal{H}} \frac{1}{\sqrt{1 - \frac{|u(w)|^2}{\gamma^2}}}, \end{aligned}$$

and

$$\sum_{j=1}^{p} |(e_j h_j)(w)| \leq \|Sk_w\|_{\mathcal{H}} \|h\|_{\mathcal{H}} \frac{1}{\sqrt{1 - \frac{|u(w)|^2}{\gamma^2}}},$$

and hence that both the series $\sum\limits_{i \in I} g_i q_i$ and $\sum\limits_{j=1}^{p} e_j h_j$ converge at each point of $u^{-1}(D(0,\gamma))$. Now from equation (4.27) we obtain

$$(h - \sum_{i \in I} g_i q_i - \sum_{j=1}^{p} e_j h_j)|_{u^{-1}(D(0,\gamma))} \in \bigcap_{m \in \mathbb{N}} u^m|_{u^{-1}(D(0,\gamma))} \mathcal{H}|_{u^{-1}(D(0,\gamma))},$$

which along with the hypothesis (4.24) implies that

$$h = \sum_{i\in I} g_i q_i + \gamma^{-1} M_u \sum_{j=1}^{p} e_j h_j \text{ on } u^{-1}(D(0,\gamma)) \quad \text{and} \quad \sum_{i\in I}\sum_{k=0}^{\infty} |c_{ki}|^2 + \sum_{j=1}^{p}\sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2.$$

(ii) Again we consider $T = \gamma_1^{-1} M_u$. Therefore by using Remark 4.3.2 and proceeding as in case (i) we obtain

$$h = \gamma^{-1} M_u \sum_{j=1}^{p} e_j h_j \text{ on } u^{-1}(D(0,\gamma)) \quad \text{and} \quad \sum_{j=1}^{p}\sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|h\|^2.$$

□

Now we are in a position to describe the nearly T_B^{-1} invariant subspaces with defect p corresponding to a finite Blaschke B in Dirichlet type spaces $\mathcal{D}_\alpha$ for $\alpha \in [-1,1]$. Now onwards we assume that B is Blaschke product of degree m and therefore for any non trivial nearly T_B^{-1} invariant subspace $\mathcal{M}$ in $\mathcal{D}_\alpha$ with defect p and $\mathcal{M} \nsubseteq T_B\mathcal{D}_\alpha$ we have

$$1 \leq r := dim(\mathcal{M} \ominus (\mathcal{M} \cap T_B\mathcal{D}_\alpha)) \leq m,$$

which follows by similar argument as in Lemma 4.2.3. In the sequel, we now endow the space $\mathcal{D}_\alpha$ with two different equivalent norms introduced by Liang and Partington (see Section 3, [39]) according to the cases $\alpha \in [-1,0)$ and $\alpha \in [0,1]$ and hence we divide the analysis into two subsections.

4.3.1 $\alpha \in [-1,0)$:

Note that we need to endow the space $\mathcal{D}_\alpha$ with a norm in such a way so that we can get a nice lower bound of the operator T_B. Keeping this information in our mind we endow the space $\mathcal{D}_\alpha$ for $\alpha \in [-1,0)$ with the modified equivalent norm introduced by Liang and Partington in [39] which we denote by $\|\cdot\|_1$, and it is as follows: for any $f = \sum_{n=0}^{\infty} f_n B^n$ with $f_n \in \mathcal{K}_B$,

$$\|f\|_1^2 := \sum_{n=0}^{G-1} G^\alpha \|f_n\|^2_{H^2_{\mathbb{C}}(\mathbb{D})} + \sum_{n=G}^{\infty} (n+1)^\alpha \|f_n\|^2_{H^2_{\mathbb{C}}(\mathbb{D})}, \tag{4.29}$$

where G is a fixed and sufficiently large positive number to be specified below and moreover we have the same lower bound of T_B, as obtained in (see Section 3,[39]), as follows:

$$\gamma_1 := \left(1 - \frac{1}{G+1}\right)^{-\alpha/2}. \tag{4.30}$$

Thus from the definition of lower bound it follows that for any $f \in \mathcal{D}_\alpha$,

$$\|T_B f\|_1^2 = \|Bf\|_1^2 \geq \gamma_1^2 \|f\|_1^2,$$

and hence the operator $T := \gamma_1^{-1} T_B : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ *satisfies*

$$\|Tf\|_1^2 = \|\gamma_1^{-1} T_B f\|_1^2 \geq \|f\|_1^2 \quad \text{for any} \quad f \in \mathcal{D}_\alpha.$$

Note that the pair $(\mathcal{D}_\alpha, T_B)$ *also satisfies conditions* (i)-(iv) *(given in the beginning of Section 4.3) with lower bound* γ_1 *given in* (4.30)*. Furthermore as in ([39]), we choose* G *large enough so that* γ_1 *satisfies* $B^{-1}(D(0, \gamma_1)) \supset s\mathbb{D}$ *with* $s\mathbb{D}$ *a disc containing all the zeros of* B *which ensures that*

$$\|\gamma_1^{-1} B\|_{H^\infty(s\mathbb{D})} < 1. \tag{4.31}$$

Moreover, the operator $T := \gamma_1^{-1} T_B$ *satisfies all the assumptions in Lemma 4.3.1 together with the fact that*

$$\bigcap_{m \in \mathbb{N}} B^m \mathcal{D}_\alpha|_{s\mathbb{D}} = \bigcap_{m \in \mathbb{N}} T^m \mathcal{D}_\alpha|_{s\mathbb{D}} = \{0\}.$$

Combining the above facts together with Theorem 4.3.3 implies the following lemma, providing a generalization of Lemma 3.6 in [39].

Lemma 4.3.4. *Let* $\mathcal{M}$ *be a non trivial nearly* T_B^{-1} *invariant subspace of* $\mathcal{D}_\alpha$ *with defect* p *for* $\alpha \in [-1, 0)$ *and let* $\mathcal{F}$ *be the corresponding* p *dimensional defect space. Let* $\{f_i\}_{i=1}^r$ *and* $\{e_j\}_{j=1}^p$ *be an orthonormal basis of* $\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)$ *and* $\mathcal{F}$ *respectively. Then for all* $f \in \mathcal{M}$*, there exist* $\{q_i\}_{i=1}^r$ *and* $\{h_j\}_{j=1}^p$ *in* $\mathcal{O}(s\mathbb{D})$ *such that*

$$f = \sum_{i=1}^r f_i q_i + \gamma_1^{-1} T_B \sum_{j=1}^p e_j h_j \quad \text{on } s\mathbb{D}, \tag{4.32}$$

for all $i \in \{1, 2, \ldots, r\}$ *and* $j \in \{1, 2, \ldots, p\}$*, and also there exist* $(a_{ki})_{k \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}}$ *and* $(b_{kj})_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ *with*

$$q_i = \sum_{k=0}^\infty a_{ki} \left(\gamma_1^{-1} B\right)^k \quad \text{on } s\mathbb{D}, \quad h_j = \sum_{k=1}^\infty b_{kj} \left(\gamma_1^{-1} B\right)^{k-1} \quad \text{on } s\mathbb{D}, \tag{4.33}$$

and

$$\sum_{i=1}^r \sum_{k=0}^\infty |a_{ki}|^2 + \sum_{j=1}^p \sum_{k=1}^\infty |b_{kj}|^2 \leq \|f\|_{\mathcal{D}_\alpha}^2. \tag{4.34}$$

Remark 4.3.5. *If the subspace $\mathcal{M} \subseteq T_B\mathcal{D}_\alpha$, then using the same notation as in Lemma 4.3.4, for all $f \in \mathcal{M}$ there exists $\{h_j\}_{j=1}^{p}$ in $\mathcal{O}(s\mathbb{D})$ such that*

$$f = \gamma_1^{-1} T_B \sum_{j=1}^{p} e_j h_j \text{ on } s\mathbb{D},$$

and also there exists $(b_{kj})_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ with

$$h_j = \sum_{k=1}^{\infty} b_{kj}\left(\gamma_1^{-1}B\right)^{k-1} \quad \text{and} \quad \sum_{j=1}^{p}\sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|f\|^2_{\mathcal{D}_\alpha}.$$

Here our main aim is to describe the nearly T_B^{-1} invariant subspaces of $\mathcal{D}_\alpha$ with finite defect for $\alpha \in [-1, 0)$ in terms of $T_{B^{-1}}$ invariant subspaces of $H^2_{\mathbb{C}^{r+p}}(s\mathbb{D})$. In order to get a connection with invariant subspaces of $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$ we introduce the same unitary mapping $U_s : H^2_{\mathbb{C}^{r+p}}(s\mathbb{D}) \to H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$ as mentioned in (see Section 3, [39]) and it is defined by

$$(U_s f)(z) = f(sz).$$

If we denote $T_s^* := U_s T_{B^{-1}} U_s^*$, then we have the following commutative diagram (4.35)

$$\begin{array}{ccc} H^2_{\mathbb{C}^{r+p}}(s\mathbb{D}) & \xrightarrow{T_B^{-1}} & H^2_{\mathbb{C}^{r+p}}(s\mathbb{D}) \\ U_s \downarrow & & \downarrow U_s \\ H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) & \xrightarrow{T_s^*} & H^2_{\mathbb{C}^{r+p}}(\mathbb{D}). \end{array} \tag{4.35}$$

Since the disc $s\mathbb{D}$ contains all the zeros of B, then the symbol B^{-1} lies in $L^\infty(s\mathbb{T})$ and therefore by using the fact $B^{-1}(sz) = \overline{B(s^{-1}z)}$ on $\mathbb{T}$ and by repeating the identical calculations as done in ([39]) we conclude

$$(T_s^* f)(z) = T_{\overline{B(s^{-1}z)}} f(z). \tag{4.36}$$

For more details about (4.36) *(see (3.18), Section 3 in [39]). Now we state our main theorem in this subsection concerning nearly T_B^{-1} invariant subspaces with defect p in $\mathcal{D}_\alpha$ spaces with $\alpha \in [-1, 0)$ based on above notations which gives a generalization of Theorem 3.7 in [39].*

Theorem 4.3.6. *Let $\mathcal{M}$ be a nearly T_B^{-1} invariant subspace of $\mathcal{D}_\alpha$ with finite defect p for $\alpha \in [-1, 0)$ and let $\mathcal{F}$ be the corresponding p dimensional defect space. Let $E_0 := [e_1, e_2, \ldots, e_p]$, where $\{e_j\}_{j=1}^{p}$ is an orthonormal basis of $\mathcal{F}$ using norm $\|.\|_1$. Then* *(i) in the case when $\mathcal{M} \nsubseteq T_B\mathcal{D}_\alpha$, if $F_0 := [f_1, f_2, \dots, f_r]$ is a matrix containing an orthonormal basis $\{f_i\}_{i=1}^r$ of $\mathcal{M} \ominus (\mathcal{M} \cap T_B\mathcal{D}_\alpha)$, then there exists a linear subspace $\mathcal{N} \subset H^2_{\mathbb{C}^{r+p}}(s\mathbb{D})$ such that*

$$\mathcal{M} = \left\{ f \in \mathcal{D}_\alpha : f = F_0 q + \gamma_1^{-1} T_B E_0 h \quad \text{on } s\mathbb{D} : (q, h) \in \mathcal{N} \right\} \quad \text{on } s\mathbb{D},$$

together with

$$\left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{1/2} \left(\|q\|^2_{H^2_{\mathbb{C}^r}(s\mathbb{D})} + \|h\|^2_{H^2_{\mathbb{C}^p}(s\mathbb{D})} \right)^{1/2} \leq \|f\|_{\mathcal{D}_\alpha}.$$

Moreover, $\mathcal{N}$ is invariant under T_B^{-1} and hence $U_s(\mathcal{N})$ is invariant under $T_s^* = U_s T_{B^{-1}} U_s^*$ in $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$.

(ii) In the case when $\mathcal{M} \subset T_B\mathcal{D}_\alpha$, then there exists a linear subspace $\mathcal{N} \subset H^2_{\mathbb{C}^p}(s\mathbb{D})$ such that

$$\mathcal{M} = \left\{ f \in \mathcal{D}_\alpha : f = \gamma_1^{-1} T_B E_0 h : h \in \mathcal{N} \right\} \quad \text{on } s\mathbb{D},$$

together with

$$\left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{1/2} \|h\|_{H^2_{\mathbb{C}^p}(s\mathbb{D})} \leq \|f\|_{\mathcal{D}_\alpha}.$$

Moreover, $\mathcal{N}$ is invariant under T_B^{-1} and hence $U_s(\mathcal{N})$ is invariant under $T_s^* = U_s T_{B^{-1}} U_s^*$ in $H^2_{\mathbb{C}^p}(\mathbb{D})$ (note that, here $U_s : H^2_{\mathbb{C}^p}(s\mathbb{D}) \to H^2_{\mathbb{C}^p}(\mathbb{D})$).

Proof. (i) For $f \in \mathcal{M} \subset \mathcal{D}_\alpha$ with $\alpha \in [-1, 0)$, the equation (4.32) in the above Lemma 4.3.4 implies

$$f = \sum_{i=1}^r f_i q_i + \gamma_1^{-1} T_B \sum_{j=1}^p e_j h_j = F_0 q + \gamma_1^{-1} T_B E_0 h, \quad \text{on } s\mathbb{D} \tag{4.37}$$

where $q = [q_1, q_2, \dots, q_r]^t$ and $h = [h_1, h_2, \dots, h_p]^t$. Using the facts (4.31) and (4.33) we obtain the following for all $i \in \{1, 2, \dots, r\}$ and $j \in \{1, 2, \dots, p\}$,

$$\begin{aligned} \|q_i\|_{H^2(s\mathbb{D})} &= \|\sum_{k=0}^\infty a_{ki} (\gamma_1^{-1}B)^k\|_{H^2(s\mathbb{D})} \leq \sum_{k=0}^\infty |a_{ki}| \|\gamma_1^{-1}B\|^k_{H^\infty(s\mathbb{D})} \\ &\leq \left(\sum_{k=0}^\infty \|\gamma_1^{-1}B\|^{2k}_{H^\infty(s\mathbb{D})} \right)^{1/2} \left(\sum_{k=0}^\infty |a_{ki}|^2 \right)^{1/2} = \left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{-1/2} \left(\sum_{k=0}^\infty |a_{ki}|^2 \right)^{1/2}, \end{aligned}$$

and

$$\|h_j\|_{H^2(s\mathbb{D})} \leq \left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{-1/2} \left(\sum_{k=1}^\infty |b_{kj}|^2 \right)^{1/2}.$$

Therefore the above estimates along with the inequality in (4.34) yields

$$\|q\|^2_{H^2_{\mathbb{C}^r}(s\mathbb{D})} = \sum_{i=1}^{r} \|q_i\|^2_{H^2(s\mathbb{D})} \leq \left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{-1} \left(\sum_{i=1}^{r}\sum_{k=0}^{\infty} |a_{ki}|^2\right)$$
$$\leq \left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{-1} \|f\|^2_{\mathcal{D}_\alpha} < +\infty,$$

and

$$\|h\|^2_{H^2_{\mathbb{C}^p}(s\mathbb{D})} \leq \left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{-1} \|f\|^2_{\mathcal{D}_\alpha} < +\infty.$$

Thus the above implies

$$q = \sum_{k=0}^{\infty} A_k(\gamma_1^{-1}B)^k \in H^2_{\mathbb{C}^r}(s\mathbb{D}), \quad \text{where } A_k = [a_{k1}, a_{k2}, \ldots, a_{kr}]^t$$

and

$$h = \sum_{k=1}^{\infty} B_k(\gamma_1^{-1}B)^{k-1} \in H^2_{\mathbb{C}^p}(s\mathbb{D}), \quad \text{where } B_k = [b_{k1}, b_{k2}, \ldots, b_{kp}]^t.$$

Moreover, the equation (4.34) implies for all $f \in \mathcal{M}$,

$$\|q\|^2_{H^2_{\mathbb{C}^r}(s\mathbb{D})} + \|h\|^2_{H^2_{\mathbb{C}^p}(s\mathbb{D})} \leq \left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{-1} \|f\|^2_{\mathcal{D}_\alpha}. \tag{4.38}$$

Now we define a linear subspace as follows:

$$\mathcal{N} := \left\{(q,h) \in H^2_{\mathbb{C}^r}(s\mathbb{D})) \times H^2_{\mathbb{C}^p}(s\mathbb{D}) : \exists f \in \mathcal{M},\ f = F_0 q + \gamma_1^{-1} T_B E_0 h \quad \text{on } s\mathbb{D}\right\},$$

satisfying for any $f \in \mathcal{M}$, $\exists\ (q,h) \in \mathcal{N}$ such that $f = F_0 q + \gamma_1^{-1} T_B E_0 h$ on $s\mathbb{D}$. Next we show that $\mathcal{N}$ is invariant under $T_{B^{-1}}$. By considering $T = \gamma_1^{-1} T_B$ in Lemma 4.3.1, the equation (4.14) with $m = 0$ implies

$$f = Qf + TRf = Qf + \gamma_1^{-1} T_B R f.$$

Moreover, on $s\mathbb{D}$, the above equation together with (4.37) yields that

$$F_0 q + \gamma_1^{-1} T_B E_0 h = Q(F_0 q + \gamma_1^{-1} T_B E_0 h) + \gamma_1^{-1} T_B R(F_0 q + \gamma_1^{-1} T_B E_0 h)$$
$$= F_0 A_0 + \gamma_1^{-1} B R(F_0 q + \gamma_1^{-1} T_B E_0 h),$$

which further satisfies

$$F_0(q - A_0) + \gamma_1^{-1} T_B E_0 h = \gamma_1^{-1} B R(F_0 q + \gamma_1^{-1} T_B E_0 h).$$

Next by using the fact T_B is injective, we conclude from the above that

$$\gamma_1^{-1}R(F_0q+\gamma_1^{-1}T_BE_0h) = F_0(\sum_{k=1}^{\infty}A_k\gamma_1^{-k}B^{k-1}) + \gamma_1^{-1}E_0h = F_0(T_{B^{-1}}q) + \gamma_1^{-1}E_0h. \tag{4.39}$$

Moreover, by using the fact that $R(F_0q+\gamma_1^{-1}T_BE_0h) \in \mathcal{M}\oplus\mathcal{F}$ we obtain

$$R(F_0q+\gamma_1^{-1}T_BE_0h) = P^{\mathcal{M}}R(F_0q+\gamma_1^{-1}T_BE_0h) + E_0B_1. \tag{4.40}$$

Thus by combining equations (4.39) and (4.40) we get

$$\begin{aligned}\gamma_1^{-1}P_{\mathcal{M}}R(F_0q+\gamma_1^{-1}T_BE_0h) &= F_0(T_{B^{-1}}q) + \gamma_1^{-1}E_0(\sum_{k=2}^{\infty}B_k(\gamma_1^{-1}B)^{k-1})\\ &= F_0(T_{B^{-1}}q) + \gamma_1^{-1}T_BE_0(T_{B^{-1}}h).\end{aligned}$$

Note that $\gamma_1^{-1}P^{\mathcal{M}}R(F_0q+\gamma_1^{-1}T_BE_0h) \in \mathcal{M}$ and hence from the definition of $\mathcal{N}$ we conclude $(T_{B^{-1}}q, T_{B^{-1}}h) \in \mathcal{N}$. Thus $\mathcal{N}$ is $T_{B^{-1}}$ invariant in $H^2_{\mathbb{C}^{r+p}}(s\mathbb{D})$. Finally, by using the diagram (4.35) we have $T_s^*(U_s(\mathcal{N})) \subset U_s(\mathcal{N})$, that is $U_s(\mathcal{N})$ is invariant under T_s^*.

(ii) If $\mathcal{M} \subset T_B\mathcal{D}_\alpha$, then by using Remark 4.3.5 and proceeding as in case (i) we obtain a linear subspace $\mathcal{N} \subset H^2_{\mathbb{C}^p}(s\mathbb{D})$ such that

$$\mathcal{M} = \left\{ f \in \mathcal{D}_\alpha : f = \gamma_1^{-1}T_BE_0h : h \in \mathcal{N}\right\} \quad \text{on } s\mathbb{D},$$

together with

$$\left(1 - \|\gamma_1^{-1}B\|^2_{H^\infty(s\mathbb{D})}\right)^{1/2} \|h\|_{H^2_{\mathbb{C}^p}(s\mathbb{D})} \le \|f\|_{\mathcal{D}_\alpha}.$$

Moreover, $\mathcal{N}$ is invariant under T_B^{-1} and $U_s(\mathcal{N})$ is invariant under $T_s = U_sT_{B^{-1}}U_s^*$ in $H^2_{\mathbb{C}^p}(\mathbb{D})$. (note that here $U_s : H^2_{\mathbb{C}^p}(s\mathbb{D}) \to H^2_{\mathbb{C}^p}(\mathbb{D})$). This completes the proof. □

4.3.2 $\alpha \in [0,1]$:

Here we consider $\mathcal{D}_\alpha$ spaces with $\alpha \in [0,1]$ and B is a finite Blaschke product of degree m. We now endow $\mathcal{D}_\alpha$ with the following equivalent norm introduced by Liang and Partington in [39] which we denote by $\|.\|_2$ and is defined by

$$\|f\|_2^2 := \sum_{n=0}^{\infty}(n+1)^\alpha \|g_n\|^2_{H^2_{\mathbb{C}}(\mathbb{D})} \tag{4.41}$$

for any $f = \sum_{n=0}^{\infty} g_n B^n$ *with* $g_n \in \mathcal{K}_B$ *(see Theorem 4.1.1). Therefore we have,*

$$\|T_B f\|_2^2 = \|Bf\|_2^2 = \sum_{n=0}^{\infty}(n+2)^\alpha \|g_n\|^2_{H^2_{\mathbb{C}}(\mathbb{D})} \geq \|f\|_2^2$$

which implies that the operator $T_B : (\mathcal{D}_\alpha, \|.\|_2) \to (\mathcal{D}_\alpha, \|.\|_2)$ *is lower bounded and the lower bound (4.13) of* T_B *relative to the norm* $\|.\|_2$ *is* $\gamma_2 := 1$*. Moreover, the pair* $(\mathcal{D}_\alpha, B)$ *also satisfies the conditions* (i)-(iv) *given in the beginning of Section 4.3. Furthermore, it is easy to check that* $B^{-1}(D(0,1)) = B^{-1}(\mathbb{D}) = \mathbb{D}$ *and* $\bigcap_{m\in\mathbb{N}} B^m \mathcal{D}_\alpha = \{0\}$ *on* $\mathbb{D}$*. These facts along with Theorem 4.3.3 (with* $\mathcal{H} = \mathcal{D}_\alpha, u = B, \gamma = \gamma_2 = 1$ *and* $I = \{1, 2, \ldots, r\}$*) gives the following lemma which is a generalization of Lemma 3.3. in [39].*

Lemma 4.3.7. *Let* $\mathcal{M}$ *be a non trivial nearly* T_B^{-1} *invariant subspace of* $\mathcal{D}_\alpha$ *for* $\alpha \in [0, 1]$ *such that* $\mathcal{M} \nsubseteq T_B\mathcal{D}_\alpha$ *and let* $\{f_i\}_{i=1}^r$ *and* $\{e_j\}_{j=1}^p$ *be an orthonormal basis of* $\mathcal{M} \ominus (\mathcal{M} \cap T_B\mathcal{D}_\alpha)$ *and the defect space* $\mathcal{F}$ *respectively. Then for any* $f \in \mathcal{M}$*, there exist* $\{q_i\}_{i=1}^r$ *and* $\{h_j\}_{j=1}^p$ *in* $\mathcal{O}(\mathbb{D})$ *such that*

$$f = \sum_{i=1}^{r} f_i q_i + T_B \sum_{j=1}^{p} e_j h_j$$

for any $i \in \{1, 2, \ldots, r\}$*,* $j \in \{1, 2, \ldots, p\}$ *and also there exist* $(c_{ki})_{k\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}}$ *and* $(d_{kj})_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ *with*

$$q_i = \sum_{k=0}^{\infty} c_{ki} B^k, h_j = \sum_{k=1}^{\infty} d_{kj} B^{k-1}, \tag{4.42}$$

and

$$\sum_{i=1}^{r}\sum_{k=0}^{\infty} |c_{ki}|^2 + \sum_{j=1}^{p}\sum_{k=1}^{\infty} |d_{kj}|^2 \leq \|f\|^2_{\mathcal{D}_\alpha}. \tag{4.43}$$

Remark 4.3.8. *If* $\mathcal{M} \subseteq T_B\mathcal{D}_\alpha$*, then using the same notation as in Lemma 4.3.7 for any* $f \in \mathcal{M}$ *there exists* $\{h_j\}_{j=1}^p$ *in* $\mathcal{O}(\mathbb{D})$ *such that*

$$f = T_B \sum_{j=1}^{p} e_j h_j,$$

and also there exists $(b_{kj})_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ *with*

$$h_j = \sum_{k=1}^{\infty} b_{kj} B^{k-1} \quad \text{and} \quad \sum_{j=1}^{p}\sum_{k=1}^{\infty} |b_{kj}|^2 \leq \|f\|^2_{\mathcal{D}_\alpha}. \tag{4.44}$$

Now we are in a position to describe the nearly T_B^{-1} invariant subspace with defect p in $\mathcal{D}_\alpha$ for $\alpha \in [0,1]$, providing a generalization of Theorem 3.4 in [39]. Due to Lemma 4.2.2 without loss of generality we assume $B(0) = 0$.

Theorem 4.3.9. *Let $\mathcal{M}$ be a nearly T_B^{-1} invariant subspace of $\mathcal{D}_\alpha$ with finite defect p for $\alpha \in [0,1]$ and let $\mathcal{F}$ be the p dimensional defect space. Let $E_0 := [e_1, e_2, \ldots, e_p]$ where $\{e_j\}_{j=1}^p$ is an orthonormal basis of $\mathcal{F}$ using norm $\|.\|_2$. Then*

(i) in the case when $\mathcal{M} \nsubseteq T_B\mathcal{D}_\alpha$, if $F_0 := [f_1, f_2, \ldots, f_r]$ is a matrix containing an orthonormal basis $\{f_i\}_{i=1}^r$ of $\mathcal{M} \ominus (\mathcal{M} \cap T_B\mathcal{D}_\alpha)$, then there exists a linear subspace $\mathcal{N} \subset H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$ such that

$$\mathcal{M} = \left\{ f \in \mathcal{D}_\alpha : f = F_0 q + T_B E_0 h : (q,h) \in \mathcal{N} \right\}$$

together with

$$\|q\|^2_{H^2(\mathbb{D},\mathbb{C}^r)} + \|h\|^2_{H^2(\mathbb{D},\mathbb{C}^p)} \le \|f\|^2_{\mathcal{D}_\alpha}.$$

Moreover, $\mathcal{N}$ is $T_{\overline{B}}$ invariant.

(ii) In the case $\mathcal{M} \subset T_B\mathcal{D}_\alpha$, there exists a linear subspace $\mathcal{N} \subset H^2_{\mathbb{C}^p}(\mathbb{D})$ such that

$$\mathcal{M} = \left\{ f \in \mathcal{D}_\alpha : f = T_B E_0 h : h \in \mathcal{N} \right\}$$

together with

$$\|h\|^2_{H^2_{\mathbb{C}^p}(\mathbb{D})} \le \|f\|^2_{\mathcal{D}_\alpha},$$

and $\mathcal{N}$ is $T_{\overline{B}}$ invariant.

Proof. (i) For $f \in \mathcal{M} \subset \mathcal{D}_\alpha$ with $\alpha \in [0,1]$, then by applying Lemma 4.3.7 we get

$$f = \sum_{i=1}^r f_i q_i + T_B \sum_{j=1}^p e_j h_j = F_0 q + T_B E_0 h, \tag{4.45}$$

where $q = [q_1, q_2, \ldots, q_r]^t$ and $h = [h_1, h_2, \ldots, h_p]^t$. Next by using the facts (4.42) and (4.43) we obtain the following norm equalities and norm estimates for any $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, p\}$:

$$\|q_i\|^2_{H^2_{\mathbb{C}}(\mathbb{D})} = \sum_{k=0}^\infty |c_{ki}|^2, \quad \|h_j\|^2_{H^2_{\mathbb{C}}(\mathbb{D})} = \sum_{k=1}^\infty |d_{ki}|^2,$$

and hence

$$\|q\|^2_{H^2_{\mathbb{C}^r}(\mathbb{D})} + \|h\|^2_{H^2_{\mathbb{C}^p}(\mathbb{D})} = \sum_{i=1}^r \|q_i\|^2_{H^2_{\mathbb{C}}(\mathbb{D})} + \sum_{j=1}^p \|h_j\|^2_{H^2_{\mathbb{C}}(\mathbb{D})} = \sum_{i=1}^r \sum_{k=0}^\infty |c_{ki}|^2 + \sum_{j=1}^p \sum_{k=1}^\infty |d_{kj}|^2 \le \|f\|^2_{\mathcal{D}_\alpha}.$$

Thus it follows that

$$q = \sum_{k=0}^{\infty} C_k B^k \in H^2_{\mathbb{C}^r}(\mathbb{D}), \quad \text{where } C_k = [c_{k1}, c_{k2}, \dots, c_{kr}]^t,$$

and

$$h = \sum_{k=1}^{\infty} D_k B^{k-1} \in H^2_{\mathbb{C}^p}(\mathbb{D}), \quad \text{where } D_k = [d_{k1}, d_{k2}, \dots, d_{kp}]^t.$$

Now we define a linear subspace as follows

$$\mathcal{N} := \left\{ (q,h) \in H^2_{\mathbb{C}^r}(\mathbb{D}) \times H^2_{\mathbb{C}^p}(\mathbb{D}) : \exists f \in \mathcal{M} \quad \text{such that} \quad f = F_0 q + T_B E_0 h \right\},$$

satisfying for any $f \in \mathcal{M}$, $\exists\ (q,h) \in \mathcal{N}$ such that

$$f = F_0 q + T_B E_0 h \quad \text{with} \quad \|f\|^2_{\mathcal{D}_\alpha} \geq \|q\|^2_{H^2_{\mathbb{C}^r}(\mathbb{D})} + \|h\|^2_{H^2_{\mathbb{C}^p}(\mathbb{D})}$$

Next we show that $\mathcal{N}$ is invariant under $T_{\overline{B}}$. Consider $T = T_B$ and $\mathcal{H} = \mathcal{D}_\alpha$ for $\alpha \in [0,1]$ in Lemma 4.3.1 and therefore the corresponding operators R, Q and S in Lemma 4.3.1 become $R = (T_B^* T_B)^{-1} T_B^* P_{\mathcal{M} \cap T_B \mathcal{D}_\alpha}, Q = P_{\mathcal{M} \ominus (\mathcal{M} \cap T_B \mathcal{D}_\alpha)}, S = P_{\mathcal{F}}$ and hence the equation (4.14) with $m = 0$ implies for any $f \in \mathcal{M}$,

$$f = Qf + TRf = Qf + T_B Rf,$$

which together with (4.45) yields

$$\begin{aligned} F_0 q + T_B E_0 h &= Q(F_0 q + T_B E_0 h) + T_B R(F_0 q + T_B E_0 h) \\ &= F_0 C_0 + BR(F_0 q + T_B E_0 h), \end{aligned}$$

which further satisfies

$$F_0(q - C_0) + T_B E_0 h = BR(F_0 q + T_B E_0 h).$$

Since T_B is injective, then from the above we conclude

$$R(F_0 q + T_B E_0 h) = F_0 (\sum_{k=1}^{\infty} C_k B^{k-1}) + E_0 h = F_0(T_{\overline{B}} q) + E_0 h. \tag{4.46}$$

On the other hand note that $R(F_0 q + T_B E_0 h) \in \mathcal{M} \oplus \mathcal{F}$ and hence

$$R(F_0 q + T_B E_0 h) = P_{\mathcal{M}} R(F_0 q + T_B E_0 h) + E_0 D_1. \tag{4.47}$$

Thus by combining (4.46) and (4.47) we get

$$P_{\mathcal{M}}R(F_0q + T_BE_0h) = F_0(T_{\overline{B}}q) + E_0(\sum_{k=2}^{\infty} D_kB^{k-1}) = F_0(T_{\overline{B}}q) + T_BE_0(T_{\overline{B}}h).$$

Since $P_{\mathcal{M}}R(F_0q+T_BE_0h) \in \mathcal{M}$, then from the definition of $\mathcal{N}$ it follows that $(T_{\overline{B}}q, T_{\overline{B}}h) \in \mathcal{N}$. Thus $\mathcal{N}$ is $T_{\overline{B}}$ invariant in $H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$.

(ii) If $\mathcal{M} \subset T_B\mathcal{D}_\alpha$, then by using Remark 4.3.8 and proceeding similarly as in case (i) we obtain a linear subspace $\mathcal{N} \subset H^2_{\mathbb{C}^p}(\mathbb{D})$ such that

$$\mathcal{M} = \left\{ f \in \mathcal{D}_\alpha : f = T_BE_0h : h \in \mathcal{N} \right\} \quad \text{together with} \quad \|h\|_{H^2_{\mathbb{C}^p}(s\mathbb{D})} \leq \|f\|_{\mathcal{D}_\alpha},$$

and $\mathcal{N}$ is $T_{\overline{B}}$ invariant in $H^2_{\mathbb{C}^p}(\mathbb{D})$. This completes the proof. □

Next we consider a special case of (4.3) *as discussed in (see [39, Section 3])*

$$\begin{array}{ccc} H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) & \xrightarrow{\ T\ } & H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \\ U \downarrow & & \downarrow U \\ H^2_{\mathbb{C}^{m(r+p)}}(\mathbb{D}) & \xrightarrow{\ S\ } & H^2_{\mathbb{C}^{m(r+p)}}(\mathbb{D}) \end{array} \tag{4.48}$$

Then $SU = UT_B$ *holds for the unilateral shift* $S : H^2_{\mathbb{C}^{m(r+p)}}(\mathbb{D}) \to H^2_{\mathbb{C}^{m(r+p)}}(\mathbb{D})$ *and* $T_B : H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \to H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$ *having multiplicity* $m(r+p)$*. Using this fact we end the section with the following remark concerning finite dimensional nearly* T_B^{-1} *invariant subspaces of* $\mathcal{D}_\alpha$ *for* $\alpha \in [0,1]$ *which is almost identical to Remark 3.5. in [39].*

Remark 4.3.10. *Note that the subspace* $\mathcal{N}$ *is not closed in general. In the above Theorem 4.3.9 if we consider* $\mathcal{M}$ *is finite dimensional, then* $\mathcal{N} \subset H^2_{\mathbb{C}^{r+p}}(\mathbb{D})$ *is also finite dimensional and hence closed. Then from Beurling-Lax-Halmos theorem and using diagram* (4.48) *we obtain that there exists a non negative integer* l *with* $l \leq m(r+p)$ *and an inner multiplier* $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}^l, \mathbb{C}^{m(r+p)})}(\mathbb{D})$ *such that*

$$\mathcal{N} = U^*\left(H^2_{\mathbb{C}^{m(r+p)}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^l}(\mathbb{D})\right), \quad \textit{and hence}$$
$$\mathcal{M} = \left\{ f \in \mathcal{D}_\alpha : f = F_0q + T_BE_0h : (q,h) \in U^*\left(H^2_{\mathbb{C}^{m(r+p)}}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^l}(\mathbb{D})\right) \right\}.$$

CHAPTER 5

Schmidt subspaces of block Hankel operators

5.1 Introduction

In the theory of operators on analytic function space, one of the significant classes of operators is the Hankel operator, as they have connections with many branches of mathematics, for example, function theory, harmonic analysis, approximation theory, moment problems, spectral theory, orthogonal polynomials, stationary Gaussian processes, etc. Hankel operators have different realizations because such a variety of realizations is essential in application. Depending on the need of the problem under consideration, one can choose a suitable realization concretely. For example, if $\Gamma = \{\gamma_{j+k}\}_{j,k=0}^{\infty}$ *is a bounded Hankel matrix on* $\ell^2(\mathbb{Z}_+)$*, then one can consider* Γ *as a bounded linear operator on the classical Hardy space* $H^2(\mathbb{T})$ *using natural identification between* $\ell^2(\mathbb{Z}_+)$ *and* $H^2(\mathbb{T})$*, noting that this is a linear realization. It is worth mentioning that Peller's book [46] is a well-known and accepted reference to the classical theory of Hankel operators and their various applications. In short, we use* H^2 *to denote the Hardy space.*

Recall that $P : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ *be the orthogonal projection (the Szegö projection) mentioned in Chapter 1. Then corresponding to a* $u \in BMOA(\mathbb{T})$ *(see (5.2)), the anti-linear Hankel operator* H_u *is define by*

$$H_u(f) = P(u\bar{f}), \quad f \in H^2(\mathbb{T}). \tag{5.1}$$

The symbol $u \in BMOA(\mathbb{T})$ *ensures that the Hankel operator* H_u *is bounded, which follows from the Nehari-Fefferman theorem [46, Section 1.1]. Such Hankel operator* H_u *is the anti-linear realization of the Hankel matrix*

$$\{\hat{u}(n+m)\}_{n,m\geq 0},$$

where $\hat{u}(\cdot)$ *are the Fourier coefficients of* u*. It is easy to check that the kernel of a Hankel operator* $\Gamma = H_u$ *is a shift invariant subspace (see, e.g., [46, Section 1.1]), and hence, due to the Beurling's Theorem (see Theorem 1.3.14), it is of the form* θH^2*, for some inner function* θ*. In [30], Gérard and Pushnitski raised the following question in scalar-valued Hardy space.*

Problem 1. *How can one characterize eigenspaces* $\ker(\Gamma^*\Gamma - s^2 I)$, $s > 0$, *as a class of subspaces in the Hardy space* $H^2_{\mathbb{C}}(\mathbb{D})$ *?*

In [30], authors solved the above-mentioned problem by showing that every such subspace (known as the Schmidt subspaces for the Hankel operator) can be identified with a subspace of the form $p\mathcal{K}_{z\theta}$*, where* θ *is an inner function,* $\mathcal{K}_{z\theta}$ *is a model space, and* p *is an isometric multiplier (will be defined later in Section 5.2) on* $\mathcal{K}_{z\theta}$*. In addition to that, they also provide a simple formula for the action of* Γ *on* $\ker(\Gamma^*\Gamma - s^2 I)$ *explicitly, which is completely determined by* s, p *and* θ*.*

It is important to note that the action of Γ *plays an important role in the description of all s-Schmidt pairs. Recall that [30], for a singular value* s *of* Γ*, a pair* $\{\xi, \eta\} \in \ell^2$ *is called a Schmidt pair or (more precisely, an s-Schmidt pair) of* Γ*, if it satisfies*

$$\Gamma\xi = s\eta, \quad \Gamma^*\eta = s\xi.$$

The space $\ker(\Gamma^*\Gamma - s^2 I)$ *is called Schmidt subspace of* Γ *and note that s-Schmidt pairs form a linear subspace of dimension* $\dim \ker(\Gamma^*\Gamma - s^2 I) \leq \infty$*. Therefore the problem of description of all s-Schmidt pairs of* Γ *is equivalent to the problem of the description of the action of* Γ *on* $\ker(\Gamma^*\Gamma - s^2 I)$*. Also there is another advantage of this action. Suppose* $\mathcal{C}$ *is an anti-linear map on* ℓ^2 *defined as* $\mathcal{C}\xi = \bar{\xi}$*, then the Hankel matrix* Γ *is* $\mathcal{C}$*-symmetric, that is* $\Gamma\mathcal{C} = \mathcal{C}\Gamma^*$*. It is easy to observe that*

$$\xi \in \ker(\Gamma^*\Gamma - s^2 I) \quad \Leftrightarrow \quad \bar{\xi} \in \ker(\Gamma\Gamma^* - s^2 I).$$

Moreover, the anti-linear map $\Gamma\mathcal{C}$ *maps* $\ker(\Gamma\Gamma^* - s^2 I)$ *onto itself and the map*

$$s^{-1}\Gamma\mathcal{C} : \ker(\Gamma\Gamma^* - s^2 I) \to \ker(\Gamma\Gamma^* - s^2 I)$$

is an involution. Therefore, using the action of Γ *on* $\ker(\Gamma^*\Gamma - s^2 I)$ *one can easily describe the involution map* $s^{-1}\Gamma\mathcal{C}$.

Recently, in 2021, Gérard and Pushnitski [31] established an excellent connection between these Schmidt subspaces with nearly S^**-invariant subspaces and using Hitt's [34] beautiful characterization of nearly* S^**- invariant subspaces, authors in [31] gave an alternative proof (indeed, a short proof of the main result of [30]) concerning the characterization of such Schmidt subspaces of scalar valued Hankel operator. In this direction, we would like to mention that recently in [43], authors studied the Schmidt subspace in scalar valued Hardy space.*

It is important to note that there are various pieces of literature (see, e.g., [29],[27][54]) related to the study of the spectrum of $\Gamma^*\Gamma$ *and self-adjoint Hankel operator (see, e.g., [28],[41]).*

In this chapter, we extend the study of Schmidt subspaces associated with a scalar-valued Hankel operator to a matrix-valued Hankel operator. It has been noted that sometimes matrix-valued analysis is much more complicated than scalar-valued analysis due to the non-commutative property of the matrix. On the other hand, sometimes, the matrix-valued technique solves long-standing open problem for scalar cases. For example, using some new matrix-valued technique and a deep analysis of nearly S^**-invariant subspaces of vector valued Hardy space, Aleman and Vukotić [2] proved that the product of finitely many Toeplitz operators on the vector valued Hardy space is zero if and only if at least one of the operators is zero, which answer a long-standing open problem regarding the zero-product of finitely many Toeplitz operators in scalar-valued Hardy space. Motivated by the work of Gérard and Pushnitski in [30, 31], we study the Problem 1 in the case of the vector valued Hardy spaces. More precisely, this chapter gives the answer to the following question completely.*

Problem 2. *For a given Hankel operator* H_U *with symmetric symbol* $U \in BMOA(\mathbb{T}, \mathbb{C}^m)$*, how can one characterize eigenspaces* $\ker(H_U^* H_U - s^2 I), s > 0$*, as a class of subspaces in the Hardy space* $H^2_{\mathbb{C}^m}(\mathbb{D})$*?*

Moreover, we also obtain the following results in the sequel.

- *In some special cases we show that Schmidt subspaces* $\ker(H_U^* H_U - s^2 I), s > 0$ *become nearly* S^**-invariant.*
- *Furthermore, we calculate precisely the action of the Hankel operator for the above mentioned cases on* $\ker(H_U^* H_U - s^2 I), s > 0$.

The novelty of our work lies in the fact that we obtain again a short proof of the characterization results concerning the structure of Schmidt subspaces in scalar-valued Hardy space (see Theorem 5.3.4) obtained by Gérard and Pushnitski as a consequence of our main results (see Theorem 5.3.2 and 5.3.3)), in an alternative way compared to [30, 31].

Let us now describe about the methodology and the new tools used in our work. Because of the non-commutativity property of the symbol of the Hankel operator on $H^2_{\mathbb{C}^m}(\mathbb{D})$, several issues arise, and the classical scalar-valued methods fail to resolve them.

(i) *Nearly S^*-invariant subspaces with finite defect: It is noted that non-trivial Schmidt subspaces $\mathbb{E}_{H_U}(s)$ of H_U are not always nearly S^*-invariant subspaces in the vector valued Hardy space. So we introduce the notion of nearly S^*- invariant subspaces with finite defect in $H^2_{\mathbb{C}^m}(\mathbb{D})$ (see [14, 44]), which is one of the new ingredients in studying such spaces. We have shown that non-trivial Schmidt subspaces $\mathbb{E}_{H_U}(s)$ of H_U are nearly S^*-invariant subspaces with defect less or equal to m (see Theorem 5.3.2).*

(ii) *Chattopadhyay, Das, Pradhan & O'Loughlin structural result for nearly S^*-invariant subspaces: The second new tool is the structural theorem of nearly S^*-invariant subspaces in vector valued Hardy space due to Chattopadhyay-Das-Pradhan (C-D-P) [14, Theorem 3.5] and O'Loughlin [44, Theorem 3.4]. Using these characterization results, we give a complete characterization of Schmidt subspaces in a vector valued setting (see Theorem 5.3.3).*

All unexplained notations used in this section are introduced and explained in the next section (i.e. Section 5.2).

We end the introduction by briefly mentioning the organization of the chapter. In Section 5.2, we describe all necessary notations, definitions and results related to Schmidt subspaces of Hankel operators. Section 5.3 deals with the study of the complete structure of the Schmidt subspaces in general. In Section 5.4, we describe the action of a specific class of Hankel operators on their Schmidt subspaces and conclude the section with a natural question.

5.2 Preliminaries and Notations

We begin this section by recalling the definition of vector valued Hardy space over the unit disk $\mathbb{D}$. In Chapter 1 we have discuss about the definition of the $\mathcal{E}$-valued Hardy space denoted by $H^2_{\mathcal{E}}(\mathbb{D})$ (see (1.6)) over $\mathbb{D}$.

Also recall the definition (1.1) of Banach space $\mathcal{X}$-valued $L^p(\mathbb{T},\mathcal{X})$-spaces as mentioned in Chapter 1. For any $f \in L^1(\mathbb{T},\mathcal{X})$, we define $\mathbb{P}^+_{\mathcal{X}}(f) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$.

Regarding this work, we will only consider $\mathcal{E} = \mathbb{C}^m$ (see (1.6)) for some fixed natural number m, that is we will focus only on $H^2_{\mathbb{C}^m}(\mathbb{D})$ (see Definition 1.5).

Now as we pointed out earlier in Chapter 1 and Chapter 2, $H^2_{\mathbb{C}^m}(\mathbb{D})$ can be embedded isometrically as a closed subspace of $L^2(\mathbb{T},\mathbb{C}^m)$ by identifying $H^2_{\mathbb{C}^m}(\mathbb{D})$ through the non-tangential boundary limits of the $H^2_{\mathbb{C}^m}(\mathbb{D})$ functions. Let P_m denotes the orthogonal projection of $L^2(\mathbb{T},\mathbb{C}^m)$ onto $H^2_{\mathbb{C}^m}(\mathbb{D})$. Let $\mathbb{P}_m = \mathbb{P}^+_{\mathbb{C}^m}$. Note that on $L^2(\mathbb{T},\mathbb{C}^m)$, $\mathbb{P}_m = P_m$. Let $r, m \in \mathbb{N}$, and $\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)$ denotes the space of all linear operators from $\mathbb{C}^r$ to $\mathbb{C}^m$. We denote the space of all holomorphic matrix valued functions by $Hol_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$, and by $H^\infty(\mathbb{D},\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m))$ as the subspace of $Hol_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$, consisting of all bounded analytic functions. A function $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$ is said to be an inner multiplier (or, inner function) if Θ is an isometry almost everywhere on the circle $\mathbb{T}$. Corresponding to an inner multiplier $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$, the model space denoted by $\mathcal{K}_\Theta$, and is defined as

$$\mathcal{K}_\Theta := H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^r}(\mathbb{D}).$$

By an isometric multiplier on the model space $\mathcal{K}_\Theta = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^r}(\mathbb{D})$, we mean an analytic function $F \in Hol_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^n)}(\mathbb{D})$ for some $n \in \mathbb{N}$, such that $FG \in H^2_{\mathbb{C}^n}(\mathbb{D})$ for every $G \in \mathcal{K}_\Theta$ and

$$\|FG\| = \|G\|.$$

Recall that in scalar valued case, the BMOA is the space of all analytic BMO (functions having bounded mean oscillation) functions in $\mathbb{D}$, in other words $BMOA = BMO \cap H^2$. It follows from [46, Theorem A2.7.] that

$$BMOA(\mathbb{T}) \overset{\text{def}}{=} \mathbb{P}^+_{\mathbb{C}} BMO(\mathbb{T}) = \mathbb{P}^+_{\mathbb{C}} L^\infty(\mathbb{T}) = BMO(\mathbb{T}) \cap H^2(\mathbb{T}). \tag{5.2}$$

For more details related to BMO and BMOA, we refer [46, 728p-731p].

Let $m \in \mathbb{N}$, as a generalization of scalar valued $BMOA(\mathbb{T})$, we consider $BMOA(\mathbb{T}, \mathcal{L}(\mathbb{C}^m))$ as the space of $\mathcal{L}(\mathbb{C}^m)$-valued BMOA-functions on $\mathbb{T}$, and it is defined as follows:

$$BMOA(\mathbb{T}, \mathcal{L}(\mathbb{C}^m)) \stackrel{\text{def}}{:=} \mathbb{P}^+_{\mathcal{L}(\mathbb{C}^m)} \mathcal{L}^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}^m)). \tag{5.3}$$

We use the matricial notation of $U \in BMOA(\mathbb{T}, \mathcal{L}(\mathbb{C}^m))$ as $U = [u_{ij}]_{m\times m}$, where each $u_{ij} \in BMOA(\mathbb{T})$. Now corresponding to the symbol $U \in BMOA(\mathbb{T}, \mathcal{L}(\mathbb{C}^m))$, we define the matrix-valued Hankel operator or block Hankel operator H_U on $H^2_{\mathbb{C}^m}(\mathbb{D})$ as follows:
For each $F = (f_1, f_2, \dots, f_m) \in H^2_{\mathbb{C}^m}(\mathbb{D})$,

$$H_U(F) := P_m(U\bar{F}), \text{ where } \bar{F} = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m). \tag{5.4}$$

So, the above definition (5.4) of H_U implies that $H_U = [H_{u_{ij}}]_{m\times m}$, where $H_{u_{ij}}$ are Hankel operators on $H^2_{\mathbb{C}}(\mathbb{D})$. Therefore H_U is a bounded anti-linear operator on $H^2_{\mathbb{C}^m}(\mathbb{D})$. Let $U = [u_{ij}]_{m\times m} \in BMOA(\mathbb{T}, \mathcal{L}(\mathbb{C}^m))$ be such that U is symmetric, that is, $U = U^t$ where U^t is the transpose of U, in other words $u_{ij} = u_{ji}$ for each i, j. Then it turns out that H_U^2 is a bounded, linear, and non-negative operator on $H^2_{\mathbb{C}^m}(\mathbb{D})$. Later, we will discuss about these properties in more detail (see Proposition 5.2.2). For such symbol U and the corresponding Hankel operator H_U, let us denote the Schmidt subspaces in vector valued Hardy spaces $H^2_{\mathbb{C}^m}(\mathbb{D})$ by $\mathbb{E}_{H_U}(s)$ and define by

$$\mathbb{E}_{H_U}(s) := \ker(H_U^2 - s^2 I), \quad s > 0. \tag{5.5}$$

As discussed earlier, our main goal is to characterize these subspaces in vector valued Hardy spaces. It is important to note that $\mathbb{E}_{H_U}(s)$ is an invariant subspace for H_U.

Hypothesis 5.2.1. *Assume that $U = [u_{ij}] \in BMOA(\mathbb{T}, \mathcal{L}(\mathbb{C}^m))$ satisfies $U = U^t$, that is $u_{ij} = u_{ji}$.*

Proposition 5.2.2. *Assume Hypothesis 5.2.1, and let H_U be a bounded Hankel anti-linear operator as defined in (5.4) on $H^2_{\mathbb{C}^m}(\mathbb{D})$. Then*

(i) $\langle H_U(F), G\rangle = \langle H_U(G), F\rangle$ for all $F, G \in H^2_{\mathbb{C}^m}(\mathbb{D})$.

(ii) H_U^2 is a self-adjoint bounded linear operator.

(iii) $H_U S = S^* H_U$, where S is the unitarel shift on $H^2_{\mathbb{C}^m}(\mathbb{D})$ and S^* is the adjoint of S.

Proof.

(i) Let $F = (f_1, f_2, \dots, f_m)$, $G = (g_1, g_2, \dots, g_m)$. Then

$$\begin{aligned}\langle H_U(F), G\rangle &= \langle U\bar{F}, G\rangle \\ &= \langle u_{11}\bar{f_1} + u_{12}\bar{f_2} + \cdots + u_{1m}\bar{f_m}, g_1\rangle + \cdots + \langle u_{m1}\bar{f_1} + u_{m2}\bar{f_2} + \cdots + u_{mm}\bar{f_m}, g_m\rangle \\ &= \langle u_{11}\bar{g_1} + u_{21}\bar{g_2} + \cdots + u_{m1}\bar{g_m}, f_1\rangle + \cdots + \langle u_{1m}\bar{g_1} + u_{2m}\bar{g_2} + \cdots + u_{mm}\bar{g_m}, f_m\rangle \\ &= \langle u_{11}\bar{g_1} + u_{12}\bar{g_2} + \cdots + u_{1m}\bar{g_m}, f_1\rangle + \cdots + \langle u_{m1}\bar{g_1} + u_{m2}\bar{g_2} + \cdots + u_{mm}\bar{g_m}, f_m\rangle \\ &= \langle U\bar{G}, F\rangle = \langle H_U(G), F\rangle.\end{aligned}$$

(ii) The boundedness of H_U and the property (i) together implies the property (ii).

(iii) For $F \in H^2_{\mathbb{C}^m}(\mathbb{D})$,

$$\begin{aligned}H_U S(F) = P_m(U\bar{z}\bar{F}) = P_m(\bar{z}U\bar{F}) &= P_m\left[\bar{z}\left(P_m + (I - P_m)\right)(U\bar{F})\right] \\ &= P_m\big(\bar{z}P_m(U\bar{F})\big) + P_m\big(\bar{z}(I - P_m)(U\bar{F})\big) \\ &= P_m\big(\bar{z}P_m(U\bar{F})\big) \\ &= S^* H_U(F).\end{aligned}$$

□

Remark 5.2.3. *Note that the last property (iii) holds true for any $U = [u_{ij}] \in BMOA$, and it is basically characterizes all the bounded Hankel operators on $H^2_{\mathbb{C}^m}(\mathbb{D})$. Then the following identity follows from the above Proposition 5.2.2.*

$$S^* H_U^2 S = H_U^2 - \sum_{i=1}^{m} \langle \cdot, U_i\rangle\, U_i,$$

where $U_i = [u_{1i}, u_{2i}, \dots, u_{mi}]^t$ for $i \in \{1, 2, \dots, m\}$.

Going further, we need the help of the following auxiliary Hankel operator K_U (in scalar-valued case the notion of such kind of operator mentioned in [30, 31]):

$$K_U := H_U S = S^* H_U = H_{S^*U} \tag{5.6}$$

on $H^2_{\mathbb{C}^m}(\mathbb{D})$, where U satisfies the Hypothesis 5.2.1. Depending on the context, we will use suitable definition of K_U. Since we are in the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$, then we have

$$SS^* = I_{H^2_{\mathbb{C}^m}(\mathbb{D})} - P_{\mathbb{C}^m}$$

$$= I_{H^2_{\mathbb{C}^m}(\mathbb{D})} - \sum_{i=1}^{m} \langle \cdot, k_0 \otimes e_i \rangle k_0 \otimes e_i,$$

where $P_{\mathbb{C}^m}$ is the orthogonal projection from $H^2_{\mathbb{C}^m}(\mathbb{D})$ onto $\mathbb{C}^m$, and $\{e_i\}_{i=1}^m$ is the standard orthonormal basis of $\mathbb{C}^m$, and k_0 is the reproducing kernel of $H^2_{\mathbb{C}}(\mathbb{D})$ at 0.

Recall that $U = [u_{ij}]_{m\times m}$ with $u_{ij} = u_{ji}$. For each $i \in \{1, 2, \ldots, m\}$, let us denote the ith column of U by

$$U_i = [u_{1i}, u_{2i}, \ldots, u_{mi}]^t.$$

Then it turns out,

$$H_U(e_i) = U_i = [u_{1i}, u_{2i}, \ldots, u_{mi}]^t.$$

From the definition (5.6) *of K_U, it follows that K_U is a bounded anti-linear operator, which further imply that K_U^2 is a bounded linear operator on $H^2_{\mathbb{C}^m}(\mathbb{D})$. Next we give a precise form of the operator K_U^2 in terms of H_U^2 as follows*

$$\begin{aligned} K_U^2 &= (H_U S)(H_U S) = H_U S S^* H_U \\ &= H_U \left(I_{H^2_{\mathbb{C}^m}(\mathbb{D})} - \sum_{i=1}^{m} \langle \cdot, k_0 \otimes e_i \rangle k_0 \otimes e_i \right) H_U \\ &= H_U^2 - \sum_{i=1}^{m} \langle \cdot, U_i \rangle \, U_i, \end{aligned}$$

where $\sum_{i=1}^{m} \langle \cdot, U_i \rangle U_i$ is a finite-rank operator generated by $\{U_1, U_2, \ldots U_m\}$ and having rank at most m. Moreover, this form of K_U^2 guarantees that K_U^2 is self-adjoint. Like the Schmidt subspaces of H_U, similarly we also define the eigenspaces related to K_U as follows:

$$\mathbb{E}_{K_U}(s) := \ker(K_U^2 - s^2 I), \quad s > 0, \tag{5.7}$$

and these eigenspaces $\mathbb{E}_{K_U}(s)$ are basically the Schmidt subspaces corresponding to the Hankel operator $K_U (= H_{S^*U})$. The next observation is crucial to characterize the Schmidt subspaces of H_U.

Lemma 5.2.4. *Let U satisfy the Hypothesis 5.2.1. Then*

$$\mathbb{E}_{H_U}(s) \cap \{U_1, U_2, \ldots, U_m\}^{\perp} = \mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \ldots, U_m\}^{\perp}.$$

Proof. The proof follows from the Definitions (5.5) and (5.7), of the corresponding subspaces. □

Now, let us recall some useful existing definition in Chapter 2 required to understand the structure of Schmidt subspaces.

Definition 5.2.5. *A closed subspace $\mathcal{M}$ of $H^2_{\mathbb{C}^m}(\mathbb{D})$ is said to be nearly S^*-invariant if every element $F \in \mathcal{M}$ with $F(0) = 0$ satisfies $S^*F \in \mathcal{M}$. As a generalization, we call a closed subspace $\mathcal{M} \subset H^2_{\mathbb{C}^m}(\mathbb{D})$ to be nearly S^*-invariant with defect p if and only if there is an p-dimensional subspace $\mathcal{F} \subset H^2_{\mathbb{C}^m}(\mathbb{D})$ (which may be taken to be orthogonal to $\mathcal{M}$) such that if $F \in \mathcal{M}, F(0) = 0$ then $S^*F \in \mathcal{M} \oplus \mathcal{F}$.*

Nearly S^*-invariant subspaces were first introduced by Hitt [34] and further studied by various authors (see, e.g., [1, 33, 49]). These subspaces were also studied in connection with kernels of Toeplitz operator (see, e.g., [8, 19, 22, 32]). Characterization of nearly S^*-invariant subspaces in the vector valued Hardy space was obtained by Chalendar, Chevrot, and Partington (C-C-P) [9], which provides a vectorial generalization of Hitt's [34] characterization.

Theorem 5.2.6. *[9, Theorem 4.4] Let $\mathcal{F}$ be a nearly S^*-invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ and let $(W_1, W_2, \ldots, W_r)$ be an orthonormal basis of $\mathcal{W} := \mathcal{F} \ominus (\mathcal{F} \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$. Let F_0 be the $m \times r$ matrix whose columns are $W_1, W_2, \ldots, W_r$. Then there exists an isometric mapping $J : \mathcal{F} \to \mathcal{F}'$ given by $F_0G \mapsto G$, where $\mathcal{F}' := \{G \in H^2_{\mathbb{C}^r}(\mathbb{D}) : \exists F \in \mathcal{F}, F = F_0G\}$. Moreover $\mathcal{F}'$ is S^* invariant.*

As a corollary of this theorem they have the following result,

Corollary 5.2.1. *Assume the notations used in the above Theorem 5.2.6. Then there exists an inner function $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^r))$, which is unique up to unitary equivalence and vanishes at zero, such that $\mathcal{F} = F_0(H^2_{\mathbb{C}^r}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{r'}}(\mathbb{D}))$.*

Recently, in [14, Theorem 3.5], the authors have provided a complete characterization of nearly S^*-invariant subspaces with finite defect in the vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$, which extends the result of Chalendar, Gallardo, Partington [11] from scalar valued setting to vector valued setting. In this connection, it is worth mentioning that, O'Loughlin [44] obtained the similar structural result for these nearly S^*-invariant subspaces independently. Note that Theorem 3.5 in [14] (see also, [44, Theorem 3.4]) is one of the key tools to solve our problem. For more applications on the characterization results of nearly S^*-invariant subspaces with finite defect $H^2_{\mathbb{C}^m}(\mathbb{D})$, we refer to [15, 12].

5.3 The Structure of $\mathbb{E}_{H_U}(s)$

The following lemma will be useful to prove one of our main result in this section. Throughout this section, let U satisfy the Hypothesis 5.2.1, and for $i \in \{1, 2, \ldots, m\}$, let $U_i = U(e_i)$, where $\{e_i : 1 \leq i \leq m\}$ is the standard orthonormal basis of $\mathbb{C}^m$. Then we have the following lemma:

Lemma 5.3.1. *Let $\mathcal{V} := \mathbb{E}_{K_U}(s) \ominus \left\{\mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \ldots, U_m\}^\perp\right\}$. Then $dim(\mathcal{V}) \leq m$.*

Proof. We first consider the set $\mathcal{M} := \left\{P_{\mathbb{E}_{K_U}(s)}(U_1), P_{\mathbb{E}_{K_U}(s)}(U_2), \ldots, P_{\mathbb{E}_{K_U}(s)}(U_m)\right\} \subseteq \mathbb{E}_{K_U}(s)$. Our claim is that $\mathcal{M}$ generates the subspace $\mathcal{V}$.

For $f \in \mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \ldots, U_m\}^\perp$ and each $i \in \{1, 2, \ldots, m\}$, we have

$$\left\langle P_{\mathbb{E}_{K_U}(s)}(U_i), f \right\rangle = \langle U_i, f\rangle = 0.$$

Therefore $P_{\mathbb{E}_{K_U}(s)}(u_i) \in \mathcal{V}$. Next, assume $g \in \mathcal{V} \cap \mathcal{M}^\perp$. Then, for each $i \in \{1, 2, \ldots, m\}$, we have

$$\begin{aligned}\left\langle g, P_{\mathbb{E}_{K_U}(s)}(u_i)\right\rangle = 0 &\implies \langle g, u_i\rangle = 0 \\ &\implies g \in \mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \ldots, U_m\}^\perp \implies g = 0.\end{aligned}$$

Hence, we obtain

$$\operatorname{span}\left\{P_{\mathbb{E}_{K_U}(s)}(U_1), P_{\mathbb{E}_{K_U}(s)}(U_2), \ldots, P_{\mathbb{E}_{K_U}(s)}(U_m)\right\} = \mathcal{V}.$$

Therefore, $dim(\mathcal{V}) \leq m$. □

Now we are in a position to state and prove the main theorem of this section.

Theorem 5.3.2. *Assume Hypothesis 5.2.1, and H_U is the bounded Hankel operator on $H^2_{\mathbb{C}^m}(\mathbb{D})$ given by (5.4). Then, for each $s > 0$, the Schmidt subspace $\mathbb{E}_{H_U}(s)$ of H_U is a nearly S^*-invariant subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ with defect at most m.*

Proof. Let $s > 0$ be fixed. Let $F \in \mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D})$. Note that $F(0) = 0$, so we can write $F = SG$ for some $G \in H^2_{\mathbb{C}^m}(\mathbb{D})$. Therefore, we have $H^2_U(F) = s^2F$, and

$$K^2_U(G) = S^*H^2_US(G) = S^*H^2_U(F) = s^2S^*(F) = s^2G,$$

which gives $G \in \mathbb{E}_{K_U}(s)$. That means $S^*(F) \in \mathbb{E}_{K_U}(s)$, which basically implies that

$$S^*(\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D})) \subseteq \mathbb{E}_{K_U}(s). \tag{5.8}$$

Let $\mathcal{V} := \mathbb{E}_{K_U}(s) \ominus \{\mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp\}$. Now we can decompose the subspace $\mathbb{E}_{K_U}(s)$ as a direct sum two subspaces as follows:

$$\mathbb{E}_{K_U}(s) = \left(\mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp\right) \oplus \mathcal{V}. \tag{5.9}$$

From Lemma 5.2.4, it follows that $\mathbb{E}_{H_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp = \mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp$. Therefore the above decomposition (5.9) can be rewritten as

$$\mathbb{E}_{K_U}(s) = \left(\mathbb{E}_{H_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp\right) \oplus \mathcal{V}.$$

Hence (5.8) finally becomes

$$S^*\left(\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D})\right) \subseteq \left(\mathbb{E}_{H_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp\right) \oplus \mathcal{V}. \tag{5.10}$$

Therefore using the inclusion (5.10) and using Lemma 5.3.1 we conclude that $\mathbb{E}_{H_U}(s)$ is a *nearly S^*-invariant* subspaces with defect at most m. □

The next theorem provides the characterization of Schmidt subspaces in $H^2_{\mathbb{C}^m}(\mathbb{D})$.

Theorem 5.3.3. *Let $\mathbb{E}_{H_U}(s)$ be the Schmidt subspaces of the bounded Hankel operator H_U with $U = U^t$. Then for each $s > 0$, there exist an orthonormal set of vectors $\{E_1, E_2, \dots, E_p\}$ in $H^2_{\mathbb{C}^m}(\mathbb{D})$ such that $\{E_1, E_2, \dots, E_p\} \perp \mathbb{E}_{H_U}(s)$ for some non-negative integer p satisfying $p \leq m$, $\mathbb{E}_{H_U}(s)$ is nearly S^*-invariant of defect p having the defect space $\mathcal{F} = \bigvee_{i=1}^p \{E_i\}$. Moreover, if $\{W_1, W_2, \dots, W_r\}$ is an orthonormal basis of $\mathcal{W} := \mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$ and let F_0 be the $m \times r$ matrix whose columns are $W_1, W_2, \dots, W_r$. Then*

(i) in the case when $\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}) \neq \mathbb{E}_{H_U}(s)$,

$$\mathbb{E}_{H_U}(s) = \left\{F : F(z) = F_0(z)K_0(z) + \sum_{j=1}^p zk_j(z)E_j(z) : (K_0, k_1, \dots, k_p) \in \mathcal{K}_\Theta\right\},$$

where $\mathcal{K}_\Theta = H^2_{\mathbb{C}^{r+p}}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^{r'}}(\mathbb{D})$, for some inner function $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^{r'}, \mathbb{C}^{r+p})}(\mathbb{D})$ and

$$\|F\|^2 = \|K_0\|^2 + \sum_{j=1}^p \|k_j\|^2.$$

(ii) In the case when $\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}) = \mathbb{E}_{H_U}(s)$,

$$\mathbb{E}_{H_U}(s) = \Big\{F : F(z) = \sum_{j=1}^{p} zk_j(z)E_j(z) : (k_1, \dots, k_p) \in \mathcal{K}_\Phi\Big\},$$

with the same notion as in (i) *except that* $\mathcal{K}_\Phi = H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^{p'}}(\mathbb{D})$ *for some inner function* $\Phi \in H^\infty_{\mathcal{L}(\mathbb{C}^{p'},\mathbb{C}^p)}(\mathbb{D})$ *and* $\|F\|^2 = \sum_{j=1}^{p}\|k_j\|^2$.

Proof. Let $s > 0$. By the above Theorem 5.3.2, it follows that $\mathbb{E}_{H_U}(s)$ is a *nearly* S^**-invariant* subspace of $H^2_{\mathbb{C}^m}(\mathbb{D})$ with defect at most m. Therefore, for each $s > 0$, there exists a non-negative integer p ($p \leq m$) such that p is the dimension of the defect space of $\mathbb{E}_{H_U}(s)$. Now by applying C-D-P theorem [14, Theorem 3.5] on $\mathbb{E}_{H_U}(s)$, and using the *Beurling-Lax-Halmos* characterization [51, Theorem 3.3], we conclude the proof. □

Representation of Schmidt subspaces given by Theorem 5.3.3 provides a vectorial generalization of recent known characterization results [30, Theorem 1.5] ([31, Theorem 1.3]) of Schmidt subspaces in scalar valued Hardy space, as one can get back the representation of such subspaces in $H^2_{\mathbb{C}}(\mathbb{D})$ *by using our results but in a different way. Indeed, we restate [30, Theorem 1.5] ([31, Theorem 1.3]) and provide a simple alternative short proof again by using Theorem 5.3.3.*

Theorem 5.3.4. *Let* $u \in BMOA(\mathbb{T})$ *and let* H_u *be the corresponding Hankel operator on* $H^2_{\mathbb{C}}(\mathbb{D})$ *given by* (5.1)*. Then each non-trivial Schmidt subspace* $\mathbb{E}_{H_u}(s)$*,* $s > 0$*, is of the form* $h\mathcal{K}_\theta$*, where* θ *is an inner function in* $H^2_{\mathbb{C}}(\mathbb{D})$ *and* h *is an isometric multiplier on the model space* $\mathcal{K}_\theta \subseteq H^2_{\mathbb{C}}(\mathbb{D})$*.*

Proof. Let $\mathbb{E}_{H_u}(s)$ be a non-trivial *Schmidt* subspace of the Hankel operator H_u for some $s > 0$. Then by Theorem 5.3.3, $\mathbb{E}_{H_u}(s)$ is a *nearly* S^**-invariant* subspaces having defect at most 1. Now we will prove the theorem by analyzing the following two cases.

Case I: Suppose $\mathbb{E}_{H_u}(s) \nsubseteq zH^2_{\mathbb{C}}(\mathbb{D})$. In this case our aim is to show that $\mathbb{E}_{H_u}(s)$ is nearly S^*-invariant with defect 0, that is just nearly S^*-invariant. To that aim, let $\mathcal{F}$ be the defect space having dimension at most 1. By mimicking the proof of Lemma 5.3.1, we conclude that $\mathcal{F} = \langle P_{\mathbb{E}_{K_u}(s)}(u)\rangle$. Now for any $f \in \mathbb{E}_{K_u}(s)$, by definition we have $K_u^2(f) = s^2 f$, that is, $H_u^2(f) - \langle f, u\rangle u = s^2 f$. Let $0 \neq g \in \mathbb{E}_{H_u}(s) \ominus (\mathbb{E}_{H_u}(s) \cap zH^2_{\mathbb{C}}(\mathbb{D}))$. Then

$$\begin{aligned} s^2\langle f, H_u(g)\rangle &= \langle s^2 f, H_u(g)\rangle \\ &= \langle H_u^2(f), H_u(g)\rangle - \langle f, u\rangle\langle u, H_u(g)\rangle \end{aligned}$$

$$= \langle f, H_u H_u^2(g)\rangle - \langle f, u\rangle\langle H_u(1), H_u(g)\rangle$$
$$= s^2\langle f, H_u(g)\rangle - \langle f, u\rangle\langle H_u^2(g), 1\rangle$$
$$= s^2\langle f, H_u(g)\rangle - s^2\langle f, u\rangle\langle g, 1\rangle,$$

which implies

$$s^2\langle f, u\rangle\langle g, 1\rangle = 0. \tag{5.11}$$

Since $g(\neq 0) \in \mathbb{E}_{H_u}(s) \ominus (\mathbb{E}_{H_u}(s) \cap zH^2_{\mathbb{C}}(\mathbb{D}))$ then $\langle 1, g\rangle \neq 0$, so (5.11) implies that $\langle f, u\rangle = 0$. Therefore, under the assumption $\mathbb{E}_{H_u}(s) \nsubseteq zH^2_{\mathbb{C}}(\mathbb{D})$ we get $u \perp \mathbb{E}_{K_u}(s)$ and hence the defect space $\mathcal{F} = \{0\}$. Consequently, $\mathbb{E}_{H_u}(s)$ is *nearly* S^**-invariant* and by Theorem 5.3.3 (i), $\mathbb{E}_{H_u}(s) = \big\{f : f(z) = f_0(z)K_0(z) : K_0 \in \mathcal{K}\big\}$ where $\mathcal{K} = \mathcal{K}_\phi \subset H^2_{\mathbb{C}}(\mathbb{D})$ for some inner function ϕ together with $\|f\|^2 = \|k_0\|^2$.

Case II: Suppose $\mathbb{E}_{H_u}(s) \subseteq zH^2_{\mathbb{C}}(\mathbb{D})$. In this case, it is important to note that $\mathbb{E}_{H_U}(s)$ is not *nearly* S^**-invariant* but by Theorem 5.3.2, it is *nearly* S^**-invariant* with defect 1. So, the defect space $\mathcal{F} \neq \{0\}$ and by Theorem 5.3.3 (ii), $\mathbb{E}_{H_u}(s) = \big\{f : f(z) = zk_1(z)E_1(z) : k_1 \in \mathcal{K}\big\}$, where $\{E_1\}$ is a basis of the defect space $\mathcal{F}$ and $\mathcal{K} = \mathcal{K}_\psi \subset H^2_{\mathbb{C}}(\mathbb{D})$ for some inner function ψ together with $\|f\|^2 = \|k_1\|^2$.

Therefore in both cases, $\mathbb{E}_{H_u}(s) = h\mathcal{K}_\theta$, where θ is an inner function and h is an isometric multiplier on $\mathcal{K}_\theta$. □

Motivated by the representation of $\mathbb{E}_{H_u}(s)$ in scalar valued Hardy space $H^2_{\mathbb{C}}(\mathbb{D})$ it is natural to investigate under what circumstances the defect space $\mathcal{F}$ will be zero in vector valued setting. In other words, we would like to ask the following question:

Problem 3. *When the Schmidt subspaces will become nearly S^*-invariant in vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$?*

Now it is trivial to check that if $\mathbb{E}_{H_U}(s) \subseteq zH^2_{\mathbb{C}^m}(\mathbb{D})$, then it cannot be nearly S^*-invariant. So, we have to investigate under the assumption $\mathbb{E}_{H_U}(s) \nsubseteq zH^2_{\mathbb{C}^m}(\mathbb{D})$. It is clear that if $\{U_1, U_2, \dots, U_m\} \perp \mathbb{E}_{K_U}(s)$, then the defect space $\mathcal{F}$ becomes trivial and hence $\mathbb{E}_{H_U}(s)$ is nearly S^*-invariant in $H^2_{\mathbb{C}^m}(\mathbb{D})$. In the following theorem we will provide another important necessary condition which makes $\mathbb{E}_{H_U}(s)$ to be nearly S^*-invariant.

Theorem 5.3.5. *Let* $\mathbb{E}_{H_U}(s) \nsubseteq zH^2_{\mathbb{C}^m}(\mathbb{D})$*. If dim*$(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) = m$*, then* $\mathbb{E}_{H_U}(s)$ *is nearly* S^**-invariant in* $H^2_{\mathbb{C}^m}(\mathbb{D})$ *and* $\mathbb{E}_{H_U}(s) = F_0(H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^r}(\mathbb{D}))$ *for some inner function* $\Phi \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$*, with* $\Phi(0) = 0$ *and* F_0 *is a* $m \times m$ *matrix whose columns are an orthonormal basis of* $\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$*.*

Proof. We consider an element $F \in \mathbb{E}_{H_U}(s)$ with $F(0) = 0$. Then it follows from the set inclusion (5.10) that $S^*(F) \in \left(\mathbb{E}_{H_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp\right) \oplus \mathcal{V}$, where

$$\mathcal{V} = \mathbb{E}_{K_U}(s) \ominus \left\{\mathbb{E}_{K_U}(s) \cap \{U_1, U_2, \dots, U_m\}^\perp\right\}.$$

Note that $\mathcal{V}$ is generated by $\left\{P_{\mathbb{E}_{K_U}(s)}(U_1), P_{\mathbb{E}_{K_U}(s)}(U_2), \dots, P_{\mathbb{E}_{K_U}(s)}(U_m)\right\}$.

By hypothesis, $\dim(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) = m$. Let $\{W_1, W_2, \dots, W_m\}$ be an orthonormal basis of $\mathcal{V}$. Then $\{W_1(0), W_2(0), \dots, W_m(0)\}$ is linearly independent in $\mathbb{C}^m$. Indeed, suppose for $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C}$,

$$\alpha_1 W_1(0) + \alpha_2 W_2(0) + \cdots + \alpha_m W_m(0) = 0.$$

Let $G = \alpha_1 W_1 + \alpha_2 W_2 + \cdots + \alpha_m W_m$, then $G \in \mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D})$. So, for each $i \in \{1, 2, \dots, m\}$, we have $\langle G, W_i \rangle = 0$ which implies $\alpha_i = 0$.

Take $J \in \mathbb{E}_{K_U}(s)$, then $K_U^2(J) = H_U^2(J) - \sum_{i=1}^{m} \langle J, U_i \rangle Ui = s^2 J$. Note that $H_U(\mathbb{E}_{H_U}(s)) \subset \mathbb{E}_{H_U}(s)$. Therefore for any $r \in \{1, 2, \dots, m\}$ we have,

$$\begin{aligned}
s^2 \langle J, H_U(W_r) \rangle &= \left\langle H_U^2(J), H_U(W_r) \right\rangle - \sum_{i=1}^{m} \langle J, U_i \rangle \langle U_i, H_U(W_r) \rangle \\
&= \left\langle J, H_U H_U^2(W_r) \right\rangle - \sum_{i=1}^{m} \langle J, U_i \rangle \langle H_U(e_i), H_U(W_r) \rangle \\
&= s^2 \langle J, H_U(W_r) \rangle - \sum_{i=1}^{m} \langle J, U_i \rangle \left\langle H_U^2(W_r), e_i \right\rangle \\
&= s^2 \langle J, H_U(W_r) \rangle - s^2 \sum_{i=1}^{m} \langle J, U_i \rangle \langle W_r, e_i \rangle,
\end{aligned}$$

where $\{e_1, e_2, \dots, e_m\}$ is the standard orthonormal basis of $\mathbb{C}^m$. So we obtain

$$\sum_{i=1}^{m} \langle J, U_i \rangle \langle W_r, e_i \rangle = 0 \quad \text{for } r \in \{1, 2, \dots, m\}. \tag{5.12}$$

Let W be an $m \times m$ matrix whose columns are $W_1, W_2, \dots, W_m$. Therefore $W(0) = [W_1(0), W_2(0), \dots, W_m(0)]_{m \times m}$ is an invertible matrix in $M_m(\mathbb{C})$. Now the equations derived in (5.12) can be rewritten as

$$[W(0)]^t_{m\times m}[\langle J,U_1\rangle,\langle J,U_2\rangle,\dots,\langle J,U_m\rangle]^t_{m\times 1}=0,$$

which implies

$$[\langle J,U_1\rangle,\langle J,U_2\rangle,\dots,\langle J,U_m\rangle]^t_{m\times 1}=0.$$

So, we have $\langle J,U_i\rangle=0,\forall i\in\{1,2,\dots,m\}$ and hence $J\perp\{U_1,U_2,\dots,U_m\}$ for $J\in\mathbb{E}_{K_U}(s)$. Therefore, $\mathbb{E}_{K_U}(s)\perp\{U_1,U_2,\dots,U_m\}$, which further implies $\mathcal{V}=\{0\}$. In other words, $\mathbb{E}_{H_U}(s)$ is *nearly S^*-invariant* in $H^2_{\mathbb{C}^m}(\mathbb{D})$ and

$$S^*(\mathbb{E}_{H_U}(s)\cap\mathbb{C}^{m\perp})\subset\mathbb{E}_{H_U}(s)\cap\{U_1,U_2,\dots,U_m\}^{\perp}. \tag{5.13}$$

Therefore, by Corollary 5.2.1, there exists an inner function $\Phi\in H^\infty(\mathbb{D},\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m))$ with $\Phi(0)=0$ such that $\mathbb{E}_{H_U}(s)=F_0(H^2_{\mathbb{C}^m}(\mathbb{D})\ominus\Phi H^2_{\mathbb{C}^r}(\mathbb{D}))$, where $F_0=[W_1,W_2,\dots,W_m]_{m\times m}$. □

In the above Theorem 5.3.5, the assumption dim$(\mathbb{E}_{H_U}(s)\ominus(\mathbb{E}_{H_U}(s)\cap zH^2_{\mathbb{C}^m}(\mathbb{D})))=m$ is quite natural, since in the scalar-valued case if $\mathbb{E}_{H_u}(s)\nsubseteq zH^2_{\mathbb{C}}(\mathbb{D})$, then we must have

$$dim(\mathbb{E}_{H_u}(s)\ominus\big(\mathbb{E}_{H_u}(s)\cap zH^2_{\mathbb{C}}(\mathbb{D})\big))=1.$$

In the vector valued Hardy space, there are plenty of examples of Schmidt subspaces satisfying the above assumption. Indeed, in particular, if we are in $H^2_{\mathbb{C}^2}(\mathbb{D})$, then the following examples serve our purpose.

Example 5.3.6. *Let $\phi\in\mathcal{H}^\infty(\mathbb{D},\mathbb{C})$ be an inner function, and consider*

(A)

$$U=\begin{bmatrix}\phi & 0\\ 0 & \phi\end{bmatrix},\quad then\quad H_U^2=\begin{bmatrix}H_\phi^2 & 0\\ 0 & H_\phi^2\end{bmatrix}.$$

Therefore the Schmidt subspace $\mathbb{E}_{H_U}(s)=\ker(H_U^2-s^2I)=\ker(H_\phi^2-s^2I)\oplus\ker(H_\phi^2-s^2I)$.

(B)

$$U=\begin{bmatrix}0 & \phi\\ \phi & 0\end{bmatrix},\quad then\quad H_U^2=\begin{bmatrix}H_\phi^2 & 0\\ 0 & H_\phi^2\end{bmatrix}.$$

In this case also $\mathbb{E}_{H_U}(s)=\ker(H_U^2-s^2I)=\ker(H_\phi^2-s^2I)\oplus\ker(H_\phi^2-s^2I)$.

Note that, in both cases $dim(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{C^2}(\mathbb{D}))) = 2.$

In the next section, we will provide some non-trivial examples of class of Schmidt subspaces satisfying such assumption (see Example 5.4.8). We end the section with the following remark:

Remark 5.3.7. *As mentioned in [30, Appendix], one can have similar results regarding the linear Hankel operator in vector valued setting as well. Suppose* J *is the linear involution in* $L^2(\mathbb{T}, \mathbb{C}^m)$ *defined as* $JF(z) = (f_1(\bar{z}), f_2(\bar{z}), \ldots, f_m(\bar{z}))$, *where* $F = (f_1, f_2, \ldots, f_m)$ *and* $z \in \mathbb{T}$. *Let* C *be the anti-linear involution (infact a conjugation) in* $H^2_{\mathbb{C}^m}(\mathbb{D})$ *such that* $CF(z) = \overline{F(\bar{z})}$, *then for* $U \in BMOA(\mathbb{T}, \mathcal{L}(\mathbb{C}^m))$ *we can similarly define the linear Hankel operator* G_U *in* $H^2_{\mathbb{C}^m}(\mathbb{D})$ *by*

$$G_U(F) = P_m(UJF).$$

Then, we have $G_U = H_U C$ *and* $G_U^* = CH_U$, *and therefore using our description given in Theorem 5.3.3, one can have the precise structure of* $\ker(G_U^* G_U - s^2 I)$ *and* $\ker(G_U G_U^* - s^2 I)$ *in* $H^2_{\mathbb{C}^m}(\mathbb{D})$. *Note that,*

$$\ker(G_U^* G_U - s^2 I) = C\mathbb{E}_{H_U}(s) \quad \text{and} \quad \ker(G_U G_U^* - s^2 I) = \mathbb{E}_{H_U}(s).$$

5.4 The action of H_U on $\mathbb{E}_{H_U}(s) \equiv F_0\mathcal{K}_\Theta$

The Schmidt subspaces $\mathbb{E}_{H_U}(s)$ *of the Hankel operator* H_U *remain invariant under the operator* H_u *in* $H^2_{\mathbb{C}^m}(\mathbb{D})$. *In [30, 31], the authors have been discussed the explicit formula for the action of* H_u *on these subspaces* $\mathbb{E}_{H_u}(s) \equiv p\mathcal{K}_\theta$ *in terms of the parameters* s, p *and* θ *only. Also we have noted that in scalar valued Hardy space* $H^2_{\mathbb{C}}(\mathbb{D})$ *for any non-trivial Schmidt subspace* $\mathbb{E}_{H_u}(s)$ *it automatically holds that* $dim(\mathbb{E}_{H_u}(s) \ominus (\mathbb{E}_{H_u}(s) \cap zH^2_{\mathbb{C}}(\mathbb{D}))) = 1$ *whenever* $\mathbb{E}_{H_u}(s) \nsubseteq zH^2_{\mathbb{C}}(\mathbb{D})$. *But in case of vector valued Hardy space* $H^2_{\mathbb{C}^m}(\mathbb{D})$, *if* $\mathbb{E}_{H_U}(s) \nsubseteq zH^2_{\mathbb{C}^m}(\mathbb{D})$, *then in general we have*

$$1 \leq dim(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) \leq m$$

for any non-zero subspace $\mathbb{E}_{H_U}(s)$. *Under the assumption* $dim(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) = m$, *in Theorem 5.3.5, we have seen that* $\mathbb{E}_{H_U}(s)$ *has compact form, that is* $\mathbb{E}_{H_U}(s) = F_0\mathcal{K}_\Theta$ *due to it's nearly* S^**-invariant property. In this section, we will obtain an explicit formula for the action of* H_u *on* $\mathbb{E}_{H_U}(s) = F_0\mathcal{K}_\Theta$ *under the assumption* $dim(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) = m$. *Before going to the main result of this section, we need some useful lemmas.*

Lemma 5.4.1. *Let* $\Theta \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$ *be an inner function with* $\Theta(0) = 0$ *and* $A \in M_m(\mathbb{C})$ *an unitary constant matrix such that* ΘA *is symmetric. Then* $S^*\Theta(A\bar{G}) \in \mathcal{K}_\Theta$ *for any* $G \in \mathcal{K}_\Theta$.

Proof. First note that ΘA is also an inner function in $H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^m))$. Since, ΘA is symmetric, then we have $(\Theta A)^t = \Theta A$. For any $F \in H^2_{\mathbb{C}^m}(\mathbb{D})$, we get

$$\begin{aligned}\left\langle S^*\Theta(A\bar{G}), \overline{zF}\right\rangle_{L^2} &= \left\langle \Theta(A\bar{G}), \bar{F}\right\rangle_{L^2}\\ &= \left\langle (\Theta A)^t F, G\right\rangle_{H^2}\\ &= \langle \Theta A F, G\rangle_{H^2} = 0.\end{aligned}$$

Also,

$$\left\langle S^*\Theta(A\bar{G}), \Theta H\right\rangle = \left\langle A\bar{G}, zH\right\rangle = \left\langle A\bar{G}, zH\right\rangle = \left\langle \bar{G}, zA^*H\right\rangle = 0,$$

for any $H \in H^2_{\mathbb{C}^m}(\mathbb{D})$. Hence $S^*\Theta(A\bar{G}) \in \mathcal{K}_\Theta$. □

Lemma 5.4.2. *For an inner function* $\Theta \in H^\infty(\mathbb{D}, \mathcal{L}(\mathbb{C}^r, \mathbb{C}^m))$,

$$S^*\left(\mathcal{K}_\Theta \cap \mathbb{C}^{m\perp}\right) = \mathcal{K}_\Theta \cap \left\{\bigvee_{i=1}^{r} S^*(\Theta e_i)\right\}^\perp,$$

where $\{e_i\}_{i=1}^r$ *is the standard orthonormal basis of* $\mathbb{C}^r$.

Proof. Since S is an isometry on $H^2_{\mathbb{C}^m}(\mathbb{D})$, then it is sufficient to proof

$$\mathcal{K}_\Theta \cap \mathbb{C}^{m\perp} = S\left[\mathcal{K}_\Theta \cap \left\{\bigvee_{i=1}^{r} S^*(\Theta e_i)\right\}^\perp\right].$$

Let $F \in \mathcal{K}_\Theta \cap \mathbb{C}^{m\perp}$, then $F = SG$ for some $G \in H^2_{\mathbb{C}^m}(\mathbb{D})$. Therefore $G = S^*F$ and hence $G \in \mathcal{K}_\Theta$. Also for each $i \in \{1, 2, \dots, r\}$,

$$\langle G, S^*(\Theta e_i)\rangle = \langle SG, \Theta e_i\rangle = \langle F, \Theta e_i\rangle = 0,$$

which implies that $G \in \{\bigvee_{i=1}^r S^*(\Theta e_i)\}^\perp$, and it further implies $F \in S\left[\mathcal{K}_\Theta \cap \{\bigvee_{i=1}^r S^*(\Theta e_i)\}^\perp\right]$.

Let $F_1 \in H^2_{\mathbb{C}^r}(\mathbb{D})$, then $F_1 = F_1(0) + SF_2$ for some $F_2 \in H^2_{\mathbb{C}^r}(\mathbb{D})$. Again for any $G_1 \in \mathcal{K}_\Theta \cap \{\bigvee_{i=1}^r S^*(\Theta e_i)\}^\perp$ we have

$$\langle SG_1, \Theta F_1\rangle = \langle SG_1, \Theta(F_1(0) + SF_2)\rangle$$

$$
\begin{aligned}
&= \langle SG_1, \Theta F_1(0)\rangle + \langle SG_1, S\Theta F_2\rangle \\
&= \left\langle SG_1, \Theta\left(\sum_{i=1}^{r}\langle F, e_i\rangle e_i\right)\right\rangle + \langle SG_1, S\Theta F_2\rangle \\
&= \sum_{i=1}^{r}\langle e_i, F\rangle\,\langle G_1, S^*\Theta e_i\rangle + \langle G_1, \Theta F_2\rangle = 0,
\end{aligned}
$$

which proves that $SG_1 \in \mathcal{K}_\Theta \cap \mathbb{C}^{m\perp}$ and hence $S^*\left(\mathcal{K}_\Theta \cap \mathbb{C}^{m\perp}\right) = \mathcal{K}_\Theta \cap \{\bigvee_{i=1}^{r} S^*(\Theta e_i)\}^{\perp}$. □

Hypothesis 5.4.3. *Let U satisfy Hypothesis 5.2.1, $s > 0$, and the Schmidt subspace $\mathbb{E}_{H_U}(s)$ of the Hankel operator H_U satisfies the condition dim$(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) = m$.*

Now we are in a position to state the main theorem in this section regarding the action of the Hankel operator H_U on $\mathbb{E}_{H_U}(s)$.

Theorem 5.4.4. *Let U and $\mathbb{E}_{H_U}(s)$ satisfy the Hypothesis 5.4.3. Let $\{W_1, W_2, \ldots, W_m\}$ be an orthonormal basis of $\mathcal{W} := \mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))$. Then $\mathbb{E}_{H_U}(s)$ has the following representation:*

$$\mathbb{E}_{H_U}(s) = F_0\mathcal{K}_\Theta,$$

where $F_0 = [W_1, W_2, \ldots, W_m]_{m\times m}$ and $\mathcal{K}_\Theta = H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^m}(\mathbb{D})$ for some inner function $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^m,\mathbb{C}^m)}(\mathbb{D})$ with $\Theta(0) = 0$. Moreover, for any $G \in \mathcal{K}_\Theta$, there exists an unitary constant matrix $A \in M_m(\mathbb{C})$ such that the action of H_U on $\mathbb{E}_{H_U}(s)$ is given by

$$H_U(F_0G) = s\,\mathbb{P}_m F_0\left[S^*\Theta(A\bar{G})\right], \quad G \in \mathcal{K}_\Theta. \tag{5.14}$$

Proof. Let $W_i = [w_{1i}, w_{2i}, \ldots, w_{mi}]^t$, $i \in \{1, 2, \ldots, m\}$. Since $\{e_1, e_2, \ldots, e_m\}$ is the standard orthonormal basis of $\mathbb{C}^m$, then for each $j \in \{1, 2, \ldots, m\}$,

$$
\begin{aligned}
P_{\mathbb{E}_{H_U}(s)}(e_j) &= \langle e_j, W_1\rangle W_1 + \langle e_j, W_2\rangle W_2 + \cdots + \langle e_j, W_m\rangle W_m \\
&= \overline{w_{j1}(0)}W_1 + \overline{w_{j2}(0)}W_2 + \cdots + \overline{w_{jm}(0)}W_m.
\end{aligned}
$$

Note that H_U commutes with H_U^2, and hence with the orthogonal projection onto $\mathbb{E}_{H_U}(s)$. For $1 \le i \le m$, let U_i^s denote the orthogonal projection of U_i onto $\mathbb{E}_{H_U}(s)$. Then we have

$$
\begin{aligned}
U_i^s = P_{\mathbb{E}_{H_U}(s)}(U_i) &= P_{\mathbb{E}_{H_U}(s)}H_U(e_i) \\
&= H_U P_{\mathbb{E}_{H_U}(s)}(e_i) \\
&= H_U[\overline{w_{i1}(0)}W_1 + \overline{w_{i2}(0)}W_2 + \cdots + \overline{w_{im}(0)}W_m]
\end{aligned}
$$

$$= w_{i1}(0)H_U(W_1) + w_{i2}(0)H_U(W_2) + \cdots + w_{im}(0)H_U(W_m). \tag{5.15}$$

Therefore we have

$$[U_1^s, U_2^s, \ldots, U_m^s]_{m\times m} = [H_U(W_1), H_U(W_2), \ldots, H_U(W_m)]_{m\times m}[w_{ij}(0)]^t_{m\times m}. \tag{5.16}$$

Now by Theorem 5.3.5, $\mathbb{E}_{H_U}(s) = F_0(H^2_{\mathbb{C}^m}(\mathbb{D}) \ominus \Phi H^2_{\mathbb{C}^r}(\mathbb{D}))$ for some inner function $\Theta \in H^\infty_{\mathcal{L}(\mathbb{C}^r,\mathbb{C}^m)}(\mathbb{D})$, with $\Theta(0) = 0$ and $F_0 = [W_1, W_2, \ldots, W_m]_{m\times m}$.

It is easy to check that

(i) $F_0\mathcal{K}_\Theta \cap \mathbb{C}^{m\perp} = F_0\mathcal{K}_\Theta \cap \mathcal{W}^\perp$, and

(ii) $F_0\mathcal{K}_\Theta \cap \{U_1, U_2, \ldots, U_m\}^\perp = F_0\mathcal{K}_\Theta \cap \{U_1^s, U_2^s, \ldots, U_m^s\}^\perp$.

Due to the nearly S^*-invariant property of $\mathbb{E}_{H_U}(s) = F_0\mathcal{K}_\Theta$ and using above two identities along with (5.13) we also have

$$S^*\left(F_0\mathcal{K}_\Theta \cap \mathcal{W}^\perp\right) \subseteq F_0\mathcal{K}_\Theta \cap \{U_1^s, U_2^s, \ldots, U_m^s\}^\perp. \tag{5.17}$$

Now by Lemma 5.4.2, we conclude

$$\begin{aligned} S^*\left(F_0\mathcal{K}_\Theta \cap \mathcal{W}^\perp\right) &= S^*\left(F_0(\mathcal{K}_\Theta \cap \mathbb{C}^{m\perp})\right) \\ &= F_0 S^*\left(\mathcal{K}_\Theta \cap \mathbb{C}^{m\perp}\right) \\ &= F_0\left(\mathcal{K}_\Theta \cap \left\{\bigvee_{i=1}^r S^*(\Theta e_i)\right\}^\perp\right) \\ &= F_0\mathcal{K}_\Theta \cap \left\{\bigvee_{i=1}^r F_0 S^*(\Theta e_i)\right\}^\perp. \end{aligned} \tag{5.18}$$

Combining (5.17) and (5.18), we get

$$F_0\mathcal{K}_\Theta \cap \left\{\bigvee_{i=1}^r F_0 S^*(\Theta e_i)\right\}^\perp \subseteq F_0\mathcal{K}_\Theta \cap \{U_1^s, U_2^s, \ldots, U_m^s\}^\perp. \tag{5.19}$$

Next we show that $\text{span}\{U_1^s, U_2^s, \ldots, U_m^s\} \subseteq \bigvee_{i=1}^r \{F_0 S^*(\Theta e_i)\} \subseteq F_0\mathcal{K}_\Theta$. Let $F \in \text{span}\{U_1^s, U_2^s, \ldots, U_m^s\}$ Therefore F can be written as

$$F = F_1 \oplus F_2,$$

where $F_1 \in \bigvee_{i=1}^r F_0 S^*(\Theta e_i)$ and $F_2 \in F_0\mathcal{K}_\Theta \cap \{\bigvee_{i=1}^r F_0 S^*(\Theta e_i)\}^\perp$. So we have

$$\langle F, F_2\rangle = \langle F_1, F_2\rangle + \langle F_2, F_2\rangle,$$

which due to (5.19) gives $\|F_2\|^2 = 0$, that is, $F_2 = 0$. In other words, $F = F_1 \in \bigvee_{i=1}^r F_0 S^*(\Theta e_i)$ implies

$$\operatorname{span}\{U_1^s, U_2^s, \ldots, U_m^s\} \subseteq \bigvee_{i=1}^{r} F_0 S^*(\Theta e_i). \tag{5.20}$$

Hence from the above set inclusion (5.20) we have

$$\begin{aligned}
U_1^s &= c_{11}F_0S^*\Theta e_1 + c_{21}F_0S^*\Theta e_2 + \cdots + c_{r1}F_0S^*\Theta e_r \\
U_2^s &= c_{12}F_0S^*\Theta e_1 + c_{22}F_0S^*\Theta e_2 + \cdots + c_{r2}F_0S^*\Theta e_r \\
&\ \vdots \qquad\qquad\qquad \vdots \\
U_m^s &= c_{1m}F_0S^*\Theta e_1 + c_{2m}F_0S^*\Theta e_2 + \cdots + c_{rm}F_0S^*\Theta e_r,
\end{aligned} \tag{5.21}$$

where $c_{ij}, 1 \le i \le r, 1 \le j \le m$, are some scalars. In matrix notation,

$$\begin{aligned}
[U_1^s, U_2^s, \ldots, U_m^s]_{m\times m} &= [F_0S^*\Theta e_1, F_0S^*\Theta e_2, \ldots, F_0S^*\Theta e_r]_{m\times r}\,[c_{ij}]_{r\times m} \\
&= F_0[S^*\Theta e_1, S^*\Theta e_2, \ldots, S^*\Theta e_r]_{m\times r}\,[c_{ij}]_{r\times m}.
\end{aligned} \tag{5.22}$$

From (5.16) and (5.22), it follows that

$$[H_U(W_1), H_U(W_2), \ldots, H_U(W_m)]_{m\times m}[w_{ij}(0)]^t_{m\times m} = F_0[S^*\Theta e_1, S^*\Theta e_2, \ldots, S^*\Theta e_r]_{m\times r}[c_{ij}]_{r\times m}. \tag{5.23}$$

In the proof of Theorem 5.3.5, we have seen that $\{W_1(0), W_2(0), \ldots, W_m(0)\}$ is linearly independent in $\mathbb{C}^m$, so $[w_{ij}(0)]_{m\times m}$ is an invertible matrix and so is $[w_{ij}(0)]^t_{m\times m}$. Therefore the equation (5.23) can be rewritten as

$$[H_U(W_1), H_U(W_2), \ldots, H_U(W_m)] = F_0[S^*\Theta e_1, S^*\Theta e_2, \ldots, S^*\Theta e_r][c_{ij}]\left([w_{ij}(0)]^t\right)^{-1}. \tag{5.24}$$

Now, for any $i, j \in \{1, 2, \ldots, m\}$, using (5.15) we have

$$\begin{aligned}
\|U_i^s\|^2 &= \langle U_i^s, U_i^s\rangle \\
&= \langle w_{i1}(0)H_U(W_1) + \cdots + w_{im}(0)H_U(W_m), w_{i1}(0)H_U(W_1) + \cdots + w_{im}(0)H_U(W_m)\rangle \\
&= |w_{i1}(0)|^2\left\langle H_U^2(W_1), W_1\right\rangle + |w_{i2}(0)|^2\left\langle H_U^2(W_2), W_2\right\rangle + \cdots + |w_{im}(0)|^2\left\langle H_U^2(W_m), W_m\right\rangle \\
&= s^2\left\{|w_{i1}(0)|^2 + |w_{i2}(0)|^2 + \cdots + |w_{im}(0)|^2\right\}.
\end{aligned} \tag{5.25}$$

Similarly for $i \neq j$,

$$\left\langle U_i^s, U_j^s\right\rangle = s^2\left\{w_{i1}(0)\overline{w_{j1}(0)} + w_{i2}(0)\overline{w_{j2}(0)} + \cdots + w_{im}(0)\overline{w_{jm}(0)}\right\}. \tag{5.26}$$

Also from (5.21), in a similar manner, we can derive

$$\|U_i^s\|^2 = |c_{1i}|^2 + |c_{2i}|^2 + \cdots + |c_{ri}|^2, \tag{5.27}$$

and for $i \neq j$,

$$\langle U_i^s, U_j^s \rangle = c_{1i}\overline{c_{1j}} + c_{2i}\overline{c_{2j}} + \cdots + c_{ri}\overline{c_{rj}}. \tag{5.28}$$

Combining (5.25), (5.26), (5.27) and (5.28) we get

$$[c_{ij}]^*[c_{ij}] = s^2 \left([w_{ij}(0)]^t\right)^* [w_{ij}(0)]^t,$$

which gives

$$\left(\left([w_{ij}(0)]^t\right)^{-1} \right)^* [c_{ij}]^* [c_{ij}] \left([w_{ij}(0)]^t\right)^{-1} = s^2 I. \tag{5.29}$$

Let, $A = \frac{1}{s}[c_{ij}] \left([w_{ij}(0)]^t\right)^{-1}$, then by (5.29) we have $A^*A = I$, that is A is an isometry from $\mathbb{C}^m$ to $\mathbb{C}^r$. Since by hypothesis, we have $r \leq m$, then we must have $r = m$.

So, A is an unitary matrix from $\mathbb{C}^m$ to $\mathbb{C}^m$ and therefore (5.24) can be rewritten as

$$[H_U(W_1), H_U(W_2), \ldots, H_U(W_m)] = s\, F_0 \left[S^*\Theta e_1, S^*\Theta e_2, \ldots, S^*\Theta e_r\right] A. \tag{5.30}$$

Next we derive the action of H_U on dense subspace of $F_0\mathcal{K}_\Theta$. So, consider any $G \equiv [g_1, g_2, \ldots, g_m]^t \in \mathcal{K}_\Theta \cap H^\infty(\mathbb{T}, \mathcal{L}(\mathbb{C}, \mathbb{C}^m))$, then

$$\begin{aligned}
H_U(F_0G) &= P_m \left(U\bar{F_0}\bar{G}\right) \\
&= P_m \left\{U(\bar{g_1}\bar{W_1} + \bar{g_2}\bar{W_2} + \cdots + \bar{g_m}\bar{W_m})\right\} \\
&= P_m \left\{H_U(W_1)\bar{g_1} + H_U(W_2)\bar{g_2} + \cdots + H_U(W_m)\bar{g_m}\right\} \\
&= P_m \left\{[H_U(W_1), H_U(W_2), \ldots, H_U(W_m)]\bar{G}\right\} \\
&= sP_m \left\{F_0[S^*\Theta e_1, S^*\Theta e_2, \ldots, S^*\Theta e_r]A\bar{G}\right\}.
\end{aligned}$$

Therefore, by standard limiting argument, we get the explicit action of H_U on $\mathbb{E}_{H_U}(s) = F_0\mathcal{K}_\Theta$ as follows:

$$H_U(F_0G) = s\,\mathbb{P}_m\Big(F_0[S^*\Theta e_1, S^*\Theta e_2, \ldots, S^*\Theta e_r]A\bar{G}\Big), \quad G \in \mathcal{K}_\Theta.$$

□

Remark 5.4.5. *In the above Theorem 5.4.4, if ΘA is symmetric, then due to Lemma 5.4.1, the action of H_U (see (5.14)) on $\mathbb{E}_{H_U}(s)$ is reduced to*

$$H_U(F_0 G) = s\, F_0\left[S^*\Theta(A\bar{G})\right], \qquad G \in \mathcal{K}_\Theta. \tag{5.31}$$

If $m = 1$, then ΘA is automatically symmetric. Therefore by (5.31), one can get back the action of the Hankel operator on the Schmidt subspace obtained in [30, 31].

Next we see some useful observations concerning Theorem 5.4.4.

Remark 5.4.6. *For the inner function Θ obtained in Theorem 5.4.4, $S^*\Theta A$ is symmetric at* 0.

Proof. If $S^*\Theta A = [\theta_{ij}]_{m\times m}$, then our claim is to show $\theta_{ij}(0) = \theta_{ji}(0)$. Recall that $S^*\Theta A e_i \in K_\Theta$. Now

$$\begin{aligned}\theta_{ij}(0) &= \langle S^*\Theta A(e_i), e_j\rangle = \langle F_0 S^*\Theta A(e_i), F_0 e_j\rangle = \frac{1}{s}\langle H_U(W_i), W_j\rangle \\ &= \frac{1}{s}\langle H_U(W_j), W_i\rangle = \langle F_0 S^*\Theta A(e_j), F_0 e_i\rangle = \langle S^*\Theta A(e_j), e_i\rangle = \theta_{ji}(0).\end{aligned}$$

Therefore we conclude that $S^*\Theta A(0)$ is symmetric. □

Remark 5.4.7. *It is surprising to note that, in Theorem 5.4.4, the inner function Θ is* **unitary** *almost everywhere on $\mathbb{T}$ which is not the case in general.*

As promised earlier, we conclude the section with an example of a class of non-trivial Schmidt subspaces satisfying the Hypothesis 5.4.3 in $H^2_{\mathbb{C}^2}(\mathbb{D})$.

Example 5.4.8. *Consider two non-constant inner functions $\phi, \psi \in \mathcal{H}^\infty(\mathbb{D}, \mathbb{C})$ with $\phi \neq \psi$. For $\theta = \phi + \psi$ and $\gamma = \phi - \psi$, let*

$$U = \begin{bmatrix} \theta & \gamma \\ \gamma & \theta \end{bmatrix}, \quad \text{then} \quad H_U = \begin{bmatrix} H_\theta & H_\gamma \\ H_\gamma & H_\theta \end{bmatrix}.$$

Then it is easy to see that,

$$\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^2}(\mathbb{D})) = \bigvee\{\frac{s}{4}e_1, \frac{s}{4}e_2\},$$

and hence $dim(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^2}(\mathbb{D}))) = 2$. The action of H_U on $\mathbb{E}_{H_U}(s)$ can be determined by the formula (5.14) given in Theorem 5.4.4. It is remarkable to mention that, the scalar formula will not help to find the action of H_U on $\mathbb{E}_{H_U}(s) \subseteq H^2_{\mathbb{C}^m}(\mathbb{D})$.

Open question: If $dim(\mathbb{E}_{H_U}(s) \ominus (\mathbb{E}_{H_U}(s) \cap zH^2_{\mathbb{C}^m}(\mathbb{D}))) < m$, then it is still unknown to us what will be the formula for the action of H_U on $\mathbb{E}_{H_U}(s)$.

5.5 Future Problem:

There is a more flexible definition of nearly invariant subspaces in scalar valued Hardy space, suggested by Aleman and Richter in [1]:

Definition 5.5.1. *A subspace M is called nearly invariant (in a wide sense), if given a point z_0 in the unit disk such that not all functions in M vanish at z_0 , we have the implication*

$$f \in M, f(z_0) = 0 \implies \frac{f(z)}{z - z_0} \in M.$$

It can be shown (for example, by a use of the conformal mapping sending z_0 to 0) that this definition is independent of the choice of z_0 and yields the same structural description of M , i.e. M is still the image of a model space by an isometric multiplier.

This point of view is adopted in [31], i.e. it is shown that even if a Schmidt subspace of a Hankel operator is not nearly invariant in the narrow sense, it is still nearly invariant in the wide sense, with the ensuing structural description. It is a very natural question whether this can be extended to the matrix-valued case, i.e.,
Open problem: Whether any Schmidt subspace of a matrix-valued Hankel operator (of the class considered in this thesis) is nearly invariant in the wide sense.

5.5.1 Difficulty and expected hope:

In scalar valued Hardy space due to Aleman and Richter [1], we have the definition of nearly invariant subspaces is independent of the choice of the base point $z = z_0$ and also the co-dimension of the subspace of the functions which vanish at $z = z_0$ is 1. But for vector valued Hardy space $H^2_{\mathbb{C}^m}(\mathbb{D})$, this fact (independent of the choice of the base point) is still not known to us for nearly invariant subspaces. Moreover, the co-dimension of the subspace of the functions which vanish at the base point $z = z_0$ is atmost m. In the study of Schmidt subspaces of matrix-valued Hankel operator, we have observed that if $z = 0$ is not a common zero for all functions of the Schmidt subspaces and also, the co-dimension of the subspace of the functions which vanish at $z = 0$ is exactly equals to m, then the Schmidt subspaces is nearly invariant (in narrow sense, for the base point $z = 0$).

So, when we try to find it's nearly invariance property in wide sense by using conformal mapping and also the standard unitary operator (with appropriate matrix-valued modification), we cannot show that the co-dimension of the subspace of the functions which vanish at $z = 0$ *is exactly equals to* m*. In other words, we are unable to find the existence of a mapping that make all the Schmidt subspaces of a matrix-valued Hankel operator into the Schmidt subspaces of another matrix-valued Hankel operator, having the following two properties:*
(i) $z = 0$ *is not a common zero for all function of that subspace and*
(ii) the co-dimension of the subspace of the functions which vanish at $z = 0$*, in the corresponding Schmidt subspace is exactly equal to* m*.*
To that aim, if we can prove that, the Schmidt subspaces of a matrix valued Hankel operator are also nearly invariant, which satisfy the following two properties:
(i) $z = 0$ *is not a common zero for all functions of that subspace, and*
(ii) the co-dimension of the subspace of the functions which vanish at $z = 0$*, in the corresponding Schmidt subspace is strictly less than* m*. Then we can conclude that the Schmidt subspaces of a matrix-valued Hankel operator (of the class considered in this thesis) are nearly invariant in the wide sense. We keep this problem for future investigations.*

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