Study of Certain Partition Functions: Arithmetic Properties, Quantitative Estimates, Asymptotic Formulae, and Partition Inequalities

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Certificate

This is to certify that the thesis entitled **"Study of Certain Partition Functions: Arithmetic Properties, Quantitative Estimates, Asymptotic Formulae, and Partition Inequalities"** submitted by **Mr. Gurinder Singh** to the **Indian Institute of Technology Guwahati**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Date: March 20, 2025
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Abstract

In this thesis, we study various properties of certain partition functions. We study arithmetic properties of some t-regular partition functions, $\mathrm{PDO}_t(n)$ partition function, and $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$ partition functions. We discover partition inequalities for the number of hooks of particular lengths in ordinary and t-regular partition functions. We obtain quantitative estimates for the distribution of some t-regular partition functions modulo 2. We prove infinitely many non-congruences for some t-regular partitions. We also derive asymptotic formulae for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$.

Firstly, we study t-regular partition functions. For a positive integer $t \geq 2$, let $b_t(n)$ denote the number of t-regular partitions of a non-negative integer n. Motivated by some recent conjectures of Keith and Zanello, we establish infinite families of congruences modulo 2 for $b_9(n)$ and $b_{19}(n)$. We prove some specific cases of two conjectures of Keith and Zanello on self-similarities of $b_9(n)$ and $b_{19}(n)$ modulo 2. For $t \in \{6, 10, 14, 15, 18, 20, 22, 26, 27, 28\}$, Keith and Zanello conjectured that there are no integers $A > 0$ and $B \geq 0$ for which $b_t(An + B) \equiv 0 \pmod 2$ for all $n \geq 0$. We prove that, for any $t \geq 2$ and prime ℓ, there are infinitely many arithmetic progressions $An+B$ for which $\sum_{n=0}^{\infty} b_t(An+B)q^n \not\equiv 0 \pmod{\ell}$. Next, we obtain quantitative estimates for the distributions of $b_6(n)$, $b_{10}(n)$ and $b_{14}(n)$ modulo 2. Keith and Zanello also discovered new infinite families of Ramanujan type congruences modulo 2 for $b_{21}(n)$ involving every prime p with $p \equiv 13, 17, 19, 23 \pmod{24}$. We investigate the parity of $b_{21}(n)$ involving the primes p with $p \equiv 1, 5, 7, 11 \pmod{24}$. We prove new infinite families of Ramanujan type congruences modulo 2 for $b_{21}(n)$ involving the odd primes p for which the Diophantine equation $8x^2 + 27y^2 = jp$ has primitive solutions for some $j \in \{1, 4, 8\}$, and we also prove that the Dirichlet density of such primes is equal to $1/6$. Recently, Yao provided new infinite families of congruences modulo 2 for $b_3(n)$ and those congruences involve every prime $p \geq 5$. Following a similar approach, we prove new infinite families of congruences modulo 2 for $b_{21}(n)$, and these congruences imply that $b_{21}(n)$ is odd infinitely often.

Secondly, we study hook lengths in ordinary partitions and t-regular partitions. We establish hook length biases for the ordinary partitions and motivated by them we find a few interesting hook length biases in 2-regular partitions. For a positive integer k, let $p_{(k)}(n)$ denote the number of hooks of length k in all the partitions of n. We prove that $p_{(k)}(n) \geq p_{(k+1)}(n)$ for all $n \geq 0$ and $n \neq k+1$; and $p_{(k)}(k+1) - p_{(k+1)}(k+1) = -1$ for $k \geq 2$. For integers $t \geq 2$ and $k \geq 1$, let $b_{t,k}(n)$ denote the number of hooks of length k in all the t-regular partitions of n. We find generating functions of $b_{t,k}(n)$ for certain values of t and k. Exploring hook length biases for $b_{t,k}(n)$, we observe that biases are opposite to those for ordinary partitions in certain cases. We prove that $b_{2,2}(n) \geq b_{2,1}(n)$ for all $n > 4$, whereas $b_{2,2}(n) \geq b_{2,3}(n)$ for all $n \geq 0$. Next, we prove some biases for $b_{t,k}(n)$ for fixed values of k. We prove that for any $t \geq 2$, $b_{t+1,1}(n) \geq b_{t,1}(n)$, for all $n \geq 0$. We also prove that $b_{3,2}(n) \geq b_{2,2}(n)$ for all $n > 3$, and $b_{3,3}(n) \geq b_{2,3}(n)$ for all $n \geq 0$.

Thirdly, we study mex-related partition functions. The minimal excludant of an integer partition is the least positive integer missing from the partition. Let $\sigma_o\text{mex}(n)$ (resp., $\sigma_e\text{mex}(n)$) denote the sum of odd (resp., even) minimal excludants over all the partitions of n. Recently, Baruah et al. proved a few congruences for these partition functions modulo 4 and 8, and asked for asymptotic formulae for the same. We find Hardy-Ramanujan type asymptotic formulae for both $\sigma_o\text{mex}(n)$ and

$\sigma_e\text{mex}(n)$. We also prove some infinite families of congruences for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$ modulo 4 and 8.

Finally, we study the partition function $\text{PDO}_t(n)$, introduced by Lin, which counts the total number of tagged parts over all the partitions of n with designated summands in which all parts are odd. Lin proved some congruences modulo 3 and 9 for $\text{PDO}_t(n)$, and conjectured certain congruences modulo 3^{k+2} for $k \geq 0$. He proved the conjecture for $k = 0$ and $k = 1$. We prove the conjecture for $k = 2$. We also study the lacunarity of $\text{PDO}_t(n)$ modulo arbitrary powers of 2 and 3. Using nilpotency of Hecke operators, we prove that there exists an infinite family of congruences modulo any power of 2 satisfied by $\text{PDO}_t(n)$. Next, we establish infinitely many congruences for $\text{PDO}_t(n)$ modulo 8 and 32. We prove several congruences modulo small powers of 2 and discuss the existence of congruences modulo arbitrary powers of 2 similar to those in Lin's conjecture.

Contents

1
Introduction

A partition λ of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ such that

$$n = \sum_{k=1}^{r} \lambda_k,$$

where λ_k's are called parts. We represent a partition by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. The number of partitions of n is denoted by $p(n)$. By convention, we take $p(0) := 1$. For example, there are 7 partitions of 5, namely

$$(5),\ (4,1),\ (3,2),\ (3,1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1).$$

Thus, $p(5) = 7$.

The first mathematician to introduce the concept of partitions was Gottfried Wilhelm Leibniz. In his 1674 letter, Leibniz asked J. Bernoulli about the number of "divulsions," a tearing or pulling apart, of integers, which means the number of partitions of integers in modern terminology [71]. Though Leibniz is credited with inventing integer partitions or the theory of partitions, it is Leonhard Euler who paved a remarkable path for research in the field of integer partitions. In 1741, Euler gave a presentation on partitions of integers to the St. Petersburg Academy, which led to the first publication in the field of integer partitions [45]. One of the greatest discoveries in this field was Euler's generating function for $p(n)$:

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}, \tag{1.1}$$

where $|q| < 1$. Euler proved an interesting and beautiful partition identity which says that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts. Euler also gave the series representation of the infinite product $\prod_{n\geq 1}(1-q^n)$ (known as Euler's infinite product), which is now known as Euler's pentagonal number theorem:

$$\prod_{n\geq 1}(1-q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}. \tag{1.2}$$

Euler derived recurrence relation for $p(n)$ using (1.1) and (1.2):

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots .$$

Percy Alexander MacMahon, in 1916, computed the values of $p(n)$ up to $n = 200$ by hand with the help of the Euler's recurrence relation. He calculated that $p(200) = 3,972,999,029,388$. One could easily realize that $p(n)$ grows rapidly. It was a challenging job to derive an explicit formula for $p(n)$. In 1918, using their celebrated circle method, G. H. Hardy and S. Ramanujan [56], established the following asymptotic formula for $p(n)$:

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty.$$

The values calculated by MacMahon were immensely useful for Hardy and Ramanujan in checking the accuracy of their formula. Further work of Hardy and Ramanujan [56] resulted in more accurate asymptotic formula for $p(n)$. Hans Rademacher worked on Hardy-Ramanujan's asymptotic formula and obtained an explicit formula for $p(n)$ [90]. Rademacher used Ford circles, Farey sequences, and modular symmetry in the proof.

In 1919, Ramanujan discovered the following three congruences satisfied by $p(n)$

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan provided the proofs of the first two congruences in [95] and also derived the following q-series identities:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6} = 5 + 30q + 135q^2 + \cdots, \tag{1.3}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q^7\prod_{n=1}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8} = 7 + 77q + \cdots. \tag{1.4}$$

Later, he proved the third congruence as well [94]. In the year 1919 itself, Ramanujan [95] proposed a more general conjecture in the direction of the above congruences. For non-negative integers a, b, and c, let $\delta = 5^a7^b11^c$ and let λ be an integer such that $24\lambda \equiv 1 \pmod{\delta}$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta}. \tag{1.5}$$

Ramanujan himself gave a proof of (1.5) for arbitrary a and $b = c = 0$, i.e., he proved that for an integer λ, such that $24\lambda \equiv 1 \pmod{\delta}$, we have

$$p(5^a \cdot n + \lambda) \equiv 0 \pmod{5^a}.$$

Ramanujan was trying to validate his conjecture for arbitrary b and $a = c = 0$, but before he could complete the proof, we lost him on 26th April 1920. Later H. Gupta extended the table of values of $p(n)$ of MacMahon up to $n = 300$. S. Chowla, in 1934, found the conjecture of Ramanujan false [32]. He examined Gupta's table and observed that $24 \times 243 \equiv 1 \pmod{7^3}$ but $p(243) = 133978259344888 \not\equiv 0 \pmod{7^3}$.

The correct form of Ramanujan's conjecture is as follows: Define $\delta' = 5^a 7^{b'} 11^c$, where $b' = b$, if $b = 0, 1, 2$ and $b' = \lfloor \frac{b+2}{2} \rfloor$, if $b > 2$. Then

$$p(n\delta + \lambda) \equiv 0 \pmod{\delta'}. \tag{1.6}$$

In 1938, Watson [108] gave a proof of (1.6) for $a = c = 0$ and also a more detailed version of Ramanujan's proof of (1.6) in the case $b = c = 0$. Finally, in 1967, Atkin [8] proved (1.6) for arbitrary c and $a = b = 0$.

It is an interesting problem to study the distribution of the partition function modulo positive integers M. In [85], Ono revolutionized the subject by developing aspects of the p-adic theory of half-integral weight modular forms, and using this he proved the existence of infinite families of partition congruences modulo every prime greater than or equal to 5. Let $F(q) := \sum_{n=0}^{\infty} a(n)q^n$ be a given integral power series and $0 \le r < M$. We define

$$\delta_r(F, M; X) := \frac{\#\{0 \le n \le X : a(n) \equiv r \pmod{M}\}}{X}.$$

An integral power series F is called *lacunary modulo M* if

$$\lim_{X\to\infty} \delta_0(F, M; X) = 1,$$

that is, almost all of the coefficients of F are divisible by M. The power series F is said to have odd density δ if the limit

$$\lim_{X\to\infty} \delta_1(F, 2; X)$$

exists and is equal to δ. A conjecture of Parkin and Shanks [88] predicts that, for $r \in \{0, 1\}$,

$$\lim_{X\to\infty} \delta_r(P, 2; X) = \frac{1}{2},$$

where P is the generating function for $p(n)$. It is still an open problem. In other words, the conjecture says that the partition function $p(n)$ is 'equally often' even and odd, i.e.,

$$\#\{0 \le n \le X : p(n) \text{ is odd (even)}\} \sim \frac{1}{2}X.$$

Mirsky [78], in 1983, found the first quantitative result:

$$\#\{0 \le n \le X : p(n) \text{ is odd (even)}\} > \frac{\log\log X}{2\log 2}.$$

After that, this lower bound has been improved in a number of works. Currently, the best known result for even values of $p(n)$ is due to Bellaïche and Nicolas [20]:

$$\#\{0 \leq n \leq X : p(n) \text{ is even}\} \gg 0.069\sqrt{X}\log\log X,$$

and that for odd values of $p(n)$ is due to Bellaïche, Green, and Soundararajan [19]:

$$\#\{0 \leq n \leq X : p(n) \text{ is odd}\} \gg \frac{\sqrt{X}}{\log\log X}.$$

Due to Newman [82] we know that $p(n)$ assumes infinitely many even, and infinitely many odd values. In 1966, Subbarao [104] studied the parity of $p(n)$ over arithmetic progressions and gave the following conjecture: For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $p(M)$ is odd, and there are infinitely many integers $N \equiv r \pmod{t}$ for which $p(N)$ is even.

Many mathematicians proved this conjecture for particular values of t. Later in 1996, using ideas from the theory of modular forms, Ono [84] proved Subbarao's conjecture completely for the even case whereas the odd case was proved with a condition.

Theorem 1.1. [84, Theorems 1 and 2]

(a) For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $N \equiv r \pmod{t}$ for which $p(N)$ is even.

(b) For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $p(M)$ is odd, provided there is one such M.

In 2012, Radu [91] proved the odd part of Subbarao's conjecture.

Theorem 1.2. [91, Theorem 1.3] *Let $\nu \in \{2, 3\}$ and A, B integers such that $A > B \geq 0$. Then there exists a non-negative integer n_0 such that*

$$p(An_0 + B) \not\equiv 0 \pmod{\nu}.$$

Theorem 1.2 together with Theorem 1.1 settles Subbarao's conjecture.

Next, we discuss the area of partition identities and partition inequalities. The first ever identity proven in this area was Euler's partition identity. After that, many mathematicians have discovered several beautiful partition identities, such as Rogers-Ramanujan identities, Göllnitz-Gordon identities, Sylvester's identity, MacMahon's identity (see for example, Page 13 and Chapter 7 of [3]). Along with the study of partition identities, a lot of works have been published involving partition inequalities. Let $p(n,k)$ denote the number of partitions of n into k parts. In 1942, motivated by the work of Erdös and Lehner [44], Auluck, Chowla, and Gupta [9] conjectured that $p(n,k)$ has a unique maximum for any given n, i.e., for a fixed n there exists an integer k_0 such that

$$p(n,k) \geq p(n,k-1), \text{ for } k \leq k_0$$
$$p(n,k) \leq p(n,k-1), \text{ for } k > k_0.$$

The conjecture was completely proved by Haselgrove and Temperley [57] in 1954 by showing that $p(n,k)$ attains its greatest value for at most two consecutive values of k, when n is large and fixed. Since then, a lot of research has been seen in the area of partition inequalities. To mention a few of them we cite [4, 21, 22, 23, 24, 48, 76].

Apart from the ordinary partition function $p(n)$, many restricted partition functions have been studied along the lines of progress in $p(n)$. In this thesis, along with the ordinary partition function, we investigate t-regular partition functions, mex-related partition functions, and $\mathrm{PDO}_t(n)$ partition function. We also discover some partition inequalities for the number of hooks of particular lengths in ordinary and t-regular partitions. We examine arithmetic properties of t-regular partition functions, mex-related partition functions, and $\mathrm{PDO}_t(n)$ partition function. For these partition functions, we discover infinite families of Ramanujan-type congruences. We also study the distribution of some of these partition functions modulo some positive integers. We use classical q-series techniques, the theory of modular forms, Radu's algorithm, and the theory of Diophantine equations to prove our results. Analogous to Ono's and Radu's theorems mentioned earlier, we prove non-congruences for certain t-regular partitions, for some values of t. Motivated by the conjecture of Parkin and Shanks, we obtain some quantitative estimates for the distribution of 6-, 10-, and 14-regular partitions. For mex-related partition functions $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$, we derive Hardy-Ramanujan type asymptotic formulae. We find the generating functions for the number of hooks of particular lengths in some t-regular partitions. Moving to partition inequalities, we completely uncover the hook length biases for the ordinary partition function. We also prove several hook length biases for t-regular partition functions. We use generating functions and several combinatorial tools and techniques for proving these partition inequalities.

Structure of the Thesis

The entire work of the thesis is organized in the following chapters.

- Chapter 1: Introduction
- Chapter 2: Preliminaries
- Chapter 3: Arithmetic properties of certain t-regular partitions
- Chapter 4: Certain Diophantine equations and new parity results for 21-regular partitions
- Chapter 5: Hook length biases in ordinary partitions
- Chapter 6: Hook length biases in t-regular partitions
- Chapter 7: Arithmetic properties and asymptotic formulae for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$

- Chapter 8: Divisibility of the partition function $\mathrm{PDO}_t(n)$ by powers of 2 and 3

In Chapter 2, we recall some prerequisites. We introduce Ramanujan's theta functions and some p-dissections of certain q-products. We also recall some definitions and results from the theory of modular forms.

In Chapter 3, motivated by some recent conjectures of Keith and Zanello, we establish infinite families of congruences modulo 2 for $b_9(n)$ and $b_{19}(n)$, where $b_t(n)$ denotes the number of t-regular partitions of n. We prove some specific cases of two conjectures of Keith and Zanello on self-similarities of $b_9(n)$ and $b_{19}(n)$ modulo 2. For $t \in \{6, 10, 14, 15, 18, 20, 22, 26, 27, 28\}$, Keith and Zanello conjectured that there are no integers $A > 0$ and $B \geq 0$ for which $b_t(An + B) \equiv 0 \pmod{2}$ for all $n \geq 0$. We prove that, for any $t \geq 2$ and prime ℓ, there are infinitely many arithmetic progressions $An + B$ for which $\sum_{n=0}^{\infty} b_t(An + B)q^n \not\equiv 0 \pmod{\ell}$. Next, we obtain quantitative estimates for the distributions of $b_6(n), b_{10}(n)$ and $b_{14}(n)$ modulo 2. We further study the odd densities of certain infinite families of eta-quotients related to the 7-regular and 13-regular partition functions.

In Chapter 4, we study 21-regular partitions. Keith and Zanello discovered new infinite families of Ramanujan type congruences modulo 2 for $b_{21}(n)$ involving every prime p with $p \equiv 13, 17, 19, 23 \pmod{24}$. In this chapter, we investigate the parity of $b_{21}(n)$ involving the primes p with $p \equiv 1, 5, 7, 11 \pmod{24}$. We prove new infinite families of Ramanujan type congruences modulo 2 for $b_{21}(n)$ involving the odd primes p for which the Diophantine equation $8x^2 + 27y^2 = jp$ has primitive solutions for some $j \in \{1, 4, 8\}$, and we also prove that the Dirichlet density of such primes is equal to $1/6$. Recently, Yao provided new infinite families of congruences modulo 2 for $b_3(n)$ and those congruences involve every prime $p \geq 5$. Following a similar approach, we prove new infinite families of congruences modulo 2 for $b_{21}(n)$, and these congruences imply that $b_{21}(n)$ is odd infinitely often.

In Chapter 5, we establish hook length biases for the ordinary partitions. For a positive integer k, let $p_{(k)}(n)$ denote the number of hooks of length k in all the partitions of n. We prove that $p_{(k)}(n) \geq p_{(k+1)}(n)$ for all $n \geq 0$ and $n \neq k+1$; and $p_{(k)}(k+1) - p_{(k+1)}(k+1) = -1$ for $k \geq 2$.

In Chapter 6, we study hook lengths for t-regular partition functions and discover some hook length biases. For integers $t \geq 2$ and $k \geq 1$, let $b_{t,k}(n)$ denote the number of hooks of length k in all the t-regular partitions of n. We find generating functions of $b_{t,k}(n)$ for certain values of t and k. Exploring hook length biases for $b_{t,k}(n)$ for fixed t, we observe that in certain cases biases are opposite to the biases for ordinary partitions. We prove that $b_{2,2}(n) \geq b_{2,1}(n)$ for all $n > 4$, whereas $b_{2,2}(n) \geq b_{2,3}(n)$ for all $n \geq 0$. We also propose some conjectures on biases among $b_{t,k}(n)$. Next, we prove some biases for $b_{t,k}(n)$ for fixed values of k. We prove that for any $t \geq 2$, $b_{t+1,1}(n) \geq b_{t,1}(n)$, for all $n \geq 0$. We also prove that $b_{3,2}(n) \geq b_{2,2}(n)$ for all $n > 3$, and $b_{3,3}(n) \geq b_{2,3}(n)$ for all $n \geq 0$. Finally, we state some problems for future works.

In Chapter 7, we study the partition function $\sigma_o\text{mex}(n)$ (resp., $\sigma_e\text{mex}(n)$) which denote the sum of odd (resp., even) minimal excludants over all the partitions of n. Recently, Baruah et al. proved a few congruences for these partition functions modulo 4 and 8, and asked for asymptotic formulae for the same. We find Hardy-Ramanujan type asymptotic formulae for both $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$. We use Ingham's Tauberian theorem to derive these formulae. Using Ramanujan's theta functions and properties of coefficients of second power of Euler's product, we also prove some infinite families of congruences for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$ modulo 4 and 8.

In Chapter 8, we study the arithmetic properties of the partition function $\text{PDO}_t(n)$, which counts the total number of tagged parts over all the partitions of n with designated summands in which all parts are odd. Motivated by the congruences conjectured by Lin on $\text{PDO}_t(n)$ modulo 3^{k+2} for $k \geq 0$, we prove certain congruences for $\text{PDO}_t(n)$ modulo powers of 2 and 3. We also study the lacunarity of $\text{PDO}_t(n)$ modulo arbitrary powers of 2 and 3. Similar to the congruences in Lin's conjecture, we conjecture an existence of analogous congruences for $\mathrm{PDO}_t(n)$ modulo arbitrary powers of 2. We establish infinitely many congruences for $\mathrm{PDO}_t(n)$ modulo small powers of 2. We also present another approach to prove Lin's conjecture.

2
Preliminaries

We first introduce some notations. For complex numbers a and q with $|q| < 1$, define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n).$$

We also make regular use of $f_k := (q^k;q^k)_\infty = \prod_{n=1}^{\infty}(1 - q^{kn})$, where k is a positive integer.

2.1 Ramanujan's theta functions

We recall Ramanujan's general theta function $f(a,b)$ and two special cases $\varphi(q)$ and $\psi(q)$ of that. For more details, see, for example [25, 26]. Ramanujan's general theta function $f(a,b)$, which is defined for $|ab|<1$, is given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}},$$

and its special cases are:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{f_2^2}{f_1}, \tag{2.2}$$

$$\varphi(-q) = f(-q,-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{f_1^2}{f_2}. \tag{2.3}$$

Last equalities in (2.1), (2.2) and (2.3) are due to Jacobi's triple product identity given by [26, p. 10]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

These theta functions are very useful in the theory of partitions.

2.2 Certain p-dissections

In this section, we recall some 2-, 3-, and 5-dissections of certain q-products, which will help us in the study of partition functions. Most of these dissections are taken from [59]. The following lemma contains some known 2- and 3-dissections of certain q-products.

Lemma 2.1. *We have*

$$f_1f_3 = \frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}, \tag{2.4}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6f_6^3}{f_2^9f_{12}^2} + 3q\frac{f_4^2f_6f_{12}^2}{f_2^7}, \tag{2.5}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}, \tag{2.6}$$

$$\frac{f_9}{f_1} = \frac{f_{12}^3f_{18}}{f_2^2f_6f_{36}} + q\frac{f_4^2f_6f_{36}}{f_2^3f_{12}}, \tag{2.7}$$

$$\frac{1}{f_1f_3} = \frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}, \tag{2.8}$$

$$f_1^3 = \frac{f_6f_9^6}{f_3f_{18}^3} - 3qf_9^3 + 4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}, \tag{2.9}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3f_{18}^2}{f_6f_9}, \tag{2.10}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}, \tag{2.11}$$

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}, \tag{2.12}$$

$$\frac{1}{f_1^3} = \frac{f_9^3}{f_3^{10}}\left(c^2(q^3) + 3qc(q^3)\frac{f_9^3}{f_3} + 9q^2\frac{f_9^6}{f_3^2}\right), \tag{2.13}$$

where $c(q) := \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2}$.

Proof. Equation (2.4) is [59, eq. (30.12.1)]. By replacing q with $-q$ in [59, eq. (22.6.2)] and using

$$(-q;-q)_\infty = \frac{f_2^3}{f_1f_4},$$

we obtain (2.5). Identity (2.6) is proved in [40, Lemma 2.1], (2.7) in [110, Lemma 3.5] and (2.8) is [59, eq. (30.12.3)]. Identity (2.9) is the 3-dissection formula for triangular numbers [59, eq. (14.8.5)]. Equation (2.10) is [59, eq. (14.3.2)]. We can deduce (2.11) from (2.10) by replacing q with ωq and $\omega^2 q$ and multiplying the two results. Here, ω is a primitive third root of unity. The identity (2.12) is proved in [59]. The last identity (2.13) is [59, eq. (39.2.8)]. ■

The next lemma contains some known 5-dissections of certain q-products.

Lemma 2.2. *We have*

$$f_1 = f_{25}\left(R(q^5)^{-1} - q - q^2R(q^5)\right), \tag{2.14}$$

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1} + 5q^4 - 3q^5R(q^5)\right.$$
$$\left. +2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right), \tag{2.15}$$

where $R(q) := \dfrac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$.

Proof. Equation (2.14) is [59, eq. (8.1.1)] and equation (2.15) is [59, eq. (8.4.4)]. ∎

Also, we will frequently use the congruences in the following lemma without explicitly mentioning them.

Lemma 2.3. *For any prime p, and for positive integers m and k, we have*

$$f_m^{p^k} \equiv f_{pm}^{p^{k-1}} \pmod{p^k}.$$

Lemma 2.3 is an easy consequence of the Binomial Theorem.

2.3 Modular forms

In this section, we recall some definitions and important facts from the theory of modular forms. For more details, see for example [70, 86].

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}$. We identify ∞ with $\frac{1}{0}$ and define $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{r}{s} := \frac{ar+bs}{cr+ds}$, where $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. The action is defined below.

Definition 2.1. *Suppose f is a meromorphic function on $\mathbb{H}$ and ℓ is an integer. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, the slash operator $|_\ell$ is defined by*

$$(f|_\ell\gamma)(z) := (\det \gamma)^{\ell/2}(cz+d)^{-\ell} f(\gamma z).$$

For a fixed positive integer N, the following matrix sets are subgroups of the group $\mathrm{GL}_2^+(\mathbb{R})$:

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\}.$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma(N) \subseteq \Gamma$ for some

N. The smallest N such that $\Gamma(N) \subseteq \Gamma$ is called the *level* of Γ. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ.

We recall the following proposition which gives a complete set of representatives for the cusps of $\Gamma_0(N)$ (see, for example [41, p. 99]).

Proposition 2.4. *Let*

$$C_0(N) := \left\{ \frac{c}{d} : d \mid N, \gcd(c, N) = 1 \right\},$$

where c runs through a complete residue system modulo $\gcd(c, \frac{N}{d})$. Then $C_0(N)$ is a complete set of representatives of the cusps on $\Gamma_0(N)$. Moreover, $C_0(N)$ is minimal.

We are now ready to define modular forms.

Definition 2.2. *Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a weakly modular form with integer weight ℓ on Γ if the following hold:*

1. *We have $f|_\ell \gamma = f$ for all $\gamma \in \Gamma$.*

2. *If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then there exists an integer n_γ such that $(f|_\ell\gamma)(z)$ has a Fourier expansion of the form*

$$(f|_\ell\gamma)(z) = \sum_{n \geq n_\gamma} a_\gamma(n) q_N^n,$$

with $a_\gamma(n_\gamma) \neq 0$. Here $q_N := e^{2\pi i z/N}$.

If $n_\gamma \geq 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then we call f a modular form of weight ℓ on Γ. For a positive integer ℓ, the complex vector space of all weakly modular forms (resp. modular forms) of weight ℓ with respect to a congruence subgroup Γ is denoted by

$M^!_\ell(\Gamma)$ (resp. $M_\ell(\Gamma)$). In addition, if $n_\gamma \geq 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $a_\gamma(0) = 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then f is called a cusp form. The complex vector space of cusp forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $S_\ell(\Gamma)$.

Definition 2.3. [86, Definition 1.15] *If χ is a Dirichlet character modulo N, then we say that $f \in M^!_\ell(\Gamma_1(N))$ has Nebentypus character χ if*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

The space of such weakly modular forms (resp. modular forms, cusp forms) is denoted by $M^!_\ell(\Gamma_0(N), \chi)$ (resp. $M_\ell(\Gamma_0(N), \chi)$, $S_\ell(\Gamma_0(N), \chi)$). The relationship between $M^!_\ell(\Gamma_1(N))$ and these subspaces is given by the following decomposition:

$$M^!_\ell(\Gamma_1(N)) = \bigoplus_\chi M^!_\ell(\Gamma_0(N), \chi),$$

where the direct sum runs over all Dirichlet characters modulo N. The spaces $M_\ell(\Gamma_1(N))$ and $S_\ell(\Gamma_1(N))$ have the similar decomposition:

$$M_\ell(\Gamma_1(N)) = \bigoplus_\chi M_\ell(\Gamma_0(N), \chi),$$
$$S_\ell(\Gamma_1(N)) = \bigoplus_\chi S_\ell(\Gamma_0(N), \chi).$$

2.3.1 Modularity of eta-quotients

The Dedekind's eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function f is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer.

We state a very useful result by Gordon, Hughes, and Newman [50, 80, 81] regarding eta-quotients. We will use the result to verify the modularity of certain eta-quotients appearing in the proofs of some of our results. The following form of the result is taken from [86, p. 18].

Theorem 2.5. [86, Theorem 1.64] *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then f satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^{\ell} f(z)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(\bullet) := \left(\frac{(-1)^{\ell} s}{\bullet}\right)$, where $s := \prod_{\delta|N} \delta^{r_\delta}$.

Suppose that f is an eta-quotient satisfying the conditions of Theorem 2.5 and that the associated weight ℓ is a positive integer. If f is holomorphic (resp. vanishes) at all of the cusps of $\Gamma_0(N)$, then $f \in M_\ell(\Gamma_0(N), \chi)$ (resp. $S_\ell(\Gamma_0(N), \chi)$). The following theorem of Ligozat (see, for example [28, 73, 77]) gives the necessary criterion for determining orders of an eta-quotient at cusps.

Theorem 2.6. [86, Theorem 1.65] *Let c, d and N be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If f is an eta-quotient satisfying the conditions of Theorem 2.5* *for N, then the order of vanishing of f at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta}.$$

Remark 2.3.1. *Note that in Theorem 2.6, c and d are taken non-zero as we can always choose cusp representatives according to Proposition 2.4.*

2.3.2 Hecke operators

The Hecke operators are natural linear transformations that act on spaces of modular forms. The Hecke operators on spaces of integer weight modular forms are defined as follows:

Definition 2.4. *Let m be a positive integer and $f(z)=\sum_{n=0}^{\infty}a(n)q^n \in M_\ell(\Gamma_0(N),\chi)$. Then the action of Hecke operator T_m on f is defined by*

$$f(z)|T_m := \sum_{n=0}^{\infty}\left(\sum_{d|\gcd(n,m)}\chi(d)d^{\ell-1}a\left(\frac{nm}{d^2}\right)\right)q^n.$$

In particular, if $m=p$ is prime, we have

$$f(z)|T_p=\sum_{n=0}^{\infty}\left(a(pn)+\chi(p)p^{\ell-1}a\left(\frac{n}{p}\right)\right)q^n.$$

We adopt the convention that $a(n/p)=0$ when $p\nmid n$. If f is an η-quotient with the properties listed in Theorem 2.5, and $p|s$ (here s is as defined in Theorem 2.5), then $\chi(p)=0$ so that the latter term vanishes and

$$f(z)|T_p=\sum_{n=0}^{\infty}a(pn)q^n. \tag{2.16}$$

Also, we have the following factorization property.

Theorem 2.7. *For a prime p, let $f(z) := \sum_{n=0}^{\infty} a(n)q^n$ and $g(z) := \sum_{n=0}^{\infty} b(n)q^{pn}$. If $fg \in M_\ell(\Gamma_0(N), \chi)$ is an η-quotient with the properties listed in Theorem 2.5, and $p|s$ (here s is as defined in Theorem 2.5), then*

$$(fg)\,|T_p = \left(\sum_{n=0}^{\infty} a(pn)q^n\right)\left(\sum_{n=0}^{\infty} b(n)q^n\right). \tag{2.17}$$

Proof. For every $n \geq 0$, define

$$b'(n) := \begin{cases} b\left(\frac{n}{p}\right) & \text{if } p \mid n; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f(z)g(z) = \left(\sum_{n=0}^{\infty} a(n)q^n\right)\left(\sum_{n=0}^{\infty} b'(n)q^n\right) = \sum_{n=0}^{\infty}\left(\sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=n}} a(n_1)b'(n_2)\right)q^n.$$

Therefore, using (2.16), we get

$$\begin{aligned}(fg)\,|T_p &= \sum_{n=0}^{\infty}\left(\sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=pn}} a(n_1)b'(n_2)\right)q^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{\substack{n_1,n_2\geq 0\\ \frac{n_1}{p}+\frac{n_2}{p}=n}} a(n_1)b'(n_2)\right)q^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{n_1'+n_2'=n} a(pn_1')b(n_2')\right)q^n \\ &= \left(\sum_{n_1'=0}^{\infty} a(pn_1')q^{n_1'}\right)\left(\sum_{n_2'=0}^{\infty} b(n_2')q^{n_2'}\right), \end{aligned} \tag{2.18}$$

where $n_1' = n_1/p$ and $n_2' = n_2/p$. Substituting n_1' and n_2' with n in (2.18), we get

(2.17). ■

2.3.3 U-operator

We next recall the definition of U-operator, see [86, p. 28]. For a positive integer d, the U-operator $U(d)$ is defined by

$$\left(\sum_{n\geq n_0} c(n)q^n\right) \mid U(d) := \sum_{n\geq n_0} c(dn)q^n.$$

The following proposition from [86] describes the behavior of U-operator.

Proposition 2.8. [86, p. 28] *Suppose that $f \in M_\ell(\Gamma_0(N), \chi)$. If $d \mid N$, then*

$$f(z) \mid U(d) \in M_\ell(\Gamma_0(N), \chi).$$

2.3.4 Congruences for modular forms

To establish several distribution and parity results for certain partition functions, we require knowledge of the divisibility properties of the coefficients of integral weight modular forms. One of the most useful tools is Serre's work on the divisibility of the coefficients of modular forms. Let $\mathcal{A}$ denote the subset of integer weight modular forms in $M_\ell(\Gamma_0(N), \chi)$ whose Fourier coefficients are in $\mathcal{O}_K$, the ring of algebraic integers in a number field K. Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$. Using p-adic Galois representations attached to certain modular forms by Deligne, Serre [97] proved the following remarkable theorem about the divisibility of Fourier coefficients of modular forms.

Theorem 2.9. [86, Theorem 2.65] *If $f \in \mathcal{A}$ has Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{O}_K[[q]],$$

then there is a constant $\alpha > 0$ *such that*

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{\mathfrak{m}}\} = O\left(\frac{X}{(\log X)^{\alpha}}\right).$$

Theorem 2.9 yields

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : a(n) \equiv 0 \pmod{\mathfrak{m}}\}}{X} = 1.$$

The study of the parity of the coefficients of eta-quotients is one of the most challenging and interesting questions, which has significant applications in the theory of partitions. In the literature, no eta-quotient is known till today whose coefficients have positive odd density. Using Serre's seminal work on the non-divisibility of coefficients of integral weight modular forms, lacunarity (defined on Page 5) of certain families of eta-quotients can be established. For example, Cotron et al. [34] proved lacunarity of some families of eta-quotients extending the work of Gordon-Ono [51]. We phrase their theorem as follows:

Theorem 2.10. [34, Theorem 1.1] *Let* $F(q) = \frac{\prod_{j=1}^{u} f_{\alpha_j}^{r_j}}{\prod_{j=1}^{t} f_{\gamma_j}^{s_j}}$, *and assume that*

$$\sum_{j=1}^{u} \frac{r_j}{\alpha_j} \geq \sum_{j=1}^{t} s_j\gamma_j.$$

Then the coefficients of F *are lacunary modulo* 2.

We now state a result of Sturm [103] which gives a criterion to test whether two modular forms are congruent modulo a given prime.

Theorem 2.11. *Let* p *be a prime number, and* $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ *and* $g(z) =$

$\sum_{n=n_1}^{\infty} b(n)q^n$ *be modular forms of weight* k *for* $\Gamma_0(N)$ *of characters* χ *and* ψ*, respectively, where* $n_0, n_1 \geq 0$*. If either* $\chi = \psi$ *and*

$$a(n) \equiv b(n) \pmod p \ \ \textit{for all} \ \ n \leq \frac{kN}{12} \prod_{d \ \textit{prime};\ d|N} \left(1 + \frac{1}{d}\right),$$

or $\chi \neq \psi$ *and*

$$a(n) \equiv b(n) \pmod p \ \ \textit{for all} \ \ n \leq \frac{kN^2}{12} \prod_{d \ \textit{prime};\ d|N} \left(1 - \frac{1}{d^2}\right),$$

then $f(z) \equiv g(z) \pmod p$ *(i.e.,* $a(n) \equiv b(n) \pmod p$ *for all* n*).*

2.3.5 Hecke nilpotency

Serre observed and Tate proved that the action of Hecke algebras on spaces of modular forms of level 1 modulo 2 is locally nilpotent (see for example [98, 99, 105]). This implies that if $f \in M_k \cap \mathbb{Z}[[q]]$, there is a positive integer w with the property that

$$f(z) \,|T_{p_1}|\, T_{p_2}| \cdots |T_{p_w} \equiv 0 \pmod 2$$

for every collection of odd primes $p_1, p_2, \ldots, p_w$. In [87], Ono and Taguchi showed that this phenomenon generalizes to higher levels. We recall the following result which is implied by a much general result of Ono and Taguchi [87, Theorem 1.3].

Theorem 2.12. *Let* n *be a non-negative integer and* k *be a positive integer. Let* χ *be a quadratic Dirichlet character of conductor* $9 \cdot 2^n$*. There is an integer* $c \geq 0$ *such that for every* $f \in M_k\left(\Gamma_0\left(9 \cdot 2^n\right), \chi\right) \cap \mathbb{Z}[[q]]$ *and every* $t \geq 1$

$$f(z) \,|T_{p_1}|\, T_{p_2}| \cdots |T_{p_{c+t}} \equiv 0 \pmod{2^t}$$

whenever the primes $p_1, \ldots, p_{c+t}$ are coprime to 6.

Remark 2.3.2. *Theorem 1.3 of Ono and Taguchi is stated for the space of cusps forms; however, there is a remark right after the theorem which guarantees that we can use their result for any modular forms. Ono and Taguchi remarked that one merely needs to verify that the conclusion holds for the subspace of Eisenstein series. This is easily done using well-known formulas for the Fourier expansions of Eisenstein series which are given in terms of generalized divisor functions.*

2.4 Radu's algorithm

In this section, we discuss Radu's technique developed in [92, 93], which we will use to prove certain congruences for eta-quotients. We recall that the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}),$$

where p denotes a prime.

For a positive integer M, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of M. If $r \in R(M)$ and $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of M, we write $r = (r_{\delta_1}, \ldots, r_{\delta_k})$. Define $c_r(n)$ by

$$\sum_{n=0}^{\infty} c_r(n) q^n := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{n\delta})^{r_\delta}.$$

The approach to prove congruences for $c_r(n)$ developed by Radu [92, 93] reduces the number of coefficients that one must check as compared with the classical method which uses Sturm's bound alone.

Let m be a positive integer. For any integer s, let $[s]_m$ denote the residue class of s in $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $\mathbb{S}_m \subseteq \mathbb{Z}_m$

be the set of all squares in $\mathbb{Z}_m^*$. For $t \in \{0, 1, \dots, m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0, 1, \dots, m-1\}$ by

$$P_{m,r}(t) := \left\{ t' : \exists [s]_{24m} \in \mathbb{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} \right\}.$$

Definition 2.5. *Suppose $m, M,$ and N are positive integers, $r = (r_\delta) \in R(M)$, and $t \in \{0, 1, \dots, m-1\}$. Let $k = k(m) := \gcd(m^2 - 1, 24)$ and write*

$$\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j,$$

where s and j are non-negative integers with j odd. The set Δ^* consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

1. *Each prime divisor of m is also a divisor of N.*
2. *$\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.*
3. *$kN \sum_{\delta|M} r_\delta mN/\delta \equiv 0 \pmod{24}$.*
4. *$kN \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$.*
5. *$\frac{24m}{\gcd\left(-24kt - k\sum_{\delta|M} \delta r_\delta, 24m\right)}$ divides N.*
6. *If $2|m$, then either $4|kN$ and $8|sN$ or $2|s$ and $8|(1-j)N$.*

Example 2.4.1. $(m, M, N, (r_\delta), t) = (25, 10, 10, (-3, 1, 1, -1), 14) \in \Delta^*$.

Let m, M, N be positive integers. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $r \in R(M)$ and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,1,\dots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta a + \delta k \lambda c, mc)}{\delta m}$$

and

$$p^*_{r'}(\gamma) := \frac{1}{24}\sum_{\delta|N} r'_\delta \frac{\gcd^2(\delta,c)}{\delta}.$$

We are now ready to state Radu's lemma which we will use to prove congruences for some of our partition functions.

Lemma 2.13. [92, Lemma 4.5] *Let u be a positive integer, $(m, M, N, r = (r_\delta), t) \in \Delta^*$ and $r' = (r'_\delta) \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a complete set of representatives of the double cosets of $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_w) + p^*_{r'}(\gamma_w) \geq 0$ for all $1 \leq w \leq n$. Let $t_{min} = \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\left\{\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] - \sum_{\delta|N}\delta r'_\delta\right\} - \frac{1}{24m}\sum_{\delta|M}\delta r_\delta - \frac{t_{min}}{m}.$$

If the congruence $c_r(mn+t') \equiv 0 \pmod{u}$ holds for all $t' \in P_{m,r}(t)$ and $0 \leq n \leq \lfloor \nu \rfloor$, then it holds for all $t' \in P_{m,r}(t)$ and $n \geq 0$.

To apply Lemma 2.13, we utilize the following result, which gives us a complete set of representatives of the double coset in $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$.

Lemma 2.14. [107, Lemma 4.3] *If N or $\frac{1}{2}N$ is a square-free integer, then*

$$\bigcup_{\delta|N}\Gamma_0(N)\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}\Gamma_\infty = \mathrm{SL}_2(\mathbb{Z}).$$

3

Arithmetic properties of certain t-regular partitions

3.1 Introduction

Let $t \geq 2$ be a fixed positive integer. A t-regular partition of a positive integer n is a partition of n such that none of its parts is divisible by t. For example, $(6, 4, 3, 2)$ is a 5-regular partition of 15. Let $b_t(n)$ denote the number of t-regular partitions of

[1] The contents of this chapter have been published in *Ann. Comb.* (2024).

n. The generating function for $b_t(n)$ is given by

$$\sum_{n=0}^{\infty} b_t(n)q^n = \frac{f_t}{f_1}. \tag{3.1}$$

In the literature, many authors have studied divisibility and distribution properties of $b_t(n)$ for certain values of t, see, for example [16, 29, 37, 38, 51, 60, 68, 69, 110, 112]. In a recent paper [69], Keith and Zanello studied t-regular partitions for certain values of $t \leq 28$, and made several conjectures on $b_t(n)$. In [100, 101], Singh and Barman have proved two conjectures of Keith and Zanello on $b_3(n)$ and $b_{25}(n)$ respectively, and certain specific cases related to their conjectures. In this chapter, we study some of the conjectures of Keith and Zanello.

3.2 Self-similarity results for $b_9(n)$ and $b_{19}(n)$

Keith and Zanello [69] proved various congruences for $b_9(n)$ and made the following conjecture regarding the self-similarity of $b_9(n)$.

Conjecture 3.1. [69, Conjecture 9] *For all primes $p \equiv \pm 1 \pmod{9}$, let $\alpha \equiv -3^{-1} \pmod{2p}$, $0 < \alpha < 2p$, and $\beta = \lfloor \frac{2p}{3} \rfloor$. Then*

$$\sum_{n=0}^{\infty} b_9(2pn+\alpha)q^n \equiv q^{\beta} \sum_{n=0}^{\infty} b_9(2n+1)q^{pn} \pmod{2}. \tag{3.2}$$

In [69, Theorem 8], Keith and Zanello proved some specific cases of Conjecture 3.1 corresponding to $p = 17, 19, 37$. In the following theorem, we prove some other specific cases of Conjecture 3.1 corresponding to $p = 53, 71, 73, 89$.

Theorem 3.2. *If $p \in \{53, 71, 73, 89\}$, then*

$$\sum_{n=0}^{\infty} b_9\left(2pn+\alpha\right) q^n \equiv q^{\beta} \sum_{n=0}^{\infty} b_9\left(2n+1\right) q^{pn} \pmod{2},$$

where $\alpha \equiv -3^{-1} \pmod{2p}$, $0 < \alpha < 2p$, *and* $\beta = \lfloor \frac{2p}{3} \rfloor$.

Proof. We first recall the following identity from [110, eq. (3.2)]:

$$\sum_{n=0}^{\infty} b_9(2n+1)q^n \equiv \frac{f_1 f_9^2}{f_3} \pmod{2}. \tag{3.3}$$

For primes $p \equiv -1 \pmod{9}$, consider

$$F_{p,1}(z) := \frac{\eta(z)\eta^2(9z)\eta(pz)\eta^2(3pz)\eta(9pz)}{\eta(3z)}$$

and

$$F_{p,2}(z) := \frac{\eta(z)\eta^2(3z)\eta(9z)\eta(pz)\eta^2(9pz)}{\eta(3pz)}.$$

By Theorems 2.5 and 2.6, we find that $F_{p,1}(z)$ and $F_{p,2}(z)$ are modular forms of weight 3, level $27p$ and the associated character is $\chi_0(\bullet) = (\frac{-3}{\bullet})$ (As per the third condition of Theorem 2.5, N can be any multiple of $27p$ but we take the least of them to make our computations easy). For primes $p \equiv -1 \pmod{9}$, we have $\alpha = \frac{2p-1}{3}$ and $\beta = \alpha$ since $\lfloor \frac{2p}{3} \rfloor = \lfloor \frac{2p-1}{3} \rfloor$.

From (3.3), we have

$$F_{p,1}(z) \equiv q^{\alpha+1} \left(\sum_{n=0}^{\infty} b_9(2n+1)q^n \right) f_p f_{3p}^2 f_{9p} \pmod{2}$$

and

$$F_{p,2}(z) \equiv q \left(q^{\beta} \sum_{n=0}^{\infty} b_9(2n+1)q^{pn} \right) f_1 f_3^2 f_9 \pmod{2}.$$

Applying Hecke operator T_p on $F_{p,1}(z)$ and then using (2.17), we find that

$$F_{p,1}(z)|T_p \equiv \left(\left(\sum_{n=0}^{\infty} b_9(2n+1)q^{n+\alpha+1} \right) f_p f_{3p}^2 f_{9p} \right) |T_p$$

$$\equiv \left(\left(\sum_{n=0}^{\infty} b_9(2n-2\alpha-1)q^n\right) f_p f_{3p}^2 f_{9p}\right) | T_p$$
$$\equiv \left(\sum_{n=0}^{\infty} b_9(2pn-2\alpha-1)q^n\right) f_1 f_3^2 f_9$$
$$\equiv \left(\sum_{n=0}^{\infty} b_9(2pn+2p-2\alpha-1)q^{n+1}\right) f_1 f_3^2 f_9$$
$$\equiv q\left(\sum_{n=0}^{\infty} b_9(2pn+\alpha)q^n\right) f_1 f_3^2 f_9 \pmod 2,$$

where the last congruence follows from the fact that $\alpha = \frac{2p-1}{3}$. Since the Hecke operator is an endomorphism on $M_3\left(\Gamma_0(27p), \chi_0\right)$, we obtain $F_{p,1}(z)|T_p \in M_3\left(\Gamma_0(27p), \chi_0\right)$. By Theorem 2.11, the Sturm bound for the space $M_3\left(\Gamma_0(27p), \chi_0\right)$ is $9(p+1)$. For each prime $p \in \{53, 71, 89\}$, we wish to verify the congruence

$$F_{p,1}(z)|T_p \equiv F_{p,2}(z) \pmod 2,$$

i.e.,

$$q\left(\sum_{n=0}^{\infty} b_9(2pn+\alpha)q^n\right) f_1 f_3^2 f_9 \equiv q^{\alpha+1}\frac{f_p f_{9p}^2}{f_{3p}} f_1 f_3^2 f_9 \pmod 2.$$

The coefficient of $q^{9(p+1)}$ on the left side involves the value $b_9(18p(p+1)+\alpha)$; thus, f_9/f_1 must be expanded at least that far, and the product on the right side must be constructed up to the $q^{9(p+1)}$ terms. Finally, using `SageMath`, we verify that all coefficients up to the desired bound are congruent modulo 2. This completes the proof of the theorem for these three primes.

For primes $p \equiv 1 \pmod 9$, we have $\alpha = \frac{4p-1}{3}$ and $\beta = \frac{\alpha-1}{2}$ since $\lfloor \frac{2p}{3} \rfloor = \lfloor \frac{2p-2}{3} \rfloor$. Let

$$G_{p,1}(z) := \frac{\eta(z)\eta^2(9z)\eta^8(pz)}{\eta(3z)}$$

and

$$G_{p,2}(z) := \frac{\eta^8(z)\eta^2(9pz)\eta(pz)}{\eta(3pz)}.$$

Again by Theorems 2.5 and 2.6, we find that $G_{p,1}(z)$ and $G_{p,2}(z)$ are modular forms of weight 5, level $27p$ and the associated character is $\chi_1(\bullet) = (\frac{-3}{\bullet})$.

From (3.3), we have

$$G_{p,1}(z) \equiv q^{(\alpha+3)/4}\left(\sum_{n=0}^{\infty} b_9(2n+1)q^n\right) f_p^8 \pmod 2$$

and

$$G_{p,2}(z) \equiv q\left(q^{\beta}\sum_{n=0}^{\infty} b_9(2n+1)q^{pn}\right) f_1^8 \pmod 2.$$

Applying Hecke operator T_p on $G_{p,1}(z)$ and then using (2.17), we find that

$$\begin{aligned} G_{p,1}(z)|T_p &\equiv \left(\left(\sum_{n=0}^{\infty} b_9(2n+1)q^{n+\frac{\alpha+3}{4}}\right) f_p^8\right)|T_p \\ &\equiv \left(\left(\sum_{n=0}^{\infty} b_9\left(2n-\frac{\alpha+1}{2}\right)q^n\right) f_p^8\right)|T_p \\ &\equiv \left(\sum_{n=0}^{\infty} b_9\left(2pn-\frac{\alpha+1}{2}\right)q^n\right) f_1^8 \\ &\equiv \left(\sum_{n=0}^{\infty} b_9\left(2pn+2p-\frac{\alpha+1}{2}\right)q^{n+1}\right) f_1^8 \\ &\equiv q\left(\sum_{n=0}^{\infty} b_9(2pn+\alpha)q^n\right) f_1^8 \pmod 2, \end{aligned}$$

where the last congruence follows from the fact that $\alpha = \frac{4p-1}{3}$. Clearly, $G_{p,1}(z)|T_p \in M_5(\Gamma_0(27p), \chi_1)$. By Theorem 2.11, the Sturm bound for this space of modular forms is $15(p+1)$. For the prime $p = 73$, we wish to verify the congruence

$$G_{p,1}(z)|T_p \equiv G_{p,2}(z) \pmod 2,$$

i.e.,

$$q\left(\sum_{n=0}^{\infty} b_9(2pn+\alpha)q^n\right) f_1^8 \equiv q^{(\alpha+1)/2}\frac{f_p f_{9p}^2}{f_{3p}} f_1^8 \pmod{2}.$$

In this case, Sturm bound is $15(p+1) = 15 \times 74$. Using `SageMath`, we verify that all coefficients up to the desired bound are congruent modulo 2 for the prime $p = 73$. This completes the proof of the theorem. ∎

Remark 3.2.1. *Infinitely many congruences for $b_9(n)$ modulo 2 can be deduced from the self-similarity in Theorem 3.2, see Corollary 24 in [69].*

Keith and Zanello [69] also studied 2-divisibility of $b_{19}(n)$ and proved new congruences for the prime $p = 5$ and made the following conjecture regarding the self-similarity of $b_{19}(n)$.

Conjecture 3.3. [69, Conjecture 11] *For a prime $p > 3$, let $\gamma \equiv -3 \cdot 8^{-1} \pmod{p}$, $0 < \gamma < p$, and $\delta = \lfloor \frac{3p}{8} \rfloor$. Then, for a positive proportion of primes p, it holds that:*

$$\sum_{n=0}^{\infty} b_{19}(2(pn+\gamma))q^n \equiv q^{\delta}\sum_{n=0}^{\infty} b_{19}(2n)q^{pn} \pmod{2}. \tag{3.4}$$

In [69, Theorem 10], Keith and Zanello proved that $p = 5$ satisfies (3.4). It is easy to check that $p = 7$ does not satisfy (3.4). In the following theorem, we prove that $p = 11$ satisfies (3.4).

Theorem 3.4. *We have*

$$\sum_{n=0}^{\infty} b_{19}\left(22n+2\right) q^n \equiv q^4 \sum_{n=0}^{\infty} b_{19}\left(2n\right) q^{11n} \pmod{2}$$

and therefore, for all $k \not\equiv 4 \pmod{11}$

$$b_{19}(242n + 22k + 2) \equiv 0 \pmod{2},$$

and by iteration,

$$b_{19}\left(2\cdot 11^{2d}n+2\cdot 11^{2d-1}k+2\cdot 11^{2d-2}+90\left(\frac{11^{2d-2}-1}{120}\right)\right)\equiv 0 \pmod 2,$$

for all $d,k\geq 1$ *with* $k\not\equiv 4 \pmod{11}$.

Proof. Taking $t=19$ in (3.1), we obtain

$$\sum_{n=0}^{\infty} b_{19}(n)q^n=\frac{f_{19}}{f_1}. \tag{3.5}$$

Let

$$H_{11,1}(z):=\frac{\eta(19z)\eta^{126}(121z)}{\eta(z)}$$

and

$$H_{11,2}(z):=\frac{\eta(209z)\eta^{126}(11z)}{\eta(11z)}.$$

Using Theorems 2.5 and 2.6, we find that $H_{11,1}(z)$ and $H_{11,2}(z)$ are modular forms of weight 63, level 2299 and the associated character is $\chi_2(\bullet)=(\frac{-19}{\bullet})$. We next calculate that

$$H_{11,1}(z)|T_2|T_{11}\equiv q^{29}\sum_{n=0}^{\infty} b_{19}(22n+2)q^n f_{11}^{63} \pmod 2$$

and

$$H_{11,2}(z)|T_2\equiv q^{33}\sum_{n=0}^{\infty} b_{19}(2n)q^{11n} f_{11}^{63} \pmod 2.$$

By Theorem 2.11, the Sturm bound for the space $M_{63}\left(\Gamma_0(2299),\chi_2\right)$ is 13860. We wish to verify the congruence

$$q^{29}\left(\sum_{n=0}^{\infty} b_{19}(22n+2)q^n\right)f_{11}^{63}\equiv q^{33}\left(\sum_{n=0}^{\infty} b_{19}(2n)q^{11n}\right)f_{11}^{63} \pmod 2.$$

Expansion with `SageMath` confirms that all coefficients up to the desired bound are congruent modulo 2, and the first part of the theorem is established.
Since only powers for which $11|n-4$ can be non-zero on the right side of the statement, therefore, for all $k \not\equiv 4 \pmod{11}$ we obtain:

$$b_{19}(22(11n+k)+2) = b_{19}(2\cdot 11^2 n + 22k + 2) \equiv 0 \pmod{2}.$$

Now, recursively applying the relation

$$b_{19}(2n) \equiv b_{19}(2\cdot 11^2 n + 90) \pmod{2},$$

we obtain

$$\begin{aligned}
&b_{19}(2\cdot 11^2 n + 22k + 2)\\
&\equiv b_{19}(2\cdot 11^2(11^2 n + 11k + 1) + 90) \pmod{2}\\
&= b_{19}(2\cdot 11^4 n + 2\cdot 11^3 k + 2\cdot 11^2 + 90)\\
&\equiv b_{19}(2\cdot 11^6 n + 2\cdot 11^5 k + 2\cdot 11^4 + 2\cdot 11^2\cdot 45 + 90) \pmod{2}\\
&\equiv \ldots\\
&\equiv b_{19}\left(2\cdot 11^{2d} n + 2\cdot 11^{2d-1} k + 2\cdot 11^{2d-2} + 90\left(\frac{11^{2d-2}-1}{120}\right)\right) \equiv 0 \pmod{2},
\end{aligned}$$

for all $d, k \geq 1$ with $k \not\equiv 4 \pmod{11}$. We note that the last line is given by a finite geometric summation. This completes the proof of the theorem. ■

In [68], Keith studied the divisibility of 9-regular partitions by 3. In [68, Theorem 1], he proved that $b_9(4n+3) \equiv 0 \pmod{3}$ using modular forms techniques. In the following theorem, we find the generating function for $b_9(4n+3)$ as given by Ramanujan for his congruences in (1.3) and (1.4). As an immediate consequence, we obtain that $b_9(4n+3) \equiv 0 \pmod{3}$.

Theorem 3.5. *We have*

$$\sum_{n=0}^{\infty} b_9(4n+3)q^n = 3\frac{f_2^2 f_6^2 f_9}{f_1^5}. \tag{3.6}$$

Proof. We rewrite (2.7) as

$$\frac{f_9}{f_1} = \frac{f_{12}^3}{f_{36}}\left(\frac{f_{18}}{f_2}\right)\left(\frac{1}{f_2 f_6}\right) + q\frac{f_4^2 f_{36}}{f_{12}}\left(\frac{f_6}{f_2^3}\right). \tag{3.7}$$

Magnifying equations (2.5), (2.7), and (2.8) by $q \to q^2$, and substituting resultants in (3.7), and then extracting terms with powers of q congruent to 3 modulo 4, we obtain (3.6). This completes the proof of the theorem. ∎

3.3 Parity of certain t-regular partitions in arithmetic progression

Keith and Zanello [69] conjectured the following non-congruences, analogous to Ono's and Radu's theorems (Theorems 1.1 and 1.2) for the ordinary partition function.

Conjecture 3.6. [69, Conjecture 15] *Let* $t \in \{6, 10, 14, 15, 18, 20, 22, 26, 27, 28\}$. *We have*

(1) For no integers $A > 0$ *and* $B \geq 0$, $b_t(An+B) \equiv 0 \pmod 2$ *for all* $n \geq 0$.

(2) The series $\frac{f_t}{f_1}$ *has odd density* $1/2$.

In the following theorem, for a given prime ℓ, we prove that there are infinitely many arithmetic progressions $An + B$ for which $\sum_{n=0}^{\infty} b_t(An+B)q^n \not\equiv 0 \pmod{\ell}$.

Theorem 3.7. *Let* ℓ *be a prime and let* t, r *be positive integers with* $t \geq 2$. *Then* *we have*

$$\sum_{n=0}^{\infty} b_t(rn+s)q^n \not\equiv 0 \pmod{\ell}$$

for all $s \in \{0,1,\dots,r-1\}$ *such that* $s \equiv (t-1)\frac{d^2-1}{24} \pmod{r}$ *for some integer* d *with* $\gcd(d,6tr)=1$.

Note that Theorem 3.7 with $\ell = 2$ proves Conjecture 3.6 (1) for infinitely many arithmetic progressions.

We will need the following theorem of Garthwaite and Jameson [47] for the proof of Theorem 3.7. Let $B \in \mathbb{Z}$, $k \in \frac{1}{2}\mathbb{Z}$, and $N \in \mathbb{Z}^+$. Set

$$\mathcal{S}(B,k,N,\chi) := \left\{\eta^B(\tau)F(\tau) : F(\tau) \in M_k^!(\Gamma_0(N),\chi)\right\},$$

where $M_k^!(\Gamma_0(N),\chi)$ is the space of weakly modular forms of weight k and level N with character χ.

Theorem 3.8. [47, Theorem 1] *Let* ℓ *be prime, let* $f(\tau) = q^{B/24}\sum_{n\geq n_0} a(n)q^n \in \mathcal{S}(B,k,N,\chi)$ *have rational* ℓ*-integral coefficients, and let* $u \in \mathbb{Z}^+$. *Let* $v_0 \in \mathbb{Z}$ *such that*

$$\ell \nmid a(v_0) \quad \text{and} \quad a(n) = 0 \text{ for all } n < v_0 \text{ with } n \equiv v_0 \pmod{u}.$$

Then for all $v \in \{0,\dots,u-1\}$ *such that*

$$v \equiv v_0 d^2 + B\frac{d^2-1}{24} \pmod{u}$$

for some integer d *with* $\gcd(d,6Nu)=1$, *we have that*

$$\sum a(un+v)q^n \not\equiv 0 \pmod{\ell}.$$

Proof of Theorem 3.7. Consider

$$f(\tau) := q^{\frac{t-1}{24}} \sum_{n=0}^{\infty} b_t(n) q^n = \eta^{t-1}(\tau) \frac{\eta(t\tau)}{\eta^t(\tau)}.$$

As per the notations in Theorem 3.8, we have $B = t-1$, $k = -\frac{t-1}{2}$, and $N = 24t$, i.e., $f(\tau) \in \mathcal{S}(t-1, -\frac{t-1}{2}, 24t, \chi)$ for some Nebentypus character χ. The proof follows applying Theorem 3.8 for $v_0 = 0$, since $b_t(0) = 1$. ■

3.4 Quantitative estimates for $b_t(n)$ modulo 2

In Conjecture 3.6 (2), Keith and Zanello claimed that the series $\frac{f_t}{f_1}$ has odd density 1/2 for $t \in \{6, 10, 14, 15, 18, 20, 22, 26, 27, 28\}$. In the following two theorems, we obtain quantitative estimates for the distributions of $b_t(n)$ for $t = 6, 10, 14$.

Theorem 3.9. *For large X and $t = 6, 10$, we have*

$$\#\left\{n \leq X : b_t(n) \text{ is even}\right\} \gg \sqrt{X}. \tag{3.8}$$

Theorem 3.10. *For large X, we have*

$$\#\left\{n \leq X : b_{14}(2n) \text{ is even}\right\} \gg \sqrt{X}.$$

In [1], Ahlgren found quantitative estimates for the distribution of parity of the ordinary partition function $p(n)$ in arithmetic progression. We follow a similar approach to prove Theorems 3.9 and 3.10.

Proof of Theorem 3.9. We write Euler's Pentagonal Number Theorem (1.2) in our terminology:

$$f_1 = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n}{2}(3n-1)} \tag{3.9}$$

and we recall Corollary 1 of [96]

$$\frac{f_3^3}{f_1} \equiv \sum_{n\in\mathbb{Z}} q^{n(3n-2)} \pmod 2. \tag{3.10}$$

Using the above identities in

$$\sum_{n=0}^{\infty} b_6(n)q^n \equiv \frac{f_3^2}{f_1} \pmod 2,$$

we find that

$$\sum_{n\in\mathbb{Z}} q^{n(3n-2)} \equiv \sum_{n=0}^{\infty} b_6(n)q^n \sum_{n\in\mathbb{Z}} q^{3n(3n-1)/2} \pmod 2. \tag{3.11}$$

Clearly for

$$\sum_{n=0}^{\infty} a(n)q^n := \sum_{n\in\mathbb{Z}} q^{n(3n-2)},$$

we have

$$\#\{n \leq X : a(n) \text{ is odd}\} = o(X). \tag{3.12}$$

Set $u_k := \frac{3}{2}k(3k-1)$, $k \in \mathbb{Z}$. For every non-negative integer n, define the set

$$\mathcal{M}_n := \{n - u_k : 0 \leq u_k \leq n, \text{ for some } k \in \mathbb{Z}\}.$$

Now comparing the coefficients of q^n on both sides of (3.11), we obtain

$$a(n) \equiv \sum_{m\in\mathcal{M}_n} b_6(m) \pmod 2. \tag{3.13}$$

Note that for $k \geq 1$, if $u_{-(k-1)} \leq n < u_k$, then $|\mathcal{M}_n| = 2k+1$ and if $u_k \leq n < u_{-k}$, then $|\mathcal{M}_n| = 2k$. Thus, $|\mathcal{M}_n|$ is odd if and only if n is in an interval of the form $I_k := \left[u_{-(k-1)}, u_k\right)$. There exists a positive constant C such that $I_k \subset [0, X]$, $0 \leq$

$k \le C\sqrt{X}$, for large X. The fact that the length of I_k is $\gg k$ implies

$$\#\{n \le X : n \in I_k \text{ for some } k\} \gg \sum_{k=0}^{C\sqrt{X}} k \gg X.$$

Therefore, $\#\{n \le X : |\mathcal{M}_n| \text{ is odd }\} \gg X$, and together with (3.12) we conclude that

$$\#\{n \le X : |\mathcal{M}_n| \text{ is odd, } a(n) \text{ is even}\} \gg X.$$

It is clear from (3.13) that for every $n \in \{n \le X : n \in I_k \text{ for some } k\}$, $b_6(m)$ is even for some $m \in \mathcal{M}_n$. This gives

$$\#\{m \le X : b_6(m) \text{ is even}\} \cdot \max_m (\#\{n \le X : m \in \mathcal{M}_n\}) \gg X.$$

We now wish to count $M_{m,X} := \#\{n \le X : m \in \mathcal{M}_n\}$. For fixed m, $M_{m,X}$ is not more than $\#\{k \in \mathbb{Z} : 0 \le u_k \le X\}$, and this number is clearly $\ll \sqrt{X}$. Therefore, $M_{m,X} \ll \sqrt{X}$, and we arrive at (3.8). This completes the proof for $t = 6$.

For $t = 10$, we employ

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}}$$

and (equation (10) from [58])

$$\frac{f_5^5}{f_1} \equiv \sum_{n=1}^{\infty} q^{n^2-1} + \sum_{n=1}^{\infty} q^{2n^2-1} + \sum_{n=1}^{\infty} q^{5n^2-1} + \sum_{n=1}^{\infty} q^{10n^2-1} \pmod 2$$

in

$$\sum_{n=0}^{\infty} b_{10}(n) q^n \equiv \frac{f_5^2}{f_1} \pmod 2$$

to obtain

$$\sum_{n=1}^{\infty} q^{n^2-1} + \sum_{n=1}^{\infty} q^{2n^2-1} + \sum_{n=1}^{\infty} q^{5n^2-1} + \sum_{n=1}^{\infty} q^{10n^2-1}$$
$$\equiv \sum_{n=0}^{\infty} b_{10}(n)q^n \sum_{n=0}^{\infty} q^{\frac{5n(n+1)}{2}} \pmod 2.$$

The rest of the proof goes along similar lines as in the case of $t = 6$, so we omit the details for reasons of brevity. This completes the proof of the theorem. ∎

Proof of Theorem 3.10. In order to prove Theorem 3.10, first we recall the following congruence from [16, eq. (3.4)]:

$$\sum_{n=0}^{\infty} b_7(2n+1)q^n \equiv f_1 f_{14} \pmod 2. \tag{3.14}$$

We know that the generating function for 14-regular partitions can be written as

$$\sum_{n=0}^{\infty} b_{14}(n)q^n \equiv \frac{f_1 f_{14}}{f_1^2} \pmod 2. \tag{3.15}$$

Invoking (3.14) in (3.15) yields

$$\sum_{n=0}^{\infty} b_{14}(n)q^n \equiv \frac{1}{f_1^2} \sum_{n=0}^{\infty} b_7(2n+1)q^n \pmod 2.$$

Taking those terms with even powers of q and then replacing q^2 with q, we obtain

$$\sum_{n=0}^{\infty} b_{14}(2n)q^n \equiv \frac{1}{f_1} \sum_{n=0}^{\infty} b_7(4n+1)q^n \pmod 2. \tag{3.16}$$

We then use (3.9) in (3.16) to obtain

$$\sum_{n=0}^{\infty} b_7(4n+1)q^n \equiv \sum_{n=0}^{\infty} b_{14}(2n)q^n \sum_{n\in\mathbb{Z}} q^{\frac{n}{2}(3n-1)} \pmod 2. \tag{3.17}$$

We define

$$\sum_{n=0}^{\infty} a(n)q^n := \sum_{n=0}^{\infty} b_7(4n+1)q^n \tag{3.18}$$

and

$$\sum_{n=0}^{\infty} c(n)q^n := \sum_{n=0}^{\infty} b_{14}(2n)q^n. \tag{3.19}$$

Note that

$$\#\{n \leq X : a(n) \text{ is odd}\} = o(X), \tag{3.20}$$

which follows from (3.14) and the fact that the coefficients of $f_1 f_{14}$ have odd density zero, which itself follows from Theorem 2.10.

Set $u_k := \frac{1}{2}k(3k-1)$, $k \in \mathbb{Z}$. For every non-negative integer n, we define the set

$$\mathcal{M}_n := \{n - u_k : 0 \leq u_k \leq n, \text{ for some } k \in \mathbb{Z}\}.$$

Now comparing the coefficients of q^n on both sides of (3.17), we obtain

$$a(n) \equiv \sum_{m \in \mathcal{M}_n} c(m) \pmod 2. \tag{3.21}$$

Note that for $k \geq 1$, if $u_{-(k-1)} \leq n < u_k$, then $|\mathcal{M}_n| = 2k-1$ and if $u_k \leq n < u_{-k}$, then $|\mathcal{M}_n| = 2k$. Thus, $|\mathcal{M}_n|$ is odd if and only if n is in an interval of the form $I_k := \big[u_{-(k-1)}, u_k\big)$. There exists a positive constant C such that $I_k \subset [0, X]$, $0 \leq k \leq C\sqrt{X}$, for large X. The fact that the length of I_k is $\gg k$ implies

$$\#\{n \leq X : n \in I_k \text{ for some } k\} \gg \sum_{k=0}^{C\sqrt{X}} k \gg X.$$

Therefore, $\#\{n \leq X : |\mathcal{M}_n| \text{ is odd} \gg X\}$, and together with (3.20) we conclude that

$$\#\{n \leq X : |\mathcal{M}_n| \text{ is odd, } a(n) \text{ is even}\} \gg X.$$

It is clear from (3.21) that for every $n \in \{n \leq X : n \in I_k \text{ for some } k\}$, $c(m)$ is even for some $m \in \mathcal{M}_n$. This gives

$$\#\left\{m \leq X : c(m) \text{ is even}\right\} \cdot \max_m \left(\#\left\{n \leq X : m \in \mathcal{M}_n\right\}\right) \gg X.$$

We now wish to count $N_{m,X} := \#\{n \leq X : m \in \mathcal{M}_n\}$. For fixed m, $N_{m,X}$ is not more than $\#\left\{k \in \mathbb{Z} : 0 \leq u_k \leq X\right\}$, and this number is clearly $\ll \sqrt{3X}$. Therefore, $\#\{n \leq X : m \in \mathcal{M}_n\} \ll \sqrt{3X}$, and we arrive at

$$\#\{m \leq X : c(m) \text{ is even}\} \gg \sqrt{X},$$

that is,

$$\#\{n \leq X : b_{14}(2n) \text{ is even}\} \gg \sqrt{X}.$$

This completes the proof of the theorem. ∎

3.5 Odd densities of 7- and 13-regular partition functions

In [69], Keith and Zanello demonstrated that an odd density may be constant over appropriate infinite families of eta-quotients. As an example, they proved that for any non-negative integer k, the odd density of $\frac{f_3^{9k+2}}{f_1^{3k+1}}$ is the same as the odd density of 6-regular partition function. In the following theorem, we prove that the odd density of $\frac{f_7^{7k+1}}{f_1^{k+1}}$ is the same as the odd density of 7-regular partition function.

Theorem 3.11. *Let $\delta_7^{(k)}$ denote the odd density of*

$$\frac{f_7^{7k+1}}{f_1^{k+1}}.$$

If $\delta_7^{(k)}$ exists for any $k \geq 0$ then $\delta_7^{(k)}$ exists for all $k \geq 0$, and its value is independent of k. In particular, if the odd density of 7-regular partitions $\delta_7^{(0)}$ exists then all of the $\delta_7^{(k)}$ exist and are equal to $\delta_7^{(0)}$.

Proof. We first recall the following identity from [69, eq. (9)]:

$$\frac{f_7}{f_1} \equiv f_1^6 + qf_1^2 f_7^4 + q^2 \frac{f_7^8}{f_1^2} \pmod{2}, \tag{3.22}$$

which yields

$$\frac{f_7^8}{f_1^2} \equiv f_1^5 f_7^7 + qf_1 f_7^{11} + q^2 \frac{f_7^{15}}{f_1^3} \pmod{2}, \tag{3.23}$$
$$\frac{f_7^{15}}{f_1^3} \equiv f_1^4 f_7^{14} + qf_7^{18} + q^2 \frac{f_7^{22}}{f_1^4} \pmod{2}, \tag{3.24}$$
$$\frac{f_7^{22}}{f_1^4} \equiv f_1^3 f_7^{21} + q\frac{f_7^{25}}{f_1} + q^2 \frac{f_7^{29}}{f_1^5} \pmod{2}, \tag{3.25}$$
$$\frac{f_7^{29}}{f_1^{15}} \equiv f_1^2 f_7^{28} + q\frac{f_7^{32}}{f_1^2} + q^2 \frac{f_7^{36}}{f_1^6} \pmod{2}, \tag{3.26}$$
$$\frac{f_7^{36}}{f_1^{16}} \equiv f_1 f_7^{35} + q\frac{f_7^{39}}{f_1^3} + q^2 \frac{f_7^{43}}{f_1^7} \pmod{2}, \tag{3.27}$$

and for $k \geq 6$

$$\frac{f_7^{7k+1}}{f_1^{k+1}} \equiv \frac{f_7^{7k}}{f_1^{k-6}} + q\frac{f_7^{7k+4}}{f_1^{k-2}} + q^2 \frac{f_7^{7k+8}}{f_1^{k+2}} \pmod{2}. \tag{3.28}$$

Since $\frac{7k}{7} \geq k-6$ and $\frac{7k+4}{7} \geq k-2$ for all $k \geq 6$, and the first two terms on the right-hand sides of equivalences (3.22)-(3.27) satisfy the hypothesis of Theorem 2.10, the first two terms on the right-hand side of every equivalence has odd density zero.

Therefore, assuming the existence of densities, the left term in each equivalence has the same density as the third term on the right-hand side. Thus, if the odd density of the eta quotients $\frac{f_7^{7k+1}}{f_1^{k+1}}$ exists for any $k \geq 0$, then it exists for all $k \geq 0$, and they all are equal to the odd density of $\frac{f_7}{f_1}$ (i.e., when $k = 0$), which is $\delta_7^{(0)}$. ■

Next, we show that the odd density of $\frac{f_{13}^{13k+1}}{f_1^{k+1}}$ is same as the odd density of 13-regular partition function. More precisely, we have the following result.

Theorem 3.12. *Let $\delta_{13}^{(k)}$ denote the odd density of*

$$\frac{f_{13}^{13k+1}}{f_1^{k+1}}.$$

If $\delta_{13}^{(k)}$ exists for any $k \geq 0$ then $\delta_{13}^{(k)}$ exists for all $k \geq 0$, and its value is independent of k. In particular, if the odd density of 13-regular partitions $\delta_{13}^{(0)}$ exists then all of the $\delta_{13}^{(k)}$ exist and are equal to $\delta_{13}^{(0)}$.

Proof. We first recall the following identity for 13-regular partition function from [29, eq. (3)]:

$$\frac{f_{13}}{f_1} \equiv f_4^3 + q f_2^5 f_{26} + q^6 f_{52}^3 + q^7 \frac{f_{26}^7}{f_2} \pmod 2. \tag{3.29}$$

Using (3.29) and proceeding along similar lines as shown in the proof of Theorem 3.11, we prove Theorem 3.12. ■

4

Certain Diophantine equations and new parity results for 21-regular partitions

4.1 Introduction

In [69, Theorem 13], Keith and Zanello proved infinite families of Ramanujan type congruences satisfied by $b_{21}(n)$ which involves every prime p satisfying $p \equiv 13, 17, 19, 23 \pmod{24}$. To be specific, if $p \equiv 13, 17, 19, 23 \pmod{24}$ is prime, then

[1]The contents of this chapter have been published in *Acta Arith.* (2023).

they proved that

$$b_{21}(4(p^2n + kp - 11 \cdot 24^{-1}) + 1) \equiv 0 \pmod{2} \tag{4.1}$$

for all $1 \leq k < p$, where 24^{-1} is taken modulo p^2.

In this chapter, we investigate the parity of $b_{21}(n)$. In our first main result, we prove infinite families of Ramanujan type congruences for the remaining classes of primes modulo 24, namely, $p \equiv 1, 5, 7, 11 \pmod{24}$. A key ingredient in our proof is to find the integral solutions of certain Diophantine equations, which has also been employed very recently by Ballantine, Merca, and Radu in [12] to study the parity of $b_3(n)$. To state our main results, we first introduce some notation. Given the Diophantine equation $ax^2 + by^2 = n$, by a primitive solution we mean a solution $(x, y) \in \mathbb{Z}^2$ satisfying $\gcd(x, y) = 1$. Let $\mathcal{Q}$ denote the set of odd primes p such that the Diophantine equation $8x^2 + 27y^2 = jp$ has primitive solutions for some $j \in \{1, 4, 8\}$. In Section 4.2, we prove the following theorem.

Theorem 4.1. *For every $p \in \mathcal{Q}$ and $n \geq 0$, we have*

$$b_{21}(4(p^2n + \beta p - 11 \cdot 24_p^{-1}) + 1) \equiv 0 \pmod{2}$$

for all β with $0 \leq \beta < p$ and $\beta \neq \dfrac{11(p^2 + 24 \cdot 24_p^{-1} - 1)}{24p}$, where 24_p^{-1} is the inverse of 24 taken modulo p such that $1 \leq -24_p^{-1} \leq p - 1$.

In Section 4.2, we show that, if $p \in \mathcal{Q}$, then $p \equiv 1, 5, 7, 11 \pmod{24}$, and $\mathcal{Q}$ contains infinitely many primes from each of these classes of primes modulo 24. To study the parity of $b_3(n)$, Ballantine, Merca, and Radu [12] considered the set $\mathcal{P}$ which contains the primes p such that the Diophantine equation $x^2 + 24 \cdot 9y^2 = jp$ has primitive solutions for some $j \in \{1, 4, 8\}$. We also show that $\mathcal{Q} = \mathcal{P}$.

We have $\mathcal{Q} = \{29, 59, 79, 103, 223, 227, 241, \ldots\}$. Putting $p = 29$ in Theorem 4.1,

we have

$$b_{21}(4(29^2n + 29\beta + 66) + 1) \equiv 0 \pmod 2, \tag{4.2}$$

for all $0 \leq \beta < 29$, $\beta \neq 11$. In [100, Theorem 1.4], Singh and Barman proved the congruence (4.2) using Radu's algorithm (Section 2.4). We note that proving Theorem 4.1 for larger values of $p \in \mathcal{Q}$ using the same approach is not computationally feasible.

In a recent paper [111], Yao provided new infinite families of congruences modulo 2 for $b_3(n)$, and those congruences involve every prime $p \geq 5$. In our next theorem, we prove similar congruences for $b_{21}(n)$. These congruences involve every prime $p \geq 5$ except $p = 11$. In addition, we show that if $p \geq 5$ with $p \neq 11$ is a prime, then there exists an integer $\mu(p) \in \{4, 6\}$ such that for all $k \geq 0$, $b_{21}(\frac{11p^{\mu(p)k}-5}{6})$ is odd. More precisely, we have the following theorem.

Theorem 4.2. *Let $p \geq 5$ with $p \neq 11$ be a prime.*

(i) *If $b_{21}\left(\frac{11p^2-5}{6}\right) \equiv 1 \pmod 2$, then for all $n, k \geq 0$,*

$$b_{21}\left(4p^{4k+4}n + 4p^{4k+3}\beta + \frac{11p^{4k+4}-5}{6}\right) \equiv 0 \pmod 2, \tag{4.3}$$

where $1 \leq \beta < p$ and for $k \geq 0$,

$$b_{21}\left(\frac{11p^{4k}-5}{6}\right) \equiv 1 \pmod 2. \tag{4.4}$$

(ii) *If $b_{21}\left(\frac{11p^2-5}{6}\right) \equiv 0 \pmod 2$, then for all $n, k \geq 0$ with $p \nmid (24n+11)$,*

$$b_{21}\left(4p^{6k+2}n + \frac{11p^{6k+2}-5}{6}\right) \equiv 0 \pmod 2 \tag{4.5}$$

and for $k \geq 0$,

$$b_{21}\left(\frac{11p^{6k}-5}{6}\right) \equiv 1 \pmod 2. \tag{4.6}$$

We prove Theorem 4.2 in Section 4.3.

4.2 Parity of $b_{21}(n)$ and Diophantine equations

4.2.1 Some Lemmas

We first recall the following identity from [69]:

$$\sum_{n=0}^{\infty} b_{21}(4n+1)q^n \equiv \frac{f_3^4}{f_1} \pmod 2. \tag{4.7}$$

Combining (3.9), (3.10), and (4.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_{21}(4n+1)q^n &\equiv \left(\frac{f_3^3}{f_1}\right) f_3 \\ &\equiv \sum_{n\in\mathbb{Z}} q^{n(3n-2)} \sum_{m\in\mathbb{Z}} q^{\frac{3m}{2}(3m-1)} \pmod 2.\end{aligned}$$

We define

$$\sum_{k=0}^{\infty} a(k)q^k := \sum_{n\in\mathbb{Z}} q^{n(3n-2)} \sum_{m\in\mathbb{Z}} q^{\frac{3m}{2}(3m-1)}.$$

Then, for all $k \geq 0$, we have

$$b_{21}(4k+1) \equiv a(k) \pmod 2, \tag{4.8}$$

where

$$a(k) = \left|\left\{(n,m) \in \mathbb{Z}^2 : \; n(3n-2) + \frac{3m}{2}(3m-1) = k\right\}\right|$$

which we can further rewrite as

$$a(k) = \left|\left\{(x,y) \in \mathbb{N}^2 : \; 8x^2 + 3y^2 = 24k + 11, \; 3 \nmid y\right\}\right|.$$

To prove Theorem 4.1, it is enough to prove that $a(p^2n+\beta p - 11\cdot 24_p^{-1}) \equiv 0 \pmod 2$, for those p and β given in the statement of the theorem. To prove this claim, we require the following lemmas.

Lemma 4.3. *Let $p \geq 29$ be a prime and $0 \leq \beta < p$, $\beta \neq \dfrac{11(p^2 + 24\cdot 24_p^{-1} - 1)}{24p}$, and $t = p^2n + \beta p - 11\cdot 24_p^{-1}$. Then $24t + 11 = pm$, for some positive integer m such that $p \nmid m$.*

Proof. Since 24_p^{-1} is the inverse of 24 modulo p, there exists an integer k such that $24\cdot 24_p^{-1} = kp + 1$. Then $24t + 11 = 24p^2n + 24\beta p - 11kp$. Clearly, $p \mid 24t + 11$, and therefore, there exists a positive integer m (since t is positive) such that $24t + 11 = pm$.

Suppose that $p \mid m$. Then $p^2 \mid 24(p^2n + \beta p - 11\cdot 24_p^{-1}) + 11$. This in turn implies that $p^2 \mid 24(\beta p - 11\cdot 24_p^{-1}) + 11$. Then, there exists $u \in \mathbb{Z}$ such that

$$24(\beta p - 11\cdot 24_p^{-1}) + 11 = up^2. \tag{4.9}$$

From the inequalities $0 \leq \beta < p$ and $1 \leq -24_p^{-1} \leq p-1$, we get

$$\frac{25\cdot 11}{p^2} \leq \frac{24(\beta p - 11\cdot 24_p^{-1}) + 11}{p^2} < 24 + \frac{11}{p^2}(24p - 23).$$

Since $24p - 23 < p^2$ for every $p \geq 29$, we have $0 \leq u < 24 + 11$. Also, (4.9) and the fact $p^2 \equiv 1 \pmod{24}$ imply that $u \equiv 11 \pmod{24}$. Therefore, $u = 11$. Thus,

$24(\beta p - 11 \cdot 24_p^{-1}) + 11 = 11p^2$ and then $\beta = \dfrac{11(p^2 + 24 \cdot 24_p^{-1} - 1)}{24p}$, which is a contradiction. ■

Now, for $j \in \{1, 4, 8\}$, we observe the behavior modulo 24 of a positive integer m, coprime to 6, when the equation

$$8x^2 + 27y^2 = jm \tag{4.10}$$

has primitive solutions.

Lemma 4.4. *If, for $j = 1$ and a positive integer m coprime to 6, (4.10) has primitive solutions, then $m \equiv 11 \pmod{24}$.*

Proof. Let m be a positive integer with $\gcd(m, 6) = 1$ and let (x_0, y_0) be a primitive solution of (4.10) with $j = 1$. We have $8x_0^2 + 27y_0^2 = m$. Then, $2 \nmid y_0$ and $3 \nmid x_0$. Thus, $y_0^2 \equiv 1, 9 \pmod{24}$ and $x_0^2 \equiv 1, 4, 16 \pmod{24}$. This yields that $m \equiv 8x_0^2 + 3y_0^2 \equiv 11 \pmod{24}$. ■

Lemma 4.5. *If, for $j = 4$ and a positive integer m coprime to 6, (4.10) has primitive solutions, then $m \equiv 5 \pmod{24}$.*

Proof. Let m be a positive integer with $\gcd(m, 6) = 1$ and let (x_0, y_0) be a primitive solution of (4.10) with $j = 4$. We have $8x_0^2 + 27y_0^2 = 4m$. Clearly, $2 \mid y_0$ and therefore, we write $y_0 = 2w$ for some $w \in \mathbb{Z}$. This gives $2x_0^2 + 27w^2 = m$. Therefore, $2 \nmid w$ and $3 \nmid x_0$. Then, $w^2 \equiv 1, 9 \pmod{24}$ and $x_0^2 \equiv 1, 4, 16 \pmod{24}$. Since $\gcd(x_0, y_0) = 1$ and y_0 is even, $x_0^2 \equiv 1 \pmod{24}$. Therefore, $m \equiv 2 + 3w^2 \equiv 5 \pmod{24}$. ■

Lemma 4.6. *If, for $j = 8$ and a positive integer m coprime to 6, (4.10) has primitive solutions, then $m \equiv 1, 7 \pmod{24}$. Let, for a positive integer n, $val_p(n)$ denote the exponent of the highest power of the prime p that divides n. For $j = 8$, if (x_0, y_0) is a primitive solution of (4.10), then*

(a) *if* $m \equiv 1 \pmod{24}$, *then* $val_2(y_0) \geq 3$;

(b) *if* $m \equiv 7 \pmod{24}$, *then* $val_2(y_0) = 2$.

Proof. Let m be a positive integer with $\gcd(m,6) = 1$ and let (x_0, y_0) be a primitive solution of (4.10) with $j = 8$. We have $8x_0^2 + 27y_0^2 = 8m$. Clearly, $4 \mid y_0$ and therefore, we write $y_0 = 4w$ for some $w \in \mathbb{Z}$. This gives $x_0^2 + 54w^2 = m$. Then, $2, 3 \nmid x_0$ and therefore, $x_0^2 \equiv 1 \pmod{24}$. Thus, $m \equiv 1 + 6w^2 \pmod{24}$.

Now, if $2 \mid w$ then $m \equiv 1 \pmod{24}$. If $2 \nmid w$ then $w^2 \equiv 1, 9 \pmod{24}$ and thus, $m \equiv 7 \pmod{24}$. Also, if $m \equiv 1 \pmod{24}$ then $y_0 = 4w$ and $2 \mid w$, thus $val_2(y_0) \geq 3$. On the other hand, if $m \equiv 7 \pmod{24}$ then $y_0 = 4w$ and $2 \nmid w$, thus $val_2(y_0) = 2$. ∎

The above three lemmas imply that if $p \in \mathcal{Q}$, then

$$j = 1 \Longrightarrow p \equiv 11 \pmod{24};$$
$$j = 4 \Longrightarrow p \equiv 5 \pmod{24};$$
$$j = 8 \Longrightarrow p \equiv 1, 7 \pmod{24}.$$

Lemma 4.7. *Let $p \in \mathcal{Q}$. Let m be a positive integer such that $p \nmid m$ and $pm \equiv 11 \pmod{24}$. Let $j \in \{1, 4, 8\}$ be such that $8x^2 + 27y^2 = jp$ has primitive solutions. If*

$$U_{p,m} := \left\{(u,v) \in \mathbb{Z}^2 : 8u^2 + 27v^2 = pm, \ \gcd(u,v) = 1\right\};$$
$$A_m := \left\{(a,b) \in \mathbb{Z}^2 : a^2 + 216b^2 = jm, \ \gcd(a,b) = 1\right\},$$

then $|U_{p,m}| = 2|A_m|$.

Proof. Let $p \in \mathcal{Q}$ and m be a positive integer coprime to p such that $pm \equiv 11 \pmod{24}$. Let $j \in \{1, 4, 8\}$ be such that $8x^2 + 27y^2 = jp$ has primitive solutions and let $x_1, \ y_1 \in \mathbb{Z}$ with $\gcd(x_1, y_1) = 1$ satisfying $8x_1^2 + 27y_1^2 = jp$. Notice that if $(u,v) \in U_{p,m}$ then $8u^2 + 27v^2 = pm$, i.e., $8u^2 \equiv -27v^2 \pmod{p}$. Also, we have

$8x_1^2 \equiv -27y_1^2 \pmod{p}$. Thus, $(8x_1u)^2 \equiv (27y_1v)^2 \pmod{p}$, which yields

$$(8x_1u - 27y_1v)(8x_1u + 27y_1v) \equiv 0 \pmod{p}.$$

This implies that either $8x_1u - 27y_1v \equiv 0 \pmod{p}$ or $8x_1u + 27y_1v \equiv 0 \pmod{p}$. Next, we observe that

$$\left(8x_1^2 + 27y_1^2\right)\left(8u^2 + 27v^2\right) = jmp^2,$$

which can be rewritten as

$$(8x_1u - 27y_1v)^2 + 8 \cdot 27\,(x_1v + y_1u)^2 = jmp^2. \tag{4.11}$$

It is evident from (4.11) that if $8x_1u - 27y_1v \equiv 0 \pmod{p}$ then $x_1v + y_1u \equiv 0 \pmod{p}$. Similarly, it follows that if $8x_1u + 27y_1v \equiv 0 \pmod{p}$ then $x_1v - y_1u \equiv 0 \pmod{p}$. Hence, using this information we construct a map $f : U_{p,m} \to A_m$ by $f((u,v)) = (a,b)$, where for $(u,v) \in U_{p,m}$, (a,b) is defined as follows.

(a) If $8x_1u - 27y_1v \equiv 0 \pmod{p}$ then $x_1v + y_1u \equiv 0 \pmod{p}$ and define

$$a := \frac{8x_1u - 27y_1v}{p}, \qquad b := \frac{x_1v + y_1u}{p}. \tag{4.12}$$

(b) If $8x_1u + 27y_1v \equiv 0 \pmod{p}$ then $x_1v - y_1u \equiv 0 \pmod{p}$ and define

$$a := \frac{8x_1u + 27y_1v}{p}, \qquad b := \frac{x_1v - y_1u}{p}. \tag{4.13}$$

In both cases, we prove that $(a,b) \in A_m$ in the following three steps.

Step 1: $(a,b) \in \mathbb{Z}^2$. It is clear from the definitions of a and b.

Step 2: $a^2 + 216b^2 = jm$. It is easy to check that in both the cases

$$a^2 + 216b^2 = \frac{(8x_1^2 + 27y_1^2)(8u^2 + 27v^2)}{p^2} = jm.$$

Step 3: $\gcd(a, b) = 1$. First, we write u and v in terms of a and b. From (4.12), we get

$$u = \frac{x_1 a + 27y_1 b}{j}, \quad v = \frac{8x_1 b - y_1 a}{j}, \tag{4.14}$$

and from (4.13), we get

$$u = \frac{x_1 a - 27y_1 b}{j}, \quad v = \frac{8x_1 b + y_1 a}{j}. \tag{4.15}$$

Now, if $j = 1$ then from (4.14) and (4.15), we have that $\gcd(a, b) \mid \gcd(u, v)$. Since $\gcd(u, v) = 1$, it follows that $\gcd(a, b) = 1$. If $j = 4, 8$, then y_1 is even and therefore, x_1 is odd, since $\gcd(x_1, y_1) = 1$. Also, v is odd as pm is odd. Thus, in both cases, a is even and b is odd. Then, $\gcd(a, b)$ is an odd number and from (4.14) and (4.15), it follows that $\gcd(a, b) \mid \gcd(u, v)$. Since $\gcd(u, v) = 1$, we have $\gcd(a, b) = 1$.

Next, we show that f is surjective and *two-to-one*. Let $(a, b) \in A_m$. Define u, v as defined in (4.14) and $\tilde{u}$, $\tilde{v}$ as defined in (4.15). We prove that $(u, v), (\tilde{u}, \tilde{v}) \in U_{p,m}$ in the following three steps.

Step 1: $(u, v), (\tilde{u}, \tilde{v}) \in \mathbb{Z}^2$. It is clear in the case when $j = 1$. If $j = 4$, we have $a^2 + 216b^2 = 4m$ and $8x_1^2 + 27y_1^2 = 4p$. Clearly, $2 \mid a, y_1$ and therefore, b, x_1 are odd. Also, $4 \nmid a, y_1$. Thus, $u, \tilde{u} \in \mathbb{Z}$. Since $4 \mid y_1 a$, we have $v, \tilde{v} \in \mathbb{Z}$.

If $j = 8$, then $8x_1^2 + 27y_1^2 = 8p$ and from Lemma 4.6 it follows that $p \equiv 1, 7 \pmod{24}$.

Case 1: $p \equiv 1 \pmod{24}$. By Lemma 4.6, we have $val_2(y_1) \geq 3$. As $pm \equiv 11 \pmod{24}$, we have $m \equiv 11 \pmod{24}$. Since $a^2 + 216b^2 = 8m$, from Lemma 2.6 of

[12], it follows that $val_2(a) \geq 3$. Therefore, $u, \tilde{u} \in \mathbb{Z}$. Since $8 \mid y_1 a$, we have $v, \tilde{v} \in \mathbb{Z}$.
Case 2: $p \equiv 7 \pmod{24}$. By Lemma 4.6, we have $val_2(y_1) = 2$. As $pm \equiv 11 \pmod{24}$, we have $m \equiv 5 \pmod{24}$. Since $a^2 + 216b^2 = 8m$, from Lemma 2.6 of [12], it follows that $val_2(a) = 2$ and therefore, b, x_1 are odd. Thus, $u, \tilde{u} \in \mathbb{Z}$. Since $8 \mid y_1 a$, we have $v, \tilde{v} \in \mathbb{Z}$.

Step 2: $8u^2 + 27v^2 = pm$ and $8\tilde{u}^2 + 27\tilde{v}^2 = pm$. It is easy to check that

$$8u^2 + 27v^2 = \frac{(8x_1^2 + 27y_1^2)(a^2 + 216b^2)}{j^2} = pm.$$

Similarly, $8\tilde{u}^2 + 27\tilde{v}^2 = pm$.

Step 3: $\gcd(u, v) = \gcd(\tilde{u}, \tilde{v}) = 1$. If we express a and b in terms of u and v, we get (4.12). We have $8u^2 + 27v^2 = pm$. Clearly, $p \nmid u$ and $p \nmid v$. Therefore, $p \nmid \gcd(u, v)$ and from (4.12) it follows that $\gcd(u, v) \mid \gcd(a, b)$. Since $\gcd(a, b) = 1$, we have $\gcd(u, v) = 1$. Similarly, $\gcd(\tilde{u}, \tilde{v}) = 1$.

From (4.12) and (4.13), we get

$$8x_1 u - 27y_1 v \equiv 0 \pmod{p}; \tag{4.16}$$

$$8x_1 u + 27y_1 v \equiv 0 \pmod{p}. \tag{4.17}$$

This shows that f is surjective and $f^{-1}\{(a, b)\} = \{(u, v),\ (\tilde{u}, \tilde{v})\}$.

Finally, we prove that $(u, v) \neq (\tilde{u}, \tilde{v})$. Suppose $(u, v) = (\tilde{u}, \tilde{v})$, then both (u, v) and $(\tilde{u}, \tilde{v})$ satisfy (4.16) and (4.17). This implies that $x_1 u \equiv 0 \pmod{p}$. If $p \mid x_1$, then from $8x_1^2 + 27y_1^2 = jp$, we get that $p \mid y_1$ which is a contradiction, as $\gcd(x_1, y_1) = 1$. Similarly, $p \nmid u$. Hence, $(u, v) \neq (\tilde{u}, \tilde{v})$. This shows that f is two-to-one. ∎

Corollary 4.2.1. *If p, m, j, and $U_{p,m}$ are as defined in Lemma 4.7, then for $m > 1$ $|U_{p,m}| \equiv 0 \pmod{8}$ and $|U_{p,1}| = 4$.*

Proof. If $m = 1$, then from Lemma 2.4 of [12], $a^2 + 216b^2 = jm$ has primitive solutions if $j = 1$. Therefore, $A_m = \{(\pm 1, 0)\}$, i.e., $|A_m| = 2$ and $|U_{p,m}| = 4$. If $m > 1$, then for every $(a, b) \in A_m$, a and b are nonzero. Thus, every $(a, b) \in A_m$ ensures four distinct elements $(\pm a, \pm b)$ in A_m. Hence, $|A_m| \equiv 0 \pmod 4$, and by Lemma 4.7, we get $|U_{p,m}| \equiv 0 \pmod 8$. ∎

4.2.2 Proof of Theorem 4.1

Having all the required lemmas proved, we are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Define

$$\begin{aligned}
M_1(u) &:= \left|\left\{(x, y) \in \mathbb{Z}^2 : 8x^2 + 3y^2 = u\right\}\right|;\\
M_2(u) &:= \left|\left\{(x, y) \in \mathbb{Z}^2 : 8x^2 + 27y^2 = u\right\}\right|;\\
N_1(u) &:= \left|\{(x, y) \in M_1(u) : \gcd(x, y) = 1\}\right|;\\
N_2(u) &:= \left|\{(x, y) \in M_2(u) : \gcd(x, y) = 1\}\right|.
\end{aligned}$$

Note that there are exactly four positive, reduced, primitive quadratic forms of discriminant -96, namely, x^2+24y^2; $3x^2+8y^2$; $5x^2+2xy+5y^2$; $4x^2+4xy+7y^2$. Let $N(-96, u)$ be the number of primitive representations of any integer u by positive, reduced, primitive quadratic forms of discriminant -96. Then, by [35, Lemma 3.25]

$$N(-96, u) = 2\prod_{p|u}\left(1 + \left(\frac{-6}{p}\right)_J\right),$$

where the product is over all the prime divisors p of u and $\left(\frac{-6}{p}\right)_J$ is the Jacobi symbol. A positive integer $u \equiv 1, 5, 7, 11 \pmod{24}$ can be represented by one and only one of the four quadratic forms above, as proved in [42, p. 84]. Therefore, if $u \equiv 11 \pmod{24}$, then

$$N_1(u) = N(-96, u). \tag{4.18}$$

Observe that if u is square-free, then $M_l(u) = N_l(u)$ for $l \in \{1, 2\}$. If $u = wv^2$, for some positive integers w and v with w square-free, then, for $l \in \{1, 2\}$

$$M_l(u) = \sum_{d|v} N_l(wd^2),$$

where the sum is over all the positive divisors d of u. From [42, p. 84], if $u \equiv 11 \pmod{24}$, then we have

$$M_1(u) = 2\sum_{d|u} \left(\frac{-6}{d}\right)_J. \tag{4.19}$$

By Lemma 4.3, for $t = p^2n + \beta p - 11 \cdot 24_p^{-1}$, we have $24t + 11 = pm$, for some positive integer m such that $p \nmid m$. If $(x_0, y_0) \in \mathbb{Z}^2$ is a solution for $8x^2 + 3y^2 = 24t + 11$, since $24t + 11 = pm$ is not a perfect square, we have $x_0, y_0 \neq 0$. Then, $M_l(pm) \equiv 0 \pmod{4}$. Similarly, $N_l(pm) \equiv 0 \pmod{4}$.

We need to prove that $b_{21}(4t + 1) \equiv 0 \pmod{2}$. Since $pm \equiv 11 \pmod{24}$ and $24t + 11 = pm$ is not a perfect square, we have

$$\begin{aligned} b_{21}(4t+1) &\equiv a(t) \\ &\equiv \frac{1}{4}\left(M_1(24t+11) - M_2(24t+11)\right) \\ &\equiv \frac{1}{4}\left(M_1(pm) - M_2(pm)\right) \pmod{2}. \end{aligned}$$

Therefore, it is enough to show that

$$M_1(pm) - M_2(pm) \equiv 0 \pmod{8}.$$

If $m = 1$, we have $p \equiv 11 \pmod{24}$. Then, from (4.19), we get $M_1(p) = 4$. From Corollary 4.2.1, it follows that $M_2(p) = N_2(p) = |U_{p,1}| = 4$. Hence, $M_1(p) - M_2(p) \equiv 0 \pmod{8}$.

If $m > 1$, then from (4.18), we have $N_1(pm) \equiv 0 \pmod 8$.

Case 1: If m is not a perfect square, then $pm = pwv^2$ for some positive integers w, v with w square-free. Therefore, for $l \in \{1, 2\}$,

$$M_l(pm) = \sum_{d|v} N_l(pwd^2).$$

Since $pm \equiv 11 \pmod{24}$, therefore, $\gcd(m, 6) = \gcd(v, 6) = 1$, and also $\gcd(d, 6) = 1$ for all $d \mid v$. Thus, $v^2 \equiv 1 \pmod{24}$ and $d^2 \equiv 1 \pmod{24}$ for all $d \mid v$. Then for all $d \mid v$, we have $pwd^2 \equiv pw \equiv pwv^2 \equiv pm \equiv 11 \pmod{24}$. By Corollary 4.2.1, $N_2(pwd^2) \equiv 0 \pmod 8$ for all $d \mid v$, and thus, $M_2(pm) \equiv 0 \pmod 8$. Also, from (4.18), $N_1(pwd^2) \equiv 0 \pmod 8$ for all $d \mid v$ and therefore, $M_1(pm) \equiv 0 \pmod 8$. Hence, $M_1(pm) - M_2(pm) \equiv 0 \pmod 8$.

Case 2: If m is a perfect square, since $\gcd(m, 6) = 1$, it follows that $m \equiv 1 \pmod{24}$, and therefore, $p \equiv 11 \pmod{24}$. Then, from (4.19), $N_1(p) = M_1(p) = 4$ and from Corollary (4.2.1), $N_2(p) = |U_{p,1}| = 4$. Let $m = v^2$, for some positive integer v. For $l \in \{1, 2\}$, we have

$$M_l(pm) = N_l(p) + \sum_{d|v,\ d>1} N_l(pd^2).$$

From Corollary 4.2.1, it follows that $N_2(pd^2) \equiv 0 \pmod 8$ for all $d \mid v$, $d > 1$. Also, by (4.18), $N_1(pd^2) \equiv 0 \pmod 8$ for all $d \mid v$, $d > 1$. Therefore, for $l \in \{1, 2\}$,

$$M_l(pm) \equiv N_l(p) \equiv 4 \pmod 8.$$

Hence, $M_1(pm) - M_2(pm) \equiv 0 \pmod 8$. This completes the proof. ■

4.2.3 Dirichlet density of $\mathcal{Q}$

Recall that $\mathcal{Q}$ denotes the set of odd primes p such that the Diophantine equation $8x^2 + 27y^2 = jp$ has primitive solutions for some $j \in \{1, 4, 8\}$. Here, we prove that Theorem 4.1 holds for infinitely many primes. To prove this claim, we show that the Dirichlet density of the set $\mathcal{Q}$ is positive. Moreover, we show that Theorem 4.1 holds for infinitely many primes congruent to k modulo 24 for each $k \in \{1, 5, 7, 11\}$. First, we introduce some definition and notation.

Definition 4.1. *Let P denote the set of all prime numbers. Then the Dirichlet density of $S \subset P$, denoted by $\delta(S)$, is defined as*

$$\delta(S) := \lim_{s \to 1^+} \frac{\sum_{p \in S} p^{-s}}{\sum_{p \in P} p^{-s}}.$$

Next, we recall a proposition from [12].

Proposition 4.8. [12, Proposition 2.9] *Let $\mathcal{P}$ denote the set of primes p such that the Diophantine equation $x^2 + 24 \cdot 9y^2 = jp$ has primitive solutions for some $j \in \{1, 4, 8\}$. For $k \in \{1, 5, 7, 11\}$, let $\mathcal{P}_k := \{p \in \mathcal{P} : p \equiv k \pmod{24}\}$. Then, $\delta(\mathcal{P}) = \frac{1}{6}$, and $\delta(\mathcal{P}_k) = \frac{1}{24}$ for each $k \in \{1, 5, 7, 11\}$.*

For $k \in \{1, 5, 7, 11\}$, let $\mathcal{Q}_k := \{p \in \mathcal{Q} : p \equiv k \pmod{24}\}$. In the following proposition, we find Dirichlet densities of the sets $\mathcal{Q}$ and $\mathcal{Q}_k$.

Proposition 4.9. *We have $\mathcal{Q} = \mathcal{P}$. Also, $\delta(\mathcal{Q}) = \frac{1}{6}$ and $\delta(\mathcal{Q}_k) = \frac{1}{24}$ for each $k \in \{1, 5, 7, 11\}$.*

Proof. We consider four cases depending on p modulo 24 and in each case, we prove that the sets $\mathcal{Q}$ and $\mathcal{P}$ contain the same set of primes.

Case I $p \equiv 1 \pmod{24}$. Let $p \in \mathcal{Q}$. Then by Lemma 4.6, $8x^2 + 27y^2 = 8p$ has primitive solutions. Let (x_0, y_0) be such a solution. Clearly, $8 \mid y_0$. Since $\gcd(x_0, y_0) = 1$, we have $\gcd(x_0, \frac{y_0}{8}) = 1$ and therefore, $(x_0, \frac{y_0}{8})$ is a primitive solution for $x^2 + 216y^2 = p$. Thus, $p \in \mathcal{P}$.

Conversely, let $p \in \mathcal{P}$. Then by [12, Lemma 2.4], $x^2 + 216y^2 = p$ has primitive solutions. Let (x_1, y_1) be such a solution. Clearly, $2 \nmid x_1$ and therefore, $\gcd(x_1, 8y_1) = 1$. Thus, $(x_1, 8y_1)$ is a primitive solution for $8x^2 + 27y^2 = 8p$. This implies that $p \in \mathcal{Q}$.
Case II $p \equiv 5 \pmod{24}$. Let $p \in \mathcal{Q}$. Then by Lemma 4.5, $8x^2 + 27y^2 = 4p$ has primitive solutions. Let (x_0, y_0) be such a solution. Clearly, $2 \mid y_0$ and $4 \nmid y_0$. This implies that $\gcd(4x_0, \frac{y_0}{2}) = 1$ and thus, $(4x_0, \frac{y_0}{2})$ is a primitive solution for $x^2 + 216y^2 = 8p$. Therefore, $p \in \mathcal{P}$.
Conversely, let $p \in \mathcal{P}$. Then, from [12, Lemma 2.6], $x^2 + 216y^2 = 8p$ has primitive solutions. Let (x_1, y_1) be such a solution. We have $4 \mid x_1$ but $8 \nmid x_1$. Since $\gcd(x_1, y_1) = 1$, we have $\gcd(\frac{x_1}{4}, 2y_1) = 1$ and therefore, $(\frac{x_1}{4}, 2y_1)$ is a primitive solution for $8x^2 + 27y^2 = 4p$. Thus, $p \in \mathcal{Q}$.
Case III $p \equiv 7 \pmod{24}$. Let $p \in \mathcal{Q}$. As seen in Case I, let (x_0, y_0) be a primitive solution of $8x^2 + 27y^2 = 8p$ such that $4 \mid y_0$ but $8 \nmid y_0$. Then, $(2x_0, \frac{y_0}{4})$ is a primitive solution for $x^2 + 216y^2 = 4p$ and therefore, $p \in \mathcal{P}$.
Conversely, let $p \in \mathcal{P}$. Then, from [12, Lemma 2.5], we have a primitive solution (x_1, y_1) of $x^2 + 216y^2 = 4p$ such that $2 \mid x_1$ and $4 \nmid x_1$, and therefore, $(\frac{x_1}{2}, 4y_1)$ is a primitive solution for $8x^2 + 27y^2 = 8p$. Thus, $p \in \mathcal{Q}$.
Case IV $p \equiv 11 \pmod{24}$. Let $p \in \mathcal{Q}$. Then by Lemma 4.4, $8x^2 + 27y^2 = p$ has primitive solutions. Let (x_0, y_0) be such a solution. Here, $2 \nmid y_0$ and therefore, $(8x_0, y_0)$ is a primitive solution for $x^2 + 216y^2 = 8p$. Hence, $p \in \mathcal{P}$.
Conversely, let $p \in \mathcal{P}$. Then, from [12, Lemma 2.6], we have a primitive solution (x_1, y_1) of $x^2 + 216y^2 = 8p$ such that $8 \mid x_1$. Therefore, $(\frac{x_1}{8}, y_1)$ is a primitive solution for $8x^2 + 27y^2 = p$. Thus, $p \in \mathcal{Q}$.

This proves that $\mathcal{P} = \mathcal{Q}$, and hence $\mathcal{P}_k = \mathcal{Q}_k$ for $k = 1, 5, 7, 11$. The Dirichlet densities of $\mathcal{Q}$ and $\mathcal{Q}_k$ follow from Proposition 4.8. ∎

Remark 4.2.1. *Proposition 4.9 can also be proved as done in [12, Proposition 2.9].*

4.3 Congruences for $b_{21}(n)$ modulo 2

Proof of Theorem 4.2. We define

$$\sum_{n=0}^{\infty} c(n)q^n := \frac{f_3^4}{f_1}. \tag{4.20}$$

Then by (4.7) and (4.20), for all $n \geq 0$, we get

$$b_{21}(4n+1) \equiv c(n) \pmod 2. \tag{4.21}$$

Now, Theorem 3 of [83] yields

$$c\left(p^2 n + 11\left(\frac{p^2-1}{24}\right)\right) - \gamma(n)c(n) + c\left(\frac{n - 11\left(\frac{p^2-1}{24}\right)}{p^2}\right) \equiv 0 \pmod 2. \tag{4.22}$$

In (4.22)

$$\gamma(n) \equiv pd(p) + \left(\frac{n - 11\left(\frac{p^2-1}{24}\right)}{p}\right)_L \pmod 2, \tag{4.23}$$

where $d(p)$ is a function of p and $\left(\frac{\cdot}{p}\right)_L$ denotes the Legendre symbol. Note that $c(0) = 1$ and $c(\ell) = 0$ for all $\ell < 0$. Put $n = 0$ in (4.22), to get

$$c\left(11\left(\frac{p^2-1}{24}\right)\right) \equiv \gamma(0) \pmod 2. \tag{4.24}$$

By (4.23), we get

$$\gamma(0) \equiv pd(p) + 1 \pmod 2. \tag{4.25}$$

Employing (4.25) in (4.24), we arrive at

$$pd(p) \equiv c\left(11\left(\frac{p^2-1}{24}\right)\right)+1 \pmod{2}. \tag{4.26}$$

Then (4.23) and (4.26) yield

$$\gamma(n) \equiv c\left(11\left(\frac{p^2-1}{24}\right)\right)+1+\left(\frac{n-11\left(\frac{p^2-1}{24}\right)}{p}\right)_L \pmod{2}. \tag{4.27}$$

Combining (4.22) and (4.27) gives

$$\begin{aligned} c\left(p^2n+11\left(\frac{p^2-1}{24}\right)\right) \equiv& \left(c\left(11\left(\frac{p^2-1}{24}\right)\right)+1+\left(\frac{n-11\left(\frac{p^2-1}{24}\right)}{p}\right)_L\right)c(n) \\ &+c\left(\frac{n-11\left(\frac{p^2-1}{24}\right)}{p^2}\right) \pmod{2}. \end{aligned} \tag{4.28}$$

Replacing n by $pn+11\left(\frac{p^2-1}{24}\right)$ in (4.28), we deduce that

$$\begin{aligned} c\left(p^3n+11\left(\frac{p^4-1}{24}\right)\right) \equiv& \left(c\left(11\left(\frac{p^2-1}{24}\right)\right)+1\right)c\left(pn+11\left(\frac{p^2-1}{24}\right)\right) \\ &+c\left(\frac{n}{p}\right) \pmod{2}. \end{aligned} \tag{4.29}$$

Case (i): $b_{21}\left(\frac{11p^2-5}{6}\right) \equiv 1 \pmod{2}$. In this case, by (4.21), $c\left(11\left(\frac{p^2-1}{24}\right)\right) \equiv 1 \pmod{2}$. Then from (4.29), we have

$$c\left(p^3n+11\left(\frac{p^4-1}{24}\right)\right) \equiv c\left(\frac{n}{p}\right) \pmod{2}. \tag{4.30}$$

Next, replace n by $pn+\beta$ in (4.30) with $1 \leq \beta < p$:

$$c\left(p^3(pn+\beta)+11\left(\frac{p^4-1}{24}\right)\right) \equiv 0 \pmod{2}. \tag{4.31}$$

Substituting n by pn in (4.30), we obtain

$$c\left(p^4n + 11\left(\frac{p^4-1}{24}\right)\right) \equiv c\,(n) \pmod{2}. \tag{4.32}$$

Using (4.32) repeatedly, we obtain that, for $n, k \geq 0$,

$$c\left(p^{4k}n + 11\left(\frac{p^{4k}-1}{24}\right)\right) \equiv c\,(n) \pmod{2}. \tag{4.33}$$

Finally, replacing n by $p^3(pn+\beta) + 11\left(\frac{p^4-1}{24}\right)$ in (4.33), we arrive at

$$c\left(p^{4k+4}n + p^{4k+3}\beta + 11\left(\frac{p^{4k+4}-1}{24}\right)\right) \equiv 0 \pmod{2}. \tag{4.34}$$

Combining (4.21) and (4.34), we deduce (4.3).

Putting $n = 0$ in (4.33), we get

$$c\left(11\left(\frac{p^{4k}-1}{24}\right)\right) \equiv 1 \pmod{2},$$

which, when combined with (4.21), yields (4.4).

<u>Case (ii):</u> $b_{21}\left(\frac{11p^2-5}{6}\right) \equiv 0 \pmod{2}$. In this case, by (4.21), $c\left(11\left(\frac{p^2-1}{24}\right)\right) \equiv 0 \pmod{2}$. Then, substituting n by np in (4.29), we get

$$c\left(p^4n + 11\left(\frac{p^4-1}{24}\right)\right) \equiv c\left(p^2n + 11\left(\frac{p^2-1}{24}\right)\right) + c(n) \pmod{2}. \tag{4.35}$$

Substituting n by $p^2n + 11\left(\frac{p^2-1}{24}\right)$ in (4.35), we arrive at the following re-occurrence relation:

$$c\left(p^6n + 11\left(\frac{p^6-1}{24}\right)\right) \equiv c(n) \pmod{2}. \tag{4.36}$$

Iterating re-occurrence relation (4.36), for $n, k \geq 0$, we get

$$c\left(p^{6k}n + 11\left(\frac{p^{6k}-1}{24}\right)\right) \equiv c(n) \pmod 2. \tag{4.37}$$

Notice that, in this case, if we take all those $n \geq 0$ for which $p \nmid (24n+11)$, then (4.28) takes the following form:

$$c\left(p^{2}n + 11\left(\frac{p^{2}-1}{24}\right)\right) \equiv 0 \pmod 2.$$

Replacing n by $p^2n + 11\left(\frac{p^2-1}{24}\right)$ in (4.37), we have

$$c\left(p^{6k+2}n + 11\left(\frac{p^{6k+2}-1}{24}\right)\right) \equiv 0 \pmod 2. \tag{4.38}$$

Finally, combining (4.21) and (4.38), we get (4.5). Putting $n = 0$ in (4.37), we get

$$c\left(11\left(\frac{p^{6k}-1}{24}\right)\right) \equiv 1 \pmod 2,$$

which, when combined with (4.21), yields (4.6). ■

4.4 An open problem

In this section, we propose the following conjecture for $b_{21}(n)$.

Conjecture 4.10. *If $p \in \mathcal{Q}$, then $b_{21}\left(\frac{11p^2-5}{6}\right) \equiv 1 \pmod 2$.*

A proof of Conjecture 4.10 will confirm that the congruences obtained in Theorem 4.1 are different from those obtained in Theorem 4.2. We add the following discussion on Conjecture 4.10.

Let $p \in \mathcal{Q}$. By taking $k = \frac{11p^2-11}{24}$ in (4.8), we get

$$b_{21}\left(\frac{11p^2-5}{6}\right) \equiv a\left(\frac{11p^2-11}{24}\right) \pmod 2,$$

where

$$a\left(\frac{11p^2-11}{24}\right)=\left|\left\{(x,y)\in\mathbb{N}^2:\ 8x^2+3y^2=11p^2,\ 3\nmid y\right\}\right|.$$

Since $11p^2\equiv 11 \pmod{24}$, (4.19) yields

$$\left|\left\{(x,y)\in\mathbb{Z}^2:\ 8x^2+3y^2=11p^2\right\}\right|=M_1(11p^2)=12.$$

Out of the 12 solutions of $8x^2+3y^2=11p^2$, four non-primitive solutions are $(\pm p,\pm p)$. We conjecture that for $p\in\mathcal{Q}$, if the remaining 8 solutions are $(\pm a,\pm b)$, $(\pm u,\pm v)$, then $3\nmid b,v$. That is, $a\left(\frac{11p^2-11}{24}\right)=3$ and thus, $b_{21}\left(\frac{11p^2-5}{6}\right)$ is odd. For example, if $p=29$ then $(\pm16,\pm49)$, $(\pm29,\pm29)$, and $(\pm34,\pm1)$ are the only integral solutions of $8x^2+3y^2=11\cdot29^2$; and hence, $a\left(\frac{11\cdot29^2-11}{24}\right)=3$. Therefore, to prove Conjecture 4.10, it is enough to prove that for any $p\in\mathcal{Q}$, there are no integral solutions to $8x^2+27y^2=11p^2$.

5

Hook length biases in ordinary partitions

5.1 Introduction

A *Young diagram* of a partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a left-justified array of boxes with the m-th row (from the top) having λ_m boxes. For example, the Young diagram of the partition $(5, 4, 2, 2, 1)$ is shown in Figure 5.1 (left). The *hook length* of a box in a Young diagram is the sum of the number of the boxes directly right to it, the number of boxes directly below it, and 1 (for the box itself). For example, see Figure 5.1 (right) for the hook lengths of each box in the Young diagram of the partition $(5, 4, 2, 2, 1)$.

[1]The contents of this chapter have been published in *J. Number Theory* (2024).

FIGURE 5.1: The Young diagram of the partition $(5, 4, 2, 2, 1)$ and its hook lengths

Hook lengths of partitions have an important connections with representation theory of the symmetric groups S_n and $GL_n(\mathbb{C})$. Hook lengths also appear in the Seiberg-Witten theory of random partitions, which gives the Nekrasov-Okounkov formula for arbitrary powers of Euler's infinite product in terms of hook numbers. For any complex number z, the Nekrasov-Okounkov [79] formula is

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right) = \prod_{k=1}^{\infty}(1-q^k)^{z-1},$$

where $\mathcal{H}(\lambda)$ is the multi-set of all hook lengths of the partition λ and $\mathcal{P}$ is the set of all partitions. For more in these directions, see e.g. [49, 64, 75, 102]. Other than the ordinary partition function, hook lengths have also been studied for several restricted partition functions, for example, partitions into odd parts, partitions into distinct parts, partitions into odd and distinct parts, self-conjugate partitions, and doubled distinct partitions, see e.g. [2, 10, 36, 54, 55, 89]. In this chapter, we study the total number of hooks of fixed length in ordinary partitions. In particular, we discover inequalities between the number of hooks of different lengths in ordinary partitions. We refer such inequalities between the partition functions related to hook lengths by hook length biases.

For a positive integer k, let $p_{(k)}(n)$ denote the number of hooks of length k in all the partitions of n. Han [53] derived a two-variable generating function for the

$n \to$	1	2	3	4	5	6	7	8	9	10
$p_{(1)}(n)$	1	2	4	7	12	19	30	45	67	97
$p_{(2)}(n)$	0	2	2	6	8	16	22	38	52	82
$p_{(3)}(n)$	0	0	3	3	6	12	18	27	45	63
$p_{(4)}(n)$	0	0	0	4	4	8	12	24	32	52
$p_{(5)}(n)$	0	0	0	0	5	5	10	15	25	40
$p_{(6)}(n)$	0	0	0	0	0	6	6	12	18	30
$p_{(7)}(n)$	0	0	0	0	0	0	7	7	14	21
$p_{(8)}(n)$	0	0	0	0	0	0	0	8	8	16
$p_{(9)}(n)$	0	0	0	0	0	0	0	0	9	9
$p_{(10)}(n)$	0	0	0	0	0	0	0	0	0	10

Table 5.1: Values of $p_{(k)}(n) : 1 \leq n, k \leq 10$

number of partitions of n with m hooks of length k, say $p_{(k)}(m,n)$:

$$\mathcal{F}_k(z;q) := \sum_{n,m\geq 0} p_{(k)}(m,n) z^m q^n = \frac{(-(z-1)q^k;q^k)_\infty^k}{(q;q)_\infty}. \tag{5.1}$$

From (5.1), we obtain the generating function for $p_{(k)}(n)$:

$$\sum_{n=0}^{\infty} p_{(k)}(n) q^n = \frac{\partial}{\partial z}\bigg|_{z=1} \mathcal{F}_k(z;q) = \frac{1}{(q;q)_\infty}\left(\frac{kq^k}{1-q^k}\right). \tag{5.2}$$

Table 5.1 exhibits the values of $p_{(k)}(n)$ for some values of k and n. Since the Young diagram of a partition λ of $n \leq k-1$ can have at most $k-1$ boxes, there is no hook of length k for λ. Hence, $p_{(k)}(n) = 0$ for all n satisfying $1 \leq n \leq k-1$. Also, observe that the values in every column of Table 5.1 are in non-increasing order, except for the last non-zero value in the column. That is, counting in all the partitions of n, the number of hooks of length k is at least the number of hooks of length $k+1$, with one exception of $n = k+1$. We confirm this in the following theorem.

Theorem 5.1. *For any positive integer k, $p_{(k)}(n) \geq p_{(k+1)}(n)$ for all $n \geq 0$ and $n \neq k+1$. For $k \geq 2$, we have $p_{(k)}(k+1) - p_{(k+1)}(k+1) = -1$.*

5.2 Biases between $p_{(k)}(n)$ and $p_{(k+1)}(n)$

In this section, we prove Theorem 5.1. We first discuss some prerequisites. We say that a series $S(q) := \sum_{n\geq 0} s_n q^n$ is *non-negative* if $s_n \geq 0$, for all n. We use the notation $S(q) \succcurlyeq 0$ to denote the non-negativity.

Let $a(n)$ denote the number of partitions of n with smallest part at least 2. Define $a(0) := 1$. Then the generating function for $a(n)$ is

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q^2;q)_\infty}.$$

In the following lemma, we prove weakly increasing nature of $a(n)$.

Lemma 5.2. *For all positive integers $n \geq 2$, $a(n+1) \geq a(n)$.*

Proof. For a positive integer $n \geq 2$, let $\mathcal{P}^1(n)$ denote the set of all partitions of n in which smallest part is at least 2. We define a map $\Psi_n : \mathcal{P}^1(n) \to \mathcal{P}^1(n+1)$ by

$$\Psi_n((\lambda_1, \lambda_2, \ldots, \lambda_r)) = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_r).$$

Clearly, Ψ_n is a one-to-one map. Thus, $|\mathcal{P}^1(n)| \leq |\mathcal{P}^1(n+1)|$ and therefore $a(n) \leq a(n+1)$. ∎

Next, we notice an immediate corollary to Lemma 5.2:

Corollary 5.2.1. *For any positive integer k, we have*

$$\frac{q^k - q^{k+1}}{(q^2;q)_\infty} = -q^{k+1} + H_k(q),$$

where $H_k(q) \succcurlyeq 0$.

Proof. We have

$$\frac{q^k - q^{k+1}}{(q^2;q)_\infty} = q^k \left(\sum_{n\geq 0} (a(n) - a(n-1))q^n \right)$$

$$= q^k - q^{k+1} + q^{k+2} + \sum_{n\geq 3}(a(n) - a(n-1))q^{n+k}.$$

The proof follows from Lemma 5.2. ∎

We will repeatedly use the fact that for positive integers $\alpha,\ \beta$ with $\beta > \alpha + 1$

$$\frac{q^\alpha - q^\beta}{(q^2;q)_\infty} \succcurlyeq 0. \tag{5.3}$$

Note that (5.3) is also an immediate consequence of Lemma 5.2.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. From (1.1)and (5.2), we have

$$\sum_{n=0}^{\infty} p_{(k)}(n)q^n = \sum_{r=0}^{\infty} p(r)q^r \left(\sum_{m=1}^{\infty} kq^{km}\right). \tag{5.4}$$

If $k > r \geq 0$, then comparing the coefficients of $k + r$ on both sides of (5.4) yields

$$p_{(k)}(k + r) = p(r) \cdot k. \tag{5.5}$$

If $k \geq 2$, then taking $r = 0, 1$ in (5.5), we find that $p_{(k)}(k+1) - p_{(k+1)}(k+1) = -1$.

Next, to prove the biases, we define

$$G_k(q) := \sum_{n=0}^{\infty} \left(p_{(k)}(n) - p_{(k+1)}(n)\right) q^n.$$

From (5.2) we have

$$\begin{aligned} G_k(q) &= \frac{1}{(q;q)_\infty}\left(\frac{kq^k}{1-q^k} - \frac{(k+1)q^{k+1}}{1-q^{k+1}}\right) \\ &= \frac{1}{(q;q)_\infty}\left(k\frac{q^k + q^{2k} + \cdots + q^{(k+1)k}}{1-q^{k^2+k}} - (k+1)\frac{q^{k+1} + q^{2(k+1)} + \cdots + q^{k(k+1)}}{1-q^{k^2+k}}\right) \\ &= \frac{1}{(1-q^{k^2+k})(q;q)_\infty}\times \end{aligned}$$

$$
\begin{aligned}
&\left[k\left(q^k+q^{2k}+\cdots+q^{(k-1)k}-q^{k+1}-q^{2(k+1)}-\cdots-q^{(k-1)(k+1)}\right)\right.\\
&\left.+\left(kq^{k^2}-q^{k+1}-q^{2(k+1)}-\cdots-q^{(k-1)(k+1)}-q^{k(k+1)}\right)\right]\\
=&\frac{1}{(1-q^{k^2+k})(q^2;q)_\infty}\times\\
&\left(k\left[\left(\frac{q^k-q^{k+1}}{1-q}\right)+\left(\frac{q^{2k}-q^{2(k+1)}}{1-q}\right)+\cdots+\left(\frac{q^{(k-1)k}-q^{(k-1)(k+1)}}{1-q}\right)\right]\right.\\
&+\left[\left(\frac{q^{k^2}-q^{k+1}}{1-q}\right)+\left(\frac{q^{k^2}-q^{2(k+1)}}{1-q}\right)+\cdots+\left(\frac{q^{k^2}-q^{(k-1)(k+1)}}{1-q}\right)\right]\\
&\left.+\left(\frac{q^{k^2}-q^{k(k+1)}}{1-q}\right)\right)\\
=&\frac{1}{(1-q^{k^2+k})(q^2;q)_\infty}\left(k\left[q^k+(q^{2k}+q^{2k+1})+(q^{3k}+q^{3k+1}+q^{3k+2})+\cdots+\right.\right.\\
&\left.(q^{(k-1)k}+q^{(k-1)k+1}+\cdots+q^{(k-1)k+k-2})\right]-\left[(q^{k+1}+q^{k+2}+\cdots+q^{k^2-1})\right.\\
&\left.\left.+(q^{2(k+1)}+q^{2(k+1)+1}+\cdots+q^{k^2-1})+\cdots+q^{k^2-1}\right]+\frac{q^{k^2}-q^{k^2+k}}{1-q}\right)\\
=&\,\mathcal{K}_k(q)+\frac{\mathcal{L}_k(q)}{(1-q^{k^2+k})(q^2;q)_\infty},
\end{aligned}
\tag{5.6}
$$

where

$$
\mathcal{K}_k(q):=\frac{q^k-q^{k+1}}{(1-q^{k^2+k})(q^2;q)_\infty}+\frac{q^{k^2}-q^{k^2+k}}{(1-q^{k^2+k})(q;q)_\infty}
$$

and

$$
\begin{aligned}
\mathcal{L}_k(q):=&\,(k-1)q^k+k\left[(q^{2k}+q^{2k+1})+(q^{3k}+q^{3k+1}+q^{3k+2})+\cdots+\right.\\
&\left.(q^{(k-1)k}+q^{(k-1)k+1}+\cdots+q^{(k-1)k+k-2})\right]\\
&-\left[(q^{k+2}+q^{k+3}+\cdots+q^{k^2-1})+\right.\\
&\left.(q^{2(k+1)}+q^{2(k+1)+1}+\cdots+q^{k^2-1})+\cdots+q^{k^2-1}\right].
\end{aligned}
\tag{5.7}
$$

Enumerating the terms in the second square bracket on the right-hand side of (5.7), we obtain

$$\begin{aligned}
\mathcal{L}_k(q) &= (k-1)q^k + k\left[(q^{2k}+q^{2k+1}) + (q^{3k}+q^{3k+1}+q^{3k+2}) + \cdots + \right. \\
&\quad \left.(q^{(k-1)k}+q^{(k-1)k+1}+\cdots+q^{(k-1)k+k-2})\right] \\
&\quad - \left[(q^{k+2}+q^{k+3}+\cdots+q^{2k+1}) + 2(q^{2k+2}+q^{2k+3}+\cdots+q^{3k+2}) + \cdots + \right. \\
&\quad (k-2)(q^{(k-2)(k+1)}+q^{(k-2)(k+1)+1}+\cdots+q^{(k-1)(k+1)-1}) \\
&\quad \left. +(k-1)q^{(k-1)(k+1)}\right] \\
&= \left[(k-1)q^k - (q^{k+2}+q^{k+3}+\cdots+q^{2k-1})\right] + \\
&\quad \left[(k-1)(q^{2k}+q^{2k+1}) - 2(q^{2k+2}+q^{2k+3}+\cdots+q^{3k-1})\right] + \\
&\quad \left[(k-2)(q^{3k}+q^{3k+1}+q^{3k+2}) - 3(q^{3k+3}+q^{3k+4}+\cdots+q^{4k-1})\right] + \cdots + \\
&\quad \left[2(q^{(k-1)k}+q^{(k-1)k+1}+\cdots+q^{(k-1)k+(k-2)}) - (k-1)(q^{(k-1)(k-2)})\right]. \qquad (5.8)
\end{aligned}$$

Using (5.3) repeatedly on the right hand side of (5.8), we obtain that

$$\frac{\mathcal{L}_k(q)}{(q^2;q)_\infty} \succcurlyeq 0. \qquad (5.9)$$

Also,

$$\begin{aligned}
\mathcal{K}_k(q) =& \frac{q^k-q^{k+1}}{(q^2;q)_\infty}\left(1+\frac{q^{k^2+k}}{1-q^{k^2+k}}\right) + \frac{q^{k^2}-q^{k^2+k}}{(1-q^{k^2+k})(q;q)_\infty} \\
=& \frac{q^k-q^{k+1}}{(q^2;q)_\infty} + \frac{1}{(1-q^{k^2+k})(q^2;q)_\infty} \times \\
& \left(q^{k^2+2k}-q^{k^2+2k+1}+(q^{k^2}+q^{k^2+1}+\cdots+q^{k^2+k-1})\right).
\end{aligned}$$

From Corollary 5.2.1 and using (5.3) on $\frac{q^{k^2}-q^{k^2+2k+1}}{(q^2;q)_\infty}$, we obtain that

$$\mathcal{K}_k(q) = -q^{k+1} + J_k(q), \qquad (5.10)$$

where $J_k(q) \succcurlyeq 0$. Invoking (5.9) and (5.10) in (5.6) yields

$$G_k(q) = -q^{k+1} + M_k(q),$$

where $M_k(q) \succcurlyeq 0$. This completes the proof. ■

6

Hook length biases in t-regular partitions

6.1 Introduction

Let $t \geq 2$ and $k \geq 1$ be integers. A t-distinct partition of a positive integer n is a partition of n such that any of its parts can occur at most $t-1$ times. Let $b_{t,k}(n)$ denote the number of hooks of length k in all the t-regular partitions of n and $d_{t,k}(n)$ denote the number of hooks of length k in all the t-distinct partitions of n. We recall another form of representation of a partition given by

$$\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}),$$

[1] Some results of this chapter have been published in *J. Number Theory* (2024) and rest have been accepted to be published in *Canad. Math. Bull.* (2025).

where m_j is the multiplicity of the part λ_j and $\lambda_1 > \lambda_2 > \cdots > \lambda_r$. For a partition λ, let $h_k(\lambda)$ denote the number of hooks of length k in the Young diagram of the partition λ.

In [10], Ballantine et al. studied hook lengths in partitions into odd parts, that is, in 2-regular partitions. They derived the generating functions of $b_{2,2}(n)$ and $b_{2,3}(n)$, respectively:

$$\sum_{n=0}^{\infty} b_{2,2}(n)q^n = \frac{1}{(q;q^2)_\infty}\left(q^2 + \sum_{n=2}^{\infty}\left(q^{2n-1} + q^{2(2n-1)}\right)\right), \tag{6.1}$$

$$\sum_{n=0}^{\infty} b_{2,3}(n)q^n = (-q^3;q)_\infty \frac{q^3(1+q^3)}{1-q^2} + (-q;q)_\infty\left(\frac{q^6}{1-q^4} + \frac{q^3}{1-q^6}\right). \tag{6.2}$$

Very recently, Craig et al. [36] obtained the generating functions for $b_{2,k}(n)$, for any k. In Section 6.2.1, using the ideas of Ballantine et al. [10], we obtain the generating functions of $b_{t,k}(n)$ for certain new values of t and k.

In [10], Ballantine et al. proved some hook length biases in 2-regular partitions and 2-distinct partitions. The authors, in [10], proved that $d_{2,1}(n) \geq b_{2,1}(n)$, for all $n \geq 0$. They conjectured [10, Conjecture 1.7] that for every $k \geq 2$, there exists an integer N_k such that $b_{2,k}(n) \geq d_{2,k}(n)$, for all $n \geq N_k$. Ballantine et al. [10, Theorem 1.8] proved the conjecture for $k = 2, 3$ and very recently Craig et al. [36] proved it for all k.

Motivated by the hook length biases in ordinary partitions and the work in [10, 36], we also study the hook length biases among t-regular partitions. We study the biases for $b_{t,k}(n)$ considering two cases: Case I: t is fixed and Case II: k is fixed. We present Case I in Section 6.2 and Case II in Section 6.3. In Table 6.1, we observe certain patterns and biases. In view of that, we prove two hook length biases for 2-regular partitions in Section 6.2.2. Our proofs use combinatorial arguments along with q-series manipulations. In Section 6.2.3, we give the exact values of $b_{t,k}(k)$ and $b_{t,k}(k+1)$ for all t and k.

$n \rightarrow$	1	2	3	4	5	6	7	8	9	10
$b_{2,1}(n)$	1	1	2	3	4	6	8	11	14	19
$b_{2,2}(n)$	0	1	2	2	4	6	8	11	15	20
$b_{2,3}(n)$	0	0	2	1	2	5	5	7	11	15
$b_{2,4}(n)$	0	0	0	2	2	3	5	5	10	13
$b_{2,5}(n)$	0	0	0	0	3	1	3	5	6	10
$b_{2,6}(n)$	0	0	0	0	0	3	2	4	5	7
$b_{2,7}(n)$	0	0	0	0	0	0	4	1	4	5
$b_{2,8}(n)$	0	0	0	0	0	0	0	4	2	5
$b_{2,9}(n)$	0	0	0	0	0	0	0	0	5	1
$b_{2,10}(n)$	0	0	0	0	0	0	0	0	0	5

Table 6.1: Values of $b_{2,k}(n) : 1 \leq n, k \leq 10$

6.2 Biases for $b_{t,k}(n)$ for fixed t

6.2.1 Generating functions for $b_{t,k}(n)$

In this section, we derive the generating functions of $b_{t,1}(n)$ for all $t \geq 2$. We also derive the generating functions for $b_{3,2}(n)$ and $b_{4,2}(n)$. First, we introduce some notations that would be useful in this section. The number of parts of a partition λ is called the *length* of λ and is denoted by $\ell(\lambda)$. For any part s of a partition λ, we denote by $m_\lambda(s)$ the multiplicity of s in λ, i.e., the number of times s appears as a part in λ. For a positive integer c, let $\ell_{>c}(\lambda)$ denote the number of part sizes greater than c in the partition λ. By $\ell_1(\lambda)$ (respectively $\ell_2(\lambda)$) we denote the number of parts λ_j of λ with $\lambda_j - \lambda_{j+1} = 1$ (respectively $\lambda_j - \lambda_{j+1} = 2$), where we take $\lambda_j = 0$ if $j > \ell(\lambda)$. Let δ_P denote the Kronecker delta symbol, which is equal to 1 if property P is true and 0 otherwise.

Let $b_{t,k}(m,n)$ denote the number of t-regular partitions of n with m hooks of length k. Then,

$$b_{t,k}(n) = \sum_{m=0}^{\infty} m b_{t,k}(m,n). \tag{6.3}$$

Firstly, we find the generating function of $b_{t,1}(n)$ for all $t \geq 2$.

Theorem 6.1. *For $t \geq 2$, we have*

$$\sum_{n=0}^{\infty} b_{t,1}(n)q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty}\left(\frac{q}{1-q} - \frac{q^t}{1-q^t}\right). \tag{6.4}$$

Proof. In the Young diagram of a partition λ, there is a box with hook length 1 if and only if that box is at the end of a row with no box directly below it. Therefore, the number of hooks of length 1 in a partition λ is the same as the number of distinct parts in λ. Thus,

$$\mathcal{B}_{t,1}(z;q) := \sum_{n,m\geq 0} b_{t,1}(m,n)z^m q^n = \prod_{\substack{n\geq 1\\ t\nmid n}}\left(1+\frac{zq^n}{1-q^n}\right).$$

From (6.3), we have

$$\sum_{n=0}^{\infty} b_{t,1}(n)q^n = \frac{\partial}{\partial z}\bigg|_{z=1}\mathcal{B}_{t,1}(z;q). \tag{6.5}$$

Using logarithmic differentiation, (6.5) yields (6.4). ∎

In the following theorem, we find the generating functions of $b_{3,2}(n)$ and $b_{4,2}(n)$.

Theorem 6.2. *We have*

$$\sum_{n=0}^{\infty} b_{3,2}(n)q^n = \frac{(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{q^2}{1-q} + \frac{q^2}{1-q^2} - \frac{2q^3}{1-q^3}\right) \tag{6.6}$$

and

$$\sum_{n=0}^{\infty} b_{4,2}(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}\left(q^2 + \sum_{\substack{n\geq 2\\ 4\nmid n}}(q^n+q^{2n}) - (q^4;q^8)_\infty \sum_{n=0}^{\infty}\frac{q^{8n+3}+q^{8n+5}}{1-q^{8n+4}}\right). \tag{6.7}$$

Proof. Let λ be a 3-regular partition. We observe that the number of hooks of length 2 in λ is equal to the number of different parts of λ that occur at least twice plus the number of parts λ_j of λ for which $\lambda_j - \lambda_{j+1} \geq 2$, where we take $\lambda_j = 0$ if $j > \ell(\lambda)$. For example, see Figure 6.1. The number of parts λ_j of λ for which

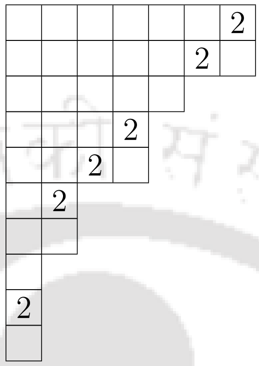

Figure 6.1: The hooks of length 2 in the Young diagram of the partition $(7, 7, 5, 4, 4, 2, 2, 1, 1, 1)$

$\lambda_j - \lambda_{j+1} \geq 2$ is nothing but

$$\ell_{>1}(\lambda) - \ell_1(\lambda) = \sum_{s\geq 2} \delta_{m_\lambda(s)\geq 1} - \ell_1(\lambda).$$

Thus, $b_{3,2}(m, n)$ is equal to the number of 3-regular partitions λ of n such that

$$\delta_{m_\lambda(1)\geq 2} + \sum_{s\geq 2} \left(\delta_{m_\lambda(s)\geq 1} + \delta_{m_\lambda(s)\geq 2}\right) - \ell_1(\lambda) = m.$$

Next, define $u(r, n)$ to be the number of 3-regular partitions λ of n such that

$$\delta_{m_\lambda(1)\geq 2} + \sum_{s\geq 2} \left(\delta_{m_\lambda(s)\geq 1} + \delta_{m_\lambda(s)\geq 2}\right) = r$$

and $v(r, n)$ to be the number of 3-regular partitions λ of n such that $\ell_1(\lambda) = r$.

Notice that

$$b_{3,2}(n) = \sum_{m=0}^{\infty} m b_{3,2}(m,n) = \sum_{r=0}^{\infty} r u(r,n) - \sum_{r=0}^{\infty} r v(r,n). \tag{6.8}$$

If we define $\mathcal{B}^1_{3,2}(z;q) := \sum_{n,r\geq 0} u(r,n) z^r q^n$ and $\mathcal{B}^2_{3,2}(z;q) := \sum_{n,r\geq 0} v(r,n) z^r q^n$, then using (6.8), we arrive at

$$\sum_{n=0}^{\infty} b_{3,2}(n) q^n = \frac{\partial}{\partial z}\bigg|_{z=1} \mathcal{B}^1_{3,2}(z;q) - \frac{\partial}{\partial z}\bigg|_{z=1} \mathcal{B}^2_{3,2}(z;q). \tag{6.9}$$

From the definition of $u(r,n)$, we have

$$\mathcal{B}^1_{3,2}(z;q) = \left(1 + q + \frac{zq^2}{1-q}\right) \prod_{\substack{n\geq 2\\ 3\nmid n}} \left(1 + zq^n + \frac{z^2q^{2n}}{1-q^n}\right).$$

Now, $v(r,n)$ counts the number of 3-regular partitions λ of n with r number of parts λ_j such that $\lambda_j - \lambda_{j+1} = 1$. If λ is a 3-regular partition, then the parts λ_j of λ for which $\lambda_j - \lambda_{j+1} = 1$ are precisely those which are congruent to 2 modulo 3 and part $\lambda_j - 1$, which is congruent to 1 modulo 3, is also a part of λ. Therefore,

$$\mathcal{B}^2_{3,2}(z;q) = \prod_{n\geq 0} \left(1 + \frac{q^{3n+1}}{1-q^{3n+1}} + \frac{q^{3n+2}}{1-q^{3n+2}} + \frac{zq^{3n+1}\cdot q^{3n+2}}{(1-q^{3n+1})(1-q^{3n+2})}\right).$$

By logarithmic differentiation, we obtain

$$\begin{aligned}
\frac{\partial}{\partial z}\bigg|_{z=1} \mathcal{B}^1_{3,2}(z;q) &= \prod_{\substack{n\geq 1\\ 3\nmid n}} \frac{1}{1-q^n} \left(q^2 + \sum_{\substack{n\geq 2\\ 3\nmid n}} \left(q^n + q^{2n}\right)\right) \\
&= \frac{(q^3;q^3)_\infty}{(q;q)_\infty} \left(\frac{q^2}{1-q} + \frac{q^2}{1-q^2} - \frac{q^3}{1-q^3} - \frac{q^6}{1-q^6}\right)
\end{aligned} \tag{6.10}$$

and

$$\frac{\partial}{\partial z}\bigg|_{z=1} \mathcal{B}^2_{3,2}(z;q) = \prod_{n\geq 0} \frac{1}{(1-q^{3n+1})(1-q^{3n+2})} \left(\sum_{n\geq 0} q^{3n+1}\cdot q^{3n+2}\right)$$
$$= \frac{(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{q^3}{1-q^6}\right). \tag{6.11}$$

Finally, from (6.9), (6.10), and (6.11), we obtain that

$$\sum_{n=0}^{\infty} b_{3,2}(n)q^n = \frac{(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{q^2}{1-q}+\frac{q^2}{1-q^2}-\frac{q^3}{1-q^3}-\frac{q^6}{1-q^6}-\frac{q^3}{1-q^6}\right). \tag{6.12}$$

We readily obtain (6.6) from (6.12).

Next, we prove (6.7), the generating function for $b_{4,2}(n)$, on the similar steps of the above proof of (6.6). In this case

$$b_{4,2}(n) = \sum_{m=0}^{\infty} m b_{4,2}(m,n) = \sum_{r=0}^{\infty} r\bar{u}(r,n) - \sum_{r=0}^{\infty} r\bar{v}(r,n), \tag{6.13}$$

where $\bar{u}(r,n)$ is the number of 4-regular partitions λ of n such that

$$\delta_{m_\lambda(1)\geq 2} + \sum_{s\geq 2}\left(\delta_{m_\lambda(s)\geq 1} + \delta_{m_\lambda(s)\geq 2}\right) = r$$

and $\bar{v}(r,n)$ to be the number of 4-regular partitions λ of n such that $\ell_1(\lambda) = r$. If we define $\mathcal{B}^1_{4,2}(z;q) := \sum_{n,r\geq 0} \bar{u}(r,n)z^r q^n$ and $\mathcal{B}^2_{4,2}(z;q) := \sum_{n,r\geq 0} \bar{v}(r,n)z^r q^n$, then using (6.13), we arrive at

$$\sum_{n=0}^{\infty} b_{4,2}(n)q^n = \frac{\partial}{\partial z}\bigg|_{z=1} \mathcal{B}^1_{4,2}(z;q) - \frac{\partial}{\partial z}\bigg|_{z=1} \mathcal{B}^2_{4,2}(z;q). \tag{6.14}$$

From the definition of $\bar{u}(r,n)$ we have

$$\mathcal{B}^1_{4,2}(z;q) = \left(1+q+\frac{zq^2}{1-q}\right)\prod_{\substack{n\geq 2\\ 4\nmid n}}\left(1+zq^n+\frac{z^2q^{2n}}{1-q^n}\right).$$

Now, $\bar{v}(r,n)$ counts the number of 4-regular partitions λ of n with r number of parts λ_j such that $\lambda_j-\lambda_{j-1}=1$. If λ is a 4-regular partition, then a part λ_j of λ for which $\lambda_j-\lambda_{j-1}=1$ can appear in these cases only:

1. $\lambda_j\equiv 3 \pmod 4$ and part $\lambda_j-1\equiv 2 \pmod 4$ is also a part of λ, but λ_j-2 is not a part of λ;
2. $\lambda_j\equiv 2 \pmod 4$ and part $\lambda_j-1\equiv 1 \pmod 4$ is also a part of λ, but λ_j+1 is not a part of λ;
3. $\lambda_j\equiv 3 \pmod 4$ and parts $\lambda_j-1\equiv 2 \pmod 4$ and λ_j-2 are also parts of λ.

Therefore,

$$\mathcal{B}^2_{4,2}(z;q) = \prod_{n\geq 0}\left(1+\sum_{j=1}^{3}\frac{q^{4n+j}}{1-q^{4n+j}}+\frac{zq^{4n+2}}{1-q^{4n+2}}\left(\frac{q^{4n+1}}{1-q^{4n+1}}+\frac{q^{4n+3}}{1-q^{4n+3}}\right)\right.$$
$$\left.+\frac{z^2q^{4n+1}\cdot q^{4n+2}\cdot q^{4n+3}}{(1-q^{4n+1})(1-q^{4n+2})(1-q^{4n+2})}\right).$$

By logarithmic differentiation, we obtain

$$\frac{\partial}{\partial z}\bigg|_{z=1}\mathcal{B}^1_{4,2}(z;q) = \prod_{\substack{n\geq 1\\ 4\nmid n}}\frac{1}{1-q^n}\left(q^2+\sum_{\substack{n\geq 2\\ 4\nmid n}}(q^n+q^{2n})\right) \tag{6.15}$$

and

$$\frac{\partial}{\partial z}\bigg|_{z=1}\mathcal{B}^2_{4,2}(z;q) = \prod_{\substack{n\geq 1\\ 4\nmid n}}\frac{1}{1-q^n}\left(\prod_{n\geq 0}(1-q^{4n+1}q^{4n+3})\right)\times$$

$$\left(\sum_{n\geq 0}\frac{q^{4n+2}(q^{4n+1}+q^{4n+3})}{1-q^{4n+1}q^{4n+3}}\right). \tag{6.16}$$

Finally, (6.14), (6.15), and (6.16) yield (6.7). ■

Remark 6.2.1. *The method used in Theorem 6.2 to obtain the generating functions for $b_{3,2}(n)$ and $b_{4,2}(n)$ can be extended to derive the generating function for $b_{t,2}(n)$, for further values of t. But as t increases, the generating functions become more complex.*

6.2.2 Hook length biases in 2-regular partitions

In Table 6.1, we observe that the values in each column, after the first entry, are in a non-increasing fashion except for the last non-zero value in the column. Unlike the case of ordinary partitions, in 2-regular partitions of n, the bias between hooks of length 1 and hooks of length 2 is the opposite. More precisely, we have the following theorem.

Theorem 6.3. *We have $b_{2,2}(n) \geq b_{2,1}(n)$, for all $n > 4$.*

Proof. For $n > 4$, let $\mathcal{B}_2(n)$ denote the set of all 2-regular or odd partitions of n. Also, let $\mathcal{R}_n \subset \mathcal{B}_2(n)$ be the set of partitions having at least one part of multiplicity more than 1; $\mathcal{S}_n \subset \mathcal{B}_2(n)$ be the set of distinct partitions with smallest part 1; and $\mathcal{T}_n \subset \mathcal{B}_2(n)$ be the set of distinct partitions with smallest part greater than 1. Then $\mathcal{B}_2(n) = \mathcal{R}_n \cup \mathcal{S}_n \cup \mathcal{T}_n$.

Case 1: Let $\lambda \in \mathcal{R}_n$. If a part λ_j of λ occurs at least twice then the rows corresponding to λ_j in the Young diagram of λ will have two hooks of length 2 and one hook of length 1. For every other part of λ which is greater than one, there is one hook of length 1 and at least one hook of length 2. For part of size 1 (if present as a part in λ), there is a hook of length 1 and there may or may not be a hook of length 2. Considering all the cases, we conclude that the number of hooks of length

1 in λ is not more than the number of hooks of length 2 in λ. Hence, $h_2(\lambda) \geq h_1(\lambda)$ for all $\lambda \in \mathcal{R}_n$.

Case 2: Let $\lambda \in \mathcal{S}_n$. Clearly, the number of hooks of length 1 in λ is exactly one more than the number of hooks of length 2 in λ. That is, $h_1(\lambda) = 1 + h_2(\lambda)$ for all $\lambda \in \mathcal{S}_n$.

Case 3: Let $\lambda \in \mathcal{T}_n$. Then, every part of λ corresponds to exactly one hook of length 1 and one hook of length 2 in the Young diagram of λ. Therefore, in this case, the number of hooks of length 1 in λ is the same as the number of hooks of length 2 in λ. Thus, $h_1(\lambda) = h_2(\lambda)$ for all $\lambda \in \mathcal{T}_n$.

In order to establish the required hook lengths bias, we next show that the extra hook of length 1 coming from Case 2 will be adjusted in Case 1. For this, we define a map $\Phi_n : \mathcal{S}_n \to \mathcal{R}_n$ by

$$\Phi_n\left((\lambda_1, \lambda_2, \ldots, \lambda_r, 1)\right) := (\lambda_2^2, \lambda_3, \ldots, \lambda_r, 1^{\lambda_1-\lambda_2+1})$$

if $r \geq 2$ and if $r = 1$ (which is possible only when n is even),

$$\Phi_n\left((\lambda_1 = n-1, 1)\right) := \begin{cases} \left((\frac{n-2}{2})^2, 1^2\right) & \text{if } n \equiv 0 \pmod 4; \\ \left((\frac{n}{2})^2\right) & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Clearly, Φ_n is a one-to-one map. For any $\lambda \in \Phi_n(\mathcal{S}_n)$, the number of hooks of length 1 in λ is exactly one less than the number of hooks of length 2 in λ. Thus, $h_1(\lambda) = h_2(\lambda) - 1$ for all $\lambda \in \Phi_n(\mathcal{S}_n)$.

Combining all the cases and the fact that Φ_n is one-to-one, we have

$$\begin{aligned} b_{2,2}(n) &= \sum_{\lambda \in \mathcal{B}_2(n)} h_2(\lambda) \\ &= \sum_{\lambda \in \mathcal{R}_n} h_2(\lambda) + \sum_{\lambda \in \mathcal{S}_n} h_2(\lambda) + \sum_{\lambda \in \mathcal{T}_n} h_2(\lambda) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda\in\mathcal{R}_n\setminus\Phi_n(\mathcal{S}_n)} h_2(\lambda) + \sum_{\lambda\in\Phi_n(\mathcal{S}_n)} h_2(\lambda) + \sum_{\lambda\in\mathcal{S}_n} h_2(\lambda) + \sum_{\lambda\in\mathcal{T}_n} h_2(\lambda)\\
&\geq \sum_{\lambda\in\mathcal{R}_n\setminus\Phi_n(\mathcal{S}_n)} h_1(\lambda) + \sum_{\lambda\in\Phi_n(\mathcal{S}_n)} (h_1(\lambda)+1) + \sum_{\lambda\in\mathcal{S}_n} (h_1(\lambda)-1) + \sum_{\lambda\in\mathcal{T}_n} h_1(\lambda)\\
&= \sum_{\lambda\in\mathcal{B}_2(n)} h_1(\lambda)\\
&= b_{2,1}(n).
\end{aligned}
$$

This completes the proof of the theorem. ■

Theorem 6.3 motivates to study the biases among $b_{2,k}(n)$ or the difference $b_{2,k}(n)-b_{2,k+1}(n)$ for $k \geq 2$. We find that the bias in Theorem 6.3 reverses for the next value of k. We prove that in 2-regular partitions of any non-negative integer n, the number of hooks of length 3 is no more than the number of hooks of length 2.

Theorem 6.4. *We have $b_{2,2}(n) \geq b_{2,3}(n)$ for all $n \geq 0$.*

Proof. Using well-known Euler's identity [3, (1.2.5)], $1/(q;q^2)_\infty = (-q;q)_\infty$ and straightforward q-series manipulations, we rewrite (6.1) as

$$
\begin{aligned}
\sum_{n=0}^{\infty} b_{2,2}(n)q^n &= (-q;q)_\infty \left(q^2 + \frac{q^3}{1-q^2} + \frac{q^6}{1-q^4}\right)\\
&= (-q;q)_\infty \left(\frac{q^3}{1-q^2} + \frac{q^2}{1-q^4}\right)\\
&= (-q^3;q)_\infty \left(\frac{q^2+2q^3+q^4+q^5+q^6}{1-q^2}\right). \qquad (6.17)
\end{aligned}
$$

Similarly, we rewrite (6.2) to obtain

$$
\sum_{n=0}^{\infty} b_{2,3}(n)q^n = (-q^3;q)_\infty \left(\frac{q^3+2q^6+q^7}{1-q^2}\right) + (-q;q)_\infty \left(\frac{q^3}{1-q^6}\right). \qquad (6.18)
$$

From (6.17) and (6.18), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}(b_{2,2}(n)-b_{2,3}(n))q^n &= (-q^3;q)_\infty\left(\frac{q^2+q^3+q^4+q^5-q^6-q^7}{1-q^2}\right)\\
&\quad-(-q;q)_\infty\left(\frac{q^3}{1-q^6}\right)\\
&=(-q^3;q)_\infty\left(\left(\frac{q^2-q^6}{1-q^2}\right)+\left(\frac{q^3-q^7}{1-q^2}\right)+\left(\frac{q^4+q^5}{1-q^2}\right)\right)\\
&\quad-(-q;q)_\infty\left(\frac{q^3}{1-q^6}\right)\\
&=(-q^3;q)_\infty\left(q^2+q^3+q^4+q^5+\frac{q^4+q^5}{1-q^2}\right)\\
&\quad-(-q;q)_\infty\left(\frac{q^3}{1-q^6}\right)\\
&=(-q^3;q)_\infty\left(\frac{q^2+q^5-q^6-q^9-q^{10}-q^{11}+q^{12}+q^{13}}{(1-q^2)(1-q^6)}\right)\\
&=(-q^3;q)_\infty\left(\frac{(1-q^2)(q^2+q^4+q^5+q^7-q^{10}-q^{11})}{(1-q^2)(1-q^6)}\right)\\
&=\frac{(-q^4;q)_\infty}{1-q^3}\left(q^2+q^4+q^5+q^7-q^{10}-q^{11}\right)\\
&=(-q^4;q)_\infty\left(\frac{q^2+q^7}{1-q^3}+\left(\frac{q^4-q^{10}}{1-q^3}\right)+\left(\frac{q^5-q^{11}}{1-q^3}\right)\right)\\
&=(-q^4;q)_\infty\left(\frac{q^2+q^7}{1-q^3}+q^4+q^5+q^7+q^8\right). \qquad (6.19)
\end{aligned}$$

Clearly, all the coefficients of the expression in (6.19), when expanded as a q-series, are non-negative. This completes the proof of Theorem 6.4. ■

6.2.3 Values of $b_{t,k}(k)$ and $b_{t,k}(k+1)$

Similar to the case of ordinary partitions, we notice certain patterns in the values in the diagonal and upper diagonal of Table 6.1. In view of that, we prove the following result, which provides the exact values of $b_{2,k}(n)$, for $n = k, k+1$ (diagonal and upper diagonal entries in Table 6.1).

Theorem 6.5. *For every integer $k \geq 1$, we have*

$$b_{2,k}(k) = \begin{cases} \frac{k+1}{2} & \text{if } k \equiv 1 \pmod 2; \\ \frac{k}{2} & \text{if } k \equiv 0 \pmod 2, \end{cases} \tag{6.20}$$

$$b_{2,k}(k+1) = \begin{cases} 1 & \text{if } k \equiv 1 \pmod 2; \\ 2 & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{6.21}$$

Proof. Let $k = 2r$ and λ be a 2-regular partition of k with a hook of length k. Then there are k boxes in the Young diagram of λ and the only box of hook of length k is at the top left corner of the Young diagram. The 2-regular partitions of k with hook of length k are precisely: $(2r-1,1)$, $(2r-3,1^3),\ldots,$ $(3,1^{2r-3})$, (1^{2r}), which are $r = k/2$ in number. If $k = 2r+1$, then the 2-regular partitions of k with hook of length k are precisely: $(2r+1)$, $(2r-1,1^2),\ldots,$ (1^{2r+1}), which are $r+1 = (k+1)/2$ in number. This proves (6.20).

Let $k = 2r$ and λ be a 2-regular partition of $k+1$ with a hook of length k. Then, the only possible partitions of $k+1$ with a hook of length k are $(2r+1)$ and (1^{2r+1}). If $k = 2r+1$, then there is only one possible 2-regular partition of $k+1$ with a hook of length k, which is (1^{2r+1}). This gives (6.21). ■

Remark 6.2.2. *From Theorem 6.5 we evaluate that for $k \geq 1$*

$$b_{2,k}(k+1) - b_{2,k+1}(k+1) = \begin{cases} -\frac{k-1}{2} & \text{if } k \equiv 1 \pmod 2; \\ -\frac{k-2}{2} & \text{if } k \equiv 0 \pmod 2, \end{cases} \tag{6.22}$$

Theorem 6.5 motivates us to extend (6.20) and (6.21) for any t-regular partitions. We find the exact values of $b_{t,k}(k)$ and $b_{t,k}(k+1)$ for any $t \geq 3$ in the following result.

Theorem 6.6. *Let $t \geq 3$ be an integer. Then*

1. *for any positive integer $k \equiv r \pmod t$*

$$b_{t,k}(k) = \frac{(t-1)k + r}{t}; \tag{6.23}$$

2. for any integer $k \geq 4$

$$b_{t,k}(k+1) = \begin{cases} \frac{(t-1)k+t}{t} & \text{if } k \equiv 0 \pmod{t}; \\ \frac{(t-1)k+r}{t} & \text{if } k \equiv r \pmod{t} \text{ for } 1 \leq r \leq t-2; \\ \frac{(t-1)k-1}{t} & \text{if } k \equiv t-1 \pmod{t}. \end{cases} \tag{6.24}$$

Also, $b_{t,1}(2) = 2$, for any $t \geq 3$; $b_{3,2}(3) = 1$; $b_{t,2}(3) = 2$, for any $t > 3$; $b_{3,3}(4) = 2$; $b_{4,3}(4) = 2$; and $b_{t,3}(4) = 3$, for any $t > 4$.

Proof. The Young diagram of a partition λ of k can have at most one hook of length k. The shape of the Young diagram of λ with the hook of length k is that of either an inverted L-shape or a vertical strip or a horizontal strip. Let $k = t\ell + r$, where $0 \leq r \leq t-1$. If $r = 0$ then the only t-regular partitions of k with the hook of length k are: $(t\ell - j, 1^j)$, where $1 \leq j \leq t\ell - 1$ and $t \nmid j$, which are $\frac{(t-1)k}{t}$ in number. If $0 < r \leq t-1$ then the corresponding partitions are: $(t\ell + r - j)$, where $0 \leq j \leq t\ell + r - 1$ with $t \nmid (r-j)$, which are $\frac{(t-1)k+r}{t}$ in number. This proves (6.23).

Next, the Young diagram of a partition λ of $k+1$ can have at most one hook of length k. In this case, the shape of the Young diagram is that of either a vertical strip or a horizontal strip or an inverted L-shape with the second part being of size 2. Let $k = t\ell + r$, where $0 \leq r \leq t-1$. If $r = 0$ then the only t-regular partitions of $k+1$ with the hook of length k are: $(t\ell - j, 2, 1^{j-1})$, where $1 \leq j \leq t\ell - 2$ and $t \nmid j$; $(t\ell + 1)$ (horizontal strip); $(1^{t\ell+1})$ (vertical strip), which are $((t-1)\ell - 1) + 2 = \frac{(t-1)k+t}{t}$ in number. If $0 < r < t-1$ then the corresponding partitions are: $(t\ell + r - j, 2, 1^{j-1})$, where $1 \leq j \leq t\ell + r - 1$ with $t \nmid (r-j)$; $(t\ell + r + 1)$ (horizontal strip); $(1^{t\ell+r+1})$ (vertical strip), which are $((t-1)\ell + r - 2) + 2 = \frac{(t-1)k+r}{t}$ in number. If $r = t-1$ then the corresponding partitions are: $(t\ell + t - 1 - j, 2, 1^{j-1})$, where $1 \leq j \leq t\ell + r - 1$ with $t \nmid (j+1)$; (no horizontal strip in this case); $(1^{t\ell+t})$ (vertical strip), which are $((t-1)\ell + r - 2) + 2 = \frac{(t-1)k+r}{t}$ in number. This proves (6.24). ∎

6.2.4 Open problems

We believe that the bias for $b_{2,k}(n)$ continues for the next values of k as obtained in Theorem 6.4. It would be interesting to study the biases between $b_{2,k}(n)$ and $b_{2,k+1}(n)$ for $k \geq 3$.

Using `SageMath` and the generating functions of $b_{3,1}(n)$ and $b_{3,2}(n)$ derived in (6.4) and (6.6), we find that:

$$\begin{aligned}\sum_{n=0}^{\infty}(b_{3,2}(n) - b_{3,1}(n))q^n &= -q - 2q^3 - 3q^5 - q^6 - 4q^7 - 2q^8 - 6q^9 - 3q^{10} - 9q^{11}\\ &- 4q^{12} - 12q^{13} - 6q^{14} - 15q^{15} - 8q^{16} - 19q^{17} - 9q^{18} - 22q^{19} - 9q^{20} - 24q^{21}\\ &- 7q^{22} - 23q^{23} - 17q^{25} + 14q^{26} - 2q^{27} + 40q^{28} + 27q^{29} + 84q^{30} + 77q^{31}\\ &+ 156q^{32} + 159q^{33} + 267q^{34} + 289q^{35} + 435q^{36} + 486q^{37} + 685q^{38} + 778q^{39}\\ &+ 1049q^{40} + 1202q^{41} + 1570q^{42} + 1809q^{43} + 2307q^{44} + 2665q^{45} + 3335q^{46}\\ &+ 3859q^{47} + 4756q^{48} + 5504q^{49} + 6701q^{50} + 7750q^{51} + 9341q^{52} + 10791q^{53}\\ &+ 12895q^{54} + 14877q^{55} + 17646q^{56} + 20326q^{57} + 23956q^{58} + 27548q^{59}\\ &+ 32286q^{60} + 37059q^{61} + 43219q^{62} + 49518q^{63} + 57494q^{64} + 65749q^{65}\\ &+ 76038q^{66} + 86796q^{67} + 100016q^{68} + 113959q^{69} + 130885q^{70} + \cdots.\end{aligned}$$

In view of the above, we conjecture the following bias.

Conjecture 6.7. *For all $n \geq 28$, $b_{3,2}(n) \geq b_{3,1}(n)$.*

This study gives rise to many interesting questions. We see in Theorem 5.1 that the number of hooks of length k is at least the number of hooks of length $k+1$ in all partitions of n, for all n except $n = k+1$. But in Theorem 6.3, we find that this bias reverses for $k = 1$ in the case of 2-regular partitions. In the direction of Conjecture 6.7 and following Theorem 6.3, it would be interesting to study the relationship between $b_{t,1}(n)$ and $b_{t,2}(n)$, for $t \geq 3$. Also, we see in Theorem 6.4 that the bias for 2-regular partitions remains the same as in the ordinary partitions for

$k = 2$. It would also be interesting to study the relationship between $b_{t,k}(n)$ and $b_{t,k+1}(n)$, for $t \geq 3$ and $k \geq 2$.

It is evident from (5.2) that $p_{(k)}(n) \equiv 0 \pmod{k}$, for every k. It would also be natural to examine the existence of such congruences for $b_{t,k}(n)$.

6.3 Biases for $b_{t,k}(n)$ for fixed k

In this section, we study biases among $b_{t,k}(n)$ for fixed k. Our first result proves that the number of hooks of length 1 in $(t+1)$-regular partitions of any non-negative integer n is greater than or equal to the number of hooks of length 1 in t-regular partitions of n. More precisely, we have the following theorem.

Theorem 6.8. *Let $t \geq 2$ be an integer. We have $b_{t+1,1}(n) \geq b_{t,1}(n)$, for all $n \geq 0$.*

For the number of hooks of length 2, we expect the same trend in t-regular partitions of any positive integer n. Our second result confirms the bias for the number of hooks of length 2 between 2- and 3-regular partitions.

Theorem 6.9. *For all integers $n > 3$, we have $b_{3,2}(n) \geq b_{2,2}(n)$.*

We observe similar biases for hooks of length 3. In particular, we have the following theorem.

Theorem 6.10. *For all non-negative integers n, we have $b_{3,3}(n) \geq b_{2,3}(n)$.*

We prove Theorem 6.8 in Section 6.3.1 and we prove Theorems 6.9 and 6.10 in Section 6.3.2.

6.3.1 Biases among $b_{t,1}(n)$ for $t \geq 2$

In this section, we prove Theorem 6.8. To prove Theorem 6.8, we first prove the following lemma.

$\tau \in \mathcal{B}_3(12)$	$\Phi_{3,12}(\tau) \in \mathcal{B}_4(12)$	$\tau \in \mathcal{B}_3(12)$	$\Phi_{3,12}(\tau) \in \mathcal{B}_4(12)$
$(8,4)$	$(6,3,2,1)$	$(8,2^2)$	$(6,2^3)$
$(8,2,1^2)$	$(6,2^2,1^2)$	$(8,1^4)$	$(6,2,1^4)$
$(7,4,1)$	$(7,3,1^2)$	$(5,4,2,1)$	$(5,3,2,1^2)$
$(5,4,1^3)$	$(5,3,1^4)$	(4^3)	$(3^3,1^3)$
$(4^2,2^2)$	$(3^2,2^2,1^2)$	$(4^2,2,1^2)$	$(3^2,2,1^4)$
$(4^2,1^4)$	$(3^2,1^6)$	$(4,2^4)$	$(3,2^4,1)$
$(4,2^3,1^2)$	$(3,2^3,1^3)$	$(4,2^2,1^4)$	$(3,2^2,1^5)$
$(4,2,1^6)$	$(3,2,1^7)$	$(4,1^8)$	$(3,1^9)$

Table 6.2: $\Phi_{t,n}$ for $t = 3$ and $n = 12$

Lemma 6.11. *Let $t \geq 2$ be an integer. We have $b_{t+1}(n) \geq b_t(n)$, for all $n \geq 0$.*

Proof. Let $\mathcal{B}_t(n)$ denote the set of all t-regular partitions of n. For fixed t and n, define a map $\Phi_{t,n} : \mathcal{B}_t(n) \to \mathcal{B}_{t+1}(n)$. For any $\tau \in \mathcal{B}_t(n)$, $\Phi_{t,n}(\tau)$ is a partition in $\mathcal{B}_{t+1}(n)$ with parts from τ which are multiple of $t+1$ changed in such a way that they are not multiple of $t+1$ and other parts remain same. A part of τ which is a multiple of $t+1$, is of the form $(t+1)(t\ell + r) = t(t+1)\ell + r(t+1)$, for some non-negative integer ℓ and $1 \leq r \leq t-1$ ($r \neq 0$, since $\tau \in \mathcal{B}_t(n)$). Under the map $\Phi_{t,n}$ the part of part size $t(t+1)\ell + r(t+1)$ of τ is changed to $(t(t+1)\ell + rt,\ r)$, which means that $t(t+1)\ell + rt$ and r are considered as two parts in $\Phi_{t,n}(\tau)$. For example, Table 6.2 shows the mapping of 3-regular partitions of 12 to 4-regular partitions of 12 under the map $\Phi_{3,12}$. The 3-regular partitions of 12 which are not listed in Table 6.2 are also 4-regular partitions of 12 and hence mapped to themselves. Next, we prove that $\Phi_{t,n}$ is an injective map. For $\tau_1, \tau_2 \in \mathcal{B}_t(n)$, let $\Phi_{t,n}(\tau_1) = \Phi_{t,n}(\tau_2)$. The parts of $\Phi_{t,n}(\tau_1)$ and $\Phi_{t,n}(\tau_2)$ which are not of the type $t(t+1)\ell + rt$ or r (for some non-negative integer ℓ and $1 \leq r \leq t-1$) are also the parts of τ_1 and τ_2. If $t(t+1)\ell + rt$ and r are the parts of $\Phi_{t,n}(\tau_1)$ and $\Phi_{t,n}(\tau_2)$ with multiplicity, say m, then $(t+1)(t\ell + r)$ is a part in both τ_1 and τ_2 with multiplicity m. This implies that $\tau_1 = \tau_2$. Therefore, $\Phi_{t,n}$ is an injective map. This proves that $|\mathcal{B}_t(n)| \leq |\mathcal{B}_{t+1}(n)|$, i.e., $b_t(n) \leq b_{t+1}(n)$. ∎

Having Lemma 6.11 proved, we are now ready to prove Theorem 6.8.

Proof of Theorem 6.8. Let $\overline{\ell}(\lambda)$ denote the number of distinct parts in a partition λ. It is easy to observe that for any partition τ the number of hooks of length 1 in the Young diagram of τ is the same as the number of distinct parts of τ, i.e., $h_1(\tau) = \overline{\ell}(\tau)$. From Lemma 6.11, we have that $b_t(n) \leq b_{t+1}(n)$, for all $n \geq 0$. Note that the number of distinct parts in $\tau \in \mathcal{B}_t(n)$ is less than or equal to the number of distinct parts in $\Phi_{t,n}(\tau) \in \mathcal{B}_{t+1}(n)$. Therefore, $b_{t,1}(n) \leq b_{t+1,1}(n)$, for all $n \geq 0$. ■

6.3.2 Biases between $b_{2,k}(n)$ and $b_{3,k}(n)$ for $k = 2, 3$

In this section, we prove Theorems 6.9 and 6.10.

We represent a partition τ from $\mathcal{B}_2(n)$ by

$$\left((6k+5)^{\alpha_{k,5}},\ (6k+3)^{\alpha_{k,3}},\ (6k+1)^{\alpha_{k,1}}\right)_{k\geq 0},$$

where $\alpha_{k,j}$ is the multiplicity of the part $6k+j$. From a partition $\tau \in \mathcal{B}_2(n)$, we define triples by

$$\tau_k = \left((6k+5)^{\alpha_{k,5}},\ (6k+3)^{\alpha_{k,3}},\ (6k+1)^{\alpha_{k,1}}\right)_k,$$

such that $\tau = (\tau_k)_{k\geq 0}$. The map $\Phi_{2,n} : \mathcal{B}_2(n) \to \mathcal{B}_3(n)$ is defined by

$$\begin{aligned}\Phi_{2,n}(\tau) &= \Phi_{2,n}\left(\left((6k+5)^{\alpha_{k,5}},\ (6k+3)^{\alpha_{k,3}},\ (6k+1)^{\alpha_{k,1}}\right)_{k\geq 0}\right)\\ &:= \left((6k+5)^{\alpha_{k,5}},\ (6k+2)^{\alpha_{k,3}},\ (6k+1)^{\alpha_{k,1}};\ 1^{\alpha_{k,3}}\right)_{k\geq 0}.\end{aligned}$$

$$\text{We take } (\Phi_{2,n}(\tau))_k = \begin{cases} \left((6k+5)^{\alpha_{k,5}},\ (6k+2)^{\alpha_{k,3}},\ (6k+1)^{\alpha_{k,1}}\right)_k & \text{if } k \geq 1;\\ \left(5^{\alpha_{0,5}},\ 2^{\alpha_{0,3}},\ 1^{\alpha_{0,1}+\sum_{j\geq 0}\alpha_{j,3}}\right) & \text{if } k = 0.\end{cases}$$

Proof of Theorem 6.9. In the Young diagram of a partition, a hook of length 2, which we call a 2-hook, may arise in two different ways.

(a) We call a 2-hook an m-2-hook if it appears due to the multiplicity of a part being greater than one.

(b) We call a 2-hook a g-2-hook if it appears in the column corresponding to a part λ_j with gap between λ_j and λ_{j+1} being more than 1.

For example, see Figures 6.2 and 6.3. Note that for $k \geq 1$, τ_k and $(\Phi_{2,n}(\tau))_k$ have

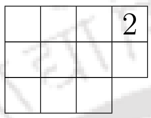

FIGURE 6.2: m-2-hook

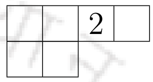

FIGURE 6.3: g-2-hook

the same number of m-2-hooks but the number of g-2-hooks for τ_k is either equal to or one more than the number of g-2-hooks for $(\Phi_{2,n}(\tau))_k$. Also, the number of 2-hooks in τ_0 and $(\Phi_{2,n}(\tau))_0$ differ by at most 1.

The idea of our proof is as follows. The number of 2-hooks in τ_k is either less than or equal to the number of 2-hooks in $(\Phi_{2,n}(\tau))_k$ or one more than the number of 2-hooks in $(\Phi_{2,n}(\tau))_k$. For the case in which a triple τ_k loses a 2-hook while going under the map $\Phi_{2,n}$, we assign a distinct triple to τ_k to compensate the loss of one 2-hook for it. For the other case, when the number of 2-hooks is the same for τ_k and $(\Phi_{2,n}(\tau))_k$, we are done. In this way, we prove that a partition $\tau \in \mathcal{B}_2(n)$ either has the number of 2-hooks less than the number of 2-hooks in $\Phi_{2,n}(\tau) \in \mathcal{B}_3(n)$, or (in the other case, when τ loses 2-hooks while going under $\Phi_{2,n}$) along with $\Phi_{2,n}(\tau)$ we associate a partition, say τ', to τ which compensates the loss.

We study triples τ_k in four cases. The cases in which $(\Phi_{2,n}(\tau))_k$ has one 2-hook fewer than τ_k, we associate a 4-tuple (a part of a partition in $\mathcal{B}_3(n)$ and different than $(\Phi_{2,n}(\tau))_k$) to τ_k, which has at least one 2-hook.

Case 1: $\alpha_{k,3} = 0$**.** In this case, the number of 2-hooks in τ_k is the same as the number of 2-hooks in $(\Phi_{2,n}(\tau))_k = \tau_k$, if $k \geq 1$. For $k = 0$, the number of 2-hooks in $(\Phi_{2,n}(\tau))_0$ is greater than or equal to the number of 2-hooks in τ_0.

Case 2: $\alpha_{k,1} = 0$**.** For $k \geq 1$, the number of 2-hooks in τ_k is the same as the number of 2-hooks in $(\Phi_{2,n}(\tau))_k$. For $k = 0$, if $\tau_0 \neq (5^{\alpha_{0,5}},\ 3)$ then the number of 2-hooks in τ_k is less than or equal to the number of 2-hooks in $(\Phi_{2,n}(\tau))_0$. If $\tau_0 = (5^{\alpha_{0,5}},\ 3)$ and $\alpha_{0,5} \neq 0$ then we cover the loss of a 2-hook by associating $\rho_0 := (5^{\alpha_{0,5}-1}, 4^2, 1^x)$ to τ_0, where x is the multiplicity of 1 coming in the scene due to other triples of $\tau = (\tau_k)_{k\geq 0}$. If $\tau_0 = (3)$ (i.e., $\alpha_{0,5} = 0$ in $\tau_0 = (5^{\alpha_{0,5}},\ 3)$) then we cover the loss of 2-hook as follows. Since $n > 3$, there is the smallest part with part size greater than or equal to 5, say λ_j. In this case, we take 5 from the part λ_j and associate (4^2) to $\tau_0 = (3)$. For the remaining part $\lambda_j - 5$, we proceed by considering it as a part of the partition under consideration and if $\lambda_j - 5 = 6r + 6$, for some $r \geq 0$, then we change it to $(6r + 5, 1)$ along with other parts while applying $\Phi_{2,n}$. In this case $\rho_0 := (5^w, 4^{2+z}, 2^y, 1^x)$, where x is the multiplicity of 1 coming due to the other triples; y, z, w are the multiplicities of parts $2, 4, 5$ (respectively), which may occur due to the part $\lambda_j - 5$. For example, if $\tau = (11, 3)$ then $\rho_0 = (5, 4^2, 1)$; if $\tau = (7, 3)$ then $\rho_0 = (4^2, 2)$; if $\tau = (9, 3)$ then $\rho_0 = (4^3)$; if $\tau = (5^2, 3)$ then $\rho_0 = (5, 4^2)$.

Case 3: $\alpha_{k,3} > 1$ **and** $\alpha_{k,1} \neq 0$**.** In this case, there is at most one loss of 2-hook in $(\Phi_{2,n}(\tau))_k$, which we cover by the following map

$$\begin{aligned} f(\tau_k) &= f\left(((6k+5)^{\alpha_{k,5}},\ (6k+3)^{\alpha_{k,3}},\ (6k+1)^{\alpha_{k,1}})_k\right) \\ &= \left((6k+5)^{\alpha_{k,5}},\ 6k+4,\ (6k+2)^{\alpha_{k,3}-1},\ (6k+1)^{\alpha_{k,1}};\ 1^{\alpha_{k,3}-2}\right)_k. \end{aligned}$$

In this case, we associate

$$\sigma_k := \left((6k+5)^{\alpha_{k,5}},\ 6k+4,\ (6k+2)^{\alpha_{k,3}-1},\ (6k+1)^{\alpha_{k,1}}\right)_k$$

to τ_k for $k \geq 1$. For τ_0, $\sigma_0 = (5^{\alpha_{0,5}},\ 4,\ 2^{\alpha_{0,3}-1},\ 1^{\alpha_{0,1}+s})$, where s is the number of 1s due to other triples.

Case 4: $\alpha_{k,3} = 1$ **and** $\alpha_{k,1} \neq 0$**.** In this case also, there is at most one loss of 2-hook in $(\Phi_{2,n}(\tau))_k$, which we cover by the following map

$$g(\tau_k) = g\left(((6k+5)^{\alpha_{k,5}},\ 6k+3,\ (6k+1)^{\alpha_{k,1}})_k\right)$$
$$= \begin{cases} ((6k+5)^{\alpha_{k,5}},\ 6k+4,\ (6k+1)^{\alpha_{k,1}-1};\ 6k-1,\ 1)_k & \text{if } k \geq 1; \\ (5^{\alpha_{0,5}},\ 4,\ 1^{\alpha_{0,1}-1}) & \text{if } k = 0. \end{cases}$$

Here, for $k \geq 1$, part $6k-1 = 6(k-1)+5$ is considered as a part of τ_{k-1}, doing which does not decrease the number of 2-hooks in τ_{k-1}. In this case, we associate

$$\delta_k := \left((6k+5)^{\alpha_{k,5}},\ 6k+4,\ (6k+1)^{\alpha_{k,1}-1}\right)_k$$

to τ_k for $k \geq 1$. For τ_0, $\sigma_0 = (5^{\alpha_{0,5}},\ 4,\ 1^{\alpha_{0,1}-1+s})$, where s is the number of 1s due to other triples.

Now, let $\tau = (\tau_k)_{k\geq 0} \in \mathcal{B}_2(n)$. We consider the following two cases.

Case A. If the number of 2-hooks in τ_k is less than or equal to the number of 2-hooks in $(\Phi_{2,n}(\tau))_k$ for all k (from Case 1 and Case 2), then we define $\tau^* := \Phi_{2,n}(\tau)$. Clearly, $h_2(\tau) \leq h_2(\tau^*)$.

Case B. If for any $k \geq 0$, the number of 2-hooks in $(\Phi_{2,n}(\tau))_k$ is one less than the number of 2-hooks in τ_k, we take τ' to be a partition in $\mathcal{B}_3(n)$ with $(\Phi_{2,n}(\tau))_k$ replaced by the required ρ_0, σ_k or δ_k, which has at least one 2-hook. In this case, we define $\tau^* := (\Phi_{2,n}(\tau), \tau')$ and $h_2(\tau^*) := h_2(\Phi_{2,n}(\tau)) + h_2(\tau')$ (Note that τ^* is a set of two partitions from $\mathcal{B}_3(n)$). In that way, in this case also we have, $h_2(\tau) \leq h_2(\tau^*)$.

Finally, since $\Phi_{2,n}$ is an injective map, all $\Phi_{2,n}(\tau)$ are distinct. Note that $(\Phi_{2,n}(\tau))_k$, ρ_0, σ_k and δ_k are all distinct as well. Therefore, τ' and $\Phi_{2,n}(\tau)$ are also distinct. For example, see Table 6.3.

Hence, we have

$$b_{2,2}(n) = \sum_{\tau\in\mathcal{B}_2(n)} h_2(\tau) = \sum_{\substack{\tau\in\mathcal{B}_2(n)\\ \text{Case A}}} h_2(\tau) + \sum_{\substack{\tau\in\mathcal{B}_2(n)\\ \text{Case B}}} h_2(\tau)$$

$\tau \in \mathcal{B}_2(13)$	$\tau^* = \Phi_{2,n}(\tau)$	$\tau^* = (\Phi_{2,n}(\tau), \tau')$	$h_2(\tau)$	$h_2(\tau^*)$
(13)	(13)		1	1
$(11, 1^2)$	$(11, 1^2)$		2	2
$(9, 3, 1)$		$((8, 2, 1^3), (8, 4, 1))$	2	2+2
$(9, 1^4)$	$(8, 1^5)$		2	2
$(7, 5, 1)$	$(7, 5, 1)$		2	2
$(7, 3^2)$	$(7, 2^2, 1^2)$		3	3
$(7, 3, 1^3)$		$((7, 2, 1^4), (7, 4, 1^2))$	3	2+3
$(7, 1^6)$	$(7, 1^6)$		2	2
$(5^2, 3)$		$((5^2, 2, 1), (5, 4^2))$	3	2+2
$(5^2, 1^3)$	$(5^2, 1^3)$		3	3
$(5, 3^2, 1^2)$		$((5, 2^2, 1^4), (5, 4, 2, 1^2))$	4	3+2
$(5, 3, 1^5)$		$((5, 2, 1^6), (5, 4, 1^4))$	3	2+2
$(5, 1^8)$	$(5, 1^8)$		2	2
$(3^4, 1)$		$((2^4, 1^5), (4, 2^3, 1^3))$	2	2+3
$(3^3, 1^4)$		$((2^3, 1^7), (4, 2^2, 1^5))$	3	2+3
$(3^2, 1^7)$		$((2^2, 1^9), (4, 2, 1^7))$	3	2+2
$(3, 1^{10})$		$((2, 1^{11}), (4, 1^9))$	2	1+2
(1^{13})	(1^{13})		1	1
Total number of 2-hooks			43	57

Table 6.3: Outline of the proof of Theorem 6.9 for $n = 13$

$$\leq \sum_{\substack{\tau \in \mathcal{B}_2(n) \\ \text{Case A}}} h_2(\Phi_{2,n}(\tau)) + \sum_{\substack{\tau \in \mathcal{B}_2(n) \\ \text{Case B}}} (h_2(\Phi_{2,n}(\tau) + h_2(\tau'))$$
$$= \sum_{\tau \in \mathcal{B}_2(n)} h_2(\tau^*) \leq \sum_{\tau \in \mathcal{B}_3(n)} h_2(\tau) = b_{3,2}(n).$$

This completes the proof of the theorem. ■

Proof of Theorem 6.10. In the Young diagram of a partition, a hook of length 3, which we call a 3-hook may arise in four different ways.

(a) We call a 3-hook an m_3-3-hook if it arises due to the multiplicity of a part being greater than two and it appears in the third last column from the columns corresponding to λ_j in the Young diagram.

(b) We call a 3-hook a g-3-hook if it appears in the column corresponding to a part λ_j with gap between λ_j and λ_{j+1} being more than 2.

(c) We call a 3-hook an m_2-3-hook if it arises due to the multiplicity of a part λ_j being at least two and it appears in the second last column from the columns corresponding to λ_j in the Young diagram.

(d) We call a 3-hook a s-3-hook if it appears in the column corresponding to a part λ_j with gap between λ_j and λ_{j+1} being exactly 1 and the part λ_{j+1} occurs once.

For example, see Figures 6.4-6.7. Similar to the case of 2-hooks, for $k \geq 1$, τ_k and

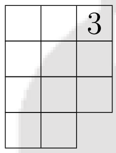

Figure 6.4: m_3-3-hook

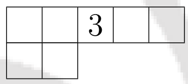

Figure 6.5: g-3-hook

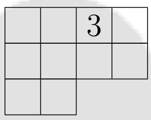

Figure 6.6: m_2-3-hook

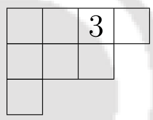

Figure 6.7: s-3-hook

$(\Phi_{2,n}(\tau))_k$ have the same number of m_3-3-hooks. Also, the number of g-3-hooks for τ_k is same as the number of g-3-hooks for $(\Phi_{2,n}(\tau))_k$, when $k \geq 1$. However, the number of m_2-3-hooks for $(\Phi_{2,n}(\tau))_k$ is either equal to or one less than the number of m_2-3-hooks for τ_k, for $k \geq 1$. Note that for a 2-regular partition, there is no s-3-hook in its Young diagram. Therefore, the number of 3-hooks in τ_k can be, at the most, one less than the number of 3-hooks in $(\Phi_{2,n}(\tau))_k$. For $k = 0$, the number of m_3-3-hooks for τ_0 is either equal to or one less than the number of m_3-3-hooks for $(\Phi_{2,n}(\tau))_0$. Whereas, the number of g-3-hooks for τ_0 is either equal to or one more than the number of g-3-hooks for $(\Phi_{2,n}(\tau))_0$ and the same is the case for m_2-3-hooks. Therefore, the number of 3-hooks in τ_0 can be, at the most, two less than the number of 3-hooks in $(\Phi_{2,n}(\tau))_0$.

The idea of the proof is similar to the proof of Theorem 6.9. A partition $\tau \in \mathcal{B}_2(n)$ either has the number of 3-hooks less than or equal to the number of 3-hooks in $\Phi_{2,n}(\tau) \in \mathcal{B}_3(n)$, or (in the other case, when τ loses 3-hooks while going under $\Phi_{2,n}$) we associate a different partition, say τ', to τ which compensates the loss.

We study the triples τ_k in two cases.

Case 1: $k \geq 1$. Note that the number of m_2-3-hooks for τ_k decreases under the map $\Phi_{2,n}$ only when $\alpha_{k,3} \geq 2$ and $\alpha_{k,1} \geq 1$. In that case, we associate a new tuple to τ_k to cover the loss of an m_2-3-hook by using the following map

$$\begin{aligned}F(\tau_k) &= F\left(((6k+5)^{\alpha_{k,5}},\ (6k+3)^{\alpha_{k,3}},\ (6k+1)^{\alpha_{k,1}})_k\right)\\ &= \left((6k+5)^{\alpha_{k,5}},\ (6k+4)^2,\ (6k+2)^{\alpha_{k,3}-2},\ (6k+1)^{\alpha_{k,1}-1};\ (6k-1),\ 1^{\alpha_{k,3}-2}\right)_k.\end{aligned}$$

In this case, we associate

$$\theta_k := \left((6k+5)^{\alpha_{k,5}},\ (6k+4)^2,\ (6k+2)^{\alpha_{k,3}-2},\ (6k+1)^{\alpha_{k,1}-1}\right)_k$$

to τ_k, which clearly has at least one 3-hook (m_2-3-hook corresponding to the parts $6k+4$) to compensate the loss. Here, part $6k-1 = 6(k-1)+5$ is considered as a part of τ_{k-1}, doing which does not decrease the number of 2-hooks in τ_{k-1}.

Case 2: $k = 0$. In this case, there might be a loss of at most two 3-hooks and that also when $\alpha_{0,3} > 0$. We have $\tau_0 = (5^{\alpha_{0,5}},\ 3^{\alpha_{0,3}},\ 1^{\alpha_{0,1}})$. Depending on the multiplicity of the part 3, $\alpha_{0,3} = 4\ell + j$, $0 \leq j \leq 3$, we consider the following two cases.

Subcase (a): $\ell > 0$. In this case, we compensate for the loss with the following map:

$$G(\tau_0) = \begin{cases} (5^{\alpha_{0,5}},\ 4^{3\ell},\ 1^{\alpha_{0,1}}) & \text{if } j=0; \\ (5^{\alpha_{0,5}},\ 4^{3\ell},\ 2,\ 1^{\alpha_{0,1}+1}) & \text{if } j=1; \\ (5^{\alpha_{0,5}},\ 4^{3\ell+1},\ 1^{\alpha_{0,1}+2}) & \text{if } j=2; \\ (5^{\alpha_{0,5}},\ 4^{3\ell+2},\ 1^{\alpha_{0,1}+1}) & \text{if } j=3. \end{cases}$$

Clearly, in each case, $G(\tau_0)$ has at least two 3-hooks. We associate θ_0 to τ_0, which is $G(\tau_0)$ including the multiplicity of part size 1 coming from the other triples τ_k.

Subcase (b): $\ell = 0$. Here, $j = 0$ can not be the case since $\alpha_{0,3} > 0$. For $j = 3$, the loss of a 3-hook can be covered by the same map G in the above subcase, i.e.,

$$G(\tau_0) = G((5^{\alpha_{0,5}},\ 3^3,\ 1^{\alpha_{0,1}})) = (5^{\alpha_{0,5}},\ 4^2,\ 1^{\alpha_{0,1}+1}).$$

We associate $\theta_0 = (5^{\alpha_{0,5}},\ 4^2,\ 1^{\alpha_{0,1}+1+\sum_{k\geq 1}\alpha_{k,3}})$ to τ_0 in this case.

For $j = 1$, $(\Phi_{2,n}(\tau))_0 = \Phi_{2,n}\left((5^{\alpha_{0,5}},\ 3,\ 1^{\alpha_{0,1}})\right) = \left(5^{\alpha_{0,5}},\ 2,\ 1^{\alpha_{0,1}+\sum_{k\geq 0}\alpha_{k,3}}\right)$. If either $\alpha_{0,1} \neq 0$ or $\sum_{k\geq 0}\alpha_{k,3} \neq 1$ then there is no loss of 3-hook under $\Phi_{2,n}$. If $\alpha_{0,1} = 0$ and $\sum_{k\geq 0}\alpha_{k,3} = 1$, then the loss of a 3-hook is covered by taking 1 from $\sum_{k\geq 0}\alpha_{k,3}$ and changing part size 3 to part size 4 as follows

$$H(\tau_0) = H((5^{\alpha_{0,5}},\ 3)) = (5^{\alpha_{0,5}},\ 4)\,.$$

In this case $\theta_0 = H(\tau_0)$.

For $j = 2$, $(\Phi_{2,n}(\tau))_0 = \Phi_{2,n}\left((5^{\alpha_{0,5}},\ 3^2,\ 1^{\alpha_{0,1}})\right) = \left(5^{\alpha_{0,5}},\ 2^2,\ 1^{\alpha_{0,1}+\sum_{k\geq 0}\alpha_{k,3}}\right)$. If either $\alpha_{0,1} \neq 0$ or $\sum_{k\geq 0}\alpha_{k,3} \neq 0$, then the loss of a 3-hook is covered by

$$I(\tau_0) = I((5^{\alpha_{0,5}},\ 3^2,\ 1^{\alpha_{0,1}})) = \left(5^{\alpha_{0,5}},\ 4,\ 1^{\alpha_{0,1}+\sum_{k\geq 0}\alpha_{k,3}}\right).$$

For $\alpha_{0,1} = 0$ and $\sum_{k\geq 0}\alpha_{k,3} = 0$, let $n > 6$. Then there is the smallest part with part size greater than or equal to 5, say λ_d. In this case, we take 4 from the part λ_d and associate $(4,\ 2^3)$ to (3^2). For the remaining part $\lambda_d - 4$, we proceed by considering it as a part of the partition and if $\lambda_d - 4 = 6r + 6$, for some $r \geq 0$, then we change it to $(6r + 5, 1)$ along with other parts while applying $\Phi_{2,n}$. If the final multiplicity of 1 is v then (1^v) is changed to $(2^{v/2})$ or $(2^{(v-1)/2}, 1)$, depending on v being even or odd, respectively. In this case $\theta_0 := (5^w, 4^{1+z}, 2^y, 1^x)$, where x is the multiplicity of 1 (either 0 or 1); y, z, w are the multiplicities of parts $2, 4, 5$ (respectively), which may also occur due to the part $\lambda_d - 4$. For example, if $\tau = (11, 3^2)$ then $\tau' = (7, 4, 2^3)$ and $\theta_0 = (4, 2^3)$; if $\tau = (7, 3^2)$ then $\tau' = \theta_0 = (4, 2^4, 1)$; if $\tau = (5^2, 3^2)$ then $\tau' = \theta_0 = (5, 4, 2^3, 1)$.

Now, let $\tau = (\tau_k)_{k\geq 0} \in \mathcal{B}_2(n)$ and $n > 6$. We consider two cases.

Case A. If the number of 3-hooks in τ_k is less than or equal to the number of 3-hooks in $(\Phi_{2,n}(\tau))_k$ for all k, then we define $\tau^* := \Phi_{2,n}(\tau)$. Clearly, $h_3(\tau) \leq h_3(\tau^*)$.

Case B. If for any $k \geq 0$, the number of 3-hooks in $(\Phi_{2,n}(\tau))_k$ is less than the number of 3-hooks in τ_k, we take τ' to be a partition in $\mathcal{B}_3(n)$ with $(\Phi_{2,n}(\tau))_k$ replaced by the required θ_k, which covers the loss of one or two 3-hooks. In this case, we define $\tau^* := (\Phi_{2,n}(\tau), \tau')$ and $h_3(\tau^*) := h_3(\Phi_{2,n}(\tau)) + h_3(\tau')$ (Note that τ^* is a set of two partitions from $\mathcal{B}_3(n)$). In this case also we have, $h_3(\tau) \leq h_3(\tau^*)$.

Since $\Phi_{2,n}$ is an injective map, all $\Phi_{2,n}(\tau)$ are distinct. Note that $(\Phi_{2,n}(\tau))_k$ and θ_k are all distinct as well. Therefore, all τ' and $\Phi_{2,n}(\tau)$ are also distinct. For example, see Table 6.4. Hence, we have for $n > 6$

$$\begin{aligned} b_{2,3}(n) &= \sum_{\tau \in \mathcal{B}_2(n)} h_3(\tau) = \sum_{\substack{\tau \in \mathcal{B}_2(n) \\ \text{Case A}}} h_3(\tau) + \sum_{\substack{\tau \in \mathcal{B}_2(n) \\ \text{Case B}}} h_3(\tau) \\ &\leq \sum_{\substack{\tau \in \mathcal{B}_2(n) \\ \text{Case A}}} h_3(\Phi_{2,n}(\tau)) + \sum_{\substack{\tau \in \mathcal{B}_2(n) \\ \text{Case B}}} (h_3(\Phi_{2,n}(\tau) + h_2(\tau')) \\ &= \sum_{\tau \in \mathcal{B}_2(n)} h_3(\tau^*) \leq \sum_{\tau \in \mathcal{B}_3(n)} h_3(\tau) = b_{3,3}(n). \end{aligned}$$

For $0 \leq n \leq 6$, it is easy to check that the inequality $b_{2,3}(n) \leq b_{3,3}(n)$ holds. This

$\tau \in \mathcal{B}_2(13)$	$\tau^* = \Phi_{2,n}(\tau)$	$\tau^* = (\Phi_{2,n}(\tau), \tau')$	$h_3(\tau)$	$h_3(\tau^*)$
(13)	(13)		1	1
$(11, 1^2)$	$(11, 1^2)$		1	1
$(9, 3, 1)$	$(8, 2, 1^3)$		1	2
$(9, 1^4)$	$(8, 1^5)$		2	2
$(7, 5, 1)$	$(7, 5, 1)$		1	1
$(7, 3^2)$		$((7, 2^2, 1^2), (4, 2^4, 1))$	3	1+2
$(7, 3, 1^3)$	$(7, 2, 1^4)$		2	2
$(7, 1^6)$	$(7, 1^6)$		2	2
$(5^2, 3)$	$(5^2, 2, 1)$		2	3
$(5^2, 1^3)$	$(5^2, 1^3)$		3	3
$(5, 3^2, 1^2)$		$((5, 2^2, 1^4), (5, 4, 1^4))$	1	2+2
$(5, 3, 1^5)$	$(5, 2, 1^6)$		1	2
$(5, 1^8)$	$(5, 1^8)$		2	2
$(3^4, 1)$		$((2^4, 1^5), (4^3, 1))$	2	2+3
$(3^3, 1^4)$		$((2^3, 1^7), (4^2, 1^5))$	3	2+3
$(3^2, 1^7)$		$((2^2, 1^9), (4, 1^9))$	2	1+2
$(3, 1^{10})$	$(2, 1^{11})$		2	2
(1^{13})	(1^{13})		1	1
Total number of 3-hooks			32	44

Table 6.4: Outline of the proof of Theorem 6.10 for $n = 13$

completes the proof. ■

6.3.3 Open problems

Let $t \geq 2$ and $k \geq 1$ be integers. The main motive of our study is to find the biases among $b_{t,k}(n)$ and $d_{t,k}(n)$, for fixed values of k. If λ is a t-distinct partition of n, then it is also a $(t+1)$-distinct partition of n. Therefore, $d_{t+1,k}(n) \geq d_{t,k}(n)$, for all $n \geq 0$. For a fixed value of k, we want to find the biases in the following diagram:

$$\begin{array}{ccc} b_{t,k}(n) & ? & d_{t,k}(n) \\ ? & & \wedge\!\backslash \\ b_{t+1,k}(n) & ? & d_{t+1,k}(n) \end{array}$$

Li and Wang while studying Beck type identities in [72, Theorem 1.6] proved that for all $t \geq 2$ and $n \geq 0$

$$\sum_{\lambda \in \mathcal{D}_t(n)} \bar{\ell}(\lambda) - \sum_{\lambda \in \mathcal{B}_t(n)} \bar{\ell}(\lambda) \geq 0,$$

where $\mathcal{D}_t(n)$ is the set of all t-distinct partitions of n. Since $h_1(\lambda) = \bar{\ell}(\lambda)$, it implies that $d_{t,1}(n) \geq b_{t,1}(n)$, for all $t \geq 2$ and $n \geq 0$. Also, from Theorem 6.8 we have $b_{t+1,1}(n) \geq b_{t,1}(n)$, for all $t \geq 2$ and $n \geq 0$. Therefore, for $k = 1$, all the biases are known and the diagram is complete for all $t \geq 2$ and $n \geq 0$:

$$\begin{array}{ccc} b_{t,1}(n) & \leqslant & d_{t,1}(n) \\ \wedge\!\!\!\!\setminus & & \wedge\!\!\!\!\setminus \\ b_{t+1,1}(n) & \leqslant & d_{t+1,1}(n) \end{array}$$

It is known due to Ballantine et al. [10] that $b_{2,2}(n) \geq d_{2,2}(n)$ for all $n \geq 0$ and $b_{2,3}(n) \geq d_{2,3}(n)$ for all $n \geq 8$. Also, we have Theorems 6.9 and 6.10. Therefore, for $k = 2, 3$, we have the following diagram for all but finitely many $n \geq 0$:

$$\begin{array}{ccc} b_{2,k}(n) & \geqslant & d_{2,k}(n) \\ \wedge\!\!\!\!\setminus & & \wedge\!\!\!\!\setminus \\ b_{3,k}(n) & ? & d_{3,k}(n) \end{array}$$

Our method of the proof of Theorem 6.9 can not be generalized to prove the biases for the number of hooks of length 2 in t-regular partitions for the next values of t. However, numerical evidence suggest that the number of hooks of length 2 in t-regular partitions increases with increasing values of t. For example, in Table 6.5 values in every column are in increasing order. In view of this, we propose the following conjecture.

Conjecture 6.12. *Let $t \geq 3$ be an integer. We have $b_{t+1,2}(n) \geq b_{t,2}(n)$, for all $n \geq 0$.*

$n \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
$b_{3,2}(n)$	0	2	1	5	5	11	12	22	28	43	53	79
$b_{4,2}(n)$	0	2	2	5	7	12	18	27	39	55	76	106
$b_{5,2}(n)$	0	2	2	6	7	15	18	33	42	67	87	129
$b_{6,2}(n)$	0	2	2	6	8	15	21	34	47	71	98	140
$b_{7,2}(n)$	0	2	2	6	8	16	21	37	48	77	101	151
$b_{8,2}(n)$	0	2	2	6	8	16	22	37	51	78	107	155
$b_{9,2}(n)$	0	2	2	6	8	16	22	38	51	81	108	161
$b_{10,2}(n)$	0	2	2	6	8	16	22	38	52	81	111	162
$b_{11,2}(n)$	0	2	2	6	8	16	22	38	52	82	111	165
$b_{12,2}(n)$	0	2	2	6	8	16	22	38	52	82	112	165
$b_{13,2}(n)$	0	2	2	6	8	16	22	38	52	82	112	166

Table 6.5: Values of $b_{t,2}(n) : 1 \leq n \leq 12$ and $3 \leq t \leq 13$

Recently, several hook length biases among t-regular and t-distinct partitions have been established with the help of generating functions, see, for example [10, 36]. The generating functions of $b_{2,2}(n)$ and $b_{3,2}(n)$ are already known. Our proof of Theorem 6.9 does not use any generating function technique. It would be interesting to prove the bias established in Theorem 6.9 with the help of generating functions. To find a similar proof of Theorem 6.10 we need to first derive the generating function of $b_{3,3}(n)$ as it is not yet known. Further, it would be very interesting to know if, for positive integers k and $t \geq 2$, there exists an integer $N_{t,k}$ such that $b_{t+1,k}(n) \geq b_{t,k}(n)$, for all $n \geq N_{t,k}$. This is true for certain values of t and k as we see in Theorems 6.8, 6.9 and 6.10. However, proving similar results for general values of t and k seems to be a hard problem.

7

Arithmetic properties and asymptotic formulae for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$

7.1 Introduction

In 2015, Fraenkel and Peled [46], working in the area of game theory, defined the term minimal excludant for any set S of positive integers as the least positive integer missing from S. Later in 2019, Andrews and Newman [6] introduced this in partition theory. They defined the minimal excludant of an integer partition π, denoted by $\mathrm{mex}(\pi)$, as the least positive integer missing from the partition. With

[1] The contents of this chapter have been published in *Ramanujan J.* (2024).

this, they also considered the sum of minimal excludants over all the partitions of n, denoted by $\sigma\text{mex}(n)$:

$$\sigma\text{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi),$$

where $\mathcal{P}(n)$ is the set of all partitions of n. For example, the values of minimal excludants for each partition of $n = 4$ are: $\text{mex}((4)) = 1$; $\text{mex}((3,1)) = 2$; $\text{mex}((2,2)) = 1$; $\text{mex}((2,1,1)) = 3$; $\text{mex}((1,1,1,1)) = 2$, with $\sigma\text{mex}(4) = 1+2+1+3+2 = 9$. The works of Andrews and Newman [6, 7] motivated great research in the theory of partitions. Many mathematicians have introduced the concept of minimal excludant parts for various restricted partition functions, see, for example, [11, 13, 14, 30, 39, 61, 62, 65, 66, 67]. Very recently, Baruah et al. [15] refined the arithmetic function $\sigma\text{mex}(n)$ by considering the sum of odd and even minimal excludants separately. More specifically, for a positive integer n, Baruah et al. [15] defined the following two arithmetic functions:

$$\sigma_o\text{mex}(n) := \sum_{\substack{\pi \in \mathcal{P}(n) \\ 2 \nmid \text{mex}(\pi)}} \text{mex}(\pi),$$

$$\sigma_e\text{mex}(n) := \sum_{\substack{\pi \in \mathcal{P}(n) \\ 2 \mid \text{mex}(\pi)}} \text{mex}(\pi).$$

For example, for $n = 4$, $\sigma_o\text{mex}(4) = 1+1+3 = 5$ and $\sigma_o\text{mex}(4) = 2+2 = 4$. Note that, for all $n \geq 0$, $\sigma\text{mex}(n) = \sigma_o\text{mex}(n) + \sigma_e\text{mex}(n)$.

7.2 Infinite families of congruences for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$

Baruah et al. [15] established two identities involving $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$. More precisely, they proved that

$$G_o(q) := \sum_{n=0}^{\infty} \sigma_o\mathrm{mex}(n)q^n = \frac{1}{2}\left((-q;q)_\infty^2 + (q;q)_\infty^2\right) = \frac{1}{2}\left(\frac{f_2^2}{f_1^2} + f_1^2\right), \tag{7.1}$$

$$G_e(q) := \sum_{n=0}^{\infty} \sigma_e\mathrm{mex}(n)q^n = \frac{1}{2}\left((-q;q)_\infty^2 - (q;q)_\infty^2\right) = \frac{1}{2}\left(\frac{f_2^2}{f_1^2} - f_1^2\right). \tag{7.2}$$

Using (7.1) and (7.2), following three Ramanujan-type congruences modulo 4 and 8 for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$ were also established in [15]:

$$\sigma_o\mathrm{mex}(2n+1) \equiv 0 \pmod{4}, \tag{7.3}$$

$$\sigma_o\mathrm{mex}(4n+1) \equiv 0 \pmod{8}, \tag{7.4}$$

$$\sigma_e\mathrm{mex}(4n) \equiv 0 \pmod{4}. \tag{7.5}$$

Very recently, Du and Tang [43] proved another two congruences for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$ modulo 8 and 16 which were conjectured by Baruah et al. in [15]:

$$\sigma_o\mathrm{mex}(8n+1) \equiv 0 \pmod{16},$$

$$\sigma_e\mathrm{mex}(8n) \equiv 0 \pmod{8}.$$

In this section, we prove the following theorem which provides two new congruences for $\sigma_e\mathrm{mex}(n)$ modulo 4. We use Ramanujan's theta functions and certain identities involving them in the proof.

Theorem 7.1. *For all $n \geq 0$, we have*

$$\sigma_e\mathrm{mex}(10n+6) \equiv 0 \pmod{4}, \tag{7.6}$$

$$\sigma_e\text{mex}(10n+8) \equiv 0 \pmod{4}. \tag{7.7}$$

We develop a relation between $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$, and use (7.3)-(7.5) to establish interesting families of infinitely many congruences for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$ modulo 4 and 8. More precisely, we have the following theorem in which we prove Ramanujan-type congruences for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$ with infinitely many primes involved in each family.

Theorem 7.2. *Let $n \geq 0$.*

1. *For all primes $p \equiv 5, 7, 11 \pmod{12}$ and odd integers $1 \leq k < p$, we have*

$$\sigma_e\text{mex}\left(2p^2n + kp + \frac{p^2-1}{12}\right) \equiv 0 \pmod{4}. \tag{7.8}$$

2. *For all primes $p \equiv 5, 7, 11 \pmod{24}$, we have*

$$\sigma_e\text{mex}\left(4p^2n + kp + \frac{p^2-1}{12}\right) \equiv 0 \pmod{8}, \tag{7.9}$$

where $1 \leq k < p$ *with* $k \equiv \begin{cases} 1 \pmod{4}, & \text{if } p \equiv 11 \pmod{24}; \\ 3 \pmod{4}, & \text{if } p \equiv 5, 7 \pmod{24}. \end{cases}$

3. *For all primes $p \equiv 5, 7, 11 \pmod{24}$, we have*

$$\sigma_o\text{mex}\left(4p^2n + kp + \frac{p^2-1}{12}\right) \equiv 0 \pmod{4}, \tag{7.10}$$

where $1 \leq k < p$ *with* $k \equiv \begin{cases} 2 \pmod{4}, & \text{if } p \equiv 5, 11 \pmod{24}; \\ 0 \pmod{4}, & \text{if } p \equiv 7 \pmod{24}. \end{cases}$

We prove Theorem 7.1 in Section 7.2.1 and Theorem 7.2 in Section 7.2.2.

7.2.1 Proof of Theorem 7.1

Firstly, we recall some results on Ramanujan's theta functions. From (2.1) and (2.3), we have

$$\varphi(-q) = \frac{\varphi^2(-q^2)}{\varphi(q)}. \tag{7.11}$$

We recall two identities from [25, p. 40] which relate $\varphi(q)$ and $\psi(q)$, namely

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{7.12}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8). \tag{7.13}$$

From (7.12) and (7.13), we deduce that

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{7.14}$$

$$\varphi(-q) = \varphi(q^4) - 2q\psi(q^8). \tag{7.15}$$

The Lambert series representation of $\varphi^2(q)$ is given by [26, p. 58]

$$\varphi^2(q) = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}. \tag{7.16}$$

We recall from [26, p. 107]

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2} = q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \tag{7.17}$$

where $\left(\frac{n}{5}\right)$ denotes the Legendre symbol. From (7.17), we arrive at

$$\sum_{\substack{n=1 \\ 5\nmid n}}^{\infty} \frac{q^n}{1+q^{2n}} \equiv q\frac{f_5^5}{f_1} \pmod{2}. \tag{7.18}$$

We are now equipped to present a proof of Theorem 7.1.

Proof of Theorem 7.1. From (7.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma_e\text{mex}(n)q^n &= \frac{1}{2}\left(\frac{f_2^2}{f_1^2}-f_1^2\right)\\ &= \frac{1}{2}f_2\left(\frac{f_2}{f_1^2}-\frac{f_1^2}{f_2}\right)\\ &= \frac{1}{2}f_2\left(\frac{1}{\varphi(-q)}-\varphi(-q)\right)\\ &= \frac{1}{2}f_2\left(\frac{\varphi(q)}{\varphi^2(-q^2)}-\varphi(-q)\right),\end{aligned} \tag{7.19}$$

where the last equality is due to (7.11). Next, invoking (7.14) and (7.15) in (7.19), we deduce that

$$\sum_{n=0}^{\infty}\sigma_e\text{mex}(n)q^n = \frac{1}{2}f_2\left(\frac{\varphi(q^4)}{\varphi^2(-q^2)}-\varphi(q^4)+2q\left(\frac{\psi(q^8)}{\varphi^2(-q^2)}+\psi(q^8)\right)\right). \tag{7.20}$$

Extracting terms with even powers of q on both sides of (7.20) and then replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma_e\text{mex}(2n)q^n &= \frac{1}{2}f_1\left(\frac{\varphi(q^2)}{\varphi^2(-q)}-\varphi(q^2)\right)\\ &= \frac{f_1\varphi(q^2)}{\varphi^2(-q)}\left(\frac{1-\varphi^2(-q)}{2}\right)\\ &= \frac{f_1\varphi(q^2)}{\varphi^2(-q)}\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^n}{1+q^{2n}}\right),\end{aligned} \tag{7.21}$$

where the last equality follows from (7.16). Reducing (7.21) modulo 4 yields

$$\sum_{n=0}^{\infty}\sigma_e\text{mex}(2n)q^n \equiv 2f_1\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}$$

$$\equiv 2f_1\left(\sum_{\substack{n=1\\5|n}}^{\infty}\frac{q^n}{1+q^{2n}}+\sum_{\substack{n=1\\5\nmid n}}^{\infty}\frac{q^n}{1+q^{2n}}\right)$$

$$\equiv 2f_1\left(\sum_{n=1}^{\infty}\frac{q^{5n}}{1+q^{10n}}+q\frac{f_5^5}{f_1}\right)\pmod 4, \tag{7.22}$$

where the last equality is due to (7.18). Finally, employing 5-dissections of f_1 and $\frac{1}{f_1}$ from Lemma 2.2 in (7.22) and extracting terms of the type q^{5n+3} on both sides, we arrive at

$$\sum_{n=0}^{\infty}\sigma_e\mathrm{mex}(10n+6)q^{5n+3}\equiv 0 \pmod 4. \tag{7.23}$$

Clearly, (7.23) implies (7.6). Similarly, employing 5-dissections of f_1 and $\frac{1}{f_1}$ from Lemma 2.2 in (7.22) and extracting terms of the type q^{5n+4} on both sides, we arrive at

$$\sum_{n=0}^{\infty}\sigma_e\mathrm{mex}(10n+8)q^{5n+4}\equiv 0 \pmod 4. \tag{7.24}$$

Clearly, (7.24) implies (7.7). This completes the proof. ■

7.2.2 Proof of Theorem 7.2

We observe the pattern in the coefficients of q-product f_1^2. Define

$$f_1^2 := \sum_{n=0}^{\infty} a(n)q^n.$$

In [33], Cooper, Hirschhorn, and Lewis studied the powers of Euler's product $(q;q)_\infty = f_1$. In particular, we use their result regarding the coefficients of f_1^2, which we write as a lemma below.

Lemma 7.3. [33, Theorem 1] *Let* $p \equiv 5, 7, 11 \pmod{12}$ *be a prime. Then*

$$a\left(pn + \frac{p^2-1}{12}\right) = \epsilon\ a\left(\frac{n}{p}\right), \tag{7.25}$$

where $\epsilon = \begin{cases} 1 & p \equiv 7, 11 \pmod{12}; \\ -1 & p \equiv 5 \pmod{12}. \end{cases}$

In (7.25), $a\left(\frac{n}{p}\right)$ is taken to be zero whenever $\frac{n}{p}$ is not an integer. Notice an easy consequence of Lemma 7.3: For any prime $p \equiv 5, 7, 11 \pmod{12}$, we have

$$a\left(p^2n + pk + \frac{p^2-1}{12}\right) = 0, \tag{7.26}$$

where k is an integer with $1 \le k < p$.

Proof of Theorem 7.2. In view of (7.1) and (7.2), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \sigma_o\mathrm{mex}(n)q^n &= \frac{1}{2}\left(\frac{f_2^2}{f_1^2} + f_1^2\right) \\ &= \frac{1}{2}\left(\frac{f_2^2}{f_1^2} - f_1^2\right) + f_1^2 \\ &= \sum_{n=0}^{\infty} \sigma_e\mathrm{mex}(n)q^n + \sum_{n=0}^{\infty} a(n)q^n. \end{aligned} \tag{7.27}$$

Now, from (7.26) and (7.27), we obtain that for any prime $p \equiv 5, 7, 11 \pmod{12}$,

$$\sigma_o\mathrm{mex}\left(p^2n + pk + \frac{p^2-1}{12}\right) = \sigma_e\mathrm{mex}\left(p^2n + pk + \frac{p^2-1}{12}\right), \tag{7.28}$$

where k is an integer with $1 \le k < p$.

1. From (7.3), we have that $\sigma_o\mathrm{mex}(n)$ is divisible by 4 for every odd integer n. Also, for any prime $p \equiv 5, 7, 11 \pmod{12}$ and an odd integer k, $2p^2n+pk+\frac{p^2-1}{12}$ is odd. Therefore, replacing n by $2n$ in (7.28) and reducing it modulo 4 yields (7.8).

2. Similarly, from (7.4), we have that $\sigma_o\text{mex}(n)$ is divisible by 8 whenever $n \equiv 1 \pmod 4$. We observe that for any prime $p \equiv 5, 7, 11 \pmod{24}$,

$$4p^2n + pk + \frac{p^2-1}{12} \equiv 1 \pmod 4,$$

for $1 \leq k < p$ with $k \equiv \begin{cases} 1 \pmod 4, & \text{if } p \equiv 11 \pmod{24}; \\ 3 \pmod 4, & \text{if } p \equiv 5, 7 \pmod{24}. \end{cases}$

Therefore, replacing n by $4n$ in (7.28) and reducing it modulo 8 yields (7.9).

3. Next, we notice from (7.5) that $\sigma_e\text{mex}(n)$ is divisible by 4 whenever $n \equiv 0 \pmod 4$. Also, for any prime $p \equiv 5, 7, 11 \pmod{24}$,

$$4p^2n + pk + \frac{p^2-1}{12} \equiv 0 \pmod 4,$$

for $1 \leq k < p$ with $k \equiv \begin{cases} 2 \pmod 4, & \text{if } p \equiv 5, 11 \pmod{24}; \\ 0 \pmod 4, & \text{if } p \equiv 7 \pmod{24}. \end{cases}$

Therefore, replacing n by $4n$ in (7.28) and reducing it modulo 4 yields (7.10).

This completes the proof. ■

7.3 Asymptotic formulae for $\sigma_o\text{mex}(n)$ and $\sigma_e\text{mex}(n)$

Grabner and Knopfmacher [52] obtained the Hardy-Ramanujan type asymptotic formula for $\sigma\text{mex}(n)$, though they were working on "smallest gap" in a partition that has exactly the same meaning as that of minimal excludant. The asymptotic formula for $\sigma\text{mex}(n)$ due to Grabner and Knopfmacher [52] is as follows:

$$\sigma\text{mex}(n) \sim \frac{1}{4\sqrt[4]{6n^3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty. \tag{7.29}$$

In [67], Kaur et al. studied the following function:

$$\sigma_d\mathrm{mex}(n) := \sum_{\pi \in \mathcal{D}(n)} \mathrm{mex}(\pi),$$

where $\mathcal{D}(n)$ denotes the collection of partitions of n into distinct parts. Moreover, they obtained the following Hardy-Ramanujan type asymptotic result for $\sigma_d\mathrm{mex}(n)$:

$$\sigma_d\mathrm{mex}(n) \sim \frac{1}{2\sqrt[4]{3n^3}} \exp\left(\pi\sqrt{\frac{n}{3}}\right) \quad \text{as } n \to \infty.$$

In [15], Baruah et al. asked for the Hardy-Ramanujan type asymptotic formulae for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$. In this section, we prove that as $n \to \infty$, $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$ behave same. More specifically, we have the following result.

Theorem 7.4. *We have*

$$\sigma_o\mathrm{mex}(n) \sim \sigma_e\mathrm{mex}(n) \sim \frac{1}{8\sqrt[4]{6n^3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

as $n \to \infty$.

We see that Theorem 7.4 immediately implies (7.29) since

$$\sigma\mathrm{mex}(n) = \sigma_o\mathrm{mex}(n) + \sigma_e\mathrm{mex}(n) \sim \frac{1}{4\sqrt[4]{6n^3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

as $n \to \infty$.

To obtain asymptotic formulae for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$, we use the Tauberian theorem of Ingham [63]. Ingham's theorem enables us to derive asymptotic formula for the coefficients $c(n)$ of certain power series $C(q) := \sum_{n=0}^{\infty} c(n)q^n$ from the behavior of its generating function $C(e^{-y})$ while $y \to 0^+$. The following result is a special case of Ingham's theorem, see, for example, [67].

Theorem 7.5. [63, Theorem 1] *Let* $C(q) := \sum_{n=0}^{\infty} c(n)q^n$ *be a power series with* *radius of convergence 1. Assume that $\{c(n)\}$ is a weakly increasing sequence of non-negative real numbers. If there are constants $\mu, \nu \in \mathbb{R}$, and $\lambda > 0$ such that*

$$C(e^{-y}) \sim \mu y^{\nu} e^{\frac{\lambda}{y}}, \quad \textit{as } y \to 0^+$$

then we have

$$c(n) \sim \frac{\mu}{2\sqrt{\pi}} \frac{\lambda^{\frac{2\nu+1}{4}}}{n^{\frac{2\nu+3}{4}}} e^{2\sqrt{\lambda n}}, \quad \textit{as } n \to \infty.$$

The next lemma is an easy representation of Theorem 1.1 of Berndt and Kim [27] in our setup. For more details on the terminology involved, see [27].

Lemma 7.6. [27, Theorem 1.1] *Let E_{2n}, $n \geq 0$, denote the $2n$-th Euler number, and let $H_n(x)$, $n \geq 0$, be the n-th Hermite polynomial. Then, as $y \to 0^+$,*

$$\sum_{n=0}^{\infty} (-1)^n e^{-(an^2+bn)y} \sim e^{\left(\frac{a-2b}{4}\right)y} \sum_{n=0}^{\infty} \frac{E_{2n} a^n}{(2n)! 2^{2n+1}} y^n \cdot H_{2n}\left(\frac{b-a}{2\sqrt{a}}\sqrt{y}\right). \tag{7.30}$$

Proof. Replacing $\frac{1-t}{1+t}(= q)$ by e^{-ay} and then b by $\frac{b}{a}$ in [27, Theorem 1.1], we can easily deduce (7.30). ∎

In the following lemma, we prove the weakly increasing nature of $\{\sigma_o\text{mex}(n)\}$ and $\{\sigma_e\text{mex}(n)\}$ using a combinatorial technique. This lemma will be used to employ Theorem 7.5 in the proof of Theorem 7.4.

Lemma 7.7. *The sequences $\{\sigma_o\text{mex}(n)\}$ and $\{\sigma_e\text{mex}(n)\}$ are weakly increasing.*

Proof. Let $\mathcal{P}_o(n)$ (resp., $\mathcal{P}_e(n)$) denote the set of all partitions π of n with $\text{mex}(\pi)$ odd (resp., even). We construct a map

$$\Phi : \mathcal{P}_o(n) \to \mathcal{P}_o(n+1),$$

where for $\pi \in \mathcal{P}_o(n)$, if $\text{mex}(\pi) \neq 1$ then $\Phi(\pi)$ is the partition of $n+1$ with 1 added as a part to π, and if mex$(\pi) = 1$ then $\Phi(\pi)$ is the partition of $n+1$ with a largest part, say λ_π, of π replaced with $\lambda_\pi + 1$. Notice that if $\pi_1, \pi_2 \in \mathcal{P}_o(n)$ are two distinct partitions then $\Phi(\pi_1) \neq \Phi(\pi_2)$. Therefore, the map Φ is injective. Also, under the map Φ, mex(π) of any partition $\pi \in \mathcal{P}_o(n)$ remains same. Thus, for $n \geq 1$

$$\sigma_o\text{mex}(n) \leq \sigma_o\text{mex}(n+1)$$

and therefore, $\{\sigma_o\text{mex}(n)\}$ is weakly increasing.

For $\{\sigma_e\text{mex}(n)\}$, the proof follows in a similar way with the map defined in this case as follows:

$$\Psi : \mathcal{P}_e(n) \to \mathcal{P}_e(n+1),$$

where for $\pi \in \mathcal{P}_e(n)$, $\Psi(\pi)$ is the partition of $n+1$ with 1 added as a part to π. ■

We recall the following modular transformation property satisfied by the Dedekind eta-function $\eta(z)$ [86, Theorem 1.61]: For $z \in \mathbb{H}$,

$$\eta(-1/z) = (-iz)^{1/2}\eta(z). \tag{7.31}$$

We are now in a position to present a proof of Theorem 7.4.

Proof of Theorem 7.4. From Euler's Pentagonal Number theorem (1.2), we have

$$\begin{aligned} U(q) := (q;q)_\infty &= -1 + \sum_{n=0}^{\infty}(-1)^n q^{\frac{3}{2}n^2+\frac{1}{2}n} + \sum_{n=0}^{\infty}(-1)^n q^{\frac{3}{2}n^2-\frac{1}{2}n} \\ &= -1 + U_1(q) + U_2(q), \end{aligned} \tag{7.32}$$

where

$$U_1(q) = \sum_{n=0}^{\infty}(-1)^n q^{\frac{3}{2}n^2+\frac{1}{2}n} \quad \text{and} \quad U_2(q) = \sum_{n=0}^{\infty}(-1)^n q^{\frac{3}{2}n^2-\frac{1}{2}n}.$$

Next, using Lemma 7.6 with $a=\frac{3}{2}$ and $b=\frac{1}{2}$, we obtain that as $y\to 0^+$

$$\begin{aligned} U_1(e^{-y}) &= \sum_{n=0}^{\infty}(-1)^n e^{-\left(\frac{3}{2}n^2+\frac{1}{2}n\right)y} \\ &= e^{y/8}\sum_{n=0}^{N}\frac{E_{2n}3^n}{(2n)!2^{3n+1}}y^n\cdot H_{2n}\left(-\sqrt{\frac{y}{6}}\right)+O(y^{N+1/2}). \end{aligned}$$

Using the fact that $E_0=1$ and $H_0(x)=1$, we arrive at

$$\lim_{y\to 0^+} U_1(e^{-y})=\frac{1}{2}. \tag{7.33}$$

By repeating a similar argument we find that

$$\lim_{y\to 0^+} U_2(e^{-y})=\frac{1}{2}. \tag{7.34}$$

Now, (7.32), (7.33), and (7.34) imply that

$$\lim_{y\to 0^+} U(e^{-y})=0,$$

which in turn implies that

$$\lim_{y\to 0^+} U^2(e^{-y})=0. \tag{7.35}$$

Using modular transformation property (7.31) of Dedekind's eta-function, it is easy to show that

$$\frac{1}{(e^{-y};e^{-y})_\infty}\sim\sqrt{\frac{y}{2\pi}}e^{\frac{\pi^2}{6y}}\quad \text{as } y\to 0^+. \tag{7.36}$$

Using an identity:

$$(-q;q)_\infty=\frac{(q^2;q^2)_\infty}{(q;q)_\infty}$$

and (7.36), we find that

$$(-e^{-y};e^{-y})_\infty^2 = \frac{(e^{-2y};e^{-2y})_\infty^2}{(e^{-y};e^{-y})_\infty^2} \sim \left(\frac{\sqrt{\frac{y}{2\pi}}e^{\frac{\pi^2}{6y}}}{\sqrt{\frac{2y}{2\pi}}e^{\frac{\pi^2}{12y}}}\right)^2 = \frac{1}{2}e^{\frac{\pi^2}{6y}} \tag{7.37}$$

as $y \to 0^+$. Finally, from (7.1), (7.35), and (7.37), we conclude that

$$\begin{aligned} G_o(e^{-y}) &= \frac{1}{2}\left((-e^{-y};e^{-y})_\infty^2 + (e^{-y};e^{-y})_\infty^2\right) \\ &\sim \frac{1}{4}e^{\frac{\pi^2}{6y}} \end{aligned} \tag{7.38}$$

as $y \to 0^+$. Note that $G_o(q)$ has real non-negative coefficients and by Lemma 7.7, $\{\sigma_o\mathrm{mex}(n)\}$ is weakly increasing. In view of (7.38), employing Theorem 7.5 with $\mu = 1/4$, $\nu = 0$, and $\lambda = \pi^2/6$, we obtain that

$$\sigma_o\mathrm{mex}(n) \sim \frac{1}{8\sqrt[4]{6n^3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty.$$

By the similar arguments, we arrive at

$$\sigma_e\mathrm{mex}(n) \sim \frac{1}{8\sqrt[4]{6n^3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty.$$

This completes the proof of Theorem 7.4. ■

8

Divisibility of the partition function $\mathrm{PDO}_t(n)$ by powers of 2 and 3

8.1 Introduction

In [5], Andrews, Lewis, and Lovejoy introduced partitions with designated summands. A partition of n with designated summands is obtained from an ordinary partition of n by tagging exactly one of each part size. They denote the number of partitions of n with designated summands by $\mathrm{PD}(n)$. For example, $\mathrm{PD}(4) = 10$ with the relevant partitions being $(4')$, $(3', 1')$, $(2', 2)$, $(2, 2')$, $(2', 1', 1)$, $(2', 1, 1')$,

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$(1',1,1,1)$, $(1,1',1,1)$, $(1,1,1',1)$, $(1,1,1,1')$. They also studied another partition function $\mathrm{PDO}(n)$ which counts the number of partitions of n with designated summands in which all parts are odd. From the above example, $\mathrm{PDO}(4) = 5$. Later, many authors studied these two partition functions, see for example [18, 31, 109].

Recently, Lin [74] introduced two new partition functions $\mathrm{PD}_t(n)$ and $\mathrm{PDO}_t(n)$ related to partitions with designated summands. The partition function $\mathrm{PD}_t(n)$ counts the total number of tagged parts over all the partitions of n with designated summands. For instance, $\mathrm{PD}_t(4) = 13$. The other partition function $\mathrm{PDO}_t(n)$ counts the total number of tagged parts over all the partitions of n with designated summands in which all parts are odd. For example, $\mathrm{PDO}_t(4) = 6$. Lin found the generating functions of $\mathrm{PD}_t(n)$ and $\mathrm{PDO}_t(n)$. The generating function of $\mathrm{PDO}_t(n)$ is given by

$$G(q) := \sum_{n=0}^{\infty} \mathrm{PDO}_t(n)q^n = \frac{qf_2f_3^2f_{12}^2}{f_1^2f_6}. \tag{8.1}$$

Lin also established many congruences modulo small powers of 3 satisfied by $\mathrm{PD}_t(n)$ and $\mathrm{PDO}_t(n)$. For example, Lin proved the following Ramanujan-type congruences modulo 9 and 27 satisfied by $\mathrm{PDO}_t(n)$: For $n \geq 0$,

$$\mathrm{PDO}_t(8n) \equiv \mathrm{PDO}_t(12n) \equiv \mathrm{PDO}_t(12n+8) \equiv 0 \pmod 9,$$
$$\mathrm{PDO}_t(24n) \equiv \mathrm{PDO}_t(36n) \equiv \mathrm{PDO}_t(36n+24) \equiv 0 \pmod{27}.$$

He further conjectured the following congruences.

Conjecture 8.1. [74, Conjecture 6.1] *For $k, n \geq 0$,*

$$\mathrm{PDO}_t(8 \cdot 3^k n) \equiv 0 \pmod{3^{k+2}}, \tag{8.2}$$
$$\mathrm{PDO}_t(12 \cdot 3^k n) \equiv 0 \pmod{3^{k+2}}. \tag{8.3}$$

Lin proved Conjecture 8.1 for $k = 0, 1$ using basic q-series techniques. In Section 8.2, we prove this conjecture for $k = 2$. The conjecture is open for general k. In Section 8.3, we prove that the generating function of $\mathrm{PDO}_t(n)$ is lacunary modulo arbitrary powers of 2 and 3.

The partition function $\mathrm{PDO}_t(n)$ has also been studied modulo small powers of 2. Lin conjectured and Baruah and Kaur [17] proved the following congruences modulo 8: For $n \geq 0$,

$$\mathrm{PDO}_t(8n+6) \equiv \mathrm{PDO}_t(8n+7) \equiv 0 \pmod{8}. \tag{8.4}$$

Using 2- and 3-dissections of certain q-products, Vandna and Kaur [106] also proved several congruences modulo 8, 16, 48, 144, 288, 1152, 6912. For example, for $n \geq 0$,

$$\begin{aligned}
\mathrm{PDO}_t(12n+6) \equiv \mathrm{PDO}_t(48n+38) &\equiv 0 \pmod{16},\\
\mathrm{PDO}_t(32n+12) \equiv \mathrm{PDO}_t(96n+76) &\equiv 0 \pmod{48},\\
\mathrm{PDO}_t(48n+44) \equiv \mathrm{PDO}_t(96n+60) &\equiv 0 \pmod{144}.
\end{aligned}$$

In Section 8.4, using the work of Ono and Taguchi [87] on nilpotency of Hecke operators we prove that there exist infinitely many congruences modulo arbitrary powers of 2. In Section 8.5, we use Radu's algorithm and q-series techniques to prove congruences for $\mathrm{PDO}_t(n)$ modulo small powers of 2. In Section 8.6, we present another approach to handle Lin's conjecture.

8.2 Proof of Lin's conjecture for $k = 2$

In this section, we prove Conjecture 8.1 for $k = 2$. In particular, we prove the following theorem.

Theorem 8.2. *For all $n \geq 0$, we have*

$$\mathrm{PDO}_t(72n) \equiv 0 \pmod{81}, \tag{8.5}$$

$$\mathrm{PDO}_t(108n) \equiv 0 \pmod{81}. \tag{8.6}$$

Proof. By (8.1), we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(n) q^n = q \frac{f_2 f_3^2 f_{12}^2}{f_1^2 f_6}.$$

Substituting the 3-dissection formula for $\frac{f_2}{f_1^2}$ from (2.11), we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(n) q^n = q \frac{f_3^2 f_{12}^2}{f_6} \left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right).$$

Extracting those terms of the form q^{3n} on both sides of the above equation and replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(3n) q^n = 4q \frac{f_2 f_4^2 f_6^3}{f_1^4}.$$

Substituting the 2-dissection formula for $\frac{1}{f_1^4}$ from (2.6) yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(3n) q^n = 4q f_2 f_4^2 f_6^3 \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \right).$$

Extracting those terms of the form q^{2n} on both sides of the above equation and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n) q^n = 16q f_2^4 f_4^4 \left(\frac{f_3}{f_1^3} \right)^3.$$

Substituting the 2-dissection formula for $\frac{f_3}{f_1^3}$ from (2.5) yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n)q^n = 16q\frac{f_4^{22}f_6^9}{f_2^{23}f_{12}^6} + 16\cdot 9q^2\frac{f_4^{18}f_6^7}{f_2^{21}f_{12}^2} + 16\cdot 27q^3\frac{f_4^{14}f_6^5f_{12}^2}{f_2^{19}} + 16\cdot 27q^4\frac{f_4^{10}f_6^3f_{12}^6}{f_2^{17}}.$$

Extracting those terms of the form q^{2n} on both sides of the above equation and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(12n)q^n = 16\cdot 9q\frac{f_2^{18}f_3^7}{f_1^{21}f_6^2} + 16\cdot 27q^2\frac{f_2^{10}f_3^3f_6^6}{f_1^{17}}. \tag{8.7}$$

Using the Binomial theorem we have

$$\frac{f_2^{10}}{f_1^{17}} \equiv \frac{f_2^9}{f_1^{18}}f_1f_2 \equiv \frac{f_6^3}{f_3^6}f_1f_2 \pmod 3.$$

Substituting the 3-dissection formula for f_1f_2 from (2.12), we obtain

$$\frac{f_2^{10}}{f_1^{17}} \equiv \frac{f_6^4f_9^4}{f_3^7f_{18}^2} - q\frac{f_6^3f_9f_{18}}{f_3^6} - 2q^2\frac{f_6^2f_{18}^4}{f_3^5f_9^2} \pmod 3. \tag{8.8}$$

Again, using the Binomial theorem we observe that

$$\frac{f_2^{18}}{f_1^{21}} \equiv \frac{f_2^{18}}{f_1^{27}}f_1^6 \equiv \frac{f_6^6}{f_3^9}\left(f_1^3\right)^2 \pmod 9.$$

Substituting the 3-dissection formula for f_1^3 from (2.9), we obtain

$$\frac{f_2^{18}}{f_1^{21}} \equiv \frac{f_6^8f_9^{12}}{f_3^{11}f_{18}^6} - 6q\frac{f_6^7f_9^9}{f_3^{10}f_{18}^3} + 9q^2\frac{f_6^6f_9^6}{f_3^9} + 8q^3\frac{f_6^5f_9^3f_{18}^3}{f_3^8} - 24q^4\frac{f_6^4f_{18}^6}{f_3^7} + 16q^6\frac{f_6^2f_{18}^{12}}{f_3^5f_9^6} \pmod 9. \tag{8.9}$$

Substituting (8.8) and (8.9) in (8.7), and then extracting the terms of the form q^{3n}

on both sides, we find that

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^{3n} \equiv 16 \cdot 81 q^3 \frac{f_6^4 f_9^6}{f_3^2} - 16 \cdot 27 q^3 \frac{f_6^9 f_9 f_{18}}{f_3^3} \pmod{81}.$$

Replacing q^3 by q, and then substituting the 2-dissection formula for $\frac{f_3}{f_1^3}$ from (2.5) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^n &\equiv -16 \cdot 27 q \frac{f_2^9 f_3 f_6}{f_1^3} \pmod{81} && (8.10) \\ &\equiv -16 \cdot 27 q f_2^9 f_6 \frac{f_3}{f_1^3} \pmod{81} \\ &\equiv -16 \cdot 27 q \frac{f_4^6 f_6^4}{f_{12}^2} - 16 \cdot 81 q^2 f_2^2 f_4^2 f_6^2 f_{12}^2 \pmod{81} \\ &\equiv -16 \cdot 27 q \frac{f_4^6 f_6^4}{f_{12}^2} \pmod{81}. && (8.11) \end{aligned}$$

Extracting the terms of the form q^{2n} on both sides of (8.11), we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(72n)q^{2n} \equiv 0 \pmod{81}.$$

This completes the proof of (8.5). We next prove (8.6). Using the Binomial theorem we have

$$\frac{f_2^9}{f_1^3} \equiv \frac{f_6^3}{f_3} \pmod{3}. \tag{8.12}$$

Combining (8.12) and (8.10), we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^n \equiv -16 \cdot 27 q f_6^4 \pmod{81}. \tag{8.13}$$

Extracting the terms of the form q^{3n} on both sides yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(108n)q^{3n} \equiv 0 \pmod{81}.$$

This completes the proof of (8.6). ■

8.3 Lacunarity of $\mathrm{PDO}_t(n)$ modulo arbitrary powers of 2 and 3

In this section, we prove that the generating function of $\mathrm{PDO}_t(n)$ is lacunary modulo arbitrary powers of 2 and 3.

It is a well-known fact that modular forms with integer Fourier coefficients are lacunary modulo any positive integer. In [74], Lin remarked that the generating function of $\mathrm{PDO}_t(n)$ is a modular form. We find that Lin's observation is not correct because this function is not holomorphic at the cusp 1. Also, the generating function of $\mathrm{PDO}_t(n)$ does not satisfy the conditions of [34, Theorem 1.1]. Therefore, it is an interesting problem to study the lacunarity of $G(q) = \sum_{n=0}^{\infty} \mathrm{PDO}_t(n)q^n$ modulo arbitrary powers of primes. In the following theorem, we prove that $G(q)$ is lacunary modulo arbitrary powers of 2 and 3.

Theorem 8.3. *For any positive integer k, we have*

$$\lim_{X\to\infty} \delta_0(G, 2^k; X) = 1, \tag{8.14}$$

$$\lim_{X\to\infty} \delta_0(G, 3^k; X) = 1. \tag{8.15}$$

Given a prime p, let

$$A_p(z) := \prod_{n=1}^{\infty} \frac{(1-q^n)^p}{(1-q^{pn})} = \frac{\eta^p(z)}{\eta(pz)}.$$

Then using the Binomial theorem we have

$$A_p^{p^k}(z) = \frac{\eta^{p^{k+1}}(z)}{\eta^{p^k}(pz)} \equiv 1 \pmod{p^{k+1}}.$$

Define $B_{p,k}(z)$ by

$$B_{p,k}(z) := \left(\frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)}\right) A_p^{p^k}(z). \tag{8.16}$$

Taking (8.16) modulo p^{k+1}, we have

$$B_{p,k}(z) \equiv \frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)} = \frac{qf_2f_3^2f_{12}^2}{f_1^2f_6}. \tag{8.17}$$

Combining (8.1) and (8.17), we obtain

$$B_{p,k}(z) \equiv \sum_{n=0}^{\infty} \mathrm{PDO}_t(n)q^n \pmod{p^{k+1}}. \tag{8.18}$$

Proof of Theorem 8.3. We put $p = 2$ in (8.16) to obtain

$$B_{2,k}(z) = \left(\frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)}\right) A_2^{2^k}(z) = \frac{\eta(2z)^{1-2^k}\eta(3z)^2\eta(12z)^2\eta(z)^{2^{k+1}-2}}{\eta(6z)}.$$

Now, $B_{2,k}$ is an eta-quotient with level $N = 144$. The cusps of $\Gamma_0(144)$ are represented by fractions $\frac{c}{d}$ where $d \mid 144$ and $\gcd(c,d) = 1$ (Proposition 2.4). By Theorem 2.6, we find that $B_{2,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned} S := {} & 12\frac{\gcd(d,1)^2}{\gcd(d,12)^2}\left(2^{k+1}-2\right) + 6\frac{\gcd(d,2)^2}{\gcd(d,12)^2}\left(1-2^k\right) + 8\frac{\gcd(d,3)^2}{\gcd(d,12)^2} \\ & - 2\frac{\gcd(d,6)^2}{\gcd(d,12)^2} + 2 \geq 0. \end{aligned}$$

To find all the possible values of S, we prepare Table 8.1 using MATLAB. Using Table 8.1, we find that $S \geq 0$ for all $d \mid 144$. Hence, $B_{2,k}(z)$ is holomorphic at every

$d \mid 144$	$\frac{\gcd(d,1)^2}{\gcd(d,12)^2}$	$\frac{\gcd(d,2)^2}{\gcd(d,12)^2}$	$\frac{\gcd(d,6)^2}{\gcd(d,12)^2}$	$\frac{\gcd(d,3)^2}{\gcd(d,12)^2}$
1	1	1	1	1
2	0.2500	1	1	0.2500
3, 9	0.1111	0.1111	1	1
4, 8, 16	0.0625	0.2500	0.2500	0.0625
6, 18	0.0278	0.1111	1	0.2500
12, 24, 36, 48, 72, 144	0.0069	0.0278	0.2500	0.0625

Table 8.1: Possible values of S

cusp $\frac{c}{d}$. Using Theorem 2.5, we find that the weight of $B_{2,k}(z)$ is $\ell = 2^{k-1}+1$. Also, the associated character for $B_{2,k}(z)$ is given by $\chi_1 = \left(\frac{(-1)^{2^{k-1}+1}2^{4-2^k}3^3}{\bullet}\right)$. Finally, Theorem 2.5 yields that $B_{2,k}(z) \in M_{2^{k-1}+1}(\Gamma_0(144), \chi_1)$ for $k \geq 1$. Since $B_{2,k}(z) \in M_{2^{k-1}+1}(\Gamma_0(144), \chi_1)$, Theorem 2.9 implies that the Fourier coefficients of $B_{2,k}(z)$ are almost always divisible by $m = 2^k$. Now, using (8.18) we complete the proof of (8.14).

We next prove (8.15). We put $p = 3$ in (8.16) to obtain

$$B_{3,k}(z) = \left(\frac{\eta(2z)\eta(3z)^2\eta(12z)^2}{\eta(z)^2\eta(6z)}\right) A_3^{3^k}(z) = \frac{\eta(2z)\eta(3z)^{2-3^k}\eta(12z)^2\eta(z)^{3^{k+1}-2}}{\eta(6z)}.$$

Now, $B_{3,k}$ is an eta-quotient with $N = 144$. The cusps of $\Gamma_0(144)$ are represented by fractions $\frac{c}{d}$ where $d \mid 144$ and $\gcd(c,d) = 1$ (Proposition 2.4). By Theorem 2.6, we find that $B_{3,k}(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned} L := {} & 12\frac{\gcd(d,1)^2}{\gcd(d,12)^2}\left(3^{k+1}-2\right) + 4\frac{\gcd(d,3)^2}{\gcd(d,12)^2}\left(2-3^k\right) + 6\frac{\gcd(d,2)^2}{\gcd(d,12)^2} \\ & - 2\frac{\gcd(d,6)^2}{\gcd(d,12)^2} + 2 \geq 0. \end{aligned}$$

Using Table 8.1, we find that $L \geq 0$ for all $d \mid 144$. As before, using Theorem 2.5 we find that $B_{3,k}(z) \in M_{3^k+1}(\Gamma_0(144), \chi_2)$, where the character χ_2 is given by $\chi_2 = \left(\frac{2^4 3^{3-3^k}}{\bullet}\right)$. Using Theorem 2.9 and (8.18), we find that $\mathrm{PDO}_t(n)$ is divisible by 3^k for almost all $n \geq 0$. This completes the proof of (8.15). ∎

8.4 Hecke nilpotency and congruences for $\mathrm{PDO}_t(n)$ modulo arbitrary powers of 2

In this section, we prove an infinite family of congruences for $\mathrm{PDO}_t(n)$ modulo arbitrary powers of 2, using nilpotency of Hecke operators. For that, we employ a result of Ono and Taguchi, Theorem 2.12, to the modular form $B_{2,k}(z)$. We observe that, for any positive integer k, the generating function of $\mathrm{PDO}_t(n)$ is congruent to an eta-quotient modulo 2^k, and the eta-quotient is a modular form whose level lands in Ono and Taguchi's list. This allows us to use a result of Ono and Taguchi to prove the following congruence for $\mathrm{PDO}_t(n)$.

Theorem 8.4. *Let n be a non-negative integer. Then there is an integer $s \geq 0$ such that for every $u \geq 1$ and distinct primes $p_1, \ldots, p_{s+u}$ coprime to 6, we have*

$$\mathrm{PDO}_t\left(p_1 \cdots p_{s+u} \cdot n\right) \equiv 0 \pmod{2^u}$$

whenever n is coprime to $p_1, \ldots, p_{s+u}$.

Proof. Taking $p = 2$ in (8.18), we have

$$B_{2,k}(z) \equiv \sum_{n=0}^{\infty} \mathrm{PDO}_t(n) q^n \pmod{2^{k+1}}.$$

Note that $B_{2,k}(z) \in M_{2^{k-1}+1}\left(\Gamma_0(144), \chi_1\right)$. Employing Theorem 2.12, we find that there is an integer $s \geq 0$ such that for any $u \geq 1$,

$$B_{2,k}(z) \,|T_{p_1}|\, T_{p_2}| \cdots |T_{p_{s+u}} \equiv 0 \pmod{2^u}$$

whenever $p_1, \ldots, p_{s+u}$ are coprime to 6. It follows from the definition of Hecke operators that if $p_1, \ldots, p_{s+u}$ are distinct primes and if n is coprime to $p_1 \cdots p_{s+u}$

then

$$\mathrm{PDO}_t\left(p_1 \cdots p_{s+u} \cdot n\right) \equiv 0 \pmod{2^u}.$$

This completes the proof of the theorem. ■

8.5 Congruences for $\mathrm{PDO}_t(n)$ modulo powers of 2

Theorem 8.4 involves primes which are coprime to 6. This left the case for the primes 2 and 3. In the following theorem, we derive infinitely many congruences for $\mathrm{PDO}_t(n)$ modulo 8 and 32, which involve the primes 2 and 3.

Theorem 8.5. *For any prime $p \equiv -1 \pmod 6$ and $n \geq 0$, we have*

$$\mathrm{PDO}_t\left(6p^2n + 6kp + 3p^2\right) \equiv 0 \pmod 8, \tag{8.19}$$

$$\mathrm{PDO}_t\left(24p^2n + 24kp + 12p^2\right) \equiv 0 \pmod{32}, \tag{8.20}$$

where $k = 1, 2, \ldots, p-1$.

Proof. We first recall the Jacobi's identity [26, Theorem 1.3.9]

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{8.21}$$

From [74, eq. (5.6)], we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(3n) q^n \equiv 4q f_2^3 f_6^3 \pmod 8. \tag{8.22}$$

Extracting the terms with odd powers of q on each side of (8.22), and dividing both sides by q, and then substituting q^2 by q, we deduce that

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n+3)q^n \equiv 4f_1^3 f_3^3 \pmod 8. \tag{8.23}$$

Invoking (8.21) in (8.23) yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n+3)q^n \equiv 4 \cdot \sum_{m=0}^{\infty} q^{\frac{m(m+1)}{2}} \cdot \sum_{n=0}^{\infty} q^{\frac{3n(n+1)}{2}} \pmod 8,$$

which we rewrite as

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n+3)q^n \equiv 4 \cdot \sum_{m,n\geq 0} q^{\frac{m(m+1)}{2}+\frac{3n(n+1)}{2}} \pmod 8. \tag{8.24}$$

For any prime $p \geq 3$, the congruence

$$\frac{m(m+1)}{2} + \frac{3n(n+1)}{2} \equiv pk + \frac{p^2-1}{2} \pmod{p^2}$$

is equivalent to

$$(2m+1)^2 + 3\,(2n+1)^2 \equiv 8pk \pmod{p^2}. \tag{8.25}$$

Since $p \geq 3$, we can rewrite (8.25) as

$$x^2 + 3y^2 \equiv 8pk \pmod{p^2}, \tag{8.26}$$

for some $x, y \in \mathbb{Z}$. From (8.26), we obtain

$$x^2 + 3y^2 \equiv 0 \pmod p. \tag{8.27}$$

We notice from (8.27) that $p \mid x$ if and only if $p \mid y$. Now, if $p \mid x$ and $p \mid y$, it follows from (8.26) that $p \mid k$. If $p \nmid x$ and $p \nmid y$, then from (8.27) we find that

$$\frac{x^2}{y^2} \equiv -3 \pmod{p},$$

which is true if -3 is a quadratic residue modulo p. Therefore, x and y in (8.26) exist only if either $p \mid k$ or -3 is a quadratic residue modulo p. In other words, no such x and y exist if -3 is a quadratic nonresidue modulo p and $1 \le k \le p-1$. Also, -3 is a quadratic nonresidue modulo p precisely for the primes $p \equiv -1 \pmod 6$. Hence, employing above analysis in (8.24), we conclude that for all $n \ge 0$,

$$\mathrm{PDO}_t\left(6\left(p^2n + pk + \frac{p^2-1}{2}\right) + 3\right) \equiv 0 \pmod 8,$$

for any prime $p \equiv -1 \pmod 6$ and $1 \le k \le p-1$. This completes the proof of (8.19).

Next, we prove (8.20). From (8.47), we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n)q^n \equiv 16qf_2^7 f_6(f_1f_3) \pmod{32}. \tag{8.28}$$

Substituting (2.4) in (8.28), we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n)q^n \equiv 16\left(qf_2^{12} + q^2 f_2^6 f_6^6\right) \pmod{32}. \tag{8.29}$$

Extracting the terms with even powers of q on each side of (8.29), and substituting q^2 by q yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(12n)q^n \equiv 16qf_2^3 f_6^3 \pmod{32}. \tag{8.30}$$

Extracting the terms with odd powers of q on each side of (8.30), and dividing both sides by q and then substituting q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(24n+12)q^n \equiv 16qf_1^3f_3^3 \pmod{32}.$$

From here, the proof of (8.20) goes on the similar steps as shown in the proof of (8.19). This completes the proof of Theorem 8.5. ■

We obtain the following corollary to Theorem 8.5.

Corollary 8.5.1. *For any prime* $p \equiv -1 \pmod 6$, *and* $n \geq 0$, $\ell \geq 1$

$$\mathrm{PDO}_t\left(3^\ell(6p^2n+6kp+3p^2)\right) \equiv 0 \pmod 8, \tag{8.31}$$

$$\mathrm{PDO}_t\left(3^\ell(24p^2n+24kp+12p^2)\right) \equiv 0 \pmod{32}, \tag{8.32}$$

where $k = 1, 2, \ldots, p-1$.

Proof. Substituting the value of f_2^3 obtained from (2.9) in (8.22), we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(3n)q^n \equiv 4qf_6^4 - 12q^3f_6^3f_{18}^3 \pmod 8. \tag{8.33}$$

Extracting those terms of the form q^{3n} on both sides of (8.33) and replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(9n)q^n \equiv 4qf_2^3f_6^3 \pmod 8. \tag{8.34}$$

From (8.22) and (8.34), we conclude that for all $n \geq 0$,

$$\mathrm{PDO}_t(3n) \equiv \mathrm{PDO}_t(9n) \pmod 8,$$

which by induction implies that for all $\ell \geq 1$ and $n \geq 0$,

$$\mathrm{PDO}_t(3n) \equiv \mathrm{PDO}_t(3^\ell n) \pmod{8}. \tag{8.35}$$

Then, (8.31) follows from (8.19) and (8.35).

Next, we prove (8.32). Substituting the value of f_2^3 obtained from (2.9) in (8.30), and then extracting those terms of the form q^{3n} on both sides, we obtain

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^{3n} \equiv 16q^3 f_6^3 f_{18}^3 \pmod{32}. \tag{8.36}$$

Replacing q^3 by q in (8.36) yields

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(36n)q^{n} \equiv 16q f_2^3 f_6^3 \pmod{32}. \tag{8.37}$$

From (8.30) and (8.37), we conclude that for all $n \geq 0$,

$$\mathrm{PDO}_t(12n) \equiv \mathrm{PDO}_t(36n) \pmod{32},$$

which by induction implies that for all $\ell \geq 1$ and $n \geq 0$,

$$\mathrm{PDO}_t(12n) \equiv \mathrm{PDO}_t(3^\ell \cdot 12n) \pmod{32}. \tag{8.38}$$

Then, (8.32) follows from (8.20) and (8.38). This completes the proof of the corollary. ■

Next, using Radu's approach (explained in Section 2.4), we derive several congruences for $\mathrm{PDO}_t(n)$ modulo different powers of 2.

Theorem 8.6. *For all $n \geq 0$, we have*

$$\mathrm{PDO}_t(6n) \equiv 0 \pmod{8}, \tag{8.39}$$

$$\mathrm{PDO}_t(12n) \equiv 0 \pmod{16}, \tag{8.40}$$

$$\mathrm{PDO}_t(24n) \equiv 0 \pmod{32}, \tag{8.41}$$

$$\mathrm{PDO}_t(48n) \equiv 0 \pmod{64}, \tag{8.42}$$

$$\mathrm{PDO}_t(96n) \equiv 0 \pmod{128}, \tag{8.43}$$

$$\mathrm{PDO}_t(192n) \equiv 0 \pmod{256}. \tag{8.44}$$

Proof. Define

$$\sum_{n=0}^{\infty} a(n)q^n := \frac{f_2 f_3^2 f_{12}^2}{f_1^2 f_6}. \tag{8.45}$$

From (8.1) and (8.45), we find that, for all $n \geq 0$,

$$\mathrm{PDO}_t(n+1) = a(n). \tag{8.46}$$

For all the values of m and t listed in Table 8.2 and $M = 12$, $N = 12$, and $r = (-2, 1, 2, 0, -1, 2)$, we verify that $(m, M, N, r, t) \in \Delta^*$. By Lemma 2.14, we know that the set $\left\{ \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} : \delta | 12 \right\}$ forms a complete set of double coset representatives of $\Gamma_0(12)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$. Let $\gamma_\delta = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$. For every triple (m, t, r') from Table 8.2, we use `SageMath` to verify that $p_{m,r}(\gamma_\delta) + p^*_{r'}(\gamma_\delta) \geq 0$ for each $\delta \mid 12$. We find that for each case the set $P_{m,r}(t) = \{t\}$. We then compute the upper bound $\lfloor \nu \rfloor$ in each case, as in Lemma 2.13. Using `SageMath`, we verify that $a(mn + t) \equiv 0 \pmod{u}$ for all m, t, u in Table 8.2 and for $n \leq \lfloor \nu \rfloor$. By Lemma 2.13, we derive that $a(mn + t) \equiv 0 \pmod{u}$ for all $n \geq 0$, and for all m, t, u mentioned in Table 8.2. Then, from (8.46), we conclude that for all $n \geq 0$ and m, t, u in Table 8.2,

$$\mathrm{PDO}_t(mn + t + 1) \equiv 0 \pmod{u}.$$

This proves all the congruences (8.39)-(8.44). ■

m	t	r'	$\lfloor \nu \rfloor$	u
6	5	$(5,0,0,0,0,0)$	6	8
12	11	$(10,0,0,0,0,0)$	10	16
24	23	$(20,0,0,0,0,0)$	20	32
48	47	$(40,0,0,0,0,0)$	39	64
96	95	$(80,0,0,0,0,0)$	77	128
192	191	$(160,0,0,0,0,0)$	154	256

Table 8.2: Values of m, t, r', $\lfloor \nu \rfloor$, and u

We remark that the congruences (8.39) and (8.40) can also be derived from [74, p. 14]

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(6n)q^n = 16q\frac{f_2^4 f_3^3 f_4^4}{f_1^9}. \tag{8.47}$$

Our main motive of collectively writing and proving congruences (8.39)-(8.44) is to look for the pattern for $\mathrm{PDO}_t(n)$ modulo powers of 2 and check the existence of congruences modulo powers of 2 similar to (8.2) and (8.3). We conjecture some congruences modulo arbitrary powers of 2 based on the congruences obtained in Theorem 8.6 and our investigation on `SageMath` for next values of k and calculable values of n.

Conjecture 8.7. *For $k, n \geq 0$,*

$$\mathrm{PDO}_t\left(3 \cdot 2^k n\right) \equiv 0 \pmod{2^{k+2}}.$$

8.6 Another approach for Lin's conjecture

In this section, we present another approach to prove the congruences in Conjecture 8.1. We study

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t\left(8 \cdot 3^k n\right) q^n \tag{8.48}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t\left(12 \cdot 3^k n\right) q^n \tag{8.49}$$

modulo 3^{k+3}. From [74, p. 16], we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(8n)\, q^n \equiv 2^2 \cdot 3^2 q f_1^2 f_2^2 f_3^2 f_6^2 \pmod{3^3} \tag{8.50}$$

and from [74, eq. (5.12)], we have

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(12n)\, q^n \equiv 3^2 q f_6^4 \pmod{3^3}. \tag{8.51}$$

From (8.50) and (8.51), we obtain (8.2) and (8.3), respectively, for $k = 0$. Using basic q-series techniques and 2- and 3-dissections of certain q-products, we prove the following theorem.

Theorem 8.8. *We have:*

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(8 \cdot 3n)\, q^n \equiv 2^3 \cdot 3^3 q f_1^2 f_2^2 f_3^2 f_6^2 \pmod{3^4}, \tag{8.52}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(12 \cdot 3n)\, q^n \equiv 2^5 \cdot 3^3 q f_6^4 \pmod{3^4}. \tag{8.53}$$

With $k = 1$, (8.2) and (8.3) readily follow from Theorem 8.8.

Proof of Theorem 8.8. From [74, p. 16], we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PDO}_t(8n)q^n &\equiv 2^2\cdot 3^2 q\frac{f_2^8f_3^7}{f_1^{13}}\\ &\equiv 2^2\cdot 3^2 q\frac{f_2^9f_3^7}{f_1^{18}}\left(\frac{f_1^5}{f_2}\right)\\ &\equiv 2^2\cdot 3^2 q f_6^3 f_3\left(f_1^3\right)\left(\frac{f_1^2}{f_2}\right) \pmod{3^4}. \end{aligned} \tag{8.54}$$

Invoking (2.9) and (2.10) in (8.54), and then extracting those terms of the form q^{3n} from both sides, we obtain

$$\sum_{n=0}^{\infty}\mathrm{PDO}_t(8\cdot 3n)q^{3n} \equiv 2^3\cdot 3^3 q^3 f_3^2 f_6^2 f_9^2 f_{18}^2 \pmod{3^4}. \tag{8.55}$$

Replacing q^3 by q in (8.55) yields (8.52).

From (8.47), we have

$$\sum_{n=0}^{\infty}\mathrm{PDO}_t(6n)q^n = 2^4 q f_2^4 f_4^4\left(\frac{f_3}{f_1^3}\right)^3. \tag{8.56}$$

Substituting (2.5) in (8.56), and extracting those terms of the form q^{2n} from both sides, we arrive at

$$\sum_{n=0}^{\infty}\mathrm{PDO}_t(12n)q^{2n} = 2^4\cdot 3^3 q^4\frac{f_4^{10}f_6^3f_{12}^6}{f_2^{17}} + 2^4\cdot 3^2 q^2\frac{f_4^{18}f_6^7}{f_2^{21}f_{12}^2}. \tag{8.57}$$

Substituting q^2 by q in (8.57) and taking modulo 3^4, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PDO}_t(12n)q^n &\equiv 2^4\cdot 3^3 q^2\frac{f_2^9f_3^3f_6^6}{f_1^{15}}\left(\frac{f_2}{f_1^2}\right) + 2^4\cdot 3^2 q\frac{f_2^{18}f_3^7}{f_1^{18}f_6^2}\left(\frac{1}{f_1^3}\right)\\ &\equiv 2^4\cdot 3^3 q^2\frac{f_6^9}{f_3^2}\left(\frac{f_2}{f_1^2}\right) + 2^4\cdot 3^2 q f_3 f_6^4\left(\frac{1}{f_1^3}\right) \pmod{3^4}. \end{aligned} \tag{8.58}$$

Using (2.11) and (2.13) in (8.58), and extracting terms of the form q^{3n}, we arrive at

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(12\cdot 3n)q^{3n} \equiv 2^5\cdot 3^3 q^3 f_6^{12} + 2^4\cdot 3^4 q^3 \frac{f_6^4 f_9^9}{f_3^{11}} \pmod{3^4}. \tag{8.59}$$

Finally, substituting q^3 by q in (8.59) yields (8.53). ■

For $k\geq 2$, the technique we use to prove Theorem 8.8 becomes tedious to analyze (8.48) and (8.49) modulo 3^{k+3}. Here, we use the theory of modular forms for the case $k=2$. In particular, we have the following theorem.

Theorem 8.9. *We have:*

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t\left(8\cdot 3^2 n\right) q^n \equiv 2^4\cdot 3^4 q f_1^2 f_2^2 f_3^2 f_6^2 \pmod{3^5}, \tag{8.60}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t\left(12\cdot 3^2 n\right) q^n \equiv 3^4 q f_6^4 \pmod{3^5}. \tag{8.61}$$

For $k=2$, Conjecture 8.1 follows from Theorem 8.9.

Proof of Theorem 8.9. From [74, p. 16], we have

$$F_1(z) := \sum_{n=0}^{\infty} \mathrm{PDO}_t(8n)q^n = 36q\frac{f_2^8 f_3^7}{f_1^{13}}.$$

For $k\geq 0$, we define

$$A_{k,1}(z) := 36\frac{\eta^{3^{k+3}-13}(z)\eta^8(2z)}{\eta^{3^{k+2}-7}(3z)}$$

and

$$B_{k,1}(z) := 2^{k+2}\cdot 3^{k+2}\frac{\eta^{3^{k+3}-13}(z)\eta^8(2z)}{\eta^{3^{k+2}-7}(3z)}.$$

For $k \geq 0$, we have

$$A_{k,1}(z) \equiv F_1(z) \pmod{3^{k+3}}$$

and

$$B_{k,1}(z) \equiv 2^{k+2} \cdot 3^{k+2} q f_1^2 f_2^2 f_3^2 f_6^2 \pmod{3^{k+3}}. \tag{8.62}$$

We prove that both $A_{k,1}(z)$ and $B_{k,1}(z)$ are modular forms of level 18 and weight $3^{k+2}+1$ with trivial character χ_0 modulo 18. By Theorem 2.5, $A_{k,1}(z)$ and $B_{k,1}(z)$ are eta-quotients of level $N = 18$. The cusps of $\Gamma_0(18)$ are represented by $\frac{c}{d}$, where $d \mid 18$ and $\gcd(c,d) = 1$ (Proposition 2.4). By Theorem 2.6, $A_{k,1}(z)$ is holomorphic at the cusp $\frac{c}{d}$ if and only if

$$(3^{k+3} - 13) + \frac{\gcd(d,2)^2}{2}(8) - \frac{\gcd(d,3)^2}{3}(3^{k+2} - 7) \geq 0,$$

which is true since

$$(3^{k+3} - 13) - \frac{\gcd(d,3)^2}{3}(3^{k+2} - 7) \geq 0,$$

for every $d \mid 18$. Similarly, $B_{k,1}(z)$ is holomorphic at every cusp. Thus, by Theorems 2.5 and 2.6, we find that $A_{k,1}(z), B_{k,1}(z) \in M_{3^{k+2}+1}(\Gamma_0(18), \chi_0)$.

Now, we define

$$F_{k,1}(z) := A_{k,1}(z) \mid U^k(3),$$

where U^k means applying U-operator k times. Since 3 divides the level of the modular form $A_{k,1}(z)$, by Proposition 2.8, $F_{k,1}(z) \in M_{3^{k+2}+1}(\Gamma_0(18), \chi_0)$. By Theorem 2.11, the Sturm bound for $M_{3^{k+2}+1}(\Gamma_0(18), \chi_0)$ is $3(3^{k+2} + 1)$.

We have

$$\begin{aligned} F_{k,1}(z) &= A_{k,1}(z) \mid U^k(3) \\ &\equiv F_1(z) \mid U^k(3) \\ &\equiv \sum_{n=0}^{\infty} \mathrm{PDO}_t(8 \cdot 3^k n) q^n \pmod{3^{k+3}}. \end{aligned} \tag{8.63}$$

From (8.62) and (8.63), we find that

$$F_{k,1}(z) \equiv B_{k,1}(z) \pmod{3^{k+3}} \tag{8.64}$$

if and only if

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(8 \cdot 3^k n) q^n \equiv 2^{k+2} \cdot 3^{k+2} q f_1^2 f_2^2 f_3^2 f_6^2 \pmod{3^{k+3}}. \tag{8.65}$$

Therefore, to prove the congruence (8.60), it is enough to establish (8.64) for $k = 2$. In view of Theorem 2.11, we verify (8.64) for $k = 2$ up to the Sturm bound using `SageMath`, and this completes the proof of (8.60).

Next, we prove (8.61). From [74, eq. (5.4)], we have

$$F_2(z) := \sum_{n=0}^{\infty} \mathrm{PDO}_t(4n) q^n = 6q \frac{f_2^3 f_3^2 f_6^3}{f_1^6}.$$

For $k \geq 0$, we define

$$A_{k,2}(z) := 6 \frac{\eta^{3^{k+2}-6}(z) \eta^3(2z) \eta^3(6z)}{\eta^{3^{k+1}-2}(3z)}$$

and

$$B_{k,2}(z) := 2^{\beta_k} \cdot 3^{k+1} \frac{\eta^{3^{k+2}-6}(z) \eta^3(2z) \eta^3(6z)}{\eta^{3^{k+1}-2}(3z)},$$

where $\beta_k := 2k+1$, for even k and $\beta_k := 0$, for odd k. For all $k \geq 0$, we have

$$A_{k,2}(z) \equiv F_2(z) \pmod{3^{k+2}},$$

and

$$B_{k,2}(z) \equiv 2^{\beta_k} \cdot 3^{k+1} q f_6^4 \pmod{3^{k+2}}. \tag{8.66}$$

We prove that both $A_{k,2}(z)$ and $B_{k,2}(z)$ are modular forms of level 36 and weight $3^{k+1}+1$ with trivial character χ_0 modulo 36. By Theorem 2.5, $A_{k,2}(z)$ and $B_{k,2}(z)$ are eta-quotients of level $N = 36$. The cusps of $\Gamma_0(36)$ are represented by $\frac{c}{d}$, where $d \mid 36$ and $\gcd(c,d) = 1$. By Theorem 2.6, $A_{k,2}(z)$ is holomorphic at the cusp $\frac{c}{d}$ if and only if

$$(3^{k+2}-6) + \frac{\gcd(d,2)^2}{2}(3) - \frac{\gcd(d,3)^2}{3}(3^{k+1}-2) + \frac{\gcd(d,6)^2}{6}(3) \geq 0,$$

which is true since

$$(3^{k+2}-6) - \frac{\gcd(d,3)^2}{3}(3^{k+1}-2) \geq 0,$$

for every $d \mid 36$. Similarly, $B_{k,2}(z)$ is holomorphic at every cusp. Thus, by Theorems 2.5 and 2.6, we find that $A_{k,2}(z), B_{k,2}(z) \in M_{3^{k+1}+1}(\Gamma_0(36), \chi_0)$.

Now, we define

$$F_{k,2}(z) := A_{k,2}(z) \mid U^k(3).$$

Since 3 divides the level of the modular form $A_{k,2}(z)$, by Proposition 2.8, $F_{k,2}(z) \in M_{3^{k+1}+1}(\Gamma_0(36), \chi_0)$. By Theorem 2.11, the Sturm bound for $M_{3^{k+1}+1}(\Gamma_0(36), \chi_0)$ is $6(3^{k+1}+1)$.

We have

$$\begin{aligned} F_{k,2}(z) &= A_{k,2}(z) \mid U^k(3) \\ &\equiv F_2(z) \mid U^k(3) \\ &\equiv \sum_{n=0}^{\infty} \mathrm{PDO}_t(4\cdot 3^k n) q^n \pmod{3^{k+2}}. \end{aligned} \tag{8.67}$$

From (8.66) and (8.67), we find that

$$F_{k,2}(z) \equiv B_{k,2}(z) \pmod{3^{k+2}} \tag{8.68}$$

if and only if

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t(4\cdot 3^k n) q^n \equiv 2^{\beta_k}\cdot 3^{k+1} q f_6^4 \pmod{3^{k+2}}. \tag{8.69}$$

Therefore, to prove the congruence (8.61), it is enough to establish (8.68) for $k=3$. In view of Theorem 2.11, we verify (8.68) for $k=3$ up to the Sturm bound using `SageMath`, and this completes the proof of (8.61). ■

Remark 8.6.1. *We can establish (8.50)-(8.53) by using smaller values of k in the above proof. Also, for higher values of k verification of (8.65) and (8.69) becomes difficult because of the large Sturm bound. We verify (8.65) and (8.69) for a few more values of k up to the computable amount and propose Conjecture 8.10.*

Observing (8.50)-(8.61) and analyzing (8.48) and (8.49) for more values of k on `SageMath`, we propose the following conjecture.

Conjecture 8.10. *For $k \geq 0$, we have*

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t\left(8\cdot 3^k n\right) q^n \equiv 2^{k+2}\cdot 3^{k+2} q f_1^2 f_2^2 f_3^2 f_6^2 \pmod{3^{k+3}}, \tag{8.70}$$

$$\sum_{n=0}^{\infty} \mathrm{PDO}_t\left(12\cdot 3^k n\right) q^n \equiv 2^{\alpha_k}\cdot 3^{k+2} q f_6^4 \pmod{3^{k+3}}, \tag{8.71}$$

where $\alpha_k := 2k+3$, for odd k and $\alpha_k := 0$, for even k.

In comparison to Conjecture 8.1, Conjecture 8.10 is more general. Also, a proof of Conjecture 8.10 will immediately settle Conjecture 8.1.

Theorem 8.11. *Conjecture 8.10 implies Conjecture 8.1.*

Proof. We prove the theorem by induction on k. Suppose (8.70) is true for all $k \geq 0$. Let (8.2) hold for some k. Then, from (8.70), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO}_t\left(8\cdot 3^{k+1}n\right)q^n &\equiv 2^{k+3}\cdot 3^{k+3} q f_1^2 f_2^2 f_3^2 f_6^2 \\ &\equiv 2\cdot 3\sum_{n=0}^{\infty}\mathrm{PDO}_t\left(8\cdot 3^k n\right)q^n \pmod{3^{k+4}}. \qquad (8.72)\end{aligned}$$

By our induction hypothesis, the right-hand side of (8.72) is divisible by 3^{k+3}. Therefore,

$$\mathrm{PDO}_t(8\cdot 3^{k+1}n) \equiv 0 \pmod{3^{k+3}}.$$

Hence, (8.2) is true for $k+1$. If (8.71) is true, (8.3) follows in a similar way. This completes the proof. ∎

Remark 8.6.2. *By Conjecture 8.10, we also understand the co-existence of congruences (8.2) and (8.3), since from (8.70) and (8.71), we have*

$$2^{\alpha_k} f_2^4\cdot\sum_{n=0}^{\infty}\mathrm{PDO}_t\left(8\cdot 3^k n\right)q^n \equiv 2^{k+2} f_1^8\cdot\sum_{n=0}^{\infty}\mathrm{PDO}_t\left(12\cdot 3^k n\right)q^n \pmod{3^{k+2}}. \qquad (8.73)$$

It is clear from (8.73) that if Conjecture 8.10 is true then (8.2) holds if and only if (8.3) holds.

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Publications

Publications from Thesis work

1. A. Singh, G. Singh, and R. Barman, *Certain Diophantine equations and new parity results for 21-regular partitions*, Acta Arith. 210 (2023), 337–351.

2. R. Barman, G. Singh, and A. Singh, *Divisibility of the partition function* $\mathrm{PDO}_t(n)$ *by powers of 2 and 3*, Bull. Aust. Math. Soc. 109 (1) (2024), 14–25.

3. R. Barman, A. Singh, and G. Singh, *Arithmetic properties of certain* t*-regular partitions*, Ann. Comb. 28 (2) (2024), 439–457.

4. R. Barman and G. Singh, *Arithmetic properties and asymptotic formulae for* $\sigma_o\mathrm{mex}(n)$ *and* $\sigma_e\mathrm{mex}(n)$, Ramanujan J., 65 (1) (2024), 125–137.

5. G. Singh and R. Barman, *Hook length biases in ordinary and* t*-regular partitions*, J. Number Theory, 264 (2024) 41–58.

6. G. Singh and R. Barman, *A note on Lin's conjecture on* $\mathrm{PDO}_t(n)$, Ramanujan J. 67, 79 (2025). https://doi.org/10.1007/s11139-025-01116-4

7. G. Singh and R. Barman, *Hook length inequalities for* t*-regular partitions in the* t*-aspect*, Canad. Math. Bull. (accepted).