INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
SHORT ABSTRACT OF THESIS

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SHORT ABSTRACT

This thesis is mainly devoted to the computation of the number of cliques of certain Cayley graphs, namely the *Paley-type graphs, Peisert graphs* and *Peisert-like graphs*. Barring the case of the Peisert graphs, the focus is on the number of cliques of orders three (triangles) and four. Let q be a prime power such that $q \equiv 1 \pmod 4$. The Paley graph of order q is the graph with vertex set as the finite field $\mathbb{F}_q$ and edges defined as, ab is an edge if and only if $a - b$ is a non-zero square in $\mathbb{F}_q$. The first part of this thesis involves defining a generalization of the Paley graph, called the Paley-type graph on the commutative ring $\mathbb{Z}_n$ for certain values of n, precisely $n = 2^s p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $s = 0$ or 1, $\alpha_i \geq 1$, where the distinct primes p_i satisfy $p_i \equiv 1 \pmod 4$ for all $i = 1, \ldots, k$. For such n, we define the graph with vertex set $\mathbb{Z}_n$ and edges defined as, ab is an edge if and only if $a - b$ is a square in the set of units of $\mathbb{Z}_n$. We look at some properties of this graph. For primes $p \equiv 1 \pmod 4$, Evans, Pulham and Sheehan computed the number of complete subgraphs of order four in the Paley graph. Recently, Dawsey and McCarthy found the number of triangles and complete subgraphs of order four in the generalized Paley graph of prime power order. We find the number of triangles and complete subgraphs of order four in the Paley-type graph successively for $n = p^\alpha$ ($p \equiv 1 \pmod 4$ being a prime and $\alpha \geq 1$) and for general n, using character sums and combinatorial methods.

A graph is called symmetric if its automorphism group acts transitively both on the vertices and edges. Another kind of symmetry occurs if a graph is isomorphic to its complement, in which case the graph is called self-complementary. It turns out that the Paley graphs are both self-complementary and symmetric (SCS). Peisert gave a full description of SCS graphs as well as their automorphism groups. He derived that there is another infinite family of SCS graphs apart from the Paley graphs, and in addition, one more graph not belonging to any of the two former families. The graphs in the infinite family that he discovered are now known as Peisert graphs. The second part of the thesis is devoted to the computation of the number of cliques in the Peisert graph. We find the number of triangles and cliques of order four and express them using finite field hypergeometric functions as developed by Greene, McCarthy and Ono, which are assembled with the well known Gauss and Jacobi sums. Then, we provide an asymptotic result on the number of cliques of order $m \geq 1$ in the graph.

The final part of the thesis involves defining a Peisert-like graph analogous to the Peisert graph and exploring some of its properties. We follow in the footsteps of Greene and define hypergeometric functions corresponding to Dirichlet characters. Then, using these functions, we find the number of triangles and cliques of order four in the Peisert-like graph.