

Quantum Vortex Lattices and Collective Modes of Rotating Bose- Einstein Condensates with Synthetic Gauge potentials

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by
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under the supervision of
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March, 2026



Declaration



This is to certify that the work contained in this thesis entitled “**Quantum Vortex Lattices and Collective Modes of Rotating Bose- Einstein Condensates with Synthetic Gauge potentials**” submitted by me to the Indian Institute of Technology, Guwahati, for the award of the degree of PhD, is a bonafide work carried out by me under the supervision of Dr. Pankaj Kumar Mishra and that this work has not been submitted elsewhere for a degree of diploma. I also wish to state that to the best of my knowledge and understanding nothing in this report amounts to plagiarism.

June 17, 2026

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Certificate



It is certified that the work contained in this thesis entitled “**Quantum Vortex Lattices and Collective Modes of Rotating Bose- Einstein Condensates with Synthetic Gauge potentials**” by **Rony Borai**, has been carried out under my supervision, and this work has not been submitted elsewhere for a PhD. degree.

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Preface



The realization of Bose–Einstein condensation in dilute atomic gases has provided an exceptional platform for exploring fundamental aspects of quantum many-body physics in a highly controllable environment. One of the most remarkable manifestations of superfluidity in these systems is the formation of quantized vortices when the condensate is subjected to rotation. At sufficiently high rotation frequencies, these vortices organize themselves into ordered lattice structures, typically forming the well-known Abrikosov lattice. The study of vortex lattices in Bose–Einstein condensates (BECs) has therefore become an important avenue for understanding the interplay between rotation, interactions, and external potentials in quantum fluids.

Recent experimental and theoretical developments have significantly expanded the scope of research in this field. Techniques such as optical lattices, disorder potentials, and synthetic gauge fields have enabled unprecedented control over ultracold atomic systems. Optical lattice potentials can pin vortices and induce structural transformations of vortex lattices, while disorder in the form of impurities can destabilize ordered configurations and lead to vortex-lattice melting and turbulent dynamics. In parallel, the engineering of synthetic gauge potentials has opened new possibilities for simulating electromagnetic effects in neutral atomic gases, leading to novel rotational phenomena and modified condensate dynamics. These developments provide a rich framework for investigating vortex structures, their stability, and their collective excitation spectra.

Another important aspect of rotating condensates is the study of their collective excitations, which provide valuable insights into the dynamical stability and physical properties of the system. The Bogoliubov–de Gennes (BdG) formalism serves as

a powerful theoretical tool for analyzing these excitations. In particular, the presence of density-dependent synthetic gauge potentials can introduce nonlinear rotational effects, significantly altering both the equilibrium structure of the condensate and the corresponding excitation spectrum.

Motivated by these developments, this thesis investigates the structural transformations and collective excitation dynamics of vortex lattices in rotating Bose–Einstein condensates under the influence of optical lattice potentials, impurities, differential rotation, and density-dependent synthetic gauge potentials. The primary objective is to understand how these external perturbations modify vortex-lattice configurations, induce lattice melting and turbulence, and influence the excitation spectrum of the condensate.

The first part of this work focuses on the structural transformation of vortex lattices in the presence of a square optical lattice potential and differential rotation, where the lattice and the condensate rotate at different angular velocities. Using quantitative measures such as the structure factor and lattice energy, we analyze the transition from the conventional Abrikosov lattice to a square vortex lattice through an intermediate coexistence phase. In certain parameter regimes, the ordered lattice structure undergoes melting, leading to irregular vortex motion accompanied by turbulence-like behavior in the kinetic energy spectra.

The second part of the thesis examines the influence of impurities on vortex-lattice stability and turbulence in rotating condensates. Both static random impurities and dynamically oscillating obstacles are considered. Our results show that sufficiently strong impurities can destroy the long-range order of the vortex lattice, driving the system into a melted regime characterized by turbulent vortex dynamics. The energy spectra in this regime display characteristic power-law scaling, indicating features consistent with quantum turbulence.

The final part of the thesis investigates the effects of density-dependent synthetic gauge potentials on the structure and collective excitation spectrum of rotating condensates. Using the Bogoliubov–de Gennes framework together with analytical and hydrodynamic approaches, we analyze how nonlinear rotation induced by such gauge potentials modifies the excitation modes of the system. The results demonstrate phenomena such as the violation of Kohn’s theorem, significant changes in vortex-displacement modes, and modifications of surface excitation frequencies. In addition, the influence of nonlinear rotation on a toroidally trapped condensate is studied, where a structural transition from a ring-shaped condensate to a giant vortex state is observed.

The investigations presented in this thesis are primarily based on numerical sim-

ulations of the Gross–Pitaevskii equation using spectral and time-splitting methods, which allow accurate determination of both the ground-state properties and the dynamical behavior of the condensate.

Overall, the work presented here contributes to a deeper understanding of vortex-lattice physics and collective excitations in rotating Bose–Einstein condensates, particularly in the presence of structured potentials, disorder, and synthetic gauge fields. The thesis is organized into seven chapters. Chapter 1 introduces the theoretical background and reviews relevant literature. Chapter 2 presents the numerical methods used for solving the Gross–Pitaevskii equation. Chapters 3 and 4 investigate vortex-lattice transformations and turbulence induced by optical lattice potentials and impurities. Chapters 5 and 6 focus on the effects of density-dependent gauge potentials on vortex-lattice structures and collective excitations in ring and Toroidal geometry, respectively. Finally, Chapter 7 summarizes the main findings of the thesis and outlines possible directions for future research.





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To my family ...





Publications

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Introduction

Vortices are the fundamental aspects of both classical and quantum fluids, which emerge as an inherent response to rotational motion. Vortices are characterized by a effective hole in the fluid density around which the fluid circulates. In classical fluids, vortices may form at arbitrary rotation , and the circulation can vary continuously. In contrast, in quantum fluids, the density vanishes at the core, and the formation of vortices is subject to a critical rotation threshold, with the circulation being quantized. Quantum vortices therefore represent fundamental excitations of quantum fluids and have attracted sustained attention over recent decades and now become a central theme in the study of quantum fluids [1, 2]. This area of research encompasses a wide range of systems [3, 4], including superfluid helium [5], liquid crystals [6], quantum magnets [7], and other topologically quantum materials [8]. Although quantum vortices were first observed in superfluid helium and superconductors, studying vortex physics in strongly interacting superfluid like He II is challenging because of complex interparticle interactions and limited tunability. On the other hand, the experimental realization of Bose–Einstein condensates (BECs) [9–11], in which below a critical temperature the atoms condense into the lowest possible energy state and form a macroscopic quantum state has significantly alleviated these challenges by providing precise control over the physical parameters that govern system dynamics. Key properties such as the confining potential [12–14], dimensionality [15], particle density, and even the interatomic interactions can be precisely tuned [16]. This capability enables the systematic exploration of a variety of phenomena, including vortex nucleation, vortex lattice formation, and elementary excitations [17].

The highly tunable and controllable nature of atomic BECs has established them as a prominent platform for quantum simulation. Building on Feynman’s suggestion,

the objective is to use the universal nature of quantum mechanics to construct controllable systems that mimic the physics of problems that are experimentally inaccessible. In that direction, many condensed matter phenomena such as Anderson localization [18, 19], the superfluid to Mott insulator transition [20], spin-Hall effect [21, 22], and the paramagnetic to antiferromagnetic phase transition under an applied magnetic field [23] have been successfully observed in Bose gases. Another interesting aspect of BECs is their ability to generate topological defects known as quantum vortices. In experiments, these vortices are created using methods such as laser stirring [24, 25], rotating magnetic traps [26, 27], and phase imprinting [28, 29]. When a condensate undergoes rapid rotation within a harmonic trap, it forms a regular array of vortices, usually arranged in a triangular or hexagonal pattern similar to the Abrikosov lattice formed in superconductors under strong magnetic fields [17]. Quantum simulators play a crucial role in investigating physically complex systems within a simple and highly controllable environment. However, the construction of such simulators often leads to new physical phenomena that arise from the techniques used in designing and controlling the underlying system. A notable example is the development of synthetic gauge fields in ultracold atomic gases, which arise from externally applied fields or engineered forces acting on the condensate, causing neutral atoms to behave as if they were moving under the influence of magnetic or electric fields [30, 31].

In general, artificial gauge fields can be classified into two categories: static and dynamic. In the static case, the internal atomic degrees of freedom remain unaffected, and the gauge field is externally imposed. Such static gauge fields can be generated using two primary methods. The first is through physical rotation of the system, where the Coriolis force in the rotating frame plays a role analogous to the Lorentz force acting on a charged particle in a magnetic field [32–35]. The second method employs Raman [36], in which two internal atomic states are coherently coupled via light-matter interactions. In this approach, an effective vector potential emerges from the adiabatic motion of optically dressed states, giving rise to artificial magnetic fields for neutral atoms. Prominent examples of static synthetic gauge potential includes the experimental observation of vortices in rotating BECs. In such systems, the vortices are associated with an effective vector potential arising from the Coriolis force in the rotating frame, which plays a role analogous to a magnetic vector potential for charged particles [32–35]. Synthetic gauge potential in ultracold gases are also connected to phenomena such as orbital magnetism [37–39] and engineered spin-orbit [40] and spin-angular-momentum couplings [41–43] in bosonic systems. These concepts are particularly significant because they contribute to the emergence of topological states of matter [44, 45], which

are otherwise challenging to realize in neutral bosonic systems without the presence of artificial gauge potentials. In dynamic case, there is time-dependent feedback between light and matter, manifested as a density-dependent gauge potential. Several theoretical studies have been carried out in this context for both continuum [46] and lattice based systems [47–49]. These dynamic gauge fields lead to a range of unusual transport phenomena trapped in double well [50] systems and harmonic traps [51, 52], as well as the violation of Kohn’s theorem [53]. Beyond theoretical investigations, density-dependent magnetism has also been demonstrated experimentally, both for bosons confined in two-dimensional optical lattices [54] and for ensembles of Rydberg atoms [55].

Within this broader framework of synthetic gauge potential, significant attention has also been devoted to the structural transformation of vortex lattices, where the static gauge potential is generated due to the rotation of the condensate [32–35]. Several studies have demonstrated transitions from triangular vortex lattices to square and other lattice geometries, with the resulting structures determined primarily by the geometry of the underlying optical lattice [56–59]. Furthermore, a variety of techniques have been used to investigate melting phenomena in vortex lattice configurations. These include anisotropic expansion [60], quantum tunneling [61], and the introduction of controlled disorder potentials [62]. More recently, nontrivial vortex configurations beyond the regular triangular lattice have been reported in the condensate confined within a harmonic trap and subjected to a density-dependent dynamic gauge potential, where the resulting vortex patterns are governed by the strength of the nonlinear rotation induced by the density-dependent gauge potential [63].

Although numerous studies have been conducted on structural transformations and melting phenomena, some important aspects remain unexplored. In particular, the effects of both static and dynamic impurities on melting have not been thoroughly investigated. Moreover, the nature of quantum turbulence in the melted regime remains largely unexplored. In addition, while structural transformations have been reported in systems with dynamic gauge potentials, a detailed study of the collective excitation spectrum and vortex displacement modes is still lacking. Finally, the role of density dependent gauge potentials in annular geometries has yet to be addressed. In this thesis, we aim to address the aforementioned gaps by utilizing the mean-field Gross-Pitaevskii formalism.

This chapter is organized to introduce the fundamental concepts and theoretical framework necessary for understanding vortex dynamics in Bose-Einstein condensates. It begins with an overview of Bose-Einstein condensation, including its ex-

perimental realization. The mean-field description of weakly interacting condensates is then discussed through the Gross–Pitaevskii Equation, followed by extensions to rotating systems. Subsequent subsections describe the Dissipative Gross–Pitaevskii Equation, dimensional reduction leading to quasi-two-dimensional condensates, the Thomas–Fermi approximation for the density profile, the hydrodynamic formulation of the condensate, and the elementary excitations described by Bogoliubov theory. The chapter then focuses on topological defects in condensates, particularly quantized vortices, discussing phase defects, vortex energy and angular momentum, and the motion of individual vortices. This is followed by a discussion of vortex states such as vortex rings, vortex reconnection processes, and sound wave emission from vortex dynamics. The concept of quantum turbulence is subsequently introduced, including energy decomposition and spectral analysis. Finally, the chapter addresses electromagnetic gauge potentials and the realization of synthetic gauge fields in ultracold atomic gases, covering light–matter interaction, the role of the Berry phase, theoretical methods for generating synthetic gauge potentials using two-level atomic models, and recent theoretical developments on vortex lattices and the collective dynamics of rotating Bose–Einstein condensates in synthetic gauge fields.

1.1 Bose-Einstein condensation

Bose-Einstein condensation (BEC) is a quantum state of matter that was first predicted by Satyendra Nath Bose in 1924. Bose derived Planck’s law of black-body radiation by applying statistical arguments based on the maximization of entropy for a collection of photons [64]. Shortly thereafter, Albert Einstein generalized this idea to an ideal gas of massive, indistinguishable particles by incorporating de Broglie’s concept of matter waves, thereby predicting the phenomenon now known as Bose-Einstein condensation [65, 66]. Particles that obey Bose-Einstein statistics are called bosons and are characterized by having integer spin. Examples of bosons include photons and atoms such as ^1H , ^4He , ^7Li , ^{23}Na , ^{85}Rb , and ^{87}Rb . For a gas of N noninteracting bosons of mass m confined in a volume V , the ensemble averaged occupation number of a single-particle state s with energy ε_s is given by the Bose–Einstein distribution [67],

$$\langle n_s \rangle = \frac{1}{\exp[(\varepsilon_s - \mu)/(k_B T)] - 1} \quad (1.1)$$

where μ is the chemical potential, k_B is the Boltzmann constant, and T is the temperature. According to Eq. 1.1, for the single-particle state with the lowest energy ϵ_0 and a fixed chemical potential $\mu < \epsilon_0$, the occupation number of the ground state, $\langle n_0 \rangle$, increases rapidly as the temperature T is lowered, but do not exceed the total number of particles in the system. This phenomenon, where a significant number of particles occupy the lowest-energy state, is referred to as Bose–Einstein condensation (BEC).

At high temperatures, particles can occupy a wide range of available energy states, and the total number of particles can be expressed as $N = N_0 + N_{\text{ex}}$, where N_0 denotes the number of atoms in the ground state and N_{ex} represents the number of atoms in the excited states. As the temperature is lowered, BEC occurs at a critical temperature at which, $N_{\text{ex}} = N$. For an ideal Bose gas confined in a volume V , the critical temperature is given by

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left(\frac{N}{2.612V} \right)^{2/3}, \quad (1.2)$$

At temperatures below the critical temperature T_c , the system does not condense completely into the ground state. Instead, a finite fraction of the particles continues to occupy excited energy levels. The fraction of particles in the condensate at a temperature $T < T_c$ is therefore given by

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c} \right)^3, \quad (1.3)$$

where N_0 denotes the number of particles in the ground state. The phenomenon of BEC can also be understood from the perspective of wave particle duality. In this picture, each boson is characterized by a thermal de Broglie wavelength, $\lambda_{\text{dB}} = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{1/2}$, which represents the spatial extent of the particle's quantum wave packet at temperature T . The onset of condensation is therefore determined by the condition

$$n\lambda_{\text{dB}}^3 \leq 2.612 \quad (1.4)$$

where $n = N/V$ denotes the particle number density of the gas. Thus, the condensate occurs when the above condition is satisfied. This means that for temperatures lower than the critical temperature $T < T_c$, the wave packets of individual bosons begin to overlap. The resulting condensate may be viewed as a single macroscopic quantum state often described as a giant matter wave, formed through the constructive interference of the de Broglie waves of the constituent bosons (see Fig. 1.1).

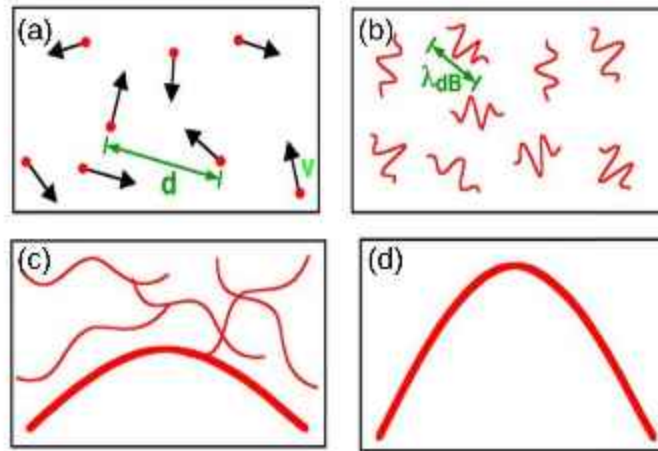


Figure 1.1: Schematic illustration of the formation of a Bose-Einstein condensate. (a) At high temperature, the atoms behave as classical particles with random thermal motion and an interparticle spacing d . (b) As the temperature decreases, the thermal de Broglie wavelength λ_{dB} increases and atoms begin to exhibit wave like behavior. (c) When the temperature approaches the critical temperature T_c , the matter waves of neighboring atoms start to overlap, signaling the onset of condensation. (d) At temperatures well below T_c , a macroscopic number of atoms occupy the same quantum ground state, forming a single coherent matter wave known as a Bose-Einstein condensate. The figure is taken from [68].

1.1.1 Experimental realization of BEC

After the theoretical prediction of BEC, the first experimental observation of such condensation phenomenon was made in superfluid helium ^4He [69, 70]. Later in 1938, London established a connection between the unusual properties of liquid helium and the concept of BEC [71]. Subsequently, Landau developed the hydrodynamic theory of superfluid helium, introducing a description in which the system is divided into superfluid and normal fluid components [72], closely resembling the earlier two fluid model proposed by Tisza [73]. Building on the ideas of Penrose [74], Onsager later emphasized the crucial role of macroscopic phase coherence in the realization of BEC [75]. A major theoretical breakthrough was then achieved by Bogoliubov [76], who demonstrated that the fundamental nature of BEC is not destroyed by the presence of weak interactions. However, he showed that interactions profoundly modify the low-lying excitation spectrum of a Bose condensed gas, giving rise to a phonon like spectrum that coincides with the form assumed by Landau. This result provided a microscopic explanation for the stability of superfluid flow. However, due to the presence of strong interatomic interactions it turns out that superfluid ^4He does not represent an ideal weakly interacting Bose gas, with only a small fraction of atoms occupying the condensed state. In the early 1970s, researchers began serious experimental efforts to re-

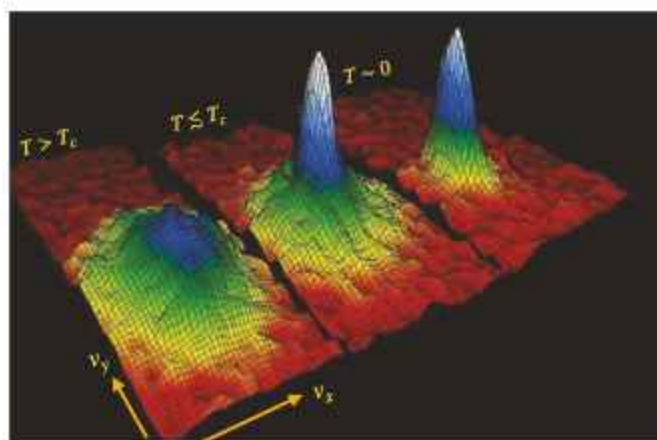


Figure 1.2: Velocity distribution of rubidium atoms after time-of-flight expansion of the atomic cloud at different temperatures (from left: $T \sim 400$ nK, middle: $T \sim 200$ nK, right: $T \sim 50$ nK). For temperatures above the critical temperature ($T > T_c$), the atoms occupy higher momentum states and the expanded cloud exhibits a broad Gaussian distribution characteristic of a thermal gas. As the temperature approaches the critical temperature ($T \sim T_c$), a narrow peak begins to emerge at $k = 0$, indicating the onset of Bose-Einstein condensation. At temperatures below the critical temperature ($T < T_c$), a pronounced central peak appears, demonstrating that a macroscopic number of atoms occupy the zero-momentum ground state, which is the hallmark of Bose-Einstein condensation, surrounded by a broader thermal cloud of atoms with random velocities. The figure is adapted from the experiment performed at JILA, Boulder [9].

alize BEC. Initial attempts focused on atomic hydrogen, which was cooled using compression and cryogenic techniques. However, further progress was hindered by significant experimental challenges, including atom losses due to sticking to the container walls and large three body recombination rates, which severely limited the achievable densities. Major advances in the cooling and trapping of neutral alkali atoms during the 1980s and early 1990s transformed the field. Techniques such as laser cooling and magnetic trapping made it possible to cool dilute atomic gases to ultralow temperatures, and alkali atoms rapidly replaced hydrogen as the most promising candidates for observing BEC in a weakly interacting system. Although several groups worldwide pursued this goal, BEC was first experimentally observed in 1995 by the group of Cornell and Wieman at JILA in Boulder, Colorado, using a gas of ^{87}Rb atoms [9]. Shortly thereafter, the group led by Ketterle at MIT reported the observation of BEC in a gas of ^{23}Na atoms [10]. Evidence for BEC was also obtained in a gas of ^7Li atoms by the group of Hulet at Rice University in Houston, Texas [77]. Eventually, BEC was realized in atomic hydrogen in 1998 by the group of Kleppner at MIT [78]. Laser cooling and trapping laid the foundation for the experimental study of ultracold quantum gases. In recognition of these accomplishments, the Nobel Prize in Physics was

awarded in 1997 to Steven Chu, Claude Cohen-Tannoudji, and William Phillips. A few years later, Eric Cornell, Carl Wieman, and Wolfgang Ketterle were honored with the Nobel Prize for their pioneering experimental observation of BEC. After the experimental observation of BEC the field has experienced remarkable growth, leading to the exploration of many fascinating quantum phenomena. These include topological defects such as quantum vortices[32, 33], quantum turbulence [2, 79], supersolidity [80], Anderson localization [81], self bound quantum droplets [82], and synthetic gauge potentials [46]. Following these experimental achievements, numerous theoretical and numerical studies have been carried out using the mean field Gross Pitaevskii equation (GPE) [17]. In the next section, we describe this framework and present the GPE that governs the condensate dynamics.

1.2 Rotating BEC

Following the first experimental realization of Bose-Einstein condensation in dilute atomic gases in 1995 [9, 10], a major direction of research has been the study of condensates under rotation. Rotation introduces angular momentum into the system and gives rise to a rich variety of quantum fluid phenomena, including quantized vortices, vortex lattices, and collective excitations, making rotating Bose-Einstein condensates (BECs) an ideal platform for exploring superfluidity in a highly controllable environment. The first experimental evidence of quantized vortices in atomic condensates was reported shortly after, where vortices were nucleated by stirring the condensate using rotating anisotropic traps [33]. These experiments demonstrated the existence of a critical rotation frequency above which vortices enter the system, confirming the superfluid nature of BECs. Soon after, the controlled creation of multiple vortices and their imaging established the fundamental properties of quantized circulation in dilute gases [17]. A major breakthrough came with the observation of highly ordered vortex lattices in rapidly rotating condensates [32]. In this experiment, vortices were found to arrange themselves into triangular Abrikosov lattices, analogous to those observed in type-II superconductors. Subsequent studies investigated the structural properties, stability, and collective dynamics of these lattices, including Tkachenko oscillations and lattice deformation modes [60, 34].

As the rotation frequency approaches the radial trapping frequency, the system enters the fast rotation regime, where centrifugal effects significantly modify the condensate structure. In this limit, the vortex number increases and the system can be driven

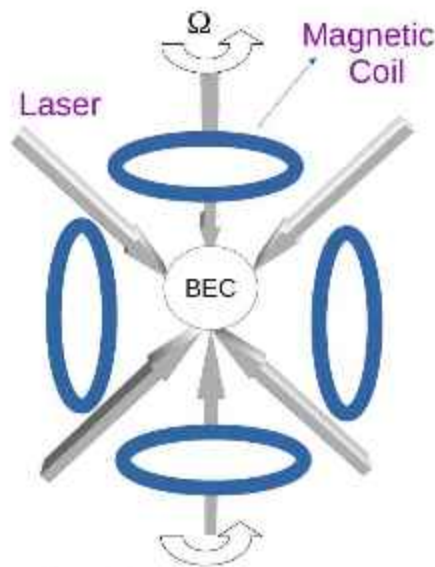


Figure 1.3: Schematic representation of a rotating BEC confined in a magnetic trap. The condensate forms at the center of the trap, where intersecting laser beams cool and confine the atoms. The surrounding magnetic coils provide the trapping potential, and rotation is induced about the $\hat{z}$ axis with angular frequency Ω , leading to the formation of quantized vortices in the condensate.

into the lowest Landau level regime, as demonstrated experimentally by Schweikhard *et al.* [35]. In this regime, the vortex lattice remains ordered, but its elastic shear strength is strongly reduced. Experimental and theoretical studies have shown that the dynamics in this limit are governed by lowest Landau level physics, establishing a link between rotating BECs and quantum Hall systems [17, 83]. Another important direction of research concerns multiply quantized vortices and giant vortex states. Experiments employing ring and anharmonic confinement have successfully generated giant vortices with large density depleted cores containing multiple units of circulation [84].

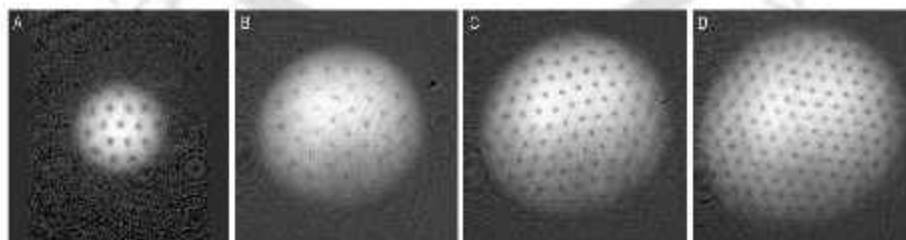


Figure 1.4: Observation of vortex lattices in a rotating Bose-Einstein condensate. The images show condensates containing approximately (A) 16, (B) 32, (C) 80, and (D) 130 vortices. As the rotation increases, the vortices arrange themselves into a highly ordered triangular (Abrikosov) lattice, indicating the crystallization of quantized vortices. Figure is adapted from [32].

1.3 Mean-Field Description of Weakly Interacting BEC: Gross Pitaevskii Equation

In the framework of second quantization, a gas of N bosonic atoms can be described using the field operator $\hat{\Psi}(\mathbf{r}, t)$, where $\mathbf{r} = (x, y, z)$ denotes the spatial coordinate and t is the time variable. The system is assumed to be fully condensed into the lowest energy state. For simplicity, we take $\hat{\Psi}(\mathbf{r}, t) = \hat{\Psi}(\mathbf{r})$

The second quantized Hamiltonian that describes a gas of bosons, which interact through binary collisions and are confined by an external trapping potential, is given by [85–88]:

$$\hat{H} = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) \right) \hat{\Psi}(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V_{\text{int}}(|\mathbf{r} - \mathbf{r}'|) \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}) \quad (1.5)$$

where $-\hbar^2 \nabla^2 / 2m$, represents the kinetic energy operator of the particles, V_{ext} is the external trapping potential and $V_{\text{int}}(\mathbf{r} - \mathbf{r}')$ denotes the interatomic interaction potential between two particles separated by a distance $(\mathbf{r} - \mathbf{r}')$. We restrict our discussion to elastic binary collisions and do not consider inelastic (internal-state-changing) processes, which would require a tensorial form of the interaction potential.

The field operators $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^\dagger(\mathbf{r})$ are the annihilation and creation operators, respectively, of a bosonic particle at position $\mathbf{r}$.

The creation and annihilation field operators obey the following canonical Bose commutation relations [85–88]:

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')] = [\hat{\Psi}^\dagger(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = 0, \quad (1.6)$$

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}'). \quad (1.7)$$

In cold and dilute alkali gases, the atomic de Broglie wavelength is much larger than the range of the interatomic interaction. Consequently, binary collisions are dominated by elastic s wave scattering and the full interatomic interaction potential can be replaced by an effective contact pseudopotential.

$$V_{\text{int}}(\mathbf{r} - \mathbf{r}') = \frac{4\pi\hbar^2 a}{m} \delta(\mathbf{r} - \mathbf{r}'). \quad (1.8)$$

where a is the s -wave scattering length and m is the atomic mass. Using the pseudo

potential approximation the Hamiltonian (Eq. 1.5) can be written as [89]

$$\hat{H} = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) \right) \hat{\Psi}(\mathbf{r}) + \frac{U_0}{2} \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}), \quad (1.9)$$

where $U_0 = 4\pi\hbar^2 a/m$ and $V_{\text{ext}}(\mathbf{r})$ is the external trapping potential. In order to describe the system at fixed chemical potential, we work in the grand canonical formalism and introduce the operator

$$\hat{\mathcal{H}} = \hat{H} - \mu \hat{N}, \quad (1.10)$$

where

$$\hat{N} = \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \quad (1.11)$$

is the total particle number operator and μ is the chemical potential.

The dynamics of the field operator are obtained from the Heisenberg equation of motion,

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = [\hat{\Psi}(\mathbf{r}, t), \hat{\mathcal{H}}]. \quad (1.12)$$

together with the bosonic commutation relations this leads to a nonlinear equation of motion for the field operator,

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\mathbf{r}, t) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) - \mu \right) \hat{\Psi}(\mathbf{r}, t) + \int d\mathbf{r}' V_{\text{int}}(\mathbf{r} - \mathbf{r}') \hat{\Psi}^\dagger(\mathbf{r}', t) \hat{\Psi}(\mathbf{r}', t) \hat{\Psi}(\mathbf{r}, t). \quad (1.13)$$

In the mean-field limit, where a macroscopic number of particles occupies a single quantum state, the bosonic field operator may be decomposed as

$$\hat{\Psi}(\mathbf{r}, t) = \sqrt{N} \psi(\mathbf{r}, t) + \delta\hat{\Psi}(\mathbf{r}, t), \quad (1.14)$$

where $\psi(\mathbf{r}, t)$ is the condensate wave function and $\delta\hat{\Psi}(\mathbf{r}, t)$ represents fluctuations about the mean field.

By substituting Eq. 1.14 into Eq. 1.13 and neglecting higher-order fluctuation terms in the limit of a large condensate fraction, we obtain the time-dependent Gross-Pitaevskii equation.

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) + g_{3D} |\psi(\mathbf{r}, t)|^2 - \mu \right) \psi(\mathbf{r}, t). \quad (1.15)$$

Here, $g_{3D} = \frac{4\pi\hbar^2 N a_0}{m}$ denotes the effective two-body interaction coefficient, where a_0 is the s -wave scattering length and m is the atomic mass. The condensate wave function $\psi(\mathbf{r}, t)$ is normalized according to $\int_{\mathbb{R}^3} |\psi(\mathbf{r}, t)|^2 d^3r = N$.

While the expectation value of the fluctuation operator vanishes, $\langle \delta\hat{\Psi} \rangle = 0$, its higher-order correlations need not. The quantity $\langle \delta\hat{\Psi}^\dagger \delta\hat{\Psi} \rangle$ quantifies the depletion of particles from the condensate due to interactions, whereas $\langle \delta\hat{\Psi} \delta\hat{\Psi} \rangle$ captures pairing correlations associated with virtual excitations out of the condensate. Terms involving three fluctuation operators $\langle \delta\hat{\Psi}^\dagger \delta\hat{\Psi} \delta\hat{\Psi} \rangle$ describe scattering processes among non-condensed particles. In the regime of a large condensate fraction, such contributions are strongly suppressed. Moreover, the leading effect of anomalous correlations is already incorporated through the renormalization of the contact interaction. Consequently, at zero temperature and for weak depletion, these higher-order terms can be consistently neglected.

1.3.1 Mean Field Description of Rotating BEC

The GPE discussed above provides a mean-field description of a BEC in the absence of rotation. However, when the condensate is set into rotation with angular velocity Ω , the external trapping potential becomes explicitly time dependent in the laboratory frame. To simplify both analytical treatment and numerical implementation, it is convenient to transform the GPE to a reference frame that co-rotates with the trapping potential. This transformation involves the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = -i\hbar\nabla$ is the canonical momentum operator. For a gas rotating with angular velocity Ω , the GPE is expressed as [87, 88]:

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) + g_{3D} |\psi(\mathbf{r}, t)|^2 - \mu - \Omega \cdot \mathbf{L} \right) \psi(\mathbf{r}, t). \quad (1.16)$$

1.3.2 Dissipative Gross-Pitaevskii Equation (DGPE)

The aforementioned time-dependent GPE provides a mean-field description of a BEC at $T = 0$, where it is assumed that the number of particles and energy are conserved. However, experimental observations indicate that there is damping in the collective excitations due to finite temperature effects, which are not captured by the standard GPE. To address this issue, Choi *et al.* proposed a model that phenomenologically incorporates the damping process into the mean-field dynamics, which has been adapted in the literature. In the following section, we present the dissipative Gross-Pitaevskii (DGPE)

equation by following the method described in [90].

In the presence of damping, the equation of motion for the condensate wavefunction ψ is written as,

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \hat{Q} \psi(\mathbf{r}, t), \quad (1.17)$$

The damping drives the system towards an equilibrium state, which is described by the GPE (1.16) at $T = 0$. The resulting dynamics therefore represent a relaxation process. Accordingly, the operator $\hat{Q}$ can be decomposed into two parts: a Hermitian operator $\hat{H}$, which governs the conservative dynamics (Eq. 1.16), and a non-Hermitian operator $\hat{\Gamma}$, which accounts for dissipation. The total operator can thus be written as the sum of these two contributions,

$$\hat{Q} = \hat{H} + \hat{\Gamma}. \quad (1.18)$$

At equilibrium, the anti-Hermitian part of $\hat{Q}\psi$ must be zero, and the anti-Hermitian operator can be expressed as.

$$\hat{Q}\psi = i\Gamma \left(-\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{r}) \psi + g_{3D} |\psi|^2 \psi - \mu \psi - \mathbf{\Omega} \cdot \mathbf{L} \right), \quad (1.19)$$

where Γ is a dimensionless parameter that is inversely proportional to the relaxation time. In this thesis, we consider $\Gamma = \gamma$ with $\gamma < 0$, where γ is the damping parameter that also provides an estimate of the rate at which the non-condensed component relaxes into the condensate. Now, from Eq. (1.17), we can write

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = (1 + i\Gamma) \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g_{3D} |\psi|^2 - \mu - \mathbf{\Omega} \cdot \mathbf{L} \right) \psi(\mathbf{r}). \quad (1.20)$$

By multiplying both sides of Eq. (1.20) by $(i - \gamma)$ and neglecting higher-order terms in γ (which is typically of order $\gamma \sim 10^{-3}$), we obtain the following DGPE

$$(i - \gamma)\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g_{3D} |\psi|^2 - \mu - \mathbf{\Omega} \cdot \mathbf{L} \right) \psi(\mathbf{r}). \quad (1.21)$$

1.3.3 Dimensional Reduction: Quasi two dimension GPE

In this thesis, our focus is limited to condensates that are effectively two-dimensional. Experimentally, such a regime is achieved by imposing an anisotropic harmonic trap with strong confinement along one spatial direction. Specifically, we consider a trap-

ping geometry where the condensate is tightly confined along the z -axis, allowing the system to be described by an effective quasi two dimensional mean-field theory with dynamics confined to the $x - y$ plane. The external trapping potential is taken to be $V_{\text{ext}} = \frac{1}{2}m\omega_{\perp}^2(x^2 + y^2) + \omega_z^2 z^2$ where $\omega_{\perp}$, and ω_z are the radial and axial trapping frequencies, respectively. In the strongly anisotropic limit, ($\lambda = \omega_z/\omega_{\perp} \gg 1$), where the condensate assumes a pancake-shaped geometry and the wave function can be factorized as [91–93]

$$\Psi(\mathbf{r}, t) = \psi(x, y, t) \frac{1}{\pi^{1/4} a_{hz}^{1/2}} \exp\left(-\frac{z^2}{2a_{hz}^2}\right), \quad (1.22)$$

where, $a_{hz} = \sqrt{\hbar/(m\omega_z)} = a_{\perp}/\sqrt{\lambda}$ is the harmonic oscillator length associated with the axial confinement. Substituting the ansatz 1.22 into the three dimensional GPE 1.21 and integrating over the axial coordinate results in an effective rescaling of the interaction strength, such that the three-dimensional coupling constant g_{3D} is reduced by a factor $1/(\sqrt{2\pi} a_{hz})$ and leads to the following two-dimensional (2D) GPE:

$$i\hbar \frac{\partial \psi(x, y, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla_{\perp}^2 + V_{\text{ext}}(x, y) - \mu + \frac{g_{3D}}{2\pi a_{hz}} |\psi(x, y, t)|^2 - \Omega \hat{L}_z \right] \psi(x, y, t) \quad (1.23)$$

Equation (3.3) can be cast into a dimensionless form by introducing the characteristic scales associated with the radial harmonic confinement. Length, time, and energy are measured in units of the harmonic oscillator length $a_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$, the inverse trapping frequency $\omega_{\perp}^{-1}$, and the energy scale $\hbar\omega_{\perp}$, respectively. Accordingly, we introduce the dimensionless variables

$$t = \omega_{\perp}^{-1} t', \quad (x, y) = a_{\perp} (x', y'), \quad \psi = \frac{\sqrt{N}}{a_{\perp}} \psi', \quad (1.24)$$

where primes denote dimensionless quantities. Substituting these scalings into Eq. (3.3) and dropping the primes for notational convenience, we obtain the dimensionless GPE

$$(i - \gamma) \frac{\partial \psi(x, y, t)}{\partial t} = \left[-\frac{1}{2} \nabla_{\perp}^2 + V_{\text{ext}}(x, y) - \mu + g |\psi(x, y, t)|^2 - \Omega L_z \right] \psi(x, y, t). \quad (1.25)$$

Here, the dimensionless nonlinear interaction parameter is given by $g = \frac{4\pi N a_s}{a_{\perp}} \sqrt{\frac{\pi\lambda}{2}}$, the harmonic trapping potential takes the form $V_{\text{ext}}(x, y) = \frac{1}{2} (x^2 + y^2)$, and the angular momentum operator is defined as $L_z = i(y \partial_x - x \partial_y)$.

1.3.4 Thomas-Fermi density profile

In this analysis, we determine the ground-state density profile of the condensate using the Thomas-Fermi approximation. In this regime, repulsive interatomic interactions are the predominant force, leading the condensate to expand to a mean radius significantly larger than the characteristic oscillator [94]. Consequently, the radial density gradients are greatly diminished, making the associated kinetic energy contribution negligible compared to the trapping and interaction energies.

The energy functional corresponding to the GPE (1.16) is given by

$$E[\psi] = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla \psi(\mathbf{r})|^2 + V_{\text{ext}}(\mathbf{r}) |\psi(\mathbf{r})|^2 + \frac{g}{2} |\psi(\mathbf{r})|^4 - \Omega \psi^*(\mathbf{r}) \hat{L}_z \psi(\mathbf{r}) - \mu |\psi(\mathbf{r})|^2 \right]. \quad (1.26)$$

To proceed, we adopt a hydrodynamic description by expressing the condensate wave function in terms of its density and phase using the Madelung decomposition [95],

$$\psi(r, \phi) = \sqrt{n(r, \phi)} \exp[iS(r, \phi)]. \quad (1.27)$$

where $n(r, \phi)$ is the condensate density and $S(r, \phi)$ is the phase. Here r and ϕ denote polar coordinates in the plane. By applying this transformation and utilizing the Thomas-Fermi approximation, we can express the energy functional (see Eq. 1.26) in a simplified form.

$$E[n, S] = \int d\mathbf{r} \left[\frac{m}{2} n \mathbf{v}^2 + V(r) n + \frac{g}{2} n^2 - \Omega n (\mathbf{r} \times \mathbf{v})_z - \mu n \right]. \quad (1.28)$$

In the rigid-body rotation limit, the superfluid velocity assumes the form

$$\mathbf{v}_{\text{sf}} = \boldsymbol{\Omega} \times \mathbf{r}. \quad (1.29)$$

Upon substituting this relation into Eq. (1.28) and performing a variational minimization with respect to the density $n(r, \phi)$, the Thomas-Fermi density profile in the rotating frame is obtained.

$$n(r) = \frac{1}{g} \left[\mu - V(r) + \frac{1}{2} m \Omega^2 r^2 \right]. \quad (1.30)$$

The chemical potential μ is determined from the normalization condition

$$\int dr d\phi n(r, \phi) = N. \quad (1.31)$$

Equation (1.30) provides the background density profile of the rotating condensate. The spatial extent of the cloud along the Cartesian directions is determined by the Thomas-Fermi radii,

$$R_{x,y}^2 = \frac{2\mu}{m(\omega_{x,y}^2 - \Omega^2)}, \quad (1.32)$$

which define the boundaries where the density vanishes. Using these radii, the normalization condition (1.31) can be evaluated straightforwardly in polar coordinates, leading to an expression for the chemical potential μ in the isotropic limit.

$$\mu = \sqrt{\frac{mgN}{\pi}} [(\omega_{x,y}^2 - \Omega^2)]^{1/4} \quad (1.33)$$

1.3.5 Hydrodynamic description of the condensate

The hydrodynamic representation of the GPE (Eq. 1.16) can be obtained using the Madelung transformation (Eq. 1.27). Substituting this transformation into the GPE and separating the resulting expression into its real and imaginary parts yields two coupled hydrodynamic equations governing the evolution of the condensate density and phase. The corresponding superfluid velocity field is defined as

$$\mathbf{v}(\mathbf{r}, t) = \frac{\hbar}{m} \nabla S. \quad (1.34)$$

Continuity equation. The imaginary part gives the equation for particle number conservation,

$$\frac{\partial n}{\partial t} + \nabla \cdot [n(\mathbf{v} - \Omega \times \mathbf{r})] = 0. \quad (1.35)$$

Euler equation. The real part leads to the equation of motion for the superfluid velocity,

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} m \mathbf{v}^2 + V_{\text{ext}}(\mathbf{r}) + g n - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right) = -m \Omega \times \mathbf{v}. \quad (1.36)$$

The last term represents the Coriolis force arising in the rotating frame, while the term proportional to $\nabla^2 \sqrt{n}$ is the quantum pressure.

Thomas-Fermi limit. In the Thomas-Fermi regime, where the density varies slowly in space, the quantum pressure term can be neglected. The Euler equation then sim-

plifies to

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} m \mathbf{v}^2 + V(\mathbf{r}) + g n \right) = -m \boldsymbol{\Omega} \times \mathbf{v}. \quad (1.37)$$

1.3.6 Elementary excitations: Bogoliubov equations

As discussed earlier, the time dependent GPE (1.16) provides a mean-field description of the dynamical evolution of the condensate. Experimental studies reveal that, in addition to the ground state, the condensate exhibits collective excitation modes. Although such excitations can be analyzed using fully quantum mechanical methods [76, 96], in this thesis we adopt a classical method based on the GPE in which, small amplitude fluctuation are introduced around the condensate ground state wave function and upon linearizing the GPE with respect to these fluctuations leads to an eigenvalue problem that is equivalent to that obtained from the corresponding quantum mechanical analysis. Within this approach, the effect of the fluctuation can be considered by writing the order parameter in the following manner

$$\psi(\mathbf{r}, t) = [\psi_0(\mathbf{r}) + \delta\psi(\mathbf{r}, t)] \exp(-i\mu t). \quad (1.38)$$

Here ψ_0 denotes the condensate ground state, and $\delta\psi$ represents a small fluctuation superimposed on this background. The nonlinear interaction term in the GPE couples waves propagating in opposite directions so the fluctuations cannot be represented by a single plane wave component. Therefore, we will seek a solution that explicitly includes both forward and backward propagation [76, 96].

$$\delta\psi(\mathbf{r}, t) = \sum_i u_i(\mathbf{r}) e^{-i\omega_i t} - v_i^*(\mathbf{r}) e^{i\omega_i^* t}, \quad (1.39)$$

where $u_i(\mathbf{r})$ and $v_i(\mathbf{r})$ are the Bogoliubov mode amplitudes and ω is the corresponding excitation frequency. By substituting Eqs. (1.38) into Eq. (1.16) and collecting terms proportional to $e^{-i\omega_i t}$ and $e^{i\omega_i^* t}$, we obtain a set of coupled linear equations for the Bogoliubov amplitudes. These equations are known as Bogoliubov-de Gennes (BdG)

equations which govern the elementary excitations of the condensate [76, 96].

$$\omega_i u_i(\mathbf{r}) = \left[-\frac{1}{2}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + 2g|\psi_0(\mathbf{r})|^2 - \mu - \Omega\hat{L}_z \right] u_i(\mathbf{r}) - g\psi_0^2(\mathbf{r})v_i(\mathbf{r}), \quad (1.40)$$

$$-\omega_i v_i(\mathbf{r}) = \left[-\frac{1}{2}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + 2g|\psi_0(\mathbf{r})|^2 - \mu + \Omega\hat{L}_z \right] v_i(\mathbf{r}) - g\psi_0^{*2}(\mathbf{r})u_i(\mathbf{r}), \quad (1.41)$$

If we consider the Bogoliubov modes in the vectorial form $\mathbf{W}_i(\mathbf{r}) = (u_i(\mathbf{r}), v_i(\mathbf{r}))^T$ the coupled BdG equations can be formulated as follows [87]:

$$\hat{\sigma}_z \hat{\mathcal{M}} \mathbf{W}_i = \omega_i \mathbf{W}_i, \quad (1.42)$$

Here, $\hat{\sigma}_z = \text{diag}(1, -1)$ and the operator $\hat{\mathcal{M}}$ is defined as

$$\hat{\mathcal{M}} = \begin{pmatrix} \hat{\mathcal{L}}_+ & -g\psi_0^2(\mathbf{r}) \\ -g\psi_0^{*2}(\mathbf{r}) & \hat{\mathcal{L}}_- \end{pmatrix}. \quad (1.43)$$

with the operators $\hat{\mathcal{L}}_{\pm}$ defined as

$$\hat{\mathcal{L}}_{\pm} = -\frac{1}{2}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + 2g|\psi_0(\mathbf{r})|^2 - \mu \mp \Omega\hat{L}_z. \quad (1.44)$$

The operator $\hat{\sigma}_z \hat{\mathcal{M}}$ is not Hermitian with respect to the conventional inner product. Rather, it obeys the relation

$$(\hat{\sigma}_z \hat{\mathcal{M}})^\dagger = \hat{\sigma}_z (\hat{\sigma}_z \hat{\mathcal{M}}) \hat{\sigma}_z^{-1}, \quad (1.45)$$

which is the condition of a pseudo Hermiticity operator associated with the metric operator $\hat{\sigma}_z$ [97–99]. Because of this pseudo hermicity the eigenvalues can be either real or complex. The existence of such complex eigenvalues can also be understood from the fact that the operator $\hat{\sigma}_z \hat{\mathcal{M}}$ is self-adjoint only in a generalized sense, namely with respect to an indefinite inner product. More precisely, the operator is self-adjoint with respect to the non-positive definite inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle_{\sigma_z} \equiv \int d\mathbf{r} \mathbf{A}^\dagger(\mathbf{r}) \hat{\sigma}_z \mathbf{B}(\mathbf{r}), \quad (1.46)$$

where $\mathbf{A}(\mathbf{r}) = (A_1(\mathbf{r}), A_2(\mathbf{r}))^T$ and $\mathbf{B}(\mathbf{r}) = (B_1(\mathbf{r}), B_2(\mathbf{r}))^T$ are arbitrary two-component Bogoliubov spinors. After writing this expression explicitly in terms of the components

yields

$$\langle \mathbf{A}, \mathbf{B} \rangle_{\sigma_z} = \int d\mathbf{r} [A_1^*(\mathbf{r}) B_1(\mathbf{r}) - A_2^*(\mathbf{r}) B_2(\mathbf{r})]. \quad (1.47)$$

As seen from Eq. (1.47), the inner product is no longer positive definite and may take negative values. The Bogoliubov spectrum could exhibit complex eigenfrequencies appearing as complex conjugate pairs, which reflects pseudo-Hermitian symmetry of the Bogoliubov–de Gennes operator. By appropriately combining Eqs. (1.40) and (1.41), one finds that the solutions of Eq. (1.42) satisfy the orthogonality condition

$$(\omega_i - \omega_j^*) \langle \mathbf{W}_i, \mathbf{W}_j \rangle_{\sigma_z} = 0. \quad (1.48)$$

This relation shows that Bogoliubov modes are orthogonal with respect to the σ_z -weighted inner product defined in Eq. (1.46).

It should be noted that the Bogoliubov modes are normalized according to

$$\langle \mathbf{W}_i, \mathbf{W}_i \rangle_{\sigma_z} = \int d\mathbf{r} [|u_i(\mathbf{r})|^2 - |v_i(\mathbf{r})|^2] = 1, \quad (1.49)$$

a condition required to ensure the correct bosonic commutation relations upon quantization of the quasiparticle excitations.

In the absence of rotation and external trapping, as in a uniform condensate, the Bogoliubov modes can be expressed as plane waves: $u_{\mathbf{k}}(\mathbf{r}) = u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}}$, $v_{\mathbf{k}}(\mathbf{r}) = v_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}}$ with $\mathbf{k}$ is the wave vector. Substituting this ansatz into the BDG equations (1.40) and (1.41) yields the Bogoliubov dispersion relation [76]:

$$E_{\mathbf{k}} = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2gn_0 \right)}. \quad (1.50)$$

From this dispersion relation, we observe that for large momenta, the spectrum follows a quadratic relationship with $\mathbf{k}$, recovering the free particle nature. Conversely, for low momenta, it behaves like sound wave propagation, where $E_{\mathbf{k}} \propto k$.

In the case of trapped condensate, its finite size leads to a discrete excitation spectrum. The dispersion relations for the collective excitations were studied analytically in [100] and were found to be in excellent agreement with experimental observations at low temperatures, confirming the accuracy of the mean-field GP description in this regime. The lowest-energy collective excitations include dipole (center of mass), monopole (breathing), and quadrupole modes. The collective modes of a trapped condensate behave like phonons at higher frequencies, i.e, the propagation of density

waves through the condensate, similar to sound waves. The interaction strength and the condensate density determine the velocity of these phonons. Even at higher energies, excitations can exhibit a single-particle character because the condensate density varies spatially, particularly in the low-density regions near the edge of the cloud.

1.4 Topological Defects in BEC: Quantized Vortices

Vortices are regions in a fluid where particles circulate around a straight or curved axis. Depending on their vorticity, vortices can be classified into two types: rotational and irrotational. In rotational vortices, the vorticity is finite within the vortex core, where azimuthal velocity increases linearly with the radial distance from the core. In contrast, in irrotational vortices, the vorticity is zero everywhere except at the vortex core, and the flow velocity is inversely proportional to the core's radius. In a normal fluid, irrotational flow tends to break down in the central region due to viscosity, causing the flow to become rotational. A common example of this phenomenon is the eye of a hurricane. In the superfluid regime, the absence of viscosity ensures that the vorticity vanishes everywhere outside the vortex core, so that the flow remains strictly irrotational. This means that, although a fluid element circulates around the vortex axis, it does not rotate about its own center, in contrast to the behavior observed in solid body rotation. The concept of quantized vortices was first theoretically predicted by Onsager [101] and subsequently developed by Feynman [102]. These predictions were later confirmed experimentally in superfluid helium (He II) by Vinen [103–105].

Similar to superfluid helium (He II), one of the striking properties observed in BECs is their ability to support quantized vortices. There are several methods through which vortices can be created in the condensate, such as laser stirring [24, 26], rotation [106, 32], phase imprinting [28], merging two different condensates [107], and the Kibble-Zurek mechanism [108]. Among these methods, the first experimental creation of vortices in BECs was achieved through rotational stirring, where a cigar-shaped condensate in a magnetic trap was perturbed by two far-detuned optical beams that copropagated along the axis of the sample and rotated around it. Later, this method is extensively used for generating vortex lattices.

1.4.1 Phase defect and Quantized Vortices

As discussed earlier, within the mean-field description, the condensate wavefunction can be expressed in terms of its amplitude and phase as [95] $\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)}$

where $n(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ denote the density and phase of the condensate, respectively. Consider an arbitrary closed contour C . The total variation of the phase accumulated along this path is given by [17]

$$\Delta S = \oint_C \nabla S \cdot d\mathbf{l}, \quad (1.51)$$

where $d\mathbf{l}$ is the infinitesimal line element along the contour. Since the initial and final points on the closed path coincide, the condensate wavefunction must remain single-valued. This requirement implies the quantization condition

$$\Delta S = 2\pi q, \quad q = 0, \pm 1, \pm 2, \dots, \quad (1.52)$$

where q is an integer winding number. When $q \neq 0$, the phase cannot be defined continuously everywhere inside the region bounded by C . Instead, there exists a point in the region around which the phase undergoes a 2π circulation. At that point, the phase is undefined, and the only way for the condensate wavefunction to remain single-valued is for the density to vanish.

The presence of a phase defect has a clear physical interpretation when the condensate is viewed as a fluid. In the hydrodynamic formulation of the mean field theory, the superfluid velocity is determined by the phase gradient according to

$$\mathbf{v}(\mathbf{r}) = \frac{\hbar}{m} \nabla S(\mathbf{r}). \quad (1.53)$$

As a consequence, the quantization of the phase implies a corresponding constraint on the circulation of the velocity field. The circulation Γ around a closed contour C enclosing a phase defect is given by

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = q \kappa, \quad \kappa = \frac{h}{m}, \quad (1.54)$$

where q is an integer winding number and κ denotes the quantum of circulation. This quantization of circulation differentiates the superfluid from ordinary viscous fluids. To elucidate the resulting flow pattern, consider the case $q \neq 0$ and choose the contour C to be a circle of radius r centered on the phase singularity. Restricting attention to two-dimensional flow in the xy plane and adopting polar coordinates (r, θ) , the line element along the contour is

$$d\mathbf{l} = r d\theta \hat{\mathbf{e}}_\theta, \quad (1.55)$$

where $\hat{\mathbf{e}}_\theta$ is the azimuthal unit vector. The circulation then takes the form

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = \int_0^{2\pi} r v_\theta(r) d\theta = 2\pi r v_\theta(r). \quad (1.56)$$

Comparison of Eqs. (1.54) and (1.56) yields the azimuthal velocity profile

$$v_\theta(r) = \frac{q\kappa}{2\pi r} = \frac{q\hbar}{mr}. \quad (1.57)$$

From Equation (1.60), it can be seen that the velocity around the vortex approaches infinity, which decreases to zero at an infinite distance. The sign of q determines the direction of the condensate around the core, and if q is increased, the velocity of the condensate around the vortex core also increases discontinuously.

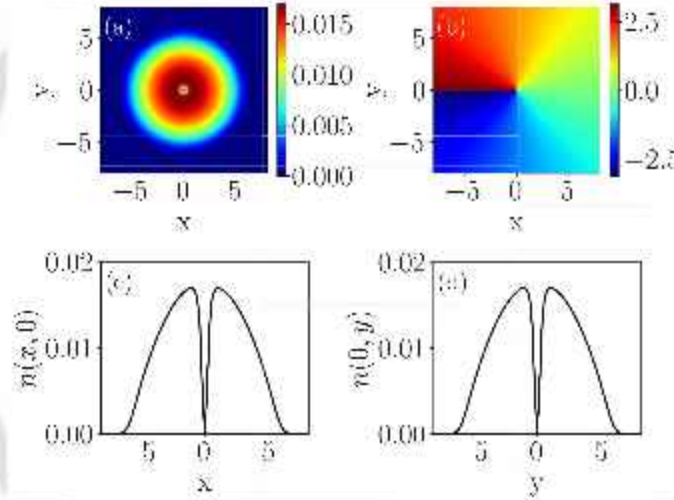


Figure 1.5: Density and phase structure of the condensate containing a single vortex. (a) Two-dimensional density distribution $n(x, y)$ showing a density depletion at the vortex core. (b) Phase profile $\arg(\psi)$ of the condensate wavefunction exhibiting the characteristic 2π phase winding around the vortex center. (c) Density profile $n(x, 0)$ along the x direction passing through the vortex core, showing a pronounced density minimum at the center. (d) Density profile $n(0, y)$ along the y direction, also indicating the density depletion associated with the vortex core.

1.4.2 Vortex energy and angular momentum

In this section, we calculate the energy of a single quantized vortex using the mean-field GPE. To do this, we consider a stationary, straight vortex whose core lies along the z -axis. Such a vortex solution of the GPE carries a well-defined angular momentum about the vortex axis.

In cylindrical coordinates (r, θ, z) , a general stationary solution of the GPE that carries angular momentum around the z -axis can be written in the form

$$\psi(r, \theta, z) = f(r) e^{i\kappa\theta}, \quad (1.58)$$

where $f(r)$ is a real radial amplitude and κ is an integer winding number that characterizes the charge of the vortex.

Within the hydrodynamic formulation the superfluid velocity is related to the phase of the condensate by

$$\mathbf{v} = \frac{\hbar}{m} \nabla S. \quad (1.59)$$

For the phase $S = \kappa\theta$, this yields a purely azimuthal velocity field

$$\mathbf{v}(r) = \frac{\hbar\kappa}{mr} \hat{\mathbf{e}}_\theta, \quad (1.60)$$

which diverges as $r \rightarrow 0$. This apparent singularity is resolved by the fact that the condensate density vanishes at the vortex axis, ensuring a smooth and physically acceptable solution of the GP equation.

Substituting the ansatz (1.58) into the stationary GPE and rescaling the amplitude as $f(r) \rightarrow \sqrt{n_0} f(r)$, one obtains the radial equation

$$\xi^2 \left(\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} \right) + \left(1 - \frac{\xi^2 \kappa^2}{r^2} \right) f - f^3 = 0, \quad (1.61)$$

where ξ is the healing length and n_0 is the bulk condensate density. This equation describes the suppression of the condensate density near the vortex core and its recovery to the uniform value far from the core.

The asymptotic behavior of Eq. (1.61) can be analyzed in limiting cases. Close to the vortex axis ($r \rightarrow 0$), the centrifugal term proportional to κ^2/r^2 dominates, leading to

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{\kappa^2}{r^2} f = 0. \quad (1.62)$$

This Cauchy–Euler equation admits the regular solution [109]

$$f(r) \propto \left(\frac{r}{\xi} \right)^{|\kappa|}, \quad (1.63)$$

which shows that the condensate density vanishes at the vortex core and that the core size is of the order of the healing length ξ . For large distances ($r \rightarrow \infty$), the amplitude

approaches its bulk value, $f(r) \rightarrow 1$.

The energy of a vortex may be computed from the GP energy functional in the absence of an external potential. The free energy $F = H - \mu N$, after integration over a cylindrical volume of radius R and length L , takes the form

$$E = \int r dr d\theta dz \left[\frac{\hbar^2}{2m} \left| \frac{\partial \psi}{\partial r} \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\mathbf{e}}_\theta \right|^2 + \frac{g}{2} |\psi|^4 - \mu |\psi|^2 \right]. \quad (1.64)$$

Subtracting the energy of the uniform condensate, $E_0 = \frac{g}{2} n_0^2 V$ where $V = \pi R^2 L$ is the volume of the cylindrical system. So, the excess energy associated with the vortex is given by

$$E_v = \frac{\kappa^2 L m n_0}{4\pi} \int_0^R r dr \left[\left(\frac{df}{dr} \right)^2 + \frac{\kappa^2 f^2}{r^2} + \frac{1}{2\xi^2} (f^2 - 1)^2 \right], \quad (1.65)$$

In the limit $R \gg \xi$, the dominant contribution to the energy arises from the azimuthal kinetic energy term proportional to κ^2/r^2 . Evaluating the integral in this regime yields

$$E_v \simeq \frac{\kappa^2 L m n_0}{4\pi} \ln \left(\frac{R}{\xi} \right), \quad (1.66)$$

which shows that the vortex energy grows logarithmically with the system size and scales quadratically with the vortex charge κ [110, 111].

While the previous analysis concentrated on the static properties of a single vortex, the energetic considerations mentioned above also have significant dynamical implications. Specifically, variations in vortex energy can generate forces that drive vortex motion, even without interactions with other vortices. In the upcoming section, we will discuss how an individual vortex moves within a condensate.

1.4.3 Motion of individual vortices

We begin our discussion by considering a straight vortex located near the boundary of a condensate confined within a rigid container. The superfluid velocity must vanish at the container wall. This boundary condition can be satisfied by introducing an image vortex of opposite circulation located outside the container. As a result, the real vortex and its image form an effective vortex-antivortex pair, causing the vortex to move parallel to the boundary and circulate around the container. In a harmonically trapped condensate, the situation is more subtle due to the smooth spatial variation of the condensate density. In this case, the image vortex picture is no longer appropriate.

Instead, the motion of an off-centre vortex can be understood in terms of the Magnus force, an effect well known in classical hydrodynamics. A vortex line may be viewed as a rotating cylinder immersed in the fluid. When displaced from the trap centre, the vortex experiences a radial force arising from the position dependence of the system's free energy.

The free energy $E(\mathbf{r}_0)$ decreases as the vortex moves away from the trap centre, where $\mathbf{r}_0$ denotes the vortex position. The resulting radial force,

$$\mathbf{F}_r = -\frac{\partial E}{\partial \mathbf{r}_0}, \quad (1.67)$$

has contributions from both the kinetic energy of the circulating flow and a buoyancy like effect associated with the low-density vortex core. This force is balanced by the Magnus force acting on the vortex line,

$$\mathbf{F}_M = -mn\boldsymbol{\kappa} \times \mathbf{v}_L, \quad (1.68)$$

where $\boldsymbol{\kappa}$ is the circulation vector and $\mathbf{v}_L$ is the velocity of the vortex line. Balancing these forces yields

$$\frac{\partial E}{\partial \mathbf{r}_0} = mn\boldsymbol{\kappa} \times \mathbf{v}_L. \quad (1.69)$$

Since the vortex velocity is perpendicular to the force acting on it, the net effect is a precession of the vortex about the trap centre. More generally, the vortex follows trajectories of constant free energy. In an axisymmetric harmonic trap, this motion corresponds to circular orbits, whereas in a non-axisymmetric trap the vortex follows elliptical paths. Experimental studies have directly observed this precessional motion using real time imaging of vortices in BEC, finding excellent agreement with theoretical predictions [112].

1.5 Vortex states :

In this section, we discuss several vortex states that arise in condensate, including vortex rings, vortex reconnections, and the most challenging regime of all, quantum turbulence.

1.5.1 Vortex Ring

A vortex ring is created when a vortex line bends and form a closed circular loop. Each part of the ring moves as a result of the flow generated by the other parts, causing the ring to travel in a straight line at a constant speed [113]. Fig. 1.6 illustrates a vortex ring approaching and interacting with a straight vortex line.



Figure 1.6: A vortex ring traveling towards a vortex line. The figure is adapted from [113]

1.5.2 Kelvin waves

A Kelvin wave is a wavelike distortion of a vortex core, where the vortex line oscillates in a sinusoidal or helical shape around its equilibrium position [113]. Fig. 1.7(left) illustrates such a wave with amplitude A and wavelength λ . For very small amplitude Kelvin waves, the disturbance rotates around the vortex line with a well defined angular frequency. This frequency increases as the wavelength becomes shorter, meaning that shorter Kelvin waves rotate faster than longer ones. Kelvin waves can be excited through vortex interactions. For instance, when a vortex ring collides with a straight vortex line, the resulting disturbance generates Kelvin waves along the line. In addition, vortex rings themselves can support Kelvin-wave excitations. However, the presence of these distortions modifies the ring dynamics, leading to a reduction in its propagation speed compared to that of an unperturbed circular vortex ring, as illustrated in Fig. 1.7 (right).

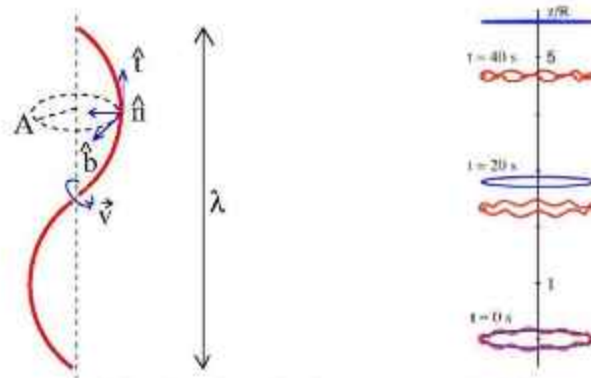


Figure 1.7: Left: Schematic illustration of Kelvin waves with amplitude A and wavelength λ on a vortex line. The three unit vectors corresponding to the tangent, normal, and binormal directions are shown. The Kelvin waves propagate along the binormal direction, opposite to the direction of the fluid flow. Right: Comparison between the motion of an unperturbed vortex ring (radius $R = 0.1$ cm, blue) and a vortex ring perturbed by Kelvin waves (relative amplitude $A/R = 0.05$, red). The figure is adapted from Ref. [114].

1.6 Vortex reconnection

When two quantum vortex lines come close to each other, they can reconnect, meaning that they break and join again in a different way. This process changes the topology of the flow, as illustrated in Fig. 1.8. Vortex reconnection have been observed experimentally in both superfluid [115] and atomic BEC [116]. As the vortices approach each other, their curvature increases near the midpoint, causing them to accelerate. They then collide, reconnect, and move apart. In two dimensions, vortex reconnection manifest as an annihilation process, where a vortex and an antivortex of opposite circulation cancel each other out. It is observed that when a third vortex is present, its interaction generates a soliton like sound wave pulse [117]. In the presence of a fourth vortex, this pulse is further transformed into dispersive sound waves that propagate away from the interaction region [118].

1.7 Sound wave emission

Vortices can lose energy even at zero temperature by transferring it to the surrounding fluid in the form of sound. This energy loss occurs when the vortices experience acceleration during their mutual motion. Another way of producing sound wave is vortex reconnection. In two-dimensional systems, vortex annihilation does not leave behind vortices but instead converts all the energy into propagating sound waves.

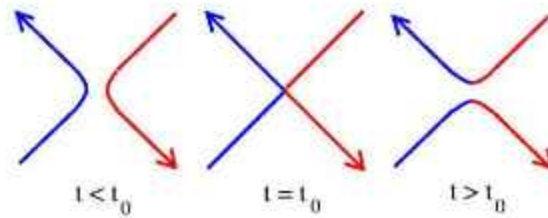


Figure 1.8: Schematic diagram illustrating the reconnection of two vortex lines. The arrows denote the direction of vorticity, representing the circulation of the fluid around the vortex axis. Left: vortex lines approaching before reconnection ($t < t_0$); Middle: reconnection occurs at $t = t_0$; Right: vortex lines separating after reconnection ($t > t_0$). The figure is taken from [119]

1.8 Quantum turbulence :

Turbulence is referred to as an irregular, chaotic, and disordered motion in fluids. In normal fluids, external perturbations inject energy into the system, which is then dissipated through [120]. This dissipation leads to the development of turbulent flow, such as the turbulence observed in a rotating bucket filled with a normal fluid. In contrast, a superfluid has no viscosity, so the usual mechanism of energy dissipation is absent, and energy is redistributed through the stretching, bending, and reconnection of vortices. These processes amplify distortions in the flow and lead to a disordered arrangement of vortices known as a vortex tangle, which is referred to as a state of quantum turbulence. In 1955, Feynman first demonstrated that in a turbulent state, the irregular motion of vortices leads to frequent reconnections. These reconnections excite Kelvin waves, which generate smaller vortex loops while emitting sound waves [102]. This process repeats in a cascade, producing progressively smaller vortex loops until their size becomes comparable to the vortex core. The turbulence state can be classified into two distinct regimes based on vortex density and energy distribution: quasi-classical turbulence and ultra-quantum turbulence [121]. In quasi classical turbulence [122], the energy spectrum follows the Kolmogorov scaling law ($k^{-5/3}$), which indicates a energy cascade transferring from large scales to smaller scales similar to the large scale flow patterns observed in normal fluid. This behavior has been confirmed by numerical simulations based on the GPE [123] and vortex filament models [124], where energy at large length scales is associated with the temporary formation of bundles of vortices. Ultra quantum turbulence appears under different conditions. In this regime, the vortex configuration is more random, and the energy spectrum does not show a classical cascade [121]. Instead, the energy is concentrated at intermediate scales and follows k^{-1} power law behavior. Here, the behavior at small scales is dominated by individual

vortices [125]. Furthermore, in the absence of forcing, this regime decays more slowly than the quasi-classical case.

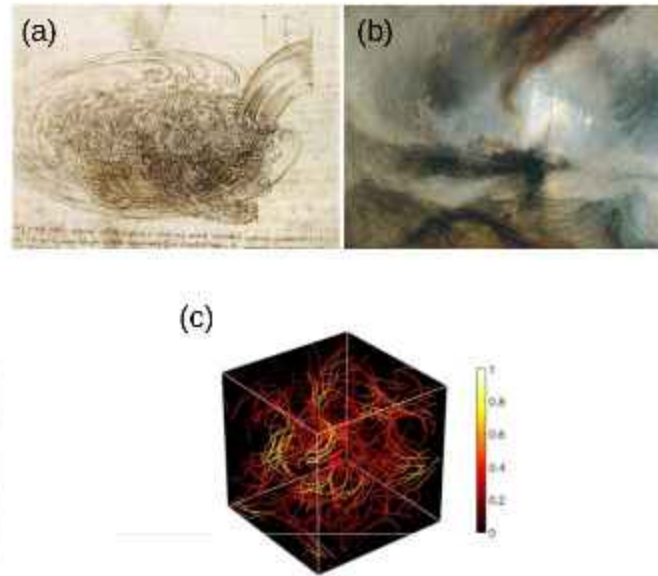


Figure 1.9: (a) Early visual representations of turbulent motion.: (a) sketch from the *Codex Leicester* by Leonardo da Vinci illustrating water flowing past obstacles; (b) *Snow Storm: Steam boat off a Harbour's Mouth* by J. M. W. Turner (1775–1851). (c) Numerical visualization of quantum turbulence in superfluid helium simulated in a periodic domain using the vortex filament method. Source: [126]

1.8.1 Energy decomposition and spectra

In this section, we describe the energy spectra used to characterize turbulent phenomena in the condensate. Starting from the GP energy functional, the kinetic energy (E_K) and the quantum pressure (E_q) of the condensate can be written as [125, 127, 79]

$$E_K = \frac{1}{2} \int d\mathbf{r} n(\mathbf{r}) v^2(\mathbf{r}), E_q = \frac{1}{2} \int d^2\mathbf{r} \left| \nabla \sqrt{n(\mathbf{r})} \right|^2, \quad (1.70)$$

where $n(\mathbf{r}) = |\psi(\mathbf{r})|^2$ is the particle density and $\mathbf{v}(\mathbf{r}) = \nabla S(\mathbf{r})$ is the superfluid velocity field. Within the hydrodynamic approach, it is convenient to introduce the density weighted velocity field

$$\mathbf{u}(\mathbf{r}) = \sqrt{n(\mathbf{r})} \mathbf{v}(\mathbf{r}), \quad (1.71)$$

The field $\mathbf{u}(\mathbf{r})$ can be decomposed using the Helmholtz decomposition,

$$\mathbf{u} = \mathbf{u}^j + \mathbf{u}^c, \quad (1.72)$$

where the incompressible and compressible components satisfy

$$\nabla \cdot \mathbf{u}^i = 0, \quad (1.73)$$

$$\nabla \times \mathbf{u}^c = 0. \quad (1.74)$$

Following Parseval's theorem, the kinetic-energy contributions associated with the different velocity components can be expressed in Fourier space as

$$E_{\text{kin}}^\zeta = \frac{m}{2} \int d^2 \mathbf{k} |\tilde{\mathbf{u}}_\zeta(\mathbf{k})|^2, \quad (1.75)$$

where $\zeta \in \{i, c, q\}$ denotes the incompressible, compressible, and quantum-pressure contributions, respectively. The Fourier transform of the corresponding velocity field is defined as

$$\tilde{\mathbf{u}}_\zeta(\mathbf{k}) = \int d^2 \mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \mathbf{u}_\zeta(\mathbf{r}). \quad (1.76)$$

The kinetic energy in cylindrical coordinates in momentum space, yielding

$$E_{\text{kin}}^\zeta = \int_0^\infty dk \varepsilon_{\text{kin}}^\zeta(k), \quad (1.77)$$

where $\varepsilon_{\text{kin}}^\zeta(k)$ denotes the kinetic-energy spectral density. In Fourier space, the velocity field can be decomposed into components parallel and perpendicular to the wavevector $\mathbf{k}$. Denoting the Fourier transform of $\mathbf{u}(\mathbf{r})$ by $\tilde{\mathbf{u}}(\mathbf{k})$, the compressible and incompressible components are obtained using projection operators,

$$\tilde{u}_\alpha^c(\mathbf{k}) = \sum_\beta \frac{k_\alpha k_\beta}{k^2} \tilde{u}_\beta(\mathbf{k}), \quad (1.78)$$

$$\tilde{u}_\alpha^i(\mathbf{k}) = \sum_\beta \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) \tilde{u}_\beta(\mathbf{k}). \quad (1.79)$$

The parallel component corresponds to compressible (longitudinal) motion, while the perpendicular component represents incompressible (transverse) motion. The corresponding spectral densities are commonly defined as

$$\varepsilon_{\text{kin}}^i(k) = |\tilde{\mathbf{u}}^i(\mathbf{k})|^2, \quad \varepsilon_{\text{kin}}^c(k) = |\tilde{\mathbf{u}}^c(\mathbf{k})|^2. \quad (1.80)$$

In practice, kinetic-energy spectra $\varepsilon_{\text{kin}}^\zeta(k)$ are commonly obtained by discretizing Fourier space into annular shells in wavenumber space and summing the Fourier amplitudes

over angular intervals. While this binning procedure is widely used, it can introduce noise in the spectrum at small wavenumbers due to the limited number of modes contributing to each shell. To address these limitations and achieve a more accurate evaluation of the spectra, we adopt the analytical method introduced by Bradley and collaborators. This approach is based on the angle-averaged Wiener-Khinchin theorem, which establishes a connection between spectral densities and with an associated correlation function. Within this framework, the kinetic-energy spectrum $\varepsilon_{\text{kin}}^{\zeta}(k)$ can be expressed as [128]

$$\varepsilon_{\text{kin}}^{\zeta}(k) = \frac{m}{2} \int d^2\mathbf{x} \Lambda_2(k, |\mathbf{x}|) C[\mathbf{u}_{\zeta}, \mathbf{u}_{\zeta}](\mathbf{x}), \quad (1.81)$$

where $\Lambda_2(k, |\mathbf{x}|) = \frac{k}{2\pi} J_0(k|\mathbf{x}|)$ is the two-dimensional kernel function involving the Bessel function J_0 . The function $C[\mathbf{u}_{\zeta}, \mathbf{u}_{\zeta}](\mathbf{x})$ denotes the two-point autocorrelation function of the velocity field $\mathbf{u}_{\zeta}$ in position space. Equation (1.81) implies that, for any position-space field $\mathbf{u}_{\zeta}$, there exists a corresponding spectral density that is equivalent to an angle-averaged two-point correlation in momentum space.

1.9 Synthetic gauge potential and artificial magnetism in ultracold atomic gases.

A synthetic gauge potential creates a way to generate artificial magnetism in neutral atomic systems, enabling neutral atoms to behave like charged particles in a magnetic field. The concept of simulating magnetic effects in neutral superfluids was first proposed by Richard Feynman in 1981 and the main goal of creating such artificial magnetic fields is to explore fundamental quantum mechanical phenomena in a highly controllable environment [129]. Many of these phenomena are challenging to observe in conventional condensed matter systems due to experimental limitations.

1.9.1 Electromagnetic Gauge Potentials:

One of the most fundamental and simplest realizations of gauge theory is found in classical electromagnetic field theory, where the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ govern the interactions of charged particles. These fields can be expressed in terms of scalar and vector potentials, and reflects the underlying gauge symmetry of the theory. The Lagrangian of a charged particle in an electromagnetic field can be

written as

$$L_{\text{em}} = \frac{1}{2} m \dot{\mathbf{r}}^2 + q \mathbf{A}(\mathbf{r}, t) \cdot \dot{\mathbf{r}} - q W(\mathbf{r}, t), \quad (1.82)$$

where $W(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ represent the electrostatic scalar potential and the magnetic vector potential, respectively. These potentials can be related to the electric and magnetic fields through the following relation:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla W(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \quad (1.83)$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t), \quad (1.84)$$

with the particle velocity $\mathbf{v} = \dot{\mathbf{r}}$. The well known Lorentz force, can then be obtained by minimizing the Lagrangian

$$\mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (1.85)$$

which demonstrates that a charged particle experiences two forces within the electromagnetic field: one along the direction of the electric field and another perpendicular to both the velocity and the magnetic field. The gauge potentials are remain invariant under gauge transformations

$$\mathbf{A}(\mathbf{r}, t) \rightarrow \mathbf{A}(\mathbf{r}, t) + \nabla F(\mathbf{r}, t), \quad (1.86)$$

$$W(\mathbf{r}, t) \rightarrow W(\mathbf{r}, t) - \frac{\partial F(\mathbf{r}, t)}{\partial t}, \quad (1.87)$$

for an arbitrary scalar function $F(\mathbf{r}, t)$. This implies that the choice of gauge potentials is not unique; since different gauge choices correspond to the same physical electromagnetic fields.

The corresponding Hamiltonian can be written as

$$\hat{H}_{\text{em}} = \frac{1}{2m} (\mathbf{p} - q\mathbf{A}(\mathbf{r}, t))^2 + q W(\mathbf{r}, t), \quad (1.88)$$

where the canonical momentum $\mathbf{p} = (p_x, p_y, p_z)$ is obtained from

$$p_x = \frac{\partial L_{\text{em}}}{\partial v_x} = mv_x + qA_x. \quad (1.89)$$

Analogously, the Hamiltonian describing a gas of neutral atoms within the mean field framework can be formulated in terms of effective vector and scalar potentials. This

formal equivalence allows us to implement gauge theories in neutral atomic systems, making it possible to simulate electromagnetic phenomena in a highly controllable environment.

In the next section, we describe the techniques through which gauge potentials can be introduced in neutral atomic systems, along with the corresponding mathematical form of the Hamiltonian. In the following, we will discuss the standard methods by which static synthetic gauge potentials can be engineered in neutral ultracold atomic gases:

1.9.2 Rotating Bose-Einstein Condensates

The simplest method for creating synthetic gauge potentials in a system of neutral atoms is through rotation, where neutral atoms are induced to rotate using laser stirring [32–35]. In the rotating frame, the Hamiltonian of a gas of neutral atoms rotating with angular velocity Ω is given by

$$H_{\text{rot}} = H - \Omega \cdot \mathbf{L} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) - \Omega \cdot \mathbf{L} + g|\Phi|^2, \quad (1.90)$$

where the angular momentum operator is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

The Hamiltonian can be rearranged into the following form

$$H_{\text{rot}} = \frac{1}{2m} (\mathbf{p} - m \Omega \times \mathbf{r})^2 + V(\mathbf{r}) - \frac{1}{2} m (\Omega \times \mathbf{r})^2 + g|\Phi|^2. \quad (1.91)$$

where the Coriolis force effectively plays the role of the Lorentz force acting on a charged particle in a magnetic field. As a result, the system acquires an effective vector potential given by

$$\mathbf{A}_{\text{eff}}(\mathbf{r}) = m (\Omega \times \mathbf{r}), \quad (1.92)$$

allowing the neutral atoms to behave analogously to charged particles subjected to a magnetic field.

1.9.3 Light–Matter Interaction:

In this approach, the internal atomic states are coupled to the center of mass motion of the atoms using Raman laser beams. This spatially dependent laser coupling induces geometric (Berry) phases in the atomic wavefunction, resulting in the emergence of an effective synthetic gauge potential that governs the center of mass dynamics of the

atoms [130, 131].

In this approach, magnetism is simulated within a neutral atomic system in a manner analogous to orbital magnetism in the Aharonov-Bohm effect [132, 133]. In that phenomenon, the wavefunction of a charged particle acquires a phase when it traverses a closed path enclosing a nonzero magnetic flux. In a similar way, magnetism can be engineered in neutral atoms through the Berry phase that arises when the Hamiltonian of a quantum system depends on one or more parameters that change adiabatically in time. If the atoms move sufficiently slowly, a Berry connection emerges in real space when the atomic Hamiltonian depends parametrically on [134]. This Berry connection acts as an effective vector potential, thereby giving rise to synthetic gauge fields. This effective potential arises from a space-dependent light-matter interaction that couples the center of mass motion to the internal dynamics of atoms.

1.9.3.1 Berry Phase

A quantum system described by the Hamiltonian $\hat{H}(\lambda)$ is assumed to depend on a set of external parameters collectively denoted by $\lambda = (\lambda_1, \lambda_2, \dots)$. The stationary states $|\psi_n(\lambda)\rangle$ of the system, which are assumed to form a discrete set, together with their corresponding energies $E_n(\lambda)$, are defined by the eigenvalue equation

$$\hat{H}(\lambda) |\psi_n(\lambda)\rangle = E_n(\lambda) |\psi_n(\lambda)\rangle. \quad (1.93)$$

For each fixed value of λ , the set $\{|\psi_n(\lambda)\rangle\}$ forms a complete orthonormal basis of the Hilbert space. Therefore, an arbitrary state of the system can be expanded as

$$|\psi\rangle = \sum_n c_n |\psi_n(\lambda)\rangle. \quad (1.94)$$

It is assumed that the parameter λ evolves along a closed path in parameter space, and satisfies the condition $\lambda(T) = \lambda(0)$ over the time interval $[0, T]$. In the adiabatic approximation, the slow variation of parameters ensures that the system remains proportional to the initial state, differing only by a phase factor. Therefore, at any time t , the state of the system can be written as $|\psi(t)\rangle = c_i(t) |\psi_i(\lambda(t))\rangle$, where $c_i(t)$ is a phase factor to be determined.

The time evolution of the state $|\psi(t)\rangle$ is governed by the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(\lambda(t)) |\psi(t)\rangle. \quad (1.95)$$

Substituting the adiabatic ansatz $|\psi(t)\rangle = c_i(t) |\psi_i(\lambda(t))\rangle$ into the Schrödinger equation and projecting onto $\langle\psi_i(\lambda(t))|$, we obtain

$$i\hbar\dot{c}_i(t) = [E_i(t) - i\hbar \dot{\lambda} \cdot \mathbf{A}_i(\lambda)] c_i(t), \quad (1.96)$$

where the Berry connection is defined as

$$\mathbf{A}_i(\lambda) = i\langle\psi_i(\lambda)|\nabla_{\lambda}\psi_i(\lambda)\rangle. \quad (1.97)$$

This first order differential equation can be integrated to yield

$$c_i(T) = \exp\left(-\frac{i}{\hbar} \int_0^T E_i(t) dt\right) \exp(i\gamma_i[\mathcal{C}]) c_i(0), \quad (1.98)$$

where the first exponential corresponds to the usual dynamical phase, while the second exponential represents an additional geometric phase given by [134]

$$\gamma_i[\mathcal{C}] = \oint_{\mathcal{C}} \mathbf{A}_i(\lambda) \cdot d\lambda. \quad (1.99)$$

The phase $\gamma_i[\mathcal{C}]$ is called geometric because it depends only on the closed path $\mathcal{C}$ traversed in parameter space and not on the rate at which the path is followed, provided the adiabatic approximation remains valid.

Eq. 1.99 shows that $\mathbf{A}_i(\lambda)$ plays a role analogous to an electromagnetic vector potential in parameter space, known as the Berry connection. The associated Berry curvature,

$$\mathbf{B}_i(\lambda) = \nabla_{\lambda} \times \mathbf{A}_i(\lambda), \quad (1.100)$$

can therefore be interpreted as an effective magnetic field in parameter space.

1.9.3.2 Theoretical Framework for Generating Synthetic Gauge Potentials via Light-Matter Interaction

As we discussed earlier, the synthetic or artificial magnetic field in neutral atomic systems arises from the coupling between the center of mass motion and the internal degrees of freedom, induced by spatially varying light-matter interaction. In ultracold atomic systems, such coupling can be implemented through different light-matter interaction schemes. One of the simplest realizations is based on a two level atomic model, in which each atom possesses a ground state and an excited state. These two levels are coupled by a monochromatic laser field [135, 31, 30]. A second widely used

configuration is the Λ -scheme, where two hyperfine ground states are coupled via Raman transitions through an excited state [31, 30]. In this subsection, we discuss the simplest scheme based on a two level atomic model. The Hamiltonian describing a two-component noninteracting atomic gas subject to light-matter interaction is given by

$$\hat{H} = \left(\frac{\hat{\mathbf{p}}^2}{2M} + V_{\text{ext}}(\mathbf{r}) \right) \otimes \mathbb{I} + \hat{H}_{\text{int}}(\mathbf{r}), \quad (1.101)$$

where $\hat{\mathbf{p}} \equiv -i\hbar\nabla$ is the momentum operator and $V_{\text{ext}}(\mathbf{r})$ represents the external trapping potential, and $\mathbb{I}$ denotes the identity operator acting on the internal Hilbert space. Here we assume that the particle is initially prepared in a specific optical dressed state. The atoms will move slowly following the adiabatic approximation, and the final state is always proportional to the initial state. We defined the eigenstates of the atom, along with their corresponding energies, through the eigenvalue equation

$$\hat{H}_{\text{int}}(\mathbf{r}) |\psi_n(\mathbf{r})\rangle = E_n(\mathbf{r}) |\psi_n(\mathbf{r})\rangle. \quad (1.102)$$

The dynamics of the atom is governed by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(\mathbf{r}, t)\rangle = \hat{H} |\psi(\mathbf{r}, t)\rangle, \quad (1.103)$$

where, $|\psi(\mathbf{r}, t)\rangle$ is the full state of the system and can be expanded in the dressed state basis as $|\psi(\mathbf{r}, t)\rangle = \sum_n \psi_n(\mathbf{r}, t) |\psi_n(\mathbf{r})\rangle$. Applying the momentum operator onto the full state vector and inserting the completeness relation $\sum_m |\psi_m(\mathbf{r})\rangle \langle \psi_m(\mathbf{r})| = \mathbb{I}$, we obtain

$$\hat{\mathbf{p}} |\psi(\mathbf{r}, t)\rangle = \sum_{m,n} [(\delta_{m,n} \hat{\mathbf{p}} + \mathbf{A}_{m,n}(\mathbf{r})) \psi_n(\mathbf{r}, t)] |\psi_m(\mathbf{r})\rangle, \quad (1.104)$$

Here, we define $\mathbf{A}_{m,n}(\mathbf{r}) \equiv \langle \psi_m(\mathbf{r}) | \hat{\mathbf{p}} | \psi_n(\mathbf{r}) \rangle$. After projecting the kinetic energy term onto a given dressed state $|\psi_i(\mathbf{r})\rangle$, we obtain:

$$\left\langle \psi_i(\mathbf{r}) \left| \frac{\hat{\mathbf{p}}^2}{2M} \right| \psi(\mathbf{r}, t) \right\rangle = \left[\frac{(\hat{\mathbf{p}} - \mathbf{A}_i(\mathbf{r}))^2}{2M} + W_i(\mathbf{r}) \right] \psi_i(\mathbf{r}, t), \quad (1.105)$$

It is observed that $\mathbf{A}_i(\mathbf{r})$ is related to the Berry connection, thus serving as an effective magnetic vector potential. Furthermore, Eq. 1.105 indicates that the atom experiences an additional effective scalar potential $W_i(\mathbf{r})$. The time evolution of the probability

amplitude $\psi_i(\mathbf{r}, t)$ is governed by $\psi_i(\mathbf{r}, t)$:

$$i\hbar \frac{\partial \psi_i(\mathbf{r}, t)}{\partial t} = \left[\frac{(\hat{\mathbf{p}} - \mathbf{A}_i(\mathbf{r}))^2}{2M} + E_i(\mathbf{r}) + W_i(\mathbf{r}) \right] \psi_i(\mathbf{r}, t). \quad (1.106)$$

Where $E_i(\mathbf{r})$ represents the energy of the dressed state.

1.9.3.3 Two-Level Atomic Model

We consider a two-level atom with transition frequency ω_0 between the ground state $|1\rangle$ and the excited state $|2\rangle$. Within the electric dipole approximation, the atom light interaction is described by [31, 30]

$$\hat{H}_{\text{dip}} = -\hat{\mathbf{d}} \cdot \mathbf{E}, \quad (1.107)$$

where $\hat{\mathbf{d}}$ is the dipole operator.

Assuming a semiclassical monochromatic electric field

$$\mathbf{E}(t) = \frac{1}{2} \left(\mathbf{E}_0 e^{-i\omega_f t} + \mathbf{E}_0^* e^{i\omega_f t} \right), \quad (1.108)$$

the total internal Hamiltonian reads

$$\hat{H}'_{\text{int}} = \hat{H}_{\text{at}} + \hat{H}_{\text{dip}} \quad (1.109)$$

where

$$\hat{H}_{\text{at}} = \frac{\hbar\omega_0}{2} |2\rangle\langle 2| - \frac{\hbar\omega_0}{2} |1\rangle\langle 1|. \quad (1.110)$$

Defining the Rabi frequency

$$\kappa = -\frac{\mathbf{d}_{eg} \cdot \mathbf{E}_0}{\hbar}, \quad (1.111)$$

with $\mathbf{d}_{21} = \langle 2|\hat{\mathbf{d}}|1\rangle$, the dipole Hamiltonian becomes

$$\hat{H}_{\text{dip}} = \frac{\hbar}{2} \left(\kappa e^{-i\omega_f t} + \kappa^* e^{i\omega_f t} \right) |2\rangle\langle 1| + c.c. \quad (1.112)$$

Rotating-Wave Approximation Under near-resonant conditions $\omega_f \approx \omega_0$, we apply the rotating-wave approximation (RWA), where rapidly oscillating counter rotating

terms are neglected. The atom light interaction Hamiltonian becomes

$$\hat{H}_{\text{dip}}^{\text{RWA}} = \frac{\hbar}{2} \left(\kappa e^{-i\omega_f t} |2\rangle\langle 1| + \kappa^* e^{i\omega_f t} |1\rangle\langle 2| \right) \quad (1.113)$$

The Hamiltonian describing the internal dynamics of the atom under the rotating-wave approximation (RWA) is given by

$$\hat{H}'_{\text{int}}{}^{\text{RWA}} = \hat{H}_{\text{at}} + \hat{H}_{\text{dip}}^{\text{RWA}}. \quad (1.114)$$

The explicit time dependence in $\hat{H}'_{\text{int}}$ can be removed by performing the unitary transformation

$$\hat{U}(t) = e^{i\omega_f t/2} |2\rangle\langle 2| + e^{-i\omega_f t/2} |1\rangle\langle 1|. \quad (1.115)$$

The transformed Hamiltonian then reads

$$\hat{H}_{\text{int}} = \hat{U} \hat{H}'_{\text{int}} \hat{U}^\dagger - i\hbar \hat{U} \frac{\partial}{\partial t} \hat{U}^\dagger, \quad (1.116)$$

which yields

$$\hat{H}_{\text{int}}^{\text{RWA}} = \frac{\hbar}{2} (\Delta |2\rangle\langle 2| - \Delta |1\rangle\langle 1| + \kappa |2\rangle\langle 1| + \kappa^* |1\rangle\langle 2|), \quad (1.117)$$

In the matrix form the Hamiltonian can be written as,

$$\hat{H}_{\text{int}}^{\text{RWA}} = \frac{\hbar}{2} \begin{pmatrix} \Delta & \kappa \\ \kappa^* & -\Delta \end{pmatrix}, \quad (1.118)$$

where $\Delta = \omega_0 - \omega_f$ is the detuning.

Generalized Rabi Frequency Introducing the generalized Rabi frequency

$$\Omega_r = \sqrt{|\kappa|^2 + \Delta^2}, \quad (1.119)$$

and defining the mixing angle θ through

$$\cos \theta = \frac{\Delta}{\Omega_r}, \quad \sin \theta = \frac{|\kappa|}{\Omega_r}, \quad (1.120)$$

with $\kappa = |\kappa|e^{-i\phi}$, the Hamiltonian can be written as

$$\hat{H}_{\text{int}} = \frac{\hbar\Omega_r}{2} \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & -\cos\theta \end{pmatrix}. \quad (1.121)$$

Dressed States The eigenstates (dressed states) are

$$|+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}, \quad (1.122)$$

with eigenvalues $\pm\hbar\Omega_r/2$.

Effective Gauge Potentials Evaluating the Berry connection for these dressed states yields the effective vector and scalar potentials:

$$\mathbf{A}_{\pm} = -\frac{\hbar}{2}(1 \mp \cos\theta) \nabla\phi, \quad (1.123)$$

$$W_{\pm} = \frac{\hbar^2}{8M} [(\nabla\theta)^2 + \sin^2\theta (\nabla\phi)^2]. \quad (1.124)$$

The corresponding synthetic magnetic field is

$$\mathbf{B}_{\pm} = \nabla \times \mathbf{A}_{\pm} = \pm \frac{\hbar}{2} \nabla(\cos\theta) \times \nabla\phi. \quad (1.125)$$

This expression shows that a nonzero synthetic magnetic field arises when both the mixing angle θ and the phase ϕ vary in space. A spatial dependence of θ can be achieved by modulating the Rabi frequency $|\kappa|$ through the laser intensity profile or by varying the detuning Δ , for example via a position dependent Zeeman shift induced by an inhomogeneous magnetic field.

After discussing the generation of a static gauge potential through rotation and light-matter interaction, it is important to emphasize that a more general and physically richer scenario arises when the gauge field itself becomes dynamical. In such cases, the effective gauge potential is no longer externally imposed but evolves self-consistently with the condensate density and phase. This leads to a time-dependent feedback manifested as a density-dependent gauge potential. The detailed formulation and derivation of this dynamical gauge potential are presented in the theoretical model developed in Chapter 5.

1.10 Vortex Lattices and Collective Dynamics of Rotating Bose–Einstein Condensates in Synthetic Gauge Fields: Recent Theoretical Developments

Since the experimental realization of BECs, ultracold atomic gases have become an ideal platform for quantum simulation. Owing to their remarkable controllability, such as tunable interactions, external trapping potentials, and optical lattice geometries, allows for the exploration of a wide range of condensed matter phenomena. Notably, various effects, such as Anderson localization [18, 19], the superfluid to Mott insulator transition [20], and the spin Hall effect [21, 22], have been successfully demonstrated in these highly controlled systems. The essence of quantum simulation lies in the ability to understand complex many body systems through controllable analog models. Interestingly, the techniques developed for these simulations often give rise to innovative methods for manipulating ultracold gases, among which the generation of synthetic gauge potentials plays a crucial role.

In this section, we present a brief overview of the existing literature on synthetic gauge potentials, focusing on the two main classes: static and dynamic gauge potentials. Their key characteristics and recent developments are discussed. We then outline the central motivation and objectives of this thesis, paying special attention to the structural transformation of vortex lattices and the associated collective excitation spectrum. Through this study, the thesis aims to address several fundamental questions regarding the influence of gauge potentials on vortex lattice structure and excitation dynamics, which may ultimately contribute to a deeper understanding of ultracold atomic systems and their potential applications in quantum simulation.

The formation of topological excitations in the form of quantized vortices is one of the most striking features of BECs. Experimentally, such vortices can be created through techniques such as laser [24, 25], rotating magnetic or optical traps [26, 106], and phase imprinting [28, 29]. In this thesis, we focus on the structural transformation of vortex lattices caused by synthetic gauge potentials, both static and dynamic. For static gauge potentials, our specific interest will be on rotating condensates. It has been observed that, analogous to type II superconductors, once the rotational frequency exceeds the critical threshold, vortices nucleate and organize into a triangular (Abrikosov) lattice structure that minimizes the system's free energy [32, 17]. Extensive theoretical and experimental studies have examined the structural transformations of vortex lattices. For example, Tung *et al.* [57] experimentally demonstrated the structural transition

of a vortex lattice induced by an optical lattice potential. Later theoretical studies within the mean field GP framework established that the resulting lattice configuration depends sensitively on the optical lattice geometry, vortex number, and lattice depth [56, 58, 136]. In the case of dipolar condensates, the situation becomes even richer, as the lattice structure additionally depends on the dipolar interaction strength and the dipole polarization angle, leading to more complex transitions [137, 138].

Beyond the structural transitions of vortex lattices induced by optical lattice potentials, considerable attention has also been devoted to the study of vortex lattice melting. Although vortex lattice melting is a well known phenomenon in type-II superconductors, where thermal fluctuations and impurities can disrupt the long range crystalline order of vortices [139–141]. Interestingly, similar behavior has also been noticed in BECs. In ultracold atomic gases, vortex lattice melting was first experimentally observed by Engels *et al.* [142] through the anisotropic expansion. Thereafter, Coddington *et al.* [60] showed that laser-induced shear can lead to significant distortion of the vortex lattice structure. Furthermore, Sharma *et al.* demonstrated that the inclusion of thermal fluctuations results in a transition to a melted vortex lattice state [143]. Several theoretical works have also been performed to complement the experimental observations. In this regard, Snoek and Stoof investigated vortex lattice melting in a stack of pancake-shaped condensates coupled via a one-dimensional optical lattice [61]. Their study revealed that interlayer tunneling of density fluctuations enhances phase fluctuations and destabilizes the vortex lattice, resulting in a melting transition. Subsequently, Mithun *et al.* analyzed vortex lattice melting using the mean field GP formalism. Their work demonstrated that both a localized Gaussian impurity and a quasiperiodic optical lattice can induce significant lattice distortion, eventually leading to melting [62]. Thereafter, numerous investigations have been carried out to examine the effects of random impurity potentials on vortex lattice melting [144, 145].

Research into quantum turbulence in ultracold atomic gases has gained significant importance, particularly when studied alongside vortex lattice melting. This intriguing field has captivated attention due to its intrinsic links to superfluid dynamics and nonequilibrium phenomena, which provide deeper insight into the behavior of complex quantum systems. Numerous experimental [146–148] and numerical investigations [125, 149] have aimed to verify the well established Kolmogorov scaling law in quantum turbulent flows, where the kinetic energy spectrum exhibits characteristic power law behavior associated with the inverse energy cascade. Several techniques have been proposed and implemented to investigate quantum turbulence in ultracold atomic gases. These techniques include stirring the condensate with a moving

Gaussian obstacle [150–152], rotating paddles [153], and merging independent condensates [154, 155].

Despite substantial progress has been made in understanding structural transformations, vortex lattice melting, and turbulence in ultracold gases, recent developments have shifted attention toward the generation of synthetic gauge fields. This area has appeared as an important subfield, as ultracold atomic systems offer an exceptionally controllable environment for engineering artificial gauge potentials, thereby providing a versatile platform for quantum simulation, enabling the exploration of several paradigmatic condensed matter phenomena [31, 30]. One notable example is the experimental observation of orbital magnetism in ultracold atomic gases [37–39]. The techniques used to generate artificial magnetism in ultracold atomic gases give rise to several novel phenomena. Depending on the implementation, these techniques can be classified into two categories: static and dynamic. As discussed earlier, in addition to rotation, static synthetic gauge potentials can also be generated through light–matter interaction. This has led to the realization of several phenomena in ultracold gases, such as spin orbit coupling in bosonic [40] systems as well as in fermionic systems [156, 157], and spin–angular-momentum coupling in bosonic gases [41–43]. In the quasidynamical scenario, the coupling between the synthetic gauge potential and the quantum state gives rise to a time dependent feedback mechanism, resulting in a density-dependent gauge potential. This concept has been theoretically explored in both continuum systems [46] and lattice based geometries [47–49]. Moreover, experimental realizations of density-dependent magnetism have been reported in lattice systems involving bosonic [158] and fermionic atoms [159], as well as in Rydberg atomic ensembles [55]. Complementary studies in continuum systems have further revealed exotic topological structures such as domain walls [160] and chiral solitons [161, 162]. This framework gives rise to a variety of unconventional phenomena. In particular, unusual transport mechanisms have been predicted in systems such as double well potentials [50], optical-lattice [54] and harmonic potentials [51, 52]. Theoretical studies have further shown that density-dependent gauge potentials lead to the violation of Kohn’s theorem [53] and can induce irregular dynamics [53] and chaotic collective behavior in two-dimensional condensates [163]. In two dimensions, it introduces an effective nonlinear rotation term that results in the formation of non-Abrikosov vortex lattices and ring like vortex structures [164–167, 63]. Subsequent studies have also demonstrated that this nonlinear rotation modifies the critical rotation frequency for vortex nucleation [168] and the appearance of ghost vortices [169].

In addition to the structural changes observed in vortex lattices, the study of collec-

tive excitation spectrum of a condensate offers important insights into the stability of these lattices. Research on such excitations, which involve vortices or vortex lattices, has been conducted in both quasi two-dimensional and three-dimensional condensates [170–172]. For instance, Smith *et al.* [173] experimentally observed a gyroscopic tilting mode of a vortex array belonging to the Kelvin–Tkachenko mode branch. Later, Simula performed a numerical analysis of the collective mode spectrum using Bogoliubov–de Gennes theory and provided a detailed characterization of vortex motion associated with Kelvin–Tkachenko waves [172]. Furthermore, studies on dipolar condensates have shown that anisotropic dipole–dipole interactions can lead to more complex excitations, including vortex-core tilting and bending [174]. In the regime of fast rotation and strong dipolar interactions, vortex lattices may even form stripe patterns, giving rise to shear motion in the low-lying Tkachenko modes, an effect absent in conventional vortex lattices with isotropic interactions [175].

Although several studies have investigated the structural transformation and melting phenomena of vortex lattices, along with the collective excitation spectrum in the presence of static and dynamic gauge potentials, several important questions remain unanswered. In particular, a detailed understanding of structural transformations under differential rotation is still lacking. Furthermore, the influence of different types of impurities on vortex-lattice structural transformations, as well as the possible emergence of turbulence in distorted vortex lattice configurations, has not been thoroughly explored. In the case of dynamic gauge potentials, studies addressing the collective excitation spectrum are also limited. Therefore, in this thesis we focus on addressing these problems through a series of investigations.

1.11 Layout of this thesis

The layout of the subsequent Chapters of the thesis is follows.

In **Chapter 2** we present the numerical framework employed to solve the Gross-Pitaevskii (GP) equation and to study the dynamical properties of the condensate. The chapter begins with the formulation of the GP equation used to model the rotating Bose–Einstein condensate. The alternative direct implicit time-splitting spectral method used in this work is then presented in detail. The underlying time-splitting strategies, namely the Lie and Strang schemes, are discussed, followed by an overview of the spectral method used for spatial discretization. Finally, the procedures for imaginary time propagation, used to obtain the ground state of the condensate, and real time

propagation, used to investigate the dynamical evolution of the system, are described.

Chapter 3 presents the first part of the research work, which investigates the structural transformation of vortex lattices in a rotating Bose–Einstein condensate subjected to a square optical lattice potential. In addition, the effect of differential rotation, where the condensate and the optical lattice rotate with different angular velocities, is examined. The structural properties of the vortex lattice are characterized using quantities such as the structure factor and lattice energy. The results show a transition from the conventional Abrikosov vortex lattice to a square lattice configuration through an intermediate coexistence phase. The influence of optical lattice parameters, including lattice strength and lattice constant, on this transition is analyzed. Furthermore, the melted regime of the vortex lattice is examined by computing the compressible and incompressible kinetic energy spectra, revealing turbulence-like behavior in the condensate.

In **Chapter 4** we investigate the influence of impurities on the structural stability and turbulence of vortex lattices in rotating Bose–Einstein condensates trapped in a square optical lattice potential. Two types of impurities are considered: static random impurities and dynamically oscillating obstacles. For static impurities, vortex-lattice melting occurs once the impurity strength exceeds a certain threshold value, which depends on the lattice constant of the impurities and the rotation frequency of the condensate. The melted regime is characterized by irregular vortex dynamics and turbulence-like behavior, confirmed through compressible and incompressible kinetic energy spectra displaying Vinen-type power-law scaling. For dynamic impurities, the melting process depends on the position and oscillation frequency of the obstacles. Interestingly, both types of impurities exhibit similar power-law behavior in the angular momentum and kinetic energy spectra, indicating universal characteristics in the turbulence-like regime.

Chapter 5 focuses on the collective excitation spectrum of a quasi-two-dimensional Bose–Einstein condensate subjected to nonlinear rotation induced by a density-dependent synthetic gauge potential. The excitation spectrum is analyzed using the Bogoliubov–de Gennes (BdG) formalism, supported by variational and hydrodynamic approaches. The study shows that the dipole-mode frequency becomes dependent on the nonlinear rotation strength, leading to a violation of Kohn’s theorem. Several vortex-displacement modes, including Tkachenko, circular, quadratic, and rational modes, are identified and analyzed. Their frequencies are found to be highly sensitive to the strength and sign of the nonlinear rotation. An analytical expression for the Tkachenko mode frequency is derived using a hydrodynamic approach and shows good agree-

ment with the BdG and Fourier analyses. The chapter also examines the splitting of excitation modes with opposite angular momenta and the behavior of surface excitation modes under positive and negative nonlinear rotation.

Chapter 6 extends the study of density-dependent synthetic gauge potentials to a toroidally trapped Bose–Einstein condensate. The chapter investigates how nonlinear rotation influences both the structural configuration and the excitation spectrum of the condensate. As the nonlinear rotation strength increases, the condensate undergoes a transition from a ring-shaped configuration to a giant vortex state at the center of the trap. The Thomas–Fermi approximation is employed to determine the parameter regime in which this transition occurs and to identify the corresponding solution region in the space defined by the nonlinear rotation strength and the toroidal radius. The collective excitation spectrum is further analyzed using both the BdG and hydrodynamic approaches to study the effect of nonlinear rotation on the frequencies of the normal modes.

Chapter 7 provides a summary of the major findings of the thesis and presents concluding remarks. The chapter highlights the key results related to vortex-lattice structural transformations, melting phenomena, turbulence-like dynamics, and collective excitations in rotating Bose–Einstein condensates. Particular emphasis is placed on the roles played by optical lattice potentials, impurities, differential rotation, and density-dependent synthetic gauge potentials in shaping the structural and dynamical properties of the system. The chapter concludes with a discussion of possible directions for future research in the study of vortex dynamics, quantum turbulence, and synthetic gauge field effects in ultracold quantum fluids.

Numerical techniques for solving the mean-field Gross-Pitaevskii equation

In this chapter, we describe the numerical method used to solve the quasi two dimensional Gross-Pitaevskii (GP) equation. There are several techniques that are used to solve this nonlinear differential equation and to obtain converged ground states and to study the dynamics of condensates. In this thesis, we primarily employ the Alternate direction implicit-time splitting pseudo spectral (ADI-TSSP) schemes for numerical simulations [176–178].

The structure of this chapter is organized as follows. We begin with a detailed description of the time-splitting spectral scheme used to solve the time-dependent GP equation in quasi-two-dimensional geometry using both imaginary and real time propagation. We then present an example demonstrating the convergence of the ground state obtained with this numerical scheme. Finally, we discuss the vortex detection technique employed in our work.

2.1 Alternate Direction Implicit-Time Splitting pseudo Spectral (ADI-TSSP) schemes

In this section, we outline the numerical approach used to solve the time dependent Gross-Pitaevskii equation (GPE). Although the GPE has been introduced in Chapter 1, we briefly restate its dimensionless form here to keep the discussion self-contained before describing the numerical implementation.

The dimensionless time-dependent GPE governing the dynamics of a quasi two

dimensional BEC can be written as

$$i \frac{\partial \psi(x, y, t)}{\partial t} = \left[-\frac{1}{2} \nabla_{\perp}^2 + V_{\text{ext}}(x, y) + g |\psi(x, y, t)|^2 - \Omega L_z \right] \psi(x, y, t), \quad (2.1)$$

where $\psi(x, y, t)$ is the condensate wave function, $V_{\text{ext}}(x, y)$ denotes the external trapping potential, g represents the interaction strength, μ is the chemical potential, and ΩL_z accounts for the rotational contribution.

To solve Eq. (2.1) numerically, we use a time-splitting pseudo spectral method combined with an alternate direction implicit (ADI) strategy. The key idea of the time splitting approach is to separate the evolution operator into parts that can be handled more efficiently. The time dependent GPE can be formally written as

$$\partial_t \psi(t, \mathbf{r}) = (A + B) \psi(t, \mathbf{r}), \quad (2.2)$$

where A and B represent the linear and nonlinear contributions to the equation, respectively. The formal solution can then be expressed as

$$\psi(t + \Delta t) = e^{(A+B)\Delta t} \psi(t). \quad (2.3)$$

The time splitting scheme approximates the evolution operator $e^{(A+B)t}$ by decomposing it into a product of operators involving e^{At} and e^{Bt} . The most commonly used splitting schemes are the Lie splitting and the Strang splitting, which are first and second order accurate in time, respectively. In the Lie splitting scheme, the evolution over one time step Δt is approximated as

$$\psi(t + \Delta t) \approx e^{A\Delta t} e^{B\Delta t} \psi(t), \quad (2.4)$$

which provides first-order accuracy in time. A higher order approximation is obtained using the Strang splitting scheme, which is second order accurate. In this approach, the evolution over one time step Δt is given by

$$\psi(t + \Delta t) \approx e^{A\Delta t/2} e^{B\Delta t} e^{A\Delta t/2} \psi(t). \quad (2.5)$$

For the GPE, the operators are chosen as

$$A = \frac{i}{2} \nabla_{\perp}^2 + i\Omega L_z, \quad (2.6)$$

$$B = -iV_{\text{ext}}(x, y) - ig|\psi|^2. \quad (2.7)$$

The nonlinear and potential contributions associated with operator B reduce to a set of ordinary differential equations that can be integrated exactly. The kinetic and rotational terms contained in operator A are treated using a pseudo spectral approach combined with the alternate direction implicit (ADI) method, where spatial derivatives are evaluated efficiently using fast Fourier transforms (FFTs). This method allows for accurate and stable time evolution of the condensate wave function while maintaining computational efficiency.

2.1.1 Lie splitting

In this approach, for a small time step Δt , the evolution can be written as

$$\psi(t + \Delta t, x) \approx e^{i(\frac{1}{2}\nabla^2 + \Omega L_z)\Delta t} e^{-i(V(x,t) + \beta|\psi|^2)\Delta t} \psi(t, x). \quad (2.8)$$

Let the total simulation time interval $[0, T]$ be discretized uniformly with time step Δt , such that $T = N\Delta t$, where N is the total number of time steps. Denoting $t_n = n\Delta t$, the Lie splitting scheme advances the solution from t_n to t_{n+1} in two stages.

First, the linear part of the equation is solved

$$i\frac{\partial\psi_1}{\partial t} = -\frac{1}{2}\nabla^2\psi_1 - \Omega L_z\psi_1, \quad \psi_1(t_n) = \psi^n. \quad (2.9)$$

Next, the nonlinear and potential contributions are treated by solving

$$i\frac{\partial\psi_2}{\partial t} = V(x, t)\psi_2 + \beta|\psi_2|^2\psi_2, \quad \psi_2(t_n) = \psi_1(t_{n+1}). \quad (2.10)$$

The solution at the next time level is then given by

$$\psi^{n+1} = \psi_2(t_{n+1}). \quad (2.11)$$

ADI treatment of the linear operator: For a two dimensional condensate $(x, y) \in \mathbb{R}^2$, the presence of the angular momentum operator $L_z = i(y\partial_x - x\partial_y)$ makes the linear operator non diagonal in Fourier space. To efficiently solve the linear step, we use an Alternate Direction Implicit (ADI) strategy. In this method, the spatial derivatives are separated into successive one-dimensional problems.

The linear evolution is therefore decomposed into two substeps:

$$i\frac{\partial\psi^{(1)}}{\partial t} = -\frac{1}{2}\partial_{xx}\psi^{(1)} - i\Omega y\partial_x\psi^{(1)}, \quad (2.12)$$

$$\psi^{(1)}(t_n) = \psi^n, \quad (2.13)$$

followed by

$$i\frac{\partial\psi^{(2)}}{\partial t} = -\frac{1}{2}\partial_{yy}\psi^{(2)} + i\Omega x\partial_y\psi^{(2)}, \quad (2.14)$$

$$\psi^{(2)}(t_n) = \psi^{(1)}(t_{n+1}). \quad (2.15)$$

Each of these one-dimensional equations can be solved efficiently using fast Fourier transforms (FFTs).

Nonlinear step Finally, the nonlinearity and potential contributions are integrated analytically. The corresponding ordinary differential equation

$$i\frac{\partial\psi}{\partial t} = V(x,t)\psi + \beta|\psi|^2\psi \quad (2.16)$$

has the exact solution

$$\psi(t,x) = \psi(t_n,x) \exp \left[-i\beta|\psi(t_n,x)|^2(t-t_n) - i \int_{t_n}^t V(s,x) ds \right]. \quad (2.17)$$

Thus the wavefunction at time t_{n+1} is obtained as

$$\psi^{n+1}(x) \approx \psi(t_{n+1},x). \quad (2.18)$$

2.1.2 Strang scheme

We now describe the Strang scheme, which is a second order accurate extension of the Lie splitting method discussed previously. In the Strang splitting approach, the time evolution operator is symmetrically decomposed so that the nonlinear and linear parts are applied in a balanced manner within each time step.

For a time step Δt , the evolution of the wave function can be approximated as

$$\psi(t + \Delta t, x) \approx e^{i(\frac{1}{2}\nabla^2 + \Omega L_z)\Delta t/2} e^{-i(V(x,t) + \beta|\psi|^2)\Delta t} e^{i(\frac{1}{2}\nabla^2 + \Omega L_z)\Delta t/2} \psi(t, x). \quad (2.19)$$

Let $t_n = n\Delta t$ denote the discrete time levels. The Strang ADI-TSSP method advances the solution from t_n to t_{n+1} through the following sequence of steps.

Step 1: Linear evolution in the x direction

$$i\frac{\partial\psi^{(1)}}{\partial t} = -\frac{1}{2}\partial_{xx}\psi^{(1)} - i\Omega y\partial_x\psi^{(1)}, \quad t_n < t \leq t_{n+\frac{1}{2}}, \quad (2.20)$$

with the initial condition

$$\psi^{(1)}(t_n) = \psi^n.$$

Step 2: Linear evolution in the y direction

$$i\frac{\partial\psi^{(2)}}{\partial t} = -\frac{1}{2}\partial_{yy}\psi^{(2)} + i\Omega x\partial_y\psi^{(2)}, \quad t_n < t \leq t_{n+\frac{1}{2}}, \quad (2.21)$$

with $\psi^{(2)}(t_n) = \psi^{(1)}(t_{n+\frac{1}{2}})$.

Step 3: Nonlinear and potential evolution The nonlinear and potential contributions are integrated by solving

$$i\frac{\partial\psi^{(3)}}{\partial t} = V(x,t)\psi^{(3)} + \beta|\psi^{(3)}|^2\psi^{(3)}, \quad t_n < t \leq t_{n+1}, \quad (2.22)$$

with $\psi^{(3)}(t_n) = \psi^{(2)}(t_{n+\frac{1}{2}})$.

Step 4: Second half-step in the y direction

$$i\frac{\partial\psi^{(4)}}{\partial t} = -\frac{1}{2}\partial_{yy}\psi^{(4)} + i\Omega x\partial_y\psi^{(4)}, \quad t_{n+\frac{1}{2}} < t \leq t_{n+1}, \quad (2.23)$$

with $\psi^{(4)}(t_{n+\frac{1}{2}}) = \psi^{(3)}(t_{n+1})$.

Step 5: Second half-step in the x direction

$$i\frac{\partial\psi^{(5)}}{\partial t} = -\frac{1}{2}\partial_{xx}\psi^{(5)} - i\Omega y\partial_x\psi^{(5)}, \quad t_{n+\frac{1}{2}} < t \leq t_{n+1}, \quad (2.24)$$

with $\psi^{(5)}(t_{n+\frac{1}{2}}) = \psi^{(4)}(t_{n+1})$. The updated wave function is then obtained as $\psi^{n+1} = \psi^{(5)}(t_{n+1})$. Each linear substep is solved efficiently using fast Fourier transforms (FFTs)

within the alternate direction implicit framework. The Strang ADI-TSSP scheme is second-order accurate in time and spectrally accurate in space, with a computational complexity of $O(M \log M)$. An advantage of this scheme is that it preserves several physical properties of the GPE, such as mass conservation and time reversibility, while remaining unconditionally stable. Although the energy is not conserved exactly, the method provides highly accurate results for the dynamical evolution of condensates.

2.1.3 Spatial discretization and implementation

In this approach, the condensate wave function is expanded in terms of Fourier modes, which allows spatial derivatives to be computed efficiently using Fast Fourier Transforms (FFTs). The computational domain is taken as a sufficiently large rectangular box $O = [-L_x, L_x] \times [-L_y, L_y]$, on which periodic boundary conditions are imposed. The domain is discretized using uniform grids in both spatial directions. Let the grid points be defined as $x_j = -L_x + jh_x$, $y_k = -L_y + kh_y$, where $j = 0, 1, \dots, J-1$ and $k = 0, 1, \dots, K-1$, and h_x and h_y are the spatial step sizes in the x and y directions, respectively. Within the pseudo-spectral framework, the wave function $\psi(x_j, y_k, t)$ is represented through discrete Fourier expansions. For example, the Fourier representation of the wave function in the x direction can be written as

$$\psi(t, x_j, y_k) = \sum_{p=-J/2}^{J/2-1} \hat{\psi}_p(t, y_k) e^{i\mu_p x_j}, \quad (2.25)$$

where $\hat{\psi}_p(t, y_k)$ are the Fourier coefficients and $\mu_p = \frac{\pi p}{L_x}$ denotes the wavenumber in the x direction. Similarly, the Fourier representation in the y direction can be expressed as

$$\psi(t, x_j, y_k) = \sum_{q=-K/2}^{K/2-1} \hat{\psi}_q(t, x_j) e^{i\lambda_q y_k}, \quad (2.26)$$

where $\hat{\psi}_q(t, x_j)$ are the Fourier coefficients in the y direction and $\lambda_q = \frac{\pi q}{L_y}$ represents the corresponding wavenumber. Using this representation, spatial derivatives can be computed spectrally. For instance, the first derivatives in the x and y directions become

$$\partial_x \psi(t, x_j, y_k) \approx \sum_{p=-J/2}^{J/2-1} i\mu_p \hat{\psi}_p(t, y_k) e^{i\mu_p x_j}, \quad (2.27)$$

$$\partial_y \psi(t, x_j, y_k) \approx \sum_{q=-K/2}^{K/2-1} i\lambda_q \hat{\psi}_q(t, x_j) e^{i\lambda_q y_k}, \quad (2.28)$$

In our work, these Fourier transforms are evaluated using the FFT and inverse FFT routines, which significantly reduce the computational cost. The linear parts of the split equations are integrated exactly in Fourier space, while the nonlinear terms are computed directly in real space. This pseudo spectral approach provides spectral accuracy in space and enables efficient computation of the condensate dynamics. For clarity, the overall numerical procedure is illustrated through the Flowchart shown below.

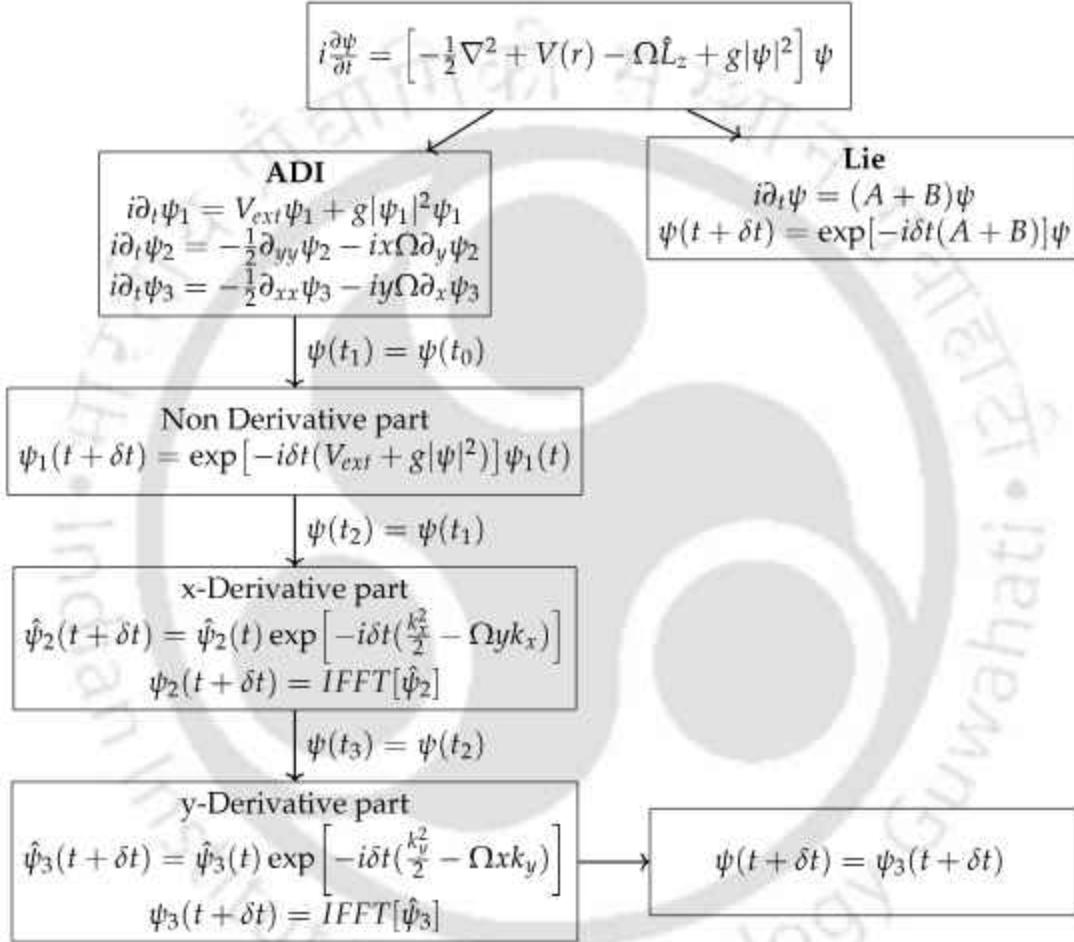


Figure 2.1: Flowchart summarizing the steps involved in the ADI-TSSP method with Lie splitting for the numerical solution of the time-dependent GPE.

Imaginary and real time propagation: In order to obtain the ground state of the condensate, we utilize the imaginary time propagation technique. In this method, the real time variable is transformed as $t \rightarrow -i\tau$. During this evolution, higher energy components of the wave function decay exponentially faster than the lowest energy state,

allowing the system to converge to the ground state after repeated normalization of the wave function. Once the ground state is obtained, the real time propagation method is used to study the dynamical evolution of the condensate. In this case, the time dependent GPE is evolved in real time using the same ADI-TSSP scheme described above.

2.1.4 Convergence of ground state

dx	N_x, N_y	dt	Energy	μ
0.2	2^6	10^{-3}	3.190598	4.349339
0.1	2^7	10^{-3}	3.190496	4.349331
0.05	2^8	10^{-3}	3.190496	4.349328
0.02	2^9	10^{-3}	3.190496	4.349325
0.01	2^{10}	10^{-3}	3.190496	4.349310

Table 2.1: Numerical convergence for a vortex state in a harmonically confined condensate obtained via imaginary time propagation. The energy and chemical potential (μ) converge to four decimal precision as the spatial grid spacing decreases ($dx \lesssim 0.05$).

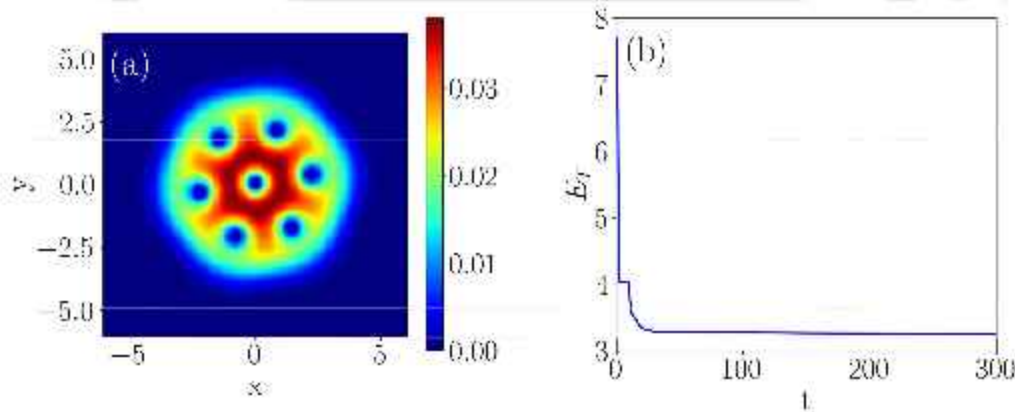


Figure 2.2: (a) Pseudocolor plot of the condensate ground state density $|\psi(x, y)|^2$ in a harmonic trapping potential for nonlinearity $g = 100$ and rotation frequency $\Omega = 0.8$. (b) Convergence of the total energy as a function of time during imaginary-time propagation toward the ground state. All lengths are expressed in units of the harmonic oscillator length a_{ho} , and time is expressed in units of ω^{-1} .

To illustrate the efficiency of the Strang time splitting spectral method combined with the alternating direction implicit (ADI-TSSP) scheme, we present a numerical example demonstrating the convergence of the ground-state solution of a rotating Bose-Einstein condensate. The method is employed to obtain the stationary state through imaginary time propagation, while the stability and accuracy of the numerical scheme are examined by considering different spatial and temporal discretizations. In

this study, the condensate is confined in a harmonic trapping potential and subjected to rotation. Starting from an appropriate initial condition (Gaussian), the system evolves toward its ground state under imaginary time propagation. For sufficiently large rotation frequencies, the condensate develops quantized vortices which self organize into a regular triangular (hexagonal) vortex lattice structure. To verify the reliability of the numerical scheme, simulations are performed for various values of the spatial grid spacing (dx) and the time step ($dt = 10^{-3}$). The convergence of the solution is analyzed through the total energy and the condensate density profile.

2.2 Vortex Detection Algorithm

It is well established that a condensate is represented by a complex wave function, and vortices within the condensate can be identified through two characteristic properties: (i) the circulation of the phase around the vortex core and (ii) a local minimum in the probability density. In this section, we describe the numerical vortex detection algorithm used to identify vortices in the mean-field wave function ψ . The algorithm is designed for efficient implementation and allows not only the localization of vortices but also the determination of their sign and multiplicity.

2.2.1 Detection using phase circulation

Vortices are detected by computing the phase circulation around each grid point of the discretized wave function. Since the phase of the wave function is represented by its principal value $\text{Arg}(\psi) \in (-\pi, \pi]$, discontinuities appear whenever the phase crosses $\pm\pi$. To recover the continuous phase variation, the phase differences between neighboring grid points are corrected by adding or subtracting appropriate multiples of 2π whenever the magnitude of the phase jump exceeds π . In practice, the phase circulation is evaluated around each grid point (x_i, y_j) by considering the eight surrounding neighboring points that form a closed loop on the computational lattice (see Fig. 2.3). The principal phase $\phi_p = \text{Arg}(\psi)$ is first calculated at each point along the loop. Moving sequentially along the closed path, the phase difference between successive points is evaluated. If the magnitude of the phase difference exceeds π , the phase of the second point is corrected by adding or subtracting an appropriate integer multiple of 2π so that the phase difference between the two points remains within the interval $(-\pi, \pi]$. After evaluating the phase differences for all eight successive pairs of points along the closed loop, the algorithm returns to the starting point. The total

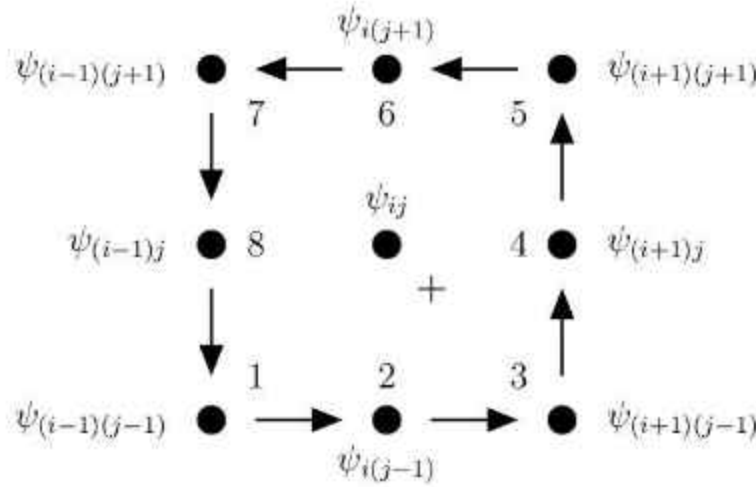


Figure 2.3: Schematic diagram illustrating the calculation of phase circulation around a grid point. The central point represents the location about which the phase circulation is evaluated using the surrounding neighboring points.

phase change accumulated along this path gives the phase circulation around the grid point. This circulation is obtained from the difference between the reconstructed phase and the original principal phase at the starting location. Finally, dividing the phase circulation by 2π yields the vortex charge (winding number) m associated with that grid point.

Alternatively, the same procedure can be interpreted as counting the phase jumps along the closed loop. A negative phase jump with magnitude greater than π contributes $+1$, while a positive phase jump with magnitude greater than π contributes -1 . The sum of these contributions gives the winding number of the vortex.

2.2.2 Detection using density curvature

The condensate density alone is not sufficient to identify vortices, since the density vanishes at the vortex core. To reduce numerical errors, we also consider the density curvature, which increases away from the vortex center over a characteristic length scale determined by the healing length. The density curvature is computed using a finite difference approximation,

$$\kappa = \frac{|\psi_{i-1,j}|^2 + |\psi_{i+1,j}|^2 - 2|\psi_{i,j}|^2}{(\Delta x)^2} + \frac{|\psi_{i,j-1}|^2 + |\psi_{i,j+1}|^2 - 2|\psi_{i,j}|^2}{(\Delta y)^2}. \quad (2.29)$$

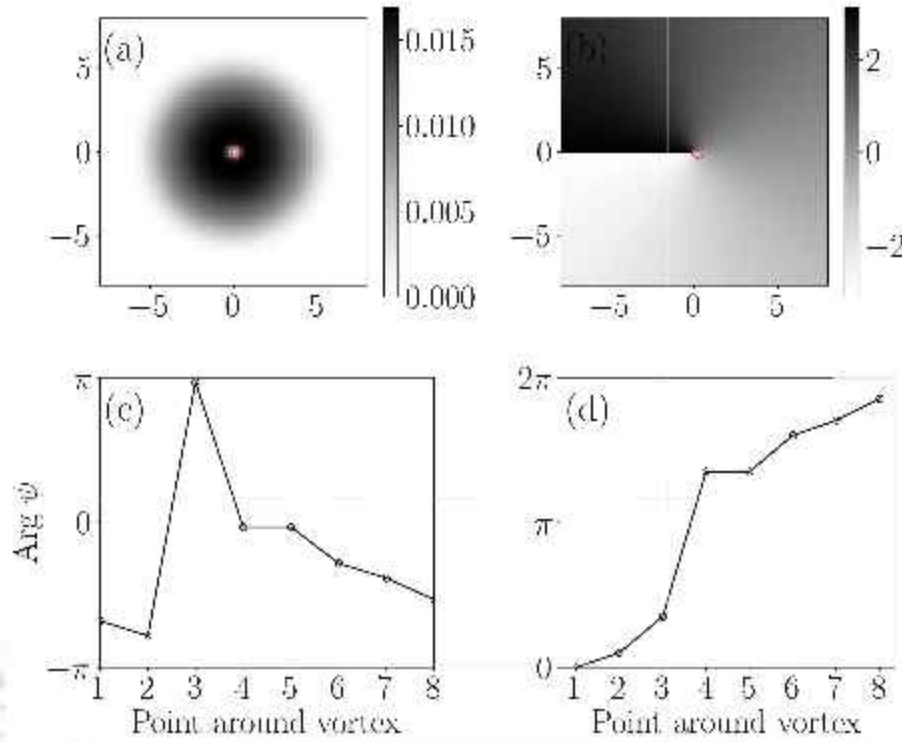


Figure 2.4: Detection of a vortex through phase circulation. (a) Condensate density $|\psi|^2$ with the detected vortex position marked. (b) Phase $\arg(\psi)$ of the wavefunction showing the phase singularity. (c) Principal phase evaluated at the eight points surrounding the vortex core. (d) Corresponding unwrapped phase, showing a continuous phase increase from 0 to 2π , indicating a quantized circulation of 2π .

Vortices with curvature less than a threshold κ_v are discarded, as they correspond to regions of very low condensate density. In practice, we use $\kappa_v \approx 10^{-3}$. Combining the density curvature criterion with the phase circulation method provides a robust vortex detection technique.

Figure 2.4 illustrates the vortex detection using phase circulation. The vortex core is identified from the density minimum, while the phase of the wave function shows a phase singularity. The principal phase around the vortex is then unwrapped, revealing a continuous phase change of 2π , which confirms the presence of a quantized vortex.

2.3 Summary and Conclusion

In this chapter, we have discussed the numerical method used to solve the dimensionless Gross-Pitaevskii equation (GPE). The equation is solved using the alternating direction implicit time-splitting spectral (ADI-TSSP) scheme, where imaginary time

propagation is used to obtain the ground state and real time propagation is used to study the condensate dynamics. We have also implemented a vortex detection method based on the phase circulation of the condensate wavefunction. A vortex is identified when the total phase winding around a closed contour equals 2π , indicating a singly quantized vortex.



Structural transformation of the vortex lattice in presence of differential rotation between the square lattice and the condensates

In this chapter, we present the structural transformation of a vortex lattice in the presence of a square lattice potential. In particular, we analyze the role of differential rotation, where the lattice potential and the condensate rotate at different angular velocities. The structural changes are quantified using the structure factor and lattice energy. Our results demonstrate that, in the absence of differential rotation, the vortex lattice transforms from an Abrikosov lattice to a square lattice through an intermediate coexistence phase, with the transition strongly dependent on the lattice parameters such as lattice strength and lattice constant. When differential rotation is introduced, a variety of distinct structural phases emerge, with the system behavior varying significantly as the relative rotation rate changes from negative to positive values. Moreover, in the melted regime, signatures of turbulence appear, characterized by power law scaling in both compressible and incompressible kinetic energy spectra.

The chapter is organized as follows. We first provide a brief introduction highlighting the motivation for the present study. This is followed by a description of the relevant physical quantities, such as the structure factor, entropy, and energy spectrum, etc., used to characterize the structural transition and turbulence phenomena. We then present detailed numerical simulation results obtained from the mean field

Gross-Pitaevskii model. A portion of the results presented in this chapter, corresponding to the case without differential rotation, has been published in *Nonlinear Dynamics and Applications: Proceedings of the ICNDA 2022* (Springer International Publishing), pp. 1251–1261 (2022). The complete results of this chapter will be communicated in a forthcoming publication.

3.1 Introduction

The structural transformation of vortex lattices in BECs has been extensively discussed in Chapter 1. However, to ensure this chapter is self contained, we present a brief review of the structural changes of the vortex lattice in presence of the optical lattice for the rotating BECs. Since the first experimental realization of vortices in BECs by the Madison group using laser stirring [33], the field has experienced significant theoretical and numerical progress aimed at understanding the mechanisms governing vortex lattice formation. Many studies analyze the structure and dynamics of vortex lattices within the framework of the mean field Gross-Pitaevskii equation. In recent years, BECs trapped in optical lattices have become a versatile platform for exploring various condensed matter phenomena, including Bloch oscillations [179], Wannier-Stark ladders [180], Josephson junction arrays [181], and the superfluid-Mott-insulator transition [20]. Several studies have also investigated the influence of the optical lattice depth and periodicity on the vortex lattice structure [182, 183, 57, 58, 56]. In this context, Tung et al. [57] experimentally observed the structural phase transition of the vortex lattice from a hexagonal to a square configuration by varying the pinning strength of the optical lattice potential. Based on the mean field GP formalism, Pu et al. [56] demonstrated that the structural transition strongly depends on the ratio of the number of vortices to the number of pinning sites, with enhanced vortex pinning occurring when the vortex density matches the lattice potential density [58]. More recently, several studies have revealed the complex nature of vortex lattice structural transitions in binary and dipolar BECs trapped in optical lattices [184, 136]. For instance, Williams et al. [27] showed that the dynamical regimes of vortex lattices depend sensitively on the depth and periodicity of the optical lattice. In dipolar systems, structural transitions between different vortex-lattice configurations, such as from Abrikosov to square lattices, can also occur by tuning the rotation frequency or the ratio of inter- and intraspecies interactions even in the absence of optical lattices [185]. Furthermore, Kumar et al. [59] reported transitions from square lattices to vortex sheet like structures in

dipolar BECs confined in square optical lattices by varying the ratio of interspecies and intraspecies interactions. On the other hand, quantum turbulence in ultracold atomic gases has emerged as an important area of research, particularly in relation to vortex lattice melting. Owing to its strong connection with superfluid hydrodynamics and nonequilibrium dynamics, this field provides a powerful framework for understanding complex physical systems. Both experimental [146–148] and numerical investigations [125, 149] have explored the appearance of Kolmogorov scaling in quantum turbulent flows, where the kinetic energy spectrum displays a characteristic power-law behavior corresponding to the inverse energy cascade. To generate and probe turbulence in ultracold gases, several approaches have been developed, such as stirring the condensate with a moving Gaussian obstacle [150–152], using rotating paddles [153], and merging independent condensates [154, 155].

In this work, we perform a detailed numerical investigation of the structural transformation of vortex lattices in a BEC within the condensate and a Gaussian distributed square lattice potential. By obtaining the ground state of the condensate for different lattice parameters, we find that, in the absence of differential rotation, the triangular (Abrikosov) vortex lattice transforms into a square lattice through an intermediate coexistence state. When differential rotation is introduced, a variety of distinct structural phases emerge. These transitions are characterized using quantities such as the structure factor, angular momentum, and lattice energy. Furthermore, by analyzing the kinetic energy spectrum, we demonstrate the emergence of turbulence in the melted regime.

3.2 Mean-field model

In a rotating frame of reference, condensate confined by an external potential and rotated with an angular velocity Ω can be modelled using the following dimensionless dissipative Gross Pitaevskii equation [184]:

$$(i - \gamma)\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) - \mu + g_{3D} |\Psi(\mathbf{r}, t)|^2 - \Omega(\mathbf{r}, t) \hat{L}_z \right] \Psi(\mathbf{r}, t), \quad (3.1)$$

where $g_{3D} = 4\pi\hbar^2 a_s / m$ represents the nonlinear interaction, with a_s being the s -wave scattering length, μ is the chemical potential, $L_z = -i\hbar (x\partial_y - y\partial_x)$ represents the an-

gular momentum, and γ is the dissipation parameter. For our analysis we consider the external potential as,

$$V_{\text{ext}} = V_{\text{ho}} + V_{\text{SL}} \quad (3.2)$$

In the above equation, $V_{\text{ho}} = \frac{1}{2}m\omega_{\perp}^2(x^2 + y^2) + \omega_z^2 z^2$ represents the harmonic potential, while V_{SL} denotes the Gaussian-distributed square lattice given by

$$V_{\text{SL}} = \sum_{n_1} \sum_{n_2} V_0 \exp \left[-\frac{|\mathbf{r} - \mathbf{r}_{n_1, n_2}|^2}{(\sigma/2)^2} \right],$$

where $\mathbf{r}_{n_1, n_2} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2$ denote the lattice vector with n_1 and n_2 being the number of lattice sites along the x - and y - direction respectively. V_0 is the strength of the Gaussian distributed square lattice potential, and σ represents the width of the Gaussian laser beam [186].

Since our work focuses on the structural transformation of vortex lattice in the quasi two-dimensional regime, we assume a pancake geometry for the condensate ($\lambda = \omega_z / \omega_{\perp} \gg 1$). Under this assumption the wave function can be decomposed as $\Psi(\mathbf{r}, t) = \psi(x, y, t) \exp(-z^2/2\sigma_z^2) / \sqrt[4]{\pi a_{hz}^2}$, with $a_{hz} = \sqrt{\hbar/(m\omega_z)} = a_{\perp} / \sqrt{\lambda}$, which rescaled the nonlinear interaction g_{3D} by a factor $1/(2\pi a_{hz})$. This reduction leads to the following two-dimensional (2D) Gross-Pitaevskii equation:

$$i\hbar \frac{\partial \psi(x, y, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla_{\perp}^2 + V_{\text{ext}}(x, y) - \mu + \frac{g_{3D}}{2\pi a_{hz}} |\psi(x, y, t)|^2 - \Omega \hat{L}_z \right] \psi(x, y, t) \quad (3.3)$$

Equation (3.3) can be rewritten in the dimensionless form by introducing the characteristic scales of the harmonic potential as length $a_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$, time $\omega_{\perp}$, and energy $\hbar\omega_{\perp}^{-1}$. The length, time, and the wave function are scaled as $t = \omega_{\perp}^{-1} t'$, $(x, y) = a_{\perp}(x', y')$, and $\psi = (\sqrt{N}/a_{\perp}) \psi'$. Applying these transformations yields the dimensionless Gross-Pitaevskii equation:

$$(i - \gamma) \frac{\partial \psi(x, y, t)}{\partial t} = \left[-\frac{1}{2} \nabla_{\perp}^2 + V_{\text{ext}}(x, y) - \mu + g |\psi(x, y)|^2 - \Omega L_z \right] \psi(x, y, t). \quad (3.4)$$

Where the nonlinear interaction parameter is given by $g = 4N\pi a_s \sqrt{\pi\lambda} / \sqrt{2} a_{\perp}$, the

harmonic trapping potential is defined as $V_{ho} = \frac{1}{2}(x^2 + y^2)$, and the angular momentum operator is represented by $L_z = i(y\partial_x - x\partial_y)$. For simplicity, the primes have been omitted from the dimensionless variables. In this equation, the coordinates (x and y), time (t), and angular velocity Ω is normalized by the units of harmonic oscillator length $a_{\perp}$, the inverse trapping frequency $\omega_{\perp}^{-1}$, and the frequency $\omega_{\perp}$, respectively. The amplitudes of the Gaussian distributed square lattice (V_0), and the chemical potential (μ), are normalized to the unit of $\hbar\omega_{\perp}$. The dissipation term γ , has been incorporated to account for the effects of temperature and the non-condensate fraction [187, 188]. Physically, the parameter γ represents the rate at which non-condensate particles get converted into the condensate one. This can be estimated using the transition probability $W^+(N)$, as calculated by Gardiner *et al.* [189], using quantum kinetic theory. Here $W^+(N)$ denotes the rate at which thermal particles above the condensate band enter the condensate due to collisions. The value of γ has been calculated using the following formula [190].

$$\gamma = \frac{W^+(N)}{\omega_{\perp}} = \frac{4g_c m (a_s k_b T)^2}{\pi \hbar^3 \omega_{\perp}}, \quad (3.5)$$

where $g_c = 3$ is the correction factor [190], a_s is the scattering length for the nonlinear interaction term k_b is Boltzmann's constant, and T is the temperature of the non condensate. Using the experimental parameters [90, 191], $T = 0.1T_c$, $T_c = 300$ nK, $a_s = 5.5$ nm, and $\omega_{\perp} = 2\pi \times 10$ Hz, we obtain $\gamma \approx 0.03$.

The Gaussian distributed square lattice potential V_{SL} can be realized in experiments through the superposition of four counter-propagating laser beams with wavelength $\lambda \approx 830$ nm [27] that consider the form as [136],

$$V_{SL} = \sum_{n_1} \sum_{n_2} V_0 \exp \left[-\frac{|\mathbf{r} - \mathbf{r}_{n_1, n_2}|^2}{(\sigma/2)^2} \right], \quad (3.6)$$

where $\mathbf{r}_{n_1, n_2} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2$ denote the lattice vector with n_1 and n_2 being the number of lattice sites along the x - and y - direction respectively. V_0 is the strength of the Gaussian distributed square lattice potential, and σ represents the width of the Gaussian laser beam [186]. For our analysis, we have considered the lattice unit vector as $\mathbf{a}_1 = a\hat{i}$, $\mathbf{a}_2 = a\hat{j}$ to obtain the square lattice, where a represents the lattice constant. In the present, we vary both the lattice constant as well as the potential strength (V_0) to obtain the vortex lattice state. Throughout the study, the Gaussian beam waist of the laser is kept at $\sigma = 0.65$.

To incorporate differential rotation, in which the condensate (Ω) and the lattice potential (ω) rotate with different angular velocities, we project the lattice potential onto the condensate frame by performing the following coordinate transformation:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(\delta\omega t) & \sin(\delta\omega t) \\ -\sin(\delta\omega t) & \cos(\delta\omega t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.7)$$

Here $\delta\omega = \omega - \Omega$ denotes the relative rotation frequency between the condensate and the square lattice (SL). Accordingly, two cases are considered: (i) when the condensate rotates faster than the pinning potential, i.e., $\Omega > \omega$, and (ii) when the lattice potential rotates faster than the condensate, i.e., $\omega > \Omega$.

The chapter is organized as follows. In Sec. 3.2, we present the mean-field formulation based on the dissipative GP equation. In Sec. 3.3, we provide a detailed account of the numerical simulations, along with a brief description of the relevant physical observables used to identify the structural phases and characterize the vortex lattice configurations. In Sec. 3.4, we present the numerical results on the structural transformations of the vortex lattice in the absence and presence of differential rotation. In both cases, we first analyze the ground state configurations, followed by a discussion of the corresponding physical quantities. Finally, we summarize our findings in Sec. 3.5.

3.3 Numerical details and different quantities to characterize the structure and dynamics of vortex lattice

We use the GPE-lab software that uses the implicit-time splitting pseudo spectral method [176] to solve the governing mean-field GP equation (4.1) with a space step $\Delta x = \Delta y = 0.1860$ and time step $\Delta t = 0.001$. To make our simulation results viable to the experiment, we choose $N = 2 \times 10^4$ atoms corresponding to the condensate of ^{87}Rb atom. The axial and transverse frequency of the harmonic trap is considered as $\omega_z = 2\pi \times 100$ Hz and $\omega_\perp = 2\pi \times 10$ Hz, respectively. Considering the scattering length of ^{87}Rb as $a_s = 5.5\text{nm}$ [191, 33] with the characteristic length scale as $a_\perp = 2.41\mu\text{m}$. We fix the other parameters as $g = 1000$ and $\gamma = 0.03$. In order to conserve the norm of the condensate wave function for a nonzero value of γ , the chemical potential has been adjusted at each time step. In our simulation, the chemical potential at each time step

is corrected using $\Delta\mu = (\Delta t)^{-1} \ln \left(\int d^2\mathbf{r} |\psi(\mathbf{r}, t)|^2 / \int d^2\mathbf{r} |\psi(\mathbf{r}, t + \Delta t)|^2 \right)$ [144].

In this work, we utilize the structure factor to characterize the structural transformation of the vortex lattice of the rotating BECs. The structure factor is defined as $S(\mathbf{k}) = \frac{1}{N_c} \sum_{i=1}^{N_v} n_i \exp(i\mathbf{k} \cdot \mathbf{r}_i)$, where n_i , $\mathbf{r}_i$, and N_c denote the winding number, position of the i th vortex, and the total winding number, respectively. N_v is the total number of vortices present in the condensate. The nature of the vortex lattice is determined based on the value of $S(k) = |S(\mathbf{k})|$. For instance, with a perfect square lattice, $S(k)$ lies between $0.5 \lesssim S(k) \lesssim 1.0$. For the situation when $S(k) < 0.5$, we characterize the state of the vortex lattice as deviated from the square structure and thus termed as in the melted state.

To elucidate the transition from the vortex lattice state to the melting state, we have also utilized the total energy of the condensate as one of the relevant physical quantities. The total energy of the system is defined as [176],

$$E_T = \int \left[\frac{1}{2} |\nabla\psi|^2 + V_{\text{ext}} |\psi|^2 + \frac{g}{2} |\psi|^4 - \Omega \psi^* L_z \psi \right] dx dy, \quad (3.8)$$

where the first, second, third, and fourth terms represent the kinetic (E_K), the potential (E_{pot}), the interaction (E_{int}), and the rotational energy (E_{rot}) of the condensate, respectively.

To characterize the melting state of the vortex lattice we compute the energy spectrum and show the presence of the turbulence-like features of the condensate. The kinetic energy spectra have two components, namely compressible and incompressible parts in which the former is related to the sound wave production and later is related to the vortices [123, 153]. The scaling laws in the kinetic energy spectrum provide us with a deeper understanding of the turbulent behaviour in a fluid [125, 127]. The kinetic-energy $\frac{1}{2} |\nabla\psi|^2$ in equation (3.8) can be written as:

$$\frac{1}{2} |\nabla\psi|^2 = \frac{1}{2} \left(\rho(\mathbf{r}) |\mathbf{u}|^2 + \left| \nabla \sqrt{\rho(\mathbf{r})} \right|^2 \right), \quad (3.9)$$

where the Madelung transformation $\psi(\mathbf{r}) = \sqrt{\rho(\mathbf{r})} e^{i\phi}$ yields the condensate density and the superfluid velocity $\mathbf{u} = \nabla\phi$. The first and second terms represent, respectively, the kinetic energy density E_K and the quantum pressure E^q . The different component

of energy are given by

$$E_K = \frac{1}{2} \int \rho(r) |\mathbf{u}|^2 d^2r, \quad (3.10)$$

$$E^q = \frac{1}{2} \int \left| \nabla \sqrt{\rho(r)} \right|^2 d^2r. \quad (3.11)$$

Here, the velocity $\mathbf{u}$ is decomposed as a sum of the incompressible $\mathbf{u}_{ic}$ and compressible $\mathbf{u}_c$, such as $\mathbf{u} = \mathbf{u}_{ic} + \mathbf{u}_c$, where $\nabla \cdot \mathbf{u}_{ic} = 0$ represents the solenoidal part and $\nabla \times \mathbf{u}_c = 0$ represents the irrotational part.

Following Parseval's theorem, different energy components can be represented as

$$E_{\text{kin}}^\zeta = \frac{m}{2} \int d^2\mathbf{k} |\hat{\mathbf{u}}^\zeta(\mathbf{k})|^2, \quad (3.12)$$

where $\zeta \in \{i, c, q\}$ being incompressible, compressible and quantum pressure components of the kinetic energy, respectively and

$$\hat{\mathbf{u}}^\zeta(\mathbf{k}) = \frac{1}{2\pi} \int d^2\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \mathbf{u}^\zeta(\mathbf{r}) \quad (3.13)$$

Transforming equation (3.12) into the cylindrical coordinates in Fourier space (k space) for a 2D condensate, we obtain

$$E_{\text{kin}}^\zeta = \int_0^\infty dk k \varepsilon_{\text{kin}}^\zeta(k). \quad (3.14)$$

In general, to compute the energy spectra $\varepsilon_{\text{kin}}^\zeta$, data are first binned in the wavenumber (k) space, and then summed over angular intervals. This approach is suitable for small- k regimes. However, to obtain a more precise spectrum, we have used the analytical evaluation of k -space integrals and its numerical implementation developed by Bradley and coworkers [128]. The scheme utilizes the angle-averaged Wiener-Khinchin theorem, which connects the spectral densities with an associated correlation function. Following this, $\varepsilon_{\text{kin}}^\zeta$ from equation (3.14) could be represented as [128]

$$\varepsilon_{\text{kin}}^\zeta(k) = \frac{m}{2} \int d^2\mathbf{x} \Lambda_2(k, |\mathbf{x}|) C[\mathbf{u}^\zeta, \mathbf{u}^\zeta](\mathbf{x}), \quad (3.15)$$

where $\Lambda_2(k, |\mathbf{x}|) = (1/2\pi)k J_0(k|\mathbf{x}|)$ is the 2D kernel function, involving the Bessel function J_0 and $C[\mathbf{u}^\zeta, \mathbf{u}^\zeta](\mathbf{x})$ represents the two-point auto-correlation function in position for a given velocity field. The above relation implies that for any of the position-space fields $\mathbf{u}^\zeta$, there exists a spectral density [see equation (3.15)], that is, equivalent to an

angle-averaged two-point correlation in k space.

To have a comprehensive understanding of the transition from an ordered vortex lattice to the melted-vortex-lattice state, we have used the total information entropy (S_{enp}) for both real space and momentum space. In real space, the entropy (S_r) can be expressed as

$$S_r = - \int \rho(\mathbf{r}) \log \rho(\mathbf{r}) d^2 \mathbf{r}, \quad (3.16)$$

where $\rho(\mathbf{r}) = |\psi(\mathbf{r})|^2$ represents probability density in the real space. The entropy in momentum space S_k is given by

$$S_k = - \int \rho(\mathbf{k}) \log \rho(\mathbf{k}) d^2 \mathbf{k}, \quad (3.17)$$

where $\rho(\mathbf{k}) = |\psi_{\mathbf{k}}|^2$ is the probability density in the Fourier space. The total entropy $S_{\text{en}} = S_r + S_k$ has the contribution of both real space and momentum space entropy [192, 59]. The finite value of the entropy characterizes the extent of disorderness present in the system. For our analysis, the S_{en} is rescaled with respect to the entropy corresponding to condensate without having an impurity (S_{en}^0).

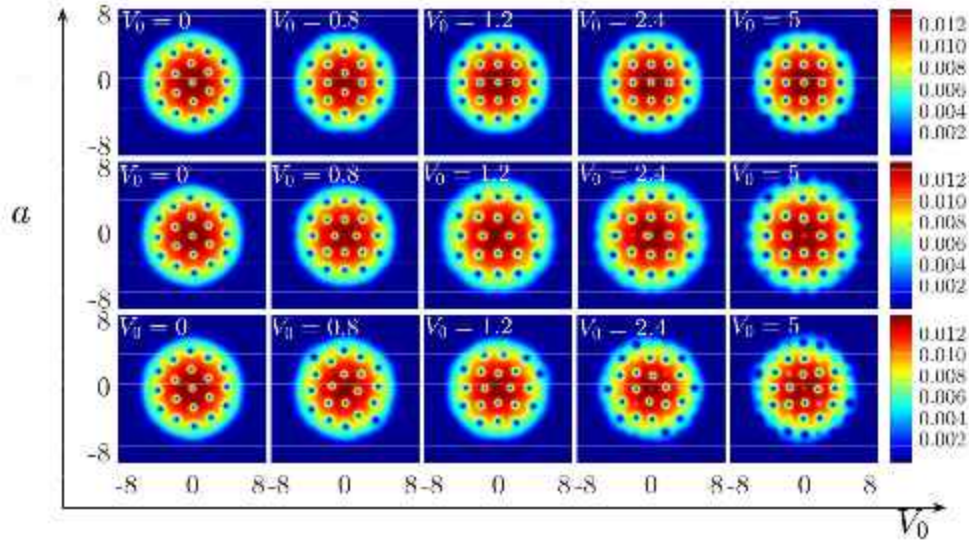


Figure 3.1: Density profile in real space with the increasing strength of the lattice potential from left to right: (a1)–(a5) for $a = 2.2$ and $V_0 = (0.0, 0.8, 1.2, 2.4, 5.0)$; (b1)–(b5) for $a = 2.0$ and $V_0 = (0.0, 0.8, 1.2, 2.4, 5.0)$; and (c1)–(c5) for $a = 1.8$ and $V_0 = (0.0, 0.8, 1.2, 2.4, 5.0)$ for $\Omega = 0.7$ and $g = 1000$. For lower V_0 vortex lattice is triangular in nature also known as Abrikosov lattice(AL). As the value of V_0 increases, the vortex lattice is transformed into a square lattice for high values of V_0 .

3.4 Numerical Results

3.4.1 Effect of square lattice potential on the vortex lattice

In this section, we present our numerical results for the vortex lattice structure of a BECs in the presence of Gaussian distributed square lattice potential. In our numerical simulations, we fix the angular velocity at $\Omega = 0.7$ and vary the potential amplitude V_0 for different lattice constants a . Figure 3.1 shows the condensate density profiles for three different lattice constants (a) at this fixed angular velocity, while gradually increasing the lattice potential strength V_0 . From the first column of Fig. 3.1, we observe that in the absence of an square lattice potential the singly quantized vortices arrange themselves in a triangular pattern. This configuration is known as the Abrikosov lattice (AL), owing to its similarity with the vortex lattice structure observed in type-II superconductors [33]. For sufficiently small values of V_0 , the triangular lattice exhibits slight distortions due to the influence of the square lattice potential. As the amplitude of the lattice potential increases, the vortices gradually align with the antinodes of the lattice potential. At sufficiently large lattice depths, all vortices become pinned to these sites, resulting in a pinned lattice (PL) configuration that exhibits square symmetry in our case.

In Fig. 3.2, we present the reciprocal lattice structure corresponding to the vortex lattices shown in Fig. 3.1 for different pinning strengths V_0 . This provides information about the periodicity of the condensate density. In the absence of the square lattice potential ($V_0 = 0$), the reciprocal lattice exhibits a regular hexagonal pattern, consistent with the triangular Abrikosov lattice observed in the density plots. For weak lattice strength ($V_0 = 0.8$), the triangular lattice becomes slightly distorted. With further increase in V_0 , a distorted square pattern emerges and eventually evolves into a well defined square lattice around $V_0 = 2.0$, becoming more pronounced at $V_0 = 5$. These results indicate a structural crossover of the vortex lattice with increasing pinning strength.

In order to quantify the potential strength at which the structural transformation in the vortex lattice takes place, in Fig. 3.3, we show the variation of structure factor profile and lattice energy with respect to V_0 for different Ω . We notice that at small rotation ($\Omega = 0.55, 0.58, 0.62$), the structure factor profile and lattice energy show there exists an intermediate state where it is difficult to determine the structure of vortex lattice. This we term as the coexistence state (CS). We find that for $\Omega = 0.55$ the CS begins at $V_0 = 0.6$ and the vortex structure is completely pinned above $V_0 = 1.4$. For

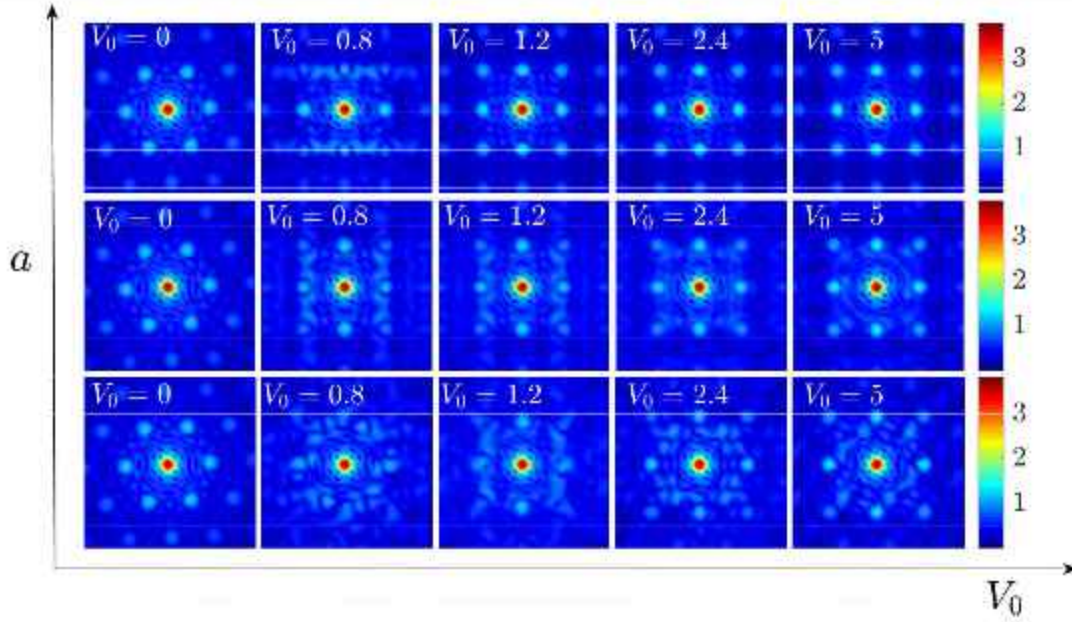


Figure 3.2: Pseudo color representation of the condensate density in momentum space validating the lattice structures shown in Fig. 3.1 for different pinning strengths: (a1)–(a5) for $a = 2.2$ and $V_0 = (0.0, 0.8, 1.2, 2.4, 5.0)$; (b1)–(b5) for $a = 2.0$ and $V_0 = (0.0, 0.8, 1.2, 2.4, 5.0)$; and (c1)–(c5) for $a = 1.8$ and $V_0 = (0.0, 0.8, 1.2, 2.4, 5.0)$, where the pinning strength increases from left to right.

$\Omega = 0.58$ and $\Omega = 0.62$, the CS starts from $V_0 = 0.4$ and 0.8 respectively and the vortex lattice gets fully pinned above $V_0 = 1.4$ and 2.2 , respectively.

The right panel of Figure 3.3 shows the lattice energy and structure factor plots for relatively higher rotations ($\Omega = 0.725, 0.8, 0.85$) for the lattice constant $a = 2.2$. For $\Omega = 0.725$ we find that the lattice energy diminishes steadily upon increasing V_0 , which suggests partial pinning of vortices. A sharp decrease in $E_{lattice}$ is noticeable at $V_0 = 0.6$ which indicates that the vortices are pinned above this pinning strength. Similarly, the structure factor $S(k)$ increases successively against V_0 and exhibits steady behavior above $V_0 = 0.6$ where all vortices get attached to the pinning sites. Similar trend has been observed for $\Omega = 0.8$ and 0.85 (See Figs. 3.3(e) and 3.2(f)).

In Fig. 3.4 we plot the phases of the vortex lattice in $\Omega - V_0$ plane for different lattice constants $a = 1.8, 2.0, 2.2$. We find that at $\Omega = 0.725$ the minimum strength of potential (V_{min}) required to get the perfectly pinned square lattice of the vortices is 0.6 . The rotation at which the minimum strength of the potential is required to perfectly pin the lattice depends upon the lattice constant of the square lattice. We vary the lattice constant between $a = 1.8 - 2.2$ and observe that the transformation of the vortex lattice structure against V_0 for different lattice constant a . The phase diagrams for other lattice constants (a) (does not present here) have the same nature as those for $a = 2.2$.

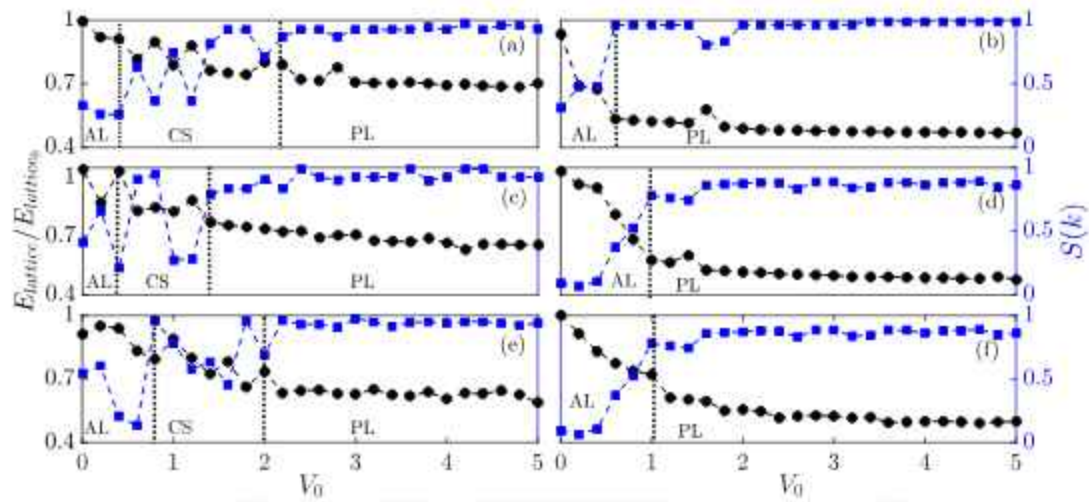


Figure 3.3: Variation of structure factor $S(k)$ and lattice energy $E_{lattice}$ with pinning strength V_0 for different rotational frequencies: (a) $\Omega = 0.55$, (b) $\Omega = 0.58$, (c) $\Omega = 0.62$, (d) $\Omega = 0.725$, (e) $\Omega = 0.8$, and (f) $\Omega = 0.85$. The three regions are defined as Abrikosov lattice(AL), coexisting state(CS), and pinned lattice(PL). The faster rotation of the condensate can decrease the coexisting region and after a threshold value of Ω vortex lattice makes a sharp transition from Abrikosov lattice to pinned lattice. Here, other parameters are same as Fig. 3.1.

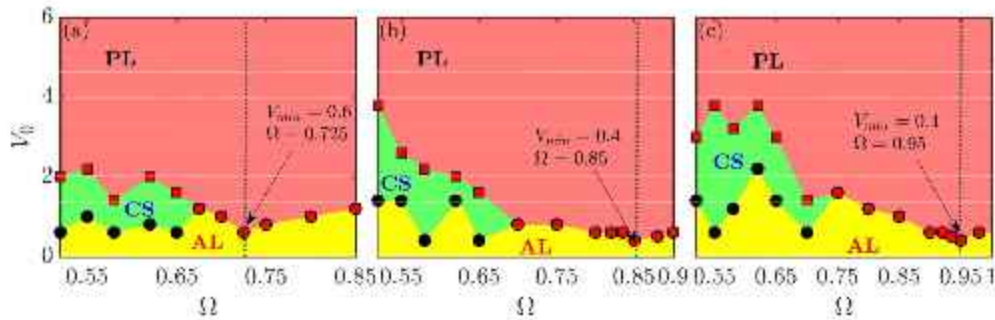


Figure 3.4: Phase diagram of different lattice structures depending upon the value of $S(k)$ for different lattice constant (a) $a = 2.2$, (b) $a = 2$, and (c) $a = 1.8$. Here (yellow, green, orange) region illustrates the AL, CS, and PL domains. Dashed line annotated for finding Ω corresponds to minimum pinning strength (V_{min}).

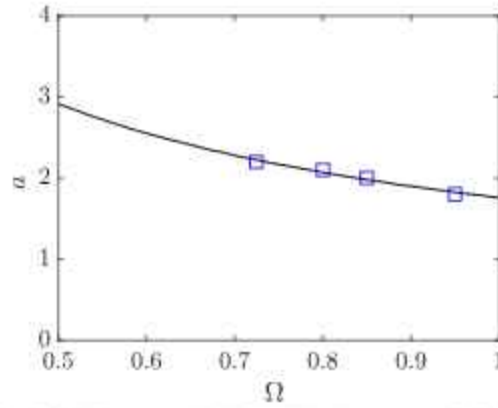


Figure 3.5: Optimum rotational frequencies Ω (obtained from phase diagram) for different lattice constant. Here, the solid line represents the fitted curve with Eq. (3.18)

In particular, there exists a rotational angular velocity at which the strength of the potential is minimum to get the pinned state. As we analyze the relation between the lattice constant a and the rotational velocity Ω at which minimum pinning strength is required to get the pinned lattice, we find that it follows

$$a = C\Omega^d \quad (3.18)$$

where constant $C = 1.76$ and d is the exponent of Ω as shown in Fig. 3.5. We find that the magnitude of d obtained for the square lattice is ~ -0.73 . This implies that the pinning strength V_0 takes the least value when a fulfills the above relation.

3.4.2 Influence of differential rotation on the structural phases of vortex lattice

Our study of the structural transformation of vortex lattices begins with the co-rotating case, where the square lattice and the condensate rotate with the same angular velocity. In this situation, the presence of an Gaussian distributed square lattice modifies the vortex lattice geometry, and the structural phase depends on several lattice parameters such as the lattice constant, lattice depth, and symmetry of the lattice potential. In this section, we extend our analysis to the case of differential rotation, where the condensate (Ω) and the lattice potential (ω) rotate with different angular velocities. To incorporate this effect, we project the lattice potential onto the condensate frame using the coordinate transformation given in Eq. 3.7. In this study, we have chosen a small value of $\gamma = 0.01$ to avoid the overdamped motion in the condensate. To prevent

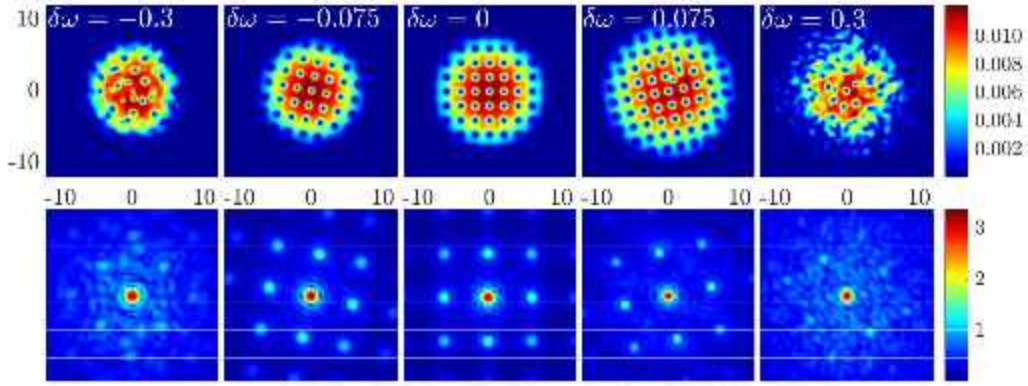


Figure 3.6: Density profile in real space and Fourier space with the variation of differential rotation: $\delta\omega = -0.3, -0.075, 0.0, 0.075, 0.3$. The density profile at $\delta\omega = 0.0$ shows the pinned lattice structure and with increment of the $\delta\omega$ leads a in tilted pinned vortex lattice structure, for sufficient high $\delta\omega$, lattice structure becomes fully disordered and exhibits melting behaviour. Here, we used lattice constant $V_0 = 9$.

the collective excitation motion, we gradually increase the rotational frequency of the square lattice (SL) by varying $\delta\omega$ adiabatically. Therefore, the relative rotational frequency is then ramped up as $\delta\omega = 0.005t\Theta(\delta\omega_f - \delta\omega)$, where $\Theta(x)$ is a step function. In our work, we investigate the time evolution of the pinned vortex lattice structure for different values of the differential rotational frequency ($\delta\omega$) and lattice depth (V_0). Such a pinned initial state is obtained by considering the following set of parameters: rotational frequency $\Omega = 0.85$, lattice constant $a = 2$, and nonlinearity $g = 1000$. In order to clearly characterize the different dynamical phases of the vortex lattice structure, we calculate the time average of structure factor and angular momentum.

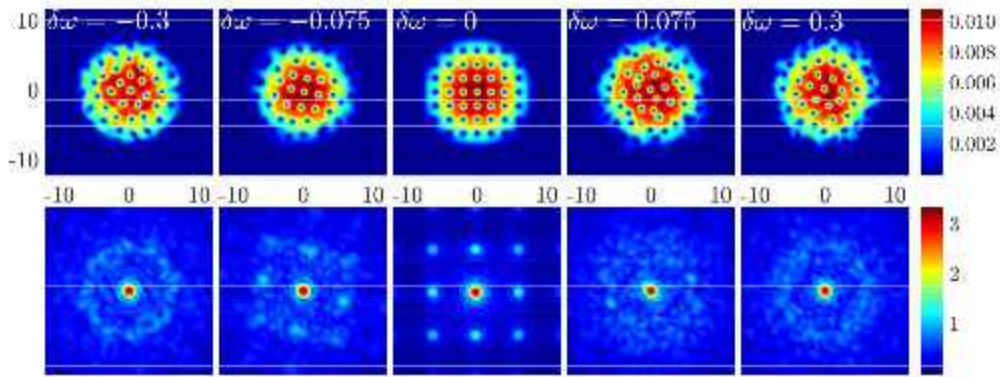


Figure 3.7: Density profile in real space and Fourier space with the variation of differential rotation: $\delta\omega = -0.3, -0.075, 0.0, 0.075, 0.3$. The density profile at $\delta\omega = 0.0$ shows the pinned lattice structure and with increment of the $\delta\omega$ leads distortion in the pinned vortex lattice, for sufficient high $\delta\omega$, lattice structure becomes fully disordered and exhibits melting behaviour. Here, we used lattice constant $V_0 = 2.5$.

Fig. 3.6 illustrates the condensate density profile in both real and Fourier space for different values of $\delta\omega$ at $V_0 = 9$. For small positive differential rotation ($\delta\omega = 0.075$), we observe the formation of a tilted square vortex lattice, which eventually melts as $\delta\omega$ increases to 0.3. A similar trend is observed for negative values of $\delta\omega$. Apart from these dynamical phases, the condensate density profiles reveal another notable feature: the size of the condensate increases for positive $\delta\omega$, while it decreases for negative $\delta\omega$. This indicates a clear asymmetry in the condensate dynamics between positive and negative differential rotation. For $\delta\omega < 0$, the optical lattice (OL) rotates slower than the condensate. When $|\delta\omega|$ is small, the OL drags the initially pinned vortices, which reduces the effective rotation of the condensate. As a result, both the size of the condensate and its angular momentum decrease. This behavior is evident in Fig. 3.9(a), where the structure factor $S(k)$ remains around 0.85. However, as $|\delta\omega|$ increases further, $S(k)$ gradually decreases due to the depinning of vortices from their pinning sites caused by the faster rotation of the condensate.

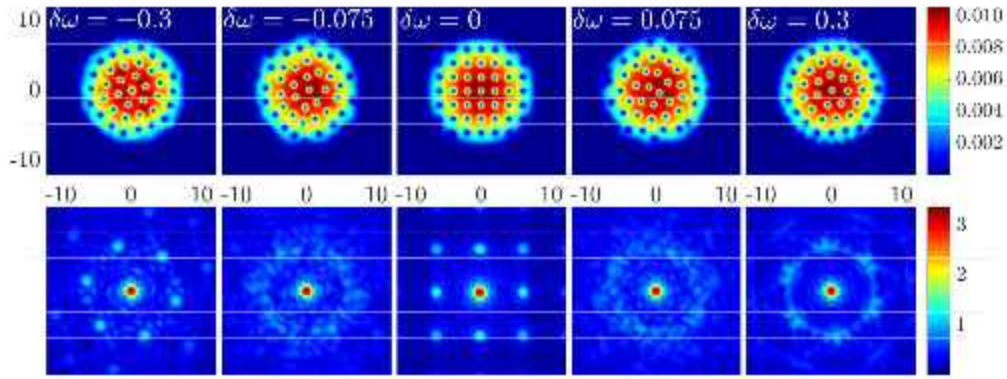


Figure 3.8: Density profile in real space and Fourier space with the variation of differential rotation: $\delta\omega = -0.3, -0.075, 0.0, 0.075, 0.3$. The density profile at $\delta\omega = 0.0$ shows the pinned lattice structure and with increment of the $\delta\omega$ leads distortion in the pinned vortex lattice, for sufficient high $\delta\omega$ Abrikosov lattice is formed. Here, we used lattice constant $V_0 = 1$.

Next, we consider the case of relatively weak pinning strength. Figure 3.7 shows the effect of differential rotation between the condensate and the pinning potential on the vortex-lattice structure for a wide range of $\delta\omega$ at $V_0 = 2.5$. We observe that the square vortex lattice gradually melts as the magnitude of $\delta\omega$ increases. Figure 3.8 presents the condensate density profiles in real and Fourier space for different values of $\delta\omega$ at $V_0 = 1.0$. With increasing positive differential rotation, the vortex lattice first becomes distorted at $\delta\omega = 0.075$, and eventually the hexagonal Abrikosov lattice reappears at larger differential rotation ($\delta\omega = 0.3$), similar to the case without a pinning potential. A similar behavior is also observed for $\delta\omega = -0.3$, indicating a symmetric dynamical

phase transformation for both positive and negative values of $\delta\omega$. This behavior can be attributed to the weak pinning strength, which becomes insufficient to hold the vortices when the pinning potential rotates differently from the condensate.

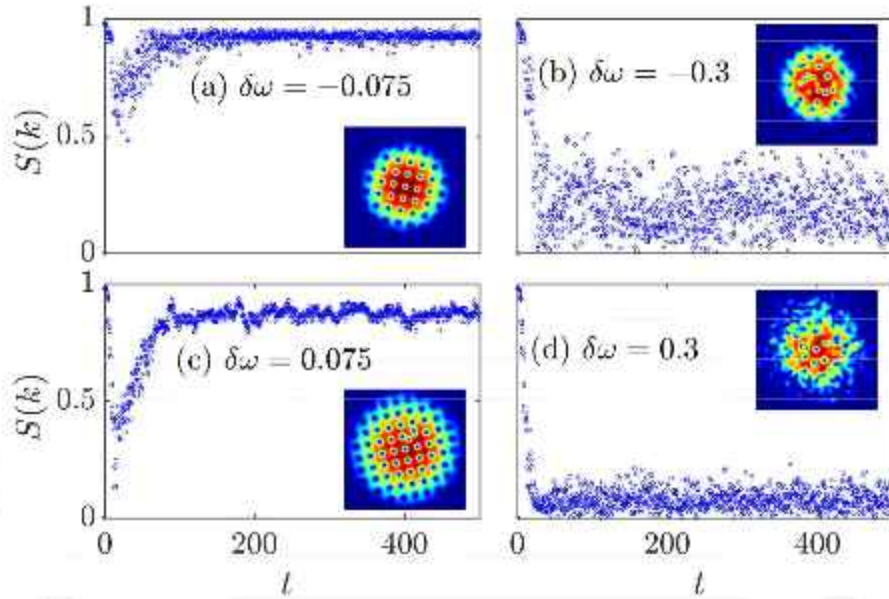


Figure 3.9: Time evolution of structure factor for $V_0 = 9$, and for different $\delta\omega$. For $\delta\omega = -0.075$, structure remains square but size of the condensate is reduced, whereas for $\delta\omega = -0.3$, lattice structure become fully disordered and structure factor shows a decreasing behaviour. Similarly for $\delta\omega = 0.075$, a tilted vortex lattice pattern is formed, and for $\delta\omega = 0.3$, lattice structure shows melting behaviour. Structure factor evaluated at the square lattice reciprocal lattice vector $\mathbf{k}_{sq} = (\frac{2\pi}{a}, 0)$.

Figures 3.9(c) and (d) illustrate the time evolution of the structure factor for $\delta\omega > 0$. In this scenario, the square lattice rotates faster than the condensate, resulting in the nucleation of additional vortices to accommodate the higher rotation rate. The emergence of these extra vortices increases both the size and angular momentum of the condensate. For small values of $\delta\omega$, the vortex lattice retains its square symmetry, and the structure factor $S(k)$ remains around 0.85. However, as $\delta\omega$ continues to increase, the square symmetry becomes distorted, and $S(k)$ begins to decrease. The interaction between the excess vortices and the pinned vortices ultimately leads to depinning, resulting in the melting of the vortex lattice.

Figure 3.10 shows the time-averaged structure factor $\langle S(k, t) \rangle$ and the angular momentum per atom $\langle l_z \rangle$ as a function of $\delta\omega$ for several values of V_0 . We observe that for small V_0 , the value of $\langle l_z \rangle$ remains close to its initial value, indicating the presence of intermediate vortices that undergo sliding motion. In contrast, for larger V_0 , $\langle l_z \rangle$ becomes greater than its initial value, suggesting that additional vortices are nucleated

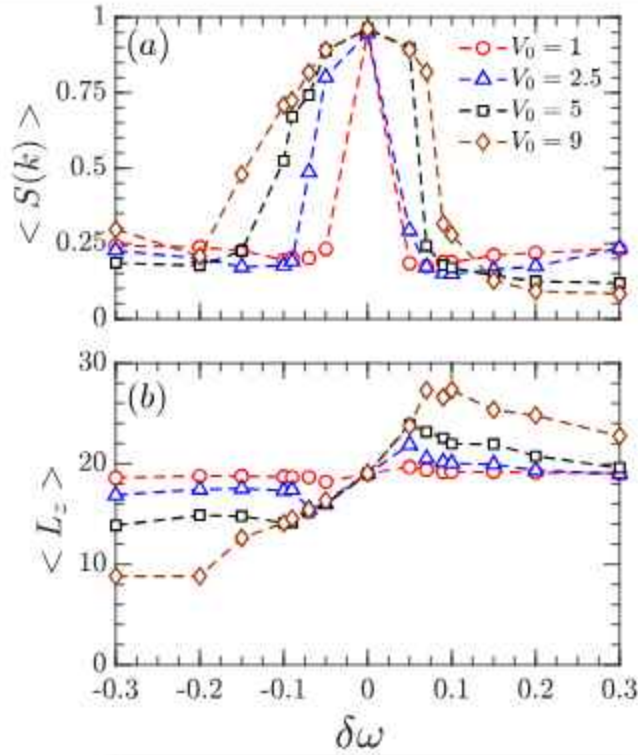


Figure 3.10: Time-averaged values of (a) structure factor $\langle S(k, t) \rangle$ and (b) angular momentum per atom $\langle l_z \rangle$ as a function of $\delta\omega$ for several values of V_0 . In (b), $\langle l_z \rangle$ is close to the initial value for low V_0 whereas, $\langle l_z \rangle$ is greater than the initial value for high V_0 . The plot of the rigid-body estimation [184] $l_z = C(\Omega + \delta\omega)/\sqrt{1 - (\Omega + \delta\omega)^2}$ is shown by the dotted curve $C = 8.9$ is the fitting parameter.

in the condensate. The presence of these excess vortices reduces the effectiveness of vortex pinning, leading to a highly distorted vortex lattice and eventual melting.

Figure 3.11 shows the variation of the lattice energy and total energy as a function of $\delta\omega$ for different values of V_0 . As shown in Fig. 3.11(a), the lattice energy is minimum at $\delta\omega = 0$ and gradually increases with increasing $\delta\omega$. This behavior arises because differential rotation causes vortices to depin from their pinning centers, and for sufficiently large $\delta\omega$ the vortex lattice becomes strongly distorted. The total energy also increases with $\delta\omega$; however, its magnitude is higher for $\delta\omega > 0$ than for $\delta\omega < 0$. This asymmetry occurs because, for $\delta\omega > 0$, the condensate expands and additional vortices are nucleated in order to accommodate the faster rotation of the square lattice. Based on these observations, we classify the vortex phases following Ref. [184]. For $\delta\omega < 0$, three distinct phases are identified: (i) a fully pinned vortex lattice for $\langle S(k, t) \rangle > 0.7$, (ii) an inner pinned and outer depinned vortex lattice for $0.25 < \langle S(k, t) \rangle < 0.7$, and (iii) a vortex-lattice sliding phase for $\langle S(k, t) \rangle < 0.25$. For $\delta\omega > 0$, four phases are observed: (i) a fully pinned vortex lattice for $\langle S(k, t) \rangle > 0.7$, (ii)

a premelting vortex lattice for $0.25 < \langle S(k, t) \rangle < 0.7$, (iii) a vortex-lattice liquid phase for $\langle S(k, t) \rangle < 0.25$ and $\langle l_z \rangle - l_z(t=0) > 2$, and (iv) a vortex-lattice sliding phase for $\langle S(k, t) \rangle < 0.25$ and $\langle l_z \rangle - l_z(t=0) < 2$.

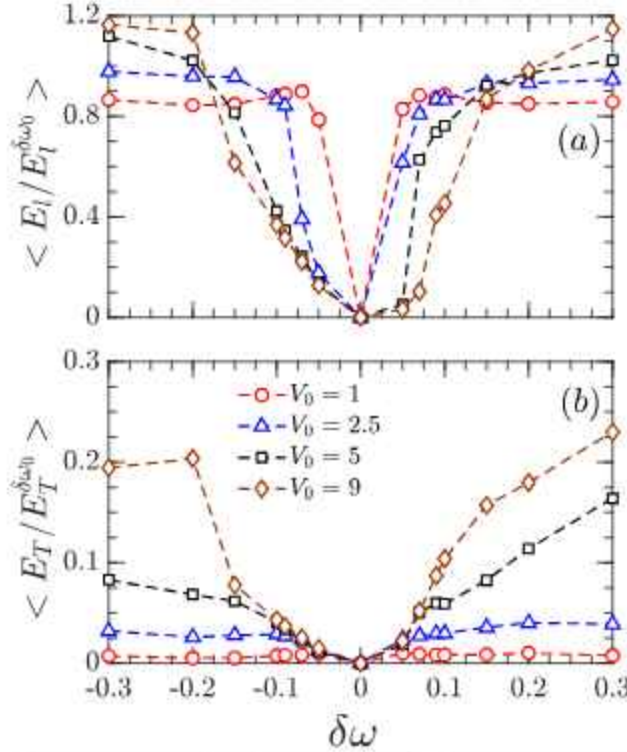


Figure 3.11: Time averaged lattice energy $\langle E_l \rangle$ (a) and total energy $\langle E_T \rangle$ (b) as functions of $\delta\omega$ for several values of V_0 . The quantities $E_l^{\delta\omega_0}$ and $E_T^{\delta\omega_0}$ denote the corresponding energies evaluated at $\delta\omega = 0$.

Figure 3.12 shows the time-averaged entropy $\langle S_{en} \rangle$ as a function of the differential rotation $\delta\omega$ for different values of the pinning strength V_0 . The entropy provides a measure of the degree of disorder in the vortex lattice structure. For small values of $\delta\omega$, the entropy remains relatively low, indicating an ordered vortex lattice configuration. As $\delta\omega$ approaches zero and increases further, the entropy rises, suggesting an increase in structural disorder within the vortex lattice. This increase in entropy is associated with vortex depinning and the subsequent distortion of the lattice structure due to differential rotation between the condensate and the lattice potential. For larger pinning strengths (e.g., $V_0 = 9$), the entropy shows a more pronounced variation with $\delta\omega$, reflecting stronger dynamical changes in the vortex lattice. In contrast, for weaker pinning strengths ($V_0 = 1$ and 2.5), the entropy varies more gradually, indicating that the vortex lattice is less sensitive to differential rotation. Overall, the increase in entropy with $\delta\omega$ signifies the transition from an ordered pinned vortex lattice to a more

disordered or melted vortex configuration.

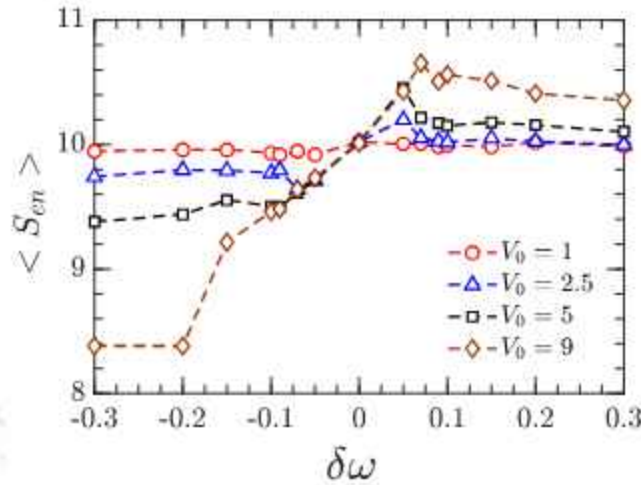


Figure 3.12: Variation of time average of total entropy $\langle S_{en} \rangle$ with $\delta\omega$ for different values of V_0 . $\langle S_{en} \rangle$, remains relatively constant and close to its initial value for smaller values of V_0 . However, it significantly increases as V_0 becomes larger.

The phase diagram in Fig. 3.13 summarizes the vortex lattice dynamics obtained for different values of $\delta\omega$ and V_0 . Within the region enclosed by the solid black line, the vortex lattice remains completely pinned to the lattice potential. In the region between the black and red solid lines, the vortex lattice enters a partially depinned phase, where vortices at the condensate periphery begin to depin from their pinning sites due to differential rotation. Above the red boundary, the vortex lattice exhibits a sliding phase. However, for $V_0 \leq 2$, the vortices reorganize into a triangular pattern, since the lattice potential is too weak to pin the vortices under large differential rotation. On the other hand, the region between the black and blue solid lines corresponds to a premelting vortex state. Beyond this boundary, the vortex lattice becomes fully disordered and exhibits a liquid like phase. In the presence of an lattice potential, a vortex experiences three main forces: the driving force (F_d), the pinning force (F_p), and the vortex-vortex interaction force (F_{vv}). Thus, the total force acting on a vortex can be written as $F = F_d + F_p + F_{vv}$. According to the force balance condition, vortices remain pinned as long as $F_p > F_d + F_{vv}$.

For $\delta\omega < 0$, the vortex-vortex interaction force has a relatively weak influence because the number of vortices, the condensate size, and the angular momentum decrease. In this regime, the driving force mainly consists of the Magnus force (F_M) and the mutual friction force (F_D). The mutual friction arises from the interaction between the vortex core and the normal fluid component, but it becomes negligible at very

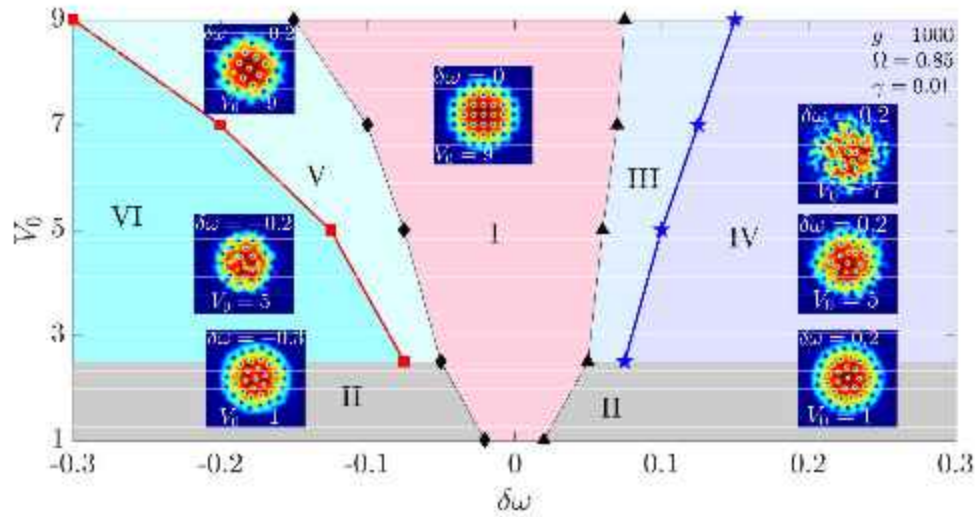


Figure 3.13: Phase diagram of different dynamical vortex states. (I) fully pinned vortex lattice. (II) triangular vortex lattice. (III) pre-melting vortex lattice. (IV) vortex liquid. (V) inner-pinned outer-depinned vortex lattice. (VI) Vortex lattice sliding. phase boundary is determined by calculating the time-averaged values of the structure factor $\langle S(k, t) \rangle$ and the angular momentum per atom $\langle l_z \rangle$ as a function of $\delta\omega$ for several values of V_0 . For $\delta\omega \leq 0$ the structure factor lies within $0.25 < \langle S(k, t) \rangle < 0.7$ bounded by black and red color where the vortex sliding occurs, the region below the red line the average value $\langle S(k, t) \rangle < 0.25$. Similarly, for $\delta\omega \geq 0$ the region between blue and black where average value of structure factor lies within $0.25 < \langle S(k, t) \rangle < 0.7$ and above the blue region average value is $\langle S(k, t) \rangle < 0.25$ where vortex lattice melts for higher V_0 . Finally, in the pinned lattice region near the $\delta\omega \approx 0$ the average value of structure factor lies $\langle S(k, t) \rangle > 0.7$.

low temperatures. Due to differential rotation, the Magnus force acting on a vortex is given by $F_M = a_z m \rho \kappa \delta\omega r \hat{z} \times \hat{\theta}$, where $\kappa = h/m$ is the quantum circulation. Since the Magnus force is proportional to the radial distance r , vortices near the condensate edge experience a larger force and tend to depin from their pinning sites. For $\delta\omega > 0$, interstitial vortices play an important role in the melting of the vortex lattice. In this regime, additional vortices are nucleated in order to accommodate the faster rotation of the lattice potential. Furthermore, the Thomas-Fermi approximation breaks down when $\Omega + \delta\omega = \omega \approx 1$, which further influences the melting behavior of the vortex lattice.

So far, we have observed that depending on the value of $\delta\omega$, the vortex lattice transforms into a melted or disordered state, and in this disordered regime, the vortices lose their regular lattice arrangement and become randomly oriented while interacting with one another. This state creates favorable conditions for the emergence of turbulent dynamics in the condensate. To characterize the melted vortex lattice state more comprehensively, we analyze the kinetic energy spectra of the condensate. We first ex-

amine the incompressible kinetic energy spectrum, ϵ_{kin}^i , shown in Fig. 3.14, followed by the compressible kinetic energy spectrum, ϵ_{kin}^c , presented in Fig. 3.15. The scaling behavior of these spectra provides deeper insight into the nature of the turbulent regime and enables a quantitative characterization of the flow.

In our spectral analysis, we identify three characteristic length scales in momentum space that are relevant to the condensate. These scales are defined as

$$k_{R_{\text{TF}}} = \frac{2\pi}{R_{\text{TF}}}, \quad k_{l_0} = \frac{2\pi}{l_0}, \quad k_{\tilde{\zeta}} = \frac{\pi}{\tilde{\zeta}}, \quad (3.19)$$

where $R_{\text{TF}} = \sqrt{2\mu}$ denotes the Thomas-Fermi radius, l_0 represents the average inter-vortex spacing, and $\tilde{\zeta}$ is the healing length of the condensate. Here n_v denotes the vortex density (number of vortices per unit area), and the intervortex spacing is estimated as $l_0 \sim 1/\sqrt{n_v}$. Based on these characteristic scales, the kinetic energy spectrum as a function of the wave number k can be divided into two distinct regimes: the ultra-violet (UV) regime, defined by $k \gg \tilde{\zeta}^{-1}$, and the infrared (IR) regime, where $k \ll \tilde{\zeta}^{-1}$. The UV region mainly reflects the internal structure of vortex cores, whereas the IR region corresponds to large scale vortex motion and collective flow dynamics.

$\delta\omega$	R_{TF}	$\tilde{\zeta}$	l_0
-0.3	3.68	0.27	1.53
0	4.4	0.22	1.43
0.3	5.19	0.19	1.49

Table 3.1: Estimation of different characteristic length scales R_{TF} , l_0 , and $\tilde{\zeta}$ for different $\delta\omega$ at $V_0 = 7$ and $\Omega = 0.8$. All the reported values are averaged over five different realizations of the random impurity potential.

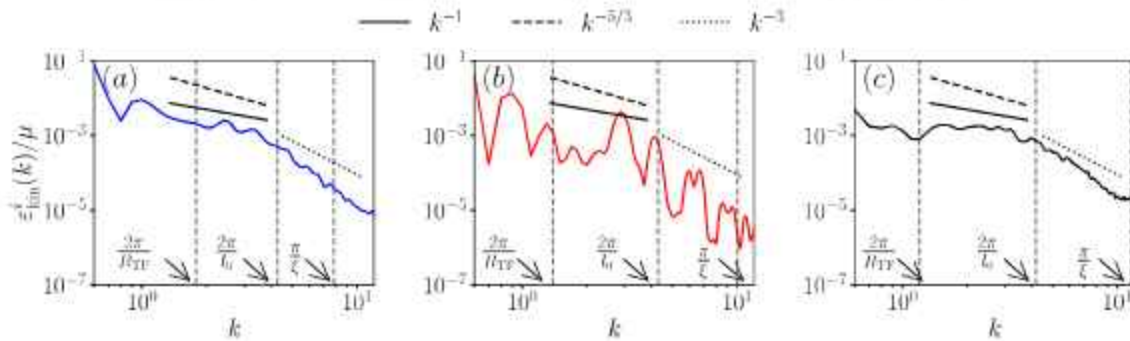


Figure 3.14: Incompressible kinetic energy spectra for different $\delta\omega$ at $V_0 = 7$: (a) $\delta\omega = -0.3$, (b) $\delta\omega = 0$, and (c) $\delta\omega = 0.3$. The spectra display k^{-1} and k^{-3} power laws in the IR and UV regions, respectively.

Table 3.1 summarizes the estimated values of the characteristic length scales R_{TF} ,

l_0 , and ξ for different differential rotation $\delta\omega$. The incompressible kinetic energy spectra for these parameters are shown in Fig. 3.14. For $\delta\omega = 0$, the spectra exhibit multiple peaks, indicating the presence of an ordered vortex lattice. As a result, no clear power-law behavior is observed in this region [see Fig. 3.14(b)]. In contrast, for differential rotation $\delta\omega = -0.3$ and 0.3 , the spectral peaks disappear and a clear power-law behavior emerges. In this case, the spectrum follows a k^{-1} scaling in the range $k_{R_{TF}} < k < k_{l_0}$ and a k^{-3} scaling in the region $k_{l_0} < k < k_\xi$ [see Fig. 3.14(a) and (c)]. The k^{-1} scaling at length scales larger than the inter vortex separation indicates a Vinen-like energy cascade driven by the random motion of vortices, suggesting the onset of turbulence in the system. Similar scaling behavior has also been reported for two-dimensional trapped condensates [154].

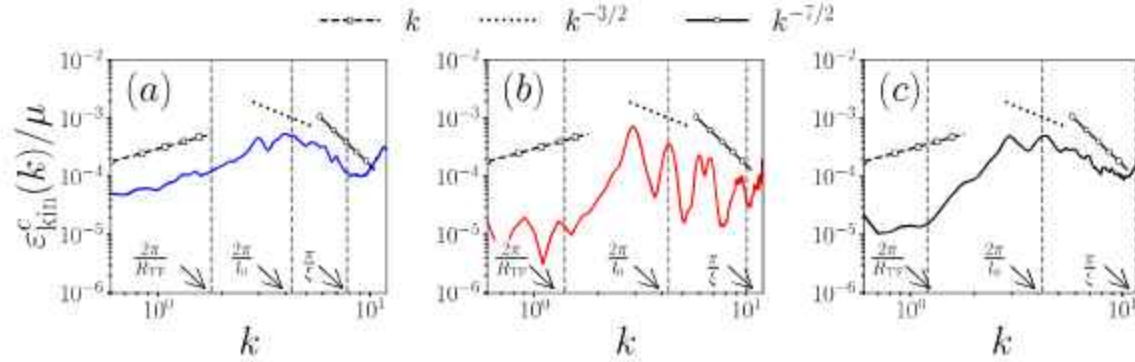


Figure 3.15: Compressible kinetic energy spectra for different $\delta\omega$ at $V_0 = 7$: (a) $\delta\omega = -0.3$, (b) $\delta\omega = 0$, and (c) $\delta\omega = 0.3$. The spectra display k^{-1} and k^{-3} power laws in the IR and UV regions, respectively.

Next, we analyze the behavior of the compressible component of the kinetic energy spectrum in the melted vortex lattice state. In Fig. 3.15, we present the compressible energy spectra ϵ_{kin}^c for the same set of parameters as in Fig. 3.14. For all impurity lattice constants, the spectra exhibit an approximate k scaling at small wave numbers, which is associated with thermal equilibrium of the energy. For $\delta\omega = 0$ [see Fig. 3.15(a)], no clear power-law behavior is observed in the intermediate region, and the presence of multiple peaks reflects the ordered vortex lattice structure of the condensate, consistent with the incompressible spectra. For $\delta\omega = -0.3$ and 0.3 , the spectrum develops a $k^{-3/2}$ scaling in a narrow region around k_{l_0} [see Fig. 3.15(a), and (d)], suggesting the presence of weak wave turbulence in the melted vortex lattice state. However, in the region $k_{l_0} < k < k_\xi$ the spectrum follows a $k^{-7/2}$ scaling, which is indicative of turbulent dynamics.

3.5 Conclusion

In this Chapter, using the mean-field model we have numerically investigated the structural transformation of the vortex lattice in a rotating BEC that is trapped in a square lattice. By varying the strength of the pinning potential and the rotation frequency of the condensate, we were able to obtain several form of the vortex lattice, including the Abrikosov lattice (AL), a pinned square lattice (PL), and a coexistence phase in which both Abrikosov and square lattice structures appear simultaneously.

Using the structure factor, we determined the minimum pinning potential strength required to completely transform the vortex lattice from the Abrikosov lattice to the pinned lattice state. The analysis was performed for different lattice constants of the square lattice potential ($a = 2.2, 2.0$, and 1.8) and for a wide range of rotation frequencies ($\Omega = 0.55 - 0.95$). Our results show that for sufficiently large angular velocities ($\Omega \gtrsim 0.725$), the vortex lattice undergoes a direct transition from the Abrikosov lattice to the pinned lattice beyond a critical pinning strength. In contrast, for lower angular velocities an intermediate coexistence lattice (CL) phase appears between the AL and PL states. Based on these results, we constructed phase diagrams in the parameter space of pinning potential strength and angular velocity for different lattice constants of the lattice potential. We find that the minimum rotation frequency required to pin the vortex lattice strongly depends on the lattice constant of the lattice potential, and it increases as the lattice constant is reduced. Furthermore, we explored the dynamics of the vortex lattice when the square lattice and the condensate rotate with different angular velocities. By introducing a differential rotation parameter $\delta\omega$, we observed several structural regimes of the vortex lattice. For small values of $\delta\omega$, the lattice structure remains ordered, while increasing $\delta\omega$ gradually distorts the lattice configuration. This structural evolution is clearly reflected in the behavior of the averaged structure factor, lattice energy, and the total energy of the condensate.

For sufficiently large positive and negative values of $\delta\omega$, the vortex lattice undergoes a melting transition, leading to a disordered vortex configuration. In this regime, the vortices become randomly oriented and interact strongly with each other, giving rise to turbulent dynamics in the condensate. The emergence of turbulence is further supported by the analysis of the kinetic energy spectra, which exhibit characteristic power law scalings associated with turbulence.

Impurity-induced vortex lattice melting and turbulence in rotating Bose–Einstein condensates

In the Chapter 3, we presented a detailed analysis of the structural transformations of vortex lattices in a rotating BEC in the presence of a Gaussian distributed square lattice potential and differential rotation. Our results revealed several distinct vortex lattice phases that depend sensitively on the lattice parameters and the magnitude of the differential rotation. In this Chapter, we extend the analysis to show the effect of impurities on the vortex lattice structure. In particular, we focus on developing a comprehensive understanding of the interplay between two different types of impurities, namely static and dynamic impurities, and their role in the melting of the vortex lattice. One of the primary objectives of this study is to explore the turbulent characteristics of the melted vortex lattice (MVL) state in the presence of both static and dynamic impurity potentials. By analyzing the kinetic energy spectra of the condensate, we demonstrate the emergence of turbulence in the vortex lattice melted state for both types of impurities. This observation suggests that the MVL state exhibits universal turbulent features, irrespective of the underlying mechanism responsible for its formation. This Chapter is based on our work published in J. Phys. B: At. Mol. Opt. Phys. [193].

4.1 Introduction

The literature on the structural transformation of vortex lattices has been thoroughly discussed in Chapter 1. However, to ensure that the current chapter is self contained, we provide a brief overview of previous studies that explain the mechanisms behind the melting of vortex lattices.

Since the realization of Bose-Einstein condensates (BECs) in laboratory experiments, it has become a robust platform to simulate, in a controlled way, many of the microscopic condensed matter phenomena like Anderson localization [18, 19], superfluid-to-mott-insulator transition [20], and spin-Hall effect [21, 22]. Another feature of BECs is the precise generation of topological defects, which manifest as quantum vortices. In laboratory experiments, these vortices are generated using laser stirring [24, 25], rotating magnetic traps [26, 106, 27], and phase imprinting [28, 29].

A rapidly rotating BEC, subject to a harmonic trap generates a periodic array of vortices generally arranged on triangular or hexagonal lattices which appear to be the same as the Abrikosov lattice generated in the high T_c superconductor in the presence of a strong magnetic field. As the rotating BECs are subject to the optical trap the vortices get pinned to the optical lattice and display a structural transition. For instance, several works show the transition from triangular to square and other complex structures that solely depend upon the nature of the optical lattice [56, 58, 185].

The melting of the vortex lattice in the presence of disorder impurities in BECs offers an intriguing insight into the interplay of rotation, interaction and disorder. The vortex lattice melting phenomenon is quite a common phenomenon in high T_c superconductors, where the competition between impurity disorder and thermal fluctuations leads to the vortex melting [139–141]. Direct observations of the melting of two-dimensional superconducting vortex lattices have been reported in [141]. Despite being vastly different systems, the vortex dynamics in superfluids (BEC and liquid helium) and superconductors exhibit similar behaviour. There have been several experimental studies on vortex lattice melting in BECs. For example, Engels [142] observed the melting phenomena due to the anisotropic expansion of the vortex lattice. Coddington *et al.* [60] experimentally demonstrated that shearing force generated from laser beams destroys the vortex lattice in the condensates. The experimental observations are duly complemented by the theoretical and numerical analysis. For example, Snoek and Stoof [61] extensively investigated the melting phenomenon in several pancake-shaped condensates coupled through tunnelling in a one-dimensional optical lattice. Using mean-field Gross-Pitaevskii (GP) equations several numerical investiga-

tions have been conducted to understand the dynamical behaviour of vortex lattices under the influence of various impurity types in rotating BECs. For instance, Mithun *et al.* [62] numerically analyzed the distortion of a vortex lattice in the condensate under the influence of a single impurity in the form of Gaussian potential and found that the vortex lattice makes a transition to the disordered arrangement of the vortex phase also termed as the vortex lattice melted state in presence of quasi-periodic impurity generated by the superposition of two optical lattices. Following this, melting of the Abrikosov vortex lattice structure was also reported in the presence of random impurity [144, 145].

On the other hand in recent years there has been a significant surge in understanding the turbulence in superfluid, particularly through the understanding of the dynamics of vortices in BECs [194]. Several experimental [146, 195, 148] and numerical [125, 149, 196] studies have been performed on the turbulent behaviour of BECs. In a two-dimensional (2D) fluid, the existence of the inverse energy cascade [197, 198], where energy flows from smaller to larger spatial length scales, plays a crucial role in understanding turbulent behaviour. Remarkably, the inverse energy cascade of turbulent flows is effectively described by the well-known Kolmogorov's $k^{-5/3}$ scaling law in the incompressible kinetic energy spectrum, which has been firmly established in this field [199, 125]. Several techniques have been used to generate the turbulence in BECs. Reeves *et al.* [127] numerically analyzed the turbulent BECs where the turbulence is generated via stirring the condensate with a Gaussian obstacle barrier. They identified three distinct regions related to the vortex dynamics and associated turbulent behaviour by analyzing the power law exponent of the kinetic energy spectrum. Neely *et al.* [151] used the Gaussian obstacle to observe vortex dipoles. Later on, Mithun *et al.* [152] used the Gaussian obstacle potential to generate turbulence in BECs. In contrast, their study demonstrated the decay of turbulent behaviour by tuning the intercomponent interaction strength. Das *et al.* [153] has rigorously explored the vortex dynamics by incorporating the rotating paddle potential [200] in a two-species binary condensate and reported the vortex clustering due to the counter-rotating of two paddle potentials. Recently, Anirudh *et al.* studied turbulence in rotating BECs and BEC mergers [154, 155]. Furthermore, the incompressible spectrum exhibited power-law behaviour with exponents of $k^{-5/3}$ and k^{-3} for low and high wavenumbers, respectively. In a subsequent study, Silva *et al.* [201] investigated the turbulent behaviour of a mass-imbalanced binary BEC by perturbing the harmonic trap using a time-dependent periodic potential. This approach is analogous to the stirring of the condensate, which leads to turbulent dynamics in the system.

So far, the turbulent behaviour in a condensate has been investigated for the condensate stirring through different approaches either without the trap or with the trap under harmonic potentials. However, the analysis of turbulence in the vortex lattice melting state, where the organized arrangement of vortices becomes completely disordered due to strong inter-vortex interactions, is still lacking. In this paper, we utilize a slightly different protocol to analyze the effect of various types of impurity potential on the square lattice pattern of vortices. In the absence of impurities, the rotating condensates exhibit a square lattice structure due to the pinning of the vortices on Gaussian distributed square lattice potential. As impurity is introduced the vortex lattice makes a transition from the ordered to a disordered state. In our analysis, we consider both dynamical and static types of impurities, specifically a Gaussian obstacle and random static impurity, respectively. In the case of static impurity, the randomness is generated by varying the strength of the impurity, whereas for dynamic impurities, the Gaussian obstacle is stirred throughout the condensate by revolving it circularly. In both scenarios, the transition from the ordered vortex lattice structure to the melted vortex lattice structure has been systematically captured by examining various quantities such as structure factor, angular momentum, information entropy, and different energies of the condensate. One of the main aims of this work is to explore the turbulent feature of the melted vortex lattice state in the presence of both static as well as dynamic impurity potential. By analyzing the kinetic energy spectrum, we establish the presence of the turbulent nature of the condensate in the vortex lattice melted state in the case of both the impurities that indicate the universality of the melted vortex lattice state irrespective of the mechanism through which it has been generated.

The paper is structured as follows. In section 4.2, we provide the governing equations and details of different static and dynamical potentials used in the paper to trigger the melting of the vortex lattices. Section 4.3 presents a brief description of the simulation details and several quantities used to characterize the different states of the vortex lattices. The description includes the definitions of structure factor, energy, angular momentum, and drag force associated to the condensate density. In section 4.4, we present the numerical simulation results in the case of random impurity potential and Gaussian obstacle potential. For each type of potential, we investigate the impact of various parameters, such as impurity strength or impurity lattice constant for random static impurity, and further, the position or oscillation frequency of the Gaussian obstacle on the vortex lattice structure. In section 4.5, we finally conclude our work.

4.2 Mean-field model

In a rotating frame of reference, condensate confined by an external potential and rotated with an angular velocity Ω can be modelled using the following non-dimensional dissipative Gross Pitaevskii equation [184]:

$$(i - \gamma) \frac{\partial \psi(x, y, t)}{\partial t} = \left[-\frac{1}{2} \nabla_{\perp}^2 + V_{\text{ext}}(x, y) - \mu + g |\psi(x, y)|^2 - \Omega L_z \right] \psi(x, y, t). \quad (4.1)$$

For our analysis we consider the external potential as,

$$V_{\text{ext}} = V_{\text{ho}} + V_{\text{SL}} + V_{\text{imp}}. \quad (4.2)$$

In the above equation, $V_{\text{ho}} = \frac{1}{2}(x^2 + y^2)$ represents the harmonic potential, while V_{SL} and V_{imp} denote the Gaussian-distributed square lattice and impurity lattice potential, respectively [see equations (4.3) and (4.4)].

Where the nonlinear interaction parameter is given by $g = 4N\pi a_s \sqrt{\pi\lambda} / \sqrt{2}a_{\perp}$, the harmonic trapping potential is defined as $V_{\text{ho}} = \frac{1}{2}(x^2 + y^2)$, and the angular momentum operator is represented by $L_z = i(y\partial_x - x\partial_y)$. For simplicity, the primes have been omitted from the dimensionless variables. In this equation, the coordinates (x and y), time (t), and angular velocity Ω is normalized by the units of harmonic oscillator length $a_{\perp}$, the inverse trapping frequency $\omega_{\perp}^{-1}$, and the frequency $\omega_{\perp}$, respectively. The amplitudes of the Gaussian distributed square lattice (V_0), and impurity lattice (V_0^{imp}), along with the chemical potential (μ), are normalized to the unit of $\hbar\omega_{\perp}$. The dissipation term $\gamma = 0.03$, has been incorporated to account for the effects of temperature and the non-condensate fraction [187, 188].

The Gaussian distributed square lattice potential V_{SL} can be realized in experiments through the superposition of four counter-propagating laser beams with wavelength $\hat{\lambda} \approx 830 \text{ nm}$ [27] that consider the form as [136],

$$V_{\text{SL}} = \sum_{n_1} \sum_{n_2} V_0 \exp \left[-\frac{|\mathbf{r} - \mathbf{r}_{n_1, n_2}|^2}{(\sigma/2)^2} \right], \quad (4.3)$$

where $\mathbf{r}_{n_1, n_2} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2$ denote the lattice vector with n_1 and n_2 being the number of lattice sites along the x - and y - direction respectively. V_0 is the strength of the

Gaussian distributed square lattice potential, and σ represents the width of the Gaussian laser beam [186]. For our analysis, we have considered the lattice unit vector as $\mathbf{a}_1 = a\hat{i}$, $\mathbf{a}_2 = a\hat{j}$ to obtain the square lattice, where a represents the lattice constant. In the present, we vary both the lattice constant as well as the potential strength (V_0) to obtain the vortex lattice state. Throughout the study, the Gaussian beam waist of the laser is kept at $\sigma = 0.65$.

The main aim of the work is to investigate the effect of the static and dynamic impurities. In the first part, we focus on the static impurity with randomly distributed amplitudes in 2D, which in the experiment can be realized through the laser speckle method [202]. The random impurity potential used in our work has the form given by [62]:

$$V_{\text{imp}} = \sum_{n_1} \sum_{n_2} V_0^{\text{imp}} \left[-\frac{|\mathbf{r} - \mathbf{r}_{n_1, n_2}^{\text{imp}}|^2}{(\sigma/2)^2} \right], \quad (4.4)$$

where $\mathbf{r}_{n_1, n_2}^{\text{imp}} = n_1 \mathbf{a}_1^{\text{imp}} + n_2 \mathbf{a}_2^{\text{imp}}$ denotes the lattice vector of impurity potential with n_1 and n_2 being the number of lattice sites along the x - and y - direction, respectively. We consider the square geometry of the random potential with two lattice unit vectors given by $\mathbf{a}_1^{\text{imp}} = a_{\text{imp}}\hat{i}$, $\mathbf{a}_2^{\text{imp}} = a_{\text{imp}}\hat{j}$, where a_{imp} is the lattice constant corresponding to the impurity potential. V_0^{imp} is the strength of the impurity potential. Here, the amplitude of the potential at different lattice sites is randomly distributed whose value lies in the range of $[-V_0^{\text{imp}}, V_0^{\text{imp}}]$. In the second case, we will investigate the impact of dynamically generated impurities, which can be realized through stirring a laser beam in the condensate [195, 203–205].

The single Gaussian obstacle is defined as,

$$V_{\text{obstacle}} = V_0^{\text{obs}} \exp \left[-\frac{(x - x_0(t))^2 + (y - y_0(t))^2}{d^2} \right], \quad (4.5)$$

where $x_0 = r_0 \cos(\omega t)$ and $y_0 = -r_0 \sin(\omega t)$ are the x - and y -coordinates of the obstacle which have oscillation frequency ω with r_0 is the distance of the obstacle from the trap center. The amplitude of the obstacle is set to be $V_0^{\text{obs}} = 80\hbar\omega_{\perp}$. We consider the size of the obstacle $d = 0.3$ for all the simulation runs.

4.3 Numerical details and different quantities to characterize the structure and dynamics of vortex lattice

We use the GPE-lab software that uses the implicit-time splitting pseudo spectral method [176] to solve the governing mean-field GP equation (4.1) with a space step $\Delta x = \Delta y = 0.1860$ and time step $\Delta t = 0.001$. To make our simulation results viable to the experiment, we choose $N = 2 \times 10^4$ atoms corresponding to the condensate of ^{87}Rb atom. The axial and transverse frequency of the harmonic trap is considered as $\omega_z = 2\pi \times 100$ Hz and $\omega_\perp = 2\pi \times 10$ Hz, respectively. Considering the scattering length of ^{87}Rb as $a_s = 5.5\text{nm}$ [191, 33] with the characteristic length scale as $a_\perp = 2.41\mu\text{m}$. We fix the other parameters as $g = 1000$ and $\gamma = 0.03$. In order to conserve the norm of the condensate wave function for a nonzero value of γ , the chemical potential has been adjusted at each time step. In our simulation, the chemical potential at each time step is corrected using $\Delta\mu = (\Delta t)^{-1} \ln \left(\int d^2\mathbf{r} |\psi(\mathbf{r}, t)|^2 / \int d^2\mathbf{r} |\psi(\mathbf{r}, t + \Delta t)|^2 \right)$ [144].

The physical quantities used in this chapter to characterize the melting of the vortex lattice are the same as those introduced in Chapter 3. Therefore, here we briefly summarize the key observables and refer the reader to Chapter 3 for a detailed description.

To quantify the structural transformation of the vortex lattice, we employ the structure factor, defined as

$$S(\mathbf{k}) = \frac{1}{N_c} \sum_{i=1}^{N_v} n_i e^{i\mathbf{k} \cdot \mathbf{r}_i}, \quad (4.6)$$

where n_i , $\mathbf{r}_i$, and N_c denote the winding number, position of the i th vortex, and the total winding number, respectively, while N_v represents the total number of vortices in the condensate. The nature of the vortex lattice is determined from the magnitude $S(k) = |S(\mathbf{k})|$. For a perfect square lattice $0.5 \lesssim S(k) \lesssim 1.0$, whereas $S(k) < 0.5$ indicates a deviation from the square lattice structure, corresponding to a melted vortex lattice state.

To further characterize the melting transition, we also analyze the total energy of the condensate, given by

$$E_T = \int \left[\frac{1}{2} |\nabla \psi|^2 + V_{\text{ext}} |\psi|^2 + \frac{g}{2} |\psi|^4 - \Omega \psi^* L_z \psi \right] dx dy, \quad (4.7)$$

where the terms represent the kinetic, potential, interaction, and rotational energy of the condensate, respectively.

In addition, to examine the turbulent features of the melted vortex lattice state, we analyze the kinetic energy spectrum. The kinetic energy can be decomposed into compressible and incompressible components, which correspond to sound wave excitations and vortex dynamics, respectively. The detailed formulation of the energy decomposition and the evaluation of the spectral density follow the same procedure described in Chapter 3.

Finally, to quantify the degree of disorder associated with the vortex lattice melting, we compute the information entropy in both real and momentum space. The total entropy, defined as $S_{\text{en}} = S_r + S_k$, provides a measure of the randomness in the system. The entropy values presented here are normalized with respect to the entropy of the condensate in the absence of impurities. The explicit expressions and numerical implementation of these quantities are discussed in detail in Chapter 3.

4.4 Numerical Results

In this section, we analyze the impact of different impurity potentials on the square lattice pattern of vortices. We have considered the effect of two kinds of impurities on the vortex lattice structure of the condensate. First, we examine the random Gaussian potential, investigating its impact on the vortex lattice structure by varying both lattice constant and strength. In the second case, we consider the oscillating Gaussian obstacle and explore its effect on the vortex lattice structure.

4.4.1 Vortex lattice melting in the presence of random static impurity

We examine the influence of a random static impurity on the vortex lattice structure of a condensate. The details of the random impurity potential (4.4) are given in Sec. 4.2. We begin the analysis of the repercussions of the impurity potential in two ways. Firstly, by changing the impurity lattice constant (a_{imp}) and secondly, by increasing the strength of the random potential (V_0^{imp}). Following this, we characterize the structural transformation induced by the impurity. In the end, we compute the kinetic energy spectra to investigate the turbulent behavior of the condensate. In our study, we introduce a dimensionless ratio $\eta = V_0^{\text{imp}} / V_0$ acts as the impurity strength to scale the strength of the disorder with respect to the amplitude of the square lattice. In figure 4.1, we illustrate a three-dimensional surface plot representing the Gaussian distributed square lattice potential and an impurity lattice potential employed in our work.

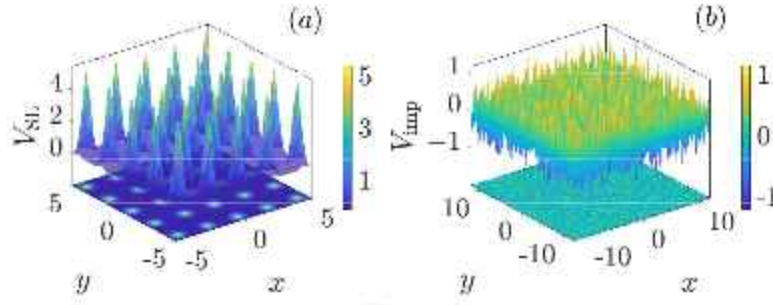


Figure 4.1: (a) Surface plot of Gaussian distributed square lattice potential with lattice constant $a = 2.2$, and (b) impurity lattice potential with impurity lattice constant $a_{\text{imp}} = 1$. The width of the Gaussian peak is 0.65, and the amplitude of the impurity potential at various lattice sites is randomly distributed over the range $[-1, 1]$.

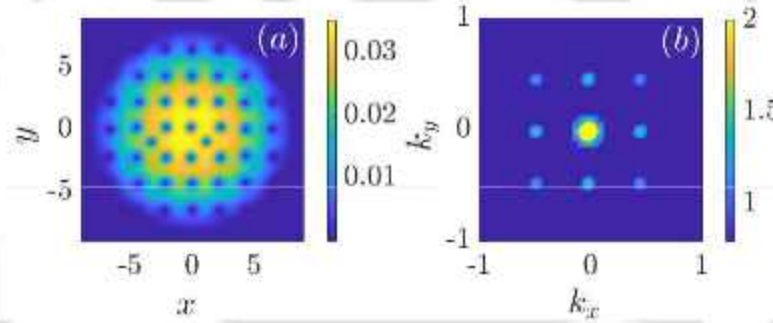


Figure 4.2: Pseudocolor representation of the condensate density in (a) real space, and (b) momentum space within a square lattice potential for lattice constant $a = 2.2$ and potential amplitude $V_0 = 5$. The other parameters are $\Omega = 0.8$ and $g = 1000$.

Figure 4.2 shows the pseudo color presentation of the condensate density in both real and momentum space, under the influence of a square lattice potential with $a = 2.2$ and $V_0 = 5$. When $\eta = 0$, vortices form a square pattern due to the pinning effect of the lattice potential. Note that due to the chosen rotational frequency of $\Omega = 0.8$, two unpinned vortices appear at $y \approx -2$, which disappear for reduced Ω .

In figure 4.3, we show the pseudo color density profile at different η and $a_{\text{imp}} = 2.2, 1.3, 1$, and 0.8 . We find that for $a_{\text{imp}} = 2.2$, the vortex lattice structure remains square even at high impurity strength (η) as shown in figure 4.3(a1)-(a4). However, as we decrease the impurity lattice constant of the random potential to $a_{\text{imp}} = 1.3$, the vortex lattice begins to distort from the square pattern at $\eta \sim 7.6$ [see figure 4.3(b2)]. The squareness of the vortex lattice is completely destroyed upon further increase of η [see figure 4.3(b3, b4)]. We observe that at $a_{\text{imp}} = 1$, obliteration of the lattice structure starts around $\eta = 3.2$ [see figure 4.3(c1)]. It implies that the critical value of η required to induce the lattice deformation decreases with a_{imp} , which is quite evident

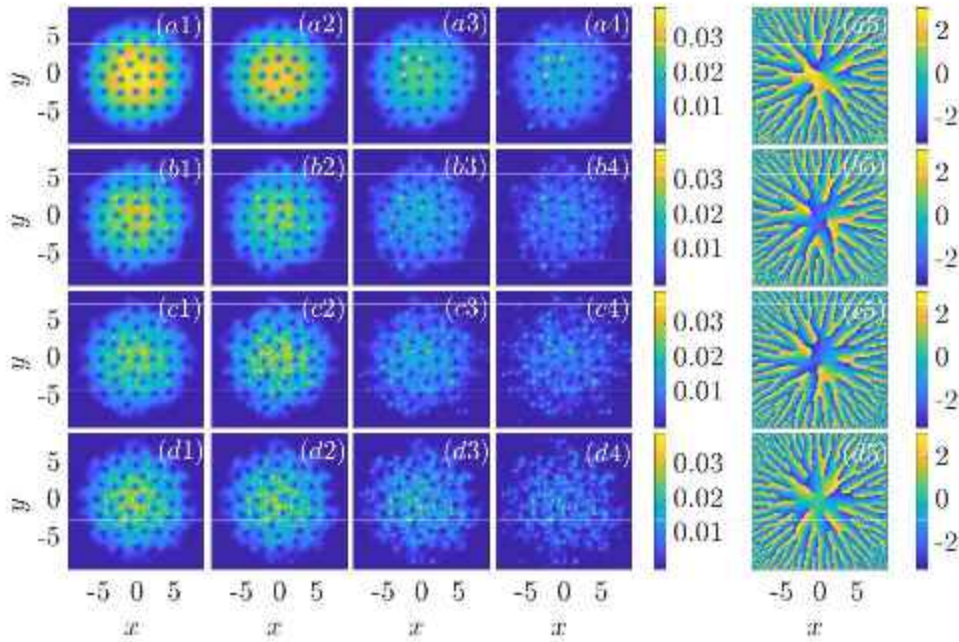


Figure 4.3: Pseudocolor representation of condensate density in real space for different a_{imp} and $\eta = V_0^{\text{imp}}/V_0$. (a1)–(a4) for fixed $a_{\text{imp}} = 2.2$ and $\eta = (3.2, 7.6, 9.2, 12)$; (b1)–(b4): $a_{\text{imp}} = 1.3$, $\eta = (3.2, 7.6, 9.2, 12)$; (c1)–(c4) at $a_{\text{imp}} = 1$, $\eta = (3.2, 7.6, 9.2, 12)$; and (d1)–(d4) at $a_{\text{imp}} = 0.8$, $\eta = (3.2, 7.6, 9.2, 12)$. The last column (a5, b5, c5, d5) displays the phase of the condensate corresponding to the melted lattice state shown in (a4, b4, c4, d4), respectively. Here, $\eta = V_0^{\text{imp}}/V_0$ is a dimensionless parameter that acts as the impurity strength. The other parameters are $a = 2.2$, $\Omega = 0.8$, $V_0 = 5$ and $g = 1000$.

from figure 4.3(d1)–(d4). The last column of the figure shows the phase of the condensate corresponding to the column (a5–d5). The phase clearly shows the presence of disordered arrays of the vortices in the melted vortex lattice state.

To confirm the periodic nature of the lattice, we also compute density in Fourier space as shown in the figure 4.4. At $\eta = 0$, the periodic peaks form a square pattern. The peaks become more irregular for the higher values of η , which reveals the melting of the vortex lattice structure. To understand the transition from the lattice vortex state to the melted vortex lattice, in figure 4.5, we show the histogram calculated for different separations of the vortices. The histogram exhibits well-separated peaks in the vortex lattice state for small η . Upon increase in η (moving along the row), the peaks become less distinguishable, indicating the destruction of long-range order and revealing the melting of the vortex lattice structure. This can be observed in the first column of figure 4.5 (a1)–(b1), where the peaks are well separated, compared to the subsequent columns where the separation becomes less pronounced.

Further, to quantitatively analyze the structural changes in the condensate, we examine the variation of the structure factor $S(k)$ with η for different impurity lattice

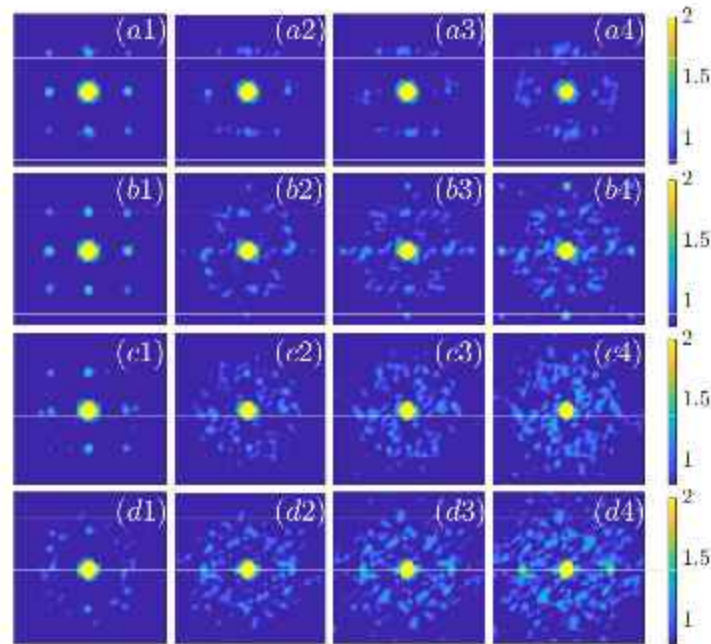


Figure 4.4: Pseudocolor representation of the condensate density in momentum space: (a1)-(a4) for fixed $a_{\text{imp}} = 2.2$ and $\eta = (3.2, 7.6, 9.2, 12)$; (b1)-(b4): For fixed $a_{\text{imp}} = 1.3$ and $\eta = (3.2, 7.6, 9.2, 12)$; (c1)-(c4) at $a_{\text{imp}} = 1$ and $\eta = (3.2, 7.6, 9.2, 12)$; and (d1-d4) at $a_{\text{imp}} = 0.8$ and $\eta = (3.2, 7.6, 9.2, 12)$. The other parameters are the same as figure 4.3.

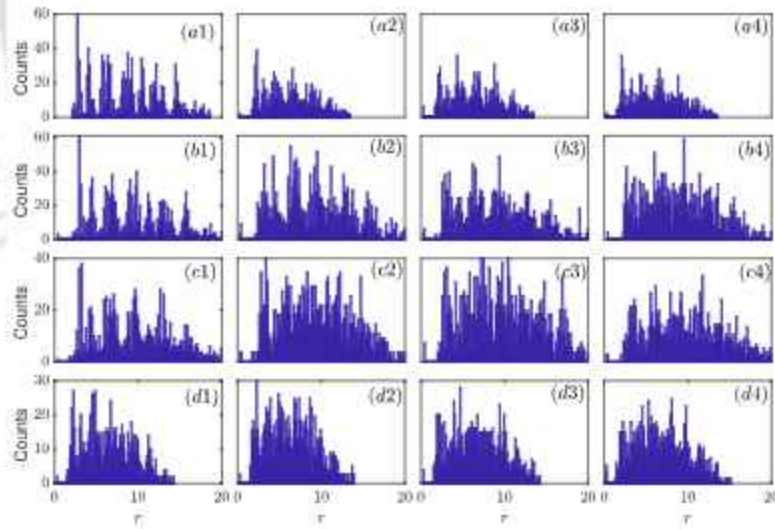


Figure 4.5: Histogram of separation between the pair of vortices (r) for different a_{imp} and η . The variation of a_{imp} and η along rows and columns are the same as that of figure 4.3. The presence of well-separated peaks indicates the presence of long-range order for the vortex lattice. High η shows the presence of the vortices at all separations, indicating the melting of the vortex lattice.

constant a_{imp} as depicted in figure 4.6(a). We observe a steady decrease in $S(k)$ as η increases and the critical value of impurity strength (η_c), at which $S(k)$ falls below 0.5 depends on a_{imp} . Note that, here, we define the range of $0.5 \lesssim S(k) \lesssim 1.0$ to represent the pinned lattice state, while $S(k) \lesssim 0.5$ indicates the melted state [56]. Specifically, at $a_{\text{imp}} = 1.3$ the vortex lattice structure exhibits melting behavior beyond $\eta = 8.0$, while for $a_{\text{imp}} = 1.0$, and 0.8 , the melting behavior commences above $\eta = 5.2$ and 4.0 , respectively. Figure 4.6(b) depicts the angular momentum ($\langle L_z \rangle$) of the vortex lattice structure

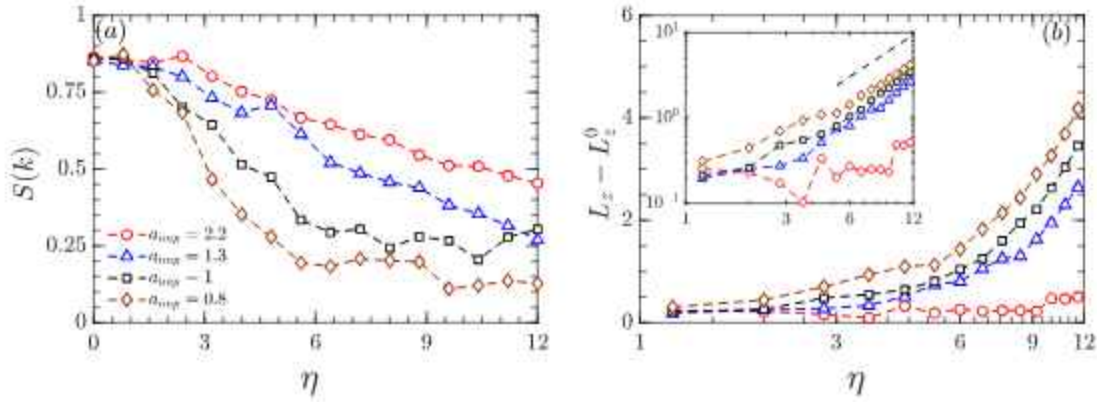


Figure 4.6: (a) Variation of structure factor ($S(k)$) and (b) angular momentum ($\langle L_z \rangle - \langle L_z^0 \rangle$) as a function of η for different impurity lattice constants (a_{imp}). The inset in (b) shows the variation $\langle L_z \rangle - \langle L_z^0 \rangle$ in the log scale. Here, $\langle L_z^0 \rangle$ is the angular momentum of the initially pinned vortex lattice. The black dashed in inset represents the power law fitting to the energy with respect to η with an exponent ~ 1.73 in the vortex lattice melted region.

as a function of η (in a semilog scale). This provides complementary information to the structure factor analysis. We find that $\langle L_z \rangle$ remains near zero for $a_{\text{imp}} = 2.2$, which indicates that no new vortices are created in the condensate at this impurity strength to distort the lattice structure. Conversely, for $a_{\text{imp}} = 1.3$, $\langle L_z \rangle$ exhibits an increasing trend beyond a certain threshold value of η , suggesting the nucleation of additional vortices. Moreover, as we decrease a_{imp} further, $\langle L_z \rangle$ shows a pronounced increasing behavior with higher magnitudes. The inset of figure 4.6(b) describes the variation of $\langle L_z \rangle$ with η in the log-log scale which reveals a power law nature of the angular momentum with η for larger values of η with a critical exponent ~ 1.73 . The power law behavior shows the completely disordered nature of the vortex arrangement. The presence of additional nucleated vortices introduces an incommensurate effect on the vortex lattice and interacts with the pinned vortices. This interaction leads to the deformation of the vortex lattice structure. To gain a more comprehensive insight into the emergence of disorder related to vortex lattice melting, we calculate the informa-

tion entropy of the system. Figure 4.7 illustrates the total information entropy S_{en} of the condensate as a function of the parameter η , considering different values of a_{imp} . For larger values of a_{imp} , such as 2.2, the entropy remains constant regardless of η . However, as we decrease a_{imp} , the entropy shows a noticeable increase with η . In the logarithmic scale, this increase follows a power-law behavior with an exponent of 1.73 in the melted region.

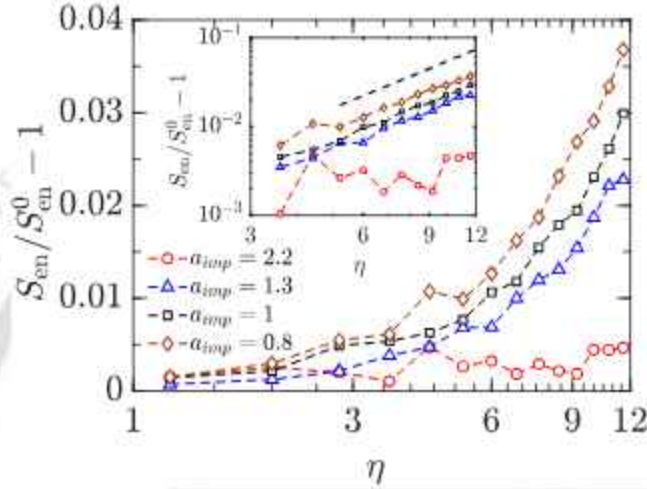


Figure 4.7: Variation of information entropy (S_{en}) with respect to η for different values of a_{imp} . The inset represents the S_{en} in logscale, which depicts a power-law behaviour of S_{en} for larger η with an exponent of 1.73. The black dashed line guides the eyes for the power-law fitting. Note that, S_{en} rescaled with entropy of the pinned lattice, S_{en}^0 at $V_0 = 5.0$.

To elucidate the cause of this structural transformation, we calculate the relevant physical contributions by decomposing the total energy into its various components, as shown in equation (3.8). In figure 4.8, we show the variation of kinetic (E_K), potential (E_{pot}), interaction (E_{int}), and total energy (E_T) with respect to η for different a_{imp} . Figure 4.8(a) illustrates that all the energy components are nearly the same order for $a_{\text{imp}} = 2.2$. This observation is evident because there is no alteration in the vortex-lattice structure at this specific lattice parameter. However, as we decrease the lattice parameter, the energy components exhibit an increasing trend above a critical value of η [see figure 4.8(b)]. In the melted vortex lattice region, kinetic energy (E_K) dominates the other energy components. In the ordered lattice configuration, vortices are arranged in a regular pattern, leading to lower kinetic energy because the motion of vortices is restricted. However, as the vortex lattice melts, the vortices move freely and are no longer confined to fixed positions. Increased disorder facilitates free movement of the vortices, which finally leads to the large kinetic energy. As the vortices are

no longer in the pinned state, they interact with each other, and this interaction leads the interaction energy (E_{int}) to be dominant over potential energy [see figure 4.8(b)]. Additionally, as the value of a_{imp} decreases, the magnitude of the energy components increases. The inset of figure 4.8(b) depicts the variation of energy components in the log-log scale and reveals a power law dependence with critical exponent 1.73 for higher values of η .

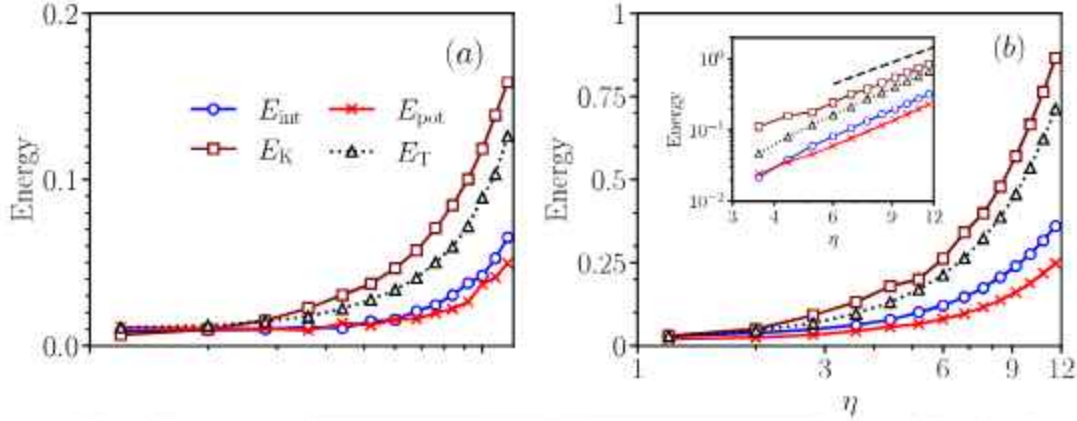


Figure 4.8: Different energies E_{int} , E_{pot} , E_K , and E_T as a function of η for lattice constants: (a) $a_{\text{imp}} = 2.2$, and (b) $a_{\text{imp}} = 0.8$. Each energy is scaled as $\text{Energy} = \text{Energy}/\text{Energy}^0 - 1$, where Energy^0 is the energy of the square vortex lattice without any impurities. Inset in (b) shows the variation of energy with respect to η in the logscale where, the black dashed line shows the power law variation of energy with respect to η with an exponent ~ 1.73 in the vortex lattice melted state.

To characterize the melted vortex lattice state more comprehensively, we examine the kinetic energy spectra of the condensate. We begin by focusing on the incompressible kinetic energy spectrum (ϵ_{kin}^i) shown in figure 4.9, followed by the compressible energy spectra (ϵ_{kin}^c) in figure 4.10. The scaling laws of these spectra provide a more profound understanding and quantification of the turbulent regions. In our spectral analysis, we identified three distinct length scales in k space that pertain to the condensate. These length scales are characterized as $k_{R_{\text{TF}}} = \frac{2\pi}{R_{\text{TF}}}$, $k_{l_0} = \frac{2\pi}{l_0}$, and $k_{\xi} = \frac{\pi}{\xi}$, where $R_{\text{TF}} = \sqrt{2\mu}$ represents the Thomas Fermi radius, $l_0 = \frac{1}{n_v}$ denotes the inter vortex separation, and n_v stands for the number of vortices per unit area. In our study we decompose the kinetic energy spectrum over wave number k into two different regions: the ultraviolet regime ($k \gg \xi^{-1}$) and the infrared regime ($k \ll \xi^{-1}$). In Table 4.1, we provide the estimated values of R_{TF} , l_0 , and ξ for different impurity lattice constant a_{imp} and the strength of the impurity potential η .

Figure 4.9 represents the incompressible energy spectra for different impurity lattice constants and the strength of the impurity potential. For $a_{\text{imp}} = 2.2$, the spectra

a_{imp}	η	R_{TF}	l_0	ξ
2.2	3.6	4.961	1.605	0.285
	12	5.083	1.594	0.277
0.8	3.6	5.323	1.618	0.265
	12	6.191	1.780	0.228

Table 4.1: Estimation of different characteristic length scales R_{TF} , l_0 , and ξ for different a_{imp} and η at $\Omega = 0.8$. All the reported values are averaged over five different realizations of random impurity potential.

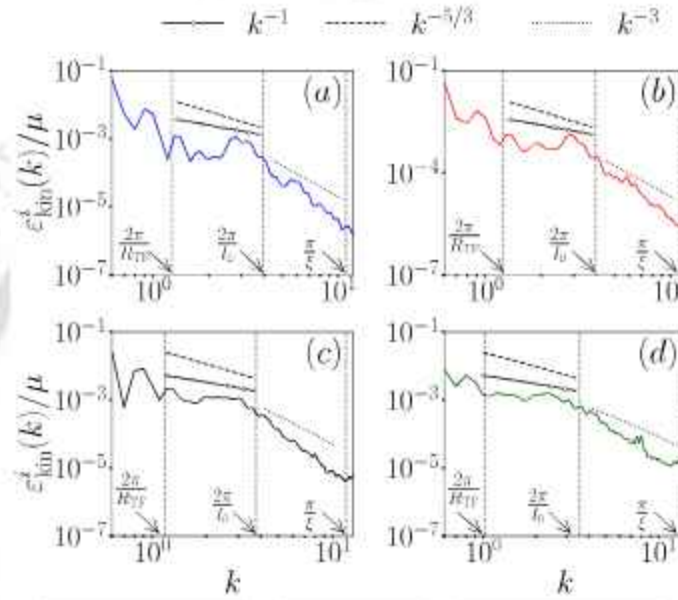


Figure 4.9: Incompressible kinetic energy spectra for different η and a_{imp} : (a) $a_{\text{imp}} = 2.2$, and $\eta = 3.6$, (b) $a_{\text{imp}} = 2.2$, and $\eta = 12$, (c) $a_{\text{imp}} = 0.8$, and $\eta = 3.6$, (d) $a_{\text{imp}} = 0.8$, and $\eta = 12$. It exhibits k^{-1} and k^{-3} power laws in the IR and UV regions, respectively.

show multiple peaks in the range $k_{R_{\text{TF}}} < k < k_{l_0}$ for $\eta = 3.6$ and 12.0 , indicating the presence of the organized vortex lattice in the condensate. Therefore, no noticeable power law behavior has been observed in that region [see figure 4.9(a) and (b)]. However, the spectrum adheres to a k^{-3} power law in the range $k_{l_0} < k < k_{\xi}$, which is associated with the presence of the vortex core. When the impurity lattice constant is further reduced to $a_{\text{imp}} = 0.8$, the peaks disappear, and the spectra exhibit power law behavior with exponents of k^{-1} in the range $k_{R_{\text{TF}}} < k < k_{l_0}$ and k^{-3} for larger length scales $k_{l_0} < k < k_{\xi}$ [see figure 4.9(c) and (d)]. The scaling behavior at length scales larger than the inter-vortex distance suggests the presence of turbulence arising from the random motion of the vortices. The presence of k^{-1} scaling of the energy spectrum indicates the Vinen-like cascade of the energy in the scale larger than the

intervortex separation. Similar kinds of scalings have been reported for the turbulent two-dimensional condensate trapped in the harmonic trap [154].

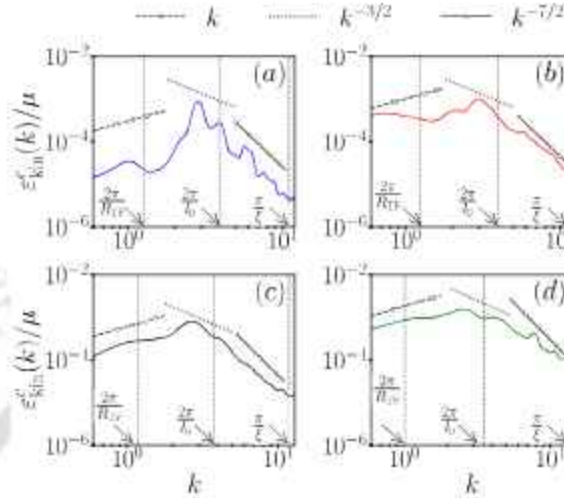


Figure 4.10: Compressible kinetic energy spectra for different η and a_{imp} : (a) $a_{\text{imp}} = 2.2$, and $\eta = 3.6$; (b) $a_{\text{imp}} = 2.2$, and $\eta = 12$; (c) $a_{\text{imp}} = 0.8$, and $\eta = 3.6$; (d) $a_{\text{imp}} = 0.8$, and $\eta = 12$. It shows k power law for small value of k . In the UV region, it follows $k^{-3/2}$, and $k^{-7/2}$ power law.

Next, we analyze the behavior of the compressible part of the kinetic energy spectrum in the vortex lattice melted state. In figure 4.10, we show the compressible energy spectra $\varepsilon_{\text{kin}}^c$ for the same set of parameters as shown in figure 4.9. We observe that compressible energy spectra for all impurity lattice constants seem to follow a k scaling for small wave numbers, and this behavior is related to the thermal equilibrium of the energy. For $a_{\text{imp}} = 2.2$ [see figure 4.10(a) and (b)], no distinct power law behavior has been observed, and the presence of multiple peaks indicates the existence of the vortex lattice structure within the condensate, as discussed earlier in the case of incompressible energy spectra. The spectrum exhibit $k^{-7/2}$ power law in the region $k_{l_0} < k < k_\xi$ which corresponds to the turbulence behavior. For high impurity strength, the spectrum develops a $k^{-3/2}$ power law for a small region around k_{l_0} [see figure 4.10(d)], indicating the weak wave turbulence in melted vortex lattice. So far, our investigation has been mainly focused on the effect of random impurity potential on the vortex lattice structure, revealing that the lattice constant and amplitude of the impurities significantly affect the structure of the vortex lattice. Further, we show the presence of Vinen-like scaling in the inertial range of the incompressible energy spectrum that suggests the presence of turbulence nature of the condensate in the vortex lattice melted state.

4.4.2 Characterization of the structural transformation of vortices in rotating BECs with different rotational frequencies in the presence of random impurity potential

In this section, we present a comprehensive analysis of the impact of various rotational frequencies on a condensate in the presence of random static impurities.

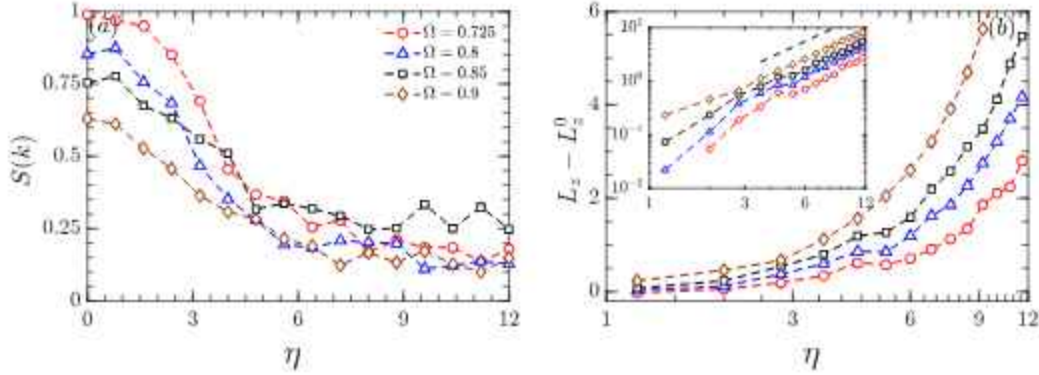


Figure 4.11: Variation of (a) the structure factor, $S(k)$, and (b) the angular momentum, $\langle L_z \rangle - \langle L_z^0 \rangle$, as a function of η for different Ω at $a_{\text{imp}} = 0.8$. Increasing Ω lowers the threshold value of η above which the vortex lattice deforms, accompanied by a drop in $S(k)$ below 0.5 in figure (a). The inset in (b) shows the variation of the same angular momentum on a log scale. The black dashed line in the log scale plot highlights the power-law behavior with respect to η . Here, $\langle L_z^0 \rangle$ represents the angular momentum of the initially pinned vortex lattice.

From Figure 4.13, we have obtained that the critical value η_c depends upon the rotational frequency Ω regardless of the impurity lattice constant a_{imp} . Here, we present the corresponding structure factor and angular momentum to obtain the threshold value of η_c for various Ω at $a_{\text{imp}} = 0.8$ as shown in figure 4.11. We find that $S(k)$ gradually decreases upon increase in η and the critical value η_c above which $S(k) < 0.5$ depends on Ω . In figure 4.11(a), we illustrate the structural deformation of the vortex lattice. We find that the structural deformation of the vortex lattice occurs at $\eta = 4$ and $\eta = 3.2$ for $\Omega = 0.725$ and 0.8 , respectively, however, for $\Omega = 0.85$ and 0.9 , it happens at $\eta = 2.8$ and 2.4 respectively. $\langle L_z \rangle$ increases upon increase in η , and the magnitude of $\langle L_z \rangle$ depends quite significantly on Ω as depicted in the figure 4.11(b). The inset shows that, in the melting region, $\langle L_z \rangle$ has a power law dependence on η with an exponent 1.73. Next, in figure 4.12, we analyze the variation of kinetic (E_K), potential (E_{pot}), interaction (E_{int}), and total energy (E_T) with respect to η for different rotational frequencies: (a) $\Omega = 0.725$, (b) $\Omega = 0.9$ for $a_{\text{imp}} = 0.8$. We observe a consistent pattern in the behavior of energy components, irrespective of the value of Ω .

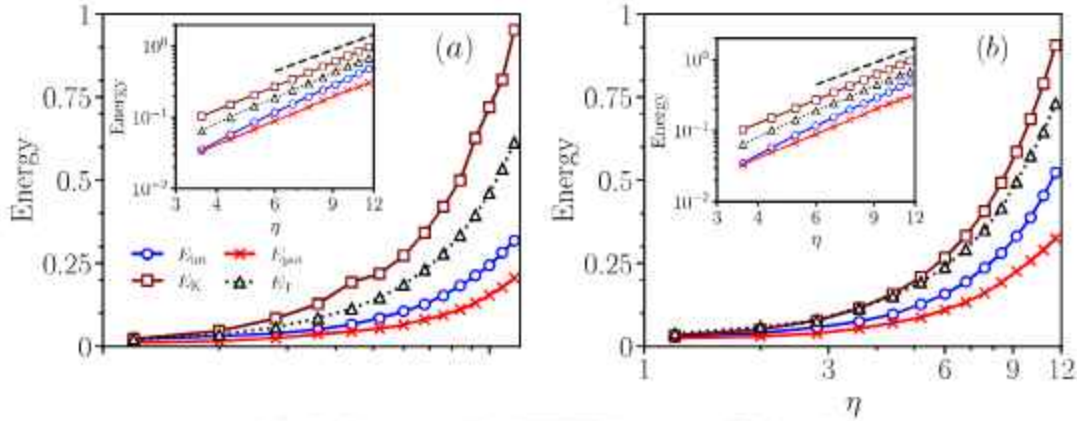


Figure 4.12: Variation of different energies, E_{int} , E_{pot} , E_K , and E_T as a function of η for impurity lattice constant $a_{imp} = 0.8$ for different Ω : (a) $\Omega = 0.725$, (b) $\Omega = 0.9$. The inset shows the same in the log scale. The black dashed line is drawn to show the power law behavior of the energy components with in the melted state.

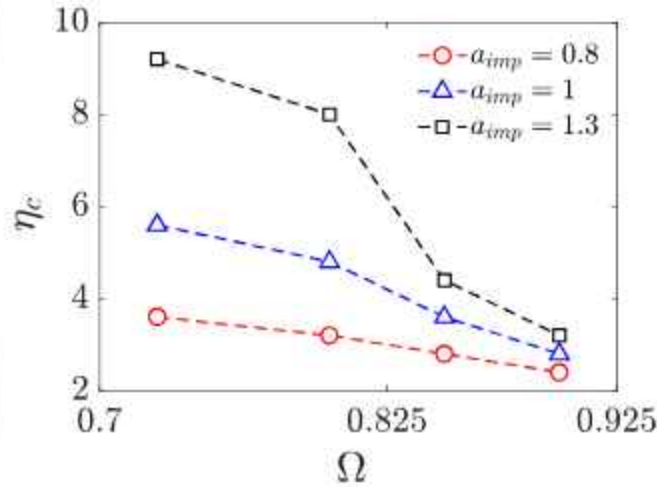


Figure 4.13: Variation of critical impurity strength (η_c) as a function of Ω for different impurity lattice constants (a_{imp}). The results show that the η_c decreases as the Ω increases.

Specifically, in the melted region, the kinetic energy (E_K) tends to dominate over the other energy components consistently, and the dominance of nonlinear energy over potential energy is observed across all frequencies. However, it is important to note that the rotational frequency (Ω) affects the critical value at which the vortex lattice structure exhibits structural abnormalities. The inset of figure 4.12(a) presents a log-log plot of the energy components and reveals a power law dependence on η with critical exponent 1.73 in the melted region.

In figure 4.13, we present a comparative analysis about the dependence of η_c on

the rotational frequency Ω for impurity lattice constants $a_{\text{imp}} = 1.3$ (black squares), 1.0 (blue triangles), and 0.8 (red circles). Our observations indicate that as the rotational frequency increases, the critical value of the strength of the impurity potential above which the vortex lattice melts decreases. The above suggests that higher rotational frequency leads to an earlier onset of vortex lattice melting. Additionally, we notice that the critical value η_c increases with the increment of a_{imp} for a lower value of Ω . Although η_c appears to be different for a low value of Ω , for higher Ω , η_c value converges towards the same order regardless a_{imp} .

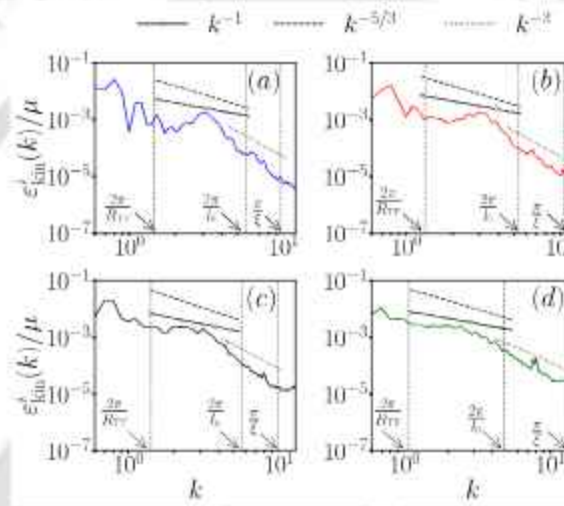


Figure 4.14: Incompressible kinetic energy spectra for $\Omega = 0.9$ for different a_{imp} , and η : (a) $a_{\text{imp}} = 2.2$, $\eta = 3.6$; (b) $a_{\text{imp}} = 2.2$, $\eta = 12$; (c) $a_{\text{imp}} = 0.8$, $\eta = 3.6$; (d) $a_{\text{imp}} = 0.8$, $\eta = 12$. It exhibits k^{-1} and k^{-3} power laws in the IR and UV regions, respectively.

Figures 4.14 and 4.15 show the incompressible and compressible spectra corresponding to the melted vortex lattice corresponding to the different a_{imp} and η . In particular (a) for $a_{\text{imp}} = 2.2$, and $\eta = 3.6$; (b) for $a_{\text{imp}} = 2.2$, and $\eta = 12$; (c) for $a_{\text{imp}} = 0.8$, and $\eta = 3.6$ and (d) $a_{\text{imp}} = 0.8$, and $\eta = 12$. Like other spectrum behavior in the melted vortex lattice state, the incompressible spectrum exhibits Vinen like spectrum ($\propto k^{-1}$) in the IR range of the wave number space for all the cases. However, the compressible spectra show the presence of the weak turbulence.

In the next part of our study, we consider the impurities as a rotating Gaussian obstacle and analyze its effect on the vortex lattice structure and the turbulence nature of the condensate in the melted vortex lattice state.

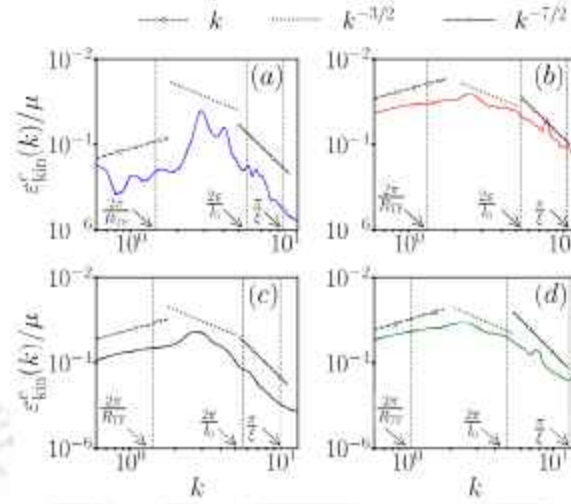


Figure 4.15: Compressible kinetic energy spectra of the condensate at different impurity lattice strength (η) at $\omega = 0.9$ for different η and a_{imp} : (a) $a_{\text{imp}} = 2.2$, and $\eta = 3.6$; (b) $a_{\text{imp}} = 2.2$, and $\eta = 12$; (c) $a_{\text{imp}} = 0.8$, and $\eta = 3.6$; (d) $a_{\text{imp}} = 0.8$, and $\eta = 12$. The spectra display a k power law for small value of k , while in the UV region it follows $k^{-3/2}$, and $k^{-7/2}$ power laws.

4.4.3 Vortex lattice melting in presence of oscillating Gaussian obstacle

In this section, we investigate the effect of the oscillating Gaussian obstacle on the structure of the vortex lattice for rotating BECs trapped under a Gaussian distributed square lattice potential. To systematically examine the vortex generation in the condensate due to an oscillating Gaussian obstacle, we first analyze the dynamics of vortex formation in the absence of both rotation and the Gaussian distributed square lattice potentials. Subsequently, we extend the analysis to investigate the impact of the rotating obstacle on the vortex lattice structure for rotating BECs trapped under a harmonic as well as square lattice potential.

As the Gaussian obstacle is stirred through the condensate, vortices are generated to minimize the energy of the system [127, 152, 153]. There are several ways to control the generation of vortices in an obstacle, either by increasing the distance of the obstacle r_0 from the center of the condensate or by increasing the oscillation frequency ω of the obstacle. Here, we aim to investigate the resultant dynamical effect of the obstacle on the vortex lattice state. To make the analysis self-contained, in the following, we concisely present the dynamics of non-rotating condensate in the presence of an oscillating obstacle for various r_0 and ω .

Following earlier work, we consider an impenetrable obstacle with $V_0/\mu > 1$ [152]

with obstacle depth $V_0 = 80$ and the chemical potential $\mu = 12.26$. The width of the obstacle has been fixed to $d = 0.3$.

In figure 4.16, we show the temporal evolution of the angular momentum $\langle L_z \rangle = \int \psi^\dagger (y\partial_x - x\partial_y) \psi dx dy$, for different r_0 and ω . For $r_0 = 0.5$, the angular momentum $\langle L_z \rangle$ remains zero even at a higher frequency ($\omega = 3$), indicating the absence of the vortices. However, as the value of r_0 is increased to 2, we observe an increase in the angular momentum, which indicates the generation of vortices within the condensate. This phenomenon is clearly visible in the inset density profile.

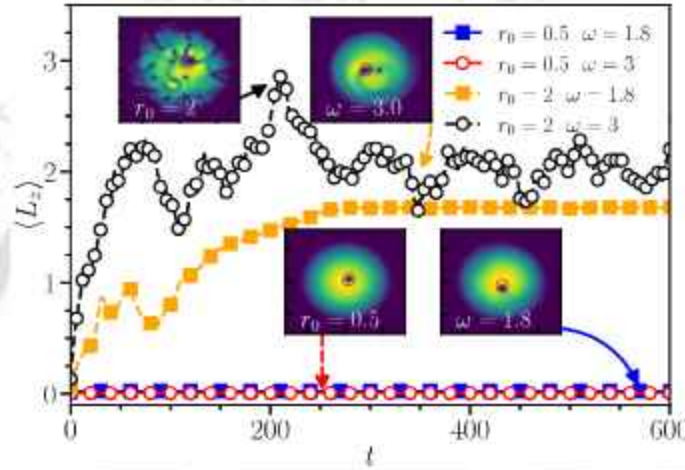


Figure 4.16: Temporal evolution of the $\langle L_z \rangle$ for $r_0 = 0.5$ and 2 at $\omega = 1.8$, and 3. The inset shows the pseudo-color presentation of the condensate density in real space at $t = 400$. The other parameters are $\Omega = 0.0$, $g = 1000$, and $V_0 = 5.0$. The red circle in the density plot (inset) indicates the obstacle's position.

So far, we have observed that vortex generation by dynamic impurities modeled as Gaussian obstacles can be influenced by two factors: the distance from the condensate center (r_0) and the oscillation frequency (ω) of the obstacle. Next, we consider the rotating BECs under the Gaussian distributed square lattice and harmonic trap and analyze the impact of the dynamic Gaussian impurities on the structure of the vortex lattice by varying its frequency and its position from the center of the condensate in the similar line as studied with the static impurities case in the section 4.4.1. In figure 4.17, we show the pseudo colour representation of the density while varying both r_0 and ω of the dynamical Gaussian obstacle. We find that at $r_0 = 0.5$, the vortex lattice structure remains pinned even for higher oscillation frequencies [see figure 4.17(a1)-(a5)]. However, we notice a small distortion in the vortex lattice structure at $\omega = 3$ upon increasing r_0 from 0.5 to 1 [see figure 4.17(b5)]. The vortex lattice structure starts getting deformed quite significantly around $\omega = 1.2$ upon considering $r_0 = 1.5$ as

shown in figure 4.17(c4). From figure 4.17(d2), we can see that an increase in r_0 from 1.5 to 2 results in deformation in the lattice structure even at lower oscillation frequency $\omega = 0.6$. These characteristics are more pronounced in the density distribution in momentum space [see figures 4.18(a1)–(a5)]. We find that the periodic peaks are arranged in a square pattern, signifying the pinned vortex lattice of the condensate density in real space. In the presence of a sufficiently high oscillation frequency and high r_0 , the translational symmetry and long-range order of the lattice structure are completely destroyed. As a result, no periodic peaks are visible, while the vortex lattice structure remains in the square lattice for smaller r_0 . Thus, it characterized the melted state of the vortex lattice structure, which appears in figures 4.18(d1)–(d5). We show the structural transformation of the vortex lattice from an ordered to a disordered state based upon the values of ω and r_0 in a more systematic way in the figures 4.19 and 4.20.

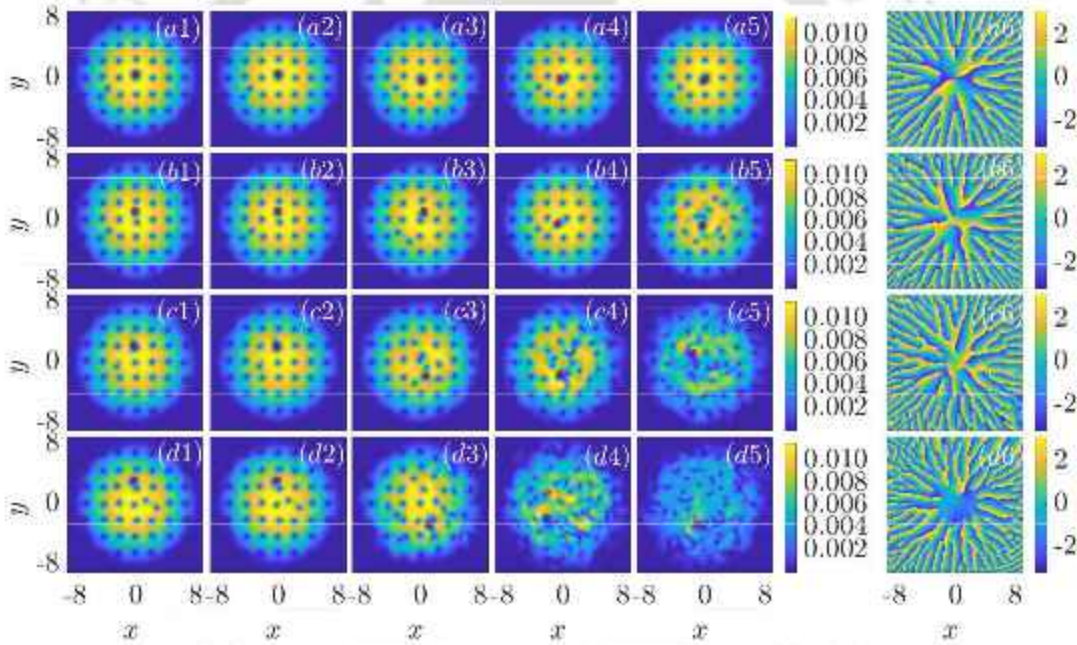


Figure 4.17: Pseudocolour representation of condensate density in real space for different r_0 and ω values of the obstacle. (a1)–(a5): For fixed $r_0 = 0.5$ and ω varies (0, 0.2, 0.6, 1.2, 2.0); (b1)–(b5): at $r_0 = 1$ and $\omega = (0, 0.2, 0.6, 1.2, 2.0)$; (c1)–(c5) at $r_0 = 1.5$ and $\omega = (0, 0.2, 0.6, 1.2, 2.0)$; and (d1)–(d5) $r_0 = 2$ and $\omega = (0, 0.2, 0.6, 1.2, 2)$, respectively. Other parameters are $a = 2.2$, $\Omega = 0.8$, $V_0 = 5$ and $g = 1000$. The red circle indicates the obstacle position. The last column (a6–d6) represents the phase of the condensate corresponding to the vortex melting state represented in (a5–d5), respectively.

Similar to the analysis of static impurities case, here also to quantify the structural transformation of the vortex lattice in the presence of the dynamic impurity, we compute the structure factor ($S(k)$) and the angular momentum ($\langle L_z \rangle$) of the condensate, taking into account the square lattice configuration. Figure 4.19 illustrates the tempo-

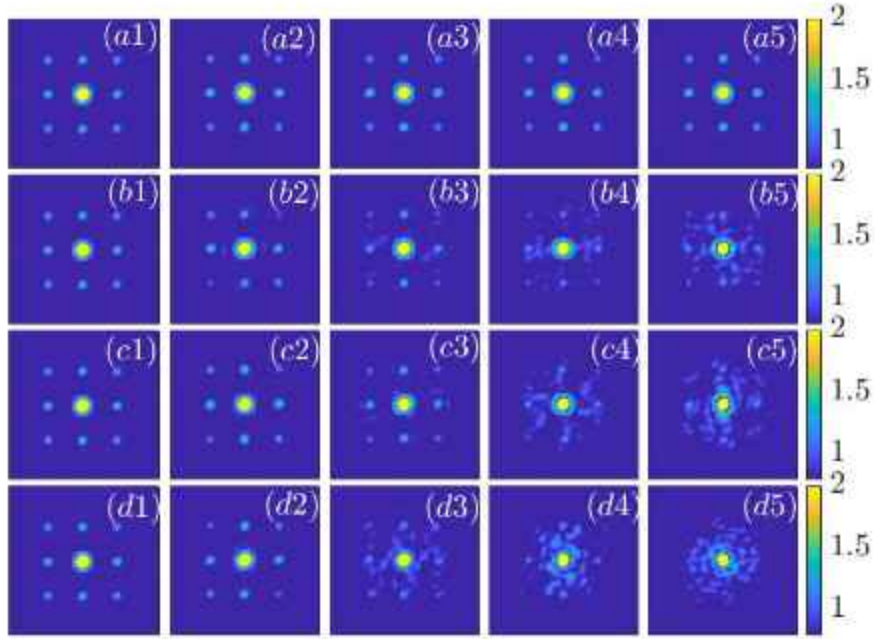


Figure 4.18: Pseudocolour representation of density in momentum space corresponding to the real space densities in figure 4.17. (a1-a5): For $r_0 = 0.5$ and $\omega = (0.2, 0.6, 1.2, 2)$; (b1-b5): For fixed $r_0 = 1.0$ and $\omega = (0.2, 0.6, 1.2, 2.0)$; (c1-c5) For $r_0 = 1.5$ and $\omega = (0.2, 0.6, 1.2, 2)$; and (d1-d5) For $r_0 = 2$ and $\omega = (0.2, 0.6, 1.2, 2.0)$. The other parameters are the same as in figure 4.17.

ral variations of $S(k)$ and $\langle L_z \rangle$ for different oscillation frequency ω at $r_0 = 2$. We find that when $\omega = 0$ (red open circles), $S(k)$ remains relatively constant around 0.8. However, increasing ω to 2.0 induces a deformation in the vortex lattice structure, leading to a significant decrease in $S(k)$ towards 0 (blue open circles). For small values of ω , no interstitial vortices are present in the condensate, thereby causing $\langle L_z \rangle$ to remain close to 0 (red open circles). Conversely, for larger values of $\omega = 2.0$, the angular momentum $\langle L_z \rangle$ exhibits an increasing trend over time as depicted in figure 4.19. This feature indicates that extra vortices are being nucleated within the condensate due to the faster oscillation of the obstacle, and these newly generated vortices interact with the vortices pinned near the maxima of the Gaussian distributed square lattice potential, which eventually results in the melting of the vortex lattice structure [152].

To have more insight into the vortex lattice melting state in the presence of the dynamical impurities, we analyze the time-averaged structure factor $\langle S(k) \rangle_t$ and angular momentum $\langle L_z \rangle_t - \langle L_z^0 \rangle_t$ as a function of ω for different r_0 . Here, $\langle L_z \rangle_t$ is rescaled with respect to the angular momentum of the initial state $\langle L_z^0 \rangle_t$ at $\omega = 0$. Notably, a significant decrease in $\langle S(k) \rangle_t \lesssim 0.5$ indicates the structural deformation of the vortex lattice,

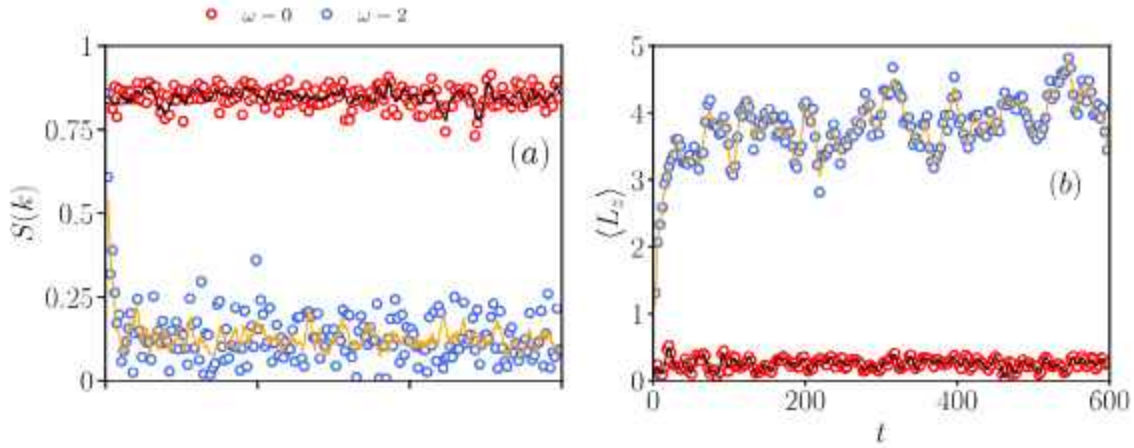


Figure 4.19: Temporal evolution of (a) the structure factor $\langle S(k) \rangle$, and (b) the angular momentum $\langle L_z \rangle$ of the condensate for different oscillation frequencies ($\omega = 0.0, 2.0$) for the obstacle located at $r_0 = 2.0$. The black and orange solid lines represent the moving average at $\omega = 0.0$ and $\omega = 2.0$, respectively.

consistent with the observed density profiles. For $r_0 = 0.5$, the $\langle S(k) \rangle_t$ remains approximately 0.85. However, as we increase r_0 to 1, $\langle S(k) \rangle_t$ decreases. Moreover, when $r_0 = 1.5$, the critical oscillation frequency at which the vortex lattice structure begins to exhibit melting behavior is $\omega \sim 1.2$, while for $r_0 = 2$, the threshold value is approximately $\omega \sim 0.8$. Thus, an increase in r_0 leads to a decrease in the critical ω beyond which the vortex lattice structure undergoes melting. In figure 4.20(b), we observe that $\langle L_z \rangle_t$ remains close to its initial value for smaller r_0 but increases as r_0 becomes larger. The increase of the angular momentum value beyond the threshold value of ω indicates the generation of additional vortices in the condensate, with the number of generated vortices depending on ω and r_0 . In the inset of figure 4.20(b), we show $\langle L_z \rangle$ in a logarithmic scale, revealing a power-law behavior with ω in the melting region, characterized by an exponent of 1.73 which is same as the case for static impurity.

To investigate the influence of vortex generation in a condensate, we analyze the normalized drag force acting on an oscillating obstacle which is defined as $\vec{f} = (\langle f_x \rangle, \langle f_y \rangle)$ $= i\partial_t \int \psi^\dagger \nabla \psi dx dy$ [206]. In figure 4.21 we show the x- and y-component of the drag force for $r_0 = 2$ at $\omega = 0$, and 2. The drag force serves as an indicator of the impact of vortices on the dynamics of an obstacle within a condensate. When vortices are immobilized at their respective positions in the Gaussian distributed square lattice, the drag force is relatively small compared to the case where vortices are randomly distributed throughout the condensate. For $\omega = 0$ [figure 4.21(a) and figure 4.21(c)], where no additional vortices are nucleated in the condensate, the drag force components $\langle f_x \rangle$ and $\langle f_y \rangle$ remain zero over time. However, when additional

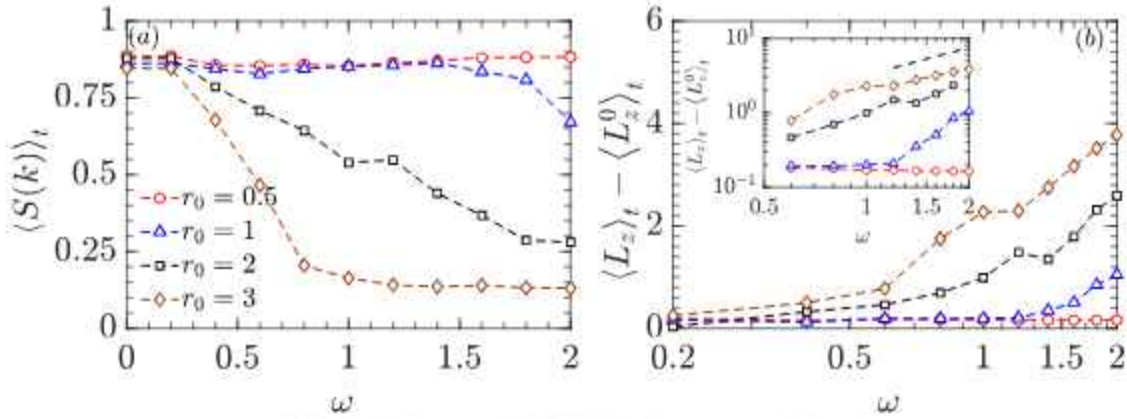


Figure 4.20: (a) Time average of structure factor $\langle S(k) \rangle_t$ and (b) angular momentum $\langle L_z \rangle_t - \langle L_z^0 \rangle_t$ as a function ω for different r_0 . The inset in (b) shows the variation $\langle L_z \rangle_t - \langle L_z^0 \rangle_t$ in the log scale. $\langle L_z^0 \rangle_t$ is the angular momentum of the initially pinned vortex lattice at $r_0 = 0$, and $\omega = 0$. The black dashed line is drawn to show the power law nature with ω in the melted region.

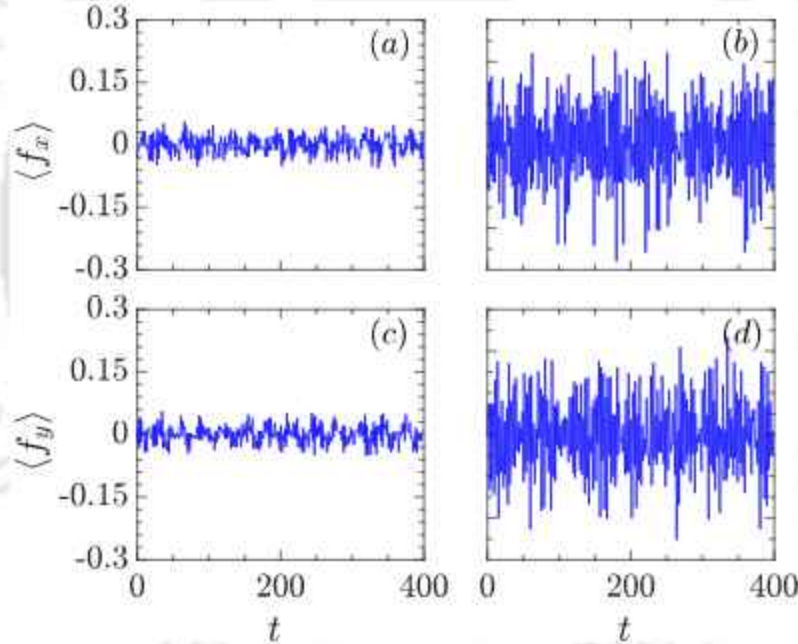


Figure 4.21: Time evolution of $\langle f_x \rangle$ and $\langle f_y \rangle$ with $r_0 = 2$ for different values of ω . Panels (a) and (b) display the evolution of $\langle f_x \rangle$, while panels (c) and (d) depict the evolution of $\langle f_y \rangle$ for $\omega = 0$ and $\omega = 2$, respectively. When $\omega = 0$, both $\langle f_x \rangle$ and $\langle f_y \rangle$ oscillate around zero. In contrast for $\omega = 2$, they increase periodically.

vortices are generated, $\langle f_x \rangle$ and $\langle f_y \rangle$ exhibit oscillatory behavior within the range of $-0.15 \lesssim (\langle f_x \rangle, \langle f_y \rangle) \lesssim 0.15$ [figure 4.21(b) and figure 4.21(d)]. Furthermore, we calculate the time average $\langle \langle f_{(x,y)} \rangle \rangle_t = \sum_{i=0}^T \langle f_{(x,y)} \rangle / T$ of $\langle f_x \rangle$ and $\langle f_y \rangle$ to analyze their variation with the oscillation frequency ω and the position of the obstacle r_0 , as de-

picted in figure 4.22. It is evident that the averaged values $\langle f_x \rangle_t$ and $\langle f_y \rangle_t$ initially increase, but at large ω , the rate of increase becomes significantly lower for low r_0 . On the other hand, $\langle f_x \rangle_t$ and $\langle f_y \rangle_t$ gradually increase as the vortex lattice structure undergoes a transition from a pinned lattice to a melted lattice structure. Additionally, for higher values of r_0 , the magnitude of the drag force dominates over lower r_0 values for a given ω . In conclusion, the drag force also complements the findings of the structural transformation by showing the dependence of both r_0 and ω . Now, we move on

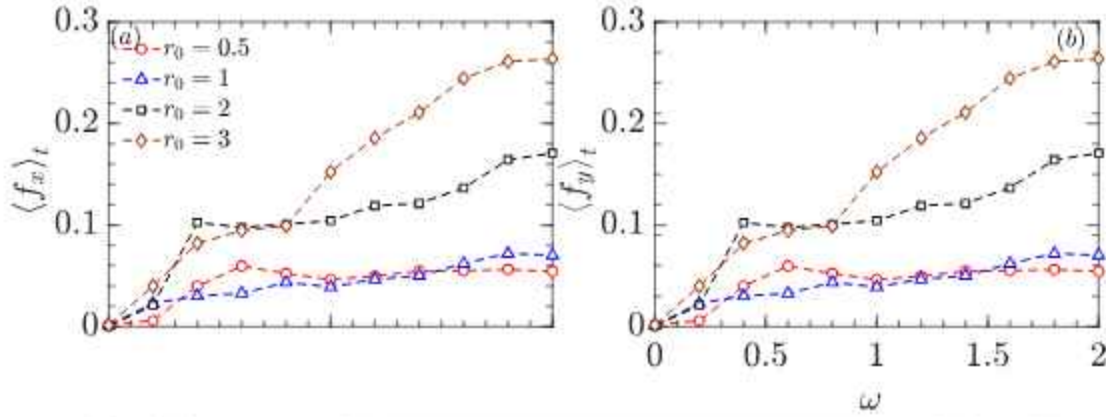


Figure 4.22: Time-average of drag force with oscillation frequency ω for (a) $\langle f_x \rangle_t$ and (b) $\langle f_y \rangle_t$ as a function of ω for various values of r_0 . Both $\langle f_x \rangle$ and $\langle f_y \rangle$ increase with ω for larger values of r_0 .

to estimate the extent of vortex disorder using entropy. In figure 4.23 we plot the time average of total information entropy ($\langle S_{en} \rangle_t$) of the condensate as a function of ω . The non-zero value of the information entropy indicates the disorderness in the system. It is easy to see that for smaller values of r_0 such as 0.5, the entropy remains nearly constant across various ω .

This particular feature suggests that the vortex lattice structure remains relatively ordered, and there is minimal disruption in its arrangement. The entropy maintains a consistent level, suggesting a well-organized state. However, as we increase r_0 to larger values, specifically 2 and 3, we notice the emergence of a distinctive trend. The entropy depicts a noticeable increase with higher ω as the larger oscillation frequency disrupts the ordered arrangement of the vortices, leading to a significant rise in the condensate entropy. In the melted region, we observe a power law behavior in $\langle S_{en} \rangle_t$ with ω similar to the angular momentum with an exponent of 1.73 as depicted in the inset of figure. In order to provide a comprehensive illustration of the structural transformation of vortices, we present a phase diagram in the $r_0 - \omega$ plane based on the structure factor analysis.

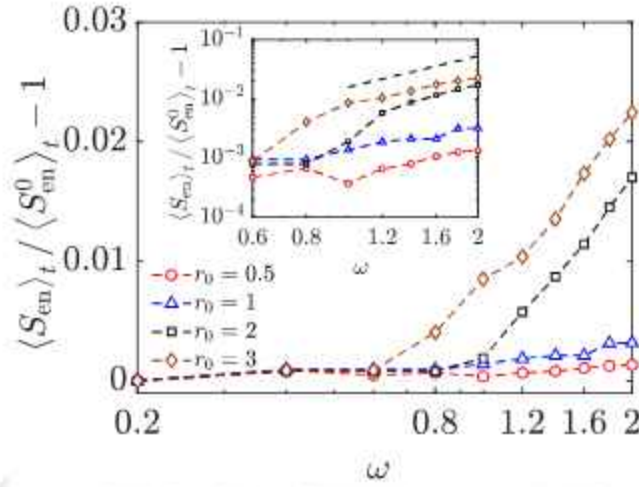


Figure 4.23: Variation of the time average of total entropy $\langle S_{en} \rangle_t$ with obstacle oscillation frequency (ω) for different values of r_0 in semilog scale. The inset figure shows the variation of $\langle S_{en} \rangle_t$ in log scale, which displays a power-law behaviour for larger ω . Here, the entropy rescaled with the entropy of the pinned lattice $\langle S_{en}^0 \rangle_t$ at $V_0 = 5$.

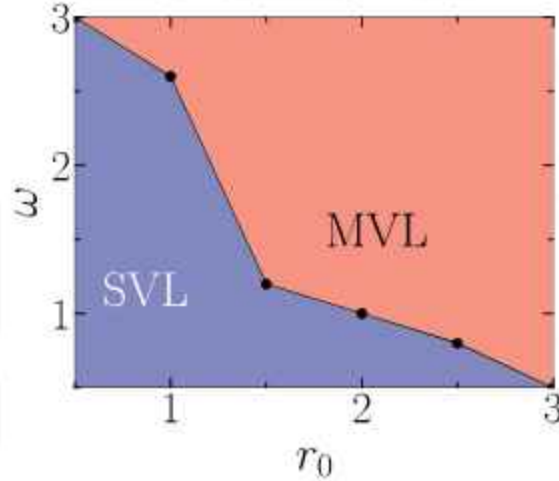


Figure 4.24: Phase diagram of vortex lattice structures in r_0 - ω plane at $\Omega = 0.8$. The blue-shaded region represents the square vortex lattice (SVL), and the region filled with red colour represents the melted vortex lattice (MVL) region. The black dashed line was obtained using the $\langle S(k) \rangle$ values to differentiate the two regions.

The phase diagram, depicted in figure 4.24, reveals two distinct regions, the square vortex lattice (SVL) and melted vortex lattice (MVL) region. It is evident that the lattice structure remains unaltered for lower values of r_0 even for higher ω , and the threshold value of ω decreases with increasing r_0 . Next, we examine the turbulent behavior associated with vortex lattice melting by incorporating the incompressible (ϵ_{kin}^i) and the compressible (ϵ_{kin}^c) energy spectrum of the condensate. Similar to the random impu-

r_0	ω	R_{TF}	l_0	ξ
2	0.6	5.003	1.567	0.282
	2	5.145	1.542	0.274
3	0.6	5.093	1.571	0.277
	2	5.221	1.515	0.271

Table 4.2: Estimation of different characteristic length scales R_{TF} , l_0 , and ξ for different r_0 and ω at $\Omega = 0.8$. All the reported values are averaged over different time intervals, specifically at $t = 0, 50, 100, 200, 300$, and 400 .

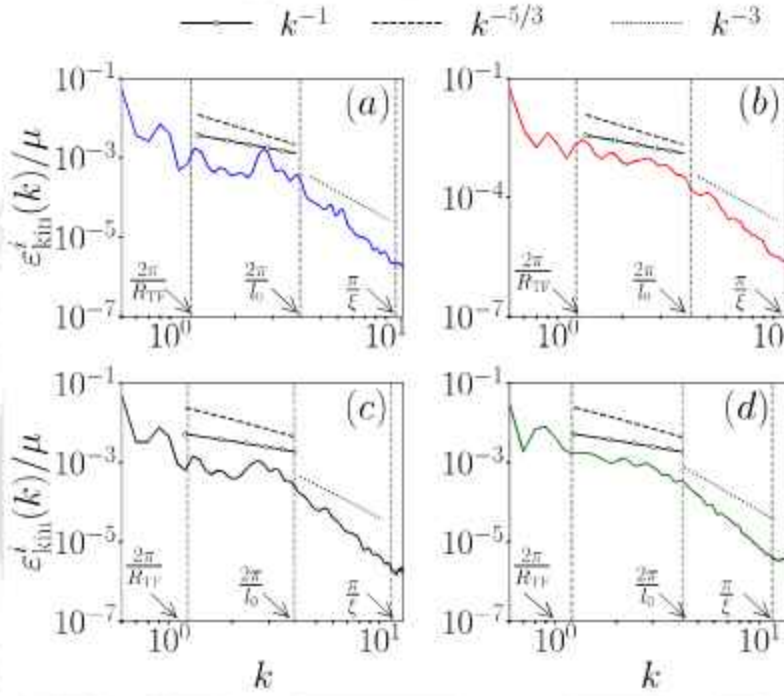


Figure 4.25: Incompressible kinetic energy spectra for different r_0 , and ω : (a) $r_0 = 2, \omega = 0.6$; (b) $r_0 = 2, \omega = 2$; (c) $r_0 = 3, \omega = 0.6$; (d) $r_0 = 3, \omega = 2$. It follows k^{-1} and k^{-3} power laws in the IR and UV regions, respectively.

urity potential discussed in the previous section, we identify three distinct length scales, namely $k_{R_{TF}}$, k_{l_0} , and k_ξ in k space. We compute both the compressible and incompressible spectra at different time intervals, specifically at $t = 0, 50, 100, 200, 300$, and 400 . We calculate the average energy spectra for both compressible and incompressible components. Table 4.2 provides the estimated values of R_{TF} , l_0 , and ξ for different r_0 and ω . In figure 4.25, we plot ϵ_{kin}^i for various r_0 and ω . Similar to the random impurity potential for all r_0 , the spectrum displays a power-law behavior k^{-3} at the length scale greater than k_ξ , primarily attributed to the vortex core. We find the presence of Vinen-like scaling ($\epsilon_{kin}^i \sim k^{-1}$) in the region $k_{R_{TF}} < k < k_{l_0}$, characterized by the clustering

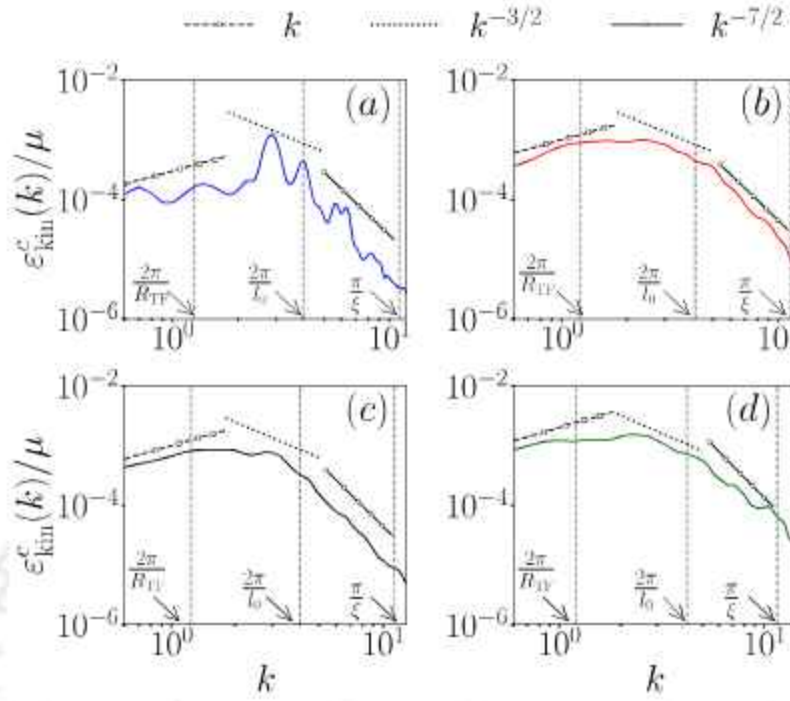


Figure 4.26: Compressible kinetic energy spectra for different r_0 , and ω : (a) $r_0 = 2$, and $\omega = 0.6$; (b) $r_0 = 2$, and $\omega = 2$; (c) $r_0 = 3$, and $\omega = 0.6$; (d) $r_0 = 3$, and $\omega = 2$. It shows k power law for small value of k . In the UV region, it exhibits $k^{-3/2}$, and $k^{-7/2}$ power law.

of vortices without combining into a larger vortex [154, 207]. In figure 4.26, we show the compressible energy spectra, utilizing the same parameters outlined previously in figure 4.25. The observed spectra demonstrate a specific trend characterized by k for small wave numbers. In the range $k_{R_{TF}} < k < k_{l_0}$, the spectra display a power law scaling of $k^{-3/2}$ a typical feature of the weak wave turbulence as reported in [154]. However, for higher wave number range, i.e., $k_{l_0} < k < k_{\xi}$, the spectra show the $k^{-7/2}$ scalings.

4.5 Conclusions and Perspective

In this Chapter, we have investigated the effect of impurities on the structural transformation, leading to the turbulence state of a vortex lattice structure appearing as a ground state for the rotating condensate trapped under the Gaussian distributed square lattice. We have considered two types of impurities, namely, random static impurities and dynamical ones generated using the oscillating Gaussian obstacles. With static impurities we have considered the effect of both the lattice constant of the im-

purity potential as well as its strength. We have found that the square vortex lattice structure gets melted upon increasing the strength of the impurities beyond a threshold strength of the potential. The threshold strength of the potential beyond which the vortex lattice melts decreases upon decreasing the lattice constant of the impurities below the lattice constant of the vortex lattice. We have also established a critical role of the rotational frequency of the condensate in influencing the critical impurity strength required for the vortex lattice melting. To characterize the melted state of the vortex lattice more concretely, we have computed the incompressible and compressible kinetic energy spectrum, which display the presence of turbulence-like features in the condensate power with the presence of Vinen-like scaling of the spectra in the infrared regime.

To show the presence of universality in the melted state of the vortex lattice structure, we have also investigated the influence of an oscillating Gaussian obstacle on the vortex lattice structure and the associated dynamics of the condensate. In this scenario, the melting of the vortex lattice occurs upon either changing the position or oscillation frequency of the obstacle, and the threshold value for melting was found to be independent of the rotational frequency of the condensate. Additionally, we have observed that the angular momentum in the melted region exhibited the same power law exponent in the logarithmic scale regardless of the type of impurity. Further, the investigations related to the kinetic energy spectra revealed a similar power law behavior in a melted state irrespective of the types of impurity. In conclusion, our analysis suggests different techniques to generate a turbulent state through vortex lattice melting in rotating BECs. These findings contribute to a deeper understanding of complex dynamics and structural transformations in rotating BECs.

Excitation spectrum of vortex-lattice modes in a rotating condensate with a density-dependent gauge potential

In Chapters 3 and 4, we considered the rotation-induced static gauge potential and examined the structural transformations and melting of vortex lattices under the influence of lattice potential and impurities. In this chapter, we extend the analysis to dynamic gauge potential and investigate the effect of this gauge potential on both the vortex lattice structure and the associated excitation spectrum. In particular we investigate the collective excitation spectrum of a quasi-2D Bose-Einstein condensate trapped in a harmonic confinement with nonlinear rotation induced by a density-dependent gauge potential. Using a Bogoliubov-de Gennes (BdG) analysis, we show that the dipole mode frequency depends strongly on the nonlinear interaction strength, violating Kohn's theorem which is also well complemented by the analytical expression for the dipole and breathing modes obtained employing a variational analysis. We identify four different vortex displacement modes - Tkachenko, circular, quadratic, and rational - whose frequencies are rotation-sensitive. An analytical Tkachenko mode frequency from a hydrodynamic approach agrees well with that computed using BdG and Fourier analysis. The excitation spectrum remains symmetric around the angular quantum number $l = 0$, but energy splitting between l and $-l$ varies with the sign and strength of the nonlinear rotation. Finally, we demonstrate that the surface mode excitation frequency increases (decreases) with an increase in the positive (negative) nonlinear rotation strength. The Chapter is based on the work published in Physical Review A [208].

5.1 Introduction

The literature related to the vortex lattice structure and corresponding excitation spectrum in presence of synthetic gauge potential has been extensively presented in Chapter 1. However, to ensure that the current chapter is self contained, we provide a brief overview of previous studies related to the vortex lattice structure in presence of synthetic gauge potential in rotating BECs. Over the past few decades, research on topological excitations in various quantum liquids has attracted significant interest [3, 4], encompassing areas such as superfluid helium [5], liquid crystals [6], quantum magnets [7], and topological systems [8]. The discovery of vortex lattice phase in rotating Bose-Einstein condensates (BECs) has opened up new possibilities for exploring different phases and excitations of topological vortex states in a more controlled setting [17]. These developments have garnered considerable interest from the ultra-cold atom community, driven by the variety of excitation modes, including Tkachenko waves, which arise from vortex displacement modes observed in both superfluid helium and BECs [209].

Several works have focused on analyzing the underlying mechanism behind the excitation of vortex lattice displacement modes in rotating BECs with short range interactions. For instance, Coddington *et al.* [60] experimentally observed the Tkachenko mode by focusing a resonant laser beam at the center of the condensate. Subsequently, there were a series of theoretical approaches that have been adopted to develop an adequate hydrodynamic description of the Tkachenko mode [210–215]. In particular, Baym [211] analytically derived the Tkachenko mode within the framework of the continuum approximation and reported that the mode frequency approaches zero in the centrifugal limit, indicating the influence of finite compressibility on the mode's dynamics. The mode frequency remains linear in the stiff mode region, while it shows a quadratic nature in the soft range. Keçeli and Oktel [216] extended the study of Tkachenko mode for two-component BECs and predicted the appearance of optical and acoustic modes.

The low-lying collective excitations of rotating BECs reveal important insights into the stability of different topological phases of vortex lattice states of rotating BECs. Collective excitations involving vortices or vortex lattices have been studied both in quasi-2D and 3D [170–172]. Smith *et al.* [173] experimentally observed a gyroscopic tilting mode of a vortex array, which belongs to the common Kelvin-Tkachenko mode branch. Following this, Simula [172] carried out a numerical analysis of the collective-mode spectra using Bogoliubov-de Gennes (BdG) theory in a three-dimensional rotat-

ing BEC confined in an oblate harmonic trap and reported a comprehensive analysis of the motion of the vortices corresponding to the Kelvin-Tkachenko waves. In other directions, the stability and excitations of more complex systems like dipolar condensates with a vortex lattice have been shown to host more complex excitations which lead to the tilting and bending of the vortex core, responsible for the local instability of the BECs [174]. Jia *et al.* [175] subsequently extended the analysis of low-lying excitations of the vortex lattice in the presence of the anisotropic dipolar interaction and reported that in the limit of fast rotation and strong dipole-dipole interactions vortices form stripe lattices which results in the appearance of a shearing motion among the low lying Tkachenko modes, a phenomenon absent in the conventional vortex lattice with isotropic interactions.

The observation of artificial electromagnetism in ultracold gas systems has attracted immense interest among researchers due to the realization of a variety of novel phenomena [31, 30]. This induced electromagnetism gives rise to two types of gauge potentials, namely static and quasi-dynamic. In the static case, where feedback between the light and the matter wave is absent, they have been experimentally shown to exhibit various interesting phenomena, such as, including orbital magnetism [37–39], spin-orbit coupling with bosons [40] and fermions [156, 157] and spin-angular-momentum coupling with bosons [41–43]. In the quasi-dynamical case however, the interaction between the gauge potential and the quantum state leads to the appearance of time-dependent feedback in the form of a density-dependent gauge potential, which has been theoretically explored in continuum systems [46] and lattice-based geometries [47–49]. Apart from this, experimental realizations of density-dependent magnetism have also been achieved in lattice-based systems for both bosonic [158] and fermionic [159] systems as well as in ensembles of Rydberg atoms [55]. Complementary to the lattice implementations, density-dependent gauge theories have also been realised in the continuum observing unusual topological effects associated with domain walls [160] and chiral solitons [161, 162].

Theoretical models have played an important role in building understanding of the phenomenology associated with synthetic gauge theories. Here interest has focused on understanding the properties of this unusual nonlinear system, such as the solitary wave [217–228] and domain-wall [229] solutions and further their transport mechanisms in double-well [50], optical-lattice [54] and harmonic potentials [51, 52]. Theoretical exploration on collective excitations indicate that the presence of gauge potentials leads to the violation of Kohn's theorem. Previously, it has been analytically and numerically demonstrated that density-dependent gauge potentials induce irreg-

ular dynamics [53] and chaotic collective behavior in two-dimensional BECs [230]. The density-dependent gauge potential leads to nonlinear rotation in the system [164–167] ultimately responsible for the formation of non-Abrikosov vortex lattices and ring vortex like arrangements [63], while subsequent studies have focused on understanding the critical rotation frequency for vortex nucleation [168] and the appearance of ghost vortices [169].

The exploration of vortex lattice displacement modes under the influence of the density-dependent gauge potential remains an open question. Building upon the formalism developed for the density-dependent gauge potentials in rotating condensates, we present a detailed analysis of the vortex displacement modes and excitations for rotating condensates in the presence of nonlinear rotation induced by the gauge potential by solving the Bogoliubov-de Gennes (BdG) equations. Our results are supported by the time evolution of the expectation values of physical observables, which are analyzed after introducing an appropriately chosen perturbation to the time-dependent Gross-Pitaevskii equation.

Through a variational analysis, we analytically calculate the frequencies of the dipole and breathing modes. In addition, we derive the frequency of the vortex displacement modes associated with the Tkachenko wave using hydrodynamic continuum theory. This analytical approach allows us to critically compare our findings with those obtained from the BdG analysis and the time evolution of physical observables. Furthermore, we show that the energy splitting between the angular quantum numbers $+l$ and $-l$ is induced by the nonlinear rotation resulting from the density-dependent gauge potential.

The structure of the paper is as follows: In Sec. 5.2, we introduce the mean-field Gross-Pitaevskii model incorporating the density-dependent gauge potential and further provide the simulation details. Sec. 5.3 presents the effect of the density-dependent gauge potential on the ground state and the low-lying excitation spectrum. Section 5.4 presents the real-time dynamics of the perturbed ground state, illustrating the dipole mode oscillation frequency and performing variational calculations to derive the equations of motion for the center of mass. Section 5.5 focuses on the effect of the density-dependent gauge potential on the vortex displacement mode, providing an analytical expression for the Tkachenko mode frequency using hydrodynamic continuum theory. In Sec. 5.6, we summarise our findings.

5.2 Theoretical Model

We consider a weakly interacting Bose-Einstein condensate composed of N two-level atoms. These atoms are coupled through a coherent light-matter interaction driven by an applied laser field. The Hamiltonian of this system, utilizing the rotating wave approximation, is given by [31]:

$$\hat{\mathcal{H}} = \left(\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}) \right) \otimes \mathbf{1} + \hat{\mathcal{H}}_{\text{int}} + \hat{\mathcal{U}}_{\text{MF}}, \quad (5.1)$$

Here, $\hat{\mathbf{p}}$ is the momentum operator, $V(\mathbf{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$ is the trapping potential and $\mathbf{1}$ represents the 2×2 identity matrix. The mean-field interactions are described by $\hat{\mathcal{H}}_{\text{int}} = (1/2)\text{diag}[\Delta_1, \Delta_2]$, where $\Delta_i = g_{ii}|\Psi_i|^2 + g_{ij}|\Psi_j|^2$ and $g_{ij} = 4\pi\hbar^2 a_{ij}/m$ with a_{ij} being the s-wave scattering lengths for collisions between atoms in internal states i and j ($i, j = 1, 2$) [30]. The light-matter interaction, denoted as $\hat{\mathcal{U}}_{\text{MF}}$, can be defined as [30]

$$\hat{\mathcal{U}}_{\text{MF}} = \frac{\hbar\Omega_r}{2} \begin{pmatrix} \cos\theta(\mathbf{r}) & e^{-i\phi(\mathbf{r})}\sin\theta(\mathbf{r}) \\ e^{i\phi(\mathbf{r})}\sin\theta(\mathbf{r}) & -\cos\theta(\mathbf{r}) \end{pmatrix}. \quad (5.2)$$

Here Ω_r represents the Rabi frequency, $\theta(\mathbf{r})$ denotes the mixing angle, and $\phi(\mathbf{r})$ indicates the phase of the incident laser beam. To introduce interactions into the dressed states, perturbed dressed states can be defined using perturbation theory [46]

$$|\Psi_{\pm}\rangle = |\pm\rangle \pm \frac{\Delta_d}{\hbar\Omega_r} |\mp\rangle \quad (5.3)$$

with $|\pm\rangle$ representing the unperturbed eigenstates of $\hat{\mathcal{U}}_{\text{MF}}$, and $\Delta_d = \sin(\theta/2)\cos(\theta/2)(\Delta_1 - \Delta_2)/2$ denotes the mean-field detuning. After projecting the atomic motion onto the dressed-state basis $|\Psi_{\pm}\rangle$, the effective Hamiltonian can be written as,

$$\hat{H}_+ = \frac{(\hat{\mathbf{p}} - \mathbf{A}_+)^2}{2m} + W_+ + \frac{\hbar\Omega_r}{2} + \Delta_+ + V(\mathbf{r}), \quad (5.4)$$

where $\mathbf{A}_+ = i\hbar\langle\Psi_+|\nabla\Psi_+\rangle$, and $W_+ = \hbar^2|\langle\Psi_+|\nabla\Psi_-\rangle|^2/2m$ are the vector gauge potential and scalar potential, respectively. $\Delta_+ = (\Delta_1\cos^2(\theta/2) + \Delta_2\sin^2(\theta/2))/2$ is the dressed mean-field detuning. The generalized Gross-Pitaevskii (GP) equation in three dimensions can be derived by employing the adiabatic approximation, as discussed

in [166]:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \Omega_n(\mathbf{r}, t) \hat{L}_z + g_{\text{eff}} |\Psi|^2 \right] \Psi. \quad (5.5)$$

In deriving Eq. (5.5), a spatially dependent Rabi coupling of the form $\Omega_r = \kappa_0 r$ is introduced, where r is the radial distance and κ_0 is a constant. The laser field generating this coupling carries an azimuthal phase profile $\phi = l\varphi$, with l representing the angular momentum quantum number and φ is the polar angle.

The density-dependent rotation, denoted as, $\Omega_n(\mathbf{r}, t)$ takes the following form [164]:

$$\Omega_n(\mathbf{r}, t) = \Omega + Cn(\mathbf{r}, t), \quad (5.6)$$

here $n(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2$ represents the condensate density and $\Omega = (\hbar l)/(4m)(\kappa_0/\Delta)^2$ is the rigid-body rotation strength, which can be controlled using the light matter interaction. Then, the strength of the nonlinear rotation can be expressed as $C = l(g_{11} - g_{22})\kappa_0^2/(2m\Delta^2)$. Note that the nonlinear rotation can be incorporated experimentally in the condensate using a laser carrying a orbital angular momentum with $l = 1$, which is similar to the creation of spin-angular-momentum-coupled Bose-Einstein condensates using Laguerre-Gaussian laser beams [41–43].

To obtain the quasi-2D Gross-Pitaevskii (GP) equation, we start by defining the wave function as $\Psi(\mathbf{r}, t) = \psi(x, y, t) \exp(-z^2/2\sigma_z^2)/\sqrt{\pi\sigma_z^2}$ and consider that the condensate is confined in a highly anisotropic trap where $\omega_z \gg \omega_{x,y}$. By integrating out the axial degrees of freedom, we obtain the following 2D GP equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(x, y) - \Omega_n \hat{L}_z + g_{2D} |\psi|^2 \right] \psi. \quad (5.7)$$

For the sake of simplicity, in what follows we use $\Omega_n(\mathbf{r}, t) \equiv \Omega_n$, and $\psi(x, y, t) \equiv \psi$. The nonlinear coefficients appearing in Eq. (5.7) are defined as $g_{2D} = g_{\text{eff}}/\sqrt{2\pi}\sigma_z$ and $C \rightarrow C/\sqrt{2\pi}\sigma_z$. Here, $n(x, y, t) = |\psi(x, y, t)|^2$ denotes the quasi-two-dimensional density of the condensate. By applying spatiotemporal scaling, the variables are redefined as $t = \omega_{\perp}^{-1} t'$, $(x, y) = a_{\perp} (x', y')$, and $\psi = \sqrt{N} \psi'/a_{\perp}$, where $a_{\perp} = \sqrt{\hbar/m\omega_{\perp}}$ is the harmonic oscillator length scale. These transformations recast Eq. (5.7) into the following dimensionless form:

$$i \frac{\partial \psi}{\partial t} = \left[-\frac{\nabla^2}{2} + V(x, y) - \Omega_n \hat{L}_z + g |\psi|^2 \right] \psi. \quad (5.8)$$

In Eq. (5.8), the scaled parameters are defined as follows: $g = g_{2D}Nm/\hbar^2$ and $\Omega_n(x, y, t) = \Omega/\omega_\perp + \tilde{C}n(x, y, t)$ where $\tilde{C} = CN/m\hbar\sqrt{2\pi}\sigma_z$.

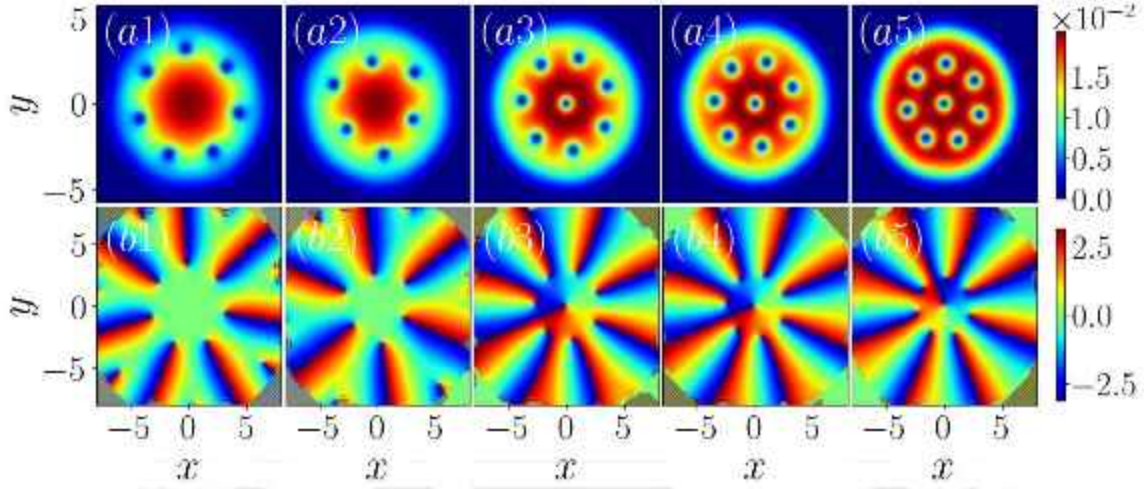


Figure 5.1: Pseudo-color representation of the condensate density for (a1) $\tilde{C} = -30$, (a2) $\tilde{C} = -20$, (a3) $\tilde{C} = 0$, (a4) $\tilde{C} = 10$, and (a5) $\tilde{C} = 20$ at $\Omega = 0.6$ and $g = 400$. Panels (b1)–(b5) show the phase profiles of density illustrated in (a1)–(a5).

In our work we consider a condensate consisting of $N = 3 \times 10^4$ Sr atoms. The s-wave scattering length of the Sr atom is $a_s = 2.83$ nm. The radial trap frequency is set to $\omega_\perp = 2\pi \times 100$ Hz, and the trap aspect ratio is $\omega_z/\omega_\perp = 10$. These parameters give a dimensionless contact interaction strength of $g \approx 400$, which is fixed for all numerical calculations.

5.3 Ground state and Collective excitation

Here we calculate and analyse the vortex states of the density-dependent gauge theory previously derived in Sec. 5.2. Depending upon the values of $\tilde{C}$, the gauge potential induces a nonlinear rotation in the condensate, which affects the vortex lattice structure in several ways. For $\tilde{C} < 0$, the vortex lattice structure shows a transition from the Abrikosov lattice structure to a ring structure. However, for $\tilde{C} > 0$ the density profile exhibits a large plateau region where the vortices are mainly concentrated near the trap center [63]. Following the ground state analysis, here, we aim to explore how the nonlinear rotation alters the low-lying collective excitation modes and how this connects to the different vortex lattice morphologies. We begin our analysis by exploring the effect of $\tilde{C}$ on the vortex lattice structure. In Fig. 5.1 we present the condensate density and phase profile of the condensate for different values of $\tilde{C}$ by keeping the

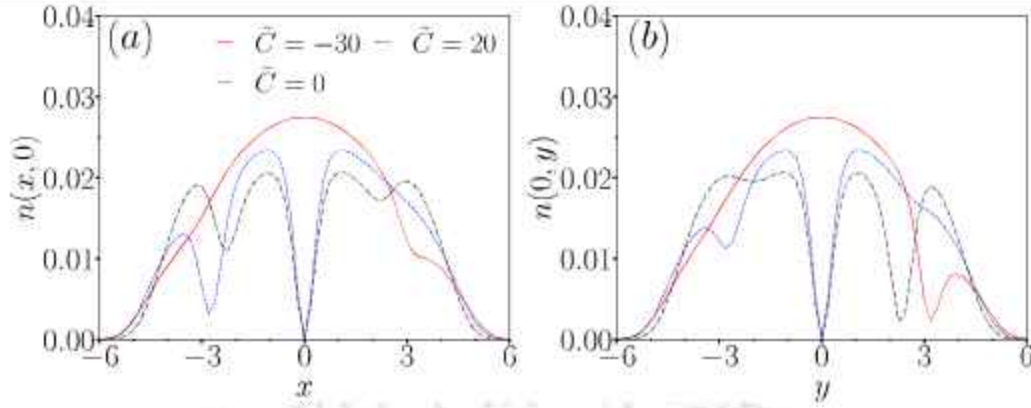


Figure 5.2: The condensate density profile cross-section along (a) x -direction $|\psi(x, 0)|^2$ and (b) along the y -direction $|\psi(0, y)|^2$ for three different non-linear rotation: $\tilde{C} = -30, 0, 20$. The other parameters are $\Omega = 0.6$ and $g = 400$. These one-dimensional profiles represents the cross-section of the 2D density of Fig. 5.1.

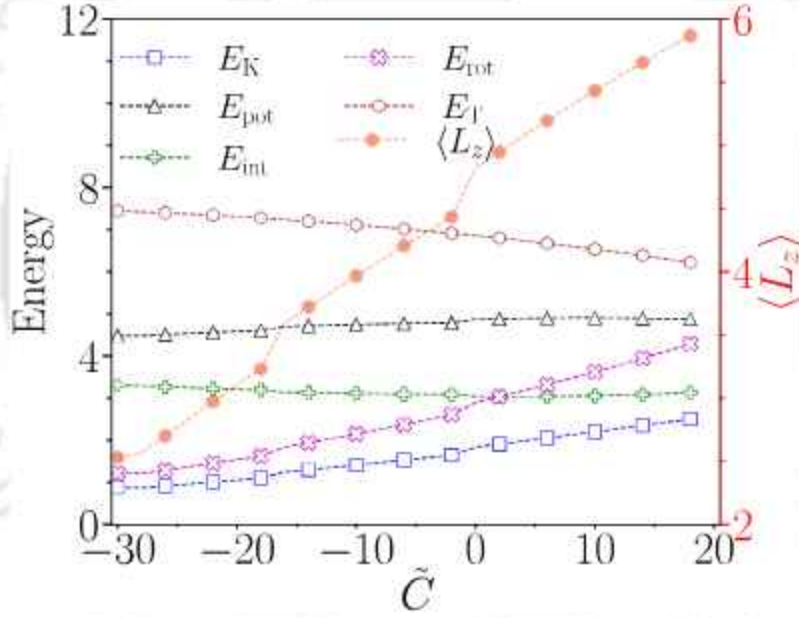


Figure 5.3: Variation of different components of the total energy, total energy E_T , kinetic energy E_K , potential energy E_{pot} , and interaction energy E_{int} as a function of $\tilde{C}$. Right vertical axis shows the variation of angular momentum $\langle L_z \rangle$. E_{int} and E_{pot} remains almost unchanged with $\tilde{C}$, while E_K show linear increment with $\tilde{C}$. The linear increment of $\langle L_z \rangle$ with $\tilde{C}$ indicates the increase in the number of vortices upon increasing $\tilde{C}$.

rigid body rotation frequency fixed at $\Omega = 0.6$, and the effective mean-field interaction strength as $g = 400$. We observe that depending on the value of $\tilde{C}$ the gauge potential-induced nonlinear rotation can have a profound impact on the vortex lattice structure. For $\tilde{C} < 0$, the vortex lattice structure is modified from the Abrikosov lattice to a ring

like structure (see Fig. 5.1(a1)-(a2)), while for $\tilde{C} > 0$ the density profile exhibits a large plateau region, reminiscent of the density plateau of a quantum droplet [231] with vortices concentrated near the trap center [see Fig. 5.1(a4)-(a5) and Fig. 5.2(a)-(b)]. The plateau region becomes more pronounced in the absence of vortices [63].

In order to understand the appearance of the different vortex lattice structures in Fig. 5.3, we compute the different contributions to the total energy, namely the kinetic energy $E_K = \frac{1}{2} \int |\nabla \psi|^2 dx dy$, potential energy $E_{\text{pot}} = \frac{1}{2} \int (x^2 + y^2) |\psi|^2 dx dy$, interaction energy $E_{\text{int}} = \frac{g}{2} \int |\psi|^4 dx dy$, rotational energy $E_{\text{rot}} = - \int \Omega_n \psi^* \hat{L}_z \psi dx dy$ and total energy $E_T = E_K + E_{\text{pot}} + E_{\text{int}} + E_{\text{rot}}$ as well as the angular momentum $\langle \hat{L}_z \rangle$ as a function of $\tilde{C}$. With the variation of $\tilde{C}$, the E_{pot} remains almost constant while E_{int} exhibits slight decreasing behavior as $\tilde{C}$ increases. In contrast, E_K increases as the nonlinear rotation parameter varies from negative to positive values. This behavior arises due to the shifting of the vortices towards the periphery of the harmonic trap, causing the minimization of the energy for negative $\tilde{C}$ values. Additionally, the angular momentum $\langle \hat{L}_z \rangle$ also exhibits an increasing trend with $\tilde{C}$ due to the possibility of the nucleation of additional vortices. Similarly, E_{rot} also follows a similar trend to the angular momentum as $\tilde{C}$ is varied. Overall, the total energy E_T decreases when $\tilde{C}$ is varied.

Next, we shift our focus to investigate the collective excitations of the vortex lattice structure using the Bogoliubov-de Gennes (BdG) theory. In order to excite the modes, we perturb the ground state ψ_0 with a fluctuation $\delta\psi$ to obtain the excited-state wave function as given by [88, 172]:

$$\psi(x, y, t) = [\psi_0(x, y) + \delta\psi(x, y, t)] \exp(-i\mu t), \quad (5.9)$$

where μ is the chemical potential of the ground state, and the perturbation wave function $\delta\psi(x, y, t)$ is of the form:

$$\delta\psi(x, y, t) = u(x, y) \exp(-i\omega t) - v^*(x, y) \exp(i\omega^* t), \quad (5.10)$$

where $u(x, y)$ and $v(x, y)$ are the Bogoliubov mode amplitudes and ω is the corresponding frequency. The Bogoliubov mode amplitudes follow the relation

$$\int [|u(x, y)|^2 - |v(x, y)|^2] dx dy = 1. \quad (5.11)$$

By substituting Eq. (5.9) into Eq. (5.8) and equating the time dependent terms according

to $\exp(-i\omega t)$ and $\exp(i\omega t)$, respectively, we obtained the set of BdG equations,

$$\begin{aligned} \omega u &= \left[\mathcal{L}_1 - \tilde{C}(|\psi_0|^2 L_z + \psi_0^* L_z \psi_0) \right] u \\ &\quad - [g\psi_0^2 - \tilde{C}\psi_0 L_z \psi_0] v, \end{aligned} \quad (5.12a)$$

$$\begin{aligned} -\omega v &= \left[\mathcal{L}_2 + \tilde{C}(|\psi_0|^2 L_z + \psi_0 L_z \psi_0^*) \right] v \\ &\quad - [g\psi_0^{*2} + \tilde{C}\psi_0^* L_z \psi_0^*] u, \end{aligned} \quad (5.12b)$$

Here $\mathcal{L}_1 = -\frac{1}{2}\nabla^2 + V + 2g|\psi_0|^2 - \mu - \Omega L_z$, and $\mathcal{L}_2 = -\frac{1}{2}\nabla^2 + V + 2g|\psi_0|^2 - \mu + \Omega L_z$. The set of BdG equations 5.12 are solved to obtain the eigenfrequencies ω and eigenmodes u, v . To visualize a normal mode, we analyze the temporal evolution of the perturbed density profile as follows:

$$n(x, y, t) = [\psi_0(x, y) + p\delta\psi(x, y, t)]^2 \quad (5.13)$$

which facilitates the study of the excitation spectrum in response to small perturbations to the nonlinear system. The parameter $p \ll 1$ is small in amplitude and used to control the population of quasiparticle excitations.

Next we present the effect of density-dependent rotation on the collective excitation spectrum. To achieve this, we first calculate the excitation frequencies by numerically solving the BdG equations 5.12. In addition to this approach, we use an alternative technique to excite the dipole mode, which is feasibly accessible in experiments. In this approach, a small linear perturbation has been added in the Hamiltonian to excite the dipole mode of the condensate ground state. We also complement these calculations with the analytical form of the dipole frequency obtained using a variational approach. Apart from the dipole mode we also analyze the relevant vortex displacement modes by imparting a linear perturbations to the ground state of the condensate density as described in the Eq. 5.13. In Fig. 5.4, we show the variation of the frequencies of the first six low-lying excitation modes as $\tilde{C}$ is varied for a fixed rotation frequency $\Omega = 0.6$ and interaction strength $g = 400$. The angular quantum number l associated with each mode is determined by constructing the complex field $n' = \psi^*(x, y) u(x, y) - \psi(x, y) v(x, y)$, as proposed in [232]. The phase S of n' is then computed, and l is determined by evaluating the phase winding around a closed contour C that encloses the condensate at the Thomas-Fermi radius: $\frac{1}{2\pi} \oint_C \nabla S \cdot dr$, where ∇S denotes the spatial gradient of the phase S . The lowest frequencies (∇) corresponding

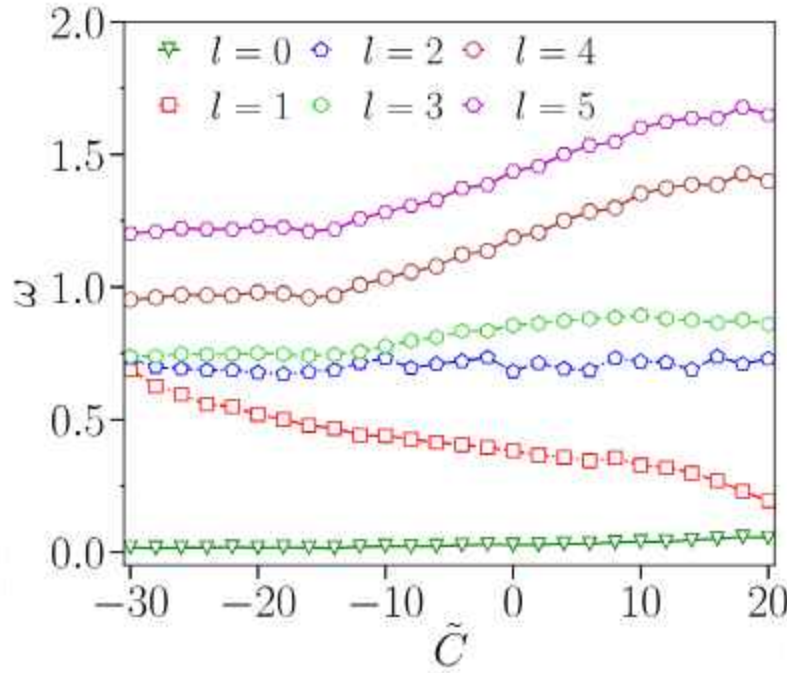


Figure 5.4: Collective excitation spectrum for various angular quantum numbers l with radial quantum number $n_r = 0$ as a function of nonlinear rotation strength $\tilde{C}$ at $\Omega = 0.6$ are presented. The triangles and squares denote the Tkachenko and dipole modes, respectively. The pentagon, hexagon, and circles represent the higher order surface modes.

to the vortex displacement excitations captures the motion of vortex cores around the equilibrium position in the rotating frame. Here, $l = 0 - 1$ denotes respectively the Tkachenko, and dipole excitation, while $l \geq 2$ represents the higher order surface excitation modes [215]. In the excitation spectrum, the frequencies marked by red squares ($\square$) represents the dipole mode oscillations corresponding to the center of mass motion of the condensate, which exhibits a decreasing trend upon increasing $\tilde{C}$. It should be noted here that as per the constraint imposed due to the Kohn theorem for condensates experiencing only rigid body rotation, usually, the dipole frequency does not show any dependency on atomic interactions. However, it can be observed that in the presence of nonlinear rotation, the dipole frequency shows a significant deviation from the Kohn theorem [163, 53]. Violation of the Kohn theorem also occurs for spin-orbit coupled condensates due to the coupling between the center of mass and the internal spin degrees of freedom, as shown in several of the works [233–236]. The decreasing trend of the dipole frequency can be attributed to the presence of a density-angular momentum coupling factor in the BdG equations, per Eqs. 5.12(a) and 5.12(b). This coupling is responsible for the centrifugal force that is influenced by both the sign and

magnitude of the nonlinear rotation parameter, $\tilde{C}$. A large value of $\tilde{C}$ breaks the symmetry of the harmonic motion of the condensate, resulting in the dipole frequency as observed in the Fig. 5.4. On the other hand, we find that the higher order modes exhibit an increasing frequency as $\tilde{C}$ is varied from negative to positive values. These modes are later classified as surface excitations of the condensate based on the time evolution of the perturbed density profile, obtained using Eq. 5.13. It can also be characterized by angular quantum numbers $l > 2$, which are arranged in ascending order according to their corresponding excitation frequencies.

5.4 Time dependent characteristics of the collective modes

In the following section we explore the time-dependent characteristics of the collective modes. In addition to the linearization technique, the low-lying collective excitations, such as dipole, monopole, and quadrupole modes, can also be excited using an additional method, in which a time-independent linear perturbation is added to the Hamiltonian [237–240]. In this work, our focus is to find the the dipole mode; thus, we only consider a perturbation associated with this specific mode. To excite the dipole mode, we consider a linear perturbative potential $V_p(x) = \lambda x$ at $t \geq 0$ along with the harmonic potential, where $0 < \lambda < 1$. This addition modifies the Gross-Pitaevskii equation by incorporating an additional term, $V_p(x)$. The generalized Gross-Pitaevskii equation Eq. (5.8) is solved by considering the rotating stationary state as the initial condition to construct time-dependent physical observables, specifically focusing on the motion of the center of mass in the case of the dipole mode. Furthermore, the frequency of this mode is determined through Fourier transformation of the temporal evolution of the center of mass of the rotating condensate.

To investigate the effect of the perturbation systematically, in Fig. 5.5 we display a pseudocolor map of the density profile corresponding to the solution of Eq. (5.8) at time $t = 0, 20, 40$, and 60 . The analysis is conducted with the parameters $\Omega = 0.6$, $\lambda = 0.1$, and $g = 400$. The first ((a1)-(a5)), second ((b1)-(b5)), and the third row ((c1)-(c5)) of Fig. 5.5 corresponds to $\tilde{C} = -30, 0$, and 20 , respectively. We find that at $t = 0$, the condensate is initially positioned at the origin (see 5.5(a1),(b1), and (c)). However, as time progresses, the perturbation causes the condensate to exhibit a dipole oscillation around the origin, independent of $\tilde{C}$ (see 5.5(a2)-(a5), (b2)-(b5), and (c2)-(c5)). To characterize the motion of the center of mass, we calculate the time-dependent

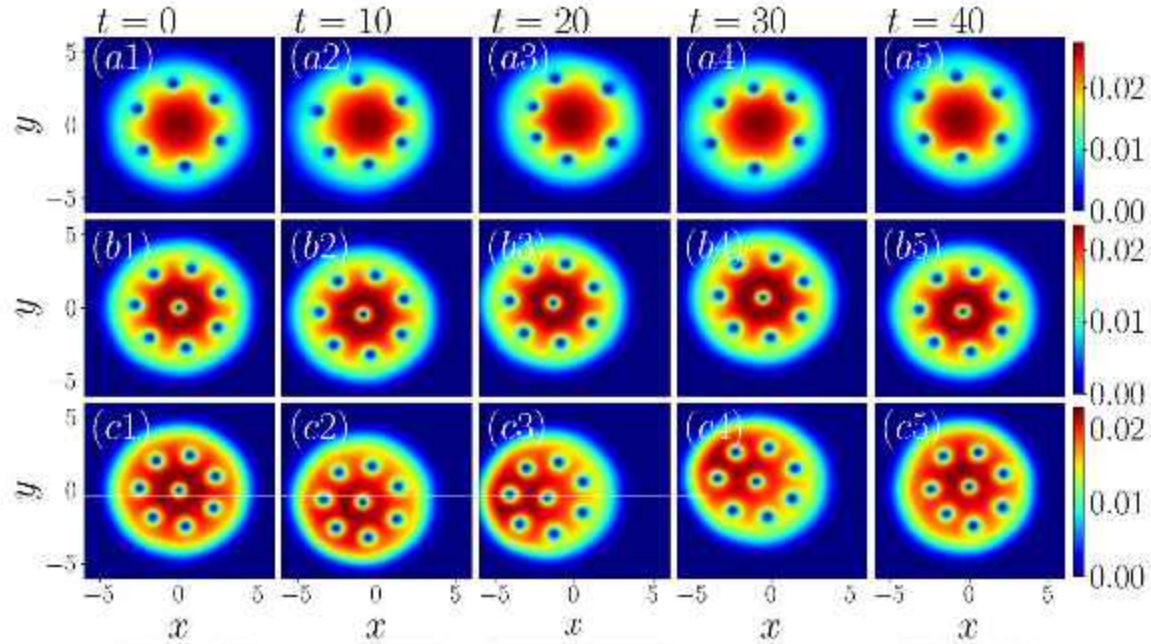


Figure 5.5: Real space condensate density at different times portraying the vortex displacement that arises due to the application of a linear perturbation on the ground state which excites mainly the dipole mode for different values of $\tilde{C}$ at $\Omega = 0.6$. (a1)-(a5); for fixed $\tilde{C} = -30$ and $t = (0, 10, 20, 30, 40)$; (b1)-(b5); for fixed $\tilde{C} = 0$ and $t = (0, 10, 20, 30, 40)$; (c1)-(c5); for fixed $\tilde{C} = 20$ and $t = (0, 10, 20, 30, 40)$. The real-time dynamics of the condensate reveal its precessional motion around the trap center with time.

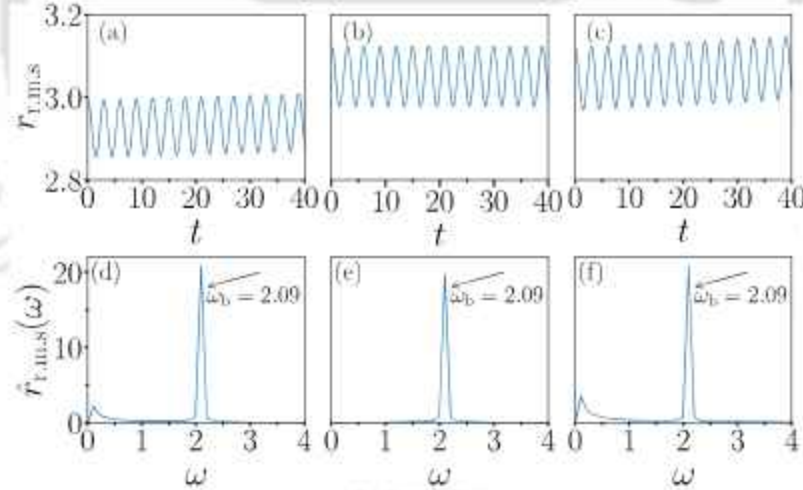


Figure 5.6: Panels (a)-(c), shows the root mean square radius $r_{r.m.s.}(t)$ of the condensate as a function of time for $\tilde{C} = -30, 0, 20$, respectively, and (d)-(f) illustrates the corresponding Fourier transform with a dominant peak at $\omega_b = 2.09$.

center of mass $x_{cm} = \langle x(t) \rangle$, which can be expressed as

$$x_{cm}(t) = \int \int x |\psi(x, y, t)|^2 dx dy. \quad (5.14)$$

To determine the breathing mode using the perturbation technique, we add a potential $V_b = \lambda(x^2 + y^2)$ to the time-dependent GP equation. We then analyze the mode by performing a Fourier transform on the time evolution of the root mean square radius $r_{\text{rms}} = \int (x^2 + y^2)^{1/2} |\psi|^2 dx dy$ of the condensate. In Fig. 5.6, we present the time evolution of r_{rms} and the corresponding Fourier transform for three different values of $\tilde{C}$. We observe that, irrespective of $\tilde{C}$, the dominant peak appears at $\omega_b = 2.09$, indicating that the breathing mode does not depend on the strength of the nonlinear rotation. In Fig. 5.7, we present the variation of x_{cm} (first row) as a function of time corresponding to $\tilde{C} = -30, 0$, and 20 , demonstrating an oscillatory behavior over time [see Fig. 5.7(a)-(c)]. Next, by doing the Fourier transform [see Fig. 5.7(d)-(f)], we identify the dominant frequency of x_{cm} . For $\tilde{C} = -30$, the dominant frequency is at $\omega_d = 0.68$, as the value of nonlinear rotation strength changes from $\tilde{C} = 0$ to $\tilde{C} = 20$, there is a noticeable shift in the dominant frequency from $\omega_d = 0.39$ to $\omega_d = 0.19$, respectively. Additionally, it is worth noting that the linear perturbation can also be given as $V_p(y) = \lambda y$ along the y direction. In which case, the dipole frequency can be obtained from y_{cm} .

Using a combination of linearization and perturbative techniques, we observe that the dipole frequency strongly depends on the nonlinear rotation strength as shown in Fig. 5.7(g). To better comprehend the behavior of the dipole frequency we employ a variational approach [241, 242]. The motivation behind using this method has two primary aspects: first, to derive an analytical expression for the frequencies of the dipole mode, and second, to establish an equation of motion that provides an accurate solution to the dynamics under nonlinear rotation. This method has been used to understand both single-component [241], binary [243] and spin-orbit coupled condensates [235, 233] to describe collective excitations. Following a similar approach, we consider the variational *ansatz* with a Gaussian profile and variational parameters as center-of-mass and width of the condensate as in the linear limit (without atomic interactions or rotation) of the ground state of the GP equation. The normalized Gaussian

ansatz appropriate for a condensate confined in a harmonic potential is given by

$$\begin{aligned} \psi(x, y, t) = \frac{1}{\sqrt{\pi\sigma(t)^2}} \exp \left[-\frac{(x - x_0(t))^2}{2\sigma(t)^2} \right. \\ + i\alpha_x(t)(x - x_0(t)) + i\beta_x(t)(x - x_0(t))^2 \\ - \frac{(y - y_0(t))^2}{2\sigma(t)^2} + i\alpha_y(t)(y - y_0(t)) \\ \left. + i\beta_y(t)(y - y_0(t))^2 \right]. \end{aligned} \quad (5.15)$$

The variational parameters play a crucial role in accurately capturing the dynamics of the condensate. The width $\sigma(t)$ characterizes the spatial extent of the condensate, while $x_0(t)$ and $y_0(t)$ denote its center-of-mass coordinates along x , and y directions, $\alpha_{x,y}(t)$ corresponds to the slope of the phase, enabling the ansatz to account for the center-of-mass motion or overall momentum of the condensate. The parameter $\beta_{x,y}(t)$ represents the chirp, introducing a quadratic phase that describes the expansion or contraction of the Gaussian wavefunction. Here we wish to emphasize that without the chirp term, the ansatz would only be able to capture the static width profile of the condensate and fails to capture the full dynamics of the condensate's breathing and expansion characteristics.

The Euler-Lagrange equations of motion are then expressed as:

$$\begin{aligned} \ddot{x}_0 - 2\dot{y}_0 \left(\Omega + \frac{\tilde{C}}{4\pi\sigma^2} \right) + x_0 \left[1 - \left(\Omega + \frac{\tilde{C}}{4\pi\sigma^2} \right)^2 \right] \\ + \frac{\tilde{C}\dot{\sigma}}{2\pi\sigma^3} y_0 = 0 \end{aligned} \quad (5.16a)$$

$$\begin{aligned} \ddot{y}_0 + 2\dot{x}_0 \left(\Omega + \frac{\tilde{C}}{4\pi\sigma^2} \right) + y_0 \left[1 - \left(\Omega + \frac{\tilde{C}}{4\pi\sigma^2} \right)^2 \right] \\ - \frac{\tilde{C}\dot{\sigma}}{2\pi\sigma^3} x_0 = 0 \end{aligned} \quad (5.16b)$$

$$\begin{aligned} \ddot{\sigma} - \frac{1}{\sigma^3} - \frac{g}{2\pi\sigma^3} + \frac{\tilde{C}}{2\pi\sigma^3} \left[(x_0\dot{y}_0 - y_0\dot{x}_0) \right. \\ \left. + \left(\Omega + \frac{\tilde{C}}{4\pi\sigma^2} \right) (x_0^2 + y_0^2) \right] = 0. \end{aligned} \quad (5.16c)$$

Equations (5.16) demonstrate that the nonlinear rotation significantly influences the

oscillation of the center of mass and the width of the condensate. It is also observed that the dynamical equations for the center of mass and width depend on x_0 and $\tilde{\sigma}$, respectively. This dependency indicates a coupling between the dynamical equations, which is not observed in the single component case [241, 242]. Consequently, the dynamics of the center of mass and the width will exhibit different behavior from those of single-component condensates due to the influence of nonlinear rotation.

To determine a mathematical expression for the dipole mode frequency, we consider the small amplitude oscillation of x_0, y_0 and σ around the equilibrium points. After linearizing Eqs. 5.16(a)-(c) around the equilibrium point $x_0 = 0, y_0 = 0$, and $\sigma = \sigma_0$, where σ_0 is the width of the Gaussian ground state, we obtain

$$\omega_d = 1 - \left(\Omega + \frac{\tilde{C}}{4\pi\sigma_0^2} \right) \quad (5.17)$$

where the dipole mode frequency decreases with increasing $\tilde{C}$, the same phenomenon which was also found from the BdG spectrum. The dipole mode obtained using the variational approach is shown as a dashed line in Fig. 5.7(g).

Additionally, to calculate the breathing mode which is not illustrated in Fig. 5.4, we performed a Fourier transform of the time evolution of the root mean square radius r_{rms} after quenching the harmonic trapping strength [239], similar to the analysis of the dipole mode. However, the observations from variational approach indicate that this mode is independent of the nonlinear rotation strength $\tilde{C}$. The frequency of the breathing mode is obtained using the variational approach as,

$$\omega_b = \sqrt{\frac{g}{2\pi\sigma_0^4} - \frac{3}{\sigma_0^4}} - 1 \quad (5.18)$$

Equation (5.18) indicates that the breathing mode frequency remains unaffected by the density-dependent gauge potential. To determine the breathing frequency, we first compute the equilibrium condensate width σ_0 by solving the steady-state equation:

$$\sigma_0^4 - \frac{g}{2\pi\sigma_0^3} - 1 = 0, \quad (5.19)$$

which yields $\sigma_0 = 2.83$ for our parameter set. Substituting this value into Eq. (5.18), we find the breathing mode frequency to be $\omega_b = 1.93$. In Fig. 5.7(g) and (h), we present a comparison of dipole and breathing mode frequencies obtained using three distinct techniques. Here the frequencies derived from the perturbative method and

the variational approach show excellent agreement with those from the Bogoliubov-de Gennes spectrum, demonstrating a strong agreement between these methods.

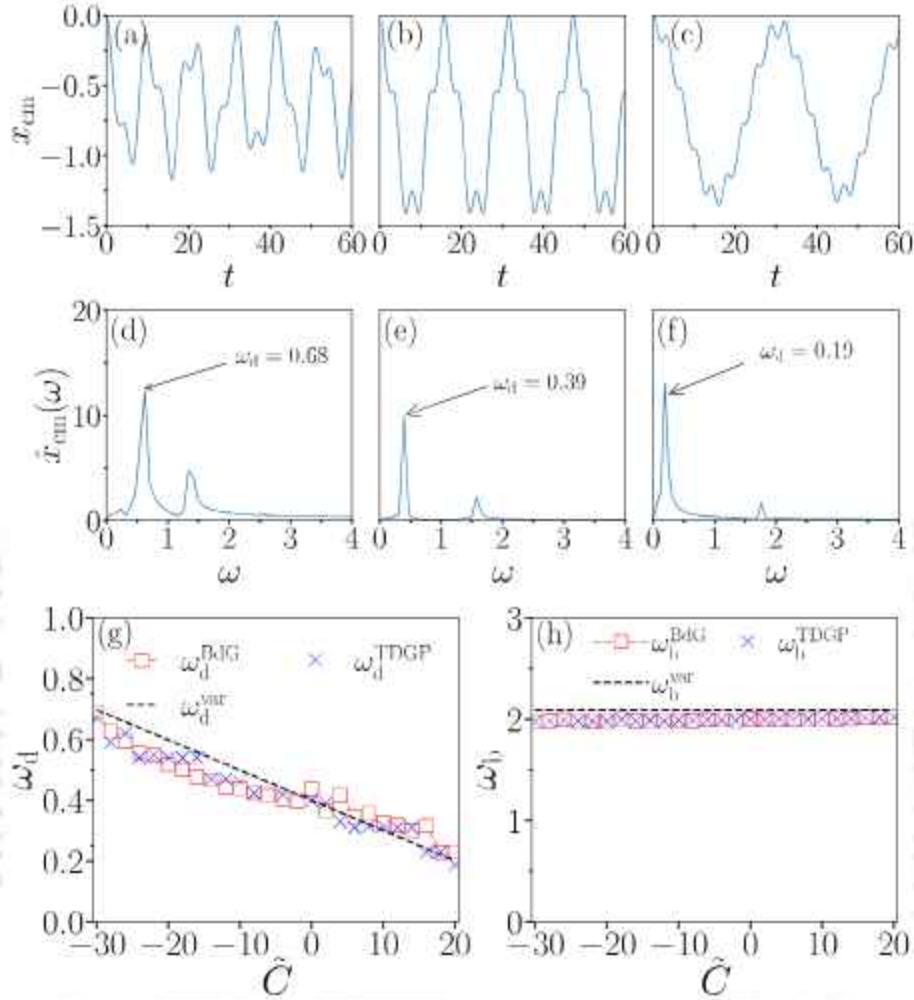


Figure 5.7: Panels (a)-(c), show the center of mass oscillation $x_{cm}(t)$ of the condensate as a function of time for $\tilde{C} = -30, 0, 20$, respectively. (d)-(f) illustrates the Fourier transform of x_{cm} . The dominant peaks observed at $\omega_d = 0.68, 0.39$, and 0.19 are the dipole modes obtained via a linear perturbation to the condensate. The variation of the dipole and breathing mode frequencies as a function of the nonlinear rotation strength $\tilde{C}$ is illustrated in (g) and (h), respectively, at $\Omega = 0.6$. The black dashed line, blue crosses ($\times$), and the red squares ($\square$) represent the dipole modes obtained using the variational analysis, linear perturbation to the condensate, and from the Bogoliubov de-Gennes equations respectively.

5.5 Vortex-displacement modes

The motion of vortices is intrinsically coupled to the superflow of the condensate, mediated through the Magnus force, which establishes a connection between the dynam-

ics of the vortices to the underlying condensate. Due to this, the degrees of freedom associated with vortices give rise to additional normal modes in the excitation spectrum. Since the nonlinear rotational strength influences both the Magnus force and the number of vortices, it becomes particularly interesting to study how this nonlinear rotation affects the mode frequencies and the collective vortex dynamics. In this section, we provide a description of different vortex displacement modes associated with few-vortex arrays in the presence of the density-dependent gauge potential using the Bogoliubov-de Gennes wave-function formalism. Within the rotating frame, the low-lying vortex displacement modes describe the motion of the vortex core around its equilibrium position and can be categorized into four different kinds, as discussed by Campbell [244] and Simula [172], which are described as follows:

1. **Tkachenko Mode (T):** The Tkachenko mode represents a fundamental vortex displacement mode characterized by a torsional oscillation of the vortex lattice, where each vortex executes an elliptical motion around its equilibrium position. It is one of the zero modes associated with the coupled Eqs. (5.12) and is related to the breaking of SO(2) symmetry. Unlike the usual orthonormalization conditions, such as $\int [u_i(x, y)u_j(x, y) - v_i(x, y)v_j(x, y)] dx dy = \delta_{ij}$ and $\int [u_i(x, y)v_j(x, y) - u_j(x, y)v_i(x, y)] dx dy = 0$, the Tkachenko mode satisfies instead the condition $\int (u_1^*(x, y)u_1(x, y) - v_1^*(x, y)v_1(x, y)) dx dy = 0$ for the appropriate eigenstates. It should be noted that in our work, we have obtained a complex eigenvalue for the Tkachenko mode, with an imaginary part in the order of $\sim 10^{-3}$. Since we have studied the collective excitations of the ground state, where dynamical instability is not anticipated, therefore this imaginary component has been neglected. It should be emphasized that for the zero-frequency Tkachenko mode, the orthonormalization condition, $(\int (|u(x, y)|^2 - |v(x, y)|^2) dx dy = 0)$ is exactly satisfied whereas for the nonzero Tkachenko mode the norm remains close to zero. In our case norm $N_T = \int (|u(x, y)|^2 - |v(x, y)|^2) dx dy$ lies in the range $10^{-4} < N_T < 10^{-3}$. Moreover, the existence of the non-zero Tkachenko mode frequencies is trivial in such systems. At this juncture, it is worth mentioning the previous investigations demonstrating the non-zero Tkachenko eigen frequencies in both two and three dimensional systems [211, 170, 171, 215].
2. **Circular Mode (C):** This mode represents the center-of-mass excitation of the vortex array. The entire vortex lattice undergoes retrograde circular motion around the trap center as a rigid body in a frame co-rotating with the trapping potential.

3. **Quadratic Modes (Q):** These modes involve a rigid-body oscillation of the central vortex while other surrounding vortices can move relative to one another.
4. **Rational Modes (R):** In these modes, the central vortex remains stationary while the other vortices exhibit relative motion within the lattice.

We begin by presenting the density distributions of the eigenfunctions associated with the Tkachenko mode. In Fig. 5.8 we illustrate $|u(x, y)|^2$ and $|v(x, y)|^2$ for specific values of $\tilde{C} = -30, 0$, and 20 . Here, the eigenfrequencies satisfy the aforementioned condition $|u(x, y)|^2 = |v(x, y)|^2$ [see middle and lower panel]. The density distribution reveals distinct peaks at the positions of the vortex cores. As a result, the perturbation described in Eq. (5.13) shifts the vortex cores from their equilibrium positions, causing them to trace elliptical trajectories.

In addition to the perturbation caused by the eigenfunctions obtained from the Bogoliubov-de Gennes analysis, the Tkachenko oscillation can also be excited experimentally through two different mechanisms. In one approach [34], a focused resonant laser beam removes atoms from the center of the condensate, creating a density dip. The condensate responds by flowing inward to fill the dip and outward to expand its radius. The Coriolis force then redirects the inward flow in the direction of the lattice rotation while the outward flow moves in the opposite direction. This shearing motion disturbs the vortices from their equilibrium positions and triggers oscillations in the lattice.

In the second method [60], a resonant laser light is employed to generate a Gaussian dip in the radial trapping potential to produce an inward flow of the condensate that resembles the dynamics observed in the first method. Motivated by this second approach, we introduce a Gaussian perturbation of the form

$V_p(x, y) = -\exp(-(x^2 + y^2)/(R_{TF}/2)^2)$, where $R_{TF} = \sqrt{2\mu/m\omega_\perp^2}$ is the Thomas-Fermi radius for Eq. (5.8). This leads to an effective potential defined as

$$V_{\text{eff}}(x, y) = V(x, y) - \exp(-(x^2 + y^2)/(R_{TF}/2)^2), \quad (5.20)$$

where $V(x, y)$ is the harmonic trapping potential. This allows us to investigate the nature of Tkachenko oscillations through the time evolution of particular physical observables [215]. We then evolve the previously obtained stationary state in real-time using the modified Gross-Pitaevskii equation with Eq. (5.20) and analyze the resulting dynamics, which helps to validate the results obtained from the BdG analysis.

Since the Tkachenko mode involves an elliptically polarized collective motion of

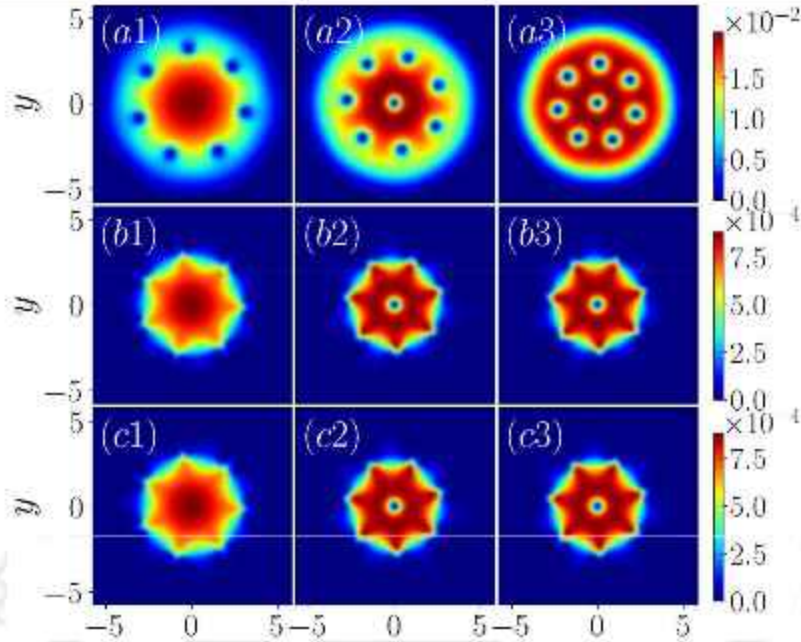


Figure 5.8: Pseudocolor representation of the condensate density for (a1) $\tilde{C} = -30$, (a2) $\tilde{C} = 0$, and (a3) $\tilde{C} = 20$. Panels (b1)–(b3), and (c1)–(c3) show the eigen mode functions $|u(x,y)|^2$, and $|v(x,y)|^2$ corresponding to the Tkachenko mode.

vortices around their equilibrium positions, it can be decomposed into transverse and longitudinal polarization components. These correspond to the azimuthal and radial oscillations of the vortices, respectively. The transverse vortex motion is characterized by the averaged angular displacement

$$\theta(t) = \sum_{i=1}^{N_v} \left(\arctan \left(\frac{y_i(t)}{x_i(t)} \right) - \arctan \left(\frac{y_0(t)}{x_0(t)} \right) \right), \quad (5.21)$$

where N_v represents the total number of vortices, excluding the central vortex. The longitudinal vortex motion, representing the radial displacement of the vortices, is given by $r_L(t) = \sum_{i=1}^{N_v} |r_i(t) - r_i(0)|$.

In Fig. 5.9, we illustrate the temporal evolution of the transverse motion of vortices for three different values of $\tilde{C}$: -20 , 0 , and 20 (see Figs. (a1)–(c1)). The observed phase jumps in the transverse motion arise as the vortices follow an elliptical trajectory. These jumps typically occur when either $x(t)$ or $y(t)$ approaches zero or undergoes a sign change, leading to a discontinuous shift in the angular phase. We also compute the Fourier transform of the motion to determine the frequency of the Tkachenko mode. Our findings indicate that at $\tilde{C} = 0$, the Tkachenko mode frequency is $\omega_T = 0.060$. This frequency increases to 0.080 for $\tilde{C} = -20$ and further to 0.085 for $\tilde{C} = 20$. In

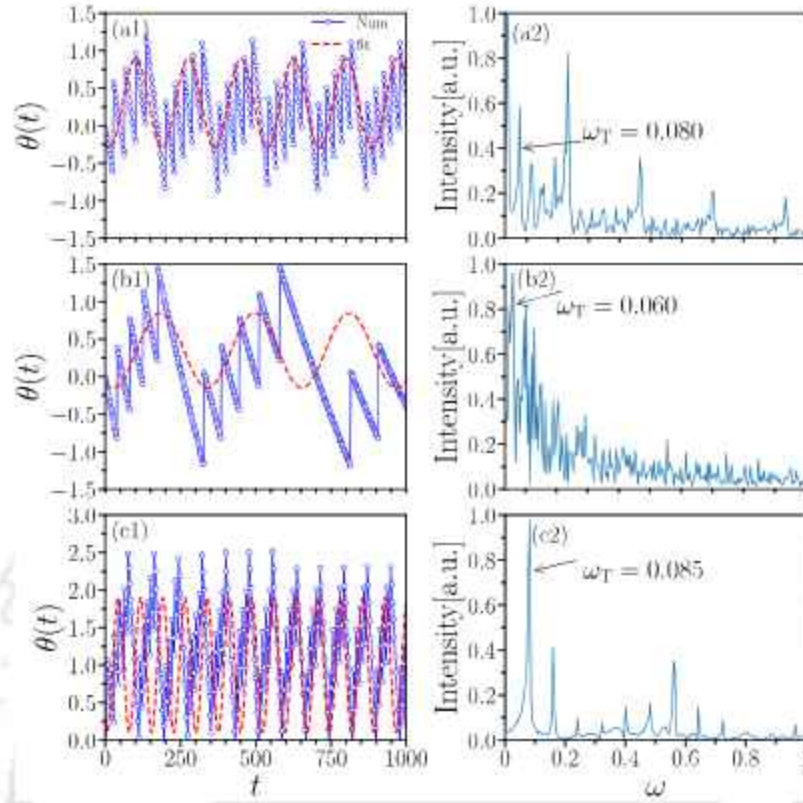


Figure 5.9: Transverse oscillations of vortices for (a1) $\tilde{C} = -20$, (b1) $\tilde{C} = 0$, and (c1) $\tilde{C} = 20$. Panels (a2)-(c2) show the corresponding Fourier analysis of these oscillation patterns. The dashed red lines represent sinusoidal fits of the form $A \sin(Bt + D)$. The fitting parameters are: $A = 0.6, 0.5$, and 0.9 ; $B = 0.036, 0.02$, and 0.085 for $\tilde{C} = -20, 0$, and 20 , respectively. The phase shift D is fixed at 4.2 .

Fig. 5.10, we present the time evolution of the longitudinal motion of vortices, along with its Fourier spectrum. The observed peaks at $\omega_L = 0.091, 0.064$, and 0.085 are close to ω_T , indicating that the vortex core exhibits elliptical motion around its equilibrium position. It is important to note that for $\tilde{C} = 20$, the vortex motion is primarily governed by the Tkachenko mode, whereas for $\tilde{C} = -20$ and 0 , additional frequency components are present. This is because, with positive values of $\tilde{C}$, the vortices tend to localize at the center, which reduces the distance between them. The vortex core size is also increases, which agrees well with the principles of continuum theory. The real-time evolution of Eq. (5.8) requires a very small time step, $\Delta t \approx 10^{-7}$, due to the density-dependent rotation term. This significantly increases the computational cost. To reduce this cost, we introduce a small dissipation parameter, $\gamma = 2 \times 10^{-3}$, in Eq. (5.8) by replacing i with $(i - \gamma)$ on the left-hand side of the equation. In order to capture the temporal oscillation of the vortex more adequately we fit the transverse $\theta(t)$ and longitudinal motion ($r_L(t)$) of the vortices with a sinusoidal curve as denoted

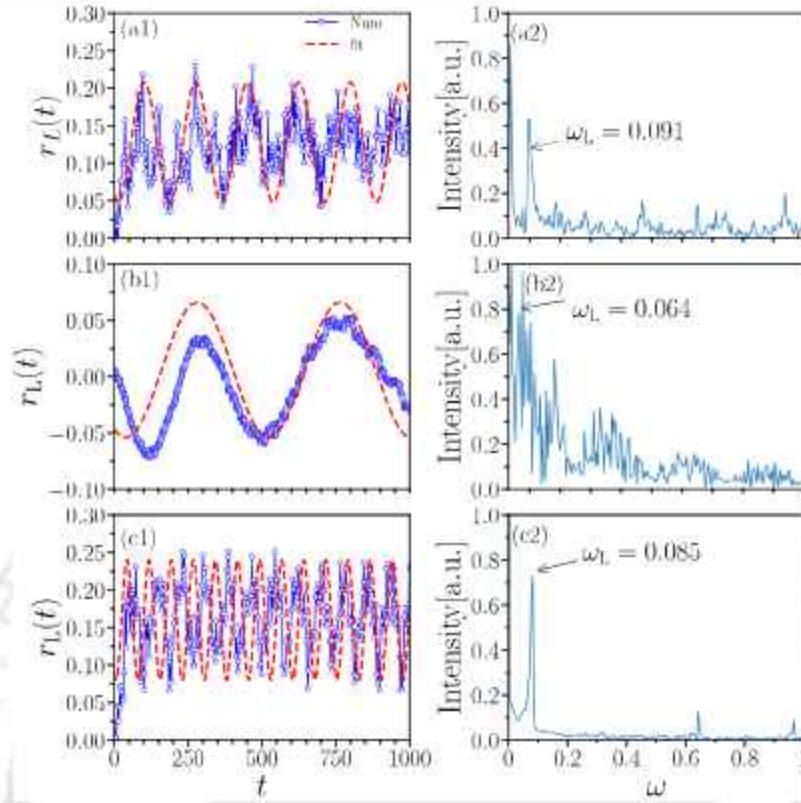


Figure 5.10: Longitudinal oscillations of vortices for (a1) $\tilde{C} = -20$, (b1) $\tilde{C} = 0$, and (c1) $\tilde{C} = 20$. Panels (a2)-(c2) show the corresponding Fourier analysis of these oscillation patterns. The dashed red lines represent sinusoidal fits of the form $A \sin(Bt + D)$. The fitting parameters are: $A = 0.08, 0.06$, and 0.08 ; $B = 0.036, 0.013$, and 0.085 for $\tilde{C} = -20, 0$, and 20 , respectively. The phase shift D is fixed at 4.2 .

with a solid black line. The fitting suggests the sinusoidal nature of the vortex oscillation as the system is perturbed by the low-lying collective excitation modes. We have provided the corresponding time evolution animations as supplementary files [see supplementary movies]. This particular feature of the vortex dynamics may be attributed to the excitation of the Tkachenko mode as also reported for the rotating BECs [215]. It should be noted that due to a structural change of the vortex lattice from the hexagonal configuration to a ring formation, the time series of the vortex evolution appears to contain some noise which also becomes manifests as an appearance of multiple peaks in the frequency spectrum. This also may be the reason that we detect fewer frequencies for $\tilde{C} \geq 0$ (see 5.9(b1)-(b2), and 5.9(c1)-(c2)) compared to those we find for $\tilde{C} < 0$ (see Fig. 5.9(a1)-(a2)). One may decrease the noise in both the time series and frequency spectrum for $\tilde{C} \geq 0$ by considering a larger vortex lattice which can be attained in the higher rigid body frequency limit ($\Omega \sim 1$) [215].

To gain a better understanding of the Tkachenko mode frequency, we derive its an-

alytical expression using the continuum hydrodynamic approach. This method has been previously applied to describe the Tkachenko mode in both single [211] and two-component [216] BECs. Following the same framework, we derive the expression for the Tkachenko mode frequency using the hydrodynamic approach. The superfluid acceleration equation for a condensate with velocity $\mathbf{v}$, vortex displacement field ϵ , and vorticity Ω_{vort} , can be expressed as [211],

$$m \left(\frac{\partial \mathbf{v}}{\partial t} + \Omega_{\text{vort}} \times \epsilon \right) = -\nabla \mu, \quad (5.22)$$

where μ is the chemical potential, and m is the mass of the superfluid. The elastic energy density of a triangular lattice in two dimensions is expressed in the following form: [211],

$$\mathcal{E} = 2C_1(\nabla \cdot \epsilon)^2 + C_2 \left[\left(\frac{\partial \epsilon_x}{\partial x} - \frac{\partial \epsilon_y}{\partial y} \right)^2 + \left(\frac{\partial \epsilon_x}{\partial y} + \frac{\partial \epsilon_y}{\partial x} \right)^2 \right], \quad (5.23)$$

where C_1 is the compressional modulus, and C_2 the shear modulus of the vortex lattice, and in terms of vorticity for an incompressible fluid, $C_2 = -C_1$, and

$$C_2 = \frac{n\Omega_{\text{vort}}}{16}. \quad (5.24)$$

where n is the density of the condensate.

The conservation of particle number and momentum results in the following continuity and momentum conservation equations.

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (5.25)$$

where $\mathbf{j} = n\mathbf{v}$ is the particle current.

$$m \left(\frac{\partial \mathbf{j}}{\partial t} + \Omega_{\text{vort}} \times \mathbf{j} \right) + \nabla P = -\sigma, \quad (5.26)$$

where P represents pressure, which is related to the chemical potential as $\nabla P = n\nabla \mu$. The stress vector σ can be derived from the total elastic energy $E_{\text{el}} = \int d^2r \mathcal{E}(r)$, given by

$$\sigma = \frac{\delta E_{\text{el}}}{\delta \epsilon} = -4C_1 \nabla (\nabla \cdot \epsilon) - 2C_2 \nabla^2 \epsilon. \quad (5.27)$$

From Eqs. (5.26) and (5.22), along with $\nabla P = n\nabla\mu$, we derive the relationship

$$\Omega_{\text{vort}} \times (\dot{\epsilon} - \mathbf{v}) = \frac{\nabla \cdot \boldsymbol{\sigma}}{mn}. \quad (5.28)$$

The curl and divergence of this equation yield

$$\nabla \cdot (\dot{\epsilon} - \mathbf{v}) = \frac{\nabla \times \boldsymbol{\sigma}}{\Omega_{\text{vort}} mn} = \frac{C_2}{\Omega_{\text{vort}} mn} \nabla^2 (\nabla \times \epsilon), \quad (5.29)$$

and

$$\nabla \times \dot{\epsilon} + \Omega_{\text{vort}} \nabla \cdot \epsilon = -\frac{\nabla \cdot \boldsymbol{\sigma}}{\Omega_{\text{vort}} mn} = \frac{C_2 + 2C_1}{\Omega_{\text{vort}} mn} \nabla^2 (\nabla \cdot \epsilon). \quad (5.30)$$

The divergence of the superfluid acceleration, Eq. (5.22) is

$$\left(-\frac{\partial^2}{\partial t^2} + s^2 \nabla^2 \right) n = n \Omega_{\text{vort}} \cdot (\nabla \times \epsilon). \quad (5.31)$$

The speed of sound s is given by $ms^2 = \partial P / \partial n$ [211]. Then by using Eq. (5.29) and eliminating $\nabla \times \epsilon$ while disregarding the terms of order ∇^4 , we derive the following equation:

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{2C_2}{mn} \nabla^2 \right) \nabla \cdot \epsilon = \frac{1}{n} \frac{\partial^2}{\partial t^2} n. \quad (5.32)$$

After linearising the coupled equations (5.31) and (5.32), we obtain the mode frequencies by solving the corresponding secular equation for a given wavevector k

$$\begin{aligned} \lambda(k, \omega) = & \omega^4 - \omega^2 \left[\Omega_{\text{vort}}^2 + \left(s^2 + \frac{4}{mn} (C_1 + C_2) \right) k^2 \right] \\ & + \frac{2s^2 C_2}{mn} k^4 = 0. \end{aligned} \quad (5.33)$$

Equation (5.33) can be simplified to

$$\omega^4 - A\omega^2 + \frac{2s^2 C_2}{mn} k^4 = 0. \quad (5.34)$$

where $A = \left[\Omega_{\text{vort}}^2 + \left(s^2 + \frac{4}{nm} (C_1 + C_2) \right) k^2 \right]$. By solving Eq. (5.34) for ω^2 we obtain

$$\omega^2 = \frac{A}{2} \left[A \pm \sqrt{A^2 - \frac{8s^2 C_2 k^4}{nm}} \right]. \quad (5.35)$$

For $2s^2 C_2 k^4 / nm \ll A$ the mode frequencies are given by

$$\omega_I^2 = \Omega_{\text{vort}}^2 + \left(s^2 + \frac{4(C_1 + C_2)}{nm} \right) k^2, \quad (5.36)$$

and

$$\omega_T^2 = \frac{2C_2}{nm} \frac{s^2 k^4}{(\Omega_{\text{vort}}^2 + (s^2 + 4(C_1 + C_2)/nm)k^2)}. \quad (5.37)$$

The frequency ω_I corresponds to the inertial mode of the condensate, which arises from the Coriolis force in the rotating frame, while ω_T denotes the Tkachenko mode. To determine the Tkachenko mode frequency from Eq. (5.37), we evaluate the sound velocity at the trap center as $s_0 = \sqrt{g n_0}$, where $n_0 = \mu/g$ represents the central density. The effective wave vector is inversely proportional to the Thomas-Fermi radius, i.e., $k = \alpha/R_{\text{Th}}$, and where α is a free parameter, which we choose as $\alpha = 7$ in our work. The vorticity is computed using the expression

$$\Omega_{\text{vort}} = \left[2\Omega + C \left(x \frac{\partial n(x,y)}{\partial x} + y \frac{\partial n(x,y)}{\partial y} \right) \right] \hat{e}_z \quad (5.38)$$

here the peak value of Ω_{vort} is used to determine the Tkachenko mode frequency.

In Fig. 5.11, we present the variation of the Tkachenko mode frequency ω_T as a function of the nonlinear rotation $\tilde{C}$ strength obtained from three distinct techniques: BdG analysis, by perturbing the time-dependent GPE using Eq. (5.20), and the continuum hydrodynamic approach. We find that ω_T has a minimum around $\tilde{C} = 0$, and exhibits a gradual increase, following a quadratic dependency as $\tilde{C}$ deviates from zero in either the positive or negative direction. We observe a strong agreement between the values of ω_T obtained from the BdG analysis and the time-dependent Gross-Pitaevskii equation (TDGP). However, the frequency predicted by the hydrodynamic approach matches quite well for positive values of $\tilde{C}$, whereas for the negative range (shown by light grey region), the ω_T^{Ana} (marked by black filled squares) deviates compared to other approaches. This discrepancy arises from the underlying assumption of hy-

$\tilde{C}$	mode	ω	label	$\tilde{C}$	mode	ω	label
-30	1	0.023	C	4	1	0.088	C
	2	0.0243	Q		2	0.078	T
	3	0.150	T		3	0.105	Q
	4	0.127	Q		4	0.140	Q
	5	0.130	Q		5	0.181	Q
	6	0.263	Q		6	0.285	Q
	7	0.268	Q		7	0.310	Q
	8	0.561	S		8	0.363	Q
	9	0.697	D		8	0.417	D
-20	1	0.019	C	8	10	0.661	S
	2	0.076	Q		1	0.059	C
	3	0.087	T		2	0.084	T
	4	0.127	Q		3	0.107	Q
	5	0.152	Q		4	0.140	Q
	6	0.294	Q		5	0.176	Q
	7	0.500	D		6	0.208	Q
	8	0.547	S		7	0.261	Q
-4	1	0.040	C	16	8	0.328	Q
	2	0.046	T		9	0.361	D
	3	0.158	R		10	0.629	S
	4	0.216	Q		1	0.079	T
	5	0.224	Q		2	0.086	C
	7	0.226	Q		3	0.171	Q
	6	0.339	Q		4	0.228	Q
	7	0.425	D		5	0.280	Q
0	8	0.655	S		6	0.316	D
	1	0.067	C		7	0.368	Q
	2	0.070	T		8	0.415	Q
	3	0.088	Q		9	0.421	Q
	4	0.120	Q		10	0.739	S
	5	0.170	Q		1	0.085	C
	6	0.179	Q		2	0.096	T
	7	0.281	Q		3	0.171	Q
	8	0.353	Q		4	0.230	D
	9	0.436	D		5	0.283	Q
	10	0.676	S		6	0.317	Q
0	1	0.067	C		7	0.367	Q
	2	0.070	T		8	0.415	Q
	3	0.088	Q		9	0.450	Q
	4	0.120	Q		10	0.769	S
	5	0.170	Q				
	6	0.179	Q				
	7	0.281	Q				
	8	0.353	Q				
	9	0.436	D				
	10	0.676	S				

Table 5.1: Frequencies of the low-lying excitations for $\Omega = 0.6, g = 400$ at different values of $\tilde{C}$. Here, the notations represent circular (C), dipole (D), quadrupole (Q), Tkachenko (T), surface (S), and rational (R) modes. Movies illustrating the dynamical behavior of all listed modes are available in the Supplemental Material (Phys. Rev. A **112**, 043308 (2025)).

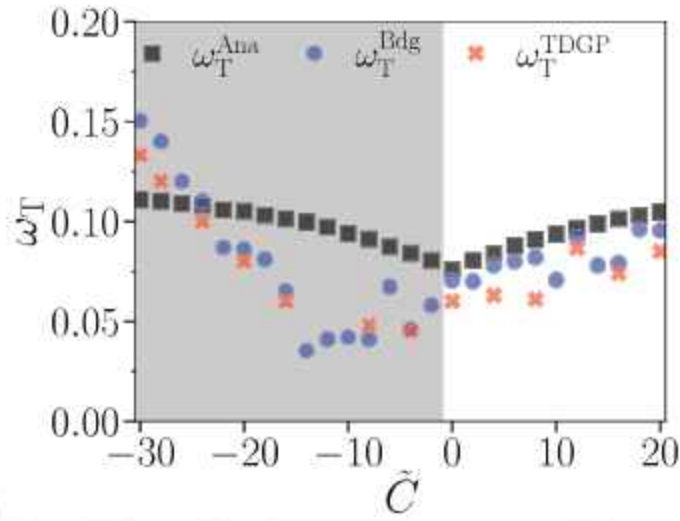


Figure 5.11: Variation of the Tkachenko mode frequency ω_T as a function of nonlinear rotation $\tilde{C}$. The filled black squares, blue circles, and crosses represent the Tkachenko frequency obtained from continuum hydrodynamic theory (Eq. (5.37)), BdG analysis, and the Gaussian perturbation applied to the time-dependent GPE, respectively.

drodynamic continuum theory, which considers the vortex core size to be significantly smaller than the intervortex distance in the lattice. Here it is instructive to mention that for $\tilde{C} < 0$, the vortices in general arrange on a ring at the periphery of the condensate and contain only a few number of vortices. As a result of this the assumption of continuum hydrodynamic theory appears not be valid for the region of $\tilde{C} < 0$ which may be linked to the reason for the apparent deviations observed between the Bogoliubov modes and the predictions from the continuum theory. However for $\tilde{C} > 0$ the Magnus force generated from the density-dependent gauge potential causes the vortices in general to become located near the center of the trap that results in an overall reduction in the inter-vortex spacing.

So far, we have analyzed the vortex displacement modes by examining the post-perturbation dynamics of the condensate density for different values of $\tilde{C}$. In Table 5.1, we characterize different vortex displacement modes associated to different excitation frequencies across a range of $\tilde{C}$ values. For negative values, we consider $\tilde{C} = -30, -20$, and -4 , while for positive values, we include $\tilde{C} = 0, 4, 8, 16$, and 20 .

Overall we observed that the number of vortex displacement modes varies systematically with the nonlinear rotational strength $\tilde{C}$. The number of modes changes with $\tilde{C}$ because the vortex number in a ground state depends upon the nonlinear rotational strength, and similarly the number of displacement modes are equal to the vortices present in the condensate. This correspondence can be understood from the role of

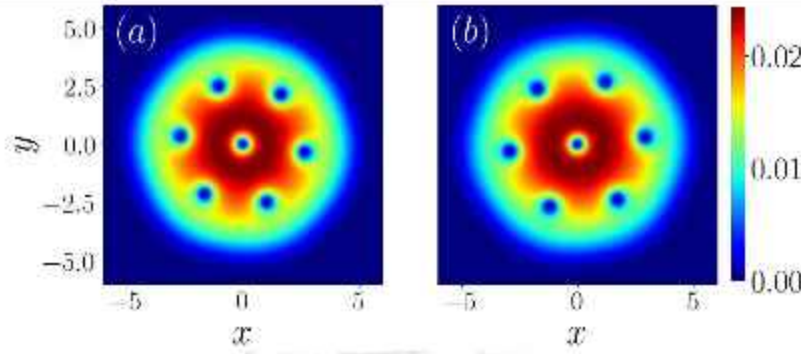


Figure 5.12: Pseudo-color representation of the condensate density for (a) $\tilde{C} = -4$, and (b) $\tilde{C} = -12$ at $\Omega = 0.6$ and $g = 400$.

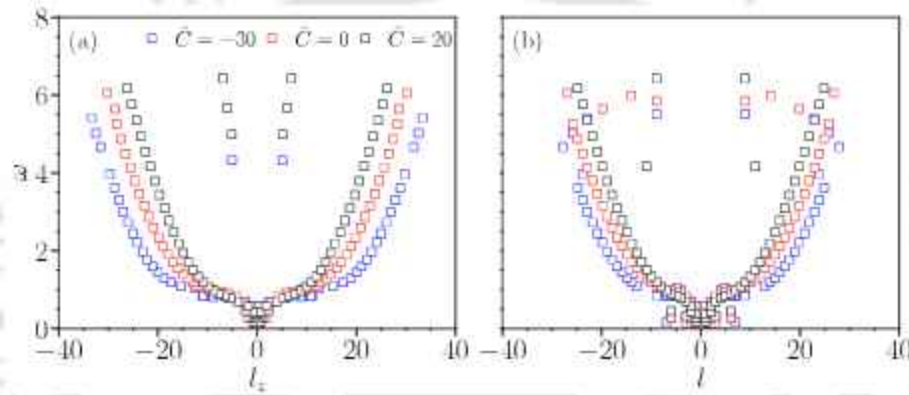


Figure 5.13: Excitation spectrum as a function of (a) angular momentum, and (b) angular quantum number for $\tilde{C} = -30, 0$, and 20 .

the Magnus force, $\mathbf{F}_M = \boldsymbol{\omega} \times \mathbf{v}$, which governs the dynamics of the vortices. Since the Magnus force is proportional to the velocity $\mathbf{v}$, the system can exhibit N_v displacement modes, where N_v represents the number of vortices. Moreover, it is observed that the energy associated with both the circular and quadrupole modes increases as $\tilde{C}$ is varied from negative to positive. This is because, for large negative values of $\tilde{C}$, the vortices are arranged at the edge of the condensate, where the tails of the density profile appear to decay. In contrast, for positive values of $\tilde{C}$, the vortices are arranged near the trap center, where the density is high. The rational mode (R) is found for $\tilde{C} = -4$ and is absent for other values of $\tilde{C}$. This occurs because, at $\tilde{C} = -4$, the arrangement of vortices follow the hexagonal symmetry with seven vortices, which is broken for other values of $\tilde{C}$. Interestingly, we also observe a rational mode at $\tilde{C}$, suggesting that similar lattice symmetry exist there, although this case is not included in the Table 5.1. In Fig. 5.12, we present the condensate density profiles for $\tilde{C} = -4$ and -12 . It is observed that a nearly perfect hexagonal symmetry exists for both parameter values.

It is important to note that neither the dipole mode (D) nor the surface mode (S) is associated with vortex displacement. The surface mode consists of excitations localized at the edge of the condensate and often leads to the nucleation of new vortices. Interestingly, the energy of the surface mode remains relatively low for $\tilde{C} < 0$ but increases significantly for $\tilde{C} > 0$. For instance, as shown in Table 5.1, the surface mode energy is $\omega = 0.561$ at $\tilde{C} = -30$, rising to $\omega = 0.769$ at $\tilde{C} = 20$. This behavior can be attributed to the underlying density distribution: for $\tilde{C} < 0$, the condensate density decreases near the boundary, making it energetically favorable to excite surface ripples. In contrast, for $\tilde{C} > 0$, the density profile exhibits a broad plateau, requiring more energy to generate such surface deformations. Figure 5.13(a) and (b) show the excitation spectrum ω as a function of angular momentum l_z and the angular quantum number l , respectively, for three different values of $\tilde{C}$: $\tilde{C} = -30, 0$, and 20 . The angular momentum of the excitations relative to the rotating condensate is defined as [232]

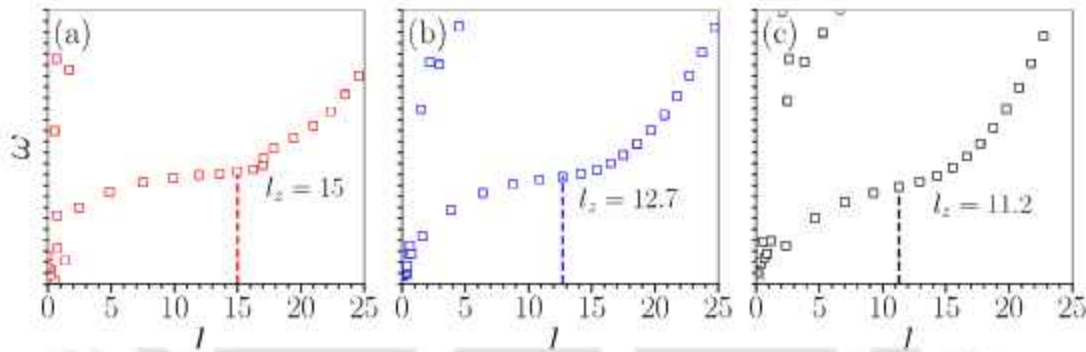


Figure 5.14: The frequencies of the low-lying excitations are shown as a function of their orbital angular momentum relative to the condensate ground state for (a) $\tilde{C} = -30$, (b) $\tilde{C} = 0$, and (c) $\tilde{C} = 20$. For $\tilde{C} = -30$, the roton-like minima shift toward higher angular momentum, and towards lower angular momentum for $\tilde{C} = 20$.

$$l_z = \frac{(l_u - l_\psi)N_u + (l_v + l_\psi)N_v}{N_u + N_v}, \quad (5.39)$$

where

$$l_\alpha = -i\hbar \frac{\int dx dy \alpha^* (x \frac{\partial \alpha}{\partial y} - y \frac{\partial \alpha}{\partial x})}{\int dx dy \alpha^* \alpha}, \quad (5.40)$$

with α being one of the wave functions $u(x, y)$, $v(x, y)$, or $\psi(x, y)$, and $N_u = \int dx dy |u(x, y)|^2$, $N_v = \int dx dy |v(x, y)|^2$. The excitation spectrum shows a high degree of symmetry around $l = 0$. Notably, the energy splitting between the modes with angular momen-

tum l and $-l$ is influenced by the nonlinear rotation parameter $\tilde{C}$. Both Figures 5.13(a) and (b) illustrate that the energy splitting, for both angular momentum and angular quantum number, is larger when $\tilde{C} = -30$ and diminishes substantially when $\tilde{C} = 20$. It has also been observed that the energy of the surface excitation of the condensate increases for $\tilde{C} = 20$, while it decreases for $\tilde{C} = -30$. In the excitation spectrum, we observe flattening of the energy below a threshold value of the angular momentum (see Fig. 5.14) which may be attributed to the roton like energy dip observed for the low number of vortices as reported in [172]. These roton-like minima may lead to the formation of additional vortices in the system by triggering instabilities at the surface of the condensate. We note that these hanging modes shift to higher angular momenta for negative values of $\tilde{C}$ and to lower angular momenta for positive values of $\tilde{C}$. This type of shifting is a direct consequence of the expansion and contraction of the Thomas-Fermi radius, which increases for negative $\tilde{C}$ and decreases for positive $\tilde{C}$.

5.6 Conclusion

In this work, we have investigated the effects of nonlinear rotation in a quasi-two-dimensional Bose-Einstein condensate, which arises from a density-dependent gauge potential on the collective excitation spectrum using Bogoliubov-de Gennes theory. We found that the strength of the nonlinear rotation parameter influences the excitation spectrum in several ways. By analyzing the time evolution of the condensate density profile, perturbed by the eigenstates of the collective mode, we identified different low-lying collective modes.

Using a variational analysis, we have found that the center of mass oscillation and the width of the condensate are coupled with each other and also depend upon the strength of the nonlinear rotation. We also derived an analytical expression for the dipole and breathing modes using this approach. Additionally, we compared the frequency of the dipole mode by examining the time evolution of center of mass and incorporating an appropriate perturbation into the time-dependent Gross-Pitaevskii equation. Our findings indicate that the nonlinear rotation leads to a violation of the Kohn's theorem, where the frequency of the dipole mode depends on the interaction strength. We have shown that the frequency of the Tkachenko mode strongly depends on the strength of the nonlinear rotation, and the frequency attains minimum at $\tilde{C} = 0$, which symmetrically increases for both negative and positive values of $\tilde{C}$. Additionally, we also derived an analytical expression for the Tkachenko mode frequency using

a continuum hydrodynamic approach which was found to agree with our numerical results. Furthermore, the frequency is comprehended through the time evolution of the transverse and longitudinal motion of the vortices when a Gaussian perturbation is applied to the Gross-Pitaevskii equation (GPE). In addition to the Tkachenko mode, we also identified other fundamental modes depending upon nonlinear rotation. Also, the excitation spectrum was presented as a function of the angular momentum, showing that the energy splitting between l and $-l$ depends on the strength of the nonlinear rotation. Subsequently, the presence of roton-like minima in the excitation spectrum explained the presence of surface mode excitations which shift to higher angular momentum for negative values of nonlinear rotation, and to lower angular momentum for positive values of nonlinear rotation.

The unusual nature of the superfluidity presented in this study presents opportunities for future studies related to synthetic nonlinear gauge potentials. It would be interesting to investigate how altering the confining geometry changes the vortex morphology [245–247]; moreover how the density-dependent gauge potential modifies the three-dimensional vortex line and ring solutions and their associated dynamics is another avenue to potentially explore. Finally, the role of superfluid turbulence in this system provides a route to understanding the interplay of how energy is distributed across length scales with a spatially varying vorticity [248–250].

Ground state properties and excitation spectrum of a toroidal condensate under a density-dependent gauge potential

After presenting the effect nonlinear rotation induced by a density-dependent gauge potential on the collective excitation spectrum of a quasi-two-dimensional Bose-Einstein condensate confined in a harmonic trap in Chapter 5, in this Chapter we extend the analysis for the condensate trapped in toroidal geometry. In particular we numerically investigate the effects of a density-dependent gauge potential which induces nonlinear rotation on a Bose-Einstein condensate confined in a toroidal trapping geometry. By focusing on the resulting vortex lattice configurations, we demonstrate that increasing the strength of the nonlinear rotation leads to a structural transition from a ring-shaped vortex lattice to a giant vortex state. The quantum circulation associated with the giant vortex is found to be highly sensitive to the strength of the nonlinear rotation. Additionally, we identify the parameter regime in which the Thomas-Fermi density profile remains valid by mapping the solution space based on the strength of the nonlinear rotation and the radius of the confining potential. Based on the Bogoliubov de Gennes analysis, we investigate the impact of nonlinear rotation on the collective excitation spectrum. Our results reveal a violation of the Kohn theorem, accompanied by changes in the breathing mode frequency that indicate radial deformation of the condensate. Our findings are further substantiated through a comprehensive hydrodynamic analysis. Finally, we analyze the stability of multiply quantized vortices under nonlinear rotation. Our findings indicate that while nonlinear rotation can enhance the global stability of these states, it does not necessarily ensure their local stability.

6.1 Introduction

The literature related to the vortex lattice structure and corresponding excitation spectrum in presence of synthetic gauge potential has been extensively presented in Chapter 1. However, to ensure that the current Chapter is self contained, we provide a brief overview of previous studies related to the vortex lattice structure in presence of synthetic gauge potential in rotating BECs in particular for the condensate trapped in the toroidal geometry. The study of topological excitations in quantum fluids has become a cornerstone of modern condensed matter physics, owing to its profound implications across a wide range of physical systems, including superfluid helium [5], liquid crystals [6], quantum magnets [7], and topological matter [8]. These excitations often realized as vortices, solitons, or other defect structures encode the nontrivial topology of the underlying order parameter and play a decisive role in determining macroscopic properties, collective dynamics, and phase transitions in many-body systems [3, 4]. Understanding their formation, stability, and dynamics is therefore essential for unraveling the complex behavior of quantum fluids both in and out of equilibrium.

Among the various experimental platforms available, rotating Bose-Einstein condensates (BECs) stand out as an exceptionally clean and versatile system for investigating topological phenomena. The controlled generation of quantized vortices and vortex lattices in BECs offers a direct and highly tunable realization of topological defects in a quantum fluid [251, 252]. These systems have enabled detailed studies of vortex nucleation, lattice formation, quantum turbulence, and non-equilibrium dynamics, providing insights that are often difficult to access in more complex condensed matter environments.

A particularly rich setting for exploring superfluid topology is provided by toroidally trapped BECs, where the multiply connected geometry supports circulating superfluid flow and long-lived persistent currents [253, 254, 13]. Such systems have attracted significant theoretical and experimental interest, leading to investigations of persistent currents [255, 245], ring dark solitons [256, 257], weak links and atomtronic analogs of Josephson junctions [258, 259], as well as spontaneous symmetry breaking and metastability [260]. Under rotation the annular geometry gives rise to an especially diverse array of vortex configurations, ranging from regular vortex lattices to giant vortices, with their stability governed by the interplay between angular velocity, interaction strength, and trap geometry [261–263].

More recently, the incorporation of spin-orbit coupling has further enriched this landscape, introducing new mechanisms for generating and stabilizing complex topo-

logical profiles. The combined effect of SOC and rotation in both spin-1/2 and spin-1 BECs has been shown to produce a variety of exotic ground states, including skyrmion lattices, necklace states, supersolid, and other vortex patterns [264–269]. These developments highlight toroidal, rotating spinor BECs as a powerful platform for exploring emergent topology, synthetic gauge fields, and novel quantum phases in highly controllable settings.

The study of collective excitations plays a central role in elucidating the mechanisms governing the stability, deformation, and dynamical response of vortex lattices [208]. In BECs, a substantial body of theoretical and experimental work has been devoted to characterizing the collective excitation spectrum [270, 271, 213, 272, 246, 273, 274]. In particular, Refs. [213, 272] present analytical predictions for collective mode frequencies in the high-angular-velocity regime, while the dependence of the normal-mode spectrum on radial and azimuthal degrees of freedom has been systematically investigated in Refs. [270, 258, 271].

Vortex dynamics and surface-mode excitations in annular BECs have been studied extensively [246], demonstrating that inner-surface modes are associated with induced vortex dipoles, in contrast to harmonically trapped condensates where surface excitations typically nucleate vortex monopoles. The collective excitation spectrum during the transition of a condensate from a filled-sphere configuration to a thin shell, along with the effect of gravity on these modes, has been examined in Refs. [273, 274]. In addition to single-component BECs, multi-component BECs also showcase distinctive excitation spectra in an annular trap and have been explored in various studies. In particular, the interplay of rotation and miscibility in the low-lying excitations is explored in [275]. While the yrast spectrum of two-component BECs in a ring geometry is analyzed in [276], Lundth *et al.* [277] discuss the excitation spectrum of spin-1 condensates and show that instabilities can arise from spin–spin interactions; subsequent works have investigated the metastability of ferromagnetic spin-1 BECs confined in quasi-one-dimensional rotating ring traps [278] and the stability of persistent flow in spinor condensates [279].

The experimental realization of artificial magnetism in quantum gases provides a valuable tool for studying a spectrum of quantum phenomena associated with condensed matter systems in ultracold atoms [31, 30]. This induced magnetism typically manifests as a static gauge potential, leading to a range of novel phenomena, such as orbital magnetism [37–39], spin-orbit coupling with both bosons and fermions [40, 156, 157], and spin-angular-momentum coupling in bosonic systems [41–43, 280]. In addition to these static schemes, several theoretical and experimental studies have in-

troduced quasi-dynamic gauge potentials, where the gauge potential directly depends on the quantum state, producing time-dependent feedback in the form of a density-dependent gauge potential. Such effects have been explored in both continuum systems [46] and lattice geometries [47–49]. Experimentally, density-dependent magnetism has been demonstrated in lattice systems for both bosons [158] and fermions [159], as well as in Rydberg ensembles [55]. Apart from lattice systems, continuum realizations have revealed unusual phenomena such as the observation of domain walls [160] and chiral solitons [161, 162]. Subsequent works have focused on understanding the unusual phenomena induced by this class of artificial gauge potentials, including solitary-wave solutions [217–228, 281, 229] and their transport mechanisms in double-well, optical-lattice, harmonic traps and ring configurations [50, 54, 51, 52, 282, 283]. Analytical and numerical works have demonstrated that density-dependent gauge potentials produce irregular dynamics [53, 163] and have revealed unusual chaotic behavior in two dimension [230]. Density-dependent gauge potentials also give rise to nonlinear rotation [164–167], which has been linked to the formation of non-Abrikosov vortex lattices and ring-like vortex arrangements [63]. Later studies have investigated the critical rotation frequency for vortex nucleation [168] and the appearance of ghost vortices [169].

Although several studies have investigated the structural transformation and collective excitation spectrum of vortex lattices under a density-dependent gauge potential in harmonic traps, the influence of such a gauge potential within a toroidal geometry remains unexplored. In this work, we provide a comprehensive analysis of the structural transformation of vortex lattices induced by a density-dependent gauge potential in a toroidal trap. We derive the corresponding Thomas-Fermi density profile and identify the parameter regime that supports physically admissible solutions. We also investigate the effect of the density-dependent gauge potential on the size of the giant vortex, which is formed utilizing different techniques such as absorption and repulsive methods. To examine the collective dynamics, we compute the excitation spectrum by solving the Bogoliubov de Gennes (BdG) equations, and further interpret the results within the hydrodynamic framework. Our results reveal that the nonlinear rotation generated by the density-dependent gauge potential induces an energy splitting between excitation modes with angular quantum numbers l and $-l$. Furthermore, we study the effect of the density-dependent gauge potential on the stability of multiply quantized vortices.

The paper is organized as follows. In Sec. 6.2, we introduce the mean-field Gross-Pitaevskii model incorporating the density-dependent gauge potential and describe

the numerical methods used. Section 6.3 presents an analysis of the ground-state properties and the corresponding Thomas-Fermi density profile. In Sec. 6.4, we examine how the density-dependent gauge potential affects the size of the giant vortex generated by various techniques, including absorption and repulsive pinning. Section 6.5 explores the influence of the density-dependent gauge potential on the collective excitation spectrum, while Section 6.6 discusses the stability of multiply quantized vortices in its presence. Finally, in Sec. 6.8, we summarize our findings.

6.2 Mean-field model

The Hamiltonian of a weakly interacting Bose-Einstein condensate, consisting of N two-level atoms coupled through a coherent light-matter interaction, can be written using the rotating wave approximation as [31]:

$$\hat{\mathcal{H}} = \left(\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{r}) \right) \otimes \mathbf{1} + \hat{\mathcal{H}}_{\text{int}} + \hat{\mathcal{U}}_{\text{MF}}, \quad (6.1)$$

where $\hat{\mathcal{U}}_{\text{MF}}$ is the light matter interaction and can be defined as [30]

$$\hat{\mathcal{U}}_{\text{MF}} = \frac{\hbar\Omega_r}{2} \begin{pmatrix} \cos\theta(\mathbf{r}) & e^{-i\phi(\mathbf{r})} \sin\theta(\mathbf{r}) \\ e^{i\phi(\mathbf{r})} \sin\theta(\mathbf{r}) & -\cos\theta(\mathbf{r}) \end{pmatrix}, \quad (6.2)$$

Here Ω_r represents the Rabi frequency, $\theta(\mathbf{r})$ denotes the mixing angle, and $\phi(\mathbf{r})$ indicates the phase of the incident laser beam. The momentum operator is given by $\hat{\mathbf{p}}$, and the trapping potential is defined as $V(\mathbf{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$, while $\mathbf{1}$ represents the 2×2 identity matrix in the dressed basis. The mean-field interactions are described by $\hat{\mathcal{H}}_{\text{int}} = (1/2)\text{diag}[\Delta_1, \Delta_2]$, where $\Delta_i = g_{ii}|\Psi_i|^2 + g_{ij}|\Psi_j|^2$ and $g_{ij} = 4\pi\hbar^2 a_{ij}/m$ with a_{ij} being the s-wave scattering lengths for collisions between atoms in internal states i and j ($i, j = 1, 2$) [30]. To construct the density-dependent gauge theory, we begin with the assumption that the mean field interaction is much weaker than the light-matter coupling. Based on this assumption, we can determine the dressed states using first-order perturbation theory [46].

$$|\Psi_{\pm}\rangle = |\pm\rangle \pm \frac{\Delta_d}{\hbar\Omega_r} |\mp\rangle \quad (6.3)$$

with $|\pm\rangle$ representing the unperturbed eigenstates of $\hat{\mathcal{U}}_{\text{MF}}$, and

$$\Delta_d = \sin(\theta/2) \cos(\theta/2) (\Delta_1 - \Delta_2) / 2$$

denotes the mean-field detuning. The effective Hamiltonian is obtained by projecting the atomic motion onto the dressed state basis $|\Psi_{\pm}\rangle$ and can be expressed as

$$\hat{H}_+ = \frac{(\hat{\mathbf{p}} - \mathbf{A}_+)^2}{2m} + W_+ + \frac{\hbar\Omega_r}{2} + \Delta_+ + V(\mathbf{r}), \quad (6.4)$$

where $\mathbf{A}_+ = i\hbar \langle \Psi_+ | \nabla \Psi_+ \rangle$, and $W_+ = (\hbar^2/2m) \times |\langle \Psi_+ | \nabla \Psi_- \rangle|^2$ are the vector and scalar potentials, respectively. $\Delta_+ = (\Delta_1 \cos^2(\theta/2) + \Delta_2 \sin^2(\theta/2)) / 2$ is the dressed mean-field detuning. By invoking the adiabatic approximation and projecting onto the dressed-state manifold $|\Psi_+\rangle$, the generalized Gross-Pitaevskii equation in three dimensions can be derived following the procedure explained in [166].

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) - \Omega_n(\mathbf{r}, t) \hat{L}_z + g_{\text{eff}} |\Psi|^2 \right] \Psi. \quad (6.5)$$

In deriving Eq. (6.5), we introduce a spatially dependent Rabi coupling of the form $\Omega_r = \kappa_0 r$, where r denotes the radial distance from the trap center and κ_0 is a constant. The laser field generating this coupling carries an azimuthal phase profile $\phi = l\varphi$, where l is the orbital angular momentum quantum number of the light and φ denotes the polar angle. The density-dependent rotation represented as $\Omega_n(\mathbf{r}, t)$ is defined in the following way [164]:

$$\Omega_n(\mathbf{r}, t) = \Omega + Cn(\mathbf{r}, t), \quad (6.6)$$

here $n(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|^2$ represents the condensate density while $\Omega = (\hbar l) / (4m) (\kappa_0 / \Delta)^2$ illustrates the controllable rigid-body rotation strength via light-matter interaction. Moreover, the nonlinear rotation strength can be expressed as $C = l(g_{11} - g_{22})\kappa_0^2 / (2m\Delta^2)$.

It should be noted that nonlinear rotation can be experimentally incorporated into the condensate using a laser that carries orbital angular momentum with $l = 1$. This approach is similar in spirit to the creation of spin-angular-momentum-coupled condensates using Laguerre-Gaussian laser beams [41–43]. To obtain the quasi-two dimensional Gross-Pitaevskii (GP) equation, we factorize the condensate wave function as $\Psi(\mathbf{r}, t) = \psi(x, y, t) \exp(-z^2/2\sigma_z^2) / (\pi\sigma_z^2)^{1/4}$, and assume that the condensate is confined in a highly anisotropic harmonic trap such that $\omega_z \gg \omega_{x,y}$. Under this con-

dition, the axial dynamics are frozen into the ground state of the harmonic oscillator. By integrating out the axial degree of freedom, we arrive at the following effective two-dimensional GP equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(x, y) - \Omega_n \hat{L}_z + g|\psi|^2 \right] \psi. \quad (6.7)$$

For notational simplicity in what follows we define $\Omega_n(\mathbf{r}, t) \equiv \Omega_n$, and $\psi(x, y, t) \equiv \psi$. The nonlinear coefficients appearing in Eq. (6.7) are defined as $g \rightarrow g_{\text{eff}}/\sqrt{2\pi}\sigma_z$ and $C \rightarrow C/\sqrt{2\pi}\sigma_z$. Here, $n(x, y, t) = |\psi(x, y, t)|^2$ denotes the quasi-two-dimensional density of the condensate. Introducing spatiotemporal scaling, we express time, length, and the order parameter in units of the transverse trapping frequency and oscillator length,

$$t \rightarrow \frac{t}{\omega_\perp}, (x, y) \rightarrow a_\perp(x, y), \text{ and } \psi \rightarrow \frac{\sqrt{N}}{a_\perp} \psi \quad (6.8)$$

where $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the harmonic oscillator length scale. With these rescalings, Eq. (6.7) can be cast into the following dimensionless form:

$$i \frac{\partial \psi}{\partial t} = \left[-\frac{\nabla^2}{2} + V(x, y) - \Omega_n \hat{L}_z + g|\psi|^2 \right] \psi. \quad (6.9)$$

In Eq. (6.9), the scaled parameters are defined as follows: $g = g_{2D}Nm/\hbar^2$ and $\Omega_n(x, y, t) = \Omega/\omega_\perp + \tilde{C}n(x, y, t)$ where $\tilde{C} = CN/m\hbar\sqrt{2\pi}\sigma_z$. The toroidal trap potential is given by $V(x, y) = \frac{1}{2}(r - r_0)^2$, where $r = \sqrt{x^2 + y^2}$ represents the radial coordinate in the xy -plane, and r_0 denotes the toroidal radius at which the potential attains its minimum.

6.3 Ground State and Thomas-Fermi Analysis

We start our analysis by examining the impact of the nonlinear strength $\tilde{C}$ on the ground state of the vortex lattice structure trapped in a toroidal potential. In Fig. 6.1, we present the condensate density and phase profile for different values of $\tilde{C}$ at $\Omega = 0.45$ and $g = 250$, with the radius of the toroidal trap potential fixed at $r_0 = 4$. At $\tilde{C} = 0$, we observe that the condensate shows a ring-shaped vortex lattice structure, featuring a central hole with a total quantum circulation of six, surrounded by a regular array of vortices distributed within the annular region see[6.1(a3)]. As $\tilde{C}$ continues to increase on the positive side, vortices that are initially situated in the annulus begin to move toward the center [6.1(a4)]. This movement leads to an increase in both

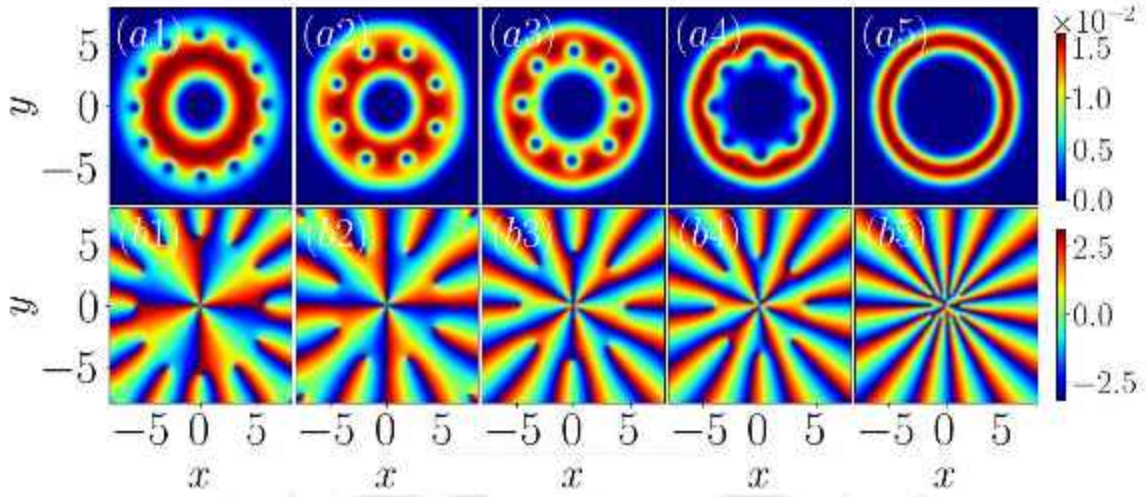


Figure 6.1: Pseudo color representations of the condensate density for $\tilde{C} = -40, -20, 0, 6$, and 12 [(a1)–(a5)] at $\Omega = 0.45$ and $g = 250$. Panels (b1)–(b5) display the corresponding phase profiles of the densities shown in (a1)–(a5).

the size of the hole and its circulation. With a further increase in $\tilde{C}$, the surrounding vortices completely fall into the central region, resulting in the formation of a giant vortex [6.1(a5)]. Conversely, when $\tilde{C}$ is negative, the behavior of the condensate contrasts with that observed for positive $\tilde{C}$. In this scenario, both the central hole region and quantum circulation gradually decrease. Meanwhile, the vortices in the annular region move away from the center, and for sufficiently large negative values of $\tilde{C}$, the vortices become arranged along the periphery of the condensate [see Figs. 6.1(a1–a2)]. Figure 6.1(b1–b5) presents the phase profiles corresponding to the density illustrated in 6.1(a1)–(a5). The phase profiles reveal that the multiply quantized central vortex circulation decreases when $\tilde{C} < 0$, and increases for $\tilde{C} > 0$. For instance, at $\tilde{C} = 0$, the circulation is 6, which falls to four for $\tilde{C} = -40$, and rises to 15 for $\tilde{C} = 12$.

After obtaining the ground state solution using the GP equation, we compute the Thomas-Fermi density distribution derived from the energy functional associated with Eq. 6.5 for further analysis. In the hydrodynamic picture, using the Madelung transformation $\psi = \sqrt{n} \exp(i\nu S)$, where n , S , and ν denote the condensate density, phase, and quantum circulation respectively, the energy functional can be expressed as [284]

$$E_T = \int dr n \left[\frac{\nu^2}{2r^2} - \frac{\Omega_n^2 r^2}{2} + \frac{gn}{2} + V(r) - \mu \right] \quad (6.10)$$

Here, $V(r) = \frac{1}{2}(r - r_0)^2$ is the toroidal trap potential and the term proportional to $\nu^2/2r^2$ represents the centrifugal contribution associated with the quantized circula-

tion in the central hole of the condensate.

To obtain the Thomas-Fermi distribution, we minimize the Eq. 6.10 with respect to the the condensate density, which leads us to the following expression:

$$V(r) - \frac{1}{2}r^2 \left(\Omega^2 + 2\Omega\tilde{C}n + \frac{3}{4}\tilde{C}^2n^2 - \frac{v^2}{r^2} \right) + gn = \mu \quad (6.11)$$

By solving the accompanying quadratic equation (Eq. 6.11), we can obtain the form of the general Thomas-Fermi density $n(r)$ given as

$$n(r) = -\frac{4}{3\tilde{C}^2r^2} \left\{ -[g - \Omega\tilde{C}r^2] + \sqrt{[g - \Omega\tilde{C}r^2]^2 + \frac{3\tilde{C}^2r^2}{2} \left[V(r) + \frac{1}{2} \left(\Omega^2 - \frac{v^2}{r^2} \right) - \mu \right]} \right\}. \quad (6.12)$$

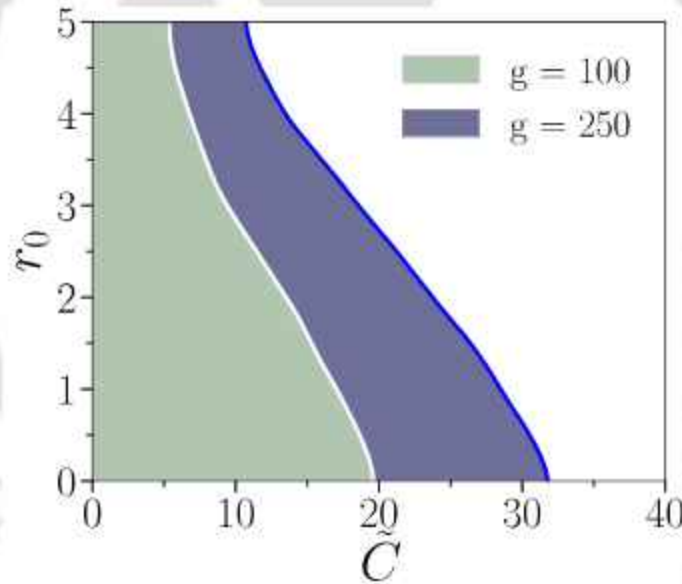


Figure 6.2: Stability diagram based on the Thomas-Fermi density distribution in the r_0 and $\tilde{C}$ parameter space for various nonlinear interaction strengths (g) at $\Omega = 0.45$.

In the presence of the ring potential, the Thomas-Fermi density defined by Eq. (6.12) possesses the non-trivial turning points (radii) given by

$$R_{\text{TF}}^{\pm} = \frac{r_0 \pm \sqrt{r_0^2 - (1 - \Omega^2)(r_0^2 + v^2 - \mu)}}{1 - \Omega^2}, \quad (6.13)$$

valid for $0 \leq \Omega < 1$, and the pair of roots given by Eq. (6.13) describe the inner radius

($R_{\text{TF}}^- < r_0$) for the negative branch, while the outer radius ($R_{\text{TF}}^+ > r_0$) is obtained from the positive branch.

Equation 6.12 demonstrates that the Thomas-Fermi density is influenced significantly by the strength of the nonlinear rotation $\tilde{C}$ and the radius of the toroidal potential r_0 . Specifically, Eq. 6.11 illustrates that the centrifugal force increases upon increasing $\tilde{C}$, which works against confinement. Therefore, depending on the values of r_0 and $\tilde{C}$, the condensate may exceed the confinement. To determine the admissible solution region and find the maximum value of $\tilde{C}$ that maintains a stable condensate, we impose the normalization condition

$$2\pi \int_{r_{\text{in}}}^{r_{\text{out}}} r n(r) dr = 1, \quad (6.14)$$

where the inner and outer radii r_{in} and r_{out} are the points at which the condensate density vanishes ($n(r) = 0$). For a fixed density-dependent coupling $\tilde{C}$, we scan integer circulation numbers ν up to a guess maximum ν_m , and for each ν , we vary the chemical potential μ within the range $\mu_{\text{min}} \leq \mu \leq \mu_{\text{max}}$ to obtain the value μ_g that satisfies the normalization condition. When a valid pair (ν, μ_g) is found, the corresponding energy $E(\nu, \mu_g)$ (see Eq. 6.10) is computed. If multiple admissible pairs exist for the same $\tilde{C}$, the relevant solution is taken to be the one that minimizes the energy.

Following the numerical procedure outlined above, in Fig. 6.2, we present the allowed numerical solutions in the $(r_0, \tilde{C})$ plane. The two shaded regions represent the allowed solutions for the interatomic interaction strengths $g = 100$ and 250, respectively. The white and blue lines indicate the maximum values of $\tilde{C}$ for which a solution can exist within the Thomas-Fermi density framework. We find that the upper bound of $\tilde{C}$ decreases with increasing r_0 , since a larger r_0 reduces the strength of the effective trapping potential [see Eq. (6.11)], thereby weakening the confinement.

To further characterize this structural transition, we calculate conserved quantities in the form of the total energy and the angular momentum as

$$E_T = \int \left(\frac{1}{2} |\nabla \psi|^2 + V + \frac{g}{2} |\psi|^4 - \Omega_n \psi^* \hat{L}_z \psi \right) dx dy, \quad (6.15a)$$

$$\langle L_z \rangle = -i \int \psi^* \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi dx dy. \quad (6.15b)$$

We then examine their variation as a function of the nonlinear rotation strength $\tilde{C}$,

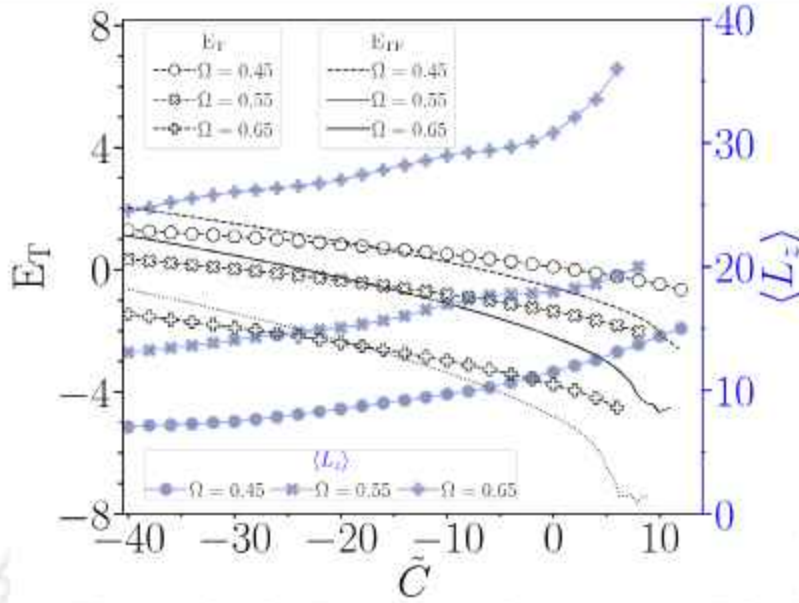


Figure 6.3: The variation of total energy E_T (left), and angular momentum $\langle L_z \rangle$ (right) as a function of non-linear rotation strength $\tilde{C}$, for rotational frequencies $\Omega = 0.45, 0.55$, and 0.65 . The black dashed, dotted, and dash-dotted curves show the energy prediction from the Thomas-Fermi density distribution. The total energy decreases as $\tilde{C}$ increases, while the angular momentum increases with $\tilde{C}$.

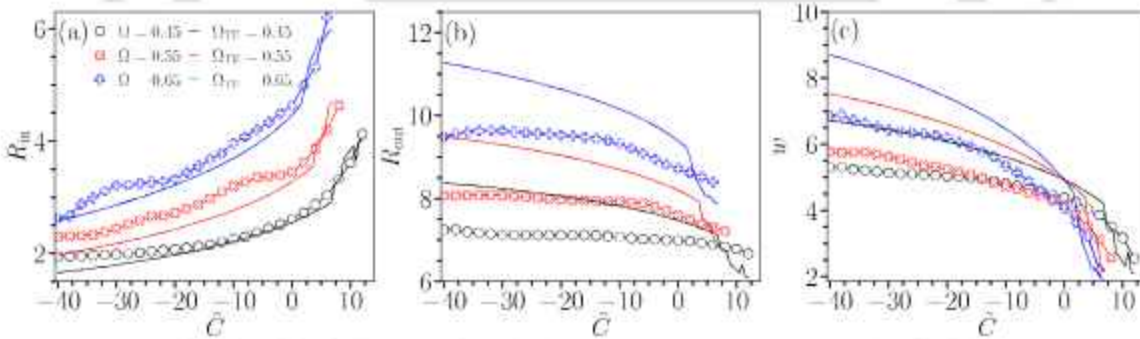


Figure 6.4: Variation of (a) inner radius R_{in} , (b) outer radius R_{out} , and (c) width $w = R_{out} - R_{in}$ of the condensate as functions of $\tilde{C}$ for $\Omega = 0.45, 0.55$, and 0.65 at $g = 250$. The black, red, and blue curves correspond to the Thomas-Fermi predictions. Increasing $\tilde{C}$ from negative to positive values reduces both R_{out} , and w , while R_{in} increases, indicating a narrowing of the annular condensate for $\tilde{C} > 0$.

as shown in Fig. 6.3. We find that as $\tilde{C}$ increases from negative to positive the energy of the system decreases, indicating a transition from a ring-shaped vortex lattice to a giant vortex, which is energetically more favorable than the ring structure. A similar decrease in energy associated with the formation of a giant vortex has been reported in the literature for a Mexican hat potential, where the structural phase transition depends on the angular velocity [272]. On the other hand depending on $\tilde{C}$, the $\langle L_z \rangle$ increases monotonically due to the modulation of the background density of the con-

condensate and the dynamics of the vortices. In Fig. 6.4 we plot the inner R_{in} and outer radii R_{out} of the condensate together with its width $w = R_{\text{out}} - R_{\text{in}}$ as a function of $\tilde{C}$ for different values of Ω . Note that the inner radius decreases as $\tilde{C}$ increases from negative to positive values, while the outer radius decreases with $\tilde{C}$. Consequently the condensate becomes narrower for $\tilde{C} > 0$ (the width w decreases) and broader for $\tilde{C} < 0$ (the width w increases).

6.4 Giant Vortex Formation: absorptive and repulsive technique

In addition to toroidal confinement, a giant vortex structure can also be generated in a rapidly rotating condensate trapped in a harmonic potential via two alternative methods: atom removal and a repulsive pinning potential [285, 34, 286]. For the case of atom-removal, a resonant laser beam is focused on the condensate center to create a density depletion. The resulting mean field pressure gradient attempts to drive atoms toward the hole, but the centrifugal potential associated with rotation deflects the inflowing atoms and instead generates a circulating flow around the depletion, which gives rise to a giant vortex and the toroidal geometry. On the other hand, a repulsive (pinning) potential at the condensate center pushes individual vortex cores outward and forced them to arrange around the barrier and thereby create ring-shaped density. In the following section, we incorporate the laser-induced potential into the GPE (Eq. 5.8) and examine the impact of gauge-induced nonlinear rotation on the formation of a giant vortex. To consider the effect of laser beam, we introduce a Gaussian perturbation in the trap, so the effective potential becomes

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + V_{\text{laser}}, \quad (6.16a)$$

$$V_{\text{laser}} = V_0 \exp \left[-\frac{(x - x_0)^2 + (y - y_0)^2}{(r_0/2)^2} \right] \quad (6.16b)$$

where V_{laser} is the Gaussian perturbation with barrier height V_0 , beam center (x_0, y_0) and width r_0 . The generalized GP equation is then evolved in real-time, using the stationary state obtained at $\Omega = 0.9$ and $g = 1000$ as the initial state. In the present work, we consider $V_0 = -100i$ to implement the atom removal potential with the other parameters $r_0 = 9$, and $(x_0 = 0, y_0 = 0)$. In the simulation the potential is applied over a

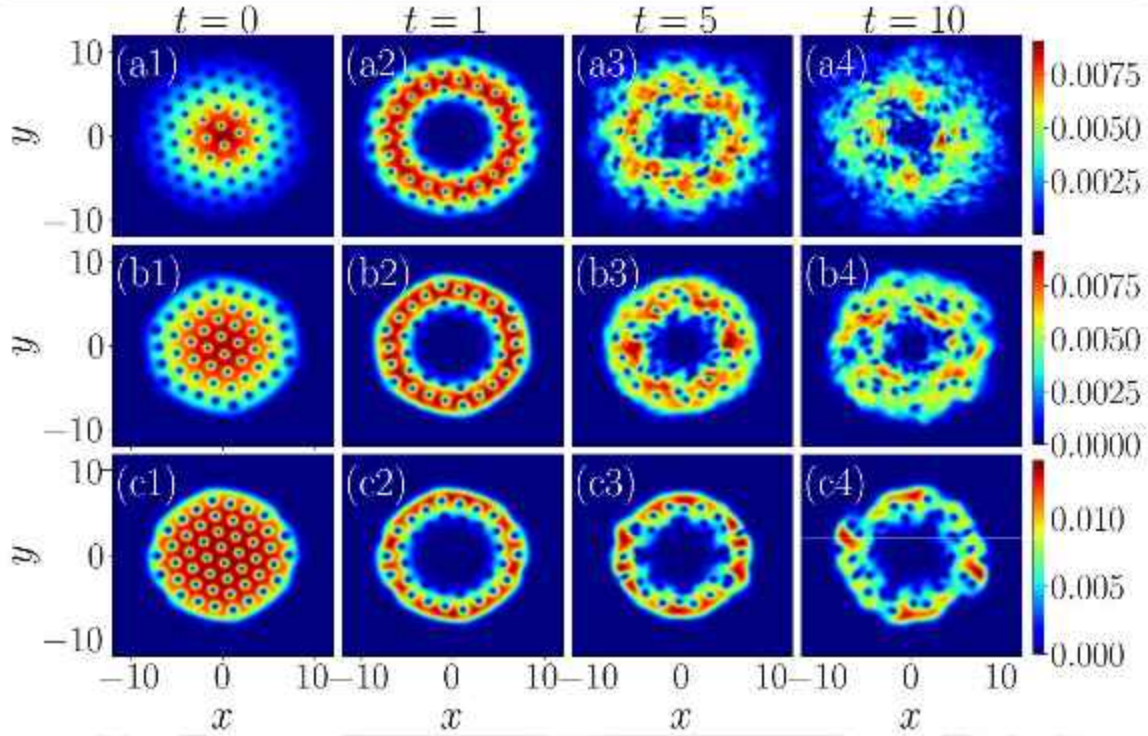


Figure 6.5: Real-space condensate density at different times illustrating the formation of a giant vortex at the centre, induced by an atom-removal Gaussian perturbation applied to the ground state for different values of $\tilde{C}$ at $\Omega = 0.9$ and $g = 1000$. (a1)–(a5): for fixed $\tilde{C} = -30$ and $t = (0, 1, 5, 10)$; (b1)–(b5): for fixed $\tilde{C} = 0$ and $t = (0, 1, 5, 10)$; (c1)–(c5): for fixed $\tilde{C} = 16$ and $t = (0, 1, 5, 10)$.

dimensionless time interval of $t = [0, 0.05]$.

To systematically examine the effect of the density-dependent gauge potential on giant-vortex formation using the atom removal technique, in Fig. 6.5, we demonstrate the snapshots of condensate density at $t = 0, 1, 5$, and 10 , where the first [(a1)–(a5)], second [(b1)–(b5)], and third [(c1)–(c5)] rows correspond to $\tilde{C} = -30, 0$, and 16 , respectively. We find that at $t = 0$ the condensate forms a hexagonal vortex-lattice structure for $\tilde{C} = 0$, and for $\tilde{C} = -30$ the vortices are pushed to the periphery of the condensate, whereas for $\tilde{C} = 16$ they concentrate near the center of the trap. This behavior is associated with $\tilde{C}$ can be attributed to the Magnus force and has been studied previously in the existing literature [63, 222]. Interestingly, at $t = 1$ the condensate density becomes strongly suppressed at the centre, leading to the formation of a giant vortex irrespective of the nonlinear-rotation strength $\tilde{C}$. We further observe the inner radius of the toroidal density expands for positive values of $\tilde{C}$ and contracts for negative $\tilde{C}$; correspondingly, the annular condensate becomes narrower for positive $\tilde{C}$. Interestingly, at $t = 1$, the central density becomes strongly suppressed, leading to the formation

of a giant vortex irrespective of the nonlinear rotation strength $\tilde{C}$. We further observe that the inner radius of the toroidal density increases for positive $\tilde{C}$ and decreases for negative $\tilde{C}$; correspondingly, the annular condensate becomes narrower for larger $\tilde{C}$. Time snapshots also show that the hole area oscillates with time. To characterize these oscillations more clearly, in Fig. 6.6(d) we show the variation of the core area with time for $\tilde{C} = -30, 0$, and 16 . The mean core area is largest for $\tilde{C} = 16$ and smallest for $\tilde{C} = -30$. This behaviour arises from the interplay between the mean-field pressure gradient, which tends to refill the density hole, and the centrifugal potential, which acts outward. The difference in the mean core area can be explained from the expression of the centrifugal potential (Eq. 6.11), which increases for positive $\tilde{C}$ and decreases for negative $\tilde{C}$. Although the mean core area depends on $\tilde{C}$, the oscillation frequency

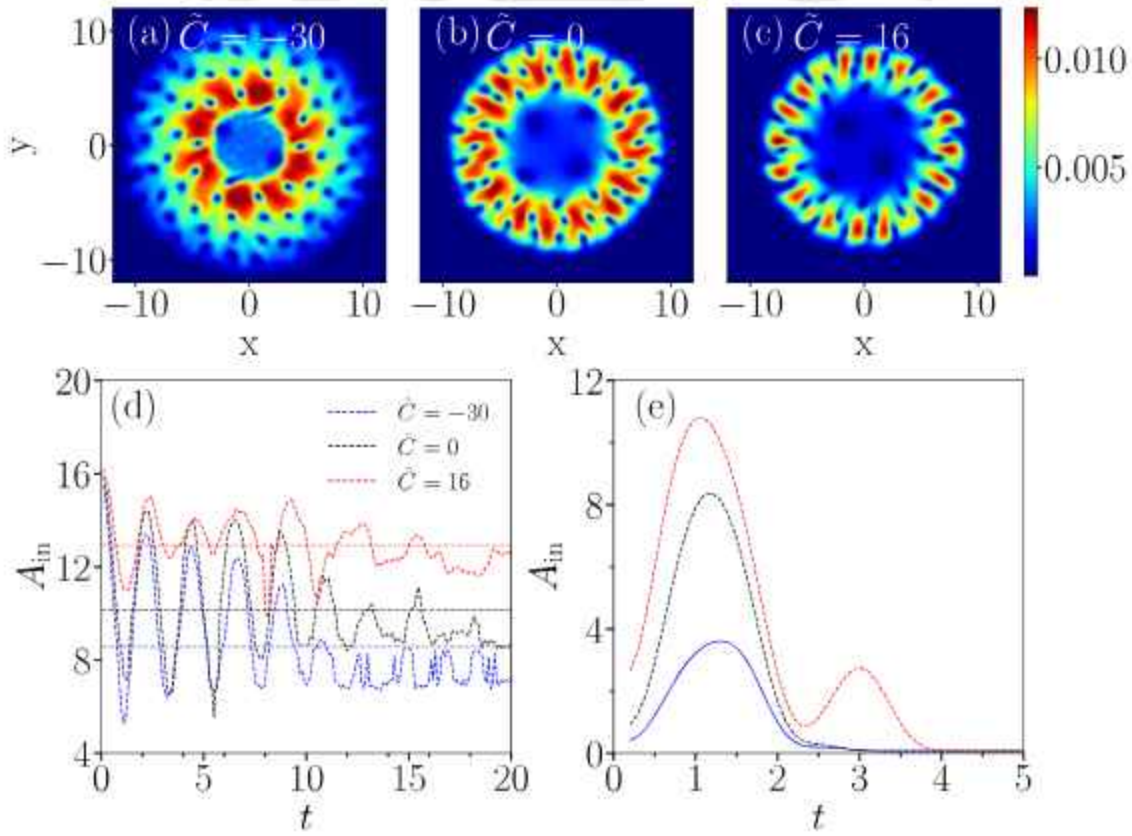


Figure 6.6: (a)-(c) show the Pseudo color presentation of the condensate density at $t = 1$ for three different values of $\tilde{C}$, illustrating the emergence of a giant vortex at the center due to the repulsive Gaussian perturbation. (d) The time evolution of the core area A_{in} for different $\tilde{C}$, where the dashed horizontal lines indicate the mean value of the core area oscillations. (e) The time evolution of the core area for different values of $\tilde{C}$. The core area attains its maximum for $\tilde{C} = 16$ and minimum for $\tilde{C} = -30$.

shows little variation. From the first two minima of the time series of the core area,

the oscillation frequencies are found to be $\omega_{A_{in}} = 2.99, 2.85$, and 2.82 for $\tilde{C} = -30, 0$, and 16 , respectively. The weak dependence of $\omega_{A_{in}}$ on $\tilde{C}$ can be understood from the contribution of the Magnus force given by

$$\mathbf{M} = \kappa \times \mathbf{v}, \quad (6.17)$$

where $\mathbf{v} = -y(\Omega + Cn)\hat{\mathbf{e}}_x + x(\Omega + Cn)\hat{\mathbf{e}}_y$, and κ is circulation of the vortex. The Magnus force acts radially inward and becomes stronger for $\tilde{C} > 0$, while it weakens for $\tilde{C} < 0$, leading to differences in the core-area amplitude but leaving the oscillation frequency nearly unchanged.

Following the analysis of atom-removal dynamics, we now extend our study to investigate the effect of a density-dependent gauge potential on the formation of a giant vortex in the presence of a repulsive Gaussian potential. In this study, we consider a repulsive Gaussian potential with a barrier height of $V_0 = 100$ and apply it for the same interval as the atom removal. All other parameters remain consistent with the previous analysis. In Fig. 6.6(a)-(c), we present the condensate density at $t = 1$ for values of $\tilde{C} = -30, 0$, and 16 . The density profile shows that, similar to the previous case, a giant vortex forms at the center, with the radius of this vortex being larger for $\tilde{C} > 0$ compared to when $\tilde{C} < 0$. The stripes observed in the density depend on the amplitude of the laser beam and have been studied previously in the literature [286]. Next, in Fig. 6.6(e), we present the time evolution of the giant vortex core area. The time series of the core area supports the findings from the density plots: the amplitude of core area is larger when $\tilde{C} > 0$ compared to when $\tilde{C} < 0$. After a certain period, the core area diminishes to zero as atoms from the outer region refill the depleted area, leading to a reduction in the size of the giant vortex.

However, no oscillation has been observed in the core area for the repulsive potential. The reason for this is that the angular momentum of a rotating lattice with radius R is proportional to $L \propto \Omega R^2$. When atoms are removed from the center, the angular momentum of the atoms within the annular region increases, resulting in a new lattice with fewer atoms and a higher angular velocity Ω . As atoms in the outer region flow inward to refill the depletion under the action of the mean-field pressure, they encounter an effective barrier created by this increased angular momentum: the condensate cannot accelerate its rotation as fast as the hole fills and leads to an oscillation of the core area.

In contrast, when a repulsive potential is applied, the total angular momentum of the condensate is conserved and no net increase of angular momentum per particle is

induced in the annulus. Although the repulsive potential expels atoms from the centre and creates a temporary depletion, the system can relax and the outer atoms are able to refill the hole once the perturbation is removed. Because there is no abrupt change in the angular-momentum distribution and core area oscillations are not observed in this case.

6.5 Collective excitation spectrum

Having established the dependence of the ground state configuration on density dependent gauge potential notably the change in the the size of the central hole due to nonlinear rotation, in this section we study the collective excitation spectrum. The excitations can be examined by using two different approach. The first is the Bogoliubov–de Gennes (BdG) framework, which performs a microscopic linear stability analysis about the stationary condensate solution and yields the complete spectrum of quasiparticle modes. The second is the hydrodynamic approach, derived from the Madelung transformation of the Gross–Pitaevskii equation, and offers a description based on density and phase fluctuations. We have applied both methods to obtain the excitation spectrum while varying the strength of the nonlinear rotation.

In the BdG approach, we perturb the ground state ψ_0 by a small fluctuation $\delta\psi$, so that the condensate wave function is written as

$$\psi(x, y, t) = e^{-i\mu t} [\psi_0(x, y) + \delta\psi(x, y, t)], \quad (6.18)$$

where μ is the chemical potential of the ground state. The fluctuation is expanded as

$$\delta\psi(x, y, t) = u(x, y)e^{i\omega t} - v^*(x, y)e^{-i\omega t}, \quad (6.19)$$

with $u(x, y)$ and $v(x, y)$ denoting the Bogoliubov mode amplitudes and ω the corresponding excitation frequency. The quasiparticle amplitudes satisfy the normalization condition

$$\int (|u(x, y)|^2 - |v(x, y)|^2) dx dy = 1. \quad (6.20)$$

Substituting Eqs. (6.18) into the time-dependent GP Eq. 6.9 and equating the terms

corresponds to $e^{i\omega t}$ and $e^{-i\omega t}$, we obtain the coupled BdG equations

$$\begin{aligned} \omega u = & \left[\mathcal{L} - \Omega L_z - \tilde{C}(|\psi_0|^2 L_z + \psi_0^* L_z \psi_0) \right] u \\ & - [g\psi_0^2 - \tilde{C}\psi_0 L_z \psi_0] v, \end{aligned} \quad (6.21a)$$

$$\begin{aligned} -\omega v = & \left[\mathcal{L} + \Omega L_z + \tilde{C}(|\psi_0|^2 L_z + \psi_0 L_z \psi_0^*) \right] v \\ & - [g\psi_0^{*2} + \tilde{C}\psi_0^* L_z \psi_0^*] u, \end{aligned} \quad (6.21b)$$

in Eqs. (6.21) we define the function

$$\mathcal{L} = -\frac{1}{2}\nabla^2 + V + 2g|\psi_0|^2 - \mu. \quad (6.22)$$

The BdG equations 6.21 are solved to determine the eigenfrequencies ω and the eigenmodes u, v .

Another approach to finding the frequencies of collective modes is through the hydrodynamic method. In this approach, the time-dependent GP equation (Eq. 6.9) is recast in terms of the condensate density and phase via the Madelung transformation. This technique has been widely used to analyze the collective excitations of annular condensates [213, 272]. Using a similar method, we derive the equations for the density n and phase S that are suitable for our system. We then linearize these equations around the equilibrium state by introducing small fluctuations in phase δS , and density δn , which leads to the following set of linearized hydrodynamic equations.

$$\begin{aligned} \frac{\partial}{\partial t} \delta n + \left[\frac{v}{2r^2} - (\Omega + \tilde{C}n_0) \right] \frac{\partial \delta n}{\partial \phi} - \tilde{C} \delta n \frac{\partial \delta n}{\partial \phi} \\ + n_0 \nabla^2 \delta S = 0 \end{aligned} \quad (6.23a)$$

$$\begin{aligned} \frac{\partial}{\partial t} \delta S + \left[\frac{v}{2r^2} - (\Omega + \tilde{C}n_0) \right] \frac{\partial \delta S}{\partial \phi} + g \delta n \\ - \frac{\tilde{C}v}{r} \delta n = 0 \end{aligned} \quad (6.23b)$$

where n_0 is the equilibrium Thomas-Fermi density, and can be obtained from Eq. 6.12.

Then by considering the density and phase modulations of the form $\delta n, \delta S \propto \exp(i(l\phi - \omega t))$,

we derive the following set of equations.

$$\left(g\delta n - \frac{\tilde{C}v}{r}\right) - i\delta S \left[\omega - \frac{lv}{r^2} + l(\Omega + \tilde{C}n_0)\right] = 0 \quad (6.24a)$$

$$i\delta n \left[-\omega + \left[\frac{v}{r^2} - l(\Omega + \tilde{C}n_0)\right]\right] + \frac{\partial}{r\partial r} \left(n_0 r \frac{\partial}{\partial r} \delta S\right) - \frac{l^2 n_0}{r^2} \delta S = 0 \quad (6.24b)$$

After substituting Eq. 6.24(a) into the Eq. 6.24(b), we obtain the following equation:

$$\delta S \left[\left(\omega - \frac{lv}{r^2} + l[\Omega + \tilde{C}n_0]\right)^2 - \left(g - \frac{\tilde{C}v}{r}\right) \frac{n_0 l^2}{r^2} \right] - \left(g - \frac{\tilde{C}v}{r}\right) \frac{\partial}{r\partial r} \left(n_0 r \frac{\partial}{\partial r} \delta S\right) = 0 \quad (6.25)$$

We solve Eq. 6.25 numerically to obtain the collective mode frequencies, imposing zero-flux boundary conditions and discretizing the annular domain $[R_{in}, R_{out}]$ using N interior grid points. We elaborate the numerical technique in the following section.

6.6 Numerical technique to solve the hydrodynamic equation.

In this section, we outline the numerical method used to solve the hydrodynamic equation and compute the frequencies of the collective excitations. We solve Eq. 6.25 by rewriting it as a quadratic eigenvalue problem of the form

$$(\lambda^2 M_2 + \lambda M_1 + M_0) f = 0, \quad (6.26)$$

where the matrices are defined as

$$M_2 = I, \quad (6.27)$$

$$M_1 = 2 \text{diag}[B(r)], \quad (6.28)$$

$$M_0 = L + \text{diag} \left[B(r)^2 - \left(g - \frac{Cv}{r}\right) \frac{l^2 n_0(r)}{r^2} \right], \quad (6.29)$$

and

$$B(r) = l(\Omega + C n_0(r)) - \frac{lv}{r^2}. \quad (6.30)$$

The linearized hydrodynamic operator in M_0 has the form

$$\begin{aligned}\mathcal{L}f(r) &= -\frac{1}{r} \left(g - \frac{Cv}{r} \right) \frac{d}{dr} \left[p(r) \frac{df}{dr} \right], \\ p(r) &= n_0(r) r.\end{aligned}\quad (6.31)$$

We employ a conservative finite-difference scheme based on midpoint fluxes to discretize the operator

$$p_{i+\frac{1}{2}} = p(r_i + h/2), \quad (6.32)$$

and denote the flux evaluated at the midpoint between r_i and r_{i+1} , with $h = (R_{\text{out}} - R_{\text{in}})/(N+1)$. The discrete operator is then given by

$$(\mathcal{L}f)_i = -\frac{1}{h^2 r_i} \left[p_{i+\frac{1}{2}}(f_{i+1} - f_i) - p_{i-\frac{1}{2}}(f_i - f_{i-1}) \right]. \quad (6.33)$$

This yields a symmetric tridiagonal matrix L with elements

$$L_{i,i-1} = -\frac{p_{i-\frac{1}{2}}}{h^2 r_i}, \quad (6.34)$$

$$L_{i,i} = \frac{p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}}}{h^2 r_i}, \quad (6.35)$$

$$L_{i,i+1} = -\frac{p_{i+\frac{1}{2}}}{h^2 r_i}. \quad (6.36)$$

At the inner and outer radii R_{in} and R_{out} we consider vanishing flux boundary conditions.

In Fig. 6.7, we present the variation of the frequencies of the low-lying excitation modes as the nonlinear rotation strength $\tilde{C}$ is varied, while keeping the rotation frequency fixed at $\Omega = 0.45$ and the interaction strength $g = 250$. The angular quantum number l associated with each mode is determined by constructing the complex field

$$n' = \psi^*(x, y) u(x, y) - \psi(x, y) v(x, y), \quad (6.37)$$

as proposed in [232]. The phase S of n' is then calculated, and l is found by evaluating the phase winding around a closed contour C that encloses the condensate at the Thomas-Fermi radius

$$l = \frac{1}{2\pi} \oint_C \nabla S \cdot d\mathbf{r}, \quad (6.38)$$

where ∇S represents the spatial gradient of the phase S . The dashed lines repre-

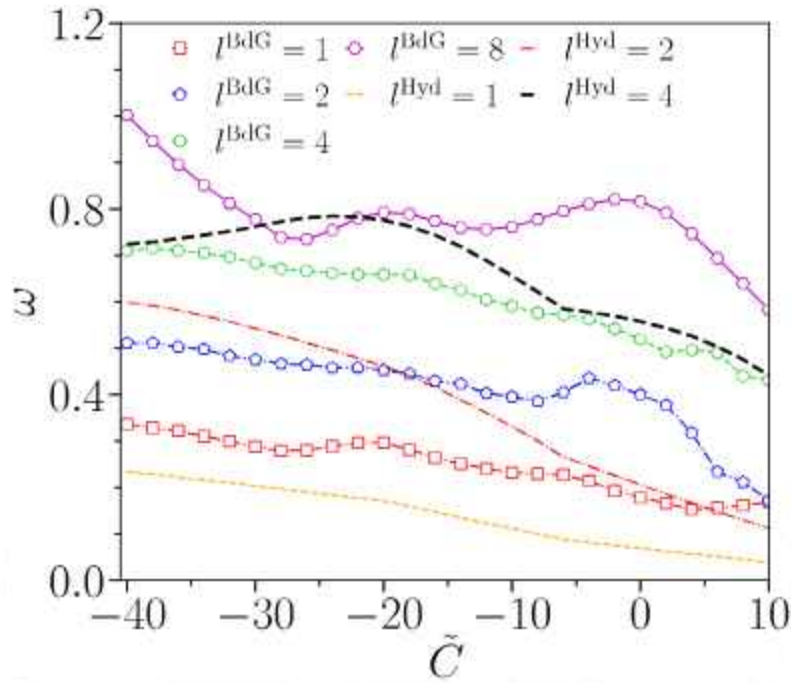


Figure 6.7: The collective excitation spectrum for different angular quantum numbers l as a function of nonlinear rotation strength $\tilde{C}$ at $\Omega = 0.45$ and $g = 250$. The symbols represent the frequencies obtained from the BdG equation, while the dashed lines illustrate the frequencies of the normal modes obtained from hydrodynamic theory.

sent the mode frequencies obtained through the hydrodynamic approach (by solving Eq. 6.25), while the symbols indicate the results from the BdG method. We observe that the dipole mode ($l = 1$) frequency decreases with increasing nonlinear rotation strength $\tilde{C}$. This behavior contradicts the Kohn theorem, which asserts that the dipole mode frequency should remain unaffected by interaction strength. However, such violations are not unique, and similar observations have been reported in other systems, such as spin-orbit coupled condensates [233, 235]. In our case, the reduction in the dipole frequency arises from the coupling between density and angular momentum, which substantially modifies the effective centrifugal force. Moreover, higher-order modes also show a decreasing trend with increasing $\tilde{C}$. This behavior indicates an underlying structural transition of the condensate, which shifts from a ring-shaped configuration to a giant vortex state as the nonlinear rotation becomes stronger.

The differences between the excitation frequencies obtained from the BdG theory and those predicted by the hydrodynamic approach arise from the underlying assumptions of the hydrodynamic description. In the hydrodynamic (Thomas-Fermi) limit, the quantum-pressure term is neglected, which implies that density variations are as-

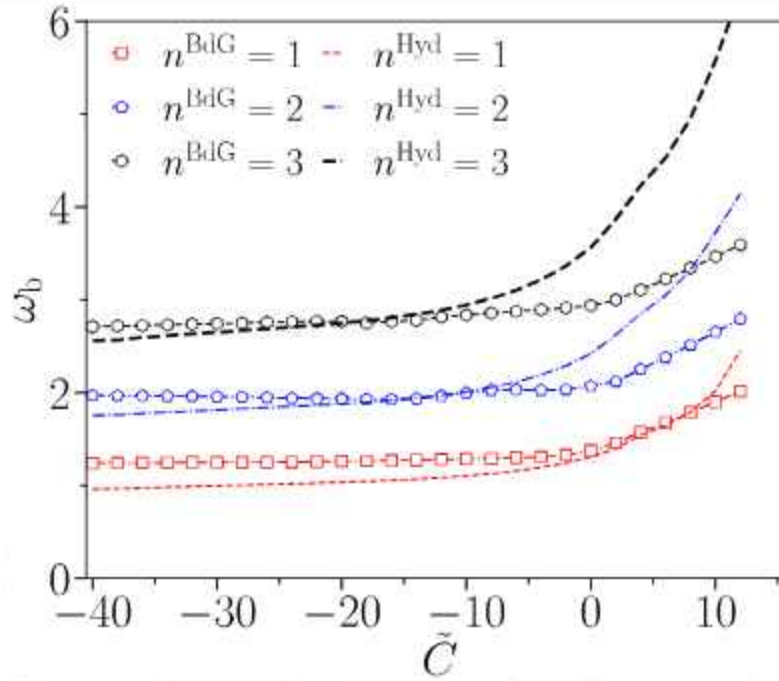


Figure 6.8: Variation of the breathing mode frequency ($l = 0$) with nonlinear rotation strength $\tilde{C}$ for three lowest radial quantum numbers $n = 1, 2$, and 3 . The markers and dashed lines represent the breathing mode frequencies obtained from the BdG and hydrodynamic approaches, respectively. The deviation of ω_b beyond a critical value of $\tilde{C}$ indicates a transition from a ring shaped vortex state to a giant vortex state.

sumed to be smooth and to occur over length scales much larger than the healing length. In contrast, the BdG theory includes the complete kinetic-energy contribution from the GP equation, allowing it to account for density variations on length scales that are comparable to the healing length. Additional discrepancies may also arise from the boundary conditions employed in the hydrodynamic model. In our analysis, a no-flux condition is imposed at both the inner and outer radii, which constrains the phase gradient at the boundaries. Such simplified boundary conditions do not fully capture the microscopic structure of the condensate near the edges, where density variations are more pronounced an effect that is included within the BdG framework. This type of mismatch between the excitation frequencies obtained from the BdG and hydrodynamic approaches has also been noted in [100].

In Fig. 6.8, we present the variations of the three lowest breathing modes ($l = 0$) corresponding to radial quantum numbers $n = 1, 2$, and 3 as a function of the nonlinear rotation strength $\tilde{C}$. We employ both the BdG and hydrodynamic approaches, and find that the behavior is consistent across both methods. Our observations indicate that the breathing frequency remains nearly constant over a broad range of negative

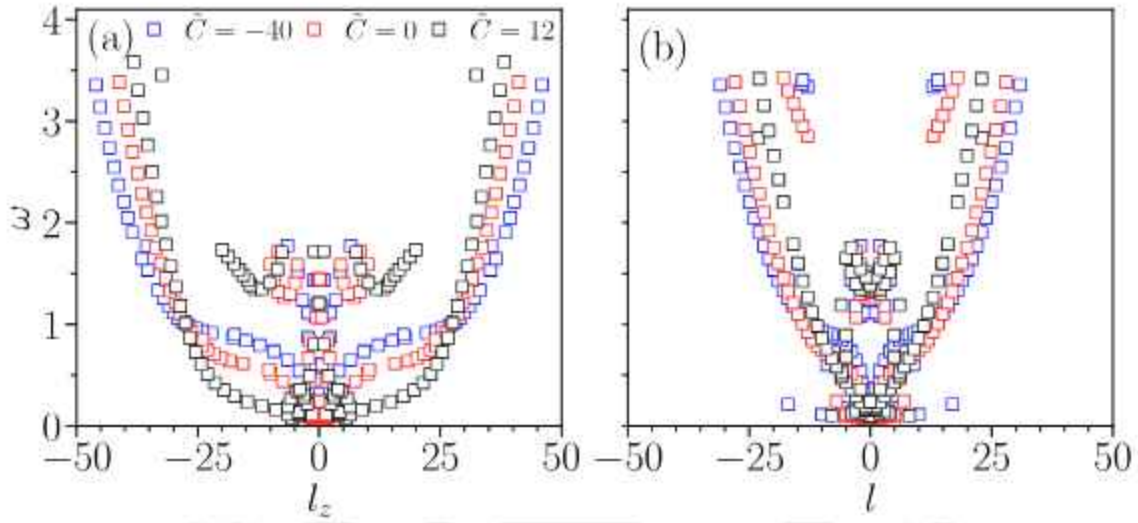


Figure 6.9: Excitation spectrum as a function of (a) angular momentum, and (b) angular quantum number for $\tilde{C} = -40, 0$, and 12 .

values of $\tilde{C}$. As $\tilde{C}$ becomes positive, however, the frequencies increase monotonically. This increase suggests a structural transformation from a ring structure to a giant vortex, accompanied by a decrease in the width of the condensate as $\tilde{C}$ increases. The observed increase in the l_b is due to the structural transformation of the condensate from a ring shaped vortex lattice to a giant vortex state. In the giant vortex regime, the condensate becomes more tightly confined in the radial direction, which results in a reduced annular width. This narrowness enhances the density gradients within the annulus, subsequently strengthening the effective restoring force against radial compression. The increased stiffness of the condensate results in a higher breathing mode frequency.

In Fig. 6.9(a) and (b), we illustrate the variation of the excitation frequency ω as a function of angular momentum l_z and the angular quantum number l , respectively for three different values of $\tilde{C}$: $\tilde{C} = -40, 0$, and 12 . The angular momentum of an excitation mode with respect to the rotating condensate is defined by

$$l_z = \frac{(l_u - l_\psi)N_u + (l_v + l_\psi)N_v}{N_u + N_v}, \quad (6.39)$$

where

$$l_\alpha = -i\hbar \frac{\int dx dy \alpha^* (x \frac{\partial \alpha}{\partial y} - y \frac{\partial \alpha}{\partial x})}{\int dx dy \alpha^* \alpha}, \quad (6.40)$$

with α being one of the wave functions $u(x, y)$, $v(x, y)$, or $\psi(x, y)$, and

$$N_u = \int dx dy |u(x, y)|^2, \quad N_v = \int dx dy |v(x, y)|^2. \quad (6.41)$$

The spectrum exhibits symmetry around $l = 0$, but the splitting between the $-l$ and l branches depends on the nonlinear rotation parameter $\tilde{C}$. For instance, at $\tilde{C} = -40$, this splitting is larger, whereas for $\tilde{C} = 12$ it is reduced. Additionally, in Fig. 6.9(a), a shallow dip appears in the excitation spectrum for small l_z and the minimum of this dip decreases further for $\tilde{C} > 0$, indicating a softening of the corresponding excitation modes. In contrast, for negative $\tilde{C}$, the modes become stiffer and the dip is less pronounced. This behaviour arises because increasing $\tilde{C} > 0$ enhances the effective Magnus force, which drives the vortex distribution from a ring-like arrangement toward a giant-vortex configuration. In this giant-vortex state, vortex displacement modes vanish, making surface modes the primary low-energy excitations, whose frequencies are impacted by the nonlinear rotation.

6.7 Stability of multiply quantized vortices

In this section, we investigate the effect of the density-dependent gauge potential on the stability of multiply quantized vortices (MQVs) for the condensates trapped in a harmonic potential. In general, MQVs possess higher energy compared to any configurations that consist of arrays of singly quantized vortices and therefore tend to split into multiple singly quantized vortices, as physical systems evolve toward lower energy states in thermal equilibrium. It has been shown previously that MQVs can be stabilized by introducing a pinning potential, with the stability controlled by the radius and the strength of the pinning potential [84].

Our objective is to examine the stability of MQVs under gauge potential induced nonlinear rotation by analyzing both the free energy landscape and the associated BdG excitation spectra. In Fig. 6.10, we present the variation of free energies of MQVs as a function of the rotational frequency Ω for different nonlinear rotation strengths $\tilde{C} = -30, 0$, and 40 . From Fig. 6.10(b), we observe that for $\tilde{C} = 0$, the vortex free state ($m = 0$) has the lowest free energy. As Ω is increased, there exists a critical frequency Ω_c where the energy of a singly quantized vortex intersects with the energy of the vortex free state. At this point, the energy of the single vortex state becomes equal to that of the vortex free state. For $\Omega > \Omega_c$, the energy of higher order ($m > 1$) vortex

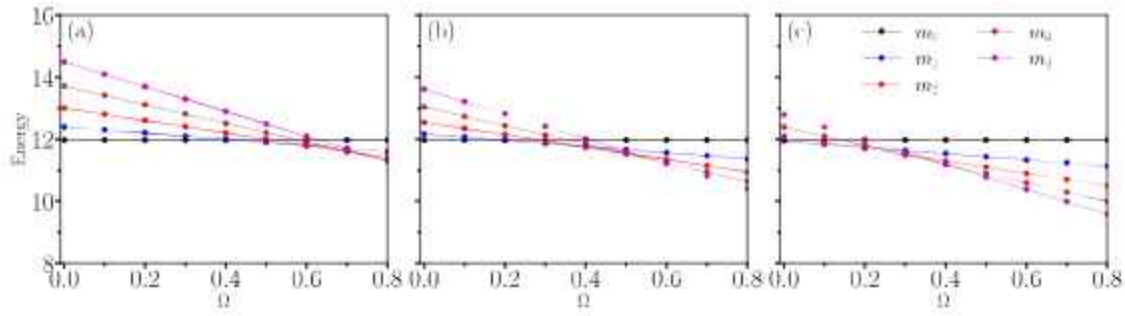


Figure 6.10: The free energy as a function of the rotation frequency Ω is shown for nonlinear rotation strengths (a) $\tilde{C} = -30$, (b) $\tilde{C} = 0$, and (c) $\tilde{C} = 40$. In the absence of nonlinear rotation ($\tilde{C} = 0$), the intersection between the free energies of the vortex-free state and the singly quantized vortex occurs at the critical rotation frequency $\Omega_c = 0.21$. For negative nonlinear rotation strength ($\tilde{C} = -30$), this intersection is shifted to higher values of Ω , whereas for positive nonlinear rotation strength ($\tilde{C} = 40$), it is shifted toward lower rotation frequencies.

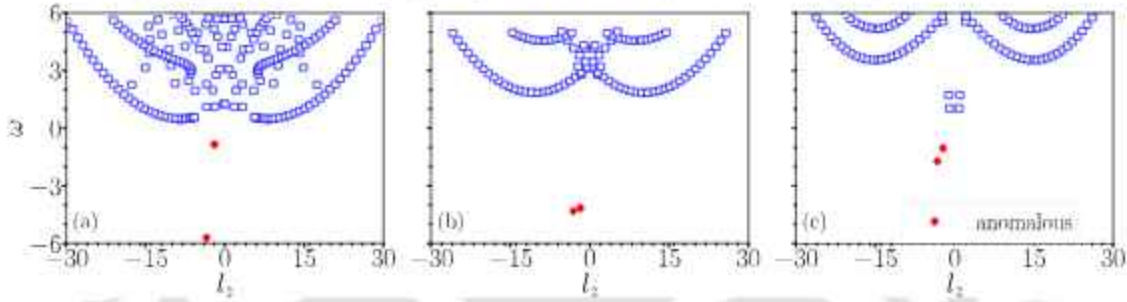


Figure 6.11: Quasiparticle excitation energies (ω) as a function of the quasiparticle angular momentum (l_z) for a triply quantized vortex ($m = 3$) at a rotation frequency $\Omega = 0.3$, illustrated for (a) $\tilde{C} = -30$, (b) $\tilde{C} = 0$, and (c) $\tilde{C} = 60$. The red marker denotes the anomalous mode. For negative nonlinear rotation strength ($\tilde{C} = -40$), the anomalous mode is shifted toward more negative energies, indicating an enhancement of the instability, whereas for positive nonlinear rotation strength ($\tilde{C} = 40$), it is shifted upward toward positive energies, reflecting a reduction of the instability.

decreases relative to the vortex free state. However, this reduction does not indicate the stabilization of MQVs, as in this region, angular momentum is accommodated through the nucleation of singly quantized vortices from the condensate boundary, which leads to nonaxisymmetric vortex configurations and vortex lattice formation. Consequently, MQVs remain energetically unstable against dissociation into single quantum vortices despite their lower free energy compared to the vortex free state. A qualitatively different behavior has been observed for positive nonlinear rotation strength $\tilde{C} = 40$. In this scenario, the free energy intersections between MQV states and the vortex free state are shifted to rotation frequencies $\Omega < \Omega_c$, where vortex nucleation and nonaxisymmetric vortex configurations are energetically unfavorable. As a result, MQVs become the lowest energy configuration and are therefore globally energetically stable. This stabil-

ity arises from the coupling between the condensate density and angular momentum, which lowers the total energy of the MQVS and energetically favors the higher order energy states for positive nonlinear rotation strength. In contrast, for $\tilde{C} < 0$, this coupling acts in opposition to the imposed rotation, shifting the stabilizing frequencies to $\Omega > \Omega_c$ and thereby suppressing the stability of MQVs.

After analyzing the global stability of multiply quantized vortices (MQVs) through the free energy landscape, we now examine their local stability via BdG analysis. In Fig. 6.11, we present the excitation spectra for a triply quantized vortex ($m = 3$) at a fixed rotation frequency $\Omega = 0.3$ for different nonlinear rotation strengths $\tilde{C} = 0, -30$, and 40. Within this framework, local stability is determined by the presence of anomalous modes, i.e., excitations with negative energy and positive norm ($\omega < 0$ and $N > 0$, see Eq. (6.20)), which indicate directions along which the condensate can lower its energy. For all values of $\tilde{C}$, we observe anomalous modes at angular momenta $l = -2$ and $l = -3$. The $l = -2$ mode corresponds to a quadrupole (elliptical) deformation of the condensate density surrounding the vortex, whereas the $l = -3$ mode is associated with the splitting of the multiply quantized vortex into three singly quantized vortices. The behavior of these two modes with nonlinear rotation is markedly different. The $l = -3$ mode becomes more negative for $\tilde{C} = -30$, which indicates a strong enhancement of the splitting instability due to the reduction in the effective rotation in the high density region. In contrast, for a positive nonlinear rotation strength ($\tilde{C} = 40$), the anomalous mode $l = -3$ shifts upward in energy, reflecting a partial reduction of the instability. Despite this upward shift, the anomalous modes do not cross into the zero energy region, even at this relatively large positive nonlinear strength. As the rotation frequency and nonlinear rotation strength are further increased, additional negative energy excitations appear at higher angular momenta, indicating the emergence of new instability channels. On the other hand, the $l = -2$ mode shows a non-monotonic behavior. For $\tilde{C} = 0$, it has a relatively large negative energy, indicating that the condensate strongly favors quadrupole deformation. However, for both $\tilde{C} = -30$ and $\tilde{C} = 40$, the energy of this mode moves close to zero. This means that the mode becomes a soft mode, meaning that, although the system can still deform in this way, there is only a very weak energetic preference for doing so. Physically, this occurs because the mechanisms that resist shape deformation are weakened. For $\tilde{C} = -30$, even though the central density increases and the vortex core becomes narrower, the effective rotational frequency is reduced in the dense region, which weakens the restoring force for the quadrupole deformation. For $\tilde{C} = 40$, the central density decreases, and reduce the density gradient that provides resistance to quadrupole deformation. In

both cases, the condensate offers little resistance to this mode, but at the same time there is no strong driving force to amplify it. As a result, while the quadrupole deformation mode remains present but soft, the dynamics are ultimately governed by the strongly unstable $l = -3$ mode, which drives the splitting of the vortex. Therefore, although positive nonlinear rotation lowers the free energy of MQVs and can render them globally stable, it is not sufficient on its own to eliminate the anomalous modes and achieve local stability.

6.8 Conclusion

In this Chapter, we have investigated the impact of density-dependent gauge potential induced nonlinear rotation on the structural transformation of a vortex lattice in a condensate confined within a toroidal geometry. We find that, depending on the strength of the nonlinear rotation, the system undergoes a transition from a ring shaped vortex lattice to a giant vortex state. In particular, the annular region of the condensate becomes narrower for positive values of the nonlinear rotation strength and wider for negative values. We observe a qualitatively similar influence of nonlinear rotation when the giant vortex state is generated through atom-removal and absorption techniques. In addition, we identify the permissible solution regime by mapping the parameter space defined by the trap radius and the nonlinear rotation strength, which represents the region where the condensate exists. We further examine the effect of nonlinear rotation on the collective excitation spectrum by employing both the BdG and hydrodynamic approaches. Our results reveal a decreasing phenomenon in the dipole mode, indicating a violation of Kohn's theorem. In contrast, the breathing mode exhibits an increasing trend beyond a critical nonlinear rotation strength, signaling radial compression of the condensate. Furthermore, we analyze the influence of nonlinear rotation on the stability of multiply quantized vortices. It was found that while it makes higher order vortices globally stable, it is unable to achieve local stability.

Beyond the analysis of the ground state and excitation spectrum under nonlinear rotation in a toroidal geometry, the problem of persistent current formation presents an interesting direction for future investigation. In particular, it would be worthwhile to explore the impact of the gauge potential induced nonlinear rotation on phase slip dynamics [287] and Hysteresis loops [288]. Furthermore, incorporating impurity effects and dissipation could provide deeper insight into the dissipative dynamics and vortex excitations in such systems [289].

Conclusions and future scope

7.1 Main conclusions of the thesis

In this thesis, we have investigated the structural transformation of vortex lattice in Bose-Einstein condensates under the influence of two different types of gauge potentials: static and dynamic. In the first two works, we considered a static gauge potential generated through rotation and studied the structural transformations and melting phenomena of the vortex lattice in the presence of differential rotation with an square lattice potential, as well as in the presence of static and dynamic impurities. In addition, we analyze the turbulent characteristics that emerge in the melted vortex lattice state. In the subsequent two studies, we focus on the dynamic gauge potential and explore the structural transformations and collective excitation spectra arising from the nonlinear rotation induced by this gauge potential. These investigations are carried out for condensates confined in both harmonic and ring shaped trapping potentials. The detailed conclusions of these studies are summarized below.

In **Chapter 3**, we numerically investigated the structural transformations of vortex lattices in rotating BECs trapped in a square lattice potential. Our study revealed several distinct lattice configurations, including the Abrikosov lattice (AL), pinned square lattice (PL), and an intermediate coexistence lattice (CL) phase. Using the structure factor as a quantitative measure, we determined the minimum pinning potential strength required for the transition from the Abrikosov lattice to the pinned lattice state. We also constructed phase diagrams for different lattice constants and rotation frequencies, showing that the critical rotation required for pinning increases as the lattice constant decreases. Furthermore, we examined the effect of differential rotation between the condensate and the lattice potential, which leads to several distinct structural phases

of the vortex lattice. For sufficiently large values of the differential rotation parameter, the lattice undergoes a melting transition, resulting in a disordered vortex configuration. In this melted regime, vortices become randomly distributed, giving rise to turbulent features in the condensate. The turbulent nature of the system is further confirmed through the analysis of the kinetic energy spectra, which exhibit characteristic power-law scalings associated with turbulence.

In **Chapter 4**, we studied the effect of impurities on the structural transformation and melting of vortex lattices in a rotating BEC confined in a Gaussian square lattice potential. We examined two types of impurities: random static impurities and oscillating Gaussian obstacles acting as dynamical impurities. For static impurities, we found that the square vortex lattice melts when the impurity strength exceeds a critical threshold, which decreases as the impurity lattice constant becomes smaller than the vortex lattice spacing. The critical impurity strength is also influenced by the rotational frequency of the condensate. The melted vortex lattice state is characterized through compressible and incompressible kinetic energy spectra, which exhibit turbulence-like features with Vinen-like scaling in the infrared regime. Similar melting and turbulent behaviour were also observed in the presence of oscillating Gaussian obstacles. Furthermore, the angular momentum and kinetic energy spectra show similar power-law behavior in the melted regime irrespective of the impurity type, indicating the universal nature of the vortex lattice melting and the resulting turbulent state in rotating BECs.

In **Chapter 5**, we investigated the effects of nonlinear rotation in a quasi two dimensional BEC arising from a density-dependent gauge potential. Using Bogoliubov-de Gennes theory, we analyzed the collective excitation spectrum and demonstrated that the nonlinear rotation parameter influences the excitation spectrum in several ways. By examining the time evolution of the condensate density perturbed by the eigenstates of the collective modes, we identified several low-lying collective excitations. Through a variational analysis, we showed that the center of mass oscillation and the width of the condensate are coupled and depend on the strength of the nonlinear rotation. We also derived an analytical expression for the dipole and breathing modes using this approach. By comparing the dipole mode frequency obtained from the time evolution of the center of mass with results from the time dependent GP equation, we demonstrated that nonlinear rotation leads to a violation of Kohn's theorem, where the dipole mode frequency becomes dependent on the interaction strength. We further showed that the Tkachenko mode frequency strongly depends on the nonlinear rotation parameter, attaining a minimum at $\tilde{C} = 0$ and increasing symmetrically for both

positive and negative values of $\tilde{C}$. An analytical expression for the Tkachenko mode frequency obtained using a continuum hydrodynamic approach was found to agree well with our numerical results. In addition to the Tkachenko mode, several other fundamental modes were identified. The excitation spectrum as a function of angular momentum reveals an energy splitting between modes with l and $-l$, which depends on the strength of nonlinear rotation. Moreover, the appearance of roton-like minima in the excitation spectrum indicates the presence of surface mode excitations, which shift toward higher angular momentum for negative nonlinear rotation and toward lower angular momentum for positive nonlinear rotation.

In **Chapter 6**, we investigated the impact of density-dependent gauge potential induced nonlinear rotation on the structural transformation of vortex lattices in a condensate confined within a toroidal geometry. Our results show that, depending on the strength of the nonlinear rotation, the system undergoes a transition from a ring-shaped vortex lattice to a giant vortex state. Specifically, the annular region of the condensate becomes narrower for positive values of the nonlinear rotation strength and wider for negative values. A qualitatively similar effect of nonlinear rotation is also observed when the giant vortex state is generated through atom removal and absorption techniques. We further determined the permissible solution regime by mapping the parameter space defined by the trap radius and the nonlinear rotation strength, which identifies the region where the condensate remains stable. The influence of nonlinear rotation on the collective excitation spectrum was examined using both BdG and hydrodynamic approaches. Our analysis shows a decrease in the dipole mode frequency, indicating a violation of Kohn's theorem. In contrast, the breathing mode frequency increases beyond a critical nonlinear rotation strength, signaling radial compression of the condensate. Moreover, we studied the stability of multiply quantized vortices and found that nonlinear rotation can make higher order vortices globally stable, although local stability is not achieved.

7.2 Scope for future work

The results presented in this thesis open several directions for future investigation in presence of gauge potentials. One possible extension would be to explore how a density-dependent gauge potential influences the behavior of three-dimensional vortex lines and their associated dynamics. Another promising direction is to investigate the role of superfluid turbulence in such systems, which may provide deeper insight

into how energy is distributed across different length scales in the presence of spatially varying vorticity [248–250]. It would also be interesting to study finite-temperature effects by employing the stochastic Gross-Pitaevskii equation, which naturally incorporates dissipation and thermal fluctuations, thereby providing a more realistic description of experimentally accessible condensates. Further, it would be worthwhile to explore the impact of gauge-potential induced nonlinear rotation on phase slip dynamics [287] and hysteresis loops [288]. In addition, incorporating impurity effects and dissipation could provide deeper insight into dissipative dynamics and vortex excitations in such systems [289]. Finally, investigating the influence of density-dependent gauge potentials on the formation and dynamics of quantum droplets in the beyond mean field regime also presents an intriguing area for further study.



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