# ON THE DARBOUX POLYNOMIALS AND SIMPLICITY OF POLYNOMIAL DERIVATIONS

by

Ashish Kumar Kesarwany

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI-781039, INDIA
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*A Thesis submitted*
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**Ashish Kumar Kesarwany**
**(Roll Number: 136123002)**

*to the*
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# *Declaration*

I hereby declare that the work contained in this thesis entitled **"On the Darboux Polynomials and Simplicity of Polynomial Derivations"** was done by me under the supervision of **Dr. Vinay Wagh**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

May, 2024

**Ashish Kumar Kesarwany**
Roll No. 136123002
Department of Mathematics
Indian Institute of Technology Guwahati

# *Certificate*

It is certified that the work contained in this thesis entitled **“On the Darboux Polynomials and Simplicity of Polynomial Derivations”** by **Ashish Kumar Kesarwany,** a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy, has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

May, 2024

**Dr. Vinay Wagh**
Associate Professor
Department of Mathematics
Indian Institute of Technology Guwahati

*Dedicated to*

*my beloved parents,*

*my lovely wife and little daughter*

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# *Abstract*

Derivations and Darboux polynomials are useful methods to study polynomial or rational differential system. In the vocabulary of differential algebra, Darboux polynomials coincides with generators of polynomial differential ideals, that is $f \in k[x_1, \ldots, x_n]$ is a Darboux polynomial iff $f \neq 0$ and the ideal $(f)$ is differential. The present thesis studies certain classes of derivations having no Darboux polynomials with monomial cofactor.

Another important notion in commutative algebra, simple derivations has also been studied in this thesis. Simple derivations play an important role in numerous problems. This thesis studies certain classes of derivations that are simple.

**Chapter 1:** In this chapter, firstly, we give the assumptions we will follow throughout this thesis. Further, we introduce the notion of derivations, module of derivations, and we recall a few results about derivations. We describe some properties of module of derivations. We discuss derivations in polynomial rings. We further describe the ring of constants, $d$-differential ideals, Darboux elements, Shamsuddin derivation, isotropy subgroups, Lie algebra, and some results based on them. We also describe some properties of simple derivations and some important applications of simple derivations.

**Chapter 2:** In this chapter, we prove results about non-existence of Darboux polynomial with monomial cofactor of some polynomial derivations of $k[x, y]$. More specifically, we show that the following class of derivations, under suitable conditions, do not have a Darboux polynomial with monomial cofactor.

- $D = y\partial_x + (y^r + gy + c)\partial_y$ (Theorem 2.1.1 for $r \neq 3$) and (Theorem 2.1.3 for $r = 3$)

**Chapter 3:** In this chapter, we prove results about the simple $k$-derivation of the polynomial ring $k[x, y]$. More precisely, in this chapter, we give two different derivations of the polynomial ring $k[x, y]$ and show that they are simple derivations of the polynomial ring $k[x, y]$. we show that the following derivations, under suitable conditions, are simple derivations of the polynomial ring $k[x, y]$.

- $\mathfrak{D}_h = y^2\partial_x + (xy + h)\partial_y$ (Theorem 3.1.1)
- $D = y\partial_x + (y^2 + xy + 1)\partial_y$ (Theorem 3.2.1)

**Chapter 4:** In this chapter, we first discuss Shamsuddin's result and then generalize some of the results from Chapter 3 to higher dimensions. More precisely, we prove that the following classes of derivations, under suitable conditions, are simple.

- $d_n = {x_2}^2\partial_{x_1} + (x_1x_2 + h)\partial_{x_2} + (h_1x_3 + t_1)\partial_{x_3} + \cdots + (h_{n-2}x_n + t_{n-2})\partial_{x_n}$, (Theorem 4.3.1)
- $d = x_2\partial_{x_1} + ({x_2}^2 + x_1x_2 + 1)\partial_{x_2} + (h_1x_3 + t_1)\partial_{x_3} + \cdots + (h_{n-2}x_n + t_{n-2})\partial_{x_n}$ (Theorem 4.3.2)

**Chapter 5:** In this chapter, We pose some problems arising out of the work carried out in this thesis.

# Contents

# List of Symbols

| | |
|---|---|
| $\mathbb{N}$ | the set of strictly positive integers |
| $\mathbb{Z}$ | the ring of integers |
| $\mathbb{Q}$ | the field of rational numbers |
| $A$ | a ring |
| $I, J$ | ideals of ring $A$ |
| $\mathfrak{m}$ | a maximal ideal of $A$ |
| $\mathfrak{p}$ | a prime ideal of $A$ |
| $\sqrt{I}$ | the ideal $\{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$ for an ideal $I$ |
| $\langle a_1, \ldots, a_n \rangle$ | ideal generated by the elements $a_1, \ldots, a_n$ |
| $\mathrm{Spec}(A)$ | the set of all prime ideals of $A$ |
| $IJ$ | the product of ideals $I$ and $J$ |
| $I + J$ | the sum of ideals $I$ and $J$ |
| $M$ | an $A$-module |
| $k$ | a field |
| $k^*$ | nonzero elements of field $k$ |
| $\otimes_k$ | tensor product of $k$-modules |
| $\mathrm{Der}(A, M)$ | set of all derivations from $A$ to $M$ |
| $\mathrm{Der}_k(A, M)$ | set of all $k$-derivations from $A$ to $M$ |
| $\mathrm{Der}(A)$ | set of all derivations from $A$ to $A$ |
| $\mathrm{Der}_k(A)$ | set of all $k$-derivations from $A$ to $A$ |
| $x, y, x_1, \ldots, x_n$ | indeterminates over $k$ |
| $\text{pol-ord}_{x=a}(f)$ | order of pole of rational function $f$ at $a$ |
| $\partial_{x_1}, \ldots, \partial_{x_n}, \partial_x, \partial_y$ | partial derivatives with respect to $x_1, \ldots, x_n, x, y$ respectively |
| $\deg_x(f)$ | the degree of polynomial $f$ in variable $x$ |
| $\mathrm{Aut}_k(k[x, y])$ | set of all $k$-automorphisms of $k[x, y]$ |

# Introduction

The module of derivations and the module of differentials play an important role in commutative algebra and algebraic geometry. They are analogous to the tangent bundles and cotangent bundles of manifolds.

Simple derivations play an important role in numerous problems, especially in the area of commutative algebra. Simple derivations can be used in the construction of simple noncommutative rings, which are not division rings. Let $A$ be a ring and $d$ be a derivation of $A$. Then it is well known that the skew polynomial ring $A[x; d]$ is a simple ring if and only if $d$ is simple (recall that a ring $R$ with unity is said to be simple if its only two-sided ideals are $\langle 0 \rangle$ and $R$) [CF08, GW04].

A significant amount of research has been conducted over the past few years in the field of simple derivations of polynomial rings in finitely many variables over a field of characteristic zero. The researchers have developed interest in subject because of its applications in various other areas, for example, application to the construction of nonholonomic irreducible module over Weyl algebra. Let $A_2$ be the second Weyl algebra, i.e., the ring of differential operators over the polynomial ring $k[x, y]$ and

$$d = \partial_x + f\partial_y,$$

be a simple $k$-derivation of $k[x, y]$ with $f \in k[x, y]$. Then, it has been proved in [DSDLR06] that there exists a polynomial $g \in k[x, y]$, such that $\dfrac{A_2}{A_2(d+g)}$ is a simple nonholonomic $A_2$- module.

Simple derivations can be used to construct simple Lie rings. Let $A$ be a commutative ring with unity and let $d$ be a nonzero derivation of $A$. Recall that a Lie ring $L$ is said to be simple if it has no Lie ideals other than $\langle 0\rangle$ and $L$. Denote by $A_0$ the Lie ring whose elements are the elements of $A$, with the product defined as following

$$[a, b] = ad(b) - bd(a),$$

for all $a, b \in A$. It has been proved by C. R. Jordan and D. A. Jordan in [JJ78, Theorem 3] that if $A$ is noetherian, then $A_0$ is simple if and only if $d$ is a simple derivation of $A$. This result was further generalized by A. Nowicki to the case of non-noetherian rings also [Now85].

Therefore, searching for simple derivations of commutative rings is very interesting. Examples of simple derivations are not readily available and are difficult to find, even in the case of polynomial rings. Indeed, it is generally challenging to check whether a given derivation is simple. The problem is complicated even for the polynomial ring in two variables over a field of characteristic zero.

Consider the polynomial ring $k[x_1, \dots, x_n]$ in $n$ variables and consider the $k$-derivation $d$ of $k[x_1, \dots, x_n]$ given by

$$d = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n}.$$

Then it would be of great interest to find necessary and sufficient conditions on $f_1, \dots, f_n$ for the derivation $d$ to be simple. In complete generality, the answer to this question is obviously very complicated. This question is evident only for $n = 1$ because all the simple derivations of $k[x_1]$ are of the form $\alpha\partial_{x_1}$ for $\alpha \in k^*$. For $n = 2$, only some sporadic examples of simple $k$-derivations of polynomial ring $k[x, y]$ are known. The problem seems to be complicated even with the extra assumption that $d(x) = 1$. Some examples and classes of simple derivations of $k[x, y]$ are available in [Jor81], [MMON01], [Leq08] [Now08], [Gav09], [Kou12], [Kou14]. The description of all the simple $k$-derivations $d$ of $k[x, y]$ such that $d(x) = 1$

and $d(y) = ay + b$, where $a(x), b(x)$ are polynomials of $k[x]$, has been given in [Now94].

In [Jor81], D. A. Jordan has proved that $k$-derivation $d$ of $k[x, y]$ of the form $y^3\partial_x + (1 - xy)\partial_y$ is simple. A derivation $d$ of $k[x_1, \cdots, x_n]$ such that

$$d = \partial_{x_1} + \sum_{i=2}^{n} (a_i x_i + b_i)\partial_{x_i},$$

with $a_i, b_i \in k[x_1]$, is called Shamsuddin derivation. In [Leq08], Y. Lequain has developed an algorithm that characterizes all the simple Shamsuddin derivations in $k[x_1, \ldots, x_n]$.

A $k$-derivation $d$ of $k[x, y]$ is called the quadratic derivation if it is of the form

$$d = \partial_x + \left(y^2 + ay + b\right)\partial_y,$$

where $a(x), b(x) \in k[x]$. In [MMON01], A. Maciejewski, J. Moulin-Ollagnier, and A. Nowicki have given an algebraic characterization of simple derivations $d$ of $k[x, y]$ such that $d = \partial_x + (y^2 + a(x)y + b(x))\,\partial_y$ with $a(x), b(x) \in k[x]$. They have shown that such types of derivations are equivalent to $\Delta_p = \partial_x + \Big(y^2 - p(x)\Big)\partial_y$ for some suitable $p(x) \in k[x]$. They have further shown that if $p(x)$ is of an odd degree, then $\Delta_p$ is a simple derivation. If $p(x)$ is a quadratic polynomial, then $p$ satisfies an arithmetic condition is a necessary and sufficient condition for $\Delta_p$ to be simple.

In [BLL03], P. Brumatti, Y. Lequain, D. Levcovitz have constructed examples of simple derivations $d$ of the local ring $k[x, y]_{\langle x,y\rangle}$ such that $d(x) = 1$ and $\deg_y d(y) = s$, where $s$ is an arbitrary positive integer. Most of the published examples of simple derivations $d$ of $k[x, y]$ with $d(x) = 1$ are of the form $d = \partial_x + f\partial_y$, where $f \in k[x, y]$ and $\deg_y f \leq 2$. In particular, there does not seem to be any example with $\deg_y f$ as an arbitrary positive integer. In [Now08], A. Nowicki has tried to answer this. He has shown that the class of $k$-derivation of the form $\partial_x + (y^s + cx)\partial_y$ with $s \geq 2$ and $c \in k^*$, is a simple derivation of $k[x, y]$. In [Gav09], V. S. Gavran has generalized A. Nowicki's result and has proved that the derivation $\partial_x + (y^m + ax^n)\partial_y$, where $m, n \in \mathbb{N}$, and $a \in k^*$, is a simple derivation of $k[x, y]$. He actually showed that the derivation $\partial_x + (y^m + ax^n)\partial_y$ is equivalent to the derivation $\partial_x + (y^m + x^n)\partial_y$ and proved the simplicity of the latter one.

In [Kou12], S. Kour has studied the simplicity of $k$-derivations of the form $y^r\partial_x + (y^s x + g)\partial_y$, where $0 \leq r < s$ and $g \in k[y]$. She has shown that the derivation $y^{s-1}\partial_x + (y^s x + g(y))\partial_y$ with $g \in k[y]$ $\deg g \leq s$ and $\gcd(y, g) = 1$ is a simple derivation of $k[x, y]$. She has also discussed the simplicity of the derivation in the case when $g$ is a nonzero constant polynomial. In this case, she has proved that the $k$-derivation $y^r\partial_x + (y^s x + c)$, where $s \geq 1$, $0 \leq r \leq s - 1$, and $c \in k^*$ is a simple $k$-derivation of polynomial ring $k[x, y]$. In [Kou14], S. Kour has described simple derivations of the form $y^r\partial_x + (y^s x^t + c)\partial_y$, where $r \geq 0$, $s, t \geq 1$, and $c \in k^*$. She has shown that the derivation $y^r\partial_x + (y^s x^t + c)\partial_y$ of $k[x, y]$, where $s, t \geq 1$, $0 \leq r < s$, and $c \in k^*$ is a simple $k$-derivation.

It is shown that certain $k$-derivations of the polynomial ring $k[x, y]$ are simple if and only if there is no corresponding Darboux polynomials (see, for example, [MMON01, Now08, Kou12]). Darboux polynomials are analogous to eigenvectors in matrix theory. Among polynomial differential equations, Darboux polynomials with nonzero cofactors are widely used; they coincide with the partial first integrals (see, for example, [Zc95, MONS95]) of the system of polynomial differential equations determined by the given derivation.

A $k$-derivation $d$ of $A[x_1, \ldots, x_n]$ is called generalized triangular derivation if $d(x_i) \in A[x_1, \ldots, x_i]$, for $1 \leq i \leq n$. It has been proved in [Kou17] that if A is a unique factorization domain, $d$ be a generalized triangular derivation of $A[x_1, \cdots, x_n]$, and if $d$ is a simple derivation of $A$, then $d$ is a simple derivation of $A[x_1, \ldots, x_n]$ if and only if $d$ has no Darboux element. Darboux polynomials have been studied in many papers under certain conditions on cofactor (see, for example, [MMON01], [Now08], [Gav09]).

A $k$-derivation of the form $z^s\partial_x + x^s\partial_y + y^s\partial_z$ for $s \geq 2$ of the polynomial ring $k[x, y, z]$ is called Jouanolou derivation. It has been proved in [Jou79] that for every $s \geq 2$, the Jouanolou derivation has no Darboux polynomial.

In [Gav09], it has been shown that the derivation $\partial_x + (y^m + x^n)\partial_y$ where $m, n \in \mathbb{N}$, $m \geq 2$ does not have a Darboux polynomial $F$ with monomial cofactor $sy^{m-1}$, where $s = \deg_y F$.

In [GMM13], an algebro-geometric description of simple derivations and Darboux polynomials can be found. R. V. Gurjar, K. Masuda, and M. Miyanishi have given an alternate way to check whether a $\mathbb{C}$-derivation $D$ of $\mathbb{C}[x, y]$ is simple. An element $f$ of affine domain $\mathbb{C}[x, y]$ is called an integral element with respect to $D$ if $D(f)$ is divisible by $f$ in $\mathbb{C}[x, y]$. Clearly, if $f$ is an integral element of a derivation $D$ of $\mathbb{C}[x, y]$ then $\langle f \rangle$ is a $D$-differential ideal, and hence $D$ can not be simple. It is shown in [GMM13, Theorem 2] that if $D$ is a surjective derivation of $\mathbb{C}[x, y]$, then there exists an integral element with respect to $D$. In this thesis, we have studied integral elements and call them Darboux elements.

## Structure of the Thesis

This thesis is divided into two parts. The first part of the thesis discusses the non-existence of Darboux polynomial with a monomial cofactor, whereas the second part discusses simple derivations of the polynomial ring. The primary goal of the first part of the thesis is to study some classes of derivations of the polynomial ring $k[x, y]$ having no Darboux polynomial with a monomial cofactor. The primary goal of the second part of the thesis is to study some classes of simple derivations of polynomial rings.

# Chapter 1

# Preliminaries

## 1.1 Introduction

Throughout this thesis, we assume that all rings are commutative with unity and $k$ denotes a field of characteristic zero unless specified otherwise. In this chapter, we introduce the basic terminology and present some preliminary concepts and facts which will be often used in the subsequent chapters. In Section 1.2, definitions and basic facts are introduced. In this section, we present the notion of derivations and the module of derivations. We describe some properties of module of derivations. We discuss derivations in polynomial rings. We further describe the ring of constants, d-differential ideals, Darboux elements, Shamsuddin derivation, isotropy subgroups, and Lie algebra and recall a few related results that are directly relevant to this thesis. In Section 1.3, we recall some properties of simple derivations. In this section, we discuss how Darboux polynomials can be used to show whether a given derivation is simple. At the end of Section 1.3, we discuss a result by S. Kour [Kou12], which will be used in Chapter 3. Section 1.4 discusses some of the important applications of simple derivations. This section shows how simple derivations can be used to construct simple noncommutative rings (Ore extension), simple Ore extension, simple Lie rings, regular rings, nonholonomic irreducible modules over Weyl algebra, etc.

## 1.2 Definitions and Basic Facts

We start this section with the definition of a derivation from [Sin11].

**Definition 1.2.1** (Derivation). *Let $A$ be a ring and $M$ be an $A$-module. Then a map $D : A \to M$ is called a* **derivation** *if $D$ is an additive group homomorphism and satisfies the* **Leibniz condition:**

$$D(ab) = aD(b) + bD(a),$$

*for all $a, b \in A$.*

By a derivation **on** a ring $A$, we mean a derivation $A \to M$ for some $A$-module $M$, and by a derivation **of** a ring $A$, we mean a derivation $A \to A$. The set of all derivations $A \to M$ is written as $\mathrm{Der}(A, M)$. The set of all derivations $A \to A$ is written as $\mathrm{Der}(A)$.

Let $R$ be a ring. If $A$ is an $R$-algebra, then $d \in \mathrm{Der}(A, M)$ is said to be an $R$-derivation if $d$ is an $R$-module homomorphism. The set of all $R$-derivations $A \to M$ is denoted by $\mathrm{Der}_R(A, M)$. We represent the set of all $R$-derivations $A \to A$ by $\mathrm{Der}_R(A)$.

### Some Properties of Module of Derivations

1. $\mathrm{Der}(A, M)$ becomes an A-module in a natural way, with $D_1 + D_2$ and $aD_1$ defined as

$$(D_1 + D_2)(a) = D_1(a) + D_2(a),$$

and

$$(aD_1)(b) = aD_1(b),$$

for all $a, b \in A$ and $D_1, D_2 \in Der(A, M)$. Clearly, $\mathrm{Der}_R(A, M)$ is an $A$-submodule of $\mathrm{Der}(A, M)$. Note that $\mathrm{Der}(A) = \mathrm{Der}_{\mathbb{Z}}(A)$.

2. If $D \in \mathrm{Der}_R(A, M)$, then by using induction on $n$, one can see that for $a \in A$ and non-negative integer $n$,

$$D(a^n) = na^{n-1}D(a).$$

More generally, for $a_1, \dots a_r, \in A$ and non-negative integers $n_1, \dots, n_r$, we have

$$D({a_1}^{n_1} \dots {a_r}^{n_r}) = \sum_{i=1}^{r} n_i {a_1}^{n_1} \dots {a_{i-1}}^{n_{i-1}} {a_i}^{n_i - 1} {a_{i+1}}^{n_{i+1}} \dots {a_r}^{n_r} D(a_i).$$

3. If $D \in \mathrm{Der}_R(A)$, then we have Leibniz formula for powers of $D$ (here, power means the composition of maps).

$$D^n(ab) = \sum_{i=0}^{n} \binom{n}{i} D^i(a) D^{n-i}(b).$$

If $A$ has characteristic $p > 0$, where $p$ is a prime, then the above expression becomes

$$D^p(ab) = bD^p(a) + aD^p(b),$$

and hence $D^p$ is also a derivation of $A$.

4. If $D_1, D_2 \in \mathrm{Der}(A)$, then $[D_1, D_2] = D_1D_2 - D_2D_1$ is also a derivation of $A$. Thus, $[D_1, D_2] \in \mathrm{Der}(A)$. It is, therefore, immediate that $\mathrm{Der}(A)$ together with [ , ] becomes a Lie algebra.

**Example 1.2.2.** *Let $x$ be an indeterminate. Define $d : A[x] \to A[x]$ by* $d\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=1}^{n} i a_i x^{i-1}$. *Then $d$ is a derivation of $A[x]$.*

*In this thesis, we shall denote it by* $\partial_x$ *and call it the partial derivative with respect to* $x$.

**Example 1.2.3.** *Consider the polynomial ring* $A[x, y]$, *in two variables* $x$ *and* $y$ *over a ring* $A$, *then the partial derivative map* $\partial_x : A[x, y] \to A[x, y]$ *is a derivation which is* $A[y]$*-linear. Hence* $\partial_x \in Der_{A[y]}(A[x, y])$.

**Remark 1.2.4.** *Let* $x$ *be an indeterminate and let* $d$ *be a derivation of* $A$. *Then* $d$ *extends to a derivation of* $A[x]$ *by assigning arbitrary value to* $d(x)$.

Next two lemmas are elementary observations.

**Lemma 1.2.5.** *Let* $d$ *be a derivation of* $A$. *Then* $d(1) = 0$.

**Lemma 1.2.6.** *Let* $d$ *be a derivation of* $A$, *and let* $\theta$ *be an automorphism of* $A$. *Then* $\theta d \theta^{-1}$ *is also a derivation of* $A$.

**Remark 1.2.7.** *Let* $k$ *be a ring and* $A$ *be a* $k$*-algebra. Assume that the set* $S$ *generates* $A$ *as a* $k$*-algebra. Let* $d$ *be a* $k$*-derivation of* $A$. *Then the derivation* $d$ *is completely determined by its values on* $S$.

## Derivations in Polynomial Rings

We now discuss $k$-derivations of the polynomial ring $k[x_1, \dots, x_n]$ and of the field of fractions $k(x_1, \dots, x_n)$ of $k[x_1, \dots, x_n]$.

Let $k[\underline{X}] = k[x_1, \dots, x_n]$ be the polynomial ring over $k$. For each $i$, the partial derivative $\partial_{x_i}$ is a $k$-derivation of $k[\underline{X}]$. It is the unique derivation of $k[\underline{X}]$ such that $d(x_i) = 1$ and $d(x_j) = 0$ for all $j \neq i$. As a consequence of the above facts, we get

**Theorem 1.2.8** ( [Now94]). *Let* $k[\underline{X}] = k[x_1, \dots, x_n]$ *be the polynomial ring over a ring* $k$.

*(1). If* $f_1, \dots, f_n \in k[\underline{X}]$, *then there exists a unique* $k$*-derivation* $d$ *of* $k[\underline{X}]$ *such that* $d(x_1) = f_1, \dots, d(x_n) = f_n$. *This* $k$*-derivation is of the form:*

$$d = f_1\partial_{x_1} + \ldots + f_n\partial_{x_n}.$$

*In this case, d is also written as* $(f_1, \ldots, f_n)$.

*(2).* $Der_k(k[\underline{X}])$ *is a free* $k[\underline{X}]$*-module of rank* $n$ *with basis* $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$.

*(3).* $\partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}$, *for all* $i, j \in \{1, \ldots, n\}$.

*(4). If* $d \in Der_k(k[\underline{X}])$ *and* $f \in k[\underline{X}]$, *then*

$$d(f) = \sum_{i=1}^{n} \partial_{x_i}(f)d(x_i).$$

The following theorem extends the above result to the quotient field of $k[x_1, \ldots, x_n]$.

**Theorem 1.2.9** ( [Now94]). *Let* $k(x_1, \ldots, x_n)$ *be the quotient field of* $k[x_1, \ldots, x_n]$.

*(1). If* $f_1, \ldots, f_n \in k(x_1, \ldots, x_n)$, *then there exists a unique k-derivation d of* $k(x_1, \ldots, x_n)$ *such that* $d(x_1) = f_1, \ldots, d(x_n) = f_n$. *This k-derivation is of the form:*

$$d = f_1\partial_{x_1} + \ldots + f_n\partial_{x_n}.$$

*In this case, d is also written as* $(f_1, \ldots, f_n)$.

*(2).* $Der_k(k(x_1, \ldots, x_n))$ *is a free* $k(x_1, \ldots, x_n)$*-module of rank* $n$ *with basis* $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$.

*(3).* $\partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}$, *for all* $i, j \in \{1, \ldots, n\}$.

*(4). If* $d \in Der_k(k(x_1, \ldots, x_n))$ *and* $f \in k(x_1, \ldots, x_n)$, *then*

$$d(f) = \sum_{i=1}^{n} \partial_{x_i}(f)d(x_i).$$

## Ring of Constants

We now define the ring of constants from [Now94].

**Definition 1.2.10** (Ring of constants)**.** *Let $\mathfrak{D}$ be a family of derivations of $A$. Then the ring of constants of $A$ with respect to $\mathfrak{D}$ is defined to be the set*

$$\{a \in A \mid d(a) = 0 \ \forall \ d \in \mathfrak{D}\},$$

*and is denoted by $A^{\mathfrak{D}}$.*

Note that this set is a subring of $A$ and is also called the $\ker(\mathfrak{D})$. If $A$ is a $k$-algebra and $\mathfrak{D}$ is a family of $k$-derivations of $A$, then $A^{\mathfrak{D}}$ is a $k$-subalgebra of $A$. If $A$ is a field, then $A^{\mathfrak{D}}$ is a subfield of $A$. If $\mathfrak{D}$ has only one element, $d$, then we write $A^d$ instead of $A^{\{d\}}$. It is evident that $A^{\mathfrak{D}} = \cap_{d\in\mathfrak{D}} A^d$.

We now recall some elementary results about the ring of constants of $A$ with respect to a family of derivations of $A$.

**Proposition 1.2.11.** ( [NN88, Lemma 2.1]) *Let $k$ be a field of characteristic zero and $A$ be a $k$-domain. If $\mathfrak{D}$ is a family of $k$-derivations of $A$, then $A^{\mathfrak{D}}$ is integrally closed in $A$.*

*Proof.* Let $a \in A$ be an integral element over $A^{\mathfrak{D}}$ and let $f(x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n \in A^{\mathfrak{D}}[x]$ be a monic polynomial such that $f(a) = 0$ that is

$$a^n + b_1a^{n-1} + \cdots + b_{n-1}a + b_n = 0.$$

We choose $n$ to be minimal among the degrees of all such elements of $A^{\mathfrak{D}}[x]$. If $d \in \mathfrak{D}$, then

$$0 = d(0) = d(f(a)) = f_x(a)d(a),$$

where $f_x(a)$ denotes the value of $\partial_x(f)$ at $x = a$. Clearly, $f_x(a)$ can not be zero (because $n$ is minimal and $\text{char}(k)$ is zero). As $A$ is a $k$-domain, we get $d(a) = 0$, that is $a \in A^d$ for every $d \in \mathfrak{D}$. Thus, $a \in \cap_{d\in\mathfrak{D}} A^d = A^{\mathfrak{D}}$.

□

**Proposition 1.2.12.** ( [Now94, Proposition 3.1.4]) *Let $k$ be a field of characteristic zero. If $\mathfrak{D}$ is a family of $k$-derivations of a field $L$ containing $k$, then the field $L^{\mathfrak{D}}$ is algebraically closed in $L$.*

**Remark 1.2.13.** *These results are not true if $A$ is an integral domain of characteristic $p > 0$. For example, take $A = k[x]$, where $k$ is a field of characteristic $p > 0$ and $d = \partial_x$, then $A^d = k[x^p]$, not integrally closed in $A$.*

## $d$-differential Ideal

**Definition 1.2.14** ($d$-differential ideal)**.** *Let $d$ be a derivation of $A$. An ideal $I$ of $A$ is called a $d$-differential ideal (or $d$-stable ideal or $d$-ideal or $d$-invariant ideal) if $d(I) \subseteq I$.*

An ideal is said to be a maximally $d$-differential ideal if it is a maximal element of the family

$$\mathscr{F} = \{J \subsetneq A \mid J \text{ is a d-differential ideal of } A\}.$$

By Zorn's Lemma, $\mathscr{F}$ has a maximal element.

We now recall the following lemma from [AM69].

**Lemma 1.2.15** ( [AM69])**.** *Let $I$ be a proper ideal of $A$. Then $\{P \in \operatorname{Spec}(A) | I \subseteq P\}$ has minimal elements.*

The lemma leads to the following definition:

**Definition 1.2.16** (Minimal Prime Ideal)**.** *Let $I$ be a proper ideal of $A$. A prime ideal $\mathfrak{p}$ of $A$ containing $I$ is said to be a minimal prime ideal of $I$ if it is a minimal element of the family $\{\mathfrak{p} \in \operatorname{Spec}(A) | I \subseteq \mathfrak{p}\}$.*

*The minimal primes of the zero ideal are called the minimal primes of $A$.*

If $A$ contains a field of characteristic zero, then Y. Lequain proved that all minimal prime ideals of a $d$-differential ideal are $d$-differential [Leq71]. Therefore, a maximally $d$-differential ideal is a prime ideal.

**Example 1.2.17.** *Ideals $\langle 0 \rangle$ and $A$ are $d$-differential ideals of $A$.*

**Example 1.2.18.** *Consider* $k[x,y]$ *and* $I = \langle x^2+y \rangle$. *Let* $d$ *be the* $k$-*derivation with* $d(x) = x$ *and* $d(y) = 2y$. *Then* $d(I) \subseteq I$ *and hence* $I$ *is a* $d$-*differential ideal of* $k[x,y]$.

**Remark 1.2.19.** (1). Let $A$ be a ring and $d$ be a derivation of $A$. If $I$ and $J$ are $d$-differential ideals of $A$, then so are $IJ$ and $I+J$. Note that the intersection of any arbitrary family of $d$-differential ideals is again a $d$-differential ideal.

(2). If $d$ is a derivation of $A$ and $I$ is an ideal of $A$, then $d(I^n) \subseteq I^{n-1}$ for all $n \geq 1$. Therefore, the ideal $\cap_{n\geq 0} I^n$ is a $d$-differential ideal.

**Lemma 1.2.20.** *Let* $(A, \mathfrak{m})$ *be a local ring and* $d$ *be a derivation of* $A$. *Then* $A$ *has a unique maximally* $d$-*differential ideal.*

*Proof.* If possible, suppose $A$ does not have a unique maximally $d$-differential ideal. Let $I_1$ and $I_2$ be two maximally $d$-differential ideals of $A$. Clearly, $I_1 + I_2$ is also $d$-differential ideal and $I_1 + I_2 \subseteq \mathfrak{m}$ as $I_1, I_2 \subseteq \mathfrak{m}$. Therefore, $I_1 + I_2$ is a proper $d$-differential ideal of $A$; hence $I_1 = I_1 + I_2 = I_2$. This is a contradiction. □

The following result gives $d$-differentiability of minimal primes of a $d$-differential ideal.

**Lemma 1.2.21.** ( [Kap76, Lemma 1.8]) *Let* $A$ *be a ring containing a field of characteristic zero and* $d$ *be a derivation of* $A$. *Let* $I$ *be an ideal of* $A$. *Then*

1. *If* $I$ *is a* $d$-*differential ideal, then* $\sqrt{I}$ *and all minimal prime ideals of* $I$ *are also* $d$-*differential.*
2. *If* $I$ *is maximally* $d$-*differential, then* $I$ *is a prime ideal of* $A$.
3. *If* $\langle 0 \rangle$ *is maximally* $d$-*differential, then* $A$ *is an integral domain.*

**Definition 1.2.22** (Simple Derivation). *If* $A$ *has no* $d$-*differential ideal other than* $\langle 0 \rangle$ *and* $A$, *then* $A$ *is called* $d$-*simple, and the derivation* $d$ *is called a simple derivation.*

**Example 1.2.23.** *Let $x$ be an indeterminate over $k$ and $A = k[x]$. Then the derivation $\partial_x$ is a simple derivation of $A$.*

**Example 1.2.24.** *If $k$ is any field of characteristic $p > 0$, and if $d$ is any derivation of $k[x_1, \ldots, x_n]$, then $d({x_1}^p) = 0$; and hence $\langle {x_1}^p \rangle$ is a $d$-differential ideal of $d$. So, $d$ can not be simple.*

*In this thesis, we shall consider the field $k$ of characteristic zero.*

**Definition 1.2.25** (Equivalent Derivations)**.** *Two $k$-derivations, $d_1$ and $d_2$ of a $k$-algebra $A$, are said to be* ***equivalent*** *if there is an automorphism $\sigma \in \mathrm{Aut}_k(A)$ such that $d_2 = \sigma d_1 \sigma^{-1}$.*

**Remark 1.2.26.** *If $d_1$ and $d_2$ are equivalent derivations, then $d_1$ is simple if and only if $d_2$ is simple.*

**Example 1.2.27.** *The derivations $\partial_x$ and $\partial_y$ of $k[x, y]$ are equivalent derivations as $\partial_y = \sigma \partial_x \sigma^{-1}$, where $\sigma \in \mathrm{Aut}_k(k[x, y])$ given by*

$$\sigma(x) = x + y, \quad \sigma(y) = x.$$

**Example 1.2.28** ( [MMON01])**.** *Let $f, g, h \in k[x]$ and let $d, \delta$ be derivations of $k[x, y]$ given by*

$$d = \partial_x + (y^2 + fy + g)\partial_y,$$

*and*

$$\delta = \partial_x + \left(y^2 + (f + 2h)y + g + h^2 + fh - h'\right) \partial_y,$$

*then $\delta = \sigma d \sigma^{-1}$, where $\sigma : k[x, y] \to k[x, y]$ is a $k$-automorphism given by $\sigma(x) = x$ and $\sigma(y) = y - h$. Hence $d$ and $\delta$ are equivalent derivations of $k[x, y]$.*

**Example 1.2.29** ( [Now94])**.** *Let $d$ and $\delta$ be two $k$-derivations of $k[x, y, z]$ given by*

$$d = x\partial_x + y\partial_y + z\partial_z,$$

*and*

$$\delta = x\partial_x + (y - x^2)\partial_y + (z - y^2 + 2x^2y)\partial_z,$$

*then* $d$ *and* $\delta$ *are equivalent derivations of* $k[x, y, z]$ *as* $\delta = \sigma d \sigma^{-1}$*, where* $\sigma \in \mathrm{Aut}_k(k[x, y, z])$ *given by*

$$\sigma(x) = x, \quad \sigma(y) = y + x^2, \quad \sigma(z) = z + y^2.$$

**Theorem 1.2.30.** *Let* $k$ *be an algebraically closed field. Consider the following derivation of* $k[x, y]$

$$D = f(x)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y},$$

*where* $f \in k[x] \setminus k$ *and* $g \in k[x, y]$*. Then* $D$ *is not a simple derivation of* $k[x, y]$*.*

*Proof.* Since $k$ is algebraically closed, so every polynomial can be expressed as a product of linear polynomials. Let $x - \alpha$ be a linear factor of $f(x)$ and $f(x) = (x - \alpha)f_1(x)$ then, we see that

$$D(x - \alpha) = f(x) = f_1(x)(x - \alpha).$$

Thus, $\langle x - \alpha \rangle$ is a $D$-differential ideal; hence $D$ is not simple. ☐

## Darboux Elements

**Definition 1.2.31** (Darboux element)**.** *A nonzero, non-unit element* $f \in A$ *is said to be a Darboux element of* $d$ *if* $d(f) = \lambda f$ *for some* $\lambda \in A$*. This element* $\lambda$ *is called a cofactor of the Darboux element* $f$*. If* $A$ *is a polynomial ring, then this* $f$ *is called a Darboux polynomial, and* $\lambda$ *is called a polynomial eigenvalue. If* $\lambda$ *is a monomial, then* $f$ *is called the Darboux polynomial of* $d$ *with a monomial cofactor.*

**Proposition 1.2.32.** *The product of two Darboux elements of a derivation* $d$ *is either zero or a Darboux element of* $d$*.*

*Proof.* Let $f$ and $g$ be two Darboux elements of a $k$-derivation $d$ with cofactors $\lambda$ and $\mu$, respectively, i.e.,

$$d(f) = \lambda f \text{ and } d(g) = \mu g.$$

Let $fg \neq 0$. Then

$$d(fg) = fd(g) + gd(f) = (\lambda + \mu)fg.$$

Hence, $fg$ is also a Darboux element of $k$-derivation $d$. $\square$

**Remark 1.2.33.** *Let $A$ be a unique factorization domain containing a field $k$ of characteristic zero and $d$ be a $k$-derivation of $A$. If $f$ is a Darboux element of $d$, then all its irreducible factors and hence all factors of $f$ are also Darboux elements of $d$. It is a consequence of the result of [Leq71]: all minimal prime ideals of a differential ideal are differential ideals.*

*Thus, looking for Darboux polynomials of a given $k$-derivation $d$ (where $k$ is a field of characteristic $0$) reduces to looking for irreducible ones.*

**Proposition 1.2.34.** ( [Now94, Proposition 2.2.2]) *Let $k$ be a field of characteristic zero and $x_1, \ldots, x_n$ be indeterminates. Let $d$ be a $k$-derivation of $k(x_1, \ldots, x_n)$ such that $d(k[x_1, \ldots, x_n]) \subseteq k[x_1, \ldots, x_n]$. Let $f$ and $g$ be non-zero coprime polynomials of $k[x_1, \ldots, x_n]$. Then $f/g \in k(x_1, \ldots, x_n)^d$ if and only if $f$ and $g$ are Darboux polynomials with the same cofactor.*

**Example 1.2.35.** (1). If $f \in A^d$ is a non-zero and non-unit in $A$, then $f$ is a Darboux element of $d$ with cofactor 0.

(2). Let $f = x^2 + y \in k[x, y]$ and $d$ be a $k$-derivation of $k[x, y]$ with $d(x) = x$ and $d(y) = 2y$. Then $d(f) = 2x^2 + 2y = 2f$, which shows that $f$ is a Darboux polynomial with eigenvalue 2.

(3). Let $f = xy^2 - yx^2 \in k[x, y]$ and let $d$ be a $k$-derivation of $k[x, y]$ such that $d(x) = x^2$ and $d(y) = y^2$, then $d(f) = 2(x + y)f$, which shows that $f$ is a Darboux polynomial with eigenvalue $2(x + y)$.

(4). [Now08] Let $d = \partial_x + (y^s + px)\partial_y$ where $p(\neq 0) \in k$ be a derivation of $k[x, y]$. If we take $s = 0$, then $d(y - x - \frac{1}{2}px^2) = 0$ so $y - x - \frac{1}{2}px^2$ is a Darboux polynomial with cofactor zero.

(5). [Now08] Let $d = \partial_x + (y^s + px)\partial_y$ where $p(\neq 0) \in k$ be a derivation of $k[x, y]$. If we take $s = 1$, then $d(y + px + p) = y + px + p$, so $y + px + p$ is a Darboux polynomial with cofactor 1.

## Shamsuddin Derivations

We quote the definition of Shamsuddin derivation and discuss some results related to Shamsuddin derivation from [Leq08].

**Definition 1.2.36** (Shamsuddin derivation). *A derivation $d$ of $k[x_1, \dots, x_n]$ is called a Shamsuddin derivation if it has the following form:*

$$d = \partial_{x_1} + (a_2 x_2 + b_2)\,\partial_{x_2} + \cdots + (a_n x_n + b_n)\,\partial_{x_n},$$

*where $a_i, b_i \in k[x_1]$, for every $i = 2, \dots, n$.*

**Remark 1.2.37.** *Not all Shamsuddin derivations are simple.*

**Example 1.2.38.** *Consider the following Shamsuddin derivation of $k[x, y]$*

$$d = \partial_x + (xy)\partial_y.$$

*Then $d$ is not simple because $\langle y \rangle$ is a $d$-differential ideal.*

**Example 1.2.39.** *Consider the following Shamsuddin derivation of $k[x, y, z]$*

$$d = \partial_x + (xy + 1)\partial_y + (xz + 1)\partial_z.$$

*Then $d$ is not simple because $\langle y - z \rangle$ is a $d$-differential ideal (as $d(y - z) = x(y - z)$).*

At this stage, we can ask the following question

**Question 1.2.40.** *Does there exist a necessary condition for a Shamsuddin derivation to be simple?*

Y. Lequain answered the question by giving an algorithmic characterization [Leq08].

**Theorem 1.2.41** ( [Leq08]). *It is possible to decide (effectively) when a Shamsuddin derivation*

$$d = \partial_{x_1} + (a_2x_2 + b_2)\,\partial_{x_2} + \cdots + (a_nx_n + b_n)\,\partial_{x_n},$$

*is simple. One has to compute some invariants of the polynomials* $a_i, b_i \in k[x_1]$.

There are many simple derivations that are not Shamsuddin. For example, consider the following derivation of $k[x_1, \ldots, x_n]$.

**Example 1.2.42.** ( [Now94, Example 13.4.3]) Let $d$ be the $k$-derivation of $k[x_1, \ldots, x_n]$ given by

$$d = \partial_{x_1} + \sum_{i=2}^{n} (1 + x_{i-1}x_i)\,\partial_{x_i}.$$

Then $d$ is simple, but not Shamsuddin.

## Isotropy Subgroup

We give the definition of the Isotropy subgroup.

**Definition 1.2.43** (Isotropy Subgroup). *We denote by* $\mathrm{Aut}_k\left(k[x,y]\right)$ *the group of* $k$*-automorphisms of* $k[x,y]$*. Let* $\mathrm{Aut}_k\left(k[x,y]\right)$ *act on* $\mathrm{Der}_k\left(k[x,y]\right)$ *by:*

$$(\sigma, d) \mapsto \sigma^{-1} \circ d \circ \sigma = \sigma^{-1} d \sigma.$$

*Fix a derivation* $d \in \mathrm{Der}_k(k[x,y])$*. The isotropy subgroup, with respect to this group action, is defined as*

$$Aut\,(k[x,y])_d = \left\{\rho \in \mathrm{Aut}_k\,(k[x,y]) \ \mid \ \rho^{-1} d\rho = d\right\}.$$

In Propositions 1.3.7 and 1.3.8, we shall discuss the results by Baltazar and Bertoncello et al., respectively, which say that the isotropy subgroup of a simple Shamsuddin derivation is trivial. In Proposition 1.3.9, we shall discuss a result by Baltazar, which states that an element of isotropy subgroup of a simple derivation of $k[x_1, \ldots, x_n]$ will be trivial if it satisfies a condition.

## Lie Algebra

We present the definition of Lie algebra.

**Definition 1.2.44** (Lie algebra)**.** *Let $M$ be an $A$-module. A map $[\ ,\ ] : M \times M \to M$ sending $(x, y) \to [x, y]$ is called a Lie bracket if it satisfies the following conditions:*

*(a).* $[\ ,\ ]$ *is bilinear,*

*(b).* $[x, x] = 0$ *for all* $x \in M$*,*

*(c).* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in M$*.*

*An $A$-module $M$ together with $[\ ,\ ]$ is called Lie algebra.*

**Example 1.2.45.** *Let $M$ be an $A$-module. The simplest example of Lie algebra is given by defining the Lie bracket $[x, y] = 0$ for all $x, y \in M$. A Lie algebra satisfying this condition is called abelian.*

**Example 1.2.46.** *Let $M_n(A)$ denote the ring of $n \times n$ matrices over $A$. Then the Lie bracket $[x, y] = xy - yx$ for all $x, y \in M_n(A)$, makes $M_n(A)$ into a Lie algebra.*

As discussed already in §1.2 on page 9, we know that $\mathrm{Der}(A)$ is also a Lie algebra with Lie bracket defined as $[D_1, D_2] = D_1D_2 - D_2D_1$ for all $D_1, D_2 \in \mathrm{Der}(A)$.

## 1.3 Properties of Simple Derivations

In this section, we present some properties of simple derivations. We also provide proof of some results for the sake of completeness.

**Proposition 1.3.1** ( [Now94]). *If $d$ is a simple $k$-derivation of a non-zero $k$-algebra $A$, then $A^d$ (the ring of constants of $d$) is a field.*

*Proof.* Let $r \in A^d \setminus \{0\}$. Then $d(r) = 0 \in \langle r \rangle$, and thus $\langle r \rangle$ is a $d$-differential ideal of $A$. As $d$ is a simple derivation of $A$, the ideal $\langle r \rangle = A$. Hence $sr = 1$, for some $s \in A$. But then

$$d(s) = 1d(s) = srd(s) = sd(rs) = sd(1) = s0 = 0.$$

Thus $s \in A^d$, so $r$ is invertible in $A^d$. □

**Proposition 1.3.2** ( [Now94]). *Assume that $\mathbb{Q} \subseteq k$. If $d$ is a simple $k$-derivation of a non-zero $k$-algebra $A$, then $A$ is a $k$-domain.*

*Proof.* Let $\mathfrak{p}$ be a minimal prime ideal of $A$. Since $\mathbb{Q} \subseteq k$, $\mathfrak{p}$ is a $d$-ideal of $A$ (Lemma 1.2.21). Therefore, $0 = \mathfrak{p}$ is a prime ideal of $A$. Hence $A/\{0\} \simeq A$ is a domain. □

**Lemma 1.3.3** ( [Now94]). *Let $k \subseteq k'$ be a field extension and $A$ be a $k$-algebra. Let $d$ be a $k$-derivation of $A$. Consider the $k'$-algebra $A \otimes_k k'$ and define a $k'$-linear map*

$$d \otimes 1 : A \otimes_k k' \to A \otimes_k k',$$

*given by $(d \otimes 1)(a \otimes \alpha) = d(a) \otimes \alpha$, for every $a \in A$ and $\alpha \in k'$. Then $d \otimes 1$ is $k'$-derivation of $A \otimes_k k'$.*

*Proof.* To show that, $d \otimes 1$ is $k'$-derivation, it suffices to show that $d \otimes 1$ satisfies Leibniz condition. Let $a = \sum_{i=1}^{s} a_i \otimes \alpha_i$ and $b = \sum_{i=1}^{t} b_i \otimes \beta_i$ be the two elements of $A \otimes_k k'$ , then $ab = \sum_{i=1}^{s} \sum_{j=1}^{t} a_i b_j \otimes \alpha_i \beta_j$. Therefore,

$$
\begin{aligned}
(d \otimes 1)(ab) &= \sum_{i=1}^{s} \sum_{j=1}^{t} d(a_i b_j) \otimes \alpha_i \beta_j \\
&= \sum_{i=1}^{s} \sum_{j=1}^{t} (a_i d(b_j) + b_j d(a_i)) \otimes \alpha_i \beta_j \\
&= \sum_{i=1}^{s} \sum_{i=j}^{t} a_i d(b_j) \otimes \alpha_i \beta_j + \sum_{i=1}^{s} \sum_{j=1}^{t} b_j d(a_i) \otimes \alpha_i \beta_j \\
&= \sum_{i=1}^{s} \sum_{i=j}^{t} (a_i \otimes \alpha_i)\,(d(b_j) \otimes \beta_j) + \sum_{i=1}^{s} \sum_{i=j}^{t} (b_j \otimes \beta_j)\,(d(a_i) \otimes \alpha_i) \\
&= \sum_{i=1}^{s} \sum_{i=j}^{t} (a_i \otimes \alpha_i)\,(d \otimes 1)\,(b_j \otimes \beta_j) + \sum_{i=1}^{s} \sum_{i=j}^{t} (b_j \otimes \beta_j)\,(d \otimes 1)\,(a_i \otimes \alpha_i) \\
&= \sum_{i=1}^{s} (a_i \otimes \alpha_i) \sum_{i=j}^{t} (d \otimes 1)\,(b_j \otimes \beta_j) + \sum_{j=1}^{t} (b_j \otimes \beta_j) \sum_{i=1}^{s} (d \otimes 1)\,(a_i \otimes \alpha_i) \\
&= a(d \otimes 1)(b) + b(d \otimes 1)(a)
\end{aligned}
$$

Thus, $(d \otimes 1)(ab) = a(d \otimes 1)(b) + b(d \otimes 1)(a)$. This shows that $d \otimes 1$ is a $k'$-derivation. □

**Proposition 1.3.4** ( [Now94]). *Let $k \subset k'$ be a field extension and $d$ is a $k$-derivation of a $k$-algebra $A$. Then*

*(1). $A$ is finitely generated over $k$ if and only if $A \otimes_k k'$ is a finitely generated algebra over $k'$.*

*(2). $A^d = k$ if and only if $(A \otimes_k k')^{d \otimes 1} = k'$.*

**Proposition 1.3.5** ( [Now94]). *Let $k \subseteq k'$ be a field extension, and $d$ be a $k$-derivation of a $k$-algebra $A$ with $A^d = k$. Consider the $k'$-derivation $d \otimes 1$ of $k'$-algebra $A \otimes_k k'$. Then $d$ is simple if and only if $d \otimes 1$ is simple.*

*Proof.* Assume that $d$ is a simple derivation of $A$ and let $I'$ be a nonzero

$d\otimes 1$-differential ideal of $A\otimes_k k'$. Consider the following set

$$I=\{a\in A \mid a\otimes 1\in I'\}.$$

It is clear that $I$ is a $d$-differential ideal of $A$. But $d$ is a simple derivation, so either $I=A$ or $I=\{0\}$. If $I=A$, then $1\otimes 1\in I'$, so $I'=A\otimes_k k'$.

Suppose now that $I=\{0\}$. Let $B'=\{w_i \mid i\in\Lambda\}$ be a basis of $k'$ over $k$ containing 1. Every nonzero element of $A\otimes_k k'$ has a unique representation of the form

$$a_1\otimes w_1+a_2\otimes w_2+\cdots+a_n\otimes w_n,$$

where $n\in\mathbb{N}$, $a_1,\dots,a_n$ are nonzero elements of $A$ and $w_1,\dots,w_n$ are pairwise different elements from the set $B'$. Since $I'$ is nonzero, there exists a nonzero $u\in I'$. Let

$$u=a_1\otimes w_1+a_2\otimes w_2+\cdots+a_n\otimes w_n,$$

with $n$ and $a_1,\dots,a_n$ as above, and assume that $n$ is minimal. If $n=1$, then

$$a_1\otimes 1=(1\otimes {w_1}^{-1})(a_1\otimes w_1)\in I',$$

and we have a contradiction as $0\neq a_1\in I=\{0\}$. Thus $n\geq 2$. Let $J$ be the smallest $d$-differential ideal of $A$ containing $a_1$. Since $d$ is simple derivation of $A$, $J=A$ and hence

$$s_0a_1+s_1d(a_1)+s_2d^2(a_1)+\cdots+s_pd^p(a_1)=1,$$

for some $s_0,s_1,\dots,s_p\in A$.

Let $v=\delta(u)$, where $\delta=\sum_{j=0}^{p} s_j(d\otimes 1)^j$. Then $v$ is a nonzero element of $I'$ of the form

$$v=1\otimes w_1+t_2\otimes w_2+\cdots+t_n\otimes w_n,$$

for some $t_2, \dots, t_n \in A$. But the element

$$w = (d \otimes 1)(v) = d(t_2) \otimes w_2 + \cdots + d(t_n) \otimes w_n,$$

is also in $I'$. Therefore, by the minimality of $n$, $w = 0$, that is $t_2, \dots, t_n \in A^d = k$. Thus, we have $v = 1 \otimes t$, where $t = w_1 + t_2 w_2 + \cdots + t_n w_n$. Clearly, $t \neq 0$. Hence $I'$ contains a unit. This completes the proof of the first part.

Assume now that $d \otimes 1$ is simple. Let $I$ be a nonzero $d$-differential ideal of $A$. Then $I \otimes_k k'$ is a nonzero $d \otimes 1$-differential ideal of $A \otimes_k k'$; therefore, $I \otimes_k k' = A \otimes_k k'$. Consider the sequence $I \to A \to 0$ of $A$-modules. Since $I \otimes_k k' \to A \otimes_k k' \to 0$ is exact and $k'$ faithfully flat over $k$, it follows that $I \to A \to 0$ is exact, so $I = A$. □

**Proposition 1.3.6.** ( [MMON01, Propoposition 10.1]) *Let $k \subset k'$ be an extension of fields (of characteristic zero), and let $d$ be a derivation of $k[x_1, \dots, x_n]$. Consider the derivation $d'$ of $k'[x_1, \dots, x_n]$ such that $d'(x_i) = d(x_i)$, for $i = 1, \dots, n$. Then $d$ is simple if and only if $d'$ is simple.*

*Proof.* We may identify $k[x_1, \dots, x_n] \otimes_k k'$ as $k'[x_1, \dots, x_n]$ and $d \otimes 1$ as $d'$. Suppose $d'$ is simple. Then by Proposition 1.3.1, $k'[x_1, \dots, x_n]^{d'}$ is a field, so $k'[x_1, \dots, x_n]^{d'} = k'$, so $k[x_1, \dots, x_n]^d = k$ by Proposition 1.3.4(2). By Proposition 1.3.5, $d$ is simple. Conversely, suppose that $d$ is simple. Then by Proposition 1.3.1, $k[x_1, \dots, x_n]^d$ is a field, so $k[x_1, \dots, x_n]^d = k$, so Proposition 1.3.5 implies that $d'$ is simple.

□

**Proposition 1.3.7** ( [Bal16]). *Let $D$ be a Samsuddin derivation of $k[x, y]$. If $D$ is a simple derivation of $k[x, y]$, then*

$$\mathrm{Aut}(k[x, y])_D = \{id\}.$$

**Proposition 1.3.8** ( [BL20]). *If $d$ is a simple Shamsuddin derivation of the polynomial ring $k[x_1, \dots, x_n], n \geq 2$, then its isotropy group is trivial.*

**Proposition 1.3.9** ( [Bal16]). *Let $D \in \mathrm{Der}_k(k[x_1, \dots, x_n])$ be a simple derivation, and $\rho \in \mathrm{Aut}(k[x_1, \dots, x_n])_D$ be an isomorphism in the isotropy* *subgroup. Suppose a maximal ideal* $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ *exists such that* $\rho(\mathfrak{m}) = \mathfrak{m}$*, then* $\rho = id$*.*

**Proposition 1.3.10** ( [Kou12]). *Let* $k$ *be an algebraically closed field of characteristic zero and let* $d$ *be a* $k$*-derivation of* $k[x, y]$ *such that* $\langle dx, dy \rangle = k[x, y]$*. Then the following statements are equivalent:*

1. $d$ *is a simple derivation of* $k[x, y]$*;*
2. $d$ *has no Darboux element.*

*Proof.* It is trivial that if $d$ is simple, then $d$ has no Darboux polynomial.

Now, assume that $d$ is not simple. Then there exists a proper nontrivial $d$-differential ideal of $k[x, y]$ and hence, by Lemma 1.2.21, a nontrivial $d$-differential prime ideal $\mathfrak{p}$ of $k[x, y]$. If $\mathfrak{p}$ is a maximal ideal, then $\mathfrak{p} = \langle x - \alpha, y - \beta \rangle$ for some $\alpha, \beta \in k$ as $k$ is an algebraically closed field. As $d(\mathfrak{p}) \subseteq \mathfrak{p}$, we have $d(x) = d(x - \alpha) \in \mathfrak{p}$ and $d(y) = d(y - \beta) \in \mathfrak{p}$ but $\langle dx, dy \rangle = k[x, y]$ and hence $\mathfrak{p} = k[x, y]$, a contradiction.

Hence $\mathfrak{p}$ is not maximal. This means that $\mathfrak{p}$ is of height one. Since in $k[x, y]$, all height one prime ideals are principal, $d$ has a Darboux element. □

**Remark 1.3.11.** *Proposition 1.3.10 remains valid when* $k$ *is not assumed to be algebraically closed. (This is a consequence of Proposition 1.3.12, see below.)*

**Lemma 1.3.12.** *Let* $k$ *be a field of characteristic zero. Let* $A$ *be a* $k$*-algebra and* $\mathfrak{m}$ *a maximal ideal of* $A$ *with the property that* $A/\mathfrak{m}$ *is an algebraic extension of* $k$*. For any* $k$*-derivation* $d$ *of* $A$*, we have* $d(\mathfrak{m}) \subseteq \mathfrak{m} \iff d(A) \subseteq \mathfrak{m}$*.*

*Proof.* It is obvious that if $d(A) \subseteq \mathfrak{m}$, then $d(\mathfrak{m}) \subseteq \mathfrak{m}$. Now, suppose that $d(\mathfrak{m}) \subseteq \mathfrak{m}$. Let $a \in A$. Since the element $\bar{a} = a + \mathfrak{m}$ of $A/\mathfrak{m}$ is algebraic over $k$, we may consider its minimal polynomial $P(x) \in k[x]$. Since $P(\bar{a}) = 0$ and $P'(\bar{a}) \neq 0$ (where $P'(x)$ is the usual derivative), we have $P(a) \in \mathfrak{m}$ and $P'(a) \notin \mathfrak{m}$. So $\mathfrak{m} \ni d(P(a)) = P'(a)d(a)$ and consequently $d(a) \in \mathfrak{m}$. □

**Proposition 1.3.13.** *Let $k$ be a field of characteristic zero. Let $A$ be a UFD and a finitely generated $k$-algebra of Krull dimension 2, and let $d$ be a $k$-derivation of $A$ such that $\langle d(A)\rangle = A$. Then $d$ is simple if and only if it has no Darboux element.*

*Proof.* It is trivial that if $d$ is simple, then $d$ has no Darboux polynomial.

Conversely, since $A$ is finitely generated $k$-algebra, every maximal ideal $\mathfrak{m}$ of $A$ has the property that $A/\mathfrak{m}$ is algebraic over $k$. So Lemma 1.3.12 implies that no maximal ideal $\mathfrak{m}$ of $A$ is $d$-differential. If $d$ is not simple, then there exist a nonzero prime ideal $\mathfrak{p}$ of $A$ that is $d$-differential, and this $\mathfrak{p}$ is not maximal by the preceding sentence. So $\mathfrak{p}$ has height 1 and hence is a principal ideal, say $\mathfrak{p} = \langle f\rangle$. Clearly, $f$ is a Darboux element of $d$. ☐

**Proposition 1.3.14.** ( [MMON01, Proposition 2.1]) *If $d : k[x, y] \to k[x, y]$ is a derivation such that d(x)=1, then $d$ is simple if and only if $d$ has no Darboux polynomial.*

## 1.4 Applications of Simple Derivations

In this section, we present some applications of simple derivations.

1. Simple derivations are useful in the construction of simple noncommutative rings which are not field.

   Let $A$ be a $k$-algebra and $d$ be a $k$-derivation on $A$. Extend $d$ to the polynomial ring $A[x]$ by setting $d(x) = 0$. Let $\partial_x$ denotes the partial derivative with respect to $x$. Using $d$, and the usual addition and multiplication in $A[x]$, we define another multiplication $*$ in $A[x]$ as follows: For $f, g \in A[x]$, let

$$f * g = \sum_{i=0}^{n} \frac{{\partial_x}^i(f)d^i(g)}{i!},$$

   where $n = \deg_x(f)$. We write $S = A[x; d]$ to mean that

(a) $(S, +, *)$ is a ring with unity, containing $A$ as a subring;

(b) $x$ is an element of $S$;

(c) $S$ is a free left $A$-module with basis $\{1, x, x^2, \ldots\}$;

(d) $x * r = rx + d(r)$ for all $r \in A$.

For $a \in A$, note that $x * a = ax + d(a)$ and $a * x = ax$. Therefore, if $d$ is nonzero, then $*$ is always noncommutative.

Such a ring $S$ is called a **skew polynomial ring** or **Ore extension** of $A$ with respect to the derivation $d$. We remark that the Ore extension also has a more general definition which involves a ring endomorphism of $A$, see [Ore33].

In the following proposition, K. R. Goodearl and R. B. Warfield, Jr. constructed a simple noncommutative ring, i.e., $A[x; d]$, whenever $d$ is a simple derivation of $A$.

**Proposition 1.4.1.** ( [GW04, Proposition 2.1]) Let $A$ be a ring and $d$ a derivation of $A$. Then the skew polynomial ring $A[x; d]$ is a simple ring (that is, $A[x; d]$ has no two-sided ideals other than 0 and $A[x; d]$) if and only if $d$ is simple.

2. Simple derivations can be used to construct simple Lie rings.

Recall that a Lie ring $L$ is said to be simple if it has no Lie ideals other than 0 and $L$. Denote by $A_0$ the Lie ring whose elements are the elements of $A$, with the product defined as following

$$[a, b] = ad(b) - bd(a),$$

for all $a, b \in A$. In this situation, C. R. Jordan and D. A. Jordan have proved the following theorem [JJ78].

**Theorem 1.4.2.** [JJ78, Theorem 3] With the notations as above, suppose $A$ is noetherian. Then $A_0$ is a simple Lie ring if and only if $A$ is $d$-simple.

A. Nowicki further extended this result to non-noetherian rings [Now85].

3. Simple derivations can be used to check the regularity of a ring.

   A. Seidenberg showed that if $A$ is a finitely generated domain and $d$ is simple, then $A$ is regular [Sei67].

   R. Hart showed that if $A$ is a finitely generated local domain, then $A$ is regular if and only if there exists a simple derivation of $A$ [Har75].

4. Simple derivations have been used intensively in constructing non-holonomic irreducible modules over Weyl algebra. Let $A_2$ be the second Weyl algebra, i.e., the ring of differential operators over the polynomial ring $k[x,y]$ and

$$d = \partial_x + f\partial_y,$$

   where $f \in k[x,y]$, be a simple $k$-derivation of $k[x,y]$. Then, it has been proved in [DSDLR06] that there exists a polynomial $g \in k[x,y]$, such that $\dfrac{A_2}{A_2(d+g)}$ is a simple nonholonomic $A_2$-module. This result is also mentioned in [Cou07]. This is a consequence of the following equivalent statements.

   **Theorem 1.4.3** ( [DSDLR06]). Let $d = \partial_x + f\partial_y$, with $f \in k[x,y]$, be a derivation of $k[x,y]$. Let $g \in k[x,y]$. Then the following statements are equivalent:

   (a) $A_2(d+g)$ is a maximal left ideal of $A_2$.

   (b) $d+g$ does not admit any Darboux operator in $k[x,y]\langle\partial_y\rangle$.

# Chapter 2

# Non-existence of Darboux Polynomial with Monomial Cofactor of Some Polynomial Derivations

In this chapter, we consider the following class of $k$-derivations

$$y\partial_x + (y^r + gy + c)\partial_y, \text{ where } r \in \mathbb{N},\ g(x), c(x) \in k[x]$$

and we prove that there is no Darboux polynomial with monomial cofactors for these derivations.

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[1] The results in this chapter are communicated for a possible publication [Kes23].

## 2.1 The Derivations $y\partial_x + (y^r + gy + c)\partial_y$ of $k[x, y]$

In this section, we consider the $k$-derivation of the form $D = y\partial_x + (y^r + gy + c)\partial_y$, where $r \in \mathbb{N}$, $g(x), c(x) \in k[x]$. We show that if $r \neq 3$, $\deg g \geq 1$, and $c \in k^*$, then $D$ does not have a Darboux polynomial with monomial cofactor. If $r = 3$, then $D$ does not have a Darboux polynomial with monomial cofactor if $\deg g \geq 2$ and $c \in k^*$. In this section, we prove the following two theorems.

**Theorem 2.1.1.** *Let $D = y\partial_x + (y^r + gy + c)\partial_y$, where $r \in \mathbb{N}$, $r \neq 3$, and $g(x), c \in k[x]$, with $\deg g \geq 1$ and $c \in k^*$, be a $k$-derivation of $k[x, y]$. Then the derivation $D$ does not have a Darboux polynomial with monomial cofactor.*

In the following remark, we give an example to show that conditions on $r$ and $g$ are necessary. For example, for $r = 3$ and $\deg g = 1$, then the following derivation has a Darboux polynomial with monomial cofactor.

**Remark 2.1.2.** *If $\deg g = 1$, then $f = x - y^2$ is a Darboux polynomial of $D = y\partial_x + \left(y^3 - xy + \frac{1}{2}\right)\partial_y$ with cofactor $2y^2$.*

$$\begin{aligned}
D(f) &= \left(y\partial_x + \left(y^3 - xy + \frac{1}{2}\right)\partial_y\right)(x - y^2) \\
&= y\partial_x(x - y^2) + \left(y^3 - xy + \frac{1}{2}\right)\partial_y(x - y^2) \\
&= y - 2y^4 + 2xy^2 - y = 2y^2(x - y^2) \\
\implies D(f) &= 2y^2 f
\end{aligned}$$

If we replace the condition $\deg g \geq 1$ with $\deg g \geq 2$, then the following theorem covers the case $r = 3$.

**Theorem 2.1.3.** *Let $D = y\partial_x + (y^3 + gy + c)\partial_y$, where $g(x), c(x) \in k[x]$, with $\deg g \geq 2$ and $c \in k^*$, be a $k$-derivation of $k[x, y]$. Then the derivation $D$ does not have a Darboux polynomial with monomial cofactor.*

### 2.1.1 The Non-existence of Darboux Polynomial of $D$

In this section, we prove Theorems 2.1.1 and 2.1.3.

*Proof of Theorem 2.1.1.* Let $f$ be a Darboux polynomial of $D$ with monomial cofactor $\lambda$. Write $f = \sum_{i=0}^{s} b_i y^i \in k[x][y]$ with $b_s \in k[x]$ be nonzero. Then, we have

$$\begin{aligned} & D(f) = \lambda f \\ \implies y\left(\sum_{i=0}^{s} b_i' y^i\right) + (y^r + gy + c)\left(\sum_{i=1}^{s} i b_i y^{i-1}\right) &= \lambda\left(\sum_{i=0}^{s} b_i y^i\right). \end{aligned} \tag{2.1.1}$$

In the following lemma, we prove that $s \geq 1$.

**Lemma 2.1.4.** *$f$ is a non-constant polynomial in $y$ over $k[x]$, in particular,* $\deg_y f \geq 1$.

*Proof.* If $f \in k[x]$ then $yf_x = \lambda f$, so $f \mid yf_x$. Since $f \in k[x]$ and $f \mid yf_x$, we obtain $f \mid f_x$, so $f_x = 0$ and hence $f \in k$, which is not possible because $f$ is a Darboux element. □

**Claim.** $b_0 \neq 0$.

Indeed, if $b_0 = 0$ then $y \mid f$, so $y$ is a Darboux polynomial of $D$ by Remark 1.2.33. However, $D(y) = y^r + gy + c$ where $c \in k^*$, so $y$ is not a Darboux polynomial, which is a contradiction and hence $b_0 \neq 0$ is proved. □

If $r = 1$, then the derivation $D$ becomes $y\partial_x + ((g+1)y + c)\,\partial_y$. For this derivation, D. Yan has already proved that $D$ does not have a Darboux polynomial [Yan19]. So we may assume that $r \geq 2$ throughout the remainder of the proof.

Substituting $r = 2$ in (2.1.1), we obtain

$$y\left(\sum_{i=0}^{s} b_i{}'y^i\right) + (y^2 + gy + c)\left(\sum_{i=1}^{s} ib_iy^{i-1}\right) = \lambda\left(\sum_{i=0}^{s} b_iy^i\right). \qquad (2.1.2)$$

The coefficient of $y^{s+1}$ in the left-hand side of (2.1.2) is $b_s{}'+sb_s$. If $b_s{}'+sb_s = 0$ then $b_s \mid b_s{}'$, so $b_s{}' = 0$, so $b_s \in k^*$ and $sb_s = b_s{}' + sb_s = 0$, which implies that $s = 0$, contradicting Lemma 2.1.4. So $b_s{}' + sb_s \neq 0$.

Equating the highest power of $y$ terms on both sides in (2.1.2), we get

$$\begin{aligned} b_s{}'y^{s+1} + sb_sy^{s+1} = \lambda b_sy^s &\implies \lambda b_s = (b_s{}' + sb_s)y \\ &\implies b_s \mid (b_s{}' + sb_s) \\ &\implies b_s \mid b_s{}' \\ &\implies b_s \in k^*. \end{aligned}$$

Therefore, for $r = 2$, we have

$$\lambda = sy = sy^{2-1}.$$

If $r \geq 3$, then equating the highest power of $y$ terms on both sides in (2.1.1), we have

$$sb_sy^{r+s-1} = \lambda b_sy^s \implies \lambda = sy^{r-1}.$$

Thus, for $r \geq 2$, we have $\lambda = sy^{r-1}$. Thus, if $D$ has a Darboux polynomial with monomial cofactor $\lambda$, then $\lambda$ is of the form $sy^{r-1}$.

Hence, (2.1.1) becomes

$$y\left(\sum_{i=0}^{s} b_i{}'y^i\right) + (y^r + gy + c)\left(\sum_{i=1}^{s} ib_iy^{i-1}\right) = sy^{r-1}\left(\sum_{i=0}^{s} b_iy^i\right). \qquad (2.1.3)$$

From (2.1.3), comparing the the term free from $y$, we obtain

$$cb_1 = 0 \implies b_1 = 0. \tag{2.1.4}$$

**Lemma 2.1.5.** (a) $s \geq 3$ and there exists $i$ such that $1 < i < s$ and $b_i \neq 0$.

(b) $\deg_y f \geq r$, i.e., $r \leq s$.

We postpone the proof of this lemma to §2.1.2.

**Lemma 2.1.6.** $\deg_y f \geq 4$, *i.e.,* $s \geq 4$.

We postpone the proof of this lemma to §2.1.2.

Again, from (2.1.3), for $1 \leq i \leq r-2$, comparing the coefficients of $y^i$, we get

$$(b_{i-1})' + ib_i g + (i+1)b_{i+1}c = 0. \tag{2.1.5}$$

And for $r-1 \leq i \leq s-1$, comparing the coefficients $y^i$, we obtain

$$(b_{i-1})' + (i-r+1)b_{i-r+1} + ib_i g + (i+1)cb_{i+1} = sb_{i-r+1}. \tag{2.1.6}$$

Again, equating the coefficients of $y^s$ and $y^{s+1}$ from (2.1.3), we get

$$\begin{aligned} (b_{s-1})' + (s-r+1)b_{s-r+1} + sb_s g &= sb_{s-r+1} \\ \implies (b_{s-1})' + sb_s g &= (r-1)b_{s-r+1} \qquad (2.1.7) \\ {b_s}' + (s-r+2)b_{s-r+2} &= sb_{s-r+2} \\ \implies {b_s}' &= (r-2)b_{s-r+2}. \qquad (2.1.8) \end{aligned}$$

If $r \geq 4$, then we compare the coefficients of $y^i$, for $s+2 \leq i \leq s+r-2$ in the (2.1.3), we obtain

$$(i-r+1)b_{i-r+1} = sb_{i-r+1} \implies (s+r-i-1)b_{i-r+1} = 0. \qquad (2.1.9)$$

We have already shown that $b_0 \neq 0$ . Thus, $b_0$ is either a nonzero constant or a non-constant polynomial. In both cases, we will obtain a contradiction. A summary of these cases is depicted in Figure 2.1.

**Case 1.** $b_0$ *is a nonzero constant.*

Using the following lemma, we obtain a contradiction in each of the sub-cases below, based on $\deg_y f$. We postpone the proof of this lemma to §2.1.2.

**Lemma 2.1.7.** *If* $b_0 \in k^*$ *then*

1. For $1 \leq i \leq r-1$, $b_i = 0$.
2. For $r \leq i \leq s$, $\deg b_i = (i-r)\deg g$.

Therefore, $f$ reduces to

$$f = b_0 + b_r y^r + b_{r+1}y^{r+1} + \cdots + b_{s-1}y^{s-1} + b_s y^s.$$

**Sub-case 1.1.** $s = r$.

Then, is this case, we have $f = b_0 + b_s y^s$. By part (a) of Lemma 2.1.5, this case does not occur.

**Sub-case 1.2.** $s \geq (r+1)$.

If $r = 2$, then from (2.1.7), we have

$$(b_{s-1})' + sb_s g = b_{s-1} \implies \deg b_{s-1} = \deg b_s + \deg g. \qquad (2.1.10)$$

From Lemma 2.1.7, we have $\deg b_{s-1} = (s-3)\deg g$ and $\deg b_s = (s-$

2) $\deg g$. Substituting these values in (2.1.10), we get

$$\begin{aligned}(s-3)\deg g &= (s-2)\deg g + \deg g\\ \implies \deg g &= 0.\end{aligned}$$

This contradicts the fact that $\deg g \geq 1$.

If $r \geq 4$, then by Lemma 2.1.7, we see that $\deg b_{s-1} = (s-1-r)\deg g \geq 0$ as $s \geq r+1$ in this case, but from (2.1.9), we have $b_{s-1} = 0$, which is a contradiction.

Summarizing, we see that if $b_0$ is a nonzero constant, then $D$ has no Darboux polynomial with a monomial cofactor.

**Case 2.** *$b_0$ is a non-constant polynomial.*

Note that (2.1.5) with $i = 1$ shows that ${b_0}' = 0 \iff b_2 = 0$, so the hypothesis of Case 2 implies that $b_2 \neq 0$. Using the following lemma, we obtain a contradiction below, based on $\deg_y f$. We postpone the proof of this lemma to §2.1.2.

**Lemma 2.1.8.** *If $b_0 \in k[x] \setminus k$ then for $3 \leq i \leq s$, $\deg b_i = \deg b_2 + (i-2)\deg g$.*

We have seen in Lemma 2.1.5 that $r \leq s$. Suppose $r = 2$. Recall that $s \geq 4$. Then, from (2.1.7)

$$(b_{s-1})' + s b_s g = b_{s-1} \implies \deg b_{s-1} = \deg b_s + \deg g. \tag{2.1.11}$$

By Lemma 2.1.8, we have

$$\deg b_{s-1} = \deg b_2 + (s-3)\deg g,$$

and

$$\deg b_s = \deg b_2 + (s-2)\deg g.$$

Putting these values in (2.1.11), we have

$$\begin{aligned} \deg b_2 + (s-3)\deg g &= \deg b_2 + (s-2)\deg g + \deg g \\ \implies \deg g &= 0. \end{aligned}$$

This contradicts the fact that $\deg g \geq 1$.

If $r \geq 4$, then from Lemma 2.1.8, we see that $\deg b_{s-1} = \deg b_2 + (s-3)\deg g \geq \deg g$, as in this case, we have $s \geq r$, but from (2.1.9), we have $b_{s-1} = 0$, which is a contradiction.

Hence, from all the above cases, we see that $D = y\partial_x + (y^r + gy + c)\partial_y$ does not have any Darboux element $f$ with cofactor $\lambda = sy^{r-1}$, where $s = deg_y f$. Recall, we have already proved that if $D$ has a Darboux polynomial with monomial cofactor $\lambda$, then $\lambda$ is of the form $sy^{r-1}$.

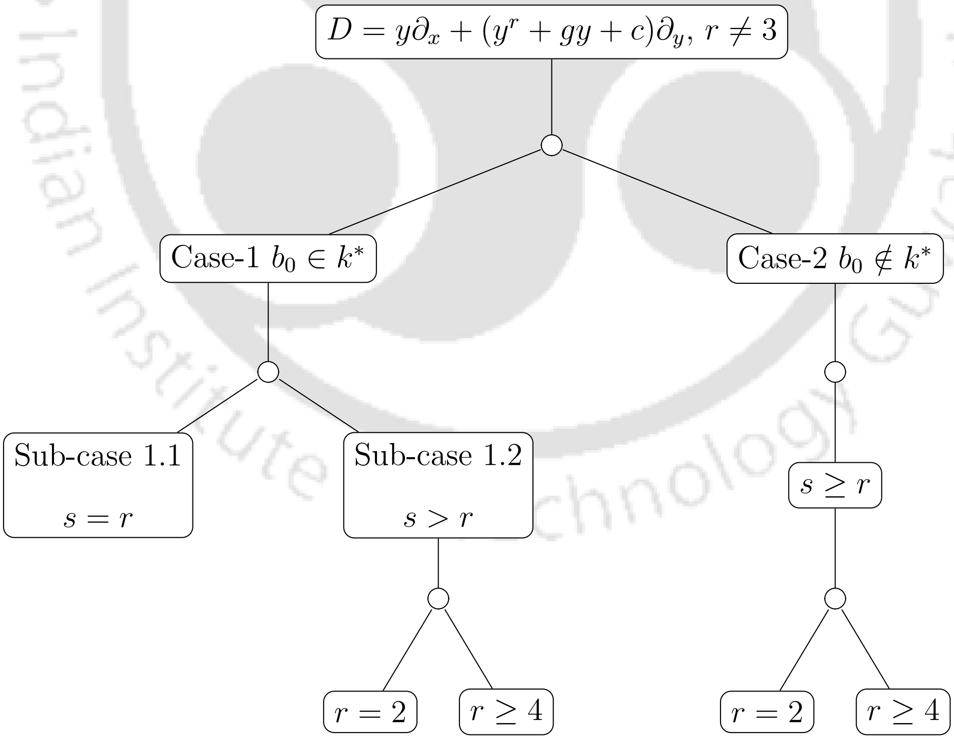

Figure 2.1: The derivation $y\partial_x + (y^r + gy + c)\partial_y$, $r \neq 3$

$\square$

*Proof of Theorem 2.1.3.* Let $f = b_0 + b_1 y + b_2 y^2 + \cdots + b_s y^s \in k[x][y]$ with $b_s \neq 0$, be a Darboux polynomial of $D$, with monomial a cofactor $\lambda$. Recall that $s \geq 4$. Also, as in the proof of Theorem 2.1.1, the cofactor $\lambda$ is of the form $sy^{r-1} = sy^2$, as $r = 3$, and $b_0 \neq 0$.

By Lemma 2.1.8, we have $\deg b_i = \deg b_2 + (i-2)\deg g$, for all $i \geq 3$. In particular, $\deg b_s = \deg b_2 + (s-2)\deg g$ and $\deg b_{s-1} = \deg b_2 + (s-3)\deg g$ which implies $\deg b_s = \deg b_{s-1} + \deg g$. Since $\deg g \geq 2$ and hence $\deg b_s \geq \deg b_{s-1} + 2$, equivalently, $\deg {b_s}' \geq \deg b_{s-1} + 1$, i.e., $\deg {b_s}' > \deg b_{s-1}$ but from (2.1.8), we have ${b_s}' = b_{s-1}$, which is a contradiction.

Hence, we see that $D = y\partial_x + (y^3 + gy + c)\partial_y$ does not have any Darboux polynomial $f$ with cofactor $\lambda = sy^2$, where $s = \deg_y f$.

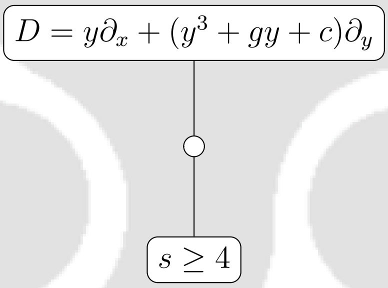

Figure 2.2: The derivation $y\partial_x + (y^3 + gy + c)\partial_y$

$\square$

### 2.1.2 Proofs of the Intermediate Lemmas

We continue to use the notations developed during the proof of the Theorem 2.1.1.

*Proof of Lemma 2.1.5.* Using $\lambda = sy^{r-1}$, the equation $Df = \lambda f$ can be written as

$$\begin{aligned} & yf_x + (y^r + gy + c)f_y = sy^{r-1}f \\ \implies & yf_x + (gy + c)f_y = sy^{r-1}f - y^r f_y = y^{r-1}(sf - yf_y) = y^{r-1}\sum_{i=0}^{s-1}(s-i)b_iy^i \\ \implies & yf_x + (gy + c)f_y = y^{r-1}H, \end{aligned} \tag{2.1.12}$$

where we define $H = \sum_{i=0}^{s-1}(s-i)b_iy^i$.

(a) Indeed, suppose the contrary. Then $f = b_0 + b_sy^s$ and $H = sb_0$, so (2.1.12) gives $y(b_0{}' + b_s{}'y^s) + (gy + c)sb_sy^{s-1} = sb_0y^{r-1}$, so

$$b_0{}'y + csb_sy^{s-1} + sb_sgy^s + b_s{}'y^{s+1} = sb_0y^{r-1}. \tag{2.1.13}$$

If $s = 1$ then (2.1.13) reads $cb_1 + (b_0{}' + b_1g)y + b_1{}'y^2 = b_0y^{r-1}$. Since $b_1 = 0$, we get $b_0{}'y = b_0y^{r-1}$, which implies that $b_0 = b_0{}'$. This is impossible because $b_0 \neq 0$.

If $s = 2$ then (2.1.13) reads $(b_0{}' + 2cb_2)y + 2b_2gy^2 + b_2{}'y^3 = 2b_0y^{r-1}$. If $b_2 = 0$ then this gives $b_0{}' = 2b_0$, which is impossible. So $2b_2g \neq 0$ and consequently $b_0{}' + 2cb_2 = 0$, $2b_2gy^2 = 2b_0y^{r-1}$ and $b_2{}' = 0$. One can see that this implies that $r = 3$ and $\deg g = 1$, which contradicts the fact that if $r = 3$ then $\deg g \geq 2$.

It follows that $s \geq 3$. But this is impossible, because the left-hand side of (2.1.13) contains at least two non-zero terms ($csb_sy^{s-1}$ and $sb_sgy^s$) whereas the right-hand side contains only one. This contradiction completes the proof of the claim.

Since there exists $i$ such that $0 < i < s$ and $b_i \neq 0$, and since $b_1 = 0$ by (2.1.4), it follows that

$$s \geq 3 \text{ and there exists } i \text{ such that } 1 < i < s \text{ and } b_i \neq 0.$$

(b) By part (a) of Lemma 2.1.5, it follows that $\deg_y H \geq 2$. So (2.1.12) gives

$$r + 1 \leq \deg_y(y^{r-1}H) = \deg_y(yf_x + (gy + c)f_y) \leq s + 1$$

which proves that $r \leq s$.

□

*Proof of Lemma 2.1.6.* By part (a) of Lemma 2.1.5, we know that $s \geq 3$. Arguing by contradiction, suppose that $s = 3$. Since $b_1 = 0$ (2.1.4), we have $f = b_0 + b_2y^2 + b_3y^3$, where $b_i \in k[x]$ and $b_3 \neq 0$. Since we proved that $r \leq s$, we have $r \in \{2, 3\}$.

If $r = 2$ then $\lambda = sy^{r-1} = 3y$, so $D(f) - \lambda f = D(b_0 + b_2y^2 + b_3y^3) - 3y(b_0 + b_2y^2 + b_3y^3)$ can be computed explicitly:

$$D(f) - \lambda f = {b_3}'y^4 + (3gb_3 + {b_2}' - b_2)y^3 + (2gb_2 + 3cb_3)y^2 + (2cb_2 + {b_0}' - 3b_0)y.$$

Since the above is the zero polynomial in $k[x][y]$, we have ${b_3}' = 0$ and $2gb_2 + 3cb_3 = 0$. The first equation implies that $b_3 \in k^*$ and the second gives $2gb_2 = -3cb_3$, so $g \in k^*$, a contradiction. So the case $r = 2$ is impossible. If $r = 3$ then $\lambda = sy^{r-1} = sy^2$ and

$$D(f) - \lambda f = ({b_3}' - b_2)y^4 + (3gb_3 + {b_2}')y^3 + (2gb_2 + 3cb_3 - 3b_0)y^2 + (2cb_2 + {b_0}')y.$$

So ${b_3}' - b_2 = 0$ and $3gb_3 + {b_2}' = 0$, which implies that $3gb_3 = -{b_2}' = -{b_3}''$, so $b_3 \mid {b_3}''$, so ${b_3}'' = 0$ and consequently $3gb_3 = 0$, a contradiction. So the case $r = 3$ too is impossible. This completes the proof that $s \geq 4$. □

*Proof of Lemma 2.1.7.* We prove these using induction on $i$. From Eqn. (2.1.4), we have $b_1 = 0$. Suppose $b_i = 0$, for all $1 \leq i \leq m < r - 1$, then from (2.1.5), we have

$$\begin{aligned}(b_{m-1})' + mb_m g + (m+1)b_{m+1}c &= 0\\ \implies b_{m+1} &= 0.\end{aligned}$$

Hence, by induction $b_i = 0$, for all $1 \le i \le r-1$. This completes the proof of the first part of the lemma.

For the second part of the lemma, substituting $i = r-1$ in (2.1.6), we have

$$\begin{aligned}(b_{r-2})' + (r-1)b_{r-1}g + rb_r c &= sb_0\\ \implies b_r &= \frac{sb_0}{rc} \text{ (constant)}.\end{aligned}$$

Thus, $\deg b_r = 0 = (r-r)\deg g$. Suppose that $\deg b_i = (i-r)\deg g$, for all $r \le i \le m < s$. Then, from (2.1.6), we have

$$(b_{m-1})' + (m-r+1)b_{m-r+1} + mb_m g + (m+1)cb_{m+1} = sb_{m-r+1}.$$

Now, since the degrees of $(b_{m-1})'$ and $b_{m-r+1}$ are less than the $\deg b_m$, we get

$$\begin{aligned}\deg b_{m+1} &= \deg b_m + \deg g\\ &= (m-r)\deg g + \deg g\\ &= (m+1-r)\deg g\end{aligned}$$

Hence, by induction $\deg b_i = (i-r)\deg g$, for all $r \le i \le s$. □

*Proof of Lemma 2.1.8.* We prove this using induction on $i$. If $r = 2$, then from (2.1.6) for $i = 2$, we have

$$2b_2 g + 3cb_3 = 0 \implies \deg b_3 = \deg b_2 + \deg g$$

If $r = 3$, then from (2.1.5) for $i = 1$, we get ${b_0}' + 2cb_2 = 0$, and from (2.1.6) for $i = 2$, we get

$$2b_2g + 3b_3c = sb_0$$

Note that, for $r = 3$, $\deg g \geq 2$ and hence $\deg b_2g = \deg {b_0}'g \geq \deg b_0 + 1$. Therefore

$$\deg b_3 = \deg b_2 + \deg g.$$

If $r > 3$, then from (2.1.5) for $i = 2$, we have

$$2b_2g + 3b_3c = 0$$

$$\implies \deg b_3 = \deg b_2 + \deg g.$$

Suppose that $\deg b_i = \deg b_2 + (i-2)\deg g$, for all $3 \leq i \leq m < s$, then from (2.1.5) and (2.1.6), we obtain

$$(b_{m-1})' + (m-r+1)b_{m-r+1} + mb_mg + (m+1)cb_{m+1} = sb_{m-r+1},$$

where $b_{m-r+1} = 0$, for all $m \leq r-2$. Now, degrees of $(b_{m-1})'$ and $b_{m-r+1}$ are less than the $\deg b_mg$, we get

$$\deg b_{m+1} = \deg b_m + \deg g = \deg b_2 + (m-2)\deg g + \deg g = \deg b_2 + (m-1)\deg g.$$

Therefore, by the induction, $\deg b_i = \deg b_2 + (i-2)\deg g$, for all $i \geq 3$. □

# Chapter 3

# Some Simple $k$-derivations of $k[x, y]$

Let $k$ denotes a field of characteristic zero. In this chapter, we consider the class of $k$-derivations $y^2\partial_x + (xy + h)\partial_y$, where $h \in k[y]$, $\gcd(y, h) = 1$ and $\deg_y h \leq 1$ and $y\partial_x + (y^2 + xy + 1)\partial_y$ of the polynomial ring $k[x, y]$, and we show that these are simple derivations of the polynomial ring $k[x, y]$. In the view of Proposition 1.3.5, we may assume that $k$ is algebraically closed.

## 3.1 The Derivation $y^2\partial_x + (xy + h)\partial_y$ of $k[x, y]$

Our discussion in this section focuses on the simplicity of the $k$-derivation $y^2\partial_x + (xy + h)\partial_y$, where $h \in k[y]$, $\gcd(y, h) = 1$, and $\deg_y h \leq 1$ of the polynomial ring $k[x, y]$. For $h \in k[y]$, let $\mathfrak{D}_h$ denote the $k$-derivation $y^2\partial_x + (xy + h)\partial_y$. In this section, we prove the following theorem.

**Theorem 3.1.1.** *Suppose* $h \in k[y]$ *be such that* $\gcd(y, h) = 1$, *and* $\deg_y h \leq 1$. *Then* $\mathfrak{D}_h$ *is a simple derivation of* $k[x, y]$.

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[1] The results in this chapter are published in [KW24]. .

We first prove the following Lemma.

**Lemma 3.1.2.** *Suppose* $h \in k[y]$ *be such that* $\gcd(y,h)=1$*, and* $\deg_y h \leq 1$*. Then* $\mathfrak{D}_h$ *is simple iff* $\mathfrak{D}_1$ *is simple.*

*Proof.* Put $h = ay+b$, with $b \neq 0$. Let $\alpha \in k$ be such that $\alpha^2 b = 1$. Consider the $k$-algebra automorphism $\sigma : k[x,y] \to k[x,y]$ given by $\sigma(x) = \alpha(x+b)$, $\sigma(y) = \alpha y$. Then $\sigma^{-1}\mathfrak{D}_h\sigma = \alpha^{-1}\left(y^2\partial_x + (xy+1)\partial_y\right) = \alpha^{-1}\mathfrak{D}_1$. Therefore, $\mathfrak{D}_h$ is simple if and only if $\mathfrak{D}_1$ is simple. ☐

Therefore, to prove the simplicity of $\mathfrak{D}_h = y^2\partial_x+(xy+h)\partial_y$, where $h \in k[y]$, $\gcd(y,h)=1$, and $\deg_y h \leq 1$, it suffices to prove that $\mathfrak{D}_1 = y^2\partial_x + (xy+1)\partial_y$ is a simple derivation of $k[x,y]$.

Throughout this section we write $d$ for the derivation $\mathfrak{D}_1 = y^2\partial_x+(xy+1)\partial_y$.

We use Proposition 1.3.10 to prove that $d$ is simple.

We now recall the following definition.

**Definition 3.1.3.** *Let* $f \in k(x)$ *be a rational function. Then a point* $x_0 \in k$ *is called a pole of order* $m$ *of* $f(x)$*, if* $\frac{1}{f}$ *has a zero of order* $m$ *at* $x_0$ *i.e. if* $(x-x_0)^m f(x) \in k[x]_{\langle x-x_0\rangle}$ *and* $(x-x_0)^{m-1}f(x) \notin k[x]_{\langle x-x_0\rangle}$*. We denote this number* $m$ *by pol-ord*$_{x=x_0} f$*.*

We now prove some lemmas.

**Lemma 3.1.4.** *Considering* $d$ *to be the derivation of* $k(y)[x]$*, let* $f = \sum_{i=0}^{n} a_i x^i$ *be a Darboux polynomial of the derivation* $d$ *in the polynomial ring* $k(y)[x]$ *with* $a_n = 1$ *and* $n \geq 1$*. Then* $a_i \in k[y]_y$ *for* $0 \leq i \leq n-1$*.*

*Proof.* Let $\mu$ be the cofactor of Darboux polynomial $f$ in the ring $k(y)[x]$, i.e., $d(f) = \mu f$. Thus,

$$y^2\sum_{i=1}^{n} i a_i x^{i-1} + (xy+1)\sum_{i=0}^{n}(a_i)_y x^i = \mu\sum_{i=0}^{n} a_i x^i, \qquad (3.1.1)$$

where $(a_i)_y$ denotes the differentiation of $a_i$ in $k(y)$ with respect to $y$.

**Claim 3.1.1.** $\mu \neq 0$

If possible suppose $\mu = 0$ then from (3.1.1), we have

$$y^2 \sum_{i=1}^{n} i a_i x^{i-1} + (xy+1) \sum_{i=0}^{n} (a_i)_y x^i = 0 \tag{3.1.2}$$

Equating coefficient of $x^n$ in (3.1.2) to 0, we get

$$y(a_{n-1})_y = 0 \implies (a_{n-1})_y = 0$$

Equating coefficient of $x^{n-1}$ in (3.1.2) to 0, we have

$$ny^2 + (a_{n-1})_y + y(a_{n-2})_y = 0 \implies a_{n-2} = -\frac{ny^2}{2} + c_{n-2}$$

where $c_{n-2} \in k$. Comparing the coefficients of $x^{n-2}$ in (3.1.2) yields

$$(n-1)a_{n-1}y^2 + (a_{n-2})_y + y(a_{n-3})_y = 0 \implies a_{n-3} = ny - \frac{(n-1)y^2 a_{n-1}}{2} + c_{n-3}$$

where $c_{n-3} \in k$. Comparing the coefficients of $x^{n-3}$ in (3.1.2), we have

$$\begin{aligned} &(n-2)a_{n-2}y^2 + (a_{n-3})_y + y(a_{n-4})_y = 0 \\ \implies &(a_{n-4})_y = \frac{n(n-2)y^3}{2} - c_{n-2}y + (n-1)a_{n-1} - \frac{n}{y} \\ \implies &a_{n-4} = \frac{n(n-2)y^4}{8} - \frac{c_{n-2}y^2}{2} + (n-1)a_{n-1}y - n\ln y + c_{n-4} \end{aligned}$$

where $c_{n-4} \in k$. This shows that $a_{n-4}$ contains a term involving $\ln y$, which is a contradiction as $a_{n-4} \in k(y)$. This completes the proof of the claim.

In (3.1.1), Since $(a_n)_y = 0$, it follows that $d(f)$ has degree at most $n$ in $x$, i.e., $\deg_x(f) \geq \deg_x(df)$. Therefore, $\mu \in k(y)$ and $\deg_x(f) = \deg_x(df)$.

Comparing the coefficient of $x^n$ in (3.1.1), we get

$$\mu = y(a_{n-1})_y.$$

It follows that

$$d(f) = y(a_{n-1})_y f,$$

and hence that, for $0 \leq i \leq n$,

$$y(a_{i-1})_y = y(a_{n-1})_y a_i - (a_i)_y - y^2(i+1)a_{i+1}, \tag{3.1.3}$$

where $a_{-1} = 0$.

By (3.1.3), it is sufficient to demonstrate that $a_{n-1} \in k[y]_y$. Assume, if possible, that $a_{n-1} \notin k[y]_y$. Then $a_{n-1}$ will have a pole at some nonzero element, say $\alpha$ of order, say $m > 0$.

**Claim.** pol-ord$_{y=\alpha} a_{n-j} = jm$, for every, $j = 1, \ldots, n$.

We shall prove the claim using induction on $j$. It is clear that the result is valid for $j = 1$. Let's suppose for $1 \leq i \leq j < n$, $a_{n-i}$ has a pole of order $im$ at $\alpha$. Substituting $i = n - j$ in (3.1.3), we get

$$y(a_{n-j-1})_y = y(a_{n-1})_y a_{n-j} - (a_{n-j})_y - (n-j+1)y^2 a_{n-j+1}. \tag{3.1.4}$$

Note that pol-ord$_{y=\alpha} y(a_{n-1})_y a_{n-j} = (j+1)m+1$, pol-ord$_{y=\alpha}(a_{n-j})_y = jm+1$, and pol-ord$_{y=\alpha} y^2 a_{n-j+1} = (j-1)m$. Thus by (3.1.4), pol-ord$_{y=\alpha} y(a_{n-j-1})_y = (j+1)m+1$. This gives pol-ord$_{y=\alpha} a_{n-j-1} = (j+1)m$. This completes the proof of the claim.

In particular, pol-ord$_{y=\alpha} a_0 = nm$. But from (3.1.3), for $i = 0$, we have

$$y(a_{n-1})_y a_0 = (a_0)_y + y^2 a_1. \tag{3.1.5}$$

Therefore, pol-ord$_{y=\alpha}(a_0)_y = nm + m + 1$, i.e., pol-ord$_{y=\alpha} a_0 = nm + m$.

This is a contradiction as $\text{pol-ord}_{y=\alpha} a_0 = nm$. ☐

**Lemma 3.1.5.** *Let $f = \sum_{i=0}^{n} a_i x^i$ be a Darboux polynomial of the derivation $d$ in the polynomial ring $k[y]_y[x]$ with $a_n = 1$ and $n \geq 1$. Then $a_{n-1} = \frac{\alpha}{y}$ for some integer $\alpha > 0$.*

*Proof.* Firstly, we shall show that $a_{n-1} \in k[1/y]$. Assume, if possible, that $a_{n-1} \notin k[1/y]$. Then $\deg_y(a_{n-1})$ is well defined and is positive. Put $\deg_y(a_{n-1}) = s \geq 1$.

**Case 1.** $s = 1$.

In this case, $a_{n-1}$ has the form $a_{n-1} = \beta y + \alpha_0 + \frac{\alpha}{y} + \frac{\alpha_2}{y^2} + \cdots + \frac{\alpha_l}{y^l}$, where $\beta, \alpha_0, \alpha, \alpha_i \in k$ for $i = 2, \ldots, l$ and $\beta \neq 0$. Then, as $a_{n-2} \in k[y]_y$, the constant term of $y(a_{n-2})_y$ will be 0. However, by (3.1.3), we have

$$y(a_{n-2})_y = y(a_{n-1})_y a_{n-1} - (a_{n-1})_y - ny^2.$$

Note that the constant term of $y(a_{n-1})_y a_{n-1}$ is 0, the constant term of $(a_{n-1})_y$ is $\beta$, and the constant term of $ny^2 = 0$. Therefore the constant term of $y(a_{n-2})_y$ is $-\beta$, which is nonzero. This is a contradiction.

**Case 2.** $s > 1$.

**Claim.** $\deg_y(a_{n-j}) = js$, for every, $j = 1, \ldots, n$.

We shall prove the claim using induction on $j$. Clearly, the result is true for $j = 1$. Suppose $\deg_y(a_{n-i}) = is$, for $1 \leq i \leq j < n$. Putting $i = n - j$ in (3.1.3), we have

$$y(a_{n-j-1})_y = y(a_{n-1})_y a_{n-j} - (a_{n-j})_y - (n-j+1)y^2 a_{n-j+1}. \tag{3.1.6}$$

Note that $\deg_y\left(y(a_{n-1})_y a_{n-j}\right) = (j+1)s$, $\deg_y\left((a_{n-j})_y\right) = js - 1$, and $\deg_y\left(y^2 a_{n-j+1}\right) = (j-1)s + 2$. Therefore by (3.1.6), $\deg_y\left(y(a_{n-j-1})_y\right) = (j+1)s$, i.e., $\deg_y\left(a_{n-j-1}\right) = (j+1)s$.

In particular, $\deg_y a_0 = ns$. But from (3.1.5), $\deg_y a_0 = ns+s+1$, which is a contradiction as $\deg_y a_0 = ns$.

Therefore, $a_{n-1}$ has the following form

$$a_{n-1} = \alpha_0 + \frac{\alpha}{y} + \frac{\alpha_2}{y^2} + \cdots + \frac{\alpha_l}{y^l},$$

where $\alpha_0, \alpha, \alpha_i \in k, \quad \forall \quad i = 2, \ldots, l$. There now follows a sequence of sublemmas that will complete the proof of Lemma 3.1.5. □

**Sublemma 1.** $\alpha_i = 0, \quad \forall \quad i > 1$.

*Proof.* If possible, suppose there exists $i > 1$ such that $\alpha_i \neq 0$. Then $a_{n-1}$ has a pole of the order, say $t > 1$, at $y = 0$.

**Claim.** $\text{pol-ord}_{y=0} a_{n-j} = jt$, for every, $j = 1, \ldots, n$.

We shall prove the claim using induction on $j$. Clearly, the claim is true for $j = 1$. Suppose $\text{pol-ord}_{y=0} a_{n-i} = it$, for $1 \leq i \leq j < n$. From (3.1.4), we have

$$\text{pol-ord}_{y=0} y(a_{n-1})_y a_{n-j} = (j+1)t,$$

$$\text{pol-ord}_{y=0} (a_{n-j})_y = jt+1,$$

$$\text{pol-ord}_{y=0} y^2 a_{n-j+1} = (j-1)t-2.$$

Thus by (3.1.4), $\text{pol-ord}_{y=0} y(a_{n-j-1})_y = (j+1)t$. This gives $\text{pol-ord}_{y=0} a_{n-j-1} = (j+1)t$. This completes the proof of the claim.

In particular, $\text{pol-ord}_{y=0} a_0 = nt$. But from (3.1.5), $\text{pol-ord}_{y=0} a_0 = nt+t-1$, which is a contradiction as $\text{pol-ord}_{y=0} a_0 = nt$.

Therefore, $a_{n-1}$ reduces to $a_{n-1} = \alpha_0 + \frac{\alpha}{y}$, where $\alpha_0, \alpha \in k$. □

**Sublemma 2.** $\alpha_0 = 0$

*Proof.* If possible, suppose $\alpha_0 \neq 0$. Then $a_{n-1} = \alpha_0 + \frac{\alpha}{y}$, and $(a_{n-1})_y = -\frac{\alpha}{y^2}$.

**Claim.** For all $j \geq 1$, we have:

- if $j \leq \left\lfloor \dfrac{n+1}{2} \right\rfloor$, then $\deg a_{n-2j} = 2j$,
- if $j \leq \left\lfloor \dfrac{n}{2} \right\rfloor$, then $\deg a_{n-(2j+1)} = 2j$.

We shall prove the claim using induction on $j$. Putting $i = n-1$ in (3.1.3), we have

$$\begin{aligned}
& y(a_{n-2})_y = y(a_{n-1})_y a_{n-1} - (a_{n-1})_y - ny^2 \\
\implies\ & y(a_{n-2})_y = y\left(-\frac{\alpha}{y^2}\right)\left(\alpha_0 + \frac{\alpha}{y}\right) + \frac{\alpha}{y^2} - ny^2 \\
\implies\ & (a_{n-2})_y = \left(-\frac{\alpha}{y^2}\right)\left(\alpha_0 + \frac{\alpha}{y}\right) + \frac{\alpha}{y^3} - ny \\
\implies\ & \deg a_{n-2} = 2.
\end{aligned}$$

Again, from (3.1.3), for $i = n-2$, we get

$$\begin{aligned}
& y(a_{n-3})_y = y(a_{n-1})_y a_{n-2} - (a_{n-2})_y - (n-1)a_{n-1}y^2 \\
\implies\ & (a_{n-3})_y = -\frac{\alpha a_{n-2}}{y^2} - \frac{(a_{n-2})_y}{y} - (n-1)\left(\alpha_0 + \frac{\alpha}{y}\right) y \\
\implies\ & (a_{n-3})_y = -\frac{\alpha a_{n-2}}{y^2} - \frac{(a_{n-2})_y}{y} - (n-1)\left(\alpha_0 y + \alpha\right) \\
\implies\ & \deg a_{n-3} = 2.
\end{aligned}$$

Thus, the claim is true for $j = 1$. Suppose $\deg a_{n-2j} = 2j$ for some $1 \leq j < \left\lfloor \dfrac{n+1}{2} \right\rfloor$ and $\deg a_{n-(2j+1)} = 2j$ for some $1 \leq j < \left\lfloor \dfrac{n}{2} \right\rfloor$. Putting

$i = n - 2j - 1$ in (3.1.3), we obtain

$$\begin{aligned} y(a_{n-2j-2})_y &= y(a_{n-1})_y a_{n-2j-1} - (a_{n-2j-1})_y - (n-2j)a_{n-2j}y^2 \\ \implies (a_{n-2j-2})_y &= -\frac{\alpha a_{n-2j-1}}{y^2} - \frac{(a_{n-2j-1})_y}{y} - (n-2j)a_{n-2j}y \\ \implies \deg a_{n-2j-2} &= \deg a_{n-2j} + 2 = 2(j+1). \end{aligned}$$

Again, from (3.1.3), for $i = n - 2j - 2$, we have

$$\begin{aligned} y(a_{n-2j-3})_y &= y(a_{n-1})_y a_{n-2j-2} - (a_{n-2j-2})_y - (n-2j-1)a_{n-2j-1}y^2 \\ \implies (a_{n-2j-3})_y &= -\frac{\alpha a_{n-2j-2}}{y^2} - \frac{(a_{n-2j-2})_y}{y} - (n-2j-1)a_{n-2j-1}y \\ \implies \deg a_{n-2j-3} &= \deg a_{n-2j-1} + 2 = 2(j+1). \end{aligned}$$

This completes the proof of the claim. In particular, the degree of $0 = a_{-1}$ is positive, which is a contradiction. Hence $\alpha_0 = 0$. Therefore, $a_{n-1}$ reduces to

$$a_{n-1} = \frac{\alpha}{y}.$$

□

**Sublemma 3.** $\alpha \neq 0$.

*Proof.* If possible, suppose $\alpha = 0$. Then we shall get a contradiction in each of the following cases, which will conclude that $\alpha \neq 0$.

**Case 1.** $n = 1$.

Then, in this case, $a_0 = 0$ and $a_1 = 1$. Putting $i = 0$ in (3.1.3), we obtain

$$y(a_{-1})_y = y(a_0)_y a_0 - (a_0)_y - y^2 \implies 0 = -y^2,$$

which is absurd.

**Case 2.** $n = 2$.

Then, in this case, $a_1 = 0$ and $a_2 = 1$. From (3.1.3), for $i = 0$, we get $(a_0)_y = 0$. Again, from (3.1.3), for $i = 1$, we have

$$y(a_0)_y = -2y^2 \implies -2y^2 = 0,$$

which is absurd.

**Case 3.** $n = 3$.

Then, in this case, $a_2 = 0$ and $a_3 = 1$. From, (3.1.3), for $i = 0, 1, 2$, we have

$$(a_0)_y = -y^2 a_1 \tag{3.1.7}$$

$$y(a_0)_y = -(a_1)_y \tag{3.1.8}$$

$$(a_1)_y = -3y \tag{3.1.9}$$

Putting the values of $(a_0)_y$ and $(a_1)_y$ from (3.1.7) and (3.1.9) into (3.1.8), we obtain $a_1 = -\frac{3}{y^2}$, which implies $(a_1)_y = \frac{6}{y^3}$, which is a contradiction as we have $(a_1)_y = -3y$ from (3.1.9).

**Case 4.** $n \geq 4$.

Then, in this case, $a_{n-1} = 0$ and $a_n = 1$. From, (3.1.3), for $i = n - 1$, we obtain

$$(a_{n-2})_y = -ny \implies a_{n-2} = -\frac{ny^2}{2} + b_{n-2}, \tag{3.1.10}$$

where $b_{n-2} \in k$. Putting $i = n - 2$ in (3.1.3), we have

$$y(a_{n-3})_y = -(a_{n-2})_y \implies (a_{n-3})_y = n \implies a_{n-3} = ny + b_{n-3}, \tag{3.1.11}$$

where $b_{n-3} \in k$. Again, from (3.1.3), for $i = n-3$, we get

$$\begin{aligned}
& y(a_{n-4})_y = -(a_{n-3})_y - (n-2)a_{n-2}y^2 \\
\implies & y(a_{n-4})_y = -n - (n-2)y^2\left(-\frac{ny^2}{2} + b_{n-2}\right) \\
\implies & (a_{n-4})_y = -\frac{n}{y} + \frac{n(n-2)y^3}{2} - (n-2)b_{n-2}y \\
\implies & a_{n-4} = -n\ln y + \frac{n(n-2)y^4}{8} + \frac{(n-2)b_{n-2}y^2}{2}.
\end{aligned} \tag{3.1.12}$$

Thus, it shows that $a_{n-4}$ contains a term involving $\ln y$, which is a contradiction as $a_{n-4} \in k[y]_y$.

Hence $\alpha \neq 0$. □

**Sublemma 4.** *$\alpha$ is a strictly positive integer.*

*Proof.* We have $a_{n-1} = \frac{\alpha}{y}$, which implies $(a_{n-1})_y = -\frac{\alpha}{y^2}$. From (3.1.3), for $i = n-1$, we have

$$\begin{aligned}
& y(a_{n-2})_y = y(a_{n-1})_y a_{n-1} - (a_{n-1})_y - ny^2 \\
\implies & y(a_{n-2})_y = y\left(-\frac{\alpha}{y^2}\right)\frac{\alpha}{y} + \frac{\alpha}{y^2} - ny^2 \\
\implies & y(a_{n-2})_y = -\frac{\alpha(\alpha-1)}{y^2} - ny^2 \\
\implies & a_{n-2} = \frac{\alpha(\alpha-1)}{2y^2} - \frac{ny^2}{2} + c_{n-2},
\end{aligned} \tag{3.1.13}$$

where $c_{n-2} \in k$. Putting $i = n-2$ in (3.1.3), we get

$$
\begin{aligned}
y(a_{n-3})_y &= y(a_{n-1})_y a_{n-2} - (a_{n-2})_y - (n-1)a_{n-1}y^2 \\
\implies y(a_{n-3})_y &= y\left(-\frac{\alpha}{y^2}\right)\left(\frac{\alpha(\alpha-1)}{2y^2} - \frac{ny^2}{2} + c_{n-2}\right) \\
&\qquad - \left(-\frac{\alpha(\alpha-1)}{y^3} - ny\right) - (n-1)\alpha y \\
\implies y(a_{n-3})_y &= -\frac{\alpha(\alpha-1)(\alpha-2)}{2y^3} + \left(n-\alpha+\frac{n\alpha}{2}\right)y - \frac{\alpha c_{n-2}}{y} \\
\implies (a_{n-3})_y &= -\frac{\alpha(\alpha-1)(\alpha-2)}{2y^4} + \left(n-\alpha+\frac{n\alpha}{2}\right) - \frac{\alpha c_{n-2}}{y^2} \\
\implies a_{n-3} &= \frac{\alpha(\alpha-1)(\alpha-2)}{6y^3} + \left(n-\alpha+\frac{n\alpha}{2}\right)y + \frac{\alpha c_{n-2}}{y} + c_{n-3},
\end{aligned}
\tag{3.1.14}
$$

where $c_{n-3} \in k$. Proceeding in this fashion, we see that the coefficient of $y^{-j}$ in $a_{n-j}$ is $\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$ for $1 \leq j \leq n+1$. Since $0 = a_{-1}$ and $\alpha \neq 0$, it follows that $\alpha$ is one of $1, 2, \ldots, n$. This completes the proof of this sublemma. □

**Lemma 3.1.6.** *d does not have a Darboux polynomial.*

*Proof.* Suppose $d$ has a Darboux polynomial say $g$. Put $d(g) = \lambda g$ for some $\lambda \in k[x,y]$. Note that $g \notin k[y]$ for if $g \in k[y]$, then $dg = (xy+1)\partial_y(g) = \lambda g$.

- If $\lambda = 0$, then $(xy+1)\partial_y g = 0 \implies \partial_y g = 0 \implies g \in k$, which is impossible.
- If $\lambda \neq 0$, then as $xy+1$ is irreducible and $\deg_x(xy+1) = 1$, we have $g \mid \partial_y(g)$, hence $g \in k$, which is impossible.

Note that $\langle g \rangle$ is a nonzero proper ideal of $k(y)[x]$, which is invariant under the derivation $d$. Hence $k(y)[x]$ is not simple. Write

$$g = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0,$$

where $b_n \neq 0,\ n \geq 1$, and $b_i \in k[y]$. Put $f = \frac{g}{b_n}$, then

$$\begin{aligned} d(f) &= d\left(\frac{g}{b_n}\right) \\ &= \frac{b_n d(g) - g d(b_n)}{b_n^2} \\ &= \frac{\lambda b_n - d(b_n)}{b_n^2} g \\ &= \left(\frac{\lambda b_n - d(b_n)}{b_n}\right) f \\ &= \mu f, \end{aligned}$$

where $\mu = \dfrac{\lambda b_n - d(b_n)}{b_n^2}$. Thus, it shows that $f$ is a Darboux polynomial of $d$ in $k(y)[x]$. Put $a_i = \dfrac{b_i}{b_n}$, then

$$f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

where $n \geq 1$, $a_n = 1$, and for every $0 \leq i \leq n$ $a_i = \frac{b_i}{b_n}$.

By Lemmas 3.1.4 and 3.1.5, for every $0 \leq i \leq n-2$, $a_i \in k[y]_y$ and $a_{n-1} = \frac{\alpha}{y}$, where $\alpha > 0$. Then we shall arrive at a contradiction in each of the following cases, showing that such a $g$ does not exist.

**Case 1.** $n = 1$.

From (3.1.13), for $n = 1$, we have

$$a_{-1} = \frac{\alpha(\alpha - 1)}{2y^2} - \frac{y^2}{2} + c_{-1} = 0.$$

Comparing the coefficients of $y^2$ on both sides, we get $-\frac{1}{2} = 0$, which is absurd.

**Case 2.** $n = 2$.

From (3.1.14), for $n = 2$, we have

$$a_{-1} = \frac{\alpha(\alpha-1)(\alpha-2)}{6y^3} + 2y + \frac{\alpha c_0}{y} + c_{-1} = 0.$$

Comparing the coefficients of $y$ on both sides, we get $2 = 0$, which is absurd.

**Case 3.** $n \geq 3$.

The constant term in $y(a_{n-4})_y$ will be 0. From, (3.1.3) for $i = n-3$, we have

$$\begin{aligned} y(a_{n-4})_y &= y(a_{n-1})_y a_{n-3} - (a_{n-3})_y - (n-2)a_{n-2}y^2 \\ \implies y(a_{n-4})_y &= -\frac{\alpha}{y}a_{n-3} - (a_{n-3})_y - (n-2)a_{n-2}y^2. \end{aligned}$$

Thus, the constant term of $y(a_{n-4})_y = -\alpha($ coefficient of $y$ in $a_{n-3})$ $-$ constant term of $(a_{n-3})_y - (n-2)($ coefficient of $\frac{1}{y^2}$ in $a_{n-2})$.
By (3.1.14), the coefficient of $y$ in $a_{n-3}$ is $\left(n - \alpha + \frac{n\alpha}{2}\right)$, the constant term of $(a_{n-3})_y$ is $\left(n - \alpha + \frac{n\alpha}{2}\right)$ and by (3.1.13), the coefficient of $\frac{1}{y^2}$ in $a_{n-2}$ is $\frac{\alpha(\alpha-1)}{2}$. Therefore, the constant term of $y(a_{n-4})_y$ is

$$\begin{aligned} -\alpha\left(n - \alpha + \frac{n\alpha}{2}\right) - \left(n - \alpha + \frac{n\alpha}{2}\right) - \frac{\alpha(\alpha-1)(n-2)}{2} &= 0 \\ \implies (2-n)\alpha^2 = n\alpha + n. \end{aligned} \tag{3.1.15}$$

Note that the L.H.S. of (3.1.15) is a negative quantity as $n \geq 3$ and $\alpha > 0$, whereas the R.H.S. is positive, which is absurd.

Therefore, $d$ does not have a Darboux polynomial. □

**Theorem 3.1.7.** *Let $x, y$ be indeterminates over $k$ and $k[x,y]$ be the polynomial ring. The derivation $d = y^2\partial_x + (xy+1)\partial_y$ is a simple derivation of the ring $k[x,y]$.*

*Proof.* Note that $\langle dx, dy \rangle = k[x,y]$. So, by Proposition 1.3.10, it suffices to show that $d$ does not have any Darboux polynomial. In Lemma 3.1.6, we have demonstrated that $d$ does not have Darboux polynomial. Therefore the polynomial ring $k[x,y]$ is d-simple. $\square$

## 3.2 The Derivation $y\partial_x+(y^2+xy+1)\partial_y$ of $k[x,y]$

In this section, we study the $k$-derivation of the form $D = y\partial_x + (y^2 + xy + 1)\partial_y$. A. Maciejewski, J. Moulin-Ollagnier, and A. Nowicki have shown that the derivation $\partial_x + (y^2 + ay + b)\partial_y$, where $a(x), b(x) \in k[x]$ is equivalent to the derivation $\Delta_p$ where, $\Delta_p = \partial_x + (y^2 - p)\partial_y$, where $p(x) \in k[x]$ [MMON01]. They have further given some conditions on $p(x)$ under which $\Delta_p$ is simple. This derivation gives us the motivation to study the derivation $y\partial_x + (y^2 + xy + 1)\partial_y$. We prove that this is a simple derivation of $k[x,y]$.

We use Proposition 1.3.10 to prove the following theorem.

**Theorem 3.2.1.** *The ring $k[x,y]$ is a $D$-simple where $D = y\partial_x + (y^2 + xy + 1)\partial_y$ represents a $k$-derivation of $k[x,y]$.*

*Proof.* By Proposition 1.3.10, it suffices to show that $\Big\langle D(x), D(y)\Big\rangle = k[x,y]$ and that the derivation $D$ has no Darboux polynomial to establish that $D$ is a simple derivation of $k[x,y]$. Now $D(x) = y$, $D(y) = y^2 + xy + 1$, then we express 1 as a combination of $D(x)$ & $D(y)$ as follows:

$$\begin{aligned} 1 &= (-y - x)\, y + (y^2 + xy + 1), \\ &= (-y - x)\, D(x) + 1 \cdot D(y). \end{aligned}$$

So, $\Big\langle D(x), D(y)\Big\rangle = \Big\langle y, y^2 + xy + 1\Big\rangle = k[x,y]$. Suppose D has a Darboux polynomial say $f$, with cofactor $\lambda$. Write $f = a_0 + a_1 y + \cdots + a_s y^s$ with

$a_i \in k[x]$ and $a_s \neq 0$. Then, we have

$$\begin{aligned} D(f) &= \lambda f \\ \implies y\left(\sum_{i=0}^{s} a_i{}' y^i\right) + (y^2 + xy + 1)\left(\sum_{i=1}^{s} i a_i y^{i-1}\right) &= \lambda\left(\sum_{i=0}^{s} a_i y^i\right) \end{aligned} \tag{3.2.1}$$

The coefficient of $y^{s+1}$ in the left-hand-side of (3.2.1) is $a_s{}' + sa_s$. If $a_s{}' + sa_s = 0$ then $a_s \mid a_s{}' \implies a_s \in k^*$ and $sa_s = 0 \implies s = 0$ and $a_0 \in k^* \implies f = a_0 \in k^*$, which contradicts the fact that $f$ is Darboux polynomial. So $a_s{}' + sa_s \neq 0$ and consequently $\lambda \neq 0$. Since $\lambda \neq 0$, we can write $\lambda = \sum_{i=0}^{t} b_i y^i$ with $b_i \in k[x]$ and $b_t \neq 0$.

Equating the highest power of $y$ terms on both sides, we have

$$\begin{aligned} (a_s' + sa_s)y^{s+1} = a_s b_t y^{s+t} &\implies t = 1 \text{ and } a_s' + sa_s = a_s b_t \\ &\implies a_s \mid a_s' + sa_s \\ &\implies a_s \mid a_s' \\ &\implies a_s \in k^*. \end{aligned}$$

Thus, we have $t = 1$ and $b_t = s(\neq 0)$. Since, $a_s \in k^*$ and therefore, we may assume without loss of generality that $a_s = 1$. Thus, (3.2.1) now becomes

$$y\left(\sum_{i=0}^{s} a_i{}' y^i\right) + (y^2 + xy + 1)\left(\sum_{i=1}^{s} i a_i y^{i-1}\right) = (b_0 + sy)\left(\sum_{i=0}^{s} a_i y^i\right). \tag{3.2.2}$$

Again, comparing the coefficients of $y^s$, we have

$$\begin{aligned} a_{s-1}' + sxa_s + (s-1)a_{s-1} &= b_0 a_s + sa_{s-1} \\ \implies b_0 &= a_{s-1}' + sx - a_{s-1}. \end{aligned}$$

Thus,

$$\lambda = a'_{s-1} + sx - a_{s-1} + sy.$$

Hence, (3.2.2), now becomes

$$\sum_{i=0}^{s} a_i' y^{i+1} + (y^2+xy+1)\left(\sum_{i=1}^{s} i a_i y^{i-1}\right) = (a'_{s-1} + sx - a_{s-1} + sy)\left(\sum_{i=0}^{s} a_i y^i\right). \tag{3.2.3}$$

From (3.2.3), comparing the terms free from $y$ on both sides, we have

$$a_1 = (a'_{s-1} + sx - a_{s-1})a_0. \tag{3.2.4}$$

Again, comparing the coefficients of $y^i$, $1 \leq i \leq s-2$ both sides in (3.2.3), we obtain

$$(i+1)a_{i+1} = (s-i+1)a_{i-1} - a'_{i-1} + (a'_{s-1} - a_{s-1} + sx)a_i - ia_i x. \tag{3.2.5}$$

Comparing the coefficients of $y^{s-1}$ both sides in (3.2.3), yields

$$a'_{s-2} - 2a_{s-2} - a_{s-1}x + s = (a'_{s-1} - a_{s-1})a_{s-1}. \tag{3.2.6}$$

Now notice that $a_0 \neq 0$ because if $a_0 = 0$ then $a_i = 0$ for all $i = 0, \ldots, s-1$, so (3.2.6) gives $s = 0$, a contradiction.

**Claim 3.2.1.** $s \geq 2$.

If $s = 1$, then $f = a_0 + y$ and $\lambda = x + y + a'_0 - a_0$. Thus,

$$\begin{aligned} Df = \lambda f &\implies a'_0 y + (y^2 + xy + 1) = (x + y + a'_0 - a_0)(a_0 + y) \\ &\implies a_0(x + a'_0 - a_0) = 1. \end{aligned}$$

This shows that both $a_0$ and $x + a'_0 - a_0$ are units; hence, comparing the coefficients of $x$ on both sides, we get $a_0 = 0$, which contradicts that $a_0 \neq 0$. Therefore $s \geq 2$.

From the above discussion, we can see that for $1 \leq i \leq s-2$, $a_i$ can be expressed in terms of $a_{s-1}$ and $a_0$. For all possible values of $a_{s-1} \in k[x]$, we prove that there is no Darboux polynomial. We prove this by considering cases $a_{s-1} \in k$ and $a_{s-1} \notin k$. A summary of these cases is depicted in Figure 3.1.

**Case 1.** $a_{s-1} \in k$.

**Claim 3.2.2.** $\deg a_i = \deg a_0 + i$, *for* $i = 1, 2, \ldots, s-1$.

We shall prove this claim using induction on $i$. From (3.2.4), we have

$$\deg a_1 = \deg a_0 + 1.$$

This shows that the claim is true for $i = 1$. Suppose that $\deg a_i = \deg a_0 + i$, for every $1 \leq i \leq p < s-1$, then from (3.2.5), for $i = p$, we have

$$(p+1)a_{p+1} = (s-p+1)a_{p-1} + a'_{p-1} + (sx - a_{s-1})a_p - pa_p x$$
$$\implies \deg a_{p+1} = \deg a_p + 1 = \deg a_0 + (p+1).$$

Thus, by induction, $\deg a_i = \deg a_0 + i$, $\forall\ i = 1, \ldots, s-1$. In particular, $\deg a_{s-1} = \deg a_0 + (s-1) \geq 1$. This contradicts the fact that $a_{s-1} \in k$. Thus, this case is not possible.

**Case 2.** $a_{s-1} \in k[x] \setminus k$.

**Lemma 3.2.2.** *If* $a_{s-1} \in k[x] \setminus k$ *then the following hold.*

*(a)* $s \geq 3$

*(b) If* $s = 3$ *and* $\deg(a_2) = 1$ *then* $\deg(a_0) > 1$.

*(c) If* $s = 4$ *and* $\deg(a_3) = 1$ *then* $\deg(a_0) > 0$.

*Proof.* (a) We have already proved that $s \geq 2$. Arguing by contradiction, assume that $s = 2$. Then (3.2.4) and (3.2.6) give:

$$a_1 = ({a_1}' + 2x - a_1)a_0 \tag{3.2.7}$$

$$a_0{}' - 2a_0 - a_1x + 2 = (a_1{}' - a_1)a_1 \tag{3.2.8}$$

We have $a_0 \mid a_1$ by (3.2.7), so (3.2.8) gives $a_0 \mid (a_0{}' + 2)$. Then $a_0{}' + 2 = 0$, so (3.2.8) gives $-2a_0 - a_1x = (a_1{}' - a_1)a_1$, so $a_1 \mid a_0$ and hence $a_1 = \mu a_0$ for some $\mu \in k^*$. By (3.2.7), $a_1{}' + 2x - a_1 = \mu \in k^*$, so $a_1 = 2x + c$ for some $c \in k$. Since $a_0{}' = -2$ and $2 = a_1{}' = \mu a_0{}'$, We get $\mu = -1$. So $-1 = \mu = a_1{}' + 2x - a_1 = 2 - c$ gives $c = 3$, i.e., $a_1 = 2x + 3$ and $a_0 = -2x - 3$. This contradicts (3.2.8), so (a) is proved.

(b) Arguing by contradiction, assume that $s = 3$, $\deg(a_2) = 1$ and $\deg(a_0) \leq 1$. Write $a_0 = px + q$ and $a_2 = ux + v$ with $p, q, u, v \in k$, $u \neq 0$ and $(p,q) \neq (0,0)$. Equation (3.2.4) gives $a_1 = ((3-u)x + u - v)(px + q)$. We also have $\lambda = a_2{}' + 3x - a_2 + 3y = (3-u)x + 3y + u - v$. With these values values of $\lambda, a_0, a_1, a_2$ (and $a_3 = 1$), we can compute the quantity $P = D(f) - \lambda f$ explicitly. We find:

$$\begin{aligned} P =& (-pu^2 + 5pu - 6p)x^3y + (2pu + u^2 - 6p - u)x^2y^2 \\ &+ (2pu^2 - 2puv - qu^2 - 5pu + 5pv + 5qu - 6q)x^2y \\ &+ (-4pu + 2pv + 2qu - u^2 + 2uv + 6p - 6q - v)xy^2 \\ &+ (-pu^2 + 2puv - pv^2 + 2qu^2 - 2quv - 5qu + 5qv - 3p + 2u)xy \\ &+ (pu - pv - 3qu + 2qv - uv + v^2 + 3q + 3)y^2 \\ &+ (-qu^2 + 2quv - qv^2 + p - 3q + 2v)y \end{aligned} \tag{3.2.9}$$

Since $P$ is the zero polynomial in $k[x,y]$, we have

$$-p(u-2)(u-3) = 0$$

$$2pu + u^2 - 6p - u = 0$$

$$2pu^2 - 2puv - qu^2 - 5pu + 5pv + 5qu - 6q = 0$$

$$-4pu + 2pv + 2qu - u^2 + 2uv + 6p - 6q - v = 0$$

$$-pu^2 + 2puv - pv^2 + 2qu^2 - 2quv - 5qu + 5qv - 3p + 2u = 0$$

$$-qu^2 + 2quv - qv^2 + p - 3q + 2v = 0$$

$$pu - pv - 3qu + 2qv - uv + v^2 + 3q + 3 = 0$$

If we set $p = 0$ then $q \neq 0$ and second and third equation simplify to $u(u-1) = 0$ and $-q(u-2)(u-3) = 0$, which have no common solutions. So there are no solutions with $p = 0$. By the first equation, all solutions have $u \in \{2, 3\}$. Setting $u = 3$ in the second equation gives "$6 = 0$", which is absurd.

If $u = 2$ then setting $u = 2$ in second equation gives $p = 1$. Again, putting $u = 2$ and $p = 1$ in third equation gives $v = 2$. Further, putting $u = 2, p = 1$ and $v = 2$ in the fourth equation gives $q = 2$. Finally, substituting $u = v = q = 2$ and $p = 1$ in sixth equation, we get $-1 = 0$, which is absurd.

So the above equations have no solutions $p, q, u, v \in k$ such that $u \neq 0$ and $(p, q) \neq (0, 0)$. This proves (b).

(c) Arguing by contradiction, assume that $s = 4$, $\deg(a_3) = 1$ and $\deg(a_0) = 0$. Write $a_0 = q \in k^*$ and $a_3 = ux + v$ with $u \in k^*$, $v \in k$. Equation (3.2.4) gives $a_1 = q((4-u)x + u - v)$. Equation (3.2.5) with i $= 1$ gives $2a_2 = 4a_0 - a_0{}' + (a_3{}' - a_3 + 3x)a_1 = 4q + q((3-u)x + u - v)((4-u)x + u - v)$. We also have $\lambda = a_0 3 + 4x - a_3 + 4y = (4-u)x + 4y + u - v$. With these values of $\lambda, a_0, a_1, a_2, a_3$ (and $a_4 = 1$), we compute the quantity $P = D(f) - \lambda f$. We find:

$$\begin{aligned}
P &= (-qu^2+7qu+u^2-12q-u)x^2y^3 \\
&\quad + (3qu^2-2quv-14qu+7qv-u^2+2uv+12q-v)xy^3 \\
&\quad + \left(\frac{q(u-2)(u-3)(u-4)}{2}\right)x^3y^2 + \Bigg(-16q+3u-3qu^2v \\
&\qquad\qquad + \frac{3quv^2}{2}+9quv+5qu-\frac{9qv^2}{2}+\frac{3qu^3}{2}-\frac{9qu^2}{2}\Bigg)xy^2 \\
&\quad - \left(\frac{q(3u^2-18u+26)(u-v)}{2}\right)x^2y^2 \\
&\quad + \left(4-4q+v^2+\frac{7qu}{2}-2qu^2-qv^2-\frac{7qv}{2}-uv+3quv\right)y^3 \\
&\quad + \left(4q+3v-\frac{qu^3}{2}+\frac{qv^3}{2}+5qv-6qu+\frac{3qu^2v}{2}-\frac{3quv^2}{2}\right)y^2
\end{aligned} \tag{3.2.10}$$

Since $P$ is the zero polynomial in $k[x,y]$, we obtain the following equations:

$$-qu^2+7qu+u^2-12q-u=0$$

$$3qu^2-2quv-14qu+7qv-u^2+2uv+12q-v=0$$

$$q(u-2)(u-3)(u-4)=0$$

$$3qu^3-6qu^2v+3quv^2-9qu^2+18quv-9qv^2+10qu-32q+6u=0$$

$$-q(3u^2-18u+26)(u-v)=0$$

$$-4qu^2+6quv-2qv^2+7qu-7qv-2uv+2v2-8q+8=0$$

$$-qu3+3qu^2v-3quv^2+qv^3-12qu+10qv+8q+6v=0$$

Since $q \neq 0$, the third equation gives $u \in \{2,3,4\}$, so the fifth equation gives $u=v$. Setting $u=v$ in second equation gives

$$qu^2-7qu+u^2+12q-u=0$$

Adding the first equation and above the above equation gives $u^2 - u = 0 \implies u = 0$ or $1$, which is a contradiction because $u \in \{2, 3, 4\}$.

So the above equations have no solutions $q, u, v \in k$ such that $q, u \neq 0$. This proves (b).

□

We shall break this case into the following sub-cases.

1. $a_{s-1} \neq (s - m)(x + \alpha)$ for any $\alpha \in k$ and $m \in \{0, 1, \ldots, s - 2\}$.
2. $a_{s-1} = (s - m)(x + \alpha)$ for some $\alpha \in k \setminus \{1\}$ and $m \in \{0, 1, \ldots, s - 2\}$.
3. $a_{s-1} = (s - m)(x + 1)$ for some $m \in \{0, 1, \ldots, s - 2\}$.

In each sub-case, we show that the Darboux polynomial does not exist.

**Sub-case 2.1.** *$a_{s-1} \neq (s - m)(x + \alpha)$ for any $\alpha \in k$ and $m \in \{0, 1, \ldots, s - 2\}$.*

**Claim 3.2.3.** *$\deg a_i = \deg a_0 + i \deg a_{s-1}$, for $i = 1, \ldots, s - 1$.*

We shall prove this claim using induction on $i$. From (3.2.4), we have

$$\deg a_1 = \deg a_0 + \deg a_{s-1}.$$

This shows that the claim is true for $i = 1$. Suppose that $\deg a_i = \deg a_0 + i \deg a_{s-1}$, for every $1 \leq i \leq p < s - 1$, then from (3.2.5), for $i = p$, we have

$$(p + 1)a_{p+1} = (s - p + 1)a_{p-1} - a'_{p-1} + (a'_{s-1} - a_{s-1} + sx)a_p - pa_px$$
$$\implies \deg a_{p+1} = \deg a_p + \deg a_{s-1} = \deg a_0 + (p + 1) \deg a_{s-1}.$$

Thus, by induction, $\deg a_i = \deg a_0 + i \deg a_{s-1}$, $\forall\ i = 1, \ldots, s - 1$. In particular,

$$\begin{aligned}\deg a_{s-1} &= \deg a_0 + (s-1)\deg a_{s-1}\\ \implies \deg a_0 &= (2-s)\deg a_{s-1}.\end{aligned}$$

Now, since $\deg a_0 \geq 0$, this implies $(2-s)\deg a_{s-1} \geq 0$, i.e., $s \leq 2$, which contradicts Lemma 3.2.2. So this sub-case is not possible.

**Sub-case 2.2.** *$a_{s-1} = (s-m)(x+\alpha)$ for some $\alpha \in k \setminus \{1\}$ and $m \in \{0,1,\ldots,s-2\}$.*

**Claim 3.2.4.** $\deg(a_i) = \begin{cases} \deg(a_0) + i & \textit{for } \ 1 \leq i \leq m, \\ \deg(a_0) + (i-1) & \textit{for } \ m < i \leq s-1 \end{cases}$

The proof of the claim can be divided into two cases $m = 0$ and $m > 1$.

(a) If $m = 0$, then $a'_{s-1} - a_{s-1} + sx = s(1-\alpha)$.

We shall prove this claim using induction on $i$. For $i = 1$, from (3.2.4), we have

$$a_1 = s(1-\alpha)a_0 \implies \deg a_1 = \deg a_0.$$

This shows that the claim is true for $i = 1$. Suppose that $\deg a_i = \deg a_0 + (i-1)$, for every $1 \leq i \leq p < s-1$, then from (3.2.5), for $i = p$, we get

$$\begin{aligned}(p+1)a_{p+1} &= (s-p+1)a_{p-1} - a'_{p-1} + (s-\alpha)a_p - pa_px\\ \implies \deg a_{p+1} &= \deg a_p + 1 = \deg a_0 + p.\end{aligned}$$

Thus, by induction, $\deg a_i = \deg a_0 + (i-1)$, for $i = 1, \ldots, s-1$.

(b) If $m > 0$ then in this case $a'_{s-1} - a_{s-1} + sx = (s-m)(1-\alpha) + mx$.

From (3.2.4) and (3.2.5), we have

$$\left.\begin{aligned}
a_1 &= ((s-m)(1-\alpha)+mx)a_0,\\
2a_2 &= sa_0 - a_0' + [(s-m)(1-\alpha)+(m-1)x]a_1,\\
3a_3 &= (s-1)a_1 - a_1' + [(s-m)(1-\alpha)+(m-2)x]a_2,\\
&\vdots \qquad \vdots\\
ma_m &= (s-m+2)a_{m-2} - a_{m-2}' + [(s-m)(1-\alpha)+x]a_{m-1},\\
(m+1)a_{m+1} &= (s-m+1)a_{m-1} - a_{m-1}' + (s-m)(1-\alpha)a_m,\\
(m+2)a_{m+2} &= (s-m)a_m - a_m' + [(s-m)(1-\alpha)-x]a_{m+1},\\
&\vdots \qquad \vdots\\
(i+1)a_{i+1} &= (s-i+1)a_{i-1} - a_{i-1}' + [(s-m)(1-\alpha)+(m-i)x]a_i.
\end{aligned}\right\} \tag{3.2.11}$$

We shall prove it using induction on $i$. From (3.2.11), we have

$$a_1 = ((s-m)(1-\alpha)+mx)a_0 \implies \deg a_1 = \deg a_0 + 1.$$

This shows that the claim is true for $i = 1$. Suppose $\deg a_i = \deg a_0 + i$, for all $1 \le i \le p < m$, then from (3.2.11), we get

$$\begin{aligned}
&(p+1)a_{p+1} = (s-p+1)a_{p-1} - a_{p-1}' + [(s-m)(1-\alpha)+(m-p)x]a_p\\
&\implies \deg a_{p+1} = \deg a_p + 1 = \deg a_0 + (p+1).
\end{aligned}$$

Thus, by the induction, $\deg a_i = \deg a_0 + i$, for $i \le m$. This completes the proof of the first part of the claim.

For the second part of the claim, putting $i = m$ in (3.2.11), we obtain

$$\begin{aligned}
&(m+1)a_{m+1} = (s-m+1)a_{m-1} - a_{m-1}' + (s-m)(1-\alpha)a_m\\
&\implies \deg a_{m+1} = \deg a_m = \deg a_0 + m.
\end{aligned}$$

Suppose $\deg a_i = \deg a_0 + (i-1)$, for all $m < i \leq p < s-1$, then from (3.2.11), for $i = p$, we obtain

$$(p+1)a_{p+1} = (s-p+1)a_{p-1} - a'_{p-1} + [(s-m)(1-\alpha) + (m-p)x]a_p$$

$$\implies \deg a_{p+1} = \deg a_p + 1 = \deg a_0 + p.$$

Thus by induction, $\deg a_i = \deg a_0 + (i-1)$, for $m < i \leq s-1$. This completes the proof of the second part of the claim.

In particular $\deg(a_{s-1}) = \deg(a_0) + (s-2) = 1$, so $\deg(a_0) = 3 - s \geq 0$,i.e., $s \leq 3$. But in part (a) of Lemma 3.2.2, we have $s \geq 3$. It follows that $s = 3$, $\deg(a_0) = 0$ and $\deg(a_2) = 1$, which contradicts the part (b) of Lemma 3.2.2. So this case is not possible.

**Sub-case 2.3.** $a_{s-1} = (s-m)(x+1)$ *for some* $m \in \{0, 1, \ldots, s-2\}$.

**Claim 3.2.5.** $\deg(a_i) = \begin{cases} \deg(a_0) + i & \text{for } 1 \leq i \leq m, \\ \deg(a_0) + (i-2) & \text{for } \max(m,1) < i \leq s-1 \end{cases}$

The proof of the claim can be divided into two cases $m = 0$ and $m > 1$.

(a) If $m = 0$ then ${a'}_{s-1} - a_{s-1} + sx = 0$. From (3.2.4), $a_1 = 0$.

We shall prove the claim using induction on $i$. From (3.2.5), for $i = 1$, we have

$$2a_2 = sa_0 - {a_0}' \implies \deg a_2 = \deg a_0.$$

This shows that the claim is true for $i = 1$. Suppose that $\deg a_i = \deg a_0 + (i-2)$, for every $2 \leq i \leq p < s-1$, then from (3.2.5), for $i = p$, we have

$$(p+1)a_{p+1} = (s-p+1)a_{p-1} - {a_{p-1}}' - pa_p x$$

$$\implies \deg a_{p+1} = \deg a_p + 1 = \deg a_0 + (p-1)$$

Thus, by induction, $\deg a_i = \deg a_0 + (i-2)$, for all $i = 2, \ldots, s-1$.

(b) If $m > 0$ then $a'_{s-1} - a_{s-1} + sx = mx$. From (3.2.4) and (3.2.5), we get the following set of equations

$$\left.\begin{aligned} a_1 &= mxa_0, \\ 2a_2 &= sa_0 - a'_0 + (m-1)xa_1, \\ 3a_3 &= (s-1)a_1 - a'_1 + (m-2)xa_2, \\ \vdots \quad & \qquad \vdots \\ ma_m &= (s-m+2)a_{m-2} - a'_{m-2} + xa_{m-1}, \\ (m+1)a_{m+1} &= (s-m+1)a_{m-1} - a'_{m-1}, \\ (m+2)a_{m+2} &= (s-m)a_m - a'_m - xa_{m+1}, \\ \vdots \quad & \qquad \vdots \\ (i+1)a_{i+1} &= (s-i+1)a_{i-1} - a'_{i-1} + (m-i)xa_i. \end{aligned}\right\} \qquad (3.2.12)$$

We shall prove it using induction on $i$. From (3.2.12), we have

$$a_1 = mxa_0 \implies \deg a_1 = \deg a_0 + 1.$$

This shows that the claim is true for $i = 1$. Suppose $\deg a_i = \deg a_0 + i$, for all $1 \leq i \leq p < m$, then from (3.2.12), we get

$$\begin{aligned} (p+1)a_{p+1} &= (s-p+1)a_{p-1} - a'_{p-1} + (m-p)xa_p \\ \implies \deg a_{p+1} &= \deg a_p + 1 = \deg a_0 + (p+1). \end{aligned}$$

Thus, by the induction, $\deg a_i = \deg a_0 + i$, for $i \leq m$. This completes the proof of the first part of the claim.

For the second part of the claim, putting $i = m$ in (3.2.12), we obtain

$$\begin{aligned} (m+1)a_{m+1} &= (s-m+1)a_{m-1} - a'_{m-1} \\ \Longrightarrow \deg a_{m+1} &= \deg a_{m-1} = \deg a_0 + m - 1. \end{aligned}$$

Suppose $\deg a_i = \deg a_0 + (i-2)$, for all $m < i \leq p < s-1$, then from (3.2.11), for $i = p$, we obtain

$$\begin{aligned} (p+1)a_{p+1} &= (s-p+1)a_{p-1} - a'_{p-1} + (m-p)xa_p \\ \Longrightarrow \deg a_{p+1} &= \deg a_p + 1 = \deg a_0 + p - 1. \end{aligned}$$

Thus by induction, $\deg a_i = \deg a_0 + (i-1)$, for $m < i \leq s-1$. This completes the proof of the second part of the claim.

In particular $\deg(a_{s-1}) = \deg(a_0) + (s-3) = 1$, so $\deg(a_0) = 4 - s \geq 0$, i.e., $s \leq 4$. But in part (a) of Lemma 3.2.2, we have $s \geq 3$. It follows that one of the following holds:

- $s = 3$, $\deg(a_0) = 1$ and $\deg(a_2) = 1$,
- $s = 4$, $\deg(a_0) = 0$ and $\deg(a_3) = 1$.

Each one of these cases contradicts Lemma 3.2.2, so this case is impossible.

Thus, in all the cases, we get a contradiction; hence, there is no $f \in k[x,y] \setminus k$ such that $Df = \lambda f$. This means $D$ does not have a Darboux polynomial in $k[x,y]$ implies $D = y\partial_x + (y^2+xy+1)\partial_y$ is a simple derivation of $k[x,y]$.

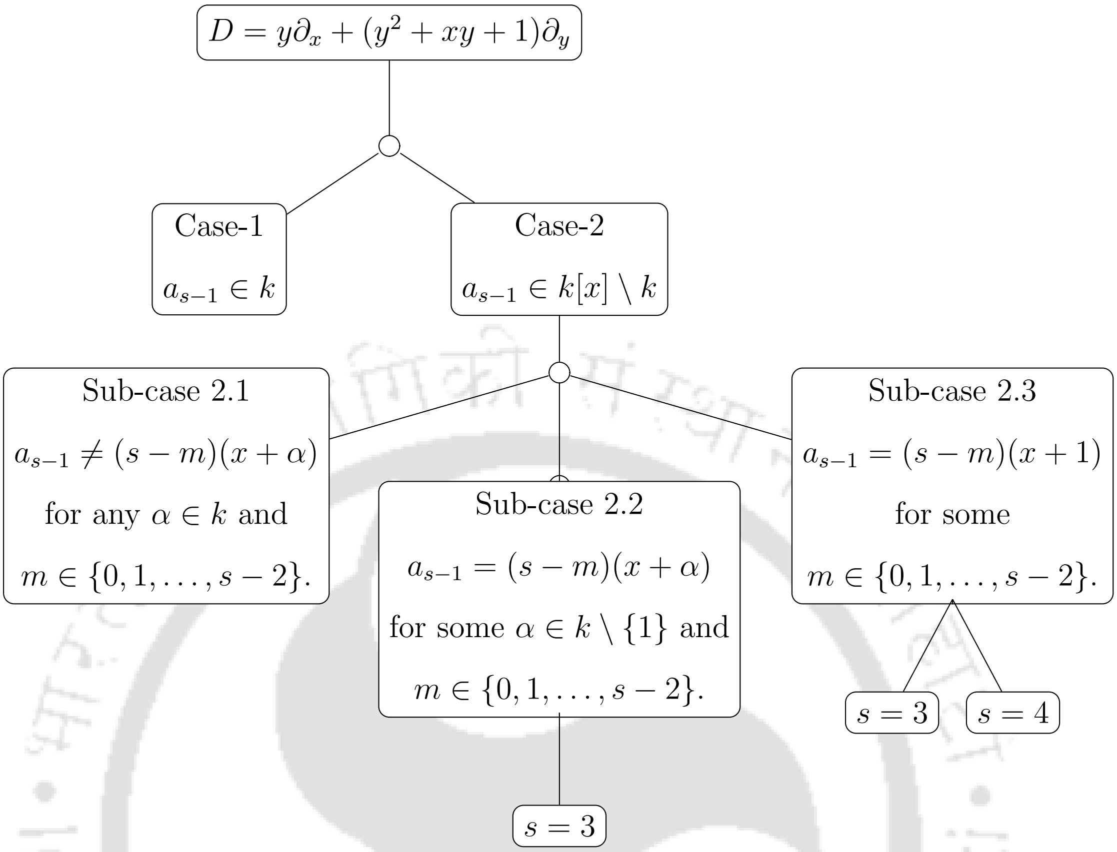

Figure 3.1: The derivation $y\partial_x + (y^2 + xy + 1)\partial_y$

$\square$

# Chapter 4

# Simple Derivations in Higher Dimensions

## 4.1 Introduction

Throughout this chapter, we assume $n \geq 2$. In this chapter, we study two different classes of simple derivations of the polynomial ring $k[x_1, \ldots, x_n]$. We use the same notations as those in the previous chapters.

There are very few known examples of simple derivations in higher dimensions. The first ever published example of simple $k$-derivation of $k[x_1, \ldots, x_n]$ was given by D. A. Jordan in [Jor81]. He showed that the following derivation is simple.

**Example 4.1.1** ( [Jor81]). *The derivation given by*

$$\delta_1 = (1 - x_1x_2)\partial_{x_1} + {x_1}^3\partial_{x_2} + x_2\partial_{x_3} + \cdots + x_{n-1}\partial_{x_n},$$

*is a simple derivation of the polynomial ring* $k[x_1, \ldots, x_n]$.

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[1] The results in this chapter are published in [KW24]. .

J. Archer, in his Ph.D. thesis, had given the following example, which he claimed is a simple derivation of $k[x_1, \dots, x_n]$ [Jor81].

$$\delta_2 = \partial_{x_1} + \sum_{i=2}^{n} x_i x_{i-1} \partial_{x_i},$$

but $\langle x_2 \rangle$ is a $\delta_2$-ideal, so $\delta_2$ is not a simple derivation. A. Nowcki has shown that a small correction:

$$(x_i x_{i-1} + 1) \text{ instead of } x_i x_{i-1},$$

gives an example of simple $k$-derivation of $k[x_1, \dots, x_n]$ [Now94].

A. Nowicki has given the following example of simple derivation of the polynomial ring $k[x_1, \dots, x_n]$ [Now94, Example 13.4.3].

**Example 4.1.2** ( [Now94]). *The $k$-derivation given by*

$$\delta_3 = \partial_{x_1} + ({x_1}^2 x_2 + x_1)\partial_{x_2} + \cdots + ({x_{n-1}}^2 x_n + x_{n-1})\partial_{x_n}.$$

*is a simple derivation of the polynomial ring $k[x_1, \dots, x_n]$.*

A. Maciejewski , J. Moulin-Ollagnier and A. Nowicki have given following two examples of simple derivations of $k[x_1, \dots, x_n]$.

**Example 4.1.3** ( [MMON01]). *The $k$-derivations $\delta_4$ and $\delta_5$ of $k[x, y, t_1, \dots, t_n]$ defined as following are simple derivations.*

$$\begin{cases} \delta_4(x) = 1 \\ \delta_4(y) = y^2 + x \\ \delta_4(t_1) = y \\ \delta_4(t_2) = t_1 t_2 + 1 \\ \delta_4(t_3) = t_2 t_3 + 1 \\ \qquad \vdots \\ \delta_4(t_n) = t_{n-1} t_n + 1 \end{cases} \quad \text{and} \quad \begin{cases} \delta_5(x) = 1 \\ \delta_5(y) = y^2 + x^5 + 2x \\ \delta_5(t_1) = x^2 t_1 + xy \\ \delta_5(t_2) = {t_1}^2 t_2 + t_1 \\ \delta_5(t_3) = {t_2}^2 t_3 + t_2 \\ \qquad \vdots \\ \delta_5(t_n) = {t_{n-1}}^2 t_n + t_{n-1}. \end{cases}$$

S. Kour has given the following classes of simple derivations of $k[x_1, \dots, x_n]$.

**Example 4.1.4** ( [Kou12]). *Let $x_1, \dots, x_n$ be indeterminates. Let $\delta_6$ be the class of $k$-derivations of $k[x_1, \dots, x_n]$ given by*

$$\delta_6 = ({x_1}^s x_2 + c)\partial_{x_1} + {x_1}^r \partial_{x_2} + g_3 \partial_{x_3} + \cdots + g_n \partial_{x_n}, \tag{4.1.1}$$

*where $s \geq 2$, $0 \leq r \leq s-1$, $c \in k \setminus \{0\}$, and $g_i \in k[x_{i-1}] \setminus k$ for $3 \leq i \leq n$. If $r \geq 1$, then $\delta_6$ is simple derivation of $k[x_1, \dots, x_n]$.*

**Example 4.1.5** ( [Kou14]). *Let $x_1, \dots, x_n$ be indeterminates. Let $\delta_7$ be a $k$-derivation of $k[x_1, \dots, x_n]$ given by*

$$\delta_7 = ({x_1}^s {x_2}^t + c)\partial_{x_1} + {x_1}^r \partial_{x_2} + g_3 \partial_{x_3} + \cdots + g_n \partial_{x_n}, \tag{4.1.2}$$

*where $r, s, t \in \mathbb{N}$, $r < s$, $c \in k^*$, and $g_i \in k[x_{i-1}] \setminus k$ for $3 \leq i \leq n$. Then $\delta_7$ is a simple derivation of $k[x_1, \dots, x_n]$.*

The simplicity of the derivations $\delta_6$ and $\delta_7$ have been proved using the following lemma and induction on $n$.

**Lemma 4.1.6.** *Let $\delta$ be a derivation as defined in Examples 4.1.4 and 4.1.5. Let $f \in k[x_1, \dots, x_m]$, for some $2 \leq m \leq n$, be such that $\delta(f) \in k[x_m]$. Then $f \in k$.*

## 4.2 Shamsuddin's Result

We recall the following Theorem from [Sha77]. Many applications of this result can be seen in [Jor81] and [Now94]. We recall the proof of this result for the sake of completeness from [Now94].

**Theorem 4.2.1** ( [Sha77]). *Let $A$ be a $k$-algebra, and let $d$ be a simple $k$-derivation of $A$. Let $t$ be an indeterminate. Extend the derivation $d$ to a derivation $\widetilde{d}$ of the polynomial ring $A[t]$ by setting $\widetilde{d}(t) = at + b$ where $a, b \in A$. Then the following two conditions are equivalent.*

*(1)* $\widetilde{d}$ *is simple.*

*(2) There exists no element* $r$ *of* $A$ *such that* $d(r) = ar + b$.

*Proof.* (1) $\implies$ (2). If there is an element $r \in A$ such that $d(r) = ar + b$, then $\langle t - r \rangle$ is a proper $\widetilde{d}$-ideal of $A[t]$.

(2) $\implies$ (1). Suppose that there exists a nonzero $\widetilde{d}$-ideal $\widetilde{I}$ of $A[t]$ such that $\widetilde{I} \neq A[t]$. Then $\widetilde{I} \cap A$ is a proper $d$-differential ideal of $A$; hence, since $d$ is simple, $\widetilde{I} \cap A = \{0\}$. Let $l = \min\{\deg(f) \mid f \neq 0,\ f \in \widetilde{I}\}$ and let

$$I = \{0\} \cup \{r \in A \mid \exists\ f \in \widetilde{I},\ \deg(f) = l, c_f = r\},$$

where $c_f$ denotes the leading coefficient of a polynomial $f$. As $\widetilde{I} \neq \{0\}$ and $\widetilde{I} \cap A = \{0\}$, we have $l \geq 1$. It is clear that $I$ is an ideal of $A$.

We now show that $I$ is a $d$-differential ideal of the $k$-algebra $A$. Let $r \in I$ and $f$ be a polynomial in $A[t]$ such that $f \in \widetilde{I}$, $\deg(f) = l$ and $c_f = r$. If $d(r) = 0$, then obviously $d(r) \in I$. Assume that $d(r) \neq 0$ and put $g = \widetilde{d}(f) - laf$. Now note that $g$ is of degree $l$ with $c_g = d(r)$. Clearly, $g \in \widetilde{I}$, that is $d(r) \in I$. Therefore $I$ is a $d$-differential ideal of $A$. The simplicity of $d$ and the fact that $I \neq \{0\}$ imply that $I = A$. Thus, there exists a monic polynomial $f \in \widetilde{I}$ such that $\deg(f) = l$. Let

$$f = t^l + r_{l-1}t^{l-1} + \cdots + r_1 t + r_0,$$

where $r_0, \ldots, r_{l-1} \in A$ and consider the polynomial $g = \widetilde{d}(f) - laf$. Then $g \in \widetilde{I}$ and

$$g = st^{l-1} + s_{l-2}t^{l-2} + \cdots + s_1 t + s_0,$$

for some $s_0, \ldots, s_{l-2} \in A$ and $s = lb + d(r_{l-1}) - ar_{l-1}$.

As $g \in \widetilde{I}$, by the minimality of $l$, we have $g = 0$. In particular, $s = 0$. We get the equality $d(r) = ar + b$, where $r = -r_{l-1}/l$. It is a contradiction with the second condition. Hence, $\tilde{d}$ is a simple derivation of $A[t]$.

□

## 4.3 Main Results

In this section, we discuss two classes of simple derivations of the polynomial ring $k[x_1, \ldots, x_n]$. In Theorem 4.3.1, we generalize Theorem 3.1.1 to higher dimensions. In Theorem 4.3.2, we give a generalization of Theorem 3.2.1 to higher dimensions. In the proof, we use Shamsuddin's result (Theorem 4.2.1).

**Theorem 4.3.1.** *Let $x_1, \ldots, x_n$ be indeterminates. Let $d$ be a $k$-derivation of $k[x_1, \ldots, x_n]$ given by*

$$d_n = {x_2}^2\partial_{x_1} + (x_1x_2 + h)\partial_{x_2} + (h_1x_3 + t_1)\partial_{x_3} + \cdots + (h_{n-2}x_n + t_{n-2})\partial_{x_n},$$

*where $h \in k[x_2], \gcd(x_2, h) = 1$ and $\deg_{x_2} h \leq 1$ and $h_i, t_i \in k[x_1, \ldots, x_{i+1}]$, $t_i \neq 0$ with $\deg_{x_{i+1}} t_i < \deg_{x_{i+1}} h_i$, $\deg_{x_2} h_1 \geq 3$ and for $i > 1$, $\deg_{x_{i+1}} h_i \geq 1$. Then $d_n$ is a simple derivation of $k[x_1, \ldots, x_n]$.*

*Proof.* We prove this using induction on the number of indeterminates. Consider the following derivation of $k[x_1, x_2]$

$$d_2 = {x_2}^2\partial_{x_1} + (x_1x_2 + h)\partial_{x_2},$$

where $h \in k[x_2]$, $\gcd(x_2, h) = 1$, and $\deg_{x_2} h \leq 1$ then by Theorem 3.1.1 , $d_2$ is simple.

Let $a = h_1(x_1, x_2)$ with $\deg_{x_2} h_1 = l_1 \geq 3$ and $b = t_1(x_1, x_2)$.

**Claim 4.3.1.** *There does not exist any $f \in k[x_1, x_2]$ such that $d_2 f = af + b$.*

Write $f = a_0 + a_1x_2 + \cdots + a_s(x_2)^s$ with $a_i \in k[x_1]$. Suppose

$$\begin{aligned} d_2 f &= af + b \\ \implies {x_2}^2\left(\sum_{i=1}^{s} {a_i}'{x_2}^i\right) + \left(x_1x_2 + h\right)\left(\sum_{i=1}^{s} ia_i{x_2}^{i-1}\right) &= a\left(\sum_{i=1}^{s} a_i{x_2}^i\right) + b. \end{aligned} \tag{4.3.1}$$

Now, note that $f \neq 0$, for if $f = 0$, then $b = t_1 = 0$, which is a contradiction since $t_i \neq 0$. Comparing the coefficients of $(x_2)^{s+l_1}$ in (4.3.1), we have

$$a_s = 0,$$

which is absurd. Thus there does not exist any $f \in k[x_1, x_2]$ such that $d_2 f = af + b$. Hence, by Theorem 4.2.1

$$d_3 = \widetilde{d_2} = {x_2}^2 \partial_{x_1} + (x_1 x_2 + h)\partial_{x_2} + (h_1(x_2)x_3 + t_1(x_1, x_2))\partial_{x_3},$$

is a simple derivation. Now let us assume that

$$\begin{aligned} d_{n-1} = {x_2}^2 \partial_{x_1} + (x_1 x_2 + h)\partial_{x_2} + (h_1 x_3 + t_1)\partial_{x_3} + \cdots \\ + (h_{n-3} x_{n-1} + t_{n-3})\partial_{x_{n-1}}, \qquad (4.3.2) \end{aligned}$$

is a simple derivation of $k[x_1, \ldots, x_{n-1}]$. Let $a = h_{n-2}(x_1, \ldots, x_{n-1})$ with $\deg_{x_{n-1}} h_{n-2} = l_{n-2} \geq 1$ and $b = t_{n-2}(x_1, \ldots, x_{n-1}) \neq 0$. Now, we want to check whether there exists $f \in k[x_1, \ldots, x_{n-1}]$ such that $d_{n-1} f = af + b$. Write $f = a_0 + a_1 x_{n-1} + \cdots + a_s (x_{n-1})^s \in k[x_1, \ldots, x_{n-2}][x_{n-1}]$ and

$$d_{n-1} f = af + b.$$

Now note that $f \neq 0$, for if $f = 0$, then $b = t_{n-2} = 0$, which is a contradiction since $t_{n-2} \neq 0$. Comparing the coefficients of $(x_{n-1})^{s+l_{n-2}}$, we have

$$a_s = 0,$$

which is absurd. Thus there does not exist any $f \in k[x_1, \ldots, x_{n-1}]$ such that $d_{n-1} f = af + b$. Hence, by Theorem 4.2.1

$$d_n = \widetilde{d_{n-1}} = {x_2}^2 \partial_{x_1} + (x_1 x_2 + h)\partial_{x_2} + (h_1 x_3 + t_1)\partial_{x_3} + \cdots + (h_{n-2} x_n + t_{n-2})\partial_{x_n},$$

is a simple derivation.

Therefore by induction, $d_n$ is simple for $n \geq 2$. $\square$

**Theorem 4.3.2.** *Let $x_1, \dots, x_n$ be indeterminates. Let $d$ be a $k$-derivation of $k[x_1, \dots, x_n]$ given by*

$$d = x_2\partial_{x_1} + ({x_2}^2 + x_1x_2 + 1)\partial_{x_2} + (h_1x_3 + t_1)\partial_{x_3} + \cdots + (h_{n-2}x_n + t_{n-2})\partial_{x_n},$$

*where for each $i$, $h_i, t_i \in k[x_1, \dots, x_{i+1}]$, $t_i \neq 0$ with $\deg_{x_{i+1}} t_i < \deg_{x_{i+1}} h_i$, $\deg_{x_2} h_1 \geq 2$ and for $i > 1$, $\deg_{x_{i+1}} h_i \geq 1$ . Then $d$ is a simple derivation of $k[x_1, \dots, x_n]$.*

*Proof.* For $n = 2$, this statement is precisely the Theorem 3.2.1. For $n \geq 2$, the proof is almost the same as that of Theorem 4.3.1. $\square$

## 4.4 Alternate Proof of Theorems 4.3.1 and 4.3.2

Discussion in this section is based on the suggestions given by one of the examiners of this thesis. Our sincere thanks to him for these constructive suggestions.

**Definition 4.4.1.** *Let $A$ be a ring. A degree function on $A$ is a set map $\deg : A \to \mathbb{N} \cup \{-\infty\}$ that satisfies the following conditions for all $f, g \in A$:*

*(i)* $\deg(f) = -\infty \iff f = 0$

*(ii)* $\deg(f + g) \leq max(\deg(f), \deg(g))$

*(iii)* $\deg(fg) = \deg(f) + \deg(g)$.

**Remark 4.4.2.** *(i) If $A$ admits a degree function then $A$ is either an integral domain or the zero ring.*

*(ii) If* $\deg$ *is a degree function on $A$ then $\deg(u) = 0$ for all units $u$ of $A$, and if $f, g \in A$ are such that $\deg(f) \neq \deg(g)$ then $\deg(f + g) = max(\deg(f), \deg(g))$.*

**Definition 4.4.3.** *Let $A$ be a domain with a degree function $\deg : A \to \mathbb{N} \cup \{-\infty\}$, and let $D : A \to A$ be a derivation. Let $S_D = \{\deg(Df) - \deg(f) \mid f \in A \setminus \{0\}\}$ and observe that $S_D$ is a nonempty subset of $\mathbb{Z} \cup \{-\infty\}$. If the set $S_D$ has a maximum element then we say that the degree of $D$ is defined, and we define $\deg(D) = max(S_D)$. If $S_D$ does not have a maximum element, we say that the degree of $D$ is undefined.*

**Example 4.4.4.** *Let $A = k[x, y]$. Then $y$-degree is a degree function, $\deg_y : A \to \mathbb{N} \cup \{-\infty\}$. Consider the derivations $D_1 : A \to A$ and $D : A \to A$ of Theorems 3.1.1 and 3.2.1:*

$$D_1 = y^2\partial_x + (xy + 1)\partial_y \qquad \text{and} \qquad D = y\partial_x + (y^2 + xy + 1)\partial_y.$$

*Then the degrees of $D_1$ and $D$ with respect to $\deg_y$ are defined: $\deg_y(D_1) = 2$ and $\deg_y(D) = 1$.*

The following is a consequence of Shamsuddin's result (Theorem 4.2.1).

**Corollary 4.4.5.** *Let $k$ be a field of characteristic $0$, let $A$ be a $k$-domain and let $\deg : A \to \mathbb{N} \cup \{-\infty\}$ be a degree function on $A$. Suppose that $d : A \to A$ is a simple $k$-derivation such that $\deg(d)$ is defined. Let $a, b$ be non-zero elements of $A$ such that $\deg(a) > \deg(d)$ and $\deg(a) > \deg(b)$, and let $\tilde{d} : A[t] \to A[t]$ be the unique derivation that extends $d$ and satisfies $\tilde{d}(t) = at + b$. Then $\tilde{d}$ is simple. Moreover, consider the degree function $\deg_t : A[t] \to \mathbb{N} \cup \{-\infty\}$. Then the degree of $\tilde{d}$ with respect to $\deg_t$ is defined and $\deg_t(\tilde{d}) = 0$.*

*Proof.* Suppose that there exists $r \in A$ such that $d(r) = ar + b$. Then $r \neq 0$ (because $b \neq 0$) and consequently $\deg(dr) - \deg(r) \in S_D$. Since $\deg(d) = max(S_D)$, we obtain

$$\deg(d) \geq \deg(dr) - \deg(r) \tag{4.4.1}$$

Note that $\deg(b) < \deg(a) \leq \deg(a) + \deg(r) = \deg(ar)$, so $\deg(ar + b) = \deg(ar)$ (by the second part of Remark 4.4.2). It follows that $\deg(d(r)) = \deg(ar + b) = \deg(ar) = \deg(a) + \deg(r)$, so $\deg(d(r)) - \deg(r) = \deg(a) >$

$\deg(d)$. This contradiction shows that no element $r$ of $A$ satisfies $d(r) = ar + b$. By Theorem 4.2.1, it follows that $\tilde{d}$ is simple. It is clear that the $t$-degree $\deg_t$ is a degree function on $A[t]$.

In order to prove that $\deg_t(\tilde{d}) = 0$, it suffices to check the following two facts:

(i) For every $f \in A[t] \setminus \{0\}$, $\deg_t \tilde{d}((f)) - \deg_t(f) \leq 0$.

(ii) There exists $f \in A[t] \setminus \{0\}$ such that $\deg_t \tilde{d}((f)) - \deg_t(f) = 0$.

Here, (i) is an obvious consequence of $\tilde{d}(t) = at + b$ and of the formula $\tilde{d}(f(t)) = f^{(d)}(t) + f'(t)\tilde{d}(t)$, and (ii) is satisfied by $f = t$.

Therefore, $\deg_t(\tilde{d}) = max(S_{\tilde{d}}) = 0$. □

**Corollary 4.4.6.** *Let $k$ be a field of characteristic $0$, let $A$ be a $k$-domain and let $\deg : A \to \mathbb{N} \cup \{-\infty\}$ be a degree function on $A$. Suppose $d : A \to A$ is a simple $k$-derivation such that $\deg(d)$ is defined. Consider the polynomial ring $A[t_1, \ldots, t_n]$ and suppose that $(a_i)_{i=0}^{n-1}$ and $(b_i)_{i=0}^{n-1}$ satisfy:*

- *$a_0, b_0 \in A \setminus \{0\}$, $\deg(a_0) > \deg(d)$ and $\deg(a_0) > \deg(b_0)$;*
- *For each $i \in \{1, \ldots, n-1\}$, $a_i, b_i \in A[t_1, \ldots, t_i] \setminus \{0\}$, $\deg_{t_i}(a_i) > 0$ and $\deg_{t_i}(a_i) > \deg_{t_i}(b_i)$.*

*Let $\tilde{d} : A[t_1, \ldots, t_n] \to A[t_1, \ldots, t_n]$ be the unique derivation that extends $d$ and satisfies*

$$\tilde{d}(t_i) = a_{i-1}t_i + b_{i-1}, \qquad \text{for all } i = 1, \ldots, n.$$

*Then $\tilde{d}$ is simple.*

*Proof.* This follows by $n$ applications of Corollary 4.4.5. □

It is clear that Theorems 4.3.1 and 4.3.2 are special cases of Corollary 4.4.6.

# Chapter 5

# Future Scope

## 5.1 Special Cases

Based on the work carried out in this thesis, we discuss the simplicity of the class of $k$-derivations of the form $y^m\partial_x + (yg+h)\partial_y$, where $m \geq 0$ and $g(x), h(x) \in k[x]$. We have proved the case $m = 0$ as follows:

**Theorem 5.1.1.** *Consider* $D = \partial_x + (yg(x)+h)\partial_y$*, where* $g(x) \in k[x]$*. If* $\deg g(x) \geq 1$*, and* $h \in k^*$*, then* $D$ *is a simple derivation of* $k[x,y]$*.*

*Proof.* We know that the derivation $d_1 = \frac{\partial}{\partial x}$ is a simple derivation of $k[x]$. Now, we claim that there does not exist any $f \in k[x]$ such that

$$d_1 f = gf + h.$$

Suppose that $f'(x) = g(x)f(x) + h$. Then $f(x) \neq 0$, because $h \in k^*$. So $\deg(f' - h) = \deg(gf) = \deg g + \deg f > \deg f$ is absurd. So $f$ does not exist.

Thus there is no $f \in k[x]$ such that $d_1 f = af + b$. Hence by Theorem 4.2.1,

$$d_2 = \widetilde{d_1} = \partial_x + (yg + h)\partial_y,$$

is a simple derivation. ☐

**Remark 5.1.2.** *The converse of Theorem 5.1.1 need not be true, i.e., if $D$ is simple then $\deg g \geq 1$ and $h \in k^*$ need not be true. For example, the derivation $\partial_x + \Big((x^2 + x)y + x^2\Big)\partial_y$, given in §13.3 of [Now94], is a simple derivation of $k[x, y]$.*

For the case $m \geq 1$, we have proved the following statement.

**Theorem 5.1.3.** *Consider $D = y^m\partial_x + (yg + h)\partial_y$, where $m \in \mathbb{N}$, and $g(x), h(x) \in k[x]$. If $D$ is simple, then $h(x) \in k^*$ and $\deg g \geq 1$.*

*Proof.* If $h(x) \notin k^*$, then $D(h) = y^m h' \in \langle h(x), y\rangle$ and $D(y) = yg + h \in \langle h(x), y\rangle$. Thus, it shows that $\langle h(x), y\rangle$ is a $D$-differential ideal; hence, $D$ is not simple. Thus, if $D$ is simple then $h \in k^*$.

If $\deg g(x) = 0$ and $g(x) \in k^*$, then $D(yg + h) = g(yg + h)$, i.e., the ideal $\langle yg(x) + h(x)\rangle$ is a $D$-differential ideal; hence, $D$ is not simple.

If $g(x) = 0$, then $D(y^{m+1} - (m + 1)xh(x)) = 0$, which shows that $\langle y^{m+1} - (m + 1)xh(x)\rangle$ is a $D$-differential ideal; hence, $D$ can not be simple. ☐

## 5.2 Some Questions

In view of Theorem 5.1.3, and generalizing Theorem 5.1.1, we ask the following question.

**Question 5.2.1.** *Consider $D = y^m\partial_x + (yg + h)\partial_y$, where $m \in \mathbb{N}$, and $g(x), h(x) \in k[x]$. Is it the case that $D$ is simple if and only if $h(x) \in k^*$ and $\deg g \geq 1$.*

If Question 5.2.1 has an affirmative answer, then many known results follow as special cases. For example:

1. If we take $m = 3$, $g(x) = -x$ and $h(x) = 1$, then the derivation $y^3\partial_x + (1 - xy)\partial_y$, is simple [Jor81, Theorem 1].
2. With $g(x) = x^t$ , this becomes the special case of Theorems 4.1 and 6.1 of [Kou14] by taking $s = 1$.
3. With $m = 0$ and $h(x) \in k^*$ in Theorem 5.1.1, this coincides with examples from §13.3 of [Now94] and gives a more general class of simple derivations.

We now propose some more problems emerging from work carried out in this thesis.

Q1. Let $(F, \lambda)_d$ denotes Darboux polynomial $F$ with cofactor $\lambda$ of the derivation $d$.

Using the technique introduced by V.S. Gavran in [Gav09] and Jiantao Li in [Li15], we wish to find $\sigma \in \mathrm{Aut}_k(k[x, y])$ and a primitive root of unity $\varepsilon$ such that $\sigma^{-i}d\sigma^i = \varepsilon^i d$, where $d$ is one of the derivations introduced in Chapter 2. Further, for $(F, \lambda)_d$, we wish to find $(\bar{F}, \bar{\lambda})_d$ with the help of $\sigma^{-i}d\sigma^i = \varepsilon^i d$, such that $\bar{\lambda}$ is a monomial.

Thus, it suffices to consider only the monomial cofactors for the existence of a Darboux polynomial. In Chapter 2, we have already proved that each of these derivations does not have a Darboux polynomial with monomial cofactor. Following Proposition 1.3.10, an affirmative answer to above question will imply that all these derivations are simple.

Q2. For almost all the results in this thesis, while proving that the derivation is simple, one needs to consider many cases. In turn the proof becomes specific to the derivation under consideration. To avoid this situation, we would like to give a generic criteria for checking the simplicity of the derivation.

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