

Loan Portfolio Management in the Paradigm of Leveraged Risk, Liquidity Risk and Limited Liability

by

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Loan Portfolio Management in the Paradigm of Leveraged Risk, Liquidity Risk and Limited Liability

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by

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Declaration

I hereby declare that the work contained in this thesis entitled “**Loan Portfolio Management in the Paradigm of Leveraged Risk, Liquidity Risk and Limited Liability**” was done by me, under the supervision of **Prof. Siddhartha Pratim Chakrabarty**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in this thesis entitled “**Loan Portfolio Management in the Paradigm of Leveraged Risk, Liquidity Risk and Limited Liability**” by **Deb Narayan Barik**, a student in Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

August, 2025

Prof. Siddhartha Pratim Chakrabarty

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To

My Family

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Abstract

In the context of a loan portfolio, banks and other financial institutions are subjected to Limited Liability protection. However, in most of the model formulation, this protection is not taken into consideration. Accordingly, in order to address this gap, we have focused on incorporation of Limited Liability in the model paradigm. We begin by considering four models, one for maximizing expected return and the other with minimization of risk (with a threshold of expected return), both for the scenarios of including and excluding Limited Liability. Our theoretical results show that the solutions of the models with Limited Liability produce better results than the others, in terms of both maximizing expected return and minimizing risk. More specifically, the portfolios that included Limited Liability are less risky as compared to the portfolios that did not include Limited Liability. An illustrative example is presented to support the theoretical results obtained.

In case of liquidity risk with Limited Liability protection, we construct a novel loan portfolio model with Limited Liability, while maintaining a threshold level of haircut in the portfolio. For the constructed three-time step loan portfolio, at the initial time, the bank raises capital via debt and equity, investing the same in several classes of loans, while at the final time, the bank either meets its liabilities or becomes insolvent. At the intermediate time step, a fraction of the deposits are withdrawn, resulting in liquidation of some of the bank's assets. We have proposed a liquidation strategy that minimizes liquidation cost and also reduces liquidation of the safe asset, thereby reducing the risk of default. Our theoretical results show that the model with the haircut constraint leads to lesser liquidity risk, as compared to the scenario of no haircut constraint being imposed. A numerical example then demonstrates these results.

Next, we consider the bank's dynamic decision problem in the intermediate time step for a discrete-time setup. We have considered a three-time-step model. Initially, the banks raise money through debt and equity and invest in different types of loans. It liquidates its assets and raises new funds at the intermediate-time step to meet the short-term debt holder's claim. Further, it has to meet specific capital requirements given by the regulators. We have theoretically studied the effect of raising new equity and debt. We show that in some cases, raising equity and debt may

increase the return on equity, and in some cases, it may decrease the return on equity. We have discussed several cases and given a bound on the capital that can be raised. We have added an equity holder's constraint, which ensures the return on equity and desists the bank from defaulting at the final time point.

Finally, we have considered the portfolio optimization model with Limited Liability in a continuous time setup. We have shown that the model with Limited Liability incorporates less riskier assets. If the risk is low, the result with and without Limited Liability is the same. We then present a numerical example to illustrate this result.



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Chapter 1

Introduction and thesis outline

1.1 Introduction

Mathematical portfolio theory pertains to the design and management (including risk management) of investment portfolios. The foundation of modern portfolio theory can be traced back to the seminal work of Harry Markowitz [54], which established return and risk as the two drivers of portfolio management. The portfolio setup mechanism, prior to this, was mostly sentiment driven. But the establishment of modern portfolio theory emphasized the importance of return-risk trade-off, a concept that has stood the test of time. While the original measure of risk was identified as the standard deviation (or equivalently the variance) of returns of assets, several subsequent risk measure have been developed, including (among others) Value-at-Risk (VaR), Conditional VaR (CVaR) and Lower Partial Moment. For this work, we focus on the study of portfolio theory in the paradigm of risk management in general, and on Leverage Ratio and Liquidity Risk, in particular, the details of which are elucidated in the following presentation.

1.1.1 Return-risk model

In the paradigm of modern portfolio theory, there are two things which are typically expected, in the design of a portfolio, namely, maximization of returns and stability of returns. Therefore a “good portfolio” means a strategy of investment that generates stable returns. One can analyze the stability by studying the “risk” of the portfolio, and better stability is achieved by minimizing this “risk” (to be discussed subsequently). So the construction of a portfolio means maximization of returns and minimization of risk. This goal can be achieved in several settings, like using different risk measures, or using a performance measure instead of return.

1.1.2 Risk measure

Risk usually refers to the uncertainty of returns, whose definition depends upon investors. So various measures of uncertainty or various definitions of risk measure, have emerged in the study

of portfolio theory. Variance played the role of measuring the uncertainty of measuring the risk in the early days of portfolio theory. Prigent [66] and Meyfredi [59] (among others) have categorized them into two classes:

(A) **Dispersion risk measure:** This type of risk measure shows the dispersive nature of returns of the portfolio.

(i) *Standard Deviation [SD]:*

This is the earliest risk measure in this domain, which was used by Markowitz [54] in his seminal work, and is defined as,

$$SD = \sqrt{E(X_P - E(X_P))^2},$$

where X_P denote the return of the portfolio P and E denotes the expectation.

(ii) *Mean Absolute Deviation [MAD]:*

This is also an popular dispersive risk measure, and is defined as,

$$MAD = E|X_P - E(X_P)|.$$

These are the two widely used examples of dispersive risk measures. Having said so, one of the most significant drawbacks of this risk measure is that it incorporates both upward and downward returns. But a speculator is worried about low returns only, because a downward return adversely affects the portfolio returns. Accordingly, in order to study the losses, or the returns below a benchmark level, the concept of downside risk measure was proposed.

(B) **Downward risk measure:**

This type of risk measures accounts for the chance of the fall of portfolio return below a benchmark level. Roy [70], Telser [74] and Kataoka [46] introduced the safety first criterion, from different perspective, while Markowitz [55] introduced the concept of semi-variance. Subsequently, the Basel Committee on Banking Supervision (BCBS) included the wide usage of VaR, which was followed by the introduction of CVaR, to address some of the shortcomings of VaR. Rockafellar and Uryasev [69] discussed in detail, about both these risk measures, namely, VaR and CVaR. Here we discuss some of the downside risk measures.

(i) *Roy's Safety First Criterion:* In this criterion, the chance of getting a portfolio return lower than R_{\min} (benchmark level), is minimized, which in mathematical terms is given by,

$$\min_{\vec{w}} \mathbb{P}(X_P < R_{\min}),$$

where \mathbb{P} , X_P and \vec{w} denote the probability measure, the return of the portfolio and the underlying portfolio, respectively.

- (ii) *Telser's Criterion*: In this criterion, the return of the portfolio is maximized, provided it satisfies the constraint that the probability of a return lower than a benchmark level is less than a pre-specified level, as fixed by the investor. So in this method, one ascertains the chances of fall as a risk, unlike dispersion risk measure. The mathematical form of this problem is,

$$\max_{\vec{w}} E[X_P] \text{ subject to } \mathbb{P}(X_P < R_{\min}) \leq \epsilon,$$

where ϵ is fixed, and all other parameter are as defined previously.

- (iii) *Kataoka's Criterion*: Unlike the previous two criterion, in this model we maximize R_{\min} so that the returns which are less than this, is less than ϵ . Accordingly, the mathematical form is given by,

$$\max_{\vec{w}} R_{\min} \text{ subject to } \mathbb{P}(X_P < R_{\min}) \leq \epsilon.$$

- (iv) *Semi-variance*: Semi-variance, introduced by Markowitz [55], considers only the negative dispersion and its mathematical form is given by,

$$\text{Semivariance} = \sqrt{E[(E(X) - X)^+]^2}.$$

- (v) *VaR*: VaR estimates how much a portfolio can lose with a given probability (in a confidence interval). The mathematical form of VaR of X at confidence level $(1 - \alpha)$ is given by,

$$VaR^\alpha(X) = -\inf\{x : \alpha < F_X(x)\}.$$

Charpentier and Oulidi [64], have discussed the VaR driven asset allocation problem with four assets, through a graphical analysis of the asset allocation problem.

1.1.3 Coherent and convex risk measure

It is clear from the above discussion that there are several risk measures, and accordingly, the next step entails the determination of the quality of the risk measure. Artzner et al. [3] focused on some drawbacks of VaR and introduced a set of axioms, based on regulation of private banks by central bank. Assuming a reference risk free asset with return r , the axioms for a risk measure ρ are given by,

- (A) **Axiom T** *Translation Invariance*:

$$\rho(X + \alpha) = \rho(X) - \alpha,$$

where α is any real number. The axiom simply means that investing an amount (α) in the secured position decreases the overall risk of the portfolio.

(B) **Axiom S** *Sub-additivity*: For two portfolios X_1 and X_2 ,

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).$$

This property can be seen as “a merger does not create extra risk”.

(C) **Axiom PH** *Positive Homogeneity*: For all $\lambda \geq 0$

$$\rho(\lambda X) = \lambda \rho(X)$$

This axiom shows that risk measure is a linear function of the size of the positions. But it neglects liquidity risk (because endogenous liquidity rises due to the size of the position).

(D) **Axiom M** *Monotonicity*: For all portfolios X_1 and X_2 ,

$$X_1 \leq X_2 \Rightarrow \rho(X_1) \geq \rho(X_2)$$

It means that the portfolio whose return almost surely dominates another one is less risky.

(E) **Axiom R** *Relevance*: For all X with $X \leq 0$,

$$\rho(X) > 0,$$

which says risk if it exists, is to be considered.

(F) **Axiom C** *Convexity*: For portfolios X_1 and X_2 , and for all $0 \leq \lambda \leq 1$,

$$\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2).$$

Definition 1.1.1 (Coherent Risk Measure). *A risk measure satisfying the axioms T, S, PH, M is called coherent.*

Definition 1.1.2 (Convex Risk Measure). *A risk measure satisfying the axioms T, M and C is called convex.*

1.2 Limited Liability

In the banking structure, the bank owner is protected by Limited Liability, which protects the owners of the bank (or by extension, any Limited Liability firm) from personal liability, in the event of bankruptcy or insolvency, that is, in such scenarios, the depositors cannot lay any claim on the owner’s personal assets, for their deposit exposure. From a historical perspective, Limited Liability, in practice is achieved by way of a private contractual setup [20]. It offers the advantage of safeguard of passive investors, from creditors, in the event of bankruptcy. In today’s economy,

the implications of Limited Liability, as a source of moral hazard, is significantly evident [35]. The authors of the work [35] strongly suggest the structural connect between moral hazard and Limited Liability, and highlighted the disruptive socio-economic consequences of the same.

In the twentieth century, the international financial community generated several financial instruments to speculate, hedge, and encourage many other business ideas. The authors in [28], [36] have shown that these instruments increase the risk of moral hazard. The articles [51], [61] discuss the new investment opportunities arising from the securization of contractual debt from other domains. And at this time, the dependence on other sectors on finance reinforces the argument “too big to fail” in subsequent crises. All these activities happened in the context of the 1980 banking deregulation. A growing market’s significant lack of regulatory activity broadens the crisis path [52].

The question of dynamic moral hazard, resulting from the un-observable effort of an agent with Limited Liability, in terms of low frequency high magnitude losses is examined in [9, 73]. The setup consisted of the agent and the principal, who (unlike the agent) has unlimited liability. An optimal approach, in terms of payments to the agent, is contingent on the good performance of the agent, in absence of which the payments are stopped. This in turn, leaves room for both the extreme possibilities of the firm size diminishing to zero or experiencing unbounded growth. Limited Liability, while acting as an incentive for investors, and facilitating economic growth, can have adverse consequences, such as risk taking tendencies, thereby causing economic loss [72]. Accordingly, the consequences of the latter can be sought to be mitigated through regulatory mechanisms (including capital requirements) and statutory insurance. An analysis of limited liability for insurance markets is studied in [15], by considering limited liability protection for non-life insurers. In particular, the case considered in the study is one where there is an exchangeable nature of insurance risk, in case of policy holders. The author then goes on to establish the existence of a partial equilibrium in the insurance sector.

The bank’s decision problem with Limited Liability, in conjunction with capital requirements mandated in Basel III, has been studied in [2]. In [5], we have demonstrated that the inclusion of Limited Liability, in the decision model (in discrete time, through the excess return function) can lead to decreased leveraged risk. With Limited Liability protection, the required amount of risk is less, as compared to the case without Limited Liability protection, when it comes to attaining a target.

1.3 Capital requirement

The Basel regulations which is applicable (today) to the global banking system, was the consequence of a large number of bank failures experienced at the onset of the twentieth century, and the con-

sequent recognition of maintaining the stability and resilience of the banking structure as a whole. Accordingly, the first of the continuously evolving Basel regulations came by the way of Basel-I, with the goal of enhancing the soundness of banks via the introduction of capital requirements, predominantly based on credit risk exposure [10]. The provisions included in the (subsequent) Basel-II regulations, not only included revised norms for capital requirements, but also for market risk and operational risk [11]. It also allowed for the banks to adopt their own Internal Ratings Based (IRB) models, subject to regulatory oversight. However, the adoption of IRB approach (not unexpectedly) opened up the possibility that banks on their part, could potentially camouflage the “real risk profile” of their portfolio, from the supervisory mechanism, such as the banks under-reporting their risk profile by adoption of off-balance sheet activities.

Bank’s leverage to a great extent may end up playing a vital role in triggering a financial crisis, as was the case with the spectacular financial collapse of 2008, which is now widely attributed to excessive leveraging by the prominent investment banks. Given the fact that many of these financial institutions which were over-leveraged, had maintained a healthy level of capital requirement compliance, the regulators, namely the BCBS were prompted by this, to set a target upper bound on the extent to which a bank can be leveraged [25]. Accordingly, they introduced a non-risk based capital measure [12], namely, the Leverage Ratio, to insulate the banks from over-leveraging, by way of capital requirements. Consequently, the Leverage Ratio (LR) is defined as,

$$\text{Leverage Ratio} := \frac{\text{Capital Measure}}{\text{Exposure Measure}}.$$

This ratio is a counter-cyclical measure, having identified that usage of pro-cyclical measures alone, is not sufficient in achieving the goals of bank stability. LR also offers the advantage of lesser regulatory arbitrage and simplicity, in terms of deployment and monitoring. Hence, imposing an LR restriction becomes simpler from a regulatory perspective. Blum in his article [14], has demonstrated that adoption of risk-based capital requirement, in conjunction with LR, serves as a better criterion for capital requirement (since the imposition of the LR restriction ensures that all banks report their risk-profile in a more truthful manner). In [33], D’Hulster has discussed about various aspects of LR and presented an analogy of the benefits of adoption of LR as a part of the regulatory framework. Hildebrand [41], in his article, has claimed that implementing risk-based capital requirements with LR leads to lowering of leverage of the bank, which in turns decreases the chances of default or insolvency of the bank.

The well established approaches for determination of capital requirements, notwithstanding, the possibility of higher returns drives banks to greater risk exposure. Further, it is possible that banks will tend to underplay the extent of the risk exposure, to the supervisors, which in the worst case, may even lead to bankruptcy. This problem and its step-by-step solution was discussed in [14]. The problem of limited ability of supervisors to decipher whether the bank disclosures are

honest or not, is shown to have an enforceable solution, by way of imposition of a risk-independent LR restriction. A World Bank report [33] has elaborated upon the concept of leverage, and also the necessity of the LR, to complement the already existent capital requirement framework, with the caveat of it (LR) being one of the several policy tools, in the paradigm of assessing the leverage buildup in a financial institution. In this context, Hildebrand [41] highlighted the benefits of a LR, while recognizing the shortcomings. The benefits include the complementary non-risk based nature and the simplicity, in terms of definition, application and monitoring, while the shortcomings include off-balance sheet exposure, profitability and pro-cyclicality. Dell’Ariccia et al. [31] in their work related the low interest rate scenario with the leverage, as well as risk-taking by the banks. The authors concluded that when banks are accorded the leeway of determining their capital structure, reduction in interest rates lead to increased leverage, and consequently, higher risk, provided the loan demand function has a diminishing slope. However, in case of fixed capital structure, the impact is contingent on the extent of the leverage. Subsequently, Smith et al. [2] worked on the bank’s decision problem, contingent on the LR restriction. In the model, the capital holding is considered, with all investment done in one risky asset, and all payments being done from the bank’s side. It was shown that every bank has a tendency to hold less capital, which is otherwise very evident. Further, one constant \hat{k} was derived for each bank, depending upon the assets of the bank, thereby establishing a relation with LR and risk taking. Finally, it was noted that relative to a solely risk-based capital framework, the imposition of the LR requirement leads to lower probabilities of bank failure.

In [48] and [68], the authors have categorized the loan types as high risk loans and low risk loans. Unlike the previous works, in this case, instead of considering the analysis of the portfolio of one bank, they have established the stability of the banks, by the expected number of bank failures, among the banks operating in the market. Further in the equilibrium (where the demand of loans and the supply of loans are equal, it is defined as the zero net value of the bank), they determined the relation between the LR and the banks’ portfolio, with low risk and high risk loans. Finally, the papers also analyzed that higher LR increases the stability of the bank. As the LR increases, the number of high risk loans decreases in the portfolio of the bank. Sale of bank loans can often be motivated by limitations resulting from regulatory requirements [19]. In case of unregulated banks, there is enough incentive to extend loans and then sell off these loans to other banks, rather than adopt a more traditional approach of accepting deposits to fund these loans. Banks can extend credit only to the extent that regulators allow for. In this scenario, it is the approach of loan sales that is attractive for such banks (thereby going beyond the permissible regulatory limits) and also for the banks which have room (from capital requirement perspective) for purchasing these loans. In practice, this amounts to capital buffer of non-local banks to support local projects routed through local banks.

The aspects of leverage and the consequent risk is intricately linked to risk-return paradigm of the portfolio of loans held by the bank [56]. Accordingly, the distribution of loan portfolios was studied sector-wise to encapsulate the cyclical characteristics of different types of loans and yields. The classical Markowitz approach is applied with a Value-at-Risk (VaR) constraint (to accommodate regulatory requirements), as well as the relationship maximization of the utility function. Further, the model was also used to capture information about default correlations. In addition, this exhaustive study also included the determination of a no-arbitrage principle driven pricing of loans, taking into account the credit risk associated with the debtor. An optimization approach to the construction of a loan portfolio using VaR and CVaR constraints is presented in [60]. To this end, the basic approach of Lagrangian algorithm is employed, in order to determine the efficient frontier. An alternative approach of minimization of credit risk, in terms of expected loss is empirically analyzed in [23]. Accordingly, a methodology based on the Large Deviation Theory (LDT) is used for portfolio optimization, by taking into account, the heterogeneity of risk characteristic across different geographical locations. The key takeaway was the demonstration of a significant improvement in the performance of this approach, vis-a-vis other benchmark portfolios, with this improvement being achieved in terms of enhanced excess return and reduced credit risk. For a detailed description of portfolio management, in presence of default risk, one may refer to the KMV document [47]. The authors of the work, enumerated the key aspects of this problem, and covered models of default risk, default correlation and value correlation. This is followed by risk contribution and its relation to optimal diversification, as well as economic capital. They concluded with an accurate and detailed description of the risk of losses experienced in loan portfolios, by considering different “sub-portfolios”, of a typically very large portfolio of banks. In our study, we have constructed the loan portfolio which satisfies the Basel III capital requirement criteria.

1.4 Liquidity risk

Managing liquidity is a very critical aspect for the smooth functioning of banks. Liquidity relates to the bank’s ability to meet its obligations at any given point of time. Therefore, well-managed and liquid assets in the portfolio enhance the bank’s ability to meet these obligations. A liquidity buffer can also help in the bank’s survival under stressed market conditions. On the other hand, liquidity helps in day-to-day operations of the bank, such as settlements, lending and withdrawal demands. Therefore, regulators have imposed several liquidity criteria for banks to enable a robust structure to absorb systemic shocks as well retaining the confidence of depositors and investors. After the 2008 financial crisis, the BCBS focused on the introduction of prudent measures for ensuring the liquidity of bank assets. In accordance with this objective, Basel III has introduced Liquidity Coverage Ratio (LCR) and Net Stable Funding Ratio (NSFR). Besides this, they have also introduced several

monitoring tools to capture specific information about a bank's cash flows, balance sheet structure, available unencumbered collateral and specific market indicators. These monitoring tools include metrics like contractual maturity mismatch, concentration of funding, available unencumbered assets, market-related monitoring tools, etc., the details of which are available in [13]. On the other hand, Basel has also proposed a frequency with which banks calculate and report the metrics. The frequency for the LCR may be monthly, weekly, or even daily, depending on the scenario. On the other hand, the NSFR is mandated be calculated and reported at least quarterly. The time lag in reporting should be as short as feasible, so as to not exceed two weeks for both LCR and NSFR.

The BCBS in its report [12], observed that many banks which had maintained good levels of capital during the 2008 crisis, suffered significant losses as a consequence of their assets being rendered illiquid. This was the genesis for the inclusion of regulatory provisions pertaining to liquidity of banks, via the introduction of LCR, which is defined as,

$$\text{LCR} := \frac{\text{High-quality liquid assets}}{\text{Total net cash outflows over the next 30 calendar days}}.$$

The regulators defined the LCR so as to achieve a certain level of highly liquid assets, to enable the bank's survival over the next 30 days, in a stress scenario. Furthermore, in order to ensure medium-to-long-term funding of assets, the BCBS introduced NSFR [13], defined as,

$$\text{NSFR} := \frac{\text{Available amount of stable funding}}{\text{Required amount of stable funding}}.$$

The regulatory provisions require that banks and financial institutions have to maintain these ratios at above of 100%. In the context of characterizing the liquidity of banks' assets, the key indicator is "haircut", which reflects the risk of the value of assets declining, as a result of immediate sale or liquidation of these assets. In our study, we will adopt this concept of "haircut" in formulating the regulation of liquidity of the loan portfolio. In [77] authors have discussed about the classification of liquid asset depending on haircuts. A relation of liquidity of bank assets vis-a-vis the stability of bank is presented in the study carried out by Wagner [79]. A continuous time dynamic model for maximization of equity return, with liquidity requirements (introduced in Basel III) is analyzed in [43], wherein the authors have demonstrated that liquidity requirements result in the decrease of loss magnitude (in the event of default), at the cost of an increased likelihood of default. For an elaborate discussion on banking economics and several policy issues, the interested reader may refer to [34].

1.5 Dynamic portfolio theory

Dynamic portfolio theory (DPT) is an extended form of classical portfolio optimization problem into a multi-period or continuous-time framework, which incorporates the temporal evolution of

asset returns, investor preferences, market conditions and regulatory requirements, unlike static portfolio theory, which considers a single-period optimization, DPT accounts for inter-temporal decision making, allowing investors to rebalance portfolios over time in response to new information or objective changes.

Continuous-time models deal with mathematical models of financial markets in which agents are allowed to trade continuously, rather than only at discrete periods. Thus, the market security prices must be modelled as continuous-time processes, usually stochastic. Since trading in modern financial markets can be executed quickly, the model reasonably approximates the real markets (at least if we ignore price changes that happen overnight and over weekends and holidays). A problem with multiperiod discrete-time models is that they quickly become untractable for computational purposes and the notation tends to be cumbersome. Moreover, continuous-time models provide an opportunity to model complex price dynamics using a small number of model parameters that frequently have nice intuitive interpretation. For these reasons scholars have studied continuous-time models, which are often limits of discrete time multi period models, when the length of each time period goes to zero [30].

Continuous-time models involve more advanced and complex mathematical tools than the case of the discrete-time models. These models open the path to deriving explicit solutions (analytic or numerical) for many pricing and investment problems. This method has become a standard method with researchers and industry professionals using continuous-time models.

The continuous-time portfolio selection problem was first established by Merton [57, 58], wherein the model was set up to optimize the portfolio selection problem, in continuous time, from the perspective of an investor. The problem formulation sought to determine the portfolio that needs to be held between the current time and the finite final time T , via maximization of the expected terminal utility of the wealth level. The portfolio under this model framework is assumed to be comprised of a risk-free asset and a finite number of risky assets, which are accessible for the purpose of investment. The optimization problem (to determine the optimal portfolio) reduces to a stochastic control problem, which again reduces to the Hamilton–Jacobi–Bellman (HJB) equation. This equation is a central result in optimal control theory, characterizing the value function of a dynamic optimization problem. Under some specific conditions, this HJB equation can have a classical solution [39]. To overcome the limitations of classical solutions, Crandall and Lions [26] developed the theory of viscosity solutions, which allows us to define and study weak solutions to HJB equations even when the value function is nonsmooth.

Continuous time portfolio theory is a fascinating research topic after Merton’s seminal work. The most natural extension of Markowitz mean-variance portfolio optimization in a dynamic set-up was studied by [53, 80]. Detemple [32] has studied DPT models under partial information. The model in this literature considers Bayesian learning for updating beliefs about expected returns.

Modern research seeks to integrate continuous-time portfolio optimization models with data-driven methods. Reinforcement Learning (RL) offers promising tools to approximate optimal dynamic policies in high-dimensional and nonlinear environments [49]. One of the most trending topics is HFT (High Frequency Trading). The strategies in this area are based on continuous time models [21].

DPT has several applications in the real world, such as asset management problems, including pension funds, insurance and other financial products that foster growth and/or stability. Besides this fund management, it is also widely used in risk management and derivative pricing (like short rate, swap price, option price, etc.). In chapter 5, we studied a continuous-time loan portfolio optimisation problem. As a result, this model generates a continuous-time asset allocation strategy.

1.6 Interest rate risk

Interest rate risk affects financial assets, derivatives, liabilities, cash flows, etc., due to interest rate fluctuation. It is an important part of market risk, mainly affecting insurance companies, pension funds, asset managers, and corporations with interest rate-sensitive exposures. Central bank policy, inflation expectations, economic growth, and liquidity conditions are the main drivers of interest rate movements. Because of its systemic impact, interest rate risk is closely monitored by regulators, especially under Basel III and the IFRS 9 framework. Interest rate risk in the banking book (IRRBB) is currently part of the Basel capital framework's Pillar 2 (Supervisory Review Process) according to the article [7]. A good example of interest rate risk is the 2022–2023 interest rate hikes by the U.S. Federal Reserve and other central banks [44]. It has created large swings in bond portfolios and has raised interest rate risk as a systemic threat.

Interest rate risk can be measured by Duration and Convexity. In the paradigm of bond portfolio management, Duration is a measure of risk. It is also used to determine immunization, necessitated by the volatility in interest rates [22]. On the other hand, Convexity adjusts for curvature in the price-yield relationship. It measures the rate at which the Duration of a bond changes as interest rates change [37].

Modelling the interest rate is necessary for the pricing and risk management of interest rate derivatives, which are financial instruments whose value is derived from the level of interest rates and corresponding characteristics. In this paradigm, short-rate models are widely used to study the evolution of interest rates. Popular approaches to model short rates are Vasicek Model, Cox–Ingersoll–Ross (CIR) Model and Hull–White Model [18]. The other approach, the Heath–Jarrow–Morton (HJM) model, involves the evolution of interest rate curves [67].

One drawback of the seminal model by Merton was that it assumed the risk-free rate of interest to be a constant. This shortcoming (pertaining to the constant risk-free interest rate) was addressed

by Vasicek [78], and Ho and Lee [42]. Both these articles consider the interest rate to be following a stochastic process, that is both free of arbitrage and matches the current yield curve (implied by the pricing of interest rate derivatives). Vasicek [78] had considered a mean-reverting process for the interest rate and provided an analytic solution for the interest rate derivatives. This, in turn, has led to the determination of the bond pricing formula. The price of the bond can be correlated with the risk-premium in case of interest rate risk. In a more recent work [50], the authors have revisited the portfolio optimization problem, this time for a portfolio comprising of a savings account and some risky bonds. In chapter 5, we have considered the loan portfolio optimization problem with a stochastic interest rate.

1.7 KMV model

Kealhofer, McQuown, and Vasicek had developed a credit risk model, which is named after their names as the KMV model. Later, this model was commercialized by Moody's KMV model. The KMV model concerns a critical area in finance that studies the possibility of a borrower defaulting on a financial obligation. Merton's structural model inspires this model to estimate firms' Expected Default Frequency (EDF) using market-based information rather than solely accounting data [27, 47]. This model treats the firm's equity as a call option on its assets. Under this setup, default occurs if the firm's asset value falls below its debt obligation at maturity. The core output of this model is the *Distance to Default (DD)*, which measures the number of standard deviations the firm is away from default. It captures credit quality in a forward-looking and market-sensitive way. KMV uses current market data (equity prices, volatility) to infer credit risk in real-time, making it more responsive than models relying on accounting ratios. Besides this, it is also grounded in economic theory (firm value, capital structure), giving the model strong interpretability.

1.8 Thesis outline

We have started with an elaborate introduction of the necessary mathematical and financial concepts used in this thesis. This discussion is treated as the first chapter of this thesis.

The introduction is followed by chapter 2. In this chapter, we have analyzed the effect of including it in the model formulation. We formulate four models, two of them involve maximizing the expected return with risk constraint, including and excluding Limited Liability, and other two involve minimization of risk with threshold level of return with and without Limited Liability. Our theoretical results show that the solutions of the models with Limited Liability produce better results than the others, in both minimizing risk and maximizing expected return. More specifically, the portfolios which included Limited Liability are less risky as compared to the portfolios which did not include Limited Liability. Finally, an illustrative example is presented to support the

theoretical results obtained.

In chapter 3, we consider the problem of a bank's loan portfolio in the context of liquidity risk, while allowing for the Limited Liability protection enjoyed by the bank. Accordingly, we construct a novel loan portfolio model with limited liability, while maintaining a threshold level of haircut in the portfolio. For the constructed three-time step loan portfolio, at the initial time, the bank raises capital via debt and equity, investing the same in several classes of loans, while at the final time, the bank either meets its liabilities or becomes insolvent. At the intermediate time step, a fraction of the deposits is withdrawn, resulting in the liquidation of some of the bank's assets. The liquidated portfolio is designed with the goal of minimizing the liquidation cost. Our theoretical results show that the model with the haircut constraint leads to lesser liquidity risk, as compared to the scenario of no haircut constraint being imposed. Finally, we present numerical results to illustrate the theoretical results which were obtained.

In chapter 4, we have studied the bank dynamic decision problem in the intermediate time step for a discrete-time setup. We have considered a three-time-step model. Initially, the banks raise money through debt and equity and invest in different types of loans. It liquidates its assets and raises new funds at the intermediate-time step to meet the short-term debt holder's claim. Further, it has to meet specific capital requirements given by the regulators. In this work, we have theoretically studied the effect of raising new equity and debt. We show that in some cases, raising equity and debt may increase the return on equity, and in some cases, it may decrease the return on equity. We have discussed several cases and given a bound on the capital that can be raised. We have added an equity holder's constraint, which ensures the return on equity and desists the bank from defaulting at the final time point.

In chapter 5, we have presented a novel approach for the bank's decision problem, incorporating Limited Liability in the objective function. Accordingly, we consider continuous time models, with and without Limited Liability. We compare the solutions of these two models to demonstrate the effect of inclusion of Limited Liability. To solve the problem with the objective function incorporating Limited Liability, we approximate the payoff function to another set of functions for which we have closed-form solutions. Then, we show that the solution with Limited Liability incorporates less risky assets, while simultaneously increasing the resilience of the bank. After that, we use the metric of DD , from the KMV Model, to analyze the bank's resiliency, by considering that the interest rate follows the Vasicek model. Finally, we illustrate the results obtained with a numerical example.

And finally, the chapter 6 discusses the concluding remarks and the future direction of the research.

Chapter 2

Does Limited Liability reduce leveraged risk?: The case of loan portfolio management

2.1 Introduction

Return-risk models form the foundation of modern portfolio theory, which is widely used in decisions process when setting up a bank's loan portfolio. Banks and other financial institutions are subjected to Limited Liability protection. However, in most of the model formulations in literature, this problem is not taken into consideration. Accordingly, to address this, we have analyzed the effect of including it in the model formulation, in the presentation made in this chapter. We formulate four models, two of which maximize the expected return with risk constraint (including and excluding Limited Liability), and the other two minimize risk with a threshold level of return (with and without Limited Liability). For this problem, we choose the risk measure as the Expected Loss. We derive the change in the Expected Loss, resulting from loan portfolio allocation (which is related to the bank or the financial institution's resilience). Our theoretical results show that the solutions of the models with Limited Liability produce better results than without it, in both the case of minimizing risk and maximizing expected return. More specifically, the portfolios that include Limited Liability are less risky than those that did not. Finally, an illustrative example is presented to support the theoretical results obtained. The organization of the rest of the chapter is as follows. In section 2.2, we provide a description of the portfolio optimization models, with and without Limited Liability. In section 2.3, we present the fundamental motivation of the work through the presentation of our novel theoretical results. In section 2.4, we present two illustrative examples in support of our mathematical results. In particular, we present plots to demonstrate the sensitivity of Expected Loss, Unexpected Loss and returns vis-a-vis the portfolio allocation of the loans. Finally, in section 2.5 we present our conclusions and main takeaways from the work. This sequence of presentation, as described, is also shown in the following flow chart in Figure 2.1.

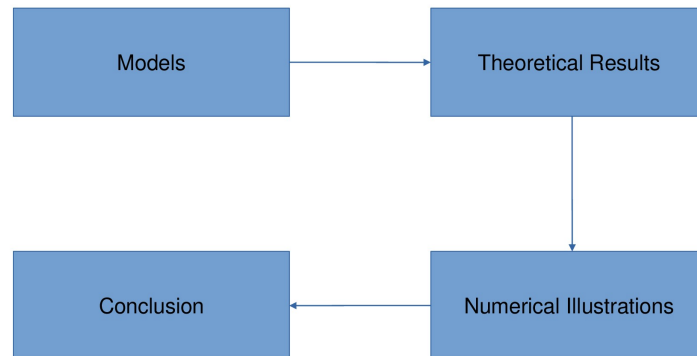


Figure 2.1: Outline of the work presented

2.2 The Models

The risk management of a loan portfolio in a bank is an increasingly complex exercise, due to several factors. Banks have been extended the legal protections emanating from the concept of Limited Liability, which effectively means that the losses cannot exceed the net value of the bank. In the event of the net worth being less than the depository liability, the depositors do not recover their entire deposit, as a result of the bank having gone bankrupt. However, if the net value at the end is more than debt, then the depositors are entitled to the remaining assets after the payout have been made to the creditors of the bank. While most of the literature on managing a loan portfolio does the modeling in the regime of the standard risk-return model, in some of the works, the notion of Limited Liability has been considered for the model setup. While the models due to [2, 14, 31, 48, 68] mention about the Limited Liability considerations, however they do not extend this discussion to a narrative on the advantages and disadvantages resulting from the usage of Limited Liability in the model paradigm. A summary of the relevant work in the risk-return framework for the loan portfolio decision problem is presented in the Table 2.1, which also highlights the novelty and advancement of this work vis-a-vis the existing literature.

In this chapter, we show that including Limited Liability plays a vital role in optimizing the expected return, as well as in reducing the expected and the unexpected loss of the portfolio. Accordingly, we formulate two sets of problems, by the consideration of expected return and risk.

2.2.1 Models without Limited Liability

Before going to the modelling setup (for a loan portfolio of size n), we introduce the parameters that are going to be used in the risk-return framework, as given in Table 2.2.

Case 1	Case 2	Case 3	Case 4	Source
No	No	Yes	No	[14], [31], [2], [48], [68].
No	Yes	No	No	[56],[60],[23].
Yes	No	No	No	[19], [63].
Yes	Yes	Yes	Yes	This work

Table 2.1: Comparative literature highlighting the novelty of the work. Case 1: Maximization of expected return without Limited Liability; Case 2: Minimization of risk without Limited Liability; Case 3: Maximization of expected return with Limited Liability; Case 4: Minimization of risk with Limited Liability

Symbol	Meaning
k_{lev}	Leverage Ratio.
$K(x)$	Internal Ratings Based (IRB) Capital Requirement for the portfolio x .
δ	Opportunity cost of the capital (equity).
X_x	Realizations for the loan portfolio x .
$\rho(x)$	Risk measure for the portfolio x (EL, UL etc.)
μ	Lower-bound on Expected Return
θ	Upper-bound on the risk.

Table 2.2: List of symbols used in the model

Problem 1. *Maximization of Expected Return without Limited Liability:*

$$\max_{x,k} [E[X_x] - (1 - k) - \delta k],$$

subject to,

$$k \geq \max(k_{lev}, K(x)), \sum_{i=1}^n x_i = 1, x_i \geq 0 \forall i = 1 : n, \rho(x) \leq \theta.$$

Here $E(X) = \sum_{i=1}^n x_i R_i$, where x_i and R_i are the weights and the expected return of the i -th loan in the portfolio. The objective functional (motivated by [48]) to be maximized involves the maximization of the expected return minus a function of k (which is the larger of the Leverage Ratio and the IRB based capital requirement), with $\delta > 1$. In addition, the usual conditions of sum of weights being equal to one, no-short selling being permissible, and an upper bound on the risk, are applicable. Finally, we denoted by $(x_1, k_1) \in \mathbb{F}_1$, the solution of the optimization problem, with \mathbb{F}_1 being the feasible region for this optimization problem.

Problem 2. *Minimization of Risk without Limited Liability:*

$$\min_{x,k} \rho(x),$$

subject to,

$$k \geq \max \{k_{lev}, K(x)\}, \sum_{i=1}^n x_i = 1, x_i \geq 0 \forall i = 1 : n, E[X_x] - (1 - k) - \delta k \geq \mu.$$

Here, the constraints on k , the sum of weights being equal to one and no-short selling being permissible, are akin to that of Problem 1. Also the expected excess return over $((1 - k) + \delta k)$ is required to be at least a threshold value, denoted by μ . Finally, we denote by $(x_2, k_2) \in \mathbb{F}_2$, the solution the problem, with \mathbb{F}_2 being the feasible region for this optimization problem.

2.2.2 Models With Limited Liability

Before going to the modelling aspects of Limited Liability, let us discuss some important assumptions.

- (A) It is observable [4] that banks typically will not invest in loans with the default probability $p \geq 0.2$ (which can also be concluded by analyzing the forms of Unexpected Loss). As reported in [4], the loans in investment grade and speculative grade have the maximum probability of default being 0.2. Also the article [38] notes that for the S&P Rating with highest default probability, namely the rating of “D-”, the default probability lies between 0.18 and 0.20.
- (B) If the bank invests all its money in a single loan, then if the expected return from risky loan, after paying the liabilities and the opportunity cost of the equity, is higher than that of the safe loan, then the bank chooses the risky loan in their portfolio, which mathematically translates to,

$$R_H - (1 - k'_H) - \delta k'_H > R_L - (1 - k'_L) - \delta k'_L.$$

Here R_H and R_L are the expected return from high risk and low risk loans, respectively, where $k'_H = \max \{k_H, k_{lev}\}$ and $k'_L = \max \{k_L, k_{lev}\}$, are the capital holdings for high risk and low risk loans, respectively. Further, k_H and k_L are the capital requirements, based on the Internal Ratings Based (IRB) approach.

- (C) The loans are uncorrelated among themselves.

Accordingly, we are now in a position to define the following two problems, involving Limited Liability. For this purpose we slightly modify the objective functional, motivated by [2], for Problems 3 and 4.

Problem 3. *Maximization of Profit with Limited Liability:*

$$\max_{x,k} E[\max((X_x - (1 - k)), 0)] - \delta k,$$

subject to

$$k \geq \max(k_{lev}, K(x)), \sum_{i=1}^n x_i = 1, x_i \geq 0 \forall i = 1 : n, \rho(x) \leq \theta.$$

Let $(x_{1L}, k_{1L}) \in \mathbb{F}_{1L}$ be the solution of the problem, with \mathbb{F}_{1L} being the feasible region for this optimization problem.

Problem 4. *Minimization of Risk with Limited Liability:*

$$\min_{x,k} \rho(x),$$

subject to

$$k \geq \max\{k_{lev}, K(x)\}, \sum_{i=1}^n x_i = 1, x_i \geq 0 \forall i = 1 : n, E[\max\{(X_x - (1 - k)), 0\}] - \delta k \geq \mu.$$

Here μ is the threshold value of the expected excess return. Let $(x_{2L}, k_{2L}) \in \mathbb{F}_{2L}$ be the solution of the problem, with \mathbb{F}_{2L} being the feasible region for this optimization problem.

2.3 Results

Here we begin with an important theorem.

Theorem 2.3.1. *Investing more in risky asset increases Expected Loss and Unexpected Loss.*

Proof. In this proof we use the formula used by [47], for the expected loss (EL) for portfolio P ,

$$EL_P = \sum_i x_i p_i \lambda_i,$$

and for the unexpected loss (UL) for portfolio P ,

$$UL_P = \sqrt{\sum_i \sum_j x_i x_j \rho_{ij} UL_i UL_j},$$

where $UL_i = \lambda_i \sqrt{p_i(1 - p_i)}$ and λ_i and p_i are the Loss Given Default (LGD) and Probability of Default (PD), respectively, for the i -th loan. So it is clear that risky loans have large λ , as well as p_i , and hence increasing either or both of them, for a risky loan, also increases both EL and UL. \square

Now we show that including Limited Liability in the model leads to out-performance of the models without Limited Liability. First we show the minimization of risk model with and without Limited Liability.

Theorem 2.3.2. *In case of minimizing risk, Problem 4 outperforms Problem 2.*

Proof. For a particular portfolio x and a particular k , we have,

$$E[\max\{(X_x - (1 - k)), 0\}] - \delta k \geq E[X_x] - (1 - k) - \delta k.$$

Therefore \mathbb{F}_{2L} contains more points than \mathbb{F}_2 . In other words, $\mathbb{F}_2 \subseteq \mathbb{F}_{2L}$. Hence,

$$\min\{\rho(x)|(x, k) \in \mathbb{F}_{2L}\} \leq \min\{\rho(x)|(x, k) \in \mathbb{F}_2\}.$$

□

Now, in order to prove that Problem 3 outperforms Problem 1, we first establish the following Lemmas.

Lemma 2.3.3. *In the solution of Problem 1, $k_1 = k'_1$ where $k'_1 = \max(K(x_1), k_{lev})$.*

Proof. $E[X_x] - (1 - k) - \delta k$ is a monotonically decreasing function of k (keeping x fixed), since as $\delta > 1$. So if $k_1 > k'_1$, then we get,

$$E[X_{x_1}] - (1 - k'_1) - \delta k'_1 > E[X_{x_1}] - (1 - k_1) - \delta k_1,$$

which is a contradiction, since this objective function attains its maximum value at (x_1, k_1) . Hence $k_1 > k'_1$ is impossible. But since, $k_1 \geq k'_1$, hence we conclude that $k_1 = k'_1$ □

Lemma 2.3.4. *$E[X_{x_1}] - (1 - k'_1) - \delta k'_1 \geq E[X_{x_{1L}}] - (1 - k'_{1L}) - \delta k'_{1L}$, where $k'_{1L} = \max\{K(x_{1L}), k_{lev}\}$.*

Proof. The proof follows from the fact that LHS in the inequality, is the maximum value of the function $E[X_x] - (1 - k) - \delta k$. □

Now let us come to the main result.

Theorem 2.3.5. *In case of maximizing expected return, Problem 3 outperforms Problem 1*

Proof. Using Lemma 2.3.4 we get,

$$E[X_{x_1}] - (1 - k'_1) - \delta k'_1 \geq E[X_{x_{1L}}] - (1 - k'_{1L}) - \delta k'_{1L}.$$

Case 1: We consider $k'_1 = k'_{1L} = k_{lev}$, so that the above inequality becomes,

$$E[X_{x_1}] \geq E[X_{x_{1L}}].$$

Therefore by the Arbitrage Pricing theory (APT), x_1 contains more risky loans than x_{1L} .

Case-2: Next we consider $k_1 = k_{lev}$ and $k_{1L} = K(x_{1L})$. Now, since $k_{1L} = K(x_{1L}) = \max\{K(x_{1L}), k_{lev}\} \geq k_{lev}$, therefore,

$$E[X_{x_1}] - (1 - k_{lev}) - \delta k_{lev} \geq E[X_{x_{1L}}] - (1 - k_{lev}) - \delta k_{lev},$$

as (x_1, k_1) is the optimal solution. Hence, this implies that,

$$E[X_{x_1}] \geq E[X_{x_{1L}}].$$

Therefore by the APT, x_1 contains more risky loans than x_{1L} .

Case-3: We now consider $k_1 = K(x_1)$ and $k_{1L} = k_{lev}$. Then,

$$E[X_{x_1}] - (1 - K(x_1)) - \delta K(x_1) \geq E[X_{x_{1L}}] - (1 - k_{lev}) - \delta k_{lev}.$$

Now,

$$E[X_{x_1}] - (1 - k_{lev}) - \delta k_{lev} \geq E[X_{x_1}] - (1 - K(x_1)) - \delta K(x_1),$$

since $k_{lev} \leq K(x_1)$ in this case. Now from the above two inequalities, we get,

$$E[X_{x_1}] - (1 - k_{lev}) - \delta k_{lev} \geq E[X_{x_{1L}}] - (1 - k_{lev}) - \delta k_{lev}.$$

Therefore,

$$E[X_{x_1}] \geq E[X_{x_{1L}}].$$

Therefore by the APT, x_1 contains more risky loans than x_{1L} .

Case-4: We finally consider the case $k_1 = K(x_1)$ and $k'_{1L} = K(x_{1L})$. Accordingly, we get,

$$E[X_{x_1}] - (1 - K(x_1)) - \delta K(x_1) \geq E[X_{x_{1L}}] - (1 - K(x_{1L})) - \delta K(x_{1L}).$$

This implies that,

$$\sum_i x_{1i} R_i - (1 - K(x_1)) - \delta K(x_1) \geq \sum_i x_{1Li} R_i - (1 - K(x_{1L})) - \delta K(x_{1L}).$$

Now we know that the function $K(x) = \sum_i x_i K_i$, where K_i is the capital requirement for the i -th loan [48]. Applying this in the inequality, we obtain,

$$\sum_i (x_{1i} - x_{1Li})(R_i - (1 - k_i) - \delta k_i) \geq 0.$$

Now since $\sum_i x_{1i} = \sum_i x_{1Li} = 1$, therefore some of $(x_{1i} - x_{1Li})$ are negative, while some are positive. As the overall sum is positive, hence the positive terms have more weightage. Hence x_1 has more risky investment than x_{1L} .

Therefore we can see that portfolio x_{1L} contains less riskier loans than portfolio x_1 , thereby producing more profit upon success of the bank. \square

2.4 An Example

Financial institutions classify the investment instruments (loans, in this case), contingent on its quality, particularly its creditworthiness. The ranking of the loan seekers is executed through various grades, with the prominent ones being the ratings of S&P and Moody's. In the context of our discussion, we construct an example to illustrate our theoretical results, presented in the preceding Section. As a model built-up, we consider a basic scenario of two kinds of loans, namely, a safe loan and a risky loan. From the perspective of credit risk management, some of the factors which play a pivotal role in characterizing the loan are PD, Exposure at Default (EAD), LGD, returns and the statutory capital requirements. The values of some of these parameters are available in articles [48, 71] and also publicly available on the websites of banks ([1] and [4], for instance). Finally, motivated from [48], we take the value of $\delta = 1.04$.

For the illustrative example, we have constructed a loan portfolio of three loans, namely, one which is completely safe while the other two are risky, with one of these two being riskier than the other owing to greater the PD and expected loss. The value of all the parameters are enumerated in Table 2.3.

Loan Type	Return	PD	LGD
Safe Loan	$r_{rf} = 3\%$	0	0
Less Risky Loan	$r_s = 9\%$	$p_s = 6.1\%$	$lgd_s = 10\%$
More Risky Loan	$r_r = 13.2\%$	$p_r = 12.2\%$	$lgd_r = 9\%$

Table 2.3: Risk parameters for the three loans

We consider another example as tabulated in Table 2.4. In this example, we take $k = 5\%$, with the decrement in Expected Loss being 2.27% and 11.79%, respectively, in case of the risk minimization model and the expected return maximization model, resulting from the incorporation of Limited Liability.

Loan Type	Return	PD	LGD
Safe Loan	$r_{rf} = 5\%$	0	0
Less Risky Loan	$r_s = 9\%$	$p_s = 5\%$	$lgd_s = 10\%$
More Risky Loan	$r_r = 12\%$	$p_r = 12\%$	$lgd_r = 9\%$

Table 2.4: Risk parameters for the three loans

Expected Loss plays the role of risk measure in our model. The formula in [75], for the capital requirement for loans as a function of probability of default and loss given default, is given by,

$$C(PD, LGD) = LGD \times (Z - PD),$$

where Z is obtained as,

$$Z = \left(\phi \left[\frac{\phi^{-1}(PD) + \sqrt{\rho}\phi^{-1}(0.999)}{\sqrt{1-\rho}} \right] \right).$$

Here ϕ is the cumulative standard normal distribution, while ρ is different for different types of loans (the details of which are available in [75]). For this work, we have taken $\rho = 0.15$. We solve the minimization of risk model by using “scipy” package of Python. Then, the model becomes,

$$\min_{x,k} p_s \times lgd_s \times x_1 + p_r \times lgd_r \times x_2,$$

subject to:

$$x_0 + x_1 + x_2 = 1, k \geq k_{lev},$$

$$k \geq \max\{C(p_s, lgd_s), k_{lev}\} \times x_1 + \max\{C(p_r, lgd_r), k_{lev}\} \times x_2$$

and

$$\max(R_4, 0)(1-p_s)(1-p_r) + \max(R_5, 0)(1-p_s)p_r + \max(R_6, 0)p_s(1-p_r) + \max(R_7, 0)p_s p_r - \delta k \geq 0.098.$$

Here,

$$R_4(x_0, x_1, x_2, k) = r_r f x_0 + (1 + r_s)x_1 + (1 + r_r)x_2 - (1 - k),$$

$$R_5(x_0, x_1, x_2, k) = r_r f x_0 + (1 + r_s)x_1 + (1 - lgd_r)x_2 - (1 - k),$$

$$R_6(x_0, x_1, x_2, k) = r_r f x_0 + (1 - lgd_s)x_1 + (1 + r_r)x_2 - (1 - k),$$

$$R_7(x_0, x_1, x_2, k) = r_r f x_0 + (1 - lgd_s)x_1 + (1 - lgd_r)x_2 - (1 - k).$$

Solving this, we get the loan portfolio allocation of (5.72%, 13.37%, 80.91%) with the Leverage Ratio of 4%. On the other hand, solving for the model of minimizing risk, without Limited Liability, we get loan portfolio allocation of (2.43%, 13.18%, 84.39%), again with Leverage Ratio 4%. So for this particular example, there is a 3.68% decrease in the expected loss, resulting from the inclusion of Limited Liability.

For the next case, we illustrate the case of maximizing return with, as well as without Limited Liability. Accordingly, for this case, the model with Limited Liability will be transformed into a smooth optimization problem. We have given an upper bound of 1.2% in Expected Loss for the problem. We have taken the Leverage Ratio to be 4%. Solving this, we get 12.35% decrease in Expected Loss for changing this objective function in this example.

Now we come to the problem of maximizing return with and without Limited Liability. Then Problem 1 is easily solved, since all the functions in its objective functional and the constraints, are smooth. However since the objective functional contains the “max”, therefore it is not differentiable. Accordingly, the steps of our methodology are as follows:

- (1) Model Formulation: The model with Limited Liability (henceforth called as Model 1-L), which is not differentiable, is given by:

$$\max_{x,k} E(\max(X_x - (1 - k)), 0) - \delta \times k,$$

subject to,

$$k \geq \max\{k_{lev}, K(x)\}, \sum_{i=0}^2 x_i = 1, EL \leq 0.012 \text{ and } x_i \geq 0, \forall i = 0, 1, 2.$$

Therefore, we transform this model into another problem which is differentiable. To do this we include some new variables, namely x_4, x_5, x_6 and x_7 , in order to handle the non-differentiability part. Accordingly, the new model (henceforth called as Model 1-L-NM) is given by,

$$\max_{x,k} [x_4(1 - p_s)(1 - p_r) + x_5(1 - p_s)p_r + x_6p_s(1 - p_r) + x_7p_s p_r - 1.04 \times k],$$

subject to,

$$k \geq \max\{k_{lev}, K(x)\}, \sum_{i=0}^2 x_i = 1, EL \leq 0.012 \text{ and } x_i \geq 0, \forall i = 0, 1, 2,$$

as well as,

$$x_4(x_4 - R_4) = 0, x_5(x_5 - R_5) = 0, x_6(x_6 - R_6) = 0 \text{ and } x_7(x_7 - R_7) = 0.$$

- (2) Transformation: The supremum of the objective functionals for the above problem has the same value. Let obj_{1L} and obj_{1LNM} be the objective functional of the above models. Therefore the objective functional of the first problem is given by,

$$\begin{aligned} obj_{1L} &= \max(R_4, 0)(1 - p_s)(1 - p_r) + \max(R_5, 0)(1 - p_s)p_r \\ &+ \max(R_6, 0)p_s(1 - p_r) + \max(R_7, 0)p_s p_r - 1.04 \times k. \end{aligned}$$

Let $X_{1L} = (x_0, x_1, x_2, k)$ be the solution of Model 1-L. It also belongs to the feasible set of the Model 1-L-NM, with suitable values of x_4, x_5, x_6 and x_7 .

Let obj_{1LNM} attain its maximum at $X_{1LNM} = (x'_0, x'_1, x'_2, k', x'_4, x'_5, x'_6, x'_7)$. Now,

$$\frac{\partial (obj_{1LNM})}{\partial x_i} \geq 0, \forall i = 4, 5, 6, 7, \quad (2.4.1)$$

and the constraint involving $x_i, i = 4, 5, 6, 7$ does not involve $x_j, j = 4, 5, 6, 7$, excluding i . So any change in x_i does not affect x_j . From equation (2.4.1), we see that obj_{1LNM} increases with $x_i, i = 4, 5, 6, 7$. Hence x_i takes the largest value in the region allowed by the constraint. Therefore, $x_i = \max(R_i(x_0, x_1, x_2, k), 0), i = 4, 5, 6, 7$. Consequently, we get,

$$obj_{1LNM}(X_{1LNM}) = obj_{1L}(x'_0, x'_1, x'_2, k').$$

As the restrictions on (x_0, x_1, x_2, k) are the same in both the problems, hence we obtain,

$$obj_{1L}(x'_0, x'_1, x'_2, k') \leq obj_{1L}(x_0, x_1, x_2, k), \quad (2.4.2)$$

since (x_0, x_1, x_2, k) maximizes the obj_{1L} . Let us calculate $R_i(X_{1L})$, $i = 4, 5, 6, 7$ and $x_i = \max(R_i(x_0, x_1, x_2, k), 0)$, $i = 4, 5, 6, 7$. Further, we denote $X'_{1L} = (x_0, x_1, x_2, k, x_4, x_5, x_6, x_7)$. As X_{1LNM} maximizes the obj_{1LNM} , hence we get,

$$obj_{1LNM}(X'_{1L}) \leq obj_{1LNM}(X_{1LNM}),$$

and,

$$obj_{1L}(x_0, x_1, x_2, k) \leq obj_{1L}(x'_0, x'_1, x'_2, k'), \quad (2.4.3)$$

as,

$$obj_{1LNM}(X'_{1L}) = obj_{1L}(x_0, x_1, x_2, k) \text{ and } obj_{1LNM}(X_{1LNM}) = obj_{1L}(x'_0, x'_1, x'_2, k').$$

Therefore, from equations (2.4.2) and (2.4.3), we get that both the problems have the same supremum. If Model 1-L has a unique solution, then we get, $(x_0, x_1, x_2, k) = (x'_0, x'_1, x'_2, k')$.

In the example, we have taken three loan portfolios, one of which is completely risk-free. One of the remaining two is less risky (say L_s) and the remaining one is more risky (say L_r). In Figure 2.2, we have shown the change in EL against the change of investment in risky loans, where the x -axis presents the investment in L_s and y axis represents the investment in L_r . Also $(1 - x - y)$ represents the investments in a completely safe loan. Therefore $\{(x + y \leq 1, x \geq 0, y \geq 0)\}$ contains all possible portfolios. From the image we see that increasing investments in risky loans increases the EL . Full investment in the completely safe loan gives an EL of 0%. In the case of full investment in L_s and L_r , the resulting EL are 0.61% and 1.10%, respectively. Now, we discuss the change in the expected loss. The sensitivity of UL is given in Figure 2.3 with the investments in risky loans. Full investment in L_s and L_r , results in UL of 2.39% and 2.95%, respectively. Next, we show the change of return with and without Limited Liability. We have shown the change of returns by fixing x (investment in L_s) fixed at 0% and 5% in two images and then another two keeping y (investment in L_r) fixed at 0% and 10%. These four cases are done by keeping Leverage Ratio (k) at 4% (Figure 2.4), 7% (Figure 2.5) and 10% (Figure 2.6).

Now there are four scenarios of realizations (denoted by $s_i, i = 1 : 4$), for the two risky loans, namely, both the loans are repaid (s_1), only L_r defaults (s_2), only L_s defaults (s_3) and both the loans default (s_4). In Figure 2.4, we have plotted the returns profile when the Leverage Ratio is 4%. In this case, let us assume that the bank has invested total wealth of L_r ($x = 0$). So, if L_r defaults, the value of bank becomes $(1 - lgd_s) - (1 - k)$, which is negative. Therefore the bank fails to meet its liabilities. In Figure 2.5, we have plotted the returns profile when the Leverage Ratio

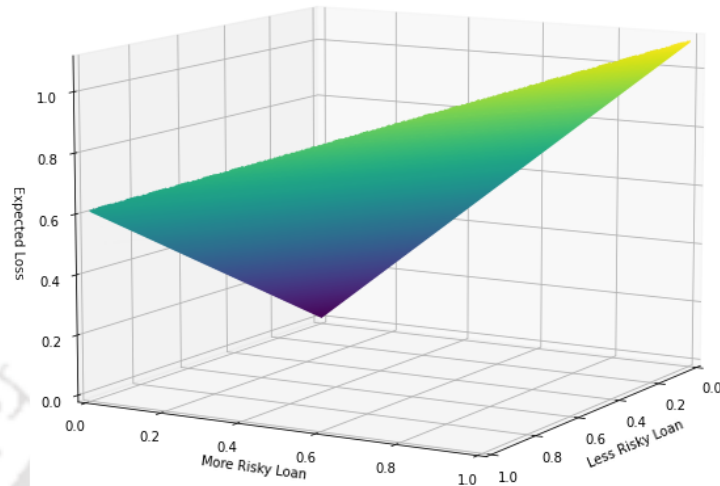


Figure 2.2: Expected Loss against investments in risky loans

is 7%. In this case, let us assume that the bank has invested total wealth to L_r . So, if L_r defaults, the value of bank becomes $(1 - lgd_s) - (1 - k)$, which is negative. Therefore, the bank again fails to meet the liabilities. The same consequences would be observed if the bank invests all its money to L_s ($y = 0$). When the Leverage Ratio is 7%, the loss in the worst case is less than that for the case in which the Leverage Ratio is 4%. If there is a penalty associated with the default and it is proportional to the amount of loss, then keeping a 7% Leverage Ratio causes less penalty in case of failure. In Figure 2.6, we have shown the change of return with keeping the Leverage Ratio at 10%. It is an interesting case, in the sense that even with the entire investments in L_s or L_r , the bank can survive the worst cases, due to the Leverage Ratio being 10%. We can see that banks can survive all worst cases for the possible portfolios for this parameter value of the loans. From the graph, we see that there is no gap between the two lines. Therefore, an increment in the Leverage Ratio increases bank stability and consequently decreases in terms of the gap between the two lines.

2.5 Summary

Banks and other significant financial institutions have Limited Liability protection. However, in the literature discussed, the decision process usually goes through the classical framework, whereby the profit is measured via the Expected Return or utility functions of the realizations, in conjunction

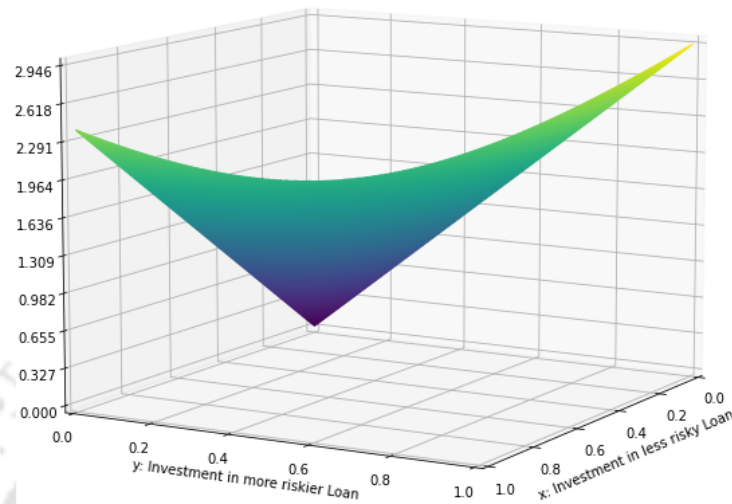


Figure 2.3: Unexpected Loss against investments in risky loans

with the risk profiles. However, typically, the notion of Limited Liability is not considered in the model which is used to make the decision. To this end, this chapter is focused on the approach of incorporating Limited Liability in the model setup. We theoretically establish the benefit of Limited Liability for both the objectives of profit maximization and risk minimization.

We have shown the comparative analysis between the models, for maximization of Expected Return, without Limited Liability (Problem 1) and with Limited Liability (Problem 3), with a same upper bound on risk for both. Further, (Problem 2) and (Problem 4) minimizes risk without and with Limited Liability, respectively, while keeping identical lower bound for the Expected Return, for both.

Moreover, the analysis shows that in the case of the first comparison, the model with Limited Liability protection has less risky loans than the model without Limited Liability, which supports the result obtained in Theorem 2.3.5. In the case of the second comparison, risk can be more minimized using the Limited Liability as the lower bound on the return with Limited Liability, as predicted in Theorem 2.3.2.

Therefore, from a practitioner's point of view, using Limited Liability in the decision model can help make the right investment decision, as Limited Liability protection reduces the amount of risk for the banks. So the construction of the portfolio, which meets the criteria of the investments (threshold level of profit or risks) and incorporates Limited Liability, is undoubtedly a more valuable

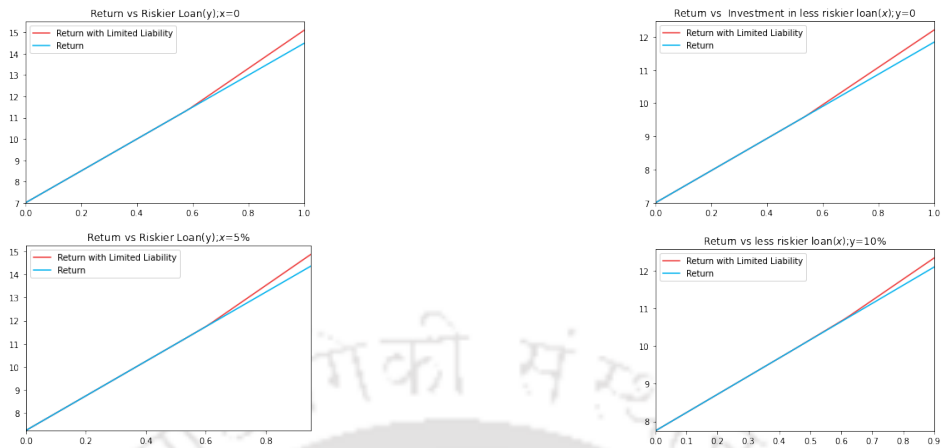


Figure 2.4: Expected Returns versus risky loans keeping the Leverage Ratio at 4%

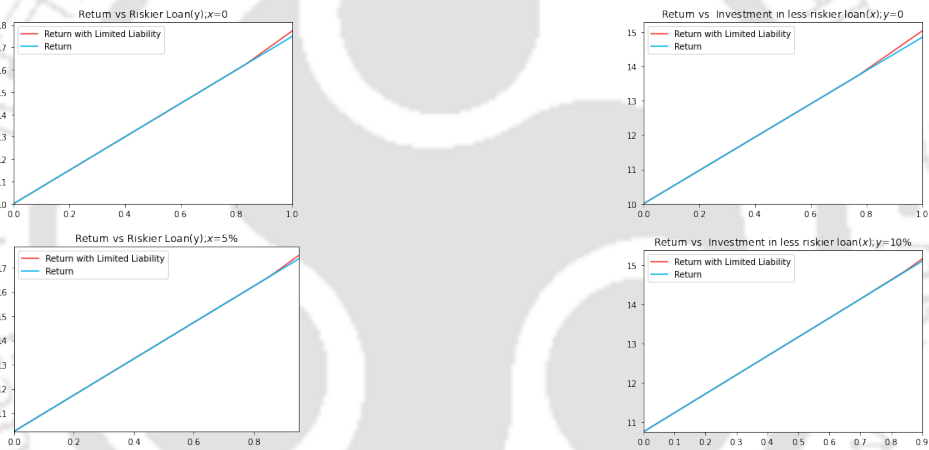


Figure 2.5: Expected Returns versus risky loans keeping the Leverage Ratio at 7%

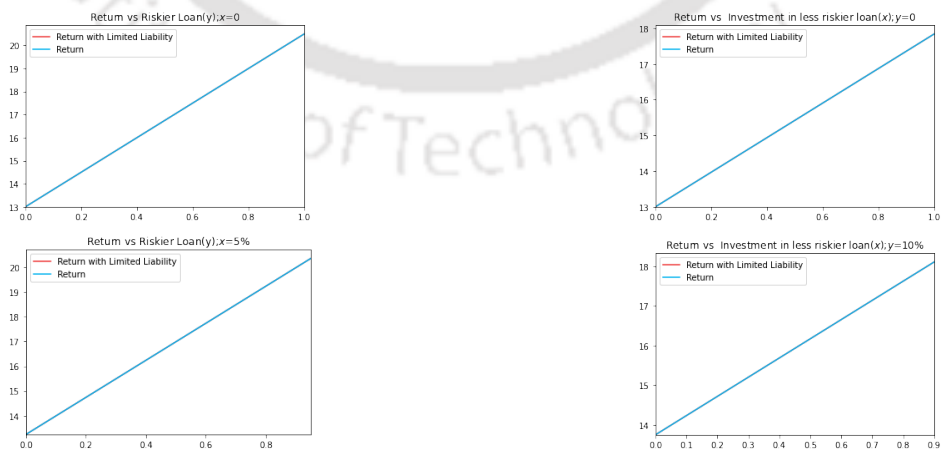


Figure 2.6: Expected Returns versus risky loans keeping the Leverage Ratio at 10%

and realistic model for the actual scenario.



Chapter 3

Loan portfolio management and liquidity risk: The impact of Limited Liability and haircut

3.1 Introduction

In chapter 2, we observed the effect of inclusion of Limited Liability in the excess return function, with capital requirements. In this chapter, we consider the problem of a bank's loan portfolio in the context of liquidity risk, while allowing for the Limited Liability protection enjoyed by the bank. Here we discuss the problem of liquidation, wherein at any intermediate time (between the initiation and maturity of a loan portfolio), the bank faces a claim from the depositor(s). In order to meet the claim, the bank makes use of its reserves, and in case of insufficiency of the same, it raises money, as suggested in [16, 45]. However, raising money in the intermediate time point can potentially be expensive (high cost, resulting from a stress scenario) or simply not possible, in which case the bank will have to resort to liquidation of some of its assets from the loan portfolio. In this context, a liquidation strategy, when a bank loan portfolio violates the capital requirements strategy, is suggested in [17]. More specifically, the authors suggested a value-minimizing strategy (value here means the total value of money raised from the sale) in order to meet capital requirement criteria. However, the case of deposit withdrawals is not discussed, in the event of which the bank (in order to meet the claim of depositor(s)) will seek to minimize the liquidation cost at an intermediate time step. Golec and Perotti [40] observe that safe assets are typically liquid assets, and therefore, the liquidity cost model recommends the liquidation of safe assets. However, this practice of liquidating safe assets (inevitably) increases the chance of default at the final time point, and it is precisely for this scenario that our work introduces a lower bound of risk for the liquidated portfolio, with the goal being the avoidance of liquidation of safe assets. Accordingly, we construct a novel loan portfolio model with Limited Liability, while maintaining a threshold level of haircut in the portfolio and capital requirements. For the constructed three-time step loan portfolio, at the initial time, the

bank raises capital via debt and equity, investing the same in several classes of loans, while at the final time, the bank either meets its liabilities or becomes insolvent. At the intermediate time step, a fraction of the deposits are withdrawn, resulting in liquidation of some of the bank's assets. We have proposed a liquidation strategy that minimizes liquidation cost and also reduces liquidation of the safe asset, thereby reducing the risk of default. Our theoretical results show that the model with the haircut constraint leads to lesser liquidity risk, as compared to the scenario of no haircut constraint being imposed. Finally, we present numerical results to illustrate the theoretical results which were obtained. The organization of the remainder of this chapter is as follows. In section 3.2, we introduce the intuitive motivation of the models which are then presented in section 3.2.1. In section 3.3 we demonstrate the advantages of our modeling approach, which is followed by an example and discussion of results in section 3.4. Finally, the conclusions and takeaways of this chapter is presented in section 3.5.

3.2 Formulation of the Model

In [5], we had discussed at length, the aspects of management of a loan portfolio, in the context of incorporating Limited Liability. It is but natural, now to bring forth the consideration of liquidity, the relevance and importance of which can be gauged from the recent events, such as the collapse of the Silicon Valley Bank. To encapsulate the dynamics of liquidity consideration, in the existent framework, we begin with the enumeration of the necessary (and important) components towards achieving this end, that is, incorporating liquidity in the model.

The incorporation of liquidity entails the construction of “at least” a three (time) setup model ($t = 0, 1, 2$), since the consideration of two time steps would not create a situation of liquidity crunch, due to the absence of the likelihood of deposit withdrawal at any intermediate time point of the loan duration (of two time points). Having considered the (aforesaid) simplest model setup, it is assumed that any liquidity requirement emerges as the result of a fraction of deposits being withdrawn at the intermediate time point $t = 1$ (which for now is assumed to be known, but may be generalized to being generated through a random process, such as the exponential distribution). As a result of withdrawal demand of a fraction of the deposits, the bank is likely to face the necessity of liquidating a commensurate fraction of the loan assets (or the loan portfolio). Accordingly, in this chapter, the goal is to develop (and of course, analyze) a two-step approach to optimize the expected return on the loan portfolio, followed by an optimal liquidation strategy. The authors in [40] have done a detailed review of safe assets, wherein they have classified the low-risk holdings in terms of their safety and liquidity. They have shown a positive correlation between safety and liquidity, though the two concepts remain distinct. In [13], the Basel Committee also agreed that less risky assets tend to have higher liquidity. On the other hand, safe assets are less risky and,

hence, produce less return. Therefore, maintaining liquidity in the portfolio affects its profitability, because of which banks tends to hold on to less liquid assets. Therefore, banks need to keep the required amount of liquid assets, as prescribed by the regulators, while the remaining funds can be invested as per the bank's requirements. At the intermediate or final time step, if the bank faces default, then it sells its asset to the secondary market. Therefore, liquidation is a post-default process. The timeline of the events on a time scale of $t = 0, 1, 2$ is illustrated below in Table 3.1.

$t = 0$	$t = 1$	$t = 2$
Bank collects money through debt (d) and equity (e) and then invests in safe assets and risky assets.	A fraction of depositors withdraw their deposits and consequently the bank has to liquidate some assets.	All the risky investments have matured. Hence the bank either pays its liabilities or faces insolvency/bankruptcy.

Table 3.1: Timeline of events for the bank's portfolio

3.2.1 Model Description

The deposit structure of the bank is assumed to follow the classical firm value assumption (due to Merton [58]), where the banks' assets are considered equal to the sum of the money raised through equity, e (from shareholders) and the money raised through debt, d (from depositors). Now, this amount is assumed to be invested in three types of loans, namely, a safe loan L_0 and two risky loans L_1 and L_2 (where L_1 is less risky than L_2).

The impact of different strategies on the occurrence of bank-runs have been discussed in [34], where it is shown that deposit insurance and suspension of deposit withdrawal, reduces the likelihood of bank-runs. In our model, we assume that in absence of deposit insurance, the banks offer a higher interest rate r_d with $(1 + r_d > 1)$ to the long-term depositors, who hold on to their deposits with the bank until the final time step ($t = 2$), in order to lower the chance of early deposit withdrawal. Wagner notes in [79] that bank loans have always been identified as being illiquid. Having said so, the introduction of credit derivatives have resulted in increased liquidity of a loan portfolio, particularly since these derivative instruments help the banks to (both) transfer as well as sell off the loan portfolios, to investors. In this context, one may distinguish between the terms "high liquidity" and "low liquidity", which correspond to "easy-to-sell" and "hard-to-sell" assets, respectively. It is evident that more (less) liquid assets have to take less (more) haircut (a term which captures the impact on the price, as a direct result of unwinding the asset).

Accordingly, we let the constant γ_i denote the haircut for loan L_i ($i = 0, 1, 2$). It is obvious that the more risky the asset is, the more haircut it will experience during the liquidation. Therefore,

$$\gamma_0 < \gamma_1 < \gamma_2. \quad (3.2.1)$$

Since the haircut represents the expected amount of reduction in the valuation of the asset, therefore, we have considered the liquidity cost of the loans to be the same as the haircut. Further, at the time $t = 1$, when the liquidity requirement emerges, we assume that a certain fraction α_w (of d) is withdrawn and another fraction α_d (of d) is deposited. This leads to the following observations:

- (1) If $\alpha_d \geq \alpha_w$, then the new deposits are enough to meet the withdrawal demand of the depositors and therefore there is no need to liquidate any part of the loan portfolio.
- (2) If $\alpha_d < \alpha_w$ then the new deposits prove to be insufficient to meet the obligations to the depositors demanding withdrawal, which will trigger the process of liquidation from the loan portfolio. Now, the dilemma faced by the bank is as follows (summarized in Table 3.2):
 - (A) If the safe asset (L_0) is liquidated, then the likelihood of default risk at $t = 2$ increases, even though this exercise means less haircut.
 - (B) In contrast, if the riskier loans (L_1 and L_2) are liquidated then the likelihood of default risk at $t = 2$ decreases, but at the cost of greater haircut.

Loans	Pros	Cons
L_0	Lesser haircut	More default risk
L_1 and/or L_2	Less default risk	Greater haircut

Table 3.2: Pros and cons for the safe asset and the risky assets

- (3) The model setup assumes that the equity values remain unchanged at time $t = 1$.

In summary, if $\alpha_d \geq \alpha_w$, the amount of new deposits is more than that of the withdrawal demand. Then, the bank does not need to liquidate its assets. On the other hand, the condition $\alpha_d < \alpha_w$, means that the withdrawal demand exceeds the new deposits and then the bank has to liquidate its assets. Therefore, we introduce an equation to present the fact that the liquidated value of the assets is equal to the additional amount of money required over and above the amount raised through the new deposits.

Suppose that we liquidate a fraction β_i for loan L_i for $0 \leq \beta_i \leq 1$ ($i = 0, 1, 2$). Then we get,

$$\sum_{i=0}^2 \beta_i (1 - \gamma_i) L_i = (\alpha_w - \alpha_d) d. \quad (3.2.2)$$

Before we present the description of the models which are being proposed in this work, we enumerate the various variables for the same in Table 3.3. Note that Expected Loss = Probability of Default \times Loss Given Default.

Variable	Description
L	The total amount of loss from liquidating the whole position
θ_1	Upper bound on risk (> 0)
θ_2	Lower bound on risk (> 0)
β_i	Liquidation strategy for the i -th asset at time $t = 1$
γ	Amount of loss due to liquidation
e	Equity component of the bank's portfolio.
η_i	Loss Given Default (LGD) for the i -th loan
EL_i	Expected Loss (EL) for the i -th loan
X	Realization of the portfolio

Table 3.3: Description of the model variables

Model 1. The goal is to solve the maximization problem:

$$\max_{x,e} [\mathbb{E} [\max (X - (1 + r_d)(1 - e), 0)] - \delta e],$$

subject to the constraints of:

(A) $0 \leq x_i \leq 1 \forall i = 0, 1, 2$ (short selling is not permissible).

(B) $\sum_{i=0}^2 x_i = 1.$

(C) $e \geq \max(k_{lev}, K(x)).$

(D) $\rho(x) \leq \theta_1$ (upper bound on risk).

(E) $\sum_{i=0}^2 x_i \gamma_i \leq L$ (upper bound on the total amount of haircut).

Model 2. Now we construct another model like the preceding one by removing the last constraint, namely, the upper bound on to the total haircut amount, in the event of liquidation of the entire portfolio. Accordingly, the goal is to solve the maximization problem:

$$\max_{x,e} [\mathbb{E} [\max (X - (1 + r_d)(1 - e), 0)] - \delta e],$$

subject to the constraints of:

(A) $0 \leq x_i \leq 1 \forall i = 0, 1, 2$ (short selling is not permissible).

(B) $\sum_{i=0}^2 x_i = 1.$

(C) $e \geq \max(k_{lev}, K(x)).$

(D) $\rho(x) \leq \theta_1$ (upper bound on risk).

The objective of the bank is to maximize the expected profit, after the pay-out to the debt holders. Considering this, we additionally incorporate Limited Liability and cost of equity, in the objective function of our model structure. In order to accomplish this, the bank has to create the portfolio (a three asset portfolio in this case) in an optimal manner. Here, x represents the portfolio of the given (three) assets. Besides this, choosing an equity and debt structure is also necessary so as to optimize the profit. Therefore, we have to maximize the objective function with respect to x (the loan portfolio) and e (equity structure). Accordingly, we represent it in the constraint (C). Since banks are subjected to capital requirements, that is, the amount of equity is subjected to a risk-adjusted capital and LR requirement. Further, constraint (D) imposes an upper bound on risk, for the initial investment decision. The constraints (A) and (B) are standard portfolio constraints, namely, restriction on short-selling and the sum of weights being equal to 1. In our modelling set-up, we have incorporated a cap on the haircut for the entire portfolio. Constraint (E) represents this condition. In order to study the effect of this constraint, we have constructed Model 2, without this constraint. We have considered that the bank raises capital through debt and equity. Therefore, we have the condition:

$$e + d = 1, \tag{3.2.3}$$

which implies that the debt is $d = 1 - e$.

Theorem 3.2.1. *Using Limited Liability in the model reduces the incorporation of risky loans in the portfolio.*

Description and Outline of Proof. Limited Liability allows investors to take more risk as it provides safety net. In the banking paradigm the authors in [19, 63] have constructed the model in the risk-return framework. Here the return is taken as the expected return from the loans minus the debt value and the cost of equity, that is,

$$[\mathbb{E}[X] - r_d(1 - e) - \delta e]. \tag{3.2.4}$$

On the other hand several authors [2, 14, 31] have suggested the bank decision problem, by including the Limited Liability aspect in the risk-return framework. This gives the expected value of realization after paying off the debt, if the net is positive and zero otherwise, minus the cost of equity. Mathematically, this translates to,

$$\mathbb{E}[\max(X - r_d(1 - e), 0)] - \delta e. \tag{3.2.5}$$

In the previous chapter, we have shown that the inclusion of Limited Liability reduces the inclusion of risky assets in the portfolio. Proof of this argument is a lengthy one, and accordingly we discuss only the outline of the proof, here. To show that the inclusion of Limited Liability reduces

incorporation of risky assets, we have to construct a model with same constraints and the objective function, but without Limited Liability. Before going to the proof we need the following lemmas from chapter 2:

- (A) Objective function without Limited Liability is decreasing function of e .
- (B) In the model without Limited Liability the optimal leverage ratio is given by

$$e = \max\{k_{lev}, K(x)\}.$$

The main proof can be done by considering four cases depending on the optimal leverage ratio of the models, with and without Limited Liability, on the lines of the similar proof in chapter 2. \square

Model 3. *The formulation of this model deals with designing the strategy of liquidating the problem of the loans portfolio. Let us denote the present value of the i -th loan at time 1 by $X_i^{(1)}$, $i = 0, 1, 2$. Then the goal is to solve the minimization problem:*

$$\min_{\beta} \left[\sum_{i=0}^2 \beta_i \gamma_i \right],$$

subject to the constraints of:

- (A) $0 \leq \beta_i \leq x_i$, $i = 0, 1, 2$.
- (B) $\sum_{i=0}^2 \beta_i (1 - \gamma_i) X_i^{(1)} = (\alpha_w - \alpha_d) d$ (payment for the withdrawals by depositors).
- (C) $\sum_{i=0}^2 \beta_i EL_i \geq \theta_2$ (lower bound on risk of liquidated portfolio).

Model 4. *Next we construct the model without the risk-lower bound. Accordingly, we get the following the minimization problem:*

$$\min_{\beta} \left[\sum_{i=0}^2 \beta_i \gamma_i \right],$$

subject to the constraints of:

- (A) $0 \leq \beta_i \leq x_i$, $i = 0, 1, 2$.
- (B) $\sum_{i=0}^2 \beta_i (1 - \gamma_i) X_i^{(1)} = (\alpha_w - \alpha_d) d$ (payment for the withdrawals by depositors).

The motivation for adding the last constraint in Model 3 is given as Remark 3.2.2 as follows.

Remark 3.2.2. Banks will (naturally) try to minimize the loss from liquidating the loan portfolio. Since the safe asset is the most liquid in our loan portfolio set-up, hence liquidating this asset leaves the riskier assets in the portfolio for the next step ($t = 2$). Consequently, in case of any unfavorable conditions, the bank has to face a significant loss, if the riskier loan is not repaid in a timely manner. Keeping this in mind, we include the constraint (C) in Model 3 to have a cap on liquidating the safe asset.

Remark 3.2.3. The present value of the i -th loan is given by:

$$X_i^{(1)} = \left(1 + \frac{r_i}{2}\right) X_i^{(0)}, \quad (3.2.6)$$

and the final value of the loan is given by:

$$X_i^{(2)} = (1 + r_i) X_i^{(0)}, \quad (3.2.7)$$

where r_i is the interest rate in the i -th loan, for $i = 0, 1, 2$. Obviously, $r_1 < r_2 < r_3$, for $i = 0, 1, 2$. The present value is useful for trading at the intermediate time step. If the bank needs money to meet depositors' claims, they will have to sell their assets in the secondary market, at the intermediate time step ($t = 1$).

While the assets or the loans all mature at time $t = 2$, in contrast, the maturity of the debts is not fixed initially. This is because, the depositors can withdraw their money at time $t = 1$. So, the maturity of the debts is not fixed initially.

3.3 Model Analysis

Theorem 3.3.1. The solution of Model 1 exists.

Proof. Let S be the region defined by its constraints in Model 1. Since the portfolio $x = (1, 0, 0)$, with equity being 100% is feasible, therefore $S \neq \phi$ (because the haircut for the safe loan is $\gamma_0 = 0$, the expected loss is zero, and 100% equity satisfies all the capital requirement conditions). Further, since S is a closed and bounded set in \mathbb{R}^3 , therefore it is a compact set. The objective function lies in $C(\mathbb{R}^3)$. So Weierstrass Theorem¹ assures that the solution to the problem exists, which we denoted by (x^*, e_0^*) , where $x^* = (x_0^*, x_1^*, x_2^*)$. \square

Theorem 3.3.2. The solution of Model 2 exists.

Proof. The proof for the existence for the solution of Model 2 follows on the lines of Theorem 3.3.1. \square

¹If $f(x)$ is continuous on a nonempty feasible set S , which is closed and bounded, then $f(x)$ has a global minimum in S .

Theorem 3.3.3. *The total amount of loss is more sensitive in case of illiquid asset.*

Proof. Total amount of loss due to liquidation is given by,

$$L = \beta\gamma^\top = \sum_{i=0}^2 \beta_i \gamma_i, \quad (3.3.1)$$

where $\beta = (\beta_0, \beta_1, \beta_2)$ and $\gamma = (\gamma_0, \gamma_1, \gamma_2)$. Now we see that $\frac{\partial L}{\partial \beta_2} \geq \frac{\partial L}{\partial \beta_1} \geq \frac{\partial L}{\partial \beta_0}$. Therefore increasing the weights of the risky loans will increase the losses at higher rate, as compared to the increase of losses resulting from increasing the weight of the safe asset. \square

Theorem 3.3.4. *Sensitivity of the solution of Model 1 with L (upper bound on the haircut).*

Proof. Recall that, the constraint (E) of Model 1 is given by,

$$\sum_{i=0}^2 x_i \gamma_i \leq L, \quad (3.3.2)$$

where L is the maximum limit on haircut. Accordingly, we consider the equation:

$$\sum_{i=0}^2 \frac{x_i \gamma_i}{L} = 1 \Rightarrow \sum_{i=0}^2 \frac{x_i}{L/\gamma_i} = 1. \quad (3.3.3)$$

It can be observed that, as L decreases, $\frac{L}{\gamma_i}$ ($\gamma_i > 0$) decreases. So the upper-bound of incorporating liquid investments reduces with the decrement of L . If the optimal portfolio for the investment is $\mathbf{x}^* = (x_0^*, x_1^*, x_2^*)$, then its corresponding haircut is L^* . Consequently, a decrease in the upper bound of the constraint by an amount greater than or equal to $(L - L^*)$ affects the solution, and the new solution of Model 1 excludes illiquid assets. \square

In our study, we have considered a case where the bank needs to liquidate its assets in order to meet the depositor's claim. The risk lower bound (θ_2) is proposed for the liquidated portfolio in the intermediate time step. As θ_2 increases, the liquidated portfolio contains more risky assets. Therefore, to raise the same amount of money, the bank has to sell more risky assets if θ_2 is increased. Therefore, this procedure reduces the liquidation of safe assets, which in turn, increases the bank's resilience at the final time step. Based on these observations we have the following Theorem.

Theorem 3.3.5. *Higher lower bound on risk in constraint (C) of Model 3 reduces the liquidation of safe assets at $t = 1$.*

Proof. The constraint is given by,

$$\sum_{i=0}^2 \beta_i \text{EL}_i \geq \theta_2. \quad (3.3.4)$$

As already observed (in the proof of Theorem 3.3.3), we have $\frac{\partial L}{\partial \beta_2} \geq \frac{\partial L}{\partial \beta_1} \geq \frac{\partial L}{\partial \beta_0}$. Therefore liquidating the most liquid asset causes the safe (risk-free) asset less haircut. Nevertheless, in the event of bankruptcy, the bank has to face an adverse scenario due to liquidation of safe assets. Therefore, if we add this constraint in the model, with the risk bound θ_2 (which is less than θ_1 , because the portfolio has risk measure less than θ_1 by Model 1), then a suitable threshold of this θ_2 will restrict the liquidation of safe assets and make it incumbent on the bank to liquidate the illiquid assets (risky loans). The mathematical justification for the preceding result is as follows. We first consider the plane given by the following equation:

$$\sum_{i=0}^2 \beta_i EL_i = \theta_2 \Rightarrow \sum_{i=0}^2 \frac{\beta_i EL_i}{\theta_2} = 1 \Rightarrow \sum_{i=0}^2 \frac{\beta_i}{\left(\frac{\theta_2}{EL_i}\right)} = 1. \quad (3.3.5)$$

We observe that the upper half of this plane contains the feasible region for Model 3. As the θ_2 increases, $\frac{\theta_2}{EL_i}$ also increases, provided $EL_i > 0$. Therefore the lower bound for liquidating the risky assets increases, and hence the bank has to liquidate the more risky asset for higher value of θ_2 .

From the constraint (B) in Model 3, it is obvious that decreasing β_0 results in increase of β_1 and β_2 . Differentiating the constraint (B) with respect to β_0 we get,

$$(1 - \gamma_0) X_0^{(1)} = -\frac{\partial}{\partial \beta_0} \left(\beta_1(1 - \gamma_1) X_1^{(1)} + \beta_2(1 - \gamma_2) X_2^{(1)} \right). \quad (3.3.6)$$

Let $b = \min \left((1 - \gamma_1) X_1^{(1)}, (1 - \gamma_2) X_2^{(1)} \right)$. Since the realizations are all positive, hence $b > 0$. Therefore, we have,

$$\frac{\partial}{\partial \beta_0} (\beta_1 b + \beta_2 b) < 0 \Rightarrow \frac{\partial}{\partial \beta_0} (\beta_1 + \beta_2) < 0. \quad (3.3.7)$$

Therefore, higher risk bound reduces the liquidation of safe assets, and consequently, this eventually reduces the risk of the remaining portfolio at the final time step. \square

3.4 An Example

In this section, we construct an example in order to illustrate the theoretical results presented in the preceding section. For this purpose, we consider three loans, comprising of one safe investment and two risky investments (one of which is riskier than the other). In this context, the term riskiness refers to the credit worthiness of the debtor, which is contingent on parametric values such as probability of default (PD) and loss given default (LGD), some of which (the parameter values) are available in [48, 71]. For the purpose of our illustrative example, we have taken the values for return, PD and LGD to be the same as in chapter 2. Further, the values of the haircuts of different types of loans are listed in the last column of the Table 3.4. It may be noted that the authors in [77] have discussed in detail about the haircuts applicable for various class of assets. Finally we

take $\delta = 1.04$, which was motivated from [48]. In summary, Table 3.4 enumerates all the details of the parameter values pertaining to the loans.

We have taken a three-loan portfolio to illustrate the theoretical results obtained. One of these three loans is safe, and the other two are risky. All the necessary parameter values are declared in Table 3.4, along with Leverage Ratio restriction of $k_{lev} = 4\%$. Furthermore, the assumption is that the bank offers 1% interest rates to depositors at the final time step ($t = 2$).

Loan Type	Return	PD	LGD	Haircut
Safe Loan	$r_{rf} = 3\%$	0	0	0%
Less Risky Loan	$r_s = 9\%$	$p_s = 6.1\%$	$lgd_s = 10\%$	10%
More Risky Loan	$r_r = 13.2\%$	$p_r = 12.2\%$	$lgd_r = 9\%$	20%

Table 3.4: Risk parameters for the three loans

We begin with the solution for Model 1, which with the inclusion of the parameter values in Table 3.4, reduces to the model:

$$\max_{x,e} [\mathbb{E} [\max (X - 1.01 \times (1 - e), 0)] - 1.04e],$$

subject to the constraints of:

(A) $0 \leq x_i \leq 1 \forall i = 0, 1, 2.$

(B) $\sum_{i=0}^2 x_i = 1.$

(C) $e \geq \max (0.04, K(x)).$

(D) $x_0 \times 0 + x_1 \times 0.0061 + x_2 \times 0.01098 (= \text{Expected Loss}) \leq 0.012 (= \theta_1)$

(E) $x_0 \times 0 + x_1 \times 0.1 + x_2 \times 0.2 \leq 0.15 (= L)$

Further, in an analogous manner, Model 2, which does not include a cap on the haircut becomes:

$$\max_{x,e} [\mathbb{E} [\max (X - 1.01 \times (1 - e), 0)] - 1.04e],$$

subject to the constraints of:

(A) $0 \leq x_i \leq 1 \forall i = 0, 1, 2.$

(B) $\sum_{i=0}^2 x_i = 1.$

(C) $e \geq \max (0.04, K(x)).$

(D) $x_0 \times 0 + x_1 \times 0.0061 + x_2 \times 0.01098 (= \text{Expected Loss}) \leq 0.012 (= \theta_1).$

Model 1 and Model 2 are continuous, non-differentiable problems (as the objective function is not differentiable). However, it can be transformed into a differentiable optimization problem, which has been elaborately discussed in chapter 2. After solving the problems, we get the portfolio (7.95%, 34.10%, 57.95%) (that is, investments of 7.95% in L_0 , 34.10% in L_1 and 57.95% in L_2) from Model 1 (with LR of 4%) and for Model 2 we get (4.36%, 25.49%, 70.15%) (LR of 4.81%). The results show that the inclusion of the cap on the haircut reduces the incorporation of illiquid assets in its portfolio. The inclusion of cap on loss of liquidation in the model reduces the amount of loss due to liquidation of the entire portfolio by 9.52%. Hence Model 1 outperforms Model 2 from the perspective of liquidity risk management ².

Next, we obtain the solution to the problem of liquidating assets, at time $t = 1$, to meet the claims of the depositors, both with and without a lower bound on the risk. For this purpose, we need the solution of investment decision at $t = 0$, which in turn plays the role of the first constraint. We first proceed with the solution of Model 1, that is, (7.95%, 34.10%, 57.95%). Here, in order to solve for the model, we have taken $\alpha_w = 10\%$ and $\alpha_d = 0$. In other words, 10% of the deposits are withdrawn and there is no further investment. Accordingly, Model 3 becomes:

$$\min_{\beta} [\beta_0 \times 0 + \beta_1 \times 0.1 + \beta_2 \times 0.2],$$

subject to the constraints:

$$(A) \quad 0 \leq \beta_i \leq x_i, \quad i = 0, 1, 2, \quad (x_0 = 0.0795, x_1 = 0.3410, x_2 = 0.5795).$$

$$(B) \quad \beta_0(1 - 0)X_0^{(1)} + \beta_1(1 - 0.1)X_1^{(1)} + \beta_2(1 - 0.2)X_2^{(1)} = 0.1 \times 0.96.$$

$$(C) \quad \beta_0 \times 0 + \beta_1 \times 0.0061 + \beta_2 \times 0.01098 \geq \theta_2$$

We solve this problem with two different values of θ_2 , first for $\theta_2 = 0.05\%$ and then for $\theta_2 = 0.1\%$. Before going to the solution of Model 3, we solve Model 4, which gives the liquidation strategy without lower bound on risk. With all these parameter values, the problem becomes:

$$\min_{\beta} [\beta_0 \times 0 + \beta_1 \times 0.1 + \beta_2 \times 0.2],$$

subject to the constraints:

$$(A) \quad 0 \leq \beta_i \leq x_i, \quad i = 0, 1, 2, \quad (x_0 = 0.0795, x_1 = 0.3410, x_2 = 0.5795).$$

$$(B) \quad \beta_0(1 - 0)X_0^{(1)} + \beta_1(1 - 0.1)X_1^{(1)} + \beta_2(1 - 0.2)X_2^{(1)} = 0.1 \times 0.96.$$

Solving Model 4 we get (7.95%, 1.63%, 0%), that is, liquidation happens for L_0 and L_1 . Model 3, with $\theta_2 = 0.05\%$ gives the liquidation portfolio of (1.86%, 8.20%, 0%) and for $\theta_2 = 0.1\%$ we get

²We have used the “scipy” package of Python to solve these models

(0%, 3.93%, 6.93%). As the value of θ_2 increases, it increases the liquidation of risky loans. Hence the remaining portfolio contains less risky loans, that is, the risk also reduces. The results show that $\theta_2 = 0.05\%$ reduces risk of the remaining portfolio by 4.80%. This supports the result established in Theorem 3.3.5, *i.e.*, a lower bound on risk for the liquidating portfolio reduces the risk for the remaining assets, and as a result of which the chance of default is also reduced. Therefore the solution from Model 3 performs better than Model 4.

Next, we take the solution of Model 2 for solving the liquidation portfolio. The formulation of the liquidation models is the same as discussed above, except for the values of $(x_i : i = 0, 1, 2)$. Solving Model 4 (without lower bound on the risk) gives the portfolio of (4.36%, 5.5%, 0%), that is, the liquidation of 4.36% and 5.5% of safe assets and less risky assets respectively. We solve Model 3, first with a risk-lower bound of $r = 0.05\%$ and then for $r = 0.1\%$, resulting in the same liquidation strategies observed from the previous Model 3, with the input being the solution of Model 1. All the results are summarized in Tables 3.5 and 3.6

Solution of	Model used as input	Portfolio with $\theta_2 = 0.05\%$	Portfolio with $\theta_2 = 0.1\%$
Model 3	Model 1	(1.86%, 8.20%, 0%)	(0%, 3.93%, 6.93%)
Model 3	Model 2	(1.86%, 8.20%, 0%)	(0%, 3.93%, 6.93%)

Table 3.5: Results for Model 3

Solution of	Model used as input	Portfolio
Model 4	Model 1	(7.95%, 1.63%, 0%)
Model 4	Model 2	(4.36%, 5.5%, 0%)

Table 3.6: Results for Model 4

Next, taking the solution of Model 1 as the first constraint of Models 3 and 4, we proceed to show the feasible region for the Models 3 and 4, respectively.

Figures 3.1 and 3.2 show the feasible regions for the Models 3 and 4, respectively. The two constraints in Model 4 are presented in Figure 3.2. The first constraint represents the available funds for liquidation. The second constraint represents the equality of the liquidated value and the depositor's claim. These are presented in Figure 3.2 in two sub-figures. The first sub-figure illustrates the feasible region inside the cube, and the second sub-figure allows the points on the surface, as a candidate for the feasible region. The intersection of these two regions is the feasible region for the problem. Similarly, the constraints of the Model 3 are presented in the Figure 3.1. The first two constraints for this Model are the same as those in Model 4. The last sub-figure presents the risk lower bound for the liquidated portfolio. The upper half of this plane is the feasible region for the problem. Therefore, the intersection of these three regions is the feasible region for the problem. Figure 3.1 visualizes the feasible region for Model 3 with lower risk. The

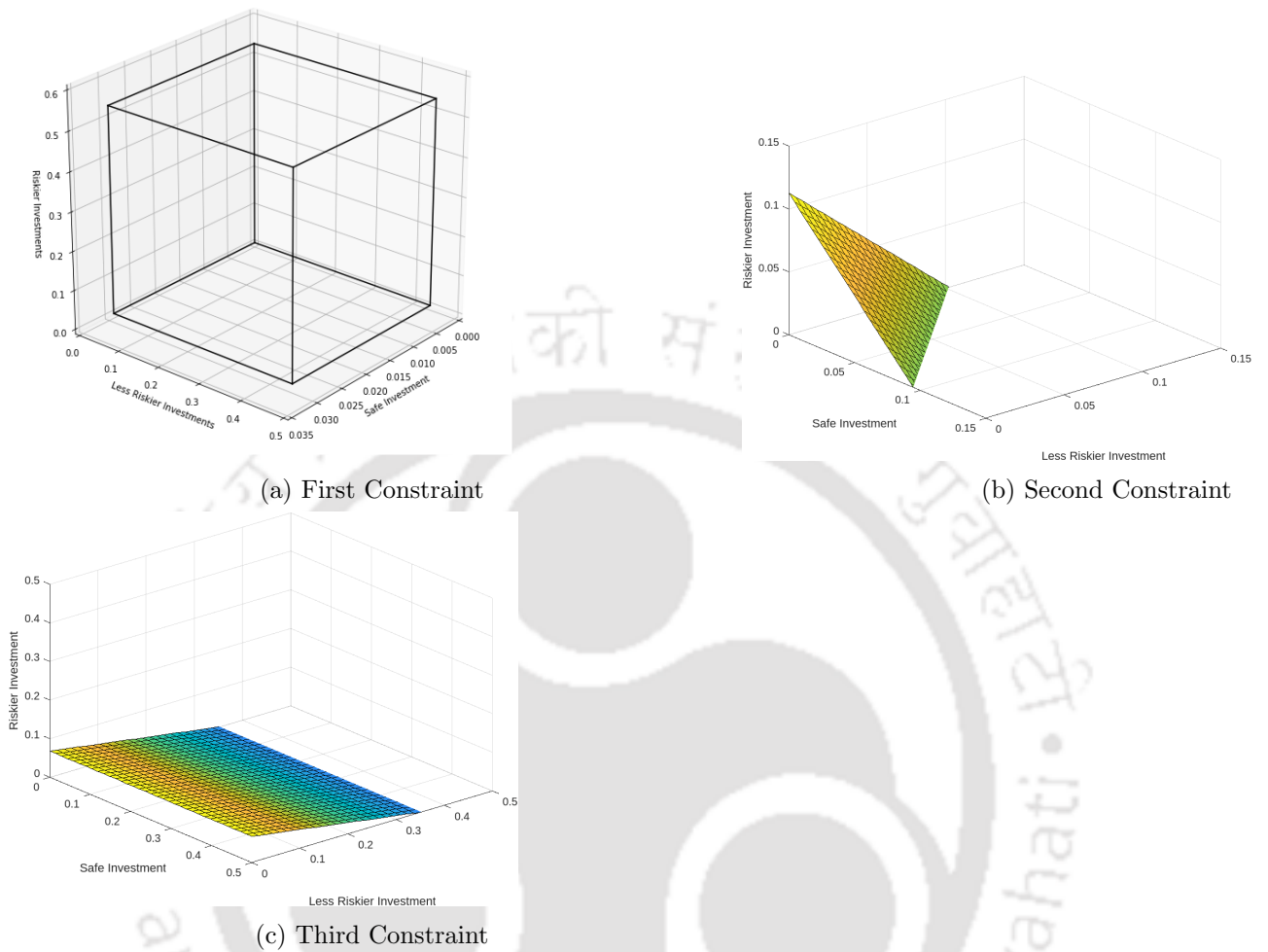


Figure 3.1: The feasible region for Model 3

purpose of these figures is to show the feasible region for the liquidation problems. In the third sub-figure of Figure 3.1, the points under the plotted surface are not admissible. From the second sub-figure of Figure 3.1, we can see that the liquidated portfolio, which lies in this area, contains more safe assets. Therefore, by eliminating this area with a lower bound on risk, the liquidation of safe assets can be reduced. The constraint (E) in Model 1 is presented in Figure 3.3. Here, A_1 represents an upper bound of 10% in haircut, that is, the lower half of this surface is feasible whereas A_2 represents the 15% upper bound of the haircut. It may be noted that in the figures, the x , y and z -axes represent L_0 , L_1 and L_2 , respectively.

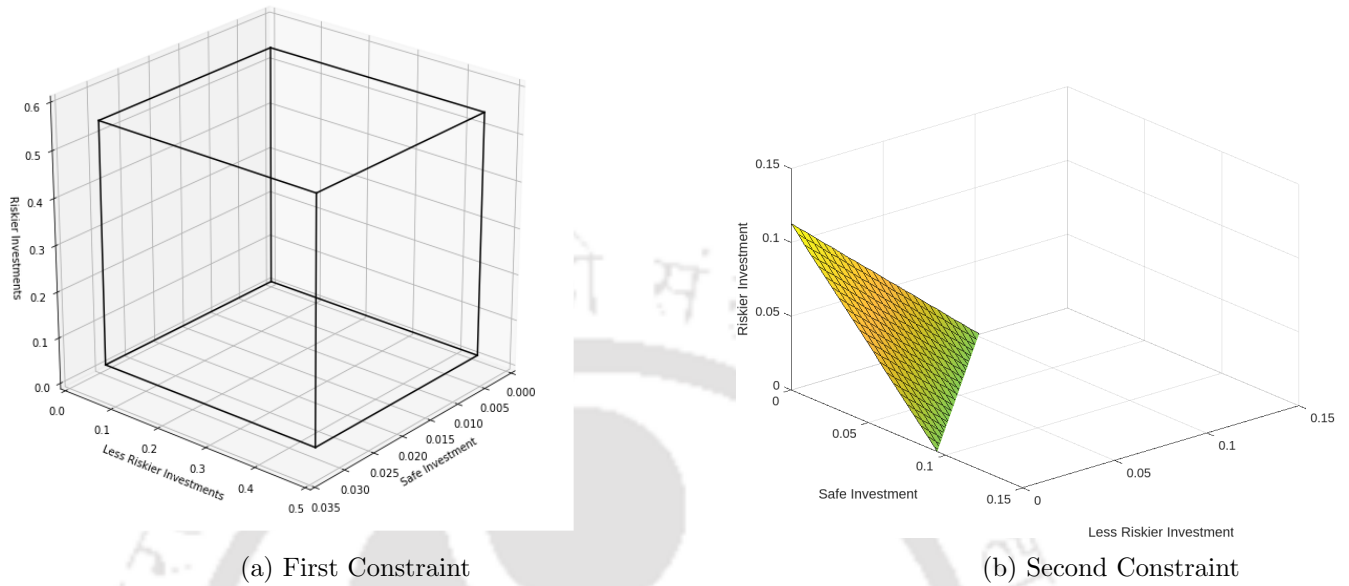


Figure 3.2: The feasible region for Model 4

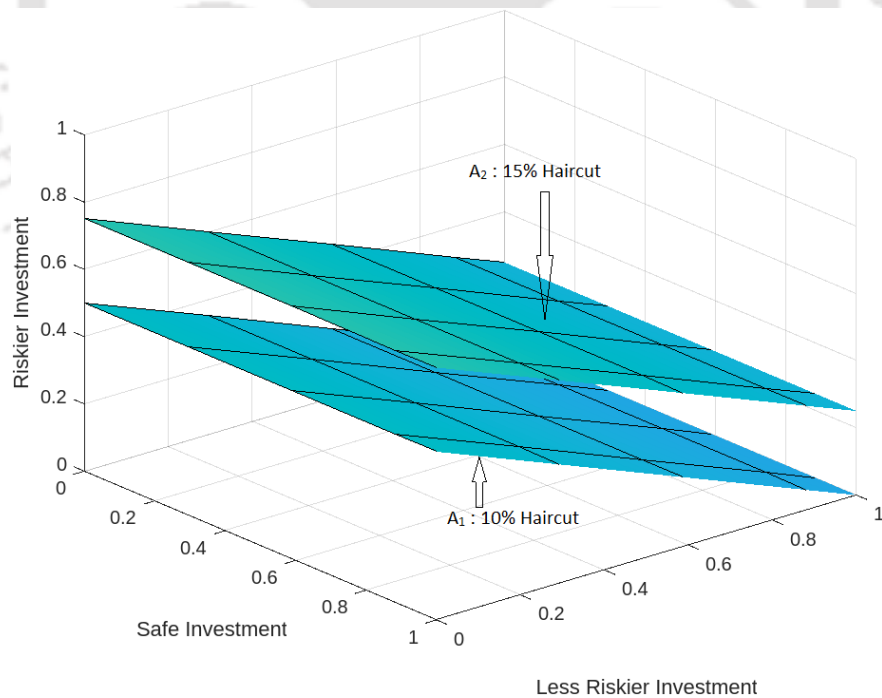


Figure 3.3: Upper Bound on Haircut

3.5 Summary

Liquidity is one of the major concerns in today's bank's portfolio decision, especially in the context of maintaining solvency and avoiding a bank-run. The inclusion of illiquid assets in the portfolio of the bank, increases the likelihood of it being unable to meet its obligations to the depositors. On the other hand, by virtue of their very structure and design, banks enjoy protections of Limited Liability. Accordingly, this chapter presents a novel approach of incorporating the liquidity constraint, while retaining Limited Liability protection, in case of the loan portfolio held by banks.

In this work, we have presented a three-time step model that optimizes profit with Limited Liability at the initial time point of $t = 0$. Here, we have established the comparison between the two models, namely, Model 1) (with a cap on the haircut) and Model 2) (without a cap on the haircut), with the other constraints remaining identical. At $t = 1$, a fraction of depositors claim their money back. Therefore the model solves this problem of liquidating a fraction of assets. Further, we compare Model 3) (which includes a risk-lower bound) and Model 4) (which excludes a risk-lower bound), with all other constraints being identical. Finally, at $t = 2$, the bank either faces bankruptcy or makes a profit after paying all the liabilities. Our analysis shows that in the first comparison, the model with a cap on the total haircut includes less illiquid assets in the portfolio, as compared to the model without a cap on the total haircut. On the other hand, Limited Liability also protects the bank in the worst condition. Again the inclusion of the Limited Liability in the model reduces the incorporation of the risky asset in the portfolio. In the case of the second comparison, a lower bound on the risk resists the liquidation of safe loans, as compared to the model without a lower bound on the risk.

Therefore, from a practitioner's point of view, the model with the cap on the haircut and Limited Liability can help to make the right investment decision because it reflects a more accurate scenario and produces better results. Incorporation of Limited Liability reduces risk and upper bound on haircut helps to survive in the stress scenario. In the intermediate time step model, the liquidation strategy is obtained by solving Model 3) via minimization of liquidation costs (with a lower bound on the risk). It resists the liquidation of safe assets, which increases the chance of default. Since it produces better results by satisfying all the investment criteria, therefore it is undoubtedly a more pertinent model from the perspective of decision-making.

In summary, the main contribution of the work reported in this chapter is the construction of models incorporating Limited Liability and liquidity constraints, whose solution demonstrates the reduction of risky and illiquid assets in the portfolio. Further, at the intermediate time step, we have suggested a liquidation model that resists the liquidation of safe assets. Finally, in the context of relevance for banks and policy makers, the suggested model is more comprehensive via the inclusion of maximization of excess expected return (incorporating Limited Liability), while

accommodating the regulatory requirements. This approach potentially enables the banks to reduce their risk exposure at the final time point, thereby leading to their greater resilience.



Chapter 4

Dynamic loan portfolio management in a three time step model

4.1 Introduction

Banks' problems in the intermediate time step are crucial for business cycles. It involves raising money, liquidating assets, meeting liabilities, etc. Besides all these cash flows, the bank has to meet the regulatory requirements. Therefore, modelling this phenomenon is an important and interesting research topic. The authors in [8] and [24] have discussed dynamic bank behaviour to comply with the regulatory requirements. In the chapter 3, we have studied the case when the bank cannot raise money and liquidates its assets to meet depositors' claims. This chapter focuses on the bank's dynamic decision problem in the intermediate time step for a discrete-time setup, involving three time steps. Initially, the banks raise money through debt and equity and invest in different types of loans. It liquidates its assets and raises new funds at the intermediate-time step to meet the short-term debt holder's claim. Further, it has to meet capital requirements given by the regulators. Here, we theoretically study the effect of raising new equity and debt. We show that in some cases, raising equity and debt may increase the return on equity, and in some cases, it may decrease the return on equity. We have discussed several cases and given a bound on the capital that can be raised. We have added an equity holder's constraint, which ensures the return on equity and desists the bank from defaulting at the final time point. Behn et al. [8] have suggested four methods to meet the capital ratio requirements. In this study, we have analyzed these four in detail. We have constructed a model which includes these four strategies, so that the bank can choose among these four methods to optimize their goals. The outline of the chapter is as follows. We begin with a detailed presentation of the models in section 4.2. Then, in section 4.3 we present all the theoretical results, which are then illustrated by an example in section 4.4. Finally, section 4.5 summarizes the main takeaways and concluding remarks of the work carried out.

4.2 Problem and Model Formulation

For the problem setup, the banks are assumed to follow the firm based model, wherein they raise capital through debt and equity. The capital raised is assumed to be invested by the bank in three type of (loan) assets, namely, one safe asset and two risky assets. For our study, we have considered the contagion risk, and have adopted the payoff structure to be the same as in [2]. The authors in [2] have discussed about two type of assets, with the condition that when the safer asset defaults, then the risky assets also default. A dynamic model for bank behaviour is discussed in [8], with the incorporation of various regulatory provisions. In particular, the model in [8] illustrates four different strategies, in order to adjust the capital ratio, so as to meet the regulatory requirements.

We begin by presenting a schematic diagram of the proposed model, in Figure 4.1. We denote the safe asset as L_0 and the risky assets by L_1 and L_2 (with L_1 being less risky than L_2). Further the suffix D indicates a default event, that is, $\{L_i D\}_{i=1,2}$ denotes the default of the asset L_i .

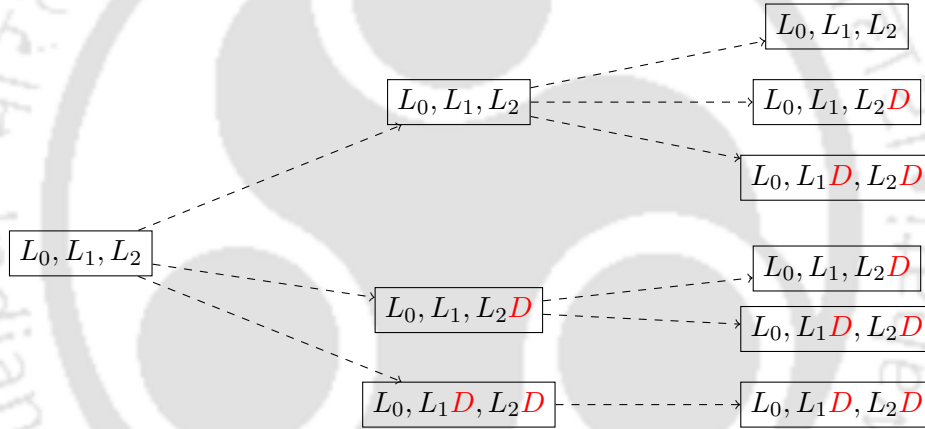


Figure 4.1: Schematic presentation of the possible scenarios

We have constructed Model 5 to determine the initial investment strategy, by considering the fact that the bank is subjected to Limited Liability [2], while at the same time meeting the regulatory capital requirements. The objective function maximizes the expected profit, after the payout to the debt-holders. It also incorporates the cost of equity. Here, the variable $X = \sum_{i=0}^2 x_i L_i^{(1)}$ is the random variable, where L_i is the value of the i -th loan asset at time $t = 1$. In case of default (the probability of which is p_i), L_i takes the value $1 - \lambda_i$. In the event of no default (the probability of which is $(1 - p_i)$), L_i takes the value $(1 + r_i)$. It may be noted that we will denote $r_0 = r$. Choosing a capital structure is also important, in the context of maximizing the payoff. Therefore, we have to maximize the objective function, with respect to x and e .

Model 5 (At $t = 0$).

$$\max_{x,e} [\mathbb{E} [\max (X - \beta_{ST}(1 - e) - (1 + r_d)\beta_{LT}(1 - e), 0)] - \delta e],$$

subject to the constraints,

1. $0 \leq x_i \leq 1 \forall i = 0, 1, 2$ (short selling is not permissible),
2. $\sum_{i=0}^2 x_i = 1$,
3. $e \geq \max(k_{lev}, K(x))$,
4. $\rho(x) \leq \theta_1$ (upper bound on risk).

Here, r_d is the interest rate being paid to the long term debt holders. This interest rate is lesser than the interest rate applicable for L_0 , because banks make profit only when the interest rate that they charge on the loans is more than the interest rate that they pay the depositors. On the other hand, in addition to safety, banks also provide liquidity and transaction convenience to the depositors. Further, β_{ST} denotes the fraction of depositors who are short term debt holders, and $\beta_{LT}(= 1 - \beta_{ST})$ is the fraction of long term debts issued by the bank. The modeling for banks decision problem at $t = 1$ is a complex task, because it involves multiple possibilities. Accordingly, we enumerate the dynamics (at $t = 1$) which needs to be incorporated into the model:

- (A) Bank can either raise money through debt and equity, or it can rebalance its assets to repay the short term debt and to meet regulatory requirements [8].
- (B) Bank will continue its business as long as the business is profitable.

We have constructed a novel model to address this issue at the intermediate time step. This model has the advantage of liquidating and rebalancing assets, as well as raising money through debt and equity. In the model, if the equity holders expect to get more return for continuing their holdings into the next time point, then they will hold on to their position until the next time point. Otherwise, they will liquidate their assets and collect the proceeds. We call this as the “equity holder’s criteria”. Let us define τ_i as the default time for the i -th loan. Our model maximizes the expected return at time $t = 1$ by fulfilling the capital requirement as well as the equity holder’s criteria. The main contribution and novelty of our model is the inclusion of all the four strategies, suggested in [8] (to maintain the Leverage Ratio) and the construction of which are enumerated below:

- (A) By issuing new equity (raised in the market or from retained earnings) to buy back debt, while keeping the assets constant.

- (B) By using new equity to fund asset growth, while keeping the debt constant.
- (C) By selling assets and using the proceeds to buy back debt, while keeping the equity constant.
- (D) By rebalancing assets towards less risky positions (thus decreasing average risk weights in the portfolios), while keeping assets and equity constant.

The last two strategies are included in the model via a term given by,

$$\mathbb{E} \left[\sum_{i=0}^2 x_i^{(1)} L_i^{(2)} \mathbb{1}_{\tau_i > 1} \right],$$

in the objective function, because it allows for rebalancing of assets in order to pay the debt holders. The variable $x^{(1)} = (x_0^{(1)}, x_1^{(1)}, x_2^{(1)})$, is the portfolio that is acquired at time $t = 1$ and held till the final time point $t = 2$. Let v_e and v_d be the amount of new equity and debt (at $t = 1$), respectively. It includes selling the assets without raising new equity and debt ($v_e = 0 = v_d$), which is incorporated in the third strategy. Further, the bank can also rebalance its assets to reduce the risk (both with and without the condition $v_e = 0 = v_d$) after paying the short term debt also, thereby incorporating the fourth strategy. Our model allows to raise new debt and equity in the intermediate time step and also allow to restructure the loan portfolio. So the bank can issue new debt, keeping the asset constant and pay the short term debt, which is represented by the first strategy. Additionally, it can also simultaneously fund asset growth, which is captured in the second strategy.

Model 6 (At $t = 1$).

$$\max_{x^{(1)}, v_e, v_d} \left[\mathbb{E} \left[\sum_{i=0}^2 x_i^{(1)} L_i^{(2)} \mathbb{1}_{\tau_i > 1} \right] - \phi_d v_d - \phi_e v_e \right],$$

subject to the constraints of:

$$1. \sum_{i=0}^2 \max\{x_i^{(1)} - x_i, 0\} L_i^{(1)} \mathbb{1}_{\tau_i > 1} + ST = \sum_{i=0}^2 \max\{x_i - x_i^{(1)}, 0\} L_i^{(1)} + (1 - \phi_e) v_e + (1 - \phi_d) v_d,$$

2. Leverage Ratio > Restriction,

$$3. \frac{\frac{1}{1+r} \mathbb{E}_{\mathbb{Q}} \left[\left(\sum_{i=0}^2 x_i^{(1)} L_i^{(2)} \mathbb{1}_{\tau_i > 1} - v_d - LT \right)^+ \right]}{e + v_e} \geq \min \left\{ 1 + r_e, \frac{\left(\sum_{i=0}^2 x_i L_i^{(1)} - Debt \right)^+}{e} \right\} \geq 0,$$

$$4. \rho(x^{(1)}) \leq \theta_2,$$

$$5. 0 \leq v_e \leq E \text{ and } 0 \leq v_d \leq D.$$

In the model formulation, ST and LT , denote the short-term and long-term debt, respectively. Also, r , r_e and θ_2 is used to denote the risk-free, the target return on equity (for the equity holders to continue with their holdings) and upper bound on risk at $t = 1$, respectively. Further, E and D represent the upper bounds (or cap) on the amount of equity and debt (issued by the bank), respectively. Finally, ϕ_e and ϕ_d , are the cost of issuing equity and debt, respectively. We now make the following observations with regards to the constraints, incorporated in the model.

- (A) The objective function maximizes the return of expected payoff at final time step and minimizes the cost of raising capital.
- (B) The first constraint represents the budget restriction at the intermediate time step. It represents that the money used to repay the short-term debt and to invest more in loans is equal to the funds raised through debt, equity and selling some of its assets. The bank is able to make an investment if the loan has not defaulted yet. Therefore we have added the term $\mathbb{1}_{\tau_i > 1}$, where τ_i is the default time for the i -th asset.
- (C) The second constraint shows that at time $t = 1$, the bank must satisfy the Leverage Ratio restriction.
- (D) The third constraint implies that the expected (with respect to risk-neutral probability measure Q) discounted return on equity at time $t = 1$ should be greater than the minimum value between the target growth factor and the return on equity at time $t = 1$. Shareholders allow for continuation of their current position, on to the next time step, provided that the constraint is satisfied. Else, they will liquidate their position. Therefore (as already noted) we call it as “equity holder’s constraint”. Here, we have compared the present return on equity with the expected discounted return on equity. For this valuation, we have considered risk-neutral probability measures motivated by the American call option pricing technique [29].
- (E) The fourth constraint creates an upper bound on the risk for the loan portfolio at time $t = 1$.
- (F) Finally, the fifth constraint represents that in any situation, there is an cap on the amount of equity and debt raised, respectively.

4.3 Theoretical Results

In this section, we discuss some theoretical results which follows from the models presented in the preceding section. In [6], we have studied the scenario where a bank cannot raise capital through debt and equity, in the intermediate time step ($t = 1$). In this chapter, we present a general setup, where banks can issue debt and equity, as well as liquidate their assets. Behn et al. [8] have

presented a dynamic bank model to study the bank's behavior in order to meet the regulatory constraints. One of the most important regulatory aspects is the capital structure'. The regulatory framework, formulated by the Basel Committee on Banking Supervision (BCBS) had introduced the "Leverage Ratio", to facilitate the resilience of the banking system. Accordingly, the authors in [8] have stated four strategies to fulfill the capital requirement. In this chapter, we have constructed a novel model by incorporating these four strategies via the simple Model 6. It maximizes the expected return and minimizes the cost of issuing new debt or equity. It also considers the capital ratio requirement and equity holders' constraints. In the following description, we describe how Model 6 incorporates all these four strategies.

According to the modeling setup, the bank can raise capital through debt, equity and liquidating assets, but the question is to ascertain as to which of these is the most preferable for a bank? Are these decisions scenario-dependent (the nodes in Figure 4.1 represent the scenario)? We have addressed these cases in the following results. Before going to the discussion of the statements of the theorems and their proofs, we have made an assumption for the brevity of the proof and to obtain a relation between the issuance of debt and equity simultaneously (the assumption is that all the realizations are non-negative at the final time step).

Assumption 1.

$$\begin{aligned} & \frac{\frac{1}{1+r} \mathbb{E}_{\mathbb{Q}} \left[\left(\sum_{i=0}^2 x_i^{(1)} L_i^{(2)} \mathbb{1}_{\tau_i > 1} - v_d - LT \right)^+ \right]}{e + v_e} \\ & \geq \frac{\frac{1}{1+r} \mathbb{E}_{\mathbb{Q}} \left[\left(\sum_{i=0}^2 x_i^{(1)} L_i^{(2)} \mathbb{1}_{\tau_i > 1} - v_d - LT \right) \right]}{e + v_e} \\ & \geq \min \left\{ 1 + r_e, \frac{\left(\sum_{i=0}^2 x_i L_i^{(1)} - Debt \right)^+}{e} \right\}. \end{aligned}$$

When all the realization are positive then, the first term in the above relation becomes positive.

Theorem 4.3.1. *Bank prefers to raise capital from that source (debt and equity) which has the lower raising cost.*

In Model 5, we have considered that there are two types of debt, namely long term debt and short term debt. Short term debt affects the solvency of the bank, in the intermediate time step.

Theorem 4.3.2. *Larger amount of short term debt at initial time point increases the chance of default at the intermediate time step.*

Proof. The bank faces default in the intermediate time step, if it is unable to meet the depositors' claim, that is, the bank is unable to return the money back to the short term debt holders. This

happens if the realization from portfolio is less than the value of short term debt. So, if R_1 is the realization of the portfolio at time $t = 1$, then the bank survives the worst case (Node-3) provided the following condition holds:

$$R_1 \text{ [For the worst case]} \geq \beta_{ST}(1 - e) \Rightarrow x_0(1 + r) + x_1(1 - \lambda_1) + x_2(1 - \lambda_2) \geq \beta_{ST}(1 - e).$$

This implies that,

$$\beta_{ST} \leq \frac{x_0(1 + r) + x_1(1 - \lambda_1) + x_2(1 - \lambda_2)}{(1 - e)}.$$

From the above inequalities, we can clearly see that for a large value of β_{ST} , the bank has lesser chance of surviving at $t = 1$. Therefore, it increases the chance of default. The other nodes also behave in a similar manner. \square

Having fewer short-term debts has some advantages for the other nodes. In Node-2, if $x_2(1 - \lambda_2) \geq \beta_{ST}(1 - e)$, then the bank can meet depositors' claims without liquidating assets or issuing new debt or equity. Also, at Node-1, a lesser amount of short-term obligation reduces the amount of liquidation or issuing new debt and equity, which, in turn reduces the cost of raising money in the intermediate time step. Before proceeding further, we introduce the variable Z_1 as the realization, after paying the short-term debt, that is, $Z_1 = R_1 - \text{Short term debt} = R_1 - \beta_{ST}(1 - e)$.

Theorem 4.3.3. *Bank can survive the worst scenario at $t = 1$, provided,*

$$x_0 = \max \left\{ 0, \frac{x_1\lambda_1 + x_2\lambda_2 + \beta_{ST}(1 - e) - 1}{r} \right\}.$$

Proof. Let us assume that the bank survives Node-3, at $t = 1$. Then the net value of the assets is more than the debt, that is,

$$\begin{aligned} x_0(1 + r) + x_1(1 - \lambda_1) + x_2(1 - \lambda_2) &\geq \beta_{ST}(1 - e) \\ \Rightarrow x_0r &\geq x_1\lambda_1 + x_2\lambda_2 + \beta_{ST}(1 - e) - 1 \\ \Rightarrow x_0 &\geq \frac{x_1\lambda_1 + x_2\lambda_2 + \beta_{ST}(1 - e) - 1}{r}. \end{aligned}$$

This relation says that the investment, x_0 satisfying the above inequality can help the bank to survive the worst scenario, if all the other conditions are satisfied. \square

The authors in [2] have discussed the effect of the Leverage Ratio on bank stability, by considering different states (s_1 and s_2) to study the relation. In our study, the scenarios are presented in Figure 4.1, and each scenario presents a state. The study in [2] shows that more equity leads to stable banking. However, having more equity reduces the return on equity, and we prove this argument in the following theorem.

Theorem 4.3.4. *In the intermediate time step, the return on equity reduces, as the amount of equity holders increases.*

Proof. Let $\mathbb{E}[R_1]$ denote the expectation of the random variable R_1 , presenting the realization of the loan portfolio at time $t = 1$. Then the expected return on equity is given by,

$$\begin{aligned} r_{\text{equity}} &:= \frac{\mathbb{E}[R_1] - \beta_{ST}(1 - e) - \beta_{LT}(1 + r_d)(1 - e)}{e} \\ &= \frac{\mathbb{E}[R_1] - (1 - e)(\beta_{ST} + \beta_{LT}(1 + r_d))}{e} \\ &= \frac{\mathbb{E}[R_1] - (1 - e)(1 + \beta_{LT}r_d)}{e} \\ &= \frac{\mathbb{E}[R_1] - (1 + \beta_{LT}r_d) + e(1 + \beta_{LT}r_d)}{e}. \end{aligned}$$

Now differentiating with respect to e , we get,

$$\begin{aligned} \frac{\partial r_{\text{equity}}}{\partial e} &= \frac{e(1 + \beta_{LT}r_d) - (\mathbb{E}[R_1] - (1 + \beta_{LT}r_d) + e(1 + \beta_{LT}r_d))}{e^2} \\ &= \frac{-(\mathbb{E}[R_1] - (1 + \beta_{LT}r_d))}{e^2}. \end{aligned}$$

Now in order to show that more amount of equity reduces the expected return on equity, we need to show that $\frac{\partial r_{\text{equity}}}{\partial e} \leq 0$, and to prove this, it only remains to show that $\mathbb{E}[R_1] \geq (1 + \beta_{LT}r_d)$. From the definition we have,

$$\begin{aligned} \mathbb{E}[R_1] &= x_0\mathbb{E}[L_0^{(1)}] + x_1\mathbb{E}[L_1^{(1)}] + x_2\mathbb{E}[L_2^{(1)}] \\ &\geq \mathbb{E}[L_0^{(1)}] \quad (\text{because } \mathbb{E}[L_2^{(1)}] \geq \mathbb{E}[L_1^{(1)}] \geq \mathbb{E}[L_0^{(1)}]) \\ &= (1 + r) \\ &= \beta_{ST}(1 + r) + \beta_{LT}(1 + r) \\ &\geq \beta_{ST} + \beta_{LT}(1 + r) \\ &\geq \beta_{ST} + \beta_{LT}(1 + r_d) \quad (\text{as } r_d \leq r) \\ &= 1 + \beta_{LT}r_d \end{aligned}$$

This completes the proof. \square

The preceding theorem discusses about the expected return on equity at the intermediate time step. Next, we study the change of return on equity with the amount raised at the intermediate time step, through equity v_e and debt v_d . Let $V_{\text{equity}}(v_e, v_d)$ denotes the expected discounted return on equity, while raising v_e amount of equity and v_d amount of debt. Using Assumption 1, we obtain,

$$V_{\text{equity}}(v_e, v_d) = \frac{Z_1 + (1 - \phi_e)v_e + (1 - \phi_d)v_d - \frac{v_d + \beta_{LT}(1 - e)(1 + r_d)^2}{1 + r}}{e + v_e}.$$

In Figure 4.2 we demonstrate the change of $V_{\text{equity}}(v_e, v_d)$ versus $v_d(v_e = 0)$ and $v_e(v_d = 0)$. In the next theorem we study the change of V_{equity} with v_d , without raising any equity ($v_e = 0$).

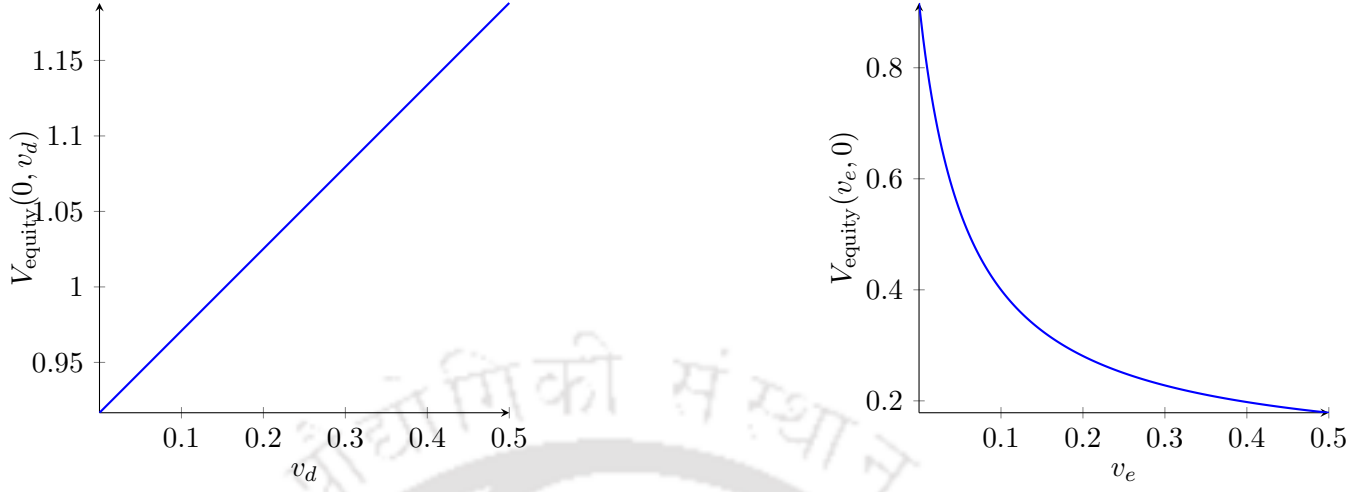


Figure 4.2: This figure shows the change of return on equity with issuing new debt and equity in the intermediate time step $\left(\phi_d \leq \frac{r}{1+r}\right)$.

Theorem 4.3.5. *Issuing fund only through debt (without equity) increase the discounted expected payoff of the equity, provided $\phi_d \leq \frac{r}{1+r}$.*

Proof. Let R_1 denote the realization of the portfolio at time $t = 1$. Let us consider that the bank has issued v_d amount of debt. Then it has paid $\phi_d v_d$ to raise this amount, and it has the same amount of equity e . Accordingly,

$$V_{\text{equity}}(0, v_d) = \frac{Z_1 + (1 - \phi_d)v_d - \frac{v_d}{1+r} + \frac{\beta_{LT}(1-e)(1+r_d)^2}{1+r}}{e}.$$

Here $Z_1 = R_1 - \beta_{ST}(1 - e)$ is the realization after paying the short term debt. We get the value of the equity by discounted expected realization of equity at the final time point, with the expectation being taken with respect to risk neutral probability measure. To get the value of equity per unit volume, we divide it by the amount of equity. Now differentiating w.r.t. v_d , we get,

$$\frac{\partial}{\partial v_d} V_{\text{equity}}(0, v_d) = \frac{1 - \phi_d - \frac{1}{1+r}}{e}.$$

Hence $\frac{\partial V_{\text{equity}}}{\partial v_d}$ will be an increasing function, provided,

$$1 - \phi_d - \frac{1}{1+r} \geq 0 \Rightarrow \frac{r}{1+r} \geq \phi_d.$$

This completes the proof. \square

At the time $t = 1$ and Node-3, for Model 4.1, the two loans have already defaulted. If the bank is able to survive this scenario, it can issue new debt to increase profit for the equity-holders, enabling them to continue their holding to the next time step.

Corollary 4.3.6. For $\phi_d \geq \frac{r}{1+r}$, $V_{equity}(0, v_d)$ is decreasing function. Then the bank has an upper bound on raising new debt from equity holders constraint.

Proof. The proof of the first part is quite obvious from the preceding theorem. Now to get the cap, we use the equity holder's constraint in the Model 6,

(A) When the expected return is less than the target, then,

$$\begin{aligned} Z_1 + (1 - \phi_d)v_d - \frac{v_d}{1+r} - \beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} &\geq Z_1 - \beta_{LT}(1-e)(1+r_d) \\ \Rightarrow \left(1 - \phi_d - \frac{1}{1+r}\right)v_d &\geq \beta_{LT}(1-e)(1+r_d)\left(\frac{1+r_d}{1+r} - 1\right) \\ \Rightarrow \left(\frac{r}{1+r} - \phi_d\right)v_d &\geq \beta_{LT}(1-e)(1+r_d)\left(\frac{r_d - r}{1+r}\right). \end{aligned}$$

From our construction, we have, $r_d < r$, and therefore multiplying by -1 , we get,

$$\begin{aligned} \left(\phi_d - \frac{r}{1+r}\right)v_d &\leq \beta_{LT}(1-e)(1+r_d)\left(\frac{r - r_d}{1+r}\right) \\ \Rightarrow v_d &\leq \frac{\beta_{LT}(1-e)(1+r_d)\left(\frac{r - r_d}{1+r}\right)}{\left(\phi_d - \frac{r}{1+r}\right)}. \end{aligned}$$

(B) For the case when the return exceeds the target return, we have,

$$\begin{aligned} \frac{Z_1 + (1 - \phi_d)v_d - \frac{v_d}{1+r} - \beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r}}{e} &\geq (1+r_e) \\ \Rightarrow Z_1 - \beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} - (1+r_e)e &\geq \left(\phi_d - \frac{r}{1+r}\right)v_d \\ \Rightarrow v_d &\leq \frac{Z_1 - \beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} - (1+r_e)e}{\left(\phi_d - \frac{r}{1+r}\right)v_d}. \end{aligned}$$

The above inequalities give an upper bound on how much debt the bank can issue when the cost of raising new debt is high. \square

Raising money through debt decreases the net debt of the bank, and hence it reduces the Leverage Ratio.

Theorem 4.3.7. Issuing money through debt decreases the Leverage Ratio.

Proof. For raising v_d amount of debt, the Leverage Ratio is given by,

$$\text{Leverage Ratio} := \frac{Z_1 + (1 - \phi_d)v_d - v_d - \beta_{LT}(1-e)(1+r_d)}{Z_1 + v_d}.$$

Partially differentiating with respect to v_d , we get,

$$\begin{aligned}\frac{\partial}{\partial v_d} (\text{Leverage Ratio}) &= \frac{-\phi_d(Z_1 + v_d) - (Z_1 - \phi_d v_d - \beta_{LT}(1-e)(1+r_d))}{(Z_1 + v_d)^2} \\ &= \frac{-Z_1(1 + \phi_d) + \beta_{LT}(1-e)(1+r_d)}{(Z_1 + v_d)^2}.\end{aligned}$$

If the bank survives, then its realization must be more than the debt, that is,

$$Z_1 \geq \beta_{LT}(1-e)(1+r_d) \Rightarrow Z_1(1 + \phi_d) \geq \beta_{LT}(1-e)(1+r_d).$$

The last inequality implies that $\frac{\partial}{\partial v_d} (\text{Leverage Ratio})$ is negative. Hence the Leverage Ratio decreases with v_d . \square

Issuing new debt increases leverage, and so the Leverage Ratio restriction creates a cap for the amount of debt issued.

Theorem 4.3.8. *Leverage Ratio restriction implements a cap on the amount of debt can be issued.*

Proof. The capital ratio constraint, given by second constraint in Model 6, is given by,

$$\begin{aligned}\frac{Z_1 + (1 - \phi_d)v_d - v_d - \beta_{LT}(1-e)(1+r_d)}{Z_1 + v_d} &\geq k_{lev} \\ \Rightarrow Z_1 + (1 - \phi_d)v_d - v_d - \beta_{LT}(1-e)(1+r_d) &\geq k_{lev}Z_1 + k_{lev}v_d \\ \Rightarrow Z_1 - k_{lev}Z_1 - \beta_{LT}(1-e)(1+r_d) &\geq v_d(k_{lev} + \phi_d) \\ \Rightarrow v_d &\leq \frac{Z_1 - k_{lev}Z_1 - \beta_{LT}(1-e)(1+r_d)}{k_{lev} + \phi_d}.\end{aligned}$$

The above inequality defines a cap through capital requirement condition. It can be easily seen that this upper bound increases as k_{lev} and ϕ_d decrease, that is, for higher value of Leverage Ratio, the restriction decreases the cap on the debt that can be issued. For higher values of Z_1 , that is, the realization of the portfolio acquired at time $t = 0$ allows more debt to be issued. \square

Therefore, from the above two results, we conclude that the model allows the bank to issue debt that satisfy both the equity holders constraint and capital requirement condition. In the next case we show the effect when the firm issue new equity only. Now, if the bank issues new equity, then the expected discounted return per unit share is,

$$\begin{aligned}V_{\text{equity}}(v_e, 0) &= \frac{\frac{1}{1+r} [\mathbb{E}_Q [(Z_1 + (1 - \phi_e)v_e)(1+r) - (1+r_d)^2\beta_{LT}(1-e)]]}{e + v_e} \\ &= \frac{Z_1 - \frac{(1+r_d)^2}{1+r}\beta_{LT}(1-e) + (1 - \phi_e)v_e}{e + v_e}.\end{aligned}$$

This definition leads us to study the change of the function, $f(v_e)$. The following theorem discusses this relation mathematically. Let $Z_{1LT} := Z_1 - \frac{(1+r_d)^2}{1+r}\beta_{LT}(1-e)$, that is, the discounted return on equity at time $t = 1$.

Theorem 4.3.9. *If $Z_{1LT} \geq (1 - \phi_e)e$, then $V_{equity}(v_e, 0)$ decreases, as v_e increases.*

Proof. We can define the following function, in terms of Z_{1LT} .

$$V_{equity}(v_e, 0) := \frac{Z_{1LT} + (1 - \phi_e)v_e}{e + v_e}.$$

Differentiating w.r.t., v_e , we have,

$$\frac{\partial}{\partial v_e} V_{equity}(v_e, 0) = \frac{(1 - \phi_e)e - Z_{1LT}}{(e + v_e)^2} \leq 0.$$

Since, $Z_{1LT} \geq e$, therefore, $Z_{1LT} \geq (1 - \phi_e)e$. □

This theorem implies that in favorable scenarios, raising new equity decreases the expected discounted return on equity.

Corollary 4.3.10. *In some adverse scenarios, the reverse can happen, that is, if $Z_{1LT} \leq (1 - \phi_e)e$ then $V_{equity}(v_e, 0)$ increases with an increase in v_e .*

Issuing new equity reduces the return on equity. Hence from the equity holders constraint, we get an upper bound for the amount of equity we can issue.

Theorem 4.3.11. *From equity holder's constraint we get an upper bound on the amount of equity issued. Equity holder's constraint is more than the minimum value of the current return or the target return.*

Proof. From the equity holder's constraint, we can say that the return on equity should be more than the present value.

(A) When the current return on equity is less than the target return, then,

$$\begin{aligned} & \frac{Z_1 + (1 - \phi_e)v_e - \beta_{LT}(1 - e)\frac{(1+r_d)^2}{1+r}}{e + v_e} \geq \frac{Z_1 - \beta_{LT}(1 - e)(1 + r_d)}{e} \\ \Rightarrow & eZ_1 + e(1 - \phi_e)v_e - e\beta_{LT}(1 - e)\frac{(1 + r_d)^2}{1 + r} \\ \geq & eZ_1 - e\beta_{LT}(1 - e)(1 + r_d) + v_eZ_1 - v_e\beta_{LT}(1 - e)(1 + r_d) \\ \Rightarrow & e\beta_{LT}(1 - e)(1 + r_d) - e\beta_{LT}(1 - e)\frac{(1 + r_d)^2}{1 + r} \geq v_e(Z_1 - \beta_{LT}(1 - e)(1 + r_d) - e(1 - \phi_e)) \\ \Rightarrow & v_e \leq \frac{\beta_{LT}e(1 - e)(1 + r_d)\frac{r - r_d}{1 + r}}{Z_1 - \beta_{LT}(1 - e)(1 + r_d) - (1 - \phi_e)e}. \end{aligned}$$

(B) When the current return crosses the target return, then,

$$\begin{aligned}
& \frac{Z_1 + (1 - \phi_e)v_e - \beta_{LT}(1 - e)\frac{(1+r_d)^2}{1+r}}{e + v_e} \geq (1 + r_e) \\
\Rightarrow & Z_1 + (1 - \phi_e)v_e - \beta_{LT}(1 - e)\frac{(1+r_d)^2}{1+r} \geq (1 + r_e)e + (1 + r_e)v_e \\
\Rightarrow & Z_1 - \beta_{LT}(1 - e)\frac{(1+r_d)^2}{1+r} - (1 + r_e)e \geq v_e(r_e + \phi_e) \\
\Rightarrow & v_e \leq \frac{Z_1 - \beta_{LT}(1 - e)\frac{(1+r_d)^2}{1+r} - (1 + r_e)e}{(r_e + \phi_e)}.
\end{aligned}$$

The above inequalities give the upper bound on the amount of capital that can be raised via equity. \square

For lower issuing cost (ϕ_e), issuing new equity raises the capital ratio.

Theorem 4.3.12. *If $\phi_e \leq \frac{\beta_{LT}(1 - e)(1 + r_d)}{Z_1}$, then the Leverage Ratio increases, with issuing new equity.*

Proof. After issuing v_e amount of capital, the Leverage Ratio is given by,

$$\text{Leverage Ratio} := \frac{Z_1 + (1 - \phi_e)v_e - \beta_{LT}(1 - e)(1 + r_d)}{Z_1 + v_e}.$$

Now differentiating it w.r.t. v_e , we get,

$$\frac{\partial}{\partial v_e} (\text{Leverage Ratio}) = \frac{\beta_{LT}(1 - e)(1 + r_d) - Z_1\phi_e}{(Z_1 + v_e)^2}.$$

Clearly if the given condition is satisfied then capital ratio is an increasing function of v_e . \square

Clearly if bank has less debt and the issuing cost is high, then it not necessary to collect fund through equity. The payment to issue new equity then becomes a new burden. The changing pattern of the Leverage Ratio with v_e and v_d is presented in Figure 4.3. The above results discuss the cap on the capital raised through a single source, namely, either debt or equity. All the cap are in terms of the variable introduced initially and the realization of the portfolio implemented at the initial stage. These caps do not contain any decision variable of the intermediate time step ($x^{(1)}, v_d, v_e$). Now, we proceed to the case where banks issue funds through both the sources, debt and equity. Here, we also find that the caps, for the variable v_e and v_d , are dependent on the decision variables of the intermediate time step. Hence, considering all the advantages, the bank must satisfy all the constraints defined in the intermediate time step.

Theorem 4.3.13. *Leverage Ratio restriction gives an upper bound on the amount of debt that can be issued, which is dependent on the amount of equity which has been raised.*

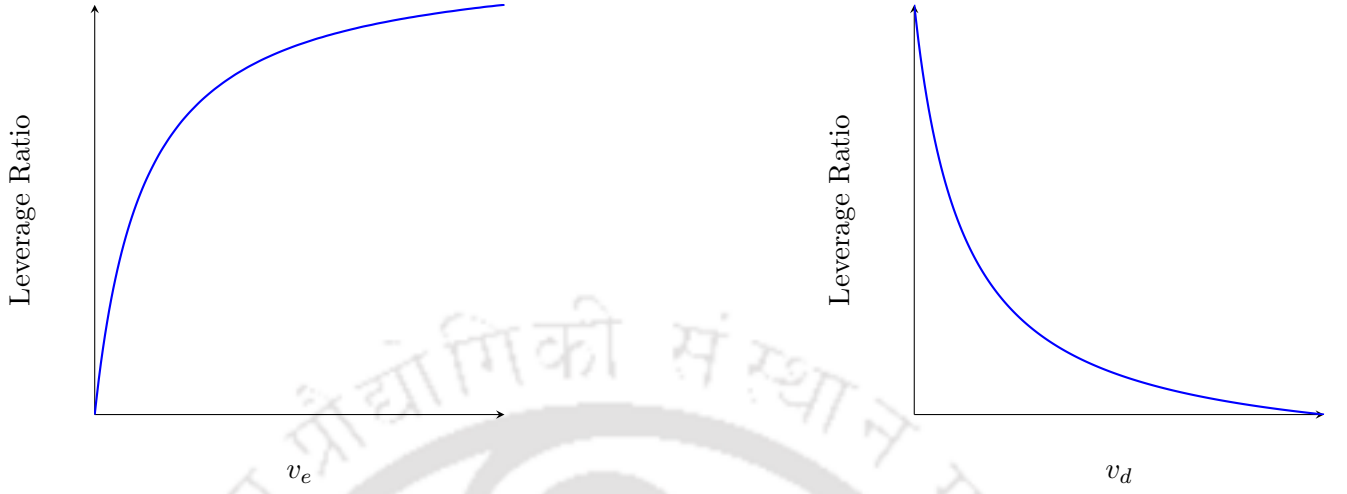


Figure 4.3: This figure shows the change of capital ratio with issuing new equity and debt in the intermediate time step.

Proof. From the Leverage Ratio requirement, we get,

$$\frac{Z_1 + (1 - \phi_d)v_d + (1 - \phi_e)v_e - v_d - \beta_{LT}(1 - e)(1 + r_d)}{Z_1 + v_d + v_e} \geq k_{lev},$$

which implies that,

$$\begin{aligned} & Z_1 + (1 - \phi_d)v_d + (1 - \phi_e)v_e - v_d - \beta_{LT}(1 - e)(1 + r_d) \geq k_{lev}Z_1 + k_{lev}v_d + k_{lev}v_e \\ \Rightarrow & Z_1 - k_{lev}Z_1 - \beta_{LT}(1 - e)(1 + r_d) + (1 - \phi_e)v_e - k_{lev}v_e \geq v_d(k_{lev} + \phi_d) \\ \Rightarrow & v_d \leq \frac{Z_1 - k_{lev}Z_1 - \beta_{LT}(1 - e)(1 + r_d) + (1 - \phi_e - k_{lev})v_e}{k_{lev} + \phi_d} \end{aligned}$$

The above inequality imposes an upper bound on v_d that can be issued when v_e amount of equity has been raised. \square

Theorem 4.3.14. *Equity holder's constraint gives an upper bound on the amount of equity that can be raised. This upper bound is dependent of the amount of debt being issued.*

Proof. The facility of raising capital through debt and equity simultaneously, affect each other. The upper bound on the amount of equity that can be raised is dependent on the amount of debt issued.

(A) When the current return on equity is less than the target return, then we have,

$$\begin{aligned}
& \frac{Z_1 + (1 - \phi_e)v_e + (1 - \phi_d)v_d - \frac{v_d + \beta_{LT}(1-e)(1+r_d)^2}{1+r}}{e + v_e} \geq \frac{Z_1 - \beta_{LT}(1-e)(1+r_d)}{e} \\
\Rightarrow & eZ_1 + e(1 - \phi_e)v_e + e(1 - \phi_d)v_d - \frac{ev_d}{1+r} - e\beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} \\
\geq & eZ_1 - e\beta_{LT}(1-e)(1+r_d) + v_eZ_1 - v_e\beta_{LT}(1-e)(1+r_d) \\
\Rightarrow & ev_d \left(1 - \phi_d - \frac{1}{1+r}\right) + e\beta_{LT}(1-e)(1+r_d) - e\beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} \\
\geq & v_e(Z_1 - \beta_{LT}(1-e)(1+r_d) - e(1 - \phi_e)) \\
\Rightarrow & v_e \leq \frac{ev_d \left(1 - \phi_d - \frac{1}{1+r}\right) + \beta_{LT}e(1-e)(1+r_d)\frac{r-r_d}{1+r}}{Z_1 - \beta_{LT}(1-e)(1+r_d) - e(1 - \phi_e)}.
\end{aligned}$$

(B) When the current return crosses the target return, we obtain,

$$\begin{aligned}
& \frac{Z_1 + (1 - \phi_e)v_e + (1 - \phi_d)v_d - \frac{v_d + \beta_{LT}(1-e)(1+r_d)^2}{1+r}}{e + v_e} \geq (1 + r_e) \\
\Rightarrow & Z_1 + (1 - \phi_e)v_e + \left(1 - \phi_d - \frac{1}{1+r}\right)v_d - \beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} \geq (1 + r_e)(e + v_e) \\
\Rightarrow & v_e(\phi_e + r_e) \leq Z_1 + \left(\frac{r}{1+r} - \phi_d\right)v_d - \beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} - (1 + r_e)e \\
\Rightarrow & v_e \leq \frac{Z_1 + \left(\frac{r}{1+r} - \phi_d\right)v_d - \beta_{LT}(1-e)\frac{(1+r_d)^2}{1+r} - (1 + r_e)e}{\phi_e + r_e}.
\end{aligned}$$

This inequalities create a cap on v_e when v_d is already issued, and the cap increases with v_d . \square

From the above inequality, it is clear that in favorable condition, the value of the cap gets reduced, due to a higher value of Z_1 . Finally, the results regarding the issuance of debt and equity are summarized in Table 4.1.

Table 4.1: Table to summarize the results. * Decreases provided $Z_{1LT} \geq (1 - \phi_e)e$. ** Increases provided $Z_{1LT} < (1 - \phi_e)e$. † No, provided $\phi_d \leq \frac{r}{1+r}$, †† Yes, provided $\phi_d > \frac{r}{1+r}$.

	Change in Capital Ratio	Cap due to Capital Ratio	Change in V_{equity}	Cap due to 'Equity holder's constraint'.
Only, v_d	Decreases	Yes	Increases	No † Yes ††
Only, v_e	Increases	No	Decreases * Increases **	Yes
Both v_e, v_d	Simultaneous effect	Cap on v_d dependent on v_e	Simultaneous effect	Cap on v_e dependent on v_d

4.4 An Example

In this section, we present an example to validate the preceding results. In particular, we consider two examples, one with parameter values of $\beta_{ST} = 0.7$, $D = 0.01$, $E = 0.02$ and the other with the parameter values $\beta_{ST} = 0.5$, $D = 0.20$, $E = 0.30$. The remaining parameters which are identical for both the examples, are given in Tables 4.2 and 4.3.

Table 4.2: Parameters pertaining to the three loans, L_0 , L_1 and L_2

	Interest	PD	LGD
L_0	3%	0%	0%
L_1	9%	6.1%	10%
L_2	13.2%	12.2%	9%

Table 4.3: The Other Parameters

$\theta_1 = 1.2\%$	$r_d = 1\%$
$k_{lev} = 4\%$	$\theta_2 = 1.0\%$

1. *Example 1:* For Example 1, the application of Model 5 results in the initial investment strategy of (0%, 59.04%, 40.96%), with the Leverage Ratio being 4%. For the intermediate time steps, we discuss the results node-wise.

(A) *Node-3:* Here we can see that the bank has very limited access to capital via new debt and equity. It can issue a maximum of 2% equity and 1% debt, of the initial total wealth. The Leverage Ratio, $\frac{Z_1 - LTD}{Z_1}$, at this point becomes negative. So, the bank will have to raise capital to meet the capital requirement. Mathematically, the bank has to collect v_e amount of capital, so that the Leverage Ratio is at least 4%, that is,

$$\frac{Z_1 - \beta_{LT}(1 - e)(1 + r_d)}{Z_1 + v_e} \geq 0.04.$$

Substituting all these values, we obtain that $v_e \geq 0.0791$. However, the bank is not in a position to issue this much equity, and hence it fails to survive. Therefore bank has to liquidate its position and goes into bankruptcy.

(B) *Node-2:* In this scenario, the more risky loan L_2 has defaulted, but the less risky loan is in good condition. Further, the Leverage Ratio constraint and the “equity holder’s constraint” are both satisfied without issuing new debt or equity. This leads to a nonempty feasible region. Solving the model, we get the result that, the bank’s portfolio is (35.33%, 00.05%) (the last one has defaulted) with $v_d = 0.01$ and $v_e = 0.012$. Here, due to low raising cost of the equity, the bank raises debt first, and then it raises equity.

Theorem 4.3.14 creates an upper bound for the amount of equity that can be raised. Substituting the values, we get the upper bound is 0.012, which is exactly same as v_e .

(C) *Node-1*: In this case all the loans have survived. Therefore bank has higher amount of realization. “Equity holder’s constraint” protects the bank from failure at the final time point. Therefore, it has a large investment in the safe asset to ensure the return of the equity holders. The portfolio at time $t = 1$ is given by (44.81%, 00.07%, 20.00%) in the three loans. As a result, the bank raises the available debt and equity ($v_d = 0.01$ and $v_e = 0.02$).

2. *Example 2*: For Example 2, the application of Model 5 results in the initial investment strategy of

(0%, 54.01%, 45.99%) with the Leverage Ratio being 4%.

(A) *Node-3*: In this node, the bank has lower short term debt, and has more access to capital via new debt and equity, for which the model has a feasible solution. Hence bank survives this stage, which means that in an unfavorable condition, the bank can raise capital, so that it can survive the unfavorable condition. We solved this model by the differential evolution method and determined that the bank invests 85.39% in the safest asset, and issues 29.94% in equity and 19.94% in debts, that is, it almost raises the amount of capital that it can raise. From Corollary 4.3.10, we see that issuing new equity increases return on equity. And to satisfy capital requirement, the bank has to increase it’s Leverage Ratio. From Theorem 4.3.12, it is evident that bank has to raise equity to increase capital requirement. So bank has to issue new equity. The solution shows that the bank issued that total amount of equity which it can raise, and from Theorem 4.3.13 the upper bound of the debt (depending on this equity) is determined to be 1.64, which is much higher than the amount that is available. Hence bank is able to issue the full amount of accessible debt.

(B) *Node-2*: It also has the same dynamics like the previous problem. In this case, the bank invests 69.87% in the safe asset and 0.04% asset in the less risky loans, with $v_d = 0.2$, that is, it issues the net amount of debt it can issue. Finally the amount of equity raised is given by, $v_e = 0.0077$ which is also same as the upper bound given by Theorem 4.3.14.

(C) *Node-1*: Here the portfolio is given by (99.90%, 5.09%, 0%). We can see that bank is able to make investments in riskier assets after ensuring the “equity holder’s constraint”. In this case also, the banks issue all the available equity and debt available. Therefore $v_e = 0.3$ and $v_d = 0.2$.

To demonstrate the bank’s behavior, we have considered another example, only for *Node-1*. In this

example we have changed only D to be 0.9, with all other parameters being the same as Example 2. It shows that the bank invests its money in safe assets to guarantee the equity holder's expectation of return. Investment in risky assets reduces the return on equity. Therefore to demonstrate this phenomenon we have considered this case with $D = 0.9$, that is, a bank can issue a large amount of debt. In that case, the result shows that 99.98% is invested in the safe asset (percentage is given with respect to the initial wealth), and 42.61% in the less riskier asset. Compared with the case ($D = 0.2$) we can say that banks invest first in safe assets, in order to ensure the constraints and then only it goes for risky assets. Here the "equity holder's constraint" shields the bank from default. When a bank satisfies this constraint it cannot fail under any scenario and so there is a large amount of investments in the safe asset. Results show that the return on equity is 2.5674 for the Example 1 and 1.1209 for Example 2, while for the last problem it becomes 1.0801. Therefore, in favorable conditions, the issuance of more equity leads to reduction in its expected return.

4.5 Summary

The theoretical results of this study suggest some crucial properties of the bank dynamic problem in the intermediate time point. We have shown that the equity holders' constraint protects banks from defaulting. In favorable scenarios, issuing new equity decreases the return on equity, but in a worst-case scenario, it increases the return on equity. Hence, the bank needs new capital to satisfy the equity holders constraint and the Leverage Ratio constraint. So, if the bank has scope to raise enough capital in a bad condition, it will raise the capital in order to survive. On the other hand, issuing new debt increases return on equity (for lower issuance cost), whereas equity return decreases with issuing new debt (when issuing cost is more considerable). The Leverage Ratio always caps the amount of debt issued. The "equity holder's constraint" ensures the return crosses the expected return, if the previous cycle return is higher than the expected return. It desists banks from risk-taking and ensures its survival. Accordingly, the reshuffling of assets is done in order to satisfy all the constraints.

Chapter 5

Can Limited Liability increase stability for banks: A dynamic portfolio approach

5.1 Introduction

This chapter presents a novel approach for the bank's decision problem, incorporating Limited Liability in the objective function. The focus of the work is to study the effect of the inclusion of Limited Liability in the model, for a continuous time setup. Accordingly, we consider two continuous time models, with and without Limited Liability, involving the maximization of the expected utility function of terminal wealth. We compare the solutions of these two models to demonstrate the effect of the inclusion of Limited Liability. To solve the problem with the objective function incorporating Limited Liability, we approximate the payoff function with another set of functions for which we have closed-form solutions for the investment portfolio. Then, we show that the solution with Limited Liability incorporates less risky assets, while simultaneously increasing the resilience of the bank.

The KMV model [76] is a credit risk model that estimates the probability of default in the framework of Merton's structural credit risk model, which treats the firm's equity as a call option on its assets. Default occurs when the market value of assets falls below the default point. This model uses a metric, namely, Distance to Default (DD), to measure the company's probability of default. DD measures how close a company is to defaulting on its debt, with a higher value of DD implying a lower risk of default.

The outline of the chapter is as follows. We present a detailed description of the models in section 5.2. Then, in section 5.3 we derive all the theoretical results, which are then illustrated by an example in section 5.4. Finally, section 5.5 summarizes the main takeaways and concluding remarks on the work reported in this chapter.

5.2 Formulation of the Model

In this section, we briefly discuss the formulation of the Hamilton-Jacobi-Bellman (HJB) equation in the context of the problem under consideration. Accordingly, two optimization problems are formulated, one with Limited Liability and the other without Limited Liability. As stated previously, we consider the DD metric in the paradigm of the KMV model, used to study the resilience of the firm. Also, the interest rate risk is taken into consideration for this model. We have considered that the interest rate follows the Vasicek model which has been described in this section.

5.2.1 HJB Equation Formulation of an Optimization Process

In this subsection, we discuss briefly how the HJB equation is formulated for an optimization process. Let $\{(W(t))\}_{t \in [0, \infty)}$ be a m -dimensional Brownian motion. Further, let $Y(t)$ be a n -dimensional state process, whose evolution is given by the stochastic differential equation (SDE) as follows:

$$dY(t) = \Lambda(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))dW(t), \quad (5.2.1)$$

with an initial condition of $Y(t_0) = y_0$. Here $u(t)$ is a d -dimensional control process, $\Lambda(\cdot)$ is the drift, and $\Sigma(\cdot)$ is the instantaneous standard deviation. Let $[t_0, t_1]$ with $0 \leq t_0 < t_1 < \infty$ be the pertinent time window. Our goal is to determine an admissible control $u(\cdot)$ such that for each initial value (t_0, y_0) , the utility functional,

$$J(t_0, y_0; u) := \mathbb{E}^{t_0, y_0} \left(\int_{t_0}^T L(s, Y^u(s), u(s))dt + \Psi(T, Y^u(T)) \right), \quad (5.2.2)$$

is to be maximized. Here we want to maximize the functional up to time T , a fixed time horizon. Therefore, we want to solve the problem:

$$\sup_{u \in \mathcal{A}(t_0, y_0)} J(t_0, y_0; u).$$

Accordingly, the value function is defined as,

$$V(t, y) := \sup_{u \in \mathcal{A}(t, y)} J(t, y; u), \quad (t, y) \in Q.$$

Here $Q = [t_0, t_1] \times O$ where $O \in \mathbb{R}^n$ is an open set. For each function $G \in C^{1,2}(Q)$ and $(t, y) \in Q$, ($v \in U$ where U is a subset of \mathbb{R}^d), we consider the following differential operator:

$$A^v G(t, y) := G_t(t, y) + 0.5 \sum_{i,j=1}^n \Sigma_{ij}^*(t, y, v) \cdot G_{y_i y_j}(t, y) + \sum_{i=1}^n \Lambda_i(t, y, v) \cdot G_{y_i}(t, y), \quad (5.2.3)$$

where $\Sigma^* := \Sigma \Sigma^\top$. From the verification theorem [65], if the problem,

$$\sup_{v \in U} [A^v G(t, y) + L(t, y, v)] = 0, \quad (t, y) \in Q \text{ and } G(t, y) = \Psi(t, y), \quad (t, y) \in \partial Q. \quad (5.2.4)$$

has a solution, then the following holds:

(A) $G(t, y) \geq J(t, y; u)$ for all $(t, y) \in Q$ and $u(\cdot) \in \mathcal{A}(t, y)$.

(B) If for $(t, y) \in Q$ there exists a control $u^*(\cdot) \in \mathcal{A}(t, y)$ with,

$$u^*(s) \in \arg \max_{v \in U} [A^v G(s, Y^*(s)) + L(s, Y^*(s), v)], \quad (5.2.5)$$

for all $s \in [t, \tau]$ (where Y^* is the solution of the controlled SDE corresponding to $u^*(\cdot)$), then we have,

$$G(t, y) = V(t, y) = J(t, y; u^*), \quad (5.2.6)$$

i.e., $u^*(\cdot)$ is an optimal control and G coincides with the value function.

5.2.2 Problem Description

We construct the problem with a Money Market Account (MMA) and a risky bond. Let $\pi(t)$ denote the amount invested in the risky bond. Our goal is to determine $\pi(t)$ so that the terminal wealth is maximized. As mentioned earlier, the interest rate $r(t)$ follows the Vasicek model (discussed elaborately in [67]) given by,

$$dr(t) = a(t)dt + bdW(t),$$

where $a(t)$ is given by,

$$a(t) = \theta(t) - \alpha r(t), \quad \alpha > 0.$$

Here α and $\frac{\theta(t)}{\alpha}$ are the mean reversal speed and level, respectively. Further, b is the instantaneous standard deviation. Accordingly, we begin with a portfolio where the bank can invest in a MMA and a bond (zero coupon) with maturity $T_1 > T$, with T being the time horizon in which we are seeking to optimize our wealth level. The bond price using risk premium is presented in [50, 62]. The dynamics of price of the risky bond is represented by,

$$dP(t, T_1) = P(t, T_1) [\mu(t)dt + \sigma(t)dW(t)].$$

We take $\mu(t) = r(t) + \sigma(t)\zeta(t)$ ($\zeta(t)$ being the risk premium), as motivated by [50]. Also, we take $\sigma(t) = \frac{b}{\alpha} (1 - \exp(-\alpha(T_1 - t)))$. Now the total wealth level follows the SDE,

$$\begin{aligned} dX(t) &= X(t) [(\pi(t)\mu(t) + (1 - \pi(t))r(t))dt + \pi(t)\sigma(t)dW(t)], \\ &= X(t) [(\pi(t)\zeta(t)\sigma(t) + r(t))dt + \pi(t)\sigma(t)dW(t)]. \end{aligned} \quad (5.2.7)$$

Now our objective is to maximize the utility of the terminal wealth $U(X(T))$. Consequently, the optimization problem becomes,

$$\max_{\pi(\cdot) \in \mathcal{A}(0, x_0)} \mathbb{E} [U(X(T))],$$

with the initial condition $X(0) = x_0 = 1$ (normalizing), and $\mathcal{A}(0, x_0)$ is the set of admissible controls corresponding to the initial condition $t = t_0$ and $x = x_0$. Now, comparing with equation (5.2.1), we get,

$$\begin{aligned}
Y(t) &= (X(t), r(t))^\top, \\
\Lambda(t, x, r, \pi) &= (x(\pi\zeta\sigma + r), a)^\top, \\
\Sigma(t, x, r, \pi) &= (x\pi\sigma, b)^\top, \\
\Sigma^*(t, x, r, \pi) &= \begin{pmatrix} x^2\pi^2\sigma^2 & bx\pi\sigma \\ bx\pi\sigma & b^2 \end{pmatrix}, \\
A^\pi G(t, x, r) &= G_t + 0.5(x^2\pi^2\sigma^2 G_{xx} + 2x\pi b\sigma G_{xr} + b^2 G_{rr}), \\
&\quad + x(\pi\zeta\sigma + r)G_x + aG_r.
\end{aligned} \tag{5.2.8}$$

Hence, in order to get the desired optimal control we need to solve the following HJB equation:

$$\sup_{|\pi| \leq \delta} A^\pi G(t, x, r) = 0; \quad G(T, x, r) = U(X(T)), \tag{5.2.9}$$

with δ being the upper-bound of the investment in the risky asset. The allocation is with an upper bound θ for the Expected Loss (EL), for the entire portfolio. For the MMA, the probability of default is zero, and it has no contribution to the EL . So the EL of the portfolio is,

$$EL = \pi(t)p\lambda \leq \theta \Rightarrow \pi(t) \leq \frac{\theta}{p\lambda} = \delta \text{ (bound on risky asset).}$$

In the above equation, p and λ denote the PD and LGD, respectively. In case we take the utility function of the form $U(x) = x^\gamma$, $\gamma \in \mathbf{R}$ then we can find a closed feedback form for $\pi(t)$, as a function of investment-instrument's characteristic. As already noted, the goal is to study the effect of inclusion of Limited Liability in the model. To make a comparison, we need a closed form solution (for the investment portfolio), and therefore, we approximate the pay-off function with some function of the form x^γ , where $\gamma \in \mathbf{R}$ in L^2 -norm. For the case without Limited Liability, the pay-off function, X is directly approximated by X^1 . Therefore, the utility function becomes,

$$U_1(X) = X,$$

which represents the objective function without Limited Liability. In this case, we know that the solution exists when the amount invested in the risky asset has a cap and the amount invested in the risky asset plateaus to the upper-bound given by the constraint. In this context, Limited Liability means that the pay-off function cannot fall below a certain level, because if that happens, then the bank faces bankruptcy. In this chapter, we denote this bankruptcy level by F , that is, the payoff function in this case becomes $\max(F, X)$. In this case, the boundary function is a Lipschitz continuous function. We can approximate it with a C^2 function in the above-mentioned functions.

Later we show that the γ for the best approximated function lies in $(0,1)$. In this case, the optimization model is the same as the model mentioned in [50]. Then we find the optimal amount invested in the risky asset. Note that the expected terminal wealth is continuously dependent on the wealth level, and in turn, the wealth level is also continuously dependent on the amount invested in the risky asset. Hence, expected terminal wealth is continuously dependent on the amount invested in the risky asset. Therefore we can compare the objective functions with their approximated versions. Now, for this approximation, we need a cap on X , which is denoted by B , that is, the maximum wealth of the portfolio in one year ($T = 1$) can be B . Finally, we get,

$$U_2(X) = X^{\gamma^*}$$

which is the best approximation of $\max(F, X)$ in the above mentioned set. Then, in order to discuss the final effect, we use the DD concept from KMV Model. The DD defined as:

$$DD(\mu(t), D, \sigma(t), t) := \frac{\ln\left(\frac{V_A}{D}\right) + (\mu - \frac{1}{2}\sigma(t)^2)t}{\sigma(t)\sqrt{t}}.$$

Here, V_A denotes the initial wealth and D is the face value of the debt. When this DD decreases, the chance of default increases and hence the stability decreases. The interested reader may refer to [27] for a detailed description of this phenomenon.

5.3 Analysis

In this section we present the theoretical results of this study. To begin with, we study the relationship between the DD and the risk of the bank portfolio.

Theorem 5.3.1. *DD reduces when the holding in riskier security increases.*

Proof. Differentiating DD with respect to σ , we get,

$$\frac{\partial}{\partial \sigma}(DD) = \frac{-rt\sqrt{t} - \frac{\sigma^2 t \sqrt{t}}{2}}{(\sigma\sqrt{t})^2}.$$

Here $\mu(t) = r(t) + \sigma(t)\zeta(t)$ and $\zeta(t)$ represents the risk premium. From the above equation, we can see that $\frac{\partial}{\partial \sigma}(DD) < 0$, and hence it is clear that DD decreases as the $\sigma(t)$ increases. \square

An interesting question pertains to how does the final portfolio vary with the γ in the utility function $U(x) = x^\gamma$. In the following theorem, we have studied the relation between this γ and the final portfolio.

Theorem 5.3.2. *When γ in x^γ increases, then the holding in the risky asset increases.*

Proof. Proceeding on the lines of [50], we get the optimal portfolio as,

$$\pi^*(t) = \max \left(\frac{1}{1-\gamma} \cdot \frac{\zeta(t) + b\beta(t)}{\sigma(t)}, \delta \right),$$

where, $\beta(t) = \frac{\gamma}{\alpha} (1 - \exp(\alpha(t-T)))$. If the optimal holding plateaus at the maximum amount, then there is nothing to prove. Accordingly, let us consider the case when,

$$\pi^*(t) = \frac{1}{1-\gamma} \cdot \frac{\zeta(t) + b\beta(t)}{\sigma(t)}.$$

Now substituting this $\beta(t)$ in the equation, we get,

$$\pi^*(t) = \frac{1}{1-\gamma} \cdot \frac{\zeta(t) + b\frac{\gamma}{\alpha}(1 - \exp(\alpha(t-T)))}{\sigma(t)}.$$

Let $k(t) := \frac{b}{\alpha}(1 - \exp(\alpha(t-T)))$. Then substituting this $k(t)$ in the preceding equation, we get,

$$\pi^*(t) = \frac{1}{1-\gamma} \cdot \frac{\zeta(t) + \gamma k(t)}{\sigma(t)}.$$

Finally, differentiating $\pi^*(t)$ w.r.t γ , we get,

$$\begin{aligned} \frac{\partial \pi^*(t)}{\partial \gamma} &= \frac{(1-\gamma)\sigma(t)k(t) + \sigma(t)(\zeta(t) + \gamma k(t))}{(1-\gamma)^2\sigma^2(t)}, \\ &= \frac{\sigma(t)k(t) + \sigma(t)\zeta(t)}{(1-\gamma)^2\sigma^2(t)} > 0. \end{aligned}$$

Therefore, the optimal holding in the risky asset increases with increase in the γ of the utility function. \square

We now prove a Lemma which will be used to prove the next theorem.

Lemma 5.3.3. $g(x) = \left(\frac{x^2}{2} - x\right) \log x - \frac{x^2 - 4x}{4}$, $x > 0$ is an increasing function.

Proof. The proof is evident, since,

$$\frac{dg}{dx} = (x-1) \log x > 0.$$

This shows that the function is increasing in the given range. \square

We have considered the approximation of the pay-off function with x^γ , where $\gamma \in \mathbf{R}$. In the following theorem, we show that the best-approximated function (for the objective function, with Limited Liability) has $\gamma \in (0, 1)$.

Theorem 5.3.4. *The optimal approximation of the payoff function, with Limited Liability, is given by x^{γ^*} where $\gamma^* \in (0, 1)$.*

Proof. The error term in L^2 norm is given by:

$$\begin{aligned}\text{Err}(\gamma) &= \int_0^B (\max(F, x) - x^\gamma)^2 dx, \\ &= \int_0^F (F - x^\gamma)^2 dx + \int_F^B (x - x^\gamma)^2 dx.\end{aligned}$$

We know that $F < 1$ (Bankruptcy level) and $B > 1$ (Upper bound for net worth in the given timeline) are constants. Now differentiating the error term w.r.t γ , we get,

$$\frac{d(\text{Err}(\gamma))}{d\gamma} = -2 \int_0^F (F - x^\gamma) x^\gamma \log x dx - 2 \int_F^B (x - x^\gamma) x^\gamma \log x dx.$$

(A) For the case $\gamma = 1$, we get,

$$\begin{aligned}\frac{d(\text{Err}(\gamma))}{d\gamma} &= -2 \int_0^F (F - x) x \log x dx - 2 \int_F^B (x - x) x \log x dx, \\ &= -2 \int_0^F (F - x) x \log x dx.\end{aligned}$$

Now, in order to evaluate the integral, we apply the formula, $\int u dv = uv - \int v du$ with $u = \log x$ and $v = \frac{Fx^2}{2} - \frac{x^3}{3}$, which renders the integral as,

$$\begin{aligned}&\int_0^F (F - x) x \log x dx, \\ &= \left[\left(\frac{Fx^2}{2} - \frac{x^3}{3} \right) \log x \right]_0^F - \int_0^F \left(\frac{Fx^2}{2} - \frac{x^3}{3} \right) dx, \\ &= \frac{F^3}{6} \log F - \frac{5}{36} F^3 < 0.\end{aligned}$$

Since $F < 1$, hence $\log x$ is negative, and therefore,

$$\frac{d(\text{Err}(\gamma))}{d\gamma} = -2 \int_0^F (F - x) x \log x dx > 0.$$

So the optimal γ satisfies the condition $\gamma^* < 1$.

(B) For the case of $\gamma = 0$, we get,

$$\begin{aligned}
\frac{d(\text{Err}(\gamma))}{d\gamma} &= -2 \int_0^F (F-1) \log x dx - 2 \int_F^B (x-1) \log x dx, \\
&= -2 \left[[(F-1)x(\log x - 1)]_0^F + \left[\left(\frac{x^2}{2} - x \right) \log x - \frac{x^2 - 4x}{4} \right]_F^B \right], \\
&= -2 [(F-1)F(\log F - 1)], \\
&- 2 \left[\left[\left(\frac{B^2}{2} - B \right) \log B - \frac{B^2 - 4B}{4} \right] - \left[\left(\frac{F^2}{2} - F \right) \log F - \frac{F^2 - 4F}{4} \right] \right].
\end{aligned}$$

Note that, $g(x) = \left(\frac{x^2}{2} - x \right) \log x - \frac{x^2 - 4x}{4}$, is an increasing function. Hence $g(B) > g(F)$.

So the second term in the last equation is positive and the first term $[(F-1)F(\log F - 1)] > 0$.

Hence, we get,

$$\frac{d(\text{Err}(\gamma))}{d\gamma} < 0.$$

So, if γ increases, the error decreases at $\gamma = 0$. Hence the optimal γ satisfies the condition $\gamma^* > 0$.

Therefore, combining both these cases, we conclude that $0 < \gamma^* < 1$. □

All these theories presented above lead to the conclusion that the incorporation of Limited Liability leads to inclusion of less risky assets in the portfolio. We state this conclusion, in the form of the following corollary.

Corollary 5.3.5. *The maximum holding in the risky asset continuously changes the value of the objective function. The approximated function for the Limited Liability has lower γ , as compared to the objective function without Limited Liability. Hence, the above theorem implies that the model with Limited Liability incorporates less risky assets than the objective function without Limited Liability.*

5.4 An Example

We solve the optimization problem stated in (5.2.9) with $U_1(X)$ and $U_2(X)$, being the utility functions, for the case with Limited Liability and without Limited Liability, respectively. The risky loan has the probability of default $p = 0.1$ and the loss given default $\lambda = 0.6$. Therefore, we get the upper bound on the investment made on the risky bond as $\delta = 0.83$, with the total bound of expected loss being $EL = 0.05$. For this example, we have taken $F = 0.75$, the threshold level, below which the bank faces bankruptcy. We choose the bound of the wealth as $B = 1.2$, that is, using the bonds, the bank can produce a maximum gain of 20%. This is quite natural because of

the lower return of the bond. Using these parameter values, the best approximation for the pay-off function with limited liability is given by,

$$U_2(X) = X^{0.1821}.$$

Therefore, we have the two utility functions. Accordingly, we solve the optimization problem and compare the solutions. To solve the optimization problem, the other parameter values are taken as, $\alpha = 0.15$, $b = 0.67$, $\theta(t) = 0.0075$, $\zeta(t) = 0.3$, $T_1 = 1.5$, $T = 1$ and the IRB based capital requirement $k = 0.2$. Therefore, the capital structure of the firm is $\text{Capital}(t) \geq \max(0.04, k\pi(t))$, with 0.04 being the minimum capital requirement mandated by the supervisor.

From Figure 5.1, we see that when we solve the optimization problem, without Limited Liability, then the portfolio holds as much as the risky loan as possible. On the other hand, with Limited Liability, the portfolio holds less risky loans up to a finite time point, after which it plateaus at the maximum amount of risky loan. This happens because, as time progresses and approaches maturity, the volatility decreases, thereby behaving like a risk free asset. Hence it can be incorporated (near maturity) in the portfolio, without increasing the risk. Thus this result is consistent from the financial point of view.

From Figure 5.2 we observe that the DD decreases, as the total risk of the portfolio increases. As the DD decreases, the chances of default increase and the firm becomes less stable. On the other hand, as the time approaches maturity, the volatility is reduced. Consequently, there is an increment in the DD , which causes better resiliency of the bank.

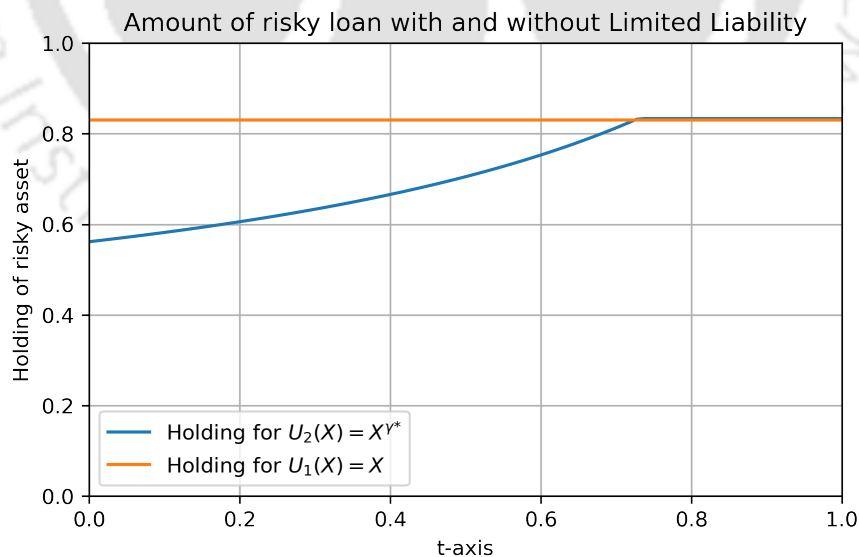


Figure 5.1: Weight of risky loan with and without Limited Liability

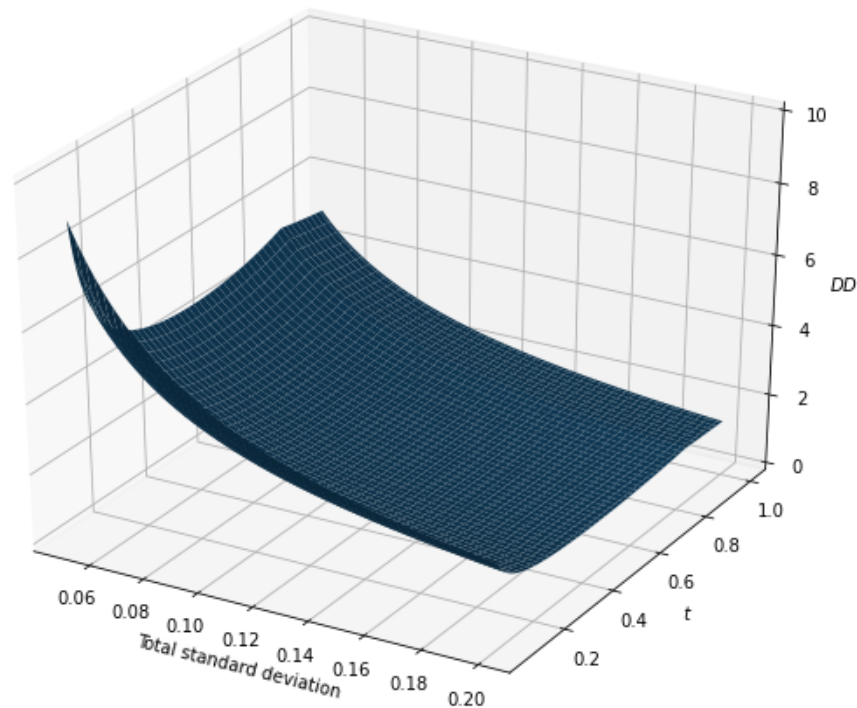


Figure 5.2: Distance of default

5.5 Summary

Bank's decision problems are usually constructed using a traditional return-risk framework. For a discrete time setup, literature shows that including Limited Liability in the decision model can reduce the risk in the final portfolio. Having said so, the effect of the inclusion of Limited Liability also needs to be considered in the continuous-time step model. To this end, this article focuses on incorporating Limited Liability into the bank's decision problem.

In this work, we have shown the benefit of combining the Limited Liability in the decision problem. In particular, we have constructed two models to maximize the utility of the terminal wealth, one of which is with Limited Liability and the other is without Limited Liability. The case with Limited Liability is a continuous but non-differentiable function. Therefore, in order to show the effect of the inclusion of Limited Liability, we have approximated it with a C^2 -function, to determine the optimal strategy. Finally, we have proved that the model with Limited Liability leads to less incorporation of risky assets. Banks and other financial firms are subjected to Limited Liability protection and hence the incorporation of Limited Liability reflects a more real-world scenario.

Chapter 6

Conclusion and Future Research

In this thesis, we have addressed the problem of constructing a bank loan portfolio incorporating Limited Liability in the reward function. We have considered reward maximization models and risk minimization problems with Limited Liability. In the modeling approach, we focused on constructing the portfolio, incorporating regulatory aspects like capital requirements, liquidity constraints, etc. We have studied these through discrete time and continuous time model setups.

We have developed optimization problems, with and without Limited Liability. We compared the results to show the effect of incorporating Limited Liability, wherein Expected Loss plays the role of the risk measure, in the problem formulation. We derive the optimized portfolio of the return maximization model and the risk minimization model and make a comparative analysis. The theoretical results show that a portfolio obtained from the model incorporating Limited Liability produces better results in terms of risk management and as well as expected excess return.

We have also considered the liquidity risk in the portfolio model construction. We have taken a three time step model to show the effect of the liquidity. The portfolio obtained from the model that maximizes excess return with a haircut constraint (loss due to liquidation) incorporates less illiquid assets than the portfolio obtained from the model without a haircut constraint. We have also proposed a liquidation strategy which minimizes the liquidation cost, with a risk lower bound on the liquidated portfolio. This, in turn, can be used to control the liquidation of the safe assets thereby facilitating an increase in the stability of the banks.

We have examined the case when a bank can simultaneously raise equity and debt, in addition to selling and buying new assets. To this end, we have developed a three time step model. In the intermediate time step, the bank raises new capital and liquidates assets. Here, we have considered the contagion risk, that is, when the riskier loan defaults, the safer loans also default. We have derived the amount of debt and equity that can be raised in a given scenario.

Finally, we have presented a continuous time model to consider the effect of the inclusion of Limited Liability. Accordingly, we have considered an approximated function for the objective

function with Limited Liability. The results show that the incorporation of Limited Liability prevents the inclusion of risky assets in the optimized portfolio.

The results from chapter 3 facilitate the broadening of the path to solve the problem for a multiple number of intermediate time steps. Instead of deterministic intermediate time steps, they can also be random. In that case, another probability distribution can be added to capture the intermediate time steps in the path of financial transactions. The protection of debt holders can also be considered in the model. Another important thing is that the fraction of deposit withdrawals is considered constant. While, in principle it can be viewed as a random variable, we must capture it in the model by some other probability distribution function. A comprehensive treatment of these aspects remains an important direction for future research.

An extension of chapter 4 can also be made, where the withdrawal time and the fraction of debt to be withdrawn can also be random. The loan can be priced using a derivative pricing technique. In this case, the amount of debt and equity raised as well as liquidated assets can depend on the withdrawal time and amount. Incorporating these properties opens another direction of research we would like to explore.

Finally, chapter 5 has multiple directions of future research possibilities. This optimization problem can be solved by using the viscosity solution technique. Considering the deposit withdrawals in conjunction with a liquidity loss (on selling the bank's assets at any intermediate time step) can lead to another possible research direction. In this case, we can consider the volatility of the bank's assets and adopt the Almgren-Chriss strategy for liquidating bank assets, in the event that liquidation is necessary to meet the depositors' claims, which also opens a path of further exploration.

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List of Accepted or Communicated Papers

Based on the work in this thesis, the following research articles have been accepted or communicated.

1. Deb Narayan Barik and Siddhartha P. Chakrabarty, Can Limited Liability Increase stability for banks: A dynamic portfolio approach. (Communicated)
2. Deb Narayan Barik and Siddhartha P. Chakrabarty, Dynamic loan portfolio management in a three time step model. (Communicated)
3. Deb Narayan Barik, and Siddhartha P. Chakrabarty, Loan portfolio management and Liquidity Risk: The impact of Limited Liability and haircut, *Journal of Quantitative Economics*, pp. 1-22, 2025.
4. Deb Narayan Barik and Siddhartha P. Chakrabarty, Does Limited Liability reduce leveraged risk?: The case of loan portfolio management, *Journal of Risk and Financial Management*, vol. 15, no. 11, pp. 519, 2022.