ON NONLOCAL PURELY CRITICAL AND SUPERCRITICAL PROBLEMS AND SUPERLINEAR SEMIPOSITONE PROBLEMS

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ON NONLOCAL PURELY CRITICAL AND SUPERCRITICAL PROBLEMS AND SUPERLINEAR SEMIPOSITONE PROBLEMS

by

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To My Family

Certificate

This is to certify that the thesis entitled **"On nonlocal purely critical and supercritical problems and superlinear semipositone problems"** submitted by **Mr. Uttam Kumar** to the Indian Institute of Technology Guwahati, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

The main objective of this thesis is to examine purely critical and supercritical exponent problems involving nonlocal operator in the symmetric domain. The nonlocal superlinear semipositone problem is also investigated.

In the first chapter, we talk about the rationale behind writing the thesis as well as the primary goals that it aims to achieve. After that, we provide a precise summary of the most important aspects of our primary difficulties as well as the significance of our works.

The preliminaries are presented in the second chapter. The fractional Sobolev spaces and embedding results are discussed. We look at how different solutions concepts, such as viscosity and weak solutions, are connected. We also examine topological techniques like genus theory used in later chapters.

In the third chapter, we establish Struwe's type global compactness result for the following critical exponent problem involving fractional p-Laplace operator in a bounded domain $\Omega \subset \mathbb{R}^N$ which is invariant under the action of a group G of orthogonal transformations

$$(P^s_{p,Q,\Omega}) \begin{cases} (-\Delta)^s_p u &= Q(x)|u|^{p^*_s-2}\,u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\qquad\quad \text{on } \Omega^c, \end{cases}$$

where $0 < s < 1$, $1 < p < \infty$ such that $sp < N$, $p^*_s := \frac{Np}{N-ps}$ is fractional critical Sobolev exponent, $Q(x)$ is continuous and strictly positive in $\overline{\Omega}$. We provide a detailed description of all G-invariant Palais Smale sequences for the energy functional associated with the problem $(P^s_{p,Q,\Omega})$.

In the fourth chapter of the thesis, we study Coron's type problems involving nonlocal operators with critical nonlinearities.
In the first part of this chapter, we consider the following problem

$$(P^s_\Omega)\begin{cases} (-\Delta)^s u &= |u|^{2^*_s-2}\,u \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad \text{on } \Omega^c, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ be a bounded annular domain which is invariant under a group G of orthogonal transformations of $\mathbb{R}^N$. We establish the existence of a positive and multiple sign-changing solutions to the problem (P^s_Ω).
In the second part of the fourth chapter, we show the existence of a positive and multiple sign-changing solutions to problem $(P^s_{p,\Omega})$

$$(P^s_{p,\Omega})\begin{cases} (-\Delta)^s_p u &= |u|^{p^*_s-2}\,u \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad \text{on } \Omega^c, \end{cases}$$

in some bounded domain $\Omega \subset \mathbb{R}^N$ with nontrivial topology under some symmetry assumptions.

The fifth chapter of the thesis is dedicated to the study of supercritical exponent problem involving fractional Laplace operator. We consider the following nonlocal problem

$$(P^s_{b,\Omega})\begin{cases} (-\Delta)^s u &= b(x)|u|^{q-2}\,u \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where Ω be a bounded domain in $\mathbb{R}^N$ with some symmetry assumptions, $N \geq 2s$, $s \in (0,1)$, $b \in C^{0,\alpha}(\overline{\Omega})$ and positive, $q > 2^*_s$. Here, we show the existence of a positive and multiple sign-changing solutions to problem $(P^s_{b,\Omega})$.

The sixth chapter deals with study of nonlocal superlinear subcritical semipositone problem. We prove the existence of a positive solution to the following Dirichlet boundary value problem

$$
\begin{cases}
(-\Delta)_p^s u &= \mu(u^r - 1) \text{ in } \Omega, \\
u &> 0 \qquad\qquad \text{in } \Omega, \\
u &= 0 \qquad\qquad \text{on } \Omega^c,
\end{cases}
$$

where Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary, $p - 1 < r < p_s^* - 1$, $\mu > 0$ is a parameter.

Finally, the last chapter of the thesis deals with study of nonlocal superlinear critical semipositone problem. We show the existence of a positive solution to the following Dirichlet boundary value problem

$$
\begin{cases}
(-\Delta)_p^s u &= u^{p_s^* - 1} - \mu \text{ in } \Omega, \\
u &> 0 \qquad\qquad \text{in } \Omega, \\
u &= 0 \qquad\qquad \text{on } \Omega^c,
\end{cases}
$$

where Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary and $\mu > 0$ is a parameter.

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List of Symbols

Symbol	**Meaning**
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^N$	N-dimesional Euclidean space
Ω	open subset in $\mathbb{R}^N$
Ω^c	Complement of Ω in $\mathbb{R}^N$
$\lvert A \rvert$	Lebesgue measure of the set A
$\overline{A}$	Closure of the set A
$o_k(1)$	$\lim\limits_{k \to +\infty} o_k(1) = 0$
$f^+(x)$	$\max\{f(x), 0\}$
p'	Conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$
$\rightharpoonup$	Weak convergence
$\hookrightarrow$	Continuous embedding
$\hookrightarrow\hookrightarrow$	Compact embedding
$B(x, r)$	Ball of radius r centered at x
C_0^∞	Set of infinitely differentiable functions with compact support
$L^p(\Omega)$	Lebesgue space
$W^{s,p}(\Omega)$, $D^{s,p}(\Omega)$	Fractional Sobolev spaces with order s and exponent p
$W_0^{s,p}(\Omega)$	Closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\mathbb{R}^N)$
DI	Derivative of the given functional I

Symbol	**Meaning**
$D_0^{s,p}(\Omega)$	Closure of $C_0^\infty(\Omega)$ in $D^{s,p}(\mathbb{R}^N)$
$W_0^{-s,p'}(\Omega)$	Topological dual of $W_0^{s,p}(\Omega)$
$D_0^{-s,p'}(\Omega)$	Topological dual of $D_0^{s,p}(\Omega)$
$\|u\|$	Norm of u in $D_0^{s,p}(\Omega)$
X^*	Topological dual space of any norm linear space X
$\|u\|_{L^p(\Omega)}$	Norm of u in $L^p(\Omega)$
$\|u\|_{W_0^{s,p}(\Omega)}$	Norm of u in $W_0^{s,p}(\Omega)$
$\|u\|_{D_0^{-s,p'}(\Omega)}$	Norm of u in $D_0^{-s,p'}(\Omega)$
$\|u\|_{W_0^{-s,p'}(\Omega)}$	Norm of u in $W_0^{-s,p'}(\Omega)$
$\|u\|_X$	Norm of u in Banach space X
$\langle \cdot, \cdot \rangle_X$	Duality pair between any norm linear space X and its dual X^*
Δ	Laplacian
Δ_p	p-Laplacian
$(-\Delta)^s$	Fractional Laplacian
$(-\Delta)_p^s$	Fractional p-Laplacian

1 Introduction

1.1 Overview

The study of variational problems using nonlocal operators has received a lot of attention in the recent years. In contrast to the standard Laplacian, which operates by pointwise differentiation, these operators are frequently specified by global integration, allowing them to explain processes such as diffusion in the presence of long-range interactions. In this context, a model operator is the fractional Laplacian $(-\Delta)^s$, for $0 < s < 1$, (see[57]) which is defined as

$$(-\Delta)^s u(x) := P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \qquad x \in \mathbb{R}^N, \tag{1.1.1}$$

up to a normalized constant. Here, $P.V.$ stands for Cauchy principal values and it is defined as

$$P.V.\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|\,x-y\,|^{N+2s}}\,dy.$$

In the literature, fractional Laplacian is defined in many different ways (see [69]). The fractional Laplacian defined in (1.1.1) is known as integral fractional Laplacian (or also as Riesz fractional Laplacian).

Another representation of (1.1.1) is the spectral fractional Laplacian. The fractional Laplacian $(-\Delta)^s$ is the fractional power of positive Laplace operator $-\Delta$, in the bounded domain Ω with zero Dirichlet boundary data defined by its spectral decomposition using the powers of eigenvalues of the original operator. Let $\{\varphi_j, \lambda_j\}_{j\in\mathbb{N}}$ be the set of eigenfunctions and eigenvalues of the following Dirichlet problem

$$-\Delta u = \lambda\, u \quad \text{in } \Omega, \quad u = 0 \quad \text{ on } \partial\Omega,$$

then $\{\varphi_j, \lambda_j^s\}_{j\in\mathbb{N}}$ is the set of eigenfunctions and eigenvalues of the corresponding fractional problem

$$(-\Delta)^s u = \lambda\, u \quad \text{in } \Omega, \quad u = 0 \quad \text{ on } \partial\Omega.$$

Let

$$\widetilde{H}_0^s(\Omega) := \left\{ u = \sum_{j\in\mathbb{N}} a_j\varphi_j \in L^2(\Omega) : \|u\|_{\widetilde{H}_0^s(\Omega)} = \left(\sum_{j\in\mathbb{N}} a_j^2\lambda_j^s\right)^{\frac{1}{2}} < \infty \right\}.$$

We denote by $\widetilde{H}^{-s}(\Omega)$ the dual space of $\widetilde{H}_0^s(\Omega)$. For $u \in \widetilde{H}_0^s(\Omega)$, $u = \sum_{j\in\mathbb{N}} a_j\varphi_j$ with $a_j = \int_\Omega u\varphi_j\,dx$, the fractional power of Dirichlet Laplacian $(-\Delta)^s$ is defined as

$$(-\Delta)^s u = \sum_{j\in\mathbb{N}} a_j\lambda_j^s\varphi_j \ \in \widetilde{H}^{-s}(\Omega). \tag{1.1.2}$$

One more way to define fractional Laplacian $(-\Delta)^s$ of $u(x)$ in $\mathbb{R}^N$ requires the solution of elliptic equation involving the standard Laplacian in $N+1$ dimensional half plane. This is called the extension method or the Dirichlet-to-Neumann operator method. It requires the solution of a degenerate elliptic equation in the half-plane using $u(x)$ as Dirichlet boundary data, followed by a sort of normal derivative of the solution. We illustrate this method which is based on the following result from [29]. Given a function $u(x)$ on $\mathbb{R}^N$, consider the extension $v(x,t)$ on $\mathbb{R}^N \times (0,\infty)$ that solves

$$\left.\begin{aligned} \operatorname{div}(t^{1-2s}\nabla v) &= 0 \\ v(x,0) &= u(x). \end{aligned}\right\} \tag{1.1.3}$$

Then

$$C(-\Delta)^s u(x) = \lim_{t\to 0} \frac{v(x,t)-v(x,0)}{t^{2s}}, \tag{1.1.4}$$

for some constant C which depends on N and s. In the bounded domain Ω, this method holds in the case of spectral fractional Laplacian for functions with zero Dirichlet boundary conditions, the half-plane being substituted by a cylinder $\mathcal{C} = \Omega \times (0,\infty)$ over the original domain Ω.

The integral and extension method representations of fractional Laplacian in $\mathbb{R}^N$ are equivalent but not in any bounded domain $\Omega \subset \mathbb{R}^N$ (see for more details [73]). The various representations of fractional Laplacian require different boundary conditions on bounded domain. We consider the following prototype problem involving fractional Laplace operator

$$(-\Delta)^s u(x) = f(x,u) \quad \text{in } \Omega, \tag{1.1.5}$$

where Ω be a bounded domain in $\mathbb{R}^N$. The Riesz fractional Laplacian requires exterior boundary conditions on $\mathbb{R}^N \setminus \Omega$, while the spectral fractional Laplacian admits local boundary condition on $\partial\Omega$. In the last decade, equations involving fractional Laplacian have been the main subject of investigation in many of the research works. Caffarelli et al. [27] studied free boundary value problems involving fractional Laplacian. Chang and González [34] investigated fractional Laplace operator in conformal geometry. Silvestre et al. [28] obtained regularity results of the obstacle problem involving fractional Laplacian. For more details one may refer to [16] and the references therein.

Over the last several decades, the problems involving quasilinear elliptic partial differential operators have been the center of attraction. One such example is the p-Laplacian operator which is nonlinear generalization of Laplace operator. For $1 < p < \infty$, p-Laplace operator (see [71]) is defined as

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u), \tag{1.1.6}$$

where the $|\nabla u|^{p-2}$ is defined as

$$|\nabla u|^{p-2} := \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \cdots + \left(\frac{\partial u}{\partial x_N} \right)^2 \right]^{\frac{p-2}{2}}.$$

The operator reduces to the Laplace operator for $p = 2$.
Let Ω be a bounded domain in $\mathbb{R}^N$. Problems of the type

$$\left. \begin{aligned} -\Delta_p u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned} \right\} \tag{1.1.7}$$

for various types of nonlinearities f, have been the main focus for researchers in the past few decades. The critical power $f(x,u) = u^{p^*-1}$, for $N > p$, and $p^* = \frac{Np}{N-p}$, is the most important cases of the problem (1.1.7) as the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is not compact and hence the standard variational techniques to investigate problems with critical nonlinearities can't be used.

For the case $p = 2$, with critical power $f(x,u) = u^{2^*-1}$ and for $N > 2$, Pohožaev [86] showed the non-existence of positive solution in a starshaped domain. In the seminal paper [25], Brézis and Nirenberg established that the critical problem with small linear perturbation yield positive solutions. In [2], Ambroseti et al. proved some results on existence and multiplicity of solutions for a sublinear perturbation of the critical power. Coron [48] showed the existence of a positive solution to critical exponent problem in the annulus having small hole. In the pioneering work [4], Bahri and Coron established the existence of at least one positive solution to critical problem in every domain Ω having nontrivial reduced homology with $\mathbb{Z}/2$-coefficients.
Critical exponent problems involving p-Laplacian have discussed in the literature [59, 63, 77] and the references therein.

On the other hand, the existence and multiplicity of sign-changing solutions or nodal solutions of classical elliptic boundary value problems also have been widely investigated in the past decades, one can see [8] and references therein. In [92], Struwe studied the existence of infinitely many solutions of the boundary value problem and existence of periodic solutions of the ordinary differential equation

$$x''(t) + f(t, x, x') = 0, \tag{1.1.8}$$

where the growth of the function f with respect to the variable x is faster than linear. Here, the author presented an example by taking $f(t, x, x') = x|x|^{k-1},\ k > 1$ in equation (1.1.8) and showed the existence of infinitely many sign-changing solutions. In [7], Bartsch studied these types of results in the case of semilinear elliptic equations with subcritical nonlinearity. The author developed the abstract critical point theory to study nodal solutions to the semilinear elliptic problems.

Some elliptic equations involving various operators such as Laplace operator ([14, 74]), p-Laplace operator ([11, 13]), and Schrödinger operator ([12]), sign-changing solutions have been achieved by using min-max method with invariant set of descending flow.

It is a well-known fact that the lack of compactness in elliptic problems, which are invariant under the translations or dilations leads to interesting phenomena. Specifically, it results in an effect of the topology of domain on the number of solutions of suitable perturbations of such problems (see for a detailed discussion [23, 95]). In [39], Clapp studied how the symmetries of the domain affect the lack of compactness. Here author assumed that the domain is invariant under the action of some groups of orthogonal transformations of $\mathbb{R}^N$. Clapp proved that lack of compactness can only occur if domain contains some finite G-orbit.

Clapp and Pacella [44] established the existence of a positive solution and multiple sign-changing solutions to purely critical exponent problems involving Laplacian operator in the domains with the arbitrary size of holes. In this paper, the authors made symmetry assumptions on the domain and demonstrated that the Palais-Smale condition holds for higher energy levels. Using this fact, they showed the existence of multiple solutions to semilinear elliptic problems with purely critical nonlinearities in the domains with the arbitrary size of the hole. In [40], the authors showed the existence of multiple solutions to the Bahri-Coron problem in some domains with nontrivial topology under some symmetry assumptions.

Pacella et al. [76] investigated the existence of multiple solutions to purely critical exponent problems involving p-Laplacian operator in an annular-shaped domain under some symmetry assumptions.

For the case $f(x,u) = u^q$ with $q > p^* - 1$, the problem (1.1.7) is termed as supercritical exponent problem. In the past decade, purely supercritical exponent problems involving semilinear elliptic operators have been widely explored in specific types of domains (for a detailed discussion see [42, 45]).

In [41], the authors studied the existence of multiple solutions to anisotropic critical and supercritical problems in symmetric domains. Here, the authors established the results for particular classes of domains, which either admit a map onto a domain of lower dimension preserving the Laplace operator or achieved by rotating some domain Θ of lower dimension around some linear subspace of $\mathbb{R}^N$. This allowed to reduce the supercritical exponent problem to anisotropic subcritical or critical problems in the domain Θ. In [46], the authors studied the existence of multiple solutions to supercritical exponent problems involving p-Laplacian operator.

Since 1967, researchers have focused on the study of nonlinear eigenvalue problem of the form

$$\left.\begin{aligned} -\Delta u &= \mu f(u) \text{ in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1.9}$$

where $\mu > 0$ is a positive parameter.

(1.1.9) is referred to as positone problem in the literature when f is positive and monotone. One such example is when one takes $f(u) = e^u$.

However, we are interested in the case where f satisfies

$$f(0) < 0, \ f \text{ is monotone and eventually positive,} \tag{1.1.10}$$

which is referred as semipositone problem in the literature. The study of positive solutions

to semipoistone problems is significantly more difficult, because a solution's range must include both negative and positive f regions. These issues were examined in [72], and it was realized that they provided significant hurdles, which was subsequently confirmed by H. Berestycki, L.A. Caffarelli and L. Nirenberg in [15]. In 1988, Castro and Shivaji (see [33]) formally introduced semipositone problems with Dirichlet boundary conditions, where various problematic differences were noticed in their investigation when compared to the study of positone problems. The initial effort of Castro and Shivaji in [33] has led to many works in the recent years.
For the recent trends in the problems with semipositone nonlinearities involving semilinear and quasilinear elliptic operators, we refer to [38, 50, 64] and the references therein.

Recently, there has been increasing attention on studying the non-linear problems involving the fractional p-Laplacian operators, both in pure mathematical research and in real-world applications, such as population dynamics, phase transition phenomenon, obstacle problem, image processing, mathematical finance, game theory and so on. We refer the reader to [3, 26, 31, 51, 58, 84, 94] and the references therein.
For $0 < s < 1$ and $1 < p < \infty$, fractional p-Laplacian (see [57]) is defined as

$$(-\Delta)_p^s u(x) := P.V. \int_{\mathbb{R}^N} \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+sp}} dy, \quad x \in \mathbb{R}^N, \tag{1.1.11}$$

up to a normalized constant. $P.V.$ is defined as

$$P.V. \int_{\mathbb{R}^N} \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+sp}} dy = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x,\epsilon)} \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+sp}} dy$$

In the case when $p = 2$, the operator reduces to fractional Laplace operator.

Consider the following prototype of nonlocal equation:

$$\left.\begin{aligned} (-\Delta)_p^s u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \Omega^c, \end{aligned}\right\} \tag{1.1.12}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. The celebrated work of Nezza et al. [57]

provides the necessary functional set up to study these nonlocal problems using variational method.

Problems involving fractional p-Laplacian are extensively studied by many researchers (see [36, 61, 80, 81]). Here the authors have studied various aspects, viz, existence, multiplicity and regularity of solutions to the problems involving fractional p-Laplacian operator.
A purely critical exponent problem involving a fractional p-Laplacian operator does not appear to have been investigated as of yet, as far as we are aware. Additionally, the purely supercritical exponent problem that involves the fractional Laplacian operator has not been explored up to this point.
Recently, Dhanya et al. [56] studied the existence of positive solutions to semipositone problem involving fractional Laplacian. As far as we know, this is the only one article available related to semipositone problem involving fractional Laplacian.

1.2 Objective and a brief outline of the thesis

The primary goal of this thesis is to investigate purely critical and supercritical exponent problems in the symmetric domain involving nonlocal operator. We also study the nonlocal superlinear semipositone problem.
The works carried out in the thesis are presented in seven chapters. Now we present a concise overview of the thesis and the significance and originality of our findings.

1.2.1 Preliminaries

In the second chapter, we present the preliminaries. We recall the fractional Sobolev spaces, which describe the solution spaces for the nonlocal problems. The various features of solution spaces and embedding results are discussed. We explore different notions of solutions such as viscosity and weak solutions to problems and how they are related. We go through the various inequalities that have been employed throughout the thesis. We also recall the important concepts and the results of topological tools like genus theory that have been used in later chapters.

1.2.2 A Global compactness result in symmetric domain

In the third chapter, we establish Struwe's type global compactness result for the following nonlocal critical exponent problem in a bounded domain $\Omega \subset \mathbb{R}^N$ under some symmetry assumptions

$$(P^s_{p,Q,\Omega}) \begin{cases} (-\Delta)^s_p u &= Q(x)|u|^{p^*_s-2}u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\qquad \text{on } \Omega^c, \end{cases}$$

where $0 < s < 1$, $1 < p < \infty$ such that $sp < N$, $p^*_s := \frac{Np}{N-ps}$ is fractional critical Sobolev exponent, Q is continuous and strictly positive in $\overline{\Omega}$.

Problems involving critical exponent have always been of immense interest for research. In his pioneering work in [93], Struwe studied the non compactness of the embedding of $H^1(\Omega)$ into $L^{2^*}(\Omega)$, arising due to the translation and dilation invariance, in details and gave the splitting of the Palais-Smale sequence for the energy functional $J : D^{1,2}_0(\Omega) \to \mathbb{R}$, given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}$$

associated with the critical exponent problem

$$-\Delta u = |u|^{2^*-2}u \text{ in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $2^* = \frac{2N}{N-2}$ is critical Sobolev exponent. This result finds application in studying ground state solutions for nonlinear Schrödinger equations, Yamabe-type equations, minimization problem etc.

The Struwe's type global compactness results are studied in [83] for the critical exponent problems involving the fractional Laplacian operator and in [22], for the critical exponent problems involving the fractional p-Laplacian operator.

The main obstacle in treating problem $(P^s_{p,Q,\Omega})$ by using standard variational techniques is the non-compactness of the embedding $D^{s,p}_0(\Omega) \hookrightarrow L^{p^*_s}(\Omega)$. In addition to this, the classification of all positive solutions to the critical problem $(P^s_{p,Q,\Omega})$ in $\mathbb{R}^N$ is also not available. Another issue is the non-existence of sign-changing solutions to problem $(P^s_{p,Q,\Omega})$ in a half-space which is not available for $p \neq 2$. This problem appears as a limiting problem to $(P^s_{p,Q,\Omega})$ and must be taken into account when describing the lack of compactness. We impose some symmetry on the domain which allow us to handle the lack of compactness of the associated energy functional.

1.2.3 Multiple solutions to nonlocal critical exponent problem in symmetric domain

The fourth chapter deals with Coron's type problem involving nonlocal operators. We say a domain $\Omega \subset \mathbb{R}^N$ is an annular shaped domain if

$$0 \notin \Omega,\ A_{R_1,R_2} := \{x \in \mathbb{R}^N : 0 < R_1 < |x| < R_2\} \subset \Omega, \quad 0 < R_1 < R_2.$$

The classical Coron's problem goes back to 1984 and says that the critical problem defined on an annular shaped domain Ω

$$\begin{aligned} -\Delta u &= u^{\frac{N+2}{N-2}} \text{ in } \Omega,\\ u &> 0 \quad\quad \text{ in } \Omega,\\ u &= 0 \quad\quad \text{ on } \partial\Omega, \end{aligned}$$

admits a positive solution provided that R_2/R_1 is too large, or we can say the size of the hole is too small.

Consider the following nonlocal purely critical exponent problem

$$(P^s_\Omega)\begin{cases} (-\Delta)^s u &= |u|^{2^*_s-2}u \quad \text{in } \Omega,\\ u &= 0 \qquad\qquad \text{on } \Omega^c, \end{cases}$$

where $2^*_s := \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent and $\Omega \subset \mathbb{R}^N$ is an annular shaped bounded domain which is invariant under a group G of orthogonal transformations of $\mathbb{R}^N$.

We observe that analogous to the local case, the topology of the domain affects the solution of the nonlocal critical exponent problem. In [88], Ros-Oton and Serra proved the non-existence of positive solutions for the purely nonlocal critical exponent problem involving the fractional Laplacian operator $(-\Delta)^s$ in the star-shaped domain.

Secchi et al. [89] studied the Coron problem for fractional Laplacian. They presented nontrivial solution to a purely critical exponent problem in an annular domain with a sufficiently small hole. Chtioui et al. [1] proved the fractional counterpart of the Bahri-Coron Theorem [4]. They showed the existence of a nontrivial solution in domains with nontrivial topology. Nevertheless, the hypothesis of nontrivial topology is not necessary to get the solution as Shioji et al. [79] got the solution in a contractible domain. So the complete characterization of the domains for which solution exists to the nonlocal critical exponent problem is not known. Even this complete characterization of domains is still unknown for the local cases.

Recently the sign-changing solutions for the problems involving fractional p-Laplacian have been explored. For sub-critical exponent, Chang et al. [35] have shown the existence of infinitely many sign-changing solutions to fractional p-Laplacian equations by using the methods of invariant sets of descending flow and min-max theory.

In [32], Capella discussed the solutions of critical exponent problem involving half-Laplacian in annular-shaped domains.

We show the existence of a positive solution and multiple sign-changing solutions to the semilinear critical exponent problem (P^s_Ω) in an annular-shaped domain $\Omega \subset \mathbb{R}^N$, invariant under a group G of orthogonal transformation of $\mathbb{R}^N$. Here, we establish the multiplicity results for the domain with a very small hole. Moreover, the multiplicity results for a domain with the arbitrary size of holes are also shown.

In the second part of this chapter, consider the following purely critical exponent problem involving fractional p-Laplacian

$$(P^s_{p,\Omega}) \begin{cases} (-\Delta)^s_p u &= |u|^{p^*_s-2}\, u \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad \text{on } \Omega^c, \end{cases}$$

where $p^*_s := \frac{Np}{N-ps}$ is fractional critical Sobolev exponent and $\Omega \subset \mathbb{R}^N$ is a bounded domain with non-trivial topology, which is invariant under a group G of orthogonal transformations of $\mathbb{R}^N$.

We establish the existence of a positive and multiple sign-changing solutions to a critical problem $(P^s_{p,\Omega})$ in domains with some non-trivial topology under some symmetry assumptions.

The main ideas to handle to the problems in this chapter are as follows. First, we use the global compactness result to show that the Palais-Smale condition holds below a certain energy level. Then, we construct the multiple sign-changing critical points by constructing different invariant sets of descending flow defined by a pseudo-gradient vector field of the associated energy functional to the problem $(P^s_{p,\Omega})$ and using the topological tool, the genus, for these invariant sets.

1.2.4 Multiple solutions to nonlocal supercritical exponent problem in symmetric domain

The fifth chapter deals with the study of purely supercritical exponent problem involving fractional Laplacian operator in domains of revolutions, under some symmetry assumptions. We consider the the following fractional supercritical exponent problem

$$(P^s_{b,\Omega}) \begin{cases} (-\Delta)^s u &= b(x)|u|^{q-2}\,u \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2s$, $s \in (0,1)$, $b \in C^{0,\alpha}(\overline{\Omega})$ and positive, $q > 2^*_s$ where $2^*_s = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent.

We consider the cylinder $\mathcal{C}_\Omega$ associated with the bounded domain Ω defined as

$$\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$$

and its lateral boundary

$$\partial_L\mathcal{C}_\Omega = \partial\Omega \times (0,\infty).$$

The points in $\mathcal{C}_\Omega$ are denoted by (x,t) where $x \in \Omega$ and $t \in (0,\infty)$.
Using the s-harmonic extension introduced by Luis Caffarelli [29], we convert the nonlocal problem $(P^s_{b,\Omega})$ into a local problem

$$(P^s_{b,\mathcal{C}_\Omega})\begin{cases} -\mathrm{div}(t^{1-2s}\nabla v) &= 0 \quad \text{in } \mathcal{C}_\Omega,\\ v &= 0 \quad \text{on } \partial_L\mathcal{C}_\Omega,\\ \partial^s_\nu v &= b(x)|u|^{q-2}u \text{ on } \Omega\times 0.\end{cases}$$

The weak solution to the problem $(P^s_{b,\mathcal{C}_\Omega})$ is the critical point of the energy functional which is defined as

$$J_\Omega(v) = \frac{c_s}{2}\int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla v|^2\,dx\,dt - \frac{1}{q}\int_\Omega b(x)|u(x)|^q\,dx.$$

If $v(x,t)$ satisfies $(P^s_{b,\mathcal{C}_\Omega})$ then $u = v(\cdot,0)$, defined in sense of traces, is a solution to problem $(P^s_{b,\Omega})$.
As the problem $(P^s_{b,\mathcal{C}_\Omega})$ is a supercritical exponent problem, we convert it into an anisotropic critical or subcritical exponent problem by rotating some domain Θ of lower dimension which is base of cylinder $\mathcal{C}_\Theta$ around some linear subspace of $\mathbb{R}^N$, preserving the extended operator. We have following anisotropic critical exponent problem

$$(P^s_{a,b,\mathcal{C}_\Theta})\begin{cases} -\mathrm{div}(t^{1-2s}a(x)\nabla v) &= 0 \quad \text{in } \mathcal{C}_\Theta,\\ v &= 0 \quad \text{on } \partial_L\mathcal{C}_\Theta,\\ \partial^s_\nu v &= b(x)|u|^{2^*_s-2}u \text{ on } \Theta\times 0.\end{cases}$$

Here, we first establish the Struwe's type global compactness result for the problem $(P^s_{a,b,\mathcal{C}_\Theta})$. Precisely, we study the splitting of G-invariant Palais-Smale sequence of the energy functional associated with the problem $(P^s_{a,b,\mathcal{C}_\Theta})$. Then using this result we establish the compactness of Palais-Smale sequence at higher energy levels. Next we show the existence of multiple nodal solutions using the mountain pass theorem for sign-changing solutions.

1.2.5 Nonlocal superlinear semipositone problem with subcritical growth

The sixth chapter deals with the study of nonlocal superlinear subcritical semipositone problem. We consider the following problem

$$\left.\begin{aligned} (-\Delta)_p^s u &= \mu(u^r - 1) \text{ in } \Omega, \\ u &> 0 \qquad\qquad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \Omega^c, \end{aligned}\right\} \tag{1.2.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p-1 < r < p_s^* - 1$ and $\mu > 0$ be a positive parameter.

In contrast to positone problems, in which the strong maximum principle ensures the positivity of non-negative solutions, a semipositone problem can admit non-negative solutions having zeros in the interior of Ω even in the case of the Laplacian, as shown in [33]. Thus, the most commonly pursued existence theory using monotone iteration which demands a positive subsolution, can be challenging.

Initially, we construct a pointwise supersolution in the neighborhood of the boundary and then using this we obtain L^∞ a priori estimates for the positive viscosity solution. Then we apply the results from the degree theory and performing delicate analysis, obtain a positive solution to the problem (1.2.1).

1.2.6 Nonlocal superlinear semipositone problem with critical growth

In the last chapter, we investigate the superlinear critical semipositone problem involving nonlocal operator. Consider the following Dirichlet boundary value problem

$$\left.\begin{aligned} (-\Delta)_p^s u &= u^{p_s^*-1} - \mu \text{ in } \Omega, \\ u &> 0 \qquad\qquad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \Omega^c, \end{aligned}\right\} \tag{1.2.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\mu > 0$ is a positive parameter.

It is easy to show that the problem (1.2.2) has no solution for any $\mu > 0$ by employing the Pohozaev type identity for fractional p-Laplacian (see [75]). Therefore, we investigate the Brezis-Nirenberg type perturbation problem to (1.2.2), that is,

$$\left.\begin{aligned} (-\Delta)_p^s u &= \lambda u^{p-1} + u^{p_s^*-1} - \mu \text{ in } \Omega, \\ u &> 0 \qquad\qquad\qquad\quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\qquad\quad \text{on } \Omega^c, \end{aligned}\right\} \tag{1.2.3}$$

where $\lambda > 0$ is a positive parameter. Here, we prove the existence of ground state positive solution to the problem (1.2.3). Since it is a critical exponent problem, we employ concentration compactness arguments to show that the Palais-Smale condition holds below certain energy level. Subsequently, by obtaining uniform $C_d^\alpha(\overline{\Omega})$ ($\alpha \in (0, s]$) a priori estimates and performing subtle analysis, we get a positive solution to the problem (1.2.3).

Note: Throughout this thesis, C, c, c_i, C_i, K_i, $i \in \mathbb{N}$, are considered to be generic positive constants, which may vary from line to line and chapter to chapter.

2

Preliminaries

This chapter reviews the different fractional Sobolev spaces that serve as solution spaces to nonlocal problems. The various properties of fractional Sobolev spaces and the embedding results have also been examined. The different notions of solutions to nonlocal problems have been discussed. We recall the concept of symmetric domain and its related terminologies. We also recall the various results related to nonlocal operators which have been used throughout the thesis.

2.1 Fractional Sobolev spaces

In this section, we discuss the various fractional Sobolev spaces and their embedding results.

2.1.1 Functional spaces

Here we collect some known results about the fractional Sobolev spaces (see [16, 57]). Let $\Omega \subseteq \mathbb{R}^N$ be any open set. Then for $0 < s < 1$ and $1 < p < \infty$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \right\}$$

endowed with norm

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}.$$

Next, we recall the following embedding result for the space $W^{s,p}(\Omega)$:

Proposition 2.1.1. ([57]) *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary. Also, let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < N$. Then $W^{s,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ continuously, for $1 \le \gamma \le p_s^* := \frac{Np}{N-sp}$. Moreover, this embedding is compact for $\gamma < p_s^*$.*

Here p_s^* denotes the Sobolev type critical exponent in the fractional framework. The space $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)})$ is separable, reflexive, and uniformly convex Banach space (see [57]). Next, we state Sobolev-type inequality in the fractional framework.

Proposition 2.1.2. ([57]) *Let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < N$. Then there exists a positive constant $C = C(N,p,s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$, we have*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \le C \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{1/p}.$$

Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$.

2.1.2 Solution spaces

In order to study the Dirichlet boundary value data of nonlocal problem we consider the following spaces. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

equipped with the norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{1/p},$$

where $Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$. The set $C_0^\infty(\Omega)$ is dense in $W_0^{s,p}(\Omega)$ with respect to the norm $\|\cdot\|_{W_0^{s,p}(\Omega)}$ (see [16, 57]).

Proposition 2.1.3. ([57]) *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary and let $\gamma \in [1, p_s^*]$. Also, let $s \in (0,1)$ and $p \in (1,\infty)$ be such that $sp < N$. Then for $u \in W_0^{s,p}(\Omega)$, there exists a positive constant $C = C(N,p,s,\gamma)$ such that*

$$\|u\|_{L^\gamma(\Omega)} \leq C\|u\|_{W_0^{s,p}(\Omega)}.$$

Moreover, this embedding is compact for each $\gamma \in [1, p_s^*)$.

The space $(W_0^{s,p}(\Omega), \|\cdot\|_{W_0^{s,p}(\Omega)})$ is separable, reflexive, and uniformly convex Banach space (see [57]). The dual of the space $W_0^{s,p}(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$ with the norm $\|\cdot\|_{W^{-s,p'}(\Omega)}$, where $p' = \frac{p}{p-1}$ is the conjugate to p.

Now we recall the functional spaces required to study nonlocal critical problem. For a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, and for $sp < N$, we consider the space

$$D^{s,p}(\mathbb{R}^N) := \left\{u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{D^{s,p}(\mathbb{R}^N)} < \infty\right\}$$

endowed with Gagliardo seminorm $[\cdot]_{D^{s,p}(\mathbb{R}^N)}$ which is defined as

$$[u]_{D^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{1/p}.$$

Let $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary. We consider the following closed linear subspace of $D^{s,p}(\mathbb{R}^N)$ as the solution space.

$$D_0^{s,p}(\Omega) := \left\{ u \in D^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\},$$

which is a Banach space endowed with the norm

$$\|\cdot\| = [\cdot]_{D^{s,p}(\mathbb{R}^N)}, \tag{2.1.1}$$

as the embedding $D_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $1 \leq r \leq p_s^*$. Moreover that embedding is also compact for $1 \leq r < p_s^*$. We can also define the space $D_0^{s,p}(\Omega)$ as completion of $C_0^\infty(\Omega)$ in the norm $[\cdot]_{D^{s,p}(\mathbb{R}^N)}$, provided Ω has smooth boundary. We note that, for $u \in D_0^{s,p}(\Omega)$ and $1 \leq r \leq p_s^*$,

$$\int_{\mathbb{R}^N} |u(x)|^r\, dx = \int_\Omega |u(x)|^r\, dx.$$

and

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy = \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy$$

For a subset $A \subset \mathbb{R}^N$, we denote the localized Gagliardo seminorm by

$$[u]_{D^{s,p}(A)} := \left(\int_{A \times A} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{1/p}.$$

We define

$$S_{s,p} := \inf_{u \in D_0^{s,p}(\mathbb{R}^N)} \left\{ [u]_{D^{s,p}(\mathbb{R}^N)} : \|u\|_{L^{p_s^*}(\mathbb{R}^N)} = 1 \right\} \tag{2.1.2}$$

which is the sharp constant for in the Sobolev inequality for $D_0^{s,p}(\mathbb{R}^N)$, namely

$$S_{s,p} \|u\|_{L^{p_s^*}(\mathbb{R}^N)} \leq [u]^p_{D^{s,p}(\mathbb{R}^N)}, \tag{2.1.3}$$

for all $u \in D_0^{s,p}(\mathbb{R}^N)$.

The space $(D_0^{s,p}(\Omega), \|\cdot\|)$ is separable, reflexive, and uniformly convex Banach space.

The dual of the space $D_0^{s,p}(\Omega)$ is denoted by $D^{-s,p'}(\Omega)$ with the norm $\|\cdot\|_{D^{-s,p'}(\Omega)}$, where $p' = \frac{p}{p-1}$ is the conjugate to p satisfying

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

2.2 Different notions of solutions

In this section, we discuss the different notions of solutions to the problems involving nonlocal operators.

2.2.1 Weak and viscosity solution

We recall the following Lebesgue space.

$$L_{sp}^{p-1}(\mathbb{R}^N) := \left\{ u \in L_{loc}^{p-1}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}}\, dx < \infty \right\}.$$

Consider the following boundary value problem

$$\left.\begin{aligned} (-\Delta)_p^s u(x) &= h(x,u) && \text{in } \Omega, \\ u(x) &= 0 && \text{on } \Omega^c. \end{aligned}\right\} \tag{2.2.1}$$

Definition 2.2.1. *A function $u \in W^{s,p}(\mathbb{R}^N) \cap L_{sp}^{p-1}(\mathbb{R}^N)$ is a weak supersolution (subsolution) of (2.2.1) if*

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\, dx\, dy \geq (\leq) \int_\Omega h(x,u)\varphi\, dx.$$

for every $\varphi \in W_0^{s,p}(\Omega)$.

We say that u is a weak solution of (2.2.1) if it is both a weak supersolution and subsolution to the problem.

As we know, in the classical p-Laplacian problem for the range $1 < p < 2$, the p-Laplacian operator becomes singular when $\nabla u = 0$. So while defining the correct viscosity solution, one has to deal with this issue. The same issue holds here. In the nonlocal setting the singular range is $1 < p \leq \frac{2}{2-s}$. To give pointwise sense to the nonlocal operator, we need to work with a more restricted class of functions [68]. Let $D \subset \Omega$ be an open set, we define

$$C^2_\beta(D) := \left\{ u \in C^2(D) : \sup_{x \in D} \left(\frac{\min\{d_u(x), 1\}^{\beta-1}}{|\nabla u(x)|} + \frac{|D^2 u(x)|}{d_u(x)^{\beta-2}} \right) < \infty \right\},$$

where d_u is the distance from the set of critical points of u denoted as N_u, that is

$$d_u(x) := dist(x, N_u), \qquad N_u := \{x \in \Omega : \nabla u(x) = 0\}.$$

Definition 2.2.2. *A function u is a viscosity super-solution (subsolution) of* (2.2.1) *if*

(i) $u < +\infty$ $(u > -\infty)$ *a.e. in* $\mathbb{R}^N$, $u > -\infty$ $(u < +\infty)$ *a.e. in* Ω.

(ii) *u is lower (upper) semicontinous in* Ω.

(iii) *If* $\phi \in C^2(B(x_0, r)) \cap L^{p-1}_{sp}(\mathbb{R}^N)$ *for some* $B(x_0, r) \subset \Omega$ *such that* $\phi(x_0) = u(x_0), \phi \leq u (\phi \geq u)$ *in* $B(x_0, r)$, $\phi = u$ *in* $\Omega \setminus B(x_0, r)$ *and one of the following conditions hold:*

  (a) $p > \frac{2}{2-s}$ *or* $\nabla \phi(x_0) \neq 0$,

  (b) $1 < p \leq \frac{2}{2-s}$, $\nabla \phi(x_0) = 0$ *such that x_0 is an isolated critical point of ϕ, and* $\phi \in C^2_\beta(B(x_0, r))$ *for some* $\beta > \frac{sp}{p-1}$, *then*

$$(-\Delta)^s_p \phi(x_0) \geq (\leq) h(x_0, \phi(x_0)).$$

(iv) $u_- := \max\{-u, 0\}$ $(u_+ := \max\{u, 0\})$ *belongs to* $L^{p-1}_{sp}(\mathbb{R}^N)$.

We say that u is a viscosity solution of (2.2.1) *if it is both a viscosity supersolution and subsolution to the problem.*

2.2.2 Equivalence of Weak and Viscosity Solutions

In [6], the authors have discussed the equivalence of weak and viscosity solutions to problems involving nonlocal operators with a more general non-homogeneous term, including fractional derivatives. Here we state the equivalence of weak and viscosity solutions for nonlocal problem (2.2.1).

Theorem 2.2.1. *Let $1 < p < \infty$. Assume that $h = h(x, u)$ is uniformly continuous in $\Omega \times \mathbb{R}$, non-increasing in t, and satisfies the growth condition*

$$|h(x,t)| \leq \gamma(|t|) + \phi(x),$$

with $\gamma \geq 0$ is a continous function and $\phi \in L^\infty_{loc}(\Omega)$. Thus, if $u \in L^\infty(\mathbb{R}^N)$ and lower semi-continuous in $\mathbb{R}^N$ is a viscosity supersolution of (2.2.1) then it is a weak supersolution of the problem (2.2.1).

Definition 2.2.3. *Let u be a weak supersolution to (2.2.1) in $D \subset \Omega$. We say that (u, h) satisfies the comparison principle (CPP) in D if for every weak subsolution v of (2.2.1) such that $u \geq v$ a.e in $\mathbb{R}^N \setminus D$ we have $u \geq v$ a.e in D.*

Theorem 2.2.2. *Let $1 < p < \infty$. Assume u is a continuous weak supersolution of problem (2.2.1) and $h = h(x, u)$ is continuous in $\Omega \times \mathbb{R}$. If (CPP) holds then u is a viscosity supersolution of problem (2.2.1).*

2.3 Symmetries

In this section, we study the concept of symmetric domain and functions and its related terminologies that have been used in subsequent chapters.

2.3.1 G-invariant domains and functions

Let $SO(N)$ be the special group of orthogonal transformation of $\mathbb{R}^N$ and G be a closed subgroup of $SO(N)$.

Definition 2.3.1. *A domain $\Omega \subset \mathbb{R}^N$ is said to be invariant under the action of G (or G-invariant) if for each $x \in \Omega, g \in G$, we have $gx \in \Omega$.*

Definition 2.3.2. *A function $u : \Omega \to \mathbb{R}$ is said to be invariant under the action of G(or G-invariant) if $u(gx) = u(x)$ for each $x \in \Omega, g \in G$.*

For $x \in \mathbb{R}^N$, the set

$$Gx := \{gx \in \mathbb{R}^N : g \in G\}$$

is known as the G-orbit x and the subgroup

$$G_x := \{g \in G : gx = x\}.$$

is called the G-isotropy subgroup x. Note that Gx is G-homeomorphic to the homogeneous G-space G/G_x with $|G/G_x| = \#Gx$, where $|G/G_x|$ denotes the index of G_x in G and is $\#Gx$ denotes the cardinality of Gx.

2.3.2 G-invariant fractional Sobolev space

We consider the problem

$$(P_\Omega^s)\begin{cases} (-\Delta)^s u &= |u|^{2_s^*-2}\, u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \Omega^c, \end{cases}$$

Weak solution to (P_Ω^s) are given by the critical points of the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2_s^*}\int_\Omega |u(x)|^{2_s^*}\,dx. \tag{2.3.1}$$

We note that I is a C^1 functional with the derivative $DI \in D^{-s,2}(\Omega)$ defined by

$$\langle DI(u), v\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u|^{2_s^*-2}uv\,dx, \tag{2.3.2}$$

where $v \in D_0^{s,2}(\Omega)$. For linear transformation $g : \mathbb{R}^N \to \mathbb{R}^N$ and $u \in D_0^{s,2}(g(\Omega))$ we define

$$u_g(x) = (det\ Dg)^{\frac{N-2s}{2N}}\, u(g(x)), \tag{2.3.3}$$

where $\det Dg$ is the Jacobian determinant of the transformation g. Next for G-invariant domain Ω, we define the orthogonal action of g on $D_0^{s,2}(\Omega)$ as $gu := u_{g^{-1}}$ for every $u \in$

$D_0^{s,2}(\Omega)$ where $u_{g^{-1}}$ is as defined in (2.3.3). Let

$$D_0^{s,2}(\Omega)^G := \left\{ u \in D_0^{s,2}(\Omega) \text{ such that } gu = u \text{ for all } g \in G \right\} \tag{2.3.4}$$

be the subspace of $D_0^{s,2}(\Omega)$ of G-invariant functions. Then using (2.3.3) and the fact that Ω is G-invariant one can check that I is G-invariant for $G \subset SO(N)$, i.e. $I(gu) = I(u)$ for all $u \in D_0^{s,2}(\Omega)$. Therefore by the principle of symmetric criticality [82], the critical points of the restriction of I to the space $D_0^{s,2}(\Omega)^G$ are G-invariant solutions of problem (P_Ω^s). We define the G-invariant Palais-Smale sequence for I as follows.

Definition 2.3.1. *A sequence $\{u_k\}_{k\in\mathbb{N}}$ such that*

$$u_k \in D_0^{s,2}(\Omega)^G, \quad I(u_k) \to c, \quad \textit{and} \quad \|DI(u_k)\| \to 0 \quad \textit{in} \quad D^{-s,2}(\Omega)$$

is called a G-invariant Palais-Smale sequence for I, or a G-PS-sequence for short.

We say that I satisfies the G-Palais-Smale condition $(PS)_c^G$ at c if every G-PS-sequence for I such that $I(u_k) \to c$ has a convergence subsequence.

2.4 Mountain pass theorem for sign-changing solutions

Mountain pass type theorem for sign-changing solutions to problems involving semilinear elliptic equations has been studied in section 3 of [44]. Here we recall the same for the nonlocal setting. Let G be a closed subgroup of $SO(N)$, Ω be a G-invariant bounded domain in $\mathbb{R}^N$ and I be as in (2.3.1). Consider the negative gradient flow $\varphi : \mathcal{G} \to D_0^{s,2}(\Omega)^G$ of I, defined by

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,u) & = -DI(\varphi(t,u)) \\ \varphi(0,u) & = u \end{cases}$$

where $\mathcal{G} := \{(t,u) : u \in D_0^{s,2}(\Omega)^G, 0 \leq t \leq T(u)\}$ and $T(u) \in (0,\infty]$ is the maximal existence time for the trajectory $t \mapsto \varphi(t,u)$. We say that a subset $\mathcal{D}$ of $D_0^{s,2}(\Omega)^G$ is strictly positively invariant under φ if

$$\varphi(t,u) \in \text{int}(\mathcal{D}) \quad \text{for every } u \in \mathcal{D} \text{ and every } t \in (0, T(u))$$

where $\text{int}(\mathcal{D})$ denotes interior of $\mathcal{D}$ in $D_0^{s,2}(\Omega)^G$. If $\mathcal{D}$ is strictly positively invariant under φ, then the set

$$\mathcal{A}(\mathcal{D}) := \{u \in D_0^{s,2}(\Omega)^G : \varphi(t,u) \in \mathcal{D} \text{ for some } t \in (0, T(u))\}$$

is open in $D_0^{s,2}(\Omega)^G$, and the entrance time map

$$e_{\mathcal{D}}(u) := \inf\{t \geq 0 : \varphi(t,u) \in \mathcal{D}\}$$

is continuous. We state the following quantitative deformation lemma, which follows using the similar arguments as in Lemma 3 in [47]. We define the sublevel set as

$$I^d := \left\{ u \in D_0^{s,2}(\Omega) \ : \ I(u) \leq d \right\}.$$

Corollary 2.4.1. *If I has no sign-changing critical point $u \in D_0^s(\Omega)^G$ with $I(u) = d$, then the set*

$$\mathcal{D}_d^G := B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G) \cup I^d$$

is strictly positively invariant under φ, and the map

$$\varrho_d : \mathcal{A}(\mathcal{D}_d^G) \to \mathcal{D}_d^G, \qquad \varrho_d(u) := \varphi(e_{\mathcal{D}_d^G}(u), u)$$

is odd and continuous, and satisfies $\varrho_d(u) = u$ for every $u \in \mathcal{D}_d^G$.

2.4.1 Genus

A subset $\mathcal{Y}$ of $D_0^{s,2}(\Omega)^G$ is called symmetric if $-u \in \mathcal{Y}$ for every $u \in \mathcal{Y}$.

Definition 2.4.1. *Let $\mathcal{D}$ and $\mathcal{Y}$ be symmetric subsets of $D_0^{s,2}(\Omega)^G$. The genus of $\mathcal{Y}$ relative to $\mathcal{D}$, denoted $\mathfrak{g}(\mathcal{Y},\mathcal{D})$, is the smallest number $m \in \mathbb{N}$ such that $\mathcal{Y}$ can be covered by $m+1$ open symmetric subsets $\mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_m$ of $D_0^{s,2}(\Omega)^G$ with the following two properties:*

- (i) *$\mathcal{Y} \cap \mathcal{D} \subset \mathcal{U}_0$ and there exists an odd and continuous map $\vartheta_0 : \mathcal{U}_0 \to \mathcal{D}$ such that $\vartheta_0(u) = u$ for $u \in \mathcal{Y} \cap \mathcal{D}$,*
- (ii) *there exist odd continuous maps $\vartheta_j : \mathcal{U}_j \to \{-1, 1\}$ for every $j = 1, ..., m$.*

If no such cover exists, we define $\mathfrak{g}(\mathcal{Y},\mathcal{D}) := \infty$.

If $\mathcal{D} = \emptyset$ we write $\mathfrak{g}(\mathcal{Y}) := \mathfrak{g}(\mathcal{Y}, \emptyset)$. This is the usual Krasnoselskii genus. $\mathcal{D}$ is called symmetric neighborhood retract if there exist a symmetric neighborhood $\mathcal{U}$ of $\mathcal{D}$ in $D_0^{s,2}(\Omega)^G$ and an odd continuous map $\varrho : \mathcal{U} \to \mathcal{D}$ such that $\varrho(u) = u$ for every $u \in \mathcal{D}$.

Definition 2.4.2. *Let $\mathcal{D} \subset \mathcal{H}$ be subsets of $D_0^{s,2}(\Omega)^G$. I satisfies $(PS)_c$ relative to $\mathcal{D}$ in $\mathcal{H}$, if every sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that*

$$u_n \notin \mathcal{D}, \; I(u_n) \to c, \; DI(u_n) \to 0,$$

has a convergent subsequence. If $\mathcal{D} = \emptyset$ we say that I satisfies $(PS)_c$ in $\mathcal{H}$.

Set $\mathcal{D}_c^G := B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G) \cup I^c$, and define

$$c_j := \inf\{c \in \mathbb{R} : \mathfrak{g}(\mathcal{D}_c^G, \mathcal{D}_0^G) \geq j\}$$

Proposition 2.4.1. *Assume that I satisfies $(PS)_{c_j}$ relative to $\mathcal{D}_0^G$ in $D_0^{s,2}(\Omega)^G$. Then the following holds:*

- (i) *There exists a sign-changing critical point $u \in D_0^{s,2}(\Omega)^G$ of I with $I(u) = c_j$.*
- (ii) *If $c_j = c_{j+1}$, then I has infinitely many sign-changing critical points $u \in D_0^{s,2}(\Omega)^G$ with $I(u) = c_j$.*

Consequently, if I satisfies $(PS)_c$ relative to $\mathcal{D}_0^G$ in $D_0^{s,2}(\Omega)^G$ for every $c \leq d$, then I has at least $\mathfrak{g}(\mathcal{D}_c^G, \mathcal{D}_0^G)$ pairs of sign-changing critical points $u \in D_0^{s,2}(\Omega)^G$ with $I(u) \leq d$.

We now state a mountain pass result for sign-changing solution.

Theorem 2.4.1. *Let W be a finite dimensional subspace of $D_0^{s,2}(\Omega)^G$ and let $d := \sup_W I$. If I satisfies $(PS)_c$ relative to $\mathcal{D}_0^G$ in $D_0^{s,2}(\Omega)^G$ for every $c \leq d$, then I has at least $\dim(W) -$ 1 pairs of sign-changing critical points $u \in D_0^{s,2}(\Omega)^G$ with $I(u) \leq d$.*

We refer to Proposition 3.6 and Theorem 3.7 of [44] for the proofs of Proposition 2.4.1 and Theorem 2.4.1.

2.5 Important results

In this section, we recall the various properties of fractional p-Laplacian and results related to nonlocal operators. We also recall the different algebraic inequalities that have been used throughout the thesis.
First we sate a vital property of fractional p-Laplace operator, which does not hold for classical p-Laplace operator for $p \neq 2$.

Lemma 2.5.1. ([36, Lemma 2.2]) *The operator $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is weak to weak continuous.*

The strong maximum principle for fractional p-Laplacian is given as follows:

Lemma 2.5.2. ([80, Lemma 2.3]) *Let $u \in W_0^{s,p}(\Omega)$ satisfy*

$$\begin{aligned} (-\Delta)_p^s u &\geq 0 \quad \text{weakly in } \Omega, \\ u &\geq 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

Then u has a lower semi-continuous representative in Ω, which is either identically 0 or positive.

Next we recall some weighted Hölder spaces. Let the distance function $d : \overline{\Omega} \to \mathbb{R}_+$ be defined by

$$d(x) := \operatorname{dist}(x, \partial\Omega), \qquad x \in \overline{\Omega}. \tag{2.5.1}$$

The weighted Hölder type spaces are defined as follows:

$$C_d^0(\overline{\Omega}) := \Big\{ u \in C^0(\overline{\Omega}) : u/d^s \text{ admits a continuous extension to } \overline{\Omega} \Big\},$$
$$C_d^{0,\alpha}(\overline{\Omega}) := \Big\{ u \in C^0(\overline{\Omega}) : u/d^s \text{ admits a } \alpha\text{-Hölder continuous extension to } \overline{\Omega} \Big\}$$

equipped with the norms

$$\|u\|_{C_d^0(\overline{\Omega})} := \|u/d^s\|_{L^\infty(\Omega)},$$
$$\|u\|_{C_d^{0,\alpha}(\overline{\Omega})} := \|u\|_{C_d^0(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x)/d^s(x) - u(y)/d^s(y)|}{|x-y|^\alpha},$$

respectively. The embedding $C_d^{0,\alpha}(\overline{\Omega}) \hookrightarrow\hookrightarrow C_d^0(\overline{\Omega})$ is compact, for all $\alpha \in (0,1)$.
Next we state some regularity results for the problems involving fractional p-Laplacian.

Consider the following boundary value nonlocal problem

$$\left.\begin{aligned} (-\Delta)_p^s u(x) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \Omega^c, \end{aligned}\right\} \tag{2.5.2}$$

where $\Omega \subset \mathbb{R}^N$ is bounded domain with $C^{1,1}$ boundary $\partial\Omega$ and $f \in L^\infty(\Omega)$.
First we recall global Hölder regularity result.

Theorem 2.5.1. ([66, Theorem 1.1]) *Let $p > 1$, there exists $\alpha \in (0,s]$ and $C_\Omega > 0$, depending only on $N, p,$ and s, with C_Ω also depending on Ω, such that, for all weak solution $u \in W_0^{s,p}(\Omega)$ of problem* (2.5.2), *$u \in C^\alpha(\overline{\Omega})$ and*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C_\Omega \|f\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}. \tag{2.5.3}$$

Next we state fine boundary regularity result.

Theorem 2.5.2. ([67, Theorem 1.1]) *Let $p \geq 2$, there exists $\alpha \in (0,s]$ and $C > 0$, depending on N, p, s and Ω, such that, for all weak solution $u \in W_0^{s,p}(\Omega)$ of problem* (2.5.2), *$u \in C_d^{0,\alpha}(\overline{\Omega})$ and*

$$\|u\|_{C_d^{0,\alpha}(\overline{\Omega})} \leq C \|f\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}. \tag{2.5.4}$$

We also collect the following discrete Picone inequality.

Proposition 2.5.1. ([19, Proposition 4.2]) *Let $1 < p < \infty$ and $1 < q \leq p$. Let u, v be two measurable functions with $v \geq 0$ and $u > 0$, then*

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{v(x)^q}{u(x)^{p-1}}-\frac{v(y)^q}{u(y)^{p-1}}\right] \leq |v(x)-v(y)|^q|u(x)-u(y)|^{p-q}. \tag{2.5.5}$$

Next we recall the following truncation lemma.

Lemma 2.5.1. ([22, Lemma A.2]) *Let $\zeta \in C_0^\infty(B(0,2))$ be a positive function such that $\zeta \equiv 1$ on $B(0,1)$. Then*

$$\lim_{n\to\infty}[v\zeta(\mu_n\cdot)-v]_{D^{s,p}(\mathbb{R}^N)} = 0 \tag{2.5.6}$$

for any $v \in D_0^{s,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with $q < p_s^*$ and $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that $\mu_n \to 0$.

We have the following Brezis-Lieb result.

Theorem 2.5.1. ([24, Theorem 1]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\{u_n\} \subset L^p(\Omega)$, $1 \leq p < \infty$ be such that $\|u_n\|_{L^p} \leq C$ for some $C > 0$ and $u_n \to u$ a.e. in Ω. Then*

$$\lim_{n\to+\infty}[\|u_n-u\|^p_{L^p(\Omega)} - \|u_n\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}] = 0.$$

We have the following inequality.

Lemma 2.5.2. ([68, Lemma 3.3]) *Let $p > 1$ and $a, b \in \mathbb{R}$. Then*

$$||a|^{p-2}a-|b|^{p-2}b| \leq c(|b|+|a-b|)^{p-2}|a-b|$$

where c depends only on p.

We also have the well known Simon's inequalities (see [91]). For all $\zeta, \xi \in \mathbb{R}^N$, we have the following:

$$\left.\begin{aligned} |\zeta-\xi|^p &\leq \tfrac{1}{p-1}\Big[\left(|\zeta|^{p-2}\zeta-|\xi|^{p-2}\xi\right).(\zeta-\xi)\Big]^{\frac{p}{2}}\left(|\zeta|^p+|\xi|^p\right)^{\frac{2-p}{2}}, 1<p<2,\\ |\zeta-\xi|^p &\leq 2^p\Big(|\zeta|^{p-2}\zeta-|\xi|^{p-2}\xi\Big).(\zeta-\xi)\ , \qquad p\geq 2. \end{aligned}\right\} \tag{2.5.7}$$

We also recall the following standard inequalities which follows from [21, Lemma A.2] for $g(t) = t^+$. For all $a \geq b$ and $1 < p < \infty$, there exists some constant $C_p > 0$ such that

$$|a^+ - b^+|^p \leq (a-b)^{p-1}(a^+ - b^+) \text{ and } (a-b)^{p-1} \leq C_p(a^{p-1} - b^{p-1}). \tag{2.5.8}$$

Another important inequality which is used repeatedly in this thesis is given below.

$$\left.\begin{aligned} (x_1+x_2)^p &\leq x_1^p + x_2^p,\ 0 < p \leq 1,\ x_1, x_2 \geq 0,\\ (x_1+x_2)^p &\leq 2^{p-1}(x_1^p + x_2^p),\ p > 1,\ x_1, x_2 \geq 0. \end{aligned}\right\} \tag{2.5.9}$$

3

A Global compactness result in symmetric domain

In this chapter, we establish Struwe's type compactness result for the following nonlocal problem with critical nonlinearity

$$(P^s_{p,Q,\Omega})\begin{cases} (-\Delta)^s_p u &= Q(x)|u|^{p^*_s-2}\,u, \quad \text{in } \Omega, \\ \quad u(x) &= 0, \qquad\qquad\qquad \text{on } \Omega^c, \end{cases}$$

where Ω is a bounded and G-invariant domain in $\mathbb{R}^N$ and Q is a positive, continuous, and G-invariant function in $\overline{\Omega}$. Here we study the splitting of Palais-Smale sequence of the functional associated with the following problem

$$(P^{s,G}_{p,Q,\Omega})\begin{cases}(-\Delta)^s_p u &= Q(x)|u|^{p^*_s-2}\,u, \quad \text{in } \Omega,\\ u(x) &= 0, \quad\quad\quad\quad\quad\ \text{on } \Omega^c,\\ u(gx) &= u(x) \quad\quad\quad\quad \text{for all } g\in G,\end{cases}$$

We show that the non-compactness is due to the solutions of the following limiting problem

$$(P^{s,K}_{p,\infty})\begin{cases}(-\Delta)^s_p u &= |u|^{p^*_s-2}\,u, \quad \text{in } \mathbb{R}^N,\\ u(x) &\to 0, \quad\quad\quad\quad \text{as } |x|\to\infty,\\ u(gx) &= u(x) \quad\quad\quad \text{for all } g\in K,\end{cases}$$

concentrating at G-orbits of Ω with orbit type G/K for some closed subgroup K of finite index in G.

3.1 Main Result

In this section we state the global compactness result for nonlocal problems with critical nonlinearities in the symmetric domain. We define the energy functional associated with the problems $(P^{s,G}_{p,Q,\Omega})$ and $(P^{s,K}_{p,\infty})$.

The energy functional $E_Q : D^{s,p}_0(\Omega)\to\mathbb{R}$ associated with the problem $(P^{s,G}_{p,Q,\Omega})$ is defined by

$$E_Q(u) := \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{1}{p^*_s}\int_\Omega Q|u(x)|^{p^*_s}\,dx.$$

We note that E_Q is a C^1 functional with the derivative $DE_Q\in D^{-s,p'}(\Omega)$ defined by

$$\langle DE_Q(u),v\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy - \int_\Omega Q|u|^{p^*_s-2}uv\,dx,$$

where $v\in D^{s,p}_0(\Omega)$.

The energy functional $E_\infty : D^{s,p}_0(\mathbb{R}^N)\to\mathbb{R}$ associated to the the problem $(P^{s,K}_{p,\infty})$ is given by

$$E_\infty(u) := \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{1}{p^*_s}\int_{\mathbb{R}^N}|u|^{p^*_s}\,dx.$$

We recall the G-invariant fractional Sobolev spaces that has been discussed in the chapter 2 for $p=2$.

For linear transformation $g:\mathbb{R}^N\to\mathbb{R}^N$ and $u\in D_0^{s,p}(g(\Omega))$, we define

$$u_g(x)=(det\ Dg)^{\frac{N-sp}{Np}}u(g(x)), \tag{3.1.1}$$

where $\det Dg$ is the Jacobian determinant of the transformation g. The orthogonal action of g on $D_0^{s,p}(\Omega)$ is defined as $gu:=u_{g^{-1}}$ for every $u\in D_0^{s,p}(\Omega)$ where $u_{g^{-1}}$ is as defined in (3.1.1). Let

$$D_0^{s,p}(\Omega)^G:=\{u\in D_0^{s,p}(\Omega)\text{ such that } gu=u \text{ for all } g\in G\}$$

be the subspace of $D_0^{s,p}(\Omega)$ of G-invariant functions. Then using (3.1.1) and the fact that Ω and Q are G-invariant, it is easy to check that E_Q is G-invariant for $G\subset SO(N)$, i.e. $E_Q(gu)=E_Q(u)$ for all $u\in D_0^{s,p}(\Omega)$. Therefore by the principle of symmetric criticality [82], the critical points of the restriction of E_Q to the space $D_0^{s,p}(\Omega)^G$ are G-invariant solutions of problem $(P^s_{p,Q,\Omega})$. The G-invariant solution of the problem $(P^{s,G}_{p,Q,\Omega})$ are the critical points of the restriction of the energy functional E_Q to the space $D_0^{s,p}(\Omega)^G$. We define the G-invariant Palais-Smale sequence for E_Q as follows.

Definition 3.1.1. *A sequence $(u_k)_{k\in\mathbb{N}}$ such that*

$$u_k\in D_0^{s,p}(\Omega)^G,\quad E_Q(u_k)\to c,\quad and\quad \|DE_Q(u_k)\|_{D^{-s,p'}(\Omega)}\to 0\quad in\quad D^{-s,p'}(\Omega)$$

is called a G-invariant Palais-Smale sequence for E_Q, or a G-PS-sequence for short.

We say that E_Q satisfies the G-Palais-Smale condition $(PS)_c^G$ at c if every G-PS-sequence for E_Q such that $E_Q(u_k)\to c$ has a convergent subsequence.

We state our main result.

Theorem 3.1.1. *Let $(u_k)_{k\in\mathbb{N}}$ be a G-invariant Palais-Smale sequence for E_Q at the level c. Then, after passing to a subsequence, there exist a (possibly trivial) G-invariant solution u to problem $(P^{s,G}_{p,Q,\Omega})$, an integer $m\geq 0$, m closed subgroups $G_1,\dots,G_m$ of finite index in G, m bounded sequences $(y_{1,k})_{k\in\mathbb{N}},\dots,(y_{m,k})_{k\in\mathbb{N}}$ in $\mathbb{R}^N$, m sequences $(\varepsilon_{1,k})_{k\in\mathbb{N}},\dots,(\varepsilon_{m,k})_{k\in\mathbb{N}}$*

in $(0,\infty)$, *and* m *non trivial solutions* $\widetilde{u}_1,\dots,\widetilde{u}_m$ *to problems*

$$(-\Delta)^s_p \widetilde{u}_i = |\widetilde{u}_i|^{p^*_s-2}\widetilde{u}_i, \qquad \widetilde{u}_i \in D^{s,p}_0(\mathbb{H}_i)$$

with the following properties:

(*i*) $G_{y_{i,k}} = G_i$ *for all* $k\in\mathbb{N}$, *and* $y_{i,k}\to y_i$ *in* $\overline{\Omega}$ *as* $k\to\infty$ *for each* $i=1,\dots,m$.

(*ii*) *If* $\varepsilon^{-1}_{i,k}\mathrm{dist}(y_{i,k},\partial\Omega)\to\infty$ *then* $\mathbb{H}_i=\mathbb{R}^N$ *and if* $(\varepsilon^{-1}_{i,k}\mathrm{dist}(y_{i,k},\partial\Omega))$ *is bounded then* $\mathbb{H}_i\neq\mathbb{R}^N$. *Also* $\varepsilon^{-1}_{i,k}|gy_{i,k}-g'y_{i,k}|\to\infty$ *as* $k\to\infty$ *for all* $[g']\neq[g]$ *in* G/G_i *and* $i=1,\dots,m$.

(*iii*) $\mathbb{H}_i$ *and* $\widetilde{u}_i$ *is* G_i*-invariant for each* $i=1,\dots,m$.

(*iv*) $\lim\limits_{k\to\infty}\left\|u_k-u-\sum_{i=1}^m\sum_{[g]\in G/G_i}\varepsilon_{i,k}^{\frac{sp-N}{p}}Q(y_i)^{\frac{sp-N}{sp^2}}\widetilde{u}_i\left(g^{-1}\left(\frac{\cdot-gy_{i,k}}{\varepsilon_{i,k}}\right)\right)\right\|=0.$

(*v*) $E_Q(u)+\sum\limits_{i=1}^m|G/G_i|\left(\frac{|G/G_i|}{Q(y_i)^{\frac{N-ps}{sp}}}\right)E_\infty(\widetilde{u}_i)=c.$

Now we discuss the implications of the Theorem 3.1.1. We write

$$\mu^{s,G}_{p,Q} := \left(\min_{x\in\overline{\Omega}}\frac{|G/G_x|}{Q(x)^{\frac{N-sp}{sp}}}\right)\frac{s}{N}S_{s,p}^{\frac{N}{sp}}.$$

where $S_{s,p}$ is the best fractional critical Sobolev constant for the embedding $D^{s,p}_0(\mathbb{R}^N)\hookrightarrow L^{p^*_s}(\mathbb{R}^N)$.

As a consequence of Theorem 3.1.1, we have the following Corollary. We refer to Theorem 2.5 in [46] for the proof.

Corollary 3.1.1. E_Q *satisfy* $(PS)^G_c$ *at every* $c<\mu^{s,G}_{p,Q}$. *In particular, if every* G*-orbit in* Ω *is infinite, then* $\mu^{s,G}_{p,Q}=\infty$ *and hence* E_Q *satisfies* $(PS)^G_c$ *at every* $c\in\mathbb{R}$.

For $p=2$, we note that $\frac{s}{N}S_s^{\frac{N}{2s}}$ is the energy of the ground state solution of the following nonlocal limiting problem (P^s_∞),

$$(P^s_\infty)\begin{cases}(-\Delta)^s u &= |u|^{2^*_s-2}u, \quad \text{in } \mathbb{R}^N,\\ u(x) &\to 0, \qquad\qquad \text{as } |x|\to\infty,\end{cases}$$

which, up to some constant, is given by

$$U_{\varepsilon,z}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x-z|^2}\right)^{\frac{N-2s}{2}}, \quad z \in \mathbb{R}^N, \varepsilon > 0.$$

They satisfy

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|U_{\varepsilon,z}(x) - U_{\varepsilon,z}(y)|^2}{|x-y|^{N+2s}}\, dx\, dy = S_s^{\frac{N}{2s}} = \int_{\mathbb{R}^N} |U_{\varepsilon,z}(x)|^{p_s^*}\, dx.$$

Next corollary gives the concentration of the non-convergent G-PS-sequences of E at the level $\mu^G_{Q,s}$. One can refer to Theorem 6.1 in [43] for the proof.

Corollary 3.1.2. *Let $\{u_k\}_{k\in\mathbb{N}}$ be a G-PS-sequence for E such that $E(u_k) \to \mu^G_{Q,s}$. Then a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ either converges strongly to a nontrivial solution of the problem $(P^{s,G}_{\Omega})$, or there exist $\nu = \pm 1$, and sequences $\{y_k\}_{k\in\mathbb{N}}$ in Ω and $\{\varepsilon_k\}_{k\in\mathbb{N}}$ in $(0,\infty)$, such that*

(i) *$y_k \to y \in \overline{\Omega}$ as $k \to \infty$, $G_{y_k} = G_y$ for all k, and*

$$\frac{|G/G_y|}{Q(y)^{\frac{N-2s}{2s}}} = \min_{x\in\overline{\Omega}} \frac{|G/G_x|}{Q(x)^{\frac{N-2s}{2s}}} < \infty,$$

(ii) *$\varepsilon_k^{-1} dist(y_k, \partial\Omega) \to \infty$ and $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ as $k \to \infty$ for all $[g] \neq [g'] \in G/G_y$,*

(iii) $$\left\| u_k - (-1)^{\nu} \sum_{[g]\in G/G_y} Q(y)^{\frac{2s-N}{4s}} U_{\varepsilon_k, gy_k} \right\| \to 0 \quad \text{in } D^{s,2}_0(\mathbb{R}^N) \text{ as } k \to \infty.$$

3.2 Proof of the main result

In this section we provide the proof of the Theorem 3.1.1. We assert that Theorem 3.1.1 follows from the iteration of this result.

Proposition 3.2.1. *Let $(u_k)_{k\in\mathbb{N}}$ be a G-invariant Palais-Smale sequence for E_Q at the level $c > 0$ such that $u_k \rightharpoonup 0$ weakly in $D^{s,p}_0(\Omega)^G$. Then, after passing to a subsequence, there exist a closed subgroup K of finite index in G, a sequence $(y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^N$, a sequence*

$(\varepsilon_k)_{k\in\mathbb{N}}$ *in* $(0,\infty)$*, a non trivial solution* $\widetilde{u}$ *to the limit problem*

$$(-\Delta)_p^s \widetilde{u} = |\widetilde{u}|^{p^*-2}\widetilde{u}, \qquad \widetilde{u} \in D_0^{s,p}(\mathbb{H}),$$

where $\mathbb{H} = \mathbb{R}^N$ *if* $\varepsilon_k^{-1}\mathrm{dist}(y_k, \partial\Omega) \to \infty$ *and* $\mathbb{H} = \{z \in \mathbb{R}^N : \nu(y_0)\cdot z > -d\}$ *or* $\mathbb{H} = \{z \in \mathbb{R}^N : \nu(y_0)\cdot z > d\}$ *if* $\lim_{k\to\infty} \varepsilon_k^{-1}\mathrm{dist}(y_k, \partial\Omega) = d \in [0,\infty)$ *where* $\nu(y_0)$ *is the interior unit normal at* $y_0 \in \partial\Omega$ *for* $y_0 = \lim_{k\to\infty} y_k$*, and a* G*-invariant Palais-Smale sequence* (v_k) *for* E_Q *with the following properties:*

(*i*) $G_{y_k} = K$ *for all* $k \in \mathbb{N}$*, and* $y_k \to y_0$ *in* $\overline{\Omega}$ *as* $k \to \infty$.

(*ii*) *If* $\varepsilon_k^{-1}\mathrm{dist}(y_k, \partial\Omega) \to \infty$ *then* $\mathbb{H} = \mathbb{R}^N$ *and if* $(\varepsilon_k^{-1}\mathrm{dist}(y_k, \partial\Omega))$ *is bounded then* $\mathbb{H} \neq \mathbb{R}^N$*. Also* $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ *as* $k \to \infty$ *if* $[g'] \neq [g]$ *in* G/K .

(*iii*) $\mathbb{H}$ *and* $\widetilde{u}$ *are* K*-invariant.*

(*iv*) $\lim_{k\to\infty} \left\| u_k - v_k - \sum_{[g]\in G/K} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u}\left(g^{-1}\left(\frac{\cdot - gy_k}{\varepsilon_k}\right)\right)\right\| = 0.$

(*v*) $E_Q(v_k) \to c - \left(\frac{|G/K|}{Q(y_0)^{\frac{N-sp}{sp}}}\right) E_\infty(\widetilde{u})$ *as* $k \to \infty$.

First we recall some results that have been used to prove the Proposition 3.2.1.

We have the following Brezis-Lieb type result for the fractional-order space $D_0^{s,p}(\mathbb{R}^N)$ from [22].

Lemma 3.2.1. *Let* $(u_k)_{k\in\mathbb{N}} \subset D_0^{s,p}(\mathbb{R}^N)$ *be such that* $u_k \rightharpoonup u$ *in* $D_0^{s,p}(\mathbb{R}^N)$ *and* $u_k \to u$ *almost everywhere, as* $k \to \infty$*. Then*

$$[u_k]^p_{D^{s,p}(\mathbb{R}^N)} - [u_k - u]^p_{D^{s,p}(\mathbb{R}^N)} = [u]^p_{D^{s,p}(\mathbb{R}^N)} + o_k(1).$$

Also we have the following Sobolev type embedding with localized seminorm from Proposition 2.3 in [21].

Lemma 3.2.2. *Let* $s \in (0,1)$ *such that* $sp < N$*. We fix* $0 < r < R$*, then for every* $u \in D_0^{s,p}(B(0,r))$ *there holds*

$$\left(\int_{B(0,r)} |u|^{p_s^*}\, dx\right)^{\frac{p}{p_s^*}} \leq \mathcal{T} \int_{B(0,R)\times B(0,R)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy$$

where the constant $\mathcal{T} = \mathcal{T}(N, s, p, R/r) > 0$ *and* $\frac{1}{\mathcal{T}}$ *goes to* 0 *as* R/r *converges to* 1.

For $1 < p < \infty$ we define the monotone function $J_p : \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$J_p(\xi) := |\xi|^{p-2}\xi, \quad \xi \in \mathbb{R}^N.$$

We also recall the following Caccioppoli inequality. One can refer Proposition 2.9 [22] for the proof.

Lemma 3.2.3. *Let* $F \in D^{-s,p'}(\Omega)$ *and let* $u \in D_0^{s,p}(\Omega)$ *with*

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy = \langle F, \varphi \rangle, \quad \textit{for any } \varphi \in D_0^{s,p}(\Omega).$$

Then for every open set Ω' *such that* $\Omega' \cap \Omega \neq \emptyset$ *and every positive* $\psi \in C_0^\infty(\Omega')$ *we have*

$$\begin{aligned} &\int_{\Omega' \times \Omega'} \frac{|u(x)\psi(x) - u(y)\psi(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \\ &\leq \mathcal{C} \int_{\Omega' \times \Omega'} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{N+sp}} \left(|u(x)|^p + |u(y)|^p\right) dx \, dy \\ &+ \mathcal{C} \left(\sup_{y \in sppt(\psi)} \int_{\mathbb{R}^N \setminus \Omega'} \frac{|u(x)|^{p-1}}{|x - y|^{N+sp}} \, dx \right) \int_{\Omega'} |u| \psi^p \, dx + \mathcal{C} \Big| \langle F, u\psi^p \rangle \Big|, \end{aligned}$$

for some constant $\mathcal{C} > 0$.

We recall the following lemma and for the proof one can refer A.1. in [22].

Lemma 3.2.4. *Let* ψ *be a Lipschitz function with compact support* K *and* $u \in D_0^{s,p}(\mathbb{R}^N)$. *Then* $\psi u \in D_0^{s,p}(\mathbb{R}^N)$ *and we have the estimate*

$$[\psi u]^p_{D^{s,p}(\mathbb{R}^N)} \leq C_1 \|\psi\|^p_{L^\infty(\mathbb{R}^N)} [u]^p_{D^{s,p}(\mathbb{R}^N)} + C_2 \|\nabla \psi\|^p_{L^\infty(\mathbb{R}^N)} \|u\|^p_{L^{p_s^*}(\mathbb{R}^N)}$$

for some $C_1 = C_1(N, s) > 0$ *and* $C_2 = C_2(N, s, K) > 0$.

Next we discuss invariance of $D^{s,p}(\mathbb{R}^N)$ and $L^{p_s^*}(\mathbb{R}^N)$ norms under translation and dilation. We have the following scaling invariance result for E_∞ from Lemma 2.5 of [22].

Lemma 3.2.5. *(Scaling invariance). For $y \in \mathbb{R}^N$ and $\lambda > 0$, we set*

$$z \in \Omega_{y,\lambda} := \left\{z \in \mathbb{R}^N : \lambda z + y \in \Omega\right\}$$

and for $u \in D_0^{s,p}(\Omega)$, $z \in \Omega_{y,\lambda}$, we define

$$\overline{u}(z) := \lambda^{\frac{N-sp}{p}} u(\lambda z + y) \in D_0^{s,p}(\Omega_{y,\lambda}) \quad \text{and} \quad \overline{Q}(z) := Q(\lambda z + y).$$

Then

$$[\overline{u}]_{D^{s,p}(\mathbb{R}^N)} = [u]_{D^{s,p}(\mathbb{R}^N)} \text{ and } \left\|\overline{Qu}\right\|_{L^{p_s^*}(\mathbb{R}^N)} = \|Qu\|_{L^{p_s^*}(\mathbb{R}^N)}.$$

Also for $w, \varphi \in D_0^{s,p}(\mathbb{R}^N)$, we define

$$\overline{w}(z) := \lambda^{\frac{sp-N}{p}} w\left(\frac{z-y}{\lambda}\right), \qquad \overline{\varphi}(z) := \lambda^{\frac{N-sp}{p}} \varphi(\lambda z + y).$$

Then $\langle DE_\infty(\overline{w}),\varphi\rangle = \langle DE_\infty(w),\overline{\varphi}\rangle$ and

$$\sup_{\varphi \in D_0^{s,p}(\Omega)} \left|\left\langle DE_\infty(\overline{w}), \frac{\varphi}{[\varphi]_{D^{s,p}(\mathbb{R}^N)}}\right\rangle\right| = \sup_{\varphi \in D_0^{s,p}(\Omega_{y,\lambda})} \left|\left\langle DE_\infty(w), \frac{\varphi}{[\varphi]_{D^{s,p}(\mathbb{R}^N)}}\right\rangle\right|.$$

We also recall Lemma 3.3 in [41] which is used to prove Proposition 3.2.1.

Lemma 3.2.6. *Given sequences $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,\infty)$ and $(\xi_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^N$, there exist a sequence $(y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^N$ and a closed subgroup K of G such that, after passing to a subsequence, the following statements hold true:*

(i) *The sequence $(\varepsilon_k^{-1}\text{dist}(G\xi_k, y_k))$ is bounded.*

(ii) *$G_{y_k} = K$ for all $k \in \mathbb{N}$.*

(iii) *If $|G/K| < \infty$ then $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ for any $g, g' \in G$ with $g'g^{-1} \notin K$.*

(iv) *If $|G/K| = \infty$ then there is a closed subgroup K' of G such that $K \subset K'$, $|G/K'| = \infty$ and $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ for any $g, g' \in G$ with $g'g^{-1} \notin K'$.*

Now we prove the Proposition 3.2.1.

Proof of Proposition 3.2.1: We prove the Proposition 3.2.1 in the following steps:

Step 1: Since PS-sequences for E_Q are bounded in $D_0^{s,p}(\Omega)$,

$$\int_\Omega Q|u_k|^{p_s^*}\,dx = \frac{N}{s}E(u_k) - \frac{N}{sp}DE(u_k)u_k \to \frac{Nc}{s} > 0.$$

Let

$$\delta := \min\left\{\frac{Nc}{sp}, \frac{1}{2\mathcal{C}\mathcal{T}}\left(\max_{\overline{\Omega}} Q\right)^{\frac{sp-N}{N}}\right\}^{\frac{N}{sp}}.$$

Let $B(x,r)$ denote the closed ball in $\mathbb{R}^N$ with centre x and radius r. Then the Levy Concentration function

$$\mathbf{\Phi}_k(r) := \sup_{x\in\mathbb{R}^N}\int_{B(x,r)} Q|u_k|^{p_s^*}$$

satisfies that $\mathbf{\Phi}_k(0) = 0$ and $\mathbf{\Phi}_k(\infty) > \delta$ for k large enough. Hence we may choose $\xi_k \in \mathbb{R}^N$ and $\varepsilon_k > 0$ such that

$$\sup_{x\in\mathbb{R}^N}\int_{B(x,\varepsilon_k)} Q|u_k|^{p_s^*} = \int_{B(\xi_k,\varepsilon_k)} Q|u_k|^{p_s^*} = \delta, \tag{3.2.1}$$

Observe that, since Ω is bounded, the sequence ξ_k is bounded. By Lemma 3.2.6, after passing to a subsequence, there exist (y_k) in $\mathbb{R}^n$, a subgroup K of G, and $C_1 > 0$ such that $G_{y_k} = K$ and $\varepsilon_k^{-1}\text{dist}(G\xi_k, y_k) < C_1$ for all $k \in \mathbb{N}$. Therefore, (y_k) is bounded and there exists $g_k \in G$ such that $B(g_k\xi_k, \varepsilon_k) \subset B(y_k, C\varepsilon_k)$ with $C := C_1 + 1$. As Q and u_k are G-invariant, this implies that

$$\delta = \int_{B(\xi_k,\varepsilon_k)} Q|u_k|^{p_s^*} = \int_{B(g_k\xi_k,\varepsilon_k)} Q|u_k|^{p_s^*} \le \int_{B(y_k,C\varepsilon_k)} Q|u_k|^{p_s^*}. \tag{3.2.2}$$

Now we claim that $|G/K| < \infty$. If not then, by Lemma 3.2.6, there exists a closed subgroup K' of G such that $K \subset K'$, $|G/K'| = \infty$ and $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ for any $[g], [g'] \in G/K'$ with $[g] \neq [g']$. Hence, for each $m \in \mathbb{N}$, we may choose $g_1, \ldots, g_m \in G$ such that $[g_i] \neq [g_j]$ in G/K' and $B(g_iy_k, C\varepsilon_k) \cap B(g_jy_k, C\varepsilon_k) = \emptyset$ for $i \neq j$ and k sufficiently large. From inequality (3.2.2) we obtain that

$$m\delta \le \sum_{i=1}^{m}\int_{B(g_iy_k,C\varepsilon_k)} Q|u_k|^{p_s^*} \le \int_\Omega Q|u_k|^{p_s^*} = \frac{Nc}{s} + o_k(1),$$

for every $m \in \mathbb{N}$. This is a contradiction. Hence $|G/K| < \infty$.

Step 2: For $z \in \Omega_k := \{z \in \mathbb{R}^N : \varepsilon_k z + y_k \in \Omega\}$ set

$$\overline{u_k}(z) := \varepsilon_k^{\frac{N-sp}{p}} u_k(\varepsilon_k z + y_k) \quad \text{and} \quad \overline{Q_k}(z) := Q(\varepsilon_k z + y_k)$$

Thus, $\overline{u_k}$ and $\overline{Q_k}$ are K-invariant, that is,

$$[\overline{u_k}]_{D_0^{s,p}(\Omega_k)} = [u_k]_{D_0^{s,p}(\Omega)} \quad and \quad \int_{\Omega_k} \overline{Q_k}|\overline{u_k}|^{p_s^*} = \int_{\Omega} Q|u_k|^{p_s^*}.$$

In particular, $\overline{u_k}$ is a bounded sequence in $D^{s,p}(\mathbb{R}^N)^K$. Hence up to a subsequence, $\overline{u_k} \rightharpoonup \overline{u}$ weakly in $D^{s,p}(\mathbb{R}^N)^K$, $\overline{u_k} \to \overline{u}$ a.e on $\mathbb{R}^N$ and $\overline{u_k} \to \overline{u}$ in $L^q_{loc}(\mathbb{R}^N)$ for every $q \in [1, p_s^*)$. Observe also that, since $\varepsilon_k^{-1}|\xi_k - y_k| < C < \infty$ for all k,

$$\begin{aligned} \delta = \int_{B(\xi_k,\varepsilon_k)} Q|u_k|^{p_s^*} &\leq \int_{B(y_k,\varepsilon_k(C+1))} Q|u_k|^{p_s^*} \\ &= \int_{B(0,C+1)} \overline{Q_k}|\overline{u}_k|^{p_s^*} \end{aligned} \tag{3.2.3}$$

and thus it implies that $|B(\xi_k, \varepsilon_k) \cap \Omega| > 0$.

We claim that $\overline{u} \neq 0$. Suppose by contradiction that $\overline{u} = 0$ a.e, then $\overline{u_k} \to 0$ in $L^q_{loc}(\mathbb{R}^N)$ for every $q \in [1, p_s^*)$. Let $h \in C_0^\infty(\mathbb{R}^N)$ be positive such that

$$supp(h) \subset B(z, 1) \subset B(0, 3/2), \quad \text{for any } z \in B(0, 1/2). \tag{3.2.4}$$

In Lemma 3.2.2, we take $r = 3/2$ and $R = 2$, then the following Sobolev inequality holds for functions in $D_0^{s,p}(B(0, 3/2))$, that is

$$\left(\int_{B(0,3/2)} |\overline{u_k}|^{p_s^*}\, dx\right)^{\frac{p}{p_s^*}} \leq \mathcal{T} \int_{B(0,2)\times B(0,2)} \frac{|\overline{u_k}(x) - \overline{u_k}(y)|^p}{|x-y|^{N+sp}}\, dx\, dy = \mathcal{T}[\overline{u_k}]^2_{D^{s,p}(B(0,2))}, \tag{3.2.5}$$

for a constant $\mathcal{T} = \mathcal{T}(N, s) > 0$. By Hölder inequality and (3.2.5), since $h\overline{u_k} \in D_0^{s,p}(B(0, 3/2))$, it follows that

$$\int_{\mathbb{R}^N} \overline{Q_k} h^p |\overline{u_k}|^{p_s^*}\,dx \le \mathcal{T}\left(\max_{\overline{\Omega}} Q\right)^{\frac{N-sp}{N}} \left(\int_{B(z,1)} \overline{Q_k}|\overline{u_k}|^{p_s^*}\,dx\right)^{\frac{sp}{N}} [h\overline{u_k}]^p_{D^{s,p}(B(0,2))} \quad (3.2.6)$$

for some positive constant $\mathcal{T}$ depending only on N, s, p. We observe that by definition of DE_∞

$$\int_{\mathbb{R}^{2N}} \frac{J_p(\overline{u_k}(x)-\overline{u_k}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$= \int_{\mathbb{R}^N} |\overline{u_k}|^{p_s^*-2}\overline{u_k}\varphi\,dx + \langle DE_\infty(\overline{u_k}),\varphi\rangle, \quad \text{for any } \varphi\in D_0^{s,p}(\Omega_k).$$

Then, by applying Lemma 3.2.3 for every $k\in\mathbb{N}$ with the choices

$$\Omega := \Omega_k, \quad \Omega' := B(0,2), \quad u := \overline{u_k}, \quad \psi := h, \quad F := \overline{Q_k}|\overline{u_k}|^{p_s^*-2}\overline{u_k} + E'_\infty(\overline{u_k}),$$

we get

$$\begin{aligned}
&\left(\int_{B(0,2)\times B(0,2)} \frac{|\overline{u_k}(x)h(x)-\overline{u_k}(y)h(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)\\
&\le \mathcal{C}\int_{B(0,2)\times B(0,2)} \frac{|h(x)-h(y)|^p}{|x-y|^{N+sp}}\left(|\overline{u_k}(x)|^p+|\overline{u_k}(y)|^p\right)dx\,dy\\
&+\mathcal{C}\left(\sup_{y\in B(0,3/2)}\int_{\mathbb{R}^N\setminus B(0,2)} \frac{|\overline{u_k}(x)|^{p-1}}{|x-y|^{N+sp}}\,dx\right)\int_{B(0,3/2)}|\overline{u_k}|h^p\,dx\\
&+\mathcal{C}\int_{B(0,3/2)} \overline{Q_k}h^p|\overline{u_k}|^{p_s^*}\,dx + \mathcal{C}\left|\langle DE_\infty(\overline{u_k}),\overline{u_k}h^p\rangle\right|. \qquad (3.2.7)
\end{aligned}$$

From (3.2.3) we know that $B(0,2)\cap\Omega_k$ is a nonempty open set. Now we will estimate the terms on the right-hand side of (3.2.7). For the first term on the right-hand side, using the change of variable $z=y-x$, and using the strong L^p convergence of $\{\overline{u_k}\}_{k\in\mathbb{N}}$ to 0 we have

$$\int_{B(0,2)\times B(0,2)} \frac{|h(x)-h(y)|^p}{|x-y|^{N+sp}}\left(|\overline{u_k}(x)|^p+|\overline{u_k}(y)|^p\right)dx\,dy$$

$$
\begin{aligned}
&= \int_{B(0,2)\times B(0,2)} \frac{|h(x)-h(z+x)|^p}{|z|^p} \frac{1}{|z|^{N+sp-p}} \left(|\overline{u_k}(x)|^p + |\overline{u_k}(z+x)|^p\right) dz\, dx \\
&\leq \int_{B(0,2)\times B(0,2)} \left(\int_0^1 \frac{|\nabla h(x+tz)|}{|z|^{\frac{N}{p}+s-1}} dt \right)^p \left(|\overline{u_k}(x)|^p + |\overline{u_k}(z+x)|^p\right) dz\, dx \\
&= \int_{B(0,2)\times B(0,2)} \left(\int_0^1 |\nabla h(x+tz)|^p \, dt \right) \frac{1}{|z|^{N+sp-p}} \left(|\overline{u_k}(x)|^p + |\overline{u_k}(z+x)|^p\right) dz\, dx \\
&\leq \|\nabla h\|_{L^\infty}^p \int_{B(0,2)} \left(\int_{B(0,2)} \frac{1}{|x-y|^{N+sp-p}} dy \right) |\overline{u_k}(x)|^p \, dx \\
&+ \|\nabla h\|_{L^\infty}^p \int_{B(0,2)} \left(\int_{B(0,2)} \frac{1}{|x-y|^{N+sp-p}} dx \right) |\overline{u_k}(y)|^p \, dy \\
&= o_k(1). \qquad (3.2.8)
\end{aligned}
$$

For the second term on the right-hand side of (3.2.7), by using Hölder inequality and for every $y \in B(0,3/2)$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^N \setminus B(0,2)} \frac{|\overline{u_k}(x)|^{p-1}}{|x-y|^{N+sp}} dx &\leq \left(\int_{\mathbb{R}^N \setminus B(0,2)} |x-y|^{(-N-sp)\frac{p_s^*}{p_s^*-p+1}} dx \right)^{\frac{p_s^*-p+1}{p_s^*}} \\
&\quad \cdot \left(\int_{\mathbb{R}^N} |\overline{u_k}|^{p_s^*} \right)^{\frac{p-1}{p_s^*}} \qquad (3.2.9) \\
&\leq M
\end{aligned}
$$

for some constant $M > 0$, and due to the strong L^p convergence of $\{\overline{u_k}\}_{k\in\mathbb{N}}$ to 0 , we have

$$
\int_{B(0,3/2)} |\overline{u_k}|^{p-1} h^p \, dx = o_k(1). \qquad (3.2.10)
$$

For the third term by recalling the definition of Levy Concentration function, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} \overline{Q_k} h^p |\overline{u_k}|^{p_s^*} dx &\leq \mathcal{T} \left(\max_{\overline{\Omega}} Q \right)^{\frac{N-sp}{N}} \left(\int_{B(z,1)} \overline{Q_k} |\overline{u_k}|^{p_s^*} dx \right)^{\frac{sp}{N}} [h\overline{u_k}]^p_{D^{s,p}(B(0,2))} \\
&\leq \mathcal{T} \left(\max_{\overline{\Omega}} Q \right)^{\frac{N-sp}{N}} \delta^{\frac{sp}{N}} \quad [h\overline{u_k}]^p_{D^{s,p}(B(0,2))}. \qquad (3.2.11)
\end{aligned}
$$

For the last term, using Lemma 3.2.5 and the fact $E'_\infty(u_k) \to 0$, first we have

$$\sup_{\varphi \in D_0^{s,p}(\Omega_k)} \left| \left\langle DE_\infty(\overline{u_k}), \frac{\varphi}{[\varphi]_{D^{s,p}(\mathbb{R}^N)}} \right\rangle \right| = o_k(1),$$

and since $\{h^p \overline{u_k}\}_{k\in\mathbb{N}}$ is bounded in $D_0^{s,p}(\Omega_k)$ in view of Lemma 3.2.4, the above equality implies that

$$|\langle DE_\infty(\overline{u_k}), h^p \overline{u_k}\rangle| = o_k(1). \tag{3.2.12}$$

Now by using (3.2.8)-(3.2.12) in (3.2.7), we have

$$\begin{aligned} [h\overline{u_k}]^p_{D^{s,p}(B(0,2))} &\leq \mathcal{C}\mathcal{T}\left(\max_{\overline{\Omega}} Q\right)^{\frac{N-sp}{N}} \delta^{\frac{sp}{N}} \quad [h\overline{u_k}]^p_{D^{s,p}(B(0,2))} + o_k(1) \\ &\leq \frac{1}{2}[h\overline{u_k}]^p_{D^{s,p}(B(0,2))} + o_k(1). \end{aligned}$$

This implies $[h\overline{u_k}]^p_{D^{s,p}(B(0,2))} = o_k(1)$ as $k \to \infty$. Thus by using Lemma 3.2.2, it implies

$$\int_{B(0,3/2)} |h\overline{u_k}|^{p_s^*}\, dx = o_k(1). \tag{3.2.13}$$

Since condition (3.2.13) holds for every $h \in C_0^\infty(B(z,1))$, we obtain that $\{\overline{u_k}\}_{k\in\mathbb{N}}$ converges to zero in $L^{p_s^*}_{loc}(B(z,1))$. Again as $z \in B(0,1/2)$ in (3.2.4) is arbitrary, we conclude that $\{\overline{u_k}\}_{k\in\mathbb{N}}$ converges to zero in $L^{p_s^*}(B(0,1))$, which contradicts (3.2.3). Hence $\overline{u} \neq 0$.

Step 3: After passing to a subsequence, we have that $y_k \to y_0$ in $\mathbb{R}^N$ and $\varepsilon_k \to \varepsilon$ in $[0,\infty)$. If $\varepsilon \neq 0$ then, as $\overline{u_k} \rightharpoonup 0$ weakly in $D_0^{s,p}(\Omega)$, we would have that $\overline{u} = 0$, a contradiction. Thus, $\varepsilon_k \to 0$. Therefore, $d_k = (\varepsilon_k^{-1} dist(y_k, \partial\Omega)) \to d \in [0,\infty]$,

$$\overline{Q_k} \to Q_0 := Q(y_0) \qquad \text{in} \quad L^\infty_{loc}(\mathbb{R}^N).$$

We consider two cases:

1) If $d \in [0,\infty)$ then $y_0 \in \partial\Omega$. If, after passing to a subsequence, (y_k) belongs to $\overline{\Omega}$ we set $d_0 := -d$, otherwise we set $d_0 := d$. Let

$$\mathbb{H} = \{z \in \mathbb{R}^N : z \cdot \nu_0 > d_0\},$$

where ν_0 is the inward pointing unit normal to $\partial\Omega$ at y_0. Since y_0 is K-invariant, ν_0 is K-invariant and so is $\mathbb{H}$. An easy geometric argument shows that, if X is compact and $X \subset \mathbb{R}^N \setminus \overline{\mathbb{H}}$, then there exists k_0 such that $X \subset \mathbb{R}^N \setminus \Omega_k$ for all $k \geq k_0$. In particular, as $\overline{u_k} = 0$ in $\mathbb{R}^N \setminus \Omega_k$ and $\overline{u_k} \to \overline{u}$ a.e. in $\mathbb{R}^N$, we have that $\overline{u} = 0$ a.e. in $\mathbb{R}^N \setminus \mathbb{H}$. Therefore, $\overline{u} \in D_0^{s,p}(\mathbb{H})$. Also, if X is compact and $X \subset \mathbb{H}$, there exists k_0 such that $X \subset \Omega_k$ for all $k \geq k_0$.

2) If $d = \infty$, we set $\mathbb{H} = \mathbb{R}^N$. It follows from (3.2.1) that $y_k \in \Omega$ and, again, if X is compact and $X \subset \mathbb{H}$, there exists k_0 such that $X \subset \Omega_k$ for all $k \geq k_0$.
Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{H})$ and set $\varphi_k(x) := \varepsilon_k^{\frac{sp-N}{p}} \varphi(\varepsilon_k^{-1}(x - y_k))$. Then, by the previous remarks, $\mathrm{supp}(\varphi) \subset \Omega_k$ for k large enough and, consequently, $\mathrm{supp}(\varphi_k) \subset \Omega$. As (φ_k) is bounded in $D_0^{s,p}(\Omega)$ we have that

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(\overline{u_k}(x) - \overline{u_k}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy - \int_{\Omega_k} \overline{Q_k} |\overline{u_k}|^{p_s^*-2} \overline{u_k} \varphi$$
$$= \langle DE_Q(\overline{u_k}), \varphi_k \rangle + o_k(1). \tag{3.2.14}$$

It is easy to check that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(\overline{u_k}(x) - \overline{u_k}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy$$
$$= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(\overline{u}(x) - \overline{u}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy \tag{3.2.15}$$

for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{H})$. Similarly,

$$\lim_{k\to\infty} \int_{\mathbb{H}} \overline{Q_k} |\overline{u_k}|^{p_s^*-2} \overline{u}_k \varphi = \int_{\mathbb{H}} Q_0 |\overline{u}|^{p_s^*-2} \overline{u} \varphi. \tag{3.2.16}$$

It follows from (3.2.14), (3.2.15) and (3.2.16) that $\overline{u}$ is a weak solution to problem

$$(-\Delta)_p^s u = Q_0 |u|^{p_s^*-2} u, \qquad u \in D_0^{s,p}(\mathbb{H}).$$

Therefore,

$$\widetilde{u} := Q_0^{\frac{N-sp}{sp^2}} \overline{u}$$

is a nontrivial, K-invariant solution to problem $(P^{s,K}_{p,\infty})$.

Step 4: We define $v_k \in D^{s,p}_0(\Omega)^G$ as follows: Let $\varphi \in C^\infty(\mathbb{R}^N)$ be radially symmetric and such that $0 \le \varphi \le 1, \varphi \equiv 1$ on $B(0,1)$ and $\varphi \equiv 0$ outside of $B(0,2)$. Let

$$4\rho_k := \min\left\{dist(y_k, \partial\Omega), |gy_k - g'y_k| \colon [g] \neq [g'] \in G/K\right\}.$$

Thus, $\varepsilon_k^{-1}\rho_k \to \infty$. Now as $G_{y_k} = K$ and $\widetilde{u}$ is K-invariant, the function

$$w_k := \sum_{[g]\in G/K} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u}(\varepsilon_k^{-1} g^{-1}(\,\cdot\, - gy_k))\, \varphi(\rho_k^{-1}(\,\cdot\, - gy_k)) \in D^{s,p}_0(\Omega)$$

is also G-invariant. Indeed, for $\tau \in G/K$, using the fact that φ is radial, we have

$$\begin{aligned}
w_k(\tau^{-1}x) &= \sum_{[g]\in G/K} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u}(\varepsilon_k^{-1} g^{-1}(\tau^{-1}x - gy_k))\, \varphi(\rho_k^{-1}(\tau^{-1}x - gy_k)) \\
&= \sum_{[g]\in G/K} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u}(\varepsilon_k^{-1} g^{-1}\tau^{-1}(x - \tau gy_k))\, \varphi(\rho_k^{-1}\tau^{-1}(x - \tau gy_k)) \\
&= \sum_{[g]\in G/K} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u}(\varepsilon_k^{-1} g^{-1}\tau^{-1}(x - \tau gy_k))\, \varphi(\rho_k^{-1}(x - \tau gy_k)) \\
&= \sum_{[\hat{g}]\in G/K} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u}(\varepsilon_k^{-1} \hat{g}^{-1}(x - \hat{g}y_k))\, \varphi(\rho_k^{-1}(x - \hat{g}y_k)) \\
&= w_k(x)
\end{aligned}$$

We set $v_k := u_k - w_k \in D^{s,p}_0(\Omega)^G$. So the result (iv) of the Proposition (3.2.1) follows from the fact that $\varepsilon_k^{-1}\rho_k \to \infty$ and using Lemma 2.5.1. Next as $|G/K| < \infty$, without loss of generality we assume that $G/K = \{[g_1], ..., [g_n]\}$. Then as $\overline{u_k} \rightharpoonup \overline{u} = Q(y)^{\frac{sp-N}{sp^2}}\, \widetilde{u}$ weakly in $D^{s,p}_0(\mathbb{R}^N)$, and u_k is G-invariant, it follows that

$$\begin{aligned}
&\left\| u_k - \sum_{i=1}^{n} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp^2}}\, \widetilde{u} g_i^{-1}\left(\frac{\cdot - g_i y_k}{\varepsilon_k}\right)\right\|^p \\
&= \left\| \varepsilon_k^{\frac{N-sp}{p}} u_k(\varepsilon_k \,\cdot\, + g_i y_k) - \sum_{i=1}^{n} Q(y_0)^{\frac{sp-N}{sp^2}}\, \widetilde{u} g_i^{-1}\left(\cdot + \frac{g_1 y_k - g_i y_k}{\varepsilon_k}\right)\right\|^p
\end{aligned}$$

$$= \left\| \overline{u}_k g_1^{-1} - Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u} g_1^{-1} - \sum_{i\neq 1} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u} g_i^{-1}\left(\cdot + \frac{g_1 y_k - g_i y_k}{\varepsilon_k}\right)\right\|^p$$

$$= \left\| \overline{u}_k g_1^{-1} - \sum_{i\neq 1} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u} g_i^{-1}\left(\cdot + \frac{g_1 y_k - g_i y_k}{\varepsilon_k}\right)\right\|^p - \left\| Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u} g_1^{-1}\right\|^p + o_k(1)$$

$$= \left\| u_k - \sum_{i\neq 1} Q(y_0)^{\frac{sp-N}{sp^2}} \widetilde{u} g_i^{-1}\left(\cdot + \frac{g_1 y_k - g_i y_k}{\varepsilon_k}\right)\right\|^p - Q(y_0)^{\frac{sp-N}{sp}} \|\widetilde{u}\|^p + o_k(1)$$

and, inductively we have

$$\|u_k\|^p = \left\| u_k - \sum_{i=1}^{n} \varepsilon_k^{\frac{sp-N}{p}} Q(y_0)^{\frac{sp-N}{sp}} \widetilde{u} g_i^{-1}\left(\frac{\cdot - g_i y_k}{\varepsilon_k}\right)\right\|^p + \frac{n}{Q(y)^{\frac{N-sp}{sp}}} \|\widetilde{u}\|^p + o_k(1)$$

That is,

$$\|u_k\|^p = \|v_k\|^p + \frac{|G/K|}{Q(y_0)^{\frac{N-sp}{sp}}} \|\widetilde{u}\|^p + o_k(1). \tag{3.2.17}$$

Also using Brézis-Lieb type Lemma 3.2.1, we obtain

$$\int_\Omega Q|u_k|^{p_s^*} = \int_\Omega Q|v_k|^{p_s^*} + \left(\frac{|G/K|}{Q(y_0)^{\frac{N-sp}{sp}}}\right) \int_{\mathbb{R}^N} |\widetilde{u}|^{p_s^*} \tag{3.2.18}$$

and thus (v) follows from (3.2.17)-(3.2.18) and hence this completes the proof of the proposition. □

Now we conclude the proof of Theorem 3.1.1 by giving induction argument as in [39].

3.2.1 Proof of Theorem 3.1.1

First we note that the G-invariant Palais-Smale sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $D_0^{s,p}(\Omega)$, and thus from

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy = \frac{N}{s} E_Q(u_k) - \frac{N-sp}{Np} DE_Q(u_k)u_k = \frac{Nc}{s} + o_k(1),$$

we infer that $c \geq 0$. Let $u_k \rightharpoonup u$ weakly in $D_0^{s,p}(\Omega)^G$ and $u_k \to u$ a.e. in Ω. Also we have $DE_Q(u) = 0$ and hence $u_k^1 := u_k - u$ is a G-PS-sequence such that $u_k^1 \rightharpoonup 0$ weakly in

$D_0^{s,p}(\Omega)^G$ and $\lim_{k\to\infty} E_Q(u_k^1) = c - E_Q(u) + o_k(1) < c$. If $c - E_Q(u) > 0$, then applying the Proposition (3.2.1) on the sequence $\{u_k^1\}_{k\in\mathbb{N}}$, we get a closed subgroup K_1 of finite index in G, a sequence $\{y_{1,k}\}_{k\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$, a sequence $\{\varepsilon_{1,k}\}_{k\in\mathbb{N}}$ in $(0,\infty)$, a K_1-invariant solution $\widetilde{u_1}$ of the limiting problem $(P_{p,\infty}^{s,K_1})$ and a G-invariant Palais-Smale sequence $\{v_k^1\}_{k\in\mathbb{N}}$ for E such that

$$\begin{aligned}\lim_{k\to\infty} E_Q(v_k^1) &= \lim_{k\to\infty} E_Q(u_k^1) - \left(\frac{|G/K|}{Q(y_0)^{\frac{N-sp}{sp}}}\right) E_\infty(\widetilde{u_1}) \\ &= c - E_Q(u) - \left(\frac{|G/K|}{Q(y_0)^{\frac{N-sp}{sp}}}\right) E_\infty(\widetilde{u_1})\end{aligned}$$

Now noticing that E_Q have G-invariant Palais-Smale sequence only at non-negative level, the above iteration of Proposition (3.2.1) stops after m number of times, and thus we have the conclusion of Theorem 3.1.1. □

3.3 Conclusion

In this chapter, we have studied Struwe's type compactness result for nonlocal critical exponent problems in the symmetric domain. The Theorem 3.1.1 provides a precise description of all G-invariant Palais-Smale sequences for the energy functional associated with the nonlocal problem $(P_{p,Q,\Omega}^{s,G})$, which relates the symmetries of the concentration points to those of the solution to the limit problem that concentrates at those points. We observe that analogous to the local case, for $p \neq 2$, the nonexistence result for sign-changing solutions to problem $(P_{p,Q,\Omega}^{s,G})$ in half-space is not available. This problem arises as a limiting problem to $(P_{p,Q,\Omega}^{s,G})$ and must be taken into account when describing the lack of compactness. The Theorem 3.1.1 allows us to use the symmetries to tackle the lack of compactness.

Corollary 3.1.1 implies that the lack of compactness can only occur if Ω contains some finite G-orbit. It also asserts that the problem $(P_{p,Q,\Omega}^{s,G})$ has infinitely many solutions if every G-orbit of Ω is infinite.

In the continuation, global compactness results for weakly coupled purely critical nonlocal systems in the bounded domain under symmetry assumptions would be an interesting problem to be investigated.

4

Multiple solutions to nonlocal critical exponent problem in symmetric domain

In this chapter, we investigate the existence of a positive solution and multiple sign-changing solutions to purely critical exponent problems involving nonlocal operators in some domains with symmetry assumptions.

4.1 Coron's type problem involving fractional Laplace operator

The first part of this chapter deals with the critical exponent problem involving fractional Laplace operator

$$(P_\Omega^s)\begin{cases} (-\Delta)^s u &= |u|^{2_s^*-2}\,u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \Omega^c, \end{cases}$$

where $2_s^* := \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent and $\Omega \subset \mathbb{R}^N$ is a bounded annular-shaped domain which is invariant under a group G of orthogonal transformations of $\mathbb{R}^N$.

We state theorems that assert the multiplicity of solutions for the problem (P_Ω^s). We assume that Ω is invariant under the action of a closed subgroup G of $SO(N)$. We denote by $Gx := \{rx : r \in G\}$ the G-orbit of $x \in \mathbb{R}^N$, by $|G/G_x|$ its cardinality and by

$$l = l(G) := \min\{|G/G_x|\colon x \in \mathbb{R}^N \setminus \{0\}\}.$$

We state the following multiplicity result for the domain with a small hole.

Theorem 4.1.1. *Given $\delta > 0$, there exists R_δ with the following property: for every closed subgroup G of $SO(N)$ with $l = l(G) \geq 2$ and every G-invariant domain Ω such that*

$$0 \notin \Omega, \quad \Omega \supset \{x \in \mathbb{R}^N : \; R_1 < |x| < R_2\} \quad \textit{and} \quad 0 < R_1/R_2 < R_\delta,$$

problem(P_Ω^s) has at least one non-negative G-invariant solution u_1 and $l-1$ pairs of G-invariant sign-changing solutions $\pm u_2, ..., \pm u_l$ which satisfy

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \leq jS_s^{\frac{N}{2s}} + \delta, \qquad j = 1, ..., l.$$

The second main result is about existence of solutions for the domains with a hole of arbitrary size.

Theorem 4.1.2. *Given $0 < R_1 < R_2$ and $m \in \mathbb{N}$, there exist a positive integer l_0, depending on m and R_2/R_1 such that, for every closed subgroup G of $SO(N)$ with $l = l(G) \geq l_0$ and every G-invariant domain Ω such that*

$$0 \notin \Omega, \quad \Omega \supset \{x \in \mathbb{R}^N : R_1 < |x| < R_2\},$$

problem (P^s_Ω) has at least one non-negative G-invariant solution u_1 and $m-2$ pairs of G-invariant sign-changing solutions $\pm u_2, ..., \pm u_{m-1}$.

Note that the domain with hole of arbitrary size can be handled by considering large value of $l = l(G)$. Indeed, if N is even, for the group of rotations $G_n = \{e^{2\pi ik/n} : k = 0, ..., n-1\}$ we have $l(G_n) = n$ where the group action is given by multiplication on each complex coordinate of $\mathbb{C}^N/2 \cong \mathbb{R}^N$.

4.1.1 Radial solution to $(P^s_{A_{R_1,R_2}})$

In this section, we show the existence of radial solution to the problem (P^s_Ω). Note that the weak solution to (P^s_Ω) are given by the critical points of the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2^*_s}\int_\Omega |u(x)|^{2^*_s}\,dx. \tag{4.1.1}$$

We observe that I is a C^1 functional with $DI \in D^{-s,2}(\Omega)$ defined as

$$\langle DI(u), v\rangle = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u|^{2^*_s-2}uv\,dx, \tag{4.1.2}$$

where $v \in D^{s,2}_0(\Omega)$. Then using (2.3.3) and the fact that Ω is G-invariant one can check that I is G-invariant, i.e. $I(gu) = I(u)$ for all $u \in D^{s,2}_0(\Omega)$ and $g \in G$. Therefore by the principle of symmetric criticality, the critical points of the restriction of I to the space $D^{s,2}_0(\Omega)^G$ are G-invariant solutions of problem (P^s_Ω). The G-invariant Nehari manifold for the functional I is defined by

$$\mathcal{N}(\Omega)^G := \left\{u \in D^{s,2}_0(\Omega)^G : u \neq 0, \|u\|^2 = \|u\|^{2^*_s}_{L^{2^*_s}(\Omega)}\right\}.$$

For $G = \{1\}$, $\mathcal{N}(\Omega)^G = \mathcal{N}(\Omega)$ is the usual Nehari manifold and we set

$$c_\infty := \inf\{I(u) : u \in \mathcal{N}(\Omega)\} = \frac{s}{N} S_s^{\frac{N}{2s}},$$

where S_s is the best constant for the Sobolev embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$.
Note that for $G = \{1\}$ and Ω bounded, S_s it is never attained but if G is nontrivial, the infimum of I on $\mathcal{N}(\Omega)^G$ might be attained.
We note that for $\Omega = A_{R_1,R_2}$ and $G = SO(N)$ then the infimum

$$c(R_1, R_2) := \inf\{I(u) : u \in \mathcal{N}(A_{R_1,R_2})^{SO(N)}\}$$

is always attained. We define radial fractional Sobolev spaces as follows:

$$D^{s,2}_{0,rad}(\Omega) := \{u \in D^{s,2}_0(\Omega) : u(x) = u(|x|)\}$$

Lemma 4.1.1. *Let $N > 2s$, $0 < R_1 < R_2$ and $\Omega = A_{R_1,R_2}$, there exist a radially symmetric, non-negative solution of (P^s_Ω).*

Proof. Consider the energy functional $I^+ : D^{s,2}_{0,rad}(\Omega) \to \mathbb{R}$, defined as

$$I^+(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2_s^*}\|u^+\|^{2_s^*}_{L^{2_s^*}(\Omega)}$$

where $u^+ := \max\{u, 0\}$. Note that I^+ is a C^1 functional with $DI^+ \in D^{-s,2}_{rad}(\Omega)$ defined by

$$\langle DI^+(u), v\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}}\, dx\, dy - \int_\Omega |u^+|^{2_s^*-2}u^+ v\, dx,$$

where $v \in D^{s,2}_{0,rad}(\Omega)$ and admits the mountain pass geometry. Now we claim that I^+ satisfies Palais-Smale condition. Indeed let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence in $D^{s,2}_{0,rad}(\Omega)$ such that

$$c := \sup_m I^+(u_m) < \infty \quad \text{and} \quad DI^+(u_m) \to 0 \quad \text{in} \quad D^{-s,2}_{rad}(\Omega),$$

where $D^{-s,2}_{rad}(\Omega)$ denotes the dual space of $D^{s,2}_{0,rad}(\Omega)$. Now

$$I^+(u_m) = \frac{1}{2}\|u_m\|^2 - \frac{1}{2^*_s}\|u_m^+\|^{2^*_s}_{L^{2^*_s}(\Omega)} \tag{4.1.3}$$

and

$$\langle DI^+(u_m), v\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_m(x)-u_m(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u_m^+|^{2^*_s-2}u_m^+ v\,dx, \tag{4.1.4}$$

where $v \in D^{s,2}_{0,rad}(\Omega)$. In particular

$$\langle DI^+(u_m), u_m\rangle = \|u_m\|^2 - \|u_m^+\|^{2^*_s}_{L^{2^*_s}(\Omega)}. \tag{4.1.5}$$

It is straight forward to prove that sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded. Thus up to a subsequence $\{u_m\}_{m\in\mathbb{N}}$ converges weakly to a limit u in $\in D^{s,2}_{0,rad}(\Omega)$. By the compact embedding $D^{s,2}_{0,rad}(\Omega) \hookrightarrow L^p(\Omega)$ for every $1 \le p < \infty$ in annular domains [22], we have strong convergence of u_m to u in $L^{2^*_s}(\Omega)$ and $|u_m|^{2^*_s-2}u_m$ to $|u|^{2^*_s-2}u$ in $L^{\frac{2^*_s}{2^*_s-1}}(\Omega)$. Passing to the limit as $m \to \infty$ in (4.1.4) we get

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u^+|^{2^*_s-2}u^+ v\,dx = 0, \tag{4.1.6}$$

$v \in D^{s,2}_{0,rad}(\Omega)$ and passing to the limit in (4.1.5) and by setting $v = u$ in (4.1.6) we get

$$\lim_{m\to\infty}\|u_m\|^2 = \|u^+\|^{2^*_s}_{L^{2^*_s}(\Omega)} = \|u\|^2. \tag{4.1.7}$$

Thus $u_m \rightharpoonup u$ weakly and $\|u_m\|^2 \to \|u\|^2$ and so $u_m \to u$ strongly in $D^{s,2}_{0,rad}(\Omega)$. This proves that I^+ satisfies the Palais-Smale condition. So by the mountain pass theorem [87], there exists a critical point u of I^+. Finally, by using Theorem 2.5.1, $u \in C^s(\bar{\Omega})$ and Lemma 2.5.2 implies that u is positive in Ω. □

In the following lemma, we construct some radially symmetric test functions with controlled energy. These functions will be used in the proof of our main results.

Lemma 4.1.2. *Given* $0 < R_1 < R_2$ *and* $m \in \mathbb{R}$, *there exist* $R_1 =: P_0 < P_1 < \cdots < P_m :=$

R_2 and non-negative radial functions $\omega_1, ..., \omega_m \in \mathcal{N}(A_{R_1,R_2})^{SO(N)}$ such that

$$supp(\omega_i) \subset A_{P_i,P_{i+1}} \quad and \quad I(\omega_i) = c(R_1^{\frac{1}{m}}, R_2^{\frac{1}{m}}), \quad i = 1, ..., m$$

where I be as in (4.1.1).

Proof. Let $\lambda > 0$ and $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be the dilation given by $\phi(x) = \lambda x$. Under this Möbius transformation, any open and bounded set Ω transforms into $\phi(\Omega) := \lambda\Omega$. For $u \in D_0^{s,2}(\phi(\Omega))$, we define $u_\phi \in D_0^{s,2}(\Omega)$ by

$$u_\phi(x) = (det\ D\phi)^{\frac{N-2s}{2N}} u(\phi(x)), \tag{4.1.8}$$

where det $D\phi = \lambda^N$ is the Jacobian determinant of the transformation ϕ. Then the map $u \mapsto u_\phi$ is a linear isometry of $D_0^{s,2}(\phi(\Omega)) \cong D_0^{s,2}(\Omega)$ and of $L^{2_s^*}(\phi(\Omega)) \cong L^{2_s^*}(\Omega)$, as

$$\begin{aligned} &\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_\phi(x) - u_\phi(y))(v_\phi(x) - v_\phi(y))}{|x-y|^{N+2s}}\, dx\, dy \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}}\, dx\, dy \end{aligned} \tag{4.1.9}$$

and

$$\int_\Omega |u_\phi(x)|^{2_s^*}\, dx = \int_{\phi(\Omega)} |u(x)|^{2_s^*}\, dx \tag{4.1.10}$$

Also from (4.1.9)-(4.1.10) we have the following dilation invariance property for I.

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_\phi(x) - u_\phi(y))^2}{|x-y|^{N+2s}}\, dx\, dy - \frac{1}{2_s^*}\int_\Omega |u_\phi(x)|^{2_s^*}\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\, dx\, dy - \frac{1}{2_s^*}\int_{\phi(\Omega)} |u(x)|^{2_s^*}\, dx. \end{aligned} \tag{4.1.11}$$

Take $\lambda = (R_2/R_1)^{1/m}$ and define $P_i = \lambda^i R_1$, for $i = 1, ..., m$. Let ϕ be the dilation by λ, that is $\phi(x) = \lambda x$. Now, fix a non-negative radial minimizer ω_1 of I on $\mathcal{N}(A_{P_0,P_1})^{SO(N)}$ and define

$$\omega_{i+1}(x) := \lambda^{\frac{N-2s}{2}} \omega_i(\lambda x).$$

Since $\phi(A_{P_{i-1},P_i}) = A_{P_i,P_{i+1}}$, the dilation invariance of I as given by (4.1.11) yields that

ω_{i+1} is a non-negative radial minimizer of I on $\mathcal{N}(A_{P_i,P_{i+1}})^{SO(N)}$, with $I(\omega_{i+1}) = I(\omega_1) = c(P_0, P_1)$. Finally, by rescaling, it follows that $c(P_0, P_1) = c(R_1^{1/m}, R_2^{1/m})$. □

4.1.2 Variational principle for sign-changing solutions for (P_Ω^s)

In this section we discuss the variational principle for the sign-changing solutions for the problem (P_Ω^s). We note that in the local case, that is for $s = 1$, if $u \in \mathcal{N}^G$ then u^+ and u_- also belong to $\mathcal{N}^G$. In general, in nonlocal setting we have

$$u \,\text{is a sign-changing solution to problem}\, (P_\Omega^s) \Rightarrow u^+ \notin \mathcal{N}^G \,\text{or}\, u^- \notin \mathcal{N}^G.$$

This is due to the nonlocal interactions between u^+ and u^- in the term $[u]_{D^{s,2}(\mathbb{R}^N)}$, given by

$$[u]^2_{D^{s,2}(\mathbb{R}^N)} - [u^+]^2_{D^{s,2}(\mathbb{R}^N)} - [u^-]^2_{D^{s,2}(\mathbb{R}^N)} > 0 \quad \text{if } u^\pm \neq 0.$$

Therefore we define the sign-changing Nehari set for the functional I as

$$\mathcal{N}_{sc}^G := \left\{u \in \mathcal{N}(\Omega)^G : u^\pm \neq 0,\ \langle I'(u), u^+\rangle = \langle I'(u), u^-\rangle = 0\right\},$$

where $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = -\min\{u(x), 0\}$ and $\langle , \rangle$ denotes the duality pairing of between $(D_0^{s,2}(\Omega))^*$ and $D_0^{s,2}(\Omega)$. Clearly, $\mathcal{N}^G$ contains all nontrivial solutions of problem (P_Ω^s) and sign-changing solutions of problem (P_Ω^s) lie on $\mathcal{N}_{sc}^G$. In the following lemma, we give some bounds for $u \in \mathcal{N}_{sc}$.

Lemma 4.1.3. *There exists $\eta_1, \eta_2 > 0$ such that*

(i) $\|u^\pm\| \geq \eta_1$ *for all* $u \in \mathcal{N}_{sc}^G$,

(ii) $\int_\Omega |u^\pm|^{2_s^*}\, dx \geq \eta_2$ *for all* $u \in \mathcal{N}_{sc}^G$.

Proof. For $u \in \mathcal{N}_{sc}^G$, we have $\langle I'(u), u^\pm\rangle = 0$. This implies

$$\int_\Omega |u|^{2_s^*-2} u u^+\, dx = \int_\Omega (u^+)^{2_s^*}\, dx.$$

For $u \in D_0^{s,2}(\Omega)$, we define $\Omega_- := \{x \in \Omega : u(x) < 0\}$ and $\Omega_+ := \{x \in \Omega : u(x) > 0\}$ and express

$$\mathbb{R}^N \times \mathbb{R}^N = ((\Omega_+ \cup \Omega_-) \times (\Omega_+ \cup \Omega_-)) \cup ((\Omega_+ \cup \Omega_-) \times \Omega^c) \cup (\Omega^c \times (\Omega_+ \cup \Omega_-)) \cup (\Omega^c \times \Omega^c).$$

We observe that

$$\begin{aligned}
&\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \int_{\Omega_+ \times \Omega_+} \frac{(u^+(x)-u^+(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\Omega_+ \times \Omega_-} \frac{(u^+(x)+u^-(y))(u^+(x))}{|x-y|^{N+2s}}\,dx\,dy \\
&+ \int_{\Omega_- \times \Omega_+} \frac{(u^-(x)+u^+(y))(u^+(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\Omega_+ \times \Omega^c} \frac{(u^+(x))^2}{|x-y|^{N+2s}}\,dx\,dy \\
&+ \int_{\Omega^c \times \Omega_+} \frac{(u^+(y))^2}{|x-y|^{N+2s}}\,dx\,dy,
\end{aligned}$$

and hence we get

$$\langle I'(u), u^+ \rangle = \langle I'(u^+), u^+ \rangle + R_1^+(u),$$

where

$$R_1^+(u) = \int_{\Omega_+ \times \Omega_-} \frac{u^-(y)u^+(x)}{|x-y|^{N+2s}}\,dx\,dy + \int_{\Omega_- \times \Omega_+} \frac{u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy.$$

As for $u \in \mathcal{N}_{sc}^G$, we have $\langle I'(u), u^\pm \rangle = 0$ and since $R_1^+(u) > 0$, we obtain $\langle I'(u^+), u^+ \rangle < 0$. This implies that

$$\|u^+\|^2 < \int_\Omega (u^+)^{2_s^*}\,dx.$$

Similarly, we obtain

$$\|u^-\|^2 < \int_\Omega (u^-)^{2_s^*}\,dx.$$

Now for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|u(x)|^{2_s^*} \leq \epsilon |u(x)|^2 + C_\epsilon |u(x)|^{2_s^*} \quad \text{for all } x \in \Omega. \tag{4.1.12}$$

So by using Sobolev inequalities, there exists $C^* > 0$ such that

$$\|u^\pm\|^2 \leq \epsilon C^* \|u^\pm\|^2 + C_\epsilon C^* \|u^\pm\|^{2_s^*}.$$

Since $2_s^* > 2$, taking $\epsilon = \frac{1}{2C^*}$, it is easy to show that (i) holds. Moreover, by using (4.1.12), we obtain

$$\eta_1^2 \leq \|u^\pm\|^2 \leq \epsilon C^* \|u^\pm\|^2 + C_\epsilon \|u^\pm\|^{2_s^*}_{L^{2_s^*}(\Omega)}.$$

Now by choosing $\epsilon = \frac{1}{2C^*}$, we get

$$\eta_2 = \frac{\eta_1^2}{2C_\epsilon} \leq \|u^\pm\|^{2_s^*}_{L^{2_s^*}(\Omega)}.$$

This concludes the proof of Lemma 4.1.3. □

Motivated by Theorem 3.2 in [9], the main idea here is to construct the multiple sign-changing critical points by constructing different invariant sets of descending flow defined by a pseudo-gradient vector field of the associated even functional I defined by (4.1.1) and using the topological tool, the genus, for these invariant sets.

We write $\mathcal{P}^G := \{u \in D_0^{s,2}(\Omega)^G : u \geq 0\}$ for the convex cone of the positive functions in $D_0^{s,2}(\Omega)^G$ and, for $\alpha \geq 0$, we set

$$B_\alpha(\mathcal{P}^G) := \{u \in D_0^{s,2}(\Omega)^G : \operatorname{dist}(u, \mathcal{P}^G) \leq \alpha\}$$

where $\operatorname{dist}(u, \mathcal{A}) := \inf_{v \in \mathcal{A}} \|u - v\|$.

Lemma 4.1.4. *There exists $\alpha > 0$ such that*

(i) $[B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G)] \cap \mathcal{N}_{sc}^G = \emptyset$, *and*

(ii) $B_\alpha(\mathcal{P}^G)$ *and* $B_\alpha(-\mathcal{P}^G)$ *are strictly positively invariant under* φ.

Proof. (i): For every u in $D_0^{s,2}(\Omega)^G$,

$$|u^-|_{2_s^*} = \min_{v \in \mathcal{P}^G} |u - v|_{2_s^*} \leq S_s^{-1/2} \min_{v \in \mathcal{P}^G} \|u - v\| = S_s^{-1/2} \operatorname{dist}(u, \mathcal{P}^G). \tag{4.1.13}$$

Therefore, using Lemma 4.1.3 (ii), there exists $\alpha > 0$ such that $\operatorname{dist}(u, \mathcal{P}^G) > \alpha$ for every $u \in \mathcal{N}_{sc}^G$. Moreover, since $\mathcal{N}_{sc}^G$ is symmetric with respect to origin, $\operatorname{dist}(u, -\mathcal{P}^G) = \operatorname{dist}(u, \mathcal{P}^G) > \alpha$, and (i) follows.

(ii): Here we follow the approach as in [47]. First we claim that $DI(u) = u - G(u)$ where

$G(u) \in D_0^{s,2}(\Omega)$ is the unique solution of the equation

$$\begin{aligned} (-\Delta)^s(G(u)) &= |u|^{2_s^*-2}u \quad \text{in } \Omega, \\ G(u) &= 0 \qquad\qquad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

Indeed note that $G(u)$ is uniquely determined by

$$\langle (G(u)), v\rangle := \int_\Omega |u|^{2_s^*-2} uv = \langle |u|^{2_s^*-2} u, v\rangle.$$

From integration by parts formula in [88], we get

$$\begin{aligned} \langle (-\Delta)^{s/2}(G(u)), (-\Delta)^{s/2} v\rangle &= \langle (-\Delta)^s(G(u)), v\rangle \\ &= \langle |u|^{2_s^*-2} uv\rangle \\ &= \int_\Omega |u|^{2_s^*-2} uv\, dx \end{aligned}$$

This implies

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{((G(u))(x) - (G(u))(y))(v(x)-v(y))}{|x-y|^{N+2s}} dx\, dy = \int_\Omega |u|^{2_s^*-2} uv\, dx.$$

Now from (4.1.2) we have

$$\begin{aligned} \langle DI(u), v\rangle &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dx\, dy - \int_\Omega |u|^{2_s^*-2} uv\, dx \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dx\, dy \\ &\quad - \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{((G(u))(x) - (G(u))(y))(v(x)-v(y))}{|x-y|^{N+2s}} dx\, dy \\ &= \langle u - (G(u)), v\rangle \end{aligned}$$

This proves the claim. Now from the maximum principle [90] we have $G(u) \in \mathcal{P}^G$ for $u \in \mathcal{P}^G$. Let $u \in D_0^{s,2}(\Omega)$. Then noticing that $dist(G(u), \mathcal{P}^G) \leq \|G(u)^-\|$ and recalling

(4.1.13), we get

$$\begin{aligned} dist(G(u), \mathcal{P}^G)\|G(u)^-\| &\leq \|G(u)^-\|^2 \\ &\leq \langle (G(u)), G(u)^- \rangle \\ &= \int_\Omega |u|^{2_s^*-2} uG(u)^- \, dx \\ &\leq \int_\Omega |u^-|^{2_s^*-1} |G(u)^-| \, dx \\ &\leq |u^-|_{2_s^*}^{2_s^*-1} \|G(u)^-\|_{2_s^*} \\ &\leq S_s^{-2_s^*/2} dist(u, \mathcal{P}^G)^{2_s^*-1} \|G(u)^-\| \end{aligned}$$

Hence $dist(G(u), \mathcal{P}^G) \leq S_s^{-2_s^*/2} dist(u, \mathcal{P}^G)^{2_s^*-1}$ for all $u \in D_0^{s,2}(\Omega)$. Thus, for $0 < \nu < 1$, there exists an $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$,

$$dist(G(u), \mathcal{P}^G) \leq \nu \; dist(u, \mathcal{P}^G) \text{ for every } u \in B_\alpha(\mathcal{P}^G).$$

Thus $G(u) \in int(B_\alpha(\mathcal{P}^G))$ for $u \in B_\alpha(\mathcal{P}^G)$. As $B_\alpha(\mathcal{P}^G)$ is closed and convex, Theorem 5.2 in [53] implies

$$u \in B_\alpha(\mathcal{P}^G) \implies \varphi(t, u) \in B_\alpha(\mathcal{P}^G) \text{ for } t \in [0, T(u)). \tag{4.1.14}$$

Now, assume that there exists $u \in B_\alpha(\mathcal{P}^G)$ and $t \in (0, T(u))$ such that $\varphi(t, u) \in \partial B_\alpha(\mathcal{P}^G)$. Then from Mazur's separation theorem, there exists a continuous linear functional $L \in D^{-s,2}(\Omega)$ and $\beta > 0$ such that $L(\varphi(t, u)) = \beta$ and $L(w) > \beta$ for $w \in intB_\alpha(\mathcal{P}^G)$. Consequently

$$\frac{\partial}{\partial r}\bigg|_{r=t} L(\varphi(r, u)) = L(-DI(\varphi(t, u))) = L(G(\varphi(t, u))) - \beta > 0.$$

Hence, there exists $\epsilon > 0$ such that $L(\varphi(r, u)) < \beta$ for $r \in (t - \epsilon, t)$. Thus $\varphi(r, u) \notin B_\alpha(\mathcal{P}^G)$ for $r \in (t - \epsilon, t)$. This contradicts (4.1.14) and hence (ii) holds. □

Note that for $\alpha > 0$ in Lemma 4.1.4, I has no sign changing solutions in $B_\alpha(\mathcal{P}^G) \cup B_\alpha(-\mathcal{P}^G)$.

4.1.3 Existence of multiple nodal solutions in annular domain

Let G be a closed subgroup of $SO(N)$, Ω be G-invariant and I be as in (4.1.1). Let

$$l := \min\{|G/G_x|: x \in \mathbb{R}^N \setminus \{0\}\}.$$

Lemma 4.1.5. *The energy functional I satisfy $(PS)_c$ in $D_0^{s,2}(\Omega)^G$ for every $c < lc_\infty$.*

Proof. First by the standard calculations we note that any $(PS)_c$ sequence $\{u_m\}_{m\in\mathbb{N}}$ of I is bounded in $D_0^{s,2}(\Omega)$. Thus

$$u_m \rightharpoonup u^0 \quad \text{in} \quad D_0^{s,2}(\Omega), \tag{4.1.15}$$

$$u_m \rightharpoonup u^0 \quad \text{in} \quad L^{2_s^*}(\Omega), \tag{4.1.16}$$

where (4.1.16) follows from the continuous embedding of $D_0^{s,2}(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$. Hence

$$v_m = u_m - u^0 \rightharpoonup 0 \quad \text{in} \quad D_0^{s,2}(\Omega). \tag{4.1.17}$$

If $v_m \to 0 \quad \text{in} \quad D_0^{s,2}(\Omega)$ then $(PS)_c$ conditions holds and thus completing the proof. So we assume that $v_m \nrightarrow 0 \quad \text{in} \quad D_0^{s,2}(\Omega)$. Now we proceed in two steps:

Step I: First we claim that the function u^0 is a weak solution of (P_Ω^s). Moreover

$$I(v_m) \to \beta \leq c - c_\infty \quad \text{and} \quad DI(v_m) \to 0.$$

In fact from (4.1.15) and (4.1.16), for any $\varphi \in C_0^\infty(\Omega)$, we obtain

$$\begin{aligned}
0 =& \lim_{m\to\infty} \langle DI(u_m),\varphi\rangle \\
=& \lim_{m\to\infty} \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_m(x)-u_m(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u_m|^{2_s^*-2}u_m\varphi\,dx\right) \\
=& \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u^0(x)-u^0(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u^0|^{2_s^*-2}u^0\varphi\,dx \\
=& \langle DI(u^0),\varphi\rangle.
\end{aligned}$$

Hence u^0 is a weak solution of (P^s_Ω). By the Brezis-Lieb lemma in [10], we have

$$\|v_m\|^2 = \|u_m\|^2 - \|u^0\|^2 + o_m(1) \quad \text{and} \; \|v_m\|^{2^*_s}_{L^{2^*_s}(\Omega)} = \|u_m\|^{2^*_s}_{L^{2^*_s}(\Omega)} - \|u^0\|^{2^*_s}_{L^{2^*_s}(\Omega)} + o_m(1),$$

where $\lim_{m\to\infty} o_m(1) = 0$. Using this we get

$$\begin{aligned} I(v_m) &= I(u_m) - I(u^0) + o_m(1) \quad \text{and} \\ DI(v_m) &= DI(u_m) - DI(u^0) + o_m(1) = o_m(1) \quad \text{in} \quad D^{-s}(\Omega). \end{aligned}$$

Therefore $I(v_m) \to \beta \leq c - c_\infty$ and $DI(v_m) \to 0$. This proves the claim.

Step II: Let u be a weak solution of (P^s_Ω), then

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u|^{2^*_s-2}u\varphi\,dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. By the density of $C_0^\infty(\Omega)$ in $D_0^{s,2}(\Omega)$ we get

$$0 = \langle DI(u), u\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega |u|^{2^*_s}\,dx.$$

Now as from the Sobolev inequality we have

$$S_s|u|^2_{2^*_s} \leq \|u\|^2_2 = \|u\|^{2^*_s}_{L^{2^*_s}(\Omega)},$$

we infer that any non-trivial critical point of I satisfies

$$I(u) = \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\|u\|^{2^*_s}_{L^{2^*_s}(\Omega)} \geq \frac{s}{N} S_s^{\frac{N}{2s}} = c_\infty > 0.$$

Now applying Propositon 3.2.1 on the sequence $\{u_m - u^0\}_{m\in\mathbb{N}}$, we have

$$I(u_m) \geq I(u^0) + l\, I_\infty(\overline{u}) + o_m(1).$$

where $\overline{u}$ is the solution of the associated limiting problem and $I_\infty(\overline{u})$ denotes the associated energy functional. Now we have

$$c - c_\infty \geq I(v_m) = I(u_m) - I(u^0) + o_m(1) \geq lc_\infty.$$

This implies $c \geq (l+1)c_\infty$ which contradicts our assumption $c < lc_\infty$. $\square$

In the following lemma, we study the level up to which the compactness holds for the sign-changing solutions of (P_Ω^s). The proof follows as in [44]. We give it here for completeness.

Lemma 4.1.6. *If $l \geq 2$ then there exists ε_0 such that I satisfy $(PS)_c$ relative to $\mathcal{D}_0^G$ in $D_0^{s,2}(\Omega)^G$ for every $c < (l+1)c_\infty + \varepsilon_0$.*

Proof. Let $\varepsilon_0 \in (0, c_\infty]$ be such that (P_Ω^s) has no solution u with $I(u) < c_\infty + \varepsilon_0$. Then, as $l \geq 2$ and $\epsilon \leq c_\infty$, this gives $2lc_\infty \geq (l+1)c_\infty + \varepsilon_0$. Equivalently, $lc_\infty \geq c_\infty + \epsilon_0$ and since the minimal energy of a solution of (P_Ω^s) is close to lc_∞, we take $c < (l+1)c_\infty + \epsilon_0$, which ensures that $2lc_\infty \geq (l+1)c_\infty + \varepsilon_0$.

Let $\{u_m\}_{m\in\mathbb{N}}$ be a $(PS)_c$ sequence relative to $\mathcal{D}_0^G$ in $D_0^{s,2}(\Omega)^G$ i.e.,

$$u_m \notin \mathcal{D}_0^G, \quad I(u_m) \to c < (l+1)c_\infty + \varepsilon_0, \quad DI(u_m) \to 0.$$

By contradiction, assume that $\{u_m\}_{m\in\mathbb{N}}$ has no convergent subsequence. Then Corollary 3.1.2 implies that $l < \infty$ and there exist sequence $\{y_m\}_{m\in\mathbb{N}}$ in Ω, $\{\varepsilon_m\}_{m\in\mathbb{N}}$ in $(0,\infty)$ and $\nu \in \{1,-1\}$ such that $|G/G_{y_m}| = l$ and

$$\left\| u_m - \nu \sum_{z\in Gy_m} U_{\varepsilon_m,z} \right\| \to 0 \text{ as } m \to \infty.$$

Now as we have

$$I(U_{\varepsilon,z}) = \frac{s}{N} S_s^{\frac{N}{2s}} =: c_\infty,$$

this implies that $dist(u_m, \mathcal{P}^G \cup -\mathcal{P}^G) \to 0$ as $m \to \infty$, contradicting that $u_m \notin \mathcal{D}_0^G$. $\square$

Now we prove Theorems 4.1.1 and 4.1.2.

Proof of Theorem 4.1.1. Let $\varepsilon_0 \in (0, c_\infty)$ be as in Lemma 4.1.6 and consider $\delta < \varepsilon_0$.

Due to the dilation invariance of I, $c(R_1, R_2) = c(R_1/R_2, 1)$ and it is easy to verify that

$$c(R,1) \to c_\infty \qquad \text{as} \quad R \to 0.$$

Therefore, there exist R_δ such that

$$c(R_1^{\frac{1}{l+1}}, R_2^{\frac{1}{l+1}}) < c_\infty + \frac{\delta}{l+1} \quad \text{if} \quad R_1/R_2 < R_\delta.$$

Let $\omega_1, ..., \omega_{l+1}$ be non-negative radial functions as in Lemma 4.1.2. Then, $\omega_1, ..., \omega_{l+1} \in \mathcal{N}(\Omega)^G$. Let $W_{k+1} := \text{span}\{\omega_1, ..., \omega_{k+1}\}$ be the vector space generated by $\omega_1, ..., \omega_{k+1}$. We claim that $\dim(W_{k+1}) = k$. Indeed, as for $i \neq j$ the functions ω_i and ω_j have disjoint support, we set $K_i :=$ supp ω_i, $K_j :=$ supp ω_j and estimate the inner product of ω_i and ω_j in $D_0^{s,2}(\Omega)$ as follows.

$$\begin{aligned} \langle \omega_i, \omega_j \rangle &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(\omega_i(x) - \omega_i(y))(\omega_j(x) - \omega_j(y))}{|x-y|^{N+2s}} \, dx\, dy \\ &= \int_{K_i \times Kj} \frac{-\omega_i(x)\omega_j(y)}{|x-y|^{N+2s}} \, dx\, dy + \int_{K_j \times K_i} \frac{-\omega_i(y)\omega_j(x)}{|x-y|^{N+2s}} \, dx\, dy \\ &< 0 \end{aligned} \tag{4.1.18}$$

Thus $\{\omega_1, ..., \omega_{k+1}\}$ forms the negative inner product set in $D_0^{s,2}(\Omega)$ and hence by Corollary 2 of [55], we infer that $\dim(W_{k+1}) = k$. Moreover, for each $k = 1, ..., l$,

$$\max_{W_{k+1}} I \leq \sum_{i=1}^{k} \max_{t>0} I(t\omega_i) = kc(R_1^{\frac{1}{l+1}}, R_2^{\frac{1}{l+1}}) \leq kc_\infty + \delta < lc_\infty + \varepsilon_0.$$

Since $l \geq 2$, we have that $I(\omega_1) \leq c_\infty + \delta < lc_\infty$. So, by Lemma 4.1.5, $\inf_{\mathcal{N}(\Omega)^G} I$ is attained at a non-negative solution at $u_1 \in \mathcal{N}(\Omega)^G$ with $I(u_1) \leq c_\infty + \delta$.
On the other hand, Theorem 2.4.1 and Lemma 4.1.6 give the existence of $l-1$ pairs of sign-changing critical points $\pm u_2, ..., \pm u_l \in \mathcal{N}(\Omega)^G$ of I with

$$I(u_k) \leq kc_\infty + \delta, \qquad k = 2, ..., l,$$

as claimed. □

Proof of Theorem 4.1.2. Let $0 < R_1 < R_2$ and $m \in \mathbb{N}$ be given. Define

$$l_0 := \frac{1}{c_\infty} mc(R_1^{1/m}, R_2^{1/m}) \quad \text{and} \quad \omega_1, ..., \omega_m \in \mathcal{N}(\Omega)^G$$

be non-negative radial function as in Lemma 4.1.2. As in the proof of Theorem 3.1.1 let $W_{k+1} := \text{span}\,\{\omega_1, ..., \omega_{k+1}\}$ be the vector space generated by $\omega_1, ..., \omega_{k+1}$ with $\dim(W_{k+1}) = k$. Since $\omega_1, ..., \omega_m \in \mathcal{N}(\Omega)^G$, for each $k = 1, ..., m-1$ we have

$$\max_{W_{k+1}} I \leq \sum_{i=1}^{k} \max_{t>0} I(t\omega_i) = kc(R_1^{1/m}, R_2^{1/m}) \leq l_0 c_\infty.$$

By assumption $l > l_0$, then $\max_{W_k} I < lc_\infty$ and Lemma 4.1.5 implies that $\inf_{\mathcal{N}(\Omega)^G} I$ is attained at a positive solution at $u_1 \in \mathcal{N}(\Omega)^G$ with $I(u_1) \leq c(R_1^{1/m}, R_2^{1/m})$. Moreover Theorem 2.4.1 and Lemma 4.1.6 give the existence of $m-1$ pairs of sign-changing critical points $\pm u_2, ..., \pm u_m \in \mathcal{N}(\Omega)^G$ of I with

$$I(u_k) \leq kc(R_1^{1/m}, R_2^{1/m}), \qquad k = 2, ..., m-1,$$

as claimed. □

4.2 Coron's type problem involving fractional p-Laplace operator

We establish the existence of a positive and multiple sign-changing solutions to fractional p-Laplacian equation with purely critical nonlinearity

$$(P^s_{p,\Omega}) \begin{cases} (-\Delta)^s_p u &= |u|^{p^*_s - 2}\, u \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad \text{on } \Omega^c, \end{cases}$$

in some bounded domain $\Omega \subset \mathbb{R}^N$ for $s \in (0,1)$, $p \in (1, \infty)$ and fractional critical Sobolev exponent $p^*_s = \frac{Np}{N-sp}$ under some symmetry assumptions. Now we fix Γ be a closed subgroup of $SO(N)$ and Γ-invariant bounded smooth domain $\mathcal{D}$ contained in Ω such that $\#\Gamma y = \infty$

for every $y \in \mathcal{D}$.

We state our main result:

Theorem 4.2.1. *There exists an increasing sequence $(\ell_m)_{m\in\mathbb{N}}$ of positive real numbers, depending only on Γ, and $\mathcal{D}$ with the following property: If Ω is a bounded smooth domain which contains $\mathcal{D}$ and if it is invariant under the action of a closed subgroup G of Γ for which the inequality*

$$\min_{x\in\overline{\Omega}}|G/G_x|> \ell_m \tag{4.2.1}$$

holds true, then problem $(P^s_{p,\Omega})$ has at least $m-1$ pairs of G -invariant solutions $\pm u_1, \dots, \pm u_{m-1}$ such that u_1 is positive, $u_2, \dots, u_{m-1}$ change sign, and

$$\int_\Omega |u_j|^{p^*_s} \leq \ell_j S_{s,p}^{\frac{N}{sp}} \qquad \text{for every } j = 1, \dots, m. \tag{4.2.2}$$

Now we illustrate the above result with an example.

Example 4.2.1. *Let $\mathcal{D}_0$ be a bounded smooth domain in $\mathbb{R}^{N-1}$, $N \geq 3$ with $\mathcal{D}_0 \subset \{(x, y) \in \mathbb{R}\times\mathbb{R}^{N-2} : x \geq \epsilon\}$.*
Set $\mathcal{D} := \{(z, x') \in \mathbb{C}\times\mathbb{R}^{N-2} \equiv \mathbb{R}^N : (|z|, y) \in \mathcal{D}_0\}$, be Γ-invariant bounded smooth domain such that $\#\Gamma y = \infty$ for every $y \in \mathcal{D}$. Let $\Gamma = \mathbb{S}^1$ of unit complex numbers acting by $e^{i\theta}(z, x') := (e^{i\theta}z, x')$ then $\mathcal{D}$ is Γ-invariant and every Γ- orbit in $\mathcal{D}$ is circle. If $G_n := \{e^{2\pi ik/n} : k = 0, \cdots, n-1\}$ then $\#G_n x = n$ for every $x \in (\mathbb{C}\setminus\{0\})\times\mathbb{R}^{N-2}$. If Ω is G_n-invariant smooth bounded domain with $\mathcal{D} \subset \Omega \subset (\mathbb{C}\setminus\{0\})\times\mathbb{R}^{N-2}$ and $\min_{x\in\overline{\Omega}}|G_n/G_{n_x}|> \ell_m$, Theorem 4.2.1 gives at least m-pair of solutions to the problem $(P^s_{p,\Omega})$.

4.2.1 Proof of main Theorem 4.2.1

A Variational principle for sign-changing solutions for $(P^s_{p,\Omega})$

In this section we first give a variational principle for the sign-changing solutions of the problem $(P^s_{p,\Omega})$. Then we prove the mountain pass type result for the energy functional

$E : D_0^{s,p}(\Omega) \to \mathbb{R}$ associated with the problem $(P^s_{p,\Omega})$ given by

$$E(u) := \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{1}{p_s^*}\int_\Omega |u(x)|^{p_s^*}\,dx.$$

We note that E is a C^1 functional with $E' \in D^{-s,p'}(\Omega)$ defined as

$$\langle DE(u),v\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy - \int_\Omega |u|^{p_s^*-2}uv\,dx,$$

where $v \in D_0^{s,p}(\Omega)$. In [96], Chang et al. obtained the sign-changing solutions of subcritical problem for $p = 2$ by converting the nonlocal problem in to a local problem via harmonic extension. As $D_0^{s,p}(\Omega)$ is not a Hilbert space if $p \neq 2$ and the functional E is only of class $\mathcal{C}^1$ if $p \in (1,2)$, the approach of [96] does not work for the problem $(P^s_{p,\Omega})$. As we know that E admits a pseudo-gradient vector field. But, in general, this vector field is not suitable to obtain sign-changing solutions. Using a result of Bartsch, Liu and Weth [13] we shall obtain a locally Lipschitz vector field, whose associated flow is a descending flow for E with the property that a small enough neighborhood of the positive and the negative cone is strictly positively invariant. Then, one can follow the proof given in [44] to obtain mountain pass type result i.e., Theorem 4.2.2. In order to prove that the assumptions of Lemma 2.1 in [13] are satisfied, we require the following lemmas.

Lemma 4.2.1. *For $p \in (1,\infty)$, there exist $C_1, C_2, C_3, C_4 > 0$ such that, for all $\xi, \eta \in \mathbb{R}^N$,*

$$\begin{aligned}
(J_p(\xi) - J_p(\eta))\cdot(\xi-\eta) &\geq C_1 \frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}} && \text{if } 1 < p \leq 2,\\
(J_p(\xi) - J_p(\eta))\cdot(\xi-\eta) &\geq C_2\, |\xi-\eta|^p && \text{if } p > 2,\\
|J_p(\xi) - J_p(\eta)| &\leq C_3\, |\xi-\eta|^{p-1} && \text{if } 1 < p \leq 2,\\
|J_p(\xi) - J_p(\eta)| &\leq C_4 \frac{|\xi-\eta|}{(|\xi|+|\eta|)^{2-p}} && \text{if } p > 2.
\end{aligned}$$

Define $\Psi : D_0^{s,p}(\Omega) \to D_0^{s,p}(\Omega)$ as $\Psi(u) = v$, where v is the unique solution of the problem

$$\begin{cases} (-\Delta)_p^s v &= |u|^{p_s^*-2}u \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \Omega^c, \end{cases} \tag{4.2.3}$$

which can be written in the weak form

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega |u|^{p_s^*-2}u\varphi\,dx \tag{4.2.4}$$

for all $\varphi \in D_0^{s,p}(\Omega)$.

Lemma 4.2.2. *There exists $C>0$ such that, for every $u \in D_0^{s,p}(\Omega)$,*

$$\langle DE(u),\, u-\Psi(u)\rangle \geq \begin{cases} C\,\|u-\Psi(u)\|^2\,(\|u\|+\|\Psi(u)\|)^{p-2} & \text{if } 1<p\leq 2, \\ C\,\|u-\Psi(u)\|^p & \text{if } p>2, \end{cases}$$

Proof. Let $u \in D_0^{s,p}(\Omega)$ and $\Psi(u)=v$. Set $\tilde{u}(x,y)=u(x)-u(y)$ and $\tilde{v}(x,y)=v(x)-v(y)$. Using Lemma (4.2.1) and utilizing the fact that v solves the problem (4.2.3), for $p>2$, we have

$$\begin{aligned} \langle DE(u),\, u-\Psi(u)\rangle &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(\tilde{u}(x,y))(\tilde{u}(x,y)-\tilde{v}(x,y))}{|x-y|^{N+sp}}\,dx\,dy - \int_\Omega |u|^{p_s^*-2}u(u-v)\,dx \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(\tilde{u}(x,y))(\tilde{u}(x,y)-\tilde{v}(x,y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\qquad - \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(\tilde{v}(x,y))(\tilde{u}(x,y)-\tilde{v}(x,y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\geq C_2 \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y)-\tilde{v}(x,y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\geq C\,\|u-\Psi(u)\|^p, \end{aligned}$$

for some $C>0$. Next, for $p\in(1,2]$, again by Lemma 4.2.1 and the fact that v solves the problem (4.2.3), we have

$$\begin{aligned} \langle DE(u),\, u-\Psi(u)\rangle &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(\tilde{u}(x,y))(\tilde{u}(x,y)-\tilde{v}(x,y))}{|x-y|^{N+sp}}\,dx\,dy - \int_\Omega |u|^{p_s^*-2}u(u-v)\,dx \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(\tilde{u}(x,y))(\tilde{u}(x,y)-\tilde{v}(x,y))}{|x-y|^{N+sp}}\,dx\,dy \end{aligned}$$

$$- \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(\tilde{v}(x,y))(\tilde{u}(x,y)-\tilde{v}(x,y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\geq C_1 \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y)-\tilde{v}(x,y)|^2|\tilde{u}(x,y)+\tilde{v}(x,y)|^{p-2}}{|x-y|^{N+sp}}\,dx\,dy.$$

Now using Holder's inequality, we get

$$\begin{aligned}
\|u-v\|^p &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y)-\tilde{v}(x,y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y)-\tilde{v}(x,y)|^p(|\tilde{u}(x,y)|+|\tilde{v}(x,y)|)^{\frac{p(p-2)}{2}}(|\tilde{u}(x,y)|+|\tilde{v}(x,y)|)^{\frac{p(2-p)}{2}}}{|x-y|^{N+sp}}\,dx\,dy \\
&\leq C_0 \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y)-\tilde{v}(x,y)|^2(|\tilde{u}(x,y)|+|\tilde{v}(x,y)|)^{p-2}}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{p}{2}} \\
&\quad \cdot \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(|\tilde{u}(x,y)|+|\tilde{v}(x,y)|)^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{2-p}{2}} \\
&\leq C_0 \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y)-\tilde{v}(x,y)|^2(|\tilde{u}(x,y)|+|\tilde{v}(x,y)|)^{p-2}}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{p}{2}} (\|u\|+\|v\|)^{\frac{p(2-p)}{2}}
\end{aligned}$$

for some $C_0 > 0$. Next, we have

$$\|u-v\|^2 \leq C_0^{\frac{2}{p}} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y)-\tilde{v}(x,y)|^2(|\tilde{u}(x,y)|+|\tilde{v}(x,y)|)^{p-2}}{|x-y|^{N+sp}}\,dx\,(\|u\|+\|v\|)^{2-p}.$$

So, for some $C > 0$, we obtain

$$\langle DE(u),\, u-\Psi(u)\rangle \geq C\,\|u-\Psi(u)\|^2\,(\|u\|+\|\Psi(u)\|)^{p-2}.$$

This proves the Lemma 4.2.2. □

Lemma 4.2.3. *There exists $C > 0$ such that, for every $u \in D_0^{s,p}(\Omega)$,*

$$\|DE(u)\|_{D^{-s,p'}(\Omega)} \leq \begin{cases} C\,\|u-\Psi(u)\|^{p-1} & \text{if } 1 < p \leq 2, \\ C\,\|u-\Psi(u)\|\,(\|u\|+\|\Psi(u)\|)^{p-2} & \text{if } p > 2. \end{cases}$$

Proof. Let $u \in D_0^{s,p}(\Omega)$ and $\Psi(u) = v$. Set $\tilde{u}(x,y) = u(x)-u(y)$ and $\tilde{v}(x,y) = v(x)-v(y)$.

Then for any $\phi \in D_0^{s,p}(\Omega)$, setting $\tilde{\phi}(x,y) = \phi(x) - \phi(y)$, we have

$$\begin{aligned}|\langle DE(u),\,\phi\rangle| &= \left|\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(\tilde{u}(x,y))(\tilde{\phi}(x,y))}{|x-y|^{N+sp}}\,dx\,dy - \int_\Omega |u|^{p_s^*-2}u\phi\,dx\right| \\ &= \left|\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(J_p(\tilde{u}(x,y)) - J_p(\tilde{v}(x,y)))\,(\tilde{\phi}(x,y))}{|x-y|^{N+sp}}\,dx\,dy\right| \\ &\leq C\left[\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|J_p(\tilde{u}(x,y)) - J_p(\tilde{v}(x,y))|^{\frac{p}{p-1}}}{|x-y|^{N+sp}}\,dx\,dy\right]^{\frac{p-1}{p}} \|\phi\|.\end{aligned}$$

Thus,

$$\|DE(u)\|_{D^{-s,p'}(\Omega)} \leq C\left[\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|J_p(\tilde{u}(x,y)) - J_p(\tilde{v}(x,y))|^{\frac{p}{p-1}}}{|x-y|^{N+sp}}\,dx\,dy\right]^{\frac{p-1}{p}}. \tag{4.2.5}$$

For $p > 2$, using Lemma 4.2.1 and Holder's inequality we obtain

$$\begin{aligned}\|DE(u)\|_{D^{-s,p'}(\Omega)} &\leq C\left[\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y) - \tilde{v}(x,y)|^{\frac{p}{p-1}}(|\tilde{u}(x,y) + \tilde{v}(x,y)|)^{\frac{p(p-2)}{p-1}}}{|x-y|^{N+sp}}\,dx\,dy\right]^{\frac{p-1}{p}} \\ &\leq C\left[\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{u}(x,y) - \tilde{v}(x,y)|^p}{|x-y|^{N+sp}}\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(|\tilde{u}(x,y) + \tilde{v}(x,y)|)^p}{|x-y|^{N+sp}}\,dx\,dy\right]^{\frac{p-2}{p}} \\ &\leq C\,\|u - \Psi(u)\|\,(\|u\| + \|\Psi(u)\|)^{p-2}.\end{aligned}$$

Next, for $1 < p \leq 2$, using Lemma (4.2.1) and (4.2.5), we obtain

$$\|DE(u)\|_{D^{-s,p'}(\Omega)} \leq C\,\|u - \Psi(u)\|^{p-1}.$$

This concludes the prove of Lemma 4.2.3. □

The above two lemmas imply that u is a critical point of E if and only if $\Psi(u) = u$. The nontrivial G-invariant critical points of E lie on the Nehari manifold

$$\mathcal{N}^G := \left\{u \in D_0^{s,p}(\Omega)^G \setminus \{0\} : \langle DE(u), u\rangle = 0\right\}.$$

Sign-changing Nehari set for the functional E is defined as

$$\mathcal{N}_{sc}^G := \left\{u \in \mathcal{N}(\Omega)^G : u^{\pm} \neq 0,\ \langle DE(u), u^+\rangle = \langle DE(u), u^-\rangle = 0\right\},$$

where $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = -\min\{u(x), 0\}$ and $\langle , \rangle$ denotes the duality pairing of between $D_0^{s,p}(\Omega)^*$ and $D_0^{s,p}(\Omega)$. Clearly, $\mathcal{N}^G$ contains all nontrivial solutions of problem $(P_{p,\Omega}^s)$ and sign-changing solutions of problem $(P_{p,\Omega}^s)$ lie on $\mathcal{N}_{sc}$.
Next we prove the following lemma which gives the bounds for $u \in \mathcal{N}_{sc}$.

Lemma 4.2.4. *There exists $\eta_1, \eta_2 > 0$ such that*

- *(i)* $\|u^{\pm}\| \geq \gamma_1$ *for all* $u \in \mathcal{N}_{sc}$,
- *(ii)* $\int_\Omega |u^{\pm}|^{p_s^*}\, dx \geq \gamma_2$ *for all* $u \in \mathcal{N}_{sc}$.

Proof. For $u \in \mathcal{N}_{sc}^G$, we have $\langle E'(u), u^{\pm}\rangle = 0$. This implies

$$\int_\Omega |u|^{p_s^*-2} u u^+\, dx = \int_\Omega (u^+)^{p_s^*}\, dx.$$

For $u \in D_0^{s,p}(\Omega)$, we define $\Omega_- := \{x \in \Omega : u(x) < 0\}$, $\Omega_+ := \{x \in \Omega : u(x) > 0\}$ and write

$$\mathbb{R}^N \times \mathbb{R}^N = ((\Omega_+ \cup \Omega_-) \times (\Omega_+ \cup \Omega_-)) \cup ((\Omega_+ \cup \Omega_-) \times \Omega^c) \cup (\Omega^c \times (\Omega_+ \cup \Omega_-)) \cup (\Omega^c \times \Omega^c).$$

We observe that

$$\begin{aligned}
&\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(u(x) - u(y))(u^+(x) - u^+(y))}{|x-y|^{N+sp}}\, dx\, dy \\
&= \int_{\Omega_+ \times \Omega_+} \frac{|u^+(x) - u^+(y)|^p}{|x-y|^{N+sp}}\, dx\, dy + \int_{\Omega_+ \times \Omega_-} \frac{J_p(u^+(x) + u^-(y))(u^+(x))}{|x-y|^{N+sp}}\, dx\, dy \\
&+ \int_{\Omega_- \times \Omega_+} \frac{J_p(u^-(x) + u^+(y))(u^+(y))}{|x-y|^{N+sp}}\, dx\, dy + \int_{\Omega_+ \times \Omega^c} \frac{|u^+(x)|^p}{|x-y|^{N+sp}}\, dx\, dy \\
&+ \int_{\Omega^c \times \Omega_+} \frac{|u^+(y)|^p}{|x-y|^{N+sp}}\, dx\, dy,
\end{aligned}$$

and hence we get

$$\langle DE(u), u^+\rangle = \langle DE(u^+), u^+\rangle + S_1^+(u),$$

where

$$S_1^+(u) = \int_{\Omega_+\times\Omega_-} \frac{J_p(u^+(x)+u^-(y))(u^+(x)) - |u^+(x)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_{\Omega_-\times\Omega_+} \frac{J_p(u^-(x)+u^+(y))(u^+(y)) - |u^+(y)|^p}{|x-y|^{N+sp}}\,dx\,dy$$

As for $u \in \mathcal{N}_{sc}^G$, we have $\langle DE(u), u^{\pm}\rangle = 0$ and since $S_1^+(u) > 0$, we obtain $\langle DE(u^+), u^+\rangle < 0$. This implies that

$$\|u^+\|^p < \int_\Omega (u^+)^{p_s^*}\,dx.$$

Similarly, we obtain

$$\|u^-\|^p < \int_\Omega (u^-)^{p_s^*}\,dx.$$

Now for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|u(x)|^{p_s^*} \le \epsilon |u(x)|^p + C_\epsilon |u(x)|^{p_s^*} \quad \text{for all } x \in \Omega. \tag{4.2.6}$$

So by using Sobolev inequalities, there exists $\overline{C} > 0$ such that

$$\|u^{\pm}\|^p \le \epsilon\overline{C}\|u^{\pm}\|^p + C_\epsilon\overline{C}\|u^{\pm}\|^{p_s^*}.$$

Since $p_s^* > p$, taking $\epsilon = \frac{1}{2\overline{C}}$, it is easy to show that (i) holds. Moreover, by using (4.2.6), we obtain

$$\gamma_1^p \le \|u^{\pm}\|^p \le \epsilon\overline{C}\|u^{\pm}\|^p + C_\epsilon \|u^{\pm}\|^{p_s^*}_{L^{p_s^*}(\Omega)}.$$

Now by taking $\epsilon = \frac{1}{2\overline{C}}$, we get

$$\gamma_2 = \frac{\eta_1^p}{2C_\epsilon} \le \|u^{\pm}\|^{p_s^*}_{L^{p_s^*}(\Omega)}.$$

This proves the Lemma 4.2.4. □

We write $\mathcal{P}^G := \{u \in D_0^{s,p}(\Omega)^G : u \ge 0\}$ for the convex cone of positive functions in

$D_0^{s,p}(\Omega)^G$ and, for $\varepsilon > 0$, we set

$$B_\varepsilon(\mathcal{P}^G) := \{u \in D_0^{s,p}(\Omega)^G : \ \mathrm{dist}(u, \mathcal{P}^G) \le \varepsilon\},$$

where $\mathrm{dist}(u, \mathcal{X}) := \inf_{v \in \mathcal{X}} \|u - v\|$.

Lemma 4.2.5. *There exists $\varepsilon > 0$ such that*

(i) $\left[B_\varepsilon(\mathcal{P}^G) \cup B_\varepsilon(-\mathcal{P}^G)\right] \cap \mathcal{N}_{sc}^G = \emptyset$,

(ii) $\Psi(B_\varepsilon(\mathcal{P}^G)) \subset \mathrm{int}\,(B_\varepsilon(\mathcal{P}^G))$ *and* $\Psi(B_\varepsilon(-\mathcal{P}^G)) \subset \mathrm{int}(B_\varepsilon(-\mathcal{P}^G))$.

Proof. (i): It follows from the uniqueness of the solution to problem (4.2.3) that $v \in D_0^{s,p}(\Omega)^G$ if $u \in D_0^{s,p}(\Omega)^G$ and it also follows that Ψ is odd. Furthermore, by the maximum principle, $\Psi(u) \in \mathcal{P}^G$ if $u \in \mathcal{P}^G$. Now, for every $u \in D_0^{s,p}(\Omega)^G$, we have

$$\|u^-\|_{L^{p_s^*}(\Omega)} = \min_{v \in \mathcal{P}^G} \|u - v\|_{L^{p_s^*}(\Omega)} \le S_{s,p}^{-1/p} \min_{v \in \mathcal{P}^G} \|u - v\| = S_{s,p}^{-1/p} \mathrm{dist}(u, \mathcal{P}^G). \tag{4.2.7}$$

Therefore, using Lemma 4.2.4 (ii), there exists $\epsilon_0 > 0$ such that $\mathrm{dist}(u, \mathcal{P}^G) > \epsilon_0$ for every $u \in \mathcal{N}_{sc}^G$. Moreover, since $\mathcal{N}_{sc}^G$ is symmetric with respect to origin, $\mathrm{dist}(u, -\mathcal{P}^G) = \mathrm{dist}(u, \mathcal{P}^G) > \epsilon_0$, and (a) holds true for every $\epsilon \in (0, \epsilon_0)$.

(ii): Next, set $\Psi(u) = v$. Take $\varphi = v^-$ in (4.2.4) and using Holder's and Sobolev inequality we get

$$\begin{aligned}
\|v^-\|^p &\le \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(v(x) - v(y))(v^-(x) - v^-(y))}{|x - y|^{N+sp}} \, dx \, dy \\
&= \int_\Omega |u|^{p_s^* - 2} u v^- \, dx \\
&= \int_\Omega |u^-|^{p_s^* - 2} u^- v^- \, dx \\
&\le \left(\int_\Omega |u^-|^{p_s^*}\right)^{\frac{p_s^* - 1}{p_s^*}} \left(\int_\Omega |v^-|^{p_s^*}\right)^{\frac{1}{p_s^*}} \\
&\le S_{s,p}^{-1/p} \left\|u^-\right\|_{L^{p_s^*}(\Omega)}^{p_s^* - 1} \left\|v^-\right\|.
\end{aligned}$$

Using this inequality with (4.2.7) we obtain

$$\mathrm{dist}(v, \mathcal{P}^G)^{p-1} \le ||v^-||^{p-1} \le S_{s,p}^{-1/p} \left\|u^-\right\|_{L^{p_s^*}(\Omega)}^{p_s^* - 1} \le S_{s,p}^{-p_s^*/p} \,\mathrm{dist}(u, \mathcal{P}^G)^{p_s^* - 1}.$$

Hence, there exists $\varepsilon \in (0, \varepsilon_0)$ such that $\Psi(B_\varepsilon(\mathcal{P}^G)) \subset \text{int}(B_\varepsilon(\mathcal{P}^G))$. Since Ψ is odd, this is also true for $-\mathcal{P}^G$. □

Proposition 4.2.1. *Let $\mathcal{K} := \{u \in D_0^{s,p}(\Omega)^G : E'(u) = 0\}$ and $\mathcal{W} := D_0^{s,p}(\Omega)^G \setminus \mathcal{K}$. Then there exists a locally Lipschitz continuous vector field $\Phi : \mathcal{W} \to D_0^{s,p}(\Omega)^G$ with the following properties:*

(i) For $\varepsilon > 0$ as in Lemma 4.2.5 $\Phi(B_\varepsilon(\mathcal{P}^G)) \subset \text{int}(B_\varepsilon(\mathcal{P}^G))$ *and* $\Phi(B_\varepsilon(-\mathcal{P}^G)) \subset \text{int}(B_\varepsilon(-\mathcal{P}^G))$.

(ii) For all $u \in \mathcal{W}$,

$$\frac{1}{2} \|u - \Phi(u)\| \leq \|u - \Psi(u)\| \leq 2 \|u - \Phi(u)\| .$$

(iii) For $C > 0$ as in Lemma 4.2.2 *and all $u \in \mathcal{W}$,*

$$\langle DE(u), u - \Phi(u)\rangle \geq \begin{cases} \frac{1}{2}C \|u - \Psi(u)\|^2 (\|u\| + \|\Psi(u)\|)^{p-2} & \text{if } 1 < p \leq 2, \\ \frac{1}{2}C \|u - \Psi(u)\|^p & \text{if } p > 2. \end{cases}$$

(iv) Φ is odd.

Proof. By using Lemma 4.2.2-4.2.5, the assumption of Lemma 2.1 in [13] is satisfied. So the proof follows. □

Combining Proposition 4.2.1 and Lemma 4.2.3 we have

$$\langle DE(u), u - \Phi(u)\rangle \geq C \|DE(u)\|_{D^{-s,p'}(\Omega)} \|u - \Psi(u)\| \quad \forall\, u \in D_0^{s,p}(\Omega)^G. \tag{4.2.8}$$

Now we prove mountain pass type theorem to obtain the sign-changing solutions.

Theorem 4.2.2. *Let V be a finite dimensional subspace of $D_0^{s,p}(\Omega)^G$. If E satisfies condition $(PS)_c^G$ in $D_0^{s,p}(\Omega)$ for every $c \leq \sup_V E$, then E has at least* $\dim V - 1$ *pairs of sign changing critical points $u \in D_0^{s,p}(\Omega)^G$ such that $E(u) \leq \sup_V E$.*

Proof. For a fixed $\epsilon > 0$ as in Lemma 4.2.5,$E : D_0^{s,p}(\Omega)^G \to \mathbb{R}$ does not have sign changing critical points in $B_\varepsilon(\mathcal{P}^G) \cup B_\varepsilon(-\mathcal{P}^G)$. For $u \in \mathcal{W}$, let $\varphi(t, u)$ be the unique solution to the

Cauchy problem

$$\begin{cases} \frac{d}{dt}\varphi(t,u) &= -\varphi(t,u) + \Phi(\varphi(t,u)), \\ \varphi(0,u) &= u, \end{cases} \tag{4.2.9}$$

with maximal existence interval $[0, T(u))$. As ϕ is odd, $\varphi(t,u)$ is odd in u. From (4.2.8), we have

$$\begin{aligned} \frac{d}{dt}E(\varphi(t,u)) &= -\langle DE(\varphi(t,u)), \varphi(t,u) - \Phi(\varphi(t,u))\rangle \\ &\leq -C\,\|DE(\varphi(t,u))\|_{D^{-s,p'}(\Omega)}\,\|\varphi(t,u) - \Psi(\varphi(t,u))\| \\ &< 0. \end{aligned}$$

Therefore, $t \mapsto E(\varphi(t,u))$ is strictly decreasing in $[0, T(u))$. If E does not have a sign changing critical point $u \in D_0^{s,p}(\Omega)^G$ with $E(u) = c$, then the closed set is

$$\mathcal{B}_c^G := B_\varepsilon(\mathcal{P}^G) \cup B_\varepsilon(-\mathcal{P}^G) \cup E^c$$

is strictly positively invariant under φ, where $E^c := \{u \in D_0^{s,p}(\Omega)^G : E(u) \leq c\}$. It is evident that

$$\mathcal{A}(\mathcal{B}_c^G) := \{u \in \mathcal{W} : \varphi(t,u) \in \mathcal{B}_c^G \text{ for some } \ t \in (0, T(u))\}$$

is open in $D_0^{s,p}(\Omega)^G$, and the entrance time map $e_c : \mathcal{A}(\mathcal{B}_c^G) \to \mathbb{R}$ is defined as

$$e_c(u) := \inf\{t \geq 0 : \varphi(t,u) \in \mathcal{B}_c^G\}$$

is continuous. In addition the map $\varrho_c : \mathcal{A}(\mathcal{B}_c^G) \to \mathcal{B}_c^G$ given by $\varrho_c(u) := \varphi(e_c(u), u)$ is odd and continuous. By noticing that, if $\|DE(\varphi(t,u))\|_{D^{-s,p'}(\Omega)} \geq \beta > 0$ for all $t \in [0, t_0]$, then from Proposition 4.2.1 and inequality (4.2.8) we get

$$\|u - \varphi(t_0, u)\| \leq \int_0^{t_0} \left\| \frac{d}{dt}\varphi(t,u) \right\| dt$$

$$
\begin{aligned}
&\leq 2\int_0^{t_0} \|\varphi(t,u) - \Psi(\varphi(t,u))\|\, dt \\
&\leq -\frac{2}{C\beta}\int_0^{t_0} \frac{d}{dt}E(\varphi(t,u))\, dt \\
&= \frac{2}{C\beta}[E(u) - E(\varphi(t_0,u))].
\end{aligned}
$$

Using the above inequality we can imitate the proof of Proposition 3.6 in [44] to conclude the proof of the Theorem 4.2.2. □

Now we proceed as in [44] to prove the Theorem 4.2.1.

Proof of main Theorem 4.2.1: Let $\mathcal{P}_1(\mathcal{D})$ be the collection of all nonempty Γ-invariant bounded smooth domains contained in $\mathcal{D}$, and define

$$\mathcal{P}_m(\mathcal{D}) := \{(\mathcal{D}_1, \ldots, \mathcal{D}_m) : \mathcal{D}_i \in \mathcal{P}_1(\mathcal{D}),\ \ \mathcal{D}_i \cap \mathcal{D}_j = \emptyset \text{ if } i \neq j\}.$$

Since $\#\Gamma x = \infty$ for all $x \in \mathcal{D}_i$, Corollary 3.1.1 asserts that E satisfies condition $(PS)_c^\Gamma$ in $D_0^{s,p}(\mathcal{D}_i)$ for every $c \in \mathbb{R}$. Hence mountain pass theorem [87] yields a nontrivial least energy Γ-invariant solution $\omega_{\mathcal{D}_i}$ to problem $(P^s_{p,\Omega})$ in $\mathcal{D}_i$ which satisfies

$$E(\omega_{\mathcal{D}_i}) = \max_{t\geq 0} E(t\,\omega_{\mathcal{D}_i}). \tag{4.2.10}$$

Clearly, $\omega_{\mathcal{D}_i} \in D_0^{s,p}(\Omega)^G$, we set $c_{\infty,p} := \frac{s}{N} S_{s,p}^{\frac{N}{sp}}$ and define

$$c_m := \inf\left\{\sum_{i=1}^m E(\omega_{\mathcal{D}_i}) : (\mathcal{D}_1, \ldots, \mathcal{D}_m) \in \mathcal{P}_m(\mathcal{D})\right\} \quad \text{and} \quad \ell_m := c_{\infty,p}^{-1} c_m.$$

Next we claim that (ℓ_m) is strictly increasing. We note that $E(\omega_{\mathcal{D}}) \geq c_1$. Since $E(\omega_{\mathcal{D}_i}) \geq c_{\infty,p}$, so for any $(\mathcal{D}_1, \ldots, \mathcal{D}_m) \in \mathcal{P}_m(\mathcal{D})$ with $m \geq 2$ we have

$$\sum_{i=1}^m E(\omega_{\mathcal{D}_i}) = \sum_{i=1}^{m-1} E(\omega_{\mathcal{D}_i}) + E(\omega_{\mathcal{D}_m}) \geq c_{m-1} + c_\infty > c_{m-1}$$

It follows that

$$c_{m-1} + c_{\infty,p} < c_m \quad \text{and} \quad \ell_{m-1} + 1 < \ell_m.$$

Let G is a closed subgroup of Γ and Ω is a G-invariant bounded smooth domain which contains $\mathcal{D}$ and satisfies (4.2.1), we choose $\varepsilon > 0$ and $(\mathcal{D}_1, \ldots, \mathcal{D}_m) \in \mathcal{P}_m(\mathcal{D})$ such that

$$c_m \leq \sum_{i=1}^{m} E(\omega_{\mathcal{D}_i}) < c_m + \varepsilon < \left(\min_{x \in \overline{\Omega}} |G/G_x| \right) c_{\infty,p}.$$

Let V be the subspace of $D_0^{s,p}(\Omega)^G$ generated by $\{\omega_{\mathcal{D}_1}, \ldots, \omega_{\mathcal{D}_m}\}$. Since $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ if $i \neq j$, we have that $\dim V = m - 1$. Using (4.2.10) we obtain that

$$\sup_V E \leq \sum_{i=1}^{m-1} E(\omega_{\mathcal{D}_i}) < \left(\min_{x \in \overline{\Omega}} |G/G_x| \right) c_{\infty,p}.$$

It follows from Corollary 3.1.1 that E satisfies condition $(PS)_c^G$ in $D_0^{s,p}(\Omega)$ for every $c \leq \sup_V E$, so the mountain pass theorem [87] yields a positive critical point u_1 of E in $D_0^{s,p}(\Omega)^G$ and Theorem 4.2.2 yields $\dim V - 1$ pairs of sign changing critical points $\pm u_2, \ldots, \pm u_m \in D_0^{s,p}(\Omega)^G$ such that $E(u_i) \leq \sup_V E$.
Now we can argue as in [41], to show that the u_i's may be suitably chosen so that (4.2.2) holds true. This completes the proof of the Theorem 4.2.1. □

4.3 Conclusion

In this chapter, we have studied the Coron's type problems involving nonlocal operators in some bounded domain under symmetry assumptions. In the first part of the chapter, we have established the existence of a positive solution and multiple sign-changing solutions to the problem involving fractional Laplacian with critical growth in the annular-shaped domain, which is invariant under a group G of orthogonal transformations. We have shown the existence results for the domains with a hole of arbitrary size as symmetry assumptions on the domains allow us to handle it by considering the large value of l.
Since we are dealing with the nonlocal operators, the sign-changing Nehari set is different from the local cases. In the local case, the radially symmetric test functions with controlled energy on disjoint concentric annular-shaped subsets of Ω are orthogonal. However, in the nonlocal case, these are not orthogonal due to the nonlocal interactions in the energy norm.

In the second part of this chapter, we have investigated the existence of multiple sign-changing solutions to the problem involving fractional p-Laplace operator with critical growth in domains with nontrivial topology under symmetry assumptions. Here, we have established these results in more general domains with nontrivial topology by exploiting symmetry assumptions on the domain.

The Coron's problem involving fractional p-Laplace operator is still open without the symmetry assumptions on the domain.

Another challenging problem in this direction is the complete characterization of the domains for which solutions exist to nonlocal purely critical exponent problems. As far as we know, this is still open even in local case.

5

Multiple solutions to nonlocal supercritical exponent problem in symmetric domain

In this chapter, we show the existence of a prescribed number of solutions to the following problem involving fractional Laplacian with supercritical growth in domains of revolutions under symmetry assumptions

$$
(P^s_{b,\Omega})\begin{cases} (-\Delta)^s u &= b(x)|u|^{q-2}u \quad \text{in } \Omega,\\ \qquad u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2s$, $s \in (0,1)$, $b \in C^{0,\alpha}(\overline{\Omega})$ and positive, $q > 2^*_s$ and $2^*_s = \frac{2N}{N-2s}$ is fractional critical Sobolev exponent.

5.1 Preliminaries and functional setting

In this section, we discuss the different representations of fractional Laplacian and corresponding energy formulations associated to the problem $(P^s_{b,\Omega})$. We also discuss solution spaces and symmetries on extended domain.

Let $\{\varphi_j, \lambda_j\}_{j\in\mathbb{N}}$ be the eigenfunctions and eigenvalues of the following Dirichlet problem

$$-\Delta u = \lambda\, u \quad \text{in } \Omega, \quad u = 0 \quad \text{ on } \partial\Omega,$$

then $\{\varphi_j, \lambda_j^s\}_{j\in\mathbb{N}}$ is the set of eigenfunctions and eigenvalues of the corresponding fractional problem

$$(-\Delta)^s u = \lambda\, u \quad \text{in } \Omega, \quad u = 0 \quad \text{ on } \partial\Omega.$$

Let

$$\widetilde{H}^s_0(\Omega) := \left\{ u = \sum_{j\in\mathbb{N}} a_j\varphi_j \in L^2(\Omega) : \|u\|_{\widetilde{H}^s_0(\Omega)} = \left(\sum_{j\in\mathbb{N}} a_j^2\lambda_j^s\right)^{\frac{1}{2}} < \infty \right\}.$$

We denote by $\widetilde{H}^{-s}(\Omega)$ the dual space of $\widetilde{H}^s_0(\Omega)$. For $u \in \widetilde{H}^s_0(\Omega)$, $u = \sum_{j\in\mathbb{N}} a_j\varphi_j$ with $a_j = \int_\Omega u\varphi_j\, dx$, the fractional power of Dirichlet Laplacian $(-\Delta)^s$ is defined as

$$(-\Delta)^s u = \sum_{j\in\mathbb{N}} a_j\lambda_j^s\varphi_j \ \in \widetilde{H}^{-s}(\Omega),$$

which is also known as spectral fractional Laplacian. We define the inner product on $\widetilde{H}^s_0(\Omega)$ by

$$\langle u, v\rangle := \int_\Omega (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v\, dx.$$

It is easy to show that $\widetilde{H}^s_0(\Omega)$ is Hilbert space. We note that $\|u\|_{\widetilde{H}^s_0(\Omega)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)}$.

Definition 5.1.1. *We say that $u \in \widetilde{H}^s_0(\Omega)$ is a solution to the problem $(P^s_{b,\Omega})$ if it satisfies*

$$\int_\Omega (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}\varphi\, dx = \int_\Omega b(x)|u|^{q-2}u\varphi\, dx$$

for all $\varphi \in \widetilde{H}_0^s(\Omega)$.

Associated to the problem $(P^s_{b,\Omega})$, we consider the energy functional

$$I(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}u|^2\,dx - \frac{1}{q}\int_\Omega b(x)|u(x)|^q\,dx.$$

This functional is not well defined in $\widetilde{H}_0^s(\Omega)$ as we are dealing with supercritical exponent problem $(P^s_{b,\Omega})$.

One of the main difficulties in the study of problem $(P^s_{b,\Omega})$ is that the fractional Laplacian is a nonlocal operator. Caffarelli et al. [29] developed a local interpretation of the fractional Laplacian in $\mathbb{R}^N$ by considering a Dirichlet to Neumann type operator in the domain $\{(x,t) \in \mathbb{R}^{N+1} : t > 0\}$.

We consider the cylinder $\mathcal{C}_\Omega$ associated with the bounded domain Ω denoted as

$$\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+ := \left\{(x,t) \subset \mathbb{R}^{N+1} \text{ such that } x \in \mathbb{R}^N, t \in (0,\infty)\right\} \tag{5.1.1}$$

and the lateral boundary of $\mathcal{C}_\Omega$ is denoted by $\partial_L\mathcal{C}_\Omega$ and is defined as

$$\partial_L\mathcal{C}_\Omega := \partial\Omega \times (0,\infty).$$

The points in $\mathcal{C}_\Omega$ are denoted by (x,t) where $x \in \Omega$ and $t \in (0,\infty)$.

We consider the s-harmonic extension $v = E_s(u)$ in the cylinder $\mathcal{C}_\Omega$ for a given function $u \in \widetilde{H}_0^s(\Omega)$ as a solution of the problem

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla v) &= 0 \quad \text{in } \mathcal{C}_\Omega, \\ v &= 0 \quad \text{on } \partial_L\mathcal{C}_\Omega, \\ v(x,0) &= u(x) \text{ on } \Omega \times 0. \end{cases}$$

Then, $(-\Delta)^s$, is given by the following Dirichlet to Neumann map of v on $\Omega \times 0$,

$$u \mapsto \partial^s_\nu v := -c_s \lim_{t\to 0} t^{1-2s}\partial_t v|_{\Omega\times 0},$$

where $\nu = (0,\cdots,0,-1) \in \mathbb{R}^{N+1}$ and $c_s := 2/[(4\pi)^{2s}\Gamma(2-2s)]$.

The extension function belongs to the space

$$X_0^s(\mathcal{C}_\Omega) = \overline{C_0^\infty(\mathcal{C}_\Omega)}^{\|\cdot\|_{X_0^s(\mathcal{C}_\Omega)}} \quad \text{with } \|w\|_{X_0^s(\mathcal{C}_\Omega)} = c_s \left(\int_{\mathcal{C}_\Omega} t^{1-2s} |\nabla w|^2 \, dx \, dt \right)^{1/2}.$$

We note that the extension operator is an isometry between $X_0^s(\mathcal{C}_\Omega)$ and $\widetilde{H}_0^s(\Omega)$ i.e.

$$\|E_s(u)\|_{X_0^s(\mathcal{C}_\Omega)} = c_s \|u\|_{\widetilde{H}_0^s(\Omega)} \quad \forall u \in \widetilde{H}_0^s(\Omega).$$

Now we can reformulate our problem $(P^s_{b,\Omega})$ using s-harmonic extension as

$$(P^s_{b,\mathcal{C}_\Omega}) \begin{cases} -\mathrm{div}(t^{1-2s} \nabla v) & = 0 \qquad \text{in } \mathcal{C}_\Omega, \\ v & = 0 \qquad \text{on } \partial_L \mathcal{C}_\Omega, \\ \partial_\nu^s v & = b(x)|u|^{q-2}u \text{ on } \Omega \times 0. \end{cases}$$

A weak solution to the problem $(P^s_{b,\mathcal{C}_\Omega})$ is a function $v \in X_0^s(\mathcal{C}_\Omega)$ such that

$$c_s \int_{\mathcal{C}_\Omega} t^{1-2s} \nabla v . \nabla \varphi \, dx \, dt = \int_\Omega b|u|^{q-2} u \varphi(\cdot, 0) \, dx, \quad \forall \ \varphi \in X_0^s(\mathcal{C}_\Omega).$$

For any weak solution $v \in X_0^s(\mathcal{C}_\Omega)$ to the problem $(P^s_{b,\mathcal{C}_\Omega})$, the function $u = v(\cdot, 0)$, defined in the sense of trace, belongs to the space $\widetilde{H}_0^s(\Omega)$ and is a weak solution to the problem $(P^s_{b,\Omega})$. The converse is also true. Hence, both formulations are equivalent.
We denote by Tr_Ω the trace operator on $\Omega \times 0$ for functions in $X_0^s(\mathcal{C}_\Omega)$, and consistently use the notation

$$u = Tr_\Omega(v) \ \text{ for } \ v \in X_0^s(\mathcal{C}_\Omega).$$

The energy functional associated with the problem $(P^s_{b,\mathcal{C}_\Omega})$ is given as

$$J_\Omega(v) = \frac{c_s}{2} \int_{\mathcal{C}_\Omega} t^{1-2s} |\nabla v|^2 \, dx \, dt - \frac{1}{q} \int_\Omega b(x) |u(x)|^q \, dx.$$

Again we observe that the functional J_Ω is not well-defined in $X_0^s(\mathcal{C}_\Omega)$ due to supercritical non-linearity terms in the problem $(P^s_{b,\mathcal{C}_\Omega})$. Next we recall the following trace inequality [5].

Theorem 5.1.1. *For every $v \in X_0^s(\mathcal{C}_\Omega)$, $1 \le r \le 2_s^*$ and $N > 2s$, we have the following trace inequality:*

$$C\left(\int_\Omega |u(x)|^r\,dx\right)^{\frac{2}{r}} \le \int_{\mathcal{C}_\Omega} t^{1-2s}|\nabla v|^2\,dx\,dt \tag{5.1.2}$$

where $u = Tr_\Omega v$ and $C = C(r,s,N,\Omega) > 0$.

Remark 5.1.1. *When $r = 2_s^*$, the best constant in (5.1.2) will be denoted by S_s. It is not achieved on any bounded domain, so we have*

$$S_s\left(\int_{\mathbb{R}^N} |u(x)|^{\frac{2N}{N-2s}}\,dx\right)^{\frac{N-2s}{N}} \le \int_{\mathbb{R}_+^{N+1}} t^{1-2s}|\nabla v|^2\,dx\,dt \quad \forall\, v \in X_0^s(\mathbb{R}_+^{N+1}).$$

However, it is achieved when $\Omega = \mathbb{R}^N$ and u takes the form

$$u(x) = u_\tau(x) = \frac{\tau^{\frac{N-2s}{2}}}{(|x-x_0|^2+\tau^2)^{\frac{N-2s}{2}}}$$

for some $x_0 \in \mathbb{R}^N, \tau > 0$ and $v = E_s(u)$.

Next we consider the following definition of group action. We denote by $O(N)$ the group of linear isometries of $\mathbb{R}^N$. We say that a closed subgroup G of $SO(N)$ acts on the base of $\mathbb{R}_+^{N+1}$ if for $g \in G$ and $(x,t) \in \mathbb{R}_+^{N+1}$

$$g(x,t) = (gx,t).$$

Therefore, $G(x,t) := (\{gx \text{ s.t } g \in G\}, t)$ denotes its G-orbit and $\#G(x,t) = \#Gx$ its cardinality. A subset X of $\mathbb{R}_+^{N+1}$ is said to be G-invariant if $G(x,t) \subset X$ for every $(x,t) \in X$ and a function $u : X \to \mathbb{R}$ is called G-invariant if it is constant on every $G(x,t)$ with $(x,t) \in X$. The G-fixed-point set of X is the set $X^G := \{(x,t) \in X : (gx,t) = (x,t)\ \forall g \in G\}$.

5.2 Main result

In this section, we demonstrate some domains of revolutions for which the supercritical exponent problem $(P^s_{b,\mathcal{C}_\Omega})$ exhibits solutions.

Take $q = 2^*_{s,N,k} := \frac{2(N-k)}{N-k-2s}$ to the problem $(P^s_{b,\mathcal{C}_\Omega})$ in the domain of the following form: Write $k := k_1 + \cdots + k_d$ with $k_1, ..., k_d \in \mathbb{N}$, $1 \le d \le N-k-d$, and we consider Ω to be the base of cylinder $\mathcal{C}_\Omega$ as

$$\Omega := \{(x^1, ..., x^d, x') \in \mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_d+1} \times \mathbb{R}^{N-k-d} : \left(|x^1|, ..., |x^d|, x'\right) \in \Theta\}, \quad (5.2.1)$$

where Θ is a bounded smooth domain in $\mathbb{R}^{N-k}$ whose closure is contained in $(0,\infty)^d \times \mathbb{R}^{N-k-d}$. We assume that b is radial in x^i, i.e. it can be written as

$$b(x^1, \dots, x^d, x') = \gamma(|x^1|, \dots, |x^d|, x'). \quad (5.2.2)$$

Note that $2^*_{s,N,k}$ is the critical exponent for the Sobolev embedding in the trace sense $X^s_0(\mathbb{R}^{N+1-k}_+) \hookrightarrow L^{2^*_{s,N,k}}(\mathbb{R}^{N-k})$.

In order to obtain solutions, we will assume that $\mathcal{C}_\Theta$ has some symmetries. We consider $O(N-k-d)$ as a subgroup of $O(N-k)$ and action of this subgroup is defined as

$$g(x'', x', t) := (x'', gx', t) \;\; \forall (x'', x', t) \in \mathbb{R}^d \times \mathbb{R}^{N-k-d} \times (0,\infty), \; g \in O(N-k-d).$$

Next, we fix a closed subgroup Γ of $O(N-k-d)$ and a Γ-invariant bounded smooth domain $\mathcal{D}$ contained in $(0,\infty)^d \times \mathbb{R}^{N-k-d}$ such that $\#\Gamma y = \infty$ for every $y \in \mathcal{D}$. Let $\varrho : (0,\infty)^d \times \mathbb{R}^{N-k-d} \to \mathbb{R}$ be the function given by

$$\varrho(y_1, ..., y_d, y') := y_1^{k_1} \cdots y_d^{k_d}, \qquad y_i \in (0,\infty), \;\; y' \in \mathbb{R}^{N-k-d}. \quad (5.2.3)$$

Under these assumptions, we obtain the following result.

Theorem 5.2.1. *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ and $\mathcal{D}$, with the following property: If Θ is a bounded smooth domain in $\mathbb{R}^{N-k}$ such that $\mathcal{D} \subset \overline{\Theta} \subset (0,\infty)^d \times \mathbb{R}^{N-k-d}$, and if Θ is invariant under the action of a closed subgroup G of Γ for which*

$$\min_{y \in \overline{\Theta}} \frac{\varrho(y)^{\frac{N-k}{2s}}}{\gamma(y)^{\frac{N-k-2s}{2s}}} \#Gy > \ell_m,$$

then the supercritical problem $(P^s_{b,\mathcal{C}_\Omega})$ *with* $q = 2^*_{s,N,k}$ *has at least* m *pairs of solutions* $\pm v_1, \dots, \pm v_m$ *in* $\mathcal{C}_\Omega$ *of the form*

$$v_j(x^1, \dots, x^d, x', t) = w_j\left(\left|x^1\right|, \dots, \left|x^d\right|, x', t\right),$$

where v_1 *is positive,* $v_2, \dots, v_m$ *change sign, and* w_j *is* G*-invariant and satisfies*

$$\int_\Theta \gamma(y) \left|Tr_\Theta w_j\right|^{2^*_{s,N,k}} \leq \ell_j S_s^{N/2s} \qquad \textit{for every } j = 1, \dots, m.$$

Theorem 5.2.1 gives sufficient conditions for the existence of a prescribed number of solutions. It does not require the cardinality of all orbits to be infinite, but it does require that orbits have large enough cardinality.

For example, we may take Γ to be the group of unit complex numbers acting by multiplication on the second factor of $\mathbb{R} \times \mathbb{C} \times \mathbb{R}^{N-k-3}$ and $\mathcal{D}$ to be a Γ-invariant domain whose closure is contained in $(0, \infty) \times (\mathbb{C} \smallsetminus \{0\}) \times \mathbb{R}^{N-k-3}$. Then, every Γ-orbit in $\mathcal{D}$ is a circle. If $G_n := \{e^{2\pi i k/n} : k = 0, ..., n-1\}$ is the cyclic subgroup of Γ of order n, then $\#G_n y = n$ for every $y \in \mathbb{R} \times (\mathbb{C} \smallsetminus \{0\}) \times \mathbb{R}^{N-k-3}$. So, if Θ is G_n-invariant, $\mathcal{D} \subset \overline{\Theta} \subset (0, \infty) \times (\mathbb{C} \smallsetminus \{0\}) \times \mathbb{R}^{N-k-3}$ and

$$n \left[\operatorname{dist}(\Theta, \{0\} \times \mathbb{R}^{N-k-1})\right]^k > \ell_m,$$

then Theorem 5.2.1 yields at least m pairs of solutions to problem $(P^s_{b,\mathcal{C}_\Omega})$ with $q = 2^*_{s,N,k}$ in $\mathcal{C}_\Omega$.

It is easy to show that the problem $(P^s_{b,\mathcal{C}_\Omega})$ with $q = 2^*_{s,N,k}$ can be reduced to an anisotropic nonlocal critical problem of the form

$$\begin{cases} -\operatorname{div}(t^{1-2s} \varrho(y) \nabla w) &= 0 \quad \text{in } \mathcal{C}_\Theta, \\ w &= 0 \quad \text{on } \partial_L \mathcal{C}_\Theta, \\ \partial^s_\nu w &= \gamma(y) |Tr_\Theta w|^{2^*_{s,N,k}-2} Tr_\Theta w \text{ on } \Theta \times 0 \end{cases} \tag{5.2.4}$$

on the domain Θ as in (5.2.1).

5.3 Anisotropic nonlocal critical problem in domain of lower dimension

In this section, we study anisotropic nonlocal critical exponent problem in domain of lower dimension.

We note that $2^*_{s,N,k} := \frac{2(N-k)}{N-k-2s}$ is the critical exponent in dimension $n := N-k = \dim\Theta$. We assume that $\mathcal{C}_\Theta$ is G-invariant bounded smooth domain in $\mathbb{R}^{n+1}_+$ in this section and $a, b \in C^0(\overline{\Theta})$ are G-invariant functions such that a and b are positive on $\overline{\Theta}$.

We consider the following problem

$$(P^s_{a,b,\mathcal{C}_\Theta}) \begin{cases} -\mathrm{div}(t^{1-2s}a(x)\nabla v) &= 0 \quad \text{in } \mathcal{C}_\Theta, \\ v &= 0 \quad \text{on } \partial_L\mathcal{C}_\Theta, \\ \partial^s_\nu v &= b(x)|u|^{2^*_s-2}u \text{ on } \Theta\times 0, \end{cases}$$

where $2^*_s = \frac{2n}{n-2s}$.

5.3.1 Dilation invariance and group action

Here we demonstrate the dilation invariance of energy functional associated with the problem $(P^s_{a,b,\mathcal{C}_\Theta})$, which is very crucial for studying global compactness result.

Let $\lambda > 0$ and $\phi : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$ denote the dilation defined by $\phi(x,t) = (\lambda x, \lambda t)$. The Mobius transformation ϕ converts any cylinder $\mathcal{C}_\Theta$ into a rescaled version of itself given by $\phi(\mathcal{C}_\Theta) = \lambda\mathcal{C}_\Theta$. We note that the Jacobian W_ϕ of ϕ satisfies

$$W^T_\phi(x,t)W_\phi(x,t) = \lambda^2 I$$

where T denotes the tranpose and I is $(n+1)\times(n+1)$ identity matrix. We also have $|\det W_\phi(x,t)| = \lambda^{n+1}$. For $v, w \in X^s_0(\mathcal{C}_\Theta)$, we define

$$v_\phi(x,t) = h\circ\phi \text{ with } h(x,t) = \lambda^{\frac{n-2s}{2}}v(x,t) \text{ and } w_\phi(x,t) = k\circ\phi \text{ with } k(x,t) = \lambda^{\frac{n-2s}{2}}w(x,t).$$

Next we observe that

$$\begin{aligned}
\int_{\mathcal{C}_\Theta} t^{1-2s}\nabla v_\phi \cdot \nabla w_\phi \, dx\, dt &= \int_{\mathcal{C}_\Theta} t^{1-2s}[W_\phi(x,t)\nabla h(\phi(x,t))]\cdot[W_\phi(x,t)\nabla k(\phi(x,t))]\, dx\, dt \\
&= \int_{\mathcal{C}_\Theta} t^{1-2s} W_\phi^T(x,t)W_\phi(x,t)\nabla h(\phi(x,t))\cdot \nabla k(\phi(x,t))\, dx\, dt \\
&= \int_{\mathcal{C}_\Theta} t^{1-2s}\lambda^2 I \nabla h(\phi(x,t))\cdot \nabla k(\phi(x,t))\, dx\, dt \\
&= \int_{\mathcal{C}_\Theta} (t\lambda)^{1-2s}\lambda^{n+1}\lambda^{2s-n}\nabla h(\phi(x,t))\cdot \nabla k(\phi(x,t))\, dx\, dt \\
&= \int_{\mathcal{C}_\Theta} \lambda^{2s-n}(t\lambda)^{1-2s}\, |\det W_\phi(x,t)|\, \nabla h(\phi(x,t))\cdot \nabla k(\phi(x,t))\, dx\, dt \\
&= \int_{\phi(\mathcal{C}_\Theta)} \lambda^{2s-n} t^{1-2s}\nabla h(x,t)\cdot \nabla k(x,t)\, dx\, dt \\
&= \int_{\phi(\mathcal{C}_\Theta)} t^{1-2s}\nabla v\cdot \nabla w\, dx\, dt.
\end{aligned} \tag{5.3.1}$$

Therefore the map $v \mapsto v_\phi$ is a linear isometry of $X_0^s(\phi(\mathcal{C}_\Theta)) \cong X_0^s(\mathcal{C}_\Theta)$. Let $\phi_x : \mathbb{R}^n \to \mathbb{R}^n$ denote the dilation defined by $\phi_x(x) = \lambda x$. It is easy to show that

$$\int_\Theta |u_{\phi_x}|^{2_s^*}\, dx = \int_{\phi_x(\Theta)} |u|^{2_s^*}\, dx. \tag{5.3.2}$$

That is the map $u \mapsto u_{\phi_x}$ is also a linear isometry of $L^{2_s^*}(\phi_x(\Theta)) \cong L^{2_s^*}(\Theta)$.
As we know that the solutions of the problem $(P^s_{a,b,\mathcal{C}_\Theta})$ are the critical points of the energy functional $J_\Theta : X_0^s(\mathcal{C}_\Theta) \to \mathbb{R}$ defined by

$$J_\Theta(v) = \frac{c_s}{2}\int_{\mathcal{C}_\Theta} t^{1-2s} a(x)|\nabla v|^2\, dx\, dt - \frac{1}{2_s^*}\int_\Theta b(x)|u(x)|^{2_s^*}\, dx. \tag{5.3.3}$$

From the invariance (5.3.1) and (5.3.2) it follows that $J_\Theta(v_\phi) = J_{\phi(\Theta)}(v)$.
We observe that J_Θ is a C^1 functional with $DJ_\Theta \in X^{-s}(\mathcal{C}_\Theta)$ defined as

$$\langle DJ_\Theta(v),\varphi\rangle = c_s\int_{\mathcal{C}_\Omega} t^{1-2s} a(x)\nabla v.\nabla\varphi\, dx\, dt - \int_\Theta b|u|^{2_s^*-2}u\varphi(\cdot,0)\, dx, \quad \forall\, \varphi \in X_0^s(\mathcal{C}_\Theta). \tag{5.3.4}$$

Let G be a closed subgroup of $SO(n)$, and assume that $\mathcal{C}_\Theta$ is G-invariant on the base Θ. The action of G on Θ induces an orthogonal G-action on $X_0^s(\mathcal{C}_\Theta)$ is given by

$$(gv)(x,t) := v(g^{-1}x, t).$$

The energy functional defined on $X_0^s(\mathcal{C}_\Theta)$ is G-invariant, that is, $J_\Theta(gv) = J_\Theta(v)$ for every $v \in X_0^s(\mathcal{C}_\Theta)$, $g \in G$.

Therefore by the principle of symmetric criticality [82], weak solution of problem $(P^s_{a,b,\mathcal{C}_\Theta})$ are the critical points of the restriction of J_Θ to the subspace of G-fixed points

$$X_0^s(\mathcal{C}_\Theta)^G := \{v \in X_0^s(\mathcal{C}_\Theta) : v(gx,t) = v(x,t) \text{ for all } g \in G\}$$

of $X_0^s(\mathcal{C}_\Theta)$. The G-invariant solution of the problem $(P^{s,G}_{a,b,\mathcal{C}_\Theta})$ defined in subsection 5.3.3 are the critical points of the restriction of the energy functional J_Θ to the space $X_0^s(\mathcal{C}_\Theta)^G$.

5.3.2 Existence result for anisotropic nonlocal critical problem $(P^s_{a,b,\mathcal{C}_\Theta})$

We fix a closed subgroup Γ of $SO(n)$ and a Γ-invariant bounded smooth domain $\mathcal{D}$ in $\mathbb{R}^n$ such that $\#\Gamma x = \infty$ for every $x \in \mathcal{D}$. We assume that the functions a and b are Γ-invariant. Then, the following result holds true.

Theorem 5.3.1. *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ, $\mathcal{D}$, a and c, with the following property: If Θ is a bounded smooth domain which contains $\mathcal{D}$ and if it is invariant under the action of a closed subgroup G of Γ for which the inequality*

$$\min_{x\in\overline{\Theta}} \frac{a(x)^{\frac{n}{2s}} \#Gx}{b(x)^{\frac{n-2s}{2s}}} > \ell_m \tag{5.3.5}$$

holds true, then problem $(P^s_{a,b,\mathcal{C}_\Theta})$ has at least m pairs of G -invariant solutions $\pm v_1, \ldots, \pm v_m$ such that v_1 is positive, $v_2, \ldots, v_m$ change sign, and

$$\int_\Theta b(x)\,|Tr_\Theta v_j|^{2_s^*} \le \ell_j S_s^{\frac{n}{2s}} \qquad \text{for every } j = 1, \ldots, m, \tag{5.3.6}$$

where S_s is the best Sobolev constant for the embedding $X_0^s(\mathbb{R}_+^{n+1}) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$.

5.3.3 Representation of G-invariant Palais-Smale sequences

We first establish Struwe's type compactness result for the following anisotropic fractional Laplace equation with critical nonlinearity

$$(P^{s,G}_{a,b,\mathcal{C}_\Theta}) \begin{cases} -\mathrm{div}(t^{1-2s}a(x)\nabla v) &= 0 \quad \text{in } \mathcal{C}_\Theta, \\ v &= 0 \quad \text{on } \partial_L\mathcal{C}_\Theta, \\ \partial_\nu^s v &= b(x)|u|^{2_s^*-2}u \text{ on } \Theta \times 0, \\ v(gx,t) &= v(x,t) \quad \text{for all } g \in G. \end{cases}$$

We study the splitting of G-invariant Palais-Smale sequence of the functional associated with the problem $(P^{s,G}_{a,b,\mathcal{C}_\Theta})$. We show that the non-compactness is due to the solutions of the following limiting problem

$$(P^{s,K}_\infty) \begin{cases} -\mathrm{div}(t^{1-2s}\nabla v) &= 0 \quad \text{in } \mathbb{R}_+^{n+1}, \\ \partial_\nu^s v &= |u|^{2_s^*-2}u \text{ on } \mathbb{R}^n \times 0, \\ v(gx,t) &= v(x,t) \quad \text{for all } g \in K \end{cases}$$

concentrating at G-orbits of Ω with orbit type G/K for some closed subgroup K of finite index in G.

Definition 5.3.1. *A sequence $\{v_k\}_{k\in\mathbb{N}}$ such that*

$$v_k \in X_0^s(\mathcal{C}_\Theta)^G, \quad J_\Theta(v_k) \to c, \quad \textit{and} \quad \|DJ_\Theta(v_k)\| \to 0 \quad \textit{in} \quad X^{-s,2}(\mathcal{C}_\Theta)$$

is called a G-invariant Palais-Smale sequence for J_Θ.

We say that J_Θ satisfies the G-Palais-Smale condition $(PS)_c^G$ at c if every G-invariant Palais-Smale sequence for J_Θ such that $J_\Theta(v_k) \to c$ has a convergence subsequence.

We write $J_\infty : X_0^s(\mathbb{R}^{n+1}_+) \to \mathbb{R}$ for the energy functional for the problem $(P^{s,K}_\infty)$, given by

$$J_\infty(v) := \frac{c_s}{2}\int_{\mathbb{R}^{n+1}_+} t^{1-2s}|\nabla v|^2\,dx\,dt - \frac{1}{2_s^*}\int_{\mathbb{R}^n}|u(x)|^{2_s^*}\,dx.$$

We shall prove the following theorem.

Theorem 5.3.2. *Let $\{v_k\}_{k\in\mathbb{N}}$ be a G-invariant Palais-Smale sequence for J_Θ at $c \in \mathbb{R}$. Then, replacing $\{v_k\}_{k\in\mathbb{N}}$ by a subsequence if necessary, there exist a solution v of problem $(P^{s,G}_{a,b,\mathcal{C}_\Theta})$, m closed subgroups $K_1, ..., K_m$ of finite index in G and, for each $i = 1, ..., m$, a sequence $\{y_{i,k}\}_{k\in\mathbb{N}}$ in Θ, a sequence $\{\varepsilon_{i,k}\}_{k\in\mathbb{N}}$ in $(0,\infty)$, a K_i-invariant solution $\widetilde{v}_i$ of the limiting problem (P^{s,K_i}_∞) such that*

- *(i)* $G_{y_{i,k}} = K_i$ *for all* $k \geq 1$, *and* $y_{i,k} \to y_i$ *as* $k \to \infty$,
- *(ii)* $\varepsilon_{i,k}^{-1}|gy_{i,k} - g'y_{i,k}| \to \infty$ *and* $\varepsilon_{i,k}^{-1}dist(y_{i,k}, \partial\Theta) \to \infty$ *as* $k \to \infty$ *for all* $[g] \neq [g'] \in G/K_i$,
- *(iii)*

$$\left\| v_k - v - \sum_{i=1}^{m}\sum_{[g]\in G/K_i} \varepsilon_{i,k}^{\frac{2s-n}{2}}\left(\frac{a(y_i)}{b(y_i)}\right)^{\frac{n-2s}{4s}} \widetilde{v}_i\left(g^{-1}\left(\frac{\cdot - gy_{i,k},\cdot}{\varepsilon_{i,k}}\right)\right)\right\|_{X_0^s(\mathbb{R}^{n+1}_+)} \to 0$$

  in $X_0^s(\mathbb{R}^{n+1}_+)$,
- *(iv)* $J_\Theta(v_k) \to J_\Theta(v) + \sum_{i=1}^m |G/K_i|\left(\frac{a(y_i)^{n/2s}}{b(y_i)^{\frac{n-2s}{2s}}}\right) J_\infty(\widetilde{v}_i)$ *as* $k \to \infty$.

For every solution v to problem $(P^{s,G}_{a,b,\mathcal{C}_\Theta})$, we have $J_\Theta(v) \geq 0$ and $J_\infty(v) \geq \frac{s}{n}S_s^{\frac{n}{2s}}$ for every non-trivial solution v to problem $(P^{s,K}_\infty)$, the following assertion follows directly from Theorem 5.3.2.

Corollary 5.3.1. *The functional J_Θ satisfies $(PS)^G_c$ for every*

$$c < \left(\min_{x\in\overline{\Omega}} \frac{a(x)^{\frac{n}{2s}}|G/G_x|}{b(x)^{\frac{n-2s}{2}}}\right)\frac{s}{n}S_s^{\frac{n}{2s}}.$$

Moreover, if $|G/G_x| = \infty$ for all $x \in \overline{\Theta}$, then J_Θ satisfies condition $(PS)^G_c$ for every $c \in \mathbb{R}$.

We recall the following Brezis-Lieb type result from [46].

Lemma 5.3.1. *Let $q \in [1,\infty)$, $\{b_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mathbb{R}^n)$ and $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^q(\mathbb{R}^n)$, such that $b_k(x) \to \bar{b}(x)$ and $u_k(x) \to u(x)$ a.e. in $\mathbb{R}^n$. Then $u \in L^q(\mathbb{R}^n)$ and*

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^n} b_k\,|u_k|^q - b_k\,|u_k-u|^q\right) = \int_{\mathbb{R}^n}\bar{b}\,|u|^q\,.$$

Next we prove the Brezis-Lieb type result for energy norm.

Lemma 5.3.2. *let $\{a_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mathbb{R}^n)$ and $\bar{a}\in L^\infty(\mathbb{R}^{n+1}_+)$ be such that $a_k \to \bar{a}$ in $L^\infty_{loc}(\mathbb{R}^{n+1}_+)$ and let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $X^s_0(\mathbb{R}^{n+1}_+)$ such that $v_k \rightharpoonup v$ weakly in $X^s_0(\mathbb{R}^{n+1}_+)$. Then*

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^{n+1}_+} t^{1-2s}\left(a_k|\nabla v_k|^2 - a_k|\nabla(v_k-v)|^2\right)\,dx\,dt\right) = \int_{\mathbb{R}^{n+1}_+} t^{1-2s}\bar{a}|\nabla v|^2\,dx\,dt$$

Proof. We proceed as in the Lemma 3.5 [46]. We observe that

$$\begin{aligned} &t^{1-2s}\Big(a_k|\nabla v_k|^2 - a_k|\nabla(v_k-v)|^2 - \bar{a}|\nabla v|^2\Big) \\ &\quad= t^{1-2s}\Big((a_k-\bar{a})\nabla(2v_k-v)\nabla v + 2\bar{a}\nabla(v_k-v)\nabla v\Big). \end{aligned} \tag{5.3.7}$$

For a fix $R,\ r>0$ and $(z,r)\in\mathbb{R}^{n+1}_+$ there exists a constant $C>0$ such that

$$\begin{aligned} &\left|\int_{\mathbb{R}^{n+1}_+} t^{1-2s}\Big((a_k-\bar{a})\nabla(2v_k-v)\nabla v\Big)\,dx\,dt\right| \\ &\leq \left|\int_{B((z,r),R)} t^{1-2s}\Big((a_k-\bar{a})\nabla(2v_k-v)\nabla v\Big)\,dx\,dt\right| \\ &\qquad + \left|\int_{\mathbb{R}^{n+1}_+\setminus B((z,r),R)} t^{1-2s}\Big((a_k-\bar{a})\nabla(2v_k-v)\nabla v\Big)\,dx\,dt\right| \\ &\leq C|a_k-\bar{a}|_{L^\infty(B((z,r),R))} + C\int_{\mathbb{R}^{n+1}_+\setminus B((z,r),R)} t^{1-2s}|\nabla v|^2\,dx\,dt. \end{aligned}$$

Consequently,

$$\limsup_{k\to\infty}\left|\int_{\mathbb{R}^{n+1}_+} t^{1-2s}\Big((a_k-\bar{a})\nabla(2v_k-v)\nabla v\Big)\,dx\,dt\right| \leq C\int_{\mathbb{R}^{n+1}_+\setminus B((z,r),R)} t^{1-2s}|\nabla v|^2\,dx\,dt$$

and letting $R \to \infty$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} t^{1-2s}\Big((a_k - \overline{a})\nabla(2v_k - v)\nabla v\Big)\, dx\, dt = 0. \tag{5.3.8}$$

On the other hand, as $v_k - v \rightharpoonup 0$ weakly in $X_0^s(\mathbb{R}^{n+1}_+)$, we conclude that

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} t^{1-2s}\overline{a}\nabla(v_k - v)\nabla v\, dx\, dt = 0. \tag{5.3.9}$$

Using equations (5.3.7),(5.3.8) and (5.3.9), we get the desired result. □

Lemma 5.3.3. *Let $\{b_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mathbb{R}^n)$ and $\overline{b} \in L^\infty(\mathbb{R}^n)$ be such that $b_k \to \overline{b}$ in $L^\infty_{loc}(\mathbb{R}^n)$. Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $X_0^s(\mathbb{R}^{n+1}_+)$ such that $v_k \rightharpoonup v$ weakly in $X_0^s(\mathbb{R}^{n+1}_+)$ with $u \in L^\infty_{loc}(\mathbb{R}^n)$. Then*

$$b_k\, |u_k|^{2^*_s-2}\, u_k - b_k\, |u_k - u|^{2^*_s-2}\, (u_k - u) \longrightarrow \overline{b}\, |u|^{2^*_s-2}\, u \quad \text{in } X^{-s,2}(\mathbb{R}^{n+1}_+).$$

Proof. The proof is similar to that of Lemma 8.9 in [95]. □

First, we establish the following proposition. Theorem 5.3.2 follows from the iteration of this result.

Proposition 5.3.1. *Let $\{v_k\}_{k\in\mathbb{N}}$ be a G-invariant Palais-Smale sequence for J_Θ such that $v_k \rightharpoonup 0$ weakly in $X_0^s(\mathcal{C}_\Theta)^G$ and $J_\Theta(v_k) \to c > 0$. Then, replacing $\{v_k\}_{k\in\mathbb{N}}$ by a subsequence if necessary, there exist a closed subgroup K of finite index in G, a sequence $\{y_k\}_{k\in\mathbb{N}}$ in Θ, a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ in $(0,\infty)$, a K-invariant solution $\widetilde{v}$ of the limiting problem $(P^{s,K}_\infty)$, and a G-invariant Palais-Smale sequence $\{w_k\}_{k\in\mathbb{N}}$ for J_Θ such that*

- *(i)* $G_{y_k} = K$ *for all k, and $y_k \to y_0$ as $k \to \infty$,*
- *(ii)* $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ *and* $\varepsilon_k^{-1} dist(y_k, \partial\Theta) \to \infty$ *as $k \to \infty$ for all $[g] \neq [g'] \in G/K$,*
- *(iii)* $w_k = v_k - \sum_{[g]\in G/K} \varepsilon_k^{\frac{2s-n}{2}} \left(\frac{a(y_0)}{b(y_0)}\right)^{\frac{n-2s}{4s}} \widetilde{v}\left(g^{-1}\left(\frac{\cdot - gy_k, \cdot}{\varepsilon_k}\right)\right) + o(1)$ *in* $X_0^s(\mathbb{R}^{n+1}_+)$,
- *(iv)* $J_\Theta(w_k) \to c - |G/K| \left(\frac{a(y_0)^{n/2s}}{b(y_0)^{\frac{n-2s}{2s}}}\right) J_\infty(\widetilde{v})$ *as $k \to \infty$.*

We also recall Lemma 3.3 in [41] which is used to prove Proposition 5.3.1.

Lemma 5.3.4. *Given sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$ in $(0,\infty)$ and $\{\xi_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$, there exist a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ and a closed subgroup K of G such that, after passing to a subsequence, the following statements hold true:*

(i) *The sequence $(\varepsilon_k^{-1}\text{dist}(G\xi_k, y_k))$ is bounded.*

(ii) *$G_{y_k} = K$ for all $k \in \mathbb{N}$.*

(iii) *If $|G/K| < \infty$ then $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ for any $g, g' \in G$ with $g'g^{-1} \notin K$.*

(iv) *If $|G/K| = \infty$ then there is a closed subgroup K' of G such that $K \subset K'$, $|G/K'| = \infty$ and $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ for any $g, g' \in G$ with $g'g^{-1} \notin K'$.*

Now we give the proof of Proposition 5.3.1.

Proof of Proposition 5.3.1:

We prove the Proposition 5.3.1 in the following steps:

Step 1: We define sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \in (0,\infty)$, $\{y_k\}_{k\in\mathbb{N}} \in \mathbb{R}^n$ and a subgroup K of finite index in G such that $G_{y_k} = K$. Let $\{v_k\}_{k\in\mathbb{N}}$ be a G-invariant Palais-Smale sequence for J_Θ at the level $c > 0$. So it is bounded in $X_0^s(\mathcal{C}_\Theta)$ and

$$\int_\Theta b|u_k|^{2_s^*}\,dx = \frac{n}{s}J_\Theta(v_k) - \frac{nc_s}{2s}\langle DJ_\Theta(v_k), v_k\rangle \to \frac{nc}{s} > 0.$$

Set

$$\delta := \min\left\{\frac{nc}{2s}, \left(\frac{S_s \min_{x\in\overline{\Theta}} a(x)}{2(\max_{x\in\overline{\Theta}} b(x))^{\frac{n-2s}{n}}}\right)^{\frac{n}{2s}}\right\} \tag{5.3.10}$$

Let $B(x,r)$ denote the closed ball in $\mathbb{R}^n$ with centre x and radius r. Then the Levy Concentration function

$$\mathbf{\Phi}_k(r) := \sup_{x\in\mathbb{R}^n}\int_{B(x,r)} b|u_k|^{2_s^*}$$

satisfies that $\mathbf{\Phi}_k(0) = 0$ and $\mathbf{\Phi}_k(\infty) > \delta$ for k large enough. Hence we may choose $\xi_k \in \mathbb{R}^n$ and $\varepsilon_k > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\int_{B(x,\varepsilon_k)} b|u_k|^{2_s^*} = \int_{B(\xi_k,\varepsilon_k)} b|u_k|^{2_s^*} = \delta. \tag{5.3.11}$$

By Lemma 5.3.4, after passing to a subsequence, there exist $\{y_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$, a subgroup K of G, and $C_1 > 0$ such that $G_{y_k} = K$ and $\varepsilon_k^{-1}\text{dist}(G\xi_k, y_k) < C_1$ for all $k \in \mathbb{N}$. Therefore, (y_k)

is bounded and there exists $g_k \in G$ such that $B(g_k\xi_k, \varepsilon_k) \subset B(y_k, C\varepsilon_k)$ with $C := C_1 + 1$. As b and u_k are G-invariant, this implies that

$$\delta = \int_{B(\xi_k,\varepsilon_k)} b|u_k|^{2_s^*} = \int_{B(g_k\xi_k,\varepsilon_k)} b|u_k|^{2_s^*} \leq \int_{B(y_k,C\varepsilon_k)} b|u_k|^{2_s^*}. \tag{5.3.12}$$

Now we claim that $|G/K| < \infty$. If not then, by Lemma 5.3.4, there exists a closed subgroup K' of G such that $K \subset K'$, $|G/K'| = \infty$ and $\varepsilon_k^{-1}|gy_k - g'y_k| \to \infty$ for any $[g], [g'] \in G/K'$ with $[g] \neq [g']$. Hence, for each $m \in \mathbb{N}$, we may choose $g_1, \ldots, g_m \in G$ such that $[g_i] \neq [g_j]$ in G/K' and $B(g_iy_k, C\varepsilon_k) \cap B(g_jy_k, C\varepsilon_k) = \emptyset$ for $i \neq j$ and k sufficiently large. From inequality (5.3.12) we obtain that

$$m\delta \leq \sum_{i=1}^{m} \int_{B(g_iy_k,C\varepsilon_k)} b|u_k|^{2_s^*} \leq \int_{\Theta} b|u_k|^{2_s^*} = \frac{nc}{s} + o_k(1),$$

for every $m \in \mathbb{N}$. This is a contradiction. Hence $|G/K| < \infty$.

Step 2: We assert that $\varepsilon_k^{-1} dist(y_k, \partial\Theta) \to \infty$ and that $y_k \in \Theta$, and we define a nontrivial K-invariant solution $\tilde{v} \in X_0^s(\mathbb{R}_+^{n+1})$ to the problem $(P_\infty^{s,K})$.

For $z \in \Theta_k := \{z \in \mathbb{R}^n : \varepsilon_k z + y_k \in \Theta\}$ and $\mathcal{C}_{\Theta_k} := \Theta_k \times [0, \infty)$, set

$\overline{v_k}(z,t) := \varepsilon_k^{\frac{n-2s}{2}} v_k(\varepsilon_k z + y_k, \varepsilon_k t)$, $\quad a_k(z) := a(\varepsilon_k z + y_k)$, $\quad b_k(z) := b(\varepsilon_k z + y_k)$ and $\overline{u_k}(z) := \varepsilon_k^{\frac{n-2s}{2}} u_k(\varepsilon_k z + y_k)$.

As $G_{y_k} = K$ and v_k, u_k, a and b are G-invariant, we have that $\overline{v_k}, \overline{u_k}, a_k$ and b_k are K-invariant. We note that

$$\int_{\mathcal{C}_\Theta} t^{1-2s} a(x) |\nabla v_k|^2 \, dx \, dt = \int_{\mathcal{C}_{\Theta_k}} t^{1-2s} a_k(z) |\nabla \overline{v_k}|^2 \, dz \, dt \quad \text{and}$$
$$\int_\Theta b(x)|u(x)|^{2_s^*} \, dx = \int_{\Theta_k} b_k(z) |\overline{u_k}|^{2_s^*} \, dz.$$

Therefore, $\{\overline{v_k}\}_{k\in\mathbb{N}}$ is bounded in $X_0^s(\mathbb{R}_+^{n+1})$. As a consequence, up to a subsequence, $\overline{v_k} \rightharpoonup \overline{v}$ weakly in $X_0^s(\mathbb{R}_+^{n+1})$, $\overline{v_k} \to \overline{v}$ strongly in $L^2_{loc}(\mathbb{R}_+^{n+1})$, $\overline{u_k} \to \overline{u}$ strongly in $L^r_{loc}(\mathbb{R}^n)$, for $1 \leq r < 2_s^*$, and $\overline{u_k} \to \overline{u}$ a.e. in $\mathbb{R}^n$. If $\overline{v} = 0$ and so $\overline{u} = 0$ then, for every $(z, r) \in \mathbb{R}_+^{n+1}$ and every $h \in C_c^\infty(B((z,r),1))$,

$$\begin{aligned}
S_s\Big(\int_{\Theta_k} |Tr_{\Theta_k}(h\overline{v_k})|^{2_s^*}\Big)^{\frac{2}{2_s^*}} &\le \int_{\mathcal{C}_{\Theta_k}} t^{1-2s}|\nabla(h\overline{v_k})|^2 \\
&= \int_{\mathcal{C}_{\Theta_k}} t^{1-2s}\nabla\overline{v_k}\cdot\nabla(h^2\overline{v_k}) + C_2\int_{\mathcal{C}_{\Theta_k}} t^{1-2s}|\nabla h|^2\overline{v_k}^2 \\
&\le \int_{\mathcal{C}_{\Theta_k}} t^{1-2s}\nabla\overline{v_k}\cdot\nabla(h^2\overline{v_k}) + o_k(1) \\
&= \Big(\max_{x\in\overline{\Theta}} a(x)\Big)^{-1}\left(\int_{\Theta_k} h^2 b_k|\overline{u_k}|^{2_s^*} - DJ_\Theta(v_k)\left(h^2\left(\frac{\cdot - y_k,\cdot}{\varepsilon_k}\right)v_k\right)\right) \\
&\quad + o_k(1) \\
&\le \Big(\max_{x\in\overline{\Theta}} a(x)\Big)^{-1}\Big(\max_{x\in\overline{\Theta}} b(x)\Big)^{\frac{n-2s}{n}}\Big(\int_{B(z,1)} b_k|\overline{u_k}|^{2_s^*}\Big)^{\frac{2s}{n}} \cdot \\
&\quad \Big(\int_{\Theta_k} |Tr_{\Theta_k}(h\overline{v_k})|^{2_s^*}\Big)^{\frac{2}{2_s^*}} + o_k(1) \\
&\le \Big(\max_{x\in\overline{\Theta}} a(x)\Big)^{-1}\Big(\max_{x\in\overline{\Theta}} b(x)\Big)^{\frac{n-2s}{n}}\delta^{\frac{2s}{n}}\Big(\int_{\Theta_k} |Tr_{\Theta_k}(h\overline{v_k})|^{2_s^*}\Big)^{\frac{2}{2_s^*}} + o_k(1) \\
&= \frac{S_s}{2}\Big(\int_{\Theta_k} |Tr_{\Theta_k}(h\overline{v_k})|^{2_s^*}\Big)^{\frac{2}{2_s^*}} + o_k(1) \qquad (5.3.13)
\end{aligned}$$

where the first inequality is Trace Sobolev's inequality, second inequality follows from $\overline{v_k} \to 0$ strongly in $L^2_{loc}(\mathbb{R}^{n+1}_+)$, third follows from the fact that $\{v_k\}_{k\in\mathbb{N}}$ is a Palais-Smale sequence and Holder's inequality, and the last one uses (5.3.10). We observe that for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$, $w \in X_0^s(\mathbb{R}^{n+1}_+)$

$$\int_{\mathbb{R}^n} |Tr_\Theta(w)|^{2_s^*}|Tr_\Theta(\varphi)|^2\,dx \le \Big(\int_{supp\,w} |Tr_\Theta(w)|^{2_s^*}\,dx\Big)^{\frac{2s}{n}}\Big(\int_{\mathbb{R}^n} |Tr_\Theta(w\varphi)|^{2_s^*}\,dx\Big)^{\frac{n-2s}{n}}. \qquad (5.3.14)$$

Using (5.3.13) and (5.3.14), we have that $Tr_{\Theta_k}(\overline{v_k}) = \overline{u_k} \to 0$ in $L^{2_s^*}_{loc}(\mathbb{R}^n)$. However, since $\varepsilon_k^{-1}|\xi_k - y_k| < C < \infty$ for all k,

$$\begin{aligned}
\delta = \int_{B(\xi_k,\varepsilon_k)} b|u_k|^{2_s^*} &\le \int_{B(y_k,\varepsilon_k(C+1))} b|u_k|^{2_s^*} \\
= \int_{B(0,C+1)} b_k|\overline{u}_k|^{2_s^*} &\le (\max_{x\in\overline{\Theta}} b(x))\int_{B(0,C+1)} |\overline{u}_k|^{2_s^*}.
\end{aligned} \qquad (5.3.15)$$

This is a contradiction. Hence, $\overline{u} \neq 0$ and so $\overline{v} \neq 0$.

As $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ are bounded, up to a subsequence, $\varepsilon_k \to \varepsilon$ in $[0,\infty)$ and $y_k \to y_0$ in $\mathbb{R}^n$. We claim $\varepsilon = 0$. Suppose it is not zero, then, as $v_k \rightharpoonup 0$ weakly in $X_0^s(\mathcal{C}_\Theta)$, we have that $\overline{v} = 0$, which is a contradiction. So, $\varepsilon = 0$. It follows that

$$a_k \to a_0 := a(y_0) \quad \text{and} \quad b_k \to b_0 := b(y_0) \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}^n).$$

Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$ be such that $\text{supp}(\varphi) \subset \mathcal{C}_{\Theta_k}$ for k large enough and set $\varphi_k(z,t) := \varepsilon_k^{\frac{2s-n}{n}} \varphi(\varepsilon_k^{-1}((z-y_k),t))$. As (φ_k) is bounded in $X_0^s(\mathcal{C}_\Theta)$ we have that

$$\int_{\mathcal{C}_{\Theta_k}} t^{1-2s} a_k(z) \nabla \overline{v}_k \cdot \nabla \varphi \, dz\, dt - \int_{\Theta_k} b_k |\overline{u}_k|^{2_s^*-2} \overline{u}_k \varphi \, dz = langle DJ_\Theta(v_k), \varphi_k\rangle = o_k(1). \tag{5.3.16}$$

It is easy to check that

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} t^{1-2s} a_k(z) \nabla \overline{v}_k \cdot \nabla \varphi \, dz\, dt = \int_{\mathbb{R}^{n+1}_+} t^{1-2s} a_0 \nabla \overline{v} \cdot \nabla \varphi \, dz\, dt. \tag{5.3.17}$$

Similarly, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} b_k |\overline{u}_k|^{2_s^*-2} \overline{u}_k \varphi \, dz = \int_{\mathbb{R}^n} b_0 |\overline{u}|^{2_s^*-2} \overline{u} \varphi \, dz. \tag{5.3.18}$$

Using (5.3.16), (5.3.17) and (5.3.18), we conclude that

$$\int_{\mathbb{R}^{n+1}_+} t^{1-2s} a_0 \nabla \overline{v} \cdot \nabla \varphi \, dz\, dt - \int_{\mathbb{R}^n} b_0 |\overline{u}|^{2_s^*-2} \overline{u} \varphi \, dz = 0$$

for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$ with $\text{supp}(\varphi) \subset \mathcal{C}_{\Theta_k}$ for k large enough. By arguing as in [93, 95], one can show that if the sequence $\{\varepsilon_k^{-1} dist(y_k, \partial\Theta)\}_{k\in\mathbb{N}}$ are bounded, then $\overline{v}$ will be the solution to the problem

$$\begin{cases} -\text{div}(t^{1-2s}\nabla v) &= 0 \qquad\qquad \text{in } \mathbb{H}^{n+1}_+, \\ \qquad\qquad \partial_\nu^s v &= \frac{b_0}{a_0}|u|^{2_s^*-2}u \text{ on } \mathbb{H}^n \times 0, \end{cases}$$

in some half-space $\mathbb{H}^{n+1}_+$ in $\mathbb{R}^{n+1}_+$, contradicting the fact that this problem does not have a non-trivial solution in a starshaped subdomain of $\mathbb{R}^{n+1}_+$(see [18]). Hence $\varepsilon_k^{-1} dist(y_k, \partial\Theta) \to \infty$. This implies that $y_k \in \Theta$, otherwise $B(y_k, C\varepsilon_k) \subset \mathbb{R}^n \setminus \Theta$, contradicting (5.3.12). This also shows that $\overline{v}$ is a solution to problem

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla v) &= 0 \qquad\qquad \text{in } \mathbb{R}^{n+1}_+, \\ \partial^s_\nu v &= \frac{b_0}{a_0}|u|^{2^*_s-2}u \text{ on } \mathbb{R}^n \times 0. \end{cases}$$

Hence,

$$\widetilde{v} := \left(\frac{b_0}{a_0}\right)^{\frac{n-2s}{4s}} \overline{v}$$

is a non-trivial K-invariant solution to the problem $(P^{s,K}_\infty)$.

Step 3: We define a sequence $\{w_k\}_{k\in\mathbb{N}}$ which satisfies (iii) and (iv) and is G-invariant Palais-Smale sequence for J_Θ.

Let $G/K := \{[g_1], ..., [g_m]\}$ and

$$r_k := \frac{1}{4}\min\{\mathrm{dist}(y_k, \partial\Theta), |g_i y_k - g_j y_k| : i, j = 1, ..., m, i \neq j\}.$$

Choose a radially symmetric function $\chi \in C_0^\infty(\mathbb{R}^{n+1}_+)$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$ and define

$$w_k(x,t) := v_k(x,t) - \sum_{i=1}^{m} \varepsilon_k^{\frac{2s-n}{2}} \overline{v}\left(g_i^{-1}\left(\frac{x - g_i y_k, t}{\varepsilon_k}\right)\right)\chi(r_k^{-1}(x - g_i y_k, t)).$$

Since $\overline{v}$ is K-invariant and $G_{y_k} = K$ for all $k \in \mathbb{N}$, we have w_k is G-invariant. Similarly, for $j = 1, ..., m$, the functions

$$l^j_k(x,t) := v_k(x,t) - \sum_{i=j}^{m} \varepsilon_k^{\frac{2s-n}{2}} \overline{v}\left(g_i^{-1}\left(\frac{x - g_i y_k, t}{\varepsilon_k}\right)\right)$$

are G-invariant in $X^s_0(\mathbb{R}^{n+1}_+)$. We note that $r_k\varepsilon_k^{-1} \to \infty$. Then one can check that

$$\lim_{k\to\infty}\left\| v_k - w_k - \sum_{i=1}^{m} \varepsilon_k^{\frac{2s-n}{2}} \overline{v}\left(g_i^{-1}\left(\frac{\cdot - g_i y_k, \cdot}{\varepsilon_k}\right)\right)\right\|_{X^s_0(\mathbb{R}^{n+1}_+)} = \|l^1_k - w_k\|_{X^s_0(\mathbb{R}^{n+1}_+)} = 0. \tag{5.3.19}$$

Thus $\{w_k\}$ satisfies (iii). Now set

$$\begin{aligned}
\bar{l}_k^j(z,t) :=& \varepsilon_k^{\frac{n-2s}{2}} l_k^j(\varepsilon_k z + g_j y_k, \varepsilon_k t) \\
&= \varepsilon_k^{\frac{n-2s}{2}} v_k(\varepsilon_k z + g_j y_k, \varepsilon_k t) - \sum_{i=j+1}^{m} \bar{v}\left(g_i^{-1}\left(z + \frac{g_j y_k - g_i y_k}{\varepsilon_k}, t\right)\right) - \bar{v}(g_j^{-1} z, t) \\
&= \bar{v}_k(g_j^{-1} z, t) - \sum_{i=j+1}^{m} \bar{v}\left(g_i^{-1}\left(z + \frac{g_j y_k - g_i y_k}{\varepsilon_k}, t\right)\right) - \bar{v}(g_j^{-1} z, t)
\end{aligned}$$

Since $\bar{v}_k \rightharpoonup \bar{v}$ weakly in $X_0^s(\mathbb{R}_+^{n+1})$ and $\varepsilon_k^{-1}|g_j y_k - g_i y_k| \to \infty$ for every $i \neq j$, we have

$$\bar{v}_k \circ g_j^{-1} - \sum_{i=j+1}^{m} \bar{v}\left(g_i^{-1}\left(\cdot + \frac{g_j y_k - g_i y_k}{\varepsilon_k}, \cdot\right)\right) \rightharpoonup \bar{v} \circ g_j^{-1} \text{ weakly in } X_0^s(\mathbb{R}_+^{n+1}).$$

Using Lemma 5.3.2 we get

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} t^{1-2s} a(x) |\nabla l_k^j|^2 \, dx\, dt &= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a_k(z) |\nabla \bar{l}_k^j|^2 \, dz\, dt \\
&= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a_k \left| \nabla \left(\bar{v}_k \circ g_j^{-1} - \sum_{i=j+1}^{m} \bar{v}\left(g_i^{-1}\left(\cdot + \frac{g_j y_k - g_i y_k}{\varepsilon_k}, \cdot\right)\right)\right)\right|^2 \\
&\quad - \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a_0 |\nabla(\bar{v} \circ g_j^{-1})|^2 + o_k(1) \\
&= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a \left| \nabla \left(v_k - \sum_{i=j+1}^{m} \varepsilon_k^{\frac{2s-N}{2}} \bar{v}\left(g_i^{-1}\left(\frac{\cdot - g_i y_k, \cdot}{\varepsilon_k}\right)\right)\right)\right|^2 \\
&\quad - \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a_0 |\nabla \bar{v}|^2 + o_k(1) \\
&= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a |\nabla l_k^{j+1}|^2 - \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a_0 |\nabla \bar{v}|^2 + o_k(1).
\end{aligned}$$

These identities for $j = 1, \dots m$, along with (5.3.19), gives

$$\int_{\mathcal{C}_\Theta} t^{1-2s} a |\nabla w_k|^2 = \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a |\nabla l_k^1|^2 = \int_{\mathcal{C}_\Theta} t^{1-2s} a |\nabla v_k|^2 - m \int_{\mathbb{R}_+^{n+1}} t^{1-2s} a_0 |\nabla \bar{v}|^2 + o_k(1). \tag{5.3.20}$$

Similarly

$$\int_\Theta b |w_k|^{2_s^*} = \int_\Theta b |\nabla u_k|^{2_s^*} - m \int_{\mathbb{R}_n} b_0 |\bar{v}|^{2_s^*} + o_k(1). \tag{5.3.21}$$

Thus from (5.3.20)-(5.3.21) we obtain

$$\begin{aligned} J_\Theta(v_k) &= J_\Theta(w_k) + m\left(\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} t^{1-2s} a_0 |\nabla \overline{v}|^2 - \frac{1}{2^*_s}\int_{\mathbb{R}^n} b_0 |\overline{v}|^{2^*_s}\right) + o_k(1) \\ &= J_\Theta(w_k) + |G/K| \left(\frac{a_0^{n/2s}}{b_0^{(n-2s)/2s}}\right) J_\infty(\widetilde{v}) + o_k(1). \end{aligned}$$

This proves (iv).

Since $DJ_\infty(\widetilde{v}) = 0$, a similar argument using Lemma 5.3.3 shows that

$$o_k(1) = DJ_\Theta(v_k) = DJ_\Theta(w_k) + o_k(1) \text{ in } X^{-s,2}(\mathbb{R}^{n+1}_+).$$

Thus $\{w_k\}$ is the required G-invariant Palais-Smale sequence for J_Θ. This completes the proof of the Proposition 5.3.1.

Proof of the Theorem 5.3.2 is similar to proof of Theorem 3.1.1.

5.4 Proof of Theorem 5.2.1 and Theorem 5.3.1

To obtain the sign-changing solutions in Theorem 5.3.1, we require the following results.

Theorem 5.4.1. *Let V be a finite dimensional subspace of $X^s_0(\mathcal{C}_\Theta)^G$. If J_Θ satisfies condition $(PS)^G_\tau$ in $X^s_0(\mathcal{C}_\Theta)^G$ for every $\tau \le \sup_V J_\Theta$, then J_Θ has at least $\dim V - 1$ pairs of sign changing critical points $v \in X^s_0(\mathcal{C}_\Theta)^G$ such that $J_\Theta(v) \le \sup_V J_\Theta$.*

We note that the above theorem has been studied for the case $s = \frac{1}{2}$ in [32]. We refer to Theorem 2.6 of [46] for the proof of Theorem 5.4.1.

For the proof of the Theorem 5.3.1, we proceed as in [41].

Proof of Theorem 5.3.1. We define ℓ_m as follows: Let $\mathcal{P}_1(\mathcal{D})$ be the collection of all nonempty Γ-invariant bounded smooth domains contained in $\mathcal{D}$, and define

$$\mathcal{P}_m(\mathcal{D}) := \{(\mathcal{D}_1, \ldots, \mathcal{D}_m) : \mathcal{D}_i \in \mathcal{P}_1(\mathcal{D}),\ \ \mathcal{D}_i \cap \mathcal{D}_j = \emptyset \text{ if } i \neq j\}.$$

Note that $\mathcal{P}_m(\mathcal{D}) \neq \emptyset$ for each $m \in \mathbb{N}$. Since $\#\Gamma x = \infty$ for all $x \in \mathcal{D}_i$, Corollary 5.3.1

asserts that J_Θ satisfies condition $(PS)^\Gamma_\tau$ in $X^s_0(\mathcal{C}_{\mathcal{D}_i})$ for every $\tau \in \mathbb{R}$. Hence, the mountain pass theorem [87, Theorem 2.2] yields a nontrivial least energy Γ-invariant solution $\omega_{\mathcal{C}_{\mathcal{D}_i}}$ to problem $(P^s_{a,b,\mathcal{C}_\Theta})$ in $\mathcal{C}_{\mathcal{D}_i}$ which satisfies

$$J_\Theta(\omega_{\mathcal{C}_{\mathcal{D}_i}}) = \max_{e\geq 0} J_\Theta(e\,\omega_{\mathcal{C}_{\mathcal{D}_i}}). \tag{5.4.1}$$

Extending $\omega_{\mathcal{C}_{\mathcal{D}_i}}$ by zero outside $\mathcal{C}_{\mathcal{D}_i}$ we have that $\omega_{\mathcal{C}_{\mathcal{D}_i}} \in X^s_0(\mathcal{C}_{\mathcal{D}_i})$. Set $\tau_\infty := \frac{s}{n} S_s^{\frac{n}{2s}}$ and define

$$\tau_m := \inf\left\{\sum_{i=1}^m J_\Theta(\omega_{\mathcal{C}_{\mathcal{D}_i}}) : (\mathcal{D}_1,\ldots,\mathcal{D}_m) \in \mathcal{P}_m(\mathcal{D})\right\} \quad \text{and} \quad \ell_m := \tau_\infty^{-1}\tau_m. \tag{5.4.2}$$

It is easy to see that (ℓ_m) is nondecreasing.

If G is a closed subgroup of Γ and Θ is a G-invariant bounded smooth domain which contains $\mathcal{D}$ and satisfies (5.3.5), we choose $\varepsilon > 0$ and $(\mathcal{D}_1,\ldots,\mathcal{D}_m) \in \mathcal{P}_m(\mathcal{D})$ such that

$$\tau_m \leq \sum_{i=1}^m J_\Theta(\omega_{\mathcal{C}_{\mathcal{D}_i}}) < \tau_m + \varepsilon < \left(\min_{x\in\overline{\Theta}} \frac{a(x)^{\frac{n}{2s}} \# Gx}{b(x)^{\frac{n-2s}{2s}}}\right)\tau_\infty.$$

Let V be the subspace of $X^s_0(\mathcal{C}_\Theta)^G$ generated by $\{\omega_{\mathcal{C}_{\mathcal{D}_1}},\ldots,\omega_{\mathcal{C}_{\mathcal{D}_m}}\}$. Since $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ if $i \neq j$, we have that $\dim V = m$. Moreover, identity (5.4.1) implies that

$$\sup_V J_\Theta \leq \sum_{i=1}^m J_\Theta(\omega_{\mathcal{C}_{\mathcal{D}_i}}) < \left(\min_{x\in\overline{\Theta}} \frac{a(x)^{\frac{n}{2s}} \# Gx}{b(x)^{\frac{n-2s}{2s}}}\right)\tau_\infty.$$

It follows from Corollary 5.3.1 that J_Θ satisfies condition $(PS)^G_\tau$ in $X^s_0(\mathcal{C}_\Theta)^G$ for every $\tau \leq \sup_V J_\Theta$, so the mountain pass theorem [87, Theorem 2.2] yields a positive critical point v_1 of J_Θ in $X^s_0(\mathcal{C}_\Theta)^G$ and Theorem 5.4.1 yields $\dim V - 1$ pairs of sign changing critical points $\pm v_2,\ldots,\pm v_m \in X^s_0(\mathcal{C}_\Theta)^G$ such that $J_\Theta(v_i) \leq \sup_V J_\Theta$. Now we argue as in [41], to show that the v_i's may be suitably chosen so that (5.3.6) holds true. □

Now we give the proof of the Theorem 5.2.1.

Proof of Theorem 5.2.1. Let Ω be as in (5.2.1). If $v(x^1,\ldots,x^d,x',t) = w\left(\left|x^1\right|,\ldots,\left|x^d\right|,x',t\right)$

with $w \in \mathcal{C}^2(\mathcal{C}_\Theta)$ then

$$-\mathrm{div}(t^{1-2s}\nabla v(x,t)) = -\frac{1}{\varrho(y)}\mathrm{div}(t^{1-2s}\varrho(y)\nabla w(y,t))$$

where $(x,t) = (x^1,\dots,x^d,x',t)$, $(y,t) = \left(\left|x^1\right|,\dots,\left|x^d\right|,x',t\right)$ and ϱ is the function defined in (5.2.3). Hence, v satisfies

$$-\mathrm{div}(t^{1-2s}\nabla v) = 0 \text{ in } \mathcal{C}_\Omega$$

if and only if w satisfies

$$-\mathrm{div}(t^{1-2s}\varrho(y)\nabla w) = 0 \text{ in } \mathcal{C}_\Theta$$

As we know that $2^*_{s,N,k} := \frac{2(N-k)}{N-k-2s}$ is the critical exponent in dimension $n := N-k = \dim\Theta$. Therefore, Theorem 5.2.1 follows immediately from Theorem 5.3.1. □

5.5 Conclusion

In this chapter, we have studied the existence of a prescribed number of solutions to the problem involving fractional Laplacian with supercritical growth in domains of revolutions under symmetry assumptions. Firstly, we convert the nonlocal problem to the local one using s-harmonic extension method. Then, by rotating some domain of lower dimension around some linear subspace of $\mathbb{R}^N$, preserving the extended operator, the supercritical exponent problem gets reduced to a critical or a subcritical anisotropic problem. Then, we have established the global compactness result for that anisotropic critical exponent problem. At last we have investigated the existence of the multiple sign-changing solutions. The salient feature of this problem is that the extended degenerate elliptic equations in $N+1$ dimensional upper-half space preserves the rotational symmetry on the base.
The existence of a prescribed number of solutions to the purely supercritical exponent problem involving fractional p-Laplacian for $p \neq 2$ is still open as s-harmonic extension method does not exist for $p \neq 2$.

6

Nonlocal superlinear semipositone problem with subcritical growth

In this chapter, we investigate the existence of a positive solution to p-superlinear semipositone problem with subcritical growth involving fractional p-Laplace operator.

6.1 Main result

We look for a positive solution to following Dirichlet boundary value problem

$$(P^\mu)\begin{cases} (-\Delta)_p^s u &= \mu(u^r-1) \text{ in } \Omega, \\ \quad u &> 0 \qquad\quad \text{ in } \Omega, \\ \quad u &= 0 \qquad\quad \text{ on } \Omega^c, \end{cases} \tag{6.1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p-1<r<p_s^*-1$ with $p_s^* := \frac{Np}{N-ps}$ is fractional critical Sobolev exponent.

Theorem 6.1.1. *For $p \geq 2$, there exists $\mu_0 > 0$ such that the problem (P^μ) admits a positive solution for $\mu \in (0,\mu_0)$ and $p-1<r<p_s^*-1$.*

Remark 6.1.1. *We remark that with slight modification the above theorem can be proved for more general Dirichlet problem like $(-\Delta)_p^s u = \lambda f(u)$ in Ω and $u=0$ in $\mathbb{R}^N \setminus \Omega$ where $f:[0,\infty)\to\mathbb{R}$ is such that $f(0)<0$, $f(s)\geq 0$ for $s>>1$ and an additional growth assumption $\lim_{s\to\infty}\frac{f(s)}{s^{q-1}} = b$ for some $b>0$ and $q\in(p-1,p_s^*-1)$.*

6.2 L^∞ a priori estimate for viscosity solutions

In this section, we give some regularity results, which are used in subsequent sections to prove the main results.

6.2.1 Removal of PV

First we make an important observation that if u is smooth enough, the principal value $P.V.$ in the definition of fractional p-Laplacian can be replaced with integral over $\mathbb{R}^N$ in the degenerate case i.e when $p > \frac{2}{2-s}$. For brevity of the notation, for any real number a, by a^{p-1} we mean $a^{p-1} = |a|^{p-2}a$. With this notation, we can also define

$$(-\Delta)_p^s u(x) = P.V. \int_{\mathbb{R}^N} \frac{(u(x)-u(x+z))^{p-1} + (u(x)-u(x-z))^{p-1}}{|z|^{N+sp}}\, dz. \tag{6.2.1}$$

The equivalence of the definitions (1.1.11) and (6.2.1) can be proved using a simple change of variable.

We recall Lemma 2.11 of [67].

Lemma 6.2.1. *If $u \in C^{1,\gamma}_{loc}(\Omega), \gamma \in [0,1]$, and $K \subset \Omega$ is compact, then there exist $C_{K,u}, R_K > 0$ such that for all $x \in K, z \in B(0, R_K)$*

$$\begin{aligned} |u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}| &\leq C_{K,u}|z|^{\gamma+p-1} && \text{if } p \geq 2,\\ |u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}| &\leq C_{K,u}|z|^{(\gamma+1)(p-1)} && \text{if } p < 2. \end{aligned}$$

Now using above estimate we argue for removing of the principal value from the expression (6.2.1).

Lemma 6.2.2. *If $p > \frac{2}{2-s}$, and $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\gamma}_{loc}(\Omega), \gamma \in [0,1]$ then*

$$(-\Delta)^s_p u(x) = \int_{\mathbb{R}^N} \frac{(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}}{|z|^{N+sp}}\, dz.$$

Proof. Let $p \geq 2$ and $x \in K$ for some K compactly contained in Ω. Using the Lemma 6.2.1,

$$\int_{B(0,R_K)} \frac{|(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}|}{|z|^{N+sp}} \leq C_{K,u}\int_{B(0,R_K)} |z|^{\gamma+p-1-N-sp} dz < \infty.$$

It can easily be seen that

$$\int_{B^c(0,R_K)} \frac{|(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}|}{|z|^{N+sp}} < \infty.$$

Therefore, $\dfrac{|(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}|}{|z|^{N+sp}} \in L^1(\mathbb{R}^N)$ and hence by dominated convergence theorem we can write,

$$\lim_{\epsilon\to 0}\int_{z\in B^c(0,\epsilon)} \frac{(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}}{|z|^{N+sp}} dz = \int_{\mathbb{R}^N} \frac{(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}}{|z|^{N+sp}} dz.$$

Now if $p \in (\frac{2}{2-s}, 2)$ then again using the second pointwise estimate in Lemma 6.2.1, and arguing as in the above case $p \geq 2$, we can prove that

$$\frac{|(u(x) - u(x+z))^{p-1} + (u(x) - u(x-z))^{p-1}|}{|z|^{N+ps}} \in L^1(\mathbb{R}^N)$$

and thus the conclusion follows as before.

□

6.2.2 Barrier function under fractional p-Laplacian

Here we study the behaviour of distance function under the fractional p-Laplacian operator, by defining an appropriate barrier function. We refer to [60] for the barrier function for fractional Laplacian.

We recall the next two Lemmas from [68].

Lemma 6.2.3. *Let e be a unit vector in $\mathbb{R}^N$, let $p > 1$, and let $a \geq 0$. Then*

$$\int_{S^N} (|e \cdot \omega| + a)^{p-2}\, d\omega \leq c(1+a)^{p-2}$$

where S^N is the unit sphere around the origin and c depends only on n and p.

Lemma 6.2.4. *Let l be an affine function and let $r \in (0, \infty)$. Then*

$$\int_{B(x,r \setminus B(x,\epsilon)} \frac{|l(x+y) - l(x)|^{p-2}(l(x+y) - l(x))}{|y|^{N+sp}}\, dy = 0$$

for all $\epsilon \in (0, r)$.

We denote by $d(x)$ by the distance of x to Ω, that is,

$$d(x) := dist(x, \partial\Omega), \qquad x \in \Omega.$$

For some $\delta > 0$, we define Ω_δ as follows:

$$\Omega_\delta = \{x \in \Omega / d(x, \partial\Omega) < \delta\}.$$

We always assume that $\delta > 0$ is small enough such that $d(x)$ is well defined and C^2 in Ω_δ. We define the barrier function as follows

$$\xi(x) = \begin{cases} d^\beta(x) & \text{if } x \in \Omega_\delta \\ r(x) & \text{if } x \in \Omega \setminus \Omega_\delta \\ 0 & \text{if } x \in \Omega^c \end{cases} \tag{6.2.2}$$

for $\beta > 0$ and a function r such that ξ is positive and C^2 in Ω with

$$r(x) \geq d^\beta(x) \text{ for } x \in \Omega \setminus \Omega_\delta. \tag{6.2.3}$$

Lemma 6.2.5. *Let $p \geq \frac{2}{2-s}$. There exist $\delta > 0$ and $C > 0$ such that for $\beta \in (0, \frac{sp}{p-1})$*

$$-(-\Delta)_p^s \xi(x) \leq -Cd(x)^{\beta(p-1)-sp} \quad in \quad \Omega_\delta.$$

Proof. Through out the proof we assume that $x \in \Omega_\delta$. For simplicity in notation, we write fractional p-Laplacian as

$$-(-\Delta)_p^s u(x) = \int_{\mathbb{R}^N} \frac{\varrho_+(u,x,y) + \varrho_-(u,x,y)}{|y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N. \tag{6.2.4}$$

where

$$\varrho_+(u,x,y) = |u(x+y) - u(x)|^{p-2}(u(x+y) - u(x))$$

and

$$\varrho_-(u,x,y) = |u(x-y) - u(x)|^{p-2}(u(x-y) - u(x)).$$

Since ξ is C^2 function, it is easy to show that there is a constant C_1 such that for $y \in B^c(0,\delta)$

$$|-(-\Delta)_p^s \xi(x)| \leq \int_{B^c(0,\delta)} \frac{|\varrho_+(\xi,x,y) + \varrho_-(\xi,x,y)|}{|y|^{N+sp}} \, dy \leq C_1. \tag{6.2.5}$$

Now we will estimate the integral over $B(0,\delta)$. We write

$$\begin{aligned} -(-\Delta)_p^s \xi(x) =& \int_{B(0,\delta)} \frac{\varrho_+(\xi,x,y)+\varrho_-(\xi,x,y)}{|y|^{N+sp}}\, dy \\ =& I_1(x)+I_{2_+}(x)+I_{2_-}(x)+I_3(x), \end{aligned} \tag{6.2.6}$$

where

$$\begin{aligned} I_1(x) =& \int_{D_1} \frac{-2|\xi(x)|^{p-2}\xi(x)}{|y|^{N+sp}}\, dy, \\ I_{2_+}(x) =& \int_{D_{2_+}} \frac{\varrho_+(\xi,x,y)-|\xi(x)|^{p-2}\xi(x)}{|y|^{N+sp}}\, dy, \\ I_{2_-}(x) =& \int_{D_{2_-}} \frac{\varrho_-(\xi,x,y)-|\xi(x)|^{p-2}\xi(x)}{|y|^{N+sp}}\, dy, \\ I_3(x) =& \int_{D_3} \frac{\varrho_+(\xi,x,y)+\varrho_-(\xi,x,y)}{|y|^{N+sp}}\, dy \end{aligned} \tag{6.2.7}$$

with the following domains of integration.

$$\begin{aligned} D_1 =& \{y \in B(0,\delta)/x+y \notin \Omega \quad \text{and} \quad x-y \notin \Omega\}, \\ D_{2_\pm} =& \{y \in B(0,\delta)/x \pm y \in \Omega \quad \text{and} \quad x \mp y \notin \Omega\}, \\ D_3 =& \{y \in B(0,\delta)/x+y \in \Omega \quad \text{and} \quad x-y \in \Omega\}. \end{aligned} \tag{6.2.8}$$

For notational simplicity we denote $d = d(x)$, whenever there is no confusion.

Estimation of $I_1(x)$: We observe that $0 \notin D_1$. With a change of variable $y = d(x)z$, we have

$$I_1(x) = -d^{\beta(p-1)-sp} \int_{d^{-1}D_1} \frac{1}{|z|^{N+sp}}\, dz$$

For some $R > 0$ we have $d^{-1}D_1 \subset B^c(0,R)$ and there exists $C_2 > 0$ such that

$$I_1(x) \leq -C_2 d^{\beta(p-1)-sp}. \tag{6.2.9}$$

Estimation of $I_{2_+}(x)$:

$$I_{2_+}(x) = \int_{D_{2_+}} \frac{\varrho_+(\xi,x,y) - |\xi(x)|^{p-2}\xi(x)}{|y|^{N+sp}}\,dy$$
$$= \int_{D_{2_+}} \frac{|d^\beta(x+y)-d^\beta(x)|^{p-2}(d^\beta(x+y)-d^\beta(x))}{|y|^{N+sp}}\,dy + \int_{D_{2_+}} \frac{-|d^\beta(x)|^{p-2}d^\beta(x)}{|y|^{N+sp}}\,dy. \tag{6.2.10}$$

The estimation of second integral in $I_{2_+}(x)$ is same as $I_1(x)$. Taking $d^{\beta(p-2)}d(x)^\beta$ outside the integral of first integral in (6.2.10), and using the fact that for $\lambda > 0$, $\lambda dist(x,\partial\Omega) = dist(\lambda x, \lambda\partial\Omega)$ and with a change of variable $y = d(x)z$ in first integral in (6.2.10), we have

$$d^{\beta(p-1)-sp}\int_{d^{-1}D_{2_+}} \frac{|(dist(d^{-1}x+z, d^{-1}\partial\Omega))^\beta - 1|^{p-2}((dist(d^{-1}x+z,d^{-1}\partial\Omega))^\beta - 1)}{|z|^{N+sp}}\,dz$$

We write

$$d^{-1}D_{2_+} = (d^{-1}D_{2_+}\cap B(0,R)) \cup (d^{-1}D_{2_+}\cap B(0,R)^c) := B_{1,R}\cup B_{2,R}.$$

We find that for $0 < \beta \le \beta_0 < \frac{sp}{p-1}$ we have

$$dist(d^{-1}x+z, d^{-1}\partial\Omega))^\beta - 1 \le |z|^{\beta_0}.$$

Therefore if R is large

$$\int_{B_{2,R}} \frac{|(dist(d^{-1}x+z, d^{-1}\partial\Omega))^\beta - 1|^{p-2}((dist(d^{-1}x+z,d^{-1}\partial\Omega))^\beta - 1)}{|z|^{N+sp}}\,dz$$
$$\le \int_{B_{2,R}} \frac{|z|^{\beta_0(p-1)}}{|z|^{N+sp}}\,dz$$
$$\le C_3$$

independent of β. To conclude, we see that for fixed R we have

$$\lim_{\beta\to 0}\int_{B_{1,R}} \frac{|(dist(d^{-1}x+z, d^{-1}\partial\Omega))^\beta - 1|^{p-2}((dist(d^{-1}x+z,d^{-1}\partial\Omega))^\beta - 1)}{|z|^{N+sp}}\,dz = 0,$$

so we conclude that for all β small, for some $C_4 > 0$ we have

$$I_{2_+}(x) \leq -C_4 d^{\beta(p-1)-sp}. \tag{6.2.11}$$

Similarly, we estimate $I_{2_-}(x)$.

Estimation of $I_3(x)$: We observe that $0 \in D_3$. For some $\epsilon > 0$ we write:

$$D_3 = B(0, \epsilon d) \cup (D_3 \setminus B(0, \epsilon d)) := B_1 \cup B_2.$$

We have

$$I_3(x) = \int_{D_3} \frac{\varrho_+(\xi, x, y) + \varrho_-(\xi, x, y)}{|y|^{N+sp}}\, dy.$$

Here we only estimate $\int_{D_3} \frac{\varrho_+(\xi, x, y)}{|y|^{N+sp}}\, dy$ in $I_3(x)$, as the estimation of $\int_{D_3} \frac{\varrho_-(\xi, x, y)}{|y|^{N+sp}}\, dy$ follows using the similar arguments. We also estimate $I_3(x)$ on B_1 only because on B_2 the estimation of $I_3(x)$ is same as $I_{2_+}(x)$. So we estimate

$$\left| \int_{B_1} \frac{|\xi(x+y) - \xi(x)|^{p-2}(\xi(x+y) - \xi(x))}{|y|^{N+sp}}\, dy \right|.$$

Since ξ is C^2 function, using Taylor's expansion, we have

$$\xi(x+y) = \xi(x) + \nabla\xi(x) \cdot y + y^\top \cdot D^2\xi(\alpha) \cdot y \quad \text{for some} \quad \alpha \in x + \theta y, \theta \in (0,1).$$

We denote

$$l(x+y) = \xi(x) + \nabla\xi(x) \cdot y,$$

so we have

$$\xi(x+y) - l(x+y) = y^\top \cdot D^2\xi(\alpha) \cdot y.$$

Similarly, we have

$$\xi(x) = \xi(x+y) + \nabla\xi(x+y) \cdot (-y) + (-y)^\top \cdot D^2\xi(\gamma) \cdot (-y) \quad \text{for some} \quad \gamma \in x+y-\theta y, \theta \in (0,1).$$

We denote

$$l(x) = \xi(x+y) + \nabla\xi(x+y)\cdot(-y),$$

so we have

$$\xi(x) - l(x) = (-y)^\top \cdot D^2\xi(\gamma)\cdot(-y)$$

and

$$l(x+y) - l(x) = \nabla\xi(x)\cdot y.$$

Let $g(t) := |t|^{p-2}t$ and using Lemma 6.2.4 and Lemma 2.5.2, we have

$$\begin{aligned}
&\left|\int_{B_1} \frac{|\xi(x+y)-\xi(x)|^{p-2}(\xi(x+y)-\xi(x))}{|y|^{N+sp}}\,dy\right| \\
&\le \int_{B_1} \frac{|g(\xi(x+y)-\xi(x)) - g(l(x+y)-l(x))|}{|y|^{N+sp}}\,dy \\
&\le C_5 \int_{B_1} \frac{(|l(x+y)-l(x)|+|\xi(x)-l(x)|)^{p-2}|\xi(x)-l(x)|}{|y|^{N+sp}}\,dy \\
&\le C_5 \int_{B_1} \frac{(|\nabla\xi(x)\cdot y|+|(-y)^\top\cdot D^2\xi(\gamma)\cdot(-y)|)^{p-2}|(-y)^\top\cdot D^2\xi(\gamma)\cdot(-y)|}{|y|^{N+sp}}\,dy.
\end{aligned}$$

Since $x \in \Omega_\delta$, $\xi(x) = d^\beta(x)$. Then

$$\begin{aligned}
&\nabla\xi(x) = \beta d^{\beta-1}\nabla d(x), \\
&\frac{\partial^2\xi(x)}{\partial x_i\partial x_j} = \beta(\beta-1)d^{\beta-2}A_{ij}(x) \quad \text{for} \quad 1\le i,j\le N,
\end{aligned}$$

where

$$A_{ij}(x) = \frac{\partial d(x)}{\partial x_i}\frac{\partial d(x)}{\partial x_j} + \frac{d(x)}{\beta-1}\frac{\partial^2 d(x)}{\partial x_i\partial x_j}.$$

We have $\gamma = x+y-\theta y$, $\theta\in(0,1)$ and $x\in\Omega_\delta, y\in B_1$ so A_{ij} is bounded. We denote

$$\sup_\gamma \beta(\beta-1)A_{ij}(\gamma) = M_{i,j},$$

so we have $|D^2\xi(\gamma)|\leq d^{\beta-2}M$ for some $M>0$. Hence we have our estimates as

$$\begin{aligned}
&\left|\int_{B_1}\frac{|\xi(x+y)-\xi(x)|^{p-2}(\xi(x+y)-\xi(x))}{|y|^{N+sp}}\,dy\right|\\
&\leq C_5\int_{B_1}\frac{(|\beta d^{\beta-1}\nabla d(x)\cdot y|+|d^{\beta-2}M|y|^2|)^{p-2}|d^{\beta-2}M|y|^2|}{|y|^{N+sp}}\,dy\\
&\leq C_5 d^{(\beta-2)(p-1)}\int_{B_1}\frac{(|\beta d\nabla d(x)\cdot y|+|M|y|^2|)^{p-2}|M|y|^2|}{|y|^{N+sp}}\,dy.
\end{aligned}$$

With a change of variable $y=d(x)z$, we have

$$\begin{aligned}
&\leq C_5 d^{(\beta-2)(p-1)}d^{2(p-2)}d^2 d^{-sp}\int_{B(0,\epsilon)}\frac{(|\beta\nabla d(x)\cdot z|+|M|z|^2|)^{p-2}|M|z|^2|}{|z|^{N+sp}}\,dz\\
&\leq C_5 d^{\beta(p-1)-sp}\int_{B(0,\epsilon)}\frac{(|\beta\nabla d(x)\cdot z|+|M|z|^2|)^{p-2}|M|z|^2|}{|z|^{N+sp}}\,dz\\
&\leq C_5 d^{\beta(p-1)-sp}\int_0^\epsilon\int_{S^N}\left(|\beta\nabla d(x)\cdot\omega|r+Mr^2\right)^{p-2}Mr^{1-sp}\,d\omega\,dr\\
&\leq C_5 d^{\beta(p-1)-sp}\int_0^\epsilon\int_{S^N}\left(\frac{|\beta\nabla d(x)\cdot\omega|}{|\beta\nabla d(x)|}+\frac{Mr}{|\beta\nabla d(x)|}\right)^{p-2}Mr^{p-1-sp}|\beta\nabla d(x)|^{p-2}\,d\omega\,dr\\
&\leq C_5 M d^{\beta(p-1)-sp}\int_0^\epsilon\int_{S^N}\left(1+\frac{Mr}{|\beta\nabla d(x)|}\right)^{p-2}r^{p-1-sp}|\beta\nabla d(x)|^{p-2}\,d\omega\,dr
\end{aligned}$$

If $p\geq 2$ using Lemma 6.2.3, we get

$$\begin{aligned}
&\left|\int_{B_1}\frac{|\xi(x+y)-\xi(x)|^{p-2}(\xi(x+y)-\xi(x))}{|y|^{N+sp}}\,dy\right|\\
&\leq C_5 M d^{\beta(p-1)-sp}\int_0^\epsilon\int_{S^N}\left(1+\frac{M^{p-2}r^{p-2}}{|\beta\nabla d(x)|^{p-2}}\right)r^{p-1-sp}|\beta\nabla d(x)|^{p-2}\,d\omega\,dr\\
&\leq C_6 M d^{\beta(p-1)-sp}\left(|\beta\nabla d(x)|^{p-2}\epsilon^{p-sp}+M^{p-2}\epsilon^{p-2+p(1-s)}\right).
\end{aligned}\tag{6.2.12}$$

If $\frac{2}{2-s}<p<2$ and using Lemma 6.2.3, we have

$$\begin{aligned}
&\left|\int_{B_1}\frac{|\xi(x+y)-\xi(x)|^{p-2}(\xi(x+y)-\xi(x))}{|y|^{N+sp}}\,dy\right|\\
&\leq C_6 M d^{\beta(p-1)-sp}\int_0^\epsilon\left(\frac{Mr}{|\beta\nabla d(x)|}\right)^{p-2}r^{p-1-sp}|\beta\nabla d(x)|^{p-2}\,dr\\
&\leq C_6 M^{p-1}d^{\beta(p-1)-sp}\epsilon^{p-2+p(1-s)}.
\end{aligned}\tag{6.2.13}$$

So, for $p \geq 2$, using (6.2.5), (6.2.9), (6.2.11), (6.2.12), we have

$$\begin{aligned}-(-\Delta)_p^s \xi(x) \leq & d(x)^{\beta(p-1)-sp}\left(C_1 d(x)^{sp-\beta(p-1)} - C_2 - C_4\right) \\ & + C_6 M d(x)^{\beta(p-1)-sp}\left(|\beta \nabla d(x)|^{p-2}\epsilon^{p-sp} + M^{p-2}\epsilon^{p-2+p(1-s)}\right).\end{aligned}$$

Thus, there exists $C > 0$ such that

$$-(-\Delta)_p^s \xi(x) \leq -C d(x)^{\beta(p-1)-sp} \quad \text{in} \quad \Omega_\delta.$$

Similarly, for $\frac{2}{2-s} < p < 2$, using (6.2.5), (6.2.9), (6.2.11), (6.2.13), we get the desired result. □

We recall the Holder estimates for viscosity solutions from Theorem 1 of [70].

Theorem 6.2.1. *For $1 < p < \infty$, assume $f \in C(B(0,2)) \cap L^\infty(B(0,2))$ and let $u \in L^\infty(\mathbb{R}^N)$ be viscosity solution of*

$$(-\Delta)_p^s u = f \qquad \text{in } B(0,2).$$

Then u is Holder continuous in $B(1,0)$ and in particular there exist $\alpha \in (0,1)$ and c depending on s,p such that

$$\|u\|_{C^\alpha(B(0,1))} \leq c\left(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B(0,2))}^{\frac{1}{p-1}}\right). \tag{6.2.14}$$

We also recall the following comparison principle for the viscosity solutions from Theorem 4.1 of [68].

Theorem 6.2.2. *Let u and v be viscosity supersolution and viscosity subsolution, respectively, in Ω. Assume further that both v and $-u$ are upper semicontinuous in Ω and $u \geq v$ on $\partial\Omega$ and almost everywhere in Ω^c. Then $u \geq v$ in Ω.*

We define for $x_0 \in \partial\Omega,\ \tau > 0$

$$\Omega_{x_0}^\tau := \left\{x \in \mathbb{R}^N : x_0 + \tau x \in \Omega\right\}.$$

Also for the function ξ defined in (6.2.2), we set $\xi^\tau_{x_0}(x) := \xi(x_0 + \tau x)$ and define

$$d_\tau(x) := dist(x, \partial\Omega^\tau_{x_0}) = \tau^{-1} dist(x_0 + \tau x, \partial\Omega). \tag{6.2.15}$$

Lemma 6.2.6. *Assume Ω is a C^2 bounded domain, $s \in (0,1)$ and $2 \le p < \infty$. Also let $\beta = sp - \theta$ in (6.2.2) for $\theta \in \left(\frac{sp(p-2)}{p-1}, sp\right)$. Then there exist $c_0, \delta > 0$ such that*

$$(-\Delta)^s_p \xi^\tau_{x_0} \ge c_0 d_\tau^{sp^2 - 2sp - \theta(p-1)} \quad in \ (\Omega^\tau_{x_0})_\delta, \ 0 < \tau < 1. \tag{6.2.16}$$

Moreover, if u satisfies $(-\Delta)^s_p u \le c_1 d_\tau^{sp^2 - 2sp - \theta(p-1)}$ in $\Omega^\tau_{x_0}$ for some $c_1 > 0$ with $u = 0$ in $\mathbb{R}^N \setminus \Omega^\tau_{x_0}$, then

$$u(x) \le c_2 \left(c_1 + \|u\|_{L^\infty(\Omega^\tau_{x_0})}\right) d_\tau^{sp-\theta}, \ x \in (\Omega^\tau_{x_0})_\delta \tag{6.2.17}$$

for some $c_2 > 0$ is only depending on s, δ, θ, p and c_0.

Proof. First, from (6.2.15), we observe that

$$\begin{aligned} x \in (\Omega^\tau_{x_0})_\delta &\Leftrightarrow \{x \in \mathbb{R}^N : x_0 + \tau x \in \Omega, \ d_\tau(x) < \delta\} \\ &\Leftrightarrow \{x \in \mathbb{R}^N : x_0 + \tau x \in \Omega, \ dist(x, \partial\Omega^\tau_{x_0}) < \delta\} \\ &\Leftrightarrow \{x \in \mathbb{R}^N : x_0 + \tau x \in \Omega, \ \tau^{-1} dist(x_0 + \tau x, \partial\Omega) < \delta\} \\ &\Leftrightarrow \{x \in \mathbb{R}^N : x_0 + \tau x \in \Omega, \ dist(x_0 + \tau x, \partial\Omega) < \tau\delta\} \\ &\Leftrightarrow x_0 + \tau x \in (\Omega)_{\tau\delta} \end{aligned} \tag{6.2.18}$$

Now as $\tau < 1$ and $(\Omega)_{\tau\delta} \subset (\Omega)_\delta$, hence for $x \in (\Omega^\tau_{x_0})_\delta$ we have $x_0 + \tau x \in (\Omega)_{\tau\delta} \subset (\Omega)_\delta$. The rest of the arguments for establishing (6.2.16) follows using the similar calculations as in the proof of the Lemma 6.2.5 by taking $\beta = sp - \theta$ and translating $\xi(x)$ to $\xi^\tau_{x_0}(x)$ and $(\Omega)_\delta$ to $(\Omega^\tau_{x_0})_\delta$.

Next we prove (6.2.17). For this, let u be a viscosity solution of

$$\begin{cases} (-\Delta)^s_p u & \le c_1 d_\tau^{sp^2 - 2sp - \theta(p-1)} & \text{in } \Omega^\tau_{x_0}, \\ \qquad u & = 0 & \text{on } (\Omega^\tau_{x_0})^c. \end{cases} \tag{6.2.19}$$

We choose $R > 0$ and let $v = R\xi^{\tau}_{x_0}$, then we have

$$(-\Delta)^s_p v \geq c_o R^{p-1} d_\tau^{sp^2-2sp-\theta(p-1)} \geq c_1 d_\tau^{sp^2-2sp-\theta(p-1)} \geq (-\Delta)^s_p u \quad \text{in } (\Omega^\tau_{x_0})_\delta$$

if we take $R > \left(\frac{c_1}{c_0}\right)^{1/(p-1)}$. We also have $u = v = 0$ in $\mathbb{R}^N \setminus \Omega^\tau_{x_0}$ and $v \geq u$ in $\Omega^\tau_{x_0} \setminus (\Omega^\tau_{x_0})_\delta$ if R is chosen so that $R \geq \|u\|_{L^\infty(\Omega^\tau_{x_0})}(\tau\delta)^{\theta-sp}$. Then by comparison principle $u \leq v$ in $(\Omega^\tau_{x_0})_\delta$. We observe that if we take

$$R = \left(\frac{c_1}{c_0}\right)^{1/(p-1)} + (\tau\delta)^{\theta-sp}\|u\|_{L^\infty(\Omega^\tau_{x_0})}$$

then

$$R > \left(\frac{c_1}{c_0}\right)^{1/(p-1)} \quad \text{and} \quad R > (\tau\delta)^{\theta-sp}\|u\|_{L^\infty(\Omega^\tau_{x_0})}$$

so we have proved

$$u(x) \leq c_2\left((c_1)^{1/(p-1)} + \|u\|_{L^\infty(\Omega^\tau_{x_0})}\right) d_\tau^{sp-\theta} \quad \text{in } (\Omega^\tau_{x_0})_\delta$$

where $c_2 = \left((\tau\delta)^{\theta-sp} + (c_0)^{-1/(p-1)}\right)$. □

6.2.3 Regularity results

In this section we give the regularity of the solutions of the semipositone subcritical exponent problem. Here we consider the following nonexistence result as considered in [22].

$(\mathcal{NA})$: Let $\mathcal{H}$ be a half-space in $\mathbb{R}^N$ or whole of $\mathbb{R}^N$ and $\mu > 0$. Then there does not exist any nontrivial $u \in C^\alpha(\mathcal{H})$ which solves the following semipositone critical exponent problem.

$$\begin{cases} (-\Delta)^s_p u &= \mu(|u|^{p^*_s-2}u - 1) \text{ in } \mathcal{H}, \\ \qquad u &= 0 \qquad\qquad \text{on } \mathbb{R}^N \setminus \mathcal{H}. \end{cases} \tag{6.2.20}$$

Theorem 6.2.3. *Assume Ω is a C^2 bounded domain, $0 < s < 1$, $2 \leq p \leq \infty$, $p-1 < q < \frac{N(p-1)+sp}{N-sp}$, and $g \in c(\overline{\Omega} \times \mathbb{R})$ satisfies $|g(x,z)| \leq c|z|^r$, $x \in \Omega$, $z \in \mathbb{R}$ where $c > 0$ and*

$0 < r < q$. *Let u is positive viscosity solution of the problem*

$$\begin{cases} (-\Delta)_p^s u(x) &= u^q + g(x,u) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \Omega^c. \end{cases} \tag{6.2.21}$$

Then there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Proof. Assume this is not true so there exists a sequence of positive solutions $\{u_k\}$ of problem (6.2.21) such that $M_k = \|u_k\|_{L^\infty(\Omega)} \to \infty$. Let $x_k \in \Omega$ be points with $u_k(x_k) = M_k$ and introduce the functions

$$v_k(y) = \frac{u_k(x_k + \mu_k y)}{M_k}, \; y \in \Omega^k,$$

where

$$\Omega^k := \left\{ y \in \mathbb{R}^N : x_k + \mu_k y \in \Omega \right\}, \; \mu_k \in \mathbb{R},$$

and functions v_k satisfies $0 < v_k \leq 1$ and $v_k(0) = 1$. Next we have

$$\frac{M_k^{p-1}}{M_k^q} \frac{1}{\mu_k^{sp}} (-\Delta)_p^s v_k(y) = v_k^q + h_k,$$

where $|h_k| \leq CM_k^{r-q}$ and $h_k \in C(\Omega^k)$. We choose $\mu_k = M_k^{\frac{-(q-(p-1))}{sp}}$ which tends to zero. Then

$$(-\Delta)_p^s v_k(y) = v_k^q + h_k \quad \text{in} \quad \Omega^k. \tag{6.2.22}$$

Next passing to subsequences, there are two cases, either $d(x_k)\mu_k^{-1} \to \infty$ or $d(x_k)\mu_k^{-1} \to d \geq 0$.

Suppose that the first case holds, then $\Omega_k \to \mathbb{R}^N$ as $k \to \infty$. As the right hand side in (6.2.22) is uniformly bounded and using estimates in (6.2.14) with an application of Arzela-Ascoli's theorem and a diagonal argument to get $v_k \to v$ locally uniformly in $\mathbb{R}^N$. On passing the limit in (6.2.22), thanks to Corollary 4.7 in [30], we obtain that v solves

$(-\Delta)_p^s v = v^q$ in viscosity sense. By using Theorem 6.2.1 we get $v \in C^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Since $v(0) = 1$, the strong maximum principle implies $v > 0$. Thus v solves strongly $(-\Delta)_p^s v = v^q$ in $\mathbb{R}^N$. However due to non-existence assumption $(\mathcal{NA})$ for $p-1 < q < \frac{N(p-1)+sp}{N-sp}$, we get a contradiction.

If the second case holds then we assume that $x_k \to x_0 \in \partial\Omega$. Without loss of generality, we assume $\nu(x_0) = -e^N$. Define

$$w_k(y) = \frac{u_k(\zeta_k + \mu_k y)}{M_k} \quad y \in D^k,$$

where $\zeta_k \in \partial\Omega$ is the projection of x_k on $\partial\Omega$ and

$$D^k := \{y \in \mathbb{R}^N : \zeta_k + \mu_k y \in \Omega\}. \tag{6.2.23}$$

Observe that

$$0 \in \partial D^k, \tag{6.2.24}$$

and

$$D^k \to \mathbb{R}^N_+ = \{y \in \mathbb{R}^N : y_N > 0\} \text{ as } k \to +\infty.$$

It is easy to show that w_k satisfies (6.2.22) in D^k with a different function h_k, but with the same bounds. Setting

$$y_k := \frac{x_k - \zeta_k}{\mu_k},$$

so that $|y_k| = d(x_k)\mu_k^{-1}$, we observe that $w_k(y_k) = 1$. Next by passing to further subsequence $y_k \to y_0$, with $|y_0| = d > 0$, we claim that y_0 is in the interior of half-space $\mathbb{R}^N_+$. For this it is sufficient to show that

$$d = \lim_{k\to\infty} d(x_k)\mu_k^{-1} > 0. \tag{6.2.25}$$

We observe that by (6.2.22), and since $r < q$, we get

$$(-\Delta)_p^s w_k \le C \le C_1 d_k^{sp^2-2sp-\theta(p-1)} \text{ in } D^k,$$

for every $\theta \in \left(\frac{sp(p-2)}{p-1}, sp\right)$, where $d_k(y) = dist(y, \partial D^k)$. By Lemma 6.2.6, for a fixed θ there exists constant $C_0 > 0$ and $\delta > 0$ such that $w_k(y) \leq C_0 d_k(y)^{sp-\theta}$ if $d_k(y) < \delta$. Clearly, since by (6.2.24) $|y_k| > d_k(y_k)$, if $d_k(y) < \delta$, then $1 \leq C_0 d_k(y)^{sp-\theta} \leq C_0 |y_k|^{sp-\theta}$, which asserts that $|y_k|$ is bounded from below so that we have (6.2.25).

Now again using (6.2.14) as in the first case to obtain $w_k \to w$ uniformly on compact set of $\mathbb{R}^N_+$, with $0 \leq w \leq 1$, $w(y_0) = 1$ and $w(y) \leq C y_N^{sp-\theta}$ for $y_N < \delta$. Thus $w \in C(\mathbb{R}^N)$ is non-negative, bounded solution of

$$\begin{cases} (-\Delta)_p^s w &= w^q \quad \text{in } \mathbb{R}^N_+, \\ \qquad w &= 0 \quad\ \ \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+. \end{cases}$$

Again using Theorem (6.2.1) we get $w \in C^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Since $w(0) = 1$, the strong maximum principle implies $w > 0$. Again this is contradiction due to non-existence assumption $(\mathcal{NA})$. So this proves the theorem. □

6.3 Proof of main result

Using the substitution $w = \gamma u$ where $\gamma^{r+1-p} = \mu$, we see that (6.1.1) is equivalent to the nonlocal problem

$$\begin{cases} (-\Delta)_p^s w &= w^r - \gamma^r \quad \text{in } \Omega, \\ \qquad w &> 0 \qquad\qquad \text{in } \Omega, \\ \qquad w &= 0 \qquad\qquad \text{on } \Omega^c. \end{cases} \tag{6.3.1}$$

We use the above observation to study the equivalent problem.

Definition 6.3.1. *We define $K : C(\overline{\Omega}) \to C(\overline{\Omega}) \cap W_0^{s,p}(\Omega)$ as $K(f) = u$ where u is the unique weak solution of $(-\Delta)_p^s u = f$ in Ω and $u = 0$ on Ω^c.*

The weak solution u can be obtained as the minimizer of the associated functional in $W_0^{s,p}(\Omega)$. Using Theorem 2.5.1, $u \in C^\alpha(\overline{\Omega})$ for $\alpha \in (0, s]$ and thus the map K is well defined. Also, from Theorem 2.2.2, we infer that the weak solution u is in fact a viscosity solution. Set

$$F(\mu,u)=\mu(|u|^r-1) \tag{6.3.2}$$

Finding a weak solution of the nonlinear problem

$$\begin{cases} (-\Delta)_p^s u &= F(\mu,u) \ \text{ in } \Omega,\\ u &> 0 \qquad\quad \text{ in } \Omega,\\ u &= 0 \qquad\quad \text{ on } \Omega^c. \end{cases} \tag{6.3.3}$$

is equivalent to finding a fixed point of the map $KF(\mu,u)$. By the regularity results for subcritical problem (see, [65]), the solution of (6.3.3) is in $L^\infty(\Omega)$. Now using the regularity result as before u is a continuous viscosity solution of (6.3.3) i.e. $u=KF(\mu,u)$. Using the rescaling argument we see that $u=KF(\mu,u)$ iff $w=K\tilde{F}(\gamma,w)$ where w satisfies

$$\begin{cases} (-\Delta)_p^s w &= |w|^r-\gamma^r \ \text{ in } \Omega,\\ w &= 0 \qquad\qquad \text{ on } \Omega^c, \end{cases} \tag{6.3.4}$$

and $\tilde{F}(\gamma,w)=|w|^r-\gamma^r$. We shall denote

$$S(\gamma,w)=w-K\tilde{F}(\gamma,w) \text{ for } 0\le\gamma<\infty. \tag{6.3.5}$$

For $\gamma=0$, the map $S(0,w)$ is denoted as $S_0(w)$. The solutions of $S_0(w)=0$ are nothing but the solutions of

$$\begin{cases} (-\Delta)_p^s w &= |w|^r \ \text{ in } \Omega,\\ w &= 0 \quad\ \ \text{ on } \Omega^c. \end{cases} \tag{6.3.6}$$

To prove Theorem 6.1.1, we determine the degree of S_0 as follows.

Lemma 6.3.1. *There exists* $0<R_1<R_2$ *such that* $S_0(w)\neq 0$ *for all* $\|w\|_{L^\infty(\Omega)}\in\{0,R_1\}$ *and* $deg(S_0,B_{R_2}-\overline{B}_{R_1},0)=-1$.

To prove this lemma we use Proposition 2.1 and Remark 2.1 of [52]. We state the result here for sake of completeness.

Proposition 6.3.1. *Let C be a cone in a Banach space X and $\Phi : C \to C$ be a compact map such that $\Phi(0) = 0$. Assume there exists $0 < R_1 < R_2$ such that*

(i) $x \neq t\Phi(x)$ *for all* $t \in [0,1]$ *and* $\|x\|_X = R_1$.

(ii) *There exists a map* $T : \overline{B}_{R_2} \times [0,\infty) \to C$ *such that* $T(x,0) = \Phi(x)$ *for all* $\|x\|_X = R_2, T(x,t) \neq x$ *for* $\|x\|_X = R_2$ *and* $0 \leq t < \infty$ *and* $T(x,t) = x$ *has no solution for* $x \in \overline{B}_{R_2}$, $t \geq t_0$.

Then if $U = \{x \in C : R_1 \leq \|x\|_X < R_2\}$ *and* $B_\rho = \{x \in C : \|x\|_X < \rho\}$, *one has* $deg(I - \Phi, B_{R_2}, 0) = 0, deg(I - \Phi, B_{R_1}, 0) = 1$ *and* $deg(I - \Phi, U, 0) = -1$.

Next we have the following lemma which follows using the strict convexity of the domain and the monotonicity of the solutions given in Theorem 1.1 in [49].

Lemma 6.3.2. *Let Ω be a strictly convex bounded smooth domain, and define $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) > \delta\}$ for $\delta > 0$. Then the following holds for a weak solution $u \in C_d^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$ of the problem* (6.3.6)

$$\begin{aligned} &\text{there exist } \gamma, \epsilon > 0, \text{ depending only on } \Omega, \text{ such that} \\ &\forall\ x \in \Omega \setminus \Omega_\epsilon \text{ there is a part of a cone } I_x \text{ with} \\ &\quad (i) u(\xi) \geq u(x)\ \forall\ \xi \in I_x, \\ &\quad (ii) I_x \subset \Omega_{\frac{\epsilon}{2}}, \\ &\quad (iii) meas(I_x) \geq \gamma. \end{aligned} \tag{6.3.7}$$

Proof of Lemma 6.3.1. Let $X = C(\overline{\Omega})$, $C = \{u \in X : u(x) \geq 0\}$ and $\Phi : K\tilde{F}(0,\cdot) : C(\overline{\Omega}) \to C(\overline{\Omega})$. By the Theorem 2.5.1 and compactness of $C(\overline{\Omega}) \to C(\overline{\Omega})$, we prove that the map Φ is compact. We shall prove first (i). We shall prove that if $t\Phi(u) = u$ for some $t \in [0,1]$, then necessarily $\|u\|_{L^\infty(\Omega)} \geq c(s,p,r)$. Let $t\Phi(u) = u$, i.e

$$\begin{cases} (-\Delta)_p^s u &= t^{p-1}|u|^r \quad \text{in } \Omega, \\ \qquad u &= 0 \qquad\qquad \text{on } \Omega^c. \end{cases} \tag{6.3.8}$$

Using the variational characterization of the principal eigenvalue and the L^∞ regularity of the weak solution of (6.3.8)

$$\lambda_{1,s}^p \int_\Omega |u|^p \leq \|u\|_{W_0^{s,p}(\Omega)} = t^{p-1} \int_\Omega |u|^{r+1} \leq \|u\|_{L^\infty(\Omega)}^{r+1-p} \int_\Omega |u|^p.$$

Now by choosing R_1 small enough we have (i) of Proposition 6.3.1. Now we shall verify (ii). Define $T(u,t) = K\tilde{F}(0, (|u|+t)^r)$. Then $T(u,0) = \Phi(u)$ and we need to verify two more conditions, viz.,

(a) $T(u,t) \neq u$ for all $\|u\|_{L^\infty(\Omega)} = R_2$ and $0 \leq t < \infty$.

(b) $T(u,t) = u$ has no solution for $u \in \overline{B}_{R_2}$ and $t \geq t_0$.

We will verify condition (b) first. We claim a stronger result:

Claim: $T(u,t) = u$ has no solution of $t \geq t_0$. Suppose that for any arbitrary t there exists a solution $u_t \in C(\overline{\Omega})$ of $T(u,t) = u_t$. Taking $\frac{\varphi_1^p}{u_t^{p-1}}$ as the test function and using Proposition 2.5.1, we have

$$\begin{aligned}\int_\Omega \frac{(u_t+t)^r \varphi_1^p}{u_t^{p-1}} &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u_t(x)-u_t(y))}{|x-y|^{N+sp}} \left(\frac{\phi_1(x)^p}{u_t(x)^{p-1}} - \frac{\varphi_1(y)^p}{u_t(y)^{p-1}} \right) \\ &\leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi_1(x)-\varphi_1(y)|^p}{|x-y|^{N+sp}} \\ &= \lambda_1 \int_\Omega |\varphi_1|^p. \end{aligned} \tag{6.3.9}$$

Now, by Lemma 6.3.2, for any $x \in \Omega \setminus \Omega_\epsilon$ we have that

$$\begin{aligned}\gamma(\inf \Omega_{\frac{\epsilon}{2}} \varphi_1^p) u_t(x)^{r-p+1} &\leq \int_{I_x} u_t^{r-p+1}(\xi)\varphi_1(\xi) \\ &\leq \int_\Omega u_t^{r-p+1}(x)\varphi_1(x)dx \\ &\leq \int_\Omega \frac{(u_t+t)^r \varphi_1^p}{u_t^{p-1}} \\ &= \lambda_1 \int_\Omega |\varphi_1|^p. \end{aligned}$$

Thus we get that $\|u_t\|_{L^\infty(\Omega \setminus \Omega_\epsilon)} \leq C$ for all t and $u_t > 0$ in Ω.

Now again using (6.3.9) we have the following estimate

$$t\int_{\Omega\setminus\Omega_\epsilon}\frac{\varphi_1^p}{C^{p-1}}\leq t\int_{\Omega\setminus\Omega_\epsilon}\frac{\varphi_1^p}{u_t^{p-1}}\leq t\int_{\Omega}\frac{\varphi_1^p}{u_t^{p-1}}=\lambda_1\int_{\Omega}|\varphi_1|^p. \tag{6.3.10}$$

Thus, from (6.3.10), we infer that $T(u,t)=u$ has no solution for $t\geq t_0$.

Next we verify claim (a). We proceed as in [62]. We show that if u solves $T(u,t)=u$ for $t\in[0,\infty)$ then $\|u\|_{L^\infty(\Omega)}\leq M$ (independent of t).

Suppose $\|u\|_{L^\infty(\Omega)}$ is not uniformly bounded, i.e., there exists $t_k\in[0,\infty)$ such that for corresponding solutions u_k, $\|u_k\|_{L^\infty(\Omega)}\to\infty$. Denote $M_k=\|u_k\|_{L^\infty(\Omega)}\to\infty$ and let $x_k\in\Omega$ be points with $M_k=u_k(x_k)$. First we show that, up to a subsequence,

$$\frac{t_k}{\|u_k\|_{L^\infty(\Omega)}}\to 0 \text{ as } k\to\infty.$$

Indeed, without loss of generality, we assume that $t_k>0$ for all k and $t_k\to\infty$. Define $w_k:=\frac{u_k}{t_k}$ and $\lambda_k:=t_k^{r-p+1}$. Then, it is easy to check that (w_k,λ_k) satisfies

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|w_k(x)-w_k(y)|^{p-2}(w_k(x)-w_k(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}=\lambda_k\int_\Omega(w_k+t)^r\phi.$$

Then from the comparison principle, we have $w_k\geq\bar{w}_k$ where $\bar{w}_k\in C^\alpha$ for $\alpha\in(0,s]$ and satisfies

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\bar{w}_k(x)-\bar{w}_k(y)|^{p-2}(\bar{w}_k(x)-\bar{w}_k(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}=\lambda_k\int_\Omega\phi, \tag{6.3.11}$$

for all $\phi\in C_0^\infty(\Omega)$. If $\sup_k\|\bar{w}_k\|_{L^\infty(\Omega)}=C$, then taking $\phi=\bar{w}_k$ in (6.3.11), we get

$$\|w_k\|^p_{W_0^{s,p}(\Omega)}\leq C\lambda_k|\Omega|. \tag{6.3.12}$$

Now for a nontrivial $\phi \geq 0$ in (6.3.11), using Holder's inequality we get

$$\begin{aligned}
0 < \int_\Omega \phi &\leq \frac{1}{\lambda_k}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\bar{w}_k(x)-\bar{w}_k(y)|^{p-2}(\bar{w}_k(x)-\bar{w}_k(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}} \\
&\leq \frac{1}{\lambda_k}\|w_k\|^{p-1}_{W_0^{s,p}(\Omega)}\|\phi\|_{W_0^{s,p}(\Omega)} \\
&\leq \lambda_k^{\frac{-1}{p}} C_1 \to 0, \qquad (6.3.13)
\end{aligned}$$

which is a contradiction. Therefore we have

$$\sup_k \frac{\|u_k\|_{L^\infty(\Omega)}}{t_k} = \sup_k \|w_k\|_{L^\infty(\Omega)} \geq \sup_k \|\bar{w}_k\|_{L^\infty(\Omega)} = \infty.$$

Now we introduce the Gidas-Spruck translated function

$$v_k(y) = \frac{u_k(x_k+\mu_k y)}{M_k}, \quad y \in \Omega^k$$

where

$$\Omega^k = \left\{y \in \mathbb{R}^N : x_k + \mu_k y \in \Omega\right\} \text{ and } \mu_k = M_k^{-\frac{r-(p-1)}{sp}}.$$

A straight forward calculation yields

$$(-\Delta)^s v_k(x) = \left(v_k(x) + \frac{t_k}{M_k}\right)^r, \qquad (6.3.14)$$

and since $t_k \in [0, t_0]$, $\frac{t_k}{M_k^p} \to 0$ as $k \to \infty$.

Then using the arguments as in the proof of Theorem 6.2.3, if we pass through the limit we get a contradiction to the non-existence results for a sub-critical problem in $\mathbb{R}^N$ or upper half plane. Hence $\|u\|_{L^\infty(\Omega)}$ is uniformly bounded.

Now if we choose $R_2 > \max\{R_1, M\}$ then we have verified the assumptions of the Proposition 6.3.1. So we have $\deg(I - K\tilde{F}(0,\cdot), B_{R_2} - B_{R_1}, 0) = -1$, or $\deg(S(0,w), B_{R_2} - \overline{B}_{R_1}, 0) = -1$. That is $deg(S_0, B_{R_2} - \overline{B}_{R_1}, 0) = -1$. □

Next we proceed as in [37]. We determine the degree of $S(\gamma, .)$ by connecting $S(\gamma, .)$ and $S(0, .)$ using the homotopy invariance of degree with respect to γ. In particular, this will imply that $S(\gamma, w)$ has a solution w satisfying $R_1 < \|w\|_{L^\infty(\Omega)} < R_2$. Then we show that solution of the rescaled problem (6.3.4) is positive in Ω for some small γ.

Theorem 6.3.1. *For $p \geq 2$, the problem* (6.3.4) *admits a positive solution for $\gamma \in [0, \gamma_0]$. Hence, problem* (6.1.1) *admits a positive solution for all $\mu \in (0, \mu_0)$.*

Proof. We prove the theorem in two steps:
STEP 1: There exists a $\gamma_0 > 0$ such that $deg(S(\gamma, \cdot), B_{R_2} - \overline{B}_{R_1}, 0) = -1$ for all $\gamma \in [0, \gamma_0]$.
STEP II: $S(\gamma, w) = 0$ for some w in $B_{R_2} - \overline{B}_{R_1}$ implies $w > 0$.
Step I follows if we can show that $S(\gamma, w) \neq 0$ for $\|w\| \in \{R_1, R_2\}$, then using the standard properties of degree we can show that $deg(S(\gamma, \cdot), B_{R_2} - \overline{B}_{R_1}, 0) = -1$. Suppose there exists a sequence (γ_n, w_n) such that $\gamma_n \to 0$ and $\|w_n\| \in \{R_1, R_2\}$. Since $K\tilde{F}(\gamma, \cdot) : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact, we can find a function $w_0 \in C(\overline{\Omega})$ with $\|w_0\| \in \{R_1, R_2\}$ and $S(0, w_0) = 0$. This contradicts the previous lemma and hence Step I is proved. Now we prove the step II. If w_0 is a non-zero solution of $S(w_0, 0) = 0$, then $w_0 > 0$ and $\inf_{x \in \Omega} \frac{w_0(x)}{d^s(x)} > 0$. Let $C_+ = \{u \in C^0_{d^s}(\Omega) : u(x) \geq 0\}$. Then w_0 is in the interior of C_+. Let $S(\gamma_n, w_n) = 0$ and $\gamma_n \to 0$. Since $\|w_n\|_{C(\overline{\Omega})}$ is bounded, using Theorem 2.5.1 and Theorem 2.5.2, we have

$$\|w_n\|_{C^\alpha(\overline{\Omega})} \leq C \text{ and } \left\| \frac{w_n}{d^s(x)} \right\|_{C^\alpha(\overline{\Omega})} \leq C.$$

Thus, by Ascoli Arzela theorem, upto a subsequence

$$w_n \to w_0 \text{ and } \frac{w_n}{d^s(x)} \to \frac{w_0}{d^s(x)} \text{ in } C^0(\overline{\Omega}).$$

Lastly, the positivity of w_0 in $\overline{\Omega}$ follows using the above by uniform convergence and the fact that $\frac{w_n(x)}{d^s(x)} > 0$ in $\overline{\Omega}$ for large n. Hence we conclude that there exists a positive solution for γ small enough and thus for (P_μ) for $\mu < \mu_0$.

□

6.4 Conclusion

In this chapter, we have studied the existence of a positive solution to the nonlocal superlinear semipositone problem with the subcritical growth term. The salient feature of this

problem is to construct a barrier function in the neighbourhood of the boundary, which is used to get L^∞ a priori bound for the viscosity solutions. Then we applied the degree theory results to obtain the positive solution to the problem (P^μ). Here the restriction on p, viz., $p \geq 2$ in Theorem 6.1.1 and Theorem 6.3.1 is due to non availability of the fine boundary regularity results involving fractional p-Laplace operator for the case $1 < p < 2$. This embarks to consider the case $\frac{2}{2-s} < p < 2$ for future exploration.
Next considering the problems involving fractional (p, q)-Laplace operators open news questions that to be explored.

7

Nonlocal superlinear semipositone problem with critical growth

In this chapter, we study the existence of a positive solution to p-superlinear semipositone problem with critical growth involving fractional p-Laplace operator.

7.1 Main results

We consider the following nonlocal semipositone problem

$$\left.\begin{aligned} (-\Delta)_p^s u &= u^{p_s^*-1} - \mu \text{ in } \Omega, \\ u &> 0 \quad\quad\quad\;\; \text{in } \Omega, \\ u &= 0 \quad\quad\quad\;\; \text{on } \Omega^c, \end{aligned}\right\} \tag{7.1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p_s^* := \frac{Np}{N-sp}$ is fractional critical Sobolev exponent, $\mu > 0$ is a positive parameter.

It is easy to show, using the Pohozaev identity (see [75]), that the problem (7.1.1) has no solution for any $\mu > 0$ in a star-shaped domain. This motivates us to study the Brezis-Nirenberg type critical semipositone fractional p-Laplacian problem

$$\left.\begin{aligned} (-\Delta)_p^s u &= \lambda u^{p-1} + u^{p_s^*-1} - \mu \text{ in } \Omega, \\ u &> 0 \quad\quad\quad\quad\quad\quad\quad\;\; \text{in } \Omega, \\ u &= 0 \quad\quad\quad\quad\quad\quad\quad\;\; \text{on } \Omega^c, \end{aligned}\right\} \tag{7.1.2}$$

where $\lambda, \mu > 0$ are parameters. For a given $\lambda > 0$, The solution of the problem (7.1.2) are the critical points of the energy functional $E_\mu : D_0^{s,p}(\Omega) \to \mathbb{R}$ defined by

$$E_\mu(u) = \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{\lambda}{p}\int_\Omega u^p\,dx - \frac{1}{p_s^*}\int_\Omega u^{p_s^*}\,dx + \int_\Omega \mu u\,dx.$$

All the weak solutions of problem (7.1.2) lie on the set

$$\mathcal{N}_\mu = \left\{ u \in D_0^{s,p}(\Omega) : u > 0 \text{ in } \Omega \text{ and } \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega (\lambda u^p + u^{p_s^*} - \mu u)\,dx \right\}.$$

A weak solution that minimizes E_μ on $\mathcal{N}_\mu$ is a ground state solution for (7.1.2). Let λ_1 denotes first Dirichlet eigenvalue of the fractional p Laplacian, which is defined as

$$\lambda_1 = \inf_{u\in D_0^{s,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy}{\int_\Omega |u(x)|^p\,dx}. \tag{7.1.3}$$

Now we state our main results.

Theorem 7.1.1. *For $p \geq 2$, if $N \geq sp^2$ and $\lambda \in (0, \lambda_1)$, then there exists $\mu^* > 0$ such that $\mu \in (0, \mu^*)$, problem (7.1.2) has a ground state solution $u_\mu \in C_d^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$.*

We observe that the scaling $u \mapsto \mu^{\frac{-1}{p_s^*-p}} u$ transforms the first equation in the following critical semipositone nonlocal problem

$$\left.\begin{aligned} (-\Delta)_p^s u &= \lambda u^{p-1} + \mu(u^{p_s^*-1} - 1) \text{ in } \Omega, \\ u &> 0 \qquad\qquad\qquad\qquad \text{ in } \Omega, \\ u &= 0 \qquad\qquad\qquad\qquad \text{ on } \Omega^c, \end{aligned}\right\} \tag{7.1.4}$$

into

$$(-\Delta)_p^s u = \lambda u^{p-1} + u^{p_s^*-1} - \mu^{\frac{p_s^*-1}{p_s^*-p}}.$$

Theorem 7.1.2. *For $p \geq 2$, if $N \geq sp^2$ and $\lambda \in (0, \lambda_1)$, then there exists $\mu^* > 0$ such that $\mu \in (0, \mu^*)$, problem (7.1.4) has a ground state solution $u_\mu \in C_d^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$.*

7.2 Proof of the main results

In this section, we give the proof of our main results. We proceed as in [85]. We consider the modified problem

$$\left.\begin{aligned} (-\Delta)_p^s u &= \lambda u_+^{p-1} + u_+^{p_s^*-1} - \mu f(u) \text{ in } \Omega, \\ u &= 0 \qquad\qquad\qquad\qquad\qquad \text{ on } \Omega^c, \end{aligned}\right\} \tag{7.2.1}$$

where $u_+(x) = \max\{u(x), 0\}$ and

$$f(t) = \begin{cases} 1 & t \geq 0, \\ 1 - |t|^{p-1} & -1 < t < 0, \\ 0 & t \leq -1. \end{cases}$$

Weak solution of this problem coincide with the critical point of the C^1-functional

$$E_\mu(u) = \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega\left(-\frac{\lambda u_+^p}{p} - \frac{u_+^{p_s^*}}{p_s^*}\right)dx + \mu\int_{\{u\geq 0\}} u\,dx$$
$$+\mu\left[\int_{\{-1<u<0\}}\left(u - \frac{u|u|^{p-1}}{p}\right)dx - \left(1-\frac{1}{p}\right)|\{u\leq -1\}|\right],$$

where $|.|$ denotes the Lebesgue measure in $\mathbb{R}^N$.

We also recall the definition of $S_{s,p}$.

$$S_{s,p} = \inf_{u\in D_0^{s,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy}{\left(\displaystyle\int_\Omega |u(x)|^{p_s^*}\,dx\right)^{p/p_s^*}} \tag{7.2.2}$$

is the best constant in Sobolev inequality, which is independent of Ω. We recall the following proposition from [20] regarding the minimization problem (7.2.2).

Proposition 7.2.1. *Let $1 < p < \infty$, $s \in (0,1)$, $N > sp$, and $S_{s,p}$ be as in (7.2.2).*

- *(i) There exists a minimizer for $S_{s,p}$.*
- *(ii) For every minimizer U, there exists $x_0 \in \mathbb{R}^N$ and a constant sign monotone function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x) = u(|x-x_0|)$.*
- *(iii) For every minimizer U, there exists $\lambda_U > 0$ such that*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(U(x)-U(y))^{p-1}(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy = \lambda_U\int_{\mathbb{R}^N} U^{p_s^*-1}v\,dx \quad \forall\, v \in D^{s,p}(\mathbb{R}^N).$$

We state the concentration compactness theorem involving fractional p-Laplace operator which is used to prove that E_μ satisfies Palais-Smale condition below certain energy level. First we recall the definition of fractional gradient.

Definition 7.2.1. *[17] (s,p) gradient of a function $v \in D_0^{s,p}(\mathbb{R}^N)$ is defined as*

$$|D^s v(x)|^p = \int_{\mathbb{R}^N}\frac{|v(x+h)-v(x)|^p}{|h|^{N+sp}}\,dh.$$

We note that (s,p) gradient is well defined a.e in $\mathbb{R}^N$ and $|D^s v| \in L^p(\mathbb{R}^N)$.

Next we recall the concentration compactness theorem [17].

Theorem 7.2.1. *Let* $(u_n) \subset D_0^{s,p}(\mathbb{R}^{\mathbb{N}})$ *be a weakly convergent subsequence with weak limit* u*. Then there exist two bounded measures* κ *and* ν*, an atmost enumerable set of indices* I*, and positive real numbers* κ_i, ν_i*, points* $x_i \in \overline{\Omega}, i \in I$*, such that the following convergence hold weakly* in the sense of measures.*

$$|D^s u_n|^p dx \rightharpoonup \kappa \geq |D^s u|^p dx + \sum_{i \in I} \kappa_i \delta_{x_i}$$

$$|u_n|^{p_s^*} dx \rightharpoonup \nu = |u|^{p_s^*} dx + \sum_{i \in I} \nu_i \delta_{x_i}$$

$$S_{s,p}^{1/p} \nu_i^{1/p_s^*} \leq \kappa_i^{1/p} \quad \text{for all} \quad i \in I$$

where $S_{s,p}$ *as in* (7.2.2).

For the proof of the Theorem 7.1.1, we use the following compactness result.

Lemma 7.2.1. *For any fixed* $\lambda, \mu > 0$*,* E_μ *satisfies the* $(PS)_c$ *conditions for all*

$$c < \frac{s}{N} S_{s,p}^{N/sp} - \left(1 - \frac{1}{p}\right) \mu |\Omega|. \tag{7.2.3}$$

Proof. Let (u_n) be a $(PS)_c$ sequence. First we claim that (u_n) is bounded. We have

$$E_\mu(u_n) = \frac{1}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + \int_\Omega \left(-\frac{\lambda u_{n+}^p}{p} - \frac{u_{n+}^{p_s^*}}{p_s^*} \right) dx + \mu \int_{\{u_n \geq 0\}} u_n \, dx$$

$$+ \mu \left[\int_{\{-1 < u_n < 0\}} \left(u_n - \frac{u_n |u_n|^{p-1}}{p} \right) dx - \left(1 - \frac{1}{p}\right) |\{u_n \leq -1\}| \right] = c + o_n(1) \tag{7.2.4}$$

and

$$\begin{aligned} \langle DE_\mu(u_n), v \rangle = & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^{N+sp}} \, dx \, dy \\ & + \int_\Omega \left(\lambda u_{n+}^{p-1} v - u_{n+}^{p_s^* - 1} v \right) dx + \mu \left[\int_{\{u_n \geq 0\}} v \, dx + \int_{\{-1 < u_n < 0\}} \left(1 - |u_n|^{p-1}\right) v \, dx \right] \\ & = o_n(1) \|v\| \quad \forall v \in D_0^{s,p}(\Omega). \end{aligned} \tag{7.2.5}$$

Taking $v = u_n$ in the equation (7.2.5), we get

$$\begin{aligned}\langle DE_\mu(u_n), u_n\rangle = & \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega \left(-\lambda u_{n+}^p - u_{n+}^{p_s^*}\right)\,dx \\ & + \mu\left[\int_{\{u_n\geq 0\}} u_n\,dx + \int_{\{-1<u_n<0\}} \left(1-|u_n|^{p-1}\right)u_n\,dx\right] = o_n(1)\|u_n\|.\end{aligned} \tag{7.2.6}$$

Dividing (7.2.6) by p and subtracting from (7.2.4), we have

$$\begin{aligned}\left(\frac{1}{p}-\frac{1}{p_s^*}\right)\int_\Omega u_{n+}^{p_s^*}\,dx = & \left(1-\frac{1}{p}\right)|\{u_n\leq -1\}| + c + o_n(1)(1+\|u_n\|) \\ & + \mu\left(\frac{1}{p}-1\right)\left[\int_{\{u_n\geq 0\}} u_n\,dx + \int_{\{-1<u_n<0\}} u_n\,dx\right] \\ \frac{s}{N}\int_\Omega u_{n+}^{p_s^*}\,dx \leq & \left(1-\frac{1}{p}\right)\mu|\Omega| + c + o_n(1)(1+\|u_n\|)\end{aligned} \tag{7.2.7}$$

We also have,

$$\begin{aligned}\int_\Omega u_{n+}^p\,dx \leq & |\Omega|^{sp/N}\left(\int_\Omega u_{n+}^{p_s^*}\,dx\right)^{p/p_s^*} \\ \leq & |\Omega|^{sp/N}\frac{N}{s}\left[\left(1-\frac{1}{p}\right)\mu|\Omega| + c + o_n(1)(1+\|u_n\|)\right]^{p/p_s^*}.\end{aligned} \tag{7.2.8}$$

Similarly we use Holder's inequality to estimate other terms in (7.2.4). So using (7.2.7) and (7.2.8) and the fact that $\frac{p}{p_s^*} < 1$, we can show that sequence (u_n) is bounded in $D_0^{s,p}(\Omega)$. Since (u_n) is bounded so is (u_{n+}), a renamed subsequence which converges to some $u_+ \geq 0$ weakly in $D_0^{s,p}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, p_s^*)$, *a.e.* in Ω and

$$|D^s u_{n+}|^p dx \rightharpoonup \kappa, \quad u_{n+}^{p_s^*} dx \rightharpoonup \nu \tag{7.2.9}$$

in the sense of measure, where κ and ν are bounded measures. Using Theorem 7.2.1 there exists a countable index set I and positive real numbers κ_i, ν_i, points $x_i \in \overline{\Omega}$, $i \in I$ such that

$$\kappa \geq |D^s u_+|^p dx + \sum_{i\in I}\kappa_i\delta_{x_i}, \quad \nu = u_+^{p_s^*} dx + \sum_{i\in I}\nu_i\delta_{x_i} \tag{7.2.10}$$

and $S_{s,p}^{1/p}\nu_i^{1/p_s^*} \leq \kappa_i^{1/p}$ for all $i \in I$. We claim that $I = \emptyset$. Suppose by contradiction that there exists $i \in I$. We fix a concentration point x_i, a smooth function $\varphi : \mathbb{R}^{\mathbb{N}} \to [0,1]$ such that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$ and for $i \in I$ and $\rho > 0$ set

$$\varphi_{i,\rho}(x) = \varphi\left(\frac{x - x_i}{\rho}\right),\ x \in \mathbb{R}^N.$$

Clearly $\varphi_{i,\rho} : \mathbb{R}^N \to [0,1]$ is smooth function such that $\varphi_{i,\rho}(x) = 1$ for $|x - x_i| \leq \rho$ and $\varphi_{i,\rho}(x) = 0$ for $|x - x_i| \geq 2\rho$. We note that the sequence $(\varphi_{i,\rho}u_{n+})$ is bounded in $D_0^{s,p}(\Omega)$. Taking $v = \varphi_{i,\rho}u_{n+}$ in (7.2.5), we get

$$\begin{aligned} &\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{i,\rho}u_{n+}(x) - \varphi_{i,\rho}u_{n+}(y))}{|x - y|^{N+sp}}\, dx\, dy \\ &= \int_\Omega \left(\lambda u_{n+}^{p-1}\varphi_{i,\rho}u_{n+} + u_{n+}^{p_s^*-1}\varphi_{i,\rho}u_{n+}\right) dx - \mu \int_\Omega \varphi_{i,\rho}u_{n+}\, dx + o_n(1). \end{aligned} \tag{7.2.11}$$

The term on the left hand side of (7.2.11) can be estimated as

$$\begin{aligned} &\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{i,\rho}u_{n+}(x) - \varphi_{i,\rho}u_{n+}(y))}{|x - y|^{N+sp}}\, dx\, dy \\ &\geq \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_{n+}(x) - u_{n+}(y)|^p}{|x - y|^{N+sp}}\varphi_{i,\rho}(x)\, dx\, dy \\ &+ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{i,\rho}(x) - \varphi_{i,\rho}(y))}{|x - y|^{N+sp}} u_{n+}(y)\, dx\, dy. \end{aligned} \tag{7.2.12}$$

For the first term in right hand side of (7.2.12), for a fixed x, taking the transformation $y - x = h$ and using the definition (7.2.1) and (7.2.9), we have

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_{n+}(x) - u_{n+}(y)|^p}{|x - y|^{N+sp}}\varphi_{i,\rho}(x)\, dx\, dy \to \int_{\mathbb{R}^N} \varphi_{i,\rho}\, d\kappa. \tag{7.2.13}$$

For the second term in right hand side of (7.2.12), using the definition of (s,p) gradient, we have

$$\begin{aligned} &\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{i,\rho}(x) - \varphi_{i,\rho}(y))}{|x - y|^{N+sp}} u_{n+}(y)\, dx\, dy \\ &\leq C\left(\int_{\mathbb{R}^N} |u_{n+}|^p |D^s\varphi_{i,\rho}|^{p\,dx}\right)^{\frac{1}{p}}. \end{aligned} \tag{7.2.14}$$

Now we consider the first term in right hand side of (7.2.11) i.e.,

$$\int_\Omega \left(\lambda u_{n+}^p \varphi_{i,\rho} + u_{n+}^{p_s^*}\varphi_{i,\rho} - \mu\varphi_{i,\rho}u_{n+}\right) dx.$$

Using (7.2.9) we have

$$\int_\Omega \varphi_{i,\rho} u_{n+}^{p_s^*}\, dx \to \int_\Omega \varphi_{i,\rho}\, d\nu. \tag{7.2.15}$$

Now we observe that

$$\begin{aligned}&\left| \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_{i,\rho}(x)-\varphi_{i,\rho}(y))}{|x-y|^{N+sp}} u_{n+}(y)\, dx\, dy \right. \\ &\qquad \left. + \int_\Omega \left(-\lambda u_{n+}^p\varphi_{i,\rho} + \mu\varphi_{i,\rho}u_{n+}\right) dx \right| \\ &\le C\left[\left(\int_{\mathbb{R}^N} |u_{n+}|^p |D^s\varphi_{i,\rho}|^p\, dx\right)^{\frac{1}{p}} + \int_{\Omega\cap B_{2\rho}(x_i)} u_{n+}^p\, dx + \mu\int_{\Omega\cap B_{2\rho}(x_i)} u_{n+}\, dx\right].\end{aligned}$$

We also have

$$\int_{\Omega\cap B_{2\rho}(x_i)} u_{n+}^p\, dx \to \int_{\Omega\cap B_{2\rho}(x_i)} u_+^p\, dx. \tag{7.2.16}$$

Next passing to the limit in (7.2.11) and using (7.2.12)-(7.2.16), we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \varphi_{i,\rho}\, d\kappa - \int_\Omega \varphi_{i,\rho}\, d\nu \\ &\le C\left[\left(\int_{\mathbb{R}^N} |u_{n+}|^p |D^s\varphi_{i,\rho}|^p\, dx\right)^{\frac{1}{p}} + \int_{\Omega\cap B_{2\rho}(x_i)} u_+^p\, dx + \mu\int_{\Omega\cap B_{2\rho}(x_i)} u_+\, dx\right].\end{aligned}$$

Now letting $\rho\to 0$, right hand side of the above inequality goes to 0. This implies $\kappa_i \le \nu_i$, which together with $\nu_i > 0$ and $S_{s,p}^{1/p}\nu_i^{1/p_s^*} \le \kappa_i^{1/p}$ gives $\nu_i \ge S_{s,p}^{N/sp}$. On the other hand, passing the limit in (7.2.7) and using (7.2.9) and (7.2.10) we get

$$\nu_i \le \frac{N}{s}\left[\left(1-\frac{1}{p}\right)\mu|\Omega| + c\right] < S_{s,p}^{N/sp}$$

a contradiction. Hence $I = \emptyset$ and

$$\int_\Omega u_{n+}^{p_s^*}\, dx \to \int_\Omega u_+^{p_s^*}\, dx. \tag{7.2.17}$$

Next passing to further subsequence, u_n converges weakly to u i.e., $u_n \rightharpoonup u$ in $D_0^{s,p}(\Omega)$, strongly in $L^r(\Omega)$ for $r \in [1, p_s^*)$ and a.e. in Ω.
We note that u weakly solves the problem (7.2.1) and using the Brezis-Lieb type lemma and (7.2.17) we can show that $u_n \to u$ in $D_0^{s,p}(\Omega)$. This complete the proof of Lemma 7.2.1. □

Next we state some results regarding the minimization problem (7.2.2) (See [78]) that is used to prove mountain pass results. For any $\epsilon > 0$, the function

$$U_\epsilon(x) = \frac{1}{\epsilon^{(N-sp)/p}} U\left(\frac{|x|}{\epsilon}\right) \tag{7.2.18}$$

is a minimizer for $S_{s,p}$ and also satisfy the following equations

$$(-\Delta)_p^s U = U^{p_s^*-1}, \text{ and } \|U\|^p = \|U\|^{p_s^*}_{L_{p_s^*}(\mathbb{R}^N)} = S_{s,p}^{N/sp}.$$

Due to absence of explicit formula for U, we have the following asymptotic estimates.

Lemma 7.2.2. *There exists $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{c_1}{r^{(N-sp)/(p-1)}} \leq U(r) \leq \frac{c_2}{r^{(N-sp)/(p-1)}}$$

and

$$\frac{U(\theta r)}{U(r)} \leq \frac{c_2}{c_1} \frac{1}{\theta^{(N-sp)/(p-1)}}.$$

We have some auxiliary estimates from [78].
For $\epsilon, \delta > 0$, and θ as in Lemma 7.2.2, set

$$m_{\epsilon,\delta} = \frac{U_\epsilon(\delta)}{U_\epsilon(\delta) - U_\epsilon(\theta\delta)}$$

and

$$g_{\epsilon,\delta}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq U_\epsilon(\theta\delta) \\ m_{\epsilon,\delta}^p (t - U_\epsilon(\theta\delta)) & \text{if } U_\epsilon(\theta\delta) \leq t \leq U_\epsilon(\delta) \\ t + U_\epsilon(\delta)(m_{\epsilon,\delta}^{p-1} - 1) & \text{if } t \geq U_\epsilon(\delta), \end{cases}$$

and let

$$G_{\epsilon,\delta}(t) = \int_0^t g'_{\epsilon,\delta}(\tau)\, d\tau = \begin{cases} 0 & \text{if } 0 \le t \le U_\epsilon(\theta\delta) \\ m_{\epsilon,\delta}(t - U_\epsilon(\theta\delta)) & \text{if } U_\epsilon(\theta\delta) \le t \le U_\epsilon(\delta) \\ t & \text{if } t \ge U_\epsilon(\delta). \end{cases} \tag{7.2.19}$$

The functions $g'_{\epsilon,\delta}$ and $G_{\epsilon,\delta}$ are nondecreasing and absolutely continuous. Consider the radially symmetric nonincreasing function

$$u_{\epsilon,\delta}(r) = G_{\epsilon,\delta}(U_\epsilon(r)),$$

which satisfies

$$u_{\epsilon,\delta}(r) = \begin{cases} (U_\epsilon(r) & \text{if } r \le \delta \\ 0 & \text{if } r \ge \theta\delta. \end{cases}$$

We have the following estimates for $u_{\epsilon,\delta}$.

Lemma 7.2.3. *There exists a constant $C = C(N, p, s) > 0$ such that for any $\epsilon \le \delta/2$,*

$$\|u_{\epsilon,\delta}\|^p \le S_{s,p}^{N/sp} + C\left(\frac{\epsilon}{\delta}\right)^{(N-sp)/(p-1)} \tag{7.2.20}$$

$$\|u_{\epsilon,\delta}\|^p_{L^p(\mathbb{R}^N)} \ge \begin{cases} \frac{1}{C}\epsilon^{sp}\log\left(\frac{\delta}{\epsilon}\right) & \text{if } N = sp^2 \\ \frac{1}{C}\epsilon^{sp} & \text{if } N > sp^2 \end{cases} \tag{7.2.21}$$

$$\|u_{\epsilon,\delta}\|^{p_s^*}_{L_{p_s^*}(\mathbb{R}^N)} \ge S_{s,p}^{N/sp} - C\left(\frac{\epsilon}{\delta}\right)^{N/(p-1)} \tag{7.2.22}$$

From Lemma 7.2.3, we get the following estimates for

$$S_{\epsilon,\delta}(\lambda) := \frac{\|u_{\epsilon,\delta}\|^p - \lambda\|u_{\epsilon,\delta}\|^p_{L^p}}{\|u_{\epsilon,\delta}\|^p_{L^{p_s^*}}}$$

there exists a constant $C = C(N, s, p) > 0$ such that for any $\epsilon \le \delta/2$,

$$S_{\epsilon,\delta}(\lambda) \le \begin{cases} S_{s,p} - \frac{\lambda}{C}\epsilon^{sp}\log\left(\frac{\delta}{\epsilon}\right) + C\left(\frac{\epsilon}{\delta}\right)^{sp} & \text{if } N = sp^2 \\ S_{s,p} - \frac{\lambda}{C}\epsilon^{sp} + C\left(\frac{\epsilon}{\delta}\right)^{(N-sp)/(p-1)} & \text{if } N > sp^2 \end{cases} \tag{7.2.23}$$

Now we show that for all sufficiently small $\mu > 0$ E_μ has a uniformly positive mountain pass level below the threshold for compactness given in Lemma 7.2.1 .

Lemma 7.2.4. *There exist $\mu_0, \rho, c_0, R > \rho$ and $\beta < \frac{s}{N} S_{s,p}^{N/sp}$ such that the following hold for all $\mu \in (0, \mu_0)$:*

(i) *$E_\mu(0) = 0$ and $E_\mu(u) \ge c_0$ for all u such that $\|u\| = \rho$,*

(ii) *$E_\mu(tu_{\epsilon,\delta}) \le 0$ for all $t \ge R$ and $\epsilon \le \delta/2$ and $\delta \in (0, 1]$,*

(iii) *denoting by $\Gamma = \{\gamma \in C([0,1], D_0^{s,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = Ru_{\epsilon,\delta}\}$ the class of paths joining the origin to $Ru_{\epsilon,\delta}$,*

$$c_0 \le c_\mu := \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} E_\mu(\gamma(t)) \le \beta - \left(1 - \frac{1}{p}\right)\mu|\Omega| \tag{7.2.24}$$

for all sufficiently small $\epsilon > 0$,

(iv) *E_μ has a critical point u_μ at the level c_μ.*

Proof. Using (7.1.3) and (7.2.2), we have

$$E_\mu(u) \ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p - \frac{S_{s,p}^{-p_s^*/p}}{p_s^*}\|u\|^{p_s^*} - \mu\left(2 - \frac{1}{p}\right)|\Omega|. \tag{7.2.25}$$

Since $\lambda < \lambda_1$, so (i) follows from (7.2.25) for sufficiently small $c_0, \mu, \rho > 0$.

Next, as $u_{\epsilon,\delta} > 0$, for $t \ge 0$, we have

$$E_\mu(tu_{\epsilon,\delta}) = \frac{t^p}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_{\epsilon,\delta}(x) - u_{\epsilon,\delta}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \frac{\lambda t^p}{p}\int_\Omega u_{\epsilon,\delta}^p\,dx - \frac{t^{p_s^*}}{p_s^*}\int_\Omega u_{\epsilon,\delta}^{p_s^*}\,dx + \mu t\int_\Omega u_{\epsilon,\delta}\,dx.$$

Using the Holder's and Young's inequalities, we obtain

$$\mu t\int_\Omega u_{\epsilon,\delta}\,dx \le \mu t|\Omega|^{1-1/p}\left(\int_\Omega u_{\epsilon,\delta}^p\,dx\right)^{1/p} \le C_\lambda \mu^{\frac{p}{p-1}} + \frac{\lambda t^p}{2p}\int_\Omega u_{\epsilon,\delta}^p\,dx$$

where

$$C_\lambda = \left(1 - \frac{1}{p}\right)\left(\frac{2}{\lambda}\right)^{\frac{p}{p-1}} |\Omega|.$$

So we get

$$E_\mu(tu_{\epsilon,\delta}) \le \frac{t^p}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_{\epsilon,\delta}(x) - u_{\epsilon,\delta}(y)|^p}{|x-y|^{N+sp}}\, dx\, dy - \frac{\lambda t^p}{2p} \int_\Omega u_{\epsilon,\delta}^p\, dx - \frac{t^{p_s^*}}{p_s^*} \int_\Omega u_{\epsilon,\delta}^{p_s^*}\, dx + C_\lambda \mu^{\frac{p}{p-1}}. \tag{7.2.26}$$

Now by (7.2.23) for $\delta, \mu \in (0,1]$ and $\epsilon \le \delta/2$ with $\|u_{\epsilon,\delta}\|_{L^{p_s^*}} = 1$, we have

$$E_\mu(tu_{\epsilon,\delta}) \le \frac{t^p}{p}(S_{s,p} + C) - \frac{t^{p_s^*}}{p_s^*} + C_\lambda \mu^{\frac{p}{p-1}},$$

from which (ii) follows for sufficiently large $R > \rho$.

The first inequality in (7.2.24) follows from (i) as $R > \rho$. Next we maximize the right hand side of (7.2.26) over $t \ge 0$ with $\|u_{\epsilon,\delta}\|_{L^{p_s^*}} = 1$ and obtain

$$c_\mu \le \frac{s}{N}\left[\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_{\epsilon,\delta}(x) - u_{\epsilon,\delta}(y)|^p}{|x-y|^{N+sp}}\, dx\, dy - \frac{\lambda}{2}\int_\Omega u_{\epsilon,\delta}^p\, dx\right]^{N/sp} + C_\lambda \mu^{\frac{p}{p-1}},$$

and (7.2.23) gives that the integral on the right hand side of the above inequality is strictly less than $S_{s,p}$ for all sufficiently small $\epsilon \le \delta/2, \delta > 0$ since $N \ge sp^2$ and $\lambda > 0$, so the second inequality in (7.2.24) holds for sufficiently small $\mu > 0$.

Clearly (iv) follows from $(i) - (iii)$, Lemma 7.2.1, and mountain pass lemma. □

Next we show that u_μ is uniformly bounded in $D_0^{s,p}(\Omega)$ and uniformly equi-integrable in $L^{p_s^*}(\Omega)$, and also uniformly bounded in $C_d^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$, for all sufficiently small $\mu \in (0, \mu_0.)$

Lemma 7.2.5. *There exists $\mu_* \in (0, \mu_0]$ such that the following hold for all $\mu \in (0, \mu_*)$:*

(i) *u_μ is uniformly bounded in $D_0^{s,p}(\Omega)$.*

(ii) *$\int_E |u_\mu|^{p_s^*}\, dx$ as $|E| \to 0$, uniformly in μ.*

(iii) *u_μ is uniformly bounded in $C_d^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$.*

Proof. We have,

$$E_\mu(u_\mu) = \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_\mu(x)-u_\mu(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega\left(-\frac{\lambda u_{\mu+}^p}{p} - \frac{u_{\mu+}^{p_s^*}}{p_s^*}\right)dx + \mu\int_{\{u_\mu\geq 0\}} u_\mu\,dx$$
$$+\mu\left[\int_{\{-1<u_\mu<0\}}\left(u_\mu - \frac{u_\mu|u_\mu|^{p-1}}{p}\right)dx - \left(1-\frac{1}{p}\right)|\{u_\mu\leq -1\}|\right] = c_\mu \tag{7.2.27}$$

and

$$\langle DE_\mu(u_\mu), v\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_\mu(x)-u_\mu(y)|^{p-2}(u_\mu(x)-u_\mu(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$+\int_\Omega\left(\lambda u_{\mu+}^{p-1}v - u_{\mu+}^{p_s^*-1}v\right)dx + \mu\left[\int_{\{u_\mu\geq 0\}} v\,dx + \int_{\{-1<u_\mu<0\}}\left(1-|u_\mu|^{p-1}\right)v\,dx\right]$$
$$= 0 \quad \forall v\in D_0^{s,p}(\Omega). \tag{7.2.28}$$

By taking $v = u_\mu$ in (7.2.28), dividing by p and subtracting from (7.2.24), we get

$$\frac{s}{N}\int_\Omega u_{\mu+}^{p_s^*}\,dx \leq \left(1-\frac{1}{p}\right)\mu|\Omega| + c_\mu \leq \beta. \tag{7.2.29}$$

Then (i) follows from (7.2.29) and (7.2.28) with $v = u_\mu$.

Suppose (ii) does not hold. Then there sequence $\mu_j \to 0$ and (E_j) with $|E_j| \to 0$ such that

$$\int_{E_j} |u_{\mu_j}|^{p_s^*}\,dx > 0. \tag{7.2.30}$$

Since (u_{μ_j}) is bounded by (i), so is (u_{μ_j+}) a renamed subsequence which converges to some $u_+ \geq 0$ weakly in $D_0^{s,p}(\Omega)$. Following the argument as in compactness Lemma 7.2.1 we get

$$\int_\Omega u_{\mu_j+}^{p_s^*}\,dx \to \int_\Omega u_+^{p_s^*}\,dx. \tag{7.2.31}$$

As in the proof of Lemma 7.2.1, upto a subsequence, (u_{μ_j}) then converges to u in $D_0^{s,p}(\Omega)$, and so also in $L^{p_s^*}(\Omega)$. Then

$$\int_{E_j} |u_{\mu_j}|^{p_s^*}\,dx \le \int_\Omega \left| |u_{\mu_j}|^{p_s^*} - |u|^{p_s^*} \right|\,dx + \int_{E_j} |u|^{p_s^*}\,dx \to 0,$$

contradicting (7.2.30). Thus (ii) holds.

The regularity assertion in (iii) follows from the following theorem. □

Theorem 7.2.2. *Let Ω be a bounded, $\mu \in (0,\mu_*)$ and $u_\mu \in D_0^{s,p}(\Omega)$ weakly solve $(-\Delta)_p^s u_\mu = f(x,u_\mu)$ in Ω for f satisfying the following growth condition*

$$|f(x,t)| \le C_1 + C_2|t|^{p_s^*-1}, \tag{7.2.32}$$

where $C_1, C_2 > 0$ are constants. Assume that u_μ is uniformly bounded in $D_0^{s,p}(\Omega)$. In addition $\int_E |u_\mu|^{p_s^*}\,dx$ as $|E| \to 0$, uniformly in μ. Then u_μ is is uniformly bounded in $C_d^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,s]$.

Proof. By [21], it is sufficient to show that $f(x,u_\mu) \in L^{\bar q}(\Omega)$ for some $\bar q > \frac{N}{sp}$. We define $g_\tau(t) := t(t_k)^\tau$, where $t_k := \min\{t,k\}$ and $k > 0, t \ge 0, \tau \ge 0$ and extend g_τ and t_k as an odd function in $\mathbb{R}$. We have

$$G_\tau(t) := \int_0^t (g_\tau'(s))^{1/p}\,ds \ge \frac{p(\tau+1)^{1/p}}{p+\tau} g_{\frac{\tau}{p}}(t). \tag{7.2.33}$$

and due to [21] and (7.2.32)

$$[G_\tau(u_\mu)]^p_{D_0^{s,p}(\Omega)} \le \langle (-\Delta)_p^s u_\mu, g_\tau(u_\mu)\rangle \le C\left(\int_\Omega |u_\mu||u_{\mu,k}|^\tau\,dx + \int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^\tau\,dx\right).$$

By using Sobolev's inequality and (7.2.33) we get

$$\left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\,dx\right)^{\frac{p}{p_s^*}} \le C_\tau\left(\int_\Omega |u_\mu||u_{\mu,k}|^\tau\,dx + \int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^\tau\,dx\right). \tag{7.2.34}$$

Now, given $\{u_\mu\}_{\mu\in(0,\mu_*)} = \mathcal{F}$ is uniformly integrable in $L^{p_s^*}(\Omega)$, so we have

$$\lim_{M\to\infty}\left(\sup_{u_\mu\in\mathcal{F}} \int_{\{|u_\mu|>M\}} |u_\mu|^{p_s^*}\,dx\right) = 0$$

i.e. for $\sigma > 0$, there exists $M_0(\sigma)$, such that for all $M > M_0(\sigma)$

$$\sup_{u_\mu \in \mathcal{F}} \int_{\{|u_\mu|>M\}} |u_\mu|^{p_s^*}\, dx < \sigma.$$

This implies that

$$\int_{\{|u_\mu|>M\}} |u_\mu|^{p_s^*}\, dx < \sigma \;\forall\; M > M_0(\sigma) \text{ and } \forall\, \mu \in (0, \mu_*).$$

Now we estimate the first term on right side of equation (7.2.34) as follows

$$\begin{aligned}
\int_\Omega |u_\mu||u_{\mu,k}|^\tau\, dx &\leq \int_{\{|u_\mu|\leq M_0\}} |u_\mu||u_{\mu,k}|^\tau\, dx + \int_{\{|u_\mu|>M_0\}} |u_\mu||u_{\mu,k}|^\tau\, dx \\
&\leq M_0^\tau \int_\Omega |u_\mu|\, dx + \int_{\{|u_\mu|>M_0\}} |u_\mu|^{p_s^*}|u_{\mu,k}|^\tau\, dx \\
&\leq M_0^\tau \int_\Omega |u_\mu|\, dx + \left(\int_{\{|u_\mu|>M_0\}} |u_\mu|^{p_s^*}\, dx\right)^{1-\frac{p}{p_s^*}} \left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\, dx\right)^{\frac{p}{p_s^*}} \\
&\leq M_0^\tau \int_\Omega |u_\mu|\, dx + \sigma^{\left(1-\frac{p}{p_s^*}\right)} \left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\, dx\right)^{\frac{p}{p_s^*}}.
\end{aligned}$$

We also estimate the second term on right side of equation (7.2.34),

$$\begin{aligned}
\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^\tau\, dx &\leq \int_{\{|u_\mu|\leq M_0\}} |u_\mu|^{p_s^*}|u_{\mu,k}|^\tau\, dx + \int_{\{|u_\mu|>M_0\}} |u_\mu|^{p_s^*}|u_{\mu,k}|^\tau\, dx \\
&\leq M_0^\tau \int_\Omega |u_\mu|^{p_s^*}\, dx + \left(\int_{\{|u_\mu|>M_0\}} |u_\mu|^{p_s^*}\, dx\right)^{1-\frac{p}{p_s^*}} \left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\, dx\right)^{\frac{p}{p_s^*}} \\
&\leq M_0^\tau \int_\Omega |u_\mu|^{p_s^*}\, dx + \sigma^{\left(1-\frac{p}{p_s^*}\right)} \left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\, dx\right)^{\frac{p}{p_s^*}}.
\end{aligned}$$

Now by collecting all these estimates, we have

$$\left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\, dx\right)^{\frac{p}{p_s^*}} \leq C_\tau M_0^\tau \left[\int_\Omega (|u_\mu|+|u_\mu|^{p_s^*})\, dx + \sigma^{\left(1-\frac{p}{p_s^*}\right)} \left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\, dx\right)^{\frac{p}{p_s^*}}\right].$$

We choose σ such that $\sigma^{\left(1-\frac{p}{p_s^*}\right)} = \frac{1}{2}$ and using the fact that u_μ is uniformly bounded in

$D_0^{s,p}(\Omega)$ we have,

$$\left(\int_\Omega |u_\mu|^{p_s^*}|u_{\mu,k}|^{\frac{\tau p_s^*}{p}}\,dx\right)^{\frac{p}{p_s^*}} \le C_\tau M_0^\tau [u_\mu]_{D_0^{s,p}(\Omega)} \le C_1$$

where C_1 is independent of μ. Now letting $k \to \infty$ we obtain

$$\left(\int_\Omega |u_\mu|^{p_s^*(1+\frac{\tau}{p})}\,dx\right)^{\frac{p}{p_s^*}} \le C_1$$

i.e., we have

$$\|u_\mu\|_{L^{p_s^*(1+\frac{\tau}{p})}(\Omega)} \le C_2. \tag{7.2.35}$$

Then our result follows from [36] and Theorem 2.5.2. □

Proof of the Theorem 7.1.1: We claim that u_μ is positive in Ω. It is sufficient to show that for every sequence $\mu_j \to 0$, a subsequence u_{μ_j} is positive in Ω. By Lemma 7.2.4, we have

$$\begin{aligned} E_{\mu_j}(u_{\mu_j}) =& \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_{\mu_j}(x)-u_{\mu_j}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega \left(-\frac{\lambda u_{\mu_j+}^p}{p} - \frac{u_{\mu_j+}^{p_s^*}}{p_s^*}\right)dx + \mu_j \int_{\{u_{\mu_j}\ge 0\}} u_{\mu_j}\,dx \\ &+ \mu_j \left[\int_{\{-1<u_{\mu_j}<0\}} \left(u_{\mu_j} - \frac{u_{\mu_j}|u_{\mu_j}|^{p-1}}{p}\right)dx - \left(1-\frac{1}{p}\right)|\{u_{\mu_j}\le -1\}|\right] \\ &= c_{\mu_j} \ge c_0 \end{aligned}$$

and also have

$$\begin{aligned} \langle DE_{\mu_j}(u_{\mu_j}), v\rangle =& \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_{\mu_j}(x)-u_{\mu_j}(y)|^{p-2}(u_{\mu_j}(x)-u_{\mu_j}(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &+ \int_\Omega \left(\lambda u_{\mu_j+}^{p-1} v - u_{\mu_j+}^{p_s^*-1} v\right)dx + \mu_j\left[\int_{\{u_{\mu_j}\ge 0\}} v\,dx + \int_{\{-1<u_{\mu_j}<0\}} \left(1-|u_{\mu_j}|^{p-1}\right) v\,dx\right] \\ &= 0 \quad \forall\, v \in D_0^{s,p}(\Omega). \end{aligned}$$

Now as for $\mu_j \to 0$, $u_{\mu_j} \to u$ in $C^\alpha(\overline{\Omega})$, so we have

$$\frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_\Omega \left(-\frac{\lambda u_+^p}{p} - \frac{u_+^{p_s^*}}{p_s^*}\right)dx \ge c_0 \tag{7.2.36}$$

and

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy+\int_\Omega \left(\lambda u_+^{p-1}v - u_+^{p_s^*-1}v\right)\,dx = 0$$

$\forall\, v\in D_0^{s,p}(\Omega)$. So u is a nontrivial weak solution of the problem

$$\begin{cases} (-\Delta)_p^s u &= \lambda u_+^{p-1} + u_+^{p_s^*-1} \text{ in } \Omega,\\ u &= 0 \qquad\qquad\quad \text{on } \Omega^c. \end{cases} \tag{7.2.37}$$

We claim $u>0$ in Ω. Since u satisfy (7.2.36) and from mountain pass Lemma 7.2.4, we have $c_0>0$ this implies that u can not be identically zero. For each $x\in\Omega$, $\lambda u_+^{p-1}(x)+u_+^{p_s^*-1}(x)\geq 0$ and using strong minimum principle and Hopf's lemma [54], we have

$$u(x)\geq cd^s(x)>0 \text{ in } \Omega.$$

Now since $u_{\mu_j}\to u$ in $C_d^{0,\alpha}(\overline{\Omega})$ for $\alpha\in(0,s]$, then $u_{\mu_j}>0$ in Ω for sufficiently large j. So we conclude that for small μ, $u_\mu>0$ is solution of the problem (7.1.2) with $DE_\mu(u_\mu)=0$ and $E_\mu(u_\mu)=c_\mu$ where c_μ is as given in Lemma 7.2.4 viz.,

$$c_\mu := \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} E_\mu(\gamma(t))$$

where $\Gamma=\{\gamma\in C([0,1],D_0^{s,p}(\Omega)) : \gamma(0)=0, \gamma(1)=Ru_{\epsilon,\delta}\}$. Now we will show that u_μ is ground state solution, that is, u_μ minimizes E_μ on $\mathcal{N}_\mu$. Let $\alpha=\inf\limits_{w\in\mathcal{N}_\mu} E_\mu(w)$ then $\alpha\leq c_\mu$. So it remains to show that $c_\mu\leq\alpha$. For $w\in\mathcal{N}_\mu$, define the fibering map $g:[0,\infty)\to\mathbb{R}$ by $g(t):=E_\mu(tw)$, so we have for $t\geq 0$

$$g(t)=E_\mu(tw)=\frac{t^p}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy-\frac{\lambda t^p}{p}\int_\Omega w^p\,dx-\frac{t^{p_s^*}}{p_s^*}\int_\Omega w^{p_s^*}\,dx+\mu t\int_\Omega w\,dx.$$

It is easy to show that g is differentiable with respect to t and for $w\in\mathcal{N}_\mu$, we have

$$\begin{aligned} g'(t) =& t^{p-1}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy-\lambda t^{p-1}\int_\Omega w^p\,dx - t^{p_s^*-1}\int_\Omega w^{p_s^*}\,dx+\mu\int_\Omega w\,dx\\ =& \left(t^{p-1}-t^{p_s^*-1}\right)\left[\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy-\lambda\int_\Omega w^p\,dx\right]+\left(1-t^{p_s^*-1}\right)\mu\int_\Omega w\,dx. \end{aligned}$$

We observe that integrals inside box bracket is positive since $\lambda < \lambda_1$ and other integral is also positive since $w > 0$, so $g'(t) > 0$ for $0 \leq t < 1$, $g'(1) = 0$ and $g'(t) < 0$ for $t > 1$. Therefore

$$\max_{t\geq 0} E_\mu(tw) = E_\mu(w) > 0$$

since $g(0) = 0$. For fixed $t_0 > \max\{1, R\}$, we define the map $\gamma : [0,1] \to D_0^{s,p}(\Omega)$ as $\gamma(t) := (t_0 w)t$ that satisfies $E_\mu(t_0 w) \leq 0$ and so $\gamma \in \Gamma$. Hence

$$c_\mu \leq \max_{0\leq t\leq 1} E_\mu(\gamma(t)) \leq \max_{t\geq 0} E_\mu(tw) = E_\mu(w).$$

Since $w \in \mathcal{N}_\mu$ is arbitrary, we get $c_\mu \leq \alpha$. This completes the proof of the Theorem 7.1.1.

7.3 Conclusion

In this chapter, we have investigated the existence of a positive solution to the semipositone problem involving a fractional p-Laplacian operator with critical growth. Here, we have studied Brezis- Nirenberg type perturbation to the problem (7.1.1). First we have proved that the energy functional associated with the modified problem satisfies the Palais-smale condition below some energy level by employing the concentration compactness theorem. Next, we have shown that energy functional has a mountain pass type geometry below some energy level. Then we have demonstrated that the modified problem has a positive solution, hence a solution to our perturbed problem for small μ.

The research in this chapter has raised some open questions for further investigation. Here we have solved our problem for the case $p \geq 2$ as the fine boundary regularity result is available only for this case. It will be interesting to study the case when $1 < p < 2$. In this case, one of the main challenges would be the requirement of fine boundary type regularity result.

Another exciting exploration will be the study of the semipositone problem involving fractional (p, q)-Laplacian operators with critical growth terms. Here the main challenges are the requirement of the concentration compactness theorem and fine boundary regularity result.

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Publications

List of Publications from Thesis work

1. U. Kumar and S. Tiwari, *Multiple sign-changing solutions of nonlocal critical exponent problem in symmetric domains. Mediterr. J. Math.* (2022) 19:189. DOI: 10.1007/s00009-022-02070-x.
2. U. Kumar and S. Tiwari, *Multiple solution to Bahri-Coron problem involving fractional p-Laplacian in some domain with non-trivial topology.* (Accepted in Topological Methods in Nonlinear Analysis)
3. U. Kumar and S. Tiwari, *Multiple sign-changing solutions of nonlocal supercritical exponent problem in symmetric domains.* (Under review)
4. *Nonlocal superlinear semipositone problem.* (With R. Dhanya, S. Tiwari, and R. Jana)(To be submitted soon)