REPRESENTATION NUMBER OF WORD-REPRESENTABLE GRAPHS

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Representation Number of Word-Representable Graphs

by

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To

My family

Certificate

This is to certify that the thesis entitled *Representation Number of Word-Representable Graphs* submitted by *Ms. Khyodeno Mozhui* to the Indian Institute of Technology Guwahati, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

This thesis is on the theory of word-representable graphs – a promising area of research lying at the intersection of graph theory and combinatorics on words.

A graph is called a word-representable graph if it can be represented by a word over its vertex set, such that two vertices are adjacent if and only if they alternate in that word. The minimum value k such that a word-representable graph can be represented by a word in which every vertex appears k times is called its representation number. Furthermore, if a word-representable graph is represented by a word which is a concatenation of permutations of its vertices, is referred to as a permutationally representable graph. The minimum number of such permutations required to represent the graph is called the permutation-representation number (prn) of the graph.

In this thesis, we study word-representable graphs, with a special focus on their representation number and permutation-representation number. While the classes of graphs with representation number or prn one and two are well-established in the literature, only few classes of graphs are known to have representation number or prn three. In fact, the problems of determining the representation number of a word-representable graph and the prn of a permutationally representable graph are NP-hard. Specifically, both the problems of deciding whether a word-representable graph has representation number three and a permutationally representable graph has *prn* three are NP-complete.

Contributing to the class of graphs with representation number or *prn* at most three, we show that the stacked book graphs, melon graphs and their line graphs belong to this class. Further, we study the representation number of bipartite graphs and validate the conjecture that the representation number of a bipartite graph on $n \geqslant 9$ vertices is at most $\lceil \frac{n}{4} \rceil$, leaving the bipartite graphs whose partite sets are of equal and even size. We also provide an upper bound for the *prn* of bipartite graphs.

It is known that word-representable graphs are characterized by semi-transitive orientations. Using semi-transitive orientations as a tool, in this thesis, we settle the open problem that whether the line graph of a non-word-representable graph is always non-word-representable. In this connection, we show that the line graphs of Mycielski graphs of odd cycles of length at least five, which are known to be non-word-representable, admit semi-transitive orientations. Further, we also establish various relations between representation number, boxicity and dimension of posets.

The results of this thesis are primarily established through various constructions of words, all of which can be executed in polynomial time.

Contents

Introduction

The topic of word-representable graphs is an emerging area of graph theory and word combinatorics, and has connections to dimension theory of partially ordered sets. This thesis constructs words representing various graphs and study their representation numbers. Further, the thesis investigates the permutation-representation number of a comparability graph, which is indeed the dimension of the associated partially ordered set.

A graph $G = (V, E)$ is said to be a word-representable if there exists a word w over its vertex set V, called a word-representant, such that two vertices a and b in G are adjacent if and only if a and b alternate in the word w. If no such word exists, then the graph is said to be non-word-representable. Suppose the word w representing a word-representable graph G is of the form $p_1p_2 \ldots p_k$, where each p_i is a permutation on the vertex set of G. We call the graph G as permutationally representable graph or permutationally k-representable graph.

The theory of word-representable graphs originated from algebra, where Kitaev and Seif [2008] used this notion to study the free spectrum of the Perkin semigroup. The class of word-representable graphs has connection to rescheduling problem as described by Graham and Zang [2008]. Moreover, the class of word-representable graphs is important from a graph-theoretical point of view as it covers several important graph classes such as circle graphs, cover graphs, comparability graphs and 3-colorable graphs. Brightwell [1993] showed that it is an NP-complete problem to recognize the class of cover graphs of posets. This implies that recognizing whether a graph is word-representable or not is an NP-complete problem. In [Halldórsson et al., 2011], it was established that the maximum clique problem is polynomially solvable in the class of word-representable graphs.

An orientation of a graph is semi-transitive if it is acyclic and for any directed path $\langle a_0, a_1, \ldots, a_n \rangle$ either there is no edge between a_0 and a_n or $\overrightarrow{a_i a_j}$ is an edge for all $0 \leqslant i < j \leqslant k$. An undirected graph is semi-transitive if it admits a semi-transitive orientation. The notion of semi-transitive orientations was introduced by Halldórsson et al. [2016], which is a generalization of transitive orientations. An important subclass of the class of word-representable graphs is the class of permutationally representable graphs. Kitaev and Seif [2008] showed that the class of comparability graphs is precisely the class of permutationally representable graphs. Note that comparability graphs are graphs which admit transitive orientations and have a close relation with partial orders. In [Halldórsson et al., 2016], it was proved that a graph is word-representable if and only if it admits a semi-transitive orientation. There is a long line of research in the literature where semi-transitive orientations were used as a powerful tool for studying word-representability of a graph, such as polyomino triangulations [Akrobotu et al., 2015], (extended-)Mycielski graphs [Hameed, 2024; Kitaev and Pyatkin, 2025], split graphs [Kitaev et al., 2021; Kitaev and Pyatkin, 2024], triangle-free graphs [Kitaev and Pyatkin, 2023] and Kneser graphs [Kitaev and Saito, 2020]. The monograph by Kitaev and Lozin [2015] gives a comprehensive introduction to the theory of word-representable graphs, their connections and contributions to various topics.

Kitaev and Pyatkin [2008] showed that for every word-representable graph there exists a k-uniform word (each letter appears k times) which represents it, for some k; in which case, we call the graph a k-word-representable graph. The minimum k such that a word-representable graph G is k-word-representable is called the representation number of G. While the complete graphs are characterized as the graphs with representation one in [Kitaev and Lozin, 2015], the circle graphs, which are not complete graphs, were shown to be precisely the word-representable graphs with representation number two in [Halldórsson et al., 2011]. Further, it was established that the problem of determining whether a graph has representation k, for fixed $k \geqslant 3$, is NP-complete. Nevertheless, for certain graph classes the representation numbers are known. For example, prisms were shown to have representation number three in [Kitaev, 2013]. The crown graphs with $2n$ vertices were established to have representation number $\lceil \frac{n}{2} \rceil$ in [Glen et al., 2018]. The Cartesian product of two complete graphs K_n and K_2 were shown to have representation number three in [Broere and Zantema, 2019]. In [Hefty et al., 2024], it was established that the k-cube has representation number $O\left(\frac{\log(k)}{\log\log(k)}\right)$. Additionally, Akgün et al. [2019] computed the representation number for graphs with up to nine vertices. Halldórsson et al. [2011] showed that every word-representable graph with n vertices is n-word-representable. Fleischmann et al. [2024] provided a characterization of k-word-representable graph in terms of k-circle representation.

For a comparability graph G, the minimum k such that G is permutationally k-representable is called the permutation-representation number (in short, *prn*) of the graph G. Thus, we see that the representation number of a permutationally representable graph is at most its *prn*. The complete graphs are the graphs with *prn* one. Every comparability graph with a transitive orientation induces a partially ordered set (poset). Halldórsson et al. [2011] showed that if the *prn* of a comparability graph is k then the dimension of the poset associated with the comparability graph is at most k. Baker et al. [1972] showed that the class of permutation graphs is exactly the class of comparability graphs whose associated posets have dimension at most two. Thus, we can see that the class of permutation graphs characterizes the class of comparability graphs with *prn* at most two. Yannakakis [1982] established that determining whether a poset has dimension at most k is NP-complete, for fixed $k \geqslant 3$. This implies that determining whether a permutationally representable graph has *prn* at most k is NP-complete, for fixed $k \geqslant 3$. Further, from the results of Yannakakis [1982] and Felsner et al. [2017], we see that determining whether a height-2 poset has dimenision at most k is NP-complete, for fixed $k \geqslant 3$. The class of bipartite graphs is exactly the class of comparability graphs whose associated posets have height-2. Thus, determining whether a bipartite graph has *prn* at most k is NP-complete, for fixed $k \geqslant 3$. However, upper bounds for the *prn* can be ascertained from dimension theory for certain comparability graphs such as planar comparability graphs [Felsner et al., 2015], split comparability graphs [Guenver and Rampon, 2004], chordal comparability graphs [Ma and Spinrad, 1991], generalized crown graphs [Trotter, 1974] and treelike comparability graphs [Trotter and Moore, 1977]. In the literature, there are series of articles on the bounds of dimension of poset, with bounds given in terms of maximum degree by Erdős et al. [1991]; Füredi and Kahn [1986]; Scott and Wood [2020]; Trotter [1992]. Moreover, Adiga et al. [2011] gave relations between the dimension of a poset and the boxicity of its underlying comparability graph.

Hefty et al. [2024] introduced the representation number of a semi-transitive orientation, which we refer to as the s-representation number. A uniform word-representant w of a word-representable graph G induces a semi-transitive orientation of G such that an edge $\overline{ab}$ is oriented from a to b if the first occurrence of a appears on the left hand side of the first occurrence of b in w. An orientation of a graph is k-word-representable if there exists a k-uniform word which gives the same orientation. The s-representation number of a semi-transitive orientation is the smallest k such that a k-uniform word yields that orientation. Note that different semi-transitive orientations of the same graph may have different s-representation numbers. However, the representation number of the graph is the minimum s-representation number of all the semi-transitive orientations of that graph. The Hasse diagram of a poset can be regarded as a graph with a semi-transitive orientation. Hefty et al. gave an upper bound on the s-representation of a Hasse diagram of a poset in terms of its width.

Contribution of the thesis

This thesis aims to further the study of the representation number of word-representable graphs and the permutation-representation number (*prn*) of comparability graphs. Focusing on the $\left(\lceil \frac{n}{4} \rceil\right)$-conjecture of Glen et al. on the representation number of bipartite graphs, the thesis proves that the conjecture is valid for all bipartite graphs, leaving the bipartite graphs with partite sets of even and equal size. For the bipartite graphs, the thesis provides a better upper bound for the *prn*. Towards contributing to the classes of graphs of representation number three and of *prn* three, the thesis completely characterizes and classifies melon graphs and their line graphs, and certain special classes of bipartite graphs, viz., extended crown graphs and stacked book graphs, with respect to their representation number and *prn*. Furthermore, utilizing the tool of semi-transitive orientations, the thesis settles one of the open problems on line graphs and shows that the line graphs of non-word-representable graphs are not always non-word-representable. The thesis also studies the relationship between the representation number defined via a semi-transitive orientation of a word-representable graph and the dimension of the poset induced by the semi-transitive orientation. In establishing the aforementioned results, the thesis constructs word-representants extensively in an efficient manner.

Organization of the thesis

The necessary preliminaries and background material are presented in Chapter 1 of the thesis. Starting from Chapter 2, the contributions of the thesis have been organized into seven chapters, which are divided into three parts as follows:

Part-I: Bipartite Graphs

Chapter 2: Permutation-Representation Number

Chapter 3: Representation Number

Chapter 4: Special Classes

Part-II: Melon Graphs

Chapter 5: (Permutation-)Representation Number

Chapter 6: Line Graphs of Melon Graphs

Part-III: Semi-Transitive Orientations

Chapter 7: Line Graphs

Chapter 8: Dimension vs Representation Number

Chapter 2. In this chapter, we study the *prn* of a bipartite graph by using the concept of a neighborhood inclusion graph. The neighborhood inclusion graph of a bipartite graph is a comparability graph. The corresponding poset indeed is a subposet of the stack of a height-2 poset (defined by Trotter [1981]). In this connection, we devise a polynomial-time procedure to construct permutations on the vertices of a given bipartite graph whose concatenation represents the bipartite graph. This gives us an upper bound for the *prn* of bipartite graphs in terms of the width of the poset mentioned above. The work contained in this chapter was presented at the 26th Japan Conference on Discrete and Computational Geometry, Graphs, and Games (JCDCG3, 2024), Tokyo, Japan, 10-12 September 2024; and subsequently accepted for publication in the "Journal of Information Processing".

Chapter 3. Glen et al. [2018] conjectured that the representation number of a bipartite graph on n vertices is at most $\lceil \frac{n}{4} \rceil$. In this chapter, we focus on validating the conjecture posed by Glen at al. In fact, we show that the conjecture holds for the bipartite graphs, leaving the bipartite graphs with partite sets of equal and even size. Consequently, we also show that the crown graphs have the highest representation number among all bipartite graphs on $2n$ vertices, when n is odd, addressing the conjecture by Akgün et al. [2019] in this regard. The work contained in this chapter was submitted to a journal.

Chapter 4. In this chapter, we study two special classes of bipartite graphs, viz., extended crown graphs and stacked book graphs. Inspired by the Kimble split operation for posets (Kimble [1973]), we construct a class of bipartite graphs defined by posets called extended crown graphs and show that their permutation-representation number attains the upper bound given in Chapter 2. Further, we prove that the extended crown graphs defined by posets of width two are circle graphs and have *prn* at most three. Next, we focus on a stacked book graph, which is the Cartesian product of a star graph and a path. We prove that the *prn* of a stacked book graph is at most three. Subsequently, we show that the representation number of a stacked book graph coincides with its *prn*, and we classify these graphs according to their *prn*. The first part of the chapter on extended crown graphs was presented at the International Conference on Graph Theory and its Applications (ICGTA 2024), Bengaluru, 20-22 June 2024; and subsequently submitted to a journal. The second part on stacked book graphs is to appear in the proceedings: Current Progress in Interdisciplinary Research; Select Papers of RIC 2024, Vol. 3, Springer.

Chapter 5 & 6. Contributing to the class of graphs with *prn* and representation number three, this chapter focuses on a subclass of 3-colorable planar graphs, viz., melon graphs. We establish that the representation number of a melon graph is at most three. Furthermore, we characterize the class of permutationally representable melon graphs and show that their *prn* is also at most three. We also classify the melon graphs with respect to their *prn* as well as representation numbers. In Chapter 6, we consider the line graph of a melon graph and characterize that it is word-representable if and only if the triangular book graph with three pages is a forbidden induced subgraph. Subsequently, we show that the representation number as well as the *prn* of the word-representable and the permutationally representable line graphs of melon graphs are at most three. The work contained in these chapters was submitted to a journal.

Chapter 7. It is known that the line graph of a word-representable graph need not be word-representable. In this chapter, first, we provide an alternative and short proof for showing that the line graphs of complete graphs K_n (for $n \geqslant 5$) are not word-representable. Furthermore, Kitaev et al. [2011a,b] posed an open problem to determine whether the line graph of a non-word-representable graph is always non-word-representable. In this chapter, we settle this problem by considering a class of non-word-representable graphs, viz., Mycielski graphs of odd cycles of length at least five, and show that their line graphs are word-representable. The work contained in this chapter was submitted to a journal.

Chapter 8. In this chapter, first we study a relationship between the s-representation number of a semi-transitive orientation of a graph and the dimension of the poset induced by the semi-transitive orientation. This gives us an upper bound for the dimension of a poset in terms of the s-representation number of its Hasse diagram. Furthermore, we also establish that the boxicity of a word-representable graph G is bounded above in terms of the representation number of G and the length of a longest directed path of a semi-transitive orientation of G. Consequently, we also customize the upper bound for the *prn* of a comparability graph.

1

Preliminaries

The word-representable graphs is an interplay between graphs and words. This concept has connections with partially ordered sets (posets). In order to make the thesis self-contained to the possible extent, the required definitions and results on graphs, words and also posets are presented in this chapter. Further, in this chapter, we also fix the notation for various fundamental concepts used throughout the thesis. Finally, we provide the necessary background material on word-representable graphs. This includes formal definitions, exploring the relationships with various classes of graphs, and presenting key results on word-representable graphs. For the concepts that are used but not defined in this thesis, if any, we recommend to refer [Golumbic, 2004; Kitaev and Lozin, 2015; Trotter, 1992; West, 1996] on the respective topics.

1.1 Words

In this section we formally define the notion of words and related concepts.

Definition 1.1.1 (Word)**.** An alphabet is a finite set and its elements are called letters. A *word* over an alphabet A is a finite sequence of elements of A, written by juxtaposing the letters in the sequence. The empty word, which contains no letters, is denoted by ε.

Definition 1.1.2 (Subword)**.** A *subword* u of a word w is a subsequence of the sequence w, denoted by $u \ll w$. If u is not a subword of w, we write $u \not\ll w$. A subword u of a word w is called a *factor* if $w = xuy$, for some (possibly empty) words x and y.

Remark 1.1.3. We extensively use the subword notation $\ll$ in the thesis.

Example 1.1.4. The word $abaa$ is a subword of $w = aabbabac$, but not a factor. Whereas, the subword $bbab$ is a factor of w.

Notation 1.1.5. Given a word w over an alphabet A and a subset $B \subseteq A$, the subword of w restricted to the letters of B is denoted by $w|_B$. For example, consider the word $w = abbcbaccb$ over the alphabet $\{a, b, c\}$. Then $w|_{\{a,b\}} = abbbab$.

Definition 1.1.6. The *reverse of a word* w, denoted by w^R, is the word obtained by writing the letters of w in the reverse order, i.e., when $w = a_1a_2 \cdots a_k$ then $w^R = a_k \cdots a_2a_1$.

Definition 1.1.7 (k-uniform word)**.** Let k be a positive integer. A word w is said to be *k-uniform* if every letter of w occurs exactly k times in w. In particular, a 1-uniform word is called a *permutation*.

Example 1.1.8. The word $bacdbadcee$ is a 2-uniform word.

Notation 1.1.9. Suppose the letter a occurs k times in a word w. For $1 \leqslant i \leqslant k$, $a^{(i)}$ denotes the ith occurrence of the letter a in w from left to right. For example, $bacaabcb = b^{(1)}a^{(1)}c^{(1)}a^{(2)}a^{(3)}b^{(2)}c^{(2)}b^{(3)}$.

Notation 1.1.10. Suppose every letter of a word w occurs at least k times in w and $\{i_1, i_2, \ldots, i_m\}$ is a non-empty subset of $\{1, 2, \ldots, k\}$. The m-uniform subword of w consisting of the i_1th, i_2th, ..., i_mth occurrences of each letter of w is denoted by $w^{(\{i_1,i_2,\ldots,i_m\})}$. For instance, if $w = bacdbadceebcaa$ then $w^{(\{1,2\})} = bacdbadcee$.

Definition 1.1.11 (Cyclic shift)**.** Let $w = uv$ be a word, for some words u and v. The word vu is called a *cyclic shift* of w.

Example 1.1.12. The cyclic shifts of the word $bacdbadcee$ are $badceebacd$ and $ebacdbadce$.

Definition 1.1.13. Suppose a and b are two distinct letters appearing in a word w. We say that a and b alternate in w if $w|_{\{a,b\}}$ is of the form $abab\cdots$ or $baba\cdots$ (of even or odd length). Otherwise, the letters a and b are said to be non-alternating in the word w.

Example 1.1.14. Consider the word $w = bacdbadce$ over the alphabet $\{a, b, c, d, e\}$. The letters b and a alternate in w because $w|_{\{a,b\}} = baba$, while the letters c and d are non-alternating in w because $w|_{\{c,d\}} = cddc$.

Definition 1.1.15 (Concatenation)**.** Let u and v be two words over an alphabet A. The *concatenation* of u and v, denoted by uv, is the word obtained by appending v after the last letter of u. For example, if $u = abc$ and $v = ca$ then their concatenation is $uv = abcca$.

1.2 Graphs

Definition 1.2.1 (Graph)**.** A *graph* is a pair (V, E), where V is a finite set called the set of *vertices* and E is a finite multi-set (having repetition of elements) of unordered pairs of vertices, called the set of *edges*. For $a, b \in V$, an edge $\{a, b\} \in E$ is denoted by $\overline{ab}$. Every graph is depicted as per the following: each vertex of a graph is represented by a think dot and each edge is represented by a line/curve connecting the corresponding two dots.

Definition 1.2.2. A directed graph (in short, digraph) is a graph in which the edges are ordered pairs of vertices. For two vertices a, b, we write $\overrightarrow{ab}$ to denote a directed edge from a to b. In the depiction of digraph, the edges are represented along with the corresponding direction.

Example 1.2.3. The graph $G = (\{1, 2, 3, 4\}, \{\overline{12}, \overline{23}, \overline{34}, \overline{34}, \overline{22}, \overline{14}\})$ is depicted in Figure 1.1. The digraph $H = (\{a, b, c, d\}, \{\overrightarrow{ab}, \overrightarrow{ab}, \overrightarrow{ba}, \overrightarrow{bc}, \overrightarrow{cd}, \overrightarrow{da}, \overrightarrow{dd}\})$ is depicted in Figure 1.2.

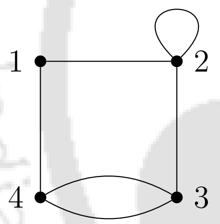


FIGURE 1.1: A graph G

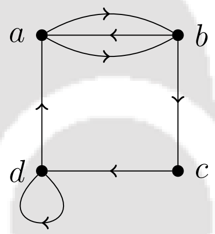


FIGURE 1.2: A digraph H

Definition 1.2.4 (Simple graph)**.** An edge in a graph of the form $\overline{aa}$ is called a *loop*. The edges that connect the same pair of vertices are called *multiple edges* or *parallel edges*. A graph without loops and without multiple edges is called a *simple graph*.

Throughout this thesis, unless stated otherwise, the term 'graph' refers to a simple and undirected graph.

Definition 1.2.5. Let $G = (V, E)$ be a graph and $a, b \in V$. If $\overline{ab} \in E$ then a and b are said to be *adjacent* in G. Otherwise, a and b are non-adjacent in G. The *neighborhood* of a, denoted by $N(a)$, is given by $N(a) = \{b \in V \mid \overline{ab} \in E\}$. The *degree* of a, denoted by $\deg(a)$, is the size of $N(a)$.

Definition 1.2.6 (Reduced graph)**.** A graph G is said to be *reduced* if no two vertices of G have the same neighborhood.

Definition 1.2.7 (Subgraph)**.** A graph $H = (V', E')$ is said to be *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A graph obtained from G by deleting some (possibly none) of its vertices (and of course, only those edges incident with these vertices) is called an *induced subgraph* of G. The induced subgraph of G on a vertex set $V' \subseteq V$ is denoted by $G[V']$.

Definition 1.2.8 (Paths and cycles)**.** A *path* in a graph is a sequence of distinct vertices $\langle a_1, a_2, \ldots, a_n \rangle$ such that there is an edge from a_i to a_{i+1}, for all $1 \leqslant i \leqslant n-1$. A path $\langle a_1, a_2, \ldots, a_n \rangle$ with an exception that $a_1 = a_n$ is called a *cycle*. The *length* of a path or cycle is the number of its edges. An n-vertex graph itself is a path, then it is denoted by P_n. Similarly, an n-vertex graph itself is a cycle, then it is denoted by C_n. In case of digraphs, aforementioned path or cycle is called a *directed path* or *directed cycle*.

Definition 1.2.9 (Clique and clique number)**.** In a graph $G = (V, E)$, a *clique* is a subset of V in which every pair of vertices are adjacent in G. The size of a maximum clique in G is called the *clique number* of G, denoted by $\omega(G)$. If $\omega(G) = |V|$ then G is called a *complete graph*, denoted by K_n, where $|V| = n$.

Definition 1.2.10 (Matching)**.** A *matching* in a graph $G = (V, E)$ is a subset of E in which no two edges share a common vertex. A matching is called a *perfect matching* if it covers all the vertices of G.

Definition 1.2.11 (Connected graph)**.** A graph is said to be *connected* if there is a path between every pair of vertices. Otherwise, it is a disconnected graph. A maximal connected subgraph of a graph G is called a *connected component* of G.

Definition 1.2.12. The *complement* of a graph $G = (V, E)$ is a graph $G' = (V, E')$ such that $\overline{ab} \in E'$ if and only if $\overline{ab} \notin E$, for all $a, b \in V$.

Definition 1.2.13 (Local complementation)**.** The *local complementation* of a graph G at vertex a is the graph obtained from G by replacing the induced subgraph $G[N(a)]$ with its complement (and the rest of the graph remains unchanged).

Definition 1.2.14 (Vertex-minor)**.** A *vertex-minor* of a graph G is the graph obtained by a sequence of vertex deletions and local complementations in G.

Definition 1.2.15. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that $\overline{ab} \in E_1$ if and only if $\overline{f(a)f(b)} \in E_2$, for all $a, b \in V_1$. Two isomorphic graphs have the same depiction (without labeling the vertices).

Here we present several important classes of graphs that appear frequently in this thesis.

Definition 1.2.16 (Edgeless graph)**.** A graph $G = (V, E)$ is called an *edgeless graph* if $E = \varnothing$. An n-vertex edgeless graph is denoted by O_n.

Definition 1.2.17 (Independent set)**.** In a graph $G = (V, E)$, an *independent set* is a subset of V in which every pair of vertices are non-adjacent.

Definition 1.2.18 (Transitive orientation)**.** An *orientation* of a graph is an assignment of direction to each edge so that the resulting graph is a directed graph. An orientation of a graph G is *transitive* if $\overrightarrow{ab}, \overrightarrow{bc}$ in G then $\overrightarrow{ac}$ in G, for all vertices a, b, c of G.

Definition 1.2.19 (Comparability graph)**.** A graph is said to be a *comparability graph* if it admits a transitive orientation.

Remark 1.2.20. Note that a comparability graph can admit multiple transitive orientations.

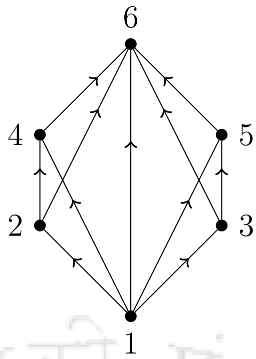


FIGURE 1.3: A comparability graph with a transitive orientation

Definition 1.2.21 (Permutation graph)**.** A graph G is called a *permutation graph* if each vertex of G can be associated with a line segment connecting two parallel lines such that two vertices are adjacent in G if and only if their associated line segments intersect. For example, a permutation graph on six vertices is given in Figure 1.4.

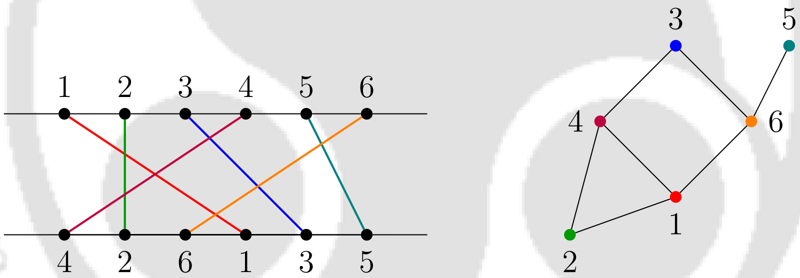


FIGURE 1.4: An example of a permutation graph

Dushnik and Miller [1941] characterized permutation graphs in terms of comparability graphs as per the following theorem.

Theorem 1.2.22 ([Dushnik and Miller, 1941])**.** *The class of permutation graphs is the intersection of the class of comparability graphs and the class of their complements.*

An important generalization of permutation graphs is the class of circle graphs.

Definition 1.2.23 (Circle graph)**.** A graph G is called a *circle graph* if each vertex of G can be associated with a chord of a circle such that two vertices are adjacent in

G if and only if their associated chords intersect. For example, a circle graph on five vertices is given in Figure 1.5.

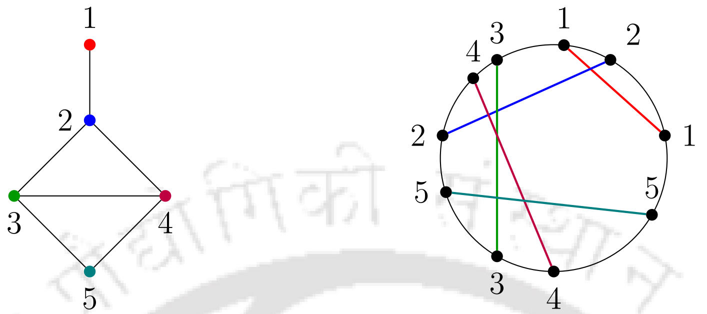


FIGURE 1.5: An example of a circle graph

Definition 1.2.24 (Bipartite graph)**.** A graph $G = (V, E)$ is a *bipartite graph* if its vertex set V can be partitioned into two disjoint subsets $V = A \cup B$ such that every edge in G connects a vertex from A to a vertex in B. If each vertex of A is adjacent to all vertices of B then the bipartite graph is said to be complete and it is denoted by $K_{m,n}$, where m and n are the sizes of the partite sets.

Definition 1.2.25 (Crown graph)**.** A *crown graph*, denoted by $H_{n,n}$, is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. Figure 1.6 depicts the crown graph on 8 vertices.

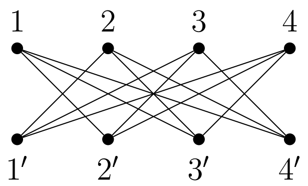


FIGURE 1.6: The crown graph $H_{4,4}$

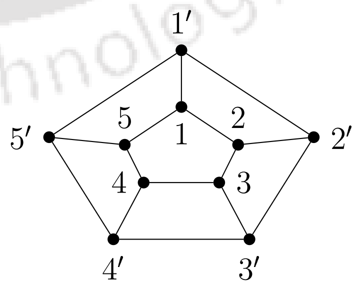


FIGURE 1.7: The prism Pr_5

Definition 1.2.26 (Prism)**.** For $n \geqslant 3$, a *prism* Pr_n is a graph consisting of two cycles $\{1, 2, \ldots, n\}$ and $\{1', 2', \ldots, n'\}$ connected by the edges $\overline{ii'}$, for $i = 1, 2, \ldots, n$. The prism Pr_5 is shown in Figure 1.7.

Definition 1.2.27 (Interval graph)**.** A graph $G = (V, E)$ is called an *interval graph* if each vertex $a \in V$ can be associated with an interval on the real line such that two vertices are adjacent in G if and only if their associated intervals intersect. We say that these intervals form an interval representation of the graph G. Figure 1.8 depicts an interval graph on five vertices along with an interval representation.

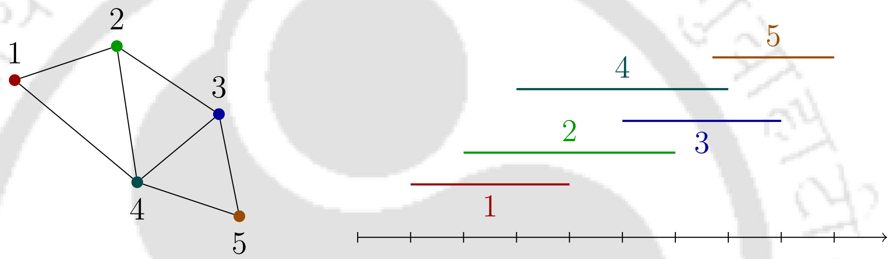


Figure 1.8: An interval graph (left) and an interval representation (right) for it

Definition 1.2.28 (Chordal graph)**.** A graph is called a *chordal graph*, also known as a *triangulated graph*, if every cycle of length at least four has a chord – an edge connecting two non-adjacent vertices of the cycle.

Definition 1.2.29 (Split graph)**.** A graph is called a *split graph* if and only if its vertex set can be partitioned into a clique and an independent set.

Definition 1.2.30 (Planar graphs)**.** A graph G is called a *planar graph* if it can be drawn on the plane such that no two edges in G cross each other.

Definition 1.2.31 (Outerplanar)**.** A planar graph is called *outerplanar* if all its vertices lie on the unbounded face of a planar drawing.

Definition 1.2.32 (Melon graph)**.** A *melon graph* is a collection of vertex-disjoint paths with two common endpoints. We call these paths as the constituent paths of the melon graph. Following the notation given in [Dissaux et al., 2023], a melon graph M is denoted by $M = (E_1, E_2, \ldots, E_m)$, where E_i's $(1 \leqslant i \leqslant m)$ are constituent paths with common endpoints 0 and $0'$. Since the graphs that we are considering are simple, there will be at most one constituent path of length one in a melon graph. Figure 1.9 is an example of a melon graph.

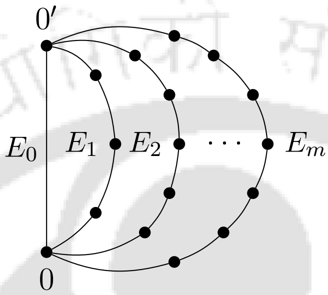


FIGURE 1.9: Melon graph

Remark 1.2.33. Melon graphs are also known as generalized theta graphs (cf. [Liu and Yang, 2023]). Note that if all the constituent paths of a melon graph are of same length then the melon graph is called a theta graph (cf. [Verstraëte and Williford, 2019]).

Definition 1.2.34 (k-colorable graph)**.** A graph G is *k-colorable* if the vertices of G can be colored with at most k colors such that no two adjacent vertices share the same color. The minimum value of k in which the graph G is k-colorable is called the *chromatic number* of the graph, denoted by $\chi(G)$.

Definition 1.2.35 (Perfect graph)**.** A graph G is called a *perfect graph* if $\chi(H) = \omega(H)$, for all induced graph H of G.

Definition 1.2.36 (Line graph)**.** The *line graph* of a graph $G = (V, E)$, denoted by $L(G)$, is a graph with E as the vertex set and two vertices in $L(G)$ are adjacent if and only if their corresponding edges share a vertex in G. For $k \geqslant 1$, we write $L^{k+1}(G) = L(L^k(G))$, where $L^1(G) = L(G)$.

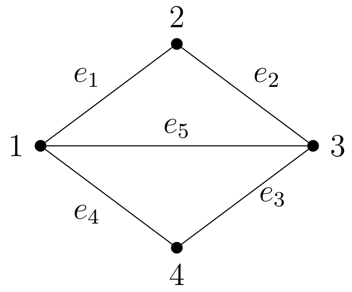


FIGURE 1.10: A graph H

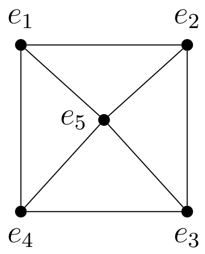


FIGURE 1.11: Line graph $L(H)$

Definition 1.2.37 (Shortcut)**.** A directed graph with the vertices $\{a_1, a_2, \ldots, a_k\}$ is called a *shortcut* if it contains a directed path, say $\langle a_1, a_2, \ldots, a_k \rangle$, and the edge $\overrightarrow{a_1 a_k}$, called the shortcutting edge, such that there is no edge $\overrightarrow{a_i a_j}$, for some i, j with $1 \leqslant i < j \leqslant k$. In particular, a shortcut must have at least four vertices, i.e., $k \geqslant 4$.

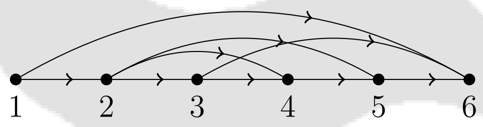


FIGURE 1.12: An example of a shortcut

Definition 1.2.38 (Semi-transitive orientation)**.** An orientation of a graph is said to be *semi-transitive* if the directed graph has no directed cycles (acyclic) and no shortcuts. Figure 1.13 is an example of a semi-transitively oriented graph.

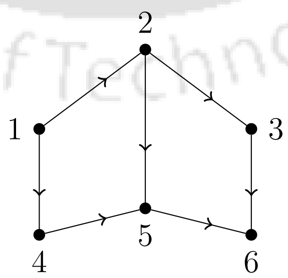


FIGURE 1.13: A graph with a semi-transitive orientation

Remark 1.2.39. Note that every transitive orientation is a semi-transitive orientation. However, the converse is not necessarily true.

The focus of this thesis is on word-representable graphs. This class of graphs belongs to a wide family of graph classes known as hereditary. In the following, we review the notion of hereditary and present some fundamental results related to this notion.

Definition 1.2.40 (Hereditary class)**.** A class $\mathcal{C}$ of graphs is *hereditary* if it contains all induced subgraphs of each graph in $\mathcal{C}$. In otherwords, $\mathcal{C}$ is closed under taking induced subgraphs.

Remark 1.2.41. Comparability graphs, permutation graphs, circle graphs, and interval graphs are hereditary classes.

One important property of hereditary classes is that they admit forbidden induced subgraph characterization.

Definition 1.2.42. Let $\mathcal{H}$ be a set of graphs. The class of graphs that do not contain any graph from $\mathcal{H}$ as an induced subgraph is called $\mathcal{H}$-free graphs, denoted by $\text{Free}(\mathcal{H})$. In this case, the graphs in $\mathcal{H}$ are called the *forbidden induced subgraphs* for the class $\text{Free}(\mathcal{H})$. If $\mathcal{H} = \{G\}$ then $\text{Free}(\mathcal{H})$ is simply denoted by $\text{Free}(G)$.

Theorem 1.2.43 ([Kitaev and Lozin, 2015])**.** *A class $\mathcal{C}$ of graphs is hereditary if and only if there is a set $\mathcal{H}$ of graphs such that $\mathcal{C} = \text{Free}(\mathcal{H})$.*

Example 1.2.44. The class of complete graphs is hereditary and can be characterized as $\text{Free}(O_2)$. Thus, O_2 is the forbidden induced subgraph for the class of complete graphs.

Note that there are several classes of graphs that are closed under induced subgraphs are also closed under subgraphs, vertex-minors, etc. In such cases, these classes can be characterized in terms of forbidden subgraphs, vertex-minors, and other related structures.

The class of circle graphs is hereditary as deleting a vertex from a circle graph is equivalent to deleting a chord from a circle. Thus, there exists a characterization for the class of circle graphs in terms of forbidden induced subgraphs. Unfortunately, the complete list of forbidden induced subgraphs is not known. Nonetheless, the class of circle graphs is closed under local complementation and hence are closed under taking vertex minors. Bouchet [1994] characterized the class of circle graphs by three forbidden vertex minors.

Theorem 1.2.45 ([Bouchet, 1994])**.** *A graph is a circle graph if and only if it does not contain the three graphs shown in Figure 1.14 as vertex minors.*

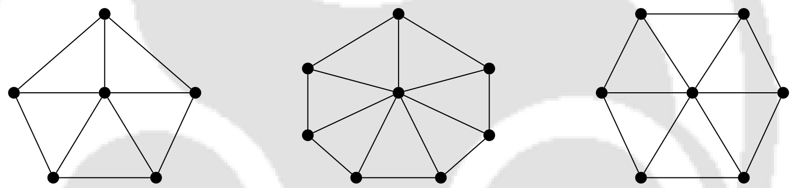


Figure 1.14: Forbidden vertex minors for the class of circle graphs

1.3 Posets

Definition 1.3.1 (Poset)**.** A *partially ordered set* (in short, a *poset*) is a pair $(P, \prec)$ consisting of a set P and a partial order $\prec$ on P. We always refer a poset by its underlying set (rather than a pair).

Definition 1.3.2. For two elements a, b of a poset P, if $a \prec b$ or $b \prec a$ then a and b are said to be *comparable*. Otherwise, they are *incomparable*.

Definition 1.3.3 (Total order, chain and height)**.** A partial order under which every pair of elements is comparable is called a *total order* or *linear order*. The totally ordered set is called a *chain*. The size of a maximal chain in a poset P is called the *height* of the P, denoted by $h(P)$.

Definition 1.3.4 (Chain cover)**.** A *chain cover* of a poset P is a collection of disjoint chains of P such that their union is the poset P.

Definition 1.3.5 (Antichain and width)**.** A set of elements of a poset P in which every pair of elements are incomparable is called an *antichain* of P. The size of a maximal antichain of P is called the *width* of P, denoted by $w(P)$.

The following theorem by Dilworth [1950] established a relationship between a chain cover and the width of a poset.

Theorem 1.3.6 (Dilworth's theorem)**.** *For a finite poset, the size of the smallest chain cover is equal to the width of the poset.*

Definition 1.3.7 (Cover graph)**.** The *cover graph* of a poset P, denoted by G_P, is a graph whose vertex set is the set P and two vertices a and b are adjacent in G_P if and only if either a covers b or b covers a in P; here, we say a covers b if $b \prec a$ and there is no element c in P such that $b \prec c \prec a$.

Definition 1.3.8 (Hasse diagram)**.** The *Hasse diagram* of a poset P is a depiction of the cover graph G_P such that an element b is placed above another element a if a covers b in P.

Definition 1.3.9 (Realizer)**.** A *realizer* of a poset $(P, \prec)$ is a set of linear orders on P, say $\{\prec_1, \prec_2, \ldots, \prec_k\}$, such that $\prec$ equals $\bigcap_{i=1}^{k} \prec_i$, i.e., for all $a, b \in P$, $a \prec b$ if and only if $a \prec_i b$, for all $1 \leqslant i \leqslant k$.

Definition 1.3.10 ([Dushnik and Miller, 1941])**.** The *dimension* of a poset P is the least positive integer k such that P has a realizer of size k, and is denoted by $\dim(P)$.

The following are some of the results on the dimension of posets.

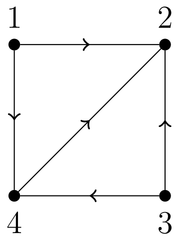


FIGURE 1.15: A comparability graph D

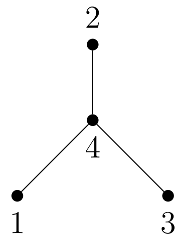


FIGURE 1.16: Hasse diagram of P_D

Theorem 1.3.11 ([Hiraguti, 1955]). *Let P be a poset with $n \geqslant 4$ elements. Then* $\dim(P) \leqslant \frac{n}{2}$.

Theorem 1.3.12 ([Dilworth, 1950]). *Let P be a poset. Then* $\dim(P) \leqslant w(P)$.

Considering the results in [Kimble, 1973; Trotter, 1975], we state the following:

Theorem 1.3.13 ([Dilworth, 1950; Kimble, 1973; Trotter, 1975]). *Let P be a poset with n elements. Then,* $\dim(P) \leqslant \min\{w(P), n - w(P)\}$.

Remark 1.3.14. Every comparability graph $G = (V, E)$ induces a partial order $\prec$ on V, i.e., $\overrightarrow{ab} \in E$ if and only if $a \prec b$.

Example 1.3.15. A comparability graph on four vertices along with a transitive orientation is given in Figure 1.15. The poset induced by the orientation is depicted in Figure 1.16.

The following theorem by Trotter et al. [1976] shows that posets with the same underlying comparability graph have equal dimension.

Theorem 1.3.16 ([Trotter et al., 1976]). *Let P and Q be posets that have the same underlying comparability graph. Then* $\dim(P) = \dim(Q)$.

Based on the underlying comparability graph of posets, Baker et al. [1972] provided a characterization of posets with dimension at most two.

Theorem 1.3.17 ([Baker et al., 1972]). *A graph is a permutation graph if and only if it is the underlying comparability graph of a poset that has dimension at most two.*

However, determining the dimension of a given poset remains computationally hard, as shown by the following theorem.

Theorem 1.3.18 ([Yannakakis, 1982]). *For any $k \geqslant 3$, it is NP-complete to determine if the dimension of a poset is at most k.*

Even when restricted to height-2 posets (also called bipartite posets), determining the dimension remains computationally hard. By combining the complexity result established in [Yannakakis, 1982] and [Felsner et al., 2017], we have the following theorem.

Theorem 1.3.19 ([Felsner et al., 2017; Yannakakis, 1982]). *For any $k \geqslant 3$, it is NP-complete to determine if the dimension of a height-2 poset is at most k.*

Definition 1.3.20 (Kimble split). The *split of a poset* $(P, \prec_P)$, denoted by $\text{split}(P)$, is a height-2 poset $(Q, \prec_Q)$ such that $Q = P \cup P'$ where $P' = \{a' \mid a \in P\}$ and for any $a, b \in Q$, $a \prec_Q b$ if

- $a \in P$ and $b \in P'$, and
- $a \prec_P c$, where $b = c'$, for some $c \in P$.

We have the following results on the dimension of specific posets established in the literature.

Theorem 1.3.21 ([Trotter, 1974]). *If the underlying comparability graph of a poset P is a crown graph $H_{n,n}$ then* $\dim(P) \leqslant n$.

Theorem 1.3.22 ([Trotter and Moore, 1977]). *If the cover graph of a poset P is a tree then* $\dim(P) \leqslant 3$.

Theorem 1.3.23 ([Ma and Spinrad, 1991]). *If the underlying comparability graph of a poset P is chordal then* $\dim(P) \leqslant 4$.

Theorem 1.3.24 ([Guenver and Rampon, 2004]). *If the underlying comparability graph of a poset P is a split graph then* $\dim(P) \leqslant 3$.

Theorem 1.3.25 ([Felsner et al., 2015]). *If the underlying comparability graph of a poset P is outerplanar and $h(P) = 2$ then* $\dim(P) \leqslant 3$.

Theorem 1.3.26 ([Felsner et al., 2015]). *If the underlying comparability graph of a poset P is planar then* $\dim(P) \leqslant 4$.

An interesting result on the Kimble split of a poset with respect to the poset dimension is as follows:

Theorem 1.3.27 ([Kimble, 1973]). *Let P be a poset. Then*

$$\dim(P) \leqslant \dim(\mathrm{split}(P)) \leqslant 1 + \dim(P).$$

1.4 Word-Representable Graphs

Definition 1.4.1 (Word-representable graph). A simple graph $G = (V, E)$ is *word-representable* if there exists a word w over V such that, for all $a, b \in V$, $\overline{ab} \in E$ if and only if a and b alternate in the word w. In this case, we say that the word w represents the graph G, and w is called a *word-representant* for G.

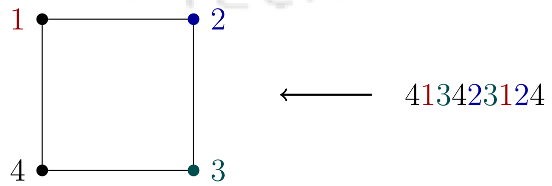


FIGURE 1.17: An example of a word-representable graph

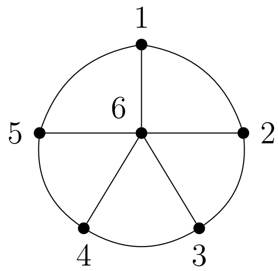


FIGURE 1.18: Wheel graph W_5

Remark 1.4.2. Not all graphs are word-representable. The wheel graph W_5 (see Figure 1.18) is the smallest non-word-representable graph [Kitaev and Pyatkin, 2008].

Proposition 1.4.3 ([Kitaev and Lozin, 2015])**.** *The class of word-representable graphs is hereditary.*

Proposition 1.4.4 ([Kitaev and Lozin, 2015])**.** *If w is a word-representant for a graph G then w^R is also a word-representant for G.*

Proposition 1.4.5 ([Kitaev and Pyatkin, 2008])**.** *Let $w = w_1aw_2aw_3$ be a word-representant for a graph $G = (V, E)$, for some (possibly empty) words w_1, w_2 and w_3, and w_2 does not contain the vertex a. If X is the set of all vertices that appear exactly once in w_2 then the possible candidates for a to be adjacent to in G are the vertices in X.*

Definition 1.4.6 (k-word-representable graph)**.** A graph G is *k-word-representable* if there exists a k-uniform word-representant for G.

Theorem 1.4.7 ([Kitaev and Pyatkin, 2008])**.** *A graph G is word-representable if and only if G is k-word-representable, for some k.*

Proposition 1.4.8 ([Kitaev and Pyatkin, 2008])**.** *If w is a k-uniform word-representant for a word-representable graph G then any cyclic shift of w also represents G.*

Remark 1.4.9. The uniformity of the word-representant in Theorem 1.4.8 is necessary. For example, the word 213243 represents the graph given in Figure 1.17. However, the word 432132 obtained by cyclic shift of 213243 does not represent the graph.

Proposition 1.4.10 ([Kitaev and Pyatkin, 2008])**.** *A k-word-representable graph G is also $(k+1)$-word-representable. Hence, each word-representable graph has infinitely many word-representants.*

Lemma 1.4.11 ([Broere and Zantema, 2019])**.** *Let w be a k-uniform word-representant for G. For $m > 1$, let $A_1, A_2, \ldots, A_m$ be non-empty subsets of $\{1, 2, \ldots, k\}$ such that, for all $j \in \{1, 2, \ldots, k-1\}$, there exists $i \in \{1, 2, \ldots, m\}$ where $\{j, j+1\} \subseteq A_i$. Then the word $w^{(A_1)}w^{(A_2)} \ldots w^{(A_m)}$ represents the graph G.*

Theorem 1.4.12 (Kitaev and Lozin [2015])**.** *A graph G is word-representable if and only if every connected component of G is word-representable.*

The following theorem, given in [Kitaev and Lozin, 2015], establishes the computational complexity of the recognition problem for word-representable graphs.

Theorem 1.4.13 ([Kitaev and Lozin, 2015])**.** *It is an NP-complete problem to recognize whether a given graph is word-representable.*

Word-representable graphs are also studied in terms of semi-transitive orientations, which is very useful in understanding word-representability of graphs.

Theorem 1.4.14 ([Halldórsson et al., 2016])**.** *A graph G is word-representable if and only if it admits a semi-transitive orientation.*

Theorem 1.4.15 ([Kitaev and Sun, 2024])**.** *Let $G = (V, E)$ be a word-representable graph and $a \in V$. Then there exists a semi-transitive orientation of G such that a is a source[1] (or a sink[2]).*

[1]A vertex in which all the edges are directed towards its adjacent vertices.
[2]A vertex in which all the edges are directed to it from its adjacent vertices.

The following results are the classes of graphs which were shown to be word-representable in the literature.

Theorem 1.4.16 ([Halldórsson et al., 2011]). *All 3-colorable graphs are word-representable.*

Theorem 1.4.17 ([Kitaev and Lozin, 2015]). *The class of triangle-free word-representable graphs is exactly the class of cover graphs of posets.*

Remark 1.4.18. If G is semi-transitively oriented and triangle-free then it is the Hasse diagram of some poset.

Theorem 1.4.19 ([Kitaev and Lozin, 2015]). *All comparability graphs are word-representable.*

1.4.1 Permutationally Representable Graphs

Definition 1.4.20 (Permutationally representable graph). A graph $G = (V, E)$ is said to be *permutationally representable* if G has a word-representant of the form $p_1p_2 \cdots p_k$, where each p_i is a permutation on V. In this case, by considering the number of permutations, the graph is called a *permutationally k-representable graph.*

Remark 1.4.21. Since each vertex of G appears exactly once in each permutation, and exhibits the alternating property, any combination of these permutations $p_1, p_2, \ldots, p_k$ also represents the graph G.

The following theorem gives a characterization for the class of permutationally representable graphs.

Theorem 1.4.22 ([Kitaev and Seif, 2008]). *A graph is permutationally representable if and only if it is a comparability graph.*

McConnell and Spinrad [1997] showed that recognizing whether a graph is a comparability graph can be done in linear time. Thus, recognizing whether a graph

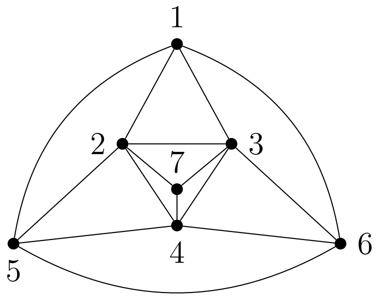


Figure 1.19: co-T_2

is permutationally representable can be done in linear time. Furthermore, since the class of comparability graphs is hereditary, Gallai [1967] provided a complete characterization of this class in terms of forbidden induced subgraphs.

Note that every bipartite graph is a comparability graph, the following is an immediate corollary of Theorem 1.4.22.

Corollary 1.4.23. *Bipartite graphs are permutationally representable.*

Theorem 1.4.24 ([Kitaev and Pyatkin, 2008])**.** *Let $G = (V, E)$ be a graph on n vertices and $a \in V$ be an all-adjacent vertex[3]. Then G is word-representable if and only if $G[V \backslash \{a\}]$ is permutationally representable.*

The following theorem provides a necessary condition for a graph to be word-representable.

Theorem 1.4.25 ([Kitaev and Pyatkin, 2008])**.** *In a word-representable graph, the neighborhood of each vertex is a comparability graph.*

A natural question is to ask whether the converse of Theorem 1.4.25 holds. The answer to this question is a no. A counterexample to this question was given in [Halldórsson et al., 2010], and it is called the co-T_2 graph (see Figure 1.19).

Theorem 1.4.26 ([Halldórsson et al., 2010])**.** *The graph co-(T_2) is not word-representable.*

[3]If $\deg(a) = n - 1$, the vertex a is called an all-adjacent vertex.

1.4.2 Representation Number

In view of Theorem 1.4.7, we have the following notion associated to word-representable graphs.

Definition 1.4.27 (Representation number)**.** The minimal k such that a graph G is k-word-representable is called the *representation number* of G, and it is denoted by $\mathcal{R}(G)$.

Remark 1.4.28. Suppose w is a word-representant for a graph $G = (V, E)$ and $V_1 \subseteq V$. Then $w|_{V_1}$ is a word-representant for the graph $G[V_1]$. Thus, for an induced subgraph G' of G, we have $\mathcal{R}(G') \leqslant \mathcal{R}(G)$.

Proposition 1.4.29 ([Kitaev et al., 2011a,b])**.** *A k-uniform word representing a complete graph K_n is a word of the form u^k, where u is a permutation on K_n's vertices.*

From Proposition 1.4.29, it is evident that the class graphs with representation number one is precisely the class of complete graphs. We have the following characterization for the class graphs with representation number two.

Theorem 1.4.30 ([Halldórsson et al., 2011])**.** *For a graph G different from a complete graph, $\mathcal{R}(G) = 2$ if and only if G is a circle graph.*

For $k \geqslant 3$, the class of graphs with representation number k is characterized through the notion of k-circle representation.

Definition 1.4.31 ([Fleischmann et al., 2024])**.** For $k \geqslant 1$, a *k-circle representation* of a graph G is a drawing in which each vertices of G is associated with a k-gon (a polygon with k sides) inscribed in a common circle. Two vertex of G are adjacent if and only if the sides of their corresponding k-gons intersect exactly $2k$ times. Figure 1.20 depicts an example of a graph and a 3-circle-representation of it.

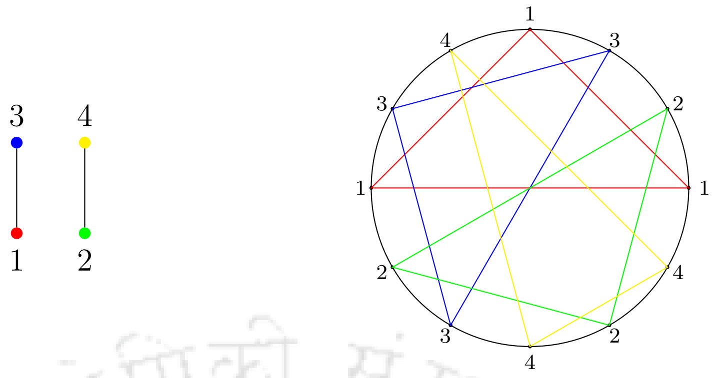


Figure 1.20: A graph and a 3-circle representation for it.

Theorem 1.4.32 ([Fleischmann et al., 2024])**.** *For $k \geqslant 3$, a graph G is k-word-representable if and only if there exists a k-circle representation of G.*

However, by the following theorem given in [Halldórsson et al., 2011], we see that determining the representation number of a word-representable graph is NP-complete, for any fixed $k \geqslant 3$.

Theorem 1.4.33 ([Halldórsson et al., 2011])**.** *Deciding whether a given graph is k-word-representable for any fixed k, $3 \leqslant k \leqslant \lceil \frac{n}{2} \rceil$, is NP-complete.*

Remark 1.4.34. Suppose $G = (V, E)$ is a disconnected graph and $G[V_1], \ldots, G[V_t]$ be the connected components of G where V is the disjoint union of $V_1, \ldots, V_t$. By Theorem 1.4.12, we observe that G is word-representable if and only if each $G[V_i]$ is word-representable, for $1 \leqslant i \leqslant t$. Thus, by Theorem 1.4.7 and Proposition 1.4.10, we observe that $\mathcal{R}(G) = \max\{2, \mathcal{R}(G[V_1]), \ldots, \mathcal{R}(G[V_t])\}$.

The following are some of the results on the representation number of word-representable graph available in the literature.

Theorem 1.4.35 ([Kitaev, 2013; Kitaev and Pyatkin, 2008])**.** *For $n \geqslant 3$, the representation number of the prism Pr_n is three.*

Theorem 1.4.36 ([Glen et al., 2018])**.** *For $n \geqslant 5$, the representation number of the crown graph $H_{n,n}$ is $\lceil \frac{n}{2} \rceil$.*

Theorem 1.4.37 ([Halldórsson et al., 2011])**.** *Let G_n be the graph obtained from a crown graph $H_{n,n}$ by adding an all-adjacent vertex. The representation number of G_n is n.*

The following are the conjectures related to the representation number of bipartite graphs.

Conjecture 1.4.38 ([Glen et al., 2018])**.** Each bipartite graph on $n \geqslant 9$ vertices has representation number at most $\lceil \frac{n}{4} \rceil$.

Conjecture 1.4.39 ([Akgün et al., 2019])**.** The crown graph $H_{n,n}$ has the highest representation number among all bipartite graphs on $2n$ vertices.

1.4.3 Permutation-Representation Number

Definition 1.4.40 (Permutation-representation number)**.** The minimal k such that a graph G is permutationally k-representable is called the *permutation-representation number* (in short, *prn*) of G, and it is denoted by $\mathcal{R}^p(G)$.

Remark 1.4.41. It is evident that $\mathcal{R}(G) \leqslant \mathcal{R}^p(G)$, for a comparability graph G.

Remark 1.4.42. Suppose w is a word that represents the graph $G = (V, E)$ permutationally, and $V_1 \subseteq V$. Then $w|_{V_1}$ is a word that represents $G[V_1]$ permutationally . Thus, for an induced subgraph G' of G, we have $\mathcal{R}^p(G') \leqslant \mathcal{R}^p(G)$.

In the following, we reconcile the results that reveal the connection between the *prn* of a comparability graph and the dimension of its induced poset, and present explicitly that they are the same. Accordingly, we present the upper bounds for the *prn* of comparability graphs, particularly for bipartite graphs.

The notion of the permutation-representation number of a comparability graph was first coined in [Broere, 2018], and remarked that the *prn* of a comparability graph coincides with the dimension of the poset induced by the graph. While the remark is evident from the original result in [Halldórsson et al., 2011], in the following we recall the result in terms of *prn* and explicitly present the relation between the notions of *prn* and the dimension.

Lemma 1.4.43 ([Halldórsson et al., 2011]). *If G is a comparability graph then $\mathcal{R}^p(G) \leqslant k$ if and only if $\dim(P_G) \leqslant k$.*

In fact, Halldórsson et al. [2011] showed that every realizer of P_G is precisely a set of permutations of vertices of G whose concatenation represents G permutationally. Hence, considering such a set of minimal size on either side, we can deduce the following corollary.

Corollary 1.4.44. *If G is a comparability graph then $\mathcal{R}^p(G) = k$ if and only if $\dim(P_G) = k$.*

Consistently, we use the term dimension when we refer to a poset, and the *prn* (in symbols $\mathcal{R}^p(G)$) when we refer to a comparability graph, while these concepts give the same value.

In view of Corollary 1.4.44, we review the results known for the dimension of posets.

McConnell and Spinrad [1997] showed that the posets of dimension at most two can be recognized in linear time. Further, Baker et al. [1972] (cf. Theorem 1.3.17) characterized these posets in terms of their underlying comparability graphs, viz., permutation graphs. Thus, permutation graphs are the graphs with *prn* at most two. Further, by Theorem 1.3.18 and Corollary 1.4.44, we can conclude that, for a given k, determining whether a comparability graph has *prn* k is NP-complete.

Note that bipartite graphs are precisely the underlying comparability graph of height-2 posets (cf. [Adiga et al., 2011; Felsner et al., 2017]). Thus, by Theorem 1.3.19 and Corollary 1.4.44, we can also conclude the following theorem.

Theorem 1.4.45. *For $k \geqslant 3$, its is NP-complete to determine if a bipartite graph has prn k.*

Further, by Theorem 1.3.13, we deduce the following corollary.

Corollary 1.4.46. *For a bipartite graph $G = (A \cup B, E)$, $\mathcal{R}^p(G) \leqslant \min\{|A|, |B|\}$.*

An upper bound on the *prn* of bipartite graphs by Broere [2018], is stated below.

Theorem 1.4.47 ([Broere, 2018])**.** *For a bipartite graph $G = (A \cup B, E)$, we have*

$$\mathcal{R}^p(G) \leqslant \min\{\alpha, \beta\},$$

where $\alpha = |\{N(a)|a \in A\}|$ and $\beta = |\{N(b)|b \in B\}|$.

1.4.4 Graph Operations

We now present few graph operations that are used in various parts of the thesis.

Definition 1.4.48 (Subdivision of an edge)**.** In a graph $G = (V, E)$, a *subdivision of an edge* $\overline{ab} \in E$ is the operation of replacing the edge $\overline{ab}$ with a path connecting a and b.

Definition 1.4.49 (k-subdivision)**.** A *subdivision of a graph* G is a graph obtained from G by replacing each edge $\overline{ab}$ in G with a path connecting a and b. A subdivision is called a *k-subdivision* if the paths replacing each edge of G is of length at least k.

Suppose G is a word-representable graph and ϕ is a semi-transitive orientation of G. If $\overrightarrow{ab} \in \phi$, replace $\overrightarrow{ab}$ with the path $\{a, a_1, \ldots, a_t, b\}$ and chose the orientation in this path to be transitive. The orientation obtained for the resultant graph is semi-transitive. Particularly, for 3-word-representable graphs we have the following lemma.

Lemma 1.4.50 ([Kitaev and Pyatkin, 2008])**.** *Let $G = (V, E)$ be a 3-word-representable graph and $a, b \in V$. The graph H obtained from G by adding a path of length at least three connecting a and b is 3-word-representable.*

The next theorem is an immediate consequence of Lemma 1.4.50.

Theorem 1.4.51 ([Kitaev and Pyatkin, 2008]). *For a graph G there exists a 3-word-representable graph H that contains G as a minor. In particular, a 3-subdivision of every graph is 3-word-representable.*

Definition 1.4.52 (Module). A subset B of the vertex set V of a graph G is called a *module* if each vertex in B have the same neighborhood in the set $V \backslash B$.

Theorem 1.4.53 ([Hefty et al., 2024; Kitaev, 2013]). *Suppose that $G = (V, E)$ is a word-representable graph and $a \in V$. Let G' be the graph obtained from G by replacing the vertex a with a module M, where M is a comparability graph. Then G' is also word-representable. Moreover, if $\mathcal{R}(G) = k$ and $\mathcal{R}^p(M) = l$ then $\mathcal{R}(G') = \max\{k, l\}$.*

Remark 1.4.54. Note that an edgeless graph O_n is a comparability graph and $\mathcal{R}^p(O_n) = 2$. From Theorem 1.4.53, we observe that if we replace a vertex of a word-representable graph G by a module O_n then $\mathcal{R}(G') = \max\{\mathcal{R}(G),\ 2\}$.

Remark 1.4.55. Suppose G is an edgeless graph and we replace a vertex of G by a module M, which is connected. Then $\mathcal{R}(G') = \max\{2,\ \mathcal{R}^p(M)\}$. Clearly, if we extend this to *prn*, we have $\mathcal{R}^p(G') = \max\{2,\ \mathcal{R}^p(M)\}$. Therefore, *prn* of a disconnected graph is the maximum *prn* among its connected components.

From the above-mentioned remark, it is sufficient to focus on the (permutation-) representation number of connected graphs. Unless stated otherwise, all graphs considered in the rest of this thesis are considered to be connected.

Definition 1.4.56 (Cartesian product of graphs). The *Cartesian product of two graph* $G = (V, E)$ and $H = (V, E)$, denoted by $G \square H$, is a graph with the vertex $V \times V'$ and any two vertices (u, u') and (v, v') are adjacent in $G \square H$ if and only if

- $u = v$ and u' and v' are adjacent in H, or

- $u' = v'$ and u and v are adjacent in G.

Definition 1.4.57 (k-cube)**.** A *k-cube*, denoted by Q_k, is the Cartesian product of K_2 with itself k times, i.e., $Q_k = \underbrace{K_2 \Box K_2 \Box \cdots \Box K_2}_{k \text{ times}}$.

The following result shows that the Cartesian product of two graphs preserves word-representability.

Theorem 1.4.58 ([Kitaev and Lozin, 2015])**.** *Let G and H be two word-representable graphs. Then the Cartesian product $G \Box H$ is also word-representable.*

However, if both the graphs G and H are permutationally representable then $G \Box H$ need not be permutationally representable. A counterexample is the prism $Pr_3 = K_3 \Box K_2$, the Cartesian product of complete graphs [Gallai, 1967]. In case, if both the graphs are bipartite then their Cartesian product is a permutationally representable graph. In fact, we have the following result.

Proposition 1.4.59 ([White, 1970])**.** *The Cartesian product of two bipartite graphs is bipartite.*

While the representation number of the Cartesian product of two word-representable graphs is not known, the following theorem provides an upper bound on the representation number.

Theorem 1.4.60 ([Srinivasan and Hariharasubramanian, 2024])**.** *Let G and H be two word-representable graphs with m and n vertices, respectively. If $\mathcal{R}(G) = k$ and $\mathcal{R}(H) = l$. Then*

$$\mathcal{R}(G \Box H) \leqslant (k + l + \min\{m, n\}).$$

In particular, the representation number of $K_n \Box K_2$, for $n \geqslant 1$, is established in the theorem below.

Theorem 1.4.61 ([Broere and Zantema, 2019])**.** *The Cartesian product $K_n \Box K_2$ has representation number n, for $n = 1, 2, 3$, and representation number 3, for all $n > 3$.*

Theorem 1.4.62 ([Kitaev, 2013]). *For $n \geqslant 2$, the Cartesian product $L_n = P_n \square P_2$, known as ladder graph, has representation number 2.*

Theorem 1.4.63 ([Hefty et al., 2024]). *The representation number of k-cube is $O\left(\frac{\log k}{\log \log k}\right)$.*

Part I

Bipartite Graphs

2 Permutation-Representation Number

In this chapter, we study the permutation-representation number (*prn*) of bipartite graphs. First, we observe the properties of permutations that participate in a word representing a given bipartite graph G. We then introduce a specific class of directed graphs called neighborhood inclusion graphs and use them to develop a polynomial-time procedure for constructing a sequence of permutations that, when concatenated, represent the graph G. As a consequence, we obtain an upper bound on the *prn* of bipartite graphs. Notably, this bound is expressed in terms of the width of a poset associated with a neighborhood inclusion graph, which is at most half the number of vertices in G. In view of Theorem 1.4.47, we focus only on the reduced bipartite graphs and estimate their *prn*.

In this chapter, G denotes a reduced bipartite graph $(A \cup B, E)$ unless stated otherwise.

2.1 General Properties of Permutations

In this section, we prove some general properties of permutations used in representing a bipartite graph. These properties will play a vital role in constructing the respective permutations.

Suppose a bipartite graph G is represented by $p_1 \cdots p_k$, where p_i $(1 \leqslant i \leqslant k)$ is a permutation of the vertices of G. First, we show that a vertex adjacent to two other vertices will not appear in between them in any permutation p_i.

Lemma 2.1.1. *If $\overline{ab}, \overline{cb} \in E$ then $abc \not\ll p_i$, for all $1 \leqslant i \leqslant k$.*

Proof. Suppose $abc \ll p_t$, for some t. By the choice of a and c, they are non-adjacent in the bipartite graph G. Hence, as $ac \ll p_t$, there exists a permutation p_s such that $ca \ll p_s$. Further, since $\overline{ab} \in E$ and $ab \ll p_t$, we have $ab \ll p_i$, for all $1 \leqslant i \leqslant k$. Consequently, $cb \ll p_s$. This is not possible, as $\overline{cb} \in E$ and $bc \ll p_t$. Hence, $abc \not\ll p_i$, for all $1 \leqslant i \leqslant k$. □

As a consequence of Lemma 2.1.1, we observe that if any vertex adjacent to a vertex a appears on the right side (or left side) of a in a permutation p_i then every other vertex adjacent to a also appears on the right side (respectively, left side) of a in the permutation p_i and hence in all permutations due to the alternating property.

Lemma 2.1.2. *For $\overline{ab} \in E$ and $1 \leqslant i \leqslant k$, if $ab \ll p_i$ then $ab' \ll p_j$, for all $b' \in N(a)$ and $1 \leqslant j \leqslant k$.*

Proof. For $b' \in N(a)$, suppose $b'a \ll p_i$. Then $b'ab \ll p_i$. But by Lemma 2.1.1, this is not possible. Hence, $ab' \ll p_i$. Consequently, as a and b' are adjacent, $ab' \ll p_j$ for all j. □

We now prove a result on appearance of neighbors of two vertices, say a and c, in a permutation, when they have common neighbors. If the neighbors of a appear on the right side (or left side) of a in a permutation p_i then the neighbors of c also appear on the right side (respectively, left side) of c in the permutation p_i.

Lemma 2.1.3. *Let a, c be two vertices of G such that $N(a) \cap N(c) \neq \varnothing$. For $1 \leqslant i \leqslant k$ and $b \in N(a)$, if $ab \ll p_i$ then $cb' \ll p_i$, for all $b' \in N(c)$. Hence, $cb' \ll p_j$, for all j.*

Proof. Suppose $ab \ll p_i$. Let $b'' \in N(a) \cap N(c)$. By Lemma 2.1.2, $ab'' \ll p_i$. If $b''c \ll p_i$ then $ab''c \ll p_i$. This is not possible by Lemma 2.1.1. Hence, $cb'' \ll p_i$. Again, by Lemma 2.1.2, $cb' \ll p_i$, for all $b' \in N(c)$. □

2.2 Constructing Permutations

Let $S = \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ be an ordered set with indices $i_1 < i_2 < \cdots < i_k$. We write $\underline{S}$ to represent the word $a_{i_k} a_{i_{k-1}} \cdots a_{i_1}$ in which the elements of S appear exactly once and are arranged in such a way that the indices are in decreasing order. Similarly, we write $\overline{S}$ to represent the word $a_{i_1} a_{i_2} \cdots a_{i_k}$ in which the elements of S appear exactly once and are arranged such that the indices are in increasing order.

Definition 2.2.1 (Neighborhood inclusion graph)**.** Let $G = (V, E)$ be a graph and $A \subseteq V$. We define a *neighborhood inclusion graph of G with respect to A* as the directed graph $\mathcal{N}_A(G) = (A, E')$, where for all $a, b \in A$ we have that $\overrightarrow{ab} \in E'$ if and only if $N(a) \subseteq N(b)$.

Remark 2.2.2. Since the set containment relation is transitive, the graph $\mathcal{N}_A(G)$ is a comparability graph and hence it induces a poset.

Notation 2.2.3. In what follows, let $G = (A \cup B, E)$ be a bipartite graph. Consider the neighborhood inclusion graph $\mathcal{N}_A(G)$ with respect to A. The poset $P_{\mathcal{N}_A(G)}$ induced by $\mathcal{N}_A(G)$ is simply denoted by P_A.

Construction

Consider a chain cover of P_A as per the following:

$$\begin{aligned} &X_1 : a_{1,1} \prec a_{1,2} \prec a_{1,3} \prec \cdots \prec a_{1,n_1} \\ &X_2 : a_{2,1} \prec a_{2,2} \prec a_{2,3} \prec \cdots \prec a_{2,n_2} \\ &\vdots \\ &X_k : a_{k,1} \prec a_{k,2} \prec a_{k,3} \prec \cdots \prec a_{k,n_k} \end{aligned}$$

For each chain X_i, we create a permutation p_i of vertices of $A \cup C$, where C is a relabeled set of B as described below.

For a set of vertices U and a vertex a, let $U_{-a} = U \backslash N(a)$ be the set of non-adjacent vertices of a in the set U.

Consider one of the chains, say X_1, and construct a permutation, say p'_1, as per the following. In X_1, since $N(a_{1,(i-1)}) \subsetneq N(a_{1,i})$, we write the *exclusive neighbors* of $a_{1,i}$, i.e., $N(a_{1,i})_{-a_{1,(i-1)}}$, on the right side of $a_{1,i}$ and on the left side of $a_{1,(i-1)}$. Then we write the vertices in B which are non-adjacent to any of the vertices in X_1, i.e., $B_{-a_{1,n_1}}$, on the left side of a_{1,n_1}. Now we write the remaining vertices of G, i.e., the vertices of the chains $X_2, \ldots, X_k$, on the left side of $B_{-a_{1,n_1}}$. Although the vertices within the sets $N(a_{1,i})_{-a_{1,(i-1)}}$ and $B_{-a_{1,n_1}}$ can be written in any order, for convenience, we will write them in the increasing order as per their indices. Thus,

$$p'_1 = x_1 \overline{B_{-a_{1,n_1}}} a_{1,n_1} \overline{N(a_{1,n_1})_{-a_{1,(n_1-1)}}} a_{1,(n_1-1)} \cdots a_{1,2} \overline{N(a_{1,2})_{-a_{1,1}}} a_{1,1} \overline{N(a_{1,1})},$$

where $x_1 = X_2^\circ \cdots X_k^\circ$. Here, for $1 \leqslant j \leqslant k$, $X_j^\circ = a_{j,1} a_{j,2} \cdots a_{j,n_j}$, the word obtained by writing the vertices of X_j in the increasing order as per the ordering of P_A.

Consider p'_{1_B} the subword of p'_1 with the vertices of B. Relabel the word p'_{1_B} as $c_1 c_2 \cdots c_m$ (where $m = |B|$) with indices increasing from left to right. Thus, $C = \{c_1, \ldots, c_m\}$ is the relabeled set of B. Now let p_1 be the permutation obtained from p'_1 by replacing the relabeled vertices from C in place of corresponding vertices of B.

For $i = 2, \ldots, k$, based on the chain X_i, we construct the permutation p_i with vertices of $A \cup C$ by

$$p_i = x_i \underline{C_{-a_{i,n_i}}} a_{i,n_i} \underline{N(a_{i,n_i})_{-a_{i,(n_i-1)}}} a_{i,(n_i-1)} \cdots a_{i,2} \underline{N(a_{i,2})_{-a_{i,1}}} a_{i,1} \underline{N(a_{i,1})},$$

where $x_i = X_1^\circ \cdots X_{i-1}^\circ X_{i+1}^\circ \cdots X_k^\circ$ and the neighborhoods are considered with the vertices of C.

Finally, set $p_0 = X_k^\circ X_{k-1}^\circ \cdots X_1^\circ c_m c_{m-1} \cdots c_1$.

Demonstration

A demonstration of this method for obtaining a word that permutationally represents a bipartite graph is given in Example 2.2.4.

Example 2.2.4. Let $G = (A \cup B, E)$ be the bipartite graph given in Figure 2.1.

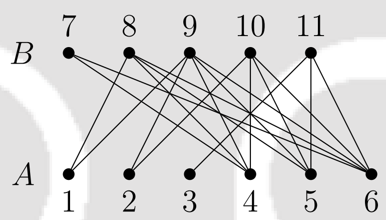


Figure 2.1: A bipartite graph G

Note that the poset P_A shown in Figure 2.2 has width three. Consider the following chain cover of P_A:

$$X_1 : 1 < 4$$
$$X_2 : 2 < 5 < 6$$
$$X_3 : 3$$

The corresponding permutations are as per the following, in which two-digit numbers (i.e., 10 and 11) are indicated in brackets.

$$p_1 = 2563(11)47(10)189$$

$$p_2 = 1436758(11)29(10)$$
$$p_3 = 1425698(10)73(11)$$
$$p_0 = 32561498(10)7(11)$$

The permutations p_1, p_2, p_3, p_0 on concatenation, i.e., the word $p_0p_1p_2p_3$, represents the graph G permutationally.

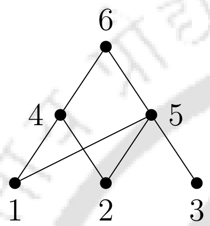


FIGURE 2.2: P_A

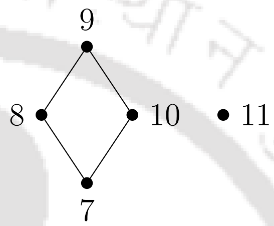


FIGURE 2.3: P_B

The poset P_B shown in Figure 2.3, has also width three. Consider the following chain cover of P_B:

$$Y_1 : 7 \prec 8 \prec 9$$
$$Y_2 : 10$$
$$Y_3 : 11$$

The corresponding permutations are as per the following:

$$q_1 = (10)(11)392851764$$
$$q_2 = 789(11)13(10)4652$$
$$q_3 = 789(10)412(11)653$$

The word $w = q_1q_2q_3$ does not represent G as the vertices 1 and 6 are non-adjacent in G while $w_{\{1,6\}} = 161616$. Also, note that

$$N_{\{7<8<9\}}(1) \subsetneq N_{\{7<8<9\}}(6)$$
$$N_{\{10\}}(1) \subsetneq N_{\{10\}}(6)$$
$$N_{\{11\}}(1) \subsetneq N_{\{11\}}(6)$$

We concatenate $q_0 = (11)(10)789461523$ to the word w so that $q_0q_1q_2q_3$ represents G permutationally.

Time Complexity

First, observe that the neighborhood inclusion graph of a graph on r vertices can be constructed in polynomial time. For instance, using a brute force method, a pair of neighborhoods can be compared in $O(r^2)$ time, and $\binom{r}{2}$ pairs are to be compared so that it can be done in $O(r^4)$ time. Next, recall that obtaining a chain cover of a poset of s elements can be done in $O(s^3)$ time (cf. [Felsner et al., 2003]). One may also refer to the algorithm given by Cáceres [2023] to obtain minimum chain cover of a poset.

Given a bipartite graph G with the partite sets A and B of sizes m and n, respectively. Let $m \leqslant n$ and consider the partite set A. We have, for all $a, b \in A$, $N(a) \subseteq N(b)$ if and only if $a \prec b$ in P_A. The time complexity for obtaining the poset P_A takes $O((mn)^2)$ time, as for each pair from m vertices, we need to compare the neighborhoods from the n-vertex set B. Further, a chain cover of P_A can be obtained in $O(m^3)$ time, which can be subsumed within $O((mn)^2)$. Assume that each chain of the chain cover of P_A is maintained in an array as per their ordering.

Note that the construction of each permutation p_i $(1 \leqslant i \leqslant k)$ involves finding the neighbors of each vertex in X_i and arranging them in order. Assume the graph G is maintained in the adjacency matrix. Note that relabeling of vertices of B takes $O(n)$ time. Except the labels, since B and C are the same, we will use the set B

for measuring the complexity in the following. Also, maintain an array of vertices of B in the increasing order of indices and another array of vertices of B in the decreasing order of indices. Note that we can prepare these arrays in $O(n^2)$ time. Every time, while constructing the permutation p_i (from its right side), take a copy of the chain X_i and a copy of arranged vertices of B, say B'. Find the neighbors of $a_{i,1}$ in $O(n)$ time from B' and arrange them in the same order for constructing p_i while deleting them in B'. Then repeat the same for each $a_{i,j}$ $(1 < j \leqslant n_i)$ to construct the permutation p_i in $O(mn)$ time, which includes arranging the prefix x_i. The permutation p_0 can be easily constructed in $O(m+n)$ time. As the number of permutations is $O(m)$, the entire process of constructing the permutations will take $O(m^2n)$ time.

Hence, the overall time complexity for constructing permutations on the vertex set of the bipartite graph G is $O((mn)^2)$, that is a polynomial in the number of vertices of G.

2.3 Correctness and Upper Bound

In continuation, we now prove the correctness of the procedure given in Section 2.2, and subsequently establish an upper bound for $\mathcal{R}^p(G)$. Further, we mention a case for which the *prn* attains the upper bound.

Theorem 2.3.1. *The word $p_0p_1\cdots p_k$ permutationally represents the bipartite graph $G' = (A \cup C, E')$, where*

$$E' = \{\overline{ac} \mid \overline{ab} \in E \text{ (for } a \in A,\ b \in B\text{), and } b \text{ is relabeled as } c \in C\}.$$

Proof. We show that two vertices a and c are adjacent in G' if and only if a and c alternate in the word $p_0p_1\cdots p_k$.

Suppose $\overline{ac} \in E'$. Let $a \in X_s$, say $a = a_{s,t}$, for some $1 \leqslant t \leqslant n_s$. Since c is in the neighborhood of a, c appears on the right side of a in p_s, i.e., $ac \ll p_s$. Further, for all $i \neq s$, as $a \in X_s$, we have $a \ll x_i$ so that $ac \ll p_i$. Clearly, $ac \ll p_0$. Hence, $ac \ll p_i$, for all $0 \leqslant i \leqslant k$, so that a and c alternate in the word $p_0 p_1 \cdots p_k$.

Conversely, suppose a and c are non-adjacent in G'. We deal with this in three cases. In each case, we identify two permutations: one with the subword ac and the other with the subword ca.

- Case 1: $a, c \in A$.
  - Subcase 1.1: Suppose $a, c \in X_s$. Then either $a < c$ or $c < a$. Without loss of generality, assume $a < c$. Then $ca \ll p_s$. Clearly, $ac \ll p_0$.
  - Subcase 1.2: Suppose $a \in X_s$ and $c \in X_t$, for $t \neq s$. Clearly, $ac \leqslant p_t$ and $ca \ll p_s$.
- Case 2: $a \in A$ and $c \in C$. Clearly, $ac \ll p_0$. Suppose $a \in X_s$. Then note that $ca \ll p_s$.
- Case 3: $a, c \in C$. Let $a = c_p$ and $c = c_q$. Without loss of generality, assume $p < q$. Then $ac = c_p c_q \ll p_1$ and $ca = c_q c_p \ll p_0$.

Hence, a and c do not alternate in the word $p_0 p_1 \cdots p_k$. □

For $0 \leqslant i \leqslant k$, let p_i' be the permutation of vertices of $A \cup B$ obtained from p_i by relabeling the vertices of C with their original labels from B. Then clearly, the word $p_0' p_1' \cdots p_k'$ represents the graph G as stated in the following corollary of Theorem 2.3.1.

Corollary 2.3.2. *The word $p_0' p_1' \cdots p_k'$ permutationally represents the graph G.*

Further, since the width of P_A is the size of a minimum chain cover (Dilworth's Theorem 1.3.6), we have the following corollary of Theorem 2.3.1.

Corollary 2.3.3. $\mathcal{R}^p(G) \leqslant 1 + w(P_A)$.

In view of the symmetry between A and B, we also have $\mathcal{R}^p(G) \leqslant 1 + w(P_B)$. Let $\kappa_0 = \min\{w(P_A), w(P_B)\}$. Since the words representing G can be obtained using chain covers with respect to A or B, we have the following result:

Theorem 2.3.4. *For a bipartite graph $G = (A \cup B, E)$, we have*

$$\mathcal{R}^p(G) \leqslant 1 + \kappa_0.$$

If a bipartite graph G has a crown graph as an induced subgraph, we call it as an induced crown graph of G. Based on the *prn* for crown graphs (see. Theorem 1.3.21), we state a lower bound on the *prn* of bipartite graphs. Since the class of bipartite graphs is hereditary, if H is an induced subgraph of G then $\mathcal{R}^p(H) \leqslant \mathcal{R}^p(G)$. Therefore, we have the following result.

Lemma 2.3.5. *If $H_{k,k}$ is an induced crown graph of a bipartite graph G then $\mathcal{R}^p(G) \geqslant k$.*

From the above results, we can conclude the *prn* of a particular type of bipartite graph.

Theorem 2.3.6. *Let $G = (A \cup B, E)$ be a bipartite graph and $H_{k,k}$ be a largest induced crown graph of G. If $k = \kappa_0$ then $\mathcal{R}^p(G) = \kappa_0$ or $1 + \kappa_0$.*

2.4 Conditions for κ_0-Representability

In this section, we provide a criterion for the chain covers of P_A (or P_B) such that the given bipartite graph $G = (A \cup B, E)$ is permutationally κ_0-representable.

Notation 2.4.1. Let X be a chain in P_A and $b \in B$. The relative neighborhood of b with respect to X, i.e., $N(b) \cap X$, is denoted by $N_X(b)$.

Lemma 2.4.2. *Suppose X is a chain in P_A. For all $b, b' \in B$, the relative neighborhoods $N_X(b)$ and $N_X(b')$ are comparable, i.e., $N_X(b) \subseteq N_X(b')$ or $N_X(b') \subseteq N_X(b)$.*

Proof. Let $X : a_1 \prec a_2 \prec \cdots \prec a_n$. Assume that $N_X(b)$ and $N_X(b')$ are incomparable, i.e., $N_X(b) \nsubseteq N_X(b')$ and $N_X(b') \nsubseteq N_X(b)$. Then $N_X(b) \neq \varnothing$ and $N_X(b') \neq \varnothing$. There exists $a_r \in N_X(b)\backslash N_X(b')$ and $a_s \in N_X(b')\backslash N_X(b)$. As $a_r, a_s \in X$, without loss of generality, assume $a_r \prec a_s$. Then $N(a_r) \subseteq N(a_s)$. This implies that $b \in N(a_s)$, which is a contradiction. Hence, $N_X(b)$ and $N_X(b')$ are comparable. □

Lemma 2.4.3. *For any chain X, if $N_X(b) \subsetneq N_X(b')$, for all $b, b' \in B$, then b occurs before b' in the permutation corresponding to the chain X.*

Proof. Let $X : a_1 \prec a_2 \prec \cdots \prec a_n$ and p be the permutation corrsponding to the chain X. Suppose $b'b \ll p$. We have the following cases:

- Case 1: $b' \in B_{-a_n}$. We have $N_X(b') = \varnothing$. Clearly, $N_X(b') \subseteq N_X(b)$, which is a contradiction.
- Case 2: For $2 \leqslant i \leqslant n$, $b' \in N(a_i)_{-a_{i-1}}$. We have $b \in N(a_j)_{-a_{j-1}}$ where $j \leqslant i$. This implies $N_X(b') \subseteq N_X(b)$, which is a contradiction.
- Case 3: $b' \in N(a_1)$. Clearly $b \in N(a_1)$ and we have $N_X(b') = N_X(b)$, which is also a contradiction.

Hence, b always occur before b' in p. □

If the chain cover has only one chain, it is evident that $\mathcal{R}^p(G) = 2$. Assume $M = \{X_1, X_2, \ldots, X_k\}$ $(k \geqslant 2)$ is a chain cover of poset P_A. For $t \geqslant 2$, let $X_t \in M$. For some $b, b' \in B$, if $N_{X_t}(b) = N_{X_t}(b') = \{a_{t,l}, a_{t,(l+1)}, \ldots, a_{t,n_t}\}$, where $1 \leqslant l \leqslant n_t$ then the permutation p_t corresponding to X_t is of the form

$$x_t\underline{B_{-a_{t,n_t}}}a_{t,n_t}\underline{N(a_{t,n_t})_{-a_{t,(n_t-1)}}}a_{t,(n_t-1)}\cdots a_{t,l}\underline{N(a_{t,l})_{-a_{t,(l-1)}}}a_{t,(l-1)}\cdots \underline{N(a_{t,1})},$$

where $b, b' \in N(a_{t,l})_{-a_{t,(l-1)}}$.

Based on this observation, we make a minor modification to the method described in Section 2.2, i.e., for all $b, b' \in B$ if $N_{X_1}(b) = N_{X_1}(b')$ and $N(b') \subsetneq N(b)$ then put b before b' in p_1 such that $b = c_p$ and $b' = c_q$, where $p < q$.

Remark 2.4.4. Note that the neighborhoods of no two vertices in G are the same. As $X_i \cap X_j = \varnothing$ for $i \neq j$, we have $N(b') \subsetneq N(b)$ if and only if $N_{X_i}(b') \subseteq N_{X_i}(b)$ for all chains X_i and $N_{X_t}(b') \subsetneq N_{X_t}(b)$ for some chain X_t.

We now show that G is permutationally κ_0-representable based on the containment between $N(b)$ and $N(b')$, for all $b, b' \in B$. If $N(b)$ and $N(b')$ are incomparable, for all $b, b' \in B$ then we show that $\mathcal{R}^p(G) \leqslant \kappa_0$. Otherwise, it is inconclusive. However, when $N(b)$ and $N(b')$ are comparable for some $b, b' \in B$, under certain conditions, we show that G is permutationally κ_0-representable.

Theorem 2.4.5. *Suppose $M = \{X_1, X_2, \ldots, X_k\}$ $(k \geqslant 2)$ is a chain cover of the poset P_A. If M satisfies the following property with respect to B then G is permutationally k-representable.*

For all $b, b' \in B$, there exist i, j with $i \neq j$
such that $N_{X_i}(b) \subseteq N_{X_i}(b')$ and $N_{X_j}(b') \subseteq N_{X_j}(b)$. $\cdots\cdots\cdots\cdots\cdots (\zeta_B)$

Proof. Let G' be the relabeled graph of G and $w' = p_1 p_2 \cdots p_k$ be the word where $p_1, p_2, \ldots, p_k$ are permutation of vertices of G' as per the construction given in Section 2.2, along with the above-mentioned modification to p_1. It is sufficient to show that w' represents G' permutationally as the vertices of G' can be relabeled back to their original labeling in w', which permutationally represents G.

We show that two vertices a and c are adjacent in G' if and only if a and c alternate in the word w'.

Suppose $\overline{ac} \in E'$. Let $a \in X_t$, say $a = a_{t,l}$, for some $1 \leqslant l \leqslant n_t$. Since c is in the neighborhood of a, c appears on the right side of a in p_t, i.e., $ac \ll p_t$. Further, for all $i \neq t$, as $a \in X_t$, we have $a \ll x_i$ so that $ac \ll p_i$. Hence, $ac \ll p_i$, for all $1 \leqslant i \leqslant k$, so that a and c alternate in the word w'.

Conversely, suppose a and c are non-adjacent in G'. We deal with this in three cases. In each case, we identify two permutations: one with the subword ac and the other with the subword ca.

- Case 1: $a, c \in A$.
  - Subcase 1.1: Suppose $a, c \in X_s$. Then either $a \prec c$ or $c \prec a$. Without loss of generality, assume $a \prec c$. Then $ca \ll p_s$. Since $a, c \in X_s$, there exists X_t $(t \neq s)$ such that $a, c \notin X_t$. Then $ac \ll X_s^{\circ} \ll x_t \ll p_t$.
  - Subcase 1.2: Suppose $a \in X_s$ and $c \in X_t$, for $t \neq s$. Then $ac \ll X_s^{\circ} c \ll x_t c \ll p_t$. Similarly, $ca \ll p_s$.
- Case 2: $a \in A$ and $c \in C$. Suppose $a \in X_s$. As $c \notin N(a)$, clearly $ca \ll p_s$. Since $a \in X_s$, there exists X_t $(t \neq s)$ such that $a \notin X_t$. Then $ac \ll X_s^{\circ} c \ll x_t c \ll p_t$.
- Case 3: $a, c \in C$. Let $a = c_p$ and $c = c_q$. Without loss of generality, assume $p < q$. Then $ac = c_p c_q \ll p_1$, which means that $N_{X_1}(a) \subseteq N_{X_1}(c)$. As $N_{X_i}(a) \subseteq N_{X_i}(c)$ and $N_{X_j}(c) \subseteq N_{X_j}(a)$ for some i and j, there exist a chain X_s such that $N_{X_s}(c) \subseteq N_{X_s}(a)$.
  - Subcase 3.1: Suppose $N_{X_s}(c) = N_{X_s}(a)$. As c_p and c_q are in the same exclusive neighborhood in p_s and are written in decreasing order, we have $c_q c_p = ca \ll p_s$.
  - Subcase 3.2: Suppose $N_{X_s}(c) \subsetneq N_{X_s}(a)$. Then there exists $a_r \in N_{X_s}(a) \backslash N_{X_s}(c)$ such that $ca \ll ca_r a \ll p_s$.

Hence, a and c do not alternate in w'. □

Theorem 2.4.6. *Suppose $M = \{X_1, X_2, \ldots, X_k\}$ $(k \geqslant 3)$ is a chain cover of the poset P_A. If M satisfies the following property with respect to B then G is permutationally k-representable.*

For any $b, b' \in B$, if $N(b) \subseteq N(b')$,

then there exists X_j such that $b \notin N(a_{jn_j})$. $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\gamma_B)$

Proof. As per the procedure described in Section 2.2, let $p_1, p_2, \ldots, p_k$ be the permutations corresponding to the chains $X_1, X_2, \ldots, X_k$, respectively. For $1 \leqslant i \leqslant k$, note that each p_i is of the form $x_i\overline{B_{-a_{i,n_i}}}w_i$, where w_i is the subword of p_i restricted to the set $X_i \cup N(a_{i,n_i})$.

We now reconstruct new permutations $q_1, q_2, \ldots, q_k$ on the set $A \cup B$ by considering certain modifications on the permutations $p_1, p_2, \ldots, p_k$.

$$
\begin{aligned}
q_1 &= x'_1\overline{D_1}w_1 \\
q_2 &= x'_2\overline{D_2}w'_2\overline{F_1} \\
&\vdots \\
q_{k-1} &= x'_{k-1}\overline{D_{k-1}}w'_{k-1}\overline{F_{k-2}} \\
q_k &= x_k\overline{D_k}w'_k\overline{F_{k-1}}
\end{aligned}
$$

Here, considering $D_0 = \varnothing$ and $X_{k+1} = \varnothing$, for $1 \leqslant i \leqslant k$, $\overline{D_i}$ is the permutation on the set $D_i = (B_{-a_{i,n_i}} \backslash D_{i-1}) \cup X_j$ (where $j = i + 1$) given by

$$\overline{D_i} = \overline{B_{-a_{j,n_j}}}a_{j,n_j}\underline{N(a_{j,n_j})_{-a_{j,(n_j-1)}}}\cdots a_{j,2}\underline{N(a_{j,2})_{-a_{j,1}}}a_{j,1}\underline{N(a_{j,1})}.$$

Note that $\overline{D_i}$ is the permutation obtained by the procedure described in Section 2.2 by considering the singleton chain cover $\{X_j\}$ and the set $B_{-a_{i,n_i}} \backslash D_{i-1}$ in place of B. Further, for $1 \leqslant i \leqslant k - 1$,

- x'_i is the subword of x_i restricted to the set $A\backslash(X_i \cup X_{i+1})$,
- w'_i is the subword of w_i restricted to the set $(N(a_{i,n_i})\backslash D_{i-1}) \cup X_i$, and
- $\overline{F_i}$ is the subword of $r(\overline{D_i})$ restricted to the set $D_i\backslash X_{i+1}$. Note that $r(\overline{D_i})$ is the reverse of the word $\overline{D_i}$.

We now show that $q_1q_2\cdots q_k$ represents G permutationally.

Suppose a and b are adjacent in the graph G. Let $a \in X_t$. For every $i \notin \{t-1, t\}$, $a \ll x'_i \ll x_i$, this implies $ab \ll q_i$. If $b \in D_{t-1}$ then $ab \ll \overline{D_{t-1}} \ll q_{t-1}$ and

$ab \ll w'_t\overline{F_{t-1}} \ll q_t$. If $b \notin D_{t-1}$ then $ab \ll \overline{D_{t-1}}w'_{t-1}F_{t-2} \ll q_{t-1}$ and $ab \ll w'_t \ll q_t$. Hence, $ab \ll q_i$ for all $1 \leqslant i \leqslant k$, so that a and b alternate in the word $q_1q_2\cdots q_k$.

Assume that a and b are non-adjacent in the graph G. We handle this in the following four cases. In each case, we identify two permutations q_s and q_t (for some $s \neq t$) such that $ab \ll q_s$ and $ba \ll q_t$.

- Case 1: $a, b \in A$.
  - Subcase 1.1: Suppose $a, b \in X_s$. Then either $a \prec b$ or $b \prec a$. Without loss of generality, assume $a \prec b$. Then $ba \ll w'_s \ll q_s$. Since $a, b \notin X_i$ (for any $i \neq s$) and $k \geqslant 3$, there exists X_t ($t \neq s, s-1$) such that $a, b \notin X_t$. Then $ab \ll x'_t \ll q_t$.
  - Subcase 1.2: Suppose $a \in X_s$ and $b \in X_t$, for $s \neq t$. Then $ba \ll q_s$ and $ab \ll q_t$.
- Case 2: $a \in A$ and $b \in B$. Suppose $a \in X_s$. If $b \in D_{s-1}$ then $ba \ll \overline{D_{s-1}} \ll q_{s-1}$ and $ab \ll w'_s\overline{F_{s-1}} \ll q_s$. Otherwise, $ab \ll q_{s-1}$ and $ba \ll w'_s \ll q_s$.
- Case 3: $a, b \in B$ and $N(a) \subseteq N(b)$. Let $a \notin N(a_{jn_j})$, for some j (by μ_B).
  - Subcase 3.1: $a \in D_{s-1}$. We have $a \in F_{s-1}$. Suppose $b \in D_{s-1}$ then either $ba \ll \overline{D_{s-1}} \ll q_{s-1}$ and $ab \ll \overline{F_{s-1}} \ll q_s$, or $ab \ll \overline{D_{s-1}}$ and $ba \ll \overline{F_{s-1}} \ll q_s$. Suppose $b \notin D_{s-1}$. Then $ab \ll q_{s-1}$ and $ba \ll q_s$.
  - Subcase 3.2: $a \notin D_{s-1}$. We have $a \in D_s$. Suppose $b \in D_s$. Then either $ab \ll \overline{D_s} \ll q_s$ and $ba \ll \overline{F_s} \ll q_{s+1}$, or $ba \ll \overline{D_s} \ll q_s$ and $ab \ll \overline{F_s} \ll q_{s+1}$. Suppose $b \notin D_s$. Then $ab \ll q_s$ and $ba \ll q_{s+1}$.
- Case 4: $a, b \in B$, and for $i \neq j$, $N_{X_i}(a) \subseteq N_{X_i}(b)$ and $N_{X_j}(b) \subseteq N_{X_j}(a)$. In this case, clearly, we have $ab \ll q_i$ and $ba \ll q_j$.

Hence, a and b do not alternate in $q_1q_2\cdots q_k$. $\square$

Corollary 2.4.7. *If $\kappa_0 = w(P_A)$ (or $= w(P_B)$) and a minimal chain cover of P_A (or P_B) satisfies one of the following conditions, ζ_B, ζ_A, γ_B or γ_A then G is permutationally κ_0-representable.*

Remark 2.4.8. Let $G = (A \cup B, E)$ be a bipartite graph such that both P_A and P_B are antichains. Then P_A (also P_B) has only one chain cover. For all $a, b \in A$ (also in B), since $N(a)$ and $N(b)$ are incomparable, the chain cover of P_A (of P_B) satisfies ζ_B (respectively, ζ_A). Hence, $\mathcal{R}^p(G) \leqslant \kappa_0$. In particular, every regular bipartite graph is permutationally κ_0-representable.

We now present a class of bipartite graphs constructed from Mycielski graphs of crown graphs and demonstrate the procedure given in Section 2.2 .

Example 2.4.9. Let G be a graph with vertex set $[n] = \{1, 2, \ldots, n\}$. The *Mycielski graph* of G, denoted by $\mu(G)$, is an undirected graph with the vertex set $[n] \cup \{1', 2', \ldots, n'\} \cup \{0\}$ and the edge set $E(G) \cup \{\overline{0i'} \mid i \in [n]\} \cup \{\overline{ji'}, \overline{ij'} \mid \overline{ij} \in E(G)\}$, where $E(G)$ is the edge set of G. Note that G is an induced subgraph of $\mu(G)$. The graph $\mu(G)^0$ is the graph obtained from $\mu(G)$ by removing the vertex 0 from it. If G is a bipartite graph with bipartition A and B then $\mu(G)^0$ is also a bipartite graph with bipartition $A \cup A'$ and $B \cup B'$, where $A' = \{i' | i \in A\}$ and $B' = \{j' | j \in B\}$.

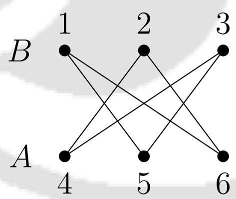


FIGURE 2.4: $H_{3,3}$

Let us consider the crown graph $H_{3,3}$ as shown in Figure 2.4 with partite sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. Construct the Mycielski graph $\mu(H_{3,3})$ and accordingly the bipartite graph $\mu(H_{3,3})^0$, as shown in Figure 7.11 and Figure 2.6, respectively. Note that the neighborhood inclusion graphs of $\mu(H_{3,3})^0$ with respect to $A \cup A'$ and $B \cup B'$ are isomorphic, as shown in Figure 2.7 and Figure 2.8.

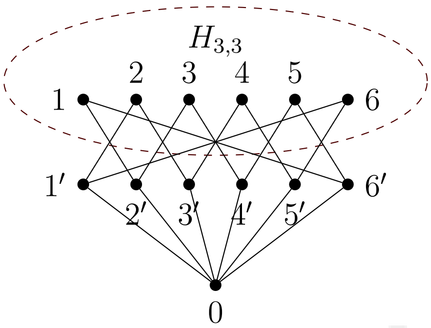


FIGURE 2.5: $\mu(H_{3,3})$

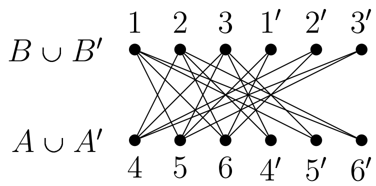


FIGURE 2.6: $\mu(H_{3,3})^0$

Consider the following chain cover of the poset $P_{A\cup A'}$ with respect to the partite set $A \cup A'$ of $\mu(H_{3,3})^0$:

$$X_1' : 1' \prec 1$$

$$X_2' : 2' \prec 2$$

$$X_3' : 3' \prec 3$$

The corresponding permutations are as per the following.

$$q_1 = 2'23'344'15'6'1'56$$

$$q_2 = 1'13'355'24'6'2'46$$

$$q_3 = 1'12'266'34'5'3'45$$

Note that the chain cover $\{X_1', X_2', X_3'\}$ satisfies Corollary 2.4.7 and hence $q_1q_2q_3$ represent the graph $\mu(H_{3,3})^0$ permutationally. Further, since $H_{3,3}$ is an induced subgraph of $\mu(H_{3,3})^0$, we have $\mathcal{R}^p(\mu(H_{3,3})^0) = 3$.

Note that for $n \geqslant 3$, the neighborhood inclusion graph with respect to a partite set of $\mu(H_{n,n})^0$ is a collection of n disjoint edges. Moreover, since $H_{n,n}$ is an induced subgraph of $\mu(H_{n,n})^0$, it is evident that $\mathcal{R}^p(\mu(H_{n,n})^0) = \kappa_0$.

Remark 2.4.10. Recall the poset P_A from Example 2.2.4. The chain cover $\{X_1, X_2, X_3\}$ of P_A satisfies Corollary 2.4.7; therefore, the word $p_1p_2p_3$ permutationally represent the graph G.

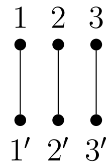


Figure 2.7: $P_{A \cup A'}$

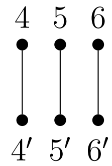


Figure 2.8: $P_{B \cup B'}$

2.5 Comparison with Broere's Result

We now show that the upper bound for *prn* of a bipartite graph obtained in Theorem 2.3.4 is an improvement of the upper bound given by Broere (Theorem 1.4.47).

Let $G = (A \cup B, E)$ be a reduced bipartite graph. Accordingly, the number of distinct neighborhoods of A is $|A|$, i.e., $\alpha = |\{N(a) | a \in A\}| = |A|$. Also, the number of distinct neighborhoods of B is $|B|$, i.e., $\beta = |\{N(b) | b \in B\}| = |B|$. Hence, since $w(P_A) \leqslant |A|$ and $w(P_B) \leqslant |B|$, we have $\kappa_0 \leqslant \min\{\alpha, \beta\}$.

In case $\mathcal{R}^p(G) = 1 + \kappa_0$, we observe that $\kappa_0 < \min\{\alpha, \beta\}$. On the contrary, suppose $\kappa_0 = \min\{\alpha, \beta\}$. We arrive at a contradiction as shown below in two cases. Without loss of generality, assume $\alpha \leqslant \beta$. Then $\kappa_0 = \alpha = |A|$.

- If $w(P_A) \leqslant w(P_B)$ then $w(P_A) = \kappa_0 = |A|$ so that P_A is an antichain. Hence, by Remark 2.4.8, G is permutationally κ_0-representable. But $\mathcal{R}^p(G) = 1 + \kappa_0$.

- Else, $w(P_B) = \kappa_0 < w(P_A) \leqslant |A| = \min\{\alpha, \beta\}$.

Hence, if $\mathcal{R}^p(G) = 1 + \kappa_0$, we have $\kappa_0 < \min\{\alpha, \beta\}$.

Theorem 2.5.1. *Let G be a bipartite graph. Then $\mathcal{R}^p(G) \leqslant 1 + \kappa_0 \leqslant \min\{\alpha, \beta\}$.*

Example 2.5.2. Consider the bipartite graph G given in Figure 2.9. By Theorem 1.4.47, G is permutationally 3-representable. However, note that the reduced graph of G satisfies Corollary 2.4.7 and $\kappa_0 = 2$. Hence, $\mathcal{R}^p(G) = 2$.

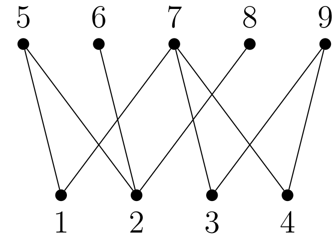


Figure 2.9: A bipartite graph

3 Representation Number

Focusing on the representation number of bipartite graphs, in this chapter, we show that every bipartite graph is $\left(1 + \lceil \frac{m}{2} \rceil\right)$-representable, where m is the size of its smallest partite set. Furthermore, we validate that the conjecture (see Conjecture 1.4.38) by Glen et al. on the representation number of bipartite graphs, case wise based on the sizes of the partite sets. In fact, we show that the conjecture holds for all bipartite graphs leaving the bipartite graphs whose partite sets are of equal and even size. In case of the bipartite graphs with partite sets of equal and even size, using the neighborhood inclusion graph approach, we prove the conjecture for certain subclasses, viz., the bipartite graphs with $h(P_A) \geqslant 3$ or P_A contains two disjoint chains of size 2.

In view of Remark 1.4.54, in this chapter, $G = (A \cup B, E)$ is a reduced bipartite graph.

3.1 Construction of Uniform Word-Representant

In this section, we construct a uniform word that represents a given bipartite graph. As a result, we obtain an upper bound on its representation number. Let G be a bipartite graph with bipartition $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where $m \leqslant n$.

Construction

Through the following five-step procedure, we construct a $\left(1 + \lceil \frac{m}{2} \rceil\right)$-uniform word-representant of G.

I. Partition the partite set A into pairs. If m is odd then consider the set $A \cup \{a_{m+1}\}$, where $a_{m+1} \notin A \cup B$ and $N(a_{m+1}) = \varnothing$. Accordingly, let m be even and partition the set A into $\frac{m}{2}$ pairs, say

$$(a_1, a_2), (a_3, a_4), \ldots, (a_{m-1}, a_m).$$

II. **Word $w_{(i,j)}$:** For any pair (a_i, a_j), construct a word $w_{(i,j)}$ over the set $\{a_i, a_j\} \cup B$ given by

$$w_{(i,j)} = B_{*i} a_i B_i a_j w'_{(i,j)} a_i B'_j a_j B'_{*j}$$

where,

i. $B_{*i} B'_{*j}$: A permutation on the vertices of B which are non-adjacent to any vertex of $\{a_i, a_j\}$.

ii. B_i: A permutation on the neighbors of a_i but not of a_j, i.e., $N(a_i) \backslash N(a_j)$.

iii. $w'_{(i,j)}$: A permutation on the common neighbors of a_i and a_j, i.e., $N(a_i) \cap N(a_j)$.

iv. B'_j: A permutation on the neighbors of a_j but not of a_i, i.e., $N(a_j) \backslash N(a_i)$.

We call the words $B_{*i}, B_i, w'_{(i,j)}, B'_j$ and B'_{*j} as *blocks* of the word $w_{(i,j)}$.

III. **Construction of $D_{(p:q)}$:** Consider two consecutive pairs, say (a_p, a_{p+1}) and (a_{q-1}, a_q), for $1 \leqslant p < m-1$; i.e., $p+2 = q-1$. If $p = m-1$ then consider $q = 2$, i.e., the last and first pairs. Avoiding these two pairs, from the remaining pairs, consider the word with second components in the sequence followed by the word with the first components in the sequence to construct the word $D_{(p:q)}$. That is, for various values of p, the words $D_{p,q}$ are as per the following:

$$D_{(p:q)} = \begin{cases} a_6a_8\cdots a_ma_5a_7\cdots a_{m-1}, & \text{if } p=1 \text{ (remove first two pairs);} \\ a_2a_4\cdots a_{m-4}a_1a_3\cdots a_{m-5}, & \text{if } p=m-3 \text{ (remove last two pairs);} \\ a_4a_6\cdots a_{m-2}a_3a_5\cdots a_{m-3}, & \text{if } p=m-1 \text{ (remove last and first pairs);} \\ a_2a_4\cdots a_{p-1}a_{q+2}\cdots a_ma_1a_3\cdots a_{p-2}a_{q+1}\cdots a_{m-1}, & \text{else.} \end{cases}$$

IV. **Construction of w_0:** Consider the reversal of the word $w_{(1,2)}$ restricted to the vertices of B. Then construct the permutation w_0 on the set $A \cup B$ by

$$w_0 = a_2a_1w^R_{(1,2)}|_B a_4a_3\cdots a_ma_{m-1}.$$

V. **Construction of w:** Finally, construct a $\left(1+\frac{m}{2}\right)$-uniform word

$$w = w_{(1,2)}D_{(1:4)}w_{(3,4)}D_{(3:6)}w_{(5,6)}\cdots w_{(m-1,m)}D_{(m-1:2)}w_0$$

over the set $A \cup B$.

Remark 3.1.1. Note that if m is odd, we construct the word w over the vertex set

$A \cup B \cup \{a_{m+1}\}$. It is evident that the word obtained by removing all the occurrences of the vertex a_{m+1} from w is a word-representant of the bipartite graph G.

Remark 3.1.2. It is straightforward to observe that the construction of uniform word-representant given in this section can be done in polynomial time.

Demonstration

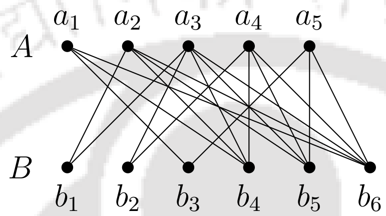


FIGURE 3.1: A bipartite graph

Example 3.1.3. Let $G = (A \cup B, E)$ be the bipartite graph given in Figure 3.1. Consider the smaller partite set $A = \{a_1, a_2, a_3, a_4, a_5\}$. Since A is of odd size, add an isolated vertex a_6 to the graph G such that $N(a_6) = \varnothing$. Let $A_1 = A \cup \{a_6\}$.

I. Partition the set A_1 into pairs, i.e., $(a_1, a_2), (a_3, a_4), (a_5, a_6)$.

II. For each pair, we construct the following words:

i. We have, $N(a_1) = \{b_3, b_4, b_6\}$ and $N(a_2) = \{b_1, b_4, b_5, b_6\}$. As per the construction, $B_{*1} = b_2$, $B_1 = b_3$, $w'_{(1,2)} = b_4b_6$, $B'_2 = b_1b_5$ and B'_{*2} is an empty word. Thus,

$$w_{(1,2)} = b_2a_1b_3a_2b_4b_6a_1b_1b_5a_2.$$

ii. We have, $N(a_3) = \{b_1, b_2, b_4, b_5, b_6\}$ and $N(a_4) = \{b_2, b_4, b_5, b_6\}$. As per the construction, $B_{*3} = b_3$, $B_3 = b_1$, $w'_{(3,4)} = b_2b_4b_5b_6$, B'_4 and B'_{*4} are empty words. Thus,

$$w_{(3,4)} = b_3a_3b_1a_4b_2b_4b_5b_6a_3a_4.$$

iii. We have, $N(a_5) = \{b_3, b_5, b_6\}$ and $N(a_6) = \varnothing$. As per construction, $B_{*5} = b_1b_2b_4$, $B_5 = b_3b_5b_6$, $w'_{(5,6)}$, $B'_6 = b_1b_5$ and B'_{*6} are empty words. Thus,

$$w_{(5,6)} = b_1b_2b_4a_5b_3b_5b_6a_6a_5a_6.$$

Note that is in this process, if $b \in B$ is not in the neighborhood of a pair (a_{2s-1}, a_{2s}), for some $s \in \{1, 2, 3\}$, we put b in $B_{*(2s-1)}$ of $w_{(2s-1,2s)}$.

III. Construct the permutations, $D_{(1:4)}, D_{(3:6)}$ and $D_{(5:2)}$, i.e.,

$$D_{(1:4)} = a_6a_5$$
$$D_{(3:6)} = a_2a_1$$
$$D_{(5:2)} = a_4a_3$$

IV. Construct the word w_0, i.e.,

$$w_0 = a_2a_1w^R_{(1,2)}|_B a_4a_3a_6a_5$$
$$w_0 = a_2a_1b_5b_1b_6b_4b_3b_2a_4a_3a_6a_5.$$

V. Finally, we have the word $w = w_{(1,2)}D_{(1:4)}w_{(3,4)}D_{(3:6)}w_{(5,6)}D_{(5:2)}w_0$ which represents $G \cup \{a_6\}$.

$$w = b_2a_1b_3a_2b_4b_6a_1b_1b_5a_2a_6a_5b_3a_3b_1a_4b_2b_4b_5b_6a_3a_4a_2a_1b_1b_2b_4a_5b_3b_5b_6a_6a_5a_6a_4a_3$$
$$a_2a_1b_5b_1b_6b_4b_3b_2a_4a_3a_6a_5.$$

VI. Remove the vertex a_6 from w, we see that

$$w|_{(A\cup B)} = b_2a_1b_3a_2b_4b_6a_1b_1b_5a_2a_5b_3a_3b_1a_4b_2b_4b_5b_6a_3a_4a_2a_1b_1b_2b_4a_5b_3b_5b_6a_5a_4a_3$$
$$a_2a_1b_5b_1b_6b_4b_3b_2a_4a_3a_5$$

is a word-representant for the bipartite graph G.

Observations and Remarks

We present some remarks on the occurrences of each vertex of G in the word w for better understanding on the construction of w. Note that $a^{(i)}$ denote the ith occurrence of a vertex a in the word w (from left to right).

Remark 3.1.4. For each vertex $b \in B$, we have, for all $1 \leqslant s \leqslant \frac{m}{2}$, $b^{(s)} \ll w_{(2s-1,2s)}$, and $b^{(1+\frac{m}{2})} \ll w_0$.

Remark 3.1.5. For all $1 \leqslant s \leqslant \frac{m}{2}$, the word $w_{(2s-1,2s)}$ contains only the vertices a_{2s-1} and a_{2s} from the partite set A. For each $a \in A$,

- $a \in \{a_1, a_2\}$. We have

$$\begin{aligned} a^{(1)}a^{(2)} &\ll w_{(1,2)} \\ \text{for all } 3 \leqslant i \leqslant \tfrac{m}{2}, \quad a^{(i)} &\ll D_{(2i-3:2i)} \\ a^{(1+\frac{m}{2})} &\ll w_0. \end{aligned}$$

- $a \in \{a_{2s-1}, a_{2s}\}$, where $2 \leqslant s \leqslant \frac{m}{2}$. We have

$$\begin{aligned} \text{for all } i \in \{1, 2, \ldots, s-2, s+1, \ldots, \tfrac{m}{2}\}, \quad a^{(i)} &\ll \begin{cases} D_{(2i-1:2i+2)}, & \text{if } i \neq \frac{m}{2}; \\ D_{(m-1:2)}, & \text{if } i = \frac{m}{2} \end{cases} \\ a^{(s-1)}a^{(s)} &\ll w_{(2s-1,2s)} \\ a^{(1+\frac{m}{2})} &\ll w_0. \end{aligned}$$

Correctness

Remark 3.1.4 and Remark 3.1.5 are very handy in proving that the word w represents G in the following theorem.

Theorem 3.1.6. *The word w represents the bipartite graph G.*

Proof. We show that for any two vertices $a, b \in A \cup B$, a and b are adjacent if and only if a and b alternate in w, so that w is a word-representant of the bipartite graph G.

Suppose a and b are adjacent in G. Let $a \in A$ and $b \in B$. We deal this in the following two cases:

- Case 1: $a \in \{a_1, a_2\}$. From Remark 3.1.4 and Remark 3.1.5, we have

$$\begin{aligned} a^{(1)}b^{(1)}a^{(2)} &\ll w_{(1,2)} \\ \text{for all } \; 3 \leqslant i \leqslant \tfrac{m}{2}, \quad b^{(i-1)} &\ll w_{(2i-3,2i-2)} \text{ and } a^{(i)} \ll D_{(2i-3:2i)} \\ b^{(\frac{m}{2})} &\ll w_{(m-1,m)} \\ a^{(1+\frac{m}{2})}b^{(1+\frac{m}{2})} &\ll w_0 \end{aligned}$$

  so that $a^{(1)}b^{(1)}a^{(2)}b^{(2)} \cdots a^{(1+\frac{m}{2})}b^{(1+\frac{m}{2})} \ll w$.

- Case 2: $a \in \{a_{2s-1}, a_{2s}\}$, for $2 \leqslant s \leqslant \frac{m}{2}$. From Remark 3.1.4 and Remark 3.1.5, we have

$$\begin{aligned} \text{for each } \; i \in \{1, \ldots, s-2, s+1, \ldots, \tfrac{m}{2}\}, \quad b^{(i)} &\ll w_{(2i-1,2i)} \text{ and} \\ a^{(i)} &\ll \begin{cases} D_{(2i-1:2i+2)}, & \text{if } i \neq \frac{m}{2}; \\ D_{(m-1:2)}, & \text{if } i = \frac{m}{2} \end{cases} \\ b^{(s-1)} &\ll w_{(2s-3,2s-2)} \\ a^{(s-1)}b^{(s)}a^{(s)} &\ll w_{(2s-1,2s)} \\ b^{(1+\frac{m}{2})}a^{(1+\frac{m}{2})} &\ll w_0 \end{aligned}$$

  so that $b^{(1)}a^{(1)}b^{(2)}a^{(2)} \cdots b^{(1+\frac{m}{2})}a^{(1+\frac{m}{2})} \ll w$.

Hence, a and b alternate in the word w.

Conversely, suppose a and b are non-adjacent in G. We show that, for some $j \geqslant 1$, one of the following words is a subword of w: $a^{(j)}b^{(j)}b^{(j+1)}a^{(j+1)}$, $b^{(j)}a^{(j)}a^{(j+1)}b^{(j+1)}$, $a^{(j)}a^{(j+1)}b^{(j)}b^{(j+1)}$, $b^{(j)}b^{(j+1)}a^{(j)}a^{(j+1)}$. We handle this in the following three cases:

- Case 1: $a, b \in A$. We have the following subcases:

  – Subcase 1.1: a, b belong to the same pair. For $1 \leqslant s \leqslant \frac{m}{2}$, let $a = a_{2s-1}$ and $b = a_{2s}$.

i) If $s = 1$ then note that $a^{(2)}b^{(2)} = a_1^{(2)}a_2^{(2)} \ll w_{(1,2)}$. Further,

$$a_2^{(3)}a_1^{(3)} \ll \begin{cases} w_0, & \text{for } m \leqslant 4; \\ D_{(3:6)}, & \text{for } m \geqslant 6 \end{cases}$$

so that $a^{(2)}b^{(2)}b^{(3)}a^{(3)} \ll w$.

ii) If $s = 2$ then note that $a^{(2)}b^{(2)} = a_3^{(2)}a_4^{(2)} \ll w_{(3,4)}$. Further,

$$a_4^{(3)}a_3^{(3)} \ll \begin{cases} w_0, & \text{for } m \leqslant 4; \\ D_{(5:2)}, & \text{for } m = 6; \\ D_{(5:8)}, & \text{for } m > 6 \end{cases}$$

so that $a^{(2)}b^{(2)}b^{(3)}a^{(3)} \ll w$.

iii) If $s > 2$ then we have,

$$b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll D_{(2s-5:2s-2)}w_{(2s-1,2s)} \ll w.$$

– Subcase 1.2: a and b belong to different pairs.

i) If one of a, b is in the first pair. Without loss of generality, let $a \in \{a_1, a_2\}$. Then as $b \notin \{a_1, a_2\}$, b does not appear in $w_{(1,2)}$. Accordingly, we have

$$a^{(1)}a^{(2)} \ll w_{(1,2)} \text{ so that } a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w.$$

ii) If $a \in \{a_{2s-1}, a_{2s}\}$, for $2 \leqslant s \leqslant \frac{m}{2}$ then consider the following cases depending on whether b is in the subsequent/preceding pair or not:

(a) If $b \in \{a_{2s+1}, a_{2s+2}\}$ then

$$\begin{aligned} b^{(s-1)} &\ll D_{(2s-3:2s)} \\ a^{(s-1)}a^{(s)} &\ll w_{(2s-1,2s)} \\ b^{(s)}b^{(s+1)} &\ll w_{(2s+1,2s+2)} \end{aligned}$$

so that $b^{(s-1)}a^{(s-1)}a^{(s)}b^{(s)} \ll w$.

(b) If $b \notin \{a_{2s-3}, a_{2s-2}, a_{2s+1}, a_{2s+2}\}$ then we have

$$b^{(s-1)}a^{(s-1)}a^{(s)}b^{(s)} \ll \begin{cases} D_{(2s-3:2s)}w_{(2s-1,2s)}D_{(2s-1:2)}, & \text{for } s = \frac{m}{2}; \\ D_{(2s-3:2s)}w_{(2s-1,2s)}D_{(2s-1:2s+2)}, & \text{for } s < \frac{m}{2} \end{cases}$$

so that $b^{(s-1)}a^{(s-1)}a^{(s)}b^{(s)} \ll w$.

- Case 2: $a \in A$ and $b \in B$. Let $a \in \{a_{2s-1}, a_{2s}\}$, for $1 \leqslant s \leqslant \frac{m}{2}$.

i) If $b \in N(a_{2s-1}) \backslash N(a_{2s})$ then $b^{(s)} \ll B_{2s-1} \ll w_{(2s-1,2s)}$.

(a) If $s = 1$ then $b^{(1)}a^{(1)}a^{(2)} \ll w_{(1,2)}$ and $b^{(2)} \ll w_{(3,4)}$. Thus, $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w$.

(b) If $s > 1$ then $b^{(s-1)} \ll w_{(2s-3,2s-2)}$ and $b^{(s)}a^{(s-1)}a^{(s)} \ll w_{(2s-1,2s)}$. Thus, $b^{(s-1)}b^{(s)}a^{(s-1)}a^{(s)} \ll w$.

ii) If $b \in N(a_{2s}) \backslash N(a_{2s-1})$ then $b^{(s)} \ll B'_{2s} \ll w_{(2s-1,2s)}$.

(a) If $s = 1$ then $a_1^{(1)}a_1^{(2)}b^{(1)} \ll w_{(1,2)}$ and $b^{(2)} \ll w_{(3,4)}$, so that $a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w$.

(b) If $s > 1$ then $b^{(s-1)} \ll w_{(2s-3,2s-2)}$ and $a^{(s-1)}a^{(s)}b^{(s)} \ll w_{(2s-1,2s)}$, so that

$$b^{(s-1)}a^{(s-1)}a^{(s)}b^{(s)} \ll w.$$

iii) If $b \notin N(a_{2s-1}) \cup N(a_{2s})$ then either $b^{(s)} \ll B_{*(2s-1)}$ or $b^{(s)} \ll B'_{*(2s)}$.

(a) If $s = 1$ then $b^{(1)}a^{(1)}a^{(2)} \ll w_{(1,2)}$ or $a^{(1)}a^{(2)}b^{(1)} \ll w_{(1,2)}$, and $b^{(2)} \ll w_{(3,4)}$. Thus,

$$b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w \text{ or } a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w.$$

(b) If $s > 1$ then $b^{(s-1)} \ll w_{(2s-3,2s-2)}$, and $b^{(s)}a^{(s-1)}a^{(s)} \ll w_{(2s-1,2s)}$ or $a^{(s-1)}a^{(s)}b^{(s)} \ll w_{(2s-1,2s)}$. Thus,

$$b^{(s-1)}b^{(s)}a^{(s-1)}a^{(s)} \ll w \text{ or } b^{(s-1)}a^{(s-1)}a^{(s)}b^{(s)} \ll w.$$

- Case 3: $a, b \in B$. Note that $w = w_{(1,2)}uw_0$, for some u. Clearly, $w_{(1,2)}|_B \ll w_{(1,2)}$, and by construction of w_0 we have, $w^R_{(1,2)}|_B \ll w_0$. Since a and b occur exactly once in $w_{(1,2)}$ (cf. Remark 3.1.4), without loss of generality, assume $ab \ll w_{(1,2)}$. Then $w|_{\{a,b\}} = ab \cdots ba$. Since w is a uniform word, both a and b must occur the same number of times. Thus, two b's occur consecutively in $w|_{\{a,b\}}$, so that a and b do not alternate in w.

Hence, a and b do not alternate in the word w. □

An upper bound on the representation number of a bipartite graph is evident from Theorem 3.1.6 as stated in the following corollary.

Corollary 3.1.7. *The representation number of a bipartite graph is at most $1 + \lceil \frac{m}{2} \rceil$, where m is the size of the smallest partite set.*

3.2 Validation of Conjecture

From results given in [Glen et al., 2018] and [Akgün et al., 2019], we observe that, for graphs with at most nine vertices, the representation number of a bipartite graph is at most $1 + \lceil \frac{m}{2} \rceil$ and the upper bound is tight. In this section, we focus on validating

Conjecture 1.4.38 that $\mathcal{R}(G) \leqslant \lceil\frac{m+n}{4}\rceil$, for $m \geqslant 5$ through various cases based the sizes of the partite sets.

Theorem 3.2.1. *Let G be a bipartite graph. Then we have the following:*

1. *If $n \geqslant m+3$ then $\mathcal{R}(G) \leqslant \lceil\frac{m+n}{4}\rceil$.*
2. *If m is even and $n \in \{m+1, m+2\}$ then $\mathcal{R}(G) \leqslant \lceil\frac{m+n}{4}\rceil$.*

Proof.

1. By Corollary 3.1.7, if $m = 2k$ or $m = 2k-1$, for some $k \geqslant 1$, we have

$$\mathcal{R}(G) \leqslant 1 + \lceil\frac{m}{2}\rceil = 1 + k.$$

On the other hand, if $m = 2k$ then $\lceil\frac{m+n}{4}\rceil \geqslant \lceil\frac{4k+3}{4}\rceil = k+1$. Similarly, if $m = 2k-1$, $\lceil\frac{m+n}{4}\rceil \geqslant \lceil\frac{4k+1}{4}\rceil = k+1$. Hence, in any case, $\mathcal{R}(G) \leqslant 1+k \leqslant \lceil\frac{m+n}{4}\rceil$.

2. Let $m = 2k$, for some $k \geqslant 1$. Note that $\mathcal{R}(G) \leqslant 1+k$. If $n = m+1$ then $\lceil\frac{m+n}{4}\rceil = \lceil\frac{4k+1}{4}\rceil = k+1$. Similarly, if $n = m+2$ then $\lceil\frac{m+n}{4}\rceil = \lceil\frac{4k+2}{4}\rceil = k+1$ so that $\mathcal{R}(G) \leqslant 1+k = \lceil\frac{m+n}{4}\rceil$.

□

3.2.1 A Partite Set of Odd Size

We now show that, for $m \geqslant 5$, a bipartite graph is $\lceil\frac{m+n}{4}\rceil$-representable if m is odd.

Theorem 3.2.2. *Let G be a bipartite graph with partite sets of sizes m and n, where $5 \leqslant m \leqslant n \leqslant m+2$. If m is odd then G is $\lceil\frac{m}{2}\rceil$-representable.*

Proof. Let $P_1 = \{b \in B|\ \deg(b) = m-1\}$ and $P_2 = B\backslash P_1$. If $P_2 = \varnothing$ then the graph G is the crown graph $H_{m,m}$ and hence G is $\lceil\frac{m}{2}\rceil$-representable (cf. Theorem 1.4.36).

Suppose $P_2 \neq \varnothing$. Let $P_1 = \{b_1, b_2, \ldots, b_r\}$ and $P_2 = \{b_{r+1}, b_{r+2}, \ldots, b_n\}$. We construct a word representing G, along the lines of the construction given in Section

3.1. For which, since m is odd, we consider the set $A \cup \{a_{m+1}\}$ with a new isolated vertex a_{m+1} and partition it into pairs.

If $|P_1| = m$, note that P_2 has at most two vertices. If any one of the vertices of P_2 is of degree $\lceil \frac{m}{2} \rceil$, in the partition of $A \cup \{a_{m+1}\}$ into pairs we must have a specific pair (c_1, c_2) chosen as per the following:

- In case $n = m + 2$ and $\deg(b_{m+1}) = \deg(b_{m+2}) = \lceil \frac{m}{2} \rceil$, if $|N(b_{m+1}) \cap N(b_{m+2})| \geqslant 2$ then choose $c_1, c_2 \in N(b_{m+1}) \cap N(b_{m+2})$. Otherwise, choose $c_1, c_2 \in N(b_{m+1}) \backslash N(b_{m+2})$; this is possible since $m \geqslant 5$, we have $\lceil \frac{m}{2} \rceil \geqslant 3$.

- In case $n \in \{m + 1, m + 2\}$ and only one $b \in P_2$ is of degree $\lceil \frac{m}{2} \rceil$, choose $c_1, c_2 \in N(b)$.

For rest of the cases, the partition of $A \cup \{a_{m+1}\}$ into pairs can be done arbitrarily.

For each $1 \leqslant i \leqslant r$, since $\deg(b_i) = m - 1$, there exists $a_i \in A$ such that b_i is non-adjacent to a_i. Further, since G is reduced (i.e., each vertex of G has distinct neighborhood), these a_i's are distinct. Clearly, the remaining vertices of A, say $a_{r+1}, \ldots, a_m$, are adjacent to each vertex of P_1. If a_m is non-adjacent to a vertex in P_1 then $|P_1| = m$. Hence, the size of P_1 is at most m.

Let $M_1 = \{(a_1, a_2), \ldots, (a_{m-2}, a_{m-1}), (a_m, a_{m+1})\}$ be a partition of $A \cup \{a_{m+1}\}$ into pairs. In case $|P_1| = m$ and $\deg(b) = \lceil \frac{m}{2} \rceil$, for some $b \in P_2$ then without loss of generality, let $(a_1, a_2) = (c_1, c_2)$, as selected above.

We say that a vertex c is adjacent to a pair (a_i, a_j) if c is adjacent to a_i or a_j. Recall that in the construction of the word $w_{(i,i+1)}$ given in Section 3.1, the arrangement of the vertices of B within each block among $B_{*i}, B'_{*(i+1)}, B_i, B'_{(i+1)}$, and $w'_{(i,i+1)}$ are in any order. For each pair (a_i, a_{i+1}), we now construct the word $w_{(i,i+1)}$ following the process given in Section 3.1, but by arranging the vertices of B in each of the blocks in a certain order.

Note that M_1 has at least three pairs, as $m \geqslant 5$. For each $b \in B$, case wise, first we implement the following steps:

1. If b is non-adjacent to (at least) two pairs in M_1, say (a_i, a_{i+1}) and (a_j, a_{j+1}), then put b in B_{*i} of $w_{(i,i+1)}$ and put b in $B'_{*(j+1)}$ of $w_{(j,j+1)}$.

2. If b is non-adjacent to only one pair (a_i, a_{i+1}) in M_1, where $i \neq m$ then put b in B_{*i} of $w_{(i,i+1)}$ if b is in B'_{t+1} of some other word $w_{(t,t+1)}$. Otherwise, put b in $B'_{*(i+1)}$ of $w_{(i,i+1)}$.

3. If b is non-adjacent to only (a_m, a_{m+1}) then put b in $B'_{*(m+1)}$ of $w_{(m,m+1)}$ if b is in B_t of some other word $w_{(t,t+1)}$. Otherwise, put b in B_{*m} of $w_{(m,m+1)}$.

Then we focus on arranging the vertices of B within each block of $w_{(i,i+1)}$, for all i. Observe that if two vertices b and b' of B occur in the same block of $w_{(i,i+1)}$ then b and b' must occur in different blocks of some word $w_{(t,t+1)}$, as $G \cup \{a_{m+1}\}$ is reduced; otherwise b and b' would have the same neighborhood.

- For every pair of vertices b and b' within the same block of $w_{(1,2)}$, if b' occurs before b in different blocks of some word $w_{(t,t+1)}$ then put b before b' in the block of $w_{(1,2)}$. Else, put b' before b in the block of $w_{(1,2)}$.

- For $i > 1$, for every pair of vertices b and b' within the same block of $w_{(i,i+1)}$, if b' occurs before b in the word $w_{(1,2)}$ then put b before b' in the block of $w_{(i,i+1)}$. Else, put b' before b in the block of $w_{(i,i+1)}$.

By Corollary 3.1.7, the following $(1 + \lceil \frac{m}{2} \rceil)$-uniform word w represents $G \cup \{a_{m+1}\}$.

$$w = w_{(1,2)} D_{(1:4)} w_{(3,4)} \cdots w_{(m,m+1)} D_{(m:2)} w_0$$

Recall that w_0 is a permutation on the set $A \cup B \cup \{a_{m+1}\}$. We show that the following $\lceil \frac{m}{2} \rceil$-uniform word w_1 obtained by dropping the suffix w_0 from w represents the graph $G \cup \{a_{m+1}\}$.

$$w_1 = w_{(1,2)} D_{(1:4)} w_{(3,4)} \cdots w_{(m,m+1)} D_{(m:2)}$$

Note that if any two vertices are adjacent in $G \cup \{a_{m+1}\}$ then they alternate in w and hence they also alternate in w_1. Suppose two vertices of $G \cup \{a_{m+1}\}$ are non-adjacent. If both are in $A \cup \{a_{m+1}\}$ or one is in $A \cup \{a_{m+1}\}$ and the other is in B, observe that they do not alternate in w so that they will not alternate in w_1, as w_0 does not have any role in the proof of non-adjacency for $m > 4$ in Corollary 3.1.7.

We only need to show that if $a, b \in B$ then a and b do not alternate in w_1. Since every vertex of B occur exactly once in each word $w_{(i,i+1)}$ of w, we conclude the result by proving the following claim.

Claim. There exist two words $w_{(r,r+1)}$ and $w_{(s,s+1)}$ such that $ab \ll w_{(r,r+1)}$ and $ba \ll w_{(s,s+1)}$.

Let a or b be non-adjacent to at least two pairs in M_1. Suppose a and b belong to the same block of some word $w_{(j,j+1)}$. If $j = 1$ then $ab \ll w_{(1,2)}$ if only if $ba \ll w_{(t,t+1)}$, for some $t \neq 1$. If $j \neq 1$ then $ab \ll w_{(j,j+1)}$ if and only if $ba \ll w_{(1,2)}$. Suppose a and b belong to different blocks in every $w_{(2s-1,2s)}$, for $1 \leqslant s \leqslant \frac{m}{2}$. Let a be non-adjacent to (a_r, a_{r+1}) and (a_s, a_{s+1}). Without loss of generality, assume that $a \ll B_{*r} \ll w_{(r,r+1)}$ then $a \ll B'_{*(s+1)} \ll w_{(s,s+1)}$. We have $ab \ll w_{(r,r+1)}$ and $ba \ll w_{(s,s+1)}$.

On the other hand, both a, b are non-adjacent to at most one pair in M_1. That is, a is adjacent to all pairs or a is non-adjacent to exactly one pair. Also, b is adjacent to all pairs or b is non-adjacent to exactly one pair. Accordingly, by symmetry between a and b, we have the following cases: (i) both a, b are adjacent to all pairs, (ii) a is non-adjacent to one pair, say (a_i, a_{i+1}), and b is adjacent to all pairs (iii) a is non-adjacent to one pair, say (a_i, a_{i+1}), and b is non-adjacent to one pair, say (a_j, a_{j+1}).

Since $N(a_{m+1}) = \varnothing$, any vertex adjacent to the pair (a_m, a_{m+1}) must be adjacent to only a_m. Depending on whether i or j equals m, aforementioned three cases will yield the following. In case (i), we have $a, b \in N(a_m)$. In case (ii), clearly $b \in N(a_m)$; and if $i \neq m$ then $a \in N(a_m)$, otherwise $a \notin N(a_m)$. In case (iii), we get all four possibilities based on $a \in N(a_m)$ or not, and $b \in N(a_m)$ or not. Accordingly, by symmetry between a and b, we prove our claim in the following three cases:

- Case 1: $a, b \in N(a_m)$. In this case, a and b belong to the same block B_m of $w_{(m,m+1)}$. Thus, if $ab \ll w_{(1,2)}$ then $ba \ll w_{(m,m+1)}$. Otherwise, $ab \ll w_{(m,m+1)}$ if $ba \ll w_{(1,2)}$.

- Case 2: $a, b \notin N(a_m)$. If a and b belong to the same block, as seen before, we have, $ab \ll w_{(m,m+1)}$ if and only if $ba \ll w_{(1,2)}$. Otherwise, without loss of generality, suppose $a \ll B_{*m}$ and $b \ll B'_{*(m+1)}$ of $w_{(m,m+1)}$ so that $ab \ll w_{(m,m+1)}$. Since $b \ll B'_{*(m+1)}$, as per our construction of placing the vertices of B (cf. step 3), the vertex b must occur in B_t of some other word $w_{(t,t+1)}$. If a and b belong to the same block of $w_{(t,t+1)}$ then $ab \ll B_t \ll w_{(t,t+1)}$ if and only if $ba \ll w_{(1,2)}$, when $t \neq 1$. However, if $t = 1$, as M_1 has at least three pairs, there exists $j \notin \{1, m\}$ such that $ba \ll w_{(j,j+1)}$ if and only if $ab \ll w_{(1,2)}$. Else, if a is not in the block B_t of $w_{(t,t+1)}$ then $ba \ll w_{(t,t+1)}$. Thus, in any case, there exist r, s such that $ab \ll w_{(r,r+1)}$ and $ba \ll w_{(s,s+1)}$.

- Case 3: $a \in N(a_m)$ and $b \notin N(a_m)$. Suppose a and b belong to the same block of some word $w_{(j,j+1)}$. If $j = 1$ then $ab \ll w_{(1,2)}$ if and only if $ba \ll w_{(t,t+1)}$, for some $t \neq 1$. If $j \neq 1$ then $ab \ll w_{(j,j+1)}$ if and only if $ba \ll w_{(1,2)}$. Suppose a and b belong to different blocks in every $w_{(2s-1,2s)}$, for $1 \leqslant s \leqslant \frac{m}{2}$. We handle this in the following two subcases.

  – Subcase 3.1: $b \ll B'_{*(m+1)}$. In this case, $ab \ll B_m B'_{*(m+1)} \ll w_{(m,m+1)}$ and, by our construction of placing the vertices of B (cf. step 3), b belongs to B_t of some word $w_{(t,t+1)}$. In the word $w_{(t,t+1)}$, if a does not belong to B_{*t} then $ba \ll w_{(t,t+1)}$. Otherwise, $ab \ll B_{*t} B_t \ll w_{(t,t+1)}$. But, since M_1 has at least three pairs, there exists another word $w_{(s,s+1)}$ with $s \notin \{m, t\}$ and $a \ll B'_{s+1} \ll w_{(s,s+1)}$ (by step 2 of our construction), so that $ba \ll w_{(s,s+1)}$.

  – Subcase 3.2: $b \ll B_{*m}$. Then $ba \ll w_{(m,m+1)}$. Note that, for all odd j and $1 \leqslant j < m$, b belongs to either $w'_{(j,j+1)}$ or $B'_{(j+1)}$ of $w_{(j,j+1)}$.

(i) If b is of degree at most $m-2$ then b belongs to B'_{t+1} of some word $w_{(t,t+1)}$. In the word $w_{(t,t+1)}$, if a does not belong to $B'_{*(t+1)}$ then $ab \ll w_{(t,t+1)}$. Otherwise, $ba \ll B'_{t+1}B'_{*(t+1)} \ll w_{(t,t+1)}$. As $a \ll B'_{*(t+1)}$ and M_1 has at least three pairs, a belongs to either B_s or $w'_{(s,s+1)}$ of some word $w_{(s,s+1)}$, where $s \notin \{t, m\}$ so that $ab \ll w_{(s,s+1)}$.

(ii) If b is of degree $m-1$ then the size of P_1 must be m. Note that if $|P_1| < m$ then a_m must be adjacent to each vertex of P_1. Also, b belongs to $w'_{(j,j+1)}$, for all odd j and $1 \leqslant j < m$. Suppose $ba \ll w_{(j,j+1)}$, for all odd j and $1 \leqslant j < m$. Then $a \ll B'_{j+1}$, for each $w_{(j,j+1)}$; otherwise if $a \ll B'_{*(t+1)}$ of some word $w_{(t,t+1)}$ then the vertex a either belongs to B_s or $w'_{(s,s+1)}$, for all odd $s \neq t$ and, hence, we arrive at a contradiction. Thus, $\deg(a) = \lceil \frac{m}{2} \rceil$. As per the partition of the set $A \cup \{a_{m+1}\}$ into pairs, a_1 and a_2 are considered to be adjacent to a. Thus, a belongs to $w'_{(1,2)}$ of $w_{(1,2)}$. This is a contradiction, as a and b are in different blocks of every $w_{(j,j+1)}$, for odd j and $1 \leqslant j \leqslant m$. Hence, there exists $t \neq m$ such that $ab \ll w_{(t,t+1)}$.

Thus, a and b do not alternate in w_1.

Hence, $w_1|_{A\cup B}$ represents the graph G so that G is $\lceil \frac{m}{2} \rceil$-representable. □

Theorem 3.2.3. *Let G be a bipartite graph with partite sets of sizes m and n, where $5 \leqslant m \leqslant n$. If m is odd then $\mathcal{R}(G) \leqslant \lceil \frac{m+n}{4} \rceil$.*

Proof. In view of Theorem 3.2.1(1), it is sufficient to verify the statement for $n \in \{m, m+1, m+2\}$. Let $m = 2k+1$, for some $k \geqslant 2$ and $n = 2k+t$, for $1 \leqslant t \leqslant 3$. Then by Theorem 3.2.2,

$$\lceil \frac{m+n}{4} \rceil = \lceil \frac{4k+t+1}{4} \rceil = k+1 = \lceil \frac{m}{2} \rceil \geqslant \mathcal{R}(G).$$

□

Through Theorem 3.2.1 and Theorem 3.2.3, apart from the bipartite graphs with partite sets of equal and even size, we proved that the conjecture by Glen et al. on the representation number of bipartite graphs is valid for all other bipartite graphs.

Remark 3.2.4. As the crown graph on $2n$ vertices has representation number $\lceil \frac{n}{2} \rceil$, in view of Corollary 3.1.7 and Theorem 3.2.2, the conjecture by Akgün et. al. (see. Conjecture 1.4.39) that the crown graph $H_{n,n}$ has the highest representation number among all bipartite graphs on $2n$ vertices holds when n is odd.

3.2.2 Partite Sets of Equal and Even Size

In this section, we focus on bipartite graphs with partite sets of equal and even size, i.e., $m = n$ and m is even. For a subclass of these graphs, we also verify Conjecture 1.4.38 by constructing a uniform word-representant using the neighborhood inclusion graphs.

Recall that P_A is the poset induced by the neighborhood inclusion graph $\mathcal{N}_A(G)$ with respect to A.

Theorem 3.2.5. *Let G be a bipartite graph with a partite set $A = \{a_1, a_2, \ldots, a_m\}$, where $m = 2k$ $(k \geqslant 3)$, and $h(P_A) \geqslant 3$. Then $\mathcal{R}(G) \leqslant k$.*

Proof. As P_A is a poset of height at least three, there exists a chain of size three, say $\{a_1, a_2, a_3\}$, such that $a_1 \prec a_2 \prec a_3$. We now construct a k-uniform word-representant for G. We follow the same method given in Section 3.1 but with the following modification. First, partition the set A into $k-1$ pairs given by:

$$(a_1 \prec a_2 \prec a_3, a_4), (a_5, a_6), \ldots, (a_{m-1}, a_m)$$

Except for the first pair, as given in Section 3.1, we construct the word $w_{(i,j)}$ for each pair (a_i, a_j), for $5 \leqslant i \leqslant m-1$ and odd i. Note that the first component of the first pair is the chain $a_1 \prec a_2 \prec a_3$. Considering the index of its first element, the word $w_{(i,j)}$ corresponding the first pair is denoted by $w_{(1,4)}$ and constructed over the set $\{a_1, a_2, a_3, a_4\} \cup B$ as per the following:

$$w_{(1,4)} = B_{*1}a_3B_3a_2B_2a_1B_1a_4C_1a_1C_2a_2C_3a_3B'_4a_4B'_{*4}$$

where,

i. $B_{*1}B'_{*4}$: A permutation on the vertices of B which are non-adjacent to any vertex of $\{a_1, a_2, a_3, a_4\}$.

ii. B_1: A permutation on the neighbors of a_1 but not of a_4, i.e., $N(a_1)\backslash N(a_4)$.

iii. B_2: A permutation on the neighbors of a_2 but not of a_1 and a_4, i.e., $N(a_2)\backslash(N(a_1) \cup N(a_4))$.

iv. B_3: A permutation on the neighbors of a_3 but not of a_2 and a_4, i.e., $N(a_3)\backslash(N(a_2) \cup N(a_4))$.

v. C_1: A permutation on the common neighbors of a_1 and a_4, i.e., $N(a_1) \cap N(a_4)$.

vi. C_2: A permutation on the common neighbors of a_4 and a_2 but not of a_1, i.e., $(N(a_2) \cap N(a_4))\backslash N(a_1)$.

vii. C_3: A permutation on the common neighbors of a_4 and a_3 but not of a_2, i.e., $(N(a_3) \cap N(a_4))\backslash N(a_2)$.

viii. B'_4: A permutation on the neighbors of a_4 but not of a_3, i.e., $N(a_4)\backslash N(a_3)$.

Similar to w_0 constructed in Section 3.1, we construct the permutation u_0 on the set $A \cup B$ by

$$u_0 = a_4a_1a_2a_3w^R_{(1,4)}|_Ba_6a_5 \cdots a_ma_{m-1}.$$

Finally, we construct a k-uniform word

$$w_2 = w_{(1,4)}D_{(1:6)}w_{(5,6)}D_{(5:8)}w_{(7,8)} \cdots w_{(m-1,m)}D_{(m-1:4)}u_0$$

over the set $A \cup B$, where the word $D_{(p,q)}$ is constructed as per the definition given in Section 3.1 for consecutive pairs, in which whenever the index 1 is referred the word $a_1a_2a_3$ will be placed together. For example, $D_{(5,8)} = a_4a_{10}a_{12}\cdots a_ma_1a_2a_3a_9a_{11}\cdots a_{m-1}$.

We show that for any two vertices $a, b \in A \cup B$, a and b are adjacent in G if and only if a and b alternate in w_2, so that w_2 is a word-representant for the bipartite graph G. The proof is similar to the proof of Theorem 3.1.6, by considering the construction given here.

Suppose a and b are adjacent in G. Let $a \in A$ and $b \in B$. We deal this in the following three cases:

- Case 1: $a \in \{a_1, a_2, a_3, a_4\}$. We have

$$\begin{aligned} a^{(1)}b^{(1)}a^{(2)} &\ll w_{(1,4)} \\ \text{for all } 3 \leqslant i \leqslant k-1, \quad b^{(i-1)} &\ll w_{(2i-1,2i)} \text{ and } a^{(i)} \ll D_{(2i-1:2i+2)} \\ b^{(k-1)} &\ll w_{(m-1,m)} \\ a^{(k)}b^{(k)} &\ll u_0 \end{aligned}$$

so that $a^{(1)}b^{(1)}a^{(2)}b^{(2)}\cdots a^{(k)}b^{(k)}$ is a subword of w_2.

- Case 2: $a \in \{a_5, a_6\}$. We have

$$\begin{aligned} b^{(1)} &\ll w_{(1,4)} \\ a^{(1)}b^{(2)}a^{(2)} &\ll w_{(5,6)} \\ \text{for all } 4 \leqslant i \leqslant k-1, \quad b^{(i-1)} &\ll w_{(2i-1,2i)} \text{ and } a^{(i-1)} \ll D_{(2i-1:2i+2)} \\ b^{(k-1)} &\ll w_{(m-1,m)} \text{ and } a^{(k-1)} \ll D_{(m-1:4)} \\ b^{(k)}a^{(k)} &\ll u_0 \end{aligned}$$

so that $b^{(1)}a^{(1)}b^{(2)}a^{(2)}\cdots b^{(k)}a^{(k)}$ is a subword of w_2.

- Case 3: $a \in \{a_{2s-1}, a_{2s}\}$, for $4 \leqslant s \leqslant k$. We have

$$
\begin{aligned}
&& b^{(1)} &\ll w_{(1,4)} \\
&& a^{(1)} &\ll D_{(1:6)} \\
\text{for all } i \in \{3, \dots, s-2, s+1, \dots, k\}, && b^{(i-1)} &\ll w_{(2i-1,2i)} \\
&& a^{(i-1)} &\ll \begin{cases} D_{(2i-1:2i+2)}, & \text{if } i \neq k; \\ D_{(m-1:4)}, & \text{if } i = k \end{cases} \\
&& b^{(s-2)} &\ll w_{(2s-3,2s-2)} \\
&& a^{(s-2)}b^{(s-1)}a^{(s-1)} &\ll w_{(2s-1,2s)} \\
&& b^{(k)}a^{(k)} &\ll u_0
\end{aligned}
$$

so that $b^{(1)}a^{(1)}b^{(2)}a^{(2)} \cdots b^{(k)}a^{(k)}$ is a subword of w_2.

Hence, a and b alternate in the word w_2.

Conversely, suppose a and b are non-adjacent in G. We show that, for some $j \geqslant 1$, one of the following words is a subword of w_2: $a^{(j)}b^{(j)}b^{(j+1)}a^{(j+1)}$, $b^{(j)}a^{(j)}a^{(j+1)}b^{(j+1)}$, $a^{(j)}a^{(j+1)}b^{(j)}b^{(j+1)}$, $b^{(j)}b^{(j+1)}a^{(j)}a^{(j+1)}$. We deal this in the following three cases:

- Case 1: $a, b \in A$. We have the following subcases:

  – Subcase 1.1: a, b belong to the same pair.

    i) Let a, b belong to the first pair. If $a, b \in \{a_1, a_2, a_3\}$, without loss of generality, assume $a = a_s$ and $b = a_t$ with $s > t$. Then clearly $a^{(1)}b^{(1)}b^{(2)}a^{(2)} \ll w_{(1,4)}$. Otherwise, let $a \in \{a_1, a_2, a_3\}$ and $b = a_4$. Then

    $$a^{(1)}b^{(1)}a^{(2)}b^{(2)} \ll w_{(1,4)}, \text{ and } b^{(3)}a^{(3)} \ll \begin{cases} u_0, & \text{for } m = 6; \\ D_{(5:8)}, & \text{for } m > 6 \end{cases}$$

    so that $a^{(2)}b^{(2)}b^{(3)}a^{(3)} \ll w_2$. Thus, in any case, $a^{(j)}b^{(j)}b^{(j+1)}a^{(j+1)} \ll w_2$, for some j.

    ii) If $a = a_5$ and $b = a_6$ then $a^{(1)}b^{(1)}a^{(2)}b^{(2)} \ll w_{(5,6)}$ and

$$b^{(3)}a^{(3)} \ll \begin{cases} u_0, & \text{for } m = 6; \\ D_{(7:4)}, & \text{for } m = 8; \\ D_{(7:10)}, & \text{for } m > 8 \end{cases}$$

so that $a^{(2)}b^{(2)}b^{(3)}a^{(3)} \ll w_2$.

iii) For $4 \leqslant s \leqslant k$, if $a = a_{2s-1}$ and $b = a_{2s}$ then

$$a^{(s-2)}b^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_{(2s-1,2s)} \text{ and}$$

$$b^{(s-3)}a^{(s-3)} \ll \begin{cases} D_{(1:6)}, & \text{for } s = 4; \\ D_{(2s-5:2s-2)}, & \text{for } s > 4 \end{cases}$$

so that $b^{(s-3)}a^{(s-3)}a^{(s-2)}b^{(s-2)} \ll w_2$.

– Subcase 1.2: a and b belong to different pairs.

i) Let one of a, b belong to the first pair $(a_1 \prec a_2 \prec a_3, a_4)$. Without loss of generality, let $a \in \{a_1, a_2, a_3, a_4\}$. As $b \notin \{a_1, a_2, a_3, a_4\}$, b does not appear in $w_{(1,4)}$. Accordingly, we have $a^{(1)}a^{(2)} \ll w_{(1,4)}$ so that $a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w_2$.

ii) If $a \in \{a_{2s-1}, a_{2s}\}$, for $3 \leqslant s \leqslant k$ then consider the following cases depending on whether b is in the subsequent/preceding pair or not:

(a) If $b \in \{a_{2s+1}, a_{2s+2}\}$ then

$$b^{(s-2)} \ll \begin{cases} D_{(1:6)}, & \text{for } s = 3; \\ D_{(2s-3:2s)}, & \text{for } s > 3 \end{cases}$$

$$a^{(s-2)}a^{(s-1)} \ll w_{(2s-1,2s)}$$

$$b^{(s-1)}b^{(s)} \ll w_{(2s+1,2s+2)}$$

so that $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_2$.

(b) If $b \notin \{a_{2s-3}, a_{2s-2}, a_{2s+1}, a_{2s+2}\}$ then

$$b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll \begin{cases} D_{(1:6)}w_{(5,6)}D_{(5:4)}, \text{for } s=3 \text{ and } m=6; \\ D_{(1:6)}w_{(5,6)}D_{(5:8)}, \text{for } s=3 \text{ and } m>6; \\ D_{(2s-3:2s)}w_{(2s-1,2s)}D_{(2s-1:4)}, \text{for } s=k; \\ D_{(2s-3:2s)}w_{(2s-1,2s)}D_{(2s-1:2s+2)}, \text{ else.} \end{cases}$$

Thus, $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_2$.

- Case 2: $a \in A$ and $b \in B$. We deal this case in the following three subcases.
  - Subcase 2.1: $a \in \{a_1, a_2, a_3, a_4\}$.
    i) If $b \in N(a_4)\backslash N(a_3)$ then $a^{(1)}a^{(2)}b^{(1)} \ll w_{(1,4)}$. This implies that $a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w_2$.
    ii) If $b \in N(a_3)\backslash N(a_4)$ then $b^{(1)}a^{(1)}a^{(2)} \ll w_{(1,4)}$ and $b^{(2)} \ll w_{(5,6)}$. Thus, $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_2$.
    iii) If $b \notin N(a_3) \cup N(a_4)$ then either $b^{(1)} \ll B_{*1}$ or $b^{(1)} \ll B'_{*4}$, and $b^{(2)} \ll w_{(5,6)}$. Thus, either $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_2$ or $a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w_2$.
  - Subcase 2.2: $a \in \{a_{2s-1}, a_{2s}\}$, for $3 \leqslant s \leqslant k$.
    i) If $b \in N(a_{2s})\backslash N(a_{(2s-1)})$ then $b^{(s-1)} \ll B'_{2s}$. Thus,

$$b^{(s-2)} \ll \begin{cases} w_{(1,4)}, & \text{for } s=3; \\ w_{(2s-3,2s-2)}, & \text{for } s>3, \end{cases} \quad \text{and } a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_{(2s-1,2s)},$$

so that $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_2$.

    ii) If $b \in N(a_{2s-1})\backslash N(a_{2s})$ then $b^{(s-1)} \ll B_{2s-1}$. Thus,

$$b^{(s-2)} \ll \begin{cases} w_{(1,4)}, & \text{for } s=3; \\ w_{(2s-3,2s-2)}, & \text{for } s>3, \end{cases} \quad \text{and } b^{(s-1)}a^{(s-2)}a^{(s-1)} \ll w_{(2s-1,2s)},$$

so that $b^{(s-2)}b^{(s-1)}a^{(s-2)}a^{(s-1)} \ll w_2$.

iii) If $b \notin N(a_{2s-1}) \cup N(a_{2s})$ then either $b^{(s-1)} \ll B_{*(2s-1)}$ or $b^{(s-1)} \ll B'_{*(2s)}$. Thus,

$$b^{(s-2)} \ll \begin{cases} w_{(1,4)}, & \text{for } s = 3; \\ w_{(2s-3,2s-2)}, & \text{for } s > 3 \end{cases}$$

so that either $b^{(s-2)}b^{(s-1)}a^{(s-2)}a^{(s-1)} \ll w_2$ or $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_2$.

- Case 3: $a, b \in B$. The word w_2 is of the form $w_{(1,4)}vu_0$. Clearly, $w_{(1,4)}|_B \ll w_{(1,4)}$, and by construction of u_0 we have, $w^R_{(1,4)}|_B \ll u_0$. Since a and b occur exactly once in $w_{(1,4)}$, without loss of generality, assume that $ab \ll w_{(1,4)}$ so that $w_2|_{\{a,b\}} = ab \cdots ba$. Since w_2 is a uniform word, both a and b must occur the same number of times. Thus, two b's occur consecutively in $w_2|_{\{a,b\}}$, so that a and b do not alternate in w_2.

Hence, w_2 represents G so that $\mathcal{R}(G) \leqslant k$. □

Remark 3.2.6. In view of Theorem 3.2.5, note that Conjecture 1.4.38 holds for all bipartite graphs considered in this section having $h(P_A) \geqslant 3$.

For another special class from the remaining graphs, we have the following theorem.

Theorem 3.2.7. *Let G be a bipartite graph with a partite set $A = \{a_1, a_2, \ldots, a_m\}$, where $m = 2k$ $(k \geqslant 3)$. If $h(P_A) = 2$ and contains at least two disjoint chains of size two then $\mathcal{R}(G) \leqslant k$.*

Proof. Let $a_1 < a_2$ and $a_4 < a_5$ be two disjoint chains in P_A. We extend the method given in Theorem 3.2.5, and construct a k-uniform word-representant for G. First, partition the set A into $k-1$ pairs given by:

$$(a_1 < a_2, a_3), (a_4 < a_5, a_6), (a_7, a_8), \ldots, (a_{m-1}, a_m)$$

Except for the first two pairs, as given in Section 3.1, we construct the word $w_{(i,j)}$ for each pair (a_i, a_j), for $7 \leqslant i \leqslant m-1$ and i odd. For the first pair $(a_1 \prec a_2, a_3)$, construct the word denoted by $w_{(1,3)}$ on the set $\{a_1, a_2, a_3\} \cup B$ as per the following:

$$w_{(1,3)} = B_{*1}a_2B_2a_1B_1a_3C_1a_1C_2a_2B'_3a_3B'_{*3}$$

where,

i. $B_{*1}B'_{*3}$: A permutation on the vertices of B which are non-adjacent to any vertex of $\{a_1, a_2, a_3\}$.

ii. B_1: A permutation on the neighbors of a_1 but not of a_3, i.e., $N(a_1)\backslash N(a_3)$.

iii. B_2: A permutation on the neighbors of a_2 but not of a_1 and a_3, i.e., $N(a_2)\backslash(N(a_1) \cup N(a_3))$.

iv. C_1: A permutation on the common neighbors of a_1 and a_3, i.e., $N(a_1) \cap N(a_3)$.

v. C_2: A permutation on the common neighbors of a_3 and a_2 but not of a_1, i.e., $(N(a_2) \cap N(a_3))\backslash N(a_1)$.

vi. B'_3: A permutation on the neighbors of a_3 but not of a_2, i.e., $N(a_3)\backslash N(a_2)$.

Similarly, for the pair $(a_4 \prec a_5, a_6)$, construct the word denoted by $w_{(4,6)}$ on the set $\{a_4, a_5, a_6\} \cup B$ as per the following:

$$w_{(4,6)} = B_{*4}a_5B_5a_4B_4a_6C_4a_4C_5a_5B'_6a_6B'_{*6}$$

where,

i. $B_{*4}B'_{*6}$: A permutation on the vertices of B which are non-adjacent to any vertex of $\{a_4, a_5, a_6\}$.

ii. B_4: A permutation on the neighbors of a_4 but not of a_6, i.e., $N(a_4)\backslash N(a_6)$.

iii. B_5: A permutation on the neighbors of a_5 but not of a_4 and a_6, i.e., $N(a_5)\backslash(N(a_4) \cup N(a_6))$.

iv. C_1: A permutation on the common neighbors of a_4 and a_6, i.e., $N(a_4) \cap N(a_6)$.

v. C_2: A permutation on the common neighbors of a_6 and a_5 but not of a_4, i.e., $(N(a_5) \cap N(a_6)) \setminus N(a_4)$.

vi. B_6': A permutation on the neighbors of a_6 but not of a_5, i.e., $N(a_6) \setminus N(a_5)$.

Now construct the permutation v_0 on the set $A \cup B$ by considering the reversal of the word $w_{(1,3)}$ restricted to the vertices of B as per the following:

$$v_0 = a_3 a_1 a_2 w^R_{(1,3)}|_B a_6 a_4 a_5 a_8 a_7 \cdots a_m a_{m-1}$$

Finally, construct the k-uniform word w_3 over $A \cup B$ as per the following:

$$w_3 = w_{(1,3)} D_{(1:6)} w_{(4,6)} D_{(4:8)} w_{(7,8)} \cdots w_{(m-1,m)} D_{(m-1:3)} v_0$$

Along the similar lines of Theorem 3.2.5, we show that the word w_3 represents the graph G. Suppose a and b are adjacent in G. Let $a \in A$ and $b \in B$. We deal this in the following four cases:

- Case 1: $a \in \{a_1, a_2, a_3\}$: We have

$$\begin{aligned} a^{(1)} b^{(1)} a^{(2)} &\ll w_{(1,3)} \\ b^{(2)} &\ll w_{(4,6)} \text{ and } a^{(3)} \ll D_{(4:8)} \\ \text{for all } \ 4 \leqslant i \leqslant k-1, \quad b^{(i-1)} &\ll w_{(2i-1,2i)} \text{ and } a^{(i)} \ll D_{(2i-1:2i+2)} \\ b^{(k-1)} &\ll w_{(m-1,m)} \\ a^{(k)} b^{(k)} &\ll v_0 \end{aligned}$$

  so that $a^{(1)} b^{(1)} a^{(2)} b^{(2)} \cdots a^{(k)} b^{(k)} \ll w_3$.

- Case 2: $a \in \{a_4, a_5, a_6\}$: We have

$$\begin{aligned} b^{(1)} &\ll w_{(1,3)} \\ a^{(1)}b^{(2)}a^{(2)} &\ll w_{(4,6)} \\ \text{for all } 4 \leqslant i \leqslant k-1, \quad b^{(i-1)} &\ll w_{(2i-1,2i)} \text{ and } a^{(i-1)} \ll D_{(2i-1:2i+2)} \\ b^{(k-1)} &\ll w_{(m-1,m)} \text{ and } a^{(k-1)} \ll D_{(m-1:3)} \\ b^{(k)}a^{(k)} &\ll v_0 \end{aligned}$$

so that $b^{(1)}a^{(1)}b^{(2)}a^{(2)}\cdots b^{(k)}a^{(k)} \ll w_3$.

- Case 3: $a \in \{a_7, a_8\}$: We have

$$\begin{aligned} b^{(1)} &\ll w_{(1,3)} \text{ and } a^{(1)} \ll D_{(1:6)} \\ b^{(2)} &\ll w_{(4,6)} \\ a^{(2)}b^{(3)}a^{(3)} &\ll w_{(7,8)} \\ \text{for all } 5 \leqslant i \leqslant k-1, \quad b^{(i-1)} &\ll w_{(2i-1,2i)} \text{ and } a^{(i-1)} \ll D_{(2i-1:2i+2)} \\ b^{(k-1)} &\ll w_{(m-1,m)} \text{ and } a^{(k-1)} \ll D_{(m-1:3)} \\ b^{(k)}a^{(k)} &\ll v_0 \end{aligned}$$

so that $b^{(1)}a^{(1)}b^{(2)}a^{(2)}\cdots b^{(k)}a^{(k)} \ll w_3$.

- Case 4: $a \in \{a_{2s-1}, a_{2s}\}$, for $5 \leqslant s \leqslant k$. We have

$$\begin{aligned} b^{(1)} &\ll w_{(1,3)} \text{ and } a^{(1)} \ll D_{(1:6)} \\ b^{(2)} &\ll w_{(4,6)} \text{ and } a^{(2)} \ll D_{(4:8)} \\ \text{for all } \ i \in \{4, \ldots, s-2, s+1, \ldots, k\}, \quad b^{(i-1)} &\ll w_{(2i-1,2i)} \text{ and} \\ a^{(i-1)} &\ll \begin{cases} D_{(2i-1:2i+2)}, & \text{if } i \neq k; \\ D_{(m-1:3)}, & \text{if } i = k \end{cases} \\ b^{(s-2)} &\ll w_{(2s-3,2s-2)} \\ a^{(s-2)}b^{(s-1)}a^{(s-1)} &\ll w_{(2s-1,2s)} \\ b^{(k)}a^{(k)} &\ll v_0 \end{aligned}$$

so that $b^{(1)}a^{(1)}b^{(2)}a^{(2)}\cdots b^{(k)}a^{(k)} \ll w_3$.

Hence, a and b alternate in the word w_3. Conversely, suppose a and b are non-adjacent in G.

- Case 1: $a, b \in A$. We have the following subcases:

  – Subcase 1.1: a and b belong to the same pair.

    i) Let a, b belong to the first pair. If $a, b \in \{a_1, a_2\}$, without loss of generality, assume $a = a_2$ and $b = a_1$. Then clearly $a^{(1)}b^{(1)}b^{(2)}a^{(2)} \ll w_{(1,3)}$. Otherwise, let $a \in \{a_1, a_2\}$ and $b = a_3$. Then $a^{(2)}b^{(2)} \ll w_{(1,3)}$ and

$$b^{(3)}a^{(3)} \ll \begin{cases} v_0, & \text{for } m = 6; \\ D_{(4:8)}, & \text{for } m > 6, \end{cases}$$

    so that $a^{(2)}b^{(2)}b^{(3)}a^{(3)} \ll w_3$. Thus, in any case, $a^{(j)}b^{(j)}b^{(j+1)}a^{(j+1)} \ll w_3$, for some j.

    ii) Let a, b belong to the second pair. If $a, b \in \{a_4, a_5\}$, without loss of generality, assume $a = a_5$ and $b = a_4$. Then clearly $a^{(1)}b^{(1)}b^{(2)}a^{(2)} \ll w_{(4,6)}$. Otherwise, let $a \in \{a_4, a_5\}$ and $b = a_6$. Then $a^{(2)}b^{(2)} \ll w_{(4,6)}$ and

$$b^{(3)}a^{(3)} \ll \begin{cases} v_0, & \text{for } m = 6; \\ D_{(7:3)}, & \text{for } m = 8; \\ D_{(7:10)} & \text{for } m > 8, \end{cases}$$

    so that $a^{(2)}b^{(2)}b^{(3)}a^{(3)} \ll w_3$. Thus, in any case, $a^{(j)}b^{(j)}b^{(j+1)}a^{(j+1)} \ll w_3$, for some j.

    iii) Let $a, b \in \{a_7, a_8\}$. Without loss of generality, assume $a = a_7$ and $b = a_8$. Then $b^{(1)}a^{(1)} \ll D_{(1:6)}$ and $a^{(2)}b^{(2)} \ll w_{(7,8)}$ so that $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_3$.

    iv) Let $a, b \in \{a_9, a_{10}\}$. Assume $a = a_9$ and $b = a_{10}$. Then $b^{(2)}a^{(2)} \ll D_{(4:8)}$ and $a^{(3)}b^{(3)} \ll w_{(9,10)}$ so that $b^{(2)}a^{(2)}a^{(3)}b^{(3)} \ll w_3$.

v) For $6 \leqslant s \leqslant k$, if $a = a_{2s-1}$ and $b = a_{2s}$ then $b^{(s-3)}a^{(s-3)} \ll D_{(2s-5:2s-2)}$ and $a^{(s-2)}b^{(s-2)} \ll w_{(2s-1,2s)}$. Thus, $b^{(s-3)}a^{(s-3)}a^{(s-2)}b^{(s-2)} \ll w_3$.

– Subcase 1.2: a and b belong to different pairs.

i) Let one of a, b be in the first pair $(a_1 \prec a_2, a_3)$. Without loss of generality, assume $a \in \{a_1, a_2, a_3\}$. Then $a^{(1)}a^{(2)} \ll w_{(1,3)}$ and $b^{(1)} \not\ll w_{(1,3)}$. Thus, $a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w_3$.

ii) Let one of a, b be in the second pair $(a_4 \prec a_5, a_6)$ and assume $a \in \{a_4, a_5, a_6\}$. Then $b^{(1)} \ll D_{(1:6)}$ and $a^{(1)}a^{(2)} \ll w_{(4,6)}$. Thus, $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_3$.

iii) For $4 \leqslant s \leqslant k$, if $a \in \{a_{2s-1}, a_{2s}\}$ then consider the following cases depending on whether b is in the subsequent/preceding pair or not:

(a) If $b \in \{a_{2s+1}, a_{2s+2}\}$ then

$$b^{(s-2)} \ll \begin{cases} D_{(4:8)}, & \text{for } s = 4; \\ D_{(2s-3:2s)}, & \text{for } s > 4 \end{cases}$$

$$a^{(s-2)}a^{(s-1)} \ll w_{(2s-1,2s)}$$

$$b^{(s-1)}b^{(s)} \ll w_{(2s+1,2s+2)}$$

so that $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_3$.

(b) If $b \notin \{a_{2s-3}, a_{2s-2}, a_{2s+1}, a_{2s+2}\}$ then

$$b^{(s-2)} \ll \begin{cases} D_{(4:8)}, & \text{for } s = 4; \\ D_{(2s-3:2s)}, & \text{for } s > 4 \end{cases}$$

$$a^{(s-2)}a^{(s-1)} \ll w_{(2s-1,2s)}$$

$$b^{(s-1)} \ll \begin{cases} D_{(2s-1:2s+2)}, & \text{for } s \neq k; \\ D_{(m-1:3)}, & \text{for } s = k \end{cases}$$

so that $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_3$.

- Case 2: $a \in A$ and $b \in B$. We deal this case in the following three subcases.

– Subcase 2.1: $a \in \{a_1, a_2, a_3\}$.

i) If $b \in N(a_3)$ then $a^{(1)}a^{(2)}b^{(1)} \ll w_{(1,3)}$ so that $a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w_3$.

ii) If $b \in N(a_2) \backslash N(a_3)$ then $b^{(1)}a^{(1)}a^{(2)} \ll w_{(1,3)}$ and $b^{(2)} \ll w_{(4,6)}$. Thus, $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_3$.

iii) If $b \notin N(a_2) \cup N(a_3)$ then either $b^{(1)} \ll B_{*1}$ or $b^{(1)} \ll B'_{*3}$, and $a^{(1)}a^{(2)} \ll w_{(1,3)}$. Thus, either $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_3$ or $a^{(1)}a^{(2)}b^{(1)}b^{(2)} \ll w_3$.

– Subcase 2.2: $a \in \{a_4, a_5, a_6\}$.

i) If $b \in N(a_6)$ then $b^{(1)} \ll w_{(1,3)}$ and $a^{(1)}a^{(2)}b^{(2)} \ll w_{(4,6)}$. Thus, $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_3$.

ii) If $b \in N(a_5) \backslash N(a_6)$ then $b^{(1)} \ll w_{(1,3)}$ and $b^{(2)}a^{(1)}a^{(2)} \ll w_{(4,6)}$. Thus, $b^{(1)}b^{(2)}a^{(1)}a^{(2)} \ll w_3$.

iii) If $b \notin N(a_5) \cup N(a_6)$ then $b^{(1)} \ll w_{(1,3)}$, and either $b^{(2)} \ll B_{*4}$ or $b^{(2)} \ll B'_{*6}$ so that either $b^{(1)}b^{(2)}a^{(1)}a^{(2)} \ll w_3$ or $b^{(1)}a^{(1)}a^{(2)}b^{(2)} \ll w_3$.

– Subcase 2.3: $a \in \{a_{2s-1}, a_{2s}\}$, for $4 \leqslant s \leqslant k$.

i) If $b \in N(a_{2s})$ then

$$b^{(s-2)} \ll \begin{cases} w_{(4,6)}, & \text{if } s = 4; \\ w_{(2s-3)(2s-2)}, & \text{if } s > 4, \end{cases} \quad \text{and}$$

$$a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_{(2s-1,2s)}$$

Thus, $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_3$.

ii) If $b \in N(a_{2s-1})$ then $b^{(s-2)} \ll \begin{cases} w_{(4,6)}, & \text{if } s = 4; \\ w_{(2s-3)(2s-2)}, & \text{if } s > 4, \end{cases}$ and $b^{(s-1)}a^{(s-2)}a^{(s-1)} \ll w_{(2s-1,2s)}$. Thus, $b^{(s-2)}b^{(s-1)}a^{(s-2)}a^{(s-1)} \ll w_3$.

iii) If $b \notin N(a_{2s-1}) \cup N(a_{2s})$ then

$$b^{(s-2)} \ll \begin{cases} w_{(4,6)}, & \text{if } s = 4; \\ w_{(2s-3)(2s-2)}, & \text{if } s > 4, \end{cases}$$

and either $b^{(s-1)} \ll B_{*(2s-1)}$ or $b^{(s-1)} \ll B'_{*(2s)}$. Thus, either $b^{(s-2)}b^{(s-1)}a^{(s-2)}a^{(s-1)} \ll w_3$ or $b^{(s-2)}a^{(s-2)}a^{(s-1)}b^{(s-1)} \ll w_3$.

- Case 3: $a, b \in B$. The word w_3 is of the form $w_{(1,3)}vv_0$. Clearly, $w_{(1,3)}|_B \ll w_{(1,3)}$, and by construction of v_0, we have $w^R_{(1,3)}|_B \ll v_0$. Since a and b occur exactly once in $w_{(1,3)}$, without loss of generality, assume $ab \ll w_{(1,3)}$ so that $w_3|_{\{a,b\}} = ab \cdots ba$. Since w_3 is a uniform word, both a and b must occur the same number of times. Thus, two b's occur consecutively in $w_3|_{\{a,b\}}$. Thus, a and b do not alternate in w_3.

Hence, w_3 represents G so that $\mathcal{R}(G) \leqslant k$. $\square$

4

Special Classes

Focusing on identification of graphs with the (permutation-)representation number three, in this chapter, we investigate two classes of bipartite graphs – the extended crown graphs and the stacked book graphs. The extended crown graphs are inspired by the Kimble split operation for posets. We show that the *prn* of extended crown graphs attains the bounds given in Theorem 2.3.6. Notably, we show that when the poset has width two then its extended crown graph belongs to the class of circle graphs and has *prn* at most three. For stacked book graphs, we prove that the *prn* is at most three, and subsequently show that the representation number and the *prn* of a stacked book graph coincide. Finally, we provide a complete classification of stacked book graphs based on their representation numbers.

4.1 Extended Crown Graphs

Definition 4.1.1 (Extended crown graph)**.** Given an n-element poset P, an *extended crown graph* defined by P is a bipartite graph $G = (A \cup B, E)$ on $2n$ vertices constructed as per the following:

i. Let A be the set of elements of P and $B = \{v' : v \in A\}$.

ii. Consider a largest antichain $M = \{a_1, a_2, \ldots, a_k\} \subseteq A$ of the poset P.

iii. Let $M' = \{a'_1, a'_2, \ldots, a'_k\} \subseteq B$.

iv. For $1 \leqslant i, j \leqslant k$ and $i \neq j$, put $\{a_i, a'_j\} \in E$, i.e., construct a crown graph of $2k$ vertices with M and M' as partite sets.

v. Then, for all $v \in A \backslash M$, put $\{v, v'\} \in E$.

vi. Further, for all $u, v \in A$, if $u \prec v$ in P then iteratively make the vertices of $N(u)$ adjacent to v, i.e., for all $w \in N(u), \{w, v\} \in E$.

vii. The resulting graph $G = (A \cup B, E)$ is an extended crown graph.

Remark 4.1.2. Note that an extended crown graph defined by a poset of width two is isomorphic to the split of the poset (for the definition of split, see Definition 1.3.20). However, this may not be true for an extended crown graph and the split of a poset of width at least three, as shown in Figure 4.1.

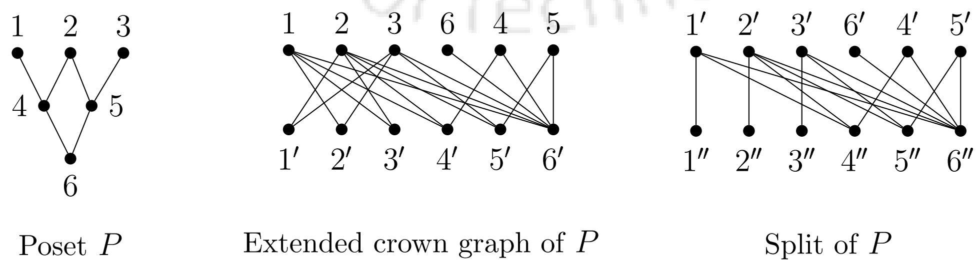


FIGURE 4.1: An extended crown graph and a split of a poset

Figure 4.2: An extended crown graph of a 4-element chain

Figure 4.3: An example for Remark 4.1.5

Remark 4.1.3. If a poset on n elements is an antichain then the corresponding extended crown graph is the crown graph $H_{n,n}$.

Remark 4.1.4. An extended crown graph of a chain will have $H_{1,1}$ as its largest induced crown graph. An example of the extended crown graph of a 4-element chain considering the largest antichain $\{1\}$ is depicted in Figure 4.2.

Remark 4.1.5. Let $G = (A \cup B, E)$ be the extended crown graph generated by a poset P of width two. It can be easily observed that P_A and P are the same, where P_A is the poset induced by the neighborhood inclusion graph of G with respect to A. This need not be true for posets of width at least three. For example, the poset P and the corresponding P_A given in Figure 4.3 are not the same.

However, in the following proposition, we observe that the widths of P and P_A are the same. Accordingly, the *prn* of extended crown graphs can be estimated as stated in Corollary 4.1.8.

Proposition 4.1.6. *Suppose $G = (A \cup B, E)$ is an extended crown graph defined by a poset P. Then $w(P_A) = w(P)$ and hence,* $\min\{w(P_A), w(P_B)\} = w(P)$.

Proof. Let M be a largest antichain of P. It can be observed from the construction of G using M that the neighborhoods of any pair of vertices in M are incomparable, as the vertices of M and M' form a crown graph. Hence, M is an antichain in P_A so that $w(P_A) \geqslant w(P)$.

On the other hand, as per Step-6 in the construction of G, if $u \prec v$ in P then $N(u) \subseteq N(v)$ in G. Thus, any two comparable elements in P are comparable in P_A. Hence, $w(P_A) \leqslant w(P)$.

Further, since G has a crown graph of size $w(P)$, we have $w(P_B) \geqslant w(P)$. Consequently, $\min\{w(P_A), w(P_B)\} = w(P)$. □

If $w(P) = k$, since $H_{k,k}$ is an induced subgraph of G, we have the following corollary of Proposition 4.1.6.

Corollary 4.1.7. *If $w(P) = k$ then $H_{k,k}$ is the largest induced crown graph of G.*

This, in turn, establishes that the extended crown graphs satisfy Theorem 2.3.4, as stated in the following corollary.

Corollary 4.1.8. *If G is an extended crown graph defined by a poset P then $\mathcal{R}^p(G) = w(P)$ or $1 + w(P)$.*

Remark 4.1.9. If G is an extended crown graph defined by a poset of width one then $\mathcal{R}^p(G) = 2$ because G is a bipartite graph.

4.1.1 Posets of Width 2

We now determine the *prn* as well as the representation number of extended crown graph defined by a poset of width two.

Let P be a poset of width two. By Corollary 4.1.8, the *prn* of an extended crown graph G defined by P is either two or three. Here, in Theorem 4.1.11, we prove that $\mathcal{R}^p(G) = 2$ if and only if the cover graph of P is a path or a disconnected graph.

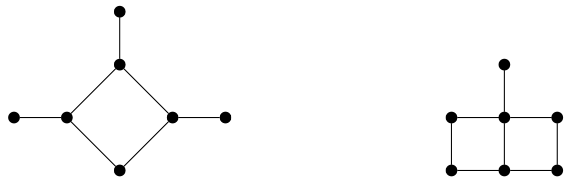


Figure 4.4: Forbidden induced subgraphs for the graphs with *prn* two

First, note that if the cover graph of P is disconnected then P is a union of two chains (in which no two elements chosen one each from the two chains are comparable). In this case, the graph G is disconnected, having two components. It can be observed that the poset induced by the neighborhood graph of each of these components is a chain. Accordingly, we can conclude $\mathcal{R}^p(G)$ as per the following remark.

Remark 4.1.10. If P is a union of two chains then $\mathcal{R}^p(G) = 2$.

If the cover graph of P is connected then one among the posets P_1, P_1', P_2, P_3, P_3', P_4 and P_5 given in Figure 4.5 is a subposet of P. Note that the cover graph of each of P_1, P_1' and P_2 is a path, but it is not in the case of P_3, P_3', P_4 and P_5. The extended crown graph defined by both P_3, P_3' is G_3, while it is G_4 for P_4 and G_5 for P_5 as shown in Figure 4.5. Recall from Gallai [1967] that S_1 and S_2 given in Figure 4.4 are forbidden induced subgraphs for the graphs with *prn* two. As S_1 is an induced subgraphs of G_3 and G_4, and S_2 is an induced subgraph of G_5, the *prn* of G_3, G_4 and G_5 are at least 3. Hence, by Corollary 4.1.8, $\mathcal{R}^p(G_3) = \mathcal{R}^p(G_4) = \mathcal{R}^p(G_5) = 3$.

Suppose the cover graph of P is a path. Then P has P_1, P_1' or P_2 as one of its subposets and hence, P can be seen as any one of the two types shown in Figure 4.6. Accordingly, as per the following cases, we prove that an extended crown graph G defined by P has *prn* two.

- P is of Type-1: Let $P = X_1 \cup X_2 \cup \{a\}$, where X_1 and X_2 are chains and none of the elements of X_1 are comparable to elements of X_2 and vice versa. Also,

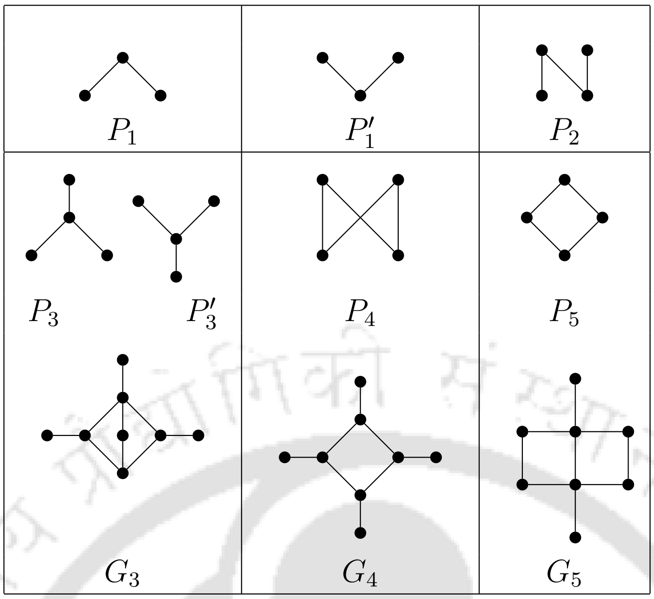


FIGURE 4.5: Posets of width two and corresponding extended crown graphs

the element a is comparable to all elements of P. Note that $\{\{a\} \cup X_1, X_2\}$ is a minimal chain cover of P and hence of P_A in view of Remark 4.1.5. The chain cover satisfies the condition (ζ_B) given in Theorem 2.4.5. Hence, $\mathcal{R}^p(G) = 2$.

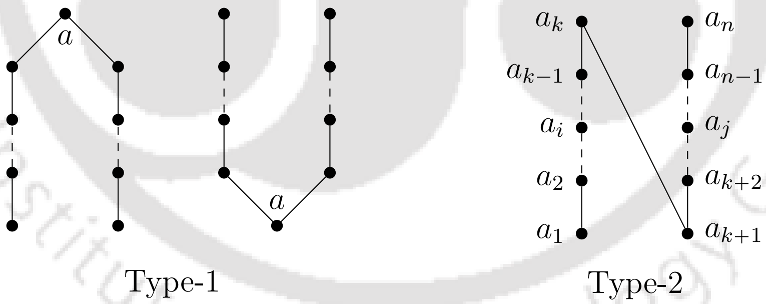


FIGURE 4.6: Posets whose cover graph is a path

- P is of Type-2: Consider the largest antichain $\{a_i, a_j\}$ of P. The respective extended crown graph G is as shown in Figure 4.7. Consider the following permutations:

$$p_1 = w_1 a_{k+1} a_k a'_{k+1} a'_k w_2 \text{ and } p_2 = w_3 w_4$$

where,

$$w_1 = a_n a'_n a_{n-1} a'_{n-1} \cdots a_j a'_i \cdots a_{k+2} a'_{k+2}$$
$$w_2 = a_{k-1} a'_{k-1} \cdots a_i a'_j \cdots a_2 a'_2 a_1 a'_1$$
$$w_3 = a_1 a_2 \cdots a_i \cdots a_k a'_1 a'_2 \cdots a'_j \cdots a'_k$$
$$w_4 = a_{k+1} a_{k+2} \cdots a_j \cdots a_n a'_{k+1} a'_{k+2} \cdots a'_i \cdots a'_n$$

It can be observed that the word $p_1 p_2$ represents G permutationally. Hence, $\mathcal{R}^p(G) = 2$.

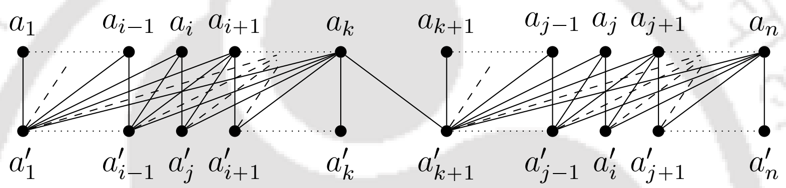


Figure 4.7: Extended crown graph for Type-2 poset

Thus, the result obtained in this section is summarized in the following theorem.

Theorem 4.1.11. *Let G be the extended crown graph defined by a poset P of width two. Then $\mathcal{R}^p(G) = 3$ if and only if the cover graph of P is connected and not a path.*

In the following, we further provide the representation number of the extended crown graphs.

Theorem 4.1.12. *Let P be a poset of width two and G is an extended crown graph defined by P then $\mathcal{R}(G) = 2$.*

Proof. Consider a minimal chain cover $M = \{X_1, X_2\}$ of the poset P, where

$$X_1 : a_1 \prec a_2 \cdots \prec a_r$$
$$X_2 : b_1 \prec b_2 \cdots \prec b_s$$

Let $\{a_{t_1}, b_{t_2}\}$, for some t_1 and t_2, be a largest antichain of P. Set $X_1' = \{a_1', a_2', \ldots, a_{t_1-1}', b_{t_2}', a_{t_1+1}', \ldots, a_r'\}$ and $X_2' = \{b_1', b_2', \ldots, b_{t_2-1}', a_{t_1}', b_{t_2+1}', \ldots, b_s'\}$. Consider the following words:

$$w_1 = a_r a_r' a_{r-1} a_{r-1}' \cdots a_i b_{t_2}' \cdots a_2 a_2' a_1 a_1' C_1 a_1 C_2 a_2 \cdots a_{r-1} C_r a_r C_0$$

$$w_2 = b_s b_s' b_{s-1} b_{s-1}' \cdots b_j a_{t_1}' \cdots b_2 b_2' b_1 b_1' D_1 b_1 D_2 b_2 \cdots b_{s-1} D_s b_s D_0$$

$$\text{where, } C_i = \begin{cases} \text{a permutation of the set } X_2' \backslash N(a_r), & i = 0 \\ \text{a permutation of the set } X_2' \cap N(a_1), & i = 1 \\ \text{a permutation of the set } (X_2' \cap N(a_i)) \backslash N(a_{i-1}), & 2 \leqslant i \leqslant r \end{cases}$$

$$D_j = \begin{cases} \text{a permutation of the set } X_1' \backslash N(b_s), & j = 0 \\ \text{a permutation of the set } X_1' \cap N(b_1), & j = 1 \\ \text{a permutation of the set } (X_1' \cap N(b_j)) \backslash N(b_{j-1}), & 2 \leqslant j \leqslant s \end{cases}$$

In each permutation, arrange the vertices in increasing indices. Additionally, b_{t_2}' must occur between a_{t_1-1}' and a_{t_1+1}. Similarly, a_{t_1}' must occur between b_{t_2-1}' and b_{t_2+1}

Note that w_1 is a word over the set $X_1 \cup B$ where vertices of X_1 occur twice and the vertices of B occur only once. Similarly, w_2 is a word over the set $X_2 \cup B$ where the vertices of X_2 occur twice and the remaining vertices occur only once. If we concatenate the words w_1 and w_2, the resultant word is a 2-uniform word over the set $A \cup B$.

We now show that the word $w_1 w_2$ represents the extended crown G, i.e., for any two vertices a and b of G, a and b are adjacent if and only if a and b alternate in $w_1 w_2$.

Suppose a and b are adjacent in G. We deal this part in the following two cases:

- $a \in X_1$: Clearly, $aba \ll w_1$ and $b \ll w_2$. We have, $abab \ll w_1 w_2$.

- $a \in X_2$: We have, $b \ll w_1$ and $aba \ll w_2$. Therefore, $baba \ll w_1w_2$.

Thus, a and b alternate in w_1w_2.

Conversely, suppose a and b are non-adjacent in G. We show that a and b do not alternate in w_1w_2 through various cases as per the following:

- $a, b \in A$:
  - $a, b \in X_1$: Let $a = a_i$ and $b = a_j$, where $1 \leqslant i < j \leqslant r$. We have $baab \ll w_1$.
  - $a, b \in X_2$: Let $a = b_i$ and $b = b_j$, where $1 \leqslant i < j \leqslant s$. We have $baab \ll w_2$.
  - $a \in X_1$ and $b \in X_2$: We have, $aa \ll w_1$ and $bb \ll w_2$. Clearly, $aabb \ll w_1w_2$.
- $a \in A$ and $b \in B$:
  - $a \in X_1$ and $b \in X_1'$: $baa \ll w_1$ and $b \ll w_2$. Clearly, $baab \ll w_1w_2$.
  - $a \in X_1$ and $b \in X_2'$: $aab \ll w_1$ and $b \ll w_2$. We have, $aabb \ll w_1w_2$.
  - $a \in X_2$ and $b \in X_1'$: $b \ll w_1$ and $aab \ll w_2$. We have, $baab \ll w_1w_2$.
  - $a \in X_2$ and $b \in X_2'$: $b \ll w_1$ and $baa \ll w_2$. Thus, $bbaa \ll w_1w_2$.
- $a, b \in B$: Note that if C_i's or D_j's are not an empty word then clearly there is an edge joining the two chains X_1 and X_2. Assume that $a_l \prec b_m$, for some l and m in P. Clearly, $\{a_1', a_2', \ldots, a_l'\} \subset N(b_m)$. Similarly, if $b_l \prec a_m$, for some l and m in P then $\{b_1', b_2', \ldots, b_l'\} \subset N(a_m)$. We observed that the subwords $C_1C_2 \cdots C_rC_0 = b_1'b_2' \cdots a_{t_1}' \cdots b_s'$ and $D_1D_2 \cdots D_sD_0 = a_1'a_2' \cdots b_{t_2}' \cdots a_r'$. We have, $a_r' \cdots b_{t_2}' \cdots a_1'b_1' \cdots a_{t_1}' \cdots b_s'b_s' \cdots a_{t_1}' \cdots b_1'a_1' \cdots b_{t_2}' \cdots a_r' \ll w_1w_2$. Without loss of generality, $abba \ll w_1w_2$.

Hence, a and b do not alternate in w_1w_2. $\square$

Corollary 4.1.13. *The extended crown graph defined by a poset of width two is a circle graph.*

4.2 Stacked Book Graphs

Definition 4.2.1. A *stacked book graph*, denoted by $B_{m,n}$, is $K_{1,m} \square P_n$, the Cartesian product of a star graph and a path (see Figure 4.8).

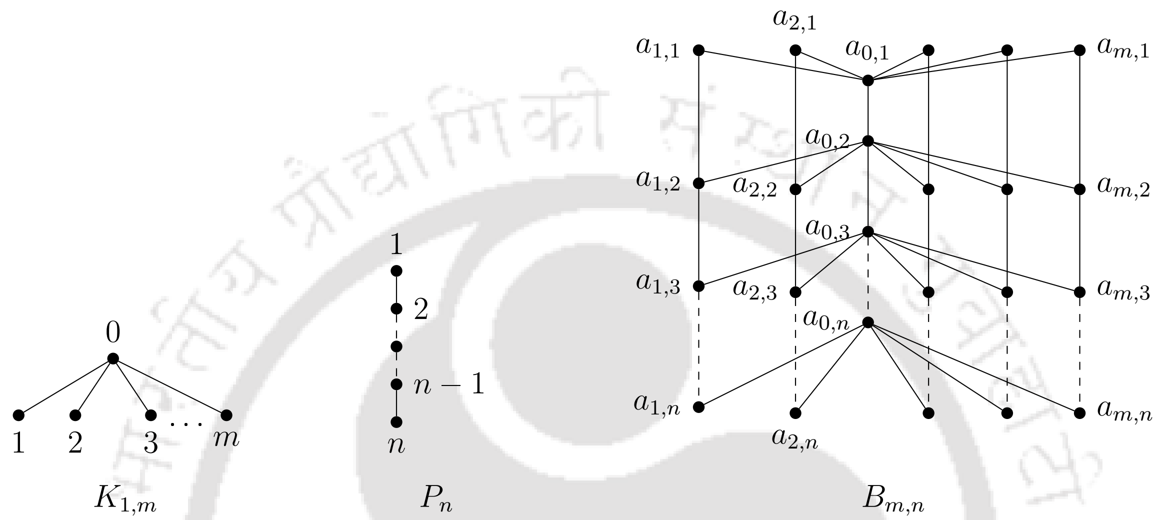


FIGURE 4.8: Stacked book graph $B_{m,n}$

Note that both the star graphs and paths are bipartite graphs, hence every stacked book graph $B_{m,n}$ is a bipartite graph. We consider the vertex sets of the star graph $K_{1,m}$ as $\{0, 1, 2, \ldots, m\}$ and the path P_n as $\{1, 2, \ldots, n\}$. In $B_{m,n} = K_{1,m} \square P_n$, the vertices are denoted as $a_{i,j}$ rather than (i, j). The set $S = \{a_{0,1}, a_{0,2}, \ldots, a_{0,n}\}$ is referred to as the spine of the stacked book graph $B_{m,n}$.

In the following, we give a characterization for the permutationally representable Cartesian products of graphs.

Theorem 4.2.2. *A graph $G \square G'$ is permutationally representable if and only if either both G and G' are bipartite graphs or one of them is permutationally representable and the other is an edgeless graph.*

Proof. Suppose G is permutationally representable and G' is an edgeless graph with n vertices. Then the graph $G \square G'$ is a disconnected graph consisting of n copies of the graph G. Since G is permutationally representable, $G\square G'$ is also permutationally representable, from Remark 1.4.54.

By Proposition 1.4.59, if G and G' are bipartite graphs then $G\square G'$ is a bipartite graph, and hence, it is permutationally representable.

Conversely, suppose G or G' is not permutationally representable. By definition of the Cartesian product, both G and G' are induced subgraphs of $G\square G'$. Hence, by hereditary property, $G\square G'$ is not permutationally representable.

Suppose G and G' are permutationally representable where G is not a bipartite graph and G' is not an edgeless graph. Note that if G is triangle-free then G contains an odd cycle and hence G is not permutationally representable. Let $a, b, c \in V$ and $a', b' \in V'$ such that $\overline{ab}, \overline{bc}, \overline{ac} \in E$ and $\overline{a'b'} \in E'$. Consider a transitive orientation ϕ on $G\square G'$. Without loss of generality, $\overrightarrow{(a,a')(b,a')}$, $\overrightarrow{(b,a')(c,a')}$), $\overrightarrow{(a,a')(c,a')}$ in $G\square G'$. Since a' and b' are adjacent, (b,a') and (b,b') are adjacent in $G\square G'$. If $\overrightarrow{(b,a')(b,b')}$ in the transitive orientation ϕ of $G\square G'$ then by transitive relation, (a,a') must be oriented to (b,b'). If $\overrightarrow{(b,b')(b,a')}$ in ϕ of $G\square G'$ then (b,b') must be oriented to (a,a'). However, (a,a') and (b,b') are non-adjacent in $G\square G'$, which is a contradiction. $\square$

4.2.1 The *prn* of $B_{m,n}$

We now establish that a stacked book graph $B_{m,n}$ is permutationally 3-representable. We achieve this by explicitly constructing three permutations over the vertices of $B_{m,n}$ such that their concatenation represents $B_{m,n}$ permutationally.

First, we consider the case when n is even and then extend the process for all n. Consider a stacked book graph $B_{m,n}$ and remove its spine. The resultant graph, denoted by $B^0_{m,n}$, has m connected components each of which is isomorphic to the path P_n as shown in Figure 4.9. It is known that a path is permutationally 2-representable (cf. [Gallai, 1967]) and hence $B^0_{m,n}$ is permutationally 2-representable, in view of Remark 1.4.55. However, in the direction of constructing three permutations for $B_{m,n}$, we indeed construct a 3-uniform word that permutationally represents $B^0_{m,n}$

and extend it to $B_{m,n}$.

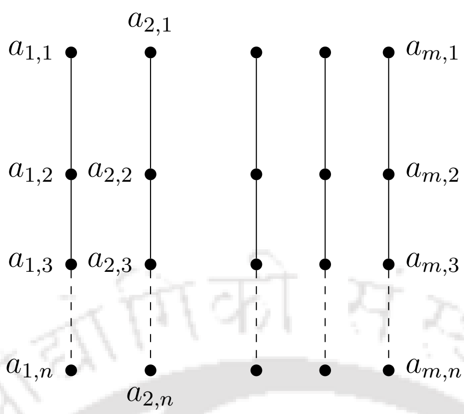


FIGURE 4.9: $B^0_{m,n}$

Consider the following three permutations over the vertices of $B^0_{m,n}$:

$$q'_1 = x(n)x(n-2)\cdots x(4)x(2)$$

$$q'_2 = y(1)y(2)\cdots y(m-1)y(m)$$

$$q'_3 = a_{m,1}a_{m-1,1}\cdots a_{1,1}z(2)z(4)\cdots z(n-2)a_{m,n}a_{m-1,n}\cdots a_{1,n}$$

where, for $2 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and $2 \leqslant k \leqslant n-2$,

$$x(i) = a_{m,i-1}a_{m,i}a_{m-1,i-1}a_{m-1,i}\cdots a_{1,i-1}a_{1,i}$$

$$y(j) = a_{j,1}a_{j,3}\cdots a_{j,n-1}a_{j,n}a_{j,n-2}\cdots a_{j,2}$$

$$z(k) = a_{m,k+1}a_{m,k}a_{m-1,k+1}a_{m-1,k}\cdots a_{1,k+1}a_{1,k}$$

We now show that the 3-uniform word $q'_1q'_2q'_3$ represents $B^0_{m,n}$.

Lemma 4.2.3. *If n is even, the word $q'_1q'_2q'_3$ represents $B^0_{m,n}$.*

Proof. Let a and b be two vertices of $B^0_{m,n}$. We prove that $\overline{ab}$ is an edge in $B^0_{m,n}$ if either *ababab* or *bababa* is a subword of $q'_1q'_2q'_3$, and conversely.

Suppose $\overline{ab}$ is an edge in $B^0_{m,n}$. Then $a = a_{j,i}$ for some $1 \leqslant i \leqslant n-1$, and $b = a_{j,i+1}$ for some $1 \leqslant j \leqslant m$ (or vice versa, which can be addressed by symmetry). We consider the following cases and prove this part:

- i is even: We have $ba \ll x(i+2)x(i) \ll q_1'$, $ba \ll y(j) \ll q_2'$ and $ba \ll z(i) \ll q_3'$. Thus, $bababa \ll q_1'q_2'q_3'$.

- i is odd: We have $ab \ll x(i+1) \ll q_1'$, $ab \ll y(j) \ll q_2'$. For the permutation q_3', we have the following subcases:

  - $i = 1$: We have, $ab \ll a_{j,1}z(2) \ll q_3'$.
  - $1 < i < n-1$: We have $ab \ll z(i-1)z(i+1) \ll q_3'$.
  - $i = n-1$: We have, $ab \ll z(n-2)a_{j,n} \ll q_3'$.

  Thus, $ababab \ll q_1'q_2'q_3'$.

Hence, a and b alternate in the word $q_1'q_2'q_3'$.

Conversely, assume that $\overline{ab}$ is not an edge in $B^0_{m,n}$. Let $a = a_{r_1,s_1}$ and $b = a_{r_2,s_2}$, where $1 \leqslant r_1 \leqslant r_2 \leqslant m$ and $1 \leqslant s_1, s_2 \leqslant n$. We consider the following cases:

- $r_1 = r_2$: Clearly, $|s_1 - s_2| \geqslant 2$. We have

$$a_{r_1,n-1}a_{r_1,n}a_{r_1,n-3}a_{r_1,n-2}\cdots a_{r_1,1}a_{r_1,2} \ll q_1'$$

  and $a_{r_1,1}a_{r_1,3}a_{r_1,2}\cdots a_{r_1,n-1}a_{r_1,n-2}a_n \ll q_3'$. Without loss of generality, assume $s_1 < s_2$ then $ba \ll q_1'$ and $ab \ll q_3'$.

- $r_1 < r_2$ and $s_1 = s_2$: We have $ba \ll x(t) \ll q_1'$, for some even $1 < t \leqslant n$ and $ab \ll y(r_1)y(r_2) \ll q_2'$.

- $r_1 < r_2$ and $|s_1 - s_2| = 1$: Either s_1 or s_2 is even. Assume s_1 is even. Then $ba \ll x(s_1) \ll q_1'$ or $ba \ll z(s_1)a_{r_2,n}a_{r_1,n} \ll q_3'$, and $ab \ll q_2'$. The case is similar when s_2 is even.

- $r_1 < r_2$ and $|s_1 - s_2| \geqslant 2$: Suppose $s_1 < s_2$. Then $ba \ll q_1'$ and $ab \ll q_2'$. If $s_1 > s_2$ then $ba \ll q_3'$ and $ab \ll q_2'$.

Hence, the result. □

As per the following, we extend the permutations q_1', q_2', q_3' constructed for the graph $B^0_{m,n}$ to the permutations q_1, q_2, q_3, respectively, over the vertex set of $B_{m,n}$ such that $q_1q_2q_3$ represents $B_{m,n}$ permutationally.

1. Set

$$q_1 = q_1' a_{0,1}$$
$$q_2 = a_{0,n}a_{0,n-2}\cdots a_{0,2}q_2' a_{0,1}a_{0,3}\cdots a_{0,n-1}$$
$$q_3 = a_{m,1}a_{m-1,1}\cdots a_{1,1}a_{0,1}z(2)\cdots z(n-2)a_{m,n}\cdots a_{1,n}$$

2. For $i = 2$ to n in the sequence, we update the permutations q_1 and q_3 with the vertex $a_{0,i}$, as described in the following steps:

   i) if i is even then replace $x(i)$ with $a_{0,i}x(i)$ in q_1, and replace $a_{0,i-1}$ with $a_{0,i}a_{0,i-1}$ in q_3.

   ii) if i is odd and $i < n-1$ then replace $a_{0,i-1}$ with $a_{0,i-1}a_{0,i}$ in q_1, and replace $z(i+1)$ with $a_{0,i}z(i+1)$ in q_3.

   iii) if $i = n-1$ then replace $a_{0,n-2}$ with $a_{0,n-2}a_{0,n-1}$ in q_1, and replace $z(n-2)$ with $z(n-2)a_{0,n-1}$ in q_3.

Theorem 4.2.4. *If n is even, the word $q_1q_2q_3$ represents $B_{m,n}$.*

Proof. The graph $B^0_{m,n}$ is an induced subgraph of $B_{m,n}$ and the word $q_1'q_2'q_3'$ representing $B^0_{m,n}$ is a subword of $q_1q_2q_3$. For word-representability, it is sufficient to verify the adjacency and non-adjacency of the vertices $a_{0,1}, a_{0,2}, \ldots, a_{0,n}$ with vertices of $B_{m,n}$.

Let $a = a_{0,i}$, where $1 \leqslant i \leqslant n$ and $b = a_{r,s}$, where $0 \leqslant r \leqslant m$ and $1 \leqslant s \leqslant n$. We claim that $\overline{ab}$ is an edge in $B_{m,n}$ if either $ababab$ or $bababa$ is a subword of $q_1q_2q_3$, and conversely.

Suppose $\overline{ab}$ is an edge in $B_{m,n}$. We have the following two cases:

- $r = 0$: We have

$$a_{0,n}a_{0,n-2}a_{0,n-1}a_{0,n-4}\cdots a_{0,5}a_{0,2}a_{0,3}a_{0,1} \ll q_1,$$

$$a_{0,n}a_{0,n-2}\cdots a_{0,2}a_{0,1}a_{0,3}\cdots a_{0,n-3}a_{0,n-1} \ll q_2,$$

$$a_{0,2}a_{0,1}a_{0,4}a_{0,3}\cdots a_{0,n-2}a_{0,n-3}a_{0,n}a_{0,n-1} \ll q_3.$$

  Clearly, if i is odd and $s = i+1$ then $ba \ll q_1$, $ba \ll q_2$ and $ba \ll q_3$. If $i < n$ is even and $s = i+1$ then $ab \ll q_1$, $ab \ll q_2$ and $ab \ll q_3$.

- $r \neq 0$ and $s = i$: If i is even then we have $ab \ll a_{0,i}x(i) \ll q_1$, $ab \ll a_{0,i}p_2 \ll q_2$ and $ab \ll a_{0,i}z(i)a_{0,n}a_{0,n-1}a_{m,n}a_{m-1,n}\cdots a_{1,n} \ll q_3$. If i is odd then we have $ba \ll x(i+1)a_{0,i} \ll q_1$, $ba \ll p_2a_{0,i} \ll q_2$ and $ba \ll a_{m,1}a_{m-1,1}\cdots a_{1,1}a_{0,1}z(i-1)a_{0,i} \ll q_3$.

Conversely, if $\overline{ab}$ is not an edge in $B_{m,n}$ then we prove the result in the following cases:

- $r = 0$: Clearly from the above case, we see that if $s < i$ then $ba \ll q_1$ and $ab \ll q_3$. Else, if $s > i$ then $ab \ll q_1$ and $ba \ll q_3$.

- $r \neq 0$ and $s \neq i$: We consider this in three subcases:

  - $i = 1$: We have $ba \ll q_1$ and $ab \ll a_{0,1}z(2)\cdots z(n-2)a_{m,n}\cdots a_{1,n} \ll q_3$.
  - i is even: We have, $ab \ll q_2$. If $i < s$ then $ba \ll q_1$. Else if, $i > s$ then $ba \ll q_3$.
  - i is odd: We have, $ba \ll q_2$. If $i < s$ then $ab \ll q_3$. Else if $i > s$ then $ab \ll q_1$.

Thus, a and b do not alternate in the word $q_1q_2q_3$. Hence, the word $q_1q_2q_3$ represents the graph $B_{m,n}$. □

Theorem 4.2.5. *Every stacked book graph is permutationally 3-representable.*

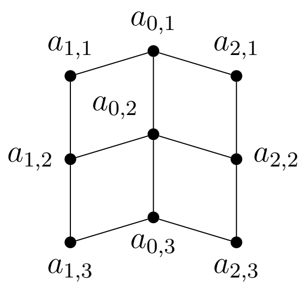


FIGURE 4.10: $B_{2,3}$

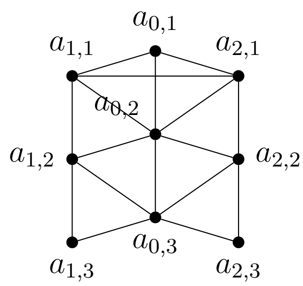


FIGURE 4.11: Local complementation at $a_{0,1}, a_{1,3}$ and $a_{2,3}$

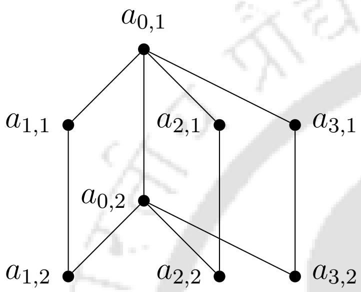


FIGURE 4.12: $B_{3,2}$

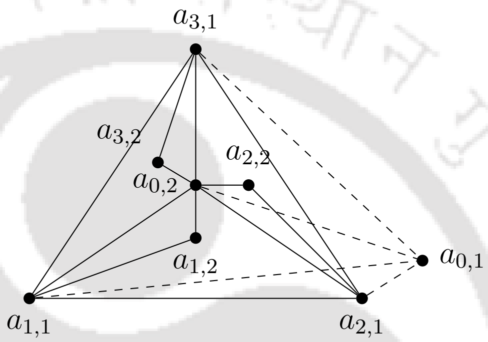


FIGURE 4.13: Local complementation of $B_{3,2}$ at $a_{0,1}$

Proof. By Theorem 4.2.4, we see that if n is even then $B_{m,n}$ is permutationally 3-representable. If n is odd then consider the stacked book graph $B_{m,n+1}$, which is permutationally 3-representable. Since $B_{m,n}$ is an induced subgraph of $B_{m,n+1}$, by hereditary property of word-representable graphs, $B_{m,n}$ is also permutationally 3-representable. $\square$

4.2.2 Classification

In this section, we give a complete classification of stacked book graphs with respect to their representation number and the *prn*.

Lemma 4.2.6. *The representation number of $B_{2,3}$ (given in Figure 4.10) is three.*

Proof. First, we show that $\mathcal{R}(B_{2,3}) \geqslant 3$. Suppose $\mathcal{R}(B_{2,3}) = 2$, i.e., $B_{2,3}$ is a circle graph. Since the class of circle graphs is closed under local complementation, consider the local complementions at vertices $a_{0,1}, a_{1,3}$ and $a_{2,3}$ of in the graph $B_{2,3}$. The resultant graph, as shown in Figure 4.11, is not a circle graph. This is evident as it contains the wheel graph W_5 as an induced subgraph, which is a non-word-representable graph. Thus, the graph $B_{2,3}$ is not a circle graph, so that $\mathcal{R}(B_{2,3}) \geqslant 3$. By Theorem 4.2.5, we see that $\mathcal{R}(B_{2,3}) \leqslant 3$. Hence, $\mathcal{R}(B_{2,3}) = 3$. □

Lemma 4.2.7. *The representation number of $B_{3,2}$ (given in Figure 4.12) is three.*

Proof. Suppose $\mathcal{R}(B_{3,2}) = 2$. Since the circle graphs are closed under local complementation, consider the graph obtained by local complementation at vertex $a_{0,1}$ of $B_{3,2}$, as shown in Figure 4.12. Note that its induced subgraph represented in thick edges in Figure 4.13 is not a word-representable graph (compare with the list of graphs in [Kitaev and Lozin, 2015, Figure 3.9]). Hence, the graph in Figure 4.13 is not a circle graph and so is $B_{3,2}$. Therefore, $\mathcal{R}(B_3) \geqslant 3$. Since $\mathcal{R}(B_{3,2}) \leqslant 3$, we have $\mathcal{R}(B_{3,2}) = 3$. □

Theorem 4.2.8. *A stacked book graph has the representation number three if and only if it contains $B_{2,3}$ or $B_{3,2}$ as an induced subgraph.*

Proof. Suppose $B_{m,n}$ does not contain $B_{2,3}$ and $B_{3,2}$ as induced subgraphs. The values for m and n are as follows:

- $m = n = 1$: $B_{m,n}$ is the complete graph K_2, which is known to have representation number one.
- $m = 1$ and $n \geqslant 2$: $B_{m,n}$ is isomorphic to the ladder graph L_n. From Theorem 1.4.62, ladder graphs are shown to have representation number two.
- $m \geqslant 2$ and $n = 1$: $B_{m,n}$ is isomorphic to the star graph $K_{1,m}$, which has representation number two.
- $m = n = 2$: $B_{2,2}$ is isomorphic to the ladder graph L_3. Thus, $\mathcal{R}(B_{m,n}) = 2$.

Thus, $\mathcal{R}(B_{m,n}) \leqslant 2$ if $B_{m,n}$ does not contain $B_{2,3}$ and $B_{3,2}$ as induced subgraphs.

From Lemma 4.2.6 and Lemma 4.2.7, we have $\mathcal{R}(B_{2,3}) = \mathcal{R}(B_{3,2}) = 3$. If $m = 2$ and $n \geqslant 3$, the stacked book graph $B_{m,n}$ contains $B_{2,3}$ as an induced subgraph. Similarly, if $m \geqslant 3$ and $n = 2$, the stacked book graph $B_{m,n}$ contains $B_{3,2}$ as an induced subgraph. If $m, n \geqslant 3$ then the stacked book graph $B_{m,n}$ contains both $B_{2,3}$ and $B_{3,2}$ as induced subgraphs. By hereditary property of word-representable graphs, $\mathcal{R}(B_{m,n}) \geqslant 3$. Hence, by Theorem 4.2.5, we conclude that the representation number of $B_{m,n}$ is three, for all $B_{m,n}$ containing $B_{2,3}$ or $B_{3,2}$ as an induced subgraph. □

Note that for a permutationally representable graph, the *prn* is at least its representation number. The following is a consequence of the above result.

Corollary 4.2.9. *The graphs $B_{2,3}$ or $B_{3,2}$ is an induced subgraph of $B_{m,n}$ if and only if $\mathcal{R}^p(B_{m,n}) = 3$.*

Thus, we have the following classification of stacked book graphs with respect to their representation number and *prn*.

Theorem 4.2.10. *For a stacked book graph $B_{m,n}$, we have,*

1. *if $m = 1$ and $n = 1$ then $\mathcal{R}(B_{m,n}) = \mathcal{R}^p(B_{m,n}) = 1$;*

2. *otherwise, if $m = 1$ or $n = 1$ or $m = n = 2$ then $\mathcal{R}(B_{m,n}) = \mathcal{R}^p(B_{m,n}) = 2$;*

3. *otherwise, $\mathcal{R}(B_{m,n}) = \mathcal{R}^p(B_{m,n}) = 3$.*

4.3 Future Directions of Part-I

In Chapter 2, we developed a polynomial-time algorithm (cf. Section 2.2) to generate permutations of the vertices of a given bipartite graph. It is natural to ask for extending this computational procedure to comparability graphs, in general. Further, one can explore the neighborhood inclusion graphs and come up with better ways to improve the bounds.

In Chapter 3, we used the neighborhood inclusion graph approach and established that the conjecture by Glen et al. [2018] holds for certain subclasses of bipartite graphs, whose partite sets are of equal and even size. Accordingly, we propose that the conjecture is also valid for the remaining bipartite graphs. In our approach, we considered a specific chain cover of the poset associated with the partite set of a bipartite graph. Considering a smallest chain cover and refining our method, one may improve the upper bound on the representation number of bipartite graphs.

In Chapter 4, for an extended crown graph G defined by a poset P, we showed that the $w(P) \leqslant \mathcal{R}^p(G) \leqslant 1 + w(P)$. We also determined the *prn* and representation number for a specific case of extended crown graphs, which are defined by posets of width two. Characterizing the *prn* of extended crown graphs defined by posets of width at least three is an open problem. Further, we conjecture that the representation number of an extended crown graph defined by a poset P equals the representation number of the crown graph with $2w(P)$ vertices.

In view of the work on stacked book graphs in Chapter 4, a general problem of determining the *prn* as well as the representation numbers of the Cartesian product of two arbitrary bipartite graphs remains open.

Part II

Melon Graphs

5

(Permutation-)Representation Number

The class of melon graphs is a subclass of planar graphs that are 3-colorable and hence all melon graphs are word-representable (cf. Theorem 1.4.16). In this chapter, we study the representation number of melon graphs and prove that it is at most three. We also provide a characterization of melon graphs with representation number three. Towards finding the *prn* of melon graphs which are permutationally representable, first we provide a characterization for the melon graphs admitting the transitive orientations in terms of the lengths of their constituent paths. In view of Theorem 1.3.26, it is known that the *prn* of permutationally representable melon graphs is at most four. However, in this chapter, we show that their *prn* is indeed at most three and characterize the melon graphs with *prn* three.

5.1 Representation Number

In this section, we show that the melon graphs are 3-word-representable and classify the graphs with representation numbers two and three. It is evident that the complete graphs K_2 and K_3 are the only melon graphs with representation number one.

Lemma 5.1.1. *Every melon graph is 3-word-representable.*

Proof. Suppose all the paths of a melon graph $M = (E_1, \ldots, E_m)$ are of length at most two. Since M is simple, there will be at most one path of length one. If M has a path of length one, say E_1, then all other paths E_i $(2 \leqslant i \leqslant m)$ are of length two, say $E_i = \langle 0, x_i, 0' \rangle$, with intermediate vertex x_i. Clearly, the 2-uniform word $x_2x_3 \ldots x_m00'x_m \ldots x_3x_200'$ represents M. If all paths are of length two in M then $M = K_{2,m}$, a complete bipartite graph, so it is a circle graph. Note that a path of length at least three is also a circle graph, as it is a tree.

Further, it can be observed that an arbitrary melon graph can be obtained by adding paths of length at least three to a circle graph described in the previous paragraph, i.e., a path of length at least three or a melon graph in which all paths are of length at most two. Since adding a path of length at least three to a 3-word-representable graph results in a 3-word-representable graph (cf. Lemma 1.4.50), we have all melon graphs are 3-word-representable. □

We now classify the melon graphs based on their number of constituent paths and give their representation numbers through a series of lemmas.

Let $M = (E_1, E_2, \ldots, E_m)$ be a melon graph. If $m \leqslant 2$ then M is either a path or a cycle; in which case, if $M \neq K_2$ or K_3 then $\mathcal{R}(M) = 2$.

Lemma 5.1.2. *For $m \geqslant 3$, if $m - 2$ constituent paths of M are of length at most two then $\mathcal{R}(M) = 2$.*

Proof. In Lemma 5.1.1, we observed that if the length of each constituent path is at most two then $\mathcal{R}(M) = 2$. If M has one or two constituent paths of length at least three, we show the result as per the following cases:

- Suppose M has a constituent path, say E_1, of length at least three and others of length two. Let $E_1 = \langle 0, a_1, a_2, \ldots, a_k, 0' \rangle$, and $E_i = \langle 0, x_i, 0' \rangle$, for all $2 \leqslant i \leqslant m$. Consider the cycle $\{E_1, E_2\}$ and construct a 2-uniform word, as per the following, representing the cycle using the technique given in [Kitaev and Lozin, 2015, Section 3.2]:

$$w = 0x_2a_10a_2a_1a_3a_2 \cdots a_ka_{k-1}0'a_kx_20'$$

  Since $x_2, x_3, \ldots, x_m$ have the same neighbourhood, i.e., $\{0, 0'\}$, replace first occurrence of x_2 by the word $x_2x_3 \cdots x_m$, and the second occurrence of x_2 by the word $x_m \cdots x_3x_2$ in w to get the following 2-uniform word, which evidently represents M:

$$0x_2x_3 \cdots x_ma_10a_2a_1a_3a_2 \cdots a_ka_{k-1}0'a_kx_m \cdots x_3x_20'$$

- Suppose M has two constituent paths, say E_1 and E_2, of length at least three and others of length two. For $k_1, k_2 \geqslant 2$, let $E_1 = \langle 0, a_1, a_2, \ldots, a_{k_1}, 0' \rangle$, $E_2 = \langle 0, b_1, b_2, \ldots, b_{k_2}, 0' \rangle$, and $E_i = \langle 0, x_i, 0' \rangle$, for all $3 \leqslant i \leqslant m$. We construct a 2-uniform word w', representing the cycle $\{E_1, E_2\}$, given by

$$w' = 0b_1a_10a_2a_1 \cdots 0'a_{k_1}b_{k_2}0'b_{k_2-1}b_{k_2} \cdots b_2b_3b_1b_2.$$

  We substitute the factors $0b_1a_10$ and $0'a_{k_1}b_{k_2}0'$ of w' by $0b_1x_3x_4 \cdots x_ma_10$ and $0'a_{k_1}x_m \cdots x_4x_3b_{k_2}0'$, respectively, to get the following 2-uniform word, which represents M:

$$0b_1x_3x_4 \cdots x_ma_10a_2a_1 \cdots 0'a_{k_1}x_m \cdots x_4x_3b_{k_2}0'b_{k_2-1}b_{k_2} \cdots b_2b_3b_1b_2$$

- Suppose M has a constituent path, say E_1, of length at least three, a constituent path, say E_2, of length one, and others of length two. Let $E_1 = \langle 0, a_1, a_2, \ldots, a_{k_1}, 0' \rangle$, $E_2 = \langle 0, 0' \rangle$, and $E_i = \langle 0, x_i, 0' \rangle$, for all $3 \leqslant i \leqslant m$. Consider the cycle $\{E_1, E_2\}$ and construct the following 2-uniform word representing the cycle:

$$w'' = 00'a_1 0 a_2 a_1 \cdots a_{k_1} a_{k_1 - 1} 0' a_{k_1}$$

Since $x_3, x_4 \ldots, x_m$ have the same neighbourhood, i.e., $\{0, 0'\}$, replace the factor $00'$ by the word $x_3 x_4 \cdots x_m 00' x_m \cdots x_4 x_3$ to get the following 2-uniform word, which represents M:

$$x_3 x_4 \cdots x_m 00' x_m \cdots x_4 x_3 a_1 0 a_2 a_1 \cdots a_{k_1} a_{k_1 - 1} 0' a_{k_1}$$

- Suppose M has two constituent paths, say E_1 and E_2, of length at least three, a constituent path, say E_3, of length one, and others of length two. Let $E_1 = \langle 0, a_1, a_2, \ldots, a_{k_1}, 0' \rangle$, $E_2 = \langle 0, b_1, b_2, \ldots, b_{k_2}, 0' \rangle$, $E_3 = \langle 0, 0' \rangle$, and $E_i = \langle 0, x_i, 0' \rangle$, for all $4 \leqslant i \leqslant m$. Then, as per the previous case, consider the 2-uniform word representing $\{E_1, E_3, E_4, \ldots, E_m\}$ as per the following:

$$x_4 \cdots x_m 00' x_m \cdots x_4 a_1 0 a_2 a_1 \cdots a_{k_1} a_{k_1 - 1} 0' a_{k_1}$$

In this word, replace the factor $00'$ by the word $b_1 0 b_2 b_1 b_3 b_2 \cdots b_{k_2} b_{k_2 - 1} 0' b_{k_2}$ to get the following 2-uniform word representing M:

$$x_4 x_5 \cdots x_m b_1 0 b_2 b_1 b_3 b_2 \cdots b_{k_2} b_{k_2 - 1} 0' b_{k_2} x_m \cdots x_5 x_4 a_1 0 a_2 a_1 \cdots a_{k_1} a_{k_1 - 1} 0' a_{k_1}$$

Hence, $\mathcal{R}(M) = 2$. □

Lemma 5.1.3. *Let M_3 be a melon graph with three constituent paths, each of length exactly three. Then $\mathcal{R}(M_3) = 3$.*

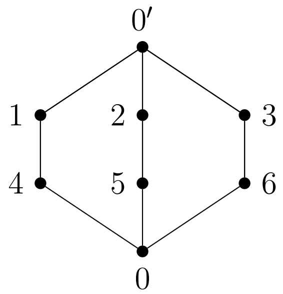


Figure 5.1: M_3

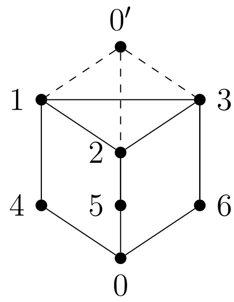


Figure 5.2: Local complementation of M_3 at 0'

Proof. Assume $\mathcal{R}(M_3) = 2$. Since the class of circle graphs is closed under local complementation, consider the graph as shown in Figure 5.2, obtained by local complementation at vertex $0'$ of M_3 (given in Figure 5.1). Note that its induced subgraph represented in thick edges has representation number three (compare with the list of graphs in [Akgün et al., 2019, Figure 4]). Hence, the graph in Figure 5.2 is not a circle graph and so is M_3. Hence, $\mathcal{R}(M_3) = 3$. □

Let $\mathcal{M}_3$ be the family of melon graphs with three constituent paths, each of length at least three.

Lemma 5.1.4. *If $M \in \mathcal{M}_3$ then $\mathcal{R}(M) = 3$.*

Proof. By Theorem 1.2.45, the class of circle graphs is closed under vertex-minor. Note that M_3 can be obtained from M by a successive operations of local complementation followed by vertex deletion on the intermediate vertices of the constituent paths. Thus, if $\mathcal{R}(M) = 2$ then M_3 must be a circle graph; a contradiction to Lemma 5.1.3. Hence, $\mathcal{R}(M) = 3$. □

Lemma 5.1.5. *The representation number of the graph B_3 (given in Figure 5.3) is three.*

Proof. The graph B_3 is isomorphic to the stacked book graph $B_{3,2}$. Hence, by Lemma 4.2.7, the representation number of B_3 is three. □

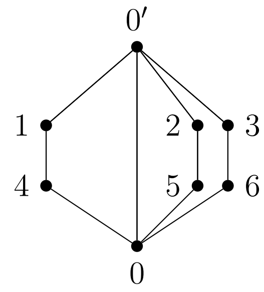


Figure 5.3: B_3

We have the following consequence of Lemma 5.1.5. Let $\mathcal{M}_4$ be the family of melon graphs with four constituent paths, of which one is of length one and the others of length at least three.

Corollary 5.1.6. *If $M \in \mathcal{M}_4$, since B_3 is a vertex-minor of M, we have $\mathcal{R}(M) = 3$.*

We now classify the melon graphs with respect to their representation numbers. Indeed, we show that the representation number of a melon graph is three if and only if it has three constituent paths, each of length at least three.

Let $M = (E_1, \ldots, E_m)$ be a melon graph. If the number of constituent paths of length at least three is at most two then by Lemma 4, $\mathcal{R}(M) = 2$. On the other hand, M has at least three constituent paths of length at least three. Suppose E_1, E_2 and E_3 are of length at least three. If M has the edge $\overline{00'}$, say E_4 then it can be observed that M has an induced subgraph from $\mathcal{M}_4$, which is obtained by deleting $E_5, \ldots, E_m$ except the endpoints. Hence, by Corollary 5.1.6, $\mathcal{R}(M) = 3$. If 0 and $0'$ are non-adjacent in M then we can observe that M has an induced subgraph from $\mathcal{M}_3$ so that $\mathcal{R}(M) = 3$. Thus, we have the following characterization of melon graphs in terms of induced subgraphs with respect to the representation number.

Theorem 5.1.7. *Let M be a melon graph. Then $\mathcal{R}(M) = 3$ if and only if it contains an induced subgraph from $\mathcal{M}_3 \cup \mathcal{M}_4$.*

Note that the melon graph M_3 is a vertex-minor of every member in $\mathcal{M}_3$. Also, the melon graphs in $\mathcal{M}_4$ contain B_3 as vertex-minor. Thus, we have the following corollary with a characterization in terms of vertex-minors.

Corollary 5.1.8. *Let M be a melon graph. Then $\mathcal{R}(M) = 3$ if and only if M contains M_3 or B_3 as a vertex-minor.*

Further, we have the following corollary for the representation number of book graphs.

Corollary 5.1.9. *Let B_m be a book graph. Then $\mathcal{R}(B_m) = 3$ if and only if it contains B_3 as an induced subgraph.*

5.2 Characterization for Comparability

All melon graphs are word-representable but not all melon graphs are comparability graphs. For example, if we consider a melon graph with two constituent paths, one of even length and the other of odd length then the melon graph is an odd cycle, which is not a comparability graph, except K_3. In this section, we characterize the melon graphs admitting the transitive orientations.

First, we prove that the melon graphs with constituent paths of the same parity are comparability graphs.

Lemma 5.2.1. *If all the constituent paths of a melon graph M have the same parity (either all of them are of even length or of odd length) then M is a bipartite graph. Hence, M is a comparability graph.*

Proof. We prove the result using induction on the number of constituent paths m in the melon graph $M = (E_1, \ldots, E_m)$.

If $m = 1$ or 2 then clearly M is a bipartite graph, as M is a path or an even cycle, respectively. Consider a melon graph M with $k + 1$ constituent paths of the same parity. By inductive hypothesis, the melon graph $(E_1, \ldots, E_k)$ is a bipartite graph with bipartition, say $\{A, B\}$, of vertex set. Without loss of generality, suppose $0 \in A$. Let $E_{k+1} = \langle a_0 = 0, a_1, \ldots, a_r = 0' \rangle$. Place the vertices of E_{k+1} such that $a_i \in A$ if i is even and $a_i \in B$ if i is odd. If all the constituent paths of M are of even length then clearly $0' \in A$; otherwise, $0' \in B$. In any case, the proposed partition of vertices of E_{k+1} ensures the bipartition so that M is a bipartite graph. $\square$

Remark 5.2.2. Suppose a melon graph M has constituent paths of lengths both even and odd. If M has a constituent path of even length at least four then this along with any odd length constituent path results in an odd cycle of length at least five so that M is not a comparability graph.

Now, suppose all constituent paths of even parity have length exactly two. If 0 and $0'$ are non-adjacent then a constituent path of length two along with an odd length constituent path forms an odd cycle of length at least five. Again, M is not a comparability graph in this case. In case 0 and $0'$ are adjacent, we observe that M is a comparability graph. First, consider the subgraph, say M', of M consisting of all odd length constituent paths. Note that M' is a bipartite graph with bipartition, say $\{A, B\}$ and $0 \in A$. Suppose the edges of M' are oriented from A to B. Let $E = \langle 0, x, 0' \rangle$ be any constituent path of length two in M. Orient the edges of E as $\overrightarrow{0x}$ and $\overrightarrow{x0'}$ and note that M' along with E is a comparability graph as $\overline{00'}$ is an edge and the transitivity is preserved. Similarly, any constituent path of length two in M can be oriented so that M is a comparability graph.

Thus, we have the following theorem to characterize the melon graphs admitting transitive orientations.

Theorem 5.2.3. *A melon graph M is a comparability graph if and only if one of the following holds:*

1. *All constituent paths of M have the same parity.*

2. *The endpoints 0 and $0'$ are adjacent, and all constituent paths of even parity should be of length two.*

5.3 Permutation-Representation Number

Let $M_c = (E_1, \ldots, E_m)$ be a melon graph that satisfies one of the criteria given in Theorem 5.2.3, i.e., M_c is a comparability graph. In the following, we show that $\mathcal{R}^p(M_c)$ is at most three.

First, we partition the vertex set of M_c into a star, say S_m, and a tree, say T_m, as shown in Figure 5.4. Note that the star graph S_m is the complete bipartite graph $K_{1,m}$ or it is $K_{1,m-1}$ when one of the E_i's is of length one, i.e., 0 and $0'$ are adjacent. The graphs S_m and T_m are induced subgraphs of M_c, where the tree $T_m = M_c \backslash S_m$ has at most m leaves. Note that if M_c has k number of $E_i's$ of lengths one or two then the number of leaves in T_m will be $m - k$.

It can be ascertained from Theorem 1.3.22 that the *prn* of trees is at most three. In the following, we construct a specific type of permutations of the vertices of T_m which will be extended to the permutations of the vertices of M_c.

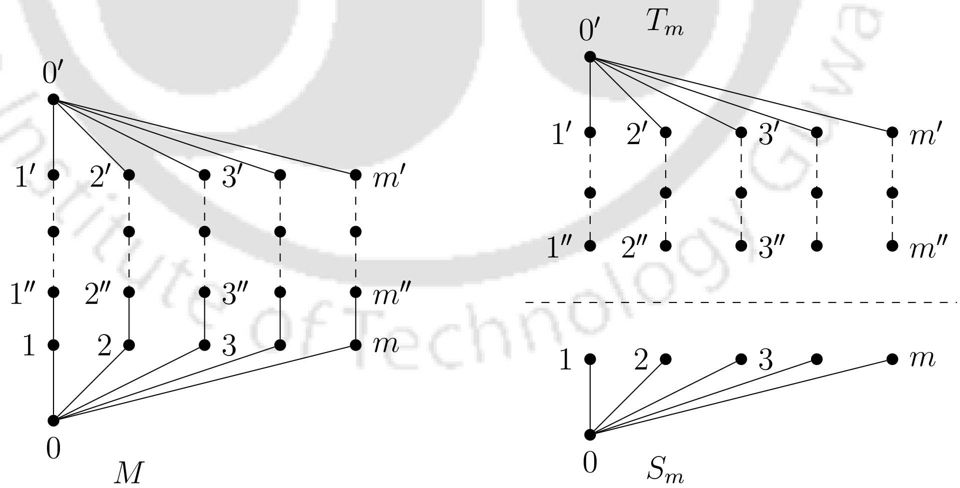


FIGURE 5.4: Melon graph and its partition

5.3.1 Permutations for T_m

Consider the tree T_m with m leaves (constructed above), while the number of leaves in T_m might be less than m. We construct a word representing T_m with m leaves permutationally. Subsequently, the leaves that were not present in T_m originally can be deleted from the word.

Considering the root $0'$ in zeroth level, we divide all the vertices of T_m into two sets based on the parity of levels they are present. Let A be the set of vertices of T_m present in odd levels and B be the set of vertices in even levels. We construct three permutations p_1, p_2 and p_3 on the vertices of T_m using Algorithm 1 and show that the word $w = p_1p_2p_3$ represents T_m permutationally.

Remark 5.3.1. Algorithm 1 follows breadth first search (BFS) for visiting the vertices of the input tree. If n is the number of vertices of T_m, visiting the vertices takes $O(n)$ time, as input being a tree (cf. [Cormen et al., 2009]). Except the vertices $0', 1', \ldots, m'$, when other vertices are visited, the permutations p_1, p_2, and p_3 are updated, which takes $O(n)$ time at each vertex. Hence, Algorithm 1 runs in $O(n^2)$ time.

Remark 5.3.2. Let a and b be two vertices of T_m. If the vertex a is inserted on the left side of b in a permutation p_i $(1 \leqslant i \leqslant 3)$, the order that they appear in p_i remains unchanged, i.e., a always appears on the left side of b in p_i.

In the following, we now prove the correctness of Algorithm 1.

Theorem 5.3.3. *The 3-uniform word $p_1p_2p_3$ represents $T_m = (A \cup B, E)$.*

Proof. We show that for any two vertices $a, b \in A \cup B$, $\overline{ab} \in E$ if and only if a and b alternate in $p_1p_2p_3$.

Suppose a and b are adjacent in T_m, i.e., $\overline{ab} \in E$. Without loss of generality, assume that $b = t(a)$ (this also means that $s(b) = a$). We consider the following two cases:

Algorithm 1: Generating permutations for T_m.

Input: $T_m = (A \cup B, E)$ with root $0'$ and A is the set of vertices present in odd level and B is the set of vertices present in even level.
Output: Three permutations p_1, p_2 and p_3 of the vertices of T_m.

Let Q be a queue.
$Q \leftarrow \{t(1'), t(2'), \dots, t(m')\}$.
$p_1 \leftarrow 1't(1')2' \cdots m'0't(2') \cdots t(m')$. ▷ $t(a)$ is the child of a
$p_2 \leftarrow m't(m') \cdots 2't(2')1'0't(1')$.
$p_3 \leftarrow 1't(1')2't(2') \cdots m't(m')0'$.
while Q *is not empty* **do**
  Remove the the first element a from Q.
  if a *is not a leaf* **then**
    if $a \in A$ **then**
      Replace a with $at(a)$ in p_1.
      Replace $as(a)$ with $as(a)t(a)$ in p_2. ▷ $s(a)$ is the parent of a
      Replace a with $at(a)$ in p_3.
    end
    else
      Replace a with $t(a)a$ in p_1.
      Replace a with $t(a)a$ in p_2.
      $p_3 \leftarrow t(a)p_3$.
    end
    Append $t(a)$ to Q.
  end
end
return The permutations p_1, p_2 and p_3.

- Case 1: $a \in A$. As per the construction and Remark 5.3.2, we have

$$ab = at(a) \ll p_1$$
$$as(a)b = as(a)t(a) \ll p_2$$
$$ab = at(a) \ll p_3$$

so that $ababab \ll p_1p_2p_3$.

- Case 2: $a \in B$. Suppose $a = 0'$. Clearly, $b \in \{1', 2', \dots, m'\}$, so that $bababa \ll p_1p_2p_3$. Suppose $a \neq 0'$. As per the construction and Remark 5.3.2, we have $ba = t(a)a \ll p_i$ for all $1 \leqslant i \leqslant 3$ so that $bababa \ll p_1p_2p_3$.

Hence, a and b alternate in the word $p_1p_2p_3$.

Conversely, suppose $\overline{ab} \notin E$. We need to show that there exist two distinct permutations such that $ab \ll pi$ and $ba \ll pj$, for some i and j. We deal this in the following two cases:

- Case 1: The path connecting a and b does not contain the vertex $0'$. Let $\langle a_0, a_1, a_2, \ldots, a_k\rangle$, where $a_0 = a$ and $a_k = b$, be the path connecting a and b. Without loss of generality, assume that a is visited before b. This means that $a_1 = t(a), a_2 = t(a_1) = t(t(a))$ and so on. Then we handle this in the following two subcases.

  – Subcase 1.1: $a \in A$. Since a_1 is the child of a, we have $aa_1 = at(a) \ll p_i$ for all $1 \leqslant i \leqslant 3$. The vertex a_1 must be present in an even level, i.e., $a_1 \in B$. As per the construction, we have

  $$aa_2 \ll aa_2a_1 = at(a_1)a_1 \ll p_1$$
  $$aa_2 \ll aa_2a_1 = at(a_1)a_1 \ll p_2$$
  $$a_2a \ll a_2aa_1 = t(a_1)aa_1 \ll p_3$$

  By Remark 5.3.2, the vertices a and a_2 do not alternate. When the vertices $a_2, a_3, \ldots a_k$ are visited sequentially, since a_0 is not the parent of a_2, for any $3 \leqslant i \leqslant k$, we have

  $$aa_i \ll p_1$$
  $$aa_i \ll p_2$$
  $$a_ia \ll p_3$$

  so that $ababba \ll p_1p_2p_3$.

  – Subcase 1.2: $a \in B$. Note that $a_1a = t(a)a \ll p_i$ for all $1 \leqslant i \leqslant 3$. The vertex a_1 must be present in an odd level, i.e., $a_1 \in A$. As per the construction, we have

$$a_2a \ll a_1a_2a = a_1t(a_2)a \ll p_1$$
$$aa_2 \ll a_1aa_2 = a_1at(a_1) \ll p_2$$
$$a_2a \ll a_1a_2a = a_1t(a_1)a \ll p_3$$

By Remark 5.3.2, the vertices a and a_2 do not alternate in $p_1p_2p_3$. When the vertices $a_2, a_3, \ldots a_k$ are visited sequentially, since a_0 is not the parent of a_2, for any $3 \leqslant i \leqslant k$, we have

$$a_ia \ll p_1$$
$$aa_i \ll p_2$$
$$a_ia \ll p_3$$

so that $baabba \ll p_1p_2p_3$.

- Case 2: The path connecting a and b contains the vertex $0'$. We handle this in the following three subcases.
  - Subcase 2.1: $a, b \in \{1', 2', \ldots, m'\}$. Without loss of generality, assume that $a = i'$ and $b = j'$ where $i < j$. By steps 3 and 4 of Algorithm 1, we have $ab \ll p_1$ and $ba \ll p_2$.
  - Subcase 2.2: $a \in \{1', 2', \ldots, m'\}$ and $b \notin \{1', 2', \ldots, m'\}$. Let $a = i'$ and consider the path $\langle i', 0', j', \ldots, b\rangle$ where $j \neq i$. If $i < j$ then $i'j' \ll p_1$ and $j'i' \ll p_2$. As per the construction and Subcase 1.1, $i'j'b \ll p_1$ and $j'ba \ll p_2$. Thus, $ab \ll p_1$ and $ba \ll p_2$. If $i > j$ then then $j'i' \ll p_1$ and $i'j' \ll p_2$. As per the construction and Subcase 1.1, $j'ba \ll p_1$ and $aj'b \ll p_2$. Thus, $ba \ll p_1$ and $ab \ll p_2$.
  - Subcase 2.2: $a, b \notin \{1', 2', \ldots, m'\}$. Let $\langle a, \ldots, i', 0, j', \ldots, b\rangle$ be the path from a to b. Without loss of generality, assume $i < j$. We have, $i'j' \ll p_1$ and $j'i' \ll p_2$. By Subcase 1.1, $i'aj'b \ll p_1$ and $j'bi'a \ll p_2$. Thus, $ab \ll p_1$ and $ba \ll p_2$.

Thus, a and b do not alternate in $p_1p_2p_3$.

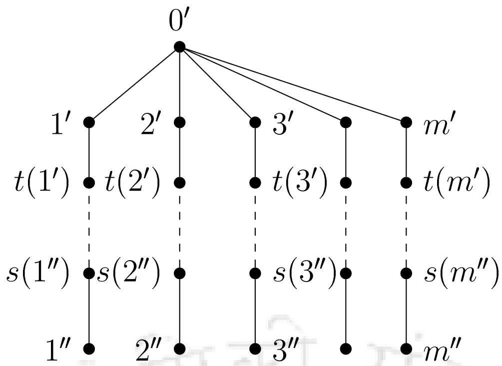


Figure 5.5: Depiction of T_m

Hence, the word $p_1p_2p_3$ permutationally represents T_m. $\square$

5.3.2 Word-Representant for M_c

In this section, we extend the 3-uniform word $p_1p_2p_3$ representing T_m to a 3-uniform word that permutationally represents M_c.

Let $1'', 2'', \ldots, m''$ be the leaves of T_m. Note that some of them are possibly non-existing, as the number of leaves in T_m may be less than m. Note that $s(a)$ is the parent of a (non-root) vertex a in T_m. With this notation, the tree T_m can be depicted as shown in Figure 5.5.

Lemma 5.3.4. *If all the constituent paths are of even length then* $\mathcal{R}^p(M_c) \leqslant 3$.

Proof. If each constituent path of M_c is of length two then M_c is a complete bipartite graph $K_{2,m}$ so that $\mathcal{R}^p(M_c) = 2$. If one of the constituent paths is of length at least four then we construct three permutations whose concatenation represents M_c.

Let $M_c = (E_1, \ldots, E_m)$ such that the constituent paths are arranged in non-increasing order by their lengths. As stated in Theorem 5.3.3, consider the word $p_1p_2p_3$ that represents the tree T_m. Since the constituent paths of M_c are of even length, the leaves of T_m are present in even levels. For reference, the permutations p_1, p_2 and p_3 will be as shown in Figure 5.6.

$$
\begin{array}{|ll|}
\hline
p_1 = & 1't(t(1'))\cdots s(1'')1''s(s(1''))\cdots t(1')2'3'\cdots m'0't(t(2'))\cdots s(2'')2''s(s(2''))\cdots t(2') \\
 & \cdots t(t(i'))\cdots s(i'')i''s(s(i''))\cdots t(i')\cdots t(t(m'))\cdots s(m'')m''s(s(m''))\cdots t(m') \\
p_2 = & m't(t(m'))t(m')\cdots s(m'')s(s(m''))m''\cdots i't(t(i'))t(i')\cdots s(i'')s(s(i''))i'' \\
 & \cdots 2't(t(2'))t(2')\cdots s(2'')s(s(2''))2''1'0't(t(1'))t(1')\cdots s(1'')s(s(1''))1'' \\
p_3 = & s(m'')m''\cdots s(i'')i''\cdots s(2'')2''s(1'')1''\cdots 1't(1')2't(2')\cdots i't(i')\cdots m't(m')0' \\
\hline
\end{array}
$$

Figure 5.6: Permutations for T_m with leaves at even levels

We now extend the permutations p_1, p_2 and p_3 to the vertices of M_c using the following 2-step process:

1. For each path E_i, from $i = 1$ to m, update the permutations p_1 and p_2 iteratively as per the following:

   i) If E_i is of length two then replace $0'$ by $i0'$ in p_1 and p_2.

   ii) If E_i is of length four and $i \neq 1$ (i.e., i' is not the leftmost child of $0'$), replace i'' by ii'' in p_1, and replace $s(i'')i''$ by $is(i'')i''$ in p_2.

   iii) In all other cases, replace $s(i'')i''$ by $is(i'')i''$ in p_1, and replace i'' by ii'' in p_2.

2. Using the resultant permutations p_1, p_2 and p_3, set $p_1' = p_1 0$, $p_2' = p_2 0$ and $p_3' = p_1'|^R_{\{1,2,\ldots,m\}} 0 p_3$. Note that $p_1'|^R_{\{1,2,\ldots,m\}}$ is the reversal of the subword of p_1' consisting of only occurrence of the letters $1, 2, \ldots, m$.

We show that the word $p_1'p_2'p_3'$ represents M_c permutationally. Since $p_1p_2p_3$ represents T_m and $p_1p_2p_3 \ll w$, it is sufficient to show that the vertices $0, 1, \ldots, m$ of S_m updated in the permutations p_1', p_2' and p_3' maintain the adjacency in M_c. We indeed prove in the following three cases that, for $1 \leqslant i \leqslant m$, the vertices 0 and i alternate, and i alternates with $0'$ or i'', depending on the length of E_i; but not with any other vertex of M_c.

1. For $1 \leqslant i \leqslant m$, we have $i0 \ll p_1'$, $i0 \ll p_2'$ and $i0 \ll p_3'$ so that i and 0 alternate in w. For each vertex b in T_m, we have $b0 \ll p_1'$ but $0b \ll p_3'$. Thus, b and 0 do not alternate in w. Further, it can be observed that $ij \ll p_1'$ then $ji \ll p_1'|^R_{\{1,2,\ldots,m\}} \ll p_3'$, for any $j \neq i$ in $S_m \backslash \{0\}$.

2. For $1 \leqslant i \leqslant m$, if E_i is of length two then the neighborhood of i, $N(i) = \{0, 0'\}$. Note that $i0'0 \ll p_1'$, $i0'0 \ll p_2'$, and $i00' \ll p_3'$ so that i alternates with 0 and $0'$ in w. For each vertex b in $T_m \backslash \{0'\}$, if $b \in E_1$ then we have $bi \ll p_1'$, and $ib \ll p_3'$; otherwise, we have $bi \ll p_2'$, and $ib \ll p_3'$. Thus, b and i do not alternate in w.

3. For $1 \leqslant i \leqslant m$, if E_i is of length at least four then $N(i) = \{0, i''\}$. Note that $ii'' \ll p_1'$, $ii'' \ll p_2'$, and $ii'' \ll p_3'$. Thus, along with Case-1, we have i alternates with 0 and i'' in w. For $b \in T_m \backslash N(i)$, since i belongs to the factor substituted in place of the factors i'' and $s(i'')i''$ in p_1 and p_2, as well as the prefix added to p_3, any vertex which does not alternate with i'' in T_m also does not alternate with i, i.e.,

   - $b = 0'$. If $i = 1$ then $i0' \ll p_1'$ and $0'i \ll p_2'$. Otherwise, if $i \neq 1$ then $0'i \ll p_1'$ and $i0' \ll p_2'$.
   - $b \in E_i \backslash \{0', s(i'')\}$. We have $bi \ll p_2'$ and $ib \ll p_3'$.
   - $b \in E_j \backslash \{0'\}$ and $j < i$. We have $bi \ll p_1'$ and $ib \ll p_3'$.
   - $b \in E_j \backslash \{0'\}$ and $j > i$. We have $bi \ll p_2'$ and $ib \ll p_3'$.

   Note that, except $s(i'')$, no vertex in T_m alternates with i'' and hence with i. Moreover, as per the construction of p_1' and p_2', it can be observed that i does not alternate with $s(i'')$. Thus, i and b do not alternate in w.

Hence, the word $p_1'p_2'p_3'$ represents the graph M_c so that $\mathcal{R}^p(M_c) \leqslant 3$. □

Lemma 5.3.5. *If all the paths are of odd length then $\mathcal{R}^p(M_c) \leqslant 3$.*

p_1	$= 1't(t(1'))\cdots s(s(1''))1''s(1'')\cdots t(1')2'3'\cdots m'0't(t(2'))\cdots s(s(2''))2''s(2'')\cdots$ $\cdots t(2')\cdots t(t(m'))\cdots s(s(m''))m''s(m'')\cdots t(m')$
p_2	$= m't(t(m'))t(m')\cdots m''s(m'')\cdots 2't(t(2'))t(2')\cdots 2''s(2'')1'0't(t(1'))t(1')\cdots 1''s(1'')$
p_3	$= m''\cdots 2''1''s(s(m''))s(m'')\cdots s(s(1''))s(1'')\cdots 1't(1')2't(2')\cdots m't(m')0'$

FIGURE 5.7: Permutations for T_m with leaves at odd levels

Proof. Let $M_c = (E_1, \ldots, E_m)$ such that the constituent paths are arranged in non-decreasing order by their lengths. As stated in Theorem 5.3.3, consider the word $p_1p_2p_3$ that represents the tree T_m. Since the constituent paths of M_c are of odd length, the leaves of T_m are in odd levels. For reference, the permutations p_1, p_2 and p_3 will be as shown in Figure 5.7.

We now extend the permutations p_1, p_2 and p_3 to the vertices of M_c using the following steps:

1. Replace $2'$ in p_1 with $012'$.
2. For each path E_i, from $i = 2$ to m, update the permutation p_1 iteratively as per the following and set the resultant permutation as p_1':
   i) If E_i is of length three then replace i'' with $i''i$. Note that in this case $i' = i''$.
   ii) Else, if E_i is of length at least five then replace $t(i')$ with $t(i')i$.
3. If 0 and $0'$ are adjacent then replace $0'$ with $00'$ in p_2; else, replace $0'$ with $0'0$ in p_2.
4. Using the resultant permutation p_2, set $p_2' = p_2m\cdots 21$.
5. Remove the vertices $1'', 2'', \ldots, m''$ from p_3 and add the prefix $m''\cdots 2''1''$, say p_3'' is the resultant permutation.

6. Iteratively update the permutation p_3'' by replacing i'' with $i''i$, for all $1 \leqslant i \leqslant m$; then set $p_3' = 0p_3''$.

We show that the word $p_1'p_2'p_3'$ permutationally represents M_c . Since the constituent paths of M_c are arranged in non-decreasing order by their lengths, if we remove the leaves $1'', 2'' \cdots, m''$ from p_3 and add the prefix $m'' \cdots 1''$ to p_3 then the word $p_1p_2p_3''$ also represents T_m permutationally. It is sufficient to show that, for $1 \leqslant i \leqslant m$, the vertex i'' alternates only with $s(i'')$ in $p_1p_2p_3''$. Clearly, $i''s(i'') \ll p_1$, $i''s(i'') \ll p_2$ and $i''s(i'') \ll p_3''$, and so, i and $s(i'')$ alternate in $p_1p_2p_3''$. For $b \in T_m \backslash \{s(i''), i''\}$, in the following cases, we observe that b and i'' do not alternate:

i) $b \in E_i$. We have $bi'' \ll p_2$ and $i''b \ll p_3''$.

ii) $b \in E_j \backslash \{j''\}$ and $j > i$. We have $bi'' \ll p_2$ and $i''b \ll p_3''$.

iii) $b = j''$ and $j > i$. We have $i''b \ll p_1$ and $bi'' \ll p_3''$.

iv) $b \in E_j$ and $j < i$. We have $bi'' \ll p_1$ and $i''b \ll p_3''$.

Now we show that the vertices $0, 1, \ldots, m$ of S_m updated in the permutations p_1', p_2' and p_3' maintain the adjacency in M_c. We indeed prove in the following two cases that, for $1 \leqslant i \leqslant m$, the vertices 0 and i alternate, and i alternates with $0'$ or i'', depending on the length of E_i; but not with any other vertex of M_c.

1. If 0 and $0'$ are non-adjacent in M_c then note that $00' \ll p_1'$ and $0'0 \ll p_2'$ so that 0 and $0'$ do not alternate in $p_1'p_2'p_3'$. Otherwise, the neighborhood of 0, $N(0) = \{0', 1, 2, \ldots, m\}$. Note that $0a \ll p_1'$, $0a \ll p_2'$ and $0a \ll p_3'$, for all $a \in N(0)$.

   For each vertex b in $T_m \backslash N(0)$, if $b \in E_1$ then note that $b0 \ll p_1'$ and $0b \ll p_3'$; otherwise, $0b \ll p_1'$ and $b0 \ll p_2'$. Hence, b and 0 do not alternate in $p_1'p_2'p_3'$.

2. For each $1 \leqslant i \leqslant m$, note that $i''i \ll p_1'$, $i''i \ll p_2'$ and $i''i \ll p_3'$. Thus, i'' and i alternate in $p_1'p_2'p_3'$.

For b in $M_c \backslash \{0, i, i''\}$, we observe that b and i do not alternate in $p'_1 p'_2 p'_3$, as per the following:

i) $b = 0'$. We have $0'i \ll p'_2$ and $i0' \ll p'_3$.

ii) $b \in E_i \backslash \{i'', 0, 0'\}$. We have $bi \ll p'_1$ and $ib \ll p'_3$.

iii) $b \in E_j \backslash \{j'', 0, 0'\}$ where $j > i$. We have $bi \ll p'_2$ and $ib \ll p'_3$.

iv) $b \in E_j \backslash \{j'', 0, 0'\}$ where $j < i$. We have $bi \ll p'_1$ and $ib \ll p'_3$.

v) $b \in \{1'', 2'', \ldots, (i-1)''\}$. We have $bi \ll p'_1$ and $ib \ll p'_3$.

vi) $b \in \{(i+1)'', \ldots, m''\}$. We have $ib \ll p'_1$ and $bi \ll p'_3$.

vii) For $b \in S_m \backslash \{0\}$. If $b < i$ then $bi \ll p'_1$ and $ib \ll p'_2$. Otherwise, $ib \ll p'_1$ and $bi \ll p'_2$.

Hence, the word $p'_1 p'_2 p'_3$ permutationally represents M_c so that $\mathcal{R}^p(M_c) \leqslant 3$. □

Lemma 5.3.6. *Suppose the endpoints* 0 *and* $0'$ *are adjacent, and each constituent path of even parity is of length two. Then* $\mathcal{R}^p(M_c) \leqslant 3$.

Proof. Without loss of generality, assume $E_1 = \langle 0, 0' \rangle$ and, for $2 \leqslant i \leqslant k$, $E_i = \langle 0, x_i, 0' \rangle$ are paths of length two in M_c. First, construct the permutations, say q_1, q_2 and q_3, for the graph obtained by considering only the odd paths in M_c (cf. Lemma 5.3.5). Then update the permutations as per the following:

1. In q_1, substitute $0(k+1)$ by $0(k+1)x_2 \cdots x_k$.

2. In q_2, substitute $00'$ by $0x_2 \cdots x_k 0'$.

3. In q_3, substitute $(k+1)'$ by $x_k \cdots x_2 (k+1)'$.

We observe that $q_1 q_2 q_3$ represents M_c.

Note that $0x_2 \cdots x_k 0' \ll q_1$, $0x_2 \cdots x_k 0' \ll q_2$, and $0x_k \cdots x_2 0' \ll q_3$. Therefore, all x_i's alternate with 0 and $0'$ in $q_1 q_2 q_3$. However, for all $b \in M_c \backslash \{0, 0', x_2, \ldots, x_k\}$, if

$b \notin E_{(k+1)}$ then $x_i b \ll q_1$ and $bx_i \ll q_2$. If $b \in E_{(k+1)} \setminus \{(k+1)'\}$ then $bx_i \ll q_1$ and $x_i b \ll q_2$. If $b = (k+1)'$ then $bx_i \leqslant q_1$ and $x_i b \ll q_3$. Thus, we have b and x_i do not alternate in $q_1 q_2 q_3$. Furthermore, for all $2 \leqslant i < j \leqslant k$, x_i and x_j do not alternate in $q_1 q_2 q_3$, as $x_i x_j \ll q_1$ and $x_j x_i \ll q_3$. □

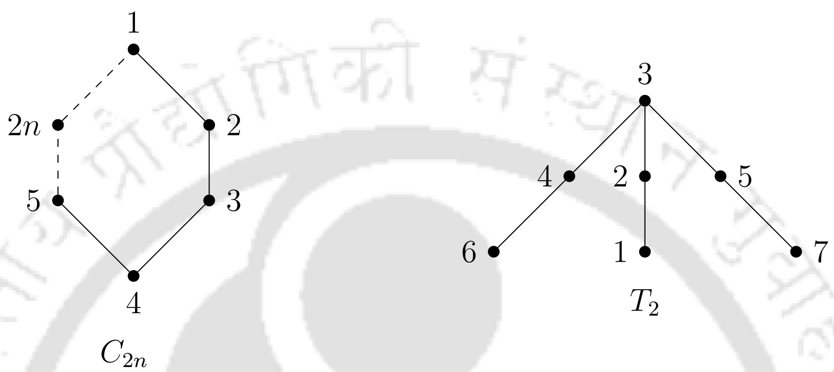


Figure 5.8: Graphs with *prn* three

We now conclude this section with the *prn* of M_c.

Theorem 5.3.7. *$\mathcal{R}^p(M_c) = 3$ if and only if it contains an even cycle C_{2n}, for $n \geqslant 3$, or T_2 (given in Figure 5.8) as an induced subgraph.*

Proof. In [Gallai, 1967], the permutation graphs, i.e., the class of graphs with *prn* at most two, are characterized in terms of forbidden induced subgraphs (see, e.g., [Limouzy, 2010, Theorem 7, Page 200]). Among the forbidden subgraphs for permutation graphs, only C_{2n} (for $n \geqslant 3$) and T_2 are the possible induced subgraphs of M_c. Note that T_2 is an induced subgraph of M_c if M_c is a book graph with at least three pages (in which C_4 is a largest induced cycle). Accordingly, in view of Lemma 5.3.4, Lemma 5.3.5 and Lemma 5.3.6, since $\mathcal{R}^p(M_c) \leqslant 3$, the result follows. □

Remark 5.3.8. Among melon graphs, while $\mathcal{R}^p(K_n) = 1$ (for $n \leqslant 3$), we have $\mathcal{R}^p(M_c) = 2$ if M_c is the book graph with two pages or the constituent paths of M_c are of length at most two or M_c is the combination of both.

5.4 Future Directions

The melon graphs are series-parallel graphs. It is known that the class of series-parallel graphs is a subclass of 3-colorable planar graphs (see [Dissaux et al., 2023]) and hence a simple series-parallel graph is word-representable. Accordingly, one may try to extend the work presented in this chapter to series-parallel graphs. More precisely, the following questions arise in this connection.

1. What is the representation number of a series-parallel graph?
2. One can ask for a classification of permutationally representable series-parallel graphs with respect to their *prn* between three and four.

In the literature, characterizing the word-representability of planar graphs is an open problem. Several authors have contributed to this problem by characterizing the word-representability of some subclass of planar graphs through operations, such as triangulations and face subdivison, on a known word-representable planar graph, as demonstrated in [Akrobotu et al., 2015; Chen et al., 2016; Glen, 2019; Glen and Kitaev, 2017]. It would be interesting to study the word-representabilty of triangulation (face-subdivision) of melon graphs.

6

Line Graphs of Melon Graphs

In this chapter, we contribute yet another class of graphs – with representation number and *prn* three – by studying the word-representability of line graphs of melon graphs. First, we prove that the line graphs of melon graphs are word-representable if and only if the melon graphs do not contain the triangular book graph with three pages as induced subgraphs. Then we show that their representation number is at most three and that the value three attains by only those melon graphs which consists of three constituent paths of length at least two. Further, we observe that the only comparability graphs arising from line graphs of melon graphs are paths and cycles of even length, along with the wheel graph W_4. Consequently, we obtain their *prn* to be at most three.

6.1 Representation Number

If a melon graph M has only two constituent paths, i.e., M is a cycle then the line graph $L(M)$ is isomorphic to M and hence it is a 2-word-representable graph. In addition, if 0 and $0'$ are adjacent in M (as shown in Figure 6.1), we construct a 2-uniform word representing $L(M)$ as per the following:

1. Using the method given in [Kitaev and Lozin, 2015, Section 3.2], we construct a 2-uniform word for the cycle $\langle a, b, \ldots, y, d, c, \ldots z, x, a\rangle$, that is,

$$axba \cdots dycd \cdots xz$$

2. By including e, consider the 2-uniform word

$$axbea \cdots dyced \cdots xz$$

   and note that it represents $L(M)$. Hence, $L(M)$ is a 2-word-representable graph.

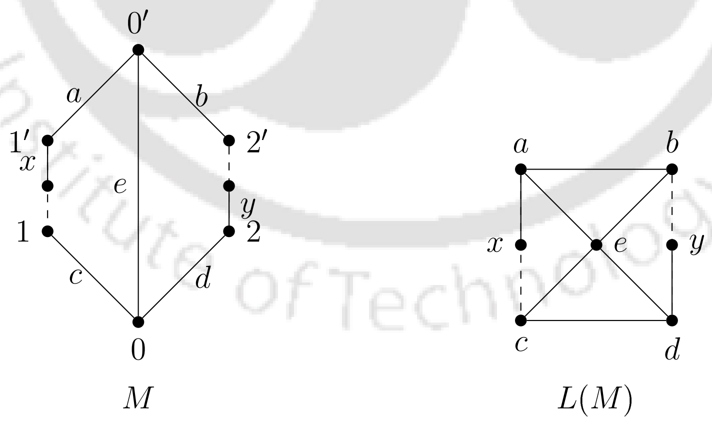
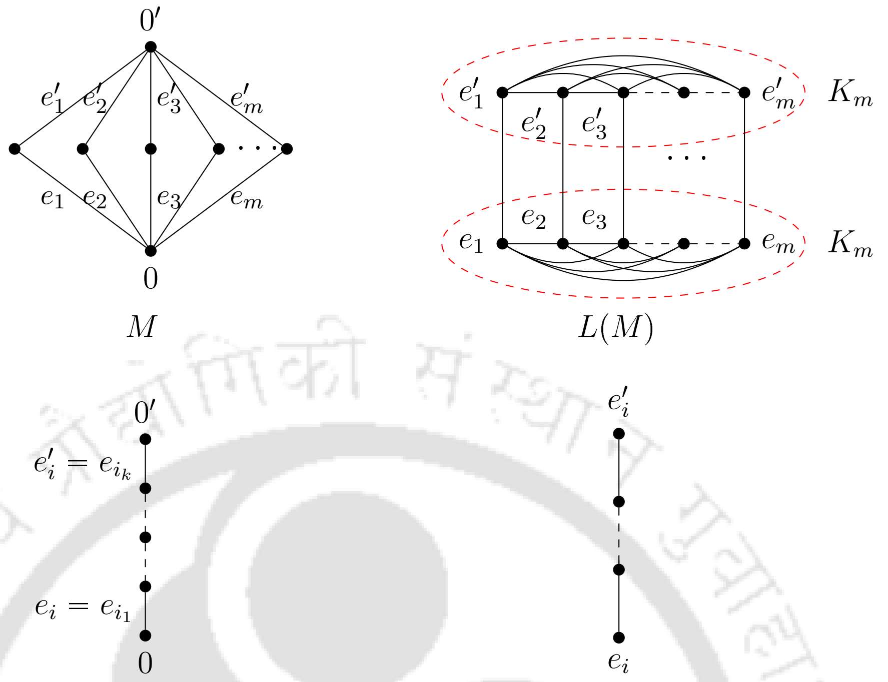


FIGURE 6.1: A melon graph with three constituent paths and its line graph

Suppose a melon graph $M = (E_1, \ldots, E_m)$ has all its constituent paths of length exactly two. Then the line graph of the melon graph is the Cartesian product



A constituent path of M The corresponding path in $L(M)$

Figure 6.2: A melon graph and its line graph

$K_m \square K_2$ of the complete graphs K_m and K_2, as shown in Figure 6.2. In Theorem 1.4.61, it was shown that the representation number of $K_m \square K_2$ is at most three.

In the following, we show that the line graphs of the melon graphs whose constituent paths are of length at least two are 3-word-representable.

Lemma 6.1.1. *Let M be a melon graph such that the endpoints 0 and $0'$ are non-adjacent. Then $L(M)$ is 3-word-representable.*

Proof. If the constituent paths of M are of length two then $L(M) = K_m \square K_2$ as shown in Figure 6.2, and its representation number is at most three. In fact, the following 3-uniform word w representing $K_m \square K_2$ can be constructed using the method given in [Broere and Zantema, 2019]:

$$w = e_1 e_2 \cdots e_m e'_1 e_1 e'_2 e_2 \cdots e'_m e_m e'_1 e'_2 \cdots e'_m e_1 e'_1 e_2 e'_2 \cdots e_m e'_m$$

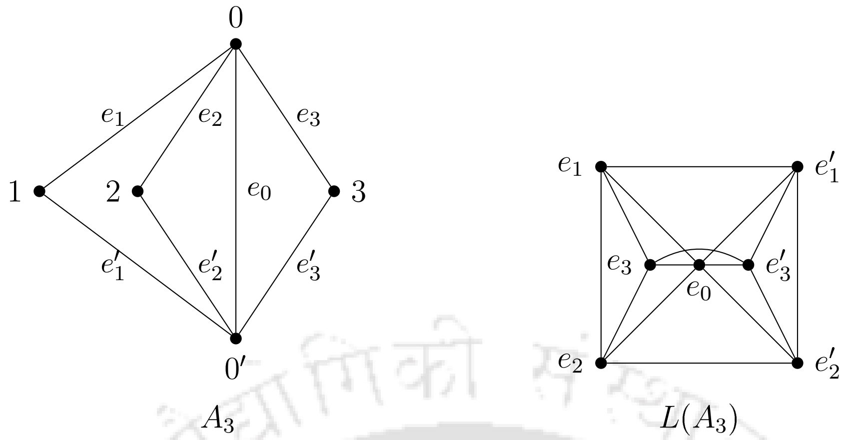


FIGURE 6.3: A triangular book graph and its line graph

Note that, as shown in Figure 6.2, the line graph $L(M)$ has the edge $\overline{e_ie'_i}$ corresponding to the two-length path $\langle e_i, e'_i \rangle$.

Suppose M has a constituent path E_i of length at least three. For $k \geqslant 3$, let $E_i = \langle e_i = e_{i_1}, \ldots, e_{i_k} = e'_i \rangle$, where e_{i_j}, $1 \leqslant j \leqslant k$, are the edges. Note that the line graph $L(M)$ has a path of length $k-1$ corresponding to E_i of M, which can be obtained by applying the subdivision (i.e., replacing an edge by a path) on the edge $\overline{e_ie'_i}$ as shown in Figure 6.2, and extend the word w representing $K_m \square K_2$.

If the edge $\overline{e_ie'_i}$ is replaced by a path of length two, say $\langle e_i, x, e'_i \rangle$ then replace the factors e'_ie_i and $e_ie'_i$ in w by $xe_ie'_ix$ and $e_ixe'_i$, respectively. Note that e_i and e'_i do not alternate in the word after the replacement. Between two consecutive occurrence of x only e_i and e'_i occur exactly once. Hence, x alternate only with e_i and e'_i.

If we replace the edge $\overline{e_ie'_i}$ by a path of length k, for $k \geqslant 3$ then first replace the edge by a path of length two (with an intermediate vertex x) and note that the resultant graph is 3-word-representable, as described above. Then delete the intermediate vertex x and add the path of length k. By Lemma 1.4.50, it can be observed that the resultant graph is 3-word-representable. $\square$

Lemma 6.1.2. *Let A_3 be the triangular book graph with three pages, as shown in Figure 6.3. The line graph $L(A_3)$ is not word-representable.*

Proof. Suppose $L(A_3)$ is word-representable. Then by Theorem 1.4.25, the subgraph induced by the neighborhood of every vertex of $L(A_3)$ must be a comparability graph. However, it can be observed that the subgraph of $L(A_3)$ induced by the neighborhood of the vertex e_0 is the prism Pr_3 (cf. Figure 6.3), which is not a comparability graph. For instance, the complement of Pr_3 is the graph C_6. Since C_6 is a comparability graph with *prn* three, Pr_3 is not a comparability graph (see [Gallai, 1967]).

Thus, $L(A_3)$ is not word-representable. □

Lemma 6.1.3. *The graph* $H = (V, E)$*, where* $V = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m, x\}$ *and* $E = \left\{\overline{a_i a_j} \mid i \neq j\right\} \cup \left\{\overline{b_i b_j} \mid i \neq j\right\} \cup \left\{\overline{xy} \mid y \in V \backslash \{x\}\right\} \cup \left\{\overline{a_1 b_1}, \overline{a_2 b_2}\right\}$*, is permutationally 3-representable.*

Proof. Consider the following three permutations over the set V:

$$q_1 = a_1 b_2 a_3 b_3 a_4 b_4 \cdots a_m b_m a_2 b_1 x$$

$$q_2 = a_1 a_3 a_4 \cdots a_m b_2 a_2 b_3 b_4 \cdots b_m b_1 x$$

$$q_3 = b_2 b_3 b_4 \cdots b_m a_1 b_1 a_3 a_4 \cdots a_m a_2 x$$

We show that the word $w = q_1 q_2 q_3$ represents H permutationally.

Note that x occurs at the end of each permutation in w so that x alternates with all a_i's and b_i's in w. Since $a_1 a_3 \cdots a_m a_2 \ll q_i$, for all $i = 1, 2, 3$, we have a_r and a_s, for $r \neq s$, alternate in w. Similarly, for each $i = 1, 2, 3$, we have $b_2 b_3 b_4 \cdots b_m b_1 \ll q_i$ so that b_r and b_s, for $r \neq s$, alternate in w. Further note that $a_1 b_1 a_1 b_1 a_1 b_1 \ll w$ and $b_2 a_2 b_2 a_2 b_2 a_2 \ll w$. Hence, any two adjacent vertices of H alternate in w.

We consider all the non-adjacent pairs of vertices in H and observe that they do not alternate in w. For all $r, s \in \{3, \ldots, m\}$, a_r and b_s are non-adjacent in H and $a_r b_s b_s a_r \ll q_2 q_3 \ll w$. Further, for $2 \leqslant r \leqslant m$, a_1 is non-adjacent to b_r and note that $a_1 b_r b_r a_1 \ll q_2 q_3 \ll w$. Also, a_2 is non-adjacent to b_s, for $s \in \{1, 3, \ldots, m\}$ and, clearly $a_2 b_s b_s a_2 \ll q_2 q_3 \ll w$. Next, note that b_1 is non-adjacent to a_r for $2 \leqslant r \leqslant m$ and we have $a_r b_1 b_1 a_r \ll q_2 q_3$. Finally, observe that b_2 is non-adjacent to a_s, for

$s \in \{1, 3, \ldots, m\}$ and $a_s b_2 b_2 a_s \ll q_2 q_3$. □

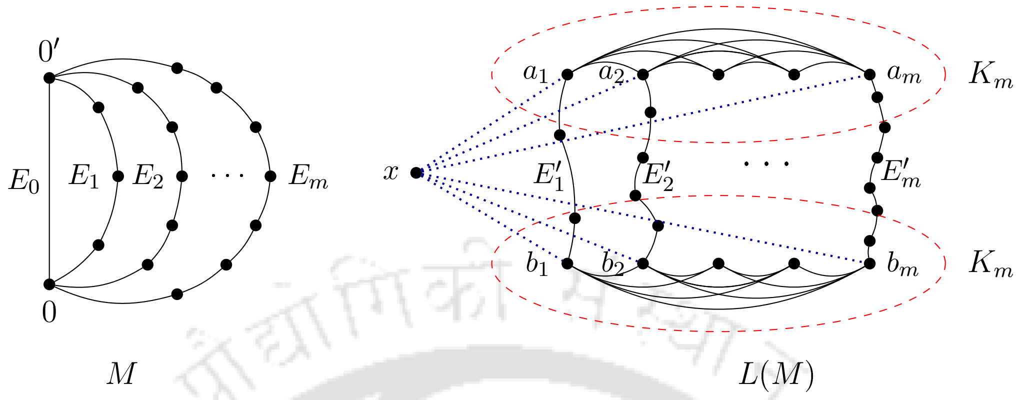


FIGURE 6.4: Melon graph with $\overline{00'}$ and its line graph

Lemma 6.1.4. *Let M be a melon graph in which the endpoints 0 and $0'$ are adjacent and does not contain A_3 as an induced subgraph. Then $L(M)$ is 3-word-representable.*

Proof. Let $M = (E_0, E_1, \ldots, E_m)$ such that the constituent paths are arranged in the non-decreasing order by their lengths. Thus, E_0 is the constituent path of length one. Further, since M does not contain A_3 as an induced subgraph, M must have at most two constituent paths each of length two, i.e., E_1 and E_2 are possibly of length two, and the remaining are of length at least three.

As shown in Figure 6.4, note that $L(M)$ has two copies of K_m, with vertex sets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$, as induced subgraphs connected by paths E'_i $(1 \leqslant i \leqslant m)$ between a_i and b_i $(1 \leqslant i \leqslant m)$ and having a vertex x adjacent to all the vertices a_i's and b_i's. Here, the vertex x corresponds to the path E_0 of M and, for $1 \leqslant i \leqslant m$, the path E'_i corresponds to the path E_i of M and of length one less than the length of E_i.

In the following, we construct a 3-uniform word representing $L(M)$ by iteratively extending the word w representing the graph H given in Lemma 6.1.3.

Case 1: Suppose the length of E_i (for $i = 1, 2$) is two. Note that H is clearly an induced subgraph of $L(M)$ in which $\overline{a_1 b_1}$ and $\overline{a_2 b_2}$ are the paths E'_1 and E'_2, respectively. Set $H' = H$ and $w' = w$.

Case 2: Suppose E_1 is of length three. Update the path E'_1 in the graph H with an intermediate vertex y_1 so that $E'_1 = \langle a_1, y_1, b_1 \rangle$ is a path of length two. Let H_1 be the resultant graph. Accordingly, extend the word w to w_1 by replacing the factor a_1b_1 with the word $y_1b_1a_1y_1$, and the first occurrence of b_1 with b_1y_1. Note that the replacement of the word $y_1b_1a_1y_1$ ensures that the vertices a_1 and b_1 do not alternate and y_1 alternates only with a_1 and b_1. Hence, the 3-uniform word w_1 represents H_1.

Case 3: Suppose E_2 is of length three. If E_1 is of length two then set $w_1 = w$ and $H_1 = H$. Otherwise, consider the graph H_1 and its word-representant w_1 from Case 2. Update the path E'_2 in the graph H_1 with an intermediate vertex y_2 so that $E'_2 = \langle a_2, y_2, b_2 \rangle$ is a path of length two. Let H_2 be the resultant graph. In the word w_1, replace the factor b_2a_2 with the word $y_2a_2b_2y_2$ and the third occurrence of a_2 with a_2y_2. Note that the resultant 3-uniform word w_2 represents H_2. Set $H' = H_2$ and $w' = w_2$.

Case 4: For $i = 1$ or 2, suppose E_i is of length k, where $k \geqslant 4$. Consider the graph H_2 from Case 3 and its word-representant w_2. Remove the intermediate vertex y_i of E'_i in H_2 and connect a_i and b_i with a path of length $k-1$. By Lemma 1.4.50, the resultant graph, say H', is 3-word-representable. Let w' be the word-representant of H'.

Suppose each of $E_3, \ldots, E_t$ (for some $t \leqslant m$) is of length three. Update the graph H' by the paths $E'_i = \langle a_i, y_i, b_i \rangle$, for $3 \leqslant i \leqslant t$. Let H'' be the resultant graph. Accordingly, extend w' to w'' by replacing the factor a_ib_i in w' with the word $y_ia_ib_iy_i$ and the second occurrence of b_i by b_iy_i. For $3 \leqslant i \leqslant t$, the factor $y_ia_ib_iy_i$ ensures that y_i alternates only with a_i and b_i in w''. Hence, the 3-uniform word w'' represents H''.

For the paths $E_{t+1}, \ldots, E_m$, which are of length at least four in M, consider the graph H'' and add paths E'_i between the vertices a_i and b_i. Note that the resultant graph is $L(M)$. Since H'' is 3-word-representable, by Lemma 1.4.50, $L(M)$ is 3-word-representable. □

Theorem 6.1.5. *Let M be a melon graph. The line graph $L(M)$ is word-representable*

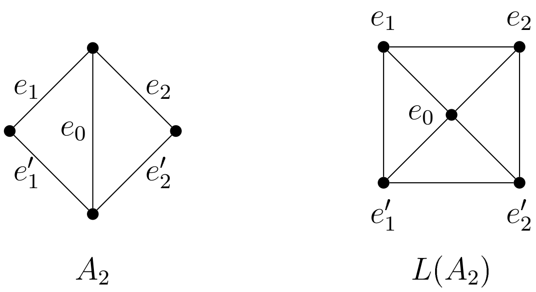


FIGURE 6.5: Triangular book graph with two pages and its corresponding line graphs

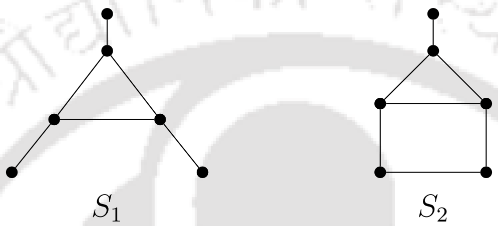


FIGURE 6.6: Non-comparability graphs

if and only if M does not contain A_3 as an induced subgraph.

Theorem 6.1.6. *For a melon graph M, if $L(M)$ is word-representable then $\mathcal{R}(L(M)) \leqslant 3$. Furthermore, $\mathcal{R}(L(M)) = 3$ if and only if M consists of three constituent paths of length at least two.*

Note that the graph $K_3 \square K_2$ is not a circle graph. Further, any melon graph M consisting of at least three constituent paths always contains $K_3 \square K_2$ as a vertex-minor. Hence, we have the following corollary of Theorem 6.1.6.

Corollary 6.1.7. *For a melon graph M, let $L(M)$ be word-representable. Then $\mathcal{R}(L(M)) = 3$ if and only if $L(M)$ contains $K_3 \square K_2$ as a vertex-minor.*

6.2 Permutation-Representation Number

Finally, we present the following result on the *prn* of the permutationally representable line graphs of melon graphs.

Theorem 6.2.1. *Among the line graphs of melon graphs, the following are the only comparability graphs: $L(P_n)$, $L(C_{2n})$, for $n \geqslant 2$, and $L(A_2)$. Here, P_n is the path on n vertices, C_{2n} is the cycle on $2n$ vertices and A_2 is the graph given in Figure 6.5. Moreover, the following on their prn are evident.*

1. $\mathcal{R}^p(L(P_2)) = 1$ *and* $\mathcal{R}^p(L(P_n)) = 2$, *for* $n \geqslant 3$.

2. $\mathcal{R}^p(L(C_4)) = 2$ *and* $\mathcal{R}^p(L(C_{2n})) = 3$, *for* $n \geqslant 3$.

3. $\mathcal{R}^p(L(A_2)) = 2$.

Proof. Suppose M is a melon graph with three constituent paths of length at least two. Clearly, the line graph of M contains $K_3\Box K_2$ as a vertex minor (see Figure 6.2). Note that $K_3\Box K_2$ is the prism Pr_3, which is not a comparability graph (see Lemma 6.1.2). Also, note that the graph obtained by applying subdivision on the edges joining two copies of K_3 in $K_3\Box K_2$ contains S_1 or S_2 given in Figure 6.6 as an induced subgraph. It is known that S_1 and S_2 are not comparability graphs (see the characterization of comparability graphs in [Gallai, 1967]). Thus, M is not a comparability graph. Hence, M can have at most two paths, or whenever M has three paths, it has a path of length one.

If M has only one constituent path then $M = P_n$, for some $n \geqslant 2$. Then $L(M) = P_{n-1}$ and hence it is a permutation graph.

Suppose $M = (E_1, E_2)$. As M is a cycle, $L(M)$ is isomorphic to M. Note that odd cycles are not comparability graphs. Hence, in this case, the line graph of M is a comparability graph if both E_1 and E_2 are of the same parity, i.e., M is an even cycle.

Suppose $M = (E_1, E_2, E_3)$ is a melon graph, where E_1 is of length one. We have the following cases:

- E_2 and E_3 are not of the same parity: The line graph of M contains an odd cycle of length at least five as an induced subgraph. Thus, $L(M)$ is not a comparability graph.

- E_2 and E_3 are of odd parity: From Figure 6.1, it can be observed that the line graph $L(M)$ contains either S_1 or S_2, given in Figure 6.6, as an induced subgraph. Thus, $L(M)$ is not a comparability graph.

- E_2 and E_3 are of even parity:

  - If E_2 or E_3 is of length at least four then M contains an odd cycle of length at least five, which includes E_1. Accordingly, the line graph of M will also contain an odd cycle as an induced subgraph so that $L(M)$ is not a comparability graph.

  - If both E_2 and E_3 are of length two then $M = A_2$, as shown in Figure 6.5. Accordingly, $L(A_2)$, which is isomorphic to the wheel graph W_4, is a permutation graph (refer to the forbidden induced subgraphs of permutation graphs in [Gallai, 1967]).

$\square$

Part III

Semi-Transitive Orientations

7

Line Graphs

In the literature, it is known that the line graph of a word-representable graph need not be word-representable. For instance, while the line graph of a cycle is word-representable, the line graph of the complete graph K_5 is not. Among the non-word-representable graphs, there are examples of non-word-representable graphs whose line graphs are non-word-representable. However, there exists no example of a non-word-representable whose line graph is word-representable. Thus, it is an open problem that whether the line graph of a non-word-representable graph is always non-word-representable or not? In this chapter, we address this open problem by considering a class of non-word-representable graphs, viz., Mycielski graphs of odd cycles of length at least five, and show that their line graphs are word-representable.

7.1 Line Graphs of Word-Representable Graphs

In this section, we discuss two key results related to the word-representability of line graphs. First, we provide an alternative proof for the word-representability of line graphs of complete graphs. Then we show that if a connected graph G on at least six vertices contains a vertex of degree at least four then the line graph operation applied at least two times to the graph G is non-word-representable.

Kitaev et al. [2011a,b] showed that the line graph of the complete graph K_n, for $n \leqslant 4$, is word-representable, whereas the line graphs of other complete graphs are non-word-representable.

Theorem 7.1.1 ([Kitaev et al., 2011a,b]). *For $n \geqslant 5$, the line graph of a complete graph K_n is non-word-representable.*

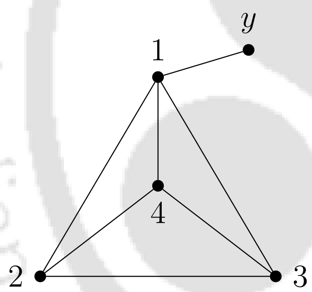


FIGURE 7.1: K_4'

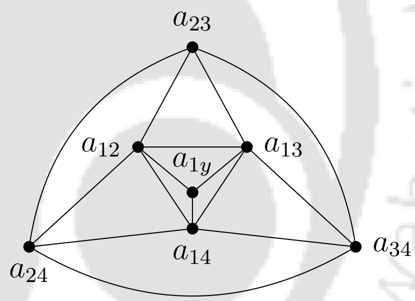


FIGURE 7.2: $L(K_4')$

Consider the graph K_4' given in Figure 7.1. Note that the line graph $L(K_4')$ is as per Figure 7.2, in which the vertices a_{ij} are corresponding to the edges $\overline{ij}$ in K_4'. It can be observed that $L(K_4')$ is isomorphic the graph co-(T_2), which is non-word-representable by Theorem 1.4.26. Hence, we have the following lemma.

Lemma 7.1.2. *The graph $L(K_4')$ is non-word-representable.*

Note that if H is a subgraph of G then $L(H)$ is an induced subgraph of $L(G)$. By hereditary property of word-representable graphs, all induced subgraphs of a word-representable graph are word-representable. Hence, by Lemma 7.1.2, we have the following corollaries.

Corollary 7.1.3. *If a graph G contains K_4' as a subgraph then its line graph $L(G)$ is non-word-representable.*

In particular, since a complete graph K_n, for $n \geqslant 5$, contains K_4' as a subgraph, the line graph $L(K_n)$ is non-word-representable. Thus, we have a shorter and alternative proof for the word-representability of line graphs of complete graphs given in Theorem 7.1.1.

Corollary 7.1.4. *If G is a connected graph on at least five vertices and contains K_4 as an induced subgraph then $L(G)$ is non-word-representable.*

The following theorem allows us to turn almost every graph into a non-word-representable graph by applying the line graph operation at most k times.

Theorem 7.1.5 ([Kitaev et al., 2011a,b])**.** *If a connected graph G is neither a cycle, a path nor the complete bipartite graph $K_{1,3}$ then the line graph $L^k(G)$ is non-word-representable, for $k \geqslant 4$.*

Using Corollary 7.1.4, we establish the following theorem.

Theorem 7.1.6. *Let G be a connected graph on six vertices which contains a vertex of degree at least four. Then $L^k(G)$ is non-word-representable, for $k \geqslant 2$.*

Proof. Note that the complete bipartite graph $K_{1,4}$ is a proper subgraph of G and hence K_4 is a proper subgraph of $L(G)$. Thus, by Corollary 7.1.4, we see that $L^2(G)$ is non-word-representable. Further, the line graph $L^k(G)$, for each $k \geqslant 2$, contains K_4 as a proper subgraph. Hence, we can conclude that $L^k(G)$ is non-word-representable, for $k \geqslant 2$. □

7.2 Line Graphs of Non-Word-Representable Graphs

In this section, we address the following open problem posed in [Kitaev et al., 2011a,b].

Problem 1. *Is the line graph of a non-word-representable graph always non-word-representable?*

Problem 1 was highlighted at multiple contexts (e.g., see [Kitaev, 2017; Kitaev and Pyatkin, 2018]) and there have been attempts to address the problem. For instance, in [Akbar et al., 2021; Akrobotu, 2021], a negative answer to Problem 1 was attempted by giving the graph A depicted in Figure 7.3 as counterexample. However, we show that the line graph of A is non-word-representable. As a result, the problem remains open. Nevertheless, we settle the problem by showing that the line graph of the Mycielski graph of odd cycle $\mu(C_{2n+1})$, for $n \geqslant 2$, is word-representable.

7.2.1 $L(W_5')$ is Non-Word-Representable

In [Akbar et al., 2021; Akrobotu, 2021], the non-word-representable graph A given in Figure 7.3 was considered and shown that $L(A)$ is word-representable. Note that the graph A contains W_5' given in Figure 7.4 as a subgraph, and thus $L(A)$ contains $L(W_5')$ as an induced subgraph. As word-representable graphs are hereditary, $L(W_5')$ must be word-representable if $L(A)$ is word-representable. However, we observe that $L(W_5')$ is non-word-representable showing that the graph in Figure 7.3 is not a counterexample to Problem 1.

Notation 7.2.1. The statement $\exists(abca)$ is true for a word w denotes that there exists at least two consecutive occurrences of the vertex a in the word w such that vertices b and c occur between them and b appears before c.

Notation 7.2.2. For the statement $\forall(abca)$ is true for the word w denotes that

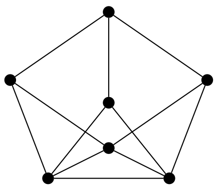


Figure 7.3: A

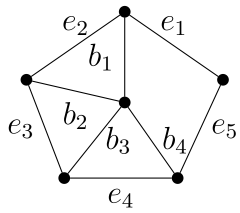


Figure 7.4: W_5'

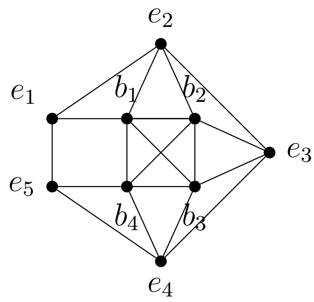


Figure 7.5: $L(W_5')$

between every two consecutive occurrences of the vertex a in w, the vertices b and c occur once, and b appears before c.

The following two results are needed to prove that the graph $L(W_5')$ is non-word-representable.

Proposition 7.2.3 ([Kitaev et al., 2011a,b])**.** *Let w represents a graph $G = (V, E)$, and $a, b, c \in V$ and $\overline{ab}, \overline{ac} \in E$. Then*

- *if $\overline{bc} \notin E$ then both the statements $\exists(abca)$ and $\exists(acba)$ are true for w, while*
- *if $\overline{bc} \in E$ then exactly one of the statement $\forall(abca)$ and $\forall(acba)$ is true for w.*

Lemma 7.2.4 ([Kitaev and Saito, 2020])**.** *Suppose that the vertices in $\{a, b, c, d\}$ induce a subgraph S in a partially oriented graph such that $\overrightarrow{ab}$ and $\overrightarrow{bc}$ are edges, $\overline{cd}$ and $\overline{da}$ are non-oriented edges, and S is different from the complete graph K_4. Then the unique way to orient $\overline{cd}$ and $\overline{da}$ in order not to create a directed cycle or a shortcut is $\overrightarrow{ad}$ and $\overrightarrow{dc}$.*

Theorem 7.2.5. *The line graph $L(W_5')$ is non-word-representable.*

Proof. Suppose $L(W_5')$ is word-representable and w is a k-uniform word that represents $L(W_5')$. Since any cyclic shift of a word representing a graph also represents the same graph (see Proposition 1.4.8), assume that the leftmost letter of w is b_1. Since $\{b_1, b_2, b_3, b_4\}$ induces a clique in $L(W_5')$, $w|_{\{b_1,b_2,b_3,b_4\}}$ is of the form u^k, by Proposition

1.4.29, where $u = b_1 b_{i_2} b_{i_3} b_{i_4}$ such that $i_2, i_3, i_4 \in \{2, 3, 4\}$. In the following, we show that either $u = b_1 b_2 b_3 b_4$ or $u = b_1 b_4 b_3 b_2$.

Suppose $i_2, i_4 \in \{3, 4\}$ so that $u = b_1 b_{i_2} b_2 b_{i_4}$. Applying Proposition 7.2.3 on the vertices b_1, b_{i_2} and b_2, the statement $\forall(b_1 b_{i_2} b_2 b_1)$ is true for w. Similarly, applying Proposition 7.2.3 on the vertices b_1, b_2 and b_{i_4}, the statement $\forall(b_1 b_2 b_{i_4} b_1)$ is true for w. As the vertex e_2 is non-adjacent to b_{i_2} but adjacent to b_1, the statements $\exists(b_1 e_2 b_{i_2} b_1)$ and $\exists(b_1 b_{i_2} e_2 b_1)$ are true for w. Also, since the vertex e_2 is non-adjacent to b_{i_4}, the statements $\exists(b_1 e_2 b_{i_4} b_1)$ and $\exists(b_1 b_{i_4} e_2 b_1)$ are true for w. From the statements $\exists(b_1 e_2 b_{i_2} b_1)$ and $\forall(b_1 b_{i_2} b_2 b_1)$, we get $\exists(b_1 e_2 b_2 b_1)$. From the statements $\exists(b_1 b_{i_4} e_2 b_1)$ and $\forall(b_1 b_2 b_{i_4} b_1)$, we get $\exists(b_1 b_2 e_2 b_1)$. Thus, we can conclude that $\exists(b_1 e_2 b_2 b_1)$ and $\exists(b_1 b_2 e_2 b_1)$ are true for w. This is a contradiction to Proposition 7.2.3 when applied to the vertices b_1, b_2 and e_2. Thus, either $u = b_1 b_2 b_{i_3} b_{i_4}$ or $u = b_1 b_{i_2} b_{i_3} b_2$.

Suppose $u = b_1 b_2 b_4 b_3$. Then applying Proposition 7.2.3 to the vertices b_2, b_3 and b_4, the statement $\forall(b_2 b_4 b_3 b_2)$ is true for w. Similarly, applying Proposition 7.2.3 to the vertices b_2, b_3 and b_1, the statement $\forall(b_2 b_3 b_1 b_2)$ is true for w. Since the vertex e_3 is non-adjacent to b_1 but adjacent to b_2, the statements $\exists(b_2 e_3 b_1 b_2)$ and $\exists(b_2 b_1 e_3 b_2)$ are true for w. Also, since the vertex e_3 is non-adjacent to b_4, the statements $\exists(b_2 e_3 b_4 b_2)$ and $\exists(b_2 b_4 e_3 b_2)$ are true for w. From the statements $\exists(b_2 b_1 e_3 b_2)$ and $\forall(b_2 b_3 b_1 b_2)$, we get $\exists(b_2 b_3 e_3 b_2)$. From the statements $\exists(b_2 e_3 b_4 b_2)$ and $\forall(b_2 b_4 b_3 b_2)$, we get $\exists(b_2 e_3 b_3 b_2)$. Thus, we can conclude that $\exists(b_2 e_3 b_3 b_2)$ and $\exists(b_2 b_3 e_3 b_2)$ are true for w. This is a contradiction to Proposition 7.2.3 when applied to the vertices b_2, b_3 and e_3. Thus, $u = b_1 b_2 b_3 b_4$.

Suppose $u = b_1 b_3 b_4 b_2$. Then, applying Proposition 7.2.3 to the vertices b_1, b_2 and b_3, the statement $\forall(b_2 b_1 b_3 b_2)$ is true for w. Similarly, applying Proposition 7.2.3 to the vertices b_2, b_3 and b_4, the statement $\forall(b_2 b_3 b_4 b_2)$ is true for w. Since the vertex e_3 is adjacent to b_2 but non-adjacent to b_1, the statements $\exists(b_2 e_3 b_1 b_2)$ and $\exists(b_2 b_1 e_3 b_2)$ are true for w. Also, since e_3 is non-adjacent to b_4, the statements $\exists(b_2 e_3 b_4 b_2)$ and $\exists(b_2 b_4 e_3 b_2)$ are true for w. From the statements $\exists(b_2 e_3 b_1 b_2)$ and $\forall(b_2 b_1 b_3 b_2)$, we get

$\exists(b_2e_3b_3b_2)$. From the statements $\exists(b_2b_4e_3b_2)$ and $\forall(b_2b_3b_4b_2)$, we get $\exists(b_2b_3e_3b_2)$. Thus, we can conclude that $\exists(b_2e_3b_3b_2)$ and $\exists(b_2b_3e_3b_2)$ are true for w. This is a contradiction to Proposition 7.2.3 when applied to the vertices b_2, b_3 and e_3. Thus, $u = b_1b_4b_3b_2$.

Note that the second case, i.e., $u = b_1b_4b_3b_2$, can be obtained from the first case, i.e., $u = b_1b_2b_3b_4$, by reversing the word w and taking a cyclic shift. Therefore, it is sufficient to prove the theorem for the first case.

Let D_0 be the orientation of $L(W_5')$ associated with the word w such that for any two adjacent vertices a and b in $L(W_5')$, $\overrightarrow{ab} \in D_0$ if and only if the first occurrence of a occurs before the first occurrence of b in w. Since w represents $L(W_5')$, in view of Theorem 1.4.14, D_0 must be a semi-transitive orientation. However, we show that D_0 cannot be semi-transitive so that $L(W_5')$ is non-word-representable.

Since the vertex b_1 is the leftmost letter of w, b_1 must be a source in D_0. Thus, we have $\overrightarrow{b_1e_1}, \overrightarrow{b_1e_2}, \overrightarrow{b_1b_2}, \overrightarrow{b_1b_3}$ and $\overrightarrow{b_1b_4}$. Further, since $w|_{\{b_1,b_2,b_3,b_4\}} = (b_1b_2b_3b_4)^k$, we have $\overrightarrow{b_2b_3}, \overrightarrow{b_2b_4}, \overrightarrow{b_3b_4}$ in D_0.

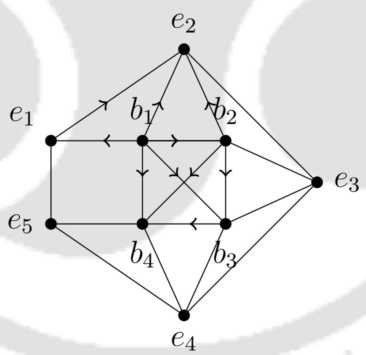


Figure 7.6: Partial orientation of $L(W_5')$

Consider the subgraph induced by $\{b_1, b_2, b_3, e_2\}$. Note that if $\overrightarrow{e_2b_2} \in D_0$ then $\overrightarrow{b_1b_3}$ becomes the shortcutting edge of the directed path $\langle b_1, e_2, b_2, b_3\rangle$, which is not possible as D_0 does not contain any shortcut. Thus, $\overrightarrow{b_2e_2} \in D_0$. Similarly, considering the subgraph induced by $\{b_1, b_2, e_1, e_2\}$, we must have $\overrightarrow{e_1e_2} \in D_0$. The partial orientation of W_5' with respect to the semi-transitive orientation D_0 is shown in Figure 7.6.

Claim 1. For $2 \leqslant i \leqslant 4$, $\overrightarrow{e_i e_{i+1}} \in D_0$.

Suppose $\overrightarrow{e_3 e_2} \in D_0$. Consider the subgraph induced by $\{b_1, b_2, e_2, e_3\}$. Note that if $\overrightarrow{b_2 e_3} \in D_0$ then $\overrightarrow{b_1 e_2}$ becomes the shortcutting edge of the directed path $\langle b_1, b_2, e_3, e_2 \rangle$, which is not possible in D_0 as it does not contain any shortcut. Thus, $\overrightarrow{e_3 b_2} \in D_0$. If $\overrightarrow{b_3 e_3} \in D_0$ then the vertices $\{b_2, b_3, e_3\}$ form a directed cycle, which is not possible as D_0 is an acyclic orientation. Thus, $\overrightarrow{e_3 b_3} \in D_0$. Now applying Lemma 7.2.4 to the vertices $\{e_3, b_3, b_4, e_4\}$, we have $\overrightarrow{e_3 e_4}$ and $\overrightarrow{e_4 b_4}$. If $\overrightarrow{b_3 e_4} \in D_0$ then $\overrightarrow{b_1 b_4}$ becomes the shortcutting edge for the directed path $\langle b_1, b_3, e_4, b_4 \rangle$, which is not possible. Thus, $\overrightarrow{e_4 b_3} \in D_0$. Again applying Lemma 7.2.4 to the vertices $\{e_4, b_3, b_4, e_5\}$, we have $\overrightarrow{e_4 e_5}, \overrightarrow{e_5 b_4} \in D_0$. Finally, if $\overrightarrow{e_1 e_5}$ then $\overrightarrow{b_1 b_4}$ becomes the shortcutting edge of the directed path $\langle b_1, e_1, e_5, b_4 \rangle$. Thus, $\overrightarrow{e_5 e_1} \in D_0$. But then, $\langle e_3, e_4, e_5, e_1, e_2 \rangle$ forms a shortcut with the shortcutting edge $\overrightarrow{e_3 e_2}$ as shown in Figure 7.7. Hence, we have $\overrightarrow{e_2 e_3} \in D_0$.

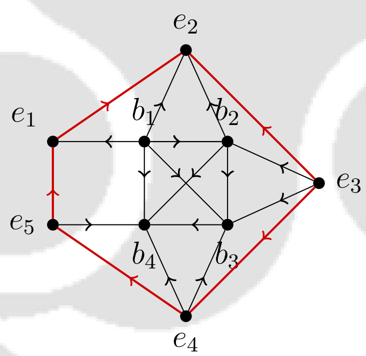
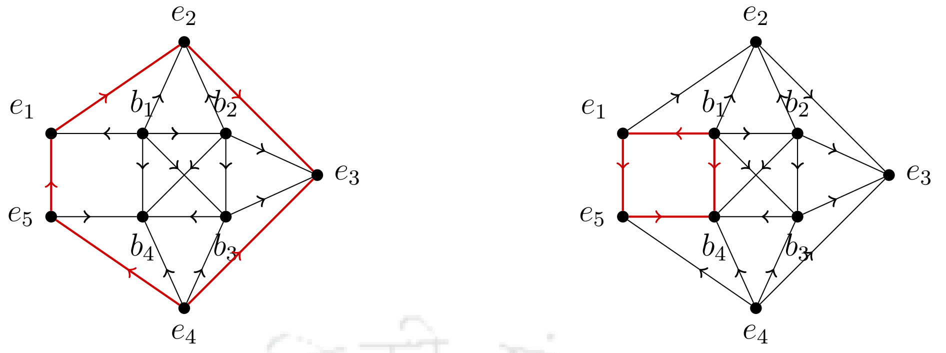


FIGURE 7.7: The directed path $\langle e_3, e_4, e_5, e_1, e_2 \rangle$ forms a shortcut

Suppose $\overrightarrow{e_4 e_3} \in D_0$. By a similar argument, considering the subgraph induced by $\{b_2, b_3, e_2, e_3\}$, we have $\overrightarrow{b_3 e_3} \in D_0$. Note that if $\overrightarrow{e_3 b_2} \in D_0$ then $\overrightarrow{b_1 b_2}$ becomes the shortcutting edge of the directed path $\langle b_1, b_3, e_3, b_2 \rangle$, which is not possible. Thus, $\overrightarrow{b_2 e_3} \in D_0$. If $\overrightarrow{b_4 e_4}$ then $\overrightarrow{b_3 e_3}$ becomes the shortcutting edge of the directed path $\langle b_3, b_4, e_4, e_3 \rangle$, which is not possible. Thus, $\overrightarrow{e_4 b_4} \in D_0$. By a similar argument, considering the subgraph induced by $\{b_2, b_3, b_4, e_4\}$, we have $\overrightarrow{e_4 b_3} \in D_0$. Applying



(a) (b)

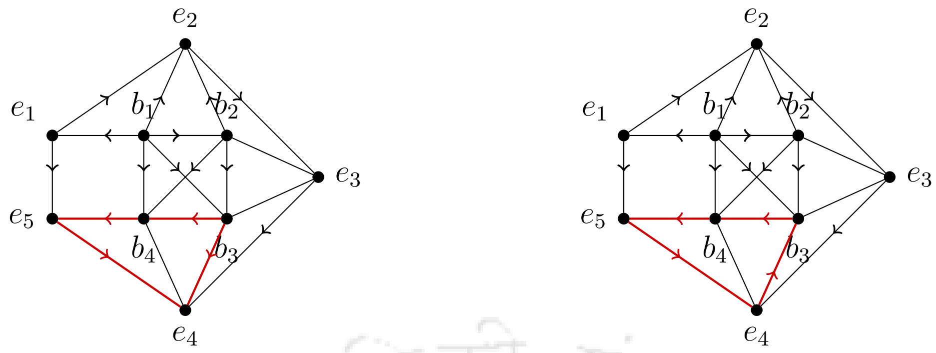
Figure 7.8: (a) $\langle e_4, e_5, e_1, e_2, e_3 \rangle$ and (b) $\langle b_1, e_1, e_5, b_4 \rangle$ form shortcuts

Lemma 7.2.4 to the vertices $\{e_4, b_3, b_4, e_5\}$, $\overrightarrow{e_4e_5}, \overrightarrow{e_5b_4} \in D_0$. Finally, if $\overrightarrow{e_5e_1} \in D_0$ then $\overrightarrow{e_4e_3}$ becomes the shortcutting edge of the directed path $\langle e_4, e_5, e_1, e_2, e_3 \rangle$ in D_0 (see Figure 7.8 (a)), which is not possible. Otherwise, if $\overrightarrow{e_1e_5} \in D_0$ then the edge $\overrightarrow{b_1b_4}$ is the shortcutting edge of the directed path $\langle b_1, e_1, e_5, b_4 \rangle$ (see Figure 7.8 (b)), which is not possible in D_0. Hence, $\overrightarrow{e_3e_4} \in D_0$.

Suppose $\overrightarrow{e_5e_4} \in D_0$. Then $\overrightarrow{e_1e_5} \in D_0$; otherwise if $\overrightarrow{e_5e_1}$ then $\overrightarrow{e_5e_4}$ becomes the shortcutting edge of the directed path $\langle e_5, e_1, e_2, e_3, e_4 \rangle$, which is not possible in D_0. In the subgraph induced by the vertices $\{b_1, b_4, e_1, e_5\}$, it is clear that $\overrightarrow{b_4e_5} \in D_0$, as the directed edge $\overrightarrow{e_5b_4}$ will create a shortcut $\langle b_1, e_1, e_5, b_4 \rangle$. As $\overrightarrow{b_4e_5} \in D_0$, if $\overrightarrow{b_3e_4} \in D_0$ then $\overrightarrow{b_3b_4}$ becomes the shortcutting edge of the directed path $\langle b_3, b_4, e_5, e_4 \rangle$ in D_0 (see Figure 7.9 (a)), which is not possible. If $\overrightarrow{e_4b_3} \in D_0$ then the vertices $\{b_3, b_4, e_5, e_4\}$ form a directed cycle (see Figure 7.9 (b)), which is also not possible in D_0. Since any orientation of the edge $\overline{b_3e_4}$ leads to a shortcut or a directed cycle, we arrive at a contradiction. Thus, $\overrightarrow{e_4e_5} \in D_0$.

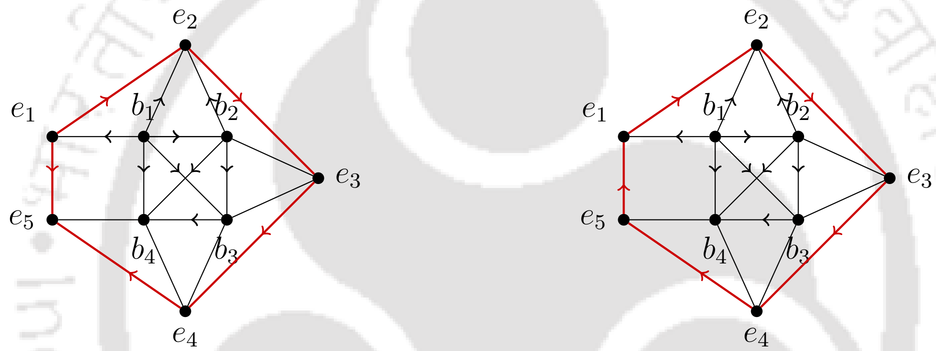
As $\overrightarrow{e_ie_{i+1}} \in D_0$, for all $1 \leqslant i \leqslant 4$. When $\overrightarrow{e_1e_5} \in D_0$ then $\overrightarrow{e_1e_5}$ is the shortcutting edge of the directed path $\langle e_1, e_2, e_3, e_4, e_5 \rangle$ (see Figure 7.10 (a)). When $\overrightarrow{e_5e_1} \in D_0$ then the vertices $\{e_1, e_2, e_3, e_4, e_5\}$ forms a directed cycle in D_0 (see Figure 7.10 (b)).

Thus, D_0 cannot be a semi-transitive orientation and hence $L(W_5')$ is non-word-representable. $\square$



(a) (b)

FIGURE 7.9: (a) $\langle b_3, b_4, e_5, e_4\rangle$ forms a shortcut and (b) $\langle b_3, b_4, e_5, e_4, b_3\rangle$ forms a directed cycle



(a) (b)

FIGURE 7.10: (a) $\langle e_1, e_2, e_3, e_4, e_5\rangle$ forms a shortcut and (b) $\langle e_1, e_2, e_3, e_4, e_5, e_1\rangle$ forms a directed cycle

7.2.2 Mycielski Graphs of Odd Cycles

In this section, we show that the line graph of a Mycielski graph of an odd cycle of length at least five is word-representable.

Definition 7.2.6 (Mycielski graph)**.** Let G be a graph with vertex set $[n] = \{1, 2, \ldots, n\}$. The *Mycielski graph* of G, denoted by $\mu(G)$, is an undirected graph with the vertex set $[n] \cup \{1', 2', \ldots, n'\} \cup \{0\}$ and the edge set $E(G) \cup \{\overline{0i'} \mid i \in [n]\} \cup \{\overline{ji'}, \overline{ij'} \mid \overline{ij} \in E(G)\}$, where $E(G)$ is the edge set of G.

Remark 7.2.7. Note that G is an induced subgraph of $\mu(G)$.

Example 7.2.8. The Mycielski graph of an odd cycle C_{2n+1} is shown in Figure 7.11.

Theorem 7.2.9 ([Kitaev and Pyatkin, 2025])**.** *The graph $\mu(C_{2n+1})$ is non-word-representable for all $n \geqslant 1$.*

Remark 7.2.10. The Mycielski graph $\mu(C_3)$ contains W_5' as a subgraph (see Figure 7.12) and hence $L(\mu(C_3))$ is non-word-representable by Theorem 7.2.5.

Remark 7.2.11. Note that a graph G is always an induced subgraph of its Mycielski graph. Thus, if G is not triangle-free then $L(\mu(G))$ is non-word-representable.

In the following, for $n \geqslant 2$, we prove that the line graph of $\mu(C_{2n+1})$ is word-representable by giving a semi-transitive orientation to $L(\mu(C_{2n+1}))$.

Theorem 7.2.12. *For $n \geqslant 2$, the line graph $L(\mu(C_{2n+1}))$ is word-representable.*

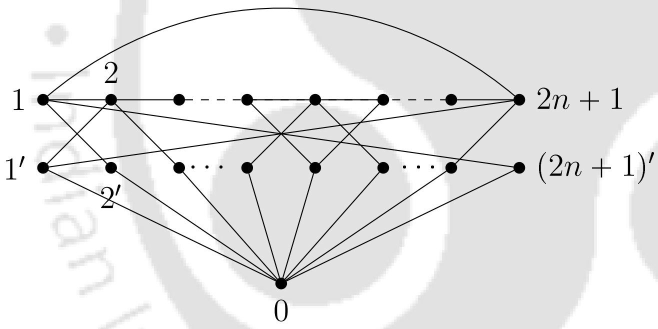


FIGURE 7.11: $\mu(C_{2n+1})$

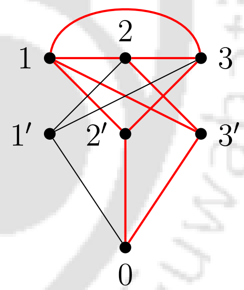


FIGURE 7.12: $\mu(C_3)$

To describe the line graph $L(\mu(C_{2n+1}))$, we give a double-indexed label for each edge of $\mu(C_{2n+1})$. For $1 \leqslant i < j \leqslant 2n+1$, let c_{ij} denote the edge of the cycle C_{2n+1} in $\mu(C_{2n+1})$ with endpoints i and j. For $0 \leqslant i \leqslant 2n+1$ and $1 \leqslant j \leqslant (2n+1)$, let $a_{ij'}$ be the edge of $\mu(C_{2n+1})$ with end points i and j'. Note that two vertices in $L(\mu(C_{2n+1}))$ are adjacent if and only if one of their indices are the same.

We consider the following two subgraphs of $\mu(C_{2n+1})$ which partition its edge set, as per the labeling c_{ij}'s and $a_{ij'}$'s given to the edges. The part with the edges c_{ij}'s is the original graph C_{2n+1} and let B_{2n+1} be the spanning subgraph with the remaining edges $a_{ij'}$'s. Observe that B_{2n+1} is a bipartite graph.

To prove our main result stated in Theorem 7.2.12, we give an orientation D to the edges of $L(\mu(C_{2n+1}))$, for $n \geqslant 2$, and show that D is a semi-transitive orientation. We consider the edge set of $L(\mu(C_{2n+1}))$ in three parts, viz., (i) the edges of $L(B_{2n+1})$, (ii) the edges of $L(C_{2n+1})$, and (iii) the remaining edges between $L(B_{2n+1})$ and $L(C_{2n+1})$, and assign the orientation D as per the following:

I. **Semi-transitive orientation assigned to** $L(B_{2n+1})$: The graph B_{2n+1} is a subgraph of the complete bipartite graph $K_{2n+2,2n+1}$, so that $L(B_{2n+1})$ is an induced subgraph of $L(K_{2n+2,2n+1})$. Note that the line graph of a complete bipartite graph $L(K_{m,n})$ is isomporphic to the Cartesian product of the complete graphs K_m and K_n (cf. de Werra and Hertz [1999]) as shown Figure 7.13. The vertices in each rectangular box form a clique, i.e., for $1 \leqslant i \leqslant m$, $L_i = \{a_{ij'} | \ 1 \leqslant j \leqslant n\}$ and, for $1 \leqslant j \leqslant n$, $L'_j = \{a_{ij'} | \ 1 \leqslant i \leqslant m\}$ form cliques in $L(K_{m,n})$. By Theorem 1.4.58, the line graph $L(K_{m,n})$ is word-representable. By hereditary property of word-representable graphs, $L(B_{2n+1})$ is word-representable and hence admits a semi-transitive orientation. In view of Theorem 1.4.58, consider the following semi-transitive orientation of $L(K_{2n+2,2n+1})$ restricted to the graph $L(B_{2n+1})$.

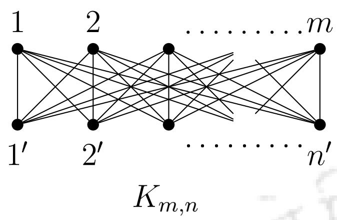
(i) For $0 \leqslant i \leqslant 2n+1$, $\overrightarrow{a_{ij'_1}a_{ij'_2}}$ if $j_1 < j_2$.

(ii) For $1 \leqslant j \leqslant 2n+1$, $\overrightarrow{a_{i_1j'}a_{i_2j'}}$ if $i_1 > i_2$.

II. **Semi-transitive orientation assigned to** $L(C_{2n+1})$: Consider the following orientation on $L(C_{2n+1})$.

(i) For $1 \leqslant i \leqslant 2n-1$, $\overrightarrow{c_{(i+1)(i+2)}c_{i(i+1)}}$.

(ii) $\overrightarrow{c_{1(2n+1)}c_{12}}$ and $\overrightarrow{c_{(2n)(2n+1)}c_{1(2n+1)}}$.



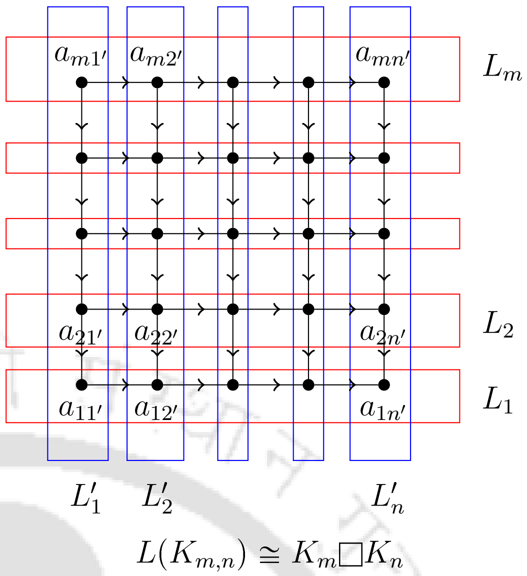


FIGURE 7.13: $K_{m,n}$ and $L(K_{m,n})$ with a semi-transitive orientation

Note that this orientation does not contain any directed cycles and shortcuts and hence a semi-transitive orientation.

III. **Orientation of the edges between** $L(B_{2n+1})$ **and** $L(C_{2n+1})$: Each edge $\overline{ac}$ in $L(\mu(C_{2n+1}))$, where a is a vertex in $L(B_{2n+1})$ and c is a vertex in $L(C_{2n+1})$, orient $\overrightarrow{ac}$ in D.

Note that the edges between $L(B_{2n+1})$ and $L(C_{2n+1})$ in $L(\mu(C_{2n+1}))$ are precisely as per the following: (i) each vertex a of L_1 is adjacent to $c_{1(2n+1)}$ and c_{12}, (ii) each vertex a of L_{2n+1} is adjacent to $c_{1(2n+1)}$ and $c_{(2n)(2n+1)}$, and (iii) for $2 \leqslant i \leqslant 2n$, each vertex a of L_i is adjacent to $c_{i(i+1)}$ and $c_{(i-1)i}$.

Remark 7.2.13. In the subgraph $L(B_{2n+1})$, the vertices in each L_i are as follows: $L_0 = \{a_{01'}, a_{02'}, \ldots, a_{0(2n+1)'}\}$, $L_1 = \{a_{12'}, a_{1(2n+1)'}\}$, $L_{2n+1} = \{a_{(2n+1)1'}, a_{(2n+1)(2n)'}\}$, and, for $2 \leqslant i \leqslant 2n$, $L_i = \{a_{i(i-1)'}, a_{i(i+1)'}\}$.

Further, the vertices in each L'_j are as follows: $L'_1 = \{a_{01'}, a_{21'}, a_{(2n+1)1'}\}$, $L'_{2n+1} = \{a_{0(2n+1)'}, a_{1(2n+1)'}, a_{(2n)(2n+1)'}\}$, and $L'_j = \{a_{0j'}, a_{(j-1)j'}, a_{(j+1)j'}\}$, for $2 \leqslant j \leqslant 2n$.

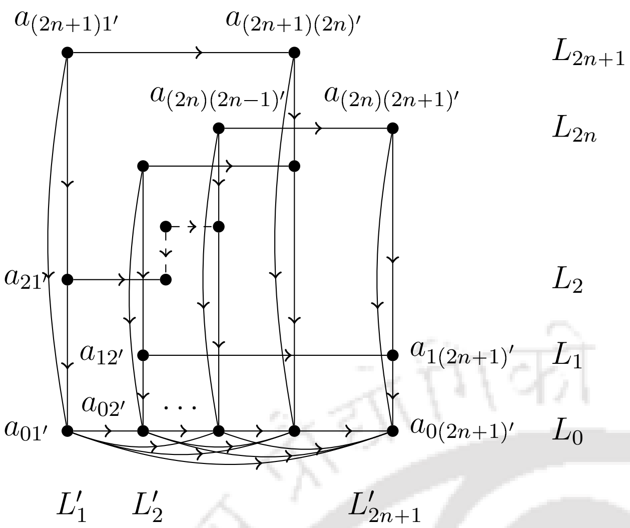


FIGURE 7.14: $L(B_{2n+1})$

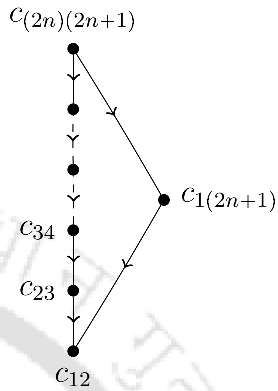


FIGURE 7.15: $L(C_{2n+1})$

Remark 7.2.14. As per the orientation D assigned to the induced subgraph $L(B_{2n+1})$ (see Figure 7.14), we have the following observations:

i) There is no directed path from any vertex of L_i to any vertex of L_j when $j > i$ in $L(B_{2n+1})$.

ii) For all $2 \leqslant i \leqslant 2n+1$, there is no directed path from a vertex of L_i to any vertex of L_{i-1} in $L(B_{2n+1})$.

iii) There is no directed path from any vertex of L'_i to any vertex of L'_j when $j < i$ in $L(B_{2n+1})$.

iv) For all $1 \leqslant i \leqslant 2n$, there is no directed path from a vertex of L'_i to any vertex of L'_{i+1} in $L(B_{2n+1})$.

Remark 7.2.15. As per the orientation D assigned to the induced subgraph $L(C_{2n+1})$ (see Figure 7.15), there is a directed path from the vertex $c_{i(i+1)}$, for $2 \leqslant i \leqslant 2n$, to a vertex $c_{j(j+1)}$ in D if $j < i$. Further, considering the orientation assigned to the edges between $L(B_{2n+1})$ and $L(C_{2n+1})$, there is a directed path from $a \in L_j$ to $c_{i(i+1)}$

if $j \geqslant i$. Note that if $j \in \{1, 2n, 2n+1\}$ then there is a directed path from $a \in L_j$ to $c_{1(2n+1)}$.

Remark 7.2.16. There is no directed edge from a vertex of $L(C_{2n+1})$ to a vertex of $L(B_{2n+1})$ in the orientation D of $L(\mu(C_{2n+1}))$. Thus, there cannot be any directed path from a vertex of $L(C_{2n+1})$ to a vertex of $L(B_{2n+1})$. Consequently, Remark 7.2.14 is valid for the entire graph $L(\mu(C_{2n+1}))$.

Example 7.2.17. The orientation D assigned to $L(\mu(C_5))$ is depicted in Figure 7.16.

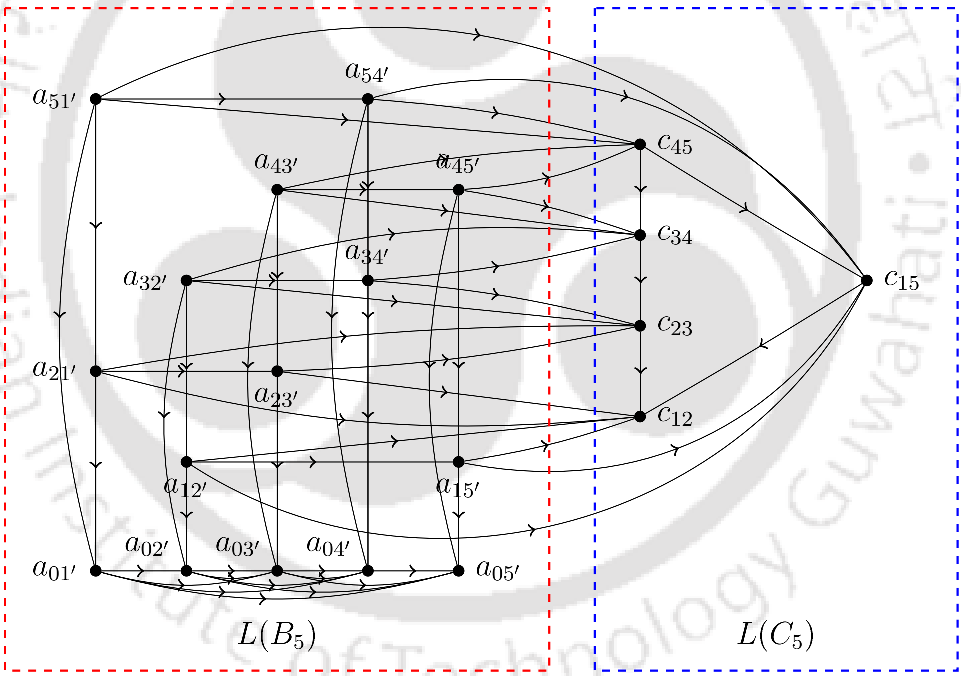


Figure 7.16: $L(\mu(C_5))$ with the semi-transitive oreintation D

To conclude that the orientation D of $L(\mu(C_{2n+1}))$ is semi-transitive, we prove the following two lemmas.

Lemma 7.2.18. *For $n \geqslant 2$, the orientation D of $L(\mu(C_{2n+1}))$ is acyclic.*

Proof. Suppose the orientation D of $L(\mu(C_{2n+1}))$ contains a directed cycle, say $\langle b_1, b_2, \ldots, b_k, b_1\rangle$. Since the orientation D restricted to the induced subgraphs $L(B_{2n+1})$ and $L(C_{2n+1})$ of $L(\mu(C_{2n+1}))$ are semi-transitive, the directed cycle $\langle b_1, b_2, \ldots, b_k, b_1\rangle$ must involve some vertices of $L(B_{2n+1})$ and some vertices of $L(C_{2n+1})$.

Suppose b_1 is a vertex of $L(C_{2n+1})$. Then there exists b_i $(2 \leqslant i \leqslant k)$ such that b_i is a vertex of $L(B_{2n+1})$. This implies that there is a directed path from b_1 to b_i, contradicting Remark 7.2.16. Similarly, if b_1 is in $L(B_{2n+1})$, then b_j is in $L(C_{2n+1})$ for some $2 \leqslant j \leqslant k$. Then there is a directed path from b_j to b_1, contradicting Remark 7.2.16. Hence, the orientation D is acyclic. □

Lemma 7.2.19. *For $n \geqslant 2$, the orientation D of $L(\mu(C_{2n+1}))$ does not contain any shortcut.*

Proof. Suppose that the orientation D contains a shortcut $X = \langle b_1, b_2, \ldots, b_k\rangle$, for $k \geqslant 4$, with the shortcutting edge $\overrightarrow{b_1 b_k}$. Since the orientation D restricted to $L(B_{2n+1})$ and $L(C_{2n+1})$ are semi-transitive, the shortcut X must involve some vertices of $L(B_{2n+1})$ and some vertices of $L(C_{2n+1})$. Further, if b_i is a vertex in $L(C_{2n+1})$, then for any $j > i$, b_j cannot be in $L(B_{2n+1})$; otherwise, there will be a directed path from b_i to b_j, which is a contradiction to Remark 7.2.16. Thus, there exists $2 \leqslant t \leqslant k$ such that $b_1, b_2, \ldots, b_{t-1}$ are vertices of $L(B_{2n+1})$ and $b_t, \ldots, b_k$ are vertices of $L(C_{2n+1})$. Note that b_1 and b_k must be a vertex of $L(B_{2n+1})$ and $L(C_{2n+1})$, respectively. However, we show that b_k is not in $L(C_{2n+1})$ so that none of the vertices of $L(C_{2n+1})$ are in X. Hence, no such shortcut is possible in D, which concludes the proof.

We show that b_k is not in $L(C_{2n+1})$ in the following four cases:

- Case 1: $b_k = c_{12}$. Since b_1 is a vertex of $L(B_{2n+1})$ and adjacent to c_{12}, the vertex b_1 must be among the vertices $a_{12'}, a_{1(2n+1)'}, a_{21'}$ and $a_{23'}$. Also, since b_{k-1} is adjacent to c_{12}, the vertex b_{k-1} must be among the vertices

$c_{1(2n+1)}, c_{23}, a_{12'}, a_{1(2n+1)'}, a_{21'}$ and $a_{23'}$. We handle this in the following four subcases.

- Subcase 1.1: $b_1 = a_{12'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{12'}$ to any of the vertices $a_{21'}, a_{23'}$ and c_{23}. Thus, the only possible shortcut is $\langle a_{12'}, a_{1(2n+1)'}, c_{1(2n+1)}, c_{12}\rangle$. However, these vertices induce a clique (see Figure 7.17) in $L(\mu(C_{2n+1}))$, and hence b_1 cannot be $a_{12'}$.

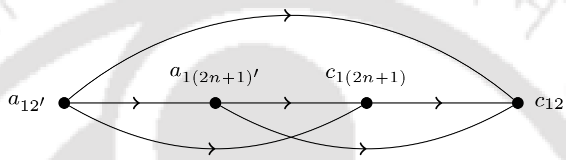


FIGURE 7.17: The subgraph of $L(\mu(C_{2n+1}))$ induced by $a_{12'}, a_{1(2n+1)'}, c_{1(2n+1)}$ and c_{12}

- Subcase 1.2: $b_1 = a_{1(2n+1)'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{1(2n+1)'}$ to any of the vertices $c_{23}, a_{12'}, a_{21'}$ and $a_{23'}$. This implies that b_{k-1} must be $c_{1(2n+1)}$. Clearly, there is no directed path of length at least two from $a_{1(2n+1)'}$ to $c_{1(2n+1)}$. Then the shortcut X is a clique of size three, which is a contradiction to the length of a shortcut. Thus, b_1 cannot be $a_{1(2n+1)'}$.
- Subcase 1.3: $b_1 = a_{21'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{21'}$ to any of the vertices $a_{12'}$, $a_{1(2n+1)'}$ and $c_{1(2n+1)}$. Thus, the only possible shortcut is $\langle a_{21'}, a_{23'}, c_{23}, c_{12}\rangle$. However, these vertices induces a clique in $L(\mu(C_{2n+1}))$ (see Figure 7.18). Hence, b_1 cannot be $a_{21'}$.

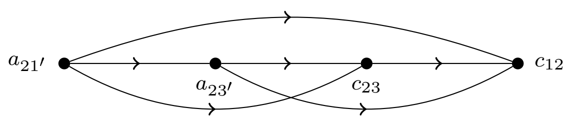


FIGURE 7.18: The subgraph of $L(\mu(C_{2n+1}))$ induced by $a_{21'}, a_{23'}, c_{23}$ and c_{12}

- Subcase 1.4: $b_1 = a_{23'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{23'}$ to any of the vertices $c_{1(2n+1)}, a_{12'}, a_{1(2n+1)'}$ and $a_{21'}$. This implies that b_{k-1} must be c_{23}. Clearly, there is no directed path of length at least two from $a_{23'}$ to c_{23}. Then the shortcut X is a clique of size three, which is a contradiction to the length of a shortcut. Thus, b_1 cannot be $a_{23'}$.

Hence, b_k cannot be c_{12}.

- Case 2: For $2 \leqslant i \leqslant 2n-1$, $b_k = c_{i(i+1)}$. The vertex b_1 must be among the vertices $a_{i(i-1)'}$, $a_{i(i+1)'}$, $a_{(i+1)i'}$ and $a_{(i+1)(i+2)'}$, and the vertex b_{k-1} must be among the vertices $c_{(i+1)(i+2)}$, $a_{i(i-1)'}$, $a_{i(i+1)'}$, $a_{(i+1)i'}$ and $a_{(i+1)(i+2)'}$. We handle this case in the following four subcases:

  - Subcase 2.1: $b_1 = a_{i(i-1)'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{i(i-1)'}$ to any of the vertices $a_{(i+1)i'}$, $a_{(i+1)(i+2)'}$ and $c_{(i+1)(i+2)}$. Thus, b_{k-1} must be $a_{i(i+1)'}$. Clearly, there is no directed path of length at least two from $a_{i(i-1)'}$ to $a_{i(i+1)'}$. Then the shortcut X is of length two, which is a contradiction. Hence, b_1 cannot be $a_{i(i-1)'}$.
  - Subcase 2.2: $b_1 = a_{i(i+1)'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{i(i+1)'}$ to any of the vertices $a_{i(i-1)'}$, $a_{(i+1)i'}$, $a_{(i+1)(i+2)'}$ and $c_{(i+1)(i+2)}$. This implies that there is no directed path from $a_{1(2n+1)'}$ to any possible vertices of b_{k-1}. Thus, b_1 cannot be $a_{i(i+1)'}$.
  - Subcase 2.3: $b_1 = a_{(i+1)i'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{(i+1)i'}$ to the vertices $a_{i(i-1)'}$, $a_{i(i+1)'}$. The only possible shortcut is $\langle a_{(i+1)i'}, a_{(i+1)(i+2)'}, c_{(i+1)(i+2)}, c_{i(i+1)} \rangle$. However, these vertices induce a clique in $L(\mu(C_{2n+1}))$ (see Figure 7.19). Thus, b_1 cannot be $a_{(i+1)i'}$.
  - Subcase 2.4: $b_1 = a_{(i+1)(i+2)'}$. By Remark 7.2.16 and Remark 7.2.15, there is no directed path from $a_{(i+1)(i+2)'}$ to any of the vertices $a_{i(i-1)'}$, $a_{i(i+1)'}$

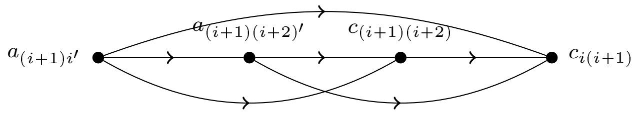


Figure 7.19: The subgraph induced by $a_{(i+1)i'}, a_{(i+1)(i+2)'}, c_{(i+1)(i+2)}$ and $c_{i(i+1)}$

and $a_{(i+1)i'}$. Thus, b_{k-1} must be $c_{(i+1)(i+2)}$. Clearly, there is no directed path of length at least two from $a_{(i+1)(i+2)'}$ to $c_{(i+1)(i+2)}$. Then the shortcut X is a clique of size three, which is a contradiction. Thus, $a_{(i+1)(i+2)'}$ cannot be b_1.

Hence, b_k cannot be $c_{i(i+1)}$, for $1 \leqslant i \leqslant 2n-1$.

- Case 3: $b_k = c_{1(2n+1)}$. The vertex b_1 must be one among the vertices $a_{12'}$, $a_{1(2n+1)'}$, $a_{(2n+1)1'}$ and $a_{(2n+1)(2n)'}$; and the vertex b_{k-1} must be among the vertices $c_{(2n)(2n+1)}$, $a_{12'}$, $a_{1(2n+1)'}$, $a_{(2n+1)1'}$ and $a_{(2n+1)(2n)'}$. If $b_1 \in \{a_{12'}, a_{1(2n+1)'}\}$ then, by Remark 7.2.16 and Remark 7.2.15, there is no directed path from the vertex b_1 to any possible vertices of b_{k-1}. Thus, b_k can neither be $a_{12'}$ nor $a_{1(2n+1)'}$. Further, if $b_1 = a_{(2n+1)1'}$ then the only possible shortcut is $\langle a_{(2n+1)1'}, a_{(2n+1)(2n)'}, c_{(2n)(2n+1)}, c_{1(2n+1)}\rangle$. However, these vertices induce a clique in $L(\mu(C_{2n+1}))$ (see. Figure 7.20). Thus, b_1 cannot be $a_{(2n+1)1'}$.

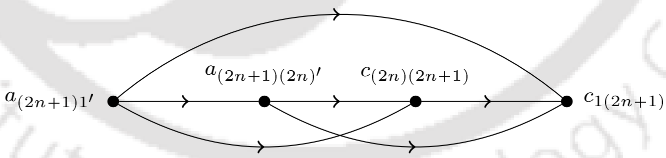


Figure 7.20: The subgraph induced by $a_{(2n+1)1'}, a_{(2n+1)(2n)'}, c_{(2n)(2n+1)}$ and $c_{1(2n+1)}$

If $b_1 = a_{(2n+1)(2n)'}$ then there is no directed path from b_1 to any of the vertices $a_{(2n+1)1'}, a_{12'}$ and $a_{1(2n+1)'}$. Thus, b_{k-1} must be $c_{(2n)(2n+1)}$. Clearly, there is no directed path of length at least two from b_1 to $c_{(2n)(2n+1)}$. Thus, the shortcut X forms a clique of size three, which is a contradiction. Hence, b_1 cannot be $a_{(2n+1)(2n)'}$.

- Case 4: $b_k = c_{(2n)(2n+1)}$. Since $c_{(2n)(2n+1)}$ is a source in the orientation D when restricted to the induced subgraph $L(C_{2n+1})$. The vertex b_{k-1} must be a vertex from $L(B_{2n+1})$. Thus, the vertices of b_1 and b_{k-1} must be among the vertices $a_{(2n)(2n-1)'}$, $a_{(2n)(2n+1)'}$, $a_{(2n+1)1'}$ and $a_{(2n+1)(2n)'}$. By Remark 7.2.16 and Remark 7.2.15, if $b_1 \in \{a_{(2n)(2n-1)'}, a_{(2n+1)1'}\}$ then the shortcut X must be a clique size three, which is a contradiction. If $b_1 \in \{a_{(2n)(2n+1)'}, a_{(2n+1)(2n)'}\}$ then there is no directed path of length at least two from b_1 to $c_{(2n)(2n+1)}$. Thus, the shortcut X forms a clique of size two, which is a contradiction. Hence, b_k cannot be $c_{(2n)(2n+1)}$.

$\square$

Proof of Theorem 7.2.12. In view of Lemma 7.2.18 and Lemma 7.2.19, there exists a semi-transitive orientation for $L(\mu(C_{2n+1}))$. Hence, by Theorem 1.4.14, $L(\mu(C_{2n+1}))$ is word-representable. $\square$

7.3 Future Directions

The class of graphs that we have considered in this work is the Mycielski graphs of odd cycles of length at least five, which are known to be triangle-free and non-word-representable. We showed that their corresponding line graphs are, in fact, word-representable. Further, in view of Corollary 7.1.4 and Remark 7.2.11, we raise the following questions:

1. Is the line graph of a triangle-free graph always word-representable?

2. Does there exist a non-word-representable graph with clique number three whose line graph is word-representable?

The answers to the above-mentioned two questions could potentially lead to a complete characterization of word-representability of line graphs.

8

Dimension vs Representation Number

Hefty et al. [2024] introduced the concept of representation number of a semi-transitive orientation (s-representation number) and established a connection with the posets. We further the relationship between the posets and word-representable graphs. First, we show that the dimension of a poset arising from a semi-transitive orientation is bounded above by a function of its s-representation number. That is, given a semi-transitive orientation with s-representation number t, the dimension of the corresponding poset is at most $3t - 2$. Further, we establish an upper bound on the boxicity of a word-representable graph G in terms of its representation number and the length of longest directed path of a semi-transitive orientation of G. Consequently, we also find an upper bound for the prn of a comparability graph.

8.1 Upper Bound for Poset Dimension

Word-representable graphs are the graphs that admit semi-transitive orientations.

Notation 8.1.1. Let $G = (V, E)$ be a graph and ϕ be a semi-transitive orientation of G. We denote the resulting directed graph as $G_\phi = (V, \phi)$. Note that the edge set is identical to the relation ϕ.

Remark 8.1.2. From the proof of Theorem 1.4.14, one can observe that if w is a word-representant of a graph $G = (V, E)$ then the semi-transitive orientation ϕ of G induced by w is as follows: for $\overline{ab} \in E$, $a^{(1)}$ occurs before $b^{(1)}$ in the word w if and only if $\overrightarrow{ab} \in \phi$. Note that the leftmost vertex in w will be a source in G_ϕ.

In view of Remark 8.1.2, Hefty et al. [2024] defined the k-word-representatbility and the s-representation number of a semi-transitive orientation as per the following.

Definition 8.1.3. A semi-transitive orientation ϕ of a graph G is said to be *k-word-representable* it there exists a k-uniform word w which induces the same orientation. We say that the word w represents the semi-transitive orientation ϕ.

Definition 8.1.4. The *s-representation number* of a semi-transitive orientation ϕ is the minimal k such that ϕ is k-word-representable. It is denoted by $\mathcal{R}^s(\phi)$.

Remark 8.1.5. The representation number of a word-representable graph is the minimum s-representation number among all semi-transitive orientations of G, i.e., $\mathcal{R}(G) = \min\{\mathcal{R}^s(\phi) \mid \phi \text{ is a semi-transitive orientation of } G\}$.

Notation 8.1.6. Let G be a word-representable graph with $\mathcal{R}(G) = k$ and $S(G) = \{\phi \mid \phi \textit{ is a semi-transitive orientation of } G \text{ and } \mathcal{R}^s(\phi) = k\}$. We define $\ell_G = \min\limits_{\phi \in S(G)} \{l(G_\phi) + 1\}$, where $l(G_\phi)$ is the length of a longest directed path in G_ϕ.

A graph $G = (V, E)$ with a semi-transitive orientation G_ϕ gives rise to a partial order $\prec$ on V defined by, for $a, b \in V$, $a \prec b$ if there is a directed path from a to b in

G_ϕ. We say that $P_{\widehat{G_\phi}}$ is the associated poset of G_ϕ and $\widehat{G_\phi} = (V, \widehat{\phi})$ is the underlying comparability graph of the poset $P_{\widehat{G_\phi}}$. The transitive orientation induced by the graph G_ϕ is denoted by $\widehat{\phi}$.

Remark 8.1.7. If ϕ is a transitive orientation of G then $G_\phi = \widehat{G_\phi}$. However, if ϕ is semi-transitive but not transitive then $G_\phi \neq \widehat{G_\phi}$.

In the following, we establish a relationship between s-representation number of ϕ and the dimension of the poset associated with G_ϕ.

Proposition 8.1.8. *Let w be a k-uniform word-representant of G that induces the semi-transitive orientation ϕ . If $\langle a, a_1, a_2, \ldots, a_t, b\rangle$ is a directed path from a to b in G_ϕ then there exists no $j \in \{1, 2, \ldots, k\}$ such that $b^{(j)}a^{(j)} \ll w$.*

Proof. We proof by induction on the length of the directed path $\langle a, a_1, a_2, \ldots, a_t, b\rangle$.

- *Base case*: Suppose the directed path is of length two, i.e., $\langle a, a_1, b\rangle$. Assume that there exists $j \in \{1, 2, \ldots, k\}$ such that $b^{(j)}a^{(j)} \ll w$. Since $\overrightarrow{a_1b} \in \phi$, $a_1^{(t)}$ must appear before $b^{(t)}$, for all $t \in \{1, 2, \ldots, k\}$. Thus, $a_1^{(j)}b^{(j)} \ll a_1^{(j)}b^{(j)}a^{(j)} \ll w$. This is a contradiction as $\overrightarrow{aa_1} \in E(G_\phi)$ and $a^{(t)}$ must appear before $a_1^{(t)}$, for all $t \in \{1, 2, \ldots, k\}$.

- *Induction hypothesis*: Assume that for the directed path $\langle a, a_1, a_2, \ldots, a_{t-1}, b\rangle$ of length t, there exists no $j \in \{1, 2, \ldots, k\}$ such that $b^{(j)}a^{(j)} \ll w$.

- *Inductive step*: Consider the directed path $\langle a, a_1, a_2, \ldots, a_t, b\rangle$ of length $t + 1$. Applying induction hypothesis on the directed path $\langle a, a_1, \ldots, a_t\rangle$, we have $a^{(i)}a_t^{(i)} \ll w$, for all $i \in \{1, 2, \ldots, k\}$. Since $\overrightarrow{a_tb} \in \phi$, we have $a_t^{(i)}b^{(i)} \ll w$, for all $i \in \{1, 2, \ldots, k\}$. Thus, $a^{(i)}b^{(i)} \ll w$, for all $i \in \{1, 2, \ldots, k\}$.

Hence, there exists no $j \in \{1, 2, \ldots, k\}$ such that $b^{(j)}a^{(j)} \ll w$. □

Theorem 8.1.9. *Let ϕ be a semi-transitive orientation of a word-representable graph $G = (V, E)$. Then* $\dim(P_{\widehat{G_\phi}}) \leqslant 3\mathcal{R}^s(\phi) - 2$.

Proof. Let $\mathcal{R}^s(\phi) = k$ and let w be a k-uniform word which represents ϕ. By Lemma 1.4.11, the word $w^{(\{1,2\})}w^{(\{2,3\})} \cdots w^{(\{k-1,k\})}$ also represents ϕ, i.e., for any two vertices $a, b \in V$,

i) if a and b are adjacent in G then $a^{(i)}b^{(i)}a^{(i+1)}b^{(i+1)} \ll w^{(\{i,i+1\})}$ for $1 \leqslant i \leqslant k-1$, or $b^{(i)}a^{(i)}b^{(i+1)}a^{(i+1)} \ll w^{(\{i,i+1\})}$ for $1 \leqslant i \leqslant k-1$.

ii) if a and b are not adjacent in G then there exists $j \in \{1, 2, \ldots, k-1\}$ such that one of the following holds:

- $a^{(j)}a^{(j+1)}b^{(j)}b^{(j+1)} \ll w^{(\{j,j+1\})}$
- $b^{(j)}b^{(j+1)}a^{(j)}a^{(j+1)} \ll w^{(\{j,j+1\})}$
- $a^{(j)}b^{(j)}b^{(j+1)}a^{(j+1)} \ll w^{(\{j,j+1\})}$
- $b^{(j)}a^{(j)}a^{(j+1)}b^{(j+1)} \ll w^{(\{j,j+1\})}$

For each $w^{(\{i,i+1\})}$, we construct an interval graph I_i, in which, for $a \in V$ we associate the interval $[(a, w^{(\{i,i+1\})})^1, (a, w^{(\{i,i+1\})})^2]$, where $(a, w^{(\{i,i+1\})})^1$ is the position of the first occurrence of a in $w^{(\{i,i+1\})}$ and $(a, w^{(\{i,i+1\})})^2$ is the position of the second occurrence of a in $w^{(\{i,i+1\})}$.

Note that each $w^{(\{i,i+1\})}$ is a 2-uniform word. Let $a, b \in V$. If $abab \ll w^{(\{i,i+1\})}$ or $baba \ll w^{(\{i,i+1\})}$ or $abba \ll w^{(\{i,i+1\})}$ or $baab \ll w^{(\{i,i+1\})}$ then the intervals assigned to a and b intersect. Thus, a and b are adjacent in I_i. If $aabb \ll w^{(\{i,i+1\})}$ or $bbaa \ll w^{(\{i,i+1\})}$ then the intervals assigned to a and b do not intersect, and hence a and b are non-adjacent in I_i.

Let f_i denote the interval representation of I_i constructed with respect to the word $w^{(\{i,i+1\})}$. We now construct two linear orders L_i^1 and L_i^2 corresponding to I_i.

In order to get the two linear orders, we define two sets C_i and D_i as follows: for every non-adjacent pair of vertices a, b in $\widehat{G_\phi}$ with respect to I_i,

$$\left.\begin{array}{l}\overrightarrow{ab} \in C_i \\ \overrightarrow{ba} \in D_i\end{array}\right\} \text{ if and only if } (a, w^{(\{i,i+1\})})^2 < (b, w^{(\{i,i+1\})})^1.$$

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, where $E_1 = \widehat{\phi} \cup C_i$ and $E_2 = \widehat{\phi} \cup D_i$.

Claim 2. G_1 and G_2 are acyclic digraphs.

We prove the claim for G_1. If $\widehat{G_\phi}$ is complete graph then $C_i = \varnothing$. Thus, G_1 is acyclic as $\widehat{G_\phi}$ is transitively oriented. Suppose $\widehat{G_\phi}$ is not a complete graph. Then $C_i \neq \varnothing$. Assume that G_1 is not acyclic. Let $D = \langle a_1, a_2, \ldots, a_t, a_1\rangle$ be a shortest directed cycle in G_1.

First, we show that $t > 2$. If $t = 2$ then there exists $a, b \in V$ such that $\overrightarrow{ab}, \overrightarrow{ba} \in E_1$. As $\widehat{G_\phi}$ is a simple graph, both the directed edges cannot be in $\widehat{\phi}$. By construction of the set C_i, both the directed edges also cannot be in C_i. Without loss of generality, assume that $\overrightarrow{ab} \in \widehat{\phi}$ and $\overrightarrow{ba} \in C_i$. But if $\overrightarrow{ab} \in \widehat{\phi}$ then a and b must be adjacent in $\widehat{G_\phi}$ and thus $\overrightarrow{ba} \notin C_i$, which is a contradiction.

Now, we show that two consecutive edges of D cannot be to $\widehat{\phi}$ or C_i. Suppose there exists two consecutive edges $\overrightarrow{a_ja_{j+1}}$ and $\overrightarrow{a_{j+1}a_{j+2}}$ in D such that $\overrightarrow{a_ja_{j+1}}, \overrightarrow{a_{j+1}a_{j+2}} \in \widehat{\phi}$. Since $\widehat{\phi}$ is a transitive orientation, we have $\overrightarrow{a_ja_{j+2}} \in \widehat{\phi}$. Thus, we get a directed cycle of length $t-1$, which is a contradiction to our assumption that D is the shortest cycle in G_1.

Suppose there exists two consecutive edges $\overrightarrow{a_ja_{j+1}}$ and $\overrightarrow{a_{j+1}a_{j+2}}$ in D such that $\overrightarrow{a_ja_{j+1}}, \overrightarrow{a_{j+1}a_{j+2}} \in C_i$. As per the interval representation f_i, $(a_j, w^{(\{i,i+1\})})^2 < (a_{j+1}, w^{(\{i,i+1\})})^1$ and $(a_{j+1}, w^{(\{i,i+1\})})^2 < (a_{j+2}, w^{(\{i,i+1\})})^1$. This means that $(a_j, w^{(\{i,i+1\})})^2 < (a_{j+2}, w^{(\{i,i+1\})})^1$ and hence a_j and a_{j+2} are non-adjacent in G_ϕ. If a_j and a_{j+2} are adjacent in $\widehat{G_\phi}$ either there is a directed path from a_j and a_{j+2} or there is a directed path from a_{j+2} and a_{j+1} in G_ϕ. Applying Proposition 8.1.8, we see that there is a directed path from a_j and a_{j+2}. Hence, $\overrightarrow{a_ja_{j+2}} \in \widehat{\phi}$, which forms a directed cycle of length $t-1$, contradicting our assumption of D. If a_j and a_{j+2} are non-adjacent in $\widehat{G_\phi}$ then clearly $\overrightarrow{a_j a_{j+2}} \in C_i$, which also forms a directed cycle of length $t-1$, which is a contradiction. Thus, the edges of D alternate between $\widehat{\phi}$ and C_i. It also follows that $t \geqslant 4$ is even.

Without loss of generality, assume that $\overrightarrow{a_1a_2}, \overrightarrow{a_3a_4}, \ldots \overrightarrow{a_{t-1}, a_t} \in C_i$. Then we have, $(a_j, w^{(\{i,i+1\})})^2 < (a_{j+1}, w^{(\{i,i+1\})})^1$ for each odd j and $1 \leqslant j \leqslant t-1$. Also, $\overrightarrow{a_2a_3}, \overrightarrow{a_4a_5}, \ldots \overrightarrow{a_{t-2}, a_{t-1}}, \overrightarrow{a_t, a_1} \in \widehat{\phi}$. Note that if $\overrightarrow{ab} \in \widehat{\phi}$ then there is a directed path a to b in G_ϕ. By Proposition 8.1.8, $(a, w^{(\{i,i+1\})})^1 < (b, w^{(\{i,i+1\})})^1$. Thus, for all odd j and $l > j+1$, we have $(a_j, w^{(\{i,i+1\})})^2 < (a_l, w^{(\{i,i+1\})})^1$ so that a_j and a_l are non-adjacent in G_ϕ. But if a_j and a_l are adjacent in $\widehat{\phi}$ then we get a directed cycle of length at most $t-1$, which is a contradiction to our assumption. Also, if $\overrightarrow{a_j a_l} \in C_i$ then, in this case also, we get a directed cycle of length at most $t-1$, which is also a contradiction.

Thus, we proved that there cannot be any directed cycles in G_1. In a similar way, we can show that G_2 is an acyclic directed graph.

As G_1 and G_2 are acyclic, we can construct the linear orders L_i^1 and L_i^2 by using topological sorting on G_1 and G_2, respectively.

Now we consider the set $\{w^{(\{i\})} \mid 1 \leqslant i \leqslant k\}$. Note that if a and b are non-adjacent in G and $a^{(i)}b^{(i)}b^{(i+1)}a^{(i+1)} \ll w^{(\{i,i+1\})}$ then $ab \ll w^{(\{i\})}$ and $ba \ll w^{(\{i+1\})}$. Thus corresponding to each word $w^{(\{i\})}$, we construct a linear order L_i' such that $ab \ll w^{(\{i\})}$ if and only if $aL_i'b$.

Thus, the set $\{L_i^1, L_i^2 \mid 1 \leqslant i \leqslant k-1\} \cup \{L_i' | 1 \leqslant i \leqslant k\}$ is a realizer of the poset $P_{\widehat{G_\phi}}$. Hence, $\dim(P_{\widehat{G_\phi}}) \leqslant 2(k-1) + k = 3k-2$ so that $\dim(P_{\widehat{G_\phi}}) \leqslant 3\mathcal{R}^s(\phi) - 2$. □

According to Theorem 1.4.30, circle graphs are the graphs whose representation number is at most two. The following corollary is an immediate consequence of Theorem 8.1.9.

Corollary 8.1.10. *If G is a circle graph and ϕ is a semi-transitive orientation that yields a 2-uniform word-representant of G then* $\dim(P_{\widehat{G_\phi}}) \leqslant 4$.

In view of Remark 1.4.18 and Theorem 8.1.9, we have obtained a relation between the s-representation number of a hasse diagram of a poset and its dimension.

Corollary 8.1.11. *Supppose H is the Hasse diagram of a poset P. Then* $\dim(P) \leqslant 3\mathcal{R}^s(H) - 2$.

8.2 Boxicity

Definition 8.2.1 (k-box)**.** A *k-box* is the Cartesian product of closed intervals on the real line, i.e., $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$.

Definition 8.2.2. A *k-box representation* of a graph G is a mapping of each vertex of G to a k-box in the k-dimensional Euclidean space such that two vertices in G are adjacent if and only if their corresponding k-boxes intersect. Note that if $k = 1$ then G is an interval graph as each 1-box is a closed interval. Figure 8.1 is an example of a 2-box-representation of a graph.

Definition 8.2.3 ([Roberts, 1969])**.** The *boxicity* of a graph, denoted by $\mathrm{box}(G)$, is the minimum integer k such that G has a k-box representation.

By definition, the boxicity of a complete graph is zero, as there are no non-adjacent vertex pairs to represent geometrically. Furthermore, it is easy to see that a graph has boxicity one if and only if it is an interval graph. Cozzens [1981] established that computing the boxicity of an arbitrary graph is NP-hard. Subsequently, Yannakakis [1982] showed that deciding whether the boxicity of a graph is at most three is NP-complete. Finally, Kratochvíl [1994] strengthened this result by proving that determining whether a graph has boxicity at most two is also NP-complete.

Let $G = (V, E)$ be any graph and $G_i = (V, E_i)$, $1 \leqslant i \leqslant t$ such that $E = E_1 \cap E_2 \cap \cdots \cap E_t$. We say that G is the intersection graph of G_i's and denoted as $G = \bigcap_{i=1}^{t} G_i$. Boxicity of a graph can be defined in terms of intersection of graphs as follows:

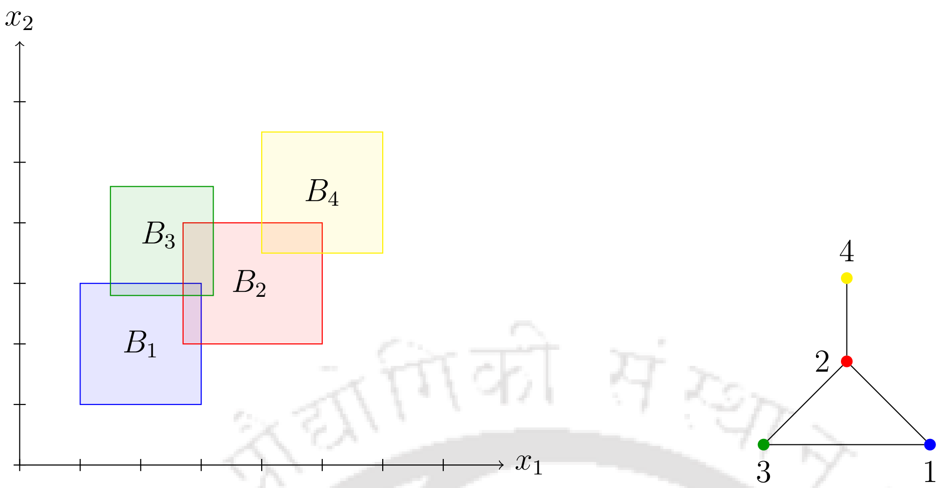


FIGURE 8.1: A 2-box representation of a graph

Lemma 8.2.4 ([Roberts, 1969]). *The boxicity of a non complete graph G is the minimum positive integer t such that G can be represented as the intersection of t interval graphs. Moreover, if $G = \bigcap_{i=1}^{n} G_i$ for some G_i, then* $\mathrm{box}(G) \leqslant \sum_{i=1}^{n} \mathrm{box}(G_i)$.

For comparability graphs, Adiga et al. [2011] established a connection between dimension of a poset and the boxicity of its underlying comparability graphs, as stated in the following theorems.

Theorem 8.2.5 ([Adiga et al., 2011]). *Let P be a poset such that* $\dim(P) > 1$. *Then* $\dim(P) \leqslant 2\mathrm{box}(G_P)$.

Theorem 8.2.6 ([Adiga et al., 2011]). *Let P be a poset, and* $\chi(G_P) > 1$. *Then* $\mathrm{box}(G_P) \leqslant (\chi(G_P) - 1)\dim(P)$.

By Theorem 8.1.9 and Theorem 8.2.6, the following theorem gives an upper bound on the boxicty of a comparability graph in terms of s-representation number and length of a longest directed path of a semi-transitive orientation associated with the comparability graph.

Theorem 8.2.7. *Let ϕ be a semi-transitive orientation of a word-representable graph G. Then*

$$\text{box}(\widehat{G_\phi}) \leqslant (3\mathcal{R}^s(\phi) - 2)l(G_\phi).$$

Proof. From Theorem 8.1.9, we have $\dim(P_{\widehat{G_\phi}}) \leqslant 3\mathcal{R}^s(\phi) - 2$. Applying Theorem 8.2.6, $\dfrac{\text{box}(\widehat{G_\phi})}{\chi(\widehat{G_\phi}) - 1} \leqslant \dim(P_{\widehat{G_\phi}})$. Since comparability graphs are perfect, we have $\chi(\widehat{G_\phi}) = \omega(\widehat{G_\phi})$. By Proposition 8.1.8, we have $\omega(\widehat{G_\phi}) = l(G_\phi) + 1$ so that

$$\text{box}(\widehat{G_\phi}) \leqslant (3\mathcal{R}^s(\phi) - 2)l(G_\phi).$$

□

As a consequence of Theorem 8.2.7, we obtained an upper bound on the boxicity of the underlying comparability graphs of a poset in terms of the s-representation number of its Hasse diagram and height. It is worth noting that the height of a poset is equal to the clique number of its underlying comparability graph.

Corollary 8.2.8. *Let H be the Hasse diagram of a poset P. Then*

$$\text{box}(G_P) \leqslant (3\mathcal{R}^s(H) - 2)h(P).$$

As word-representable graphs are a generalization of comparability graphs, in the following theorem we provide an upper bound on the boxicty of a word-representable graph G in terms of its representation number and length of a longest directed path in a semi-transitive orientation of G.

Theorem 8.2.9. *Let $G = (V, E)$ be a word-representable graph with $\mathcal{R}(G) = k$. Then,*

$$\text{box}(G) \leqslant \ell_G k - 1.$$

Proof. Let w be a k-uniform word-representant for G and ϕ be a semi-transitive orientation induced by w such that $l(G_\phi) = \ell_G$ - 1. For $1 \leqslant i \leqslant k$, let $w^{(i)}$ be the subword consisting of ith occurrence of each vertex of G in w. The word $u = w^{(\{1\})}w^{(\{2\})}\cdots w^{(\{k\})}$, being concatenation of permutations on V, represents a comparability graph, say G_w, over the vertex set V. For all $a, b \in V$, note the following:

1. If $\overrightarrow{ab} \in \phi$ then $a^{(i)}b^{(i)} \ll w$ for all $1 \leqslant i \leqslant k$. Thus, $a^{(i)}b^{(i)} \ll w^{(i)}$ for all $1 \leqslant i \leqslant k$, and hence a and b are adjacent in G_w.

2. If $\overline{ab} \notin E$ and $a^{(j)}b^{(j)}b^{(j+1)}a^{(j+1)} \ll w$, for some $1 \leqslant j < k$, then $a^{(j)}b^{(j)} \ll w^{(j)}$ and $b^{(j+1)}a^{(j+1)} \ll w^{(j+1)}$ so that a and b are not adjacent in G_w.

3. If $\overline{ab} \notin E$ and $b^{(j)}a^{(j)}a^{(j+1)}b^{(j+1)} \ll w$, for some $1 \leqslant j < k$, then $b^{(j)}a^{(j)} \ll w^{(j)}$ and $a^{(j+1)}b^{(j+1)} \ll w^{(j+1)}$ so that a and b are not adjacent in G_w.

However, when $\overline{ab} \notin E$, having $a^{(j)}a^{(j+1)}b^{(j)}b^{(j+1)} \ll w$ or $b^{(j)}b^{(j+1)}a^{(j)}a^{(j+1)} \ll w$, for some j, does not guarantee the non-adjacency of a and b in G_w. In order to tackle these cases, we construct $k-1$ interval graphs as per the following.

For $1 \leqslant i \leqslant k-1$, let $w^{(i,i+1)}$ be the 2-uniform subword of w consisting of the ith and $(i+1)$th occurrence of each vertex of G. For each word $w^{(\{i,i+1\})}$, we construct an interval graph I_i over the vertex set V, say $\{a_1, a_2, \ldots, a_n\}$, by assigning the interval

$$[(a_i, w^{(\{i,i+1\})})^1, (a_i, w^{(\{i,i+1\})})^2]$$

to each vertex a_i, for $1 \leqslant i \leqslant n$. Note that $(a_i, w^{(\{i,i+1\})})^1$ is the position of 1st occurrence of a_i in $w^{(\{i,i+1\})}$ and $(a_i, w^{(\{i,i+1\})})^2$ is the position of 2nd occurrence of a_i in $w^{(\{i,i+1\})}$.

Now, if $\overline{ab} \notin E(G)$ and $a^{(j)}a^{(j+1)}b^{(j)}b^{(j+1)} \ll w$ or $b^{(j)}b^{(j+1)}a^{(j)}a^{(j+1)} \ll w$, for some j, then in the interval graph I_j constructed based on $w^{(j,j+1)}$, the corresponding intervals of a and b do not intersect. Thus, a and b are not adjacent in I_j. Hence, the graph G can be written as the intersection of the graphs given by

$$G = G_w \cap I_1 \cap I_2 \cap \cdots \cap I_{k-1}.$$

Thus, by Lemma 8.2.4, we have $\mathrm{box}(G) \leqslant \mathrm{box}(G_w) + k - 1$.

Since G_w is a comparability graph, in view of Theorem 8.2.6, we have $\mathrm{box}(G_w) \leqslant (\chi(G_w) - 1)\dim(P_{G_w})$. However, since comparability graphs are perfect, we have $\chi(G_w) = \omega(G_w)$. Also, note that $\dim(P_{G_w}) = \mathcal{R}^p(G_w) \leqslant k$. Hence,

$$\begin{aligned}
\mathrm{box}(G) &\leqslant \mathrm{box}(G_w) + k - 1 \\
&\leqslant (\omega(G_w) - 1)k + k - 1 \\
&= \omega(G_w)k - 1 \\
&= \ell_G k - 1,
\end{aligned}$$

as $\omega(G_w)$ is equal to number of vertices in a longest directed path in G_ϕ, by Proposition 8.1.8. □

Corollary 8.2.10. *For a circle graph G,* $\mathrm{box}(G) \leqslant 2\ell_G - 1$.

From Corollary 1.4.44, the *prn* of a comparability graph and dimension of a poset associated to it have the same value. In view of Theorem 8.2.5 and Theorem 8.2.9, we have the following corollary.

Corollary 8.2.11. *Let G be a comparability graph with $\mathcal{R}(G) = k$. Then*

$$k \leqslant \mathcal{R}^p(G) \leqslant 2(\ell_G k - 1).$$

Note that the class of complete graphs characterizes the class of graphs with (permutation)-representation number one. In the following, we provide a bound on the *prn* of comparability graphs with representation number two and show that this bound is tight.

8.3 Future Directions

An interesting direction for future research is to investigate lower bounds on the dimension of a poset in terms of the s-representation number of its Hasse diagram. Specifically, given a s-representation number t of semi-transitive orientation that induces a poset P, is there a function $g(t)$ such that $g(t) \leqslant \dim(P)$? Establishing such a bound would provide a new perspective and help in characterizing the dimension of a poset more precisely in terms of the s-representation number of its Hasse diagram. Similarly, it would be worthwhile to examine whether lower bounds exist for the boxicity of word-representable graphs in terms of representation numbers.

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