Spherical mean on Métivier groups and uniqueness results for quaternion Weyl transform

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DECLARATION

I do hereby declare that this thesis entitled **“Spherical mean on Métivier groups and uniqueness results for quaternion Weyl transform”** is a presentation of my original research work done under the supervision of **Dr. Rajesh Kumar Srivastava**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of doctor of philosophy. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

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CERTIFICATE

This is certified that the work contained in the thesis entitled **“Spherical mean on Métivier groups and uniqueness results for quaternion Weyl transform”** by **Mr. Rupak Kumar Dalai** (Roll No. 156123020) has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

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Abstract

Let $Z_{r,R}$ be the space of all continuous functions on the annulus $B_{r,R}$ in $\mathbb{C}^n$ whose λ-twisted spherical mean, in the setup of the Métivier groups, vanishes over the spheres $S_s(z) \subset B_{r,R}$ with ball $B_r(0) \subseteq B_s(z)$. We characterize the spherical harmonic coefficients of functions in $Z_{r,R}$, eventually, in terms of polynomial growth, by which we infer a support theorem. Additionally, we prove that non-harmonic complex cones are sets of injectivity for the λ-twisted spherical mean.

We notice that there are some results, which could not be settled for the Métivier groups, hold for H-type groups, such as sufficient condition for a function to be in $Z_{r,R}$ and the support theorem for continuous functions. Moreover, we derive an analogue of the Heche-Bochner identity. And we show that the boundary of bounded domains are sets of injectivity for λ-twisted spherical mean in the H-type groups.

Further, we study the injectivity of the spherical mean for continuous functions on the Métivier groups. The spherical mean is injective for $f(z,.) \in L^p(\mathbb{R}^m),\ 1 \leq p \leq 2$, with tempered growth in the z-variable. This result is also true for a class of functions in $L^p(\mathbb{C}^n)$, $1 \leq p \leq \infty$, without tempered growth. Then, we obtain a two-radii theorem for functions, which are tempered in the z-variable and periodic in the centre variable.

Afterwards, we study the boundedness and several properties of the quaternion Wigner transform. Using the quaternion Wigner transform as a tool, we define the quaternion Weyl transform (QWT) and prove that the QWT is compact for a certain class of symbols in $L^r(\mathbb{R}^4, \mathbb{Q})$ with $1 \leq r \leq 2$. Moreover, it cannot be extended as a bounded operator for symbols in $L^r(\mathbb{R}^4, \mathbb{Q})$ for $2 < r < \infty$. To show this, we give an example of a square-integrable and compactly supported quaternion valued function having infinite $L^{r'}$-norm, where r' is the conjugate index of r.

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In addition, we prove a rank analogue of the Benedicks-Amrein-Berthier theorem for the QWT. In other words, if an integrable function is supported on a product of finite measure sets in $\mathbb{R}^2$ and has finite rank Weyl transform, then it must be trivial.

The quaternion Fourier transform motivated us to define an analogue of twisted spherical mean in the quaternion setup. However, we later realized that further developments proceed likely with twisted spherical mean on the Heisenberg group. In the end, we concluded with some remarks on injectivity and Helgason's support theorem for the quaternion twisted spherical mean.

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Chapter 1

Introduction

Let f be a function on $\mathbb{R}^n$. The spherical mean of f is defined as the convolution

$$f * \mu_\iota(x) = \int_{|y|=\iota} f(x-y)d\mu_\iota(y), \tag{1.0.1}$$

where μ_ι is the normalized surface measure on the sphere $S_\iota^{n-1} = \{x \in \mathbb{R}^n : |x| = \iota\}$. This is nothing but the average of f over the sphere of radius ι centred at x. The above transform (1.0.1) has been intensively studied in [2, 7, 17, 19, 23, 34, 36], due to its applications to PDEs, approximation theory, mathematical physics, medical imaging, etc, see [3–5, 20, 21, 38, 48]. Here we mention some problems related to the spherical mean $f * \mu_\iota$, which have received significant interest.

- *Support theorem.* For a continuous function f on $\mathbb{R}^n$, under which conditions, f is supported in the ball $B_r(0)$ if and only if $f * \mu_\iota(x) = 0$ for $\iota > |x| + r$ and for all $x \in \mathbb{R}^n$. For details, see [34].
- *Sets of injectivity.* Finding the set $S \subseteq \mathbb{R}^n$ for $\mathcal{C} \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$ such that $f \in \mathcal{C}$ and $f * \mu_\iota(x) = 0$ for all $\iota \geq 0$ and $x \in S$, implies $f = 0$, see [7].
- *Injectivity of the spherical mean operator.* When does the operator f into $f * \mu_\iota$ for a fixed $\iota > 0$, turn out to be injective [67].

- *Inversion formula.* Finding a closed form inversion formula for $\mathcal{R}_S(f)$, where $\mathcal{R}_S(f)$ is the restriction of $f * \mu_\iota$ on some set $S \subseteq \mathbb{R}^n$ and $\iota \geq 0$. This problem has been solved when S is a sphere, cylinder, hyperplane, or some other special hypersurface, we refer to [20, 21, 32, 39, 47, 49].

- *Range description.* To characterize the range of $\mathcal{R}_S(f)$ for some set $S \subseteq \mathbb{R}^n$. For the case, when S is a sphere, this characterization has been determined, see [3–5, 48].

Considerable efforts have been made by many authors to generalize the above mentioned investigations on various spaces such as Euclidean space, Heisenberg group and spaces of constant curvature. The aim of this thesis is to look into the above listed first three problems on the Métivier groups.

In a remarkable result, Helgason proved a support theorem for continuous functions having polynomial growth whose spherical mean vanishes over the spheres surrounding a ball. In other words, let μ_ι be the normalized surface measure on the sphere S_ι^{n-1}. If f is a continuous function on $\mathbb{R}^n, (n \geq 2)$, such that $|x|^k f(x)$ is bounded for each non-negative integer k, then f is supported in the ball $B_r(0)$ if and only if $f * \mu_\iota(x) = 0$ for $\iota > |x| + r$ and for all $x \in \mathbb{R}^n$, (see [34]).

Later in [19], Epstein and Kleiner significantly generalized the Helgason's support theorem by characterizing the space of all continuous functions on $\mathbb{R}^n$, whose spherical mean vanishes over all the spheres surrounding a ball, in terms of spherical harmonic coefficients having polynomial growth. This result was first proved by Globevnik [29] in the plane.

Let H_k be the restriction of the space of homogeneous harmonic polynomials of degree k to the unit sphere S^{n-1}, and $\{Y_k^l : l = 1, \ldots, d_k\}$ is an orthonormal basis for

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H_k. Then any $f \in C(\mathbb{R}^n)$ can be expressed as

$$f(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} a_{kl}(\rho)\, Y_k^l(\omega),$$

where $x = \rho\omega$ and $\rho = |x|$. In [19], authors had shown that $f * \mu_\iota(x) = 0$ for all $x \in \mathbb{R}^n$ and $\iota > |x| + B$ as long as $a_{kl} \in \operatorname{span}\{\rho^{k-n-2i} : i = 0, 1, \ldots, k-1\}$, whenever $\rho > B$.

Consequently, the support theorem is an immediate corollary of the above result [19]. For other related work, we refer to [12, 17, 53, 71, 72].

Further, in the article [46], Thangavelu and Narayanan proved an analogue of the Helgason's support theorem for the twisted spherical mean (TSM) for certain Schwartz class functions on $\mathbb{C}^n$. In [54], the authors have characterized the space of all continuous functions on $\mathbb{C}^n$ having TSM vanishes over the spheres surrounding a ball, and proved an exact analogue of the Helgason's support theorem for the twisted spherical mean on $\mathbb{C}^n$ $(n \geq 2)$. For $n = 1$, the authors [54] have proved a stronger result relaxing decay condition. On Riemannian two-point homogeneous spaces, analogues of the Globevnik problem have been established by V. V. Volchkov and Vit. V. Volchkov [73].

Now, consider the Métivier group denoted by $G \simeq \mathbb{C}^n \times \mathbb{R}^m$, with the group law

$$(z,t) \cdot (w,\tau) = \left(\begin{array}{c} z + w \\ t_j + \tau_j + \frac{1}{2} \operatorname{Re}(z \cdot \overline{U^{(j)} w}),\ j = 1, \ldots, m \end{array} \right),$$

where $U^{(j)}$'s are skew-symmetric matrices. We prove necessary conditions for a function to be in $Z^*_{r,R}$, a subspace of ceratin smooth functions on $B_{r,R}$. This result gives a support theorem for the functions, which transform according to the unitary irreducible representation of the unitary group $U(n)$. The later functions are known as type functions. A detailed explanation and definition of as yet undefined symbols is provided in Chapter 2.

Moreover, we derive that the non-harmonic complex cones in $\mathbb{C}^n$ are sets of injectivity for the λ-twisted spherical mean for the class of continuous functions on Métivier groups. For a brief history of work related to sets of injectivity for TSM on the Heisenberg group, we refer to [6, 45, 58–61].

Further, we study some results on H-type groups, a special case of Métivier groups. We prove sufficient conditions for a function to be in $Z_{r,R}$ and derive a support theorem. Then we notice that an analogue of the Heche-Bochner identity is true, and the boundary of a bounded domain is a set of injectivity for λ-twisted spherical mean.

We would like to mention that the results in the case of Métivier groups are of restrictive nature with those in the Heisenberg group due to the fact that the symplectic bilinear form appears in the group action of the Métivier groups need not be $U(n)$-invariant. This happens because of the higher dimensional centre of the Métivier groups, and the distinct eigenvalues of symplectic matrix V_λ, in general. However, we prove that the symplectic bilinear form for H-type groups is similar to that of the Heisenberg group up to an orthogonal transformation.

The third problem, known as one radius theorem, concerning the injectivity of the spherical mean operator, has recently attracted significant attention. Particularly, does the operator f into $f * \mu_\iota$ for a fixed $\iota > 0$ turn out to be injective. In general, the answer to this is negative, since there are non-trivial bounded continuous functions, e.g. Bessel functions φ, for which $\varphi * \mu_\iota = 0$, when ι is a zero of the Bessel function. The injectivity of the spherical mean is an ever interesting question and studied by several authors, including [1, 56, 67, 68, 76]. Thangavelu [67] has shown that the one radius theorem is true for $L^p(\mathbb{R}^n)$, when $1 \leq p \leq 2n/(n-1)$, by exploiting the spectral decomposition of the Laplacian.

One radius theorem has also been considered for the Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$. Indeed, in [67], it has been shown that if $f \in L^p(\mathbb{H}^n), 1 \leq p < \infty$, then $f * \mu = 0$ implies $f = 0$, where μ is a compactly supported rotation invariant probability measure with no mass at the centre. The proof of this result is based on a summability result due to Strichartz [63] for sub-Laplacian on $\mathbb{H}^n$. Recently, this summability result has been extended for the H-type groups, see [44].

Although, in Métivier groups, the analogue to summability result [63] is yet to settle due to the appearance of a multi-parameter singular integral, whose kernel is not radial, because of the higher dimensional centre. However, we show that the mean operator f into $f * \mu$ is injective, when $f(z,.) \in L^p(\mathbb{R}^m), 1 \leq p \leq 2$, and f is of tempered growth in the z-variable. This result is obtained by employing the simplified λ-twisted spherical mean on the Métivier groups, which we introduced in Chapter 2 and the special Hermite expansion as discuss in Chapter 3. Moreover, when $\mu = \mu_s$, the normalized surface measure on the set $\{(z,0) : \ |z| = s\}$, we prove one radius theorem for continuous functions f, when $f(z,.) \in L^p(\mathbb{R}^m), 1 \leq p \leq 2$, and $f^\lambda(z)e^{\frac{1}{4}\left|J'_\lambda z_\lambda\right|^2}$ is in $L^q(\mathbb{C}^n), 1 \leq q \leq \infty$. In the end, we obtain a two-radii theorem for the tempered continuous functions in the z-variable and 2π-periodic in the t-variable.

Since the symplectic bilinear form appears in the group action of the Métivier groups is far from $U(n)$-invariance, the λ-twisted spherical mean cannot be radialised as in the case of the Heisenberg group. However, it is elliptical up to a rotation. We connect this elliptical mean to the twisted spherical mean of a Lie group having $3n$-dimensional step two nilpotent Lie algebra. This fact unfolds many tools for studying the spherical mean in the Métivier groups setup. We obtain the spectral decomposition for L^2-functions in terms of eigenfunctions of sub-Laplacian on this particular Lie group. This reduction eases towards proving an analogue of one radius theorem on the general Métivier groups.

The next portion of this thesis is devoted to the quaternion Weyl transform (QWT), a generalized form of the classical Weyl transform. We underline that the generalization is non-trivial due to the multiplication of quaternions being non-commutative. Over the last few years, the quaternion approach has been widely explored in various aspects of data analysis, such as colour image processing, flow visualization and spoken word recognition, etc.

It is interesting to mention that, while the study of quantization problem in quantum mechanics, a certain pseudo-differential operator was anticipated by Hermann Weyl [74]. This operator is later known as Weyl transform, which is useful in various areas of mathematics and physics, especially in harmonic analysis, PDE, time-frequency analysis, etc. The compactness property of the Weyl transform was studied on $L^2(\mathbb{R}^n)$ with the symbol in $L^p(\mathbb{R}^{2n})$, when $1 \leq p \leq 2$, see [75]. In a sharp contrast, the Weyl transform fails to be even bounded for $2 < p < \infty$, see [57]. The boundedness property of the Weyl transform has been considered in different setups, including the Heisenberg group, quaternion Heisenberg group, upper half-plane, Euclidean and Heisenberg motion groups [16, 28, 51, 52].

In recent times, the uncertainty principle for quaternion Fourier transform (QFT) has received significant attention [10, 15, 33, 40]. The non-commutativity of the quaternion multiplication and the Fourier kernel make QFT different from the classical Fourier transform. The QFT has an important application in data analysis, particularly in colour image processing, etc. Since the quaternion algebra decomposes into two complex planes, the QFT can be split into two Euclidean Fourier transforms, which makes QFT accessible. For instance, a version of Hardy's theorem studied in [35] can be generalized to the QFT, see Section 4.1. But that approach cannot be extended to quaternion Fourier-Wigner transform and QWT. In Section 4.2, we study the boundedness of quaternion Fourier-Wigner transform that arises due to the QFT. Consequently, we consider the boundedness of the QWT for the symbol in $L^p(\mathbb{R}^4, \mathbb{Q})$, the space of quaternion valued L^p functions.

In [11], Benedicks proved that if $f \in L^1(\mathbb{R}^n)$, then both the sets $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ and $\{\xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0\}$ cannot have finite Lebesgue measure, unless $f = 0$. Concurrently, in [8], Amrein-Berthier reached the same conclusion via the Hilbert space theory. This result further extended to certain unimodular groups in the form of the qualitative uncertainty principle (QUP). A group G is said to satisfy QUP if for each $f \in L^2(G)$ with $m\{x \in G : f(x) \neq 0\} < m(G)$ and

$$\int_{\hat{G}} \operatorname{rank}\hat{f}(\lambda)\, d\hat{m}(\lambda) < \infty, \tag{1.0.2}$$

implies $f = 0$, see [9]. For the Heisenberg group $\mathbb{H}^n$, the condition (1.0.2) of QUP implies $\hat{f}$ should be supported on a set of finite Plancherel measure as well as $\operatorname{rank}\hat{f}(\lambda)$ is finite for almost all λ.

In [43], Narayanan and Ratnakumar proved that if $f \in L^1(\mathbb{H}^n)$ is supported on $B \times \mathbb{R}$, where B is a compact subset of $\mathbb{C}^n$, and $\hat{f}(\lambda)$ has finite rank for each λ, then $f = 0$. Thereafter, the compact set B is replaced by a set of finite measure [25, 70]. An analogous result is also true for the step two nilpotent Lie groups with MW condition [14, 25], and for the quaternion Heisenberg group [27]. Further, a non trivial extension of the result for the Heisenberg motion group is established in [26].

In Section 4.3, we consider a quaternion analogue of the Heisenberg group Weyl transform, and we prove a version of the Benedicks-Amrein-Berthier theorem. In other words, if $g \in L^1(\mathbb{R}^4, \mathbb{Q})$ is non-zero and supported on a product of two dimensional finite measure sets, then the Weyl transform $W(g)$ cannot have finite rank.

Since the TSM on the Heisenberg group $\mathbb{H}^2$ and the Fourier transform on $\mathbb{C}^2$ are closely related, we can similarly theorize the quaternion twisted spherical mean (QTSM) as

$$f \times_{\mathbb{Q}} \mu_\iota(p,q) = \int_{|(u,v)|=\iota} e^{\pi i(u_1q_1 - p_1v_1)} f(p-u, q-v) e^{\pi j(u_2q_2-p_2v_2)} d\mu_\iota(u,v),$$

where μ_ι is the normalized surface measure on the sphere in $\mathbb{R}^4$ centred at origin and radius $\iota > 0$. Then we can look at the above listed problems for QTSM. Due to the fact that the decomposition of quaternion algebra into two complex planes, the QTSM also split into two Heisenberg group TSMs. Hence, some results directly extend in this setup as well. Chapter 4 is concluded by a remark about the sets of injectivity and Helgason's support theorem for the QTSM.

This thesis is organized as follows:

In Chapter 2, we consider the λ-twisted spherical mean on the Métivier groups. We give necessary conditions for a function to be in $Z^*_{r,R}$. These conditions are in terms of spherical harmonic coefficients. As a consequence, we get a support theorem for type functions. Further, we derive that non-harmonic complex cones are uniqueness sets for the λ-twisted spherical mean. Chapter 2 ends with some results on H-type groups such as sufficient conditions for a function to be in $Z_{r,R}$, Heche-Bochner identity and boundary of bounded domains are sets of injectivity for twisted spherical mean in the H-type setup.

In Chapter 3, we investigate the injectivity of the spherical mean for continuous functions on the Métivier groups using the spectral decomposition of L^2-functions. We acquire certain classes of functions for which the spherical mean is injective. Moreover, we obtain a version of the two-radii theorem.

In Chapter 4, we begin with QFT and observe that it can be reduced to the Euclidean Fourier transform, which is not the case for quaternion Wigner transform and QWT. We study the boundedness and several properties of the quaternion Wigner transform. Further, we define QWT corresponding to quaternion valued symbols and look into its boundedness. The next half of this chapter contains a rank analogue of the Benedicks-Amrein-Berthier theorem for the QWT and a version of Beurling's theorem for the quaternion Fourier-Weyl transform. We end with some comments regarding quaternion twisted spherical mean.

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Chapter 2

Spherical mean on Métivier groups and support theorem

This chapter splits into two parts. In the first part, we study the λ-twisted spherical mean on the Métivier group $G \simeq \mathbb{C}^n \times \mathbb{R}^m$ and review some important aspects of its vector fields. Then we prove necessary conditions for a function to be in $Z^*_{r,R}$ followed by a support theorem for type functions. Further, we show that non-harmonic complex cones are uniqueness sets for the λ-twisted spherical mean.

The second part of this chapter deals with some results on H-type groups, which could not be settled for Métivier groups. We prove sufficient condition for a function to be in $Z_{r,R}$. Additionally, we state an analogue of Heche-Bochner identity and see that the boundary of bounded domains are sets of injectivity for λ-twisted spherical mean.

2.1 Preliminaries

Let G be a connected, simply connected Lie group with real step two nilpotent Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$. We can choose an inner product on $\mathfrak{g}$ to make the above decomposition orthogonal. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp : \mathfrak{g} \to G$ is surjective, and hence G can be

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parameterized by $\mathfrak{g}$, endowed with the exponential coordinates. Now, we can identify $X+T \in \mathfrak{b} \oplus \mathfrak{z}$ with $\exp(X+T)$ and denote it by $(X,T) \in \mathbb{R}^d \times \mathbb{R}^m$. Since $[\mathfrak{b},\mathfrak{b}] \subseteq \mathfrak{z}$ and $[\mathfrak{b},[\mathfrak{b},\mathfrak{b}]] = 0$, by the Baker-Campbell-Hausdorff formula, the group law on G can be expressed as

$$(X,T)\cdot(Y,S) = (X+Y,\ T+S+\frac{1}{2}[X,Y]),$$

where $X,Y \in \mathfrak{b}$ and $T,S \in \mathfrak{z}$. Now, for $\omega \in \mathfrak{z}^*$, consider the skew-symmetric bilinear form B_ω on $\mathfrak{b}$ by $B_\omega(X,Y) = \omega\left([X,Y]\right)$. Let m_ω be the orthogonal complement of $r_\omega = \{X \in \mathfrak{b} : B_\omega(X,Y) = 0,\ \forall\ Y \in \mathfrak{b}\}$ in $\mathfrak{b}$. Then B_ω is called a non-degenerate bilinear form when r_ω is trivial.

In this thesis, we discuss some particular types of step two nilpotent Lie groups known as Métivier groups.

Métivier groups. We say group G is Métivier group if B_ω is non-degenerate for all non-zero $\omega \in \mathfrak{z}^*$. In this case, $d = 2n$, even. Let $B_1,\dots,B_{2n}$ and $Z_1,\dots,Z_m$ be orthonormal bases for $\mathfrak{b}$ and $\mathfrak{z}$, respectively. Since $[\mathfrak{b},\mathfrak{b}] \subseteq \mathfrak{z}$, there exist scalars $U_{j,l}^{(k)}$ such that

$$[B_j,B_l] = \sum_{k=1}^{m} U_{j,l}^{(k)} Z_k, \quad 1 \le j,l \le 2n.$$

For $1 \le k \le m$, define $2n \times 2n$ skew-symmetric matrices by $U^{(k)} = (U_{j,l}^{(k)})$. Then the group law for the Métivier group can be expressed as

$$(x,t)\cdot(\xi,\tau) = \begin{pmatrix} x_i+\xi_i,\ i=1,\dots,2n \\ t_j+\tau_j+\frac{1}{2}\langle x, U^{(j)}\xi\rangle,\ j=1,\dots,m \end{pmatrix}, \tag{2.1.1}$$

where $x,\xi \in \mathbb{R}^{2n}$ and $t,\tau \in \mathbb{R}^m$. For $x = (x_1,\dots,x_n,y_1,\dots,y_n) \in \mathbb{R}^{2n}$, write $z = (x_1+iy_1,\dots,x_n+iy_n) = (z_1,\dots,z_n)$ and say, z be the complexification of x. Let $z,w \in \mathbb{C}^n$ be the complexification of $x,\xi \in \mathbb{R}^{2n}$. If we fix the notation $U^{(j)}w$ for the complexification of $U^{(j)}\xi$, then (2.1.1) can be simplified to

$$(z,t)\cdot(w,\tau)=\begin{pmatrix} z+w \\ t_j+\tau_j+\frac{1}{2}\,\mathrm{Re}\,(z\cdot\overline{U^{(j)}w}),\ j=1,\ldots,m \end{pmatrix}. \tag{2.1.2}$$

H-type groups. Suppose $\mathfrak{g}$ is endowed with an inner product $\langle\cdot,\cdot\rangle$ such that for each $Z\in\mathfrak{z}$, the map $J_Z:\mathfrak{b}\to\mathfrak{b}$ defined by $\langle J_Z(X),Y\rangle=\langle Z,[X,Y]\rangle$ for $X,Y\in\mathfrak{b}$, satisfies $J_Z^T=-J_Z$. We say that $\mathfrak{g}$ is H-type if $J_Z^2=-|Z|^2I$, whenever $Z\in\mathfrak{z}$. Hence it follows that

$$J_ZJ_{Z'}+J_{Z'}J_Z=-2\langle Z,Z'\rangle I \text{ for all } Z,Z'\in\mathfrak{z},$$

where I denote the identity mapping. A connected, simply connected Lie group G with H-type Lie algebra is called Heisenberg type (or H-type) group. The H-type groups, introduced by A. Kaplan [37], are examples of Métivier groups. However, there are Métivier groups that differ from the H-type groups. For more details, see [41, 42].

Theorem 2.1.1. [13] *Let G be a connected, simply connected Lie group with real step two nilpotent Lie algebra $\mathfrak{g}$. Then G is a H-type group if and only if G is isomorphic to $\mathbb{R}^{2n+m}$ with the group law (2.1.1) and the matrices $U^{(1)},\ldots,U^{(m)}$ satisfy the following conditions:*

(a) $U^{(j)}$ is skew-symmetric $2n\times 2n$ orthogonal matrix, for $j=1,\ldots,m$.

(b) $U^{(j)}U^{(l)}+U^{(l)}U^{(j)}=0$ for all $1\le j,l\le m$ with $j\ne l$.

We identify $\mathfrak{z}^*$ with $\mathbb{R}^m$ and denote $\mathbb{R}^m\setminus\{0\}=\mathbb{R}^m_*$. For $\lambda\in\mathbb{R}^m_*$, it follows from Theorem 2.1.1 that

$$\sum_{j=1}^{m}\lambda_jU^{(j)}=|\lambda|V,$$

where V is an orthogonal matrix. This fact will enable us to deduce that λ-twisted spherical mean on H-type groups is similar to the $|\lambda|$-twisted spherical mean on the Heisenberg group.

Let μ_s be the normalized surface measure on the set $\{(z,0):\ |z|=s\}\subset G$. Then the partial spherical mean of a function $F\in L^1_{loc}(G)$ can be defined by

$$F*\mu_s(z,t)=\int_{|w|=s}F\left((z,t)\cdot(-w,0)\right)\ d\mu_s(w). \tag{2.1.3}$$

Let

$$F^\lambda(z)=\int_{\mathbb{R}^m}F(z,t)e^{i\lambda\cdot t}dt$$

be the inverse Fourier transform of F in the t-variable. Then

$$(F*\mu_s)^\lambda(z)=\int_{|w|=s}F^\lambda(z-w)e^{\frac{i}{2}\sum_{j=1}^m\lambda_j\mathrm{Re}\,(z\cdot\overline{U^{(j)}w})}\ d\mu_s(w). \tag{2.1.4}$$

Define the λ-twisted spherical mean of $f\in L^1(\mathbb{C}^n)$ by

$$f\times_\lambda\mu_s(z)=\int_{|w|=s}f(z-w)\ e^{\frac{i}{2}\sum_{j=1}^m\lambda_j\mathrm{Re}\,(z\cdot\overline{U^{(j)}w})}\ d\mu_s(w). \tag{2.1.5}$$

From (2.1.4) we get $(F*\mu_s)^\lambda=F^\lambda\times_\lambda\mu_s$. Thus, the partial spherical mean $F*\mu_s$ on the Métivier group G can be thought of the λ-twisted spherical mean $F^\lambda\times_\lambda\mu_s$. Note that λ-twisted spherical mean (2.1.5) is the complexification of the mean

$$f\times_\lambda\mu_s(x)=\int_{|\xi|=s}f(x-\xi)\ e^{\frac{i}{2}\sum_{j=1}^m\lambda_j\langle x,U^{(j)}\xi\rangle}\ d\mu_s(\xi). \tag{2.1.6}$$

Definition 2.1.2. *Let $B_{r,R}=\{z\in\mathbb{C}^n: r<|z|<R\}$ be an open annulus in $\mathbb{C}^n$, where $0\le r<R\le\infty$. Let $Z_{r,R}$ be the space of all continuous functions f on $B_{r,R}$ such that $f\times_\lambda\mu_s(z)=0$ on the spheres $S_s(z)\subset B_{r,R}$ and the ball $B_r(0)\subseteq B_s(z)$.*

Let $Z^\infty_{r,R}$ be the space of all smooth functions in $Z_{r,R}$. Consider a smooth non-negative radial function ϕ on $\mathbb{C}^n$, supported in $B_1(0)$ and $\int_{\mathbb{C}^n}\phi=1$.

When $\epsilon > 0$, write $\phi_\epsilon(z) = \epsilon^{-2n}\phi(\frac{z}{\epsilon})$. For $f \in Z^\infty_{r,R}$, define $S_\epsilon(f)$ by

$$S_\epsilon(f)(z) = \int_{\mathbb{C}^n} f(z-w)\phi_\epsilon(w)e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \mathrm{Re}\,(z\cdot\overline{U^{(j)}w})}dw.$$

Then we can deduce that $S_\epsilon(f) \in Z^\infty_{r+\epsilon,R-\epsilon}$. Since supp $\phi_\epsilon \subseteq B_\epsilon(0)$, and

$$\begin{aligned} S_\epsilon(f)(z) - f(z) = &\int_{|w|\le\epsilon} \phi_\epsilon(w)e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \mathrm{Re}\,(z\cdot\overline{U^{(j)}w})}(f(z-w)-f(z))dw \\ &+ \int_{|w|\le\epsilon}(e^{\frac{i}{2}\sum_{j=1}^m \lambda_j \mathrm{Re}\,(z\cdot\overline{U^{(j)}w})} - 1)f(z)\phi_\epsilon(w)dw, \end{aligned}$$

together with f is continuous, letting ϵ goes to 0, it follows that $S_\epsilon(f)$ converges to f locally uniformly. Thus, without loss of generality, we can assume the functions in $Z_{r,R}$ are smooth.

Since $Z_{r,R}$ is closed under small translation, it follows that $Z_{r,R}$ will be invariant under the action of appropriate vector fields on G.

The left-invariant vector fields on G are

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^m\left(\sum_{l=1}^n\left(x_l U^{(k)}_{l,j} + y_l U^{(k)}_{n+l,j}\right)\right)\frac{\partial}{\partial t_k}, \\ Y_j &= \frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{k=1}^m\left(\sum_{l=1}^n\left(x_l U^{(k)}_{l,n+j} + y_l U^{(k)}_{n+l,n+j}\right)\right)\frac{\partial}{\partial t_k}, \\ T_k &= \frac{\partial}{\partial t_k}, \text{ where } k = 1,\dots,m,\ j = 1,\dots,n, \end{aligned}$$

and $(x_1,\dots,x_n,y_1,\dots,y_n,t_1,\dots,t_m) \in \mathbb{R}^{2n}\times\mathbb{R}^m$. In fact, they generate a basis of the Lie algebra of the Métivier group G. Given that $U^{(s)}$'s are skew-symmetry, so we obtain the following commutation relations

$$[X_i, X_j] = \sum_{k=1}^m U^{(k)}_{i,j}\frac{\partial}{\partial t_k},\ [X_{n+i}, X_{n+j}] = \sum_{k=1}^m U^{(k)}_{n+i,n+j}\frac{\partial}{\partial t_k},\ [X_i, X_{n+j}] = \sum_{k=1}^m U^{(k)}_{i,n+j}\frac{\partial}{\partial t_k},$$

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for $i, j = 1, \ldots, n$. Since $U^{(1)}, \ldots, U^{(m)}$ are linearly independent, the dimension of the space spanned by $\{(U^{(1)}_{i,j}, \ldots, U^{(m)}_{i,j}) : \ i, j = 1, \ldots, n\}$ will be m.

Now, for $1 \leq j \leq n$, define

$$\begin{aligned}
Z_j &= \frac{1}{2}(X_j - iY_j)\\
&= \frac{\partial}{\partial z_j} + \frac{1}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\left\{x_l\left(U^{(k)}_{l,j} - iU^{(k)}_{l,n+j}\right) + y_l\left(U^{(k)}_{n+l,j} - iU^{(k)}_{n+l,n+j}\right)\right\}\frac{\partial}{\partial t_k},\\
\bar{Z}_j &= \frac{1}{2}(X_j + iY_j)\\
&= \frac{\partial}{\partial \bar{z}_j} + \frac{1}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\left\{x_l\left(U^{(k)}_{l,j} + iU^{(k)}_{l,n+j}\right) + y_l\left(U^{(k)}_{n+l,j} + iU^{(k)}_{n+l,n+j}\right)\right\}\frac{\partial}{\partial t_k}.
\end{aligned}$$

Consider the function F on $G = \mathbb{C}^n \times \mathbb{R}^m$ of type $F(z,t) = e^{i\lambda \cdot t} f(z)$, where $\lambda \in \mathbb{R}^m_*$. Then the vector fields Z_j and $\bar{Z}_j$ reduce to

$$\begin{aligned}
Z^\lambda_j &= \frac{\partial}{\partial z_j} + \frac{1}{4}\sum_{l=1}^{n}\left\{\left(\beta^\lambda_l + i\alpha^\lambda_l\right) z_l + \left(-\beta^\lambda_l + i\alpha^\lambda_l\right)\bar{z}_l\right\}\\
&= \frac{\partial}{\partial z_j} + \frac{1}{4}\nu_j \bar{z}_j + \frac{1}{4}\sum_{\substack{l=1\\(l\neq j)}}^{n}(\eta_l z_l + \nu_l \bar{z}_l), \qquad (2.1.7)\\
\bar{Z}^\lambda_j &= \frac{\partial}{\partial \bar{z}_j} + \frac{1}{4}\sum_{l=1}^{n}\left\{\left(\bar{\beta}^\lambda_l + i\bar{\alpha}^\lambda_l\right) z_l + \left(-\bar{\beta}^\lambda_l + i\bar{\alpha}^\lambda_l\right)\bar{z}_l\right\}\\
&= \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4}\bar{\nu}_j z_j - \frac{1}{4}\sum_{\substack{l=1\\(l\neq j)}}^{n}(\bar{\nu}_l z_l + \bar{\eta}_l \bar{z}_l), \qquad (2.1.8)
\end{aligned}$$

since $\eta_j = 0$, where we denote

$$\alpha^\lambda_l = \frac{1}{2}\sum_{k=1}^{m}\lambda_k\left(U^{(k)}_{l,j} - iU^{(k)}_{l,n+j}\right), \ \beta^\lambda_l = \frac{1}{2}\sum_{k=1}^{m}\lambda_k\left(U^{(k)}_{n+l,j} - iU^{(k)}_{n+l,n+j}\right)$$

and $\eta_l = \beta^\lambda_l + i\alpha^\lambda_l$, $\nu_l = -\beta^\lambda_l + i\alpha^\lambda_l$ for $1 \leq l \leq n$.

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The differential operators Z_j^λ and $\bar{Z}_j^\lambda$ play a role of left-invariant vector fields for λ-twisted convolution on $\mathbb{C}^n$. That is,

$$Z_j^\lambda(f \times_\lambda \mu_s) = Z_j^\lambda f \times_\lambda \mu_s \text{ and } \bar{Z}_j^\lambda(f \times_\lambda \mu_s) = \bar{Z}_j^\lambda f \times_\lambda \mu_s.$$

As an effect, if $f \in Z_{r,R}$, then $Z_j^\lambda f$ and $\bar{Z}_j^\lambda f$ both are in $Z_{r,R}$.

2.1.1 Bi-graded spherical harmonics

To make this thesis self-content, we discuss the bi-graded spherical harmonic expansion of continuous function on $\mathbb{C}^n$. See [18, 30, 55, 69] for details.

For $p, q \in \mathbb{Z}_+$, the set of all non-negative integers, let $P_{p,q}$ denote the space of all polynomials P in z and $\bar{z}$ of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta}\, z^\alpha \bar{z}^\beta.$$

Write $H_{p,q} = \{P \in P_{p,q} : \Delta P = 0\}$, where Δ stands for the Laplacian on $\mathbb{C}^n$. The elements of $H_{p,q}$ restricted to the unit sphere S^{2n-1} are called bi-graded spherical harmonics. We identify $H_{p,q}$ as the space of bi-graded spherical harmonics on S^{2n-1}. Let $\{Y_j^{p,q} : 1 \leq j \leq d_{p,q}\}$ be an orthonormal basis of $H_{p,q}$. By the Peter-Weyl theorem, the set $\{Y_j^{p,q} : 1 \leq j \leq d_{p,q},\ p, q \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{2n-1})$, and hence a continuous function f on $\mathbb{C}^n$ can be expressed as

$$f(\rho\omega) = \sum_{p,q} \sum_{j=1}^{d_{p,q}} a_j^{p,q}(\rho)\, Y_j^{p,q}(\omega), \tag{2.1.9}$$

where $\rho > 0,\ \omega \in S^{2n-1}$, and $a_j^{p,q}$ are called the spherical harmonic coefficients of f.

The $(p,q)^{th}$ projection of f is given by

$$\Pi_{p,q}(f)(\rho,\omega) = \sum_{j=1}^{d_{p,q}} a_j^{p,q}(\rho)\ Y_j^{p,q}(\omega). \tag{2.1.10}$$

We need the following lemma to decompose a homogeneous polynomial into homogeneous harmonic polynomials.

Lemma 2.1.3. [69] *Every* $P \in P_{p,q}$ *can be uniquely expressed as* $P(z) = P_0(z) + |z|^2 P_1(z) + \cdots + |z|^{2l} P_l(z)$, *where* $P_k \in H_{p-k,q-k}$ *and* $l \leq min(p,q)$.

Corollary 2.1.4. [54] *Let* $P \in H_{p,q}$. *Then it follows that*

$$\bar{z}_j P(z) = P_0(z) + \gamma_{p,q}|z|^2 \frac{\partial P}{\partial z_j},\ z_j P(z) = P_0'(z) + \gamma_{p,q}|z|^2 \frac{\partial P}{\partial \bar{z}_j},$$

where $\gamma_{p,q} = \frac{1}{(n+p+q-1)}$, $P_0 \in H_{p,q+1}$ *and* $P_0' \in H_{p+1,q}$.

2.2 Some results on Métivier groups

2.2.1 Characterization of certain continuous functions

As we know that the λ-twisted spherical mean of the Métivier groups need not be $U(n)$-invariant, we require to modify the space $Z_{r,R}$ appropriately.

We first recall the following fact from Geller [24]. The operator analogue of a bi-graded harmonic polynomial can be identical to the polynomial itself. Now for our purpose, we assume $P_j^{p,q}(z) = |z|^{p+q} Y_j^{p,q}(\frac{z}{|z|})$ contains the term $z^\alpha \bar{z}^\beta$, for some multi-index $\alpha, \beta \in \mathbb{Z}_+^n$ with $|\alpha| = p$ and $|\beta| = q$. By abuse of notation, we denote

$$P_j^{p,q}(Z) = Z^\alpha \bar{Z}^\beta, \tag{2.2.1}$$

where $Z^\alpha = (Z_1^\lambda)^{\alpha_1}\cdots(Z_n^\lambda)^{\alpha_n}$ and $\bar{Z}^\beta = (\bar{Z}_1^\lambda)^{\beta_1}\cdots(\bar{Z}_n^\lambda)^{\beta_n}$. Let $\tilde{a}_j^{p,q}(\rho) = \rho^{-(p+q)}a_j^{p,q}(\rho)$, where $a_j^{p,q}$ as appears in (2.1.9). Let $Z^*_{r,R}$ be the space of smooth functions f on $B_{r,R}$ satisfying the conditions

$$\Pi_{0,0}\left(P_j^{0,q}(Z)\left(\Pi_{0,q}\left(P_j^{p,0}(Z)\left(\tilde{a}_j^{p,q}P_j^{p,q}\right)\right)\right)\right) \in Z_{r,R}$$

for all $p, q \in \mathbb{Z}_+$ and $1 \leq j \leq d_{p,q}$.

Now, we fix some notations for our convenience. Denote $D_j = \rho\frac{\partial}{\partial\rho} + \frac{\nu_j}{2}\rho^2$ and $\bar{D}_j = \rho\frac{\partial}{\partial\rho} - \frac{\bar{\nu}_j}{2}\rho^2$, where ν_j is defined in (2.1.8). For multi-index $\alpha, \beta \in \mathbb{Z}_+^n$, define

$$D^\alpha = \prod_{i_1=1}^{\alpha_1}(\kappa_{1,i_1}D_1 + 2)\cdots\prod_{i_n=1}^{\alpha_n}(\kappa_{n,i_n}D_n + 2) \text{ and } \bar{D}^\beta = \prod_{j_1=1}^{\beta_1}(\tilde{\kappa}_{1,j_1}\bar{D}_1 + 2)\cdots\prod_{j_n=1}^{\beta_n}(\tilde{\kappa}_{n,j_n}\bar{D}_n + 2),$$

where $\kappa_{l,i_l}, \tilde{\kappa}_{k,j_k} \in \{\gamma_{p',q'} = \frac{1}{(n+p'+q'-1)} : 0 \leq p' \leq p,\ 0 \leq q' \leq q\}$.

In order to characterize spherical harmonic coefficients for the functions in $Z^*_{r,R}$, it would be enough to consider the following theorem.

Theorem 2.2.1. *Let $f(z) = \tilde{a}(\rho)P_{p,q}(z)$, where $\rho = |z|$ and $P_{p,q} \in H_{p,q}$. Then a necessary condition for $f \in Z^*_{r,R}$ is that $\tilde{a}$ satisfies the ODE*

$$\left(\sum_{|\beta|+k=q} d_{\beta,k}\rho^{2k}\bar{D}^\beta\right)\left(\sum_{|\alpha|+l=p} c_{\alpha,l}\rho^{2l}D^\alpha\right)\tilde{a} = 0$$

for some scalars $c_{\alpha,l}, d_{\beta,k} \in \mathbb{C}$.

In particular, if $P_{p,q}(z) = z_{l_1}^p\bar{z}_{l_2}^q$ for some $1 \leq l_1, l_2 \leq n$, then there exist $A_i, B_k \in \mathbb{C}$ such that

$$\tilde{a}(\rho) = \sum_{i=0}^{p} A_i e^{-\frac{\nu_{l_1}}{4}\rho^2}\rho^{-2(p+q+n-i)} + \sum_{k=0}^{q} B_k e^{\frac{\nu_{l_2}}{4}\rho^2}\rho^{-2(p+q+n-k)}, \qquad (2.2.2)$$

where $r < \rho < R$ and $A_0 = B_0 = 0$.

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Proof. For $p = q = 0$, we have $\tilde{a}(\rho) = \tilde{a} \times_\lambda \mu_\rho(0) = 0$, whenever $r < \rho < R$. To proceed for the other cases, we need to apply the operator Z_j^λ to f. That is,

$$Z_j^\lambda f = \frac{\partial f}{\partial z_j} + \frac{1}{4}\nu_j \bar{z}_j f + \frac{1}{4}\sum_{\substack{l=1\\(l\neq j)}}^{n}(\eta_l z_l + \nu_l \bar{z}_l) f.$$

Since $f = \tilde{a}P$, the above equation will take the form

$$Z_j^\lambda f = \frac{1}{2\rho^2}(D_j\tilde{a})\bar{z}_j P(z) + \tilde{a}\frac{\partial P}{\partial z_j} + \frac{1}{4}\sum_{l\neq j}(\eta_l z_l + \nu_l \bar{z}_l)\tilde{a}P(z). \tag{2.2.3}$$

Substituting the values of $\bar{z}_j P(z)$ and $z_j P(z)$ from Corollary 2.1.4, we have

$$\begin{aligned} Z_j^\lambda f = {} & \frac{1}{2\rho^2} D_j \tilde{a}\left(P_0 + \gamma_{p,q}|z|^2\frac{\partial P}{\partial z_j}\right) + \tilde{a}\frac{\partial P}{\partial z_j} \\ & + \frac{1}{4}\sum_{l\neq j}\left[\eta_l\tilde{a}\left(P_0' + \gamma_{p,q}|z|^2\frac{\partial P}{\partial \bar{z}_l}\right) + \nu_l\tilde{a}\left(P_0 + \gamma_{p,q}|z|^2\frac{\partial P}{\partial z_l}\right)\right]. \end{aligned}$$

After rearranging the terms, we get

$$\begin{aligned} Z_j^\lambda f = {} & \frac{1}{2\rho^2} D_j\tilde{a}P_0 + \frac{1}{4}\sum_{l\neq j}\nu_l\tilde{a}P_0 + \frac{1}{4}\sum_{l\neq j}\eta_l\tilde{a}P_0' \\ & + \frac{1}{2}\left(\gamma_{p,q}D_j + 2\right)\tilde{a}\frac{\partial P}{\partial z_j} + \frac{1}{4}\rho^2\gamma_{p,q}\sum_{l\neq j}\left(\eta_l\frac{\partial P}{\partial \bar{z}_l} + \nu_l\frac{\partial P}{\partial z_l}\right)\tilde{a}. \end{aligned}$$

Now, the projection $\Pi_{p-1,q}$ of $Z_j^\lambda f$ is given by

$$\Pi_{p-1,q}\, Z_j^\lambda f = \frac{1}{2}\left(\gamma_{p,q}\, D_j + 2\right)\tilde{a}\frac{\partial P}{\partial z_j} + \frac{1}{4}\rho^2\gamma_{p,q}\sum_{l\neq j}\nu_l\frac{\partial P}{\partial z_l}\tilde{a}.$$

If $p = 1$ and $q = 0$, then $\dfrac{\partial P}{\partial z_j}$ is a non-zero constant for some j, say ζ_j. Thus,

$$\begin{aligned}\Pi_{0,0}\ Z_j^\lambda f &= \left\{\frac{1}{2n}\left(\rho\frac{\partial}{\partial\rho} + \frac{\nu_j}{2}\rho^2\right) + 1\right\}\tilde{a}\ \zeta_j + \frac{1}{4}\sum_{l\neq j}\frac{\rho^2\nu_l}{n}\tilde{a}\ \zeta_l \qquad (2.2.4)\\ &= \zeta_j\left\{\frac{1}{2n}\left(\rho\frac{\partial}{\partial\rho} + \left(\frac{\nu_j}{2} + \sum_{l\neq j}\frac{\nu_l}{2}\frac{\zeta_l}{\zeta_j}\right)\rho^2\right) + 1\right\}\tilde{a}\\ &= \zeta_j\left\{\frac{1}{2n}\left(\rho\frac{\partial}{\partial\rho} + \frac{d_1}{2}\rho^2\right) + 1\right\}\tilde{a}(\rho),\end{aligned}$$

where $d_1 = (\nu_j + \sum_{l\neq j}\nu_l\frac{\zeta_l}{\zeta_j})$. By definition of $Z^*_{r,R}$, it is clear that $\Pi_{0,0}(Z_j^\lambda f) \in Z_{r,R}$. Evaluating λ-twisted spherical mean of $\Pi_{0,0}(Z_j^\lambda f)$ at $z = 0$, we get

$$\left\{\frac{1}{2n}\left(\rho\frac{\partial}{\partial\rho} + \frac{d_1}{2}\rho^2\right) + 1\right\}\tilde{a}(\rho) = 0.$$

By replacing $\tilde{a}(\rho) = e^{-\frac{d_1}{4}\rho^2}\tilde{a}'(\rho)$ in the above equation, we get

$$e^{-\frac{d_1}{4}\rho^2}\left\{\frac{1}{2n}\rho\frac{\partial}{\partial\rho} + 1\right\}\tilde{a}'(\rho) = 0.$$

Thus, for $p = 1$ and $q = 0$, we infer that

$$\tilde{a}(\rho) = A_1 e^{-\frac{d_1}{4}\rho^2}\rho^{-2n}.$$

However, it would be difficult to solve the ODE for the case $q = 0$ and $p \geq 2$. For instance, consider $P(z) = z_1 z_2$. Then, applying $Z_1 Z_2$ to $\tilde{a}P$, we get

$$\left\{(\gamma_{1,0}D_1 + 2)(\gamma_{2,0}D_2 + 2) + c_1c_2\rho^4\right\}\tilde{a} = 0,$$

which is yet to be solved.

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For $p \geq 2$ and $q = 0$, the function $\Pi_{0,0}\left(P^{p,0}(Z)\left(\tilde{a}^{p,0}P^{p,0}\right)\right) \in Z_{r,R}$, by evaluating its λ-twisted spherical mean at $z = 0$, we can infer that $\tilde{a}$ satisfies

$$\sum_{|\alpha|+l=p} c_{\alpha,l}\rho^{2l}D^{\alpha}\tilde{a} = 0.$$

By similar argument for $p = 0$ and $q \geq 1$, we get

$$\sum_{|\beta|+k=q} d_{\beta,k}\rho^{2k}\bar{D}^{\beta}\tilde{a} = 0.$$

In general, while $p, q \geq 1$, we conclude that

$$\left(\sum_{|\beta|+k=q} d_{\beta,k}\rho^{2k}\bar{D}^{\beta}\right)\left(\sum_{|\alpha|+l=p} c_{\alpha,l}\rho^{2l}D^{\alpha}\right)\tilde{a} = 0.$$

However, if $P(z)$ is of the form $z_{l_1}^{p}\,\bar{z}_{l_2}^{q}$, then we can express $\tilde{a}$ explicitly as in (2.2.2). For showing this, first consider the case $q = 0$ and $p \geq 1$. Since $\tilde{a}\,z_{l_1}^{p} \in Z_{r,R}^{*}$, it follows that $\Pi_{0,0}Z_{l_1}^{p}(\tilde{a}P) \in Z_{r,R}$. Thus, evaluating λ-twisted spherical mean of $\Pi_{0,0}Z_{l_1}^{p}(\tilde{a}P)$ at $z = 0$, we get

$$\prod_{i=1}^{p}\left(\gamma_{p-(i-1),0}\ D_{l_1} + 2\right)\tilde{a} = 0.$$

This, in turn, implies that

$$\tilde{a}(\rho) = \sum_{i=1}^{p} A_i e^{-\frac{\nu_{l_1}}{4}\rho^2}\rho^{-2(n+p-i)}.$$

Similarly, for $p = 0$ and $q \geq 1$, considering the operator $\bar{Z}_{l_2}^{\lambda}$, we can derive that

$$\tilde{a}(\rho) = \sum_{k=1}^{q} B_k e^{\frac{\bar{\nu}_{l_2}}{4}\rho^2}\rho^{-2(n+q-k)}.$$

If $p, q \geq 1$, by evaluating λ-twisted spherical mean of $\Pi_{0,0}\bar{Z}_{l_2}\Pi_{0,q}Z_{l_1}^{p}(\tilde{a}P)$ at $z = 0$, we obtain

$$\prod_{k=1}^{q}\left(\gamma_{p,q+(k-1)}\ \bar{D}_{l_2}+2\right)\prod_{i=1}^{p}\left(\gamma_{p-(i-1),q}\ D_{l_1}+2\right)\tilde{a}=0.$$

Hence, a solution to the above equation can be expressed as

$$\tilde{a}(\rho)=\sum_{i=1}^{p} A_i\ e^{-\frac{\nu_{l_1}}{4}\rho^2}\rho^{-2(n+p+q-i)}+\sum_{k=1}^{q} B_k\ e^{\frac{\nu_{l_2}}{4}\rho^2}\rho^{-2(n+p+q-k))}.$$

This completes the proof. □

Remark 2.2.2. *In the definition of $Z^*_{r,R}$ we have assumed that, for all $p,q\in\mathbb{Z}_+$ and $1\leq j\leq d_{p,q}$,*

$$\Pi_{0,0}\left(P_j^{0,q}(Z)\left(\Pi_{0,q}\left(P_j^{p,0}(Z)\left(\tilde{a}_j^{p,q}P_j^{p,q}\right)\right)\right)\right)\times_\lambda \mu_s(z)=0 \tag{2.2.5}$$

for all $z\in\mathbb{C}^n$ and $s>0$ with $S_s(z)\subseteq B_{r,R}$ and $B_r(0)\subseteq B_s(z)$. However, for a proof of Theorem 2.2.1, it is enough to assume that (2.2.5) holds for $z=0$, whenever $r<s<R$. Consequently, sufficient part of Theorem 2.2.1, at $z=0$, is obviously true.

Further, as compared to the Heisenberg group, it would be a reasonable question to consider $e^{\frac{c}{4}|z|^2}|z|^{-2(n+p+q-i)}P(z)$ to be in $Z_{r,\infty}$ for the appropriate choice of c and i, where $P\in H_{p,q}$. In general, the matrix $\sum_{j=1}^{m}\lambda_jU^{(j)}$ arises from the symplectic form has distinct eigenvalues make it difficult to find out the constant c. However, in the case of H-type groups, all the eigenvalues are identical. We have such a result in Section 2.3, Theorem 2.3.1.

2.2.2 Sets of injectivity and support theorem

In this subsection, we simplify the λ-twisted spherical mean on the Métivier groups to another mean, similar to the TSM on the Heisenberg group. This will help proving a support theorem for the λ-twisted spherical mean on Métivier groups for the type functions. Further, we prove that non-harmonic complex cones, which are aligned with one of the coordinate axes in $\mathbb{C}^n$, are sets of injectivity for the λ-twisted spherical mean on the Métivier groups.

For $\lambda \in \mathbb{R}^m_*$, the skew-symmetric matrix $V_\lambda = \sum_{j=1}^m \lambda_j U^{(j)}$ is non-singular (see [42]). Let $u_1 \pm iv_1, \ldots, u_n \pm iv_n$ be the eigenvectors of V_λ with corresponding eigenvalues $\pm i\mu_{\lambda,1}, \ldots, \pm i\mu_{\lambda,n}$, where $\mu_{\lambda,1} \geq \cdots \geq \mu_{\lambda,n} > 0$. Define

$$A_\lambda = (\sqrt{2}\, v_1, \ldots, \sqrt{2}\, v_n, \sqrt{2}\, u_1, \ldots, \sqrt{2}\, u_n).$$

Then A_λ is an orthogonal matrix and satisfies $V_\lambda A_\lambda = A_\lambda U_\lambda$, where

$$U_\lambda = \begin{pmatrix} 0_n & -J_\lambda \\ J_\lambda & 0_n \end{pmatrix} \tag{2.2.6}$$

with $J_\lambda = \mathrm{diag}(\mu_{\lambda,1}, \ldots, \mu_{\lambda,n})$ and 0_n is zero matrix of order n. Thus, in view of (2.2.6), we have

$$\sum_{j=1}^m \lambda_j \langle x, U^{(j)}\xi\rangle = \langle x, V_\lambda \xi\rangle = \langle A_\lambda^t x, U_\lambda A_\lambda^t \xi\rangle,$$

where $A_\lambda A_\lambda^t = I$. That is,

$$\sum_{j=1}^m \lambda_j \mathrm{Re}\left(z \cdot \overline{U^{(j)} w}\right) = \sum_{j=1}^n \mu_{\lambda,j}\ \mathrm{Im}\left((z_\lambda)_j \cdot (\bar{w}_\lambda)_j\right), \tag{2.2.7}$$

where z_λ and w_λ are complexification of $A_\lambda^t x$ and $A_\lambda^t \xi$, respectively.

Let $f \in L^1(\mathbb{C}^n)$, then define

$$f_\lambda(z) = f(\tilde{z}_\lambda), \tag{2.2.8}$$

where $z, \tilde{z}_\lambda \in \mathbb{C}^n$ be the complexification of $x, A_\lambda x \in \mathbb{R}^{2n}$, respectively. The following lemma would simplify the λ-twisted spherical mean defined by (2.1.5) on the Métivier groups.

Lemma 2.2.3. *Let $f \in L^1(\mathbb{C}^n)$ and f_λ be as in (2.2.8). Then $f \times_\lambda \mu_s(\tilde{z}_\lambda) = f_\lambda \tilde{\times}_\lambda \mu_s(z)$, where*

$$f_\lambda \tilde{\times}_\lambda \mu_s(z) = \int_{|w|=s} f_\lambda(z-w)\, e^{\frac{i}{2}\sum_{j=1}^{n} \mu_{\lambda,j}\; Im(z_j \cdot \bar{w}_j)}\, d\mu_s(w). \tag{2.2.9}$$

Proof. In view of (2.1.6), we can write

$$\begin{aligned} f \times_\lambda \mu_s(A_\lambda x) &= \int_{|\xi|=s} f(A_\lambda x - \xi)\; e^{\frac{i}{2}\langle A_\lambda x, A_\lambda U_\lambda A_\lambda^t \xi\rangle}\; d\mu_s(\xi) \\ &= \int_{|\xi|=s} f_\lambda(x - A_\lambda^t \xi)\; e^{\frac{i}{2}\langle x, U_\lambda A_\lambda^t \xi\rangle}\; d\mu_s(\xi) \\ &= \int_{|\xi|=s} f_\lambda(x - \xi)\; e^{\frac{i}{2}\langle x, U_\lambda \xi\rangle}\; d\mu_s(\xi) \\ &= \int_{|w|=s} f_\lambda(z - w)\; e^{\frac{i}{2}\sum_{j=1}^{n} \mu_{\lambda,j}\; \mathrm{Im}(z_j \cdot \bar{w}_j)}\; d\mu_s(w) \\ &= f_\lambda \tilde{\times}_\lambda \mu_s(z). \end{aligned}$$

□

To deal with the modified λ-twisted spherical mean $f_\lambda \tilde{\times}_\lambda \mu_s$, defined in (2.2.9), it is required to study the function f_λ. In particular, we need to find out those polynomials P such that $P_\lambda \in H_{p,q}$ for some p, q. Let $P_\lambda \in H_{p,q}$. By identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we get $P_\lambda \in H_l$, where $l = p + q$, and H_l is the space of all homogeneous harmonic polynomials of degree l on $\mathbb{R}^{2n}$. Since $P(x) = P_\lambda(A_\lambda^t x)$ and the Laplacian is rotation invariant, we get $P \in H_l$, and hence

$$P \in \bigoplus_{p'+q'=l} H_{p',q'}.$$

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With the above observation, we define

$$H_{p,q}^{\lambda} = \{P \in \bigoplus_{p'+q'=p+q} H_{p',q'} : P_{\lambda} \in H_{p,q}\}. \tag{2.2.10}$$

Next, we prove a similar result to the support theorem for the Métivier groups. Consider the following left-invariant differential operators for the λ-twisted spherical mean (2.2.9),

$$\tilde{Z}_j^{\lambda} = \frac{\partial}{\partial z_j} - \frac{\mu_{\lambda,j}}{4}\bar{z}_j \text{ and } \tilde{Z}_j^{*^{\lambda}} = \frac{\partial}{\partial \bar{z}_j} + \frac{\mu_{\lambda,j}}{4} z_j, \;\; j = 1, 2, ..., n.$$

Since $P_{\lambda} \in H_{p,q}$, define $P_{\lambda}^{p,q}(\tilde{Z})$ as in (2.2.1), replacing Z by $\tilde{Z}$.

Theorem 2.2.4. *Let $f = \tilde{a}P$, where $P \in H_{p,q}^{\lambda}$, be a smooth function on $\mathbb{C}^n$ and $|z|^k e^{\frac{\mu_{\lambda,1}}{4}|z|^2} f(z)$ is bounded for each $k \in \mathbb{Z}_+$. Then $\Pi_{0,0}\left(P_{\lambda}^{p,q}(\tilde{Z}) f_{\lambda}\right) \tilde{\times}_{\lambda} \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > r + |z|$ if and only if f is supported in $|z| \leq r$.*

Proof. If $f = \tilde{a}P$, then $f_{\lambda} = \tilde{a}P_{\lambda}$. It is clear that for $p = q = 0$, $\tilde{a} = 0$. Let $p \geq 1$, then applying $\tilde{Z}_j^{\lambda}$ to f_{λ}, we have

$$\tilde{Z}_j^{\lambda} f_{\lambda} = \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{\mu_{\lambda,j}}{2}\right)\tilde{a}\bar{z}_j P_{\lambda} + \tilde{a}\frac{\partial P_{\lambda}}{\partial z_j}.$$

Since $P_{\lambda} \in H_{p,q}$, substituting the value of $\bar{z}_j P_{\lambda}$ from corollary 2.1.4, we have

$$\tilde{Z}_j^{\lambda} f_{\lambda} = \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{\mu_{\lambda,j}}{2}\right)\tilde{a}P_{\lambda,0} + \left[\frac{\gamma_{p,q}}{2}\left(\rho\frac{\partial}{\partial \rho} - \rho^2\frac{\mu_{\lambda,j}}{2}\right) + 1\right]\tilde{a}\frac{\partial P_{\lambda}}{\partial z_j},$$

where $\gamma_{p,q} = \frac{1}{(n+p+q-1)}$.

Consider $q = 0$ and $p = 1$. Then there exists a j_o such that $\dfrac{\partial P_{\lambda}}{\partial z_{j_o}} \neq 0$. Thus, from the given condition that

$$\Pi_{0,0}(\tilde{Z}_{j_o}^{\lambda} f_{\lambda}) \tilde{\times}_{\lambda} \mu_s(0) = 0$$

for all $s > r$, we arrived at

$$\left[\frac{1}{2n}\left(\rho\frac{\partial}{\partial\rho} - \rho^2\frac{\mu_{\lambda,j_o}}{2}\right) + 1\right]\tilde{a} = 0,$$

whenever $\rho > r$. This leads to a solution

$$\tilde{a}(\rho) = A_1 e^{-\frac{\mu_{\lambda,j_o}}{4}\rho^2}\rho^{-2n}.$$

By using an induction argument, for $q = 0$ and $p \geq 1$, from

$$\Pi_{0,0}\left(P_\lambda^{p,0}(\tilde{Z})f_\lambda\right)\tilde{\times}_\lambda\,\mu_s(0) = 0$$

for all $s > r$, it follows that

$$\prod_{i=1}^{p}\left\{\frac{1}{2(n+p-i)}\left(\rho\frac{\partial}{\partial\rho} - \rho^2\frac{\mu_{\lambda,j_i}}{2}\right) + 1\right\}\tilde{a} = 0.$$

Solving the above equation, we get

$$\tilde{a}_{p,0}(\rho) = \sum_{i=1}^{p} A_i e^{-\frac{c_i}{4}\rho^2}\rho^{-2(p+n-i)},$$

where $c_i \in \{\mu_{\lambda,j} : 1 \leq j \leq n\}$. Similar conclusion holds true for $p = 0, q \geq 1$.

In general, for $p, q \geq 1$, $\tilde{a}$ satisfies the ODE

$$\prod_{k=1}^{q}\left\{\frac{\gamma_{p,q+1-k}}{2}\left(\rho\frac{\partial}{\partial\rho} + \rho^2\frac{d_k}{2}\right) + 1\right\}\prod_{i=1}^{p}\left\{\frac{\gamma_{p+1-i,q}}{2}\left(\rho\frac{\partial}{\partial\rho} - \rho^2\frac{c_i}{2}\right) + 1\right\}\tilde{a} = 0,$$

and be expressed as

$$\tilde{a}_{p,q}(\rho) = \sum_{i=1}^{p} A_i e^{-\frac{c_i}{4}\rho^2}\rho^{-2(p+q+n-i)} + \sum_{k=1}^{q} B_k e^{\frac{d_k}{4}\rho^2}\rho^{-2(p+q+n-k)} \tag{2.2.11}$$

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for all $\rho > r$, where $c_i, d_k \in \{\mu_{\lambda,j} : 1 \leq j \leq n\}$ and A_i, B_k are constants. Since $\mu_{\lambda,1} \geq \mu_{\lambda,j} > 0$ for all j, by the given growth conditions, we infer that $f_\lambda(z) = 0$ for all $|z| > r$. Thus, we conclude that f is supported in $|z| \leq r$. $\square$

A set $K \subset \mathbb{C}^n$ $(n \geq 2)$, closed under complex scaling, is known as a complex cone. Further, a complex cone that does not intersect the zero set of any bi-graded homogeneous harmonic polynomial is called *non-harmonic.* The zero set of the polynomial $H(z) = az_1\bar{z}_2 + |z|^2$, where $a \neq 0$ and $z \in \mathbb{C}^n$ is a non-harmonic complex cone, (see [61]).

Let $z \in \mathbb{C}^n$ be the complexification of $x \in \mathbb{R}^{2n}$, and $\tilde{z}_\lambda$ be the complexification of $A_\lambda x$. For a complex cone K, define $K_\lambda = \{z \in \mathbb{C}^n : \tilde{z}_\lambda \in K\}$. Then K_λ is also a complex cone, and K is non-harmonic if and only if K_λ is non-harmonic.

Theorem 2.2.5. *Suppose K is a non-harmonic complex cone such that K_λ is aligned with one of the coordinate axes in $\mathbb{C}^n$ $(n \geq 2)$. Let f be a continuous function on $\mathbb{C}^n$ such that $f \times_\lambda \mu_r(z) = 0$, for all $r > 0$ and $z \in K$. Then $f = 0$.*

Proof. In view of Lemma 2.2.3, $f \times_\lambda \mu_r = 0$ on K implies $f_\lambda \tilde{\times}_\lambda \mu_r = 0$ on K_λ. By the hypothesis, without loss of generality, we can assume $z = (z_1, 0, ..., 0) \in K_\lambda$ for all $z_1 \in \mathbb{C}$. Thus,

$$\int_{|w|\leq r} f_\lambda(z+w)e^{-\frac{i}{2}\sum_{j=1}^{n}\mu_{\lambda,j}\ \mathrm{Im}(z_j\cdot\bar{w}_j)}dw = \int_0^r f_\lambda \tilde{\times}_\lambda\ \mu_s(z)s^{2n-1}ds = 0$$

for all $r > 0$ and $z \in K_\lambda$. Applying $2\partial_{z_1}$ to the above equation, we get

$$\int_{|w|\leq r} \frac{\partial}{\partial w_1}\Big(f_\lambda(z+w)e^{-\frac{i}{2}\sum_{j=1}^{n}\mu_{\lambda,j}\ \mathrm{Im}(z_j\cdot\bar{w}_j)}\Big)dw$$
$$- \frac{\mu_{\lambda,1}}{2}\int_{|w|\leq r} \bar{w}_1 f_\lambda(z+w)e^{-\frac{i}{2}\sum_{j=1}^{n}\mu_{\lambda,j}\ \mathrm{Im}(z_j\cdot\bar{w}_j)}ds = 0.$$

It follows by an application of Green's theorem that

$$\int_{|w|=r} \frac{\bar{w}_1}{r} \left(f_\lambda(z+w) e^{-\frac{i}{2}\sum_{j=1}^{n} \mu_{\lambda,j} \operatorname{Im}(z_j \cdot \bar{w}_j)} \right) dw$$
$$= \frac{\mu_{\lambda,1}}{2} \int_{|w|\leq r} \bar{w}_1 f_\lambda(z+w) e^{-\frac{i}{2}\sum_{j=1}^{n} \mu_{\lambda,j} \operatorname{Im}(z_j \cdot \bar{w}_j)} dw.$$

Let $F(t) = t^{2n-1} g \tilde{\times}_\lambda \mu_t(z)$, where $g(z) = \bar{z}_1 f_\lambda(z)$. Then we have

$$\frac{F(r)}{r} = \frac{\mu_{\lambda,1}}{2} \int_0^r F(s) ds. \tag{2.2.12}$$

It is easy to see that (2.2.12) satisfies the ODE

$$F^{'}(r) = \left(\frac{\mu_{\lambda,1} r}{2} + \frac{1}{r} \right) F(r)$$

with the general solution

$$F(r) = \frac{c(z)}{r} e^{\frac{\mu_{\lambda,1} r^2}{4}}.$$

That is,

$$r^{2n-2} g \tilde{\times}_\lambda \mu_r(z) = c(z) e^{\frac{\mu_{\lambda,1} r^2}{4}}.$$

Letting r tends to 0, we get $c(z) = 0$. Hence

$$\bar{z}_1 f_\lambda \tilde{\times}_\lambda \mu_r(z) = 0,$$

for all $r > 0$ and $z \in K_\lambda$. By replicating the above procedure, we get

$$(P f_\lambda) \tilde{\times}_\lambda \mu_r(z) = 0$$

for arbitrary polynomial $P(z_1, \bar{z}_1)$. By a similar argument as in ([61], Theorem 3.1), we can conclude that $f_\lambda = 0$ and hence $f = 0$. $\square$

Remark 2.2.6. *If we consider H-type groups instead of the general Métivier groups, then the restriction on the cone to align with one of the coordinates axes could be relaxed.*

2.3 Some results on H-type groups

In this section, we see that the λ-twisted spherical mean on the H-type groups can be related to the $|\lambda|$-twisted spherical mean on the Heisenberg group. Although the λ-twisted spherical mean on H-type groups is not $U(n)$-invariant, we can prove sufficient condition for a function to be in $Z_{r,\infty}$, and an analogue of Helgason's support theorem together with Heche-Bochner identity for the H-type groups. Further, we prove that the boundary of bounded domains are sets of injectivity for λ-twisted spherical mean on the H-type groups.

We know that for the H-type groups, $\mu_{\lambda,j} = |\lambda|$ for all j, due to the fact that $\sum_{j=1}^{m} \lambda_j U^{(j)} = |\lambda| V$. Thus (2.2.7) becomes

$$\sum_{j=1}^{m} \lambda_j \mathrm{Re}\,(z \cdot \overline{U^{(j)} w}) = |\lambda|\ \mathrm{Im}\,(z_\lambda \cdot \bar{w}_\lambda)$$

and from Lemma 2.2.3, we have

$$f \times_\lambda \mu_s(z) = f_\lambda \tilde{\times}_\lambda\, \mu_s(z_\lambda) = f_\lambda \times_{|\lambda|} \mu_s(z_\lambda), \tag{2.3.1}$$

where $f_\lambda \times_{|\lambda|} \mu_s$ denotes the $|\lambda|$-twisted spherical mean on the Heisenberg group. Similarly, the λ-twisted convolution on the H-type groups can be related to the twisted convolution on the Heisenberg group by

$$f \times_\lambda g(z) = f_\lambda \times_{|\lambda|} g_\lambda(z_\lambda). \tag{2.3.2}$$

Next, we present the sufficient condition for functions to be in $Z_{r,\infty}$, which was mentioned in Remark 2.2.2.

Theorem 2.3.1. *Let* $P \in H^{\lambda}_{p,q}$ *and*

$$h(z) = \frac{e^{\frac{|\lambda|}{4}|z|^2} P(z)}{|z|^{2(n+p+q-i)}},$$

where $1 \leq i \leq p$ *and* $H^{\lambda}_{p,q}$ *is defined in (2.2.10). Then* $h \in Z_{r,\infty}$*.*

Proof. To prove the result, it needs to verify that $h \times_{\lambda} \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > |z| + r$. From (2.3.1), it is enough to show that $h_{\lambda} \times_{|\lambda|} \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > |z| + r$.

Let $\eta = n + p + q$ and consider

$$h_{\lambda} \times_{|\lambda|} \mu_s(z) = \int_{|w|=s} \frac{e^{\frac{|\lambda|}{4}|z+w|^2} P_{\lambda}(z+w)}{|z+w|^{2(\eta-i)}} e^{-\frac{i}{2}|\lambda| \operatorname{Im}(z\cdot\bar{w})} d\mu_s(w).$$

Simplifying the exponential terms, it is enough to show the following integral is zero

$$\int_{|w|=s} \frac{e^{\frac{|\lambda|}{2}\bar{z}\cdot w} P_{\lambda}(z+w)}{|z+w|^{2(\eta-i)}} d\mu_s(w).$$

Again, if we expand the exponential term, the above integral will reduce to

$$\int_{|w|=s} \frac{w^{\alpha} P_{\lambda}(z+w)}{|z+w|^{2(\eta-i)}} d\mu_s(w).$$

Hence we arrived at the same Euclidean situation, proved in ([54], Theorem 3.3). Thus, it follows that $h_{\lambda} \times_{|\lambda|} \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > |z| + r$. □

Next, we shall prove a support theorem for the λ- twisted spherical mean on the H-type groups, for which we need to recall the following support theorem for the TSM on the Heisenberg group.

Theorem 2.3.2. [54] *Let g be a continuous function on $\mathbb{C}^n$ such that for each $k \in \mathbb{Z}_+$, $|z|^k e^{\frac{|\lambda|}{4}|z|^2} g(z)$ is bounded. Then g is supported in $|z| \leq r$ if and only if $g \times_{|\lambda|} \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > r + |z|$.*

Using (2.3.1), we can prove the following support theorem for the H-type groups.

Theorem 2.3.3. *Suppose f is a continuous function on $\mathbb{C}^n$ such that for each $k \in \mathbb{Z}_+$, $|z|^k e^{\frac{|\lambda|}{4}|z|^2} f(z)$ is bounded. Then f is supported in $|z| \leq r$ if and only if $f \times_\lambda \mu_s(z) = 0$ for all $z \in \mathbb{C}^n$ and $s > r + |z|$.*

Proof. We know that $f_\lambda(z_\lambda) = f(z)$, where z and z_λ are the complexification of x and $A_\lambda^t x$, respectively. Since $|z|^k e^{\frac{|\lambda|}{4}|z|^2} f(z)$ is bounded, it follows from $|z| = |z_\lambda|$ that $|z|^k e^{\frac{|\lambda|}{4}|z|^2} f_\lambda(z)$ is bounded. Now, f is supported in $|z| \leq r$ if and only if f_λ is supported in $|z| \leq r$. Hence, we get the desired result from (2.3.1) and Theorem 2.3.2. □

Now, we state the Heche-Bochner identity for the twisted convolution on the Heisenberg group. Recall the Laguerre functions on $\mathbb{C}^n$,

$$\varphi_k^{n-1}(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2}.$$

For $\lambda \in \mathbb{R}_*^m$, define $\varphi_{k,\lambda}^{n-1}(z) = \varphi_k^{n-1}(|\lambda|^{1/2} z)$.

Theorem 2.3.4. [69] *Let $f \in L^1(\mathbb{C}^n)$ be of the form $f = Pg$, where g is radial and $P \in H_{p,q}$. Then for $\lambda > 0$,*

$$f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) = \begin{cases} (2\pi)^{-n} \lambda^{p+q} P(z) g \times_\lambda \varphi_{k-p,\lambda}^{n+p+q-1}(z), & \text{if } k \geq p \\ 0, & \text{otherwise} \end{cases}$$

and for $\lambda < 0$,

$$f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) = \begin{cases} (2\pi)^{-n}|\lambda|^{p+q} P(z) g \times_\lambda \varphi_{k-q,\lambda}^{n+p+q-1}(z), & \text{if } k \geq q \\ 0, & \text{otherwise,} \end{cases}$$

where convolution on the right-hand side is on $\mathbb{C}^{n+p+q}$.

An analogue of the above Heche-Bochner identity for H-type groups can be stated as follows.

Theorem 2.3.5. *Let* $f \in L^1(\mathbb{C}^n)$ *be of form* $f = gP$ *where* g *is radial and* $P \in H_{p,q}^\lambda$, *where* $H_{p,q}^\lambda$ *defined in (2.2.10). Then for* $\lambda \in \mathbb{R}_*^m$,

$$f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) = \begin{cases} (2\pi)^{-n}|\lambda|^{p+q} P(z) g \times_\lambda \varphi_{k-p,\lambda}^{n+p+q-1}(z'), & \text{if } k \geq p \\ 0, & \text{otherwise,} \end{cases}$$

where $z' \in \mathbb{C}^{n+p+q}$ *be such that* $|z| = |z'|$ *and convolution on the right is on* $\mathbb{C}^{n+p+q}$.

Proof. Since $\varphi_{k,\lambda}^{n-1}$ is radial, by (2.3.2) and Theorem 2.3.4 we get

$$\begin{aligned} f \times_\lambda \varphi_{k,\lambda}^{n-1}(z) &= f_\lambda \times_{|\lambda|} \varphi_{k,\lambda}^{n-1}(z_\lambda) \\ &= (2\pi)^{-n}|\lambda|^{p+q} P_\lambda(z_\lambda) g \times_{|\lambda|} \varphi_{k-p,\lambda}^{n+p+q-1}(z'_\lambda) \\ &= (2\pi)^{-n}|\lambda|^{p+q} P(z) g \times_\lambda \varphi_{k-p,\lambda}^{n+p+q-1}(z'), \end{aligned}$$

where $z'_\lambda, z' \in \mathbb{C}^{n+p+q}$ such that $|z_\lambda| = |z'_\lambda| = |z'|$. □

Next, we deduce an injectivity result for the H-type groups, which is known for the Heisenberg group.

Theorem 2.3.6. [6] *Let $\partial\Omega$ be the boundary of a bounded domain Ω in $\mathbb{C}^n$. Let f be such that $f(z)e^{\left(\frac{1}{4}+\epsilon\right)|z|^2} \in L^p\left(\mathbb{C}^n\right)$, for some $\epsilon > 0$ and $1 \leq p \leq \infty$. Suppose that $f \times \mu_s(z) = 0$ for all $z \in \partial\Omega$ and $s > 0$. Then $f = 0$.*

Now, we state an analogue of the above result for the H-type groups.

Theorem 2.3.7. *Let $\partial\Omega$ be the boundary of a bounded domain Ω in $\mathbb{C}^n$. Let f be such that $f(z)e^{\left(\frac{1}{4}+\epsilon\right)|\lambda||z|^2} \in L^p\left(\mathbb{C}^n\right)$, for some $\epsilon > 0$ and $1 \leq p \leq \infty$. Suppose that $f \times_\lambda \mu_s(z) = 0$ for all $z \in \partial\Omega$ and $s > 0$. Then $f = 0$.*

Proof. From (2.3.1), we have

$$f \times_\lambda \mu_s(z) = f_\lambda \times_{|\lambda|} \mu_s(z_\lambda).$$

Define $\Omega' = \{z_\lambda \in \mathbb{C}^n : z \in \Omega\}$. Since the boundary of bounded domain Ω' is $\partial\Omega' = \{z_\lambda : \ z \in \partial\Omega\}$, by Theorem 2.3.6, we can conclude that $f_\lambda = 0$ and hence $f = 0$. □

Concluding remark: We know that in the case of Métivier groups, the symplectic bilinear form

$$\sum_{j=1}^{m} \lambda_j \mathrm{Re}\,(z \cdot \overline{U^{(j)}w})$$

cannot be made $U(n)$-invariant, due to the fact that all of $\mu_{\lambda,j}$ need not be identical. Hence we require more assumptions on the functions to prove similar results as to the Heisenberg group. However, in the case of the H-type groups, all $\mu_{\lambda,j}$ are identical, so we do not require further assumptions to prove the results for H-type groups.

Chapter 3

Injectivity of spherical mean on Métivier groups

In this chapter, we perceive that there is a Lie group with real $3n$-dimensional step two nilpotent Lie algebra, whose twisted spherical mean is similar to $f_\lambda \tilde{\times}_\lambda \mu_r$, is defined in Lemma 2.2.3. We look for eigenfunctions of sub-Laplacian on this particular group, and via that, obtain the spectral decomposition for L^2-functions. We derive some auxiliary results related to the special Hermite functions. Then we procure certain classes of functions for which spherical mean operator is injective. Further, we obtain a version of the two-radii theorem.

3.1 Twisted spherical mean and spectral decomposition

Consider the group $\tilde{G} \simeq \mathbb{R}^{2n} \times \mathbb{R}^n$ as $\{(x, y, t) : x, y, t \in \mathbb{R}^n\}$ equipped with the group law

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(x'y - y'x)\right),$$

where the notation $xy := (x_1y_1, \cdots, x_ny_n)$ for $x, y \in \mathbb{R}^n$ denotes the coordinatewise multiplication. Note that group $\tilde{G}$ is not a Métivier group but the direct product of n copies of one dimensional Heisenberg group $\mathbb{H}^1$ with a basis of left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t_j}, \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t_j} \quad \text{and } T_j = \frac{\partial}{\partial t_j}, \tag{3.1.1}$$

where $j = 1, \ldots, n$. The sub-Laplacian on $\tilde{G}$ is

$$\mathcal{L} = -\sum_{j=1}^{n} \left(X_j^2 + Y_j^2 \right).$$

For each $\lambda' \in \mathbb{R}^n_*$, we can see that the operator $\pi_{\lambda'}(x, y, t)$ acting on $L^2(\mathbb{R}^n)$ by

$$\pi_{\lambda'}(x, y, t)\phi(\xi) = e^{i \sum_{j=1}^{n} {\lambda'}_j t_j + i \sum_{j=1}^{n} {\lambda'}_j (x_j \xi_j + \frac{1}{2} x_j y_j)} \phi(\xi + y) \tag{3.1.2}$$

are all possible irreducible unitary representations of $\tilde{G}$, where $\phi \in L^2(\mathbb{R}^n)$. If $\pi_{\lambda'}(z) = \pi_{\lambda'}(z, 0)$, then $\pi_{\lambda'}(z, t) = e^{i\lambda' t} \pi_{\lambda'}(z)$. Identifying $\tilde{G}$ with $\mathbb{C}^n \times \mathbb{R}^n$, let $L_{\lambda'}$ be the operator defined by

$$\mathcal{L}\left(e^{i\lambda' \cdot t} f(z) \right) = e^{i\lambda' \cdot t} L_{\lambda'} f(z),$$

where $z = x + iy$. Then $L_{\lambda'}$ can precisely be expressed as

$$L_{\lambda'} = -\Delta_z + \frac{1}{4} \sum_{j=1}^{n} {{\lambda'}_j}^2 |z_j|^2 + i\mathcal{N}_{\lambda'}, \quad \text{where } \mathcal{N}_{\lambda'} = \sum_{j=1}^{n} {\lambda'}_j \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right). \tag{3.1.3}$$

Let $f \in L^1(\tilde{G})$ and

$$f^{\lambda'}(z) = \int_{\mathbb{R}^n} f(z, t) e^{i\lambda' \cdot t} dt$$

be the inverse Fourier transform of f in the t-variable. Then, for this particular group

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$\tilde{G}$, the λ'-twisted spherical mean can be explicitly calculated as

$$f^{\lambda'} \times_{\lambda'} \mu_r(z) = \int_{|w|=r} f^{\lambda'}(z-w)e^{\frac{i}{2}\sum_{j=1}^{n}\lambda'_j \text{ Im}(z_j\cdot\bar{w}_j)}d\mu_r(w), \tag{3.1.4}$$

where μ_r is the normalized surface measure on the set $\{(z,0): \ |z|=r\} \subset \tilde{G}$. Similarly, if $f, g \in L^1(\tilde{G})$, then we can also define the λ'-twisted convolution as

$$f^{\lambda'} \times_{\lambda'} g^{\lambda'}(z) = \int_{\mathbb{C}^n} f^{\lambda'}(z-w)g^{\lambda'}(w)e^{\frac{i}{2}\sum_{j=1}^{n}\lambda'_j Im(z_j.\bar{w}_j)}dw. \tag{3.1.5}$$

Remark 3.1.1. *For any $\lambda \in \mathbb{R}^m_*, m \geq 2$, the modified λ-twisted spherical mean (2.2.9) coincides with the λ'-twisted spherical mean (3.1.4), where $\lambda' \in \mathbb{R}^n_+$ and each coordinate of λ' can be identified with the imaginary part of an eigenvalue of V_λ. Therefore, studying the injectivity of spherical mean on an arbitrary Métivier group G is enough to consider the spherical mean on $\tilde{G}$.*

For $\alpha \in \mathbb{Z}^n_+$, let $\Phi_\alpha(x) = \Pi_{j=1}^n h_{\alpha_j}(x_j)$, where h_{α_j} are normalized Hermite functions on $\mathbb{R}$. Then Φ_α is an eigenfunction of Hermite operator $H = -\Delta + |x|^2$ with eigenvalue $(2|\alpha|+n)$. For more details, see [69]. Moreover, for $\lambda' = (\lambda'_1, \ldots, \lambda'_n) \in \mathbb{R}^n_*$, if we define

$$\Psi^{\lambda'}_\alpha(x) = \prod_{j=1}^{n} |\lambda'_j|^{\frac{1}{4}} h_{\alpha_j}\left(\sqrt{|\lambda'_j|}x_j\right),$$

then $\Psi^{\lambda'}_\alpha$ are the eigenfunctions of the elliptic Hermite operator

$$H_{\lambda'} = -\Delta + \sum_{j=1}^{n}(\lambda'_j x_j)^2$$

with eigenvalues $\sum\limits_{j=1}^{n}(2\alpha_j+1)|\lambda'_j|$. Thus,

$$L_{\lambda'}\left(\pi_{\lambda'}(z)\Psi^{\lambda'}_\alpha, \Psi^{\lambda'}_\beta\right) = \sum_{j=1}^{n}(2\alpha_j+1)\,|\lambda'_j|\left(\pi_{\lambda'}(z)\Psi^{\lambda'}_\alpha, \Psi^{\lambda'}_\beta\right).$$

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For $\alpha, \beta \in \mathbb{Z}_+^n$, define the function as

$$\Psi_{\alpha\beta}^{\lambda'}(z) = \left(\prod_{j=1}^{n} \sqrt{\frac{|\lambda'_j|}{2\pi}}\right) \left(\pi_{\lambda'}(z)\Psi_\alpha^{\lambda'}, \Psi_\beta^{\lambda'}\right).$$

Then $\Psi_{\alpha\beta}^{\lambda'}$ are eigenfunctions of the operator

$$-\Delta_z + \frac{1}{4}\sum_{j=1}^{n} {\lambda'_j}^2 |z_j|^2.$$

The set $\{\Psi_{\alpha\beta}^{\lambda'} : \alpha, \beta \in \mathbb{Z}_+^n\}$ form a complete orthonormal set for $L^2(\mathbb{C}^n)$.

Next, we come up with some identities for $\Psi_{\alpha\beta}^{\lambda'}$, which can be derived by a suitable change of variables in the special Hermite function. Thus, for every $f \in L^2(\mathbb{C}^n)$, we have the expansion

$$f(z) = \sum_\alpha \sum_\beta \left(f, \Psi_{\alpha\beta}^{\lambda'}\right) \Psi_{\alpha\beta}^{\lambda'}(z).$$

Recall that the Laguerre function φ_k^{n-1} on $\mathbb{C}^n$ is given by

$$\varphi_k^{n-1}(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2},$$

where L_k^{n-1} are Laguerre polynomials of type $(n-1)$. For $\lambda' \in \mathbb{R}_+^n$, define $\vartheta_{k,\lambda'}^{n-1}(z) = \varphi_k^{n-1}(\sqrt{|\lambda'|}z)$, where the notation $\sqrt{|\lambda'|}z$ is fixed by

$$\sqrt{|\lambda'|}z := \left(\sqrt{|\lambda'_1|}z_1, \ldots, \sqrt{|\lambda'_n|}z_n\right). \tag{3.1.6}$$

As similar to the special Hermite function, $\Psi_{\alpha\alpha}^{\lambda'}$ can be expressed in terms of Laguerre functions as

$$\Psi_{\alpha\alpha}^{\lambda'}(z) = \left(\prod_{j=1}^{n} \frac{|\lambda'_j|}{2\pi}\right) \prod_{j=1}^{n} L_{\alpha_j}^0\left(\frac{1}{2}|\lambda'_j||z_j|^2\right) e^{-\frac{1}{4}|\lambda'_j||z_j|^2}. \tag{3.1.7}$$

Then we can derive the formula

$$\left(\prod_{j=1}^{n}\sqrt{|\lambda'_j|}\right)\sum_{|\alpha|=k}\Psi^{\lambda'}_{\alpha\alpha}(z)=(2\pi)^{-\frac{n}{2}}\vartheta^{n-1}_{k,\lambda'}(z). \tag{3.1.8}$$

Let $f\in L^2(\mathbb{C}^n)$, then in view of (3.1.8), and the completeness of $\Psi^{\lambda'}_{\alpha\beta}$'s in $L^2(\mathbb{C}^n)$, f will satisfy the identity

$$\sum_{|\alpha|=k}\sum_{\beta}\left(f,\Psi^{\lambda'}_{\alpha\beta}\right)\Psi^{\lambda'}_{\alpha\beta}(z)=\prod_{j=1}^{n}\frac{|\lambda'_j|}{2\pi}\int_{\mathbb{C}^n}f(w)\vartheta^{n-1}_{k,\lambda'}(z-w)e^{\frac{i}{2}\sum_{j=1}^{n}\lambda'_j Im(z_j.\bar{w}_j)}dw.$$

Since, the right-hand side is simply $\left(\prod_{j=1}^{n}\frac{|\lambda'_j|}{2\pi}\right)\vartheta^{n-1}_{k,\lambda'}\times_{\lambda'}f(z)$ and $\vartheta^{n-1}_{k,\lambda'}\times_{\lambda'}f(z)=f\times_{\lambda'}\vartheta^{n-1}_{k,\lambda'}(z)$, we have

$$f(z)=\left(\prod_{j=1}^{n}\frac{|\lambda'_j|}{2\pi}\right)\sum_{k=0}^{\infty}f\times_{\lambda'}\vartheta^{n-1}_{k,\lambda'}(z). \tag{3.1.9}$$

The following Proposition shows that the λ'-twisted spherical mean of $\vartheta^{n-1}_{k,\lambda'}$ will satisfy the following functional relation.

Proposition 3.1.2. *Denote $\vartheta^{n-1}_{k,\lambda'}(r)=\vartheta^{n-1}_{k,\lambda'}(w)$ for $|w|=r$. Then*

$$\vartheta^{n-1}_{k,\lambda'}\times_{\lambda'}\mu_r(z)=\left(\prod_{j=1}^{n}\frac{1}{\sqrt{\lambda'_j}}\right)\frac{k!(n-1)!}{(k+n-1)!}\vartheta^{n-1}_{k,\lambda'}(z)\vartheta^{n-1}_{k,\lambda'}(r). \tag{3.1.10}$$

Proof. From ([65], Theorem 2.1), it is known that the twisted spherical mean of φ^{n-1}_k with respect to the Heisenberg group can be written as

$$\int_{|w|=r}\varphi^{n-1}_k(z-w)e^{\frac{i}{2}Im(z\cdot\bar{w})}d\mu_r(w)=\frac{k!(n-1)!}{(k+n-1)!}\varphi^{n-1}_k(z)\varphi^{n-1}_k(r). \tag{3.1.11}$$

Now, by a change of variable, we can rewrite

$$\begin{aligned}\vartheta_{k,\lambda'}^{n-1} \times_{\lambda'} \mu_r(z) &= \int_{|w|=r} \vartheta_{k,\lambda'}^{n-1}(z-w)e^{\frac{i}{2}\sum_{j=1}^{n}\lambda'_j\ \mathrm{Im}(z_j\cdot\bar{w}_j)}d\mu_r(w)\\ &= \int_{|w|=r} \varphi_k^{n-1}(\sqrt{\lambda'}z-\sqrt{\lambda'}w)e^{\frac{i}{2}Im\left(\sqrt{\lambda'}z\cdot\overline{\sqrt{\lambda'}w}\right)}d\mu_r(w) \qquad (3.1.12)\\ &= \int_{|\ ^{-1}\!\sqrt{\lambda'}w|=r} \varphi_k^{n-1}(z'-w)e^{\frac{i}{2}Im(z'\cdot\bar{w})}d\mu_r(w),\end{aligned}$$

where $z' = \sqrt{\lambda'}z$. By a suitable change of variable in (3.1.11), we can write the above equation (3.1.12) as

$$\int_{|\ ^{-1}\!\sqrt{\lambda'}w|=r} \varphi_k^{n-1}(z'-w)e^{\frac{i}{2}Im(z'\cdot\bar{w})}d\mu_r(w) = \left(\prod_{j=1}^{n}\frac{1}{\sqrt{\lambda'_j}}\right)\frac{k!(n-1)!}{(k+n-1)!}\varphi_k^{n-1}(z')\varphi_k^{n-1}(\sqrt{\lambda'}w).$$

Hence the identity (3.1.10) is followed. □

Let $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. A function f on $\mathbb{C}^n$ is called m-homogeneous if it satisfies $f(e^{i\theta}z) = f(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) = e^{im\cdot\theta}f(z)$, where $\theta = (\theta_1, \ldots, \theta_n)$. For a function g on $\mathbb{C}^n$ define m-radialization $R_m g$ by

$$R_m g(z) = (2\pi)^{-n}\int_{[0,2\pi)^n} g\left(e^{i\theta}z\right)e^{-im\cdot\theta}d\theta. \qquad (3.1.13)$$

Then $R_m f$ is m-homogeneous and we have

$$f(z) = \sum_m R_m f(z)e^{im\cdot\theta}. \qquad (3.1.14)$$

The series in the right-hand side of (3.1.14) converges in the topology of Schwartz class function $\mathcal{S}(\mathbb{C}^n)$, see [67].

Since $\Psi_{\alpha\beta}^{\lambda'}$ is $(\beta-\alpha)$-homogeneous, we can see that

$$\left(f, \Psi_{\alpha\beta}^{\lambda'}\right) = \int_{\mathbb{C}^n} f(z)\overline{\Psi_{\alpha\beta}^{\lambda'}}(z)dz$$

is non-zero only when $\beta = \alpha + m$. Thus, if f is m-homogeneous, we can write

$$f \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'} = \left(\prod_{j=1}^{n} \frac{|\lambda'_j|}{2\pi}\right) \sum_{|\beta|=k} \left(f, \Psi^{\lambda'}_{\beta-m\,\beta}\right) \Psi^{\lambda'}_{\beta-m\,\beta}. \tag{3.1.15}$$

In [67], it has been proved that the special Hermite series of an m-homogeneous function converges in the topology of $\mathcal{S}(\mathbb{C}^n)$ for the Heisenberg group. By imitating the prove in the Métivier Group case, we have the following result.

Lemma 3.1.3. *If f is a Schwartz class function and m-homogeneous, then the series (3.1.9) of f converges in the topology of $\mathcal{S}(\mathbb{C}^n)$.*

3.2 Injectivity of Spherical mean

This section deals with the injectivity of spherical mean $f * \mu$, where $\mu \in X_P(G)$, the space of compactly supported rotation invariant probability measure with no mass at the centre of Métivier group G.

Proposition 3.2.1. *Let $1 \leq p_i \leq 2$ for $i = 1, 2$. Let $f \in C(G)$ be such that $f(z, \cdot) \in L^{p_1}(\mathbb{R}^m)$ and $f^\lambda \in L^{p_2}(\mathbb{C}^n)$ for a.e. $\lambda \in \mathbb{R}^m_*$. If f satisfies $f * \mu = 0$ for some $\mu \in X_P(G)$, then $f = 0$.*

Proof. For $\lambda \in \mathbb{R}^m_*$, let f^λ and μ^λ be the partial Fourier transform of f and μ in the t-variable, respectively. Then applying λ-twisted convolution, we get $f^\lambda \times_\lambda \mu^\lambda = 0$. Since $\mu \in X_P(G)$, by the Choquet's Theorem, we get

$$f^\lambda \times_\lambda \mu^\lambda = \int_E f^\lambda \times_\lambda \mu_r \, dM, \tag{3.2.1}$$

where $E = \{\mu_r\}$, μ_r is the normalized surface measure on the sphere of radius r centred at the origin in $\mathbb{C}^n$, and M is the measure on E. For more details, refer to [62]. From

Lemma 2.2.3, using the modified λ-twisted spherical mean, we can rewrite (3.2.1) as

$$f^\lambda \times_\lambda \mu^\lambda = \int_E (f^\lambda)_\lambda \; \tilde{\times}_\lambda \; \mu_r \, dM, \tag{3.2.2}$$

where $(f^\lambda)_\lambda(x) = f^\lambda(A_\lambda x)$ defined as in Lemma 2.2.3. For fixed λ, and considering Remark 3.1.1, we can write

$$(f^\lambda)_\lambda \; \tilde{\times}_\lambda \; \mu_r = (f^\lambda)_\lambda \times_{\lambda'} \mu_r \tag{3.2.3}$$

for some $\lambda' \in \mathbb{R}^n_+$. On the right-hand side, the λ'-twisted spherical mean is with respect to $\tilde{G}$ as defined in (3.1.4). By an appropriate approximation identity, we may assume that $(f^\lambda)_\lambda \in L^2(\mathbb{C}^n)$. Then applying the spectral decomposition (3.1.9), $(f^\lambda)_\lambda$ can be expressed in terms of $\vartheta^{n-1}_{k,\lambda'}$ as

$$(f^\lambda)_\lambda = \left(\prod_{j=1}^{n} \frac{\lambda'_j}{2\pi} \right) \sum_{k=0}^{\infty} (f^\lambda)_\lambda \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'}, \tag{3.2.4}$$

where the series converges in $L^2(\mathbb{C}^n)$. Now, it is enough to prove that each spectral projection $(f^\lambda)_\lambda \times_{\lambda'} \; \vartheta^{n-1}_{k,\lambda'} = 0$. From (3.2.2) and (3.2.4), we have

$$\sum_{k=0}^{\infty} \int_E (f^\lambda)_\lambda \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'} \times_{\lambda'} \mu_r \, dM = 0.$$

In view of Proposition 3.1.2, we get

$$\sum_{k=0}^{\infty} \mu^\lambda \left(\vartheta^{n-1}_{k,\lambda'} \right) (f^\lambda)_\lambda \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'} = 0,$$

where

$$\mu^\lambda \left(\vartheta^{n-1}_{k,\lambda'} \right) = \int_{\mathbb{C}^n} \vartheta^{n-1}_{k,\lambda'} \mu^\lambda = \int_E \vartheta^{n-1}_{k,\lambda'} dM.$$

Since for each $k \in \mathbb{Z}_+$, $\mu^\lambda \left(\vartheta^{n-1}_{k,\lambda'} \right)$ vanishes only for countable many values of λ. Hence

$(f^\lambda)_\lambda \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'} = 0$ implies $(f^\lambda)_\lambda \, \tilde{\times}_\lambda \, \mu_r = 0$ for a.e. λ and each k. Thus, $(f^\lambda)_\lambda = 0$ for a.e. λ, which concludes $f = 0$. □

In the following result, we relax the integrability condition of f^λ in the z-variable with tempered growth with the help of some approximation lemmas from Section 3.1.

Theorem 3.2.2. *Let f be a continuous function on G with $f(z, \cdot) \in L^p(\mathbb{R}^m)$, $1 \leq p \leq 2$ and f^λ has tempered growth in $\mathbb{C}^n$ for a.e. $\lambda \in \mathbb{R}^m_*$. If f satisfies $f * \mu = 0$ for some $\mu \in X_P(G)$, then $f = 0$.*

Proof. Since f is integrable in the second variable, applying λ-twisted convolution on $f * \mu = 0$, we get $f^\lambda \times_\lambda \mu^\lambda = 0$ for a.e. λ. Hence we claim $f^\lambda = 0$ for almost all λ. But tempered growth of f^λ reduces to show that

$$\int_{\mathbb{C}^n} f^\lambda(z) g(z) dz = 0$$

for every g in $\mathcal{S}(\mathbb{C}^n)$. Since g admits an m-radialization expansion, we can replace both g and f^λ with their m-radialization. Therefore, it is enough to consider

$$\int_{\mathbb{C}^n} R_m f^\lambda(z) g(z) dz = 0 \tag{3.2.5}$$

for all m-homogeneous $g \in \mathcal{S}(\mathbb{C}^n)$. If we fix λ, then there exists $\lambda' \in \mathbb{R}^n_+$ as in (3.2.3), and by Lemma 3.1.3, we can reciprocate g with $g \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'}$ in (3.2.5). Hence it is enough to examine

$$\int_{\mathbb{C}^n} R_m f^\lambda(z) g \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'}(z) dz = 0$$

for every k, which is equivalent to

$$\int_{\mathbb{C}^n} \vartheta^{n-1}_{k,\lambda'} \times_{\lambda'} R_m f^\lambda(z) g(z) dz = 0.$$

From (3.1.15), it is clear that $\vartheta^{n-1}_{k,\lambda'} \times_{\lambda'} R_m f^\lambda \in L^2(\mathbb{C}^n)$. And we also have

$$\vartheta^{n-1}_{k,\lambda'} \times_{\lambda'} R_m f^\lambda \times_\lambda \mu^\lambda = 0.$$

Using Proposition 3.2.1, we conclude that $\vartheta^{n-1}_{k,\lambda'} \times_{\lambda'} R_m f^\lambda = 0$. This proves the theorem.

□

In the previous results, we have considered $\mu \in X_P(G)$. In the following result, we replace μ with μ_r, the normalized surface measure on $\{z \in \mathbb{C}^n : |z| = r\}$, which requires more decay in the z-variable. Let denote J'_λ be a diagonal $n \times n$ matrix such that $(J'_\lambda)^2 = J_\lambda$, where J_λ is defined in (2.2.6).

Theorem 3.2.3. *Let f be a continuous function on G such that $f(z,\cdot) \in L^p(\mathbb{R}^m)$, $1 \leq p \leq 2$, and $f^\lambda(z)e^{\frac{1}{4}\left|J'_\lambda z_\lambda\right|^2}$ is in $L^q(\mathbb{C}^n), 1 \leq q \leq \infty$, for a.e. $\lambda \in \mathbb{R}^m_*$. If f satisfies $f * \mu_r = 0$ for some $r > 0$, then $f = 0$.*

Proof. Applying λ-twisted convolution on $f * \mu_r = 0$, we get $f^\lambda \times_\lambda \mu_r = 0$ for a.e. λ. For fix λ, there exists $\lambda' \in \mathbb{R}^n_+$ as in (3.2.3), and then using Lemma 2.2.3, it follows that

$$(f^\lambda)_\lambda \times_{\lambda'} \mu_r = 0. \tag{3.2.6}$$

Consider the λ'-twisted convolution of equation (3.2.6) and $\vartheta^{n-1}_{k,\lambda'}$. Then using Proposition 3.1.2, we get

$$\vartheta^{n-1}_{k,\lambda'}(r)\,(f^\lambda)_\lambda \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'} = 0$$

for all $k \in \mathbb{Z}_+$. Since the zero sets of Laguerre polynomials are disjoint, $\vartheta^{n-1}_{k,\lambda'}(r) \neq 0$ for all k except one, say $k = l$. That is, $(f^\lambda)_\lambda \times_{\lambda'} \vartheta^{n-1}_{k,\lambda'} = 0$ for all $k \neq l$. Hence we get $(f^\lambda)_\lambda = C(f^\lambda)_\lambda \times_{\lambda'} \vartheta^{n-1}_{l,\lambda'}$ for some non-zero constant C. Since $R_m(f^\lambda)_\lambda$ is m-

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homogeneous, in view of (3.1.15), we get

$$R_m(f^\lambda)_\lambda = C R_m(f^\lambda)_\lambda \times'_\lambda \vartheta^{n-1}_{l,\lambda'} = C\left(\prod_{j=1}^{n}\frac{|\lambda'_j|}{2\pi}\right)\sum_{|\beta|=l}\left((f^\lambda)_\lambda, \Psi^{\lambda'}_{\beta-m\,\beta}\right)\Psi^{\lambda'}_{\beta-m\,\beta}.$$

Replacing $R_m(f^\lambda)_\lambda$ with $R_m((f^\lambda)_\lambda\, e^{-\frac{1}{4}|\lambda' z|^2})$, we have

$$R_m((f^\lambda)_\lambda\, e^{\frac{1}{4}|\lambda' z|^2}) = C\left(\prod_{j=1}^{n}\frac{|\lambda'_j|}{2\pi}\right)\sum_{|\beta|=l}\left((f^\lambda)_\lambda, \Psi^{\lambda'}_{\beta-m\,\beta}\right)\Psi^{\lambda'}_{\beta-m\,\beta}\, e^{\frac{1}{4}|\lambda' z|^2}.$$

By the hypothesis $f^\lambda(z)e^{\frac{1}{4}\left|J'_\lambda z_\lambda\right|^2} \in L^q(\mathbb{C}^n)$, it follows that the left-hand side is in $L^q(\mathbb{C}^n)$. Since the right-hand side is a polynomial, we conclude that $f = 0$. □

Now, we prove a version of the two-radii theorem for the class of tempered continuous functions on the Métivier group G, which are periodic in the centre variable.

Theorem 3.2.4. *Let f be a tempered continuous function in the z-variable and 2π-periodic in the centre variable of G. If f satisfies $f * \mu_{r_i} = 0,\ i = 1, 2,$ then $f = 0$ as long as*

(i) $\frac{r_1^2}{r_2^2}$ *is not a quotient of zeros of Laguerre polynomials L_k^{n-1} for any k.*

(ii) $\frac{r_1}{r_2}$ *is not a quotient of zeros of Bessel functions J_{n-1}.*

Proof. For $l \in \mathbb{Z}^m$, define the l-th Fourier coefficient of f by

$$f^l(z) = \int_{[0,2\pi]^m} f(z,t)e^{il\cdot t}dt.$$

It follow from Lemma 3.1.3 that $f^l \in L^2(\mathbb{C}^n)$. Further, taking the l-twisted spherical mean of $f * \mu_{r_i} = 0$ by using uniqueness of the Fourier series, we get $f^l \times_l \mu_{r_i} = 0$ for $i = 1, 2$. Let fix $l \neq 0$, then using Lemma 2.2.3 and Equation (3.2.3), we can write

$$(f^l)_l \times_{\lambda'} \mu_{r_i} = 0 \ \text{ for some } \lambda' \in \mathbb{R}^n_+ \text{ and } i = 1, 2,$$

where $(f^l)_l = f^l(A_l x)$ as defined in Lemma 2.2.3. Then using Proposition 3.1.2, we get

$$\vartheta_{k,\lambda'}^{n-1}(r_i)\,(f^l)_l \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1}(z) = 0 \text{ for } i = 1, 2.$$

Since for each k, either $\vartheta_{k,\lambda'}^{n-1}(r_1) \neq 0$ or $\vartheta_{k,\lambda'}^{n-1}(r_2) \neq 0$, we have $(f^l)_l \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1} = 0$. Hence $(f^l)_l = 0$. When $l = 0$, the l-twisted spherical mean conditions $f * \mu_{r_i} = 0, \ i = 1, 2$ led to two radii theorem on $\mathbb{C}^n$. Hence each Fourier coefficient of f is zero, and thus $f = 0$. □

Remark 3.2.5. *Since the twisted spherical mean of a Métivier group is coherence with the direct product of* n *copies of one dimensional Heisenberg group, we could show one radius theorem for* $f \in L^p(G), \ 1 \leq p \leq 2$. *However, when we approach as similar to the Heisenberg group [67] for* $f \in L^p(G), \ 2 < p < \infty$, *based on a summability result, we seek* L^p*- boundedness of a multi-parameter singular integral, whose kernel may fail to be a Calderón-Zygmund kernel.*

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Chapter 4

Boundedness and uniqueness of Quaternion Weyl transform

In this chapter, we review some important properties of the quaternion Fourier transform and quaternion Wigner transform. Then we define quaternion Weyl transform (QWT) via quaternion Wigner transform and discuss its boundedness. Further, we prove a variant of the Benedicks-Amrein-Berthier theorem for the QWT.

4.1 Preliminaries and some auxiliary results

The quaternion algebra. The quaternion algebra was discovered by W. R. Hamilton in 1843. It is an extension of complex numbers to a four dimensional algebra, which is denoted by $\mathbb{Q}$. Every element of $\mathbb{Q}$ is a linear combination of a real scalar and three orthogonal imaginary units i, j and k with real coefficients. In other words,

$$\mathbb{Q} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where the imaginary units i, j and k follow the Hamilton's multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

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The conjugate of q is defined by $\bar{q} = q_0 - iq_1 - jq_2 - kq_3$, which satisfies

$$\bar{\bar{q}} = q, \quad \overline{p+q} = \bar{p} + \bar{q}, \quad \overline{pq} = \bar{q}\bar{p}, \quad \text{for all} \quad p, q \in \mathbb{Q}.$$

The module $|q|$ is given by

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Since $\mathbb{Q}$ is non-commutative in multiplication, various results on the complex field cannot directly extend to the quaternion algebra. To be in control of it, $\mathbb{Q}$ can be split into two planes spanned by $\{i - j, 1 + ij\}$ and $\{i + j, 1 - ij\}$. Precisely for $q \in \mathbb{Q}$,

$$q = q_+ + q_-, \quad \text{where } q_\pm = \frac{1}{2}(q \pm iqj). \tag{4.1.1}$$

Writing in terms of the real components q_0, q_1, q_2, $q_3 \in \mathbb{R}$, we have

$$q_\pm = \{q_0 \pm q_3 + i(q_1 \mp q_2)\} \frac{1 \pm k}{2} = \frac{1 \pm k}{2} \{q_0 \pm q_3 + j(q_1 \pm q_2)\}. \tag{4.1.2}$$

Then, it satisfy the modulus identity $|q|^2 = |q_+|^2 + |q_-|^2$. The following commutator relations give a justification for the above decomposition (4.1.2),

$$(1+k)e^{\pm aj} = e^{\mp ai}(1+k) \text{ and } (1-k)e^{\pm aj} = e^{\pm ai}(1-k), \text{ where } a \in \mathbb{R}. \tag{4.1.3}$$

A quaternion valued function $f : \mathbb{R}^2 \to \mathbb{Q}$ can be expressed as

$$f(x_1, x_2) = f_0(x_1, x_2) + if_1(x_1, x_2) + jf_2(x_1, x_2) + kf_3(x_1, x_2),$$

where each $f_l(x_1, x_2)$ is a real valued function. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $|\alpha| = \alpha_1 + \alpha_2$, where $\mathbb{Z}_+$ denotes the set of all non-negative integers. Define the differential operator

$$D^\alpha = \left(-i\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(-i\frac{\partial}{\partial x_2}\right)^{\alpha_2}.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^2, \mathbb{Q})$ is the set of smooth functions from $\mathbb{R}^2$ to $\mathbb{Q}$ satisfying

$$\sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^2} \left(1 + |x|^2\right)^N |(D^\alpha f)(x)| < \infty,$$

for every $N \in \mathbb{Z}_+$. For $1 \le r < \infty$, $L^r(\mathbb{R}^2, \mathbb{Q})$ is the space of all quaternion valued functions such that

$$\|f\|_r = \left(\int_{\mathbb{R}^2} |f(x_1, x_2)|^r \, dx_1 dx_2\right)^{1/r} < \infty,$$

and for $r = \infty$, $L^\infty(\mathbb{R}^2, \mathbb{Q})$ is the space of all essentially bounded measurable functions with $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^2} |f(x)| < \infty$. For $r = 2$, define the inner product of $f, g \in L^2(\mathbb{R}^2, \mathbb{Q})$ by

$$\langle f, g \rangle = \int_{\mathbb{R}^2} f(x)\overline{g(x)} dx.$$

Quaternion Fourier transform. For $f \in L^1(\mathbb{R}^2, \mathbb{Q})$, the quaternion Fourier transform $\mathcal{F}(f) : \mathbb{R}^2 \to \mathbb{Q}$ is defined by

$$\mathcal{F}(f)(y_1, y_2) = \int_{\mathbb{R}^2} e^{-2\pi i x_1 y_1} f(x_1, x_2) e^{-2\pi j x_2 y_2} dx_1 dx_2. \tag{4.1.4}$$

This is also known as two-sided quaternion Fourier transform. Further, f can be reconstructed from the QFT.

Theorem 4.1.1. [15,33](Inversion formula) *If f and $\mathcal{F}(f)$ both are in $L^1(\mathbb{R}^2,\mathbb{Q})$, then*

$$f(x_1,x_2)=\int_{\mathbb{R}^2} e^{2\pi i y_1 x_1}\mathcal{F}(f)(y_1,y_2)e^{2\pi j y_2 x_2}dy_1dy_2.$$

Due to the fact that $ie^{aj}=e^{-aj}i$, $a\in\mathbb{R}$, the QFT of $f\in L^2(\mathbb{R}^2,\mathbb{Q})$ can be reframed by

$$\begin{aligned}\mathcal{F}(f)(y_1,y_2)&=\int_{\mathbb{R}^2} e^{-2\pi i x_1 y_1} f(x_1,x_2)\, e^{-2\pi j x_2 y_2} dx_1dx_2\\ &=\int_{\mathbb{R}^2} e^{-2\pi i x_1 y_1} e^{-2\pi j x_2 y_2}\left(f_0(x_1,x_2)+jf_2(x_1,x_2)\right)dx_1dx_2\\ &+\int_{\mathbb{R}^2} e^{-2\pi i x_1 y_1} e^{2\pi j x_2 y_2}\left(if_1(x_1,x_2)+\mathrm{k}f_3(x_1,x_2)\right)dx_1dx_2\\ &=\int_{\mathbb{R}^2} e^{-2\pi i x_1 y_1} e^{-2\pi j x_2 y_2}\tilde{f}^j(x_1,x_2)\,dx_1dx_2,\end{aligned}\tag{4.1.5}$$

where

$$\tilde{f}^j(x_1,x_2)=f_0(x_1,x_2)+\mathrm{i}f_1(x_1,-x_2)+\mathrm{j}f_2(x_1,x_2)+\mathrm{k}f_3(x_1,-x_2).\tag{4.1.6}$$

Similarly,

$$\mathcal{F}(f)(y_1,y_2)=\int_{\mathbb{R}^2}\tilde{f}^i(x_1,x_2)\,e^{-2\pi i x_1 y_1}e^{-2\pi j x_2 y_2}dx_1dx_2,\tag{4.1.7}$$

where

$$\tilde{f}^i(x_1,x_2)=f_0(x_1,x_2)+\mathrm{i}f_1(x_1,x_2)+\mathrm{j}f_2(-x_1,x_2)+\mathrm{k}f_3(-x_1,x_2).\tag{4.1.8}$$

The right-hand side of (4.1.5) is known as the left-sided quaternion Fourier transform of $\tilde{f}^j$, where the right-hand side of (4.1.7) is known as the right-sided quaternion Fourier transform of $\tilde{f}^i$. Note that $\|f\|_2=\|\tilde{f}^i\|_2=\|\tilde{f}^j\|_2$. Further, if $f(x_1,-x_2)=f(x_1,x_2)$, then $\tilde{f}^j=f$ and if $f(-x_1,x_2)=f(x_1,x_2)$, then $\tilde{f}^i=f$.

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Theorem 4.1.2. (Parseval's theorem) *Let* $f, g \in L^1(\mathbb{R}^2, \mathbb{Q}) \cap L^2(\mathbb{R}^2, \mathbb{Q})$, *then*

$$\int_{\mathbb{R}^2} \mathcal{F}(f)(y)\overline{\mathcal{F}(g)(y)}dy = \int_{\mathbb{R}^2} \tilde{f}^i(x)\overline{\tilde{g}^i(x)}dx, \tag{4.1.9}$$

where $\tilde{f}^i$ *and* $\tilde{g}^i$ *are defined as in (4.1.8). In particular, if* f *and* g *are even in first variable, then* $\langle \mathcal{F}(f), \mathcal{F}(g)\rangle = \langle f, g\rangle$.

Proof. The left integral $\int_{\mathbb{R}^2} \mathcal{F}(f)(y)\overline{\mathcal{F}(g)(y)}dy$ of (4.1.9) is equal to

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i x_1 y_1} f\left(x_1, x_2\right) e^{-2\pi j x_2 y_2} e^{2\pi j x_2' y_2} \overline{g\left(x_1', x_2'\right)} e^{2\pi i x_1' y_1} dx dx' dy.$$

By adapting the method used in (4.1.7), the above integral becomes

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{f}^i(x) e^{-2\pi i x_1 y_1} e^{-2\pi j x_2 y_2} e^{2\pi j x_2' y_2} e^{2\pi i x_1' y_1} \overline{\tilde{g}^i(x')} dx dx' dy \\ = &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{f}^i(x)\delta(x'-x)\overline{\tilde{g}^i(x')}dx' dx = \int_{\mathbb{R}^2} \tilde{f}^i(x)\overline{\tilde{g}^i(x)}dx. \end{aligned}$$

□

Theorem 4.1.3. (Plancherel theorem for QFT) *If* $f \in L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$, *then*

$$\|f\|_2 = \|\mathcal{F}(f)\|_2.$$

Proof. For $f \in L^1(\mathbb{R}^2, \mathbb{Q}) \cap L^2(\mathbb{R}^2, \mathbb{Q})$, Plancherel theorem will follow from (4.1.9), as $\|f\|_2 = \|\tilde{f}^i\|_2$. For arbitrary $f \in L^2(\mathbb{R}^2, \mathbb{Q})$, since $L^1(\mathbb{R}^2, \mathbb{Q}) \cap L^2(\mathbb{R}^2, \mathbb{Q})$ is dense in $L^2(\mathbb{R}^2, \mathbb{Q})$, there exists a sequence $\{f_n\}$ in $L^1(\mathbb{R}^2, \mathbb{Q}) \cap L^2(\mathbb{R}^2, \mathbb{Q})$ such that $\{f_n\}$ converges to f in $L^2(\mathbb{R}^2, \mathbb{Q})$. Then $\{\mathcal{F}(f_n)\}$ is a Cauchy sequence and will converge to some $g \in L^2(\mathbb{R}^2, \mathbb{Q})$. We shall define $\mathcal{F}(f) = g$. Thus,

$$\|\mathcal{F}(f)\|_2 = \lim_{n\to\infty} \|\mathcal{F}(f_n)\|_2 = \lim_{n\to\infty} \|f_n\|_2 = \|f\|_2.$$

□

As in (4.1.1), $f \in L^1(\mathbb{R}^2, \mathbb{Q})$ can be decomposed by $f = f_+ + f_-$. Then the QFT becomes

$$\mathcal{F}(f)(y_1, y_2) = \mathcal{F}(f_+)(y_1, y_2) + \mathcal{F}(f_-)(y_1, y_2). \tag{4.1.10}$$

Using (4.1.3), the QFT of $f_\pm$ reduces to the Euclidean Fourier transform, i.e.,

$$\mathcal{F}(f_\pm)(y_1, y_2) = \int_{\mathbb{R}^2} e^{-2\pi i(x_1y_1 \mp x_2y_2)} f_\pm(x_1, x_2)\, dx_1 dx_2. \tag{4.1.11}$$

Due to the modulus identity $|q|^2 = |q_+|^2 + |q_-|^2$, we have the following two relations for $f \in L^1(\mathbb{R}^2, \mathbb{Q})$,

$$\begin{aligned} |f(x_1, x_2)|^2 &= |f_+(x_1, x_2)|^2 + |f_-(x_1, x_2)|^2, \\ |\mathcal{F}(f)(y_1, y_2)|^2 &= |\mathcal{F}(f_+)(y_1, y_2)|^2 + |\mathcal{F}(f_-)(y_1, y_2)|^2. \end{aligned} \tag{4.1.12}$$

Hardy's theorem and rotations. In [35], an extension of Hardy's classical characterization of real Gaussians to the case of complex Gaussians proved for the Euclidean Fourier transform, and later generalized to the unitary space $\mathbb{C}^n$, see [64]. We notice that the fundamental result in [35] can be extended to QFT with the help of (4.1.10). We state the result in [64] for $n = 2$.

Theorem 4.1.4. [64] *Let $f \in L^1(\mathbb{R}^2)$ and $\psi^0 = (\psi_1^0, \psi_2^0) \in (-\pi/2, \pi/2)^2$ such that the integral*

$$\widehat{f}\left(e^{i\psi^0} s\right) = \int_{\mathbb{R}^2} f(x) \exp\left(-2\pi i x \left(e^{i\psi^0} s\right)\right) dx$$

converges for all $s \in \mathbb{R}^2$ and satisfies

$$\left|\widehat{f}\left(e^{i\psi^0} s\right)\right| \leq C_1 e^{-\pi |s|^2/\alpha}, \tag{4.1.13}$$

where C_1 and α are positive constants. Then f has an analytic extension to $\mathbb{C}^2$. Furthermore, suppose $\theta^0 = (\theta_1^0, \theta_2^0) \in \mathbb{R}^2$ is such that the extension of f satisfies

$$\left|f\left(e^{i\theta^0} r\right)\right| \leq C_2 e^{-\pi\alpha |r|^2} \tag{4.1.14}$$

for some $C_2 > 0$ *and all* $r \in \mathbb{R}^2$*, where* α *is as above. Then* f *is a rotation of a multiple of* $e^{-\pi\alpha|x|^2}$ *through the angle* $-\theta_j^0$ *with respect to* x_j *in the* z_j*-plane* $(j = 1, 2)$:

$$f(z) = C \exp\left(-\pi\alpha\left(e^{-2i\theta_1^0} z_1^2 + e^{-2i\theta_2^0} z_2^2\right)\right) = C \exp\left(-\pi\alpha\left(e^{-i\theta^0} z\right)^2\right).$$

Moreover, $-\theta_j^0 \equiv \psi_j^0 \,(\mathrm{mod}\pi)$ *and* $\left|\psi_j^0\right| < \frac{\pi}{4}$, $j = 1, 2$.

Now, consider a function $g \in L^1(\mathbb{R}^2, \mathbb{Q})$ satisfying the hypotheses of Theorem 4.1.4 in terms of QFT. Using (4.1.2) and (4.1.11), $\mathcal{F}(g)$ can be split into the Euclidean Fourier transform of $\tilde{g}_\pm$, where $\tilde{g}_\pm$ are complex valued functions and $g = \tilde{g}_+ \frac{1+k}{2} + \tilde{g}_- \frac{1-k}{2}$. Hence both $\tilde{g}_\pm$ will satisfy the hypotheses of Theorem 4.1.4. Therefore, $\tilde{g}_\pm$ are rotations of multiple of $e^{-\pi\alpha|x|^2}$ through the angle $-\theta_j^0$ with respect to x_j in the z_j-plane. Thus, we have the following version of Hardy's theorem associated with QFT.

Suppose $g \in L^1(\mathbb{R}^2, \mathbb{Q})$ *such that* $\mathcal{F}(g)(e^{i\psi^0} s)$ *converges for all* s*, and satisfies* $|\mathcal{F}(g)(e^{i\psi^0} s)| \le C_1 e^{-\pi|s|^2/\alpha}$*, where* ψ^0, C_1 *and* α *are as in Theorem 4.1.4. Then* g *has an analytic extension to* $\mathbb{C}^2$*. Furthermore, suppose* $\theta^0 \in \mathbb{R}^2$ *such that the extension of* g *satisfies* $|g(e^{i\theta^0} r)| \le C_2 e^{-\pi\alpha|r|^2}$ *for some* $C_2 > 0$ *and all* $r \in \mathbb{R}^2$*. Then* g *is a rotation of a multiple of* $e^{-\pi\alpha|x|^2}$ *through the angle* $-\theta^0$ *on* $\mathbb{C}^2$*. That is*

$$g(z) = C \exp\left(-\pi\alpha\left(e^{-i\theta^0} z\right)^2\right) + C' \exp\left(-\pi\alpha\left(e^{-i\theta^0} z\right)^2\right) k.$$

Moreover, $-\theta_j^0 \equiv \psi_j^0 \,(\mathrm{mod}\pi)$ *and* $\left|\psi_j^0\right| < \frac{\pi}{4}$, $j = 1, 2$.

4.2 The Fourier-Wigner transform and Weyl transform

4.2.1 Boundedness of Fourier-Wigner transform.

Before exploring the Weyl transform, define a related transform known as Fourier-Wigner transform, a helpful tool for studying the Weyl transform. For simplicity, throughout the Section, we use the terminology Fourier-Wigner transform, Wigner transform and Weyl transform instead of quaternion Fourier-Wigner transform, quaternion Wigner transform and quaternion Weyl transform, respectively, since it will be clear from the context.

Definition 4.2.1. *Let f and g be in $\mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$. Then the Fourier-Wigner transform of f and g is defined by*

$$V(f,g)(q,p) = \int_{\mathbb{R}^2} e^{2\pi i\left(q_1x_1+\frac{1}{2}q_1p_1\right)} f(x+p)\overline{g(x)} e^{2\pi j\left(q_2x_2+\frac{1}{2}q_2p_2\right)} dx, \tag{4.2.1}$$

where $q = (q_1,\, q_2),\;\; p = (p_1,\, p_2) \in \mathbb{R}^2.$

Using a simple change of variable, we can rewrite (4.2.1) as

$$V(f,g)(q,p) = \int_{\mathbb{R}^2} e^{2\pi i q_1 y_1} f\left(y+\frac{p}{2}\right) \overline{g\left(y-\frac{p}{2}\right)} e^{2\pi j q_2 y_2} dy. \tag{4.2.2}$$

Note that $V : \mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right) \times \mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right) \to \mathcal{S}\left(\mathbb{R}^4, \mathbb{Q}\right)$ is a bilinear map.

Now, we regard some properties of the Wigner transform. For studying the Weyl transform, we required the notion of the Wigner transform of two functions from $L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$. Consequently, we begin with the QFT of the Fourier-Wigner transform.

Theorem 4.2.2. *Let f and g be in $\mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$, then for $x, \xi \in \mathbb{R}^2$,*

$$\mathcal{F}\left(V(f,g)\right)(x,\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \xi_1 p_1} f\left(x+\frac{p}{2}\right) \overline{g\left(x-\frac{p}{2}\right)} e^{-2\pi j \xi_2 p_2} dp. \tag{4.2.3}$$

Proof. For $\varepsilon > 0$, define the function W_ε on $\mathbb{R}^4$ by

$$W_\varepsilon(x,\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\varepsilon^2\pi|q|^2} e^{-2\pi i x_1 q_1 - 2\pi i \xi_1 p_1} V(f,g)(q,p) e^{-2\pi j x_2 q_2 - 2\pi j \xi_2 p_2} dq dp. \tag{4.2.4}$$

By Fubini's theorem and the fact that Fourier transform of

$$\varphi(x) = e^{-\pi|x|^2} \quad \text{for } x \in \mathbb{R}^2 \tag{4.2.5}$$

is equal to φ, we get

$$\begin{aligned} W_\varepsilon(x,\xi) = {} & \int_{\mathbb{R}^2} e^{-2\pi i \xi_1 p_1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} e^{-\varepsilon^2 \pi |q_1|^2} e^{-2\pi i (x_1 - y_1) q_1} dq_1 \right) f\left(y+\frac{p}{2}\right) \overline{g\left(y-\frac{p}{2}\right)} \\ & \left(\int_{\mathbb{R}} e^{-2\pi j (x_2 - y_2) q_2} e^{-\varepsilon^2 \pi |q_2|^2} dq_2 \right) dy \, e^{-2\pi j \xi_2 p_2} dp \\ = {} & \int_{\mathbb{R}^2} e^{-2\pi i \xi_1 p_1} \left(\int_{\mathbb{R}^2} \varepsilon^{-2} e^{\frac{\pi|x-y|^2}{-\varepsilon^2}} f\left(y+\frac{p}{2}\right) \overline{g\left(y-\frac{p}{2}\right)} dy \right) e^{-2\pi j \xi_2 p_2} dp. \end{aligned}$$

Now, for each $p \in \mathbb{R}^2$, define the function F_p by

$$F_p(y) = f\left(y+\frac{p}{2}\right) \overline{g\left(y-\frac{p}{2}\right)}. \tag{4.2.6}$$

Using (4.2.6), we get

$$W_\varepsilon(x,\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \xi_1 p_1} \left(\varphi_\varepsilon * F_p\right)(x) e^{-2\pi j \xi_2 p_2} dp,$$

where $\varphi_\varepsilon(x) = \varepsilon^{-2}\varphi\left(\frac{x}{\varepsilon}\right)$. For fixed $p \in \mathbb{R}^2$, from (4.2.6), it follows that $\varphi_\varepsilon * F_p$ converges to F_p uniformly on compact subsets of $\mathbb{R}^2$. Let N be any positive integer. Then there exists a positive constant C_N such that

$$|(\varphi_\varepsilon * F_p)(x)| \leq \sup_{x\in\mathbb{R}^2} \left| f\left(x+\frac{p}{2}\right) g\left(x-\frac{p}{2}\right)\right| \leq C_N \left(1+|p|^2\right)^{-N}, \tag{4.2.7}$$

for all $\varepsilon > 0$. So, by (4.2.7) and the dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0} W_\varepsilon(x,\xi) = \int_{\mathbb{R}^2} e^{-2\pi i\xi_1 p_1} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)} e^{-2\pi i\xi_2 p_2} dp.$$

But, applying the dominated convergence theorem in (4.2.4), we can conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0} W_\varepsilon(x,\xi) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i x_1 q_1 - 2\pi i \xi_1 p_1} V(f,g)(q,p) e^{-2\pi j x_2 q_2 - 2\pi j \xi_2 p_2} dq dp \\ &= \mathcal{F}\left(V(f,g)\right)(x,\xi).\end{aligned}$$

□

In view of Theorem 4.2.2, the Wigner transform of $f,\ g \in \mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$ can be defined by

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^2} e^{-2\pi i\xi_1 p_1} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)} e^{-2\pi j\xi_2 p_2} dp. \tag{4.2.8}$$

Some of its properties are obtained in the following.

Proposition 4.2.3. *Let $f,\ g \in \mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$ and $2 \leq r \leq \infty$, then the Wigner transform $W(f,g) \in L^r\left(\mathbb{R}^4, \mathbb{Q}\right)$. Moreover,*

$$\|W(f,g)\|_r \leq \|f\|_2 \|g\|_2.$$

It follows from Proposition 4.2.3 that W can extend to $L^2\left(\mathbb{R}^2, \mathbb{Q}\right) \times L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$.

Proof. Let $f,\ g \in \mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$. By Cauchy-Schwartz inequality, we get

$$\begin{aligned} |W(f,g)(x,\xi)| &= \left| \int_{\mathbb{R}^2} e^{-2\pi i \xi_1 p_1} f\left(x+\frac{p}{2}\right) \overline{g\left(x-\frac{p}{2}\right)} e^{-2\pi j \xi_2 p_2} dp \right| \\ &\leq \int_{\mathbb{R}^2} \left| f\left(x+\frac{p}{2}\right) \right| \left| \overline{g\left(x-\frac{p}{2}\right)} \right| dp \leq \|f\|_2 \|g\|_2. \end{aligned}$$

Hence

$$\|W(f,g)\|_\infty \leq \|f\|_2 \|g\|_2.$$

Now for $r = 2$, the Plancherel theorem gives

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W(f,g)(x,\xi)|^2 \, dx d\xi = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \left| f\left(x\right) \overline{g\left(x-p\right)} \right|^2 dp \right) dx \quad = \|f\|_2 \|g\|_2.$$

The result follows from the Riesz-Thorin interpolation. □

Though we show that the Wigner transform is L^2 bounded for all f and g in $L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$, we could only prove the orthogonality relation for a sub-class of functions.

Theorem 4.2.4. (The Moyal Identity) *Let* $f_1,\ g_1,\ f_2,\ g_2 \in \mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$ *such that* $f_l(x_1, x_2) = f_l(-x_1, x_2)$ *and* $g_l(x_1, x_2) = g_l(-x_1, x_2)$ *for* $l = 1, 2$. *Then*

$$\left\langle W\left(f_1, g_1\right), W\left(f_2, g_2\right) \right\rangle = \left\langle f_1,\ f_2 \left\langle \bar{g_2},\ \bar{g_1} \right\rangle \right\rangle.$$

Proof. Consider

$$\left\langle W\left(f_1, g_1\right), W\left(f_2, g_2\right) \right\rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} W\left(f_1, g_1\right)(x,\xi) \overline{W\left(f_2, g_2\right)(x,\xi)} dx d\xi \tag{4.2.9}$$

$$\begin{aligned} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} & e^{-2\pi i \xi_1 p_1} f_1\left(x+\frac{p}{2}\right) \overline{g_1\left(x-\frac{p}{2}\right)} e^{-2\pi j \xi_2 p_2} \\ & \times e^{2\pi j \xi_2 p_2'} g_2\left(x-\frac{p'}{2}\right) \overline{f_2\left(x+\frac{p'}{2}\right)} e^{2\pi i \xi_1 p_1'} dp dp' dx d\xi. \end{aligned}$$

We can write $f_1 = \frac{1-k}{2} f_{11} + \frac{1+k}{2} f_{12}$, $\bar{f}_2 = f_{21}\frac{1-k}{2} + f_{22}\frac{1+k}{2}$ from (4.1.2) and $\bar{g}_1 = ig_{11} - g_{12}$, $g_2 = ig_{21} - g_{22}$, where f_{lm} and g_{lm} for $1 \leq l, m \leq 2$ are combinations of two real valued functions with imaginary unit j, for example $h_1 + jh_2$. Then we have

$$f_1\left(x+\frac{p}{2}\right)\overline{g_1\left(x-\frac{p}{2}\right)} = \sum_{1\leq l,m\leq 2} \frac{1+(-1)^l k}{2} f_{1,l}\left(x+\frac{p}{2}\right) i^m g_{1,m}\left(x-\frac{p}{2}\right),$$

$$g_2\left(x-\frac{p'}{2}\right)\overline{f_2\left(x+\frac{p'}{2}\right)} = \sum_{1\leq l',m'\leq 2} i^{l'} g_{2,l'}\left(x-\frac{p'}{2}\right) f_{2,m'}\left(x+\frac{p'}{2}\right)\frac{1+(-1)^{m'} k}{2}.$$

Replacing these values in (4.2.9) and then using the commutator rules (4.1.3) together with the assumptions of functions, the integral (4.2.9) can be phrased as

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sum_{1\leq l,m,l',m'\leq 2} \frac{1+(-1)^l k}{2} f_{1,l}\left(x+\frac{p}{2}\right) i^m g_{1,m}\left(x-\frac{p}{2}\right) \\
&\qquad\qquad \times i^{l'} g_{2,l'}\left(x-\frac{p}{2}\right) f_{2,m'}\left(x+\frac{p}{2}\right)\frac{1+(-1)^{m'} k}{2} dxdp \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_1\left(x+\frac{p}{2}\right)\overline{g_1\left(x-\frac{p}{2}\right)} g_2\left(x-\frac{p}{2}\right)\overline{f_2\left(x+\frac{p}{2}\right)} dxdp.
\end{aligned}$$

By an appropriate change of variables we can conclude that

$$\langle W(f_1, g_1), W(f_2, g_2)\rangle = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_1(u)\overline{g_1(v)} g_2(v)\overline{f_2(u)} dudv = \langle f_1,\ f_2\langle \bar{g}_2,\ \bar{g}_1\rangle\rangle.$$

□

Zeros of the Wigner transform. The zero set of the Wigner transform is useful in studying the injectivity of a general Berezin transform and the generalized Berezin quantization problem. In [31], the authors studied under which conditions the Euclidean Wigner transform never vanishes. It is shown that when f and g are generalized Gaussian, then the Euclidean Wigner transform is never vanishing. Moreover, some pairs are obtained from the basic example of the one-sided exponential function $e^{-at}\mathbf{1}_{(0,\infty)}$. Now, we produce some examples of pairs (f, g) for which $W(f, g)$ is never vanishing.

Let $f, g \in L^2(\mathbb{R}^2)$ be such that the Euclidean Wigner transform does not vanish. Then consider

$$W(f,(1+k)g)(x,\xi)=\int_{\mathbb{R}^2} e^{-2\pi i\xi_1 p_1} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}(1-k)e^{-2\pi j\xi_2 p_2}dp,$$

where $a, b \in \mathbb{R}$. By using (4.1.3), we get

$$W\left(f,(1+k)g\right)(x,\xi)=\int_{\mathbb{R}^2} e^{-2\pi i\xi\cdot p} f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}(1-k)dp.$$

Hence we obtain that the pair $(f,(a+bk)g)$ makes Wigner transform into zero free. It is also clear that Gaussian φ defined in (4.2.5) makes so. Further, we give some examples of such pairs, generalized from the Gaussian. For $a, b, c, d \in \mathbb{R}^2$ define

$$\pi^i_{b_1-d_1}(a)f(x)=e^{2\pi i(b_1-d_1)x_1}f(x-a),\ \pi^j_{b_2-d_2}(c)f(x)=e^{2\pi j(b_2-d_2)x_2}f(x-c).$$

Then $W\left(\pi^i_{b_1-d_1}(a)f,\pi^j_{b_2-d_2}(c)g\right)(x,\xi)$ is equal to

$$\int_{\mathbb{R}^2} e^{2\pi i\left((b_1-d_1)\left(x_1+\frac{p_1}{2}\right)-\xi_1 p_1\right)} f\left(x-a+\frac{p}{2}\right)\overline{g\left(x-c-\frac{p}{2}\right)}e^{2\pi j\left((b_2-d_2)\left(-x_2+\frac{p_2}{2}\right)-\xi_2 p_2\right)}dp.$$

Replacing p by $p+a-c$, the above integral can be written as

$$e^{2\pi i\left((b_1-d_1)x_1-(a_1-c_1)\xi_1+\frac{1}{2}(a_1-c_1)(b_1-d_1)\right)}W(f,g)(x',\xi')e^{2\pi j\left(-(b_2-d_2)x_2-(a_2-c_2)\xi_2+\frac{1}{2}(a_2-c_2)(b_2-d_2)\right)},$$

where $x'=x-\frac{a+c}{2}$ and $\xi'=\xi-\frac{b-d}{2}$. By suitable change of variables, we can conclude that $W(f,g)\neq 0$ if only if $W\left(\pi^i_{b_1}(a)f,\pi^j_{b_2}(c)g\right)\neq 0$.

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4.2.2 The Weyl transform.

In this subsection, we introduce the Weyl transform and look for its connection with the Wigner transform. Let $\sigma \in \mathcal{S}\left(\mathbb{R}^4, \mathbb{Q}\right)$ be a symbol. Then define the Weyl transform W_σ on $\mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$ corresponding to σ by

$$\langle W_\sigma f, g\rangle = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma\left(x,\xi\right) W\left(f,g\right)\left(x,\xi\right) dx d\xi, \tag{4.2.10}$$

where $f, g \in \mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$. It can be seen that W_σ is continuous on $\mathcal{S}\left(\mathbb{R}^2, \mathbb{Q}\right)$.

Theorem 4.2.5. *Let $\sigma \in L^r\left(\mathbb{R}^4, \mathbb{Q}\right)$ for $1 \leq r \leq 2$. Then W_σ is a bounded operator on $L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$ and satisfies $\|W_\sigma\| \leq \|\sigma\|_r$.*

Proof. The proof will follow from Proposition 4.2.3. □

The Weyl transform (4.2.10) is defined involving the Wigner transform. The following result illuminates the Weyl transform and later on describes the compactness of the Weyl transform for a sub-class of symbols.

Theorem 4.2.6. *Let $\sigma \in \mathcal{S}\left(\mathbb{R}^4, \mathbb{Q}\right)$ such that $\sigma(x,\xi) = \sigma(x,-\xi)$. Then for $\varphi \in L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$,*

$$\left(W_\sigma \varphi\right)(v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i \xi_1 (v_1 - u_1)} \sigma\left(\frac{u+v}{2}, \xi\right) e^{-2\pi j \xi_2 (v_2 - u_2)} \varphi(u) du d\xi.$$

Proof. Let φ, $\psi \in L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$. From (4.2.10), we have

$$\begin{aligned}\langle W_\sigma \varphi, \psi\rangle &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma\left(x,\xi\right) W\left(\varphi,\psi\right)\left(x,\xi\right) dx d\xi \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma\left(x,\xi\right) e^{-2\pi i \xi_1 p_1} \varphi\left(x+\frac{p}{2}\right) \overline{\psi\left(x-\frac{p}{2}\right)} e^{-2\pi j \xi_2 p_2} dp\, dx\, d\xi.\end{aligned}$$

By the change of variables $u = x + \frac{p}{2}$ and $v = x - \frac{p}{2}$, the above integral becomes

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma\left(\frac{u+v}{2}, \xi\right) e^{-2\pi i \xi_1 (v_1 - u_1)} \varphi(u)\overline{\psi(v)} e^{-2\pi j \xi_2 (v_2 - u_2)} du dv d\xi.$$

Using arguments like (4.1.5), (4.1.7) and the fact that $\sigma(x, -\xi) = \sigma(x, \xi)$, we get

$$\langle W_\sigma \varphi, \psi\rangle = \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i \xi_1 (v_1 - u_1)} \sigma\left(\frac{u+v}{2}, \xi\right) e^{-2\pi j \xi_2 (v_2 - u_2)} \varphi(u) du d\xi\right) \overline{\psi(v)} dv.$$

Hence, we can conclude that

$$\left(W_\sigma \varphi\right)(v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i \xi_1 (v_1 - u_1)} \sigma\left(\frac{u+v}{2}, \xi\right) e^{-2\pi j \xi_2 (v_2 - u_2)} \varphi(u) du d\xi.$$

□

Theorem 4.2.7. *Let $\sigma \in L^1\left(\mathbb{R}^4, \mathbb{Q}\right)$ such that $\sigma(x,\xi) = \sigma(x, -\xi)$. Then W_σ is a trace class operator on $L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$.*

Proof. From Theorem 4.2.6, we get that W_σ is an integral operator with the kernel

$$K(u, v) = \int_{\mathbb{R}^2} e^{-2\pi i \xi_1 (v_1 - u_1)} \sigma\left(\frac{u+v}{2}, \xi\right) e^{-2\pi j \xi_2 (v_2 - u_2)} d\xi.$$

For $\sigma \in L^1\left(\mathbb{R}^4, \mathbb{Q}\right)$, we have

$$\|W_\sigma\|_{S_1} = \int_{\mathbb{R}^2} |K(u,u)| du \leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\sigma\left(u, \xi\right)\right| du d\xi = \|\sigma\|_1.$$

Hence, we conclude that W_σ is a trace class operator. □

Theorem 4.2.8. *Let $\sigma \in L^r\left(\mathbb{R}^4, \mathbb{Q}\right), 1 \leq r \leq 2$, such that $\sigma(x,\xi) = \sigma(x, -\xi)$. Then W_σ is a compact operator on $L^2\left(\mathbb{R}^2, \mathbb{Q}\right)$.*

Proof. In view of Theorem 4.2.6, we obtain that W_σ is an integral operator with the kernel

$$K(u,v)=\int_{\mathbb{R}^2} e^{-2\pi i\xi_1(v_1-u_1)}\sigma\left(\frac{u+v}{2},\xi\right)e^{-2\pi j\xi_2(v_2-u_2)}d\xi.$$

Suppose $\sigma\in L^2\left(\mathbb{R}^4,\mathbb{Q}\right)$, then the Hilbert-Schmidt norm of W_σ will be

$$\begin{aligned}\|W_\sigma\|_{S_2}^2&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|K(u,v)|^2dudv=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|K\left(x-\frac{p}{2},x+\frac{p}{2}\right)\right|^2dxdp\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2}e^{-2\pi i\xi_1p_1}\sigma\left(x,\xi\right)e^{-2\pi j\xi_2p_2}d\xi\right|^2dxdp\qquad(4.2.11)\\&=\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|\left(\mathcal{F}_2\sigma(x,\cdot)\right)(p)\right|^2dp\right)dx,\end{aligned}$$

where $\mathcal{F}_2\sigma$ denotes the QFT in the second variable. Now by the Plancherel theorem 4.1.3, we can conclude that $\|W_\sigma\|_{S_2}^2=\|\sigma\|_2^2$.

Let $\sigma\in L^r\left(\mathbb{R}^4,\mathbb{Q}\right), 1\leq r\leq 2$. Then we can choose a sequence σ_k in $\mathcal{S}\left(\mathbb{R}^4,\mathbb{Q}\right)$ converges to $\sigma\in L^r\left(\mathbb{R}^4,\mathbb{Q}\right)$. Therefore, each W_{σ_k} is a Hilbert-Schmidt operator, and hence compact. By Theorem 4.2.5, W_σ is the limit of W_{σ_k} in the space of bounded linear operators on $L^2\left(\mathbb{R}^2,\mathbb{Q}\right)$. Thus, W_σ is compact. □

The following results will give necessary and sufficient conditions for boundedness of the Weyl transform for $r>2$.

Proposition 4.2.9. *The following two statements are equivalent:*

1. *The Weyl transform W_σ is a bounded linear operator on $L^2\left(\mathbb{R}^2,\mathbb{Q}\right)$ for all $\sigma\in L^r\left(\mathbb{R}^4,\mathbb{Q}\right)$, where $2<r<\infty$.*
2. *There exists a positive constant C such that $\|W(f,g)\|_{r'}\leq C\|f\|_2\|g\|_2$ for all $f,g\in L^2\left(\mathbb{R}^2,\mathbb{Q}\right)$, where r and r' are conjugate indices.*

We skip the proof of Proposition 4.2.9 as it will be similar to the Euclidean case ([75], page no. 66-69).

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Proposition 4.2.10. *Let* $f \in L^2(\mathbb{R}^2, \mathbb{Q})$ *be a compactly supported function such that* $\int_{\mathbb{R}^2} f(x)dx \neq 0$. *If* W_σ *is a bounded operator on* $L^2(\mathbb{R}^2, \mathbb{Q})$ *for every* $\sigma \in L^r(\mathbb{R}^4, \mathbb{Q})$, *where* $2 < r < \infty$, *then* $\|\mathcal{F}(f)\|_{r'} < \infty$.

Proof. Let $f \in L^2(\mathbb{R}^2, \mathbb{Q})$ be supported on the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Consider the function $\tilde{f}^i$ defined in (4.1.8). Then $\tilde{f}^i \in L^2(\mathbb{R}^2, \mathbb{Q})$ and is supported on $\mathbb{D}$. Let $\bar{\bar{f}}(x) = \overline{f(-x)}$. Then, $W(\tilde{f}^i, \bar{\bar{f}})(x,\xi) = 0$ for all $\xi \in \mathbb{R}^2$, if $|x| > 1$. By Proposition 4.2.9, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|W\left(\tilde{f}^i, \bar{\bar{f}}\right)(x,\xi)\right|^{r'} dx d\xi < \infty.$$

Therefore, by Minkowshki's integral inequality and Hölder's inequality, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2}\left|\int_{|x|\leq 1} e^{4\pi i \xi_1 x_1} W\left(\tilde{f}^i, \bar{\bar{f}}\right)(x,\xi) e^{4\pi j \xi_2 x_2} dx\right|^{r'} d\xi\right)^{\frac{1}{r'}} \\
&\leq \int_{|x|\leq 1}\left(\int_{\mathbb{R}^2}\left|W\left(\tilde{f}^i, \bar{\bar{f}}\right)(x,\xi)\right|^{r'} d\xi\right)^{\frac{1}{r'}} dx \\
&\leq \left(\int_{|x|\leq 1} dx\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|W\left(\tilde{f}^i, \bar{\bar{f}}\right)(x,\xi)\right|^{r'} dx d\xi\right)^{\frac{1}{r'}} < \infty.
\end{aligned}$$

Hence,

$$I(\xi) = \int_{|x|\leq 1} e^{4\pi i \xi_1 x_1} W\left(\tilde{f}^i, \bar{\bar{f}}\right)(x,\xi) e^{4\pi j \xi_2 x_2} dx \in L^{r'}\left(\mathbb{R}^2, \mathbb{Q}\right). \tag{4.2.12}$$

But

$$\begin{aligned}
I(\xi) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{4\pi i \xi_1\left(x_1 - \frac{p_1}{2}\right)} \tilde{f}^i\left(x + \frac{p}{2}\right) \tilde{f}\left(x - \frac{p}{2}\right) e^{4\pi j \xi_2\left(x_2 - \frac{p_2}{2}\right)} dx dp \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{4\pi i \xi_1 v_1} \tilde{f}^i(u) \tilde{f}(v) e^{4\pi j \xi_2 v_2} du dv,
\end{aligned}$$

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where the last equality follows by a change of variables. A similar calculation as in (4.1.7) leads to

$$I(\xi)=\int_{\mathbb{R}^2} f(u)\int_{\mathbb{R}^2} e^{-4\pi i\xi_1 v_1} f(v) e^{-4\pi j\xi_2 v_2}\,dv\,du=\mathcal{F}\left(f\right)(0)\mathcal{F}\left(f\right)(2\xi).$$

From (4.2.12), it follows that $\|\mathcal{F}(f)\|_{r'}<\infty$. ☐

Now, we illustrate that QWT is not a bounded operator for $2<r<\infty$. Contrary to Proposition 4.2.10, for each $2<r<\infty$, we give example of $f\in L^2\left(\mathbb{R}^2,\mathbb{Q}\right)$, which is non-negative and compactly supported, satisfying $\|\mathcal{F}(f)\|_{r'}=\infty$.

Example 4.2.1. *First recall the example of such function for the Euclidean Fourier transform, which was produced by Simon [57]. Consider the cube*

$$Q=\{x\in\mathbb{R}^n:-a\le x_j\le a,\ j=1,2,\ldots,n\} \tag{4.2.13}$$

lying inside $\{x\in\mathbb{R}^n:|x|\le 1\}$. *For* $\alpha\in\left(0,\frac{1}{2}\right)$ *define the function* g_α *on* $\mathbb{R}^n$ *by*

$$g_\alpha(x)=\begin{cases}\prod_{j=1}^n |x_j|^{-\alpha}, & x\in Q\setminus\{0\}\\ 0, & \textit{otherwise.}\end{cases} \tag{4.2.14}$$

Here g_α *is positive, square-integrable and compactly supported real valued function on* $\mathbb{R}^n$. *For each* $1<r'<2$, *there exists* $\alpha\in\left(0,\frac{1}{2}\right)$ *such that* $\int_{\mathbb{R}^n}|\hat{g}_\alpha(\xi)|^{r'}d\xi=\infty$, *where* $\hat{g}_\alpha$ *is the Euclidean Fourier transform of* g_α.

Now, we will modify the above example for the quaternion Fourier transform on $\mathbb{R}^2$. *Let* Q *be the cube on* $\mathbb{R}^2$ *as in (4.2.13). For* $\alpha\in\left(0,\frac{1}{2}\right)$ *define*

$$f_\alpha(x)=g_\alpha(x)\frac{1-k}{2}, \tag{4.2.15}$$

where $g_\alpha(x)$ *is a function on* $\mathbb{R}^2$ *defined as in (4.2.14). From (4.1.11), we get*

$$\mathcal{F}(f_\alpha)(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i(x_1\xi_1+x_2\xi_2)} f_\alpha(x_1,x_2)dx_1dx_2 = \hat{g}_\alpha(\xi)\frac{1-k}{2}.$$

Hence from the Euclidean setup it follows that for each $1 < r' < 2$*, there exists* $\alpha \in \left(0, \frac{1}{2}\right)$ *such that* $\int_{\mathbb{R}^2} |\mathcal{F}(f_\alpha)(\xi)|^{r'} d\xi = \infty$.

4.3 Uniqueness results

4.3.1 Benedicks-Amrein-Berthier theorem for the Weyl transform.

A fruitful method of working on certain problems in the Heisenberg group is to consider the Weyl transform instead of group Fourier transform. For the Heisenberg group, the Weyl transform of $h \in L^1(\mathbb{C}^n)$ is defined by

$$\mathcal{W}(h) = \int_{\mathbb{C}^n} h(z)\pi(z)dz, \tag{4.3.1}$$

where $z = x + iy \in \mathbb{C}^n$ and $\pi(z)$ is an operator on $L^2(\mathbb{R}^n)$ defined by

$$\pi(z)\psi(\xi) = e^{2\pi i(x\cdot\xi + \frac{1}{2}x\cdot y)}\psi(\xi + y). \tag{4.3.2}$$

Further, for $\psi_1, \psi_2 \in L^2(\mathbb{R}^n)$, the following square integrability relation holds

$$\int_{\mathbb{C}^n} |\langle \pi(z)\psi_1, \psi_2\rangle|^2 dz = \|\psi_1\|^2\|\psi_2\|^2. \tag{4.3.3}$$

In addition, the inversion formula for the Weyl transform holds (see [68]),

$$h(z) = \operatorname{tr}\left(\mathcal{W}(h)\pi(-x,-y)\right) \tag{4.3.4}$$

and satisfies the Plancherel formula

$$\|\mathcal{W}(h)\|_{HS} = \|h\|_2. \tag{4.3.5}$$

It appeared in [25, 43, 70] that a non-trivial function $h \in L^1(\mathbb{C}^n)$ supported on a set of finite measure cannot have finite rank Weyl transform $\mathcal{W}(h)$. In this section, we prove an analogous result for the Weyl transform, which is defined in terms of QFT.

Consider the Weyl transform defined in (4.2.10) and assume that σ is even. From (4.2.3) and (4.2.8), we have

$$\langle W_\sigma\varphi, \psi\rangle = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma(x,\xi) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i(x_1q_1+\xi_1p_1)} V(\varphi,\psi)(q,p) e^{-2\pi j(x_2q_2+\xi_2p_2)} dqdp\, dxd\xi.$$

By proceeding through similar techniques as in (4.1.5), (4.1.7) and then using $\sigma(-x,-\xi) = \sigma(x,\xi)$, we get

$$\begin{aligned}\langle W_\sigma\varphi, \psi\rangle &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i(x_1q_1+\xi_1p_1)}\sigma(x,\xi)e^{-2\pi j(x_2q_2+\xi_2p_2)} dxd\xi\right) V(\varphi,\psi)(q,p)dqdp\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\mathcal{F}\sigma)(q,p) \int_{\mathbb{R}^2} e^{2\pi i\left(q_1x_1+\frac{1}{2}q_1p_1\right)} \varphi(x+p)\overline{\psi(x)} e^{2\pi j\left(q_2x_2+\frac{1}{2}q_2p_2\right)} dx\, dqdp,\end{aligned}$$

where the last equality holds from (4.2.1). Again using an argument like (4.1.5) and the fact that $\sigma(-x,-\xi) = \sigma(x,\xi)$ implies $(\mathcal{F}\sigma)(-q,-p) = (\mathcal{F}\sigma)(q,p)$, leads to

$$\langle W_\sigma\varphi, \psi\rangle = \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\mathcal{F}\sigma)(q,p) e^{2\pi i\left(q_1x_1+\frac{1}{2}q_1p_1\right)} \varphi(x+p) e^{2\pi j\left(q_2x_2+\frac{1}{2}q_2p_2\right)} dqdp\right) \overline{\psi(x)} dx. \tag{4.3.6}$$

For $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$, consider the operator $\rho(x,y)$ defined on $L^2(\mathbb{R}^2, \mathbb{Q})$ by

$$\rho(x,y)\varphi(\xi) = e^{2\pi i(x_1\xi_1+\frac{1}{2}x_1y_1)}\varphi(\xi+y)e^{2\pi j(x_2\xi_2+\frac{1}{2}x_2y_2)}, \tag{4.3.7}$$

where $\varphi \in L^2(\mathbb{R}^2, \mathbb{Q})$. Let $x, y, u, v \in \mathbb{R}^2$, then the following relation holds

$$\rho(u,v)\rho(x,y)\varphi(\xi) = e^{\pi i(x_1v_1 - u_1y_1)}\rho(u+x, v+y)\varphi(\xi)e^{\pi j(x_2v_2 - u_2y_2)}, \tag{4.3.8}$$

where $\varphi \in L^2(\mathbb{R}^2, \mathbb{Q})$ and $\xi \in \mathbb{R}^2$.

For $g \in L^1(\mathbb{R}^4, \mathbb{Q})$, in view of (4.3.1) and (4.3.6), define Weyl transform of g by

$$W(g) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(x,y)\rho(x,y)dxdy.$$

Consider the subspaces $L^2_{\pm}(\mathbb{R}^2, \mathbb{Q}) = \{\varphi_{\pm} : \varphi \in L^2(\mathbb{R}^2, \mathbb{Q})\}$ and let $P_{\pm}$ be the projections of $L^2(\mathbb{R}^2, \mathbb{Q})$ onto $L^2_{\pm}(\mathbb{R}^2, \mathbb{Q})$, respectively. Since $(1-k)(a+jb) = (a+ib)(1-k)$ for $a, b \in \mathbb{R}$, we have

$$\rho(u,v)\rho(x,y)\varphi_- = \pi(u,v)\pi(x,y)\varphi_-. \tag{4.3.9}$$

Next, we shall show that g can be retrieved from the following so-called inversion formula for the Weyl transform.

Theorem 4.3.1. (Inversion formula) *Let $g \in L^1(\mathbb{R}^4, \mathbb{Q})$. Then*

$$g(x,y) = tr\left(W(g)\rho(-x,-y)P_-\right).$$

Proof. Let $\{\tilde{e}_l : l \in \Lambda\}$ be an orthonormal basis for $L^2(\mathbb{R}^2)$. Then, it follows that $\mathsf{B} = \{e_l = \tilde{e}_l(\frac{1-k}{\sqrt{2}}) : l \in \Lambda\}$ is an orthonormal basis of $L^2_-(\mathbb{R}^2, \mathbb{Q})$. We extend B to an orthonormal basis $\{e_l : l \in \Lambda'\}$ of $L^2(\mathbb{R}^2, \mathbb{Q})$, where $\Lambda \subset \Lambda'$. Now,

$$\begin{aligned} \operatorname{tr}\left(W(g)\rho(-x,-y)P_-\right) &= \sum_{l\in\Lambda'} \langle W(g)\rho(-x,-y)P_-e_l, e_l\rangle = \sum_{l\in\Lambda} \langle W(g)\rho(-x,-y)e_l, e_l\rangle \\ &= \sum_{l\in\Lambda} \left(\langle W(h_1)\rho(-x,-y)e_l, e_l\rangle + j\,\langle W(h_2)\rho(-x,-y)e_l, e_l\rangle\right), \end{aligned}$$

where $g = h_1 + jh_2$ and h_1, h_2 are complex valued functions. In view of (4.3.1) and

(4.3.9), we have

$$\begin{aligned}\operatorname{tr}\left(W(g)\rho(-x,-y)P_-\right) &= \sum_{l\in\Lambda}\left(\langle\mathcal{W}(h_1)\pi(-x,-y)e_l, e_l\rangle + j\,\langle\mathcal{W}(h_2)\pi(-x,-y)e_l, e_l\rangle\right)\\ &= \sum_{l\in\Lambda}\left(\langle\mathcal{W}(h_1)\pi(-x,-y)\tilde{e}_l, \tilde{e}_l\rangle + j\,\langle\mathcal{W}(h_2)\pi(-x,-y)\tilde{e}_l, \tilde{e}_l\rangle\right),\end{aligned}$$

where the last equality holds as $(\frac{1-k}{\sqrt{2}})(\frac{1+k}{\sqrt{2}}) = 1$. Hence,

$$\operatorname{tr}\left(W(g)\rho(-x,-y)P_-\right) = \operatorname{tr}\left(\mathcal{W}(h_1)\pi(-x,-y)\right) + j\ \operatorname{tr}\left(\mathcal{W}(h_2)\pi(-x,-y)\right).$$

Thus, by (4.3.4) we get

$$\operatorname{tr}\left(W(g)\rho(-x,-y)P_-\right) = h_1(x,y) + jh_2(x,y) = g(x,y).$$

□

Let $g \in L^2(\mathbb{R}^4, \mathbb{Q})$ and write $g = h_1 + jh_2$ for complex valued functions h_1 and h_2. If we follow the proof of Theorem 4.3.1, then we can have

$$\|W(g)P_-\|_{HS}^2 = \|\mathcal{W}(h_1)\|_{HS}^2 + \|\mathcal{W}(h_2)\|_{HS}^2 = \|g\|_2^2, \tag{4.3.10}$$

where the last equality obeys from the Plancherel formula (4.3.5).

Let $A \subset \mathbb{R}^4$ be of the form

$$A = A^{(1,3)} \times A^{(2,4)}, \tag{4.3.11}$$

where $A^{(1,3)} = \{(x_1, y_1) \in \mathbb{R}^2 : (x_1, x_2, y_1, y_2) \in A\}$ and $A^{(2,4)} = \{(x_2, y_2) \in \mathbb{R}^2 : (x_1, x_2, y_1, y_2) \in A\}$ are the projections of A in X_1Y_1-plane and X_2Y_2-plane, respectively. Then the following is the main result of the section.

Theorem 4.3.2. *Let* $g \in L^1(\mathbb{R}^4, \mathbb{Q})$ *and* $\{(x, y) \in \mathbb{R}^4 : g(x, y) \neq 0\} \subseteq A$, *where* $A = A^{(1,3)} \times A^{(2,4)}$ *is defined as in (4.3.11). Suppose* $A^{(1,3)}$ *and* $A^{(2,4)}$ *have finite* 2*-dimensional Lebesgue measure. If* $W(g)$ *has finite rank, then* $g = 0$.

Let $g \in L^2(\mathbb{R}^4, \mathbb{Q})$, and $W(g)$ be a finite rank operator. Then there exists an orthonormal basis $\{\varphi_1, \varphi_2, \ldots\}$ of $L^2(\mathbb{R}^2, \mathbb{Q})$ such that $\mathcal{R}(W(g)) = S$, where $S = \operatorname{span}\{\varphi_1, \ldots, \varphi_N\}$ and $\mathcal{R}$ stands for the range. Define the orthogonal projection P_S of $L^2(\mathbb{R}^2, \mathbb{Q})$ onto S. Let A be a measurable subset of $\mathbb{R}^4$. Define a pair of orthogonal projections E_A and F_S of $L^2(\mathbb{R}^4, \mathbb{Q})$ by

$$E_A g = \chi_A g \qquad \text{and} \qquad W(F_S g) = P_S W(g), \tag{4.3.12}$$

where χ_A denotes the characteristic function of A. Then $\mathcal{R}(E_A) = \{g \in L^2(\mathbb{R}^4, \mathbb{Q}) : g = \chi_A g\}$ and $\mathcal{R}(F_S) = \{g \in L^2(\mathbb{R}^4, \mathbb{Q}) : \mathcal{R}(W(g)) \subseteq S\}$.

Now we compute the Hilbert-Schmidt norm of $E_A F_S$, when $m(A)$ is finite.

Lemma 4.3.3. *The operator* $E_A F_S$ *is an integral operator with kernel*

$$K(x, y, u, v) = \chi_A(x, y) tr\left(P_S \rho(u, v) \rho(-x, -y) P_-\right),$$

where $x, y, u, v \in \mathbb{R}^2$.

Proof. For $g \in L^2(\mathbb{R}^4, \mathbb{Q})$, we have $W(F_S g) = P_S W(g)$. Then by Theorem 4.3.1, the inversion formula for the Weyl transform, we have

$$\begin{aligned}(F_S g)(x, y) &= \operatorname{tr}\left(W(F_S g) \rho(-x, -y) P_-\right) = \operatorname{tr}\left(P_S W(g) \rho(-x, -y) P_-\right) \\ &= \int_{\mathbb{R}^4} g(u, v) \operatorname{tr}\left(P_S \rho(u, v) \rho(-x, -y) P_-\right) du dv.\end{aligned}$$

Hence, we can write

$$\begin{aligned}(E_AF_Sg)(x,y) &= \chi_A(x,y)(F_Sg)(x,y)\\ &= \chi_A(x,y)\int_{\mathbb{R}^4} g(u,v)\mathrm{tr}\left(P_S\rho(u,v)\rho(-x,-y)P_-\right)dudv\\ &= \int_{\mathbb{R}^4} g(u,v)K(x,y,u,v)dudv,\end{aligned}$$

where $K(x,y,u,v)=\chi_A(x,y)\mathrm{tr}\left(P_S\rho(u,v)\rho(-x,-y)P_-\right).$ □

Lemma 4.3.4. *E_AF_S is Hilbert-Schmidt satisfying $\|E_AF_S\|_{HS}^2\leq m(A)N^2$, where $N=$ $\dim(S)$.*

Proof. It is proved in Lemma 4.3.3 that E_AF_S is an integral operator with kernel $K(x,y,u,v)$. Therefore,

$$\begin{aligned}\|E_AF_S\|_{HS}^2 &= \int_{\mathbb{R}^4}\int_{\mathbb{R}^4}|K(x,y,u,v)|^2dudvdxdy\\ &= \int_{\mathbb{R}^4}|\chi_A(x,y)|^2\left(\int_{\mathbb{R}^4}\left|\mathrm{tr}\left(P_S\rho(u,v)\rho(-x,-y)P_-\right)\right|^2dudv\right)dxdy\\ &= \int_{\mathbb{R}^4}\chi_A(x,y)\left(\int_{\mathbb{R}^4}\left|\sum_{j=1}^N\langle\rho(u,v)\rho(-x,-y)P_-\varphi_j,\varphi_j\rangle\right|^2dudv\right)dxdy.\end{aligned} \tag{4.3.13}$$

In view of (4.3.9), the above inner integral reduces to

$$\int_{\mathbb{R}^4}\left|\sum_{j=1}^N\langle\pi(u,v)\pi(-x,-y)P_-\varphi_j,\varphi_j\rangle\right|^2dudv=\int_{\mathbb{R}^4}\left|\sum_{j=1}^N\langle\pi(u,v)\psi_j^{x,y},\varphi_j\rangle\right|^2dudv,$$

where $\psi_j^{x,y}=\pi(-x,-y)P_-\varphi_j$ and $\|\psi_j^{x,y}\|_2\leq\|\varphi_j\|_2=1$ for all $x,y\in\mathbb{R}^2$. Then using (4.3.3), we get

$$\int_{\mathbb{R}^4}\left|\sum_{j=1}^N\langle\pi(u,v)\psi_j^{x,y},\varphi_j\rangle\right|^2dudv\leq N\sum_{j=1}^N\int_{\mathbb{R}^4}\left|\langle\pi(u,v)\psi_j^{x,y},\varphi_j\rangle\right|^2dudv\leq N^2.$$

Hence from (4.3.13), we have $\|E_AF_S\|_{HS}^2\leq m(A)N^2$. □

We need the following result from [8], which describes an interesting property of measurable sets with finite measure. Let denote $uB = \{x \in \mathbb{R}^n : x - u \in B\}$.

Lemma 4.3.5. [8] *Let B be a measurable set in $\mathbb{R}^n$ with $0 < m(B) < \infty$. If B_0 is a measurable subset of B with $m(B_0) > 0$, then for $\epsilon > 0$, there exists $u \in \mathbb{R}^n$ such that*

$$m(B) < m(B \cup uB_0) < m(B) + \epsilon.$$

Let E and F be orthogonal projections on a Hilbert space $\mathcal{H}$. Denote $E \cap F$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(E) \cap \mathcal{R}(F)$. Then, we have the relation

$$\|E \cap F\|_{HS}^2 = \dim \mathcal{R}(E \cap F) \leq \|EF\|_{HS}^2. \tag{4.3.14}$$

Proposition 4.3.6. *Let $A = A^{(1,3)} \times A^{(2,4)}$ be defined as in (4.3.11). Suppose $A^{(1,3)}$ and $A^{(2,4)}$ have finite 2-dimensional Lebesgue measure. Then the projection $E_A \cap F_S = 0$.*

Proof. Assume towards contrary that there exists a non-zero function $g_0 \in \mathcal{R}(E_A \cap F_S)$. Then $\mathcal{R}(W(g_0)) \subseteq S$. Let

$$\begin{aligned} A_0^{(1,3)} = \{(x_1, y_1) \in A^{(1,3)} : \exists\, I_{(x_1,y_1)} \subseteq A^{(2,4)} \text{ with } m(I_{(x_1,y_1)}) > 0 \\ \text{and } g_0(x_1, x_2, y_1, y_2) \neq 0\, \forall\, (x_2, y_2) \in I_{(x_1,y_1)}\}. \end{aligned} \tag{4.3.15}$$

Clearly, $0 < m(A_0^{(1,3)}) < \infty$. Choose $s \in \mathbb{N}$ such that

$$s > 2\, m(A_0^{(1,3)}) m(A^{(2,4)}) N^2,$$

where $N = \dim(S)$. Now, we construct an increasing sequence of measurable sets $\{A_l^{(1,3)} : l = 1, \dots, s\}$ containing $A_0^{(1,3)}$. Let $B_0 = A_0^{(1,3)}$ and $B = A_{l-1}^{(1,3)}$, by Lemma 4.3.5, for $\epsilon = \frac{1}{2m(A^{(2,4)})N^2}$ there exists $u_l = (u_l^{(1)}, u_l^{(2)}) \in \mathbb{R}^2$ such that

$$m(A_{l-1}^{(1,3)}) < m(A_{l-1}^{(1,3)} \cup u_l A_0^{(1,3)}) < m(A_{l-1}^{(1,3)}) + \frac{1}{2m(A^{(2,4)})N^2}.$$

Write $A_l^{(1,3)} = A_{l-1}^{(1,3)} \cup u_l A_0^{(1,3)}$ and $A_s = A_s^{(1,3)} \times A^{(2,4)}$. In view of (4.3.14) and Lemma 4.3.4, we obtain

$$\begin{aligned}\dim \mathcal{R}(E_{A_s} \cap F_S) &\leq m(A_s)N^2 = m(A_s^{(1,3)})m(A^{(2,4)})N^2 \\ &\leq \left(m(A_0^{(1,3)}) + \frac{s}{2m(A^{(2,4)})N^2} \right) m(A^{(2,4)})N^2 < s. \qquad (4.3.16)\end{aligned}$$

Next, we shall produce $s+1$ linearly independent functions in $\mathcal{R}(E_{A_s} \cap F_S)$. Let

$$g_l(x,y) = g_0(x_1 - u_l^{(1)}, x_2, y_1 - u_l^{(2)}, y_2)e^{\pi i(y_1 u_l^{(1)} - x_1 u_l^{(2)})}.$$

We shall show that $g_l \in \mathcal{R}(F_S)$ for each $l = 1, \ldots, s$. To do so, let $\varphi \in L^2(\mathbb{R}^2, \mathbb{Q})$ and $j > N$. Then

$$\begin{aligned}\langle W(g_l)\varphi, \varphi_j \rangle &= \int_{\mathbb{R}^4} g_l(x_1, x_2, y_1, y_2)\langle \rho(x_1, x_2, y_1, y_2)\varphi, \varphi_j \rangle dx_1 dx_2 dy_1 dy_2 \\ &= \int_{\mathbb{R}^4} g_0(x_1 - u_l^{(1)}, x_2, y_1 - u_l^{(2)}, y_2)e^{\pi i(y_1 u_l^{(1)} - x_1 u_l^{(2)})}\langle \rho(x,y)\varphi, \varphi_j \rangle dx dy \\ &= \int_{\mathbb{R}^4} g_0(x,y)e^{\pi i(y_1 u_l^{(1)} - x_1 u_l^{(2)})}\langle \rho(x_1 + u_l^{(1)}, x_2, y_1 + u_l^{(2)}, y_2)\varphi, \varphi_j \rangle dx dy.\end{aligned}$$

From (4.3.8), we have

$$\rho(x_1, x_2, y_1, y_2)\rho(u_l^{(1)}, 0, u_l^{(2)}, 0) = e^{\pi i(y_1 u_l^{(1)} - x_1 u_l^{(1)})}\rho(x_1 + u_l^{(1)}, x_2, y_1 + u_l^{(2)}, y_2).$$

Thus,

$$\begin{aligned}\langle W(g_l)\varphi, \varphi_j \rangle &= \int_{\mathbb{R}^4} g_0(x,y)\langle \rho(x_1, x_2, y_1, y_2)\rho(u_l^{(1)}, 0, u_l^{(2)}, 0)\varphi, \varphi_j \rangle dx dy \\ &= \int_{\mathbb{R}^4} g_0(x,y)\langle \rho(x,y)\psi, \varphi_j \rangle dx dy = \langle W(g_0)\psi, \varphi_j \rangle = 0.\end{aligned}$$

Hence $\mathcal{R}(W(g_l)) \subseteq S$. Since, $A_m = (A_0^{(1,3)} \cup u_1 A_0^{(1,3)} \cup \cdots \cup u_m A_0^{(1,3)}) \times A^{(2,4)}$ and $g_l = 0$ on $(u_l A_0^{(1,3)} \times A^{(2,4)})^c$, we have $E_{A_m} g_l = g_l$ for $l = 1, \ldots, m$. Furthermore, $E_{A_m \setminus A_{m-1}} g_l = 0$ for $l = 1, \ldots, m-1$ and, in view of (4.3.15), $E_{A_m \setminus A_{m-1}} g_m \neq 0$. Therefore g_m cannot be written as a linear combination of $g_0, \ldots, g_{m-1}$. Thus, $g_0, \ldots, g_s \in \mathcal{R}(E_{A_s} \cap F_S)$, are linearly independent functions, which contradicts (4.3.16). $\square$

Proof of Theorem 4.3.2. For $g \in L^2(\mathbb{R}^4, \mathbb{Q})$, the result follows from Proposition 4.3.6. Further, if $g \in L^1(\mathbb{R}^4, \mathbb{Q})$, then (4.3.10) and the finite rank assumption on $W(g)$ give $g \in L^2(\mathbb{R}^4, \mathbb{Q})$. This completes the proof. $\square$

Beurling's theorem. The version of Beurling's theorem for the Fourier-Weyl transform on step two nilpotent Lie groups proved in [50] can be generalized for the quaternion Fourier-Weyl transform.

For $\xi = (\xi', \xi'')$, where $\xi', \xi'' \in \mathbb{R}^2$, define the quaternion Fourier-Weyl transform of $f \in L^1(\mathbb{R}^4, \mathbb{Q})$ by

$$\tilde{\mathcal{F}}(W(f))(\xi) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2\pi i(x_1\xi_1'' - y_1\xi_1')} f(x,y)\rho(x,y) e^{2\pi j(x_2\xi_2'' - y_2\xi_2')} dxdy.$$

Then, we have the following version of Beurling's theorem.

Theorem 4.3.7. *Let $f \in L^1(\mathbb{R}^4, \mathbb{Q})$ be such that*

$$\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} |f(x,y)| \left|\langle \tilde{\mathcal{F}}(W(f))(\xi)\varphi, \varphi\rangle\right| e^{2\pi|x\cdot\xi'' - y\cdot\xi'|} dxdyd\xi' d\xi'' < \infty, \tag{4.3.17}$$

where $\varphi(x) = e^{-\pi|x|^2}$ is the Gaussian. Then $f = 0$.

Proof. Let $g(x,y) = f(x,y)\langle \rho(x,y)\varphi, \varphi\rangle$, then $\langle \tilde{\mathcal{F}}(W(f))(\xi)\varphi, \varphi\rangle$ is the Fourier transform of $g(x,y)$ at $(-\xi'', \xi')$. Now from (4.3.17), we get

$$\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} |g(x,y)|\, |\mathcal{F}(g)(-\xi'', \xi')| e^{2\pi|x\cdot\xi'' - y\cdot\xi'|} dxdyd\xi' d\xi'' < \infty.$$

Applying Beurling's theorem [33] for the QFT, we get $g(x,y) = 0$. Since $\langle \rho(x,y)\varphi, \varphi\rangle = e^{-\frac{\pi}{2}(|x|^2+|y|^2)}$, we can conclude that $f = 0$. $\square$

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4.3.2 A remark on injectivity sets for the QTSM

Let recall the spherical mean and associated set of injectivity problems discussed in Chapter 2. Suppose μ_ι be the normalized surface measure on the sphere S_ι^{k-1} of radius ι centred at the origin in $\mathbb{R}^n$. Consider $S \subseteq \mathbb{R}^n$ and $\mathcal{C} \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$. Then S is a set of injectivity for the spherical mean for $\mathcal{C}$ if for every $g \in \mathcal{C}$,

$$g * \mu_\iota(x) = \int_{|y|=\iota} g(x-y)\,d\mu_\iota(y) = 0$$

for all $(\iota, x) \in (0,\infty) \times S$ implies $g = 0$. The injectivity problem was studied for the Heisenberg group in terms of the twisted spherical mean (TSM). A subset $\Gamma \subseteq \mathbb{C}^n$ is a set of injectivity for the TSM for $\mathcal{G} \subseteq L^1_{\text{loc}}(\mathbb{C}^n)$ if for every $g \in \mathcal{G}$,

$$g \times \mu_\iota(z) = \int_{|w|=\iota} g(z-w)e^{\pi i\,\mathrm{Im}(z.\bar{w})}\,d\mu_\iota(w) = 0$$

for all $(\iota, z) \in (0,\infty,) \times \Gamma$ implies $g = 0$. A considerable amount of work has been done in this direction, see [45, 59, 61, 71].

In particular, let μ_ι be the normalized surface measure of the sphere centre at origin and radius ι in $\mathbb{R}^4$. Now, consider $\Lambda \subseteq \mathbb{R}^4$ and $\mathcal{G}_{\mathbb{Q}} = \mathcal{G}\cdot(\frac{1+k}{2}) + \mathcal{G}\cdot(\frac{1-k}{2}) \subseteq L^1_{\text{loc}}(\mathbb{R}^4, \mathbb{Q})$. We say Λ is a set of injectivity for the quaternion twisted spherical mean (QTSM) in $\mathcal{G}_{\mathbb{Q}}$, if for every $f \in \mathcal{G}_{\mathbb{Q}}$,

$$f \times_{\mathbb{Q}} \mu_\iota(p,q) = \int_{|(u,v)|=\iota} e^{\pi i(u_1q_1-p_1v_1)} f(p-u, q-v) e^{\pi j(u_2q_2-p_2v_2)}\,d\mu_\iota(u,v) = 0$$

for all $\iota > 0$ and for all $(p,q) \in \Lambda$ implies $f = 0$. *In view of* (4.1.3), *it can be inferred that* Λ *is a set of injectivity for the QTSM for* $\mathcal{G}_{\mathbb{Q}}$ *if and only if* Λ *and* $\tilde{\Lambda}$ *are set of injectivity for the TSM for* $\mathcal{G}$, *where* $\tilde{\Lambda} = \{(p_1, p_2, q_1, q_2) : (p_1, -p_2, q_1, -q_2) \in \Lambda\}$.

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Helgason's support theorem. In a remarkable result, Helgason proved the following support theorem (see [34]). If g is a continuous function on $\mathbb{R}^n, (n \geq 2)$ such that $|x|^l g(x)$ is bounded for each $l \in \mathbb{Z}_+$, then g is supported in the ball $B_r(0)$ if and only if $g * \mu_\iota(x) = 0, \forall\ x \in \mathbb{R}^n$ and $\forall \iota > |x| + r$. Together with different analogues in various setups, Helgason's support theorem extended for TSM, see [19, 46, 54]. In view of the above discussion, the following is holds true.

Let f be a quaternion valued continuous function on $\mathbb{R}^4$ such that $|(p,q)|^l f(p,q)$ is bounded for each $l \in \mathbb{Z}_+$. Then f is supported in the ball $B_r(0)$ if and only if $f \times_{\mathbb{Q}} \mu_\iota(p,q) = 0, \forall\ (p,q) \in \mathbb{R}^4$ and $\forall \iota > |(p,q)| + r$.

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Chapter 5

Concluding Remarks

In Chapter 2, we prove the necessary conditions for a function to be in $Z^*_{r,R}$, and some results on the H-type groups. It would be interesting to establish the sufficient conditions for a function to be in $Z^*_{r,R}$, and extension of the results for H-type groups to Métivier groups.

In the recent article [44], it has been investigated the injectivity results for three different spherical means on H-type groups. The first one is the standard spherical mean, the average of a function over the spheres in the complement of the centre, which is similar to the spherical mean considered in this thesis. The second one is bi-spherical mean, the average over the product of spheres in the centre and its complement, i.e.,

$$f * \mu_{r,s}(z,t) = \int_{|w|=r} \int_{|u|=s} f\left(z-w, t-u-\frac{1}{2}[z,w]\right) d\mu_r \ d\nu_s,$$

where $\mu_{r,s} = \mu_r \times \nu_s$, ν_s is the normalized surface measure on the sphere $\{t \in \mathfrak{z} : |t| = s\}$. The third one is the homogeneous spherical mean. The H-type group admits a family of dilations acting as automorphisms by

$$\delta_r(z,t) = \left(rz, r^2 t\right), \ r > 0.$$

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This family of dilations make the H-type group a homogeneous Lie group whose homogeneous dimension is $Q = 2n + 2m$. Then the Korányi norm is defined as

$$|(z,t)| = \left(|z|^4 + |t|^2\right)^{1/4}, \ |\delta_r(z,t)| = r|(z,t)|.$$

There exists a unique Radon measure σ on the unit sphere $\Sigma = \{(z,t) : |(z,t)| = 1\}$ such that integral of integrable function can be written in polar coordinates as

$$\int_G f(z,t)dzdt = \int_0^\infty \int_\Sigma f\left(\delta_r(z,t)\right) d\sigma r^{Q-1} dr.$$

For $r > 0$, define $\sigma_r = \delta_r(\sigma)$ by

$$\int_G f(z,t)d\sigma_r = \int_G f\left(\delta_r(z,t)\right) d\sigma.$$

Then the homogeneous spherical mean of a function f is defined as

$$f * \sigma_r(z,t) = \int_\Sigma f\left((z,t)\delta_r(w,s)^{-1}\right) d\sigma.$$

Therefore, it is natural to ask whether the results in this thesis proved for spherical mean can be generalized for bi-spherical mean and homogeneous spherical mean.

In the article [63], it has been proved that for $f \in L^p(\mathbb{H}^n), \ 1 < p < \infty$,

$$\lim_{r\to 1} \sum_{k=0}^{\infty} r^k \int_{-\infty}^{\infty} f * e_k^\lambda d\mu(\lambda) = f,$$

where the limit exists in the L^p-norm. Recently, this summability result has been extended to the H-type groups in [44]. We would like to investigate a similar summability result for functions in the Métivier groups. Then it will help to extend the injectivity of spherical mean operator for $p > 2$, which was proved for $1 \leq p \leq 2$ in chapter 3.

In chapter 4, we have posed some restrictions on symbols for some results. That is we consider symbols, which are even in some variables. We would like to explore if such restrictions can remove for those results.

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List of communicated papers

1. R. K. Dalai, S. Ghosh and R. K. Srivastava, *Spherical means on Métivier groups and support theorems,* arXiv:2108.11744.
2. R. K. Dalai and R. K. Srivastava, *Injectivity of Spherical Mean on the Métivier Group,* arXiv:2108.12729.
3. R. K. Dalai, S. Ghosh and R. K. Srivastava, *Boundedness and uniqueness of quaternion Weyl transform,* **J. Pseudo-Differ. Oper. Appl. (in press)**.

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