On Nonnegative Matrices and Generalized M-matrices

by

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by

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Declaration

I do hereby declare that the work contained in this thesis entitled "**On Non-negative Matrices and Generalized M-matrices**" has been carried out by me, under the supervision of Dr. Sriparna Bandopadhyay, Assistant Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in this thesis entitled "**On Nonnegative Matrices and Generalized M-matrices**" by Ms. Manideepa Saha, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

July 18, 2013

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Manideepa Saha
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To my Parents....

More than ever.

There are men who struggle for a day, and they are good.

There are men who struggle for a year, and they are better.

There are some who struggle many years, and they are better still.

But there are those who struggle all their lives, and these are the indispensable ones.

Life of Galileo by Bertolt Brecht

Abstract

This dissertation deals mainly with analyzing and characterizing certain classes of matrices, nonnegative matrices and generalizations of M-matrices. More specifically, this involves a detailed study of the structure of the generalized eigenspace of nonnegative matrices and the generalized nullspace of generalized M-matrices. We first use preferred and quasi-preferred bases of generalized eigenspaces associated with the spectral radius of nonnegative matrices to analyze the existence and uniqueness of a variant of the Jordan canonical form which we call a *Frobenius-Jordan form*. Based on the Frobenius-Jordan form, spectral and combinatorial properties of nonnegative matrices are discussed. We also consider graph representations of nonnegative bases for nonnegative matrices and derived necessary conditions for the existence of such graph bases.

Next we consider two types of generalized M-matrices based on the generalization of nonnegative matrices, namely the class of GM-matrices and $M_\vee$-matrices. In this thesis we attempt to generalize the combinatorial properties of singular M-matrices to the class of singular GM-matrices and singular $M_\vee$-matrices. We prove the existence of a preferred basis for a subclass of $M_\vee$-matrices and obtain similar equivalent conditions for the equality of the height and level characteristics of $M_\vee$-matrices. In an attempt to obtain similar results for the class of GM-matrices, we give a complete answer regarding the existence of a preferred basis in terms of the order of such matrices. We also provide some characterizations of nonsingular $M_\vee$-matrix involving positivity of the sums of principal minors and stability. We show that some of the important properties, such as inverse positivity, do not carry over to the entire class of $M_\vee$-matrices, but to a subclass of these matrices. We also extend the inverse-positivity property of nonsingular $M_\vee$-matrices to generalized inverses of singular $M_\vee$-matrices. Motivated by interesting characterizations of singular M-matrices, we then introduce the concepts of eventually monotonicity and eventually nonnegativity on subsets of $\mathbb{R}^n$, which are used to characterize a subclass of $M_\vee$-matrices.

Contents

List of Symbols

Important symbols are listed below:

$\|S\|$	:	cardinality of the set S.
$\mathbb{N}$	:	the set of all natural numbers.
$\mathbb{R}$	:	the set of all real numbers.
$\mathbb{R}^{n,n}$	:	the set of $n \times n$ real matrices A.
$\mathbb{R}^n$	:	the set of $n \times 1$ vectors x.
a_{ij}	:	$(i, j)th$ entry of the matrix A.
x_i	:	ith coordinate of the vector x.
$\langle n \rangle$	:	the set $\{1, 2, \ldots, n\}$.
$\Gamma(A)$	:	graph of the matrix A.
$R(A)$	:	reduced graph of the matrix A.
$N(A)$	:	the null space of A.
$n(A)$	:	nullity of A.
range(A)	:	the range space of A.
$\sigma(A)$	:	spectrum of A.
$\rho(A)$	:	spectral radius of A.

Chapter 1

Introduction

1.1 Prologue

This dissertation deals mainly with analyzing and characterizing certain classes of matrices which are essentially generalizations of the class of M-matrices. The class of M-matrices introduced in 1937 by Ostrowski [42, 43], are of the form $sI - B$ where B is a nonnegative matrix with spectral radius $\rho(B)$ and $s > 0$ satisfies $s \geqslant \rho(B)$. Some of the most beautiful and elegant applications of the classical theory of M-matrices arise in the study of Markov chains in probability and statistics. The theory of Markov process comprises the largest and one of the most important part of the theory of stochastic processes. This importance is further enhanced by many applications it has found in the physical, biological, and social sciences as well as in engineering and commerce. In addition, many of the modern methods for analyzing these chains make strong use of recent developments in the theory of singular M-matrices and generalized matrix inverse. Another important discipline in mathematical science in which theory of M-matrices find elegant applications is economics. In particular, theory of M-matrices are used to greatly simplify the construction and the analysis of Leontief's input-output models, which basically deals with the particular question: *What level of output should each of n industries in a particular economic situation produce, in order that it will just be sufficient to satisfy the total demand of economy for that product?* M-matrices also arise very frequently in investigations concerning the convergence of iteration processes in matrix computations in relation to systems of linear equations

$Ax = b$. The reason for this is that very often in practical situations, A has positive diagonal entries and nonpositive off-diagonal entries, so that the iteration matrices for the Jacobi, Gauss-Seidel and S.O.R. (for the relaxation parameter ω satisfying $0 < \omega < 1$) methods are necessarily nonnegative.

Since the concept of an M-matrix is applied in such diverse areas of science, there has been extensive research carried out on this class mainly in two directions. One is in determining and analyzing the properties of singular reducible M-matrices, the relationship between the spectral and graph theoretic properties of these matrices. The otherone is the study of nonsingular M-matrices in trying to obtain equivalent characterizations of the class based on the entrywise nonnegativity property of the inverse.

The study of the connection between the graph theoretic and the spectral properties of a matrix was motivated by Perron and Frobenius's work on the spectral radius of a nonnegative matrix. It was shown by Perron [6] and then by Frobenius [14–16] that (entry-wise) positive matrices and irreducible nonnegative matrices have the spectral radius as one of its eigenvalues. Furthermore, it is a simple eigenvalue, called the Perron root, and the corresponding eigenspace is spanned by a positive vector called the Perron eigenvector. Subsequently, attempts were made to generalize some of these results to the class of general nonnegative matrices which may be reducible, but the results obtained were considerably weaker.

In his Ph.D dissertation [46], Schneider studied the combinatorial structure of the generalized eigenspace corresponding to the spectral radius of a reducible nonnegative matrix or equivalently the generalized nullspace of a reducible singular M-matrix, which was initiated by Frobenius [16]. He observed that only under certain extreme conditions, the height characteristic of an M-matrix, which describes the analytic structure of the generalized nullspace of the matrix, is equal to its level characteristic, which is determined by the singular graph of the matrix. Since for an arbitrary M-matrix the height and the level characteristics are not equal, he questioned the nature of the relationship between them for the class of M-matrices and the possible conditions under which the two characteristics are equal. These questions were answered after a majorization relation between the two characteristics was established for the class of M-matrices and some equivalent conditions for the equality of the two were obtained. However, these results could only be proved after the existence of preferred bases was established for the generalized nullspace of singular M-matrices. A preferred basis is a nonnegative basis of the generalized eigenspace having certain combinatorial properties determined by the graph theoretic structure of the matrix written in Frobenius normal form. The initial contribution in this direction was made by Rothblum [45], Richman and Schneider [44]. Rothblum combined the combinatorial structure of nonnegative matrices, which was developed by Schneider [47] and Carlson [5], with the Perron-Frobenius theory of nonnegative matrices to obtain a nonnegative basis having similar combinatorial properties as the preferred basis for the generalized eigenspace of the spectral radius. Later Richman and Schneider [44] proved the existence of a preferred basis for the generalized nullspace of an M-matrix.

As the eigenvalues and the entries of the eigenvectors of a matrix are continuous functions of the entries of the matrix, there may exist matrices having small negative entries, but for which the spectral radius is an eigenvalue and there is a nonnegative eigenvector associated with this eigenvalue. This observation motivates many researchers, see [8,29,39], to study this class of matrices. This class of matrices is said to satisfy the Perron-Frobenius property.

Recently, Noutsos [39] obtained necessary and sufficient conditions for a matrix to have a Perron eigenvector and extended some of the results of Perron and Frobenius for nonnegative matrices to the class of matrices satisfying the Perron-Frobenius property. In [33], Naqvi and McDonald introduced another generalization of the class of nonnegative matrices, known as eventually nonnegative matrices. They studied the combinatorial properties of the generalized eigenspace associated with the spectral radius of this class of matrices.

1.2 The relevance and aim of the topic of research

In this dissertation we focus on studying the combinatorial properties of nonnegative matrices and of generalized M-matrices. An extensive theory on the properties of M-matrices is developed due to their role in numerical analysis (e.g., splittings in iterative methods and discretization of differential equations), modelling of the economy, optimization and Markov chains [1, 2]. Researchers are continuously interested to generalize the notion of M-matrices based on the generalizations of nonnegative matrices. For a general overview on this, we refer the reader to [7,8,29,33,39–41]. As mentioned earlier, in this dissertation we consider two types of generalizations of M-matrices, namely GM-matrices and $M_\vee$-matrices, and study the combinatorial structure of their generalized nullspace.

In [39], Noutsos generalized the class of nonnegative matrices based on the well known Perron-Frobenius theory of this class. It is obvious from the continuity of the eigenvalues and the entries of the eigenvectors as functions of the entries of matrices that the Perron-Frobenius property may also hold for matrices having some small negative entries. In [49], Tarazaga et al. presented a sufficient condition that guarantees the existence of a Perron-Frobenius eigenpair for the class of real symmetric matrices with some negative entries. Motivated by their work, Noutsos in [39], generalized the class of nonnegative matrices to the class of matrices A such that both A and A^T possess the Perron-Frobenius property and in [8], Elhashash and Szyld termed this class of matrices as $WPFn$. Based on this generalization of nonnegative matrices, Elhashash and Szyld in [7], generalized the class of M-matrices and called it GM-matrices. They extended some analogous results known to be true for M-matrices to the class of GM-matrices. In particular, they proved that if A is a nonsingular GM-matrix, then $A^{-1} \in WPFn$.

Among the other generalizations of nonnegative matrices, eventually nonnegative matrices are the focus of study in recent years by many researchers, partly because they generalize and shed new light on the class of nonnegative matrices (see [29, 33, 34, 39, 53, 54]) and partly because of their applications in systems theory (see [40,50–52]). A matrix A is called an eventually nonnegative matrix if there exists a $k_0 \in \mathbb{N}$ such that A^k is a nonnegative matrix for all $k \geqslant k_0$. In dynamical systems, one is frequently interested in qualitative information regarding state evaluation. In particular, due to physical and modelling constraints arising in engineering, biological, medical and economic applications, it is of common interest to impose condition for nonnegative states [1]. Such applications typically arise in the theory of linear systems,

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n,n}, \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geqslant 0. \tag{1.1}$$

The solution is given by $x(t) = e^{tA}x_0, \quad t \geqslant 0$ and the set

$$\{x(t) = e^{tA}x_0 \mid t \in [0, \infty)\}$$

is known as the trajectory emanating from x_0. Then the trajectory emanating from x_0 becomes nonnegative and remains nonnegative if there exists a $t_0 \geqslant 0$ such that $e^{tA} \geqslant 0$ for all $t \geqslant t_0$ and such a matrix A is termed as an eventually exponentially nonnegative matrix. In [40], Noutsos and Tstatsomeros proved that eventually nonnegative matrices having the largest Jordan block of size at most 1, corresponding to the eigenvalue 0 are eventually exponentially nonnegative matrices. Thus the combination of such a matrix with the initial point x_0 results in trajectories that reach and stay in the nonnegative orthant.

The relation between the combinatorial structure of nonnegative matrices and their spectrums, eigenvectors, and Jordan structure is interesting as well as useful and an overview of this can be found in [2, 21, 48]. Frieldland [13], Handelman [20], Zaslavsky and Tam [54] and Zaslavsky and McDonad [53] looked at extending these results on the combinatorial structure to eventually nonnegative matrices, but in their study they found that this relationship was inconsistent. For example, the eventually nonnegative matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is irreducible with the spectral radius $\rho(A) = 2$ appearing as an eigenvalue with multiplicity two. Again the irreducible nonnegative matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

has its spectral radius 4 as a simple eigenvalue, but the associated eigenvector $x = [0\ 0\ 1\ 1]^T$ is not positive. But in [33], Naqvi and McDonald showed that the nilpotent part of an eventually nonnegative matrix contributes significantly in determining the combinatorial structure for the same. Many combinatorial properties, including the above mentioned properties of nonnegative matrices carried over to the class of eventually nonnegative matrices if the matrix is nonnilpotent.

Olesky, Tsatsomeros and Driessche [41] generalized the class of M-matrices based on eventually nonnegative matrices and termed them as $M_\vee$-matrices, that is, an $M_\vee$-matrix has the form $sI - B$ where $0 < \rho(B) \leqslant s$ and B is eventually nonnegative. They showed that $M_\vee$-matrices satisfied many properties analogous to that of M-matrices. As mentioned earlier Schneider and Hershkowitz, in [23–25], studied extensively the combinatorial structure of the generalized nullspace of M-matrices and the relation between the spectral and graph theoretic properties of these matrices. Motivated by their work, in this thesis, we try to generalize the combinatorial properties of singular M-matrices to the class of GM-matrices and $M_\vee$-matrices. In this dissertation, we study the Preferred Basis Theorem (see [45]) for a subclass of $M_\vee$-matrices and try to obtain similar equivalent conditions for the equality of the height and level characteristics of $M_\vee$-matrices. We also study the combinatorial properties of these matrices. In an attempt to obtain similar results for the class of GM-matrices, we give a complete answer regarding the existence of a preferred basis in terms of the order of such matrices. We use the characteristics of preferred and quasi-preferred bases of generalized eigenspaces associated with the spectral radius of nonnegative matrices to analyze the existence and the uniqueness of a variant of the Jordan canonical form which we call the *Frobenius-Jordan form*. Based on this Frobenius-Jordan form, interesting results related to the spectral and combinatorial properties of nonnegative matrices are obtained.

The study of nonsingular M-matrices was initiated by Ostrowski and was later continued and consolidated by various others like Fan [9], and Fiedler and Ptak [12]. They presented some useful characterizations of this class in terms of some easily verifiable conditions. Schneider later tried to obtain analogous results for the class of singular M-matrices. We in this dissertation try to extend some of these results to the class of GM and $M_\vee$-matrices and obtain some interesting results on the generalized inverses of these matrices.

1.3 Thesis overview

The proposed thesis attempts to contribute to the theory of nonnegative matrices and generalized M-matrices. The contents of the chapters in the remaining part of the thesis are described briefly as follows:

Chapter 2:

In this chapter we discuss the connection between the spectral and the combinatorial properties of nonnegative matrices. Because of their nice combinatorial properties,preferred basis and quasi-preferred basis have special significance on the study of nonnegative matrices. In this chapter we use these bases to analyze the existence and uniqueness of a variant of the Jordan canonical form called the *Frobenius-Jordan form.* We also investigate some graph theoretic properties of nonnegative matrices with the help of its Frobenius-Jordan form. Furthermore, we obtain permuted graph representations of nonnegative bases for nonnegative matrices. We derive necessary conditions for the existence of such graph bases.

Chapter 3:

In this chapter we study the connection between the combinatorial and the spectral structure of the generalized null space of generalized M-matrices, namely the class of GM-matrices and the class of $M_\vee$-matrices. We prove the Preferred Basis Theorem for a subclass of $M_\vee$-matrices and obtain similar equivalent conditions as that of M-matrices for the equality of the height and level characteristics of this class of matrices.

Chapter 4:

In this chapter, we present some useful characterizations of nonsingular $M_\vee$-matrices in terms of stability and positivity of sums of principal minors. We also obtain some results for the class of $M_\vee$-matrices similar to those obtained for M-matrices, for example the inverse-positivity property of nonsingular M-matrices. Next we study a subclass of singular $M_\vee$-matrices, and obtain interesting results on the generalized inverses of these matrices. Lastly, we present some characteristics of matrices of this subclass with reference to their generalized inverses.

Chapter 5:

We conclude by giving a brief summary and analysis of the work reported in the previous chapters. We also discuss the possible future directions of our research.

Chapter 2

The Frobenius-Jordan form of nonnegative matrices

2.1 Introduction

In this chapter we discuss the relation between the spectral and combinatorial properties of nonnegative matrices. It is well-known that for the invariant subspace associated with the spectral radius of a nonnegative matrix, there exist several types of nonnegative bases which have a nice combinatorial structure, see [3, 22–25, 30, 36–38, 44, 45, 48], called *preferred bases and quasi-preferred bases*. We use these bases to analyze the existence and uniqueness of a variant of the Jordan canonical form named *Frobenius-Jordan form*. The said form is a combination of the classical Jordan canonical form [17, 26] in the part associated with the eigenvalues that are different from the spectral radius; while it is like the Frobenius normal form [15] in the part associated with the spectral radius.

The chapter proceeds as follows: Section 2.2 contains some notation and preliminary results mostly introduced in [24]. In Section 2.3 we introduce the *Frobenius-Jordan form* of a matrix and show the existence and uniqueness (up to similarity transformation) of such a form for nonnegative matrices. Furthermore, we investigate some graph theoretic properties of nonnegative matrices with the help of the Frobenius-Jordan form. In Section 2.4 we obtain permuted graph representations of nonnegative bases for nonnegative matrices. We derive necessary conditions for the existence of such graph bases and show that they need not always exist. We conclude with a summary.

2.2 Notation and Preliminaries

This section contains basic notations and some preliminary results, mostly from [24]. We denote the set $\{1, 2, \ldots, n\}$ by $\langle n \rangle$. For a real $n \times m$ matrix $A = [a_{i,j}]$ we use the following terminology and notation.

- $A \geqslant 0$ (A is nonnegative) if $a_{i,j} \geqslant 0$, for all $i \in \langle n \rangle$, $j \in \langle m \rangle$.
- $A > 0$ (A is strictly positive) if $a_{i,j} > 0$, for all $i \in \langle n \rangle$, $j \in \langle m \rangle$.

If $n = m$, then we denote by

- $\sigma(A)$ the spectrum of A.
- $\rho(A) = \max_{\lambda \in \sigma(A)}\{|\lambda|\}$, the spectral radius of A.
- $N(A)$ the nullspace of A, and by $n(A)$ the nullity of A.
- $\text{index}_\lambda(A)$ the size of the largest Jordan block associated with the eigenvalue λ, and if A is singular we simply write $\text{index}_0(A)$ as $\text{index}(A)$.
- $E_\lambda(A)$, the generalized eigenspace of A corresponding to the eigenvalue λ, i. e., $N((\lambda I - A)^n)$. In case A is a singular matrix, we simply write $E(A)$ for $E_0(A)$.

Definition 2.1. *An $n \times n$ matrix A is said to be* reducible *if there exists a permutation matrix Π such that*

$$\Pi A \Pi^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{2.1}$$

where B and D are square matrices or $A = 0$ in case $n = 1$. Otherwise A is called irreducible.

If A is reducible and of the form (2.1), and if a diagonal block is reducible, then this block can be reduced further via permutation similarity. If this process is continued, then finally there exists a suitable permutation matrix Π such that A is in the block triangular form

$$\Pi A \Pi^T = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ 0 & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{pp} \end{bmatrix}, \tag{2.2}$$

where each block A_{ii} is square and is either irreducible or a 1×1 null matrix. This block triangular form is called a *Frobenius normal form* of A. An irreducible matrix consists of one block, is in Frobenius normal form.

If $A = [A_{ij}]$ is an $n \times n$ matrix in Frobenius normal form with p block rows and columns, and when discussing matrix-vector multiplication with A or the structure of eigenvectors of A, we partition vectors b analogously in p vector components b_i conformably with A, and we define, $\text{supp}(b) := \{i \in \langle p \rangle : b_i \neq 0\}$, as the *support* of b.

For the matrix A, the (directed) graph of A denoted by $\Gamma(A)$ is a graph with vertices $1, 2, \dots, n$ in which (i, j) is an edge if and only if $a_{ij} \neq 0$. A *path* from vertex j to vertex m of *length* t is a sequence of t vertices $v_1, v_2, \dots, v_t$ such that (v_l, v_{l+1}) is an edge in $\Gamma(A)$ for $l = 1, 2, \dots, t-1$ where $v_1 = j$ and $v_t = m$. We say a vertex j has *access* to m, if $i = j$ or there is a path from j to m in $\Gamma(A)$ and in this case we write $j \rightarrow m$. We write $j \nrightarrow m$ if j does not have access to m. The *transitive closure* of $\Gamma(A)$ denoted by $\overline{\Gamma(A)}$ is the graph with the same vertex set as that of $\Gamma(A)$ and (i, j) is an edge in $\overline{\Gamma(A)}$ if i has access to j in $\Gamma(A)$. If j has access to m and m has access to j, we say j and m *communicate.* The communication relation is an equivalence relation on $\{1, 2, \dots, n\}$ and an equivalence class α is called a *class* of A. For any two classes α and β of A, we say that α has access to β in $\Gamma(A)$ if there are vertices $i \in \alpha$ and $j \in \beta$ such that i has access to j in $\Gamma(A)$.

The *reduced graph* of A, denoted by $R(A)$ is the graph with vertex set consisting of all the classes in A and (i, j) is an edge in $R(A)$ if and only if i has access to j in $\Gamma(A)$.

For any $\alpha, \beta \subset \{1, 2, \dots, n\}$, $A_{\alpha\beta}$ denotes the submatrix of A whose rows are indexed by α and whose columns are indexed by β. If α is a class of A, then we say that α is a *basic class* if $\rho(A_{\alpha\alpha}) = \rho(A)$, a *singular class* if $A_{\alpha\alpha}$

is singular, an *initial class* if it is not accessed by any other class of A and a *final class* if it does not have access to any other class of A.

A *chain* of classes is a collection of classes such that each class in the collection has access to or from every other class in the collection. A chain of classes with initial class J and final class K is called a chain from J to K. The *length of a chain* is the number of singular classes it contains. We say J has access to K in n *steps* if the length of the longest chain from J to K is n.

Definition 2.2. *For a set W of vertices in the vertex set $V(A)$ of $R(A)$ we introduce the following sets.*

$$\begin{aligned}
\text{below}(W) &= \{i \in V(A) : \textit{ there exists } j \in W \textit{ such that } i \to j\};\\
\text{above}(W) &= \{i \in V(A) : \textit{ there exists } j \in W \textit{ such that } j \to i\};\\
\text{top}(W) &= \{i \in W : j \in W, i \to j \textit{ imply } i = j\};\\
\text{bottom}(W) &= \{i \in W : j \in W, j \to i \textit{ imply } i = j\}.
\end{aligned}$$

Definition 2.3. *[24, 25] Let A be an $n \times n$ singular matrix in Frobenius normal form (2.2), and let $H(A)$ be the collection of all singular vertices in $R(A)$.*

(*i*) *We define the* singular graph $S(A)$ *associated with* $R(A)$ *as the graph with vertex set* $H(A)$ *and* (i, j) *is an edge if and only if* i *has access to* j *in* $R(A)$.

(*ii*) *The* level of a vertex i *in* $R(A)$, *denoted by* level(i), *is the maximal number of singular vertices on a path in* $R(A)$ *that terminates at* i.

(*iii*) *Let* x *be a block-vector with* p *blocks, partitioned according to the Frobenius normal form of* A. *The* level of x, *denoted by* level(x), *is defined to be* $\max\{\text{level}(i) : i \in \text{supp}(x)\}$.

(*iv*) *For a nonzero vector* x *in the generalized nullspace* $E(A)$, *we define the* height of x, *denoted by* height(x), *to be the minimal nonnegative integer* k *such that* $A^k x = 0$.

The other essential objects in our analysis are appropriately chosen sets of basis vectors for the generalized eigenspace associated with the spectral radius.

Definition 2.4. *[23] Let A be a singular matrix in Frobenius normal form (2.2), and let $H(A) = \{\alpha_1, \ldots, \alpha_q\}$, with $\alpha_1 < \ldots < \alpha_q$ being the set of singular vertices in $R(A)$.*

A set of vectors $x^1 = [x_j^1], \ldots, x^q = [x_j^q] \geqslant 0$ is called a quasi-preferred set *for A if*

$$x_j^i > 0 \text{ if } j \to \alpha_i, \text{ and } x_j^i = 0 \text{ if } j \nrightarrow \alpha_i$$

for all $i = 1, \ldots, q$ and $j = 1, \ldots, p$.

If in addition we have

$$-Ax^i = \sum_{k=1}^{q} c_{k,i} x^k, \; i = 1, \ldots, q,$$

where the $c_{k,i}$ satisfy

$$c_{k,i} > 0 \text{ if } \alpha_k \to \alpha_i, i \neq k; \text{ and } c_{k,i} = 0 \text{ if } \alpha_k \nrightarrow \alpha_i \text{ or } i = k$$

then the set of vectors $x^1, \ldots, x^q$ is said to be a preferred set *for A. A (quasi-)preferred set that forms a basis for $E(A)$ is called* (quasi-)preferred basis *for A.*

Important objects that we use to combine the spectral and combinatorial structure of nonnegative matrices are the level and height characteristics.

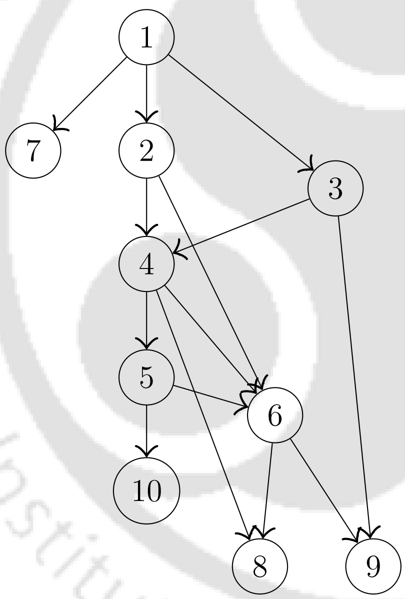
Definition 2.5. *[24, 25] Let $t = \text{index}(A)$. For $i \in \langle t \rangle$, let $\eta_i(A) = n(A^i) - n(A^{i-1})$. The sequence $(\eta_1(A), \ldots, \eta_t(A))$ is called the* height characteristic *of A, and is denoted by $\eta(A)$.*

Definition 2.6. *[25] The cardinality of the jth level of $S(A)$ is denoted by $\lambda_j(A)$. If $S(A)$ has m levels, then the sequence $(\lambda_1(A), \ldots, \lambda_m(A))$ is called a* level characteristic *of A, and is denoted by $\lambda(A)$.*

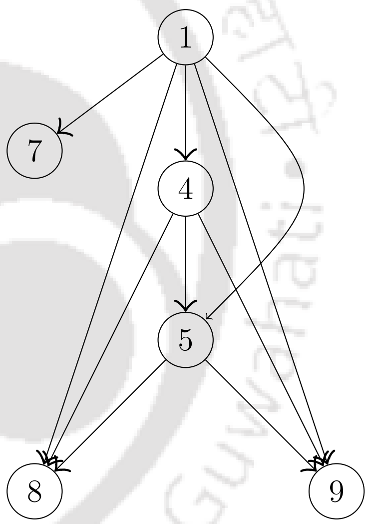
Example 2.1. Let

$$A = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The graph $\Gamma(A)$ of A is:



The singular graph $S(A)$ of A is:



If Δ_i is the collection of all vertices of level i, then $\Delta_1 = \{1,2,3\}, \Delta_2 = \{4,7\}, \Delta_3 = \{5,6,10\}, \Delta_4 = \{8,9\}$.

If we take $x = [0\ -1\ -1\ 1\ 1\ 2\ 0\ 0\ 0\ 0\ 0]^T \in E(A)$, then $\text{level}(x) = 3 = \text{height}(x)$.

Take $W = \{3,4,5,8,10\}$, then $\text{top}(W) = \{8,10\}$ and $\text{bottom}(W) = \{3\}$.

Also we have $\lambda(A) = (1,2,1,2)$ and $\eta(A) = (3,1,1,1)$. □

In the following sections we make frequent use of the interplay of a nonnegative matrix B and the M-matrix $A = \rho(B)I - B$.

Definition 2.7. *An $n \times n$ matrix A is called an* M-matrix *if it can be written as $A = sI - B$, where $B \geqslant 0$ and $s \geqslant \rho(B)$.*

The following results are well-known.

Theorem 2.1. *[24] Let A be an M-matrix. If x is a nonnegative vector in $E(A)$, then* $\text{height}(x) = \text{level}(x)$.

Theorem 2.2. *[24, 45] (Preferred Basis Theorem) If A is an M-matrix, then there exists a preferred basis for the generalized nullspace $E(A)$ of A.*

In the next section we introduce the Frobenius-Jordan form of a nonnegative matrix.

2.3 The Frobenius Jordan Form of a Nonnegative matrix

In this section we prove the existence of a Frobenius-Jordan form for a nonnegative matrix and discuss the combinatorial properties.

Lemma 2.1. *[11, 28] If two non-negative matrices have no zero rows then their product (when defined) has no zero rows either.*

Theorem 2.3. *Let B be an $n \times n$ nonnegative matrix with spectral radius ρ. Then there exists a nonsingular matrix $T = [T_1\ T_2]$ such that the columns of T_1 form a quasi-preferred basis of $\rho I - B$ and*

$$T^{-1}BT = \begin{bmatrix} Z_F & Z_{12} \\ 0 & Z_J \end{bmatrix} = Z, \tag{2.3}$$

where Z_F is nonnegative, in block upper-triangular form

$$Z_F = \begin{bmatrix} \rho I_{n_1} & Z_{1,2} & \dots & Z_{1,t} \\ 0 & \rho I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & Z_{t-1,t} \\ 0 & \dots & \dots & \rho I_{n_t} \end{bmatrix}, \tag{2.4}$$

$\sigma(Z_F) = \{\rho\}$, $\rho \notin \sigma(Z_J)$, and Z_J is in Jordan canonical form. If, furthermore, for $j = 1, \ldots, t-1$, none of the blocks $Z_{j,j+1}$ has a zero column, then the block-sizes $n_1, \ldots, n_t$ are invariant.

Proof. Consider the M-matrix $A = \rho I - B$. Without loss of generality, we may assume that A is in Frobenius normal form (2.2), and let $\alpha_1 < \alpha_2 < \ldots < \alpha_q$ be the singular vertices of A. Since A is an M-matrix, by Theorem 2.2 it follows that A has a preferred basis $\{x^1, x^2, \ldots, x^q\}$ for the generalized nullspace $E(A)$, with $Ax^i = -\sum_{k=1}^{q} \hat{c}_{k,i} x^k$, so that

$$Bx^i = \rho x^i + \sum_{\substack{k=1 \\ k \neq i}}^{q} \hat{c}_{k,i} x^k, \tag{2.5}$$

where the $\hat{c}_{k,i}$ satisfy $\hat{c}_{k,i} > 0$ if $\alpha_k \to \alpha_i$, and $\hat{c}_{k,i} = 0$ if $\alpha_k \nrightarrow \alpha_i$ for $i, k \in \{1, \ldots, q\}$, $i \neq k$. If we set $T_1 = [x^1, \ldots, x^q]$, then equation (2.5) implies that $BT_1 = T_1 Z_F$ with

$$Z_F = \begin{bmatrix} \rho & \hat{c}_{1,2} & \ldots & \hat{c}_{1,q} \\ 0 & \rho & \ldots & \hat{c}_{2,q} \\ \vdots & \vdots & \ddots & \hat{c}_{q-1,q} \\ 0 & 0 & \ldots & \rho \end{bmatrix},$$

nonnegative. Then Z_F can always be partitioned as in (2.4). Since the columns of T_1 are linearly independent, we can extend them to a basis of $\mathbb{R}^n$ to form $T' = [T_1\ T_1']$, i.e., we have

$$BT' = T' \begin{bmatrix} Z_F & Z_{12} \\ 0 & Z_2 \end{bmatrix}$$

for some matrices Z_{12} and Z_2, where Z_2 and Z_F have no common eigenvalue. Then there exists a unique solution W to the Sylvester equation $Z_F W - W Z_2 = -Z_{12}$, (see [17]), and with

$$T = T' \begin{bmatrix} I & W \\ 0 & I \end{bmatrix} \operatorname{diag}(I, V_2) = [T_1\ T_2],$$

where V_2 is a nonsingular matrix such that $V_2^{-1} Z_2 V_2 = Z_J$ is in Jordan canonical form, we have that $T^{-1}BT$ is as in (2.3).

It remains to show that the block-sizes $n_1, \ldots, n_t$ of Z_F are invariant if none of the blocks $Z_{i,i+1}$ in (2.4) has a zero column. Set $m_0 = 0, m_i = n_1 + \ldots + n_i$ and $X^i = [x^{m_{i-1}+1}, \ldots, x^{m_i}]$, for $i = 1, 2, \ldots, t$. Let $(\lambda_1, \ldots, \lambda_\ell)$ be the level characteristic of A, with ℓ being the length of the longest chain in A.

We first prove by induction on i that for $i \in \{1, \ldots, t\}$ we have height$(x^j) = i$, for all $j \in \{m_{i-1} + 1, \ldots, m_i\}$ with the convention $n_0 = 0$. For $j \in \langle n_1 \rangle$, we have $Ax^j = 0$, due to (2.3) and the fact that the columns $x^1, \ldots, x^q$ of T_1 form a quasi-preferred basis for $E(A)$. This shows that height$(x^j) = 1$, for $j \in \langle n_1 \rangle$.

Now assume that for any i with $i < k \leqslant \ell$, we have height$(x^j) = i$, for all $j \in \{m_{i-1} + 1, \ldots, m_i\}$. Thus, we have $A^i X^i = 0$ and the columns of $A^{i-1}X^i$ are nonzero. But then $BT_1 = T_1 Z_F$ implies that $-AX^k = X^1 Z_{1,k} + \ldots + X^{k-1} Z_{k-1,k}$. Multiplying with A^{k-1} and A^{k-2} respectively from the left, we obtain $A^k X^k = 0$ and $A^{k-1} X^k = \left(A^{k-2} X^{k-1}\right) Z_{k-1,k} \neq 0$, by Lemma 2.1 as each column of both the matrices $Z_{k-1,k}$ and $A^{k-2}X^{k-1}$ is nonzero, $Z_{k-1,k}$ is a nonnegative matrix and either $A^{k-2}X^{k-1}$ or $-A^{k-2}X^{k-1}$ must be nonnegative. This shows that height$(x^j) = k$, for all $j \in \{m_{k-1} + 1, \ldots, m_k\}$. As a consequence of this and Theorem 2.1, we conclude that for $i \in \{1, \ldots, t\}$, level$(x^j) = i$ for all $j \in \{m_{i-1} + 1, \ldots, m_i\}$. Thus, we have $n_i = \lambda_i$ and $t = \ell$. □

We call a matrix Z, as defined in Theorem 2.3, a *Frobenius-Jordan form of* B and Z_F the *leading diagonal block* of this form.
The following example shows that the block sizes need not always be invariant.

Example 2.2. Consider the matrix $A = 2I - B$ where

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ & 2 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & 1 & 1 & 0 \\ & & & & 2 \end{bmatrix}.$$

The singular vertices of A are ordered according to their access level here, level$(1) = 1$, level$(2) = 2 =$ level(3), level$(4) = 3$. For the quasi-preferred basis $x^1 = [1\ 0\ 0\ 0\ 0]^T$, $x^2 = [1\ 1\ 0\ 0\ 0]^T$, $x^3 = [1\ 0\ 1\ 1\ 0]^T$, $x^4 = [1\ 1\ 0\ 0\ 1]^T$

we have

$$A[x^1\ x^2\ x^3\ x^4] = [x^1\ x^2\ x^3\ x^4]\begin{bmatrix} 2 & 1 & 2 & 1 \\ & 2 & 0 & 1 \\ & & 2 & 0 \\ & & & 2 \end{bmatrix} = \begin{bmatrix} 2 & [1\ \ 2] & 0 \\ & 2I_2 & \begin{bmatrix}1\\0\end{bmatrix} \\ & & 2 \end{bmatrix} =: Z_{F_1}$$

$$A[x^1\ x^2\ x^3\ x^4] = \begin{bmatrix} 2 & 1 & [2\ \ 1] \\ & 2 & [0\ \ 1] \\ & & 2I_2 \end{bmatrix} =: Z_{F_2}$$

So, the block sizes are not unique.

Remark 2.1. If we consider a particular Frobenius normal form of A with the number of diagonal blocks is equal to the number of levels in $R(A)$ and the ith diagonal block is of order λ_i, then the block-sizes in (2.4) are always invariant and $n_j = \lambda_j$ for all j.

The following example shows that in general the Frobenius-Jordan form of a matrix may not be unique.

Example 2.3. Let

$$B = \left[\begin{array}{cc|cc|cc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{array}\right].$$

Then $\rho(B) = 4$ and if $A = 4I - B$, then index$(A) = 3$ and

$$E(A) = N((4I - B)^3) = \left\{x = [x_i] \in \mathbb{R}^{6,1} : x_1 = x_2,\ x_3 = x_4,\ x_5 = x_6\right\}.$$

Consider the quasi-preferred bases spanned by the columns of $X = [x^1\ x^2\ x^3]$ and $Y = [y^1\ y^2\ y^3]$, with $x^1 = [1\ 1\ 0\ 0\ 0\ 0]^T$; $x^2 = [1\ 1\ 1\ 1\ 0\ 0]^T$; $x^3 = [1\ 1\ 1\ 1\ 1\ 1]^T$; and $y^1 = x^1$; $y^2 = 2x^2$; $y^3 = 4x^3$. Then we have

$$BX = X\begin{bmatrix} 4 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} = XZ_{F_1},\ BY = Y\begin{bmatrix} 4 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix} = YZ_{F_2}.$$

Thus, the leading diagonal block of a Frobenius-Jordan form (and hence the Frobenius-Jordan form) of a nonnegative matrix is not unique.

Our next theorem shows that the leading diagonal blocks of any two Frobenius-Jordan forms of a matrix B are related by a similarity transformation via a block upper triangular matrix.

Theorem 2.4. *Let*

$$Z_F = \begin{bmatrix} \rho I_{n_1} & Z_{1,2} & \dots & Z_{1,t} \\ 0 & \rho I_{n_2} & \dots & Z_{2,t} \\ \vdots & \vdots & \ddots & Z_{t-1,t} \\ 0 & 0 & \dots & \rho I_{n_t} \end{bmatrix}$$

and

$$\tilde{Z}_F = \begin{bmatrix} \rho I_{m_1} & \tilde{Z}_{1,2} & \dots & \tilde{Z}_{1,t} \\ 0 & \rho I_{m_2} & \dots & \tilde{Z}_{2,t} \\ \vdots & \vdots & \ddots & \tilde{Z}_{t-1,t} \\ 0 & 0 & \dots & \rho I_{m_s} \end{bmatrix}$$

be the leading diagonal blocks corresponding to two Frobenius-Jordan forms of a nonnegative matrix B with $n_1 + \ldots + n_t = m_1 + \ldots + m_s = q$. Then there exists an upper triangular matrix F with positive diagonal entries such that

$$\tilde{Z}_F = F^{-1} Z_F F.$$

In particular, if the block-sizes are invariant, that is, $t = s$ and $n_i = m_i = \lambda_i$, then the matrix $F = [F_{i,j}]$ has the same block structure and the diagonal blocks $F_{1,1}, \ldots, F_{t,t}$ are positive diagonal matrices.

Proof. Let $A = \rho I - B$ and let $H(A) = \{\alpha_1, \ldots, \alpha_q\}$, with $\alpha_1 < \alpha_2 < \ldots < \alpha_q$ be the set of singular vertices in $R(A)$ with $A = \rho I - B$, where q is the algebraic multiplicity of the eigenvalue 0 of A. Let Z_F and $\tilde{Z}_F$ be the leading diagonal blocks of two Frobenius-Jordan forms of B corresponding to the nonsingular matrices $T = [T_1\ T_2]$ and $\tilde{T} = [\tilde{T}_1\ \tilde{T}_2]$, so that the columns of $T_1 = [x^1 \ldots x^q]$ and $\tilde{T}_1 = [\tilde{x}^1 \ldots \tilde{x}^q]$, both form quasi-preferred bases, respectively, with $BT_1 = T_1 Z_F$ and $B\tilde{T}_1 = \tilde{T}_1 \tilde{Z}_F$. Since the columns of T_1 and $\tilde{T}_1$ are bases for the generalized nullspace of A, there exists a nonsingular matrix $F \in \mathbb{R}^{q,q}$ such that $\tilde{T}_1 = T_1 F$ with $F = [f_{i,j}]$. Thus, for any $i \in \langle q \rangle$ we have

$$\tilde{x}^i = f_{1,i} x^1 + f_{2,i} x^2 + \ldots + f_{q,i} x^q. \tag{2.6}$$

Let $i \in \langle q\rangle$ and consider the set $V_i = \{\alpha_j \in H(A) : f_{j,i} \neq 0\}$. We now show that $V_i \subseteq \text{below}(\alpha_i)$.

Suppose first that $\alpha_j \in \text{top}(V_i)$ but $\alpha_j \notin \text{below}(\alpha_i)$. Then $x^i_{\alpha_j} = 0 = \tilde{x}^i_{\alpha_j}$, but $\alpha_j \in \text{top}(V_i)$ implies that $f_{j,i} \neq 0$, and if $f_{r,i} \neq 0$ and $\alpha_j \to \alpha_r$ (in which case $x^r_{\alpha_j} > 0$), then $r = j$.

Thus from equation (2.6) we obtain $\tilde{x}^i_{\alpha_j} = f_{j,i} x^j_{\alpha_j}$ which implies that $f_{j,i} = 0$, which is a contradiction. Hence, we have $\text{top}(V_i) \subseteq \text{below}(\alpha_i)$.

Suppose next that $\alpha_j \in V_i \backslash \text{top}(V_i)$. Then there exists $\alpha_r \in \text{top}(V_i)$ such that $\alpha_j \to \alpha_r$ and $j \neq r$, which implies that $\alpha_j \in \text{below}(\alpha_i)$, because $\text{top}(V_i) \subseteq \text{below}(\alpha_i)$. This shows that $V_i \subseteq \text{below}(\alpha_i)$, i. e., $f_{j,i} = 0$ if $\alpha_j \nrightarrow \alpha_i$. Thus it follows that F is an upper triangular matrix with nonzero diagonals. Since $A\tilde{T}_1 = \tilde{T}_1\tilde{Z}_F$, $AT_1 = T_1 Z_F$ and $\tilde{T}_1 = T_1 F$, it follows that $T_1 Z_F F = T_1 F \tilde{Z}_F$, which implies that $Z_F F = F\tilde{Z}_F$.

If $\alpha_r \in V_i$ and $\alpha_i \to \alpha_r$, then $\alpha_r \in \text{below}(\alpha_i)$ and thus $i = r$. This shows that $\alpha_i \in \text{top}(V_i)$ and hence equation (2.6) implies that $\tilde{x}^i_{\alpha_i} = f_{i,i} x^i_{\alpha_i}$. Thus $f_{ii} > 0$.

The last part of the theorem follows from the fact that for all $i \in \langle q\rangle, V_i \subset \text{below}(\alpha_i)$. □

One may raise the question whether every possible Jordan form as in (2.3) with a nonnegative basis T_1 stems from a quasi-preferred basis. This is not the case as the following example shows.

Example 2.4. The matrix

$$B = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

has $\rho = \rho(B) = 2$. Take $A = \rho I - B$. Consider the nonnegative basis of $E(A)$ spanned by the columns of $T = [x_1\ x_2]$, with $x_1 = [1\ 1\ 1]^T, x_2 = [2\ 3\ 3]^T$. Then $B[x_1\ x_2] = [x_1\ x_2]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = TZ$, where $Z = 2I$ is the leading block of Frobenius-Jordan form of B, but the columns of T do not form a quasi-preferred basis for A. Note that in this example $\text{index}_2(B) = 1$.

Since not every nonnegative basis with columns that satisfy condition (2.3) in Theorem 2.3 is a quasi-preferred basis one may ask whether there is a weaker relation.

Example 2.5. The matrix

$$B = \left[\begin{array}{c|cc|c} 2 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

has $\rho(B) = 2$ and $E_2(B) = \{x = [x_i] \in \mathbb{R}^4 : x_2 = x_3\}$.

Consider the nonnegative basis of $E_2(B)$ spanned by the columns of $T = [x^1\ x^2\ x^3]$ with $x^1 = [1\ 0\ 0\ 0]^T$, $x^2 = [1\ 1\ 1\ 0]^T$, $x^3 = [0\ 1\ 1\ 1]^T$. Then,

$$BT = T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} =: TZ_F,$$

where Z_F is the leading diagonal block of a Frobenius-Jordan form of B. Here we have $x_1^3 = 0$ but $1 \to 3$, $x_1^2 > 0$ but $1 \not\to 2$.

In Theorem 2.4 we have shown that Frobenius-Jordan forms of a nonnegative matrix may not be unique, but the leading diagonal blocks of any two Frobenius-Jordan forms are related via a block upper triangular similarity transformation with diagonal blocks are positive diagonal matrices.

In order to characterize different Frobenius-Jordan forms of the same matrix B, we study a Frobenius-Jordan form where the leading block has the maximal number of nonzeros. We now denote this leading block by $Z_{F,\max}$.

Example 2.6. In Example 2.3, Z_{F_1} and Z_{F_2} both contain maximal number of nonzeros and they are not permutationally similar, whereas they are diagonally similar.

Unfortunately not all Frobenius-Jordan forms with a maximal number of nonzeros are diagonally similar as the following example shows.

Example 2.7. Let

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

It is easy to check that $\rho(B) = 2$ and for $A = 2I - B$ we have $\text{index}(A) = 5$. Consider the two preferred bases spanned by the columns of $X = [x^1\ x^2\ x^3\ x^4\ x^5\ x^6]$ and $Y = [y^1\ y^2\ y^3\ y^4\ y^5\ y^6]$, where

$$\begin{aligned} x^1 &= [1\ 0\ 0\ 0\ 0\ 0\ 0]^T, & y^1 &= x^1, \\ x^2 &= [1\ 1\ 0\ 0\ 0\ 0\ 0]^T, & y^2 &= x^2, \\ x^3 &= [2\ 2\ 1\ 1\ 0\ 0\ 0]^T, & y^3 &= [3\ 3\ 2\ 2\ 0\ 0\ 0]^T, \\ x^4 &= [2\ 2\ 0\ 0\ 1\ 0\ 0]^T, & y^4 &= [5\ 5\ 0\ 0\ 4\ 0\ 0]^T, \\ x^5 &= [4\ 4\ 2.5\ 2\ 0\ 1\ 0]^T, & y^5 &= x^5, \\ x^6 &= [18\ 18\ 10.25\ 8\ 1\ 5\ 1]^T, & y^6 &= x^6. \end{aligned}$$

We have $BX = XZ_{F_X}$ and $BY = YZ_{F_Y}$ with

$$Z_{F_X} = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 & 9 \\ 0 & 2 & 1 & 1 & 1 & 2.5 \\ 0 & 0 & 2 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad Z_{F_Y} = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 & 9 \\ 0 & 2 & 2 & 4 & 1.25 & 3.375 \\ 0 & 0 & 2 & 0 & 0.25 & 0.125 \\ 0 & 0 & 0 & 2 & 0 & 0.25 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

If Z_{F_X} and Z_{F_Y} were diagonally similar and $D = \text{diag}(d_1, d_2, d_3, d_4, d_5, d_6)$ such that $Z_{F_X}D = DZ_{F_Y}$, then D would have to satisfy the homogeneous linear system $d_1 = d_2 = d_3 = d_5 = d_6$, $d_5 = 1.25\ d_2$, which however only has the trivial solution, hence Z_{F_X} and Z_{F_Y} are not diagonally similar.

In our next theorem we show that the subgraph of B corresponding to the leading block of any leading block in a Frobenius-Jordan form is a subgraph of $Z_{F,\max}$. For this we make use of the following lemma.

Lemma 2.2. *[22] Let $B \in \mathbb{R}^{n,n}$ be in Frobenius normal form (2.2) and let $x \in \mathbb{R}^n$ be partitioned analogously. Then* $\operatorname{supp}(Bx) \subseteq \operatorname{below}(\operatorname{supp}(x))$.

Lemma 2.3. *Let $B \in \mathbb{R}^{n,n}$ be a nonnegative matrix in Frobenius-Jordan form (2.3) with leading block $Z_F = [z_1, \ldots, z_q]$ that corresponds to the quasi-preferred basis spanned by the columns of $T_1 = [x^1, \ldots, x^q]$. Let $A = \rho I - B$ with $\rho = \rho(B)$ and let $H(A) = \{\alpha_1, \ldots, \alpha_q\}$, with $\alpha_1 < \ldots < \alpha_q$ be the set of singular vertices in $R(A)$. Then, for any $i \in \langle q \rangle$,* $\operatorname{supp}(z_i) \subseteq \operatorname{below}(\alpha_i)$.

Proof. Let $Z_F = [z_{i,j}]$. Then $BT_1 = T_1 Z_F$ implies that

$$Bx^i = \rho x^i + \sum_{k=1}^{i-1} z_{k,i} x^k, \; i = 1, \ldots, q. \tag{2.7}$$

We have to show that for every $i \in \langle q \rangle$ the inclusion $\operatorname{supp}(z_i) \subseteq \operatorname{below}(\alpha_i)$ holds, which is equivalent to $\operatorname{top}(\operatorname{supp}(z_i)) \subseteq \operatorname{below}(\alpha_i)$.

Let $\alpha_k \in \operatorname{top}(\operatorname{supp}(z_i))$. Then (2.7) implies that

$$(Bx^i)_{\alpha_k} = \rho x^i_{\alpha_k} + z_{k,i} x^k_{\alpha_k}. \tag{2.8}$$

If $(Bx^i)_{\alpha_k} = \rho x^i_{\alpha_k}$, then equation (2.8) implies that $z_{k,i} = 0$, which is a contradiction. So we must have $\alpha_k \in \operatorname{supp}(Ax^i)$ and so by Lemma 2.2 we have $\alpha_k \in \operatorname{supp}(x^i)$. Then from the definition of the quasi-preferred basis it follows that $\alpha_k \in \operatorname{below}(\alpha_i)$. □

Using this lemma we can now prove the following theorem.

Theorem 2.5. *All possible graphs associated with a leading block of a Frobenius-Jordan form of B are subgraphs of the graph of $Z_{F,\max}$.*

Proof. Consider the M-matrix $A = \rho I - B$ with $\rho = \rho(B)$. Suppose that $BT_{\max} = T_{\max} Z_{F,\max}$, with $Z_{F,\max} = [\hat{z}_1, \ldots, \hat{z}_q]$, such that the columns of $T_{\max}$ form a quasi-preferred basis. By Theorem 2.2, there exist a preferred basis spanned by the columns of $Y = [y^1, \ldots, y^q]$ for $E(A)$ and let $BY = YZ_F$ with $Z_F = [z_1, \ldots, z_q]$ be the corresponding part of the Frobenius-Jordan form. Then by definition, for $i \in \langle q \rangle$ we have $\operatorname{supp}(z_i) = \operatorname{below}(\alpha_i)$. But by Lemma 2.3, $\operatorname{supp}(\hat{z}_i) \subseteq \operatorname{below}(\alpha_i) = \operatorname{supp}(z_i)$. Since $Z_{F,\max}$ contain maximal nonzero entries, we must have $\operatorname{supp}(\hat{z}_i) = \operatorname{below}(\alpha_i)$. □

As a consequence of Theorem 2.5 we have that every leading block with a maximal number of nonzeros is associated with a preferred basis, while all the leading blocks with fewer nonzeros only are related to quasi-preferred bases.

In this section we introduce Frobenius-Jordan forms and analyzed the relationship between different such forms. In the next section we discuss about permuted graph bases for the generalized eigenspace associated with $\rho(B)$.

2.4 Nonnegative permuted graph basis for nonnegative matrices

In this section we present some partial results associated with a special choice of the nonnegative basis T_1 for the invariant subspace associated with spectral radius of a nonnegative matrix B. This topic is of interest in the solution of nonsymmetric algebraic Riccati equations with elementwise nonnegative solution, (see [18, 19, 32]), where for a block matrix, one considers the invariant subspace equation

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} Z_1$$

and X is the elementwise nonnegative solution of the Riccati equation $B_{21} + B_{22}X - XB_{11} - XB_{12}X = 0$. If the matrix Z_1 is the matrix associated with the spectral radius of a nonnegative matrix, then the Frobenius-Jordan form yields a nonnegative basis, but it is not necessarily of this special form. So one may ask whether such a special basis exists for a given nonnegative matrix or at least for permutationally similar matrix.

Another motivation to study this topic is the initialization of discrete-time dynamical systems. Consider an iteration $x_{k+1} = Bx_k$ with an $n \times n$ nonnegative matrix B of spectral radius 1, and suppose that the columns of a matrix T_1 span the invariant subspace of B associated with the eigenvalues of modulus equal to 1, i.e., $BT_1 = T_1 Z_{1,1}$ and $Z_{1,1}$ is associated with all the eigenvalues of B of modulus 1. Completing T_1 to a nonsingular matrix T we obtain

$$T^{-1}BT = \begin{bmatrix} Z_{1,1} & Z_{1,2} \\ 0 & Z_{2,2} \end{bmatrix}, \; y_k = \begin{bmatrix} y_k^1 \\ y_k^2 \end{bmatrix} = T^{-1}x_k.$$

Then for arbitrary y_0^2, the iterates are bounded if and only if y_0^1 is in the invariant subspace spanned by the eigenvectors associated with eigenvalues of modulus 1, while they grow unbounded if y_0^1 has a component in the direction of a generalized eigenvector. Furthermore, the iteration becomes stationary if and only if y_0^1 is in the invariant subspace spanned by the eigenvectors associated with the eigenvalue 1. In many applications such as Markov chains or positive systems, (see [2, 10]), the iterates describe positive quantities such as probabilities or concentrations and then also the initial vectors must be nonnegative. Such initial vectors can be easily constructed via the Frobenius-Jordan form. In the transformation matrix to Frobenius-Jordan form the matrix T_2 that completes T_1 to a nonsingular matrix is special (to create the Jordan structure), but for the analysis and for carrying out such iterations it is not necessary that the second diagonal block is in Jordan form. A particularly convenient choice of T_2 is obtained if the columns of T_1 form a *nonnegative row permuted graph basis* [31] for the generalized eigenspace of B associated with the spectral radius of B, i.e., if it is of the form

$$T_1 = \Pi \begin{bmatrix} I \\ Y \end{bmatrix}$$

with a permutation matrix Π and a nonnegative matrix Y, which again solves an algebraic Riccati equation associated with the nonnegative matrix $\Pi B \Pi^T$, partitioned in appropriate block form.

If the columns of T_1 is a nonnegative permuted graph basis, then choosing $T_2 = \Pi \begin{bmatrix} 0 \\ I \end{bmatrix}$ we have that

$$T = [T_1, T_2] = \Pi \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}, \ T^{-1} = \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix} \Pi^T,$$

and it can be checked that there is a nonnegative starting vectors

$$x_0 = Ty_0 = \Pi \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} y_0^1 \\ y_0^2 \end{bmatrix} = \Pi \begin{bmatrix} y_0^1 \\ y_0^2 + Y y_0^1 \end{bmatrix}$$

in such a way that the iteration converges. If all other eigenvalues of B except for the real eigenvalues associated with the spectral radius have modulus less than 1, then y_0^2 can be chosen arbitrarily and if y_0^1 is in the eigenspace of Z_{11}, then the iteration will converge to a stationary point.

Thus a nonnegative permuted graph basis would be really helpful, but the following example shows that such a nonnegative permuted graph basis of the invariant subspace associated with the spectral radius does not always exist for every nonnegative matrix.

Definition 2.8. *[31] Let $\mathcal{U}$ be an n-dimensional subspace of $\mathbb{C}^{m+n}$. Then, there exists a permutation matrix Π, and a square matrix X such that*

$$\mathcal{U} = \operatorname{Im} \Pi^T \begin{bmatrix} I \\ X \end{bmatrix}$$

where the entries of X satisfy $|x_{ij}| < 1$. It follows that a subspace can be represented with a basis that has an identity in selected rows and norm-bounded entries in the remaining ones. We call such a form a permuted graph representation *and the basis as* permuted graph basis.

Example 2.8. Let

$$B = \left[\begin{array}{cc|cc|ccc} 2 & 2 & 2 & 0 & 2 & 1 & 0 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 \end{array}\right].$$

Then $\rho(B) = 4$ and for $A = 4I - B$ we have $\operatorname{index}(A) = 2$ so that

$$E(A) = \left\{ x = [x_i] \in \mathbb{R}^{7,1} \mid 3x_4 = 4x_3, x_5 = x_6 = x_7, 4(x_2 - x_1) = 2x_4 - x_5 \right\}.$$

If x^1, x^2, x^3 is any quasi-preferred basis for $E(A)$, then we have

$$\begin{aligned} x^1 &= [p,\ p,\ 0,\ 0,\ 0,\ 0,\ 0]^T, \\ x^2 &= [u,\ u + \frac{w}{2},\ \frac{3w}{4},\ w,\ 0,\ 0,\ 0]^T, \\ x^3 &= [z,\ z + \frac{y}{2} - \frac{x}{4},\ \frac{3y}{4},\ y,\ x,\ x,\ x]^T \end{aligned}$$

with nonnegative p, u, w, x, y, z. For $X = [x^1\ x^2\ x^3]$ all possible submatrices of X that may contribute rows to the identity matrix of the permuted graph basis are

$$\begin{aligned} X_1 &= [X(1,:),\ X(3,:),\ X(5,:)]^T \\ X_2 &= [X(1,:),\ X(4,:),\ X(5,:)]^T, \\ X_3 &= [X(2,:),\ X(3,:),\ X(5,:)]^T, \\ X_4 &= [X(2,:),\ X(4,:),\ X(5,:)]^T \end{aligned}$$

But it is easily seen that $(XX_1^{-1})_{23}, (XX_2^{-1})_{12}, (XX_3^{-1})_{12}, (XX_4^{-1})_{23}$ entries are all negative. Hence, there does not exist any nonnegative permuted graph basis for B. Note that in this example, the level and height characteristic are different, since $\lambda(A) = (2,2) \neq \eta(A) = (3,1)$.

Example 2.9. Consider the matrix

$$B = \left[\begin{array}{cc|cc|ccc|cc} 2 & 2 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{array}\right],$$

with $\rho(B) = 4$ and for $A = 4I - B$ we have index$(A) = 3$, and

$$\begin{aligned} E(A) = \ & \{x = [x_i] \in \mathbb{R}^{9,1} \mid x_8 = x_9,\ x_5 = x_6 = x_7,\ 4x_3 = 3x_3 + \tfrac{4x_6}{7}, \\ & 4(x_2 - x_1) = 2x_4 - \tfrac{9x_5}{28}\}. \end{aligned}$$

Here again, as in Example 2.8, B does not possess any nonnegative permuted graph basis, whereas level and height characteristic are equal, $\lambda(A) = \eta(A) = (2,1,1)$.

To obtain a criteria for the existence of nonnegative permuted graph bases we have the following result.

Lemma 2.4. *Let a nonnegative matrix B, partitioned as* (2.2)*, having q basic classes, possess a nonnegative permuted graph basis for the generalized* *eigenspace* $E_\rho(B)$ *with* $\rho = \rho(B)$ *and let* $\mathcal{I}_i = \{k :\ k$ *belongs to the class* γ *such that* $\gamma \in \text{below}(\alpha_i)$ *and* $k \notin \text{below}(\alpha_j),\ j \leqslant i-1\}$. *Then each block of* X *with columns that form a quasi-preferred basis corresponding to the partitioned*

$$X = \begin{bmatrix} X_{\mathcal{I}_1} \\ X_{\mathcal{I}_2} \\ \vdots \\ X_{\mathcal{I}_q} \end{bmatrix}, \tag{2.9}$$

will contribute one row to the identity.

Proof. Let $A = \rho I - B$ and let $H(A) = \{\alpha_1, \ldots, \alpha_q\}$ with $\alpha_1 < \ldots < \alpha_q$ be the singular classes of A and $\text{level}(\alpha_{i-1}) \leqslant \text{level}(\alpha_i)$, for all $2 \leqslant i \leqslant q$. Without loss of generality we may assume that B is in Frobenius normal form with a spectral radius of algebraic multiplicity q . Since A is an M-matrix, it has a quasi-preferred basis, given by the columns of $X = [x^1, \ldots, x^q]$ where each x_i is partitioned as (2.9). By convention we assume that $\mathcal{I}_q$ contains all other nonbasic classes γ which does not have access to any basic class.

We show that each set $\mathcal{I}_i$ will contribute a row(in particular the ith row) to the identity of the nonnegative permuted graph basis.

Suppose that $\beta := \{\beta_1, \ldots, \beta_q\}$ is the set of indices that are associated with the identity, i. e., if $\bar{\beta} = \langle n\rangle \backslash \beta$, then there exists a permutation $\Pi = [\Pi_\beta^T, \Pi_{\bar{\beta}}^T]^T$ defined by the indices in β and $\bar{\beta}$ such that $X_\beta := \Pi_\beta X$ is invertible, and

$$\Pi X X_\beta^{-1} = \begin{bmatrix} X_\beta \\ X_{\bar{\beta}} \end{bmatrix} X_\beta^{-1} = \begin{bmatrix} I \\ Y \end{bmatrix}, \tag{2.10}$$

with $Y \geqslant 0$.

If our assumption is not true, then there exists an index j such that $\beta_i \notin \mathcal{I}_j$, for all $i \in \langle q\rangle$. Let $\tilde{j}$ be the least such index. Then $\tilde{j} \neq 1$ because otherwise X_β would have a zero column, which is a contradiction.

Let $\tilde{j} > 1$. Consider a row k of $X_{\bar{\beta}}$ such that $k \in \mathcal{I}_{\tilde{j}}$ and $(X_{\bar{\beta}})_{k,\tilde{j}} > 0$, in particular we choose k to be in the $\tilde{j}$th basic class $\alpha_{\tilde{j}}$. If y_k^T is the kth row of Y and e_k represents the kth row of the identity matrix, then $y_k^T X_\beta = (e_k^T X_{\bar{\beta}} X_\beta^{-1}) X_\beta = e_k^T X_{\bar{\beta}}$, and $(y_k^T X_\beta)_{\tilde{j}} = (X_{\bar{\beta}})_{k,\tilde{j}} > 0$. Thus $(e_k^T X_{\bar{\beta}})_l = 0$ for all $l \in \langle \tilde{j}-1\rangle$ imply that $y_{ki} x_{\beta_i,l} = 0$ for all $i \in \langle q\rangle$. Thus for $i \in \langle q\rangle$, $y_{ki} \neq 0$ implies that $x_{\beta_i,l} = 0$ for all $l \in \langle\tilde{j}-1\rangle$. Consider the set $Q = \{i \in \langle q\rangle \mid y_{ki} \neq 0\}$. Then for any $i \in Q$, $x_{\beta_i,l} = 0$ for all $l \in \langle\tilde{j}-1\rangle$, that is, $\beta_i \nrightarrow \alpha_l$, for all $l \in \langle\tilde{j}-1\rangle$.

Since $(y_k^T X_\beta)_{\tilde{j}} = \sum_{i\in Q} y_{ki} x_{\beta_i,\tilde{j}} > 0$, there is an $i \in Q$ such that both y_{ki} and $x_{\beta_i,\tilde{j}}$ are positive. Then $\beta_i \to \alpha_{\tilde{j}}$ and hence $\beta_i \in \mathcal{I}_{\tilde{j}}$, as $\beta_i \nrightarrow \alpha_l$, for all $l \in \langle\tilde{j}-1\rangle$. This contradicts the assumption that $\mathcal{I}_{\tilde{j}}$ does not contribute any row to the identity of the permuted graph basis. Thus each $\mathcal{I}_i$ will contribute a row to the identity of the permuted graph basis. □

Lemma 2.4 implies that no block in a Frobenius-Jordan form can contribute more than one row to the identity. Thus the identity cannot be larger than the number of blocks. However, as we have seen, there may not exist a nonnegative permuted graph basis, which means that some blocks do not at all contribute rows to the identity.

However, if each block is to contribute exactly one row to the identity, then we must have the following relation.

Corollary 2.1. *Suppose that B is a nonnegative matrix having a nonnegative permuted graph basis for the generalized eigenspace $E_\rho(B)$. If it has a quasi-preferred basis $\{x^1, \ldots, x^q\}$ with $x^i = [x^i_j]$, partitioned as (2.9), such that there exist unique $k_1, \ldots, k_q$ with $\min_j \frac{(x_i^{i+1})_j}{(x_i^i)_j} = \frac{(x_i^{i+1})_{k_i}}{(x_i^i)_{k_i}}$, then each of $k_1, \ldots, k_q$ will contribute a row to the identity of the nonnegative permuted graph basis.*

Proof. Consider the matrix $X = [x^1 \ \ldots \ x^q]$. Thus there exists indices $j_1, \ldots, j_q$ from each block that contribute rows to the identity of the non-negative permuted graph basis. We now show that for each $i = 1, \ldots, q$

$$\frac{(x_i^{i+1})_{k_i}}{(x_i^i)_{k_i}} \geqslant \frac{(x_i^{i+1})_{j_i}}{(x_i^i)_{j_i}}$$

Then the result will follow from the uniqueness of the k_i. Clearly for each $i = 1, \ldots, q$ both the indices k_i and j_i are from the same block. Since the columns of the matrices X and $\Pi \begin{bmatrix} I \\ Y \end{bmatrix}$ with some nonnegative Y both are bases for the generalized eigenspace, there exists a matrix $C \in \mathbb{R}^{q,q}$ such that $X = \Pi \begin{bmatrix} I \\ Y \end{bmatrix} C$. Note that Y is also block a upper triangular matrix such that the indices of the ith block corresponds to the set $\mathcal{I}_i \backslash \{j_i\}$. It can be easily seen that,

$$C = \begin{bmatrix} (x_1^1)_{j_1} & (x_1^2)_{j_1} & \dots & (x_1^q)_{j_1} \\ 0 & (x_2^2)_{j_2} & \dots & (x_2^q)_{j_2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (x_p^q)_{j_q} \end{bmatrix},$$

which implies that for each $i = 1, \dots, q$ there exist nonnegative scalars (the elements of Y) α_i, β_i such that

$$\begin{aligned} \left(x_i^i\right)_{k_i} &= \alpha_i \left(x_i^i\right)_{j_i} \\ \left(x_i^{i+1}\right)_{k_i} &= \beta_i \left(x_{i+1}^{i+1}\right)_{j_{i+1}} + \alpha_i \left(x_i^{i+1}\right)_{j_i}. \end{aligned} \tag{2.11}$$

Since all β_i, α_i,$(x_i^i)_{j_i}$ and $\left(x_{i+1}^{i+1}\right)_{j_{i+1}}$ are nonnegative, the claim follows from the equations in (2.11). □

Corollary 2.1 gives a computational criterion to check the existence of permuted graph basis. One computes a preferred bases and checks the inequalities and their uniqueness. If this holds then a permuted graph basis exists, if not then it is an open problem to guarantee the existence.

We have seen that nonnegative permuted graph bases are a very convenient tool for the initialization of nonnegative dynamical systems, but not every nonnegative matrix possesses a nonnegative permuted graph basis even if they possess the same level and height characteristic. It is an open problem to characterize the class of nonnegative matrices that have a nonnegative permuted graph basis.

2.5 Conclusion

We present a variant of the Jordan canonical form for nonnegative matrices and show the uniqueness of such a canonical form up to block triangular similarity transformation. We also study some combinatorial properties of nonnegative matrices with the help of this canonical form. Finally we present some necessary conditions for the existence of nonnegative permuted graph basis for nonnegative matrices and we demonstrat the fact that not every nonnegative matrix has such bases by an example.

Chapter 3

Combinatorial structure of generalized M-matrices

3.1 Introduction

In this chapter we consider two types of generalizations of M-matrices, namely the class of GM-matrices [7] and $M_\vee$-matrices [41]. We show that Preferred Basis Theorem and the Index Theorem for M-matrices are not true for GM-matrices of order greater than 2, whereas we prove the existence of a preferred basis for a subclass of $M_\vee$-matrices. We also present a procedure to obtain a preferred basis from a quasi-preferred basis for the generalized null space for a certain subclass of $M_\vee$-matrices. The existence of quasi-preferred basis for this class of matrices was shown by Naqvi and McDonald in [33]. The existence of quasi-preferred and preferred bases for M-matrices was shown by Rothblum, Schneider and Hershkowitz in [45] and [23]. Their proof was accomplished by induction on the diagonal blocks of an M-matrix in Frobenius normal form. Here, by using similar techniques we give a constructive method to obtain a preferred basis from a given quasi-preferred basis for a subclass of $M_\vee$-matrices. Also the procedure proves the existence of a preferred basis for this subclass of $M_\vee$-matrices.

One interesting problem is to study the relation between the height and level characteristic of $M_\vee$-matrices. In [33], it was proved that the height characteristic is always majorized by its level characteristic for a specific subclass of $M_\vee$-matrices. In this chapter we give some necessary and sufficient conditions for the equality of these two characteristics. Later we describe the concepts of anchored chains and well structured graphs and give a sufficient condition for the reduced graph of a subclass of $M_\vee$-matrices to be well structured.

We now describe the chapter in more detail. In section 3.3 we consider the class of GM-matrices, which has the form, $A = sI - B$, where B and B^T possess the Perron-Frobenius property and $\rho(B) \leqslant s$. We show that a quasi-preferred basis and hence a preferred basis for the generalized null space of these matrices of order exceeding 2, need not exist. But we prove that Preferred Basis Theorem and the Index Theorem hold if the order is 2.

Next we consider another generalization of M-matrices, called $M_\vee$-matrices which has the form $A = sI - B$ where B is an eventually nonnegative matrix and $\rho = \rho(B) \leqslant s$. In subsection 3.4.1 we give a procedure to obtain a preferred basis from a given quasi-preferred basis for M-matrices and $M_\vee$-matrices with $\text{index}_\rho(A) \leqslant 1$, written in block triangular form. We summarize the whole procedure in an algorithm. To end with we illustrate how the algorithm works through some examples.

Subsection 3.4.2 splits into two subsections. One of them deals with height and level characteristics and also gives some necessary and sufficient conditions for their equality, and the other one gives a sufficient condition for the reduced graph of $M_\vee$-matrices to be well structured.

3.2 Convention

Let A be a square matrix of order n. Throughout this chapter we assume that the matrix A is in *Frobenius normal form* (see 2.2), namely a block (upper) triangular form with p diagonal blocks where the diagonal blocks are irreducible matrices and we denote the $(i,j)th$ block of the Frobenius normal form of A by A_{ij}. Every x with n entries is assumed to be partitioned into p vector components x_i conformably with A.

3.3 Combinatorial structure of GM-matrices

In this section we consider one of the generalizations of M-matrices known as GM-matrices. We extend some results on the combinatorial spectral properties of M-matrices to this class. In particular, it is shown that the *Preferred Basis Theorem*(see Theorem 2.2 in Chap.1) and the *Index Theorem* do not hold good for the class of GM-matrices of order $n \geqslant 3$, whereas the theorems are true for $n < 3$.

Definition 3.1. *A matrix $A \in \mathbb{R}^{n,n}$ is said to have* Perron-Frobenius property *if the spectral radius is an eigenvalue associated with a nonnegative eigenvector. By $WPFn$ we denote the collection of all $n \times n$ matrices A, for which both A and A^T possess the Perron-Frobenius property.*

Definition 3.2. *A matrix $A \in \mathbb{R}^{n,n}$ is said to be a GZ-matrix if it can be expressed in the form $A = sI - B$, where s is a positive scalar and $B \in WPFn$. Moreover, if $A = sI - B$ is a GZ-matrix such that $\rho(B) \leqslant s$, then A is called a GM-matrix.*

Example 3.1. Consider the matrix

$$A = 2I - B = 2I - \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

The eigenvalues of B are $2, 2, -2$. As $[1\ 0\ 0]^T$ and $[0\ 1\ 1]^T$ are nonnegative left and right eigenvectors of B corresponding to 2 respectively, so A is a GM-matrix.

We begin by stating the *Index Theorem* for M-matrices.

Theorem 3.1. *[45](Index Theorem) If A is a singular M-matrix, then* $\operatorname{index}(A)$ *is equal to maximum level of a vertex in $R(A)$.*

We will show that the size of the largest Jordan block associated with 0, in the Jordan form of a GM-matrix of order 2, is combinatorially determined, but the *Index Theorem* need not be true if the size of the matrix exceeds 2.

Lemma 3.1. For any $A \in \mathbb{R}^{2,2}$ with the spectral radius $\rho(A) \in \sigma(A)$, the length of the longest chain of A is always less than or equal to $\text{index}_{\rho(A)}(A)$.

Proof. If $\text{index}_{\rho(A)}(A) = 2$, then the result is obviously true. Suppose

$$\text{index}_{\rho(A)}(A) = 1 \text{ and length of the longest chain} = 2.$$

So there are exactly two basic classes $\{1\}$ and $\{2\}$ such that either $\{1\} \to \{2\}$ or $\{2\} \to \{1\}$ and hence either A or A^T is of the form $\begin{pmatrix} \rho(A) & * \\ 0 & \rho(A) \end{pmatrix}$, where $*$ is nonzero.

In each of the cases $\text{index}_{\rho(A)}(A) = 2$, a contradiction to our assumption. Hence the result follows. □

The following example shows that the above result does not hold good if the order of the matrix exceeds 2.

Example 3.2. Consider the GM-matrix A in Example 3.1. Note that $[0, 1, 1]^T$ and $[2, 0, 1]^T$ are two linearly independent eigenvectors of A corresponding to the eigenvalue 0, so $\text{index}(A) = 1$. But the maximal level of a vertex in $\Gamma(A)$ is 2.

We now give an example of a 2×2 matrix that satisfies the hypothesis of the above theorem and for which $\text{index}_{\rho(A)}(A) <$ length of the longest chain in $\Gamma(A)$. Hence even for 2×2 matrices, the condition $\rho(A) \in \sigma(A)$ is not sufficient for their equality.

Example 3.3. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\rho(A) = 1 \in \sigma(A)$ and $\text{index}_{\rho(A)}(A) = 2$ whereas the length of the longest chain in $\Gamma(A)$ is 1.

In the next lemma we give a subclass of 2×2 matrices for which $\text{index}_{\rho(A)}(A) =$ Length of the longest chain in $\Gamma(A)$.

Lemma 3.2. *If $A = (a_{ij}) \in WPF2$ but not a nonnegative matrix, then the following statements are equivalent:*

(i) $\text{index}_{\rho(A)}(A) = 2$.

(ii) $a_{ij} < 0$ *for some* $i \neq j$.

(iii) *A is in triangular form with diagonal entries equal to $\rho(A)$.*

Proof. Since $A \in WPF2$, so there exists nonnegative vectors $x = [x_j]$ and $y = [y_j]$ such that,

$$Ax = \rho(A)x \text{ and } y^T A = \rho(A)y^T$$

which imply,

$$(a_{11} - \rho(A))x_1 + a_{12}x_2 = 0 \tag{3.1}$$

$$a_{21}x_1 + (a_{22} - \rho(A))x_2 = 0 \tag{3.2}$$

$$(a_{11} - \rho(A))y_1 + a_{21}y_2 = 0 \tag{3.3}$$

$$a_{12}y_1 + (a_{22} - \rho(A))y_2 = 0. \tag{3.4}$$

(ii) ⇒ (iii): Suppose that (ii) holds and assume that $a_{12} < 0$. We claim that x_2 cannot be positive. If $x_2 > 0$, then from equation (3.1) we must have $x_1 > 0$ and,

$$a_{11} > \rho(A). \tag{3.5}$$

Since $a_{11}+a_{22} = \lambda+\rho(A)$ where λ is the other eigenvalue of A, so $a_{22}-\rho(A) < 0$. Thus equation (3.4) implies that $a_{12} \geqslant 0$, a contradiction. So $x_2 = 0$. Then (3.1) and (3.2) imply that $a_{11} = \rho(A)$ and $a_{21} = 0$. Hence it follows that, A is an upper triangular matrix. Similarly one can show that $y_1 = 0$ and $a_{22} = \rho(A)$. Thus it shows that $\lambda = \rho(A)$.

(iii) ⇒ (i) is obvious.

(i) ⇒ (ii): Assume that A satisfies condition (i). Since the eigenvalues of A are $\dfrac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2}$ and $\text{index}_{\rho(A)}(A) = 2$, so $(a_{11} - a_{22})^2 + 4a_{12}a_{21} = 0$ which implies that a_{12} and a_{21} are of opposite sign, in which case the result holds or one of them must be zero. If one of $a_{ij} = 0$, then $a_{11} = a_{22} = \rho(A)$ and since A is not a nonnegative matrix, so the other one must be negative. Hence (ii) holds. □

Corollary 3.1. *Suppose $A \in WPF2$. Then* $\text{index}_{\rho(A)}(A) = 2$ *if and only if either A or A^T is of the form* $\begin{pmatrix} \rho(A) & * \\ 0 & \rho(A) \end{pmatrix}$, *where $*$ is nonzero.*

Corollary 3.2. *If $A \in WPF2$, then* $\text{index}_{\rho(A)}(A) =$ *length of the longest chain in $\Gamma(A)$.*

Proof. If $A \geqslant 0$, then the result is known to be true. Suppose A is not a nonnegative matrix. If $\text{index}_{\rho(A)}(A) = 1$, then either A is a diagonal matrix or has two distinct eigenvalues, but in both the cases length of the longest chain in $\Gamma(A)$ is 1. If $\text{index}_{\rho(A)}(A) = 2$, then the result follows from Lemma 3.2. □

Index Theorem for GM-matrices of order 2 is an immediate consequence of the above corollary.

Theorem 3.2. *If $A = \rho I - B$ is a singular GM-matrix of order 2, then* $\text{index}(A) = \max\{\text{level}(i) : i \in V(R(A))\}$, *where $V(R(A))$ is the vertex set of $R(A)$.*

We next show that there is a nonnegative basis for the generalized nullspace $E(A)$ and the positive entries are combinatorially determined.

Lemma 3.3. *Suppose $A \in WPF2$ and let $\alpha_1 \ldots \alpha_M (M = 1$ or $2)$ be the basic classes for A. Then there always exists a nonnegative basis $\{x^1, \ldots, x^M\}$ for $E_{\rho(A)}(A)$ such that $x^i_j > 0$ if and only if j has access to the ith basic class α_i.*

Proof. The result is known to be true if A is a nonnegative matrix. Hence assume that A has at least one negative entry. We consider two cases:

Case I: Suppose that $\text{index}_{\rho(A)}(A) = 1$. Then by the Corollary 3.2, length of the longest chain in $\Gamma(A)$ is 1. If A has two basic classes then A is a nonnegative diagonal matrix with diagonal entries equal to $\rho(A)$ in which case the result follows. Let A has only one basic class. By Lemma 3.2 both a_{12} and a_{21} are nonnegative. Let one of a_{12}, a_{21} is 0. Suppose that $a_{12} = 0$, then A has two different diagonal entries, $\rho(A)$ and say, λ. If $a_{11} = \rho(A)$, then $x^1 = [1, \frac{a_{21}}{\rho(A)-\lambda}]^T$ will be the required vector, and if $a_{22} = \rho(A)$, then $x^1 = [0, 1]^T$ will be the required vector.

Suppose that both a_{12} and a_{21} are positive. Then the only basic class of A will be $\{1, 2\}$. Since $A \in WPF2$, so there is a nonnegative vector $x^1 = [x^1_1, x^1_2]^T \neq 0$ such that $Ax^1 = \rho(A)x^1$ which implies

$$(a_{11} - \rho(A))x^1_1 + a_{12}x^1_2 = 0 \tag{3.6}$$

$$a_{21}x^1_1 + (a_{22} - \rho(A))x^1_2 = 0, \tag{3.7}$$

and hence $x_j^1 > 0 \ \forall j = 1, 2$.

Case II: Suppose that $\text{index}_{\rho(A)}(A) = 2$. If A is a nonnegative matrix, then the result follows from Theorem 2.4. If A is not a nonnegative matrix, then by Lemma 3.2, A has two basic classes $\{1\}, \{2\}$ such that either $2 \to 1$ or $1 \to 2$. If $2 \to 1$, then the required generalized eigenvectors are $x^1 = [1, 1]^T$ and $x^2 = [0, 1]^T$ that satisfy $x_j^i > 0$ if and only if j has access to the ith basic class, for $i, j \in \{1, 2\}$. □

We now prove Preferred Basis Theorem for GM-matrices of order 2.

Theorem 3.3. *If A is a singular GM-matrix of order 2, then there exists a preferred basis for the generalized null space $E(A)$ of A.*

Proof. The result is known to be true if A is an M-matrix, hence let A be a GM-matrix which is not an M-matrix. The existence of a quasi-preferred basis for $E(A)$ is an immediate consequence of Lemma 3.3. We now show that every quasi-preferred basis of $E(A)$ is a preferred basis.

Let the columns of X form a quasi-preferred basis for $E(A)$. If $\text{index}(A) = 1$, then $AX = 0$ and hence the columns of X form a preferred basis for A. Suppose that $\text{index}(A) = 2$ then $X = [x^1 \ x^2]$, where x^2 is a positive vector and $x_1^1 > 0$. By Lemma 3.2 exactly one of a_{12} or a_{21} must be zero. Assume that $a_{12} \neq 0$. Then $a_{21} = 0 = a_{11} = a_{22}$. Then $AX = X \begin{bmatrix} 0 & \frac{a_{12}x_2^2}{x_1^1} \\ 0 & 0 \end{bmatrix}$. Thus the columns of X form a preferred basis for $E(A)$. □

The following examples show that the conclusions of Theorem 3.2 and Theorem 3.3 do not hold for $WPFn$ matrices if the order of the matrix exceeds 2.

Example 3.4. Let,

$$A = 3I - B = 3I - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Clearly $B \in WPF3$ and hence A is a GM-matrix of order 3. Note that $\text{index}(A) = 2$ whereas $\max\{\text{level}(i) : i \in V(R(A))\} = 1$, where $V(R(A))$ is the vertex set of $R(A)$

Example 3.5. Consider the matrix

$$A = 2I - B = 2I - \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Clearly A is a singular GM-matrix with the singular classes $\{1, 2\}$ and $\{3\}$. Suppose that there is a preferred basis $\{x^1, x^2\}$ for $E(A)$ such that $x_i^j > 0$ if and only if i has access to the jth singular class. So by the assumption $x_i^1 > 0$, $\forall i = 1, 2$. But $Ax^1 = 0$ implies that $x_1^1 + x_2^1 = 0$ which cannot happen by our assumption. Hence Theorem 3.3 is not true for $n = 3$.

Remark 3.1. By taking $\tilde{A} = \mathrm{diag}(A, B) \in \mathbb{R}^{n,n}$, where A is as in Example 3.5 or in Example 3.4 and any matrix B having $\rho(B) < \rho(A)$, we can conclude that Theorems 3.2 and 3.3 do not hold good for any $n > 3$.

3.4 Combinatorial structure of $M_\vee$-matrices

3.4.1 Preferred basis for $M_\vee$-matrices

In this section we first prove some results on the combinatorial properties of quasi-preferred bases of a subclass of $M_\vee$-matrices which will be used subsequently to give a constructive method for obtaining a preferred basis from a quasi-preferred basis.

Definition 3.3. *Let $A \in \mathbb{R}^{n,n}$. For any two vertices i and j of $R(A)$, we define* $\mathrm{hull}(i, j) := \mathrm{above}(i) \bigcap \mathrm{below}(j)$.

Definition 3.4. *A square matrix A is called an* eventually nonnegative (positive) *matrix if there is a positive integer n_0 such that $A^k \geqslant 0 (A^k > 0)$ for all $k \geqslant n_0$.*

Definition 3.5. *A square matrix A is called an* $M_\vee$-matrix *if it can be expressed as $A = sI - B$ with eventually nonnegative B and $s \geqslant \rho(B)$.*

Throughout the remaining two sections we assume that a singular $M_\vee$-matrix A has the form $A = \rho I - B$, where B is an eventually nonnegative matrix with $\rho = \rho(B)$ and A has q singular classes with $H(A) = \{\alpha_1, \ldots, \alpha_q\}$ as the set of singular classes of A, where $\alpha_1 < \ldots < \alpha_q$.

The following results are well known.

Theorem 3.4. *[22] Let A be a square matrix in block triangular form and let x be a vector. Then* $\mathrm{supp}(Ax) \subseteq \mathrm{below}(\mathrm{supp}(x))$.

Theorem 3.5. *[33] Suppose that A is an eventually nonnegative matrix with* $\mathrm{index}(A) \leqslant 1$ *and* $D_A = \{d \mid \theta - \alpha = \frac{c}{d},$ *where* $re^{2\pi i\theta} \in \sigma(A), re^{2\pi i\alpha} \in \sigma(A),\ r > 0, c \in \mathbb{Z}^+, d \in \mathbb{Z}\backslash\{0\}, gcd(c,d) = 1\}$. *Let g be a prime number such that $g \notin D_A$ and $A^k \geqslant 0$ for all $k \geqslant g$. Then $\overline{R(A)} = \overline{R(A^g)}$.*

Lemma 3.4. *[33] Let $A \in \mathbb{C}^{n,n}$ and $\lambda \in \sigma(A), \lambda \neq 0$. Then for all $k \notin D_A$ we have $N(\lambda I - A) = N(\lambda^k I - A^k)$ and the Jordan blocks of λ^k in $J(A^k)$ are obtained from the Jordan blocks of λ in $J(A)$ by replacing λ with λ^k.*

Theorem 3.6. *[33] Let A be an eventually nonnegative matrix with* $\mathrm{index}(A) \leqslant 1$. *Then A has a quasi-preferred basis for $E_\rho(A)$.*

Lemma 3.5. *Given any two vertices i, j of $\Gamma(A)$ if for some positive integer k, $(A^k)_{ij} \neq 0$, then there is a path from i to j in $\Gamma(A)$.*

Proof. The proof of the result follows from the fact that

$$(A^k)_{ij} = \sum_{i_1}\sum_{i_2}\cdots\sum_{i_{k-1}} A_{ii_1}A_{i_1i_2}\ldots A_{i_{k-1}j}$$

and $A_{ii_1}A_{i_1i_2}\ldots A_{i_{k-1}j} \neq 0$ for some $i_1, i_2, .., i_{k-1}$ only if there is path from i to j through $i_1, i_2, ..., i_{k-1}$. □

Lemma 3.6. *Let A be a singular matrix and let X be such that its columns form a quasi-preferred basis of $E(A)$. If Z is such that $AX = XZ$, then*

$$Z_{ij} = 0 \text{ if } \alpha_i \nrightarrow \alpha_j \tag{3.8}$$

In particular, Z is triangular with all its diagonal entries equal to 0.

Proof. Since $AX = XZ$ and $X = [x^1 \ \ldots \ x^q]$, we have

$$Ax^j = \sum_{\substack{i=1\\ i\neq j}}^{q} Z_{ij}x^i + Z_{jj}x^j \text{ for all } j = 1, \ldots, q. \tag{3.9}$$

Take any $\alpha_j \in H(A)$ and consider the set $Q = \{\alpha_i \in H(A) | Z_{ij} \neq 0\}$. To show $Q \subseteq \mathrm{below}(\alpha_j)$, it is enough to show, $\mathrm{top}(Q) \subseteq \mathrm{below}(\alpha_j)$.

Consider any $\alpha_k \in \text{top}(Q)$. If $\alpha_k \notin \text{below}(\alpha_j)$ then $[(A - Z_{jj}I)x^j]_{\alpha_k} = 0$, since $\text{supp}\left((A - Z_{jj}I)x^j\right) \subseteq \text{below}(\text{supp}(x^j))$. Then equation (3.9) gives $\sum_{\substack{i\in Q\\ i\neq j}} Z_{ij}x^i_{\alpha_k} = 0$. But $\alpha_k \in \text{top}(Q)$ implies $Z_{kj}x^k_{\alpha_k} = 0$ which is not possible, hence $\alpha_k \in \text{below}(\alpha_j)$.

Since $AX = XZ$ and $\{x^1, \ldots, x^q\} \subseteq E(A)$, $A^nX = 0 = XZ^n$. Since Z is triangular and X is of full column rank, all the diagonal entries of Z must be equal to 0. □

Lemma 3.7. *Let A be an M-matrix and X be such that the columns of X form a quasi-preferred basis for $E(A)$. Let Z be the matrix satisfying the condition $-AX = XZ$. If α_i and α_j are two singular classes with* $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$, *then $Z_{ij} > 0$.*

Proof. Let there exist a pair of singular classes α_i, $\alpha_j \in H(A)$ such that $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$ and $Z_{ij} \leqslant 0$. Since $X = [x^1 \ \ldots \ x^q]$, by Lemma 3.6,

$$\left(-Ax^j\right)_{\alpha_i} = x^i_{\alpha_i}Z_{ij} + \ldots + x^{j-1}_{\alpha_i}Z_{j-1,j}. \tag{3.10}$$

As $\{x^1, \ldots, x^q\}$ is a quasi-preferred basis for A and $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$, equation (3.10) gives $(-Ax^j)_{\alpha_i} = x^i_{\alpha_i}Z_{ij} \leqslant 0$. Also since A is an M-matrix and $(Ax^j)_{\alpha_i} = A_{\alpha_i,\alpha_i}x^j_{\alpha_i} + \sum_{k=\alpha_i+1}^{\alpha_j} A_{\alpha_i,k}x^j_k$, it follows that $A_{\alpha_i,\alpha_i}x^j_{\alpha_i} \geqslant 0$. Since A_{α_i,α_i} is an irreducible singular M-matrix, $A_{\alpha_i,\alpha_i}x^j_{\alpha_i} \geqslant 0$ which implies $A_{\alpha_i,\alpha_i}x^j_{\alpha_i} = 0$([2],pg.156). Hence it follows that $\sum_{k=\alpha_i+1}^{\alpha_j} A_{\alpha_i,k}x^j_k = 0$ and for any $k = \alpha_i + 1, \ldots, \alpha_j$, if $A_{\alpha_i,k} < 0$ then $x^j_k = 0$. This contradicts $\alpha_i \to \alpha_j$, hence $Z_{ij} > 0$. □

Lemma 3.8. *Let A be an $M_\vee$-matrix with* $\text{index}_\rho(A) \leqslant 1$ *and X be such that its columns form a quasi-preferred basis in $E(A)$. Let Z be the matrix satisfying the condition $-AX = XZ$. If α_i and α_j are two singular classes with* $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$, *then $Z_{ij} > 0$.*

Proof. Let $A = \rho I - B$, where B is an eventually nonnegative matrix with $\text{index}(B) \leqslant 1$ and $\rho = \rho(B)$. Thus there exists a prime number g such that

$g \notin D_B$ and $B^l \geqslant 0$ for all integer $l \geqslant g$, where D_B is as defined in Theorem 3.5. Since $-AX = XZ$, $B^kX = X\bar{Z}^k$ for any positive integer k, where $\bar{Z} = Z + \rho I$. Take $\tilde{B} = B^g$ and $\tilde{Z} = \bar{Z}^g$, then $\tilde{B} \geqslant 0$ and since by Theorem 3.5 the accessibility relations in B and $\tilde{B}$ are same, columns of X will also be a quasi-preferred basis for $E(\rho^g I - \tilde{B})$. If α_i, α_j are singular classes of A with $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$ then by Lemma 3.7, $\tilde{Z}_{ij} > 0$. We will use induction on l to show that $\bar{Z}^l_{ij} = l\rho^{l-1}Z_{ij}$ for any integer $l \geqslant 2$, hence $\tilde{Z}_{ij} > 0$ will imply $Z_{ij} > 0$.

For $l = 2$, $(\bar{Z}^2)_{ij} = 2\rho\bar{Z}_{ij} + \sum\limits_{l=i+1}^{j-1} \bar{Z}_{il}\bar{Z}_{lj} = 2\rho Z_{ij} + \sum\limits_{l=i+1}^{j-1} Z_{il}Z_{lj}$.

Since $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$, from Lemma 3.6 it follows that $Z_{il}Z_{lj} = 0$ for all l, $i + 1 \leqslant l \leqslant j - 1$. Thus $(\bar{Z}^2)_{ij} = 2\rho Z_{ij}$. Let $\bar{Z}^l_{ij} = l\rho^{l-1}Z_{ij}$ for all $l < k$ and $k > 2$.
Now,

$$
\begin{aligned}
(\bar{Z}^k)_{ij} &= \bar{Z}_{ii}(\bar{Z}^{k-1})_{ij} + \sum_{l=i+1}^{j-1} \bar{Z}_{il}(\bar{Z}^{k-1})_{lj} + \bar{Z}_{ij}(\bar{Z}^{k-1})_{jj} \\
&= \rho(k-1)\rho^{k-2}Z_{ij} + \sum_{l=i+1}^{j-1} Z_{il}(\bar{Z}^{k-1})_{lj} + Z_{ij}\rho^{k-1} \qquad (3.11) \\
&= k\rho^{k-1}Z_{ij} + \sum_{l=i+1}^{j-1} Z_{il}(\bar{Z}^{k-1})_{lj}. \qquad (3.12)
\end{aligned}
$$

From Lemma 3.5, if $Z_{il}(\bar{Z}^{k-1})_{lj} \neq 0$ for some l, $i + 1 \leqslant l \leqslant j - 1$ then there is a path from i to l in $\Gamma(Z)$ and from l to j in $\Gamma(\bar{Z})$. Hence by Lemma 3.6, there is a path from i to j in $\Gamma(A)$ through at least 3 singular classes i, l and j of A, which contradicts the fact that $\text{hull}(i, j) \bigcap H(A) = \{i, j\}$. Thus $\sum\limits_{l=i+1}^{j-1} Z_{il}(\bar{Z}^{k-1})_{lj} = 0$, or $(\bar{Z}^k)_{ij} = k\rho^{k-1}Z_{ij}$. Hence $\tilde{Z}_{ij} = g\rho^{g-1}Z_{ij} > 0$, which implies $Z_{ij} > 0$ and the result follows. □

If B is an eventually nonnegative matrix with $\text{index}(B) \leqslant 1$, it is known from [33] that B and hence $A = \rho I - B$ has a quasi-preferred basis. We next give a procedure to obtain a preferred basis from a quasi-preferred basis for any $M_\vee$-matrix A, where $A = \rho I - B$ with $\text{index}(B) \leqslant 1$.

Procedure 3.1. ***Constructive method of obtaining a preferred basis from a quasi-preferred basis:***

Let $A = \rho I - B$ be an $M_\vee$-matrix with $\text{index}_\rho(A) \leqslant 1$ and $X = [x^1\ x^2\ \ldots\ x^q]$ be an $n \times q$ matrix whose columns form a quasi-preferred basis for $E(A)$ and let Z be the matrix satisfying $-AX = XZ$.

We now construct a preferred basis (from the given quasi-preferred basis X) $\tilde{X}$ such that $-A\tilde{X} = \tilde{X}\tilde{Z}$ for some nonnegative matrix $\tilde{Z}$.

If the columns of X already give a preferred basis for $E(A)$, then we are done.

Let X be such that its columns form a quasi-preferred basis but not a preferred basis for $E(A)$, then there exist indices i and j such that $\alpha_i \to \alpha_j$ and $Z_{ij} \leqslant 0$. Consider the set $\mathcal{I} = \{j \in \langle q\rangle \mid Z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle q\rangle \mid \alpha_i \to \alpha_j \text{ and } Z_{ij} = 0 \text{ for some } i\}$, then $\mathcal{I} \neq \emptyset$. Let j be the least index in $\mathcal{I}$, then the first $j-1$ columns of X forms a preferred set for $E(A)$. To find an $\tilde{x}^j$ such that if $\tilde{X}$ is the matrix obtained by replacing the jth column x_j of X by $\tilde{x}^j$, then the first j columns of $\tilde{X}$ will be a preferred set of $E(A)$. Finally we show that it can be done for every $j \geqslant 2$.

Let

$$\begin{array}{rcl} Q &=& \{i \in \langle j-1\rangle \mid Z_{ij} < 0\} \\ R &=& \{i \in \langle j-1\rangle \mid Z_{ij} = 0, \alpha_i \to \alpha_j\} \\ S &=& Q \bigcup R \\ \bar{Q} &=& \langle j-1\rangle \backslash Q \\ \bar{R} &=& \langle j-1\rangle \backslash R \\ \bar{S} &=& \bar{Q} \bigcap \bar{R}. \end{array}$$

By assumption $S \neq \emptyset$. Since for all $i \in S$, $\alpha_i \to \alpha_j$, there exists an $l(i) \in H(A)$ such that $\alpha_{l(i)} \to \alpha_j$ and $\text{hull}(\alpha_i, \alpha_{l(i)}) \bigcap H(A) = \{\alpha_i, \alpha_{l(i)}\}$. Since for all $i \in S$, $Z_{ij} \leqslant 0$ and from Lemma 3.8, $Z_{i,l(i)} > 0$, $l(i) < j$ for all $i \in S$.

Case I: $Q = \emptyset$. Let $\tilde{x}^j = x^j + \sum_{i \in R} x^{l(i)}$.

Since $-Ax^{l(i)} = Z_{i,l(i)}x^i + \sum_{\substack{k=1\\k\neq i}}^{l(i)-1} Z_{k,l(i)}x^k$ and $-Ax^j = \sum_{i \in \bar{R}} Z_{ij}x^i$,

$$-A\tilde{x}^j = \sum_{i \in R} Z_{i,l(i)}x^i + \sum_{i \in R} \sum_{\substack{k=1\\k\neq i}}^{l(i)-1} Z_{k,l(i)}x^k + \sum_{i \in \bar{R}} Z_{ij}x^i. \tag{3.13}$$

Since the first $j-1$ columns of X formed a preferred set for $E(A)$ and $Z_{i,l(i)} > 0$ for all $i \in R$, $\{x^1, \dots, x^{j-1}, \tilde{x}^j\}$ forms a preferred set for $E(A)$.

Case II: $Q \neq \emptyset$. Let $\tilde{x}^j = x^j + \beta \sum_{i \in S} x^{l(i)}$, then

$$-A\tilde{x}^j = \beta \sum_{i\in R} Z_{i,l(i)} x^i + \sum_{i \in Q} \left(\beta Z_{i,l(i)} + Z_{ij}\right) x^i + \sum_{i\in S} \sum_{\substack{k=1 \\ k\neq i}}^{l(i)-1} \beta Z_{k,l(i)} x^k + \sum_{i \in \bar{S}} Z_{ij} x^i. \tag{3.14}$$

For $\beta > \max_{i \in Q} \left\{ \dfrac{-Z_{ij}}{Z_{i,l(i)}} \right\} > 0$, $\{x^1, \dots, x^{j-1}, \tilde{x}^j\}$ forms a preferred set for $E(A)$. Hence in both cases if we take $\tilde{X} = [x^1 \dots \tilde{x}^j \dots x^q]$ and if $\bar{Z}$ is the matrix satisfying the condition $-A\tilde{X} = \tilde{X}\bar{Z}$, then the leading j columns of $\tilde{X}$ form a preferred set for $E(A)$. The above process is repeated with X replaced by $\tilde{X}$. Since at every stage at least one more column is included in the preferred set, after at most $q-j$ steps we will get a preferred basis for $E(A)$.

The following theorem is an immediate consequence of the above procedure.

Theorem 3.7. *If $A = \rho I - B$ is an $M_\vee$-matrix with* $\text{index}_\rho(A) \leqslant 1$*, then there is a preferred basis for $E(A)$.*

Remark 3.2. Procedure 3.1 can also be used to obtain a preferred basis from a given quasi-preferred basis for M-matrices.

We summarize the entire procedure below.

Algorithm 1 Given $A \in \mathbb{R}^{n,n}, X \in \mathbb{R}^{n,q}$

$H(A) = \{\alpha_1, \ldots, \alpha_q\}$ basis classes of A
$Z = X^+AX$ (X^+ is the pseudo inverse of X)
$\mathcal{I} = \{j \in \langle q\rangle \mid Z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle q\rangle \mid Z_{ij} = 0, \alpha_i \to \alpha_j, \text{ for some } i\}$
while $\mathcal{I} \neq \emptyset$ **do**
  $j = \min \mathcal{I}$
  $Q = \{i \in \langle j-1\rangle | Z_{ij} < 0\} = \{i_1, \ldots, i_m\}$
  $R = \{i \in \langle j-1\rangle | Z_{ij} = 0, \alpha_i \to \alpha_j\} = \{i_{m+1}, \ldots, i_t\}$
  if $Q = \emptyset$ **then**
    for $k = m+1 : t$ **do**
      $l(k) \longleftarrow \text{hull}(\alpha_{i_k}, \alpha_{l(k)}) \bigcap H(A) = \{\alpha_{i_k}, \alpha_{l(k)}\}$ and $\alpha_{l(k)} \to \alpha_j$
    end for
    for $r = 1 : n$ **do**

$$X_{rj} \longleftarrow X_{rj} + \sum_{k=m+1}^{t} X_{rl(k)}$$

    end for
  else
    for $k = 1 : t$ **do**
      $l(k) \longleftarrow \text{hull}(\alpha_{i_k}, \alpha_{l(k)}) \bigcap H(A) = \{\alpha_{i_k}, \alpha_{l(k)}\}$ and $\alpha_{l(k)} \to \alpha_j$
    end for
    Choose $\beta > \max\limits_{1 \leqslant k \leqslant m} \left\{ \dfrac{-Z_{i_k j}}{Z_{i_k l(k)}} \right\}$
    for $r = 1 : n$ **do**

$$X_{rj} \longleftarrow X_{rj} + \beta \sum_{k=1}^{t} X_{rl(k)}$$

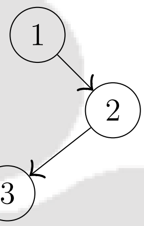
    end for
  end if
  $Z = X^+AX$
  $\mathcal{I} = \{j \in \langle q\rangle \mid Z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle q\rangle \mid Z_{ij} = 0, \alpha_i \to \alpha_j, \text{ for some } i\}$
end while

We illustrate the above procedure with the help of the following example.

Example 3.6. Let

$$B = \begin{bmatrix} 2 & 2 & 4 & -1 & 0 & 0 \\ 2 & 2 & -1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 6 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

Then $B^k \geqslant 0$ for all $k \geqslant 7$ with $\rho(B)$=4. Consider the $M_\vee$ matrix $A = 4I - B$ so that $E(A) = N(A^3)$ and $\text{index}_4(A) = 1$. The reduced graph of A is given by,



Consider the quasi-preferred basis for $E(A)$ given by,

$$\begin{aligned} x^1 &= [2 \;\; 2 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \\ x^2 &= [271 \;\; 241 \;\; 36 \;\; 12 \;\; 0 \;\; 0]^T \\ x^3 &= [3.0625 \;\; 1 \;\; 2.8 \;\; 1 \;\; 1.5 \;\; 1]^T \end{aligned}$$

Take $X = [x^1\ x^2\ x^3]$. Then $-AX = XZ$ implies that

$$Z = \begin{bmatrix} 0 & 36 & -0.35 \\ 0 & 0 & 0.25 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the set $\mathcal{I} = \{j \in \langle 4 \rangle \mid Z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle 4 \rangle \mid Z_{ij} = 0, \alpha_i \to \alpha_j, \text{ for some } i\} = \{3\} \bigcup \varnothing$. So 3 is the least index in $\mathcal{I}$. Now consider the set $Q = \{i \mid Z_{i3} < 0\} = \{1\}$. Again we have $\text{hull}(1,2) \bigcap H(A) = \{1, 2\}$. Define the vector $x^3_{new} = x^3 + x^2$ so that

$$-Ax^3_{new} = 35.65x^1 + 0.25x^2 + 4x^3_{new}$$

Then,

$$-A[x^1\ x^2\ x^3_{new}] = [x^1\ x^2\ x^3_{new}]\begin{bmatrix} 0 & 36 & 35.65 \\ 0 & 0 & 0.25 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we have the preferred basis $\{x^1, x^2, x^3_{new}\}$ for $E(A)$ such that if $X_{final} = [x^1\ x^2\ x^3_{new}]$, then

$$-AX_{final} = X_{final}\begin{bmatrix} 0 & 36 & 35.65 \\ 0 & 0 & 0.25 \\ 0 & 0 & 0 \end{bmatrix} := X_{final}Z_{final}.$$

3.4.2 Height and level characteristics of $M_\vee$-matrices and well structured graphs

In this subsection we will extend many of the results from by Schneider and Hershkowitz in [21, 24, 25] obtained for singular M-matrices, to the class of singular $M_\vee$-matrices. This subsection essentially deals with height characteristic and level characteristic. We give some necessary and sufficient conditions for their equality. Later we obtain a sufficient condition for the reduced graph of an $M_\vee$-matrix to be a well structured graph.

3.4.2.1 Height and level characteristics of $M_\vee$-matrices

Definition 3.6. *[24, 25] Let* $t = \text{index}(A)$. *For* $i \in \langle t\rangle$, *let* $\eta_i(A) = n(A^i) - n(A^{i-1})$. *The sequence* $(\eta_1(A), \ldots, \eta_t(A))$ *is called the* height characteristic *of* A, *and is denoted by* $\eta(A)$. *Normally we write* η_i *for* $\eta_i(A)$ *where no confusion should result.*

The height characteristic is also known as the Weyr Characteristic.

Definition 3.7. *[25] Let A be a singular matrix and let* $\text{index}(A) = t$.

(i) *Let S be a collection of vectors in $E(A)$, and let $\eta_k(S)$ be the number of vectors in S of height k. We define the* height signature *$\eta(S)$ of S as the t-tuple $(\eta_1(S), \ldots, \eta_t(S))$.*

(ii) *A basis $\mathcal{B}$ for $E(A)$ is said to be a* height basis *for $E(A)$ if $\eta(\mathcal{B}) = \eta(A)$.*

Definition 3.8. *[25] Let A be a singular matrix.*

(i) *The Segré Characteristic $j(A)$ of A is defined to be the nonincreasing sequence of sizes of the jordan blocks of A associated with the eigenvalue* 0.

(ii) *A sequence $(x^1, \ldots, x^s)$ of vectors in $E(A)$ is said to be a* Jordan chain *for A if $Ax^i = x^{i-1}$, $i \in \{2, \ldots, s\}$, and $Ax^1 = 0$. We call x^s the* top *of the chain $(x^1, \ldots, x^s)$.*

(iii) *A basis for $E(A)$ that consists of disjoint Jordan chains for A is said to be a* Jordan basis *for $E(A)$.*
As is well known, $E(A)$ always has a Jordan basis.

Remark 3.3. Observe that every Jordan basis for A is height basis, but clearly a height basis need not be a Jordan basis.

Definition 3.9. *[21] Let $a = (a_1, \ldots, a_r)$ be a nonincreasing sequence of positive integers. Consider the diagram formed by r columns of stars such that the jth column has a_j stars. The sequence $a^\star$* dual to *a is defined to be the sequence of row lengths of the diagram, reordered in a nonincreasing order.*

It is well known that the height characteristic and the *Segré* characteristic are dual sequences (see [48]).

Definition 3.10. *[25] The cardinality of the jth level of $S(A)$ is denoted by $\lambda_j(A)$. If $S(A)$ has m levels, then the sequence $(\lambda_1(A), \ldots, \lambda_m(A))$ is called the* level characteristic *of A, and is denoted by $\lambda(A)$. Normally we write λ_i for $\lambda_i(A)$ where no confusion should result.*

Convention 3.1. We will always assume, the level characteristic and the height characteristic of A to be $(\lambda_1, \ldots, \lambda_m)$ and $(\eta_1, \ldots, \eta_t)$ respectively.

Remark 3.4. *[33]* If A is an $M_\vee$-matrix with $\text{index}_\rho(A) \leqslant 1$ then m and t in the above two definitions are equal.

Definition 3.11. *[25] Let A be a square matrix.*

(i) *Let S be a collection of vectors in $E(A)$, and let $\lambda_k(S)$ be the number of vectors in S of level k. We define the* level signature *$\lambda(S)$ of S as the m-tuple $(\lambda_1(S), \ldots, \lambda_m(S))$.*

(ii) *A basis* $\mathcal{B}$ *for* $E(A)$ *is said to be a* level basis *for* $E(A)$ *if* $\lambda(\mathcal{B}) = \eta(A)$.

(iii) *A basis* $\mathcal{B}$ *for* $E(A)$ *is said to be a* height-level basis *for* $E(A)$ *if* $\mathcal{B}$ *is both a height and a level basis.*

Definition 3.12. *Let* A *be an* $n \times n$ *singular matrix and let* $\mathcal{B} = \{x^1, \ldots, x^q\}$ *be a basis for* $E(A)$. *Denote* $X = [x^1 \ldots x^q] \in \mathbb{R}^{n,q}$. *Then there exists a unique matrix* $C \in \mathbb{R}^{q,q}$ *such that* $AX = XC$. *We call the matrix* C*, the* induced matrix for A by $\mathcal{B}$*, and we denote it by* $C(A, \mathcal{B})$.

Example 3.7. Consider the matrix A in Example 2.1. Then $\lambda(A) = (1, 2, 1, 2)$, $\eta(A) = (3, 1, 1, 1)$ and $j(A) = (4, 1, 1)$.

The following concepts were introduced in [24].

Definition 3.13. *Let* $\mathcal{P}$ *be the set of p-tuples of nonnegative integers.* $\mathcal{P}$ *is partially ordered in the following way: If* $a = (a_1, \ldots, a_p)$ *and* $b = (b_1, \ldots, b_p)$ *are in* $\mathcal{P}$*, then we define* $a \preccurlyeq b$ *if*

$$\begin{cases} \sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i, & k \in \langle p-1 \rangle \\ \sum_{i=1}^{p} a_i = \sum_{i=1}^{p} b_i \end{cases}$$

If $a \preccurlyeq b$*, then we say that* a *is majorized by* b*. If* $a \preccurlyeq b$ *and* $a \neq b$*, then we write* $a \prec b$.

Remark 3.5. Let $\mathcal{B}$ be a basis of $E(A)$. If $\eta(\mathcal{B}) = (\eta_1(\mathcal{B}), \ldots, \eta_t(\mathcal{B}))$ is the height signature of $\mathcal{B}$, then for any $k \in \langle t \rangle$, $\mathcal{B}$ has $\eta_1(\mathcal{B}) + \ldots + \eta_k(\mathcal{B})$ elements of height at most k and hence $\eta_1(\mathcal{B}) + \ldots + \eta_k(\mathcal{B}) \leqslant \eta_1 + \ldots + \eta_k$, so $\eta(\mathcal{B}) \preccurlyeq \eta(A)$.

By a similar argument, $\lambda(\mathcal{B}) \preccurlyeq \lambda(A)$ for any basis of $E(A)$.

Lemma 3.9. *[24] Given* A*, let* y *be a linear combination of the* n*-component vectors* $x^1, \ldots, x^r$*. Then,* $\text{level}(y) \leqslant \max\{\text{level}(x^i) : i \in \langle r \rangle\}$.

Lemma 3.10. *[24] Given* A*, let* y *be a linear combination of the* n*-component vectors* $x^1, \ldots, x^r$*. Then,* $\text{height}(y) \leqslant \max\{\text{height}(x^i) : i \in \langle r \rangle\}$.

Lemma 3.11. *If $\mathcal{B}$ is a preferred basis of an $M_\vee$-matrix A with* $\text{index}_\rho(A) \leqslant 1$, *then* $\text{level}(A^k x) \leqslant \text{level}(x) - k$ *for all $x \in \mathcal{B}$ and $k \geqslant 1$.*

Proof. Let $\mathcal{B} = \{x^1, \ldots, x^q\}$. Since

$$(-1)^k A^k x^i = \sum_{i_1}\sum_{i_2}\ldots\sum_{i_k} c_{i_1 i_2}\ldots c_{i_k i} x^{i_1}, \qquad i_1 \neq \ldots \neq i_k \neq i,$$

and $c_{i_1 i_2}\ldots c_{i_k i} > 0$ for some $i_1 \neq \ldots \neq i_k \neq i$ if and only if there is a chain of length k from i_1 to i, it follows that $\text{level}(x^{i_1}) \leqslant \text{level}(x^i) - k$, for all i_1. Hence by Lemma 3.9 the result follows.

□

Corollary 3.3. *For any preferred basis $\mathcal{B}$ of A,* $\text{height}(x) \leqslant \text{level}(x)$, *for all $x \in \mathcal{B}$.*

Proof. Follows from the above lemma. □

Lemma 3.12. *Let A be any $M_\vee$- matrix with* $\text{index}_\rho(A) \leqslant 1$ *and $x \in E(A)$. Then* $\text{height}(x) \leqslant \text{level}(x)$.

Proof. Let $\mathcal{B} = \{x^1, \ldots, x^q\}$ be a preferred basis for A, then $x = \sum_{i=1}^{q} c_i x^i$ for some c_i's. Let $Q = \{i \mid c_i \neq 0\}$, then clearly, $l = \text{level}(x) = \max\{\text{level}(x^i) \mid i \in \text{top}(Q)\}$. From Corollary 3.3 it follows that for all $i \in Q$, $\text{height}(x^i) \leqslant \text{level}(x^i) \leqslant l$, or $A^l x^i = 0$. So it follows that $A^l x = 0$ and therefore, $\text{height}(x) \leqslant l = \text{level}(x)$.

□

Remark 3.6. From the above lemma we can easily conclude that if A is any $M_\vee$- matrix with $\text{index}_\rho(A) \leqslant 1$ and $\mathcal{B}$ is any basis for $E(A)$ then $\lambda(B) \preceq \eta(B)$.

Remark 3.7. If A is any $M_\vee$- matrix with $\text{index}_\rho(A) \leqslant 1$ then from Lemma 3.9, the set $\Lambda_k(A)$ consisting of all vectors in $E(A)$ with level less than or equal to k form a vector space and in view of the above lemma, $\Lambda_k(A) \subseteq N(A^k)$, hence $\lambda(A) \preccurlyeq \eta(A)$.

Lemma 3.13. *Let A be an $M_\vee$-matrix with* $\text{index}_\rho(A) \leqslant 1$. *Then for any nonnegative vector x in $E(A)$,* $\text{height}(x) = \text{level}(x)$.

Proof. It suffices to show that level$(x) \leqslant$ height(x). Let $\{x^1, \ldots, x^q\}$ be a preferred basis for $E(A)$ then $x = \sum_{i=1}^{q} c_i x^i$ for some scalars c_i, and $l = \text{level}(x) = \max\{\text{level}(x^i) \mid i \in \text{top}(Q)\}$ where Q is as defined in Lemma 3.12. Clearly since x is nonnegative, for any $i \in \text{top}(Q)$, $c_i > 0$. In view of the above argument it is enough to show $A^{l-1}x \neq 0$.

Let $-Ax^i = \sum_{k=1}^{q} c_{ki} x^k$ where the c_{ki}'s are as in the definition of a preferred basis, then

$$(-1)^{l-1} A^{l-1} x = (-1)^{l-1} \left(\sum_{i \in Q} c_i A^{l-1} x^i \right).$$

From Lemma 3.12, height$(x) \leqslant$ level(x) and hence it follows that

$$\begin{aligned}(-1)^{l-1} \left(\sum_{i \in Q} c_i A^{l-1} x^i \right) &= (-1)^{l-1} \sum_{\substack{i \in Q \\ \text{level}(x_i) = l}} c_i A^{l-1} x^i \\ &= \sum_{i_1} \sum_{i_2} \cdots \sum_{i_l - 1} \sum_{\substack{i \in Q \\ \text{level}(x_i) = l}} c_{i_1 i_2} \ldots c_{i_{l-1} i} c_i x^{i_1}.\end{aligned}$$

Since for every $i \in Q$ with level$(x^i) = l$, $c_i > 0$ and there is a sequence of distinct indices $i_i, i_2, \ldots, i_{l-1}$ such that $c_{i_1 i_2} \ldots c_{i_{l-1} i} > 0$, it follows that $A^{l-1} x \neq 0$. □

Remark 3.8. From the above lemma it is clear that for any nonnegative level basis of $E(A)$ and in particular for a preferred basis $\mathcal{B}$ of $E(A)$, $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \lambda(A)$.

Remark 3.9. If $\mathcal{B}$ is a nonnegative height basis, then $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \eta(A)$ and this together with Remark 3.8 and 3.5 imply that $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \eta(A) = \lambda(A)$. Hence $\mathcal{B}$ is also a level basis.

In [33], it was shown that the level characteristic of an eventually nonnegative matrix B with index$(B) \leqslant 1$ is majorized by the height characteristic which implies that the level characteristic of an $M_\vee$- matrix A with index$_\rho(A) \leqslant 1$, is also majorized by the height characteristic. Motivated by the necessary and sufficient conditions obtained by Schneider and Hershkowitz in [24] for the equality of these two characteristics for singular M-matrices, we independently try to obtain similar conditions for the equality of these two characteristics for the class of $M_\vee$-matrices.

Theorem 3.8. *Let A be an $M_\vee$-matrix with $\text{index}_\rho(A) \leqslant 1$. Then the following are equivalent:*

(i) $\eta(A) = \lambda(A)$.

(ii) *For all $x \in E(A)$, $\text{height}(x) = \text{level}(x)$.*

(iii) *For every basis $\mathcal{B}$ of $E(A)$, $\text{height}(x) = \text{level}(x)$ for all $x \in \mathcal{B}$.*

(iv) *For some height basis $\mathcal{B}$ of $E(A)$, $\text{height}(x) = \text{level}(x)$ for all $x \in \mathcal{B}$.*

(v) *Every height basis for A is a level basis for A.*

(vi) *Every level basis for A is a height basis for A.*

(vii) *Some preferred basis for A is a height basis for A.*

(viii) *There exists a nonnegative height-level basis for A.*

(ix) *There is a nonnegative height basis for A.*

(x) *For all $j \in \langle t \rangle$, there exists a nonnegative basis for $N(A^j)$.*

(xi) *For every level basis $\mathcal{B}$ for A with induced matrix $C = C(A, \mathcal{B})$, the block $C_{k-1,k}$ has full column rank for all $k \in \langle t \rangle$.*

(xii) *There exists a level basis $\mathcal{B}$ for A with induced matrix $C = C(A, \mathcal{B})$, such that for all $k \in \langle t \rangle$, the blocks $C_{k-1,k}$ have full column rank.*

Proof. (i) $\Rightarrow$ (ii): Condition (i) implies that for any k, $\dim(\Lambda_k(A)) = \lambda_1 + \ldots + \lambda_k = \eta_1 + \ldots + \eta_k = \dim(N(A^k))$. So from Remark 3.7 it follows that $\Lambda_k(A) = N(A^k)$ and hence (ii) follows.
(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious.
(iv) $\Rightarrow$ (v) : By assumption we have a height basis $\mathcal{B}$ such that for each $x \in \mathcal{B}$, $\text{height}(x) = \text{level}(x)$, hence it follows that $\eta(A) = \eta(\mathcal{B}) = \lambda(\mathcal{B})$. Since $\lambda(\mathcal{B}) \leqslant \lambda(A)$ from Remark 3.5, and $\eta(A) \geqslant \lambda(A)$, it follows that $\eta(A) = \lambda(A)$, hence (i) and (iii) hold.

If $\mathcal{B}'$ is any height basis, then (iii) and (i) imply $\lambda(\mathcal{B}') = \eta(\mathcal{B}') = \eta(A) = \lambda(A)$. Thus $\mathcal{B}'$ is a level basis.

(v) $\Rightarrow$ (vi) : Consider a Jordan basis $\mathcal{B}$ for $E(A)$ derived from the set $T = \{y^1, \ldots, y^{\bar{t}}\}$ and let $\max\{\text{height}(y^k) \mid k \in \langle \bar{t} \rangle\} = l$. Since A is an $M_\vee$-matrix with $\text{index}_\rho(A) \leqslant 1$, index$(A)$ is equal to the length of the longest chain in A and hence it follows that $\max\{\text{level}(y_k) \mid k \in \langle \bar{t} \rangle\} = l$.

Since $\mathcal{B}$ is a height basis, $\eta(\mathcal{B}) = \eta(A)$, hence by assumption $\lambda(\mathcal{B}) = \lambda(A)$. Also for any basis $\mathcal{B}'$, $\lambda(\mathcal{B}') \preccurlyeq \eta(\mathcal{B}') \preccurlyeq \eta(A)$ and $\lambda(\mathcal{B}') = \lambda(A) \preccurlyeq \eta(A)$ if it is a level basis, hence to show that every level basis is a height basis, it is enough to show $\eta(A) = \lambda(A)$ or $\eta(\mathcal{B}) = \lambda(\mathcal{B})$.

Any y^i for which $\text{height}(y^i) = \text{level}(y^i)$, $\text{height}(A^k y^i) = \text{height}(y^i) - k = \text{level}(y^i) - k \geqslant \text{level}(A^k y^i)$, hence it follows that $\text{height}(A^k y^i) = \text{level}(A^k y^i)$ for any $k \leqslant \text{height}(y^i)$. Then for y^i for which $\text{height}(y^i) = l$, $\text{height}(A^k y^i) = \text{level}(A^k y^i)$ for any $k \leqslant l$. From the above argument it follows that if $\lambda(\mathcal{B}) \neq \eta(\mathcal{B})$, then there exists a $y^i \in T$ with $\text{height}(y^i) < l$ such that $\text{height}(y^i) < \text{level}(y^i)$. Let $\text{height}(y^i) = s$ and $\text{level}(y^i) = p$. Consider any $y^j \in T$ with $\text{height}(y^j) = l = \text{level}(y^j)$, then there exists an r such that $\text{height}(A^r y^j) = \text{height}(y^i) = s$. Consider the element $z = y^i + A^r y^j$, and the new basis $\bar{\mathcal{B}}$ obtained from $\mathcal{B}$ by replacing $A^r y^j$ with z. Since $\mathcal{B}$ is a height basis, the new basis $\bar{\mathcal{B}}$ so constructed will also be a height basis and since $\text{level}(A^r y^j) = \text{height}(A^r y^j) = s$, $\text{level}(z) = p > s$. Hence $\lambda(\bar{\mathcal{B}}) \prec \lambda(\mathcal{B}) = \lambda(A)$ which contradicts the assumption that every height basis is a level basis. Hence for any $y^i \in T$, $\text{height}(y^i) = \text{level}(y^i)$ which implies $\eta(\mathcal{B}) = \lambda(\mathcal{B})$.

(vi) $\Rightarrow$ (vii) and (vii) $\Rightarrow$ (viii) follow from the fact that every preferred basis is a level basis.

(viii) $\Rightarrow$ (ix) : is obvious.

(ix) $\Rightarrow$ (x) : Let $\mathcal{B}$ be a nonnegative height basis for A. Then $\eta(\mathcal{B}) = \eta(A) = (\eta_1, \ldots, \eta_p)$. Thus for any $j \in \langle p \rangle$, there are $\eta_1 + \ldots + \eta_j$ elements in $\mathcal{B}$ of height at most j and since $\dim(N(A^j)) = \eta_1 + \ldots + \eta_j$ these elements will form a nonnegative basis for $N(A^j)$.

(x) $\Rightarrow$ (xi) : Suppose that for each $j \in \langle p \rangle$, there exists a nonnegative basis for $N(A^j)$. Let $\mathcal{B}$ be a level basis for A with the induced matrix $C = C(A, \mathcal{B})$. To show that for each k, $C_{k-1,k}$ has full column rank.

Suppose that there is a k such that $C_{k-1,k}$ does not have full column rank and we assume that k is the least of such indices. We have,

$$A[X^{(1)}\dots X^{(t)}] = [X^{(1)}\dots X^{(t)}]\begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1t} \\ & 0 & C_{23} & \dots & C_{2t} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & C_{t-1,t} \\ & & & & 0 \end{bmatrix} \quad (3.15)$$

with $X^{(i)} = [x_1^i \dots x_{\lambda_i}^i]$ in which the columns give the elements of $\mathcal{B}$ having level i.

Since $C_{k-1,k} = [C_1^{k-1,k} \dots C_{\lambda_k}^{k-1,k}]$ does not have full column rank, so there is a column say $C_j^{k-1,k}$ in $C_{k-1,k}$ which is a linear combination of its preceding columns. Since every column of $C_{k-1,k}$ is a nonzero vector, there exists scalars $\beta_1, \dots, \beta_{j-1}$, not all zeros, such that $C_j^{k-1,k} = \sum_{r=1}^{j-1} \beta_r C_r^{k-1,k}$. Then from equation (3.15) for $r = 1, \dots, j-1$ we have,

$$Ax_r^k = X^{(1)}C_r^{1,k} + X^{(2)}C_r^{2,k} + \dots + X^{(k-1)}C_r^{k-1,k}, \quad (3.16)$$

and

$$Ax_j^k = X^{(1)}C_j^{1,k} + X^{(2)}C_j^{2,k} + \dots + X^{(k-1)}\left(\sum_{r=1}^{j-1} \beta_r C_r^{k-1,k}\right). \quad (3.17)$$

If $z = x_j^k - \sum_{r=1}^{j-1} \beta_r x_r^k$, then it follows that height$(z) \leqslant k-1$. Let height$(z) = h$, then by assumption $N(A^h)$ has a nonnegative basis, say, $\{y^1, \dots, y^m\}$. Then $z = d_1 y^1 + \dots + d_m y^m$ for some scalars d_i. Let level$(z) = l$, then since level$(y^i) =$ height$(y^i) \leqslant h$, $l \leqslant h < k$. Construct a new basis $\tilde{\mathcal{B}}$ from $\mathcal{B}$ by replacing x_j^k with z in $\mathcal{B}$. Then $\lambda_i(\tilde{\mathcal{B}}) = \lambda_i$ for all $i \notin \{l, k\}$; $\lambda_l(\tilde{\mathcal{B}}) = \lambda_l + 1$; $\lambda_k(\tilde{\mathcal{B}}) = \lambda_k - 1$, hence it follows that $\lambda(\tilde{\mathcal{B}}) \succ \lambda(A)$, which is a contradiction. Thus (xii) holds.

(xi) $\Rightarrow$ (xii) : is obvious.

(xii) $\Rightarrow$ (i) : Let there exist a level basis $\mathcal{B}$ for A with the induced matrix $C = C(A, \mathcal{B})$ such that for all $k \in \langle t \rangle$ the block $C_{k-1,k}$ has full column rank. To show that $\lambda(A) = \eta(A)$.

From equation (3.15) we have $A^{k-1}X^{(k)} = X^{(1)}C_{12}C_{23}\dots C_{k-1,k}$. Since $C_{j-1,j}$'s are of full column rank so height$(x_i^k) = k$ for all $i \in \langle\lambda_k\rangle$. Hence we have height$(x)$ = level(x) for all $x \in \mathcal{B}$ and, $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \lambda(A)$.

If $\eta(A) \succ \lambda(A)$ then there exists a k for which $\lambda_k > \eta_k$. Since $A^{k-1}X^{(k)} = X^{(1)}C_{12}C_{23}\dots C_{k-1,k}$ and each of the matrices $X_1, C_{12}, C_{23}, \dots, C_{k-1,k}$ is of full column rank, rank$(A^{k-1}X^{(k)}) = \lambda_k(\mathcal{B}) = \lambda_k$, which is equal to the number of columns in $X^{(k)}$. Hence no linear combination of the columns in $X^{(k)}$ can belong to $N(A^{k-1})$. Also since $A^kX^{(k)}$ is the $\mathbf{0}$ matrix, $\eta_k = n(A^k) - n(A^{k-1}) \geqslant \lambda_k$, which is a contradiction. Hence it follows that $\eta(A) = \lambda(A)$.

□

Theorem 3.9. *Let A be an $M_\vee$-matrix with* index$_\rho(A) \leqslant 1$. *Then $\eta(A) = \lambda(A)$ if and only if there exists a nonnegative Jordan basis for $-A$.*

Proof. Since every nonnegative Jordan basis for $-A$ is a nonnegative height basis for A, the 'if' part follows from Theorem 3.8(ix).

The 'only if' part can be obtained by proceeding as in Theorem 6.10 of [24]. □

We next consider two extreme cases: (i) Each path in $R(A)$ has at most one singular vertex, (ii) all singular vertices lie on a single path.

Theorem 3.10. *[47] Let A be an M-matrix. Then the following are equivalent:*

(i) The Segré characteristic of A is $(1, 1, \dots, 1)$.

(ii) The level characteristic of A is (t).

Theorem 3.11. *[47] Let A be an M-matrix. Then the following are equivalent:*

(i) The Segré characteristic of A is (t).

(ii) The level characteristic of A is $(1, 1, \dots, 1)$.

Theorems 3.10 and 3.11 are also true for an $M_\vee$-matrix A with index$_\rho(A) \leqslant 1$, due to Theorem 3.5 and Lemma 3.4.

Remark 3.10. Let A be an $M_\vee$-matrix with $\text{index}_\rho(A) \leqslant 1$. Then in Theorem 3.10 since 0 is a simple eigenvalue of every singular block in the Frobenius normal form of A, t is the algebraic multiplicity of 0. Also, the number of 1's in the *Segré* characteristic in Theorem 3.10 is t. Therefore Theorem 3.10 states that in the extreme case (i) we have that $\lambda(A) = j(A)^\star = \eta(A)$.

Similarly for the other extreme case (ii), considered in Theorem 3.11, $\lambda(A) = j(A)^\star = \eta(A)$.

The following examples show that the results in Theorem 3.10 and Theorem 3.11 need not be true for an $M_\vee$-matrix A having $\text{index}_\rho(A) > 1$.

Example 3.8. Consider the $M_\vee$-matrix

$$A = 4I - B = 4I - \left[\begin{array}{cc|cc|cc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{array}\right].$$

The matrix A has $\text{index}_4(A) = 2 > 1$; $t = 2$ and A is in Frobenius normal form having irreducible diagonal blocks A_{11},A_{22}, A_{33} so that the singular vertices in $R(A)$ are 1 and 3. The *Segré characteristic* is $(1, 1)$ since it has two jordan blocks of size 1 corresponding to the eigenvalue 0, whereas the *level characteristic* is $(1, 1)$.

Example 3.9. Consider the $M_\vee$ matrix A given by,

$$A = 4I - B = 4I - \begin{bmatrix} 2 & 2 & 0.5 & 0.5 \\ 2 & 2 & 0.5 & 0.5 \\ 1 & -1 & 2 & 2 \\ -1 & 1 & 2 & 2 \end{bmatrix}.$$

Then the matrix A has $\text{index}_4(A) = 2 > 1$. The *Segré characteristic* $j(A)$ of A is (2) since it has only one jordan block corresponding to 0 of size 2, whereas the level characteristic is (1) since it has only one irreducible block, the matrix itself.

3.4.2.2 Hall condition and well structured graphs

In this section we show that the reduced graph of an $M_\vee$-matrix is a well structured graph with the help of the Hall Marriage condition.

We first state Hall's theorem essentially as it is found in [4].

Theorem 3.12. *[4] Let $E_1, \ldots, E_h$ be subsets of a given set E. Then the following are equivalent:*

(i) *We have,*

$$\left|\bigcup_{i\in\alpha} E_i\right| \geqslant |\alpha|, \qquad \text{for all } \alpha \subseteq \langle h\rangle. \tag{3.18}$$

(ii) *There exist distinct elements $e_1, \ldots, e_h$ of E such that $e_i \in E_i$, $i \in \langle h\rangle$.*

The condition (3.18) is often referred to as the Hall Marriage condition.

Definition 3.14. *[25] Let S be an acyclic graph. A chain $(i_1, \ldots, i_m)$ is called an anchored chain if the level of i_k is k, $k \in \langle m\rangle$.*

Definition 3.15. *[25] Let S be an acyclic graph.*

(i) *A set κ of chains in S is said to be a* chain decomposition *of S if each vertex of S belongs to exactly one chain in κ.*

(ii) *A chain decomposition κ of S is said to be an* anchored chain decomposition *of S if every chain in κ is anchored.*

(iii) *S is said to be* well structured *if there exists an anchored chain decomposition of S.*

The following result is due to [25].

Theorem 3.13. *[25] Let S be an acyclic graph with levels $L_1, \ldots, L_t$. Then the following are equivalent:*

(i) *The sets $E_i = \mathrm{below}(i) \bigcap L_k$, $i \in L_{k+1}$, satisfy the Hall Marriage Condition for all $k \in \langle t-1\rangle$.*

(ii) *S is well structured.*

In the next theorem we show that reduced graph of an $M_\vee$-matrix is well structured.

Theorem 3.14. *Let A be an $M_\vee$-matrix with* $\text{index}_\rho(A) \leqslant 1$. *If* $\eta(A) = \lambda(A)$, *then the reduced graph $R(A)$ is well structured.*

Proof. Let $\{\alpha_1, \ldots, \alpha_q\}$ be the set of all singular vertices of A ordered according to levels. $L_1, \ldots, L_t$ mentioned in the above lemma are the levels of $R(A)$, i.e., L_i is the collection of singular vertices of level i and t is the length of the longest chain in $R(A)$. It suffices to show that, $E_i = \text{below}(\alpha_i) \bigcap L_k$ with $\alpha_i \in L_{k+1}$ satisfies the condition (i) of Theorem 3.13, for all $k \in \langle t-1 \rangle$.

Suppose if the E_i's as defined above do not satisfy the Hall marriage condition for all $k \in \langle t-1 \rangle$, then there exists a k_0 and an $\alpha \subseteq \langle \lambda_{k_0+1} \rangle$ such that $|\bigcup_{i \in \alpha} E_i| < |\alpha|$. Without loss of generality let $\alpha = \{1, 2, \ldots, r\}$.

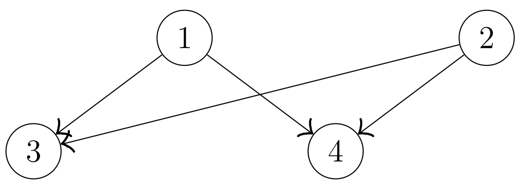
Consider a preferred basis $\mathcal{B}$. Since $\eta(A) = \lambda(A)$ it is also a height basis. If X is the matrix, the columns of which give the elements of $\mathcal{B}$ and C is the corresponding induced matrix, then since $\eta(A) = \lambda(A)$, $C_{k,k+1}$'s are of full column rank, for all $k \in \langle t-1 \rangle$. Since $\mathcal{B}$ is a preferred basis, $C_{ij} \neq 0$ if and only if $\alpha_i \to \alpha_j$. Hence $|\bigcup_{i=1}^{r} E_i| < r$ implies that in the submatrix of C_{k_0,k_0+1} of order $\lambda_{k_0} \times r$ formed by taking only the first r columns of C_{k_0,k_0+1}, there are less than r nonzero rows, which contradicts the fact that the r columns are linearly independent.

□

Remark 3.11. Note that $\eta(A) = \lambda(A)$ is a sufficient condition for the reduced graph $R(A)$ to be well structured, which need not be a necessary condition. For example, consider the M-matrix $A = I - B$ where,

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The reduced graph $R(A)$ of A is given by,



Then $\{(1,3),(2,4)\}$ is an anchored chain decomposition for A and hence $R(A)$ is well structured. But note that $\lambda(A) = (2,2)$ whereas $\eta(A) = (3,1)$.

3.5 Conclusion

We consider two types of generalizations of M-matrices, namely the GM-matrices and the $M_\vee$-matrices. Initially we show the existence of a preferred basis for GM-matrices of order 2 and also demonstrate with the help of an example, the fact that a quasi-preferred (and hence a preferred) basis need not exist if the order of the matrix exceeds 2.

Next we consider another generalization of M-matrices, known as $M_\vee$-matrices and show the existence of preferred basis for a subclass of these matrices. In particular we give a method to obtain a preferred basis for M-matrices and $M_\vee$-matrices, from a quasi-preferred basis. We next consider height, level and *Segré* characteristics of $M_\vee$-matrices and try to understand their mutual relationship. Based on results obtained for M-matrices in [24], we state and prove some equivalent conditions for the equality of the height characteristic and the level characteristic for a subclass of $M_\vee$-matrices. We also present a sufficient condition for the reduced graph of $M_\vee$-matrices to be well structured.

Chapter 4

Characterization of Generalized M-matrices

4.1 Introduction

Nonsingular M-matrices can be characterized in various ways, in terms of positivity of principal minors, stability and the inverse positivity property (see [27]). In subsection 4.2.1 of this chapter we give some useful characterizations of nonsingular $M_\vee$-matrices in terms of positivity of sums of principal minors and stability. Also we show that some of the important properties, such as inverse positivity, do not carry over to the entire class of $M_\vee$-matrices, but to a subclass of these matrices. In subsection 4.2.2 we generalize the inverse-positivity property for a subclass of nonsingular $M_\vee$-matrices, which was given by Elhashash and Szyld in [7], to singular $M_\vee$-matrices. Next we introduce the concepts of eventually monotonicity and eventually nonnegativity on a subset of $\mathbb{R}^n$, to characterize a subclass of $M_\vee$-matrices. In addition, this subclass of $M_\vee$-matrices A with index$(A) \leqslant 1$, is also characterized in terms of some special types of generalized inverses.

4.2 Characterization of $M_\vee$-matrices

In this section we consider the class of $M_\vee$-matrices and also extend some results known for M-matrices to the class of $M_\vee$-matrices.

Definition 4.1. *[8] For any real matrix A, we define a set of integers D_A*

(the denominator set *of the matrix A) as follows:*

$$D_A \;:=\; \Big\{ d \mid \theta - \alpha = \frac{c}{d}, \;\; where \;\; re^{2\pi i\theta}, re^{2\pi i\alpha} \in \sigma(A), r > 0, c \in \mathbb{Z}^+, \\ d \in \mathbb{Z}\backslash\{0\}, (c,d) = 1, \;\; and \; |\theta - \alpha| \notin \{1, 2 \ldots\} \Big\}.$$

$$P_A := \quad \{kd \mid d > 0, k \in \mathbb{Z} \; and \; d \in D_A\} \quad (Problematic\ Powers\ of\ A).$$

$$N_A := \quad \{1, 2 \ldots\} \backslash P_A \qquad\qquad (Nice\ Powers\ of\ A).$$

The set D_A consists of the denominators of the reduced fractions that represent the argument difference, normalized by $\frac{1}{2\pi}$, of two distinct eigenvalues of A that lie on the same circle with center at the origin. Note that D_A is always a finite set and is empty if and only if one of the following conditions hold:

1. *A has no distinct eigenvalues lying on the same circle with center at the origin.*

2. *The argument difference of two distinct eigenvalues lying on the same circle with center at the origin are irrational multiples of 2π.*

Definition 4.2. *[8] A matrix $A \in \mathbb{R}^{n,n}$ is said to have the* strong Perron-Frobenius property *if the spectral radius $\rho(A)$ is a simple positive eigenvalue that is strictly larger in modulus than any other eigenvalue and there exists a positive eigenvector corresponding to $\rho(A)$. By PFn we denote the collection of all $n \times n$ matrices A such that A is an eventually positive matrix.*

Remark 4.1. *[39]* It turns out that $A \in PFn$ if and only if both A and A^T possess the strong Perron-Frobenius property.

4.2.1 Nonsingular $M_\vee$-matrices

In this subsection we consider the class of nonsingular $M_\vee$-matrices. We show that in general the inverse of a nonsingular $M_\vee$-matrix need not be in $WPFn$, but under certain additional conditions, the inverse is in $WPFn$. We also obtain some new characterizations of nonsingular $M_\vee$-matrices, in terms of the positivity of sums of the principal minors and stability.

Theorem 4.1. *[33] Let A be an eventually nonnegative matrix with* $\text{index}(A) \leqslant 1$. *Let g be a prime number such that $g \notin D_A$ and $A^s \geqslant 0$ for all $s \geqslant g$. Let $\kappa = (K_1, \ldots, K_k)$ be the order partition of $\langle n \rangle$ such that A^g_κ is in Frobenius-normal form. Then A_κ is in Frobenius-normal form.*

Lemma 4.1. *Let B be a nonnilpotent, eventually nonnegative matrix. Then there exists a nonnegative vector x such that $Bx = \rho x$ with $\rho = \rho(B)$. Furthermore, if* $\text{index}(B) \leqslant 1$ *and B is irreducible, then the vector x is positive and $\rho(B)$ is simple.*

Proof. Since B is eventually nonnegative, we can always choose a prime number $k_0 \in N_B$ such that $B^k \geqslant 0$ for all $k \geqslant k_0$. Since $\rho(B^{k_0}) = \rho(B)^{k_0} = \rho^{k_0}$ and $B^{k_0} \geqslant 0$, there exists a nonnegative vector x such that $B^{k_0}x = \rho^{k_0}x$ and hence by Lemma 3.4 of Chapter 3, $Bx = \rho x$.

As B is irreducible with $\text{index}(B) \leqslant 1$, so Theorem 4.1 implies that B^{k_0} is irreducible and hence x(or $-x$) must be a positive vector. Also ρ is a simple positive eigenvalue follows from the fact that for all $k \geqslant k_0$, B^k is an irreducible nonnegative matrix and Lemma 3.4. □

Note that the above result is not true if B is nilpotent or if $\text{index}(B) > 1$, as the following two examples illustrate.

Example 4.1. Consider the matrix $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Then B is a nilpotent, eventually nonnegative matrix, and there does not exist any nonnegative eigenvector corresponding to its spectral radius 0.

Example 4.2. Consider the matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Then B is an irreducible, nonnilpotent, eventually nonnegative matrix with $\text{index}(B) = 2$. If $x = [x_i]$ is any eigenvector of B associated with its spectral radius 2, then $x_4 = 0$ and thus there can not exist a positive eigenvector corresponding to $\rho(B)$.

Theorem 4.2. *[39] Let $A \in \mathbb{R}^{n,n}$ be a nonnilpotent eventually nonnegative matrix. Then both A and A^T possess the Perron-Frobenius property.*

Theorem 4.3. *Let A written as $sI - B$, for some B, be a matrix in $\mathbb{R}^{n,n}$ whose eigenvalues (counting multiplicity) are arranged in the following manner: $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$. If B is an irreducible, nonnilpotent, eventually nonnegative matrix with* $\text{index}(B) \leqslant 1$*, then the following statements are equivalent:*

(*i*) *A is a nonsingular $M_\vee$-matrix;*

(*ii*) *A^{-1} is an eventually positive matrix and $0 < \lambda_n < Re(\lambda_i)$ for all $i \neq n$.*

Proof. (i) $\Rightarrow$ (ii): From assumption, $A = sI - B$ is an $M_\vee$-matrix, with $s > \rho(B) = \rho$ and B an irreducible, nonnilpotent, eventually nonnegative matrix. Hence ρ is positive and is a simple eigenvalue of B. We first show that $0 < \lambda_n < Re(\lambda_i)$ for all $i \neq n$.
Note that for every i, $\lambda_i = s - \lambda_i(B)$ for a corresponding $\lambda_i(B) \in \sigma(B)$. Hence $s - \rho$ is a simple eigenvalue of A and since $0 < s - \rho = |s - \rho| \leqslant |s - |\lambda_i(B)|| \leqslant |s - \lambda_i(B)|$, $\lambda_n = s - \rho$ and $\lambda_n > 0$.
For any $i \neq n$, $\lambda_n = s - \rho < s - Re(\lambda_i(B))$ if $Re(\lambda_i(B)) \geqslant 0$ and $\lambda_n = s - \rho < s < s - Re(\lambda_i(B))$ if $Re(\lambda_i(B)) < 0$.

Lastly we need to show that A^{-1} is an eventually positive matrix. Since B (and hence B^T) is an irreducible, nonnilpotent, eventually nonnegative matrix, by Lemma 4.1 ρ is simple and there exist positive vectors x, y such that $Ax = \lambda_n x$ and $y^T A = \lambda_n y^T$. Without loss of generality we may assume that $X = \begin{bmatrix} x & X^{(1)} \end{bmatrix}$, then

$$A^{-1} = \begin{bmatrix} x & X^{(1)} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_n} & 0 \\ 0 & J_{n-1} \end{bmatrix} \begin{bmatrix} y^T \\ Y^{(1)} \end{bmatrix}$$

where $X^{(1)}, {Y^{(1)}}^T \in \mathbb{R}^{n,n-1}$ and $J_{n-1} \in \mathbb{R}^{n-1,n-1}$ with $\frac{1}{\lambda_n} \notin \sigma(J_{n-1})$. Thus for any positive integer k we have that,

$$A^{-k} = \begin{bmatrix} x & X^{(1)} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_n^k} & 0 \\ 0 & J_{n-1}^k \end{bmatrix} \begin{bmatrix} y^T \\ Y^{(1)} \end{bmatrix}$$

$$i.e., \quad \lambda_n^k A^{-k} = \begin{bmatrix} x & X^{(1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D_{n-1}^{(k)} \end{bmatrix} \begin{bmatrix} y^T \\ Y^{(1)} \end{bmatrix} \tag{4.1}$$

where $D_{n-1}^{(k)} = \lambda_n^k J_{n-1}^k$ and eigenvalues of $D_{n-1}^{(k)}$ are $\frac{\lambda_n^k}{\lambda_i^k}$ with $|\frac{\lambda_n^k}{\lambda_i^k}| < 1$ and thus $\lim_{k\to\infty} D_{n-1}^{(k)} = 0$. Hence the equation (4.1) implies that

$$\lim_{k\to\infty} \lambda_n^k A^{-k} = xy^T > 0.$$

So, there exists a positive integer k_0 such that $(A^{-1})^k > 0$ for all $k \geqslant k_0$, i.e., A^{-1} is eventually positive.

(ii) $\Rightarrow$ (i): Since B is nonnilpotent, eventually nonnegative matrix, so $B \in WPFn$ by Theorem 4.2 and so the assumption implies that A is an invertible GM-matrix(see [7]). Thus it follows that $s > \rho(B)$, and A is an invertible $M_\vee$-matrix. □

Since the inverse of a reducible matrix is reducible, so the irreducibility of B (and hence of A) cannot be relaxed. One may ask whether there is a weaker relation.

Example 4.3. Consider the matrix,

$$A = 5I - B = 5I - \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}.$$

Note that A is an invertible $M_\vee$-matrix as $B^k \geqslant 0$ for all $k \geqslant 2$ and $\rho(B) = 4$. Also the eigenvalues of A are $5, 5, 3, 1$ and,

$$A^{-1} = \begin{bmatrix} 0.2667 & 0.0667 & 0.04 & -0.04 \\ 0.0667 & 0.2667 & -0.04 & 0.04 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} = E + F(\text{say}),$$

$$\text{where } E = \begin{bmatrix} 0.2667 & 0.0667 & 0 & 0 \\ 0.0667 & 0.2667 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} \text{ and } F = A^{-1} - E.$$

Note that $EF = FE = \frac{1}{5}F$ and $F^2 = 0$. Then by induction argument it can be easily checked that for any positive integer k,

$$A^{-k} = E^k + k5^{1-k}F.$$

Thus for any positive integer k, the $(1,4)th$ and the $(2,3)th$ entries of A^{-k} are always negative, hence A^{-1} is not *eventually positive* although B is a nonnilpotent, eventually nonnegative matrix with index$(B) = 1$. Note that the matrix B is reducible.

The following example show that the condition index$(B) \leqslant 1$ in Theorem 4.3 cannot be relaxed.

Example 4.4. *[7]* Consider the invertible $M_\vee$-matrix

$$A = 3I - B = 3I - \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that index$(B) = 2$ and $A^{-1} = 3^{-2}(E + F)$ where

$$E = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 9 & 9 & 6 & 3 \\ 9 & 9 & 3 & 6 \end{bmatrix} \text{ and } F = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As $EF = FE = 3F$ and $F^2 = 0$, it can be easily seen that $(A^{-1})^k = 3^{-2k}E^k + k3^{-k-1}F$ and hence A^{-1} is not an eventually nonnegative matrix although B is an irreducible, nonnilpotent, eventually nonnegative matrix.

We now give equivalent characterizations of nonsingular $M_\vee$-matrices in terms of stability and positivity of sums of principal minors. We first state some equivalent characterizations of nonsingular M-matrices, which could be found in [2, 26].

Theorem 4.4. *Let $A = sI - B$ be an $n \times n$ matrix, where $B \geqslant 0$. If A is nonsingular, then the following statements are equivalent:*

(i) *A is an invertible M-matrix.*

(ii) *A is positive stable.*

(iii) $s > \rho(B)$.

(iv) *All the real eigenvalues of A are positive.*

(v) *The sum of the all principal minors of size k are positive, for all $k \in \langle n \rangle$.*

(vi) *$A + tI$ is nonsingular for all $t > 0$.*

(vii) *$A + I$ is nonsingular and $G = (A + I)^{-1}(A - I)$ is convergent, i.e., $\rho(G) < 1$.*

(viii) *There exists an $x > 0$ such that $Ax > 0$.*

(ix) *$A + D$ is nonsingular, for every nonnegative diagonal matrix D.*

Olesky, Tsatsomeros and Driessche [41] in an attempt to extend the above theorem to the class of $M_\vee$-matrices, obtained the following result.

Theorem 4.5. *[41] Let $A = sI - B$ be an $n \times n$ matrix, where B is a nonnilpotent, eventually nonnegative matrix with power index $k_0 \geqslant 0$, that is, k_0 is the least positive integer such that $B^k \geqslant 0$ for all $k \geqslant k_0$. Let K be the cone defined as $K = B^{k_0}\mathbb{R}^n_+$. Then the following statements are equivalent:*

(*a*) *A is an invertible $M_\vee$-matrix.*

(*b*) $s > \rho(B)$.

(*c*) *A^{-1} exists and $A^{-1}K \subseteq \mathbb{R}^n_+$.*

(*d*) *$Ax \in K$ implies $x \geqslant 0$.*

In the next theorem we show that the conditions (ii) – (vii) in Theorem 4.4 are also equivalent characterizations for $M_\vee$-matrices, whereas it is shown that the conditions (viii) and (ix) need not be true for $M_\vee$-matrices.

Theorem 4.6. *Let $A = sI - B$ be an $n \times n$ matrix, where B is a nonnilpotent, eventually nonnegative matrix with $\rho = \rho(B)$. If A is nonsingular, then the following statements are equivalent:*

(i) *A is an invertible $M_\vee$-matrix.*

(ii) *A is positive stable.*

(iii) *$s > \rho$.*

(iv) *All the real eigenvalues of A are positive.*

(v) *The sum of the all principal minors of size k are positive, for all $k \in \langle n \rangle$.*

(vi) *$A + tI$ is nonsingular for all $t > 0$.*

(vii) *$A + I$ is nonsingular and $G = (A + I)^{-1}(A - I)$ is convergent, i.e., $\rho(G) < 1$.*

Proof. It is obvious from the definition of an $M_\vee$-matrix, that the conditions (i) – (iii) are equivalent.

(iii) $\Rightarrow$ (iv): For any real eigenvalue λ of A, $s - \lambda$ is a real eigenvalue of B and thus $s - \lambda \leqslant \rho$. Since $s > \rho$, so $\lambda > 0$.

(iv) $\Rightarrow$ (iii): As ρ is an eigenvalue of B, so $s - \rho$ is a real eigenvalue of A and hence by our assumption $s > \rho$.

(ii) $\Rightarrow$ (v) is true, [see, Problem no. 2 of Chapter 2.1, pg 94 in [27]].

(v) $\Rightarrow$ (vi): The expansion of $\det(A + tI)$ as a polynomial in t is a monic polynomial of degree n whose coefficients are $E_k(A)$, for $k \in \langle n \rangle$, where $E_k(A)$ is the sum of all $k \times k$ minors of A. Since the $E_k(A)$'s are positive for all $k \in \langle n \rangle$, $\det(A + tI)$ is nonsingular for all $t > 0$.

(vi) $\Rightarrow$ (iv): If λ is any real eigenvalue of A and $\lambda < 0$, then by condition (vi), $A - \lambda I$ is nonsingular, which is a contradiction. Hence $\lambda > 0$.

(vi) $\Leftrightarrow$ (vii): $A + I$ is nonsingular by assumption. If μ is any eigenvalue of $G = (A + I)^{-1}(A - I)$, then there is an $x \neq 0$ such that $Gx = \mu x$, i.e., $(A - I)x = \mu(A + I)x$, which implies $\mu \neq 1$ and $Ax = \left(\frac{1+\mu}{1-\mu}\right) x$. Then by condition (ii), $Re\left(\frac{1+\mu}{1-\mu}\right) > 0$, i.e., $Re\left((1 + \mu)(1 - \bar{\mu})\right) > 0$. Thus, $|\mu| < 1$ and hence $\rho(G) < 1$.

Conversely, if λ is any real eigenvalue of A, then by condition (vii), $\lambda \neq -1$. So there is an $x \neq 0$ such that $Gx = \left(\frac{\lambda-1}{\lambda+1}\right) x$ and thus by assumption $|\lambda - 1| < |\lambda + 1|$ which implies that $\lambda > 0$. $\square$

The following examples show that conditions (viii) and (ix) of Theorem 4.4 do not in general hold good for $M_\vee$-matrices.

Example 4.5. Consider the $M_\vee$-matrix $A = 12.5I - B$ where

$$B = \begin{bmatrix} 9.5 & 1 & 1.5 \\ -14.5 & 16 & 10.5 \\ 10.5 & -3 & 4.5 \end{bmatrix}.$$

Consider the nonnegative diagonal matrix $D = \text{diag}(2, 0, \frac{35}{8})$. Then it can be seen that the matrix $A + D$ is singular.

Example 4.6. Consider the $M_\vee$-matrix $A = 0.75I - B$ where

$$B = \begin{bmatrix} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

For any positive vector $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ we have that,

$$Ax = \begin{bmatrix} 1.75x_1 + x_2 + x_3 \\ -x_1 - 0.25x_2 - x_3 \\ 0.75x_3 \\ 0.25x_4 \end{bmatrix}.$$

As $x > 0$, so the second entry of Ax is always negative. Thus Ax can not be a positive vector.

4.2.2 Singular $M_\vee$-matrices

In this section we study a subclass of $M_\vee$-matrices. We try to extend the notion of inverse-eventually positivity to this subclass and show that any matrix in this class will have a generalized inverse which is eventually positive. Various types of generalized inverses have been defined and studied by several authors. The important class of generalized inverses for our purpose are those that leave the subspace $V_A = \text{range}(A^m)$ invariant. The following definitions are due to [35].

Definition 4.3. *Let* $A \in \mathbb{R}^{n,n}$ *with* $m = \text{index}(A)$. *Then each* $Y \in \mathbb{R}^{n,n}$ *satisfying the condition,*

$$YAx = x \text{ for all } x \in V_A \text{ with } V_A = \bigcap_{k=0}^{\infty} \text{range}(A^k) = \text{range}(A^m)$$

is called a generalized left inverse of *A*. *Similarly, each* $Z \in \mathbb{R}^{n,n}$ *satisfying the condition,*

$$x^T AZ = x^T \text{ for all } x \in V_A$$

is called a generalized right inverse of *A*.

Some equivalent definitions of generalized left inverses are given in the following lemma.

Lemma 4.2. *[2, 35] Let* $A \in \mathbb{R}^{n,n}$. *Then the following statements are equivalent for* $Y \in \mathbb{R}^{n,n}$:

(i) *Y is a generalized left inverse of A.*

(ii) $YA^{m+1} = A^m$, *where* $m = \text{index}(A)$.

(iii) $YA^{k+1} = A^k$, *where* $k \geqslant \text{index}(A)$.

(iv) $YA^{k+1} = A^k$, *for some* $k \geqslant 0$.

Similar characterization can obviously be obtained for generalized right inverses.

Definition 4.4. *Let* $A, Y \in \mathbb{R}^{n,n}$. *Consider the following conditions:*

(1) $AYA = A$.

(2) $YAY = Y$.

(3) $AY = (AY)^T$.

(4) $YA = (YA)^T$.

(5) $AY = YA$.

(6) $YA^{m+1} = A^m, \qquad m = \text{index}(A)$.

Let λ be any subset of $\{1,2,3,4\}$, which contains 1*. Then a* λ-inverse *of A is a matrix Y which satisfies the condition* i*, for each $i \in \lambda$.*

*The*Drazin inverse *of A is a matrix Y which satisfies the conditions* $(2), (5)$ *and* (6)*, hence it is a generalized left inverse.*

In our next theorem we extend Theorem 4.3 for a subclass of singular $M_\vee$-matrices.

Theorem 4.7. *If $A = \rho I - B$ is a singular $M_\vee$-matrix where $\rho(B) = \rho$, B is an irreducible, nonnilpotent, eventually nonnegative matrix with* $\text{index}(B) \leqslant 1$*, then there always exists an eventually positive generalized left inverse of A.*

Proof. Let $A = XJX^{-1}$ be the Jordan canonical form of A. Then there exists positive vectors x and y such that $Ax = 0$ and $y^T A = 0$. Without loss of generality we may assume that $X = \begin{bmatrix} x & X^{(1)} \end{bmatrix}$, then

$$A = \begin{bmatrix} x & X^{(1)} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} y^T \\ Y^{(1)} \end{bmatrix},$$

where $D \in \mathbb{R}^{n-1,n-1}$ is the nonsingular part of the Jordan canonical form J of A.
Choose a large positive number s such that $s > \frac{1}{|\lambda|}$ for all $\lambda(\neq 0) \in \sigma(A)$. Consider the matrix

$$Y_1 = \begin{bmatrix} s & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

Take $Y = XY_1X^{-1}$, so for any positive integer k, $Y^k = XY_1^kX^{-1}$. Then

$$\frac{1}{s^k}Y^k = \begin{bmatrix} x & X^{(1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{D}(k) \end{bmatrix} \begin{bmatrix} y^T \\ Y^{(1)} \end{bmatrix},$$

where $\tilde{D}(k) = \frac{1}{s^k}\left(D^{-1}\right)^k$ and any eigenvalue $\lambda(k)$ of $\tilde{D(k)}$ is of absolute value less than 1. Hence it follows that, $\lim_{k\to\infty} \tilde{D(k)} = 0$ and,

$$\lim_{k\to\infty} \frac{1}{s^k}Y^k = xy^T > 0.$$

This shows that there exists a positive integer k_0 such that $Y^k > 0$ for all $k \geqslant k_0$, that is, Y is an eventually positive matrix. We now show that Y is a generalized left inverse of A. Let $m = \text{index}(A)$. Then,

$$Y_1 J^{m+1} = \begin{bmatrix} s & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & D^{m+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D^m \end{bmatrix} = J^m.$$

Thus $XY_1X^{-1}XJ^{m+1}X^{-1} = XJ^mX^{-1}$, or, $YA^{m+1} = A^m$. □

Remark 4.2. Note that from the above theorem we cannot conclude that every generalized left inverse of A is eventually positive. Consider the matrix

$$A = 8I - B = 8I - \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix}.$$

Then $B^k > 0$ for all $k \geqslant 3$ and so A is an $M_\vee$-matrix satisfying the conditions of the previous theorem. Let $A = XJX^{-1}$ be the Jordan canonical form of A where

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0.25 & 2 & 0.75 \\ 0.25 & -2 & -0.25 \\ 0.25 & 2 & -0.25 \end{bmatrix}.$$

Consider the generalized left (Drazin) inverse,

$$\begin{aligned} Y &= X\tilde{J}X^{-1} = X \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.25 & -0.0625 \\ 0 & 0 & 0.25 \end{bmatrix} X^{-1} \\ &= \begin{bmatrix} 0.0625 & -0.125 & 0.0625 \\ 0.0625 & 0.125 & -0.1875 \\ -0.1875 & -0.125 & 0.3125 \end{bmatrix}. \end{aligned}$$

Then for any positive integer k, $Y^k = X\tilde{J}^kX^{-1}$. Using induction, it can be easily checked that

$$\tilde{J}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4^k} & -\frac{k}{4^{k+1}} \\ 0 & 0 & \frac{1}{4^k} \end{bmatrix},$$

which implies

$$\begin{aligned} Y^k &= X\tilde{J}^kX^{-1} \\ &= \begin{bmatrix} \frac{1}{4} & 2 & \frac{3}{4} \\ \frac{1}{4} & -2 & -\frac{1}{4} \\ \frac{1}{4} & 2 & -\frac{1}{4} \end{bmatrix} \tilde{J}^kX^{-1} \\ &= \begin{bmatrix} 0 & \frac{2}{4^k} & \frac{3-2k}{4^{k+1}} \\ 0 & -\frac{2}{4^k} & \frac{2k-1}{4^{k+1}} \\ 0 & \frac{2}{4^k} & -\frac{2k+1}{4^{k+1}} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & -1 \end{bmatrix}. \end{aligned}$$

This shows that for any positive integer k, the $(1,2)th$ entry of Y^k is always negative. Hence Y is not an eventually positive matrix.

The following theorem gives some characterizations of singular M-matrices in terms of generalized left inverses, obtained by Neumann and Plemmons in [35].

Theorem 4.8. *[35] Let $A = sI - B$ where $B \geqslant 0$ and $s > 0$. Then the following statements are equivalent:*

(i) *A is an M-matrix.*

(ii) *A has a generalized left inverse Y with $Y \geqslant 0$.*

(iii) *A has a generalized left inverse Y, which is nonnegative on V_A. That is,*

$$x \geqslant 0 \text{ and } x \in V_A \Rightarrow Yx \geqslant 0.$$

(iv) *A is monotone on V_A. That is,*

$$Ax \geqslant 0 \text{ and } x \in V_A \Rightarrow x \geqslant 0.$$

Motivated by these characterizations of singular M-matrices, we now introduce some new definitions and concepts, in order to give some interesting characterizations of a subclass of singular $M_\vee$-matrices.

Definition 4.5. *Let $A \in \mathbb{R}^{n,n}$ and $S \subseteq \mathbb{R}^n$. Then we say that A is* eventually nonnegative on S, *if $x \in S$ and $x \geqslant 0$ imply that there exists a positive integer k_0, such that $A^k x \geqslant 0$, for all $k \geqslant k_0$.*

Remark 4.3. Note that if A is an eventually nonnegative matrix such that $A^k \geqslant 0$ for all $k \geqslant g$, then we can choose $k_0 = g$.

Definition 4.6. *Let $A \in \mathbb{R}^{n,n}$ and $S \subseteq \mathbb{R}^n$. Then we say that A is* eventually monotone on S, *if there exists a positive integer k_0, such that for any $x \in S$, $A^k x \geqslant 0$, for all $k \geqslant k_0$, implies $x \geqslant 0$.*

The following is an example of a matrix which is eventually monotone on a subspace S of $\mathbb{R}^4$.

Example 4.7. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Take $S = \mathbb{R}^2$. Let $x \in S$ and there exists k_0 such that $A^k x \geqslant 0$ for all $k \geqslant k_0$ which imply that

$$\begin{bmatrix} 1 & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geqslant 0 \text{ for all } k \geqslant k_0,$$

which implies that $x_2 = 0$ and $x_1 \geqslant 0$. Hence $x \geqslant 0$ on S and thus the matrix satisfies eventually monotone property on S.

Next theorem gives some equivalent characterizations for a subclass of $M_\vee$-matrices.

Theorem 4.9. *Let $A = sI - B$ where $s > 0$ and B is an eventually nonnegative matrix satisfying the conditions of Theorem 4.7. Then the following statements are equivalent:*

(i) *A is an $M_\vee$-matrix.*

(ii) *A has a generalized left inverse Y, which is eventually positive.*

(iii) *A has a generalized left inverse Y, which is eventually nonnegative.*

(iv) *A has a generalized left inverse Y, which is eventually nonnegative on V_A.*

(v) *A is eventually monotone on V_A.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 4.7 and (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Leftrightarrow$ (v): Let Y be a generalized left inverse of A such that if $x \in V_A$ and $x \geqslant 0$, then there exists a $\tilde{k}$ such that $Y^k x \geqslant 0$ for all $k \geqslant \tilde{k}$. To show that A is eventually monotone on V_A.

Let $x \in V_A$ and k_1 be a positive integer such that $A^k x \geqslant 0$ for all $k \geqslant k_1$ and let $m = \text{index}(A)$. Choose k_2 such that $k_2 \geqslant \max\{k_1, m\}$. Since $x \in V_A = R(A^m)$, there exists a z such that $x = A^m z$. Thus for any $k \geqslant k_2$,

$A^k x = A^{m+k} z \in V_A$ and $A^k x \geqslant 0$. Again by assumption (iv), there exists a k_3 such that $Y^s A^k x \geqslant 0$ for all $s \geqslant k_3$ and for all $k \geqslant k_2$. Choose $k_0 \geqslant \max\{k_2, k_3\}$. Then for any $k \geqslant k_0$, $Y^k A^k x \geqslant 0$, i.e., $Y^k A^{m+k} z \geqslant 0$. Thus $YA^{k+1} = A^k$ for all $k \geqslant m$ implies that $YA^{m+1} z \geqslant 0$, or, $x = A^m z = YA^{m+1} z \geqslant 0$.

Conversely let A be eventually monotone on V_A, i.e., if $x \in V_A$ and there exists a k_0 such that $A^k x \geqslant 0$, for all $k \geqslant k_0$, then $x \geqslant 0$. Let Y be the Drazin inverse of A. To show that Y is eventually nonnegative on V_A.

Let $x \in V_A$, $x \geqslant 0$ and $x = A^m z$ for some z. For any $k \geqslant m$ we have that $A^k(Y^k x) = Y^k(A^k x) = Y^k A^{m+k} z = YA^{m+1} z = A^m z \geqslant 0$. Hence by our assumption $Y^k x \geqslant 0$ for all $k \geqslant m$. Thus (iv) holds.

(iv) $\Rightarrow$ (i): Let Y be a generalized left inverse of A which is eventually nonnegative on V_A. To show that $s \geqslant \rho$, where $\rho = \rho(B)$.

Suppose that $s \neq \rho$. Choose a nonnegative vector x such that $Bx = \rho x$ and thus for any positive integer k, $A^k x = (s - \rho)^k x$ and hence $x \in V_A$. So, by (iv) there exists a k_0 such that $Y^k x \geqslant 0$ for all $k \geqslant k_0$. Take $\tilde{k} = \max(k_0, m)$, where $m = \text{index}(A)$. Then for any $k \geqslant \tilde{k}$,

$$\begin{aligned} (s-\rho)^{m+k} x &= A^{m+k} x \\ &= YA^{m+k+1} x \\ &= Y^k A^{m+2k} x \\ &= (s-\rho)^{m+2k} Y^k x \end{aligned}.$$

Thus $(s-\rho)^k x = (s-\rho)^{2k} Y^k x$ for all $k \geqslant \tilde{k}$. As x and $Y^k x$ with $k \geqslant \tilde{k}$ are all nonnegative vectors, so we must have $s > \rho$. □

Lemma 4.3. *Let A be any real square matrix of order n. If Y is a $\{1\}$-inverse of A with* $\text{range}(YA) = \text{range}(A)$, *then*

(i) $YAx = x$, *for all* $x \in \text{range}(A)$.

(ii) $YA^{k+1} = A^k$, *for all* $k \geqslant 1$. *In particular,* $\text{index}(A) \leqslant 1$ *and Y is a generalized left inverse of A.*

Proof. (i) Since range(YA) = range(A), so any $x \in \text{range}(A)$ can be written as $x = YAz$, for some z and hence $YAx = YAYAz = YAz = x$.

(ii) We prove it by induction on k. Let $k = 1$ and $x \in \mathbb{R}^n$. Then $Ax \in \text{range}(A)$ and hence by the given hypothesis there exists a z such that

$Ax = YAz$, which implies that $YA^2x = YAYAz = YAz = Ax$. Thus $YA^2 = A$. Now suppose that $k > 1$ and $YA^{t+1} = A^t$, for all $t < k$. Then $YA^{k+1} = YA^k \cdot A = A^{k-1} \cdot A = A^k$.

Suppose that, $m = \text{index}(A) > 1$. Then there exists an x, such that $x \in \text{range}(A)$ and $x \notin \text{range}(A^2)$. Hence $x = Ay$ for some $y \in \mathbb{R}^n$. Take $y = u + v$ with $u \in \text{range}(A^m)$ and $A^m v = 0$. So, $A^m y = A^m u$, or, $YA^m y = YA^m u$ and since $m > 1$, so $A^{m-1}y = A^{m-1}u$. Repeating this process upto $(m-1)$ steps, we get $x = Ay = Au$, hence $x \in \text{range}(A^{m+1})$ and $m > 1$ imply that $x \in \text{range}(A^2)$, a contradiction. Thus $\text{index}(A) \leqslant 1$.

□

Lemma 4.4. *Let A be any real square matrix of order n. If Z is a $\{1\}$-inverse of A with $\text{range}(Z^T A^T) = \text{range}(A^T)$, then*

(i) $x^T AZ = x^T$, *for all* $x \in \text{range}(A)$.

(ii) $A^{k+1}Z = A^k$, *for all* $k \geqslant 1$. *In particular,* $\text{index}(A) \leqslant 1$ *and Z is a generalized right inverse of A.*

Proof. Proof is similar to that of Lemma 4.3. □

The following result gives characterizations of M-matrices in terms of some special types of generalized inverses.

Theorem 4.10. *[35] Let $A = sI - B$ where $B \geqslant 0$, $s > 0$ and $\rho(B) = \rho$. Then for $S = \text{range}(A)$, the following statements are equivalent:*

(i) *A is an M-matrix with* $\text{index}(A) \leqslant 1$.

(ii) *A has a $\{1\}$-inverse Y which is a nonnegative matrix and* $\text{range}(YA) = S$.

(iii) *A has a $\{1\}$-inverse Y with* $\text{range}(YA) = S$, *such that Y is nonnegative on S.*

(iv) *A has a $\{1, 2\}$-inverse Z with* $\text{range}(Z) = S$, *such that Z is nonnegative on S.*

(v) *A is monotone on S.*

We conclude this chapter by giving similar characterizations in terms of generalized inverses of a singular $M_\vee$-matrix, $A = \rho I - B$, where B is eventually positive, $\text{index}(A) \leqslant 1$ and $\rho(B) = \rho$.

Theorem 4.11. *Let $A = sI - B$ where B is an eventually positive matrix, $s > 0$ and $\rho(B) = \rho$. Then for $S = \text{range}(A)$, the following statements are equivalent:*

(i) *A is an $M_\vee$-matrix with* $\text{index}(A) \leqslant 1$.

(ii) *A has a $\{1\}$-inverse Y which is an eventually nonnegative matrix and* $\text{range}(YA) = S$.

(iii) *A has a $\{1\}$-inverse Y with* $\text{range}(YA) = S$, *such that Y is eventually nonnegative on S.*

(iv) *A has a $\{1, 2\}$-inverse Z with* $\text{range}(Z) = S$, *such that Z is eventually nonnegative on S.*

(v) *A is eventually monotone on S.*

Proof. We first show that if A is a matrix with $\text{index}(A) \leqslant 1$, then Y is a generalized left inverse if and only if Y is a $\{1\}$-inverse with range(YA) = range(A).

If $\text{index}(A) < 1$, that is, A is nonsingular, Then the result is obviously true, hence assume that $\text{index}(A) = 1$. If The 'if' part follows from Lemma 4.3. Now for the 'only if part', let us assume that Y is a generalized left inverse of A. Any $x \in \mathbb{R}^n$ can be written as $x = u + v$ with $u \in \text{range}(A)$ and $Av = 0$. Then $Ax = Au$. Since Y is a left inverse and $\text{index}(A) = 1$, so we have $YAu = u$ and $AYAx = AYAu = Au = Ax$ and hence $AYA = A$.

Next if $x \in \text{range}(YA)$, then as in the earlier case, x can be written as $x = YAy$ for some $y \in \text{range}(A)$. Since Y is a left inverse, $x = y \in \text{range}(A)$. Conversely if $x \in \text{range}(A)$, then $x = YAx$ and so $x \in \text{range}(YA)$. Hence $\text{range}(YA) = \text{range}(A)$.

From Theorem 4.9 it follows that if index$(A) \leqslant 1$, then the conditions (ii), (iii), (v) are equivalent to the statement that "A is an $M_\vee$-matrix". Thus we have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v). To complete the proof it is enough to show (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (iv): Let Y be a $\{1\}$-inverse of A such that range(YA) = range(A) and Y is eventually nonnegative on range(A). Take $Z = YAY$. Then it can be easily checked that Z is a $\{1,2\}$-inverse of A. Since $Z = YAY$ and range(YA) = range(A), so range$(Z) \subseteq$ range(A). Again if $x \in$ range(A), then by Lemma 4.3(i), $x = YAx = ZAx$ and hence range(Z) = range(A). In order to show that Z is eventually nonnegative on range(A), it suffices to show that $Z^k x = Y^k x$ for all $x \in$ range(A), and for all positive integer k.
Let $x = Au$ for some $u \in \mathbb{R}^n$, then $Zx = YAYx = YAYAu = YAu = Yx$. Now assume that $k > 1$, and $Z^t x = Y^t x$, for all $x \in$ range(A) and for all $t < k$. Then $Z^k x = Z^{k-1}(Zx) = Z^{k-1}(Yx) = Y^{k-1}(Yx) = Y^k x$ and by induction on k, $Z^k x = Y^k x$ for all positive integer k.

(iv) $\Rightarrow$ (i): Suppose that Z is a $\{1,2\}$-inverse of A such that range(Z) = range(A) and Z is eventually nonnegative on $R(A)$. Then Z is a $\{1\}$-inverse implies that range(ZA) = range(A) and hence by Lemma 4.3, index$(A) \leqslant 1$ and Z is a generalized left inverse of A. Hence the generalized left inverse Z is eventually nonnegative on $V_A = R(A)$ and (i) follows from Theorem 4.9.
Thus the conditions (i) $-$ (v) are equivalent. □

Remark 4.4. Similar results can be obtained for generalized right inverses Z, with S = range(A^T) and range(YA) replaced by range$(Z^T A^T)$ in the above statements.

4.3 Conclusion

We present some useful characterizations of nonsingular $M_\vee$-matrices in terms of stability and positivity of sums of principal minors. We also obtain some results for the class of $M_\vee$-matrices analogous to those obtained for M-matrices, for example the inverse-positivity property of nonsingular M-matrices. Next we study a subclass of singular $M_\vee$-matrices, and obtain interesting results on the generalized inverses of these matrices. Lastly, we present some characterizations of this subclass, in terms of generalized inverses.

Chapter 5

Conclusion and Future works

In this dissertation we study the combinatorial properties of nonnegative matrices and generalized M-matrices. To begin with, we consider several types of nonnegative bases which have interesting combinatorial structures, namely the quasi-preferred and preferred bases. We use these bases to introduce a variant of the Jordan canonical form for nonnegative matrices and proved the uniqueness of such a canonical form up to block triangular similarity transformation. We also study some combinatorial properties of nonnegative matrices with the help of this canonical form. Moreover, we consider permuted graph representations of nonnegative bases for nonnegative matrices and derived some necessary conditions for the existence of such bases. Nonnegative permuted graph basis is a very convenient tool for the initialization of nonnegative dynamical systems, but not every nonnegative matrix possesses a nonnegative permuted graph basis. It is an open problem to characterize the class of nonnegative matrices that have a nonnegative permuted graph basis.

Next we consider two types of generalizations of M-matrices based on generalizations of nonnegative matrices, called GM-matrices and $M_\vee$-matrices and try to extend the combinatorial properties of singular M-matrices to the class of GM-matrices and $M_\vee$-matrices. We prove the existence of a preferred basis for a subclass of $M_\vee$-matrices and obtain similar equivalent conditions as those obtained for singular M-matrices, for the equality of the height and level characteristics for this class of matrices. It is interesting to study combinatorial properties for the entire class of $M_\vee$-matrices. We also try to obtain similar results for the class of GM-matrices and show the existence of a preferred basis for this class, if the order of the matrix does not exceed 2. We give a counterexample to show that we cannot conclude the same if the order of the matrix exceeds 2.

Motivated by results obtained for the characterization of nonsingular M-matrices we try to characterize the class of nonsingular $M_\vee$-matrices in terms of stability and positivity of sums of principal minors. In addition to this, we prove that for a subclass of nonsingular $M_\vee$-matrices the inverse of these matrices has eventually positivity property, analogous to the inverse-positivity property for nonsingular M-matrices. We also demonstrate, with the help of examples, that the above property is not carried over to the entire class of $M_\vee$-matrices. We also extend the notion of eventually nonnegativity property of the inverse, for a subclass of nonsingular $M_\vee$-matrices to the eventually nonnegativity property of generalized left and right inverses of a subclass of singular $M_\vee$-matrices. We could not, however, establish this property for the entire class of singular $M_\vee$-matrices. We also find some interesting characterizations of a subclass of $M_\vee$-matrices in terms of some other types of generalized inverses. It would be interesting to study all these properties for the entire class of $M_\vee$-matrices.

List of papers communicated/to be communicated

Based on the work in this thesis, the following research articles are communicated/to be communicated.

1. M. Saha, V. Mehrmann, *The Frobenius-Jordan form of nonnegative matrices*, communicated.
2. M. Saha, S. Bandopadhyay, *Combinatorial structure of generalized M-matrices*, communicated.
3. M. Saha, S. Bandopadhyay, *Characterization of Generalized M-matrices*, under preparation.

References

[1] A. Berman, M. Neumann, and R. J. Stern. *Nonnegative Matrices in Dynamic Systems*. Wiley-Interscience, New York, 1989.

[2] A. Berman and R. Plemmons. *Nonnegative matrices in the Mathematical Sciences*. SIAM, 1994.

[3] R. Bru and M. Neumann. Nonnegative Jordan bases. *Lin. Multilin. Algebra*, 23:95–109, 1988.

[4] R. A. Brualdi. *Introductory Combinatorics*. North-Holland, 1977.

[5] D. Carlson. A note on M-matrix equations. *J. Soc. nd. Appl. Math.*, 11:1027–1033, 1963.

[6] G. Debreu and I. N. Herstein. Nonnegative square matrices. *Econometrica*, 21:597–607, 1953.

[7] A. Elhashash and D. B. Szyld. Generalizations of M-matrices which may not have a nonnegative inverse. *Lin. Algebra Appl.*, 429:2435–2450, 2008.

[8] A. Elhashash and D. B. Szyld. On general matrices having the Perron-Frobenius property. *ELA*, 17:389–413, 2008.

[9] K. Fan. Topological proofs for certain theorems on matrices with nonnegative elements. *Monatsh Math.*, 62:219–237, 1958.

[10] L. Farina and S. Ronaldi. *Positive Linear Systems: Theory and its Applications*. John Wiley and Sons Inc., New York, 2000.

[11] B. Fernandez and N. Muriel. Regular variation and related results for the multivariate $GARCH(p, q)$ model with constant conditional correlations. *J. of Multivariate Analysis*, 100:1538–1550, 2009.

[12] M. Fiedler and V. Ptak. On matrices with nonpositive off-diagonal elements and positive principal minors. *Czech. Math. J.*, 12:382–400, 1962.

[13] S. Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics*, 29:43–60, 1978.

[14] G. F. Frobenius. Über Matrizen aus positiven Elementen,. *Sitzungsber. Dsch. Akad. Wiss. Berl.*, pages 471–476, 1908.

[15] G. F. Frobenius. Über Matrizen aus positiven Elementen, II. *Sitzungsber. Dsch. Akad. Wiss. Berl.*, pages 514–518, 1909.

[16] G. F. Frobenius. Über Matrizen aus nicht negativen Elementen,. *Sitzungsber. Kön. Preuss. Akad. Wiss.*, pages 456–477, 1912.

[17] F. R. Gantmacher. *The Theory of Matrices, Volume I.* Chelsea Publishing company, 1959.

[18] C.-H. Guo. A note on the minimal nonnegative solution of a nonsymmetric algebraic Riccati equation. *Linear Algebra Appl.*, 357:299–302, 2002.

[19] X.-X. Guo, W.-W. Lin, and S.-F. Xu. A structure-preserving doubling algorithm for nonsymmetric algebraic Riccati equation. *Numer. Math.*, 103:393–412, 2006.

[20] D. Handelman. Positive Matrices and Dimension Groups affiliated to C^*-algebras and topological Markov Chains. *J. Operator Theory*, 6:55–74, 1981.

[21] D. Hershkowitz. The combinatorial structure of generalized eigenspaces-from nonnegative matrices to general matrices. *Lin. Algebra Appl.*, 302-303:173–191, 1999.

[22] D. Hershkowitz, U. Rothblum, and H. Schneider. The combinatorial structure of the generalized nullspace of a block triangular matrix. *Lin. Algebra Appl.*, 116:9–26, 1989.

[23] D. Hershkowitz and H. Schneider. On the generalized nullspace of M-matrices and Z-matrices. *Lin. Algebra Appl.*, 106:5–23, 1988.

[24] D. Hershkowitz and H. Schneider. Height bases, level bases and the equality of the height and level characteristics of an M-matrix. *Lin. Multilin. Algebra*, 25:149–171, 1989.

[25] D. Hershkowitz and H. Schneider. Combinatorial bases, derived Jordan sets and the equality of the height and level characteristics of an M-matrix. *Lin. Multilin. Algebra*, 29:21–42, 1991.

[26] R. A. Horn and C. Johnson. *Matrix Analysis.* Cambridge University Press, New York, 1985.

[27] R. A. Horn and C. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, New York, 1991.

[28] S. K. Jain. Nonnegative rectangular matrices having certain nonnegative W-weighted group inverses. *Proc. Amer. Math. Soc.*, 85(1):1–9, 1982.

[29] C. R. Johnson and P. Tarazaga. On matrices with Perron-Frobenius Properties and some nonnegative entries. *Positvity*, 8:327–338, 2004.

[30] J. J. McDonald, M. Neumann, and H. Schneider. Resolvents of minus M-matrices and splittings of M-matrices. *Lin. Algebra Appl.*, 195:17–33, 1993.

[31] V. Mehrmann and F. Poloni. Doubling Algorithms With Permuted Lagrangian Graph Bases. *SIAM J. Matrix. Anal. Appl.*, 30:1339–1357, 2012.

[32] V. Mehrmann and X. Xu. Explicit Solutions for a Riccati Equation from Transport Theory. *SIAM J. Matrix. Anal. Appl.*, 2008.

[33] S. C. Naqvi and J. J. Mcdonald. The combinatorial structure of eventually nonnegative matrices. *ELA*, 9:255–269, 2002.

[34] S. C. Naqvi and J. J. McDonald. Eventually nonnegative matrices are similar to seminonnegative matrices. *Lin. Algebra Appl.*, 381:245–258, 2004.

[35] M. Neumann and R. J. Plemmons. Generalized Inverse-positivity and splitting of M-matrice. *Lin. Algebra Appl.*, 17:208–234, 1978.

[36] M. Neumann and H. Schneider. Principal components of M-matrices. *Lin. Multilin. Algebra*, 32:131–148, 1992.

[37] M. Neumann and H. Schneider. Corrections and Additions to "Principal Components of Minus M-matrices. *Lin. Multilin. Algebra*, 36:147–149, 1993.

[38] M. Neumann and H. Schneider. Algorithms for computing bases for the Perron eigenspace with prescribed nonnegativity and combinatorial properties. *SIAM J. Matrix. Anal. Appl.*, 15:578–591, 1994.

[39] D. Noutsos. On Perron-Frobenius property of matrices having some negative entries. *Lin. Algebra Appl.*, 412:132–153, 2006.

[40] D. Noutsos and M. J. Tsatsomeros. Reachability and Holdability of nonnegative states. *Siam J. Matrix Anal. Appl.*, 30(2):700–712, 2008.

[41] D. D. Olesky, M. J. Tsatsomeros, and P. Van Den Driessche. $M_{\vee}$-matrices: a generalization of M-matrices based on eventually nonnegative matrices. *ELA*, 18:339–351, 2009.

[42] A. M. Ostrowski. On the eigenvector belonging to the maximal root of a nonnegative matrix. *Proc. Edinburgh Math. Soc.*, 12:107–112, 1960/61.

[43] A. M. Ostrowski. On positive matrices. *Math. Annalen*, 150:276–284, 1963.

[44] D. J. Richman and H. Schneider. On the singular graph and the Weyr characteristics of an M-matrix. *Aequationes Math.*, 17:208–234, 1978.

[45] U. Rothblum. Algebraic eigenspace of nonnegative matrices. *Lin. Algebra Appl.*, 12:281–292, 1975.

[46] H. Schneider. *Matrices with non-negative elements*. PhD thesis, Edinburgh University, 1952.

[47] H. Schneider. The elementary divisors, associated with 0, of nonsingular M-matrices. *Proc. Edinburgh Math. Soc.*, 10:108–122, 1956.

[48] H. Schneider. The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties : A survey. *Lin. Algebra Appl.*, 23:21–35, 1979.

[49] P. Tarazaga, M. Raydan, and A. Hurman. Perron-Frobenius theorem for matrices with some negative entries. *Lin. Algebra Appl.*, 328:57–68, 2001.

[50] B. G. Zaslavsky. Controllability of quasi monotone systems in a positive cone. *Auto. Remote Control*, 48:292299, 1987.

[51] B. G. Zaslavsky. Eventually nonnegative realization of difference control systems . *Dynamical Systems and Related Topics, Adv. Ser. Dynam. Systems*, 9:573602, 1991.

[52] B. G. Zaslavsky. Nullcontrollability ofd iscretetime positive control systems. *Int. J. Control*, 58:65–78, 1993.

[53] B. G Zaslavsky and J. J. McDonald. A characterization of Jordan canonical forms which are similar to eventually nonnegative matrices. *Lin. Algebra Appl.*, 372:253–285, 2003.

[54] B. G. Zaslavsky and B. S. Tam. On the Jordan form of an irreducible matrix with eventually nonnegative powers. *Lin. Algebra Appl.*, 302/303:303330, 1999.