On the bipartite distance matrix and the bipartite Laplacian matrix

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November 2021

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A Thesis Submitted
in Fulfilment of the Requirements
for the Degree of

Doctor of Philosophy

by

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November 2021

DECLARATION

I do hereby declare that the work contained in this thesis entitled **"On the bipartite distance matrix and the bipartite Laplacian matrix"** has done by me, under the supervision of **Dr. Sukanta Pati**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

Guwahati
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CERTIFICATE

This is to certify that the thesis entitled **"On the bipartite distance matrix and the bipartite Laplacian matrix"** submitted by **Rakesh Jana**, a student of Department of Mathematics, Indian Institute of Technology Guwahati to the Indian Institute of Technology Guwahati, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Guwahati
November 2021

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Dedicated to

My Father Rakhal Ch. Jana

and

My Mother Chandana Jana

Acknowledgments

With gratitude, I acknowledge the efforts of well-wishers who have supported in their special ways to accomplish and complete this dissertation.

It is a genuine pleasure to express my deep sense of thanks and gratitude to my mentor, philosopher and guide, Prof. Sukanta Pati. His Socratic questioning, constructive criticism and incredible experience have helped me to enrich my growth not only as a researcher in Mathematics but also as a good human being. I am fortunate to have him as my supervisor, who truly cared about my work, was eager to listen to my queries and concerns, and kept insisting on top quality work in all of my endeavours. He has always been there for me in all my academic as well as non-academic circumstances. I cherish every moment I spent with him in the last few years, and I will carry these beautiful memories for the rest of my life.

I want to convey my sincere thanks to the doctoral committee members Prof. Bhaba Kumar Sarma, Dr. Bikash Bhattacharjya and Dr. Sriparna Bandopadhyay for reviewing my research work periodically and giving valuable suggestions for the improvements of the same. I want to express my warm and sincere thanks to Prof. Kalpesh Kapoor, Head of the Department of Mathematics and Prof. M. Guru Prem Prasad, Prof. N. Selvaraju, Prof. Swaroop Nandan Bora, former Heads of the Department of Mathematics, for providing excellent research facilities during my PhD programme. I express my humble gratitude to the teachers at IIT Guwahati for their teachings and various discussions during the MA101 and MA102 courses, which improve my knowledge of the subjects. My hearty thanks to all other faculty members of the Mathematics department.

I sincerely acknowledge the Indian Institute of Technology Guwahati for providing me with various facilities necessary to carry out my research. I am most grateful to the Ministry of Human and Resource Development, Government of India, for providing me financial assistance to complete my thesis work. I express my sincere thanks to all the department staff for helping me out in official and technical related matters.

I express my sincere thanks to Professor R. B. Bapat for their helpful comments and suggestions on my work. I would also like to thank him for allowing me to include these results here.

I express my special and heartiest thanks to Dr. Swarup Kumar Panda for his support, suggestion and helping hands during this period. I owe my thanks to Somnath, Sunil, Sumit,

Rupak, Shantiram, Ayan, Abhijeet, Shyam, Shamik, Ankita and many others with whom I have shared some of the best moments of my life.

My heartiest thanks to Atasi for her valuable friendship, care, encouragement and helpful suggestions. She was fundamental in supporting me during these stressful and challenging moments. She will always share a part of my soul.

Finally, it is impossible for me to adequately express my gratitude towards my parents for their unequivocal support, unwavering love, quiet patience, and, most importantly, allowing me to be as ambitious as I wanted. I am grateful for their unconditional love, which surrounds me and gives me strength at every moment of my life. This accomplishment would not have been possible without them. They have been a source of inspiration throughout my life, and I dedicate this thesis to them. My thanks and love to my younger sister Payel for all her affection and care that I can not forget in my whole life. I also convey my love to Sithun, Ratan da, Manu di, Sourav, Urmi and Tanuruchi for their support in every aspect of my life.

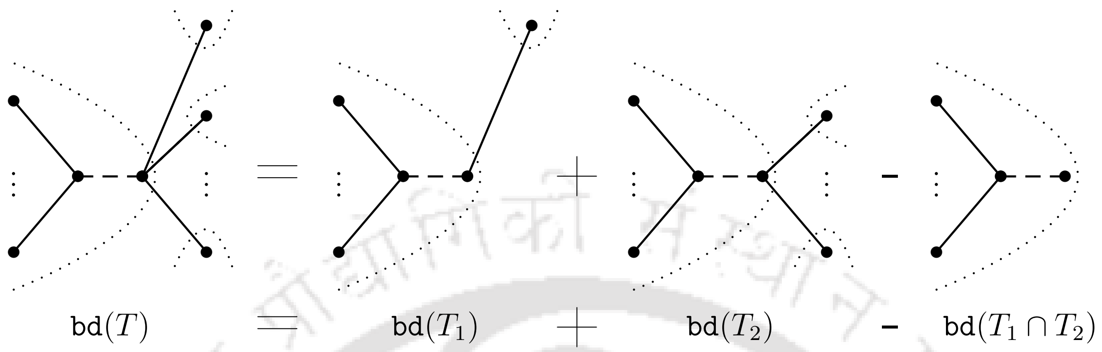
Abstract

The study of the properties of graph via matrices is widely studied subject that ties together two seemingly unrelated branches of mathematics; graph theory and linear algebra. Graham and Pollak in [GP71] proved a remarkable result which tells that the determinant of the distance matrix of a tree only depends on the number of vertices in the tree. They showed that for a tree T on n vertices, the determinant of the distance matrix of T is given by $\det D(T) = (-1)^{n-1}2^{n-2}(n-1)$. This impressive result created a lot of interest among the researchers. In the same paper, Graham and Pollak established a relationship between loop addressing problem in data communication systems and the number of negative eigenvalues of the distance matrix. Since then many generalizations have been proposed in order to understand the distance matrix better. Yet, the understanding seems to be far from complete. We present one such point of view here showing how many more combinatorial objects are linked together

All graphs considered here are simple and finite. For a bipartite graph, we do not require the complete adjacency matrix to display the adjacency information in the graph. Godsil in [God85], used a smaller size matrix to display this information. Later on this matrix was named the bipartite adjacency matrix. The bipartite adjacency matrix [BM08, pp. 7] has appeared in many places in the literature and it is used widely in the study the inverse of the bipartite graph with unique perfect matching. In this thesis, similar to the bipartite adjacency matrix, we define the bipartite distance matrix of a bipartite graph with unique perfect matching. That is, the *bipartite distance matrix* $\mathfrak{B}(G)$ of a bipartite graph G with a unique perfect matching on $2p$ vertices is a $p \times p$ matrix whose (i, j)-th entry is the distance between vertices l_i and r_j, where $L := \{l_1, \ldots, l_p\}$, $R := \{r_1, \ldots, r_p\}$ is a vertex bipartition of G. Although the size of the bipartite distance matrix is half of the size of the graph, but we observe that it still provides much information about the underlying graph. We observe that $\det \mathfrak{B}(G)$ is always a multiple of 2^{p-1}. This is similar to the well known result of Graham and Pollak which tells that the determinant of the usual distance matrix of a tree on n vertices is a multiple of 2^{n-2}. We define the bipartite distance index of G as $\mathtt{bd}(G) := \det \mathfrak{B}(G)/(-2)^{p-1}$.

It is known that if a tree has a perfect matching, then the perfect matching is unique. We use the word 'nonsingular tree' to mean a tree with a (unique) perfect matching. As a tree is a bipartite graph, the study of the properties of the bipartite distance matrix of a nonsingular tree is naturally starting point. We show that the bipartite distance index

of a nonsingular tree T satisfies an interesting inclusion-exclusion type of principle at any matching edge of the tree which is explained below (the shaded line is the matching edge under consideration). This gives us a recursive formula to compute $\mathtt{bd}(T)$.



Even more interestingly, we show that the bipartite distance index of a nonsingular tree T can be completely characterized by the structure of T via what we call the f-alternating sums. By the *f-alternating sum* $f_S(T)$ of a nonsingular tree T, we define the following sum

$$f_S(T) := \sum_{\substack{P \in \mathcal{A}_T \\ P=[u,\cdots,v]}} [d(u)-2][d(v)-2]S\left(\frac{|P|}{2}\right),$$

where $S : \mathbb{N} \to \mathbb{R}$ is a sequence, $\mathcal{A}_T$ is the set of all alternating paths (a path which has the starting edge, each alternate edge thereafter and the last edge from the matching) in T. We show that the bipartite distance index of a nonsingular tree is actually the f-alternating sum corresponding to the sequence $S = (1, 1, 3, 3, 5, 5, \ldots)$. We also show that, in general, for any sequence S, the f-alternating sum of a nonsingular tree satisfies the above described inclusion-exclusion type of principle.

A well known result by Graham, Hoffman and Hosoya [GHH77] is that the determinant of the distance matrix of a graph only depends on its blocks and is independent of how they are assembled. In a similar way, we identify some basic elements and a merging operation and show that each of the trees that can be constructed from a given set of elements, sequentially using this operation, have the same bipartite distance index, independent of the order in which the sequence is followed. For the class of trees that can be obtained in this way, we give a surprisingly simple way to evaluate the determinant of the bipartite distance matrix.

We observe that the bipartite distance matrix of a nonsingular tree is always invertible and its determinant can be described using the structure of the tree. What about the inverse? Can the entries of the inverse be described combinatorially? We supply answer to these questions in the thesis. However, the answer to these questions leads us to an unexpected generalization of the usual Laplacian matrix of a graph. We call it the bipartite Laplacian matrix. This generalized Laplacian matrix is usually not symmetric, but it still has many properties like the usual Laplacian matrix. It turns out that the usual Laplacian matrix of any tree is a very special case of the bipartite Laplacian matrix. We study some of the fundamental properties of the bipartite Laplacian matrix and compare them with those of the usual Laplacian matrix. We discuss how a multiplicity of an eigenvalue of the bipartite Laplacian matrix of a nonsingular tree is related to the tree structure. Further, we provide a formula for the inverse of the bipartite distance matrix of a nonsingular tree with the help of its bipartite Laplacian matrix.

A combinatorial description of all the minors of the usual Laplacian matrix of a graph was supplied by Chaiken in [Cha82] Quite similar to that, we also provide a combinatorial description of all the minors of the bipartite Laplacian matrix of a nonsingular tree. In particular we observe that a minor of the bipartite Laplacian matrix of a nonsingular tree T obtained by deleting k many rows in $X \subseteq R$ and k many columns in $Y \subseteq L$ (with $|X| = |Y| = k$) enumerates the number of spanning forests in T that have (a) k trees, (b) each tree contains exactly one vertex in X and exactly one vertex in Y. As size of the bipartite Laplacian matrix is half of the size of that tree, it provides a faster way to enumerate the number of the above described spanning forests compare to Chaiken's all minor matrix tree result on the usual Laplacian matrix.

There are many studies of distance-like matrices available in the literature. We consider two of them. The first one is the q-distance matrix which is a generalization of the usual distance matrix. We show that the determinant of the q-bipartite distance matrix of a tree with a unique perfect matching can still be expressed as a combinatorial sum over the set of alternating paths with respect to a suitable sequence. The second distance-like matrix we study is the exponential distance matrix, which has a very simple expression for the determinant. We show that an extremely similar result hold here too. This is unexpected, as our matrix is of half the order. Recall that, the determinant of the usual distance matrix of a tree is just dependent on the number of vertices and is independent of the tree structure and the determinant of the bipartite distance matrix of a tree with a unique perfect matching was dependent on the tree structure and on some smaller classes it was independent of the tree structure. We give a similar results for the q-bipartite distance matrix. However, as another surprise, we show that the determinant of the exponential bipartite distance matrix of a tree with a unique perfect matching is independent of the tree structure.

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List of Symbols

$[n]$	$\{1, 2, \ldots, n\}$
$\mathbb{1}$	column vector of all ones of an appropriate size
$\mathfrak{B}(G)$	bipartite distance matrix of G
$\texttt{bd}(G)$	bipartite distance index of G
$\texttt{bd}_{\texttt{q}}(G)$	q-bipartite distance index of G
$\boldsymbol{\tau}_T^l$	restriction of $\boldsymbol{\tau}_T$ on L
$\boldsymbol{\tau}_T^r$	restriction of $\boldsymbol{\tau}_T$ on R
$\mathcal{L}(G)$	Laplacian matrix of G
$\mathcal{A}_{v,T}$	set of all alternating path in T that starts at v
$\mathcal{A}_{v,T}^+$	set of all even alternating path in T that starts at v
$\mathcal{A}_{v,T}^-$	set of all odd alternating path in T that starts at v
$\texttt{diff}_T(v)$	quantity $\lvert\mathcal{A}_{v,T}^+\rvert - \lvert\mathcal{A}_{v,T}^-\rvert$
$\texttt{dist}_G(u,v)$ or $\texttt{dist}(u,v)$	distance between the vertices u and v in G
$\boldsymbol{e}_k$	column vector of all entries zero except the kth entry which is 1 whose dimension will be understood from the context
$\mathfrak{L}(G)$	bipartite Laplacian matrix of G
$\mathfrak{L}(r_i, l_j)$	The (i,j)th entry of the bipartite Laplacian matrix of $\mathfrak{L}$
$\lceil x \rceil$	ceiling function that maps x to the least integer greater than or equal to x
$\mathcal{M}$	set of all matching edges
$\boldsymbol{\{}k\boldsymbol{\}}$	$1 + q + q^2 + \cdots + q^{k-1}$, k is nonnegative integer
$\boldsymbol{\mu}_v$	signed degree vector at v
$d_G(v)$ or $d(v)$	degree of the vertex v in G
$f_S(T)$	f-alternating sum of T with respect to the sequence S

$G - v$	graph obtained by deleting the vertex v from G and all edges that are incident on v
$M(X, Y)$	submatrix of M determined by the rows corresponding to X and the columns corresponding to Y
$M(X \mid Y)$	submatrix of M obtained by deleting the rows in X and the columns in Y
$M_{:,i}$	i-th column of the matrix M
$M_{i,:}$	i-th row of the matrix M
P_{2p}	path on $2p$ vertices
T	Tree
$T \circ K_1$	corona tree obtained by attaching a pendant vertex at every edge of T

1 Introduction

The study of the properties of a graph via matrices associated with it is known as *algebraic graph theory*. The linear algebraic graph theory is a fascinating subject that ties together two seemingly unrelated branches of mathematics; graph theory and linear algebra. It is a rich area of research and finds its application in several subjects like chemistry [GR79, Mer89], computer Science [CS11], electrical networks [Kir47], social networks [CS97], statistical design [Con87], etc. There is a large literature on algebraic graph theory, well documented in several books, see for example [Bap14, Big74, BH11, BP94, BR91, CCDS80, GR01].

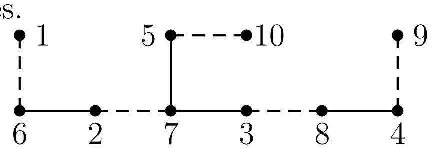
Throughout this thesis, all graphs are assumed to be simple, connected, undirected, unweighted and finite. Let G be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and the edge set E. An edge between vertices u and v is denoted by $[u, v]$. We shall use the notation $u \sim v$ (resp. $u \nsim v$) to mean 'u is adjacent to v' (resp. u is not adjacent to v). The degree of a vertex v is denoted by $d_G(v)$ or simply by $d(v)$. The distance between two vertices u and v is denoted by $\texttt{dist}_G(u, v)$ or simply by $\texttt{dist}(u, v)$.

We shall now recall a few terminologies and some related facts. We refer the reader to the classical texts [BM08, CCDS80] for other notations and for further clarifications.

- A graph G is called a bipartite graph if the vertex set V can be partitioned into two subsets L and R, such that no two vertices from the same set are adjacent. We shall call such a partition (L, R) a *vertex bipartition*, or simply *bipartition* of the graph G. The vertex bipartition (L, R) of G is unique if and only if G is connected. If G is a bipartite graph, then we shall always denote its vertex bipartition by (L, R). It is well-known that *a graph G is a bipartite graph if and only if G has no cycle of odd length.*
- A matching in graph G is a set of edges such that no two edges have a common end vertex. A perfect matching of a graph G is a matching that covers every vertex of graph G. Thus, no graph on an odd number of vertices can have a perfect matching. For a graph G with a unique perfect matching, we shall use $\mathcal{M}$ to denote the unique perfect matching of G.
- An alternating path in a graph G with a perfect matching $\mathcal{M}$ is a path with the starting edge, each alternate edge thereafter and the last edge from the matching (so,

the remaining edges are nonmatching edges). Note that our notion of an alternating path is a special case of an $\mathcal{M}$-alternating path as defined in [BM08, pp. 415].

▶ **Example 1.0.1.** Consider the following nonsingular tree T. Here the dashed edges are the matching edges.

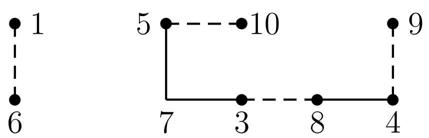


Note that, by our definition of an alternating path, $[1, 6]$ and $[1, 6, 2, 7]$ are alternating paths but $[6, 2, 7]$ and $[10, 5, 7, 3]$ are not an alternating paths. ◀

- A tree is a special type of connected graph that contains no cycles. In a tree, between any two vertices, there always exists a unique path. Let T be a tree and u, v be two vertices in T. By $[u, \cdots, v]$ we mean the unique u-v path in the tree T. It is well known that a tree is always a bipartite graph. Also, when a tree T has a perfect matching, the perfect matching is unique.
- Let G be a graph. A vertex $v \in G$ is called a *pendant vertex* of G if the degree of the vertex v is one in G. In the case of tree, we call such vertex a *leaf* of the tree.

  A vertex $v \in G$ is called a *quasi-pendant* vertex of G if it is adjacent to a pendant vertex of G.
- Let e be an edge of $G = (V, E)$. We define the graph $G - e = (V', E')$, where $V' = V$ and $E' = E \setminus \{e\}$). It is called the graph obtained deleting the edge e from G.
- Let v be a vertex of G. We define the graph $G - v = (V', E')$, where $V' = V \setminus \{v\}$ and $E' = \{e \in E \mid e \text{ is not incident on } v\}$. It is called the graph obtained deleting the vertex v from G. In pictures, deletion of a vertex v means removing the vertex v from G together with all edges incident with v.
- Let v be a vertex of G. A *branch* at v of G is a connected component of $G - v$. If B is a branch at v of G, then by $G - B$, we denote the graph obtained by deleting all the vertices of B from G. (Pictorially all the edges incident with those vertices are also removed).

▶ **Example 1.0.2.** Consider the tree T as described in Example 1.0.1. Take $G = T$ and $v = 2$. The graph $G - v$ is shown below.



Notice that there are two connected components in the graph $G - v$. Therefore, paths $[1, 6]$ and $[10, 5, 7, 3, 8, 4, 9]$ are the two branches of G at 2. ◀

1.1 Nonsingular Trees

Among the various matrices associated with a graph, the adjacency matrix of a graph is probably the most popular and extensively investigated one.

Let G be a graph on n vertices. The *adjacency matrix* $A(G)$ of G is the symmetric square matrix of size n whose (i, j)th entry a_{ij} is given by

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

F. Harary [Har62] in 1962 and independently H. Sachs [Sac64] in 1964 gave a beautiful combinatorial formula for the determinant of the adjacency matrix of a graph.

► **Theorem 1.1.1.** *Let G be a graph on n vertices. Let $\mathcal{H}$ be the set of all spanning subgraphs H of G whose connected components are some paths on two vertices and some cycles. Then*

$$\det A(G) = \sum_{H \in \mathcal{H}} (-1)^{n-c_p(H)} (-2)^{c(H)},$$

where $c_p(H)$ denotes the number of components in H which are paths and $c(H)$ denotes the number of components in H which are cycles. ◄

The adjacency matrix of a graph may or may not be singular. For example, for a tree with an odd number of vertices, the set $\mathcal{H}$ in Theorem 1.1.1 turns out to be an empty set. Hence $\det A(T) = 0$ for each tree T with an odd number of vertices.

For a bipartite graph with a unique perfect matching $\mathcal{M}$, the set $\mathcal{H}$ in Theorem 1.1.1 turns out to be $\mathcal{M}$. This leads us following well-known observation.

► **Corollary 1.1.2.** *Let G be a bipartite graph with a unique perfect matching. Then* $\det A(G) = \pm 1$. ◄

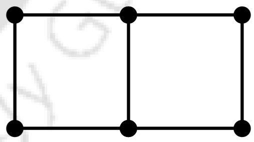


Figure 1.1: The bipartite graph G with three perfect matchings and $\det A(G) = -1$.

Here we remark that the converse of Corollary 1.1.2 is not valid in general. For example, see Figure 1.1. However, the converse is true for trees.

► **Corollary 1.1.3.** *Let T be a tree. Then* $\det A(T) \in \{0, 1, -1\}$. *Also* $\det A(T) \neq 0$ *if and only if T has a perfect matching.* ◄

As we shall be discussing these type of trees, let us use the word '*nonsingular tree*' to mean a tree with a (unique) perfect matching.

1.2 Bipartite distance matrix

For a bipartite graph, we do not require the complete adjacency matrix to display the adjacency information in the graph. Godsil [God85] in 1985, used a smaller size matrix to display this information. Later on this matrix was named the *bipartite adjacency matrix*, see [BM08, YY17].

▶ **Definition 1.2.1 (Bipartite adjacency matrix).** Let G be a bipartite graph with a vertex bipartition (L, R), where $L := \{l_1, \ldots, l_r\}$ and $R := \{r_1, \ldots, r_s\}$. The *bipartite adjacency matrix* $\mathbb{A}(G)$ of G is a $r \times s$ matrix whose (i, j)th entry $\mathbb{A}_{ij}$ is given by

$$\mathbb{A}_{ij} = \begin{cases} 1 & \text{if } l_i \text{ is adjacent to } r_j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that rows of $\mathbb{A}(G)$ are indexed by $l_1, \ldots, l_r$ and columns of $\mathbb{A}(G)$ are indexed by $r_1, \ldots, r_s$. ◀

For a bipartite graph G, the adjacency matrix $A(G)$ is permutationally similar to

$$\begin{bmatrix} \mathbf{0} & \mathbb{A}(G) \\ \mathbb{A}(G)^t & \mathbf{0} \end{bmatrix}.$$

Therefore, in order to study the properties of the adjacency matrix of a bipartite graph G, it is enough to study that the same for the bipartite adjacency matrix of G.

Godsil [God85] in 1985 showed that for a bipartite graph with a unique perfect matching, the bipartite adjacency matrix is similar to a lower triangular matrix with all diagonal entries equal to one.

▶ **Theorem 1.2.2.** [God85, Lemma 2.1] *Let G be a bipartite with a vertex bipartition (L, R). Then G has a unique perfect matching if and only if the vertices in L and R can be ordered such a way that $\mathbb{A}(G)$ is a lower triangular matrix, with all its diagonal entries equal to one.* ◀

Let G be a bipartite graph with a unique perfect matching. In view of Corollary 1.1.2, we can see that $\mathbb{A}(G)$ is always invertible and the inverse of $\mathbb{A}(G)$ is an integral matrix. If $\mathbb{A}(G)^{-1}$ is nonnegative (that is, all entries are nonnegative), then $\mathbb{A}(G)$ can be viewed as a bipartite adjacency matrix of another bipartite multigraph. Recently, Yang and Ye [YY17] provided a complete solution to a long standing problem posed by Godsil in [God85] on characterizing the bipartite graphs with a unique perfect matching G such that $\mathbb{A}(G)^{-1}$, the inverse of the bipartite adjacency matrix, is diagonally similar to a nonnegative matrix.

Significant work on the inverse of bipartite graphs with a unique perfect matching can be found in [HM76, God85, TK09, MM13, PP15b, PP15a, YY17, BPP17, PP17, YY17] and references therein.

In this thesis, similar to the bipartite adjacency matrix, we define the bipartite distance matrix of a bipartite graph with unique perfect matching. That is, the *bipartite distance matrix* of a bipartite graph with a unique perfect matching on $2p$ vertices is a $p \times p$ matrix whose (i, j)-th entry is the distance between vertices l_i and r_j, where (L, R) is a vertex bipartition of G. We will define this matrix in a more rigorous way in Chapter 2. Although the size of the bipartite distance matrix is half of the size of the graph, but we observe that it still provides much information about the underlying graph.

▶ **Definition 1.2.3 (Distance matrix).** Let G be a graph on n vertices. The *distance matrix* $D(G)$ of G is the symmetric square matrix of size n whose (i, j)th entry d_{ij} is the distance between i and j. ◀

Let us recall one of the striking results about the determinant of the distance matrix of a tree given by Graham and Pollack in 1971.

▶ **Theorem 1.2.4.** [GP71] *Let T be a tree on n vertices. Then the determinant of the distance matrix is $(-1)^{n-1}2^{n-2}(n-1)$.* ◀

This result tells us that the determinant of the distance matrix of a tree only depends on the number of vertices and does not depend on the tree's structure. It also tells us that the determinant is a multiple of 2^{n-2}.

Also, note that, by using the results by Stevanović and Indulal [SI09], one can easily find the determinant of the distance matrix of a complete bipartite graph.

▶ **Theorem 1.2.5.** [SI09, Corollary 3] *Let $K_{a,b}$ be a complete bipartite graph on n vertices, where $n = a + b$. Then the determinant of the distance matrix of $K_{a,b}$ is given by*

$$\det D(K_{a,b}) = (-2)^{n-2}[4(a-1)(b-1) - ab]. \quad ◀$$

Here also, we see that the determinant of the distance matrix is a multiple of 2^{n-2}.

It is natural to ask the following question. Let G be a bipartite graph with a unique perfect matching on $2p$ vertices. Since the size of the bipartite distance matrix is half of that of the usual distance matrix, whether the determinant of the bipartite distance matrix is a multiple of 2^{p-1}? It is surprising and easy to see that the answer is in the affirmative.

We found that the determinant of the bipartite distance matrix of a bipartite graph with a unique perfect matching on $2p$ vertices is always divisible by 2^{p-1}.

It is now natural to consider the number obtained by dividing the determinant of the bipartite distance matrix by $(-2)^{p-1}$. How does it relate to the structure? We call this number the *bipartite distance index* of the graph. By $\texttt{bd}(G)$, we denote the bipartite distance index of G.

As a tree is a bipartite graph, the study of the properties of a bipartite distance matrix of a nonsingular tree is naturally a starting point. We show that the bipartite distance index of a nonsingular tree satisfies an interesting inclusion-exclusion type of principle at any matching edge of the underlying tree, which is explained in Figure 1.2. This gives us a recursive formula to compute the bipartite distance index of the underlying tree.

Even more interestingly, we show that the determinant of the bipartite distance matrix of a nonsingular tree can be described by the combinatorial objects that we call f-alternating sums. Let $S : \mathbb{N} \to \mathbb{R}$ be a sequence. By the f-alternating sum of a nonsingular tree T, we mean the following sum

$$\sum_{P=[u,\cdots,v]} [d(u)-2][d(v)-2]S\left(\frac{|P|}{2}\right),$$

where the summation is taken over all alternating paths P of T. We show that the bipartite distance index of a nonsingular tree is actually the f-alternating sum corresponding to the sequence $S = (1, 1, 3, 3, 5, 5 \ldots)$.

Let us recall another fascinating result on the usual distance matrix by Graham, Hoffman and Hosoya, which states that the determinant of the distance matrix of a graph only depends on the blocks and it is independent of how they are assembled; thus, it is explaining Theorem 1.2.4 why the determinant of the distance matrix of a tree only depends on the number of vertices.

▶ **Theorem 1.2.6.** [GHH77] *If G is a strongly connected graph with blocks* [1] $G_1, G_2, \ldots, G_r$,

1 A block is a maximal connected subgraph which has no cut vertex.

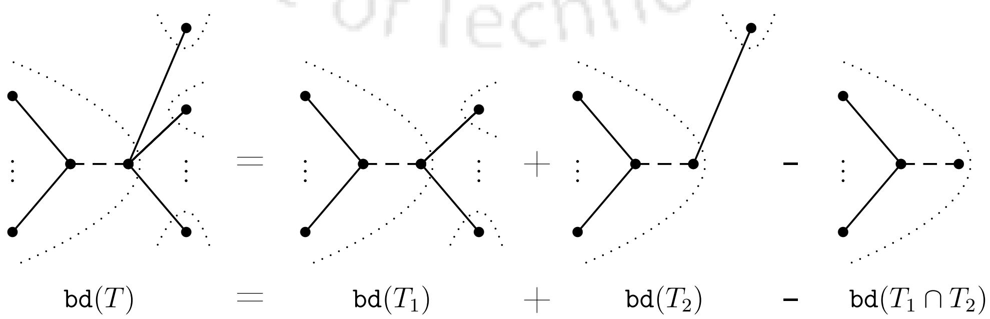


Figure 1.2: Illustration of Theorem 2.3.1, the inclusion-Exclusion principle at a matching edge (the shaded line is the matching edge under consideration).

then

$$\texttt{cof}\, D(G) = \prod_{i=1}^{r} \texttt{cof}\, D(G_i); \qquad \det D(G) = \sum_{i=1}^{r} \det D(G_i) \prod_{j \neq i} \texttt{cof}\, D(G_j),$$

where cof $D(G)$ *means the sum of all the cofactors of the matrix* $D(G)$. ◀

The above two results, Theorem 1.1.1 and Theorem 1.2.6, are very attractive due to their simplicity and many generalizations have been proposed in order to understand them better and better; for example, see [BKN05, BLP06, ZD16, BS16, Bap19].

It is natural to ask, are there some nonsingular trees on $2p$ vertices with the same bipartite distance index (hence the same determinant of their bipartite distance matrices)?

One can easily find examples where these numbers are different. However, to our surprise, we found some classes of trees which were structurally related to each other having the same bipartite distance index. It turns out that they are all created from the same set of basic elements using a merging operation, sequentially. Therefore on this class, the bipartite distance index remains the same, independent of the structures of the trees. For this class of trees, one can give a straightforward way to evaluate the determinant of their bipartite distance matrix.

Let T be a nonsingular tree. For an even natural number k, an alternating path $[u_1, \ldots, u_k]$ is called a *pendant path*, if $d(u_1) = 1$, $d(u_i) = 2$, $i = 2, \ldots, k-1$ and $d(u_k) \geq 2$. Let $P = [v_1, v_2, \ldots, v_k]$ and $P' = [v'_1, v'_2, \ldots, v'_k]$ be two pendant paths in two nonsingular trees T and T', respectively. Then by *merging* T *and* T' *along the pendant paths*, we mean the new tree obtained by identifying the vertices v_i with v'_i for each $i \in \{1, \ldots, k\}$, while keeping the rest of the vertices distinct.

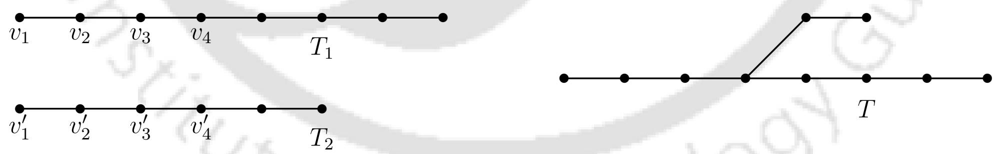


Figure 1.3: Merging of two trees along pendant paths of four vertices.

Note that if we want to merge T and T' along some pendant paths of k vertices, then there may be more than one way to do it, and the resulting trees could be nonisomorphic. For example, if we take T and T_2 from Figure 1.3 and try to merge them along pendant paths of 4 vertices, then there are two ways to do it, and the results are nonisomorphic (see Figure 1.4).

Let $T + T' - P_k$ denote the set of all trees that can be obtained by merging trees T and T' along a pendant path of k vertices. This set will be treated as empty if one of T or T' does not have a pendant path of k vertices. Let $\mathcal{F} = \{P_{k_1}, \ldots, P_{k_n}\}$ and $\mathcal{G} = \{P_{r_1}, \ldots, P_{r_{n-1}}\}$ be

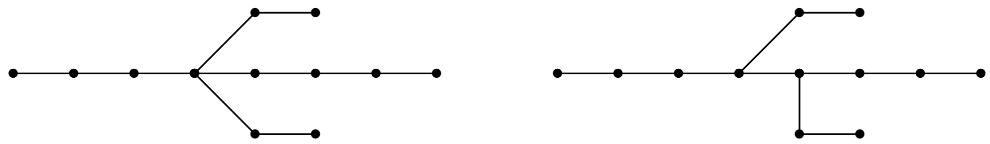


Figure 1.4: Two nonisomorphic tree obtained by merging of T and T_2 from Figure 1.3 along pendant paths of 4 vertices.

two multisets consisting paths of even order, $n \geq 2$. Then by

$$P_{k_1} + \cdots + P_{k_n} - P_{r_1} - \cdots - P_{r_{n-1}}$$

we denote the class of trees that can be obtained by repeated application of the following rules in all possible ways, while making $\mathcal{G} = \emptyset$.

a) Select a path P_{r_i} from $\mathcal{G}$. Select two trees T and T' from $\mathcal{F}$ that have pendant paths of r_i vertices.

b) Merge T and T' along some pendant paths of r_i vertices. Let the resulting tree be T_r.

c) Delete T and T' from $\mathcal{F}$ and put T_r in $\mathcal{F}$. Delete P_{r_i} from $\mathcal{G}$.

Note that when $\mathcal{G} = \emptyset$, the set $\mathcal{F}$ will contain exactly one tree. If step a) fails at some stage, then we immediately set $\mathcal{F} = \emptyset$ and break the loop. This means, we just collect the trees for those cases, for which we could reach $\mathcal{G} = \emptyset$. For brevity, let us also use the notation $\mathcal{F} - \mathcal{G}$ to denote the above class. We observe that the bipartite distance index remains the same for each tree in the class $\mathcal{F} - \mathcal{G}$, independent of how it has been created, and provide a simpler formula to calculate that common value. These results are presented in Chapter 2.

1.3 Bipartite Laplacian matrix and the inverse of the bipartite distance matrix

In Chapter 2, we observe that the bipartite distance matrix of a nonsingular tree is always invertible, and its determinant can be described using the structure of T. What about the inverse? Can the entries of the inverse be described combinatorially? For the usual distance matrix, such a description can be found in [GL78].

▶ **Definition 1.3.1 (Laplacian matrix).** Let G be a graph on n vertices. Let $\mathsf{Deg}(G)$ be the diagonal matrix of order n with the ith diagonal entry equal to the degree of the vertex i in G. The *Laplacian matrix* $\mathcal{L}(G)$ of G is defined as $\mathcal{L}(G) = \mathsf{Deg}(G) - A(G)$. ◀

Let us recall that the inverse of the usual distance matrix of a tree has a definite relationship with the usual Laplacian matrix.

▶ **Theorem 1.3.2.** [BKN05] *Let T be a tree on n vertices. For each $i = 1, \ldots, n$, let $\delta_i = 2 - d(i)$, and put $\boldsymbol{\delta}^t = [\delta_1, \ldots, \delta_n]$. Then*

$$D(T)^{-1} = -\frac{1}{2}\mathcal{L}(T) + \frac{1}{2(n-1)}\boldsymbol{\delta}\boldsymbol{\delta}^t.$$ ◀

Can we have a similar statement for the bipartite distance matrix too? In that case, what is an appropriate replacement for the Laplacian matrix? We call it the *bipartite Laplacian matrix*. Let T be a nonsingular tree with the vertex bipartition $(L = \{l_1, \ldots, l_p\}, R = \{r_1, \ldots, r_p\})$. Then the bipartite Laplacian matrix of T is defined as $\mathbb{D} - \mathbb{A}^t$, where $\mathbb{A}$ is the bipartite adjacency matrix of T and $\mathbb{D}$ is a square matrix whose (i, j)th is either $\pm d(r_i)d(l_j)$ or 0 depending on the nature of r_i-l_j path, see Definition 3.1.1 for more details. We first observe that the bipartite Laplacian matrix is usually not a symmetric whereas the Laplacian matrix is always a symmetric matrix. Interestingly, it turns out that the usual Laplacian matrix of any tree (need not be nonsingular) is a very special case of the bipartite Laplacian matrix. Although the bipartite Laplacian is usually not a symmetric matrix but this matrix has many properties similar to that of a Laplacian matrix, see Theorem 3.2.3. Another motivation to study the bipartite Laplacian matrix is that even though its size is half of the number of vertices (of the nonsingular tree), it still gives us some information about the tree. It may save us some time. With the help of the bipartite Laplacian matrix, we present a formula for the inverse of the bipartite distance matrix of a nonsingular tree which is quite similar to Theorem 1.3.2. These are the content of Chapter 3.

For a graph G, Chaiken in 1982 [Cha82] gave a combinatorial interpretation of all minor of usual Laplacian matrix of G as a sum of nonsingular substructures. A similar combinatorial interpretation of all minor of Laplacian matrix has been discussed for a mixed graph by Bapat, Grossman, and Kulkarni in [BGK99]. Merris [Mer89] in 1989 considered the edge version of the Laplacian matrix and established a bridge between its cofactors with the Wiener index, which has several application in chemistry. Bapat, Grossman, and Kulkarni [BGK00] in 2000 gave a combinatorial interpretation of the minors of the edge version of the Laplacian matrix when the underlying graph is a tree. The study of finding all minor of the usual Laplacian matrix finds their use in many different areas, see for example [Kir47, CS97, Con87, Mer89], etc. and have been subject to many independent studies. Let G be a graph and $X, Y \subseteq V(G)$ such that $|X| = |Y| = k$. Recall that Chaiken's [Cha82] all minor matrix tree results on the usual Laplacian matrix tells that the absolute value of a minor of the usual Laplacian matrix obtained by removing its k many rows in X and k many columns in Y is the number of spanning forest in G that have (a) k trees, (b) each tree contains exactly one vertex in X and exactly one vertex in Y. Therefore, in order to find the number of spanning forest of G of the above type, we have to find the determinant of a matrix of size $n - k$, where n is the size of the graph. In Chapter 4 we observe that a minor of the bipartite Laplacian matrix of a nonsingular tree T also enumerates the number of spanning forests of T of the above type. In particular, we show that a minor of the bipartite Laplacian matrix of a nonsingular tree T obtained by deleting k many rows in X and k many columns in Y enumerates the number of spanning forests in T that have (a) k trees, (b) each tree contains exactly one vertex in X and exactly one vertex in Y. As the size of the bipartite Laplacian matrix is half of the order of that tree, it may provide a faster way to enumerate the number of the above described spanning forest of a nonsingular tree compare to Chaiken's all minor matrix tree result on the usual Laplacian matrix, see for Example 4.5.2.

1.4 q-analogue versions of the bipartite distance matrix

Bapat, Lal, and Pati [BLP06] in 2006 introduced two types of q-analogue version of distance matrix, namely, q-distance matrix and exponential distance matrix. Indeed, in the same paper [BLP06], Bapat *et al.* generalized many concepts of the distance matrix of a tree to that of q-distance matrix of a tree. Since then, the q-analogue version of the distance matrix generated a considerable interest and has been studied by many researchers (see, for example, [BLP06, Siv10, BS11, LSZ14]). In Chapter 5, we consider q-analogue version of the bipartite distance matrix. In particular, we consider two types of q-analogue versions of the bipartite distance matrix, namely the q-bipartite distance matrix and the exponential bipartite distance matrix. We then completely characterized the determinant of those two types of q-analogue versions of the bipartite distance matrix of nonsingular tree. Although the determinant of q-bipartite distance matrix depends on the tree structure (as expected) but surprisingly, the determinant of the exponential bipartite distance matrix of a nonsingular tree does not depend on the tree structure. Similar to the bipartite distance index, we also define q-bipartite distance index of a nonsingular tree and observe that it also satisfies an inclusion-exclusion type principle at any matching edge of the underlying tree. Finally, with the help of that inclusion-exclusion principle we show that the q-bipartite distance index of a nonsingular tree is nothing but a f-alternating sum corresponding to the sequence $(1, 1, q+2, q+2, 2q+3, 2q+3, \ldots)$.

1.5 Organization of the thesis

The thesis is organized as follows. There are six chapters in the thesis. Chapter 1 contains a brief introduction of the thesis and a few lines for motivation.

In Chapter 2, we study the bipartite distance matrix of a bipartite graph with unique perfect matching. We show that the determinant of the bipartite distance matrix of a bipartite graph with a unique perfect matching on $2p$ vertices is always divisible by 2^{p-1}. Based on this observation, we define the bipartite distance index as a quotient obtained by diving the determinant of the bipartite distance matrix by $(-2)^{p-1}$. We show that for a nonsingular tree on $2p$ vertices, 2^{p-1} is the highest power of 2 that can divide the determinant of its bipartite distance matrix. Henceforth, we provide a recursive formula to calculate the bipartite distance index of a nonsingular tree and using that we provide a combinatorial description of the determinant of the bipartite distance matrix of a nonsingular tree relating the tree structure. We identify some basic elements and a merging operation and show that each of the trees that can be constructed from a given set of elements, sequentially using this operation, have the same bipartite distance index, independent of the order in which the sequence is followed. For the class of trees that can be obtained in this way, we give a surprisingly simple way to evaluate the determinant of their bipartite distance matrix.

In Chapter 3, we supply a combinatorial description of the inverse of the bipartite distance matrix of a nonsingular tree and establish identities that are similar to some well known identities. The study leads us to an unexpected generalization of the usual Laplacian matrix of a graph. We call this bipartite Laplacian matrix. This generalized Laplacian matrix is usually not symmetric, but it still has many properties like the usual Laplacian matrix. Further we discuss how a multiplicity of an eigenvalue of the bipartite Laplacian matrix of a nonsingular tree is related to the tree structure. We also obtain a lower bound on the geometric multiplicity of the eigenvalue -2 of the bipartite distance matrix of a nonsingular tree.

Chapter 4, deals with an arbitrary minor of the bipartite Laplacian matrix of a nonsingular tree. We discuss under what condition a square submatrix of the bipartite Laplacian matrix of a nonsingular tree becomes singular. Further, we provide a complete combinatorial formula of any arbitrary minor of the bipartite Laplacian matrix of a nonsingular tree. It turns out that each minor of the bipartite Laplacian matrix of nonsingular tree counts a particular type of spanning forest in the underlying tree, depending on the choice of rows and columns are used to obtain that minor.

In Chapter 5, we study two types of q-analogues versions of the bipartite distance matrix of a bipartite graph with unique perfect matching, namely exponential bipartite distance matrix and q-bipartite distance matrix. We extends our results of the bipartite distance matrix that we obtained in Chapter 2 to that of a q-analogue version of the bipartite distance matrix. We observe that although the determinant of q-bipartite distance matrix depends on the tree structure, surprisingly, the determinant of the exponential bipartite distance matrix of a nonsingular tree does not depend on the tree structure.

In Chapter 6, we outline some future directions of this thesis.

2 The bipartite distance matrix

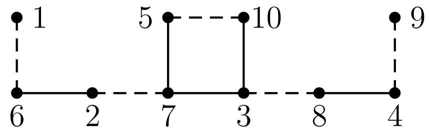
Since this thesis focuses on the bipartite distance matrices, we dedicate this chapter to the foundations of the bipartite distance matrices. We begin by defining the bipartite distance matrix for any bipartite graph with a unique perfect matching and then we observe that the determinant of the bipartite distance matrix of a bipartite graph with a unique perfect matching on $2p$ vertices is always a multiple of 2^{p-1}. Based on this observation, we define the bipartite distance index of a bipartite graph with a unique perfect matching as a quotient obtained by dividing the determinant of the bipartite distance matrix by $(-2)^{p-1}$. It turns out that for a nonsingular tree on $2p$ vertices, 2^{p-1} is the highest power of 2 that can divide the determinant of its bipartite distance matrix. We provide a combinatorial description of the bipartite distance index of nonsingular tree via a combinatorial object, known as f-alternating sum. Finally, we identify some basic elements and a merging operation and show that each of the trees that can be constructed from a given set of elements, sequentially using this operation, have the same bipartite distance index, independent of the order in which the sequence is followed.

2.1 Preliminaries

By the definition of a bipartite graph, we can partition the vertex of a bipartite graph into two vertex sets L and R so that no vertices belonging to the same set are adjacent to each other. By following the proof of [God85, Lemma 2.1], we can see that Godsil labels the vertices of L and R in a particular way in order to get a lower triangular form in the bipartite adjacency matrix. We define a 'canonical' way to order the sets L and R of a bipartite graph with unique perfect matching and we such a bipartition, a standard vertex bipartition.

▶ **Definition 2.1.1 (Standard vertex bipartition).** Let G be a labeled, connected, bipartite graph on $2p$ vertices with a unique perfect matching $\mathcal{M}$. A vertex bipartition $(L = \{l_1, \ldots, l_p\}, R = \{r_1, \ldots, r_p\})$ of the vertex set of G is called a *standard vertex bipartition* of G if $\mathcal{M} := \{[l_1, r_1], \ldots, [l_p, r_p]\}$. ◀

▶ **Example 2.1.2.** Consider the following bipartite graph G with a unique perfect matching. Here the dashed edges are the matching edges.



Note that ($L = \{l_1 = 1, \ldots, l_5 = 5\}$, $R = \{r_1 = 6, \ldots, r_5 = 10\}$) is a standard vertex bipartition of G. ◀

Let us define the bipartite distance matrix of a bipartite graph with a unique perfect matching.

▶ **Definition 2.1.3 (The bipartite distance matrix).** Let G be a labeled, connected, bipartite graph on $2p$ vertices with a unique perfect matching. Let ($L = \{l_1, \ldots, l_p\}, R = \{r_1, \ldots, r_p\}$) be a standard vertex bipartition of G. The *bipartite distance matrix* of G, denoted by $\mathfrak{B}(G)$ or simply by $\mathfrak{B}$, is a square matrix of order p whose rows are indexed by $l_1, \ldots, l_p$ and columns are indexed by $r_1, \ldots, r_p$ such that the (i, j)th entry of $\mathfrak{B}(G)$ is the distance from l_i to r_j in G. ◀

Note that $\mathfrak{B}(G) = D(L, R)$ where D is the usual distance matrix of G and $D(L, R)$ is the submatrix of D induced by the rows corresponding to L and the columns corresponding to R. If the rows and columns of the usual distance matrix D of G are indexed by $l_1, \ldots, l_p, r_1, \ldots, r_p$, then the bipartite distance matrix of G is nothing but the upper right block of D.

▶ **Example 2.1.4.** For the graph G in Example 2.1.2, the bipartite distance matrix $\mathfrak{B}(G)$ is given below along with the usual distance matrix $D(G)$.

$$D(G) = \left[\begin{array}{ccccc|ccccc} 0 & 2 & 4 & 6 & 4 & 1 & 3 & 5 & 7 & 5 \\ 2 & 0 & 2 & 4 & 2 & 1 & 1 & 3 & 5 & 3 \\ 4 & 2 & 0 & 2 & 2 & 3 & 1 & 1 & 3 & 1 \\ 6 & 4 & 2 & 0 & 4 & 5 & 3 & 1 & 1 & 3 \\ 4 & 2 & 2 & 4 & 0 & 3 & 1 & 3 & 5 & 1 \\ \hline 1 & 1 & 3 & 5 & 3 & 0 & 2 & 4 & 6 & 4 \\ 3 & 1 & 1 & 3 & 1 & 2 & 0 & 2 & 4 & 2 \\ 5 & 3 & 1 & 1 & 3 & 4 & 2 & 0 & 2 & 2 \\ 7 & 5 & 3 & 1 & 5 & 6 & 4 & 2 & 0 & 4 \\ 5 & 3 & 1 & 3 & 1 & 4 & 2 & 2 & 4 & 0 \end{array}\right], \quad \mathfrak{B}(G) = \begin{bmatrix} 1 & 3 & 5 & 7 & 5 \\ 1 & 1 & 3 & 5 & 3 \\ 3 & 1 & 1 & 3 & 1 \\ 5 & 3 & 1 & 1 & 3 \\ 3 & 1 & 3 & 5 & 1 \end{bmatrix}.$$

Here we note that $\mathfrak{B}(G)$ is the upper right block of $D(G)$. ◀

If G is a bipartite graph with a unique perfect matching, then we can have many standard vertex bipartitions depending on the labeling of its vertices. However, we can see that the bipartite distance matrices corresponding to them are similar to each other.

▶ **Remark 2.1.5.** Let G be a labeled, connected, bipartite graph on $2p$ vertices with a unique perfect matching. Let (L, R) be a standard vertex bipartition of G.

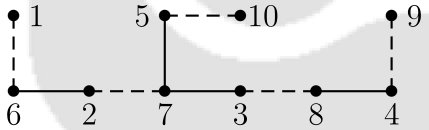
(a) Let G_1 be the graph obtained by interchanging the labels of l_i with r_i for all $i = 1, \ldots, p$ in G. Then $\mathfrak{B}(G) = \mathfrak{B}(G_1)^t$. Therefore, $\mathfrak{B}(G)$ and $\mathfrak{B}(G_1)$ are similar.

(b) Let G_2 be obtained by relabeling the vertices within the part L, keeping it a standard bipartition (that is, we also relabel the respective vertices in R accordingly). Then $\mathfrak{B}(G_2)$ is similar to $\mathfrak{B}(G)$.

Hence, in either case, $\det \mathfrak{B}(G) = \det \mathfrak{B}(G_1) = \det \mathfrak{B}(G_2)$. So $\det \mathfrak{B}(G)$ remains unchanged under isomorphism. ◀

Let us look at an example of the bipartite distance matrix when the underlying graph is a nonsingular tree.

▶ **Example 2.1.6.** Consider the following nonsingular tree T. Here the dashed edges are the matching edges.



Note that $L = \{l_1 = 1, \ldots, l_5 = 5\}$, $R = \{r_1 = 6, \ldots, r_5 = 10\}$ is the standard vertex bipartition of T.

The bipartite distance matrix $\mathfrak{B}(T)$ is shown below.

$$\mathfrak{B}(T) = \begin{bmatrix} 1 & 3 & 5 & 7 & 5 \\ 1 & 1 & 3 & 5 & 3 \\ 3 & 1 & 1 & 3 & 3 \\ 5 & 3 & 1 & 1 & 5 \\ 3 & 1 & 3 & 5 & 1 \end{bmatrix}.$$

Here we note that $\det \mathfrak{B}(T) = 2^4 \times 5$. ◀

Recall Theorem 1.2.4, the determinant formula of Graham and Pollak for a tree T on n vertices, showed that $\det D(T)$ is always a multiple of 2^{n-2}. Let T be a nonsingular tree on $2p$ vertices. Since the size of $\mathfrak{B}(T)$ is half of that of the usual distance matrix $D(T)$ and $\det D(T)$ is a multiple of $2^{2(p-1)}$, the first (not so logical) question we ask is whether $\det \mathfrak{B}(T)$ is always a multiple of 2^{p-1}? It is surprising and easy to see that the answer is in the affirmative.

▶ **Theorem 2.1.7.** *Let G be a connected, bipartite graph on $2p$ vertices with a unique perfect matching. Then, for any standard vertex bipartition, $\det \mathfrak{B}(G)$ is a multiple of 2^{p-1}.* ◀

Proof. The proof follows by observing that, each entry of $\mathfrak{B}$ is odd; replacing the ith column $\mathfrak{B}_{:,i}$ with $\mathfrak{B}_{:,i} - \mathfrak{B}_{1,i}\mathfrak{B}_{:,1}$, for $i = 2, 3, \ldots, p$; and noting that the lower-right submatrix of order $p - 1$ has all entries even. ■

By Theorem 2.1.7, we can see that $\det \mathfrak{B}(G)/2^{p-1}$ is an interesting object to study. This leads us to give following definition.

▶ **Definition 2.1.8.** Let G be a connected, bipartite graph on $2p$ vertices with a unique perfect matching. The *bipartite distance index* of G is denoted by $\mathtt{bd}(G)$ and it is defined as $\mathtt{bd}(T) := \det \mathfrak{B}(T)/(-2)^{p-1}$. ◀

▶ **Example 2.1.9.** Let G be the bipartite graph as defined in Example 2.1.2. We can see that $\det \mathfrak{B}(G) = 2^{5-1} \times 5$. Therefore, the bipartite distance index of G is 5.

In a similar way, we can see that $\mathtt{bd}(T) = 5$, for the tree T as defined in Example 2.1.6. ◀

2.1.1 The bipartite distance index

It is natural to wonder whether the bipartite index of a bipartite graph G with a unique perfect matching is related to the structure of the given graph G. In this context, following operation plays a crucial role in determining the bipartite distance index of G.

▶ **Definition 2.1.10 (Attaching a new P_2 at u).** (a) Let G be a connected graph and v be a vertex. Let $\widehat{G}$ be the graph obtained from G by introducing two new vertices u, w and adding the edges $[v, u]$, $[u, w]$. We refer to this operation as *attaching a new P_2 at the vertex v*.

(b) Let G be a connected bipartite graph on $2p$ vertices with a unique perfect matching. Let (L, R) be a standard vertex bipartition of G. Suppose $\widehat{G}$ is the graph obtained by attaching a new P_2 at some vertex $v \in G$. To compute the bipartite distance matrix of $\widehat{G}$ we need to label the new vertices. We adopt the following procedure for that:

i) if $v \in L$, then we put $u = r_{p+1}$, $w = l_{p+1}$, and

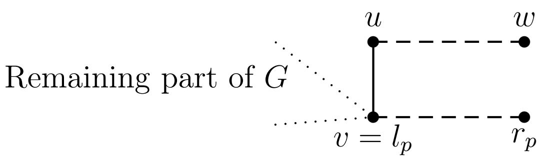
ii) if $v \in R$, then we put $u = l_{p+1}$, $w = r_{p+1}$. ◀

The following result shows that bipartite distance index of a bipartite graph G with a unique perfect matching remain same after adding a new P_2 at any quasi-pendant vertex of G.

▶ **Lemma 2.1.11.** *Let G be a connected, bipartite graph on $2p$ vertices with a unique perfect matching. Let v be a quasi-pendant vertex of G. Let $\widehat{G}$ be the bipartite graph on $2p + 2$ vertices obtained by attaching a new P_2 at v. Then*

$$\det \mathfrak{B}(\widehat{G}) = -2 \det \mathfrak{B}(G), \qquad \textit{and} \qquad \mathtt{bd}(\widehat{G}) = \mathtt{bd}(G). \quad ◀$$

Proof. Let $L = \{l_1, \ldots, l_p\}$, $R = \{r_1, \ldots, r_p\}$ be a standard vertex bipartition of G and assume without loss of generality that $v = l_p$. Let the graph $\widehat{G}$ be constructed from G by taking two new vertices u, w and adding the edges $[v, u]$, $[u, w]$.



Clearly, $\widehat{G}$ is a bipartite graph with a unique perfect matching. In view of item (b) of Definition 2.1.10, we take $u = r_{p+1}$ and $w = l_{p+1}$. It follows that

$$\mathfrak{B}(\widehat{G}) = \begin{bmatrix} \mathfrak{B}(G) & \mathfrak{B}(G)\boldsymbol{e}_p \\ \boldsymbol{e}_p^t\mathfrak{B}(G) + 2\mathbb{1}^t & 1 \end{bmatrix}.$$

In $\mathfrak{B}(\widehat{G})$, subtracting the column p from the column $p+1$, we get the following matrix

$$C = \begin{bmatrix} \mathfrak{B}(G) & \mathbf{0} \\ \boldsymbol{e}_p^t\mathfrak{B}(G) + 2\mathbb{1}^t & -2 \end{bmatrix}.$$

By expanding the determinant of C along the last column we see that $\det \mathfrak{B}(\widehat{G}) = \det C = -2\ \det \mathfrak{B}(G)$ and so $\texttt{bd}(\widehat{G}) = \texttt{bd}(G)$. ■

A nonsingular tree has a beautiful structure. It can be generated from a P_2 by repeatedly attaching a new P_2 at some vertex.

▶ **Remark 2.1.12.** The largest of the distances between pairs of vertices in T is called the *diameter* of T. We know that P_2 is the only nonsingular tree of diameter one and there are no nonsingular trees of diameter two. Thus any other nonsingular tree will have a diameter at least three. Let $T \neq P_2$ be a nonsingular tree and $P = [v_1, \ldots, v_k]$ be a path corresponding to the diameter of T. Thus $k \geq 4$. It follows that v_1 must be a leaf and the degree $d(v_2) = 2$. Thus, T can be seen to be generated from another tree by attaching a new P_2 at the vertex v_3. ◀

The trees that can be generated from a P_2 by repeatedly attaching a new P_2 at quasi-pendant vertices are called *corona trees*. Corona trees can also be obtained by taking a tree T and attaching a new leaf at each vertex of T, see Figure 2.1.

As $\texttt{bd}(P_2) = 1$, in view of Lemma 2.1.11, we see that the bipartite distance index of every corona tree is 1.

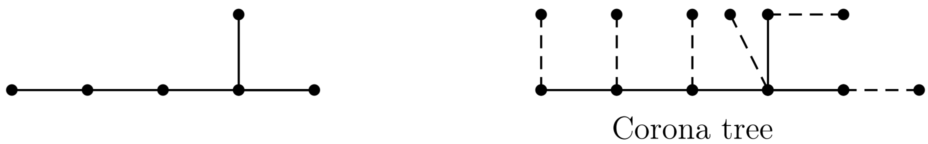


Figure 2.1: Construction of corona tree from a tree.

▶ **Corollary 2.1.13.** *Let T be a corona tree on $2p$ vertices. Then* $\det \mathfrak{B}(T) = (-1)^{p-1}2^{p-1}$ *and* $\mathtt{bd}(T) = 1$. ◀

The converse of this result is not true. The tree T_3 in Figure 2.2 is an example.

Since a nonsingular tree is a bipartite graph with a unique perfect matching, the study of the properties of a bipartite distance matrix of a nonsingular tree is naturally a starting point. From now onward we focus on characterizing the bipartite distance index of a nonsingular tree.

2.2 The bipartite distance index of a tree

In this section our main aim is to provide a recursive formula to calculate the bipartite distance index of a nonsingular tree. Note that if we could supply a result similar to Lemma 2.1.11 where we allow attaching a new P_2 at any vertex (not just a quasi-pendant one), then we would obtain a recursive formula to compute $\mathtt{bd}(T)$. In this section we shall do that. But for that we need to know the bipartite distance indices of the paths P_{2p}.

▶ **Proposition 2.2.1.** *Let $p \in \mathbb{N}$. Then*

$$\mathtt{bd}(P_{2p}) = \begin{cases} p & \text{if } p \text{ is odd} \\ p-1 & \text{if } p \text{ is even.} \end{cases}$$ ◀

Proof. Let $T = [l_1, r_1, l_2, r_2, \ldots, l_p, r_p]$ be a path on $2p$ vertices.

First notice that the bipartite distance matrix of T can be expressed as

$$\mathfrak{B}(T) = \begin{bmatrix} 1 & 3 & 5 & \cdots & 2p-3 & 2p-1 \\ 1 & 1 & 3 & \cdots & 2p-5 & 2p-3 \\ 3 & 1 & 1 & \cdots & 2p-7 & 2p-5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 2p-5 & 2p-7 & 2p-9 & \cdots & 1 & 3 \\ 2p-3 & 2p-5 & 2p-7 & \cdots & 1 & 1 \end{bmatrix}$$

For $i \geq 2$, let E_i denote the matrix $I - \boldsymbol{e}_i\boldsymbol{e}_{i-1}^t$, where I is the identity matrix and $\boldsymbol{e}_i$ is the standard basis vector (whose all entries are zero except the i-th entry which is one) of dimension p. First notice that for $i > 1$

$$\texttt{dist}(l_i, r_j) = \begin{cases} \texttt{dist}(l_{i-1}, r_j) + 2 & \text{for } j < i-1, \\ 0 & \text{for } j = i-1,. \\ \texttt{dist}(l_{i-1}, r_j) - 2 & \text{for } j > i-1 \end{cases}$$

Let $P = E_2E_3\cdots E_p$. Then it follows that

$$P\mathfrak{B}(T) = E_2E_3\cdots E_{p-1}\mathfrak{B}(T) = \begin{bmatrix} 1 & 3 & 5 & \cdots & 2p-3 & 2p-1 \\ 0 & -2 & -2 & \cdots & -2 & -2 \\ 2 & 0 & -2 & \cdots & -2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 2 & 2 & 2 & \cdots & -2 & -2 \\ 2 & 2 & 2 & \cdots & 0 & -2 \end{bmatrix}.$$

Thus it can be verified that

$$P\mathfrak{B}(T)P^t = \left[\begin{array}{c|ccccc} 1 & 2 & 2 & 2 & \cdots & 2 \\ \hline 0 & -2 & 0 & 0 & \cdots & 0 \\ 2 & -2 & -2 & 0 & \cdots & 0 \\ 2 & 0 & -2 & -2 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 2 & 0 & \cdots & 0 & -2 & -2 \end{array}\right].$$

Write $P\mathfrak{B}(T)P^T = \begin{bmatrix} 1 & 2\mathbb{1}^t \\ 2(\mathbb{1} - \boldsymbol{e}_1) & C \end{bmatrix}$. Note that $\det C = (-2)^{p-1}$. Let $\boldsymbol{x}$ be a vector whose i-th entry is $-1/2$ if i is odd and 0 otherwise. Let $\boldsymbol{y}$ be a vector whose i-th entry is

$-1/2$ if i is odd and $1/2$ otherwise. As $C\boldsymbol{x} = \mathbb{1}$ and $C\boldsymbol{y} = e_1$, we see that

$$4\mathbb{1}^t C^{-1}(\mathbb{1} - e_1) = 4\mathbb{1}^t C^{-1}(C\boldsymbol{x} - C\boldsymbol{y}) = 4\mathbb{1}^t(\boldsymbol{x} - \boldsymbol{y}) = \begin{cases} -(p-1) & \text{if } p \text{ is odd;} \\ -(p-2) & \text{if } p \text{ is even.} \end{cases}$$

By applying the *Schur complement* formula [HJ12, p. 24] for the determinant, we get

$$\det \mathfrak{B}(T) = \det(P\mathfrak{B}(T)P^T) = (-2)^{p-1}\big(1 - 2\mathbb{1}^t C^{-1} 2(\mathbb{1} - e_1)\big).$$

Using $\texttt{bd}(T) = \det \mathfrak{B}(T)/(-2)^{p-1}$, the conclusion follows. ■

Note that the previous result along with Lemma 2.1.11 can help us evaluate the bipartite distance index of many nonsingular trees. Consider P_6, a path on 6 vertices. By Proposition 2.2.1, $\texttt{bd}(P_6) = 3$. Now consider the tree T_1 obtained from P_6 as show Figure 2.2. By Lemma 2.1.11, $\texttt{bd}(T_1) = 3$. Similarly, again by Lemma 2.1.11, $\texttt{bd}(T_2) = 3$. Thus, multiplying $(-2)^{p-1}$ we get the value of $\det \mathfrak{B}(T)$ for these trees, for example, $\det \mathfrak{B}(T_2) = (-2)^4 \times 3$.

However, there are trees whose bipartite distance index cannot be computed from these two results alone. For example, the tree T_3 in Figure 2.2 is such a tree. In the next subsection we shall develop a technique that will help us to compute the bipartite index of any nonsingular tree.

2.2.1 Recursive formula to calculate the bipartite distance index of a tree

We need to consider a few combinatorial objects that will helps us to determine the relationship between the bipartite distance indices of a nonsingular tree T with the tree $\widehat{T}$ is obtained by attaching a new P_2 at any vertex of T.

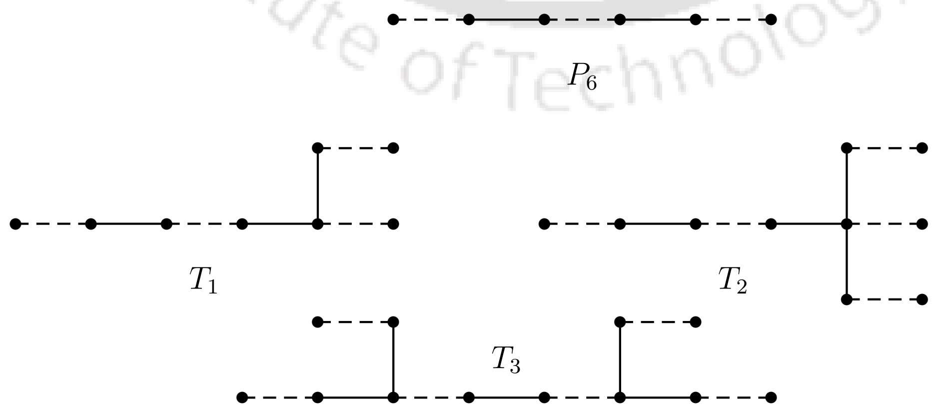


Figure 2.2: The bipartite distance indices of T_1 and T_2 can be be computed only by Lemma 2.1.11 and Proposition 2.2.1, but that of T_3 cannot be computed. Dashed lines are matching edges.

The following initial observation will play a crucial role to prove many results.

▶ **Remark 2.2.2.** Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $[u, v]$ be an edge in T. Let B_1 be the branch at v that contains the vertex u and B_2 be the branch at u that contains the vertex v.

(a) If $u \in L$ and $[u, v]$ is a matching edge then there does not exist any alternating path from a vertex in $r_i \in B_1 \cap R$ to a vertex in $l_j \in B_2 \cap L$.

(b) If $u \in R$ and $[u, v]$ is not a matching edge then there does not exist any alternating path from a vertex in $r_i \in B_1 \cap R$ to a vertex in $l_j \in B_2 \cap L$. ◀

Let us classify alternating paths depending on number of its matching edges.

▶ **Definition 2.2.3.** Let G be a connected, graph with a unique perfect matching. An alternating path in a graph G is called an *odd (even) alternating path* if it has an odd (even) number of matching edges. ◀

Let T be a nonsingular tree and v be a vertex in T. Let us denote the set of all even alternating paths in T that start at v by $\mathcal{A}^+_{T,v}$ or simply by $\mathcal{A}^+_v$, and the set of all odd alternating paths in T which start at v by $\mathcal{A}^-_{T,v}$ or simply by $\mathcal{A}^-_v$.

▶ **Definition 2.2.4.** Let T be a nonsingular tree and v be a vertex in T. Let us define the quantity $\mathtt{diff}_T(v)$ by

$$\mathtt{diff}_T(v) := |\mathcal{A}^+_{T,v}| - |\mathcal{A}^-_{T,v}|,$$

that is, $\mathtt{diff}_T(v)$ is obtained by subtracting the number of odd alternating paths at v from the number of even alternating paths at v in T. ◀

▶ **Example 2.2.5.** Let us consider the tree T in Example 2.1.6. Note that the set of all odd alternating paths of T that starts at the vertex 1 is given by

$$\mathcal{A}^-_1 = \{[1, 6], [1, 6, 2, 7, 5, 10], [1, 6, 2, 7, 3, 8]\}.$$

Similarly, the set of all even alternating paths of T that starts at the vertex 1 is given by

$$\mathcal{A}^+_1 = \{[1, 6, 2, 7], [1, 6, 2, 7, 3, 8, 9, 4]\}.$$

It follows that

$$\mathtt{diff}_T(1) = |\mathcal{A}^+_1| - |\mathcal{A}^-_1| = 2 - 3 = -1.$$

Let $\mathtt{diff}_T$ be the vector of size 10 such that the i-th entry of $\mathtt{diff}_T$ corresponds to $\mathtt{diff}_T(i)$. Then we have

$$\mathtt{diff}_T = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 \end{bmatrix}^t.$$ ◀

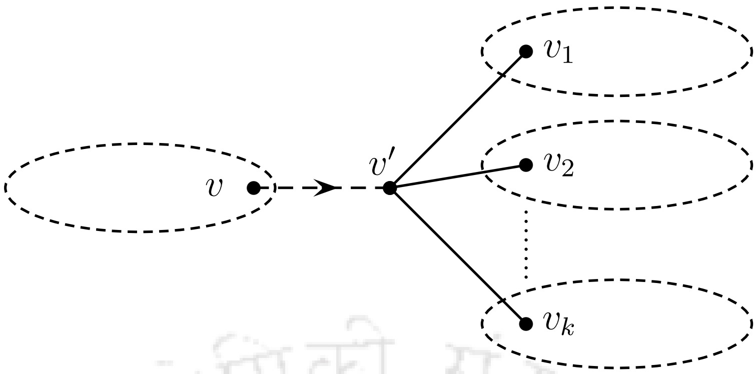


Figure 2.3: Figure for the proof of Lemma 2.2.6.

We can observe that if v is a quasi-pendant vertex of T then $\texttt{diff}_T(v) = -1$ as there is a single odd alternating path starting at v. The quantity $\texttt{diff}_T(v)$ plays a crucial role on understanding the bipartite distance index of a nonsingular tree. The following result supplies a recursive process to compute $\texttt{diff}_T(v)$ at any vertex $v \in T$.

► **Lemma 2.2.6.** *Let T be a nonsingular tree and v be a vertex of T. Suppose v' is the matching partner of v and $N(v')$ is the set of neighbors of v'. Then*

$$\texttt{diff}_T(v) = -1 - \textstyle\sum_{x \in N(v'), x \neq v} \texttt{diff}_T(x).$$ ◄

Proof. Let us assume that $d(v') = k+1$, $k \geq 0$. Suppose $N(v') = \{v, v_1, \ldots, v_k\}$. For an illustration see Figure 2.3. Note that all alternating path starting at the vertex v must pass through the vertex v' and so such paths contains a unique vertex from the set $N(v') \setminus \{v\}$ except the alternating path $[v, v']$. Therefore, an alternating path starting at the vertex v other than the path $[v, v']$ corresponds to an alternating path starting at some vertex v_i, $i = 1, \ldots, k$. Further note that an alternating path starting at v_i for some $i = 1, \ldots, k$ cannot pass through the vertex v' and so each alternating path starting at v_i also corresponds to an alternating path starting at v. Therefore there is a bijection map from $\mathcal{A}_{T,v} \setminus \{[v, v']\}$ to $\bigcup\limits_{i=1}^{k} \mathcal{A}_{T,v_i}$.

By P_i we mean a path starting at the vertex v_i and by $[v, v', P_i]$ we mean the path constructed from P_i by extending it to v through the vertex v'. It is easy to note that $P_i \in \mathcal{A}^-_{T,v_i}$ for some i if and only if $[v, v', P_i] \in \mathcal{A}^+_{T,v}$. Similarly, $P_i \in \mathcal{A}^+_{T,v_i}$ for some i if and

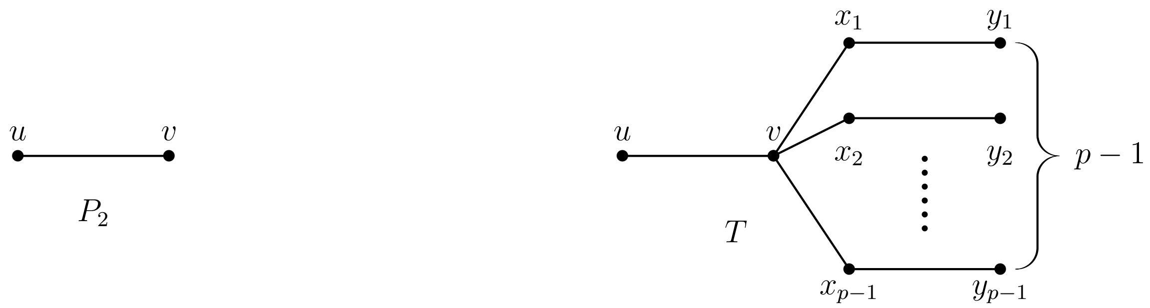


Figure 2.4: Tree constructed by attaching $p-1$ many P_2s at an end vertex of P_2.

only if $[v, v', P_i] \in \mathcal{A}^-_{T,v} \setminus \{[v, v']\}$. Since $\mathcal{A}_{T,v_i} \cap \mathcal{A}_{T,v_j} = \emptyset$ for each $i \neq j$, it follows that

$$|\mathcal{A}^+_{T,v}| = \sum_{i=1}^{k} |\mathcal{A}^-_{T,v_i}| \quad \text{and} \quad |\mathcal{A}^-_{T,v}| = \sum_{i=1}^{k} |\mathcal{A}^+_{T,v_i}| + 1$$

This completes the proof. ■

By using Lemma 2.2.6 we can easily compute the quantity $\texttt{diff}_T(v)$ starting from the quasi-pendant vertices because each vertex v is connected to some leaf of the tree by an alternating path starting at v. In the following example we illustrate Lemma 2.2.6.

▶ **Example 2.2.7.** Let $p > 1$. Consider the path $P_2 = [u, v]$. Now construct a tree T on $2p$ vertices by attaching $p-1$ new P_2 at the vertex v, as shown Figure 2.4.

Since x_i is a quasi-pendant vertex in T, it follows that $\texttt{diff}_T(x_i) = -1$, for each $i = 1, \ldots, p-1$. In a similar way we can conclude that $\texttt{diff}_T(v) = -1$. It is only remain to calculate $\texttt{diff}_T(u)$. Clearly, v is the matching partner of u and $N(v) = \{u, x_1, \ldots, x_{p-1}\}$. By Lemma 2.2.6, it follows that

$$\texttt{diff}_T(u) = -1 - \textstyle\sum_{i=1}^{p-1} \texttt{diff}_T(x_i) = -1 + (p-1) = p-2.$$

In a similar way, by applying Lemma 2.2.6, we see that

$$\texttt{diff}_T(y_i) = -1 - \sum_{z \in N(x_i), z \neq y_i} \texttt{diff}_T(z) = -1 - \texttt{diff}_T(v) = 0,$$

for each $i = 1, \ldots, p-1$. ◀

Let us consider the tree T as shown in Figure 2.4. We can note that $\texttt{diff}_T(u) = p-2$. Therefore, by taking a suitable value of p we can construct a tree T with $|\texttt{diff}_T(u)| > 1$. It follows that, for a nonsingular tree T and $v \in T$, the quantity $|\texttt{diff}_T(v)|$ can be more than one.

▶ **Definition 2.2.8.** Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). The vector $\boldsymbol{\tau}_T$, or simply $\boldsymbol{\tau}$, of size $2p$ is defined by

$$\boldsymbol{\tau}_T(v) := 1 - d(v)[1 + \mathtt{diff}_T(v)], \qquad \text{for each } v \text{ in } T. \tag{2.1}$$

The entries of $\boldsymbol{\tau}_T$ are ordered according to $l_1, \ldots, l_p, r_1, \ldots, r_p$. The restriction of $\boldsymbol{\tau}_T$ on L is denoted by $\boldsymbol{\tau}_T^l$ or simply by $\boldsymbol{\tau}^l$ and the restriction of $\boldsymbol{\tau}_T$ on R is denoted by $\boldsymbol{\tau}_T^r$ or simply by $\boldsymbol{\tau}^r$.

Note that $\boldsymbol{\tau}_T^r$ and $\boldsymbol{\tau}_T^l$ are vectors of size p whose i-th component is $\boldsymbol{\tau}_T(r_i)$ and $\boldsymbol{\tau}_T(l_i)$, respectively. ◀

▶ **Example 2.2.9.** Let us consider the tree T in Example 2.1.6. Note that $d(7)$, the degree of the vertex 7, is 3. By Example 2.2.5, we know that $\mathtt{diff}_T(7) = 0$. It follows that

$$\boldsymbol{\tau}_T(7) = 1 - d(7)[1 + \mathtt{diff}(7)] = 1 - 3(1 + 0) = -2$$

By Example 2.1.6, we know that $L = \{l_1 = 1, \ldots, l_5 = 5\}$, $R = \{r_1 = 6, \ldots, r_5 = 10\}$ is a standard vertex bipartition of T. Therefore, the restriction of $\boldsymbol{\tau}_T$ on L is given by

$$\boldsymbol{\tau}_T^l = \begin{bmatrix} \boldsymbol{\tau}_T(1) & \boldsymbol{\tau}_T(2) & \boldsymbol{\tau}_T(3) & \boldsymbol{\tau}_T(4) & \boldsymbol{\tau}_T(5) \end{bmatrix}^t = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \end{bmatrix}^t.$$

In a similar way, we can calculate the the restriction of $\boldsymbol{\tau}_T$ on R as

$$\boldsymbol{\tau}_T^r = \begin{bmatrix} \boldsymbol{\tau}_T(6) & \boldsymbol{\tau}_T(7) & \boldsymbol{\tau}_T(8) & \boldsymbol{\tau}_T(9) & \boldsymbol{\tau}_T(10) \end{bmatrix}^t = \begin{bmatrix} 1 & -2 & 1 & 0 & 1 \end{bmatrix}^t. \quad ◀$$

The understanding of $\boldsymbol{\tau}^r$ is crucial in our discussion.

▶ **Remark 2.2.10.** (a) Consider the tree T in Figure 2.5. It has a matching edge $[l_{k_1}, r_{k_1}]$. Ignore the arrow on it, for now. The other vertices that are adjacent to r_{k_1} are $l_{k_1+1}, l_{k_2+1}, \ldots, l_{k_{s-1}+1}$. When we delete the edges between r_{k_1} and these vertices, we obtain the following nonsingular trees: T_1 with vertex set $\{l_1, \ldots, l_{k_1}, r_1, \ldots, r_{k_1}\}$, T_2 with vertex set $\{l_{k_1+1}, \ldots, l_{k_2}, r_{k_1+1}, \ldots, r_{k_2}\}$, and so on up to T_s with vertex set $\{l_{k_{s-1}+1}, \ldots, l_{k_s}, r_{k_{s-1}+1}, \ldots, r_{k_s}\}$. For $i = 2, \ldots, s$, let F_i be the subtree of T obtained by taking T_1 and T_i and by inserting the edge $[r_{k_1}, l_{k_{i-1}+1}]$. These are nonsingular trees too.

To understand $\boldsymbol{\tau}^r$, let us put an arrow on the edge $[l_{k_1}, r_{k_1}]$ from r_{k_1} to l_{k_1}. This arrow indicates that, from a vertex r_i in T_2, we may have an alternating path to a vertex in T_1 or T_2 but we do not have an alternating path to a vertex in $T_3, \ldots, T_s$. Similarly, from a vertex r_i in T_3, we may have an alternating path to a vertex in T_1 or T_3 but we do not have an alternating path to a vertex in $T_2, T_4, T_5, \ldots, T_s$. Similar statements are true for vertices r_i

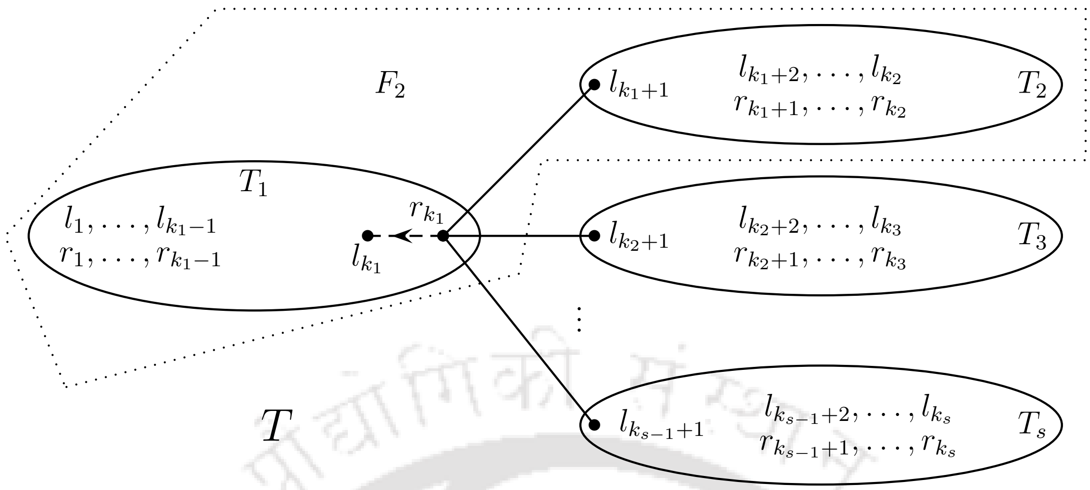


Figure 2.5: Understanding $\boldsymbol{\tau}_T^r$ at any matching edge.

in $T_4, \dots, T_s$. Also, from a vertex r_i in T_1, we only have alternating paths to vertices in T_1 but not to a vertex in $T_2, \dots, T_s$.

In this set up, for any $r_i \in F_2$, $r_i \neq r_{k_1}$, we see that $d_T(r_i) = d_{F_2}(r_i)$ and $\texttt{diff}_T(r_i) = \texttt{diff}_{F_2}(r_i)$. Hence,

$$\boldsymbol{\tau}_T(r_i) = \boldsymbol{\tau}_{F_j}(r_i) \qquad \text{if } r_i \in F_j,\ r_i \neq r_{k_1}.$$

(b) Recall that $A_{i,:}$ means the ith row of A. Let $\overline{\mathfrak{B}(F_2)_{k_1,:}}$ denote the row induced by $\mathfrak{B}(F_2)_{k_1,:}$ on $r_1, \dots, r_{k_s}$. That is, $\overline{\mathfrak{B}(F_2)_{k_1,:}}$ is obtained by inserting 0 entries into $\mathfrak{B}(F_2)_{k_1,:}$ at the places corresponding to $r_i \notin F_2$. Notice that $\overline{\mathfrak{B}(F_2)_{k_1,:}}(r_i) = \mathfrak{B}(F_2)_{k_1,:}(r_i)$ if $r_i \in F_2$ and $\overline{\mathfrak{B}(F_2)_{k_1,:}}(r_i) = 0$ if $r_i \notin F_2$. Then it is immediate from the structure that

$$\mathfrak{B}(T)_{k_1,:} = \overline{\mathfrak{B}(F_2)_{k_1,:}} + \cdots + \overline{\mathfrak{B}(F_s)_{k_1,:}} - (s-2)\overline{\mathfrak{B}(T_1)_{k_1,:}}. \qquad ◀$$

Let us illustrate Remark 2.2.10 with an example.

▶ **Example 2.2.11.** Consider the tree T, as shown in Figure 2.6. Edges in the perfect matching are shown as dashed lines.

Note that $[l_3, r_3]$ is a matching edge and the other vertices that are adjacent to r_3 is l_4, l_7. For the given tree T, we take $k_1 = 3$ and $k_2 = 6$ in Remark 2.2.10. Note that T_1, T_2, and T_3 are nonsingular tree with the vertex set $\{l_1, l_2, l_3, r_1, r_2, r_3\}$, $\{l_4, l_5, l_6, r_4, r_5, r_6\}$, and $\{l_7, l_8, r_7, r_8\}$, respectively. Further, F_2 is a subtree tree of T induced by $\{l_1, \dots, l_6, r_1, \dots, r_6\}$ and F_3 is a subtree of T induced by $\{l_1, l_2, l_3, l_7, l_8, r_1, r_2, r_3, r_7, r_8\}$.

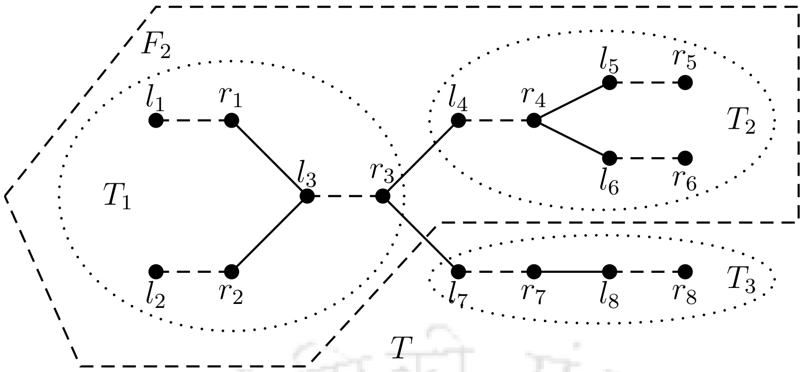


Figure 2.6: Understanding Remark 2.2.10.

a) By direct computation we get

$$\begin{aligned}
\begin{bmatrix}\boldsymbol{\tau}_T(r_1) & \cdots & \boldsymbol{\tau}_T(r_8)\end{bmatrix} &= \begin{bmatrix}1 & 1 & -5 & 4 & -1 & -1 & 3 & -1\end{bmatrix}\\
\begin{bmatrix}\boldsymbol{\tau}_{F_2}(r_1) & \cdots & \boldsymbol{\tau}_{F_2}(r_6)\end{bmatrix} &= \begin{bmatrix}1 & 1 & -3 & 4 & -1 & -1\end{bmatrix}\\
\begin{bmatrix}\boldsymbol{\tau}_{F_3}(r_1) & \boldsymbol{\tau}_{F_3}(r_2) & \boldsymbol{\tau}_{F_3}(r_3) & \boldsymbol{\tau}_{F_3}(r_7) & \boldsymbol{\tau}_{F_3}(r_8)\end{bmatrix} &= \begin{bmatrix}1 & 1 & -3 & & & & 3 & -1\end{bmatrix}
\end{aligned}$$

It follows that

$$\boldsymbol{\tau}_T(r_i) = \boldsymbol{\tau}_{F_j}(r_i) \qquad \text{if } r_i \in F_j,\ r_i \neq r_3.$$

b) First note that $\mathfrak{B}(F_2)$ is a bipartite distance matrix of F_2 and the size of $\mathfrak{B}(F_2)$ is 6×6. We create a row vector $\overline{\mathfrak{B}(F_2)_{3,:}}$ of size 1×8 whose entries are indexed by $r_1, \ldots, r_8$ with $\overline{\mathfrak{B}(F_2)_{3,:}}(r_i) = \mathfrak{B}(F_2)_{3,:}(r_i)$ if $r_i \in F_2$ and $\overline{\mathfrak{B}(F_2)_{3,:}}(r_i) = 0$ if $r_i \notin F_2$. In the similar way we calculate $\overline{\mathfrak{B}(F_3)_{3,:}}$ and $\overline{\mathfrak{B}(T_1)_{3,:}}$. It follows that

$$\begin{aligned}
\overline{\mathfrak{B}(F_2)_{3,:}} &= \begin{bmatrix}1 & 1 & 1 & 3 & 5 & 5 & 0 & 0\end{bmatrix}\\
\overline{\mathfrak{B}(F_3)_{3,:}} &= \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0 & 3 & 5\end{bmatrix}\\
\overline{\mathfrak{B}(T_1)_{3,:}} &= \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\\
\overline{\mathfrak{B}(T)_{3,:}} &= \begin{bmatrix}1 & 1 & 1 & 3 & 5 & 5 & 3 & 5\end{bmatrix}
\end{aligned}$$

Since $s = \deg(r_3) = 3$, from above it follows that

$$\mathfrak{B}(T)_{3,:} = \overline{\mathfrak{B}(F_2)_{3,:}} + \overline{\mathfrak{B}(F_3)_{3,:}} - (s-2)\overline{\mathfrak{B}(T_1)_{3,:}}. \qquad \blacktriangleleft$$

The following result helps us to describe the changes in $\mathfrak{B}(T)$ and $\boldsymbol{\tau}_T^r$ under the operation of attaching a new P_2 at some vertex of a nonsingular tree T.

▶ **Lemma 2.2.12.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Fix a vertex r_k. Let $\widehat{T}$ be obtained by adding two new vertices l_{p+1} and r_{p+1} and the edges $[v = r_k, l_{p+1}]$, $[l_{p+1}, r_{p+1}]$ to T.*

a) Then we have

$$\boldsymbol{\tau}^r_{\widehat{T}} = \left[\begin{array}{c} \\ \boldsymbol{\tau}^r_T \\ \\ \hline 0 \end{array}\right] - \left[\begin{array}{c} 0 \\ \vdots \\ 1 + \mathtt{diff}_T(v) \\ \vdots \\ 0 \\ \hline -1 - \mathtt{diff}_T(v) \end{array}\right] \begin{array}{c} \\ \\ \textit{position } k \\ \\ \\ \\ \end{array}$$

b) Consider the component of T containing l_k after deleting the edge $[l_k, r_k]$. Assume that the vertex set of this component is $\{l_1, r_1, \ldots, l_{k-1}, r_{k-1}, l_k\}$. Then

$$\begin{aligned} \mathfrak{B}(\widehat{T})_{p+1,:} \;\; = \;\; & \left[\begin{array}{ccccccc|c} & & & \mathfrak{B}(T)_{k,:} & & & & 0 \end{array}\right] \\ & + \left[\begin{array}{ccccccc|c} 2 & \cdots & 2 & 0 & 0 & \cdots & 0 & 1 \end{array}\right], \end{aligned}$$

where the entries 2 in the last vector are for the vertices $r_1, \ldots, r_{k-1}$. ◀

Proof. For a concrete illustration of Lemma 2.2.12, consider the tree T in Figure 2.5 and $v = r_k = r_{k_1}$.

a) We already know that $\boldsymbol{\tau}_{\widehat{T}}(r_i) = \boldsymbol{\tau}_T(r_i)$ if $r_i \neq v, r_{p+1}$. We also have $\mathtt{diff}_{\widehat{T}}(v) = \mathtt{diff}_T(v)$ and $d_{\widehat{T}}(v) = d_T(v) + 1$. Hence,

$$\boldsymbol{\tau}_{\widehat{T}}(v) = \boldsymbol{\tau}_T(v) - [1 + \mathtt{diff}_T(v)].$$

Furthermore, we have, $\mathtt{diff}_{\widehat{T}}(r_{p+1}) = -\,\mathtt{diff}_T(v) - 1$ and $d_{\widehat{T}}(r_{p+1}) = 1$. Hence,

$$\boldsymbol{\tau}_{\widehat{T}}(r_{p+1}) = 0 - [-1 - \mathtt{diff}_T(v)].$$

b) This follows, as $\mathtt{dist}(l_{p+1}, r_i) = \mathtt{dist}(l_k, r_i) + 2$ for $i = 1, \ldots, k-1$, $\mathtt{dist}(l_{p+1}, r_i) = \mathtt{dist}(l_k, r_i)$ for $i = k, k+1, \ldots, p$ and $\mathtt{dist}(l_{p+1}, r_{p+1}) = 1$. ■

▶ **Lemma 2.2.13.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Then $\mathbb{1}^t \boldsymbol{\tau}^r_T = 1$.* ◀

Proof. We proceed by induction. The base case follows from the fact that $\boldsymbol{\tau}^r_T(r_1) = 1$ for $T = P_2$. Let the result be true for all nonsingular trees with less than $2p$ vertices. Let $\widehat{T}$ be a nonsingular trees on $2p$ vertices. Then we can view the tree $\widehat{T}$ obtained from T by attaching a new P_2 at some vertex $v \in T$. By Lemma 2.2.12, we know that the sum of the entries of $\boldsymbol{\tau}_T^r$ is preserved when going from T to $\widehat{T}$. Therefore, the result follows by applying Lemma 2.2.12. ■

The following result will be used as the induction step for our main result of the section.

► **Lemma 2.2.14.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Fix the vertex $v = r_k$. Let $\widehat{T}$ be obtained by adding two new vertices l_{p+1} and r_{p+1} and the edges $[r_k, l_{p+1}]$, $[l_{p+1}, r_{p+1}]$ to T. Assume that $\mathtt{bd}(T) \neq 0$ and $\mathfrak{B}(T)\boldsymbol{\tau}_T^r = \mathtt{bd}(T)\mathbb{1}$. Then we have*

$$\mathfrak{B}(\widehat{T})\boldsymbol{\tau}_{\widehat{T}}^r = \mathtt{bd}(\widehat{T})\mathbb{1} \quad \text{and} \quad \mathtt{bd}(\widehat{T}) = \mathtt{bd}(T) + 2[1 + \mathtt{diff}_T(v)]. \qquad ◄$$

Proof. Taking $y(i) = \mathtt{dist}(l_{p+1}, r_i)$ and $x(i) = \mathtt{dist}(l_i, r_{p+1})$ for each $i = 1, 2, \ldots, p$, we have

$$\mathfrak{B}(\widehat{T}) = \begin{bmatrix} \mathfrak{B}(T) & x \\ y^t & 1 \end{bmatrix} \quad \text{and} \quad \mathfrak{B}(\widehat{T})(e_{p+1} - e_k) = \begin{bmatrix} 2\mathbb{1} \\ 0 \end{bmatrix}. \tag{2.2}$$

Let us consider the matrix $E_1 := I - e_k e_{p+1}^t$. Note that $\det(E_1) = 1$ and $\mathfrak{B}(\widehat{T})E_1 = \begin{bmatrix} \mathfrak{B}(T) & 2\mathbb{1} \\ y^t & 0 \end{bmatrix}$. By hypothesis, we have $\mathtt{bd}(T) \neq 0$ and $\mathfrak{B}(T)\boldsymbol{\tau}_T^r = \mathtt{bd}(T)\mathbb{1}$. Further, consider the matrix $E_2 := I - \frac{2}{\mathtt{bd}(T)} \begin{bmatrix} \boldsymbol{\tau}_T^r \\ 0 \end{bmatrix} e_{p+1}^t$. Then we have $\det(E_2) = 1$ and

$$\mathfrak{B}(\widehat{T})E_1E_2 = \begin{bmatrix} \mathfrak{B}(T) & 2\mathbb{1} \\ y^t & 0 \end{bmatrix} - \frac{2}{\mathtt{bd}(T)} \begin{bmatrix} \mathfrak{B}(T)\boldsymbol{\tau}_T^r \\ y^t\boldsymbol{\tau}_T^r \end{bmatrix} e_{p+1}^t = \begin{bmatrix} \mathfrak{B}(T) & 0 \\ y^t & -\frac{2}{\mathtt{bd}(T)} y^t \boldsymbol{\tau}_T^r \end{bmatrix}.$$

Hence $\det(\mathfrak{B}(\widehat{T})) = -\frac{2}{\mathtt{bd}(T)} y^t \boldsymbol{\tau}_T^r \det \mathfrak{B}(T) = (-2)^p y^t \boldsymbol{\tau}_T^r$, by definition. It follows that $\mathtt{bd}(\widehat{T}) = y^t \boldsymbol{\tau}_T^r$.

Consider the component H of T containing l_k after deleting all the edges incident at r_k except the edge $[l_k, r_k]$. (Thus, if r_k has degree one, then H would be T.) Assume that the vertex set of this component is $\{l_1, r_1, \ldots, l_{s-1}, r_{s-1}, l_k, r_k\}$. Then, by Lemma 2.2.12, we obtain

$$\begin{aligned} y^t = & \begin{bmatrix} & \mathfrak{B}(T)_{k,:} & \end{bmatrix} \\ & + \begin{bmatrix} 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{bmatrix}, \end{aligned} \tag{2.3}$$

where the entries 2 in the last vector are for the vertices $r_1, \ldots, r_{s-1}$.

By Remark 2.2.10, we note that $\tau_H(r_i) = \tau_T(r_i)$ for each $i = 1, \ldots, s-1$. Further note that $d_H(r_k) = 1$ and $\texttt{diff}_H(r_k) = \texttt{diff}_T(r_k)$. By Equation (2.1), we get

$$\tau_H(r_k) = 1 - d_H(r_k)[1 + \texttt{diff}_H(r_k)] = -\texttt{diff}_T(r_k).$$

Hence, by Lemma 2.2.13 and Equation (2.3), we have

$$\begin{aligned} \texttt{bd}(\widehat{T}) = y^t \boldsymbol{\tau}_T^r &= \texttt{bd}(T) + 2\mathbb{1}^t \boldsymbol{\tau}_H^r - 2\boldsymbol{\tau}_H(r_k) \\ &= \texttt{bd}(T) + 2 - 2(-\texttt{diff}_H(r_k)) \\ &= \texttt{bd}(T) + 2[1 + \texttt{diff}_H(r_k)]. \end{aligned}$$

As $\texttt{diff}_H(r_k) = \texttt{diff}_T(r_k)$, this establishes the second identity.

To show the first identity note that

$$\begin{aligned} \mathfrak{B}(\widehat{T})\boldsymbol{\tau}_{\widehat{T}}^r &= \begin{bmatrix} \mathfrak{B}(T) & x \\ y^t & 1 \end{bmatrix} \left(\begin{bmatrix} \boldsymbol{\tau}_T^r \\ \hline 0 \end{bmatrix} - \begin{bmatrix} [1 + \texttt{diff}_T(r_k)]\boldsymbol{e}_k \\ \hline -1 - \texttt{diff}_T(r_k) \end{bmatrix} \right) \\ &= \begin{bmatrix} \texttt{bd}(T) \\ \vdots \\ \texttt{bd}(T) \\ \hline \texttt{bd}(\widehat{T}) \end{bmatrix} - [1 + \texttt{diff}_T(r_k)] \begin{bmatrix} \mathfrak{B}(T) & x \\ y^t & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_k \\ \hline -1 \end{bmatrix} \\ &= \begin{bmatrix} \texttt{bd}(T) \\ \vdots \\ \texttt{bd}(T) \\ \hline \texttt{bd}(\widehat{T}) \end{bmatrix} - [1 + \texttt{diff}_T(r_k)]\mathfrak{B}(\widehat{T})(\boldsymbol{e}_k - \boldsymbol{e}_{p+1}) \\ &= \begin{bmatrix} \texttt{bd}(T) \\ \vdots \\ \texttt{bd}(T) \\ \hline \texttt{bd}(\widehat{T}) \end{bmatrix} + [1 + \texttt{diff}_T(r_k)] \begin{bmatrix} 2 \\ \vdots \\ 2 \\ 2 \\ \hline 0 \end{bmatrix} \quad \text{by Equation (2.2)} \\ &= \texttt{bd}(\widehat{T})\mathbb{1}, \end{aligned}$$

by the second identity. This completes the proof. ∎

This brings us to the main result of this section.

▶ **Theorem 2.2.15.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R).*

a) Then $\texttt{bd}(T)$ is an odd number and $\mathfrak{B}(T)\boldsymbol{\tau}_r(T) = \texttt{bd}(T)\mathbb{1}$.

b) Fix a vertex v and let $\widehat{T}$ be obtained by attaching a new P_2 at v. Then $\mathtt{bd}(\widehat{T}) = \mathtt{bd}(T) + 2[1 + \mathtt{diff}_T(v)]$. ◀

Proof. We proceed by induction. Let $p = 1$ and $T = P_2$. Let v be any arbitrary vertex. Then $\mathtt{diff}_v = -1$. It follows that $\boldsymbol{\tau}_T^r = [1]$. Since $\mathtt{bd}(T) = 1$, we see that $\mathfrak{B}(T)\boldsymbol{\tau}_T^r = \mathtt{bd}(T)\mathbb{1}$. So a) holds. In view of Remark 2.1.5, without loss assume that $v = r_p$ and $\widehat{T}$ be obtained by adding two new vertices l_{p+1} and r_{p+1} and the edges $[r_p, l_{p+1}]$, $[l_{p+1}, r_{p+1}]$ to T. So, by Lemma 2.2.14, b) holds. (Alternately, one can directly verify this from a P_4.)

Now suppose that the statements hold for p. Consider a nonsingular tree T' on $2p+2$ vertices. Then it is obtained from some tree T on $2p$ vertices by attaching a new P_2 at some vertex v, see Remark 2.1.12. Let (L, R) be a standard vertex bipartition of T such that $v = r_p$. Thus, T' is obtained by adding two new vertices l_{p+1} and r_{p+1} and the edges $[r_p, l_{p+1}]$, $[l_{p+1}, r_{p+1}]$ to T. By induction hypothesis, we have $\mathfrak{B}(T)\boldsymbol{\tau}_T^r = \mathtt{bd}(T)\mathbb{1}$. Hence by Lemma 2.2.14, we see that both a) and b) hold for T'. The proof is complete by induction. ■

By item (a) of Theorem 2.2.15, we know that the bipartite distance index of a nonsingular tree is always an odd number. In particular, the bipartite distance matrix of a nonsingular tree is always invertible.

► **Corollary 2.2.16.** *Let T be a nonsingular tree on $2p$ vertices. Then the bipartite distance index of T is an odd number and so the bipartite distance matrix of T is invertible.* ◀

► **Corollary 2.2.17.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\boldsymbol{\tau}_T^l$ be the restriction of $\boldsymbol{\tau}_T$ on L. Then* $(\boldsymbol{\tau}_T^l)^t\mathfrak{B}(T) = \mathtt{bd}(T)\mathbb{1}^t$. ◀

Proof. Let F be the tree obtained by interchanging l_i with r_i for each i. Then $\mathfrak{B}(F) = \mathfrak{B}(T)^t$ and $\boldsymbol{\tau}_F^r = \boldsymbol{\tau}_T^l$. Hence $\mathfrak{B}(F)\boldsymbol{\tau}_F^r = \mathtt{bd}(F)\mathbb{1} = \mathtt{bd}(T)\mathbb{1}$ and taking transpose, we get $(\boldsymbol{\tau}_T^l)^t\mathfrak{B}(T) = \mathtt{bd}(T)\mathbb{1}^t$. ■

The following is an immediate corollary to Theorem 2.2.15. It explains why the corona trees have $\mathtt{bd}(T) = 1$.

► **Corollary 2.2.18.** *Let T be a nonsingular tree on $2p$ vertices and v be a vertex in T such that $\mathtt{diff}_T(v) = -1$. If F is obtained by attaching a new P_2 at v in T, then* $\mathtt{bd}(F) = \mathtt{bd}(T)$.

In particular, since the corona trees are obtained from P_2 by attaching new P_2's at quasi-pendant vertices, repeatedly, and $\mathtt{bd}(P_2) = 1$, we see that the bipartite distance index of each corona tree is 1. ◀

► **Remark 2.2.19.** From Corollary 2.2.18, we know that the bipartite distance index of each corona tree is one. But the converse of Corollary 2.2.18 is not true in general. For example, consider the tree T_3 as shown in Figure 2.2. Clearly, T_3 is not a corona tree but $\texttt{bd}(T_3) = 1$, see Example 2.3.2. ◀

This is a comparison of the vector $\boldsymbol{\tau}^r$ with an already known vector μ in the literature.

▶ **Remark 2.2.20.** Consider a tree T on vertices $1, \dots, n$ and the classical distance matrix $D(T)$. Let μ denote the vector whose i-th entry is $\mu(i) = 2 - d(i)$, where $d(i)$ means the degree of i. A well known identity for distance matrix which is similar to Theorem 2.2.15 has been observed by Bapat, Kirkland and Neumann [BKN05]). It says $D(T)\mu = (n-1)\mathbb{1}$. So $D(T)(-\mu) = (1-n)\mathbb{1}$. Notice that, $1 - n = \det D(T)/(-2)^{n-2}$. This similar to defining $\texttt{bd}(T) = \det B(T)/(-2)^{p-1}$. ◀

▶ **Remark 2.2.21.** Theorem 2.2.15 explains why the bipartite distance index of P_{2k} were like $1, 1, 3, 3, 5, 5, \dots$. Assume k is odd and imagine a new P_2 at an end vertex v of P_{2k}. Notice that $\texttt{diff}(v) = -1$. Hence, by $\texttt{bd}(P_{2k+2}) = \texttt{bd}(P_{2k})$. If k is even, then we would get $\texttt{diff}(v) = 0$ which is even. Hence, $\texttt{bd}(P_{2k+2}) = \texttt{bd}(P_{2k}) + 2$ happens in this case. ◀

2.3 The inclusion-exclusion principle

In this section we first show the bipartite distance index $\texttt{bd}(T)$ of a nonsingular tree T satisfies an inclusion-exclusion type of principle at each matching edge. This in turn gives us a much simpler recursive formula to calculate the bipartite distance index of a nonsingular tree T from its subtrees. Before that let us recall that a *branch* at a vertex v of a graph G is a connected component of $G - v$. We do not include v in a branch at v. A branch at v is called a trivial branch if it has exactly one vertex. If B is a branch at v in G, then by $G - B$, we denote the graph obtained by deleting all the vertices of B from G (and so the edges incident with those vertices are also removed).

The following result shows that $\det \mathfrak{B}(T)$ satisfies an inclusion-exclusion type of principle at any matching edge of the tree T. Figure 1.2 illustrates why we refer Theorem 2.3.1 as an inclusion-exclusion principle.

▶ **Theorem 2.3.1 (Inclusion-Exclusion).** *Let T be a nonsingular tree on $2p$ vertices with a matching edge $[u, v]$ such that $s = d(v) \geq 3$. Let $w_1, \dots, w_{s-1}$ be the other vertices adjacent to v. Let B_{w_i} be the branches at v containing the vertex w_i, $i = 1, \dots, s-1$. Then we have*

$$\texttt{bd}(T) = \texttt{bd}(T - B_{w_1}) + \texttt{bd}(T - B_{w_2} - \cdots - B_{w_{s-1}}) - \texttt{bd}(T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}}). \blacktriangleleft$$

Proof. In view of Remark 2.1.5, let us assume that our tree is as shown in Figure 2.5 and that $u = l_{k_1}$, $v = r_{k_1}$, $w_i = l_{k_i+1}$, for $i = 1, \dots, s-1$. By Theorem 2.2.15, we know that

$\mathfrak{B}(T)\boldsymbol{\tau}_T^r = \mathtt{bd}(T)\mathbb{1}$. Thus
$$B(T)_{k_1,:}\boldsymbol{\tau}_T^r = \mathtt{bd}(T).$$

By Remark 2.2.10 b), we have
$$\mathfrak{B}(T)_{k_1,:} = \overline{\mathfrak{B}(F_2)_{k_1,:}} + \cdots + \overline{\mathfrak{B}(F_s)_{k_1,:}} - (s-2)\overline{\mathfrak{B}(T_1)_{k_1,:}}.$$

By Remark 2.2.10 a), we have $\boldsymbol{\tau}_T^r(r_i) = \boldsymbol{\tau}_{F_j}^r(r_i)$ if $r_i \in F_j, r_i \neq r_{k_1}$. Thus we have the complete information about $\boldsymbol{\tau}_T^r$ except at the vertex $v = r_{k_1}$.

Now consider r_{k_1}. We have $d_T(r_{k_1}) = s$, $d_{T_1}(r_{k_1}) = 1$ and for $j = 2, 3, \ldots, s$,
$$d_{F_j}(r_{k_1}) = 2, \qquad \mathtt{diff}_T(r_{k_1}) = \mathtt{diff}_{F_j}(r_{k_1}) = \mathtt{diff}_{T_1}(r_{k_1}).$$

It follows that
$$\begin{aligned} \boldsymbol{\tau}_T(r_{k_1}) &= -\big(s-1+s\ \mathtt{diff}_{T_1}(r_{k_1})\big) \\ \boldsymbol{\tau}_{F_j}(r_{k_1}) &= -\big(1+2\ \mathtt{diff}_{T_1}(r_{k_1})\big) \qquad j=2,\ldots,s \\ \boldsymbol{\tau}_{T_1}(r_{k_1}) &= -\big(\mathtt{diff}_{T_1}(r_{k_1})\big). \end{aligned}$$

Hence, for $j = 2, \ldots, s$, we have
$$\boldsymbol{\tau}_T(r_{k_1}) - \boldsymbol{\tau}_{F_j}(r_{k_1}) = -(s-2)(1+\mathtt{diff}_{T_1}(r_{k_1})) = (s-2)M,$$
where $M = -(1+\mathtt{diff}_{T_1}(r_{k_1}))$. We also have
$$\boldsymbol{\tau}_T(r_{k_1}) - \boldsymbol{\tau}_{T_1}(r_{k_1}) = (s-1)M.$$

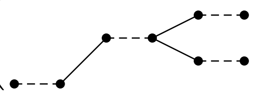
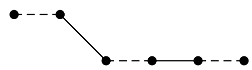
Therefore, the bipartite distance index of T is given by
$$\begin{aligned} \mathtt{bd}(T) &= \mathfrak{B}(T)_{k_1,:}\ \boldsymbol{\tau}_T^r = \overline{\mathfrak{B}(F_2)_{k_1,:}}\ \boldsymbol{\tau}_T^r + \cdots + \overline{\mathfrak{B}(F_s)_{k_1,:}}\ \boldsymbol{\tau}_T^r - (s-2)\overline{\mathfrak{B}(T_1)_{k_1,:}}\ \boldsymbol{\tau}_T^r \\ &= \mathfrak{B}(F_2)_{k_1,:}\ \boldsymbol{\tau}_{F_2}^R + (s-2)M + \cdots + \mathfrak{B}(F_s)_{k_1,:}\ \boldsymbol{\tau}_{F_s}^R + (s-2)M \\ &\qquad -(s-2)\Big(\mathfrak{B}(T_1)_{k_1,:}\ \boldsymbol{\tau}_{T_1}^R + (s-1)M\Big) \\ &= \mathtt{bd}(F_2) + \cdots + \mathtt{bd}(F_s) - (s-2)\,\mathtt{bd}(T_1). \end{aligned}$$

In a similar argument, we have $\mathtt{bd}(T - B_{w_1}) = \mathtt{bd}(F_3) + \cdots + \mathtt{bd}(F_s) - (s-3)\,\mathtt{bd}(T_1)$. Noting that $T - B_{w_2} - \cdots - B_{w_{s-1}} = F_2$ and $T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}} = T_1$, the conclusion follows. ∎

Let us illustrate Theorem 2.3.1 with an example.

▶ **Example 2.3.2.** Consider the nonsingular tree T, as shown in Figure 2.7. Edges in the perfect matching are shown as dashed lines.

Consider the vertex v in T. Note that $d(v) = 3$. By applying Theorem 2.3.1 we have

$\mathtt{bd}(T) = \mathtt{bd}\Big($  $\Big) + \mathtt{bd}\Big($  $\Big) - \mathtt{bd}\Big($ $\Big)$

$$= \Big(\mathtt{bd}(P_6) + \mathtt{bd}(P_6) - \mathtt{bd}(P_4)\Big) + \mathtt{bd}(P_6) - \mathtt{bd}(P_2)$$
$$= 3\,\mathtt{bd}(P_6) - \mathtt{bd}(P_4) - \mathtt{bd}(P_2) = 7.$$

Note also that using Theorem 2.3.1, we can see that the bipartite index of the tree T_3 in Figure 2.2 is 1 and the bipartite index of the tree T in Figure 2.6 is 3. ◀

Theorem 2.3.1 implies that the bipartite distance index of a nonsingular tree can be expressed as an integer combination of the bipartite distance indices of some paths of the form P_{2k} in a *realizable* way.

▶ **Corollary 2.3.3.** *Let T be a nonsingular tree on $2p$ vertices. Then there exist $a_i \in \mathbb{Z}$, for $i = 1, \ldots, p$ such that*

$$\mathtt{bd}(T) = a_1\,\mathtt{bd}(P_2) + a_2\,\mathtt{bd}(P_4) + \cdots + a_p\,\mathtt{bd}(P_{2p}).$$ ◀

In the below result we express the bipartite distance index of a corona tree as an integer combination of the bipartite distance indices of paths P_4 and P_2.

▶ **Lemma 2.3.4.** *Let T be a corona tree on $2p$ vertices ($p > 1$). Then*

$$\mathtt{bd}(T) = (p-1)\,\mathtt{bd}(P_4) - (p-2)\,\mathtt{bd}(P_2).$$ ◀

Proof. We prove the statement by induction on p. Clearly the statement holds for $p \le 2$. Assume the statement to be true for $p > 2$ and consider a corona tree T on $2p$ vertices. Let $P = [v_1, v_2, \ldots, v_{m-1}, v_m]$ be a path corresponding to the diameter of T. It follows by Lemma 2.1.11 that

$$\mathtt{bd}(T) = \mathtt{bd}(T - v_1 - v_2)$$

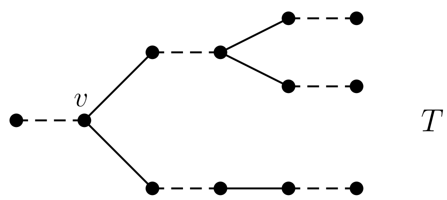


Figure 2.7: Understanding Theorem 2.3.1

We now apply induction hypothesis on the tree $T - v_1 - v_2$ to obtain the required result. ■

2.4 The f-alternating sums

In this section we study a new type of combinatorial object, known as f-alternating sum. Recall that $\mathcal{A}_G$ denotes the class of all alternating paths in G.

▶ **Definition 2.4.1 (f-alternating sum).** Let $S : \mathbb{N} \to \mathbb{R}$ be a sequence. Let G be bipartite graph with a unique perfect matching. The f-alternating sum $f_S(G)$ of G (with respect to the sequence S) is defined as

$$f_S(G) := \sum_{\substack{P \in \mathcal{A}_G \\ P=[u,\cdots,v]}} [d(u) - 2][d(v) - 2] S\Big(\frac{|P|}{2}\Big). \quad ◀$$

Note that from the definition it follows that the f-alternating sum of a path P is $f_S(P) = S(|P|/2)$ if P is an alternating path and otherwise it is 0. We first show that the function $f_S(T)$ satisfies the same recurrence relation as that of $\mathtt{bd}(T)$ in Theorem 2.3.1.

▶ **Theorem 2.4.2.** *Let $S : \mathbb{N} \to \mathbb{R}$ be a sequence and T be a nonsingular tree on $2p$ vertices with a matching edge $[u, v]$ such that $s = d(v) \geq 3$. Let $w_1, \ldots, w_{s-1}$ be the other vertices adjacent to v. Let B_{w_i} be the branches at v containing the vertex w_i, $i = 1, \ldots, s-1$. Then we have*

$$f_S(T) = f_S(T - B_{w_1}) + f_S(T - B_{w_2} - \cdots - B_{w_{s-1}}) - f_S(T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}}). \quad ◀$$

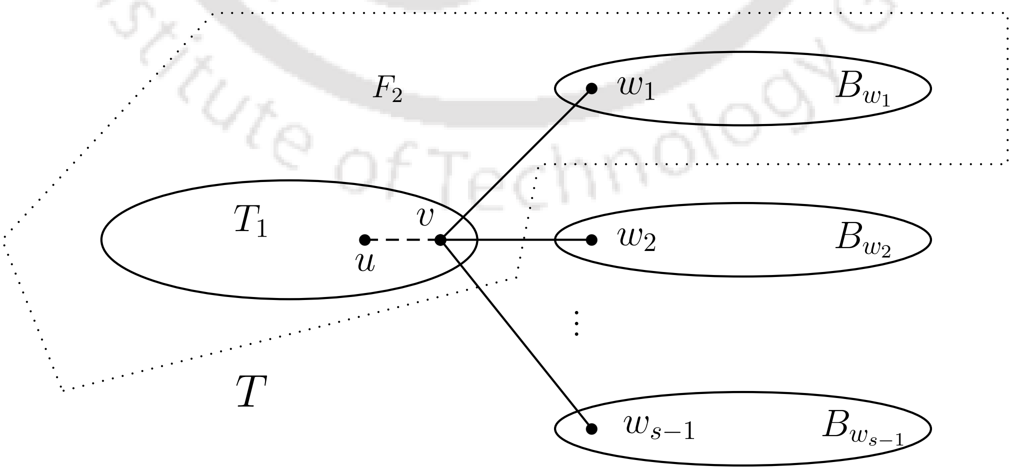


Figure 2.8: For the proof of Theorem 2.4.2.

Proof. Let $T_1 = T - B_{w_1} + \cdots - B_{w_{s-1}}$. For $j = 2, 3, \ldots, s$, by F_j denote the tree obtained from T_1 and $B_{w_{j-1}}$ by adding the edge $[v, w_{j-1}]$. For a reference, see Figure 2.8.

Notice that, if $j \neq l$, then F_j and F_l have T_1 in common. It follows that $\mathcal{A}_T = \bigcup_{j=2}^{s} \mathcal{A}_{F_j}$. By P_{ij}, let us denote the unique path between vertices i and j in T. Hence we have

$$\begin{aligned} f_S(T) &= \sum_{\substack{P\in\mathcal{A}_T \\ P=[i,\ldots,j]}} [d(i)-2][d(j)-2] f_S(P_{i,j}) \\ &= \sum_{\substack{P\in\mathcal{A}_{F_2} \\ P=[i,\ldots,j]}} [d(i)-2][d(j)-2] f_S(P_{i,j}) + \cdots + \sum_{\substack{P\in\mathcal{A}_{F_s} \\ P=[i,\ldots,j]}} [d(i)-2][d(j)-2] f_S(P_{i,j}) \\ &\quad -(s-2) \sum_{\substack{P\in\mathcal{A}_{T_1} \\ P=[i,\ldots,j]}} [d(i)-2][d(j)-2] f_S(P_{i,j}) \\ &= f_S(F_2) + \cdots + f_S(F_s) - (s-2) f_S(T_1). \end{aligned}$$

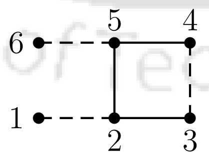
In a similar way, $f_S(T - B_{w_1}) = f_S(F_3) + \cdots + f_S(F_s) - (s-3) f_S(T_1)$. The proof now follows by noting that $T - B_{w_2} - \cdots - B_{w_{s-1}} = F_2$. ■

In the following remark we see that how the f-alternating sum of a bipartite graph G with a unique perfect matching gives structural information of the graph G.

▶ **Remark 2.4.3.** a) Take $S = (1, 1, 1, \ldots)$. Then, by definition, $f_S(P_{2p}) = 1$ and by applying Theorem 2.4.2, we see that $f_S(T) = 1$ for each T.

b) Take $S = (2, 4, 6, \ldots)$. Then $f_S(P_{2p}) = 2p$ and by applying Theorem 2.4.2, we see that $f_S(T) = |T|$ is the order of T.

c) Let G be a graph with a unique perfect matching. Although we can define the f-alternating sum of G but the identities obtained in a) and b) need not remain valid. For example consider the graph G below.



G

The only paths that contribute to the alternating sums are $[1,2], [5,6], [1,2,3,4,5,6]$. Hence, for $S = (1, 1, 1, \ldots)$ we have $f_S(G) = -1$ and for $S = (1, 2, 3, \ldots)$ we have $f_S(G) = 1$.

d) It would be interesting to know the class of graphs G with unique perfect matching that have $f_S(G) = 1$ in a) and $f_S(G) = |G|$ in b).

e) It would be interesting to know the properties of the f-alternating sums of nonsingular trees, for some more sequences S. ◀

2.4.1 Combinatorial formula for the bipartite distance index

In this subsection we prove that $\mathtt{bd}(T)$ is nothing but the f-alternating sum for the sequence $S = (1, 1, 3, 3, 5, 5, \ldots)$.

▶ **Theorem 2.4.4.** *Let T be a nonsingular tree on $2p$ vertices and $S = (1, 1, 3, 3, 5, 5, \ldots)$ be a sequence. Then*

$$\mathtt{bd}(T) = f_S(T).$$ ◀

Proof. We prove the result by induction on the number of vertices in T. For $p = 1$ or $p = 2$, the only possible nonsingular tree is a path on $2p$ vertices. This case follows from Proposition 2.2.1, as the nonzero contributions to $f_S(T)$ will come only from the end vertices.

Now suppose that the result is true for every nonsingular tree with number of vertices less than $2p$. Let T be a nonsingular tree on $2p$ vertices. Note that if T is a path then the result follows by Proposition 2.2.1. Suppose that T is not a path. Then there exists a vertex v in T such that $d(v) \geq 3$. Let $[u, v]$ be the matching edge involving v. Let $w_1, \ldots, w_{s-1}$ be the other vertices adjacent to v. Let B_{w_i} be the branches at v containing the vertex w_i, $i = 1, \ldots, s-1$. Then by Theorems 2.3.1 and 2.4.2 we have

$$\begin{aligned}
\mathtt{bd}(T) &= \mathtt{bd}(T - B_{w_1}) + \mathtt{bd}(T - B_{w_2} - \cdots - B_{w_{s-1}}) - \mathtt{bd}(T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}}) \\
&= f_S(T - B_{w_1}) + f_S(T - B_{w_2} - \cdots - B_{w_{s-1}}) - f_S(T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}}) \\
&= f_S(T).
\end{aligned}$$

The result now follows by induction. ■

2.5 Structure independence of the bipartite distance index

A well known result by Graham, Hoffman and Hosoya [GHH77] is that the determinant of the distance matrix of a graph only depends on the blocks and is independent of how they are assembled. In a similar way, we want to construct classes of nonsingular trees such that $\mathtt{bd}(T)$ remains the same for each class.

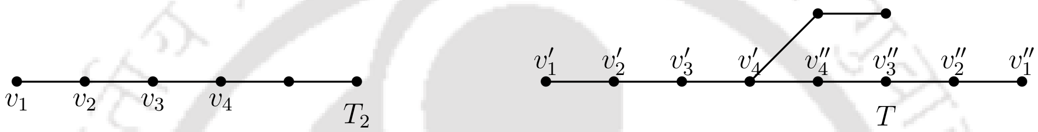
▶ **Definition 2.5.1 (Pendant Path).** Let k be an even natural number. An alternating path $[u_1, \ldots, u_k]$ in a nonsingular tree T is called a *pendant path*, if $d(u_1) = 1$, $d(u_i) = 2$, $i = 2, \ldots, k-1$ and $d(u_k) =\geq 2$. ◀

▶ **Definition 2.5.2 (Merging of two trees).** Let T and T' be two nonsingular tree with the pendant paths $P = [v_1, v_2, \ldots, v_k]$ and $P' = [v'_1, v'_2, \ldots, v'_k]$, respectively. By the merging T

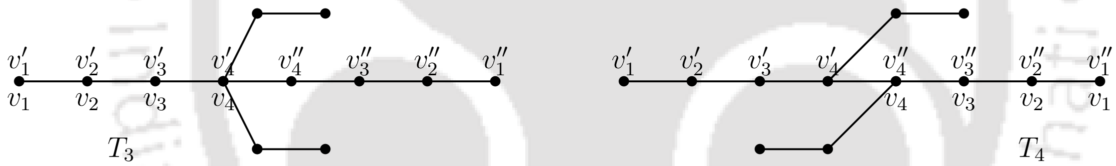
and T' along these pendant paths, we mean the new tree obtained by identifying the the vertices v_i with v_i' for each $i = 1, \ldots, k$, while keeping the rest of the vertices distinct. ◀

An example of merging is provided in Figure 2.2. If we want to merge two trees T and T' along pendant paths of k vertices, we may be able to do so in more than one way and the resulting trees could be nonisomorphic.

▶ **Example 2.5.3.** Consider the trees T and T_2 from Figure 2.2 below. Let us label the pendant paths on four vertices in both the trees. Clearly $[v_1, v_2, v_3, v_4]$ is a pendant path in T_2 on 4 vertices. Also note that $[v_1', v_2', v_3', v_4']$ and $[v_1'', v_2'', v_3'', v_4'']$ are pendant paths in T on 4 vertices.



If we want to merge T and T_2 along pendant paths of 4 vertices, then we have two possible choices and we obtain two trees T_3 and T_4, shown below.



◀

▶ **Definition 2.5.4.** Let T and T' be two nonsingular trees on the same number of vertices. Let k be an even natural number. By $T + T' - P_k$, we denotes the set of all trees that can be obtained by merging trees T and T' along pendant paths of k vertices. This set will be treated empty if one of T or T' does not have a pendant path of k vertices. ◀

In Example 2.5.3, we see that $T + T_2 - P_4$ contains two nonsingular trees T_3 and T_4.

▶ **Definition 2.5.5.** Let $\mathcal{F} = \{P_{k_1}, \ldots, P_{k_n}\}$ and $\mathcal{G} = \{P_{r_1}, \ldots, P_{r_{n-1}}\}$ be two multisets consisting path of even order, $n \geq 2$. Then by

$$P_{k_1} + \cdots + P_{k_n} - P_{r_1} - \cdots - P_{r_{n-1}}$$

we denote all the trees that can be obtained by repeated application of the following rules in all possible ways and reaching $\mathcal{G} = \emptyset$.

a) Select a path P_{r_i} from $\mathcal{G}$. Select two trees T and T' from $\mathcal{F}$ that have pendant paths of r_i vertices.

b) Merge T and T' along some pendant paths of r_i vertices. Let the resulting tree be T_r.

c) Delete T and T' from $\mathcal{F}$ and put T_r in $\mathcal{F}$. Delete P_{r_i} from $\mathcal{G}$.

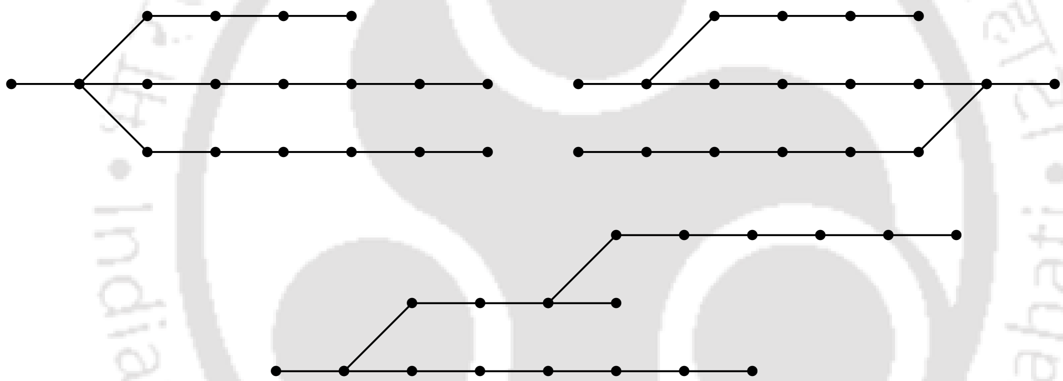
Note that when $\mathcal{G} = \emptyset$, the set $\mathcal{F}$ will contain exactly one tree which is an element of the class $\mathcal{F} - \mathcal{G}$.

If step a) fails at some stage, then we immediately set $\mathcal{F} = \emptyset$ and break the loop. This means, we just collect the trees for those cases, for which we could reach $\mathcal{G} = \emptyset$.

For brevity, let us also use the notation $\mathcal{F} - \mathcal{G}$ to denote the class $P_{k_1} + \cdots + P_{k_n} - P_{r_1} - \cdots - P_{r_{n-1}}$. Also, we allow $\mathcal{F} = \{P_{k_1}\}, \mathcal{G} = \emptyset$, in which case $\mathcal{F} - \mathcal{G} = \{P_{k_1}\}$. ◀

Let us look at an example below.

▶ **Example 2.5.6.** Let us take $\mathcal{F} = \{P_8, P_8, P_6\}$ and $\mathcal{G} = \{P_2, P_2\}$. The expression $\mathcal{F} - \mathcal{G} = P_8 + P_8 + P_6 - P_2 - P_2$ means the class of the following trees:



◀

Note that the class $\mathcal{F} - \mathcal{G}$ can be empty. By construction, each tree in $\mathcal{F} - \mathcal{G}$ has the same order and has a perfect matching. The following result supplies a condition for $\mathcal{F} - \mathcal{G}$ to be nonempty.

▶ **Proposition 2.5.7.** *Let $\mathcal{F} = \{P_{k_1}, \ldots, P_{k_n}\}$ and $\mathcal{G} = \{P_{r_1}, \ldots, P_{r_{n-1}}\}$ be two sets of even ordered paths, $n \geq 2$. Assume that $k_1 \leq k_2 \leq \cdots \leq k_n$ and $r_1 \leq r_2 \leq \cdots \leq r_{n-1}$. Then the class $\mathcal{F} - \mathcal{G}$ is nonempty if and only if $k_i > r_i$ for $i = 1, \ldots, n-1$.* ◀

Proof. Suppose that $k_i > r_i$ for $i = 1, \ldots, n-1$. Let $T_{n-1} \in P_{k_n} + P_{k_{n-1}} - P_{r_{n-1}}$. Notice that T_{n-1} has a pendant path of r_{n-1} vertices. Let $T_{n-2} \in T_{n-1} + P_{k_{n-2}} - P_{r_{n-2}}$. Notice that T_{n-2} has a pendant path of r_{n-2} vertices. Repeating the process a few times, we obtain $T_1 \in \mathcal{F} - \mathcal{G}$, where $T_1 \in T_2 + P_{k_1} - P_{r_1}$.

Conversely, suppose that $\mathcal{F} - \mathcal{G} \neq \emptyset$. Let $T \in \mathcal{F} - \mathcal{G}$. Suppose that $\mathcal{G}$ contains t paths $g_1, \ldots, g_t$ of number of vertices at least s and suppose that these paths are used in that order while generating T. At the stage of g_t, we must have used two trees in $\mathcal{F}$ which had pendant path copies of g_t in them. So at this stage, we have at least two trees in $\mathcal{F}$ that have pendant paths of order at least s. Similarly, at the stage of g_{t-1}, we must have used two trees in $\mathcal{F}$ which had pendant path copies of g_{t-1} in them. So at this stage, we have at least three trees in $\mathcal{F}$ that have pendant paths of order at least s. Continuing, at the stage of g_1, we must have at least $t+1$ trees in $\mathcal{F}$ that have pendant paths of order at least s. This is possible if from the beginning we have at least $t+1$ paths in $\mathcal{F}$ that have order at least $s+1$. The conclusion follows by putting $s = r_{n-1}, r_{n-2}, \ldots, r_1$. ■

▶ **Remark 2.5.8.** Let T_1 be a nonsingular tree with a pendant path $P = [u_1, \ldots, u_{2k}]$, $d_{T_1}(u_1) = 1$, and T_2 be a nonsingular tree with a pendant path $P' = [u'_1, \ldots, u'_{2k}]$, $d_{T_2}(u'_1) = 1$. Let T be formed by merging T_1 and T_2 along P and P'. The following statements are immediate from the definition.

a) The number of leaves (pendant vertices) in T equals to those in T_1 plus those in T_2 minus one.

b) The degree of any vertex v in T_1, $v \neq u_{2k}$, remains unchanged in the process and the the degree of any vertex v in T_2, $v \neq u'_{2k}$, remains unchanged in the process. Also $d_T(u_{2k}) = d_{T_1}(u_{2k}) + d_{T_2}(u'_{2k}) - 1$.

c) Consider the class X of all alternating paths in T that have pendant endvertices. Let F^*_T be the multiset of paths obtained by removing the labels from the paths in X. Then

$$F^*_T = F^*_{T_1} \dot{\cup} F^*_{T_2},$$

where $\dot{\cup}$ means the multiset union, that is, if $H \in F^*_{T_1}$ three times and $H \in F^*_{T_2}$ two times then $H \in F^*_T$ five times.

d) Consider the class Y of all alternating paths in T that have one endvertex pendant and the other of degree more than 2. Let G^*_T be the multiset of paths obtained by replacing each path P in Y that has an endvertex of degree $d > 2$ with $d-2$ copies of unlabeled P. Then

$$G^*_T = G^*_{T_1} \dot{\cup} G^*_{T_2} \dot{\cup} \{P_{2k}\}.$$ ◀

▶ **Lemma 2.5.9.** *Let $n \in \mathbb{N}$ and $\mathcal{F} = \{P_{k_1}, \ldots, P_{k_n}\}$ and $\mathcal{G} = \{P_{r_1}, \ldots, P_{r_{n-1}}\}$ be two sets consisting paths of even order and $T \in \mathcal{F} - \mathcal{G}$. Then $F^*_T = \mathcal{F}$ and $G^*_T = \mathcal{G}$.* ◀

Proof. The statement is true for $n = 1$. Assume that it is true for each $n < m$. Let $\mathcal{F} = \{P_{k_1}, \ldots, P_{k_m}\}$ and $\mathcal{G} = \{P_{r_1}, \ldots, P_{r_{m-1}}\}$ be two sets consisting paths of even order and $T \in \mathcal{F} - \mathcal{G}$.

Suppose that T was created by merging T_1 and T_2 along pendant copies of P_{2k}. Since T_1 was already created, it must have been created using some sub-multisets $\mathcal{F}_1$ and $\mathcal{G}_1$. That is, $T_1 \in \mathcal{F}_1 - \mathcal{G}_1$. Similarly, let $T_2 \in \mathcal{F}_2 - \mathcal{G}_2$. Since, by the time of the creation of T, $\mathcal{F}$ is singleton and $\mathcal{G}$ is empty, we see that

$$\mathcal{F} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2 \quad \text{and} \quad \mathcal{G} = \mathcal{G}_1 \dot{\cup} \mathcal{G}_2 \dot{\cup} \{P_{2k}\}.$$

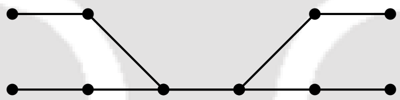
By induction hypothesis, $F^*_{T_1} = \mathcal{F}_1$ and $F^*_{T_2} = \mathcal{F}_2$. By Remark 2.5.8 d), we have

$$F^*_T = F^*_{T_1} \dot{\cup} F^*_{T_2} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2 = \mathcal{F}.$$

Similarly, $G^*_T = \mathcal{G}$. The proof is complete by induction. ■

2.5.1 Path decomposition

Let us first ask the following question: *Given a nonsingular tree T, can we find multisets $\mathcal{F}$ and $\mathcal{G}$ of even order paths such that T is in the class $\mathcal{F} - \mathcal{G}$?* The answer is in the negative. The following nonsingular tree is an example. Notice that, in the below tree, there are two degree three vertices and there is no pendant path having an end vertex is one of those degree three vertex. Therefore, we can not find any $\mathcal{F}$ and $\mathcal{G}$ of even order paths such that the below tree is in the class $\mathcal{F} - \mathcal{G}$.



Now focus on the question: *Given a nonsingular tree T, suppose that we can we find multisets $\mathcal{F}$ and $\mathcal{G}$ of even order paths such that $T \in \mathcal{F} - \mathcal{G}$. Are $\mathcal{F}$ and $\mathcal{G}$ unique?* The answer is in the affirmative. We shall call $(\mathcal{F}, \mathcal{G})$ (when it exists), the *path decomposition* of T.

▶ **Proposition 2.5.10 (Path decomposition).** *Let $\mathcal{F}, \mathcal{G}$ be multisets consisting paths of even order and $T \in \mathcal{F} - \mathcal{G}$. Then, the decomposition $(\mathcal{F}, \mathcal{G})$ is unique.* ◀

Proof. Suppose that $T \in \mathcal{F} - \mathcal{G}$ and $T \in \mathcal{F}' - \mathcal{G}'$. Then by Lemma 2.5.9, we have $\mathcal{F} = F^*_T = \mathcal{F}'$ and $\mathcal{G} = G^*_T = \mathcal{G}'$. ■

In the following result we observe that the f-alternating sum of all trees belongs to the class $\mathcal{F} - \mathcal{G}$ are same. That is, if the class $\mathcal{F} - \mathcal{G}$ is nonempty, then each tree in the class has the same f-value and this value can also be given by another expression using the f-values of the paths in $\mathcal{F}$ and $\mathcal{G}$.

▶ **Proposition 2.5.11.** *Let $\mathcal{F} = \{P_{k_1}, \dots, P_{k_n}\}$ and $\mathcal{G} = \{P_{r_1}, \dots, P_{r_{n-1}}\}$ be two sets consisting paths of even order, $n \geq 2$. Let $f : \mathbb{N} \to \mathbb{R}$. Then $f(T)$ remains the same for each $T \in \mathcal{F} - \mathcal{G}$. In fact, this common value can be given by a simpler expression*

$$f_S(T) = \sum_{P_i \in \mathcal{F}} f_S(P_i) - \sum_{P_j \in \mathcal{G}} f_S(P_j). \qquad ◀$$

Proof. Let $T \in \mathcal{F} - \mathcal{G}$. Suppose that T was created by merging T_1 and T_2 along pendant copies of P_{2k}. Then in view of Theorem 2.4.2, we have

$$f_S(T) = f_S(T_1) + f_S(T_2) - f_S(P_{2k}).$$

Since T_1 was already created, it must have been created using some submultisets $\mathcal{F}_1$ and $\mathcal{G}_1$. That is, $T_1 \in \mathcal{F}_1 - \mathcal{G}_1$. Similarly, let $T_2 \in \mathcal{F}_2 - \mathcal{G}_2$. Since, by the time of the creation of T, $\mathcal{F}$ is singleton and $\mathcal{G}$ is empty, we see that

$$\mathcal{F} = \mathcal{F}_1 \dot\cup \mathcal{F}_2 \quad \text{and} \quad \mathcal{G} = \mathcal{G}_1 \dot\cup \mathcal{G}_2 \dot\cup \{P_{2k}\}.$$

Applying induction hypothesis on the size of $\mathcal{F}$, we see that

$$f_S(T_1) = \sum_{P_i \in \mathcal{F}_1} f_S(P_i) - \sum_{P_j \in \mathcal{G}_1} f(P_j), \qquad f_S(T_2) = \sum_{P_i \in \mathcal{F}_2} f_S(P_i) - \sum_{P_j \in \mathcal{G}_2} f(P_j).$$

Hence

$$f_S(T) = f_S(T_1) + f_S(T_2) - f_S(P_{2k}) = \sum_{P_i \in \mathcal{F}} f(P_i) - \sum_{P_j \in \mathcal{G}} f_S(P_j).$$

The proof is complete. ■

In view of Proposition 2.5.11, we shall write $f_S(T_1 + T_2 - P_k) = f_S(T_1) + f_S(T_2) - f_S(P_k)$, with the understanding that the identity is true for each tree in the class $T_1 + T_2 - P_k$.

As an immediate corollary, we obtain a simpler formula for the determinant $\det \mathfrak{B}(T)$ for the trees that are in some $\mathcal{F} - \mathcal{G}$.

▶ **Corollary 2.5.12.** *Let $\mathcal{F} = \{P_{k_1}, \dots, P_{k_n}\}$ and $\mathcal{G} = \{P_{r_1}, \dots, P_{r_{n-1}}\}$ be two sets consisting paths of even order, $n \geq 2$. Then $\mathtt{bd}(T)$ and $\det \mathfrak{B}(T)$ remain the same for each $T \in \mathcal{F} - \mathcal{G}$. These common values are given by*

$$\mathtt{bd}(T) = \sum_{P_i \in \mathcal{F}} f_S(P_i) - \sum_{P_j \in \mathcal{G}} f(P_j), \qquad \det \mathfrak{B}(T) = (-2)^{p-1}\, \mathtt{bd}(T),$$

where $2p = k_1 + \cdots + k_n - r_1 - \cdots - r_{n-1}$. ◀

We conclude the section with an illustration.

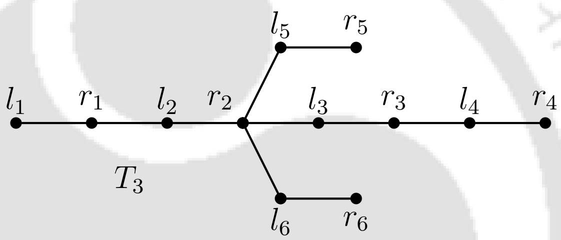
▶ **Example 2.5.13.** Consider the the tress T_3 and T_4 in Example 2.5.3. Note that both the trees T_3 and T_4 belong to the class $\mathcal{F} - \mathcal{G}$, where $\mathcal{F} = \{P_8, P_6, P_6\}$ and $\mathcal{G} = \{P_4, P_4\}$. Hence by Corollary 2.5.12,

$$\texttt{bd}(T_3) = \texttt{bd}(T_4) = f(4) + f(3) + f(3) - f(2) - f(2) = 3 + 3 + 3 - 1 - 1 = 7,$$

as $S = (1, 1, 3, 3, 5, 5, \ldots)$. Since the order $2p = 12$, we have $p - 1 = 5$. Hence

$$\det \mathfrak{B}(T_3) = \det \mathfrak{B}(T_4) = (-2)^5 \times 7 = -224.$$

We can compute $\det \mathfrak{B}(T_3)$ by first taking a standard vertex bipartition. We have taken one as shown in the picture.



For this standard bipartition the matrix $\mathfrak{B}(T_3)$ is

$$\begin{bmatrix} 1 & 3 & 5 & 7 & 5 & 5 \\ 1 & 1 & 3 & 5 & 3 & 3 \\ 3 & 1 & 1 & 3 & 3 & 3 \\ 5 & 3 & 1 & 1 & 5 & 5 \\ 3 & 1 & 3 & 5 & 1 & 3 \\ 3 & 1 & 3 & 5 & 3 & 1 \end{bmatrix}.$$

It can be checked that the determinant is indeed -224. Of course, one could verify the same for T_4, too. ◀

3 The bipartite Laplacian matrix

In Chapter 2, we proved that the bipartite distance matrix of a nonsingular tree is always invertible and its determinant can be described using the structure of T. What about the inverse of the bipartite distance matrix of a nonsingular tree? Can the entries of the inverse be described combinatorially?. In this chapter we supply a combinatorial description of the inverse of the bipartite distance matrix and establish identities that are similar to some well known identities. The study leads us to an unexpected generalization of the usual Laplacian matrix of a graph. This generalized Laplacian matrix is usually not a symmetric matrix but it still has many properties like the usual Laplacian matrix.

We introduce the bipartite Laplacian matrix of a nonsingular tree in Section 3.1. In Section 3.2 we study elementary properties of the bipartite Laplacian matrix and compare them with those of the usual Laplacian matrix. We then continue our study in Section 3.3 by finding upper bounds and lower bounds on the multiplicities of eigenvalues of the bipartite Laplacian matrix. The inverse of the bipartite distance matrix has been discussed in Section 3.4.

3.1 Preliminaries

We start the discussion by defining the bipartite Laplacian matrix of a nonsingular tree.

▶ **Definition 3.1.1 (The bipartite Laplacian matrix).** Let T be a nonsingular tree on $2p$ vertices and (L, R) be a standard vertex bipartition of T. The *bipartite Laplacian matrix* of T, denoted by $\mathfrak{L}(T)$ or simply by $\mathfrak{L}$ is the $p \times p$ matrix whose rows are indexed by $r_1, \ldots, r_p$ and the columns are indexed by $l_1, \ldots, l_p$. The (i, j)th entry of $\mathfrak{L}(T)$ is defined as

$$\mathfrak{L}(i,j) = \begin{cases} d(r_i)d(l_i) - 1 & \text{if } i = j; \\ d(r_i)d(l_j) & \text{if } i \neq j \text{ and the } r_i\text{-}l_j \text{ path is an odd alternating path;} \\ -d(r_i)d(l_j) & \text{if } i \neq j \text{ and the } r_i\text{-}l_j \text{ path is an even alternating path;} \\ -1 & \text{if } i \neq j \text{ and } r_i \sim l_j; \\ 0 & \text{otherwise.} \end{cases}$$

Sometimes we use $\mathfrak{L}(r_i, l_j)$ to denote the (i, j)th entry of $\mathfrak{L}$. ◀

Here we want to recall that the rows of the bipartite distance matrix $\mathfrak{B}(T)$ of T are indexed by $l_1, \ldots, l_p$ and the columns of $\mathfrak{B}(T)$ are indexed by $r_1, \ldots, r_p$ whereas the rows of the bipartite Laplacian matrix $\mathfrak{L}(T)$ of T are indexed by $r_1, \ldots, r_p$ and the columns of $\mathfrak{L}(T)$ are indexed by $l_1, \ldots, l_p$. [2]

▶ **Remark 3.1.2.** Let T be a nonsingular tree with a standard vertex bipartition (L, R). The bipartite Laplacian matrix of T can also be defined as $\mathbb{D} - \mathbb{A}^t$, where $\mathbb{A}$ is the bipartite adjacency matrix of T and $\mathbb{D}$ be the square matrix of order p whose (i, j)th entry is defined as

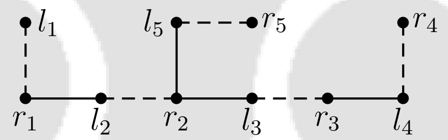
$$\mathbb{D}(i,j) = \begin{cases} d(r_i)d(l_j) & \text{if the } r_i\text{-}l_j \text{ path is an odd alternating path;} \\ -d(r_i)d(l_j) & \text{if the } r_i\text{-}l_j \text{ path is an even alternating path;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that here we consider $\mathbb{A}^t$ because by Definition 1.2.1, we indexed rows of $\mathbb{A}$ by $l_1, \ldots, l_p$ and indexed columns of $\mathbb{A}$ by $r_1, \ldots, r_p$. ◀

Let us first look at an example below.

▶ **Example 3.1.3.** Consider the nonsingular tree T as shown below.

T



Here the dashed edges are the matching edges. Clearly, $L = \{l_1, \ldots, l_5\}$, $R = \{r_1, \ldots, r_5\}$ is the standard vertex bipartition. The bipartite distance matrix $\mathfrak{B}$ and the bipartite Laplacian matrix $\mathfrak{L}$ are given by

$$\mathfrak{B} = \begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{array} \begin{array}{c} \begin{array}{ccccc} r_1 & r_2 & r_3 & r_4 & r_5 \end{array} \\ \begin{bmatrix} 1 & 3 & 5 & 7 & 5 \\ 1 & 1 & 3 & 5 & 3 \\ 3 & 1 & 1 & 3 & 3 \\ 5 & 3 & 1 & 1 & 5 \\ 3 & 1 & 3 & 5 & 1 \end{bmatrix} \end{array}, \qquad \mathfrak{L} = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{array} \begin{array}{c} \begin{array}{ccccc} l_1 & l_2 & l_3 & l_4 & l_5 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -3 & 5 & -1 & 0 & -1 \\ 2 & -4 & 3 & -1 & 0 \\ -1 & 2 & -2 & 1 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Note that here $\mathfrak{L}$ is not a symmetric matrix. ◀

2 We adopt this type of indexing for the bipartite Laplacian matrix so that we do not have to use transpose while giving an expression to the inverse of the bipartite distance matrix.

For a nonsingular tree T, we can have many standard vertex bipartitions depending on the labeling of its vertices. However, we can see that the bipartite Laplacian matrices corresponding to them are similar to each other.

▶ **Remark 3.1.4.** Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R).

(a) Let F be the tree obtained by interchanging the labels of l_i with r_i for all $i = 1, \ldots, p$ in T. Then $\mathfrak{L}(T) = \mathfrak{L}(F)^t$. Therefore, $\mathfrak{L}(T)$ and $\mathfrak{L}(F)$ are similar matrices.

(b) Let F be the tree obtained by relabeling the vertices within the part L, keeping it a standard vertex bipartition (that is, we also relabel the respective vertices in R accordingly). Then $\mathfrak{L}(T)$ is permutation similar to $\mathfrak{L}(F)$. ◀

We now introduce the concept of a signed degree vector at a vertex which is required to relate the structure of the bipartite Laplacian of the new tree with that of the old one.

▶ **Definition 3.1.5.** Let T be a nonsingular tree on $2p$ vertices with the standard vertex bipartition (L, R) and v be a vertex. Then the *signed degree vector* $\boldsymbol{\mu}_v$ at v is defined in the following way.

1. If $v \in L$, then for $i = 1, \ldots, p$, we define
   i) $\boldsymbol{\mu}_v(i) = d_T(r_i)$ if the v-r_i path is an odd alternating path,
   ii) $\boldsymbol{\mu}_v(i) = -d_T(r_i)$ if the v-r_i path is an even alternating path, and
   iii) $\boldsymbol{\mu}_v(i) = 0$ if the v-r_i path is not an alternating path.
2. In a similar way, if $v \in R$, then for $i = 1, \ldots, p$, we define
   i) $\boldsymbol{\mu}_v(i) = d_T(l_i)$ if the v-l_i path is an odd alternating path,
   ii) $\boldsymbol{\mu}_v(i) = -d_T(l_i)$ if the v-l_i path is an even alternating path, and
   iii) $\boldsymbol{\mu}_v(i) = 0$ if the v-l_i path is not an alternating path. ◀

▶ **Example 3.1.6.** Consider the tree T in Example 3.1.3. Then the signed degree vector of T at l_2 and at r_3 is given by

$$\boldsymbol{\mu}_{l_2} = \begin{bmatrix} 0 & 3 & -2 & 1 & -1 \end{bmatrix}^t \quad \text{and} \quad \boldsymbol{\mu}_{r_3} = \begin{bmatrix} 1 & -2 & 2 & 0 & 0 \end{bmatrix}^t. \qquad ◀$$

The signed degree vector at a vertex v has the property that the sum of its entries is always one.

▶ **Lemma 3.1.7.** *Let T be a nonsingular tree on $2p$ vertices with a standard bipartition (L, R). Let u be any vertex in T and $\boldsymbol{\mu}_u$ be the signed degree vector at u. Then $\mathbb{1}^t \boldsymbol{\mu}_u = 1$.* ◀

Proof. We proceed by induction on $p \geq 1$. For $p = 1$ the result is trivial. Assume the result to be true for nonsingular trees with less than $2p$ vertices. Let T be a a nonsingular tree on $2p$ vertices with a standard bipartition (L, R). Let $u \in R$. (The case of $u \in L$ can be dealt similarly.) Let $\boldsymbol{\mu}$ be the signed degree vector of u in T.

Suppose $[v_0, v_1, \ldots, v_k]$ is a longest path in T. As $p > 1$, we have $k \geq 3$ and so we may assume that $v_0, v_1 \neq u$. As T is nonsingular and this is a longest path, we have $d(v_0) = 1$ and $d(v_1) = 2$. Without loss of any generality, let us assume $v_0, v_1 \in \{l_p, r_p\}$. Let $\widehat{T} = T - \{v_0, v_1\}$ be the tree obtained from T by removing the vertices v_0 and v_1. Clearly, $u \in \widehat{T}$. Let $\widehat{\boldsymbol{\mu}}$ be the signed degree vector of u in $\widehat{T}$. Note that $\widehat{\boldsymbol{\mu}}$ is vector of size $p-1$. Clearly, $d_T(v) = d_{\widehat{T}}(v)$ for each $v \in \widehat{T} - v_2$ and $d_T(v_2) = d_{\widehat{T}}(v_2) + 1$. It follows that $\boldsymbol{\mu}(i) = \widehat{\boldsymbol{\mu}}(i)$ for each $l_i \in L \setminus \{v_2\}$.

If either $v_2 \in R$ or the u-v_2 path is not an alternating path then $\boldsymbol{\mu}^t = \begin{bmatrix} \widehat{\boldsymbol{\mu}}^t & 0 \end{bmatrix}$ and the result follows by induction. Now we assume that $v_2 \in L$ and the u-v_2 path is an alternating path. Then $v_2 \sim r_p$ and $d_T(l_p) = 1$. Let $v_2 = l_k$ for some $1 \leq k < p$. Note that the u-l_p path is also an alternating path and so we have $\widehat{\boldsymbol{\mu}}(k) = (-1)^t d_{\widehat{T}}(v_2)$ for some t and $\boldsymbol{\mu}(p) = (-1)^{t+1}$. Since $\boldsymbol{\mu}(k) = (-1)^t d_T(v_2) = \widehat{\boldsymbol{\mu}}(k) + (-1)^t$, it follows that

$$\boldsymbol{\mu}^t = \begin{bmatrix} \widehat{\boldsymbol{\mu}}^t & (-1)^{t+1} \end{bmatrix} + (-1)^t \begin{bmatrix} \boldsymbol{e}_k^t & 0 \end{bmatrix}.$$

Hence the result follows by induction. ■

3.2 Basic properties

In the following result we relate the structure of the bipartite Laplacian $\mathfrak{L}$ of the new tree with that of the old one.

► **Lemma 3.2.1.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\widehat{T}$ be the tree obtained from T by attaching a new P_2 at v. Let $\boldsymbol{\mu}_v$ be the signed degree vector at v of T.*

(a) If $v = l_k$ for some k, then $\mathfrak{L}(\widehat{T}) = \begin{bmatrix} \mathfrak{L}(T) + \boldsymbol{\mu}_v \boldsymbol{e}_k^t & -\boldsymbol{\mu}_v \\ -\boldsymbol{e}_k^t & 1 \end{bmatrix}$.

(b) If $v = r_k$ for some k, then $\mathfrak{L}(\widehat{T}) = \begin{bmatrix} \mathfrak{L}(T) + \boldsymbol{e}_k \boldsymbol{\mu}_v^t & -\boldsymbol{e}_k \\ -\boldsymbol{\mu}_v^t & 1 \end{bmatrix}$. ◄

Proof. We only provide the proof of item (a) as the proof of item (b) can be dealt in a similar way. Without loss of any generality, let us assume that $\widehat{T}$ obtained from T by adding a new path $[l_k, r_{p+1}, l_{p+1}]$ for some $1 \leq k \leq p$. Clearly $\widehat{L} = L \cup \{l_{p+1}\}$ and $\widehat{R} = R \cup \{r_{p+1}\}$ is a standard vertex bipartition of $\widehat{T}$. Let $\mathfrak{L}$ and $\widehat{\mathfrak{L}}$ be the bipartite Laplacian matrix of T and $\widehat{T}$, respectively. Since $[r_{p+1}, l_{p+1}]$ is the only alternating path that starts at r_{p+1} and $d_{\widehat{T}}(r_{p+1}) = 2$

with $[r_{p+1}, l_k]$ is not a matching edge, it follows that the all entries of the $(p+1)$th row of $\widehat{\mathfrak{L}}$ is zero except $\widehat{\mathfrak{L}}(p+1, p+1) = d_{\widehat{T}}(r_{p+1})d_{\widehat{T}}(l_{p+1}) - 1 = 1$ and $\widehat{\mathfrak{L}}(p+1, k) = -1$. Hence $\widehat{\mathfrak{L}}(p+1, :) = \begin{bmatrix} -\boldsymbol{e}_k^t & 1 \end{bmatrix}$.

Let us take $i = 1, \ldots, p$. Then $r_i \nsim l_{p+1}$. Note that the l_k-r_i path is an odd alternating path if and only if the l_{p+1}-r_i path is an even alternating path. Similarly, the l_k-r_i path is an even alternating path if and only if the l_{p+1}-r_i path is an odd alternating path. Since $d_{\widehat{T}}(l_{p+1}) = 1$, it follows that $\widehat{\mathfrak{L}}(\{1, \ldots, p\}, p+1) = -\boldsymbol{\mu}_{l_k}$, where $\boldsymbol{\mu}_{l_k}$ is the signed degree vector of T at l_k.

Since $d_T(u) = d_{\widehat{T}}(u)$ for each $u \in T$ other than l_k, it follows that $\widehat{\mathfrak{L}}(i, j) = \mathfrak{L}(i, j)$ for each $i = 1, \ldots, p$ and $j = 1, \ldots, k-1, k+1, p$.

Finally, notice that $d_{\widehat{T}}(l_k) = d_T(l_k) + 1$. Therefore, for $i = 1, \ldots, p$, we have

$$\widehat{\mathfrak{L}}(i, k) = \begin{cases} d_T(r_i)(d_T(l_k)+1) - 1 & \text{if } i = k; \\ d_T(r_i)(d_T(l_k)+1) & \text{if } i \neq k \text{ and the } r_i\text{-}l_k \text{ path is an odd alternating path;} \\ -d_T(r_i)(d_T(l_k)+1) & \text{if } i \neq k \text{ and the } r_i\text{-}l_k \text{ path is an even alternating path;} \\ -1 & \text{if } i \neq k \text{ and } r_i \sim l_k; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\widehat{\mathfrak{L}}(\{1, \ldots, p\}, k) = \mathfrak{L}(\{1, \ldots, p\}, k) + \boldsymbol{\mu}_{l_k}$. This completes the proof. ■

We now recall a well known result. It can be found in [Bap14, Lemma 4.2], for example.

▶ **Lemma 3.2.2.** *Let M be a square matrix of order n with zero row and column sums. Then the cofactors of any two elements of M are equal.* ◀

Proof. We shall first show that $(-1)^{i+j}M(i|j) = (-1)^{i+k}M(i|k)$ for all i, j, and k. Without loss of generality, let us assume $j > k$. Since each row sum of M is zero, it follows that $M(x, k) = \sum_{\substack{t=1 \\ t \neq k}}^{n} M(x, t)$ for each $x = 1, \ldots, n$. Let $\overline{M(i|j)}$ be the matrix obtained from $M(i|j)$ by adding all columns of $M(i|j)$ to the column k of $M(i|j)$. Notice that if $k = j-1$ then $\det \overline{(M(i|j))} = -\det M(i|k)$, otherwise by shifting column k to the right of column $i-1$ we can see that

$$\det M(i|j) = (-1)^{j-1-k}(-1) \det M(i|k) = (-1)^{j+k} \det M(i|k).$$

Now by applying a similar argument to M^t we can show that $(-1)^{i+j}M(i|j) = (-1)^{i+k}M(k|j)$ for all i, j, and k. This completes the proof. ■

Below we list some elementary properties of the bipartite Laplacian matrix and compare them with those of the usual Laplacian matrix, whenever possible.

Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Suppose $\mathfrak{L}$ is the bipartite Laplacian matrix of T. Then the following assertions hold.

(a) The row and the column sums of $\mathfrak{L}$ are zero. (A similar property also holds for the usual Laplacian matrix of any graph.)

(b) The cofactors of any two elements of $\mathfrak{L}$ are equal to one. (For the usual Laplacian matrix of a graph, the cofactors of any two elements are equal to the number of spanning trees.)

(c) The rank of $\mathfrak{L}$ is $n-1$. (A similar result is also true for the usual Laplacian matrix of a connected graph.)

(d) The algebraic multiplicity of 0 as an eigenvalue of $\mathfrak{L}$ is one. (A similar result is also true for the usual Laplacian matrix of a connected graph.)

(e) If $\boldsymbol{u}$ is an eigenvector of $\mathfrak{L}$ corresponding to an eigenvalue $\lambda \neq 0$ then $\mathbb{1}^t\boldsymbol{u} = 0$. (A similar property also holds for the usual Laplacian matrix of any graph.)

(f) If $T = F \circ K_1$ for some tree F on p vertices. Then $\mathcal{L}(F) = \mathfrak{L}(T)$ where $\mathcal{L}$ is the usual Laplacian matrix of F. (That is, the usual Laplacian matrix of a tree F can be seen as a bipartite Laplacian matrix of another tree T.)

(g) The matrix $\mathfrak{L}$ is a symmetric matrix if and only if T is a corona tree, that is, if $T = F \circ K_1$ for some tree F. (Whereas, the usual Laplacian matrix is always a symmetric matrix.) ◀

Proof of Item (a). The result follows from Lemma 3.2.1 and Lemma 3.1.7.

Proof of Item (b). By Lemma 3.2.2, all cofactors of $\mathfrak{L}$ are equal. We only need to show that they are equals to one. We proceed by induction on $p \geq 2$. The statement is valid for $p = 2$. Assume it holds for all nonsingular trees with vertices less than $2p$. Let T be a nonsingular tree with $2p$ vertices. Without loss of any generality, we assume that l_p is a pendant vertex in T and $d(r_p) = 2$. Let r_p be adjacent to l_k, for some $1 \leq k < p$. Let T' be the tree obtained from T by deleting vertices l_p and r_p from T. Let $\mathfrak{L}'$ be the bipartite Laplacian matrix of T'. Consider a vertex r_i such that $1 \leq i < p$. By Lemma 3.2.1, it follows that

$$\det \mathfrak{L}(i \mid p) = (-1)^{p+k} \det \mathfrak{L}'(i \mid k) = (-1)^{i+p}.$$

The last identity follows by induction hypothesis. Hence the result follows.

Proof of Item (c). By item (a), 0 is an eigenvalue of $\mathfrak{L}$. The remaining part of the proof follows from item (b).

Proof of Item (d). By item (a), the characteristic polynomial $\chi(x) := \det(xI - \mathfrak{L}(T)) = xf(x)$, where $f(x)$ is some polynomial with integer coefficients. By item (b) it follows that

$$\begin{aligned} f(0) &= \text{coefficient of } x \text{ in } \chi(x) \\ &= (-1)^{p-1}\times \text{ (sum of the principal minors of } \mathfrak{L} \text{ of size } p-1) \\ &= (-1)^{p-1}p. \end{aligned}$$

This shows that the algebraic multiplicity of 0 is 1.

Proof of Item (e). It directly follows from the fact that $\mathbb{1}^t\mathfrak{L} = \mathbf{0}$.

Proof of Item (f). Let us assume that $T = F \circ K_1$ for some tree F on p vertices. Note that for each $v \in F$ we have $d_T(v) > 1$ and there exist a leaf $u \notin F$ adjacent to v in T. Therefore for each $i = 1, \ldots, p$, exactly one of l_i and r_i is in F. We label the vertex $v \in F$ by v_i if v is adjacent to a pendant vertex l_i or r_i in T. Now note that the (i,i)th entry of $\mathfrak{L}$ is $d_T(l_i)d_T(r_i) - 1 = d_T(v_i) - 1 = d_F(v_i)$.

Now consider $i \neq j$. Since each edge in F is a nonmatching edge in T, all v_i-v_j paths in T are not an alternating path in T. Therefore, if $v_i \nsim v_j$ then both the r_i-l_j path and the r_j-l_i path are not an alternating path and so $\mathfrak{L}(i,j) = \mathfrak{L}(j,i) = 0$. Suppose $v_i \sim v_j$. Without loss of any generality, let us assume $v_i = l_i$ then $v_j = r_j$. It follows that the r_i-l_j path is an alternating path of length three and $r_j \sim l_i$. Therefore, $\mathfrak{L}(i,j) = -d_T(r_i)d_T(l_j) = -1$ and $\mathfrak{L}(j,i) = -1$. This shows that $\mathfrak{L} = \mathcal{L}$.

Proof of Item (g). Let T be not a corona tree. Then there exist an alternating path $P = [v_0, v_1, v_2, v_3]$ such that $d(v_0) = 1$, $d(v_3) > 1$. By taking $v_0 = l_1$ and $v_2 = l_2$ we see that the $(1,2)$th entry of $\mathfrak{L}$ is -1 but $(2,1)$th entry of $\mathfrak{L}$ is $-d(v_3) \neq -1$. This completes the proof. ■

3.3 Spectra of the bipartite Laplacian matrix

By item (f) of Theorem 3.2.3, all eigenvalues of the bipartite Laplacian matrix of a corona tree are nonnegative real numbers. An interesting question now arises: *Is it true that all eigenvalues of the bipartite Laplacian of each nonsingular tree are nonnegative real numbers?* Let us investigate it for the tree T as in Example 3.1.3. Let $\mathfrak{L}$ be the bipartite Laplacian matrix of the tree T as in Example 3.1.3. Let $f(x) = \det(xI - \mathfrak{L})$ be the characteristic polynomial of $\mathfrak{L}$. By a direct computation, we see that $f(x) = x(x^4 - 11x^3 + 31x^2 - 24x + 5)$. Clearly 0 is an eigenvalue of $\mathfrak{L}$. We can further observe that $f(1/4) > 0$, $f(1/2) < 0$, $f(1) > 0$, $f(3) < 0$, and $f(10) > 0$. Therefore, all eigenvalues of $\mathfrak{L}$ are nonnegative real numbers whereas $\mathfrak{L}$ is not a symmetric matrix. This is quite interesting. It seems that it is true for the bipartite Laplacian matrix of each nonsingular tree.

► **Conjecture 3.3.1.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Then all eigenvalues of the bipartite Laplacian matrix are nonnegative real numbers.* ◄

In Appendix, we show that the above conjecture is true for all tree on at most 10 vertices.

Let G be a graph and $\mathcal{L}(G)$ be its usual Laplacian matrix. By $p(G)$ and $q(G)$, denote the number of pendant vertices and the number of quasipendant vertices of G, respectively. Faria [Far85] in 1985 observed that the multiplicity of 1 as an eigenvalue of $\mathcal{L}(G)$ at least $p(G) - q(G)$. (This number was called the star degree of a graph G in [Far85].)

▶ **Theorem 3.3.2.** [Far85] *Let G be a connected graph and $\mathcal{L}(G)$ be the Laplacian matrix of G. Then the multiplicity of* 1 *as an eigenvalue of $\mathcal{L}(G)$ is at least $p(G) - q(G)$.* ◀

In 1990, Grone, Merris and Sunder gave an upper bound on the multiplicity of any eigenvalue of the usual Laplacian matrix of a tree.

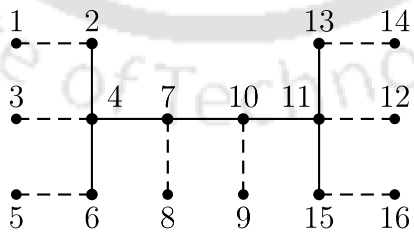
▶ **Theorem 3.3.3.** [GMS90, Theorem 2.3] *Let T be an any tree on $n \geq 2$ vertices and $\mathcal{L}(T)$ be the usual Laplacian matrix of T. Then the multiplicity of any eigenvalue of $\mathcal{L}(T)$ is at most $p(T) - 1$.* ◀

Interestingly, such bounds can also be provided for the bipartite Laplacian matrix of a nonsingular tree. In order to describe that, we need some more terminologies.

▶ **Definition 3.3.4.** Let T be a nonsingular tree. By a *pendant-two-path* at w, we mean a path $[u, v, w]$ in T such that $d_T(u) = 1$ and $d_T(v) = 2$. (As T is nonsingular, it follows that $d_T(w) > 1$.) If $[u, v, w]$ is a pendant-two-path at w then the vertex w is referred as the root of that pendant-two-path.

By $P(T)$ let us denote the total number of pendant-two-paths (at all vertices) in T. By $Q(T)$ let us denote the total number of vertices at which at least one pendant-two-paths are available (that is, number of roots of pendant-two-paths). ◀

Let us illustrate Definition 3.3.4 by considering the tree T as shown below.



Notice that $[1, 2, 4]$ and $[5, 6, 4]$ are two pendant-two-paths at 4. Similarly, $[14, 13, 11]$ and $[16, 15, 11]$ are two pendant-two-paths at 11. These are the only four pendant-two-paths in T. It follows that $P(T) = 4$. As 4 and 11 are only two vertices at which some pendant-two-path exist in T, it follows that $Q(T) = 2$.

Below we supply a result similar to the previously mentioned Faria's result for the bipartite Laplacian matrix of a nonsingular tree. Note that we do not yet know whether the bipartite

Laplacian matrix is diagonalizable or not. In Appendix, we show that the bipartite Laplacian matrix of nonsingular tree on at most 10 vertices is diagonalizable.

▶ **Theorem 3.3.5.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R) and $\mathfrak{L}$ be the bipartite Laplacian matrix of T. Then the geometric multiplicity of 1 as an eigenvalue of $\mathfrak{L}$ is at least $P(T) - Q(T)$, where $P(T)$ and $Q(T)$ are defined above.* ◀

Proof. Note that if $P(T) = 0$, then the result holds trivially. So, we assume that $P(T) \geq 1$. Let w be a vertex in T at which the available pendant-two-paths are $[u_i, v_i, w]$, $i = 1, \ldots, k$. If $k = 1$ then the contribution of w to $P(T) - Q(T)$ is zero. So assume that $k > 1$.

Let $\bar{T}$ be the tree obtained from T by removing the vertices $u_1, v_1, \ldots, u_k, v_k$. For $i = 1, \ldots, k$, let T_i be the tree obtained from T by removing just the two vertices u_i and v_i from T. Let $\boldsymbol{\mu}_i$ be the signed degree vector at v of T_i and let $\boldsymbol{\mu}$ be the signed degree vector at v of $\bar{T}$. Without loss of any generality, let us assume that the vertex set of $\bar{T}$ is $\{l_1, \ldots, l_n, r_1, \ldots, r_n\}$. Note that $\boldsymbol{\mu}_i = \begin{bmatrix} \boldsymbol{\mu}^t & 0 & \cdots & 0 \end{bmatrix}^t$ for each $i = 1, \ldots, k$, as the edges joining w and any of $u_1, v_1, \ldots, u_k, v_k$ are nonmatching.

Note that for $1 \leq i \leq k$, $u_i, z_i \in \{r_{j_i}, l_{j_i}\}$ for some $j_i > n$. By Lemma 3.2.1, we have $\mathfrak{L} = \left[\begin{array}{c|c} * & -\boldsymbol{\mu}\mathbb{1}^t \\ \hline * & I \end{array}\right]$ if $w \in L$ and $\mathfrak{L} = \left[\begin{array}{c|c} * & -\boldsymbol{e}_q\mathbb{1}^t \\ \hline * & I \end{array}\right]$ if $w \in R$, where I is an identity matrix and $1 \leq q \leq n$. It follows that $\mathfrak{L}(\boldsymbol{e}_{j_i} - \boldsymbol{e}_{j_t}) = \boldsymbol{e}_{j_i} - \boldsymbol{e}_{j_t}$ for $1 \leq i < t \leq k$. This completes the proof. ■

Next, we supply an upper bound on the geometric multiplicity of an eigenvalue of the bipartite Laplacian matrix of a nonsingular tree. The proof is similar to the proof of Theorem 3.3.3 given in [GMS90].

▶ **Theorem 3.3.6.** *Let T be a nonsingular tree on $2p$ vertices, $p > 1$, with a standard vertex bipartition (L, R). Let $\mathfrak{L}$ be the bipartite Laplacian matrix of T and let λ be an eigenvalue of $\mathfrak{L}$. Then the geometric multiplicity of λ is at most $P(T) - 1$.* ◀

Proof. We first prove the result for $P(T) = 2$. Let us assume $P(T) = 2$. Then T is a path. Let $T = [r_1, l_1, \ldots, r_p, l_p]$ be the path on $2p$ vertices. Let $\mathfrak{L}(T)$ be the bipartite Laplacian matrix of T. Let $L(\bar{T})$ be the bipartite Laplacian matrix of $\bar{T} = [r_1, l_1, \ldots, r_{p-1}, l_{p-1}]$. We claim that if $\boldsymbol{x}$ is an eigenvector of $\mathfrak{L}(T)$ corresponding to an eigenvalue $\lambda \neq 0$ then $\boldsymbol{x}(p) \neq 0$. In order to prove the claim, suppose $\lambda \neq 0$ is an eigenvalue of the bipartite Laplacian matrix of T with a corresponding eigenvector x. Let $\bar{x}$ be obtained from $\boldsymbol{x}$ by deleting the pth entry. From $\mathfrak{L}(T)\boldsymbol{x} = \lambda\boldsymbol{x}$, by Lemma 3.2.1, note that

$$\mathfrak{L}(\bar{T})\bar{\boldsymbol{x}} + \boldsymbol{\mu}(\boldsymbol{x}(p-1) - \boldsymbol{x}(p)) = \lambda\bar{\boldsymbol{x}}, \quad \text{and} \quad -\boldsymbol{x}(p-1) + \boldsymbol{x}(p) = \lambda\boldsymbol{x}(p),$$

where $\boldsymbol{\mu}$ is a signed degree vector at l_{p-1} of $\bar{T}$. If possible let, $x(p) = 0$. Then $\boldsymbol{x}(p-1) = 0$ and $\mathfrak{L}(\bar{T})\bar{\boldsymbol{x}} = \lambda\bar{\boldsymbol{x}}$. Therefore, $\bar{x}$ is an eigenvector of $\bar{T}$ corresponding to the eigenvalue λ with $\boldsymbol{x}(p-1) = 0$. With a repeated argument we see that $\boldsymbol{x} = 0$, which is a contradiction as $\boldsymbol{x}$ is an eigenvector. Therefore $\boldsymbol{x}(p) \neq 0$.

Suppose that the geometric multiplicity of λ is at least 2. Let $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ be a two linearly independent eigenvectors of $\mathfrak{L}(T)$ corresponding λ. Then by a linear combination of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ we may find $\boldsymbol{x}$ such that $\boldsymbol{x}(p) = 0$, which is a contradiction. This shows that if $P(T) = 2$ then the geometric of λ is one.

Now suppose $P(T) = k > 2$ and $l_{p-k+1}, r_{p-k+1}, \dots, l_p, r_p$ are the vertices corresponding to these k pendant-two-paths (excluding their roots). Since T is not a path, in view of Remark 3.1.4, by relabeling the vertices if necessary, we may assume that $d(l_p) = d(l_{p-1}) = 1$.

Let $\boldsymbol{x}$ be an eigenvector for λ. We shall show that among $\boldsymbol{x}(p-k+1), \dots, \boldsymbol{x}(p)$, at least two coordinates must be nonzero. If both $\boldsymbol{x}(p)$ and $\boldsymbol{x}(p-1)$ are nonzero then there is nothing to prove. If possible let, $\boldsymbol{x}(p) = 0$. Let $\bar{T}$ be obtained from T by removing the vertices r_p and l_p. Let $\bar{\boldsymbol{x}}$ be obtained from $\boldsymbol{x}$ by deleting the pth entry.

Note that $d(l_p) = 1$. Let l_q, $q \leq p-k$, be the root of the pendant-two-path $[l_q, r_p, l_p]$. Then, by item (a) of Lemma 3.2.1, looking at the last row of $\mathfrak{L}$, we see that $\boldsymbol{x}(q) = 0$. It follows that $\mathfrak{L}(\bar{T})$ has λ as an eigenvalue with $\bar{\boldsymbol{x}}$ as an eigenvector. Therefore, by induction, there exist at least two coordinates among $\boldsymbol{x}(p-k+1), \dots, \boldsymbol{x}(p-1)$ must be nonzero. This establishes that if $\boldsymbol{x}$ is an eigenvector of $\mathfrak{L}$, then among $\boldsymbol{x}(p-k+1), \dots, \boldsymbol{x}(p)$, at least two coordinates must be nonzero.

If possible, let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ be linearly independent eigenvectors of L corresponding to λ. Since for each $i = 1, \dots, k$, among the last k coordinates of $\boldsymbol{x}_i$ at least two coordinates must be nonzero, it follows that by taking an appropriate linear combinations of the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ it would possible to create an eigenvector $\boldsymbol{z}$ such that at most one coordinates among last k coordinates of $\boldsymbol{z}$ is nonzero. Thus we arrived at a contradiction. Hence the geometric multiplicity of λ is at most $P(T) - 1$. ■

We remark here that the above two results applied to corona trees give us the respective known results for the usual Laplacian matrix, as special cases.

In the following result we discuss how the bipartite Laplacian matrix of a nonsingular tree can be obtained from some of its nonsingular subtrees.

► **Remark 3.3.7.** Consider the tree T with a matching edge $[l_{k_1}, r_{k_1}]$, see Figure 3.1. Let the degree of l_{k_1} be s, $s \geq 1$. Let $r_{k_1+1}, r_{k_2+1}, \dots, r_{k_{s-1}+1}$ be some distinct vertices other than r_{k_1} that are adjacent to l_{k_1}. Note that when we delete the edges $[l_{k_1}, r_{k_1+1}]$, $\dots$, $[l_{k_1}, r_{k_{s-1}+1}]$, we obtain s many smaller nonsingular trees, say $T_1, \dots, T_s$. Assume that the vertex set of T_1 is $\{l_1, \dots, l_{k_1}, r_1, \dots, r_{k_1}\}$, the vertex set of T_2 is $\{l_{k_1+1}, \dots, l_{k_2}, r_{k_1+1}, \dots, r_{k_2}\}$, and so on up

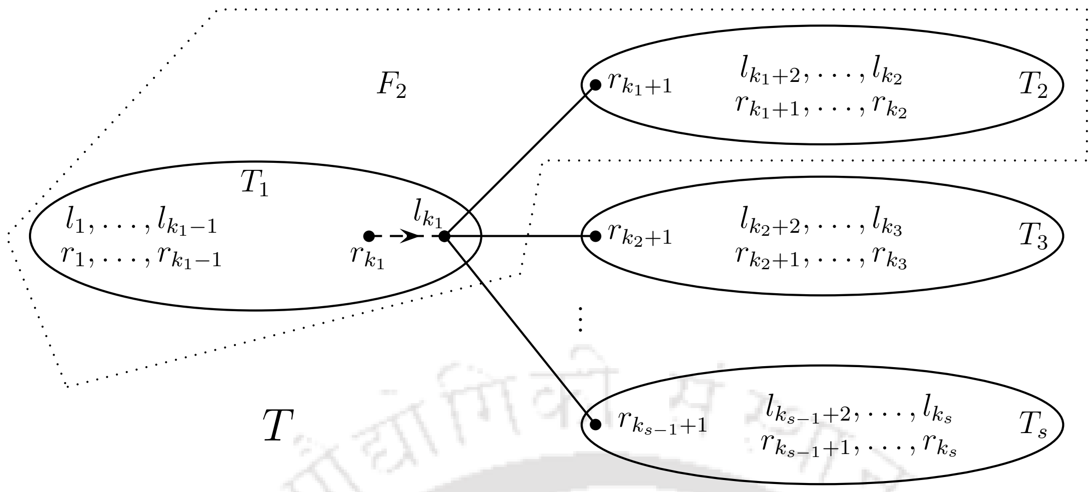


Figure 3.1: Understanding $\mathfrak{L}(T)$.

to the vertex set of T_s is $\{l_{k_{s-1}+1}, \ldots, l_{k_s}, r_{k_{s-1}+1}, \ldots, r_{k_s}\}$. Let us put an arrow on the edge $[l_{k_1}, r_{k_1}]$ from r_{k_1} to l_{k_1}. This arrow indicates that, from a vertex r_i in T_2, we do not have an alternating path to a vertex in T_1. Similarly, from a vertex r_i in T_3, we do not have an alternating path to a vertex in $T_1, T_2, T_4, T_5, \ldots, T_s$. Similar statements are true for vertices r_i in $T_4, \ldots, T_s$. Also, from a vertex l_i in T_1, we only have alternating paths to vertices in T_1 but not to a vertex in $T_2, \ldots, T_s$. Let us take F_1 be the tree T_1. For $i = 2, \ldots, s$, let F_i be the subtree of T obtained by taking F_{i-1} and T_i and by inserting the edge $[l_{k_1}, r_{k_{i-1}+1}]$. Clearly F_s is the original tree T.

a) Let $\boldsymbol{\mu}_{l_{k_1}}$ be the signed degree vector at l_{k_1} of T_1 and let $\boldsymbol{\mu}$ be the signed degree vector at l_{k_1} of T. Then $\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^t_{l_{k_1}} & \mathbf{0} \end{bmatrix}^t$.

b) Let $\mathfrak{L}(F_i)$ be the bipartite Laplacian matrix of F_i for $i = 1, \ldots, s$ and $\mathfrak{L}(T_i)$ be the bipartite Laplacian matrix of T_i for $i = 1, \ldots, s$. Clearly, $\mathfrak{L}(T_1) = \mathfrak{L}(F_1)$. Let $\boldsymbol{\mu}_{r_{k_i+1}}$ be the signed degree vector at r_{k_i+1} of T_{i+1}, for $i = 1, \ldots, s-1$. By $\boldsymbol{E}^{ij}$ we denote the matrix of an appropriate size with 1 at position (i, j) and zero elsewhere. Then, for $i = 2, \ldots, s$, we have

$$\mathfrak{L}(F_i) = \left[\begin{array}{c|c|c|c} \mathfrak{L}(T_1) + (i-1)\boldsymbol{\mu}_{l_{k_1}} \boldsymbol{e}^t_{k_1} & -\boldsymbol{\mu}_{l_{k_1}} \boldsymbol{\mu}^t_{r_{k_1+1}} & \cdots & -\boldsymbol{\mu}_{l_{k_1}} \boldsymbol{\mu}^t_{r_{k_{i-1}+1}} \\ \hline -\boldsymbol{E}^{1k_1} & \mathfrak{L}(T_2) + \boldsymbol{e}_1 \boldsymbol{\mu}^t_{r_{k_1+1}} & \cdots & \mathbf{0} \\ \hline \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \hline -\boldsymbol{E}^{1k_1} & \mathbf{0} & \cdots & \mathfrak{L}(T_i) + \boldsymbol{e}_1 \boldsymbol{\mu}^t_{r_{k_{i-1}+1}} \end{array}\right]$$

In particular,

$$\mathfrak{L}(T)=\left[\begin{array}{c|c|c|c}
\mathfrak{L}(T_1)+(s-1)\boldsymbol{\mu}_{l_{k_1}}\boldsymbol{e}_{k_1}^t & -\boldsymbol{\mu}_{l_{k_1}}\boldsymbol{\mu}_{r_{k_1+1}}^t & \cdots & -\boldsymbol{\mu}_{l_{k_1}}\boldsymbol{\mu}_{r_{k_{s-1}+1}}^t \\ \hline
-\boldsymbol{E}^{1k_1} & \mathfrak{L}(T_2)+\boldsymbol{e}_1\boldsymbol{\mu}_{r_{k_1+1}}^t & \cdots & \mathbf{0} \\ \hline
\vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \hline
-\boldsymbol{E}^{1k_1} & \mathbf{0} & \cdots & \mathfrak{L}(T_s)+\boldsymbol{e}_1\boldsymbol{\mu}_{r_{k_{s-1}+1}}^t
\end{array}\right]. \quad ◀$$

We illustrate the above remark by the following example.

▶ **Example 3.3.8.** Consider the tree T, as shown in Figure 3.2. Edges in the perfect matching are shown as dashed lines.

Note that $[l_3, r_3]$ is a matching edge and the other vertices that are adjacent to l_3 are r_4, r_7. Consider $k_1 = 3$, $k_2 = 6$ and $k_3 = 8$ in Remark 3.3.7. Note that T_1, T_2, and T_3 are nonsingular trees with the vertex set $\{l_1, l_2, l_3, r_1, r_2, r_3\}$, $\{l_4, l_5, l_6, r_4, r_5, r_6\}$, and $\{l_7, l_8, r_7, r_8\}$, respectively. Further, F_2 is a subtree tree of T induced by $\{l_1, \ldots, l_6, r_1, \ldots, r_6\}$ and F_3 is a subtree of T induced by $\{l_1, l_2, l_3, l_7, l_8, r_1, r_2, r_3, r_7, r_8\}$. We set $F_1 = T_1$.

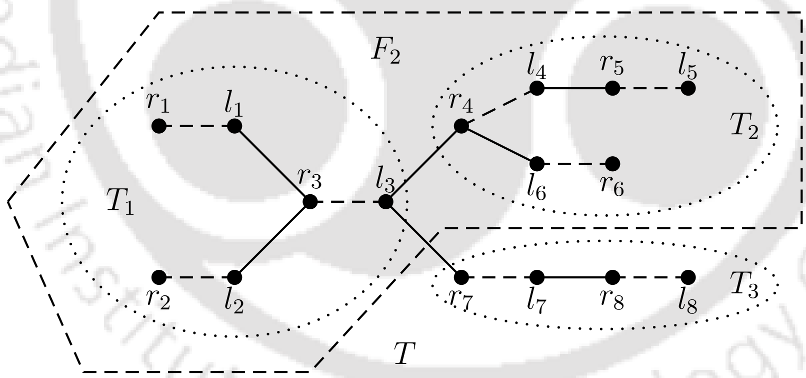


Figure 3.2: Illustration of Remark 3.3.7

Let $\boldsymbol{\mu}_{l_3}$ be the signed degree vector at l_3 of T_1, let $\boldsymbol{\mu}_{r_4}$ be the signed degree vector at r_4 of T_2, and let $\boldsymbol{\mu}_{r_7}$ be the signed degree vector at r_7 of T_3. Notice that

$$\boldsymbol{\mu}_{l_3}=\begin{bmatrix}-1 & -1 & 3\end{bmatrix}^t,\quad \boldsymbol{\mu}_{r_4}=\begin{bmatrix}2 & -1 & 0\end{bmatrix}^t,\quad \text{and}\quad \boldsymbol{\mu}_{r_7}=\begin{bmatrix}2 & -1\end{bmatrix}^t$$

Let $\boldsymbol{\mu}$ be the signed degree vector at l_3 of T. Then

$$\boldsymbol{\mu}=\begin{bmatrix}-1 & -1 & 3 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^t=\begin{bmatrix}\boldsymbol{\mu}_{l_3} & \mathbf{0}\end{bmatrix}^t$$

Let $\mathfrak{L}(T_i)$ be the bipartite Laplacian matrix of T_i for $i = 1, 2, 3$. Then we see that

$$\mathfrak{L}(T_1) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ and } \mathfrak{L}(T_2) = \begin{bmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \text{ and } \mathfrak{L}(T_3) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Notice that

$$\mathfrak{L}(F_2) = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ -1 & -1 & 5 & -6 & 3 & 0 \\ \hline 0 & 0 & -1 & 5 & -3 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{L}(T_1) + \boldsymbol{\mu}_{l_3} e_3^t & \boldsymbol{\mu}_{l_3} \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \\ \hline -\boldsymbol{E}^{13} & \mathfrak{L}(T_2) + \boldsymbol{e}_1 \boldsymbol{\mu}_{r_4}^t \end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathfrak{L}(T_1) + \boldsymbol{\mu}_{l_3} e_3^t & -\boldsymbol{\mu}_{l_3} \boldsymbol{\mu}_{r_4}^t \\ \hline -\boldsymbol{E}^{13} & \mathfrak{L}(T_2) + e_1 \boldsymbol{\mu}_{r_4}^t \end{array}\right]$$

In a similar way, we can see that

$$\mathfrak{L}(F_3) = \left[\begin{array}{ccc|ccc|cc} 1 & 0 & -3 & 2 & -1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 & -1 & 0 & 2 & -1 \\ -1 & -1 & 8 & -6 & 3 & 0 & -6 & 3 \\ \hline 0 & 0 & -1 & 5 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} \mathfrak{L}(T_1) + 2\boldsymbol{\mu}_{l_3} e_3^t & -\boldsymbol{\mu}_{l_3} \boldsymbol{\mu}_{r_4}^t & -\boldsymbol{\mu}_{l_3} \boldsymbol{\mu}_{r_7}^t \\ \hline -\boldsymbol{E}^{13} & \mathfrak{L}(T_2) + \boldsymbol{e}_1 \boldsymbol{\mu}_{r_4}^t & \mathbf{0} \\ \hline -\boldsymbol{E}^{13} & \mathbf{0} & \mathfrak{L}(T_2) + \boldsymbol{e}_1 \boldsymbol{\mu}_{r_7}^t \end{array}\right]. \qquad \blacktriangleleft$$

Grone, Merris, and Sunder [GMS90, Theorem 2.1] have talked about the integer eigenvalues $\lambda > 1$ of the usual Laplacian matrix of tree T. The following is an extension of it to the bipartite Laplacian matrix of a nonsingular tree.

▶ **Theorem 3.3.9.** *Let T be a nonsingular tree on $2p$ vertices. Suppose λ is an integer* *eigenvalue of the $\mathfrak{L}(T)$ with a corresponding eigenvector $\boldsymbol{u}$. Then the following assertions hold.*

(a) If $\lambda \neq 0, \pm 1$ then λ divides p.

(b) If $\lambda \neq \pm 1$ then no coordinate of $\boldsymbol{u}$ is zero.

(c) If $\lambda \neq \pm 1$ then the geometric multiplicity of λ is one. ◀

Proof of Item (a). By part (a) of Theorem 3.2.3, the characteristic polynomial $\chi(x) := \det(xI - \mathfrak{L}(T)) = xf(x)$, where $f(x)$ is some polynomial with integer coefficients. By item (b) of Theorem 3.2.3,

$$\begin{aligned} f(0) &= \text{coefficient of } x \text{ in } \chi(x) \\ &= (-1)^{p-1} \times \text{ (sum of the principal minors of } \mathfrak{L}(T) \text{ of size } p-1) \\ &= (-1)^{p-1} p. \end{aligned}$$

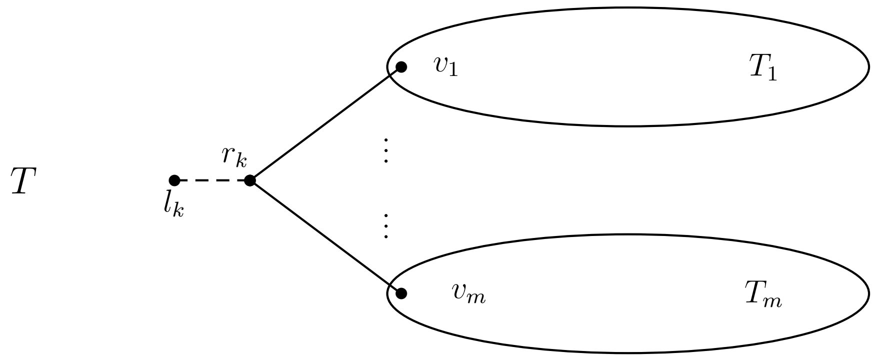
Suppose $\lambda \neq 0, \pm 1$. It follows that, $\chi(\lambda) = 0$ and hence $f(\lambda) = 0$. As each term of $f(\lambda)$ is a multiple of λ except $f(0)$, we see that λ divides $f(0)$. The result now follows.

Proof of Item (b). Suppose $\lambda \neq \pm 1$. Notice that if $T = P_2$ then $\mathbf{u} = [1]$ and the result holds for $p = 1$. Let us assume T be a tree on $2p$ vertices, $p > 1$. By way of contradiction, assume that some $\boldsymbol{u}(k) = 0$. In that case, either i) $d(l_k) \geq 2$ or ii) $d(l_k) = 1$ and $d(r_k) \geq 2$.

Let us first assume that case i) holds. In this case, we locate the matching edge $[l_k, r_k]$ in T. Imagine deleting all nonmatching edges at the vertex l_k. That will create at least two components, one of which contains the matching edge $[l_k, r_k]$.

Recall that, if we do a relabeling of vertices inside L, and call the resulting tree T', then $\mathfrak{L}(T')$ is permutation similar to $L(T)$ and hence λ will remain an eigenvalue of $\mathfrak{L}(T')$ with a corresponding eigenvector $\boldsymbol{u}'$ having a zero entry.

In view of this, we assume that T is already labeled as shown in Remark 3.3.7 and that $\boldsymbol{u}(k_1) = 0$. That is, when we delete the nonmatching edges at l_{k_1} we have s many components $T_1, T_2, \ldots, T_s$ where the component T_1 has the vertex set $\{l_1, r_1, \ldots, l_{k_1}, r_{k_1}\}$, the component T_2 has the vertex set $\{l_{k_1+1}, r_{k_1+1}, \ldots, l_{k_2}, r_{k_2}\}$, and so on.



Now, we partition the eigenvector conformally $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_s \end{bmatrix}^t$. Observe that $\boldsymbol{u}_1(k_1) = \boldsymbol{u}(k_1) = 0$ and

$$\left[\begin{array}{c|c|c|c} \mathfrak{L}(T_1) + (s-1)\boldsymbol{\mu}_{l_{k_1}}\boldsymbol{e}_{k_1}^t & -\boldsymbol{\mu}_{l_{k_1}}\boldsymbol{\mu}_{r_{k_1+1}}^t & \cdots & -\boldsymbol{\mu}_{l_{k_1}}\boldsymbol{\mu}_{r_{k_{s-1}+1}}^t \\ \hline -\boldsymbol{E}^{1k_1} & \mathfrak{L}(T_2) + \boldsymbol{e}_1\boldsymbol{\mu}_{r_{k_1+1}}^t & \cdots & \mathbf{0} \\ \hline \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \hline -\boldsymbol{E}^{1k_1} & \mathbf{0} & \cdots & \mathfrak{L}(T_s) + \boldsymbol{e}_1\boldsymbol{\mu}_{r_{k_{s-1}+1}}^t \end{array}\right] \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_s \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_s \end{bmatrix}.$$

If all $\boldsymbol{u}_i = \mathbf{0}$ for $i = 2, \ldots, s$ then $\boldsymbol{u}_1 \neq \mathbf{0}$. It follows that $\mathfrak{L}(T_1)\boldsymbol{u}_1 = \lambda\boldsymbol{u}_1$. We may use the induction hypothesis to conclude that no coordinate of $\boldsymbol{u}_1$ is zero. This is a contradiction as $\boldsymbol{u}_1(k_1) = 0$. Therefore, at least one of $\boldsymbol{u}_2, \ldots, \boldsymbol{u}_s$ is nonzero, say $\boldsymbol{u}_2 \neq 0$. Then we have $Z\boldsymbol{u}_2 = \lambda\boldsymbol{u}_2$, where $Z = \mathfrak{L}(T_2) + \boldsymbol{e}_1\boldsymbol{\mu}_{r_{k_1}+1}^t$. Let C be the matrix obtained from $\mathfrak{L}(T_2)$ by replacing the first row by $\boldsymbol{\mu}_{r_{k_1}+1}^t$. Note that $\det Z = \det \mathfrak{L}(T_2) + \det C$. By expanding the determinant of C along the first row we see that

$$\det C = \sum_i (-1)^{1+i}\boldsymbol{\mu}_{r_{k_1}+1}(i) \det \mathfrak{L}(T_2)(1|i).$$

By part (b) of Theorem 3.2.3, $(-1)^{1+i} \det \mathfrak{L}(T_2)(1|i) = 1$. Therefore, by Lemma 3.1.7, $\det C = \mathbb{1}^t\boldsymbol{\mu}_{r_{k_1}+1} = 1$. Since $\det \mathfrak{L}(T_2) = 0$, it follows that $\det Z = 1$. Let $f(x) = \det(xI - Z)$ be the characteristic polynomial of Z. Then $f(x)$ is a monic polynomial (with integer coefficients) such that $f(0) = \pm 1$. Therefore, the only possible rational eigenvalues of Z are 1 and -1. Thus we arrived at a contradiction as $\lambda \neq \pm 1$.

Now we consider the case ii). Assume that $d(r_k) = m + 1 \geq 2$ and $d(l_k) = 1$. Let $v_1, \ldots, v_m$ be the vertices other than l_k that are adjacent to r_k. Let $T_1, \ldots, T_m$ be nonsingular subtrees of T obtained by deleting the edges $[r_k, v_1], \ldots, [r_k, v_m]$ and that does not contain the edge $[l_k, r_k]$. Let us assume that the vertex of T_1 is $\{l_1, \ldots, l_{k-1}, r_1, \ldots, r_{k-1}\}$, the vertex set of T_2 is $\{l_{k+1}, \ldots, l_{k+t_1}, r_{k+1}, \ldots, r_{k+t_1}\}$,. and so on up to the vertex set of T_m is $\{l_{k+t_{m-2}+1}, \ldots, l_{k+t_{m-1}}, r_{k+t_{m-2}+1}, \ldots, r_{k+t_{m-1}}\}$.

Let us partition $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}(k) & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_m \end{bmatrix}^t$. Since $\boldsymbol{u}$ is an eigenvector and $\boldsymbol{u}(k) = 0$, it follows among $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ at least one them is a nonzero vector, say $\boldsymbol{u}_1 \neq 0$. Without loss of any generality, let us assume that $v_1 = l_1$. Notice that from a vertex r_i in T_1, we do not have an alternating path to a vertex in $T_2, \ldots, T_m$. Further note that $d_T(v) = d_{T_1}(v)$ for each $v \in T_1 - l_1$. It follows that

$$\left[\begin{array}{c|c|c} \mathfrak{L}(T_1)+\boldsymbol{\mu}_{l_1}\boldsymbol{e}_1^t & -\boldsymbol{\mu}_{l_1} & \mathbf{0} \\ \hline -\boldsymbol{e}_1^t & m & * \\ \hline * & * & * \end{array}\right] \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}(k) \\ \bar{\boldsymbol{u}} \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}(k) \\ \bar{\boldsymbol{u}} \end{bmatrix},$$

where $\bar{\boldsymbol{u}} = [\ \boldsymbol{u}_2\ \cdots\ \boldsymbol{u}_m\ ]^t$. Notice that $X\boldsymbol{u}_1 = \lambda\boldsymbol{u}_1$, where $X = \mathfrak{L}(T_1) + \boldsymbol{\mu}_{l_1}e_1^t$. Let Y be the matrix obtained from $\mathfrak{L}(T_1)$ by replacing the first column by $\boldsymbol{\mu}_{l_1}^t$. Note that $\det X = \det \mathfrak{L}(T_1) + \det Y$. By expanding the determinant of Y along the first column we see that

$$\det Y = \sum_i (-1)^{1+i}\boldsymbol{\mu}_{l_1}(i) \det \mathfrak{L}(T_1)(i|1).$$

By part (b) of Theorem 3.2.3, $(-1)^{1+i}\det \mathfrak{L}(T_1)(i|1) = 1$. Therefore, by Lemma 3.1.7, $\det X = \mathbb{1}^t\boldsymbol{\mu}_{l_1} = 1$. Since $\det \mathfrak{L}(T_1) = 0$, it follows that $\det X = 1$. Let $g(x) = \det(xI - X)$ be the characteristic polynomial of X. Then $g(x)$ is a monic polynomial (with integer coefficients) such that $g(0) = \pm 1$. Therefore, the only possible rational eigenvalues of X are 1 and -1. Thus we arrived at a contradiction as $\lambda \neq \pm 1$. This completes the proof.

Proof of Item (c). It directly follows from item (b). ■

We close this section by supplying two observations about the eigenvalues of the bipartite Laplacian matrix of a path.

► **Lemma 3.3.10.** *Let T be a path on $2p$ vertices with a standard vertex bipartition (L, R). Let λ be an eigenvalue of the bipartite Laplacian matrix of T. Then the following assertions hold.*

(a) The geometric multiplicity of λ is one.

(b) $\lambda = 2$ if and only if $p = 2k$, for some k. ◄

Proof of Item (a). It follows from Theorem 3.3.6.

Proof of Item (b). Suppose $p = 2k$. Consider the vector $\boldsymbol{u}$ as follows.

$$\boldsymbol{u} = [\ 1\ \ -1\ \ -3\ \ 3\ \ 3^2\ \ -3^2\ \ \cdots\ \ (-1)^{k-1}3^{k-1}\ \ (-1)^k 3^{k-1}\ ]^t.$$

Then $\mathfrak{L}(T)\boldsymbol{u} = 2\boldsymbol{u}$. The converse of the result follows from Theorem 3.3.9. ■

3.4 The inverse of the bipartite distance matrix

In this section, our aim is to find a formula for the inverse of the bipartite distance matrix of a nonsingular tree.

Let us start the discussion by relating the structures the $\boldsymbol{\tau}^r_T$ vectors of the new tree with that of the old one under attaching a new P_2 at a vertex

▶ **Lemma 3.4.1.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\widehat{T}$ be the tree obtained from T by attaching a new P_2 at v.*

(a) If $v = r_k$ for some k, then $\boldsymbol{\tau}^r_{\widehat{T}} = \begin{bmatrix} \boldsymbol{\tau}^r_T \\ 0 \end{bmatrix} - (1 + \mathtt{diff}_T(v)) \begin{bmatrix} \boldsymbol{e}_k \\ -1 \end{bmatrix}$.

(b) If $v = l_k$ for some k, then $\boldsymbol{\tau}^r_{\widehat{T}} = \begin{bmatrix} \boldsymbol{\tau}^r_T \\ -1 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\mu}_v(T) \\ 0 \end{bmatrix}$, *where $\boldsymbol{\mu}_v(T)$ is the signed degree vector at v of T.* ◀

Proof of Item (a). It directly follows from item (a) of Lemma 2.2.12 .

Proof of Item (b). Let $\widehat{T}$ be the tree obtained from T by introducing two new vertices $l_{p+1}, r_{p+1} \notin T$ and adding the edges $[l_k, r_{p+1}]$,$[r_{p+1}, l_{p+1}]$.

Since $[r_{p+1}, l_{p+1}]$ is the only alternating path that starts at r_{p+1}, it follows that $\mathtt{diff}_{\widehat{T}}(r_{p+1}) = -1$ and so $\boldsymbol{\tau}^r_{\widehat{T}}(p+1) = 1$.

Let us take $i = 1, \ldots, p$. Clearly, $d_T(r_i) = d_{\widehat{T}}(r_i)$. Note that if the r_i- l_k path is not an alternating path then $\mathcal{A}_{r_i,\widehat{T}} = \mathcal{A}_{r_i,T}$. It follows that $\boldsymbol{\tau}^r_T(i) = \boldsymbol{\tau}^r_{\widehat{T}}(i)$.

Now assume that the r_i- l_k path is an odd alternating path. Then the r_i- l_{p+1} path is an even alternating path. It follows that $\mathcal{A}^-_{r_i,\widehat{T}} = \mathcal{A}^-_{r_i,T}$ and

$$\mathcal{A}^+_{r_i,\widehat{T}} = \mathcal{A}^+_{r_i,T} \cup \{[r_i, \ldots, l_k, r_{p+1}, l_{p+1}]\}.$$

Therefore, $\mathtt{diff}_{\widehat{T}}(r_i) = \mathtt{diff}_T(r_i) + 1$ and so we have

$$\boldsymbol{\tau}^r_{\widehat{T}}(i) = 1 - d_{\widehat{T}}(r_i)\left[1 + \mathtt{diff}_{\widehat{T}}(r_i)\right] = \boldsymbol{\tau}^r_T(i) - d_T(r_i).$$

In a similar way, we can argue that if the r_i- l_k path is an even alternating path then we have

$$\boldsymbol{\tau}^r_{\widehat{T}}(i) = \boldsymbol{\tau}^r_T(i) + d_T(r_i).$$

Hence the result follows by applying the definition of the signed degree vector at l_k of T. ■

In the below result we discuss the relationship between the bipartite distance matrix and the bipartite Laplacian matrix of a nonsingular tree.

► **Lemma 3.4.2.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\mathfrak{B}(T)$ and $\mathfrak{L}(T)$ be the bipartite distance matrix and the bipartite Laplacian matrix of T, respectively. Let $\boldsymbol{\tau}_T^r$ be the restriction of $\boldsymbol{\tau}_T$ on R. Then*

$$-\mathfrak{L}(T)\mathfrak{B}(T) + 2\boldsymbol{\tau}_T^r \mathbb{1}^t = 2I.$$ ◄

Proof. We proceed by induction on p. Let $p = 1$. Then $T = P_2$, $\mathfrak{L}(T) = [0]$ and $\boldsymbol{\tau}_T^r = [1]$. It follows that $-\mathfrak{L}(T)\mathfrak{B}(T) + 2\boldsymbol{\tau}_T \mathbb{1}^t = 2$. Let $p = 2$. Then $T = P_4$. Let $P_4 = [l_1, r_1, l_2, r_2]$. Then $\mathfrak{L}(T) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\boldsymbol{\tau}_T^r = \begin{bmatrix} 1 & 0 \end{bmatrix}^t$. It follows that

$$-\mathfrak{L}(T)\mathfrak{B}(T) + 2\boldsymbol{\tau}_T^r \mathbb{1}^t = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = 2I.$$

Assume the result is true for p. Let $\widehat{T}$ be the nonsingular tree on $2p+2$ vertices. Then $\widehat{T}$ is obtained from some nonsingular tree T on $2p$ vertices by attaching a new P_2 at some vertex v. Let $\boldsymbol{\mu}$ be the signed degree vector at v of T. Notice that either $v \in L$ or $v \in R$.

Let us first consider that $v = l_k$ for some $1 \le k \le p$. By item (a) of Lemma 3.2.1, we have

$$\mathfrak{L}(\widehat{T}) = \begin{bmatrix} \mathfrak{L}(T) + \boldsymbol{\mu}\boldsymbol{e}_k^t & -\boldsymbol{\mu} \\ -\boldsymbol{e}_k^t & 1 \end{bmatrix}.$$

Let $\boldsymbol{x}$ be a vector of size p such that $\boldsymbol{x}(i) = \texttt{dist}(l_i, r_{p+1})$ for each $i = 1, \cdots, p$. Then the bipartite distance matrix of $\widehat{T}$ can be written as

$$\mathfrak{B}(\widehat{T}) = \begin{bmatrix} \mathfrak{B}(T) & \boldsymbol{x} \\ \boldsymbol{e}_k^t\mathfrak{B}(T) + 2\mathbb{1}^t & 1 \end{bmatrix}.$$

Now note that

$$\begin{aligned} \mathfrak{L}(\widehat{T})\mathfrak{B}(\widehat{T}) &= \begin{bmatrix} \mathfrak{L}(T) + \boldsymbol{\mu}\boldsymbol{e}_k^t & -\boldsymbol{\mu} \\ -\boldsymbol{e}_k^t & 1 \end{bmatrix}\begin{bmatrix} \mathfrak{B}(T) & x \\ \boldsymbol{e}_k^t\mathfrak{B}(T) + 2\mathbb{1}^t & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{L}(T)\mathfrak{B}(T) + \boldsymbol{\mu}\boldsymbol{e}_k^t\mathfrak{B}(T) - \boldsymbol{\mu}\boldsymbol{e}_k^t\mathfrak{B}(T) - 2\boldsymbol{\mu}\mathbb{1}^t & \mathfrak{L}(T)x + \boldsymbol{\mu}\boldsymbol{e}_k^t\boldsymbol{x} - \boldsymbol{\mu} \\ -\boldsymbol{e}_k^t\mathfrak{B}(T) + \boldsymbol{e}_k^t\mathfrak{B}(T) + 2\mathbb{1}^t & -\boldsymbol{e}_k^t\boldsymbol{x} + 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{L}(T)\mathfrak{B}(T) - 2\boldsymbol{\mu}\mathbb{1}^t & \mathfrak{L}(T)\boldsymbol{x} \\ 2\mathbb{1}^t & 0 \end{bmatrix}. \end{aligned} \tag{3.1}$$

The last equality follows from the fact that $\boldsymbol{e}_k^t\boldsymbol{x} = \boldsymbol{x}(k) = \texttt{dist}(l_k, r_{p+1}) = 1$. By item (b) of Lemma 3.4.1, we have

$$\boldsymbol{\tau}^r_{\widehat{T}} = \begin{bmatrix} \boldsymbol{\tau}^r_T \\ 1 \end{bmatrix} + \begin{bmatrix} -\boldsymbol{\mu} \\ 0 \end{bmatrix}.$$

By the induction hypothesis we get

$$\mathfrak{L}(T)\mathfrak{B}(T) = 2\boldsymbol{\tau}^r_T \mathbb{1}^t - 2I. \tag{3.2}$$

It follows from (3.1) that

$$\begin{aligned} -\mathfrak{L}(\widehat{T})\mathfrak{B}(\widehat{T}) + 2\boldsymbol{\tau}^r_{\widehat{T}}\mathbb{1}^t &= -\begin{bmatrix} 2\boldsymbol{\tau}^r_T\mathbb{1}^t - 2I - 2\boldsymbol{\mu}\mathbb{1}^t & \mathfrak{L}(T)\boldsymbol{x} \\ 2\mathbb{1}^t & 0 \end{bmatrix} + \begin{bmatrix} 2(\boldsymbol{\tau}^r_T - \boldsymbol{\mu})\mathbb{1}^t \\ 2\mathbb{1}^t \end{bmatrix} \\ &= \begin{bmatrix} 2I & -\mathfrak{L}(T)\boldsymbol{x} + 2(\boldsymbol{\tau}^r_T - \boldsymbol{\mu}) \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

Therefore in order to complete the proof for the $v = l_k$ case, we only need to show that $\mathfrak{L}(T)\boldsymbol{x} = 2(\boldsymbol{\tau}^r_T - \boldsymbol{\mu})$.

Let the degree of l_k in T be s, $(s \geq 1)$. Let T_1 be the tree obtained from T by removing all vertices adjacent to l_k except the vertex r_k. If $s = 1$ then T_1 is the same as T. Without loss of any generality, let us assume that T_1 has the vertex set $\{l_1, r_1, \ldots, l_{k-1}, r_{k-1}, l_k, r_k\}$. Let $\widehat{\boldsymbol{\mu}}$ be the signed degree vector at v of T_1. By Remark 3.3.7, $\boldsymbol{\mu} = \begin{bmatrix} \widehat{\boldsymbol{\mu}} & \mathbf{0} \end{bmatrix}^t$. Further note that

$$\boldsymbol{x} = \mathfrak{B}(T)\boldsymbol{e}_k + \begin{bmatrix} 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{bmatrix}^t, \tag{3.3}$$

where the entries 2 in the last vector are for the vertices $l_1, \ldots, l_{k-1}$. Let $\boldsymbol{z}$ be a vector of size $(p-k)$ defined as follows. For $i = 1, \ldots, (p-k)$, $\boldsymbol{z}(i) = -1$ if r_{k+i} adjacent to l_k and $\boldsymbol{z}(i) = 0$ otherwise. Hence, by Remark 3.3.7, we have

$$\begin{aligned} \mathfrak{L}(T)\begin{bmatrix} 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{bmatrix}^t &= \begin{bmatrix} \mathfrak{L}(T_1) + (s-1)\widehat{\boldsymbol{\mu}}\boldsymbol{e}_k^t & * \\ \boldsymbol{z}\boldsymbol{e}_k^t & * \end{bmatrix} \begin{bmatrix} 2(\mathbb{1} - \boldsymbol{e}_k) \\ \mathbf{0} \end{bmatrix} \\ &= 2\begin{bmatrix} \mathfrak{L}(T_1)\mathbb{1} + (s-1)\widehat{\boldsymbol{\mu}} \\ \boldsymbol{z} \end{bmatrix} - 2\begin{bmatrix} \mathfrak{L}(T_1)\boldsymbol{e}_k + (s-1)\widehat{\boldsymbol{\mu}} \\ \boldsymbol{z} \end{bmatrix} \\ &= -2\begin{bmatrix} \mathfrak{L}(T_1)\boldsymbol{e}_k \\ \mathbf{0} \end{bmatrix} \\ &= -2\begin{bmatrix} \widehat{\boldsymbol{\mu}} - \boldsymbol{e}_k \\ \mathbf{0} \end{bmatrix} \\ &= -2\boldsymbol{\mu} + 2\boldsymbol{e}_k \qquad \text{[By Remark 3.3.7]} \end{aligned} \tag{3.4}$$

From (3.2), (3.3) and (3.4), it follows that $\mathfrak{L}(T)\boldsymbol{x} = 2(\boldsymbol{\tau}_T^r - \boldsymbol{\mu})$. This completes the proof of the case $v = l_k$.

Now we consider the case $v = r_k$ for some $1 \leq k \leq p$. By item (b) of Lemma 3.2.1, we have

$$\mathfrak{L}(\widehat{T}) = \begin{bmatrix} \mathfrak{L}(T) + \boldsymbol{e}_k\boldsymbol{\mu}^t & -\boldsymbol{e}_k \\ -\boldsymbol{\mu}^t & 1 \end{bmatrix}$$

Let $\boldsymbol{y}$ be a vector of size p such that $\boldsymbol{y}(i) = \mathtt{dist}(r_i, l_{p+1})$ for each $i = 1, \cdots, p$. Then the bipartite distance matrix of $\widehat{T}$ can be written as

$$\mathfrak{B}(\widehat{T}) = \begin{bmatrix} \mathfrak{B}(T) & \mathfrak{B}(T)\boldsymbol{e}_k + 2\mathbb{1} \\ \boldsymbol{y}^t & 1 \end{bmatrix}.$$

By using the facts $\mathfrak{L}(T)\mathbb{1} = \mathbf{0}$ and $\boldsymbol{\mu}^t\mathbb{1} = 1$ we get

$$\begin{aligned} \mathfrak{L}(\widehat{T})\mathfrak{B}(\widehat{T}) &= \begin{bmatrix} \mathfrak{L}(T) + \boldsymbol{e}_k\boldsymbol{\mu}^t & -\boldsymbol{e}_k \\ -\boldsymbol{\mu}^t & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{B}(T) & \mathfrak{B}(T)\boldsymbol{e}_k + 2\mathbb{1} \\ \boldsymbol{y}^t & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{L}(T)\mathfrak{B}(T) + \boldsymbol{e}_k(\boldsymbol{\mu}^t\mathfrak{B}(T) - \boldsymbol{y}^t) & \mathfrak{L}(T)\mathfrak{B}(T)\boldsymbol{e}_k + \boldsymbol{e}_k\boldsymbol{\mu}^t\mathfrak{B}(T)\boldsymbol{e}_k + \boldsymbol{e}_k \\ -(\boldsymbol{\mu}^t\mathfrak{B}(T) - \boldsymbol{y}^t) & -\boldsymbol{\mu}^t\mathfrak{B}(T)\boldsymbol{e}_k - 1 \end{bmatrix} \end{aligned} \tag{3.5}$$

We claim that the following identity is true

$$\boldsymbol{y}^t - \boldsymbol{\mu}^t\mathfrak{B}(T) = 2[1 + \mathtt{diff}_T(v)]\mathbb{1}^t. \tag{3.6}$$

In order to verify our claim, suppose the degree of r_k in T is m, $(m \geq 1)$. Let T_0 be the tree obtained from T by removing all vertices adjacent to l_k except the vertex r_k. If $m = 1$ then T_0 is same as T.

Without loss of any generality assume that T_0 has the vertex set $\{l_1, r_1, \ldots, l_{k-1}, r_{k-1}, l_k, r_k\}$. Now notice that

$$\boldsymbol{y}^t = \boldsymbol{e}_k^t\mathfrak{B}(T) + \begin{bmatrix} 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{bmatrix}^t, \tag{3.7}$$

where the entries 2 in the last vector are for the vertices $r_1, \ldots, r_{k-1}$.

Let $l_{k_1}, \ldots, l_{k_{m-1}}$ be the vertices other than l_k that are adjacent to r_k. Let T_i be the component of $T - r_k$ that contain the vertex l_{k_i} but does not contain the vertex l_k, for $i = 1, \ldots, m-1$. For $i = 1, \ldots, m-1$, we have

$$e_k^t\mathfrak{B}(T) - e_{k_i}^t\mathfrak{B}(t) = \begin{bmatrix} -2 & \cdots & -2 & 0 & 0 & \cdots & 0 \end{bmatrix}^t + \boldsymbol{z}_i,$$

where the entries -2 in the first vector are for the vertices $r_1, \ldots, r_{k-1}$, and $\boldsymbol{z}_i$ is a vector of size p such that $\boldsymbol{z}_i(j) = 2$ if $r_j \in T_i$ and $\boldsymbol{z}_i(j) = 0$ otherwise. Therefore, it follows that

$$(m-1)e_k^t\mathfrak{B}(T) - \sum_{i=1}^{m-1} e_{k_i}^t\mathfrak{B}(t) = \begin{bmatrix} -2(m-1) & \cdots & -2(m-1) & 0 & 2 & \cdots & 2 \end{bmatrix}^t.$$

Now notice that

$$e_k^t\mathfrak{L}(T) = m\boldsymbol{\mu}^t - \boldsymbol{e}_k^t - \boldsymbol{e}_{k_1}^t - \cdots - \boldsymbol{e}_{k_{m-1}}^t.$$

Therefore we get

$$\begin{aligned} e_k^t\mathfrak{L}(T)\mathfrak{B}(T) + me_k^t\mathfrak{B}(T) &= m\boldsymbol{\mu}^t\mathfrak{B}(T) + (m-1)\boldsymbol{e}_k^t\mathfrak{B}(T) - \sum_{i=1}^{m-1} \boldsymbol{e}_{k_i}^t\mathfrak{B}(t) \\ &= m\boldsymbol{\mu}^t\mathfrak{B}(T) + \begin{bmatrix} -2(m-1) & \cdots & -2(m-1) & 0 & 2 & \cdots & 2 \end{bmatrix}^t \\ &= m\boldsymbol{\mu}^t\mathfrak{B}(T) + 2\mathbb{1}^t - 2\boldsymbol{e}_k^t - m\begin{bmatrix} 2 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{bmatrix}^t, \quad (3.8) \end{aligned}$$

where the entries 2 in the last vector are for the vertices $r_1, \ldots, r_{k-1}$. By (3.7) and (3.8), we see that

$$e_k^t\mathfrak{L}(T)\mathfrak{B}(T) = m\boldsymbol{\mu}^t\mathfrak{B}(T) + 2\mathbb{1}^t - 2\boldsymbol{e}_k^t - m\boldsymbol{y}^t.$$

Let $b = 1 + \mathtt{diff}_T(r_k)$. Then $\boldsymbol{\tau}_T^r(k) = 1 - mb$. By applying induction hypothesis we get

$$e_k^t\mathfrak{L}(T)\mathfrak{B}(T) = 2\boldsymbol{\tau}_T^r(k)\mathbb{1}^t - 2\boldsymbol{e}_k^t = 2\mathbb{1}^t - 2\boldsymbol{e}_k^t - 2mb\mathbb{1}^t.$$

It follows that $m\boldsymbol{\mu}^t\mathfrak{B}(T) = m\boldsymbol{y}^t - 2mb\mathbb{1}^t$, and our claim is established.

Now we will use the identity (3.6) to complete the remaining part of the proof. First note that $\boldsymbol{\mu}^t\mathfrak{B}(T)\boldsymbol{e}_k = \boldsymbol{y}(k) - 2b = 1 - 2b$. By using the identities (3.6) and (3.2) in (3.5) we get

$$\begin{aligned} \mathfrak{L}(\widehat{T})\mathfrak{B}(\widehat{T}) &= \begin{bmatrix} \mathfrak{L}(T)\mathfrak{B}(T) - 2b\boldsymbol{e}_k\mathbb{1}^t & \mathfrak{L}(T)\mathfrak{B}(T)\boldsymbol{e}_k + (1-2b)\boldsymbol{e}_k + \boldsymbol{e}_k \\ -2b\mathbb{1}^t & -2+2b \end{bmatrix} \\ &= \begin{bmatrix} 2\boldsymbol{\tau}_T^r\mathbb{1}^t - 2I - 2b\boldsymbol{e}_k\mathbb{1}^t & 2\boldsymbol{\tau}_T^r\mathbb{1}^t\boldsymbol{e}_k - 2b\boldsymbol{e}_k \\ 2b\mathbb{1}^t & -2+2b \end{bmatrix} \quad (3.9) \end{aligned}$$

From item (a) of Lemma 3.4.1,

$$2\boldsymbol{\tau}_{\widehat{T}}^r\mathbb{1}^t = \begin{bmatrix} 2\boldsymbol{\tau}_T^r\mathbb{1}^t - 2b\boldsymbol{e}_k^t\mathbb{1}^t \\ 2b\mathbb{1}^t \end{bmatrix}.$$

Hence the result follows by using the above identity in (3.9). ∎

The following result is an immediate consequence of Lemma 3.4.2.

▶ **Lemma 3.4.3.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\mathfrak{B}(T)$ and $\mathfrak{L}(T)$ be the bipartite distance matrix and the bipartite Laplacian matrix of T, respectively. Let $\boldsymbol{\tau}_T^l$ be the restriction of $\boldsymbol{\tau}_T$ on L. Then*

$$-\mathfrak{B}(T)\mathfrak{L}(T) + 2\mathbb{1}(\boldsymbol{\tau}_T^l)^t = 2I.$$ ◀

We are now in a position to supply a formula for the inverse of the bipartite distance matrix of a nonsingular tree T.

▶ **Theorem 3.4.4.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\mathfrak{B}(T)$ and $\mathfrak{L}(T)$ be the bipartite distance matrix and the bipartite Laplacian matrix of T, respectively. Let $\boldsymbol{\tau}_T^r$ and $\boldsymbol{\tau}_T^l$ be the restriction of $\boldsymbol{\tau}_T$ on R and L, respectively. Then*

$$\mathfrak{B}(T)^{-1} = -\frac{1}{2}\mathfrak{L}(T) + \frac{1}{\mathtt{bd}(T)}\boldsymbol{\tau}_T^r(\boldsymbol{\tau}_T^l)^t.$$ ◀

Proof. By using Lemma 3.4.1 and Lemma 3.4.2 we have

$$\left(-\frac{1}{2}\mathfrak{L}(T) + \frac{1}{\mathtt{bd}(T)}\boldsymbol{\tau}_T^r(\boldsymbol{\tau}_T^l)^t\right)\mathfrak{B}(T) = \frac{1}{2}\left(-\mathfrak{L}(T)\mathfrak{B}(T) + 2\boldsymbol{\tau}_T^r\mathbb{1}^t\right) = I.$$

This completes the proof. ■

Recall that, for the usual distance matrix D of a tree T, Russell Merris in [Mer90, Corollary 2] showed that the multiplicity of the eigenvalue -2 of D is at least $p(T) - q(T) - 1$. In the following result we observe that a similar fact is also true for the bipartite distance matrix of a nonsingular tree.

▶ **Corollary 3.4.5.** *Let T be a nonsingular tree on $2p$ $(p > 1)$ vertices with a standard vertex bipartition (L, R). Let $\mathfrak{B}$ be the bipartite distance matrix of T. If -2 is an eigenvalue of $\mathfrak{B}$ then the geometric multiplicity of -2 is at least $P(T) - Q(T)$.* ◀

Proof. Let $\mathfrak{L}$ be the bipartite Laplacian matrix of T. If $P(T) - Q(T) = 0$ then there is nothing to prove. Let us assume $P(T) - Q(T) > 1$. Let w be a root of at least two pendant-two-paths. Without loss of any generality, let us assume that $w \in L$. Let $[l_{k_1}, r_{k_1}, w], \ldots, [l_{k_m}, r_{k_m}, w]$ be pendant-two-paths at w. By Theorem 3.3.5, we noticed that $\boldsymbol{e}_{k_i} - \boldsymbol{e}_{k_j}$, $1 \le i < j \le m$, are eigenvectors of $\mathfrak{L}$ corresponding to the eigenvalue 1. Now notice that, for $i, j = 1, \ldots, m$, we have

$$\boldsymbol{\tau}_T^l(k_i) = \boldsymbol{\tau}_T^l(k_j) \qquad \text{and} \qquad \boldsymbol{\tau}_T^r(k_i) = \boldsymbol{\tau}_T^r(k_j).$$

The remaining part of the proof follows from Theorem 3.4.4. ■

4 All minor of the bipartite Laplacian matrix

The classical matrix-tree theorem was first proved by G. Kirchhoff [Kir47] in 1847 that relates the principal minor of the Laplacian matrix of G with the number of spanning trees of G. In the case of a complete graph on n vertices; the classical matrix tree theorem tells that number of labelled trees on n vertices is n^{n-2}, which is a well known Cayley's formula [Cay89]. Tutte [Tut48] in 1948 extended the matrix tree theorem to a loopless directed graph. Chaiken [Cha82] in 1982 gave a combinatorial interpretation of all minor of Laplacian matrix as a sum of nonsingular substructures. A similar combinatorial interpretation of all minor of Laplacian matrix has been discussed for a mixed graph by Bapat, Grossman, and Kulkarni in [BGK99]. Merris [Mer89] in 1989 considered the edge version of the Laplacian matrix and established a bridge between its cofactors with the Wiener index, which has several applications in chemistry. A combinatorial interpretation of the minors of the edge version of the Laplacian matrix of a tree was given by Bapat, Grossman, and Kulkarni in [BGK00]. The all minor matrix tree theorem find their use in many different areas like electrical networks [Kir47], social networks [CS97], statistical design [Con87], chemistry [Mer89]. and have been the subject of many independent studies.

Recall that a combinatorial description of all minor of the usual Laplacian matrix of a graph was supplied by Chaiken in [Cha82]. Quite similar to that, we also provide a combinatorial description of all minor of the bipartite Laplacian matrix of a nonsingular tree. In particular, we observe that a minor of the bipartite Laplacian matrix of a nonsingular tree T obtained by deleting k many rows in X and k many columns in Y enumerates the number of spanning forests in T that have (a) k trees, (b) each tree contains exactly one vertex in X and exactly one vertex in Y. Our all minor result on the bipartite Laplacian matrix (that is, Theorem 4.5.1) provides an alternative way to compute the number of the above mentioned spanning forest in a nonsingular tree T.

4.1 Preliminaries

The following is a well-known result, and proof of that can be found in existing literature (see, for example [Bap14, Lemma 2.8]).

▶ **Lemma 4.1.1.** *Let M be a square matrix of order n such that M has a zero submatrix of order $r \times s$ where $r+s \geq n+1$. Then* $\det M = 0$. ◀

Let M be a $m \times n$ and $X \subseteq \{1,\dots,m\}$, $Y \subseteq \{1,\dots,n\}$. Recall that the submatrix of M obtained by deleting the rows in X and the columns in Y is denoted by $M(X \mid Y)$. The submatrix of M determined by the rows corresponding to X and columns corresponding to Y is denoted by $M(X,Y)$.

Let us recall the following result due to Chaiken [Cha82] which provides a combinatorial description of a minor of the usual Laplacian matrix of an undirected graph G.

▶ **Theorem 4.1.2** ([Cha82]). *Let G be a graph and $\mathcal{L}$ be its usual Laplacian matrix. Suppose X,Y are two subsets of $V(G)$ such that $|X| = |Y|$. Then the absolute value of the the determinant of the submatrix $\mathcal{L}(X \mid Y)$ of $\mathcal{L}$ is the number of the spanning forest F in G such that*

(a) F has exactly $|X| = |Y|$ trees,

(b) each tree in F contains exactly one vertex in X and one vertex in Y. ◀

In Section 4.5, we provide a faster way to enumerate the number of spanning forests as described in Theorem 4.1.2 when the underlying graph is a nonsingular tree.

Let T be a nonsingular tree with a standard vertex bipartition (L,R). Let $\mathfrak{L}$ be the bipartite Laplacian matrix of T. Suppose $X \subseteq R$, $Y \subseteq L$. By the notation $\mathfrak{L}(X \mid Y)_{r_i,:}$, we mean the row in $\mathfrak{L}(X \mid Y)$ corresponding to the vertex $r_i \notin X$. In a similar way, by the notation $\mathfrak{L}(X \mid Y)_{:,l_i}$, we mean the column in $\mathfrak{L}(X \mid Y)$ corresponding to the vertex $l_i \notin Y$. The symbols $\mathfrak{L}(X,Y)_{r_j,:}$ and $\mathfrak{L}(X,Y)_{:,l_k}$ have a similar meaning, where $r_j \in X$ and $l_k \in Y$.

▶ **Lemma 4.1.3.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L,R) and $\mathfrak{L}$ be its bipartite Laplacian matrix. Suppose $X \subseteq R$, $Y \subseteq L$ with $|X| = |Y|$. Let $v \in T$ and $\widehat{T}$ be the tree obtained from T by attaching a new P_2 at v in T. Suppose $\widehat{\mathfrak{L}}$ is the bipartite Laplacian matrix of the tree $\widehat{T}$. Then we have*

$$\det \mathfrak{L}(X \mid Y) = \det \widehat{\mathfrak{L}}(X \mid Y).$$ ◀

Proof. With out loss of any generality, assume that $v = l_s$ for some $s = 1,\dots,p$. Let $\widehat{T}$ be the tree obtained from T by adding the path $[l_{p+1}, r_{p+1}, l_s]$ in T where $l_{p+1}, r_{p+1} \notin T$. (See Figure 4.1.) Clearly $d(l_{p+1}) + d(r_{p+1}) = 3$ with $r_{p+1} \notin X$ and $l_{p+1} \notin Y$. Let $r_s, r_{k_1}, \dots, r_{k_m}$ be the vertices other than r_p that are adjacent to l_s. We consider following two cases.

Case 1. Suppose $l_s \in Y$. By $\widehat{\mathfrak{L}}(X \mid Y)_{r_{p+1},:}$ we mean the row in $\widehat{\mathfrak{L}}(X \mid Y)$ that corresponds to the vertex r_{p+1}. By Lemma 3.2.1, it follows that $\widehat{\mathfrak{L}}(X \mid Y)_{r_{p+1},:} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$. By expanding the determinant of $\widehat{\mathfrak{L}}(X \mid Y)$ along the last row yields us

$$\det \widehat{\mathfrak{L}}(X \mid Y) = \det \widehat{\mathfrak{L}}(X \cup \{r_{p+1}\} | Y \cup \{l_{p+1}\}).$$

Clearly $d_T(v) = d_{\widehat{T}}(v)$ for each $v \in T - l_s$. Since $l_s \in Y$, it follows that

$$\widehat{\mathfrak{L}}(X \cup \{r_{p+1}\} | Y \cup \{l_{p+1}\}) = \mathfrak{L}(X \mid Y).$$

Therefore, we get

$$\det \mathfrak{L}(X \mid Y) = \det \widehat{\mathfrak{L}}(X \mid Y).$$

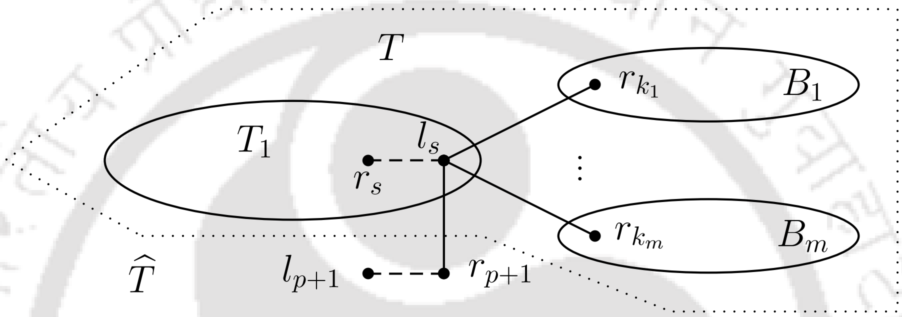


Figure 4.1: Minor of the bipartite Laplacian matrix does not change on adding a new P_2.

Case 2. Suppose $l_s \notin Y$. Let $B_1, \ldots, B_m$ be the branches of $\widehat{T}$ at l_s that do not contain r_s and r_{p+1}. Let T_1 be the tree obtained from $\widehat{T}$ by deleting the vertices l_{p+1}, r_{p+1} and the branches $B_1, \ldots, B_m$. Clearly $d_{\widehat{T}}(l_s) = d_T(l_s) + 1$ and $d_{\widehat{T}}(v) = d_T(v)$ for each $v \in T - l_s$. By Remark 3.3.7, it follows that $\widehat{\mathfrak{L}}(r_{P+1} | l_{p+1}, l_s) = \mathfrak{L}(: | l_s)$.

Let $i \in \{1, \ldots, p\}$. Clearly the l_s-r_i path is an alternating path if and only the l_{p+1}-r_i path is an alternating path. First notice that,

$$\widehat{\mathfrak{L}}(r_s, l_s) + \widehat{\mathfrak{L}}(r_s, l_{p+1}) = [d_{\widehat{T}}(r_s) d_{\widehat{T}}(l_s) - 1] + [-d_{\widehat{T}}(r_s)] = d_T(r_s) d_T(l_s) - 1 = \mathfrak{L}(r_s, l_s).$$

If $i \neq s$ and the r_i-l_s path is an alternating path and $i \neq s$ then we have

$$\begin{aligned} \widehat{\mathfrak{L}}(r_i, l_s) + \widehat{\mathfrak{L}}(r_i, l_{p+1}) &= (-1)^t [d_{\widehat{T}}(r_i) d_{\widehat{T}}(l_s)] + (-1)^{t+1} [d_{\widehat{T}}(r_i)] \\ &= (-1)^t d_T(r_i) d_T(l_s) = \mathfrak{L}(r_i, l_s), \end{aligned}$$

where $t = (\mathtt{dist}(r_i, l_s) - 1)/2$. Further note that if $r_i \sim l_s$, $i \neq s$, then $\widehat{\mathfrak{L}}(r_i, l_{p+1}) = 0$ and so $\widehat{\mathfrak{L}}(r_i, l_s) + \widehat{\mathfrak{L}}(r_i, l_{p+1}) = \mathfrak{L}(r_i, l_s)$. It follows that

$$\widehat{\mathfrak{L}}(r_i, l_s) + \widehat{\mathfrak{L}}(r_i, l_{p+1}) = \mathfrak{L}(r_i, l_s) \quad \text{for each } i \neq p+1.$$

By Lemma 3.2.1, $\widehat{\mathfrak{L}}(r_{p+1}, :) = e^t_{p+1} - e^t_s$. Therefore, by adding the column of $\widehat{\mathfrak{L}}(X \mid$

Y) corresponds to l_{p+1} to the column of $\widehat{\mathfrak{L}}(X \mid Y)$ corresponds to l_s and expanding the determinant along the last row yields us

$$\det \widehat{\mathfrak{L}}(X \mid Y) = \det \mathfrak{L}(X \mid Y).$$

This completes the proof. ■

▶ **Definition 4.1.4 (Vertex disjoint paths).** Let G be a connected graph. Two paths P_1 and P_2 are said to be *vertex disjoint paths* in G if they do not have any vertex in common, including the end vertices. ◀

Following result can be found in the literature but for the sake of completeness, we give the proof here.

▶ **Lemma 4.1.5.** *Let T be a tree (not necessarily to be a nonsingular) with the vertex set V. Let X and Y be the disjoint subset of V with $|X| = |Y|$. Then there exist at most one bijection $\sigma : X \to Y$ such that all x-$\sigma(x)$ path in T are vertex disjoint paths, for $x \in X$.* ◀

Proof. Let $\sigma : X \to Y$ be the bijection such that all x-$\sigma(x)$ path in T are vertex disjoint paths, for $x \in X$. If possible let, there exist another bijection $\gamma : X \to Y$ such that all x-$\gamma(x)$ path in T are vertex disjoint paths, for $x \in X$. Let us order the sets $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ such that $\sigma(x_i) = y_i$ for $i = 1, \ldots, k$. Note that $x_i \neq y_j$ for each i, j.

Construct a graph G with the vertex set $V(G) = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ and $x_i \sim y_j$ in G if and only if either $\sigma(x_i) = y_j$ or $\gamma(x_i) = y_j$. Clearly, G is a bipartite graph with the maximum possible degree of the vertex $v \in G$ is two. Notice that each edge in G represents a unique path in the tree T.

We claim that $d_G(x_i) = 2$ if and only if $d_G(y_i) = 2$. In order to verify our claim, let us assume $d_G(x_i) = 2$. It follows that, $\gamma(x_i) \neq y_i$. Since γ is a bijection, there exist $j \neq k$ such that $\gamma(x_j) = y_i$. It follows that $d_G(y_i) = 2$. In a similar way, we can argue that if $d_G(y_i) = 2$ then $d_G(x_i) = 2$.

Let $J = \{i \in [k] \ : \ d_G(x_i) = d_G(y_i) = 2\}$. Since $\sigma \neq \gamma$, J is nonempty. With out loss of any generality, let us assume that $J = \{1, 2, \ldots, t\}$, $t \leq k$. Let $G[J]$ be the induced subgraph of G induced by the vertex set $\{x_1, \ldots, x_t, y_1, \ldots, y_t\}$. Since degree of each vertex in $G[J]$ is two, all connected component of $G[J]$ is a cycle.

Now note that each edge in $G[J]$ corresponds to a path in the tree T. It follows that the cycle in $G[J]$ corresponds a cycle in the tree T, which is a contradiction. This completes the proof. ■

4.2 Vertex-weighted tree

Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$. We have seen in Lemma 4.1.5 that if there exist a bijection $\sigma : X \to Y$ such that all x-$\sigma(x)$ paths are vertex disjoint, for $x \in X$, then σ is unique. Our next aim is to find a necessary and sufficient condition that guarantees an existence of such bijection.

▶ **Definition 4.2.1.** Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$. A vertex $v \in T$ is called a *selected vertex* if $v \in X \cup Y$.

By the X-Y *path system* in T we mean the bijection $\sigma : X \to Y$ such that all x-$\sigma(x)$ paths in T are vertex disjoint paths, for $x \in X$. ◀

We can construct a vertex weighted tree from T depending on the choice of $X \subseteq R$ and $Y \subseteq L$ which is defined below.

▶ **Definition 4.2.2.** Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$. We assign a weight $w(v)$ to each vertex $v \in T$, where $w(v)$ is defined as

$$w(v) = \begin{cases} +1 & \text{if } v \in X, \\ -1 & \text{if } v \in Y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

In through out of this chapter, by a weight of a vertex in T, we mean the weight of the vertex v as defined in (4.1). ◀

The weight of a branch of T can be defined in the following way.

▶ **Definition 4.2.3.** Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$. Let B be a branch of T (at some vertex v). By $w(B)$ we mean the sum of the weight of all vertices in B, that is,

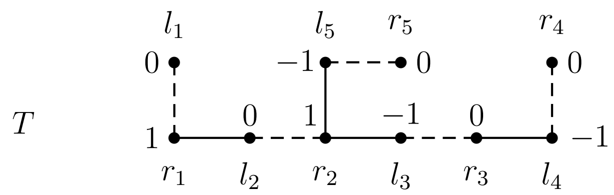
$$w(B) := \sum_{v \in B} w(v).$$

Let $[u, v]$ be an edge in T. By $w_u([u, v])$ we mean the weight of the branch at v that contains the vertex u. The notation $w_v([u, v])$ has a similar meaning. The weight $w([u, v])$ of an edge $[u, v]$ in T is defined as the product of $w_u([u, v])$ and $w_v([u, v])$, that is

$$w([u, v]) := w_u([u, v]) w_v([u, v]).$$ ◀

In order to illustrate the above mentioned definitions we consider the following example.

▶ **Example 4.2.4.** Let T be the tree as shown below. Here the dashed edges are the matching edges. Here we consider $X = \{r_1, r_2\}$ and $Y = \{l_3, l_4, l_5\}$.



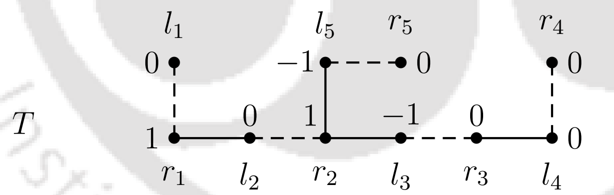
The numerical value near each vertex represents the weight of that vertex in the tree T. Consider the edge $[l_2, r_2]$. Note that the branch at l_2 that contains the vertex r_2 is the subtree of T induced by the vertex set $\{r_2, r_3, r_4, r_5, l_3, l_4, l_5\}$. We can see that

$$w_{r_2}([l_2, r_2]) = w(r_2) + \sum_{i=3}^{5}(w(r_i) + w(l_i)) = w(r_2) + w(l_3) + w(l_4) + w(l_5) = -2.$$

In a similar way, we can see that $w_{l_2}([l_2, r_2]) = w(l_1) + w(r_1) + w(l_2) = 1$. Therefore, the weight of the edge $[l_2, r_2]$ is given by

$$w([l_2, r_2]) = w_{l_2}([l_2, r_2])w_{r_2}([l_2, r_2]) = -2.$$ ◀

▶ **Example 4.2.5.** Let T be the tree as shown below. Here the dashed edges are the matching edges. Here we consider $X = \{r_1, r_2\}$ and $Y = \{l_3, l_5\}$.



The numerical value near each vertex represents the weight of that vertex in the tree T. Consider the edge $[l_2, r_2]$. Note that the branch at l_2 that contains the vertex r_2 is the subtree of T induced by the vertex set $\{r_2, r_3, r_4, r_5, l_3, l_4, l_5\}$. We can see that

$$w_{r_2}([l_2, r_2]) = w(r_2) + \sum_{i=3}^{5}(w(r_i) + w(l_i)) = w(r_2) + w(l_3) + w(l_5) = -1.$$

In a similar, way we can see that

$$w_{l_2}([l_2, r_2]) = w(l_1) + w(r_1) + w(l_2) = 1.$$

Therefore, $w([l_2, r_2])$, the weight of the edge $[l_2, r_2]$ is -1. ◀

In Example 4.2.5, we observe that $|X| = |Y|$ and $w_{l_2}([l_2, r_2]) = -w_{r_2}([l_2, r_2])$. This observation leads us following remark.

▶ **Remark 4.2.6.** Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$.

(a) If $|X| = |Y|$ then $w(T) = 0$ and so $w_u([u, v]) = -w_v([u, v])$, that is, $w([u, v]) < 0$ for each edge $[u, v]$ in T.

(b) If $X = L$ and $Y = R$ then the weight of all matching edges in T are -1 and the weight of all non-matching edges are zero. ◀

Out of the curiosity, one would ask that whether the existence of a X-Y path system implies $w([u, v]) \in \{0, -1\}$ for each edge $[u, v]$ in T. The following result supplies an affirmative answer to that question.

▶ **Lemma 4.2.7.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$. Suppose that there exist a X-Y path system in T then the following assertions hold.*

(a) $w([u, v]) \in \{0, -1\}$ for each $[u, v]$ in T.

(b) $w(B) \in \{0, 1, -1\}$ for each branch B (at some vertex v) in T. ◀

Proof of item (a). Let $[u, v]$ be an edge in T and σ be a X-Y path system in T. Since σ is a bijection from X to Y, it follows that $|X| = |Y|$. It follows that $w_u([u, v]) = -w_v([u, v])$ and so $w([u, v]) \leq 0$ for each $[u, v] \in T$. If possible, let us assume that there exist an edge $[u, v] \in T$ such that $w([u, v]) = -t$ where $t \geq 2$. With out loss of any generality, we assume that $w_v([u, v]) = t_1$ and so $w_u([u, v]) = -t_1$ where t_1 is a positive integer such that $t = t_1^2$. Clearly $t_1 > 1$. Let T_u and T_v be the components of $T - [u, v]$ that contains the vertex u and v, respectively. Since $w_v([u, v]) = t_1 > 1$, there exist $x_i \in X$ in the tree T_v such that $\sigma(x_i) \in T_u$ for each $i = i_1, \ldots, i_{t_1}$. It follows that all x_i-$\sigma(x_i)$ paths in T must contains the edge $[u, v]$ for each $i = i_1, \ldots, i_{t_1}$. Since $t_1 > 1$, it contradicts that σ is a X-Y path system in T. This completes the proof.

Proof of item (b). Let B be a branch at some vertex $v \in T$ and σ be a X-Y path system in T. If possible, let us assume that $|w(B)| \geq 2$. We take $w(B) \geq 2$. The case $w(B) \leq -2$ can be dealt in a similar way. Since $w(B) \geq 2$, there exist $x_1, x_2 \in B \cap X$ such that $\sigma(x_i) \notin B$ for $i = 1, 2$. It follows that both the x_1-$\sigma(x_1)$ path and the x_2-$\sigma(x_2)$ path passes through the vertex v. This contradicts that σ is a X-Y path system in T. This completes the proof. ■

Note that the converse of Lemma 4.2.7 is not true in general, see Figure 4.2 for a counter-example.

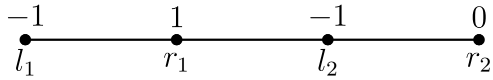


Figure 4.2: Here $X = \{r_1\}$, $Y = \{l_1, l_2\}$. The weight of each branch is either 0, 1 or -1 but there does not exist any bijection between X and Y as $|X| \neq |Y|$.

Figure 4.3 tells that in order to get the converse of Lemma 4.2.7 to be true, only including the condition $|X| = |Y|$ is not sufficient. We need something more to supply a sufficient condition that guarantees the existence of the X-Y path system in T.

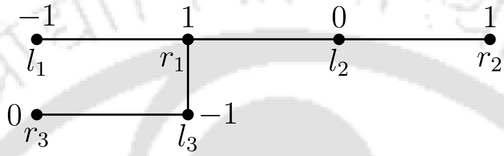


Figure 4.3: Here $X = \{r_1, r_2\}$, $Y = \{l_1, l_3\}$. The weight of each branch is either 0, 1 or -1 but there does not exist any X-Y path system as all l_1-r_x paths and all l_3-r_x paths passes through the vertex r_1 for each $r_x \in X$.

In the following result we supplies a necessary and sufficient condition that guaranties an existence of the X-Y path system in T.

▶ **Theorem 4.2.8.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$. Then there exist a X-Y path system in T if and only if the following conditions hold:*

(i) $|X| = |Y|$.

(ii) If $v \notin X \cup Y$ then either the weights of all branches at v are zero or there exist exactly two branches B_1 and B_2 at v with $w(B_1) = -w(B_2) = 1$ and the weights of all other branches at v are zero.

(iii) If $v \in X$ then there exist an unique branch B at v with $w(B) = -1$ and the weights of all other branches at v are zero.

(iv) If $v \in Y$ then there exist an unique branch B at v with $w(B) = 1$ and the weights of all other branches at v are zero. ◀

Proof. Let us first assume that σ be a X-Y path system in T. Clearly, $|X| = |Y|$. Let v be a vertex in T with $d(v) = m \geq 1$. Suppose v is adjacent to $v_1, \ldots, v_m$. Let B_i be the branch at v that contains the vertex v_i, for each $i = 1, \ldots, m$.

First assume that $v \notin X \cup Y$. If all $w(B_i) = 0$ for $i = 1, \ldots, m$ then there is nothing to prove. Let $w(B_1) \neq 0$. Then $w(B_1) = \pm 1$, by Lemma 4.2.7. We consider $w(B_1) = 1$. The case $w(B_1) = -1$ can be dealt in a similar way. Since $w(v) = 0$ and $w(T) = 0$, it follows that

$\sum_{i=2}^{m} w(B_i) = -1$ and so there exist $i = 2, \ldots, m$ such that $w(B_i) = -1$, say $w(B_2) = -1$. Therefore, there exist $r_x \in B_1 \cap X$ and $l_y \in B_2 \cap Y$ such that the r_x-l_y path contains the vertex v and so $\sigma(r_x) = l_y$. Since all x-$\sigma(x)$, $x \in X$, paths are vertex disjoint, it follows that $\sigma(x) \in B_i$ if $x \in B_i \cap X$ for each $i = 3, \ldots, m$. Therefore, $w(B_i) = 0$ for each $i = 3, \ldots, m$.

Let $v \in X$. Since $w(T) = 0$, it follows that $\sum_{i=1}^{m} w(B_i) = -1$. Therefore, there exist $i = 1, \ldots, m$ such that $w(B_i) = -1$, say $w(B_1) = -1$. It follows that $\sigma(v) \in B_1$. Therefore, x-$\sigma(x)$ path does not contains the vertex v, for $x \in X - v$. It follows that $w(B_i) = 0$ for each $i = 2, \ldots, m$. In a similar way we can argue that if $v \in Y$ then there exist $i = 1, \ldots, m$ with $w(B_i) = 1$ and $w(B_j) = 0$ for $j \neq i$.

Conversely, assume that the conditions (i)-(iv) are hold in T. Since $w(B) \in \{0, 1, -1\}$ for each branch B in T, it follows that $w(e) \in \{0, -1\}$ for each edge $e \in T$.

Our first aim is to define a suitable bijection from X to Y. Let v be an arbitrary vertex in X. By (iii), there exist an unique vertex u adjacent to v such that $w([u, v]) = -1$. Let P_v be a longest path that originated from v with weights of all of its edge is -1, that is, $w(e) = -1$ for each edge $e \in P_v$. Let y be an end vertex of P_v other than v. We define $\sigma(v) = y$.

In order to verify σ is well defined we need to show that $y = \sigma(v) \in Y$. Notice that if $v^\circ \in P_v$ is an internal vertex of P_v, that is $v^\circ \neq v, y$ then there exist two edges incidence to v° with the weight -1 and so by (ii) we have $v^\circ \notin X \cup Y$. Furthermore, by (iii), the weights of all branches at v° that does not contain any vertex of P_v are zero. It follows that if $[v', v''] \in P_v$ with $\mathtt{dist}(v, v') < \mathtt{dist}(v, v'')$ then $w_{v'}([v', v'']) = 1$. Since P_v is a longest path that originated from v with weights of all of its edge is -1, there is exactly one branch B at y with $w(B) \neq 0$ and the weights of all branch at y other than B are zero. Let $[u, y] \in P_v$. Then $w_u([u, y]) = 1$. It follows that $u \in B$ and $w(B) = 1$. Therefore, by (iv), we have $y \in Y$.

Since $|X| = |Y|$, in order to conclude σ is a bijection it is enough to argue that σ is an one-one map. Let $\sigma(x_1) = \sigma(x_2)$ for some $x_1, x_2 \in X$. Let P_{x_i} be the x_i-$\sigma(x_i)$ path for $i = 1, 2$. Clearly, the weight of all edges of P_{x_i} are -1 for $i = 1, 2$. Let u be the closest vertex from x_1 such that u is in the both P_{x_1} and P_{x_2}. Notice that $u \neq \sigma(x_1)$, otherwise it contradict that P_{x_1} is the longest path that originated from x_1 with weights of all of its edge is -1. Also note that if u is an internal vertex in P_{x_1} then there exist at least three branches at u with the weight of each of those branches are nonzero. This is a contradiction to our assumption. So the only possibility is $u = x_1$. Therefore $u \in X$ and so by (iii), there exist an unique edge with nonzero weight that incidence to the vertex x_1, it follows that $u = x_2$. This shows that σ is a one-one map. In a similar way, we can conclude that all the x-$\sigma(x)$, $x \in X$, paths in T are vertex disjoint. This completes the proof. ∎

4.3 Singular submatrices of the bipartite Laplacian matrix

In this section we shall discuss how the existence of a X-Y path system plays a crucial role on determining the value of $\det \mathfrak{L}(X,Y)$. In particular, we show that if there does not exist any X-Y path system in T then the matrix $\mathfrak{L}(X,Y)$ is singular.

By Theorem 4.2.8, we know that if any one of the condition among (i)-(iv) does not hold in Theorem 4.2.8 then we can not have any X-Y path system. In order to talk the determinant of $\mathfrak{L}(X,Y)$ we must need $|X| = |Y|$. So, we only focus on the cases on which one of the condition among (ii)-(iv) does not hold in T.

The following remark is directly follows from Theorem 4.2.8.

► **Remark 4.3.1.** Let T be a nonsingular tree with a standard vertex bipartition (L,R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$. Then there does not exist any X-Y path system in T if one of the following condition holds.

(i) There exist a branch B of T (at some vertex v) such that $|w(B)| \geq 2$.

(ii) If $v \notin X \cup Y$ and there exist two branches B_1 and B_2 at the vertex v such that $|w(B_1) + w(B_2)| \geq 2$.

(iii) If $v \in X \cup Y$ and there exist a branch B at the vertex v such that $|w(v) + w(B)| \geq 2$. ◄

Now our main objective is to argue that for each of the item of Remark 4.3.1 the determinant of the corresponding submatrix of the bipartite Laplacian matrix is zero.

► **Theorem 4.3.2.** *Let T be a nonsingular tree with a standard vertex bipartition (L,R) and $\mathfrak{L}$ be its bipartite Laplacian matrix. Let $X \subseteq R$ and $Y \subseteq L$ with $|X| = |Y|$. Suppose that there exist a branch B at some vertex $u \in T$ such that $|w(B)| \geq 2$. Then* $\det \mathfrak{L}(X \mid Y) = 0$. ◄

Proof. Let T be a nonsingular tree on $2p$ vertices. Let B be a branch at some vertex $u \in T$ such that $|w(B)| \geq 2$. Consider $w(B) = t \geq 2$. The case $w(B) \leq -2$ can be dealt in similar way. Let $v \in B$ be adjacent to u. Let F be the branch at v that contains the vertex u. (See Figure 4.4.) Suppose $|F \cap X| = m$ and $|B \cap X| = n$. Then $|F \cap Y| = m + t$ and $|B \cap Y| = n - t$. We denote the set $B \cap L$ by B^L and $B \cap R$ by B^R. The notations F^L and F^R have similar meaning. Let $r = |F^R - X|$ and $s = |B^L - Y|$. Note that the size of the square matrix $\mathfrak{L}(X \mid Y)$ is $p - m - n$.

We divide the proof into the following two cases.

Case 1. Suppose $u \in R$. (Clearly, $v \in L$.) First assume that $[u,v]$ is not a matching edge. Then there does not exist any alternating path starting from a vertex in F^R to a vertex in

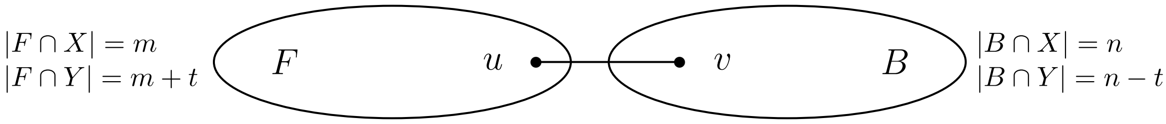


Figure 4.4: For the edge $[u, v]$, B be the branch at u that contains v and F be the branch at v that contains u.

B^L. This shows that $\mathfrak{L}(r_i, l_j) = 0$ for each $r_i \in F^R - X - u$ and $l_i \in B^L - Y$. Therefore, $\mathfrak{L}(X \mid Y)$ has a zero submatrix of order $(r-1) \times s$.

Since $[u, v]$ is not a matching edge, both B and F are nonsingular trees. Let $|B^L| = k$. Then $|F^R| = p - k$. Now notice that

$$\begin{aligned} r + s - 1 &= |F^R - X| + |B^L - Y| - 1 \\ &= [(p-k) - m] + [k - (n-t)] - 1 \\ &= p - m - n + (t-1). \end{aligned}$$

Since $t - 1 \geq 1$, by Lemma 4.1.1, it follows that $\det \mathfrak{L}(X \mid Y) = 0$.

Now we assume that $[u, v]$ is a matching edge. Let $\boldsymbol{\mu}_v$ be the signed degree vector of T at v. Suppose $\boldsymbol{y}$ be the restriction of $\boldsymbol{\mu}_v$ on the set $F^R - X - u$. Note that, for $r_i \in F^R - X - u$, we have

$$\boldsymbol{y}(r_i) = \begin{cases} d(r_i) & \text{if the } v\text{-}r_i \text{ path is an odd alternating path,} \\ -d(r_i) & \text{if the } v\text{-}r_i \text{ path is an even alternating path,} \\ 0 & \text{otherwise.} \end{cases}$$

For a vertex $z \in B^L$, we define $\epsilon(z) = (-1)^{\texttt{dist}(v,z)/2}$ if the u-z path is an alternating path and 0 otherwise.

Now notice that for each $z \in B^L$ we have

$$\mathfrak{L}(r_i, z) = \epsilon(z) d(z) \boldsymbol{y}(r_i) \quad \text{for each } r_i \in F^R - X - u.$$

It follows that

$$\mathfrak{L}(F^R - X - u, z) = \epsilon(z) d(z) \boldsymbol{y} \quad \text{for each } z \in B^L.$$

Let Z be the matrix obtained from $\mathfrak{L}(X \mid Y)$ by applying a sequence of column operation (of type adding a multiple of a column to another) that transform each columns of the submatrix $\mathfrak{L}(F^R - X - u, B^L - Y)$ to a zero column except possibly one column. Note that $\det Z = \det \mathfrak{L}(X \mid Y)$. Clearly, the matrix Z has a zero submatrix of order $(r-1) \times (s-1)$.

Let us assume $|B^R| = k$. Since $[u, v]$ is a matching edge and $u \in R$, it follows that

$|B^L| = k+1$ and $|F^R| = p-k$. Therefore, we get

$$\begin{aligned} r+s-2 &= |F^R - X| + |B^L - Y| - 2 \\ &= [(p-k)-m] + [(k+1)-(n-t)] - 2 \\ &= p-m-n+(t-1). \end{aligned}$$

Since $t-1 \geq 1$, by Lemma 4.1.1, it follows that $\det \mathfrak{L}(X \mid Y) = 0$.

Case 2. Suppose $u \in L$. (Clearly, $v \in R$.) First assume that $[u,v]$ is a matching edge. Then there does not exist any alternating path starting from a vertex in F^R to a vertex in B^L. This shows that $\mathfrak{L}(r_i, l_j) = 0$ for each $r_i \in F^R - X$ and $l_i \in B^L - Y$. Therefore, $\mathfrak{L}(X \mid Y)$ has a zero submatrix of order $r \times s$. Let $|B^R| = k$. Since $[u,v]$ is a matching edge and $v \in R$, it follows that $|B^L| = k-1$ and $|F^R| = p-k$. Therefore, we get

$$r+s = [(p-k)-m] + [(k-1)-(n-t)] = p-m-n+(t-1).$$

Since $t-1 \geq 1$, by Lemma 4.1.1, it follows that $\det \mathfrak{L}(X \mid Y) = 0$.

Finally, we assume that $[u,v]$ is not a matching edge. Let $\boldsymbol{\mu}_u$ be the signed degree vector of T at u. Suppose $\bar{\boldsymbol{y}}$ be the restriction of $\boldsymbol{\mu}_u$ on the set $F^R - X$. Note that, for $r_i \in F^R - X$, we have

$$\bar{\boldsymbol{y}}(r_i) = \begin{cases} d(r_i) & \text{if the } u\text{-}r_i \text{ path is an odd alternating path,} \\ -d(r_i) & \text{if the } u\text{-}r_i \text{ path is an even alternating path,} \\ 0 & \text{otherwise.} \end{cases}$$

For a vertex $z \in B^L$, we define $\epsilon(z) = (-1)^{\mathtt{dist}(u,z)/2}$ if the u-z path is an alternating path and 0 otherwise.

Now notice that for each $z \in B^L$ we have

$$\mathfrak{L}(r_i, z) = \epsilon(z) d(z) \bar{\boldsymbol{y}}(r_i) \quad \text{for each } r_i \in F^R - X.$$

It follows that

$$\mathfrak{L}(F^R - X, z) = \epsilon(z) d(z) \bar{\boldsymbol{y}} \quad \text{for each } z \in B^L.$$

Let $\bar{Z}$ be the matrix obtained from $\mathfrak{L}(X \mid Y)$ by applying a sequence of column operation (of type adding a multiple of a column to another) that transform each columns of the submatrix $\mathfrak{L}(F^R - X, B^L - Y)$ to a zero column except possibly one column. Note that $\det \bar{Z} = \det \mathfrak{L}(X \mid Y)$. Clearly, the matrix $\bar{Z}$ has a zero submatrix of order $r \times (s-1)$.

Let us assume $|B^R| = k$. Since $[u,v]$ is not a matching edge, both the trees F and B are nonsingular. It follows that $|B^L| = k$ and $|F^R| = p - k$. Therefore, we get

$$\begin{aligned} r + s - 1 &= |F^R - X| + |B^L - Y| - 1 \\ &= [(p-k) - m] + [k - (n-t)] - 1 \\ &= p - m - n + (t-1). \end{aligned}$$

Since $t - 1 \geq 1$, by Lemma 4.1.1, it follows that $\det \mathfrak{L}(X \mid Y) = 0$. This completes the proof. ■

The idea of the proof of the below result is similar to that we used in the proof of Theorem 4.3.2.

▶ **Theorem 4.3.3.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R) and $\mathfrak{L}$ be its bipartite Laplacian matrix. Let $X \subseteq R$ and $Y \subseteq L$ with $|X| = |Y|$. If there exists a vertex $u \notin X \cup Y$ such that $|w(B') + w(B'')| \geq 2$ for some branches B', B'' of T at the vertex u, then* $\det \mathfrak{L}(X \mid Y) = 0$. ◀

Proof. Let T be a nonsingular tree on $2p$ vertices. If either $|w(B')| \geq 2$ or $|w(B'')| \geq 2$ then the result follows by Theorem 4.3.2. With out loss of any generality, we assume that $w(B') = \pm 1$ and $w(B'') = \pm 1$. Since $|w(B') + w(B'')| \geq 2$, it follows that either $w(B') = w(B'') = 1$ or $w(B') = w(B'') = -1$. We consider $w(B') = w(B'') = 1$. The other case can be dealt in a similar way.

Let $v' \in B'$ and $v'' \in B''$ such that $u \sim v'$ and $u \sim v''$. Let F_1 be the component of $T - \{v', v''\}$ that contains the vertex u. Take $F_2 = B' \cup B''$. (See Figure 4.5.)

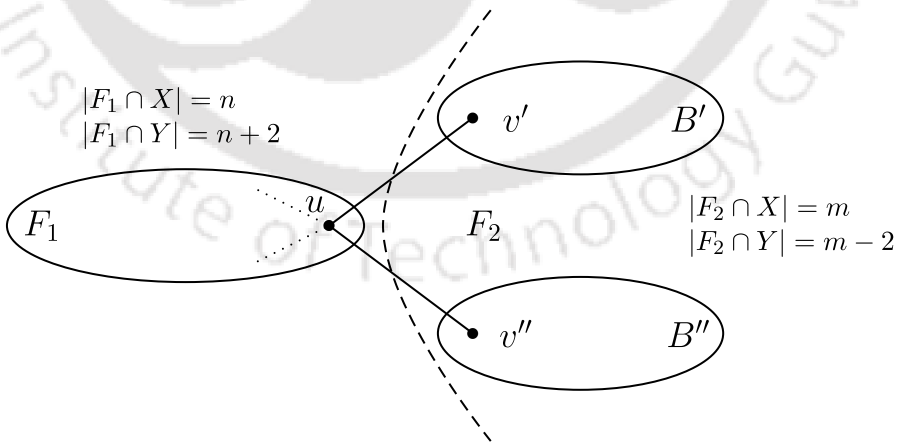


Figure 4.5: For the edge $[u, v]$, B be the branch at u that contains v and F be the branch at v that contains u.

Let $|B' \cap X| = m_1$, $|B' \cap Y| = m_1 - 1$, $|B'' \cap X| = m_2$, $|B'' \cap Y| = m_2 - 1$. Take $m = m_1 + m_2$. Then we have $|F_2 \cap X| = m$ and $|F_2 \cap Y| = m - 2$. Let $|F_1 \cap X| = n$ then

$|F_1 \cap Y| = n+2$. Suppose $r = |F_1^R - X|$ and $s = |F_2^L - Y|$. Clearly, the size of the matrix $\mathfrak{L}(X \mid Y)$ is $p-m-n$. The set $F_1 \cap L$ is denoted by F_1^L and the set $F_1 \cap R$ is denoted by F_1^R. The notations F_2^L and F_2^R have similar meaning. Now note that either $u \in L$ or $u \in R$. We divide the proof into the following two cases.

Case 1: Let us first consider $u \in R$. First assume that $[u, v']$ and $[u, v'']$ both are not matching edges. Then there does not exist any alternating path starting from a vertex in F_1^R to a vertex in F_2^L. This shows that $\mathfrak{L}(r_i, l_j) = 0$ for each $r_i \in (F_1^R - u) - X$ and $l_j \in F_2^L - Y$. Therefore, $\mathfrak{L}(X \mid Y)$ has a zero submatrix of order $(r-1) \times s$. Since both $[u, v']$ and $[u, v'']$ are not matching edges, F_1 is a nonsingular tree and F_2 is a forest consisting of two nonsingular trees. Let $|F_1^R| = k$. Then $|F_2^L| = p-k$. It follows that

$$\begin{aligned} r+s-1 &= |F_1^R - X| + |F_2^L - Y| - 1 \\ &= (k-n) + [(p-k) - (m-2)] - 1 \\ &= p-m-n+1. \end{aligned}$$

By Lemma 4.1.1, it follows that $\det \mathfrak{L}(X \mid Y) = 0$.

Now consider $[u, v']$ is a matching edge. (The case $[u, v'']$ is a matching edge can be dealt in a similar way.) Let $\boldsymbol{\mu}_{v'}$ be the signed degree vector of T at v'. Suppose $\boldsymbol{y}$ be the restriction of $\boldsymbol{\mu}_{v}$ on the set $F_1^R - X - u$. Note that, for $r_i \in F_1^R - X - u$, we have

$$\boldsymbol{y}(r_i) = \begin{cases} d(r_i) & \text{if the } v'\text{-}r_i \text{ path is an odd alternating path,} \\ -d(r_i) & \text{if the } v'\text{-}r_i \text{ path is an even alternating path,} \\ 0 & \text{otherwise.} \end{cases}$$

For a vertex $z \in F_2^L$, we define $\epsilon(z) = (-1)^{\mathtt{dist}(v',z)/2}$ if the u-z path is an alternating path and 0 otherwise. It is easy to note that $\epsilon(z) = 0$ for each $z \in B'' \cap L$. Now notice that, for each $z \in F_2^L$, we have

$$\mathfrak{L}(r_i, z) = \epsilon(z) d(z) \boldsymbol{y}(r_i) \quad \text{for each } r_i \in F_1^R - X - u.$$

Therefore, we get

$$\mathfrak{L}(F_1^R - X - u, z) = \epsilon(z) d(z) \boldsymbol{y} \quad \text{for each } z \in F_2^L.$$

Let Z be the matrix obtained from $\mathfrak{L}(X \mid Y)$ by applying a sequence of column operation (of type adding a multiple of a column to another) that transforms each columns of the submatrix $\mathfrak{L}(F_1^R - X - u, F_2^L - Y)$ to a zero column except possibly one column. Note that $\det Z = \det \mathfrak{L}(X \mid Y)$. Clearly, the matrix Z has a zero submatrix of order $(r-1) \times (s-1)$.

Let us assume $|F_1^R| = k$. Since $[u, v']$ is a matching edge and $u \in R$, it follows that $|F_1^L| = k - 1$ and so $|F_2^L| = p - (k - 1)$.

$$\begin{aligned} r + s - 2 &= (k - n) + [(p - (k - 1)) - (m - 2)] - 2 \\ &= p - m - n + 1. \end{aligned}$$

Hence, by Lemma 4.1.1, $\det \mathfrak{L}(X \mid Y) = 0$.

Case 2: Now we consider $u \in L$. First we assume that $[u, v']$ and $[u, v'']$ both are not matching edges. Let $\bar{\boldsymbol{y}}$ be the restriction of $\boldsymbol{\mu}_u$ on the set $F_1^R - X$. Then for each $r_i \in F_1^R - X$ we have

$$\bar{\boldsymbol{y}}(r_i) = \begin{cases} d(r_i) & \text{if the } u\text{-}r_i \text{ path is an odd alternating path,} \\ -d(r_i) & \text{if the } u\text{-}r_i \text{ path is an even alternating path,} \\ 0 & \text{otherwise.} \end{cases}$$

For a vertex $z \in B' \cap L$, we define $\epsilon(z) = (-1)^{(\texttt{dist}(v', z)+1)/2}$ if the v'-z path is an alternating path and 0 otherwise. Similarly, for a vertex $z \in B'' \cap L$, we define $\epsilon(z) = (-1)^{(\texttt{dist}(v'', z)+1)/2}$ if the v''-z path is an alternating path and 0 otherwise. Therefore ϵ is defined on F_2^L.

Therefore, for each $z \in F_2^L - Y$, we have

$$\mathfrak{L}(r_i, z) = \epsilon(z) d(z) \bar{\boldsymbol{y}}(r_i) \quad \text{for each } r_i \in F_1^R - X.$$

It follows that

$$\mathfrak{L}(F_1^R - X, z) = \epsilon(z) d(z) \bar{\boldsymbol{y}} \quad \text{for each } z \in F_2^L - Y.$$

Let $\bar{Z}$ be the matrix obtained from $\mathfrak{L}(X \mid Y)$ by applying a sequence of column operation (of type adding a multiple of a column to another) that transform each columns of the submatrix $\mathfrak{L}(F^R - X, F_2^L - Y)$ to a zero column except possibly one column. Note that $\det \bar{Z} = \det \mathfrak{L}(X \mid Y)$. Clearly, the matrix $\bar{Z}$ has a zero submatrix of order $r \times (s - 1)$.

Let $|F_1^R| = k$. Since $[u, v']$ and $[u, v'']$ are not matching edges, $|F_2^L| = p - k$. It follows that

$$r + s - 1 = (k - n) + [(p - k - (m - 2)] - 1 = p - m - n + 1.$$

Hence, by Lemma 4.1.1, $\det \mathfrak{L}(X \mid Y) = 0$.

Finally, we assume that $[u, v']$ is a matching edge. (The case $[u, v'']$ is a matching edge can be dealt in a similar way.) Notice that for each $r_i \in F_1^R$, the u-r_i path is not a alternating path. Furthermore, there is no alternating path from a vertex in $r_i \in F_1^R$ to a vertex in $l_j \in F_2^L$. This follows that $\mathfrak{L}(r_i, l_j) = 0$ for each $r_i \in F_1^R$ and $l_j \in F_2^L$. Therefore, $\mathfrak{L}(X \mid Y)$ has a zero submatrix of order $r \times s$. Let $|F_1^R| = k$. Since $[u, v']$ is a matching edge and $u \in R$, it follows that $|F_1^L| = k+1$ and so $|F_2^L| = p-k-1$. Therefore, we get

$$\begin{aligned} r+s &= (k-n) + [(p-k-1) - (m-2)] \\ &= p-m-n+1 \end{aligned}$$

By Lemma 4.1.1, it follows that $\det \mathfrak{L}(X \mid Y) = 0$. This completes the proof. ■

▶ **Theorem 4.3.4.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R) and $\mathfrak{L}$ be its bipartite Laplacian matrix. Let $X \subseteq R$ and $Y \subseteq L$ with $|X| = |Y|$. Suppose that there exists a vertex $u \in X \cup Y$ such that $|w(u) + w(B)| \geq 2$ for some branch B of T at the vertex u. Then $\det \mathfrak{L}(X \mid Y) = 0$.* ◀

Proof. Let $d(u) = m \geq 1$. Let $B_1, \ldots, B_m$ be the branches at u. With out loss of any generality, let us assume that $|w(u) + w(B_1)| \geq 2$. Note that if $|w(B_i)| \geq 2$ for some $i = 1, \ldots, m$ then the result directly follows by Theorem 4.3.2. Therefore, we assume that $|w(B_i)| < 2$ for each $i = 1, \ldots, m$.

Now note that the inequality $|w(u) + w(B_1)| \geq 2$ yields us either $w(u) = w(B_1) = 1$ or $w(u) = w(B_1) = -1$. Since $w(T) = 0$, it follows that either $\sum_{i=2}^{m} w(B_i) = -2$ or $\sum_{i=2}^{m} w(B_i) = 2$. In either of these cases there exist B_i and B_j with $2 \leq i < j \leq m$ such that $|w(B_i) + w(B_j)| = 2$. Hence the result follows by Theorem 4.3.3. ■

In view of Remark 4.3.1, we can unify Theorems 4.3.2, 4.3.3, and 4.3.4 into the following result.

▶ **Theorem 4.3.5.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R) and $\mathfrak{L}$ be its bipartite Laplacian matrix. Take subsets $X \subseteq R$ and $Y \subseteq L$ with $|X| = |Y|$. Suppose that there is no X-Y path system in T. Then $\det \mathfrak{L}(X \mid Y) = 0$.* ◀

By Theorem 4.3.5, note that, in order to get a nonzero minor of $\mathfrak{L}(T)$, we must choose $X \subseteq R$ and $Y \subseteq L$ such that T has a X-Y path system. In view of this, a general study of such path systems and their structural properties is required. We do that in the next section.

4.4 The class $\mathcal{F}_T^{X,Y}$

In this section, we introduce a class of spanning forests of a nonsingular tree T with a standard vertex bipartition (L, R), depending on the choice of $X \subseteq R$ and $Y \subseteq L$. The study of this class is crucial to obtain a formula for any minor of the bipartite Laplacian matrix. We shall see how the number of elements in this class changes under various operations done on the underlying tree. We start the discussion by defining this class $\mathcal{F}_T^{X,Y}$.

Let T be a nonsingular tree with a standard vertex bipartition (L, R). Suppose that $X \subseteq R$ and $Y \subseteq L$ are nonempty sets with $|X| = |Y| = k$ (say). Consider a spanning forest F of T with k components in which each component contains one vertex from X and one from Y. The collection of all such spanning forests F of T is denoted by $\mathcal{F}_T^{X,Y}$. ◀

The following is an immediate observation.

▶ **Lemma 4.4.2.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R). Suppose $X \subseteq R$ and $Y \subseteq L$ are nonempty sets with $|X| = |Y|$. The class $\mathcal{F}_T^{X,Y}$ is nonempty if and only if T has an X-Y path system.* ◀

Proof. Let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$. First assume that $F \in \mathcal{F}_T^{X,Y}$. Let $F_1, \ldots, F_k$ be the components of F. Without loss, assume that F_i contains the points $x_i \in X$ and $y_i \in Y$. Thus the x_i-y_i-path is contained in F_i. So T has a X-Y path system.

Conversely, assume that T has a X-Y path system. Without loss, assume that the path system contains the vertex disjoint paths P_i, $i = 1, \ldots, k$, where P_i is the x_i-y_i-path.

Let V_1 be the set of all those vertices of T from which there is a path to a vertex of P_1 which does not pass through any vertex of $P_2, P_3, \ldots, P_k$. Denote the induced graph $\langle V_1 \rangle$ by T_1.

In general, for $i > 1$, let V_i be the set of all those vertices of $T - V_1 - \cdots - V_{i-1}$ from which there is a path to a vertex of P_i which does not pass through any vertex of $P_{i+1}, \ldots, P_k$. Denote the induced graph $\langle V_i \rangle$ by T_i.

Then the forest F that contains $T_1, \ldots, T_k$ is in $\mathcal{F}_T^{X,Y}$. ■

▶ **Remark 4.4.3.** Let T be a nonsingular tree with a standard vertex bipartition (L, R). Suppose $X \subseteq R$ and $Y \subseteq L$ with $|X| = |Y|$. Let $\mathcal{F}_T^{X,Y} \neq \emptyset$. Then, from the proof of Lemma 4.4.2, we can see that each spanning forest $F \in \mathcal{F}_T^{X,Y}$ corresponds to a X-Y path system. By Lemma 4.1.5, we know that there can have at most one X-Y path system in T. It follows that all elements of $\mathcal{F}_T^{X,Y}$ corresponds to a unique X-Y path system. ◀

Now our aim is to see how the number of element in the class $\mathcal{F}_T^{X,Y}$ changes under some operation in the underlying tree T.

▶ **Lemma 4.4.4.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Suppose $X \subseteq R$, $Y \subseteq L$ with $|X| = |Y|$. Let $v \in T$ and $\widehat{T}$ be the tree obtained from T by attaching a new P_2 at v in T. Then we have $\mathcal{F}_T^{X,Y} = \mathcal{F}_{\widehat{T}}^{X,Y}$.* ◀

Proof. Let $u, z \notin T$ and let $\widehat{T}$ be the tree obtained from T by attaching a P_2, $[v, u, z]$, at v in T.

Let $F \in \mathcal{F}_T^{X,Y}$. Then there exist a component T_v of F that contain the vertex v. Let us construct a spanning forest $\widehat{F}$ of $\widehat{T}$ by attaching a new path P_2, $[v, u, z]$ at v in T_v. Clearly, $\widehat{F} \in \mathcal{F}_{\widehat{T}}^{X,Y}$.

Conversely, note that for each forest $F_1 \in \mathcal{F}_{\widehat{T}}^{X,Y}$, the vertices v, u, z always lies in a same component of F_1. Therefore, from a spanning forest $F_1 \in \mathcal{F}_{\widehat{T}}^{X,Y}$ of $\widehat{T}$, we can construct a forest $\bar{F}_1 \in \mathcal{F}_T^{X,Y}$ by deleting the vertices u and z. This completes the proof. ■

▶ **Lemma 4.4.5.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$. Suppose $\mathcal{F}_T^{X,Y} \neq 0$ and $w \in X \cup Y$ be a vertex in T. Let $\widehat{T}$ be the tree obtained from T by introducing two new vertices $u, v \notin T$ and adding the path $[w, v, u]$ in T. Let $(\widehat{L}, \widehat{R})$ be the standard vertex bipartition of $\widehat{T}$ obtained by extending the standard vertex bipartition (L, R) of T to $\widehat{T}$. Then we have*

$$\left|\mathcal{F}_T^{X,Y}\right| = \left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right|,$$

where $\widehat{X} = (X \cup \{u, v\}) \cap \widehat{R}$ and $\widehat{Y} = (Y \cup \{u, v\}) \cap \widehat{L}$. ◀

Proof. With out loss of any generality, let us assume $w \in X$. Then $\widehat{L} = L \cup \{v\}$ and $\widehat{R} = R \cup \{u\}$. It follows that $\widehat{X} = X \cup \{u\}$ and $\widehat{Y} = Y \cup \{v\}$.

Let σ be the X-Y path system in T. Let us define $\widehat{\sigma} : \widehat{X} \to \widehat{Y}$ such that $\widehat{\sigma}(x) = \sigma(x)$ for $x \in X$ and $\widehat{\sigma}(u) = v$. It follows that $\widehat{\sigma}$ is the $\widehat{X}$-$\widehat{Y}$ path system in $\widehat{T}$.

Now we establish one-one corresponding between the elements of the sets $\mathcal{F}_T^{X,Y}$ and $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ as follows.

- Let $F \in \mathcal{F}_T^{X,Y}$ be a spanning forest of T. Construct a spanning forest $\widehat{F}$ of $\widehat{T}$ by adding a new component in F which is a path $[u, v]$. Clearly, $\widehat{F} \in \mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$.
- Let $H \in \mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ be a spanning forest of $\widehat{T}$. Since $\widehat{\sigma}(u) = v$ and w is selected vertex, it follows that H has a component T_1 which is a path $[u, v]$. By removing T_1 from H we see that the remaining part is an element of $\mathcal{F}_T^{X,Y}$.

This completes the proof. ■

▶ **Lemma 4.4.6.** *Let T be a nonsingular tree with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$. Suppose σ is the X-Y path system in T. Let $[u, v]$ be an edge in a x-$\sigma(x)$ path, for some $x \in X$, such that either x is adjacent to u or $\sigma(x)$ is adjacent to v. Then*

$$\left|\mathcal{F}_T^{X,Y}\right| = \left|\mathcal{F}_T^{X\cup\{v\},Y\cup\{u\}}\right|.$$ ◀

Proof. We only consider the case x is adjacent to u. The other case can be dealt in a similar way.

Clearly, $u \in L$ and $v \in R$. Let $\widehat{X} = X \cup \{v\}$ and $\widehat{Y} = Y \cup \{u\}$. We define the bijection $\widehat{\sigma} : \widehat{X} \to \widehat{Y}$ by $\widehat{\sigma}(x) = u$, $\widehat{\sigma}(v) = \sigma(x)$ and $\widehat{\sigma}(z) = \sigma(z)$ if $z \in X \setminus \{x\}$. Note that $\widehat{\sigma}$ is a $\widehat{X}$-$\widehat{Y}$ path system in T.

Let us define a map $\Gamma : \mathcal{F}_T^{X,Y} \to \mathcal{F}_T^{\widehat{X},\widehat{Y}}$ as follows. For each $F \in \mathcal{F}_T^{X,Y}$, there is a component, say T_F, that contains the x-$\sigma(x)$ path. Let us construct a new spanning forest $\widehat{F}$ of T from F by deleting the edge $[u, v]$ from F. Clearly $\widehat{F}$ has $|\widehat{X}| = |X| + 1$ components. We define $\Gamma(F) = \widehat{F}$. It is easy to note that Γ is a bijection. This completes the proof. ■

▶ **Lemma 4.4.7.** *Let T be a nonsingular tree on $2p$ with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$ and $\mathcal{F}_T^{X,Y} \neq \emptyset$. Suppose $u \in X \cup Y$, $v \notin X \cup Y$ such that v is a leaf and u is adjacent to v. Let $\widehat{T}$ be the tree obtained from T by attaching k-many paths $[z_i, u_i, v]$ at the leaf v where $u_i, z_i \notin T$ for $i = 1, \dots, k$, see Figure 4.6. Let $(\widehat{L}, \widehat{R})$ be the standard vertex bipartition of $\widehat{T}$ obtained by extending the standard vertex bipartition (L, R) of T to $\widehat{T}$. Then*

$$\left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right| = (k+1)\left|\mathcal{F}_T^{X,Y}\right|,$$

where $\widehat{X} = (X \cup \{u_1, \dots, u_k, z_1, \dots, z_k\}) \cap \widehat{R}$, $\widehat{Y} = (Y \cup \{u_1, \dots, u_k, z_1, \dots, z_k\}) \cap \widehat{L}$. ◀

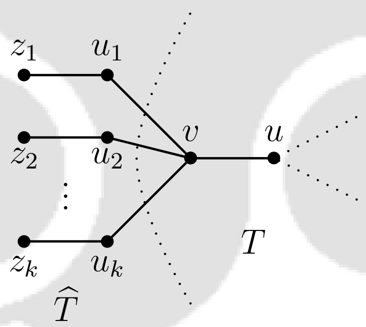


Figure 4.6: Attaching k-many P_2 at a pendant vertex.

Proof. With out loss of any generality, assume that $v \in R$. Then we have

$$\widehat{L} = L \cup \{u_1, \dots, u_k\}, \quad \text{and} \quad \widehat{R} = R \cup \{v_1, \dots, v_k\}.$$

Clearly, $\widehat{X} = X \cup \{z_1, \dots, z_k\}$ and $\widehat{Y} = Y \cup \{u_1, \dots, u_k\}$. Let σ be the X-Y path system in T. We define the map $\widehat{\sigma} : \widehat{X} \to \widehat{Y}$ by $\widehat{\sigma}(x) = \sigma(x)$ for each $x \in X$ and $\widehat{\sigma}(z_i) = u_i$ for $i = 1, \dots, k$. Clearly, $\widehat{\sigma}$ is a $\widehat{X}$-$\widehat{Y}$ path system in $\widehat{T}$.

Let F in $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$. Since $u, u_1, \dots, u_k \in \widehat{Y}$, the degree of the vertex v in F is always one. Therefore, in each $F \in \mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$, either the vertex v is adjacent to u or the vertex v is adjacent to u_i for some i.

Let us define a relation $\circledast$ on the set $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ as follows. Two forests $F_1, F_2 \in \mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ are related to each other if and only if the vertex v has same neighbor in both F_1 and F_2. Clearly, $\circledast$ defines an equivalence relation on the set $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$. Notice that $\circledast$ partitioned the set $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ into $(k+1)$ equivalence class and each class has equal number of elements which is $|\mathcal{F}_T^{X,Y}|$. Therefore, $\left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right| = (k+1)\left|\mathcal{F}_T^{X,Y}\right|$. ■

► **Lemma 4.4.8.** *Let T be a nonsingular tree on $2p$ with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$ and $\mathcal{F}_T^{X,Y} \neq \emptyset$. Suppose $u, v \notin X \cup Y$ such that v is a leaf and u is adjacent to v. Let $\widehat{T}$ be the tree obtained from T by attaching k-many paths $[z_i, u_i, v]$ at the leaf v where $u_i, z_i \notin T$ for $i = 1, \ldots, k$, see Figure 4.6. Let $(\widehat{L}, \widehat{R})$ be the standard vertex bipartition of $\widehat{T}$ obtained by extending the standard vertex bipartition (L, R) of T to $\widehat{T}$. Then*

$$\left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right| = (k+1)\left|\mathcal{F}_T^{X,Y}\right| + k\left|\mathcal{F}_T^{X',Y'}\right|,$$

where $\widehat{X} = (X \cup \{u_1, \ldots, u_k, z_1, \ldots, z_k\}) \cap \widehat{R}$, $\widehat{Y} = (Y \cup \{u_1, \ldots, u_k, z_1, \ldots, z_k\}) \cap \widehat{L}$ and $X' = (X \cup \{u, v\}) \cap R$, $Y' = (Y \cup \{u, v\}) \cap L$. ◄

Proof. With out loss of any generality, assume that $v \in R$. Then we have

$$\widehat{L} = L \cup \{u_1, \ldots, u_k\}, \quad \text{and} \quad \widehat{R} = R \cup \{v_1, \ldots, v_k\}.$$

Clearly, $X' = X \cup \{v\}$, $Y' = Y \cup \{u\}$ and $\widehat{X} = X \cup \{z_1, \ldots, z_k\}$, $\widehat{Y} = Y \cup \{u_1, \ldots, u_k\}$. Let σ be the X-Y path system in T. Let us define $\sigma' : X' \to Y'$ and $\widehat{\sigma} : \widehat{X} \to \widehat{Y}$ by $\widehat{\sigma}(x) = \sigma'(x) = \sigma(x)$ for each $x \in X$ and $\sigma'(v) = u$, $\widehat{\sigma}(z_i) = u_i$ for $i = 1, \ldots, k$. Clearly, σ' and $\widehat{\sigma}$ are X'-Y' path system and $\widehat{X}$-$\widehat{Y}$ path system, respectively.

Let F be an element of $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$. Since $u_1, \ldots, u_k \in \widehat{Y}$, the degree of the vertex v in F is either one or two. Further note that if $d_F(v) = 2$ then u is adjacent to v in F.

Let us define a relation $\circledast$ on the set $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ as follows. Two forests $F_1, F_2 \in \mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ are related to each other if and only if $d_{F_1}(v) = d_{F_2}(v)$ and $N_{F_1}(v) = N_{F_2}(v)$, where $N_G(v)$ denotes the set of all vertices of G that are adjacent to v. Clearly $\circledast$ defines a equivalence relation on $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$.

Let $F \in \mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$. Then note that if $d_F(v) = 1$ then there are $k+1$ choices available for $N_F(v)$, which are $\{u\}, \{u_1\}, \ldots, \{u_k\}$. Also, if $d_F(v) = 2$ then there are k choices available for $N_F(v)$, which are $\{u, u_1\}, \ldots, \{u, u_k\}$. Therefore, the equivalence relation $\circledast$ partitioned the set $\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}$ into $2k+1$ equivalence classes. Let $E_1, \ldots, E_{k+1}$ be the equivalence classes that are corresponds to $d_F(v) = 1$ and $\bar{E}_1, \ldots, \bar{E}_k$ be the equivalence classes that are corresponds to $d_F(v) = 2$.

For $i = 1, \ldots, k+1$, let us define a map $\Gamma_i : E_i \to \mathcal{F}_T^{X,Y}$ by $\Gamma_i(F) = \widehat{F}$, where $\widehat{F}$ is a spanning forest of T obtained from F by adding the edge $[u, v]$ (if they are not adjacent) and removing the vertices $u_1 \ldots, u_k, z_1, \ldots, z_k$. Clearly Γ_i defines a bijection. Therefore, $|E_i| = |\mathcal{F}_T^{X,Y}|$ for each $i = 1, \ldots, k+1$.

Similarly, for $i = 1, \ldots, k$, let us define a map $\bar{\Gamma}_i : \bar{E}_i \to \mathcal{F}_T^{X',Y'}$ by $\bar{\Gamma}_i(F) = \bar{F}$, where $\bar{F}$ is a spanning forest of T obtained from F by removing the vertices $u_1 \ldots, u_k, z_1, \ldots, z_k$. Clearly Γ_i defines a bijection. Therefore, $|\bar{E}_i| = |\mathcal{F}_T^{X',Y'}|$ for each $i = 1, \ldots, k$.

Hence, it follows that $\left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right| = (k+1)\left|\mathcal{F}_T^{X,Y}\right| + k\left|\mathcal{F}_T^{X',Y'}\right|$. This completes the proof. ■

The proof of the next result is easy and is omitted.

▶ **Lemma 4.4.9.** *Let T be a nonsingular tree on $2p$ with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$. Suppose there exist a X-Y path system in T. Let $v \notin X \cup Y$ be a leaf in T and $u \in X \cup Y$ be adjacent to v. Let $\widehat{T}$ be a tree obtained from T by attaching a new P_2 at the vertex v, say $[v, u_1, z_1]$, where $u_1, z_1 \notin T$. Then $\left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right| = \left|\mathcal{F}_T^{X,Y}\right|$, where $\widehat{X} = (X \cup \{u_1, v\}) \cap R$, $Y' = (Y \cup \{u_1, v\}) \cap L$.* ◀

The proof of the next result is easy and is omitted.

▶ **Lemma 4.4.10.** *Let T be a nonsingular tree on $2p$ with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$. Suppose there exist a X-Y path system in T. Let $u \in X \cup Y$ be a leaf in T and v be adjacent to u. Let $\widehat{T}$ be a tree obtained from T by attaching a P_2 at the vertex v, say $[v, z_1, v_1]$, where $v_1, z_1 \notin T$. Then $\left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right| = \left|\mathcal{F}_T^{X,Y}\right|$, where $\widehat{X} = ((X \setminus u) \cup \{z_1\}) \cap R$, $Y' = ((Y \setminus u) \cup \{z_1\}) \cap L$.* ◀

The proof of the next result is very similar to the proof of Lemma 4.4.8 and so it is omitted.

▶ **Lemma 4.4.11.** *Let T be a nonsingular tree on $2p$ with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$. Suppose there exist a X-Y path system in T. Let $u \notin X \cup Y$ be a leaf in T and $v \notin X \cup Y$ be adjacent to u. Let $\widehat{T}$ be a tree obtained from T by attaching a P_2 at the vertex v, say $[v, z_1, v_1]$, where $v_1, z_1 \notin T$. Then $\left|\mathcal{F}_{\widehat{T}}^{\widehat{X},\widehat{Y}}\right| = \left|\mathcal{F}_T^{X,Y}\right| + \left|\mathcal{F}_T^{\bar{X},\bar{Y}}\right|$, where $\widehat{X} = (X \cup \{v_1, z_1\}) \cap R$, $\widehat{Y} = (Y \cup \{v_1, z_1\}) \cap L$ and $\bar{X} = (X \cup \{u, v\}) \cap R$, $Y' = (Y \cup \{u, v\}) \cap L$* ◀

4.5 Formula to calculate any minor of the bipartite Laplacian matrix

Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Note that $L = \{l_1, \ldots, l_p\}$ and $R = \{r_1, \ldots, r_p\}$ are linearly ordered sets with the partial order "$<$" defined as $l_i < l_j$ and $r_i < r_j$ if and only if $i < j$.

Let $X \subseteq R$ and $Y \subseteq L$ with $|X| = |Y|$. Let $\pi : X \to Y$ be a bijection. The pair $\{r_i, r_j\} \subseteq X$ is called an *inversion* in π if $r_i < r_j$ and $\pi(r_i) > \pi(r_j)$. Let $n(\pi)$ denotes the number of inversions in π. The *sign* $\epsilon(\pi)$ of π is defined by $\epsilon(\pi) := (-1)^{n(\pi)}$. The *sign* $\epsilon(X)$ of X and the *sign* $\epsilon(Y)$ of Y are defined by

$$\epsilon(X) := (-1)^{\sum\limits_{r_i \in X} i} \qquad \text{and} \qquad \epsilon(Y) := (-1)^{\sum\limits_{l_i \in Y} i}.$$

We define the notation $\epsilon(X, Y)$ by

$$\epsilon(X, Y) = \begin{cases} 1 & \text{if there does not exist any } X\text{-}Y \text{ path system,} \\ \epsilon(\sigma) & \text{if } \sigma \text{ is the } X\text{-}Y \text{ path system.} \end{cases}$$

Let U be a set and A be a subset of U. Suppose $a_1, \cdots, a_k$ and $b_1, \cdots, b_\ell$ are some distinct elements in U. By the notation $A - a_1 - \cdots - a_k + b_1 + \cdots + b_\ell$, we shall mean the following set operation,

$$A - a_1 - \cdots - a_k + b_1 + \cdots + b_\ell := (A \setminus \{a_1, \ldots, a_k\}) \cup \{b_1, \ldots, b_\ell\}$$

▶ **Theorem 4.5.1 (All Minor Matrix Tree Theorem).** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $X \subseteq R$ and $Y \subseteq L$ such that $|X| = |Y|$ Then*

$$\det \mathfrak{L}(X \mid Y) = \epsilon(X)\epsilon(Y)\epsilon(X, Y)|\mathcal{F}_T^{X,Y}|.$$ ◀

Proof. We prove the result by induction on p. Note that the base case follows by item (b) of Theorem 3.2.3. Let the result be true for all nonsingular trees with less than $2p$ vertices. Suppose T is a nonsingular tree on $2p$ vertices. Note that if there does not exist any X-Y path system, by Theorem 4.3.5, $\det \mathfrak{L}(X \mid Y) = 0$ and so the result follows by Lemma 4.4.2.

Let σ be the X-Y path system. In view of Lemmas 4.1.3 and 4.4.4, we may assume that there does not exist $\{l_k, r_k\} \subseteq V(T) \setminus (X \cup Y)$ such that $d(l_k) + d(r_k) = 3$. Let $P = [v_0, v_1, v_2, \ldots, v_k]$ be a path corresponding to the diameter of T. With out loss of any generality, we assume that $v_0 = r_p$ and $v_2 = r_{p-1}$. Since $d(v_0) + d(v_1) = 3$, both v_0 and v_1 can not be an element of $V(T) \setminus (X \cup Y)$. Therefore, either $v_0 \in X$ or $v_1 \in Y$ or $v_0, v_1 \in X \cup Y$.

Let $\widehat{T}$ be the tree obtained from T by removing the vertices v_0 and v_1. Let $\widehat{\mathfrak{L}}$ be the bipartite Laplacian matrix of $\widehat{T}$. We divide the proof into two main cases.

Case I: Suppose there exist a leaf, say u, is adjacent to the vertex v_2. (See Figure 4.7.) Clearly $[u, v_2]$ is a matching edge and so $u = l_{p-1}$. It is easy to note that

$$\mathfrak{L}(r_p \mid l_p) = \widehat{\mathfrak{L}} + \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \tag{4.2}$$

• Let us first assume $v_2 \in X$. It follows that $v_1 \in Y$, because if $v_1 \notin Y$ then $v_0 \in X$, this contradict that σ is a X-Y path system.

Clearly, if $v_0 \in X$, then $\mathfrak{L}(X \mid Y) = \widehat{\mathfrak{L}}(X - v_0 \mid Y - v_1)$ and so by the induction hypothesis and Lemma 4.4.5, the result follows. Now consider $v_0 \notin X$. It follows that $u \notin Y$. Therefore, all entries of the last row of $\mathfrak{L}(X \mid Y)$ are zero except the last entry which is -1. Therefore, by expanding the determinant of $\mathfrak{L}(X \mid Y)$ along its last row, we get $\det \mathfrak{L}(X \mid Y) = -\det \widehat{\mathfrak{L}}(X \mid Y - l_p + l_{p-1})$. It is easy to note that $\epsilon(X, Y) = \epsilon(X, Y - l_p + l_{p-1})$. By applying induction hypothesis and Lemma 4.4.10, we see that

$$\det \mathfrak{L}(X \mid Y) = -\epsilon(X, Y)\epsilon(X)\epsilon(Y - l_p + l_{p-1})\left|\mathcal{F}_{\widehat{T}}^{X, Y - l_p + l_{p-1}}\right| = \epsilon(X, Y)\epsilon(X)\epsilon(Y)\left|\mathcal{F}_T^{X,Y}\right|.$$

Hence the result follows.

• Now we assume $v_2 \notin X$. We divide the proof of this part into two sub cases. Let us first consider $u \in Y$.

Clearly, if $v_1 \in Y$ then $v_0 \in X$ (otherwise it will contradict that $\mathcal{F}_T^{X,Y} \neq \emptyset$) and so $\mathfrak{L}(X \mid Y) = \widehat{\mathfrak{L}}(X - v_0 \mid Y - v_1)$. Therefore, if $v_1 \in Y$, then the result follows by the induction hypothesis. Let us take $v_1 \notin Y$. Then $v_0 \in X$ and so $\sigma(v_0) = u$. Notice that all entries of the last column of $\mathfrak{L}(X \mid Y)$ are zero except the last entry which is -1. It follows

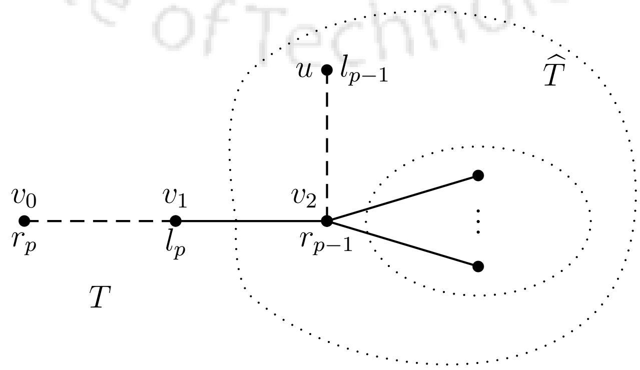


Figure 4.7: Case I: there exist a leaf u adjacent to the vertex v_2. (Here dashed edges are matching edges.)

that $\det \mathfrak{L}(X \mid Y) = -\det \widehat{\mathfrak{L}}(X - r_p + r_{p-1} \mid Y)$. By Lemma 4.4.5 and 4.4.6, we get

$$\mathcal{F}_T^{X,Y} \equiv \mathcal{F}_T^{X+r_{p-1},Y+l_p} \equiv \mathcal{F}_{\widehat{T}}^{X-r_p+r_{p-1},Y}.$$

It is easy to note that $\epsilon(X,Y) = \epsilon(X - r_p + r_{p-1}, Y)$. Therefore, by the induction hypothesis, the result follows.

Now we consider $u \notin Y$, that means, $u, v_2 \notin X \cup Y$. First notice that if $v_0, v_1 \in X \cup Y$, then by Equation (4.2), it follows that

$$\det \mathfrak{L}_T(X \mid Y) = \det \widehat{\mathfrak{L}}(X - r_p \mid Y - l_p) + \det \mathfrak{L}_{\widehat{T}}(X - r_p + r_{p-1} \mid Y - l_p + l_{p-1}).$$

It is easy to note that $\epsilon(X,Y) = \epsilon(X - r_p, Y - l_p) = \epsilon(X - r_p + r_{p-1}, Y - l_p + l_{p-1})$. By Lemma 4.4.11, it follows that

$$|\mathcal{F}_T^{X,Y}| = \left|\mathcal{F}_{\widehat{T}}^{\prime X-r_p,Y-l_p}\right| + \left|\mathcal{F}_{\widehat{T}}^{\prime X-r_p+r_{p-1},Y-l_p+l_{p-1}}\right|.$$

Hence, by the induction hypothesis, the result follows.

Now we assume either $v_0 \in X, v_1 \notin Y$ or $v_0 \notin X, v_1 \in Y$. Notice that in these cases we have either the all entries of the last column or the all entries of the last row of $\mathfrak{L}(X \mid Y)$ are zero except the last entry which is -1. It follows that

$$\det \mathfrak{L}(X \mid Y) = \begin{cases} -\det \widehat{\mathfrak{L}}(X \mid Y - l_p + l_{p-1}) & \text{if } v_0 \notin X, v_1 \in Y, \\ -\det \widehat{\mathfrak{L}}(X - r_p + r_{p-1} \mid Y) & \text{if } v_0 \in X, v_1 \notin Y. \end{cases}$$

Note that, if $v_0 \notin X$ and $v_1 \in Y$, then by Lemma 4.4.10, we get $\mathcal{F}_T^{X,Y} \equiv \mathcal{F}_{\widehat{T}}^{X,Y-l_p+l_{p-1}}$. For $v_0 \in X$ and $v_1 \notin Y$, we apply Lemma 4.4.6 and 4.4.5 to get

$$\mathcal{F}_T^{X,Y} \equiv \mathcal{F}_T^{X+r_{p-1},Y+l_{p-1}} \equiv \mathcal{F}_{\widehat{T}}^{X-r_p+r_{p-1},Y}.$$

Therefore, by applying induction hypothesis, the result follows.

Case II: Suppose there does not exist any leaf in T that is adjacent to v_2. Then $[v_2, v_3]$ is a matching edge. Therefore, $v_3 = l_{p-1}$. Let $d(v_2) = m + 2$ where $m \geq 0$ and $B_1, \ldots, B_m$ be the branches at v_2 that does not contain vertices v_1 and v_3. Since no leaf is adjacent to v, the branches B_i have exactly two vertices, say $u_i, z_i \in B_i$ with $u_i \sim v_2$, for $i = 1, \ldots, m$. (See Figure 4.8)

We divide the proof of this case into three main parts depending upon whether $v_0 \in X$ or $v_1 \in Y$.

Sub-case 1: Let us first consider $v_0 \in X$ and $v_1 \in Y$. For $x \in X$, we shall use P_x to denote

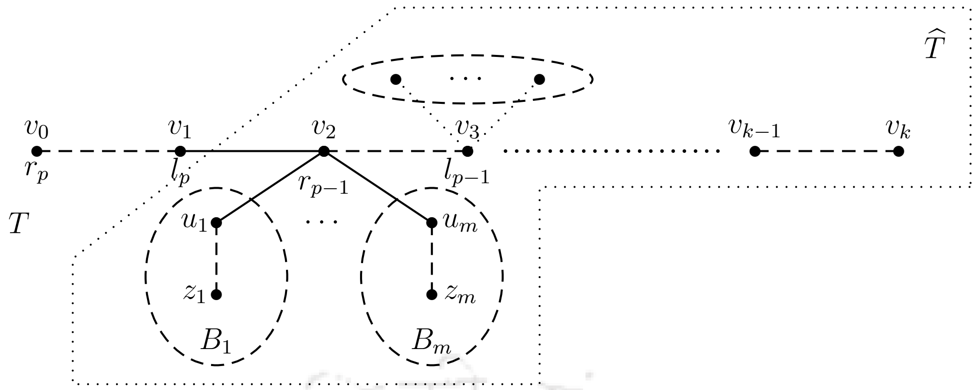


Figure 4.8: Case II: there does not exist any leaf in T that adjacent to v_2. (Here dashed edges are matching edges.

the x-$\sigma(x)$ path, $x \in X$. Clearly, if $v_2 \in X$, then the result directly follows by applying induction hypothesis and Lemma 4.4.5. Therefore, we take $v_2 \notin X$.

• Suppose $v_2 \in P_x$ for some $x \in X$. Then, there exist two branches at v_2, say B and B', such that $w(B), w(B') \neq 0$. Without loss of any generality, let us assume $B' = B_1$. Note that either $v_3 \in B$ or $v_3 \notin B$.

Let us first assume that $v_3 \in B$. By part (ii) of Theorem 4.2.8, $u_2, \ldots, u_m \in Y$ and $z_2, \ldots, z_m \in X$.

Note that if $w(B_1) = 1$, then $z_1 \in X$ and $u_1 \notin Y$. Without loss of any generality, let us assume that $u_1 = l_{p-2}$. Note that all entries of $\mathfrak{L}(X \mid Y)$ in the column corresponds to u_1 are zero except the last entry which is -1. Therefore $\det \mathfrak{L}(X \mid Y) = -\det \widehat{\mathfrak{L}}(X - r_p + r_{p-1} \mid Y - l_p + l_{p-2})$ if $v_3 \in Y$ and $\det \mathfrak{L}(X \mid Y) = \det \widehat{\mathfrak{L}}(X - r_p + r_{p-1} \mid Y - l_p + l_{p-2})$ if $v_3 \notin Y$. It is easy to note that $n(X, Y) = n(X - r_p + r_{p-1}, Y - l_p + l_{p-2})$ if $v_3 \in Y$ and $n(X, Y) = n(X - r_p + r_{p-1}, Y - l_p + l_{p-2}) - 1$ if $v_3 \notin Y$. Further, by Lemma 4.4.5 and 4.4.6, we have $\mathcal{F}_{\widehat{T}}^{\prime X - r_p + r_{p-1}, Y - l_p + l_{p-2}} \equiv \mathcal{F}_{\widehat{T}}^{\prime X - r_p, Y - l_p} \equiv \mathcal{F}_T^{X,Y}$. Therefore, by the induction hypothesis, it follows that $\det \mathfrak{L}(X \mid Y) = \epsilon(X)\epsilon(Y)\epsilon(X, Y)|\mathcal{F}_T^{X,Y}|$.

Further note that if $w(B_1) = -1$, then, $u_1 \in Y$ and $z_1 \notin X$. Since $v_2 \notin X$, it follows that $v_3 \notin Y$. Clearly, $\mathfrak{L}(r_{p-1}, l_i) + d_T(r_{p-1})\mathfrak{L}(r_{p-2}, l_i) = 0$ for each $l_i \notin Y \cup \{l_{p-1}\}$ and $\mathfrak{L}(r_{p-1}, l_{p-1}) + d_T(r_{p-1})\mathfrak{L}(r_{p-2}, l_{p-1}) = -1$. Therefore, $\det \mathfrak{L}(X, Y) = -\det \widehat{\mathfrak{L}}(X - r_p + r_{p-1} \mid Y - l_p + l_{p-1})$. Clearly, $n(X, Y) = n(X - r_p + r_{p-1}, Y - l_p + l_{p-1}) - 1$. By Lemma 4.4.6 and Lemma 4.4.5, we have $\mathcal{F}_{\widehat{T}}^{X - r_p + r_{p-1}, Y - l_p + l_{p-1}} = \mathcal{F}_{\widehat{T}}^{X - r_P, Y - l_p} = \mathcal{F}_T^{X,Y}$. Hence, by the induction hypothesis, it follows that $\det \mathfrak{L}(X \mid Y) = \epsilon(X)\epsilon(Y)\epsilon(X, Y)|\mathcal{F}_T^{X,Y}|$.

Now we shall assume $v_3 \notin B$. Then $w(B_i) \neq 0$ for some $i \geq 2$. Without loss of any generality, let us assume $w(B_2) \neq 0$. By part (iv) of Theorem 4.2.8, $w(B_1) = -w(B_2) = \pm 1$ and $w(B_i) = 0$ for each $i > 3$. Without loss of any generality, we assume $w(B_1) = -1$ and $z_2 = r_{p-3}$. Therefore, $u_1 \in Y$ and $z_2 \in X$. It follows that $\mathfrak{L}(r_{p-1}, l_{p-3}) = -1$ and

$\mathfrak{L}(r_i, l_{p-3}) = 0$ for each $i = 1, \ldots, p-4, p-2$. Therefore, $\det \mathfrak{L}(X \mid Y) = -\det \widehat{\mathfrak{L}}(X - r_p + r_{p-1} \mid Y - l_p + l_{p-3})$ if $v_3 \in Y$ and $\det \mathfrak{L}(X \mid Y) = \det \widehat{\mathfrak{L}}(X - r_p + r_{p-1} \mid Y - l_p + l_{p-3})$ if $v_3 \notin Y$. Notice that $n(X,Y) = n(X - r_p + r_{p-1}, Y - l_p + l_{p-3}) - 1$ if $v_3 \in Y$ and $n(X,Y) = n(X - r_p + r_{p-1}, Y - l_p + l_{p-3})$ if $v_3 \notin Y$. Further note that $\epsilon(X) = (-1)^{p+(p-1)}\epsilon(X - v_0 + v_2)$ and $\epsilon(Y) = (-1)^{p+(p-3)}\epsilon(Y - v_1 + u_2)$. By Lemma 4.4.6 and Lemma 4.4.5, we have $\mathcal{F}_{\widehat{T}}^{X-r_p+r_{p-1}, Y-l_p+l_{p-3}} = \mathcal{F}_{\widehat{T}}^{X-r_p, Y-l_p} = \mathcal{F}_T^{X,Y}$. Therefore, by induction hypothesis, $\det \mathfrak{L}(X \mid Y) = \epsilon(X)\epsilon(Y)\epsilon(X,Y)|\mathcal{F}_T^{X,Y}|$.

- Suppose $v_2 \notin P_x$ for each $x \in X$. Clearly $u_i, z_i \in X \cup Y$ for each $i = 1, \ldots, m$. Consider the tree $\bar{T}$ obtained from T by removing v_0, v_1 and $B_1, \ldots, B_m$. Suppose $\bar{\mathfrak{L}}$ is the bipartite Laplacian matrix of $\bar{T}$. Notice that $d_{\bar{T}}(v_2) = 1$ and $\mathfrak{L}(r_{p-1}, l_i) = d_T(r_{p-1})\bar{\mathfrak{L}}(r_{p-1}, l_i)$ for each $l_i \notin Y \cup v_3$.

If $v_3 \in Y$ then we see that $\det \mathfrak{L}(X \mid Y) = (m+2)\det \bar{\mathfrak{L}}(X - v_0 - z_1 - \cdots - z_m | Y - v_1 - u_1 - \cdots - u_m)$ and the result follows by applying induction hypothesis together with Lemma 4.4.7.

Now we take $v_3 \notin Y$. Take $X' = X + v_2 - v_0 - z_1 - \cdots - z_m$ and $Y' = Y + v_3 - v_1 - u_1 - \cdots - u_m$. It follows that

$$\mathfrak{L}(X \mid Y) = \begin{bmatrix} \bar{\mathfrak{L}}(X' \mid Y') & \bar{\mathfrak{L}}(R - X', v_3) \\ d_T(v_2)\bar{\mathfrak{L}}(v_2, L - Y') & d_T(v_2)d_T(v_3) - 1 \end{bmatrix},$$

where $(\bar{L}, \bar{R})$ is the standard vertex bipartition of $\bar{T}$. Therefore we get

$$\det \mathfrak{L}(X \mid Y) = d_T(v_2)\det \bar{\mathfrak{L}}(X' - v_2 \mid Y' - v_1) + (d_T(v_2) - 1)\det \bar{\mathfrak{L}}(X' \mid Y')$$

By Lemma 4.4.8, it follows that

$$\left|\mathcal{F}_T^{X,Y}\right| = (m+2)\left|\mathcal{F}_{\bar{T}}^{X'-v_2, Y'-v_1}\right| + (m+1)\left|\mathcal{F}_{\bar{T}}^{X',Y'}\right|$$

Hence the result follows by induction hypothesis.

Sub-case 2: Now consider the case $v_0 \in X$ but $v_1 \notin Y$. Clearly $v_2 \notin X$. Notice that all entries of the last column of $\mathfrak{L}(X \mid Y)$ are zero except the last entry which is -1. It follows that

$$\det \mathfrak{L}(X \mid Y) = -\det \mathfrak{L}(X - r_p + r_{p-1}|Y).$$

By Lemma 4.4.6 and Lemma 4.4.5, we have

$$|\mathcal{F}_T^{X,Y}| = |\mathcal{F}_T^{X+r_{p-1}, Y+l_p}| = |\mathcal{F}_{\widehat{T}}^{X-r_p+r_{p-1}, Y}|.$$

Hence the result follows by induction hypothesis.

Sub-case 3: Now consider the case $v_0 \notin X$ but $v_1 \in Y$. First note that if there exist any

B_i such that $w(B_i) = 0$ then $u_i, z_i \in X \cup Y$ and so the result follows from sub-case 1. If there exist $z_i \in B_i$, for some i, such that $z_i \in X$ and $\sigma(z_i) = v_1$, then the result follows from sub-case 2. Thus we only need to argue the case for which $d_T(v_2) = 2$. Let us assume $d_T(v_2) = 2$. Notice that

$$\begin{aligned} \mathfrak{L}(r_p, l_i) &= \widehat{\mathfrak{L}}(r_{p-1}, l_i) \qquad \text{for each } i = 1, \ldots, p-2 \\ \mathfrak{L}(r_p, l_{p-1}) &= -d_T(l_{p-1}) = -\widehat{\mathfrak{L}}(r_{p-1}, l_{p-1}) - 1 \end{aligned} \tag{4.3}$$

• Let us first assume $v_3 \in Y$. It follows that $v_2 \in X$. Clearly $\sigma(v_1) = v_2$. Since $v_3 \in Y$ and $v_2 \in X$, it follows that $\mathfrak{L}(X \mid Y) = Z$, where Z is the matrix obtained from $\widehat{L}(X - v_2 \mid Y - v_1)$ by multiplying its last row (that is, the row corresponds to v_2) by -1. Therefore,

$$\det \mathfrak{L}(X \mid Y) = -\det \widehat{L}(X - v_2 \mid Y - v_1).$$

Hence the result follows by applying induction hypothesis and Lemma 4.4.9.

• Now we consider $v_3 \notin Y$. We divide this part into two cases. First we assume $v_2 \notin X$. Notice that

$$\begin{aligned} \mathfrak{L}(r_{p-1}, l_i) &= -2\mathfrak{L}(r_p, l_i) \qquad \text{for each } i = 1, \ldots, p-2 \\ \mathfrak{L}(r_{p-1}, l_{p-1}) &= 2d_T(l_{p-1}) - 1 = -2\mathfrak{L}(r_p, l_{p-1}) - 1 \end{aligned}$$

By using elementary row operation on $\mathfrak{L}(X \mid Y)$ we see that

$$\det \mathfrak{L}(X \mid Y) = \det \mathfrak{L}(X + v_2 \mid Y + v_3) = \det \widehat{\mathfrak{L}}(X \mid Y - v_1 + v_3).$$

By Lemma 4.4.6, $|\mathcal{F}_T^{X,Y}| = |\mathcal{F}_T^{X+v_2,Y+v_3}|$. Again, by Lemma 4.4.9, $|\mathcal{F}_T^{X+v_2,Y+v_3}| = |\mathcal{F}_{\widehat{T}}^{X,Y-v_1+v_3}|$. Hence the result follows by applying induction hypothesis.

Finally we assume $v_2 \in X$ (and $v_3 \notin Y$). Clearly $\sigma(v_2) = v_1$. Let K be the matrix obtained from $\widehat{L}(X - v_2 \mid Y - v_1)$ by multiplying its last row (that is, the row corresponds to v_2) by -1. By (4.3), we observe that

$$\mathfrak{L}(X \mid Y) = K + \begin{bmatrix} & & \mathbf{0} & \\ 0 & \cdots & 0 & -1 \end{bmatrix}.$$

It follows that

$$\det \mathfrak{L}(X \mid Y) = -\det \widehat{L}(X - v_2 \mid Y - v_1) - \det \widehat{\mathfrak{L}}(X \mid Y - v_1 + v_3)$$

Let $X_1 = X - v_2$, $Y_1 = Y - v_1$ and $X_2 = X$, $Y_2 = Y - v_1 + v_3$. Notice that $\epsilon(X_1)\epsilon(Y_1) =$

$-\epsilon(X)\epsilon(Y) = \epsilon(X_2)\epsilon(Y_2)$. Further, it is easy to note that $\epsilon(X,Y) = \epsilon(X_1,Y_1) = \epsilon(X_2,Y_2)$ Now it's only remains to show that $|\mathcal{F}_T^{X,Y}| = |\mathcal{F}_{\widehat{T}}^{X_1,Y_1}| + |\mathcal{F}_{\widehat{T}}^{X_2,Y_2}|$.

Notice that there are two types of forests are there in the class $\mathcal{F}_T^{X,Y}$; one type which contain the edge $[v_2, v_3]$ and an another type which does not contain the edge $[v_2, v_3]$. Let $F \in \mathcal{F}_T^{X,Y}$. Then there exist a tree $T^* \in F$ that contains the vertex v_2. Since $\sigma(v_2) = v_1$, it follows that $v_0, v_1 \in T^*$.

i) If $[v_2, v_3] \notin F$ then $T^* = [v_0, v_1, v_2]$, a path on three vertices. Let us construct a new forest F_1 from F as follows. First remove vertices v_0, v_1 from F and then add the edge $[v_2, v_3]$. Note that $F_1 \in \mathcal{F}_{\widehat{T}}^{X_1,Y_1}$.

ii) If $[v_2, v_3] \in F$ then we construct a new forest F_2 from F by removing the vertices v_0, v_1. It is easy to note that $F_2 \in \mathcal{F}_{\widehat{T}}^{X_2,Y_2}$.

Therefore, it follows that

$$|\mathcal{F}_T^{X,Y}| \leq |\mathcal{F}_{\widehat{T}}^{X_1,Y_1}| + |\mathcal{F}_{\widehat{T}}^{X_2,Y_2}|$$

Conversely, let $F_1 \in \mathcal{F}_{\widehat{T}}^{X_1,Y_1}$ and $F_2 \in \mathcal{F}_{\widehat{T}}^{X_2,Y_2}$. Clearly $[v_2, v_3] \in F_1, F_2$. Let $T_1 \in F_1$ and $T_2 \in F_2$ such that $[v_2, v_3] \in T_1, T_2$. Now we construct two new spanning forests F_1^* and F_2^* of T as follows.

i) First delete the edge $[v_2, v_3]$ from T_1 to construct two new trees, say T_1^1 and T_1^2. Let $v_2 \in T_1^1$. Now construct a new tree $\overline{T_1^1}$ from T_1^1 by introducing two new vertices v_0, v_1 and then adding the path $[v_0, v_1, v_2]$. Let $F_1^* = \{F_1 - T_1, \overline{T_1^1}, T_1^2\}$. Clearly $[v_2, v_3] \notin F_1^*$ and $F_1^* \in \mathcal{F}_T^{X,Y}$.

ii) Construct a new tree $\overline{T_2}$ from T_2 by introducing two new vertices v_0, v_1 and then adding the path $[v_0, v_1, v_2]$. Let $F_2^* = \{F_2 - T_2, \overline{T_2}\}$. Clearly $[v_2, v_3] \in F_2^*$ and $F_2^* \in \mathcal{F}_T^{X,Y}$. Therefore, we get

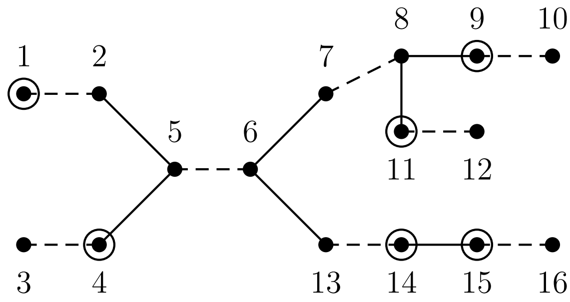
$$|\mathcal{F}_T^{X,Y}| \geq |\mathcal{F}_{\widehat{T}}^{X_1,Y_1}| + |\mathcal{F}_{\widehat{T}}^{X_2,Y_2}|$$

This completes the proof. ■

Let us discuss one application of Theorem 4.5.1 in the following example.

► **Example 4.5.2.** Let us consider the tree T as shown below. Clearly $(L = \{1, 3, \ldots, 15\}, R = \{2, 4, \ldots, 16\})$ is a standard vertex bipartition of T. Let $\mathfrak{L}$ and $\mathcal{L}$ be the bipartite Laplacian matrix and the usual Laplacian matrix of T, respectively. We consider $X = \{4, 12, 14\}$ and $Y = \{1, 9, 15\}$. Recall that by using Chaiken's result, that is, Theorem 4.1.2 we get $|\mathcal{F}_T^{X,Y}| = |\mathcal{L}(X \mid Y)|$. Also, by using Theorem 4.5.1, we know that $|\mathcal{F}_T^{X,Y}| = |\mathfrak{L}(X \mid Y)|$.

That means, if we apply Chaiken's result to calculate $|\mathcal{F}_T^{X,Y}|$ then we have to find the



T

Figure 4.9: An application of Theorem 4.5.1. Here circled vertices are selected vertices.

determinant of the following 13×13 submatrix of the usual Laplacian matrix of T

$$\mathcal{L}(X \mid Y) = \begin{bmatrix} -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Whereas if we apply Theorem 4.5.1 to calculate $|\mathcal{F}_T^{X,Y}|$ then we need to find the determinant of the following 5×5 submatrix of $\mathfrak{L}$

$$\mathfrak{L}(X \mid Y) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -3 & 8 & -1 & 0 & -1 \\ 3 & -9 & 5 & -1 & 0 \\ -1 & 3 & -2 & 0 & 0 \\ -1 & 3 & 0 & 0 & -2 \end{bmatrix}.$$

In both the above mentioned cases we see that $|\mathcal{F}_T^{X,Y}| = |\mathcal{L}(X \mid Y)| = \mathfrak{L}(X \mid Y) = 8$. Therefore, Theorem 4.5.1 provide an alternate way to calculate $\mathcal{F}_T^{X,Y}$ on which we have to calculate a determinant of a 5×5 matrix instead of 13×13. ◀

5 q-analogues of the bipartite distance matrix

Bapat, Lal, and Pati [BLP06] in 2006 introduced two types of q-analogue version of distance matrix, namely, q-distance matrix and exponential distance matrix. Indeed, in the same paper [BLP06], they generalized many concept of the distance matrix of a tree to that of q-distance matrix of a tree. The q-analogue version of the distance matrix generated a considerable interest and has been studied by many researcher (see, for example, [BLP06, Siv10, BS11, LSZ14]). In this chapter we extended almost all result of the bipartite distance matrix of nonsingular tree to that of a q-analogue version of the bipartite distance matrix.

5.1 Preliminaries

Let q be an indeterminate. For a positive integer k, we define $\{k\} := 1 + q + q^2 + \cdots + q^{k-1}$. We set $\{0\} = 0$. Let us recall the definitions of q-distance matrix and exponential distance matrix of a graph G.

▶ **Definition 5.1.1.** Let G be a connected graph on n vertices. The *q-distance matrix* $D_q(G)$ of G is the square matrix of order n whose (i, j)th entry is $\{\mathtt{dist}_G(i, j)\}$.

The *exponential distance matrix* $D_e(G)$ of G is the square matrix of order n whose (i, j)th entry is $q^{\mathtt{dist}_G(i,j)}$. ◀

If $q = 1$, then $D_1(G) = D(G)$ and so the distance matrix $D(G)$ is a particular case of the q-distance matrix $D_q(G)$.

The formulas for the determinants of the q-distance matrix and exponential distance matrix of a tree can be found in [BLP06, YY07].

▶ **Theorem 5.1.2.** *Let T be a tree on n vertices. Let D_q and D_e be the q-distance matrix and the exponential distance matrix of T, respectively. Then* $\det(D_q) = (-1)^{n-1}(n-1)(1+q)^{n-2}$ *and* $\det(D_e) = (1 - q^2)^{n-1}$. ◀

In the following definition we define two types of q-analogues of the bipartite distance matrix of bipartite graph with a unique perfect matching.

► **Definition 5.1.3.** Let G be connected, bipartite graph with a unique perfect matching on $2p$ vertices with a standard vertex bipartition (L, R). The *q-bipartite distance matrix* of G, denoted by $\mathfrak{B}_q(G)$, is the square matrix of order p such that the (i, j)th entry of $\mathfrak{B}_q(G)$ is given by $\{\mathtt{dist}_G(l_i, r_j)\}$. Note that each entry of $\mathfrak{B}_q(G)$ is a polynomial in q.

The *exponential bipartite distance matrix* of G, denoted by $\mathfrak{B}_\epsilon(G)$, is the square matrix of order p such that the (i, j)th entry of $\mathfrak{B}_\epsilon(G)$ is given by $q^{\mathtt{dist}_G(l_i, r_j)}$.

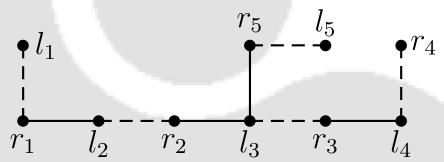
Note that the rows of $\mathfrak{B}_q(G)$ and $\mathfrak{B}_\epsilon(G)$ are indexed by $l_1, \ldots, l_p$ and the columns of $\mathfrak{B}_q(G)$ and $\mathfrak{B}_\epsilon(G)$ are indexed by $r_1, \ldots, r_p$.

If $q = 1$ then $\mathfrak{B}_q(G) = \mathfrak{B}(G)$ and $\mathfrak{B}_\epsilon(G) = \mathbb{1}\mathbb{1}^t$. Therefore, the q-bipartite distance matrix of G is a generalization of the bipartite distance matrix of G. ◄

Let us first look at the following example.

► **Example 5.1.4.** Consider the nonsingular tree T as shown below. Here the dashed edges are the matching edges.

T



Clearly, $L = \{l_1, \ldots, l_6\}$, $R = \{r_1, \ldots, r_6\}$ is a standard vertex bipartition of T. The q-bipartite distance matrix $\mathfrak{B}_q$ of the tree T is given by

$$\mathfrak{B}_q = \begin{bmatrix} 1 & q^2+q+1 & q^4+q^3+q^2+q+1 & q^6+q^5+q^4+q^3+q^2+q+1 & q^4+q^3+q^2+q+1 \\ 1 & 1 & q^2+q+1 & q^4+q^3+q^2+q+1 & q^2+q+1 \\ q^2+q+1 & 1 & 1 & q^2+q+1 & 1 \\ q^4+q^3+q^2+q+1 & q^2+q+1 & 1 & 1 & q^2+q+1 \\ q^4+q^3+q^2+q+1 & q^2+q+1 & q^2+q+1 & q^4+q^3+q^2+q+1 & 1 \end{bmatrix}$$

The exponential bipartite distance matrix $\mathfrak{B}_\epsilon$ of the tree T is shown below along with the bipartite distance matrix $\mathfrak{B}$ of T.

$$\mathfrak{B}_\epsilon = \begin{bmatrix} q & q^3 & q^5 & q^7 & q^5 \\ q & q & q^3 & q^5 & q^3 \\ q^3 & q & q & q^3 & q \\ q^5 & q^3 & q & q & q^3 \\ q^5 & q^3 & q^3 & q^5 & q \end{bmatrix}, \qquad \mathfrak{B} = \begin{bmatrix} 1 & 3 & 5 & 7 & 5 \\ 1 & 1 & 3 & 5 & 3 \\ 3 & 1 & 1 & 3 & 1 \\ 5 & 3 & 1 & 1 & 3 \\ 5 & 3 & 3 & 5 & 1 \end{bmatrix}.$$ ◄

For a bipartite graph G with a unique perfect matching, we can have many standard vertex bipartitions depending on the labeling of its vertices. However, we can see that the q analogues of the bipartite distance matrices corresponding to them are similar to each other.

Let G be a connected, bipartite graph with a unique perfect matching. Let (L, R) be a standard vertex bipartition of G.

a) Let $\widehat{G}$ be the graph obtained by interchanging the labels of l_i with r_i for all $i = 1, \ldots, p$ in G. Then $\mathfrak{B}_q(G) = \mathfrak{B}_q(\widehat{G})^t$ and both the matrices are similar. Also $\mathfrak{B}_\epsilon(G) = \mathfrak{B}_\epsilon(\widehat{G})^t$ and both the matrices are similar.

b) Let $\widehat{G}$ be the graph obtained by relabeling the vertices within the part L, keeping it a standard vertex bipartition (that is, we also relabel the respective vertices in R accordingly). Then $\mathfrak{B}_q(G)$ is permutation similar to $\mathfrak{B}_q(\widehat{G})$. Also $\mathfrak{B}_\epsilon(G)$ is permutation similar to $\mathfrak{B}_\epsilon(\widehat{G})$. ◀

The q-bipartite distance matrix and the exponential bipartite distance matrix of a nonsingular tree T are closely related to each other by the following identity.

▶ **Lemma 5.1.6.** *Let G be a connected, bipartite graph with a unique perfect matching. Then for each standard vertex bipartition we have*

$$(1-q)\mathfrak{B}_q(G) = \mathbb{1}\mathbb{1}^t - \mathfrak{B}_\epsilon(G). \qquad ◀$$

Proof. The proof follows from the identity

$$(1-q)\{k\} = 1 - q^k, \qquad \text{for } k \in \mathbb{N}. \qquad ■$$

The following result tells that if k is even then -1 is always a zero of $\{k\}$.

▶ **Lemma 5.1.7.** *Let k be an even integer. Then there exist a polynomial g in q of integer coefficient such that*

$$\{k\} = (q+1)g(q). \qquad ◀$$

Proof. Let $h(q) = 1 + q + \cdots + q^{k-1}$. Since $k-1$ is an odd integer, $h(-1) = 0$. This completes the proof. ■

Let T be a tree on $2p$ vertices. By Theorem 5.1.2, we see that $\det D_q(T)$ is always a multiple of $(1+q)^{2(p-1)}$. Since the size of $\mathfrak{B}_q(T)$ is half of that of the $D_q(T)$, one would ask whether $\det B_q(T)$ is a multiple of $(1+q)^{p-1}$. In the following result we see the answer is in the affirmative.

▶ **Theorem 5.1.8.** *Let G be a connected, bipartite graph on $2p$ vertices with a unique perfect matching. Let (L, R) be the standard vertex bipartition of G. Then $\det \mathfrak{B}_q(G)$ is divisible by $q^{p-1}(1+q)^{p-1}$.* ◀

Proof. Clearly the distance between $l_i \in L$ and $r_j \in R$ in G is a odd number. Therefore, each entry of $\mathfrak{B}_q(G)$ is of the form $\{\!\!\{k\}\!\!\}$, for some odd number k.

Let E_i denote the matrix $I - \boldsymbol{e}_1\boldsymbol{e}_i^t$, for $i = 2, \ldots, p$. Let $P = E_2E_3\cdots E_p$. Then we have

$$\mathfrak{B}_q(G)P = \begin{bmatrix} 1 & \boldsymbol{x}^t \\ * & Q \end{bmatrix},$$

for some vector $\boldsymbol{x}$. Since $\texttt{dist}(l_i, r_j)$ is odd for each i, j, all entries of Q are of the following form

$$q^t + q^{t+1} + \cdots + q^{s-1}, \qquad \text{for some even numbers } t \text{ and } s.$$

Let $k_i = \texttt{dist}(l_i, r_1)$ and $k_j = \texttt{dist}(l_i, r_j)$ where $i = 1, \ldots, p$ and $j = 2, \ldots, p$. Let $t_{i,j} = \min\{k_i, k_j\} \geq 1$. Notice that

$$\{\!\!\{k_j\}\!\!\} - \{\!\!\{k_i\}\!\!\} = \pm q^{t_{i,j}}\{\!\!\{|k_i - k_j|\}\!\!\}.$$

By Lemma 5.1.7, it follows that $\det \mathfrak{B}_q(G)$ has a factor $q^{p-1}(1+q)^{p-1}$. This completes the proof. ■

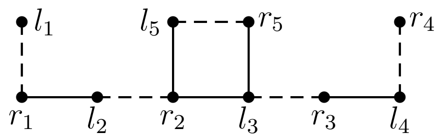
From the above result, it is easy to note that $\det \mathfrak{B}_q(G) = 0$ for $q = 0, -1$. As our main aim is to find the determinant of q-distance matrix of a nonsingular tree, so $q = 0, -1$ are not in our interest. For $q = 1$, the above result gives us Theorem 2.1.7.

In view of Theorem 5.1.8, we see that in order to find the determinant of the q-bipartite distance matrix it is enough to calculate the quantity $\det \mathfrak{B}_q(T)/q^{p-1}(1+q)^{p-1}$. This motivate us to define q-bipartite distance index of T.

► **Definition 5.1.9.** Let G be a connected, bipartite graph on $2p$ vertices with a unique perfect matching. The q-bipartite distance index of G is denoted by $\texttt{bd}_\texttt{q}(G)$ and is defined by

$$\texttt{bd}_\texttt{q}(G) := \frac{\det \mathfrak{B}_q(G)}{q^{p-1}(1+q)^{p-1}}, \qquad q \neq 0, -1.$$ ◄

► **Example 5.1.10.** Consider the following bipartite graph G with a unique perfect matching. Here the dashed edges are the matching edges.



Let $\mathfrak{B}_q(G)$ be the q-bipartite distance matrix of G. Then we can see that $\det \mathfrak{B}_q(G) = q^4(1+q)^4(2q+3)$. It follows that $\texttt{bd}_\texttt{q}(G) = 2q + 3$.

In a similar way, we can check that the q-bipartite distance index of T as defined in Example 5.1.4, is also $2q+3$. ◀

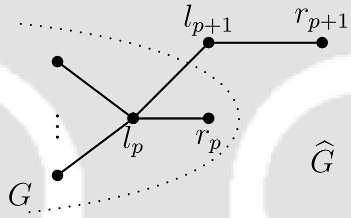
5.2 The q-bipartite distance index

The following result tells us that the $\mathtt{bd_q}(G)$ remains same if we attach a new P_2 at any quasi-pendant vertex of G. Note that for $q=1$, the result below gives us Lemma 2.1.11.

▶ **Theorem 5.2.1.** *Let G be a connected, bipartite graph on $2p$ vertices with a unique perfect matching. Let v be a quasi-pendant vertex of G. Let $\widehat{G}$ be the bipartite graph on $2p+2$ vertices obtained by attaching a new P_2 at v. Then*

$$\det \mathfrak{B}_q(\widehat{G}) = -q(1+q)\det \mathfrak{B}(G), \qquad \text{and} \qquad \mathtt{bd_q}(\widehat{G}) = \mathtt{bd_q}(G). \quad ◀$$

Proof. Let $L=\{l_1,\ldots,l_p\}$, $R=\{r_1,\ldots,r_p\}$ be a standard vertex bipartition of G and assume without loss of generality that $v=l_p$. Let the graph $\widehat{G}$ be constructed from G by taking two new vertices l_{p+1}, r_{p+1} and adding the edges $[l_p, r_{p+1}]$, $[r_{p+1}, l_{p+1}]$.



The q-bipartite distance matrix of $\widehat{G}$ can be written as

$$\mathfrak{B}_q(\widehat{G}) = \begin{bmatrix} \mathfrak{B}_q(G) & \mathfrak{B}_q(T)\boldsymbol{e}_p \\ (1+q)\mathbb{1}^t + q^2\boldsymbol{e}_p^t\mathfrak{B}_q(G) & 1 \end{bmatrix}.$$

By subtracting the column p from the column $p+1$ in $\mathfrak{B}_q(\widehat{G})$ we get

$$Q = \begin{bmatrix} \mathfrak{B}_q(G) & \mathbf{0} \\ (1+q)\mathbb{1}^t + q^2\boldsymbol{e}_p^t\mathfrak{B}_q(G) & -(q+q^2) \end{bmatrix}$$

Hence $\det \mathfrak{B}_q(\widehat{G}) = \det Q = -q(1+q)\ \det \mathfrak{B}(G)$ and $\mathtt{bd_q}(\widehat{G}) = \mathtt{bd_q}(G)$. ■

Since a corona tree can be obtained from P_2 by repeatedly attaching a new P_2 at quasi-pendant vertices and $\mathtt{bd_q}(P_2)=1$, the following corollary is immediate from Theorem 5.2.1 which tells that the q-bipartite distance index of a corona tree is always one.

▶ **Corollary 5.2.2.** *Let T be a corona tree on $2p$ vertices. Then,* $\mathtt{bd_q}(T) = 1$*. In particular,*

$$\det \mathfrak{B}_q(T) = (-1)^{p-1} q^{p-1} (1+q)^{p-1}.$$ ◀

We want to remark that the converse of Corollary 5.2.2 is not true. Let T be the tree as shown in Figure 5.1. By computation, we see that $\mathtt{bd_q}(T) = 1$ whereas T is not a corona tree.

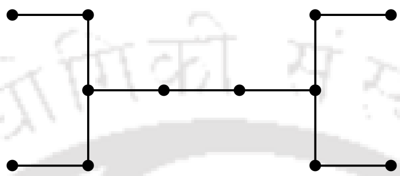


Figure 5.1: A non-corona tree T with $\mathtt{bd_q}(T) = 1$.

Note that Theorem 5.2.1 helps us to calculate the q-bipartite distance index of certain type of tree. In the next section we shall supply a recursive formula to compute $\mathtt{bd_q}(T)$ of any nonsingular tree T similar to Theorem 5.2.1 where we allow attaching a new P_2 at any vertex.

5.3 The q-bipartite distance index of a tree

Since a nonsingular tree is a bipartite graph with a unique perfect matching, the study of the properties of a bipartite distance matrix of a nonsingular tree is naturally a starting point. From now onward we mainly focus on characterize the bipartite distance index of a nonsingular tree.

In the below result we calculate the q-bipartite distance indices of the paths P_{2p}.

▶ **Theorem 5.3.1.** *Let P_{2p} be a path on $2p$ vertices. Then* $\det \mathfrak{B}_\epsilon(P_{2p}) = q^p(1-q^2)^{p-1}$ *and for $q \neq 0, -1$,*

$$\mathtt{bd_q}(P_{2p}) = \left\lceil \frac{p-2}{2} \right\rceil q + \left\lceil \frac{p}{2} \right\rceil,$$

where $\lceil x \rceil$ is the ceiling function that maps x to the least integer greater than or equal to x. ◀

Proof. Let $T = [l_1, r_1, l_2, r_2, \ldots, l_p, r_p]$ be a path on $2p$ vertices.

Our first aim is to prove $\det \mathfrak{B}_\epsilon(T) = q^p(1-q^2)^{p-1}$.

Let $T' = [l_1, r_1, \ldots, l_{p-1}, r_{p-1}]$. That is, T' is a path on $2p-2$ vertices. Notice that the matrix $\mathfrak{B}_\epsilon(T)$ can be written as

$$\mathfrak{B}_\epsilon(T) = \begin{bmatrix} \mathfrak{B}_\epsilon(T') & q^2\mathfrak{B}_\epsilon(T')_{:,p-1} \\ \boldsymbol{x}^t & q \end{bmatrix},$$

where $\boldsymbol{x} = [\ q^{2p-3} \quad q^{2p-5} \quad \cdots \quad q\ ]^t$. Take $E = I - q^2\boldsymbol{e}_{p-1}\boldsymbol{e}_p^t$. It is easy to note that

$$\mathfrak{B}_\epsilon(T)E = \begin{bmatrix} \mathfrak{B}_\epsilon(T') & \mathbf{0} \\ \boldsymbol{x}^t & q - q^3 \end{bmatrix}.$$

Therefore, by applying induction, we see that $\det \mathfrak{B}_\epsilon(T) = q^p(1-q^2)^{p-1}$.

Notice that for $q = 1$ the result follows directly follows from Proposition 2.2.1. Thus our aim is to prove the remaining part of the result for $q \neq 0, \pm 1$. By Lemma 5.1.6 know that

$$(1-q)\mathfrak{B}_q(T) = \mathbb{1}\mathbb{1}^t - \mathfrak{B}_\epsilon(T).$$

Since $q \neq 0, \pm 1$, $\mathfrak{B}_\epsilon(T)$ is invertible. It follows that

$$(1-q)^p \det \mathfrak{B}_q(T) = (-1)^p \det(\mathfrak{B}_\epsilon(T))\big(1 - \mathbb{1}^t\mathfrak{B}_\epsilon(T)^{-1}\mathbb{1}\big). \tag{5.1}$$

Let us consider the vector $\boldsymbol{x}$ of size p defined as follows.

$$\boldsymbol{x}(i) = \begin{cases} 1/q & \text{if } i \text{ is odd,} \\ -q & \text{if } i \text{ is even but } i \neq p \\ 0 & \text{if } i = p \text{ and } i \text{ is even.} \end{cases}$$

It is easy to verify that $\mathfrak{B}_\epsilon(T)\boldsymbol{x} = \mathbb{1}$ and so $\mathfrak{B}_\epsilon(T)^{-1}\mathbb{1} = \boldsymbol{x}$. Further note that

$$\mathbb{1}^t\boldsymbol{x} = \left\lceil \frac{p}{2} \right\rceil \frac{1}{q} - \left\lceil \frac{p-2}{2} \right\rceil q.$$

Let $k = \lceil \frac{p-2}{2} \rceil$. Then $k+1 = \lceil \frac{p}{2} \rceil$. By (5.1), we get

$$\begin{aligned} (1-q)\det \mathfrak{B}_q(T) &= (-1)^p q^p (1+q)^{p-1}\left[1 - (k+1)\frac{1}{q} + kq\right] \\ &= (-1)^p q^{p-1}(1+q)^{p-1}[(q-1) + k(1+q)(q-1)] \\ &= (-1)^{p-1} q^{p-1}(1+q)^{p-1}[kq + (k+1)]. \end{aligned}$$

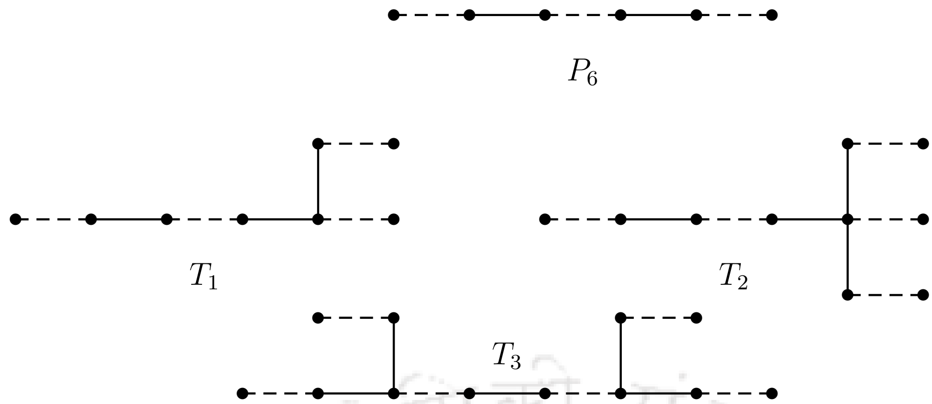


Figure 5.2: The q-bipartite distance indices of T_1 and T_2 can be be computed only by Theorem 5.2.1 and Proposition 5.3.1, but that of T_3 cannot be computed. Dashed lines are matching edges.

This completes the proof. ■

Note that the previous result along with Theorem 5.2.1 can help us evaluate the bipartite distance index of many nonsingular trees. Consider P_6, a path on 6 vertices. By Proposition 5.3.1, $\mathtt{bd_q}(P_6) = q + 2$. Now consider the tree T_1 obtained from P_6 as show Figure 2.2. By Theorem 5.2.1, $\mathtt{bd_q}(T_1) = q + 2$. Similarly, again by Theorem 5.2.1, $\mathtt{bd_q}(T_2) = q + 2$. Thus, multiplying $(-q)^{p-1}(1 + q)^{p-1}$ we get the value of $\det \mathfrak{B}_q(T)$ for these trees, for example, $\det \mathfrak{B}_q(T_2) = (-1)^{p-1}(q)^{p-1}(1 + q)^{p-1}(q + 2)$.

However, there are trees whose bipartite distance index cannot be computed from these two results alone. For example, the tree T_3 in Figure 5.2 is such a tree. In the next section we shall develop a technique that will help us to compute the bipartite index of any nonsingular tree.

5.4 Recursive formula to calculate the q-bipartite distance index of a tree

In this section we prove that the determinant of the exponential bipartite distance matrix of a nonsingular tree depends only on the number of vertices but not on the structure of the tree and provide a recursive formula to calculate the q-bipartite distance index of any nonsingular tree. Before that let us define the following vector which is a q-analogue version of the $\boldsymbol{\tau}_T$ vector, as defined in Definition 2.2.8.

▶ **Definition 5.4.1.** Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). The vector ${}^q\boldsymbol{\tau}_T$, or simply ${}^q\boldsymbol{\tau}$, of size $2p$ is defined by

$$ {}^q\boldsymbol{\tau}_T(v) := -\left[(d_T(v) - 1)q^2 + \left(1 + (d_T(v) - 1)q^2\right) \mathtt{diff}_T(v)\right], \quad \text{for each } v \text{ in } T. \tag{5.2}$$

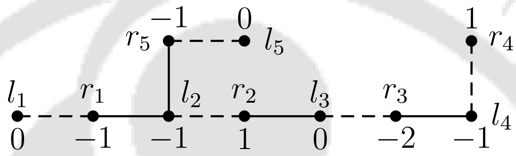
The entries of ${}^q\boldsymbol{\tau}_T$ are ordered according to $l_1, \ldots, l_p, r_1, \ldots, r_p$.

The restriction of ${}^q\boldsymbol{\tau}_T$ on L is denoted by ${}^q\boldsymbol{\tau}_T^l$ or simply by ${}^q\boldsymbol{\tau}^l$ and the restriction of ${}^q\boldsymbol{\tau}_T$ on R is denoted by ${}^q\boldsymbol{\tau}_T^r$ or simply by ${}^q\boldsymbol{\tau}^r$.

Note that ${}^q\boldsymbol{\tau}_T^r$ and ${}^q\boldsymbol{\tau}_T^l$ are vectors of size p whose i-th component is ${}^q\boldsymbol{\tau}_T(r_i)$ and ${}^q\boldsymbol{\tau}_T(l_i)$, respectively. ◀

Note that if $q = 1$ then ${}^q\boldsymbol{\tau}_T(v) = 1 - d_T(v)[1 + \texttt{diff}_T(v)]$ for each v in T. Therefore, the vector ${}^q\boldsymbol{\tau}_T$ is a generalization of the vector $\boldsymbol{\tau}_T$ as defined in Definition 2.2.8.

▶ **Example 5.4.2.** Consider the tree T as shown below. Here the dashed edges are matching edges. The numerical value near at the vertex v represent the value $\texttt{diff}_T(v)$.



Note that $d(l_2)$, the degree of the vertex l_2, is 3 and $\texttt{diff}_T(l_2) = -1$. It follows that

$$ {}^q\boldsymbol{\tau}_T(l_2) = -(3-1)q^2 - \big(1 + (3-1)q^2\big) \times (-1) = 1. $$

In a similar way, we can see that

$$ {}^q\boldsymbol{\tau}_T(r_2) = -(2-1)q^2 - \big(1 + (2-1)q^2\big) \times 1 = -1 - 2q^2. $$

Therefore, ${}^q\boldsymbol{\tau}_T^l$ and ${}^q\boldsymbol{\tau}_T^r$ of T is given by

$$ {}^q\boldsymbol{\tau}_T^l = \begin{bmatrix} 0 & 1 & -q^2 & 1 & 0 \end{bmatrix}^t, \quad \text{and} \quad {}^q\boldsymbol{\tau}_T^r = \begin{bmatrix} 1 & -1-2q^2 & 2+q^2 & -1 & 1 \end{bmatrix}^t. $$ ◀

The following theorem gives us insight to how the values of ${}^q\boldsymbol{\tau}^r$ and ${}^q\boldsymbol{\tau}^l$ changes under the operation attaching a new P_2 at a vertex.

▶ **Lemma 5.4.3.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\widehat{T}$ be the tree obtained from T by attaching a new P_2 at v.*

(a) If $v = r_k$ for some k, then ${}^q\boldsymbol{\tau}_{\widehat{T}}^r = \begin{bmatrix} {}^q\boldsymbol{\tau}_T^r \\ 0 \end{bmatrix} - (1 + \texttt{diff}_T(r_k)) \begin{bmatrix} q^2\boldsymbol{e}_k \\ -1 \end{bmatrix}$.

(b) If $v = l_k$ for some k, then ${}^q\boldsymbol{\tau}_{\widehat{T}}^l = \begin{bmatrix} {}^q\boldsymbol{\tau}_T^l \\ 0 \end{bmatrix} - (1 + \texttt{diff}_T(l_k)) \begin{bmatrix} q^2\boldsymbol{e}_k \\ -1 \end{bmatrix}$. ◀

Proof. We prove only item (a) as the proof of item (b) can dealt in a similar way. Without loss of any generality, let us assume that $\widehat{T}$ be the tree obtained from T by taking two new vertices l_{p+1}, r_{p+1} and adding the edges $[r_k, l_{p+1}]$, $[l_{p+1}, r_{p+1}]$. Note that if $r_i \in T$ then the alternating path starting from r_i lies inside the tree T. This shows that $\mathtt{diff}_T(r_i) = \mathtt{diff}_{\widehat{T}}(r_i)$ for each $i = 1, \ldots, p$. Now, each odd alternating path starting from r_k corresponds to an even alternating path starting from r_{p+1} and each even alternating path starting from r_k corresponds to an odd alternating path starting from r_{p+1}. This shows that $\mathtt{diff}_{\widehat{T}}(r_{p+1}) = -1 - \mathtt{diff}_T(r_k)$.

Since $d_{\widehat{T}}(v) = d_T(v)$ for each $v \in T - \{r_k\}$, it follows that ${}^q\boldsymbol{\tau}_{\widehat{T}}(r_i) = {}^q\boldsymbol{\tau}_T(r_i)$ for each $i \neq k$ and $i = 1, \ldots, p$. Note that $d_{\widehat{T}}(r_k) - 1 = d_T(r_k)$. Therefore, we get

$${}^q\boldsymbol{\tau}_{\widehat{T}}(r_k) = -\left[d_T(r_k)q^2 + \left(1 + d_T(r_k)q^2\right)\mathtt{diff}_T(r_k)\right] = {}^q\boldsymbol{\tau}_T(r_k) - q^2(1 + \mathtt{diff}_T(r_k)).$$

Since $d_{\widehat{T}}(r_{p+1}) = 1$, it follows that ${}^q\boldsymbol{\tau}_{\widehat{T}}(r_{p+1}) = 1 + \mathtt{diff}_T(r_k)$. This completes the proof. ■

5.4.1 The exponential bipartite distance matrix

The following result will be used as inductive step to calculate the determinant of the exponential bipartite distance matrix.

► **Lemma 5.4.4.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\widehat{T}$ be the tree obtained by attaching a new P_2 at v.*

(a) If $v = r_k$ for some k and $\mathfrak{B}_\epsilon(T){}^q\boldsymbol{\tau}_T^r = q\mathbb{1}$, then $\mathfrak{B}_\epsilon(\widehat{T}){}^q\boldsymbol{\tau}_{\widehat{T}}^r = q\mathbb{1}$.

(b) If $v = l_k$ for some k and $({}^q\boldsymbol{\tau}_T^l)^t\mathfrak{B}_\epsilon(T) = q\mathbb{1}$, then $({}^q\boldsymbol{\tau}_{\widehat{T}}^l)^t\mathfrak{B}_\epsilon(\widehat{T}) = q\mathbb{1}^t$.

Furthermore, $\det \mathfrak{B}_\epsilon(\widehat{T}) = q(1 - q^2)\det \mathfrak{B}_\epsilon(T)$. ◄

Proof. We only provide the proof of item (a) as the proof of item (b) can be dealt in a similar way. Clearly, the standard vertex bipartition of T is $(L = \{l_1, \ldots, l_p\}, R = \{r_1, \ldots, r_p\})$. Let $\widehat{T}$ be the tree obtained from T by introducing two new vertices l_{p+1}, r_{p+1} and adding the edges $[r_k, l_{p+1}]$, $[l_{p+1}, r_{p+1}]$.

Note that the matrix $\mathfrak{B}_\epsilon(\widehat{T})$ can be written as

$$\mathfrak{B}_\epsilon(\widehat{T}) = \begin{bmatrix} \mathfrak{B}_\epsilon(T) & q^2\mathfrak{B}_\epsilon(T)\boldsymbol{e}_k \\ \boldsymbol{y}^t & q \end{bmatrix},$$

where $\boldsymbol{y}(i) = q^{\mathtt{dist}(l_{p+1}, r_i)}$ for $i = 1, \ldots, p$. By item (a) of Lemma 5.4.3, we know that

$${}^q\boldsymbol{\tau}_{\widehat{T}}^r = \begin{bmatrix} {}^q\boldsymbol{\tau}_T^r \\ 0 \end{bmatrix} - (1 + \mathtt{diff}_T(r_k))\begin{bmatrix} q^2\boldsymbol{e}_k \\ -1 \end{bmatrix}.$$

First note that

$$\mathfrak{B}_\epsilon(\widehat{T})\begin{bmatrix}-q^2\boldsymbol{e}_k\\1\end{bmatrix} = -q^2\mathfrak{B}_\epsilon(\widehat{T})\boldsymbol{e}_k+\mathfrak{B}_\epsilon(\widehat{T})\boldsymbol{e}_{p+1} = -q^2\begin{bmatrix}\mathfrak{B}_\epsilon(T)\boldsymbol{e}_k\\q\end{bmatrix}+\begin{bmatrix}q^2\mathfrak{B}_\epsilon(T)\boldsymbol{e}_k\\q\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\-(q^3-q)\end{bmatrix}.$$

Therefore, by applying $\mathfrak{B}_\epsilon(T)^q\boldsymbol{\tau}_T^r = q\mathbb{1}$, we have

$$\mathfrak{B}_\epsilon(\widehat{T})^q\boldsymbol{\tau}_{\widehat{T}}^r = \begin{bmatrix}q\mathbb{1}\\\boldsymbol{y}^{t\,q}\boldsymbol{\tau}_T^r-(q^3-q)(1+\texttt{diff}_T(r_k))\end{bmatrix}. \tag{5.3}$$

Let F be the component of T obtained by removing all edges incident at r_k except the edge $[l_k, r_k]$, that contain the vertex l_k. It is easy to note that if $d_T(r_k) = 1$ then F is T itself. Further note that, an alternating path starts at $r_i \in F$ lies inside the tree F. It follows that

$$\texttt{diff}_{\widehat{T}}(r_i) = \texttt{diff}_T(r_i) = \texttt{diff}_F(r_i) \quad \text{for each } r_i \in F.$$

Clearly, $d_T(v) = d_F(v)$ for each $v \in F - r_k$. Therefore,

$${}^q\boldsymbol{\tau}_T^r(r_i) = {}^q\boldsymbol{\tau}_F^r(r_i), \qquad \text{for each } r_i \in F - r_k.$$

Notice that $\boldsymbol{y}(i) = q^{\texttt{dist}(l_k,r_i)}$ if $r_i \in (T-F)\cup\{r_k\}$ and $\boldsymbol{y}(i) = q^2q^{\texttt{dist}(l_k,r_i)}$ if $r_i \in F - r_k$. Therefore we get

$$\begin{aligned}\boldsymbol{y}^{t\,q}\boldsymbol{\tau}_T^r &= \sum_{i=1}^{p} q^{\texttt{dist}(l_k,r_i)\,q}\boldsymbol{\tau}_T^r(r_i) + (q^2-1)\sum_{\substack{r_i\in F\\ i\neq k}} q^{\texttt{dist}(l_k,r_i)\,q}\boldsymbol{\tau}_T^r(r_i)\\ &= e_k^t\mathfrak{B}_\epsilon(T)^q\boldsymbol{\tau}_T^r + (q^2-1)\left[\sum_{r_i\in F} q^{\texttt{dist}(l_k,r_i)\,q}\boldsymbol{\tau}_F^r(r_i)\right] - q(q^2-1)^q\boldsymbol{\tau}_F^r(r_k)\\ &= q + (q^2-1)e_k^t\mathfrak{B}_\epsilon(F)^q\boldsymbol{\tau}_F^r + q(q^2-1)\,\texttt{diff}_F(r_k)\\ &= q + q(q^2-1)(1+\texttt{diff}_T(r_k)).\end{aligned} \tag{5.4}$$

By (5.3) and (5.4), it follows that $\mathfrak{B}_\epsilon(\widehat{T}) = q\mathbb{1}$.

If $q = 0$ then $\det\mathfrak{B}_\epsilon(T) = \det\mathfrak{B}_\epsilon(\widehat{T}) = 0$ and so $\det\mathfrak{B}_\epsilon(\widehat{T}) = q(1-q^2)\det\mathfrak{B}_\epsilon(T)$ holds trivially. Let us assume $q \neq 0$. Take $E = I - q^2\boldsymbol{e}_k\boldsymbol{e}_{p+1}^t$. Note that all entries in the last column of $\mathfrak{B}_\epsilon(\widehat{T})E$ are zero except the $p+1$-th entry is $q-q^3$. Hence, it follows that $\det\mathfrak{B}(\widehat{T}) = q(1-q^2)\det\mathfrak{B}(\widehat{T})$. This completes the proof. ∎

▶ **Theorem 5.4.5.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Then $\mathfrak{B}_\epsilon(T)^q\boldsymbol{\tau}_T^r = q\mathbb{1}$ and $\det\mathfrak{B}_\epsilon(T) = q^p(1-q^2)^{p-1}$.* ◀

Proof. We prove the statement by induction on p. Clearly the statement holds for $p = 1$, as in this case $\mathfrak{B}_\epsilon(T) = [q]$ and ${}^q\boldsymbol{\tau}_T^r = [1]$. Assume the statement to be true for p. Let $\widehat{T}$ be a nonsingular tree on $2p+2$ vertices. Then $\widehat{T}$ is obtained from some nonsingular tree T on $2p$ by attaching a new P_2 at some vertex v. Let (L, R) be the standard vertex bipartition of T such that $v = r_p$. By the induction hypothesis, $\mathfrak{B}_\epsilon(T)^q\boldsymbol{\tau}_T^r = q\mathbb{1}$ and $\det \mathfrak{B}_\epsilon(T) = q^p(1-q^2)^{p-1}$. Therefore, by Lemma 5.4.4, $\mathfrak{B}_\epsilon(\widehat{T})^q\boldsymbol{\tau}_{\widehat{T}}^r = q\mathbb{1}$ and $\det \mathfrak{B}_\epsilon(\widehat{T}) = q(1-q^2)\det \mathfrak{B}_\epsilon(T)$. This completes the proof. ■

Here we remark that if $q = 1$ then the above result gives us Lemma 2.2.13. Hence, Theorem 5.4.5 is a generalization of Lemma 2.2.13.

The following corollary is immediate from Theorem 5.4.5.

▶ **Corollary 5.4.6.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Then $({}^q\boldsymbol{\tau}_T^l)^t\mathfrak{B}_\epsilon(T) = q\mathbb{1}^t$ and $\det \mathfrak{B}_\epsilon(T) = q^p(1-q^2)^{p-1}$.* ◀

5.4.2 The q-bipartite distance matrix

The following result will be used as inductive step in the proof of Theorem 5.4.8.

▶ **Lemma 5.4.7.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Let $\widehat{T}$ be the tree obtained by attaching a new P_2 at v. Suppose $q \neq 0, -1$.*

(a) If $v = r_k$ for some k and $\mathfrak{B}_q(T)^q\boldsymbol{\tau}_T^r = \mathtt{bd_q}(T)\mathbb{1}$ then

$$\mathfrak{B}_q(\widehat{T})^q\boldsymbol{\tau}_{\widehat{T}}^r = (\mathtt{bd_q}(T) + (1+q)[1 + \mathtt{diff}_T(v)])\mathbb{1}.$$

(b) If $v = l_k$ for some k and $({}^q\boldsymbol{\tau}_T^l)^t\mathfrak{B}_q(T) = \mathtt{bd_q}(T)\mathbb{1}$, then

$$({}^q\boldsymbol{\tau}_{\widehat{T}}^l)^t\mathfrak{B}_\epsilon(\widehat{T}) = (\mathtt{bd_q}(T) + (1+q)[1 + \mathtt{diff}_T(v)])\mathbb{1}^t.$$

Furthermore, $\mathtt{bd_q}(\widehat{T}) = \mathtt{bd_q}(T) + (1+q)[1 + \mathtt{diff}_T(v)]$. ◀

Proof. We only provide the proof of item (a) as the proof of item (b) can be dealt in a similar way. Clearly, the standard vertex bipartition of T is $(L = \{l_1, \ldots, l_p\}, R = \{r_1, \ldots, r_p\})$. Let $\widehat{T}$ be the tree obtained from T by introducing two new vertices l_{p+1}, r_{p+1} and adding the edges $[r_k, l_{p+1}]$, $[l_{p+1}, r_{p+1}]$.

Note that the matrix $\mathfrak{B}_\epsilon(\widehat{T})$ can be written as

$$\mathfrak{B}_q(\widehat{T}) = \begin{bmatrix} \mathfrak{B}_q(T) & (1+q)\mathbb{1} + q^2\mathfrak{B}_\epsilon(T)_{:,k} \\ \boldsymbol{y}^t & 1 \end{bmatrix},$$

where $\boldsymbol{y}(i) = \{\texttt{dist}(l_{p+1}, r_i)\}$ for $i = 1, \ldots, p$. By item (a) of Lemma 5.4.3, we know that

$$^{q}\boldsymbol{\tau}^{r}_{\widehat{T}} = \begin{bmatrix} ^{q}\boldsymbol{\tau}^{r}_{T} \\ 0 \end{bmatrix} - (1 + \texttt{diff}_T(r_k)) \begin{bmatrix} q^2 \boldsymbol{e}_k \\ -1 \end{bmatrix}.$$

First note that

$$\mathfrak{B}_q(\widehat{T}) \begin{bmatrix} -q^2 \boldsymbol{e}_k \\ 1 \end{bmatrix} = -q^2 \mathfrak{B}_q(\widehat{T})\boldsymbol{e}_k + \mathfrak{B}_q(\widehat{T})\boldsymbol{e}_{p+1} = \begin{bmatrix} (1+q)\mathbb{1} \\ 1 - q^2 \end{bmatrix}. \tag{5.5}$$

Therefore, by applying $\mathfrak{B}_\epsilon(T)^{q}\boldsymbol{\tau}^{r}_{T} = \texttt{bd}_\texttt{q}(T)\mathbb{1}$, we have

$$\mathfrak{B}_\epsilon(\widehat{T})^{q}\boldsymbol{\tau}^{r}_{\widehat{T}} = \begin{bmatrix} [\texttt{bd}_\texttt{q}(T) + (1+q)(1 + \texttt{diff}_T(r_k))]\mathbb{1} \\ \boldsymbol{y}^{t\,q}\boldsymbol{\tau}^{r}_{T} + (1-q^2)(1 + \texttt{diff}_T(r_k)) \end{bmatrix}. \tag{5.6}$$

Let F be the component of T obtained by removing all edges incident at r_k except the edge $[l_k, r_k]$, that contain the vertex l_k.

It is easy to note that if $d_T(r_k) = 1$ then F is T itself. Further note that, an alternating path starts at $r_i \in F$ lies inside the tree F. It follows that

$$\texttt{diff}_{\widehat{T}}(r_i) = \texttt{diff}_T(r_i) = \texttt{diff}_F(r_i) \quad \text{ for each } r_i \in F.$$

Clearly $d_T(v) = d_F(v)$ for each $v \in F - r_k$. Therefore,

$$^{q}\boldsymbol{\tau}^{r}(T)(r_i) = {}^{q}\boldsymbol{\tau}^{r}_{F}(r_i), \qquad \text{ for each } r_i \in F - r_k.$$

Notice that $\boldsymbol{y}(i) = \{\texttt{dist}(l_k, r_i)\}$ if $r_i \in (T - F) \cup \{r_k\}$ and $\boldsymbol{y}(i) = \{\texttt{dist}(l_k, r_i) + 2\}$ if $r_i \in F - r_k$. Therefore we get

$$\begin{aligned}
\boldsymbol{y}^{t\,q}\boldsymbol{\tau}^{r}_{T} &= \sum_{i=1}^{p} y(i)^{q}\boldsymbol{\tau}^{r}_{T}(r_i) \\
&= \sum_{i=1}^{p} \{\texttt{dist}(l_k, r_i)\}^{q}\boldsymbol{\tau}^{r}_{T}(r_i) + \sum_{\substack{r_i \in F \\ i \neq k}} (\{\texttt{dist}(l_k, r_i) + 2\} - \{\texttt{dist}(l_k, r_i)\})^{q}\boldsymbol{\tau}^{r}_{T}(r_i) \\
&= e_k^t \mathfrak{B}_q(T)^{q}\boldsymbol{\tau}^{r}_{T} + \sum_{\substack{r_i \in F \\ i \neq k}} (1+q) q^{\texttt{dist}(l_k, r_i)\,q}\boldsymbol{\tau}^{r}_{F}(r_i) \\
&= \texttt{bd}_\texttt{q}(T) + (1+q) \left[\sum_{r_i \in F} q^{\texttt{dist}(l_k, r_i)\,q}\boldsymbol{\tau}^{r}_{F}(r_i) \right] - (1+q) q^{q}\boldsymbol{\tau}^{r}_{F}(r_k) \\
&= \texttt{bd}_\texttt{q}(T) + (1+q) e_k \mathfrak{B}_\epsilon(F)^{q}\boldsymbol{\tau}^{r}_{F} - (1+q) q^{q}\boldsymbol{\tau}^{r}_{F}(r_k)
\end{aligned}$$

By Lemma 5.4.4, we know that $\mathfrak{B}_\epsilon(F)^q\boldsymbol{\tau}_F^r = q\mathbb{1}$. Since ${}^q\boldsymbol{\tau}_F^r(r_k) = -\,\texttt{diff}_F(r_k) = -\,\texttt{diff}_T(r_k)$, it follows that

$$\boldsymbol{y}^{tq}\boldsymbol{\tau}_T^r = \texttt{bd}_\texttt{q}(T) + q(1+q)(1+\texttt{diff}_T(r_k)) \tag{5.7}$$

From the identity (5.7) we note that

$$\boldsymbol{y}^{tq}\boldsymbol{\tau}_T^r + (1-q^2)(1+\texttt{diff}_T(r_k)) = \texttt{bd}_\texttt{q}(T) + (1+q)(1+\texttt{diff}_T(r_k)) \tag{5.8}$$

By substituting (5.8) in (5.6) we see that

$$\mathfrak{B}_\epsilon(\widehat{T})^q\boldsymbol{\tau}_{\widehat{T}} = [\texttt{bd}_\texttt{q}(T) + (1+q)(1+\texttt{diff}_T(r_k))]\mathbb{1}.$$

Therefore, in order to establish the result it only remains to show that

$$\texttt{bd}_\texttt{q}(\widehat{T}) = \texttt{bd}_\texttt{q}(T) + (1+q)(1+\texttt{diff}_T(r_k)). \tag{5.9}$$

Let us first assume that $\texttt{bd}_\texttt{q}(T) \neq 0$. Clearly, $q \neq 0, -1$. Consider the matrix $E_1 = I - q^2\boldsymbol{e}_k\boldsymbol{e}_{p+1}^t$ and $E_2 = I - \frac{1+q}{\texttt{bd}_\texttt{q}(T)}\begin{bmatrix}{}^q\boldsymbol{\tau}_T^r\\ 0\end{bmatrix}\boldsymbol{e}_{p+1}^t$. Notice that

$$\mathfrak{B}_q(\widehat{T})E_1E_2 = \begin{bmatrix}\mathfrak{B}_q(T) & (1+q)\mathbb{1}\\ \boldsymbol{y}^t & 1-q^2\end{bmatrix}E_2 = \begin{bmatrix}\mathfrak{B}_q(T) & \boldsymbol{0}\\ \boldsymbol{y}^t & (1-q^2) - \frac{1+q}{\texttt{bd}_\texttt{q}(T)}\boldsymbol{y}^{tq}\boldsymbol{\tau}_T^r\end{bmatrix}.$$

It follows that

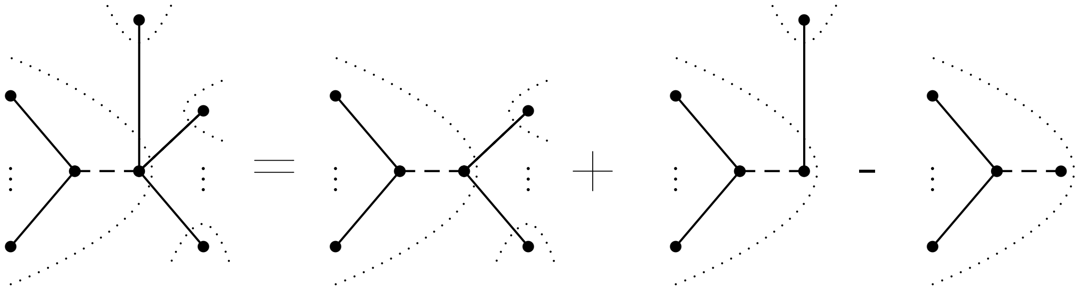
$$\texttt{bd}_\texttt{q}(\widehat{T}) = \frac{1}{q}\big[\boldsymbol{y}^{tq}\boldsymbol{\tau}_T^r + (1-q)\,\texttt{bd}_\texttt{q}(T)\big] \tag{5.10}$$

Using the identity (5.7) in (5.10) we get

$$\texttt{bd}_\texttt{q}(\widehat{T}) = \texttt{bd}_\texttt{q}(T) + (1+q)(1+\texttt{diff}_T(r_k)) \tag{5.11}$$

By substituting (5.8), (5.9) in (5.6), it establishes that if $\texttt{bd}_\texttt{q}(T) \neq 0$ then $\mathfrak{B}_q(\widehat{T})^q\boldsymbol{\tau}_{\widehat{T}} = \texttt{bd}_\texttt{q}(\widehat{T})\mathbb{1}$ and $\texttt{bd}_\texttt{q}(\widehat{T}) = \texttt{bd}_\texttt{q}(T) + (1+q)(1+\texttt{diff}_T(r_k))$.

Now we assume the case $\texttt{bd}_\texttt{q}(T) = 0$ for some $q^* \in \mathbb{C}$. Let $\mathfrak{B}^*$ is a matrix obtained from $\mathfrak{B}_q(T)$ by substituting $q = q^*$. Since $\det \mathfrak{B}_q(T)$ is a polynomial in q, it has finitely many zeros. Therefore, there exist an sequence $\{q_n\}$ converges to q^* such that $\det B_n \neq 0$ for each n where B_n is the matrix obtained from $\mathfrak{B}_q(T)$ by substituting $q = q_n$. Since each entry of $\mathfrak{B}_q(T)$ is a polynomial in q, it follows that B_n converges to B^*. Let $f(q_n) = \det(B_n)/(-q)^{p-1}(1+q)^{p-1}$. Since determinant is a continuous function, it follows that $f(q_n)$ tends to 0. Therefore, $f(q_n) + (1+q_n)(1+\texttt{diff}_T(r_k))$ converges to $(1+q^*)(1+\texttt{diff}_T(r_k))$. This establishes that $\texttt{bd}_\texttt{q}(\widehat{T}) = (1+q)(1+\texttt{diff}_T(r_k))$ for $q = q^*$. Hence the proof is complete. ■



$$\mathtt{bd_q}(T) = \mathtt{bd_q}(T_1) + \mathtt{bd_q}(T_2) - \mathtt{bd_q}(T_1 \cap T_2)$$

Figure 5.3: The inclusion-exclusion principle.

Now we are ready to supply our main result of this section which gives us a recursive way to calculate q-bipartite distance index of any nonsingular tree T.

▶ **Theorem 5.4.8.** *Let T be a nonsingular tree on $2p$ vertices with a standard vertex bipartition (L, R). Suppose $q \neq 0, -1$. Then following assertions hold.*

(a) $\mathfrak{B}_q(T)\,{}^q\boldsymbol{\tau}_T^r = \mathtt{bd_q}(T)\mathbb{1}$.

(b) Let v be a vertex and let $\widehat{T}$ be the tree obtained from T by attaching a new P_2 at v. Then

$$\mathtt{bd_q}(\widehat{T}) = \mathtt{bd_q}(T) + (1+q)(1 + \mathtt{diff}_T(v)).$$ ◀

Proof. We prove both the statements by induction on p. Clearly the item (a) holds for $p = 1$, as in this case $\mathfrak{B}_q(T) = [1]$ and ${}^q\boldsymbol{\tau}_T^r = [1]$. By Proposition 5.3.1, the item (b) holds for $p = 1$. (One can also verify the item (b) for $p = 1$ by taking P_2 and P_4.) Assume the statement to be true for p. Let $\widehat{T}$ be a nonsingular tree on $2p + 2$ vertices. Then $\widehat{T}$ is obtained from some nonsingular tree T on $2p$ by attaching a new P_2 at some vertex v. Let (L, R) be the standard vertex bipartition of T such that $v = r_p$. By the induction hypothesis, $\mathfrak{B}_q(T)\,{}^q\boldsymbol{\tau}_T^r = \mathtt{bd_q}(\widehat{T})\mathbb{1}$. Therefore, by Lemma 5.4.4, we have $\mathfrak{B}_q(\widehat{T})\,{}^q\boldsymbol{\tau}_{\widehat{T}}^r = \mathtt{bd_q}(\widehat{T})\mathbb{1}$ and $\mathtt{bd_q}(\widehat{T}) = \mathtt{bd_q}(T) + (1+q)(1 + \mathtt{diff}_T(r_k))$. This completes the proof. ■

For $q = 1$, Theorem 5.4.8 gives us Theorem 2.2.15.

5.5 The inclusion-exclusion principle

In this section our first aim is to establish that the q-bipartite distance index of a nonsingular tree satisfies an inclusion-exclusion type of principle at each matching edge and then using that result we show that the q-bipartite distance index of a nonsingular tree is nothing but a f-alternating sum with respect to the sequence $S = (1, 1, q + 2, q + 2, 2q + 3, 2q + 3, \ldots)$.

▶ **Theorem 5.5.1.** *Let T be a nonsingular tree on $2p$ vertices with a matching edge $[u, v]$ such that $s = d(v) \geq 3$. Let $w_1, \ldots, w_{s-1}$ be the other vertices adjacent to v. Let B_{w_i} be the branches at v containing the vertex w_i, $i = 1, \ldots, s-1$. Suppose $q \neq 0, -1$. Then we have*

$$\mathtt{bd_q}(T) = \mathtt{bd_q}(T - B_{w_1}) + \mathtt{bd_q}(T - B_{w_2} - \cdots - B_{w_{s-1}}) - \mathtt{bd_q}(T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}}). \blacktriangleleft$$

Figure 5.3 illustrates why we refer the above result as an inclusion-exclusion principle.

Proof of Theorem 5.5.1. Let (L, R) be the standard vertex bipartition of T such that $v = l_k$. Let the degree of r_k be s, $s \geq 1$. We set $k_1 = k$. Let $l_{k_1+1}, l_{k_2+1}, \ldots, l_{k_{s-1}+1}$ be some distinct vertices other than l_{k_1} that are adjacent to r_{k_1}, see Figure 5.4. Let B_i be the branches at v containing the vertex l_{k_i+1}, $i = 1, \ldots, s-1$. Let $B_0 = T - B_1 - B_2 - \cdots - B_{s-1}$. Note that each B_i is a nonsingular tree for $i = 0, \ldots, s-1$. Assume that the vertex set of B_0 is $\{l_1, \ldots, l_{k_1}, r_1, \ldots, r_{k_1}\}$, the vertex set of B_1 is $\{l_{k_1+1}, \ldots, l_{k_2}, r_{k_1+1}, \ldots, r_{k_2}\}$, and so on up to the vertex set of B_{s-1} is $\{l_{k_{s-1}+1}, \ldots, l_{k_s}, r_{k_{s-1}+1}, \ldots, r_{k_s}\}$. Let us put an arrow on the edge $[l_{k_1}, r_{k_1}]$ from r_{k_1} to l_{k_1}. This arrow indicates that, from a vertex r_i in B_1, we do not have an alternating path to a vertex in B_0. Similarly, from a vertex r_i in B_2, we do not have an alternating path to a vertex in $B_0, B_1, B_3, B_4, \ldots, B_{s-1}$. Similar statements are true for vertices r_i in $B_3, \ldots, B_{s-1}$. Also, from a vertex l_i in B_0, we only have alternating paths to vertices in B_0 but not to a vertex in $B_1, \ldots, B_{s-1}$. Let us take T_1 be the tree B_0. For $i = 2, \ldots, s$, let T_i be the subtree of T obtained by taking T_1 and B_i and by inserting the edge $[l_{k_1}, r_{k_{i-1}+1}]$.

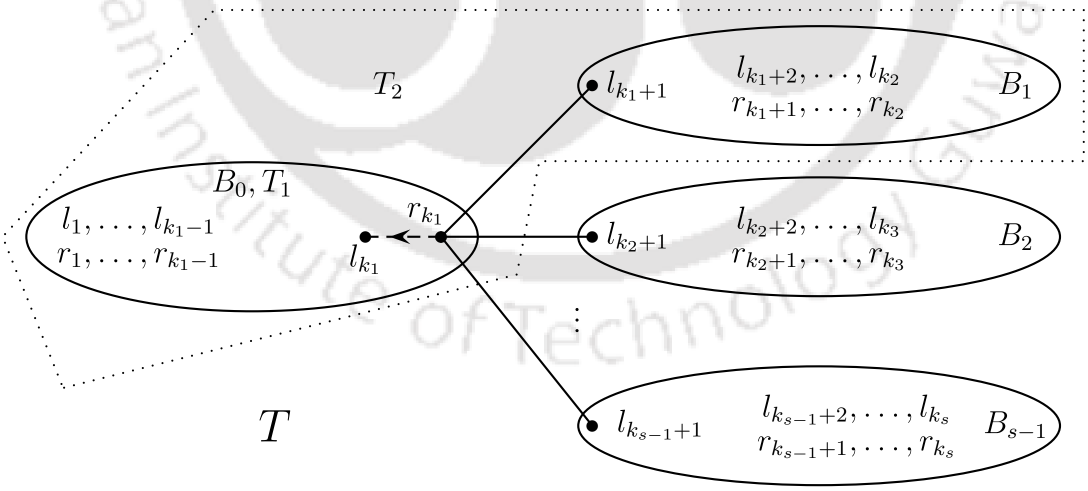


Figure 5.4: Branching of a nonsingular tree into its nonsingular subtrees.

Let $\overline{\mathfrak{B}_q(T_2)_{k_1,:}}$ denote the row induced by $\mathfrak{B}_q(T_2)_{k_1,:}$ on $r_1, \ldots, r_{k_s}$. That is, $\overline{\mathfrak{B}_q(T_2)_{k_1,:}}$ is obtained by inserting 0 entries into $\mathfrak{B}_q(T_2)_{k_1,:}$ at the places corresponding to $r_i \notin F_2$. Notice that $\overline{\mathfrak{B}_q(T_2)_{k_1,:}}(r_i) = \mathfrak{B}_q(T_2)_{k_1,:}(r_i)$ if $r_i \in T_2$ and $\overline{\mathfrak{B}_q(T_2)_{k_1,:}}(r_i) = 0$ if $r_i \notin T_2$. Then it is immediate from the structure that

$$\mathfrak{B}_q(T)_{k_1,:} = \overline{\mathfrak{B}_q(T_2)_{k_1,:}} + \cdots + \overline{\mathfrak{B}_q(T_s)_{k_1,:}} - (s-2)\overline{\mathfrak{B}_q(T_1)_{k_1,:}}. \tag{5.12}$$

Note that for any $r_i \in T_j$, $r_i \neq r_{k_1}$, we see that $d_{T_j}(r_i) = d_T(r_i)$ and $\mathtt{diff}_{T_j}(r_i) = \mathtt{diff}_T(r_i)$ for $j = 1, \ldots, s$. Hence,

$$^q\boldsymbol{\tau}^r_{T_j}(r_i) = {}^q\boldsymbol{\tau}^r_T(r_i) \qquad \text{if } r_i \in T_j,\ r_i \neq r_{k_1}.$$

Clearly, $d_T(r_{k_1}) = s$, $d_{T_1}(r_{k_1}) = 1$ and for $j = 2, 3, \ldots, s$, we have

$$d_{T_j}(r_{k_1}) = 2, \quad \text{and} \quad \mathtt{diff}_T(r_{k_1}) = \mathtt{diff}_{T_j}(r_{k_1}) = \mathtt{diff}_{T_1}(r_{k_1}).$$

It follows that

$$\begin{aligned} {}^q\boldsymbol{\tau}^r_T(r_{k_1}) &= -\big((s-1)q^2 + (1+(s-1)q^2)\ \mathtt{diff}_{T_1}(r_{k_1})\big) \\ {}^q\boldsymbol{\tau}^r_{T_j}(r_{k_1}) &= -\big(q^2 + (1+q^2)\ \mathtt{diff}_{T_1}(r_{k_1})\big) \qquad j = 2, \ldots, s \\ {}^q\boldsymbol{\tau}^r_{T_1}(r_{k_1}) &= -\big(\mathtt{diff}_{T_1}(r_{k_1})\big). \end{aligned}$$

Therefore, for $j = 2, \ldots, s$, we get

$$^q\boldsymbol{\tau}^r_T(r_{k_1}) - {}^q\boldsymbol{\tau}^r_{F_j}(r_{k_1}) = -(s-2)q^2(1 + \mathtt{diff}_{T_1}(r_{k_1})) = (s-2)C,$$

where $C = -q^2(1 + \mathtt{diff}_{T_1}(r_{k_1}))$. Further note that,

$$^q\boldsymbol{\tau}^r_T(r_{k_1}) - {}^q\boldsymbol{\tau}^r_{T_1}(r_{k_1}) = (s-1)C.$$

By Theorem 5.4.8, we know that $\mathfrak{B}_q(T)\,{}^q\boldsymbol{\tau}^r_T = \mathtt{bd_q}(T)\mathbb{1}$. In particular, we have $\mathfrak{B}_q(T)_{k_1,:}\,{}^q\boldsymbol{\tau}^r_T = \mathtt{bd_q}(T)$. It follows that

$$\begin{aligned} \mathtt{bd_q}(T) &= \mathfrak{B}_q(T)_{k_1,:}\ {}^q\boldsymbol{\tau}^r_T = \overline{\mathfrak{B}_q(T_2)_{k_1,:}}\ {}^q\boldsymbol{\tau}^r_T + \cdots + \overline{\mathfrak{B}_q(T_s)_{k_1,:}}\ {}^q\boldsymbol{\tau}^r_T - (s-2)\overline{\mathfrak{B}_q(T_1)_{k_1,:}}\ {}^q\boldsymbol{\tau}^r_T \\ &= \mathfrak{B}_q(T_2)_{k_1,:}\ {}^q\boldsymbol{\tau}^r_{T_2} + (s-2)C + \cdots + \mathfrak{B}_q(T_s)_{k_1,:}\ {}^q\boldsymbol{\tau}^r_{T_2} + (s-2)C \\ &\qquad -(s-2)\Big(\mathfrak{B}_q(T_1)_{k_1,:}\ {}^q\boldsymbol{\tau}^r_{T_2} + (s-1)C\Big) \\ &= \mathtt{bd_q}(T_2) + \cdots + \mathtt{bd_q}(T_s) - (s-2)\,\mathtt{bd_q}(T_1). \end{aligned}$$

In a similar argument, we have $\mathtt{bd_q}(T - B_1) = \mathtt{bd_q}(T_3) + \cdots + \mathtt{bd_q}(T_s) - (s-3)\,\mathtt{bd}(T_1)$. The conclusion follows by noting that $T - B_2 - \cdots - B_{s-1} = T_2$ and $T - B_1 - B_2 - \cdots - B_{s-1} = T_1$. ■

Now we illustrate Theorem 5.5.1 with the following example to see how we can calculate q-bipartite distance index of T from its (nonsingular) subtrees.

▶ **Example 5.5.2.** Consider the nonsingular tree T, as shown Figure 5.5. Edges in the perfect matching are shown as dashed lines.

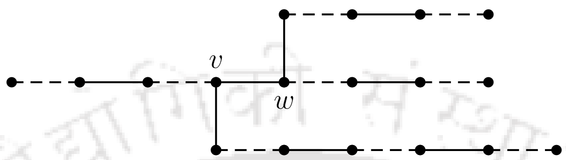
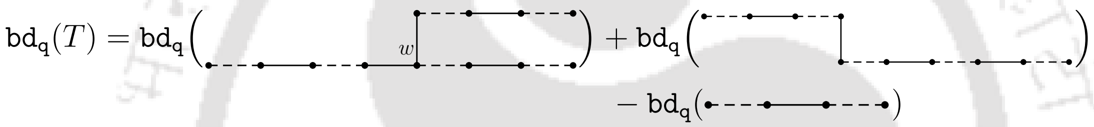


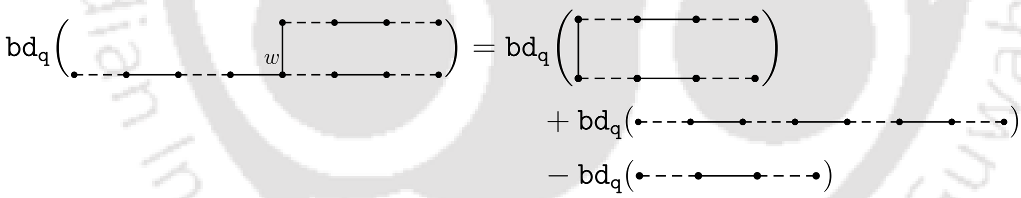
Figure 5.5: Illustration of Theorem 5.5.1

Consider the vertex v in T. Note that $d(v) = 3$. By applying Theorem 5.5.1 at v, we have



(5.13)

Further, by applying Theorem 5.5.1 at w, in the first tree of the right hand side of the above equation yields us



(5.14)

$$= 2\,\mathtt{bd_q}(P_8) - \mathtt{bd_q}(P_4)$$

Therefore, by (5.13) and (5.14), it follows that

$$\begin{aligned}\mathtt{bd_q}(T) &= (2\,\mathtt{bd_q}(P_8) - \mathtt{bd_q}(P_4)) + \mathtt{bd_q}(P_{10}) - \mathtt{bd_q}(P_4)\\ &= 2\,\mathtt{bd_q}(P_8) + \mathtt{bd_q}(P_{10}) - 2\,\mathtt{bd_q}(P_4)\\ &= 2(q+2) + (2q+3) - 2 = 4q+5.\end{aligned}$$

Hence, $\det \mathfrak{B}_q(T) = (-1)^{p-1}q^{p-1}(1+q)^{p-1}(4q+5)$. ◀

By applying Theorem 5.5.1, we can check that $\mathtt{bd_q}(T) = 1$ for the tree T as defined in Figure 5.1, $\mathtt{bd_q}(T) = 2q+3$ for the tree T as defined in Example 5.1.4, and $\mathtt{bd_q}(T) = q+2$ for the tree T as defined in Example 5.4.2.

5.6 f-alternating sum

Let T be a nonsingular tree. In Chapter 2, we define f-alternating sum of T with respect to a sequence S and observe that the f-alternating sum of T satisfies the same recurrence relation as that of $\mathtt{bd_q}(T)$ in Theorem 5.5.1, see Theorem 2.4.2.

In the following result we argue that $\mathtt{bd_q}(T)$ is nothing but the f-alternating sum with respect to the sequence $S = (1, 1, q+2, q+2, 2q+3, 2q+3, \ldots)$.

▶ **Theorem 5.6.1.** *Let T be a nonsingular tree on $2p$ vertices and $S = (1, 1, q+2, q+2, 2q+3, 2q+3, \ldots)$ be a sequence. Then $\mathtt{bd_q}(T) = f_S(T)$. In particular,*

$$\det \mathfrak{B}_q(T) = (-1)^{p-1} q^{p-1} (1+q)^{p-1} \sum_{\substack{P \in \mathcal{A}_T \\ P=[u,\cdots,v]}} [d(u)-2][d(v)-2] S\Big(\frac{|P|}{2}\Big). \quad ◀$$

Proof. We prove the statement by induction on p. Note that for $p = 1$ and $p = 2$, the only possible nonsingular tree is a path on $2p$ vertices and for a path the nonzero contributions to $f(T)$ will come only from the end vertices. Therefore, by Proposition 5.3.1, the base step holds.

For the inductive hypothesis, let us assume that the statement hold for each nonsingular tree on less than $2p$ vertices. Let T be a nonsingular tree on $2p$ vertices. If T is a path then by Proposition 5.3.1, the result holds. Suppose that T is not a path. Then there exists a vertex v in T such that $d(v) \geq 3$. Let $[u, v]$ be the matching edge involving v. Let $w_1, \ldots, w_{s-1}$ be the other vertices adjacent to v. Let B_{w_i} be the branches at v containing the vertex w_i, $i = 1, \ldots, s-1$. Then by Theorems 2.3.1 and Theorem 2.4.2 we have

$$\begin{aligned}
\mathtt{bd_q}(T) &= \mathtt{bd_q}(T - B_{w_1}) + \mathtt{bd_q}(T - B_{w_2} - \cdots - B_{w_{s-1}}) - \mathtt{bd_q}(T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}}) \\
&= f_S(T - B_{w_1}) + f_S(T - B_{w_2} - \cdots - B_{w_{s-1}}) - f_S(T - B_{w_1} - B_{w_2} - \cdots - B_{w_{s-1}}) \\
&= f_S(T).
\end{aligned}$$

This completes the proof. ■

6 Conclusion and Future Work

This dissertation investigated the bipartite distance matrix and the bipartite Laplacian matrix. We also studied minor of the bipartite Laplacian matrix and explore q-analogue versions of the bipartite distance matrix. The work performed in this dissertation provides basis for future research in several areas. These areas include:

- The bipartite distance index of a bipartite graph with a unique perfect matching.
- Classifying f-alternating sum for some other sequence S.
- Path decomposition of a nonsingular tree.
- Spectra of the bipartite Laplacian matrix.
- Further study on the q-bipartite distance matrix.

The bipartite distance index

In Chapter 2 we define bipartite distance matrix of a bipartite graph with a unique perfect matching. Although we completely characterized the determinant of a nonsingular matrix but we did not talked much about the determinant of an arbitrary bipartite graph with a unique perfect matching. We shall try to characterize the bipartite distance index of an arbitrary bipartite graph with a unique perfect matching.

f-alternating sum

In Chapter 2 and Chapter 5 we observed that the bipartite distance index and the q-bipartite distance index of a nonsingular tree are a particular type of a f-alternating sum. Further we noticed in Theorem 2.4.2 that the f-alternating sum satisfies an inclusion-exclusion type of principle at any of its matching edge. We also discussed few properties of f-alternating sums of a tree corresponding to some sequence S. It would be interesting to know properties of the f-alternating sums of nonsingular trees, for some more sequences S. We believe that there are many interesting facts lying there to be observed. In particular, it would be interesting to know the class of graphs G with a unique perfect matching that have $f_S(G) = 1$ or $f_S(G) = |G|$.

Path Decomposition

By Theorem 2.3.1, one could argue that the bipartite distance index of a nonsingular tree can be expressed as an integer combination of bipartite distance indices of some paths P_{2k}. One could ask the following question: from a given set of paths P_{2k} is it possible to construct a nonsingular tree T (by using some kind of merging operation) such that the bipartite distance index of T can be obtained as a integer combination of bipartite distance indices of given paths? We partially answered this question in Section 2.5.

Spectra of the bipartite Laplacian matrix

In Chapter 3, we have introduced bipartite Laplacian matrix which is a generalization of the usual Laplacian matrix of a tree. It was shown that it enjoys many proprieties that are true for usual Laplacian matrix. There are two properties that we believed are true but we do not have complete proofs. We mention them below.

Conjecture 1. Let $\mathfrak{L}$ be a bipartite Laplacian matrix of a nonsingular tree T. Then each eigenvalues of $\mathfrak{L}$ is a nonnegative real number.

Conjecture 2. Let $\mathfrak{L}$ be a bipartite Laplacian matrix of a nonsingular tree T. Then $\mathfrak{L}$ is diagonalizable.

Further study on the q-bipartite distance matrix

In Chapter 5, we studied two types of q-analogue versions of the bipartite distance matrix of nonsingular tree, namely, the q-bipartite distance matrix and the exponential bipartite distance matrix. We observe that although the determinant of the q-bipartite distance matrix depends on the tree structure but the determinant of the exponential bipartite distance matrix is independent of the tree structure. We need a further study in order to understand why only the determinant of the q-bipartite distance matrix depends on the tree structure.

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Publication

Based on the work carried out in this thesis, the following articles have resulted:

Accepted Articles in Refereed Journals

1. RB Bapat, Rakesh Jana, and S Pati. The bipartite distance matrix of a nonsingular tree. *Linear Algebra and its Application*, 631:254-281, 2021.

Communicated Articles in Refereed Journals

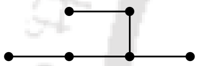
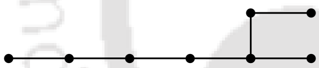
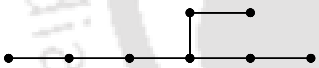
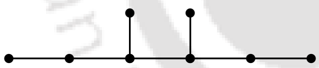
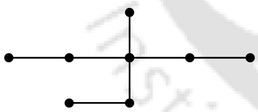
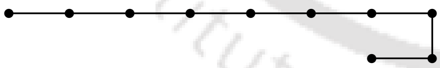
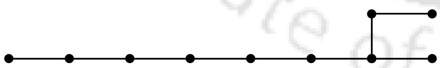
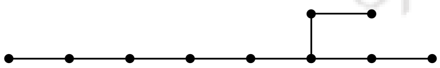
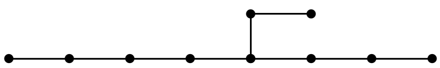
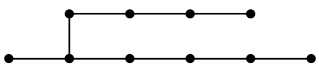
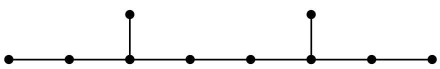
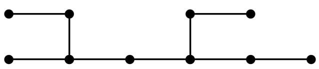
1. RB Bapat, Rakesh Jana, and S Pati. The bipartite Laplacian matrix of a nonsingular tree.
2. Rakesh Jana. A q-analogue of the bipartite distance matrix of a nonsingular tree.

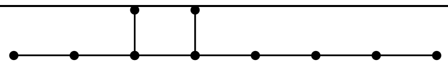
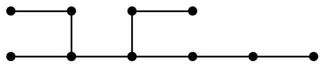
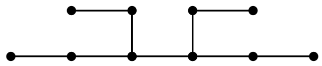
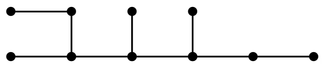
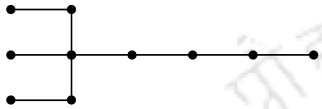
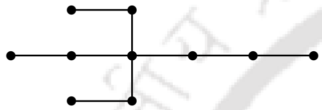
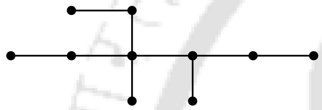
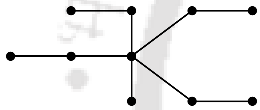
Preprint Article

1. RB Bapat, Rakesh Jana, and S Pati. All minor of the bipartite Laplacian matrix of a nonsingular tree. *(to be communicated)*

Appendix

In the below table we give a list of all nonsingular trees on at most 10 vertices with their bipartite distance index, eigenvalues of their bipartite Laplacian matrix and whether their bipartite Laplacian matrix is diagonalizable.

p	Tree T on $2p$ vertices	$\texttt{bd}(T)$	Eigenvalues of $\mathfrak{L}(T)$	Is $\mathfrak{L}(T)$ diagonalizable?
1		1	0	Yes
2		1	$0, 2$	Yes
3		3	$0, 0.7, 4.30$	Yes
3		1	$0, 1, 3$	Yes
4		3	$0, 0.35, 2, 5.65$	Yes
4		3	$0, 0.44, 2, 4.56$	Yes
4		5	$0, 0.63, 1, 6.37$	Yes
4		1	$0, 0.58, 2, 3.41$	Yes
4		1	$0, 1, 1, 4$	Yes
5		5	$0, 2.2, 1.1, 3.28, 6.40$	Yes
5		3	$0, 2.4, 1.33, 2.68, 5.74$	Yes
5		3	$0, 0.29, 1, 2.32, 7.38$	Yes
5		5	$0, 0.34, 0.73, 2.84, 7.09$	Yes
5		5	$0, 0.26, 1, 3.73, 5$	Yes
5		3	$0, 0.28, 1.52, 2.44, 4.76$	Yes
5		5	$0, 0.36, 1, 2.13, 6.50$	Yes

p	Tree T on $2p$ vertices	$\mathtt{bd}(T)$	Eigenvalues of $\mathfrak{L}(T)$	Is $\mathfrak{L}(T)$ diagonalizable?
5		3	$0, 0.31, 1.22, 2.87, 4.60$	Yes
5		5	$0, 0.43, 0.76, 2.35, 6.46$	Yes
5		9	$0, 0.53, 1, 1, 9.47$	Yes
5		1	$0, 0.38, 1.38, 2.62, 3.62$	Yes
5		3	$0, 0.38, 1, 2.62, 5$	Yes
5		7	$0, 0.59, 1, 1, 8.40$	Yes
5		1	$0, 0.52, 1, 2.31, 4.17$	Yes
5		1	$0, 1, 1, 1, 5$	Yes

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