Nash Game Based Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Model Predictive Control Applied With Laguerre-Wavelet Network Model

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Certificate

This is to certify that the thesis entitled, **Nash Game Based Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Model Predictive Control Applied With Laguerre-Wavelet Network Model**, being submitted by **P. Aadaleesan** for the award of degree of Doctor of Philosophy, is an authentic record of the research carried out by him in the Department of Chemical Engineering, Indian Institute of Technology - Guwahati, India, under my supervision. The work documented in this thesis has not been submitted to any other university or institute for award of any degree or diploma.

(Signature of Thesis Supervisor)

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Date:

Dedicated to

My mother, **Karuni**

&

My wife, **Poornima**

Abstract

Model predictive control (MPC) is one of the most successful approaches for controlling constrained processes. As MPC design explicitly uses a process model in the computation of the optimal control input, an efficient model, the closed-loop stability and robustness are the major issues in this controller design approach. The scope of this thesis broadly spans two areas: nonlinear system identification and robust MPC design. In nonlinear system identification, a newer kind of Wiener-type nonlinear model, namely *Laguerre-Wavelet network model* has been developed: the linear dynamic part is formed by the Laguerre filters and the static nonlinear part by the wavelet-network. The performance of the developed model is compared with suitable examples against a similar model of the same class.

Various forms of robust MPC approaches have been reported in the literature. On the part of robust MPC design in the present thesis, a novel robust MPC approach is addressed with a game theoretic interpretation. A Nash game approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Model Predictive Control (NGM-MPC) for linear dynamic systems with actuator saturation is proposed. Two-player non-cooperative game strategy is adopted by solving two separate objective functions, *viz.*, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures, subject to constraints of the system dynamics and the bounded constraints on its control input. The problem ultimately reduces to solving a pair of cross-coupled Riccati equations. Although solving coupled Riccati equations resulting from linear-quadratic games for their exact solution by itself is of theoretical importance, in the present work it has been extended, for the first of its kind, to the regime of receding horizon control.

The efficacy of the proposed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design is demonstrated with suitable examples by comparing its closed-loop performance and conservativeness with already existing mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design approach. The infinite horizon state feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC designs are analysed using set-theoretic approach for reasoning out their closed-loop performance and the associated conservativeness issues.

The design issues of output feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design are addressed, where three coupled algebraic Riccati equations are solved simultaneously. Furthermore, the output feedback controller is used with the Laguerre Wavelet Network model for controlling processes with parametric uncertainty, by considering it as disturbance rejection problem. The limitations of the proposed strategy are also discussed.

The proposed system identification and robust control techniques are demonstrated on a benchmark process *viz.*, continuous bioreactor.

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Notations

Symbols in System Identification

a	Time scale parameter of Laguerre filter
a_m	dilation parameter in wavelet network
b_n	translation parameter in wavelet network
b	bias term in wavelet network
$\mathcal{B}$	closed unit ball
c_i	Laguerre filter coefficients
C_ψ	admissibility condition of wavelet
l_k	Laguerre filter states
$w^{a,b}$	wavelet coefficient
ψ	wavelet function or wavelon
Θ, Φ	wavelet frame bounds

Symbols in Systems, Signals and Control

A	System State Matrix
B_0	Input distribution matrix of a stochastic signal, w^0
B_1	Input distribution matrix of an unknown-but-bounded signal, w
B_2	Input distribution matrix of control input signal, u.
C_0	Output matrix maps state (x) to controlled output $(\tilde{z})$
C_1	Output matrix maps state (x) to controlled output (z)

C_2	Output matrix maps state (x) to measured output (y)
C_z	Output matrix maps state (x) to controlled output (z)
D_{01}	Feedthrough matrix maps w to $\tilde{z}$
D_{01}	Feedthrough matrix maps u to $\tilde{z}$
D_{10}	Feedthrough matrix maps w^0 to z
D_{11}	Feedthrough matrix maps w to z
D_{12}	Feedthrough matrix maps u to z
D_{20}	Feedthrough matrix maps w^0 to y
D_{21}	Feedthrough matrix maps w to y
Q_i	State weighing matrix with index i
R	Input weighing matrix
u_k	Manipulated variable/control input
w_k	Disturbance input/process disturbance
w_k^0	Stochastic/random process signal
x_k	State variable
y_k	Process variable/measured output
z_k	Controlled output
$W \succ 0$	W is a positive definite matrix, if and only if $x^T W x > 0, \forall x \neq 0$
$W \succeq 0$	W is a positive semi-definite matrix, if and only if $x^T W x \geq 0, \forall x \neq 0$
$\|\cdot\|_p$	$p^{th} - Norm$
γ	Disturbance attenuation factor

Sets and Spaces

$\mathcal{H}_p$	Hardy space of norm p
$\mathscr{H}$	Hilbert space
$\mathcal{L}_p$	Square integrable space or Lebesgue space of norm p
$\mathbb{R}$	Real values
$\mathbb{R}^+$	Positive Real values

$\mathbb{R}^n$	Real Euclidean space of dimension n
$\mathbb{R}^{n\times n}$	Real Euclidean space of dimension $n \times n$
$\mathbb{Z}$	Set of Integers
$\mathbb{Z}^+$	Set of non-negative integers
$\emptyset$	Empty/null set
$\setminus$	Complement of two sets, such that $U \setminus V = \{v \in U \vert v \notin V\}$
$\partial\mathcal{U}$	Boundary of the set $\mathcal{U}$
$\oplus$	Set Addition, such that $A \oplus B = \{a + b \mid a \in A, b \in B\}$
Ω	Used to denote arbitrary subsets of a given Euclidean space

Operations

det	Determinant
Dom	Domain
inf	Infimum
int	Interior
sup	Supremum
$\overline{\mu}$	Closure of the set μ
$x_{k+i\vert k}$	A priori estimate/prediction of x $\forall i \geq 0$ given all data up to k

Abbrevations

cARE	coupled Algebraic Riccati Equations
CLF	Control Lyapunov Function
eqn.	equation
LQ	Linear Quadratic
LWN	Laguerre Wavelet Network model
MPC	Model Predictive Control

NGM-MPC	Nash Game based Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC
iff	if and only if
RCI	Robust Control Invariant
RCLF	Robust Control Lyapunov Function
RPI	Robust Positive Invariant
w.r.t.	with respect to

Chapter 1

Introduction

Advanced control system designs are mostly model-based strategies; in which model predictive control (MPC) is one of the most important techniques that has attracted the interests of control engineers and researchers, since its introduction in 1980's. It has enjoyed significant contribution both from theory and practice over the past three decades. However, there are still a few gap areas that demand a thorough investigation and further development to fulfill the gap that prevails. This thesis is one such attempt to fill the gap by bringing together the different concepts/ideas that prevail in different aspects of control engineering.

MPC, being a model based control strategy, always seeks a better model in order to predict the actual system dynamics that has to be controlled. This model development is achievable, in general, by three ways: *(i)* by first principles method, which needs thorough knowledge of the process characteristics (physical, chemical, thermodynamical, etc) *(ii)* the *grey-box* modeling method, which uses partial system information to fix appropriate values of some model parameters, and *(iii)* the *black-box* modeling method, which mostly uses the input-output data of the process only. Due to the non-availability of complete process knowledge and/or the inappropriate structure for controller design, the first-principles based models are not always easily available. Grey-box type of models utilize the partial information/process knowledge and the model parameters (at least few of them) are fixed using the process input-output data. However, they are rarely preferred, because development of such models is usually cumbersome and tedious. Black-box models, on the other hand, are rather preferred as their appropriate model structure is suitable for model-based controller design techniques and can be pre-defined. Also, complete process knowledge is not required making them particularly advantageous when control engineers are from various disciplines.

Using a good model (in some restricted sense) with a model predictive controller does not necessarily solve every issue of controlling a constrained dynamic system. As the very principle of MPC relies on a model which depicts/replicates the actual system dynamics, the accuracy of that model plays a predominant role in the overall closed-loop performance. Any model for that matter is not fully capable of representing the exact system dynamics; particularly, when there is an uncertainty and/or acute nonlinearity in the process. Even though there are scopes for adopting methods such as *robust system identification*, their effectiveness is very limited. The limitations may be due to, but not limited to, one or more of the following reasons:

1. all possible process uncertainty/nonlinearity might be difficult to realise.

2. model parsimony might have to be compromised.

3. online adaptation of model parameters is not always possible due to heavy computational demand and/or the complexity of model structure may not permit such quick online adaptation.

For the above reasons robust system identification is not always at the easy disposal. In such circumstances, the issue of closed-loop robustness has to be put on the controller design technique. The scope of such robust controller design has reached its height in the field of conventional feedback control system design, with the advancements in robust control theory. However, due to its inherent open-loop structure, MPC suffers from such robustness issues. It is one of the obvious technical reasons why MPC has not yet completely replaced the conventional feedback PID controller in many industries, despite other technical merits of MPC such as handling constraints, multi-variable (MIMO) systems, etc. So robust MPC design has been a topic of research for more than two decades in the control community. Later, these robustness and stability issues of MPC have been addressed by means of *terminal equality constraint*, *terminal constraint set* and/or *terminal cost* [111]. With such terminal conditions fixed *a priori*, the optimisation problem is solved online and the optimal control input value is determined at every time instant. Further details in this regard are deferred until section 1.2.2.

All the above developments, in terms of the stability of MPC, were observed in the most inevitable and often encountered dynamical system of the form,

$$x_{k+1} = f_k(x_k, u_k, w_k) \tag{1.1}$$

$$y_k = h_k(x_k, u_k, v_k) \tag{1.2}$$

subject to the constraints,

$$x_k \in \mathbb{X} \subset \mathbb{R}^n; \;\; u_k \in \mathbb{U} \subset \mathbb{R}^m; \;\; w_k \in \mathbb{W} \subset \mathbb{R}^d; \;\; y_k \in \mathbb{Y} \subset \mathbb{R}^p$$

and v_k is some Gaussian noise, resulting from the high-gain measurement devices. Any such uncertain system, $f_k(\cdot)$ and $h_k(\cdot)$ and/or a system perturbed by unknown but bounded disturbance, $w_k \in \mathbb{W}$, are of serious concern both in terms of robust stability and robust performance.

Achieving an optimal controller design for one such critical problem is one of the main goals of this thesis. However, undoubtedly, the need for a suitable model that represents the process dynamics as accurately as possible, at least in the nominal sense, is of paramount importance. Mostly, all the processes in real world are inherently nonlinear. The tractability of practical design of control systems for linear processes and the rich theoretical development associated with it makes the designer to approximate the nonlinear process by a local linear model about its nominal operating point. However, when there is a change in the operating point of the process, the local linear model cannot serve as an appropriate model of the process anymore. In such cases, the need for a model which approximates a larger operating range, incorporating the nonlinearity of the process becomes necessary. Hence the demand for design and development of suitable nonlinear models becomes inevitable.

There are different classes of nonlinear system models, of which interest with regard to the present thesis lies on those models wherein the nonlinearity can be represented as static nonlinear gain. Of such nonlinear models, the well known models are Hammerstein model and Wiener model. Basically, both Hammerstein and Wiener models are made up of a linear dynamic part (L) and a nonlinear static part (N). The linear dynamic part follows a nonlinear static part in the former, while the linear dynamic part is followed by a nonlinear static part in the latter. Nonetheless, a combination of the above two models is also possible.

In general, these classes of nonlinear models are advantageous due to their simplicity in construction and capability to handle even severe nonlinear processes, *e.g.,* pH-neutralization process. Moreover, when there is any arbitrary nonlinearity which is governed by some non-dynamic relation in terms of the process variables and parameters - such a class of nonlinear processes are the right choice to model using Hammerstein or Wiener models.

The parametric identification of Hammerstein model have been studied by many researchers; see [116, 33, 135, 164, 79, 151, 17]. In the literature, the extension of the Hammerstein and Wiener models is often in the form of an L-N-L model where a nonlinear block is embedded between two linear blocks and it is called a Wiener-Hammerstein model; see [19], [20], [131] and [28]. A recursive estimation of a parametric Wiener-Hammerstein model is proposed in [29]. Zhu has developed new algorithms for the identification of Wiener model [169] and Hammerstein model [170]. In [53] and [171] a model with a linear block embedded between two static nonlinear gains called N-L-N Hammerstein-Wiener model is used. Chernyshov has proposed an output error method for the identification of nonparametric N-L-N models [38].

In this thesis, the robust control of a nonlinear process represented by Wiener type nonlinear model is addressed. For this purpose, emphasis is paid both on the system identification part and in the design of a novel robust controller.

1.1 Nonlinear System Identification

The theory of system identification plays a significant role in many fields of science and engineering including simulation, automatic control, fault tolerant analysis, prediction, etc [21, 36, 63, 100, 134, 159]. Black box modeling is an elegant system identification technique which uses input-output data of the process to be modeled and does not need any physical insight of the process. Many linear system identification and/or function approximation methods have been developed in the past decades [100, 159] and it has attained a state of maturity. However they may not be useful when applied for modeling nonlinear systems. This demands an extensive study for the development of nonlinear system identification techniques.

1.1.1 Brief Historical Background

Over the years, many successful nonlinear modeling techniques have been developed, such as NARMAX (Nonlinear Auto Regressive Moving Average with eXogenous input) model [21, 63, 100, 117], artificial neural networks [36, 100], fuzzy-logic based models [52, 100], some combinations of them like neuro-fuzzy models [5], support vector machine and kernel methods of modeling [100], and wavelet decomposition based models [21, 42, 165, 166], to name a few. Some common issues, considered while developing any type of models listed above, include; *(i)* model parsimony, *(ii)* ease of development, and *(iii)* the accuracy. A good model is expected to be parsimonious in size, easy to develop and have high level of accuracy. None of the modeling techniques, listed above, possess all these merits simultaneously. Some tradeoff is always resorted between them while choosing a particular technique for building a model. The successful use of polynomial models (NARMAX) for nonlinear system identification has been demonstrated elsewhere [21, 63]. These polynomial models can gather global information of the system dynamics quite efficiently, while local information couldn't be approximated parsimoniously. Neural network is another successful class of nonlinear models that have been used widely over the years for nonlinear system modeling [36, 100, 117]. Although neural network is capable of identifying the process dynamics efficiently, there is no structured construction procedure for determining the optimal number of neurons and hidden layers in the network model. Fuzzy-logic based models [52, 100, 155] are also found to be very useful for modeling severe nonlinearities such as discontinuities and jumps, saturation nonlinearity and many such non-smooth nonlinearities, however, they do not possess all the merits stated above. A detailed description of the different types of nonlinear models are reported in [86, 148].

In the recent years, wavelet decomposition and multi-resolution wavelet decomposition has become a popular tool in the field of signal processing and numerical analysis [46, 104]. Wavelet decomposition is basically decomposition of any function using appropriate dilated and translated version of a basis function, called the *mother wavelet*, in some functional space, $\mathcal{L}_2$. Wavelet basis functions have the property of localization in both time and frequency domains [46, 104, 165], which is useful in approximating even severe nonlinearities in an efficient manner. However, this property is not effective in approximating linear or low-order nonlinearity in a function. In [21], this shortcoming was overcome by utilising the global approximation property of polynomial models and local approximation property of wavelets in a single structure, where the polynomial model is clubbed with the multi-resolution wavelet representation. The redundant terms are however removed using orthogonal least squares (OLS) method to make the model parsimonious in nature.

The similarity between single-hidden layer neural network and wavelet decomposition is addressed in [165] and [166], and *wavelet network* was developed. In this wavelet network, non-orthogonal wavelets *viz.*, wavelet frames, are used in discrete wavelet transformation [46, 104], to approximate functions with less number of terms [165, 166] by exploiting the associated redundancy. By using non-orthogonal wavelet frames in lieu of traditional orthogonal wavelet basis functions, they slightly compromise on orthogonality, however, gain more liberty in the choice of the wavelet basis function. In [166], the wavelet frames are generated with radial construction and they are used to develop wavelet network. Even though wavelet network could be used to approximate any static nonlinearity in an efficient manner, the developed model has to represent the entire process dynamics. This dynamical part can be incorporated by using the wavelet network either in Wiener or Hammerstein model structure [100]. In both these structures, the linear dynamic part of the model should be represented by a suitable means. The model would also have to have the information about the dimension of the process with some orthogonal representation that spans a subspace.

1.1.2 Wiener Model

Of the different classes of models representing the nonlinear systems, Wiener-type of models are of particular importance. They form a major class of nonlinear models especially in process control applications. An equally important and similar class by its constituting parts yet different in its formulation are the *Hammerstein models*.

The list of literature of using Wiener models for system identification is exhaustive. However, to name a few; [20, 136, 137, 88, 67, 156] may be cited. Wigren [160] proposed a recursive algorithm. In process control literature Wiener model usually had case studies on pH neutralization process and CSTR. In [88, 97, 102] the use of Wiener model for pH neutralization process has been demonstrated. In [27] Wiener model was used to identify distillation column. Wiener model was demonstrated on polymerization reactor using piecewise linear Wiener models in [143]. The process model parameters were estimated optimally by solving a constrained nonlinear optimisation procedure in [83]. Very recently, in [153] particle swarm optimisation is used in the identification process of the Wiener-type model.

In most of the approaches, when the Wiener model is used for control purpose, the static nonlinear portion is invertible. Hence, an inverse of the nonlinear static block is used to estimate the system states from the process output measurement. Thereby, the process can be controlled using the state-space model based control algorithms in state-feedback framework.

1.2 Model Predictive Control

Model Predictive Control, or MPC in short, is one of the major advanced control paradigms that has enjoyed huge amount of research especially in academia since its inception in 1980 [43]. However, the idea of using a control strategy which is in close connection with MPC was developed as early as in early 1960s [124]; but was left unnoticed for almost two decades. MPC has found its application in industries, apart from its academic interest, with more than 4600 industrial applications as reported in the survey by Qin and Badgwell [125].

It is worth mentioning that the actual inception of MPC took place in Shell Oil industry and was reported as Model Predictive Heuristic Control (MPHC) [130] and as Dynamic Matrix Control (DMC) [43]. There are also other frameworks of MPC: Identification and Command (IDCOM), Model Algorithmic Control (MAC), Extended Horizon Adaptive Control (EHAC), Generalised Minimum Variance (GMV), Predictive Functional Control (PFC), Generalised Predictive Control (GPC), Extended Prediction Self Adaptive Control (EPSAC), Multistep Multivariable Adaptive Control (MUSMAR), Unified Predictive Control (UPC), to name a few, where they used different kinds of process model (impulse, step, state space, transfer function, etc.), disturbance (constant, decaying, filtered white noise, etc.) or capability to adapt to time-varying models (See [31] for details). The aspect that motivates different versions of predictive controllers is the concern of the closed-loop stability, which was earlier considered to be an issue of proper tuning of the controller parameters. However, as the above frameworks were all algorithmically similar, the difference between these versions eventually become insignificant. In recent times, MPC generally refers to "a control strategy in which the current control action is determined by solving on-line an optimal control problem" [111].

The capability of MPC in handling constraints had made it very useful in the industry. By the mid-nineties, the theory of MPC had reached significant maturity. The use of state-space models finds its way into MPC owing to its well-known capability of simplicity and generalization. When the system states are not directly measurable, Kalman filter can be used in estimating the system states, which is again mostly used in state-space form. The issue of stability is also well addressed in the state-space form with the most celebrated Lyapunov stability theory. The use of linear quadratic (LQ) controller with the quadratic performance measure is well documented in [4]; extension of which in the framework of receding horizon gives much insight to the design and analysis of MPC [22].

The literature on MPC is quite extensive, right from theoretical developments to process control, where its adoption into real-time industrial use are documented. In the following sections, rather than providing an exhaustive review, a distilled review of the significant developments in MPC that fall under the scope and/or vicinity of the present thesis' investigation, and a brief account of them (whenever necessary), is furnished.

1.2.1 Basic MPC Controller

In MPC, the system states or output is predicted at every sampling time instance over some future time horizon (prediction horizon) using a dynamic model of the process to be controlled. Current measurement value of the process is used as the initial condition for the above purpose. A suitable control input is thereby computed on solving an optimal control problem that aims to optimise some performance objective. The prediction horizon may be finite or infinite; likewise, the control input may be a sequence of control values or control law, called the control horizon. With the availability of the newer measurement data of the plant the optimisation problem is repeated for subsequent control actions. Thus the feedback mechanism is achieved, but control strategy works in a receding horizon approach.

The use of quadratic performance measure (*i.e.*, based on ℓ_2-norms) results in efficient optimisation problem and simplified mathematical analysis. Consider a dynamic system described as,

$$x_{k+1} = f(x_k, u_k) \tag{1.3}$$

$$y_k = h(x_k, u_k) \tag{1.4}$$

subject to the constraints,

$$x_k \in \mathbb{X} \subset \mathbb{R}^n; \quad u_k \in \mathbb{U} \subset \mathbb{R}^m; \quad y_k \in \mathbb{Y} \subset \mathbb{R}^p$$

where x_k, u_k are the system state(s) and control input and $f(\cdot)$ and $h(\cdot)$ are the system's state and output mapping functions, respectively, for all $k \geq 0$. The ultimate goal is to find an optimal control input that minimizes an infinite horizon performance measure, with positive (semi-) definite weighing matrices, $Q = Q^T \succ (\succeq) 0$ and $R = R^T \succ 0$, $\forall\, k \geq 0$, as

$$J = \sum_{i=0}^{\infty} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} \tag{1.5}$$

$$= \sum_{i=0}^{\infty} \|Q^{1/2} x_{k+i|k}\|_2 + \|R^{1/2} u_{k+i|k}\|_2 \tag{1.6}$$

where $(\cdot)_{k+i|k}$ denotes the predicted/future values w.r.t. the time instant k. Alternatively, performance measures based on other norms, say, ℓ_1 and ℓ_∞ are also used.

$$l_1 : \quad J = \sum_{i=0}^{\infty} \|Q^{1/2} x_{k+i|k}\|_1 + \|R^{1/2} u_{k+i|k}\|_1 \tag{1.7}$$

$$l_\infty : \quad J = \sum_{i=0}^{\infty} \|Q^{1/2} x_{k+i|k}\|_\infty + \|R^{1/2} u_{k+i|k}\|_\infty \tag{1.8}$$

However, it has been reported that some unexpected/strange closed-loop behavior is observed when ℓ_1 and ℓ_∞ are used as performance measures [128]. For unconstrained case, the solution to such infinite horizon problem with a quadratic performance measure is well known as LQR [4].

Owing to the associated computational cost and the infinite-dimensional constrained optimisation problem to be solved online for an infinite horizon MPC problem, it is often defined by a finite horizon problem, for some $N > 0$, as

$$J = \sum_{i=0}^{N} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} \tag{1.9}$$

Now the optimisation problem reduces to a finite dimensional problem in terms of both finite number of decision variables and finite number of constraints. Nevertheless, for an infinite horizon problem, there may exist a piece-wise affine state feedback law [10] or a time-varying feedback control law [96].

Even though, the use of finite horizon control (eqn.(1.9)) makes the constrained control problem to a solvable finite dimensional problem, it brings in issues regarding stability.

1.2.2 Stability of MPC

The nominal closed-loop stability is ensured once an infinite horizon MPC is used [129]. Unlike the unconstrained MPC, the global stability cannot be guaranteed for constrained system with quadratic cost function, even for a linear case. For instance, an unstable input constrained system cannot be globally stabilized [132]. Moreover, imposing of constraints leads to nonlinearity in the problem that makes the stability analysis much more complex. It is also shown that the different combinations/choices of the weighing matrices (Q and R) and the length of the finite horizon problem (N) (commonly used as *tuning parameters* for MPC controller) affect the stability in a complicated way and they are in general non-convex and disconnected [101]. These bottlenecks were addressed with different proposals by various researchers in late 80s and early 90s. A thorough survey in this regard can be found in [111]. However, a brief note on the major ideas proposed to address the stability issue in MPC is given below.

(a) Terminal Equality Condition

The terminal equality condition is given as $x_N = 0$, $\forall k \geq N(> 0)$. Although this condition forces the system states to the nominal equilibrium, the origin, it can be too ambitious and sometimes too expensive in terms of control cost. At times, even satisfying the control constraint might be highly unlikely under such terminal condition.

(b) Terminal Constraint Set

When satisfying such terminal equality condition would be unfavorable, the use of terminal constraint set is an effective alternative. For the system given in (1.3), the closed-loop stability is ensured even when the system states reach a constraint set,

$\mathcal{X}_f$, such that for some $N > 0$

$$x_N \in \mathcal{X}_f \subset \mathbb{R}^n$$

and for all $k \geq N$. The state constraint set $\mathcal{X}_f$ is a closed, compact set, in the neighbourhood of the origin and contains the origin in its interior, $0 \in \text{int}(\mathcal{X}_f)$.

Although, the use of terminal (inequality) constraint set appears to be impressive, the closed-loop stability cannot be strictly ensured with the use of the terminal constraint set alone. There should be a proper way by which the intermediate states are driven to the terminal constraint set, satisfying both constraints and the monotonically decreasing *value function.* Such conditions are necessary and also sufficient (in some sense) to manifest stability of the system in the sense of the most celebrated Lyapunov Theory.

(c) Terminal Cost

Nominal stability is generally achievable for infinite horizon control problem. However, solving an infinite horizon open-loop control problem is computationally forbidden due to the resulting infinite-dimensional problem. To overcome this bottleneck, the infinite horizon problem is approximated by a finite horizon problem with the inclusion of a terminal cost. The cost associated with the system performance beyond a finite horizon of control calculation is approximated by the terminal cost and thereby preserves the nominal stability of infinite horizon problem.

Consider an infinite horizon cost functional to be minimised w.r.t. the control input be defined by,

$$\min_{\mathbf{u}} J_0^{\infty} := \sum_{i=0}^{\infty} (x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}) \tag{1.10}$$

This cost functional can be split into two parts;

$$\min_{\mathbf{u}} J_0^{\infty} = \min_{\mathbf{u}} \left(J_0^{N-1} + J_N^{\infty} \right) \tag{1.11}$$

Now, to approximate the infinite horizon cost functional, the second term can be upper approximated by a *terminal cost/penalty term*, such as $x_N^T P x_N$, for some

$N > 0$ and $P \succ 0$. Thus the second term in eqn.(1.11) can be given as,

$$\sum_{i=N}^{\infty} J_N^{\infty} \leq x_N^T P x_N. \tag{1.12}$$

Substituting eqn.(1.12) in eqn.(1.11) we obtain

$$\min_{\mathbf{u}} J_0^{\infty} \leq J_N^{\infty} + \sum_{i=0}^{N-1} (x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}) \tag{1.13}$$

$$= x_{N|k}^T P x_{N|k} + \sum_{i=0}^{N-1} (x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}) \tag{1.14}$$

where P is the solution to the discrete-time algebraic Riccati equation solved off-line.

In [34, 126] such terminal cost has been used along with the terminal constraint set and called it a *quasi-infinite horizon MPC*. Other interesting works with this approach were reported in [40] and [141], for constrained linear discrete-time systems with both terminal cost and terminal constraint set. They extended the infinite linear quadratic control (LQR) for constrained case, known as *Constrained Infinite Horizon Linear Quadratic Regulator* (CIHLQR). The general CIHLQR problem [40, 68, 107, 141]is given as

---

Problem 1.1 Solve

$$J_N = \min_{\mathbf{u}} \sum_{i=0}^{N-1} (x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}) + x_N^T P x_N \quad \forall k \geq 0, \tag{1.15}$$

subject to,

$$x_{k+1} = A x_k + B u_k, \tag{1.16}$$

$$x_k \in \mathbb{X}, \quad 0 \leq k \leq N-1 \tag{1.17}$$

$$u_k \in \mathbb{U}, \quad 0 \leq k \leq N-1 \tag{1.18}$$

$$x_N \in \mathcal{X}_f \tag{1.19}$$

where $\mathbf{u} = \{u_{k|k}, u_{k+1|k}, \cdots, u_{k+N-1|k}\}$ and $u_k = K x_k$, $\forall k \geq N$.

The particular interest in these works lies in finding the right choice of the horizon length such that after this time instant, the system states will ever remain within the terminal set. Once the closed-loop trajectories remain within the neighbourhood of the origin for the time interval, $[N+1, \infty)$, then the control is switched to a local unconstrained state feedback LQR, $u_k = Kx_k$, $\forall k \geq N$, which will asymptotically stabilise the closed-loop system by steering the states to the origin. This is sometimes called the dual-mode control [113]. Finding the horizon length plays a crucial role in the problem size and also on the resulting computational time. Sometimes this horizon length may be forbiddingly high for real time applicability. Various methods have been proposed to find an optimal horizon length [168]. In [106, 107], the terminal constraint set ($x_N \in \mathcal{X}_f$) has been relaxed and the controller performance has been found to be better than that of finite horizon MPC and CIHLQR.

Stability is also advocated by using the value function as the Lyapunov function [89, 113], whose monotonically decreasing property assures stability [22, 47]. The approach given in Problem 1.1 is widely used due to stabilizing property of resulting MPC.

1.3 Robust Model Predictive Control

The control of constrained systems is an important and demanding issue in control theory, which MPC addresses effectively. However, being a model based open-loop control scheme, MPC is not meant for robustness against uncertainties (due to plant-model mismatch) explicitly, albeit the presence of a feedback mechanism in the form of plant measurement at every time instant. Hence, the optimal performance of MPC for a fixed model may be poor in presence of a plant-model mismatch. Thus controlling a system with both constraints and model uncertainty is a challenging task. Nevertheless, addressing one such issue is the major goal of this thesis.

Robustness is achievable with a properly designed controller such that the overall closed-loop system is robust to a specified bound of uncertainty. One possible way is to incorporate this information (model uncertainty) in the controller design, but it is not always possible to capture all the uncertainties or disturbances that affect the system performance. Even if all the uncertainties and/or disturbances are unmeasurable, knowledge of their typical bounds can be used in the controller design.

1.3.1 Modeling Uncertainty

In principle, in the face of uncertainty for a discrete-time dynamic system, as often encountered in practice, be given in the form,

$$x_{k+1} = f_k(x_k, u_k, w_k) \tag{1.20}$$

$$y_k = h_k(x_k, u_k, v_k) \tag{1.21}$$

subject to the constraints,

$$x_k \in \mathbb{X} \subset \mathbb{R}^n; \;\; u_k \in \mathbb{U} \subset \mathbb{R}^m; \;\; y_k \in \mathbb{Y} \subset \mathbb{R}^p$$

and v_k is some Gaussian noise resulting from the high-gain measurement devices. The adversary in terms of uncertainty could be given as $w_k \in \mathbb{W}(x_k, u_k)$, where $\mathbb{W}(\cdot) \subset \mathbb{R}^d$ is a closed, compact set and contains the origin in its interior. Let the bounded uncertainty on f and h be $\mathscr{F} := f(x_k, u_k, \mathbb{W}(x_k, u_k))$ and $\mathscr{H} := h(x_k, u_k, \mathbb{W}(x_k, u_k), v_k)$, respectively. Therefore, hereforth let

$$x_{k+1} = f(x_k, u_k, \mathbb{W}(x_k, u_k)) \tag{1.22}$$

$$y_k = h(x_k, u_k, \mathbb{W}(x_k, u_k), v_k) \tag{1.23}$$

As $f(\cdot)$ depends on w, let $w_{(\cdot)}$ denote a disturbance sequence and $x^{\mathbf{u},\mathbf{w}}(\cdot; x_k)$ be the state trajectory resulting from an initial state x_k at some time $k \geq 0$, control sequence $\mathbf{u}$ and disturbance sequence $\mathbf{w}$. In situations where the uncertainty due to state estimations is time varying, it could better be modeled by $w_k \in \mathbb{W}_k$, where $\mathbb{W}_k$ varies appropriately with time k.

1.3.2 Open-loop & Feedback robust MPC

For almost the past four decades, the control problem of driving the states to a target set (preferably in the neighbourhood of the origin), for a constrained system subject to persistent disturbance, was solved by minimising some worst case cost [161, 49, 15, 16, 64]. In such a problem, also known as *minimax feedback control*, the major issue had been how the system states' evolution could be kept inside the target set. This has been addressed in [112]. They considered the target set to be robustly invariant. Once the states are inside the target set, the control input is determined by a pre-computed control law which ensures that the state trajectories never leaves the target set. In [15],[64], [23] and [112], the solution to this problem was provided on the basis of set-theory and set invariance property. A detailed account of the set invariance theory and its role in robust control problem can be found in [25, 26, 90].

Although satisfying the terminal conditions (c.f. Section 1.2.2 a-c) are necessary for robust and stable control of an uncertain system (eqns. (1.22) and (1.23)), the computational cost associated with it for obtaining the solution on-line (even for a finite-horizon approximation of the infinite-horizon problem) might be too expensive. In such a case, it is necessary to consider all possible realizations of $x^{\mathbf{u},\mathbf{w}}(\cdot; x_k) \in \mathscr{F}$ in the optimal control problem and it has to be ensured that each realization satisfies the constraints and the terminal conditions [113]. To address this computational issue, as an alternative approach, an explicit expression of control law using multi-parametric approach was proposed [8, 9, 10]. Here, a piecewise affine expression of the control law is computed off-line. There are also methods to address this robustness issue which rely on the min-max optimisation methods of the predicted performance [101].

As an appreciation of the strength of feedback in the light of robustness for MPC design, Mayne [109] proposed the concept of *feedback min-max MPC*. Generally in MPC design, even min-max MPC, a single control input sequence is computed with an aim to minimise the worst case cost and only the first control value is applied to the plant and the same control action is retained until the next new measurement comes. This is called *open-loop min-max MPC*. On the other hand, in feedback min-max MPC, the notion of feedback present in the receding-horizon implementation of the control, is used [139]. One such feedback MPC methodology is used in this thesis but in an entirely different setting such that the problem is posed and solved using Nash game approach, which are detailed the subsequent sections.

(a) Open-loop min-max MPC

Consider a nominal system, whose stable closed-loop is given by $x_{k+1} = f(x_k, u_k)$. If there exists a positive invariant set and $x_k \in \mathcal{X}_N \subset \mathbb{R}^n$, then the condition $x_{k+1} \in \mathcal{X}_{N-1} \subset \mathcal{X}_N$ also prevails. However, this property is no longer valid in presence of some uncertainty (eqn. (1.22)). Nevertheless, this property can be recovered by considering all the possible realizations of the $x_k \in \mathscr{F}$ that satisfy the state and/or control constraints and terminal conditions, for all admissible set of disturbances in open-loop, $\mathbb{W}^{ol}$, for computing the optimal control input sequences.

The problem to be solved in an open-loop robust MPC is given below;

Problem 1.2 *Solve*

$$\min_{\mathbf{u}_k^N} \max_{\{w_k \in \mathbb{W}^{ol}\}_0^{N-1}} \left\{ F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}) \right\} \tag{1.24}$$

subject to,

$$x_{i+1|k} = f(x_{i|k}, u_{i|k}, w_{i|k}), \qquad x_{0|k} = x_k, \ \forall k \geq 0 \tag{1.25}$$

$$x_{i|k} \in \mathbb{X}, \ \ u_{i|k} \in \mathbb{U}, \qquad i = 0, ..., N-1 \tag{1.26}$$

$$x_{N|k} \in \mathcal{X}_f \tag{1.27}$$

The decision variable $\mathbf{u}_k^N$ *is*

$$\mathbf{u}_k^N \triangleq [u_{0|k}^T, u_{1|k}^T, ..., u_{N-1|k}^T]^T \tag{1.28}$$

where $F(\cdot)$ and $L(\cdot)$ represents the terminal and stage costs, respectively. Thus, in open-loop min-max MPC a single control input sequence is used to minimise the worst case cost. Let, for all $i \geq 0$, $\mathbb{X}_i^{ol} \subset \mathcal{X}_i$ is the set of states that can be steered to $\mathcal{X}_f$, in a finite number of steps i or less for all admissible open-loop control sequence $\mathbf{u}_k^N$. If there exists a finite $N > 0$, such that $x_{N|k} \in \mathcal{X}_f$, then $\mathbb{X}_N^{ol}$ is the domain of attraction of the open-loop robust MPC for the set of admissible control sequence

$\mathbf{u}_k^N \in \mathbb{U}_N^{ol} \neq 0$.

However, in general, the open-loop formulation is too conservative and may often under-estimate the set of feasible trajectories [93]. This is due to the reason, that a single control sequence is computed such that for all allowable disturbance sequences ($w_k \in \mathbb{W}^{ol}$) the constraints are satisfied. Moreover, the open-loop min-max MPC assumes that the control input sequence ($\mathbf{u}_k^N$) computed at time k will be applied in N steps in the future, without considering the fact that the states will be measured and a new control sequence will be recomputed at each of the subsequent time steps.

(b) Feedback robust MPC

In an open-loop min-max MPC there are always chances that the trajectories satisfying $x_k \in \mathscr{F}$ may diverge, as the uncertainty may spread over the horizon, causing the set $\mathbb{X}_N^{ol}$ to be small or even empty for a reasonable $N > 0$. On the other hand, incorporating the notion of feedback available in MPC in the form of feedback control law ($\kappa_k(x_k) : \mathbb{X} \mapsto \mathbb{U},\ \forall k \geq 0$), instead of a sequence of control actions as in open-loop control problem, prevents the trajectories from diverging once $x_{k+1} = f(x_k, \kappa_k(\cdot)) + w_k$ is stable. For this reason, the idea of feedback MPC was proposed by Mayne [109] and later such idea was used in [96, 139, 111].

The problem of feedback robust MPC is given as follows [111]

Problem 1.3 *Solve*

$$\min_{\pi_k^N} \max_{\{w_{i|k} \in \mathbb{W}^{fb}\}_0^{N-1}} \left\{ F(x_{N|k}) + \sum_{i=0}^{N} L(x_{i|k}, u_{i|k}) \right\} \tag{1.29}$$

subject to,

$$x_{i+1|k} = f(x_{i|k}, \pi_k^N, w_{i|k}), \qquad x_{0|k} = x_k,\ \forall k \geq 0 \tag{1.30}$$

$$x_{i|k} \in \mathbb{X}, \qquad i = 0, ..., N-1 \tag{1.31}$$

$$\{u_{0|k}, \kappa_{i|k}\} \in \mathbb{U}, \qquad i = 1, ..., N-1 \tag{1.32}$$

$$x_{N|k} \in \mathcal{X}_f \tag{1.33}$$

The decision variable π_k^N *is*

$$\pi_k^N := [u_{0|k}, \kappa_{1|k}(x_{1|k}), ..., \kappa_{N-1|k}(x_{N-1|k})] \tag{1.34}$$

where the sequence of control actions in open-loop robust MPC (1.28) is replaced by a *control policy* (1.34).

Note that $\kappa_{i|k}(\cdot)$ for every $i \in \{1, 2, \ldots, N-1\}$ are control laws but $u_{0|k}$ is a control action, since there is only one deterministic initial state if all the states are completely measurable. Let $\mathbb{W}^{fb}$ be the set of admissible disturbance if the control policy π_k^N is used and Π_N be the set of all admissible control policies (π_k^N) of length N satisfying the constraints and terminal conditions. Then, for all $i \geq 0$, let $\mathbb{X}_i^{fb}$ denote the set of states that can be steered to the robust invariant set $\mathcal{X}_f$, in a finite number of steps i or less, by the admissible control policy π_k^N such that $\Pi_i \neq 0$; $\mathbb{X}_i^{fb} \subset \mathcal{X}_i$. Thus for some $N > 0$ if $x_N \in \mathcal{X}_f$, and $\mathbb{X}_N^{fb}$ is positive invariant, for the given system, then $\mathbb{X}_N^{fb}$ is the domain of attraction.

As the open-loop min-max MPC is more conservative than the feedback robust MPC, the domain of attraction of the later is often much larger than that of the former, i.e.,

$$\mathbb{X}_N^{ol} \subset\subset \mathbb{X}_N^{fb}$$

However, such a min-max feedback MPC scheme is difficult to implement due to its prohibitive computational cost. Moreover, it may become intractable as the horizon length is increased. To address this issue, inclusion of an additional robustness constraint is proposed in [39].

1.3.3 Feedback MPC using LMI

Apart from min-max feedback MPC, another kind of feedback robust MPC was proposed in [96] that made use of the development of Semi-Definite Programming methods [30], in terms of polytopic/ellipsoidal set. The availability of such computationally efficient methods using Linear Matrix Inequalities (LMI) simplifies the design of robust and stable MPC schemes.

In general, the convex quadratic optimisation over semi-definite cones could be given as LMIs; and, corresponding optimisation over the semi-definite cone is known as semi-definite programming. In control problems, the quadratic form of the stable Lyapunov equations could be converted into equivalent LMIs using Schur complements.

Definition 1.1 (Linear Matrix Inequality, LMI) *A linear matrix inequality (LMI) is a matrix inequality of the form,*

$$M(x) = M_0 + \sum_{i=1}^{q} x_i M_i \succ (\succeq) 0, \quad x \in \mathbb{R}^n,\ M_i = M_i^T \in \mathbb{R}^{n\times n}$$

where, x_i are the variables and M_i are positive (semi-) definite matrices.

Definition 1.2 (Schur Complement) *Given any $P(x) = P^T(x)$, $R(x) = R^T(x)$ and $Q(x)$ that affinely depend on x, then for*

$$P(x) \succ 0, \quad R(x) - Q^T(x)(P(x))^{-1}Q(x) \succ 0$$

or,

$$R(x) \succ 0, \quad P(x) - Q(x)(R(x))^{-1}Q^T(x) \succ 0$$

the Schur complement gives the equivalent LMI as

$$\begin{bmatrix} P(x) & Q(x) \\ Q^T(x) & R(x) \end{bmatrix} \succ 0.$$

The LMI based MPC proposed by Kothare et al. [96] computes a single Lyapunov function for the polytopic uncertain system. It was improved in [44, 105] by using different Lyapunov functions for each vertex of the uncertainty's polytope, thus making the approach less conservative.

The LMIs can easily be solved, even online, by virtue of the symmetric nature of the inequalities. The necessary (and sometimes even sufficient) condition for the resulting Lyapunov candidate function matrix would be *symmetric positive* *definite*. These positive definite matrices[†] have nice convergence and stability properties. However, there are also conditions in LQ problems, where non-symmetric conditions could arise, which eventually results in non-symmetric solution matrices. The existence of non-symmetric Lyapunov domains are shown in [11].

1.3.4 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

Generally, in a control system design the most important objective is to achieve a good or desired performance even in presence of any external disturbance or system parameter variations or noise or change in the operating point of the system. A good performance is quantified as that having a good transient response; whereas, the stability of a closed-loop system in the face of external disturbance and/or process uncertainties is called the robust stability. Usually a controller which is robust against any disturbance or process uncertainties may yield a very poor transient response. In contrast, the closed-loop system with a good transient performance may be very sensitive w.r.t. any external disturbance or process uncertainty. Thus these two are conflicting objectives, such that one could be achieved at the expense of the other. So control system design is basically a trade-off between these two conflicting objectives.

In robust control theory, the $\mathcal{H}_2$ control problem is aimed at finding a causal, stabilising controller, which minimises the $\mathcal{H}_2$ norm of the closed-loop transfer function between the control input and controlled output. This $\mathcal{H}_2$ design can also be accomplished by minimising a quadratic cost function associated with a linear dynamic system. The $\mathcal{H}_2$ design basically improves the closed-loop transient performance. However, as $\mathcal{H}_2$ design assumes a perfect process model information, which is practically not the case, the closed-loop stability cannot always be ensured. To address the uncertainty issue against process-model mismatch, $\mathcal{H}_\infty$ control design was developed. In $\mathcal{H}_\infty$ design the supremum of the induced norm of the transfer function between control input and controlled output is kept at a minimum possible value,

[†] Generally, positive definite matrix implies that it is a symmetric positive definite matrix

ensuring closed-loop stability against bounded uncertainty or disturbance. However, $\mathcal{H}_\infty$ design usually results in a closed-loop system of increased order.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ type robust controller, on the other hand, has been incepted by combining the celebrated robust controller designs *viz.*, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller designs. The combination is aimed at exploiting the merits of the individual controller design in order to achieve robust stability without compromising the closed-loop performance, since these two are conflicting objectives in a feedback control design. Ultimately, this is a multi-objective control problem comprising of quantities with opposite/conflicting objectives. The concept of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ was introduced in [14], by incorporating the merits of $\mathcal{H}_\infty$ robust control into LQG control. The $\mathcal{H}_2$ method ensures robust performance in terms of minimization of control effort while the $\mathcal{H}_\infty$ part of the controller ensures robust stability against process/model uncertainty and/or arbitrary disturbances, within a prescribed bound. Since then, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design has received much attention in the control research community in the past two decades [14, 94, 98, 138, 163, 145, 146]. Mostly these $\mathcal{H}_2/\mathcal{H}_\infty$ designs are optimal controllers whose performance measures have both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ representations. Thus they can be referred as multi-objective control problems.

Major developments in $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be broadly classified based on the controller structure or optimisation method employed or by the very approach of the controller design (See also [37, 158]). In [14, 74, 73] fixed-order controller design by minimizing an auxiliary cost functional is dealt, such that $\mathcal{H}_2$ performance upper bound is subjected to $\mathcal{H}_\infty$ norm constraints. The use of convex optimisation methods and LMIs for addressing this type of controller design is explored to a great extent owing to the availability of efficient and powerful convex optimisation algorithms [61, 75, 94, 60, 138, 163, 145, 146]. In [114, 115] the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller is designed by minimising the closed-loop entropy which provides an upper bound for the $\mathcal{H}_2$ performance satisfying $\mathcal{H}_\infty$ constraint. This approach is equivalent to the approach of minimising the auxiliary cost functional in [14]. It could be noted that in all these references, generally, the problem formulation has been such that it minimises a $\mathcal{H}_2$ cost subject to $\mathcal{H}_\infty$ performance constraint.

In [98, 152] the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design is considered as a two-player Nash game whose corresponding individual objective functions are taken as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures. In this game theoretic interpretation of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design, the control input and the disturbance are the minimising and maximising players of the game, respectively. Thus the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is characterised very clearly in a Nash equilibrium strategy interpretation. This approach has been adopted for nonlinear systems in [99].

However, the use of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control strategy in the MPC paradigm, has not been attempted until recently in [121]. Their work was an extension of Kothare et al. [96] for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design. In [121], for a fixed/given upper bound on either the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance measure, the upper bound of the other performance measure is minimised. Ultimately, the problem evolves in the form of LMI constraints, which can be solved using semi-definite programming tools to compute the admissible state feedback gain matrix.

Thus, the design of robust MPC based on mixed $\mathcal{H}_2/\mathcal{H}_\infty$ approach with Nash equilibrium strategy interpretation (motivated from the idea used in [98]) seems to be more interesting and meaningful, when dealing with quantities with conflicting objectives of different performance measures. Nevertheless, in general, solving a decision making problem involving two or more individuals/agents is well handled in game theory. As control system design is basically one such decision making problem, controllers can and are being designed using game theoretic approaches.

1.3.5 Game theory based Controller Design

Game theory is a field of applied mathematics, which deals with making decisions under given conditions, and finds applications in various fields such as games, war, economics, biology to name a few. The game theory was established by the pioneering work of John von Neumann in his book *Theory of Games and Economic Behavior* in 1944 [157]. In game theory, the decision making elements are considered to be players of a game, which is governed by certain defined objectives and by a set of rules (conditions). So the players have to make the right decision so as to achieve their collective or individual objectives. The optimal strategy set of the player for the given conditions, called the *Nash equilibria*, can be calculated using the standard techniques of optimal control, such as minimum principle for open-loop systems or *dynamic programming* for feedback systems. Thus in game theory, Nash equilibrium is the resultant outcome of a game, for the optimal moves of the individual players. At Nash equilibrium, one player cannot improve his outcome of the game by altering his decision unilaterally.

Game theory also finds applications in control theory. Robust control theory, especially $\mathcal{H}_\infty$ control, which deals with making corrective control action against any arbitrary but bounded adverse conditions, finds direct analogy in game theory [6, 7, 162].

In terms of game theory, games could be classified broadly into two types, *viz.*, static and dynamic games. The games whose characteristics are such that it happens in a single moment of time and depending on the choice of the decision made, which is made once and simultaneously, by the players such that it could be represented in the form of a matrix are classified as *static games*. Since in such kind of games, the issues like order of the play, information available to the players and the evolution of games are insignificant and are therefore usually suppressed. When the players/agents of the game act in a dynamic environment, which is usually represented by a set of differential or difference equations, with their corresponding objectives, the games are called *dynamic games* or *differential games*. In the context of systems and control theory, which is the main focus of this thesis, the interest lies with dynamic games only. Dynamic games are in turn classified into two major types, *viz.*, zero-sum games and non-zero sum games. A brief account of these two type of games is given in the following.

(a) Zero-Sum Games In a differential/difference game, when the result of the game is such that the pay-off of a player due to his/her decision is equal to the gain of the opponent player, such games are called Zero-sum games.

Robust control design with partial/imperfect information structure for nonlinear systems is addressed as zero-sum differential games in [13] using certainty equivalence principle. In terms of game theory, when the two players of the game have some understanding or cooperation between them, such type of games are called *cooperative games*. Moreover, cooperative games can be reduced to an optimal control problem, by reducing to a single cost function for all the players of the game. However, the problem of designing a better control system with the aim to simultaneously achieve both better transient performance and robust stability against disturbance and/or process uncertainty is well known be a problem of conflicting objectives, thus the game is an non-cooperative game.

(b) Non-Zero-Sum Games

When the pay-off is not equal to the gain of the game, between two individual players, then it is assumed that there exists an arbitrary player, usually considered to the Nature, which has its role in the game. This type of games where the profit is not equal to the loss of the players involved in the game, is called non-zero-sum games. When the players of the game have conflicting objectives then the game is a non-cooperative game too.

The analogy between robust control theory and dynamic games for linear dynamic systems with quadratic objective functions are well documented in [7]. However, the nonlinear counterpart of it remains a challenge due to the resulting *infinite-dimensional* control problem to be solved [50]. This is also known as *Hamilton-Jacobi-Issacs* (HJI) equation. On the other hand, the LQ dynamic games, especially Nash games, have been investigated in the control community for more than two decades [6, 7].

MPC, which finds its way into the interest of control community since early 90s, got its hands into the game theoretic approach as early as that time onwards [41]. In [35] game theoretic approach is extended conceptually to MPC for constrained nonlinear dynamical systems. In [150] the game theory based MPC is dealt and is demonstrated with suitable examples.

1.4 MPC based on Wiener models

Among various classes of nonlinear models, Wiener type of models have special importance as they represent many systems in process industries. There are different techniques adopted to develop such models (c.f. Section 1.1). The use of MPC control for such processes has also drawn much interest in the process control literature. In [119, 122, 97, 32, 97, 103] Wiener models are used with predictive control to control the pH neutralization process. Many others have used Wiener models with MPC for other processes too: [120] for industrial C2-splitter, [27] for distillation column, [134] for bioreactor, [3] for ALSTROM gasifier, [143] for polymerization reactor. Nonlinear MPC has been designed using both Hammerstein and Wiener models [122]. In both regulatory and tracking control problems involving such Wiener models, the static nonlinear part of it is invertible, such that the linear dynamic states are directly available for control. However, when the process is represented using a complete black-box type of models, the use of such methods become forbidden. This is due to the reason that the black-box kind of models are developed using the mere input-output data collected from the process over the region of interest of its operation. The dynamic states of the model may not be a real representation of the actual process states. So the control based on such black-box oriented Wiener model cannot support the regular idea of using the static nonlinearity inversion.

1.5 Objectives of the Thesis

The overall aim of this thesis is to develop a novel approach towards robust MPC technique using Nash game approach. To achieve this overall aim, the thesis finds the following measureable objectives:

1. Even though wavelets have found their place already in system identification no effort has been observed so far to use this technique in Wiener structure of modeling. In the present thesis, wavelets are combined with the orthogonal Laguerre filters in the form of a Wiener type model, such that the Laguerre

filters form the linear dynamic portion and wavelets form the static nonlinear portion, called *Laguerre-Wavelet Network* (LWN) model. Evaluating the efficiency of such structure in comparison to the similar structure in the literature [133], is one of the major objectives of this thesis.

2. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control has been used in MPC framework recently [121]. It should be noted that almost at the same time when mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control was introduced [14], the robust controller design based on dynamic games [7] also got its defined shape. In [98], these two methods have been bridged, as both have the right formulations to fit well with each other. One of the objectives in the present thesis is to develop an explicit *Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ model predictive controller* design technique, where a robust infinite horizon approach is adopted for systems subjected to input saturation constraint and bounded disturbance.

3. The solvability of non-symmetric algebraic Riccati equations are of both theoretical and numerical importance. The Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC developed in the present thesis results in solving such a non-symmetric coupled algebraic Riccati equations (cAREs). It is aimed to find a suitable relaxation technique in the present thesis to solve the non-symmetric cAREs.

4. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC proposed in [121] is found to be relatively conservative w.r.t. the game theoretic approach proposed in the present work for actuator saturation constrained dynamic systems. It is desired in the present thesis to give reasoning for such conservativeness of both the approaches. Even though, the dynamics of a system is linear, the presence of hard constraints makes the control problem nonlinear. Hence, using set-theoretic approach for the analysis for such non-linear control problem is also an aim of this thesis.

5. It is also desired to extend the Nash game based $\mathcal{H}_2/\mathcal{H}_\infty$ MPC for output feedback case. This technique would be useful for those systems where direct state measurement is not possible.

6. Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC is a linear controller design technique whereas the Laguerre-Wavelet Network is inherently a nonlinear modeling technique. Hence it is desired in the present thesis to suggest a suitable linearising technique that would retain the local information of LWN model at the region of operation and also would serve the linearity requirement of MPC design technique.

7. It is also aimed to apply the above techniques for a benchmark chemical engineering process *viz.*, continuous bioreactor. Bioreactor is inherently a nonlinear process with input multiplicity nonlinearity i.e., different inputs to the process may give the same output. This nonlinear gain of the process cannot be properly approximated by a simple linear model and cannot be efficiently controlled by a linear controller as well. Thus the bioreactor example chosen in the present thesis posses enough control challenge in terms of nonlinearity. The change in the process gain also has a serious effect on the closed loop stability when regular PID type controllers are used to regulate the process. Thus bioreactor process which shows change in its gain over its operating region is a potential problem with enough control challenges is chosen to demonstrate the developments made in the present thesis both on system identification and control.

8. Finally, it is also desired to explore and present a cautionary note on the fragility of output feedback $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design for the closed loop sensitivity to controller's design parameter(s) that may lead to instability.

1.6 Outline of the Thesis

This thesis is conceptually divided into two parts *viz.*, system identification and robust model predictive control design. A brief summary of each of the following chapters is furnished below.

Chapter 2 deals with the identification of a class of nonlinear systems using a new species of Wiener type model, namely *Laguerre-Wavelet Network* (LWN) model. In this model, the linear dynamic part of the Wiener structure is formed by orthogonal Laguerre filters and the static nonlinear portion is formed by wavelet network structure. The efficacy of the proposed model is demonstrated with two case studies: (i) simulation of continuous bioreactor - which exhibits input-multiplicity nonlinearity, and (ii) real time operation of a lab scale pasteurization process.

A newer kind of model predictive control is proposed in Chapter 3 which is based on the combination of the celebrated control methods, namely, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control and Nash game theory. From the problem formulation to the numerical approach of solving the desired control problem, the design of a robust controller has been dealt in detail in this chapter. The performance of the proposed robust MPC design is compared with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ based MPC design developed in [121].

In Chapter 4, a complete and thorough theoretical analysis of the closed-loop stability and robustness of the proposed Nash game based $\mathcal{H}_2/\mathcal{H}_\infty$ MPC is given. The concepts of set theory have been used to provide a theoretical analysis of the robustness of this newer kind of feedback model predictive control. For the sake of completeness, the analysis is also carried out for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design given in [121]. Some simulation results are also furnished in support of the theoretical claims made in this chapter.

Chapter 5 deals with the extension of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC controller for the output feedback case. In order to make use of the LWN model furnished in chapter 2 for control application, a suitable linearising technique is advocated that would retain the local information of LWN model at the region of operation and also would serve the linearity requirement of controller design. The linearised LWN model is used with the output feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC controller design and has been demonstrated for a benchmark chemical engineering process *viz.*, continuous bioreactor.

Finally, based on the works furnished in the chapters 2 through 5, some concluding remarks are drawn in Chapter 6. Also, based on experience and knowledge gained by the author while working for this thesis, few recommendations for future research directions are provided.

Chapter 2

Wiener type Laguerre-Wavelet Network model

2.1 Introduction

There has been tremendous progress in the field of process model development (also called System Identification) over the years, though they were mainly limited to linear models. However, nonlinear system identification enjoyed much attention in the research community in the recent years [19, 42, 134]. There are many orthonormal basis filters used in system identification like Finite Impulse Response (FIR) model, Laguerre filter, Kautz filter and Generalized Orthonormal Basis Filter (GOBF) [123, 134, 159]. With these orthonormal basis filters it is possible to obtain efficient mathematical model of linear systems even with time delays, with some *a priori* information about the process dynamics. Among the recent model developments, the models developed by [134], namely Laguerre polynomial and Laguerre ANN, are found to be interesting. These models are developed in Wiener model structure, wherein the polynomial and ANN components form the static nonlinear part of the Wiener model and Laguerre filters form the linear dynamic part. Thus, the models do not require any external feedback for representing dynamic systems.

Both polynomials and ANN provide static global nonlinear approximations. However, they have limitations for local approximations. Such local approximations are highly required for representing processes with severe nonlinearities. Wavelets are found to provide such local approximations parsimoniously to an appreciable level of accuracy.

Wavelets, which basically find applications in the field of signal processing, are getting attention into the system modeling in the recent years [19, 42] for their property of multi-scale decomposition of signals. Representation of real time signals at multiple scales serves as a very efficient and useful way to analyze them. This can be achieved by expressing the data as a weighted sum of orthonormal basis functions, which are defined in both time and frequency, such as wavelets. The weights become the wavelet coefficients. Wavelets are a computationally efficient family of multi-scale basis functions. A signal can be represented at multiple resolutions by decomposing the signal on a family of wavelets and scaling functions.

The same properties of wavelets that are exploited in signal processing such as orthogonality and multi-scale resolution, make the wavelets a suitable system approximation tool too. The feature extraction abilities of multi-scale representation of data are utilized to construct multi-scale nonlinear models that are less affected by the presence of noise in the data. The main idea is to decompose the input-output data at multiple scales and construct a nonlinear model using them. It is in this sense wavelets are being used in the present work, exploiting the structural property of discrete-time wavelet transform, which is called *Wavelet Network*.

Wavelet network, a blend of idea of neural networks and wavelets, is reported by [166] to give a good static nonlinear mapping. In the present work, utilising the wavelets' static nonlinear mapping into the Wiener model proposed by [134] a newer kind of nonlinear model is developed, namely, *Laguerre-Wavelet Network* model for representing a class of nonlinear systems. In the proposed model the static nonlinear part, basically formed in terms of wavelets, is expected to give an efficient local and global nonlinear approximation parsimoniously. Thereby, the model can give a better performance than that of Saha et al [134].

Generally, Laguerre filter models can represent any linear dynamical systems efficiently [159]. Even for approximating mildly nonlinear systems, Laguerre models could be used as a piecewise linear models. But it fails when used for modeling severely nonlinear systems. On the other hand, wavelet network model could represent any static nonlinear function with appreciable accuracy [166] but fails miserably when used for approximating linear or mildly nonlinear systems. These demerits of Laguerre models and wavelet network can be overcome when they are used together in a model structures like Wiener or Hammerstein.

2.2 Laguerre Representation

Filter design finds an important role not only in signal processing but also in the area of modeling and control of dynamic systems. Filters are broadly classified according to their operating or rejecting ability of frequencies, such as low-pass filters, band-pass filters, high-pass filters, all-pass filters, band-stop filters, etc. Earlier the interest of scientific community lied with design/synthesis of linear filters. However, the design criteria are dictated by the band of frequencies of interest and the orthogonality requirement of the filter basis functions.

Often the filters are designed and/or analysed based on their frequency response or their time-scale response to some standard test signals, such as impulse, step or ramp signals. This is achieved by using delay operation at each stage to get the required response. Transverse filters are those which have the ability to adapt its parameters so as to model a linear system. Usually filters are also classified based on their impulse response *viz.*, finite-impulse response (FIR) or infinite-impulse response (IIR) filters. The problem with FIR filters, such as transverse filters - used to model linear dynamic systems, is that they cannot be used efficiently to model systems with longer impulse response. With FIR filters this is only possible at the expense of a larger number of delay units. Ultimately this has a serious effect of increasing the order of FIR filters. To overcome this issue infinite impulse response (IIR) filters are used in place of FIR filters. IIR filters have better system modeling ability by providing infinite impulse response nature with finite order filters.

In 1935 Wiener and Lee developed a filter design consisting of a cascade of all-pass filters (lattice filters), whose outputs are summed to get the final filter response. Of these all-pass filters some may have anti-phase with each other, so that some of the corresponding frequencies are attenuated while summation.

Laguerre filters on the other hand represent a compromise between FIR and IIR filters [108]. With the proper, in fact optimal, choice of the Laguerre pole the modeling error can be reduced. When the Laguerre pole takes the value of zero, it reduces to FIR filter. Laguerre -based model is thereby considered as constrained IIR, since all the poles of Laguerre filter models are equal.

Consider a function $G(z)$ to be strictly proper ($G(\infty) = 0$), analytic in $|z| > 1$, and continuous in $|z| \geq 1$. Let $-1 < a < 1$, then there exists a sequence c_i such that the z-transfer function of the unknown system can be represented as [133, 134, 159],

$$G(z) = \sum_{i=1}^{n} c_i L_i(z) \tag{2.1}$$

The z-domain representation of the Laguerre filters with tapped coefficients is given as

$$L_i(z) = \beta \frac{(1 - az)^{k-1}}{(z - a)^k} \tag{2.2}$$

where $\beta = \sqrt{(1 - a^2)T}$ and $a = e^{-pT}$; p is the time scale parameter of the Laguerre filter which is to be chosen close to the inverse of the dominant time constant of the system, so as to give a fast rate of convergence, and T is the sampling interval.

Consider a single input single output (SISO) system modeled using a Laguerre filter network. Let $l_i(k)$ represent the output of the i^{th} order filter in the Laguerre filter network. Defining an n dimensional state vector at sampling instant k as

$$l(k) = [l_1(k),\ l_2(k), \cdots,\ l_n(k)]^T \tag{2.3}$$

It can readily be shown that a discrete time state space representation of the Laguerre network can be written in the standard state space form [134],

$$l(k+1) = Al(k) + bu(k) \tag{2.4}$$

where $u(k) \in \mathbb{R}$ is the input, and defining

$$\tau_1 = e^{-pT} \tag{2.5}$$

$$\tau_2 = T + \frac{2}{p(e^{-pT} - 1)} \tag{2.6}$$

$$\tau_3 = -\frac{Te^{-pT} - 2}{p(e^{-pT} - 1)} \tag{2.7}$$

$$\tau_4 = \frac{\sqrt{2p}(1 - \tau_1)}{p} \tag{2.8}$$

We have the input distribution matrix $b \in \mathbb{R}^{n\times 1}$,

$$b = [\tau_4 \;\; (-\tau_2/T)\tau_4 \;\; \cdots \;\; (-\tau_2/T)^{n-1}\tau_4]^T \tag{2.9}$$

and $A \in \mathbb{R}^{n\times n}$ which is a lower triangular matrix such that,

$$A = \begin{pmatrix} \tau_1 & 0 & \cdots & 0 \\ -(\tau_1\tau_2 + \tau_3)/T & \tau_1 & \cdots & 0 \\ \cdots & -(\tau_1\tau_2 + \tau_3)/T & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ (-1)^{n-1}\tau_2^{n-2}(\tau_1\tau_2 + \tau_3)/T^{n-1} & \cdots & \cdots & \tau_1 \end{pmatrix}.$$

For a linear model given by eqn.(2.1), the model output can be expressed as weighted sum of the states

$$\hat{y}(k) = c^T l(k) \tag{2.10}$$

where elements of 'c' are Laguerre filter coefficients, i.e., $c = [c_1 \;\; c_2 \;\; \cdots \;\; \cdots \;\; c_n]^T$. To make use of the Laguerre filters in nonlinear system identification, the Laguerre model could be extended for nonlinear function mapping, as shown in [134] in Wiener structure as,

$$\hat{y}(k) = \Psi(l(k)) \tag{2.11}$$

where $\Psi(\cdot)$ is a static nonlinear function, which is nothing but a state-to-output nonlinear function mapping. In [134], the function approximation is achieved using polynomial and neural networks, whereas in this thesis we address wavelet network for this purpose.

2.3 Wavelet Transforms

Wavelet theory has influenced the scientific community at a large since its inception in 1980s. It is the result of bringing together the concepts that already existed in various forms in different domains. Thus wavelets has interdisciplinary origin.

In wavelet transformation, a signal or a function is decomposed into its different frequency components such that each component has resolution matched to its scale. In the attempt to overcome the bottleneck of Fourier Transform, which is cumbersome while studying the transient response of a signal or some severe nonlinearity of a function, the concept of time-frequency localization technique was introduced in 1946 by Gabor, called the *Windowed Fourier Transform*. In the windowed Fourier Transform, a real symmetric window $h(t) = h(-t)$ is translated by b and dilated or scaled by a, such that

$$h^{a,b} = e^{ibt} h(t - a).$$

A normalised g, *i.e.*, $\|g\| = 1$, for any $(a, b) \in \mathbb{R}^2$ gives $\|g^{a,b}\| = 1$. Hence, the windowed Fourier Transform of a signal/function $f \in \mathcal{L}_2(\mathbb{R})$ is given by

$$\langle f, g^{a,b} \rangle = \int_{-\infty}^{+\infty} f(t) g(t - b) e^{-iat} dt.$$

The choice of the window size (*i.e.*, its support) is a major concern in achieving higher accuracy with the windowed Fourier Transform technique. Moreover, the window function is expected to have compact support (*i.e.*, to decay to zero on both sides centered at a specified point). The issue of the proper choice of window function was later overcome by wavelet transformation.

In wavelet transform time-frequency window of different time supports are used. A bases function, called *wavelet*, satisfying the property of compact supportness is appropriately chosen to decompose the signal/function over dilated and translated wavelets. A wavelet is a function $\psi \in \mathcal{L}_2(\mathbb{R})$ with a zero average [104]:

$$\int_{-\infty}^{+\infty} \psi(x)dx = 0.$$

It is normalised $\|\psi\| = 1$ and centered in the neighbourhood of $x = 0$. A family of time-frequency windows can be obtained by scaling/dilating ψ by a and translating it by b:

$$\psi^{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right).$$

A brief overview of some basic concepts of wavelet transform is given in this section to make the thesis self-contained.

2.3.1 Continuous Wavelet Transform

As continuous wavelet transform is first studied, some important results are summarized below. Continuous wavelet transform of any function $f \in \mathcal{L}^2(\mathbb{R}^n)$ is given by [46, 104],

$$\mathrm{w}^{a,b} = \int_{\mathbb{R}^n} f(x)|a|^{-1/2}\psi\left(\frac{x-b}{a}\right)dx \qquad (2.12)$$

where $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R}^n)$, which is radial and satisfies the admissibility condition, with $\hat{\psi}(\xi)$ being the Fourier transform, as

$$C_\psi = (2\pi)^n \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi} d\xi < \infty \qquad (2.13)$$

and the function $f(x)$ can be reconstructed back by using the inverse wavelet transform,

$$f(x) = C_\psi^{-1} \int_0^\infty a^{-(n+1)} \int_{\mathbb{R}^n} \mathrm{w}^{a,b} a^{-n/2}\psi\left(\frac{x-b}{a}\right)dadb \qquad (2.14)$$

$a \in \mathbb{R}^n$ and $b \in \mathbb{R}^+$ are the dilation and translation parameters, respectively.

2.3.2 Wavelet Frames

The continuous wavelet transform described in Section 2.3.1 is not implementable in digital computers without discretisation. So the inverse wavelet transform (2.14) when discretised, takes the following form:

$$f(x) = \sum_i \mathrm{w}_i^{a,b} a_i^{-n/2} \psi\left(\frac{x - b_i}{a_i}\right) \tag{2.15}$$

Some conditions like compactness and regularity are required to hold, when wavelet transformation is discretised using orthogonal wavelet bases as above, for the proper reconstruction of the function $f(x)$ [46, 87, 104, 165]. However, by relaxing the orthogonality of the wavelet basis function, making what is called *wavelet frames*, it is possible to reconstruct back the function. According to the distribution of data, the reconstruction of $f(x)$ with countable number of wavelet terms, $(a_i,\ b_i)$, which corresponds to a family of dilated and translated wavelets:

$$\left\{ a_i^{-n/2} \psi\left(\frac{x - b_j}{a_i}\right);\ \ i,\ j \in \mathbb{Z}^+ \right\} \tag{2.16}$$

where $a_i = a_0^i,\ b_j = j a_0^i b_0$ with the scalar parameters a_0 and b_0 which define the step sizes of dilation and translation discretisation, respectively (typically, $a_0 = 2$ and $b_0 = 1$). Out of the many solutions to discretise wavelet transforms and to possibly reconstruct the function in discrete domain, wavelet frames is one in which it is achieved by relaxation on the orthogonality of the wavelet basis function [46, 104].

Definition 2.1 *A family of functions $\{\psi_i\}_{i \in I}$ in Hilbert space $\mathscr{H}$ is called a frame if there exist $0 < \Theta \leq \Phi < \infty$ so that, for all f in $\mathscr{H}$,*

$$\Theta \|f\|^2 \leq \sum_{i \in I} |\langle f,\ \psi_i \rangle|^2 \leq \Phi \|f\|^2 \tag{2.17}$$

where Θ and Φ are the frame bounds [46].

With this proper definition of wavelet frames, the following theorems state the properties of the wavelet frames.

Theorem 2.1 *[46, 104] If* $\{\psi^{a,b}\}_{a,b\in\mathbb{Z}}$ *is a frame of* $\mathcal{L}^2(\mathbb{R})$ *then the frame bounds satisfy, with the admissibility condition* $C_\psi = \int_0^\infty \frac{|\hat{\psi}(\xi)^2|}{\xi} d\xi < +\infty$,

$$\Theta \leq \frac{C_\psi}{b_0 \log_e a} \leq \Phi$$

$$\forall \xi \in \mathbb{R} \setminus \{0\},\ \Theta \leq \frac{1}{b_0} \sum_{j=-\infty}^{+\infty} |\hat{\psi}(a^j \xi)|^2 \leq \Phi \tag{2.18}$$

□

Theorem 2.2 *[46, 104] If there exist two constants,* $0 < \Theta \leq \Phi < \infty$ *such that*

$$\forall \xi \in \mathbb{R} \setminus \{0\},\ \Theta \leq \frac{1}{b_0} \sum_{j=-\infty}^{+\infty} |\hat{\psi}(a^j \xi)|^2 \leq \Phi \tag{2.19}$$

then,

$$\Theta \|f\|^2 \leq \sum_{i \in I} |\langle f,\ \psi_i \rangle|^2 \leq \Phi \|f\|^2 \tag{2.20}$$

□

Thus wavelet frames are bases functions that satisfies the property of *boundedness.*

2.3.3 Wavelet Network

As proposed in [165, 166], equation (2.15) could be viewed as one-hidden-layer neural network. By optimally choosing the parameters w_i, a_i and b_i in (2.15), with $\psi(\cdot)$ as the hidden layer activation function and a linear function in the output layer, the construction becomes similar to that of neural networks, called *wavelet network* [166]. This method is used for static nonlinear function approximation. Although wavelet series is an infinite series, all the wavelet functions will not have data within their support [165]. However, only a finite wavelet functions is enough to map the data of finite domain such as,

$$\Psi = \left\{ \psi_{i,j} : (i,j) \in \bigcup_{q=1}^{Q} I_q \right\} \tag{2.21}$$

where I_q are the index set of wavelet functions, whose supports lie within the given data set. All the Laguerre states are fed as input to all the wavelet functions. The wavelet coefficients, which are originally the inner product of the states and the wavelet functions, are equivalent to the hidden-to-output layer weight in neural networks. The optimal choice of the parameters, called the network training, could be achieved as that of neural networks using algorithms like backpropagation.

2.4 Laguerre-Wavelet Network Model

Of the different types of block-oriented nonlinear models that are used to represent different classes of nonlinear systems, in this work, an approach to the design of a Wiener structure is proposed. The approach is based on the use of Laguerre filter banks to approximate the linear subsystem and of a wavelet network to approximate the memoryless non-linearity. As mentioned in Section 2.3.3, if trained by a suitable training algorithm (such as neural network), the discrete inverse wavelet transform structure can be used for static nonlinear mapping with the optimal selection of the network parameters.

2.4.1 Wiener Model Construction

In the present work, Wiener model structure is used. The linear dynamic part is formed by Laguerre basis filters and the wavelet network forms the static nonlinear part, referring (2.4) and (2.11),

$$l(k+1) = Al(k) + bu(k) \tag{2.22}$$

$$\hat{y}(k;\theta) = \Psi[l(k);\theta] + c^T l(k) + d \tag{2.23}$$

where $\psi \in \Psi(\cdot) \subseteq \mathcal{L}^2(\mathbb{R}^n) : \mathbb{R}^n \to \mathbb{R}$ and θ is the representation of the parameters in the wavelet network, whose optimal values will be chosen by training the wavelet network. The theta (θ) includes the wavelet network parameters; $\theta := \{a_i, b_i, \mathrm{w}_i^{a,b}\}$, to form the nonlinear wavelet network.

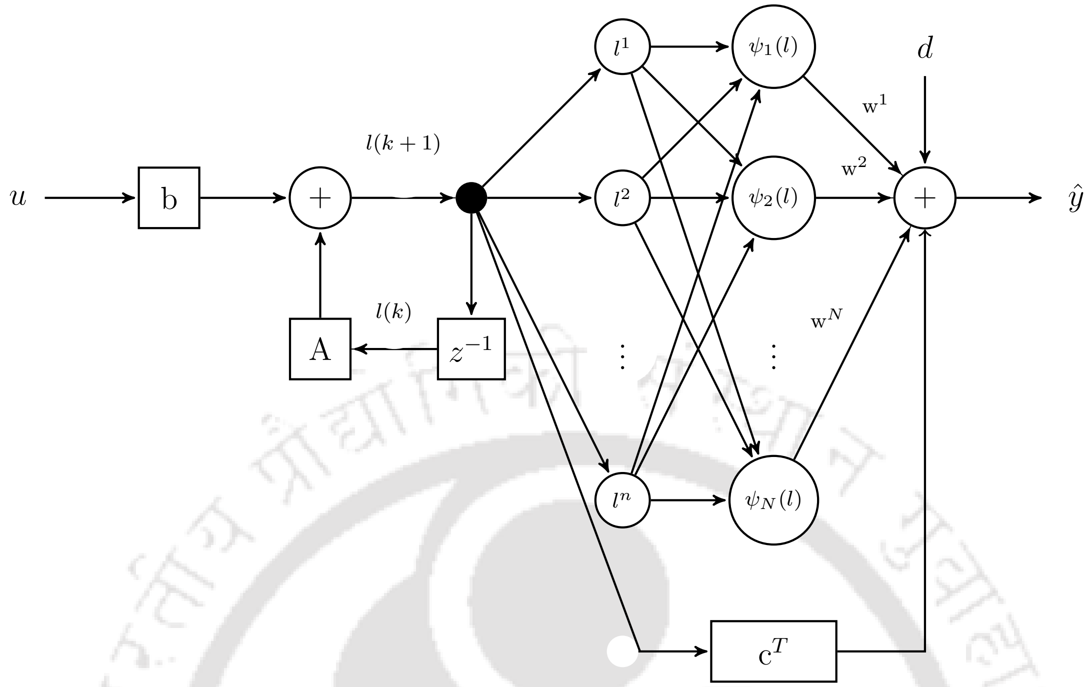

Figure 2.1: Schematic Diagram of Laguerre Wavelet Network Model

As wavelet frames form a regular lattice, they are able to estimate functions belonging to a certain functional space by using wavelet networks that contain the wavelet frames as their activation function. In [165] the construction of wavelet frames of $\mathcal{L}^2(\mathbb{R}^n)$ for multidimensional case was shown. Following the procedure described by [166], a radial wavelet frame, which has multi-scaling, is used for the construction of wavelet network.

The inputs to the wavelet network are the Laguerre filters' states, $l(k)$[‡], as given by the equation (2.4) - (2.9). The adjustable Laguerre filter parameters are the pole position within the unit circle viz. time scale parameter, p, and the number of filters, n. By analyzing the open loop response of the process a suitable time scale parameter is chosen, such that $|p| \leq 1$. With this choice of the Laguerre filter pole position the stability of the Laguerre model is ensured. Moreover, the number of filters gives the enough representation of the model to approximate process dimension. The resulting structure of Wiener type Laguerre-Wavelet Network model is shown in

‡ The variable x used in section 2.3 is nothing but the Laguerre states $l(k)$ in the LWN model.

Figure 2.1. The model parameters are chosen sequentially such that Laguerre filter parameters are initially chosen and are followed by the wavelet network parameters by recursively iterating them, until they satisfy a performance criterion.

Remark 2.1 *It should be noted that improper choice of the Laguerre parameters, indeed leads to poor model training. The convergence of the wavelet network to the prescribed level of accuracy is possible only if the total model response comes close to the process data (i.e., output target data for the network training) subject to an arbitrary tolerance value. The target data has been so chosen that it has both the dynamic and steady state information of the process. The dynamic response of the model is again due to the Laguerre filter states; whereas the wavelet network is trained to capture the nonlinear gain of the process. Thus the convergence of the network training is affected by the choice of the Laguerre parameters.*

2.4.2 Model Training

The Laguerre parameters are usually chosen with some a priori information of the process, mostly the dominant time constant [159]. In [159], where the use of Laguerre filters for system identification was proposed, it was suggested that if the choice of the Laguerre filters pole (p) is made such that, $p = 1/\tau$, where τ is the dominant time constant of the system, then the approximation of the model would be better. However, other methods were suggested in the literature [133, 134] using some statistical information criteria such as Akaike Information Criterion (AIC) and Pure Prediction Error (PPE). In the present work, the Laguerre poles (p) were chosen by making use of the dominant time constant of the process, determined from the open loop step response of the process for a small step input given at the nominal operating region.

Generally, wavelet transformation is carried out by dyadic discretisation. However, such discretisation essentially carries redundant terms due to data sparseness and thereby decreases model parsimony. This phenomenon is overcome by adaptive discretisation of the inverse wavelet transform [165]. Thus the wavelet network parameters $w_i^{a,b}$, a_i, b_i are determined through training the network by a regressive algorithm, namely the backpropagation algorithm [76, 166, 133]. In the present work, "Mexican hat function" [165],

$$\psi(x) = n - \|x\|^2 e^{\frac{\|x\|^2}{2}}, \ \ \|x\|^2 = x^T x$$

was used as the mother wavelet function, where n is the dimension of the input space of the network (*i.e.*, Laguerre states in the present study). The parameters of the wavelet network are optimized subject to the performance criteria,

$$F(k;\theta) = \frac{1}{K}\sum_{k=1}^{K} y(k) - \hat{y}(k;\theta)^2 \leq \epsilon \tag{2.24}$$

where 'ϵ' is a small positive value representing the error tolerance value and K is the number of data used.

Remark 2.2 *The convergence rate and time for training of LWN model depend on many factors such as number of input-output data, range of input perturbation, sparseness of the data set apart from the proper choice of the Laguerre parameters chosen a priori.*

2.5 Stability Analysis

The stability of the model has to be ensured over the entire range of the modeling without compromising model parsimony. Although wavelet network approximates the static nonlinear function parsimoniously it is also required to ensure the model stability. The stability can be ensured only if the model approximates a continuous and smooth static nonlinear function of the process everywhere. In case it demonstrates some spike-like sharp nonlinearity at a certain portion of the static nonlinear function and is smooth and continuous everywhere otherwise, then the stability becomes an issue w.r.t. the model. In the LWN model, however, the stability of the Laguerre filters is guaranteed by $|p| < 1$. Hence, the stability of the wavelet network is discussed in this section.

Proposition 2.1 *For any $x \in \Omega \subset \mathcal{L}^2(\mathbb{R}^n)$ mapped in a functional space $\mathcal{L}^2(\mathbb{R}^n)$ using a finite number of dilated and translated wavelet frames,* $\{\psi_i^{a,b} : a, b \in \mathbb{Z},\ i = 1, ..., Q\}$, *such that* $\Theta\|x\| \leq |\langle x, \psi_i^{a,b}\rangle| \leq \Phi\|x\|$, *for some frame bounds* Θ *and* Φ, $0 < \Theta \leq \Phi < \infty$, *then* $\left\{ \sup|\Psi(x)| \,\middle|\, \Psi(x) = \sum_i \langle x, \psi_i^{a,b}\rangle \psi_i^{a,b} \right\}$ *is bounded.*

Proof: Proof is given by contradiction. Assume that $\Psi(x)$ is unbounded,

$$\sup|\Psi(x)| = +\infty \tag{2.25}$$

With the above assumption it could be said as,

$$\sup|\langle x, \psi_i^{a,b}\rangle| = +\infty \tag{2.26}$$

By Theorem 2.1 and Theorem 2.2, it could be proved that if a radial wavelet function satisfies eqn. (2.26), then it is not a wavelet frame. But the chosen wavelet function is a wavelet frame, satisfying $\Theta\|x\| \leq |\langle x, \psi_i^{a,b}\rangle| \leq \Phi\|x\|$, which contradicts the assumption. ∎

With these model properties, the stability of the model is well established, which in turn ensures bounded input bounded output (BIBO) stability, as usually demonstrated for nonlinear models elsewhere [100, 134].

2.6 Numerical Example

To illustrate the efficacy of the LWN model a simulation example of a continuous bioreactor is given. Modeling of bioreactor is an interesting research problem. It is the nonlinearity of the process that gives the challenge for the development of a suitable model and so forth the controller design. To give a real life feel for the capability of the LWN model, it is also tested on the input-output data from real-time process, namely, ARMFIELD® Pasteurization process. All tests are done in a Windows Vista PC with Intel Core-2 Duo processor running at 1.66 GHz.

2.6.1 Bioreactor - A simulation case study

Mathematical modeling of bioreactor is of much interest due to its highly nonlinear behaviour. At the low concentration of input feed substrate, the growth rate of the cell-biomass dominates the reaction rate. Thus, increase in feed concentration results in an increase in productivity. However, beyond a certain value of the input feed concentration, the product and substrate inhibition prevails and thereupon productivity shows a negative gradient with respect to the input feed concentration. Thus, there exists an optimum value of the bioreactor productivity. The necessary condition for optimality implies that the steady-state gain (product concentration vs. manipulated feed substrate concentration) changes its sign across the optimum point and the steady state gain is zero at the optimum. Thus, the system exhibits input multiplicity and there exists a singularity at the optimum operating point.

The mechanistic model of a bioreactor is given by the following differential equations [77, 134],

$$\frac{dX}{dt} = -DX + \mu X \tag{2.27}$$

$$\frac{dS}{dt} = D(S_f - S) - \frac{1}{Y_{X/S}}\mu X \tag{2.28}$$

$$\frac{dP}{dt} = -DP + (\alpha\mu + \beta)X \tag{2.29}$$

where X represents effluent cell-mass or biomass concentration, S represents substrate concentration and P denotes product concentration. Product concentration (P) and the cell-mass concentration (X) are measured process outputs, while dilution rate (D) and the feed substrate concentration (S_f) are the process inputs. $Y_{X/S}$ represents the cell-mass yield and α and β are the yield parameters for the product. The specific growth rate, μ, is given by,

$$\mu = \frac{\mu_m(1 - \frac{P}{P_m})}{K_m + S + \frac{S^2}{K_i}} \tag{2.30}$$

where μ_m represents maximum specific growth rate, P_m, K_m and K_i are the product saturation constant, substrate saturation constant and substrate inhibition constant, respectively. The nominal model parameters are given in Table 2.1.

Table 2.1: Nominal parameters for bioreactor

Parameter	Nominal Value
$Y_{X/S}$	0.4 g/l
α	2.2 g/l
β	0.2 h^{-1}
μ_m	0.48 h^{-1}
P_m	50 g/l
K_m	1.2 g/l
K_i	22 g/l

For the present case the input-output data of SISO system of product concentration P against the feed substrate concentration S_f is taken for developing the empirical model. The operating point, around which the perturbation is given, has been chosen judiciously. In [134], the process optima has been chosen as the operating point and the perturbation had been given in the vicinity of the process optima. While the Laguerre Polynomial (LP) model, developed in [133], was able to capture the process nonlinearity around a narrow range of operating zone around the optima, it failed in the sub-optimal zone. Nevertheless, it can be argued that in real life scenario, the process data is generated as a random sequence and there is no guarantee that the data will be obtained around a specific point of one's choice. An ideal model should be robust enough to capture the entire process behaviour with whatever data available at one's disposal. With that argument, a sub-optimal point, obtained from [77], has been chosen in this work around which the perturbation is provided to collect the process data. The data required for model parameter estimation is obtained by giving a random signal in S_f at its mean value of 20 and with variance of 0.1, perturbing the mechanistic model, eqn. (2.27)-(2.30), in open-loop. Two sets of data are obtained, using one for parameter estimation and another for model validation. The Laguerre filter parameters are chosen such as, $n = 3$, $p = 0.1$

from some a priori knowledge about the system, *i.e.*, the open-loop step response and the sampling time of 0.1 hr. The wavelet parameters are estimated using backpropagation algorithm. An error tolerance, $\epsilon = 0.001$ is given as the stopping criterion for the network training.

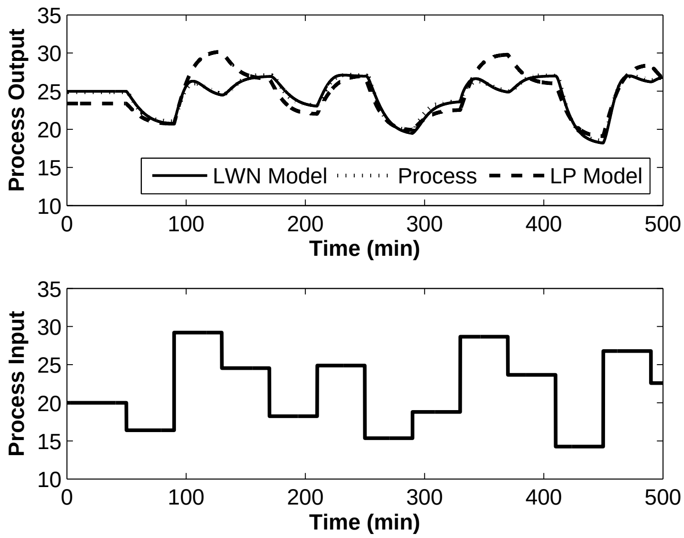

Figure 2.2: Dynamic response of the Wiener type Laguerre models for bioreactor process.

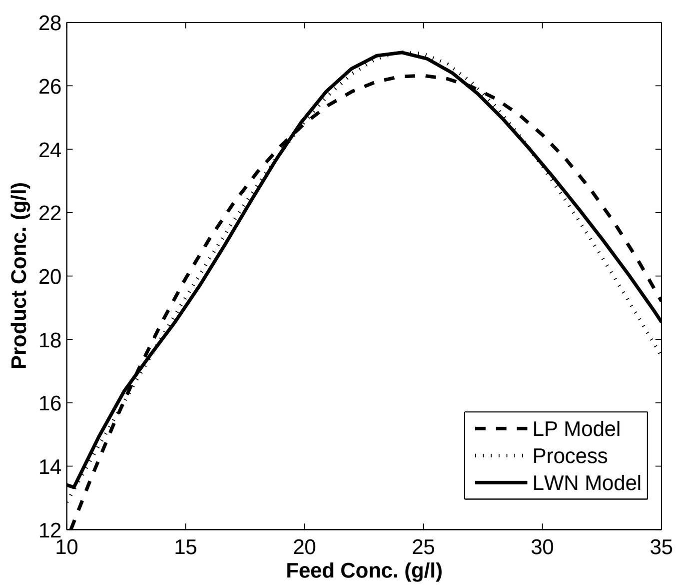

Figure 2.3: Steady-state response of the Wiener type Laguerre models for bioreactor process.

The wavelet parameter after training is shown in Appendix A (Table I). The resulting model is validated with another data set generated, and the result is shown in Figure 2.2. It could be noted that initially the process was run for 50 hours in the nominal steady state values as given in Table 2.1. Then the process was excited with a random signal about the steady state value. It could be observed from Figure 2.2 that the model is capable of capturing the process dynamics very well. Using the same Laguerre parameters, Laguerre polynomial proposed in [134] is developed with the maximum polynomial order as two. From Figure 2.2 it could be observed that Laguerre wavelet network model outperforms LP model, by closely following the response of the process.

The model is also validated for the steady state response against that of the process. Generally, model validation is performed using statistical methods such as Residual Analysis for finding out whether the developed model has missed out any necessary information. However, residual analysis cannot be used for black-box models, such as the Wiener type model developed in this thesis, as the system states and the process states are not equivalent for incorporating the replacement of the model's one-step ahead predicted value with that of the process measurement. The feed substrate concentration S_f is varied from 10 to 35 and the corresponding steady state product concentration P is observed. From Figure 2.3, it could be found that the model is able to give better and very close steady state response matching with that of the process when compared against that of the Laguerre polynomial, used in [134]. Thus the model could efficiently capture the input multiplicity nonlinearity of the process over a wide range of operation. The prominent mismatches between process and LP model at 125 hr and 350 hr in Figure 2.2 are the indication that the LP model fails to capture the process steady state characteristics. This phenomenon is also supported by Figure 2.3. The Integral Squared Error (ISE) values of LP and LWN model are 120.9904 and 23.4225, respectively, which proves that the LWN model outperforms its polynomial counterpart. Although the training of the model with 10,000 input-output data points takes 54.098 seconds, once the model is trained it takes only 6.323×10^{-5} seconds to get the output from the model for a given input.

This is one of the most welcomed results as it makes the model suitable even for fast processes in various applications.

2.6.2 Pasteurization process

To provide a realistic test for the proposed model, it was tested with lab-scale equipment, ARMFIELD^{TM}Pasteurization process (Model No. PCT23 MKII). In this process, the product stream has to be kept at a pre-determined temperature for a minimum time. Basically the process consists of a feed tank, a water heater with an electric coil for heating and a heat exchanger unit, which forms the main component of the process. The heat exchanger consists of three sections, namely, cooling unit, heating unit and a regeneration unit for preheating and/or pre-cooling. Various loops are available with this equipment. It could be operated either by employing all the control loops or with a single loop. In a loop, a single input is varied and its effect on other variables is noted. All the output variables are measurable with the available appropriate sensors calibrated in the range 0 to 100% of its operating range. The Figure 2.4 shows the schematic of the equipment.

For the present study, the feed pump rate is varied keeping the heater power and water pump rate at a constant value. In this setup, the hot water pump speed ($N2$) and heater power rate are fixed at 40% and 15%, respectively, and feed pump rate ($N1$) is being varied. Heated feed temperature ($T4$) or the output product is observed process variable in this experiment. The feed pump speed is carefully chosen to be more than 50% so that for the above operating conditions of $N2$ and heater power rate, the temperature $T4$ does not hit the saturation limit and trip the heater power for safety purpose. Sampling time of 6 seconds (0.1 min) was chosen. The input-output data are recorded using a computerized data-logging system. Two sets of data were recorded, and used as like the previous example, *i.e.*, one set for model development and another for validation.

It was observed that only two Laguerre filters ($n = 2$) are sufficient to capture the dynamics of the process and the Laguerre parameters were taken as $T = 0.09$ min

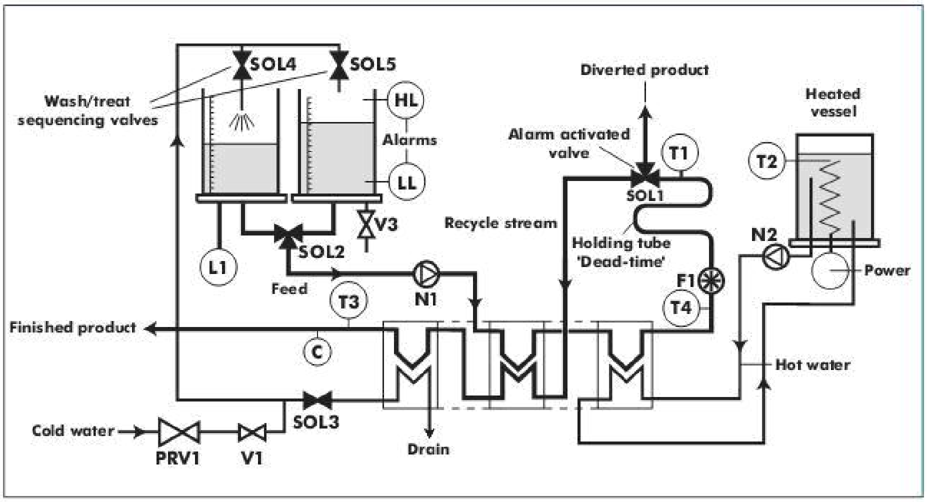

Figure 2.4: Schematic diagram of ARMFIELD™Pasteurization process.

and $p = 0.08$. Then the wavelet network was trained to map the process nonlinearity until the stopping criteria of $\epsilon = 0.001$ was reached. The final model parameters are listed in Appendix A (Table II). After the model is trained, it is validated with the other set of data and the performance efficiency of the model is shown in Figure 2.5. The model predicts the process response efficiently, except at few regions such as that near 360-380 time units. It should be noted that feed pump speed was very high at around 380 min. It is also true that the model is unable to capture the process dynamics properly for any input value above 80%. This is due to the fact that there was not enough information in the training data corresponding to the input

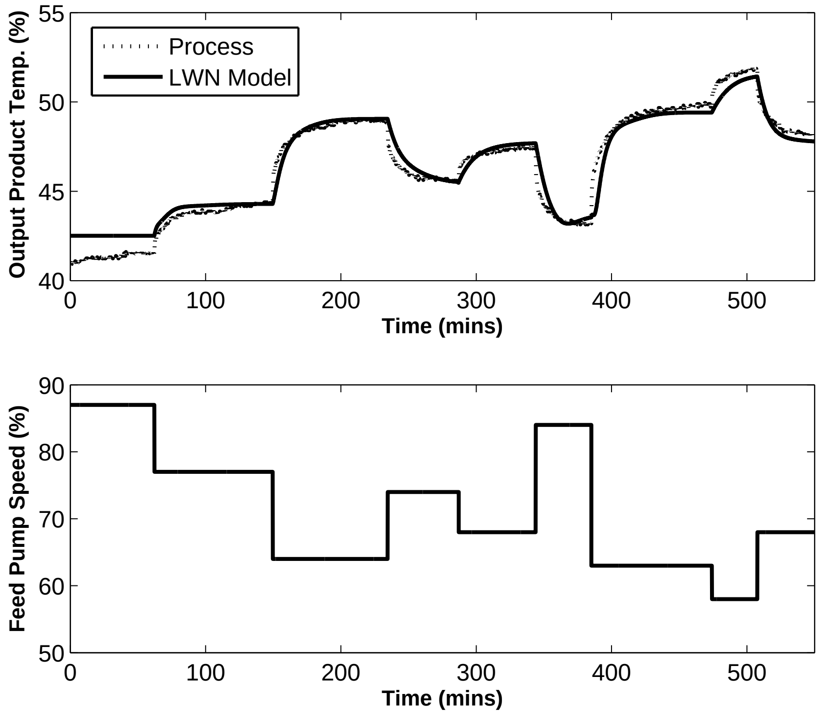

Figure 2.5: Dynamic response of LWN model for ARMFIELD^{TM}Pasteurization process.

above 80%. The comparative result depicting the efficiency of the Laguerre wavelet network against that of the Laguerre polynomial model, as that of the bioreactor example, is not given here. It was observed that Laguerre polynomial model is not capable enough to give an acceptably good performance even with the training data that is worth enough to compare it with Laguerre wavelet network model. The LWN model therefore captures the process dynamics appreciably well. It could be observed from the Figure 2.5 that there is some plant-model mismatch during the initial minutes, which is due to the initial condition mismatch in the Laguerre part of the model. Here the initial condition is chosen arbitrarily, as there is no *priori* information about it. But again the model could overcome the error due to the initial condition mismatch in a reasonable time due to the state feedback mechanism. The other discrepancies between the model and the process measurement may be due to some other dynamics in the process, which the model is not aware of.

It should also be noted that using noisy process measurement data for modeling using Laguerre-Wavelet Network model does not affect the model response, as Laguerre filters are essentially low-pass filters, which assert the high frequency noise effect in the data.

2.7 Summary

Wiener type Laguerre-Wavelet Network (LWN) model has been developed for the first time to represent a class of nonlinear system as a black-box model. The ability of wavelets to represent the spatio-temporal morphology of any arbitrary function efficiently has been exploited to approximate the nonlinear systems in a parsimonious manner.

It should also be noted although in the present study, a Wiener type system *viz.*, bioreactor, has been taken for investigation of the efficacy of the LWN model, it is not limited only for Wiener type processes. It could also be easily used as a Hammerstein type model or a L-N-L or N-L-N type model too.

Chapter 3

Nash game based Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC - State Feedback Case

3.1 Introduction

Design of robust MPC is one of the most demanding and active areas of research amongst control engineers, researchers and practitioners since the past three decades. MPC, in general, being an open-loop optimal control strategy, the problem of addressing the robustness of the system, against model uncertainties and/or unknown disturbance, are of much concern in MPC design.

There are various approaches proposed so far to address this robustness issue, of which the feedback MPC design proposed in [96] is an interesting proposal, which recognises the need for feedback in the constrained optimal control problem solved online. Very recently, the first attempt has been made to bring the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design into receding horizon control strategy in the seminal paper [121], by extending the LMI based robust MPC control algorithm [96] solved online.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design has its own significance in the control community since its inception in [14]. The multi-objective optimal control design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design motivates different approaches to solve the control problem. In general, the desire to achieve conflicting objectives of the desired closed-loop performance specifications and robustness against external disturbances or uncertainty due to plant-model mismatch and/or noises, can be better handled by mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design. Nash game based approach is a well known strategy to obtain an optimal equilibrium (Nash equilibrium) in such conflicting objective decision making problems. Thus the approach in [98] seems to be more appropriate for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design with the Nash game interpretation. Motivated by the work in [98], in the present work, the idea has been extended for the first time to the MPC paradigm. However, this extension is not straight forward, since in [98] the design has been made for continuous time systems. Furthermore, it is known from LQ dynamic game theory [58, 82] that such an approach results in cross-coupled Riccati equations as that in [98]. Although, the conversion from continuous to discrete domain is possible, in general, using bilinear transformation, to the best of the author's knowledge, there is no literature for such a conversion for cross-coupled Riccati equations resulting from linear-quadratic (LQ) Nash game into its equivalent discrete form. An LQ Nash game usually results in coupled non-symmetric Riccati equations which have to be solved to obtain the optimal solution or Nash equilibria. Nevertheless, the problem of solving non-symmetric Riccati equations is itself of much theoretical importance and it is still an active research area. A good survey of the methods to handle such non-symmetric Riccati equations can be found in [56, 57, 82]. The pair of discrete time non-symmetric cross-coupled algebraic Riccati equations (cAREs) obtained while solving an open-loop Nash game is dealt in [58]; where the invariant subspace method is adopted, for which necessary conditions are provided. A method of solving the cAREs is proposed in [58], where sufficient conditions are not provided and not elsewhere too to the best of the author's knowledge. Hence, obtaining a unique stabilizing solution cannot be ensured.

On the other hand, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control that are usually available in the literature [98, 37] have dealt with the conventional feedback control systems and not in the setting of MPC. However, the development in the present work is for the discrete time systems in the MPC framework. The present work differs from [121] by exploring the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC from a LQ game theory perspective and addressing its related issues. So once a two player LQ Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust controller for discrete time system has to be designed, a suitable means to solve the resulting discrete time non-symmetric coupled algebraic Riccati equations would be of much significance. One such suitable means, to obtain the optimal solution of the control problem, is dealt in the present work. In [82], a similar kind of problem is dealt as a soft constrained Nash game. However, the use of hard constraints, such as input actuator saturation constraints, which is quite common practically due to the actuator limitations, has not been dealt in [82]. Optimal control problem of linear systems subjected to hard constraints such as actuator constraints makes the control problem in hand nonlinear.

Optimal control for nonlinear systems are often designed as open-loop control framework. Achieving the required closed-loop stability and performance of non-linear systems make their control critical. Generally, in receding horizon control framework, which is an open-loop control strategy, finite horizon control input that satisfies the stability conditions is computed at every time instance. Only the first control value of the computed control law is implemented while the other values are discarded. However, on the other hand, infinite horizon closed-loop optimal control satisfies the closed-loop nominal stability. Hence, it is always preferred that an infinite horizon optimal control problem is solved online in order to ensure guaranteed stability. Owing to the resulting infinite-dimensionality, such infinite horizon control problem becomes computationally intractable. In contrast, the robust MPC design for saturated control input systems as in [96, 121] basically solves an infinite horizon control problem, where the feedback gain is calculated online at every time instance in the receding horizon framework. This results in a newly computed optimal state feedback gain matrix at every time instance. The location of the closed-loop state transition matrix poles in the stable region at every time instance ensures both closed-loop stability and optimal performance. Moreover, the feedback MPC, which satisfies the stability requirements implicitly, overcomes the dimensionality issues too, making it quite promising.

Solution of an infinite horizon two player non-zero sum open-loop LQ Nash game problem for robust control of discrete time systems results in a pair of cross coupled-algebraic Riccati equations (cAREs) [58]. In the present work, it is aimed to solve such a problem in the receding horizon framework. Hence, the cAREs resulting from this game will be solved online, along with some additional conditions and the updated current state of the system, to calculate an optimal control law consisting of time-varying linear feedback gain matrix at every time instance. The above receding horizon control strategy is hence termed as *Nash game approach to mixed* $\mathcal{H}_2/\mathcal{H}_\infty$ *model predictive (or receding horizon) control*. A similar approach of solving open-loop Nash game with available state information has been adopted in [147], which is called Sampled-data Nash game control. The present approach also works in a similar fashion but in a different control setting *i.e.*, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ based design. From engineering perspective, it is aimed to find a suitable means to get the solution, possibly with some minimal error, even if not for its exact solution, so as to bring the intended controller design into use.

3.2 Problem Formulation

3.2.1 Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC

Consider the discrete-time LTI system affected with process disturbance:

$$\Sigma: \quad x_{k+1} = Ax_k + B_1 w_k + B_2 u_k, \quad x(0) = x_0 \tag{3.1}$$

$$z_k = \begin{bmatrix} C_z x_k \\ D_{zu} u_k \end{bmatrix} \tag{3.2}$$

where, $x_k = [x_k^1, x_k^2, \cdots, x_k^n]^T \in \mathbb{R}^n$ is the system state, $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input (where $\mathcal{U}$ is a compact and convex set, containing the origin in its interior); w_k is an unknown disturbance contained inside a compact, bounded set,$w_k \in \mathbb{W}$. z_k denotes the controlled output. Also, A, B_1 and B_2 are matrices of appropriate dimensions.

Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design for a LTI system subject to process disturbance can be considered as a multi-objective optimal control problem; as against using *single* min-max objective function in the conventional $\mathcal{H}_\infty$ control problems [6, 7]. The maximizing and minimizing players of the Nash game (namely, w_k^* and u_k^*)[§] are assigned with their corresponding individual objective functions, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures, respectively, given below in equations (3.3) and (3.4) as,

$$J_2(u_k, w_k) := \frac{1}{2}\sum_{k=0}^{\infty}\left(x_k^T Q_1 x_k + u_k^T R u_k\right) \tag{3.3}$$

$$J_\infty(u_k, w_k) := \frac{1}{2}\sum_{k=0}^{\infty}\left(x_k^T Q_2 x_k - \gamma^2 w_k^T w_k\right) \tag{3.4}$$

where γ^2 is the upper bound of the worst case performance or the attenuation factor for the disturbance. Here $Q_i \succeq 0$ and $R \succ 0$, $i = 1, 2$. Note, since it is a two player game with conflicting objectives the state weighing matrices Q_1 and Q_2 need not be equal.

In the present work, the objective is to find minimal control effort u_k (minimizing player), within its prescribed constraint bound $u_k \in \mathcal{U}$, by minimizing its quadratic objective function ($\mathcal{H}_2$ problem), against the allowable process disturbance $w_k \in \mathbb{W}$ (maximizing player) which tries to maximize the system's response by maximizing its corresponding objective function ($\mathcal{H}_\infty$ problem). The upper bound of the allowable disturbance is given in terms of fixed γ^* for which the controller is designed, such that at least a suboptimal control ($\gamma \geq \gamma^*$) is achieved ($\mathcal{H}_\infty$ problem). Thus, it becomes a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ model predictive control problem with two-player Nash game interpretation.

The feedback MPC in the present work is adopted by repeatedly solving online an infinite horizon open-loop optimal control problem at every time instance, with the notion of feedback. A new steady-state optimal feedback control law is calculated for the corresponding current system's state information of the process. The objective

[§] f^* represents the optimal value of f.

function of the proposed MPC scheme is given in the following;

$$F(x_k,\, u_k,\, w_k) := \begin{cases} \min\limits_{u_k \in \mathcal{U}} \sum\limits_{i=0}^{\infty} \frac{1}{2}(x_{k+i|k}^T Q_1 x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k}) \\ \max\limits_{w_k \in \mathbb{W}} \sum\limits_{i=0}^{\infty} \frac{1}{2}(x_{k+i|k}^T Q_2 x_{k+i|k} - \gamma^2 w_{k+i|k}^T w_{k+i|k}) \end{cases} \quad \forall k \geq 0 \qquad (3.5)$$

Thus the above robust control problem becomes a two player Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ model predictive control. So the optimisation problem of the MPC is a multi-objective feedback min-max problem so as to find a minimal control value (u_k^*), in terms of the optimal feedback policy against that of the worst case disturbance, w_k^* (within a prescribed upper bound γ).

Before going into the details of solving the open-loop optimal control problem, the following required definition and assumption are in order, as they would be used in further derivations.

Definition 3.1 (Positive definite matrix) *In the general terminology positive definiteness is associated with symmetric matrices or Hermitian matrices. However, say a non-symmetric matrix $Z \in \mathbb{R}^{n\times n}$, could be rewritten as,*

$$Z = \underbrace{\frac{1}{2}(Z + Z^T)}_{Z_S} + \underbrace{\frac{1}{2}(Z - Z^T)}_{Z_K}$$

where, Z_S represents the symmetric part of Z and Z_K represents the skew-symmetric part of it. Then, the matrix Z is a positive definite if (and only if) its symmetric part (Z_S) is positive definite, such that $x^T Z x > 0,\ x \in \mathbb{R}^n \backslash 0$. So once the symmetric part of a real, square matrix is positive definite, then the matrix is positive definite. These matrices are called non-symmetric positive definite matrices (c.f. [78], Section 10.4) to be more specific. If the matrix Z is symmetric (i.e., $Z_K = 0$), then the results of symmetric positive definite matrices are recovered.

The term positive definiteness of a matrix is defined in this context in the present thesis, unless its associated symmetric property is stated explicitly.

3.3 Solving the $\mathcal{H}_2/\mathcal{H}_\infty$ problem

In the following two subsections the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ parts of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control are solved. For the infinite horizon open-loop problem, eqn. (3.5), to be solved at every time instance.

3.3.1 Solving the $\mathcal{H}_2$ problem - Minimizing Player

The Hamiltonian function (H_2) is given as,

$$\begin{aligned} H_2 = \min_{u_k \in \mathcal{U}} \sum_{i=0}^{\infty} \frac{1}{2} & \left(x_{k+i|k}^T Q_1 x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} \right) \\ & + \bar{p}_{k+1+i|k}^T \left[(A x_{k+i|k} + B_1 w_{k+i|k} + B_2 u_{k+i|k}) - x_{k+1+i|k} \right] \quad \forall k \geq 0 \end{aligned} \tag{3.6}$$

where $\bar{p}_{(\cdot)}$ is the Lagrange multiplier. To minimize the function H_2, we need to differentiate H_2 w.r.t. each component *viz.*, $x_{(\cdot)}$, $u_{(\cdot)}$ and $\bar{p}_{(\cdot)}$ and equate them to zero.

$$\frac{\partial H_2}{\partial x_{k+i|k}} = 0 : \quad Q_1 x_{k+i|k} + A^T \bar{p}_{k+1+i|k} - \bar{p}_{k+i|k} = 0 \tag{3.7}$$

$$\frac{\partial H_2}{\partial u_{k+i|k}} = 0 : \quad R u_{k+i|k} + B_2^T \bar{p}_{k+1+i|k} = 0 \tag{3.8}$$

$$\frac{\partial H_2}{\partial \bar{p}_{k+i|k}} = 0 : \quad A x_{k-1+i} + B_1 w_{k-1+i} + B_2 u_{k-1+i} - x_{k+i|k} = 0 \tag{3.9}$$

From equations (3.7) and (3.8) we have

$$\bar{p}_{k+i|k} = Q_1 x_{k+i|k} + A^T \bar{p}_{k+1+i|k} \tag{3.10}$$

$$u_{k+i|k} = -R^{-1} B_2^T \bar{p}_{k+1+i|k} \tag{3.11}$$

Assume that $\bar{p}_{(\cdot)}$ can be expressed in the form,

$$\bar{p}_{k+i|k} = P_{k+i|k}^I x_{k+i|k} \tag{3.12}$$

where $P_{(\cdot)}^I$ is an $n \times n$ real positive definite (Lyapunov) matrix. Using equation (3.12) into equation (3.11) gives

$$u_{k+i|k} = -R^{-1} B_2^T P_{k+1+i|k}^I x_{k+1+i|k} \tag{3.13}$$

3.3.2 Solving the $\mathcal{H}_\infty$ problem - Maximizing Player

The Hamiltonian function (H_∞) is given as,

$$H_\infty = \max_{w_k \in \mathbb{W}} \sum_{i=0}^{\infty} \frac{1}{2} \left(x_{k+i|k}^T Q_2 x_{k+i|k} \; - \; \gamma^2 w_{k+i|k}^T w_{k+i|k} \right) + \bar{\bar{p}}_{k+1+i|k}^T \left[(A x_{k+i|k} + B_1 w_{k+i|k} + B_2 u_{k+i|k}) - x_{k+1+i|k} \right] \; \forall k \geq 0 \quad (3.14)$$

where $\bar{\bar{p}}_{(\cdot)}$ is again a Lagrange multiplier as used in the $\mathcal{H}_2$ part of the problem.

Differentiating the Hamiltonian function (H_∞) w.r.t. each of its components, *viz.*, $x_{(\cdot)}$, $u_{(\cdot)}$ and $\bar{\bar{p}}_{(\cdot)}$ and equate them to zero.

$$\frac{\partial H_\infty}{\partial x_{k+i|k}} = 0 : \quad Q_2 x_{k+i|k} + A^T \bar{\bar{p}}_{k+1+i|k} - \bar{\bar{p}}_{k+i|k} = 0 \quad (3.15)$$

$$\frac{\partial H_\infty}{\partial w_{k+i|k}} = 0 : \quad -\gamma^2 w_{k+i|k} + B_1^T \bar{\bar{p}}_{k+1+i|k} = 0 \quad (3.16)$$

$$\frac{\partial H_\infty}{\partial \bar{\bar{p}}_{k+i|k}} = 0 : \quad A x_{k-1+i} + B_1 w_{k-1+i} + B_2 u_{k-1+i} - x_{k+i|k} = 0 \quad (3.17)$$

From equations (3.15) and (3.16) we have,

$$\bar{\bar{p}}_{k+i|k} = Q_2 x_{k+i|k} + A^T \bar{\bar{p}}_{k+1+i|k} \quad (3.18)$$

$$w_{k+i|k} = \gamma^{-2} B_1^T \bar{\bar{p}}_{k+1+i|k} \quad (3.19)$$

Again considering the similar assumption as in equation (3.12),

$$\bar{\bar{p}}_{k+i)|k} = P_{k+i|k}^{II} x_{k+i|k} \quad (3.20)$$

Using equation (3.20) in (3.19) we get,

$$\begin{aligned} w_{k+i|k} &= \gamma^{-2} B_1^T \bar{\bar{p}}_{k+1+i|k} \\ &= \gamma^{-2} B_1^T P_{k+1+i|k}^{II} x_{k+1+i|k} \end{aligned} \quad (3.21)$$

Now, from equations (3.13) and (3.21) we could write the system equation in (3.1) as

$$x_{k+1+i|k} = [I + B_2 R^{-1} B_2^T P_{k+1+i|k}^{I} - B_1 \gamma^{-2} B_1^T P_{k+1+i|k}^{II}]^{-1} A x_{k+i|k} := \Phi_{k+i|k} x_{k+i|k} \quad (3.22)$$

where $\Phi_{(\cdot)}$ is the non-singular closed-loop state transition matrix, if the inverse exists. Using the above closed loop equation again in equations (3.13) and (3.21), we get,

$$u^*_{k+i|k} = -R^{-1}B_2^T P^I_{k+1+i|k}\Phi_{k+i|k}x_{k+i|k} := K^u_{k+i|k}x_{k+i|k} \tag{3.23}$$

as the optimal control law and,

$$w^*_{k+i|k} = \gamma^{-2}B_1^T P^{II}_{k+1+i|k}\Phi_{k+i|k}x_{k+i|k} := K^w_{k+i|k}x_{k+i|k} \tag{3.24}$$

as the worst case disturbance affecting the plant, despite the control action.

By using equations (3.12) and (3.22) in (3.10) and equations (3.20) and (3.22) in (3.18), we obtain (after omitting the $x_{k|k}$ on both sides) the following pair of coupled algebraic Riccati equations†,

$$P^I_k = Q_1 + A^T P^I_k[I + B_2R^{-1}B_2^T P^I_k - B_1\gamma^{-2}B_1^T P^{II}_k]^{-1}A \tag{3.25}$$
$$P^{II}_k = Q_2 + A^T P^{II}_k[I + B_2R^{-1}B_2^T P^I_k - B_1\gamma^{-2}B_1^T P^{II}_k]^{-1}A \tag{3.26}$$

The corresponding optimal control input and the worst case disturbance for the given controller parameter triple $\{\gamma, Q_i, R\}_{i=1,2}$ are given in eqns.(3.23 & 3.24). Equations (3.25) and (3.26) are the non-symmetric coupled algebraic Riccati equations resulting from the two player Nash game robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control.

The optimal values of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cost functions are $\frac{1}{2}x^T_{k|k}P^I_k x_{k|k}$ and $\frac{1}{2}x^T_{k|k}P^{II}_k x_{k|k}$, respectively, which could be shown easily.

3.4 Saddle point solution

Due to the two-player game theory nature of the problem, there will exist a saddle point solution for this multi-objective control problem [82]. This saddle point solution, also called the *Nash equilibria*, for the two-player non-zero sum game is given by,

$$\frac{1}{2}x^T_{k|k}P^I_k x_{k|k} \geq \beta_k \geq \frac{1}{2}x^T_{k|k}P^{II}_k x_{k|k}, \quad \text{if } \| Q_1 \| \geq \| Q_2 \| \tag{3.27}$$
$$\text{or,} \quad \frac{1}{2}x^T_{k|k}P^I_k x_{k|k} \leq \beta_k \leq \frac{1}{2}x^T_{k|k}P^{II}_k x_{k|k}, \quad \text{if } \| Q_1 \| \leq \| Q_2 \| \tag{3.28}$$

†At steady state $x_{k|k} = x_{k+1|k}$ and hence $P^{(\cdot)}_k = P^{(\cdot)}_{k+1}$.

where β_k is the bound of the sub-optimal solution (saddle point) of the two performance functions of the LQ Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC. This bound emphasises the non-zero sum nature of the Nash game based control problem, which would be determined online, every time instant, for the admissible control input $u_k \in \mathcal{U}$, by the optimisation algorithm itself.

3.5 Terminal Condition

For the infinite horizon problem solved in the present work, the terminal conditions are as given below, even though they are not mentioned explicitly while solving the optimal control problem;

$$\lim_{i\to\infty} x_{k+i|k} = 0$$

when there is no disturbance acting on the system or the disturbance decays w.r.t. time. Therefore,

$$\lim_{i\to\infty} \{\bar{p}_{k+i|k},\, \bar{\bar{p}}_{k+i|k}\} \to 0$$

In the classical optimal control this is known as the *transversality condition*.

In the presence of the disturbance the system states cannot be steered to the origin. However the system states will lie inside the robust control invariant set (residual set) $\mathcal{X}_\infty$. A more detailed discussion on robust control invariant set and stability issues are discussed in Chapter 4.

Remark 3.1 *The solution matrices* (P_k^I, P_k^{II}) *of the cAREs are, in general, non-symmetric. However, to obtain the solution of the optimal control problem and to ensure the closed-loop stability according to Lyapunov stability criteria, it is enough if these matrices are positive definite as defined in Definition 3.1 above. Moreover, a positive definite function,* $V_k(x_k)$ *defined on some positive definite set, with monotonically decreasing property, such that, for,*

$$\begin{aligned} V_k(x_k) &:= x_{k|k}^T P_k x_{k|k} \le c_k,\ c_k > 0, \\ \Delta V_k(x_k) &:= x_{k+1|k}^T P_{k+1} x_{k+1|k} - x_{k|k}^T P_k x_{k|k} \le 0, \quad \forall k \ge 0, \end{aligned}$$

is a Lyapunov function (See Definition 2.2 of [12]), where $\Delta V_k(\cdot)$ denotes the Lyapunov difference w.r.t. time k. Existence of such non-symmetric Lyapunov candidate function with the corresponding non-symmetric stability domains for constrained control linear systems is already established in the literature (See [11, 12] and the references therein).

Remark 3.2 *As the control law is computed based on the present state of the system and is implemented only for present time instance, even while solving for an infinite horizon problem, hereforth, let us consider $x_k \equiv x_{k|k}$ and similarly $u_k \equiv u_{k|k}$ $\forall k \geq 0$.*

3.6 Solving the cross-coupled-AREs

Solution of the pair of cAREs to get the values of the Lyapunov function matrices, P_k^I and P_k^{II} would fetch the optimal (saddle point) solution of the Nash game approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC defined in eqn.(3.5). Here an infinite horizon is adopted for the following reasons - *(i)* nominal stability is guaranteed, *(ii)* the resulting coupled non-symmetric Riccati equations from the LQ game problem would be algebraic, which can be solved atleast by some numerical means[‡]. Freiling et al. [58], has given the necessary condition(s) that need to be satisfied, so as to get the solution pair of the cAREs. A special matrix is constructed in [58] from the cAREs given in equations (3.25) and (3.26) and the closed loop system equation (3.22), whose invariant subspace spanning generalised vectors form the solution of the cAREs.

Assuming that the matrix A is regular, the equations (3.25) and (3.26) and the equation (3.22) can be written using index transformation as,

$$\begin{pmatrix} \tilde{x}_k \\ \bar{p}_k \\ \bar{\bar{p}}_k \end{pmatrix} (q+1) = M \begin{pmatrix} \tilde{x}_k \\ \bar{p}_k \\ \bar{\bar{p}}_k \end{pmatrix} (q) \tag{3.29}$$

‡ Solving the coupled non-symmetric difference Riccati equations from a finite horizon LQ game, even numerically, is intractable.

where $q := N - k$, N is some finite horizon in future w.r.t. the current time instant k, $\tilde{x}_q = x(N+1-q)$. Let us define $S_1 := \gamma^{-2}B_1B_1^T$, $S_2 := B_2R^{-1}B_2^T$, and

$$M = \begin{pmatrix} A^{-1} & A^{-1}S_1 & A^{-1}S_2 \\ Q_1A^{-1} & A^T + Q_1A^{-1}S_1 & Q_1A^{-1}S_2 \\ Q_2A^{-1} & Q_2A^{-1}S_2 & A^T + Q_2A^{-1}S_1 \end{pmatrix} \quad (3.30)$$

Although the above construction is meant for finite horizon open-loop game, the asymptotic behaviour for an infinite horizon problem, resulting in cAREs, can be obtained as $q \to \infty$.

Theorem 3.1 *[58] (i) If* $\mathscr{S}(P^I, P^{II}) :=$ *span* $\begin{pmatrix} I \\ P^I \\ P^{II} \end{pmatrix} \subset \mathbb{R}^{3n\times n}$ *is an invariant subspace of* M *with* $det(I + S_1P^I + S_2P^{II}) \neq 0$ *then* $\begin{pmatrix} P^I \\ P^{II} \end{pmatrix}$ *is a solution of the cAREs (3.25) and (3.26).*

(ii) If $\begin{pmatrix} P^I \\ P^{II} \end{pmatrix} \in \mathbb{R}^{2n\times n}$ *is a solution of the cAREs, then* $\mathscr{S}(P^I, P^{II}) \subset \mathbb{R}^{3n\times n}$ *is an invariant subspace of* M*. Moreover, the inverse in the closed-loop matrix, i.e.,* Φ*, is the matrix of restriction of* M *to* $\mathscr{S}(P^I, P^{II})$ *w.r.t. the basis defined by the columns of* $\begin{pmatrix} I \\ P^I \\ P^{II} \end{pmatrix}$. □

Then the generalised eigenvectors of M give the solutions of the cAREs, such that; Let span$(v_1, \cdots, v_n)$ be an M-invariant subspace such that $\det(X) \neq 0$ for $\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} := (v_1, \cdots, v_n)$, then $\begin{pmatrix} P^I \\ P^{II} \end{pmatrix} := \begin{pmatrix} Y_1X^{-1} \\ Y_2X^{-1} \end{pmatrix}$ is a solution of the cAREs if $\det(I + S_1P^I + S_2P^{II}) \neq 0$.

However, in [58] only necessary conditions are given but not the sufficient conditions. Nevertheless, the choice of the appropriate generalized eigenvector which forms the solution of the cAREs is also not clear enough. However, in [59] the sufficient conditions are given by assuming $Q_1 = 0$ and for some $Q_0 \geq 0$ for continuous time system open-loop games. This makes the $3n \times 3n$ matrix M into a $4n \times 4n$ matrix. Nevertheless, solving the open-loop Nash game using the approach in [59] involves fixing many parameter matrices, which can be a cumbersome trial and error process, and the result, however, thus finally obtained is a symmetric solution.

There doesn't exist any solid analytical method for finding the solution of the cAREs, other than those given in [58, 82], which are not complete. However, bringing the cAREs into applicability at least by some means of numerical solvability is of more concern from a control engineer's perspective than finding the exact solution(s). Following propositions and the subsequent theorem will yield the numerical solution of the cAREs.

Let us denote the time-varying state feedback control law, with controller gain $K_k^u = R^{-1}B_2^T P_k^I \Phi_k$, as $\kappa_k(x_k) := \{K_k^u x_k | u_k = K_k^u x_k \in \mathcal{U}\}$.

Proposition 3.1 (Input constraint) *For the input constraint, $u_k \in \mathcal{U}$, if given in terms of the inequality bounds such that $u_{min} \leq u_k \leq u_{max}$ can be normalized to lie within the bound $-\bar{u} \leq \tilde{u}_k \leq +\bar{u}$ for the entire horizon. Then the input constraint condition in the optimisation problem is given as, $\|\tilde{u}_k\|_2^2 = \|\kappa_k(x_k)\|_2^2 \leq \bar{u}^2$.* □

The above input constraint condition is used in the optimisation problem while solving for the solution of the cAREs.

Proposition 3.2 *For an optimal controller using two-player LQ Nash games for the system given by (3.1) results in a pair of cross coupled algebraic Riccati equations as given in (3.25) and (3.26), to achieve a stable closed loop system using the control law (3.23). The stable and locally unique solution set $(P_k^{I^*},\ P_k^{II^*})$ for the cAREs is also achievable even if the cAREs satisfy*

$$0 = \epsilon_k^1 + Q_1 - P_k^I + A^T P_k^I [I + B_2 R^{-1} B_2^T P_k^I - B_1 \gamma^{-2} B_1^T P_k^{II}]^{-1} A \quad (3.31)$$

$$0 = \epsilon_k^2 + Q_2 - P_k^{II} + A^T P_k^{II} [I + B_2 R^{-1} B_2^T P_k^I - B_1 \gamma^{-2} B_1^T P_k^{II}]^{-1} A \quad (3.32)$$

for some solution set in the open-hyper balls of radius $\{\epsilon_k^1, \epsilon_k^2\} \in \mathbb{R}^{n\times n}$ *around and containing the origin.* □

Theorem 3.2 *For a system defined by* $\Sigma : (A, B_1, B_2)$*, with initial condition* x_0*, the two-player Nash game based mixed* $\mathcal{H}_2/\mathcal{H}_\infty$ *MPC yields an (sub-)optimal, stable and locally unique solution, for the pair of cAREs,*

$$0 = \epsilon_k^1 + Q_1 - P_k^I + A^T P_k^I [I + B_2 R^{-1} B_2^T P_k^I - B_1 \gamma^{-2} B_1^T P_k^{II}]^{-1} A \tag{3.33}$$

$$0 = \epsilon_k^2 + Q_2 - P_k^{II} + A^T P_k^{II} [I + B_2 R^{-1} B_2^T P_k^I - B_1 \gamma^{-2} B_1^T P_k^{II}]^{-1} A \tag{3.34}$$

satisfying the input constraint condition,

$$(\|\kappa_k(x_k)\|_2^2 - |\bar{u}|^2) + s_k^2 = 0 \tag{3.35}$$

where $s_k \in \mathbb{R}$ *and along with the saddle point optimal condition bounds (assuming* $\|Q_1\|_2 > \|Q_2\|_2$*),*

$$x_k^T P_k^I x_k \geq \beta_k \tag{3.36}$$

$$x_k^T P_k^{II} x_k \leq \beta_k \tag{3.37}$$

if it exists, such that $(P_k^{I*}, P_k^{II*}) \succ 0$*, for some open-hyper balls of radius* $\{\epsilon_k^1, \epsilon_k^2\} \in \mathbb{R}^{n\times n}$ *around and containing the origin, such that the closed loop system (3.22) is stable, using the control law (3.23), satisfying the input constraint,* $u_k \in \mathcal{U}, \forall k \geq 0$*.*

Proof Let P_k^{I*} and P_k^{II*} be the (sub-)optimal, stable and locally unique solution set of the cAREs, (3.33) and (3.34), satisfying the conditions (3.36) & (3.37) and $u_k \in \mathcal{U}$ (*i.e.*, satisfying equation (3.35)), with $x_0 \in \mathcal{X}_0$, for some open hyper balls of radius $\{\epsilon_k^{1^*}, \epsilon_k^{2^*}\} \in \mathbb{R}^{n\times n}$, respectively, in the linear space. Now assume that there exists another solution set, $\tilde{P}_k^{I*}$ and $\tilde{P}_k^{II*}$, which again satisfies (3.33) and (3.34) for some other hyper balls of radius $\{\tilde{\epsilon}_k^{1^*}, \tilde{\epsilon}_k^{2^*}\} \in \mathbb{R}^{n\times n}$, respectively, such that $\epsilon_k^{j^*} = \tilde{\epsilon}_k^{j^*} + \delta_j \in \mathbb{R}^{n\times n}$ and $\delta_j \ll \tilde{\epsilon}_k^{j^*}$ $j = 1, 2$.

Likewise, there may exist a different solution set for a different choice of $\delta_{(.)}$ for every other choice of solution set. Therefore, as $|\delta_j| \to 0$, in the linear space the open hyper balls converges to the (sub-)optimal solution,

$$\lim_{|\delta_j|\to 0} {\epsilon^{j*}}_k = \tilde{\epsilon}_k^{j^*} + \delta_j = {\tilde{\epsilon}^{j*}}_k, \quad j = 1, 2.$$

which in turn gives $P_k^{I*} = \tilde{P}_k^{I*}$ and $P_k^{II*} = \tilde{P}_k^{II*}$, thus giving an (sub-)optimal, stable and locally unique solution set to the cAREs. ■

Considering that there exists an invariant set Ω and the input constraint set $\mathcal{U}$ for the system Σ, the following definitions are in order:

Definition 3.2 (Domain of Attraction/Performance) *Given a feedback control law* $u_k : \mathbb{R}^n \to \mathbb{R}^m$*, let*

$$\mathcal{X} \triangleq \{x_k \in \Omega \mid u_k \in \mathcal{U}\} \tag{3.38}$$

be the domain of attraction/performance of the system.

Definition 3.3 (Initial condition set) *The set of initial states* x_0 *for which an admissible control policy exists is defined as*

$$\mathcal{X}_0 := \left\{x_0 \in \mathcal{X} \subset \mathbb{R}^n \middle| \, \mathcal{U} \neq \emptyset\right\} \tag{3.39}$$

Also note that, in the present case it is assumed that $\mathcal{X}_0 \subseteq \mathcal{X}$

Corollary 3.1 *An (sub-)optimal stabilizing controller (3.23), such that* $u_k \in \mathcal{U}, \forall k \geq 0$*, for any given* $x_0 \in \mathcal{X}_0$ *and* $x_k \in \mathcal{X}, \forall k > 0$*, exists, for the given controller parameter* $\{\gamma, Q_i, R\}_{i=1,2}$*, iff the locally unique, optimal matrix solution pair of the cAREs* (P_k^{I*}, P_k^{II*}) *are positive definite.*

Note, that the non-symmetric coupled algebraic Riccati equations, in general, will have non-symmetric matrices as their solutions. However, by relaxing the equality of the cAREs and with the definition of positive definite matrix (c.f. Definition 3.1) solutions, the slack variables $\{\epsilon_k^1, \epsilon_k^2\} \in \mathbb{R}^{n\times n}$ will share the burden of the non-symmetricalness of the exact solution matrices. As one of the objectives of the present work is to find some suitable numerical means to bring the cAREs resulting from LQ Nash games into applicability from an engineer's perspective rather than to look for an exact solution, this relaxation method is adopted in the present work. Moreover, from the results of [59], although they are for continuous domain case, the results in the present case are also expected to be symmetric. However, it is a known fact that this relaxation technique will only give a sub-optimal solution. Due to the calculation of a new solution at every time instant in a receding horizon fashion, even though the solution is sub-optimal, the existence of a feasible stabilisable solution is sufficient to ensure stability *i.e.*, feasibility implies stability, (c.f. [140]), when the sub-optimal solution is nearer to the optimal solution. Furthermore, as any non-singular symmetric matrix can be diagonalised, without losing the matrix property, in the present task of solving the cAREs, the solution matrices are forcefully taken as diagonal matrices. On the other hand, considering the solution matrices as diagonal matrices, the number of decision variables (entries of the matrix) reduces from $\frac{n(n+1)}{2}$ to simply n for a $n \times n$ symmetric matrix. Thus this approach ultimately yields the result as desired *i.e.*, performance of the closed loop both in terms of its transient response and stability. That is, by using a LQ-Nash game approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC, the closed-loop Lyapunov stability with monotonically decreasing values of cost function is satisfied, if positive definite matrices exists by solving equations (3.33)-(3.37).

The algorithm used to solve the Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC is given below.

Algorithm

1. Measure the system states x_k

2. $$\mathbf{vec}\begin{pmatrix} \text{Equation(3.25)} \\ \text{Equation(3.26)} \\ (\| - R^{-1}B_2^T P^I \Phi_k x_k\|_2^2 - \bar{u}^2) + s_k^2 \\ \frac{1}{2}x_k^T P_k^I x_k - \beta_k \\ \beta_k - \frac{1}{2}x_k^T P_k^{II} x_k \end{pmatrix} = \mathbf{vec}(\mathbf{0})$$

3. Obtain the solution matrices (P^I, P^{II}) by minimising $\epsilon_k^1, \epsilon_k^2, s_k, \beta_k$.

4. Compute the value of u_k from equation (3.23) and apply it to the process and model.

5. Goto step 1.

Remark 3.3 *When proportional state weighing matrices are used i.e.,*$Q_1 = \alpha Q_2$*,* $\alpha > 0$*, the two algebraic Riccati equations in (3.25) and (3.26) reduce into a single Riccati equation (c.f. [1], [2]) resulting in a symmetric solution matrix.*

Remark 3.4 *The choice of the two state weighing matrices, which offers additional degree of freedom in the controller design, is a criterion that can be tuned according to the designer's need on the closed-loop performance, such that for the given choice of* $\{\gamma,\ Q_i,\ R\}_{i=1,2}$ *the closed-loop system is stable. In general, with several numerical experiments, it is found that* $\|Q_1\|_2 > \|Q_2\|_2$ *gives smooth closed-loop transient response and* $\|Q_2\|_2 > \|Q_1\|_2$ *gives aggressive transient response.*

In the present work, an open-loop Nash game is solved online, subject to pointwise constraints, at every time instant with the new process measurement and with the notion of feedback. Thus, when such an open-loop constrained optimal control problem is solvable at every time instance for the desired performance measures, the generally used MPC approach of approximating the infinite horizon problem as a finite horizon control problem along with the suitable terminal conditions to ensure stability and robustness, as described in Section 1.2.2, is not required. In the present work, the entire domain of performance acts as the terminal set of regular MPC schemes (c.f. Chapter 4). This relaxes the designer from the burden of enlarging the domain of attraction of the terminal constraint set associated with the local terminal controller and/or the computation of the optimal length of control horizon of the MPC design.

3.7 Numerical Examples

To illustrate the efficacy of the proposed Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC against the control techniques proposed in [121], few numerical examples are considered. The closed loop performance analysis has been carried out for these systems using two types of controller algorithms - Controller-A: Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC proposed in this thesis, Controller-B: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC proposed by Orukpe et al. [121]. The present NGM-MPC robust control problem, given in equations (3.33)-(3.37), is solved by Gauss-Newton method, using the MATLAB's Optimization Toolbox command `fsolve`. In all the cases, the system is subjected to a constant (bounded) disturbance, $w_k = 0.1$, $\forall k \geq 0$.

Example 3.1 In the first example, simple first order system has been considered, that had been used in [121].

$$A = 1 \quad B_1 = 1 \quad B_2 = 1 \quad C_z = 1; \quad D_{zu} = 1$$

The γ^2 value is taken as 2.12 and initial condition $x_0 = 1$, as used in [121], with $|u_k| \leq 1$. For controller A, the values taken are $Q_1 = 3$, $Q_2 = 2$ and $R = 1$. The comparative performance of all these controllers are shown in Figure 3.1. Time-integrated performance criteria of both these controllers have been quantified using Integral Absolute Error (IAE) values. The IAE values of closed loop response under the effect of controllers A and B are 1.9368 and 2.0720, respectively. From the simulation results we could understand that the proposed method is marginally better than [121].

However the chosen system is too simple to assess the true competence of the controller A, as suggested in this thesis. More complex processes are studied to realise the effectiveness of controller A.

The next examples are furnished with an intention to show that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust MPC proposed in the present work is less conservative and gives a better performance than that of [121].

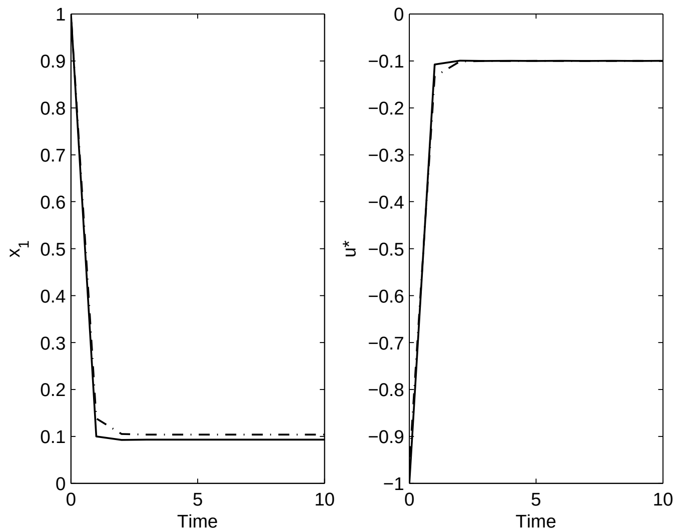

Figure 3.1: Comparison of the proposed controller A (solid) against the controller B (dash-dot) for the example used in [121].

Example 3.2 A second order stable system, shown below, with initial condition $x_0 = [1\ 0]^T$, subject to an input constraint of $|u_k| \leq 0.2$ is taken for this purpose. The minimal workable γ^2 value for which the robust MPC algorithms used in the present work and [121] are 3.0 and 24.5 respectively.

$$A = \begin{bmatrix} 0.3500 & 0.2300 \\ 1.0000 & 0.0000 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_z = [\sqrt{10}\ \ 0] \ \ D_{zu} = 1$$

The sampling time of the system is 0.1 sec and the weighing matrices are chosen as $Q_1 = \text{diag}\{7,\ 1\}$, $Q_2 = \text{diag}\{5,\ 2\}$ and $R = I$. From the closed loop performance of controllers A and B (See Figure 3.2) and their corresponding IAE values of 4.1086 and 5.8697, respectively, Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC (controller A) gives a better performance and is less conservative than the other controller.

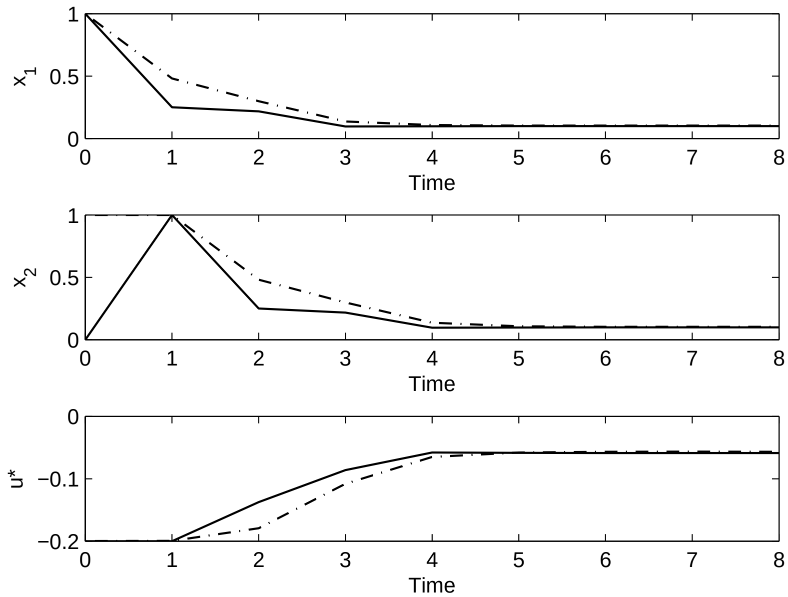

Figure 3.2: Comparison of performance of the controller A (solid) against the controllers B (dash-dot) for stable system of Example 3.2.

Example 3.3 The discrete time state-space model of a still more challenging system is taken *i.e.*, unstable system, with sampling time of 0.1 sec,

$$A = \begin{bmatrix} 2.2000 & -1.2000 \\ 1.0000 & 0.0000 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C_z = [\sqrt{10} \;\; 0]; \;\; D_{zu} = 1$$

The system has an initial condition of $x_0 = [0.2 \;\; 0]^T$, with a tight input constraint of $|u_k| \leq 0.3$. With $\gamma^2 = 5$ and 46.5, for controllers A and B, respectively, (which are their minimal workable values). For controller A, $Q_1 = \text{diag}\{10, 10\}$, $Q_2 = \text{diag}\{5, 1\}$ and $R = 1$. The performance of Nash game approach is compared against Orukpe et al.'s [121] algorithm and the results are furnished in Figure 3.3. It could be well understood that the present algorithm is again less conservative and gives a better response than the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust MPC proposed in [121], with their IAE values of 2.6521 and 3.8755, respectively.

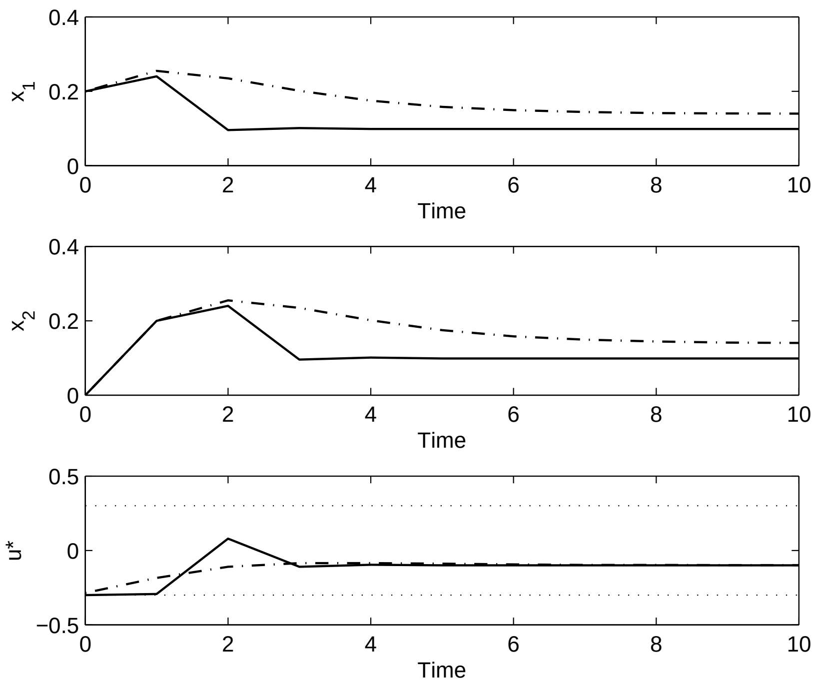

Figure 3.3: Comparison of performance of the controller A (solid) against the controller B (dash-dot) for unstable system of Example 3.3.

Chapter 4

Stability and Robustness of Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC Designs

This chapter is a note on the stability and robustness of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Model Predictive Controllers for linear state feedback systems addressed in chapter 3. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC proposed in Chapter 3 and that developed in [121] are compared in this chapter. The issues of stability and robustness of the multi-criterion optimal control are dealt in this chapter using set theoretic concepts.

4.1 Introduction

The art of analysis and synthesis of controller design using set theoretic ideas has been in practice for many years (c.f. [16, 25, 26, 45, 48, 71, 90, 127] and the references therein). It is proven to be a more elegant way of dealing with robust control design [26, 127]. It easily incorporates the state and/or output convergence results both in the framework of unconstrained [16, 25, 48] and constrained cases [26, 71, 90]. The core idea lies in the nature of the sets taken under investigation, such as invariance, boundedness, compactness and convexity. With various concepts of system dynamics and set theory, the whole realization of the controller design is easily and efficiently envisaged using set theoretic ideas.

Steering the system states to a specified target/final set, even while the system is subjected to input constraints and bounded-but-unknown disturbance has been an interesting control problem in the modern control literature. The reachability and controllability of such control problems in the perspective of set theory dates back to 1960's [48, 161]. The corresponding robust control concepts were realized both in terms of reachability of target sets and target tubes in [16]. The maximal constrained sets with constraints on both control input and states for discrete-time linear dynamic systems were treated in [45, 71, 72], with an algorithm to find the same [72]. The target set was said to be "strongly reachable" from a given state if an admissible control satisfying the constraints exists.

The theory and computation of the maximal admissible sets for linear system with both state and control constraints were given in [62], for output feedback case. One of the milestones in the applicability of set theoretic concepts for robust control problems appeared in the paper [23], that influenced the celebrated stability concepts of Lyapunov functions with set theoretic ideas. Later [26] gave a thorough discussion on these concepts. In [90], the set invariance property has been extended to model predictive control strategy. Computation of the minimal robust invariant set and maximal robust invariant sets were of much interest in the recent past [127]. Finding such robust invariant sets and reachable sets *a priori* give the designer the necessary information about the possibility of designing a suitable robust and optimal controller. Nevertheless, these computations are limited to an approximation (both from inner and outer) to the actual sets with a possibly minimal error.

The scope of this thesis, however, is limited to conceptually extend the set theoretic ideas available in the literature to analyze (but not to synthesize) the newly developed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC designs. It is a fact worth mentioning that the synthesis of the robust controller, furnished in Chapter 3, is not based on the set theoretic concepts explicitly; but it could easily be understood with better insight in the language of set theory for its robustness, stability and convergence, which is the basic intention of this chapter. The numerical means of solving the non-symmetric coupled Riccati equations by relaxation and its effect on the controller design and/or performance are extended in the following sections using set theory. Saturation constraint, which is mostly associated with actuator design constraint, along with the presence of unknown-but-bounded disturbance limits the size of the maximal invariant set of the closed loop system w.r.t. the nominal system. Furthermore, the demand on the controller to incorporate larger disturbance, makes the control problem much more complex. Enlarging the domain of attraction for such actuator saturation control problem is dealt in great deal in [80]. In [80], the designed controllers are of linear time-invariant type. A problem of this kind is of much importance in receding horizon framework too. However, the treatment in this thesis is restricted specifically to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC designs.

4.2 Preliminaries

Before going into the details of the set invariance analysis for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC designs, some basic notations and preliminary definitions are in order for the sake of completeness and self-containment. For a thorough treatment of such concepts refer [26, 84, 90] and the references therein.

The discussions in this chapter assumes the following open-loop discrete-time dynamic system:

$$\Sigma \quad : \quad x_{k+1} = f(x_k, u_k, w_k) \tag{4.1}$$

where $k \in \mathbb{Z}^+$, x_k is the system state, u_k is the control input and

$$w_k \in \mathbb{W} \subset \mathbb{R}^d \tag{4.2}$$

is some unknown, bounded disturbance. The system is subject to pointwise-in-time constraints on the control input:

$$u_k \in \mathbb{U} \subset \mathbb{R}^m \tag{4.3}$$

Let the set $\mathbb{X}$, such that $x_k \in \mathbb{X} \subset \mathbb{R}^n$ be the mapping of the corresponding control constraint set $\mathbb{U}$ onto the state-space such that $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to \mathbb{X}$, assuming that exact state measurement is available. The set $\mathbb{X}$ is compact, while $\mathbb{U}$ and $\mathbb{W}$ are closed too. Hereforth, the constraint on the control input is assumed to be time-invariant.

The disturbance sequence is said to be *allowable disturbance* if it is contained in $\mathbb{W}$ and the *control input*, *sequence* or *law* is said to be an *allowable control*, if it satisfies the input constraints $\mathbb{U}$. The set of all states $x_k \in \mathbb{X}$ are the *admissible states* for the given system.

For a perturbed system, the following set invariance definitions will be useful, which are in order.

4.2.1 Invariant Sets

Given a set Ω and an initial state $x_0 \in \Omega \subseteq \mathbb{X}$ for a dynamic system, it is of interest to determine whether the unfolding states of the system, $x_{k+1} = f(x_k, w_k)$, whose solution is given as $\phi_k(x_k, w_k)$ remains inside the set $\mathbb{X}$ for all time.

Definition 4.1 (Positive Invariant set) *The set $\Omega \subset \mathbb{R}^n$ is positively invariant for the system $x_{k+1} = f(x_k)$ iff $\forall x_0 \in \Omega$ the system states satisfy $x_k \in \Omega$, $\forall k \in \mathbb{Z}^+$.*

Definition 4.2 (Robust Positively Invariant set) *[90] The set $\Omega \subset \mathbb{R}^n$ is robust positively invariant (RPI) for the system $x_{k+1} = f(x_k, w_k)$ iff $\forall x_0 \in \Omega$ the system states satisfy $x_k \in \Omega$, $\forall w_k \in \mathbb{W}$, $\forall k \in \mathbb{Z}^+$.*

The following are the immediate results of the above definition.

Proposition 4.1 *The union of two or more RPI sets is also an RPI set.* □

Remark 4.1 *A similar statement as above of RPI set cannot be made about the intersection of RPI sets, even for the case when the disturbance is absent.*

Definition 4.3 (Robust Control Invariant set) *For a system of the form (4.1), a set $\Omega \subset \mathbb{R}^n$ is robust control invariant (RCI) set, iff $\forall x_0 \in \Omega$ there exits a control input (or a feedback control law, $u_k := \kappa(x_k)$) $u_k \in \mathbb{U}$ such that $x_{k+1} = f(x_k, u_k) \oplus \mathbb{W} \in \Omega$.*

Definition 4.4 (Maximal Robust Control Invariant set) *[90] The set $\mathcal{C}_\infty(\Omega)$ is the maximal robust control invariant (MRCI) set contained in Ω for the system of the form (4.1) iff $\mathcal{C}_\infty(\Omega)$ is RCI and contains all the RCI sets contained in Ω.*

For the RCI set of the system (4.1) and the bounded disturbance $\mathbb{W}$, any control law $\kappa(x_k) : \Omega \rightarrow \mathbb{U}$, given as

$$\kappa(x_k) = \{u_k \in \mathbb{U} \mid f(x_k, u_k) \oplus \mathbb{W} \subseteq \Omega, \ \forall x_k \in \Omega\} \tag{4.4}$$

ensures the construction of the closed-loop system such that $x_{k+1} = f(x_k, \kappa(x_k)) \oplus \mathbb{W} \subseteq \Omega, \ \forall x_k \in \Omega$.

Definition 4.5 (Constraint-admissible set) *The set $\Omega \subset \mathbb{R}^n$ is a constraint-admissible set if it is contained in $\mathbb{X} \subset \mathbb{R}^n$.*

Remark 4.2 *The set Ω is a constraint-admissible RPI set if Ω is RPI and is contained in $\mathbb{X}$.*

There are two other important positive invariant sets, which are worth mentioning and will be useful in the subsequent discussions. They are *minimal-robust positive invariant set* and *maximal-robust positive invariant set*. The following definitions are consistent with the references [90, 127].

Definition 4.6 (m-RPI set) *[127] A constraint-admissible RPI set, $\mathcal{X}_\infty$, of the system $x_{k+1} = f(x_k, w_k)$ is called its minimal robust positive invariant (m-RPI) set if it is contained in every closed RPI set of the system.*

Definition 4.7 (M-RPI set) *[127] A constraint-admissible RPI set, $\mathcal{O}_\infty$, of the system $x_{k+1} = f(x_k, w_k)$ is called its maximal robust positive invariant (M-RPI) set, if it contains every constraint-admissible RPI set of the system.*

Furthermore, it is of natural interest to determine which subset of the given set is compatible with the input constraints.

Definition 4.8 (Input admissible set) *For a given control law $u_k = \kappa(x_k)$, the input admissible set (i.e., subset) of $\Omega \subseteq \mathbb{R}^n$ is given by,*

$$\mathcal{X} \triangleq \{x_k \in \Omega \mid \kappa(x_k) \in \mathbb{U}\}.$$

It should be noted that if the input constraint $\mathbb{U}$ is given as a hyper-rectangle and the control law is given by an appropriate saturation function, sat$(\cdot)$, such that $u_k =$sat$(\kappa_k(x_k))$, as is often the case in most practical processes, then $\mathcal{X} = \Omega$.

Definition 4.9 (Feasible Input Set) *For a given x_k, that satisfies eqns. (4.1-4.3) and Definition 4.3 and 4.8, the set of all feasible control inputs is defined as*

$$\mathcal{U} := \left\{u_k \in \mathbb{U} \mid x_k \in \mathcal{X},\ w_k \in \mathbb{W},\ \forall k \geq 0\right\}$$

Remark 4.3 *It should be noted that feasible input set $\mathcal{U}$ differs from the admissible input $\mathbb{U}$, such that, in general, $\mathcal{U} \subseteq \mathbb{U}$, $\forall x_k \in \mathcal{X}$, i.e., the feasible input set $\mathcal{U}$ is the subset of admissible input set $\mathbb{U}$ such that $x_k \in \mathcal{X}$, $\forall k \geq 0$ and the control input satisfies the given input constraint for any disturbance within the bounded set, $\mathbb{W}$, for closed-loop system $x_{k+1} = f(x_k, u_k) \oplus \mathbb{W}$.*

Definition 4.10 (Feasible Initial condition set) *The set of all initial states x_0, for which a feasible control policy exists is defined as Feasible Initial condition set, i.e.,*

$$\mathcal{X}_0 := \left\{x_0 \in \mathcal{X} \subset \mathbb{R}^n \middle| \mathcal{U} \neq \emptyset\right\}. \tag{4.5}$$

For a system perturbed with some persistent arbitrary but bounded disturbance as given in eqn.(4.1), it is obvious that the system states cannot be steered to the origin. However, the states can only be steered to a target set in the neighbourhood of the origin. The problem is finding a control law such that the system states reach a target set in a finite number of steps, despite the disturbances on the state, is fundamentally a problem of finding the *robust controllable sets*.

Definition 4.11 (Robust Controllable set) *[90] The i-step robust controllable set $\mathcal{Q}_i(\Omega, \mathcal{T})$ is the largest set of states in Ω for which there exists an admissible* *"time-varying" state feedback control law such that an arbitrary terminal set $\mathcal{T} \subset \mathbb{R}^n$ is reached in "exactly" i-steps, while keeping the evolution of the system states inside Ω for the first i-1 steps, for all allowable disturbances, i.e.,*

$$\mathcal{Q}_i(\Omega, \mathcal{T}) \triangleq \Big\{ x_0 \in \mathcal{X}_0 \subset \mathbb{R}^n |\ \exists \{u_k := \kappa_k(x_k) \in \mathcal{U}\}_0^{i-1} : \{x_k \in \Omega\}_0^{i-1}, x_i \in \mathcal{T},\ \forall \{w_k \in \mathbb{W}\}_0^{i-1} \Big\} \tag{4.6}$$

It is of interest to find the maximum possible set of initial states for which a time-varying feedback control law will exist, such that the closed-loop system states reach a target set for all admissible disturbance. However, the above definition may give a more conservative problem of finding set of states if an open-loop *sequence of control inputs* drive the system states to the target set, irrespective of the disturbance sequence, given as

$$\mathcal{Q}_i^{ol}(\Omega, \mathcal{T}) \triangleq \Big\{ x_0 \in \mathcal{X}_0 \subset \mathbb{R}^n |\ \exists \{u_k \in \mathcal{U}\}_0^{i-1} \ :\ \{x_k \in \Omega\}_0^{i-1},\ x_i \in \mathcal{T},\ \forall \{w_k \in \mathbb{W}\}_0^{i-1} \Big\}. \tag{4.7}$$

The inclusion of the constraint that the control input be dependent on the state as well as time makes the fundamental difference between the feedback and open-loop robust MPC design (c.f. Section 1.3.2). Interestingly, in the absence of any disturbance $\mathcal{Q}_i(\Omega, \mathcal{T}) = \mathcal{Q}_i^{ol}(\Omega, \mathcal{T})$.

Remark 4.4 *It should be noted that if the target set $\mathcal{T}$ is RCI, then a time-invariant feedback control law will ensure that the states lie inside $\mathcal{T}$ for all time, after the states reach $\mathcal{T}$ in i-steps. Moreover, by definitions (4.2) and (4.3), if a time-invariant control law is chosen then $\mathcal{T}$ will be a RPI set, such that the closed-loop states enter $\mathcal{T}$ in exactly i-steps.*

Definition 4.12 (Robust Stabilisable Set) *[90] The set $\mathcal{S}_k(\Omega, \mathcal{T})$ is robust stabilisable set contained in Ω for the perturbed system $x_{k+1} = f(x_k, \kappa_k(x_k)) + w_k,\ \forall w_k \in \mathbb{W}$ iff $\mathcal{T}$ is a RCI subset of Ω and $\mathcal{S}_k(\Omega, \mathcal{T})$ contains all the states in Ω for which there exists an admissible time-varying feedback control law $\kappa_k(x_k) \in \mathcal{U}$ which will* *drive the system states to $\mathcal{T}$ in i steps or less, while keeping the evolution of the state inside Ω for all allowable disturbance sequences, i.e.,*

$$\mathcal{S}_k(\Omega, \mathcal{T}) \triangleq \{x_0 \in \mathcal{X}_0 \subset \mathbb{R}^n | \; \exists u_k := \kappa_k(x_k) \in \mathcal{U}, x_k \in \Omega : x_{\bar{k}} \in \mathcal{T} \subset \Omega, \; \forall \bar{k} \geq i\} \quad (4.8)$$

The difference between robust *stabilisable set* and robust *controllable set* arise w.r.t. the nature of the target set. In many control problems, the target set is either a robust control invariant set or a robust positively invariant set for a Lyapunov-stable closed-loop system. If the initial state is contained inside a robust stabilisable set then a control law guaranteeing that the target set $\mathcal{T}$ is reached by the closed-loop system states in a finite number of steps can be designed. Once the states are inside the target set one can switch to a time-invariant Lyapunov-stable controller, as used in dual-mode MPC design [113].

Definition 4.13 (Maximal Robust Stabilisable Set) *[90] The set $\mathcal{S}_\infty(\Omega, \mathcal{T})$ is the maximal robust stabilisable set contained in Ω for the system $x_{k+1} = f(x_k, \kappa_k(x_k)) + w_k$ iff $\mathcal{S}_\infty(\Omega, \mathcal{T})$ is the union of all i-step robust stabilisable sets contained in Ω.*

Remark 4.5 *From the above discussions, it can be understood that the largest possible region of attraction to the target set is equal to the maximal robust stabilisable set.*

In general, the maximal robust stabilisable set $\mathcal{S}_\infty(\Omega, \mathcal{T})$ is not equal to the maximal control invariant set $\mathcal{C}_\infty(\Omega)$, even for linear systems. $\mathcal{S}_\infty(\Omega, \mathcal{T}) \subseteq \mathcal{C}_\infty(\Omega)$ for all RCI set $\mathcal{T}$. The set $\mathcal{C}_\infty(\Omega) \setminus \mathcal{S}_\infty(\Omega, \mathcal{T})$ includes all the initial states from which it is not possible to robustly steer the system states to the stabilisable region $\mathcal{S}_\infty(\Omega, \mathcal{T})$ (and hence to $\mathcal{T}$) [90].

Definition 4.14 (Optimally Feasible set of States) *The feasible set of states of the optimal control problem, with the optimal control law $u_k = \kappa_k(x_k) \in \mathcal{U}$, is defined as*

$$\mathcal{X}_F := \left\{x_k \in \mathcal{S}_\infty(\Omega, \mathcal{T}) | \exists \kappa_k : \mathcal{S}_\infty(\cdot) \to \mathcal{U}\right\}.$$

Remark 4.6 *It should be noted that the set of all feasible states may differ from the maximal robust stabilisable set,* $\mathcal{S}_\infty$*. In general,* $x_k \in \mathcal{X}_F \subseteq \mathcal{S}_\infty$, $\forall k \geq 0$.

With the above definitions on the feasible states and control inputs, the proposition given below follows immediately.

Proposition 4.2 *In the design of robust feedback MPC for the discrete-time dynamic system given in eqns. (4.1)-(4.3), the following statements are equivalent;*

1. *when the system states lie in the feasible set as* $x_k \in \mathcal{X}_F$*, then there exists a control law* $(u_k := \kappa_k(x_k))$*, such that* $u_k \in \mathcal{U}$*, for the given state measurement.*
2. *when the control input* u_k *for the given state measurement is feasible,* $u_k \in \mathcal{U}$*, then it implies that the states lie in the feasible set,* $x_k \in \mathcal{X}_F$*.* □

With these set theoretic properties in hand, the concepts of Lyapunov theory is briefly revised to subsequently make the proper relation of the set theoretic ideas in the light of Lyapunov function to establish the stability and robustness of the closed-loop system using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC designs.

4.2.2 Lyapunov theory

Lyapunov theory is fundamentally based on the concept of energy and energy-dissipation of a dynamic system. The core idea is that an equilibrium point of a dynamic system is said to be (locally) *stable* if the Lyapunov function is a local minimum energy function of the system and the system dissipates energy w.r.t. time. To satisfy stability condition, the corresponding Lyapunov function needs to be positive definite. The very natural question that would immediately rise is that : Is there any relation between positive definiteness of a Lyapunov function and a positive invariant set? The answer could not be given in general terms. To be specific, for certain class of systems such as those with linear and affine dynamics, when there is a positive invariant set for the dynamic system, then a Lyapunov function could be derived. However, it is not true for certain systems such as positive systems.

The relationship between Lyapunov function and positive invariant sets for dynamic systems are discussed in great detail in [26]. Although, the contents of this section are consistent with that of [26], the present section is not to duplicate the results given in [26] but to furnish the required details in brief for the sake of completeness and for further discussions in the later part of the chapter.

For a system given in the form

$$\Sigma \ : \quad x_{k+1} = f(x_k, u_k, w_k), \qquad k \geq 0 \tag{4.9}$$

where the function $f(\cdot)$ is continuous, let $\mathcal{V}(\mathcal{X})$ denote the set of all C^1 functions $V_k(x_k) : \mathcal{X} \times \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ such that there exists class $\mathcal{K}_\infty$ functions‡ α_1 and α_2 satisfying

$$\alpha_1(\|x_k\|) \leq V_k(x_k) \leq \alpha_2(\|x_k\|) \tag{4.10}$$

for all $x_k \in \mathcal{X},\ k \in \mathbb{Z}^+$. Then the set $\mathcal{V}(\mathcal{X})$ is the set of all candidate Lyapunov functions for testing the robust stability of the system Σ [55]. The set $\mathcal{V}(\mathcal{X})$ is contained in the broader set $\mathcal{F}(\mathcal{X})$ of functions which need not be differentiable or radially unbounded. Given $V_k(x_k) \in \mathcal{V}(\mathcal{X})$ and the system Σ then the Lyapunov derivative $\Delta V_k := V_{k+1}(x_{k+1}) - V_k(x_k)$ is assumed to be continuous and it follows from eqn. (4.10) that $\Delta V_{(\cdot)}(0) \equiv 0$. Let us next define Lipschitz continuity - a smoothness condition stronger than the regular continuity, which will be used in the rest of the thesis.

Definition 4.15 (Lipschitz Continuity) *Given an open set $O \subseteq \mathbb{R}^l$, we say that a function $f:\ \mathbb{R}^l \rightarrow \mathbb{R}^l$ is Lipschitz-continuous on the open subset O if there exists a constant $L \in \mathbb{R}^+$ (called the Lipschitz constant of f on O) such that*

$$\|f(\alpha) - f(\beta)\| \leq L\|\alpha - \beta\|, \ \ \alpha, \beta \in O.$$

‡A continuous function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be a $\mathcal{K}$ function, if it is continuous, strictly increasing and $\varphi(0) = 0$ [26].

A function φ is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded.[118]

Definition 4.16 (Global uniform stability) *[26] The system Σ is said to be Globally Uniformly Asymptotically Stable, if it is :*

Locally Stable: For all $\nu > 0$ there exists $\delta > 0$ such that if $\|x_0\| \leq \delta$ then

$$\|x_k\| \leq \nu, \ \ \forall k \geq 0 \ \ \textit{and} \ \ w_k \in \mathbb{W} \tag{4.11}$$

Globally Attractive: For all $\nu > 0$ and $\delta > 0$ there exists some $\tau_{\nu,\delta} > 0$ such that if $\|x_0\| \leq \delta$ then

$$\|x_k\| \leq \nu, \ \ \forall k \geq \tau_{\nu,\delta} \ \ \textit{and} \ \ w_k \in \mathbb{W}. \tag{4.12}$$

The corresponding Global Lyapunov function would be a locally Lipschitz positive definite function. However for a practical control system design, the requirement of a global stability is too ambitious, as,

1. it cannot be expected to have convergence for any arbitrary initial conditions, and
2. in presence of disturbance, if persistent for all $k \geq 0$, the system states cannot be expected to converge to the origin.

For these reasons, it is enough if the Lyapunov theory holds for at least local stability and the system states converge to the set $\mathcal{X}_\infty$ in the neighborhood of the origin.

In a general setting, as the concept of Lyapunov theory and its associated positive definite Lyapunov function are meant only for dynamic systems of the form $x_{k+1} = f(x_k)$, without any external input, extending them for a controlled system with the control input u_k with/without the control input constraint $u_k \in \mathbb{U}$, is of more interest, and one such Lyapunov function is called a *control Lyapunov function.*

Definition 4.17 (Control Lyapunov Function) *[26] A positive definite function, $V_k(x_k)$ inside a convex set $\mathcal{F}(\mathcal{X})$ is a Control Lyapunov Function (CLF) for a dynamic system $x_{k+1} = f(x_k, u_k)$, iff there exists an admissible control input $u_k \in \mathbb{U}$, such that*

$$\Delta V_k(x_k) + \phi_k^V(x_k) < 0$$

for proper class-$\mathcal{K}$ functions $\phi_k^V(\cdot)$, $\forall k \in \mathbb{Z}^+$.

In other words, a CLF is simply a candidate Lyapunov function whose derivative can be made negative *pointwise* by the choice of control values.

However, when the system has the form $x_{k+1} = f(x_k, u_k, w_k)$, with $u_k \in \mathcal{U}$ perturbed by some bounded disturbance $w_k \in \mathbb{W}$, then for such system the corresponding control Lyapunov function is called a *Robust Control Lyapunov Function.*

Definition 4.18 (Robust Control Lyapunov Function) *Given a convex set of functionals $\mathcal{F}$, a function $V_k(x_k) \in \mathcal{F}$ is called a Robust Control Lyapunov Function (RCLF), for the system $x_{k+1} = f(x_k, u_k, w_k)$, when there exists a class-$\mathcal{K}$ function, $\alpha_k^V(x_k)$, and some $c^V > 0$, such that,*

$$\inf_{u_k \in \mathcal{U}} \sup_{w_k \in \mathbb{W}} \{\Delta V_k(x_k) + \alpha_k^V(x_k)\} < 0 \tag{4.13}$$

with $V_k(x_k) > c^V$ for all $k \in \mathbb{Z}^+$.

It is straight forward to conclude that when there is no disturbance affecting the system ($\mathbb{W} = \emptyset$), but control input has constraint $u_k \in \mathcal{U} \neq \emptyset$ then the robust control Lyapunov function in (4.13) reduces to simply control Lyapunov function with $c^V = 0$.

4.2.3 Lyapunov function and Invariant set

In general, Lyapunov function of a dynamic stable system is a positive definite function with the property of decreasing along the system trajectory. For a dynamic system, if a function $V_k(x_k)$ of the state variables is non-increasing along the system trajectory, then there would consequently exists a set, $\mathcal{F}$, such that (with some abuse of notation)

$$\mathcal{F}(V_k, \nu) = \{x_k : \; V_k \leq \nu\}$$

which is positive invariant and to be more precise, if $x_{\bar{k}} \in \mathcal{F}(V_{\bar{k}}, \nu)$, then $x_k \in \mathcal{F}(V_k, \nu))$ for all $k \geq \bar{k}$. Moreover, if the function is decreasing and bounded (may be bounded away from zero due to the presence of persistent disturbance, if $\mathbb{W} \neq \emptyset$), for a system given in equation (4.1), and $0 \leq \alpha < \beta < +\infty$, to a set of the form

$$\mathcal{F}(V_k, \alpha, \beta) = \{x_k : \;\; \alpha \leq V_k \leq \beta\}$$

then $x_{\bar{k}} \in \mathcal{F}(V_k, \alpha, \beta)$ implies (besides $x_k \in \mathcal{F}(V, \cdot, \beta)$, $\forall k \in \mathbb{Z}^+$) that x_k reaches the smaller set $\mathcal{F}(V_k, \alpha, \cdot)$ in some finite time instants.

The above properties necessarily means that for a given $\mu > 0$ and some arbitrary small $\epsilon > 0$ all the solutions that originate from a μ-ball of some norm with $x_0 \in \mathcal{F}(\|\cdot\|, \mu)$, would eventually reach and confine in a ϵ-ball $x_k \in \mathcal{F}(\|\cdot\|, \epsilon)$, for all $k \geq \bar{k}$. In short,

$$\mathcal{F}(V, \alpha^*) \subseteq \mathcal{F}(\|\cdot\|, \epsilon) \subset \mathcal{F}(\|\cdot\|, \mu) \subseteq \mathcal{F}(V, \beta^*)$$

for some optimal values of α^* and β^*.

It could be easily understood that the above said properties are analogous to the m-RPI set, with

$$\mathcal{X}_\infty \equiv \mathcal{F}(V, \alpha^*) \tag{4.14}$$

and the M-RPI set given in Definition (4.7) above, with $x_0 \in \mathcal{X}_0 \subset \mathcal{X}_F$ could be given as,

$$\mathcal{O}_\infty \equiv \mathcal{F}(V, \beta^*) \tag{4.15}$$

such that $\mathcal{X}_F \subseteq \mathcal{O}_\infty$ for the dynamic system given in (4.9) and with the Lyapunov function difference $\Delta V_k(x_k)$ satisfying

$$\Delta V_k(x_k) + \varphi(\|x_k\|) \leq 0 \tag{4.16}$$

for some class-$\mathcal{K}$ function, $\varphi(\cdot)$.

Some conditions that are used for analysis of stability and robustness are given in the following sections for clarity and completeness of the further discussions of the chapter. The proofs of the following propositions can be found in [70, 84].

Proposition 4.3 *If the closed loop system $x_{k+1} = f(x_k, \kappa_k(x_k))$ has a Lyapunov function $V_k(x_k) \in \mathcal{F}$ with $x_k \in \mathcal{X}$, $0 \in int(\mathcal{X})$ and $\mathcal{F}$ is forward invariant* [‡], *then the origin is asymptotically stable with basin of attraction (containing) $\mathcal{X}$.* □

[‡] For a mapping function g, a set S is a forward invariant set (resp. backward invariant set) if $v \in S$ implies that $g(v) \in S$ (resp. $g^{-1}(v)$). S is invariant if it is both forward invariant and backward invariant.

For nominal stable dynamic systems the states converge to the origin. For persistently perturbed system, on the other hand, the system states cannot be steered to the origin. However, for a perturbed system with continuous right-hand side of its dynamic state equation, the closed loop solutions stay close to the nominal solutions in the vicinity of the origin for small perturbations, which brings in robustness as given in Proposition 4.4 below.

For a feedback MPC design when the feasible set is RCI, then the closed-loop solution of the perturbed system $x_{k+1} = f(x_k, \kappa_k(x_k)) \oplus \mathbb{W}$, with a time-varying feedback law, is given as $\phi_k(x_k, \kappa_k, w_k) \in \mathcal{X}$, $\forall w_k \in \mathbb{W}$ and $\kappa_k(x_k) \in \mathcal{U}$. The solution will be Lipschitz continuous and converges asymptotically to a compact *residual set* $\mathcal{T} \subset \mathcal{X}_F$ containing a stable equilibrium point (ideally need to be the origin) inside the feasible set.

In general, for admitting unique closed-loop solutions, local Lipschitz continuous feedback control laws are sought. This is achieved by Lipschitz continuity for the set-valued maps between metric spaces. Set-valued map is the map in the metric space from one nonempty set Ξ to the subsets of another nonempty set Λ. Moreover, let $\vartheta \in \Xi$ and $\lambda \in \Lambda$. Writing $F : \Xi \rightsquigarrow \Lambda$ to represent a set-valued map F from Ξ to the subsets of Λ[§], the definition of Lipschitz continuity is rewritten below, in the present context of set valued map, without any lose of generality.

Definition 4.19 (Lipschitz Continuity) *A set-valued map $F : \Xi \rightsquigarrow \Lambda$ is Lipschitz continuous on $Z \subset \Xi$ when there exists $L \in \mathbb{R}^+$ such that for all $\vartheta, \upsilon \in Z$,*

$$F(\vartheta) \subset \left\{ \lambda \in \Lambda \, : \, d(\lambda, F(\upsilon)) \leq L d(\vartheta, \upsilon) \right\}$$

where L is the Lipschitz constant and $d(\cdot)$ represents the Euclidean distance. We say that F is ***locally Lipschitz continuous*** *when it is Lipschitz continuous on a neighbourhood of every point in Ξ.*

Definition 4.20 *[55] For a given control for the system Σ, let $\mathcal{T} \subset \mathcal{X}$ be a compact set containing $0 \in \mathcal{X}$. The solutions to Σ are robust asymptotically stable w.r.t. $\mathcal{T}$*

[§]A set-valued map $F : \Xi \rightsquigarrow \Lambda$ can also be regarded as a single-valued map $F : \Xi \rightarrow 2^{\Lambda}$ (where 2^{Λ} denotes the power set of all the subsets of Λ).

(RAS-$\mathcal{T}$) when there exists $\beta \in \mathcal{KL}$[†] such that for all admissible control, admissible disturbance and feasible initial condition set, all solutions $\phi_k(x_k, \kappa_k, w_k)$ exist for all $k \geq k_0$ and satisfy [‡]

$$|\phi_k(x_k, \kappa_k, w_k)|_{\mathcal{T}} \leq \beta(|x_k|_{\mathcal{T}}, k - k_0) \tag{4.17}$$

for all $k > k_0$.

The RAS-$\mathcal{T}$ implies that the residual set $\mathcal{T}$ is (robustly) positively control invariant. In particular, RAS-$\mathcal{T}$ (with $\mathcal{T} = \{0\}$) implies that $0 \in \mathcal{X}$ is an equilibrium point. Thus the robust stabilization problem for the system Σ is to construct an admissible control such that the solutions to Σ are RAS-$\mathcal{T}$ for some residual set $\mathcal{T}$. It is clear from these arguments that the robust asymptotic stability in turn implies robust stabilisability and the ultimate goal for a better performance is to reduce the residual set $\mathcal{T}$ to be arbitrarily small by a proper choice of the admissible control.

Proposition 4.4 (Asymptotic stability of a perturbed system) *For the system given by $f(x_k, u_k) : \mathbb{R}^n \to \mathbb{R}^n$ the following are equivalent;*

1. *the closed loop system $f(\cdot)$ satisfy a Lipschitz condition in a neighbourhood of the origin with $V_k(0) = 0$. If the origin is an exponentially stable fixed point of $x_{k+1} = f(x_k, \kappa(x_k))$, then it is an asymptotically stable fixed point of the perturbed system $x_{k+1} = f(x_k, \kappa_k(x_k)) + w_k$, for a time-varying control law [90].*

2. *the closed loop system $f(\cdot)$ is said to be robust asymptotically stable on $\text{int}(\mathcal{X})$ with respect to disturbance (w_k), if there exists a class-$\mathcal{KL}$ function β for some $\epsilon > 0$ and a compact set $\mathcal{T} \subset \text{int}(\mathcal{X})$, and there exists $\delta > 0$ such that for all w_k satisfying: (i) $\|w_k\| \leq \delta$, (ii) $|\phi_k(x_k, w_k)| \in \mathcal{T}$ for all $k \geq 0$, we have*

[†] A continuous function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a class- $\mathcal{KL}$ function, if $\varphi(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\varphi(s, \cdot)$ is monotonically decreasing to zero for each $s > 0$ [118].

[‡] The notation $|\cdot|_{\mathcal{T}}$ represents the Euclidean point-to-set distance function, that is, $|\cdot|_{\mathcal{T}} := d(\cdot, \mathcal{T})$.

$|\phi_k(x_k, w_k)| \leq \beta(\|x_k\|, k)+\epsilon$ for all $k \geq 0$ and the origin is robust asymptotically stable with respect to additive disturbance if the conditions above are satisfied with $\|w_k\| = 0$ [70]. □

In general, robust asymptotic stability of a closed loop system on $\text{int}(\mathcal{X})$ can be ensured iff [70]

1. the value function is continuous on $\text{int}(\mathcal{X})$;
2. the set of control moves or the feedback control law is locally bounded and continuous for a convex and bounded constraint set $\mathcal{U}$;

With the definitions (4.18) and (4.20) the relation between robust control Lyapunov function and robust asymptotically stabilisable set could be easily given.

Lemma 4.1 *For a dynamic system with input constraint $u_k \in \mathcal{U}$, perturbed with persistent disturbance $w_k \in \mathbb{W} \neq \emptyset$, $x_{k+1} = f(x_k, u_k) + w_k$, the existence of a robust control Lyapunov function (as defined in Definition 4.18) implies robust asymptotic stability of the perturbed system in terms of Definition 4.20.*

Proof: It is known that for a perturbed system when the states lie inside a positive invariant set $\mathcal{F}$, with its closed-loop eigenvalues lying inside the stable region, due to the optimal feedback control law $\kappa_k(x_k)$, it ensures closed-loop asymptotic stability when there exits a Lyapunov function $V_k(x_k) \in \mathcal{F}$ for all $k \geq 0$. However, due to the presence of persistent disturbance $w_k \in \mathbb{W}$ the states cannot be steered to the origin. For the closed-loop system to be robustly stable, it is enough if the system states reach a positively control invariant set, *viz.*, $\mathcal{T}$, at some $\bar{k}$ and lie inside the set for all $k \geq \bar{k} > 0$. The corresponding Lyapunov function will then reach a value $c^V > 0$, such that $V_k(x_k) \geq c^V$ for all $k \geq \bar{k} > 0$. Thus the robust asymptotic stability results in the closed-loop system states to reach an invariant set ($\mathcal{T}$) (as in Definition 4.20) which in turn implies the the existence of a robust control Lyapunov function, $V_k(x_k) \geq c^V > 0$, which is Lipschitz continuous, for some $w_k \in \mathbb{W} \neq \emptyset$. ■

4.3 Feasibility and continuity implies robust stability

For the stabilizing robust control Lyapunov function to exist, it is assumed that the set of ordered pairs (x_k, κ_k), which satisfy equations (3.33)-(3.37) in Theorem (3.2), is non-empty. The feasible set $\mathcal{X}_F$ is the set for which a feasible control law $\kappa_k \in \mathcal{U}$ exists. The necessity of continuity and feasibility for ensuring robust stability is discussed below. The following definitions and theorem are the extensions for the feasibility results given in [91] for nonlinear MPC problem and for disturbance perturbed systems given in [92].

4.3.1 Continuity of MPC control law and Value function

The continuity of the control law of feedback MPC within the admissible input set and the continuity of the value function to ensure robust asymptotic stability for the closed-loop system $x_{k+1} = f(x_k, \kappa_k(x_k)) + w_k$, in presence of admissible disturbance $w_k \in \mathbb{W}$ is necessary (and sometimes sufficient) for the robust stability of the closed-loop system.

Proposition 4.5 *If the admissible control and admissible disturbance are closed, convex and compact sets, then the MPC control law $\kappa_k(\cdot)$ is unique and Lipschitz continuous on $\mathcal{X}_F$. Furthermore, the value function $V_k(\cdot)$ is strictly convex and Lipschitz continuous on $\mathcal{X}_F$.* □

It only remains to show that for the given conditions, the existence of robust control Lyapunov function such that ΔV is locally Lipschitz implies the existence of a locally Lipschitz robustly stabilizing admissible control. In order to show this claim in Theorem (4.1) below, some required foundations are in the order.

Let Ξ and Λ be some topological spaces in which the parameter ϑ and the unknown variable λ take their respective values such that, $\vartheta \in \Xi$ and $\lambda \in \Lambda$.

Definition 4.21 (Marginal Functions) *[55] Consider an objective function $\psi : \Xi \times \Lambda \to \mathbb{R}$ and a set-valued constraint $F : \Xi \rightsquigarrow \Lambda$, then we define the optimal* *solution* $g : \Xi \to \mathbb{R}$ *to be*

$$g(\vartheta) := \sup_{\lambda \in F(\vartheta)} \psi(\vartheta, \lambda) \tag{4.18}$$

which is called a ***marginal function*** *of the parameter* ϑ*.*

The following proposition (given as Proposition 2.10 of [55]) describes the condition for the marginal function g when it inherits the properties of continuity of the objective function ψ and the set-valued map F. The proofs of the propositions below can be found in [55].

Proposition 4.6 *[55] Suppose* Ξ *is a locally compact metric space, and let* $F : \Xi \rightsquigarrow \mathbb{R}^n$ *be locally Lipschitz continuous with nonempty closed convex values. Then there exists a locally Lipschitz continuous function* $g : \Xi \to \mathbb{R}^n$ *such that* $g(\vartheta) \in F(\vartheta)$ *for all* $\vartheta \in \Xi$*.* □

Before proceeding further, consider the following assumptions regarding the structure of the function f and on the constraints $\mathcal{U}$ and $\mathbb{W}$:

- **A1** for each fixed $(x, w, k) \in \mathcal{X} \times \mathbb{W} \times \mathbb{Z}^+$, the mapping $u \mapsto f(x_k, u_k, w_k; k)$ is affine,
- **A2** the control input constraint $\mathcal{U}$ is locally Lipschitz with nonempty closed convex values,
- **A3** the disturbance $\mathbb{W}$ is locally Lipschitz with nonempty compact values, and,
- **A4** the disturbance $\mathbb{W}$ is independent of the control input u.

Also, given the Lyapunov function $V \in \mathcal{V}(\mathcal{X})$ and the system Σ, let us define the complete Lyapunov difference $L_f V : \mathcal{X} \times \mathcal{U} \times \mathbb{W} \times \mathbb{Z}^+ \to \mathbb{R}$ by the equation

$$L_f V(x, u, w; k) := V^k(x; k) + V^x(x; k).f(x, u, w; k) \tag{4.19}$$

where V^k and V^x denote the partial differences of V w.r.t. k and x, respectively. Moreover $L_f V$ is continuous and it follows immediately from eqn.(4.10) that $L_f V(0, \cdot, \cdot, \cdot) = 0$.

Proposition 4.7 *[55] Let us define* $D : \mathcal{X} \times \mathbb{R}^+ \times \mathcal{U} \times \mathbb{Z}^+ \to \mathbb{R}$ *by*

$$D(x_k, c_k, u_k; k) := \sup_{w_k \in \mathbb{W}} \left[L_f V(x, u, w; k) + \alpha_k^V(x_k) \right] \tag{4.20}$$

where D *is convex in* $u_k \in \mathcal{U}$. *If there exists the RCLF* $V(\cdot)$ *for the dynamic system* Σ *and the set-valued regulation map* $\mu(\cdot) : \mathcal{X} \times \mathbb{R}^+ \times \mathbb{Z}^+ \rightsquigarrow \mathcal{U}$ *as follows*

$$\mu(x_k, c_k) = \left\{ u_k \in \mathcal{U} \;:\; D(x_k, c_k, u_k; k) < 0,\ \forall k \in \mathbb{Z}^+ \right\} \tag{4.21}$$

where μ *is nonempty and convex in* $\mathcal{U}$ *for all* $x_k \in \mathcal{X}$ *all* $k \geq 0$ *and all* $c > c^V$, *then the system* Σ *is robustly (asymptotically) stabilisable.*

Proof It could easily be proven that D is convex in $u_k \in \mathcal{U}$ (c.f. [55]. Then for any $c_k > c^V$, there would exist an admissible (in fact, feasible) control $u_k(x_k) \in \mathcal{U}$ for all $(x, k) \in \mathcal{X} \times \mathbb{Z}^+$. The mapping $u_k \mapsto D(x_k, c_k, u_k; k)$ is the pointwise supremum of a family of finite convex functions. Therefore, $D \leq 0$ for all $(x, k) \in \mathcal{X} \times \mathbb{Z}^+$.

Let x_k denote a local solution of Σ for some initial condition in $\mathcal{X}_0$, for some admissible disturbance. Suppose $V_k(x_k) \geq c_k$ at some time $k \in \mathbb{Z}^+$ for which $\phi_k(x_k, u_k, w_k)\ \forall k \in \mathbb{Z}^+$ is defined. Then the Lyapunov derivative ΔV satisfies

$$\begin{aligned} \Delta V_k(x_k) + \alpha_k^V(x_k) &\leq D(x_k, c_k, u_k; k) \\ &\leq 0 \end{aligned} \tag{4.22}$$

Thus $V_k(\cdot)$ is strictly decreasing along trajectories whenever $V_k(\cdot) \geq c_k$. It follows from the above Lyapunov arguments that the solutions to Σ are RAS-$\mathcal{T}$ with $\mathcal{T} = \alpha_1^{-1}(c_k)\mathcal{B}$, where α_1 is the class $\mathcal{K}_\infty$ function in eqn.(4.10) and $\mathcal{B}$ is a closed unit ball.

If $c^V = 0$, then the residual set $\mathcal{T}$ could be made arbitrarily small by an appropriate choice of $c_k \geq 0$, $\forall k \geq 0$. ■

Theorem 4.1 *[55] Let the dynamic system* Σ *satisfy the assumptions* $A1 - A4$. *If there exists a robust control Lyapunov function* $V_k(x_k)$ *for the system* Σ *such that* $\Delta L_f V(\cdot) + \alpha^V$ *is locally Lipschitz, then* Σ *is robustly stabilisable via a locally Lipschitz feedback control law* μ. *If furthermore* $c^V = 0$, *then* Σ *is robustly practically stabilisable via local Lipschitz time-invariant feedback.*

Proof From the proposition 4.8 it is known that D is convex in u at points where $c_k = V_k(x_k)$. From the assumption $A3$ and proposition 4.7, it follows that D is locally Lipschitz. As defined in (4.21) let

$$\mu(x_k) := \{u_k \in \mathcal{U} \ : \ D(x_k, u_k) < 0\}$$

Now, from assumption A2 and Proposition (2.17) of [55] there exists a locally Lipschitz function $\varrho_k : \text{Dom}(\mu(\cdot)) \to \mathbb{R}^+$ such that the set-valued map $x_k \rightsquigarrow \mu(x_k) \cap \varrho_k(x_k)\mathcal{B}$ is locally Lipschitz with nonempty compact convex values on $\text{Dom}(\mu(\cdot))$. It then follows from proposition (4.7) that there exists a locally Lipschitz function $\kappa : \text{Dom}(\mu(\cdot)) \to \mathcal{U}$ such that $\kappa_k(x_k) \in \overline{\mu(x_k)}$ for all $x_k \in \text{Dom}(\mu(\cdot))$. For $c_k > c^V$ and the definition of RCLF as in eqn.(4.13) we can see that $V^{-1}[c_k, \infty) \subset \text{Dom}(\mu(\cdot))$. Then it follows from assumption A2 and proposition 4.7 that there exists a locally Lipschitz feasible control $u_k \in \mathcal{U}$ such that $u_k = \kappa_k(x_k)$ whenever $V_k(x_k) \geq c_k$. Thus $D(x_k, u_k) \leq 0$ whenever $V_k(x_k) \geq c_k$ and the rest follows from the proof of proposition 4.7ψ. ■

4.3.2 Nominal and Robust Feasibility

Definition 4.22 (Feasible for all time) *The feedback MPC control problem is feasible for all time $k > 0$ iff the initial state x_0 belongs to the feasible set and all future evolutions of the state belong to the feasible set, i.e., $x_k \in \mathcal{X}_F$, $\forall k > 0$.*

The input constraint limits the set-valued map from the input space to the state-space, $\mathcal{U} \rightsquigarrow \mathcal{X}_F$, such that it gives the necessary and sufficient condition for guaranteeing the feasibility for all time.

Theorem 4.2 *The MPC problem is feasible for all time iff $x_0 \in \mathcal{X}_F \cap \mathcal{C}_\infty(\mathbb{X})$ and $\forall x_k \in \mathcal{X}_F \cap \mathcal{C}_\infty(\mathbb{X})$ the solution to the MPC problem results in $x_{i|k} \in \mathcal{X}_F \cap \mathcal{C}_\infty(\mathbb{X}), \forall i = \{k, ..., \infty\}$, $\forall k \geq 0$.* □

Definition 4.23 (Strongly feasible) *[92] The MPC problem is strongly feasible iff for all $x_0 \in \mathcal{X}_F$ and feasible control inputs, the MPC problem is feasible for all time.*

The following theorem is also straight forward from the above concepts of feasibility.

Theorem 4.3 *The MPC problem is strongly feasible iff (i) the feasible set is positively invariant set, with a feedback control law* $\kappa_k(x_k) \in \mathcal{U}$*, for the closed-loop system* $x_{k+1} = f(x_k, \kappa_k(x_k))$*; (ii) the feasible set is control invariant set with a control input sequence* $u_k \in \mathcal{U}$ *for the closed-loop system* $x_{k+1} = f(x_k, u_k)$*.* □

However, the presence of disturbance $w_k \in \mathbb{W}$ makes the situation more complex. To address this issue, for the given MPC controller μ, the feasible set and the robust feasibility for the closed-loop system need to be modified accordingly.

Definition 4.24 (Robust feasibility) *The MPC controller is robustly feasible if and only if the given states of the system lie inside the feasible set and for all disturbance inside* $\mathbb{W}$*, the state of the system at the next time instant lies inside the feasible set, i.e.,* μ *is robustly feasible* $\Leftrightarrow\ \forall x_k \in \mathcal{X}_F : f(x_k, \mu) \oplus \mathbb{W} \in \mathcal{X}_F$*.*

For the feasible input set of the MPC controller design, the closed-loop system is now given as $x_{k+1} = f(x_k,\mathcal{U}) \oplus \mathbb{W}$. Hence, if the MPC controller is robustly feasible for all optimal and sub-optimal control inputs, it will be said to be robustly *strongly* feasible [92].

Definition 4.25 (Robust strong feasibility) *The MPC controller is robustly strongly feasible iff* $\forall x_k \in \mathcal{X}_F,\ w_k \in \mathbb{W}\ :\ f(x_k,\mathcal{U}) \oplus \mathbb{W} \in \mathcal{X}_F,\ \forall k \geq 0$*.*

Moreover, the robust feasible set needs to be a control invariant set. Thus it could be concluded that the control invariance is only a necessary condition for strong feasibility and the robust feasibility is an important factor to be considered for the controller design. The condition for robust feasibility for feedback MPC, which is the main concern of the present investigation, is given by the following proposition.

Proposition 4.8 *(i) The feedback MPC is strongly feasible iff the feasible set is positively invariant set for the closed-loop system* $x_{k+1} = f(x_k, \kappa(x_k))$*; (ii) The feedback MPC is robustly strongly feasible iff the feasible set is control invariant for the closed-loop system* $x_{k+1} = f(x_k, u_k)+w_k$ *with* $u_k \in \mathcal{U}$ *and* $w_k \in \mathbb{W}$ *and the closed-loop system states lie inside* $\mathcal{X}_F\ \forall k \geq 0$*.* □

4.3.3 Input-to-State Stability (ISS) for MPC

For dynamic systems with persistent disturbance, where the system states cannot be steered to the origin, the origin cannot be guaranteed to be asymptotically stable. To address such issue for systems in presence of input constraints, Sontag introduced the idea of *input-to-state stability* (ISS) in his benchmark paper [149], in a general setting for nonlinear system.

Definition 4.26 (ISS) *For a nonlinear, time-invariant, discrete time system of the form, $x_{k+1} = f(x_k, w_k)$, with the region of attraction $\mathcal{X} \subseteq \mathbb{R}^n$, which contains the origin in its interior, when there exists a class-$\mathcal{KL}$ function $\beta(\cdot)$ and a $\mathcal{K}$-function $\xi(\cdot)$ such that for every initial condition $x_0 \in \mathcal{X}$ and every bounded disturbance $w_k \in \mathbb{W}$, the solution of the system satisfies $\phi_k(x_k, w_k) \in \mathcal{X}$ and for all $k \geq 0$,*

$$|\phi_k(x_k, w_k)| \leq \beta(|x_k|, k) + \xi(\|w_k\|_\infty) \tag{4.23}$$

It could be easily understood from the above definition that ISS implies that the origin is asymptotically stable point for the undisturbed system, $x_{k+1} = f(x_k, 0)$, in the given region of attraction $\mathcal{X}$. Moreover, every solution/trajectory $\phi_k(x_k, w_k) \to 0$, if $w_k \to 0$ as $k \to \infty$.

The above definition of ISS stability is meant for closed-loop systems, which can be useful for closed-loop stability analysis of the MPC designs such as [96, 121]. The treatment of the closed-loop stability with origin as the asymptotically stable point with a corresponding domain of attraction is treated for the MPC controllers in [95, 66]. Although an explicit reference is not made to ISS stability in the following stability analysis of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC controllers, the existence of a Lyapunov function implies that it is a ISS-Lyapunov function, as given in [85].

Thus it could be summarised that the continuity and feasibility are the important properties to be ensured for the closed-loop system with feedback control to be robustly asymptotically stable. The existence of these properties for a given system with the required conditions in turn implies robust stability of the closed-loop system.

4.4 Robustness of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC

Robustness and closed-loop stability with the feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC algorithms proposed in explicit Nash game based approach in the present thesis and that in Orukpe et al. [121] are compared in this section in the context of set theoretic concepts furnished so far. As given in Section 4.3.2, robust feasibility at every time instance is a necessary condition for the existence of a stabilizing control law that satisfies the control objectives. Moreover, structure preserving continuous control law is preferred both for its practical applicability and also for serving as a sufficient condition along with the feasibility for the closed-loop robustness [70].

In both the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC designs, ellipsoidal sets are used for achieving the performance of the closed-loop system with the given constraints and controller parameters. Before going into the details of the robustness issue, the following definition of contractive invariant set in terms of ellipsoidal set is given below.

Let $P \in \mathbb{R}^{n\times n}$ be a positive-definite matrix. Denote ellipsoid as,

$$\mathcal{E}(P,c) = \{x \in \mathbb{R}^n \;:\; x^T P x \le c\}.$$

Definition 4.27 *[80] Let $V_k(x_k) = x_k^T P_k x_k$. The ellipsoid $\mathcal{E}(P_k, c_k)$ is said to be contractive invariant if*

$$\Delta V_k(x_k) \;=\; \left\{ \begin{array}{c} x_k^T(P_{k+1} - P_k)x_k + 2x_k^T P_{k+1}(Ax_k + B\text{sat}(K_k^u x_k)) \\ +(B\text{sat}(K_k^u x_k))^T P_{k+1}(B\text{sat}(K_k^u x_k)) \end{array} \right\} < 0$$

for all $x_k \in \mathcal{E}(P_k, c_k)^{\ddagger}$.

Clearly, if $\mathcal{E}(P_k, c_k)$ is contractive invariant, then it is inside the domain of attraction.

Proposition 4.9 *For robust controller with ellipsoidal invariant set, $\mathcal{E}(P_k, c_k) := \{x_k \;:\; x_k^T P_k x_k \le c_k\}$, as the states of a persistently disturbed system reach an invariant residual set $\mathcal{T}$ in the neighbourhood of the origin or $0 \in$int$(\mathcal{T})$ after some finite $\bar{k}$, and for all $k \ge \bar{k}$, where the constraints are assumed to be inactive, then the controller gain $\kappa_k(\cdot) \to \bar{\kappa}(\cdot)$, such that $\kappa_k(\cdot) \le \bar{\kappa}(\cdot)$, $\forall k < \bar{k}$.* □

‡ $sat(\cdot)$ represents the saturation nonlinearity. Note, in the present context of the thesis, where input constraint is considered, $sat(\cdot) \equiv \partial\mathcal{U}$.

Corollary 4.1 *For all $k \geq \bar{k} > 0$ as the closed-loop solution $\phi_k(x_k, u_k, w_k)$ reaches the minimal robust invariant set $\mathcal{X}_\infty$, the state-dependent ellipsoidal sets $\mathcal{E}(P_k, c_k)$ need to reach a minimal size for better closed-loop performance.* □

Thus the performance is improved when the controller gain of the feedback system shifts from the low gain to high gain as the system states approach towards origin (or, an invariant set containing the origin, if the system is excited with persistent disturbance). Furthermore, the following proposition, showing existence of the robust feasible sets within a defined maximal stabilisable set, paves the way to ensure robustness of the above mentioned two control algorithms taken under investigation.

Proposition 4.10 *For any $x_0 \in \mathcal{X}_0$ such that $\mathcal{X}_0 \subseteq \mathcal{S}_\infty(\cdot)$ and if there exists input $u_k \in \mathcal{U}$ and $w_k \in \mathbb{W}$, satisfying the condition that the closed-loop solution, $x_k \in \mathcal{S}_\infty(\cdot, \mathcal{T})$, at every $k \geq 0$, then for all $x_0 \in \mathcal{X}_0$ there would exist a feasible set $\mathcal{X}_F$ under the mapping $f : \mathcal{X}_F \times \mathcal{U} \times \mathbb{W} \rightarrow \mathcal{X}_F$ satisfying the condition, $\mathcal{X}_F \subseteq \mathcal{S}_\infty(\cdot, \mathcal{T})$.*

Proof Considering $\mathcal{X}_0 \subseteq \mathcal{X}_F$ and any $x_0 \in \mathcal{X}_0$, if there exists a robustly feasible input $u_k \in \mathcal{U}$, satisfying the condition that the closed-loop solution at the next time instance, $x_{k+1} \in \mathcal{X}_F$ under the mapping $f : \mathcal{X}_F \times \mathcal{U} \times \mathbb{W} \rightarrow \mathcal{X}_F$, then let us define *reachable set* $\mathcal{R}_k$ at every $k \geq 0$ as,

$$\mathcal{R}_k := \left\{ x_{k+1} \in \mathcal{X}_F \;\middle|\; \exists u_k \in \mathcal{U},\ w_k \in \mathbb{W},\ x_{k+1} = f(x_k, u_k, w_k),\ x_0 \in \mathcal{X}_0, \quad \forall k \geq 0 \right\} \tag{4.24}$$

and also,

$$\mathcal{R}_k = \zeta_k \mathcal{X}_F, \quad \forall\, \zeta_k \in [0, 1), \forall k \geq 0. \tag{4.25}$$

Then with $u_k \in \mathcal{U}$, and satisfying the condition that closed-loop solution $\phi(\cdot)$ would eventually reach a control invariant set Θ, which contains the origin in its interior, such that

$$\mathcal{R}_{k+i} \subseteq \mathcal{R}_k, \;\; \forall i \geq 0 \;\text{ at every }\; k \geq 0.$$

Moreover, if $\zeta_k \rightarrow 1$, then $\mathcal{R}_k \rightarrow \mathcal{X}_F$. As $\mathcal{X}_F$ is the maximal feasible set of states for the given system and the controller requirements/specifications, it implies that as $\zeta_k \rightarrow 1$, $\mathcal{R}_k \rightarrow \mathcal{X}_F \rightarrow \mathcal{S}_\infty(\cdot)$. This in turn implies $\mathcal{X}_F \subseteq \mathcal{S}_\infty(\cdot)$.

On the other hand, as ζ_k reaches its minimal value, say ζ_k^{min}, then $\mathcal{R}_{k+i} = \mathcal{R}_k$ and the solution $\phi(\cdot)$ would reach the set Θ. From these arguments, it could be straight forwardly understood by induction that as $k \to \infty$, or more precisely for some finite $\bar{k} > 0$, and $\forall k \geq \bar{k}$, $\zeta_k \to \zeta_k^{min}$ and the closed-loop solution $\phi_k(x_k, u_k, w_k) \in \Theta$ as $\mathcal{R}_k \subseteq \Theta$, which in turn implies that $\Theta \equiv \mathcal{T}$. ■

Proposition 4.11

$$\mathcal{T} \subseteq \mathcal{X}_\infty$$

Proof From Corollary (4.1) and the proof of Proposition (4.10) given above, it could be straight forwardly understandable that for input constrained dynamic system with persistent disturbance, bounded inside $\mathbb{W}$, there would always exist a minimal robust invariant set $\mathcal{X}_\infty$ reachable by the closed-loop solution $\phi(\cdot)$ at some finite time $\bar{k}$ and lies there for all $k \geq \bar{k} > 0$. Assume there exist a control invariant set $\Theta \subseteq \mathcal{X}_\infty$, in the neighbourhood of the origin, such that $0 \in$int(Θ). Moreover, as the feedback controller gain cannot be unboundedly increased, to make the closed-loop solution to reach the origin (the nominal equilibrium), it could only be driven to an invariant residual set $\mathcal{T}$ for all $k \geq \bar{k} > 0$. When $\mathcal{T} \equiv \Theta$ for the given controller gain, then $\mathcal{T} \subseteq \mathcal{X}_\infty$. ■

In general, the performance of the closed-loop control at every time instance could be evaluated in terms of the speed of response and the degree of closeness of the stable steady-state value of the perturbed system w.r.t. the steady-state of the nominal system, *viz.*, the origin. However, the region of the state space (when there is no constraint on the states) of a dynamic system - with hard actuator saturation input constraint and persistent disturbance bounded within a given set, which is stabilisiable by a suitable control law, is a bounded subset, $\mathcal{X}_F \subset \mathbb{R}^n$. With *a priori* prescribed performance bound for robustness in terms of the disturbance attenuation factor γ, the control problem becomes involved. The factor γ affects the size/volume of the feasible set of states $\mathcal{X}_F$ and thus in turn affects the closed-loop performance. For a less conservative controller design, γ is expected to be smaller, this makes the control problem stringent.

Hence, for the disturbance rejection optimal control problem for input constrained system, the controller is expected to be less conservative with smaller value of γ and the domain of attraction w.r.t. the final residual set ($\mathcal{T}$) is increased. The closed-loop performance of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC in [121] and that in the present work based on explicit Nash game approach is decided by ellipsoidal sets for representing the level set of the Lyapunov function as discussed in the subsequent part of this section.

As ellipsoidal sets are being used to represent the various invariant sets used in the present analysis, the following lemma needs to hold good to achieve the required closed-loop performance. Moreover, the solution matrices of the cAREs (P^I, P^{II}) are obtained by using relaxation technique in Chapter 3. Hence, in the following lemma, these solution matrices are considered as a function of the slack variable [‡] (ϵ), which are used to construct the ellipsoidal set.

Lemma 4.2 *For the perturbed system $x_{k+1} = f(x_k, sat(u_k)) + w_k$, denote,*

$$\mathcal{X}_p(\epsilon) := \mathcal{E}(P_k(\epsilon), c_0) = \{x_k \ : \ x_k^T P_k(\epsilon) x_k \leq c_0\} \tag{4.26}$$

and,

$$\mathcal{X}_\infty(\epsilon) := \mathcal{E}(P_k(\epsilon), c_\infty) = \{x_k \ : \ x_k^T P(\epsilon) x_k \leq c_\infty\} \tag{4.27}$$

Then $\mathcal{X}_p(\epsilon)$ and $\mathcal{X}_\infty(\epsilon)$ are invariant sets and for any $w_k \in \mathbb{W}$, $x_0 \in \mathcal{X}_p(\epsilon)$, by solving the cAREs and saddle point eqns.(3.36 & 3.37), the closed loop solution $\phi_k(x_0, w_k) \in \mathcal{X}_p(\epsilon)$ will enter $\mathcal{X}_\infty(\epsilon)$ in some finite time $\bar{k} > 0$ and remain in it for all $k \geq \bar{k}$. □

Theorem 4.4 *For a given perturbed system, with $x_0 \in \mathcal{X}_0$, with input constraint $u_k \in \mathcal{U}, \forall k \geq 0$, the closed loop solution $\phi_k(x_0, w_k)$ is said to be robust asymptotically stable and satisfies the following conditions simultaneously.*

1. *for any $x_0 \in \mathcal{X}_0$, there exists a time-varying feedback control law $\kappa_k(x_k) =: u_k \in \mathcal{U}$ such that $x_k \in \mathcal{X}_\infty \subset \mathcal{X}_0 \subseteq \Omega$ at some finite time $\bar{k}$, and will stay there for all $k \geq \bar{k}$.*

[‡] Recall the use of slack variable $\{\epsilon_k^1, \epsilon_k^2\}$.

2. *the Lyapunov function, which is in turn the value function,* $V_k(x_k)$ *satisfies the condition,* $\Delta V_k(x_k) < 0$ *and is continuous, such that for class-*$\mathcal{KL}_\infty$ *function* $\xi(\|x_k\|, \epsilon)$*, the condition* $\Delta V_k(x_k) + \xi(\|x_k\|, \epsilon) \leq 0$ *satisfies, for all* $x_k \in \text{int}(\mathcal{S}_\infty(\cdot))$.

3. *the convexity of the ellipsoid* $\mathcal{E}(P_k(\epsilon), c_k) = \{x_k : \ x_k^T P_k(\epsilon) x_k \leq c_k\}$*, such that it obeys Lemma (4.2), and also being it a contractive invariant set, ensures robust asymptotic stability.* □

4.4.1 Robustness of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC in Orukpe *et al* (2007)

Consider the discrete-time LTI system, with a given initial condition x_0;

$$x_{k+1} = Ax_k + B_1 w_k + B_2 u_k \tag{4.28}$$

$$z_k = \begin{bmatrix} C_z x_k \\ D_{zu} u_k \end{bmatrix} \tag{4.29}$$

where $z_k \in \mathbb{R}^{n_z}$ and other matrices and vectors are of appropriate dimensions. Assume the pair (A, B_2) is stabilisable and the disturbance is bounded:

$$\|w\|_2 := \sqrt{\sum_{k=0}^{\infty} w_k^T w_k} \leq \bar{w} \tag{4.30}$$

where $\bar{w} > 0$ is known. Then the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC control problem as in [121] is as follows:

Problem 4.1 *Find a state feedback control law,* $u_k = Fx_k$*, where* $F \in \mathbb{R}^{m \times n}$*, such that the following constraints are satisfied:*

1. *The transfer matrix from* w *to* z*, denoted as* T_{zw} *is stable and for given* $\gamma > 0$ *the* $\mathcal{H}_\infty$ *constraint*

$$\|T_{zw}\|_\infty < \gamma \tag{4.31}$$

is satisfied.

2. *For given $\alpha > 0$ the $\mathcal{H}_2$ constraint*

$$\|z\|_2 := \sqrt{\sum_{k=0}^{\infty} z_k^T z_k} < \alpha \tag{4.32}$$

is satisfied.

3. *For given $H_1, ..., H_{m_u} \in \mathbb{R}^{h\times m}$ and $\bar{u}_1, ..., \bar{u}_{m_u} > 0$ the input constraints*

$$u_k^T H_j^T H_j u_k \leq \bar{u}_j^2,\ \forall k \geq 0\ \ for\ \ j = 1, ..., m_u,$$

are satisfied.

4. *For given $E_1, ..., E_{m_x} \in \mathbb{R}^{e\times n}$ and $\bar{x}_1, ..., \bar{x}_{m_x} > 0$ the state/output constraints*

$$x_{k+1}^T E_j^T E_j x_{k+1} \leq \bar{x}_j^2,\ \forall k \geq 0\ \ for\ \ j = 1, ..., m_x,$$

are satisfied.

For the above robust control problem the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC algorithm as proposed in Orukpe et al. (2007) [121] is furnished below for the sake of completeness.

Theorem 4.5 *There exists an admissible state feedback gain matrix F if there exists solutions $Q = Q^T \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ to the following LMIs*

$$\begin{bmatrix} -Q_k & \star & \star & \star & \star \\ 0 & -\alpha^2\gamma^2 I & \star & \star & \star \\ AQ_k + B_2Y_k & \alpha^2 B_1 & Q_k & \star & \star \\ C_zQ_k & 0 & 0 & -\alpha^2 I & \star \\ D_{zu}Y_k & 0 & 0 & 0 & -\alpha^2 I \end{bmatrix} \leq 0, \tag{4.33}$$

$$\begin{bmatrix} 1 & \star & \star \\ \gamma^2\bar{w}^2 & \alpha^2\gamma^2\bar{w} & \star \\ x_k & 0 & Q_k \end{bmatrix} \geq 0, \tag{4.34}$$

$$\begin{bmatrix} \bar{u}_j^2 I & \star \\ Y_k^T H_j^T & Q_k \end{bmatrix} \geq 0, \quad j = 1, ..., m_u \tag{4.35}$$

$$\begin{bmatrix} \bar{x}_j^2 I - (1+\bar{w}^2)E_jB_1B_1^TE_j^T & \star \\ Q_kA^TE_j^T + Y_k^TB_2^TE_j^T & \frac{Q_k}{(1+\bar{w}^2)} \end{bmatrix} > 0, \quad j = 1, ..., m_x \tag{4.36}$$

where $\star$ represents terms readily inferred from symmetry. If such solutions exist, then

$$F_k = Y_k Q_k^{-1}$$

Proof Refer Orukpe et al. (2007) [121] ■

For a fixed γ^2 (or α^2), the value of α^2 (or γ^2) is minimised in the above LMI optimisation problem. Let us denote the invariant sets, e.g., $\mathcal{S}_\infty(\cdot)$, $\mathcal{X}_F(\cdot)$, $\mathcal{X}_\infty(\cdot)$, etc., defined so far, for Orukpe et al.'s algorithm [121] as $\mathcal{S}_\infty^O(\cdot)$ $\mathcal{X}_F^O(\cdot)$, $\mathcal{X}_\infty^O(\cdot)$, etc., and so forth, where the superscript 'O' stands for representing the corresponding algorithm.

In order to achieve a better performance, it is expected, as given in Corollary 4.1, that the norm of the matrix Q^{-1} needs to be decreased as k increases and the closed-loop solution $\phi(\cdot)$ reaches a minimal invariant set $\mathcal{X}_\infty$. Moreover, the controller gain at every time instant need to be increased (c.f. Proposition 4.10), as the closed-loop solution approaches $\mathcal{X}_\infty$.

Lemma 4.3 *For the LMI optimisation problem in Theorem 4.5, if there exists an ellipsoidal invariant set $\mathcal{E}(P_k, \alpha^2)$ for the saturated LTI model with non-zero bounded disturbance, then this can always be scaled to be small enough so that constraints are always satisfied.*

Proof The input is given by the feedback $u_k = sat(-K_k x_k) = sat(-Y_k Q_k^{-1} x_k)$ and the system state is restricted to satisfy $x_k^T Q_k^{-1} x_k \leq 1$. Hence Orukpe et al [121] algorithm always take $\|Q^{-1}\|$ large enough, so that the feasible values of x, such that $x_k \in \mathcal{X}_F^O \subset \mathbb{R}^n$, ensure $K_k x_k$ is smaller than the input limits. ■

Corollary 4.2 *For the system given in eqns. (4.1)-(4.3), solving the robust control problem using the robust MPC algorithm in [121] with larger $\|Q^{-1}\|$ reduces the size of the maximal feasible set $\mathcal{X}_F^O(\cdot)$*

Proof If $\|Q^{-1}\|$ is large, then the set of possible values of initial condition, $x_0 \in \mathcal{X}_0$ that satisfy the inequality $x_k^T Q_k^{-1} x_k \leq 1$, inturn, becomes small. This reduces the size of $\mathcal{X}_0^O$. However, as the controller being a cautious controller, as mentioned in Lemma 4.3, such that it restricts the controller gain from hitting the saturation limit, then the size/volume of the corresponding maximal robust stabilizable set $\mathcal{S}_\infty^O$ is also reduced.

From the above arguments, and considering the general fact that $\mathcal{X}_F \subseteq \mathcal{S}_\infty(\cdot)$, the maximal feasible set $\mathcal{X}_F^O$ is also reduced. ■

Feasibility and Continuity : For a set of properly chosen values of $\{C_z, D_{zu}, \gamma\}$ the interior point optimisation method of solving the LMI constraints in eqns.(4.33)-(4.36) are expected to ensure feasibility and thereby achieves continuity of the control law.

It could be noted from Theorem (4.5) that in the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC of [121] the Lyapunov function used is $V_k(x_k) = x_k^T P_k x_k$ and an ellipsoidal set corresponding to the level sets of the Lyapunov function at every $k \geq 0$ is given by eqn. (4.34).

Proposition 4.12 *Consider the full state feedback control problem as given in eqns. (4.1)-(4.3), with state-dependent time-varying linear feedback controller,* $u_k = \kappa_k(x_k)$*, whose closed-loop Lyapunov value function is given by,* $V_k(x_k) := \left\{x_k^T P_k x_k : \ P_k \succ 0, \ \forall k \geq 0\right\}$*. Then the state-dependent ellipsoids,* $\mathcal{E}(P_k, c_k)$*, given as,*

$$\mathcal{E}(P_k, c_k) = \left\{x_k^T P_k x_k \leq c_k\right\} \tag{4.37}$$

represents the Lyapunov level sets. □

Using the above Proposition, the existence of the relation between the size of the ellipsoidal set $\mathcal{E}(\cdot)$ and the convergence of the closed-loop system response to a stable equilibrium with a suitable RCLF could be established as given in the following theorem.

Theorem 4.6 *For a state-feedback robust control problem, when ellipsoidal sets represents the Lyapunov level sets as given in Proposition (4.13), then the existence of RCLF (Definition (4.18)), for a class-*$\mathcal{K}$ *function* $\alpha_k^V(x_k)$*, such that*

$$\Delta V_k(x_k) + \alpha_k^V(x_k) < 0$$

with $V_k(x_k) > c^V$ is Lipschitz continuous for all $k \in \geq 0$ implies that the ellipsoidal set $\mathcal{E}(P_k, c_k)$ is contractive when $\|x_{k+1}\| < \|x_k\|$. □

Corollary 4.3 *Smaller ellipsoids allow high gain control laws. Hence, the smaller ellipsoidal sets guarantee less conservative and better performing closed-loop control system design.* □

Furthermore, when the states grow *i.e.*, the norm of the state vector increases w.r.t. time, the controller gain changes to the one corresponding to bigger ellipsoids. The choice of controller gain at every time instance is thus a function of the system response. As a result, the guaranteed performance bound ($\mathcal{L}_2$ gain bound) is parameter dependent as well. Thus if the choice of controller gain depends only on the system response and not any *a priori* estimate of the worst case disturbance, then the solution to the control problem will be less conservative. Moreover, in the presence of actuator capacity constraints, the performance needs to be an explicit function of the actuator capacity.

Thus from all these arguments, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC control algorithm of Orukpe et al [121], given as LMIs (4.33)-(4.36) in Theorem 4.5, which is based on worst-case disturbance bound $\bar{w}$, makes the resulting controller relatively more conservative. This can be better understood from the definition of marginal function given in Definition 4.21 and the Proposition 4.6. When the controller gain is computed using the LMIs, which are meant for the worst-case disturbance ($\bar{w}$) always, then apart from the fact that there would exist Lipschitz continuous value function and control law, the design will be for the marginal functions of the objective function always, even though the states are steered to the interior of the control invariant feasible set $\mathcal{X}_F^O$.

As the MPC control law μ is for the worst-case disturbance at all times, the state dependent ellipsoid, $x_k^T P_k x_k + \gamma \bar{w} \leq \alpha_k^2$ will be enlarged. Thus each ellipsoid will encompass the actual and the possible worst-case deviation of states in its neighbourhood. Furthermore, when the ellipsoidal sets are larger, then the reachable set of states $\mathcal{R}_k$ will fall within the same ellipsoid, thereby making it a cautious controller within the actuator limitation.

4.4.2 Robustness of Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC

The robustness and stability of the Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC (NGM-MPC) is addressed in this section. The proof for the closed loop stability and less conservative property of the NGM-MPC is given. The following (valid) assumption will be helpful for the proof of the subsequent theorem on closed loop stability.

Assumption 4.1 *For the given linear, time-invariant discrete time system the pair* (A, B_2) *is controllable (stabilisable) and the pair* $(A, \sqrt{Q_i})$ *is observable (detectable).*

Note that this is a standing assumption valid throughout this chapter.

Definition 4.28 (Feasible Initial Condition Set) *Any initial state condition* x_0 *is said to be in the feasible initial condition set* $\mathcal{X}_0$ *(i.e.,* $x_0 \in \mathcal{X}_0$*), which is defined, for the given* $\{\gamma, Q_i, R\}_{i=1,2}$ *as*

$$\mathcal{X}_0 \equiv \{x(0) \in \mathbb{R}^n \mid \exists\, J_2(u_k, w_k) < +\infty,\ J_\infty(u_k, w_k) < +\infty\}.$$

Denoting the closed loop solution of the system

$$x_{k+1} = Ax_k + B_2 sat(\kappa_k(x_k)) + B_1 w_k, \quad \forall k \geq 0 \tag{4.38}$$

as $\phi_k(x_k, \kappa_k, w_k) \in \mathcal{X}$, where $\kappa_k(x_k) := K_k^u x_k$. Then the maximal robust stabilisable set for the NGM-MPC controller is defined as follows.

Definition 4.29 *Let us define the maximal robust stabilisable set,* $\mathcal{S}_\infty^{NG}(\cdot, \mathcal{T})$ *as,*

$$\mathcal{S}_\infty^{NG}(\cdot, \mathcal{T}) = \{x_k \in \mathcal{X}_F \ :\ \phi_k(x_k, \kappa_k, w_k) \in \mathcal{X}, \forall k \geq 0\}$$

where, $\mathcal{X}_F$ *is the robust feasible set of states, such that for the given initial condition* $x_0 \in \mathcal{X}_0 \subseteq \mathcal{X}_F$ *and controller parameters* $\{Q_i, R, \gamma\}_{i=1,2}$ *stabilizing solution of the non-symmetric cross-coupled algebraic Riccati equations exists for all* $k \geq 0$.

Let us denote the Lyapunov value function, associated with the energy of the system be $V_k^*(x_k) := J_2(u_k^*, w_k^*), \forall k \geq 0$. Then the following lemmas which satisfy the stability criterion are in order.

Lemma 4.4 (Monotonicity of the cost function) *If $V_1(x_1) \leq V_0(x_0)$ for all $x_0 \in \mathcal{X}_0 \subseteq \mathcal{S}_\infty^{NG}(\cdot, \mathcal{T})$, then $V_{k+1}^*(x_{k+1}) \leq V_k^*(x_k)$ for all $x_k \in \mathcal{X}_F$, all $k \geq 0$, such that there exists positive definite solution set (P_k^{I*}, P_k^{II*}) for the cAREs.* □

The bounded real lemma, which is generally used to prove the asymptotic stability of a $\mathcal{H}_\infty$ controller based closed-loop system, is extended here for the present case involving the solution of cAREs in the following lemma, called Extended Bounded Real Lemma (EBRL), so as to prove the asymptotic stability of the closed loop system.

Lemma 4.5 (Extended Bounded Real Lemma) *For the given LTI system Σ, with the cost function (3.3), if there exists a class-$\mathcal{KL}$ function $\xi(\|w_k\|_2^2, k)$ such that $J_\infty(u_k, w_k) \leq -\xi(\|w_k\|_2^2, k)$ for all $w_k \in \mathbb{W}$, then, for the stabilizing solution pair $\{P_k^{I*}, P_k^{II*}\} \succ 0$ satisfy (3.33) and (3.34), such that the closed-loop solution, $\phi_k(x_k, \kappa_k, w_k), \forall k \geq 0$, is robust asymptotically stable.* □

Theorem 4.7 *If the closed loop solution of the system given in eqn. (4.38), with $x_0 \in \mathcal{X}_0$, has every element of its state trajectory lying in the positively invariant subset of $\mathcal{S}_\infty(\cdot)$ for all $k \geq 0$, when the system is perturbed by an unknown persistent disturbance of bounded energy, $w_k \in \mathbb{W}$, then the system states will asymptotically reach the robust control invariant set, $\mathcal{X}_\infty$, in some finite time $\bar{k} > 0$, for the given controller parameters $\{\gamma, Q_i, R\}_{i=1,2}$ and will lie there for all $k \geq \bar{k}$.*

Proof The proof proceeds by showing that, when the value of the cost function, $J_2(u_k^*, w_k^*)$, decreases monotonically as $k \to \infty$ (c.f. Lemma 4.4) and as well satisfies Lemma 4.5 for the closed-loop system (4.38) and reaches a finite value, say $\alpha^{V^*} > 0$ as given in equation (4.13), in the open neighborhood of the origin, in some finite time $\bar{k} > 0$. Then the closed loop solution of such persistently disturbed actuator saturated system, $\phi_k(x_k, \kappa_k, w_k; c^V)$, reaches a residual set $\mathcal{T}$ according to Lemma (4.1); and such system is called RAS-$\mathcal{T}$ (c.f. Definition 4.20). This implies that, as $\sqrt{Q_i}x_k \equiv \phi(\cdot)$, $\phi(\cdot) \in \mathcal{T} \subseteq \mathcal{X}_\infty$. Interestingly, as $\kappa(\cdot) \to \infty$ then $u_k \to 0$ (for specifically $k \geq \bar{k}$) as the system is both controllable and observable according to the assumption 4.1. ■

Corollary 4.4 *A robust optimal stabilizing controller exists, if there exists positive definite solution matrices, P_k^I and P_k^{II}, for the cross coupled algebraic Riccati equations, such that the closed-loop system solution satisfies $\phi(\cdot) \in \mathcal{X}_F$, and will asymptotically reach the robust control invariant set, $\mathcal{X}_\infty$, in some finite time $k = \bar{k} > 0$, for the given controller parameter $\{\gamma, Q_i, R\}_{i=1,2}$.*

Proof It is enough to prove that the solution matrices $\{P_k^I,\ P_k^{II}\}$ should be positive definite to ensure stability, satisfying monotonicity property. Again, the necessity for these matrices to be positive definite comes directly in the sense of Lyapunov, which completes the proof. ■

The only remaining part that has to be shown is the continuity of the Lyapunov function to ensure robustness of the closed-loop system as emphasised in [70]. So the subsequent part of this section is meant to prove this claim for the NGM-MPC controller.

The extended Bounded Real lemma is not a sufficient condition to ensure robust closed-loop stability for input constrained systems, especially when the plant is operated at the boundary of the input constrained set $\partial\mathcal{U}$ *i.e.*, as the control input hits the saturation limits of the actuator. Thus the set theoretic ideas discussed so far could be used to deal with such constrained systems. In the following, using set theoretic concepts, the feasibility of the control law and continuity of the value function is shown to exist for the Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state feedback control, under such typical situations.

The continuity of the value function and the control law, when the states belong to the feasible invariant set, $x_k \in \mathcal{X}_F$ could be established by extending the results found in [65].

Theorem 4.8 (Continuity of NGM-MPC control law and its Value function) *For the given admissible control and admissible disturbance sets, if they belong to a*

C set[‡], then the NGM-MPC control law $\mu(\cdot)$ is unique and Lipschitz continuous on $\mathcal{X}$. Moreover, the value function $V_k^{NG*}(\cdot)$ is strictly convex and Lipschitz continuous on $\mathcal{X}$.

Proof Let us first define

$$\mu = \left\{ \kappa_k(x_k) =: u_k, \; : \; u_k \in \mathcal{U}, \; \forall k \geq 0 \right\} \tag{4.39}$$

then, from the cost function of the $\mathcal{H}_2$, we get

$$\mathcal{V}(x) = \left\{ V_k^{NG^*}(x_k) \middle| \min_{u_k \in \mathcal{U}} J_2(x_k, u_k; k), \;\; \forall k \geq 0 \right\}. \tag{4.40}$$

If the admissible control and admissible disturbance sets are convex, compact, closed sets, which implies $\mathcal{V}(x)$ is also a convex, compact, closed set, since there will be a projection of μ onto a subspace. It is easy to verify that $(x_k, u_k) \mapsto V_k(x_k)$ is strictly convex function. This mapping in turn depends on the solution of cAREs. By using the general idea of continuity of the Riccati equation and thereof for the solution pair of the cAREs, it could easily be understood that $\mathscr{P} \mapsto \mu$ is continuous[†]. Eventually, it follows that the time-varying control law $\kappa_k(x_k)$, $\forall k \geq 0$ is continuous, piecewise affine function on $\mathcal{X}$ and that the corresponding set of $V_k^{NG^*}(x_k)$ is strictly convex and piecewise quadratic function of $\mathcal{X}$. Lipschitz continuity follows immediately from the assumption that the feasible set $\mathcal{X}_F$ is compact. ■

The continuity established using the above theorem is valid as long as the control input u_k lies inside the feasible control set $\mathcal{U}$. This region of the control set do not activate the saturation nonlinearity. So the control input lies at the linear region of the actuator saturation *viz.*, let $\mathscr{L}(\mathcal{U})$ represent the linear region of the actuator saturation. The closed-loop system performance using such controllers could be given by *linear analysis*. The corresponding feasible state set $\mathcal{X}_F$ for the given $\mathcal{U}$ (c.f. Proposition 4.2) is restricted to the linear region *i.e.*, $\kappa \; : \mathrm{Dom}(\mu) \mapsto \mathscr{L}(\mathcal{U})$,

‡ A C-Set is a convex and compact set containing the origin [24].

† $\mathscr{P}$ represents the set of all solution pair $\{P_k^I, P_k^{II}\}$ that satisfies cAREs, for the given system conditions and controller design objectives.

such that $\mathcal{X}_F \subseteq \text{Dom}(\mu)$. Such linear analysis to find a suitable controller gain is seemingly more conservative.

If the saturation nonlinearity is captured in a sector-bound condition, then guaranteed stability and LQ performance is achieved when *circle criterion* is applied. In [81] both linear analysis and circle analysis for such actuator saturated system is investigated for better $\mathcal{H}_2$ performance. However, that work was limited to linear systems without any disturbance affecting the system states. When $u_k \in \partial\mathcal{U}$ (*i.e.*, when the control input is at the boundary of the input constraint set, in other words, when the control input hits the actuator saturation), for a system subjected to bounded disturbance, then for robust stabilisability of the closed-loop system the system states need to be inside the robust control invariant set $\mathcal{X}_F$.

To establish robust stability, when the control input hits the actuator saturation, first the existence of the control input at $\partial\mathcal{U}$ need to exist. For proving this claim in Theorem 4.9, the following valid assumptions are made:

- **A5** $\mathcal{S}_\infty(\cdot)$ is a convex set with non-empty interior.
- **A6** $\mathcal{U}$ is a closed, convex and compact set.

Theorem 4.9 *When the control input hits the actuator saturation, i.e.,* $u_k \in \partial\mathcal{U}$ *and the closed-loop solution* $\phi_k(x_k, \kappa_k, w_k)$ *remains in a closed contractive-set,* $\mathcal{X}_F$*, then for all* $w_k \in \mathbb{W}$ *the system is robustly stabilisable.*

Before proving the theorem, the required condition for robust stability *i.e.*, Lipschitz continuity of the Lyapunov function need to be ensured. This is given in the following lemma.

Lemma 4.6 *The value function* $V_k(x_k; w_k)$ *is at least locally Lipschitz continuous when* $u_k \in \partial\mathcal{U}$.

Proof For the system mapping given as $f : \mathcal{X}_F \times \mathcal{U} \times \mathbb{W} \to \mathcal{X}_F$, which is continuous, and for the proper choice of the controller parameter triple $\{Q_i, R, \gamma\}_{i=1,2}$, when $u_k \in \partial\mathcal{U}$, there exists some reachable sets $\mathcal{R}_k$ for the closed-loop system, called the extremal reachable sets:

$$\text{Ext}(\mathcal{R}_k) := \left\{x_{k+1} \in \mathcal{S}_\infty(\cdot, \mathcal{T}) \;\middle|\; 0 \neq w_k \in \mathbb{W},\; x_{k+1} = f(x_k, \kappa_k) \oplus \mathbb{W}),\; u_k \in \partial\mathcal{U}\right\} \tag{4.41}$$

and

$$\text{Ext}(\mathcal{R}_k) \subset \mathcal{X}_F \subseteq \mathcal{S}_\infty(\cdot, \mathcal{T}). \tag{4.42}$$

From the Definition 4.21 and Proposition 4.6, one could say for a closed set $\mathcal{U}$ there exists a robust control Lyapunov function $V(x_k; u_k, w_k)$, for the set-valued constraint $u_k \in \partial\mathcal{U}$ for all $x_k \in \text{Ext}(\mathcal{R}_k)$ such that equation (4.42) holds†. It then follows immediately from Proposition 4.2 and Proposition 4.7 that $\mathcal{X}_F \Leftrightarrow \mathcal{U}$ and the corresponding RCLF is locally Lipschitz continuous. ■

Using the results of the above lemma, Theorem 4.9 could be easily proved.

Proof of Theorem 4.9 The results of Lemma 4.6 shows the existence of Lipschitz continuous value function which supports the robust stabilisability of the system when $u_k \in \partial\mathcal{U}$ and $\mathbb{W} \neq \emptyset$.

From the above assumptions, the feasible input set $\mathcal{U}$ is both closed as well as convex. Moreover, the subset $\mathcal{X}_F$ of $\mathcal{S}_\infty(\cdot)$ is also convex. When there exist a locally Lipschitz continuous RCLF at $u_k \in \partial\mathcal{U}$ such that $\lim_{k\to\infty} x_k \in \text{int}(\mathcal{X}_F)$, then $\lim_{k\to\infty} u_k \in \text{int}(\mathcal{U})$, as $\mathcal{U}$ is a convex set. This eventually ensures robust stability of the closed-loop system as $k \to \infty$. ■

Remark 4.7 *It should also be noted that if the control input signal from the controller is too large which hits actuator saturation limit far away from the point of deflection from its linear region then nothing could be said about the robust stability of the closed-loop. So it implies that the control algorithm should take due care of the actuator capacity but at the same time it need not restrict the control signal always within the linear region of actuator operation. Furthermore, the maximal allowable*

† The control input $u_k \in \partial\mathcal{U}$ which is an admissible control such that the response x_k of the system is in $\text{Ext}(\mathcal{R}_k)$ is called the *Extremal Control*.

distance away from the point of deflection of the actuator linearity to ensure robust stability is still unexplored.

Thus designing time-varying feedback control laws, allowing the control input to operate at the actuator saturation level (without compromising the closed-loop stability) provides more liberty in the design than limiting them to lie within the linear region of the actuator saturation nonlinearity.

4.5 Simulation Results

The following example shows the performance of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC algorithms of the proposed method and that in [121] in the perspective of the above arguments.

Example 4.1 The robust control of unstable constrained system found in Example 3.3 is revisited here to show the comparison in the light of the context of the present chapter.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC in Orukpe et. al. For the unstable system given in example 3.3, the ellipsoidal sets corresponding to the level sets of Lyapunov function, by solving the LMI constraints in Problem 4.1 online at every time instance, is given in Figure 4.1.

From Figure 4.1 it could be noted that the ellipsoidal sets at each time instance contains the system states at its center. The size of the ellipsoidal set represents the maximal subset of the deviation of system states in the state-space for which the corresponding Lyapunov function stabilises the system in the presence of input saturation constraint and the bounded disturbance.

Nash game based Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC The simulation result of unstable system given in Example 3.3 using NGM-MPC is given in Figure 4.2. The ellipsoidal set at every time instance of the simulation as shown in Figure 4.2, give a convincing proof

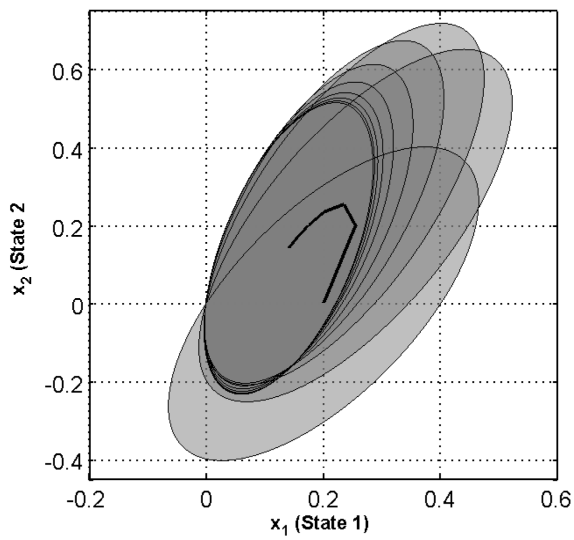

Figure 4.1: Ellipsoidal sets of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC in Orukpe et. al. for Example 3.3.

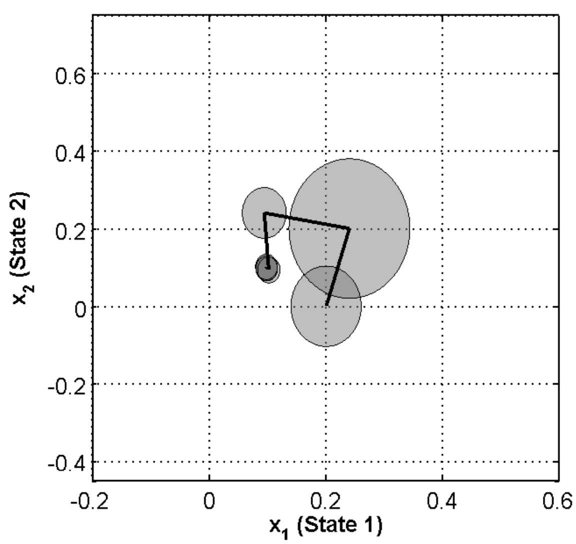

Figure 4.2: Ellipsoidal sets of Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC for Example 3.3.

for the claims made in Section 4.4.2 regarding the relation between the size of the ellipsoidal set and controller conservatism. The value of $\gamma = 7$, which is smaller than that for Orukpe et al., again confirms the analysis furnished earlier in this chapter.

In support of the claim made through Theorem 4.9, that robust stability can be ensured if the value function is Lipschitz continuous and the system states remain inside the feasible set of states (within which the system can be stabilised), the following numerical example for an unstable system is furnished below.

Example 4.2 Consider the following actuator saturation constrained discrete-time unstable system, whose system matrices are

$$A = \begin{bmatrix} 1.0070 & -1.0017 \\ 0.0050 & 1.0000 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0.0050 \\ 0.0000 \end{bmatrix};$$
$$C_z = [0 \;\; 0.6670] \; D_{zu} = 0$$

The control input is constrained $|u| \leq 10$ with the initial condition $x_0 = [1 \;\; 0]^T$. For the NGM-MPC the weighing matrices are taken as

$$Q_1 = \begin{bmatrix} 140 & 0 \\ 0 & 10 \end{bmatrix}; \quad Q_2 = \begin{bmatrix} 10 & 0 \\ 0 & 50 \end{bmatrix}; \quad R = 0.01$$

Figure 4.3 shows the closed-loop performance of NGM-MPC against Orukpe et al's for the above unstable system, whose open-loop poles are $\{1.2, 1\}$. It could be observed that the continuity and stability are not lost even though the control signal operates at the actuator saturation limit. This supports the claim made in Theorem 4.9 above. Moreover, the overall performance of the closed-loop system using the proposed control algorithm ($\gamma = 2.4$) is far better than that of Orukpe et al ($\gamma = \sqrt{43}$). Orukpe et al.'s algorithm limits the control signal to operate well within the linear region of actuator nonlinearity. Moreover, it should also be noted that the region within the bounded input set is not fully exploited by the controller, thus making it more conservative.

It should be also noted that Orukpe et al.'s algorithm was also tried with some non-zero value of D_{zu} for this example to check whether the choice might give a better performance. However, for any non-zero value for D_{zu}, Orukpe et al.'s algorithm either finds a feasible solution for larger values of γ or the algorithm fails due to lose

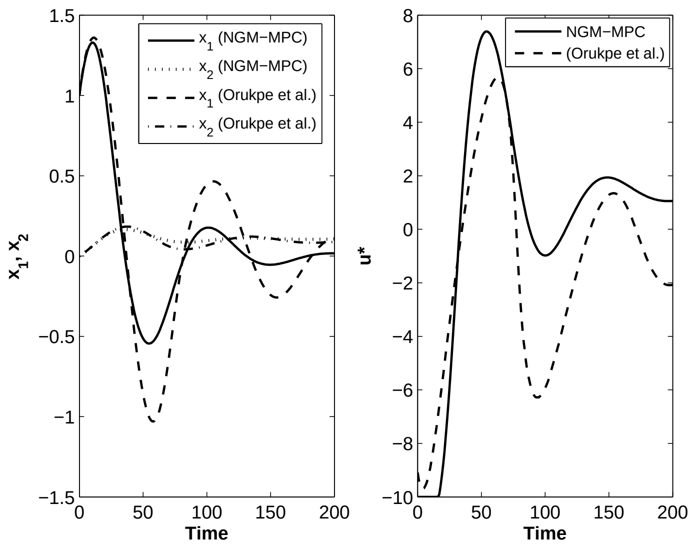

Figure 4.3: Continuity & Stability of Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC at actuator saturation.

of continuity of feasible solution (c.f. Section 4.3). However, NGM-MPC remains a successful technique in such scenario too.

4.6 Summary

The existence, feasibility, continuity and thereby robust asymptotic stability of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC control algorithms are analyzed using set theoretic concepts. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC of Orukpe et al. [121] is more conservative than the proposed Nash game based approach as the former finds the feedback control law always considering the worst-case disturbance. Moreover restricting the control signal well within the linear region of actuator nonlinearity - to ensure closed-loop stability, makes their controller overly cautious. The proposed controller that gives smaller size of the ellipsoidal sets corresponding to the time-varying feedback gain matrices computed at each time instant, is found to be less conservative. For the state feedback controller design considered in the present work, the disturbance is not explicitly measurable as that in full information case. However, its effect on the system can be implicitly inferred from the state measurement. So the controller designed based on this implicit system information (as in the present work) will be less conservative than that designed to work *always* against the worst-case disturbance.

Chapter 5

Nash game based Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC - Output Feedback Case

MPC algorithms require full knowledge of the system states while using state feedback framework. However, it is not possible to control a process directly using state feedback technique when the states of the process are not available for measurement. In such cases, the controller has to be designed with output feedback framework i.e., to control the process using the output measurements of the process. Output feedback control is accomplished using either direct output-to-input mapping or using the state feedback control by estimating the states from the process output measurement.

Linear systems, not affected by any disturbance, can be controlled by linear control techniques that employ an observer/estimator. In such cases separation principle holds good and stability can be ensured. However, stability of the closed-loop cannot be ensured, in general, by simply combining a stable estimator with a stable state feedback controller, when the system or its controller is nonlinear (as is the case with MPC for constrained systems) and the state and output disturbances are present [110]. To overcome this drawback some special care has to be taken in the robust controller design to deal with the estimation error along with modeling error.

In the present chapter, the state-feedback NGM-MPC design has been extended for output feedback case. However, the extension is not straightforward, as it is an observer based controller.

5.1 Problem Formulation

Consider the dynamic system,

$$x_{k+1} = Ax_k + B_0 w_k^0 + B_1 w_k + B_2 u_k \quad (5.1)$$

$$\tilde{z}_k = C_0 x_k + D_{01} w_k + D_{02} u_k \quad (5.2)$$

$$z_k = C_1 x_k + D_{10} w_k^0 + D_{11} w_k + D_{12} u_k \quad (5.3)$$

$$y_k = C_2 x_k + D_{20} w_k^0 + D_{21} w_k \quad (5.4)$$

where $\tilde{z}_k$ and z_k are the controlled output corresponding to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures, w_k^0 is the stochastic/random process noise (*e.g.*, due to high-gain sensors), w_k is the unknown-but-bounded disturbance (*e.g.*, the state estimation error), y_k is process measured output - satisfying the following assumptions

1. (A, B_2) is stabilisable and (C_2, A) is detectable.
2. $w_k \in \mathcal{P} \subset \ell_2$ is a bounded power signal[†] and w_k^0 is a normalized Gaussian white noise with zero mean and unit variance. Moreover $w_k^0 \perp w_j, \forall k \geq j$.
3. the initial condition $x_0 \sim \mathcal{N}(\bar{x}_0, R_0)$[‡] is independent of $w_k^0, \forall k \geq 0$.

In $\mathcal{H}_\infty$ control problem, the $\mathcal{H}_\infty$ norm $\|T_{z_k,w_k}\|_\infty^2$ can also be given in terms of the semi-norm $\|\cdot\|_\mathcal{P}$ for a disturbance bound γ as

$$\|T_{z_k,w_k}\|_\infty^2 = \sup_{w_k \in \mathcal{P}} \frac{\|z_k\|_\mathcal{P}^2}{\|w_k\|_\mathcal{P}^2} < \gamma^2, \quad (5.5)$$

[†] The set of sequences $\mathcal{P} \triangleq x : \|x\|_\mathcal{P} < \infty$ is called the set of signals with bounded power. It should be noted that $\|\cdot\|_\mathcal{P}$ is not necessarily a norm, since all the signals in $\ell_2 \subset \mathcal{P}$ have zero power. We assume $w \in \mathcal{P}$.

[‡] $\bar{x}_0$ represents the mathematical mean value of x_0 and R_0 represents its variance of the normal distribution.

whose equivalent form is

$$\sup_{w_k \in \mathcal{P}} \{ \|z_k\|^2_{\mathcal{P}} - \gamma^2 \|w_k\|^2_{\mathcal{P}} \}. \tag{5.6}$$

Being a multi-objective control problem, where the two performance functions, namely, $\mathcal{H}_2$ and $\mathcal{H}_\infty$, are given, respectively, as follows

$$J_2(w_k, u_k) \;:=\; \arg\min_{u \in \mathcal{U}} \|T_{\tilde{z}_k, w_k^0}\|_2^2 \tag{5.7}$$

$$J_\infty(w_k, u_k) \;:=\; \sup_{w_k \in \mathcal{P}} \{ \|z_k\|^2_{\mathcal{P}} - \gamma^2 \|w_k\|^2_{\mathcal{P}} \} \tag{5.8}$$

where $T_{\tilde{z}_k, w_k^0}$ represents the transfer function from w_k^0 to $\tilde{z}_k$.

As the state feedback problem is solvable, the output feedback problem is also solvable once the states are estimated by an observer. Then the resulting controller will be given as,

$$u_k^* = K_k \hat{x}_k \tag{5.9}$$

where $\hat{x}_k$ is the estimated state of the observer.

5.1.1 Solving the $\mathcal{H}_2$ problem

Let $w_k^* = H_k x_k$ be the worst case disturbance for the full information problem. On substituting this in the original system equation, one can safely eliminate the $\mathcal{H}_\infty$ part from the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, *i.e.*, eqn (5.3) will not be considered in this part of solving the $\mathcal{H}_2$ problem.

$$x_{k+1} = (A + B_1 H_k) x_k + B_0 w_k^0 + B_2 u_k \tag{5.10}$$

$$\tilde{z}_k = (C_0 + D_{01} H_k) x_k + D_{02} u_k \tag{5.11}$$

$$y_k = (C_2 + D_{21} H_k) x_k + D_{20} w_k^0 \tag{5.12}$$

Now the problem to be solved is

$$u_k^* = \arg\min_{u} \| T_{\tilde{z}_k, w_k^0} \|_2^2 \tag{5.13}$$

Generally, for a discrete time linear system given by

$$x_{k+1} = \bar{A} x_k + \bar{B} u_k + \bar{E} w_k^0, \tag{5.14}$$

$$y_k = \bar{C}_1 x_k + \bar{D}_1 w_k^0, \tag{5.15}$$

$$z_k = \bar{C}_2 x_k + \bar{D}_2 u_k \tag{5.16}$$

the solution of discrete-time $\mathcal{H}_2$ optimal control problem is given by the following two discrete-time algebraic Riccati equations (DAREs) [154];

$$P = \bar{A}^T P \bar{A} + \bar{C}_2^T \bar{C}_2 - (\bar{C}_2^T \bar{D}_2 + \bar{A}^T P \bar{B})(\bar{D}_2^T \bar{D}_2 + \bar{B}^T P \bar{B})^{-1}(\bar{D}_2^T \bar{C}_2 + \bar{B}^T P \bar{A}) \tag{5.17}$$

$$Q = \bar{\bar{A}} Q \bar{\bar{A}}^T + \bar{E}\bar{E}^T - (\bar{\bar{A}} Q \bar{C}_1^T + \bar{E}\bar{D}_1^T)(\bar{D}_1 \bar{D}_1^T + \bar{C}_1 Q \bar{C}_1^T)^{-1}(\bar{D}_1 \bar{E}^T + \bar{C}_1 Q \bar{\bar{A}}^T) \tag{5.18}$$

whose brief derivation is given in Appendix B. The above pair of DAREs could also be rewritten using matrix inversion identity† as,

$$P = \bar{A}^T P[I + \bar{B}(\bar{D}_2^T \bar{D}_2)^{-1}\bar{B}^T P]^{-1}\bar{A} + \bar{C}_2^T[I - \bar{D}_2^T(\bar{D}_2 \bar{D}_2^T)^{-1}\bar{D}_2^T]\bar{C}_2 \tag{5.19}$$

$$Q = \bar{\bar{A}} Q[I - \bar{C}_1^T(\bar{D}_1 \bar{D}_1^T)^{-1}\bar{C}_1 Q]^{-1}\bar{\bar{A}} + \bar{E}[I - \bar{D}_1^T(\bar{D}_1 \bar{D}_1^T)^{-1}\bar{D}_1]\bar{E}^T. \tag{5.20}$$

Using the analogy of the above general $\mathcal{H}_2$ control problem in eqns.(5.14)-(5.16) with the given system information in eqns. (5.10)-(5.12) and by assigning

$$\begin{aligned}
\bar{A} &= (A_C + (B_1 - B_2 R_{02}^{-1} D_{02} D_{01}) H_k) \\
\bar{\bar{A}} &= (A_O + (B_1 - B_0 D_{20}^T R_{20}^{-1} D_{21}) H_k) \\
\bar{B} &= B_2 \\
\bar{C} &= C_0 + D_{21} H_k \\
\bar{E} &= B_0 \\
I - \bar{D}_2^T(\bar{D}_2 \bar{D}_2^T)^{-1}\bar{D}_1 &= \tilde{R}_{02} \\
I - \bar{D}_1^T(\bar{D}_1 \bar{D}_1^T)^{-1}\bar{D}_1 &= \tilde{R}_{20}
\end{aligned}$$

we can get the solution for the $\mathcal{H}_2$ part of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem by solving the following two algebraic Riccati equations;

$$Ric1 \ : \quad 0 = \begin{cases} (A_C + (B_1 - B_2 R_{02}^{-1} D_{02} D_{01}) H_k)^T P_k^1 (I + B_2 R_{02}^{-1} B_2^T P_k^1)^{-1} \\ \times (A_C + (B_1 - B_2 R_{02}^{-1} D_{02} D_{01}) H_k) - P_k^1 \\ + (C_0 + D_{21} H_k)^T \tilde{R}_{02} (C_0 + D_{21} H_k) \end{cases}$$

† $I_{n\times n} - \tilde{B}(\tilde{C}\tilde{B} + \tilde{D})^{-1}\tilde{C} = (I_{n\times n} + \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}$, where $\tilde{B} \in \mathbb{R}^{n\times 1}$, $\tilde{C} \in \mathbb{R}^{1\times n}$, $\tilde{D} \in \mathbb{R}^{1\times 1}$.

(5.21)

$$Ric2 \ : \quad 0 = \begin{cases} (A_O + (B_1 - B_0 D_{20}^T R_{20}^{-1} D_{21}) H_k) P_k^2 (I - C_2^T R_{20}^{-1} C_2 P_k^2)^{-1} \\ \times (A_O + (B_1 - B_0 D_{20}^T R_{20}^{-1} D_{21}) H_k)^T - P_k^2 + B_0 \tilde{R}_{20} B_0^T \end{cases} \tag{5.22}$$

where,

$$\begin{aligned} A_C &= A - B_2 R_{02}^{-1} D_0 2^T C_0, \\ A_O &= A - B_0 D_{02}^T R_{20}^T C_2, \\ R_{02} &= D_{02}^T D_{02}, && (5.23) \\ \tilde{R}_{02} &= I - D_{02} R_{02}^{-1} D_{02}^T, && (5.24) \\ R_{20} &= D_{20} D_{20}^T, && (5.25) \\ \tilde{R}_{20} &= I - D_{20} R_{20}^{-1} D_{20}, && (5.26) \end{aligned}$$

and the state feedback controller and observer gains are

$$\begin{aligned} K_k &= -[D_{02}^T D_{02} + B_2^T P_k^1 B_2]^{-1} \\ &\quad \times [B_2^T P_k^1 (A + B_1 H_k) + D_{02}^T (C_0 + D_{01} H_k)], && (5.27) \\ L_k &= -[(A + B_1 H_k) P_k^2 (C_2 + D_{21} H_k)^T + B_0 D_{20}^T] \\ &\quad \times [D_{20} D_{20}^T + (C_2 + D_{21} H_k) P_k^2 (C_2 + D_{21} H_k)^T]^{-1} && (5.28) \end{aligned}$$

where P_k^1 and P_k^2 are the positive semi-definite matrices giving the stabilising solution to the $\mathcal{H}_2$ problem.

This part of the control problem finds out the optimal controller such that $y \mapsto u$, which has the realisation given by

$$\begin{aligned} \hat{x}_{k+1} &= (A + B_1 H_k + B_2 K_k + L_k C_2 + L_k D_{21} H_k) \hat{x}_k - L_k y_k && (5.29) \\ u_k^* &= K_k \hat{x}_k && (5.30) \end{aligned}$$

The above two equations represent the output-feedback controller, the former is an observer and the latter is a state feedback controller.

5.1.2 Solving the $\mathcal{H}_\infty$ Problem

Now, with the optimal control law $u_k^* = K_k \hat{x}_k$, the performance function $J_\infty(w,u)$ for the system given in eqns. (5.1) and (5.3), is given as

$$\begin{aligned} x_{k+1} &= (A + B_2 K_k)x_k + B_0 w_k^0 + B_1 w_k, && (5.31) \\ z_k &= (C_1 + D_{12} K_k)x_k + D_{10} w_k^0 + D_{11} w_k. && (5.32) \end{aligned}$$

Using the Stochastic Bounded Real Lemma [167] for a general discrete-time system

$$\begin{aligned} x_{k+1} &= \bar{A}x_k + \bar{B}_0 w_k^0 + \bar{B}_1 w_k \\ z_k &= \bar{C}_1 x_k + \bar{D}_{10} w_k^0 + \bar{D}_{11} w_k \end{aligned} \quad (5.33)$$

the corresponding general discrete-time $\mathcal{H}_\infty$ algebraic Riccati equation is given as,

$$\bar{A}^T X \bar{A} - X + (\bar{B}_1^T X \bar{A} + \bar{D}_{11}^T \bar{C}_1)^T (\gamma^2 I - \bar{D}_{11}^T \bar{D}_{11} - \bar{B}_1^T X \bar{B}_1)^{-1} (\bar{B}_1^T X \bar{A} + \bar{D}_{11}^T \bar{C}_1) + \bar{C}_1^T \bar{C}_1 = 0 \quad (5.34)$$

where $X \succ 0$. For some $\gamma > 0$, the worst case bounded disturbance signal is given by

$$w_k^* = (\gamma^2 I - \bar{D}_{11}^T \bar{D}_{11} - \bar{B}_1^T X \bar{B}_1)^{-1} (\bar{B}_1^T X \bar{A} + \bar{D}_{11}^T \bar{C}_1) x_k \quad (5.35)$$

In the light of the analogy of structure of the above given general discrete-time system with the system given by eqns. (5.31-5.32), one can get,

$$\begin{aligned} \bar{A} &= A + B_2 K_k \\ \bar{B}_0 &= B_0 \\ \bar{B}_1 &= B_1 \\ \bar{C}_1 &= C_1 \\ \bar{D}_{10} &= D_{10} \\ \bar{D}_{11} &= D_{11} \end{aligned}$$

and the corresponding discrete-time $\mathcal{H}_\infty$ algebraic Riccati equation as,

Ric3:

$$\left.\begin{aligned}(A+B_2K_k)^TP_k^3(A+B_2K_k)-P_k^3+[B_1^TP_k^3(A+B_2K_k)+D_{11}^T(C_1+D_{12}K_k)]^T\\ \times(\gamma^2I-D_{11}^TD_{11}-B_1^TP_k^3B_1)^{-1}[B_1^TP_k^3(A+B_2K_2)+D_{11}^T(C_1+D_{12}K_k)]\\ +(C_1+D_{12}K_k)^T(C_1+D_{12}K_k)\end{aligned}\right\}=0 \tag{5.36}$$

where $P_k^3 \succ 0$ is the stabilizing solution of the algebraic Riccati equation, with the worst-case disturbance gain

$$H_k=(\gamma^2I-D_{11}^TD_{11}-B_1^TP_k^3B_1)^{-1}(B_1^TP_k^3A+D_{11}^TC_1) \tag{5.37}$$

and

$$w_k^*=H_k\hat{x}_k. \tag{5.38}$$

Thus the equations (5.21), (5.22) and (5.36) are the three coupled algebraic Riccati equations (3-cAREs) of the optimal output feedback control problem. The coupling is attributed due to the presence of the $\mathcal{H}_\infty$ solution term, H_k, in eqns. (5.21 and (5.22); similarly, the solution of the $\mathcal{H}_2$ part of the problem, K_k, in the eqn. (5.36).

5.1.3 Saddle point solution

The saddle point solution, *viz.* Nash equilibria, for the output feedback case is similar to that of the state feedback case, except that the state information being that estimated from the observer:

$$\frac{1}{2}\hat{x}_k^TP_k^1\hat{x}_k \geq \beta_k \tag{5.39}$$

$$\frac{1}{2}\hat{x}_k^TP_k^3\hat{x}_k \leq \beta_k. \tag{5.40}$$

With this bound, updating the control law using the current state of the system, the optimal values of the stabilizing matrices *viz.* P_k^1, P_k^2 and P_k^3 are found out.

5.2 Solving the 3-cAREs

This is again similar to the case of state feedback control problem. However, it involves 3-cAREs in this case of output feedback optimal control problem.

Theorem 5.1 *For a system defined by* $\Sigma : (A, B_0, B_1, B_2)$ *with initial condition* x_0*, the two-player Nash game based mixed* $\mathcal{H}_2/\mathcal{H}_\infty$ *MPC yields an (sub-)optimal, stable solution, for the three cross-coupled algebraic Riccati equations (3-cAREs),*

$$Ric1 + \epsilon_k^1 = \boldsymbol{0} \tag{5.41}$$

$$Ric2 + \epsilon_k^2 = \boldsymbol{0} \tag{5.42}$$

$$Ric3 + \epsilon_k^3 = \boldsymbol{0} \tag{5.43}$$

and along with the saddle point optimal condition bounds,

$$\hat{x}_k^T P_k^1 \hat{x}_k \geq \beta_k \tag{5.44}$$

$$\hat{x}_k^T P_k^3 \hat{x}_k \leq \beta_k \tag{5.45}$$

if it exists, such that $(P_k^{1^*}, P_k^{2^*}, P_k^{3^*}) \succ 0$*, for some open-hyper balls of radius* $\{\epsilon_k^{1^*}, \epsilon_k^{2^*}, \epsilon_k^{3^*}\} \in \mathbb{R}^{n\times n}$ *around and containing the origin, such that the closed loop system is stable.*

Proof Let $P_k^{1^*}, P_k^{2^*}$ and $P_k^{3^*}$ be the (sub-)optimal, stable and locally unique solution set of the 3-cAREs, (5.21), (5.22) and (5.36), satisfying the conditions (5.44) & (5.45) and $u_k \in \mathcal{U}$ (i.e., satisfying equation (3.35)), with $x_0 \in \mathcal{X}_0$, for some open hyper balls of radius $\{\epsilon_k^{1^*}, \epsilon_k^{2^*}, \epsilon_k^{3^*}\} \in \mathbb{R}^{n\times n}$, respectively, in the linear space. Now assume that there exists another solution set, $\tilde{P}_k^{1*}$, $\tilde{P}_k^{2*}$ and $\tilde{P}_k^{3*}$, which again satisfies (5.21), (5.22) and (3.34) for some other hyper balls of radius $\{\tilde{\epsilon}_k^{1^*}, \tilde{\epsilon}_k^{2^*}, \tilde{\epsilon}_k^{3^*}\} \in \mathbb{R}^{n\times n}$, respectively, such that $\epsilon_k^{j^*} = \tilde{\epsilon}_k^{j^*} + \delta_j \in \mathbb{R}^{n\times n}$ and $\delta_j \ll \tilde{\epsilon}_k^{j^*}$ $j = 1, 2, 3$.

Likewise, there may exist a different solution set for a different choice of δ_j for every other choice of solution set. Therefore, as $|\delta_j| \to 0$, in the linear space the open hyper balls converges to the (sub-)optimal solution,

$$\lim_{|\delta_j|\to 0} \epsilon_k^{j^*} = \tilde{\epsilon}_k^{j^*} + \delta_j = \tilde{\epsilon}_k^{j^*}, \quad j = 1, 2, 3.$$

which in turn gives $P_k^{1^*} = \tilde{P}_k^{1^*}$, $P_k^{2^*} = \tilde{P}_k^{2^*}$ and $P_k^{3^*} = \tilde{P}_k^{3^*}$, thus giving an (sub-) optimal, stable and locally unique solution set to the 3-cAREs. ■

One of the main objectives of the thesis is to extend the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC for the output feedback case to use it with the Laguerre-Wavelet network (LWN) model. For doing so, as the static nonlinear wavelet network portion cannot be inverted, it has to be linearised online, using some efficient numerical techniques such as *Automatic Differentiation* (AD). The use of tools such as AD for linearisation not only reduces the computational demand of numerical differentiation but also gives error-free differentiation values, usually up to the machine precision level.

5.3 Successive Linearisation Model

Model predictive control relies explicitly on the process model to make an optimal decision on the control input/law to be implemented at every time instance. Hence the correctness of the process model at the prevailing operating condition, is a major factor for efficient performance of MPC. The Wiener type Laguerre-Wavelet Network model (c.f. Chapter 2) is a data-driven nonlinear model. Usually the input-output data collected for identifying nominal stable systems are obtained by exciting the process about its nominal equilibrium point. Hence the efficient control of the process by linearising the model at the nominal operating point and controlling the process using linear controller is possible only in the close vicinity of the nominal operating point of the process. However, in situations where the process operating point is shifted far away from the previous value, the above mentioned model (linearised at the nominal operating point)-based control strategy can fail miserably.

Successive linearisation approach [142] could be adopted to circumvent the above shortcoming. The successive linearisation of the static nonlinear gain by finding the Jacobian linearisation of the nonlinearity around the nominal state trajectory is made to obtain a local linear equivalent model of the state-to-output mapping. So linearising the wavelet network about the nominal value of the current Laguerre states gives a local linear state-to-output gain. The necessary condition to ensure a smooth linearized representation is that the static nonlinear gain Ψ (c.f. Chapter 2) should be Lipschitz continuous. Thus instead of *a priori* offline piecewise linearisation over the entire range of the nonlinear gain and storing the models to represent the system as a piecewise affine system, the online successive linearisation approach demands lesser memory requirement. Although the associated computational cost and the approximation error may be forbidden with earlier numerical techniques to find the Jacobian, the numerical derivative techniques such as *Automatic Differentiation* [69, 54, 144] too, could be used to overcome these issues. So automatic differentiation method is used for the successive linearisation using the MAD Module of TOMLAB® Version 6.1 in the present work.

5.3.1 Automatic Differentiation

From the standard reference for the subject [69], it is stated that,

Algorithmic or Automatic Differentiation (AD) is concerned with the accurate and efficient evaluation of derivatives for functions defined by computer programs.

AD is a numerical differentiation technique that uses chain-rule of calculus (differentiation) for the floating point evaluation of a function and/or its derivatives [144]. The principal difference between AD and the usual numerical finite difference methods is that in AD there is no discretisation or cancellation errors; moreover, the results (derivative values) are accurate to the machine precision or within the round-off used. Symbolic differentiation methods makes a function derivative into a single long expression at the point where the function's derivative has to be evaluated. This drastically increases the memory requirement for evaluating the derivatives. However, on the other hand, AD uses the common control structures such as loops, branches, sub-functions, etc., unlike symbolic differentiation methods. Also, AD works on floating point numbers making the results more accurate. Thus AD is more efficient than symbolic differentiation.

Usually AD is implemented in either of two ways: (i) *Operator Overloading* (ii) *Source Transformation*. In operator overloading the existing classes or types are redefined, initialised with appropriate values for the function and its derivative, invoke the function, and find the values of the derivatives. Source Transformation is a method that requires suitable sophisticated compiler-like software to read the computer program containing the function and determine which statements for the function's program requires differentiation. Accordingly, a new version of the given/original program is generated augmenting with the statements for which derivatives need to be calculated.

There are fundamentally two approaches for computing the derivatives using AD.
(i) *Forward Mode* : The function is evaluated for its derivative in the forward direction starting from the input (or independent) variable. In this forward mode of calculation, the sensitivities for finding the derivatives of the intermediate stages are propagated forward until the final result.
(ii) *Reverse Mode* : This works in two stages. In the first stage the original code is run and augments it with the statements to store data. Then the derivative is evaluated starting from the output (dependent variable) towards the input using all the sensitivities (also called adjoints) in the reverse direction in its second run.

There are many AD packages that works either in forward and/or reverse mode by implementing usually operator and function overloading techniques. ADOL C, ADIC, Sacado etc. are developed for working in C/C++. ADF, OpenAD, ADIFOR runs with Fortran. Python based AD packages are pyadolc, algopy, etc. MATLAB compatible version of AD is developed by TOMLAB® Inc. called MAD.

In the present work, TOMLAB®-MAD has been used for the purpose of linearisation, so as to reduce the error, incurred due to the evaluation of derivative values computationally, by methods such as finite difference. The use of AD for finding the gradient or Jacobian using MAD software package is efficient and fast enough to implement it online with a regular PC.

5.4 Numerical Example

To demonstrate the performance of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback MPC, a simple stable discrete time LTI system is taken. Choice of one such example could be justified with the claim that the developed controller will be used with the proposed Laguerre Wavelet Network model, where the linear dynamic part of the of the model, i.e., Laguerre filter will always have stable poles. A second order stable system, shown below, with initial condition $x_0 = [1 \;\; 0]^T$ is taken for this purpose. The minimal workable γ value for the present example using the proposed robust MPC algorithm is 5.

$$A = \begin{bmatrix} 0.80 & -0.09 \\ 1.00 & 0.00 \end{bmatrix} \quad B_0 = \begin{bmatrix} 0.1 \\ 1.0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0.1 \\ 0.0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}$$

$$C_0 = [1 \;\; 1] \;\; C_1 = [1 \;\; 0] \;\; C_2 = [1.10 \;\; -0.07] \;\; D_{00} = 1.0 \;\; D_{01} = 0.1$$

$$D_{02} = 0.001 \;\; D_{10} = 0.01 \;\; D_{11} = 1.0 \;\; D_{12} = 0.1 \;\; D_{20} = 0.1 \;\; D_{21} = 0.0$$

The system is perturbed with the deterministic disturbance, $w_k = 0.1$ and the stochastic gaussian white noise (w_k^0) has mean $= 0$ and standard deviation $= 0.1$ $\forall k \geq 0$. The closed loop performance is given in Figures 5.1 and 5.2. It can be observed that the effect of the measurement noise component w_k^0 does not affect that controller stability. Moreover, the presence of persistent bounded power signal w_k has its effect on the state estimation, which does not allow the estimation error of one of states (e_2) to decay to zero. However, the presence of such a non-zero bounded power disturbance does not affect the closed-loop stability of the process.

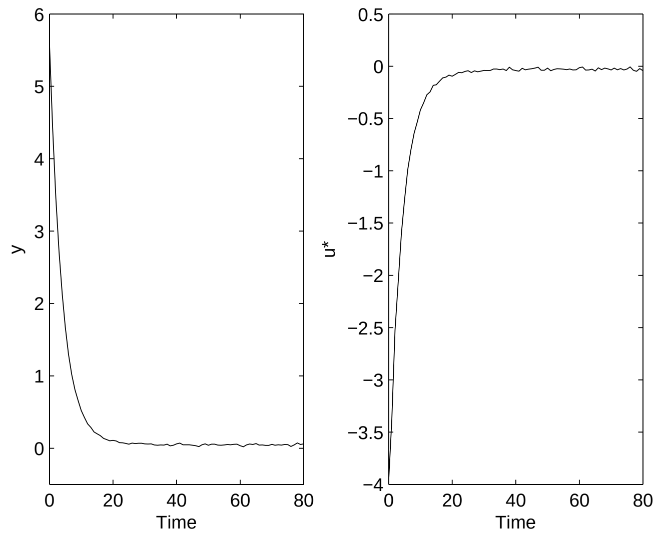

Figure 5.1: Input and output profiles of the Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC.

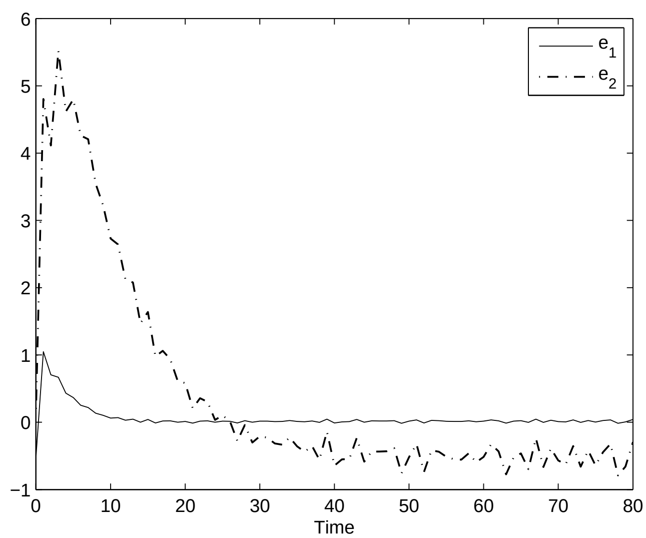

Figure 5.2: State estimation error profiles of the Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC.

5.5 Bioreactor Control

The robust control of the benchmark process *viz.*, bioreactor, is one of the objectives of the thesis. For achieving this, the dynamics of bioreactor system, which exhibits input-multiplicity nonlinearity, is efficiently captured through the nonlinear system identification method using LWN model from the process input-output data, as described in Chapter 2. The mechanistic model of the bioreactor is assumed to be the actual process for the simulation case studies.

5.5.1 Control of Bioreactor using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Output feedback MPC

In the present study on the bioreactor process, the closed loop robustness due to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback MPC is manifested against system parametric uncertainty.

For any data-driven model developed using a suitable system identification technique, *e.g.*, the LWN model for the bioreactor process under consideration, such discrepancies eventually leads to plant-model mismatch. Had there been any adaptive mechanism as adopted in [133], this plant-model mismatch can be circumvented in an efficient way. However, this adaptive procedure is not always feasible in real-time application mainly due to the complexity of the model and/or the computational cost associated with it. In such cases, the burden comes to the controller design, to achieve the desired plant operation. In the present bioreactor example, the nonlinearity of the process is due to state-to-output nonlinear static gain, which is represented by the wavelet network in the Wiener type LWN model.

For some processes the plant-model mismatch may result in deterioration of the performance or may even lead to catastrophic effects, depending on the nature of the process. The state-feedback controller design, such as those presented in Chapter 3 cannot be deployed for such processes, where the system states are not directly measurable for control, or, for model based control cases, where the states of the model do not reflect the actual process states. For any black-box model, usually, the model states fall under the later category. In such cases, it is wise to use *Successively linearized deviation* (SLD) model, than simply employing successive linearisation technique as given in section 5.3. State-Deviation (SD) model is one in which the model, whose states represent deviation of the states from a reference value (usually the previous value of the model state) as $\tilde{x}_k := x_k - x_{k-1}$, developed using the identification technique. Then w.r.t. the deviation state, at every time instance, the static nonlinear portion of the Wiener model is linearized. This will result in a *Successive linearized Deviation* (SLD) model. SLD model is sought because the NGM-MPC control algorithms (both state-feedback and output feedback) consider the origin as the nominal equilibrium point, *i.e.*, $\{0, 0\} \in \text{int}\{\mathbb{X} \times \mathbb{U}\}$. The response of the SLD-LWN model for the bioreactor process, excited by a sinusoidal input signal about the nominal input feed concentration of 20 g/l is shown in Figure 5.3. It could be observed that there is not much distortion in the performance of the LWN model due to successively linearized deviation approach. Hence the approximation error of the nominal model due to the linearisation approach is minimal and thereby the controller's capacity is maximally exploited to handle plant-model mismatch, due to approximation or worst-case disturbance.

The plant-model mismatch can occur due to the parametric change in the process. For instance, the maximum specific growth rate of the microbes (μ_m) is taken as 0.48 in the present nominal case. The nominal LWN model is developed using the input-

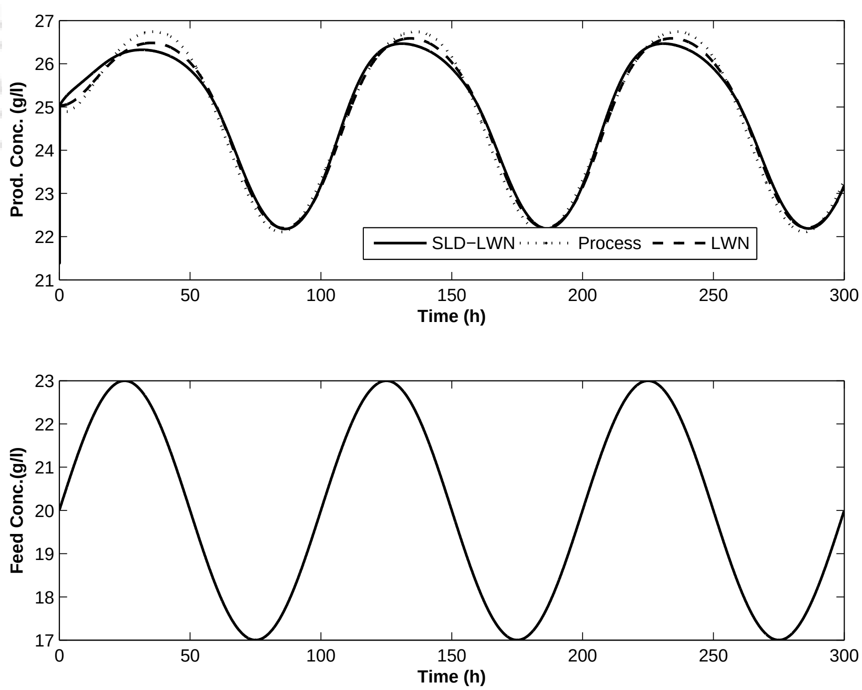

Figure 5.3: SLD-LWN model response for sinusoidal input signal.

output data of the process only for this value of μ_m. However, there could always exist the possibilities of change in the maximum specific growth of the microbes (μ_m) due to external reasons such us change in ambient temperature, etc. So the LWN model developed using the nominal value cannot represent the actual process in those circumstances. This will cause a shift in the nominal operating point of the process and hence introduces plant-model mismatch.

The closed-loop performance of the Output feedback NGM-MPC controller for step change in μ_m from its nominal value of 0.48 to 0.44 and 0.46 are given in Figures 5.4 and 5.5, respectively. The value of γ taken in both the cases for the controller design is 1.55. Change in the process parameter is also considered as disturbance rejection control problem and the process is controlled using the output-feedback NGM-MPC given in the present chapter.

5.5.2 Comparison of Output Feedback MPC Schemes

The efficacy of the proposed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output-feedback MPC (Controller A) is compared with another output feedback MPC design technique [51] (Controller B), for a class of nonlinear systems represented similarly by Wiener type nonlinear model. Although the model considered in [51] is Hammerstein-Wiener type, the Hammerstein portion at the input side of the model is removed by means of a suitable inverse function. The static nonlinearity at the output side of the model is interestingly represented by suitable linear inclusion as polytopic descriptions for controlling a class of constrained nonlinear systems. Moreover, the referred paper deals with an infinite horizon min-max output feedback MPC problem to guarantee closed-loop stability. These are some common design similarities of reference [51] with that of the output feedback controller developed in this thesis, that makes comparison of their performances reasonable. In this thesis, comparison of the closed-loop controller performance for controlling an uncertain nonlinear process *viz.*, continuous bioreactor, with the proposed controller design and that in [51] is made.

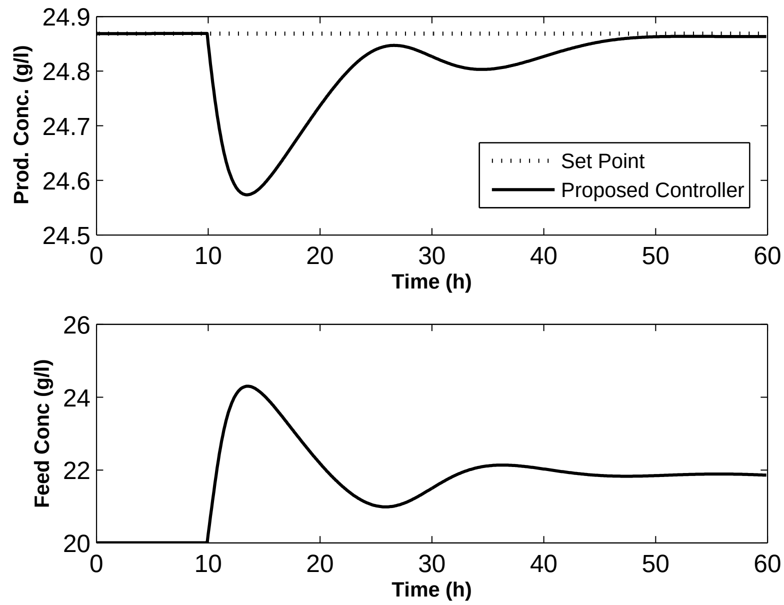

Figure 5.4: Parametric Uncertainty - Case (i) : Output and input profiles of the Bioreactor using Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC for a step change ($\mu_m = 0.44$).

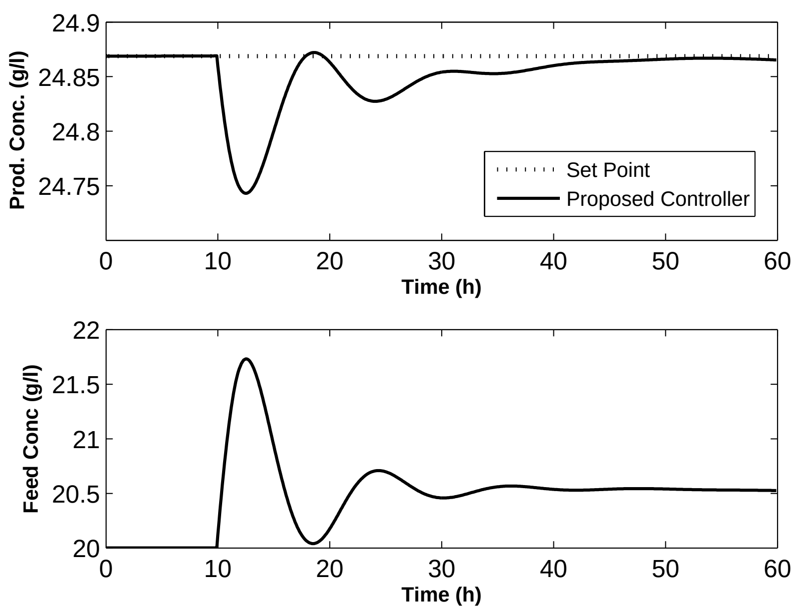

Figure 5.5: Parametric Uncertainty - Case (ii) : Output and Input profiles of the Bioreactor using Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC for a step change ($\mu_m = 0.46$).

The comparison is made for a parametric change of μ_m from its nominal value of 0.48 to 0.46 from 10th hour. Surprisingly, there is hardly any control action by the Controller B given in [51]. However, the proposed Controller A stabilizes the process and bring the process variable to its nominal value despite the parametric uncertainty. Although the failure of Controller B is surprising, the reason for this failure could be attributed to the fact that the linearising technique used by the authors of Controller B are never mentioned explicitly and it is perhaps the fragility of the linearisation that fails the controller. Nevertheless, in this thesis, the linearising technique used for controller A & B are the same *viz.*, Automatic differentiation based Successive Linearisation technique. Thus the effect of the comparison is fairly justified. A detailed MATLAB code for Controller-B is added in the Appendix C for the benefit of the reader.

5.6 Fragility

The fragility of the robust output mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Model Predictive Controller is addressed in this section. Although the developed controller is robust with respect to the disturbances and/or the plant-model mismatch as discussed in the previous sections, it shifts the sensitivity in the closed loop from the plant parameters to the controller parameters, which is termed as the *fragility* of the controller.

5.6.1 Fragility of robust controller designs

It is an usual and mostly wide presumption that in any robust controller design, the uncertainty in the plant is the most significant type of uncertainty in a control system. This is often relatively valid, as controllers are usually implemented with high precision hardware. However, it is also relatively necessary that any controller which is a part of the control system should be tolerant to some level of uncertainty in its coefficients. Thus any analytic design of a robust controller should have sufficient room for readjustment of its coefficients, as a simple scalar index in the controller design cannot capture all the performance objectives of a real world control system. This in turn needs the existence of a non-zero margin of tolerance of the designed nominal controller to ensure adequate stability and performance measures around its design coefficients. Such an issue of controller sensitivity is popularly referred to as fragility of robust controller design.

It has been demonstrated in the literature that even a small perturbation in the coefficients of the nominally designed controller can destabilize the closed-loop system [18] for various robust controller formulations *viz.*, $\mathcal{H}_2$, $\mathcal{H}_\infty$, μ and ℓ_1 design. On the other hand, to benefit from the combined effect of the two important robust controller design, *viz.*, $\mathcal{H}_2$, $\mathcal{H}_\infty$, the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller was introduced in [14]. Subsequently many version of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller have been developed. The issue of fragility is not confined only to high order controllers but also to low order controllers.

Design of robust controller for constrained dynamic systems is a demanding area of research, in the regime of model predictive control (MPC) design. Of the many different control algorithms introduced so far, a game theoretic version of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller in the pursuit of model predictive control has been proposed in chapter 5 of this thesis for the output feedback version of the NGM-MPC of chapter 3. It should be noted that the state feedback case of NGM-MPC is found to be robust both in terms of the plant uncertainty and the uncertainty of controller coefficients to a prescribed level. However, the output feedback version of the NGM-MPC robust controller does not escape the net of the class of above mentioned fragile robust controllers.

This section serves as a note on the robustness and fragility of the output feedback Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC, which is demonstrated by benchmark bioreactor example used in this thesis. Although no remedy to overcome this bottleneck in the controller design could be recommended, the aim is to give a cautionary note on the present output controller design, adding to those listed in Bhattacharrya et al.[18].

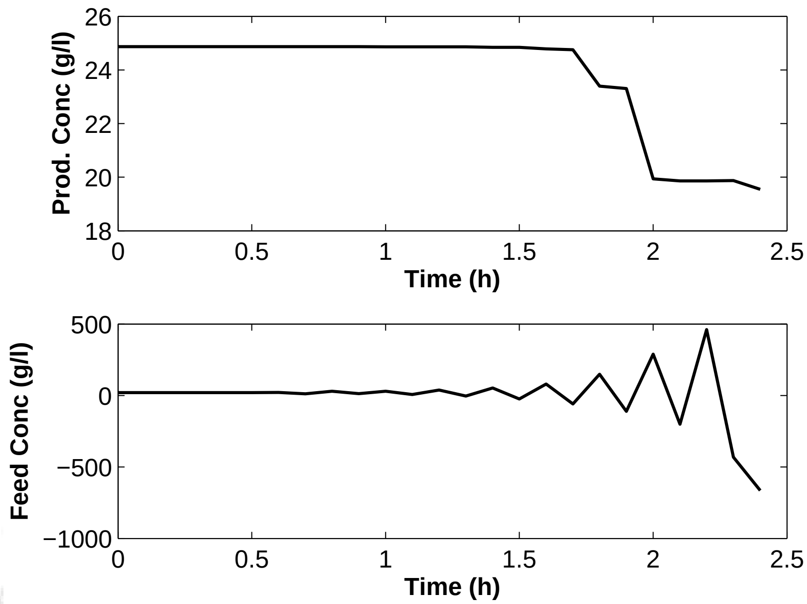

Figure 5.6: Fragility of the Output Feedback Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC for $\sim 3\%$ change in the γ value.

5.6.2 Fragility of Output Feedback Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC

Let us revisit the control of bioreactor process using the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design given in Chapter 5. To demonstrate the fragility of the controller w.r.t. the controller parameters, let us change the controller's important design parameter γ from earlier used value of 1.55 to 1.6, i.e., approximately a change of 3%. This small change in the controller parameter causes instability of the closed-loop system (Figure 5.6).

So it could be understood that there are some zones in the controller parameter space and if the controller design parameters are in the proximity of such zones, this can lead to instability of the system.

5.7 Summary

In the present chapter, an observer based infinite-horizon output feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC is designed, for those cases where the actual process states are not directly measurable. The control problem results in solving three coupled algebraic Riccati equations. Moreover, as the output feedback MPC is designed for linear dynamic processes, to use it with a class of nonlinear processes represented by Wiener type models, successive linearized deviation (SLD) model is used. For reducing the error due to numerical method of linearisation, Algorithmic/Automatic Differentiation technique is advocated. The LWN model developed in Chapter 2 is used in the output feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design for controlling the benchmark process (bioreactor). Finally, the fragility associated with the output feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design is demonstrated with a cautionary note.

Chapter 6

Conclusions & Suggestions for Future Work Direction

From the present work on nonlinear system identification and game theory based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ model predictive control the following conclusions are drawn.

1. Orthogonal basis filters, especially Laguerre filters, are effectively combined with wavelet network in Wiener type structure, which form Laguerre-Wavelet network (LWN) model. The LWN models are used for modeling of processes exhibiting input multiplicity, such as bioreactor, by constructing them in Wiener type structure. The Wiener type LWN is found to outperform the other Laguerre filter based Wiener models reported in the literature and is also parsimonious.

2. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ robust optimal controller, having the combined merits of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers, has been recently extended to the MPC paradigm [121], as a closed-loop MPC. As in mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design, where two conflicting objectives *viz.*, minimization of control effort ($\mathcal{H}_2$ problem) against the disturbance effect ($\mathcal{H}_\infty$ problem) is sought, in the present work a MPC algorithm has been designed for the first time using an explicit Nash game based approach.

3. The proposed Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC control problem is solved as an infinite horizon control problem, assuring nominal closed-loop stability. This control problem results in ultimately solving a pair of cross-coupled algebraic Riccati equations at every time instant, along with saddle point conditions. Moreover, the proposed controller is found to be less conservative than the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC by Orukpe et al [121].

4. Set theory based analysis is given for the performance of both mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC controller design, with the reasoning of relative conservatism in both designs.

5. The state feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC based on game theoretic approach cannot be directly used along with a suitable state observer when system states are not directly measurable. Moreover, it does not always guarantee closed-loop stability while controlling a nonlinear process. Hence, the output feedback version of Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC is developed which ultimately results in solving three coupled algebraic Riccati equations for an infinite horizon control problem.

6. Nash game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback MPC is applied to control bioreactor system with a LWN model of the bioreactor. This is achieved by using the LWN model as a deviation model and by successively linearising the deviation model along the state trajectory. For reducing the approximation error and to improve the controller performance, symbolic derivation algorithm, namely, automatic differentiation is employed.

7. Finally, a cautionary note is given on the fragility of the output feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC controller due to its sensitivity to its own parametric changes.

There are still some unsolved issues in connection with the contents of this thesis, which are briefly described below.

- The Laguerre-Wavelet network model has been developed for a SISO system. This could be extended for a $n \times m$-MIMO system as m-MISO model, where

m represents the number of outputs and n the number of inputs of the MIMO system. The real bottleneck found in extending the LWN model to MIMO systems is the right technique to find the optimal values of the Laguerre poles.

- The LWN model can be extended for representing other classes of nonlinear process using Hammerstein and/or Hammerstein-Wiener type structures.
- Although the Nash-game based mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC could operate at the actuator saturation limits without significant lose in the performance, the maximal tolerable limit of saturation away from the linear region is still unknown.
- The output feedback mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC design is found to be fragile to its parameter changes. Hence, an alternative way of designing output feedback mixed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ MPC can be approached.
- The controller design can be extended for other interesting and demanding processes such as integrating processes.

An interested researcher may venture into the above issues while performing research in the related area.

Appendix

A Laguerre-Wavelet Network model parameters

The LWN model parameters for the bioreactor and Pasteurization process examples are furnished in the following (See next page) Table I and II, respectively.

Table I : LWN model parameters for bioreactor

Translation parameter									
102.69	153.99	51.801	22.751	-187.68	19.776	22.628	9.055	4.847	24.087
-2.106×10^{5}	1.7553×10^{5}	8.6637×10^{5}	2.4475×10^{5}	2.8461×10^{5}	2.0058×10^{5}	2.5093×10^{5}	0.1313×10^{5}	6.1368×10^{5}	1.4858×10^{5}
Dilation parameter									
0.030193	0.028542	0.034665	0.10503	0.0072464	0.11148	0.41246	0.44146	0.027143	0.08527
5.9612×10^{-7}	1.8962×10^{-6}	4.9623×10^{-6}	7.1622×10^{-8}	5.8409×10^{-6}	1.0508×10^{-5}	3.6367×10^{-5}	3.7851×10^{-5}	4.4493×10^{-5}	4.7636×10^{-6}
5.8188×10^{-11}	4.7607×10^{-11}	1.1311×10^{-10}	3.1043×10^{-10}	2.2261×10^{-10}	6.8331×10^{-10}	2.1361×10^{-9}	2.3862×10^{-9}	2.3101×10^{-9}	2.8067×10^{-10}
Weights									
-4.8292	-5.5492	-12.59	-16.249	-12.613	12.839	-0.33935	-21.197	-11.048	-12.415
Linear Co-efficient			Output bias	Laguerre Co-efficient					
				No. of filters (N)	Time scale (p)	Sampling Time (T)			
-0.079592	6.8168×10^{-06}	-4.4967×10^{-10}	22.024	3	0.1	0.1 min			

Table II : LWN model parameters for Pasteurization process

Translation parameter						
299.00	280.74	290.35	167.61	122.54	59.071	432.07
3.2918×10^{6}	4.4616×10^{6}	4.2192×10^{6}	-5.7415×10^{6}	4.106×10^{6}	5.6566×10^{6}	3.8732×10^{5}
Dilation parameter						
0.005229	0.0067281	0.019234	0.0090542	0.033023	0.015978	0.035793
1.9733×10^{-7}	3.4307×10^{-7}	7.3500×10^{-7}	9.7443×10^{-8}	2.9049×10^{-7}	2.3522×10^{-7}	1.0664×10^{-6}
Weights						
-19.018	-25.334	-55.376	-8.6193	-10.157	141.07	-1.475
Linear Co-efficient		Output bias	Laguerre Co-efficient			
			No. of filters (N)	Time scale (p)	Sampling Time (T)	
0.033183	-8.2105×10^{-06}	21.092	2	0.08	0.09 min	

B Discrete-time Algebraic Riccati Equations of $\mathcal{H}_2$ optimal control

For a discrete time linear system given by

$$x_{k+1} = \bar{A}x_k + \bar{B}u_k + \bar{E}w_k, \tag{B1}$$

$$y_k = \bar{C}_1 x_k + \bar{D}_1 w_k, \tag{B2}$$

$$z_k = \bar{C}_2 x_k + \bar{D}_2 u_k \tag{B3}$$

let the energy (Lyapunov) function associated with it is defined as $V_k(x_k) := x_k^T P_k x_k$.

As the horizon tends towards infinity in an infinite-horizon control problem, it is known that

$$\Delta V_k(x_k) := V_k(x_{k+1}) - V_k(x_k) = 0. \tag{B4}$$

Moreover, the controlled output z_k is assumed to reach their nominal values, so that

$$z_k^T z_k = 0. \tag{B5}$$

Using the definition of the Lyapunov function and the dynamic state equation, $x_{k+1} = Ax_k + Bu_k$, for part of the controller design in $\mathcal{H}_2$ optimal control problem, we get

$$\Delta V_k(x_k) = (\bar{A}x_k + \bar{B}u_k)^T P_k (\bar{A}x_k + \bar{B}u_k) - x_k^T P_k x_k \tag{B6}$$

$$= x_k^T \bar{A}^T P \bar{A} x_k + x_k^T \bar{A}^T P_k \bar{B} u_k + u_k^T + u_k^T \bar{B}^T P_k \bar{B} u_k - x_k^T P_k x_k \tag{B7}$$

On the other hand, eqn. (B5) gives,

$$z_k^T z_k = [x_k^T \;\; u_k^T] \begin{bmatrix} \bar{C}_2^T \\ \bar{D}_2^T \end{bmatrix} [\bar{C}_2 \;\; \bar{D}_2] \begin{bmatrix} x_k \\ u_k \end{bmatrix} = 0. \tag{B8}$$

Therefore, the summation of eqns. (B7) and (B8) also equates to zero *i.e.*, $S := \Delta V(x_k) + z_k^T z_k$,

$$\begin{aligned} S &= [x_k^T \;\; u_k^T] \left(\begin{bmatrix} \bar{A}^T P \bar{A} - P & A^T P B \\ \bar{B}^T P \bar{A} & \bar{B}^T P \bar{B} \end{bmatrix} + \begin{bmatrix} \bar{C}_2^T \bar{C}_2 & \bar{C}_2^T \bar{D}_2 \\ \bar{D}_2^T P \bar{C}_2 & \bar{D}_2^T P \bar{D}_2 \end{bmatrix} \right) \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ &= [x_k^T \;\; u_k^T] \begin{bmatrix} \bar{A}^T P \bar{A} - X + \bar{C}_2^T \bar{C}_2 & \bar{A}^T P \bar{B} + \bar{C}_2^T \bar{D}_2 \\ \bar{B}^T P \bar{A} + \bar{D}_2^T P \bar{C}_2 & \bar{B}^T P \bar{B} + \bar{D}_2^T P \bar{D}_2 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ &= 0. \end{aligned} \tag{B9}$$

From the above, we get the following algebraic Riccati equation for the controller part, as given below:

$$P = \bar{A}^T P \bar{A} + \bar{C}_2^T \bar{C}_2 - (\bar{C}_2^T \bar{D}_2 + \bar{A}^T P \bar{B})(\bar{D}_2^T \bar{D}_2 + \bar{B}^T P \bar{B})^{-1}(\bar{D}_2^T \bar{C}_2 + \bar{B}^T P \bar{A}) \tag{B10}$$

Defining,

$$\mathcal{M} := \begin{bmatrix} \bar{A}^T P \bar{A} - P + \bar{C}_2^T \bar{C}_2 & \bar{A}^T P \bar{B} + \bar{C}_2^T \bar{D}_2 \\ \bar{B}^T P \bar{A} + \bar{D}_2^T P \bar{C}_2 & \bar{B}^T P \bar{B} + \bar{D}_2^T P \bar{D}_2 \end{bmatrix} \tag{B11}$$

the stabilising controller $\bar{K} \in \mathbb{R}^{1 \times n}$, from

$$\mathcal{M} \begin{bmatrix} 0 \\ I \end{bmatrix} = 0 \tag{B12}$$

for a positive definite matrix $P \in \mathbb{R}^{n \times n}$, is given by

$$\bar{K} := (\bar{B}^T P \bar{B} + \bar{D}_2^T P \bar{D}_2)^{-1}(\bar{B}^T P \bar{A} + \bar{D}_2^T P \bar{C}_2). \tag{B13}$$

Following the property of *duality* is utilised for the estimation part of the $\mathcal{H}_2$ optimal problem.

Controller	Estimator
$\bar{A}$	$\bar{\bar{A}}^T$
$\bar{B}$	$\bar{C}_1^T$
$\bar{C}_2$	$\bar{E}^T$
$\bar{D}_2$	$\bar{D}_1^T$
P	Q

where $Q \in \mathbb{R}^{n\times n}$ is a positive (semi-)definite matrix.

Accordingly, the algebraic Riccati equation for the estimation part is obtained from eqn. (B10) and the above given duality, as follows

$$Q = \bar{\bar{A}}Q\bar{\bar{A}}^T + \bar{E}\bar{E}^T - (\bar{\bar{A}}Q\bar{C}_1^T + \bar{E}\bar{D}_1^T)(\bar{D}_1\bar{D}_1^T + \bar{C}_1Q\bar{C}_1^T)^{-1}(\bar{D}_1\bar{E}^T + \bar{C}_1Q\bar{\bar{A}}^T) \quad \text{(B14)}$$

and the corresponding estimator gain $\bar{L} \in \mathbb{R}^{n\times n}$ is given as

$$\bar{L} := (\bar{\bar{A}}Q\bar{C}_1^T + \bar{E}\bar{D}_1^T)(\bar{D}_1\bar{D}_1^T + \bar{C}_1Q\bar{C}_1^T)^{-1}. \quad \text{(B15)}$$

C MATLAB code for bioreactor example using MPC given in Ding and Huang (2007) [51]

```
function DiangHuangBio

clc

clear all


load biopara.mat;%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

 load LWParambio_DevF.mat;


[A,B2] = Laguerre_filter_Matrices(p,T,N);

B1 = 1*[1;01;01]; B0 = 01*[0.1;0.5;0.1];

C0 = 10*[1 1 1]; D00 = 1; D01 = 01; D02 =01;

D20 = 001; D21 = 100;gamma = 1.5;


%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


u_opt=biopara.Feeds;

x = inv(eye(N)-A)*B2*(biopara.Feeds-u_opt);

xE=x;


[n,n]=size(A);

Time=0*0.1;

Dilu=0.15;

Feed=20;U=[Feed];

u_opt=biopara.Feeds;

x = inv(eye(N)-A)*B2*(biopara.Feeds-u_opt);

xE=x;

Y = biosim(Feed,Dilu,biopara,0);

w=zeros(1,1000);
```

```
y = Y.Prod;
Y1=[y];Ym=y;
[p,q] = size(y);

[C2,xE,ym]=Model_DH(biopara.Feeds-u_opt,xE,1);

% No. of iterations
N=500;

Am=A;E = 1;
xEst=x;
cnt=0;
TIME=[];X=[];U=[];Y1=[];Ym=[];
%% INITALIZATION OF THE OUTPUT FEEDBACK CONTROLLER
rho = 0.01;
L=rho*[0.01 0.5 0.01]';
%%
for i=1:N

    [F,ACap] = LMI_Calc(A,B1,B2,C2,gamma,xEst,xE,L,n)

    xEst = ACap*xEst + L*((y-biopara.Prods));
    EigC=eig(A-B2*F+L*C2)
    u_opt = F*xEst;

    if u_max~=0
        if abs(u_opt)>= (u_max)
            u_opt = sign(u_opt)*u_max;
        end
    end
```

```
    [C2,xE,ym] = Model_DH(biopara.Feeds-u_opt,xEst,i);
    u_opt;
    u_opt=u_opt+biopara.Feeds;
    Time=i*0.1;
    TIME=[TIME Time];
    Dilu=0.15;
    if i>=100
        u_opt=u_opt;
    end
    Feed=u_opt;
    Y = biosim(u_opt,Dilu,biopara,i);
    y = Y.Prod;

    cnt=cnt+1;
    X = [X xEst];
    U = [U u_opt];
    Y1 = [Y1 y];
    Ym = [Ym ym+biopara.Prods];

%%
    if cnt==10
        figure(1),clf,
        plot(TIME(1:end-1),X(1,1:end-1),'r'),hold on,
        plot(TIME(1:end-1),X(2,1:end-1),'r-.'),hold on,
        plot(TIME(1:end-1),X(3,1:end-1),'r:'),drawnow,
        figure(3),clf,
        subplot(212),hold on,plot(TIME(1:end-1),U(1:end-1),'r'),
        hold on,grid on,
        ylabel('Feed Conc (g/l)'),xlabel('Time (h)'),
        subplot(211),hold on,plot((0:0.1:(i/10)-0.1),24.8687*ones(1,i),'k')
```

```
        subplot(211),hold on,plot(TIME(1:end-1),Y1(1:end-1),'r'),
        xlabel('Time (h)'),

        xlabel('Time (h)'),ylabel('Prod Conc (g/l)'),xlabel('Time (h)'),
        box on,grid on,drawnow,
        pause(0)
        cnt=0;
    end
%%
end
        figure(5),
        subplot(212),hold on,plot(TIME(1:end-1),U(1:end-1),'r'),
        hold on,grid on,
        subplot(211),hold on,plot((0:0.1:(i/10)-0.1),24.8687*ones(1,i),'k')
        subplot(211),hold on,plot(TIME(1:end-1),Y1(1:end-1),'r'),
        xlabel('Time (h)'),
end

function [F,ACap] = LMI_Calc(A,B1,B2,C2,gamma,x,xm,L,n)

Q=1;R=1;
nu = 0.994;

E = 1;
W1=-1;W2=1;

setlmis([ ])

beta = lmivar(1,[1 1]);    % Defining the variables
G = lmivar(1,[n 1]);
```

```
G12 = lmivar(1,[n 1]);

G2 = lmivar(1,[n 1]);

Q11 = lmivar(1,[n 1]);

Q12 = lmivar(1,[n 1]);

Q22 = lmivar(1,[n 1]);

M = lmivar(1,[n 1]);

V = lmivar(1,[n 1]);

GAMMA = lmivar(1,[n 1]);

U = lmivar(1,[1 1]);

Yy = lmivar(1,[n 1]);

Y = lmivar(2,[1 n]);

Z = lmivar(1,[1 1]);


% Eqn.(16)-a

lmiterm([-1 1 1 0],(1-nu).^2);

lmiterm([-1 2 1 0],L*E*W1);

lmiterm([-1 2 2 Q11],1,1);

lmiterm([-1 3 1 0],(B1-L*E)*W1);

lmiterm([-1 3 2 Q12],1,1);

lmiterm([-1 3 3 Q22],1,1);



% Eqn.(17)

lmiterm([-2 1 1 G],nu.^2,1);

lmiterm([-2 1 1 G'],nu.^2,1);

lmiterm([-2 1 1 -Q11],nu.^2,1);

lmiterm([-2 2 1 G12],nu.^2,1);

lmiterm([-2 2 1 -Q12],nu.^2,1);

lmiterm([-2 2 2 G2],nu.^2,1);

lmiterm([-2 2 2 G2'],nu.^2,1);
```

```
lmiterm([-2 2 2 -Q22],nu.^2,1);
lmiterm([-2 3 1 G],L*C2,1);
lmiterm([-2 3 1 G12],L*C2,1);
lmiterm([-2 3 1 M],1,1);
lmiterm([-2 3 2 G2],L*C2,1);
lmiterm([-2 3 3 Q11],1,1);
lmiterm([-2 4 1 G],(A-L*C2),1);
lmiterm([-2 4 1 G12],(A-L*C2),1);
lmiterm([-2 4 1 -M],1,1);
lmiterm([-2 4 1 Y],B2,1);
lmiterm([-2 4 2 G2],(A-L*C2),1);
lmiterm([-2 4 3 Q12],1,1);
lmiterm([-2 4 4 Q22],1,1);

% Eqn.(23)
lmiterm([-3 1 1 G],1,1);
lmiterm([-3 1 1 G'],1,1);
lmiterm([-3 1 1 -Q11],1,1);
lmiterm([-3 2 1 G12],1,1);
lmiterm([-3 2 1 -Q12],1,1);
lmiterm([-3 2 2 G2],1,1);
lmiterm([-3 2 2 G2'],1,1);
lmiterm([-3 2 2 -Q22],1,1);
lmiterm([-3 3 1 G],(A-L*C2),1);
lmiterm([-3 3 1 G12],(A-L*C2),1);
lmiterm([-3 3 1 -M],1,1);
lmiterm([-3 3 1 Y],B2,1);
lmiterm([-3 3 2 G2],(A-L*C2),1);
lmiterm([-3 3 3 V],1,1);
```

```
% Eqn.(24)
lmiterm([-4 1 1 G],1,1);
lmiterm([-4 1 1 G'],1,1);
lmiterm([-4 1 1 -Q11],1,1);
lmiterm([-4 2 1 G12],1,1);
lmiterm([-4 2 1 -Q12],1,1);
lmiterm([-4 2 2 G2],1,1);
lmiterm([-4 2 2 G2'],1,1);
lmiterm([-4 2 2 -Q22],1,1);
lmiterm([-4 3 1 Y],1,1);
lmiterm([-4 3 3 U],1,1);

lmino = 4;

lmino = lmino + 1;

% Eqn.(26)
lmiterm([-lmino 1 1 G],1,1);
lmiterm([-lmino 1 1 G'],1,1);
lmiterm([-lmino 1 1 -Q11],1,1);
lmiterm([-lmino 2 1 G12],1,1);
lmiterm([-lmino 2 1 -Q12],1,1);
lmiterm([-lmino 2 2 G2],1,1);
lmiterm([-lmino 2 2 G2'],1,1);
lmiterm([-lmino 2 2 -Q22],1,1);
lmiterm([-lmino 3 1 M],1,1);
lmiterm([-lmino 3 1 G],L*C2,1);
lmiterm([-lmino 3 1 G12],L*C2,1);
lmiterm([-lmino 3 2 G2],L*C2,1);
lmiterm([-lmino 3 3 Yy],1,1);
```

```
lmino=lmino+1;

% Eqn.(27)
lmiterm([-lmino 1 1 G],1,1);
lmiterm([-lmino 1 1 G’],1,1);
lmiterm([-lmino 1 1 -Q11],1,1);
lmiterm([-lmino 2 1 G12],1,1);
lmiterm([-lmino 2 1 -Q12],1,1);
lmiterm([-lmino 2 2 G2],1,1);
lmiterm([-lmino 2 2 G2’],1,1);
lmiterm([-lmino 2 2 -Q22],1,1);
lmiterm([-lmino 3 1 G],A,1);
lmiterm([-lmino 3 1 G12],A,1);
lmiterm([-lmino 3 1 Y],B2,1);
lmiterm([-lmino 3 2 G2],A,1);
lmiterm([-lmino 3 3 GAMMA],1,1);

lmino=lmino+1;

% Eqn. (31)
lmiterm([-lmino 1 1 G],1,1);
lmiterm([-lmino 1 1 G’],1,1);
lmiterm([-lmino 1 1 -Q11],1,1);
lmiterm([-lmino 2 1 G12],1,1);
lmiterm([-lmino 2 1 -Q12],1,1);
lmiterm([-lmino 2 2 G2],1,1);
lmiterm([-lmino 2 2 G2’],1,1);
lmiterm([-lmino 2 2 -Q22],1,1);
lmiterm([-lmino 3 1 G],L*C2,1);
```

```
lmiterm([-lmino 3 1 G12],L*C2,1);
lmiterm([-lmino 3 1 M],1,1);
lmiterm([-lmino 3 2 G2],L*C2,1);
lmiterm([-lmino 3 3 Q11],1,1);
lmiterm([-lmino 4 1 G],(A-L*C2),1);
lmiterm([-lmino 4 1 G12],(A-L*C2),1);
lmiterm([-lmino 4 1 -M],1,1);
lmiterm([-lmino 4 1 Y],B2,1);
lmiterm([-lmino 4 2 G2],(A-L*C2),1);
lmiterm([-lmino 4 3 Q12],1,1);
lmiterm([-lmino 4 4 Q22],1,1);
lmiterm([-lmino 5 1 G],Q.^(0.5)*C2*A,1);
lmiterm([-lmino 5 1 G12],Q.^(0.5)*C2*A,1);
lmiterm([-lmino 5 1 Y],Q.^(0.5)*C2*B2,1);
lmiterm([-lmino 5 2 G2],Q.^(0.5)*C2*A,1);
lmiterm([-lmino 5 5 beta],1,1);
lmiterm([-lmino 6 1 Y],R.^(0.5),1);
lmiterm([-lmino 6 6 beta],1,1);

lmino=lmino+1;

% Eqn. (33)
lmiterm([-lmino 1 1 0],1);
lmiterm([-lmino 2 1 0],x);
lmiterm([-lmino 2 2 Q11],1,1);
lmiterm([-lmino 3 1 0],xm-x);
lmiterm([-lmino 3 2 Q12],1,1);
lmiterm([-lmino 3 3 Q22],1,1);

lmino=lmino+1;
```

```
% Eqn.(16)-b
lmiterm([-lmino\; 1\; 1\; 0],(1-nu).^2);
lmiterm([-lmino\; 2\; 1\; 0],L*E*W2);
lmiterm([-lmino\; 2\; 2\; Q11],1,1);
lmiterm([-lmino\; 3\; 1\; 0],(B1-L*E)*W2);
lmiterm([-lmino\; 3\; 2\; Q12],1,1);
lmiterm([-lmino\; 3\; 3\; Q22],1,1);

lmino=lmino+1;

% Eqn. (25)
lmiterm([-lmino 1 1 G],1,1);
lmiterm([-lmino 1 1 G’],1,1);
lmiterm([-lmino 1 1 -Q11],1,1);
lmiterm([-lmino 2 1 G12],1,1);
lmiterm([-lmino 2 1 -Q12],1,1);
lmiterm([-lmino 2 2 G2],1,1);
lmiterm([-lmino 2 2 G2’],1,1);
lmiterm([-lmino 2 2 -Q22],1,1);
lmiterm([-lmino 3 1 G],A,1);
lmiterm([-lmino 3 1 G12],A,1);
lmiterm([-lmino 3 1 Y],B2,1);
lmiterm([-lmino 3 2 G2],A,1);
lmiterm([-lmino 3 3 Z],1,1);

MPC\_LMI = getlmis;
n\_lmi = decnbr(MPC\_LMI);
c = [1 zeros(1,n\_lmi-1)];
[copt, Xopt]= mincx(MPC\_LMI, c, [0 1000 0 0 0]);
```

```
if isempty(copt)
    F = Y_opt*inv(G\_opt); % Calculate the optimal feedback gain matrix
    ACap = M*inv(G\_opt);
else
G\_opt = dec2mat(MPC\_LMI, Xopt, G);
Y\_opt = dec2mat(MPC\_LMI, Xopt, Y);
M\_opt = dec2mat(MPC\_LMI, Xopt, M);
F = Y\_opt*inv(G\_opt); % Calculate the optimal feedback gain matrix
ACap = M*inv(G\_opt);
end
return;
end


function [C_WN,X,y_m] = Model_DH(u,Xin,mode)
persistent x;
load LWParambio_DevF.mat;

m = size(t,2);

if mode==1
x_1=Xin;
else
    x_1=x;
end

%% LAGUERRE FILTERS
[A,B] = Laguerre_filter_Matrices(p,T,N);
[n,n]=size(A);
```

```
x = A*x_1 + B*u;
X=x;
%% LINEARISATION OF THE WAVELET NETWORK MODEL
Cwn2=w*(dWN(x_1,1,m,t,d));
D2 = w*wavelon(x_1,t,d)-w*(dWN(x_1,1,m,t,d)*x_1);
C_WN = (Cwn2+c+D2/x)+b;
y_m = wavenet(x,t,d,w,c,b);
if mode==2
    WN_Apx = C_WN*x;
end

end
%%  AUTOMATIC DIFFERENTIATION
function [Apx] = dWN(x,n,m,t,d)
N=length(x);
z=[];Apx=[];
for k = 1 : m
    z = fmad(x,n*ones(N,1));
    A = (((x-t(:,k)).*d(:,k)).^2-N).*exp(-0.5.*((x-t(:,k)).*d(:,k)).^2);
    Apx(k,:) = A';
end

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Out = biosim(Feed,Dilu,biopara,cnt)
persistent X0
%
% Simulation of a fermenter under input perturbation
```

```
%

if cnt==0
X0 = [biopara.Cells biopara.Subss biopara.Prods] ;
end
if cnt>=100
    biopara.Mum=0.44;
end

sampT  = 0.1;

[t,X] = ode45(@bioeqn,[0 sampT],X0,[],biopara,Dilu,Feed) ;
X0 = X(end,:);
Out.Cell = X0(1);
Out.Subs = X0(2);
Out.Prod = X0(3);
return


function [A,B] = Laguerre_filter_Matrices(p,T,N)

clear A B
tau1 = exp(-p*T);
tau2 = T + ((2/p)*(tau1-1));
tau3 = - T * tau1 - (2/p) * (tau1-1);
tau4 = sqrt(2*p)*(1-tau1)/p;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

filters = N;
```

```
for j = 1 : filters,
   A(j,j) = tau1;
end

for j1 = 1 : (filters-1),
   k=1 ;
   for j2 = (j1+1) : filters,
      k=k+1;
      A(j2,j1) = (-1)^(k-1) * tau2^(k-2) * (tau1*tau2 + tau3) / T^(k-1);
   end
end

for j = 1 : filters,
   B(j,1) = (-tau2/T)^(j-1) * tau4 ;
end
```

D Automatic Differentiation

The intension of this appendix is to give in brief the operation of automatic differentiation in both of its modes viz., forward mode and reverse mode. This is accomplished by demonstrating them with a simple example. Consider a function $f(x_1, x_2) = x_1^2 x_2 + \cos(x_2)$ which is evaluated for its derivates in both the forward and reverse modes, using the chain rule of calculus.

Forward Mode In this mode the evaluation of the derivative is executed in the regular fashion i.e., right to left of the chain rule. So the interpretation of the evaluation is easy. The variable w.r.t. which the derivative is to be evaluated depends upon the seeding of the variables. Moreover, the number of sweeps the evaluation is carried out is determined by the number of independent variables. Hence for functions of the type $f: \mathbb{R} \mapsto \mathbb{R}^n,\ n > 1$ forward mode of evaluation can be executed in a single sweep. The computational complexity associated with the evaluation is in turn dependent on the complexity of the code. The table below shows the forward mode of evaluation of the derivative of the example function $f(x_1, x_2)$ w.r.t. the independent variable x_1 (c.f the seeding used in the example).

Table D1: Forward Mode of AD

Original code statement	Derivative statement
w_1	$w_1^{'} = 1$ (seed)
w_2	$w_2^{'} = 0$ (seed)
$w_3 = w_1^2$	$w_3^{'} = 2w_1 = 2x_1$
$w_4 = w_3 w_2$	$w_4^{'} = w_3^{'} w_2 + w_3 w_2^{'} = 2x_1 x_2 + x_1^2.1$
$w_5 = \cos(w_2)$	$w_5^{'} = \sin(w_2) w_2^{'} = \sin(w_2).0$
$w_6 = w_4 + w_3$	$w_6^{'} = w_4^{'} + w_5^{'} = 2x_1 x_2 + x_1^2$

Reverse Mode The reverse mode of derivative evaluation in automatic differentiation is carried out from left to right of the chain rule of calculus i.e., in reverse direction to that of forward mode. For function s of the type $f: \mathbb{R}^m \mapsto \mathbb{R},\ m > 1$ reverse mode of achieves the evaluation of derivatives in lesser number of sweeps. However, this mode increases the memory requirement as it requires the storage of the operating elements (w_i).

For the example consider the reverse mode of evaluating the derivative/gradient is shown by the computational graph given below.

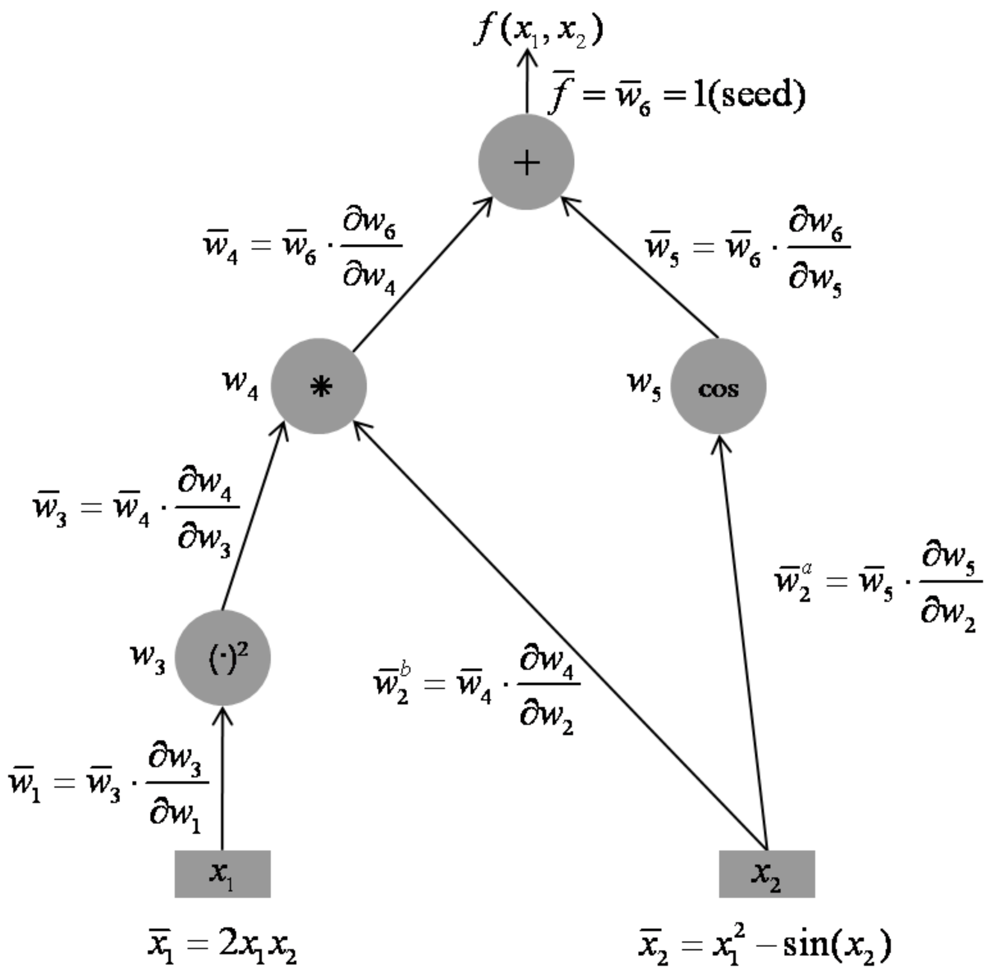

Figure D1: Computational graph depicting the Reverse Mode of AD

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Author's Response to Questions by Examiner 1

The answers to the queries/doubts raised by the reviewer has been presented below.

QUESTION 1
Page 2: At the bottom: The reason why MPC has not completely replaced PID controllers is not as simple as the thesis likes us to believe. Industry looks at payback and value addition and not along purely technical terms.

RESPONSE The statement is now altered as, "It is one of the obvious technical reasons why MPC has not yet completely replaced the conventional feedback PID controller in many industries,.".

QUESTION 2
Page 4: Paragraph 2: The Hammerstein and Wiener model configuration are wrongly explained (the order of the linear dynamic part and the static nonlinear part in Hammerstein and Wiener models).

RESPONSE The mistake has been corrected in the revised version of the thesis.

QUESTION 3
Key references are missing when there is a discussion about Hammerstein and Wiener models and its use for MPC etc. Some early works are given below and the candidate must be aware of this.

RESPONSE The following references are cited at appropriate places in the present version of the thesis.

1. Patwardhan,R.S, Lakshminarayanan.S and Shah.S.L, Constrained Nonlinear Model Predictive Control using Hammerstein and Wiener models - A Partial Least Squares Framework,AIChE Journal,44(7),pp.1611–1622,1998.

2. Lakshminarayanan.S, Shah.S.L and Nandhakumar.K, Modelling and Control of Multivariable processes: The Dynamic Projection to Latent Structures approach,AIChE Journal,43,pp.2307–2322,1997.

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QUESTION 4
Page 11 : NOTE: Performance measure is not to be equated with cost function or value function because the area of control loop performance assessment has moved to the performance assessment of MPC. So, performance measure of a MPC is no longer the same as the cost function or the value function.

RESPONSE The NOTE has been removed in the present version of the thesis.

QUESTION 5
Page 12: end of b): There is no explanation of why the closed loop stability cannot be guaranteed by use of the terminal constraint alone.

RESPONSE The terminal constraint set, which is generally an invariant set and lie in the interior of the feasible region, ensures closed loop stability when the state(s) of the system at the end of the prediction horizon (N) lie within it. However, this does not ensure that the intermediate states - from the current time instant (k) to that at $(N-1)^{(}th)$ time instant - are within the feasible region (i.e., satisfying the constraints). Moreover, a stable closed-loop performance can be ensured if there is a monotonically decreasing value function. These two being important traits in the optimal control of constrained systems, some additional measure that takes care of these traits has to be included.

In such case, the use of *terminal cost function* along with the terminal constraint set is an effective solution that satisfies the closed loop stability in the sense of Lyapunov theory. Now the optimal control problem to be solved becomes,

$$J_N = \min_u [x_N^T P x_N + \sum_{i=0}^{N_1} (x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k})], \forall k \geq 0$$

subject to

$$x_{k+1} = Ax_k + Bu_k$$
$$x_k \in \mathbb{X}; 0 \leq k \leq N-1$$
$$u_k \in \mathbb{U}; 0 \leq k \leq N-1$$
$$x_n \in \mathcal{X}_f.$$

where $x_k \in \mathbb{R}^n$ is the state variable $u_k \in \mathbb{R}^m$ in the manipulated input signal. A and B are the system and input matrices of the dynamic system. P and R are real positive definite matrices of dimension $n \times n$ whereas $Q \in \mathbb{R}^{m \times m}$ is a real positive semi-definite matrix. For a finite horizon of length N the problem is solved for the state and input constraint sets $\mathbb{X}$ and $\mathbb{U}$, respectively, with a terminal constraint set $\mathcal{X}_f$.

Such a type of MPC problem for linear cases it is called Constrained Infinite horizon Linear Quadratic Regulator (CIHLQR) [40,107,136] and for nonlinear cases it is called quasi-infinite horizon MPC [34,126].

The above explanation was already given in the paragraph following Eqn. (1.14). However, in order to enhance the clarity, the paragraph has been moved and appended to Section 1.2.2 (b) in the present version of the thesis.

QUESTION 6
Page 17: The paragraph ending with ".entirely different setting" is vague. It is important to elaborate what the "different setting" is!

RESPONSE In the revised version of the thesis, the meaning of "..entirely different setting" is elaborated as follows;

"On the other hand, in feedback min-max MPC, the notion of feedback present in the receding-horizon implementation of the control, is used [135]. One such feedback MPC methodology is used in this thesis but in an entirely different setting such that the problem is posed and solved using Nash game approach, which are detailed in the subsequent sections."

QUESTION 7
Page 43: In section 2.5, it is stated that "The stability of the model has to be ensured". Will this make sense when one is leading with open loop unstable systems (globally unstable or unstable in certain operating regime)?

RESPONSE For systems which are globally unstable or unstable in certain operating regime, the strategy used in open loop system identification technique cannot be applied directly. Generally, some sort of feedback compensation is employed in such cases in order to place all the poles in the stable region. The resulting closed loop process is thereby perturbed as required by the system identification technique. This kind of identification technique is known as closed-loop identification.The following are few of the references in this direction.

1. Liu,T., and Gao,F. Closed-loop step response identification of integrating and unstable processes, Chemical Engineering Science, 65(10), pp. 2884-2895, 2010.

2. Cheres.E, Parameter estimation of an unstable system with a PID controller in a closed loop configuration, Journal of the Franklin Institute, 343(2), pp. 204-209, 2006.

3. Sun,L.; Ohmori,H., Sano,A. Direct closed-loop identification of unstable system by output inter-sampling scheme, Proceedings of the American Control Conference, vol.1, pp. 321 - 325, 1999.

Closed loop system identification, by itself, is another interesting area of research. However, it does not fit into the scope of the present thesis (and it has been included in the final chapter recommending for future research direction). The (discrete-time) Laguerre filters, being low pass filters, always have their poles in the stable region (within the unit circle). So it can be used only for modeling stable systems. Hence stability w.r.t. the Laguerre filters is ensured inherently. However, there are chances (due to modeling error) that the static nonlinear portion of the Wiener type model can have some adverse spike-like discontinuity at some portion even if it is intended to approximate a continuous and smooth static nonlinear function. This may in-turn affect the stability. For such an adverse case it is required to ensure stability of the model due to nonlinear function in terms of continuity and smoothness.

QUESTION 8
It is not clear why the bioreactor is a benchmark problem? This must be properly motivated by bringing out the control challenges (nonlinearity, stability, non-minimum phase behavior etc.).

RESPONSE Bioreactor is inherently a nonlinear process with input multiplicity nonlinearity i.e., different inputs to the process may yield the same output. This nonlinearity of the process cannot be properly approximated by a simple linear model and cannot be efficiently controlled by a linear controller as well. Thus, bioreactor poses enormous challenge in terms of nonlinear control. Further, the process gain changes its sign across the optimal point of its operation and this has a serious limitation in terms of controllability whereby regular PID type controllers cannot be used to regulate the process. Thus choice of bioreactor is surely justifiable in the present thesis both for system identification and control.

QUESTION 9
Page 47: Is availability of 10,000 data points realistic?

RESPONSE One of the major practical concerns in collecting the process data of large volume is the operator's convenience. Moreover, it is also dependent more on the sampling time and the process time constant. In general, the data-logging is done using some software and when the process's open-loop time constant is high then the process takes larger time to settle to a new steady state value (assuming only open-loop stable processes) for any change in the process input. Mostly the chemical processes are relatively slow processes when compared to electrical or mechanical systems. In such a case the operator may have enough time in between each step change in the process input. Moreover, in the present case with the bioreactor example, the sampling time is 6 seconds, so in almost 17 hours 10,000 data points can be obtained.

Moreover, with the sophistication of the software implementation in industries, a small algorithm including a timer can be programmed into the data logging system to do the pre-defined changes in the process input at pre-defined time intervals (assuming the process is inherently stable).

QUESTION 10 :
Page 47: Not clear if noise was added to the simulations.

RESPONSE In the current study no noise was added to the simulations. However, even if noise is added to the simulations, the Laguerre filter which is essentially a low pass filter or a band pass filter in the lower frequency range, the usual high frequency noise will always be arrested. So the model output will usually be a filtered/smoothened signal even if the process output used for identification is corrupted by noise. The following demonstration, where a noise corrupted process variable is used for identification, is used to explain this phenomenon of the Laguerre filters. It can be observed in Figure 1 that the Laguerre filter does not get affected by the noise even though it is present in the process output which is used for training the model.

Moreover to emphasize the fact that the Laguerre filters do not allow the noise to propagate through it, pure noise with zero mean and unit variance is considered as input to the Laguerre filter. See Figure 2. Here it can be observed that the Laguerre filter essentially being a low-pass filter can arrest the noise, which can be inferred from it output. The bode plot of the Laguerre filter used in the bioreactor example, shown in Figure 3, aids the above claim.

QUESTION 11
Page 48: 5 seconds is not 0.1 min.

RESPONSE The mistake has been corrected in the present version of the thesis.

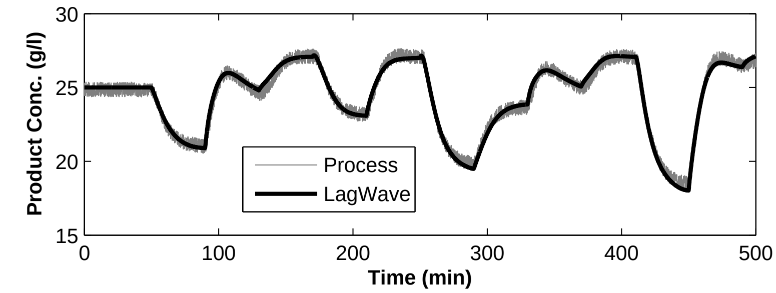


Figure 1: Response of the LWN model for noisy process data.

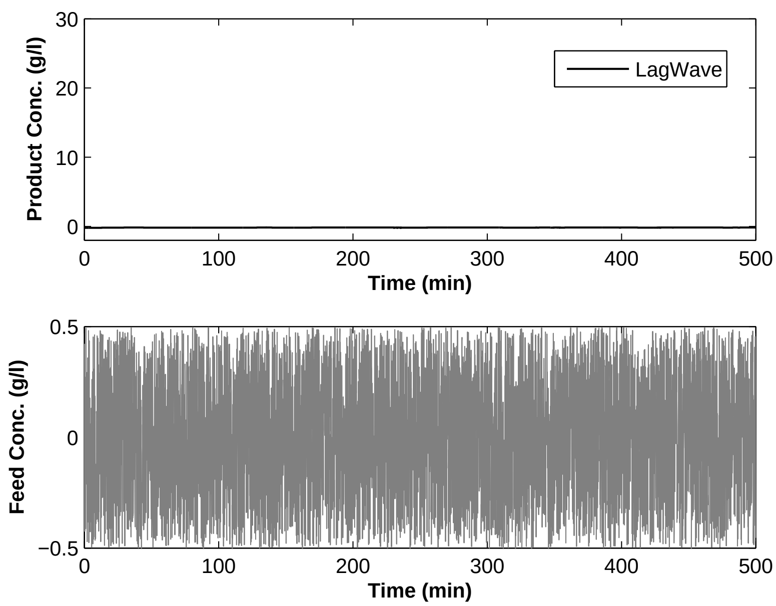


Figure 2: Response of the LWN model for pure noise as its input.

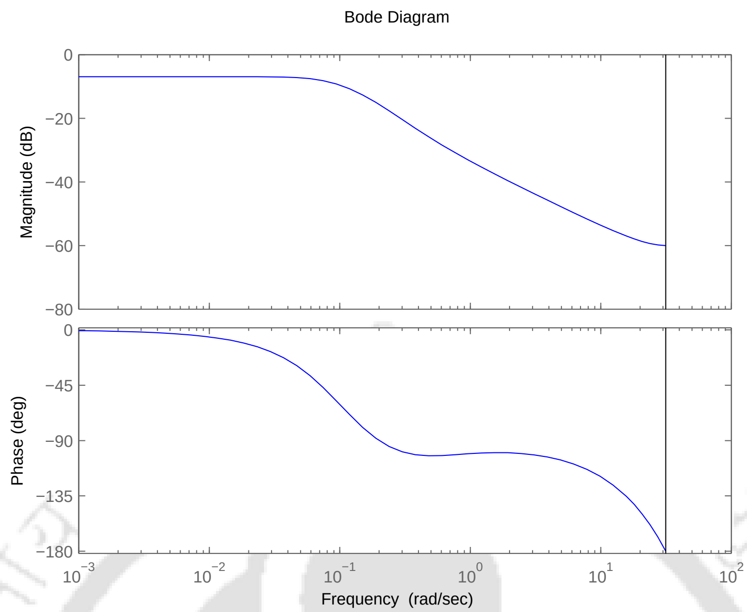


Figure 3: Bode diagram of the Laguerre filter used in LWN model of the bioreactor.

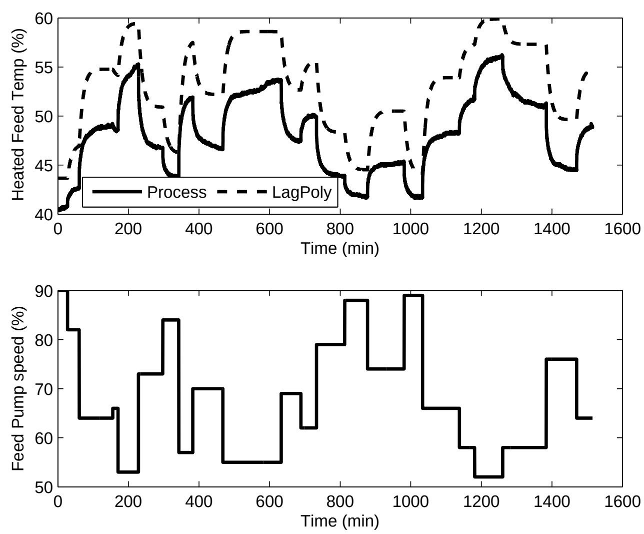


Figure 4: Response of the LP model for training data for Pasteurization process.

QUESTION 12
Page 50: LP results are not provided. This is unfortunate. Including that will show how bad LP is!

RESPONSE The response of the LP model for training data for Pasteurization process is shown in Figure 4. As the response of Laguerre Polynomial (LP) was found to be poorer even with the training data it was not shown in the thesis.

QUESTION 13
The modeling results in Chapter 2 are interesting but there is no analysis of residuals etc. The analysis of residuals is important for model building. Also, proof of model goodness must be provided using multiple test signals.

RESPONSE Residual Analysis essentially defines the analysis of the residuals i.e., the difference between the one-step predicted output from the model and the measured output from the validation data set. At each point of one-step ahead prediction in residual analysis, apart from obtaining the residual, the predicted value of the model is replaced with the actual measurement to go for the next one-step ahead prediction. Consider a dynamic system, $y_{k+1} = f(u_k, y_k)$; which is modeled as $\hat{y} = \hat{f}(u_k, \hat{y}_k)$. Then the residual of the one-step ahead prediction is given as

$$e_k = y_{k+1} - \hat{y}_{k+1} = f(u_k, y_k) - \hat{f}(u_k, \hat{y}_k)$$

At the next time instance, the one-step ahead prediction of the model is given, such that the previous output value of the model is replaced with the actual measurement, i.e.,

$$\hat{y}_{k+2} = \hat{f}(u_{k+1}, y_{k+1})$$

Hence essentially in residual analysis the output of the process which is used to obtain the output at the next time instance causing the dynamics in the process is replaced with the actual measurement to obtain the time-series data.

The residuals are analyzed to validate the model. Residuals represent either a random variable sequence or a non-random sequence, which can be easily inferred by performing auto-correlation on the residuals. If the residual is found to be a random sequence, then the model can be considered to be of good fit to the process. Otherwise, it means that some significant portion of the model is not represented properly by the model. The model developed in the present thesis work is a Wiener model, which is given as,

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = g(x_k)$$

where x is the state, u is the input,y is the output. A and B are matrices of appropriate dimensions and $g(.)$ is a static nonlinear function mapping from the states to the output space. In Wiener model, the (linear) dynamic portion is constituted by the state variables. Hence if the replacement with the actual measurement for computing the next one-step ahead prediction has to be made, it has to be with the state variable, which contributes to the dynamics involved. However, it is a known fact that the Wiener model is developed to mimic the process's mapping from the input to the output space but not the intermediate states. The intermediate states of the model and the actual process are usually unrelated. Moreover, all the process states are not always measurable too. Hence the replacement of model states with the process states is not possible in Wiener type models. And thus, the residual analysis cannot be performed to validate the Wiener type models.

QUESTION 14

The treatment in Chapter 2 is also restricted to SISO systems. The candidate alludes to the need to extend it to MIMO systems. However, one wonders if this could have been done in this thesis itself- this would have led nicely to part 2 of the thesis which is on MPC (a highly studied technique for MIMO systems).

RESPONSE The real bottleneck in extending the model for MIMO systems is the proper/optimal choice of the Laguerre poles. Although there are methods proposed in the literature to optimally choose the Laguerre poles (For example, refer: Oliveria e Silva, T. On the determination of the optimal pole position if Laguerre Filters, IEEE Trans. Signal Processing, Vol. 43, No. 9, 2079-2087, 1995.), it cannot be implemented in the present case. This is because to utilize such approaches the model has to be basically linear. Even though generally in a Wiener model if the static nonlinear portion is inverted the model becomes linear. However, in the present case the static nonlinear portion of Laguerre Wavelet Network cannot be inverted. This prevents the way to obtain a proper (in deed optimal) choice of the Laguerre poles for MIMO case, which is otherwise cumbersome to do manually. As no effective method can be advocated to resolve the issue as of now, it has been recommended as one of the future research direction from this thesis.

QUESTION 15

Chapter 3, 4 and 5 contain several interesting theoretical developments and results. However, the balance between the theory and illustrative examples has not been maintained. Looks like too strong theory has been applied on simple examples. Some comments:

Comment (a) The thesis if from ChE Department and so the candidate could have chosen examples such as high purity distillation columns, polymer reactors, pulp and paper processes, furnaces etc. for which MATLAB/SIMULINK codes are available. These processes are nonlinear, unstable, show non-minimum phase behavior and also multivariable in nature.

RESPONSE The input multiplicity nonlinearity in bioreactor process, which causes a change in the sign of the process gain over its operating region, itself carries a potential challenge. Hence bioreactor is considered as a benchmark example problem to demonstrate the efficiency of the developed theory in the present thesis. Moreover, the technical development of the thesis can also find applications in other fields of engineering. So the examples are chosen to be generic and simple enough so that the efficacy of the developments of the thesis is rightly highlighted.

Comment (b) Disturbance rejection and set point tracking are both very important aspects of process control. The examples illustrated in the thesis do not do enough justice to these aspects.

RESPONSE Although disturbance rejection and set point tracking are very important aspects of process control, in the present thesis the scope is restricted to address disturbance rejection case. This is because most of the practical chemical processes are not subjected to frequent set-point changes. They are mostly regulatory control problem, where disturbance rejection is the major concern. Moreover, the MPC developed in the present thesis can also address the set-point tracking problem since the control action is decided based on the deviation (error) of the system states (process variable) from the nominal value (set-point). Consider the system

$$x_{k+1} = Ax_k + B^2 u_k + B^1 w_k$$

whose initial condition is at origin, $x^0 = [0 \;\; 0]^T$. Let the non-zero set point is given as

$$x_k^{SP} = \begin{bmatrix} 0.2 \\ 0.0 \end{bmatrix}$$

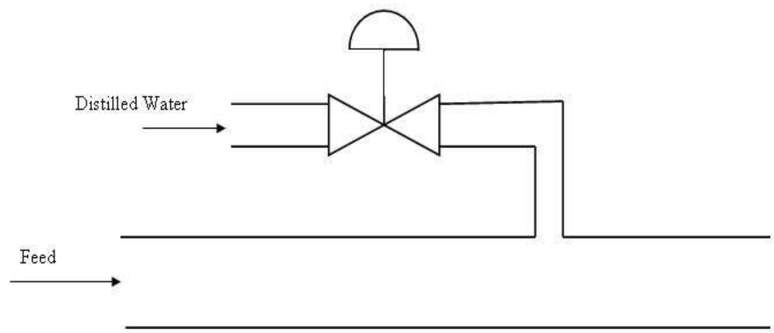
Then let the error $e_k = x_k^{SP} - x_k$. Now the set point tracking problem reduces to

$$e_{k+1} = Ae_k + B^2 u_k + B^1 w_k$$

such that $\lim_{k \longrightarrow \infty} e_k = 0$. It could be noted that as like the state, which is usually a deviation variable, the error between the set point and the state variable is also an deviation variable. Hence in such a situation, the problem solved in the present thesis inherently addresses both disturbance rejection and set point tracking control problem.

Comment (c) In the bioreactor example, use of feed concentration as a manipulated variable is less than desirable as it is to be considered as a disturbance. Flow rates must instead be used as manipulated variables.

RESPONSE In bioreactor the change in feed concentration is usually considered as disturbance, as rightly noted by the reviewer. However, the effect of change in feed concentration can be counter acted or controlled by manipulating flow rate of the feed. So the effective feed concentration to the bioreactor can be decreased/increased by decreasing/increasing the feed flow rate. The following figure illustrates how feed concentration can be varied.


Distilled Water
Feed

Author's Response to Questions by Examiner 2

The answers to the queries/doubts raised by the reviewer has been presented below.
Chapter 1

QUESTION 1
Well -written and provided a good overview and the context of the $\mathcal{H}_2$ / $\mathcal{H}_\infty$ problem, and motivates the direction of the thesis research well. Literature review is adequate and all issues that are relevant from the thesis focus are well presented. There are however works in literature that also talk of the identification of Wiener/Hammerstein models that can perhaps be included from a completion viewpoint.

RESPONSE The salient works that closely related to the present work are cited in the thesis. Other few works that are also closely related to the present work is included in the new version of the thesis.

1. Patwardhan,R.S, Lakshminarayanan.S and Shah.S.L, Constrained Nonlinear Model Predictive Control using Hammerstein and Wiener models - A Partial Least Squares Framework,AIChE Journal,44(7),pp.1611–1622,1998.

2. Lakshminarayanan.S, Shah.S.L and Nandhakumar.K, Modelling and Control of Multivariable processes: The Dynamic Projection to Latent Structures approach,AIChE Journal,43,pp.2307–2322,1997.

3. Lakshminarayanan.S, Shah.S.L and Nandhakumar.K, Identification of Hammerstein using Multivariate Statistical Tools,Chemical Engineering Science, 50(22), pp.3599–3613, 1995.

4. Lakshminarayanan.S, Shah.S.L and Nandhakumar.K, A Case Study on Nonlinear Modelling and Control using PLS,Journal of The Institution of Engineers,Singapore, 37(2),pp.21–28,1997.

Even though the literature review is made extensively to the best of the author's knowledge and ability, there may be chances of some omissions which are rather unintentional.

QUESTION 2
Has it ever been the intent in academia/practice to replace PIDs with MPCs? Also, are

PIDs necessarily more robust than MPCs?

RESPONSE It has never been intent either in academia or in practice to replace PID with MPC. The feedback structure of PID controllers with the process brings in inherent closed-loop robustness when the controller is properly tuned unlike MPC, where the control signal for the present time instant is computed based on the prediction of the process dynamics over some future time horizon with the explicit use of a process model. Hence extra care/attention has to be given for MPC to ensure the closed-loop robustness.

Chapter 2

QUESTION 3
The chapter focuses on nonlinear identification and specifically on the Laguerre-Wave(net) structure. The motivation for the wavenet structure can be better made - commonly wavelets are tools for multi-scale decomposition. What role would they play in nonlinear system identification needs to be explained clearly.

RESPONSE Wavelets based tools are widely used in signal processing applications, for their property of multi-scale decomposition of signals. However, compactly supported orthogonal wavelet systems have certain properties that can be exploited by using wavelets for system identification. Representation of real time signals at multiple scales serves as a very efficient and useful way to analyze them. This can be achieved by expressing the data as a weighted sum of orthonormal basis functions, which are defined in both time and frequency, such as wavelets. The weights become the wavelet coefficients. Wavelets are a computationally efficient family of multi-scale basis functions. A signal can be represented at multiple resolutions by decomposing the signal on a family of wavelets and scaling functions. The same properties of wavelets that are exploited in signal processing such as orthogonality and multi-scale resolution, make the wavelets a suitable system approximation tool too. The feature extraction abilities of multi-scale representation of data are utilized to construct multi-scale nonlinear models that are less affected by the presence of noise in the data. The main idea is to decompose the input-output data at multiple scales and construct a nonlinear model using them. It is in this sense wavelets are being used in the present work.

The above explanation has been incorporated at Section 2.1 of the revised thesis.

QUESTION 4
Model parsimony and model stability is discussed briefly. It is not clear how these two concepts are related. Generally parsimony is associated with compactness in structure, and usually needs to be compromised with explanatory ability vis--vis noise. The context of the stability statement is not clear.

RESPONSE The statement is given with the intention in addition to the fact that wavelet network approximates the static nonlinear function parsimoniously it is also required to ensure the model stability. The stability can be ensured only if the model approximates a continuous and smooth static nonlinear function of the process everywhere. In case it demonstrates some spike-like sharp nonlinearity at a certain portion of the static nonlinear function and is smooth and continuous everywhere otherwise, then the stability becomes an issue w.r.t. the model. Hence it is stated that, " The stability of the model has to be ensured over the entire range of the modeling without compromising model parsimony".

The above explanation has been incorporated at Sec 2.5 of the revised thesis.

QUESTION 5
Case study: Is it fare to compare LWN with LP? Would a more realistic comparison with the LW model of [133]?

RESPONSE Laguerre-Polynomial (LP) model is proposed in [133] as a Wiener type model structure. Hence the Wiener type Laguerre Wavelet Network (LWN) model proposed in the present thesis is compared against the response of Wiener type LP model. Moreover, out of the two Laguerre filter based Wiener type models in [133] namely Laguerre ANN and LP, LP model proves to be better suited for bioreactor. Hence, in the present work, where bioreactor is chosen as the major case study, the performance of proposed LWN model is compared against that of LP model.

QUESTION 6
Also experimental results on the pasteurization process needs to be highlighted more elaboately. What is the extent of nonlinearity? Are disturbances affecting the process? Can you explain the prediction errors seen in some regions?

RESPONSE The pasteurization process is a basically multi-variable process. For the present study only a SISO case is considered i.e., the feed pump speed (N1) Vs heated feed temperature (T4), by fixing the other process variables such as heater power rate and the hot water pump speed (N2) at pre-determined values. The feed pump speed is carefully chosen to be more than 50%, so that for the pre-determined values of the other process variables, the feed temperature will not hit the saturation limit and make the heater power to trip - an arrangement made for safety reasons.

For the above process setup, the steady state values at the interval of 10% is observed, as shown in the following table. It can be observed that there is change in gain over the operating region of the process, making the process nonlinear.

Table 1: Steady state response of ARMFIELD Pasteurization process

Feed Pump Speed N1 (in %)	Heated Feed Temperature T4 (in %)	Gain
50	51.8	
60	46.7	0.51
70	49.2	0.38
80	41.8	0.11
90	40.0	0.18

Although it is a pasteurization process, only water is used as the feed instead of milk. The feed is tap water obtained from the over-head tank and the process takes about 100 minutes to reach a new steady state for a given step change in the input. Hence, depending on the ambient temperature the feed water temperature may be varied, which is a source of disturbance to the process. Moreover, heat exchange is taking place in the process through a heat exchanger unit. The dissipation of the heat from the heat exchanger to atmosphere may also get affected by the room temperature (while the experiment is carried out) due to the temperature gradient. This can also affect the heat exchange rate to the feed from the hot water supply at the heat exchanger unit of the process.

Chapter 3

QUESTION 7

The results mentioned in this chapter are interesting. Firstly, the formulation of the robust control design problem as a min-max problem mentioned in Equations 3.3 and 3.4 need a little more discussion and interpretation, although an earlier reference is cited. The solution of the cARE's is well presented. The following questions may be relevant from a results perspective.

RESPONSE Equations 3.3 and 3.4 are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures. Equation 3.3 represents the total energy of the associated signals i.e., the system state and the input. As the stable equilibrium of any system lies at its minimal energy level, solving the $\mathcal{H}_2$ problem becomes a minimization problem. Equation 3.4 on the other hand is a maximization problem. Consider the transfer function from the input G to w the controlled output . The $\mathcal{H}_\infty$ norm is defined in frequency domain as

$$\|G\|_\infty = \max_\omega |G(j\omega)|$$

By Parseval's theorem the $\mathcal{H}_\infty$ norm can be characterized in time domain as

$$\|G\|_\infty = \sup\left\{\frac{\|z\|_2}{\|w\|_2} : v \neq 0\right\}$$

It follows that for any $\gamma > 0$, $\|G\|_\infty < \gamma$ if and only if

$$\begin{aligned} J_\infty(G,\gamma) &:= \max_w \left[\|z\|_2^2 - \gamma^2\|w\|_2^2\right] \\ &= \max_w \sum_{k=0}^{\infty} \left[z^T(k)z(k) - \gamma^2 w^T(k)w(k)\right] \\ &< \infty \end{aligned}$$

Thus the above is a maximization problem in linear quadratic form, which gives the induced norm of the transfer gain of G for the maximum value of w. This is considered to the worst case performance measure for the maximal allowable value of the input (considered to be the worst case causing disturbance) w.

QUESTION 8
Page 56: Is B_1 assumed to be known? What would be the approach if the disturbance is not modeled as in the case of general disturbances that affect the process.

RESPONSE In the present case it is always assumed that the disturbance is always known. For the cases where the disturbances affecting the process are not known, the distribution matrix of the disturbance B_1 is taken as the following; For a dynamic system of dimension $'n'$, the un-modeled disturbance is given as

$$B_1 = \mathbf{1}$$

where $\mathbf{1}$ is a matrix of ones with $'n'$ number of rows and let $'q'$ be the number of columns equal to the number of independent disturbances affecting the system states. The cAREs and the MPC control law are correspondingly modified and solved with $B_1 = \mathbf{1}$.

QUESTION 9
Numerical Examples: Is any mismatch introduced between the plant and the model for the 3 cases presented in this section?

RESPONSE The mismatch between the plant and the model is considered to be observed in the disturbance distribution matrix B_1 and is solved as a robust MPC control design for disturbance rejection case. Note that the mismatch is considered to be time-invariant always. This has been already elaborated in Section 1.3.1 of the thesis.

Chapter 4

QUESTION 10
The stability and robustness results developed in this chapter are good - however it is not clear if the candidate has considered the case of constraints

on the manipulate variables (Equation 4.3 needs elaboration). What about equality constraints on some of the states and the inputs? Would these results generalize for that case as well?

RESPONSE The constraints on the input $u \in \mathbb{U}$ is considered in the stability and robustness results using set theoretic concepts as given in the definitions of Robust Controllable/Stabilizable sets. Moreover, the system states are always guaranteed to lie within the robust feasible set (see Theorem 4.3). In all of the above discussions the input constraint is always considered. If there are equality constraints on the system states which are dynamic in nature, then it will be included in the state equation while solving the cAREs. Equality constraint on the system input has not been explored in the present thesis, which needs further investigation.

Chapter 5

QUESTION 11

The results of mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ formulation are extended to the output feedback case. The linearized version of the LWN model was used to demonstrate the robustness aspects. The significance of the notion of controller fragility can be discussed more clearly. As indicated in the general comments, it would be interesting to see if possible how a nonlinear MPC would perform on the bioreactor case study. However, this is not mandatory and the results of this chapter are as such acceptable.