INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
SHORT ABSTRACT OF THESIS

Name of the Student	:	Shyam Swarup Mondal
Roll Number	:	166123103
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Name of Thesis Supervisor(s)	:	Dr. Jitendriya Swain
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SHORT ABSTRACT

Abstract

This thesis focuses on certain classical problems in harmonic analysis in connection with mathematical physics. We begin with the Fourier analysis on the Euclidean space, discuss some well known results, basic definitions, and review of recent developments that motivates us to consider the problems discussed in the thesis.

We prove a restriction theorem for the Fourier-Hermite transform and obtain a Strichartz estimate for systems of orthonormal functions associated with the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ for the range $1 \leq q < \frac{n+1}{n-1}$ as an application. Besides, we show an optimal behavior of the constant in the Strichartz estimate as limit of a large number of functions.

Further, we prove a restriction theorem for the special Hermite transform and establish a Strichartz inequality as a by-product for the range $1 \leq q \leq 1 + \frac{1}{n}$, for systems of orthonormal functions associated with the special Hermite operator $\mathcal{L}$ on $\mathbb{C}^n$.

Next, we consider the Schrödinger operator $\mathcal{H} = -\Delta_{\mathbb{H}} + V$ on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}}$ is the full laplacian on $\mathbb{H}^n$ and V is a positive smooth potential grows like $|g|^\kappa, \kappa > 0$, for large value of $|g|$. We prove Szegö type limit theorem for $\mathcal{H}$ with respect to the multiplication operator $M_{\mathbf{b}}$, where $\mathbf{b}$ is a bounded real valued integrable function on $\mathbb{H}^n$. More preciously, we prove that, for any $f \in C(\mathbb{R})$,

$$\lim_{r\to\infty} \frac{\operatorname{Tr} f(\mathcal{P}_r M_{\mathbf{b}} \mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)} = \int_{\mathbb{H}^n} f(\mathbf{b}(g))\, dg,$$

where $\mathcal{P}_r$ denote the orthogonal projection of $L^2(\mathbb{H}^n)$ onto the space of eigenfunctions of $\mathcal{H}$ with eigenvalue less than or equal to r. Further, we generalize the above result by taking a 0-th order self-adjoint pseudo-differential operator A on $L^2(\mathbb{H}^n)$ with symbol $a(g,\lambda)$ relative to the operator $1+|\lambda|H+V(g)$, where H is the Hermite operator on $L^2(\mathbb{R}^n)$ and $(g,\lambda)\in\mathbb{H}^n\times\mathbb{R}^*$, in place of the multiplication operator M_b, and obtain the following Szegö type limit theorem:

$$\lim_{r\to\infty}\frac{\operatorname{Tr} f(\mathcal{P}_r A\mathcal{P}_r)}{\operatorname{Tr}(\mathcal{P}_r)}=\lim_{r\to\infty}\frac{\int_{G^r} f(a_{g,\lambda}(\xi,x))\,d\xi\,dx\,dg\,d\mu(\lambda)}{\int_{G^r} d\xi\,dx\,dg\,d\mu(\lambda)}, \tag{0.0.1}$$

where $G^r=\{(g,\lambda,\xi,x)\in\mathbb{H}^n\times\mathbb{R}^*\times\mathbb{R}^n\times\mathbb{R}^n : |\lambda|(1+|\xi|^2+|x|^2)+V(g)\le r\}$, $a(g,\lambda)=Op^W(a_{g,\lambda})$, and $\mu(\lambda)$ is the Plancherel measure on the Heisenberg group, assuming one limit exists. We show that the above Szegö type limit theorem also holds under a perturbation of the Schrödinger operator $\mathcal{H}$ by a bounded self-adjoint operator on $L^2(\mathbb{H}^n)$. Further, we show that the right hand limit of (0.0.1) remains unaltered under a compact perturbation of the pseudo-differential operator A.

For a given compact (Hausdorff) group G and a closed subgroup H of G, we present symbolic criteria for pseudo-differential operators on compact homogeneous space G/H characterizing the Schatten-von Neumann classes $S_r(L^2(G/H))$, for all $0<r\le\infty$. We provide a symbolic characterization for r-nuclear pseudo-differential operators with $0<r\le 1$, on $L^p(G/H), 1\le p<\infty$, along with applications to adjoint, product and trace formulae. Finally, as an application of the aforementioned results, we derive a trace formula and provide a criteria for the heat kernel to be r-nuclear on $L^p(G/H), 1\le p<\infty$.