HYPERGEOMETRIC FUNCTIONS AND THEIR RELATIONS TO ALGEBRAIC VARIETIES AND MODULAR FORMS

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JULY 2025

Hypergeometric Functions and their relations to Algebraic Varieties and Modular Forms

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submitted in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

Indian Institute of Technology Guwahati
Guwahati - 781039, India

July 2025

"We are what our thoughts have made us; so take care about what you think. Words are secondary. Thoughts live; they travel far."

-Swami Vivekananda

This work is dedicated

to

My Family and Thesis Supervisor

for

encouraging me to chase my dreams!

Certificate

This is to certify that the thesis entitled **"Hypergeometric functions and their relations to algebraic varieties and modular forms"** submitted by **Ms. Sulakashna** to the **Indian Institute of Technology Guwahati**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Date: July 03, 2025
Guwahati, India

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Acknowledgments

"A journey of a thousand miles begins with a single step"
-Lao Tzu

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First and foremost, I would like to use this opportunity to convey my heartfelt thanks to my supervisor *Prof. Rupam Barman* for his constant encouragement, invaluable guidance and uplifting support throughout my research period. I am deeply thankful for the precious time and efforts that he dedicated to reviewing this thesis and providing insightful suggestions and advice which helped me make this thesis worthwhile. He inspired me in every challenging phase of this journey. I am also grateful for the warmth and kindness of his family, who made me feel at home during my stay in Guwahati.

I want to convey my sincere gratitude to members of my doctoral committee *Prof. Anupam Saikia*, *Prof. K. V. Krishna*, and *Dr. P. A. S. Sree Krishna* for reviewing my research work periodically and giving valuable suggestions for improvements of the same. I want to express my deepest gratitude to *Prof. Sukhanta Pati*, *Dr. A. K. Chakrabarty*, *Dr. Vinay Wagh*, *Dr. Sriparna Bandopadhyay*, and *Dr. Jitendriya Swain* for their support, priceless advice, and enormous help during my M.Sc and PhD.

I sincerely acknowledge the Indian Institute of Technology Guwahati for providing me various facilities necessary to carry out my research. I also want to express my gratitude to all my teachers and non-academic staff of the department for their assistance and valuable support in various ways during my research journey.

I want to express my heartfelt gratitude to my dear friends Dr. Ajit Singh and H Laxmi for their unwavering support, kindness, encouragement, and help in each phase of my journey at IIT Guwahati. I wish to express my warmest gratitude to my seniors, Dr. Mohit, Dr. Anwita, and Deepa, as well as research group members Gurinder, Sipra, and Alapan, who were always there for me and helped me in various ways. I am also thankful to Jaspreet, Promila, Riddhi, Rajni, and Sushmita for becoming such awesome friends and for offering me support, when needed. I want to thank everyone who taught me something. I have forgotten most of it, but I do know that thanks to you, I am here today.

Finally, I express my heartfelt gratitude to my family. I thank my mother *Smt. Anita Rani* and my father *Shri Narender Kumar* for being the most important part of my life, my backbone, my source of happiness, and my inspiration. I am grateful for their support, endless love, faith in me, and everything. I am thankful to my sister *Rekha Rani* and my brother *Manoj* for being a constant pillar of moral support and joy which motivated me to keep going at difficult times. I am also grateful to my maternal uncle *Shri Balvinder Singh* and maternal aunt *Smt. Archana* for their love, support, and inspiration during this journey.

Finally, I am deeply grateful to everyone who contributed in many ways to the successful completion of my PhD and making an unforgettable experience for me.

Date: July 03, 2025

Guwahati, India Sulakashna

Abstract

This thesis studies hypergeometric functions and their relations to algebraic varieties and modular forms. We study p-adic hypergeometric functions, hypergeometric functions over finite fields, and their relations to elliptic curves and diagonal hypersurfaces. We also express the Fourier coefficients of certain weight three newforms as special values of p-adic hypergeometric functions, and obtain a new proof of some supercongruences conjectured by Rodriguez-Villegas related to modular $K3$ surfaces.

Let ${}_nG_n[\cdots]_q$ denote McCarthy's p-adic hypergeometric functions. Firstly, we establish two transformations for the p-adic hypergeometric function, which can be described as analogues of a transformation of Euler and a transformation of Clausen. We use a character sum identity proved by Ahlgren, Ono, and Penniston to deduce the p-adic Clausen's transformation. We also deduce special values of certain p-adic hypergeometric functions. Next, we derive an identity expressing a ${}_4G_4[\cdots]_q$ hypergeometric function as a sum of two ${}_2G_2[\cdots]_q$ hypergeometric functions. This identity generalizes some known identities satisfied by the finite field hypergeometric functions. We also prove a transformation that relates ${}_{n+2}G_{n+2}[\cdots]_q$ and ${}_nG_n[\cdots]_q$ hypergeometric functions.

Secondly, we study relationships between p-adic hypergeometric functions and the number of points on diagonal hypersurfaces over a finite field. Let D_λ^d denote the family of monomial deformations of diagonal hypersurface over a finite field $\mathbb{F}_q$ given by

$$D_\lambda^d : X_1^d + X_2^d + \cdots + X_n^d = \lambda d X_1^{h_1} X_2^{h_2} \cdots X_n^{h_n},$$

where $d, n \geq 2$, $h_i \geq 1$, $\sum_{i=1}^n h_i = d$, and $\gcd(d, h_1, h_2, \ldots, h_n) = 1$. The Dwork hypersurface is the case when $d = n$, that is, $h_1 = h_2 = \cdots = h_n = 1$. Formulas for the number of $\mathbb{F}_q$-points on the Dwork hypersurfaces in terms of McCarthy's p-adic hypergeometric functions are known. We provide a formula for the number of $\mathbb{F}_q$-points on D_λ^d in terms of McCarthy's p-adic hypergeometric function which holds for $d \geq n$. Let $D_\lambda^{d,k}$ denote a subfamily of the diagonal hypersurface over a finite field $\mathbb{F}_q$ with $n = 2$, $h_1 = k$, and $h_2 = d - k$. Let $\#D_\lambda^{d,k}$ denote the number of points on $D_\lambda^{d,k}$ in $\mathbb{P}^1(\mathbb{F}_q)$. It is easy to see that $\#D_\lambda^{d,k}$ is equal to the number of distinct zeros of the polynomial $y^d - d\lambda y^k + 1 \in \mathbb{F}_q[y]$ in $\mathbb{F}_q$. We prove that $\#D_\lambda^{d,k}$ is also equal to the number of distinct zeros of the polynomial $y^{d-k}(1-y)^k - (d\lambda)^{-d}$ in $\mathbb{F}_q$.

Thirdly, we express the trace of Frobenius of elliptic curves in terms of special values of ${}_4G_4[\cdots]_q$ and ${}_6G_6[\cdots]_q$ hypergeometric functions. These results extend the recent works of Tripathi and Meher on the finite field hypergeometric functions to wider classes of primes. We then derive summation identities for the p-adic hypergeometric functions appearing in the expressions for $\#D_\lambda^{d,k}$. As an application of the summation identities, we prove identities for the trace of Frobenius endomorphism on certain families of elliptic curves.

Finally, we study relationships between p-adic hypergeometric functions and modular forms. We prove p-adic analogues of certain classical hypergeometric identities, and using these identities we express the p-th Fourier coefficient of certain weight three newforms in terms of special values of ${}_3G_3[\cdots]_p$. Rodriguez-Villegas conjectured certain supercongruences between values of truncated hypergeometric

series and the p-th Fourier coefficients of these newforms. As a consequence of our main results, we obtain another proof of these supercongruences which were earlier proved by Mortenson and Sun.

Contents

Introduction

In this thesis, we study p-adic analogues of certain classical hypergeometric identities, relations of p-adic hypergeometric functions to algebraic varieties and modular forms, and find certain special values of the p-adic hypergeometric functions. Gauss defined ${}_2F_1$ classical hypergeometric series in his work [32] in 1812. Relations of hypergeometric series with different mathematical objects, such as differential equations, modular forms, algebraic curves, etc., have been explored by many mathematicians. For instance, there are certain results relating hypergeometric series to the periods of abelian varieties, such as elliptic curves, Calabi-Yau manifolds, and certain $K3$ surfaces. Consider the Legendre family and the Clausen form of elliptic curves given by $E_\lambda : y^2 = x(x-1)(x-\lambda)$ and $E_t : y^2 = (x-1)(x^2+t)$ for $t \in \mathbb{R}\backslash\{0,-1\}$, respectively. Then the real period of E_λ and E_t, for $t > 0$, can be expressed in terms of ${}_2F_1$-hypergeometric series and ${}_3F_2$-hypergeometric series, respectively.

The arithmetic properties of Gauss and Jacobi sums have a very long history in number theory. Number theorists have obtained finite field analogues of classical hypergeometric series using these sums, and these functions have significant applications in arithmetic geometry. It seems that hypergeometric functions over a finite field first appeared in Koblitz's work [44]. There are other definitions of hypergeometric functions over finite fields. For example, see the works of Greene [35, 36], Katz [41], McCarthy [52], Fuselier et al. [30], and Otsubo [58]. In 1987, Greene [36] introduced hypergeometric functions over finite fields as analogues of classical hypergeometric series. These functions are known as *Gaussian hypergeometric series*. He explored the properties of these series and found that they satisfy numerous transformation and summation identities that are analogous to their classical counterparts.

Some of the biggest motivations for studying Gaussian hypergeometric series have been their connections with Fourier coefficients and eigenvalues of modular forms and with counting points on certain algebraic varieties. For example, see [2, 8, 9, 26, 28, 29, 46, 47, 48, 51, 55, 56, 57, 63, 71]. Gaussian hypergeometric series have recently led to applications in graph theory, see for example [21, 22, 23].

By definition, the arguments in Gaussian hypergeometric series are the multiplicative characters, and hence, the results involving hypergeometric functions over finite fields are often restricted to primes in certain congruence classes to facilitate the existence of the multiplicative characters of specific orders; for example, see [8, 9, 28, 29, 47, 48, 63]. To overcome these restrictions, McCarthy [50, 53] introduced a function ${}_nG_n[\cdots]_q$, known as *p-adic hypergeometric function*, in terms of the p-adic gamma function with parameters from rational numbers and argument is an element of a finite field $\mathbb{F}_q$. This function can best be described as an analogue to the classical hypergeometric series in the p-adic setting. McCarthy showed how results involving hypergeometric functions over a finite field can be extended to all but finitely many primes using ${}_nG_n[\cdots]_q$. Using McCarthy's work, one can rewrite the results involving hypergeometric functions over finite fields in terms of ${}_nG_n[\cdots]_q$. However, such results for ${}_nG_n[\cdots]_q$ will only be valid for primes q where the original multiplicative characters existed over the finite field $\mathbb{F}_q$, and hence restricted to primes in certain congruence classes. Therefore, it is a non-trivial problem to extend these results to all but finitely many primes. Several authors have extended the results involving Gaussian hypergeometric series to a broader class of primes using ${}_nG_n[\cdots]_q$, see for example [11, 12, 13, 14, 17, 53, 54, 61, 62].

Numerous transformation formulas exist for the finite field hypergeometric functions, but very few exist for ${}_nG_n[\cdots]_q$ in full generality. It is an interesting problem to find p-adic analogues of classical identities. Certain transformation and summation identities for p-adic hypergeometric functions are known for some specific parameters, for example, see [11, 12, 13, 14, 15, 16, 17, 31, 61, 62]. In this thesis, we find certain summation and transformation formulas for the p-adic hypergeometric functions. We also find some special values of these functions. We prove two transformations, which can be described as p-adic analogues of a transformation of Euler and a transformation of Clausen. We derive an identity where the sum of two ${}_2G_2[\cdots]_q$ hypergeometric functions is expressed in terms of a ${}_4G_4[\cdots]_q$ hypergeometric function. We also establish a transformation that relates ${}_{n+2}G_{n+2}[\cdots]_q$ and ${}_nG_n[\cdots]_q$ hypergeometric functions.

In [53], McCarthy expressed the trace of Frobenius of elliptic curves in terms of a special value of ${}_2G_2[\cdots]_p$ hypergeometric function for all primes $p > 3$. Later, Barman and Saikia [12] represented the trace of Frobenius of elliptic curves in terms of another special value of ${}_2G_2[\cdots]_q$ hypergeometric function. Several authors have expressed the number of points on Dwork hypersurfaces over a finite field in terms of the p-adic hypergeometric functions under certain conditions, see for example [10, 34, 54]. Let D_λ^d denote the family of *monomial deformations of diagonal hypersurface* over a finite field $\mathbb{F}_q$. These families are of the form:

$$D_\lambda^d : X_1^d + X_2^d + \cdots + X_n^d = \lambda d X_1^{h_1} X_2^{h_2} \cdots X_n^{h_n},$$

where $d, n \geq 2$, $h_i \geq 1$, $\sum_{i=1}^n h_i = d$, and $\gcd(d, h_1, h_2, \ldots, h_n) = 1$. We express the number of points on D_λ^d over a finite field in terms of the p-adic hypergeometric functions under the condition $\gcd(d, q-1) = 1$. We also study some results where the sum of traces of Frobenius of a family of elliptic curves can be written in terms of special values of ${}_4G_4[\cdots]_q$ and ${}_6G_6[\cdots]_q$. As an application of summation identities, we establish relations involving trace of Frobenius of elliptic curves and ${}_3G_3[\cdots]_q$.

In [60], Rodriguez-Villegas examined 18 supercongruences where he related the truncated hypergeometric series to the Fourier coefficients of modular form of weight three and four. It was Beukers [19] who first observed supercongruences of this type in connection with the Apéry numbers used in the proof of the irrationality of $\zeta(3)$. Ahlgren and Ono [2] proved Beukers' supercongruence conjecture relating Apéry numbers to the coefficients of a certain weight four newform. All the 14 supercongruences of Rodriguez-Villegas associated with the modular form of weight four are proved, see, for example, [31, 42, 49, 51]. For a nice survey and more conjectural supercongruences, one can also see [24]. Mortenson [56] found relations between Gaussian hypergeometric series and Fourier coefficients of modular form of weight three. Consequently, he proved that the remaining four supercongruences are true for certain congruence classes of primes, and for the remaining classes he proved the supercongruences upto sign. Later, Sun [67] proved these four supercongruences for the remaining classes with a different approach involving Schröder polynomials and the Zeilberger algorithm. In this thesis, we prove that the p-th Fourier coefficients of these weight three newforms are related to the p-adic hypergeometric functions for all but finitely many primes. In other words, we extend the result of Mortenson from the Gaussian hypergeometric series to the p-adic hypergeometric functions, and obtain a new proof of the Rodriguez-Villegas conjectures .

Organization of Thesis

We intend to present the entire work of this thesis in six chapters, as mentioned below:

- Chapter 1: Preliminaries
- Chapter 2: Certain transformations of p-adic hypergeometric functions

- Chapter 3: Diagonal hypersurfaces and p-adic hypergeometric functions
- Chapter 4: Elliptic curves and p-adic hypergeometric functions
- Chapter 5: Summation identities and their applications
- Chapter 6: Weight three newforms and p-adic hypergeometric functions

In Chapter 1, we introduce classical hypergeometric series, elliptic curves, and modular forms. We then recall Gauss and Jacobi sums, and their properties. We next introduce the Gross-Koblitz formula. Finally, we define hypergeometric functions over finite fields and McCarthy's p-adic hypergeometric functions.

In Chapter 2, we derive certain transformation formulas for the p-adic hypergeometric functions. To derive the transformation formulas, we evaluate certain character sums and use some properties of the p-adic gamma function.

In Chapter 3, we relate the p-adic hypergeometric function to the number of $\mathbb{F}_q$-points on diagonal hypersurfaces. We find a formula for the number of distinct zeros of a polynomial over a finite field in terms of the p-adic hypergeometric functions which also establishes a relation to the number of points on the diagonal hypersurfaces over a finite field.

In Chapter 4, we express the sum of two traces of Frobenius endomorphism on certain families of elliptic curves in terms of special values of ${}_4G_4[\cdots]_q$ and ${}_6G_6[\cdots]_q$ hypergeometric functions. These results extend the recent works of Tripathi and Meher on the finite field hypergeometric functions to wider classes of primes.

In Chapter 5, we establish certain summation identities for ${}_nG_n[\cdots]_q$ hypergeometric functions. We also derive certain transformations and expressions involving a sum of p-adic hypergeometric functions. As an application of these identities, we find some expressions involving the sum of traces of Frobenius endomorphism on elliptic curves and p-adic hypergeometric function ${}_3G_3[\cdots]_q$.

In Chapter 6, we study relations between certain weight three newforms and the p-adic hypergeometric functions. Firstly, we derive some transformation identities for the p-adic hypergeometric functions by counting points on certain families of elliptic curves and their twists over a finite field. Employing the transformation identities, we find special values of certain p-adic hypergeometric functions, which turn out to be the p-th Fourier coefficients of weight three newforms. Using these special values, we find a new proof of the supercongruences related to $K3$ surfaces conjectured by Rodriguez-Villegas.

1
Preliminaries

In this chapter, we first introduce classical hypergeometric series, elliptic curves, modular forms, characters over finite fields, and hypergeometric functions over finite fields. We then recall the p-adic gamma function, the Gross-Koblitz formula, and McCarthy's p-adic hypergeometric function. Finally, we recall and prove certain lemmas which will be used to prove our results in the subsequent chapters.

1.1 Classical hypergeometric series

For a non-negative integer r, and $a_i, b_i \in \mathbb{C}$ with $b_i \notin \{\ldots, -3, -2, -1, 0\}$, the classical hypergeometric series ${}_{r+1}F_r$ is defined by

$$ {}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_r \end{matrix} \mid \lambda\right] := \sum_{k=0}^{\infty} \frac{(a_0)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} \cdot \frac{\lambda^k}{k!}, \tag{1.1} $$

where, for a complex number a, the rising factorial or the Pochhammer symbol $(a)_k$ is defined as $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$, $k \geq 1$. If $\Gamma(x)$ denotes the Gamma function, then we have $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$. It is well-known that the classical hypergeometric series ${}_{r+1}F_r$ converges absolutely for $|\lambda| < 1$. Also, series (1.1) converges for $\lambda = 1$ if $\mathrm{Re}(\sum b_i - \sum a_i) > 0$ and for $\lambda = -1$ if $\mathrm{Re}(\sum b_i - \sum a_i + 1) > 0$. More details on hypergeometric series can be found in [5, 7, 65].

When we truncate the infinite sum (1.1) at $k = n$, it is known as a truncated hypergeometric series. We denote the truncated hypergeometric series using a subscript for the notation as follows:

$$ {}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_r \end{matrix} \mid \lambda\right]_n := \sum_{k=0}^{n} \frac{(a_0)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} \cdot \frac{\lambda^k}{k!}. $$

If one of the a_i's is a negative integer, then the hypergeometric series (1.1) terminates.

1.2 Multiplicative characters on finite fields

For an odd prime p, let $\mathbb{F}_q$ denote the finite field having q elements, where $q = p^r$, $r \geq 1$. Let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ be the ring of p-adic integers and the field of p-adic numbers, respectively. Let $\overline{\mathbb{Q}_p}$ denote the algebraic closure of $\mathbb{Q}_p$, and let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$. Let $\mathbb{Z}_q$ denote the ring of integers of the unique unramified extension of $\mathbb{Q}_p$

with residue field $\mathbb{F}_q$. More precisely, $\mathbb{Z}_q$ is the ring of integers of $\mathbb{Q}_p(\zeta_{q-1})$, the unique unramified extension of $\mathbb{Q}_p$, where ζ_{q-1} is a primitive $(q-1)$-th root of unity in $\overline{\mathbb{Q}_p}$.

A *multiplicative character* on $\mathbb{F}_q^{\times}$ is a group homomorphism $\chi : \mathbb{F}_q^{\times} \to \mathbb{C}^{\times}$. Let $\widehat{\mathbb{F}_q^{\times}}$ be the set of all the multiplicative characters on $\mathbb{F}_q^{\times}$, and it forms a cyclic group of order $q-1$ under the pointwise multiplication: $(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x)$. Let T be a generator of the cyclic group $\widehat{\mathbb{F}_q^{\times}}$. Let $\overline{\chi}$ denote the inverse of χ. For every $x \in \mathbb{F}_q^{\times}$, we know that $\chi(x) \in \mu_{q-1}$, where μ_{q-1} is the group of all $(q-1)$-th root of unity in $\mathbb{C}^{\times}$, and $\overline{\chi}(x) = \overline{\chi(x)}$. We extend each character $\chi \in \widehat{\mathbb{F}_q^{\times}}$ from $\mathbb{F}_q^{\times}$ to $\mathbb{F}_q$ by setting $\chi(0) := 0$ including the trivial character ε. Let φ denote the quadratic character on $\mathbb{F}_q$.

It is well known that all the $(q-1)$-th roots of unity are contained inside $\mathbb{Z}_q^{\times}$. Therefore, we can consider multiplicative characters on $\mathbb{F}_q^{\times}$ to be maps $\chi : \mathbb{F}_q^{\times} \to \mathbb{Z}_q^{\times}$. We recall the *orthogonality relations* for multiplicative characters in the following lemma.

Lemma 1.1. ([39, Chapter 8]). *We have*

(1) $$\sum_{x\in\mathbb{F}_q} \chi(x) = \begin{cases} q-1 & \text{if } \chi = \varepsilon; \\ 0 & \text{if } \chi \neq \varepsilon. \end{cases}$$

(2) $$\sum_{\chi\in\widehat{\mathbb{F}_q^{\times}}} \chi(x) = \begin{cases} q-1 & \text{if } x = 1; \\ 0 & \text{if } x \neq 1. \end{cases}$$

We now recall Gauss and Jacobi sums and some of their elementary properties. For further details, see [18]. Let ζ_p denote a fixed primitive p-th root of unity in $\overline{\mathbb{Q}_p}$. The trace map $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is given by

$$\mathrm{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{r-1}}.$$

Then, the additive character $\theta : \mathbb{F}_q \to \mathbb{Q}_p(\zeta_p)$ is defined by $\theta(\alpha) := \zeta_p^{\mathrm{tr}(\alpha)}$. It is easy to see that $\theta(a+b)=\theta(a)\theta(b)$ and

$$\sum_{x\in\mathbb{F}_q}\theta(x)=0. \tag{1.2}$$

For $\chi\in\widehat{\mathbb{F}_q^\times}$, the *Gauss sum* is defined by

$$g(\chi):=\sum_{x\in\mathbb{F}_q}\chi(x)\theta(x).$$

If ζ_{q-1} is a primitive $(q-1)$-th root of unity in $\overline{\mathbb{Q}_p}$, then $g(\chi)$ lies in $\mathbb{Q}_p(\zeta_p,\zeta_{q-1})$. Using (1.2), we can easily get that

$$g(\varepsilon)=-1.$$

Let δ denote the function on multiplicative characters defined by

$$\delta(A)=\begin{cases}1, & \text{if } A \text{ is the trivial character;}\\ 0, & \text{otherwise.}\end{cases}$$

We also denote by δ the function defined on $\mathbb{F}_q$ by

$$\delta(x):=\begin{cases}1, & \text{if } x=0\text{ ;}\\ 0, & \text{otherwise.}\end{cases}$$

The following results on Gauss sums will be useful in the proof of our main results.

Lemma 1.2. ([36, Eq. 1.12]). *For $\chi\in\widehat{\mathbb{F}_q^\times}$, we have*

$$g(\chi)g(\overline{\chi})=q\cdot\chi(-1)-(q-1)\delta(\chi).$$

Theorem 1.3. ([18], Davenport-Hasse Relation). *Let m be a positive integer and let $q=p^r$ be a prime power such that $q\equiv 1(mod\ m)$. For multiplicative characters*

$\chi, \psi \in \widehat{\mathbb{F}_q^{\times}}$, *we have*

$$\prod_{\chi^m=\varepsilon} g(\chi\psi) = -g(\psi^m)\psi(m^{-m}) \prod_{\chi^m=\varepsilon} g(\chi).$$

The orthogonality relation for multiplicative characters relates θ with Gauss sums as given in the following lemma.

Lemma 1.4. ([29, Lemma 2.2]). *For* $\alpha \in \mathbb{F}_q^{\times}$,

$$\theta(\alpha) = \frac{1}{q-1}\sum_{m=0}^{q-2} g(T^{-m})T^m(\alpha).$$

Definition 1.1. *For multiplicative characters A and B over $\mathbb{F}_q$, the Jacobi sum $J(A,B)$ is defined by*

$$J(A,B) := \sum_{x\in\mathbb{F}_q} A(x)B(1-x).$$

The following lemma gives a relation between Jacobi and Gauss sums.

Lemma 1.5. ([36, Eq. 1.14]). *For* $A, B \in \widehat{\mathbb{F}_q^{\times}}$ *we have*

$$J(A,B) = \frac{g(A)g(B)}{g(AB)} + (q-1)B(-1)\delta(AB).$$

For multiplicative characters A and B on $\mathbb{F}_q$, the binomial coefficient $\binom{A}{B}$ is defined by

$$\binom{A}{B} := \frac{B(-1)}{q}J(A,\overline{B}) = \frac{B(-1)}{q}\sum_{x\in\mathbb{F}_q} A(x)\overline{B}(1-x), \tag{1.3}$$

It is easy to see that the Jacobi sum satisfies the following identity:

$$J(A,B) = A(-1)J(A,\overline{AB}). \tag{1.4}$$

We recall the following properties of the binomial coefficients from [36]:

$$\binom{A}{\varepsilon} = \binom{A}{A} = -\frac{1}{q} + \frac{q-1}{q}\delta(A). \tag{1.5}$$

1.3 Elliptic curves

Let K be a field. An elliptic curve over K is a smooth, cubic projective algebraic curve in three variables with a specified point, $O = [0 : 1 : 0]$. An elliptic curve over K is defined by the equation

$$E : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where $a_i \in K$ for all i. By de-homogenizing the equation, we get the Weierstrass form of an elliptic curve given by the equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with a point at infinity O and $a_1, a_2, a_3, a_4, a_6 \in K$. If char $(K) \neq 2$, then E can be written as

$$E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6,$$

where $b_2 = a_1^2 + 4a_2, b_4 = 2a_4 + a_1a_3$, and $b_6 = a_3^2 + 4a_6$. Employing $y \mapsto 2y$ yields

$$E : y^2 = x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4}. \tag{1.6}$$

We also define the terms given below:

$$b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,$$
$$c_4 = b_2^2 - 24b_4,$$

$$\Delta(E) = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6,$$
$$j(E) = \frac{c_4^3}{\Delta(E)},$$

where $\Delta(E)$ and $j(E)$ are called discriminant and j-invariant of the elliptic curve E, respectively.

Let E be an elliptic curve defined over $\mathbb{Q}$ and p be a prime. Let $\widetilde{E}$ denote the reduction of $E \pmod p$. If $\widetilde{E}$ is also an elliptic curve over the finite field $\mathbb{F}_p$, then p is said to be a prime of good reduction. Recall that E has good reduction at the prime p if and only if p does not divide the discriminant of E, i.e., $p \nmid \Delta(E)$. We now define the integer $a_p(E)$ by

$$a_p(E) := p + 1 - \#\widetilde{E}(\mathbb{F}_p),$$

where $\#\widetilde{E}(\mathbb{F}_p)$ is the number of $\mathbb{F}_p$-points on $\widetilde{E}$ including the point at infinity. If p is a prime of good reduction, then $a_p(E)$ is called the trace of Frobenius as it can be interpreted as the trace of the Frobenius endomorphism on E. For more details, see [64, 72]. Next, we recall the notion of a quadratic twist. Let E be an elliptic curve given by

$$E : y^2 = x^3 + ax^2 + bx + c,$$

where $a, b, c \in \mathbb{F}_p$. If $D \in \mathbb{F}_p^\times$, then the D-quadratic twist of E, denoted by E^D, is an elliptic curve given by the equation

$$E^D : y^2 = x^3 + Dax^2 + D^2bx + D^3c.$$

It is known that the traces of Frobenius of E and E^D satisfy the following relation:

$$a_p(E) = \left(\frac{D}{p}\right) a_p(E^D), \tag{1.7}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Next, we recall the definition of an isogeny of elliptic curves.

Definition 1.2. ([72, p. 386]). *Let E_1 and E_2 be two elliptic curves defined over a field K. An isogeny from E_1 to E_2 is a nonconstant homomorphism $\phi : E_1(\overline{K}) \to E_2(\overline{K})$ that is given by rational functions, where $\overline{K}$ is the algebraic closure of K.*

1.4 Modular forms

In this section, we introduce modular forms and recall some definitions and results of modular forms. For more details, see [25, 45].

Let $\mathrm{SL}_2(\mathbb{Z})$ be the special linear group, consisting of all invertible 2×2 matrices having entries from $\mathbb{Z}$ and determinant 1. For a positive integer N, we consider the following subgroups of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{\mathrm{N}} \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0, d \equiv 1 \pmod{N} \right\}.$$

A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subset \Gamma$, for some N. The *level* of Γ is the smallest N such that $\Gamma(N) \subset \Gamma$. Let $\mathbb{H}$ denote the upper half plane,

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\} \subseteq \mathbb{C}.$$

We define an action of the group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - cb > 0 \right\}$$

on $\mathbb{H}$ via the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z := \frac{az+b}{cz+d}.$$

We define the extended upper half plane and denote it by $\mathbb{H}^*$, by adjoining all the rational numbers and $i\infty$, as

$$\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}.$$

The elements in $\mathbb{Q} \cup \{i\infty\}$ are called the cusps of $\mathrm{SL}_2(\mathbb{Z})$. It is easy to see that every cusp of $\mathrm{SL}_2(\mathbb{Z})$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to $i\infty$, i.e., for any $r \in \mathbb{Q}$ there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(i\infty) = r$.

Definition 1.3. *If Γ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{F} \subset \mathbb{H}$ is a closed set with connected interior, we say that $\mathcal{F}$ is a fundamental domain for Γ (or $\Gamma\backslash\mathbb{H}$) if*
a) any $z \in \mathbb{H}$ is Γ-equivalent to a point in $\mathcal{F}$;
b) no two interior points of $\mathcal{F}$ are Γ-equivalent;
c) the boundary of $\mathcal{F}$ is a finite union of smooth curves.

Consider a function $f : \mathbb{H} \to \mathbb{C}$ for which

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{1.8}$$

where $k \in \mathbb{Z}$. We use the topology of the extended upper half plane to define the holomorphicity of f at $i\infty$. Let $q = e^{2i\pi z}$. If $f(z)$ is holomorphic on $\mathbb{H}$, then $f(q)$ will be holomorphic on the punctured unit disc. Hence, we have the *Laurent series* expansion of f centered at $q = 0$, as

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n$$

which is also called the q-series associated with f. Hence, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi izn}$$

as the *Fourier series* of f at $i\infty$. We say f is holomorphic at $i\infty$ if $a_n = 0$ for all negative integers n.

Definition 1.4. *A modular form of weight $k \in \mathbb{Z}$ for the full modular group* $\mathrm{SL}_2(\mathbb{Z})$ *is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies* (1.8) *and is holomorphic at $i\infty$. A modular form f is called a cusp form if $a_0 = 0$, that is, f vanishes at $i\infty$.*

Next, we study the modular forms of level N. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If f is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then the slash operator $|_\ell$ is defined as

$$(f|_\ell \gamma)(z) := (\det \gamma)^{\ell/2}(cz + d)^{-\ell} f(\gamma z).$$

Definition 1.5. *Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight ℓ for Γ if the following hold:*

1. *For all $z \in \mathbb{H}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$,*

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^\ell f(z).$$

2. *If* $\gamma \in SL_2(\mathbb{Z})$*, then* $(f|_\ell \gamma)(z)$ *has a Fourier series expansion of the form*

$$(f|_\ell \gamma)(z) = \sum_{n \geq 0} a_\gamma(n) q_N^n,$$

where $q_N := e^{2\pi i z/N}$.

The $\mathbb{C}$-vector space of modular forms of weight k for Γ is denoted by $M_k(\Gamma)$. If $f \in M_k(\Gamma)$ such that it vanishes at every cusp of Γ, we say f is a cusp form of weight k for Γ and the $\mathbb{C}$-vector space of such forms is denoted by $S_k(\Gamma)$. To express $M_k(\Gamma_1(N))$ as a direct sum decomposition of certain interesting subspaces, one needs to attach Dirichlet character to modular forms.

Definition 1.6. *A Dirichlet character modulo* N *is a homomorphism* $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

For any Dirichlet character χ modulo N, we define the following vector subspace of $M_k(\Gamma_1(N))$:

$$M_k(\Gamma_0(N), \chi) := \left\{ f \in M_k(\Gamma_1(N)) : f|_k \gamma = \chi(d) f, \ \forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \right\}.$$

For $M_k(\Gamma_1(N))$, we have the following direct sum decomposition

$$M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(\Gamma_0(N), \chi),$$

where the direct sum is over all the Dirichlet characters modulo N. Let χ be a Dirichlet character modulo N and n be a positive integer then we define a map $T_n : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(N), \chi)$ by

$$T_n(f) := n^{\frac{k}{2}-1} \sum_{ad=n, a>0} \chi(a) \sum_{b=0}^{d-1} f| \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

These maps are linear operators and are known as Hecke operators. These operators also map $S_k(\Gamma_0(N),\chi)$ to itself. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathcal{F}$ be any fundamental domain for Γ. Let $\overline{\Gamma}=\frac{\Gamma}{\{\pm I\}}$. Then for $f,g\in S_k(\Gamma)$, we define the Petersson inner product as

$$(f,g)=\frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\overline{\Gamma}]}\iint_{\mathcal{F}} y^k f(z)\overline{g(z)}\frac{dxdy}{y^2}, \qquad \text{where } z=x+iy.$$

Theorem 1.6. *Petersson inner product satisfies the following properties:*

1. (f,g) *is linear in* f*;*
2. (f,g) *is conjugate symmetric i.e.,* $(f,g)=\overline{(g,f)}$*;*
3. $(f,f)>0$ *for* $f\neq 0$*.*

Therefore, it defines a hermitian inner product on $S_k(\Gamma)$*.*

Definition 1.7. *Let* N *be a positive integer and* d *be a divisor of* N*, then*

1. *If* $f\in S_k(\Gamma_1(d))$*, then* $f\in S_k(\Gamma_1(N))$*.*
2. *If* $N=cd$*,* $f\in S_k(\Gamma_1(d))$*, and* $g(z):=f(cz)$*, then* $g\in S_k(\Gamma_1(N))$*.*

The subspace of $S_k(\Gamma_1(N))$ *spanned by the forms obtained in these two ways as we take the range over proper divisor* d *of* N*, is called the space of oldforms, denoted by* $S_k^{old}(\Gamma_1(N))$*.*

Definition 1.8. *The orthogonal complement of this space with respect to the Petersson inner product is called the space of newforms, denoted by* $S_k^{new}(\Gamma_1(N))$*.*

1.5 Hypergeometric functions over finite fields

In this section, we recall hypergeometric functions over finite fields as introduced by Greene, McCarthy, and Fuselier et al. Firstly, we recall hypergeometric functions over finite fields as introduced by Greene in [35, 36]. Greene established these functions as finite field analogues of classical hypergeometric series by proving finite field analogues of summation, product, and transformation formulas satisfied by the classical hypergeometric series.

Definition 1.9. ([36, Definition 3.10]). *Let n be a positive integer and $x \in \mathbb{F}_q$. For multiplicative characters $A_0, A_1, \ldots, A_n, B_1, B_2, \ldots, B_n$ on $\mathbb{F}_q^{\times}$, Greene's ${}_{n+1}F_n$-hypergeometric functions over $\mathbb{F}_q$ is defined by*

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right)_q := \frac{q}{q-1}\sum_{\chi \in \widehat{\mathbb{F}_q^{\times}}} \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x). $$

In [30], Fuselier et al. gave another definition of hypergeometric function over finite fields. Firstly, they defined a period function as follows.

Definition 1.10. ([30, p. 28]). *Let n be a positive integer and $x \in \mathbb{F}_q$. For multiplicative characters $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n$ on $\mathbb{F}_q$, the ${}_{n+1}\mathbb{P}_n$ period function over $\mathbb{F}_q$ is defined by*

$$ \begin{aligned} {}_{n+1}\mathbb{P}_n\left[\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} |x\right]_q &:= \delta(x)\prod_{i=1}^{n} J(A_i, \overline{A_i}B_i) \\ &+ \frac{q^{n+1}}{q-1}\left(\prod_{i=1}^{n} A_iB_i(-1)\right)\sum_{\chi \in \widehat{\mathbb{F}_q^{\times}}} \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x). \end{aligned} $$

We note that the binomial coefficient defined in [30] is equal to $(-q)$ times the binomial coefficient defined by Greene [36]. Since we have used Greene's definition of binomial coefficient, an extra factor of $(-q)^{n+1}$ appears in Definition 1.10. The following definition of hypergeometric functions over finite fields is given by Fuselier et al.

Definition 1.11. ([30, Eq. 4.9]). *Let n be a positive integer and $x \in \mathbb{F}_q$. For* *multiplicative characters* $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n$ *on* $\mathbb{F}_q$*, the* ${}_{n+1}\mathbb{F}_n$*-hypergeometric function over* $\mathbb{F}_q$ *is defined by*

$$
{}_{n+1}\mathbb{F}_n\left[\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right]_q
:= \frac{1}{\prod_{i=1}^{n} J(A_i, \overline{A_i}B_i)} {}_{n+1}\mathbb{P}_n\left[\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right]_q.
$$

Both the definitions of hypergeometric functions over finite fields given by Greene and Fuselier et al. use binomial coefficients of multiplicative characters. The binomial coefficients are defined using Jacobi sums. In [52], McCarthy gave another definition of hypergeometric function over finite fields using the Gauss sums.

Definition 1.12. ([52, Definition 1.4]). *Let* n *be a positive integer. For* $x \in \mathbb{F}_q$ *and multiplicative characters* $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n$ *on* $\mathbb{F}_q$*, McCarthy's* ${}_{n+1}F_n{}^*$*-hypergeometric function over* $\mathbb{F}_q$ *is defined by*

$$
{}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right)_q^* := \frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\prod_{i=0}^{n}\frac{g(A_i\chi)}{g(A_i)}
\times \prod_{j=1}^{n}\frac{g(\overline{B_j\chi})}{g(\overline{B_j})} g(\overline{\chi})\chi(-1)^{n+1}\chi(x).
$$

The following proposition relates the Greene's and McCarthy's hypergeometric functions when certain conditions on the parameters are satisfied.

Proposition 1.7. ([52, Proposition 2.5]). *If* $A_0 \neq \varepsilon$ *and* $A_i \neq B_i$ *for each* $1 \leq i \leq n$ *then*

$$
{}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right)_q^* = \left[\prod_{i=1}^{n}\binom{A_i}{B_i}^{-1}\right] {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right)_q.
$$

1.6 p-adic Gamma function and Gross-Koblitz formula

In this section, we first define p-adic gamma function and then recall the Gross-Koblitz formula. For further details, see [43, 59]. For a positive integer n, the p-adic gamma function $\Gamma_p(n)$ is defined as

$$\Gamma_p(n) := (-1)^n \prod_{0<j<n, p\nmid j} j$$

and one extends it to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) := 1$ and

$$\Gamma_p(x) := \lim_{x_n \to x} \Gamma_p(x_n)$$

for $x \neq 0$, where x_n runs through any sequence of positive integers p-adically approaching x. This limit exists, is independent of how x_n approaches x, and determines a continuous function on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p^{\times}$.

We now state the Gross-Koblitz formula which gives us a relation between the Gauss sum and the p-adic gamma function. Let $\omega : \mathbb{F}_q^{\times} \to \mathbb{Z}_q^{\times}$ be the Teichmüller character. For $a \in \mathbb{F}_q^{\times}$, the value $\omega(a)$ is just the $(q-1)$-th root of unity in $\mathbb{Z}_q$ such that $\omega(a) \equiv a \pmod{p}$. We denote by $\overline{\omega}$ the inverse of ω. We note that $\omega|_{\mathbb{F}_p^{\times}}$ is the Teichmüller character on $\mathbb{F}_p^{\times}$ with values in $\mathbb{Z}_p^{\times}$. Also, $\widehat{\mathbb{F}_q^{\times}} = \{\omega^j : 0 \leq j \leq q-2\}$. Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ which satisfies $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. For $x \in \mathbb{Q}$, we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x, i.e., $x - \lfloor x \rfloor$, satisfying $0 \leq \langle x \rangle < 1$. Then, we have the following result.

Theorem 1.8. ([37], Gross-Koblitz). *For $a \in \mathbb{Z}$ and $q = p^r$,*

$$g(\overline{\omega}^a) = -\pi^{(p-1)\sum_{i=0}^{r-1} \langle \frac{ap^i}{q-1} \rangle} \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \frac{ap^i}{q-1} \right\rangle\right).$$

1.7 McCarthy's p-adic hypergeometric functions

As mentioned in the introduction, results involving hypergeometric functions over finite fields will often be restricted to primes in certain congruence classes. To overcome these restrictions, McCarthy introduced a function ${}_nG_n[\cdots]_q$ which can best be described as an analogue of hypergeometric functions in the p-adic setting. In this section, we recall the definition of p-adic hypergeometric functions introduced by McCarthy. McCarthy's p-adic hypergeometric function ${}_nG_n[\cdots]_q$ is defined using the p-adic gamma function as follows.

Definition 1.13. ([53, Definition 5.1]). *Let p be an odd prime and $q = p^r$, $r \geq 1$. Let $t \in \mathbb{F}_q$. For positive integer n and $1 \leq k \leq n$, let a_k, $b_k \in \mathbb{Q} \cap \mathbb{Z}_p$. Then the function ${}_nG_n[\cdots]_q$ is defined by*

$$\begin{aligned}
&{}_nG_n\left[\begin{matrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{matrix} \mid t\right]_q \\
&:= \frac{-1}{q-1}\sum_{a=0}^{q-2}(-1)^{an}\ \overline{\omega}^a(t)\prod_{k=1}^{n}\prod_{i=0}^{r-1}(-p)^{-\lfloor\langle a_kp^i\rangle-\frac{ap^i}{q-1}\rfloor-\lfloor\langle -b_kp^i\rangle+\frac{ap^i}{q-1}\rfloor} \\
&\quad\times\frac{\Gamma_p(\langle(a_k-\frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle a_kp^i\rangle)}\frac{\Gamma_p(\langle(-b_k+\frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle -b_kp^i\rangle)}.
\end{aligned}$$

McCarthy proved the following lemma which establishes a relationship between ${}_{n+1}G_{n+1}[\cdots]_q$ and ${}_{n+1}F_n(\cdots)^*_q$. In [53], he proved this lemma for $\mathbb{F}_p$, but it also holds for $\mathbb{F}_q$.

Lemma 1.9. ([53, Lemma 3.3]). *For a fixed odd prime power q, let $A_i, B_k \in \widehat{\mathbb{F}_q^{\times}}$ be given by $\overline{\omega}^{a_i(q-1)}$ and $\overline{\omega}^{b_k(q-1)}$ respectively, where ω is the Teichmüller character. Then*

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right)^*_q = {}_{n+1}G_{n+1}\left[\begin{matrix} a_0, & a_1, & \ldots, & a_n \\ 0, & b_1, & \ldots, & b_n \end{matrix} \mid \frac{1}{x}\right]_q. $$

The function ${}_nG_n[\cdots]_q$ often allows results involving finite field hypergeometric functions to be extended to a wider class of primes. In [53], McCarthy gave an expression for the number of $\mathbb{F}_p$-points on certain elliptic curves in terms of ${}_2G_2[\cdots]_p$ hypergeometric functions for all but finitely many primes. His result extended the work of Fuselier [29] and Lennon [47] which were valid for primes in certain congruence classes. Later, Barman and Saikia [12, 13] extended his work to prime powers and found some new relations of these functions to the trace of Frobenius endomorphism on elliptic curves.

1.8 Some Lemmas

In this section, we recall some lemmas that will be used to prove our results. We first state a product formula for the p-adic gamma function, which is a reformulation of multiplication formula for Gauss sums, follows from [37, Theorem 3.1]. If $m \in \mathbb{Z}^+$, $p \nmid m$ and $x(q-1) \in \mathbb{Z}$, then we have

$$\prod_{i=0}^{r-1}\prod_{h=0}^{m-1}\Gamma_p\left(\left\langle\left(\frac{x+h}{m}\right)p^i\right\rangle\right) = \omega(m^{(1-x)(1-q)})\prod_{i=0}^{r-1}\Gamma_p\left(\langle xp^i\rangle\right)\prod_{h=1}^{m-1}\Gamma_p\left(\left\langle\frac{hp^i}{m}\right\rangle\right). \tag{1.9}$$

The following lemmas relate certain products of values of the p-adic gamma function. We will use these lemmas to prove our main results.

Lemma 1.10. ([12, Eq. 6]). *Let p be a prime and $q = p^r, r \geq 1$. For $0 \leq a \leq q-2$ and $t \geq 1$ with $p \nmid t$, we have*

$$\omega(t^{ta})\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{tp^ia}{q-1}\right\rangle\right)\prod_{h=1}^{t-1}\Gamma_p\left(\left\langle\frac{hp^i}{t}\right\rangle\right) = \prod_{i=0}^{r-1}\prod_{h=0}^{t-1}\Gamma_p\left(\left\langle\frac{p^ih}{t}+\frac{p^ia}{q-1}\right\rangle\right).$$

Lemma 1.11. ([12, Eq. 7]). *Let p be a prime and $q = p^r, r \geq 1$. For $0 \leq a \leq q-2$*

and $t \geq 1$ with $p \nmid t$, we have

$$\omega(t^{-ta})\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{-tp^ia}{q-1}\right\rangle\right)\prod_{h=1}^{t-1}\Gamma_p\left(\left\langle\frac{hp^i}{t}\right\rangle\right)=\prod_{i=0}^{r-1}\prod_{h=0}^{t-1}\Gamma_p\left(\left\langle\frac{p^i(1+h)}{t}-\frac{p^ia}{q-1}\right\rangle\right).$$

The following lemma is the reformulation of the reflection formula for the Gauss sums using Gross-Koblitz formula:

Lemma 1.12. ([17, Lemma 3.4]). *Let p be an odd prime and $q = p^r, r \geq 1$. For $0 < a \leq q-2$, we have*

$$\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(1-\frac{a}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\frac{ap^i}{q-1}\right\rangle\right)=(-1)^r\overline{\omega}^a(-1). \tag{1.10}$$

For $0 \leq a \leq q-2$ such that $a \neq \frac{q-1}{2}$, we have

$$\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle(\frac{1}{2}-\frac{a}{q-1})p^i\rangle)\Gamma_p(\langle(\frac{1}{2}+\frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{2}\rangle)}=\overline{\omega}^a(-1). \tag{1.11}$$

Finally, we prove two lemmas relating to fractional and integral parts of certain rational numbers, which will be used to simplify certain products of the p-adic gamma function.

Lemma 1.13. *Let p be an odd prime and $q = p^r, r \geq 1$. Let $d \geq 2$ be an integer such that $p \nmid d$. Then, for $1 \leq a \leq q-2$ and $0 \leq i \leq r-1$, we have*

$$\left\lfloor\frac{ap^i}{q-1}\right\rfloor+\left\lfloor\frac{-dap^i}{q-1}\right\rfloor=\sum_{h=1}^{d-1}\left\lfloor\left\langle\frac{hp^i}{d}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-1. \tag{1.12}$$

Proof. Let $\left\lfloor\frac{-dap^i}{q-1}\right\rfloor = dk+s$, where $k, s \in \mathbb{Z}$ and $0 \leq s \leq d-1$. This yields

$$dk+s \leq \frac{-dap^i}{q-1} < dk+s+1,$$

which implies

$$k + \frac{s}{d} \leq \frac{-ap^i}{q-1} < k + \frac{s}{d} + \frac{1}{d} \tag{1.13}$$

and

$$-k - \frac{s}{d} - \frac{1}{d} < \frac{ap^i}{q-1} \leq -k - \frac{s}{d}. \tag{1.14}$$

Since $a \neq 0$, by (1.14) we obtain $\left\lfloor \frac{ap^i}{q-1} \right\rfloor = -k-1$ and hence the left hand side of (1.12) becomes $(d-1)k + s - 1$. Since $p \nmid d$, we observe that

$$\sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{hp^i}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor = \sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{h}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor.$$

For $1 \leq h \leq d-1$, (1.13) gives

$$k + \frac{s}{d} + \frac{h}{d} \leq \frac{h}{d} - \frac{ap^i}{q-1} < k + \frac{s}{d} + \frac{1}{d} + \frac{h}{d}. \tag{1.15}$$

If $s = 0$, then $\left\lfloor \left\langle \frac{h}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor = k$ for $h = 1, \ldots, d-1$ and hence

$$\sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{h}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor = (d-1)k.$$

If $1 \leq s \leq d-1$, then (1.15) yields

$$\left\lfloor \left\langle \frac{h}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor = \begin{cases} k, & \text{if } 1 \leq h \leq d-s-1; \\ k+1, & \text{if } d-s \leq h \leq d-1. \end{cases} \tag{1.16}$$

From (1.16), we obtain

$$\sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{h}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor = (d-1)k + s.$$

Thus, for $0 \leq s \leq d-1$, we have

$$\sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{h}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - 1 = (d-1)k + s - 1.$$

Therefore, the right hand side of (1.12) becomes $(d-1)k+s-1$. This completes the proof of the lemma. ■

Lemma 1.14. *Let p be an odd prime and $q = p^r, r \geq 1$. Let l be a positive integer such that $p \nmid l$. Then, for $0 \leq a \leq q-2$ and $0 \leq i \leq r-1$, we have*

$$\left\lfloor \frac{lap^i}{q-1} \right\rfloor = \sum_{h=0}^{l-1} \left\lfloor \left\langle \frac{-hp^i}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor . \tag{1.17}$$

Proof. Let $\left\lfloor \frac{lap^i}{q-1} \right\rfloor = lk + s$, where $k, s \in \mathbb{Z}$ and $0 \leq s \leq l-1$. Then

$$lk + s \leq \frac{lap^i}{q-1} < lk + s + 1,$$

which implies

$$k + \frac{s}{l} \leq \frac{ap^i}{q-1} < k + \frac{s}{l} + \frac{1}{l}. \tag{1.18}$$

Using (1.18), we have $\left\lfloor \frac{ap^i}{q-1} \right\rfloor = k$. Since $p \nmid l$, we observe that

$$\sum_{h=1}^{l-1} \left\lfloor \left\langle \frac{-hp^i}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor = \sum_{h=1}^{l-1} \left\lfloor \left\langle \frac{-h}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor .$$

For $1 \leq h \leq l-1$, (1.18) gives

$$k + \frac{s}{l} - \frac{h}{l} + 1 \leq \left\langle \frac{-h}{l} \right\rangle + \frac{ap^i}{q-1} < k + \frac{s}{l} + \frac{1}{l} - \frac{h}{l} + 1. \tag{1.19}$$

If $s = 0$, then $\left\lfloor \left\langle \frac{-h}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor = k$ for all $1 \le h \le l-1$ and hence

$$\sum_{h=1}^{l-1} \left\lfloor \left\langle \frac{-h}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor = (l-1)k.$$

If $1 \le s \le l-1$, then (1.19) yields

$$\left\lfloor \left\langle \frac{-h}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor = \begin{cases} k+1, & \text{if } 1 \le h \le s; \\ k, & \text{if } s+1 \le h \le l-1. \end{cases} \tag{1.20}$$

From (1.20), we obtain

$$\sum_{h=1}^{l-1} \left\lfloor \left\langle \frac{-h}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor = (l-1)k + s.$$

Thus, for $0 \le s \le l-1$, we have

$$\sum_{h=0}^{l-1} \left\lfloor \left\langle \frac{-h}{l} \right\rangle + \frac{ap^i}{q-1} \right\rfloor = k + (l-1)k + s = lk + s.$$

This completes the proof of the lemma. ■

2

Certain Transformations of p-adic Hypergeometric Functions

2.1 Introduction

In [36], Greene proved finite field analogues of several transformation, summation, and reduction formulas satisfied by the classical hypergeometric series. Since then, numerous transformations and summation identities have been proved for Gaussian hypergeometric series by several authors. These transformations for hypergeometric functions over finite fields can be re-written in terms of ${}_nG_n[\cdots]_q$,

[1]The contents of this chapter have been published in *Res. Number Theory* (2022) and *Int. J. Number Theory* (2024)

and these results will hold for all q where the original characters existed over $\mathbb{F}_q$. Therefore, it is a non-trivial and important problem to find p-adic analogues of identities satisfied by the classical hypergeometric series which hold for all but finitely many primes. Only a few such results are known to date. For example, see [11, 12, 13, 14, 15, 16, 31, 61, 62].

In this chapter, we study certain transformation identities and special values of McCarthy's p-adic hypergeometric functions, which are valid for all but finitely many primes. First, we prove p-adic analogue of a transformation of Euler and a transformation of Clausen. For this, we first relate a character sum to the p-adic hypergeometric functions. Then, by finding the zeros of two p-adic hypergeometric functions ${}_2G_2[\cdots]_q$, we deduce the transformation of Euler. Then we use a character sum identity proved by Ahlgren, Ono, and Penniston to deduce the p-adic Clausen's transformation. Finally, we prove another two identities for the p-adic hypergeometric functions which will be helpful in the subsequent chapters.

2.2 p-adic analogue of a transformation of Euler

The transformation due to Euler [65, p. 10] is given by

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \mid x\right] = (1-x)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a, & c-b \\ & c \end{matrix} \mid x\right]. \tag{2.1}$$

Greene proved a finite field analogue of (2.1) in [36, Theorem 4.4 (iv)]. In this section, we prove a p-adic analogue (2.1) for some specific values of the parameters. We first prove the following lemmas.

Lemma 2.1. *Let p be an odd prime and $q = p^r, r \geq 1$. Then, for $0 \leq a \leq q-2$ such that $a \neq \frac{q-1}{2}$ and $0 \leq i \leq r-1$, we have*

$$ -2\left\lfloor \frac{2ap^i}{q-1} \right\rfloor - \left\lfloor \frac{-6ap^i}{q-1} \right\rfloor + \left\lfloor \frac{ap^i}{q-1} \right\rfloor + \left\lfloor \frac{-3ap^i}{q-1} \right\rfloor $$

$$= -\left\lfloor \left\langle \frac{p^i}{6} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{5p^i}{6} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{p^i}{2} \right\rangle + \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \frac{ap^i}{q-1} \right\rfloor . \quad (2.2)$$

Proof. If $a = 0$, then it readily follows that (2.2) is true. For $a \neq 0$, using Lemma 1.13 with $d = 6$ and $d = 3$, Lemma 1.14 with $l = 2$, and the fact that $\lfloor x \rfloor + \lfloor 1-x \rfloor = 0$ if $x \notin \mathbb{Z}$, we readily obtain (2.2). ■

Lemma 2.2. *Let p be an odd prime and $q = p^r, r \geq 1$. Then, for $0 < a \leq q-2$ and $0 \leq i \leq r-1$, we have*

$$\begin{aligned} &-\left\lfloor \frac{2ap^i}{q-1} \right\rfloor - \left\lfloor \frac{-3ap^i}{q-1} \right\rfloor \\ &= 1 - \left\lfloor \left\langle \frac{p^i}{3} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{2p^i}{3} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{p^i}{2} \right\rangle + \frac{ap^i}{q-1} \right\rfloor . \end{aligned}$$

Proof. Using Lemma 1.13 with $d = 3$ and Lemma 1.14 with $l = 2$, we obtain the desired result. ■

We now prove two propositions which enable us to express certain character sums in terms of the p-adic hypergeometric functions.

Proposition 2.3. *Let $p \geq 5$ be a prime and $q = p^r, r \geq 1$. For $x \in \mathbb{F}_q^\times$, we have*

$$\frac{1}{q(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} g(\chi)g(\chi^2)g(\overline{\chi}^3)\chi\left(\frac{-27}{4x}\right) = \frac{1}{q} + {}_2G_2\left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} \Big| \frac{1}{x}\right]_q .$$

Proof. Let T be a generator of the cyclic group $\widehat{\mathbb{F}_q^\times}$. Then we have

$$\begin{aligned} A &:= \frac{1}{q(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} g(\chi)g(\chi^2)g(\overline{\chi}^3)\chi\left(\frac{-27}{4x}\right) \\ &= \frac{1}{q(q-1)} \sum_{a=0}^{q-2} g(T^a)g(T^{2a})g(T^{-3a})T^a\left(\frac{-27}{4x}\right). \end{aligned}$$

Now, taking $T=\overline{\omega}$ and then applying Gross-Koblitz formula we deduce that

$$\begin{aligned}A&=\frac{1}{q(q-1)}\sum_{a=0}^{q-2}g(\overline{\omega}^a)g(\overline{\omega}^{2a})g(\overline{\omega}^{-3a})\overline{\omega}^a\left(\frac{-27}{4x}\right)\\&=-\frac{1}{q(q-1)}\sum_{a=0}^{q-2}\overline{\omega}^a\left(\frac{-27}{4x}\right)\prod_{i=0}^{r-1}(-p)^{\alpha_{i,a}}\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{2ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{-3ap^i}{q-1}\rangle),\end{aligned}$$

where $\alpha_{i,a}=\langle\frac{ap^i}{q-1}\rangle+\langle\frac{2ap^i}{q-1}\rangle+\langle\frac{-3ap^i}{q-1}\rangle$. Taking out the term for $a=0$ gives

$$\begin{aligned}A=&-\frac{1}{q(q-1)}\sum_{a=1}^{q-2}\overline{\omega}^a\left(\frac{-27}{4x}\right)\prod_{i=0}^{r-1}(-p)^{\alpha_{i,a}}\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{2ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{-3ap^i}{q-1}\rangle)\\&-\frac{1}{q(q-1)}.\end{aligned}$$

Using Lemma 1.11 with $t=3$, Lemma 1.10 with $t=2$ and Lemma 2.2, we deduce that

$$\begin{aligned}A=&-\frac{1}{q(q-1)}\sum_{a=1}^{q-2}\overline{\omega}^a\left(\frac{-1}{x}\right)\prod_{i=0}^{r-1}(-p)^{1+s_{i,a}}\frac{\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{p^i}{2}+\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{p^i}{3}-\frac{ap^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{3}\rangle)\Gamma_p(\langle\frac{2p^i}{3}\rangle)}\\&\times\Gamma_p(\langle\frac{2p^i}{3}-\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle p^i-\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)-\frac{1}{q(q-1)},\end{aligned}$$

where $s_{i,a}=-\left\lfloor\langle\frac{p^i}{3}\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\langle\frac{2p^i}{3}\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\langle\frac{p^i}{2}\rangle+\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\frac{ap^i}{q-1}\right\rfloor$.
Employing (1.10), we find that

$$\begin{aligned}A=&-\frac{1}{q-1}\sum_{a=1}^{q-2}\overline{\omega}^a\left(\frac{1}{x}\right)\prod_{i=0}^{r-1}(-p)^{s_{i,a}}\frac{\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{p^i}{2}+\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{p^i}{3}-\frac{ap^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{3}\rangle)\Gamma_p(\langle\frac{2p^i}{3}\rangle)}\\&\times\Gamma_p(\langle\frac{2p^i}{3}-\frac{ap^i}{q-1}\rangle)-\frac{1}{q(q-1)}\\=&\,{}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & \frac{1}{2}\end{matrix}\;|\frac{1}{x}\right]_q-\frac{1}{q(q-1)}+\frac{1}{q-1}\end{aligned}$$

$$= {}_2G_2\left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} \mid \frac{1}{x}\right]_q + \frac{1}{q}.$$

This completes the proof of the proposition. ■

Proposition 2.4. *Let $p \geq 5$ be a prime and $q = p^r, r \geq 1$. For $x \in \mathbb{F}_q^{\times}$, we have*

$$\frac{1}{q(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^{\times}}} g(\chi)g(\chi^2)g(\overline{\chi}^3)\chi\left(\frac{-27}{4x}\right) = \frac{1}{q} + \varphi(3x) \cdot {}_2G_2\left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & \frac{1}{2} \end{matrix} \mid \frac{1}{x}\right]_q.$$

Proof. Let T be a generator of the cyclic group $\widehat{\mathbb{F}_q^{\times}}$. As in Proposition 2.3, we have

$$A = \frac{1}{q(q-1)} \sum_{a=0}^{q-2} g(T^a)g(T^{2a})g(T^{-3a})T^a\left(\frac{-27}{4x}\right).$$

Replacing a by $a - \frac{q-1}{2}$, we obtain

$$A = \frac{1}{q(q-1)} \sum_{a=0}^{q-2} g(T^a\varphi)g(T^{2a})g(T^{-3a}\varphi)T^a\left(\frac{-27}{4x}\right)\varphi(-3x). \tag{2.3}$$

Using Davenport-Hasse relation for $m = 2$, $\psi = T^a$ and $m = 2$, $\psi = T^{-3a}$, we have

$$g(T^a\varphi) = \frac{g(T^{2a})g(\varphi)T^a(2^{-2})}{g(T^a)},$$
$$g(T^{-3a}\varphi) = \frac{g(T^{-6a})g(\varphi)T^{-3a}(2^{-2})}{g(T^{-3a})}.$$

Substituting these values in (2.3), we deduce that

$$A = \frac{\varphi(-3x)}{q(q-1)} \sum_{a=0}^{q-2} \frac{g^2(T^{2a})g(\varphi)T^a(2^{-2})g(T^{-6a})g(\varphi)T^{-3a}(2^{-2})}{g(T^a)g(T^{-3a})}T^a\left(\frac{-27}{4x}\right).$$

Lemma 1.2 yields

$$A=\frac{\varphi(3x)}{q-1}\sum_{a=0}^{q-2}\frac{g^2(T^{2a})g(T^{-6a})}{g(T^a)g(T^{-3a})}T^a\left(\frac{-27\times 4}{x}\right).$$

Replacing T by $\overline{\omega}$ and then applying Gross-Koblitz formula, we obtain

$$A=-\frac{\varphi(3x)}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a\left(\frac{-27\times 4}{x}\right)\prod_{i=0}^{r-1}p^{\beta_{i,a}}\frac{\Gamma_p^2(\langle\frac{2ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{-6ap^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{-3ap^i}{q-1}\rangle)},$$

where $\beta_{i,a}=2\langle\frac{2ap^i}{q-1}\rangle+\langle\frac{-6ap^i}{q-1}\rangle-\langle\frac{-3ap^i}{q-1}\rangle-\langle\frac{ap^i}{q-1}\rangle$. Using Lemma 1.11 with $t=3$ and $t=6$ and Lemma 1.10 with $t=2$, and then employing (1.11), we deduce that

$$\begin{aligned}A&=-\frac{\varphi(3x)}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a\left(\frac{-1}{x}\right)\prod_{i=0}^{r-1}p^{\beta_{i,a}}\frac{\Gamma_p(\langle\frac{p^i}{6}-\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{5p^i}{6}-\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{6}\rangle)\Gamma_p(\langle\frac{5p^i}{6}\rangle)}\\&\quad\times\frac{\Gamma_p(\langle\frac{p^i}{2}+\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{p^i}{2}+\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{p^i}{2}-\frac{ap^i}{q-1}\rangle)}{\Gamma_p^2(\langle\frac{p^i}{2}\rangle)}\\&=-\frac{\varphi(3x)}{q-1}\sum_{a=0,a\neq\frac{q-1}{2}}^{q-2}\overline{\omega}^a\left(\frac{1}{x}\right)\prod_{i=0}^{r-1}p^{\beta_{i,a}}\frac{\Gamma_p(\langle\frac{p^i}{6}-\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{5p^i}{6}-\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{6}\rangle)\Gamma_p(\langle\frac{5p^i}{6}\rangle)}\\&\quad\times\Gamma_p(\langle\frac{p^i}{2}+\frac{ap^i}{q-1}\rangle)-\frac{\varphi(3)}{q(q-1)}\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle\frac{p^i}{3}\rangle)\Gamma_p(\langle\frac{2p^i}{3}\rangle)}{\Gamma_p(\langle\frac{p^i}{6}\rangle)\Gamma_p(\langle\frac{5p^i}{6}\rangle)}.\end{aligned}$$

Putting $a=\frac{q-1}{2}$ and $t=3$ in Lemma 1.11 yields

$$\prod_{i=0}^{r-1}\frac{\Gamma_p(\langle\frac{p^i}{3}\rangle)\Gamma_p(\langle\frac{2p^i}{3}\rangle)}{\Gamma_p(\langle\frac{p^i}{6}\rangle)\Gamma_p(\langle\frac{5p^i}{6}\rangle)}=\varphi(3). \tag{2.4}$$

Using (2.4) and Lemma 2.1, we obtain

$$\begin{aligned}A=-\frac{\varphi(3x)}{q-1}\sum_{a=0,a\neq\frac{q-1}{2}}^{q-2}\overline{\omega}^a\left(\frac{1}{x}\right)\prod_{i=0}^{r-1}(-p)^{u_{i,a}}\frac{\Gamma_p(\langle\frac{p^i}{6}-\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{5p^i}{6}-\frac{ap^i}{q-1}\rangle)}{\Gamma_p(\langle\frac{p^i}{2}\rangle)\Gamma_p(\langle\frac{p^i}{6}\rangle)\Gamma_p(\langle\frac{5p^i}{6}\rangle)}\\\times\Gamma_p(\langle\frac{ap^i}{q-1}\rangle)\Gamma_p(\langle\frac{p^i}{2}+\frac{ap^i}{q-1}\rangle)-\frac{1}{q(q-1)},\end{aligned}$$

where $u_{i,a} = -\left\lfloor \langle \frac{p^i}{6} \rangle - \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \langle \frac{5p^i}{6} \rangle - \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \langle \frac{p^i}{2} \rangle + \frac{ap^i}{q-1} \right\rfloor - \left\lfloor \frac{ap^i}{q-1} \right\rfloor$. Adding and subtracting the term for $a = \frac{q-1}{2}$, we deduce that

$$A = \frac{1}{q} + \varphi(3x) \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & \frac{1}{2} \end{matrix} | \frac{1}{x} \right]_q .$$

This completes the proof of the proposition. ∎

Next, we classify the zeros of the function ${}_2G_2 \left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} | \frac{1}{x} \right]_q$ in the following theorem.

Theorem 2.5. *Let $p \geq 5$ be a prime and $q = p^r$, $r \geq 1$. Let $x \in \mathbb{F}_q$ be such that $x \neq 0, 1$. Then*

$${}_2G_2 \left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} | \frac{1}{x} \right]_q = {}_2G_2 \left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & \frac{1}{2} \end{matrix} | \frac{1}{x} \right]_q = 0$$

if and only if $\varphi(3x(1-x)) = -1$.

Proof. Here, $x \in \mathbb{F}_q$ is such that $x \neq 0, 1$. From Proposition 2.3, we have

$$A = \frac{1}{q(q-1)} \sum_{\chi \in \widehat{\mathbb{F}_q^{\times}}} g(\chi) g(\chi^2) g(\overline{\chi}^3) \chi\left(\frac{-27}{4x}\right) = \frac{1}{q} + {}_2G_2 \left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} | \frac{1}{x} \right]_q . \tag{2.5}$$

Also, from Proposition 2.4, we have

$$A = \frac{1}{q} + \varphi(3x) \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & \frac{1}{2} \end{matrix} | \frac{1}{x} \right]_q . \tag{2.6}$$

Now,

$$\begin{aligned}A &= \frac{1}{q(q-1)}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} g(\chi)g(\chi^2)g(\overline{\chi}^3)\chi\left(\frac{-27}{4x}\right)\\&= \frac{1}{q(q-1)}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} g(\overline{\chi})g(\overline{\chi}^2)g(\chi^3)\overline{\chi}\left(\frac{-27}{4x}\right)\\&= \frac{1}{q(q-1)}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} g(\chi)g(\overline{\chi})\frac{g(\overline{\chi}^2)g(\chi^3)}{g(\chi)}\overline{\chi}\left(\frac{-27}{4x}\right).\end{aligned}$$

By using Lemma 1.2 and Lemma 1.5, we obtain

$$A = \frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} J(\overline{\chi}^2,\chi^3)\overline{\chi}\left(\frac{27}{4x}\right) - 1 - \frac{J(\varepsilon,\varepsilon)}{q} + \frac{q-1}{q}. \tag{2.7}$$

Using (1.3) and (1.5), we have $J(\varepsilon,\varepsilon) = q-2$. Putting the value of $J(\varepsilon,\varepsilon)$ in (2.7) and then using (1.4), we obtain

$$\begin{aligned}A &= \frac{1}{q} - 1 + \frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \overline{\chi}\left(\frac{27}{4x}\right) J(\overline{\chi}^2,\chi^3)\\&= \frac{1}{q} - 1 + \frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \overline{\chi}\left(\frac{27}{4x}\right) J(\overline{\chi}^2,\overline{\chi})\\&= \frac{1}{q} - 1 + \frac{1}{q-1}\sum_{y\in\mathbb{F}_q}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \overline{\chi}\left(\frac{27y^2(1-y)}{4x}\right).\end{aligned} \tag{2.8}$$

By using Lemma 1.1, we know that the inner summation in (2.8) gives nonzero value only if the cubic equation $27y^2(1-y)-4x=0$ has a solution in $\mathbb{F}_q$. It is well-known that a cubic polynomial has exactly one root in $\mathbb{F}_q$ if and only if its discriminant is a non square in $\mathbb{F}_q$. The discriminant of the polynomial $27y^2(1-y)-4x$ is equal to $16\times 27^3\times x(1-x)$. Hence, from (2.8) we deduce that $A = \frac{1}{q}$ if and only if $\varphi(3x(1-x)) = -1$. Using (2.5) and (2.6), we complete the proof. ■

In the following theorem, we prove a transformation for the p-adic hypergeometric function which can be described as a p-adic analogue of (2.1) for certain particular values of a, b, and c.

Theorem 2.6. *Let $p \geq 5$ be a prime and $q = p^r$, $r \geq 1$. Then, for $x \in \mathbb{F}_q$ such that $x \neq 0, 1$, we have*

$$ {}_2G_2\left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} \mid \frac{1}{x}\right]_q = \varphi(1-x)\cdot {}_2G_2\left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & \frac{1}{2} \end{matrix} \mid \frac{1}{x}\right]_q . \tag{2.9} $$

Furthermore,

$$ {}_2G_2\left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} \mid 1\right]_q = \varphi(3)\cdot {}_2G_2\left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & \frac{1}{2} \end{matrix} \mid 1\right]_q . \tag{2.10} $$

Proof. If $x \neq 0$, then combining Proposition 2.3 and Proposition 2.4, we obtain

$$ {}_2G_2\left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} \mid \frac{1}{x}\right]_q = \varphi(3x)\cdot {}_2G_2\left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & \frac{1}{2} \end{matrix} \mid \frac{1}{x}\right]_q . \tag{2.11} $$

By taking $x = 1$, we obtain (2.10). If $x \neq 0, 1$, then $\varphi(3x(1-x)) = \pm 1$. If $\varphi(3x(1-x)) = 1$, then $\varphi(3x) = \varphi(1-x)$, and we readily obtain (2.9) from (2.11). If $\varphi(3x(1-x)) = -1$, then we obtain (2.9) by using Theorem 2.5. This completes the proof of the theorem. ■

2.3 p-adic analogue of Clausen's classical identity

The following transformation for classical hypergeometric series is a special case of Clausen's famous classical identity [7, p. 86, Eq. 4].

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & 1, & 1 \end{matrix} | x\right] = (1-x)^{-1/2}\ {}_2F_1\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ & 1 \end{matrix} | \frac{x}{x-1}\right]^2. \tag{2.12} $$

A finite field analogue of (2.12) was studied by Greene [35, Proposition 6.14]. In [27], Evans and Greene also gave a finite field analogue of the Clausen's classical identity. The p-adic analogue of (2.12) is proven in [14, Theorem 1.3], but over $\mathbb{F}_p$. In the following theorem, we prove a p-adic analogue of (2.12) over a finite field $\mathbb{F}_q$. We use a character sum identity of Ahlgren, Ono, and Penniston [3, Theorem 2.1] to deduce the following transformation over $\mathbb{F}_q$.

Theorem 2.7. *Let p be an odd prime and $q = p^r$, $r \geq 1$. Then, for $x \in \mathbb{F}_q$ such that $x \neq 0, 1$, we have*

$$ {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0 \end{matrix} | \frac{1}{x}\right]_q = \varphi(1-x)\cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} | \frac{x-1}{x}\right]_q^2 - q\cdot\varphi(1-x). $$

Proof. In [3, Theorem 2.1], Ahlgren et al. proved that if $\lambda \in \mathbb{F}_q$ such that $\lambda \neq 0, -1$, then

$$ A(\lambda, q) = \varphi(\lambda+1)(a(\lambda, q)^2 - q), \tag{2.13} $$

where $a(\lambda, q)$ and $A(\lambda, q)$ are defined as

$$ a(\lambda, q) := \sum_{x\in\mathbb{F}_q} \varphi\left((x-1)\left(x^2 - \frac{1}{\lambda+1}\right)\right), $$
$$ A(\lambda, q) := \sum_{x,y\in\mathbb{F}_q} \varphi(xy(x+1)(y+1)(x+\lambda y)). $$

We now define

$$h(\lambda) := \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi\left(\frac{1}{\lambda}\right) J(\overline{\chi}\varphi, \chi)^3. \tag{2.14}$$

Then using (1.4) in (2.14), we have

$$\begin{aligned} h(\lambda) &= \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi\left(\frac{1}{\lambda}\right) \chi(-1) J(\chi, \varphi)^3 \\ &= \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi\left(\frac{-xyz}{\lambda}\right) \sum_{x,y,z \in \mathbb{F}_q^\times} \varphi(1-x)\varphi(1-y)\varphi(1-z). \end{aligned}$$

Using Lemma 1.1, we obtain

$$\begin{aligned} h(\lambda) &= \sum_{x,y \in \mathbb{F}_q^\times} \varphi(1-x)\varphi(1-y)\varphi\left(1 + \frac{\lambda}{xy}\right) \\ &= \sum_{x,y \in \mathbb{F}_q^\times} \varphi((1+x)(1+y)xy(xy+\lambda)) \\ &= \sum_{x,y \in \mathbb{F}_q^\times} \varphi((1+x)(1+y)xy(x+y\lambda)) \\ &= A(\lambda, q). \end{aligned} \tag{2.15}$$

Now applying Lemma 1.5 in (2.14) and observing that $\overline{\omega}^a(-1) = (-1)^a$ and $3\langle \frac{p^i}{2} - \frac{ap^i}{q-1}\rangle + 3\langle \frac{ap^i}{q-1}\rangle - 3\langle \frac{p^i}{2}\rangle = -3\lfloor \langle \frac{p^i}{2}\rangle - \frac{ap^i}{q-1}\rfloor - 3\lfloor \frac{ap^i}{q-1}\rfloor$, we have

$$\begin{aligned} h(\lambda) &= \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi\left(\frac{1}{\lambda}\right) \frac{g^3(\overline{\chi\varphi})g^3(\chi)}{g^3(\varphi)} \\ &= -\frac{1}{q-1} \sum_{a=0}^{q-2} \overline{\omega}^a\left(\frac{1}{\lambda}\right) \prod_{i=0}^{r-1} (-p)^{3\langle \frac{p^i}{2} - \frac{ap^i}{q-1}\rangle + 3\langle \frac{ap^i}{q-1}\rangle - 3\langle \frac{p^i}{2}\rangle} \\ &\qquad \times \frac{\Gamma_p^3(\langle \frac{p^i}{2} - \frac{ap^i}{q-1}\rangle)\Gamma_p^3(\langle \frac{ap^i}{q-1}\rangle)}{\Gamma_p^3(\langle \frac{p^i}{2}\rangle)} \end{aligned}$$

$$= -\frac{1}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a\left(\frac{-1}{\lambda}\right)\overline{\omega}^a(-1)\prod_{i=0}^{r-1}(-p)^{-3\lfloor\langle\frac{p^i}{2}\rangle-\frac{ap^i}{q-1}\rfloor-3\lfloor\frac{ap^i}{q-1}\rfloor}$$
$$\times\frac{\Gamma_p^3(\langle\frac{p^i}{2}-\frac{ap^i}{q-1}\rangle)\Gamma_p^3(\langle\frac{ap^i}{q-1}\rangle)}{\Gamma_p^3(\langle\frac{p^i}{2}\rangle)}$$
$$= {}_3G_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0\end{matrix}\;|\frac{-1}{\lambda}\right]_q. \tag{2.16}$$

Combining (2.15) and (2.16), we have

$$A(\lambda,q) = {}_3G_3\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0\end{matrix}\;|\frac{-1}{\lambda}\right]_q.$$

We now consider a character sum which is related to $a(\lambda,q)$.

$$B := \frac{q^2\varphi(-2)}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\left(\frac{\lambda}{4(\lambda+1)}\right).$$

Using (1.3), we have

$$B = \frac{\varphi(-2)}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}J(\varphi\chi^2,\overline{\chi})J(\varphi\chi,\overline{\chi})\chi\left(\frac{\lambda}{4(\lambda+1)}\right)$$
$$= \frac{\varphi(-2)}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}J(\overline{\chi},\overline{\varphi\chi})J(\overline{\chi},\overline{\varphi})\chi\left(\frac{\lambda}{4(\lambda+1)}\right).$$

By [3, Lemma 2.2], we have

$$B = -\varphi\left(\frac{2\lambda}{\lambda+1}\right) - \varphi(-1)a(\lambda,q). \tag{2.17}$$

Employing [14, Proposition 1], we have

$$B = -\varphi\left(\frac{2\lambda}{\lambda+1}\right) - \varphi(-2)\cdot {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ 0, & 0\end{matrix} \mid \frac{\lambda+1}{\lambda}\right]_q. \tag{2.18}$$

Combining (2.17) and (2.18), we have

$$a(\lambda, q) = \varphi(2)\cdot {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ 0, & 0\end{matrix} \mid \frac{\lambda+1}{\lambda}\right]_q.$$

Substituting the values of $a(\lambda, q)$ and $A(\lambda, q)$ in (2.13), we obtain

$${}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0\end{matrix} \mid \frac{-1}{\lambda}\right]_q = \varphi(1+\lambda)\cdot {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ 0, & 0\end{matrix} \mid \frac{\lambda+1}{\lambda}\right]_q^2 - q\cdot\varphi(1+\lambda).$$

Putting $\lambda = -x$ we obtain the required identity. ■

Remark 2.3.1. *In [3], Ahlgren et al. proved* (2.13) *for* $\lambda \in \mathbb{Q}$ *such that* $\lambda \not\equiv 0, -1 \pmod{p}$. *It is easy to check that their proof works if* $\lambda \in \mathbb{F}_q \setminus \{0, -1\}$.

2.4 Some more transformations for ${}_nG_n[\cdots]_q$

In a recent paper [68], Barman and Tripathi found an identity expressing a ${}_4F_3$-Gaussian hypergeometric series as a sum of two ${}_2F_1$-Gaussian hypergeometric series. In a very recent paper [69], Tripathi and Meher proved finite field analogues of certain classical identities that relate Appell series to classical hypergeometric series. As an application, they also found another such summation identity over finite fields. In this section, we prove two identities for McCarthy's p-adic hypergeometric functions. Our first summation identity generalizes the summation identities proved in [68, 69]. The second identity relates p-adic hypergeometric functions ${}_{n+2}G_{n+2}[\cdots]_p$ and

${}_nG_n[\cdots]_p$. These identities will be used to prove some of our main results in Chapter 4. We first prove the following lemma relating to fractional and integral parts of certain rational numbers.

Lemma 2.8. *Let p be an odd prime and $q = p^r$, $r \geq 1$. Let $x = \frac{m}{d}$ be a rational number such that* $\gcd(m, d) = 1$. *For $0 \leq j \leq q-2$ and $0 \leq i \leq r-1$, we have*

$$\left\lfloor \langle xp^i \rangle - \frac{2jp^i}{q-1} \right\rfloor = \left\lfloor \left\langle \frac{xp^i}{2} \right\rangle - \frac{jp^i}{q-1} \right\rfloor + \left\lfloor \left\langle \frac{(1+x)p^i}{2} \right\rangle - \frac{jp^i}{q-1} \right\rfloor, \tag{2.19}$$

$$\left\lfloor \langle xp^i \rangle + \frac{2jp^i}{q-1} \right\rfloor = \left\lfloor \left\langle \frac{xp^i}{2} \right\rangle + \frac{jp^i}{q-1} \right\rfloor + \left\lfloor \left\langle \frac{(1+x)p^i}{2} \right\rangle + \frac{jp^i}{q-1} \right\rfloor. \tag{2.20}$$

Proof. Let $y := \langle xp^i \rangle$. Then $\frac{y}{2}$ is equal to either $\left\langle \frac{xp^i}{2} \right\rangle$ or $\left\langle \frac{xp^i}{2} \right\rangle - \frac{1}{2}$. If $y = \langle xp^i \rangle$ and $\frac{y}{2} = \left\langle \frac{xp^i}{2} \right\rangle$, then we have $\left\langle \frac{(1+x)p^i}{2} \right\rangle = \frac{y+1}{2}$. If $y = \langle xp^i \rangle$ and $\frac{y}{2} = \left\langle \frac{xp^i}{2} \right\rangle - \frac{1}{2}$, then we have $\left\langle \frac{(1+x)p^i}{2} \right\rangle = \frac{y}{2}$. Therefore, to obtain (2.19) we need to prove that

$$\left\lfloor y - \frac{2jp^i}{q-1} \right\rfloor = \left\lfloor \frac{y}{2} - \frac{jp^i}{q-1} \right\rfloor + \left\lfloor \frac{1+y}{2} - \frac{jp^i}{q-1} \right\rfloor. \tag{2.21}$$

We can write $\left\lfloor y - \frac{2jp^i}{q-1} \right\rfloor = 2k + s$, where $k \in \mathbb{Z}$ and $s \in \{0, 1\}$. This yields

$$2k + s \leq y - \frac{2jp^i}{q-1} < 2k + s + 1,$$

and hence

$$k + \frac{s}{2} \leq \frac{y}{2} - \frac{jp^i}{q-1} < k + \frac{s+1}{2}, \tag{2.22}$$

$$k + \frac{s+1}{2} \leq \frac{y+1}{2} - \frac{jp^i}{q-1} < k + 1 + \frac{s}{2}. \tag{2.23}$$

From (2.22) and (2.23), we have $\left\lfloor \frac{y}{2} - \frac{jp^i}{q-1} \right\rfloor = k$ and $\left\lfloor \frac{y+1}{2} - \frac{jp^i}{q-1} \right\rfloor = k + s$, respectively. Therefore, both the sides of (2.21) are equal to $2k + s$. This completes the proof of (2.19). Now, to obtain (2.20) we need to prove that

$$\left\lfloor y+\frac{2jp^i}{q-1}\right\rfloor=\left\lfloor \frac{y}{2}+\frac{jp^i}{q-1}\right\rfloor+\left\lfloor \frac{1+y}{2}+\frac{jp^i}{q-1}\right\rfloor. \tag{2.24}$$

We can write $\left\lfloor y+\frac{2jp^i}{q-1}\right\rfloor=2k+s$, where $k\in\mathbb{Z}$ and $s\in\{0,1\}$. Clearly, both the sides of (2.24) are equal to $2k+s$. This completes the proof of (2.20). ■

We now prove an identity for the p-adic hypergeometric functions. In the following theorem, we prove a general identity expressing a ${}_4G_4[\cdots]_q$ hypergeometric function as a sum of two ${}_2G_2[\cdots]_q$ hypergeometric functions.

Theorem 2.9. *For $k=1,\ldots,4$, let $a_k=\frac{m_k}{d_k}$ be rational numbers such that* $\gcd(m_k,d_k)=1$*. Let p be an odd prime such that $p\nmid d_1d_2d_3d_4$. Let $q=p^r$, $r\geq 1$ such that $q\equiv 1 \pmod{d}$, where $d=\operatorname{lcm}\{d_1,d_2,d_3,d_4\}$. Then, for $x\in\mathbb{F}_q$, we have*

$${}_2G_2\left[\begin{matrix}a_1, & a_2\\ a_3, & a_4\end{matrix}\,|x\right]_q+{}_2G_2\left[\begin{matrix}a_1, & a_2\\ a_3, & a_4\end{matrix}\,|-x\right]_q={}_4G_4\left[\begin{matrix}\frac{a_1}{2}, & \frac{1+a_1}{2}, & \frac{a_2}{2}, & \frac{1+a_2}{2}\\ \frac{a_3}{2}, & \frac{1+a_3}{2}, & \frac{a_4}{2}, & \frac{1+a_4}{2}\end{matrix}\,|x^2\right]_q.$$

Proof. For $x\in\mathbb{F}_q$, let

$$A_x:={}_2G_2\left[\begin{matrix}a_1, & a_2\\ a_3, & a_4\end{matrix}\,|x\right]_q+{}_2G_2\left[\begin{matrix}a_1, & a_2\\ a_3, & a_4\end{matrix}\,|-x\right]_q.$$

Then,

$$A_x=-\frac{1}{q-1}\sum_{j=0}^{q-2}(\overline{\omega}^j(x)+\overline{\omega}^j(-x))(-p)^{-\sum_{i=0}^{r-1}\left(\sum_{k=1}^{2}\left\lfloor\langle a_kp^i\rangle-\frac{jp^i}{q-1}\right\rfloor+\sum_{k=3}^{4}\left\lfloor\langle -a_kp^i\rangle+\frac{jp^i}{q-1}\right\rfloor\right)}$$
$$\times\prod_{i=0}^{r-1}\frac{\prod_{k=1}^{2}\Gamma_p\left(\left\langle a_kp^i-\frac{jp^i}{q-1}\right\rangle\right)\prod_{k=3}^{4}\Gamma_p\left(\left\langle -a_kp^i+\frac{jp^i}{q-1}\right\rangle\right)}{\Gamma_p(\langle a_1p^i\rangle)\Gamma_p(\langle a_2p^i\rangle)\Gamma_p(\langle -a_3p^i\rangle)\Gamma_p(\langle -a_4p^i\rangle)}.$$

Since $\overline{\omega}^j(-1)=-1$ if j is odd and $\overline{\omega}^j(-1)=1$ if j is even, we have

$$A_x=-\frac{2}{q-1}\sum_{j=0}^{\frac{q-3}{2}}\overline{\omega}^{2j}(x)(-p)^{-\sum_{i=0}^{r-1}\left(\sum_{k=1}^{2}\left\lfloor\langle a_kp^i\rangle-\frac{2jp^i}{q-1}\right\rfloor+\sum_{k=3}^{4}\left\lfloor\langle -a_kp^i\rangle+\frac{2jp^i}{q-1}\right\rfloor\right)}$$
$$\times\prod_{i=0}^{r-1}\frac{\prod_{k=1}^{2}\Gamma_p\left(\left\langle a_kp^i-\frac{2jp^i}{q-1}\right\rangle\right)\prod_{k=3}^{4}\Gamma_p\left(\left\langle -a_kp^i+\frac{2jp^i}{q-1}\right\rangle\right)}{\Gamma_p(\langle a_1p^i\rangle)\Gamma_p(\langle a_2p^i\rangle)\Gamma_p(\langle -a_3p^i\rangle)\Gamma_p(\langle -a_4p^i\rangle)}. \tag{2.25}$$

We have $d_k|(q-1)$ for $k=1,\ldots,4$. Using (1.9) with $m=2$ and $x=a_k-\frac{2j}{q-1}$ for $k=1,2$, we obtain

$$\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(a_k-\frac{2j}{q-1}\right)p^i\right\rangle\right)=\overline{\omega}\left(2^{(1-a_k+\frac{2j}{q-1})(1-q)}\right)\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle\left(\frac{a_k}{2}-\frac{j}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)}$$
$$\times\Gamma_p\left(\left\langle\left(\frac{1+a_k}{2}-\frac{j}{q-1}\right)p^i\right\rangle\right). \tag{2.26}$$

Again, using (1.9) with $m=2$ and $x=-a_k+\frac{2j}{q-1}$ for $k=3,4$, we obtain

$$\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(-a_k+\frac{2j}{q-1}\right)p^i\right\rangle\right)=\overline{\omega}\left(2^{(1+a_k-\frac{2j}{q-1})(1-q)}\right)\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle\left(\frac{-a_k}{2}+\frac{j}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)}$$
$$\times\Gamma_p\left(\left\langle\left(\frac{1-a_k}{2}+\frac{j}{q-1}\right)p^i\right\rangle\right). \tag{2.27}$$

Using (2.19) with $x=a_k$ for $k=1,2$ and (2.20) with $x=-a_k$ for $k=3,4$, and then adding the obtained equations, we deduce that

$$\sum_{k=1}^{2}\left\lfloor\langle a_kp^i\rangle-\frac{2jp^i}{q-1}\right\rfloor+\sum_{k=3}^{4}\left\lfloor\langle -a_kp^i\rangle+\frac{2jp^i}{q-1}\right\rfloor$$
$$=\sum_{k=1}^{2}\left(\left\lfloor\left\langle\frac{a_kp^i}{2}\right\rangle-\frac{jp^i}{q-1}\right\rfloor+\left\lfloor\left\langle\frac{(1+a_k)p^i}{2}\right\rangle-\frac{jp^i}{q-1}\right\rfloor\right)$$
$$+\sum_{k=3}^{4}\left(\left\lfloor\left\langle\frac{-a_kp^i}{2}\right\rangle+\frac{jp^i}{q-1}\right\rfloor+\left\lfloor\left\langle\frac{(1-a_k)p^i}{2}\right\rangle+\frac{jp^i}{q-1}\right\rfloor\right). \tag{2.28}$$

Substituting the expressions (2.26), (2.27), and (2.28) in (2.25), and then using the fact that $\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)^4=1$, we obtain

$$\begin{aligned}A_x=&-\frac{2\alpha}{q-1}\sum_{j=0}^{\frac{q-3}{2}}\overline{\omega}^{2j}(x)(-p)^{-\sum_{i=0}^{r-1}s_{i,j}}\\&\times\prod_{i=0}^{r-1}\frac{N_{i,j}}{\Gamma_p(\langle a_1p^i\rangle)\Gamma_p(\langle a_2p^i\rangle)\Gamma_p(\langle -a_3p^i\rangle)\Gamma_p(\langle -a_4p^i\rangle)},\end{aligned}$$

where $\alpha=\overline{\omega}(2^{(a_1+a_2-a_3-a_4)(q-1)})$ and

$$\begin{aligned}s_{i,j}=&\sum_{k=1}^{2}\left(\left\lfloor\left\langle\frac{a_kp^i}{2}\right\rangle-\frac{jp^i}{q-1}\right\rfloor+\left\lfloor\left\langle\frac{(1+a_k)p^i}{2}\right\rangle-\frac{jp^i}{q-1}\right\rfloor\right)\\&+\sum_{k=3}^{4}\left(\left\lfloor\left\langle\frac{-a_kp^i}{2}\right\rangle+\frac{jp^i}{q-1}\right\rfloor+\left\lfloor\left\langle\frac{(1-a_k)p^i}{2}\right\rangle+\frac{jp^i}{q-1}\right\rfloor\right),\\N_{i,j}=&\prod_{k=1}^{2}\Gamma_p\left(\left\langle\left(\frac{a_k}{2}-\frac{j}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{1+a_k}{2}-\frac{j}{q-1}\right)p^i\right\rangle\right)\\&\times\prod_{k=3}^{4}\Gamma_p\left(\left\langle\left(\frac{-a_k}{2}+\frac{j}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{1-a_k}{2}+\frac{j}{q-1}\right)p^i\right\rangle\right).\end{aligned}$$

We can rewrite A_x as

$$\begin{aligned}A_x&=-\frac{2\alpha}{q-1}\sum_{j=0}^{\frac{q-3}{2}}B_j\\&=-\frac{\alpha}{q-1}\sum_{j=0}^{\frac{q-3}{2}}B_j-\frac{\alpha}{q-1}\sum_{j=0}^{\frac{q-3}{2}}B_j\\&=-\frac{\alpha}{q-1}\sum_{j=0}^{\frac{q-3}{2}}B_j-\frac{\alpha}{q-1}\sum_{j=\frac{q-1}{2}}^{q-2}B_{j-\frac{q-1}{2}},\end{aligned}$$

where $B_j=\overline{\omega}^{2j}(x)(-p)^{-\sum_{i=0}^{r-1}s_{i,j}}\prod_{i=0}^{r-1}\frac{N_{i,j}}{\Gamma_p(\langle a_1p^i\rangle)\Gamma_p(\langle a_2p^i\rangle)\Gamma_p(\langle -a_3p^i\rangle)\Gamma_p(\langle -a_4p^i\rangle)}$. We can easily check that $N_{i,j-\frac{q-1}{2}} = N_{i,j}$. Also,

$$s_{i,j-\frac{q-1}{2}} = \sum_{k=1}^{2}\left(\left\lfloor\left\langle\frac{a_kp^i}{2}\right\rangle - \frac{jp^i}{q-1} + \frac{1}{2}\right\rfloor + \left\lfloor\left\langle\frac{(1+a_k)p^i}{2}\right\rangle - \frac{jp^i}{q-1} + \frac{1}{2}\right\rfloor\right)$$
$$+ \sum_{k=3}^{4}\left(\left\lfloor\left\langle\frac{-a_kp^i}{2}\right\rangle + \frac{jp^i}{q-1} - \frac{1}{2}\right\rfloor + \left\lfloor\left\langle\frac{(1-a_k)p^i}{2}\right\rangle + \frac{jp^i}{q-1} - \frac{1}{2}\right\rfloor\right).$$

Let

$$y_k = \begin{cases} \langle a_kp^i\rangle, & \text{if } k = 1, 2; \\ \langle -a_kp^i\rangle, & \text{if } k = 3, 4. \end{cases}$$

Then,

$$\begin{aligned} s_{i,j-\frac{q-1}{2}} &= \sum_{k=1}^{2}\left(\left\lfloor\frac{y_k}{2} - \frac{jp^i}{q-1} + \frac{1}{2}\right\rfloor + \left\lfloor\frac{1+y_k}{2} - \frac{jp^i}{q-1} + \frac{1}{2}\right\rfloor\right) \\ &+ \sum_{k=3}^{4}\left(\left\lfloor\frac{y_k}{2} + \frac{jp^i}{q-1} - \frac{1}{2}\right\rfloor + \left\lfloor\frac{1+y_k}{2} + \frac{jp^i}{q-1} - \frac{1}{2}\right\rfloor\right) \\ &= s_{i,j}. \end{aligned}$$

Hence, $B_{j-\frac{q-1}{2}} = B_j$. Therefore, we have

$$A_x = -\frac{\alpha}{q-1}\sum_{j=0}^{q-2} B_j. \tag{2.29}$$

Using (1.9) with $m = 2$ and $x = a_k$ for $k = 1, 2$; and with $m = 2$ and $x = -a_k$ for $k = 3, 4$, and then multiplying all the obtained equations, we have

$$\begin{aligned} &\prod_{i=0}^{r-1}\left(\prod_{k=1}^{2}\Gamma_p\left(\langle a_kp^i\rangle\right)\right)\left(\prod_{k=3}^{4}\Gamma_p\left(\langle -a_kp^i\rangle\right)\right) \\ &= \alpha\prod_{i=0}^{r-1}\prod_{k=1}^{2}\Gamma_p\left(\left\langle\frac{a_kp^i}{2}\right\rangle\right)\Gamma_p\left(\left\langle\frac{(1+a_k)p^i}{2}\right\rangle\right) \end{aligned}$$

$$\times \prod_{k=3}^{4} \Gamma_p\left(\left\langle\frac{(-a_k)p^i}{2}\right\rangle\right)\Gamma_p\left(\left\langle\frac{(1-a_k)p^i}{2}\right\rangle\right). \tag{2.30}$$

Substituting (2.30) in (2.29), we obtain

$$A_x = {}_4G_4\left[\begin{matrix}\frac{a_1}{2}, & \frac{1+a_1}{2}, & \frac{a_2}{2}, & \frac{1+a_2}{2} \\ \frac{a_3}{2}, & \frac{1+a_3}{2}, & \frac{a_4}{2}, & \frac{1+a_4}{2}\end{matrix}|x^2\right]_q.$$

This completes the proof of the theorem. ■

If we substitute $a_2 = 1 - a_1, a_3 = 0$, and $a_4 = 0$ in Theorem 2.9, then we obtain a p-adic analogue of [69, Theorem 1.3] for $q \equiv 1, d_1 + 1 \pmod{2d_1}$. If we substitute $a_3 = 0$ and $a_2 = \frac{a_4}{2}$, then for q satisfying $q \equiv 1 \pmod{d_1}$ and $q \equiv 1 \pmod{2d_4}$, we obtain a p-adic analogue of [68, Theorem 1.8].

We now recall a property of the p-adic gamma function in the following proposition.

Proposition 2.10. ([59, p. 369]). *Let $x \in \mathbb{Z}_p$. We have $\Gamma_p(1-x)\Gamma_p(x) = (-1)^{a_0(x)}$, where $a_0(x) \in \{1, 2, \ldots, p\}$ such that $x \equiv a_0(x) \pmod{p}$.*

Lemma 2.11. *Let $p \equiv -1 \pmod{d}$. Then, for $0 \leq n \leq p-2$, we have*

$$\frac{\Gamma_p\left(\left\langle\frac{-1}{d}+\frac{n}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-(d-1)}{d}+\frac{n}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{1}{d}-\frac{n}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{d-1}{d}-\frac{n}{p-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{1}{d}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{d-1}{d}\right\rangle\right)^2} = 1.$$

Proof. Suppose that $d = 2$. Then for $n \neq \frac{p-1}{2}$, using (1.11) with $r = 1$, we obtain the required result. If $n = \frac{p-1}{2}$, then using Proposition 2.10, we obtain the required result. Now, we consider $d \neq 2$. Let $x := \frac{1}{d} + \frac{n}{p-1}$ and $y := \frac{d-1}{d} + \frac{n}{p-1}$. Then $1 - x = \frac{d-1}{d} - \frac{n}{p-1}$ and $1 - y = \frac{1}{d} - \frac{n}{p-1}$. Using Proposition 2.10, we obtain $\Gamma_p\left(\left\langle\frac{1}{d}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{d-1}{d}\right\rangle\right)^2 = 1$. Note that for $d \neq 2$, x and y are not integers and hence

$\langle 1-x\rangle = 1-\langle x\rangle$ and $\langle 1-y\rangle = 1-\langle y\rangle$. Thus, we have

$$\begin{aligned}\Gamma_p(\langle x\rangle)\Gamma_p(\langle 1-x\rangle)\Gamma_p(\langle y\rangle)\Gamma_p(\langle 1-y\rangle) &= \Gamma_p(\langle x\rangle)\Gamma_p(1-\langle x\rangle)\Gamma_p(\langle y\rangle)\Gamma_p(1-\langle y\rangle)\\ &= (-1)^{a_0(\langle x\rangle)+a_0(\langle y\rangle)}.\end{aligned}$$

Hence, to complete the proof of the lemma, we need to prove that

$$(-1)^{a_0(\langle x\rangle)+a_0(\langle y\rangle)} = 1.$$

Case 1: $0 \le n < \frac{p-1}{d}$.

Clearly, $\langle x\rangle = \frac{1}{d}+\frac{n}{p-1}$ and $\langle y\rangle = \frac{d-1}{d}+\frac{n}{p-1}$. Also, we have

$$\begin{aligned}\frac{1}{d}+\frac{n}{p-1}-\left(\frac{p+1}{d}-n\right) &\equiv 0 \pmod p,\\ \frac{d-1}{d}+\frac{n}{p-1}-\left(\frac{(d-1)(p+1)}{d}-n\right) &\equiv 0 \pmod p.\end{aligned}$$

Furthermore, $\frac{p+1}{d}-n, \frac{(d-1)(p+1)}{d}-n \in \{1,2,\ldots,p\}$. Thus, $a_0(\langle x\rangle) = \frac{p+1}{d}-n$ and $a_0(\langle y\rangle) = \frac{(d-1)(p+1)}{d}-n$. Therefore, $a_0(\langle x\rangle)+a_0(\langle y\rangle)$ is an even number and hence, we are done for this case.

Case 2: $\frac{p-1}{d} < n < \frac{(d-1)(p-1)}{d}$.

Clearly, $\langle x\rangle = \frac{1}{d}+\frac{n}{p-1}$ and $\langle y\rangle = -\frac{1}{d}+\frac{n}{p-1}$. Also, we have

$$\begin{aligned}\frac{1}{d}+\frac{n}{p-1}-\left(\frac{(d+1)p+1}{d}-n\right) &\equiv 0 \pmod p,\\ -\frac{1}{d}+\frac{n}{p-1}-\left(\frac{(d-1)p-1}{d}-n\right) &\equiv 0 \pmod p.\end{aligned}$$

Furthermore, $\frac{(d+1)p+1}{d}-n, \frac{(d-1)p-1}{d}-n \in \{1,2,\ldots,p\}$. Thus, $a_0(\langle x\rangle) = \frac{(d+1)p+1}{d}-n$ and $a_0(\langle y\rangle) = \frac{(d-1)p-1}{d}-n$. Therefore, $a_0(\langle x\rangle)+a_0(\langle y\rangle)$ is an even number and hence, we are done for this case.

Case 3: $\frac{(d-1)(p-1)}{d} < n < p-1$.

Clearly, $\langle x\rangle = -\frac{d-1}{d} + \frac{n}{p-1}$ and $\langle y\rangle = -\frac{1}{d} + \frac{n}{p-1}$. Also, we have

$$-\frac{d-1}{d} + \frac{n}{p-1} - \left(\frac{(d+1)p-(d-1)}{d} - n\right) \equiv 0 \pmod p,$$
$$-\frac{1}{d} + \frac{n}{p-1} - \left(\frac{(2d-1)p-1}{d} - n\right) \equiv 0 \pmod p.$$

Furthermore, $\frac{(d+1)p-(d-1)}{d} - n, \frac{(2d-1)p-1}{d} - n \in \{1,2,\ldots,p\}$. Thus, $a_0(\langle x\rangle) = \frac{(d+1)p-(d-1)}{d} - n$ and $a_0(\langle y\rangle) = \frac{(2d-1)p-1}{d} - n$. Therefore, $a_0(\langle x\rangle) + a_0(\langle y\rangle)$ is an even number. This completes the proof of the lemma. ■

Next, we prove a transformation between ${}_{n+2}G_{n+2}[\cdots]_p$ and ${}_nG_n[\cdots]_p$ hypergeometric functions.

Theorem 2.12. *Let p be an odd prime. For a positive integer n, let $a_k, b_k \in \mathbb{Q}\cap\mathbb{Z}_p$ for $k = 1,\ldots,n$. Let d be a positive integer. If $p \equiv -1 \pmod d$ then, for $t \in \mathbb{F}_p$, we have*

$${}_{n+2}G_{n+2}\left[\begin{matrix} a_1, & \ldots, & a_n, & \frac{1}{d}, & \frac{d-1}{d} \\ b_1, & \ldots, & b_n, & \frac{1}{d}, & \frac{d-1}{d}\end{matrix} \Big| t\right]_p = {}_nG_n\left[\begin{matrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n\end{matrix} \Big| t\right]_p.$$

Proof. We have

$$\begin{aligned}&{}_{n+2}G_{n+2}\left[\begin{matrix} a_1, & \ldots, & a_n, & \frac{1}{d}, & \frac{d-1}{d} \\ b_1, & \ldots, & b_n, & \frac{1}{d}, & \frac{d-1}{d}\end{matrix} \Big| t\right]_p \\ &= -\frac{1}{p-1}\sum_{j=0}^{p-2}(-1)^{(n+2)j}\overline{\omega}^j(t)(-p)^{\alpha_j+\beta_j}M_jN_j, \end{aligned} \tag{2.31}$$

where

$$\alpha_j = -\sum_{k=1}^{n}\left(\left\lfloor \langle a_k\rangle - \frac{j}{p-1}\right\rfloor + \left\lfloor \langle -b_k\rangle + \frac{j}{p-1}\right\rfloor\right),$$
$$\beta_j = -\left\lfloor \frac{1}{d} - \frac{j}{p-1}\right\rfloor - \left\lfloor \left\langle -\frac{1}{d}\right\rangle + \frac{j}{p-1}\right\rfloor - \left\lfloor \frac{d-1}{d} - \frac{j}{p-1}\right\rfloor$$

$$- \left\lfloor \left\langle -\frac{d-1}{d} \right\rangle + \frac{j}{p-1} \right\rfloor,$$

$$M_j = \prod_{k=1}^{n} \frac{\Gamma_p\left(\left\langle a_k - \frac{j}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle -b_k + \frac{j}{p-1} \right\rangle\right)}{\Gamma_p\left(\langle a_k \rangle\right) \Gamma_p\left(\langle -b_k \rangle\right)},$$

$$N_j = \frac{\Gamma_p\left(\left\langle \frac{1}{d} - \frac{j}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle -\frac{1}{d} + \frac{j}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{d-1}{d} - \frac{j}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle -\frac{d-1}{d} + \frac{j}{p-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{1}{d} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{d-1}{d} \right\rangle\right)^2}.$$

Let $a := \left\lfloor \frac{p-1}{d} \right\rfloor$ and $b := \left\lfloor \frac{(d-1)(p-1)}{d} \right\rfloor$. Taking j in the intervals $[0, a]$, $[a+1, b]$, and $[b+1, p-2]$ respectively, we obtain $\beta_j = 0$. Using Lemma 2.11, we have $N_j = 1$. Substituting these values in (2.31), we complete the proof of the theorem. ■

3

Diagonal Hypersufaces and p-adic Hypergeometric Functions

3.1 Introduction

Let D_λ^d denote the family of *monomial deformations of diagonal hypersurface* over a finite field $\mathbb{F}_q$. These families are of the form:

$$D_\lambda^d : X_1^d + X_2^d + \cdots + X_n^d = \lambda d X_1^{h_1} X_2^{h_2} \cdots X_n^{h_n}, \tag{3.1}$$

[1] The contents of this chapter have been published in *Int. J. Number Theory* (2024) and *Finite Fields Appl.* (2024).

where $d, n \geq 2$, $h_i \geq 1$, $\sum_{i=1}^{n} h_i = d$, and $\gcd(d, h_1, h_2, \ldots, h_n) = 1$. Let $\#D_\lambda^d(\mathbb{F}_q)$ denote the number of points on the hypersurface (3.1) in $\mathbb{P}^{n-1}(\mathbb{F}_q)$. The Dwork hypersurface is the case when $d = n$, that is, $h_1 = h_2 = \cdots = h_n = 1$.

Let d be an odd integer, and let $q \equiv 1 \pmod{d}$. In [33], Goodson expressed the number of points on the Dwork hypersurface over $\mathbb{F}_q$ as $(q^{d-1} - 1)/(q - 1)$ plus a sum of finite field hypergeometric functions as defined by Greene [36]. On the other hand, formulas for the number of points on the Dwork hypersurface over a finite field $\mathbb{F}_q$ in terms of McCarthy's p-adic hypergeometric function $G[\cdots]_q$ are known in the general case (see, for example [10, 34, 54]).

In this chapter, we provide a formula for the number of $\mathbb{F}_q$-points on the hypersurface (3.1) which holds for $d > n$ as well (the non Dwork case). In [44], Koblitz expressed $\#D_\lambda^d(\mathbb{F}_q)$ in terms of Gauss and Jacobi sums under the condition that $q \equiv 1 \pmod{d}$. Using Koblitz's formula, Salerno [63] expressed $\#D_\lambda^d(\mathbb{F}_q)$ as a sum of hypergeometric functions over finite fields as defined by Katz [41], under the condition that $dh_1 \cdots h_n \mid q - 1$. We find a formula for $\#D_\lambda^d(\mathbb{F}_q)$ in terms of McCarthy's p-adic hypergeometric function when $\gcd(d, q-1) = 1$ and $n \in \mathbb{N}$. We then show that the same result holds for $n = 2$ without assuming the condition $\gcd(d, q-1) = 1$. We also provide a result that proves the number of distinct zeros of a polynomial over $\mathbb{F}_q$ is equal to the number of points on a subfamily of diagonal hypersurfaces. Before we state our main results, we define rational numbers $b_1, b_2, \ldots, b_{d-1}$ as follows.

Notation 3.1.1. *We have two cases according to $d = n$ or $d > n$.*

- *($d = n$) In this case $h_1 = \cdots = h_n = 1$. We take $b_1 = b_2 = \cdots = b_{d-1} = 0$.*
- *($d > n$) In this case one or more h_i is greater than 1. We take $b_1 = b_2 = \cdots = b_{n-1} = 0$; and corresponding to each $h_j > 1$, we take $b_{j_1} = \frac{1}{h_j}, b_{j_2} = \frac{2}{h_j}, \ldots, b_{j_{h_j-1}} = \frac{h_j-1}{h_j}$ contributing to the remaining $d - n$ values of b_i ($i = n, n+1, \ldots, d-1$). For example, if $h_1 = h_3 = 1, h_2 = 2, h_4 = 3$, then $n = 4$,*

$d = h_1 + h_2 + h_3 + h_4 = 7$, and hence $b_1 = b_2 = b_3 = 0$, $b_4 = \frac{1}{2}$ (corresponding to h_2), and $b_5 = \frac{1}{3}, b_6 = \frac{2}{3}$ (corresponding to h_4).

3.2 Number of $\mathbb{F}_q$-points on the diagonal hypersurfaces and p-adic hypergeometric functions

We first need to prove a lemma which will be used in the proof of our result.

Lemma 3.1. *Let p be an odd prime and $q = p^r, r \geq 1$. Let $d \geq 2$ be an integer such that $\gcd(d, p(q-1)) = 1$. Let $h_1, \ldots, h_n$ be positive integers such that $\sum_{k=1}^{n} h_k = d$ and $\gcd(h_1 \cdots h_n, p) = 1$ Then, for $1 \leq a \leq q-2$ and $0 \leq i \leq r-1$, we have*

$$\left\lfloor \frac{-dap^i}{q-1} \right\rfloor + \sum_{k=1}^{n} \left\lfloor \frac{h_k ap^i}{q-1} \right\rfloor$$
$$= \sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{hp^i}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor + \sum_{k=1}^{n} \sum_{h=0}^{h_k-1} \left\lfloor \left\langle \frac{-hp^i}{h_k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor - 1 - \left\lfloor \frac{ap^i}{q-1} \right\rfloor.$$

Proof. Putting $l = h_k$ for $k = 1, 2, \ldots, n$ in (1.17), and then adding them we have

$$\sum_{k=1}^{n} \left\lfloor \frac{h_k ap^i}{q-1} \right\rfloor = \sum_{k=1}^{n} \sum_{h=0}^{h_k-1} \left\lfloor \left\langle \frac{-hp^i}{h_k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor. \tag{3.2}$$

We now readily obtain the required identity by adding (1.12) and (3.2). ∎

We now prove our theorem which expresses the number of $\mathbb{F}_q$-points on the diagonal hypersurfaces in terms of p-adic hypergeometric functions.

Theorem 3.2. *Let p be an odd prime and $q = p^r$, $r \geq 1$. Let $d \geq 2$ be an integer such that $\gcd(d, q-1) = 1$. If D_λ^d is the diagonal hypersurface given by (3.1) such that $\lambda \neq 0$ and $p \nmid dh_1 \cdots h_n$, then the number of points on D_λ^d in $\mathbb{P}^{n-1}(\mathbb{F}_q)$ is given* *by*

$$\#D_\lambda^d(\mathbb{F}_q)=\frac{q^{n-1}-1}{q-1}+(-1)^n\cdot {}_{d-1}G_{d-1}\left[\begin{array}{cccc}\frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-1}{d}\\ b_1, & b_2, & \ldots, & b_{d-1}\end{array}|\lambda^d h_1^{h_1}\cdots h_n^{h_n}\right]_q,$$

where the b_i's are as given in Notation 3.1.1.

Proof. Let $N_q^d(\lambda)$ denote the number of points on the diagonal hypersurface D_λ^d in $\mathbb{A}^n(\mathbb{F}_q)$. Then we have

$$\#D_\lambda^d(\mathbb{F}_q)=\frac{N_q^d(\lambda)-1}{q-1}. \tag{3.3}$$

Let $\overline{x}=(x_1,x_2,\ldots,x_n)$ and $f(\overline{x})=x_1^d+x_2^d+\cdots+x_n^d-d\lambda x_1^{h_1}\cdots x_n^{h_n}$ and using the identity

$$\sum_{z\in\mathbb{F}_q}\theta(zf(\overline{x}))=\begin{cases}q, & \text{if } f(\overline{x})=0;\\ 0, & \text{if } f(\overline{x})\neq 0,\end{cases}$$

we obtain

$$\begin{aligned}q\cdot N_q^d(\lambda)&=\sum_{z\in\mathbb{F}_q}\sum_{x_i\in\mathbb{F}_q}\theta(zf(\overline{x}))\\ &=q^n+\sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf(\overline{x}))+\sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf(\overline{x})).\end{aligned} \tag{3.4}$$

Consider a polynomial $f_1(\overline{x})=x_1^d+\cdots+x_n^d$ and let $N_q^{'}$ be the number of solutions of the equation $f_1(\overline{x})=0$ in $\mathbb{A}^n(\mathbb{F}_q)$. Since d is an integer such that $\gcd(d,q-1)=1$, therefore $x\mapsto x^d$ is an automorphism of $\mathbb{F}_q^\times$. This gives that $N_q^{'}=q^{n-1}$. Also, repeating the same process for $f_1(\overline{x})$ as done in (3.4) for $f(\overline{x})$, we deduce that

$$q\cdot N_q^{'}=q^n+\sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf_1(\overline{x}))+\sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf_1(\overline{x})).$$

Thus,

$$\sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf_1(\overline{x})) = -\sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf_1(\overline{x})). \tag{3.5}$$

Also,

$$\sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf_1(\overline{x})) = \sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf(\overline{x})). \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), we obtain

$$\begin{aligned} q\cdot N_q^d(\lambda) &= q^n + \sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf(\overline{x})) - \sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf_1(\overline{x}))\\ &= q^n + A - B, \end{aligned} \tag{3.7}$$

where $A := \sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf(\overline{x}))$ and $B := \sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf_1(\overline{x}))$. We now evaluate the values of A and B, respectively. Firstly, we calculate B. We have

$$B = \sum_{z,x_i\in\mathbb{F}_q^\times}\theta(z(x_1^d+\cdots+x_n^d)) = \sum_{z,x_i\in\mathbb{F}_q^\times}\theta(z(x_1^d))\cdots\theta(z(x_n^d)).$$

Lemma 1.4 yields

$$\begin{aligned} B &= \frac{1}{(q-1)^n}\sum_{z,x_i\in\mathbb{F}_q^\times}\sum_{a_1,\ldots,a_n=0}^{q-2} g(T^{-a_1})\cdots g(T^{-a_n})T^{a_1}(zx_1^d)\cdots T^{a_n}(zx_n^d)\\ &= \frac{1}{(q-1)^n}\sum_{a_1,\ldots,a_n=0}^{q-2} g(T^{-a_1})\cdots g(T^{-a_n})\\ &\quad\times\sum_{z\in\mathbb{F}_q^\times}T^{a_1+\cdots+a_n}(z)\sum_{x_1\in\mathbb{F}_q^\times}T^{a_1d}(x_1)\cdots\sum_{x_n\in\mathbb{F}_q^\times}T^{a_nd}(x_n). \end{aligned}$$

The inner sums are nonzero only if $a_1+\cdots+a_n \equiv 0 \pmod{q-1}$ and $a_1d, \ldots, a_nd \equiv 0 \pmod{q-1}$. Since $\gcd(d, q-1) = 1$, all the congruences simultaneously hold only if $a_1 = \cdots = a_n = 0$. Using the fact that $g(\varepsilon) = -1$, we obtain $B = (-1)^n(q-1)$. Next, we calculate the value of A. We have

$$\begin{aligned} A &= \sum_{z,x_i\in\mathbb{F}_q^\times} \theta(zf(\overline{x})) = \sum_{z,x_i\in\mathbb{F}_q^\times} \theta(z(x_1^d+\cdots+x_n^d-\lambda d x_1^{h_1}\cdots x_n^{h_n})) \\ &= \sum_{z,x_i\in\mathbb{F}_q^\times} \theta(zx_1^d)\cdots\theta(zx_n^d)\theta(-z\lambda d x_1^{h_1}\cdots x_n^{h_n}). \end{aligned}$$

By using Lemma 1.4, we obtain

$$\begin{aligned} A =& \frac{1}{(q-1)^{n+1}} \sum_{a_1,\ldots,a_{n+1}=0}^{q-2} g(T^{-a_1})\cdots g(T^{-a_n})g(T^{-a_{n+1}}) \\ &\times \sum_{z,x_i\in\mathbb{F}_q^\times} T^{a_1}(zx_1^d)\cdots T^{a_n}(zx_n^d)T^{a_{n+1}}(-\lambda d z x_1^{h_1}\cdots x_n^{h_n}) \\ =& \frac{1}{(q-1)^{n+1}} \sum_{a_1,\ldots,a_{n+1}=0}^{q-2} g(T^{-a_1})\cdots g(T^{-a_{n+1}})T^{a_{n+1}}(-\lambda d) \\ &\times \sum_{z\in\mathbb{F}_q^\times} T^{a_1+\cdots+a_{n+1}}(z) \sum_{x_1\in\mathbb{F}_q^\times} T^{a_1d+a_{n+1}h_1}(x_1)\cdots \sum_{x_n\in\mathbb{F}_q^\times} T^{a_nd+a_{n+1}h_n}(x_n). \end{aligned}$$

The inner sums are nonzero only if $a_1+\cdots+a_{n+1} \equiv 0 \pmod{q-1}$ and $a_1d + a_{n+1}h_1, \ldots, a_nd + a_{n+1}h_n \equiv 0 \pmod{q-1}$. For $0 \leq a \leq q-2$, we have $a_i \equiv h_ia \pmod{q-1}$ for $i = 1, \ldots, n$ and $a_{n+1} \equiv -da \pmod{q-1}$ as $\gcd(d, q-1) = 1$. Thus

$$A = \sum_{a=0}^{q-2} g(T^{-ah_1})\cdots g(T^{-ah_n})g(T^{ad})T^{-ad}(-\lambda d). \tag{3.8}$$

Taking $T = \omega$, and then using Gross-Koblitz formula, we obtain

$$A = (-1)^{n+1}\sum_{a=0}^{q-2} \overline{\omega}^{ad}(-\lambda d)(-p)^{\sum_{i=0}^{r-1}\left(\sum_{k=1}^{n}\left\langle\frac{h_kap^i}{q-1}\right\rangle+\left\langle\frac{-dap^i}{q-1}\right\rangle\right)}$$

$$\times \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \frac{-dp^i a}{q-1} \right\rangle\right) \prod_{k=1}^{n} \Gamma_p\left(\left\langle \frac{h_k p^i a}{q-1} \right\rangle\right). \tag{3.9}$$

Applying Lemma 1.11 with $t = d$ and Lemma 1.10 with $t = h_k$, we have

$$\omega(d^{-ad}) \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \frac{-dp^i a}{q-1} \right\rangle\right) = \prod_{i=0}^{r-1} \frac{\prod_{h=0}^{d-1} \Gamma_p\left(\left\langle \frac{p^i(1+h)}{d} - \frac{p^i a}{q-1} \right\rangle\right)}{\prod_{h=1}^{d-1} \Gamma_p\left(\left\langle \frac{hp^i}{d} \right\rangle\right)},$$

$$\omega(h_k^{h_k a}) \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle \frac{h_k p^i a}{q-1} \right\rangle\right) = \prod_{i=0}^{r-1} \frac{\prod_{h=0}^{h_k-1} \Gamma_p\left(\left\langle \frac{p^i h}{h_k} + \frac{p^i a}{q-1} \right\rangle\right)}{\prod_{h=1}^{h_k-1} \Gamma_p\left(\left\langle \frac{hp^i}{h_k} \right\rangle\right)}.$$

Substituting these values in (3.9) yields

$$A = (-1)^{n+1} \sum_{a=0}^{q-2} \overline{\omega}^{ad}(-\lambda d)\, \overline{\omega}(d^{-ad})\, \overline{\omega}^{a}(h_1^{h_1} \cdots h_n^{h_n})(-p)^{\sum_{i=0}^{r-1} M_{a,i}}$$
$$\times \prod_{i=0}^{r-1} \prod_{h=0}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^i(1+h)}{d} - \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d} \right\rangle\right)} \prod_{k=1}^{n} \left(\prod_{h=0}^{h_k-1} \frac{\Gamma_p\left(\left\langle \frac{p^i h}{h_k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{h_k} \right\rangle\right)} \right),$$

where $M_{a,i} := \sum_{k=1}^{n} \left\langle \frac{h_k a p^i}{q-1} \right\rangle + \left\langle \frac{-dap^i}{q-1} \right\rangle = -\sum_{k=1}^{n} \left\lfloor \frac{h_k a p^i}{q-1} \right\rfloor - \left\lfloor \frac{-dap^i}{q-1} \right\rfloor$. We now have

$$A = (-1)^{n+1} + (-1)^{n+1} \sum_{a=1}^{q-2} \overline{\omega}^{a}((-1)^d \lambda^d h_1^{h_1} \cdots h_n^{h_n})(-p)^{\sum_{i=0}^{r-1} M_{a,i}}$$
$$\times \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle p^i - \frac{p^i a}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{p^i a}{q-1} \right\rangle\right) \Gamma_p^{n-1}\left(\left\langle \frac{p^i a}{q-1} \right\rangle\right)$$
$$\times \prod_{h=1}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^i h}{d} - \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d} \right\rangle\right)} \prod_{\substack{k=1 \\ h_k>1}}^{n} \prod_{h=1}^{h_k-1} \frac{\Gamma_p\left(\left\langle \frac{p^i h}{h_k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{h_k} \right\rangle\right)}.$$

Employing Lemma 3.1 and (1.10), we obtain

$$A=(-1)^{n+1}+(-1)^{n+1}q\sum_{a=1}^{q-2}\overline{\omega}^a((-1)^{d-1}\lambda^d h_1^{h_1}\cdots h_n^{h_n})(-p)^{\sum_{i=0}^{r-1}N_{a,i}}$$
$$\times\prod_{i=0}^{r-1}\Gamma_p^{n-1}\left(\left\langle\frac{p^ia}{q-1}\right\rangle\right)\prod_{h=1}^{d-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{d}-\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{hp^i}{d}\right\rangle\right)}\prod_{\substack{k=1\\h_k>1}}^{n}\prod_{h=1}^{h_k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{h_k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{hp^i}{h_k}\right\rangle\right)},$$

where $N_{a,i}:=-\sum_{h=1}^{d-1}\left\lfloor\left\langle\frac{hp^i}{d}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\sum_{k=1}^{n}\sum_{h=0}^{h_k-1}\left\lfloor\left\langle\frac{-hp^i}{h_k}\right\rangle+\frac{ap^i}{q-1}\right\rfloor+\left\lfloor\frac{ap^i}{q-1}\right\rfloor$. For $0\leq a\leq q-2$, it is observed that

$$\prod_{h=1}^{h_k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{h_k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{hp^i}{h_k}\right\rangle\right)}=\prod_{h=1}^{h_k-1}\frac{\Gamma_p\left(\left\langle\frac{-p^ih}{h_k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{-hp^i}{h_k}\right\rangle\right)}. \tag{3.10}$$

Adding and subtracting the term under summation for $a=0$, and then using (3.10) yield

$$A=(-1)^{n+1}(1-q)+(-1)^{n+1}q\sum_{a=0}^{q-2}\overline{\omega}^a((-1)^{d-1}\lambda^d h_1^{h_1}\cdots h_n^{h_n})(-p)^{\sum_{i=0}^{r-1}N_{a,i}}$$
$$\times\prod_{i=0}^{r-1}\Gamma_p^{n-1}\left(\left\langle\frac{p^ia}{q-1}\right\rangle\right)\prod_{h=1}^{d-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{d}-\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{hp^i}{d}\right\rangle\right)}\prod_{\substack{k=1\\h_k>1}}^{n}\prod_{h=1}^{h_k-1}\frac{\Gamma_p\left(\left\langle\frac{-p^ih}{h_k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{-hp^i}{h_k}\right\rangle\right)}.$$

Since $\overline{\omega}^a(-1)=(-1)^a$, we obtain

$$A=(-1)^{n+1}\left(1-q-q(q-1)\cdot{}_{d-1}G_{d-1}\left[\begin{matrix}\frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-1}{d}\\ b_1, & b_2, & \ldots, & b_{d-1}\end{matrix}\ |\lambda^d h_1^{h_1}\cdots h_n^{h_n}\right]_q\right),$$

where b_i's are as given in Notation 3.1.1. Substituting the expressions for A and B in (3.7), and then using the relation given by (3.3), we complete the proof of the theorem. ■

If d is a prime, then the condition $\gcd(d, q-1) = 1$ is equivalent to $q \not\equiv 1 \pmod{d}$. The condition $\gcd(d, q-1) = 1$ is very crucial in the proof of Theorem 3.2. Finding a formula for $\#D_\lambda^d(\mathbb{F}_q)$ in terms of McCarthy's p-adic hypergeometric function without assuming the condition $\gcd(d, q-1) = 1$ seems to be a difficult problem. In the next theorem, we find such a formula without assuming the condition $\gcd(d, q-1) = 1$ for a particular family of diagonal hypersurfaces, namely

$$D_\lambda^{d,k} : x_1^d + x_2^d = d\lambda x_1^k x_2^{d-k}, \tag{3.11}$$

where $d \geq 2$, $k \geq 1$, and $\gcd(d, k, d-k) = 1$. Barman and Saikia [15] expressed the number of points on (3.11) over $\mathbb{F}_p$ in terms of McCarthy's p-adic hypergeometric function when $k = 1$. In the following theorem, we express the number of points on the hypersurface (3.11) over $\mathbb{F}_q$ for any $k \geq 1$ and without assuming the condition $\gcd(d, q-1) = 1$.

Theorem 3.3. *Let p be an odd prime and $q = p^r$, $r \geq 1$. Let $d \geq 2$ and $k \geq 1$ be integers, and let $D_\lambda^{d,k}$ be the diagonal hypersurface given in (3.11) such that $p \nmid dk(d-k)$. Then, for $\lambda \neq 0$, the number of points on $D_\lambda^{d,k}$ in $\mathbb{P}^1(\mathbb{F}_q)$ is given by*

$$\#D_\lambda^{d,k}(\mathbb{F}_q) = 1 + {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{d-1}{d} \\ b_1, & b_2, & \ldots, & b_{d-1} \end{matrix} \Big| \lambda^d k^k (d-k)^{d-k}\right]_q,$$

where b_i's are as given in Notation 3.1.1 with $h_1 = k$ and $h_2 = d-k$.

Proof. Let $N_q^d(\lambda)$ denote the number of points on the diagonal hypersurface $D_\lambda^{d,k}$ in $\mathbb{A}^2(\mathbb{F}_q)$. Then we have

$$\#D_\lambda^{d,k}(\mathbb{F}_q) = \frac{N_q^d(\lambda) - 1}{q-1}. \tag{3.12}$$

Let $f(x_1,x_2)=x_1^d+x_2^d-d\lambda x_1^k x_2^{d-k}$ and using the identity

$$\sum_{z\in\mathbb{F}_q}\theta(zf(x_1,x_2))=\begin{cases} q, & \text{if } f(x_1,x_2)=0;\\ 0, & \text{if } f(x_1,x_2)\neq 0,\end{cases}$$

we obtain

$$\begin{aligned} q\cdot N_q^d(\lambda) &= \sum_{z\in\mathbb{F}_q}\sum_{x_i\in\mathbb{F}_q}\theta(zf(x_1,x_2))\\ &= q^2+\sum_{z,x_1,x_2\in\mathbb{F}_q^\times}\theta(zf(x_1,x_2))+\sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf(x_1,x_2))\\ &= q^2+A_1+B_1, \end{aligned}\tag{3.13}$$

where $A_1:=\sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf(x_1,x_2))$ and $B_1:=\sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf(x_1,x_2))$. Firstly, we calculate B_1. We have

$$\begin{aligned} B_1 &= \sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf(x_1,x_2))\\ &= q-1+\sum_{z\in\mathbb{F}_q^\times}\sum_{x_1\in\mathbb{F}_q^\times}\theta(zx_1^d)+\sum_{z\in\mathbb{F}_q^\times}\sum_{x_2\in\mathbb{F}_q^\times}\theta(zx_2^d). \end{aligned}\tag{3.14}$$

Using Lemma 1.4, we rewrite (3.14) as

$$\begin{aligned} B_1 &= q-1+\frac{2}{q-1}\sum_{z\in\mathbb{F}_q^\times}\sum_{x\in\mathbb{F}_q^\times}\sum_{a=0}^{q-2}g(T^{-a})T^a(zx^d)\\ &= q-1+\frac{2}{q-1}\sum_{a=0}^{q-2}g(T^{-a})\sum_{z\in\mathbb{F}_q^\times}T^a(z)\sum_{x\in\mathbb{F}_q^\times}T^{ad}(x)\\ &= -(q-1). \end{aligned}$$

We obtain the last equality by using the fact that the inner sum is nonzero only if

$a = 0$ and putting $g(\varepsilon) = -1$. Next, we simplify the expression for A_1. We have

$$\begin{aligned} A_1 &= \sum_{z,x_1,x_2\in\mathbb{F}_q^\times} \theta(zx_1^d + zx_2^d - d\lambda z x_1^k x_2^{d-k}) \\ &= \sum_{z,x_1,x_2\in\mathbb{F}_q^\times} \theta(zx_1^d)\theta(zx_2^d)\theta(-d\lambda z x_1^k x_2^{d-k}). \end{aligned}$$

Applying Lemma 1.4, we obtain

$$\begin{aligned} A_1 &= \frac{1}{(q-1)^3}\sum_{l,m,n=0}^{q-2} g(T^{-l})g(T^{-m})g(T^{-n})\times \\ &\quad \sum_{z,x_1,x_2\in\mathbb{F}_q^\times} T^l(zx_1^d)T^m(zx_2^d)T^n(-d\lambda z x_1^k x_2^{d-k}) \\ &= \frac{1}{(q-1)^3}\sum_{l,m,n=0}^{q-2} g(T^{-l})g(T^{-m})g(T^{-n})T^n(-d\lambda)\times \\ &\quad \sum_{z\in\mathbb{F}_q^\times} T^{l+m+n}(z) \sum_{x_1\in\mathbb{F}_q^\times} T^{dl+kn}(x_1) \sum_{x_2\in\mathbb{F}_q^\times} T^{dm+(d-k)n}(x_2). \end{aligned}$$

The sum over $z \in \mathbb{F}_q^\times$ is nonzero only when $n = -m-l$ by Lemma 1.1. Making this substitution, we deduce that

$$\begin{aligned} A_1 &= \frac{1}{(q-1)^2}\sum_{l,m=0}^{q-2} g(T^{-l})g(T^{-m})g(T^{l+m})T^{-l-m}(-d\lambda)\times \\ &\quad \sum_{x_1\in\mathbb{F}_q^\times} T^{(d-k)l-km}(x_1) \sum_{x_2\in\mathbb{F}_q^\times} T^{km-(d-k)l}(x_2). \end{aligned} \tag{3.15}$$

Inner sums in (3.15) are nonzero only when

$$km \equiv (d-k)l \pmod{q-1}. \tag{3.16}$$

We now solve (3.16) for l and m and find all its possible solutions. Consider $\gcd(q-1,k) = k_0$ and $\gcd(q-1,d-k) = d_0$. Take $k_1 = \frac{k}{k_0}$ and $d_1 = \frac{d-k}{d_0}$. We have

$\gcd(k, d-k) = 1$ as $\gcd(d, k, d-k) = 1$. Since $km \equiv (d-k)l \pmod{q-1}$, hence $k_0 \mid l$ which implies $l = \alpha k_0$ for $0 \leq \alpha \leq \frac{q-1}{k_0} - 1$. Also, $\gcd(\frac{q-1}{k_0}, k_1) = 1$, and hence we obtain $l = k\alpha$ where $0 \leq \alpha \leq \frac{q-1}{k_0} - 1$. Corresponding to each l depending on α, we have k_0 values of m given by $m = (d-k)\alpha + r_1\frac{q-1}{k_0}$ where $r_1 = 0, 1, \ldots, k_0 - 1$. Therefore, the total number of solutions for (3.16) is $q-1$.

Claim: All the solutions are given by $l \equiv ka \pmod{q-1}$ and $m \equiv (d-k)a \pmod{q-1}$ for $0 \leq a \leq q-2$.

Proof of the Claim. Since $\gcd(k_0, d-k) = 1$, there exists a z_1 such that $(d-k)z_1 \equiv 1 \pmod{k_0}$. For any α and $0 \leq r \leq k_0 - 1$, one can easily check that $l = k\alpha \equiv ka \pmod{q-1}$ and $m = (d-k)\alpha + r\frac{q-1}{k_0} \equiv (d-k)a \pmod{q-1}$ for $a = \alpha + \frac{q-1}{k_0}rz_1$. We now need to check if any solution is repeating or not. Suppose $ka \equiv kb \pmod{q-1}$ for some a and b. Then $k_1 a = k_1 b + \frac{q-1}{k_0}x_0$ for some x_0. Also, suppose that $(d-k)a \equiv (d-k)b \pmod{q-1}$. Then

$$d_0 d_1 a = d_0 d_1 b + (q-1)y_0, \tag{3.17}$$

for some y_0. We now have $k_1 d_0 d_1 a = k_1 d_0 d_1 b + k_1(q-1)y_0$. Substituting the value of $k_1 a$ from above, we get $d_0 d_1 x_0 = y_0 k_0 k_1$ and hence $x_0 = kz$ and $y_0 = (d-k)z$ for some z. Putting the value of y_0 in (3.17), we get $(d-k)a - (d-k)b = (q-1)(d-k)z$ which gives $a \equiv b \pmod{q-1}$. Therefore, the solutions are given by $l \equiv ka \pmod{q-1}$ and $m \equiv (d-k)a \pmod{q-1}$ for $a = 0, 1, \ldots, q-2$. This completes the proof of our claim.

Hence, the inner sums in (3.15) are nonzero only if $l \equiv ka \pmod{q-1}$ and $m \equiv (d-k)a \pmod{q-1}$, where $a = 0, 1, \ldots, q-2$. Substituting these values of l and m in (3.15) yields

$$A_1 = \sum_{a=0}^{q-2} g(T^{-ka})g(T^{-(d-k)a})g(T^{da})T^{-da}(-d\lambda).$$

Considering $n = 2$, $h_1 = k$, and $h_2 = d - k$ in (3.8), and proceeding along similar lines as shown in the proof of Theorem 3.2, we deduce that

$$A_1 = q - 1 + q(q-1) \cdot {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d}, & \frac{2}{d}, & \dots, & \frac{d-1}{d} \\ b_1, & b_2, & \dots, & b_{d-1} \end{matrix} \Big| \lambda^d k^k (d-k)^{d-k}\right]_q,$$

where b_i's are as given in Notation 3.1.1 with $h_1 = k$ and $h_2 = d - k$. Substituting the values of A_1 and B_1 in (3.13) and then using the relation (3.12), we complete the proof of the theorem. ■

3.3 Polynomials and diagonal hypersurfaces over finite fields

Let $N_q^d(\lambda)$ denote the number of points on the diagonal hypersurface $D_\lambda^{d,k}$ in $\mathbb{A}^2(\mathbb{F}_q)$. Then we have

$$\#D_\lambda^{d,k} = \frac{N_q^d(\lambda) - 1}{q - 1}. \tag{3.18}$$

Let $f(x_1, x_2) := x_1^d + x_2^d - d\lambda x_1^k x_2^{d-k}$. One can dehomogenize $f(x_1, x_2)$ to obtain $g_\lambda(y) := y^d - d\lambda y^k + 1$. Also, if θ is a root of $g_\lambda(y)$, then $(\alpha\theta, \alpha)$ lies on $f(x_1, x_2) = 0$ for all $\alpha \in \mathbb{F}_q^\times$. Suppose $g_\lambda(y)$ has $r'_q(\lambda)$ distinct zeros in $\mathbb{F}_q$. Then, $N_q^d(\lambda) = (q-1)r'_q(\lambda) + 1$ and hence,

$$\#D_\lambda^{d,k} = r'_q(\lambda). \tag{3.19}$$

In this section, we relate $\#D_\lambda^{d,k}$ to the number of zeros of another polynomial over $\mathbb{F}_q$. In the following theorem, we prove that $\#D_\lambda^{d,k}$ is equal to the number of distinct zeros of the polynomial $f_\lambda(y) = y^{d-k}(1-y)^k - (d\lambda)^{-d}$.

Theorem 3.4. *Let $d > k \geq 1$ be integers such that $\gcd(k, d) = 1$. Let p be an* *odd prime such that* $p \nmid dk(d-k)$. *Let* $q = p^r$, $r \geq 1$. *For* $\lambda \in \mathbb{F}_q^\times$, *let* $f_\lambda(y) = y^{d-k}(1-y)^k - (d\lambda)^{-d} \in \mathbb{F}_q[y]$, *and let* $r_q(\lambda)$ *be the number of distinct zeros of* $f_\lambda(y)$ *in* $\mathbb{F}_q$. *Let* $\#D_\lambda^{d,k}$ *be the number of points on* $D_\lambda^{d,k}$ *in* $\mathbb{P}^1(\mathbb{F}_q)$. *Then, we have* $r_q(\lambda) = \#D_\lambda^{d,k}$.

Proof. For $\alpha \in \mathbb{F}_q^\times$, we consider a character sum defined by

$$C(d,k,\alpha) := \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(\chi^d)g(\overline{\chi}^{d-k})\chi(\alpha)}{g(\chi^k)}.$$

We note that if $k_1 = \gcd(k, q-1)$ and $k = k_1 k_2$ such that $\gcd(k_2, q-1) = 1$ and $\chi^k = \varepsilon$ for some $\chi \in \widehat{\mathbb{F}_q^\times}$, then $\chi^{k_1} = \varepsilon$. We first write $C(d,k,\alpha)$ in terms of the number of distinct zeros of the polynomial $h_\alpha(y) = y^{d-k}(1-y)^k - (-1)^{d-k}\alpha \in \mathbb{F}_q[y]$. Applying Lemma 1.5 and then using (1.4), we have

$$\begin{aligned}
C(d,k,\alpha) &= \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(\chi^d)g(\overline{\chi}^{d-k})\chi(\alpha)}{g(\chi^k)} \\
&= \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} (J(\overline{\chi}^{d-k}, \chi^d)\chi(\alpha) - (q-1)\chi^d(-1)\chi(\alpha)\delta(\chi^k)) \\
&= \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \overline{\chi}^{d-k}(-1)J(\overline{\chi}^{d-k}, \overline{\chi}^k)\chi(\alpha) - \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} (q-1)\chi^d(-1)\chi(\alpha)\delta(\chi^k) \\
&= \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \chi^{d-k}(-1)J(\chi^{d-k}, \chi^k)\overline{\chi}(\alpha) - \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} (q-1)\chi((-1)^d\alpha) \\
&= \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \sum_{y \in \mathbb{F}_q} \chi\left(\frac{y^{d-k}(1-y)^k}{(-1)^{d-k}\alpha}\right) - \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} (q-1)\chi((-1)^d\alpha). \qquad (3.20)
\end{aligned}$$

By Lemma 1.1, the inner sum of the first summation of (3.20) is nonzero only if $h_\alpha(y)$ has a solution in $\mathbb{F}_q$. Let $n_q(\alpha)$ be the number of distinct zeros of $h_\alpha(y)$ in $\mathbb{F}_q$.

Then

$$C(d,k,\alpha) = (q-1)n_q(\alpha) - (q-1)\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} \chi((-1)^d\alpha). \tag{3.21}$$

We can also write $C(d,k,\alpha)$ as

$$\begin{aligned} C(d,k,\alpha) &= \sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \frac{g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi(\alpha)}{g(\chi^k)g(\overline{\chi}^k)} \\ &= \sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}\neq\varepsilon}} \frac{g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi(\alpha)}{g(\chi^k)g(\overline{\chi}^k)} + \sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} \frac{g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi(\alpha)}{g(\chi^k)g(\overline{\chi}^k)}. \end{aligned}$$

Using Lemma 1.2 in the first summation and the fact that $g(\varepsilon) = -1$ in the second summation, we obtain

$$C(d,k,\alpha) = \frac{1}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}\neq\varepsilon}} g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi((-1)^k\alpha) - \sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} g(\chi^d)g(\overline{\chi}^d)\chi(\alpha).$$

Adding and subtracting the terms in the first summation for $\chi \in \widehat{\mathbb{F}_q^\times}$ such that $\chi^{k_1} = \varepsilon$, we deduce that

$$\begin{aligned} C(d,k,\alpha) &= \frac{1}{q}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi((-1)^k\alpha) - \sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} g(\chi^d)g(\overline{\chi}^d)\chi(\alpha) \\ &\quad + \frac{1}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} g(\chi^d)g(\overline{\chi}^d)\chi(\alpha) \\ &= A + \frac{1-q}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} g(\chi^d)g(\overline{\chi}^d)\chi(\alpha), \end{aligned} \tag{3.22}$$

where $A = \frac{1}{q}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi((-1)^k\alpha)$. Using Lemma 1.2, we obtain

$$C(d,k,\alpha) = A - \frac{q-1}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} q\chi((-1)^d\alpha) + \frac{(q-1)^2}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} \chi(\alpha)\delta(\chi^d).$$

Since $\gcd(d,k)=1$, so we have $\gcd(d,k_1)=1$. If $\chi^{k_1}=\varepsilon$ and $\chi^d=\varepsilon$ then the order of χ divides $\gcd(k_1,d)$. Since $\gcd(k_1,d)=1$, we have $\chi=\varepsilon$. Thus

$$A = C(d,k,\alpha) - \frac{(q-1)^2}{q} + (q-1)\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}} \chi((-1)^d\alpha). \tag{3.23}$$

Combining (3.21) and (3.23), we deduce that

$$A = (q-1)n_q(\alpha) - \frac{(q-1)^2}{q}. \tag{3.24}$$

Taking $\alpha = (-1)^{d+k}(\lambda d)^{-d}$, we have $h_\alpha(y) = f_\lambda(y)$. Hence, $r_q(\lambda) = n_q(\alpha)$. Let $f(x_1,x_2) = x_1^d + x_2^d - d\lambda x_1^k x_2^{d-k}$ and using the identity

$$\sum_{z\in\mathbb{F}_q}\theta(zf(x_1,x_2)) = \begin{cases} q, & \text{if } f(x_1,x_2)=0;\\ 0, & \text{if } f(x_1,x_2)\neq 0,\end{cases}$$

we obtain

$$\begin{aligned} q\cdot N_q^d(\lambda) &= \sum_{z\in\mathbb{F}_q}\sum_{x_i\in\mathbb{F}_q}\theta(zf(x_1,x_2))\\ &= q^2 + \sum_{z,x_1,x_2\in\mathbb{F}_q^\times}\theta(zf(x_1,x_2)) + \sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf(x_1,x_2))\\ &= q^2 + A_1 + B_1, \end{aligned} \tag{3.25}$$

where $A_1 = \sum_{z,x_i\in\mathbb{F}_q^\times}\theta(zf(x_1,x_2))$ and $B_1 = \sum_{z\in\mathbb{F}_q^\times}\sum_{\substack{\text{some}\\ x_i=0}}\theta(zf(x_1,x_2))$. Using Lemma

1.4 and the orthogonality relation given by Lemma 1.1, we calculated B_1 in Theorem 3.3 and found that $B_1 = 1 - q$. Combining (3.18) and (3.25), we have

$$\begin{aligned}\#D_\lambda^{d,k} &= \frac{N_q^d(\lambda) - 1}{q-1} = \frac{q - 1 + \frac{A_1}{q} - \frac{q-1}{q}}{q-1} \\ &= \frac{q-1}{q} + \frac{A_1}{q(q-1)}.\end{aligned}$$

In Theorem 3.3, using Lemma 1.4 and the condition that $\gcd(d,k) = 1$, we simplified the character sum A_1 and found that

$$A_1 = \sum_{a=0}^{q-2} g(T^{-ka})g(T^{-(d-k)a})g(T^{da})T^{-da}(-d\lambda).$$

For $\alpha = (-1)^{d+k}(\lambda d)^{-d}$, we observe that $A = \frac{A_1}{q}$. Hence,

$$\#D_\lambda^{d,k} = \frac{q-1}{q} + \frac{A}{q-1}.$$

Using (3.24) and the fact that $r_q(\lambda) = n_q(\alpha)$, we complete the proof. ■

If $k = 1$, then $g_\lambda(y)$ can be obtained from $f_\lambda(y)$ by a change of the variable. However, for $k > 1$, we do not know if $g_\lambda(y)$ can be obtained from $f_\lambda(y)$ in a similar way. Combining (3.19) and Theorem 3.4, we obtain the following corollary.

Corollary 3.3.1. *Let $d > k \geq 1$ be integers. Let p be an odd prime such that $p \nmid dk(d-k)$. Let $q = p^r$, $r \geq 1$. For $\lambda \in \mathbb{F}_q^\times$, let $r_q(\lambda)$ and $r'_q(\lambda)$ be the number of distinct zeros of $f_\lambda(y) = y^{d-k}(1-y)^k - (d\lambda)^{-d}$ and $g_\lambda(y) = y^d - d\lambda y^k + 1$ in $\mathbb{F}_q$, respectively. If $\gcd(d,k) = 1$, then we have $r'_q(\lambda) = r_q(\lambda)$.*

Theorem 3.4 relates $r_q(\lambda)$, the number of distinct zeros of the polynomial $f_\lambda(y) = y^{d-k}(1-y)^k - (d\lambda)^{-d}$ in $\mathbb{F}_q$, to $\#D_\lambda^{d,k}$ for integers $d > k \geq 1$ with $\gcd(k,d) = 1$. In the following theorem, we express $r_q(\lambda)$ in terms of the p-adic hypergeometric functions without the condition $\gcd(k,d) = 1$. To be specific, we prove the following theorem.

Theorem 3.5. *Let $d > k \geq 1$ be integers. Let p be an odd prime such that $p \nmid dk(d-k)$. Let $q = p^r$, $r \geq 1$. For $\lambda \in \mathbb{F}_q^\times$, let $r_q(\lambda)$ denote the number of distinct zeros of $f_\lambda(y) = y^{d-k}(1-y)^k - (d\lambda)^{-d} \in \mathbb{F}_q[y]$ in $\mathbb{F}_q$. We have*

$$r_q(\lambda) = 1 + {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d} \\ 0, & \frac{1}{k}, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k} \end{matrix} \,|k^k(d-k)^{d-k}\lambda^d\right]_q$$
$$- \frac{1-q}{q} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times}, \\ \chi^{k_1}=\varepsilon, \chi\neq\varepsilon}} \overline{\chi}^d(-\lambda d)\delta(\chi^d),$$

where $k_1 = \gcd(q-1, k)$.

Proof. We first recall that for $\alpha \in \mathbb{F}_q^\times$, we have

$$C(d,k,\alpha) = \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(\chi^d)g(\overline{\chi}^{d-k})\chi(\alpha)}{g(\chi^k)}.$$

We now write $C(d,k,\alpha)$ in terms of p-adic hypergeometric functions. From (3.22), we have

$$C(d,k,\alpha) = A + \frac{1-q}{q} \sum_{\substack{\chi \in \widehat{\mathbb{F}_q^\times}, \\ \chi^{k_1}=\varepsilon}} g(\chi^d)g(\overline{\chi}^d)\chi(\alpha), \tag{3.26}$$

where $A = \frac{1}{q}\sum_{\chi \in \widehat{\mathbb{F}_q^\times}} g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi((-1)^k\alpha)$. Now, we simplify the expression for A. We have

$$A = \frac{1}{q}\sum_{\chi \in \widehat{\mathbb{F}_q^\times}} g(\chi^d)g(\overline{\chi}^{d-k})g(\overline{\chi}^k)\chi((-1)^k\alpha).$$

Replacing χ with ω^a and then applying Gross-Koblitz formula, we obtain

$$\begin{aligned}A &= \frac{1}{q}\sum_{a=0}^{q-2} g(\overline{\omega}^{-ad})g(\overline{\omega}^{a(d-k)})g(\overline{\omega}^{ak})\overline{\omega}^a((-1)^k\alpha^{-1})\\ &= -\frac{1}{q}\sum_{a=0}^{q-2}(-p)^{\sum_{i=0}^{r-1}\left(\left\langle\frac{-adp^i}{q-1}\right\rangle+\left\langle\frac{a(d-k)p^i}{q-1}\right\rangle+\left\langle\frac{akp^i}{q-1}\right\rangle\right)}\overline{\omega}^a((-1)^k\alpha^{-1})\\ &\times\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{-adp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{akp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{a(d-k)p^i}{q-1}\right\rangle\right).\end{aligned}$$

Applying Lemma 1.11 with $t=d$ and Lemma 1.10 with $t=k$ and $t=d-k$, we deduce that

$$\begin{aligned}A &= -\frac{1}{q}\sum_{a=0}^{q-2}(-p)^{\sum_{i=0}^{r-1}\left(-\left\lfloor\frac{-adp^i}{q-1}\right\rfloor-\left\lfloor\frac{akp^i}{q-1}\right\rfloor-\left\lfloor\frac{a(d-k)p^i}{q-1}\right\rfloor\right)}\overline{\omega}^a((-1)^kd^{-d}\alpha^{-1}k^k(d-k)^{d-k})\\ &\times\prod_{i=0}^{r-1}\prod_{h=0}^{k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{k}\right\rangle\right)}\prod_{h=0}^{d-k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{d-k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{d-k}\right\rangle\right)}\prod_{h=0}^{d-1}\frac{\Gamma_p\left(\left\langle\frac{p^i(1+h)}{d}-\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{d}\right\rangle\right)}\\ &= -\frac{1}{q}-\frac{1}{q}\sum_{a=1}^{q-2}(-p)^{\sum_{i=0}^{r-1}\left(-\left\lfloor\frac{-adp^i}{q-1}\right\rfloor-\left\lfloor\frac{akp^i}{q-1}\right\rfloor-\left\lfloor\frac{a(d-k)p^i}{q-1}\right\rfloor\right)}\overline{\omega}^a((-1)^dx)\\ &\times\prod_{i=0}^{r-1}\prod_{h=0}^{k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{k}\right\rangle\right)}\prod_{h=0}^{d-k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{d-k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{d-k}\right\rangle\right)}\prod_{h=0}^{d-1}\frac{\Gamma_p\left(\left\langle\frac{p^i(1+h)}{d}-\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{d}\right\rangle\right)},\end{aligned}$$

where $x:=\frac{(-1)^{d+k}k^k(d-k)^{d-k}}{\alpha d^d}$. Using Lemma 1.13, and Lemma 1.14 with $l=k$ and $l=d-k$ yield

$$\begin{aligned}A &= -\frac{1}{q}-\frac{1}{q}\sum_{a=1}^{q-2}\overline{\omega}^a((-1)^dx)(-p)^{\sum_{i=0}^{r-1}v_{a,i}+1}\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{p^ia}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle p^i-\frac{p^ia}{q-1}\right\rangle\right)\\ &\times\prod_{h=0}^{k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{k}\right\rangle\right)}\prod_{h=1}^{d-k-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{d-k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{d-k}\right\rangle\right)}\prod_{h=1}^{d-1}\frac{\Gamma_p\left(\left\langle\frac{p^ih}{d}-\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^ih}{d}\right\rangle\right)},\end{aligned}\tag{3.27}$$

where

$$v_{a,i} = -\sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{hp^i}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \sum_{h=0}^{k-1} \left\lfloor \left\langle \frac{-hp^i}{k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor - \sum_{h=1}^{d-k-1} \left\lfloor \left\langle \frac{-hp^i}{d-k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor.$$

For a positive integer l, one can easily check that the following equality holds

$$\prod_{h=0}^{l-1} \frac{\Gamma_p\left(\left\langle \frac{p^ih}{l} + \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^ih}{l} \right\rangle\right)} = \prod_{h=0}^{l-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{l} + \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-p^ih}{l} \right\rangle\right)}. \tag{3.28}$$

Substituting (3.28) with $l = k$ and $l = d-k$ in (3.27) and using (1.10), we deduce that

$$A = -\frac{1}{q} - \frac{(-p)^r}{q} \sum_{a=1}^{q-2} \overline{\omega}^a((-1)^d(-x))(-p)^{\sum_{i=0}^{r-1} v_{a,i}} (-1)^r \prod_{i=0}^{r-1} \prod_{h=0}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k} + \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-p^ih}{k} \right\rangle\right)}$$
$$\times \prod_{h=1}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} + \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} \right\rangle\right)} \prod_{h=1}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^ih}{d} - \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^ih}{d} \right\rangle\right)}.$$

Adding and subtracting the term under summation for $a = 0$ and using $\overline{\omega}^a(-1) = (-1)^a$, we obtain

$$A = \frac{q-1}{q} - \sum_{a=0}^{q-2} (-1)^{a(d-1)} \overline{\omega}^a(x)(-p)^{\sum_{i=0}^{r-1} v_{a,i}} \prod_{i=0}^{r-1} \prod_{h=0}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k} + \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-p^ih}{k} \right\rangle\right)}$$
$$\times \prod_{h=1}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} + \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} \right\rangle\right)} \prod_{h=1}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^ih}{d} - \frac{p^ia}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^ih}{d} \right\rangle\right)}$$
$$= \frac{q-1}{q} + (q-1) \cdot {}_{d-1}G_{d-1} \left[\begin{matrix} \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d} \\ 0, & \frac{1}{k}, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k} \end{matrix} \Big| x \right]_q.$$

Substituting the expression for A in (3.26) and then using Lemma 1.2 yield

$$C(d,k,\alpha)=\frac{q-1}{q}+(1-q)\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}}\chi((-1)^d\alpha)-\frac{1-q}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}}(q-1)\chi(\alpha)\delta(\chi^d)$$
$$+(q-1)\cdot {}_{d-1}G_{d-1}\left[\begin{matrix}\frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d}\\ 0, & \frac{1}{k}, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k}\end{matrix}\Big| x\right]_q. \tag{3.29}$$

From (3.21), we have

$$C(d,k,\alpha)=(q-1)n_q(\alpha)-(q-1)\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times},\\ \chi^{k_1}=\varepsilon}}\chi((-1)^d\alpha), \tag{3.30}$$

where $n_q(\alpha)$ is the number of distinct zeros of the polynomial $h_\alpha(y)$ in $\mathbb{F}_q$. Taking $\alpha=(-1)^{d-k}(\lambda d)^{-d}$, we have $n_q(\alpha)=r_q(\lambda)$. Combining (3.29) and (3.30), and taking $\alpha=(-1)^{d-k}(\lambda d)^{-d}$, we obtain the required expression for $r_q(\lambda)$. ■

Remark 3.3.1. *We remark that Theorem 3.3 follows from Theorem 3.4 and Theorem 3.5. Suppose that* $\gcd(k,d)=1$*. Then, we have* $\gcd(d,k_1)=1$*. Now, if* $\chi^{k_1}=\varepsilon$ *and* $\chi^d=\varepsilon$ *then the order of* χ *divides* $\gcd(k_1,d)$*. Since* $\gcd(k_1,d)=1$*, we have* $\chi=\varepsilon$*. Therefore, the last summation in Theorem 3.5 is empty, and hence*

$$r_q(\lambda)=1+{}_{d-1}G_{d-1}\left[\begin{matrix}\frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d}\\ 0, & \frac{1}{k}, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k}\end{matrix}\Big| k^k(d-k)^{d-k}\lambda^d\right]_q.$$

Now, by Theorem 3.4, we readily obtain Theorem 3.3.

Remark 3.3.2. *If we combine* (3.29) *and* (3.30)*, and take* $d=3n$*,* $k=2$*, and* $r=1$*, then we obtain [62, Theorem 1.2].*

4

Elliptic curves and p-adic Hypergeometric Functions

4.1 Introduction

It is well-known that the trace of Frobenius of elliptic curves can be expressed as special values of ${}_2F_1$-Gaussian hypergeometric series, see for example [8, 9, 29, 46, 47, 48, 57]. It was Ono [57] who first expressed the trace of Frobenius of elliptic curves in terms of ${}_3F_2$-Gaussian hypergeometric series. Very recently, Tripathi and Meher [69] expressed the trace of Frobenius and the sum of traces of Frobenius of

[1] The contents of this chapter have been published in *Int. J. Number Theory* (2024).

certain families of elliptic curves in terms of ${}_4F_3$-Gaussian hypergeometric series for primes in some congruence classes.

The function ${}_nG_n[\cdots]_q$ often allows results involving Gaussian hypergeometric series to be extended to a wider class of primes. In [53], McCarthy expressed the trace of Frobenius of elliptic curves in terms of a special value of ${}_2G_2[\cdots]_p$ hypergeometric function for all primes $p > 3$. Later, Barman and Saikia [12] expressed the trace of Frobenius of elliptic curves in terms of another special value of ${}_2G_2[\cdots]_q$ hypergeometric function. However, there is no formula for the trace of Frobenius of elliptic curves in terms of special values of ${}_nG_n[\cdots]_q$ hypergeometric functions with $n \geq 3$ which holds for all but finitely many primes. In this chapter, we find several expressions for the traces of Frobenius endomorphism of certain families of elliptic curves in terms of special values of ${}_4G_4[\cdots]_q$ and ${}_6G_6[\cdots]_q$ hypergeometric functions which hold for all but finitely many primes.

4.2 The trace of Frobenius of elliptic curves defined over $\mathbb{F}_q$

In the following lemma, we evaluate an expression containing certain values of the p-adic gamma function.

Lemma 4.1. *Let p be an odd prime and $q = p^r, r \geq 1$ such that $q \equiv 1 \pmod 4$. For $0 \leq n \leq q-2$ such that $n \notin \{\frac{3(q-1)}{4}, \frac{q-1}{4}\}$, we have*

$$(-p)^{\sum_{i=0}^{r-1} s_{i,n}} \prod_{i=0}^{r-1} \frac{\Gamma_p\left(\left\langle\left(\frac{1}{4}+\frac{n}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{3}{4}-\frac{n}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{3p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{p^i}{4}\right\rangle\right)}$$
$$\times \frac{\Gamma_p\left(\left\langle\left(\frac{1}{4}-\frac{n}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{3}{4}+\frac{n}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{3p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{p^i}{4}\right\rangle\right)} = 1,$$

where $s_{i,n} = -\left\lfloor \frac{3}{4} - \frac{np^i}{q-1} \right\rfloor - \left\lfloor \frac{1}{4} + \frac{np^i}{q-1} \right\rfloor - \left\lfloor \frac{3}{4} + \frac{np^i}{q-1} \right\rfloor - \left\lfloor \frac{1}{4} - \frac{np^i}{q-1} \right\rfloor$.

Proof. Since $q \equiv 1 \pmod 4$, there exists a character χ_4 of order 4, and the inverse of χ_4 is $\overline{\chi_4} = \chi_4^3$. Let $\psi \in \widehat{\mathbb{F}_q^\times}$ be such that $\psi \neq \chi_4, \overline{\chi_4}$. Then, using Lemma 1.2, we have

$$\frac{g(\chi_4\psi)g(\overline{\chi_4\psi})g(\overline{\chi_4}\psi)g(\chi_4\overline{\psi})}{g(\chi_4)g(\overline{\chi_4})g(\chi_4)g(\overline{\chi_4})} = \frac{q^2\overline{\chi_4}\psi\chi_4\psi(-1)}{q^2\overline{\chi_4}\chi_4(-1)} = 1.$$

Taking $\chi_4 = \overline{\omega}^{\frac{q-1}{4}}$ and $\psi = \overline{\omega}^n$ for some n such that $0 \leq n \leq q-2$ and $n \neq \frac{3(q-1)}{4}, \frac{q-1}{4}$, and then employing Gross-Koblitz formula, we obtain the desired result. ■

In the following theorem we express the sum of traces of Frobenius of elliptic curves as a special value of ${}_4G_4[\cdots]_{p^r}$ hypergeometric function for all odd prime p and $r \geq 1$. This gives a p-adic analogue of [69, Theorem 1.4]. We note that [69, Theorem 1.4] holds only for $q = p^r \equiv 1 \pmod 4$.

Theorem 4.2. *Let p be an odd prime and $q = p^r, r \geq 1$. Let $E_\lambda : y^2 = x(x-1)(x-\lambda)$ and $E_{-\lambda} : y^2 = x(x-1)(x+\lambda)$ be elliptic curves over $\mathbb{F}_q$ such that $\lambda \notin \{0, \pm 1\}$. Then we have*

$$a_q(E_\lambda) + a_q(E_{-\lambda}) = \varphi(-1) \cdot {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4} \end{matrix} \middle| \lambda^2 \right]_q.$$

Proof. In [46], Koike proved that

$$a_q(E_\lambda) = -q \cdot \varphi(-1) \cdot {}_2F_1\left(\begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \middle| \lambda \right)_q.$$

For $\lambda \neq 0$, using Proposition 1.7 we obtain

$$a_q(E_\lambda) = -q \cdot \varphi(-1) \binom{\varphi}{\varepsilon} {}_2F_1\left(\begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \middle| \lambda \right)_q^*.$$

Now, Lemma 1.9 and (1.5) yield

$$a_q(E_\lambda) = \varphi(-1) \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 0, & 0 \end{matrix} | \frac{1}{\lambda} \right]_q . \tag{4.1}$$

Employing Theorem 2.9 with $a_1 = a_2 = \frac{1}{2}$ and $a_3 = a_4 = 0$, we have, for $q \equiv 1 \pmod 2$,

$${}_2G_2 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 0, & 0 \end{matrix} | \frac{1}{\lambda} \right]_q + {}_2G_2 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 0, & 0 \end{matrix} | -\frac{1}{\lambda} \right]_q = {}_4G_4 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2}, & 0, & \frac{1}{2} \end{matrix} | \frac{1}{\lambda^2} \right]_q . \tag{4.2}$$

Then, (4.1) and (4.2) yield

$$\begin{aligned} a_q(E_\lambda) + a_q(E_{-\lambda}) &= \varphi(-1) \cdot {}_4G_4 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2}, & 0, & \frac{1}{2} \end{matrix} | \frac{1}{\lambda^2} \right]_q \\ &= \varphi(-1) \cdot {}_4G_4 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4} \end{matrix} | \lambda^2 \right]_q . \end{aligned}$$

This completes the proof of the theorem. ■

The following theorem gives a p-adic analogue of [69, Theorem 1.6]. Our result holds for all $q = p^r, r \geq 1$ with $p > 3$, whereas [69, Theorem 1.6] holds only for $q = p^r \equiv 1 \pmod 3$.

Theorem 4.3. *Let $p > 3$ be a prime and $q = p^r, r \geq 1$. Let $E_{a_1,a_3} : y^2 + a_1xy + a_3y = x^3$ and $E_{a_1,-a_3} : y^2 + a_1xy - a_3y = x^3$ be elliptic curves over $\mathbb{F}_q$ such that $a_1, a_3 \in \mathbb{F}_q^\times$. Then we have*

$$a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) = {}_4G_4 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6} \end{matrix} | \frac{729a_3^2}{a_1^6} \right]_q .$$

Proof. From the proof of Theorem 1.1 of [48], we have

$$a_q(E_{a_1,a_3}) = -\frac{1}{q} - \frac{1}{q(q-1)} \sum_{l=0}^{q-2} g(T^{-l})^3 g(T^{3l}) T^l \left(\frac{-a_3}{a_1^3}\right).$$

Now, we can write

$$a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) = -\frac{2}{q} - \frac{1}{q(q-1)} \sum_{l=0}^{q-2} g(T^{-l})^3 g(T^{3l}) T^l \left(\frac{a_3}{a_1^3}\right) (T^l(-1)+1).$$

We know that $T^l(-1) = -1$ if l is odd and $T^l(-1) = 1$ if l is even. Thus, we deduce that

$$a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) = -\frac{2}{q} - \frac{2}{q(q-1)} \sum_{l=0}^{\frac{q-3}{2}} g(T^{-2l})^3 g(T^{6l}) T^l \left(\frac{a_3^2}{a_1^6}\right).$$

Taking out the term under the summation for $l = 0$, we obtain

$$a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) = -\frac{2}{q-1} - \frac{2}{q(q-1)} \sum_{l=1}^{\frac{q-3}{2}} g(T^{-2l})^3 g(T^{6l}) T^l \left(\frac{a_3^2}{a_1^6}\right).$$

Taking $T = \overline{\omega}$ and then using Gross-Koblitz formula, we have

$$\begin{aligned}
& a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) \\
&= -\frac{2}{q-1} - \frac{2}{q(q-1)} \sum_{l=1}^{\frac{q-3}{2}} \overline{\omega}^l \left(\frac{a_3^2}{a_1^6}\right) (-p)^{\sum_{i=0}^{r-1}\left(3\left\langle -\frac{2lp^i}{q-1}\right\rangle + \left\langle \frac{6lp^i}{q-1}\right\rangle\right)} \\
&\times \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle -\frac{2lp^i}{q-1}\right\rangle\right)^3 \Gamma_p\left(\left\langle \frac{6lp^i}{q-1}\right\rangle\right) \\
&= -\frac{2}{q-1} - \frac{2}{q(q-1)} \sum_{l=1}^{\frac{q-3}{2}} \overline{\omega}^l \left(\frac{a_3^2}{a_1^6}\right) (-p)^{\sum_{i=0}^{r-1}\left(-3\left\lfloor -\frac{2lp^i}{q-1}\right\rfloor - \left\lfloor \frac{6lp^i}{q-1}\right\rfloor\right)} \\
&\times \prod_{i=0}^{r-1} \Gamma_p\left(\left\langle -\frac{2lp^i}{q-1}\right\rangle\right)^3 \Gamma_p\left(\left\langle \frac{6lp^i}{q-1}\right\rangle\right).
\end{aligned}$$

Using Lemma 1.11 with $t = 2$, Lemma 1.10 with $t = 6$, Lemma 1.13 with $d = 2$, Lemma 1.14 with $l = 6$ and the fact that

$$\prod_{h=1,h\neq 3}^{5} \Gamma_p\left(\left\langle \frac{hp^i}{6} + \frac{lp^i}{q-1} \right\rangle\right) = \prod_{h=1,h\neq 3}^{5} \Gamma_p\left(\left\langle \frac{-hp^i}{6} + \frac{lp^i}{q-1} \right\rangle\right),$$

we have

$$\begin{aligned}
&a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) \\
&= -\frac{2}{q-1} - \frac{2}{q(q-1)} \sum_{l=1}^{\frac{q-3}{2}} \overline{\omega}^l\left(\frac{a_3^2}{a_1^6}\right) \overline{\omega}^l(3^6) \\
&\times (-p)^{\sum_{i=0}^{r-1}\left(s_{i,l} - \left\lfloor \left\langle \frac{p^i}{2}\right\rangle - \frac{lp^i}{q-1}\right\rfloor + 1 + \left\lfloor \frac{lp^i}{q-1}\right\rfloor - \left\lfloor \left\langle \frac{-p^i}{2}\right\rangle + \frac{lp^i}{q-1}\right\rfloor - \left\lfloor \frac{lp^i}{q-1}\right\rfloor\right)} \\
&\times \prod_{i=0}^{r-1} M_{i,l} \frac{\Gamma_p\left(\left\langle \frac{p^i}{2} - \frac{lp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{-p^i}{2} + \frac{lp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle p^i - \frac{lp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{lp^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)^2},
\end{aligned}$$

where

$$\begin{aligned}
M_{i,l} = &\frac{\Gamma_p\left(\left\langle \frac{p^i}{2} - \frac{lp^i}{q-1}\right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-lp^i}{q-1}\right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-p^i}{6} + \frac{lp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{-p^i}{3} + \frac{lp^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-p^i}{6}\right\rangle\right)\Gamma_p\left(\left\langle \frac{-p^i}{3}\right\rangle\right)\Gamma_p\left(\left\langle \frac{-2p^i}{3}\right\rangle\right)\Gamma_p\left(\left\langle \frac{-5p^i}{6}\right\rangle\right)} \\
&\times \Gamma_p\left(\left\langle \frac{-2p^i}{3} + \frac{lp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{-5p^i}{6} + \frac{lp^i}{q-1}\right\rangle\right)
\end{aligned}$$

and

$$\begin{aligned}
s_{i,l} &= -2\left\lfloor \left\langle \frac{p^i}{2}\right\rangle - \frac{lp^i}{q-1}\right\rfloor + 2 + 2\left\lfloor \frac{lp^i}{q-1}\right\rfloor - \left\lfloor \left\langle \frac{-p^i}{6}\right\rangle + \frac{lp^i}{q-1}\right\rfloor \\
&- \left\lfloor \left\langle \frac{-p^i}{3}\right\rangle + \frac{lp^i}{q-1}\right\rfloor - \left\lfloor \left\langle \frac{-2p^i}{3}\right\rangle + \frac{lp^i}{q-1}\right\rfloor - \left\lfloor \left\langle \frac{-5p^i}{6}\right\rangle + \frac{lp^i}{q-1}\right\rfloor \\
&= -2\left\lfloor \left\langle \frac{p^i}{2}\right\rangle - \frac{lp^i}{q-1}\right\rfloor - 2\left\lfloor \frac{-lp^i}{q-1}\right\rfloor - \left\lfloor \left\langle \frac{-p^i}{6}\right\rangle + \frac{lp^i}{q-1}\right\rfloor - \left\lfloor \left\langle \frac{-p^i}{3}\right\rangle + \frac{lp^i}{q-1}\right\rfloor \\
&- \left\lfloor \left\langle \frac{-2p^i}{3}\right\rangle + \frac{lp^i}{q-1}\right\rfloor - \left\lfloor \left\langle \frac{-5p^i}{6}\right\rangle + \frac{lp^i}{q-1}\right\rfloor.
\end{aligned}$$

Using (1.10), (1.11), and the fact that $\left\lfloor\left\langle \frac{p^i}{2}\right\rangle - \frac{lp^i}{q-1}\right\rfloor + \left\lfloor\left\langle \frac{-p^i}{2}\right\rangle + \frac{lp^i}{q-1}\right\rfloor = 0$ for $l \neq \frac{q-1}{2}$, we have

$$\begin{aligned}
a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) &= -\frac{2}{q-1} - \frac{2}{q-1}\sum_{l=1}^{\frac{q-3}{2}} \overline{\omega}^l\left(\frac{729a_3^2}{a_1^6}\right)(-p)^{\sum_{i=0}^{r-1} s_{i,l}} \prod_{i=0}^{r-1} M_{i,l}\\
&= -\frac{2}{q-1}\sum_{l=0}^{\frac{q-3}{2}} \overline{\omega}^l\left(\frac{729a_3^2}{a_1^6}\right)(-p)^{\sum_{i=0}^{r-1} s_{i,l}} \prod_{i=0}^{r-1} M_{i,l}\\
&= -\frac{1}{q-1}\sum_{l=0}^{\frac{q-3}{2}} \overline{\omega}^l\left(\frac{729a_3^2}{a_1^6}\right)(-p)^{\sum_{i=0}^{r-1} s_{i,l}} \prod_{i=0}^{r-1} M_{i,l}\\
&\quad - \frac{1}{q-1}\sum_{l=0}^{\frac{q-3}{2}} \overline{\omega}^l\left(\frac{729a_3^2}{a_1^6}\right)(-p)^{\sum_{i=0}^{r-1} s_{i,l}} \prod_{i=0}^{r-1} M_{i,l}\\
&= -\frac{1}{q-1}\sum_{l=0}^{\frac{q-3}{2}} \overline{\omega}^l\left(\frac{729a_3^2}{a_1^6}\right)(-p)^{\sum_{i=0}^{r-1} s_{i,l}} \prod_{i=0}^{r-1} M_{i,l}\\
&\quad - \frac{1}{q-1}\sum_{l=\frac{q-1}{2}}^{q-2} \overline{\omega}^l\left(\frac{729a_3^2}{a_1^6}\right)(-p)^{\sum_{i=0}^{r-1} s_{i,l-\frac{q-1}{2}}} \prod_{i=0}^{r-1} M_{i,l-\frac{q-1}{2}}.
\end{aligned}$$

We can easily check that $s_{i,l-\frac{q-1}{2}} = s_{i,l}$ and $M_{i,l-\frac{q-1}{2}} = M_{i,l}$. Hence, we obtain

$$\begin{aligned}
a_q(E_{a_1,a_3}) + a_q(E_{a_1,-a_3}) &= -\frac{1}{q-1}\sum_{l=0}^{q-2} \overline{\omega}^l\left(\frac{729a_3^2}{a_1^6}\right)(-p)^{\sum_{i=0}^{r-1} s_{i,l}} \prod_{i=0}^{r-1} M_{i,l}\\
&= {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6}\end{matrix} \middle| \frac{729a_3^2}{a_1^6}\right]_q.
\end{aligned}$$

This completes the proof of the theorem. ∎

In the following theorem, we prove a p-adic analogue of [69, Theorem 1.5]. Our result holds for all $q = p^r, r \geq 1$ with p odd, but [69, Theorem 1.5] holds only for $q = p^r \equiv 1 \pmod 8$.

Theorem 4.4. *Let p be an odd prime and $q = p^r, r \geq 1$. Let $E_{f,h} : y^2 = x^3 + fx^2 + hx$*

and $E_{f,-h}: y^2 = x^3 + fx^2 - hx$ *be elliptic curves over* $\mathbb{F}_q$ *such that* $f, h \in \mathbb{F}_q^{\times}$. *Then we have*

$$a_q(E_{f,h}) + a_q(E_{f,-h}) = \varphi(f) \cdot {}_4G_4 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{8}, & \frac{3}{8}, & \frac{5}{8}, & \frac{7}{8} \end{matrix} \mid \frac{16h^2}{f^4} \right]_q .$$

Proof. From [9, Eq. 3.8], we have

$$q(\#E_{f,h}(\mathbb{F}_q) - 1) = q^2 + \frac{g(\varphi)\varphi(-f)}{q-1} \sum_{l=0}^{q-2} g(T^{-l}) g(T^{2l+\frac{q-1}{2}}) g(T^{-l}) T^{-2l}(f) T^l(h). \tag{4.3}$$

Note that in [9, Eq. 3.8], we have $\varphi(f)$ in place of $\varphi(-f)$. This is because in [9, Eq. 3.8] it is assumed that $q \equiv 1 \pmod 4$, and hence $\varphi(f) = \varphi(-f)$. Using Davenport-Hasse relation with $m = 2$ and $\psi = T^{2l}$, we have

$$g(T^{2l+\frac{q-1}{2}}) = \frac{g(T^{4l}) g(\varphi) T^{-l}(16)}{g(T^{2l})}. \tag{4.4}$$

Substituting (4.4) in (4.3), we obtain

$$\begin{aligned} q(\#E_{f,h}(\mathbb{F}_q) - 1) &= q^2 + \frac{g(\varphi)^2 \varphi(-f)}{q-1} \sum_{l=0}^{q-2} \frac{g(T^{-l})^2 g(T^{4l})}{g(T^{2l})} T^l \left(\frac{h}{16f^2} \right) \\ &= q^2 + \frac{q\varphi(f)}{q-1} \sum_{l=0}^{q-2} \frac{g(T^{-l})^2 g(T^{4l})}{g(T^{2l})} T^l \left(\frac{h}{16f^2} \right), \end{aligned}$$

where the last equality follows from Lemma 1.2. Using the relation $a_q(E_{f,h}) = q + 1 - \#E_{f,h}(\mathbb{F}_q)$, we have

$$a_q(E_{f,h}) = -\frac{\varphi(f)}{q-1} \sum_{l=0}^{q-2} \frac{g(T^{-l})^2 g(T^{4l})}{g(T^{2l})} T^l \left(\frac{h}{16f^2} \right).$$

Hence, we obtain

$$a_q(E_{f,h}) + a_q(E_{f,-h}) = -\frac{\varphi(f)}{q-1}\sum_{l=0}^{q-2}\frac{g(T^{-l})^2 g(T^{4l})}{g(T^{2l})}T^l\left(\frac{h}{16f^2}\right)(1+T^l(-1)).$$

We know that $T^l(-1)=1$ if l is even and $T^l(-1)=-1$ if l is odd. Thus, we obtain

$$a_q(E_{f,h}) + a_q(E_{f,-h}) = -\frac{2\varphi(f)}{q-1}\sum_{l=0}^{\frac{q-3}{2}}\frac{g(T^{-2l})^2 g(T^{8l})}{g(T^{4l})}T^l\left(\frac{h^2}{16^2f^4}\right).$$

Taking $T=\overline{\omega}$ and then using Gross-Koblitz formula, we obtain

$$\begin{aligned} a_q(E_{f,h}) + a_q(E_{f,-h}) &= -\frac{2\varphi(f)}{q-1}\sum_{l=0}^{\frac{q-3}{2}}\overline{\omega}^l\left(\frac{h^2}{16^2f^4}\right)(-p)^{\sum_{i=0}^{r-1}\left(2\left\langle-\frac{2lp^i}{q-1}\right\rangle+\left\langle\frac{8lp^i}{q-1}\right\rangle-\left\langle\frac{4lp^i}{q-1}\right\rangle\right)} \\ &\quad\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle-\frac{2lp^i}{q-1}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{8lp^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{4lp^i}{q-1}\right\rangle\right)} \\ &= -\frac{2\varphi(f)}{q-1}\sum_{l=0}^{\frac{q-3}{2}}\overline{\omega}^l\left(\frac{h^2}{16^2f^4}\right)(-p)^{\sum_{i=0}^{r-1}\left(-2\left\lfloor-\frac{2lp^i}{q-1}\right\rfloor-\left\lfloor\frac{8lp^i}{q-1}\right\rfloor+\left\lfloor\frac{4lp^i}{q-1}\right\rfloor\right)} \\ &\quad\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle-\frac{2lp^i}{q-1}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{8lp^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{4lp^i}{q-1}\right\rangle\right)}. \end{aligned}$$

Taking out the term under the summation for $l=0$ and then using Lemma 1.11 with $t=2$, Lemma 1.10 with $t=8$ and $t=4$, Lemma 1.13 with $d=2$, Lemma 1.14 with $l=8$ and $l=4$ and the fact that

$$\prod_{h=0}^{3}\Gamma_p\left(\left\langle\frac{(2h+1)p^i}{8}+\frac{lp^i}{q-1}\right\rangle\right)=\prod_{h=0}^{3}\Gamma_p\left(\left\langle\frac{-(2h+1)p^i}{8}+\frac{lp^i}{q-1}\right\rangle\right),$$

we obtain

$$a_q(E_{f,h}) + a_q(E_{f,-h}) = -\frac{2\varphi(f)}{q-1}-\frac{2\varphi(f)}{q-1}\sum_{l=1}^{\frac{q-3}{2}}\overline{\omega}^l\left(\frac{16h^2}{f^4}\right)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,l}}\prod_{i=0}^{r-1}N_{i,l},$$

where

$$N_{i,l}=\frac{\Gamma_p\left(\left\langle\frac{p^i}{2}-\frac{lp^i}{q-1}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{-lp^i}{q-1}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{-p^i}{8}+\frac{lp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-3p^i}{8}+\frac{lp^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{-p^i}{8}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-3p^i}{8}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-5p^i}{8}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-7p^i}{8}\right\rangle\right)}$$
$$\times\Gamma_p\left(\left\langle\frac{-5p^i}{8}+\frac{lp^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-7p^i}{8}+\frac{lp^i}{q-1}\right\rangle\right)$$

and

$$\alpha_{i,l}=-2\left\lfloor\left\langle\frac{p^i}{2}\right\rangle-\frac{lp^i}{q-1}\right\rfloor-2\left\lfloor\frac{-lp^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{-p^i}{8}\right\rangle+\frac{lp^i}{q-1}\right\rfloor$$
$$-\left\lfloor\left\langle\frac{-3p^i}{8}\right\rangle+\frac{lp^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{-5p^i}{8}\right\rangle+\frac{lp^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{-7p^i}{8}\right\rangle+\frac{lp^i}{q-1}\right\rfloor.$$

Thus, we can write

$$\begin{aligned}a_q(E_{f,h})+a_q(E_{f,-h})&=-\frac{2\varphi(f)}{q-1}\sum_{l=0}^{\frac{q-3}{2}}\overline{\omega}^l\left(\frac{16h^2}{f^4}\right)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,l}}\prod_{i=0}^{r-1}N_{i,l}\\
&=-\frac{\varphi(f)}{q-1}\sum_{l=0}^{\frac{q-3}{2}}\overline{\omega}^l\left(\frac{16h^2}{f^4}\right)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,l}}\prod_{i=0}^{r-1}N_{i,l}\\
&\quad-\frac{\varphi(f)}{q-1}\sum_{l=0}^{\frac{q-3}{2}}\overline{\omega}^l\left(\frac{16h^2}{f^4}\right)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,l}}\prod_{i=0}^{r-1}N_{i,l}\\
&=-\frac{\varphi(f)}{q-1}\sum_{l=0}^{\frac{q-3}{2}}\overline{\omega}^l\left(\frac{16h^2}{f^4}\right)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,l}}\prod_{i=0}^{r-1}N_{i,l}\\
&\quad-\frac{\varphi(f)}{q-1}\sum_{l=\frac{q-1}{2}}^{q-2}\overline{\omega}^l\left(\frac{16h^2}{f^4}\right)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,l-\frac{q-1}{2}}}\prod_{i=0}^{r-1}N_{i,l-\frac{q-1}{2}}.\end{aligned}$$

It is easy to see that $N_{i,l-\frac{q-1}{2}}=N_{i,l}$ and $\alpha_{i,l-\frac{q-1}{2}}=\alpha_{i,l}$. Hence, we obtain

$$a_q(E_{f,h})+a_q(E_{f,-h})=-\frac{\varphi(f)}{q-1}\sum_{l=0}^{q-2}\overline{\omega}^l\left(\frac{16h^2}{f^4}\right)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,l}}\prod_{i=0}^{r-1}N_{i,l}$$

$$= \varphi(f) \cdot {}_4G_4 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{8}, & \frac{3}{8}, & \frac{5}{8}, & \frac{7}{8} \end{matrix} \mid \frac{16h^2}{f^4} \right]_q .$$

This completes the proof of the theorem. ■

In our next theorem, we express the sum of traces of Frobenius of a family of elliptic curves in terms of ${}_6G_6[\cdots]_q$ for all $q = p^r, r \geq 1$, where $p > 3$.

Theorem 4.5. *Let $p > 3$ be a prime and $q = p^r, r \geq 1$. Let $E_{c,d} : y^2 = x^3 + cx^2 + d$ and $E_{c,-d} : y^2 = x^3 + cx^2 - d$ be elliptic curves over $\mathbb{F}_q$ such that $c, d \in \mathbb{F}_q^{\times}$. Then we have*

$$a_q(E_{c,d}) + a_q(E_{c,-d}) = \varphi(c) \cdot {}_6G_6 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{12}, & \frac{1}{4}, & \frac{5}{12}, & \frac{7}{12}, & \frac{3}{4}, & \frac{11}{12} \end{matrix} \mid \frac{729d^2}{16c^6} \right]_q$$
$$- \varphi(d) - \varphi(-d).$$

Proof. From [9, Eq. 3.4], we obtain

$$q(\#E_{c,d}(\mathbb{F}_q) - 1) = q^2 - 1 + B + C, \tag{4.5}$$

where

$$B = 1 + q\varphi(d),$$
$$C = \frac{\varphi(-c)g(\varphi)}{q-1} \sum_{n=0}^{q-2} g(T^{-2n}) g(T^{3n+\frac{q-1}{2}}) g(T^{-n}) T^n \left(\frac{d}{c^3}\right).$$

Using Davenport-Hasse relation with $m = 2$ and $\psi = T^{3n}$, we have

$$g(T^{3n+\frac{q-1}{2}}) = \frac{g(T^{6n}) g(\varphi) T^{-3n}(4)}{g(T^{3n})}. \tag{4.6}$$

Substituting (4.6) in the expression of C yields

$$\begin{aligned} C &= \frac{\varphi(-c)g(\varphi)}{q-1}\sum_{n=0}^{q-2}\frac{g(T^{-n})g(T^{-2n})g(T^{6n})g(\varphi)}{g(T^{3n})}T^n\left(\frac{d}{4^3c^3}\right) \\ &= \frac{q\varphi(c)}{q-1}\sum_{n=0}^{q-2}\frac{g(T^{-n})g(T^{-2n})g(T^{6n})}{g(T^{3n})}T^n\left(\frac{d}{4^3c^3}\right), \end{aligned}$$

where the last equality follows from Lemma 1.2. Now, substituting the values of B and C in (4.5), we have

$$q(\#E_{c,d}(\mathbb{F}_q)-1) = q^2 + q\varphi(d) + \frac{q\varphi(c)}{q-1}\sum_{n=0}^{q-2}\frac{g(T^{-n})g(T^{-2n})g(T^{6n})}{g(T^{3n})}T^n\left(\frac{d}{4^3c^3}\right).$$

Using the relation $a_q(E_{c,d}) = q+1-\#E_{c,d}(\mathbb{F}_q)$, we obtain

$$a_q(E_{c,d}) = -\varphi(d) - \frac{\varphi(c)}{q-1}\sum_{n=0}^{q-2}\frac{g(T^{-n})g(T^{-2n})g(T^{6n})}{g(T^{3n})}T^n\left(\frac{d}{4^3c^3}\right).$$

Hence, we have

$$\begin{aligned} a_q(E_{c,d}) + a_q(E_{c,-d}) = &-\frac{\varphi(c)}{q-1}\sum_{n=0}^{q-2}\frac{g(T^{-n})g(T^{-2n})g(T^{6n})}{g(T^{3n})}T^n\left(\frac{d}{4^3c^3}\right)(T^n(-1)+1) \\ &-\varphi(d)-\varphi(-d). \end{aligned}$$

Using the fact that $T^n(-1) = -1$ if n is odd and $T^n(-1) = 1$ if n is even, we deduce that

$$\begin{aligned} a_q(E_{c,d}) + a_q(E_{c,-d}) = &-\frac{2\varphi(c)}{q-1}\sum_{n=0}^{\frac{q-3}{2}}\frac{g(T^{-2n})g(T^{-4n})g(T^{12n})}{g(T^{6n})}T^n\left(\frac{d^2}{4^6c^6}\right) \\ &-\varphi(d)-\varphi(-d). \end{aligned}$$

Taking $T = \overline{\omega}$ and then using Gross-Koblitz formula, we obtain

$$\begin{aligned}
&a_q(E_{c,d}) + a_q(E_{c,-d})\\
&= -\frac{2\varphi(c)}{q-1}\sum_{n=0}^{\frac{q-3}{2}}(-p)^{\sum_{i=0}^{r-1}\left(\left\langle -\frac{2np^i}{q-1}\right\rangle+\left\langle -\frac{4np^i}{q-1}\right\rangle+\left\langle \frac{12np^i}{q-1}\right\rangle-\left\langle \frac{6np^i}{q-1}\right\rangle\right)}\overline{\omega}^n\left(\frac{d^2}{4^6c^6}\right)\\
&\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle -\frac{2np^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle -\frac{4np^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{12np^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{6np^i}{q-1}\right\rangle\right)} - \varphi(d) - \varphi(-d)\\
&= -\frac{2\varphi(c)}{q-1} - \frac{2\varphi(c)}{q-1}\sum_{n=1}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{d^2}{4^6c^6}\right)(-p)^{\sum_{i=0}^{r-1}\left(-\left\lfloor -\frac{2np^i}{q-1}\right\rfloor-\left\lfloor -\frac{4np^i}{q-1}\right\rfloor-\left\lfloor \frac{12np^i}{q-1}\right\rfloor+\left\lfloor \frac{6np^i}{q-1}\right\rfloor\right)}\\
&\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle -\frac{2np^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle -\frac{4np^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{12np^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{6np^i}{q-1}\right\rangle\right)} - \varphi(d) - \varphi(-d).
\end{aligned}$$

Using Lemma 1.11 with $t = 2$ and $t = 4$, Lemma 1.10 with $t = 12$ and $t = 6$, Lemma 1.13 with $d = 2$ and $d = 4$, Lemma 1.14 with $l = 12$ and $l = 6$, and the fact that

$$\prod_{h=0}^{5}\Gamma_p\left(\left\langle\frac{(2h+1)p^i}{12}+\frac{np^i}{q-1}\right\rangle\right) = \prod_{h=0}^{5}\Gamma_p\left(\left\langle\frac{-(2h+1)p^i}{12}+\frac{np^i}{q-1}\right\rangle\right),$$

we obtain

$$\begin{aligned}
a_q(E_{c,d}) + a_q(E_{c,-d}) = &-\frac{2\varphi(c)}{q-1}\sum_{n=1}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}(\beta_{i,n}+\gamma_{i,n})}\prod_{i=0}^{r-1}A_{i,n}B_{i,n}\\
&-\varphi(d) - \varphi(-d) - \frac{2\varphi(c)}{q-1},
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
\beta_{i,n} = -2&\left\lfloor\left\langle\frac{p^i}{2}\right\rangle-\frac{np^i}{q-1}\right\rfloor - 2\left\lfloor\frac{-np^i}{q-1}\right\rfloor - \left\lfloor\left\langle\frac{-p^i}{12}\right\rangle+\frac{np^i}{q-1}\right\rfloor\\
&-\left\lfloor\left\langle\frac{-5p^i}{12}\right\rangle+\frac{np^i}{q-1}\right\rfloor - \left\lfloor\left\langle\frac{-7p^i}{12}\right\rangle+\frac{np^i}{q-1}\right\rfloor - \left\lfloor\left\langle\frac{-11p^i}{12}\right\rangle+\frac{np^i}{q-1}\right\rfloor,
\end{aligned}$$

$$\gamma_{i,n} = -\left\lfloor \left\langle \frac{-p^i}{4} \right\rangle + \frac{np^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{3p^i}{4} \right\rangle - \frac{np^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{p^i}{4} \right\rangle - \frac{np^i}{q-1} \right\rfloor$$
$$- \left\lfloor \left\langle \frac{-3p^i}{4} \right\rangle + \frac{np^i}{q-1} \right\rfloor,$$
$$A_{i,n} = \frac{\Gamma_p\left(\left\langle \frac{p^i}{2} - \frac{np^i}{q-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-np^i}{q-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-p^i}{12} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-5p^i}{12} + \frac{np^i}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{2} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-p^i}{12} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-5p^i}{12} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-7p^i}{12} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-11p^i}{12} \right\rangle\right)}$$
$$\times \Gamma_p\left(\left\langle \frac{-7p^i}{12} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-11p^i}{12} + \frac{np^i}{q-1} \right\rangle\right),$$
$$B_{i,n} = \frac{\Gamma_p\left(\left\langle \frac{-p^i}{4} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-3p^i}{4} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{p^i}{4} - \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{3p^i}{4} - \frac{np^i}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{4} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{3p^i}{4} \right\rangle\right)^2}.$$

Adding and subtracting the term under the summation for $n = 0$ yields

$$a_q(E_{c,d}) + a_q(E_{c,-d})$$
$$= -\frac{2\varphi(c)}{q-1} \sum_{n=0}^{\frac{q-3}{2}} \overline{\omega}^n \left(\frac{729d^2}{16c^6}\right) (-p)^{\sum_{i=0}^{r-1}(\beta_{i,n}+\gamma_{i,n})} \prod_{i=0}^{r-1} A_{i,n}B_{i,n}$$
$$- \varphi(d) - \varphi(-d)$$
$$= -\frac{\varphi(c)}{q-1} \sum_{n=0}^{\frac{q-3}{2}} \overline{\omega}^n \left(\frac{729d^2}{16c^6}\right) (-p)^{\sum_{i=0}^{r-1}(\beta_{i,n}+\gamma_{i,n})} \prod_{i=0}^{r-1} A_{i,n}B_{i,n} - \varphi(d) - \varphi(-d)$$
$$- \frac{\varphi(c)}{q-1} \sum_{n=0}^{\frac{q-3}{2}} \overline{\omega}^n \left(\frac{729d^2}{16c^6}\right) (-p)^{\sum_{i=0}^{r-1}(\beta_{i,n}+\gamma_{i,n})} \prod_{i=0}^{r-1} A_{i,n}B_{i,n}$$
$$= -\frac{\varphi(c)}{q-1} \sum_{n=0}^{\frac{q-3}{2}} \overline{\omega}^n \left(\frac{729d^2}{16c^6}\right) (-p)^{\sum_{i=0}^{r-1}(\beta_{i,n}+\gamma_{i,n})} \prod_{i=0}^{r-1} A_{i,n}B_{i,n} - \varphi(d) - \varphi(-d)$$
$$- \frac{\varphi(c)}{q-1} \sum_{n=\frac{q-1}{2}}^{q-2} \overline{\omega}^n \left(\frac{729d^2}{16c^6}\right) (-p)^{\sum_{i=0}^{r-1}\left(\beta_{i,n-\frac{q-1}{2}}+\gamma_{i,n-\frac{q-1}{2}}\right)} \prod_{i=0}^{r-1} A_{i,n-\frac{q-1}{2}}B_{i,n-\frac{q-1}{2}}.$$

We can easily check that $A_{i,n-\frac{q-1}{2}} = A_{i,n}, B_{i,n-\frac{q-1}{2}} = B_{i,n}, \beta_{i,n-\frac{q-1}{2}} = \beta_{i,n}$, and

$\gamma_{i,n-\frac{q-1}{2}} = \gamma_{i,n}$. Thus, we can write

$$a_q(E_{c,d}) + a_q(E_{c,-d}) = -\frac{\varphi(c)}{q-1}\sum_{n=0}^{q-2}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}(\beta_{i,n}+\gamma_{i,n})}\prod_{i=0}^{r-1}A_{i,n}B_{i,n}$$
$$- \varphi(d) - \varphi(-d)$$
$$= \varphi(c)\cdot {}_6G_6\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{12}, & \frac{1}{4}, & \frac{5}{12}, & \frac{7}{12}, & \frac{3}{4}, & \frac{11}{12}\end{matrix} \mid \frac{729d^2}{16c^6}\right]_q$$
$$- \varphi(d) - \varphi(-d).$$

This completes the proof of the theorem. ■

In [69, Theorem 1.7], the sum of traces of Frobenius of the elliptic curves $E_{c,d}: y^2 = x^3+cx^2+d$ and $E_{c,-d}: y^2 = x^3+cx^2-d$ is expressed as a special value of a ${}_4F_3$-Gaussian hypergeometric series under the condition that $q = p^r \equiv 1 \pmod{12}$. In Theorem 4.5, we have expressed the sum of traces of Frobenius for the same families of elliptic curves in terms of a ${}_6G_6[\cdots]_q$ hypergeometric function. In the following theorem, we express the sum in terms of a ${}_4G_4[\cdots]_q$ hypergeometric function which extends [69, Theorem 1.7].

Theorem 4.6. *Let $p > 3$ be a prime and $q = p^r, r \geq 1$. Let $E_{c,d}: y^2 = x^3 + cx^2 + d$ and $E_{c,-d}: y^2 = x^3 + cx^2 - d$ be elliptic curves over $\mathbb{F}_q$ such that $c, d \in \mathbb{F}_q^\times$. Then the following statements hold:*

1. *If $q \equiv 1, 7 \pmod{12}$, then we have*

$$a_q(E_{c,d}) + a_q(E_{c,-d}) = \varphi(-3c)\cdot {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{12}, & \frac{5}{12}, & \frac{7}{12}, & \frac{11}{12}\end{matrix} \mid \frac{729d^2}{16c^6}\right]_q.$$

2. If $q \equiv 5 \pmod{12}$*, then we have*

$$a_q(E_{c,d}) + a_q(E_{c,-d}) = \varphi(c) \cdot {}_4G_4 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{12}, & \frac{5}{12}, & \frac{7}{12}, & \frac{11}{12} \end{matrix} \mid \frac{729d^2}{16c^6} \right]_q .$$

3. If $p \equiv 11 \pmod{12}$ *and* $r = 1$*, then we have*

$$a_q(E_{c,d}) + a_q(E_{c,-d}) = \varphi(c) \cdot {}_4G_4 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{12}, & \frac{5}{12}, & \frac{7}{12}, & \frac{11}{12} \end{matrix} \mid \frac{729d^2}{16c^6} \right]_p .$$

Proof. Part (1): Here, $q \equiv 1, 7 \pmod{12}$. Let χ_6 be a character of order 6. Then, by [9, Theorem 3.1] and then using Proposition 1.7 and Lemma 1.9, we have

$$\begin{aligned} a_q(E_{c,d}) &= -q \cdot \varphi(-3c) \cdot {}_2F_1 \left(\begin{matrix} \chi_6, & \chi_6^5 \\ & \varepsilon \end{matrix} \mid -\frac{27d}{4c^3} \right)_q \\ &= -q \cdot \varphi(-3c) \binom{\chi_6^5}{\varepsilon} {}_2F_1 \left(\begin{matrix} \chi_6, & \chi_6^5 \\ & \varepsilon \end{matrix} \mid -\frac{27d}{4c^3} \right)_q^* \\ &= \varphi(-3c) \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & 0 \end{matrix} \mid -\frac{4c^3}{27d} \right]_q \\ &= \varphi(-3c) \cdot {}_2G_2 \left[\begin{matrix} 0, & 0 \\ \frac{1}{6}, & \frac{5}{6} \end{matrix} \mid -\frac{27d}{4c^3} \right]_q . \end{aligned}$$

Hence,

$$\begin{aligned} &a_q(E_{c,d}) + a_q(E_{c,-d}) \\ &= \varphi(-3c) {}_2G_2 \left[\begin{matrix} 0, & 0 \\ \frac{1}{6}, & \frac{5}{6} \end{matrix} \mid -\frac{27d}{4c^3} \right]_q + \varphi(-3c) {}_2G_2 \left[\begin{matrix} 0, & 0 \\ \frac{1}{6}, & \frac{5}{6} \end{matrix} \mid \frac{27d}{4c^3} \right]_q . \end{aligned}$$

Using Theorem 2.9 with $a_1 = a_2 = 0$, $a_3 = \frac{1}{6}$, and $a_4 = \frac{5}{6}$, we deduce that

$$a_q(E_{c,d}) + a_q(E_{c,-d}) = \varphi(-3c)_4 G_4 \left[\begin{array}{cccc} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{12}, & \frac{5}{12}, & \frac{7}{12}, & \frac{11}{12} \end{array} | \frac{729d^2}{16c^6}\right].$$

This completes the proof of (1).

Part (2): Here, $q \equiv 5 \pmod{12}$. From (4.7), we have

$$\begin{aligned} a_q(E_{c,d}) + a_q(E_{c,-d}) = &-\frac{2\varphi(c)}{q-1} \sum_{n=1}^{\frac{q-3}{2}} \overline{\omega}^n \left(\frac{729d^2}{16c^6}\right) (-p)^{\sum_{i=0}^{r-1}(\beta_{i,n}+\gamma_{i,n})} \prod_{i=0}^{r-1} A_{i,n} B_{i,n} \\ &- \varphi(d) - \varphi(-d) - \frac{2\varphi(c)}{q-1}, \end{aligned}$$

where

$$\begin{aligned} \beta_{i,n} = &-2\left\lfloor \left\langle \frac{p^i}{2} \right\rangle - \frac{np^i}{q-1} \right\rfloor - 2\left\lfloor \frac{-np^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{-p^i}{12} \right\rangle + \frac{np^i}{q-1} \right\rfloor \\ &- \left\lfloor \left\langle \frac{-5p^i}{12} \right\rangle + \frac{np^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{-7p^i}{12} \right\rangle + \frac{np^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{-11p^i}{12} \right\rangle + \frac{np^i}{q-1} \right\rfloor, \\ \gamma_{i,n} = &-\left\lfloor \left\langle \frac{-p^i}{4} \right\rangle + \frac{np^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{3p^i}{4} \right\rangle - \frac{np^i}{q-1} \right\rfloor - \left\lfloor \left\langle \frac{p^i}{4} \right\rangle - \frac{np^i}{q-1} \right\rfloor \\ &- \left\lfloor \left\langle \frac{-3p^i}{4} \right\rangle + \frac{np^i}{q-1} \right\rfloor, \\ A_{i,n} = &\frac{\Gamma_p\left(\left\langle \frac{p^i}{2} - \frac{np^i}{q-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-np^i}{q-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-p^i}{12} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-5p^i}{12} + \frac{np^i}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{2} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{-p^i}{12} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-5p^i}{12} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-7p^i}{12} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-11p^i}{12} \right\rangle\right)} \\ &\times \Gamma_p\left(\left\langle \frac{-7p^i}{12} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-11p^i}{12} + \frac{np^i}{q-1} \right\rangle\right), \\ B_{i,n} = &\frac{\Gamma_p\left(\left\langle \frac{-p^i}{4} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{-3p^i}{4} + \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{p^i}{4} - \frac{np^i}{q-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{3p^i}{4} - \frac{np^i}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{4} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{3p^i}{4} \right\rangle\right)^2}. \end{aligned}$$

We can easily check that

$$\begin{aligned}\beta_{i,\frac{q-1}{4}}+\gamma_{i,\frac{q-1}{4}}&=0,\\ \prod_{i=0}^{r-1}A_{i,\frac{q-1}{4}}B_{i,\frac{q-1}{4}}&=\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle\frac{p^i}{6}\right\rangle\right)\Gamma_p\left(\left\langle\frac{5p^i}{6}\right\rangle\right)\Gamma_p\left(\left\langle\frac{p^i}{3}\right\rangle\right)\Gamma_p\left(\left\langle\frac{2p^i}{3}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{12}\right\rangle\right)\Gamma_p\left(\left\langle\frac{5p^i}{12}\right\rangle\right)\Gamma_p\left(\left\langle\frac{7p^i}{12}\right\rangle\right)\Gamma_p\left(\left\langle\frac{11p^i}{12}\right\rangle\right)}\\ &=\prod_{i=0}^{r-1}\frac{\varphi(6)\Gamma_p\left(\left\langle\frac{p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3p^i}{4}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)^2}\\ &=\varphi(6)\overline{\omega}^{\frac{q-1}{4}}(-1),\end{aligned}$$

where we used Lemma 1.10 with $t=6$ and $a=\frac{q-1}{4}$; and (1.10) with $a=\frac{q-1}{4}$ and $a=\frac{q-1}{2}$ to deduce the last two equations, respectively. Taking out the term under the summation for $n=\frac{q-1}{4}$ and then using Lemma 4.1 yield

$$\begin{aligned}a_q(E_{c,d})+a_q(E_{c,-d})=&-\frac{2\varphi(c)}{q-1}\sum_{n=1,n\neq\frac{q-1}{4}}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n}}\prod_{i=0}^{r-1}A_{i,n}\\ &-\varphi(d)-\varphi(-d)-\frac{2\varphi(c)}{q-1}-\frac{2\varphi(2d)}{q-1}\overline{\omega}^{\frac{q-1}{4}}(-1).\end{aligned}$$

Since $q\equiv 5 \pmod{12}$, so we have $p\equiv 5 \pmod{12}$ and hence either $p\equiv 1 \pmod 8$ or $p\equiv 5 \pmod 8$. Since $p\equiv 1 \pmod 4$, therefore there exists an element $y\in\mathbb{F}_p$ such that $y^2=-1$. If $p\equiv 1 \pmod 8$, then 2 and y are squares in $\mathbb{F}_p$. If $p\equiv 5 \pmod 8$, then 2 and y are nonsquares in $\mathbb{F}_p$. Hence, $\varphi(2y)=1$ for primes $p\equiv 5 \pmod{12}$. Thus, we have

$$\begin{aligned}a_q(E_{c,d})+a_q(E_{c,-d})=&-\frac{2\varphi(c)}{q-1}\sum_{n=1,n\neq\frac{q-1}{4}}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n}}\prod_{i=0}^{r-1}A_{i,n}\\ &-\varphi(d)-\varphi(-d)-\frac{2\varphi(c)}{q-1}-\frac{2\varphi(d)}{q-1}.\end{aligned}$$

We can easily check that $\beta_{i,\frac{q-1}{4}}=1$. Using Lemma 1.10 with $t=6$ and $a=\frac{q-1}{4}$, we obtain

$$\prod_{i=0}^{r-1} A_{i,\frac{q-1}{4}} = \varphi(6)\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle\frac{p^i}{4}\right\rangle\right)^3\Gamma_p\left(\left\langle\frac{3p^i}{4}\right\rangle\right)^3}{\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)^4}.$$

Using Proposition 2.10 and (1.10) with $a=\frac{q-1}{4}$, we deduce that

$$\prod_{i=0}^{r-1} A_{i,\frac{q-1}{4}} = \varphi(6)(-1)^r\overline{\omega}^{\frac{q-1}{4}}(-1).$$

Adding and subtracting the term under the summation for $n=0$ and $n=\frac{q-1}{4}$ and then using the fact that $\varphi(-1)=1$ yield

$$\begin{aligned}
&a_q(E_{c,d})+a_q(E_{c,-d})\\
&=-\frac{2\varphi(c)}{q-1}\sum_{n=0}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n}}\prod_{i=0}^{r-1}A_{i,n}-2\varphi(d)-\frac{2\varphi(d)}{q-1}\\
&\quad+\frac{2q\varphi(2d)}{q-1}\overline{\omega}^{\frac{q-1}{4}}(-1)\\
&=-\frac{\varphi(c)}{q-1}\sum_{n=0}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n}}\prod_{i=0}^{r-1}A_{i,n}\\
&\quad-\frac{\varphi(c)}{q-1}\sum_{n=0}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n}}\prod_{i=0}^{r-1}A_{i,n}-\frac{2q\varphi(d)}{q-1}+\frac{2q\varphi(2d)}{q-1}\overline{\omega}^{\frac{q-1}{4}}(-1)\\
&=-\frac{\varphi(c)}{q-1}\sum_{n=0}^{\frac{q-3}{2}}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n}}\prod_{i=0}^{r-1}A_{i,n}\\
&\quad-\frac{\varphi(c)}{q-1}\sum_{n=\frac{q-1}{2}}^{q-2}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n-\frac{q-1}{2}}}\prod_{i=0}^{r-1}A_{i,n-\frac{q-1}{2}}.
\end{aligned}$$

We can easily check that $\beta_{i,n-\frac{q-1}{2}}=\beta_{i,n}$ and $A_{i,n-\frac{q-1}{2}}=A_{i,n}$. Thus, we obtain

$$a_q(E_{c,d})+a_q(E_{c,-d})=-\frac{\varphi(c)}{q-1}\sum_{n=0}^{q-2}\overline{\omega}^n\left(\frac{729d^2}{16c^6}\right)(-p)^{\sum_{i=0}^{r-1}\beta_{i,n}}\prod_{i=0}^{r-1}A_{i,n}$$

$$= \varphi(c) \cdot {}_4G_4 \left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{12}, & \frac{5}{12}, & \frac{7}{12}, & \frac{11}{12} \end{matrix} \Big| \frac{729d^2}{16c^6} \right]_q .$$

This completes the proof of (2).

Part (3): Here, $p \equiv 11 \pmod{12}$ and $r = 1$, i.e., $q = p$. Using Theorem 2.12 with $d = 4$ and the fact that $\varphi(-1) = -1$ for $p \equiv 3 \pmod 4$, we obtain the required result. ■

4.3 The trace of Frobenius of elliptic curves defined over $\mathbb{Q}$

In Theorems 4.2, 4.3, 4.4, 4.5, and 4.6, we expressed the sum of traces of Frobenius of certain families of elliptic curves defined over $\mathbb{F}_q$ as special values of ${}_4G_4[\cdots]_q$ and ${}_6G_6[\cdots]_q$ hypergeometric functions. For certain elliptic curves defined over $\mathbb{Q}$, it is possible to find the value of one of the traces of Frobenius appearing in the sums explicitly. This allows us to write the other trace of Frobenius appearing in the sums as special values of ${}_4G_4[\cdots]_q$ and ${}_6G_6[\cdots]_q$ hypergeometric functions as stated in the theorems of this section.

Before we state our results, we recall a definition. For a nonzero integer n, we can write $n = p^m k$, where $\gcd(p, k) = 1$. We then define $\text{ord}_p(n) = m$ and $\text{ord}_p(0) = \infty$. If $a = \frac{x}{y} \in \mathbb{Q}$, then we define $\text{ord}_p(a) = \text{ord}_p(x) - \text{ord}_p(y)$.

We first recall a recurrence relation satisfied by the trace of the Frobenius endomorphism of elliptic curves. Let E be an elliptic curve defined over $\mathbb{Q}$ in the Weierstrass form. If E is an elliptic curve over defined $\mathbb{Q}$ with conductor N_E, then by modularity theorem, there exists a newform f of weight two and level N_E whose Fourier coefficients are given by the coefficients of the Hasse-Weil L-series $L(E, s)$

of E given by

$$L(E,s)=\sum_{n=1}^{\infty}\frac{a_n(E)}{n^s}.$$

The p-th trace of the Frobenius endomorphism $a_p(E)$ is the p-th coefficient of $L(E,s)$. Let $\mathbf{1}_E(p)$ be the trivial character modulo conductor N_E, that is, $\mathbf{1}_E(p)$ is 1 for primes of good reduction and 0 for primes of bad reduction. Then, the p^r-th trace of Frobenius endomorphism satisfies the following recurrence relation [25, Eq. 8.21]

$$a_{p^r}(E)=a_p(E)a_{p^{r-1}}(E)-p\mathbf{1}_E(p)a_{p^{r-2}}(E), \tag{4.8}$$

where $r\geq 2$ and $a_1(E)=1$.

The following theorem gives a p-adic analogue of [69, Theorem 1.8]. We note that [69, Theorem 1.8] is holds for all p^r with r even and $p\geq 5$ satisfying $p\equiv 3 \pmod 4$. Our result holds for odd values of r as well.

Theorem 4.7. *Let $E_{-\lambda}: y^2=x(x-1)(x+\lambda)$ be an elliptic curve over $\mathbb{Q}$, where $\lambda\in\{2,\frac{1}{2}\}$. If $p\geq 5$ is a prime such that $p\equiv 3 \pmod 4$, then*

$$a_{p^r}(E_{-\lambda})=\begin{cases}-{}_4G_4\left[\begin{matrix}0, & \frac{1}{2}, & 0, & \frac{1}{2}\\ \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4}\end{matrix}\Big|\lambda^2\right]_{p^r}, & \text{if } r \text{ is odd;}\\ -(-p)^{\frac{r}{2}}+{}_4G_4\left[\begin{matrix}0, & \frac{1}{2}, & 0, & \frac{1}{2}\\ \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4}\end{matrix}\Big|\lambda^2\right]_{p^r}, & \text{if } r \text{ is even.}\end{cases}$$

Proof. Let $E_\lambda: y^2=x(x-1)(x-\lambda)$ be an elliptic curve over $\mathbb{Q}$ such that $\lambda\neq 0,\pm 1$. From [69, Eq. 11.3], for $p\equiv 3 \pmod 4$, we have

$$a_p(E_{\frac{1}{2}})=0=a_p(E_2).$$

Using (4.8) with the fact that $a_1 = 1$, for $\lambda = 2, \frac{1}{2}$, we obtain

$$a_{p^r}(E_\lambda) = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ (-p)^{\frac{r}{2}}, & \text{if } r \text{ is even.} \end{cases} \tag{4.9}$$

Now, using (4.9) and Theorem 4.2, we obtain the desired result. ■

The following theorem gives a p-adic analogue of [69, Theorem 1.10] which extends [69, Theorem 1.10] to all the odd values of r.

Theorem 4.8. *Let $E_{\alpha,\frac{-\alpha^3}{24}} : y^2 + \alpha xy - \frac{\alpha^3}{24}y = x^3$ be an elliptic curve over $\mathbb{Q}$, where $\alpha \neq 0$. For primes $p \equiv 5, 11 \pmod{12}$ such that $p \neq 17$ and $ord_p(\alpha) = 0$, we have*

$$a_{p^r}(E_{\alpha,-\frac{\alpha^3}{24}}) = \begin{cases} {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6} \end{matrix} \mid \frac{81}{64}\right]_{p^r}, & \text{if } r \text{ is odd;} \\ -(-p)^{\frac{r}{2}} + {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6} \end{matrix} \mid \frac{81}{64}\right]_{p^r}, & \text{if } r \text{ is even.} \end{cases}$$

Proof. Let $E_{\alpha,\frac{\alpha^3}{24}} : y^2 + \alpha xy + \frac{\alpha^3}{24}y = x^3$ be an elliptic curve over $\mathbb{Q}$, where $\alpha \neq 0$. From [69, Eq. 13.1], for $p \equiv 5, 11 \pmod{12}$, we have

$$a_p(E_{\alpha,\frac{\alpha^3}{24}}) = 0.$$

Using (4.8) with the fact that $a_1 = 1$, we obtain

$$a_{p^r}(E_{\alpha,\frac{\alpha^3}{24}}) = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ (-p)^{\frac{r}{2}}, & \text{if } r \text{ is even.} \end{cases} \tag{4.10}$$

Now, using (4.10) and Theorem 4.3, we obtain the desired result. ■

The following theorem gives a p-adic analogue of [69, Theorem 1.9] which extends [69, Theorem 1.9] to all the odd values of r.

Theorem 4.9. *Let $E_{\alpha,\frac{-\alpha^2}{3}}: y^2 = x^3 + \alpha x^2 - \frac{\alpha^2}{3}x$ be an elliptic curve over $\mathbb{Q}$, where $\alpha \neq 0$. For primes $p \equiv 5, 11 \pmod{12}$ such that $ord_p(\alpha) = 0$, we have*

$$a_{p^r}(E_{\alpha,\frac{-\alpha^2}{3}}) = \begin{cases} \varphi(\alpha) \cdot {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{8}, & \frac{3}{8}, & \frac{5}{8}, & \frac{7}{8} \end{matrix} | \frac{16}{9}\right]_{p^r}, & \text{if } r \text{ is odd;} \\ -(-p)^{\frac{r}{2}} + \varphi(\alpha) \cdot {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{8}, & \frac{3}{8}, & \frac{5}{8}, & \frac{7}{8} \end{matrix} | \frac{16}{9}\right]_{p^r}, & \text{if } r \text{ is even.} \end{cases}$$

Proof. Let $E_{\alpha,\frac{\alpha^2}{3}}: y^2 = x^3 + \alpha x^2 + \frac{\alpha^2}{3}x$ be an elliptic curve over $\mathbb{Q}$, where $\alpha \neq 0$. From [69, Eq. 12.1], for $p \equiv 5, 11 \pmod{12}$, we have

$$a_p(E_{\alpha,\frac{\alpha^2}{3}}) = 0.$$

Using (4.8) with the fact that $a_1 = 1$, we obtain

$$a_{p^r}(E_{\alpha,\frac{\alpha^2}{3}}) = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ (-p)^{\frac{r}{2}}, & \text{if } r \text{ is even.} \end{cases} \tag{4.11}$$

Now, using (4.11) and Theorem 4.4, we obtain the desired result. ∎

In the following theorem, we express the trace of Frobenius of certain elliptic curves as a special value of a ${}_6G_6[\cdots]_q$ hypergeometric function.

Theorem 4.10. *Let $E_{\alpha,\frac{2\alpha^3}{27}}: y^2 = x^3 + \alpha x^2 + \frac{2\alpha^3}{27}$ be an elliptic curve over $\mathbb{Q}$ such that $\alpha \neq 0$. Let $p \equiv 7, 11 \pmod{12}$ be a prime such that $ord_p(\alpha) = 0$.*

1. *If r is odd, then we have*

$$a_{p^r}(E_{\alpha,\frac{2\alpha^3}{27}}) = \varphi(\alpha) \cdot {}_6G_6\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{12}, & \frac{1}{4}, & \frac{5}{12}, & \frac{7}{12}, & \frac{3}{4}, & \frac{11}{12} \end{matrix} | \frac{1}{4}\right]_{p^r}.$$

2. *If r is even, then we have*

$$a_{p^r}(E_{\alpha,\frac{2\alpha^3}{27}}) = \varphi(\alpha)\cdot {}_6G_6\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{12}, & \frac{1}{4}, & \frac{5}{12}, & \frac{7}{12}, & \frac{3}{4}, & \frac{11}{12} \end{matrix} |\frac{1}{4}\right]_{p^r} - 2\varphi(6\alpha) - (-p)^{\frac{r}{2}}.$$

Proof. Let $E_{\alpha,-\frac{2\alpha^3}{27}} : y^2 = x^3 + \alpha x^2 - \frac{2\alpha^3}{27}$ be an elliptic curve over $\mathbb{Q}$ such that $\alpha \neq 0$. From [69, Eq. 14.1], for $p \equiv 7, 11 \pmod{12}$, we have

$$a_p(E_{\alpha,\frac{-2\alpha^3}{27}}) = 0.$$

Using (4.8) with the fact that $a_1 = 1$, we obtain

$$a_{p^r}(E_{\alpha,\frac{-2\alpha^3}{27}}) = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ (-p)^{\frac{r}{2}}, & \text{if } r \text{ is even.} \end{cases} \tag{4.12}$$

Now, using (4.12) and Theorem 4.6, we obtain the desired result. ■

It is evident that the p-adic hypergeometric functions appearing in Theorems 4.7, 4.8, 4.9 and 4.10 have integer values. Using our main results, one can find special values of some of the p-adic hypergeometric functions. We state some of the values in the following corollary though many such values can be obtained.

Corollary 4.3.1. *We have:*

1.

$${}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4} \end{matrix} |4\right]_{1331} = 24.$$

2.

$$ {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{6}, & \frac{1}{3}, & \frac{2}{3}, & \frac{5}{6} \end{matrix} | \frac{81}{64}\right]_{1331} = -39. $$

3.

$$ {}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{8}, & \frac{3}{8}, & \frac{5}{8}, & \frac{7}{8} \end{matrix} | \frac{16}{9}\right]_{125} = 12\varphi(3). $$

4.

$$ {}_6G_6\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{12}, & \frac{1}{4}, & \frac{5}{12}, & \frac{7}{12}, & \frac{3}{4}, & \frac{11}{12} \end{matrix} | \frac{1}{4}\right]_{1331} = -36\varphi(3). $$

Proof. Taking $\lambda = 2$, $p = 11$, and $r = 3$ in Theorem 4.7, we obtain

$$ a_{11^3}(E_{-2}) = -{}_4G_4\left[\begin{matrix} 0, & \frac{1}{2}, & 0, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4}, & \frac{1}{4}, & \frac{3}{4} \end{matrix} | 4\right]_{1331}. \tag{4.13} $$

Using `SageMath`, we can easily find that $a_{11}(E_{-2}) = 4$ and then using (4.8) and (4.13), we obtain the required result (1). Taking $\alpha = 2$, $p = 11$ and $r = 3$ in Theorem 4.8 and following similar steps as shown in the proof of (1), we obtain (2). Similarly, taking $\alpha = 3$, $p = 5$ and $r = 3$ in Theorem 4.9, we prove (3). Finally, taking $\alpha = 3$, $p = 11$ and $r = 3$ in Theorem 4.10, we prove (4). ■

5

Summation Identities and Their Applications

5.1 Introduction

In view of the significant presence of Gaussian and p-adic hypergeometric functions in arithmetic geometry, it is an interesting problem to find transformation formulas and special values of these hypergeometric functions. In recent times, several transformation formulas and special values of the p-adic hypergeometric functions were found, see for example [11, 12, 13, 14, 17, 61, 62]. In this chapter, we prove

[1]The contents of this chapter have been published in *Finite Fields Appl.* (2024).

two summation identities for the p-adic hypergeometric functions appearing in the expressions for the number of $\mathbb{F}_q$-points on the diagonal hypersurface $D_\lambda^{d,k}$. As an application of the summation identities, we prove identities for the trace of Frobenius endomorphism on certain families of elliptic curves and p-adic hypergeometric functions.

5.2 Summation identities

In this section, we prove the summation identities for McCathy's p-adic hypergeometric functions. To prove these identities, we need two lemmas which connect the product of certain values of the p-adic gamma function to some character sums.

Lemma 5.1. *For $0 \leq a \leq q-2$, we have*

$$\prod_{i=0}^{r-1} \frac{\Gamma_p\left(\left\langle p^i - \frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} + \frac{ap^i}{q-1}\right\rangle\right)}{(-p)^{\left\lfloor \frac{1}{2}+\frac{ap^i}{q-1}\right\rfloor}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)} = -\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))\overline{\omega}^a(-t)\prod_{i=0}^{r-1}(-p)^{\left\lfloor \frac{-ap^i}{q-1}\right\rfloor}.$$

Proof. We have

$$\begin{aligned}
&\prod_{i=0}^{r-1} \frac{\Gamma_p\left(\left\langle p^i - \frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} + \frac{ap^i}{q-1}\right\rangle\right)}{(-p)^{\left\lfloor \frac{1}{2}+\frac{ap^i}{q-1}\right\rfloor}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)} \\
&= \prod_{i=0}^{r-1} \frac{(-p)^{\left\langle \frac{1}{2}+\frac{ap^i}{q-1}\right\rangle}\Gamma_p\left(\left\langle p^i - \frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} + \frac{ap^i}{q-1}\right\rangle\right)}{(-p)^{\frac{1}{2}+\frac{ap^i}{q-1}}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)} \\
&= \prod_{i=0}^{r-1} \frac{(-p)^{\left\langle \frac{p^i}{2}+\frac{ap^i}{q-1}\right\rangle}\Gamma_p\left(\left\langle p^i - \frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} + \frac{ap^i}{q-1}\right\rangle\right)}{(-p)^{\left\langle \frac{p^i}{2}\right\rangle}(-p)^{\frac{ap^i}{q-1}}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)} \\
&= \prod_{i=0}^{r-1} \frac{(-p)^{\left\langle \frac{p^i}{2}+\frac{ap^i}{q-1}\right\rangle}(-p)^{\left\langle -\frac{ap^i}{q-1}\right\rangle}\Gamma_p\left(\left\langle p^i - \frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} + \frac{ap^i}{q-1}\right\rangle\right)}{(-p)^{\left\langle \frac{p^i}{2}\right\rangle}(-p)^{\frac{ap^i}{q-1}+\left\langle -\frac{ap^i}{q-1}\right\rangle}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)}.
\end{aligned}$$

Using Gross-Koblitz formula, we obtain

$$\prod_{i=0}^{r-1} \frac{\Gamma_p\left(\left\langle p^i - \frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} + \frac{ap^i}{q-1}\right\rangle\right)}{(-p)^{\left\lfloor \frac{1}{2}+\frac{ap^i}{q-1}\right\rfloor}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)} = \frac{-g(\varphi\overline{\omega}^a)g(\omega^a)}{g(\varphi)}\prod_{i=0}^{r-1}(-p)^{\left\lfloor \frac{-ap^i}{q-1}\right\rfloor}. \tag{5.1}$$

Lemma 1.5 and (1.4) yield

$$\begin{aligned}\frac{-g(\varphi\overline{\omega}^a)g(\omega^a)}{g(\varphi)} &= -J(\varphi\overline{\omega}^a,\omega^a)\\ &= -\varphi\overline{\omega}^a(-1)J(\varphi\overline{\omega}^a,\varphi)\\ &= -\varphi\overline{\omega}^a(-1)\sum_{t\in\mathbb{F}_q}\varphi(1-t)\varphi\overline{\omega}^a(t)\\ &= -\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))\overline{\omega}^a(-t). \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we obtain the required result. ■

Lemma 5.2. *For* $0 \leq a \leq q-2$, *we have*

$$\prod_{i=0}^{r-1} \frac{\Gamma_p\left(\left\langle \frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} - \frac{ap^i}{q-1}\right\rangle\right)}{(-p)^{\left\lfloor \frac{1}{2}-\frac{ap^i}{q-1}\right\rfloor}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)} = -\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))\omega^a(-t)\prod_{i=0}^{r-1}(-p)^{\left\lfloor \frac{ap^i}{q-1}\right\rfloor}.$$

Proof. The proof is similar to that of Lemma 5.1. ■

Having Lemma 5.1 and Lemma 5.2 showed, we are ready to prove the summation identities.

Theorem 5.3. *Let* $d > k \geq 1$ *be odd integers. Let* p *be an odd prime such that* $p \nmid dk(d-k)$. *Let* $q = p^r, r \geq 1$. *Then, for* $x \in \mathbb{F}_q^{\times}$, *we have*

$$1 + q \cdot {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d} \\ 0, & \frac{1}{k}, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k} \end{matrix} \middle| x\right]_q$$

$$= -\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))\times$$

$$ {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{k+\frac{d-k}{2}-1}{d}, & \frac{k+\frac{d-k}{2}}{d}, & \ldots, & \frac{d-2}{d}, & \frac{d-1}{d} \\ 0, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{\frac{d-k}{2}-1}{d-k}, & \frac{\frac{d-k}{2}+1}{d-k}, & \ldots, & \frac{d-k-1}{d-k}, & 0 \end{matrix} \mid xt\right]_q. $$

Proof. For $x \in \mathbb{F}_q^{\times}$, we consider

$$ \begin{aligned} A_x &:= q \cdot {}_{d-1}G_{d-1}\left[\begin{matrix} \frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d} \\ 0, & \frac{1}{k}, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k} \end{matrix} \mid x\right]_q \\ &= -\frac{q}{q-1} - \frac{q}{q-1}\sum_{a=1}^{q-2}\overline{\omega}^a(x)(-p)^{\sum_{i=0}^{r-1} v_{a,i}} \prod_{i=0}^{r-1}\prod_{h=1}^{d-1}\frac{\Gamma_p\left(\left\langle \frac{p^ih}{d} - \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d}\right\rangle\right)} \\ &\times \prod_{h=0}^{k-1}\frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k}\right\rangle\right)} \prod_{h=1}^{d-k-1}\frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k}\right\rangle\right)}, \end{aligned} \tag{5.3} $$

where $v_{a,i} = -\sum_{h=1}^{d-1}\left\lfloor\left\langle \frac{hp^i}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor - \sum_{h=0}^{k-1}\left\lfloor\left\langle \frac{-hp^i}{k}\right\rangle + \frac{ap^i}{q-1}\right\rfloor - \sum_{h=1}^{d-k-1}\left\lfloor\left\langle \frac{-hp^i}{d-k}\right\rangle + \frac{ap^i}{q-1}\right\rfloor$. Since d and k are odd integers, so $d-k$ is even. Now, using (1.10) and the fact that $\frac{d-k}{2} \in \mathbb{Z}$, we rewrite (5.3) as

$$ \begin{aligned} A_x &= -\frac{(-1)^r q}{q-1}\sum_{a=1}^{q-2}\overline{\omega}^a(-x)(-p)^{\sum_{i=0}^{r-1} T_{a,i}} \prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle p^i - \frac{p^ia}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{p^i}{2} + \frac{p^ia}{q-1}\right\rangle\right)}{(-p)^{\left\lfloor \frac{1}{2} + \frac{ap^i}{q-1}\right\rfloor}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)} \\ &\times \Gamma_p^2\left(\left\langle \frac{p^ia}{q-1}\right\rangle\right)\prod_{h=1}^{d-1}\frac{\Gamma_p\left(\left\langle \frac{p^ih}{d} - \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d}\right\rangle\right)}\prod_{h=1}^{k-1}\frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k}\right\rangle\right)} \\ &\times \prod_{\substack{h=1 \\ h\neq \frac{d-k}{2}}}^{d-k-1}\frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k}\right\rangle\right)} - \frac{q}{q-1}, \end{aligned} $$

where $T_{a,i} = -\sum_{h=1}^{d-1}\left\lfloor\left\langle \frac{hp^i}{d}\right\rangle - \frac{ap^i}{q-1}\right\rfloor - \sum_{h=0}^{k-1}\left\lfloor\left\langle \frac{-hp^i}{k}\right\rangle + \frac{ap^i}{q-1}\right\rfloor - \sum_{\substack{h=0 \\ h\neq \frac{d-k}{2}}}^{d-k-1}\left\lfloor\left\langle \frac{-hp^i}{d-k}\right\rangle + \frac{ap^i}{q-1}\right\rfloor +$

$\left\lfloor \frac{ap^i}{q-1} \right\rfloor$. Lemma 5.1 yields

$$A_x = -\frac{q}{q-1} + \frac{(-1)^r q}{q-1} \sum_{a=1}^{q-2} \overline{\omega}^a(-x)(-p)^{\sum_{i=0}^{r-1} T_{a,i}} \sum_{t\in\mathbb{F}_q} \varphi(t(t-1))\overline{\omega}^a(-t) \prod_{i=0}^{r-1} (-p)^{\left\lfloor \frac{-ap^i}{q-1} \right\rfloor}$$
$$\times \Gamma_p^2\left(\left\langle \frac{p^i a}{q-1} \right\rangle\right) \prod_{h=1}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^i h}{d} - \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d} \right\rangle\right)} \prod_{h=1}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^i h}{k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k} \right\rangle\right)}$$
$$\times \prod_{\substack{h=1\\ h\neq \frac{d-k}{2}}}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^i h}{d-k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k} \right\rangle\right)}$$
$$= -\frac{q}{q-1} + \frac{(-1)^r q}{q-1} \sum_{a=1}^{q-2} \sum_{t\in\mathbb{F}_q} \varphi(t(t-1))\overline{\omega}^a(xt)(-p)^{\sum_{i=0}^{r-1} T_{a,i} + \left\lfloor \frac{-ap^i}{q-1} \right\rfloor} \prod_{i=0}^{r-1} \Gamma_p^2\left(\left\langle \frac{p^i a}{q-1} \right\rangle\right)$$
$$\times \prod_{h=1}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^i h}{d} - \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d} \right\rangle\right)} \prod_{h=1}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^i h}{k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k} \right\rangle\right)} \prod_{\substack{h=1\\ h\neq \frac{d-k}{2}}}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^i h}{d-k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k} \right\rangle\right)}.$$

Since $1 \le a \le q-2$ and $\gcd(p^i, q-1) = 1$, therefore $\frac{ap^i}{q-1}$ is not an integer. Also, $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ if $x \notin \mathbb{Z}$, and hence $\lfloor \frac{ap^i}{q-1} \rfloor + \lfloor \frac{-ap^i}{q-1} \rfloor = -1$. Thus,

$$A_x = -\frac{q}{q-1} + \frac{1}{q-1} \sum_{a=1}^{q-2} \sum_{t\in\mathbb{F}_q} \varphi(t(t-1))\overline{\omega}^a(xt)(-p)^{\sum_{i=0}^{r-1} u_{a,i}} \prod_{i=0}^{r-1} \Gamma_p^2\left(\left\langle \frac{p^i a}{q-1} \right\rangle\right)$$
$$\times \prod_{h=1}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^i h}{d} - \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d} \right\rangle\right)} \prod_{h=1}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^i h}{k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k} \right\rangle\right)} \prod_{\substack{h=1\\ h\neq \frac{d-k}{2}}}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^i h}{d-k} + \frac{p^i a}{q-1} \right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k} \right\rangle\right)},$$

where $u_{a,i} = -\sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{hp^i}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \sum_{h=0}^{k-1} \left\lfloor \left\langle \frac{-hp^i}{k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor - \sum_{\substack{h=0\\ h\neq \frac{d-k}{2}}}^{d-k-1} \left\lfloor \left\langle \frac{-hp^i}{d-k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor$.

Adding and subtracting the term under the summation for $a = 0$ gives

$$A_x = -\frac{q}{q-1} - \frac{1}{q-1} \sum_{t\in\mathbb{F}_q} \varphi(t(t-1)) + \frac{1}{q-1} \sum_{a=0}^{q-2} \sum_{t\in\mathbb{F}_q} \varphi(t(t-1))\overline{\omega}^a(xt)(-p)^{\sum_{i=0}^{r-1} u_{a,i}}$$

$$\times \prod_{i=0}^{r-1}\prod_{h=1}^{d-1}\frac{\Gamma_p\left(\left\langle \frac{p^ih}{d}-\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d}\right\rangle\right)}\prod_{h=0}^{k-1}\frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k}\right\rangle\right)}\prod_{\substack{h=0\\ h\neq\frac{d-k}{2}}}^{d-k-1}\frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k}+\frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k}\right\rangle\right)} \tag{5.4}$$

$$= -1-\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))\times$$

$${}_{d-1}G_{d-1}\left[\begin{matrix}\frac{1}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{k+\frac{d-k}{2}-1}{d}, & \frac{k+\frac{d-k}{2}}{d}, & \ldots, & \frac{d-2}{d}, & \frac{d-1}{d}\\ 0, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{\frac{d-k}{2}-1}{d-k}, & \frac{\frac{d-k}{2}+1}{d-k}, & \ldots, & \frac{d-k-1}{d-k}, & 0\end{matrix}\ |xt\right]_q.$$

Using (1.3) and (1.5), we obtain $\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))=-1$ and substituting this value in the first summation of (5.4) gives the last equality. This completes the proof of the theorem. ■

Theorem 5.4. *Let $d>k\geq 2$ be integers with d even. Let p be an odd prime such that $p\nmid dk(d-k)$. Let $q=p^r, r\geq 1$. Then, for $x\in\mathbb{F}_q$, we have*

$$\sum_{t\in\mathbb{F}_q}\varphi(1-t)\,{}_{d-2}G_{d-2}\left[\begin{matrix}\frac{1}{d}, & \ldots, & \frac{\frac{d}{2}-1}{d}, & \frac{\frac{d}{2}+1}{d}, & \ldots, & \frac{d-1}{d}\\ c_1, & \ldots, & c_{\frac{d}{2}-1}, & c_{\frac{d}{2}}, & \ldots, & c_{d-2}\end{matrix}\ |xt\right]_q$$

$$= -{}_{d-1}G_{d-1}\left[\begin{matrix}\frac{1}{d}, & \frac{2}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d}\\ 0, & \frac{1}{k}, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k}\end{matrix}\ |x\right]_q,$$

where $\{c_1,\ldots,c_{d-2}\}=\{\frac{1}{k},\ldots,\frac{k-1}{k},\frac{1}{d-k},\ldots,\frac{d-k-1}{d-k}\}$.

Proof. For $x\in\mathbb{F}_q^\times$, we consider

$$B_x := {}_{d-1}G_{d-1}\left[\begin{matrix}\frac{1}{d}, & \ldots, & \frac{k}{d}, & \frac{k+1}{d}, & \ldots, & \frac{d-1}{d}\\ 0, & \ldots, & \frac{k-1}{k}, & \frac{1}{d-k}, & \ldots, & \frac{d-k-1}{d-k}\end{matrix}\ |x\right]_q$$

$$= -\frac{1}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a(-x)(-p)^{\sum_{i=0}^{r-1}N_{a,i}}\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{p^ia}{q-1}\right\rangle\right)$$

$$\times \prod_{h=1}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^ih}{d} - \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d}\right\rangle\right)} \prod_{h=1}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k}\right\rangle\right)} \prod_{h=1}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k}\right\rangle\right)}, \tag{5.5}$$

where $N_{a,i} = -\sum_{h=1}^{d-1} \left\lfloor \left\langle \frac{hp^i}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \sum_{h=0}^{k-1} \left\lfloor \left\langle \frac{-hp^i}{k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor - \sum_{h=1}^{d-k-1} \left\lfloor \left\langle \frac{-hp^i}{d-k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor$. Since d is an even integers, $\frac{d}{2} \in \mathbb{Z}$. We rewrite (5.5) as

$$B_x = -\frac{1}{q-1} \sum_{a=0}^{q-2} \overline{\omega}^a(-x)(-p)^{\sum_{i=0}^{r-1} N_{a,i} + \left\lfloor \left\langle \frac{p^i}{2} \right\rangle - \frac{ap^i}{q-1} \right\rfloor} \prod_{i=0}^{r-1} \frac{\Gamma_p\left(\left\langle \frac{p^ia}{q-1}\right\rangle\right)}{(-p)^{\left\lfloor \left\langle \frac{p^i}{2} \right\rangle - \frac{ap^i}{q-1} \right\rfloor}} \frac{\Gamma_p\left(\left\langle \frac{p^i}{2} - \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)}$$
$$\times \prod_{\substack{h=1\\ h\neq \frac{d}{2}}}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^ih}{d} - \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d}\right\rangle\right)} \prod_{h=1}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k}\right\rangle\right)} \prod_{h=1}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k}\right\rangle\right)}.$$

Using Lemma 5.2, we obtain

$$B_x = \frac{1}{q-1} \sum_{a=0}^{q-2} \sum_{t\in\mathbb{F}_q^\times} \varphi(t(t-1))\overline{\omega}^a\left(\frac{x}{t}\right)(-p)^{\sum_{i=0}^{r-1} v_{i,a}}$$
$$\times \prod_{\substack{h=1\\ h\neq \frac{d}{2}}}^{d-1} \frac{\Gamma_p\left(\left\langle \frac{p^ih}{d} - \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{hp^i}{d}\right\rangle\right)} \prod_{h=1}^{d-k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{d-k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{d-k}\right\rangle\right)} \prod_{h=1}^{k-1} \frac{\Gamma_p\left(\left\langle \frac{-p^ih}{k} + \frac{p^ia}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{-hp^i}{k}\right\rangle\right)},$$

where $v_{i,a} = -\sum_{\substack{h=1\\ h\neq \frac{d}{2}}}^{d-1} \left\lfloor \left\langle \frac{hp^i}{d} \right\rangle - \frac{ap^i}{q-1} \right\rfloor - \sum_{h=1}^{k-1} \left\lfloor \left\langle \frac{-hp^i}{k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor - \sum_{h=1}^{d-k-1} \left\lfloor \left\langle \frac{-hp^i}{d-k} \right\rangle + \frac{ap^i}{q-1} \right\rfloor$.

Thus, we have

$$B_x = -\sum_{t\in\mathbb{F}_q^\times} \varphi(t(t-1))_{d-2}G_{d-2}\left[\begin{matrix} \frac{1}{d}, & \ldots, & \frac{\frac{d}{2}-1}{d}, & \frac{\frac{d}{2}+1}{d}, & \ldots, & \frac{d-1}{d} \\ c_1, & \ldots, & c_{\frac{d}{2}-1}, & c_{\frac{d}{2}}, & \ldots, & c_{d-2} \end{matrix} \mid \frac{x}{t}\right]_q,$$

where c's are as defined in the statement of the theorem. Now, using the translation $t \mapsto t^{-1}$, we obtain the required identity. ■

For example, if we take $d = 5, k = 3$ and $d = 6, k = 3$ in Theorems 5.3 and 5.4, respectively, then for $p > 5$ and $x \in \mathbb{F}_q^{\times}$, we obtain the following identities.

$$-\sum_{t\in\mathbb{F}_q} \varphi(t(t-1))_4G_4\left[\begin{matrix} \frac{1}{5}, & \frac{2}{5}, & \frac{3}{5}, & \frac{4}{5} \\ 0, & 0, & \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| xt\right]_q = 1 + q \cdot {}_4G_4\left[\begin{matrix} \frac{1}{5}, & \frac{2}{5}, & \frac{3}{5}, & \frac{4}{5} \\ \frac{1}{3}, & \frac{2}{3}, & 0, & \frac{1}{2} \end{matrix} \Big| x\right]_q,$$

$$\sum_{t\in\mathbb{F}_q} \varphi(1-t)_4G_4\left[\begin{matrix} \frac{1}{6}, & \frac{2}{6}, & \frac{4}{6}, & \frac{5}{6} \\ \frac{1}{3}, & \frac{2}{3}, & \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| xt\right]_q = -{}_5G_5\left[\begin{matrix} \frac{1}{6}, & \frac{2}{6}, & \frac{3}{6}, & \frac{4}{6}, & \frac{5}{6} \\ \frac{1}{3}, & \frac{2}{3}, & 0, & \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| x\right]_q.$$

For $r = 1$, taking $k = 1$ in Theorem 5.3 and $k = d - 1$ in Theorem 5.4, we obtain [15, Theorem 1.2] and [15, Theorem 1.3], respectively.

5.3 Applications to elliptic curves

There are many significant relations of hypergeometric functions to elliptic curves over finite fields. For example, see [8, 9, 29, 46, 47, 48, 57]. In this section, we provide certain identities for the trace of Frobenius of elliptic curves, which serve as applications of the summation identities that we derived in the previous section. Firstly, we prove a lemma that gives a relation between ${}_3G_3[\cdots]_q$ and ${}_2G_2[\cdots]_q$, and this lemma will be used to prove our main results.

Lemma 5.5. *Let p be an odd prime and $q = p^r$, $r \geq 1$. For $x \in \mathbb{F}_q$, we have*

$${}_3G_3\left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{2}, & \frac{1}{2} \end{matrix} \Big| x\right]_q = {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \Big| x\right]_q + \frac{\varphi(x)}{q}.$$

Proof. For $x \in \mathbb{F}_q$, we have

$${}_3G_3\left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{2}, & \frac{1}{2} \end{matrix} \Big| x\right]_q$$

$$= -\frac{1}{q-1}\sum_{a=0}^{q-2}(-1)^{3a}\overline{\omega}^a(x)(-p)^{\sum_{i=0}^{r-1}\beta_{i,a}}\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{ap^i}{q-1}\right\rangle\right)M_{i,a}$$
$$\times\frac{\Gamma_p\left(\left\langle\frac{p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}+\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}\right\rangle\right)},$$

where $\beta_{i,a} = -\left\lfloor\left\langle\frac{p^i}{4}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{3p^i}{4}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{p^i}{2}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{-p^i}{2}\right\rangle+\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{-p^i}{2}\right\rangle+\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\frac{ap^i}{q-1}\right\rfloor$ and $M_{i,a}=\frac{\Gamma_p\left(\left\langle\frac{p^i}{2}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}+\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}\right\rangle\right)}$. Taking out the term for $a=\frac{q-1}{2}$ and then using (1.11) and the fact that $\left\lfloor\left\langle\frac{p^i}{2}\right\rangle-\frac{ap^i}{q-1}\right\rfloor+\left\lfloor\left\langle\frac{-p^i}{2}\right\rangle+\frac{ap^i}{q-1}\right\rfloor=0$ for $0\leq a\leq q-2$ with $a\neq\frac{q-1}{2}$, we obtain

$${}_3G_3\left[\begin{matrix}\frac{1}{4}, & \frac{1}{2}, & \frac{3}{4}\\ 0, & \frac{1}{2}, & \frac{1}{2}\end{matrix}\,|x\right]_q = -\frac{1}{q-1}\sum_{\substack{a=0,\\ a\neq\frac{q-1}{2}}}^{q-2}(-1)^{3a}\overline{\omega}^a(-x)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,a}}\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{ap^i}{q-1}\right\rangle\right)$$
$$\times\frac{\Gamma_p\left(\left\langle\frac{p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}+\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}\right\rangle\right)}$$
$$-\frac{\varphi(-x)}{(q-1)(-p)^r}\prod_{i=0}^{r-1}\frac{1}{\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)^2},$$

where $\alpha_{i,a}=-\left\lfloor\left\langle\frac{p^i}{4}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{3p^i}{4}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle\frac{-p^i}{2}\right\rangle+\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\frac{ap^i}{q-1}\right\rfloor$. Using Gross-Koblitz formula for $g(\varphi)^2$, we obtain $\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)^2=(-1)^r\varphi(-1)$. Adding and subtracting the term under the summation for $a=\frac{q-2}{2}$ and using $\overline{\omega}^a(-1)=(-1)^a$, we deduce that

$${}_3G_3\left[\begin{matrix}\frac{1}{4}, & \frac{1}{2}, & \frac{3}{4}\\ 0, & \frac{1}{2}, & \frac{1}{2}\end{matrix}\,|x\right]_q = -\frac{1}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a(x)(-p)^{\sum_{i=0}^{r-1}\alpha_{i,a}}\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{ap^i}{q-1}\right\rangle\right)$$
$$\times\frac{\Gamma_p\left(\left\langle\frac{p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}+\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle\frac{-p^i}{2}\right\rangle\right)}$$

$$-\frac{\varphi(x)}{q(q-1)}+\frac{\varphi(x)}{q-1}.$$

This completes the proof of the lemma. ■

Next, we prove another lemma involving certain transformations.

Lemma 5.6. *Let p be an odd prime and $q=p^r, r\geq 1$. We have the following transformations.*

1. *For $t\in\mathbb{F}_q^\times$, we have*

$${}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ \frac{1}{2}, & \frac{1}{2}\end{matrix}\ |t\right]_q={}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{1}{4}, & \frac{3}{4}\end{matrix}\ |\frac{1}{t}\right]_q.$$

2. *For $t\in\mathbb{F}_q$, we have*

$${}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ \frac{1}{2}, & \frac{1}{2}\end{matrix}\ |t\right]_q=\frac{\varphi(-t)}{q}\cdot{}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\ |t\right]_q.$$

3. *For $t\in\mathbb{F}_q^\times$, we have*

$${}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}\ |t\right]_q=\varphi(-3t)\cdot q\cdot{}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{1}{6}, & \frac{5}{6}\end{matrix}\ |\frac{1}{t}\right]_q.$$

Identity (3) *is true for $p>3$.*

Proof. We first prove (1). For $t\in\mathbb{F}_q^\times$, we consider

$$A_t:={}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ \frac{1}{2}, & \frac{1}{2}\end{matrix}\ |t\right]_q$$

$$= \frac{-1}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a(t)(-p)^{\sum_{i=0}^{r-1} b_{i,a}}\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle \frac{p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{3p^i}{4}-\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{4}\right\rangle\right)\Gamma_p\left(\left\langle \frac{3p^i}{4}\right\rangle\right)}$$
$$\times\frac{\Gamma_p\left(\left\langle \frac{-p^i}{2}+\frac{ap^i}{q-1}\right\rangle\right)^2}{\Gamma_p\left(\left\langle \frac{-p^i}{2}\right\rangle\right)^2}, \tag{5.6}$$

where $b_{i,a}=-\left\lfloor\left\langle \frac{p^i}{4}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle \frac{3p^i}{4}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-2\left\lfloor\left\langle \frac{-p^i}{2}\right\rangle+\frac{ap^i}{q-1}\right\rfloor$. Replacing a by $-a$, we obtain the required identity.

For (2), we replace a by $a+\frac{q-1}{2}$ in (5.6) and then using the fact that

$$\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle \frac{p^i}{2}\right\rangle\right)^2=(-1)^r\varphi(-1),$$

we obtain the required identity.

We now prove (3). For $p>3$ and $t\in\mathbb{F}_q^{\times}$, we consider

$$B_t:={}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}\ |t\right]_q$$
$$=\frac{-1}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a(t)(-p)^{\sum_{i=0}^{r-1} s_{i,a}}\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle \frac{p^i}{3}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{2p^i}{3}-\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{3}\right\rangle\right)\Gamma_p\left(\left\langle \frac{2p^i}{3}\right\rangle\right)}$$
$$\times\Gamma_p\left(\left\langle \frac{ap^i}{q-1}\right\rangle\right)^2,$$

where $s_{i,a}=-\left\lfloor\left\langle \frac{p^i}{3}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-\left\lfloor\left\langle \frac{2p^i}{3}\right\rangle-\frac{ap^i}{q-1}\right\rfloor-2\left\lfloor\frac{ap^i}{q-1}\right\rfloor$. Replacing a by $a+\frac{q-1}{2}$, we deduce

$$B_t=\frac{-\varphi(t)}{q-1}\sum_{a=0}^{q-2}\overline{\omega}^a(t)(-p)^{\sum_{i=0}^{r-1} t_{i,a}+1}\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle \frac{p^i}{6}-\frac{ap^i}{q-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{5p^i}{6}-\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle \frac{p^i}{3}\right\rangle\right)\Gamma_p\left(\left\langle \frac{2p^i}{3}\right\rangle\right)}$$
$$\times\Gamma_p\left(\left\langle \frac{-p^i}{2}+\frac{ap^i}{q-1}\right\rangle\right)^2, \tag{5.7}$$

where $t_{i,a} = -\left\lfloor\left\langle\frac{p^i}{6}\right\rangle - \frac{ap^i}{q-1}\right\rfloor - \left\lfloor\left\langle\frac{5p^i}{6}\right\rangle - \frac{ap^i}{q-1}\right\rfloor - 2\left\lfloor\left\langle\frac{-p^i}{2}\right\rangle + \frac{ap^i}{q-1}\right\rfloor$. Using the fact that $\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{p^i}{2}\right\rangle\right)^2 = (-1)^r\varphi(-1)$ and Lemma 1.10 with $t = 3$ and $a = \frac{q-1}{2}$, we obtain

$$\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{p^i}{3}\right\rangle\right)\Gamma_p\left(\left\langle\frac{2p^i}{3}\right\rangle\right) = (-1)^r\varphi(-3) \times\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\frac{-p^i}{2}\right\rangle\right)^2\Gamma_p\left(\left\langle\frac{p^i}{6}\right\rangle\right)\Gamma_p\left(\left\langle\frac{5p^i}{6}\right\rangle\right). \quad (5.8)$$

Substituting (5.8) in (5.7) and replacing a by $-a$, we obtain identity (3). ■

Corollary 5.3.1. *Let $p \geq 3$ be a prime and $q = p^r, r \geq 1$.*

1. *We have*

$$\sum_{t\in\mathbb{F}_q}\varphi(1-t)_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2}\end{matrix}|t\right]_q = -\frac{1}{q} - \varphi(2).$$

2. *Let $x \neq 0, 1$ and $\frac{x-1}{x}$ a square in $\mathbb{F}_q^\times$. If $\frac{x-1}{x} = a^2$ for some $a \in \mathbb{F}_q^\times$, then*

$$\sum_{t\in\mathbb{F}_q}\varphi(1-t)_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2}\end{matrix}|xt\right]_q = -\frac{\varphi(x)}{q} - \varphi(2)(\varphi(1+a) + \varphi(1-a)).$$

3. *Let $x \neq 0$. If $\frac{x-1}{x}$ is not a square in $\mathbb{F}_q^\times$, then*

$$\sum_{t\in\mathbb{F}_q}\varphi(1-t)_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2}\end{matrix}|xt\right]_q = -\frac{\varphi(x)}{q}.$$

Proof. Taking $d = 4$ and $k = 2$ in Theorem 5.4, we have the following identity

$$\sum_{t\in\mathbb{F}_q} \varphi(1-t)_2 G_2 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2} \end{matrix} |xt\right]_q = -_3G_3 \left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{2}, & \frac{1}{2} \end{matrix} |x\right]_q.$$

Using Lemma 5.5, we obtain

$$\sum_{t\in\mathbb{F}_q} \varphi(1-t)_2 G_2 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2} \end{matrix} |xt\right]_q = -_2G_2 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} |x\right]_q - \frac{\varphi(x)}{q}.$$

Using [61, Theorem 1.2], we obtain the required result. ■

Remark 5.3.1. *Using Lemma 5.6 (2) in Corollary 5.3.1 and replacing x by $\frac{1}{x}$, we obtain [61, Theorem 1.7].*

Corollary 5.3.2. *Let $p > 3$ be a prime and $q = p^r, r \geq 1$.*

1. *We have*

$$\sum_{t\in\mathbb{F}_q} \varphi(t(t-1))_2 G_2 \left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & 0 \end{matrix} |t\right]_q = -1 - q.$$

2. *Let $x \neq 0, 1$ be such that $\varphi(3x(1-x)) = -1$. Then, we have*

$$\sum_{t\in\mathbb{F}_q} \varphi(t(t-1))_2 G_2 \left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & 0 \end{matrix} |\frac{t}{x}\right]_q = -1.$$

Proof. Taking $d = 3$ and $k = 1$ in Theorem 5.3, we find that

$$-\sum_{t\in\mathbb{F}_q} \varphi(t(t-1))_2 G_2 \left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & 0 \end{matrix} |\frac{t}{x}\right]_q = 1 + q \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ 0, & \frac{1}{2} \end{matrix} |\frac{1}{x}\right]_q.$$

We obtain (1) and (2) using [62, Corollary 1.3] and Theorem 2.5, respectively. ■

Corollary 5.3.3. *Let $p > 3$ be a prime and $q = p^r, r \geq 1$. Let $q \equiv 1 \pmod 3$ and χ_3 be a character of order 3.*

1. *We have*

$$\sum_{t\in\mathbb{F}_q^\times} \varphi(t(t-1))_2F_1\left(\begin{matrix}\chi_3, & \overline{\chi_3} \\ & \varepsilon\end{matrix} \mid \frac{1}{t}\right)_q = 1 + \frac{1}{q}.$$

2. *Let $x \neq 0, 1$ be such that $\varphi(3x(1-x)) = -1$. Then, we have*

$$\sum_{t\in\mathbb{F}_q^\times} \varphi(t(t-1))_2F_1\left(\begin{matrix}\chi_3, & \overline{\chi_3} \\ & \varepsilon\end{matrix} \mid \frac{x}{t}\right)_q = \frac{1}{q}.$$

Proof. For $x, t \in \mathbb{F}_q^\times$, Lemma 1.9 and Proposition 1.7 yield

$$\begin{aligned}
{}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{t}{x}\right]_q &= {}_2F_1\left(\begin{matrix}\chi_3, & \overline{\chi_3} \\ & \varepsilon\end{matrix} \mid \frac{x}{t}\right)_q^* \\
&= \binom{\overline{\chi_3}}{\varepsilon}^{-1} {}_2F_1\left(\begin{matrix}\chi_3, & \overline{\chi_3} \\ & \varepsilon\end{matrix} \mid \frac{x}{t}\right)_q \\
&= -q \cdot {}_2F_1\left(\begin{matrix}\chi_3, & \overline{\chi_3} \\ & \varepsilon\end{matrix} \mid \frac{x}{t}\right)_q,
\end{aligned}$$

where we obtain the last equality using (1.5). Now,

$$\begin{aligned}
\sum_{t\in\mathbb{F}_q} \varphi(t(t-1))_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{t}{x}\right]_q &= \sum_{t\in\mathbb{F}_q^\times} \varphi(t(t-1))_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{t}{x}\right]_q \\
&= -q \cdot \sum_{t\in\mathbb{F}_q^\times} \varphi(t(t-1))_2F_1\left(\begin{matrix}\chi_3, & \overline{\chi_3} \\ & \varepsilon\end{matrix} \mid \frac{x}{t}\right)_q.
\end{aligned}$$

Using Corollary 5.3.2, we obtain the desired result. ■

Remark 5.3.2. *Combining* (3.29) *and* (3.30) *and taking* $d = 4$ *and* $k = 2$, *we obtain*

$${}_3G_3\left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{2}, & \frac{1}{2} \end{matrix} \mid \frac{1}{16\alpha}\right]_q = n_q(\alpha) - 1 + \frac{(1-q)\varphi(\alpha)}{q},$$

where $n_q(\alpha)$ *is the number of distinct zeros of the polynomial* $h_\alpha(y) = y^4 - 2y^3 + y^2 - \alpha$. *We can check that if* α *is not a square in* $\mathbb{F}_q$ *then* $h_\alpha(y)$ *has no zero in* $\mathbb{F}_q$ *and hence,* $n_q(\alpha) = 0$. *Applying Lemma* 5.5, *we deduce that*

$${}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \mid \frac{1}{16\alpha}\right]_q = 0,$$

which partially gives [61, Theorem 1.2].

We have proved all the required lemmas and corollaries to prove the identities for the trace of Frobenius of elliptic curves. Let E_1 and E_2 be the elliptic curves over $\mathbb{F}_q$ given by

$$E_1 : y^2 = x^3 + fx^2 + gx, f \neq 0,$$
$$E_2 : y^2 = x^3 + ax + b \text{ with } j(E_2) \neq 0, 1728.$$

In [53], McCarthy expressed the trace of the Frobenius endomorphism on E_2 as a special value of the function ${}_2G_2[\cdots]$ as given in the following theorem.

Theorem 5.7. ([53, Theorem 1.2]). *Let* $p > 3$ *be prime. Consider an elliptic curve* $E_2/\mathbb{F}_p$ *of the form* $E_2 : y^2 = x^3 + ax + b$ *with* $j(E_2) \neq 0, 1728$. *Then*

$$a_p(E_2) = p \cdot \varphi(b) \cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \mid \frac{-27b^2}{4a^3}\right]_p.$$

In [12], Barman and Saikia did the same for the elliptic curve E_1.

Theorem 5.8. ([12, Theorem 3.5]). *Let p be an odd prime and $q = p^r, r \geq 1$. The trace of Frobenius on E_1 is given by*

$$a_q(E_1) = q \cdot \varphi(-fg) \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4} \end{matrix} \Big| \frac{4g}{f^2} \right]_q .$$

Remark 5.3.3. *McCarthy proved Theorem 5.7 for $\mathbb{F}_p$ for all primes $p > 3$. In [12], it was verified that Theorem 5.7 is true for $\mathbb{F}_q$, where $q = p^r, r \geq 1$ and $p > 3$.*

Theorem 5.9. *Let $p > 3$ be a prime and $q = p^r, r \geq 1$ such that $q \not\equiv 1 \pmod 3$. For $b, t \in \mathbb{F}_q^{\times}$, consider the elliptic curve $E_{t,b} : y^2 = x^3 + tx + b$ over $\mathbb{F}_q$. Then we have*

$$\sum_{t \in \mathbb{F}_q^{\times}} \varphi(t(t^3 - 1)) a_q(E_{t,b}) = -q \cdot \varphi(b) \cdot {}_3G_3 \left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| \frac{-27b^2}{4} \right]_q .$$

Proof. Taking $d = 4$ and $k = 3$ in Theorem 5.4, we have

$$\sum_{t \in \mathbb{F}_q} \varphi(1-t) {}_2G_2 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| x_0 t \right]_q = \sum_{t \in \mathbb{F}_q^{\times}} \varphi(1-t) {}_2G_2 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| x_0 t \right]_q$$
$$= -{}_3G_3 \left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| x_0 \right]_q .$$

Since $q \not\equiv 1 \pmod 3$, therefore $t \mapsto t^{-3}$ is an automorphism on $\mathbb{F}_q^{\times}$ and we obtain

$$\sum_{t \in \mathbb{F}_q^{\times}} \varphi(t(t^3 - 1)) {}_2G_2 \left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| \frac{x_0}{t^3} \right]_q = -{}_3G_3 \left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{3}, & \frac{2}{3} \end{matrix} \Big| x_0 \right]_q .$$

Taking $x_0 = \frac{-27b^2}{4}$ with $b \in \mathbb{F}_q^\times$ yields

$$\sum_{t\in\mathbb{F}_q^\times} \varphi(t(t^3-1))_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix} | \frac{-27b^2}{4t^3}\right]_q = -_3G_3\left[\begin{matrix} \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4} \\ 0, & \frac{1}{3}, & \frac{2}{3}\end{matrix} | \frac{-27b^2}{4}\right]_q.$$

Note that for $t, b \in \mathbb{F}_q^\times$, we have $j(E_{t,b}) \neq 0, 1728$. Hence, using Theorem 5.7, we obtain the desired result. ■

Theorem 5.10. *Let $p \geq 3$ be a prime and $q = p^r, r \geq 1$. For $f, t \in \mathbb{F}_q^\times$, consider the elliptic curve $E_{t,f} : y^2 = x^3 + fx^2 + \frac{x}{t}$ over $\mathbb{F}_q$. Then we have*

$$\sum_{t\in\mathbb{F}_q^\times} \varphi(t(t-1))a_q(E_{t,f})$$
$$= \begin{cases} -\varphi(f) - q\cdot\varphi(2f), & \text{if } f^2 = 4; \\ -\varphi(f), & \text{if } f^2-4 \text{ is not a square;} \\ -\varphi(f) - q\cdot\varphi(2f)(\varphi(1+\frac{a}{f}) + \varphi(1-\frac{a}{f})), & \text{if } f^2-4 = a^2. \end{cases}$$

Proof. We consider a character sum A_{x_0} and then using Lemma 5.6 (1), we have

$$\begin{aligned} A_{x_0} &:= \sum_{t\in\mathbb{F}_q} \varphi(1-t)_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2}\end{matrix} | x_0t\right]_q \\ &= \sum_{t\in\mathbb{F}_q^\times} \varphi(1-t)_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2}\end{matrix} | x_0t\right]_q \\ &= \sum_{t\in\mathbb{F}_q^\times} \varphi(1-t)_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4}\end{matrix} | \frac{1}{x_0t}\right]_q. \end{aligned} \tag{5.9}$$

Let $E_{f,t} : y^2 = x^3 + fx^2 + \frac{x}{t}$, where $f, t \in \mathbb{F}_q^\times$. Then by Theorem 5.8, we have

$$a_q(E_{f,t}) = q\cdot\varphi(-ft)\cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4}\end{matrix} | \frac{4}{f^2t}\right]_q. \tag{5.10}$$

Taking $x_0 = \frac{f^2}{4}$ in (5.9) and using (5.10), we have

$$A_{\frac{f^2}{4}} = \frac{\varphi(f)}{q} \sum_{t \in \mathbb{F}_q^{\times}} \varphi(t(t-1)) a_q(E_{f,t}). \tag{5.11}$$

Combining (5.11) and the expression for the character sum $A_{\frac{f^2}{4}}$ from Corollary 5.3.1, we obtain the required result. ■

Next, we recall the Hessian form of elliptic curve over $\mathbb{F}_q$. Let $a \in \mathbb{F}_q$ be such that $a^3 \neq 1$. Then the Hessian curve over $\mathbb{F}_q$ is given by the following cubic equation

$$C_a(\mathbb{F}_q) : x^3 + y^3 + 1 = 3axy.$$

Let $\#C_a(\mathbb{F}_q)$ denote the number of $\mathbb{F}_q$-points on $C_a(\mathbb{F}_q)$. In [13], Barman and Saikia expressed the number of $\mathbb{F}_q$-points on $C_a(\mathbb{F}_q)$ in terms of ${}_2G_2[\cdots]_q$ as given in the following theorem.

Theorem 5.11. ([13, Theorem 3.3]). *Let $a \in \mathbb{F}_q^{\times}$ such that $a^3 \neq 1$. Let $p > 3$ be a prime and $q = p^r, r > 1$. Then we have*

$$\#C_a(\mathbb{F}_q) = \alpha - 1 + q - q \cdot \varphi(-3a) \cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{6}, & \frac{5}{6} \end{matrix} \middle| \frac{1}{a^3}\right]_q,$$

where $\alpha = \begin{cases} 5 - 6\varphi(-3), & \text{if } q \equiv 1 \pmod 3; \\ 1, & \text{if } q \not\equiv 1 \pmod 3). \end{cases}$

In the following theorem, we prove an identity for $\#C_a(\mathbb{F}_q)$.

Theorem 5.12. *Let $p > 3$ be a prime and $q = p^r, r \geq 1$ such that $q \not\equiv 1 \pmod 3$. For $t \in \mathbb{F}_q$, $t \neq 0, 1$, consider the Hessian curve $C_t : x^3 + y^3 + 1 = 3txy$ over $\mathbb{F}_q$. If $\#C_t(\mathbb{F}_q)$ denotes the number of $\mathbb{F}_q$-points on $C_t(\mathbb{F}_q)$, then we have*

$$\sum_{t \in \mathbb{F}_q^{\times}, t \neq 1} \varphi(t(t^3-1)) \#C_t(\mathbb{F}_q) = 1.$$

Proof. Corollary 5.3.2 (1) gives

$$\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}|t\right]_q=\sum_{t\in\mathbb{F}_q^\times}\varphi(t(t-1))_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}|t\right]_q$$
$$=-1-q.$$

Using Lemma 5.6 (3),

$$\sum_{t\in\mathbb{F}_q^\times}\varphi(t(t-1))_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}|t\right]_q=q\cdot\sum_{t\in\mathbb{F}_q^\times}\varphi(3(1-t))_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{1}{6}, & \frac{5}{6}\end{matrix}|\frac{1}{t}\right]_q$$
$$=-1-q.$$

Since $q\not\equiv 1\pmod 3$, therefore $t\mapsto t^3$ is an automorphism on $\mathbb{F}_q^\times$. This yields

$$q\cdot\sum_{t\in\mathbb{F}_q^\times}\varphi(3(1-t^3))_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{1}{6}, & \frac{5}{6}\end{matrix}|\frac{1}{t^3}\right]_q=q\cdot\sum_{t\in\mathbb{F}_q^\times,t\neq1}\varphi(3(1-t^3))_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{1}{6}, & \frac{5}{6}\end{matrix}|\frac{1}{t^3}\right]_q$$
$$=-1-q.$$

Let $C_t: x^3+y^3+1=3txy$, where $t\in\mathbb{F}_q^\times$ such that $t\neq1$. Using Theorem 5.11, we obtain

$$\begin{aligned}-1-q&=\sum_{t\in\mathbb{F}_q^\times,t\neq1}\varphi(t(t^3-1))(q-\#C_t(\mathbb{F}_q))\\&=q\cdot\sum_{t\in\mathbb{F}_q}\varphi(t(t^3-1))-\sum_{t\in\mathbb{F}_q^\times,t\neq1}\varphi(t(t^3-1))\#C_t(\mathbb{F}_q)\\&=q\cdot\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))-\sum_{t\in\mathbb{F}_q^\times,t\neq1}\varphi(t(t^3-1))\#C_t(\mathbb{F}_q).\end{aligned}$$

Using the fact that $\sum_{t\in\mathbb{F}_q}\varphi(t(t-1))=-1$, we obtain the desired result. ∎

6

Weight Three Newforms and p-adic Hypergeometric Functions

6.1 Introduction

Rodriguez-Villegas [60] studied the relationship between the number of points over $\mathbb{F}_p$ on certain Calabi-Yau manifolds and truncated hypergeometric series which corresponds to a particular period of the manifold. In the same article, he examined 18 supercongruences where he related the truncated hypergeometric series to the

[1]The contents of this chapter have been published in *J. Math. Anal. Appl.* (2025).

Fourier coefficients of modular forms of weight three and four. All the 14 supercongruences of Rodriguez-Villegas associated with the modular form of weight four are proved, see for example, [31, 42, 49, 51]. For a nice survey and more conjectural supercongruences, one can also see [24].

Dedekind's eta function $\eta(z)$ is defined by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$. The integers $a(n)$, $b(n)$, and $c(n)$ are defined by

$$\sum_{n=1}^{\infty} a(n)q^n := \eta^6(4z) \in S_3\left(\Gamma_0(16), \left(\frac{-4}{d}\right)\right), \tag{6.1}$$

$$\sum_{n=1}^{\infty} b(n)q^n := \eta^3(6z)\eta^3(2z) \in S_3\left(\Gamma_0(12), \left(\frac{-3}{d}\right)\right), \tag{6.2}$$

$$\sum_{n=1}^{\infty} c(n)q^n := \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z) \in S_3\left(\Gamma_0(8), \left(\frac{-2}{d}\right)\right). \tag{6.3}$$

These weight three newforms are related to modular $K3$ surfaces. Rodriguez-Villegas [60] conjectured that for any prime $p > 3$ we have

$$\sum_{n=0}^{p-1} \frac{(2n)!^3}{n!^6} 64^{-n} \equiv a(p) \pmod{p^2}, \tag{6.4}$$

$$\sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n} \equiv b(p) \pmod{p^2}, \tag{6.5}$$

$$\sum_{n=0}^{p-1} \frac{(4n)!}{n!^4} 256^{-n} \equiv c(p) \pmod{p^2}, \tag{6.6}$$

$$\sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!^3} 1728^{-n} \equiv \gamma(p)a(p) \pmod{p^2}, \tag{6.7}$$

where $\gamma(p) := -1$ if $p \equiv 5 \pmod{12}$ and $\gamma(p) := 1$ otherwise.

Supercongruence (6.4) has already been proved by several authors including

Ahlgren [1], Ishikawa [40], Mortenson [56], and Van Hamme [70]. The supercongruences (6.5)-(6.7) were studied by Mortenson in [56]. Using finite field hypergeometric functions, Mortenson proved (6.5) for $p \equiv 1 \pmod 3$, (6.6) for $p \equiv 1 \pmod 4$ and (6.7) for $p \equiv 1 \pmod 6$. When $p \equiv -1 \pmod d$, where $d = 3, 4, 6$, Mortenson's approach only allowed him to show the supercongruences up to sign. For example, for $p \equiv -1 \pmod 3$, he proved that

$$\left(\sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n}\right)^2 \equiv b(p)^2 \pmod{p^2}.$$

Sun [67] was the first to prove the remaining cases of (6.5)-(6.7). He used another approach, namely Schröder polynomials and the Zeilberger algorithm to complete the proof of (6.5)-(6.7). In [4], a general congruence result relating the hypergeometric functions defined by Beukers et al. [20] and truncated classical hypergeometric series was also studied.

In this chapter, we study the supercongruences (6.5)-(6.7) via McCarthy's p-adic hypergeometric functions involving the p-adic Gamma function and extend Mortenson's approach to give a complete proof of (6.5)-(6.7). Firstly, we establish certain transformation identities for McCarthy's p-adic hypergeometric function ${}_nG_n[\cdots]_p$. We use these transformation identities to find the special values of p-adic hypergeometric functions which come out to be precisely the p-th Fourier coefficients of weight three newforms.

6.2 Transformations for ${}_2G_2[\cdots]_p$ and ${}_3G_3[\cdots]_p$

The following transformation for classical hypergeometric series is due to Kummer [7, p. 4, Eq. 1].

$$\begin{aligned}{}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,|x\right] &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1\left[\begin{matrix} a, & b \\ & a+b+1-c \end{matrix}\,|1-x\right] \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}{}_2F_1\left[\begin{matrix} c-a, & c-b \\ & 1+c-a-b \end{matrix}\,|1-x\right]. \end{aligned} \tag{6.8}$$

Barman and Saikia [14] found a p-adic analogue of (6.8) when $a=\frac{1}{4}, b=\frac{3}{4}$, and $c=1$. We prove a p-adic analogue of (6.8) for two more cases. We first prove a proposition which gives a relation between the traces of Frobenius of two elliptic curves. This proposition plays an essential role in the proof of a p-adic analogue of (6.8).

Proposition 6.1. *Let $p>3$ be a prime and $E_t: y^2+3xy+ty=x^3$ be a family of elliptic curves where $t\in\mathbb{F}_p$ such that $t\neq 0,1$. Then we have*

$$a_p(E_t)=\left(\frac{-3}{p}\right)a_p(E_{1-t}).$$

Proof. We have $E_t: y^2+3xy+ty=x^3$ where $t\in\mathbb{F}_p$ such that $t\neq 0,1$. Clearly, $P=(0,0)$ is a point of order 3 on E_t. Using [72, Theorem 12.16], we can find an isogeny α from E_t to E_t' such that $\ker(\alpha)=\{O,P,-P\}$ and E_t' is given by

$$E_t': y^2+3xy+ty=x^3-15tx+(-27t-7t^2).$$

We reduce E_t' to the form (1.6) and obtain

$$E_t': y^2=x^3+\frac{9}{4}x^2-\frac{27t}{2}x-\frac{27}{4}(t^2+4t).$$

We know that trace of Frobenius is invariant under isogeny. Therefore,

$$a_p(E_t) = a_p(E'_t). \tag{6.9}$$

Also, $E_{1-t} : y^2 + 3xy + (1-t)y = x^3$. We write E_{1-t} in the form (1.6) as follows.

$$E_{1-t} : y^2 = x^3 + \frac{9}{4}x^2 + \frac{3(1-t)}{2}x + \frac{(1-t)^2}{4}.$$

The (-3)-quadratic twist of E_{1-t} is given by

$$y^2 = x^3 - \frac{27}{4}x^2 + \frac{27(1-t)}{2}x - \frac{27}{4}(1-t)^2. \tag{6.10}$$

Employing $x \mapsto x+3$ in (6.10) yields

$$y^2 = x^3 + \frac{9}{4}x^2 - \frac{27t}{2}x - \frac{27}{4}(t^2+4t),$$

which is the elliptic curve E'_t. Therefore, E'_t is a (-3)-quadratic twist of E_{1-t}. Hence, (1.7) yields

$$a_p(E_{1-t}) = \left(\frac{-3}{p}\right) a_p(E'_t). \tag{6.11}$$

Combining (6.9) and (6.11), we obtain the desired result. ■

In the following theorem, we prove a p-adic analogue of (6.8) when $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $c = 1$ with the help of Proposition 6.1.

Theorem 6.2. *Let $p > 3$ be a prime and $t \in \mathbb{F}_p$ such that $t \neq 0, 1$. We have*

$${}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{1}{t}\right]_p = \varphi(-3) \cdot {}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{1}{1-t}\right]_p.$$

Proof. Let $E_{a_1,a_3} : y^2 + a_1xy + a_3y = x^3$ be a family of elliptic curves where $a_1, a_3 \in$

$\mathbb{F}_p^{\times}$. We let T denote a generator of $\widehat{\mathbb{F}_p^{\times}}$. From the proof of Theorem 1.1 of [48], we have

$$a_p(E_{a_1,a_3}) = -\frac{1}{p} - \frac{1}{p(p-1)}\sum_{l=0}^{p-2} g(T^{-l})^3 g(T^{3l})T^l\left(\frac{-a_3}{a_1^3}\right).$$

Taking $T=\omega$ and then using Gross-Koblitz formula we obtain

$$\begin{aligned} a_p(E_{a_1,a_3}) &= -\frac{1}{p-1} - \frac{1}{p(p-1)}\sum_{l=1}^{p-2}\overline{\omega}^l\left(\frac{-a_1^3}{a_3}\right)(-p)^{3\langle\frac{l}{p-1}\rangle+\langle-\frac{3l}{p-1}\rangle} \\ &\quad\times\Gamma_p\left(\frac{l}{p-1}\right)^3\Gamma_p\left(\left\langle\frac{-3l}{p-1}\right\rangle\right) \\ &= -\frac{1}{p-1} - \frac{1}{p(p-1)}\sum_{l=1}^{p-2}\overline{\omega}^l\left(\frac{-a_1^3}{a_3}\right)(-p)^{-\lfloor\frac{-3l}{p-1}\rfloor} \\ &\quad\times\Gamma_p\left(\frac{l}{p-1}\right)^3\Gamma_p\left(\left\langle\frac{-3l}{p-1}\right\rangle\right). \end{aligned}$$

Using Lemma 1.11 with $t=3$ and Lemma 1.13 with $d=3$, we deduce that

$$\begin{aligned} a_p(E_{a_1,a_3}) &= -\frac{1}{p-1} - \frac{1}{p(p-1)}\sum_{l=1}^{p-2}\overline{\omega}^l\left(\frac{-a_1^3}{27a_3}\right)(-p)^{-\lfloor\frac{1}{3}-\frac{l}{p-1}\rfloor-\lfloor\frac{2}{3}-\frac{l}{p-1}\rfloor+1} \\ &\quad\times\frac{\Gamma_p\left(\frac{l}{p-1}\right)^3\Gamma_p\left(\left\langle\frac{1}{3}-\frac{l}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{2}{3}-\frac{l}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle 1-\frac{l}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{2}{3}\right)}. \end{aligned}$$

Using (1.10) and then adding and subtracting the term under the summation for $l=0$, we have

$$\begin{aligned} a_p(E_{a_1,a_3}) &= -\frac{1}{p-1}\sum_{l=0}^{p-2}\overline{\omega}^l\left(\frac{a_1^3}{27a_3}\right)(-p)^{-\lfloor\frac{1}{3}-\frac{l}{p-1}\rfloor-\lfloor\frac{2}{3}-\frac{l}{p-1}\rfloor} \\ &\quad\times\frac{\Gamma_p\left(\frac{l}{p-1}\right)^2\Gamma_p\left(\left\langle\frac{1}{3}-\frac{l}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{2}{3}-\frac{l}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{2}{3}\right)} \end{aligned}$$

$$= {}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{a_1^3}{27a_3}\right]_p.$$

Taking $a_1 = 3, a_3 = t$ and $a_1 = 3, a_3 = 1-t$, we obtain

$$a_p(E_{3,t}) = {}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{1}{t}\right]_p,$$

$$a_p(E_{3,1-t}) = {}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid \frac{1}{1-t}\right]_p.$$

Now using Proposition 6.1, we complete the proof of the theorem. ∎

The following theorem is a p-adic analogue of (6.8) when $a = \frac{1}{6}$, $b = \frac{5}{6}$, and $c = 1$.

Theorem 6.3. *Let $p > 3$ be a prime and $t \in \mathbb{F}_p$ such that $t \neq 0, 1$. We have*

$${}_2G_2\left[\begin{matrix}\frac{1}{6}, & \frac{5}{6} \\ 0, & 0\end{matrix} \mid \frac{1}{t}\right]_p = \varphi(-1) \cdot {}_2G_2\left[\begin{matrix}\frac{1}{6}, & \frac{5}{6} \\ 0, & 0\end{matrix} \mid \frac{1}{1-t}\right]_p.$$

Proof. Let $E_t : y^2 = x^3 - 3x^2 + 4t$ be a family of elliptic curves where $t \in \mathbb{F}_p$ such that $t \neq 0, 1$. From [12, Theorem 3.4], we obtain

$$\begin{aligned} a_p(E_t) &= p \cdot \varphi(t) \cdot {}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix} \mid t\right]_p \\ &= -\frac{p \cdot \varphi(t)}{p-1} \sum_{l=0}^{p-2} \overline{\omega}^l(t) (-p)^{-\lfloor \frac{1}{3} + \frac{l}{p-1} \rfloor - \lfloor \frac{2}{3} + \frac{l}{p-1} \rfloor - 2\lfloor \frac{1}{2} - \frac{l}{p-1} \rfloor} \\ &\times \frac{\Gamma_p\left(\left\langle \frac{1}{2} - \frac{l}{p-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{1}{3} + \frac{l}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{2}{3} + \frac{l}{p-1} \right\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right)^2 \Gamma_p\left(\frac{1}{3}\right) \Gamma_p\left(\frac{2}{3}\right)}. \end{aligned}$$

Letting $l = \frac{p-1}{2} - k$, we have

$$a_p(E_t) = -\frac{p}{p-1} \sum_{k=1-\frac{p-1}{2}}^{\frac{p-1}{2}} \overline{\omega}^k \left(\frac{1}{t}\right) (-p)^{-\lfloor \frac{1}{6} - \frac{k}{p-1} \rfloor - \lfloor \frac{5}{6} - \frac{k}{p-1} \rfloor - 2\lfloor \frac{k}{p-1} \rfloor - 1} \times \frac{\Gamma_p\left(\left\langle \frac{k}{p-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{1}{6} - \frac{k}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{5}{6} - \frac{k}{p-1} \right\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right)^2 \Gamma_p\left(\frac{1}{3}\right) \Gamma_p\left(\frac{2}{3}\right)}.$$

Using Lemma 1.11 with $t = 3$ and $a = \frac{p-1}{2}$, and the fact that $\Gamma_p\left(\frac{1}{2}\right)^2 = -\varphi(-1)$, we deduce that

$$a_p(E_t) = -\frac{\varphi(-3)}{p-1} \sum_{k=1-\frac{p-1}{2}}^{\frac{p-1}{2}} \overline{\omega}^k \left(\frac{1}{t}\right) (-p)^{-\lfloor \frac{1}{6} - \frac{k}{p-1} \rfloor - \lfloor \frac{5}{6} - \frac{k}{p-1} \rfloor - 2\lfloor \frac{k}{p-1} \rfloor} \times \frac{\Gamma_p\left(\left\langle \frac{k}{p-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{1}{6} - \frac{k}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{5}{6} - \frac{k}{p-1} \right\rangle\right)}{\Gamma_p\left(\frac{1}{6}\right) \Gamma_p\left(\frac{5}{6}\right)}.$$

Since the above sum is invariant under $k \mapsto k + (p-1)$ so we have

$$\begin{aligned} a_p(E_t) &= -\frac{\varphi(-3)}{p-1} \sum_{k=0}^{p-2} \overline{\omega}^k \left(\frac{1}{t}\right) (-p)^{-\lfloor \frac{1}{6} - \frac{k}{p-1} \rfloor - \lfloor \frac{5}{6} - \frac{k}{p-1} \rfloor - 2\lfloor \frac{k}{p-1} \rfloor} \\ &\quad \times \frac{\Gamma_p\left(\left\langle \frac{k}{p-1} \right\rangle\right)^2 \Gamma_p\left(\left\langle \frac{1}{6} - \frac{k}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{5}{6} - \frac{k}{p-1} \right\rangle\right)}{\Gamma_p\left(\frac{1}{6}\right) \Gamma_p\left(\frac{5}{6}\right)} \\ &= \varphi(-3) \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & 0 \end{matrix} \mid \frac{1}{t} \right]_p . \end{aligned} \tag{6.12}$$

Replacing t with $1-t$, we obtain

$$a_p(E_{1-t}) = \varphi(-3) \cdot {}_2G_2 \left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ 0, & 0 \end{matrix} \mid \frac{1}{1-t} \right]_p . \tag{6.13}$$

Now, taking $x \mapsto x+1$ in E_t and E_{1-t} gives the following curves, respectively,

$$E_t : y^2 = x^3 - 3x + 4t - 2,$$
$$E_{1-t} : y^2 = x^3 - 3x + 2 - 4t.$$

It is easy to verify that E_{1-t} is a (-1)-quadratic twist of E_t. Therefore,

$$a_p(E_t) = \left(\frac{-1}{p}\right) a_p(E_{1-t}). \tag{6.14}$$

Combining (6.12), (6.13), and (6.14) we complete the proof of the theorem. ■

Remark 6.2.1. *In [38, Lemma 4], authors also proved the identities given by Theorems 6.2 and 6.3.*

We prove another identity for McCarthy's p-adic hypergeometric functions. We recall Bailey's cubic transformation [6, Eq. 4.06]:

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & 2b-a-1, & a+2-2b \\ & b, & a-b+\frac{3}{2} \end{matrix} \mid 4x\right] \\
&= (1-x)^{-a} \cdot {}_3F_2\left[\begin{matrix} \frac{a}{3}, & \frac{a+1}{3}, & \frac{a+2}{3} \\ & b, & a-b+\frac{3}{2} \end{matrix} \mid \frac{27x^2}{4(1-x)^3}\right].
\end{aligned} \tag{6.15}$$

In the following theorem we prove a p-adic analogue of (6.15) when $a = \frac{1}{2}$ and $b = 1$.

Theorem 6.4. *Let $p > 3$ be a prime and $x \in \mathbb{F}_p$ such that $x \neq 0, 1$. Then we have*

$$\begin{aligned}
p^2 \cdot {}_3F_2\left(\begin{matrix} \varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon \end{matrix} \mid 4x\right)_p &= {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0 \end{matrix} \mid \frac{1}{4x}\right]_p \\
&= \varphi(1-x) \cdot {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ 0, & 0, & 0 \end{matrix} \mid \frac{-4(x-1)^3}{27x^2}\right]_p
\end{aligned}$$

$$+\delta(x+2)\cdot\varphi(-1)\cdot p.$$

We prove Theorem 6.4 using a transformation of Fuselier et al. [30, Theorem 9.14]. We remark that there is a typo in [30, Theorem 9.14]. The factor $\frac{1}{(q-1)^2}$ should be $\frac{1}{q-1}$, and we recall the result below with this correction.

Proof of Theorem 6.4. For $x\neq 0,1$, using [30, Theorem 9.14] with $A=B=\varphi$ and Lemma 1.2 for $\chi=\varphi$ and $\chi=\varepsilon$, we have

$$A_x := {}_3\mathbb{F}_2\left[\begin{matrix}\varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon\end{matrix}\Big| 4x\right]_p = \frac{\varphi(1-x)}{p-1}\sum_{\chi\in\widehat{\mathbb{F}_p^\times}}\frac{g(\varphi\chi^3)g(\overline{\chi})^3}{g(\varphi)}\chi\left(\frac{x^2}{4(x-1)^3}\right)$$
$$+\delta(x+2)\varphi(-1)p. \quad (6.16)$$

Using Davenport-Hasse relation with $\psi=\chi^3$ and $m=2$, we have

$$g(\varphi\chi^3)=\frac{g(\chi^6)g(\varphi)\overline{\chi}^3(4)}{g(\chi^3)}. \quad (6.17)$$

Substituting (6.17) in (6.16), we obtain

$$A_x=\frac{\varphi(1-x)}{p-1}\sum_{\chi\in\widehat{\mathbb{F}_p^\times}}\frac{g(\chi^6)g(\overline{\chi})^3}{g(\chi^3)}\chi\left(\frac{x^2}{4^4(x-1)^3}\right)+\delta(x+2)\varphi(-1)p.$$

Taking $\chi=\omega^a$ yields

$$A_x=\frac{\varphi(1-x)}{p-1}\sum_{a=0}^{p-2}\frac{g(\overline{\omega}^{-6a})g(\overline{\omega}^{a})^3}{g(\overline{\omega}^{-3a})}\overline{\omega}^a\left(\frac{4^4(x-1)^3}{x^2}\right)+\delta(x+2)\varphi(-1)p.$$

Using Gross-Koblitz formula, we have

$$A_x=-\frac{\varphi(1-x)}{p-1}\sum_{a=0}^{p-2}\overline{\omega}^a\left(\frac{4^4(x-1)^3}{x^2}\right)(-p)^{3\langle\frac{a}{p-1}\rangle+\langle\frac{-6a}{p-1}\rangle-\langle\frac{-3a}{p-1}\rangle}$$

$$\times \frac{\Gamma_p\left(\frac{a}{p-1}\right)^3 \Gamma_p\left(\left\langle\frac{-6a}{p-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{-3a}{p-1}\right\rangle\right)} + \delta(x+2)\varphi(-1)p$$

$$= -\frac{\varphi(1-x)}{p-1}\sum_{a=1}^{p-2}\overline{\omega}^a\left(\frac{4^4(x-1)^3}{x^2}\right)(-p)^{-\lfloor\frac{-6a}{p-1}\rfloor+\lfloor\frac{-3a}{p-1}\rfloor}\frac{\Gamma_p\left(\frac{a}{p-1}\right)^3 \Gamma_p\left(\left\langle\frac{-6a}{p-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{-3a}{p-1}\right\rangle\right)}$$
$$-\frac{\varphi(1-x)}{p-1} + \delta(x+2)\varphi(-1)p.$$

Employing Lemma 1.11 with $t=6$ and $t=3$, and Lemma 1.13 with $d=3$ and $d=6$, we deduce that

$$A_x = -\frac{\varphi(1-x)}{p-1} - \frac{\varphi(1-x)}{p-1}\sum_{a=1}^{p-2}\overline{\omega}^a\left(\frac{4(x-1)^3}{27x^2}\right)(-p)^{-\lfloor\frac{1}{6}-\frac{a}{p-1}\rfloor-\lfloor\frac{5}{6}-\frac{a}{p-1}\rfloor-\lfloor\frac{1}{2}-\frac{a}{p-1}\rfloor}$$
$$\times\frac{\Gamma_p\left(\frac{a}{p-1}\right)^3\Gamma_p\left(\left\langle\frac{1}{6}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{5}{6}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{1}{2}-\frac{a}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{6}\right)\Gamma_p\left(\frac{5}{6}\right)\Gamma_p\left(\frac{1}{2}\right)}$$
$$+\delta(x+2)\varphi(-1)p.$$

Since $\overline{\omega}^a(-1)=(-1)^a$, we have $\overline{\omega}^a(-1)(-1)^a=1$. Substituting this value and then adding and subtracting the term under the summation for $a=0$, we deduce that

$$A_x = -\frac{\varphi(1-x)}{p-1}\sum_{a=0}^{p-2}\overline{\omega}^a\left(\frac{-4(x-1)^3}{27x^2}\right)(-1)^a(-p)^{-\lfloor\frac{1}{6}-\frac{a}{p-1}\rfloor-\lfloor\frac{5}{6}-\frac{a}{p-1}\rfloor-\lfloor\frac{1}{2}-\frac{a}{p-1}\rfloor}$$
$$\times\frac{\Gamma_p\left(\frac{a}{p-1}\right)^3\Gamma_p\left(\left\langle\frac{1}{6}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{5}{6}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{1}{2}-\frac{a}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{6}\right)\Gamma_p\left(\frac{5}{6}\right)\Gamma_p\left(\frac{1}{2}\right)}$$
$$+\delta(x+2)\varphi(-1)p$$
$$=\varphi(1-x)\cdot {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{6}, & \frac{5}{6}\\ 0, & 0, & 0\end{matrix}\mid\frac{-4(x-1)^3}{27x^2}\right]_p + \delta(x+2)\varphi(-1)p. \tag{6.18}$$

Also, using Definitions 1.9, 1.10, and 1.11, we obtain

$$A_x = {}_3\mathbb{F}_2\left[\begin{matrix}\varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon\end{matrix} \mid 4x\right]_p = \frac{p^2}{J(\varphi,\varepsilon)^2}\cdot {}_3F_2\left(\begin{matrix}\varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon\end{matrix} \mid 4x\right)_p. \tag{6.19}$$

Using (1.3) and (1.5), we have $J(\varphi,\varepsilon)^2 = 1$. Now, Proposition 1.7, Lemma 1.9, and (1.5) yield

$$A_x = {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0\end{matrix} \mid \frac{1}{4x}\right]_p. \tag{6.20}$$

Combining (6.18), (6.19), and (6.20), we obtain the required identity. ■

For $p \equiv 1 \pmod 3$, Greene proved the finite field analogue of (6.8) in [36, Theorem 4.4 (i)]. We have $\varphi(-3) = -1$ when $p \equiv 2 \pmod 3$ and $\varphi(-1) = -1$ when $p \equiv 3 \pmod 4$. Hence, using Theorems 6.2 and 6.3 for $t = \frac{1}{2}$, we obtain the following special values of ${}_2G_2[\cdots]_p$.

Corollary 6.2.1. *Let $p > 3$ be a prime.*

1. *For $p \equiv 2 \pmod 3$, we have*

$${}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ 0, & 0\end{matrix} \mid 2\right]_p = 0.$$

2. *For $p \equiv 3 \pmod 4$, we have*

$${}_2G_2\left[\begin{matrix}\frac{1}{6}, & \frac{5}{6} \\ 0, & 0\end{matrix} \mid 2\right]_p = 0.$$

Employing Theorem 6.4, we find certain special values of ${}_3G_3[\cdots]_p$ as listed in the following corollary. For $t \in \mathbb{F}_p$, let

$$\tilde{G}(t,p) := {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ 0, & 0, & 0 \end{matrix} \middle| t\right]_p .$$

Corollary 6.2.2. *Let $p > 3$ be a prime. We have*

$$(i)\quad \tilde{G}\left(\frac{1331}{8}, p\right) = \begin{cases} \varphi(33)(4x^2 - p), & \text{if } p \equiv 1 \pmod 4,\ x^2 + y^2 = p,\ x \text{ is odd;} \\ -p\varphi(33), & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

$$(ii)\quad \tilde{G}\left(\frac{125}{27}, p\right) = \begin{cases} \varphi(10)(4x^2 - p), & \text{if } p \equiv 1,3 \pmod 8, \text{ and } x^2 + 2y^2 = p; \\ -p\varphi(10), & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

$$(iii)\quad \tilde{G}\left(\frac{125}{4}, p\right) = \begin{cases} \varphi(5)(4x^2 - p), & \text{if } p \equiv 1 \pmod 3, \text{ and } x^2 + 3y^2 = p; \\ -p\varphi(5), & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

$$(iv)\quad \tilde{G}\left(-\frac{125}{64}, p\right) = \begin{cases} \varphi(105)(4x^2 - p), & \text{if } p \equiv 1,2,4 \pmod 7,\ x^2 + 7y^2 = p; \\ -p\varphi(105), & \text{if } p \equiv 3,5,6 \pmod 7. \end{cases}$$

$$(v)\quad \tilde{G}\left(\frac{614125}{64}, p\right) = \begin{cases} \varphi(1785)(4x^2 - p), & \text{if } p \equiv 1,2,4 \pmod 7,\ x^2 + 7y^2 = p; \\ -p\varphi(1785), & \text{if } p \equiv 3,5,6 \pmod 7. \end{cases}$$

Proof. (*i*) Taking $x = -\frac{1}{32}$ in Theorem 6.4 and then employing [57, Theorem 6 (ii)], we obtain the required values.

(*ii*) Taking $x = -\frac{1}{4}$ in Theorem 6.4 and then employing [57, Theorem 6 (iii)], we obtain the required values.

(*iii*) Taking $x = \frac{1}{16}$ in Theorem 6.4 and then employing [57, Theorem 6 (v)], we obtain the required values.

(*iv*) Taking $x = 16$ in Theorem 6.4 and then employing [57, Theorem 6 (vi)], we obtain the required values.

(*v*) Taking $x = \frac{1}{256}$ in Theorem 6.4 and then employing [57, Theorem 6 (vii)], we obtain the required values. ∎

6.3 Weight three newforms and ${}_3G_3[\cdots]_p$

In this section, we use the transformations listed in Theorems 6.2 and 6.4, and express the p-th Fourier coefficients of the modular forms defined in (6.1)-(6.3) in terms of special values of ${}_3G_3[\cdots]_p$. We first state a theorem of Beukers and Stienstra that describes the Fourier coefficients of the three modular forms given by (6.1)-(6.3).

Theorem 6.5. ([66, Eq. 14.2]). *If we define $\Phi_4(p) := a(p)$, $\Phi_3(p) := b(p)$, and $\Phi_2(p) := c(p)$, then the p-th Fourier coefficients of the modular forms are given by*

$$\Phi_M(p) = \begin{cases} 0 & \text{if } \left(\frac{-M}{p}\right) = -1; \\ 4a^2 - 2p & \text{if } \left(\frac{-M}{p}\right) = 1, \quad p = a^2 + Mb^2. \end{cases}$$

The following lemma expresses certain product of values of p-adic gamma functions in terms of a character sum.

Lemma 6.6. ([31, Lemma 3.4]). *For p an odd prime and $a \in \mathbb{Z}$, with $0 < a < p-1$, we have*

$$\frac{\Gamma_p\left(\left\langle \frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{1}{2} - \frac{a}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right)}(-p)^{-\left\lfloor \frac{1}{2} - \frac{a}{p-1}\right\rfloor} = -\sum_{t=2}^{p-1}\omega^a(-t)\varphi(t(t-1)).$$

The following classical identity is a quadratic transformation due to Whipple [5, p. 130, Eq. 3.1.15]:

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c \end{matrix} \mid z\right] \\ &= (1-z)^{-a} \cdot {}_3F_2\left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2}, & 1+a-b-c \\ & 1+a-b, & 1+a-c \end{matrix} \mid -\frac{4z}{(1-z)^2}\right]. \end{aligned} \tag{6.21}$$

In the following theorem, we recall a p-adic analogue of (6.21) when $a = b = c = \frac{1}{2}$.

Theorem 6.7. ([31, Theorem 2.5]). *Let p be an odd prime and define* $s(p) := \Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\lfloor\frac{p-1}{4}\rfloor+\lfloor\frac{p-1}{2}\rfloor}$. *For* $1 \neq x \in \mathbb{F}_p^{\times}$,

$$ {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0\end{matrix} \mid \frac{1}{x}\right]_p = s(p)\cdot\varphi(2(1-x))\cdot {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ 0, & 0, & 0\end{matrix} \mid -\frac{(1-x)^2}{4x}\right]_p $$
$$ + \delta(x+1)\cdot\varphi(-1)\cdot p. $$

We now express the p-th Fourier coefficients of the modular forms defined in (6.1)-(6.3) in terms of special values of certain p-adic hypergeometric functions.

Theorem 6.8. *Let p be an odd prime. Then we have*

$$ {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ 0, & 0, & 0\end{matrix} \mid 1\right]_p = c(p). $$

Proof. Using Theorem 6.7 with $x=-1$, we obtain

$$ {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0\end{matrix} \mid -1\right]_p = s(p)\cdot {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ 0, & 0, & 0\end{matrix} \mid 1\right]_p + p\cdot\varphi(-1). $$

It is easy to check that $s(p)=\varphi(2)$. Therefore, we have

$$ {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ 0, & 0, & 0\end{matrix} \mid 1\right]_p = \varphi(2)\cdot {}_3G_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0\end{matrix} \mid -1\right]_p - p\cdot\varphi(-2) $$
$$ = \varphi(2)\cdot p^2\cdot {}_3F_2\left(\begin{matrix}\varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon\end{matrix} \mid -1\right)_p - p\cdot\varphi(-2). \quad (6.22) $$

From [57, Theorem 6 (iii)], we obtain

$$
{}_3F_2\left(\begin{matrix} \varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon \end{matrix} \Big| -1\right)_p = \begin{cases} -\frac{\varphi(2)}{p}, & \text{if } p \equiv 5,7 \pmod 8; \\ \frac{\varphi(2)(4x^2-p)}{p^2}, & \text{if } p \equiv 1,3 \pmod 8 \text{ and } x^2+2y^2=p. \end{cases} \tag{6.23}
$$

Combining (6.22), (6.23), and Theorem 6.5 with the fact that $\varphi(-2)=1$ if $p \equiv 1,3 \pmod 8$ and $\varphi(-2)=-1$ if $p \equiv 5,7 \pmod 8$, we obtain the required result. ■

Theorem 6.9. *Let $p>3$ be a prime . Then we have*

$$
{}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3} \\ 0, & 0, & 0 \end{matrix} \Big| 1\right]_p = b(p).
$$

Proof. Let $p \equiv 1 \pmod 3$. From Lemma 1.9, Proposition 1.7, and (1.5), we have

$$
{}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3} \\ 0, & 0, & 0 \end{matrix} \Big| 1\right]_p = p^2 \cdot {}_3F_2\left(\begin{matrix} \varphi, & \chi_3, & \chi_3^2 \\ & \varepsilon, & \varepsilon \end{matrix} \Big| 1\right)_p,
$$

where χ_3 is a character on $\mathbb{F}_p$ of order 3. Employing [56, Proposition 4.2] with $d=3$, we obtain the desired result. Next, we prove the result for $p \equiv 2 \pmod 3$. Consider

$$
\begin{aligned}
A &:= {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3} \\ 0, & 0, & 0 \end{matrix} \Big| 1\right]_p \\
&= -\frac{1}{p-1} - \frac{1}{p-1}\sum_{a=1}^{p-2}(-1)^a(-p)^{-\left\lfloor \frac{1}{3}-\frac{a}{p-1}\right\rfloor - \left\lfloor \frac{2}{3}-\frac{a}{p-1}\right\rfloor - \left\lfloor \frac{1}{2}-\frac{a}{p-1}\right\rfloor} \\
&\times \frac{\Gamma_p\left(\frac{a}{p-1}\right)^3 \Gamma_p\left(\left\langle \frac{1}{3}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{2}{3}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle \frac{1}{2}-\frac{a}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{2}{3}\right)\Gamma_p\left(\frac{1}{2}\right)}.
\end{aligned}
$$

Using the fact that $\overline{\omega}^a(-1) = (-1)^a$ and Lemma 6.6, we obtain

$$A = -\frac{1}{p-1} + \frac{1}{p-1}\sum_{t=2}^{p-1}\varphi(t(t-1))\sum_{a=1}^{p-2}\overline{\omega}^a\left(\frac{1}{t}\right)(-p)^{-\lfloor\frac{1}{3}-\frac{a}{p-1}\rfloor-\lfloor\frac{2}{3}-\frac{a}{p-1}\rfloor}$$
$$\times\frac{\Gamma_p\left(\frac{a}{p-1}\right)^2\Gamma_p\left(\left\langle\frac{1}{3}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{2}{3}-\frac{a}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{2}{3}\right)}.$$

Adding and subtracting the term under the summation for $a = 0$, we have

$$A = -\frac{1}{p-1} + \frac{1}{p-1}\sum_{t=2}^{p-1}\varphi(t(t-1))\sum_{a=0}^{p-2}\overline{\omega}^a\left(\frac{1}{t}\right)(-p)^{-\lfloor\frac{1}{3}-\frac{a}{p-1}\rfloor-\lfloor\frac{2}{3}-\frac{a}{p-1}\rfloor}$$
$$\times\frac{\Gamma_p\left(\frac{a}{p-1}\right)^2\Gamma_p\left(\left\langle\frac{1}{3}-\frac{a}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{2}{3}-\frac{a}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{3}\right)\Gamma_p\left(\frac{2}{3}\right)} - \frac{1}{p-1}\sum_{t=2}^{p-1}\varphi(t(t-1))$$
$$= -\sum_{t=2}^{p-1}\varphi(t(t-1))\cdot {}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}\mid\frac{1}{t}\right]_p,$$

where the last equality is obtained by using the fact that $\sum_{t=2}^{p-1}\varphi(t(t-1)) = -1$. Employing Theorem 6.2 with the fact that $\varphi(-3) = -1$ for $p \equiv 2 \pmod 3$, we obtain

$$A = \sum_{t=2}^{p-1}\varphi(t(t-1))\cdot {}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}\mid\frac{1}{1-t}\right]_p.$$

Taking $t \mapsto 1-t$, we obtain

$$A = \sum_{t=2}^{p-1}\varphi(t(t-1))\cdot {}_2G_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}\\ 0, & 0\end{matrix}\mid\frac{1}{t}\right]_p = -A.$$

This yields $2A = 0$ and hence $A = 0$. Using Theorem 6.5, we complete the proof of the theorem. ■

Theorem 6.10. *Let $p > 3$ be a prime. Then we have*

$$ {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ 0, & 0, & 0 \end{matrix} |1\right]_p = \gamma(p)a(p), $$

where $\gamma(p) := -1$ if $p \equiv 5 \pmod{12}$ and $\gamma(p) := 1$ otherwise.

Proof. Substituting $x = -2$ in Theorem 6.4, we obtain

$$ p^2 \cdot {}_3F_2\left(\begin{matrix} \varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon \end{matrix} | -8\right)_p = \varphi(3) \cdot {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ 0, & 0, & 0 \end{matrix} |1\right]_p + \varphi(-1)p. \tag{6.24} $$

From [57, Theorem 6 (i)], we obtain

$$ {}_3F_2\left(\begin{matrix} \varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon \end{matrix} | -8\right)_p = \begin{cases} -\frac{1}{p}, & \text{if } p \equiv 3 \pmod{4}; \\ \frac{4x^2-p}{p^2}, & \text{if } p \equiv 1 \pmod{4}, \quad x^2 + y^2 = p, \ x \text{ is odd.} \end{cases} \tag{6.25} $$

Combining (6.24), (6.25), and Theorem 6.5 with the fact that $\varphi(3) = -1$ if $p \equiv 5, 7 \pmod{12}$ and $\varphi(3) = 1$ if $p \equiv 1, 11 \pmod{12}$, we complete the proof of the theorem. ∎

6.4 Proof of the Rodriguez-Villegas conjectures

In this section, we prove the Rodriguez-Villegas conjectures (6.5)-(6.7). We first recall a theorem that gives a congruence relation between p-adic hypergeometric functions ${}_3G_3[\cdots]_p$ and truncated hypergeometric series ${}_3F_2[\cdots]_{p-1}$.

Theorem 6.11. ([50, Theorem 2.5]). *Let $2 \leq d \in \mathbb{Z}$ and let p be an odd prime such* *that* $p \equiv \pm 1 \pmod{d}$. *Then*

$$
{}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{d}, & \frac{d-1}{d} \\ 0, & 0, & 0 \end{matrix} \Big| 1\right]_p \equiv {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{d}, & \frac{d-1}{d} \\ & 1, & 1 \end{matrix} \Big| 1\right]_{p-1} \pmod{p^2}.
$$

In the following theorem, we prove the Rodriguez-Villegas conjectures (6.5)-(6.7).

Theorem 6.12. *Let* $p \geq 5$ *be a prime. We have*

$$
\sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n} \equiv b(p) \pmod{p^2}, \tag{6.26}
$$

$$
\sum_{n=0}^{p-1} \frac{(4n)!}{n!^4} 256^{-n} \equiv c(p) \pmod{p^2}, \tag{6.27}
$$

$$
\sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!^3} 1728^{-n} \equiv \gamma(p)a(p) \pmod{p^2}, \tag{6.28}
$$

where $\gamma(p) := -1$ *if* $p \equiv 5 \pmod{12}$ *and* $\gamma(p) := 1$ *otherwise.*

Proof. We have

$$
{}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3} \\ & 1, & 1 \end{matrix} \Big| 1\right]_{p-1} = \sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n}.
$$

Now, Theorem 6.11 with $d = 3$ and Theorem 6.9 readily yields (6.26).

Similarly, we have

$$
{}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ & 1, & 1 \end{matrix} \Big| 1\right]_{p-1} = \sum_{n=0}^{p-1} \frac{(4n)!}{n!^4} 256^{-n}.
$$

Combining Theorem 6.11 with $d = 4$ and Theorem 6.8, we obtain (6.27).

Finally, we have

$${}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ & 1, & 1\end{matrix} \mid 1\right]_{p-1} = \sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!^3} 1728^{-n}.$$

Combining Theorem 6.11 with $d = 6$ and Theorem 6.10, we obtain (6.28). ■

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1. Sulakashna and R. Barman, *Certain transformations and values of p-adic hypergeometric functions*, Res. Number Theory 8 (2022), no. 4, Paper No. 93, 14 pp.

2. Sulakashna and R. Barman, *Number of $\mathbb{F}_q$-points on diagonal hypersurfaces and hypergeometric function*, Int. J. Number Theory 20 (2024), no. 10, 2575–2589.

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