A POSTERIORI ERROR ANALYSIS OF FINITE ELEMENT METHOD FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

by

G. Murali Mohan Reddy

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI-781039, INDIA

February, 2014

A POSTERIORI ERROR ANALYSIS OF FINITE ELEMENT METHOD FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

A thesis submitted

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

by

G. Murali Mohan Reddy

(Roll No. 08612303)

Department of Mathematics

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

February, 2014

To

my mother Sai and brother Tiru

Certificate

It is certified that the work contained in this thesis entitled **"A posteriori error analysis of finite element method for parabolic integro-differential equations"** by **G. Murali Mohan Reddy,** a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

February, 2014

Dr. Rajen Kumar Sinha
Professor
Department of Mathematics
Indian Institute of Technology Guwahati

Acknowledgement

In my research journey, from first step into IIT Guwahati till this time of departure, I have came across with several co-passengers. This dissertation would not be possible without their contribution and support.

It would be very unfair if I did not begin this note of acknowledgement without mentioning my supervisor Prof. Rajen Kumar Sinha for making the experience of working on this thesis worthwhile. I am deeply indebted to him for his invaluable guidance, utmost care and constant encouragement throughout this work. My queries, many times indiscernibly put, have been listened with patience and answered with profound clarity. I would like to thank him for carefully reading my thesis, providing useful feedbacks and posing interesting questions. I am also thankful to him for making me feel free to express my views and for sharing an excellent rapport with me.

I owe my thanks to the doctoral committee members Prof. R. Alam, Prof. D. C. Dalal and Prof. S. Natesan for their insightful comments and suggestions on my work. I would also like to thank Indian Institute of Technology Guwahati (IITG) for the facilities provided to me during my research work and to the Ministry of Human and Resource Development, Govt. of India, for offering me financial assistance for the completion of my thesis work. I offer my thanks to our lab assistant Mr. Santanu Majumdar and Mr. Pranpratim Borgohain for technical support. Sridhar Samal, Manoj Boro and Phatik Kumar of Department of Mathematics deserve special thanks for their assistance in all official matters.

I owe my sincere thanks to Dr. A. K. Chakrabarty and Dr. S. Bandopadhyay for clearing my countless queries with patience. They along with Prof. S. Natesan and Prof. S. N. Bora also deserve special thanks for their friendly nature.

I wish to thank all my friends and well-wishers for their love and encouragement during this period. I owe my thanks to Rajbhavan, Kiran, Himu, Deva, Biswajitda, Dinesh, Arnab, Kaushik, Santu, Abhishek, Anirban, jhuma and Kalyan as well as many others with whom I have spent many lively moments. Barun, Gowri, Jitu, Somu and

Chaitanya deserve a special thank for being with me all the time. I want to thank my colleagues for their help during these years. I would also like to thank my friends with whom I played cricket in IIT Guwahati. The passing clouds over the hills of IIT Guwahati, Sunday morning cricket, short cricket in afternoons, rigorous discussions on Mathematics and mild earthquakes in midnights are memorable addendum to my IIT Guwahati experience. These instances are heartily acclaimed.

I finish with Kharagpur, where the most basic source of my life energy resides: my family. I have an amazing family, perfect and unique in many ways. Their support has been unconditional all these years; they have given up many things for me to be at IIT Guwahati; they have cherished with me every great moment and supported me whenever I needed it.

February, 2014

With regards,

G. Murali Mohan Reddy

Department of Mathematics

Indian Institute of Technology Guwahati

Abstract

The aim of the thesis is to study *a posteriori* error analysis of finite element method for linear parabolic integro-differential equations (PIDEs) in a convex polygonal or polyhedral domain. PIDEs and their variants arise in various applications, such as heat conduction in material with memory, the compression of poro-viscoelasticity media, nuclear reactor dynamics and the epidemic phenomena in biology. Since PIDE may be thought of as a perturbation of the purely parabolic problem, it is therefore, natural to expect how the *a posteriori* error analysis of parabolic problems can be extended to PIDEs. Such an extension is not straightforward in the presence of Volterra integral term.

In isotropic settings, we derive *a posteriori* error estimators for both the spatially semidiscrete and the fully discrete (backward Euler and Crank-Nicolson) schemes for PIDEs. A novel space-time reconstruction operator is introduced which is an *a posteriori* counterpart of the Ritz-Volterra projection. Moreover, this reconstruction operator is a generalization of the elliptic reconstruction operator and we call it as Ritz-Volterra reconstruction operator. The Ritz-Volterra reconstruction operator is used in a crucial way to derive optimal order a posteriori error estimates in the $L^\infty(L^2)$-norm. The related *a posteriori* error estimates for the Ritz-Volterra reconstruction error are also established. For the Crank-Nicolson scheme, the derivation of the *a posteriori* error estimator relies essentially on the Ritz-Volterra reconstruction operator and a novel space-time quadratic (in time) reconstruction operator.

To reduce the number of degrees of freedom and computational effort to achieve the same convergence as compared to the isotropic meshes, we also consider the *a posteriori* error analysis for PIDE in an anisotropic framework. We derive *a posteriori* error estimators for both fully discrete backward Euler and Crank-Nicolson schemes for PIDEs in the $L^2(H^1)$-norm in a two dimensional convex polygonal domain. The *a posteriori* error indicators corresponding to space discretizations are derived using the anisotropic interpolation estimates in conjunction with a Zienkiewicz-Zhu error estimator to approach the error gradient. The error due to time discretization is derived using continuous, piecewise linear polynomial in time in case of backward Euler scheme, whereas to recover optimality for Crank-Nicolson scheme we introduce continuous, piecewise quadratic time reconstructions, namely, Crank-Nicolson memory reconstruction and three point reconstruction.

To observe the behaviour of the various estimators on isotropic mesh, we have carried out extensive numerical simulations to verify our theoretical findings. The estimators are shown to be of optimal order that matches with the error's norm.

Contents

Nomenclature **xiv**

List of Figures **xxii**

1 Introduction **1**

1.1 Problem description 1

1.2 Notations and preliminaries 2

1.3 Background and motivation 11

1.4 Organization of the thesis 23

2 Spatially Semidiscrete Error Analysis **27**

2.1 Introduction 27

2.2 Ritz-Volterra reconstruction 30

2.3 Error analysis 31

3 Fully Discrete Backward Euler Error Analysis **41**

3.1 Introduction 41

3.2 Error analysis 44

4 Fully Discrete Crank Nicolson Error Analysis **67**

4.1 Introduction 67

4.2 Quadratic space-time reconstruction 71

4.3 Error analysis 72

5 Backward Euler Anisotropic Error Analysis **97**

5.1 Introduction 97

5.2 Error analysis 99

6 Crank-Nicolson Anisotropic Error Analysis **115**

6.1 Introduction 115

6.2 Quadratic reconstructions for PIDE 118

6.3 Error analysis 121

7 Numerical Experiments **141**

8 Conclusions and Extensions **173**

8.1 Critical review of the results 173

8.2 Extensions and remarks 175

List of Publications **190**

NOMENCLATURE

PIDE	Parabolic Integro-Differential Equation
$\mathcal{A}$	Self-adjoint, uniformly positive definite second order linear elliptic partial differential operator of the form $-\nabla \cdot (A\nabla)$
$A = \{a_{ij}(x)\}$	Coefficient matrix corresponding to the operator $\mathcal{A}$
$\mathcal{B}(t,s)$	Second order partial differential operator of the form $-\nabla \cdot (B(t,s)\nabla)$
$B(t,s) = \{b_{ij}(x;t,s)\}$	Coefficient matrix corresponding to the operator $\mathcal{B}(t,s)$
$\mathbb{R}^n$	n-th dimensional Euclidean space
Ω	Bounded convex polygonal or polyhedral domain
$\partial\Omega$	Boundary of Ω
u	Exact solution of the PIDE
f	Non-homogenous term
u_0	Initial function
$L^p(\omega), 1 \leq p \leq \infty$	Standard Lebesgue space of order p over the measurable set ω
$\|\cdot\|_{L^p(\omega)}$	Norm on $L^p(\omega)$
$\langle \cdot, \cdot \rangle_\omega$	Standard $L^2(\omega)$ inner product
$W^{m,p}(\omega)$	Standard Sobolev space of order (m,p) over the measurable set ω
$\|\cdot\|_{m,p}$	Norm on $W^{m,p}(\Omega)$
$H^m(\Omega)$	Hilbert space $W^{m,2}(\Omega)$
$H_0^1(\Omega)$	Space of functions in $H^1(\Omega)$ that vanish on the boundary of Ω
$C^m(\overline{\Omega})$	Space of functions with continuous derivatives up to and including order m in $\overline{\Omega}$
$C_0^m(\Omega)$	Space of all $C^m(\Omega)$ functions with compact support in Ω

$C_0^\infty(\Omega)$	Space of all infinitely differentiable functions with compact support in Ω
$\text{supp}(\phi)$	Support of ϕ
$a(\cdot,\cdot)$	Bilinear form corresponding to the elliptic operator $\mathcal{A}$
$b(t,s;\cdot,\cdot)$	Bilinear form corresponding to the operator $\mathcal{B}(t,s)$
$b_s(t,s;\cdot,\cdot)$	Bilinear form corresponding to the operator $\mathcal{B}_s(t,s)$
$b_{ss}(t,s;\cdot,\cdot)$	Bilinear form corresponding to the operator $\mathcal{B}_{ss}(t,s)$
$b_t(t,s;\cdot,\cdot)$	Bilinear form corresponding to the operator $\mathcal{B}_t(t,s)$
$B_s(t,s)$	Matrix obtained by differentiating the matrix $\mathcal{B}(t,s)$ with respect to s
$B_{ss}(t,s)$	Matrix obtained by differentiating the matrix $\mathcal{B}(t,s)$ with respect to s twice
$B_t(t,s)$	Matrix obtained by differentiating the matrix $\mathcal{B}(t,s)$ with respect to t
α	Coercivity constant for $a(\cdot,\cdot)$
β	Continuity constant for $a(\cdot,\cdot)$
γ	Continuity constant for $b(t,s;\cdot,\cdot)$
γ'	Continuity constant for $b_s(t,s;\cdot,\cdot)$
γ''	Continuity constant for $b_{ss}(t,s;\cdot,\cdot)$
γ'''	Continuity constant for $b_t(t,s;\cdot,\cdot)$
$\mathcal{T}_h$	Shape regular, conforming triangulations of Ω for spatially semidiscrete finite element discretizations
$\mathcal{T}_n$	Shape regular, conforming triangulations of Ω for fully discrete finite element discretizations
$\mathcal{T}_h^A$	Conforming triangulations of Ω for anisotropic finite element discretizations
$\text{diam}(K)$	Longest side of K
h_K	$\text{diam}(K)$
$\mathcal{E}_h$	Set of internal sides of $\mathcal{T}_h$

$\mathcal{S}_n$	Set of internal sides of $\mathcal{T}_n$
$\sum_h$	Union of all internal sides $\bigcup_{E\in\mathcal{E}_h} E$
$\sum_n$	Union of all internal sides $\bigcup_{E\in\mathcal{S}_n} E$
$\mathcal{A}_{el}v$	Regular part of the distribution $-\text{div}(A\nabla v)$
$J_1[v]$	Spatial jump of the field $A\nabla v$ across an element side
$\mathcal{B}_{el}(t,s)v(s)$	Regular part of the distribution $-\text{div}(B(t,s)\nabla v(s))$
$J_2[v(s)]$	Spatial jump of the field $-\text{div}(B(t,s)\nabla v(s))$ across an element side
$\mathcal{A}_h$	Discrete operator corresponding to $\mathcal{A}$
$\mathcal{B}_h(t,s)$	Discrete operator corresponding to $\mathcal{B}(t,s)$
W_h	Ritz-Volterra projection operator
$\mathcal{R}$	Elliptic reconstruction operator
I_n	n-th subinterval of $[0,T]$
τ_n	Time step
h_n	Local meshsize function corresponds to each given triangulation $\mathcal{T}_n$
$\mathbb{V}_h$	Finite element space for spatially semidiscrete isotropic triangulations
$\mathbb{V}^n$	Finite element space for fully discrete isotropic triangulations
$\mathbb{V}_h^A$, $\mathbb{V}_h^{A,0}$	Finite element spaces for anisotropic triangulations
$\mathbb{P}_l$	Space of polynomials of degree $\leq l$
Π_h	Clément-type interpolation operator for isotropic mesh
T_K	Standard invertible affine map which maps the reference triangle $\hat{K}$ into the general element K
$r_{1,K}$, $r_{2,K}$	Stretching directions
$\lambda_{1,K}$, $\lambda_{2,K}$	Magnitude of stretching
$\lambda_{1,K}/\lambda_{2,K}$	Stretching factor
$U(t)$	Continuous, piecewise linear approximation in time for the fully discrete finite element solution on isotropic mesh
σ^n	Quadrature rule to approximate the Volterra integral term

P_h	L^2 projection operator for the spatially semidiscrete scheme
P_0^n	L^2 projection operator for the fully discrete scheme
R_h	Ritz projection operator
$\rho(t)$	Parabolic error
$\mathcal{R}_w$	Ritz-Volterra reconstruction operator
$\epsilon(t)$	Ritz-Volterra reconstruction error
$\mathcal{B}^*(t,s)$	Formal adjoint of the operator $\mathcal{B}(t,s)$
$\omega(t)$	Linear interpolant associated with $\mathcal{R}_w^{n-1}U^{n-1}$ and $\mathcal{R}_w^n U^n$ for the fully discrete backward Euler scheme
$\hat{\omega}_I(t)$	Linear interpolant associated with the vectors $\hat{\omega}(t_{n-1})$ and $\hat{\omega}(t_n)$
$\mathfrak{R}[u_h]$	Inner residuals for spatially semidiscrete scheme
$\mathfrak{J}[u_h]$	Jump residuals for spatially semidiscrete scheme
$\beta_S(u_h(t))$	Ritz-Volterra reconstruction error estimator in the L^2-norm for the spatially semidiscrete scheme
$\lambda_S(g(t))$	Oscillations of g in the L^2-norm for the spatially semidiscrete scheme
$\beta_{S,t}(u_h(t))$	Estimator for the time derivative of the Ritz-Volterra reconstruction error for the spatially semidiscrete scheme
$\mathfrak{R}^n[U]$	Inner residuals for fully discrete scheme
$\mathfrak{J}^n[U]$	Jump residuals for fully discrete scheme
$\alpha_{BE,n}(v)$	Ritz-Volterra reconstruction error estimator in the H^1-norm for the fully discrete backward Euler scheme
$\beta_{BE,n}(v)$	Ritz-Volterra reconstruction error estimator in the L^2-norm for the fully discrete backward Euler scheme
$\mathcal{Q}_{BE,1,n}(v)$, $\mathcal{Q}_{BE,2,n}(v)$	Quadrature error estimator for the fully discrete

	backward Euler scheme
$\sigma_{BE,1,m}$, $\sigma_{BE,2,m}$	Total estimators for the parabolic error ρ for the fully discrete backward Euler scheme
$\xi_{BE,n}$	Linear interpolation error estimator for the Volterra integral term for the fully discrete backward Euler scheme
$\zeta_{BE,n}$	Space error estimator for the fully discrete backward Euler scheme
$\eta_{BE,n}$	Time error estimator for the fully discrete backward Euler scheme
$\mu_{BE,n}$	Mesh modification error estimator for the fully discrete backward Euler scheme
$\lambda_{BE,n}$	Data oscillation error estimator for the fully discrete backward Euler scheme
$\hat{\rho}(t)$	Parabolic error for the fully discrete Crank-Nicolson scheme
$\varepsilon(t)$	Reconstruction error for the fully discrete Crank-Nicolson scheme
$\sigma(t)$	Time reconstruction error for the fully discrete Crank-Nicolson scheme
$\hat{\mathcal{U}}_{I,1}(t)$	Linear interpolant associated with $\hat{\mathcal{U}}(t_n)$, $\hat{\mathcal{U}}(t_{n-1})$
$\hat{\mathcal{U}}_{I,2}(t)$	Linear interpolant associated with $\hat{\mathcal{U}}(t_{n-1/2})$
$\sigma_{CN,1,m}$, $\sigma_{CN,2,m}$	Total estimators corresponding to parabolic error $\hat{\rho}(t)$ for the fully discrete Crank-Nicolson scheme
$\nu_{CN,n}$, $\Lambda_{CN,n}$	Time reconstruction error estimators for the fully discrete Crank-Nicolson scheme
$\alpha_{CN,n}(v)$	Ritz-Volterra reconstruction error estimator in the H^1-norm for the fully discrete Crank-Nicolson scheme
$\beta_{CN,n}(v)$	Ritz-Volterra reconstruction error estimator in the L^2-norm for

	the fully discrete Crank-Nicolson scheme
$\eta_{CN,n}$	Time error estimator for the fully discrete Crank-Nicolson scheme
$\zeta_{CN,n,1}$, $\zeta_{CN,n,2}$	Space error estimators for the fully discrete Crank-Nicolson scheme
$\xi_{CN,n}$	Linear interpolation error estimator for the fully discrete Crank-Nicolson scheme
$\mathcal{Q}_{CN,n}$	Quadrature error estimator for the fully discrete Crank-Nicolson scheme
$\mu_{CN,n}$	Mesh change estimator for the fully discrete Crank-Nicolson scheme
$\lambda_{CN,n,1}$, $\lambda_{CN,n,2}$	Data approximation error estimators for the fully discrete Crank-Nicolson scheme
$U_h(t)$	Continuous, piecewise linear approximation in time for the fully discrete finite element solution on anisotropic mesh
$\mathcal{Q}^A_{BE,1,n,K}$, $\mathcal{Q}^A_{BE,2,n,K}$	Quadrature error estimators for the fully discrete backward Euler scheme on anisotropic mesh
$\check{U}_h$	*Crank-Nicolson memory reconstruction*
$\hat{U}_h$	*Three-point reconstruction*
$\check{U}_{h,I,1}(t)$	Linear interpolant associated with the integral vector $\check{U}_{h,1}(t_{n-1/2})$
$\check{U}_{h,I,2}(t)$	Linear interpolant associated with the integral vectors $\check{U}_{h,2}(t_n)$ and $\check{U}_{h,2}(t_{n-1})$
$\hat{U}_{h,I,1}(t)$	Linear interpolant associated the integral vectors $\hat{U}_{h,1}(t_{n-1/2})$ and $\hat{U}_{h,1}(t_{n-3/2})$
$\hat{U}_{h,I,2}(t)$	Linear interpolant associated with the integral vectors $\hat{U}_{h,2}(t_{n-1/2})$ and $\hat{U}_{h,2}(t_{n-3/2})$

$\check{\mathcal{Q}}^{A}_{CN,n,K}$, $\hat{\mathcal{Q}}^{A}_{CN,n,K}$	Quadrature error estimators for the fully discrete Crank-Nicolson scheme on anisotropic mesh
ZZ	Zienkiewicz-Zhu
$\mathcal{G}_K(v_h)$	Post processing matrix
I_h	Lagrange's interpolant corresponding to $\mathbb{V}_h^{A,0}$
I_h^A	Approximate L^2 projection operator onto $\mathbb{V}_h^A$

List of Figures

1.1 The affine transformation $T_K : \hat{K} \to K$, where $\hat{K}$ is the equilateral reference triangle and K is an isosceles triangle. The unit circle is mapped into an ellipse with directions $r_{1,K}$ and $r_{2,K}$, the magnitude of stretching being $\lambda_{1,K}$ and $\lambda_{2,K}$. . . . 9

1.2 Pictorial representation of the thesis 25

2.1 Establishing optimality in the L^2-norm: An interconnection between projections and reconstructions 32

5.1 Example of an acceptable patch, where the size of the reference patch $\Delta_{\hat{K}}$ does not depends upon the aspect ratio $\frac{H}{h}$. 101

5.2 Example of an unacceptable patch, where the size of the reference patch $\Delta_{\hat{K}}$ depends upon the aspect ratio $\frac{H}{h}$. 101

7.1a This figure shows the behaviour of the error $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ for $\mathbb{P}_1$ elements. 145

7.1b For the coupling $\tau \approx h^2$, $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ decreases with second order. 145

7.2a This figure shows the behaviour of the error $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ for $\mathbb{P}_1$ elements. 146

7.2b For the coupling $\tau \approx h^2$, $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ decreases with first order. 146

7.3a This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the L^2-norm i.e., $\max_{0\le n\le m} \beta_{BE,n}(U^n)$ for $\mathbb{P}_1$ elements. 147

7.3b For the coupling $\tau \approx h^2$, $\max_{0\le n\le m} \beta_{BE,n}(U^n)$ decreases with second order. . . . 147

7.4a This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the H^1-norm i.e., $\left(\sum_{n=1}^{m} \tau_n \alpha_{BE,n}^2(U^n)\right)^{1/2}$ for $\mathbb{P}_1$ elements. 148

7.4b For the coupling $\tau \approx h^2$, $\left(\sum_{n=1}^{m} \tau_n \alpha_{BE,n}^2(U^n)\right)^{1/2}$ decreases with first order. . . 148

7.5a This figure shows the behaviour of the space estimator $\sum_{n=1}^{m}\tau_n\zeta_{BE,n}$ for $\mathbb{P}_1$ elements. 149
7.5b For the coupling $\tau \approx h^2$, $\sum_{n=1}^{m}\tau_n\zeta_{BE,n}$ decreases with second order. 149
7.6a This figure shows the behaviour of the time estimator $\sum_{n=1}^{m}\tau_n\eta_{BE,n}$ for $\mathbb{P}_1$ elements. 150
7.6b For the coupling $\tau \approx h^2$, $\sum_{n=1}^{m}\tau_n\eta_{BE,n}$ decreases with second order. 150
7.7a This figure shows the behaviour of the total estimator in $L^\infty(L^2(\Omega))$ for $\mathbb{P}_1$ elements. 151
7.7b This figure shows the inverse effectivity index of the total estimator in $L^\infty(L^2(\Omega))$ for the coupling $\tau \approx h^2$. 151
7.8a This figure shows the behaviour of the total estimator in $L^2(H^1(\Omega))$ for $\mathbb{P}_1$ elements. 152
7.8b This figure shows the inverse effectivity index of the total estimator in $L^2(H^1(\Omega))$ for the coupling $\tau \approx h^2$. 152
7.9a This figure shows the behaviour of the error $\|e\|_{L^\infty(L^2(\Omega))}$ for $\mathbb{P}_1$ elements. . . 153
7.9b For the coupling $\tau \approx h$, $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ decreases with first order. 153
7.10a This figure shows the behaviour of the error $\|e\|_{L^2(H^1(\Omega))}$ for $\mathbb{P}_1$ elements. . . 154
7.10b For the coupling $\tau \approx h$, $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ decreases with first order. 154
7.11a This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the L^2-norm i.e., $\max_{0\le n\le m}\beta_{BE,n}(U^n)$ for $\mathbb{P}_1$ elements. 155
7.11b For the coupling $\tau \approx h$, $\max_{0\le n\le m}\beta_{BE,n}(U^n)$ yields superconvergence. 155
7.12a This figure shows the behaviour of Ritz-Volterra reconstruction error estimator for the H^1-norm i.e., $\left(\sum_{n=1}^{m}\tau_n\alpha^2_{BE,n}(U^n)\right)^{1/2}$ for $\mathbb{P}_1$ elements. 156
7.12b For the coupling $\tau \approx h$, $\left(\sum_{n=1}^{m}\tau_n\alpha^2_{BE,n}(U^n)\right)^{1/2}$ decreases with first order. . . 156
7.13a This figure shows the behaviour of space estimator $\sum_{n=1}^{m}\tau_n\zeta_{BE,n}$ for $\mathbb{P}_1$ elements.157
7.13b For the coupling $\tau \approx h$, $\sum_{n=1}^{m}\tau_n\zeta_{BE,n}$ decreases with at least first order. 157
7.14a This figure shows the behaviour of time estimator $\sum_{n=1}^{m}\tau_n\eta_{BE,n}$ for $\mathbb{P}_1$ elements. 158
7.14b For the coupling $\tau \approx h$, $\sum_{n=1}^{m}\tau_n\eta_{BE,n}$ decreases with first order. 158
7.15a This figure shows the behaviour of the $L^\infty(L^2(\Omega))$ total estimator for $\mathbb{P}_1$ elements. 159

7.15b *This figure shows the inverse effectivity index of the $L^\infty(L^2(\Omega))$ total estimator for the coupling $\tau \approx h$.* . . . 159
7.16a This figure shows the behaviour of the $L^2(H^1(\Omega))$ total estimator for $\mathbb{P}_1$ elements. . . . 160
7.16b *This figure shows the inverse effectivity index of the $L^2(H^1(\Omega))$ total estimator for the coupling $\tau \approx h$.* . . . 160
7.17a This figure shows the behaviour of the error $\|e\|_{L^\infty(L^2(\Omega))}$ for $\mathbb{P}_1$ elements. 163
7.17b *We observe that $L^\infty(L^2(\Omega))$ error is of $O(h^2+\tau^2)$.* . . . 163
7.18a *This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the L^2-norm i.e., $\max_{0\le n\le m} \beta_{CN,n}$ for $\mathbb{P}_1$ elements.* . . . 164
7.18b *This figure shows that $\max_{0\le n\le m} \beta_{CN,n}$ decreases with second order.* . . . 164
7.19a *This figure shows the behaviour of the space estimator $\sum_{n=1}^{m} \tau_n \zeta_{CN,n}$ for $\mathbb{P}_1$ elements.* . . . 165
7.19b *This figure shows that $\sum_{n=1}^{m} \tau_n \zeta_{CN,n}$ decreases with second order.* . . . 165
7.20a *This figure shows the behaviour of the time reconstruction estimator $\left(\sum_{n=1}^{m} \tau_n \Lambda^2_{CN,n}\right)^{1/2}$ for $\mathbb{P}_1$ elements.* . . . 166
7.20b *This figure shows that $\left(\sum_{n=1}^{m} \tau_n \Lambda^2_{CN,n}\right)^{1/2}$ decreases with second order.* . . . 166
7.21a *This figure shows the behaviour of the time reconstruction estimator $\max_{0\le n\le m} \nu_{CN,n}$ for $\mathbb{P}_1$ elements.* . . . 167
7.21b *This figure shows that $\max_{0\le n\le m} \nu_{CN,n}$ decreases with second order.* . . . 167
7.22a *This figure shows the behaviour of the $L^\infty(L^2(\Omega))$ total estimator for $\mathbb{P}_1$ elements.* . . . 168
7.22b *This figure shows the inverse effectivity index of the $L^\infty(L^2(\Omega))$ total estimator.* 168
7.23 *Exact Solution of the PIDE (7.1) at $T = 0.1$.* . . . 169
7.24 *The backward Euler FEM solution for the PIDE (7.1) is simulated using $\mathbb{P}_1$ elements and computed using 33025 free nodes at $T = 0.1$ corresponding to $\tau = .003125$.* . . . 170
7.25 *The Crank-Nicolson FEM solution for the PIDE (7.1) is simulated using $\mathbb{P}_1$ elements and computed using 33025 free nodes at $T = 0.1$ corresponding to $\tau = .003125$.* . . . 171

1

Introduction

The main objective of this thesis is to study *a posteriori* error analysis of finite element method for linear parabolic integro-differential equations (PIDEs). This chapter introduces the problem and it contains the notations and preliminary materials to be used in the thesis. It also provides the survey for relevant literature and motivation for the present study. The chapter-wise description of the thesis is presented in the last section of this chapter.

1.1 Problem description

Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be a bounded convex polygonal or polyhedral domain with boundary $\partial\Omega$. In this thesis, we shall consider linear PIDEs of the form

$$u_t(x,t) + \mathcal{A}u(x,t) \;=\; \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \;\; (x,t) \in \Omega \times (0,T] \qquad (1.1)$$

subject to the initial condition

$$u(x,0) = u_0(x), \;\; x \in \Omega \qquad (1.2)$$

and the homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \;\; (x,t) \in \partial\Omega \times [0,T], \qquad (1.3)$$

where $u(x,t)$ is a real-valued function of x and t, $u_t(x,t) = \frac{\partial u}{\partial t}(x,t)$ and $T < \infty$. The operator $\mathcal{A}$ is a self-adjoint, uniformly positive definite second order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u)$$

and the operator $\mathcal{B}(t,s)$ is a second order partial differential operator of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient. The coefficient matrices $A = \{a_{ij}(x)\}$ and $B(t,s) = \{b_{ij}(x;t,s)\}$, $0 \leq s \leq t$ are two $n \times n$ real-valued matrices assumed to be in $L^\infty(\Omega)^{n\times n}$ in space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The initial function $u_0(x)$ and the nonhomogeneous term $f(x,t)$ are real-valued functions and assumed to be smooth for our purpose.

Such problems and variants of them arise in several physical phenomena such as heat conduction in material with memory, the compression of poro-viscoelasticity media, nuclear reactor dynamics and the epidemic phenomena in biology. Rigorous discussions on models for heat conduction in materials with memory can be found in Belleni-Morante [14], Coleman [26], Coleman and Gurtin [27], Gurtin [44], Gurtin and Pipkin [45], Miller [66], Nohel [67] and the references quoted therein. Further applications of the theory of PIDEs include the compression of poro-viscoelastic media (cf. Habetler and Schiffman [46]), nuclear reactor dynamics (cf. Pachpatte [69] and Pao [72, 73, 74]), the compartment model of a double-porosity system (cf. Hornung and Showalter [50]) and the epidemic phenomena in biology (cf. Capasso [20]).

PIDEs are often referred to as the parabolic partial differential equations with memory term or the Volterra integral term i.e. $\int_0^t \mathcal{B}(t,s)u(x,s)ds$. The presence of the Volterra integral term helps to accurately describe several physical phenomena, which causes some new difficulties in both theoretical analysis and numerical computation. Despite being the importance of the PIDEs and their variants in the modeling of several physical phenomena, the topic of *a posteriori* error analysis for such kind of equations remains unexplored. An attempt has been made in this thesis to study the *a posteriori* error analysis for PIDE (1.1).

1.2 Notations and preliminaries

In this section, we shall introduce some standard notations and preliminary materials to be used in this thesis. All functions considered here are real valued. Let Ω be a bounded domain in $\mathbb{R}^n$ (n-dimensional Euclidean space) and let $\partial\Omega$ denote the boundary of Ω. For $x = (x_1, x_2, \ldots, x_n) \in \Omega$, set $dx = dx_1 \ldots dx_n$. Further, let $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_n)$ be an n-tuple with nonnegative integer components. Denote the order of Υ as $|\Upsilon| = \Upsilon_1 + \Upsilon_2 + \ldots + \Upsilon_n$. Then, by $D^\Upsilon\phi$, we shall mean the Υth derivative of ϕ defined by

$$D^\Upsilon \phi = \frac{\partial^{|\Upsilon|}\phi}{\partial {x_1}^{\Upsilon_1} \ldots \partial {x_n}^{\Upsilon_n}}.$$

We shall make frequent reference to the following well-known function spaces. Given a Lebesgue measurable set $\omega \subset \mathbb{R}^n$, we denote by $L^p(\omega), 1 \leq p < \infty$, the linear space of equivalence classes of measurable functions ϕ on ω such that $\int_\omega |\phi(x)|^p dx$ exists and is finite. The norm on $L^p(\omega)$ is given by

$$\|u\|_{L^p(\omega)} = \left(\int_\omega |\phi(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

For $p = \infty$, $L^\infty(\omega)$ denotes the space of functions ϕ on ω such that

$$\|\phi\|_{L^\infty(\omega)} = \operatorname*{ess\,sup}_{x \in \omega} |\phi(x)| < \infty.$$

When $p = 2$, $L^2(\omega)$ is a Hilbert space with respect to the inner product

$$\langle \phi, \psi \rangle_\omega = \int_\omega \phi(x)\psi(x)dx.$$

Whenever $\omega = \Omega$, we remove the subscripts of $\|\cdot\|_{L^2(\omega)}$ and $\langle \cdot, \cdot \rangle_\omega$.

By support of a function ϕ, denoted by supp(ϕ), we mean the closure of all points x with $\phi(x) \neq 0$, i.e.,

$$\text{supp}(\phi) = \overline{\{x : \phi(x) \neq 0\}}.$$

For any nonnegative integer m, $C^m(\overline{\Omega})$ denotes the space of functions with continuous derivatives upto and including order m in $\overline{\Omega}$. $C_0^m(\Omega)$ is the space of all $C^m(\Omega)$ functions with compact support in Ω and $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω.

We now introduce the notion of Sobolev spaces. Let m be a nonnegative integer and let p be such that $1 \leq p < \infty$. The Sobolev space of order (m, p) on Ω, denoted by $W^{m,p}(\Omega)$, is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives upto order m are also in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) = \{\phi : \; D^\Upsilon \phi \in L^p(\Omega) \text{ for } 0 \leq |\Upsilon| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\begin{aligned} \|\phi\|_{m,p} &= \left(\int_\Omega \sum_{0 \leq |\Upsilon| \leq m} |D^\Upsilon \phi(x)|^p dx \right)^{\frac{1}{p}} \\ &= \left(\sum_{0 \leq |\Upsilon| \leq m} \|D^\Upsilon \phi\|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \end{aligned}$$

When $p = \infty$, the norm on the space $W^{m,\infty}(\Omega)$ is defined by

$$\|\phi\|_{m,\infty} = \max_{0 \leq |\Upsilon| \leq m} \|D^\Upsilon \phi(x)\|_{L^\infty(\Omega)}.$$

For $p = 2$, $W^{m,p}(\Omega) = H^m(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space with natural inner product defined by

$$\langle \phi, \psi \rangle_m = \sum_{0 \le |\Upsilon| \le m} \int_\Omega D^\Upsilon \phi D^\Upsilon \psi dx, \quad \phi, \psi \in H^m(\Omega).$$

The Sobolev space $H^m(\Omega)$ (respectively, $H_0^m(\Omega)$) is also defined as the closure of $C^\infty(\Omega)$ (respectively, $C_0^\infty(\Omega)$) with respect to the norm $\|\phi\|_m = \|\phi\|_{m,2}$. This result is true under some smoothness assumption on the boundary $\partial\Omega$. For a complete discussion on Sobolev spaces, see Adams and Fournier [2].

We shall also use the following space-time function spaces in our error analysis. For a given Banach space $\mathfrak{B}$, we define the space

$$L^2(0,T;\mathfrak{B}) = \left\{ \phi(t) \in \mathfrak{B} \text{ for a.e. } t \in (0,T) \text{ and } \int_0^T \|\phi(t)\|_{\mathfrak{B}}^2 dt < \infty \right\}$$

equipped with the norm

$$\|\phi\|_{L^2(0,T;\mathfrak{B})} = \left(\int_0^T \|\phi(t)\|_{\mathfrak{B}}^2 dt \right)^{\frac{1}{2}}.$$

Similarly, for a Banach space $\mathfrak{B}$, we define the space

$$L^\infty(0,T;\mathfrak{B}) = \left\{ \phi(t) \in \mathfrak{B} \text{ for a.e. } t \in (0,T) \text{ and } \operatorname*{ess\ sup}_{t \in [0,T]} \|\phi(t)\|_{\mathfrak{B}} < \infty \right\}$$

equipped with the norm

$$\|\phi\|_{L^\infty(0,T;\mathfrak{B})} = \operatorname*{ess\ sup}_{t \in [0,T]} \|\phi(t)\|_{\mathfrak{B}}.$$

When no risk of confusion exists we shall write $L^2(\mathfrak{B})$ for $L^2(0,T;\mathfrak{B})$ and $L^\infty(\mathfrak{B})$ for $L^\infty(0,T;\mathfrak{B})$.

Below, we shall discuss some preliminary materials and basic results to be used in the subsequent chapters.

Let $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the bilinear form corresponding to the elliptic operator $\mathcal{A}$ defined by

$$a(\phi,\psi) := \langle A\nabla\phi, \nabla\psi \rangle, \qquad \forall\ \phi,\psi \in H_0^1(\Omega).$$

Similarly, let $b(t,s;\cdot,\cdot)$ be the bilinear form corresponding to the operator $\mathcal{B}(t,s)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$b(t,s;\phi(s),\psi) := \langle B(t,s)\nabla\phi(s), \nabla\psi \rangle, \qquad \forall\ \phi(s),\psi \in H_0^1(\Omega).$$

Let $b_s(t,s;\cdot,\cdot)$ and $b_{ss}(t,s;\cdot,\cdot)$ be the bilinear forms corresponding to the operators $\mathcal{B}_s(t,s)$ and $\mathcal{B}_{ss}(t,s)$, respectively defined on $H_0^1(\Omega)\times H_0^1(\Omega)$ by

$$b_s(t,s;\phi(s),\psi) \ := \ \langle B_s(t,s)\nabla\phi(s),\nabla\psi\rangle, \qquad \forall\ \phi(s),\psi\in H_0^1(\Omega)$$

and

$$b_{ss}(t,s;\phi(s),\psi) \ := \ \langle B_{ss}(t,s)\nabla\phi(s),\nabla\psi\rangle, \qquad \forall\ \phi(s),\psi\in H_0^1(\Omega),$$

where $B_s(t,s)$ and $B_{ss}(t,s)$ are obtained by differentiating $B(t,s)$ partially with respect to s once and twice respectively. Further, the bilinear form $b_t(t,s;\cdot,\cdot)$, corresponding to the operator $\mathcal{B}_t(t,s)$, defined on $H_0^1(\Omega)\times H_0^1(\Omega)$ is given by

$$b_t(t,s;\phi(s),\psi) \ := \ \langle B_t(t,s)\nabla\phi(s),\nabla\psi\rangle, \qquad \forall\ \phi(s),\psi\in H_0^1(\Omega),$$

where $B_t(t,s)$ is obtained by differentiating $B(t,s)$ partially with respect to t. We assume that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi,\phi)\geq\alpha\|\phi\|_1^2 \quad \text{and} \quad |a(\phi,\psi)|\leq\beta\|\phi\|_1\|\psi\|_1, \qquad \forall\ \phi,\psi\in H_0^1(\Omega) \tag{1.4}$$

with $\alpha,\beta\in\mathbb{R}^+$. Further, we assume that the bilinear forms $b(t,s;\cdot,\cdot)$, $b_s(t,s;\cdot,\cdot)$, $b_{ss}(t,s;\cdot,\cdot)$ and $b_t(t,s;\cdot,\cdot)$ are continuous on $H_0^1(\Omega)$ i.e.,

$$|b(t,s;\phi(s),\psi)|\leq\gamma\|\phi(s)\|_1\|\psi\|_1, \qquad \forall\ \phi(s),\psi\in H_0^1(\Omega), \tag{1.5}$$

$$|b_s(t,s;\phi(s),\psi)|\leq\gamma'\|\phi(s)\|_1\|\psi\|_1, \qquad \forall\ \phi(s),\psi\in H_0^1(\Omega), \tag{1.6}$$

$$|b_{ss}(t,s;\phi(s),\psi)|\leq\gamma''\|\phi(s)\|_1\|\psi\|_1, \qquad \forall\ \phi(s),\psi\in H_0^1(\Omega) \tag{1.7}$$

and

$$|b_t(t,s;\phi(s),\psi)|\leq\gamma'''\|\phi(s)\|_1\|\psi\|_1, \qquad \forall\ \phi(s),\psi\in H_0^1(\Omega) \tag{1.8}$$

with $\gamma,\gamma',\gamma'',\gamma'''\in\mathbb{R}^+$. The weak formulation of the problem (1.1) may be stated as follows: Find $u:[0,T]\rightarrow H_0^1(\Omega)$ such that

$$\begin{aligned} \langle u_t,\phi\rangle + a(u,\phi) &= \int_0^t b(t,s;u(s),\phi)ds + \langle f,\phi\rangle, \ \ \forall\phi\in H_0^1(\Omega), \ \ t\in(0,T], \\ u(0) &= u_0. \end{aligned} \tag{1.9}$$

For the purpose of finite element procedure we consider two types of spatial discretizations in this thesis namely, *isotropic finite element discretization* and *anisotropic finite* *element discretization.* We first describe the standard isotropic finite element discretization as follows.

Isotropic finite element discretization. Let $h(x) = \text{diam}(K)$, where $K \in \mathcal{T}_h$ and $x \in K$, be a positive piecewise constant meshsize function corresponding to $\mathcal{T}_h = \{K\}$, a shape regular, conforming family of triangulations of Ω. Let $\mathcal{E}_h = \{E\}$ be the set of internal sides of $\mathcal{T}_h$. These internal sides are edges in $n = 2$ and faces in $n = 3$. Let $\sum_h$ denotes the union of all internal sides of $\mathcal{T}_h$ i.e., $\sum_h := \bigcup_{E \in \mathcal{E}_h} E$.

We now associate the following finite element space corresponding to $\mathcal{T}_h$:

$$\mathbb{V}_h := \{\chi \in H_0^1(\Omega) : \chi|_K \in \mathbb{P}_l(K), \text{ for all } K \in \mathcal{T}_h\}, \tag{1.10}$$

where $\mathbb{P}_l$ is the space of polynomials of degree $\leq l$ with $l \in \mathbb{Z}^+$. Let $P_h : L^2(\Omega) \to \mathbb{V}_h$ be the L^2 projection operator defined by

$$\langle P_h w, \chi \rangle = \langle w, \chi \rangle, \quad \forall \chi \in \mathbb{V}_h. \tag{1.11}$$

We define the discrete operators $\mathcal{A}_h : H_0^1(\Omega) \to \mathbb{V}_h$ and $\mathcal{B}_h(t,s) : H_0^1(\Omega) \to \mathbb{V}_h$ by

$$\langle \mathcal{A}_h w, \chi \rangle = a(w, \chi) \ \text{ and } \ \langle \mathcal{B}_h(t,s) w(s), \chi \rangle = b(t, s; w(s), \chi) \tag{1.12}$$

for all $\chi \in \mathbb{V}_h, 0 \leq s \leq t$, respectively. The spatially semidiscrete finite element approximation to (1.9) is then stated as follows: Find $u_h : [0,T] \to \mathbb{V}_h$ such that

$$\begin{aligned} \langle u_{h,t}, \chi \rangle + a(u_h, \chi) &= \int_0^t b(t, s; u_h(s), \chi) ds + \langle f, \chi \rangle, \quad \forall \chi \in \mathbb{V}_h, \\ u_h(.,0) &= P_h u_0. \end{aligned} \tag{1.13}$$

The following representation of the bilinear forms $a(\cdot,\cdot)$ and $b(t,s;\cdot,\cdot)$ will be of frequent use in the error analysis.
Representation of the bilinear forms. For a function $v \in \mathbb{V}_h$, the bilinear form $a(u,v)$ can be represented as

$$a(v, \phi) = \sum_{K \in \mathcal{T}_h} \langle -\text{div}(A \nabla v), \phi \rangle_K + \sum_{E \in \mathcal{E}_h} \langle J_1[v], \phi \rangle_E, \quad \forall \phi \in H_0^1(\Omega).$$

Here, $J_1[v]$ denotes the spatial jump of the field $A\nabla v$ across an element side $E \in \mathcal{E}_h$ defined as

$$J_1[v]|_E(x) = [A\nabla v]_E(x) := \lim_{\varepsilon \to 0} (A\nabla v(x + \varepsilon \mathbf{n}_E) - A\nabla v(x - \varepsilon \mathbf{n}_E)) \cdot \mathbf{n}_E, \tag{1.14}$$

where $\mathbf{n}_E$ is a unit normal vector to E at the point x. For $v \in \mathbb{V}_h$, let $\mathcal{A}_{el}v$ be the regular part of the distribution $-\text{div}(A\nabla v)$, which is defined as a piecewise continuous function such that

$$\langle \mathcal{A}_{el}v, \phi\rangle = \sum_{K\in\mathcal{T}_h} \langle -\text{div}(A\nabla v), \phi\rangle, \quad \forall\phi \in H_0^1(\Omega). \tag{1.15}$$

Thus, we can represent our bilinear form $a(\cdot,\cdot)$ as

$$a(v,\phi) = \langle \mathcal{A}_{el}v, \phi\rangle + \langle J_1[v], \phi\rangle_{\sum_h}, \quad \forall\phi \in H_0^1(\Omega). \tag{1.16}$$

Similarly, one can represent the bilinear form $b(t,s;\cdot,\cdot)$ as

$$b(t,s;v(s),\phi) = \langle \mathcal{B}_{el}(t,s)v(s), \phi\rangle + \langle J_2[v(s)], \phi\rangle_{\sum_h}, \quad \forall\phi \in H_0^1(\Omega), \tag{1.17}$$

where $\mathcal{B}_{el}(t,s)v(s)$ is the regular part of the distribution $-\text{div}(B(t,s)\nabla v(s))$, which is defined as a piecewise continuous function such that

$$\langle \mathcal{B}_{el}(t,s)v(s), \phi\rangle = \sum_{K\in\mathcal{T}_h} \langle -\text{div}(B(t,s)\nabla v(s)), \phi\rangle, \quad \forall\phi \in H_0^1(\Omega) \tag{1.18}$$

and $J_2[v(s)]$ is the spatial jump of the field $-\text{div}(B(t,s)\nabla v(s))$ across an element side $E \in \mathcal{E}_h$ as defined in (1.14) with $B(t,s)$ replacing A.

Now, we shall turn to introduce some notations for the fully discrete schemes. While dealing with the fully discrete schemes, we shall use the same symbols introduced for the spatially semidiscrete scheme by dropping the subscript index h and using the index n. Whenever there is some exception in the notations for the fully discrete schemes, we shall mention it explicitly.

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of $[0,T]$. For $n \in [1:N]$, let $\tau_n := t_n - t_{n-1}$ be the time step and $I_n := (t_{n-1}, t_n]$. We set $f^n(\cdot) = f(\cdot, t_n)$ for $t = t_n$, $n \in [0:N]$. Let $(\mathcal{T}_n)_{n\in[0:N]}$ be a family of conforming triangulations of the domain Ω. Let $h_n(x) = \text{diam}(K)$, where $K \in \mathcal{T}_n$ and $x \in K$, denotes the local meshsize function corresponds to each given triangulation $\mathcal{T}_n$. Let $\mathcal{S}_n$ denotes the set of internal sides of $\mathcal{T}_n$ and $\sum_n$ denotes the union of all internal sides i.e., $\sum_n := \bigcup_{E\in\mathcal{S}_n} E$. Each triangulation $(\mathcal{T}_n)$, for $n \in [1:N]$, is a refinement of a macro-triangulation $\mathcal{T}_0$ of the domain Ω that satisfies the same conformity and shape-regularity assumptions (cf. Brenner and Scott [15]) on its refinements. We assume the following admissible criteria on the mesh (cf. Lakkis and Makridakis [56]):

($A1$) The refined triangulation is conforming.

($A2$) The shape-regularity of an arbitrary refinement depends only on the shape-regularity of the macro-triangulation $\mathcal{T}_0$.

We allow only nested refinement of the space meshes at each time level $t = t_n$, $n \in [0 : N]$ i.e., for $0 \leq j \leq i \leq N$, $\mathcal{S}_i \cap \mathcal{S}_j = \mathcal{S}_j$. Now, we associate with these triangulations the finite element spaces:

$$\mathbb{V}^n := \{\phi \in H_0^1(\Omega) : \ \phi|_K \in \mathbb{P}_l, \ \forall K \in \mathcal{T}_n\}, \tag{1.19}$$

where $\mathbb{P}_l$ is as defined in (1.10).
Let $P_0^n : L^2(\Omega) \to \mathbb{V}^n$ denotes the L^2 projection operator and is given by

$$\langle P_0^n w, \chi_n \rangle = \langle w, \chi_n \rangle, \ \ \forall \chi_n \in \mathbb{V}^n. \tag{1.20}$$

Further, the fully discrete operators $\mathcal{A}^n : H_0^1(\Omega) \to \mathbb{V}^n$ and $\mathcal{B}^{n-r}(s) : H_0^1(\Omega) \to \mathbb{V}^n$, $0 \leq r < 1$ are defined by

$$\langle \mathcal{A}^n w, \chi_n \rangle = a(w, \chi_n), \ \ \forall \chi_n \in \mathbb{V}^n$$

and

$$\langle \mathcal{B}^{n-r}(s) w(s), \chi_n \rangle = b(t_{n-r}, s; w(s), \chi_n), \ \ \forall \chi_n \in \mathbb{V}^n.$$

Throughout this dissertation, the following notations will be used. For $n = 1, 2, \cdots, N$, set

$$\partial v^n := \frac{v^n - v^{n-1}}{\tau_n}, \qquad \bar{\partial} v^n = P_0^n \partial v^n := \frac{v^n - P_0^n v^{n-1}}{\tau_n},$$
$$t_{n-1/2} = \frac{t_n + t_{n-1}}{2} \qquad \text{and} \qquad v^{n-1/2} := \frac{v^n + v^{n-1}}{2}.$$

We now turn to describe anisotropic finite element discretization.
Anisotropic finite element discretization. In this thesis, residuals constitute the main building blocks of the *a posteriori* estimators obtained. In this context, anisotropic mesh plays an instrumental role to reduce the computational effort. The basic idea behind anisotropic mesh is to use a small meshsize in the direction of larger residual value and a larger meshsize in the direction of less residual value. To discretize the domain in an anisotropic framework, we first assume that Ω is a bounded convex polygonal domain of $\mathbb{R}^2$. Let $\mathcal{T}_h^A$ $(0 < h < 1)$ be a conforming triangulation of $\bar{\Omega}$ into triangles K (not necessarily satisfying the minimum angle condition) with $h_K \leq h$. Here, h_K denotes the diameter of the triangle K and h denotes the meshsize function corresponding to $\mathcal{T}_h^A$. We define by $\mathbb{V}_h^A$ the usual finite element space of continuous, piecewise linear functions on $\mathcal{T}_h^A$:

$$\mathbb{V}_h^A := \left\{ v_h \in C(\bar{\Omega}) : v_h|_K \in \mathbb{P}_1(K), \ \forall K \in \mathcal{T}_h^A \right\} \tag{1.21}$$

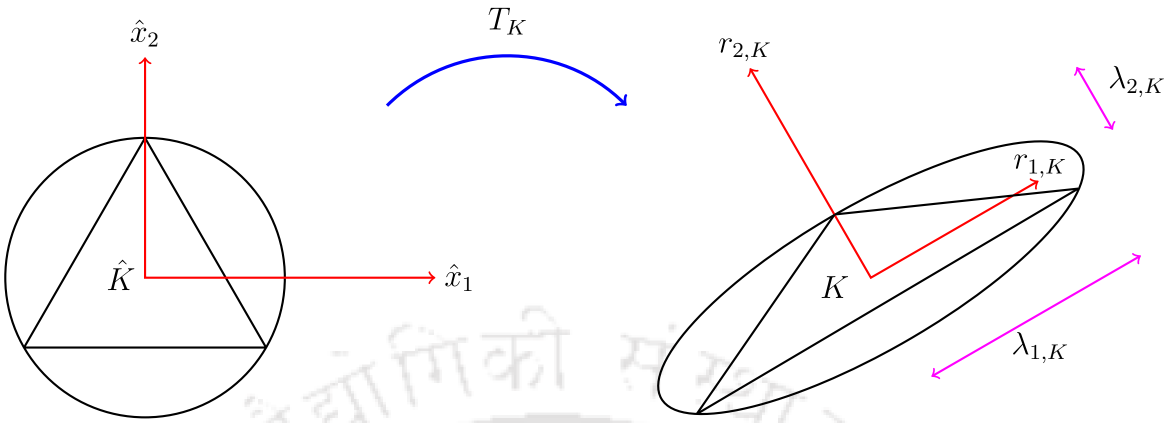

Figure 1.1: *The affine transformation $T_K : \hat{K} \to K$, where $\hat{K}$ is the equilateral reference triangle and K is an isosceles triangle. The unit circle is mapped into an ellipse with directions $r_{1,K}$ and $r_{2,K}$, the magnitude of stretching being $\lambda_{1,K}$ and $\lambda_{2,K}$.*

and we set

$$\mathbb{V}_h^{A,0} := \mathbb{V}_h^A \cap H_0^1(\Omega). \tag{1.22}$$

Below, we shall introduce some definitions to understand the mesh anisotropy. Let $T_K : \hat{K} \to K$ denotes the standard invertible affine map which maps the reference triangle $\hat{K}$ into the general element K of the triangulation $\mathcal{T}_h^A$. Let $P_K \in \mathbb{R}^{2\times 2}$ denotes the affine transformation matrix corresponding to T_K i.e.,

$$x = T_K(\hat{x}) = P_K\hat{x} + t_K, \quad \forall \hat{x} \in \mathbb{R}^2,$$

where $t_K \in \mathbb{R}^2$. Here, we study the spectral properties of the map T_K as it will be helpful in obtaining the anisotropic information about the size and orientation of the mesh element K. Now, invertibility of P_K ensures that it has singular value decomposition

$$P_K = R_K^T \Lambda_K S_K,$$

where R_K and S_K are both orthogonal matrices and Λ_K is a diagonal matrix with positive entries. Let us denote

$$\Lambda_K = \begin{pmatrix} \lambda_{1,K} & 0 \\ 0 & \lambda_{2,K} \end{pmatrix} \quad \text{and} \quad R_K = \begin{pmatrix} r_{1,K}^T \\ r_{2,K}^T \end{pmatrix} \tag{1.23}$$

with the choice $\lambda_{1,K} \geq \lambda_{2,K}$.

Thus, the deformation of any triangle $K \in \mathcal{T}_h^A$ with respect to $\hat{K}$ can be measured in terms of the stretching factor $\lambda_{1,K}/\lambda_{2,K} (\geq 1)$. For examples of such kind of transformation, we refer to Picasso [77, 78].

From time to time we shall also use the following inequalities (see Hardy *et al.* [47]):
(*i*) *Young's inequality.* For $a, b \geq 0$ and $\epsilon > 0$, the following inequality

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$$

holds.
(*ii*) *Cauchy-Schwarz inequality.* For all $a, b \geq 0, 1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

In integral form, if ϕ and ψ are both real valued functions with $\phi \in L^p(\Omega)$ and $\psi \in L^q(\Omega)$, then

$$\int_\Omega \phi\psi dx \leq \|\phi\|_p \|\psi\|_q.$$

For $p = q = 2$, the above inequality is known as *Schwarz's inequality.* The discrete version of the Schwarz's inequality may be stated as follows:
(*iii*) Let $\phi_j, \psi_j, j = 1, 2, \ldots, n$ be positive real numbers. Then

$$\sum_{j=1}^{n} \phi_j \psi_j \leq \left(\sum_{j=1}^{n} \phi_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \psi_j^2 \right)^{\frac{1}{2}}.$$

Now, we state Poincaré inequality which we shall use frequently in this thesis.

Lemma 1.2.1 (Poincaré inequality)**.** *Let Ω be a bounded open domain in $\mathbb{R}^n$. Then there exists a positive constant $C = C(\Omega)$ such that*

$$\|\phi\| \leq C\|\nabla\phi\| \quad \textit{for every} \;\; \phi \in H_0^1(\Omega).$$

In view of the Poincaré inequality, $\|\nabla(\cdot)\|$ defines a norm on $H_0^1(\Omega)$. Below, we state without proof, the following continuous version of Grownwall's lemma. For a proof, see Rao [81].

Lemma 1.2.2 (Gronwall's lemma)**.** *Let $G(t)$ be a continuous function and $H(t)$ a non-negative continuous function on its interval $t_0 \leq t \leq t_0 + a$. If a continuous function $F(t)$ has the property*

$$F(t) \leq G(t) + \int_{t_0}^{t} F(s)H(s)ds \quad \textit{for} \;\; t \in [t_0, t_0 + a],$$

then

$$F(t) \leq G(t) + \int_{t_0}^{t} G(s)H(s)exp\left[\int_{s}^{t} H(\tau)d\tau\right]ds \quad for\, t \in [t_0, t_0 + a].$$

In particular, when $G(t) = C$ a nonnegative constant, we have

$$F(t) \leq C exp\left[\int_{t_0}^{t} H(s)ds\right] \quad for\, t \in [t_0, t_0 + a].$$

The following lemma is proved to be convenient for later use (see Lakkis and Makridakis [56]).

Lemma 1.2.3. *For $a = (a_0, a_1, \ldots, a_n)$, $b = (b_0, b_1, \ldots, b_n) \in \mathbb{R}^{n+1}$ and $c \in \mathbb{R}$, if*

$$|a|^2 \leq c^2 + a \cdot b,$$

then

$$|a| \leq |c| + |b|,$$

where $|v| := \left(\sum_{j=0}^{n} v_j^2\right)^{\frac{1}{2}}$, $v = (v_0, v_1, \cdots, v_n)$.

1.3 Background and motivation

This section presents a brief survey of the relevant literature concerning PIDEs and their numerical solutions. It also elucidates the motivation for the present study.

Existence and uniqueness results. PIDE may be thought of as a perturbation of purely parabolic problem. This makes it possible to adapt the well-established existence and uniqueness theory of parabolic problems to such equations. Friedman and Shinbrot [36] have first investigated existence and uniqueness of solution for a PIDE in a general Banach space setting by constructing the resolvent operator when $\mathcal{A} \equiv \mathcal{A}(t)$ has a constant dense domain and $\mathcal{B}(t, s)$ is of convolution type operator. The related results are proved by Chen and Grimmer [25] under very reasonable assumptions which allows kernel $\mathcal{B}(t, s)$ to be optimally unbounded relative to $\mathcal{A}$. The existence of resolvent operator for non-autonomous integro-differential equation in a Banach space setting is examined by Grimmer [41]. Later, Acquistapace and Terreni [1] have studied the existence and uniqueness results in an abstract Banach space for non-autonomous PIDEs. However, their approach was different. They treated the problem as a perturbation of the purely parabolic problem and then used a fixed point argument. Subsequently, Cannon and Lin [18] have used the classical Picard iteration method to prove the existence and uniqueness results for time dependent PIDEs by treating the integral term as a perturbation of parabolic problem. Semigroup theoretic approach has been employed by Thomée and Zhang [95] to discuss the existence and uniqueness results for PIDE (1.1). Among the works concerning nonlinear versions of PIDEs, one may refer to Aizicovici [4], Bahuguna and Raghavendra [11], Heard and Rankin [48, 49], Lunardi and Sinestrari [61], Nohel [68], Sinestrari [86], Slodička [89, 90], Tanabe [92], Vrabie [99] and Webb [100, 101].

The main focus of this thesis is to derive *a posteriori* error estimators for the problem (1.1). In general, analytical solutions for (1.1) do not exist, hence numerical procedures such as *finite difference methods* and *finite element methods* are employed. At the outset we give a brief account of the literature concerning the finite difference methods for PIDEs.

Finite difference methods. The first contribution to the numerical solution of PIDEs is made by Douglas and Jones [30] using the finite difference method. They have formulated backward difference and Crank-Nicolson type methods for a nonlinear PIDE in one space variable subject to homogeneous Dirichlet boundary conditions and derived the convergence results. The proofs are based on finite-difference energy inequalities for the pure difference equation. Later, Pavlov [75] has obtained estimates for PIDE in one space dimension. Subsequently, Habetler and Schiffman [46] have discussed θ-method with stability estimates. The stability and convergence of difference schemes for quasi-linear PIDEs in two space variables is investigated by Dĵakonov [28]. In particular, he has formulated and analyzed two-level difference schemes with a split difference operator. In [93], Tavernini has considered a quasi-linear PIDE in a general Banach space settings and obtained convergence results using semigroup theoretic approach. Galeone *et al.* [37] have studied the numerical solution of a reaction-diffusion system involving a reaction term of integral type arising from biological models. They have shown the existence and the asymptotic behaviour of nonnegative numerical solutions by introducing upper and lower solutions. Further results in this direction are summarized by Brunner [16] and Yanik and Fairweather [105].

Finite element methods. In recent years, the error analysis of finite element methods for different class of problems has become one of the active research areas for applied mathematicians. One of the main reasons for the popularity of this method over the other numerical procedures in different fields of science and engineering is its ability to solve a wide class of problems with complicated structures in a simple and systematic way. The error analysis of finite element method is grouped into two categories: *A priori error analysis* and *a posteriori error analysis.*

A priori error analysis. There are wide range of articles available in literature regarding the *a priori* error estimates for PIDE (1.1) and their variants by finite element methods. The first contribution in this direction is given by Yanik and Fairweather [105]. Assuming the exact solution is smooth, they derived optimal order *a priori* error estimates for fully discrete Crank-Nicolson scheme for nonlinear PIDEs with $\mathcal{A} \equiv \mathcal{A}(t)$ and $\mathcal{B}(t,s)$ is a first-order partial differential operator. Subsequently, spatially semi-discrete scheme for PIDE (1.1) is thoroughly examined by Thomée and Zhang in [95]. They have obtained optimal order *a priori* error estimates in the L^2-norm for both smooth and non-smooth initial data by extending the spatially semidiscrete error analysis for linear parabolic equations [94] to PIDEs with an integral kernel consisting of a partial differential operator of order ≤ 2. The proof is based on the following decomposition of the main error $e = u - u_h$ as

$$e = (u - R_h u) + (R_h u - u_h),$$

where u_h and u denote the semidiscrete finite element solution and the exact solution of the PIDE, respectively. Here, $R_h : H_0^1(\Omega) \to \mathbb{V}_h$ is the Ritz projection introduced by Wheeler in [102] and is defined by

$$a(R_h v - v, \chi) = 0, \quad \forall \chi \in \mathbb{V}_h, \quad v \in H_0^1(\Omega). \tag{1.24}$$

A simple alternative approach is proposed by Cannon and Lin [17, 19] and is further developed by Lin *et al.* in [58]. The key technical tool used in these works is a generalization of the Ritz projection operator R_h, namely the nonlocal projection [17, 19] or the Ritz-Volterra projection operator [58] $W_h : H_0^1(\Omega) \to \mathbb{V}_h$, and is defined by

$$a((W_h u - u)(t), \chi) = \int_0^t b(t, s; (W_h u - u)(s), \chi)\, ds, \ \ \forall \chi \in \mathbb{V}_h, \ \ t \in [0, T]. \qquad (1.25)$$

The nonlocal projection or the Ritz-Volterra projection is used as an intermediate solution to obtain optimal error estimates for the problem (1.1) and its nonlinear variant with smooth initial data in [17, 19, 58]. Subsequently, stability estimates for Ritz-Volterra projections are thoroughly studied by Lin and Zhang [59]. In order to reduce the storage requirements during the time stepping of a general PIDE, Sloan and Thomée [88] have first proposed the application of quadrature rules with relatively higher order truncation error. Later on, several researchers have given valuable contributions towards the convergence analysis of the finite element Galerkin solution to the solution of PIDEs and its variants in the *a priori* framework. We refer to Cannon and Lin [17], Le Roux and Thomée [57], Thomée and Zhang [96], Chen *et al.* [24], Pani *et al.* [71], Pani and Sinha [70], McLean and Thomée [64], Chen and Shih [23], Zhang [106] and Sinha *et al.* [87] for further works in this direction.

A priori error analysis yields bounds of the form

$$\|u - U\|_X \leq C(u)h^r, \qquad (1.26)$$

where u and U are the exact and numerical solutions of a given problem. $C(u)$ is a positive constant depends on the exact solution u, h denotes the mesh parameter and X denotes a specified norm. The estimate (1.26) is not realistic in general as it depends on the exact solution which is unknown for most of the partial differential equations. Moreover, estimates of the form (1.26) can give asymptotic rates of convergence as the mesh parameters goes to zero, but are not designed to give an actual error estimate for a given mesh. The question of quantifying the error brings attention to a new error estimation method which is able to characterize explicitly the accuracy of approximate solutions and is known as *a posteriori* error estimation technique.

A posteriori error analysis. An *a posteriori* error estimate is a computable quantity in terms of the finite element solution and data of the given problem i.e., *a posteriori* error estimators employ the finite element solution and the data of the problem itself to derive estimates on the actual errors. On the contrary to *a priori* error analysis, *a posteriori* error analysis predicts bounds of the form

$$\|u - U\| \leq \eta(U, data), \tag{1.27}$$

where (i) the estimator $\eta(U, data)$ is a computable quantity which depends on the numerical solution U and the data of the problem; (ii) $\eta(U, data)$ decreases with optimal order with respect to the mesh parameters requiring the lowest possible regularity permitted by the problem. Estimates of the form (1.27) are of practical significance to physicists and engineers particularly for providing bounds on the errors and they are the basis for efficient adaptive meshing procedures designed to control and minimize the error.

In order to put the results of this thesis in a proper perspective, we give some relevant literature concerning *a posteriori* error analysis. Interest in *a posteriori* error estimation of finite element methods for two point boundary value problems has began with the pioneering work of Babuška and Rheinboldt [10]. A relatively complete theory for the derivation of *a posteriori* estimators regarding elliptic problems has been developed in Ainsworth and Oden [3], Babuška and Rheinboldt [9], Repin [82], Eriksson and Johnson [32], Grätsch and Bathe [39] and Verfürth [97] and the references quoted therein. However, *a posteriori* error analysis of the finite element method for parabolic problems has been a topic of investigation for the past two decades. The first significant contribution towards *a posteriori* error analysis of parabolic problems has been given by Eriksson and Johnson [33]. They have established quasi optimal error estimates in the $L^\infty(L^2(\Omega))$-norm via duality technique. However, this technique hinges on the parabolic regularizing effect which is not valid for estimates in the $L^2(H^1(\Omega))$-norm which motivates one to consider the standard energy technique. Several contributions over the last few years are devoted to *a posteriori* error estimates that are based on the energy approach. Picasso [76] has derived optimal *a posteriori* error estimates of residual type in the $L^2(H^1(\Omega))$-norm. Subsequently, optimal order estimates in the $L^2(H^1(\Omega))$-norm and suboptimal estimates in the $L^\infty(L^2(\Omega))$-norm are obtained by Verfürth [98] using energy argument for the heat equation. It is observed in [76, 98] that the energy technique for *a posteriori* error analysis of finite element discretizations of parabolic problems yields suboptimal rates in the $L^\infty(L^2(\Omega))$-norm.

Since the energy method is the most elementary technique for estimating the error in the *a priori* analysis, the question of whether this technique can be successfully applied in the *a posteriori* error analysis for the parabolic problems to obtain optimal bounds in the $L^\infty(L^2(\Omega))$-norm is very natural which is successfully addressed by Makridakis and Nochetto in [63]. They have gifted elliptic reconstruction operator $\mathcal{R}$ which restores optimality in the *a posteriori* error estimation for the purely parabolic problems in the $L^\infty(L^2(\Omega))$-norm for the spatially semidiscrete case. Earlier, a similar function to the elliptic reconstruction has been used by de Frutos and Novo [29] to prove *a posteriori* error estimates of the p-version of spatially discrete schemes for parabolic equations and its improved approximation properties are used by García-Archilla and Titi [38]. Nevertheless, this approach has gained popularity later on after the introduction of elliptic reconstruction operator $\mathcal{R}$ (cf. [12, 56, 63]) which is defined as follows: For a function $v \in H_0^1(\Omega)$, the elliptic reconstruction operator $\mathcal{R} : H_0^1(\Omega) \to H_0^1(\Omega)$ is defined by

$$a(\mathcal{R}v, \phi) = \langle \mathcal{A}_h v, \phi \rangle \quad \forall \phi \in H_0^1(\Omega), \tag{1.28}$$

where $\mathcal{A}_h$ is the discrete operator corresponding to the elliptic operator $\mathcal{A}$. Note that the elliptic reconstruction operator is an *a posteriori* dual to Wheeler's elliptic projection operator [102]. To restore the optimality, the usual idea is to split the error $e = u - u_h$ into two parts $e := u - u_h := (u - \mathcal{R}u_h) + (\mathcal{R}u_h - u_h)$ such that

- well established theory of *a posteriori* error estimation techniques for elliptic problems can be applied to bound the reconstruction (spatial) error $\mathcal{R}u_h - u_h$,

- the parabolic (temporal) error $u - \mathcal{R}u_h$ satisfies a variant of the original partial differential equation with a right-hand side that can be controlled *a posteriori* in an optimal way.

Since PIDEs may be thought of as a perturbation of the purely parabolic problems, it is therefore, natural to see how the *a posteriori* error analysis of parabolic problems can be extended to PIDEs. We would like to emphasize the fact that such an extension is not straightforward in the presence of Volterra integral term.

We first study *a posteriori* error analysis for the spatially semidiscrete finite element method for PIDE (1.1). More precisely, an attempt has been made to extend the analysis for the spatially semidiscrete scheme for the parabolic problems (cf. Makridakis and Nochetto [63]) to PIDE (1.1). A novel reconstruction operator is introduced, which is a generalization of the elliptic reconstruction operator $\mathcal{R}$ [63]. This operator is the partial right inverse of the Ritz-Volterra projection operator [17, 58] and hence, we call it as the Ritz-Volterra reconstruction operator $\mathcal{R}_w$. Using Ritz-Volterra reconstruction of the finite element solution u_h as an intermediate object, we split the main error e as Ritz-Volterra reconstruction error $\epsilon := \mathcal{R}_w u_h - u_h$ and parabolic error $\rho := u - \mathcal{R}_w u_h$. Thus, the *a posteriori* error bound on the main error e in the $L^\infty(L^2(\Omega))$-norm (see Theorem 2.3.1) is obtained by combining the error bounds on the Ritz-Volterra reconstruction error ϵ (see Lemma 2.3.1) and the parabolic error ρ (see Lemma 2.3.3). However, it is noteworthy that unlike for the parabolic problems [63], we don't have any *a posteriori* error estimators readily available in the literature to control the reconstruction error ϵ. Duality technique is used to derive bound on the reconstruction error ϵ, whereas the bound on the parabolic error ρ hinges on the energy argument. On the other hand, bound on the parabolic error in turn relies on the *a posteriori* error bound on the time derivative of the reconstruction error i.e., $\epsilon_t := (\mathcal{R}_w u_h - u_h)_t$ (see Lemma 2.3.2) which again needs duality technique into consideration.

Our next objective is to study the fully discrete backward Euler time discretization scheme for PIDE (1.1). In the absence of memory term i.e., when $\mathcal{B}(t,s)=0$, *a posteriori* error analysis for linear parabolic problems for fully discrete backward Euler scheme has been investigated by Picasso [76], Verfürth [98], and Lakkis and Makridakis [56]. Picasso [76] and Verfürth [98] have employed piecewise linear elements for the space discretizations and backward Euler scheme for the time discretizations to obtain optimal *a posteriori* error bounds in the $L^2(H^1(\Omega))$-norm but are valid without mesh change effect. Later, Lakkis and Makridakis [56] have considered the effect of mesh change and they have obtained optimal error estimate in the $L^\infty(L^2(\Omega))$-norm for the fully discrete backward Euler scheme. Their analysis is based on energy technique in the reconstruction framework.

In the present work, we have extended the spatially semidiscrete *a posteriori* error analysis of the finite element method for PIDE (1.1) to the fully-discrete backward Euler scheme. We decompose the main error $e=u-U$ into what we call as the Ritz-Volterra reconstruction error $\epsilon=U-\mathcal{R}_wU$ and the parabolic error $\rho=\mathcal{R}_wU-u$, where U is the continuous, piecewise linear time approximation of the fully discrete finite element solution. To obtain the *a posteriori* error bounds for the main error e in the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms, we first derive *a posteriori* error bounds on the Ritz-Volterra reconstruction error ϵ (see Lemma 3.2.2) which in turn relies on quadrature error bounds (see Lemma 3.2.1). An energy argument is used to bound the parabolic error ρ (see Lemma 3.2.3) which in turn relies essentially on the bounds of several errors namely, *space discretization error* (see Lemma 3.2.6), *time discretization error* (see Lemma 3.2.7), *mesh change error* (see Lemma 3.2.8) and *data oscillation error* (see Lemma 3.2.9). Finally, we estimate the main error e (see Theorem 3.2.1) by combining the error bounds on the Ritz-Volterra reconstruction error ϵ and the parabolic error ρ. Moreover, the error estimators obtained for PIDE (1.1) concerning the fully discrete backward Euler scheme generalizes the results of Lakkis and Makridakis [56] for the purely parabolic problems to PIDE (1.1).

To study higher order scheme in time, our next aim is to introduce fully discrete

Crank-Nicolson scheme for PIDE (1.1). When $\mathcal{B}(t,s) \equiv 0$, *a posteriori* error analysis concerning the Crank-Nicolson method for the parabolic problems is studied by Akrivis *et al.* [5], Bänsch *et al.* [12], Lozinski *et al.* [60], Picasso and Prachittham [79], and Verfürth [98]. For a continuous, piecewise linear approximation in time, Verfürth [98] has derived suboptimal (with respect to time steps) *a posteriori* error bound for the heat equation for the fully discrete Crank-Nicolson scheme using the standard energy technique. A continuous, piecewise quadratic polynomial so-called Crank-Nicolson reconstruction is then introduced by Akrivis *et al.* in [5] to restore the second order of convergence for the time discretization of a general parabolic problem. Later, Lozinski *et al.* [60] have introduced the reconstructions based on approximations on one time level (two-point reconstruction) as well as on two time levels (three-point reconstruction) to obtain estimators in the $L^2(H^1(\Omega))$-norm for the fully discrete Crank-Nicolson scheme. Subsequently, the effect of mesh change has been considered by Bänsch *et al.* in [12] for the analysis for the fully discrete Crank-Nicolson scheme for parabolic problems. They have employed the energy technique to establish optimal order *a posteriori* error estimate in the $L^\infty(L^2(\Omega))$-norm.

In this thesis, an effort has been made to extend the analysis of the fully discrete Crank-Nicolson finite element method for the parabolic problems of Bänsch *et al.* [12] to PIDE (1.1). Following the works [5, 12, 60], we have introduced a quadratic (in time) space-time reconstruction $\hat{U}$ to obtain the second order convergence in time. The function $\hat{U}$ is problem dependent, and is such that

- the difference $\hat{U} - \mathcal{R}_w U$ can be estimated in an *a posteriori* manner and is of $O(\tau^2)$,
- $\hat{U}$ agrees with U at the nodal time points.

This quadratic space-time reconstruction operator along with the Ritz-Volterra reconstruction operator are used in a crucial way to obtain *a posteriori* error bounds in the $L^\infty(L^2(\Omega))$-norm of the error (see Theorem 4.3.1) for the PIDE (1.1). This is accomplished by decomposing the error in a traditional way

$$e := u - U := \left(u - \hat{U}\right) + \left(\hat{U} - \mathcal{R}_w U\right) + \left(\mathcal{R}_w U - U\right) := \hat{\rho} + \sigma + \epsilon,$$

where $\hat{\rho} = u - \hat{U}$ denotes the parabolic error, $\sigma = \hat{U} - \mathcal{R}_w U$ denotes the time reconstruction error and $\epsilon = \mathcal{R}_w U - U$ denotes the Ritz-Volterra reconstruction error. We have used energy argument to bound the parabolic error $\hat{\rho}$ which includes contributions from other source of errors namely, *space discretization error* (see Lemma 4.3.7), *time discretization error* (see Lemma 4.3.5), *mesh change error* (see Lemma 4.3.6) and *data oscillation error* (see Lemma 4.3.8). The *a posteriori* error bounds on the reconstruction error ϵ are also established (see Lemma 4.3.1).

Next, we turn our attention to study *a posteriori* error analysis using anisotropic finite element discretizations. In an isotropic finite element method the aspect ratio (ratio of the diameters of the circumscribed and inscribed circles of a finite element) is bounded by a constant. But, the recent literature survey reveals that, this restriction on mesh can be relaxed and one can achieve a given level of accuracy with fewer vertices using anisotropic mesh [34, 35, 60, 65, 77, 79]. Anisotropic mesh reduces the number of degrees of freedom and computational effort leading to the reduction of memory to achieve the same convergence as compared to the isotropic mesh. For a detailed discussions on the anisotropic mesh, we refer to Apel [6] and Grosman [43], and the references mentioned therein.

First mathematical consideration of anisotropic elements go back to the fifties [91] and seventies [8, 40]. Nevertheless, the majority of works on the finite element method excludes such elements. Recently, error estimates have been proposed for anisotropic meshes, see Apel [6], Apel and Dobrowolski [7], Formaggia and Perotto [34, 35], Křížek [51], Lozinski *et al.* [60], Micheletti and Perotto [65], Picasso [77, 78], Picasso and Prachittham [79] and Prachittham [80]. An *a posteriori* error estimator is developed for anisotropic refinement of finite element grids in connection with second order linear elliptic boundary value problems in two and three dimensions by Siebert [85]. Subsequently, Formaggia and Perotto [35] have obtained *a posteriori* estimates for the elliptic partial differential equations in two dimensions on triangular meshes. In [78], Picasso has established anisotropic *a posteriori* error estimates for the Laplace equation in $H^1(\Omega)$ semi-norm. Later, Picasso [77] himself extended the results to elliptic and parabolic equations using the famous Zienkiewicz-Zhu (ZZ) estimator [108]. Among the various *a posteriori* error estimation techniques available in the literature, ZZ estimator is one of the most popular in practice. The idea behind this estimator is as follows. Since the gradient ∇u_h is less accurate than the solution, we recover an improved gradient, say $\nabla^* u_h$, by suitably fitting ∇u_h over some patches of elements. The discrepancy $\|\nabla^* u_h - \nabla u_h\|$ then identifies an estimator for the $H^1(\Omega)$ semi-norm of the discretization error $u - u_h$. Earlier, Křížek and Neittaanmäki [52] have studied some averaging techniques though the idea is nearly as old as the finite element method itself. Later on, the pioneering work by Zienkiewicz and Zhu [108], and further papers [109, 110] by them have motivated several researchers to study theoretically the amazingly good properties of the ZZ estimator for various problems. Serious efforts have been made to understand the approximation properties of this estimator and subsequently it has been realized that this ZZ estimator is effective under some smoothness assumption on the solution and the domain (see Durán *et al.* [31], Křížek and Neittaanmäki [52, 53], Lakhany *et al.* [55]). Theoretical properties of different types of ZZ like error estimators are also considered in [13, 21, 54, 104, 107]. Other than the ZZ estimator, interpolation error estimates [6, 7, 34, 35, 51] are proved to be the essential ingredients for the *a posteriori* error analysis of different physical problems on anisotropic mesh in the past decade.

In this thesis, we have extended the results of Picasso [77] concerning the fully discrete backward Euler discretizations for the parabolic problem to the PIDE (1.1) in an anisotropic framework. The emphasis is on the theoretical aspect of the anisotropic error analysis. We have derived two optimal order residual based anisotropic *a posteriori* error bounds in the $L^2(H^1(\Omega))$-norm (see Theorem 5.2.1 and Theorem 5.2.2). We first relate the error to the equation residual. Then, by localizing the residual term over each of the elements and the edges of the triangulation, we use the anisotropic interpolation error estimates [34, 35]. Finally, a ZZ estimator [108, 109, 110] is used to approach the error gradient matrix. A linear approximation of the Volterra integral term is used to estimate the quadrature error in the second estimator.

Our next and final objective is to study the fully-discrete Crank-nicolson scheme for the PIDEs on the anisotropic mesh. When $\mathcal{B}(t, s) = 0$, the literature concerning the anisotropic error analysis of the fully discrete Crank-Nicolson finite element method for the parabolic problems seems to be very limited. Inspired from Akrivis *et al.* [5], first optimal anisotropic error estimators in the $L^2(H^1(\Omega))$-norm are obtained by Lozinski *et al.* [60] for the purely parabolic problem. Subsequently, anisotropic *a posteriori* error estimates are obtained for the time-dependent convection-diffusion problem for the fully-discrete Crank-Nicolson scheme by Picasso and Prachittham [79].

In the present work, an effort has been made to generalize the fully discrete Crank-Nicolson anisotropic results of Lozinski *et al.* [60] for the purely parabolic problem to PIDE (1.1). We have derived two optimal order residual based *a posteriori* error estimates (see Theorem 6.3.1 and Theorem 6.3.2) for the PIDE (1.1) in the $L^2(H^1(\Omega))$-norm. For the time discretization error, we introduce a continuous, piecewise quadratic polynomial function so called Crank-Nicolson memory reconstruction in time which is a direct transposition of the two point reconstruction introduced by Lozinski *et al.* [60]. While analyzing with the Crank-Nicolson memory reconstruction, a linear approximation of the Volterra integral term is used in a crucial way to estimate the quadrature error. However, due to the presence of the memory term this reconstruction depends on all the previous time levels and therefore, it is not locally defined in time. Thus, we define a local time reconstruction (based on two subintervals) by considering an analogue of the Crank-nicolson memory reconstruction so called three point reconstruction based on finite difference approximation. Further, an extended linear approximation of the Volterra integral term is used to estimate the quadrature error while analyzing with the later reconstruction.

1.4 Organization of the thesis

This thesis consists of eight chapters, and is organized as follows. Chapter 1 introduces the problem and it contains the basic notations and preliminary materials to be used throughout this thesis. The motivation for the present study is also discussed.

Chapter 2 deals with the error analysis for the spatial semidiscretization of the PIDE (1.1). An attempt has been made to extend known results for the spatially semidiscrete error analysis for the parabolic problems by Makridakis and Nochetto [63] to PIDE of the form (1.1). A novel space-time reconstruction operator is introduced, which is a generalization of the elliptic reconstruction operator, and we call it as Ritz-Volterra reconstruction operator. The Ritz-Volterra reconstruction operator is used in a crucial way to derive optimal order *a posteriori* error estimates in the $L^\infty(L^2(\Omega))$-norm. The related *a posteriori* error estimates for the Ritz-Volterra reconstruction error are also established.

Chapter 3 is devoted to the *a posteriori* error analysis for the fully discrete backward Euler scheme for the problem (1.1). We derive optimal order *a posteriori* error estimates in the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms of the error using energy argument. The proof of these estimates requires a careful introduction of the fully discrete Ritz-Volterra reconstructions. Moreover, the results of this chapter generalize the results of Lakkis *et al.* [56] for the parabolic problems to PIDEs.

In Chapter 4, we derive *a posteriori* error estimate concerning fully discrete Crank-Nicolson scheme for the problem (1.1) in the $L^\infty(L^2(\Omega))$-norm. A quadratic space-time reconstruction is introduced to derive estimator which is second order accurate in time. Moreover, these results generalize the results of Bansch et al. [12] from purely parabolic problems to PIDEs.

In Chapter 5, we study *a posteriori* error analysis for the fully discrete backward Euler scheme on anisotropic mesh. We derive *a posteriori* error estimators in the $L^2(H^1(\Omega))$-norm. Moreover, the results presented in this chapter generalize the results of Picasso [77] from parabolic problems to PIDEs.

Chapter 6 is devoted to study *a posteriori* error analysis in the $L^2(H^1(\Omega))$-norm of the error for the PIDE (1.1) concerning the fully discrete Crank-Nicolson scheme on anisotropic mesh. The key technical tools used in deriving the error estimators are continuous, piecewise quadratic reconstructions namely, Crank-Nicolson memory reconstruction and three point reconstruction. While dealing with the Crank-Nicolson memory reconstruction, a linear approximation of the Volterra integral term is used in a crucial way to estimate the quadrature error. Moreover, an extended linear approximation of the Volterra integral term is used to estimate the error due to the quadrature approximation of the memory term while analyzing with the three point reconstruction.

Chapter 7 presents some numerical experiments to study the asymptotic behaviour of the proposed error estimators for two dimensional test problems on isotropic mesh.

Finally, Chapter 8 discusses the critical evaluation of the results highlighting the contributions made by the thesis. It also provides information for the scope of future investigations.

For clarity of presentation we have repeatedly mentioned the equation (1.1) and the relevant preliminary stuffs at the beginning of every chapter. Moreover, the constants $C_i, i = 1, 2, \cdots$ appeared in different chapters are not necessarily the same. The pictorial representation of the thesis is shown as follows.

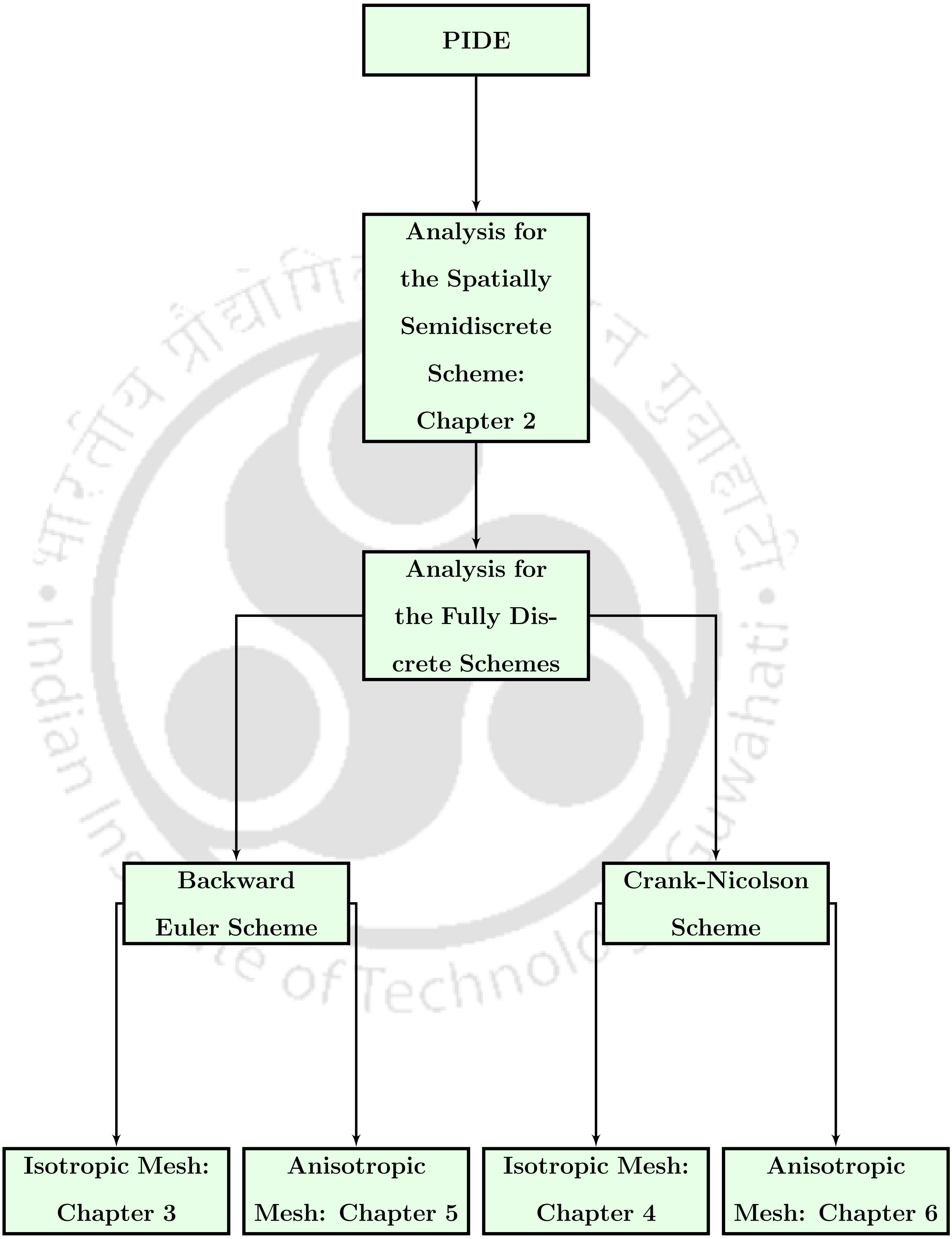

Figure 1.2: *Pictorial representation of the thesis*

2

Spatially Semidiscrete Error Analysis

In this chapter, we derive *a posteriori* error estimator for the spatially semidiscrete scheme for PIDE (1.1). A novel space-time reconstruction operator is introduced, which is a generalization of the elliptic reconstruction operator [12, 56, 63], and we call it as Ritz-Volterra reconstruction operator. Further, this reconstruction operator is shown to be the partial right inverse of the Ritz-Volterra projection [17, 58] introduced in the *a priori* analysis for PIDEs. The Ritz-Volterra reconstruction operator is used in a crucial way to derive optimal order *a posteriori* error estimate in the $L^\infty(L^2(\Omega))$-norm. Moreover, these results generalize the results of purely parabolic problems (see Makridakis and Lakkis [63]) to PIDE (1.1).

2.1 Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be a bounded convex polygonal or polyhedral domain with boundary $\partial\Omega$ and $T < \infty$. We now recall the following PIDE:

$$u_t(x,t) + \mathcal{A}u(x,t) = \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T] \qquad (2.1)$$

subject to the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T]$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

Published online in *IMA J. Numer. Anal.*, *doi:10.1093/imanum/drt059.*

The operator $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u),$$

and the operator $\mathcal{B}(t,s)$ is of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient and $A = \{a_{ij}(x)\}$ and $B(t,s) = \{b_{ij}(x;t,s)\}$ are two $n \times n$ matrices assumed to be in $L^\infty(\Omega)^{n\times n}$ in space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The initial function $u_0 = u_0(x)$ and the nonhomogeneous term f are assumed to be smooth for our purpose.

For the purpose of spatially semidiscrete scheme, we define $H_0^1(\Omega) = \{\phi \in H^1(\Omega)\big|\phi = 0 \text{ on } \partial\Omega\}$. Further, we shall call back the bilinear forms $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, $b(t,s;\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and $b_t(t,s;\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ corresponding to the operators $\mathcal{A}$, $\mathcal{B}(t,s)$ and $\mathcal{B}_t(t,s)$, respectively i.e.,

$$\begin{aligned} a(v,\psi) &:= \langle A\nabla v, \nabla\psi\rangle, \quad \forall\, v,\psi \in H_0^1(\Omega), \\ b(t,s;v(s),\psi) &:= \langle B(t,s)\nabla v(s), \nabla\psi\rangle, \quad \forall\, v(s),\psi \in H_0^1(\Omega) \end{aligned}$$

and

$$b_t(t,s;\phi(s),\psi) := \langle B_t(t,s)\nabla\phi(s), \nabla\psi\rangle, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega).$$

We assume that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi,\phi) \ge \alpha\|\phi\|_1^2 \quad \text{and} \quad |a(\phi,\psi)| \le \beta\|\phi\|_1\|\psi\|_1, \quad \forall\, \phi,\psi \in H_0^1(\Omega) \tag{2.2}$$

with $\alpha, \beta \in \mathbb{R}^+$. Further, we assume that the bilinear form $b(t,s;\cdot,\cdot)$ is continuous on $H_0^1(\Omega)$ i.e.,

$$|b(t,s;\phi(s),\psi)| \le \gamma\|\phi(s)\|_1\|\psi\|_1, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{2.3}$$

with $\gamma \in \mathbb{R}^+$.

Now, we recall the weak formulation of the problem (2.1) and the spatially semidiscrete scheme. The weak formulation is stated as follows: Find $u : [0,T] \to H_0^1(\Omega)$ such that

$$\begin{aligned} \langle u_t, \phi\rangle + a(u,\phi) &= \int_0^t b(t,s;u(s),\phi)ds + \langle f,\phi\rangle, \;\; \forall \phi \in H_0^1(\Omega), \;\; t\in(0,T], \\ u(0) &= u_0. \end{aligned} \tag{2.4}$$

Recall the following finite element space corresponding to $\mathcal{T}_h$:

$$\mathbb{V}_h := \{\chi \in H_0^1(\Omega) : \chi|_K \in \mathbb{P}_l(K), \;\; \text{for all } K \in \mathcal{T}_h\}, \tag{2.5}$$

where $\mathbb{P}_l$ is the space of polynomials of degree $\leq l$ with $l \in \mathbb{Z}^+$. The spatially semidiscrete finite element approximation $u_h : [0,T] \to \mathbb{V}_h$ of u is defined by

$$\begin{aligned} \langle u_{h,t}, \chi\rangle + a(u_h,\chi) &= \int_0^t b(t,s;u_h(s),\chi)ds + \langle f,\chi\rangle, \;\; \forall \chi \in \mathbb{V}_h, \\ u_h(.,0) &= P_h u_0, \end{aligned} \tag{2.6}$$

where P_h is the L^2-projection operator as defined in (1.11). We now recall the representations for the bilinear forms from Chapter 1. The bilinear forms $a(\cdot,\cdot)$ and $b(t,s;\cdot,\cdot)$ can be represented as

$$a(v,\phi) = \langle \mathcal{A}_{el} v, \phi\rangle + \langle J_1[v], \phi\rangle_{\sum_h}, \quad \forall \phi \in H_0^1(\Omega), \tag{2.7}$$

and

$$b(t,s;v(s),\phi) = \langle \mathcal{B}_{el}(t,s)v(s), \phi\rangle + \langle J_2[v(s)], \phi\rangle_{\sum_h}, \quad \forall \phi \in H_0^1(\Omega), \tag{2.8}$$

where $\mathcal{A}_{el}$ and $\mathcal{B}_{el}(t,s)$ are given by (1.15) and (1.18), respectively. $J_1[v]$ and $J_2[v(s)]$ are the spatial jumps of the fields $A\nabla v$ and $-\text{div}(B(t,s)\nabla v(s))$ across an element side $E \in \mathcal{E}_h$.

In the absence of the memory term i.e., when $\mathcal{B}(t,s) = 0$, *a posteriori* error analysis for spatially semidiscrete scheme has been carried out by Makridakis and Nochetto [63]. They have used elliptic reconstruction operator in combination with energy techniques to derive optimal order *a posteriori* error estimates for the parabolic problem in the

$L^\infty(L^2(\Omega))$-norm. In this chapter, we extend the *a posteriori* error analysis of parabolic problems [63] to PIDE (2.1). We derive *a posteriori* error estimator for PIDE (2.1) in the $L^\infty(L^2(\Omega))$-norm. The notion of Ritz-Volterra reconstruction operator introduced in this chapter is the key technical tool for deriving the estimator.

We organize this chapter as follows. Ritz-Volterra reconstruction operator is introduced in Section 2.2. Section 2.3 is devoted to study *a posteriori* error analysis for the spatially semidiscrete scheme.

2.2 Ritz-Volterra reconstruction

Following [63], we now recall the elliptic reconstruction operator which is defined as follows.

Definition 2.2.1 (Elliptic reconstruction)**.** *For a given* $v \in H_0^1(\Omega)$*, the elliptic reconstruction* $\mathcal{R} : H_0^1(\Omega) \to H_0^1(\Omega)$ *of* v *associated with the bilinear form* $a(\cdot,\cdot)$ *is defined by*

$$a(\mathcal{R}v, \phi) = \langle \mathcal{A}_h v, \phi\rangle, \quad \forall \phi \in H_0^1(\Omega). \tag{2.9}$$

The following definition is a generalization of the elliptic reconstruction operator and we call it as Ritz-Volterra reconstruction operator.

Definition 2.2.2 (Ritz-Volterra reconstruction)**.** *For* $v \in H_0^1(\Omega)$*, we define the Ritz-Volterra reconstruction* $\mathcal{R}_w : H_0^1(\Omega) \to H_0^1(\Omega)$ *of* v *by*

$$a(\mathcal{R}_w v, \phi) - \int_0^t b(t,s;\mathcal{R}_w v(s), \phi)ds = \langle \mathcal{A}_h v, \phi\rangle - \int_0^t \langle \mathcal{B}_h(t,s)v(s), \phi\rangle ds, \tag{2.10}$$

for all $\phi \in H_0^1(\Omega)$*.*

The function $\mathcal{R}_w v$ is referred to as the Ritz-Volterra reconstruction of v. Note that in the absence of the memory term, this definition coincides with the definition of the elliptic reconstruction operator (2.9). Although, we define the domain of definition of Ritz-Volterra reconstruction to be $H_0^1(\Omega)$ but we will use it effectively on the finite element spaces only. To motivate the definition of Ritz-Volterra reconstruction, we first consider the following elliptic Volterra equation in the weak form

$$a(\mathcal{W}_R(t),\chi) = \langle g_h,\chi\rangle + \int_0^t b(t,s;\mathcal{W}_R(s),\chi)ds, \quad \forall \chi \in H_0^1 \text{ and } t \in (0,T], \tag{2.11}$$

where g_h is given by

$$g_h = \mathcal{A}_h v - \int_0^t \mathcal{B}_h(t,s)v(s)ds, \quad v \in H_0^1(\Omega).$$

The Ritz-Volterra reconstruction $\mathcal{W}_R(t) = \mathcal{R}_w v(t) \in H_0^1(\Omega)$ of v is the solution of the problem (2.11). The existence and uniqueness of $\mathcal{W}_R$ follows from the theory of elliptic Volterra equations.

Remarks. (i) For $t \in [0,T]$, an important property of the Ritz-Volterra reconstruction operator $\mathcal{R}_w$ is that for $v \in H_0^1(\Omega)$, $v - \mathcal{R}_w v$ is orthogonal to $\mathbb{V}_h$ with respect to $a(\cdot,\cdot) - \int_0^t b(t,s;\cdot,\cdot)ds$, i.e.,

$$a(\mathcal{R}_w v - v,\phi) - \int_0^t b(t,s;(\mathcal{R}_w v - v)(s),\phi)ds = 0, \quad \forall \phi \in \mathbb{V}_h. \tag{2.12}$$

This property is known as Galerkin orthogonality and is important in the sense that it allows us to obtain *a posteriori* error estimates.

(ii) Recall from [58], the following Ritz-Volterra projection $W_h : H_0^1(\Omega) \to \mathbb{V}_h$ defined by

$$a(W_h u - u,\chi) = \int_0^t b(t,s;(W_h u - u)(s),\chi)ds, \quad \forall \chi \in \mathbb{V}_h,\ t \in [0,T]. \tag{2.13}$$

Note that Ritz-Volterra reconstruction operator $\mathcal{R}_w$ defined by (2.10) is a partial right inverse of the Ritz-Volterra projection W_h defined by (2.13). Let $\mathcal{W}_R$ denote the Ritz-Volterra reconstruction of the finite element solution u_h. Then, by Galerkin orthogonality property (2.12)

$$\mathcal{W}_R = \mathcal{R}_w u_h \Rightarrow W_h \mathcal{W}_R = u_h.$$

2.3 Error analysis

In order to derive *a posteriori* error bounds, we decompose the main error $e := u - u_h$ as follows:

$$e := \epsilon - \rho, \quad \text{where } \epsilon := \mathcal{R}_w u_h - u_h, \quad \rho := \mathcal{R}_w u_h - u. \tag{2.14}$$

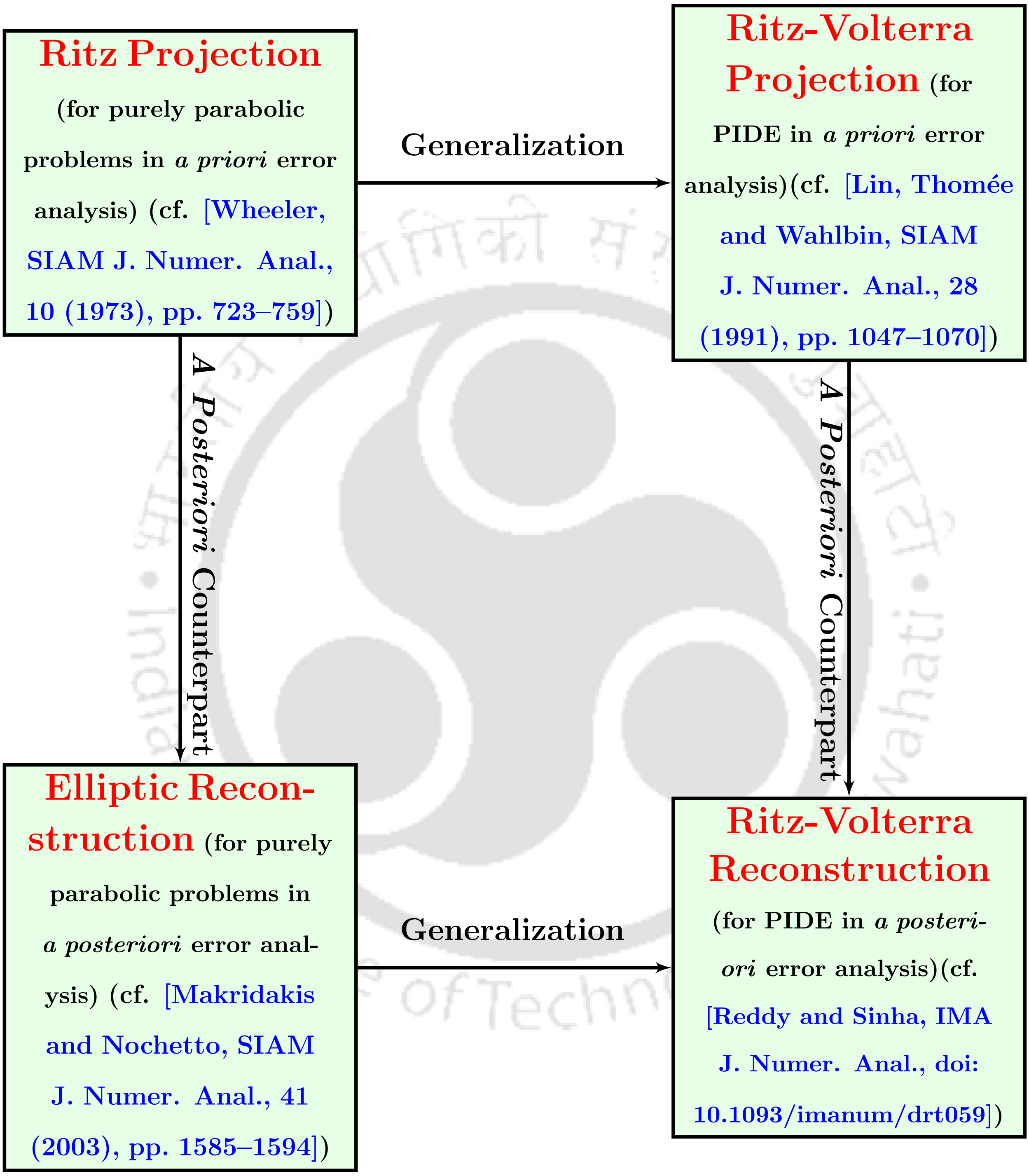

Figure 2.1: *Establishing optimality in the L^2-norm: An interconnection between projections and reconstructions*

Here, ϵ is referred to as the reconstruction error (Ritz-Volterra reconstruction error) whereas the time approximation error information is conveyed by ρ, which will be referred to as the parabolic error.

We now recall from [83] the following interpolation error estimates.

Proposition 2.3.1. *Let $\Pi_h : H_0^1(\Omega) \to \mathbb{V}_h$ be the Clément-type interpolation operator. Then, for sufficiently smooth ψ and finite element polynomial space of degree l, there exist constants $C_{1,j}$ and $C_{2,j}$ depending only upon the shape-regularity of the family of triangulations such that for $j \le l+1$*

$$\|h^{-j}(\psi - \Pi_h\psi)\| \le C_{1,j}\|\psi\|_j,$$

and

$$\|h^{1/2-j}(\psi - \Pi_h\psi)\|_{\sum_h} \le C_{2,j}\|\psi\|_j.$$

In this chapter, we shall derive traditional residual type *a posteriori* error estimators.

Residuals. Using the definitions of the discrete operators $\mathcal{A}_h$ and $\mathcal{B}_h(t,s)$ and the distributional form of semidiscrete equation (2.6), we have

$$\mathcal{A}_h u_h - \int_0^t \mathcal{B}_h(t,s)u_h(s)ds - \mathcal{A}_{el}u_h + \int_0^t \mathcal{B}_{el}(t,s)u_h(s)ds = \mathfrak{R}[u_h] + (f_h - f),$$

where

$$\mathfrak{R}[u_h] = f - u_{h,t} - \mathcal{A}_{el}u_h + \int_0^t \mathcal{B}_{el}(t,s)u_h(s)ds$$

is the inner residual and $f_h = P_h f$. Further, we define

$$\mathfrak{J}[u_h] = J_1[u_h] - \int_0^t J_2[u_h(s)]ds$$

as the jump residual.

Unlike for the parabolic problem [56, 63], we don't have any *a posteriori* error estimators available in the literature to control the reconstruction error ϵ. In this section, we first derive *a posteriori* error estimate for the reconstruction error (ϵ) in the $L^2(\Omega)$-norm which will be used to obtain *a posteriori* error estimates for the main error e.

Lemma 2.3.1 (Ritz-Volterra reconstruction error estimate)**.** *For any $v \in \mathbb{V}_h$, the following estimate holds:*

$$\begin{aligned} \|(\mathcal{R}_w v - v)(t)\| &\leq C_1 h^2 \|\mathcal{A}_h v - \mathcal{A}_{el} v - \int_0^t \mathcal{B}_h(t,s)v(s)ds + \int_0^t \mathcal{B}_{el}(t,s)v(s)ds\| \\ &\quad + C_2 h^{3/2} \|J_1[v] - \int_0^t J_2[v(s)]ds\|_{\sum_h}, \end{aligned}$$

where $C_j, j = 1, 2$ are positive constants independent of the mesh parameter but depend upon the interpolation constants and the final time T.

Proof. The proof will proceed by the duality technique. For $v \in \mathbb{V}_h$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$\begin{aligned} \mathcal{A}\psi &= \mathcal{R}_w v - v \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \Omega, \end{aligned} \tag{2.15}$$

satisfying the following regularity estimate (Ω is convex) with the constant C_Ω depending on the domain Ω:

$$\|\psi\|_2 \leq C_\Omega \|\mathcal{R}_w v - v\|. \tag{2.16}$$

We first multiply (2.15) by $\mathcal{R}_w v - v$ and integrate over Ω. Now, using Galerkin orthogonality (2.12), we obtain

$$\begin{aligned} \|\mathcal{R}_w v - v\|^2 &= a(\mathcal{R}_w v - v, \psi - \Pi_h \psi) + a(\mathcal{R}_w v - v, \Pi_h \psi) \\ &= a(\mathcal{R}_w v - v, \psi - \Pi_h \psi) - \int_0^t b(t,s;(\mathcal{R}_w v - v)(s), \psi - \Pi_h \psi)ds \\ &\quad + \int_0^t b(t,s;(\mathcal{R}_w v - v)(s), \psi)ds \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{2.17}$$

Using (2.7), (2.8) and (2.10), we arrive at

$$\begin{aligned} \mathcal{I}_1 + \mathcal{I}_2 &:= a(\mathcal{R}_w v - v, \psi - \Pi_h \psi) - \int_0^t b(t,s;(\mathcal{R}_w v - v)(s), \psi - \Pi_h \psi)ds \\ &= \langle \mathcal{A}_h v, \psi - \Pi_h \psi \rangle - \int_0^t \langle \mathcal{B}_h(t,s)v(s)ds, \psi - \Pi_h \psi \rangle \\ &\quad - a(v, \psi - \Pi_h \psi) + \int_0^t b(t,s;v(s), \psi - \Pi_h \psi)ds \end{aligned}$$

$$= \langle \mathcal{A}_h v - \int_0^t \mathcal{B}_h(t,s)v(s)ds - \mathcal{A}_{el} v + \int_0^t \mathcal{B}_{el}(t,s)v(s)ds, \psi - \Pi_h \psi\rangle$$
$$-\langle J_1[v] - \int_0^t J_2[v(s)]ds, \psi - \Pi_h\psi\rangle_{\sum_h}. \tag{2.18}$$

To bound $\mathcal{I}_3$, using the fact

$$b(t,s;(\mathcal{R}_w v - v)(s),\psi) := \langle (\mathcal{R}_w v - v)(s), \mathcal{B}^*(t,s)\psi\rangle, \tag{2.19}$$

where $\mathcal{B}^*(t,s)$ is the formal adjoint of the operator $\mathcal{B}(t,s)$ and $\|\mathcal{B}^*(t,s)\psi\| \leq C_{\mathcal{B}_1^*}\|\psi\|_2$, we obtain

$$|\mathcal{I}_3| \leq C_{\mathcal{B}_1^*}\|\psi\|_2 \int_0^t \|(\mathcal{R}_w v - v)(s)\|ds. \tag{2.20}$$

We use (2.18) and (2.19) in (2.17). Then apply Proposition 2.3.1 with the interpolation constants as $C_{i,2}, i = 1,2$ and (2.20) to obtain

$$\begin{aligned}\|\mathcal{R}_w v - v\|^2 \leq \; & \|\psi\|_2\Big\{C_{1,2}h^2\|\mathcal{A}_h v - \mathcal{A}_{el}v - \int_0^t \mathcal{B}_h(t,s)v(s)ds \\ & + \int_0^t \mathcal{B}_{el}(t,s)v(s)ds\| + C_{2,2}h^{3/2}\|J_1[v] - \int_0^t J_2[v(s)]ds\|_{\sum_h} \\ & + C_{\mathcal{B}_1^*}\int_0^t \|(\mathcal{R}_w v - v)(s)\|ds\Big\}.\end{aligned}$$

And hence, with an aid of (2.16), we have

$$\begin{aligned}\|\mathcal{R}_w v - v\| \leq \; & C_{1,2}C_\Omega h^2\|\mathcal{A}_h v - \mathcal{A}_{el}v - \int_0^t \mathcal{B}_h(t,s)v(s)ds + \int_0^t \mathcal{B}_{el}(t,s)v(s)ds\| \\ & + C_{2,2}C_\Omega h^{3/2}\|J_1[v] - \int_0^t J_2[v(s)]ds\|_{\sum_h} \\ & + C_\Omega C_{\mathcal{B}_1^*}\int_0^t \|(\mathcal{R}_w v - v)(s)\|ds.\end{aligned}$$

Finally, an application of the Gronwall's lemma yields the desired estimate with $C_i = C_{1,G}(T)C_{i,2}C_\Omega, i = 1,2$, where $C_{1,G}$ is a constant appears due to Gronwall's lemma. □

We now define the following error estimators.

$$\beta_S(u_h(t)) := C_1 h^2\|\mathfrak{R}[u_h(t)]\| + C_2 h^{3/2}\|\mathfrak{J}[u_h(t)]\|, \tag{2.21}$$
$$\lambda_S(g(t)) := \|g_h(t) - g(t)\|. \tag{2.22}$$

Here, $\beta_S(u_h(t))$ denotes the Ritz-Volterra reconstruction error estimator in the L^2-norm and $\lambda_S(g(t))$ denotes the oscillations of g in the L^2-norm, where $g_h(t) = P_h g(t)$.

The following lemma yields a bound for the time derivative of the reconstruction error.

Lemma 2.3.2 (*A posteriori* error estimate for time derivative of the Ritz-Volterra reconstruction error)**.** *For any $v \in \mathbb{V}_h$, the following bound holds in terms of the reconstruction error:*

$$\begin{aligned}\|(\mathcal{R}_w v - v)_t\| \;\leq\; & C_3 h^2 \|\frac{d}{dt}\Big\{\mathcal{A}_h v - \mathcal{A}_{el} v - \int_0^t \mathcal{B}_h(t,s)v(s)ds + \int_0^t \mathcal{B}_{el}(t,s)v(s)ds\Big\}\| \\ & + C_4 h^{3/2} \|\frac{d}{dt}\Big\{J_1[v] - \int_0^t J_2[v(s)]ds\Big\}\|_{\sum_h} \\ & + C_5 \int_0^t \|(\mathcal{R}_w v - v)(s)\| ds + C_6 \|\mathcal{R}_w v - v\|. \qquad (2.23)\end{aligned}$$

In particular, for finite element solution u_h, the following a posteriori error bound holds:

$$\begin{aligned}\|(\mathcal{R}_w u_h - u_h)_t\| \;\leq\; & C_3 h^2 \|\frac{d}{dt}\Big\{\mathcal{A}_h u_h - \mathcal{A}_{el} u_h - \int_0^t \mathcal{B}_h(t,s)u_h(s)ds \\ & + \int_0^t \mathcal{B}_{el}(t,s)u_h(s)ds\Big\}\| + C_5 C_1 \int_0^t h^2 \lambda_S(f(s))ds \\ & + C_5 \int_0^t \beta_S(u_h)(s)ds + C_4 h^{3/2} \|\frac{d}{dt}\Big\{J_1[u_h] - \int_0^t J_2[u_h(s)]ds\Big\}\|_{\sum_h} \\ & + C_6 \beta_S(u_h) + C_6 C_1 h^2 \lambda_S(f(t)), \qquad (2.24)\end{aligned}$$

where $C_j, j = 1,2,3,4,5,6$ are positive constants independent of the mesh parameter but depend upon the interpolation constants and the final time T.

Proof. Differentiating (2.12) with respect to t, for all $\phi \in \mathbb{V}_h$, we have

$$\begin{aligned}a((\mathcal{R}_w v - v)_t, \phi) \; - \; & b(t,t;(\mathcal{R}_w v - v)(t), \phi) \\ - \; & \int_0^t b_t(t,s;(\mathcal{R}_w v - v)(s), \phi)ds = 0. \qquad (2.25)\end{aligned}$$

Consider the dual elliptic problem with the forcing function to be $(\mathcal{R}_w v - v)_t$. For $v \in \mathbb{V}_h$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$\begin{aligned}\mathcal{A}\psi &= (\mathcal{R}_w v - v)_t \quad \text{in } \Omega, \qquad (2.26)\\ \psi &= 0 \quad \text{on } \Omega,\end{aligned}$$

satisfying the following regularity estimate (Ω is convex) with the constant $\bar{C}_\Omega$ depending on the domain Ω:

$$\|\psi\|_2 \leq \bar{C}_\Omega \|(\mathcal{R}_w v - v)_t\|. \tag{2.27}$$

We first multiply (2.26) by $(\mathcal{R}_w v - v)_t$ and integrate over Ω. Then, rearranging terms and using (2.25), we obtain

$$\begin{aligned} \|(\mathcal{R}_w v - v)_t\|^2 &= a((\mathcal{R}_w v - v)_t, \psi - \Pi_h\psi) - \int_0^t b_t(t, s; (\mathcal{R}_w v - v)(s), \psi - \Pi_h\psi) ds \\ &\quad -b(t, t; (\mathcal{R}_w v - v)(t), \psi - \Pi_h\psi) + \int_0^t b_t(t, s; (\mathcal{R}_w v - v)(s), \psi) ds \\ &\quad +b(t, t; (\mathcal{R}_w v - v)(t), \psi) := \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5. \end{aligned} \tag{2.28}$$

In order to handle the terms $\mathcal{J}_i$ $(i = 1, 2, 3)$, arguing analogously as in the proof of Lemma 2.3.1, we have

$$\begin{aligned} & a(\mathcal{R}_w v - v, \psi - \Pi_h\psi) - \int_0^t b(t, s; (\mathcal{R}_w v - v)(s), \psi - \Pi_h\psi) ds \\ = \ & \langle \mathcal{A}_h v - \int_0^t \mathcal{B}_h(t, s) v(s) ds - \mathcal{A}_{el} v + \int_0^t \mathcal{B}_{el}(t, s) v(s) ds, \psi - \Pi_h\psi \rangle \\ & -\langle J_1[v] - \int_0^t J_2[v(s)] ds, \psi - \Pi_h\psi \rangle_{\sum_h}. \end{aligned}$$

We differentiate both sides of the above equation with respect to t. Then, use of Cauchy-Schwarz inequality and Proposition 2.3.1 with the interpolation constants $C_{i,2}$, $i = 1, 2$ leads to

$$\begin{aligned} & |a((\mathcal{R}_w v - v)_t, \psi - \Pi_h\psi) - \int_0^t b_t(t, s; (\mathcal{R}_w v - v)(s), \psi - \Pi_h\psi) ds \\ & -b(t, t; (\mathcal{R}_w v - v)(t), \psi - \Pi_h\psi)| \\ = \ & |\langle \frac{d}{dt}\Big\{\mathcal{A}_h v - \mathcal{A}_{el} v - \int_0^t \mathcal{B}_h(t, s) v(s) ds + \int_0^t \mathcal{B}_{el}(t, s) v(s) ds\Big\}, \psi - \Pi_h\psi \rangle \\ & -\langle \frac{d}{dt}\Big\{J_1[v] - \int_0^t J_2[v(s)] ds\Big\}, \psi - \Pi_h\psi \rangle_{\sum_h}| \\ \leq \ & \|\psi\|_2 \Big\{ C_{1,2} h^2 \|\frac{d}{dt}\Big\{\mathcal{A}_h v - \mathcal{A}_{el} v - \int_0^t \mathcal{B}_h(t, s) v(s) ds + \int_0^t \mathcal{B}_{el}(t, s) v(s) ds\Big\}\| \\ & + C_{2,2} h^{3/2} \|\frac{d}{dt}\{J_1[v] - \int_0^t J_2[v(s)] ds\}\|_{\sum_h}\Big\}. \end{aligned}$$

For the terms $\mathcal{J}_4$ and $\mathcal{J}_5$, use the fact

$$b_t(t,s;(\mathcal{R}_w v - v)(s),\psi) := \langle (\mathcal{R}_w v - v)(s), \mathcal{B}_t^*(t,s)\psi\rangle,$$

and (2.19) together with $\|\mathcal{B}_t^*(t,s)\psi\| \le C_{\mathcal{B}_2^*}\|\psi\|_2$ and $\|\mathcal{B}^*(t,t)\psi\| \le C_{\mathcal{B}_3^*}\|\psi\|_2$, where $\mathcal{B}_t^*(t,s)$ is obtained by differentiating the coefficient of the operator $\mathcal{B}^*(t,s)$ with respect to t, to obtain

$$\begin{aligned}
\|(\mathcal{R}_w v \;-\; v)_t\|^2 & \\
\le\;& \|\psi\|_2\Big\{C_{1,2}h^2\|\frac{d}{dt}\Big\{\mathcal{A}_h v - \mathcal{A}_{el} v - \int_0^t \mathcal{B}_h(t,s)v(s)ds + \int_0^t \mathcal{B}_{el}(t,s)v(s)ds\Big\}\| \\
&+C_{2,2}h^{3/2}\|\frac{d}{dt}\Big\{J_1[v] - \int_0^t J_2[v(s)]ds\Big\}\|_{\sum_h} \\
&+C_{\mathcal{B}_2^*}\int_0^t \|(\mathcal{R}_w v - v)(s)\|ds + C_{\mathcal{B}_3^*}\|\mathcal{R}_w v - v\|\Big\}.
\end{aligned}$$

Using (2.27), the desired estimate (2.23) follows with constants $C_3 = C_{1,2}\bar{C}_\Omega$, $C_4 = C_{2,2}\bar{C}_\Omega$, $C_5 = C_{\mathcal{B}_2^*}\bar{C}_\Omega$ and $C_6 = C_{\mathcal{B}_3^*}\bar{C}_\Omega$. The second estimate (2.24) follows immediately from Lemma 2.3.1 (with u_h replacing v), (2.21) and (2.22). □

The first two terms on the right of (2.24) can be handled in the following way.

$$\begin{aligned}
&\|\frac{d}{dt}\Big\{\mathcal{A}_h u_h - \mathcal{A}_{el} u_h - \int_0^t \mathcal{B}_h(t,s)u_h(s)ds + \int_0^t \mathcal{B}_{el}(t,s)u_h(s)ds\Big\}\| \\
&= \|\frac{d}{dt}[\mathfrak{R}[u_h] + (f_h - f)]\| \le \|\mathfrak{R}_t[u_h]\| + \lambda_S(f_t(t)),
\end{aligned}$$

and

$$\|\frac{d}{dt}\Big\{J_1[u_h] - \int_0^t J_2[u_h(s)]ds\Big\}\|_{\sum_h} = \|\mathfrak{J}_t[u_h]\|_{\sum_h}.$$

Now, we define the estimator for the time derivative of the reconstruction error by

$$\begin{aligned}
\beta_{S,t}(u_h(t)) \;:=\;& C_3h^2\|\mathfrak{R}_t[u_h(t)]\| + C_3h^2\ \lambda_S(f_t(t)) + C_4h^{3/2}\|\mathfrak{J}_t[u_h]\|_{\sum_h} \\
&+C_5\int_0^t \beta_S(u_h)(s)ds + C_5C_1\int_0^t h^2\ \lambda_S(f(s))ds \\
&+C_6\beta_S(u_h) + C_6C_1h^2\ \lambda_S(f(t)). \qquad (2.29)
\end{aligned}$$

We now derive *a posteriori* estimate for the parabolic error ρ in the following lemma.

Lemma 2.3.3 (*A posteriori* error estimate for the parabolic error)**.** *The following estimate holds for the parabolic error* ρ*:*

$$\|\rho(t)\| \leq C_7 \left[\|\rho(0)\| + 2\int_0^t \{\beta_{S,t}(u_h(t)) + \lambda_S(f(t))\}\, ds \right],$$

where $\beta_{S,t}(u_h(t))$ *is given by (2.29) and* C_7 *is a positive constant independent of the mesh parameter but depending on the final time* T*.*

Proof. Using (2.6) and the definition of the Ritz-Volterra reconstruction, we have the following error equation for $\rho(t)$

$$\begin{aligned}\langle \rho_t, \phi\rangle \quad + \quad & a(\rho,\phi) - \int_0^t b(t,s;\rho(s),\phi)ds \\ & = \langle \mathcal{R}_w u_{h,t}, \phi\rangle + a(\mathcal{R}_w u_h, \phi) - \int_0^t b(t,s;\mathcal{R}_w u_h(s),\phi)ds - \langle f,\phi\rangle \\ & = \langle \mathcal{R}_w u_{h,t}, \phi\rangle + \langle \mathcal{A}_h u_h - \int_0^t \mathcal{B}_h(t,s)u_h(s)ds, \phi\rangle - \langle f,\phi\rangle \\ & = \langle \epsilon_t, \phi\rangle + \langle f_h - f, \phi\rangle, \qquad \forall \phi \in H_0^1(\Omega),\ \mathrm{t} \in [0,T]. \end{aligned} \tag{2.30}$$

Set $\phi = \rho$ in the error equation (2.30). Apply Cauchy-Schwarz inequality and Young's inequality together with (2.3) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho\|^2 \quad + \quad & a(\rho,\rho) = \langle \epsilon_t, \rho\rangle + \langle f_h - f, \rho\rangle + \int_0^t b(t,s;\rho(s),\rho)ds \\ \leq \quad & \frac{1}{2}\alpha\|\rho\|_1^2 + \frac{\gamma^2}{2\alpha}\left(\int_0^t \|\rho(s)\|_1 ds\right)^2 + \Big(\|\epsilon_t\| + \|f_h - f\|\Big)\|\rho\|. \end{aligned} \tag{2.31}$$

Integrate (2.31) from 0 to t. Then, use coercivity property of $a(\cdot,\cdot)$ and Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\|\rho(t)\|^2 + \alpha\int_0^t \|\rho\|_1^2 \quad \leq \quad & \|\rho(0)\|^2 + \frac{C'(T)\gamma^2}{\alpha}\int_0^t\int_0^s \|\rho(\tau)\|_1^2 d\tau ds \\ & +2\int_0^t \Big(\|\epsilon_t\| + \|f_h - f\|\Big)\|\rho\| ds,\end{aligned}$$

where $C'(T)$ is a positive constant depending on the final time T. Applying Gronwall's lemma and letting $\|\rho(\bar{t})\| = \sup_{s\leq t}\|\rho(s)\|,\ 0 \leq \bar{t} \leq t$, the desired estimate follows. □

The semidiscrete *a posteriori* error estimate in the $L^\infty(L^2)$-norm is presented in the following theorem.

Theorem 2.3.1 (Semidiscrete *a posteriori* error estimate)**.** *Let u and u_h satisfy (2.1) and (2.6), respectively. Then the following a posteriori error bound holds:*

$$\max_{0\leq t\leq T}\|(u-u_h)(t)\| \;\leq\; C_7\Bigg[\|u(0)-u_h(0)\| + \beta_S(u_h(0)) + C_3h^2\lambda_S(f(0)) + 2\int_0^t\Bigg(\beta_{S,t}(u_h(t)) + \lambda_S(f(t))\Bigg)ds\Bigg] + \beta_S(u_h(t)),$$

where $\beta_S(u_h(t))$, $\beta_{S,t}(u_h(t))$ are given by (2.21), (2.29), respectively and C_7 is a positive constant as defined in Lemma 2.3.3.

Proof. Choosing $\mathcal{R}_w u_h \in H_0^1(\Omega)$ as the comparison function, we express the error e as

$$e(t) = u_h(t) - u(t) = (\mathcal{R}_w u_h(t) - u(t)) - (\mathcal{R}_w u_h(t) - u_h(t)) = \rho(t) - \epsilon(t). \tag{2.32}$$

Also, we note that

$$\|\rho(0)\| \leq \|u(0)-u_h(0)\| + \|\mathcal{R}_w u_h(0) - u_h(0)\| = \|u(0)-u_h(0)\| + \|\epsilon(0)\|.$$

Now, apply triangle inequality to (2.32) and Lemmas 2.3.1 and 2.3.3 to complete the rest of the proof. □

Remarks. (i) The *a posteriori* error estimator obtained in Theorem 2.3.1 generalizes the result of purely parabolic problem to PIDE. In the absence of the memory term (i.e., $\mathcal{B}(t,s)=0$), our error estimator for PIDE is similar to that of the parabolic problem [63].

(ii) Theorem 2.3.1 gives the dual *a posteriori* analogue of *a priori* error estimate for semi-discrete finite element approximations to PIDE (cf. [58]).

3

Fully Discrete Backward Euler Error Analysis

This chapter is concerned with *a posteriori* error analysis of fully discrete backward Euler finite element method for the PIDE (1.1). We extend the spatially semidiscrete *a posteriori* error analysis (cf. Chapter 2) to the fully discrete backward Euler method. The Ritz-Volterra reconstruction operator again plays a key role in obtaining *a posteriori* error bounds in the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms. Moreover, the results obtained in this chapter generalize the results of parabolic problems (see Lakkis and Makridakis [56]) to PIDE (1.1).

3.1 Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be a bounded convex polygonal or polyhedral domain with boundary $\partial\Omega$ and $T < \infty$. We now recall the following PIDE

$$u_t(x,t) + \mathcal{A}u(x,t) = \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T] \qquad (3.1)$$

subject to the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T]$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

Published online in *IMA J. Numer. Anal., doi:10.1093/imanum/drt059.*

The operator $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u),$$

and the operator $\mathcal{B}(t,s)$ is of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient and $A = \{a_{ij}(x)\}$ and $B(t,s) = \{b_{ij}(x;t,s)\}$ are two $n \times n$ matrices assumed to be in $L^\infty(\Omega)^{n\times n}$ in space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The initial function $u_0 = u_0(x)$ and the nonhomogeneous term f are assumed to be smooth.

Let $H_0^1(\Omega) = \{\phi \in H^1(\Omega) \big| \phi = 0 \text{ on } \partial\Omega\}$. Recall the bilinear forms $a(\cdot,\cdot)$, $b(t,s;\cdot,\cdot)$, $b_s(t,s;\cdot,\cdot)$ and $b_t(t,s;\cdot,\cdot)$ corresponding to the operators $\mathcal{A}$, $\mathcal{B}(t,s)$, $\mathcal{B}_s(t,s)$ and $\mathcal{B}_t(t,s)$, respectively i.e.,

$$\begin{aligned} a(v,\psi) &:= \langle A\nabla v, \nabla\psi\rangle, \qquad \forall\, v,\psi \in H_0^1(\Omega), \\ b(t,s;v(s),\psi) &:= \langle B(t,s)\nabla v(s), \nabla\psi\rangle, \qquad \forall\, v(s),\psi \in H_0^1(\Omega), \\ b_s(t,s;\phi(s),\psi) &:= \langle B_s(t,s)\nabla \phi(s), \nabla\psi\rangle, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega) \end{aligned} \tag{3.2}$$

and

$$b_t(t,s;\phi(s),\psi) := \langle B_t(t,s)\nabla \phi(s), \nabla\psi\rangle, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega).$$

We assume that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi,\phi) \geq \alpha\|\phi\|_1^2 \quad \text{and} \quad |a(\phi,\psi)| \leq \beta\|\phi\|_1\|\psi\|_1, \qquad \forall\, \phi,\psi \in H_0^1(\Omega) \tag{3.3}$$

with $\alpha,\beta \in \mathbb{R}^+$. Further, we assume that the bilinear forms $b(t,s;\cdot,\cdot)$ and $b_s(t,s;\cdot,\cdot)$ are continuous on $H_0^1(\Omega)$ i.e.,

$$|b(t,s;\psi(s),\phi)| \leq \gamma\|\psi(s)\|_1\|\phi\|_1, \qquad \forall\, \psi(s),\phi \in H_0^1(\Omega), \tag{3.4}$$

$$|b_s(t,s;\psi(s),\phi)| \leq \gamma'\|\psi(s)\|_1\|\phi\|_1, \qquad \forall\, \psi(s),\phi \in H_0^1(\Omega), \tag{3.5}$$

with $\gamma, \gamma' \in \mathbb{R}^+$.

The weak formulation of the problem (3.1) is stated as follows: Find $u : [0, T] \to H_0^1(\Omega)$ such that

$$\begin{aligned} \langle u_t, \phi\rangle + a(u, \phi) &= \int_0^t b(t, s; u(s), \phi) ds + \langle f, \phi\rangle, \quad \forall \phi \in H_0^1(\Omega), \quad t \in (0, T], \\ u(0) &= u_0. \end{aligned} \tag{3.6}$$

The fully discrete backward Euler scheme may be stated as follows: Given $U^0 = P_0^0 u(0)$, find $U^n \in \mathbb{V}^n, n \in [1 : N]$ such that

$$\tau_n^{-1}\langle U^n - U^{n-1}, \phi_n\rangle + a(U^n, \phi_n) = \sigma^n(b(t_n; U, \phi_n)) + \langle f^n, \phi_n\rangle, \quad \forall \phi_n \in \mathbb{V}^n, \tag{3.7}$$

where $\mathbb{V}^n$ is given by (1.19) and P_0^n is the L^2 projection operator defined by (1.20). σ^n is the quadrature rule used to discretize the Volterra integral term. To be consistent with the backward Euler scheme, we use the left rectangular rule given by

$$\sigma^n(y) = \sum_{j=0}^{n-1} \tau_{j+1} y(t_j) \approx \int_0^{t_n} y(s) ds, \tag{3.8}$$

so that

$$\sigma^n(b(t_n; v, \phi)) = \langle \sigma^n(B(t_n)\nabla v), \nabla\phi\rangle = \langle \sum_{j=0}^{n-1} \tau_{j+1} B(t_n, t_j)\nabla v(t_j), \nabla\phi\rangle.$$

Representation of the bilinear forms. For a function $v \in \mathbb{V}^n$, we can represent our bilinear form $a(\cdot, \cdot)$ as

$$a(v, \phi) = \langle \mathcal{A}_{el} v, \phi\rangle + \langle J_1[v], \phi\rangle_{\sum_n}, \quad \forall \phi \in H_0^1(\Omega), \tag{3.9}$$

where

$$\langle \mathcal{A}_{el} v, \phi\rangle = \sum_{K \in \mathcal{T}_n} \langle -\text{div}(A\nabla v), \phi\rangle, \quad \forall \phi \in H_0^1(\Omega)$$

is the regular part of the distribution $-\text{div}(A\nabla v)$ and

$$J_1[v]|_E(x) = [A\nabla v]_E(x) := \lim_{\varepsilon \to 0} (A\nabla v(x + \varepsilon \mathbf{n}_E) - A\nabla v(x - \varepsilon \mathbf{n}_E)) \cdot \mathbf{n}_E \tag{3.10}$$

is the spatial jump of the field $A\nabla v$ across an element side $E \in \mathcal{S}_n$, where $\mathbf{n}_E$ is a unit normal vector to E at the point x.

But, the representation of the bilinear form $b(t_n;\cdot,\cdot)$ needs a little modification. For a function $v \in H_0^1(\Omega)$, we represent the bilinear form $b(t_n;\cdot,\cdot)$ as

$$\sigma^n(b(t_n;v,\phi)) = \langle \sigma^n(\mathcal{B}_{el}v), \phi\rangle + \langle \sigma^n(J_2[v]), \phi\rangle_{\sum_n}, \quad \forall \phi \in H_0^1(\Omega), \tag{3.11}$$

where $\sigma^n(\mathcal{B}_{el}v)$ is the regular part of the distribution $-\sigma^n(\mathrm{div}(B(t_n)\nabla v))$ and $\sigma^n(J_2[v])$ is the spatial jump of the field $-\sigma^n(\mathrm{div}(B(t_n)\nabla v))$ across an element side $E \in \mathcal{S}_n$ as defined in (3.10) with $B(t_n, t_j)$ replacing A.

In the absence of the memory term, i.e., when $\mathcal{B}(t,s) = 0$, *a posteriori* error analysis for linear parabolic problems concerning fully discrete backward Euler scheme has been investigated by Lakkis and Makridakis [56], Picasso [76] and Verfürth [98]. For the purely parabolic problems, Picasso [76] and Verfürth [98] have obtained optimal order estimates in the $L^2(H^1(\Omega))$-norm and suboptimal estimates in the $L^\infty(L^2(\Omega))$-norm using the standard energy method. In [56], the authors have used the elliptic reconstruction operator in the *a posteriori* framework to recover optimality for the parabolic problems in the $L^\infty(L^2(\Omega))$-norm using energy techniques. In this chapter, an attempt has been made to carry over *a posteriori* error analysis of purely parabolic problems [56] to the PIDE (3.1). Optimal *a posteriori* error estimates in both the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms are derived.

The rest of the chapter is organized as follows. In Section 3.2, we introduce Ritz-Volterra reconstruction for the fully discrete scheme. Further, *a posteriori* error estimators for the fully discrete backward Euler scheme are established in this section.

3.2 Error analysis

For the purpose of fully discrete error analysis, we now define the Ritz-Volterra reconstruction operator.

Definition 3.2.1 (Ritz-Volterra reconstruction)**.** *We define the Ritz-Volterra reconstruction $\mathcal{W}_R^n \in H_0^1(\Omega)$ of $v \in H_0^1(\Omega)$ to be a solution of the following elliptic Volterra* *integral equation in the weak form*

$$a(\mathcal{W}_R^n, \chi) = \langle g^n, \chi \rangle + \int_0^{t_n} b(t_n, s; \mathcal{W}_R(s), \chi) ds, \quad \forall \chi \in H_0^1, \tag{3.12}$$

where g^n *is given by*

$$g^n = \mathcal{A}^n v - \int_0^{t_n} \mathcal{B}^n(s) v(s) ds, \quad v \in H_0^1(\Omega).$$

Remark. The Ritz-Volterra reconstruction satisfies the Galerkin orthogonality relation

$$a(\mathcal{W}_R^n - v, \phi_n) - \int_0^{t_n} b(t_n, s; (\mathcal{W}_R - v)(s), \phi_n) = 0, \quad \forall \phi_n \in \mathbb{V}^n. \tag{3.13}$$

In this section, we shall derive *a posteriori* error estimates for the error $e = u - U$ where $U(t), t \in I_n$ is defined by

$$U(t) := l_{n-1}(t) U^{n-1} + l_n(t) U^n \text{ for } n \in [1 : N]. \tag{3.14}$$

The functions $l_{n-1}(t)$ and $l_n(t)$ are given by

$$l_n(t) := \frac{t - t_{n-1}}{\tau_n} \text{ and } l_{n-1}(t) := \frac{t_n - t}{\tau_n}. \tag{3.15}$$

The Ritz-Volterra reconstruction of $U(t)$, denoted by $\omega(t) = \mathcal{R}_w U(t)$, is given by

$$\omega(t) := l_{n-1}(t)\omega^{n-1} + l_n(t)\omega^n = l_{n-1}(t)\mathcal{R}_w^{n-1} U^{n-1} + l_n(t) \mathcal{R}_w^n U^n,$$

where $l_{n-1}(t)$ and $l_n(t)$ are defined by (3.15).

We shall use Ritz-Volterra reconstruction of U as the intermediate object to decompose the main error $e := u - U$ into two parts: the Ritz-Volterra reconstruction error ϵ and the parabolic error ρ. While the Ritz-Volterra reconstruction error depends on the *a posteriori* error bounds of the quadrature error, the parabolic error ρ satisfies a variant of PIDE (3.6) with a right hand side that can be controlled *a posteriori* in an optimal way.

We now recall from [83] the following interpolation error estimates.

Proposition 3.2.1. *Let* $\Pi^n : H_0^1(\Omega) \to \mathbb{V}^n$ *be the Clément-type interpolation operator. Then, for sufficiently smooth* ψ *and finite element polynomial space of degree* l*, there* *exist constants* $C_{1,j}$ *and* $C_{2,j}$ *depending only upon the shape-regularity of the family of triangulations such that for* $j \leq l+1$

$$\|h_n^{-j}(\psi - \Pi^n\psi)\| \leq C_{1,j}\|\psi\|_j,$$

and

$$\|h_n^{1/2-j}(\psi - \Pi^n\psi)\|_{\sum_n} \leq C_{2,j}\|\psi\|_j.$$

In order to deal with the time discretization error, we introduce

$$\hat{\omega}(t) := \int_0^t B(t,s)\nabla\omega(s)ds. \tag{3.16}$$

Further, for $t \in I_n$, let $\hat{\omega}_I(t)$ be the linear interpolant associated with the vectors $\hat{\omega}(t_{n-1})$ and $\hat{\omega}(t_n)$ and is defined as

$$\hat{\omega}_I(t) := l_{n-1}(t)\hat{\omega}(t_{n-1}) + l_n(t)\hat{\omega}(t_n). \tag{3.17}$$

Fully discrete scheme in distributional form. Using (3.7), for any $\phi \in H_0^1(\Omega)$, we have

$$\begin{aligned}
\langle\bar{\partial}U^n \quad + \quad & \mathcal{A}^nU^n - \sigma^n(\mathcal{B}^nU) - P_0^nf^n, \phi\rangle = \langle\bar{\partial}U^n + \mathcal{A}^nU^n - \sigma^n(\mathcal{B}^nU) - P_0^nf^n, P_0^n\phi\rangle \\
&= \langle\bar{\partial}U^n, P_0^n\phi\rangle + a(U^n, P_0^n\phi) - \sigma^n(b(t_n;U, P_0^n\phi)) - \langle P_0^nf^n, P_0^n\phi\rangle \\
&= \tau_n^{-1}\langle U^n - P_0^nU^{n-1}, P_0^n\phi\rangle + a(U^n, P_0^n\phi) - \sigma^n(b(t_n;U, P_0^n\phi)) - \langle P_0^nf^n, P_0^n\phi\rangle \\
&= \tau_n^{-1}\langle U^n - U^{n-1}, P_0^n\phi\rangle + a(U^n, P_0^n\phi) - \sigma^n(b(t_n;U, P_0^n\phi)) - \langle f^n, P_0^n\phi\rangle \\
&= 0.
\end{aligned}$$

Thus, the fully discrete scheme can be written in the following distributional form:

$$\bar{\partial}U^n + \mathcal{A}^nU^n(x) = \sigma^n(\mathcal{B}^nU(x)) + P_0^nf^n(x), \quad \forall x \in \Omega. \tag{3.18}$$

Residuals. For $n \in [0:N]$, we now define the inner residual as

$$\begin{aligned}
\mathfrak{R}^n(U) \quad &:= \quad \mathcal{A}_{el}U^n - \sigma^n(\mathcal{B}_{el}U) - \mathcal{A}^nU^n + \sigma^n(\mathcal{B}^nU) \\
&= \quad \mathcal{A}_{el}U^n - \sigma^n(\mathcal{B}_{el}U) - P_0^nf^n + \bar{\partial}U^n, \qquad (3.19) \\
\mathfrak{R}^0(U) \quad &:= \quad \mathcal{A}_{el}U^0 - \mathcal{A}^0U^0,
\end{aligned}$$

and the jump residual as

$$\begin{aligned} \mathfrak{J}^n[U] &:= J_1[U^n] - \sigma^n(J_2[U]), \\ \mathfrak{J}^0[U] &:= J_1[U^0]. \end{aligned} \tag{3.20}$$

The inner residual terms can also be written in the following form:

$$< \mathfrak{R}^n(U), \phi >:= \sum_{K\in\mathcal{T}_n} \langle -\text{div}(A\nabla U^n) + \sigma^n(\text{div}(B(t_n)\nabla U)) - P_0^n f^n + \frac{U^n - P_0^n U^{n-1}}{\tau_n}, \phi\rangle_K.$$

We first obtain the following bounds on the quadrature error which will be used in our subsequent analysis.

Lemma 3.2.1 (Quadrature error estimates)**.** *Assume that* $v(t) \in H_0^1(\Omega),\ t \in [0,T]$ *with* $v(t) = l_{n-1}(t)v^{n-1} + l_n(t)v^n,\ t \in I_n,$ *where* $l_{n-1}(t)$ *and* $l_n(t)$ *are given by (3.15). Then, for all* $\phi \in H_0^1(\Omega)$

$$\begin{aligned} (i)\ \ |\sigma^n(b(t_n; v, \phi)) \ &- \int_0^{t_n} b(t_n, s; v(s), \phi)ds| \\ &\le C_{Q_1}\hat{\tau}_n \left[\sum_{j=0}^{n} \tau_j \|v^j\|_1 + \sum_{j=1}^{n} \tau_j \|\partial v^j\|_1\right] \|\phi\|_1 \end{aligned}$$

and

$$\begin{aligned} (ii)\ \ |\langle \int_0^{t_n} \mathcal{B}^n(s)v(s)ds, \phi\rangle \ &- \ \langle \sigma^n(\mathcal{B}^n v), \phi\rangle| \\ &\le C_{Q_2}\hat{\tau}_n \left[\sum_{j=0}^{n} \tau_j \|\Delta^n v^j\| + \sum_{j=1}^{n} \tau_j \|\Delta^n \partial v^j\|\right] \|\phi\| \end{aligned}$$

hold true, where $\sigma^n(\cdot)$ *is given by (3.8) and* $\hat{\tau}_n = \max_{j=1}^{n} \tau_j$. C_{Q_1} *and* C_{Q_2} *are two positive constants depending on the continuity constants of the bilinear forms* $b(t, s, \cdot, \cdot)$ *and* $b_s(t, s; \cdot, \cdot)$ *but independent of the mesh parameter.*

Proof. For $\psi_{1n}(s) = (t_n - s)$, we have the following elementary fact

$$\int_{t_{n-1}}^{t_n} y(s)ds - \tau_n y(t_{n-1}) = \int_{t_{n-1}}^{t_n} \psi_{1n}(s)\frac{dy}{ds}ds. \tag{3.21}$$

By setting $y(s) = B(t_n, s)\nabla v(s),\ s \in I_n, v(s) \in H_0^1(\Omega)$, we have

$$\begin{aligned}\frac{dy}{ds} &= B_s(t_n, s)\nabla v(s) + B(t_n, s)\nabla v_s(s)\\ &= B_s(t_n, s)\nabla v(s) + B(t_n, s)\nabla \partial v^n\\ &= l_{n-1}(s)B_s(t_n, s)\nabla v^{n-1} + l_n(s)B_s(t_n, s)\nabla v^n + B(t_n, s)\nabla\partial v^n,\end{aligned}$$

where we have used $v(s) = l_{n-1}(s)v^{n-1} + l_n(s)v^n$ and $v_s(s) = \partial v^n$.

For all $\phi \in H_0^1(\Omega)$, using (3.21) we have

$$\begin{aligned}&\sigma^n(b(t_n; v, \phi)) - \int_0^{t_n} b(t_n, s; v(s), \phi)ds\\ &= \langle -\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \psi_{1j}(s)\frac{\partial\Big\{B(t_n, s)v(s)\Big\}}{\partial s}ds, \nabla\phi\rangle\\ &= -\sum_{j=1}^n \int_{t_{j-1}}^{t_j} b_s(t_n, s; \psi_{1j}(s)v(s), \phi)ds - \sum_{j=1}^n \int_{t_{j-1}}^{t_j} b(t_n, s; \psi_{1j}(s)\partial v^j, \phi)ds.\end{aligned}$$

Noting the fact $\forall t \in I_n,\ l_n(t) \le 1,\ l_{n-1}(t) \le 1$, we use continuity of $b(t, s, \cdot, \cdot)$ and $b_s(t, s; \cdot, \cdot)$ to obtain

$$\begin{aligned}|\sigma^n(b(t_n; v, \phi)) &- \int_0^{t_n} b(t_n, s; v(s), \phi)ds|\\ &\le \left[\frac{\gamma'\hat{\tau}_n}{2}\sum_{j=1}^n \tau_j\Big\{\|v^j\|_1 + \|v^{j-1}\|_1\Big\} + \gamma\hat{\tau}_n\sum_{j=1}^n \tau_j\|\partial v^j\|_1\right]\|\phi\|_1\\ &\le \left[\gamma'\hat{\tau}_n\sum_{j=0}^n \tau_j\|v^j\|_1 + \gamma\hat{\tau}_n\sum_{j=1}^n \tau_j\|\partial v^j\|_1\right]\|\phi\|_1.\end{aligned}$$

Thus, the proof of Lemma 3.2.1 (i) follows with $C_{Q_1} = \max\{\gamma', \gamma\}$.

Next, we proceed to prove Lemma 3.2.1 (ii). For all $\phi \in H_0^1(\Omega)$, using (3.21) we have

$$\begin{aligned}&\langle\int_0^{t_n} \mathcal{B}^n(s)v(s)ds, \phi\rangle - \langle\sigma^n(\mathcal{B}^n v), \phi\rangle\\ &= \langle\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \psi_{1j}(s)\frac{\partial\Big\{\mathcal{B}^n(s)v(s)\Big\}}{\partial s}ds, \phi\rangle\\ &= \langle\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \psi_{1j}(s)\frac{\partial\Big\{l_{j-1}(s)\mathcal{B}^n(s)v^{j-1} + l_j(s)\mathcal{B}^n(s)v^j\Big\}}{\partial s}ds, \phi\rangle\end{aligned}$$

Noting the fact $\forall t \in I_n,\ l_n(t) \leq 1,\ l_{n-1}(t) \leq 1$, we obtain

$$
\begin{aligned}
&|\langle \int_0^{t_n} \mathcal{B}^n(s)v(s)ds, \phi\rangle - \langle \sigma^n(\mathcal{B}^n v), \phi\rangle| \\
&\leq \left[\frac{\gamma'\hat{\tau}_n}{2}\sum_{j=1}^{n}\tau_j\Big\{\|\Delta^n v^j\|_1 + \|\Delta^n v^{j-1}\|_1\Big\} + \gamma\hat{\tau}_n\sum_{j=1}^{n}\tau_j\|\Delta^n \partial v^j\|_1\right]\|\phi\| \\
&\leq \left[\gamma'\hat{\tau}_n\sum_{j=0}^{n}\tau_j\|\Delta^n v^j\|_1 + \gamma\hat{\tau}_n\sum_{j=1}^{n}\tau_j\|\Delta^n \partial v^j\|_1\right]\|\phi\|.
\end{aligned}
$$

Taking $C_{Q_2} = \max\{\gamma', \gamma\}$, the desired result follows. □

The following lemma gives bounds on the Ritz-Volterra reconstruction error.

Lemma 3.2.2 (Ritz-Volterra reconstruction error estimates)**.** *For any $v \in \mathbb{V}^n$, the following estimates hold:*

$$
\begin{aligned}
\|\mathcal{R}_w^n v - v\|_1 &\leq \alpha_{BE,n}(v), \\
\|\mathcal{R}_w^n v - v\| &\leq \beta_{BE,n}(v),
\end{aligned}
$$

where $\alpha_{BE,n}(v)$ and $\beta_{BE,n}(v)$ are Ritz-Volterra reconstruction error estimators and are given by

$$\alpha_{BE,n}(v) := C_1\Big\{\alpha'_n(v) + h_n\mathcal{Q}_{BE,2,n}(v)\Big\}, \quad n \in [0:N], \tag{3.22}$$

$$\beta_{BE,n}(v) := C_2\Big\{\beta'_n(v) + h_n\mathcal{Q}_{BE,1,n}(v) + h_n^2\mathcal{Q}_{BE,2,n}(v)\Big\}, \quad n \in [0:N], \tag{3.23}$$

where $\alpha'_n(v)$ and $\beta'_n(v)$ are given by

$$
\begin{aligned}
\alpha'_n(v) &:= h_n\|\mathfrak{R}^n(v)\| + h_n^{1/2}\|\mathfrak{J}^n[v]\|_{\sum_n}, \quad n \in [0:N], \\
\beta'_n(v) &:= h_n^2\|\mathfrak{R}^n(v)\| + h_n^{3/2}\|\mathfrak{J}^n[v]\|_{\sum_n}, \quad n \in [0:N].
\end{aligned}
$$

$C_j, j = 1, 2$ are positive constants independent of the mesh parameters but depend upon the interpolation constants and the final time T. Further, $\mathcal{Q}_{BE,1,n}(v)$ and $\mathcal{Q}_{BE,2,n}(v)$ are the quadrature error estimators and are given by

$$\mathcal{Q}_{BE,1,n}(v) := C_{Q_1}\hat{\tau}_n\left[\sum_{j=0}^{n}\tau_j\|v^j\|_1 + \sum_{j=1}^{n}\tau_j\|\partial v^j\|_1\right], \tag{3.24}$$

$$\mathcal{Q}_{BE,2,n}(v) := C_{Q_2}\hat{\tau}_n\Bigg[\sum_{j=0}^{n}\tau_j\|\Delta^n v^j\| + \sum_{j=1}^{n}\tau_j\|\Delta^n\partial v^j\|\Bigg]. \tag{3.25}$$

Proof. For $v \in \mathbb{V}^n$, using (3.9), (3.11) and (3.12) we have

$$\begin{aligned}
a(\mathcal{R}_w^n v - v, \phi) \;&-\; \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \phi)ds \\
&= \langle \mathcal{A}^n v, \phi\rangle - \int_0^{t_n}\langle \mathcal{B}^n(s)v(s)ds, \phi\rangle - a(v,\phi) + \int_0^{t_n} b(t_n, s; v(s), \phi)ds \\
&= \langle \mathcal{A}^n v - \sigma^n(\mathcal{B}^n v) - \mathcal{A}_{el}v + \sigma^n(\mathcal{B}_{el}v), \phi\rangle - \langle J_1[v] - \sigma^n(J_2[v]), \phi\rangle_{\sum_n} \\
&\quad -\langle \int_0^{t_n}\mathcal{B}^n(s)v(s)ds, \phi\rangle - \langle\sigma^n(\mathcal{B}^n v), \phi\rangle \\
&\quad + \int_0^{t_n} b(t_n, s; v(s), \phi)ds - \sigma^n(b(t_n; v, \phi)), \;\; \forall \phi \in H_0^1(\Omega).
\end{aligned}$$

An application of the Galerkin orthogonality (3.13) yields

$$\begin{aligned}
a(\mathcal{R}_w^n v - v, \phi) \;&-\; \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \phi)ds \\
&= \langle \mathcal{A}^n v - \sigma^n(\mathcal{B}^n v) - \mathcal{A}_{el}v + \sigma^n(\mathcal{B}_{el}v), \phi - \Pi^n\phi\rangle \\
&\quad -\langle J_1[v] - \sigma^n(J_2[v]), \phi - \Pi^n\phi\rangle_{\sum_n} \\
&\quad -\langle \int_0^{t_n}\mathcal{B}^n(s)v(s)ds - \sigma^n(\mathcal{B}^n v), \phi - \Pi^n\phi\rangle \\
&\quad + \int_0^{t_n} b(t_n, s; v(s), \phi - \Pi^n\phi)ds - \sigma^n(b(t_n; v, \phi - \Pi^n\phi)).
\end{aligned}$$

Now, using the definition $b(t_n, s; v(s), \phi - \Pi^n\phi) := \langle\mathcal{B}(t_n, s)v(s), \phi - \Pi^n\phi\rangle$, for the last two terms above, we have

$$\begin{aligned}
\int_0^{t_n} b(t_n, s; v(s), \phi - \Pi^n\phi)ds \;&-\; \sigma^n(b(t_n; v, \phi - \Pi^n\phi)) \\
&:= \langle \int_0^{t_n}\mathcal{B}^n(s)v(s)ds - \sigma^n(\mathcal{B}^n v), \phi - \Pi^n\phi\rangle
\end{aligned}$$

Using Proposition 3.2.1 with $C_{i,1}, i = 1, 2$ as interpolation constants, we obtain

$$\begin{aligned}
|a(\mathcal{R}_w^n v - v, \phi)| \;&\leq\; C_{1,1}h_n\|\phi\|_1\|\mathcal{A}^n v - \sigma^n(\mathcal{B}^n v) - \mathcal{A}_{el}v + \sigma^n(\mathcal{B}_{el}v)\| \\
&\quad + C_{2,1}h_n^{1/2}\|\phi\|_1\|J_1[v] - \sigma^n(J_2[v])\|_{\sum_n} \\
&\quad + \int_0^{t_n}|b(t_n, s; (\mathcal{R}_w v - v)(s), \phi)|ds \\
&\quad + 2C_{1,1}h_n\|\int_0^{t_n}\mathcal{B}^n(s)v(s)ds - \sigma^n(\mathcal{B}^n v)\|\|\phi\|_1.
\end{aligned}$$

Taking $\phi = \mathcal{R}_w v - v$ and using (3.4), we have

$$
\begin{aligned}
& |a(\mathcal{R}_w v - v, \mathcal{R}_w v - v)| \\
\leq\ & \|\mathcal{R}_w v - v\|_1 \Big\{ C_{1,1} h_n \|\mathcal{A}^n v - \mathcal{A}_{el} v - \sigma^n(\mathcal{B}^n v) + \sigma^n(\mathcal{B}_{el} v)\| \\
& + C_{2,1} h_n^{1/2} \|J_1[v] - \sigma^n(J_2[v])\|_{\sum_n} + \gamma \int_0^{t_n} \|(\mathcal{R}_w v - v)(s)\|_1 ds \\
& + 2C_{1,1} h_n \mathcal{Q}_{BE,2,n}(v) \Big\},
\end{aligned}
$$

where we have used Lemma 3.2.1. Now, coercivity property of $a(\cdot,\cdot)$ and an application of the Gronwall's lemma yield the first inequality with $C_1 = \max\{2C_{1,1}C_{1,G}(T), C_{2,1}C_{1,G}(T)/\alpha\}$, where $C_{1,G}$ is a constant appear due to the application of Gronwall's lemma.

The proof for the L^2-error estimate will proceed by the duality technique. For $v \in \mathbb{V}^n$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$
\begin{aligned}
\mathcal{A}\psi &= \mathcal{R}_w^n v - v \quad \text{in } \Omega, \\
\psi &= 0 \quad \text{on } \Omega,
\end{aligned}
\tag{3.26}
$$

satisfying the following regularity estimate (Ω is convex) with the constant C_Ω depending on the domain Ω:

$$
\|\psi\|_2 \leq C_\Omega \|\mathcal{R}_w^n v - v\|. \tag{3.27}
$$

Multiplying (3.26) by $\mathcal{R}_w^n v - v$ and integrating over Ω and using Galerkin orthogonality (3.13), we obtain

$$
\begin{aligned}
\|\mathcal{R}_w^n v - v\|^2 &= a(\mathcal{R}_w^n v - v, \psi - \Pi^n \psi) + a(\mathcal{R}_w^n v - v, \Pi^n \psi) \\
&= a(\mathcal{R}_w^n v - v, \psi - \Pi^n \psi) - \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \psi - \Pi^n \psi) ds \\
&\quad + \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \psi) ds \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}
$$

Using (3.12), (3.9) and (3.11), we arrive at

$$\begin{aligned}\mathcal{I}_1+\mathcal{I}_2 &= \langle \mathcal{A}^n v-\sigma^n(\mathcal{B}^n v)-\mathcal{A}_{el}v+\sigma^n(\mathcal{B}_{el}v),\psi-\Pi^n\psi\rangle\\ &-\langle J_1[v]-\sigma^n(J_2[v]),\psi-\Pi^n\psi\rangle_{\sum_n}\\ &-\langle\int_0^{t_n}\mathcal{B}^n(s)v(s)ds-\sigma^n(\mathcal{B}^n v),\psi-\Pi^n\psi\rangle.\\ &+\int_0^{t_n} b(t_n,s;v(s),\psi-\Pi^n\psi)ds-\sigma^n(b(t_n;v,\psi-\Pi^n\psi)).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathcal{I}_1+\mathcal{I}_2| &= \|\mathcal{A}^n v-\sigma^n(\mathcal{B}^n v)-\mathcal{A}_{el}v+\sigma^n(\mathcal{B}_{el}v)\|\|\psi-\Pi^n\psi\|\\ &+\|J_1[v]-\sigma^n(J_2[v])\|_{\sum_n}\|\psi-\Pi^n\psi\|_{\sum_n}\\ &+\|\int_0^{t_n}\mathcal{B}^n(s)v(s)ds-\sigma^n(\mathcal{B}^n v)\|\|\psi-\Pi^n\psi\|.\\ &+|\int_0^{t_n} b(t_n,s;v(s),\psi-\Pi^n\psi)ds-\sigma^n(b(t_n;v,\psi-\Pi^n\psi))|.\end{aligned}$$

Now, using the fact

$$b(t_n,s;(\mathcal{R}_w v-v)(s),\psi) := \langle(\mathcal{R}_w v-v)(s),\mathcal{B}^*(t_n,s)\psi\rangle, \tag{3.28}$$

where $\mathcal{B}^*(t_n,s)$ is the formal adjoint of the operator $\mathcal{B}(t_n,s)$ and $\|\mathcal{B}^*(t_n,s)\psi\|\le C_{\mathcal{B}_1^*}\|\psi\|_2$, we obtain

$$|\mathcal{I}_3|\le C_{\mathcal{B}_1^*}\|\psi\|_2\int_0^{t_n}\|(\mathcal{R}_w v-v)(s)\|ds.$$

The above bounds on $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, and an application of Proposition 3.2.1, Lemma 3.2.1 with the interpolation constants as $C_{i,2}, i=1,2$, yields

$$\begin{aligned}\|\mathcal{R}_w^n v-v\|^2 &\le \|\psi\|_2\Big\{C_{1,2}h_n^2\|\mathcal{A}^n v-\mathcal{A}_{el}v-\sigma^n(\mathcal{B}^n v)+\sigma^n(\mathcal{B}_{el}v)\|\\ &+C_{2,2}h_n^{3/2}\|J_1[v]-\sigma^n(J_2[v])\|_{\sum_n}+C_{\mathcal{B}_1^*}\int_0^{t_n}\|(\mathcal{R}_w v-v)(s)\|ds\\ &+C_{1,2}h_n\mathcal{Q}_{BE,1,n}(v)+C_{1,2}h_n^2\mathcal{Q}_{BE,2,n}(v)\Big\}.\end{aligned}$$

And hence, with an aid of (3.27), we have

$$\begin{aligned}\|\mathcal{R}_w^n v-v\| &\le C_{1,2}C_\Omega h_n^2\|\mathcal{A}^n v-\mathcal{A}_{el}v-\sigma^n(\mathcal{B}^n v)+\sigma^n(\mathcal{B}_{el}v)\|\\ &+C_{2,2}C_\Omega h_n^{3/2}\|J_1[v]-\sigma^n(J_2[v])\|_{\sum_n}+C_\Omega C_{\mathcal{B}_1^*}\int_0^{t_n}\|(\mathcal{R}_w v-v)(s)\|ds\\ &+C_{1,2}C_\Omega h_n\mathcal{Q}_{BE,1,n}(v)+C_{1,2}C_\Omega h_n^2\mathcal{Q}_{BE,2,n}(v).\end{aligned}$$

Finally, an application of the Gronwall's lemma yields the desired estimate with $C_2 = \max\{C_{2,G}(T)C_{1,2}C_\Omega, C_{2,G}(T)C_{2,2}C_\Omega\}$, where $C_{2,G}$ is a constant appear due to the application of Gronwall's lemma. □

The next lemma yields a bound on the parabolic error $\rho(t)$.

Lemma 3.2.3 ($L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$ *a posteriori* error estimate for the parabolic error)**.** *For each $m \in [1:N]$, the following estimate holds:*

$$\left(\max_{[0,t_m]} \|\rho(t)\|^2 + \alpha \int_0^{t_m} \|\rho(t)\|_1^2 dt\right)^{1/2} \leq \|\rho(t_0)\| + 2C(t_m)(\sigma^2_{BE,1,m} + \sigma^2_{BE,2,m})^{1/2},$$

where

$$\sigma_{BE,1,m} = \sum_{n=1}^{m} (\zeta_{BE,n} + \eta_{BE,n} + \lambda_{BE,n})\tau_n,$$

$$\sigma^2_{BE,2,m} = \sum_{n=1}^{m} (\xi_{BE,n} + \mu_{BE,n})^2 (\tau_n/\alpha),$$

and $C(t_m)$ is a positive constant depends upon the time t_m. Moreover,

$$\xi_{BE,n} := \left(\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt\right)^{1/2}, \quad n \in [1:N] \tag{3.29}$$

is the linear interpolation error estimator for the Volterra integral term.

$$\begin{aligned}\zeta_{BE,n} := \; & C_\Omega C_3 \Bigg(\left(\frac{\hat{\tau}_n}{\tau_n}\right) \left[h_n^2 \|\partial \mathfrak{R}^n(U)\| + h_n^{3/2} \|\partial \mathfrak{J}^n[U]\|_{\Sigma_n} + \sum_{j=0}^{n-1} \beta_{BE,j}(U^j) \right] \\ & + h_n \mathcal{Q}_{BE,1,n}(U) + h_n \mathcal{Q}_{BE,1,n-1}(U) + h_n^2 \mathcal{Q}_{BE,2,n}(U) \\ & + h_n^2 \mathcal{Q}_{BE,2,n-1}(U) \Bigg), \quad n \in [1:N]\end{aligned} \tag{3.30}$$

is the space error estimator, where $\beta_{BE,n}(U^n)$ is given by (3.23). Let

$$\eta'_n := \begin{cases} \frac{1}{2}\|P_0^1 f^1 - \bar{\partial} U^1 - \mathcal{A}^0 U^0\|, & \text{for } n = 1, \\ \frac{1}{2}\tau_n \|\partial(P_0^n f^n - \bar{\partial} U^n)\|, & \text{for } n \in [2:N]. \end{cases} \tag{3.31}$$

Then, the time error estimator $\eta_{BE,n}$ is given by

$$\eta_{BE,n} = \eta'_n + \mathcal{Q}_{BE,2,n-1}(U) + \mathcal{Q}_{BE,2,n}(U), \quad n \in [1:N], \tag{3.32}$$

where $\mathcal{Q}_{BE,2,n}$ *is given by (3.25).*

$$\mu_{BE,n} := C_4 h_n \|(P_0^n - I)(f^n + \frac{U^{n-1}}{\tau_n})\|, \qquad n \in [1:N] \tag{3.33}$$

is the mesh modification error estimator and

$$\lambda_{BE,n} := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \|f^n - f(t)\| dt, \qquad n \in [1:N] \tag{3.34}$$

is the data oscillation error estimator.

The proof of the Lemma 3.2.3 relies on several auxiliary results which we shall discuss in details below. We begin with the following error equation for $\rho(t)$.

Lemma 3.2.4. *For* $t \in I_n$ *and* $\phi \in H_0^1(\Omega)$, *we have the following parabolic error equation*

$$\begin{aligned}
\langle \rho_t, \phi\rangle \;&+\; a(\rho,\phi) - \int_0^t b(t,s;\rho(s),\phi)ds \\
&= \langle \epsilon_t, \phi\rangle + a(\omega - \omega^n, \phi) - \int_0^t b(t,s;\omega(s),\phi)ds \\
&\quad + \int_0^{t_n} b(t_n,s;\omega(s),\phi)ds + \langle P_0^n f^n - f, \phi\rangle + \tau_n^{-1}\langle P_0^n U^{n-1} - U^{n-1}, \phi\rangle \\
&\quad - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^n U), \phi\rangle.
\end{aligned} \tag{3.35}$$

Proof. For $t \in I_n$, using (3.6), (3.12) and (3.18), we have $\forall \phi \in H_0^1(\Omega)$

$$\begin{aligned}
\langle \rho_t, \phi\rangle \;&+\; a(\rho,\phi) - \int_0^t b(t,s;\rho(s),\phi)ds \\
&= \langle \omega_t, \phi\rangle + a(\omega,\phi) - \int_0^t b(t,s;\omega(s),\phi)ds - \langle f,\phi\rangle \\
&= \langle \omega_t, \phi\rangle + a(\omega,\phi) - \int_0^t b(t,s;\omega(s),\phi)ds - \langle f,\phi\rangle - \langle \partial U^n, \phi\rangle \\
&\quad + \tau_n^{-1}\langle P_0^n U^{n-1} - U^{n-1}, \phi\rangle - \langle \mathcal{A}^n U^n - \sigma^n(\mathcal{B}^n U), \phi\rangle + \langle P_0^n f^n, \phi\rangle \\
&= \langle \omega_t, \phi\rangle + a(\omega,\phi) - \int_0^t b(t,s;\omega(s),\phi)ds - \langle f,\phi\rangle - \langle \partial U^n, \phi\rangle \\
&\quad + \tau_n^{-1}\langle P_0^n U^{n-1} - U^{n-1}, \phi\rangle - a(\omega^n,\phi) + \int_0^{t_n} b(t_n,s;\omega(s),\phi)ds + \langle P_0^n f^n, \phi\rangle \\
&\quad - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^n U), \phi\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \epsilon_t, \phi \rangle + a(\omega - \omega^n, \phi) - \int_0^t b(t, s; \omega(s), \phi) ds + \int_0^{t_n} b(t_n, s; \omega(s), \phi) ds \\
&\quad + \langle P_0^n f^n - f, \phi \rangle + \tau_n^{-1} \langle P_0^n U^{n-1} - U^{n-1}, \phi \rangle \\
&\quad - \langle \int_0^{t_n} \mathcal{B}^n(s) U(s) ds - \sigma^n(\mathcal{B}^n U), \phi \rangle,
\end{aligned}
$$

where we have used the fact that $\partial U^n = U_t(t), \ \ \forall t \in I_n$. □

Remark. We observe that in the absence of time-discretization error, mesh change error and quadrature error (i.e., in the absence of the second, third, fourth, sixth and seventh terms on right of (3.35)), the fully-discrete parabolic error equation (3.35) reduces to that of the parabolic error equation for the semidiscrete scheme (see (2.30), Chapter 2). This shows that space-time discretizations are properly adapted to the space discretizations. Moreover, when $\mathcal{B}(t, s) = 0$, our error equation (3.35) coincides with the error equation of purely parabolic problems (Lakkis and makridakis [56], Lemma 1.4).

The next result gives a clear picture of the terms to be estimated in order to obtain a bound for the error $\rho(t)$ in different norms.

Lemma 3.2.5. *The following bound holds for $\rho(t)$:*

$$
\|\rho(t_m^*)\|^2 + \alpha \int_0^{t_m} \|\rho(t)\|_1^2 dt \leq \|\rho(t_0)\|^2 + 2C(t_m)\mathcal{I}_m,
$$

where

$$
\|\rho(t_m^*)\| := \|\rho_*^m\| := \max_{t \in [0, t_m]} \|\rho(t)\|, \quad t_m^* \in [0, t_m]
$$

and $C(t_m)$ is a positive constant appeared due to the application of Gronwall's lemma. Moreover, $\mathcal{I}_m, m = 1, 2, \dots, N$ are defined by

$$
\mathcal{I}_m := \sum_{n=1}^{m} (\mathcal{I}_1^n + \mathcal{I}_2^n + \mathcal{I}_3^n + \mathcal{I}_4^n)
$$

with $\mathcal{I}_1^n$ denotes the spatial error and is given by

$$
\mathcal{I}_1^n := \int_{t_{n-1}}^{t_n} |\langle \epsilon_t(t), \rho(t) \rangle| dt,
$$

$\mathcal{I}_2^n$ *denotes the time discretization error and is given by*

$$\begin{aligned}\mathcal{I}_2^n \;:=\; & \int_{t_{n-1}}^{t_n} |a(\omega(t)-\omega^n,\rho(t)) - \int_0^t b(t,s;\omega(s),\rho(t))ds + \int_0^{t_n} b(t_n,s;\omega(s),\rho(t))ds \\ & -\langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^nU),\rho(t)\rangle|dt,\end{aligned}$$

$\mathcal{I}_3^n$ *denotes the mesh change error and is given by*

$$\mathcal{I}_3^n := \int_{t_{n-1}}^{t_n} |\langle P_0^n f^n - f^n + \tau_n^{-1}(P_0^nU^{n-1} - U^{n-1}),\rho(t)\rangle|dt$$

and $\mathcal{I}_4^n$ *denotes the data oscillation error, and is given by*

$$\mathcal{I}_4^n := \int_{t_{n-1}}^{t_n} |\langle f^n - f(t),\rho(t)\rangle|dt.$$

Proof. Set $\phi = \rho(t)$ in the error equation (3.35) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho(t)\|^2 \;&+\; a(\rho(t),\rho(t)) = \int_0^t b(t,s;\rho(s),\rho(t))ds + \langle \epsilon_t(t),\rho(t)\rangle \\ &+\; a(\omega(t)-\omega^n,\rho(t)) - \int_0^t b(t,s;\omega(s),\rho(t))ds + \int_0^{t_n} b(t_n,s;\omega(s),\rho(t))ds \\ &+\; \langle P_0^n f^n - f(t),\rho(t)\rangle + \tau_n^{-1}\langle P_0^nU^{n-1} - U^{n-1},\rho(t)\rangle \\ &-\; \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^nU),\rho(t)\rangle.\end{aligned}$$

Using (3.3), (3.4) and Young's inequality, a standard kickback argument yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho(t)\|^2 \;&+\; \frac{\alpha}{2}\|\rho(t)\|_1^2 \le \frac{\gamma^2}{2\alpha}\left(\int_0^t \|\rho(s)\|_1 ds\right)^2 + |\langle \epsilon_t(t),\rho(t)\rangle| \\ &+\; |a(\omega(t)-\omega^n,\rho(t)) - \int_0^t b(t,s;\omega(s),\rho(t))ds \\ &+\; \int_0^{t_n} b(t_n,s;\omega(s),\rho(t))ds - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^nU),\rho(t)\rangle| \\ &+\; |\langle P_0^n f^n - f(t),\rho(t)\rangle| + |\tau_n^{-1}\langle P_0^nU^{n-1} - U^{n-1},\rho(t)\rangle|.\end{aligned}$$

We now integrate with respect to t from t_{n-1} to t_n and then take summation over $n = 1:m$. Finally, an application of Gronwall's lemma leads to the desired result. □

We now proceed to estimate the terms $\mathcal{I}_i^n, i = 1,2,3,4$ appeared in Lemma 3.2.5. First, we provide *a posteriori* error bounds on the spatial error in the following lemma.

Lemma 3.2.6 (Spatial error estimate)**.** *With $\mathcal{I}_1^n$ as in Lemma 3.2.5, the following a posteriori bound holds for the spatial discretization error:*

$$\sum_{n=1}^{m} \mathcal{I}_1^n \leq \|\rho(t_*^m)\| \sum_{n=1}^{m} \tau_n \zeta_{BE,n},$$

where $\zeta_{BE,n}$ is given by (3.30).

Proof. To estimate the term $\mathcal{I}_1^n$, for $n \in [1:N]$, we note that

$$\begin{aligned}\mathcal{I}_1^n &= \int_{t_{n-1}}^{t_n} |\langle \epsilon_t(t), \rho(t)\rangle| dt \\ &= \tau_n^{-1} \int_{t_{n-1}}^{t_n} |\langle \omega^n - \omega^{n-1} - U^n + U^{n-1}, \rho(t)\rangle| dt. \end{aligned} \tag{3.36}$$

Since $\omega^n - U^n$ is orthogonal to $\mathbb{V}^n$ with respect to $a(\cdot,\cdot) - \int_0^{t_n} b(t_n, s; \cdot, \cdot)$, the first term in the inner product is orthogonal to $\mathbb{V}^n \cap \mathbb{V}^{n-1}$. We use the orthogonality property of the Ritz-Volterra reconstructions under the nested refinement condition. To estimate (3.36), we shall use duality technique.

For $t \in (0, T)$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the following dual elliptic problem in the weak form

$$a(\chi, \psi(t)) = \langle \chi, \rho(t)\rangle$$

satisfying the following regularity estimate:

$$\|\psi\|_2 \leq C_\Omega \|\rho\|, \ \forall \chi \in H_0^1(\Omega), \tag{3.37}$$

where the constant C_Ω depends on the domain Ω. Now, using the definition of the Ritz-Volterra reconstruction and making adjustment of the terms, we have

$$\begin{aligned}\langle \omega^n - \omega^{n-1} - U^n + U^{n-1}, \rho(t)\rangle &= a(\omega^n - \omega^{n-1} - U^n + U^{n-1}, \psi(t)) \\ &= a(\omega^n - \omega^{n-1} - U^n + U^{n-1}, \psi(t) - \Pi^n \psi(t)) \\ &\quad + \int_0^{t_n} b(t_n, s; (\omega - U)(s), \Pi^n \psi(t)) ds \\ &\quad - \int_0^{t_{n-1}} b(t_{n-1}, s; (\omega - U)(s), \Pi^n \psi(t)) ds.\end{aligned}$$

where Π^n is the Clément-type interpolation operator. Using (3.12), (3.9) and (3.11), a simple calculation leads to

$$
\begin{aligned}
& \langle \omega^n - \omega^{n-1} - U^n + U^{n-1}, \rho(t) \rangle \\
= \; & a(\omega^n - \omega^{n-1} - U^n + U^{n-1}, \psi(t) - \Pi^n \psi(t)) \\
& - \int_0^{t_n} b(t_n, s; (\omega - U)(s), \psi(t) - \Pi^n \psi(t)) ds \\
& + \int_0^{t_{n-1}} b(t_{n-1}, s; (\omega - U)(s), \psi(t) - \Pi^n \psi(t)) ds \\
& + \int_0^{t_n} b(t_n, s; (\omega - U)(s), \psi(t)) ds - \int_0^{t_{n-1}} b(t_{n-1}, s; (\omega - U)(s), \psi(t)) ds \\
= \; & \langle \mathcal{A}^n U^n - \sigma^n(\mathcal{B}^n U) - \mathcal{A}_{el} U^n + \sigma^n(\mathcal{B}_{el} U), \psi(t) - \Pi^n \psi(t) \rangle \\
& - \langle \mathcal{A}^{n-1} U^{n-1} - \sigma^{n-1}(\mathcal{B}^{n-1} U) - \mathcal{A}_{el} U^{n-1} + \sigma^{n-1}(\mathcal{B}_{el} U), \psi(t) - \Pi^n \psi(t) \rangle \\
& + \langle \sigma^n(J_2[U]) - J_1[U^n] - \sigma^{n-1}(J_2[U]) + J_1[U^{n-1}], \psi(t) - \Pi^n \psi(t) \rangle_{\Sigma_n} \\
& - \langle \int_0^{t_n} \mathcal{B}^n(s) U(s) ds - \sigma^n(\mathcal{B}^n U), \psi(t) - \Pi^n \psi(t) \rangle \\
& + \langle \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s) U(s) ds - \sigma^{n-1}(\mathcal{B}^{n-1} U), \psi(t) - \Pi^n \psi(t) \rangle \\
& + \int_0^{t_n} b(t_n, s; U(s), \psi(t) - \Pi^n \psi(t)) ds - \sigma^n(b(t_n; U, \psi(t) - \Pi^n \psi(t))) \\
& - \Big\{ \int_0^{t_{n-1}} b(t_{n-1}, s; U(s), \psi(t) - \Pi^n \psi(t)) ds - \sigma^{n-1}(b(t_{n-1}; U, \psi(t) - \Pi^n \psi(t))) \Big\} \\
& + \int_0^{t_n} b(t_n, s; (\omega - U)(s), \psi(t)) ds - \int_0^{t_{n-1}} b(t_{n-1}, s; (\omega - U)(s), \psi(t)) ds.
\end{aligned}
$$

Using (3.18) and (3.19), on each interval I_n, we have

$$
\begin{aligned}
\mathcal{A}^n U^n \; & - \; \mathcal{A}^{n-1} U^{n-1} + \sigma^{n-1}(\mathcal{B}^{n-1} U) - \sigma^n(\mathcal{B}^n U) \\
& + \; \mathcal{A}_{el} U^{n-1} - \mathcal{A}_{el} U^n + \sigma^n(\mathcal{B}_{el} U) - \sigma^n(\mathcal{B}_{el} U) \\
& \quad = \mathfrak{R}^{n-1}(U) - \mathfrak{R}^n(U) = -\tau_n \partial \mathfrak{R}^n(U).
\end{aligned}
$$

We now use (3.20) together with $\mathfrak{J}^n[U] - \mathfrak{J}^{n-1}[U] = \tau_n \partial \mathfrak{J}^n[U]$ to obtain

$$
\begin{aligned}
&|\langle \omega^n - \omega^{n-1} - U^n + U^{n-1}, \rho(t)\rangle| \\
\leq\ & \tau_n \|\partial \mathfrak{R}^n(U)\| \|\psi(t) - \Pi^n \psi(t)\| + \tau_n \|\partial \mathfrak{J}^n[U]\|_{\Sigma_n} \|\psi(t) - \Pi^n\psi(t)\|_{\Sigma_n} \\
&+ |\langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^n U), \psi(t) - \Pi^n\psi(t)\rangle| \\
&+ |\langle \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds - \sigma^{n-1}(\mathcal{B}^{n-1}U), \psi(t) - \Pi^n\psi(t)\rangle| \\
&+ |\int_0^{t_n} b(t_n, s; U(s), \psi(t) - \Pi^n\psi(t))ds - \sigma^n(b(t_n; U, \psi(t) - \Pi^n\psi(t)))| \\
&+ |\int_0^{t_{n-1}} b(t_{n-1}, s; U(s), \psi(t) - \Pi^n\psi(t))ds - \sigma^{n-1}(b(t_{n-1}; U, \psi(t) - \Pi^n\psi(t)))| \\
&+ |\int_0^{t_n} b(t_n, s; (\omega - U)(s), \psi(t))ds - \int_0^{t_{n-1}} b(t_{n-1}, s; (\omega - U)(s), \psi(t))ds|. \qquad (3.38)
\end{aligned}
$$

For the last term above, use (3.28), Cauchy-Schwarz inequality and $\|\mathcal{B}^*(t_n, s)\psi\| \leq C_{\mathcal{B}_1^*}\|\psi\|_2$ to obtain

$$
\begin{aligned}
&|\int_0^{t_n} b(t_n, s; (\omega - U)(s), \psi(t))ds - \int_0^{t_{n-1}} b(t_{n-1}, s; (\omega - U)(s), \psi(t))ds| \\
&\leq |\int_0^{t_n} \langle (\omega - U)(s), \mathcal{B}^*(t_n, s)\psi(t)\rangle ds - \int_0^{t_{n-1}} \langle (\omega - U)(s), \mathcal{B}^*(t_{n-1}, s)\psi(t)\rangle ds| \\
&\leq \|\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (\omega - U)(s)\| \|\mathcal{B}^*(t_n, s)\psi(t)\| ds \\
&+ \|\sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (\omega - U)(s)\| \|\mathcal{B}^*(t_{n-1}, s)\psi(t)\| ds \\
&\leq 2C_{\mathcal{B}_1^*} \left[\|\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\{ l_{j-1}(s)(\omega^{j-1} - U^{j-1}) + l_j(s)(\omega^j - U^j) \right\} ds\| \right] \|\psi(t)\|_2 \\
&\leq C_{\mathcal{B}_1^*} \left[\hat{\tau}_n \sum_{j=1}^{n} \beta_{BE,j}(U^j) + \hat{\tau}_{n-1} \sum_{j=0}^{n-1} \beta_{BE,j}(U^j) \right] \|\psi(t)\|_2 \\
&\leq C_{\mathcal{B}_4^*} \hat{\tau}_n \left[\sum_{j=0}^{n} \beta_{BE,j}(U^j) \right] \|\psi(t)\|_2, \qquad (3.39)
\end{aligned}
$$

where $C_{\mathcal{B}_4^*} = 2C_{\mathcal{B}_1^*}$. Using (3.39) in (3.38) and applying Proposition 3.2.1, Lemma 3.2.1 with $C_3 = \max\left(C_{\mathcal{B}_4^*}, C_{1,2}, C_{2,2}\right)$, we obtain

$$
\begin{aligned}
&|\langle \omega^n - \omega^{n-1} - U^n + U^{n-1}, \rho(t)\rangle| \\
\leq\ & C_3\|\psi\|_2\Bigg(\hat{\tau}_n\Big[h_n^2\|\partial R^n\| + h_n^{3/2}\|\partial\mathfrak{J}^n[U]\|_{\Sigma_n} + \sum_{j=0}^{n}\beta_{BE,j}(U^j)\Big] \\
&+h_n\mathcal{Q}_{BE,1,n}(U) + h_n\mathcal{Q}_{BE,1,n-1}(U) + h_n^2\mathcal{Q}_{BE,2,n}(U) + h_n^2\mathcal{Q}_{BE,2,n-1}(U)\Bigg), \quad (3.40)
\end{aligned}
$$

where $\mathcal{Q}_{BE,1,n}(U)$ and $\mathcal{Q}_{BE,2,n}(U)$ are given by (3.24) and (3.25) respectively.

From (3.40) and (3.36), we arrive at

$$
\begin{aligned}
\mathcal{I}_1^n \ \leq\ & C_3\tau_n^{-1}\int_{t_{n-1}}^{t_n}\|\psi(t)\|_2 dt\ \Bigg(\hat{\tau}_n\Big[h_n^2\|\partial R^n\| + h_n^{3/2}\|\partial\mathfrak{J}^n[U]\|_{\Sigma_n} + \sum_{j=0}^{n}\beta_{BE,j}(U^j)\Big] \\
&+h_n\mathcal{Q}_{BE,1,n}(U) + h_n\mathcal{Q}_{BE,1,n-1}(U) + h_n^2\mathcal{Q}_{BE,2,n}(U) + h_n^2\mathcal{Q}_{BE,2,n-1}(U)\Bigg) \\
\leq\ & \max_{t\in I_n}\|\rho(t)\|\tau_n\zeta_{BE,n},
\end{aligned}
$$

where we have used (3.30) and the regularity result (3.37). Summing from $n = 1 : m$, the desired result is obtained. □

The next lemma gives the information on the *a posteriori* error contributions due to time discretizations.

Lemma 3.2.7 (Time error estimate)**.** *With $\mathcal{I}_2^n$ as in Lemma 3.2.5, the following a posteriori bound holds for the time discretization error:*

$$
\sum_{n=1}^{m}\mathcal{I}_2^n \leq \|\rho_*^m\|\sum_{n=1}^{m}\tau_n\eta_{BE,n} + \sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n}\|\rho(t)\|_1^2 dt\right)^{1/2}\xi_{BE,n}\tau_n^{1/2},
$$

where $\xi_{BE,n}$ and $\eta_{BE,n}$ are given by (3.29) and (3.32), respectively.

Proof. Taking L^2-inner product with $\nabla\rho(t)$ on both sides of (3.17), we obtain

$$
\begin{aligned}
\langle\hat{\omega}_I(t), \nabla\rho(t)\rangle \ =\ & l_{n-1}(t)\int_0^{t_{n-1}} b(t_{n-1}, s; \omega(s), \rho(t))ds \\
+\ & l_n(t)\int_0^{t_n} b(t_n, s; \omega(s), \rho(t))ds.
\end{aligned}
$$

Using (3.16) and noting that $\omega(t) = l_{n-1}(t)\omega^{n-1} + l_n(t)\omega^n,\ t\in I_n$, we write $\mathcal{I}_2^n$ as

$$
\begin{aligned}
\mathcal{I}_2^n &= \int_{t_{n-1}}^{t_n} |a(l_{n-1}(t)\omega^{n-1} + l_n(t)\omega^n - \omega^n, \rho(t)) - \langle \hat{\omega}(t), \nabla \rho(t)\rangle \\
&\quad + \int_0^{t_n} b(t_n, s; \omega(s), \rho(t))ds\Big] - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^n U), \rho(t)\rangle |dt \\
&= \int_{t_{n-1}}^{t_n} |a(l_{n-1}(t)\omega^{n-1} + l_n(t)\omega^n - \omega^n, \rho(t)) - \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla \rho(t)\rangle \\
&\quad - \Big[l_{n-1}(t) \int_0^{t_{n-1}} b(t_{n-1}, s; \omega(s), \rho(t))ds + l_n(t) \int_0^{t_n} b(t_n, s; \omega(s), \rho(t))ds \\
&\quad - \int_0^{t_n} b(t_n, s; \omega(s), \rho(t))ds\Big] - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^n U), \rho(t)\rangle |dt \\
&= \int_{t_{n-1}}^{t_n} |l_{n-1}(t)\Big\{a(\omega^{n-1}, \rho(t)) - \int_0^{t_{n-1}} b(t_{n-1}, s; \omega(s), \rho(t))ds\Big\} \\
&\quad + (l_n(t) - 1)\Big\{a(\omega^n, \rho(t)) - \int_0^{t_n} b(t_n, s; \omega(s), \rho(t))ds\Big\} \\
&\quad - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^n U), \rho(t)\rangle - \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla \rho(t)\rangle |dt.
\end{aligned}
$$

Now, using the identity $(l_n(t) - 1)/l_{n-1}(t) = -1, t \in I_n$, and (3.12), we obtain

$$
\begin{aligned}
\mathcal{I}_2^n &= \int_{t_{n-1}}^{t_n} |l_{n-1}(t)\Big\{\langle \mathcal{A}^{n-1}U^{n-1}, \rho(t)\rangle - \langle \sigma^{n-1}(\mathcal{B}^{n-1}U), \rho(t)\rangle \\
&\quad - l_{n-1}(t)\Big\{\langle \mathcal{A}^n U^n, \rho(t)\rangle - \langle \sigma^n(\mathcal{B}^n U), \rho(t)\rangle\Big\} \\
&\quad - l_n(t)\langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \sigma^n(\mathcal{B}^n U), \rho(t)\rangle - \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla \rho(t)\rangle \\
&\quad + l_{n-1}(t)\langle \sigma^{n-1}(\mathcal{B}^{n-1}U) - \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds, \rho(t)\rangle |dt.
\end{aligned}
$$

An application of Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&\mathcal{I}_2^n \le \int_{t_{n-1}}^{t_n} l_{n-1}(t)\|\mathcal{A}^{n-1}U^{n-1} - \sigma^{n-1}(\mathcal{B}^{n-1}U) - \mathcal{A}^n U^n + \sigma^n(\mathcal{B}^n U)\| \|\rho(t)\| dt \\
&+ \int_{t_{n-1}}^{t_n} \Big[l_{n-1}(t)\mathcal{Q}_{BE,2,n-1}(U) + l_n(t)\mathcal{Q}_{BE,2,n}(U)\Big] \|\rho(t)\| dt \\
&+ \int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\| \|\rho(t)\|_1 dt \\
&\le 1/2\ \tau_n \max_{t \in I_n} \|\rho(t)\| \Big[\|\mathcal{A}^{n-1}U^{n-1} - \sigma^{n-1}(\mathcal{B}^{n-1}U) - \mathcal{A}^n U^n + \sigma^n(\mathcal{B}^n U)\| \\
&+ \mathcal{Q}_{BE,2,n-1}(U) + \mathcal{Q}_{BE,2,n}(U)\Big] + \left(\int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt\right)^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\rho(t)\|_1^2 dt\right)^{1/2}.
\end{aligned}
$$

The desired result now follows from (3.29) and (3.32). □

The next lemma captures *a posteriori* contributions due to mesh change.

Lemma 3.2.8 (Mesh change error estimate)**.** *With $\mathcal{I}_3^n$ as in Lemma 3.2.5, the following a posteriori error bound holds on the mesh change error:*

$$\sum_{n=1}^{m} \mathcal{I}_3^n \leq \sum_{n=1}^{m} \left(\int_{t_{n-1}}^{t_n} \|\rho(t)\|_1^2 dt \right)^{1/2} \tau_n^{1/2} \mu_{BE,n},$$

where $\mu_{BE,n}$ is given by (3.33).

Proof. The term $\mathcal{I}_3^n$ can be estimated using the orthogonality property of the L^2-projection operator. Since $\mathbb{V}^n \subset \ker(P_0^n - I)$, we have

$$\langle (P_0^n - I)(f^n + \tau_n^{-1} U^{n-1}), \phi_n \rangle = 0, \quad \forall \phi_n \in \mathbb{V}^n.$$

Using Proposition 3.2.1 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathcal{I}_3^n &= \int_{t_{n-1}}^{t_n} |\langle (P_0^n - I)(f^n + \tau_n^{-1} U^{n-1}), \rho(t) - \Pi^n \rho(t) \rangle| dt \\
&\leq C_4 h_n \int_{t_{n-1}}^{t_n} \|(P_0^n - I)(f^n + \tau_n^{-1} U^{n-1})\| \|\rho(t)\|_1 dt \\
&\leq C_4 h_n \tau_n^{1/2} \|(P_0^n - I)(f^n + \tau_n^{-1} U^{n-1})\| \left(\int_{t_{n-1}}^{t_n} \|\rho(t)\|_1^2 dt \right)^{1/2}.
\end{aligned}$$

Thus, the desired estimate follows by using (3.33). □

We consider the estimation of the data approximation error in the next lemma.

Lemma 3.2.9 (Data oscillation error estimate)**.** *With $\mathcal{I}_4^n$ as in Lemma 3.2.5, we have the following bound on the data approximation error:*

$$\sum_{n=1}^{m} \mathcal{I}_4^n \leq \|\rho_*^m\| \sum_{n=1}^{m} \tau_n \lambda_{BE,n},$$

where $\lambda_{BE,n}$ is given by (3.34).

Proof. We have

$$\mathcal{I}_4^n \leq \int_{t_{n-1}}^{t_n} \|f^n - f(t)\| \|\rho(t)\| dt \leq \left(\max_{t \in I_n} \|\rho(t)\| \right) \int_{t_{n-1}}^{t_n} \|f^n - f(t)\| dt.$$

Using (3.34), the proof of the lemma is completed. □

Proof of Lemma 3.2.3. Combining the estimates of Lemmas 3.2.6 - 3.2.9, we arrive at

$$\|\rho_*^m\|^2 + \alpha \int_0^{t_m} \|\rho(t)\|_1^2 dt \leq \|\rho(t_0)\|^2 + 2C(t_m)\Bigg[\|\rho_*^m\| \sum_{n=1}^m (\zeta_{BE,n} + \eta_{BE,n} + \lambda_{BE,n})\tau_n + \sum_{n=1}^m \left(\int_{t_{n-1}}^{t_n} \|\rho(t)\|_1^2 dt\right)^{1/2} \tau_n^{1/2}(\xi_{BE,n} + \mu_{BE,n})\Bigg].$$

For $n = [1:m]$, taking

$$a_0 = \|\rho_*^m\|, \quad a_n = \left(\alpha \int_{t_{n-1}}^{t_n} \|\rho(t)\|_1^2 dt\right)^{1/2}, \quad c = \|\rho(t_0)\|,$$

$$b_0 = 2C(t_m)\sum_{n=1}^m (\zeta_{BE,n} + \eta_{BE,n} + \lambda_{BE,n})\tau_n, \quad b_n = 2C(t_m)(\tau_n/\alpha)^{1/2}(\xi_{BE,n} + \mu_{BE,n}),$$

we apply Lemma 1.2.3 to obtain the desired result. □

The main results concerning fully discrete *a posteriori* error estimates in the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms are stated in the following theorem.

Theorem 3.2.1 (*A posteriori* error estimate in the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms**).** *Let u satisfies (3.1) and U be given by (3.14). Then, for each $m \in [1:N]$, the following error estimates hold:*

$$\max_{[0,t_m]} \|u(t) - U(t)\| \leq \|R_w^0 U^0 - u(0)\| + \max_{n\in[0:m]} \beta_{BE,n}(U^n) + 2C(t_m)(\sigma_{BE,1,m}^2 + \sigma_{BE,1,m}^2)^{1/2},$$

$$\left(\int_0^{t_m} \|u(t) - U(t)\|_1^2\right)^{1/2} \leq \alpha^{-1/2}\Big[\|R_w^0 U^0 - u(0)\| + 2C(t_m)(\sigma_{BE,1,m}^2 + \sigma_{BE,2,m}^2)^{1/2}\Big] + \left(\sum_{n=1}^m \tau_n \alpha_{BE,n-1}^2(U^{n-1})\right)^{1/2} + \left(\sum_{n=1}^m \tau_n \alpha_{BE,n}^2(U^n)\right)^{1/2},$$

where $\sigma_{BE,1,m}$, $\sigma_{BE,2,m}$ and $C(t_m)$ are defined as in Lemma 3.2.3.

Proof. We decompose the error with Ritz-Volterra reconstruction of U as an intermediate solution and obtain

$$\|u(t) - U(t)\| \leq \|\rho(t)\| + \|\epsilon(t)\|, \tag{3.41}$$

where $\rho(t) := \omega(t) - u(t)$ and $\epsilon(t) := \omega(t) - U(t)$.

Also, we know that for $t \in I_n$,

$$\|\epsilon(t)\| = \|l_{n-1}(t)\epsilon^{n-1} + l_n(t)\epsilon^n\| \leq \max\Big(\|\epsilon^{n-1}\|, \|\epsilon^n\|\Big).$$

Therefore, for $t \in [0, t_m]$, using Lemma 3.2.2 and (3.23) we have

$$\|\epsilon(t)\| \leq \max_{n\in[0,m]} \Big(\|\epsilon^{n-1}\|, \|\epsilon^n\|\Big) \leq \max_{n\in[0,m]} \beta_{BE,n}(U^n). \tag{3.42}$$

Then, the first estimate follows from (3.41), (3.42) and Lemma 3.2.3.

To prove the second estimate, using Lemma 3.2.2 and (3.22) we obtain

$$\begin{aligned}\left(\int_0^{t_m} \|\epsilon(t)\|_1^2\right)^{1/2} &= \left(\int_0^{t_m} \|l_{n-1}(t)\epsilon^{n-1} + l_n(t)\epsilon^n\|_1^2\right)^{1/2} \\ &\leq \left(\sum_{n=1}^{m} \tau_n \alpha^2_{BE,n-1}(U^{n-1})\right)^{1/2} + \left(\sum_{n=1}^{m} \tau_n \alpha^2_{BE,n}(U^n)\right)^{1/2}.\end{aligned}$$

The rest of the proof follows from Lemma 3.2.3. □

Remarks. (*i*) The *a posteriori* error estimators in Theorem 3.2.1 are formally of optimal order. Since PIDE (3.1) may be thought of as a perturbation to the parabolic problem, it is natural to expect that our *a posteriori* error estimators should reflect the contributions to the error coming from the approximation of the memory term. This fact can easily be observed through the estimators $\mathcal{Q}_{BE,1,n}(U)$ and $\mathcal{Q}_{BE,2,n}(U)$ which are of $O(\tau)$. Further, in the absence of the memory term (i.e., $\mathcal{B}(t,s) = 0$), the error estimators obtained in Theorem 3.2.1 are similar to that for the parabolic problems [56].

(*ii*) It is noteworthy that the term

$$\left(\frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt\right)^{1/2} \tag{3.43}$$

appeared in Lemma 3.2.3 is not a traditional *a posteriori* quantity, where $\hat{\omega}(t)$ and $\hat{\omega}_I(t)$ are given by (3.16) and (3.17) respectively. We know the following elementary fact that for $t \in I_n$, the error in linear interpolation is bounded as

$$\|\hat{\omega}(t) - \hat{\omega}_I(t)\| \leq C\tau_n^2 \max_{t\in I_n} \|\frac{d^2}{dt^2}(\hat{\omega}(t))\|.$$

Now, the quantity $\frac{d^2}{dt^2}(\hat{\omega}(t))$ in turn depends upon the quantities $\nabla\omega_t(t)$ and $\nabla\omega(t)$. The term $\|\nabla\omega_t(t)\|$ can be estimated as

$$\begin{aligned}\|\nabla\omega_t(t)\| &= \|\nabla\epsilon_t(t) - \nabla U_t(t)\| \\ &\leq \|\nabla\epsilon_t(t)\| + \|\nabla U_t(t)\| \\ &\leq \frac{1}{\tau_n}\Big(\|\nabla\epsilon^n\| + \|\nabla\epsilon^{n-1}\|\Big) + \|\nabla\partial U^n\|.\end{aligned}$$

Moreover, the term $\|\nabla\omega(t)\|$ can be handled as

$$\begin{aligned}\|\nabla\omega(t)\| &\leq \|\nabla\epsilon(t)\| + \|\nabla U(t)\| \\ &\leq \|l_{n-1}(t)\nabla\epsilon^{n-1} + l_n(t)\nabla\epsilon^n\| + \|l_{n-1}(t)\nabla U^{n-1} + l_n(t)\nabla U^n\| \\ &\leq \max\Big(\|\nabla\epsilon^{n-1}\|, \|\nabla\epsilon^n\|\Big) + \max\Big(\|\nabla U^{n-1}\|, \|\nabla U^n\|\Big).\end{aligned}$$

Thus, the term (3.43) is now a meaningful *a posteriori* quantity by observing the fact that bounds available for estimating the terms of the form $\|\nabla\epsilon^n\|$ (see Lemma 3.2.2).

(*iii*) If the domain Ω is not convex in particular having reentrant corners, then we don't have the full regularity of the solution (cf. Grisvard [42]) due to the singularity caused by the reentrant corners. Thus, applying duality technique will lead to suboptimal bounds due to regularity issues. However, suboptimality of the bounds can be avoided by the use of mesh refinement near the corners (cf. Chatzipantelidis *et al.* [22]).

4

Fully Discrete Crank Nicolson Error Analysis

In this chapter, we study *a posteriori* error analysis for the fully discrete Crank-Nicolson method for PIDE (1.1). We introduce a problem dependent quadratic (in time) space-time reconstruction to obtain optimal order convergence in time. This quadratic space-time reconstruction along with the Ritz-Volterra reconstruction operator are used to derive *a posteriori* error bound in the $L^\infty(L^2(\Omega))$-norm. Moreover, the results obtained in this chapter generalize the results of parabolic problems (see Bänsch *et al.* [12]) to PIDE (1.1).

4.1 Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 1$ be a bounded convex polygonal or polyhedral domain with boundary $\partial\Omega$ and $T < \infty$. Recall the following PIDE

$$u_t(x,t) + \mathcal{A}u(x,t) = \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T] \tag{4.1}$$

subject to the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T]$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

The operator $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u),$$

and the operator $\mathcal{B}(t,s)$ is of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient and $A = \{a_{ij}(x)\}$ and $B(t,s) = \{b_{ij}(x;t,s)\}$ are two $n \times n$ matrices assumed to be in $L^\infty(\Omega)^{n\times n}$ in space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The initial function $u_0 = u_0(x)$ and the nonhomogeneous term f are assumed to be smooth for our purpose.

We assume that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi,\phi) \geq \alpha\|\phi\|_1^2 \quad \text{and} \quad |a(\phi,\psi)| \leq \beta\|\phi\|_1\|\psi\|_1, \qquad \forall\, \phi,\psi \in H_0^1(\Omega) \tag{4.2}$$

with $\alpha, \beta \in \mathbb{R}^+$.

Further, we assume that the bilinear forms $b(t,s;\cdot,\cdot)$, $b_s(t,s;\cdot,\cdot)$ and $b_{ss}(t,s;\cdot,\cdot)$ are continuous on $H_0^1(\Omega)$ i.e.,

$$|b(t,s;\phi(s),\psi)| \leq \gamma\|\phi(s)\|_1\|\psi\|_1, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega), \tag{4.3}$$

$$|b_s(t,s;\phi(s),\psi)| \leq \gamma'\|\phi(s)\|_1\|\psi\|_1, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{4.4}$$

and

$$|b_{ss}(t,s;\phi(s),\psi)| \leq \gamma''\|\phi(s)\|_1\|\psi\|_1, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{4.5}$$

with $\gamma, \gamma', \gamma'' \in \mathbb{R}^+$.

The weak formulation of the problem (4.1) may be stated as follows: Find $u : [0,T] \to H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega u_t\phi dx + a(u,\phi) &= \int_0^t b(t,s;u(s),\phi)ds + \int_\Omega f\phi dx, \quad \forall \phi \in H_0^1(\Omega), \quad t \in (0,T], \\ u(.,0) &= u_0. \end{aligned} \tag{4.6}$$

Let σ^n be the quadrature rule used to discretize the Volterra integral term. To be consistent with the Crank-Nicolson scheme, we use the trapezoidal rule given by

$$\begin{aligned}\sigma^n(y) \quad &:= \sum_{j=0}^{n-2} \frac{\tau_{j+1}}{2}\big(y(t_j)+y(t_{j+1})\big) + \frac{\tau_n}{4}\big(y(t_{n-1})+y(t_{n-1/2})\big) \\ &\approx \int_0^{t_{n-1/2}} y(s)ds.\end{aligned} \tag{4.7}$$

The fully discrete Crank-Nicolson scheme may be stated as follows: Given $U^0 = P_0^0 u(0)$, find $U^n \in \mathbb{V}^n, n \in [1:N]$ such that

$$\begin{aligned}\tau_n^{-1}\langle U^n - U^{n-1}, \phi_n\rangle \quad &+ \quad a(U^{n-1/2}, \phi_n) \\ &= \quad \sigma^n(b(t_{n-1/2}; U, \phi_n)) + \langle f^{n-1/2}, \phi_n\rangle, \quad \forall \phi_n \in \mathbb{V}^n.\end{aligned} \tag{4.8}$$

Here, the quadrature rule σ^n is defined by (4.7) and $\mathbb{V}^n$ is given by (1.19).

Modified Crank-Nicolson scheme. In the case of parabolic problem, the authors in [12] have observed that during refinements the discrete Laplace operator on the finer mesh when applied to coarse grid function leads to the oscillatory behaviour. Since PIDE may be thought of as the perturbation to the parabolic problem, the same oscillatory behaviour is expected concerning the classical Crank-Nicolson scheme for the PIDE (4.1). Therefore, we consider the following modified Crank-Nicolson scheme instead of the classical Crank-Nicolson scheme.

For $n, 1 \le n \le N$, find $U^n \in \mathbb{V}^n$ such that

$$\bar{\partial} U^n + \frac{1}{2}\mathcal{A}^n U^n + \frac{1}{2} P_0^n \mathcal{A}^{n-1} U^{n-1} - P_0^n(\sigma^n(\mathcal{B}^{n-1/2}U)) - P_0^n f^{n-1/2} = 0. \tag{4.9}$$

Representation of the bilinear forms. For a function $v \in \mathbb{V}^n$, we can represent our bilinear form $a(\cdot,\cdot)$ as

$$a(v,\phi) = \langle \mathcal{A}_{el} v, \phi\rangle + \langle J_1[v], \phi\rangle_{\sum_n}, \quad \forall \phi \in H_0^1(\Omega), \tag{4.10}$$

where

$$\langle \mathcal{A}_{el} v, \phi\rangle = \sum_{K\in\mathcal{T}_n} \langle -\text{div}(A\nabla v), \phi\rangle, \quad \forall \phi \in H_0^1(\Omega)$$

is the regular part of the distribution $-\text{div}(A\nabla v)$ and

$$J_1[v]|_E(x) = [A\nabla v]_E(x) := \lim_{\varepsilon\to 0}(A\nabla v(x+\varepsilon\mathbf{n}_E) - A\nabla v(x-\varepsilon\mathbf{n}_E))\cdot\mathbf{n}_E \qquad (4.11)$$

is the spatial jump of the field $A\nabla v$ across an element side $E \in \mathcal{S}_n$, where $\mathbf{n}_E$ is a unit normal vector to E at the point x.

Similarly, for all $\phi \in H_0^1(\Omega)$, we represent the bilinear form $b(t_n, s; \cdot, \cdot)$ as

$$\int_0^{t_n} b(t_n, s; v(s), \phi)ds = \langle \int_0^{t_n} \mathcal{B}_{el}(t_n, s)v(s)ds, \phi\rangle + \langle \int_0^{t_n} J_2[v(s)]ds, \phi\rangle_{\sum_n}, \qquad (4.12)$$

where $\mathcal{B}_{el}(t_n, s)v(s)$ is the regular part of the distribution $-\text{div}(B(t_n, s)\nabla v(s))$ and $J_2[v(s)]$ is the spatial jump of the field $-\text{div}(B(t_n, s)\nabla v(s))$ across an element side $E \in \mathcal{S}_n$ as defined in (4.11) with $B(t_n, s)$ replacing A.

In the absence of the memory term (i.e., when $\mathcal{B}(t, s) = 0$), *a posteriori* error analysis for linear parabolic problems concerning Crank-Nicolson scheme have been investigated by Akrivis *et al.* [5], Bänsch *et al.* [12], Lozinski *et al.* [60] and Verfürth [98]. For the heat equation, a continuous, piecewise linear approximation in time is used to obtain suboptimal (with respect to time step) *a posteriori* error bounds using standard energy techniques in [98] by Verfürth. A continuous, piecewise quadratic polynomial so-called Crank-Nicolson reconstruction is then introduced by Akrivis *et al.* [5] to restore the second order of convergence for semidiscrete time discretization of a general parabolic problem. Subsequently, Lozinski *et al.* [60] have introduced the reconstruction based on approximations on one time level (two-point reconstruction) as in [5] as well as the reconstructions based on approximations on two time levels (three-point reconstruction) in the $L^2(H^1(\Omega))$-norm. Recently, Bänsch *et al.* [12] have used the elliptic reconstruction in combination with energy techniques to derive optimal order *a posteriori* error estimate for the Crank-Nicolson method in the $L^\infty(L^2(\Omega))$-norm. In this chapter, an attempt has been made to carry over *a posteriori* error analysis of parabolic problems [12] to PIDE (4.1).

The rest of the chapter is organized as follows. In Section 4.2, we introduce quadratic space-time reconstruction for PIDE. *A posteriori* error estimator for the fully discrete

Crank-Nicolson scheme in the $L^\infty(L^2(\Omega))$-norm is derived in Section 4.3.

4.2 Quadratic space-time reconstruction

We shall use the following notations for the introduction of quadratic (in time) space-time reconstruction.

Let $\Theta : [0,T] \to H_0^1(\Omega)$ be defined by

$$\Theta(t) \; := \; l_{n-1}(t)P_0^n\mathcal{A}^{n-1}U^{n-1} + l_n(t)\mathcal{A}^nU^n - P_0^n(\sigma^n(\mathcal{B}^{n-1/2}U)),\; t \in I_n, \qquad (4.13)$$

where $l_n(t)$ and $l_{n-1}(t)$ are given by

$$l_n(t) \; := \; \frac{(t-t_{n-1})}{\tau_n} \quad \text{and} \quad l_{n-1}(t) := \frac{(t_n - t)}{\tau_n}. \qquad (4.14)$$

Similarly, define $\hat{F} : [0,T] \to H_0^1(\Omega)$ by

$$\hat{F}(t) \; := \; \Theta(t) - P_0^n\varphi(t),\; t \in I_n, \qquad (4.15)$$

where $\varphi(t) := \hat{I}f(t)$. Here, $\hat{I}$ is a piecewise linear interpolant chosen such that

$$\hat{I}(\phi) \in \mathbb{P}_1(I_n),\;\; \hat{I}(\phi)(t_{n-1}) = \phi^{n-1} \;\text{ and }\; \hat{I}(\phi)(t_{n-1/2}) = \phi^{n-1/2}. \qquad (4.16)$$

We now define the quadratic space-time reconstruction $\hat{U} : [0,T] \to H_0^1(\Omega)$ as follows:

$$\begin{aligned}\hat{U}(t) \; &:= \; \mathcal{R}_w^{n-1}U^{n-1} - \mathcal{R}_w^n \int_{t_{n-1}}^t \hat{F}(s)ds \\ &\quad +(t-t_{n-1})\frac{\mathcal{R}_w^nP_0^nU^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}}{\tau_n},\;\; t \in I_n, \end{aligned} \qquad (4.17)$$

where $\mathcal{R}_w^nU^n$ is defined in (3.12). This definition (4.17) is motivated by the fact that $\hat{U}(t)$ satisfies the following relation:

$$\hat{U}_t(t) + \mathcal{R}_w^n\hat{F}(t) \; = \; \frac{\mathcal{R}_w^nP_0^nU^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}}{\tau_n}. \qquad (4.18)$$

Observe that

$$\hat{U}(t_{n-1}) = \mathcal{R}_w^{n-1}U^{n-1}$$

and

$$\hat{U}(t_n) = \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^n \tau_n \Big[\frac{P_0^n \mathcal{A}^{n-1}U^{n-1} + \mathcal{A}^n U^n}{2} - P_0^n f^{n-1/2} \\ -P_0^n \sigma^n (\mathcal{B}^{n-1/2}U)\Big],$$

where the integral is evaluated using the mid-point rule. Now, we use (4.9) to obtain

$$\hat{U}(t_n) = \mathcal{R}_w^n U^n.$$

Equivalently, the modified Crank-Nicolson scheme can be written as

$$\bar{\partial}U^n + \Theta^{n-1/2} = P_0^n f^{n-1/2}. \tag{4.19}$$

In view of (4.15)

$$\bar{\partial}U^n + F^{n-1/2} = 0, \tag{4.20}$$

where $\hat{F}(t_{n-1/2}) := F^{n-1/2}$.

4.3 Error analysis

In this section, we shall derive *a posteriori* error estimates for the error $e := u - U$, where u is the exact solution of the PIDE (4.1) and U is defined by

$$U(t) := l_n(t)U^n + l_{n-1}(t)U^{n-1}, \quad t \in I_n, \tag{4.21}$$

where $l_{n-1}(t)$ and $l_n(t)$ are defined by (4.14). Moreover, for $t \in I_n$, we define the Ritz-Volterra reconstructions of $U(t)$ by

$$\mathcal{R}_w U(t) := l_{n-1}(t)\mathcal{R}_w^{n-1}U^{n-1} + l_n(t)\mathcal{R}_w^n U^n. \tag{4.22}$$

We now decompose the total error e as $e := \hat{\rho} + \varepsilon$, where $\hat{\rho} := u - \hat{U}$ denotes the parabolic error and $\varepsilon := \hat{U} - U$ denotes the reconstruction error. Further, we decompose the reconstruction error as $\varepsilon := \epsilon + \sigma$, where $\epsilon := \mathcal{R}_w U - U$ is the Ritz-Volterra reconstruction error and $\sigma := \hat{U} - \mathcal{R}_w U$ is the time reconstruction error. With the above decompositions, the error e may be expressed as

$$e := \hat{\rho} + \sigma + \epsilon. \tag{4.23}$$

The idea behind the above error decomposition is as follows: (i) optimal order *a posteriori* error estimates for Ritz-Volterra reconstruction error ϵ in standard norms like L^2 and H^1 can be obtained; (ii) the parabolic error $\hat{\rho}$ satisfies a variant of the original PIDE (4.1) with a right hand side that can be controlled *a posteriori* in an optimal way; (iii) the time reconstruction $\hat{U}$ is chosen in such a way that the difference $\hat{U} - \mathcal{R}_w U$ can be estimated *a posteriori* and will be of $O(\tau^2)$.

The following linear interpolations are useful in the error analysis. Set

$$\hat{\omega}(t) := \int_0^t B(t,s)\nabla \mathcal{R}_w U(s)ds. \tag{4.24}$$

For $t \in I_n$, define $\hat{\omega}_I(t)$ to be the linear interpolant associated with the integral vectors $\hat{\omega}(t_{n-1})$, $\hat{\omega}(t_n)$ and is given by

$$\hat{\omega}_I(t) := l_{n-1}(t)\hat{\omega}(t_{n-1}) + l_n(t)\hat{\omega}(t_n). \tag{4.25}$$

Further, let

$$\hat{\mathcal{U}}(t) := \int_0^t \mathcal{B}(t,s)U(s)ds, \quad t \in I_n. \tag{4.26}$$

For $t \in I_n$, we define $\hat{\mathcal{U}}_{I,1}(t)$ to be the linear interpolant associated with $\hat{\mathcal{U}}(t_n)$, $\hat{\mathcal{U}}(t_{n-1})$ and $\hat{\mathcal{U}}_{I,2}(t)$ to be the linear interpolant associated with $\hat{\mathcal{U}}(t_{n-1/2})$ as follows:

$$\hat{\mathcal{U}}_{I,1}(t) \;\; := \;\; l_{n-1}(t)\hat{\mathcal{U}}(t_{n-1}) + l_n(t)\hat{\mathcal{U}}(t_n) \tag{4.27}$$

and

$$\begin{aligned} \hat{\mathcal{U}}_{I,2}(t) \;\; &:= \;\; \hat{\mathcal{U}}(t_{n-1/2}) + (t - t_{n-1/2})\frac{d}{dt}\hat{\mathcal{U}}(t)\big|_{t=t_n}, \\ &:= \;\; \hat{\mathcal{U}}(t_{n-1/2}) + (t - t_{n-1/2})\mathcal{Y}_n, \end{aligned} \tag{4.28}$$

where

$$\mathcal{Y}_n = \frac{d}{dt}\hat{\mathcal{U}}(t)\big|_{t=t_n}. \tag{4.29}$$

We define the jump residual, for $n \in [0:N]$, as

$$\begin{aligned} \mathfrak{J}^n[U] \;\; &:= \;\; J_1[U^n] - \int_0^{t_n} J_2[U(s)]ds, \\ \mathfrak{J}^0[U] \;\; &:= \;\; J_1[U^0]. \end{aligned} \tag{4.30}$$

To begin with, we shall derive the following *a posteriori* error estimates for the Ritz-Volterra reconstruction error. The proofs are essentially relies on the arguments in the Lemma 3.2.2 but with different inner residual structures. However, for the clarity of presentation we provide the proof below.

Lemma 4.3.1 (Ritz-Volterra reconstruction error estimates)**.** *For any $v \in \mathbb{V}^n$, the following estimates hold*

$$\|\mathcal{R}_w^n v - v\|_1 \leq \alpha_{CN,n}(v)$$

and

$$\|\mathcal{R}_w^n v - v\| \leq \beta_{CN,n}(v),$$

where

$$\begin{aligned}\alpha_{CN,n}(v) &:= C_1 h_n \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds \\ &\quad + \int_0^{t_n} \mathcal{B}_{el}(t_n,s)v(s)ds\| + C_2 h_n^{1/2} \|\mathfrak{J}^n[v]\|_{\sum_n} \end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\beta_{CN,n}(v) &:= C_3 h_n^2 \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds \\ &\quad + \int_0^{t_n} \mathcal{B}_{el}(t_n,s)v(s)ds\| + C_4 h_n^{3/2} \|\mathfrak{J}^n[v]\|_{\sum_n} \end{aligned} \tag{4.32}$$

are the Ritz-Volterra reconstruction error estimators. Moreover, the constants appeared in the estimators are positive constants depend upon the interpolation constants and the final time T.

Proof. For $v \in \mathbb{V}^n$, using (4.10), (4.12) and (3.12) we have

$$\begin{aligned} a(\mathcal{R}_w^n v - v, \phi) &- \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \phi)ds \\ &= \langle \mathcal{A}^n v, \phi\rangle - \int_0^{t_n} \langle \mathcal{B}^n(s)v(s)ds, \phi\rangle - a(v,\phi) + \int_0^{t_n} b(t_n, s; v(s), \phi)ds \\ &= \langle \mathcal{A}^n v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds - \mathcal{A}_{el} v + \int_0^{t_n} \mathcal{B}_{el}(t_n,s)v(s)ds, \phi\rangle \\ &\quad - \langle J_1[v] - \int_0^{t_n} J_2[v(s)]ds, \phi\rangle_{\sum_n}, \quad \forall \phi \in H_0^1(\Omega). \end{aligned}$$

An application of the Galerkin orthogonality (3.13) yields

$$\begin{aligned} a(\mathcal{R}_w^n v - v, \phi) &- \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \phi) ds \\ &= \langle \mathcal{A}^n v - \int_0^{t_n} \mathcal{B}^n(s) v(s) ds - \mathcal{A}_{el} v + \int_0^{t_n} \mathcal{B}_{el}(t_n, s) v(s) ds, \phi - \Pi^n \phi \rangle \\ &\quad - \langle J_1[v] - \int_0^{t_n} J_2[v(s)] ds, \phi - \Pi^n \phi \rangle_{\sum_n}, \end{aligned}$$

where $\Pi^n : H_0^1(\Omega) \to \mathbb{V}^n$ is the Clément-type interpolation operator satisfying Proposition 3.2.1. Using Proposition 3.2.1 with $C_{i,1}, i = 1, 2$ as interpolation constants, we obtain

$$\begin{aligned} |a(\mathcal{R}_w^n v - v, \phi)| \leq\ & C_{1,1} h_n \|\phi\|_1 \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s) v(s) ds \\ & + \int_0^{t_n} \mathcal{B}_{el}(t_n, s) v(s) ds\| + \int_0^{t_n} |b(t_n, s; (\mathcal{R}_w v - v)(s), \phi)| ds \\ & + C_{2,1} h_n^{1/2} \|\phi\|_1 \|J_1[v] - \int_0^{t_n} J_2[v(s)] ds\|_{\sum_n}. \end{aligned}$$

Taking $\phi = \mathcal{R}_w^n v - v$ and using (4.3), we have

$$\begin{aligned} & |a(\mathcal{R}_w^n v - v, \mathcal{R}_w^n v - v)| \\ \leq\ & \|\mathcal{R}_w^n v - v\|_1 \Big\{ C_{1,1} h_n \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s) v(s) ds + \int_0^{t_n} \mathcal{B}_{el}(t_n, s) v(s) ds\| \\ & + C_{2,1} h_n^{1/2} \|J_1[v] - \int_0^{t_n} J_2[v(s)] ds\|_{\sum_n} + \gamma \int_0^{t_n} \|(\mathcal{R}_w v - v)(s)\|_1 ds \Big\}. \end{aligned}$$

Now, coercivity property of $a(\cdot, \cdot)$ and an application of the Gronwall's lemma yields the first inequality with $C_i = C_{1,G}(T) C_{i,1}/\alpha, i = 1, 2$, where $C_{1,G}$ is a constant appearing due to the application of Gronwall's lemma.

The proof of L^2 error estimate will proceed by the duality technique. For $v \in \mathbb{V}^n$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$\begin{aligned} \mathcal{A}\psi &= \mathcal{R}_w^n v - v \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \Omega, \end{aligned} \tag{4.33}$$

satisfying the following regularity estimate (Ω is convex) with the constant C_Ω depending on the domain Ω:

$$\|\psi\|_2 \leq C_\Omega \|\mathcal{R}_w^n v - v\|. \tag{4.34}$$

Multiplying (4.33) by $\mathcal{R}_w^n v - v$ and integrating over Ω and using Galerkin orthogonality (3.13), we obtain

$$\begin{aligned}\|\mathcal{R}_w^n v - v\|^2 &= a(\mathcal{R}_w^n v - v, \psi - \Pi^n\psi) + a(\mathcal{R}_w^n v - v, \Pi^n\psi)\\ &= a(\mathcal{R}_w^n v - v, \psi - \Pi^n\psi) - \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \psi - \Pi^n\psi)ds\\ &\quad + \int_0^{t_n} b(t_n, s; (\mathcal{R}_w v - v)(s), \psi)ds\\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.\end{aligned}$$

For $\mathcal{I}_1$ and $\mathcal{I}_2$, we use (4.10), (4.12) and (3.12) to obtain

$$\begin{aligned}\mathcal{I}_1 + \mathcal{I}_2 &= \langle \mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds + \int_0^{t_n} \mathcal{B}_{el}(t_n, s)v(s)ds, \psi - \Pi^n\psi\rangle\\ &\quad -\langle J_1[v] - \int_0^{t_n} J_2[v(s)]ds, \psi - \Pi^n\psi\rangle_{\sum_n}.\end{aligned}$$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathcal{I}_1 + \mathcal{I}_2| &\leq \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds + \int_0^{t_n} \mathcal{B}_{el}(t_n, s)v(s)ds\|\|\psi - \Pi^n\psi\|\\ &\quad +\|J_1[v] - \int_0^{t_n} J_2[v(s)]ds\|_{\sum_n}\|\psi - \Pi^n\psi\|_{\sum_n}.\end{aligned}$$

Noting the fact

$$b(t_n, s; (\mathcal{R}_w v - v)(s), \psi) := \langle (\mathcal{R}_w v - v)(s), \mathcal{B}^*(t_n, s)\psi\rangle, \tag{4.35}$$

where $\mathcal{B}^*(t_n, s)$ is the formal adjoint of the operator $\mathcal{B}(t_n, s)$ and $\|\mathcal{B}^*(t_n, s)\psi\| \leq C_{\mathcal{B}_1^*}\|\psi\|_2$, we get

$$|\mathcal{I}_3| \leq C_{\mathcal{B}_1^*}\|\psi\|_2 \int_0^{t_n} \|(\mathcal{R}_w v - v)(s)\|ds.$$

Combining the bounds on $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and applying Proposition 3.2.1 with the interpolation constants as $C_{i,2}, i = 1, 2$, it now follows that

$$\begin{aligned}\|\mathcal{R}_w^n v - v\|^2 &\leq \|\psi\|_2\Big\{C_{1,2}h_n^2\|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds\\ &\quad + \int_0^{t_n} \mathcal{B}_{el}(t_n, s)v(s)ds\| + C_{2,2}h_n^{3/2}\|J_1[v] - \int_0^{t_n} J_2[v(s)]ds\|_{\sum_n}\\ &\quad +C_{\mathcal{B}_1^*}\int_0^{t_n} \|(\mathcal{R}_w v - v)(s)\|ds\Big\}.\end{aligned}$$

With an aid of (4.34), we have

$$\begin{aligned}\|\mathcal{R}_w^n v - v\| &\leq C_{1,2}C_\Omega h_n^2 \|\mathcal{A}^n v - \mathcal{A}_{el} v - \int_0^{t_n} \mathcal{B}^n(s)v(s)ds + \int_0^{t_n} \mathcal{B}_{el}(t_n,s)v(s)ds\| \\ &+ C_{2,2}C_\Omega h_n^{3/2}\|J_1[v] - \int_0^{t_n} J_2[v(s)]ds\|_{\sum_n} \\ &+ C_\Omega C_{\mathcal{B}_1^*}\int_0^{t_n} \|(\mathcal{R}_w v - v)(s)\|ds.\end{aligned}$$

Finally, an application of the Gronwall's lemma yields the desired estimate with $C_3 = C_{2,G}(T)C_{1,2}C_\Omega$ and $C_4 = C_{2,G}(T)C_{2,2}C_\Omega$, where $C_{2,G}$ is a constant appearing due to the application of Gronwall's lemma. □

We now state the main result of this section concerning fully discrete Crank-Nicolson *a posteriori* error estimate in the $L^\infty(L^2(\Omega))$-norm.

Theorem 4.3.1 ($L^\infty(L^2(\Omega))$ *a posteriori* error estimate). *Let $u(t)$ be the exact solution of (4.1) and $U(t)$ be as defined in (4.21). Then, for each $m \in [1:N]$, the following error estimate hold:*

$$\begin{aligned}\max_{t\in[0,t_m]} \|u(t) - U(t)\| &\leq \Big[\|\hat{\rho}(t_0)\|^2 + C_7\sum_{n=1}^{m} \tau_n \Lambda_{CN,n}^2\Big]^{1/2} + \Big(\sigma_{CN,1,m}^2 + \sigma_{CN,2,m}^2\Big)^{1/2} \\ &+ \max_{0\leq n\leq m} \beta_{CN,n}(U^n) + \max_{0\leq n\leq m} \nu_{CN,n},\end{aligned}$$

where $\nu_{CN,n}$ and $\Lambda_{CN,n}$ are the time reconstruction error estimators and are defined by

$$\nu_{CN,n} := \tau_n^2\Big[\beta_{CN,n}(\mathcal{W}_n) + \|\mathcal{W}_n\|\Big] \tag{4.36}$$

and

$$\Lambda_{CN,n} := \frac{\beta\tau_n^2}{\sqrt{30\alpha}}\Big[\alpha_{CN,n}(\mathcal{W}_n) + \|\mathcal{W}_n\|_1\Big]. \tag{4.37}$$

Here, $\alpha_{CN,n}(\mathcal{W}_n)$ and $\beta_{CN,n}(\mathcal{W}_n)$ are given by (4.31) and (4.32) respectively. $\mathcal{W}_n$ is an a posteriori quantity given by

$$\mathcal{W}_n := \left[\frac{1}{2}\bar{\partial}\mathcal{A}^n U^n - \frac{P_0^n\Big(f^{n-1/2} - f^{n-1}\Big)}{\tau_n}\right]. \tag{4.38}$$

$\sigma^2_{CN,1,m}$ *and* $\sigma^2_{CN,2,m}$ *are the total estimators corresponding to parabolic error* $\hat{\rho}(t)$ *and are defined by*

$$\sigma^2_{CN,1,m} := \left(C_7 \sum_{n=1}^{m} \tau_n \Big[\eta_{CN,n} + \zeta_{CN,n,2} + \zeta_{CN,n,1} + \lambda_{CN,n,1} \Big] \right)^2 \tag{4.39}$$

and

$$\sigma^2_{CN,2,m} := \frac{4C_7^2}{\alpha} \sum_{n=1}^{m} \tau_n \Big[\mu_{CN,n} + \xi_{CN,n} + \lambda_{CN,n,2} \Big]^2. \tag{4.40}$$

Here, $\eta_{CN,n}$ *is the time estimator which captures quadrature error and linear approximation errors, and is defined by*

$$\eta_{CN,n} := C_8 \Bigg[\mathcal{Q}_{CN,n} + \|\hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t)\| + \|\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t)\| \Bigg], \tag{4.41}$$

where $\hat{\mathcal{U}}(t)$, $\hat{\mathcal{U}}_{I,1}(t)$ *and* $\hat{\mathcal{U}}_{I,2}(t)$ *are given by (4.26), (4.27) and (4.28), respectively and* $\mathcal{Q}_{CN,n}$ *is given by*

$$\mathcal{Q}_{CN,n} := \sum_{j=0}^{n} \tau_j^2 \Bigg[\tau_j \|\Delta^n U^j\| + \tau_j \|\Delta^n \partial U^j\| \Bigg]. \tag{4.42}$$

$$\begin{aligned} \zeta_{CN,n,1} \;\; := \;\; & C_9 C_\Omega \left(\frac{\hat{\tau}_n}{\tau_n} \right) \Bigg[h_n^2 \|\tau_n^{-1} \Big(\mathcal{A}^n U^n - \mathcal{A}^{n-1} U^{n-1} + \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s) U(s) ds \\ & - \int_0^{t_n} \mathcal{B}^n(s) U(s) ds + \mathcal{A}_{el} U^{n-1} - \mathcal{A}_{el} U^n + \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1}, s) U(s) ds \\ & - \int_0^{t_n} \mathcal{B}_{el}(t_n, s) U(s) ds \Big) \| + h_n^{3/2} \|\partial \mathfrak{J}^n[U]\|_{\Sigma_n} + \sum_{j=0}^{n-1} \beta_{CN,j}(U^j) \Bigg] \end{aligned} \tag{4.43}$$

and

$$\zeta_{CN,n,2} \;\; := \;\; \frac{\tau_n}{2} \beta_{CN,n}(\mathcal{W}_n) \tag{4.44}$$

are the space error estimators, where $\beta_{CN,n}(U^n)$ *and* $\beta_{CN,n}(\mathcal{W}_n)$ *are given by* (4.32).

$$\xi_{CN,n} \;\; := \;\; \left(\frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt \right)^{1/2}, \tag{4.45}$$

is the linear interpolation error estimator for the Volterra integral term, where $\hat{\omega}(t)$ and $\hat{\omega}_I(t)$ are given by (4.24) and (4.25), respectively.

$$\mu_{CN,n} := C_{1,1}h_n\Bigg[\frac{1}{\sqrt{3}}\|(P_0^n - I)\mathcal{A}^{n-1}U^{n-1}\| + \|(P_0^n - I)\sigma^n(\mathcal{B}^{n-1/2}U)\| + \tau_n^{-1}\|(P_0^n - I)U^{n-1}\|\Bigg], \tag{4.46}$$

is the mesh change estimator.

$$\lambda_{CN,n,1} := \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \|f(t) - \hat{I}f(t)\|dt \tag{4.47}$$

and

$$\lambda_{CN,n,2} := 2C_{1,1}h_n \max\left\{\|(I - P_0^n)f^{n-1}\|, \|(I - P_0^n)f^{n-1/2}\|\right\} \tag{4.48}$$

are the data approximation error estimators, where $\hat{I}f$ is a piecewise linear interpolant of f. Moreover, the constants appeared in the estimators are positive constants independent of the discretization parameters but depend upon the interpolation constants and the final time T.

The proof of Theorem 4.3.1 needs some preparations. We first proceed to estimate $\hat{\rho}(t)$ which is a cumbersome task.

Lemma 4.3.2 (*A posteriori* error estimate for the parabolic error)**.** *For each $m \in [1 : N]$, the following estimate holds for $\hat{\rho}(t)$:*

$$\left(\max_{t\in[0,t_m]} \|\hat{\rho}(t)\|^2 + \frac{\alpha}{4}\int_0^{t_m} \|\hat{\rho}(t)\|_1^2 dt\right)^{1/2} \leq \left[\|\hat{\rho}(t_0)\|^2 + C_7\sum_{n=1}^{m}\tau_n\Lambda_{CN,n}^2\right]^{1/2} + \left(\sigma_{CN,1,m}^2 + \sigma_{CN,2,m}^2\right)^{1/2},$$

where $\Lambda_{CN,n}$, $\sigma_{CN,1,m}^2$ and $\sigma_{CN,2,m}^2$ are given by (4.37), (4.39) and (4.40), respectively and C_7 is a positive constant independent of the discretization parameters but depends upon the interpolation constants and the final time T.

The proof of the above lemma in turn hinges essentially on several auxiliary results which we shall discuss in detail below. We shall use the notation ρ for the error $u-\mathcal{R}_wU$ in the subsequent error analysis. We begin with the following error equation for $\hat{\rho}$.

Lemma 4.3.3. *For $t \in I_n, n \in [1:N]$ and for each $\phi \in H_0^1(\Omega)$, we have the following error equation for $\hat{\rho}(t)$:*

$$\langle \hat{\rho}_t, \phi\rangle + a(\rho,\phi) - \int_0^t b(t,s;\rho(s),\phi)ds = \langle G, \phi\rangle, \tag{4.49}$$

where G is defined by

$$\langle G,\phi\rangle := \langle G_1,\phi\rangle + (t-t_{n-1/2})\langle \mathcal{Y}_n, \phi\rangle$$

with

$$\begin{aligned}
\langle G_1,\phi\rangle \;:=\; & \langle (P_0^n - I)\Big\{l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U) - \tau_n^{-1}U^{n-1}\Big\},\phi\rangle \\
& + \langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}),\phi\rangle + \langle \hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t),\phi\rangle + \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla\phi\rangle \\
& + \langle \hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t),\phi\rangle + \langle \int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2}U(s)ds - \sigma^n(\mathcal{B}^{n-1/2}U),\phi\rangle \\
& + \langle \hat{F}(t) - \Theta(t) + f(t),\phi\rangle - \langle \tau_n^{-1}\Big[(\mathcal{R}_w^n - I)U^n - (\mathcal{R}_w^{n-1} - I)U^{n-1}\Big],\phi\rangle
\end{aligned}$$

and $\mathcal{Y}_n$ is given by (4.29).

Proof. For $t \in I_n$ and $\forall \phi \in H_0^1(\Omega)$, we first multiply (4.18) by ϕ and integrate over Ω. Then, subtract the resulting equation from (4.6) to obtain

$$\begin{aligned}
\langle \hat{\rho}_t(t),\phi\rangle \;+\; & a(u(t),\phi) - \int_0^t b(t,s;u(s),\phi)ds \\
=\; & \langle f,\phi\rangle + \langle \mathcal{R}_w^n \hat{F}(t),\phi\rangle - \tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1},\phi\rangle.
\end{aligned}$$

Using (4.22)-(4.25), we obtain

$$\begin{aligned}
& \langle \hat{\rho}_t(t),\phi\rangle + a(\rho(t),\phi) - \int_0^t b(t,s;\rho(s),\phi)ds \\
=\; & -l_{n-1}(t)\Big[a(\mathcal{R}_w^{n-1}U^{n-1},\phi) - \int_0^{t_{n-1}} b(t_{n-1},s;\mathcal{R}_wU(s),\phi)ds\Big] \\
& -l_n(t)\Big[a(\mathcal{R}_w^n U^n,\phi) - \int_0^{t_n} b(t_n,s;\mathcal{R}_wU(s),\phi)ds\Big] + \langle f(t),\phi\rangle + \langle \mathcal{R}_w^n\hat{F}(t),\phi\rangle \\
& -\tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1},\phi\rangle + \langle \hat{\omega}(t) - \hat{\omega}_I(t),\nabla\phi\rangle.
\end{aligned}$$

By the definition of the Ritz-Volterra reconstruction (3.12), it follows that

$$\begin{aligned}
&\langle \hat{\rho}_t(t), \phi\rangle + a(\rho(t), \phi) - \int_0^t b(t, s; \rho(s), \phi)ds \\
= \;& -l_{n-1}(t)\Big[\langle \mathcal{A}^{n-1}U^{n-1}, \phi\rangle - \langle \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds, \phi\rangle\Big] \\
& -l_n(t)\Big[\langle \mathcal{A}^n U^n, \phi\rangle - \langle \int_0^{t_n} \mathcal{B}^n(s)U(s)ds, \phi\rangle\Big] + \langle f(t), \phi\rangle + \langle \mathcal{R}_w^n \hat{F}(t), \phi\rangle \\
& -\tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle + \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla\phi\rangle.
\end{aligned}$$

Using (4.13) and (4.26)-(4.28), we get

$$\begin{aligned}
&\langle \hat{\rho}_t(t), \phi\rangle + a(\rho(t), \phi) - \int_0^t b(t, s; \rho(s), \phi)ds \\
= \;& \langle (P_0^n - I)\Big\{l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U)\Big\}, \phi\rangle + \langle \hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t), \phi\rangle \\
& +\langle \hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t), \phi\rangle + \langle \int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2}U(s)ds - \sigma^n(\mathcal{B}^{n-1/2}U), \phi\rangle \\
& +(t - t_{n-1/2})\langle \mathcal{Y}_n, \phi\rangle + \langle f(t), \phi\rangle + \langle \hat{\omega}(t) - \hat{\omega}_I(t), \nabla\phi\rangle \\
& +\langle \mathcal{R}_w^n \hat{F}(t) - \Theta(t), \phi\rangle - \tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle. \qquad (4.50)
\end{aligned}$$

For the last two terms on the right hand side of (4.50), an application of (4.9) yields

$$\begin{aligned}
&\langle \mathcal{R}_w^n \hat{F}(t) - \Theta(t), \phi\rangle - \tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle \\
= \;& \langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \phi\rangle + \langle \hat{F}(t) - \Theta(t), \phi\rangle \\
& +\langle (\mathcal{R}_w^n - I)F^{n-1/2}, \phi\rangle - \tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle \\
= \;& \langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \phi\rangle + \langle \hat{F}(t) - \Theta(t), \phi\rangle \\
& -\tau_n^{-1}\langle (\mathcal{R}_w^n - I)(U^n - P_0^n U^{n-1}), \phi\rangle - \tau_n^{-1}\langle \mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle \\
= \;& \langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \phi\rangle + \langle \hat{F}(t) - \Theta(t), \phi\rangle \\
& -\tau_n^{-1}\langle \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1}U^{n-1}, \phi\rangle + \tau_n^{-1}\langle U^n - P_0^n U^{n-1}, \phi\rangle \\
= \;& \langle (\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \phi\rangle + \langle \hat{F}(t) - \Theta(t), \phi\rangle \\
& -\tau_n^{-1}\langle (\mathcal{R}_w^n - I)U^n - (\mathcal{R}_w^{n-1} - I)U^{n-1}, \phi\rangle - \tau_n^{-1}\langle P_0^n U^{n-1} - U^{n-1}, \phi\rangle. \qquad (4.51)
\end{aligned}$$

Thus, the error equation (4.49) for $\hat{\rho}$ now follows using (4.50) and (4.51). □

The next lemma presents a clear picture of the terms to be estimated in order to obtain a bound for $\hat{\rho}$.

Lemma 4.3.4. *The following estimate holds for $\hat{\rho}(t)$:*

$$\max_{t\in[0,t_m]} \|\hat{\rho}(t)\|^2 + \frac{\alpha}{2}\int_0^{t_m} \Big[2\|\rho(t)\|_1^2 + \|\hat{\rho}(t)\|_1^2\Big]dt \le \|\hat{\rho}(0)\|^2 + C_7\mathcal{I}_m,$$

where

$$\begin{aligned}\mathcal{I}_m &:= \sum_{n=1}^{m}\Big(\mathcal{I}_n^{T,1} + \mathcal{I}_n^{T,2} + \mathcal{I}_n^{M,3} + \mathcal{I}_n^{S,4} + \mathcal{I}_n^{S,5} + \mathcal{I}_n^{D,6}\Big)\\ &:= \mathcal{I}_m^1 + \mathcal{I}_m^2 + \mathcal{I}_m^3 + \mathcal{I}_m^4 + \mathcal{I}_m^5 + \mathcal{I}_m^6\end{aligned}$$

with

$$\mathcal{I}_n^{T,1} := \int_{t_{n-1}}^{t_n} \|\hat{\rho}(t) - \rho(t)\|_1^2 dt, \tag{4.52}$$

$$\begin{aligned}\mathcal{I}_n^{T,2} &:= \int_{t_{n-1}}^{t_n} \Bigg[\Big|\langle\int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2}U(s)ds - \sigma^n(\mathcal{B}^{n-1/2}U), \hat{\rho}(t)\rangle\Big|\\ &\quad + \Big|\langle\hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t), \hat{\rho}(t)\rangle\Big| + \Big|\langle\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t), \hat{\rho}(t)\rangle\Big|\\ &\quad + |\langle\hat{\omega}(t) - \hat{\omega}_I(t), \nabla\hat{\rho}(t)\rangle|\Bigg]dt,\end{aligned} \tag{4.53}$$

$$\begin{aligned}\mathcal{I}_n^{M,3} &:= \int_{t_{n-1}}^{t_n} \Bigg[\Big|\Big\langle(P_0^n - I)\Big\{l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U)\\ &\quad -\tau_n^{-1}U^{n-1}\Big\}, \hat{\rho}(t)\Big\rangle\Big|\Bigg]dt,\end{aligned} \tag{4.54}$$

$$\mathcal{I}_n^{S,4} := \int_{t_{n-1}}^{t_n} \Big|\Big\langle(\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \hat{\rho}(t)\Big\rangle\Big|dt, \tag{4.55}$$

$$\mathcal{I}_n^{S,5} := \tau_n^{-1}\int_{t_{n-1}}^{t_n} \Big|\Big\langle(\mathcal{R}_w^n - I)U^n - (\mathcal{R}_w^{n-1} - I)U^{n-1}, \hat{\rho}(t)\Big\rangle\Big|dt \tag{4.56}$$

and

$$\mathcal{I}_n^{D,6} := \int_{t_{n-1}}^{t_n} \Big|\Big\langle\hat{F}(t) - \Theta(t) + f(t), \hat{\rho}(t)\Big\rangle\Big|dt. \tag{4.57}$$

Proof. Set $\phi = \hat{\rho}(t)$ in (4.49) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{\rho}(t)\|^2 + a(\rho(t), \hat{\rho}(t)) = \int_0^t b(t, s; \rho(s), \hat{\rho}(t))ds + \langle G, \hat{\rho}(t)\rangle.$$

We integrate from t_{n-1} to t_n and use the fact

$$a(\rho(t),\hat{\rho}(t)) = \frac{1}{2}a(\rho(t),\rho(t)) + \frac{1}{2}a(\hat{\rho}(t),\hat{\rho}(t)) - \frac{1}{2}a(\hat{\rho}(t)-\rho(t),\hat{\rho}(t)-\rho(t))$$

to obtain

$$\begin{aligned}\frac{1}{2}\Big\{\|\hat{\rho}(t_n)\|^2 &- \|\hat{\rho}(t_{n-1})\|^2\Big\} + \frac{1}{2}\int_{t_{n-1}}^{t_n} a(\rho(t),\rho(t))dt + \frac{1}{2}\int_{t_{n-1}}^{t_n} a(\hat{\rho}(t),\hat{\rho}(t))dt \\ &= \frac{1}{2}\int_{t_{n-1}}^{t_n} a(\hat{\rho}(t)-\rho(t),\hat{\rho}(t)-\rho(t))dt + \int_{t_{n-1}}^{t_n}\int_0^t b(t,s;\rho(s),\hat{\rho}(t))dsdt \\ &\quad + \int_{t_{n-1}}^{t_n} \langle G,\hat{\rho}(t)\rangle dt.\end{aligned}$$

Using the coercivity of $a(\cdot,\cdot)$, and continuity of $a(\cdot,\cdot)$, $b(t,s;\cdot,\cdot)$ together with a standard kickback argument, we obtain

$$\begin{aligned}&\frac{1}{2}\Big\{\|\hat{\rho}(t_n)\|^2 - \|\hat{\rho}(t_{n-1})\|^2\Big\} + \frac{\alpha}{2}\int_{t_{n-1}}^{t_n}\|\rho(t)\|_1^2 dt + \frac{\alpha}{4}\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)\|_1^2 dt \\ \leq\ & \frac{\beta}{2}\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)-\rho(t)\|_1^2 dt + C_5(T)\int_{t_{n-1}}^{t_n}\int_0^t\|\rho(s)\|_1^2 dsdt + \int_{t_{n-1}}^{t_n}|\langle G_1,\hat{\rho}(t)\rangle|dt,\end{aligned}$$

where we have used the fact that

$$\int_{t_{n-1}}^{t_n}(t - t_{n-1/2})dt = 0.$$

Summing from $n = 1 : m$ with an application of Gronwall's lemma gives the desired result with $C_7 = \max\{\beta C_6(T), 2C_6(T)\}$, where $C_6(T)$ is a Gronwall's constant. □

Now, we proceed to estimate the terms appeared in Lemma 4.3.4. We start with providing *a posteriori* error bounds on the time discretization error.

Lemma 4.3.5 (Time error estimators)**.** *The following a posteriori bounds hold for the time discretization error terms* $\mathcal{I}_m^1$ *and* $\mathcal{I}_m^2$*:*

$$\mathcal{I}_m^1 \leq \sum_{n=1}^m \tau_n \Lambda_{CN,n}^2 \tag{4.58}$$

and

$$\mathcal{I}_m^2 \leq \sum_{n=1}^m \tau_n \max_{[0,t_m]}\|\hat{\rho}(t)\|\eta_{CN,n} + \sum_{n=1}^m \tau_n^{1/2}\xi_{CN,n}\Big(\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)\|_1^2 dt\Big)^{1/2}, \tag{4.59}$$

where $\Lambda_{CN,n}$*,* $\eta_{CN,n}$ *and* $\xi_{CN,n}$ *are given by (4.37), (4.41) and (4.45), respectively.*

Proof. We know that

$$\hat{\rho}(t) - \rho(t) = -(\hat{U}(t) - \mathcal{R}_w U(t)). \tag{4.60}$$

Thus, to estimate the term $I_n^{T,1}$, we have to first estimate $\hat{U}(t) - \mathcal{R}_w U(t)$. Using (4.22), (4.18) and (4.20), we have

$$\begin{aligned}
\hat{U}_t(t) - (\mathcal{R}_w U)_t(t) \quad &:= \frac{\mathcal{R}_w^n P_0^n U^{n-1} - \mathcal{R}_w^{n-1} U^{n-1}}{\tau_n} - \mathcal{R}_w^n \hat{F}(t) - \frac{\mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1}}{\tau_n} \\
&= \mathcal{R}_w^n \Big[\frac{P_0^n U^{n-1} - U^n}{\tau_n} \Big] - \mathcal{R}_w^n \hat{F}(t) \\
&= \mathcal{R}_w^n \Big[F^{n-1/2} - \hat{F}(t) \Big].
\end{aligned}$$

We integrate from t_{n-1} to t and use the fact that $\hat{U}(t)$ interpolates with $\mathcal{R}_w U(t)$ at t_{n-1} to obtain

$$\hat{U}(t) - \mathcal{R}_w U(t) \quad = \quad -\mathcal{R}_w^n \int_{t_{n-1}}^{t} \Big\{ \hat{F}(s) - F^{n-1/2} \Big\} ds. \tag{4.61}$$

Using (4.15), (4.13) and the identity $l_{n-1}(t) + l_n(t) = 1$, $t \in I_n$, we have

$$\begin{aligned}
\hat{F}(t) - F^{n-1/2} \quad &= \Theta(t) - \Theta(t_{n-1/2}) - P_0^n[\varphi(t) - \varphi(t_{n-1/2})] \\
&= l_{n-1}(t) P_0^n \mathcal{A}^{n-1} U^{n-1} + l_n(t) \mathcal{A}^n U^n - \frac{1}{2} P_0^n \mathcal{A}^{n-1} U^{n-1} \\
&\quad - \frac{1}{2} \mathcal{A}^n U^n - P_0^n[\varphi(t) - \varphi(t_{n-1/2})] \\
&= \frac{1}{2} \Big[l_n(t) - l_{n-1}(t) \Big] \Big[\mathcal{A}^n U^n - P_0^n \mathcal{A}^{n-1} U^{n-1} \Big] \\
&\quad - P_0^n[\varphi(t) - \varphi(t_{n-1/2})], \\
&= 2(t - t_{n-1/2}) \mathcal{W}_n,
\end{aligned} \tag{4.62}$$

where $\mathcal{W}_n$ is given by (4.38).
Substituting (4.62) in (4.61), we get

$$\begin{aligned}
\hat{U}(t) - \mathcal{R}_w U(t) \quad &= \quad -2 \mathcal{R}_w^n \mathcal{W}_n \int_{t_{n-1}}^{t} (s - t_{n-1/2}) ds \\
&= \quad (t_n - t)(t - t_{n-1}) \mathcal{R}_w^n \mathcal{W}_n.
\end{aligned} \tag{4.63}$$

Using the coercivity and the continuity of the bilinear form $a(\cdot,\cdot)$, it follows that

$$\begin{aligned}
\alpha\|\hat{\rho}(t)-\rho(t)\|_1^2 &\leq a(\hat{\rho}(t)-\rho(t),\hat{\rho}(t)-\rho(t)) \\
&= (t_n-t)(t-t_{n-1})a(\mathcal{R}_w^n\mathcal{W}_n,\hat{\rho}(t)-\rho(t)) \\
&\leq (t_n-t)(t-t_{n-1})\beta\|\mathcal{R}_w^n\mathcal{W}_n\|_1\|\hat{\rho}(t)-\rho(t)\|_1,
\end{aligned}$$

where we have used (4.60) and (4.63).

Thus, we deduce that

$$\|\hat{\rho}(t)-\rho(t)\|_1 \leq \frac{\beta(t_n-t)(t-t_{n-1})}{\alpha}\Big[\alpha_{CN,n}(\mathcal{W}_n)+\|\mathcal{W}_n\|_1\Big]. \tag{4.64}$$

Finally, with an aid of (4.64), we obtain

$$\mathcal{I}_n^{T,1} := \int_{t_{n-1}}^{t_n}\|\hat{\rho}(s)-\rho(s)\|_1^2 ds \leq \frac{\beta^2\tau_n^5}{30\alpha^2}\Big[\alpha_{CN,n}(\mathcal{W}_n)+\|\mathcal{W}_n\|_1\Big]^2. \tag{4.65}$$

Thus, the first inequality (4.58) follows by taking summation over n and using (4.37).

Next, to prove the inequality (4.59), we first note that

$$\begin{aligned}
\mathcal{I}_n^{T,2} &:= \int_{t_{n-1}}^{t_n}\Bigg[\Big|\langle\int_0^{t_{n-1/2}}\mathcal{B}^{n-1/2}U(s)ds-\sigma^n(\mathcal{B}^{n-1/2}U),\hat{\rho}(t)\rangle\Big| + \Big|\langle\hat{\mathcal{U}}_{I,1}(t)-\hat{\mathcal{U}}(t),\hat{\rho}(t)\rangle\Big| \\
&\quad + \Big|\langle\hat{\mathcal{U}}(t)-\hat{\mathcal{U}}_{I,2}(t),\hat{\rho}(t)\rangle\Big| + |\langle\hat{\omega}(t)-\hat{\omega}_I(t),\nabla\hat{\rho}(t)\rangle|\Bigg]dt, \\
&:= \int_{t_{n-1}}^{t_n}\Big[|\mathcal{J}_1|+|\mathcal{J}_2|+|\mathcal{J}_3|+|\mathcal{J}_4|\Big]dt.
\end{aligned}$$

We start with estimating the term $\mathcal{J}_1$. A standard Trapezoidal rule argument for a sufficiently smooth function $g(s)$ yields

$$\int_a^b g(s)ds - \frac{(b-a)}{2}(g(a)+g(b)) = \frac{1}{2}\int_a^b (s-a)(s-b)g''(s)ds.$$

If we define

$$\psi_{2j}(s) := \begin{cases} (s-t_{j-1})(s-t_j) & \text{for } s\in[t_{j-1},t_j] \text{ and } 1\leq j\leq n-1, \\ (s-t_{j-1})(s-t_{j-1/2}) & \text{for } s\in[t_{j-1},t_{j-1/2}] \text{ and } j=n, \end{cases}$$

then

$$\int_{t_{j-1}}^{t_j} g(s)ds - \frac{\tau_j}{2}[g(t_j)+g(t_{j-1})] = \frac{1}{2}\int_{t_{j-1}}^{t_j}\psi_{2j}(s)g''(s)ds \tag{4.66}$$

and

$$\int_{t_{n-1}}^{t_{n-1/2}} g(s)ds - \frac{\tau_n}{4}[g(t_{n-1}) + g(t_{n-1/2})] = \frac{1}{2}\int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s)g''(s)ds. \tag{4.67}$$

Using (4.21), (4.66) and (4.67), we obtain

$$\begin{aligned}
&\int_0^{t_{n-1/2}} \langle \mathcal{B}^{n-1/2}(s)U(s)ds, \hat{\rho}(t)\rangle - \langle \sigma^n(\mathcal{B}^{n-1/2}U), \hat{\rho}(t)\rangle \\
= \;& \langle \int_0^{t_{n-1/2}} \mathcal{B}^{n-1/2}(s)U(s)ds - \sum_{j=0}^{n-2}\frac{\tau_{j+1}}{2}\Big[\mathcal{B}^{n-1/2}(t_j)U^j + \mathcal{B}^{n-1/2}(t_{j+1})U^{j+1}\Big] \\
& -\frac{\tau_n}{4}\Big[\mathcal{B}^{n-1/2}(t_{n-1})U^{n-1} + \mathcal{B}^{n-1/2}(t_{n-1/2})U^{n-1/2}\Big], \hat{\rho}(t)\rangle \\
= \;& \frac{1}{2}\langle \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} \psi_{2j}(s)\frac{d^2}{ds^2}\big\{\mathcal{B}^{n-1/2}(s)U(s)\big\}ds \\
& + \int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s)\frac{d^2}{ds^2}\big\{\mathcal{B}^{n-1/2}(s)U(s)\big\}ds, \hat{\rho}(t)\rangle \\
= \;& \frac{1}{2}\langle \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} \psi_{2j}(s)\left\{\frac{d^2(\mathcal{B}^{n-1/2}(s))}{ds^2}U(s) + \frac{d(\mathcal{B}^{n-1/2}(s))}{ds}\frac{d(U(s))}{ds}\right\}ds \\
& + \int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s)\left\{\frac{d^2(\mathcal{B}^{n-1/2}(s))}{ds^2}U(s) + \frac{d(\mathcal{B}^{n-1/2}(s))}{ds}\frac{d(U(s))}{ds}\right\}ds, \hat{\rho}(t)\rangle \\
\leq \;& \frac{1}{2}\left(\sum_{j=1}^{n-1}\tau_j^2\left[\frac{\gamma''}{2}\tau_j\Big(\|\Delta^n U^{j-1}\| + \|\Delta^n U^j\|\Big) + \gamma'\tau_j\|\Delta^n\partial U^j\|\right]\right. \\
& \left. +\tau_n^2\left[\frac{\gamma''}{2}\tau_j\Big(\|\Delta^n U^{n-1}\| + \|\Delta^n U^n\|\Big) + \gamma'\tau_j\|\Delta^n\partial U^n\|\right]\right)\|\hat{\rho}(t)\| \\
\leq \;& \bar{\gamma}\mathcal{Q}_{CN,n}\|\hat{\rho}(t)\|,
\end{aligned} \tag{4.68}$$

where $\mathcal{Q}_{CN,n}$ is given by (4.42) and

$$\bar{\gamma} = \max\left\{\frac{\gamma''}{2}, \frac{\gamma'}{2}\right\}.$$

Thus, in view of (4.68) we have the following bound on $\mathcal{J}_1$

$$|\mathcal{J}_1| \;\leq\; \bar{\gamma}\mathcal{Q}_{CN,n}\|\hat{\rho}(t)\|.$$

Moreover, an application of Cauchy-Schwarz inequality gives

$$|\mathcal{J}_2| \leq \|\hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t)\|\|\hat{\rho}(t)\|,$$

$$|\mathcal{J}_3| \leq \|\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t)\|\|\hat{\rho}(t)\|$$

and

$$|\mathcal{J}_4| \leq \|\hat{\omega}(t) - \hat{\omega}_I(t)\|\|\nabla\hat{\rho}(t)\|.$$

Combine the bounds on $|\mathcal{J}_1|$, $|\mathcal{J}_2|$, $|\mathcal{J}_3|$ and $|\mathcal{J}_4|$ to obtain

$$\begin{aligned}\mathcal{I}_n^{T,2} \quad \leq \quad & \int_{t_{n-1}}^{t_n} \left[\bar{\gamma}\mathcal{Q}_{CN,n} + \|\hat{\mathcal{U}}_{I,1}(t) - \hat{\mathcal{U}}(t)\| + \|\hat{\mathcal{U}}(t) - \hat{\mathcal{U}}_{I,2}(t)\|\right]\|\hat{\rho}(t)\|dt \\ & + \left(\int_{t_{n-1}}^{t_n} \|\hat{\omega}(t) - \hat{\omega}_I(t)\|^2 dt\right)^{1/2} \left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\right)^{1/2}.\end{aligned}$$

Thus, we obtain

$$\mathcal{I}_n^{T,2} \leq \tau_n \max_{[0,t_m]} \|\hat{\rho}(t)\|\eta_{CN,n} + \tau_n^{1/2}\xi_{CN,n}\left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\right)^{1/2},$$

where $\eta_{CN,n}$, $\xi_{CN,n}$ are given by (4.41), (4.45), respectively and

$$C_8 := \max\{\bar{\gamma}, 1\}.$$

The desired estimate now follows by taking summation over n. □

The next lemma gives information on the *a posteriori* contributions due to mesh change.

Lemma 4.3.6 (Mesh change estimate)**.** *We have the following bound on the mesh change error term $\mathcal{I}_m^3$:*

$$\mathcal{I}_m^3 \leq \sum_{n=1}^{m} \tau_n^{1/2}\mu_{CN,n}\left(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\right)^{1/2}, \tag{4.69}$$

where $\mu_{CN,n}$ is given by (4.46).

Proof. The orthogonality property of P_0^n now leads to

$$\begin{aligned}\mathcal{I}_n^{M,3} \quad := \quad & \int_{t_{n-1}}^{t_n} \left[\left\langle (P_0^n - I)\left\{l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U) - \tau_n^{-1}U^{n-1}\right\}, \hat{\rho}(t)\right\rangle\right] dt \\ = \quad & \int_{t_{n-1}}^{t_n} \left[\left\langle (P_0^n - I)\left\{l_{n-1}(t)\mathcal{A}^{n-1}U^{n-1} - \sigma^n(\mathcal{B}^{n-1/2}U)\right.\right.\right. \\ & \left.\left.\left. -\tau_n^{-1}U^{n-1}\right\}, \hat{\rho}(t) - \Pi^n\hat{\rho}(t)\right\rangle\right] dt.\end{aligned}$$

An application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}\mathcal{I}_n^{M,3} &\leq C_{1,1}h_n\Bigg[\Big\{\int_{t_{n-1}}^{t_n} l_{n-1}^2(t)dt\Big\}^{1/2}\|(P_0^n - I)\mathcal{A}^{n-1}U^{n-1}\| \\ &\quad +\|(P_0^n - I)\sigma^n(\mathcal{B}^{n-1/2}U)\| + \tau_n^{-1/2}\|(P_0^n - I)U^{n-1}\|\Bigg]\Big\{\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\Big\}^{1/2} \\ &\leq \tau_n^{1/2}\mu_{CN,n}\Big\{\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\Big\}^{1/2},\end{aligned}$$

where $\mu_{CN,n}$ is given by (4.46). Taking summation over n completes the proof. ☐

The next lemma captures contributions due to the spatial discretizations.

Lemma 4.3.7 (Spatial error estimates). *The following a posteriori error bound holds on the spatial discretization error term $\mathcal{I}_m^4$:*

$$\mathcal{I}_m^4 \leq \sum_{n=1}^{m} \tau_n \max_{[0,t_m]} \|\hat{\rho}(t)\|\zeta_{CN,n,2}, \tag{4.70}$$

where $\zeta_{CN,n,2}$ is given by (4.44). Moreover, the following error bound holds for $\mathcal{I}_m^5$ corresponds to the spatial discretization error due to mesh change:

$$\mathcal{I}_m^5 \leq \max_{t\in[0,t_m]} \|\hat{\rho}(t)\| \sum_{n=1}^{m} \tau_n\zeta_{CN,n,1}, \tag{4.71}$$

where $\zeta_{CN,n,1}$ is given by (4.43).

Proof. Using (4.62) and (4.32), we have

$$\mathcal{I}_n^{S,4} := \int_{t_{n-1}}^{t_n} \Big|\Big\langle(\mathcal{R}_w^n - I)(\hat{F}(t) - F^{n-1/2}), \hat{\rho}(t)\Big\rangle\Big| dt \leq \frac{\tau_n^2}{2} \max_{[0,t_m]} \|\hat{\rho}(t)\|\beta_{CN,n}(\mathcal{W}_n).$$

Hence, we obtain

$$\mathcal{I}_n^{S,4} \leq \tau_n \max_{[0,t_m]} \|\hat{\rho}(t)\|\zeta_{CN,n,2}, \tag{4.72}$$

where $\zeta_{CN,n,2}$ is given by (4.44) and the first estimate (4.70) follows by taking the summation over n. Next, to estimate $\mathcal{I}_n^{S,5}$ as given in (4.56), we exploit the orthogonality property of the Ritz-Volterra reconstructions and use the standard duality technique.

For $t \in (0,T)$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the following elliptic problem in the weak form

$$a(\chi, \psi(t)) = \langle \chi, \hat{\rho}(t) \rangle, \quad \forall \chi \in H_0^1(\Omega) \tag{4.73}$$

satisfying the following regularity estimate:

$$\|\psi(t)\|_2 \le C_\Omega \|\hat{\rho}(t)\|, \tag{4.74}$$

where the constant C_Ω depends on the domain Ω. Setting $\chi = \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}$ in (4.73) and using (3.12), (4.10) and, (4.12), we obtain

$$\begin{aligned}
& \langle \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}, \hat{\rho}(t) \rangle \\
= \;& a(\mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}, \psi(t) - \Pi^n \psi(t)) \\
& - \int_0^{t_n} b(t_n, s; (\mathcal{R}_w U - U)(s), \psi(t) - \Pi^n \psi(t)) ds \\
& + \int_0^{t_{n-1}} b(t_{n-1}, s; (\mathcal{R}_w U - U)(s), \psi(t) - \Pi^n \psi(t)) ds \\
& + \int_0^{t_n} b(t_n, s; (\mathcal{R}_w U - U)(s), \psi(t)) ds - \int_0^{t_{n-1}} b(t_{n-1}, s; (\mathcal{R}_w U - U)(s), \psi(t)) ds \\
= \;& \langle \mathcal{A}^n U^n - \int_0^{t_n} \mathcal{B}^n(s) U(s) ds - \mathcal{A}_{el} U^n + \int_0^{t_n} \mathcal{B}_{el}(t_n, s) U(s) ds, \psi(t) - \Pi^n \psi(t) \rangle \\
& - \langle \mathcal{A}^{n-1} U^{n-1} - \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s) U(s) ds - \mathcal{A}_{el} U^{n-1} \\
& + \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1}, s) U(s) ds, \psi(t) - \Pi^n \psi(t) \rangle \\
& + \langle \int_0^{t_n} J_2[U(s)] ds - J_1[U^n] - \int_0^{t_{n-1}} J_2[U(s)] ds + J_1[U^{n-1}], \psi(t) - \Pi^n \psi(t) \rangle_{\Sigma_n} \\
& + \int_0^{t_n} b(t_n, s; (\mathcal{R}_w U - U)(s), \psi(t)) ds - \int_0^{t_{n-1}} b(t_{n-1}, s; (\mathcal{R}_w U - U)(s), \psi(t)) ds.
\end{aligned}$$

We now use (4.30) together with $\mathfrak{J}^n[U] - \mathfrak{J}^{n-1}[U] = \tau_n \partial \mathfrak{J}^n[U]$ to obtain

$$|\langle \mathcal{R}_w^n U^n - \mathcal{R}_w^{n-1} U^{n-1} - U^n + U^{n-1}, \hat{\rho}(t) \rangle|$$

$$\begin{aligned}
&\leq \|\mathcal{A}^n U^n - \int_0^{t_n} \mathcal{B}^n(s)U(s)ds - \mathcal{A}_{el}U^n + \int_0^{t_n} \mathcal{B}_{el}(t_n,s)U(s)ds \\
&\quad -\mathcal{A}^{n-1}U^{n-1} + \int_0^{t_{n-1}} \mathcal{B}^{n-1}(s)U(s)ds + \mathcal{A}_{el}U^{n-1} - \int_0^{t_{n-1}} \mathcal{B}_{el}(t_{n-1},s)U(s)ds\| \\
&\quad \|\psi(t) - \Pi^n\psi(t)\| + \tau_n\|\partial\mathfrak{J}^n[U]\|_{\Sigma_n}\|\psi(t) - \Pi^n\psi(t)\|_{\Sigma_n} \\
&\quad +|\int_0^{t_n} b(t_n,s;(\mathcal{R}_wU - U)(s),\psi(t))ds \\
&\quad - \int_0^{t_{n-1}} b(t_{n-1},s;(\mathcal{R}_wU - U)(s),\psi(t))ds|. \qquad (4.75)
\end{aligned}$$

To handle the last term above, we use the fact

$$b(t_n,s;(\mathcal{R}_wU - U)(s),\psi(t)) := \langle(\mathcal{R}_wU - U)(s), \mathcal{B}^*(t_n,s)\psi(t)\rangle, \qquad (4.76)$$

where $\mathcal{B}^*(t_n,s)$ is the formal adjoint of the operator $\mathcal{B}(t_n,s)$. Now, we apply Cauchy-Schwarz inequality together with $\|\mathcal{B}^*(t_n,s)\psi(t)\| \leq C_{\mathcal{B}_1^*}\|\psi(t)\|_2$ to obtain

$$\begin{aligned}
|\int_0^{t_n} b(t_n,s;(\mathcal{R}_wU &- U)(s),\psi(t))ds - \int_0^{t_{n-1}} b(t_{n-1},s;(\mathcal{R}_wU - U)(s),\psi(t))ds| \\
&\leq |\int_0^{t_n} \langle(\mathcal{R}_wU - U)(s), \mathcal{B}^*(t_n,s)\psi(t)\rangle ds \\
&\quad - \int_0^{t_{n-1}} \langle(\mathcal{R}_wU - U)(s), \mathcal{B}^*(t_{n-1},s)\psi(t)\rangle ds| \\
&\leq \|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} (\mathcal{R}_wU - U)(s)\|\|\mathcal{B}^*(t_n,s)\psi(t)\|ds \\
&\quad +\|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j} (\mathcal{R}_wU - U)(s)\|\|\mathcal{B}^*(t_{n-1},s)\psi(t)\|ds \\
&\leq 2C_{\mathcal{B}_1^*}\Big[\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\Big\{l_{j-1}(s)(\mathcal{R}_w^{j-1}U^{j-1} - U^{j-1}) \\
&\quad +l_j(s)(\mathcal{R}_w^jU^j - U^j)\Big\}ds\|\Big]\|\psi(t)\|_2 \\
&\leq C_{\mathcal{B}_1^*}\Big[\hat{\tau}_n\sum_{j=1}^{n}\beta_{CN,j}[U] + \hat{\tau}_{n-1}\sum_{j=0}^{n-1}\beta_{CN,j}[U]\Big]\|\psi(t)\|_2 \\
&\leq 2C_{\mathcal{B}_1^*}\hat{\tau}_n\Big[\sum_{j=0}^{n}\beta_{CN,j}[U]\Big]\|\psi(t)\|_2. \qquad (4.77)
\end{aligned}$$

Using (4.77) in (4.75) and applying Proposition 3.2.1 with $C_9 = \max\Big(2C_{\mathcal{B}_1^*}, 1\Big)$, we obtain

$$
\begin{aligned}
|\langle \mathcal{R}_w^n U^n &- \mathcal{R}_w^{n-1}U^{n-1} - U^n + U^{n-1}, \hat{\rho}(t)\rangle| \\
&\leq C_9\|\psi\|_2\Bigg(\hat{\tau}_n\Bigg[h_n^2\|\tau_n^{-1}(\mathcal{A}^nU^n - \int_0^{t_n}\mathcal{B}^n(s)U(s)ds - \mathcal{A}_{el}U^n \\
&\quad + \int_0^{t_n}\mathcal{B}_{el}(t_n,s)U(s)ds - \mathcal{A}^{n-1}U^{n-1} + \int_0^{t_{n-1}}\mathcal{B}^{n-1}(s)U(s)ds \\
&\quad + \mathcal{A}_{el}U^{n-1} - \int_0^{t_{n-1}}\mathcal{B}_{el}(t_{n-1},s)U(s)ds)\| + h_n^{3/2}\|\partial\mathfrak{J}^n[U]\|_{\Sigma_n} \\
&\quad + \sum_{j=0}^{n}\beta_{CN,j}[U]\Bigg]\Bigg). \qquad (4.78)
\end{aligned}
$$

Combining (4.56) and (4.78), we arrive at

$$
\begin{aligned}
\mathcal{I}_n^{S,5} &\leq C_9\tau_n^{-1}\int_{t_{n-1}}^{t_n}\|\psi(t)\|_2dt\,\Bigg(\hat{\tau}_n\Bigg[h_n^2\|\tau_n^{-1}(\mathcal{A}^nU^n - \int_0^{t_n}\mathcal{B}^n(s)U(s)ds - \mathcal{A}_{el}U^n \\
&\quad + \int_0^{t_n}\mathcal{B}_{el}(t_n,s)U(s)ds - \mathcal{A}^{n-1}U^{n-1} + \int_0^{t_{n-1}}\mathcal{B}^{n-1}(s)U(s)ds \\
&\quad + \mathcal{A}_{el}U^{n-1} - \int_0^{t_{n-1}}\mathcal{B}_{el}(t_{n-1},s)U(s)ds)\| + h_n^{3/2}\|\partial\mathfrak{J}^n[U]\|_{\Sigma_n} + \sum_{j=0}^{n}\beta_{CN,j}[U]\Bigg]\Bigg) \\
&\leq \max_{t\in I_n}\|\hat{\rho}(t)\|\tau_n\zeta_{CN,n,1},
\end{aligned}
$$

where we have used (4.43) and the regularity result (4.74). Summing from $n = 1 : m$, the desired result is obtained. □

The data approximation error is estimated in the following lemma.

Lemma 4.3.8 (Data approximation error estimate)**.** *The following bound holds on the data approximation error term* $\mathcal{I}_m^6$*:*

$$
\mathcal{I}_m^6 \leq \sum_{n=1}^{m}\left[\tau_n\lambda_{CN,n,1}\max_{t\in[0,t_m]}\|\hat{\rho(t)}\| + \tau_n^{1/2}\lambda_{CN,n,2}\left(\int_{t_{n-1}}^{t_n}\|\hat{\rho}(t)\|_1^2dt\right)^{1/2}\right], \qquad (4.79)
$$

where $\lambda_{CN,n,1}$ *and* $\lambda_{CN,n,2}$ *are given by (4.47) and (4.48).*

Proof. Using (4.13) and (4.15), We have

$$
\begin{aligned}
\mathcal{I}_n^{D,6} &:= \int_{t_{n-1}}^{t_n} \Big|\Big\langle \hat{F}(t) - \Theta(t) + f(t), \hat{\rho}(t)\Big\rangle\Big| dt \\
&= \int_{t_{n-1}}^{t_n} \Big|\Big\langle f(t) - P_0^n\varphi(t), \hat{\rho}(t)\Big\rangle\Big| dt \\
&\leq \int_{t_{n-1}}^{t_n} \Big|\Big\langle f(t) - \varphi(t), \hat{\rho}(t)\Big\rangle\Big| dt + \int_{t_{n-1}}^{t_n} \Big|\Big\langle (I - P_0^n)\varphi(t), \hat{\rho}(t)\Big\rangle\Big| dt \\
&:= \mathfrak{J}_1 + \mathfrak{J}_2.
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we obtain

$$
\mathfrak{J}_1 \leq \max_{t\in[0,t_m]} \|\hat{\rho}(t)\| \int_{t_{n-1}}^{t_n} \|f(t) - \varphi(t)\| dt.
$$

For $\mathfrak{J}_2$, we use orthogonality property of P_0^n to have

$$
\begin{aligned}
\mathfrak{J}_2 &= \int_{t_{n-1}}^{t_n} \Big|\Big\langle (I - P_0^n)\varphi(t), \hat{\rho}(t)\Big\rangle\Big| dt \\
&= \int_{t_{n-1}}^{t_n} \Big|\Big\langle (I - P_0^n)\varphi(t), \hat{\rho}(t) - \Pi^n\hat{\rho}(t)\Big\rangle\Big| dt \\
&\leq C_{1,1}h_n \int_{t_{n-1}}^{t_n} \|(I - P_0^n)\varphi(t)\| \|\hat{\rho}(t)\|_1 dt \\
&\leq 2C_{1,1}h_n\tau_n^{1/2} \max\Big\{\|(I - P_0^n)f^{n-1}\|, \|(I - P_0^n)f^{n-1/2}\|\Big\}\bigg(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\bigg)^{1/2}.
\end{aligned}
$$

Therefore,

$$
\mathcal{I}_n^{D,6} \leq \tau_n\lambda_{CN,n,1} \max_{t\in[0,t_m]} \|\hat{\rho(t)}\| + \tau_n^{1/2}\lambda_{CN,n,2}\bigg(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\bigg)^{1/2},
$$

where $\lambda_{CN,n,1}$ and $\lambda_{CN,n,2}$ are given by (4.47) and (4.48), respectively. Now, taking summation over n, we obtain the desired result. □

Proof of Lemma 4.3.2. Application of Lemmas 4.3.5-4.3.8 in Lemma 4.3.4 yields

$$
\begin{aligned}
&\max_{t\in[0,t_m]} \|\hat{\rho}(t)\|^2 + \frac{\alpha}{2}\int_0^{t_m} \Big[2\|\rho(t)\|_1^2 + \|\hat{\rho}(t)\|_1^2\Big] dt \\
&\leq \|\hat{\rho}(0)\|^2 + C_7\bigg[\sum_{n=1}^m \tau_n\Lambda_{CN,n}^2 + \max_{t\in[0,t_m]} \|\hat{\rho}(t)\| \sum_{n=1}^m \tau_n\Big[\eta_{CN,n} + \zeta_{CN,n,2} + \zeta_{CN,n,1} \\
&\quad + \lambda_{CN,n,1}\Big] + \sum_{n=1}^m \tau_n^{1/2}\Big[\mu_{CN,n} + \xi_{CN,n} + \lambda_{CN,n,2}\Big]\bigg(\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\bigg)^{1/2}\bigg],
\end{aligned}
$$

For $n = [1 : m]$, taking

$$a_0 = \max_{t\in[0,t_m]} \|\hat{\rho}(t)\|, \qquad a_n = \left(\frac{\alpha}{2}\int_{t_{n-1}}^{t_n} \|\hat{\rho}(t)\|_1^2 dt\right)^{1/2},$$

$$c = \left[\|\hat{\rho}(t_0)\|^2 + C_7 \sum_{n=1}^{m} \tau_n \Lambda_{CN,n}^2\right]^{1/2},$$

$$b_0 = C_7 \sum_{n=1}^{m} \tau_n \Big[\eta_{CN,n} + \zeta_{CN,n,2} + \zeta_{CN,n,1} + \lambda_{CN,n,1}\Big],$$

$$b_n = C_7 (2\tau_n/\alpha)^{1/2}\Big[\mu_{CN,n} + \xi_{CN,n} + \lambda_{CN,n,2}\Big]$$

in Lemma 1.2.3, we obtain the desired result. □

Proof of Theorem 4.3.1. In view of (4.23), we apply triangle inequality to have

$$\|u(t) - U(t)\| \leq \|\hat{\rho}(t)\| + \|\sigma(t)\| + \|\epsilon(t)\|. \tag{4.80}$$

For $t \in I_n$,

$$\|\epsilon(t)\| = \|l_{n-1}(t)\epsilon^{n-1} + l_n(t)\epsilon^n\| \leq \max\Big(\|\epsilon^{n-1}\|, \|\epsilon^n\|\Big).$$

Therefore, for $t \in [0, t_m]$, using Lemma 4.3.1, we have

$$\|\epsilon(t)\| \leq \max_{n\in[0,m]} \Big(\|\epsilon^{n-1}\|, \|\epsilon^n\|\Big) \leq \max_{n\in[0,m]} \beta_{CN,n}[U]. \tag{4.81}$$

Also,

$$\begin{aligned} \|\hat{U}(t) - \mathcal{R}_w U(t)\| &\leq (t - t_{n-1})(t_n - t)\|\mathcal{R}_w^n \mathcal{W}_n\| \\ &\leq (t - t_{n-1})(t_n - t)\Big[\|(\mathcal{R}_w^n - I)\mathcal{W}_n\| + \|\mathcal{W}_n\|\Big] \\ &\leq \nu_{CN,n}, \end{aligned} \tag{4.82}$$

where $\nu_{CN,n}$ is given by (4.36).

Finally, we use (4.80)-(4.82) and Lemma 4.3.2 to obtain the desired result. □

Remarks. (i) We observe that the *a posteriori* bound in Theorem 4.3.1 is formally of optimal order. Since PIDE (4.1) may be thought of as a perturbation to the parabolic problem, it is natural to expect that *a posteriori* error estimator for PIDE should reflect the contributions to the error coming from the approximation of the memory term. This fact can easily be seen through the estimator $\eta_{CN,n}$ which is of $O(\tau^2)$. Further, in the absence of the memory term (i.e., $\mathcal{B}(t,s)=0$), the error estimator obtained in Theorem 4.3.1 is similar to that for the parabolic problems (cf. Bänsch *et al.* [12]).

(*ii*) It is noteworthy that the mesh change error term $\mathcal{I}_m^3$ can alternatively be estimated as

$$\mathcal{I}_m^3 \leq \sum_{n=1}^{m} \tau_n \max_{t\in[0,t_m]} \|\hat{\rho}(t)\|\mu'_{CN,n},$$

where $\mu'_{CN,n}$ is given by

$$\begin{aligned}\mu'_{CN,n} &= C_{1,1}h_n\Bigg[\frac{1}{2}\|(P_0^n - I)\mathcal{A}^{n-1}U^{n-1}\| + \|(P_0^n - I)\sigma^n(\mathcal{B}^{n-1/2}U)\| \\ &\quad + \tau_n^{-1}\|(P_0^n - I)U^{n-1}\|\Bigg].\end{aligned}$$

This estimate for mesh change error will lead to an alternative *a posteriori* error estimate for the error $e(t)$. In Lemma 4.3.2, the terms $\sigma^2_{CN,1,m}$ and $\sigma^2_{CN,2,m}$ take the form

$$\sigma^2_{CN,1,m} := \left(C_7\sum_{n=1}^{m}\tau_n\Big[\eta_{CN,n} + \mu'_{CN,n} + \zeta_{CN,n,2} + \zeta_{CN,n,1} + \lambda_{CN,n,1}\Big]\right)^2$$

and

$$\sigma^2_{CN,2,m} := \frac{4C_7^2}{\alpha}\sum \tau_n(\lambda_{CN,n,2} + \xi_{Cn,n})^2.$$

The corresponding changes take place in the Theorem 4.3.1.

(*iii*) The term

$$\left(\int_{t_{n-1}}^{t_n}\frac{1}{\tau_n}\|\hat{\omega}(t) - \hat{\omega}_I(t)\|dt\right)^{1/2} \tag{4.83}$$

appeared in Theorem 4.3.1 (see (4.45)) is not a traditional *a posteriori* quantity, where $\hat{\omega}(t)$ and $\hat{\omega}_I(t)$ are given by (4.24) and (4.25), respectively. Since, the error in linear interpolation is bounded as

$$\|\hat{\omega}(t) - \hat{\omega}_I(t)\| \leq C\tau_n^2 \max_{t\in I_n}\|\frac{d^2}{dt^2}(\hat{\omega}(t))\|, \quad t\in I_n,$$

where $\frac{d^2}{dt^2}(\hat{\omega}(t))$ depends upon the quantities $\nabla(\mathcal{R}_w U)_t(t)$ and $\nabla\mathcal{R}_w U(t)$. The term $\|\nabla(\mathcal{R}_w U)_t(t)\|$ can be estimated as

$$\begin{aligned}\|\nabla(\mathcal{R}_w U)_t(t)\| &= \|\nabla\epsilon_t(t) + \nabla U_t(t)\| \\ &\leq \|\nabla\epsilon_t(t)\| + \|\nabla U_t(t)\| \\ &\leq \frac{1}{\tau_n}\Big(\|\nabla\epsilon^n\| + \|\nabla\epsilon^{n-1}\|\Big) + \|\nabla\partial U^n\|,\end{aligned}$$

and for the term $\|\nabla\mathcal{R}_w U(t)\|$, we have

$$\begin{aligned}\|\nabla\mathcal{R}_w U(t)\| &\leq \|\nabla\epsilon(t)\| + \|\nabla U(t)\| \\ &\leq \|l_{n-1}(t)\nabla\epsilon^{n-1} + l_n(t)\nabla\epsilon^n\| + \|l_{n-1}(t)\nabla U^{n-1} + l_n(t)\nabla U^n\| \\ &\leq \max\Big(\|\nabla\epsilon^{n-1}\|, \|\nabla\epsilon^n\|\Big) + \max\Big(\|\nabla U^{n-1}\|, \|\nabla U^n\|\Big).\end{aligned}$$

This shows that (4.83) is now a meaningful *a posteriori* quantity by noting the fact that $\|\nabla\epsilon^n\|$ is bounded and is of $O(h)$ (see Lemma 4.3.1). Taking $\tau \approx h$, it is easy to see that the term (4.83) is optimal order.

(*iv*) The *a posteriori* error analysis of the classical Crank-Nicolson scheme leads to one additional term $\frac{1}{2}\|(P_0^n - I)\mathcal{A}^{n-1}U^{n-1}\|$ in the error bounds. However, it does not affect the optimality of the estimator.

5

Backward Euler Anisotropic Error Analysis

In this chapter, we derive two optimal *a posteriori* error estimators for fully discrete backward Euler space-time discretizations for PIDE (1.1) in an anisotropic framework. The *a posteriori* error indicator corresponding to spatial discretization is derived using the anisotropic interpolation estimates in conjunction with a ZZ error estimator to approach the error gradient. The error due to time discretization is derived using continuous, piecewise linear polynomial in time. We use the linear approximation of the Volterra integral term to estimate the quadrature error in the second estimator. The emphasis is on theoretical study of the error estimators.

5.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$ and $T < \infty$. Recall the following PIDE

$$u_t(x,t) + \mathcal{A}u(x,t) = \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T] \tag{5.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{5.2}$$

and the homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T]. \tag{5.3}$$

The operator $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u),$$

and the operator $\mathcal{B}(t,s)$ is of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient and $A = \{a_{ij}(x)\}$ and $B(t,s) = \{b_{ij}(x;t,s)\}$ are two 2×2 matrices assumed to be in $L^\infty(\Omega)^{2\times 2}$ in space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The initial function $u_0 = u_0(x)$ and the nonhomogeneous term f are assumed to be smooth for our purpose.

We assume that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi,\phi) \geq \alpha\|\nabla\phi\|^2 \quad \text{and} \quad |a(\phi,\psi)| \leq \beta\|\nabla\phi\|\|\nabla\psi\|, \qquad \forall\, \phi,\psi \in H_0^1(\Omega) \tag{5.4}$$

with $\alpha, \beta \in \mathbb{R}^+$. Here $\|\nabla(\cdot)\|$ defines a norm on $H_0^1(\Omega)$ in view of the Poincaré inequality. Further, we assume that the bilinear forms $b(t,s;\cdot,\cdot)$ and $b_s(t,s;\cdot,\cdot)$ are continuous on $H_0^1(\Omega)$ i.e.,

$$|b(t,s;\phi(s),\psi)| \leq \gamma\|\nabla\phi(s)\|\|\nabla\psi\|, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{5.5}$$

and

$$|b_s(t,s;\phi(s),\psi)| \leq \gamma'\|\nabla\phi(s)\|\|\nabla\psi\|, \quad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{5.6}$$

with $\gamma, \gamma' \in \mathbb{R}^+$.

The weak formulation of the problem (5.1) may be stated as follows: Find $u : [0,T] \to H_0^1(\Omega)$ such that

$$\begin{aligned} \langle u_t, \phi\rangle + a(u,\phi) &= \int_0^t b(t,s;u(s),\phi)ds + \langle f,\phi\rangle, \quad \forall\phi \in H_0^1(\Omega), \quad t \in (0,T], \qquad (5.7)\\ u(\cdot,0) &= u_0. \end{aligned}$$

Let I_h denotes the Lagrange's interpolant corresponding to $\mathbb{V}_h^{A,0}$, where $\mathbb{V}_h^{A,0}$ is the usual finite element subspace of $H_0^1(\Omega)$ given by (1.22) corresponding to the triangulation $\mathcal{T}_h^A$. The backward Euler scheme may be stated as follows: Given U_h^0, where $U_h^0 = I_h u_0$, find $U_h^n \in \mathbb{V}_h^{A,0}$, $n \in [1:N]$ such that

$$\langle \partial U_h^n, \phi\rangle + a(U_h^n, \phi) = \sigma^n(b(t_n; U_h, \phi)) + \langle f^n, \phi\rangle, \ \ \forall \phi \in \mathbb{V}_h^{A,0}, \tag{5.8}$$

where

$$\sigma^n(b(t_n; v, \phi)) = \langle \sum_{j=0}^{n-1} \tau_{j+1} B(t_n, t_j)\nabla v(t_j), \nabla\phi\rangle.$$

Here, the left rectangular rule is used to discretize the Volterra integral term.

One feature of the classical finite element method is that the aspect ratio is bounded by a constant for isotropic meshes. Since PIDE may be defined in narrow or irregular domain, the computational cost will be very high when the regular partition (isotropic mesh) is used (cf. Shi and Wang [84]). In such cases, a better choice is to employ anisotropic mesh. Using the anisotropic mesh, one can reduce the computational cost to achieve the same convergence as compared to the isotropic mesh. In the absence of the memory term, i.e., when $\mathcal{B}(t,s) = 0$, *a posteriori* error analysis for parabolic problem has been investigated by Picasso [77] on anisotropic mesh by employing fully discrete backward Euler method. In [77], Picasso has derived optimal order estimate in the $L^2(H^1(\Omega))$-norm for the heat equation. In this chapter, an attempt has been made to carry over anisotropic *a posteriori* error analysis of parabolic problems [77] to PIDE (5.1).

The rest of the chapter is organized as follows. Optimal order *a posteriori* error estimates in the $L^2(H^1(\Omega))$ norm are derived for the fully discrete backward Euler scheme in Section 5.2.

5.2 Error analysis

In this section, we estimate the error $e := u - U_h$, where u satisfies PIDE (5.1) and U_h is a continuous, piecewise linear approximation in time defined by

$$U_h(x,t) \ := \ l_n(t)U_h^n + l_{n-1}(t)U_h^{n-1}, \quad \forall t \in I_n, \tag{5.9}$$

where $l_n(t)$ and $l_{n-1}(t)$ are given by

$$l_n(t) \quad := \quad \frac{(t - t_{n-1})}{\tau_n}, \qquad\qquad l_{n-1}(t) := \frac{(t_n - t)}{\tau_n}, \tag{5.10}$$

respectively. For $t \in I_n$, we observe that

$$\frac{\partial U_h(x,t)}{\partial t} = \partial U_h^n. \tag{5.11}$$

We use the standard energy argument to derive two optimal order *a posteriori* upper bounds for the error in the $L^2(H^1(\Omega))$-norm. In both the estimators, we first relate the error to the equation residual and introduce Clément interpolant. Then by localizing the residual term over each of the elements and the edges of the triangulation, we use the anisotropic interpolation error estimates. The *a posteriori* error analysis for the second estimator differs from the first estimator in the sense that we use the linear approximation of the Volterra integral term to estimate the quadrature error in the second estimator.

We now recall some results on anisotropic interpolation properties proved in [34, 35]. Recall the standard invertible affine map $T_K : \hat{K} \to K$ from Chapter 1. Let Δ_K be the union of the neighboring triangles sharing a vertex with the triangle K. Then the following two assumptions must be satisfied by the mesh: (i) the number of triangles belonging to Δ_K must be bounded above, uniformly with respect to h; (ii) the diameter of the reference patch $\Delta_{\hat{K}} = T_K^{-1}(\Delta_K)$ should be bounded above, uniformly with respect to h. The second assumption on the mesh prevents the stretching directions $r_{1,K}$ and $r_{2,K}$ from changing too abruptly between the adjacent triangles of the mesh though the classical minimum angle condition is not required in this context.

Proposition 5.2.1. *Let $\Pi_h : H^1(\Omega) \to \mathbb{V}_h^{A,0}$ be the standard Clément interpolation operator. There is a constant C independent of the meshsize and aspect ratio such that, for any $v \in H^1(\Omega)$ and any $K \in \mathcal{T}_h^A$, we have*

$$\|v - \Pi_h v\|_{L^2(K)} + \lambda_{2,K}\|\nabla(v - \Pi_h v)\|_{L^2(K)} + \lambda_{2,K}^{1/2}\|v - \Pi_h v\|_{L^2(\partial K)} \leq C\omega_K(v). \tag{5.12}$$

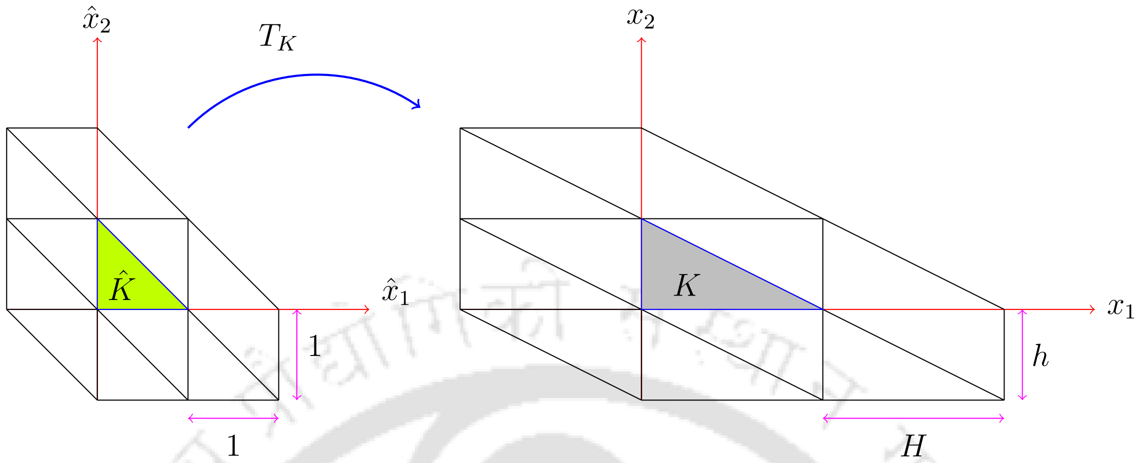

Figure 5.1: *Example of an acceptable patch, where the size of the reference patch* $\Delta_{\hat{K}}$ *does not depends upon the aspect ratio* $\frac{H}{h}$.

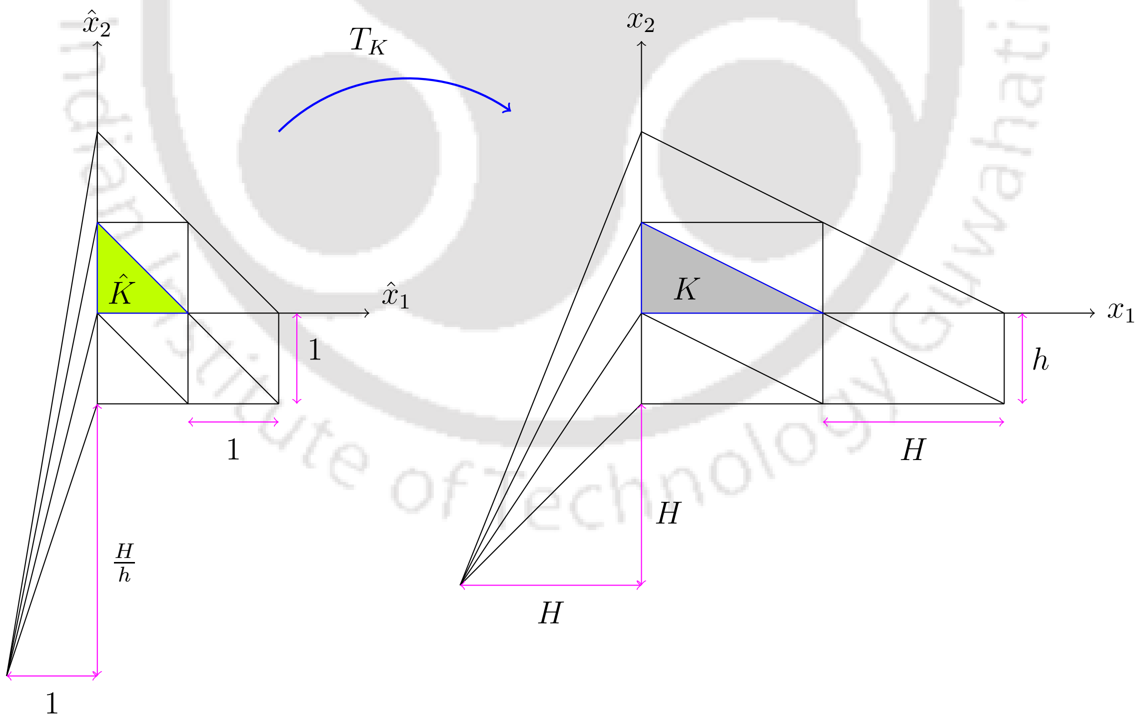

Figure 5.2: *Example of an unacceptable patch, where the size of the reference patch* $\Delta_{\hat{K}}$ *depends upon the aspect ratio* $\frac{H}{h}$.

Here, $\omega_K(v)$ is defined by

$$\omega_K^2(v) = \lambda_{1,K}^2(r_{1,K}^T G_K(v) r_{1,K}) + \lambda_{2,K}^2(r_{2,K}^T G_K(v) r_{2,K}),$$

where $\lambda_{i,K}$ and $r_{i,K}$ are as given by (1.23) and $G_K(v)$ is the following 2×2 matrix

$$G_K(v) = \sum_{K\in\Delta_K} \begin{pmatrix} \int_K \left(\frac{\partial v}{\partial x_1}\right)^2 dx & \int_K \frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} dx \\ \int_K \frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} dx & \int_K \left(\frac{\partial v}{\partial x_2}\right)^2 dx \end{pmatrix}.$$

.

Theorem 5.2.1. *Suppose the mesh satisfies that there exists a constant c independent of the time step, meshsize and aspect ratio such that*

$$\lambda_{1,K}^2(r_{1,K}^T G_K(e) r_{1,K}) \le c\lambda_{2,K}^2(r_{2,K}^T G_K(e) r_{2,K}), \quad \forall K \in \mathcal{T}_h^A. \tag{5.13}$$

Then there exists a constant C depends on the interpolation constants of Proposition 5.2.1 (hence independent of the time step, meshsize and aspect ratio) and the final time T such that the following a posteriori error bound holds.

$$\begin{aligned}
&\|e(\cdot,T)\|^2 + \alpha\int_0^T \|\nabla e\|^2 dt \le \|e(\cdot,0)\|^2 \\
&+C\sum_{n=1}^N \sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f-\partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\boldsymbol{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\boldsymbol{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(e)dt \\
&+\int_{t_{n-1}}^{t_n}\|f-f^n\|_{L^2(K)}^2 dt + \tau_n^3\|\nabla\partial U_h^n\|_{L^2(K)}^2 + \tau_n\Big(\mathcal{Q}_{BE,1,n,K}^A\Big)^2\Bigg],
\end{aligned}$$

where $\mathcal{Q}_{BE,1,n,K}^A$ is defined by

$$\begin{aligned}
\mathcal{Q}_{BE,1,n,K}^A \quad = \quad &\Bigg\{\sum_{j=1}^n \tau_j^2\Big[\|\nabla U_h^j\|_{L^2(K)} + \|\nabla U_h^{j-1}\|_{L^2(K)} + \|\nabla\partial U_h^j\|_{L^2(K)}\Big] + \tau_n\|\nabla U_h^n\|_{L^2(K)} \\
&+\tau_n\|\nabla U_h^{n-1}\|_{L^2(K)} + \sum_{j=1}^n \tau_j\Big(\|\nabla U_h^j\|_{L^2(K)} + \|\nabla U_h^{j-1}\|_{L^2(K)}\Big)\Bigg\}.
\end{aligned} \tag{5.14}$$

Here $[\cdot]$ denotes the jump of the bracketed quantity across an internal edge, $[\cdot]=0$ for an edge on the boundary $\partial\Omega$ and $\boldsymbol{n}$ is the unit edge normal.

For $t \in I_n$, set

$$\tilde{U}(t) := \int_0^t B(t,s)\nabla U_h(s)ds. \tag{5.15}$$

Let $\tilde{U}_I(t)$ be a linear approximation of $\tilde{U}(t)$ defined by

$$\tilde{U}_I(t) = l_{n-1}(t)\tilde{U}(t_{n-1}) + l_n(t)\tilde{U}(t_n), \tag{5.16}$$

where $l_{n-1}(t)$ and $l_n(t)$ are given by (5.10).

The following result is based on the linear approximation of the Volterra integral term.

Theorem 5.2.2. *Let the error equidistribution inequality (5.13) holds. Moreover, for $t \in I_n$, let $\tilde{U}(t)$ and $\tilde{U}_I(t)$ be given by (5.15) and (5.16), respectively. Then there exists a constant C depends on the interpolation constants of Proposition 5.2.1 (hence independent of the time step, meshsize and aspect ratio) and the final time T such that the following a posteriori error bound holds.*

$$\begin{aligned}
&\|e(\cdot,T)\|^2 + \alpha\int_0^T \|\nabla e\|^2 dt \le \|e(\cdot,0)\|^2 \\
&+C\sum_{n=1}^{N}\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f-\partial U_h^n+\nabla\cdot(A\nabla U_h)-\int_0^t\nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\boldsymbol{n}]\|_{L^2(\partial K)}+\frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\boldsymbol{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(e)dt \\
&+\int_{t_{n-1}}^{t_n}\|f-f^n\|_{L^2(K)}^2dt+\tau_n^3\|\nabla\partial U_h^n\|_{L^2(K)}^2+\tau_n\Big[(\mathcal{Q}_{BE,2,n,K}^A)^2+(\mathcal{Q}_{BE,2,n-1,K}^A)^2 \\
&+\delta_{n,K}^2+\|\tilde{U}(t)-\tilde{U}_I(t)\|_{L^2(K)}^2\Big]\Bigg],
\end{aligned}$$

where $\mathcal{Q}_{BE,2,n,K}^A$ and $\delta_{n,K}$ are defined by

$$\mathcal{Q}_{BE,2,n,K}^A = \sum_{j=1}^{n}\tau_j^2\Big[\|\nabla U_h^j\|_{L^2(K)}+\|\nabla U_h^{j-1}\|_{L^2(K)}+\|\nabla\partial_j U_h^j\|_{L^2(K)}\Big], \tag{5.17}$$

$$\delta_{n,K} = \sum_{j=0}^{n-1}\tau_{j+1}\|\nabla U_h^j\|_{L^2(K)}. \tag{5.18}$$

The proofs of Theorem 5.2.1 and Theorem 5.2.2 require some preparations. We shall first proceed to prove Theorem 5.2.1. For this purpose, we need to prove the following lemma which gives a bound on the quadrature error.

Lemma 5.2.1 (Quadrature error estimate for Theorem 5.2.1)**.** *Let U_h be defined by (5.9) and $\mathcal{Q}^A_{BE,1,n,K}$ be given by (5.14). Then for $t \in I_n$, the following bound holds for the quadrature error*

$$|\int_0^t b(t,s;U_h,\Pi_h e)ds - \sigma^n(b(t_n;U_h,\Pi_h e))| \le \hat{\gamma}\sum_{K\in\mathcal{T}_h^A}\mathcal{Q}^A_{BE,1,n,K}\|\nabla\Pi_h e\|,$$

where

$$\hat{\gamma} = \max\{\bar{\gamma},\gamma\}. \tag{5.19}$$

Proof. We know that for a sufficiently smooth function $y(t,s)$

$$\int_0^t y(t,s)ds = \int_0^{t_n} y(t,s)ds - \int_t^{t_n} y(t,s)ds.$$

Therefore, for $t \in I_n$

$$\begin{aligned}
&|\langle\int_0^t B(t,s)\nabla U_h(s)ds,\nabla\Pi_h e\rangle - \langle\sum_{j=0}^{n-1}\tau_{j+1}B(t_n,t_j)\nabla U_h^j,\nabla\Pi_h e\rangle| \\
\le\; & |\langle\int_0^{t_n} B(t_n,s)\nabla U_h(s)ds - \sum_{j=0}^{n-1}\tau_{j+1}B(t_n,t_j)\nabla U_h^j,\nabla\Pi_h e\rangle| \\
&+|\langle\int_t^{t_n} B(t,s)\nabla U_h(s)ds,\nabla\Pi_h e\rangle| \\
&+|\langle\int_0^{t_n}(B(t,s)\nabla U_h(s) - B(t_n,s)\nabla U_h(s))ds,\nabla\Pi_h e\rangle| \\
&:= |\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3|.
\end{aligned} \tag{5.20}$$

Thus, to obtain an upper bound for the quadrature error due to the approximation of the Volterra integral term, it suffices to estimate the terms $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$. To estimate $\mathcal{I}_1$, we know for a smooth function $g(s)$

$$\int_{t_{n-1}}^{t_n} g(s)ds - \tau_n g(t_{n-1}) = \int_{t_{n-1}}^{t_n}(t_n - s)\frac{dg}{ds}ds. \tag{5.21}$$

In view of (5.21), the term $\mathcal{I}_1$ may be written as

$$\begin{aligned}\mathcal{I}_1 &:= \langle \int_0^{t_n} B(t_n,s)\nabla U_h(s)ds, \nabla\Pi_h e\rangle - \langle \sum_{j=0}^{n-1} \tau_{j+1} B(t_n,t_j)\nabla U_h^j, \nabla\Pi_h e\rangle \\ &= \langle \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (t_j - s)\frac{\partial}{\partial s}\{B(t_n,s)\nabla U_h(s)\}ds, \nabla\Pi_h e\rangle.\end{aligned}$$

Thus, over each $K \in \mathcal{T}_h^A$

$$\begin{aligned}\mathcal{I}_1 &:= \sum_{K\in\mathcal{T}_h^A}\sum_{j=1}^{n}\langle \int_{t_{j-1}}^{t_j} (t_j - s)\frac{\partial B(t_n,s)}{\partial s}\nabla U_h(s)ds \\ &\quad + \int_{t_{j-1}}^{t_j} (t_j - s)B(t_n,s)\frac{\partial \nabla U_h(s)}{\partial s}ds, \nabla\Pi_h e\rangle_K.\end{aligned}$$

Now, using (5.5), (5.6), (5.9) and (5.21) we get

$$\begin{aligned}|\mathcal{I}_1| &\le \gamma' \sum_{K\in\mathcal{T}_h^A}\sum_{j=1}^{n} \tau_j \Big[\int_{t_{j-1}}^{t_j} \|\nabla U_h(s)\|_{L^2(K)}ds\Big]\|\nabla\Pi_h e\|_{L^2(K)} \\ &\quad +\gamma \sum_{K\in\mathcal{T}_h^A}\sum_{j=1}^{n} \tau_j \Big[\int_{t_{j-1}}^{t_j} \|\frac{\partial\nabla U_h(s)}{\partial s}\|_{L^2(K)}ds\Big]\|\nabla\Pi_h e\|_{L^2(K)} \\ &\le \sum_{K\in\mathcal{T}_h^A} \bar{\gamma}\mathcal{Q}^A_{BE,2,n,K}\|\nabla\Pi_h e\|_{L^2(K)}, \end{aligned} \tag{5.22}$$

where $\mathcal{Q}^A_{BE,2,n,K}$ is an *a posteriori* quantity and is given by (5.17) and $\bar{\gamma} = \max\{\frac{\gamma'}{2}, \gamma\}$.

Now, the continuity of the bilinear form $b(t,s;\cdot,\cdot)$ in conjunction with the definition (5.9) gives

$$\begin{aligned}|\mathcal{I}_2| &\le \gamma \sum_{K\in\mathcal{T}_h^A} \int_{t_{n-1}}^{t_n} \|\nabla U_h(s)\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)}ds \\ &\le \sum_{K\in\mathcal{T}_h^A} \frac{\tau_n\gamma}{2}\Big[\|\nabla U_h^n\|_{L^2(K)} + \|\nabla U_h^{n-1}\|_{L^2(K)}\Big]\|\nabla\Pi_h e\|_{L^2(K)}\end{aligned}$$

and

$$|\mathcal{I}_3| \le \gamma \sum_{K\in\mathcal{T}_h^A}\sum_{j=1}^{n} \tau_j\Big(\|\nabla U_h^j\|_{L^2(K)} + \|\nabla U_h^{j-1}\|_{L^2(K)}\Big)\|\nabla\Pi_h e\|_{L^2(K)}.$$

Substituting the above estimates in (5.20), we obtain

$$\begin{aligned}&|\langle \int_0^{t} B(t,s)\nabla U_h(s)ds, \nabla\Pi_h e\rangle - \langle \sum_{j=0}^{n-1} \tau_{j+1} B(t_n,t_j)\nabla U_h^j, \nabla\Pi_h e\rangle| \\ &\le \sum_{K\in\mathcal{T}_h^A} \hat{\gamma}\mathcal{Q}^A_{BE,1,n,K}\|\nabla\Pi_h e\|,\end{aligned} \tag{5.23}$$

where $\mathcal{Q}^A_{BE,1,n,K}$ and $\hat{\gamma}$ are given by (5.14) and (5.19), respectively. Here, the term $\mathcal{Q}^A_{BE,1,n,K}$ is an *a posteriori* quantity evolved due to the quadrature approximation of the Volterra integral term and is defined on each triangle K in $\mathcal{T}_h^A$ and on each time interval I_n. Thus, it gives the information for the quadrature error contributions coming from each of the triangle K of $\mathcal{T}_h^A$. This completes the proof. □

Proof of Theorem 5.2.1. Using the weak formulation (5.7), $\forall \phi \in H_0^1(\Omega)$ we have

$$\begin{aligned}&\int_\Omega \frac{\partial e}{\partial t}\phi dx + a(e,\phi) - \int_0^t b(t,s;e(s),\phi)ds \\ &= \int_\Omega f\phi dx - \int_\Omega \frac{\partial U_h}{\partial t}\phi dx - a(U_h,\phi) + \int_0^t b(t,s;U_h(s),\phi)ds.\end{aligned}$$

Taking $\phi = e$ in the above error equation and using (5.11), we obtain

$$\begin{aligned}&\int_\Omega \frac{\partial e}{\partial t} edx + a(e,e) - \int_0^t b(t,s;e(s),e)ds \\ &= \int_\Omega (f - \partial U_h^n)(e - \Pi_h e)dx - a(U_h, e - \Pi_h e) + \int_0^t b(t,s;U_h(s), e - \Pi_h e)ds \\ &+ \int_\Omega (f - \partial U_h^n)\Pi_h edx - a(U_h, \Pi_h e) + \int_0^t b(t,s;U_h(s),\Pi_h e)ds.\end{aligned}$$

Using (5.8) together with the definition (5.9) and the identity $l_{n-1}(t) + l_n(t) \equiv 1,\ t \in I_n$, we get

$$\begin{aligned}&\int_\Omega \frac{\partial e}{\partial t} edx + a(e,e) - \int_0^t b(t,s;e(s),e)ds \\ =\ &\int_\Omega (f - \partial U_h^n)(e - \Pi_h e)dx - a(U_h, e - \Pi_h e) + \int_0^t b(t,s;U_h(s), e - \Pi_h e)ds \\ &+ \int_\Omega (f - f^n)\Pi_h edx + (t_n - t)a(\partial U_h^n, \Pi_h e) \\ &+ \int_0^t b(t,s;U_h,\Pi_h e)ds - \sigma^n(b(t_n;U_h,\Pi_h e)).\end{aligned} \quad (5.24)$$

Integrating by parts on each triangle K of $\mathcal{T}_h^A$ and using Lemma 5.2.1, continuity of the bilinear forms $a(\cdot,\cdot)$, $b(t,s;\cdot,\cdot)$ together with Cauchy-Schwarz and Poincaré inequalities, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 + a(e,e) \leq \gamma\|\nabla e(t)\| \int_0^t \|\nabla e(s)\|ds$$
$$+ \sum_{K\in\mathcal{T}_h^A} \Bigg[\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}\|e-\Pi_h e\|_{L^2(K)}$$
$$+\frac{1}{2}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}\|e-\Pi_h e\|_{L^2(\partial K)} + C_2\|f-f^n\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)}$$
$$+\frac{1}{2}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\|e-\Pi_h e\|_{L^2(\partial K)}$$
$$+|t_n - t|\beta\|\nabla\partial U_h^n\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)} + \hat{\gamma}\mathcal{Q}^A_{BE,1,n,K}\|\nabla\Pi_h e\|_{L^2(K)}\Bigg],$$

where C_2 is the constant in the Poincaré inequality.

Using coercivity of the bilinear form $a(\cdot,\cdot)$, Young's inequality and Proposition 5.2.1, it now leads to

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 + \alpha\|\nabla e\|^2 \leq \frac{\gamma}{2\nu}\|\nabla e\|^2 + \frac{\gamma C_2(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h^A}\Bigg[C_1\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(e)$$
$$+C_2C_3\|f-f^n\|_{L^2(K)}\|\nabla e\|_{L^2(K)} + C_3\beta|t_n-t|\|\nabla\partial U_h^n\|_{L^2(K)}\|\nabla e\|_{L^2(K)}$$
$$+C_3\hat{\gamma}\mathcal{Q}^A_{BE,1,n,K}\|\nabla e\|_{L^2(K)}\Bigg],$$

where we have used $\|\nabla\Pi_h e\|_{L^2(K)} \leq C_3\|\nabla e\|_{L^2(K)}$ and C_1 is the interpolation constant. Again, we apply Young's inequality for the last three terms to obtain

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 + \alpha\|\nabla e\|^2 \leq \frac{\gamma}{2\nu}\|\nabla e\|^2 + \frac{\gamma C_2(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h^A}\Bigg[C_1\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(e)$$
$$+\frac{(C_2^2C_3^2 + \beta^2C_3^2 + C_3^2\hat{\gamma}^2)}{2\nu}\|\nabla e\|^2_{L^2(K)} + \frac{\nu}{2}\|f-f^n\|^2_{L^2(K)} + \frac{\nu}{2}(t_n-t)^2\|\nabla\partial U_h^n\|^2_{L^2(K)}$$
$$+\frac{\nu}{2}\left(\mathcal{Q}^A_{BE,1,n,K}\right)^2\Bigg].$$

Now, choose $\nu = \frac{(C_2^2C_3^2+\beta^2C_3^2+C_3^2\tilde{\gamma}^2+\gamma)}{\alpha}$ and integrate the above equation from t_{n-1} to t_n. Then, an application of Gronwall's lemma yields

$$\|e(t_n)\|^2 + \alpha\int_{t_{n-1}}^{t_n}\|\nabla e\|^2 dt \le \|e(t_{n-1})\|^2$$
$$+C\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f-\partial U_h^n+\nabla\cdot(A\nabla U_h)-\int_0^t\nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}+\frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(e)dt$$
$$+\int_{t_{n-1}}^{t_n}\|f-f^n\|^2_{L^2(K)}dt+\tau_n^3\|\nabla\partial U_h^n\|^2_{L^2(K)}+\tau_n\Big(\mathcal{Q}^A_{BE,1,n,K}\Big)^2\Bigg],$$

where C(T) is the Gronwall's constant, $C = \max\{2C_1C(T), C(T)\nu\}$ and we have used the fact that

$$\int_{t_{n-1}}^{t_n}(t_n-t)^2dt = \frac{\tau_n^3}{3}.$$

Taking summation from $n = 1$ to N, we complete the rest of the proof. □

Next, to prove Theorem 5.2.2, the following lemma proves to be convenient.

Lemma 5.2.2 (Quadrature error estimate for Theorem 5.2.2)**.** *Let U_h be defined by (5.9). Let $\tilde{U}(t)$, $\tilde{U}_I(t)$, $\mathcal{Q}^A_{BE,2,n,K}$ and $\delta_{n,K}$ be given by (5.15), (5.16), (5.17) and (5.18), respectively. Then for $t\in I_n$, we have*

$$\Big|\int_0^t b(t,s;U_h(s),\Pi_he)ds-\sigma^n(b(t_n;U_h,\Pi_he))\Big|$$
$$\le\sum_{K\in\mathcal{T}_h^A}\tilde{\gamma}\Big[l_n(t)\mathcal{Q}^A_{BE,2,n,K}+l_{n-1}(t)\mathcal{Q}^A_{BE,2,n-1,K}+l_{n-1}(t)\delta_{n,K}$$
$$+\|\tilde{U}(t)-\tilde{U}_I(t)\|_{L^2(K)}\Big]\|\nabla\Pi_he\|_{L^2(K)}, \tag{5.25}$$

where

$$\tilde{\gamma} = \max\{\bar{\gamma}, 2\gamma, 1\}. \tag{5.26}$$

Proof. Taking L^2-inner product with $\nabla\Pi_he$ over Ω in (5.16) we obtain

$$\langle\tilde{U}_I(t),\nabla\Pi_he\rangle = l_{n-1}(t)\int_0^{t_{n-1}}b(t_{n-1},s;U_h(s),\Pi_he)ds$$
$$+l_n(t)\int_0^{t_n}b(t_n,s;U_h(s),\Pi_he)ds.$$

Now by virtue of the identity $l_{n-1}(t)+l_n(t)\equiv 1,\ t\in I_n$, it follows that

$$
\begin{aligned}
&\int_0^t b(t,s;U_h(s),\Pi_h e)ds-\sigma^n(b(t_n;U_h,\Pi_h e))\\
=\ & l_n(t)\int_0^{t_n} b(t_n,s;U_h(s),\Pi_h e)ds+l_{n-1}(t)\int_0^{t_{n-1}} b(t_{n-1},s;U_h(s),\Pi_h e)ds\\
&-(l_{n-1}(t)+l_n(t))\sigma^n(b(t_n;U_h,\Pi_h e))+\langle\tilde{U}(t)-\tilde{U}_I(t),\nabla\Pi_h e\rangle\\
=\ & l_n(t)\Big[\int_0^{t_n} b(t_n,s;U_h(s),\Pi_h e)ds-\sigma^n(b(t_n;U_h,\Pi_h e))\Big]\\
&+l_{n-1}(t)\Big[\int_0^{t_{n-1}} b(t_{n-1},s;U_h(s),\Pi_h e)ds-\sigma^{n-1}(b(t_{n-1};U_h,\Pi_h e))\Big]\\
&-l_{n-1}(t)\Big[\sigma^n(b(t_n;U_h,\Pi_h e)-\sigma^{n-1}(b(t_{n-1};U_h,\Pi_h e))\Big]\\
&+\langle\tilde{U}(t)-\tilde{U}_I(t),\nabla\Pi_h e\rangle. \qquad (5.27)
\end{aligned}
$$

From (5.22), we have

$$
\begin{aligned}
&|\langle\int_0^{t_n} B(t_n,s)\nabla U_h(s)ds,\nabla\Pi_h e\rangle-\langle\sum_{j=0}^{n-1}\tau_{j+1}B(t_n,t_j)\nabla U_h^j,\nabla\Pi_h e\rangle|\\
&\le\sum_{K\in\mathcal{T}_h^A}\bar{\gamma}\mathcal{Q}^A_{BE,2,n,K}\|\nabla\Pi_h e\|_{L^2(K)}
\end{aligned}
$$

and

$$
\begin{aligned}
&|\langle\int_0^{t_{n-1}} B(t_{n-1},s)\nabla U_h(s)ds,\nabla\Pi_h e\rangle-\langle\sum_{j=0}^{n-2}\tau_{j+1}B(t_{n-1},t_j)\nabla U_h^j,\nabla\Pi_h e\rangle|\\
&\le\sum_{K\in\mathcal{T}_h^A}\bar{\gamma}\mathcal{Q}^A_{BE,2,n-1,K}\|\nabla\Pi_h e\|_{L^2(K)}.
\end{aligned}
$$

Using (5.5) we obtain

$$
\begin{aligned}
&|\sigma^n(b(t_n;U_h,\Pi_h e)-\sigma^{n-1}(b(t_{n-1};U_h,\Pi_h e))|\\
&=|\langle\sum_{j=0}^{n-1}\tau_{j+1}B(t_n,t_j)\nabla U_h^j-\sum_{j=0}^{n-2}\tau_{j+1}B(t_{n-1},t_j)\nabla U_h^j,\nabla\Pi_h e\rangle|\\
\le\ & 2\gamma\sum_{K\in\mathcal{T}_h^A}\sum_{j=0}^{n-1}\tau_{j+1}\|\nabla U_h^j\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)}\le\sum_{K\in\mathcal{T}_h^A}2\gamma\delta_{n,K}\|\nabla\Pi_h e\|_{L^2(K)},
\end{aligned}
$$

where $\delta_{n,K}=\sum_{j=0}^{n-1}\tau_{j+1}\|\nabla U_h^j\|_{L^2(K)}$ is an *a posteriori* quantity. Moreover,

$$
|\langle\tilde{U}(t)-\tilde{U}_I(t),\nabla\Pi_h e\rangle|\le\sum_{K\in\mathcal{T}_h^A}\|\tilde{U}(t)-\tilde{U}_I(t)\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)}.
$$

Thus, the desired result follows from (5.27) in view of the above estimates. □

Proof of Theorem 5.2.2. We use Lemma 5.2.2 in (5.24) and integrate by parts on each triangle K of $\mathcal{T}_h^A$. Then, using continuity of the bilinear forms $b(t,s;\cdot,\cdot)$ and $a(\cdot,\cdot)$ together with Cauchy-Schwarz and Poincaré inequalities leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 + a(e,e) \leq \gamma\|\nabla e(t)\| \int_0^t \|\nabla e(s)\|ds \\
&+ \sum_{K\in\mathcal{T}_h^A} \Bigg[\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}\|e-\Pi_h e\|_{L^2(K)} \\
&+\frac{1}{2}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}\|e-\Pi_h e\|_{L^2(\partial K)} \\
&+\frac{1}{2}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\|e-\Pi_h e\|_{L^2(\partial K)} \\
&+C_2\|f-f^n\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)} + |t_n - t|\beta\|\nabla\partial U_h^n\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)} \\
&+\tilde{\gamma}\Big[l_n(t)\mathcal{Q}^A_{BE,2,n,K} + l_{n-1}(t)\mathcal{Q}^A_{BE,2,n-1,K} + l_{n-1}(t)\delta_{n,K} \\
&+\|\tilde{U}(t)-\tilde{U}_I(t)\|_{L^2(K)}\Big]\|\nabla\Pi_h e\|_{L^2(K)}\Bigg].
\end{aligned}
$$

We use coercivity of the bilinear form $a(\cdot,\cdot)$ and Young's inequality to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 + \alpha\|\nabla e\|^2 \leq \frac{\gamma}{2\nu}\|\nabla e\|^2 + \frac{\gamma C_2(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds \\
&+ \sum_{K\in\mathcal{T}_h^A} \Bigg[\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}\|e-\Pi_h e\|_{L^2(K)} \\
&+\frac{1}{2}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}\|e-\Pi_h e\|_{L^2(\partial K)} \\
&+\frac{1}{2}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\|e-\Pi_h e\|_{L^2(\partial K)} \\
&+C_2\|f-f^n\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)} + |t_n - t|\beta\|\nabla\partial U_h^n\|_{L^2(K)}\|\nabla\Pi_h e\|_{L^2(K)} \\
&+\tilde{\gamma}\Big[l_n(t)\mathcal{Q}^A_{BE,2,n,K} + l_{n-1}(t)\mathcal{Q}^A_{BE,2,n-1,K} + l_{n-1}(t)\delta_{n,K} \\
&+\|\tilde{U}(t)-\tilde{U}_I(t)\|_{L^2(K)}\Big]\|\nabla\Pi_h e\|_{L^2(K)}\Bigg].
\end{aligned}
$$

An application of Proposition 5.2.1 together with $\|\nabla\Pi_h e\|_{L^2(K)} \leq C_3\|\nabla e\|_{L^2(K)}$ yields

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 + \alpha\|\nabla e\|^2 \leq \frac{\gamma}{2\nu}\|\nabla e\|^2 + \frac{\gamma C_2(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h^A}\Bigg[C_1\Bigg\{\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg\}\omega_K(e)$$
$$+C_2C_3\|f - f^n\|_{L^2(K)}\|\nabla e\|_{L^2(K)} + C_3\beta|t_n - t|\|\nabla\partial U_h^n\|_{L^2(K)}\|\nabla e\|_{L^2(K)}$$
$$+C_3\tilde{\gamma}\Big[l_n(t)\mathcal{Q}^A_{BE,2,n,K} + l_{n-1}(t)\mathcal{Q}^A_{BE,2,n-1,K} + l_{n-1}(t)\delta_{n,K}$$
$$+\|\tilde{U}(t) - \tilde{U}_I(t)\|_{L^2(K)}\Big]\|\nabla e\|_{L^2(K)}\Bigg].$$

Using Young's inequality for the last three terms, we have

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|^2 + \alpha\|\nabla e\|^2 \leq \frac{\gamma}{2\nu}\|\nabla e\|^2 + \frac{\gamma C_2(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h^A}\Bigg[C_1\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(e)$$
$$+\frac{(C_2^2C_3^2 + \beta^2C_3^2 + 4\tilde{\gamma}^2C_3^2)}{2\nu}\|\nabla e\|^2_{L^2(K)} + \frac{\nu}{2}\|f - f^n\|^2_{L^2(K)} + \frac{\nu}{2}(t_n - t)^2\|\nabla\partial U_h^n\|^2_{L^2(K)}$$
$$+\frac{\nu}{2}\Big[l_n^2(t)(\mathcal{Q}^A_{BE,2,n,K})^2 + l_{n-1}^2(t)(\mathcal{Q}^A_{BE,2,n-1,K})^2 + l_{n-1}^2(t)\delta^2_{n,K} + \|\tilde{U}(t) - \tilde{U}_I(t)\|^2_{L^2(K)}\Big]\Bigg].$$

Choose $\nu = \frac{(C_2^2C_3^2+\beta^2C_3^2+4\tilde{\gamma}^2C_3^2+\gamma)}{\alpha}$ and integrate the above equation from t_{n-1} to t_n. Then, an application of Gronwall's lemma yields

$$\|e(t_n)\|^2 + \alpha\int_{t_{n-1}}^{t_n}\|\nabla e\|^2 dt \leq \|e(t_{n-1})\|^2$$
$$+C\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(e)dt$$
$$+\int_{t_{n-1}}^{t_n}\|f - f^n\|^2_{L^2(K)}dt + \tau_n^3\|\nabla\partial U_h^n\|^2_{L^2(K)} + \tau_n\Big[(\mathcal{Q}^A_{BE,2,n,K})^2 + (\mathcal{Q}^A_{BE,2,n-1,K})^2$$
$$+\delta^2_{n,K} + \|\tilde{U}(t) - \tilde{U}_I(t)\|^2_{L^2(K)}\Big]\Bigg],$$

where $C(T)$ is the Gronwall's constant and $C = \max\{2C_1C(T), C(T)\nu\}$. Taking sum from $n = 1$ to N gives the required result. □

Remarks. (i) Due to the presence of the term $\omega_K(e)$ (and hence the gradient of the exact solution u) in both the error estimators (see Theorems 5.2.1-5.2.2), the *a posteriori* error bounds appear in Theorem 5.2.1 and Theorem 5.2.2 are not traditional *a posteriori* error estimates. From [60, 77, 108, 109] and references therein we know that for a certain class of meshes and for smooth solutions, in particular for the elliptic and parabolic problems, ZZ like error estimators are asymptotically exact. Thus, in case of PIDE to approximate the term $\omega_K(e)$ in an efficient way, we introduce ZZ error estimator as proposed in [60, 77]. Consider the following ZZ error estimator [108, 109]

$$\zeta^{ZZ}(U_h) = \begin{pmatrix} \zeta_1^{ZZ}(U_h) \\ \zeta_2^{ZZ}(U_h) \end{pmatrix} = \begin{pmatrix} (I - I_h^A)(\frac{\partial U_h}{\partial x_1}) \\ (I - I_h^A)(\frac{\partial U_h}{\partial x_2}) \end{pmatrix},$$

where I_h^A is an approximate L^2 projection operator onto $\mathbb{V}_h^A$ and is defined by its values at each vertex P as

$$I_h^A(\nabla U_h)(P) = \begin{pmatrix} I_h^A(\frac{\partial U_h}{\partial x_1})(P) \\ I_h^A(\frac{\partial U_h}{\partial x_2})(P) \end{pmatrix} = \frac{1}{\sum\limits_{P\in K, K\in\mathcal{T}_h} |K|} \begin{pmatrix} \sum\limits_{P\in K, K\in\mathcal{T}_h} |K|(\dfrac{\partial U_h}{\partial x_1})|_K \\ \sum\limits_{P\in K, K\in\mathcal{T}_h} |K|(\dfrac{\partial U_h}{\partial x_2})|_K \end{pmatrix}. \tag{5.28}$$

Therefore, in Theorems 5.2.1-5.2.2, we replace the matrix $G_K(e)$ in the term $\omega_K(e)$ by the matrix $\mathcal{G}_K(U_h)$ to recover usual *a posteriori* error estimators. For any $v_h \in \mathbb{V}_h^A$, $\mathcal{G}_K(v_h)$ is defined by

$$\mathcal{G}_K(v_h) = \begin{pmatrix} \displaystyle\int_K \left(\zeta_1^{ZZ}(v_h)\right)^2 dx & \displaystyle\int_K \zeta_1^{ZZ}(v_h)\zeta_2^{ZZ}(v_h)dx \\ \displaystyle\int_K \zeta_1^{ZZ}(v_h)\zeta_2^{ZZ}(v_h)dx & \displaystyle\int_K \left(\zeta_2^{ZZ}(v_h)\right)^2 dx \end{pmatrix}. \tag{5.29}$$

(ii) One can recover the isotropic *a posteriori* error estimates from that of anisotropic *a posteriori* error estimates of Theorems 5.2.1-5.2.2. So, in case of isotropic mesh ($\lambda_{1,K} \simeq$

$\lambda_{2,K} \simeq h_K$), Theorem 5.2.1 and Theorem 5.2.2 respectively, take the form

$$\begin{aligned}
&\|e(\cdot,T)\|^2 + \alpha \int_0^T \|\nabla e\|^2 dt \le \|e(\cdot,0)\|^2 \\
&+C \sum_{n=1}^N \sum_{K\in\mathcal{T}_h^A} \Bigg[\int_{t_{n-1}}^{t_n} \Bigg(h_K^2 \|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+h_K\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + h_K\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)dt \\
&+\int_{t_{n-1}}^{t_n} \|f-f^n\|_{L^2(K)}^2 dt + \tau_n^3\|\nabla\partial U_h^n\|_{L^2(K)}^2 + \tau_n\Big(\mathcal{Q}_{BE,1,n,K}^A\Big)^2\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
&\|e(\cdot,T)\|^2 + \alpha \int_0^T \|\nabla e\|^2 dt \le \|e(\cdot,0)\|^2 \\
&+C \sum_{n=1}^N \sum_{K\in\mathcal{T}_h^A} \Bigg[\int_{t_{n-1}}^{t_n} \Bigg(h_K^2 \|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+h_K\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + h_K\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)dt \\
&+\int_{t_{n-1}}^{t_n} \|f-f^n\|_{L^2(K)}^2 dt + \tau_n^3\|\nabla\partial U_h^n\|_{L^2(K)}^2 \\
&+\tau_n\Big[(\mathcal{Q}_{BE,2,n,K}^A)^2 + (\mathcal{Q}_{BE,2,n-1,K}^A)^2 + \delta_{n,K}^2 + \|\tilde{U}(t) - \tilde{U}_I(t)\|_{L^2(K)}^2\Big]\Bigg],
\end{aligned}$$

where the constant C depends on the mesh aspect ratio and the contributions $\mathcal{Q}_{BE,1,n,K}^A$, $\mathcal{Q}_{BE,2,n,K}^A$ and $\delta_{n,K}$ are as given by (5.14), (5.17) and (5.18) respectively.

(iii) When u is smooth enough, the error e in the $L^2(H^1(\Omega))$-norm is $O(h+\tau)$. Thus, the terms related to the inner residual and the jump residual in Theorems 5.2.1-5.2.2 are of the optimal order. The error due to the time discretization is estimated by the last three terms in the error estimates for both the error estimators. In particular, the last term in both the estimators account for the contributions coming from each of the element K due to quadrature time approximation of the Volterra integral term. The common term involving $\tau_n^3\|\nabla\partial U_h^n\|_{L^2(K)}^2$ in both the estimators is of optimal order provided $\sum_{n=1}^N \tau_n\|\nabla\partial U_h^n\|_{L^2(K)}^2$ is bounded with respect to τ. A little more simplification by virtue of Young's inequality reveals that $\tau_n\big(\mathcal{Q}^A_{BE,1,n,K}\big)^2$ is of optimal order if we assume that $\sum_{n=1}^N \Big(\sum_{j=1}^n \tau_j\|\nabla U_h^j\|^2_{L^2(K)}\Big)$, $\sum_{n=1}^N \Big(\sum_{j=1}^n \tau_j\|\nabla U_h^{j-1}\|^2_{L^2(K)}\Big)$, $\sum_{n=1}^N \Big(\sum_{j=1}^n \tau_j\|\nabla\partial_j U_h^j\|^2_{L^2(K)}\Big)$, $\sum_{n=1}^N \tau_n\|\nabla U_h^n\|^2_{L^2(K)}$ and $\sum_{n=1}^N \tau_n\|\nabla U_h^{n-1}\|^2_{L^2(K)}$ are bounded with respect to τ. Similarly, the contributions coming from taking quadrature approximation of the Volterra integral term in Theorem 5.2.2 is of optimal order if we assume that $\sum_{n=1}^N \Big(\sum_{j=1}^n \tau_j\|\nabla U_h^j\|^2_{L^2(K)}\Big)$, $\sum_{n=1}^N \Big(\sum_{j=1}^n \tau_j\|\nabla U_h^{j-1}\|^2_{L^2(K)}\Big)$ and $\sum_{n=1}^N \Big(\sum_{j=1}^n \tau_j\|\nabla\partial U_h^j\|^2_{L^2(K)}\Big)$ are bounded with respect to τ. Moreover, the term $\|\tilde{U}(t)-\tilde{U}_I(t)\|^2_{L^2(K)}$ is an *a posteriori* quantity and is naturally of optimal order as $\tilde{U}_I(t)$ is a linear approximation of the term $\tilde{U}(t)$.

(iv) The results of this chapter extend the results of parabolic problem presented in [77] to PIDE (5.1). It is known (cf. [77]) that *a posteriori* error estimators for parabolic problems in the $L^2(H^1(\Omega))$-norm are of optimal order. Since the PIDE (5.1) may be thought of as a perturbation to the parabolic problem, it is natural to expect that our *a posteriori* error estimators should reflect back the quadrature error coming from taking the approximation of the memory term. We have already observed in previous remark that the contributions comes due to quadrature approximation of the Volterra integral term leads to optimality. When $\mathcal{B}(t,s)=0$, our estimators are similar to that of the parabolic problems [77].

6

Crank-Nicolson Anisotropic Error Analysis

In this chapter, we derive two anisotropic error estimators for PIDE (1.1) in a two dimensional convex polygonal domain. A continuous, piecewise linear finite element space is employed for the space discretization and the time discretization is based on the Crank-Nicolson method. The *a posteriori* contributions corresponding to space discretization is derived using the anisotropic interpolation estimates together with the ZZ error estimator to approach the error gradient. Two different continuous, piecewise quadratic reconstructions are used to obtain the error due to time discretization. Moreover, linear approximations of the Volterra integral term are used in a crucial way to estimate the quadrature error in the approximation of the Volterra integral term. The emphasis is on the theoretical aspect of the error estimators.

6.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary $\partial\Omega$, and let $(0, T]$ be a finite interval. We now recall the following PIDE

$$u_t(x,t) + \mathcal{A}u(x,t) = \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T] \tag{6.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{6.2}$$

and the homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T]. \tag{6.3}$$

The operator $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator of the form

$$\mathcal{A}u = -\nabla \cdot (A\nabla u),$$

and the operator $\mathcal{B}(t,s)$ is of the form

$$\mathcal{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u),$$

where "∇" denotes the spatial gradient and $A = \{a_{ij}(x)\}$ and $B(t,s) = \{b_{ij}(x;t,s)\}$ are two 2×2 matrices assumed to be in $L^\infty(\Omega)^{2\times 2}$ in space variable. Moreover, the elements of $B(t,s)$ are assumed to be smooth in both t and s. The initial function $u_0 = u_0(x)$ and the nonhomogeneous term f are assumed to be smooth.

We assume that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on $H_0^1(\Omega)$ i.e.,

$$a(\phi,\phi) \geq \alpha\|\nabla\phi\|^2 \quad \text{and} \quad |a(\phi,\psi)| \leq \beta\|\nabla\phi\|\|\nabla\psi\|, \qquad \forall\, \phi,\psi \in H_0^1(\Omega) \tag{6.4}$$

with $\alpha,\beta \in \mathbb{R}^+$. Here, $\|\nabla(\cdot)\|$ defines a norm on $H_0^1(\Omega)$ in view of the Poincaré inequality. Further, we assume that the bilinear forms $b(t,s;\cdot,\cdot)$, $b_s(t,s;\cdot,\cdot)$, $b_{ss}(t,s;\cdot,\cdot)$ and $b_t(t,s;\cdot,\cdot)$ are continuous on $H_0^1(\Omega)$ i.e.,

$$|b(t,s;\phi(s),\psi)| \leq \gamma\|\nabla\phi(s)\|\|\nabla\psi\|, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega), \tag{6.5}$$

$$|b_s(t,s;\phi(s),\psi)| \leq \gamma'\|\nabla\phi(s)\|\|\nabla\psi\|, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega), \tag{6.6}$$

$$|b_{ss}(t,s;\phi(s),\psi)| \leq \gamma''\|\nabla\phi(s)\|\|\nabla\psi\|, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{6.7}$$

and

$$|b_t(t,s;\phi(s),\psi)| \leq \gamma'''\|\nabla\phi(s)\|\|\nabla\psi\|, \qquad \forall\, \phi(s),\psi \in H_0^1(\Omega) \tag{6.8}$$

with $\gamma,\gamma',\gamma'',\gamma''' \in \mathbb{R}^+$.

The weak formulation of the problem (6.1) may be stated as follows: Find $u : (0,T] \to H_0^1(\Omega)$ such that

$$\begin{aligned}\int_\Omega u_t\phi dx + a(u,\phi) &= \int_0^t b(t,s;u(s),\phi)ds + \int_\Omega f\phi dx, \quad \forall \phi \in H_0^1(\Omega),\ t\in(0,T], \quad (6.9)\\ u(.,0) &= u_0.\end{aligned}$$

We now state the Crank-Nicolson scheme as follows: Given U_h^0, where $U_h^0 = I_h u_0$, find $U_h^n \in \mathbb{V}_h^{A,0}$, $n \in [1:N]$ such that

$$\begin{aligned}\int_\Omega \partial U_h^n \phi_h dx + a(U_h^{n-1/2},\phi_h) &= \sigma^n(b(t_{n-1/2};U_h,\phi_h))\\ &\quad + \int_\Omega f^{n-1/2}\phi_h dx, \quad \forall \phi_h \in \mathbb{V}_h^{A,0}, \qquad (6.10)\end{aligned}$$

where

$$\begin{aligned}&\sigma^n(b(t_{n-1/2};U_h,\phi_h))\\ :=\ &\langle\sum_{j=0}^{n-2}\frac{\tau_{j+1}}{2}\Big[B(t_{n-1/2},t_j)\nabla U_h^j + B(t_{n-1/2},t_{j+1})\nabla U_h^{j+1}\Big],\nabla\phi_h\rangle\\ &+\langle\frac{\tau_n}{4}\Big[B(t_{n-1/2},t_{n-1})\nabla U_h^{n-1} + B(t_{n-1/2},t_{n-1/2})\nabla U_h^{n-1/2}\Big],\nabla\phi_h\rangle.\end{aligned}$$

Here, the trapezoidal rule is used to discretize the integral term in order to be consistent with the Crank-Nicolson scheme and I_h denotes the Lagrange's interpolant corresponding to $\mathbb{V}_h^{A,0}$.

In the absence of the Volterra integral term (when $\mathcal{B}(t,s)=0$), *a posteriori* error estimates concerning the Crank-Nicolson method for the parabolic problems are thoroughly studied in Akrivis *et al.* [5], Bänsch *et al.* [12], Lozinski *et al.* [60], Picasso and Prachittham [79], Verfürth [98]. For a continuous, piecewise linear approximation in time, Verfürth [98] has derived suboptimal (with respect to time steps) error bound for the heat equation in the $L^2(H^1(\Omega))$-norm using the standard energy techniques. Subsequently, a continuous, piecewise quadratic polynomial function in time so-called Crank-Nicolson reconstruction has been introduced by Akrivis *et al.* [5] to restore the appropriate second order of convergence for the semidiscrete time discretization of a general parabolic problem. The results of Akrivis *et al.* [5] are then extended by Lozinski *et al.* [60] to the fully discrete Crank-Nicolson method by introducing the quadratic reconstructions based on approximations on one time level (two-point reconstruction) and two time levels (three-point reconstruction). In this chapter, we extend the anisotropic *a posteriori* error analysis for parabolic problems of Lozinski *et al.* [60] to PIDE (6.1).

We organize the chapter as follows. Section 6.2 is devoted to quadratic reconstructions for PIDE. Further, *a posteriori* error estimators in the $L^2(H^1(\Omega))$-norm are derived for the Crank-Nicolson method in Section 6.3.

6.2 Quadratic reconstructions for PIDE

Let U_h be a continuous, piecewise linear approximation in time defined for all $t \in I_n$ by

$$\begin{aligned} U_h(x,t) &:= l_n(t)U_h^n + l_{n-1}(t)U_h^{n-1} \\ &:= U_h^{n-1/2} + (t - t_{n-1/2})\partial U_h^n, \ \ 1 \le n \le N, \end{aligned} \tag{6.11}$$

where

$$l_n(t) := \frac{(t-t_{n-1})}{\tau_n}, \qquad l_{n-1}(t) := \frac{(t_n - t)}{\tau_n}. \tag{6.12}$$

We now introduce quadratic time reconstructions which play an instrumental role in deriving *a posteriori* error bounds with second order accuracy in time for the Crank-Nicolson scheme.

Crank-Nicolson memory reconstruction. Let $\check{U}_h$ be a continuous, piecewise quadratic memory dependent reconstruction in time defined for all $t \in I_n$, $1 \le n \le N$ by

$$\begin{aligned} \check{U}_h(x,t) &:= U_h(x,t) + \frac{1}{2}(t-t_{n-1})(t-t_n)\check{w}_h^n, \\ &:= U_h^{n-1/2} + (t - t_{n-1/2})\partial U_h^n + \frac{1}{2}(t-t_{n-1})(t-t_n)\check{w}_h^n, \end{aligned} \tag{6.13}$$

where $\check{w}_h^n \in \mathbb{V}_h^{A,0}$ is defined by

$$\begin{aligned} \langle \check{w}_h^n, \phi_h \rangle &:= \langle \partial f^n, \phi_h \rangle - a(\partial U_h^n, \phi_h) + \int_0^{t_n} \partial b(t_n, s; U_h(s), \phi_h) ds \\ &\quad + b(t_n, t_n; U_h^n, \phi_h). \end{aligned} \tag{6.14}$$

Note that the quadratic time approximation $\check{U}_h$ is a generalization to the two point reconstruction introduced by Lozinski *et al.* [60]. For $t \in I_n$, $1 \leq n \leq N$, we note that

$$\frac{\partial \check{U}_h(x,t)}{\partial t} = \partial U_h^n + (t - t_{n-1/2})\check{w}_h^n, \tag{6.15}$$

and

$$\frac{\partial^2 \check{U}_h(x,t)}{\partial t^2} = \check{w}_h^n. \tag{6.16}$$

The following definition will be useful in estimating the quadrature error in approximating the Volterra integral term. Set

$$\check{U}_{h,1}(t) := \int_0^t B(t,s)\nabla \check{U}_h(s)ds. \tag{6.17}$$

For $t \in I_n$, $1 \leq n \leq N$, we define $\check{U}_{h,I,1}(t)$ to be the linear interpolant associated with the integral vector $\check{U}_{h,1}(t_{n-1/2})$, and is given by

$$\check{U}_{h,I,1}(t) \;\; := \;\; \check{U}_{h,1}(t_{n-1/2}) + (t - t_{n-1/2})\frac{d}{dt}\check{U}_{h,1}(t)\big|_{t=t_n}. \tag{6.18}$$

In order to handle the time reconstruction error, we introduce the following definition. Set

$$\check{U}_{h,2}(t) := \int_0^t B(t,s)(\nabla U_h(s) - \nabla \check{U}_h(s))ds. \tag{6.19}$$

For $t \in I_n$, we define $\check{U}_{h,I,2}(t)$ to be the linear interpolant associated with the integral vectors $\check{U}_{h,2}(t_n)$ and $\check{U}_{h,2}(t_{n-1})$, and is given by

$$\check{U}_{h,I,2}(t) := l_n(t)\check{U}_{h,2}(t_n) + l_{n-1}(t)\check{U}_{h,2}(t_{n-1}), \;\; 1 \leq n \leq N, \tag{6.20}$$

where $l_n(t)$ and $l_{n-1}(t)$ are given by (6.12).
In the quadratic reconstruction (6.13), we observe that the integral term
$\int_0^{t_n} \partial B(t_n,s)\nabla U_h(s)ds$ appears due to the application of the Leibnitz formula on the memory term. Further, we notice that, by applying any basic quadrature approximation to the memory term which is consistent with the Crank-Nicolson scheme, the quadratic reconstruction needs to be evaluated at all the previous time levels and thus, it is not

locally defined in time. Therefore, in addition to the above memory reconstruction, we define a local reconstruction by considering an analogue of it.

Three-point reconstruction. Let $\hat{U}_h$ be a three point continuous, piecewise quadratic reconstruction to (6.13) in time defined for all $t \in I_n$, $2 \leq n \leq N$ by

$$\begin{aligned} \hat{U}_h(x,t) &:= U_h(x,t) + \frac{1}{2}(t-t_{n-1})(t-t_n)\hat{w}_h^n, \\ &:= U_h^{n-1/2} + (t-t_{n-1/2})\partial U_h^n + \frac{1}{2}(t-t_{n-1})(t-t_n)\hat{w}_h^n, \end{aligned} \tag{6.21}$$

where

$$\hat{w}_h^n := \partial^2 U_h^n \quad := \quad \frac{2(\partial U_h^n - \partial U_h^{n-1})}{(\tau_n + \tau_{n-1})}.$$

Here, U_h, $\check{U}_h$ and $\hat{U}_h$ coincide at $t_1, t_2, \cdots, t_N$ and this property will later play a crucial role in obtaining *a posteriori* estimate for quadrature approximation of the Volterra integral term. An argument similar to [60] (see Remark 4.2) ensures that $\hat{U}_h$ vanishes on the boundary so that $\hat{U}_h \in \mathbb{V}_h^{A,0}$.

For $t \in I_n$, $2 \leq n \leq N$, we observe that

$$\frac{\partial \hat{U}_h(x,t)}{\partial t} = \partial U_h^n + (t - t_{n-1/2})\partial^2 U_h^n, \tag{6.22}$$

and

$$\frac{\partial^2 \hat{U}_h(x,t)}{\partial t^2} = \partial^2 U_h^n. \tag{6.23}$$

In order to account for the time discretization error due to quadrature approximation of the Volterra integral term in the context of three point reconstruction, set

$$\hat{U}_{h,1}(t) := \int_0^t B(t,s)\nabla \hat{U}_h(s)ds. \tag{6.24}$$

For $t \in I_n$, $2 \leq n \leq N$, define $\hat{U}_{h,I,1}(t)$ to be the extended piecewise linear interpolant associated with the integral vectors $\hat{U}_{h,1}(t_{n-1/2})$ and $\hat{U}_{h,1}(t_{n-3/2})$, and is given by

$$\hat{U}_{h,I,1}(t)(t) = l_{n-1/2}(t)\hat{U}_{h,1}(t_{n-1/2}) + l_{n-3/2}(t)\hat{U}_{h,1}(t_{n-3/2}), \tag{6.25}$$

where

$$l_{n-1/2}(t) = 1 - \frac{2(t_{n-1/2} - t)}{(\tau_{n-1} + \tau_n)} \quad \text{and} \quad l_{n-3/2}(t) = \frac{2(t_{n-1/2} - t)}{(\tau_{n-1} + \tau_n)}. \tag{6.26}$$

Set

$$\hat{U}_{h,2}(t) := \int_0^t B(t,s)(\nabla U_h(s) - \nabla \hat{U}_h(s))ds. \tag{6.27}$$

For $t \in I_n$, we define $\hat{U}_{h,I,2}(t)$ to be the extended linear interpolant associated with the integral vectors $\hat{U}_{h,2}(t_{n-1/2})$ and $\hat{U}_{h,2}(t_{n-3/2})$, and is given by

$$\hat{U}_{h,I,2}(t) := l_{n-1/2}(t)\hat{U}_{h,2}(t_{n-1/2}) + l_{n-3/2}(t)\hat{U}_{h,2}(t_{n-3/2}), \;\; 2 \le n \le N, \tag{6.28}$$

where $l_{n-1/2}(t)$ and $l_{n-3/2}(t)$ are given by (6.26).

6.3 Error analysis

In this section, we derive optimal order *a posteriori* upper bounds for the error $e := u - U_h$ in the $L^2(H^1(\Omega))$-norm, where u satisfies (6.1) and U_h is defined by (6.11), respectively. We used the standard energy argument in the error analysis. First, we relate the error to the equation residual and introduce Clément interpolant. Then by localizing the residual term over each of the elements and the edges of the triangulation, we use the anisotropic interpolation error estimates. In the first estimator (see, Theorem 6.3.1), a linear approximation of the Volterra integral term is used in a crucial way to estimate the quadrature error for the approximation of the memory term. Moreover, an extended linear approximation of the Volterra integral term is used to estimate the error due to the quadrature approximation of the memory term in the the later estimator (see Theorem 6.3.2).

For $t \in I_n$, set

$$\check{f}(\cdot,t) = l_{n-1}(t)f^{n-1} + l_n(t)f^n, \;\; 1 \le n \le N, \tag{6.29}$$

$$\hat{f}(\cdot,t) = f^{n-1/2} + (t - t_{n-1/2})\left(\frac{f^n - f^{n-2}}{\tau_{n-1} + \tau_n}\right), \;\; 2 \le n \le N. \tag{6.30}$$

Set $\check{e} := u - \check{U}_h$ and $\hat{e} := u - \hat{U}_h$. We now state the main results of this section in the following theorems.

Theorem 6.3.1. *Suppose that the mesh satisfies that there exists a constant c independent of the time step, meshsize and aspect ratio such that*

$$\lambda_{1,K}^2(r_{1,K}^T G_K(\check{e})r_{1,K}) \le c\lambda_{2,K}^2(r_{2,K}^T G_K(\check{e})r_{2,K}), \quad \forall K \in \mathcal{T}_h^A. \tag{6.31}$$

For any $1 \le n \le N$, $t \in I_n$, let $\check{U}_{h,1}(t)$, $\check{U}_{h,I,1}(t)$, $\check{U}_{h,2}(t)$ and $\check{U}_{h,I,2}(t)$ be given by (6.17), (6.18), (6.19) and (6.20) respectively. Then there exists a constant C depending on the interpolation constants of Proposition 5.2.1 (hence independent of the time step, meshsize and aspect ratio) and the final time T such that the following a posteriori error bound holds:

$$\begin{aligned}
&\|e(\cdot,T)\|^2 + \alpha\int_0^T \|\nabla e\|^2 dt \le \|e(\cdot,0)\|^2 \\
&+C\sum_{n=1}^N \sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f-\partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot \boldsymbol{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot \boldsymbol{n} ds]\|_{L^2(\partial K)}\Bigg)\omega_K(\check{e})dt \\
&+\tau_n^3\lambda_{2,K}^2\|\breve{w}_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n}\|f-\check{f}\|_{L^2(K)}^2 dt + \tau_n(\check{\mathcal{Q}}_{CN,n,K}^A)^2 + \tau_n^3\breve{\eta}_{n,K}^2 \\
&+\tau_n^5\|\nabla\breve{w}_h^n\|_{L^2(K)}^2 + \tau_n\Big[\breve{\eta}_{n,K}^2 + \breve{\eta}_{n-1,K}^2 + \|\check{U}_{h,1}(t)-\check{U}_{h,I,1}(t)\|_{L^2(K)}^2 \\
&+\|\check{U}_{h,2}(t)-\check{U}_{h,I,2}(t)\|_{L^2(K)}^2\Big]\Bigg],
\end{aligned}$$

where $\check{\mathcal{Q}}_{CN,n,K}^A$ and $\breve{\eta}_{n,K}$ are defined by

$$\begin{aligned}
\check{\mathcal{Q}}_{CN,n,K}^A &= \sum_{j=1}^n \tau_j^3\Big[\|\nabla U_h^j\|_{L^2(K)} + \|\nabla U_h^{j-1}\|_{L^2(K)} + \tau_j^2\|\nabla\breve{w}_h^j\|_{L^2(K)}\Big] \\
&+\sum_{j=1}^n \tau_j^3\Big[\|\nabla\partial U_h^j\|_{L^2(K)} + \tau_j\|\nabla\breve{w}_h^j\|_{L^2(K)}\Big] + \sum_{j=1}^n \tau_j^3\|\nabla\breve{w}_h^j\|_{L^2(K)}
\end{aligned} \tag{6.32}$$

and

$$\breve{\eta}_{n,K} := \sum_{j=1}^n \tau_j^3\|\nabla\breve{w}_h^j\|_{L^2(K)} \tag{6.33}$$

with $\breve{w}_h^n$ and $\check{f}$ (the linear interpolant of f) are given by (6.14) and (6.29), respectively. Here $[\cdot]$ denotes the jump of the bracketed quantity across an internal edge, $[\cdot]=0$ for an edge on the boundary $\partial\Omega$ and $\boldsymbol{n}$ is the unit edge normal.

Theorem 6.3.2. *Suppose that the mesh satiesfies that there exists a constant c independent of the time step, meshsize and aspect ratio such that*

$$\lambda_{1,K}^2(r_{1,K}^T G_K(\hat{e}) r_{1,K}) \le c\lambda_{2,K}^2(r_{2,K}^T G_K(\hat{e}) r_{2,K}), \quad \forall K \in \mathcal{T}_h^A. \tag{6.34}$$

For any $2 \le n \le N$ and $t \in I_n$, let $\hat{U}_{h,1}(t)$, $\hat{U}_{h,I,1}(t)(t)$, $\hat{U}_{h,2}(t)$ and $\hat{U}_{h,I,2}(t)$ be given by (6.24), (6.25), (6.27) and (6.28) respectively. Then there exists a constant C depending on the interpolation constants of Proposition 5.2.1 (hence independent of the time step, meshsize and aspect ratio) and the final time T such that the following a posteriori error bound holds:

$$\begin{aligned}
&\|e(\cdot,T)\|^2 + \alpha\int_{t_1}^{T}\|\nabla e\|^2 dt \le \|e(\cdot,t_1)\|^2 \\
&+C\sum_{n=2}^{N}\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f-\partial U_h^n+\nabla\cdot(A\nabla U_h)-\int_0^t\nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\boldsymbol{n}]\|_{L^2(\partial K)}+\frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\boldsymbol{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(\hat{e})dt \\
&+\tau_n^3\lambda_{2,K}^2\|\partial^2 U_h^n\|_{L^2(K)}^2+\int_{t_{n-1}}^{t_n}\|f-\hat{f}\|_{L^2(K)}^2 dt \\
&+\tau_n\Big[(\hat{\mathcal{Q}}_{CN,n,K}^A)^2+(\hat{\mathcal{Q}}_{CN,n-1,K}^A)^2+\|\hat{U}_{h,1}(t)-\hat{U}_{h,I,1}(t)(t)\|_{L^2(K)}^2\Big] \\
&+\Big\{\tau_{n-1}^2\tau_n^3+\tau_n^5\Big\}\|\nabla\partial^2 U_h^n\|_{L^2(K)}^2+\tau_n\Big[\eta_{n,K}^2+\eta_{n-1,K}^2+\|\hat{U}_{h,2}(t)-\hat{U}_{h,I,2}(t)\|_{L^2(K)}^2\Big]\Bigg],
\end{aligned}$$

where $\hat{\mathcal{Q}}_{CN,n,K}^A$, $\eta_{n,K}$ are defined by

$$\begin{aligned}
\hat{\mathcal{Q}}_{CN,n,K}^A &= \sum_{j=1}^{n}\tau_j^3\Big[\|\nabla U_h^j\|_{L^2(K)}+\|\nabla U_h^{j-1}\|_{L^2(K)}+\tau_j^2\|\nabla\partial^2 U_h^j\|_{L^2(K)}\Big] \\
&+\sum_{j=1}^{n}\tau_j^3\Big[\|\nabla\partial U_h^j\|_{L^2(K)}+\tau_j\|\nabla\partial^2 U_h^j\|_{L^2(K)}\Big]+\sum_{j=1}^{n}\tau_j^3\|\nabla\partial^2 U_h^j\|_{L^2(K)},
\end{aligned} \tag{6.35}$$

$$\eta_{n,K} := \sum_{j=1}^{n}\tau_j^3\|\nabla\partial^2 U_h^j\|_{L^2(K)} \tag{6.36}$$

and $\hat{f}$ is given by (6.30). Here $[\cdot]$ denotes the jump of the bracketed quantity across an internal edge, $[\cdot]=0$ for an edge on the boundary $\partial\Omega$ and $\boldsymbol{n}$ is the unit edge normal.

The proofs of Theorem 6.3.1 and Theorem 6.3.2 requires some preparations. We shall first proceed to prove Theorem 6.3.2. For this purpose, we need to prove a sequence of lemmas.

Lemma 6.3.1. *Let U_h and $\hat{U}_h$ be as defined by (6.11) and (6.21), respectively. Then, for any $2 \le n \le N$, $t \in I_n$ and for all $\phi_h \in \mathbb{V}_h^{A,0}$, we have*

$$\int_\Omega \frac{\partial \hat{U}_h}{\partial t}\phi_h dx + a(U_h, \phi_h) = \int_\Omega \hat{f}\phi_h dx + \frac{\tau_{n-1}(t - t^{n-1/2})}{2} a(\partial^2 U_h^n, \phi_h)$$
$$+ l_{n-1/2}(t)\sigma^n(b(t_{n-1/2}; U_h, \phi_h)) + l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2}; U_h, \phi_h)),$$

where $\hat{f}$ is given by (6.30), $l_{n-1/2}(t)$ and $l_{n-3/2}(t)$ are given by (6.26).

Proof. Let $t \in I_n$ with $2 \le n \le N$. For all $\phi_h \in \mathbb{V}_h^{A,0}$, we use (6.11) and (6.10) to have

$$\int_\Omega \partial U_h^n \phi_h dx + a(U_h, \phi_h) = \sigma^n(b(t_{n-1/2}; U_h, \phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx$$
$$+ (t - t_{n-1/2}) a(\partial U_h^n, \phi_h). \tag{6.37}$$

Using (6.22), we rewrite (6.37) as

$$\int_\Omega \frac{\partial \hat{U}_h}{\partial t}\phi_h dx + a(U_h, \phi_h) = \sigma^n(b(t_{n-1/2}; U_h, \phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx$$
$$+ (t - t_{n-1/2})\left(a(\partial U_h^n, \phi_h) + \int_\Omega \partial^2 U_h^n \phi_h dx \right). \tag{6.38}$$

With $t = t_{n-1}$, (6.10) becomes

$$\int_\Omega \partial U_h^{n-1}\phi_h dx + a(U_h^{n-3/2}, \phi_h) = \sigma^{n-1}(b(t_{n-3/2}; U_h, \phi_h)) + \int_\Omega f^{n-3/2}\phi_h dx. \tag{6.39}$$

A little simplification after subtracting (6.39) from (6.10) and then multiplying both sides by the term $2/(\tau_n + \tau_{n-1})$ we obtain

$$\int_\Omega \partial^2 U_h^n \phi_h dx \; + \; a\left(\frac{U_h^n - U_h^{n-2}}{\tau_{n-1} + \tau_n}, \phi_h \right) = \int_\Omega \frac{(f^n - f^{n-2})}{(\tau_{n-1} + \tau_n)}\phi_h dx$$
$$+ \; \frac{2}{(\tau_{n-1} + \tau_n)}\Big[\sigma^n(b(t_{n-1/2}; U_h, \phi_h)) - \sigma^{n-1}(b(t_{n-3/2}; U_h, \phi_h)) \Big]. \tag{6.40}$$

Noting the fact

$$\frac{U_h^n - U_h^{n-2}}{\tau_n + \tau_{n-1}} = \partial U_h^n - \frac{\tau_{n-1}}{2}\partial^2 U_h^n,$$

we have from (6.40)

$$\int_\Omega \partial^2 U_h^n \phi_h dx + a(\partial U_h^n, \phi_h) = \int_\Omega \frac{(f^n - f^{n-2})}{(\tau_{n-1}+\tau_n)}\phi_h dx + \frac{\tau_{n-1}}{2} a(\partial^2 U_h^n, \phi_h)$$
$$+\frac{2}{(\tau_{n-1}+\tau_n)}\Big[\sigma^n(b(t_{n-1/2}; U_h, \phi_h)) - \sigma^{n-1}(b(t_{n-3/2}; U_h, \phi_h))\Big]. \quad (6.41)$$

Substituting (6.41) on the right of (6.38), we complete the rest of the proof. □

The following lemma yields a bound on the quadrature error.

Lemma 6.3.2 (Quadrature error estimate)**.** *Let $\hat{U}_h$ and $\hat{\mathcal{Q}}^A_{CN,n,K}$ be as defined by (6.21) and (6.35), respectively. Moreover, let $\hat{U}_{h,1}(t)$ and $\hat{U}_{h,I,1}(t)(t)$ be given by (6.24) and (6.25), respectively. Then for any $\phi_h \in \mathbb{V}_h^{A,0}$ and $t \in I_n$ with $2 \le n \le N$, the following bound holds:*

$$\begin{aligned}
|\int_0^t b(t,s;\hat{U}_h(s),\phi_h)ds \;&-\; l_{n-1/2}(t)\sigma^n(b(t_{n-1/2};\hat{U}_h,\phi_h)) \\
&\quad -l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2};\hat{U}_h,\phi_h))| \\
&\le \bar{\gamma}\sum_{K\in\mathcal{T}_h^A}\Big[|l_{n-1/2}(t)|\hat{\mathcal{Q}}^A_{CN,n,K} + |l_{n-3/2}(t)|\hat{\mathcal{Q}}^A_{CN,n-1,K} \\
&\quad +\|\hat{U}_{h,1}(t) - \hat{U}_{h,I,1}(t)(t)\|_{L^2(K)}\Big]\|\nabla\phi_h\|_{L^2(K)},
\end{aligned}$$

where $\bar{\gamma} = \max\left\{\frac{\gamma''}{4}, \gamma', \frac{\gamma}{2}, 1\right\}$.

Proof. We choose any $2 \le n \le N$ and $t \in I_n$. Taking L^2 inner product with $\nabla\phi_h$ on both sides of (6.25), we have

$$\langle \hat{U}_{h,I,1}(t)(t), \nabla\phi_h\rangle = l_{n-1/2}(t)\int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h)ds$$
$$+l_{n-3/2}(t)\int_0^{t_{n-3/2}} b(t_{n-3/2}, s; \hat{U}_h(s), \phi_h)ds, \quad \forall \phi_h \in \mathbb{V}_h^{A,0}. \quad (6.42)$$

Since $l_{n-1/2}(t) + l_{n-3/2}(t) \equiv 1,\ t \in I_n$, it follows that

$$\int_0^t b(t,s;\hat{U}_h(s),\phi_h)ds - l_{n-1/2}(t)\sigma^n(b(t_{n-1/2};\hat{U}_h,\phi_h))$$
$$-l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2};\hat{U}_h,\phi_h))$$

$$
\begin{aligned}
&= l_{n-1/2}(t)\Big[\int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h)ds - \sigma^n(b(t_{n-1/2}; \hat{U}_h, \phi_h))\Big] \\
&\quad + l_{n-3/2}(t)\Big[\int_0^{t_{n-3/2}} b(t_{n-3/2}, s; \hat{U}_h(s), \phi_h)ds - \sigma^{n-1}(b(t_{n-3/2}; \hat{U}_h, \phi_h))\Big] \\
&\quad + \langle \hat{U}_{h,1}(t) - \hat{U}_{h,I,1}(t)(t), \nabla\phi_h\rangle \\
&:= l_{n-1/2}(t)\mathcal{I}_1 + l_{n-3/2}(t)\mathcal{I}_2 + \mathcal{I}_3. \qquad (6.43)
\end{aligned}
$$

To estimate the quadrature error for approximating the Volterra integral term, we first need to estimate $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$. A standard Trapezoidal rule argument for a sufficiently smooth function $g(s)$ yields

$$
\int_a^b g(s)ds - \frac{(b-a)}{2}(g(a)+g(b)) \;=\; \frac{1}{2}\int_a^b (s-a)(s-b)g''(s)ds.
$$

If we define

$$
\psi_{2j}(s) := \begin{cases} (s-t_{j-1})(s-t_j) & \text{for } s \in [t_{j-1}, t_j] \text{ and } 1 \le j \le n-1, \\ (s-t_{j-1})(s-t_{j-1/2}) & \text{for } s \in [t_{j-1}, t_{j-1/2}] \text{ and } j = n, \end{cases}
$$

then

$$
\int_{t_{j-1}}^{t_j} g(s)ds - \frac{\tau_j}{2}[g(t_j) + g(t_{j-1})] = \frac{1}{2}\int_{t_{j-1}}^{t_j} \psi_{2j}(s)g''(s)ds, \qquad (6.44)
$$

and

$$
\int_{t_{n-1}}^{t_{n-1/2}} g(s)ds - \frac{\tau_n}{4}[g(t_{n-1}) + g(t_{n-1/2})] = \frac{1}{2}\int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s)g''(s)ds. \qquad (6.45)
$$

Using (6.44) and (6.45), we obtain

$$
\begin{aligned}
\mathcal{I}_1 &:= \int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h)ds - \sigma^n(b(t_{n-1/2}; \hat{U}_h, \phi_h)) \\
&= \langle \int_0^{t_{n-1/2}} B(t_{n-1/2}, s)\nabla\hat{U}_h(s)ds, \nabla\phi_h\rangle \\
&\quad - \langle \sum_{j=0}^{n-2} \frac{\tau_{j+1}}{2}\Big[B(t_{n-1/2}, t_j)\nabla U_h^j + B(t_{n-1/2}, t_{j+1})\nabla U_h^{j+1}\Big], \nabla\phi_h\rangle \\
&\quad - \langle \frac{\tau_n}{4}\Big[B(t_{n-1/2}, t_{n-1})\nabla U_h^{n-1} + B(t_{n-1/2}, t_{n-1/2})\nabla U_h^{n-1/2}\Big], \nabla\phi_h\rangle \\
&= \frac{1}{2}\langle \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \psi_{2j}(s)\frac{d^2}{ds^2}\{B(t_{n-1/2}, s)\nabla\hat{U}_h(s)\}ds, \nabla\phi_h\rangle \\
&\quad + \frac{1}{2}\langle \int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s)\frac{d^2}{ds^2}\{B(t_{n-1/2}, s)\nabla\hat{U}_h(s)\}ds, \nabla\phi_h\rangle.
\end{aligned}
$$

Thus, over each $K \in \mathcal{T}_h^A$, we have

$$
\begin{aligned}
\mathcal{I}_1 &:= \int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h) ds - \sigma^n(b(t_{n-1/2}; \hat{U}_h, \phi_h)) \\
&= \frac{1}{2} \sum_{K \in \mathcal{T}_h^A} \Bigg[\langle \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \psi_{2j}(s) \Bigg\{ \frac{d^2(B(t_{n-1/2}, s))}{ds^2} \nabla \hat{U}_h(s) \\
&\quad + 2 \frac{d(B(t_{n-1/2}, s))}{ds} \frac{d(\nabla \hat{U}_h(s))}{ds} + B(t_{n-1/2}, s) \frac{d^2(\nabla \hat{U}_h(s))}{ds^2} \Bigg\} ds, \nabla \phi_h \rangle_K \\
&\quad + \langle \int_{t_{n-1}}^{t_{n-1/2}} \psi_{2n}(s) \Bigg\{ \frac{d^2(B(t_{n-1/2}, s))}{ds^2} \nabla \hat{U}_h(s) + 2 \frac{d(B(t_{n-1/2}, s))}{ds} \frac{d(\nabla \hat{U}_h(s))}{ds} \\
&\quad + B(t_{n-1/2}, s) \frac{d^2(\nabla \hat{U}_h(s))}{ds^2} \Bigg\} ds, \nabla \phi_h \rangle_K \Bigg].
\end{aligned}
$$

Using (6.21) together with (6.5), (6.6) and (6.7), we obtain

$$
\begin{aligned}
|\mathcal{I}_1| &:= \Big| \int_0^{t_{n-1/2}} b(t_{n-1/2}, s; \hat{U}_h(s), \phi_h) ds - \sigma^n(b(t_{n-1/2}; \hat{U}_h, \phi_h)) \Big| \\
&\leq \frac{1}{2} \sum_{K \in \mathcal{T}_h^A} \Bigg\{ \Bigg(\sum_{j=1}^{n-1} \tau_j^2 \gamma'' \Bigg[\frac{\tau_j}{2} \|\nabla U_h^j\|_{L^2(K)} + \frac{\tau_j}{2} \|\nabla U_h^{j-1}\|_{L^2(K)} \\
&\quad + \frac{\tau_j^3}{12} \|\nabla \partial^2 U_h^j\|_{L^2(K)} \Bigg] + \sum_{j=1}^{n-1} \tau_j^2 \gamma' \Bigg[2\tau_j \|\nabla \partial U_h^j\|_{L^2(K)} + \frac{\tau_j^2}{2} \|\nabla \partial^2 U_h^j\|_{L^2(K)} \Bigg] \\
&\quad + \sum_{j=1}^{n-1} \tau_j^3 \gamma \|\nabla \partial^2 U_h^j\|_{L^2(K)} + \tau_n^2 \gamma'' \Bigg[\frac{3\tau_n}{8} \|\nabla U_h^n\|_{L^2(K)} + \frac{\tau_n}{8} \|\nabla U_h^{n-1}\|_{L^2(K)} \\
&\quad + \frac{\tau_n^3}{24} \|\nabla \partial^2 U_h^n\|_{L^2(K)} \Bigg] + \tau_n^2 \gamma' \Bigg[\tau_n \|\nabla \partial U_h^n\|_{L^2(K)} + \frac{\tau_n^2}{4} \|\nabla \partial^2 U_h^n\|_{L^2(K)} \Bigg] \\
&\quad + \frac{\tau_n^3}{2} \gamma \|\nabla \partial^2 U_h^n\|_{L^2(K)} \Bigg) \|\nabla \phi_h\|_{L^2(K)} \Bigg\} \\
&\leq \bar{\gamma}_1 \sum_{K \in \mathcal{T}_h^A} \hat{\mathcal{Q}}_{CN,n,K}^A \|\nabla \phi_h\|_{L^2(K)}, \qquad (6.46)
\end{aligned}
$$

where $\bar{\gamma}_1 = \max\left\{\frac{\gamma''}{4}, \gamma', \frac{\gamma}{2}\right\}$ and $\hat{\mathcal{Q}}_{CN,n,K}^A$ is given by (6.35).
Similarly,

$$
\begin{aligned}
|\mathcal{I}_2| &:= \Big| \int_0^{t_{n-3/2}} b(t_{n-3/2}, s; \hat{U}_h(s), \phi_h) ds - \sigma^n(b(t_{n-3/2}; \hat{U}_h, \phi_h)) \Big| \\
&\leq \bar{\gamma}_1 \sum_{K \in \mathcal{T}_h^A} \hat{\mathcal{Q}}_{CN,n-1,K}^A \|\nabla \phi_h\|_{L^2(K)}. \qquad (6.47)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|\mathcal{I}_3| &:= |\langle \hat{U}_{h,1}(t) - \hat{U}_{h,I,1}(t)(t), \nabla\phi_h\rangle| \\
&\leq \sum_{K\in\mathcal{T}_h^A} \|\hat{U}_{h,1}(t) - \hat{U}_{h,I,1}(t)(t)\|_{L^2(K)}\|\nabla\phi_h\|_{L^2(K)}. \qquad (6.48)
\end{aligned}
$$

Putting the estimates (6.46)-(6.48) in (6.43), we obtain the desired result and this completes the proof. □

The following lemma reveals the contributions for the time reconstruction error.

Lemma 6.3.3 (Time reconstruction error estimate)**.** *Let U_h and $\hat{U}_h$ be as defined by (6.11) and (6.21), respectively. Let $\eta_{n,K}$ be given by (6.36). Further, let $\hat{U}_{h,2}(t)$ and $\hat{U}_{h,I,2}(t)$ be given by (6.27) and (6.28), respectively. Then for any $\phi_h \in \mathbb{V}_h^{A,0}$ and $t \in I_n$ with $2 \leq n \leq N$, we have*

$$
\begin{aligned}
&|\int_0^t b(t,s;U_h(s) - \hat{U}_h(s),\phi_h)ds| \\
&\leq \sum_{K\in\mathcal{T}_h^A} \Bigg[\frac{\gamma}{12}\Big\{|l_{n-1/2}(t)|\eta_{n,K}\|\nabla\phi_h\|_{L^2(K)} + |l_{n-3/2}(t)|\eta_{n-1,K}\|\nabla\phi_h\|_{L^2(K)}\Big\} \\
&+\|\hat{U}_{h,2}(t) - \hat{U}_{h,I,2}(t)\|_{L^2(K)}\|\nabla\phi_h\|_{L^2(K)}\Bigg].
\end{aligned}
$$

Proof. We choose any $2 \leq n \leq N$ and $t \in I_n$. Taking L^2 inner product with $\nabla\phi_h$ on both sides of (6.27) and using (6.28), we obtain for all $\phi_h \in \mathbb{V}_h^{A,0}$

$$
\begin{aligned}
|\int_0^t b(t,s;U_h(s) &- \hat{U}_h(s),\phi_h)ds| \\
&\leq |l_{n-1/2}(t)|\int_0^{t_{n-1/2}} |b(t_{n-1/2},s;U_h(s) - \hat{U}_h(s),\phi_h)|ds \\
&\quad+|l_{n-3/2}(t)|\int_0^{t_{n-3/2}} |b(t_{n-3/2},s;U_h(s) - \hat{U}_h(s),\phi_h)|ds \\
&\quad+|\langle \hat{U}_{h,2}(t) - \hat{U}_{h,I,2}(t), \nabla\phi_h\rangle|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{2}|l_{n-1/2}(t)|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}|b(t_{n-1/2},s;(s-t_{j-1})(s-t_j)\partial^2 U_h^j,\phi_h)|ds \\
&\quad +\frac{1}{2}|l_{n-1/2}(t)|\int_{t_{n-1}}^{t_{n-1/2}}|b(t_{n-1/2},s;(s-t_{n-1})(s-t_n)\partial^2 U_h^n,\phi_h)|ds \\
&\quad +\frac{1}{2}|l_{n-3/2}(t)|\sum_{j=1}^{n-2}\int_{t_{j-1}}^{t_j}|b(t_{n-3/2},s;(s-t_{j-1})(s-t_j)\partial^2 U_h^j,\phi_h)|ds \\
&\quad +\frac{1}{2}|l_{n-3/2}(t)|\int_{t_{n-2}}^{t_{n-3/2}}|b(t_{n-3/2},s;(s-t_{n-2})(s-t_{n-1})\partial^2 U_h^n,\phi_h)|ds \\
&\quad +|\langle \hat{U}_{h,2}(t)-\hat{U}_{h,I,2}(t),\nabla\phi_h\rangle| \\
&:= |\mathcal{J}_1|+|\mathcal{J}_2|+|\mathcal{J}_3|+|\mathcal{J}_4|+|\mathcal{J}_5|. \qquad (6.49)
\end{aligned}$$

To estimate $|\mathcal{J}_1|$ over each element $K\in\mathcal{T}_h^A$, we use (6.5) to obtain

$$\begin{aligned}
|\mathcal{J}_1| &:= \frac{1}{2}|l_{n-1/2}(t)|\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}|b(t_{n-1/2},s;(s-t_j)(s-t_{j+1})\partial^2 U_h^j,\phi_h)|ds \\
&\leq \frac{\gamma|l_{n-1/2}(t)|}{12}\sum_{K\in\mathcal{T}_h^A}\sum_{j=1}^{n-1}\tau_j^3\|\nabla\partial^2 U_h^j\|_{L^2(K)}\|\nabla\phi_h\|_{L^2(K)}.
\end{aligned}$$

Similarly,

$$|\mathcal{J}_2| \leq \frac{\gamma|l_{n-1/2}(t)|}{24}\sum_{K\in\mathcal{T}_h^A}\tau_n^3\|\nabla\partial^2 U_h^n\|_{L^2(K)}\|\nabla\phi_h\|_{L^2(K)},$$

$$|\mathcal{J}_3| \leq \frac{\gamma|l_{n-3/2}(t)|}{12}\sum_{K\in\mathcal{T}_h^A}\sum_{j=1}^{n-2}\tau_j^3\|\nabla\partial^2 U_h^j\|_{L^2(K)}\|\nabla\phi_h\|_{L^2(K)},$$

$$|\mathcal{J}_4| \leq \frac{\gamma|l_{n-3/2}(t)|}{24}\sum_{K\in\mathcal{T}_h^A}\tau_{n-1}^3\|\nabla\partial^2 U_h^n\|_{L^2(K)}\|\nabla\phi_h\|_{L^2(K)}$$

and

$$|\mathcal{J}_5| \leq \sum_{K\in\mathcal{T}_h^A}\|\hat{U}_{h,2}(t)-\hat{U}_{h,I,2}(t)\|_{L^2(K)}\|\nabla\phi_h\|_{L^2(K)}.$$

We now combine these estimates with (6.49) to complete the proof. □

Proof of Theorem 6.3.2. Using the weak formulation (6.9), we have for any $2\leq n\leq N$ and $t\in I_n$

$$\begin{aligned}
&\int_\Omega \frac{\partial\hat{e}}{\partial t}\phi dx + a(e,\phi) - \int_0^t b(t,s;e(s),\phi)ds \\
&= \int_\Omega f\phi dx - \int_\Omega \frac{\partial \hat{U}_h}{\partial t}\phi dx - a(U_h,\phi) + \int_0^t b(t,s;U_h(s),\phi)ds, \quad \forall\phi\in H_0^1(\Omega).
\end{aligned}$$

Choosing $\phi = \hat{e}$ in the above error equation and using (6.22) with some rearrangement of terms gives

$$
\begin{aligned}
&\int_\Omega \frac{\partial \hat{e}}{\partial t}\hat{e}dx + a(e,\hat{e}) - \int_0^t b(t,s;e(s),\hat{e})ds \\
= &\int_\Omega (f - \partial U_h^n)(\hat{e} - \Pi_h\hat{e})dx - a(U_h, \hat{e} - \Pi_h\hat{e}) + \int_0^t b(t,s;U_h(s),\hat{e} - \Pi_h\hat{e})ds \\
&-(t - t_{n-1/2})\int_\Omega \partial^2 U_h^n(\hat{e} - \Pi_h\hat{e})dx + \int_\Omega f\Pi_h\hat{e}dx + \int_0^t b(t,s;\hat{U}_h(s),\Pi_h\hat{e})ds \\
&-a(U_h,\Pi_h\hat{e}) - \int_\Omega \frac{\partial \hat{U}_h}{\partial t}\Pi_h\hat{e}dx + \int_0^t b(t,s;U_h(s) - \hat{U}_h(s),\Pi_h\hat{e})ds,
\end{aligned}
$$

where $\Pi_h : H^1(\Omega) \to \mathbb{V}_h^{A,0}$ is the *Clément* interpolation operator satisfying Proposition 5.2.1. Using Lemma 6.3.1, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{\partial \hat{e}}{\partial t}\hat{e}dx + a(e,\hat{e}) - \int_0^t b(t,s;e(s),\hat{e})ds \\
= &\int_\Omega (f - \partial U_h^n)(\hat{e} - \Pi_h\hat{e})dx - a(U_h, \hat{e} - \Pi_h\hat{e}) + \int_0^t b(t,s;U_h(s),\hat{e} - \Pi_h\hat{e})ds \\
&-(t - t_{n-1/2})\int_\Omega \partial^2 U_h^n(\hat{e} - \Pi_h\hat{e})dx + \int_\Omega (f - \hat{f})\Pi_h\hat{e}dx \\
&-\frac{\tau_{n-1}(t - t_{n-1/2})}{2}a(\partial^2 U_h^n,\Pi_h\hat{e}) + \Bigg[\int_0^t b(t,s;\hat{U}_h(s),\Pi_h\hat{e})ds \\
&-l_{n-1/2}(t)\sigma^n(b(t_{n-1/2};U_h,\Pi_h\hat{e})) - l_{n-3/2}(t)\sigma^{n-1}(b(t_{n-3/2};U_h,\Pi_h\hat{e}))\Bigg] \\
&+\int_0^t b(t,s;U_h(s) - \hat{U}_h(s),\Pi_h\hat{e})ds. \qquad (6.50)
\end{aligned}
$$

Now, we integrate (6.50) on each element K of $\mathcal{T}_h^A$ and use the estimates of Lemmas 6.3.2-6.3.3. Then, using continuity of the bilinear forms $b(t,s;\cdot,\cdot)$ and $a(\cdot,\cdot)$ together with Cauchy-Schwarz inequality and Poincaré inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + a(e,\hat{e}) \le \gamma\|\nabla\hat{e}(t)\|\int_0^t \|\nabla e(s)\|ds \\
&+\sum_{K\in\mathcal{T}_h^A}\Bigg[\|f-\partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}\|\hat{e}-\Pi_h\hat{e}\|_{L^2(K)} \\
&+\frac{1}{2}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}\|\hat{e}-\Pi_h\hat{e}\|_{L^2(\partial K)} \\
&+\frac{1}{2}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\|\hat{e}-\Pi_h\hat{e}\|_{L^2(\partial K)} \\
&+|t-t_{n-1/2}|\|\partial^2 U_h^n\|_{L^2(K)}\|\hat{e}-\Pi_h\hat{e}\|_{L^2(\partial K)} + C_2\|f-\hat{f}\|_{L^2(K)}\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \\
&+\frac{|\tau_{n-1}||t-t_{n-1/2}|}{2}\beta\|\nabla\partial^2 U_h^n\|_{L^2(K)}\|\nabla\Pi_h\hat{e}\|_{L^2(K)} + \bar{\gamma}\Big[|l_{n-1/2}(t)|\hat{\mathcal{Q}}^A_{CN,n,K} \\
&+|l_{n-3/2}(t)|\hat{\mathcal{Q}}^A_{CN,n-1,K} + \|\hat{U}_{h,1}(t)-\hat{U}_{h,I,1}(t)(t)\|_{L^2(K)}\Big]\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \\
&+\frac{\gamma}{12}\Big[|l_{n-1/2}(t)|\eta_{n,K} + |l_{n-3/2}(t)|\eta_{n-1,K}\Big]\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \\
&+\|\hat{U}_{h,2}(t)-\hat{U}_{h,I,2}(t)\|_{L^2(K)}\|\nabla\Pi_h\hat{e}\|_{L^2(K)}\Bigg],
\end{aligned}
$$

where C_2 is a constant in the Poincaré inequality.

Now, we use the fact $a(e,\hat{e}) = \frac{1}{2}\Big[a(e,e)+a(\hat{e},\hat{e})-a(e-\hat{e},e-\hat{e})\Big]$. Then, using (6.4), Proposition 5.2.1 and Young's inequality together with $\|\nabla\Pi_h\hat{e}\|_{L^2(K)} \le C_3\|\nabla\hat{e}\|_{L^2(K)}$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \frac{\alpha}{2}\|\nabla e\|^2 + \frac{\alpha}{2}\|\nabla\hat{e}\|^2 \le \frac{\gamma}{2\nu}\|\nabla\hat{e}\|^2 + \frac{\gamma C_Y(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds \\
&+\sum_{K\in\mathcal{T}_h^A}\Bigg[C_1\bigg(\|f-\partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\bigg)\omega_K(\hat{e}) \\
&+C_1|t-t_{n-1/2}|\|\partial^2 U_h^n\|_{L^2(K)}\omega_K(\hat{e}) + C_2C_3\|f-\hat{f}\|_{L^2(K)}\|\nabla\hat{e}\|_{L^2(K)} \\
&+\frac{C_3\beta}{2}|\tau_{n-1}||t-t_{n-1/2}|\|\nabla\partial^2 U_h^n\|_{L^2(K)}\|\nabla\hat{e}\|_{L^2(K)} \\
&+\beta\|\nabla(e-\hat{e})\|^2 + C_3\bar{\gamma}\Big[|l_{n-1/2}(t)|\hat{\mathcal{Q}}^A_{CN,n,K} \\
&+|l_{n-3/2}(t)|\hat{\mathcal{Q}}^A_{CN,n-1,K} + \|\hat{U}_{h,1}(t)-\hat{U}_{h,I,1}(t)(t)\|_{L^2(K)}\Big]\|\nabla\hat{e}\|_{L^2(K)} \\
&+\frac{\gamma C_3}{12}\Big[|l_{n-1/2}(t)|\eta_{n,K} + |l_{n-3/2}(t)|\eta_{n-1,K}\Big]\|\nabla\hat{e}\|_{L^2(K)} \\
&+C_3\|\hat{U}_{h,2}(t)-\hat{U}_{h,I,2}(t)\|_{L^2(K)}\|\nabla\hat{e}\|_{L^2(K)}\Bigg],
\end{aligned}
$$

where C_1 is the constant in the Proposition 5.2.1 and $C_Y(T)$ is the constant appeared due to the application of the Young's inequality.

Now, an application of Young's inequality along with $\omega_K(\hat{e}) \leq C_4\lambda_{2,K}\|\nabla\hat{e}\|_{L^2(K)}$, which follows trivially from Proposition 5.2.1 under the error equidistribution assumption (6.34), yields

$$\frac{1}{2}\frac{d}{dt}\|\hat{e}(t)\|^2 + \frac{\alpha}{2}\|\nabla e\|^2 + \frac{\alpha}{2}\|\nabla\hat{e}\|^2 \leq \frac{\gamma}{2\nu}\|\nabla\hat{e}\|^2 + \frac{\gamma C_Y(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h^A}\Bigg[C_1\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(\hat{e})$$
$$+\frac{\nu}{2}(t-t_{n-1/2})^2\lambda_{2,K}^2\|\partial^2 U_h^n\|^2_{L^2(K)} + \frac{\nu}{2}\|f-\hat{f}\|^2_{L^2(K)}$$
$$+\frac{1}{2\nu}(C_1^2C_4^2 + C_2^2C_3^2 + \beta^2C_3^2 + 2C_3^2\bar{\gamma}^2 + 2C_3^2\gamma^2 + C_3^2)\|\nabla\hat{e}\|^2_{L^2(K)}$$
$$+\frac{\nu}{2}\Big[l^2_{n-1/2}(t)(\hat{\mathcal{Q}}^A_{CN,n,K})^2 + l^2_{n-3/2}(t)(\hat{\mathcal{Q}}^A_{CN,n-1,K})^2 + \|\hat{U}_{h,1}(t) - \hat{U}_{h,I,1}(t)(t)\|^2_{L^2(K)}\Big]$$
$$+\frac{\nu}{8}\tau_{n-1}^2(t-t_{n-1/2})^2\|\nabla\partial^2U_h^n\|^2_{L^2(K)} + \frac{\beta}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\partial^2U_h^n\|^2_{L^2(K)}$$
$$+\frac{\nu}{288}\Big[l^2_{n-1/2}(t)\eta^2_{n,K} + l^2_{n-3/2}(t)\eta^2_{n-1,K}\Big] + \frac{\nu}{2}\|\hat{U}_{h,2}(t) - \hat{U}_{h,I,2}(t)\|^2_{L^2(K)}\Bigg],$$

where we have used the relation $\|\nabla(e-\hat{e})\|^2 = \|\nabla(\hat{U}_h - U_h)\|^2 = \frac{1}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\partial^2U_h^n\|^2$. Choosing $\nu = \frac{1}{\alpha}(C_1^2C_4^2 + C_2^2C_3^2 + \beta^2C_3^2 + 2C_3^2\bar{\gamma}^2 + 2C_3^2\gamma^2 + C_3^2 + \gamma)$ and integrating the above equation from t_{n-1} to t_n, we apply Gronwall's lemma to obtain

$$\|\hat{e}(t_n)\|^2 + \alpha\int_{t_{n-1}}^{t_n}\|\nabla e\|^2 dt \leq \|\hat{e}(t_{n-1})\|^2$$
$$+C\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(\hat{e})dt$$
$$+\tau_n^3\lambda_{2,K}^2\|\partial^2U_h^n\|^2_{L^2(K)} + \int_{t_{n-1}}^{t_n}\|f-\hat{f}\|^2_{L^2(K)}dt + \tau_{n-1}^2\tau_n^3\|\nabla\partial^2U_h^n\|^2_{L^2(K)}$$
$$+\tau_n\Big[(\hat{\mathcal{Q}}^A_{CN,n,K})^2 + (\hat{\mathcal{Q}}^A_{CN,n-1,K})^2 + \|\hat{U}_{h,1}(t) - \hat{U}_{h,I,1}(t)(t)\|^2_{L^2(K)}\Big]$$
$$+\tau_n^5\|\nabla\partial^2U_h^n\|^2_{L^2(K)} + \tau_n\Big[\eta^2_{n,K} + \eta^2_{n-1,K} + \|\hat{U}_{h,2}(t) - \hat{U}_{h,I,2}(t)\|^2_{L^2(K)}\Big]\Bigg],$$

where $C = \max\Big\{2C_1C(T), 4C(T)\nu, \beta C(T)\Big\}$ with $C(T)$ denoting the Gronwall's constant. Noting the fact $e(t_n) = \hat{e}(t_n)$, $\forall n$ and summing over $n = 2$ to N, we complete the rest of the proof. □

The proof of Theorem 6.3.1 will follow the arguments of the proof of Theorem 6.3.2. However, for the clarity of presentation, we present the proof which again relies on a sequence of following lemmas.

Lemma 6.3.4. *Let U_h and $\check{U}_h$ be as defined by (6.11) and (6.13), respectively. Then for any $t \in I_n$ with $1 \le n \le N$, and for all $\phi_h \in \mathbb{V}_h^{A,0}$, we have*

$$\int_\Omega \frac{\partial \check{U}_h}{\partial t}\phi_h dx + a(U_h, \phi_h) = \int_\Omega \check{f}\phi_h dx + \sigma^n(b(t_{n-1/2}; U_h, \phi_h))$$
$$+(t - t_{n-1/2})\int_0^{t_n} \partial b(t_n, s; U_h(s), \phi_h)ds + (t - t_{n-1/2})b(t_n, t_n; U_h^n, \phi_h),$$

where $\check{f}$ is given by (6.29).

Proof. For $\phi_h \in \mathbb{V}_h^{A,0}$, we use (6.11) and (6.10) to have

$$\int_\Omega \partial U_h^n \phi_h dx + a(U_h, \phi_h) = \sigma^n(b(t_{n-1/2}; U_h, \phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx$$
$$+(t - t_{n-1/2})a(\partial U_h^n, \phi_h). \qquad (6.51)$$

In view of (6.15), the equation (6.51) becomes

$$\int_\Omega \frac{\partial \check{U}_h}{\partial t}\phi_h dx + a(U_h, \phi_h) = \sigma^n(b(t_{n-1/2}; U_h, \phi_h)) + \int_\Omega f^{n-1/2}\phi_h dx$$
$$+(t - t_{n-1/2})\left(a(\partial U_h^n, \phi_h) + \int_\Omega \check{w}_h^n \phi_h dx\right). \qquad (6.52)$$

Now, an application of (6.14) gives the desired result. □

The following lemma shows the contributions for the time reconstruction error.

Lemma 6.3.5 (Time reconstruction error estimate)**.** *Let U_h and $\check{U}_h$ be defined by (6.11) and (6.13), respectively and let $\check{\eta}_{n,K}$ be given by (6.33). Then, for any $\phi_h \in \mathbb{V}_h^{A,0}$ and $t \in I_n$, $1 \le n \le N$, the following bound holds:*

$$|\int_0^t b(t, s; U_h(s) - \check{U}_h(s), \phi_h)ds|$$
$$\le \sum_{K \in \mathcal{T}_h^A}\left[\frac{\gamma}{12}\Big\{l_n(t)\check{\eta}_{n,K} + l_{n-1}(t)\check{\eta}_{n-1,K}\Big\} + \|\check{U}_{h,2}(t) - \check{U}_{h,I,2}(t)\|_{L^2(K)}\right]\|\nabla\phi_h\|_{L^2(K)}.$$

Proof. The result follows analogously to the proof of Lemma 6.3.3 by using the definition (6.20) instead of (6.28). □

Proof of Theorem 6.3.1. Let $t \in I_n$ and $1 \le n \le N$. We use the weak formulation (6.9) to have

$$\begin{aligned}
&\int_\Omega \frac{\partial \check{e}}{\partial t}\phi dx + a(e,\phi) - \int_0^t b(t,s;e(s),\phi)ds \\
&= \int_\Omega f\phi dx - \int_\Omega \frac{\partial \check{U}_h}{\partial t}\phi dx - a(U_h,\phi) + \int_0^t b(t,s;U_h(s),\phi)ds, \quad \forall \phi \in H_0^1(\Omega).
\end{aligned}$$

Choosing $\phi = \check{e}$ in the above error equation and using (6.22) with some rearrangement of terms gives

$$\begin{aligned}
&\int_\Omega \frac{\partial \check{e}}{\partial t}\check{e} dx + a(e,\check{e}) - \int_0^t b(t,s;e(s),\check{e})ds \\
= \;& \int_\Omega (f - \partial U_h^n)(\check{e} - \Pi_h\check{e})dx - a(U_h, \check{e} - \Pi_h\check{e}) + \int_0^t b(t,s;U_h(s),\check{e} - \Pi_h\check{e})ds \\
& -(t - t_{n-1/2}) \int_\Omega \check{w}_h^n(\check{e} - \Pi_h\check{e})dx + \int_\Omega f\Pi_h\check{e}dx + \int_0^t b(t,s;\check{U}_h(s),\Pi_h\check{e})ds \\
& -a(U_h,\Pi_h\check{e}) - \int_\Omega \frac{\partial \check{U}_h}{\partial t}\Pi_h\check{e}dx + \int_0^t b(t,s;U_h(s) - \check{U}_h(s),\Pi_h\check{e})ds.
\end{aligned}$$

Using (6.18) and Lemma 6.3.4, the above equation leads to

$$\begin{aligned}
&\int_\Omega \frac{\partial \check{e}}{\partial t}\check{e} dx + a(e,\check{e}) - \int_0^t b(t,s;e(s),\check{e})ds \\
= \;& \int_\Omega (f - \partial U_h^n)(\check{e} - \Pi_h\check{e})dx - a(U_h, \check{e} - \Pi_h\check{e}) + \int_0^t b(t,s;U_h(s),\check{e} - \Pi_h\check{e})ds \\
& -(t - t_{n-1/2}) \int_\Omega \check{w}_h^n(\check{e} - \Pi_h\check{e})dx + \int_\Omega (f - \check{f})\Pi_h\check{e}dx \\
& + \left[\int_0^{t_{n-1/2}} b(t_{n-1/2},s;\check{U}_h(s),\Pi_h\check{e})ds - \sigma^n(b(t_{n-1/2};U_h,\Pi_h\check{e}))\right] \\
& + \int_0^t b(t,s;U_h(s) - \check{U}_h(s),\Pi_h\check{e})ds + \langle \check{U}_{h,1}(t) - \phi_1(t), \nabla\Pi_h\check{e}\rangle \\
& +(t - t_{n-1/2}) \int_0^{t_n} \partial b(t_n,s;\check{U}_h(s) - U_h(s),\Pi_h\check{e})ds. \qquad (6.53)
\end{aligned}$$

Now, we integrate (6.53) on each element K of $\mathcal{T}_h^A$ and plug back the estimates of (6.46) with $\check{U}$ replacing $\hat{U}$. Then, using Lemma 6.3.5, (6.5), (6.8) and (6.4) together with

Cauchy-Schwarz and Poincaré inequalities leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\check{e}(t)\|^2 + a(e,\check{e}) \leq \gamma\|\nabla\check{e}(t)\|\int_0^t \|\nabla e(s)\|ds \\
&+\sum_{K\in\mathcal{T}_h^A}\Big[\|f-\partial U_h^n+\nabla\cdot(A\nabla U_h)-\int_0^t\nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}\|\check{e}-\Pi_h\check{e}\|_{L^2(K)} \\
&+\frac{1}{2}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}\|\check{e}-\Pi_h\check{e}\|_{L^2(\partial K)} \\
&+\frac{1}{2}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\|\check{e}-\Pi_h\check{e}\|_{L^2(\partial K)} \\
&+|t-t_{n-1/2}|\|\check{w}_h^n\|_{L^2(K)}\|\check{e}-\Pi_h\check{e}\|_{L^2(\partial K)}+C_2\|f-\check{f}\|_{L^2(K)}\|\nabla\Pi_h\check{e}\|_{L^2(K)} \\
&+\bar{\gamma}_1\check{\mathcal{Q}}_{CN,n,K}^A\|\nabla\Pi_h\check{e}\|_{L^2(K)}+\frac{\gamma'''}{12}|t-t_{n-1/2}|\check{\eta}_{n,K}\|\nabla\Pi_h\check{e}\|_{L^2(K)} \\
&+\frac{\gamma}{12}\Big[|l_{n-1}(t)|\check{\eta}_{n-1,K}+|l_n(t)|\check{\eta}_{n,K}\Big]\|\nabla\Pi_h\check{e}\|_{L^2(K)} \\
&+\|\check{U}_{h,2}(t)-\check{U}_{h,I,2}(t)\|_{L^2(K)}\|\nabla\Pi_h\check{e}\|_{L^2(K)} \\
&+\|\check{U}_{h,1}(t)-\check{U}_{h,I,1}(t)\|_{L^2(K)}\|\nabla\Pi_h\check{e}\|_{L^2(K)}\Big],
\end{aligned}$$

where C_2 is a constant in the Poincaré inequality.

Using (6.4) and Proposition 5.2.1 with an application of Young's inequality altogether with $\|\nabla\Pi_h\check{e}\|_{L^2(K)}\leq C_3\|\nabla\check{e}\|_{L^2(K)}$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\check{e}(t)\|^2+\frac{\alpha}{2}\|\nabla e\|^2+\frac{\alpha}{2}\|\nabla\check{e}\|^2 \leq \frac{\gamma}{2\nu}\|\nabla\check{e}\|^2+\frac{\gamma C_Y(T)\nu}{2}\int_0^t\|\nabla e(s)\|^2ds \\
&+\sum_{K\in\mathcal{T}_h^A}\Big[C_1\Big(\|f-\partial U_h^n+\nabla\cdot(A\nabla U_h)-\int_0^t\nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)} \\
&+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}+\frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Big)\omega_K(\check{e}) \\
&+C_1|t-t_{n-1/2}|\|\check{w}_h^n\|_{L^2(K)}\omega_K(\check{e})+C_2C_3\|f-\check{f}\|_{L^2(K)}\|\nabla\check{e}\|_{L^2(K)} \\
&+\frac{\beta}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\check{w}_h^n\|^2_{L^2(K)}+C_3\bar{\gamma}_1\check{\mathcal{Q}}_{CN,n,K}^A\|\nabla\check{e}\|_{L^2(K)} \\
&+\frac{C_3\gamma'''}{12}|t-t_{n-1/2}|\check{\eta}_{n,K}\|\nabla\check{e}\|_{L^2(K)}+\frac{\gamma C_3}{12}\Big[|l_{n-1}(t)|\check{\eta}_{n-1,K}+|l_n(t)|\check{\eta}_{n,K}\Big]\|\nabla\check{e}\|_{L^2(K)} \\
&+C_3\|\check{U}_{h,2}(t)-\check{U}_{h,I,2}(t)\|_{L^2(K)}\|\nabla\check{e}\|_{L^2(K)}+C_3\|\check{U}_{h,1}(t)-\check{U}_{h,I,1}(t)\|_{L^2(K)}\|\nabla\check{e}\|_{L^2(K)}\Big],
\end{aligned}$$

where we have used the facts $a(e,\check{e})=\frac{1}{2}\Big[a(e,e)+a(\check{e},\check{e})-a(e-\check{e},e-\check{e})\Big]$ and $\|\nabla(e-\check{e})\|^2=\|\nabla(\check{U}_h-U_h)\|^2=\frac{1}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\check{w}_h^n\|^2$. Now, an application of Young's inequality along with $\omega_K(\check{e}) \leq C_4\lambda_{2,K}\|\nabla\check{e}\|_{L^2(K)}$, which follows trivially from Proposition 5.2.1 under the error equidistribution assumption (6.31), yields

$$\frac{1}{2}\frac{d}{dt}\|\check{e}(t)\|^2 + \frac{\alpha}{2}\|\nabla e\|^2 + \frac{\alpha}{2}\|\nabla\check{e}\|^2 \leq \frac{\gamma}{2\nu}\|\nabla\check{e}\|^2 + \frac{\gamma C_Y(T)\nu}{2}\int_0^t \|\nabla e(s)\|^2 ds$$
$$+ \sum_{K\in\mathcal{T}_h^A}\Bigg[C_1\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{2\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{2\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(\check{e})$$
$$+\frac{\nu}{2}(t-t_{n-1/2})^2\lambda_{2,K}^2\|\breve{w}_h^n\|_{L^2(K)}^2 + \frac{\nu}{2}\|f-\check{f}\|_{L^2(K)}^2$$
$$+\frac{(C_1^2C_4^2 + C_2^2C_3^2 + C_3^2(\gamma''')^2 + C_3^2\bar{\gamma}_1^2 + 2C_3^2\gamma^2 + 2C_3^2)}{2\nu}\|\nabla\check{e}\|_{L^2(K)}^2 + \frac{\nu}{2}(\check{\mathcal{Q}}_{CN,n,K}^A)^2$$
$$+\frac{\nu}{288}(t-t_{n-1/2})^2\breve{\eta}_{n,K}^2 + \frac{\beta}{4}(t-t_{n-1})^2(t-t_n)^2\|\nabla\breve{w}_h^n|_{L^2(K)}^2$$
$$+\frac{\nu}{288}\Big[l_n^2(t)\breve{\eta}_{n,K}^2 + l_{n-1}^2(t)\breve{\eta}_{n-1,K}^2\Big] + \frac{\nu}{2}\|\check{U}_{h,2}(t) - \check{U}_{h,I,2}(t)\|_{L^2(K)}^2$$
$$+\frac{\nu}{2}\|\check{U}_{h,1}(t) - \check{U}_{h,I,1}(t)\|_{L^2(K)}^2\Bigg].$$

Thus, choosing $\nu = \frac{(C_1^2C_4^2+C_2^2C_3^2+2C_3^2\bar{\gamma}_1^2+2C_3^2\gamma^2+C_3^2+\gamma)}{\alpha}$ and integrating the above equation from t_{n-1} to t_n with an application of Gronwall's lemma leads to

$$\|\check{e}(t_n)\|^2 + \alpha\int_{t_{n-1}}^{t_n}\|\nabla e\|^2 dt \leq \|\check{e}(t_{n-1})\|^2$$
$$+C\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(\|f - \partial U_h^n + \nabla\cdot(A\nabla U_h) - \int_0^t \nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}$$
$$+\frac{1}{\lambda_{2,K}^{1/2}}\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)} + \frac{1}{\lambda_{2,K}^{1/2}}\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)\omega_K(\check{e})dt$$
$$+\tau_n^3\lambda_{2,K}^2\|\breve{w}_h^n\|_{L^2(K)}^2 + \int_{t_{n-1}}^{t_n}\|f-\check{f}\|_{L^2(K)}^2 dt + \tau_n(\check{\mathcal{Q}}_{CN,n,K}^A)^2 + \tau_n^3\breve{\eta}_{n,K}^2 + \tau_n^5\|\nabla\breve{w}_h^n\|_{L^2(K)}^2$$
$$+\tau_n\Big[\breve{\eta}_{n,K}^2 + \breve{\eta}_{n-1,K}^2 + \|\check{U}_{h,1}(t) - \check{U}_{h,I,1}(t)\|_{L^2(K)}^2 + \|\check{U}_{h,2}(t) - \check{U}_{h,I,2}(t)\|_{L^2(K)}^2\Big]\Bigg],$$

where $C = \max\Big\{2C_1C(T), C(T)\nu, \beta C(T)\Big\}$ with $C(T)$ denoting the Gronwall's constant. Noting the fact $e(t_n) = \check{e}(t_n)$, $\forall n$ and summing over $n = 1$ to N, we complete the proof.

□

Remarks. (i) Note that the error estimators derived in Theorems 6.3.1-6.3.2 are not traditional *a posteriori* error estimates because they contain the terms $\omega_K(\check{e})$ and $\omega_K(\hat{e})$ which involves the gradient of the exact solution u. It is known that ZZ like error estimators (cf. [108, 109]) are asymptotically exact for smooth solutions, see [60, 65, 77, 79] for elliptic and parabolic problems. For the purpose of approximating the terms $\omega_K(\check{e})$ and $\omega_K(\hat{e})$, we now recall from [108, 109] the following ZZ error estimator:

$$\eta^{ZZ}(U_h) = \begin{pmatrix} \eta_1^{ZZ}(U_h) \\ \eta_2^{ZZ}(U_h) \end{pmatrix} = \begin{pmatrix} (I - I_h^A)(\frac{\partial U_h}{\partial x_1}) \\ (I - I_h^A)(\frac{\partial U_h}{\partial x_2}) \end{pmatrix},$$

where I_h^A is an approximate L^2 projection operator onto $\mathbb{V}_h^A$, and is defined by its values at each vertex P as

$$I_h^A(\nabla U_h)(P) = \begin{pmatrix} I_h^A(\frac{\partial U_h}{\partial x_1})(P) \\ I_h^A(\frac{\partial U_h}{\partial x_2})(P) \end{pmatrix} = \frac{1}{\sum\limits_{P\in K, K\in\mathcal{T}_h} |K|} \begin{pmatrix} \sum\limits_{P\in K, K\in\mathcal{T}_h} |K|(\dfrac{\partial U_h}{\partial x_1})\,|_K \\ \sum\limits_{P\in K, K\in\mathcal{T}_h} |K|(\dfrac{\partial U_h}{\partial x_2})\,|_K \end{pmatrix}.$$

Therefore, in Theorems 6.3.1-6.3.2, we replace the matrices $G_K(\check{e})$ and $G_K(\hat{e})$ appearing in the terms $\omega_K(\check{e})$ and $\omega_K(\hat{e})$ respectively, by the matrix $\mathcal{G}_K(U_h)$ to recover usual *a posteriori* error estimators. For any $v_h \in \mathbb{V}_h^A$, the matrix $\mathcal{G}_K(U_h)$ is defined by

$$\mathcal{G}_K(v_h) = \begin{pmatrix} \int_K \left(\eta_1^{ZZ}(v_h)\right)^2 dx & \int_K \eta_1^{ZZ}(v_h)\eta_2^{ZZ}(v_h)dx \\ \int_K \eta_1^{ZZ}(v_h)\eta_2^{ZZ}(v_h)dx & \int_K \left(\eta_2^{ZZ}(v_h)\right)^2 dx \end{pmatrix}.$$

(ii) One can recover the isotropic *a posteriori* error estimators from the anisotropic *a posteriori* error estimators of Theorems 6.3.1-6.3.2. In case of isotropic mesh ($\lambda_{1,K} \simeq \lambda_{2,K} \simeq h_K$), Theorems 6.3.1-6.3.2 take the form

$$
\begin{aligned}
&\|e(\cdot,T)\|^2+\alpha\int_0^T\|\nabla e\|^2dt\le\|e(\cdot,0)\|^2\\
&+C\sum_{n=1}^{N}\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(h_K^2\|f-\partial U_h^n+\nabla\cdot(A\nabla U_h)-\int_0^t\nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}\\
&+h_K\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}+h_K\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)dt\\
&+\tau_n^3h_K^2\|\breve{w}_h^n\|^2_{L^2(K)}+\int_{t_{n-1}}^{t_n}\|f-\check{f}\|^2_{L^2(K)}dt+\tau_n(\breve{\mathcal{Q}}^A_{CN,n,K})^2+\tau_n^3\breve{\eta}^2_{n,K}\\
&+\tau_n^5\|\nabla\breve{w}_h^n\|^2_{L^2(K)}+\tau_n\Big[\breve{\eta}^2_{n,K}+\breve{\eta}^2_{n-1,K}+\|\check{U}_{h,1}(t)-\check{U}_{h,I,1}(t)\|^2_{L^2(K)}\\
&+\|\check{U}_{h,2}(t)-\check{U}_{h,I,2}(t)\|^2_{L^2(K)}\Big]\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\|e(\cdot,T)\|^2+\alpha\int_{t_1}^T\|\nabla e\|^2dt\le\|e(\cdot,t_1)\|^2\\
&+C\sum_{n=2}^{N}\sum_{K\in\mathcal{T}_h^A}\Bigg[\int_{t_{n-1}}^{t_n}\Bigg(h_K^2\|f-\partial U_h^n+\nabla\cdot(A\nabla U_h)-\int_0^t\nabla\cdot(B(t,s)\nabla U_h(s))ds\|_{L^2(K)}\\
&+h_K\|[A\nabla U_h\cdot\mathbf{n}]\|_{L^2(\partial K)}+h_K\|[\int_0^t B(t,s)\nabla U_h(s)\cdot\mathbf{n}ds]\|_{L^2(\partial K)}\Bigg)dt\\
&+\tau_n^3h_K^2\|\partial^2U_h^n\|^2_{L^2(K)}+\int_{t_{n-1}}^{t_n}\|f-\hat{f}\|^2_{L^2(K)}dt\\
&+\tau_n\Big[(\hat{\mathcal{Q}}^A_{CN,n,K})^2+(\hat{\mathcal{Q}}^A_{CN,n-1,K})^2+\|\hat{U}_{h,1}(t)-\hat{U}_{h,I,1}(t)(t)\|^2_{L^2(K)}\Big]\\
&+\Big\{\tau_{n-1}^2\tau_n^3+\tau_n^5\Big\}\|\nabla\partial^2U_h^n\|^2_{L^2(K)}+\tau_n\Big[\eta^2_{n,K}+\eta^2_{n-1,K}+\|\hat{U}_{h,2}(t)-\hat{U}_{h,I,2}(t)\|^2_{L^2(K)}\Big]\Bigg],
\end{aligned}
$$

where the contributions $\breve{\mathcal{Q}}^A_{CN,n,K}$, $\breve{\eta}_{n,K}$, $\hat{\mathcal{Q}}^A_{CN,n,K}$ and $\eta_{n,K}$ are given by (6.32), (6.33), (6.35) and (6.36), respectively.

(iii) When u is smooth enough, the error e in the $L^2(H^1(\Omega))$-norm is $O(h+\tau^2)$ for the Crank-Nicolson scheme. Thus, the terms related to the data oscillation, inner residual and the jump residual in Theorems 6.3.1-6.3.2 are of the optimal order. The rest of the terms along with the data oscillation term estimate the error due to the time discretization in both the error estimators. In particular, the error indicators $\breve{\mathcal{Q}}^A_{CN,n,K}$ and $\hat{\mathcal{Q}}^A_{CN,n,K}$ in the estimators account for the contributions coming from each of the

element K due to quadrature approximation of the Volterra integral term whereas the terms $\breve{\eta}_{n,K}$ and $\eta_{n,K}$ convey the contributions for the reconstruction errors coming from each of the element K. Since the error indicators $\breve{\mathcal{Q}}^A_{CN,n,K}$, $\hat{\mathcal{Q}}^A_{CN,n,K}$, $\breve{\eta}_{n,K}$ and $\eta_{n,K}$ are formally of optimal order, the terms associated with these indicators give the desired rate of convergence. The term $\left\{\tau^2_{n-1}\tau^3_n + \tau^5_n\right\}\|\nabla\partial^2 U^n_h\|^2_{L^2(K)}$ in the second estimator is of optimal order provided $\sum_{n=1}^N \tau_n\|\nabla\partial^2 U^n_h\|^2_{L^2(K)}$ is bounded with respect to τ. Similar argument holds for the term $\tau^5_n\|\nabla\breve{w}^n_h\|^2_{L^2(K)}$ appeared in Theorem 6.3.1. Also, in view of the error behaviour which is of $O(h+\tau^2)$, if we take h proportional to τ^2, the common term $\tau^3_n h^2_K\|\partial^2 U^n_h\|^2_{L^2(K)}$ in both the estimators is of higher order. Moreover, the terms $\|\breve{U}_{h,1}(t)-\breve{U}_{h,I,1}(t)\|^2_{L^2(K)}$, $\|\hat{U}_{h,1}(t)-\hat{U}_{h,I,1}(t)(t)\|^2_{L^2(K)}$, $\|\breve{U}_{h,2}(t)-\breve{U}_{h,I,2}(t)\|^2_{L^2(K)}$ and $\|\hat{U}_{h,2}(t)-\psi(t)\|^2_{L^2(K)}$ are *a posteriori* quantities and they are eventually of optimal order due to linear interpolations.

(iv) The introduction of the ZZ error estimator in Theorem 6.3.2 is not sufficient to provide a meaningful *a posteriori* error estimator due to the presence of the term $\|e(.,t_1)\|^2$ on right hand side of Theorem 6.3.2. Since the error bound in Theorem 6.3.1 is of optimal order in the $L^2(H^1(\Omega))$-norm, the *a posteriori* error estimate in Theorem 6.3.2 is thus justified by using the estimate of $\|e(.,t_1)\|^2$ from Theorem 6.3.1.

(v) When $\mathcal{B}(t,s)=0$, the error estimators in Theorems 6.3.1-6.3.2 are similar to that of the purely parabolic problems [60]. Since PIDE (6.1) may be thought of as a perturbation to the parabolic problem, it is natural to expect that our *a posteriori* error estimators should reflect back the quadrature error coming from the approximation of the memory term. The terms $\breve{\mathcal{Q}}^A_{CN,n,K}$ and $\hat{\mathcal{Q}}^A_{CN,n,K}$ are indicators for the quadrature errors which are of optimal order. Therefore, our results obtained in Theorems 6.3.1-6.3.2 generalize the results of the parabolic problems [60] to PIDE.

7

Numerical Experiments

In this chapter, we shall present numerical results for two dimensional test problems to illustrate our theoretical findings. Our main emphasis here is to understand the asymptotic behaviour of the estimators. We perform numerical tests on uniform meshes with uniform time steps. All computations are carried out using the software MATLAB-7.8. Bisection algorithm (cf. Wissgott [103]) is used to generate the refined triangulations. The behaviour of the *a posteriori* estimator of Chapter 3 are presented in Example 7.1. Example 7.2 shows the behaviour of the *a posteriori* error estimator of Chapter 4.

Example 7.1.

This example is subjected to study the behaviour of the fully discrete backward Euler *a posteriori* error estimators presented in Theorem 3.2.1. We consider the PIDE (1.1) with the coefficient matrices $A = I$ and $B(t,s) = I$, where I denotes the identity matrix. Thus, we consider PIDE of the form

$$\begin{aligned} u_t(x,y,t) - \nabla^2 u(x,y,t) &= f(x,y,t) - \int_0^t \nabla^2 u(x,y,s)ds, \ (x,y,t) \in \Omega \times (0,T], \quad (7.1) \\ u(x,y,t) &= 0, \ \ (x,y,t) \in \partial\Omega \times [0,T], \\ u(x,y,0) &= \sin(x)\sin(y), \ \ (x,y) \in \Omega, \end{aligned}$$

where "∇" denotes the spatial gradient. We consider a square domain $\Omega = (0,\pi)^2 \subset \mathbb{R}^2$. The forcing term f is chosen such that the exact solution is given by

$$u(x,y,t) = \exp(-t/2)\sin(x)\sin(y). \quad (7.2)$$

Coupling	$h(1)$	$\tau(1)$	l(runs)	c_0	Figures
$\tau \approx h^2$	$\pi/8$	0.025	3	$1.6/\pi^2$	7.1a-7.8b
$\tau \approx h$	$\pi/8$	0.05	4	$.4/\pi$	7.9a-7.16b

Table 7.1: *Choice of parameters for the problem (7.1)*

Let $(\mathcal{T}_n)_{n\in[0:N]}$ be a family of conforming triangulations $\bar{\Omega}$ into triangles K. We define $\mathbb{V}^n$ to be the usual finite element spaces of continuous, piecewise linear functions on $\mathcal{T}_n$:

$$\mathbb{V}^n := \{\phi \in H_0^1(\Omega) \cap C(\bar{\Omega}) : \ \phi|_K \in \mathbb{P}_1, \ \forall K \in \mathcal{T}_n\},$$

where $\mathbb{P}_1$ is the space of polynomials of degree ≤ 1.

To understand the asymptotic behaviour of the estimators, we choose a sequence of meshsizes $(h(i) : i \in [1 : l])$ to which we couple a sequence of time steps $(\tau(i) : i \in [1 : l])$, $\tau(i) = c_0 h(i)^k$, with k equal either 1 or 2. Here, l denotes the number of runs, ranging from 3 to 4 in each of the experiment. For each run, the spatial meshsize becomes half of the previous meshsize. For example, for the first run we have meshsize $h_1 = \pi/8$, for the second run meshsize $h_2 = \pi/16$ and so on. Table 7.1 summarizes the choice of the various parameters for each of the coupling.

For each run $i \in [1 : l]$, we compute the following quantities of interest for each time $t_m \in [0 = t_0 : \tau(i) : t_N = .1]$.

- The error norms:

$$\|e\|_{L^\infty(0,t_m,L^2(\Omega))} \ \text{ and } \ \|e\|_{L^2(0,t_m;H^1(\Omega))}.$$

- The reconstruction error estimators:

$$\left(\sum_{n=1}^{m} \tau_n \alpha_{BE,n}^2(U^n)\right)^{1/2} \quad \text{and} \quad \max_{0\leq n\leq m} \beta_{BE,n}(U^n).$$

- The space error estimators:

$$\sum_{n=1}^{m} \tau_n \zeta_{BE,n}.$$

- The time error estimators:

$$\sum_{n=1}^{m} \tau_n \eta_{BE,n}.$$

We drop the linear interpolation estimator for the approximation of the memory term, data approximation estimator and mesh change estimator from study.

For each quantity of interest, we observe its experimental order of convergence (EOC). The EOC is defined as follows: For a given finite sequence of successive runs (indexed by i), the EOC of the corresponding sequence of quantities of interest $E(i)$ (error, estimator or part of an estimator) itself is a sequence defined by

$$EOC(E(i)) = \frac{log(E(i+1)/E(i))}{log(h(i+1)/h(i))},$$

where $h(i)$ denotes the meshsize of the run i. It is important to note that, in order to exhibit the optimality of the estimators for different norms, a different coupling of the meshsize h and the step size τ must be chosen. Since, all the simulations are carried out using the $\mathbb{P}_1$ elements, we have taken the coupling $\tau \approx h^2$ to see that the $L^\infty(L^2(\Omega))$ error norm has second order of convergence while the $L^2(H^1(\Omega))$ error norm has first order of convergence for the backward Euler method.

Moreover, we look at inverse effectivity index (IEI) for each error-estimator pair, defined by

$$IEI(L^\infty(L^2(\Omega))\ Estimator) = \frac{\|e\|_{L^\infty(0,t_m;L^2(\Omega))}}{\max\limits_{0\le n\le m} \beta_{BE,n} + \sum_{n=1}^{m} \tau_n(\zeta_{BE,n} + \eta_{BE,n})},$$

and

$$IEI(L^2(H^1(\Omega))\ Estimator) = \frac{\|e\|_{L^2(0,t_m;H^1(\Omega))}}{\left(\sum_{n=1}^{m} \tau_n \alpha_{BE,n}^2\right)^{1/2} + \sum_{n=1}^{m} \tau_n(\zeta_{BE,n} + \eta_{BE,n})}.$$

All the constants involved in the estimators are taken to be 1 except Gronwall's constant which is taken to be $\exp(T)$. This is of course not true and a fine tuning of constants should be performed, but since our purpose is to check the asymptotic behaviour of the estimators, the IEI is to be understood only qualitatively.

Here, in each plot of the Figures 7.1a-7.16b, the abscissa represents time. We plot the quantities of interest (errors and estimators) and below them the corresponding EOC's for the coupling $\tau \approx h^2$ in Figures 7.1a-7.6b whereas for the coupling $\tau \approx h$, the behaviour of the quantities of interest and their EOC's are presented in Figures 7.9a-7.14b. The value of the EOC of a given quantity of interest indicates its order. The behaviour of the total estimators for the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms of the error along with their IEI's are shown in Figures 7.7a-7.8b for the coupling $\tau \approx h^2$, and in Figures 7.15a-7.16b for the coupling $\tau \approx h$. While observing the behaviour of the quantities of interest and total estimators, each curve of the plots corresponds to a given run. The most coarse grid corresponds to curve with the largest error value and the finest grid corresponds to curve with the smallest error value.

Observations

From the Figures 7.1a-7.4b, it is apparent that for the coupling $\tau \approx h^2$, the errors $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ and $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ have EOC's 2 and 1, respectively so are the estimators $\max_{n\in[0:m]} \beta_{BE,n}$ and $\left(\sum_{n=1}^m \tau_n \alpha_{BE,n}^2\right)^{1/2}$, respectively. Moreover, for the coupling $\tau \approx h^2$, we observe that both the space estimator $(\sum_{n=1}^m \tau_n \zeta_{BE,n})$ and time estimator $(\sum_{n=1}^m \tau_n \eta_{BE,n})$ decrease with second order.

Similarly, for the coupling $\tau \approx h$, the estimator $\max_{n\in[0:m]} \beta_n$ has EOC 2 whereas the errors $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$, $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ and the estimator $\left(\sum_{n=1}^m \tau_n \alpha_n^2\right)^{1/2}$ have EOC 1. Moreover, it is also evident from simulations that the time error estimator and the space error estimator decrease with at least first order for $\tau \approx h$.

Our numerical experiment shows that the estimators have the optimal rate of convergence which matches with that of the error's norm.

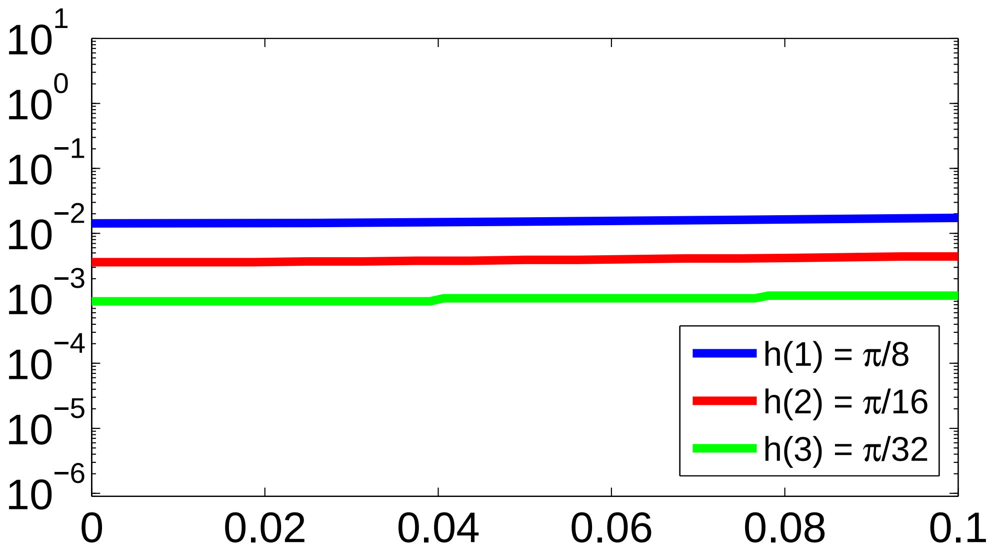

Figure 7.1a: *This figure shows the behaviour of the error* $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ *for* $\mathbb{P}_1$ *elements.*

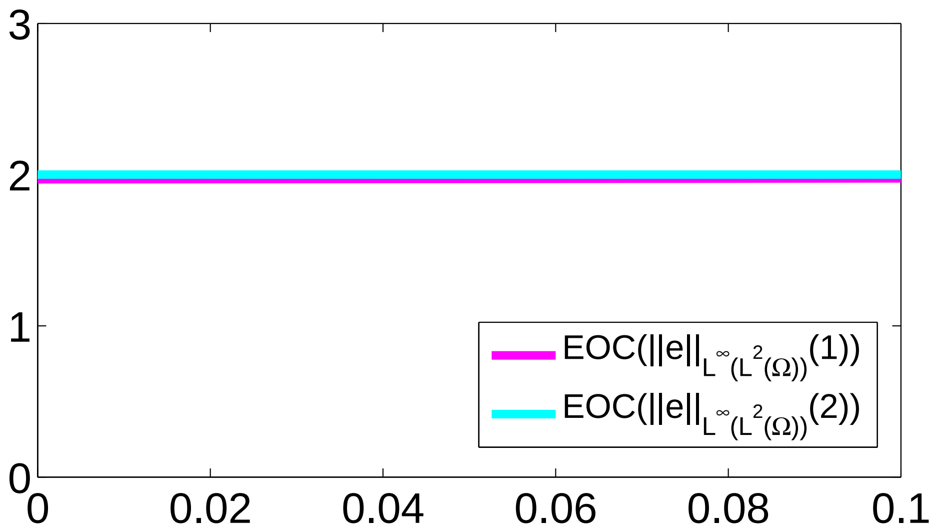

Figure 7.1b: *For the coupling* $\tau \approx h^2$, $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ *decreases with second order.*

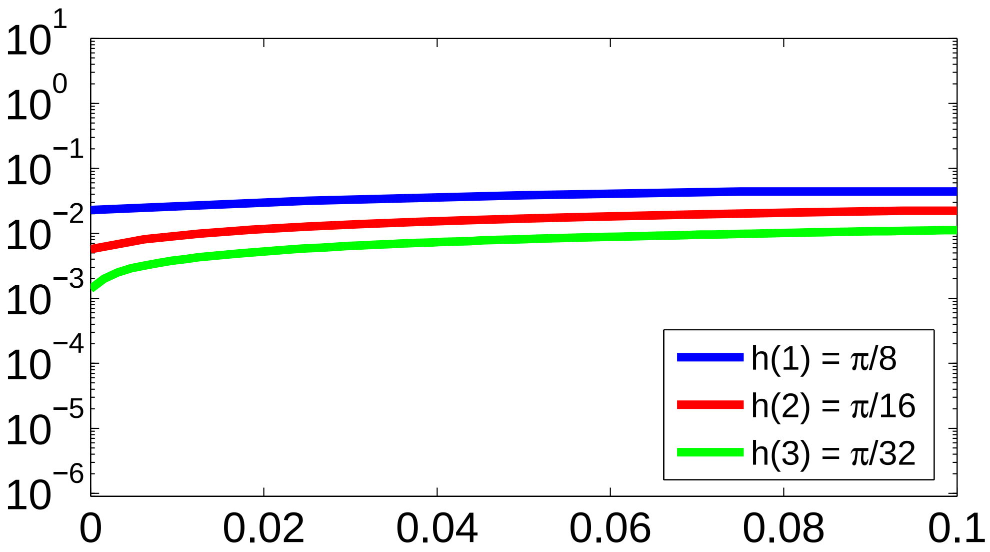

Figure 7.2a: *This figure shows the behaviour of the error* $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ *for* $\mathbb{P}_1$ *elements.*

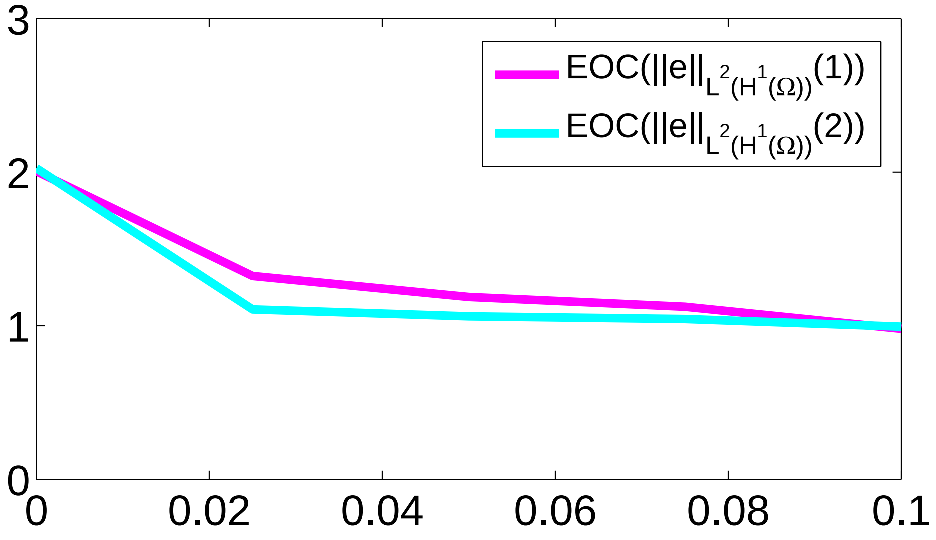

Figure 7.2b: *For the coupling* $\tau \approx h^2$, $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ *decreases with first order.*

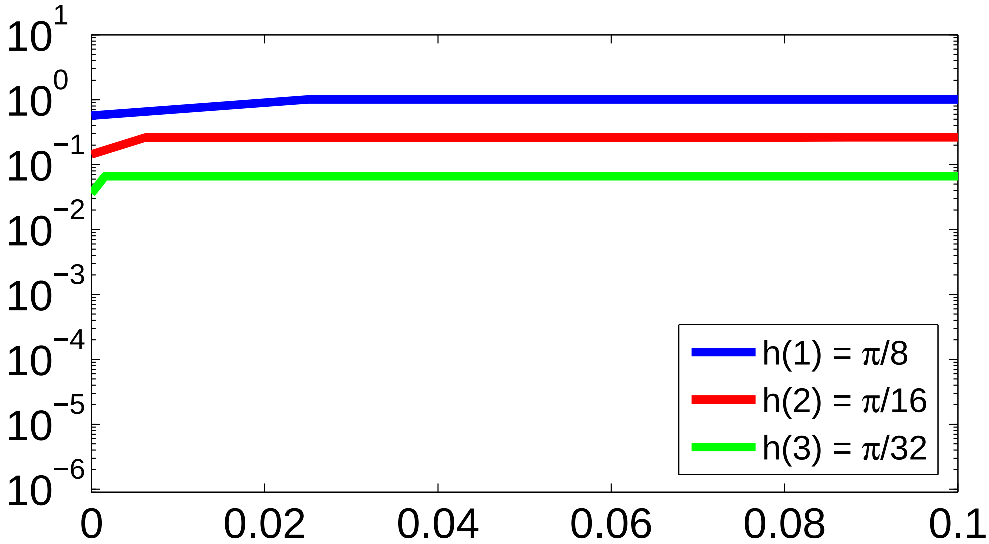

Figure 7.3a: *This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the L^2-norm i.e.,* $\max_{0\le n\le m} \beta_{BE,n}(U^n)$ *for* $\mathbb{P}_1$ *elements.*

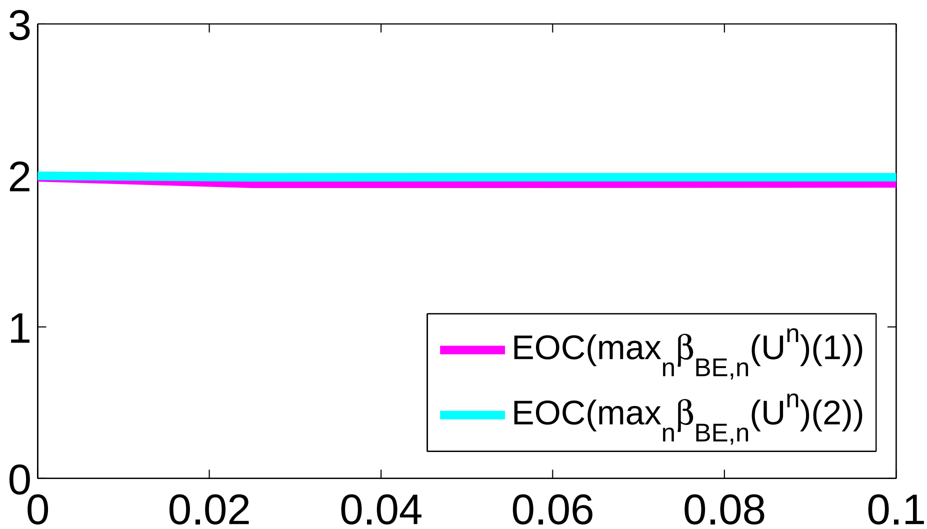

Figure 7.3b: *For the coupling* $\tau \approx h^2$, $\max_{0\le n\le m} \beta_{BE,n}(U^n)$ *decreases with second order.*

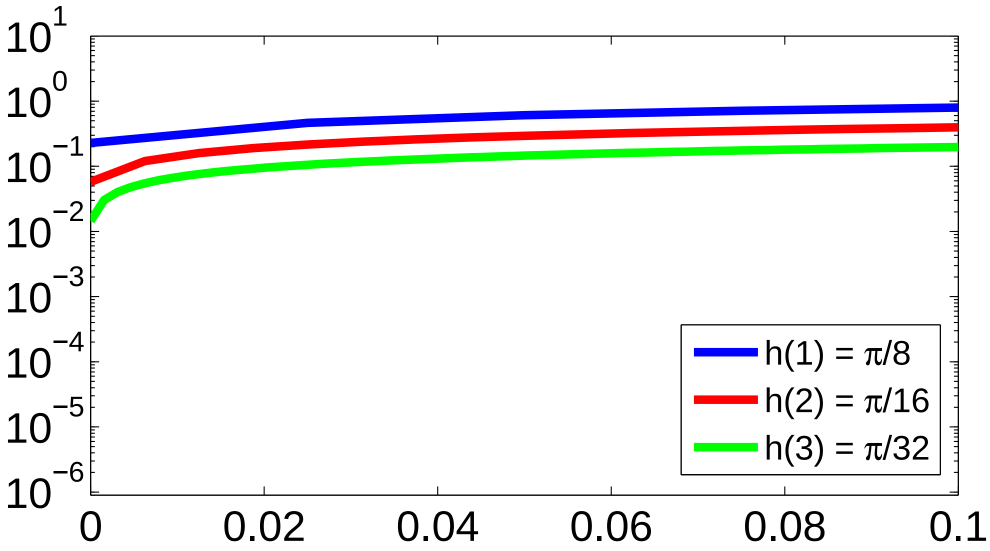

Figure 7.4a: *This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the* H^1*-norm i.e.,* $\left(\sum_{n=1}^{m}\tau_n\alpha^2_{BE,n}(U^n)\right)^{1/2}$ *for* $\mathbb{P}_1$ *elements.*

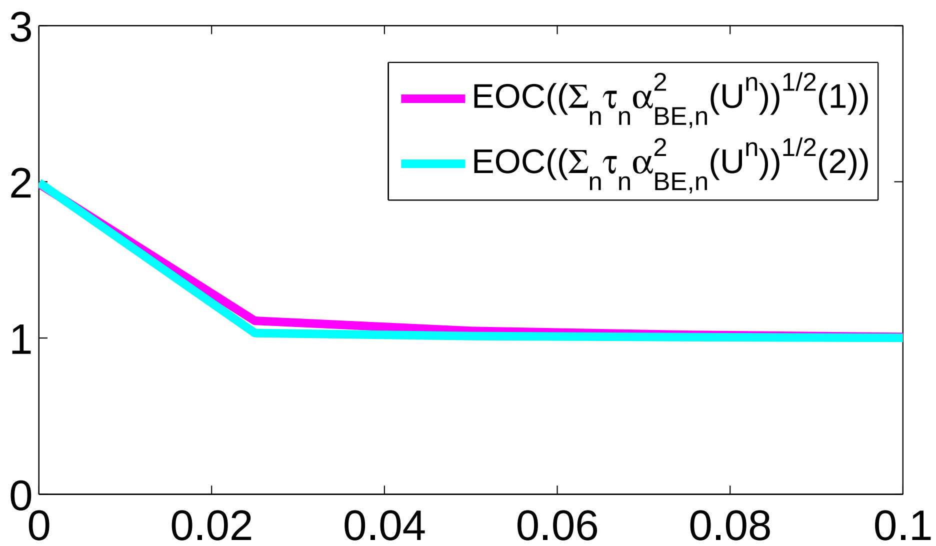

Figure 7.4b: *For the coupling* $\tau \approx h^2$, $\left(\sum_{n=1}^{m}\tau_n\alpha^2_{BE,n}(U^n)\right)^{1/2}$ *decreases with first order.*

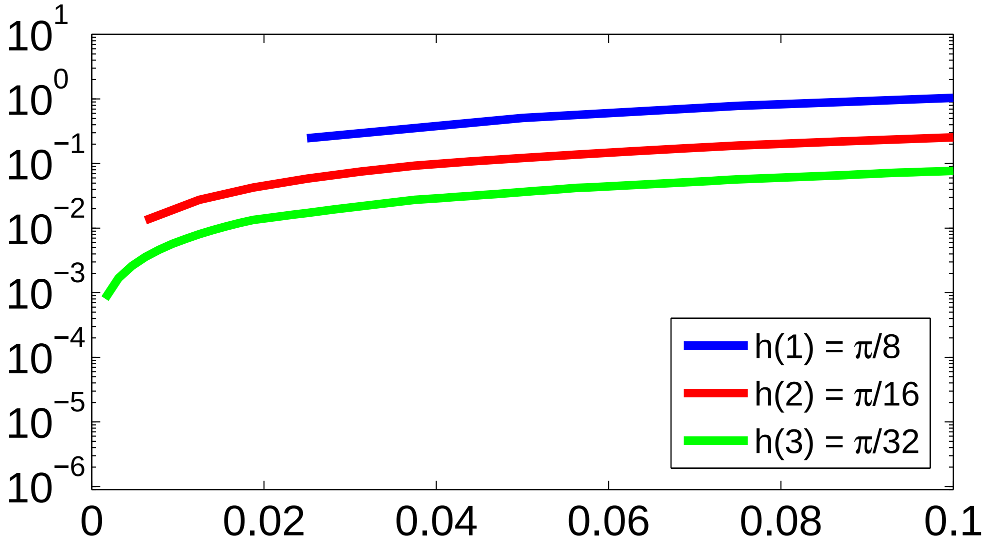

Figure 7.5a: *This figure shows the behaviour of the space estimator* $\sum_{n=1}^{m} \tau_n \zeta_{BE,n}$ *for* $\mathbb{P}_1$ *elements.*

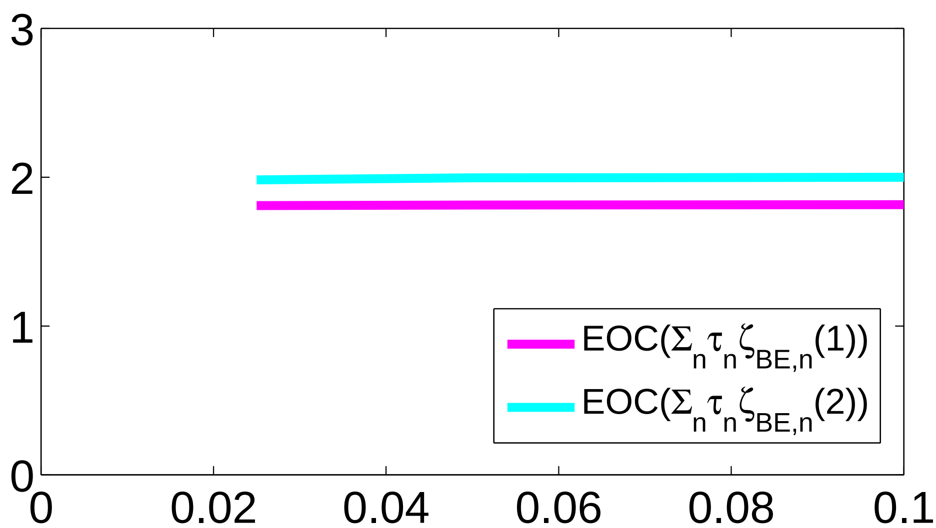

Figure 7.5b: *For the coupling* $\tau \approx h^2$, $\sum_{n=1}^{m} \tau_n \zeta_{BE,n}$ *decreases with second order.*

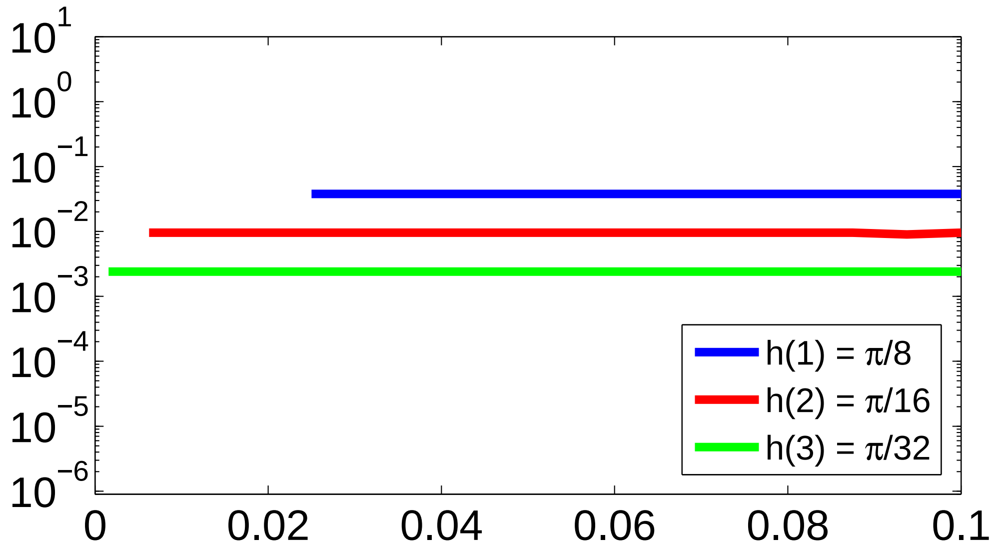

Figure 7.6a: *This figure shows the behaviour of the time estimator* $\sum_{n=1}^{m} \tau_n \eta_{BE,n}$ *for* $\mathbb{P}_1$ *elements.*

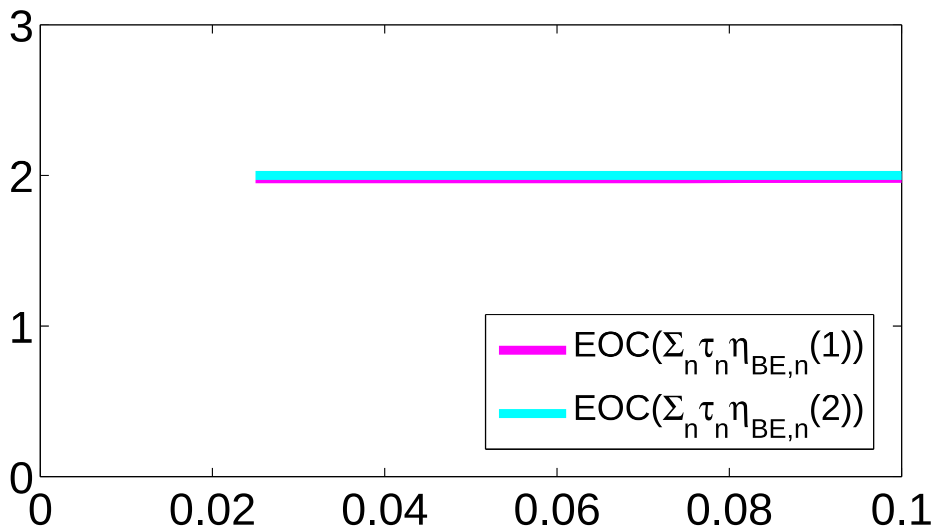

Figure 7.6b: *For the coupling* $\tau \approx h^2$, $\sum_{n=1}^{m} \tau_n \eta_{BE,n}$ *decreases with second order.*

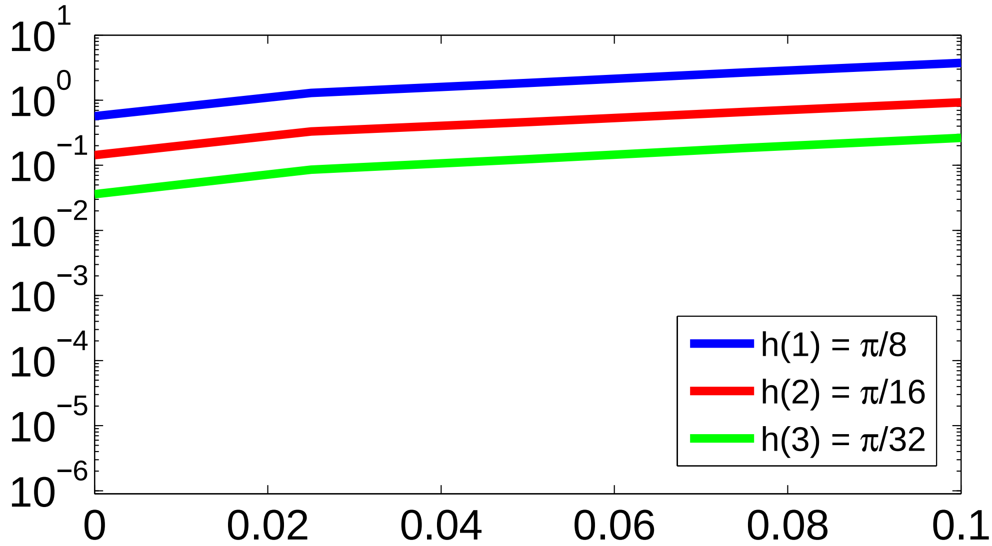

Figure 7.7a: *This figure shows the behaviour of the total estimator in* $L^\infty(L^2(\Omega))$ *for* $\mathbb{P}_1$ *elements.*

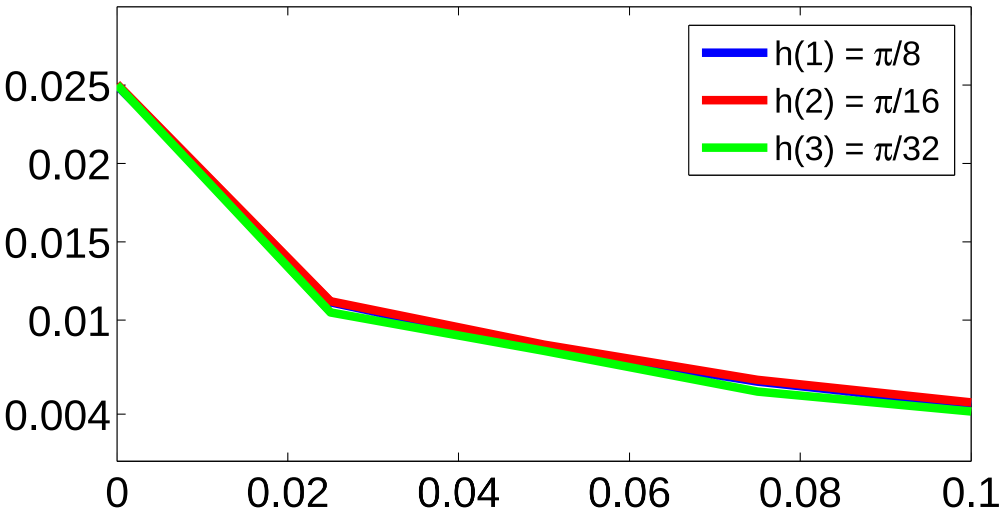

Figure 7.7b: *This figure shows the inverse effectivity index of the total estimator in* $L^\infty(L^2(\Omega))$ *for the coupling* $\tau \approx h^2$.

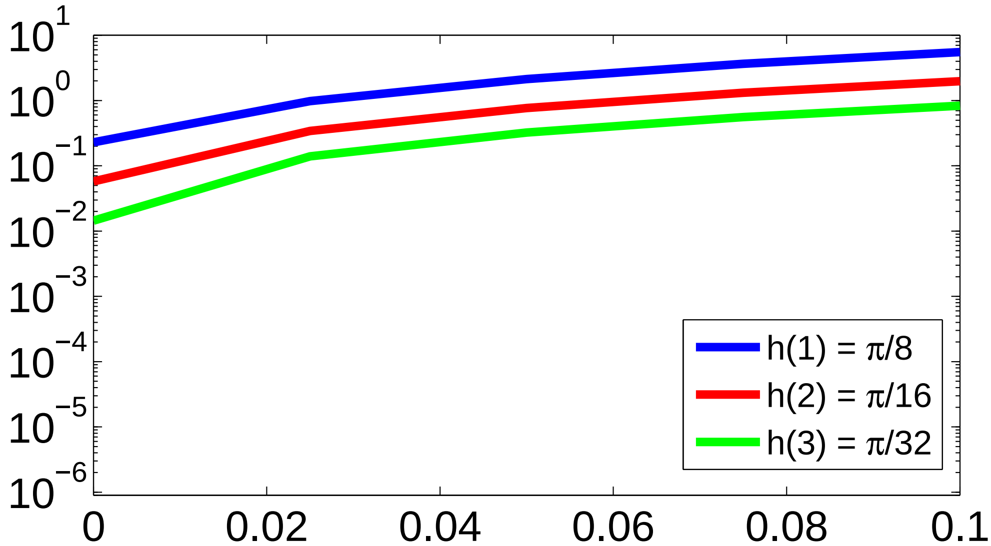

Figure 7.8a: *This figure shows the behaviour of the total estimator in* $L^2(H^1(\Omega))$ *for* $\mathbb{P}_1$ *elements.*

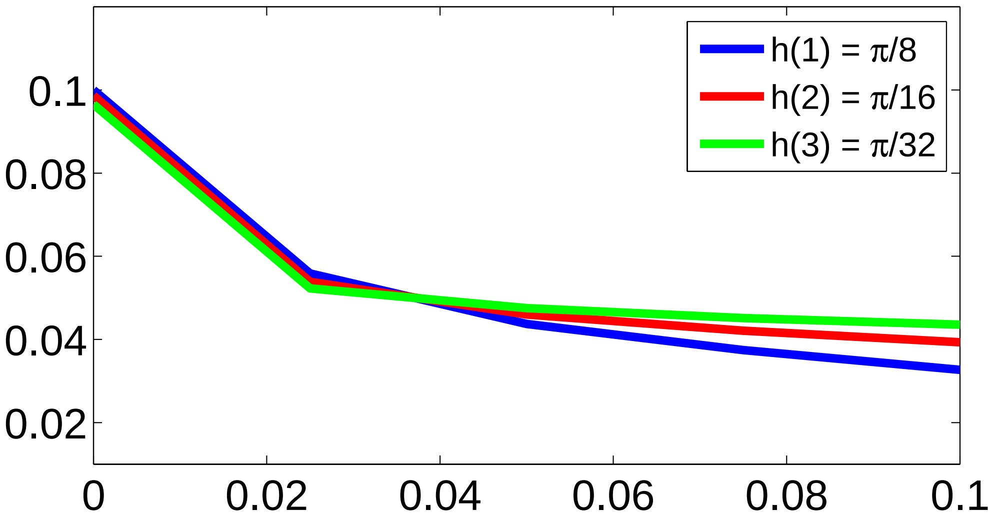

Figure 7.8b: *This figure shows the inverse effectivity index of the total estimator in* $L^2(H^1(\Omega))$ *for the coupling* $\tau \approx h^2$.

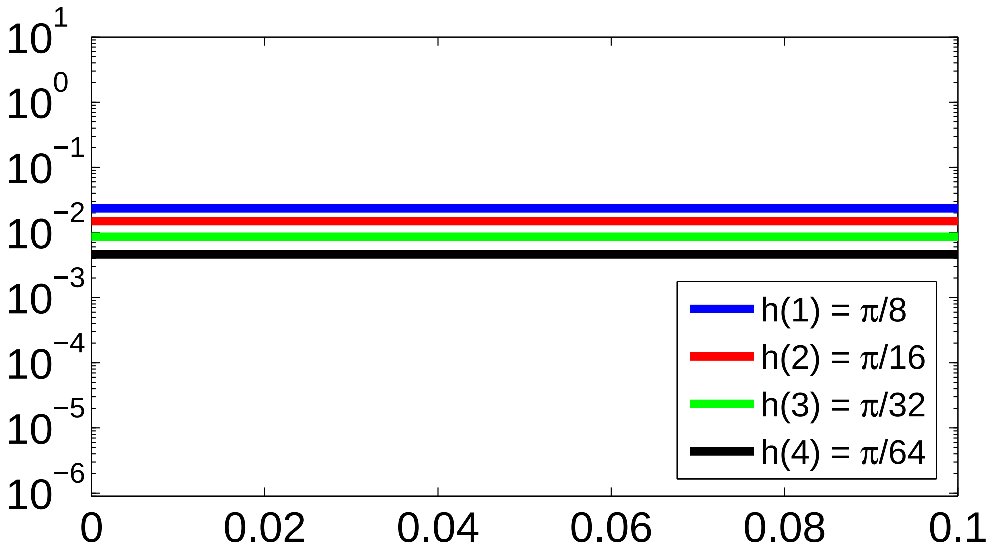

Figure 7.9a: *This figure shows the behaviour of the error* $\|e\|_{L^\infty(L^2(\Omega))}$ *for* $\mathbb{P}_1$ *elements.*

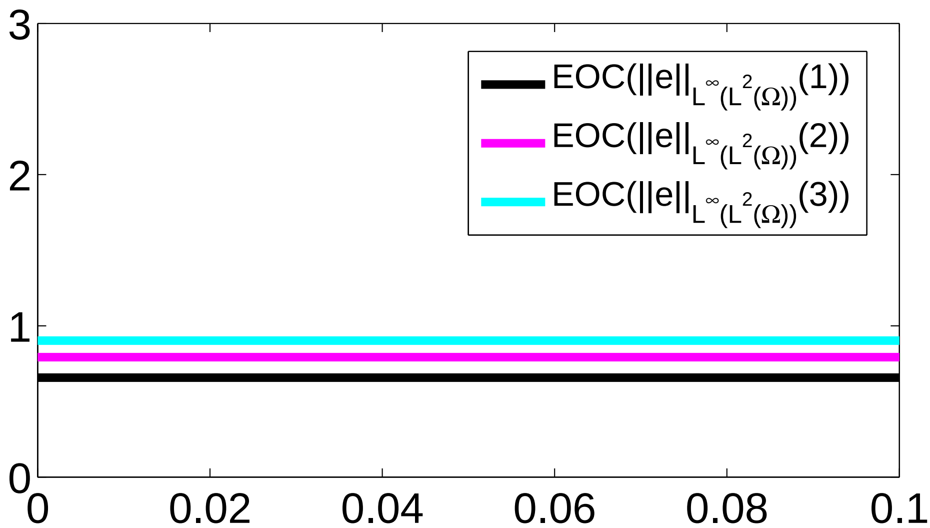

Figure 7.9b: *For the coupling* $\tau \approx h$, $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ *decreases with first order.*

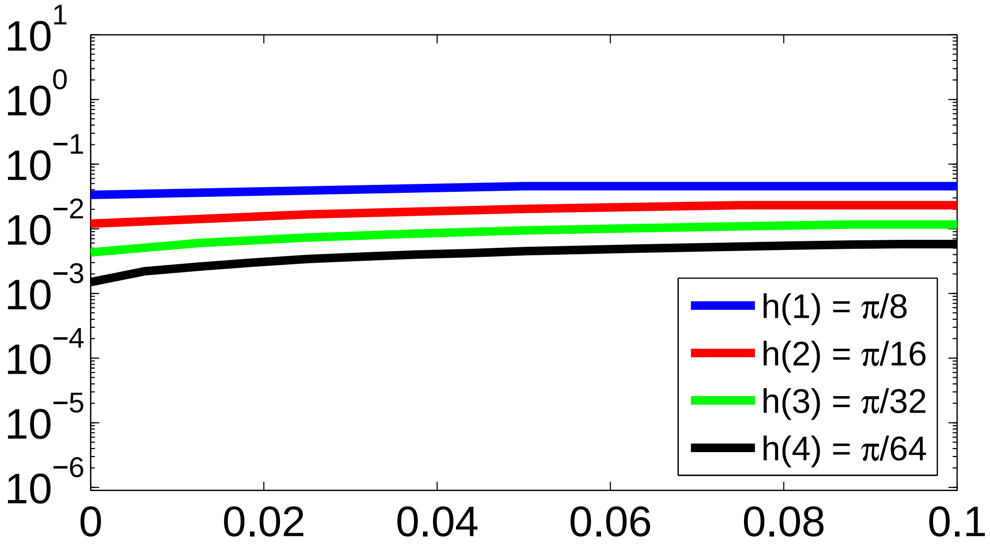

Figure 7.10a: *This figure shows the behaviour of the error* $\|e\|_{L^2(H^1(\Omega))}$ *for* $\mathbb{P}_1$ *elements.*

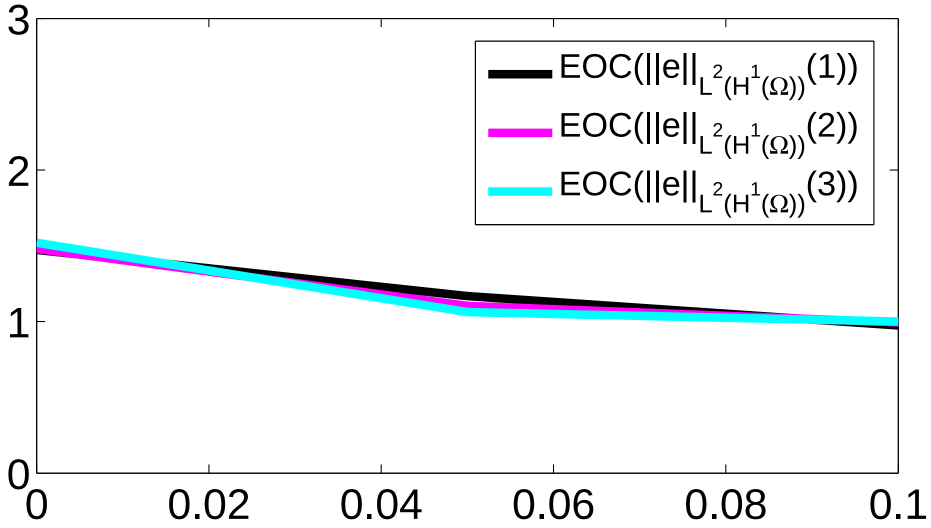

Figure 7.10b: *For the coupling* $\tau \approx h$, $\|e\|_{L^2(0,t_m;H^1(\Omega))}$ *decreases with first order.*

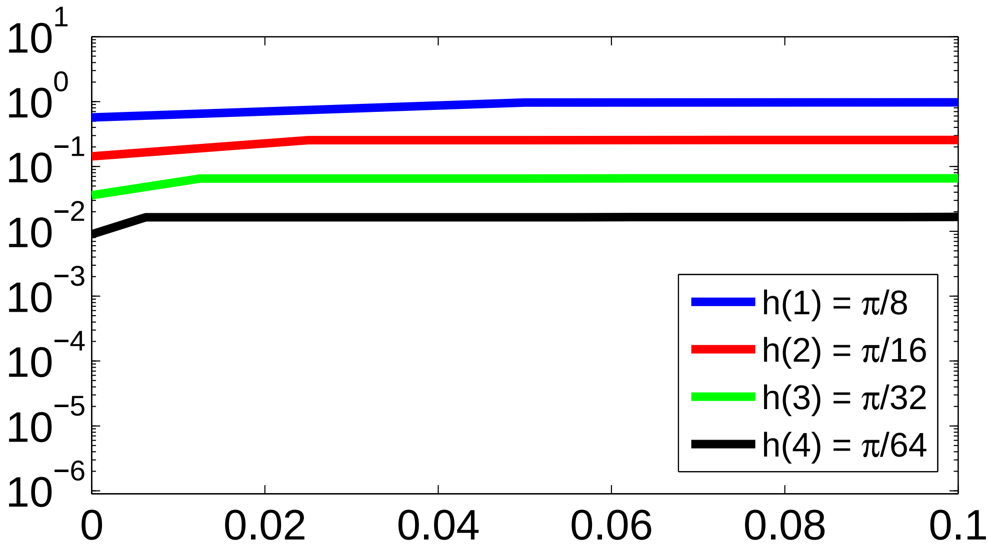

Figure 7.11a: *This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the L^2-norm i.e.,* $\max_{0\leq n\leq m} \beta_{BE,n}(U^n)$ *for* $\mathbb{P}_1$ *elements.*

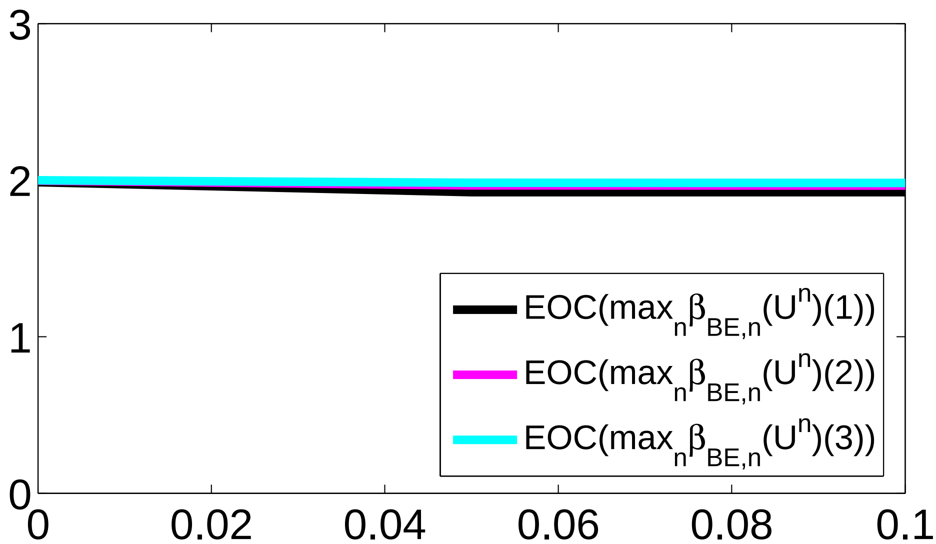

Figure 7.11b: *For the coupling* $\tau \approx h$, $\max_{0\leq n\leq m}\beta_{BE,n}(U^n)$ *yields superconvergence.*

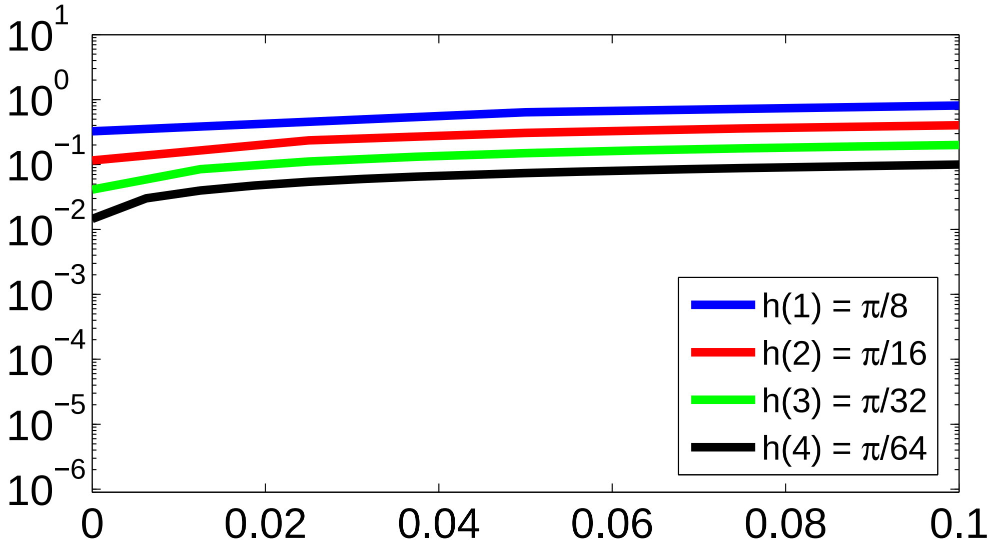

Figure 7.12a: *This figure shows the behaviour of Ritz-Volterra reconstruction error estimator for the H^1-norm i.e.,* $\left(\sum_{n=1}^{m} \tau_n \alpha^2_{BE,n}(U^n)\right)^{1/2}$ *for $\mathbb{P}_1$ elements.*

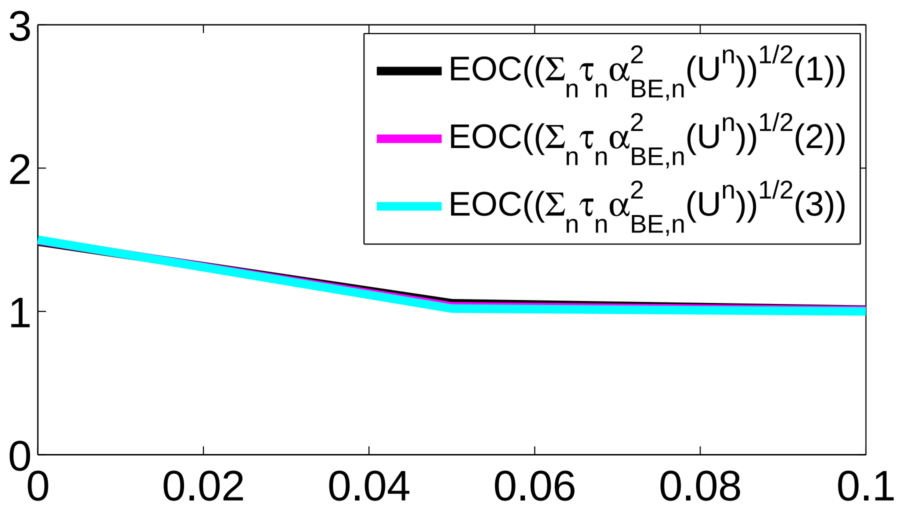

Figure 7.12b: *For the coupling $\tau \approx h$,* $\left(\sum_{n=1}^{m} \tau_n \alpha^2_{BE,n}(U^n)\right)^{1/2}$ *decreases with first order.*

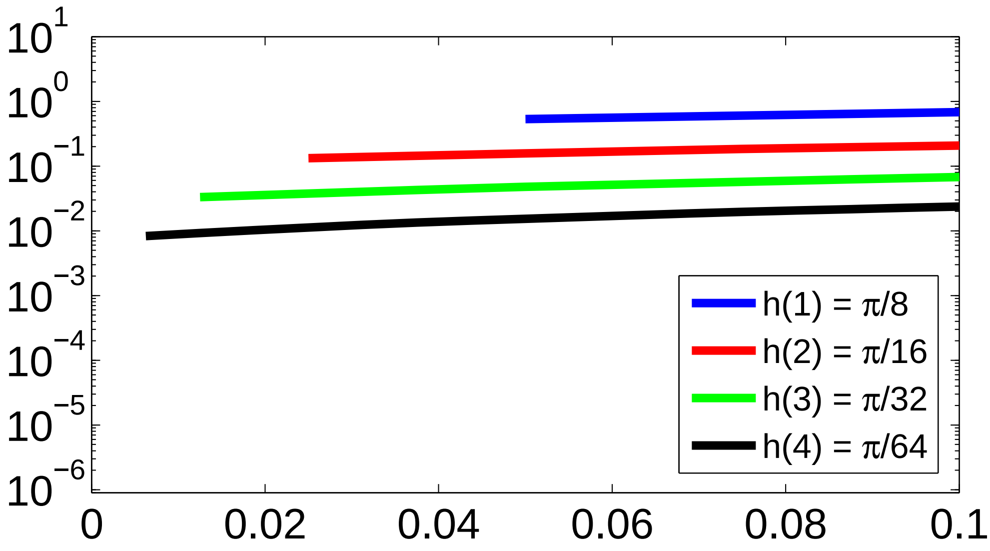

Figure 7.13a: *This figure shows the behaviour of space estimator* $\sum_{n=1}^{m} \tau_n \zeta_{BE,n}$ *for* $\mathbb{P}_1$ *elements.*

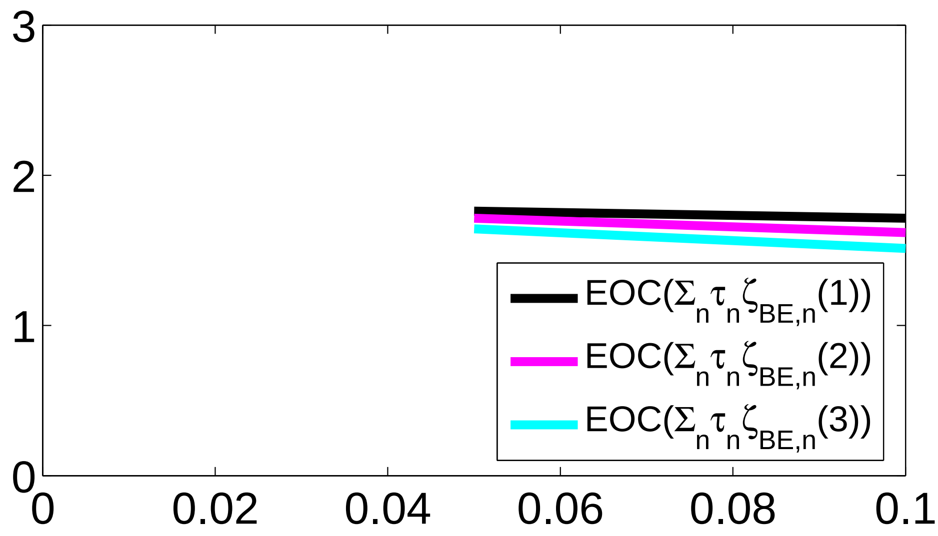

Figure 7.13b: *For the coupling* $\tau \approx h$, $\sum_{n=1}^{m} \tau_n \zeta_{BE,n}$ *decreases with at least first order.*

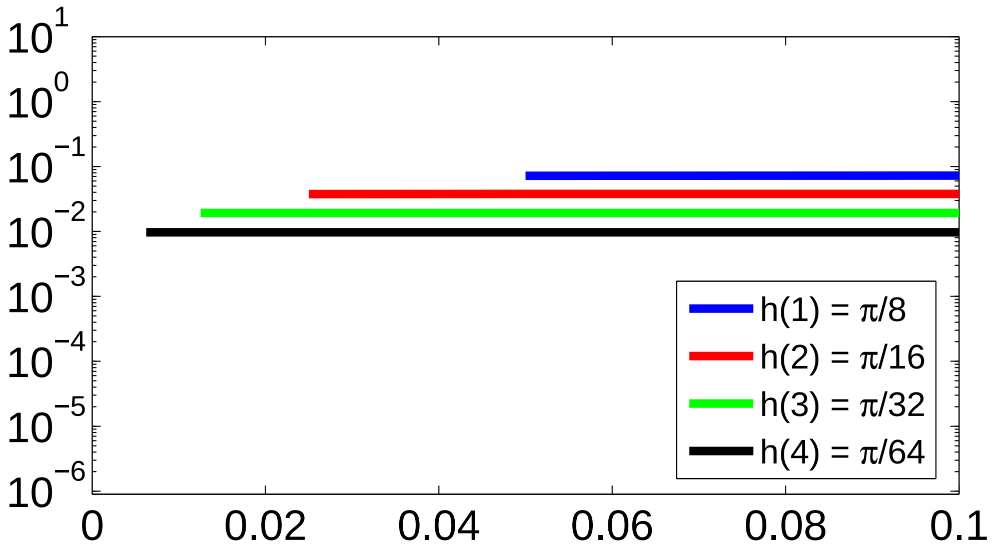

Figure 7.14a: *This figure shows the behaviour of time estimator* $\sum_{n=1}^{m} \tau_n \eta_{BE,n}$ *for* $\mathbb{P}_1$ *elements.*

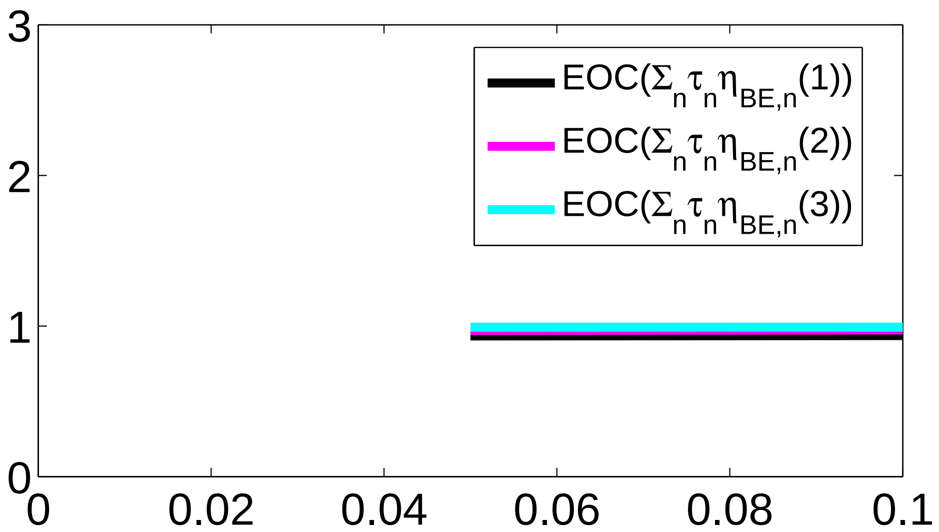

Figure 7.14b: *For the coupling* $\tau \approx h$, $\sum_{n=1}^{m} \tau_n \eta_{BE,n}$ *decreases with first order.*

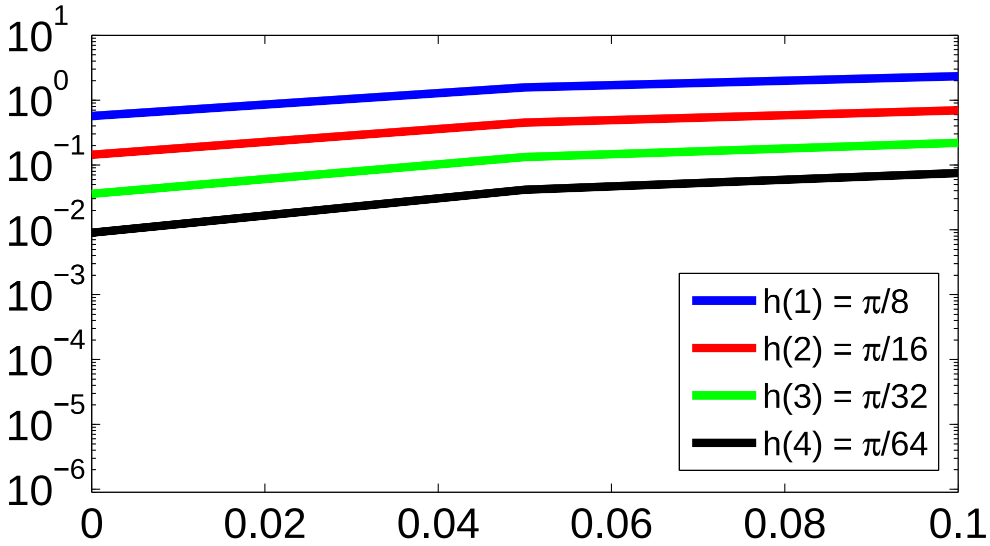

Figure 7.15a: *This figure shows the behaviour of the $L^\infty(L^2(\Omega))$ total estimator for $\mathbb{P}_1$ elements.*

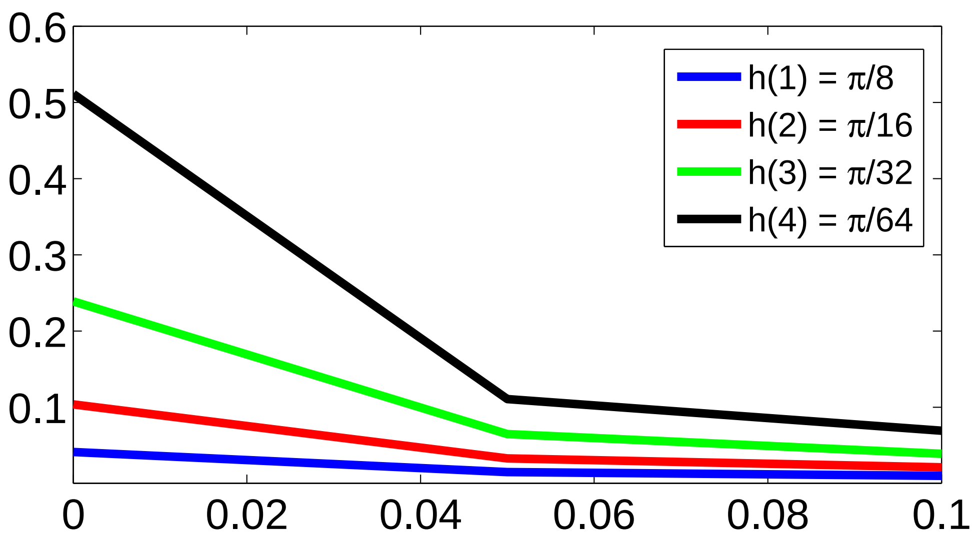

Figure 7.15b: *This figure shows the inverse effectivity index of the $L^\infty(L^2(\Omega))$ total estimator for the coupling $\tau \approx h$.*

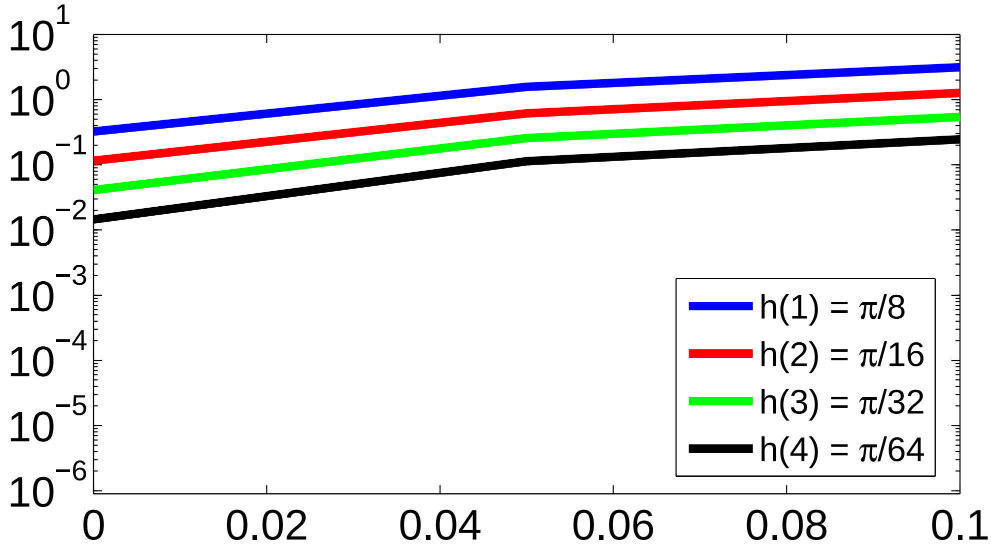

Figure 7.16a: This figure shows the behaviour of the $L^2(H^1(\Omega))$ total estimator for $\mathbb{P}_1$ elements.

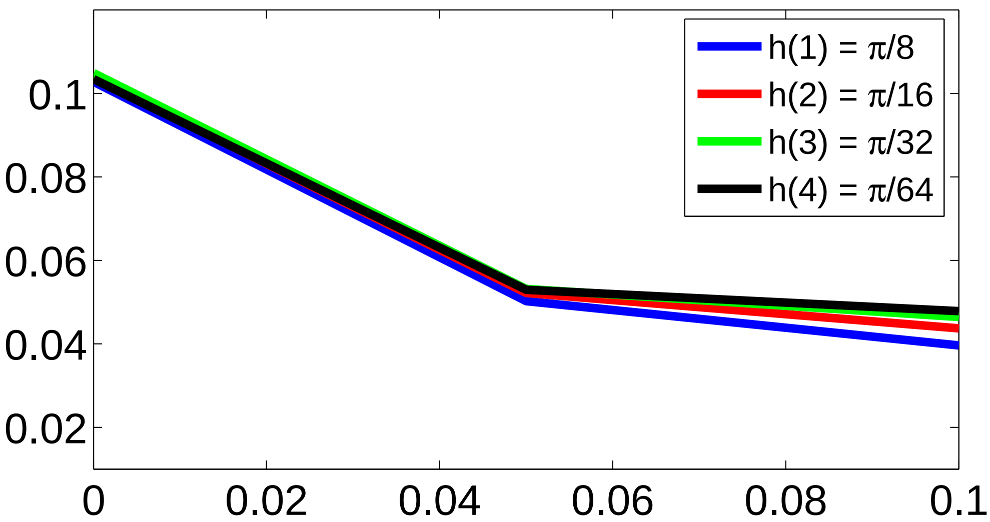

Figure 7.16b: *This figure shows the inverse effectivity index of the* $L^2(H^1(\Omega))$ *total estimator for the coupling* $\tau \approx h$.

Example 7.2.

This example is devoted to study the behaviour of the *a posteriori* error estimator presented in Theorem 4.3.1 for the fully discrete Crank-Nicolson scheme. We consider the same PIDE (7.1) with homogeneous Dirichlet boundary conditions in a square domain $\Omega = (0,\pi)^2 \subset \mathbb{R}^2$ with the exact solution is given by (7.2). We choose a sequence of meshsizes $(h(i) : i \in [1 : l])$, to which we couple a sequence of step sizes $(\tau(i) : i \in [1 : l])$ i.e., $\tau(i) = c_1 h(i)$, where c_1 is taken to be $\frac{0.4}{\pi}$. The initial mesh size $h(1)$ and the time step $\tau(1)$ are chosen to be $\frac{\pi}{8}$ and 0.05.

Here, we compute the following quantities of interest:

- The error norm:

$$\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$$

- The Ritz-Volterra reconstruction error estimator:

$$\max_{0\le n\le m} \beta_{CN,n}$$

- The space error estimator:

$$\sum_{n=1}^{m} \tau_n \zeta_{CN,n}$$

- The time reconstruction error estimators:

$$\left(\sum_{n=1}^{m} \tau_n \Lambda^2_{CN,n}\right)^{1/2} \quad \text{and} \quad \max_{0\le n\le m} \nu_{CN,n}$$

for each time $t_m \in [0 = t_0 : \tau(i) : t_N = .1]$. Here, the time error estimator i.e., $\sum_{n=1}^{m} \tau_n \eta_{CN,n}$ is zero for $\mathbb{P}_1$ elements. The estimators for linear approximation of the memory term, data approximation and mesh change are dropped from study. For each quantity of interest, we observe its EOC. Moreover, IEI is computed for the total estimator for the $L^\infty(L^2(\Omega))$-norm of Theorem 4.3.1, which is defined by

$$IEI(L^\infty(L^2(\Omega))\ Estimator) = \frac{\|e\|_{L^\infty(0,t_m;L^2(\Omega))}}{\max\limits_{0\le n\le m} \beta_{CN,n} + \max\limits_{0\le n\le m} \nu_{CN,n} + \sum_{n=1}^{m} \tau_n \zeta_{CN,n} + \left(\sum_{n=1}^{m} \tau_n \Lambda^2_{CN,n}\right)^{1/2}}.$$

The constants involved in the estimators are taken to be 1 except Gronwall's constant which is taken to be $\exp(T)$.

Here, in each of the plot in the Figures 7.17a-7.22b, the abscissa represents time. The first plot in each of the Figure 7.17-7.22 shows the error or estimator behaviour to a given run. The most coarse grid corresponds to curve with the largest error value and the finest grid corresponds to curve with the smallest error value. The second plot in Figures 7.17-7.21 corresponds to EOC of the quantity of interest and Figure 7.22b corresponds to the IEI of the total estimator for the $L^\infty(L^2(\Omega))$-norm. The value of EOC of a given quantity of interest indicates its order.

From the Figures 7.17-7.21, it is apparent that the error $\|e\|_{L^\infty(0,t_m;L^2(\Omega))}$ has second order of convergence and the estimators have the optimal rate of convergence which matches with that of the error's norm. Moreover, for this example the estimator $\max_{0\le n\le m}\beta_{CN,n}$ dominates the other estimators, namely space estimator, time estimator and time reconstruction estimators.

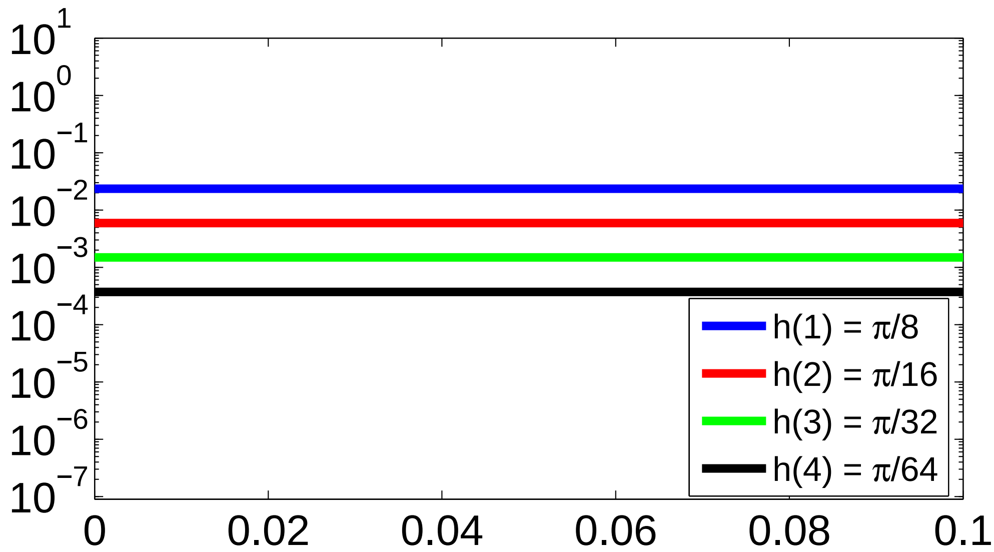

Figure 7.17a: This figure shows the behaviour of the error $\|e\|_{L^\infty(L^2(\Omega))}$ for $\mathbb{P}_1$ elements.

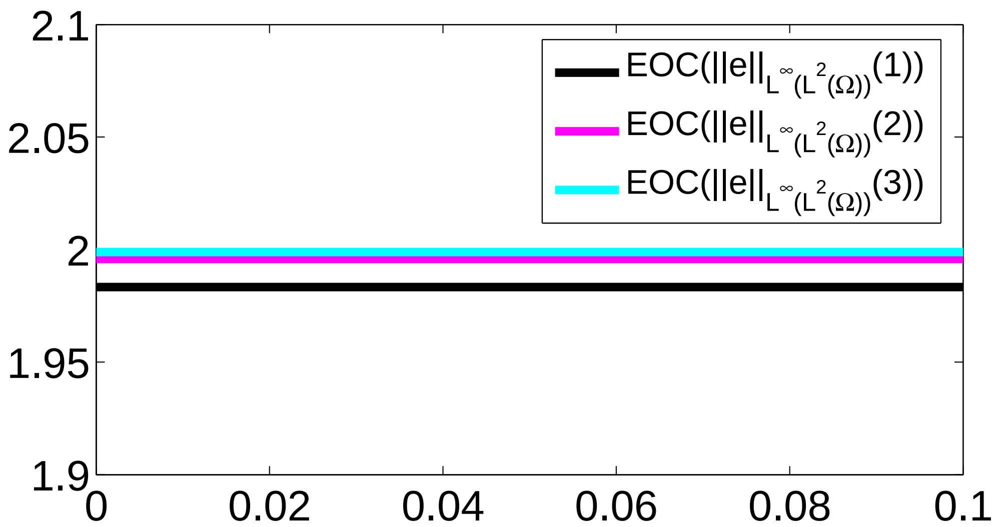

Figure 7.17b: *We observe that* $L^\infty(L^2(\Omega))$ *error is of* $O(h^2 + \tau^2)$.

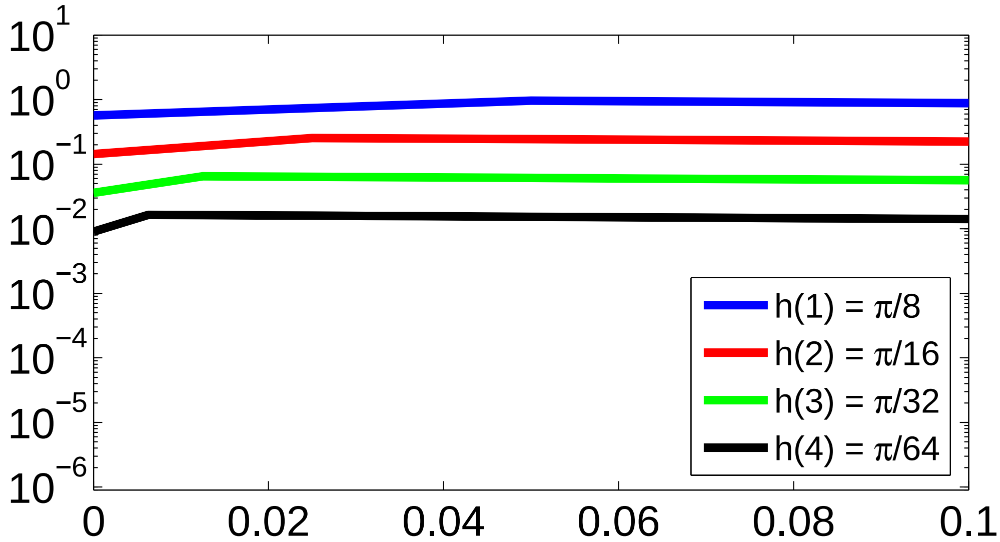

Figure 7.18a: *This figure shows the behaviour of the Ritz-Volterra reconstruction error estimator for the* L^2*-norm i.e.,* $\max_{0\leq n\leq m} \beta_{CN,n}$ *for* $\mathbb{P}_1$ *elements.*

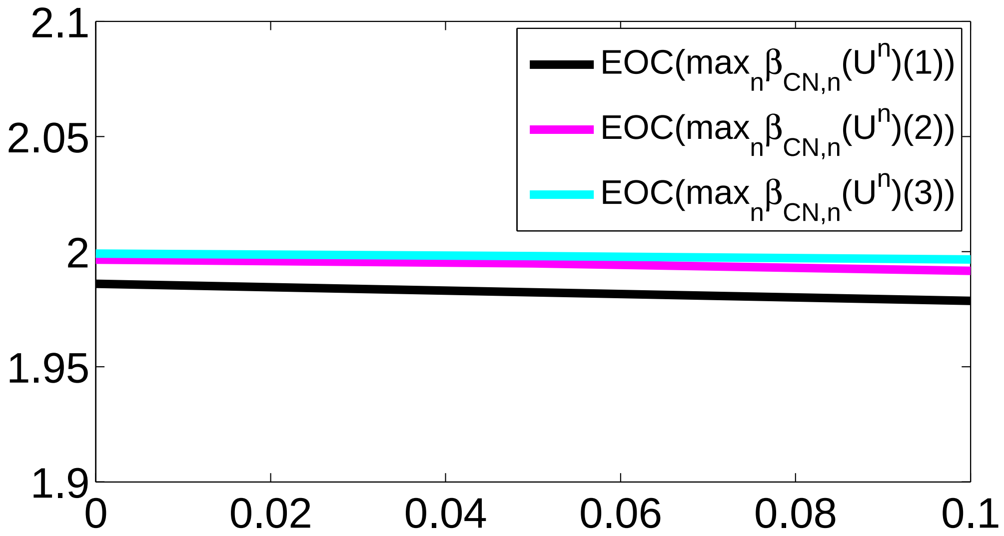

Figure 7.18b: *This figure shows that* $\max_{0\leq n\leq m} \beta_{CN,n}$ *decreases with second order.*

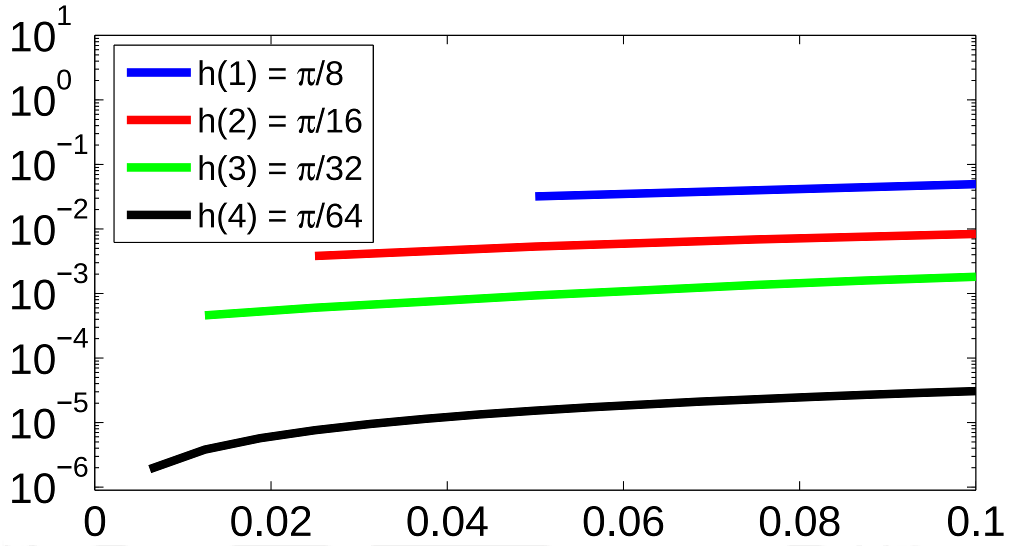

Figure 7.19a: *This figure shows the behaviour of the space estimator* $\sum_{n=1}^{m} \tau_n \zeta_{CN,n}$ *for* $\mathbb{P}_1$ *elements.*

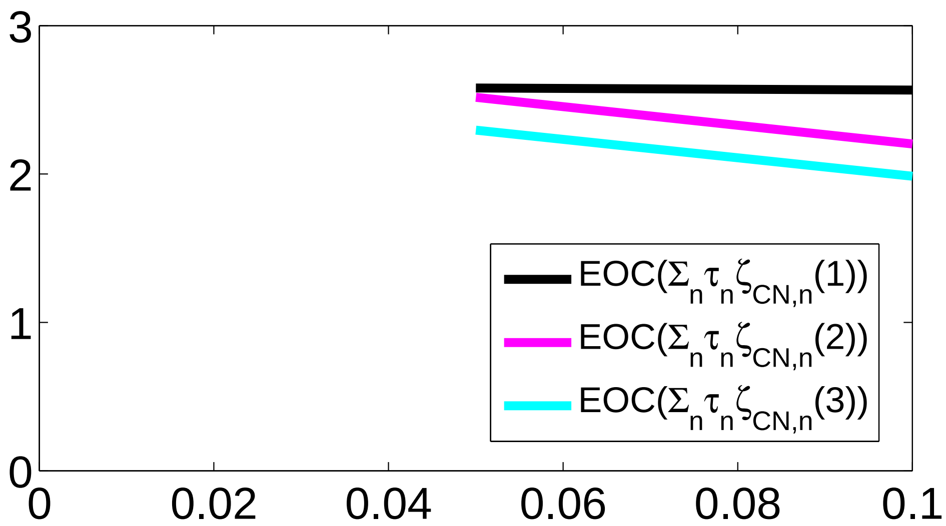

Figure 7.19b: *This figure shows that* $\sum_{n=1}^{m} \tau_n \zeta_{CN,n}$ *decreases with second order.*

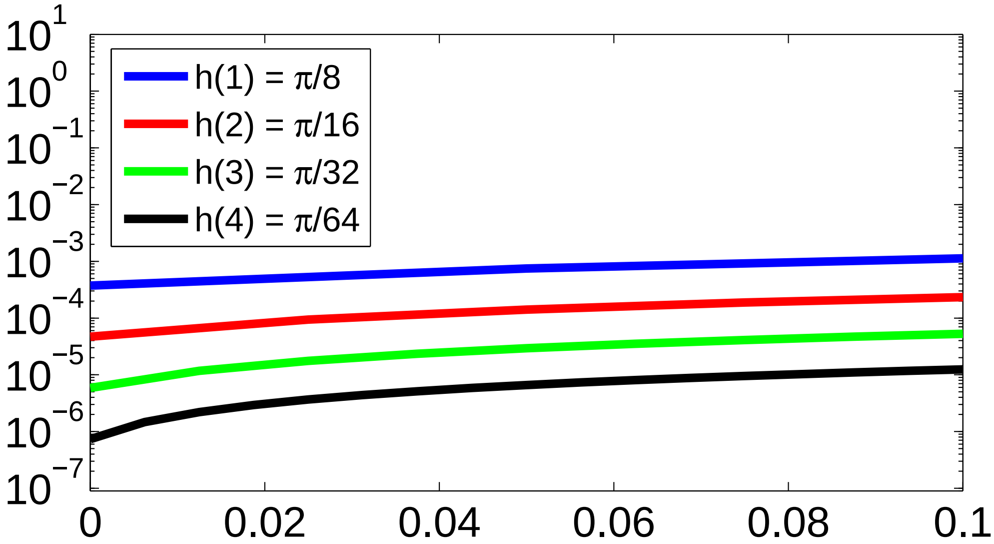

Figure 7.20a: *This figure shows the behaviour of the time reconstruction estimator* $\left(\sum_{n=1}^{m} \tau_n \Lambda_{CN,n}^2\right)^{1/2}$ *for* $\mathbb{P}_1$ *elements.*

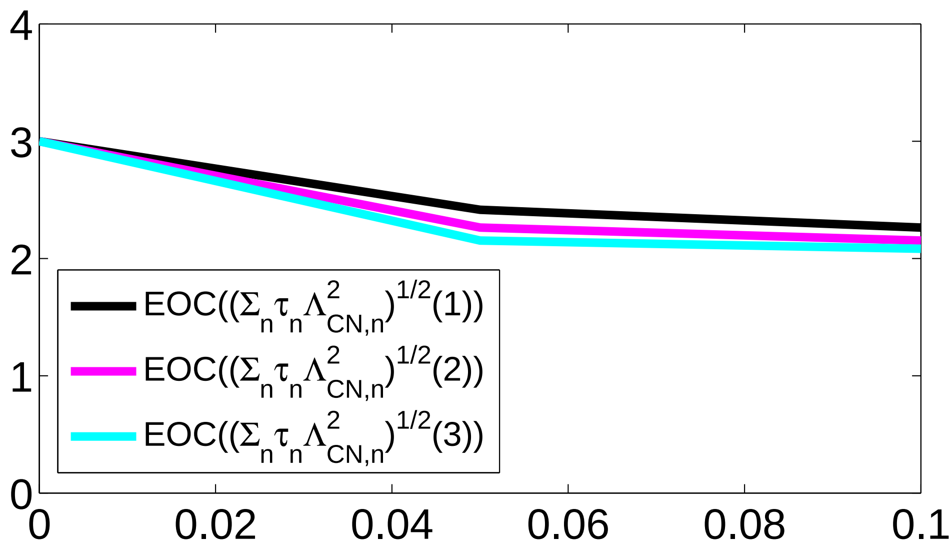

Figure 7.20b: *This figure shows that* $\left(\sum_{n=1}^{m} \tau_n \Lambda_{CN,n}^2\right)^{1/2}$ *decreases with second order.*

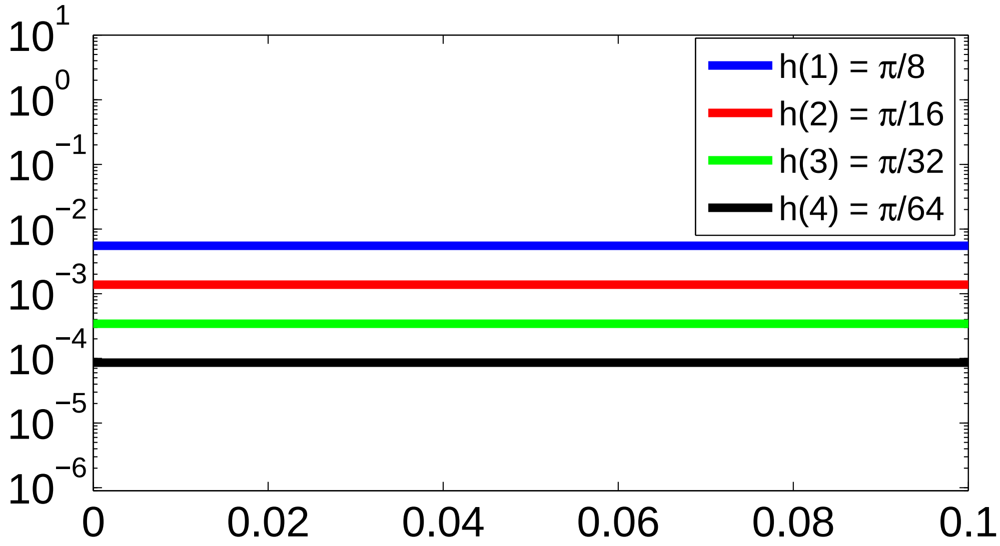

Figure 7.21a: *This figure shows the behaviour of the time reconstruction estimator* $\max_{0\leq n\leq m} \nu_{CN,n}$ *for* $\mathbb{P}_1$ *elements.*

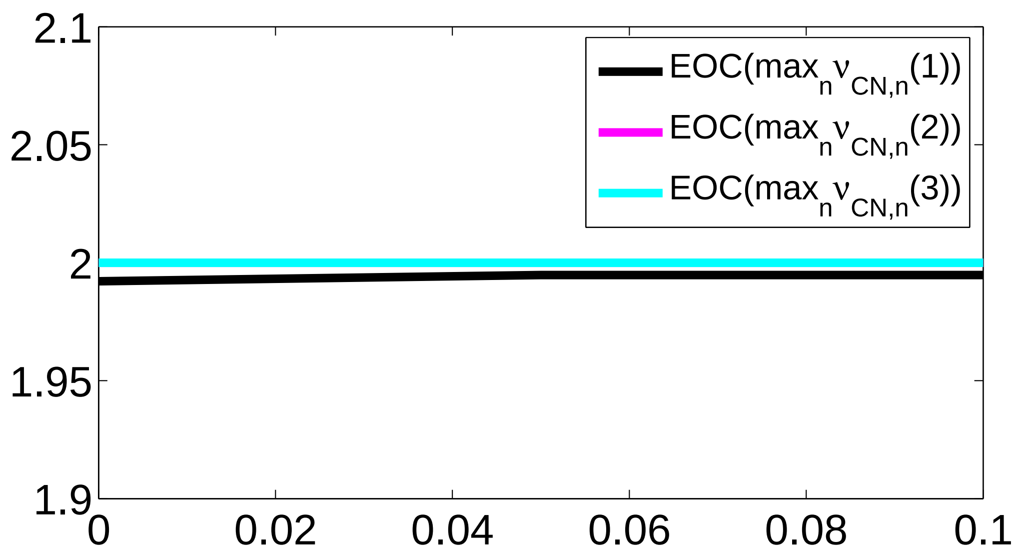

Figure 7.21b: *This figure shows that* $\max_{0\leq n\leq m} \nu_{CN,n}$ *decreases with second order.*

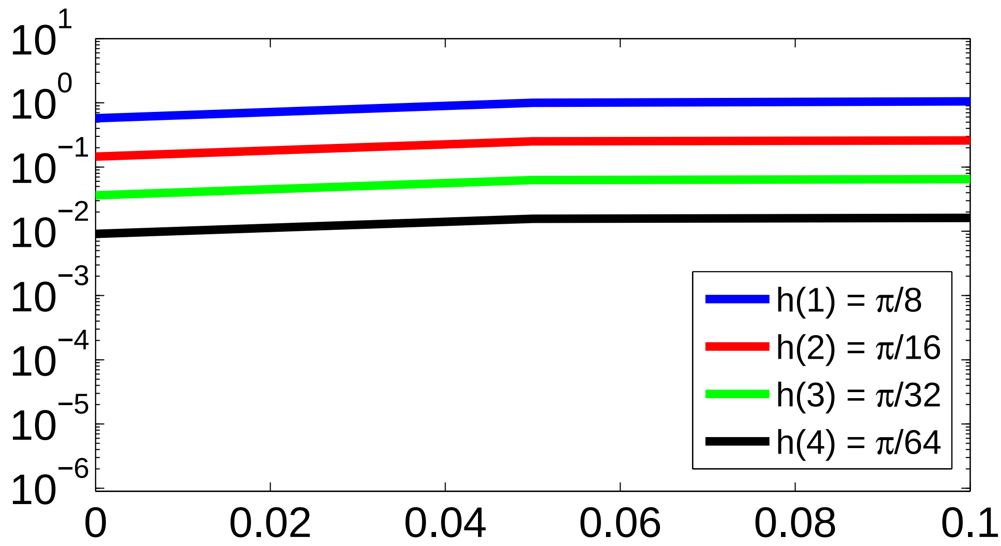

Figure 7.22a: *This figure shows the behaviour of the* $L^\infty(L^2(\Omega))$ *total estimator for* $\mathbb{P}_1$ *elements.*

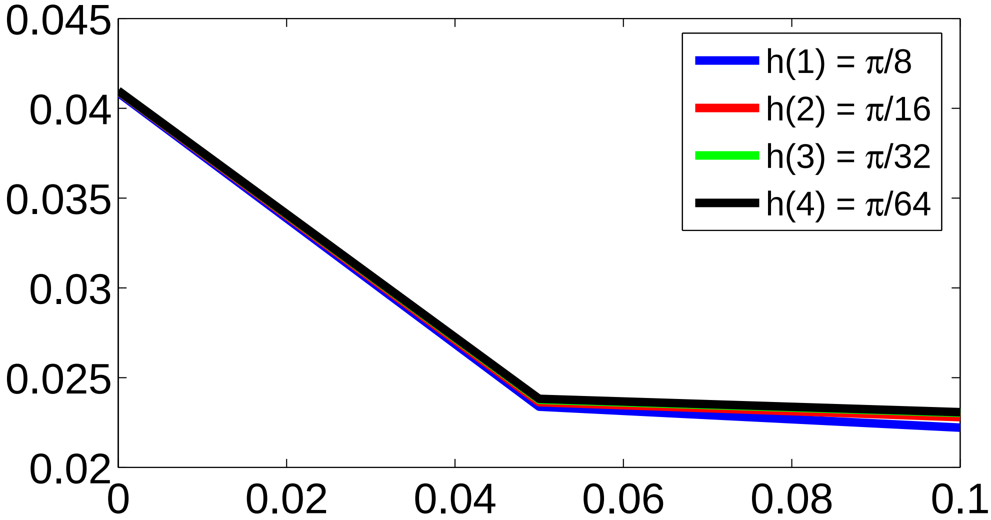

Figure 7.22b: *This figure shows the inverse effectivity index of the* $L^\infty(L^2(\Omega))$ *total estimator.*

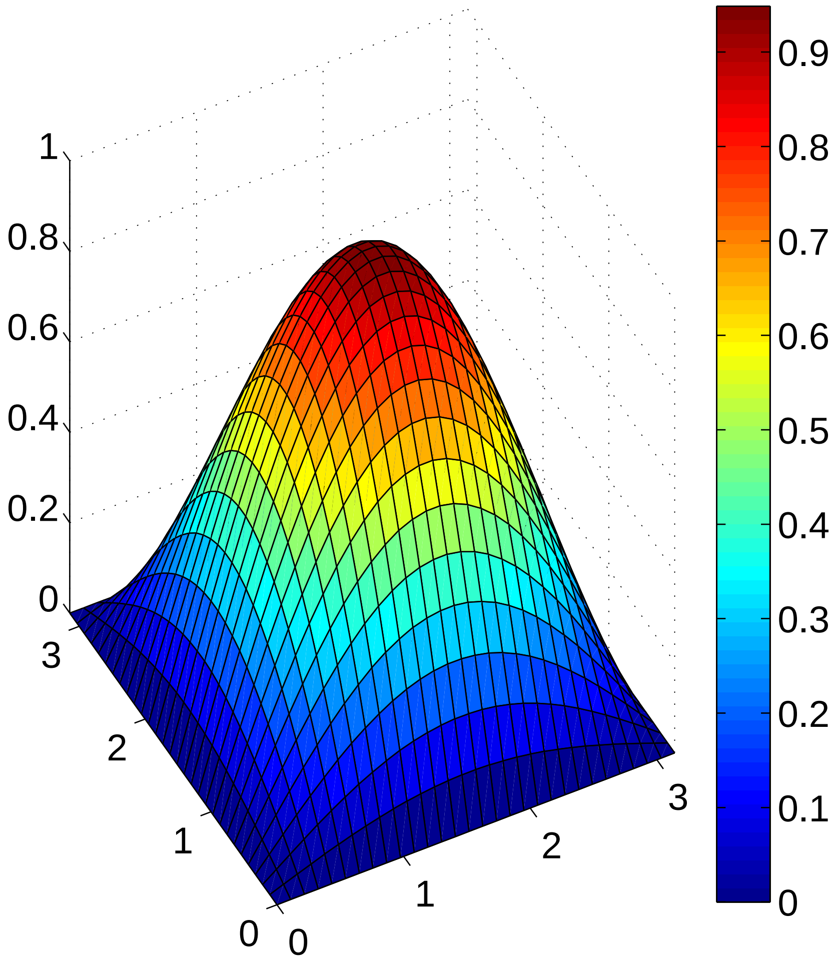

Figure 7.23: *Exact Solution of the PIDE (7.1) at $T = 0.1$.*

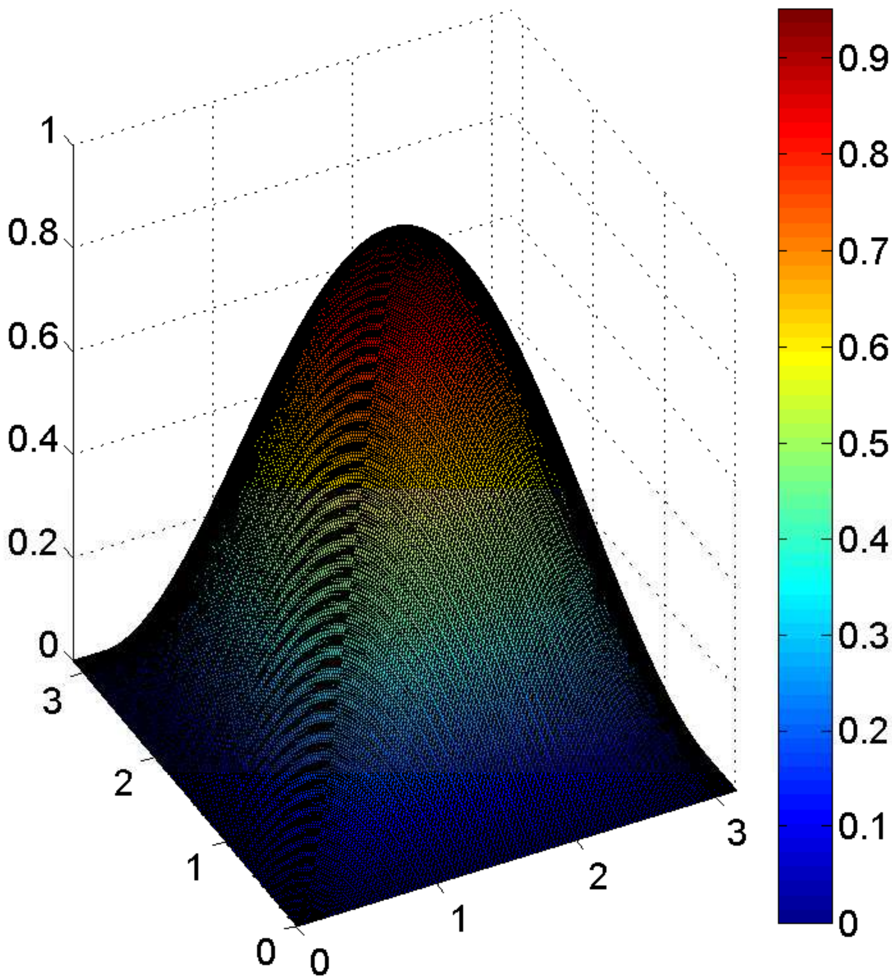

Figure 7.24: *The backward Euler FEM solution for the PIDE (7.1) is simulated using* $\mathbb{P}_1$ *elements and computed using 33025 free nodes at* $T = 0.1$ *corresponding to* $\tau = .003125$.

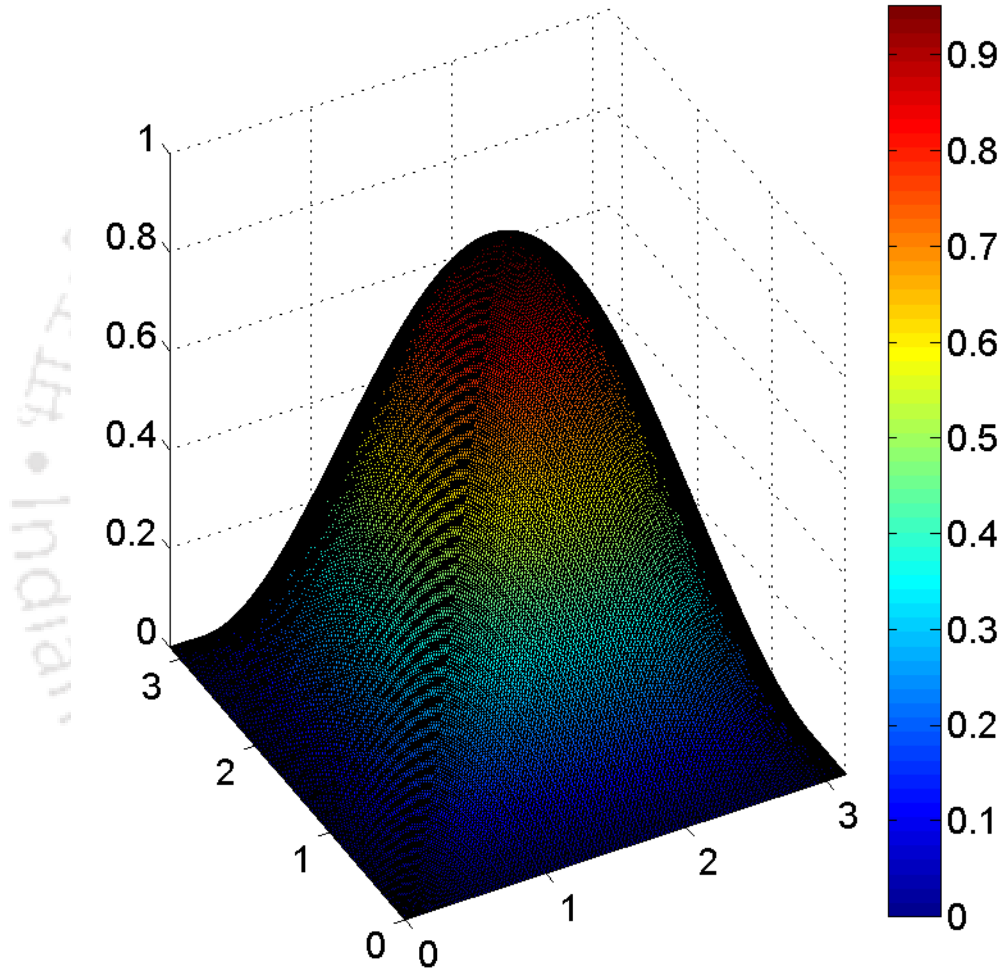

Figure 7.25: *The Crank-Nicolson FEM solution for the PIDE (7.1) is simulated using* $\mathbb{P}_1$ *elements and computed using 33025 free nodes at* $T = 0.1$ *corresponding to* $\tau = .003125$.

8 Conclusions and Extensions

This chapter is devoted to the critical assessment of the results highlighting the contributions made by this thesis and the techniques used in deriving these. It also provides information for the scope of possible extensions and future investigations.

8.1 Critical review of the results

In this thesis, the study was set out to explore the *a posteriori* error analysis of finite element methods for PIDEs of the form (1.1). *A posteriori* error bounds are derived on both the isotropic and anisotropic meshes. We believe that the work presented in this thesis could be a first step towards the development of various space-time adaptive algorithms for PIDEs. The Ritz-Volterra reconstruction introduced in this thesis unifies *a posteriori* approach from parabolic problems to PIDEs. The critical review of the results for each chapter is presented below.

Chapter 2 is devoted to *a posteriori* error analysis for the spatially semidiscrete scheme for the PIDE (1.1). The basic essential tool used for the error analysis is the Ritz-Volterra reconstruction operator, originated naturally from the theory of elliptic Volterra equations. The Ritz-Volterra reconstruction of the finite element solution is used as an intermediate solution to derive *a posteriori* error estimates in the $L^\infty(L^2(\Omega))$-norm (cf. Theorem 2.3.1). The *a posteriori* error estimates for the main error (e) in turn depends upon the *a posteriori* bounds on the parabolic error and the Ritz-Volterra reconstruction error. We use energy argument to derive an estimate for the related parabolic error (cf. Lemma 2.3.3) which again involves the time derivative of the Ritz-Volterra reconstruction error. The *a posteriori* error bounds on the Ritz-Volterra reconstruction error (cf. Lemma 2.3.1) and its temporal derivative (cf. Lemma 2.3.2) are established using the duality technique.

The *a posteriori* error analysis for the fully discrete backward Euler time discretization scheme for PIDE (1.1) is discussed in Chapter 3. Optimal *a posteriori* error bounds for the main error in the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms (cf. Theorem 3.2.1) are established. The *a posteriori* error bounds on the Ritz-Volterra reconstruction error is obtained in both the H^1 and L^2-norms (cf. Lemma 3.2.2). We use energy technique to estimate the parabolic error (cf. Lemma 3.2.3). Moreover, the estimation for the parabolic error relies on the estimation for the *space discretization error* (cf. Lemma 3.2.6), *time discretization error* (cf. Lemma 3.2.7), *mesh change error* (cf. Lemma 3.2.8) and *data oscillation error* (cf. Lemma 3.2.9).

Chapter 4 deals with *a posteriori* error analysis for PIDE (1.1) concerning fully discrete Crank-Nicolson scheme. The task of getting second order convergence in time entails a thought of carefully introducing appropriate space-time quadratic (in time) reconstruction operator. The choice of such an operator is non-trivial and is highly problem dependent. The ancestral idea of splitting the main error using space-time reconstruction operator together with the Ritz-Volterra reconstruction operator yields the *a posteriori* error estimator for PIDE (1.1) in the $L^\infty(L^2(\Omega))$-norm (cf. Theorem 4.3.1).

Chapter 5 deals with anisotropic *a posteriori* error analysis of PIDE (1.1). We derive two estimators (cf. Theorem 5.2.1 and Theorem 5.2.2) for the fully discrete backward Euler scheme. The residual based *a posteriori* error bounds in the $L^2(H^1(\Omega))$-norm are obtained. The *a posteriori* error indicator corresponding to space discretization is derived using the anisotropic interpolation estimates in conjunction with a ZZ error estimator to approach the error gradient. The error due to time discretization is derived using a continuous, piecewise linear polynomial in time. A linear approximation of the

Volterra integral term is used to estimate the quadrature error in the second estimator.

Chapter 6 is devoted to study the anisotropic *a posteriori* error estimates (cf. Theorems 6.3.1 and 6.3.2) for PIDE (1.1) concerning fully discrete Crank-Nicolson scheme. The anisotropic interpolation estimates and ZZ error estimator are used to obtain *a posteriori* error contributions corresponding to space discretization. However, for the time discretization error, a continuous, piecewise quadratic Crank-Nicolson memory reconstruction is introduced. While analyzing with the Crank-Nicolson memory reconstruction, a linear approximation of the Volterra integral term is used in a crucial way to estimate the quadrature error for the approximation of the memory term. However, due to the presence of the memory term this reconstruction depends on all the previous time levels and therefore, it is not locally defined in time. Three point reconstruction (based on two subintervals) is then introduced. Moreover, an extended linear approximation of the Volterra integral term is used to estimate the error due to the quadrature approximation of the memory term while analyzing with the three point reconstruction. One can recover optimal order isotropic error estimators through the anisotropic error estimators obtained in Chapters 5-6.

Chapter 7 is concerned with numerical experiments for two dimensional test problems to illustrate the theoretical findings of Chapters 3-4. All computations are carried out using the software MATLAB 7.8. Bisection algorithm is used to generate the refined triangulations. The main emphasis is to observe the asymptotic convergence of the estimators so that the inverse effectivity index is to be understood only qualitatively. Numerical experiments show the performance of the estimators. Moreover, the estimators decrease with optimal order which matches with that of the error's norm. The different terms in the estimators capture the spatial and the temporal errors separately.

8.2 Extensions and remarks

Despite being so rich in the *a priori* analysis and in spite of the importance of PIDE (1.1), and their variants in the modelling of several physical phenomena, the topic of *a posteriori* analysis for such kind of equations remains unexplored. To the best of our knowledge, this is the first instance that *a posteriori* error estimates of finite element method for the space-time discretization of PIDEs are revealed. Moreover, the techniques developed in this thesis unlock several new research directions and interesting open problems for future investigation. Below, we shall briefly outline some interesting problems to be persuaded in future.

Reducing storage cost while time stepping. During a time stepping scheme, one of the practical difficulties associated with PIDEs is that all the values of the discrete solution U^n need to be stored at all previous time levels when the integral term is approximated with some quadrature rule (see Chapters 3-6). Thus, it leads to a great demand of data storage and the computations. To reduce the memory and computational requirements during time stepping, Sloan and Thomée [88] have first proposed the application of quadrature rules with relatively higher-order truncation error. Moreover, they have analyzed this issue in the *a priori* settings using the higher regularity of the exact solution u in time. Since the *a posteriori* error estimates relies on the finite element solution U, it would be interesting to see how the ideas in [88] can be accomplished in an *a posteriori* framework.

Non-smooth data error analysis. In Chapters 2-6, the *a posteriori* results were examined under the hypothesis that the initial data is smooth. It is a known fact that the solution of a homogenous linear parabolic equation (i.e., when $\mathcal{B}(t,s)=0$) is smooth for positive time t, even when the initial data is not (cf. [62]). In quantitative form, it can be expressed by the inequality

$$\|u(t)\|_\alpha \leq Ct^{-\alpha/2}\|u_0\|, \quad t \in (0,T], \tag{8.1}$$

which is valid for any $\alpha \geq 0$. However, this is not the case with PIDEs as they have a limited smoothing property due to the presence of the integral term. It has been shown in [95] that the inequality (8.1) is valid only for $\alpha \leq 2$.

Thus, it would be a challenging problem to study the convergence analysis of the proposed method for semidiscrete and fully-discrete schemes for both homogenous and non-homogeneous PIDEs with non-smooth initial data i.e., when the initial function $u_0 \in L^2(\Omega)$. We strongly believe that the Ritz-Volterra reconstruction introduced in Chapters 2-3 will play a crucial role to tackle this issue.

Weakly singular PIDEs. Consider PIDEs of the form

$$u_t(x,t) + \mathcal{A}u(x,t) = \int_0^t K(t-s)\mathcal{B}u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{8.2}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{8.3}$$

and boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{8.4}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$ is a smooth domain with boundary $\partial\Omega$ and $T < \infty$. Here, $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator and $\mathcal{B}$ is a time independent second-order linear partial differential operator. Further, $K(t)$ is a weakly singular kernel such that

$$K(t) \leq Ct^{-\alpha} \text{ with } 0 \leq \alpha < 1, \quad t \in (0,T].$$

The main difficulty in case of weakly singular kernel is that the regularity of the solution with respect to time is limited, which makes higher order quadrature formulas less attractive. This in turn makes the problem (8.2) more difficult and requires more complicated techniques to deal with (see Chen, Thomée and Wahlbin [24]).

It would be interesting to see how the analysis presented in Chapters 2-6 can be extended to the PIDEs with weakly singular kernel.

Long time estimates. While deriving *a posteriori* error estimates in Chapters 2-6, we have used the Gronwall's lemma. Consequently, the constants appeared in the *a posteriori* error bounds depend on the final time T and will grow exponentially with T.

We would like to study the problem of obtaining *a posteriori* error estimates with the constant appeared in the bounds be independent of the final time T and hence they can serve as long-time estimates.

Extension of the results to hyperbolic integro-differential equations. Consider the initial boundary value problem for the hyperbolic integro-differential equation:

$$\begin{aligned} u_{tt}(x,t) + \mathcal{A}u(x,t) &= \int_0^t \mathcal{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T], \qquad (8.5) \\ u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) &= u_0(x), \quad x \in \Omega \\ u_t(x,0) &= u_1(x), \quad x \in \Omega. \end{aligned}$$

Here, $\Omega \subset \mathbb{R}^n, n \geq 1$ is a smooth domain with boundary $\partial\Omega$, $u_{tt}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$ and $T < \infty$. Further, $\mathcal{A}$ is a self-adjoint, uniformly positive definite second-order linear elliptic partial differential operator and the operator $\mathcal{B}(t,s)$ is an arbitrary partial differential operator of second order. The functions u_0, u_1 and f are assumed to be smooth.

The problem of obtaining *a posteriori* error analysis for hyperbolic integro-differential equations in the $L^\infty(L^2(\Omega))$ and $L^2(H^1(\Omega))$-norms is still widely open. It would be interesting to see how the *a posteriori* error analysis of Chapter 2, Chapter 4 and Chapter 6 can be adopted to obtain *a posteriori* error estimates for hyperbolic integro-differential equation (8.5).

Computational issues. The only omission within the realm of this thesis is numerical study of the behaviour of the anisotropic error estimators. Due to the difficulty of generating the anisotropic mesh, we have not discussed the numerical behaviour of the estimators presented in Chapters 5-6. The numerical implementation of the proposed error bounds to study the behaviour of the estimators by designing the adaptive algorithms is a challenging work which deserves special attention and will be taken up in future.

Bibliography

[1] P. Acquistapace and B. Terreni, *Existence and sharp regularity results for linear parabolic nonautonomous integro-differential equations*, Israel J. Math., 53 (1986), pp. 257–303.

[2] R. A. Adams and J. F. Fournier, *Sobolev Spaces*, Elsevier/Academic Press, Amsterdam, 2003.

[3] M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, Wiley-Interscience, New York, 2000.

[4] S. Aizicovici, *Time-dependent Volterra integro-differential equations*, J. Integral Equations, 10 (1985), pp. 45–60.

[5] G. Akrivis, C. Makridakis, and R. H. Nochetto, *A posteriori error estimates for the Crank-Nicolson method for parabolic equations*, Math. Comp., 75 (2006), pp. 511–531.

[6] T. Apel, *Anisotropic Finite Elements: Local Estimates and Applications*, B. G. Teubner, Stuttgart, 1999.

[7] T. Apel and M. Dobrowolski, *Anisotropic interpolation with applications to the finite element method*, Computing, 47 (1992), pp. 277–293.

[8] I. Babuška and A. K. Aziz, *On the angle condition in the finite element method*, SIAM J. Numer. Anal., 13 (1976), pp. 214–226.

[9] I. Babuška and W. C. Rheinboldt, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[10] ——, *A posteriori error estimates for the finite element method*, Int. J. Numer. Meth. Engng., 12 (1978), pp. 1597–1615.

[11] D. Bahuguna and V. Raghavendra, *Rothe's method to parabolic integro-differential equations via abstract integro-differential equations*, Appl. Anal., 33 (1989), pp. 153–167.

[12] E. Bänsch, F. Karakatsani, and C. Makridakis, *A posteriori error control for fully discrete Crank-Nicolson schemes*, SIAM J. Numer. Anal., 50 (2012), pp. 2845–2872.

[13] S. Bartels and C. Carstensen, *Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. II. Higher order FEM*, Math. Comp., 71 (2002), pp. 971–994.

[14] A. Belleni-Morante, *An integro-differential equation arising from the theory of heat conduction in rigid materials with memory*, Boll. Un. Mat. Ital. B (5), 15 (1978), pp. 470–482.

[15] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, New York, 2008.

[16] H. Brunner, *A survey of recent advances in the numerical treatment of Volterra integral and integro-differential equations*, J. Comput. Appl. Math., 8 (1982), pp. 213–229.

[17] J. R. Cannon and Y. Lin, *Nonclassical H^1 projection and Galerkin methods for nonlinear parabolic integro-differential equations*, Calcolo, 25 (1988), pp. 187–201.

[18] J. R. Cannon and Y. P. Lin, *Smooth solutions for an integro-differential equation of parabolic type*, Differential Integral Equations, 2 (1989), pp. 111–121.

[19] ——, *A priori L^2 error estimates for finite-element methods for nonlinear diffusion equations with memory*, SIAM J. Numer. Anal., 27 (1990), pp. 595–607.

[20] V. Capasso, *Asymptotic stability for an integro-differential reaction-diffusion system*, J. Math. Anal. Appl., 103 (1984), pp. 575–588.

[21] C. Carstensen and S. Bartels, *Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. I. Low order conforming, nonconforming, and mixed FEM*, Math. Comp., 71 (2002), pp. 945–969.

[22] P. Chatzipantelidis, R. D. Lazarov, V. Thomée, and L. B. Wahlbin, *Parabolic finite element equations in nonconvex polygonal domains*, BIT, 46 (2006), pp. S113–S143.

[23] C. Chen and T. Shih, *Finite Element Methods for Integro-Differential Equations*, World Scientific Publishing Co. Inc., 1998.

[24] C. Chen, V. Thomée, and L. B. Wahlbin, *Finite element approximation of a parabolic integro-differential equation with a weakly singular kernel*, Math. Comp., 58 (1992), pp. 587–602.

[25] G. Chen and R. Grimmer, *Semigroups and integral equations*, J. Integral Equations, 2 (1980), pp. 133–154.

[26] B. D. Coleman, *On thermodynamics, strain impulses, and viscoelasticity*, Arch. Rational Mech. Anal., 17 (1964), pp. 230–254.

[27] B. D. Coleman and M. E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys., 18 (1967), pp. 199–208.

[28] E. G. D'Jakonov, *On the stability of difference schemes for some nonstationary problems*, in Topics in Numerical Analysis, Academic Press, London, 1973, pp. 63–87.

[29] J. de Frutos and J. Novo, *A posteriori error estimation with the p-version of the finite element method for nonlinear parabolic differential equations*, Comput. Methods Appl. Mech. Engrg., 191 (2002), pp. 4893–4904.

[30] J. Douglas, Jr. and B. F. Jones, Jr., *Numerical methods for integro-differential equations of parabolic and hyperbolic types*, Numer. Math., 4 (1962), pp. 96–102.

[31] R. Durán, M. A. Muschietti, and R. Rodríguez, *On the asymptotic exactness of error estimators for linear triangular finite elements*, Numer. Math., 59 (1991), pp. 107–127.

[32] K. Eriksson and C. Johnson, *An adaptive finite element method for linear elliptic problems*, Math. Comp., 50 (1988), pp. 361–383.

[33] ——, *Adaptive finite element methods for parabolic problems. I. A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[34] L. Formaggia and S. Perotto, *New anisotropic a priori error estimates*, Numer. Math., 89 (2001), pp. 641–667.

[35] ——, *Anisotropic error estimates for elliptic problems*, Numer. Math., 94 (2003), pp. 67–92.

[36] A. Friedman and M. Shinbrot, *Volterra integral equations in Banach space*, Trans. Amer. Math. Soc., 126 (1967), pp. 131–179.

[37] L. Galeone, C. Mastroserio, and M. Montrone, *Asymptotic stability of the numerical solution for an integro-differential reaction-diffusion system*, Numer. Methods Partial Differential Equations, 5 (1989), pp. 79–86.

[38] B. García-Archilla and E. S. Titi, *Postprocessing the Galerkin method: the finite-element case*, SIAM J. Numer. Anal., 37 (2000), pp. 470–499.

[39] T. Grätsch and K. J. Bathe, *A posteriori error estimation techniques in practical finite element analysis*, Comput. & Structures, 83 (2005), pp. 235–265.

[40] J. A. Gregory, *Error bounds for linear interpolation on triangles*, in The Mathematics of Finite Elements and Applications, II, Academic Press, London, 1976, pp. 163–170.

[41] R. C. Grimmer, *Resolvent operators for integral equations in a Banach space*, Trans. Amer. Math. Soc., 273 (1982), pp. 333–349.

[42] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[43] S. Grosman, *Adaptivity in Anisotropic Finite Element Calculations*, PhD thesis, TU Chemnitz, 2006.

[44] M. E. Gurtin, *On the themodynamics of materials with memory*, Arch. Rational Mech. Anal., 28 (1968), pp. 40–50.

[45] M. E. Gurtin and A. C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal., 31 (1968), pp. 113–126.

[46] G. J. Habetler and R. L. Schiffman, *A finite difference method for analyzing the compression of poro-viscoelastic media*, Computing, 6 (1970), pp. 342–348.

[47] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1952.

[48] M. L. Heard and S. M. Rankin, III, *A semilinear parabolic Volterra integro-differential equation*, J. Differential Equations, 71 (1988), pp. 201–233.

[49] ——, *Weak solutions for a class of parabolic Volterra integro-differential equations*, J. Math. Anal. Appl., 139 (1989), pp. 78–109.

[50] U. Hornung and R. E. Showalter, *Diffusion models for fractured media*, J. Math. Anal. Appl., 147 (1990), pp. 69–80.

[51] M. Křížek, *On the maximum angle condition for linear tetrahedral elements*, SIAM J. Numer. Anal., 29 (1992), pp. 513–520.

[52] M. Křížek and P. Neittaanmäki, *Superconvergence phenomenon in the finite element method arising from averaging gradients*, Numer. Math., 45 (1984), pp. 105–116.

[53] ——, *On a global superconvergence of the gradient of linear triangular elements*, J. Comput. Appl. Math., 18 (1987), pp. 221–233.

[54] G. Kunert and S. Nicaise, *Zienkiewicz-Zhu error estimators on anisotropic tetrahedral and triangular finite element meshes*, M2AN Math. Model. Numer. Anal., 37 (2003), pp. 1013–1043.

[55] A. M. Lakhany, I. Marek, and J. R. Whiteman, *Superconvergence results on mildly structured triangulations*, Comput. Methods Appl. Mech. Engrg., 189 (2000), pp. 1–75.

[56] O. Lakkis and C. Makridakis, *Elliptic reconstruction and a posteriori error estimates for fully discrete linear parabolic problems*, Math. Comp., 75 (2006), pp. 1627–1658.

[57] M. N. Le Roux and V. Thomée, *Numerical solution of semilinear integro-differential equations of parabolic type with nonsmooth data*, SIAM J. Numer. Anal., 26 (1989), pp. 1291–1309.

[58] Y. P. Lin, V. Thomée, and L. B. Wahlbin, *Ritz-Volterra projections to finite-element spaces and applications to integro-differential and related equations*, SIAM J. Numer. Anal., 28 (1991), pp. 1047–1070.

[59] Y. P. Lin and T. Zhang, *The stability of Ritz-Volterra projection and error estimates for finite element methods for a class of integro-differential equations of parabolic type*, Appl. Math., 36 (1991), pp. 123–133.

[60] A. Lozinski, M. Picasso, and V. Prachittham, *An anisotropic error estimator for the Crank-Nicolson method: application to a parabolic problem*, SIAM J. Sci. Comput., 31 (2009), pp. 2757–2783.

[61] A. Lunardi and E. Sinestrari, *C^α-regularity for nonautonomous linear integro-differential equations of parabolic type*, J. Differential Equations, 63 (1986), pp. 88–116.

[62] M. Luskin and R. Rannacher, *On the smoothing property of the Galerkin method for parabolic equations*, SIAM J. Numer. Anal., 19 (1982), pp. 93–113.

[63] C. Makridakis and R. H. Nochetto, *Elliptic reconstruction and a posteriori error estimates for parabolic problems*, SIAM J. Numer. Anal., 41 (2003), pp. 1585–1594.

[64] W. McLean and V. Thomée, *Numerical solution of an evolution equation with a positive-type memory term*, J. Austral. Math. Soc. Ser. B, 35 (1993), pp. 23–70.

[65] S. Micheletti and S. Perotto, *Reliability and efficiency of an anisotropic Zienkiewicz-Zhu error estimator*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 799–835.

[66] R. K. Miller, *An integro-differential equation for rigid heat conductors with memory*, J. Math. Anal. Appl., 66 (1978), pp. 313–332.

[67] J. A. Nohel, *Some problems in nonlinear Volterra integral equations*, Bull. Amer. Math. Soc., 68 (1962), pp. 323–329.

[68] ——, *Nonlinear Volterra equations for heat flow in materials with memory*, in Integral and Functional Differential Equations, Dekker, New York, 1981, pp. 3–82.

[69] B. G. Pachpatte, *On a nonlinear diffusion system arising in reactor dynamics*, J. Math. Anal. Appl., 94 (1983), pp. 501–508.

[70] A. K. Pani and R. K. Sinha, *Error estimates for semidiscrete Galerkin approximation to a time dependent parabolic integro-differential equation with nonsmooth data*, Calcolo, 37 (2000), pp. 181–205.

[71] A. K. Pani, V. Thomée, and L. B. Wahlbin, *Numerical methods for hyperbolic and parabolic integro-differential equations*, J. Integral Equations Appl., 4 (1992), pp. 533–584.

[72] C. V. Pao, *Solution of a nonlinear integro-differential system arising in nuclear reactor dynamics*, J. Math. Anal. Appl., 48 (1974), pp. 470–492.

[73] ——, *Bifurcation analysis on a nonlinear diffusion system in reactor dynamics*, Appl. Anal., 9 (1979), pp. 107–119.

[74] ——, *On nonlinear reaction-diffusion systems*, J. Math. Anal. Appl., 87 (1982), pp. 165–198.

[75] A. P. Pavlov, *Stability and convergence of the method of nets for an integro-differential equation of parabolic type*, Trudy Irkutsk. Gos. Univ., 26 (1968), pp. 213–224.

[76] M. Picasso, *Adaptive finite elements for a linear parabolic problem*, Comput. Methods Appl. Mech. Engrg., 167 (1998), pp. 223–237.

[77] ——, *An anisotropic error indicator based on Zienkiewicz-Zhu error estimator: application to elliptic and parabolic problems*, SIAM J. Sci. Comput., 24 (2003), pp. 1328–1355.

[78] ——, *Numerical study of the effectivity index for an anisotropic error indicator based on Zienkiewicz-Zhu error estimator*, Comm. Numer. Methods Engrg., 19 (2003), pp. 13–23.

[79] M. Picasso and V. Prachittham, *An adaptive algorithm for the Crank-Nicolson scheme applied to a time-dependent convection-diffusion problem*, J. Comput. Appl. Math., 233 (2009), pp. 1139–1154.

[80] V. Prachittham, *Space-time adaptive algoritms for parabolic problems: a posteriori error estimates and application to microfluidics*, PhD thesis, EPFL Lausanne, 2009.

[81] M. Rama Mohana Rao, *Ordinary Differential Equations: Theory and Applications*, East-West Press Pvt. Ltd., New Delhi, 1980.

[82] S. Repin, *A Posteriori Estimates for Partial Differential Equations*, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[83] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.

[84] D. Shi and L. Wang, *An anisotropic nonconforming finite element scheme with moving grids for parabolic integro-differential equations*, J. Syst. Sci. Complex., 24 (2011), pp. 1020–1032.

[85] K. G. Siebert, *An a posteriori error estimator for anisotropic refinement*, Numer. Math., 73 (1996), pp. 373–398.

[86] E. Sinestrari, *Time-dependent integro-differential equations in Banach spaces*, in Nonlinear Phenomena in Mathematical Sciences, Academic Press, New York, 1982, pp. 939–946.

[87] R. K. Sinha, R. E. Ewing, and R. D. Lazarov, *Some new error estimates of a semidiscrete finite volume element method for a parabolic integro-differential equation with nonsmooth initial data*, SIAM J. Numer. Anal., 43 (2006), pp. 2320–2343.

[88] I. H. Sloan and V. Thomée, *Time discretization of an integro-differential equation of parabolic type*, SIAM J. Numer. Anal., 23 (1986), pp. 1052–1061.

[89] M. Slodička, *Application of Rothe's method to evolution integro-differential systems*, Comment. Math. Univ. Carolin., 30 (1989), pp. 57–70.

[90] ——, *Smoothing effect and regularity for evolution integro-differential systems*, Comment. Math. Univ. Carolin., 30 (1989), pp. 303–316.

[91] J. L. Synge, *The Hypercircle in Mathematical Physics: A Method for the Approximate Solution of Boundary Value Problems*, Cambridge University Press, New York, 1957.

[92] H. Tanabe, *Note on nonlinear Volterra integral equation in Hilbert space*, Proc. Japan Acad. Ser. A Math. Sci., 56 (1980), pp. 9–11.

[93] L. Tavernini, *Finite difference approximations for a class of semilinear Volterra evolution problems*, SIAM J. Numer. Anal., 14 (1977), pp. 931–949.

[94] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, Berlin, 2006.

[95] V. Thomée and N. Y. Zhang, *Error estimates for semidiscrete finite element methods for parabolic integro-differential equations*, Math. Comp., 53 (1989), pp. 121–139.

[96] ——, *Backward Euler type methods for parabolic integro-differential equations with nonsmooth data*, in Contributions in Numerical Mathematics, World Sci. Publ., River Edge, NJ, 1993, pp. 373–388.

[97] R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, New York, 1996.

[98] ——, *A posteriori error estimates for finite element discretizations of the heat equation*, Calcolo, 40 (2003), pp. 195–212.

[99] I. I. Vrabie, *The nonlinear version of Pazy's local existence theorem*, Israel J. Math., 32 (1979), pp. 221–235.

[100] G. F. Webb, *An abstract semilinear Volterra integro-differential equation*, Proc. Amer. Math. Soc., 69 (1978), pp. 255–260.

[101] ——, *Abstract Volterra integro-differential equations and a class of reaction-diffusion equations*, in Volterra Equations, Springer, Berlin, 1979, pp. 295–303.

[102] M. F. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.

[103] P. Wissgott, *A Space-Time Adaptive Algorithm for Linear Parabolic Problems*, Diploma Thesis, TU Vienna, 2007.

[104] N. Yan and A. Zhou, *Gradient recovery type a posteriori error estimates for finite element approximations on irregular meshes*, Comput. Methods Appl. Mech. Engrg., 190 (2001), pp. 4289–4299.

[105] E. G. Yanik and G. Fairweather, *Finite element methods for parabolic and hyperbolic partial integro-differential equations*, Nonlinear Anal., 12 (1988), pp. 785–809.

[106] N. Y. Zhang, *On fully discrete Galerkin approximations for partial integro-differential equations of parabolic type*, Math. Comp., 60 (1993), pp. 133–166.

[107] Z. Zhang and J. Z. Zhu, *Superconvergence of the derivative patch recovery technique and a posteriori error estimation*, in Modeling, Mesh Generation, and Adaptive Numerical Methods for Partial Differential Equations, Springer, New York, 1995, pp. 431–450.

[108] O. C. Zienkiewicz and J. Z. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis*, Internat. J. Numer. Methods Engrg., 24 (1987), pp. 337–357.

[109] ——, *The superconvergent patch recovery and a posteriori error estimates. I. The recovery technique*, Internat. J. Numer. Methods Engrg., 33 (1992), pp. 1331–1364.

[110] ——, *The superconvergent patch recovery and a posteriori error estimates. II. Error estimates and adaptivity*, Internat. J. Numer. Methods Engrg., 33 (1992), pp. 1365–1382.

List of accepted and communicated papers

1. G. M. M. Reddy and R. K. Sinha, *Ritz-Volterra reconstructions and a posteriori error analysis of finite element method for parabolic integro-differential equations*, IMA J. Numer. Anal., DOI:10.1093/imanum/drt059.

2. G. M. M. Reddy and R. K. Sinha, *A posteriori error analysis of Crank–Nicolson finite element method for parabolic integro-differential equations*, Submitted for publication.

3. G. M. M. Reddy and R. K. Sinha, *Anisotropic a posteriori error analysis for parabolic integro-differential equations*, Submitted for publication.

4. G. M. M. Reddy and R. K. Sinha, *On the Crank-Nicolson anisotropic a posteriori error analysis for parabolic integro-differential equations*, Submitted for publication.