ABSTRACT

In spectral graph theory, mixed graphs are crucial because they give a structure in which directed and undirected edges may coexist. If every edge of a mixed graph is undirected, then such a graph is known as an undirected graph.

An important problem in spectral graph theory is to understand when all eigenvalues of the $(0,1)$-adjacency matrix of an undirected graph are integers. An undirected graph with this property is called integral. The notion of integral undirected graph was first introduced by Harary and Schwenk in 1974, and they raised the problem of determining undirected integral graphs. Many researchers attempted to solve this problem in the last few decades, yet it remains unsolved completely. In general, the problem of characterizing integral undirected graphs seems challenging to answer. Many researchers investigated some special classes of graphs such as trees, graphs with restricted degrees, regular graphs and undirected Cayley graphs for their integrality.

A *mixed graph* G is a pair $(V(G), E(G))$ of sets, where $V(G) \neq \emptyset$ and

$$E(G) \subseteq (V(G) \times V(G)) \setminus \{(u,u) : \ u \in V(G)\}.$$

In spectral graph theory, various types of adjacency matrices of graphs are defined and studied. In the thesis, we consider the following three adjacency matrices of a mixed graph G.

(i) The (0,1)-*adjacency matrix* of G, denoted $\mathcal{A}(G)$, is the matrix $[a_{uv}]$, where a_{uv} is given by

$$a_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The *Hermitian-adjacency matrix* of G, denoted $\mathcal{H}(G)$, is the matrix $[h_{uv}]$, where h_{uv} is given by

$$h_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \text{ and } (v,u) \in E(G) \\ \mathbf{i} & \text{if } (u,v) \in E(G) \text{ and } (v,u) \notin E(G) \\ -\mathbf{i} & \text{if } (u,v) \notin E(G) \text{ and } (v,u) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

(iii) The *Hermitian-adjacency matrix of second kind* of G, denoted $\mathcal{K}(G)$, is the matrix $[k_{uv}]$, where k_{uv} is given by

$$k_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \text{ and } (v,u) \in E(G) \\ \frac{1+\mathbf{i}\sqrt{3}}{2} & \text{if } (u,v) \in E(G) \text{ and } (v,u) \notin E(G) \\ \frac{1-\mathbf{i}\sqrt{3}}{2} & \text{if } (u,v) \notin E(G) \text{ and } (v,u) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

In this thesis, we discuss the following four types of integrality of mixed graphs.

(i) H-integral mixed graph: A mixed graph is said to be H-*integral* if the eigenvalues of its Hermitian-adjacency matrix are integers.

(ii) HS-integral mixed graph: A mixed graph is said to be HS-*integral* if the eigenvalues of its Hermitian-adjacency matrix of second kind are integers.

(iii) Gaussian integral mixed graph: A mixed graph is said to be *Gaussian integral* if the eigenvalues of its $(0,1)$-adjacency matrix are Gaussian integers.

(iv) Eisenstein integral mixed graph: A mixed graph is said to be *Eisenstein integral* if the eigenvalues of its $(0,1)$-adjacency matrix are Eisenstein integers.

We first characterize H-integral and HS-integral mixed Cayley graphs over abelian groups. Thereafter, we generalize these characterizations to normal mixed Cayley graphs. We also show that a normal mixed Cayley graph is Gaussian integral (respectively Eisenstein integral) if and only if it is H-integral (respectively HS-integral). Further, we introduce two sums that are equal to an integer multiple of the Ramanujan sum. Indeed, the eigenvalues of the Hermitian-adjacency matrix (respectively the Hermitian-adjacency matrix of second kind) of a mixed circulant graph can be expressed in terms of these sums. We also express these sums in terms of the generalized Möbius function.