NOVEL BOSONIZATION TECHNIQUES IN ONE DIMENSION

A thesis submitted for the degree of
Doctor of Philosophy

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December 2024

Dedicated to my family

Declaration

The work in this thesis entitled "Novel Bosonization techniques in one dimension" has been carried out by me under the supervision of Prof. Girish S. Setlur, Department of Physics, Indian Institute of Technology Guwahati. No part of this thesis has been submitted elsewhere for award of any other degree or qualification. The original research works in this thesis have been carried out in the period from August 2018 to December 2023.

In keeping with the general practice of reporting scientific observations, due acknowledgements have been made wherever the work described is based on the findings of other investigations.

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Certificate

This is to certify the research work contained in this thesis entitled "Novel Bosonization techniques in one dimension" by Mr. Nikhil Danny Babu, a PhD student of the Department of Physics, IIT Guwahati was carried out under my supervision. This work is original and has not been submitted elsewhere for award of any degree.

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Acknowledgements

I will forever be grateful and indebted to my supervisor Prof. Girish S. Setlur for introducing me to the incredible world of condensed matter physics. His innovative ideas and constant guidance and support were crucial in helping me progress with my thesis work. His courage to challenge established notions in the subject with out of the box ideas is an inspiration to me.

I extend my sincere gratitude to my doctoral committee members both past and present - Prof. Subhradip Ghosh, Prof. Pankaj Kumar Mishra, Prof. Rajaram Swaminathan and Prof. Tapan Mishra for periodically reviewing my progress and providing their invaluable feedback and suggestions that have enabled me to improve and present my work in the shape that it is today.

I would like to express my heartfelt thanks to all the faculties and all the staff members of the Physics Department of IIT Guwahati for their support and for creating a friendly work environment. I also extend my thanks to the Physics HODs during my time here - Prof. Perumal Alagarsamy and Prof. Subhradip Ghosh for the facilities in the department and for timely addressal of the concerns of the research scholars even during the height of the Covid outbreak.

I'm incredibly grateful to my senior and collaborator Dr. Joy Prakash Das for his help and guidance. I would also like to thank Prof. F.D.M. Haldane (Princeton University), Prof. C.J. Bolech (University of Cincinnati) and Prof. Enrico Perfetto (University of Rome Tor Vergata) for their valuable comments and feedback on my work.

I thank all my friends and colleagues at IIT Guwahati.

I'm deeply grateful to my parents and my brother for their constant moral support, love and encouragement and I dedicate this thesis to my family.

Abstract

In this work, the most singular contribution to the density density correlation functions (DDCF) of strongly inhomogeneous Luttinger liquids is derived and is shown to be expressible as compact analytical functions of position and time with second order poles and involving the scale independent bare reflection and transmission coefficients. The results are validated on comparison with standard fermionic perturbation theory. The DDCF is a crucial input to the powerful non-chiral bosonization technique (NCBT) that has been successfully used to obtain the correlation functions of inhomogeneous systems in one dimension whilst treating the impurity backscattering non-perturbatively, unlike conventional methods. The exact dynamical non-equilibrium Green functions (NEGF) for a system of noninteracting chiral quantum wires coupled through a point-contact is obtained analytically. The system considered is isomorphic to integer quantum Hall (IQHE) edge states coupled through a point-contact constriction. The tunneling I-V characteristics is obtained for an arbitrary time-dependent bias in the case of infinite bandwidth in the point-contact. The case of finite bandwidth in the point-contact is also studied and non-Markovian transients in the tunneling current is observed upon sudden switch on of a bias voltage. The transient phenomena is consistent with numerical simulations and is observed to be a consequence of the appearance of a short distance cutoff in the problem when a finite bandwidth is considered. In a subsequent work, an unconventional bosonization procedure similar to NCBT is introduced, and is used to reproduce the exact NEGF of noninteracting chiral quantum wires coupled through a point-contact driven out of equilibrium by application of a bias. The novel unconventional bosonization scheme is shown to be internally consistent with Wick's theorem used to obtain four-point functions. The proposed bosonization procedure can be extended to the case of fractional quantum Hall (FQHE) edge states with a point-contact wherein interparticle interactions become important. The FQHE edge states with single channel edge modes like in the Laughlin series, are modelled as chiral Luttinger liquids. In subsequent works, the DDCF for chiral Luttinger liquids with an impurity is computed using a generating functional method and is shown to be consistent with fermionic perturbation theory. The obtained interacting DDCF is used in conjunction with the unconventional bosonization procedure to derive the tunneling density of states (TDOS) at the point-contact for electron tunneling and quasiparticle tunneling cases, and the results agree with the accepted literature, thereby demonstrating the utility of the novel bosonization procedure in obtaining non-perturbatively, the correlation functions (most singular part) of inhomogeneous systems in one dimension.

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Chapter 1

Physics of fermions in one dimension: An introduction

Mutual interaction among the particles in a one-dimensional system gives rise to several physically interesting and special properties. The behavior of interacting electrons in higher dimensional metallic systems has been successfully explained by Landau's Fermi liquid theory [2]. The central result of the Fermi liquid theory is that the low energy excitations consist of creating quasiparticles (and quasiholes) with momenta just above (below) the Fermi surface. The quasiparticles (quasi-free excitations) are interpreted as electrons dressed by the particle-hole excitations of the electron gas. The quasiparticle excitations behave as fermions and are approximated as essentially free particles. So the qualitative behavior of the Fermi liquid is similar to a free electron gas. The occupation of a state with momentum k still has a discontinuity at the Fermi surface but with a reduced amplitude ($Z < 1$). The spectral function $A(k, \omega)$ is characterized by Lorentzian peaks centered at $\omega = E(k)$, where $E(k)$ is the linearized dispersion (a reasonable approximation close to the Fermi surface)

$$E(k) \sim E(k_F) + \frac{k_F}{m*}(k - k_F) \tag{1.1}$$

and $m*$ is the renormalized mass of the electron in the Fermi liquid. For free electrons $m* = m$. The total weight of the peaks in the spectral function is Z and it describes the fraction of the excitations present in the quasiparticle state. Whereas the spectral function is simply a delta function peak for the free electron gas. The success of Landau's Fermi liquid theory can be attributed to the fact that it is not restricted to weak coupling and works well even in the regime of strong interactions. But the situation is drastically different in one dimension. Electron interactions have a more profound effect in 1D systems than in 2D or 3D systems. Even weak interactions invalidate a Fermi liquid description of the 1D electron system. One dimensional interacting fermions are described by the Luttinger liquid model, where the low energy excitations are not weakly dressed quasiparticles, but are collective density waves.

1.1 Breakdown of Fermi liquid theory in one dimension

In higher dimensional interacting systems, nearly free quasiparticle excitations exist. However in one dimension in presence of interactions, individual excitations are not possible, this is because an individual propagating electron pushes all other electrons also along with it. This is a rough way of saying that only collective excitations are possible in 1D contrary to higher dimensions. Since only collective excitations exist, this means that for fermions with spin, a single fermionic excitation splits into a collective excitation carrying charge (holons) and a collective excitation carrying spin (spinons). These excitations in general have different velocities and this phenomenon in which the electron breaks into two elementary excitations is called spin-charge separation. This means that well defined nearly free quasiparticle excitations do not exist in 1D. These properties are strikingly different from that of Fermi liquids and form the essence of the Luttinger liquid model for interacting fermions in one-dimension [3]. The drastic departure of the physics of interacting electrons in one dimension from the physics of free electrons is evident from the fact that a perurbative treatment of the interactions results in singularities. The perturbation theory in one dimension is plagued by logarithmic divergences at low momenta (infra-red divergences), owing to the fact that the Fermi surface (which consists of only two points) is totally nested. Another striking feature of a one dimensional system is the existence of particle-hole excitations with well defined momentum and energy. The excitations of an electron gas constitutes particle-hole pairs, where an electron is destroyed below the Fermi level leaving behind a hole, and is created above the Fermi level. In higher dimensions ($D \geq 2$) it is possible to create particle-hole pairs of arbitrarily low energy leading to a continuum of excitations upto zero energy with momenta $q < 2k_F$. But in one dimension, the Fermi surface is reduced to just two points and it is possible to create low-energy particle-hole excitations only for momenta $q = 0$ and $q = 2k_F$. These excitations in 1D can be shown to have well defined momentum and energy with a finite lifetime that increases when the energy tends to zero. Similar to fermionic quasiparticles in higher dimensions, robust particle-hole excitations are present in 1D and these excitations are bosonic in nature since they constitute the destruction and creation of a fermion. This is a key reason why the method of bosonization is so effective in solving the 1D interacting problem.

1.2 Tomonaga model

The breakdown of Fermi liquid theory and the failure of the perturbative approach meant that radical new ideas were needed to solve the problem of interacting fermions in one dimension. A pioneering step towards this was made by Tomonaga [4] in 1950, who showed that fermions in 1D can be described by collective degrees of freedom that behave approximately as bosons. The Hamiltonian considered in the Tomonaga model is

$$H = v_F \sum_{p,\sigma} |p| c^{\dagger}_{p,\sigma} c_{p,\sigma} + \frac{1}{2L} \sum_{p} V_p \rho(p)\rho(-p) \tag{1.2}$$

where the density operator is defined as

$$\rho(p) = \sum_{k,\sigma} c^{\dagger}_{k-p/2,\sigma} c_{k+p/2,\sigma} \tag{1.3}$$

with system size L and where v_F is the Fermi velocity. The label $\sigma = \pm 1$ denotes spin and V_p represents interactions between the fermions. The idea of Tomonaga was to decompose the density operator into two parts such that $\rho(p) = \rho_+(p) + \rho_-(p)$, where

$$\begin{aligned} \rho_+(p) &= \sum_{k>0,\sigma} c^{\dagger}_{k-p/2,\sigma} c_{k+p/2,\sigma} \\ \rho_-(p) &= \sum_{k<0,\sigma} c^{\dagger}_{k-p/2,\sigma} c_{k+p/2,\sigma} \end{aligned} \tag{1.4}$$

The density operators at different wavevectors commute with each other. But ρ_+ and ρ_- do not commute with themselves for different wavevectors. The two parts satisfy the following commutation relations for $p < 2k_F$,

$$\begin{aligned} [\rho_+(p), \rho_+(-p')] &= \delta_{p,p'} \frac{pL}{\pi} \\ [\rho_-(p), \rho_-(-p')] &= -\delta_{p,p'} \frac{pL}{\pi} \\ [\rho_+(p), \rho_-(-p')] &= 0 \end{aligned} \tag{1.5}$$

The expectation value of the above commutators are exact and hence this is a reasonable approximation. It is possible to construct certain bosonic creation and annihilation operators using these density operators by making the following definitions

$$\begin{aligned} \rho_+(p) = b_p \sqrt{\frac{pL}{\pi}} \ &; \ \rho_+(-p) = b^{\dagger}_p \sqrt{\frac{pL}{\pi}} \\ \rho_-(p) = b^{\dagger}_{-p} \sqrt{\frac{pL}{\pi}} \ &; \ \rho_-(-p) = b_{-p} \sqrt{\frac{pL}{\pi}} \end{aligned} \tag{1.6}$$

for positive p and $[b_p, b^\dagger_{p'}] = \delta_{p,p'}$. The Hamiltonian in Eq.1.2 in terms of these boson operators is

$$H = \sum_p \left(\omega_p b^\dagger_p b_p + \frac{|p|V_p}{2\pi}(b_p + b^\dagger_{-p})(b^\dagger_p + b_{-p}) \right) \tag{1.7}$$

The original Hamiltonian which was quartic in the fermion operators is now quadratic in the bosonic operators and becomes exactly diagonalizable now. The main features of this model are the linear dispersion and the bosonic commutation relations, but both these features are approximate. The Tomonaga model is applicable only for sufficiently long-range interactions. Nevertheless, the introduction of the Tomonaga model was a significant revelation in our understanding of interacting fermions in one dimension.

1.3 Luttinger liquid

The next major development in this field came with the proposal of the Luttinger model [5]. This is similar to the Tomonaga model but uses lesser number of approximations. In this model a linear spectrum is considered from the beginning, with two species of fermions - right movers ($\epsilon_p = pv_F$) and left movers ($\epsilon_p = -pv_F$), unlike the Tomonaga model which involved only one type of fermion. In the Luttinger model there is a negative energy sea of particles extending to negative infinity. The fermion operators anticommute

$$\{c_{i,p,\sigma}, c^\dagger_{j,p',\sigma'}\} = \delta_{i,j}\delta_{p,p'}\delta_{\sigma,\sigma'} \tag{1.8}$$

where $i, j = R$ for right movers and L for left movers. The density and spin operators are defined as follows,

$$\begin{aligned} \rho_i(p) &= \sum_{k,\sigma} c^\dagger_{i,k+p,\sigma} c_{i,k,\sigma} \\ \rho_i(-p) &= \sum_{k,\sigma} c^\dagger_{i,k,\sigma} c_{i,k+p,\sigma} \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} s_i(p) &= \sum_{k,\sigma} \sigma \ c^\dagger_{i,k+p,\sigma} c_{i,k,\sigma} \\ s_i(-p) &= \sum_{k,\sigma} \sigma \ c^\dagger_{i,k,\sigma} c_{i,k+p,\sigma} \end{aligned} \tag{1.10}$$

They have the property

$$\rho_i(-p) = \rho^\dagger_i(p) \ ; \ s_i(-p) = s^\dagger_i(p) \tag{1.11}$$

The density operators in the Luttinger model have similar commutation relations as in the Tomonaga model but now they are valid for all values of p,

$$\begin{aligned} [\rho_R(p), \rho_R(-p')] &= \delta_{p,p'} \frac{pL}{\pi} \\ [\rho_L(p), \rho_L(-p')] &= -\delta_{p,p'} \frac{pL}{\pi} \\ [\rho_R(p), \rho_L(-p')] &= 0 \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} [s_R(-p), s_R(p')] &= \delta_{p,p'} \frac{pL}{\pi} \\ [s_L(-p), s_L(p')] &= -\delta_{p,p'} \frac{pL}{\pi} \\ [s_R(p), s_L(p')] &= 0 \end{aligned} \tag{1.13}$$

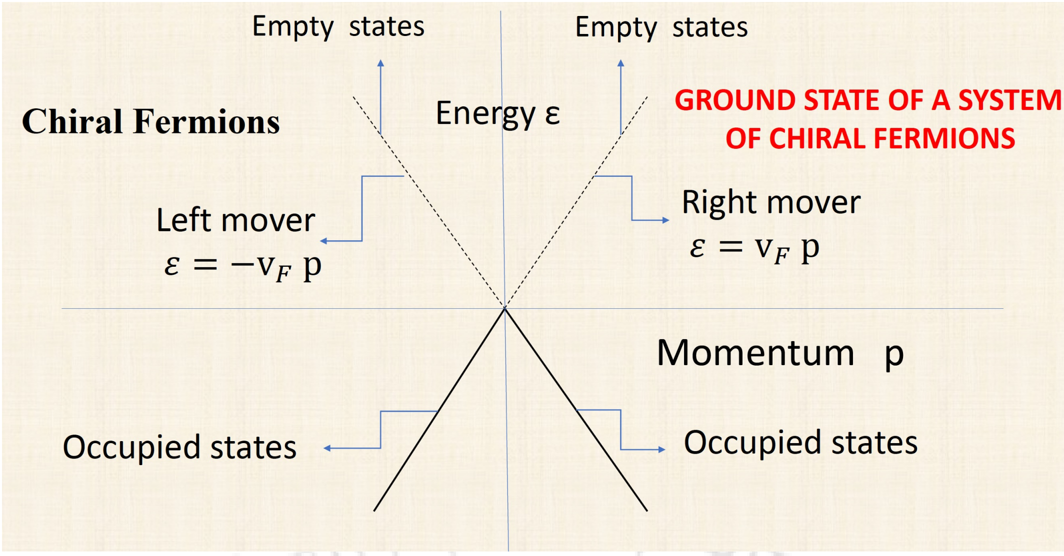

Figure 1.1: Linearized dispersion of the Luttinger model showing two species of chiral fermions.

and also

$$[s_i(p), \rho_j(p')] = 0 \tag{1.14}$$

Bosonic operators are constructed out of the densities,

$$\begin{aligned}\rho_R(p) &= b_{R,p}\sqrt{\frac{pL}{\pi}} \;;\; \rho_R(p) = b^{\dagger}_{R,p}\sqrt{\frac{pL}{\pi}}\\ \rho_L(-p) &= b^{\dagger}_{L,-p}\sqrt{\frac{pL}{\pi}} \;;\; \rho_L(p) = b_{L,-p}\sqrt{\frac{pL}{\pi}}\end{aligned} \tag{1.15}$$

$$\begin{aligned}s_R(p) &= a_{R,p}\sqrt{\frac{pL}{\pi}} \;;\; s_R(p) = a^{\dagger}_{R,p}\sqrt{\frac{pL}{\pi}}\\ s_L(-p) &= a^{\dagger}_{L,-p}\sqrt{\frac{pL}{\pi}} \;;\; s_L(p) = a_{L,-p}\sqrt{\frac{pL}{\pi}}\end{aligned} \tag{1.16}$$

The interaction terms become quadratic in the bosons and the Hamiltonian becomes exactly solvable. This description of interacting fermions in one dimension is also called Tomonaga-Luttinger liquid theory. But the disadvantage is that it invokes the presence of an infinite sea of negative energy particles. Yet this model captures the low-energy physics of a wide class of conducting systems in 1D, including spin chains, carbon nanotubes, edge states of quantum Hall systems [6] and recently quantum-wire networks in twisted bilayer Graphene [7, 8, 9, 10]. Mattis and Lieb [11] provided a correct solution of the original model proposed by Luttinger. Following several important contributions [12, 13, 14] the theory was nurtured to its present form by Haldane [15]. Our understanding of interacting one-dimensional systems is centered around the Luttinger liquid paradigm.

A good number of analytical [16, 17, 18] and numerical approaches [19, 20, 21, 22, 23] have made their mark in the long history of strongly correlated 1D systems.

1.4 Bosonization

Bosonization is the technique of expressing a fermionic operator as a function of bosonic operators. It is a very efficient method to study strongly correlated fermions in one dimension. Here we provide a brief overview of this technique.

The idea of bosonization initially started circulating in the particle physics community thanks to Coleman and Mandelstam [24]. Alan Luther and Daniel Mattis showed that the Fermi field operator is expressible in terms of bosonic fields. In the context of one dimensional fermions it was Tomonaga who showed first that the fermions could be described by a quantized field of sound waves that obey Bose statistics. Mattis and Lieb provided the correct exact solution to the Luttinger model [11]. The introduction of fermion number lowering Klein factors to the bosonization description was due to Heidenreich et al. [14] and Haldane [25]. Transport through a weak link was studied in the pioneering work of Kane and Fisher [26] in 1992 and they showed with the help of bosonization and renormalization group (RG) techniques that the nature of the interactions (attractive or replusive) played a key role in the transport. Recently attempts have been made to use bosonization to study Luttinger liquids beyond the low energy limit wherein the curvature in the dispersion becomes important [27, 28]. The mapping between fermions and bosons is exact in one dimension but works have been done to extend this technique to higher dimensions as well [29, 30, 31], but this is a challenging problem and still the utility of this technique is best realized in 1D systems. Numerous works have been done on Luttinger liquid theory and bosonization over the past 50 years or so and this subject has been extensively discussed in several review articles available in the literature [32, 33, 34].

1.5 General considerations

The essential idea of bosonization is to write down an explicit formula for the fermion field operator in terms of currents and densities. The density $\rho(x)$ and the current $J(x)$ are defined as

$$\rho(x) = \psi^{\dagger}(x)\psi(x) \ ; \ J(x) = Im[\psi^{\dagger}(x)\partial_x\psi(x)] \tag{1.17}$$

These are bilinears in the field operator ψ. The aim is to invert these relations and express $\psi(x)$ as a function of ρ and J, that is $\psi(x) = F(\rho, J; x)$. The field operator $\psi(x)$ annihilates a particle but the right-hand side constitutes a function of number conserving operators. If this relation is considered to be an operator identity then it becomes necessary to introduce Klein factors on the right-hand side that remove a particle. This is the usual point of view adopted by the bosonization community. Another perspective is to not consider the relation as a strict operator identity but rather as a mnemonic to obtain the correlation functions, then there is no need to enforce Klein factors if we are interested in obtaining only the correlation functions which are of course number conserving. The conjugate of the density is introduced

$$J(x,t) = -\rho(x,t)\partial_x\Pi(x,t) \tag{1.18}$$

The currents and densities satisfy the current algebra:

$$\begin{aligned}[][\rho(x,t), \rho(x',t)] = 0 \ ; \ [\Pi(x,t), \Pi(x',t)] = 0 \\ [\Pi(x,t), \rho(x',t)] = i\ \delta(x-x')\end{aligned} \tag{1.19}$$

and this can be used to achieve the inversion of interest. The general claim is that the field maybe written as

$$\psi(x) = U([\rho]; x)\ e^{-i\Pi(x)}\sqrt{\rho(x)} \tag{1.20}$$

The current algebra obeyed by the current and density does not distinguish between bosons and fermions, so as shown in [35] the additional object U is necessary to capture the statistics of the underlying particle. Also it is useful to write the density as $\rho(x,t) = \rho_0 + \tilde{\rho}(x,t)$, where ρ_0 is the average density which is nonzero and $\tilde{\rho}$ is the fluctuation in the density from the uniform average. The Fermi-Bose correspondence in Eq.1.20 should be seen as a mnemonic to obtain the correlation functions instead of as a strict operator identity. The object U has the following properties (since $\rho(x) = \psi^{\dagger}(x)\psi(x)$ and $-\rho(x,t)\partial_x\Pi(x,t) = Im[\psi^{\dagger}(x)\partial_x\psi(x)]$ and $\Pi(x)$ and $\rho(x)$ are Hermitian),

$$U^{\dagger}([\rho]; x)U([\rho]; x) = 1 \tag{1.21}$$

and

$$U^{\dagger}([\rho]; x)(\partial_x U([\rho]; x)) = 0 \tag{1.22}$$

This would imply that U is independent of x if $U^\dagger$ is the inverse of U. This in turn would result in $\psi(x)$ lose its meaning as a Fermi field. We can overcome this inconsistency by enforcing that the inverse of U shouldn't exist. This means that U is not unitary but is in fact a partial isometry.
The following choice of the Fermi-Bose correspondence reproduces the correct expressions for the current and density and also the fermion commutation rules in addition to the correct correlation functions that give the proper noninteracting limit [35],

$$\psi(x) = \frac{1}{\sqrt{N_0}} \sum_p n_F(p) e^{i\xi(p)} e^{i\pi sgn(p) \int_{-\infty}^{x} \rho(y)dy} e^{-i\Pi(x)} \sqrt{\rho(x)} \tag{1.23}$$

where $n_F(p) = \theta(k_F - |p|)$ and $N_0 = \sum_p n_F(p)$ and consistency checks with the expressions for density and current gives the identity $e^{i\xi(p)}e^{-i\xi(p')} = \delta_{p,p'}$.

1.6 Bosonization of a homogeneous Luttinger liquid

The Fermi-Bose correspondence in Eq.1.23 is generally valid and in principle can be applied to models with non-linear dispersion as well. But here we use it to study the low energy physics of the Luttinger model with linear dispersion and with short-range forward scattering mutual interactions. The asymptotic correlation functions are obtained by working in the random phase approximation (RPA) limit ($k_F, m \to \infty$ and $v_F = \frac{k_F}{m} < \infty$). In this limit the Fermi fields are peaked in momentum $k = \pm k_F$ and close to the Fermi surface we can decompose the field to write

$$\psi(x,t) = e^{ik_Fx}\psi_R(x,t) + e^{-ik_Fx}\psi_L(x,t) \tag{1.24}$$

Which means the harmonic analysis of the density operator can be written as (making the spin indices explicit)

$$\rho(x,\sigma,t) = \rho_0 + \rho_s(x,\sigma,t) + e^{2ik_Fx}\rho_f(x,\sigma,t) + e^{-2ik_Fx}\rho_f^*(x,\sigma,t) \tag{1.25}$$

where $\rho_s = \rho_R + \rho_L$ denotes the slow part and ρ_f is the fast oscillating part of the density. Using Haldane's harmonic analysis [36] we can express the slow part of the density in terms of the fast parts

$$\rho_f(x,\sigma,t) \sim e^{2i\pi \int_{-\infty}^{x} \rho_s(y,\sigma,t)dy} \tag{1.26}$$

This expression is used in the harmonic analysis of the density operator and the result is substituted into Eq.1.23 whilst restricting to only the slow part of the density to obtain the standard bosonization identity

$$\psi_\nu(x,\sigma,t) \sim e^{i\theta_\nu(x,\sigma,t)} \tag{1.27}$$

where the local phase is given by

$$\theta_\nu(x,\sigma,t) = \pi \int_{sgn(x)\infty}^{x} \left(\nu\rho_s(y,\sigma,t) - \int_{sgn(y)\infty}^{y} \frac{1}{v_F}\partial_t\rho_s(y',\sigma,t)dy' \right) dy \tag{1.28}$$

where $\nu = 1$ and -1 for R and L movers respectively. This conventional bosonization prescription is used to calculate the correlation functions of the Luttinger liquid, and it is obtained by using

$$< \psi_\nu(x,\sigma,t)\psi^\dagger_{\nu'}(x',\sigma',t') > \;\sim\; < e^{i\theta_\nu(x,\sigma,t)}e^{-i\theta_{\nu'}(x',\sigma',t')} > \tag{1.29}$$

Model dependent prefactors present in the Green functions are not obtainable by the bosonization method, only the dynamical part of the Green function is calculable using the formula for bosonization. The standard bosonization prescription in Eq.1.27 gives the correlation functions only for a homogeneous system i.e. with unbroken translation invariance. The Green functions are evaluated using a version of the Baker-Campbell-Hausdorff cumulant expansion

$$< \psi_\nu(x,\sigma,t)\psi^\dagger_{\nu'}(x',\sigma',t') > \;\sim\; e^{\frac{1}{2}<(i\theta_\nu(x,\sigma,t))^2>}e^{\frac{1}{2}<(-i\theta_{\nu'}(x',\sigma',t'))^2>}e^{<(i\theta_\nu(x,\sigma,t))(-i\theta_{\nu'}(x',\sigma',t'))>} \tag{1.30}$$

All higher order (beyond quadratic) moments of the density are zero for the homogeneous case. In the strongly inhomogeneous case two possibilities present themselves - a) We use the unmodified Luther-Haldane Fermi-Bose correspondence in which case we are stuck with having to include all higher even moments of the density beyond quadratic (the odd ones vanish identically) and equally unfortunately we are stuck with having to deal with a highly nonlinear sine-Gordon theory. b) We modify the Fermi-Bose correspondence as shown in this thesis so that the most singular parts of the Green functions are correctly captured by the second order cumulant, most crucially even when mutual interactions between the fermions are included (mutual forward scattering). This novel method allows us the luxury of treating the action of this strongly inhomogeneous theory as being purely quadratic in the bosons even when mutual interactions are present - a feature made possible by the observation that we are only interested in the most singular parts of the correlation functions. We naturally prefer the latter approach in this thesis. The asymptotic Green functions are characterized by a power-law behavior with exponents that depend on the forward scattering interaction parameter,

$$< \psi_{\nu}(x)\psi^{\dagger}_{\nu'}(x') > \;\; \sim \frac{1}{|x-x'|^{\gamma}} \tag{1.31}$$

The exponent $\gamma = 1$ in the noninteracting case. The standard bosonization procedure solves fully the low-energy physics of the problem of interacting fermions in one dimension without impurities in the system. Impurities break the translational invariance and the number of right movers and number of left movers are not independently conserved due to backscattering from the impurity. The conventionally used standard bosonization procedure is not well suited to deal with impurities. It is only useful in the homogeneous limit ($|R| = 0$) and in the half-line ($|R| = 1$) limit, where $|R|$ is the reflection amplitude of the impurity. Even in the absence of interactions this procedure doesn't reproduce the correct Green functions in presence of an impurity. For a noninteracting Fermi gas with an arbitrary impurity ($0 < |R| < 1$) at the origin the RL Green function obtained using standard Fermi algebra is of the form

$$< T\psi_R(x,t)\psi^{\dagger}_L(x',t') > \;\; \sim \frac{1}{(x+x'-v_F(t-t'))} \tag{1.32}$$

Whereas using Eqs.1.27 and 1.28 the bosonized Green function obtained is of the form

$$< T\psi_R(x,t)\psi^{\dagger}_L(x',t') > \;\; \sim \frac{1}{(x+x'-v_F(t-t'))^{|R|^2}} \tag{1.33}$$

which is obviously incorrect due to the nontrivial exponent $|R|^2$. This shows that the standard bosonization technique is not suitable to deal with impurity backscattering even when the fermions are non-interacting.

1.7 Bosonization of an inhomogeneous Luttinger liquid

Let us consider a Luttinger liquid with a localised impurity modelled by a potential at the origin. The usual bosonization formulae are not well suited to study this problem. The bosonized action is no longer quadratic due to the appearance of cosine (sine-Gordon) terms and is not exactly solvable. The presence of impurities in one dimensional systems continues to be a challenging problem in spite of several decades of pathbreaking theoretical efforts [37, 38, 39, 40, 41, 26, 42]. The conventional method is to use a renormalization group (RG) approach to deal with the impurities. This is reflected in several seminal works in the literature [37, 17, 43]. Kane and Fisher [37] treated the impurity as a perturbation and showed that for attractive interactions the weak barrier limit is a stable RG fixed point and that for repulsive interactions the weak link limit is a stable RG fixed point. This observation is popularly dubbed as 'healing the chain' and 'cutting the chain' phenomena. In [42] they studied the effect of electron interactions on resonant tunneling in presence of a double barrier and showed that the resonances are of non-Lorentzian line shapes with a width that vanishes as $T \to 0$, in striking contrast to the noninteracting one dimensional electron gas. Standard bosonization is effective in both the weak barrier and weak link limits. In [17] Matveev et al. used an RG technique to calculate conductance in a Luttinger liquid with impurity but in the limit of weak mutual interactions. The functional

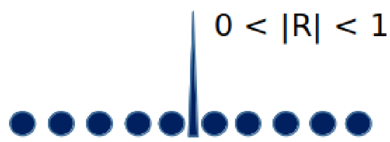

Figure 1.2: For $(0 < |R| < 1)$ conventional bosonization gives an incorrect exponent even for the noninteracting Green's functions

renormalization group (fRG) [43, 44] has proved useful in dealing with impurities and boundaries in Luttinger liquids. Although these methods have provided valuable insights in dealing with impurities in 1D systems, they cannot be used to calculate the Green functions for an arbitrary impurity strength. All these methods in some sense treat the impurity merely as an afterthought, however an arbitrary impurity significantly affects the correlation functions and this can only be gleaned by using an unconventional bosonization method that treats the impurity in a non-perturbative manner circumventing the use of RG. Such an alternative approach is the non chiral bosonization technique (NCBT) [45] proposed by our group. NCBT uses a modified version of the Fermi-Bose correspondence that can be used to calculate the most singular part of the asymptotic Green functions of inhomogeneous Luttinger liquids [46]. This technique has been successfully used to study transport properties of Luttinger liquids with impurities [47, 48, 49, 50, 51].
The idea behind NCBT is to use a non-standard harmonic analysis [45] that takes into account impurity backscattering. Haldane's standard harmonic analysis in Eq.1.26 is replaced with the following one

$$\rho_f(x,\sigma,t) \sim e^{2i\pi \int_{-\infty}^{x} (\rho_s(y,\sigma,t)+\lambda\rho_s(-y,\sigma,t))dy} \tag{1.34}$$

Here the parameter λ only takes values 0 or 1. The field operator derived from the non-standard harmonic analysis is

$$\psi_\nu(x,\sigma,t) \sim e^{i\theta_\nu(x,\sigma,t)+2\pi i\lambda\nu \int_{sgn(x)\infty}^{x} \rho_s(-y,\sigma,t)dy} \tag{1.35}$$

where θ_ν is given by Eq.1.28. The expression for the field operator in Eq.1.35 is a mnemonic to obtain the Green functions and is not a strict operator identity. The fermion commutation rules are satisfied by this field operator. While calculating the Green functions, either the annihilation or the creation operator takes $\lambda = 1$ while the other operator should have $\lambda = 0$, this is to satisfy certain point-splitting constraints as shown in [45]. This modified Fermi-Bose correspondence of NCBT gives the correct single particle Green functions for free fermions with impurity of arbitrary strength in terms of the noninteracting density-density correlations. When mutual interactions are present, the density-density correlation functions (DDCF) with interactions are obtained using the generating functional method [52] and this is used in the NCBT formula to obtain the many-body Green functions (most singular part) exactly. The Green functions (most singular part) of a strongly inhomogeneous Luttinger liquid obtained using NCBT in [46] is given below:

Case 1 - Both x_1 and x_2 on the same side of origin

$$\left\langle T\psi_R(x_1,t_1)\psi_R^\dagger(x_2,t_2)\right\rangle \sim \frac{(4x_1x_2)^{\gamma_1}}{(x_1-x_2-v_h\tau_{12})^P(-x_1+x_2-v_h\tau_{12})^Q(x_1+x_2-v_h\tau_{12})^X} \frac{1}{(-x_1-x_2-v_h\tau_{12})^X(x_1-x_2-v_F\tau_{12})^{0.5}} \tag{1.36}$$

$$\left\langle T\psi_L(x_1,t_1)\psi_L^\dagger(x_2,t_2)\right\rangle \sim \frac{(4x_1x_2)^{\gamma_1}}{(x_1-x_2-v_h\tau_{12})^Q(-x_1+x_2-v_h\tau_{12})^P(x_1+x_2-v_h\tau_{12})^X} \frac{1}{(-x_1-x_2-v_h\tau_{12})^X(-x_1+x_2-v_F\tau_{12})^{0.5}} \tag{1.37}$$

$$\left\langle T\psi_R(x_1,t_1)\psi_L^{\dagger}(x_2,t_2)\right\rangle \sim \frac{(2x_1)^{1+\gamma_2}(2x_2)^{\gamma_1}}{2(x_1-x_2-v_h\tau_{12})^{S}(-x_1+x_2-v_h\tau_{12})^{S}(x_1+x_2-v_h\tau_{12})^{Y}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{Z}(x_1+x_2-v_F\tau_{12})^{0.5}} + \frac{(2x_1)^{\gamma_1}(2x_2)^{1+\gamma_2}}{2(x_1-x_2-v_h\tau_{12})^{S}(-x_1+x_2-v_h\tau_{12})^{S}(x_1+x_2-v_h\tau_{12})^{Y}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{Z}(x_1+x_2-v_F\tau_{12})^{0.5}} \tag{1.38}$$

$$\left\langle T\psi_L(x_1,t_1)\psi_R^{\dagger}(x_2,t_2)\right\rangle \sim \frac{(2x_1)^{1+\gamma_2}(2x_2)^{\gamma_1}}{2(x_1-x_2-v_h\tau_{12})^{S}(-x_1+x_2-v_h\tau_{12})^{S}(x_1+x_2-v_h\tau_{12})^{Z}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{Y}(-x_1-x_2-v_F\tau_{12})^{0.5}} \frac{(2x_1)^{\gamma_1}(2x_2)^{1+\gamma_2}}{2(x_1-x_2-v_h\tau_{12})^{S}(-x_1+x_2-v_h\tau_{12})^{S}(x_1+x_2-v_h\tau_{12})^{Z}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{Y}(-x_1-x_2-v_F\tau_{12})^{0.5}} \tag{1.39}$$

Case 2 - x_1 and x_2 on opposite sides of the origin

$$\left\langle T\psi_R(x_1,t_1)\psi_R^{\dagger}(x_2,t_2)\right\rangle \sim \frac{(2x_1)^{1+\gamma_2}(2x_2)^{\gamma_1}(x_1+x_2)^{-1}(x_1+x_2+v_F\tau_{12})^{0.5}}{2(x_1-x_2-v_h\tau_{12})^{A}(-x_1+x_2-v_h\tau_{12})^{B}(x_1+x_2-v_h\tau_{12})^{C}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{D}(x_1-x_2-v_F\tau_{12})^{0.5}} + \frac{(2x_1)^{\gamma_1}(2x_2)^{1+\gamma_2}(x_1+x_2)^{-1}(x_1+x_2-v_F\tau_{12})^{0.5}}{2(x_1-x_2-v_h\tau_{12})^{A}(-x_1+x_2-v_h\tau_{12})^{B}(x_1+x_2-v_h\tau_{12})^{D}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{C}(x_1-x_2-v_F\tau_{12})^{0.5}} \tag{1.40}$$

$$\left\langle T\psi_L(x_1,t_1)\psi_L^{\dagger}(x_2,t_2)\right\rangle \sim \frac{(2x_1)^{1+\gamma_2}(2x_2)^{\gamma_1}(x_1+x_2)^{-1}(x_1+x_2-v_F\tau_{12})^{0.5}}{2(x_1-x_2-v_h\tau_{12})^{A}(-x_1+x_2-v_h\tau_{12})^{B}(x_1+x_2-v_h\tau_{12})^{D}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{C}(-x_1+x_2-v_F\tau_{12})^{0.5}} + \frac{(2x_1)^{\gamma_1}(2x_2)^{1+\gamma_2}(x_1+x_2)^{-1}(x_1+x_2+v_F\tau_{12})^{0.5}}{2(x_1-x_2-v_h\tau_{12})^{B}(-x_1+x_2-v_h\tau_{12})^{A}(x_1+x_2-v_h\tau_{12})^{C}} \frac{1}{(-x_1-x_2-v_h\tau_{12})^{D}(-x_1+x_2-v_F\tau_{12})^{0.5}} \tag{1.41}$$

$$\left\langle T\psi_R(x_1,t_1)\psi_L^{\dagger}(x_2,t_2)\right\rangle \sim 0 \tag{1.42}$$

$$\left\langle T\psi_L(x_1,t_1)\psi_R^{\dagger}(x_2,t_2)\right\rangle \sim 0 \tag{1.43}$$

The impurity at the origin breaks the translational invariance of the system, hence the form of the two-point Green functions depends on whether the two points are on the same side or on the opposite side of the origin. In writing down the above expressions for the Green functions the weak equality '$\sim$' symbol is used to imply that bosonization does not yield the possible prefactors in the Green functions that may in general be spatially inhomogeneous. The prefactors of the Green functions are not obtainable through bosonization, only the dynamical parts are obtained. The power-law exponents of the Green functions are in general dependent on the bare reflection amplitude of the impurity. The

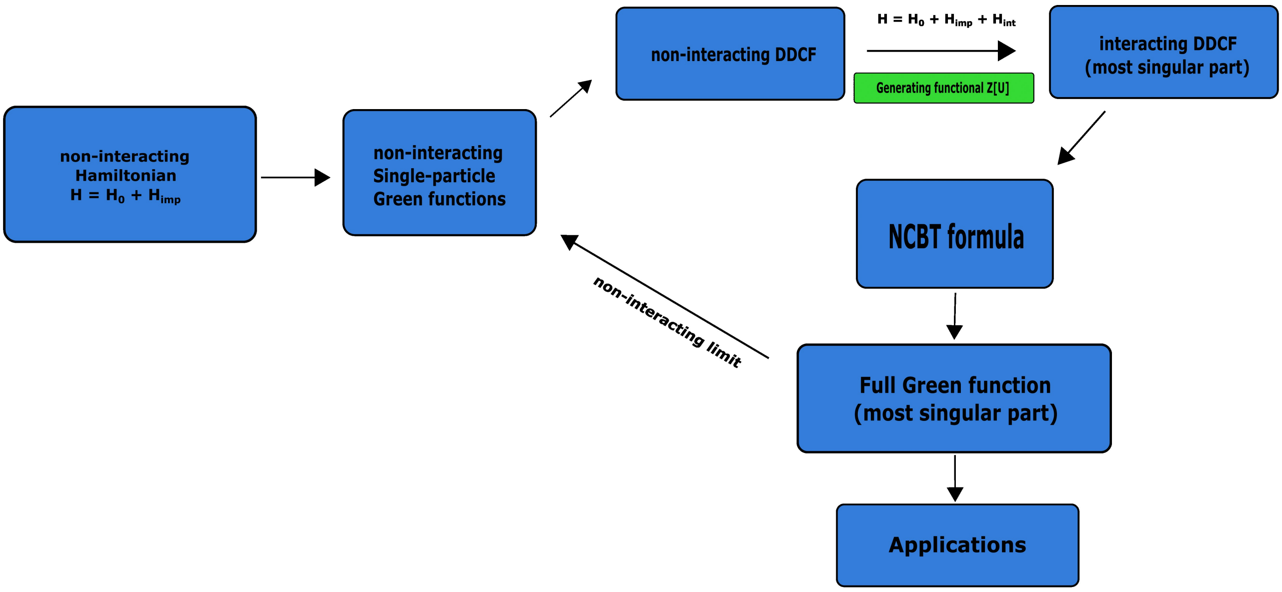

Figure 1.3: Schematic flowchart of steps involved in NCBT

explicit expressions for the exponents are:

$$Q=\frac{(v_h-v_F)^2}{8v_hv_F}\ ;\ X=\frac{|R|^2(v_h-v_F)(v_h+v_F)}{8v_h(v_h-|R|^2(v_h-v_F))}\ ;\ C=\frac{v_h-v_F}{4v_h}$$

where the holon velocity is $v_h=\sqrt{v_F^2+\frac{2v_0v_F}{\pi}}$. The other exponents can be expressed in terms of the above exponents

$$P=\frac{1}{2}+Q\ ;\ S=\frac{Q}{C}(\frac{1}{2}-C)\ ;\ Y=\frac{1}{2}+X-C$$
$$Z=X-C\ ;\ A=\frac{1}{2}+Q-X\ ;\ B=Q-X$$
$$D=-\frac{1}{2}+C\ ;\ \gamma_1=X\ ;\ \gamma_2=-1+X+2C \tag{1.44}$$

This is an important result as the most singular parts of the asymptotic Green functions are obtained exactly in terms of simple functions of position and time. Fig.1.3 is a schematic flowchart showing all the steps involved in NCBT.

1.8 Comparison of NCBT with DMRG

It is not easy to compare the results of NCBT exactly with numerical techiques like the Density Matrix Renormalization Group (DMRG). First of all the results of bosonization methods are valid in the thermodynamic limit and it is difficult to access both numerically and experimentally, the system sizes where the low energy physics described by bosonization could be probed. The computational resources needed to extract Luttinger liquid physics using DMRG is considerable. Nevertheless notable attempts have been made to compare DMRG simulations with the predictions of standard bosonization. Of interest to us are the works [53, 54], where the local spectral weight and boundary exponents were calculated using DMRG and functional renormalization group (fRG) methods for different values of impurity strengths and compared with the universal values predicted by standard bosonization. The Green functions obtained through NCBT generally have non universal power-law exponents that depend on the impurity. For a given interaction strength, the local spectral density exponent (boundary

exponent) predicted by NCBT is not a universal constant but changes with impurity strength, unlike standard bosonization that predicts a constant universal power-law exponent that depends only on the interparticle interaction strength. Another point to note is that existing DMRG studies for Luttinger liquids tend to include both the forward and backward scattering interactions but in NCBT we include only short-range forward scattering interactions.
Consider a model with dual species of fermions (R and L) without spin

$$H = H_0 + H_{imp} + H_{int} \tag{1.45}$$

where the kinetic energy term is

$$H_0 = \sum_k v_F k \; c^\dagger_R(k) c_R(k) - \sum_k v_F k \; c^\dagger_L(k) c_L(k) \tag{1.46}$$

and the impurity term in the Hamiltonian is

$$H_{imp} = \frac{V_0}{L} \sum_{k,k'} (c^\dagger_R(k) c_R(k') + c^\dagger_L(k) c_L(k')) + \frac{V_1}{L} \sum_{k,k'} c^\dagger_R(k) c_L(k') + \frac{V_1^*}{L} \sum_{k,k'} c^\dagger_L(k) c_R(k') \tag{1.47}$$

and the forward scattering fermion-fermion interaction is

$$H_{int} = \frac{v_0}{2L} \sum_q \rho(q)\rho(-q) \tag{1.48}$$

where the density fluctuation operator is defined as

$$\rho(q) = \sum_k (c^\dagger_R(k+q) c_R(k) + c^\dagger_L(k+q) c_L(k)) \tag{1.49}$$

This model has to be mapped to a lattice system in order to be able to do DMRG simulations. So let us consider a lattice such that $k \to \frac{\sin(ka)}{a}$ with $lim_{a\to 0}$ and define

$$b_\nu(n) = \frac{1}{\sqrt{N}} \sum_k e^{ikna} \; c_\nu(k) \tag{1.50}$$

where n is the site index. Now we can convert this effectively to a lattice model with

$$H_0 = \frac{v_F}{a} \frac{i}{2} \sum_{n,n'} (\delta_{n',n+1} - \delta_{n',n-1})(b^\dagger_R(n') b_R(n) - b^\dagger_L(n') b_L(n)) \tag{1.51}$$

and

$$H_{imp} = \frac{V_0}{a}(b^\dagger_R(0) b_R(0) + b^\dagger_L(0) b_L(0)) + \frac{V_1}{a} b^\dagger_R(0) b_L(0) + \frac{V_1^*}{a} b^\dagger_L(0) b_R(0) \tag{1.52}$$

and

$$H_{int} = \frac{v_0}{2a} \sum_n (b^\dagger_R(n) b_R(n) + b^\dagger_L(n) b_L(n))^2 \tag{1.53}$$

This model with dual species spinless fermions can be converted to one with a single species of spinful fermions (with $\uparrow$ as R and $\downarrow$ as L) to which the DMRG codes are well suited. This gives

$$H_0 = \frac{v_F}{a}\frac{i}{2} \sum_j (b^\dagger_\uparrow(j+1) b_\uparrow(j) - b^\dagger_\downarrow(j+1) b_\downarrow(j)) - \frac{v_F}{a}\frac{i}{2} \sum_j (b^\dagger_\uparrow(j-1) b_\uparrow(j) - b^\dagger_\downarrow(j-1) b_\downarrow(j))$$

and

$$H_{imp} = \frac{V_0}{a}(b^\dagger_\uparrow(0) b_\uparrow(0) + b^\dagger_\downarrow(0) b_\downarrow(0)) + \frac{V_1}{a} b^\dagger_\uparrow(0) b_\downarrow(0) + \frac{V_1^*}{a} b^\dagger_\downarrow(0) b_\uparrow(0)$$

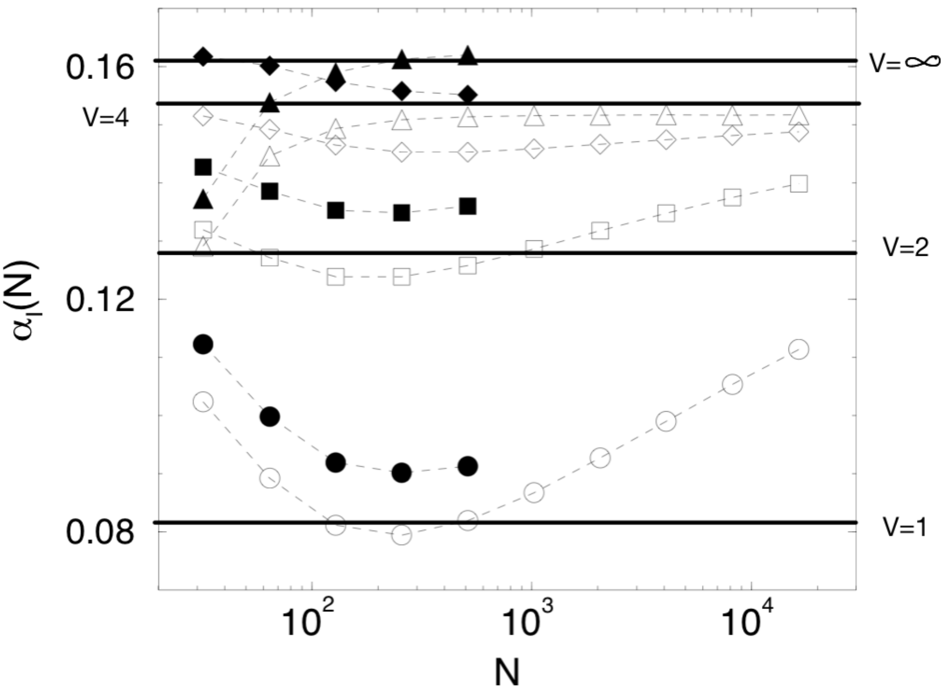

Figure 1.4: Local spectral density exponent near a impurity for a spinless fermion chain with fixed interaction strength $U = 0.5$ and various impurity strengths V: $V = 1$ circles, $V = 2$ squares, $V = 4$ diamonds, $V = \infty$ triangles. Filled symbols are DMRG results and empty symbols are fRG results. The thick lines show the results predicted by NCBT. The line marked $V = \infty$ is the half-line limit where NCBT matches exactly with conventional bosonization.

and

$$H_{int} = \frac{v_0}{2a} \sum_j (n_\uparrow(j) + n_\downarrow(j) + 2n_\uparrow(j)n_\downarrow(j))$$

where $n_\nu(j) \equiv b_\nu^\dagger(j)b_\nu(j)$. The universal physics (universal power-law exponents) predicted by bosonization only become evident at extremely long and practically irrelevant length scales. We make a comparison of the results in [53] for the local spectral density exponent near the impurity with that obtained using NCBT. The local density of states of spinless fermions near the impurity obtained using conventional bosonization is of the form

$$D(\epsilon) \sim |\epsilon|^\alpha \tag{1.54}$$

where $\alpha = \frac{1}{K} - 1$ with K being the Luttinger interaction parameter. The local density of states for an interacting many-body system is related to the Green functions of the system by the formula

$$D(\epsilon) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{it(\epsilon+E_F)} < \{\psi(x,t), \psi^\dagger(x,0)\} > \tag{1.55}$$

Using the NCBT Green functions to calculate the local density of states near the impurity we obtain the following exponent

$$\alpha = \frac{(g+1)(1-g)|R|^2}{2(1-(1-g)|R|^2)} + \frac{(1-g)^2}{2g} \tag{1.56}$$

For the half-line limit ($|R| = 1$) the exponent reduces to

$$\alpha = \frac{1}{g} - 1 \tag{1.57}$$

where $g = K = \frac{v_F}{v_h}$, which matches with the conventional bosonization result. However NCBT is more general as it captures the $|R|$ dependence of the exponent for arbitrary impurity strengths as well. The drawback of standard bosonization is that the impurity is treated perturbatively and the linearization and continuous limit processes eliminate all relevant length scales in the problem. The authors of [53, 54] consider a simple system of spinless fermions with nearest neighbour hopping and Coulomb repulsion that shows Luttinger liquid behavior

$$H = \sum_i \ -t(c_i^\dagger c_{i+1} + h.c) + U \ n_i n_{i+1} + V \ n_j \tag{1.58}$$

with $U > 0$ and an on-site impurity at site j. Fig.1.4 shows the local spectral density exponent near the impurity for a fixed interaction strength for various impurity strengths, showing the DMRG and fRG results obtained in [53] along with the NCBT and standard bosonization results valid in the thermodynamic limit. The DMRG computation is limited by system size (only upto 768 sites) and is far from the thermodynamic limit, however the DMRG exponents appear to be closer to the NCBT predictions at different impurity strengths than to the universal value predicted by standard bosonization. The fRG computations extend to larger system sizes but involve certain approximations. These observations though inconclusive nevertheless make a strong case for the applicability of unconventional bosonization methods such as NCBT for inhomogeneous systems.

1.9 Overview of the thesis

The work contained in this thesis clarifies certain technical aspects of the previously developed non chiral bosonization technique and extends this formalism to chiral Luttinger liquids driven out of equilibrium. The density-density correlation functions (DDCF) of inhomogeneous Luttinger liquids forms a crucial input to the NCBT procedure. In Chapter 2 we show the detailed derivation of the DDCF by including only the most singular moments of the density in the RPA generating functional. The DDCF obtained is compared with the results of standard perturbation theory and is found to match term by term. In Chapter 3 we study the problem of two noninteracting chiral quantum wires coupled through a point-contact driven out of equilibrium. We derive the nonequilibrium Green functions (NEGF) for the problem for infinite bandwidth as well as for finite bandwidth in the point contact and study the transport properties. In Chapter 4 we develop an unconventional bosonization ansatz that reproduces the NEGF of the noninteracting system in presence of a point-contact. The noninteracting chiral quantum wires are isomorphic to the edge states of an integer quantum Hall system with filling fraction $\nu = 1$. The edge states of a fractional quantum Hall effect (FQHE) system are equivalent to chiral Luttinger liquids where the fermions are interacting. Bosonization is an invaluable tool to study edge state transport in a FQHE system. In order to use the unconventional bosonization technique, it is necessary to compute the density-density correlations for chiral Luttinger liquids with a point contact and this is discussed in Chapter 5. The tunneling density of states (TDOS) at a point-contact for a FQHE edge is calculated using unconventional bosonization in Chapter 6 and the TDOS exponents are shown to match with earlier works.

Chapter 2

Density-density correlation functions of strongly inhomogeneous Luttinger liquids

2.1 Introduction

Analytical expressions of the correlation functions of Luttinger liquids with arbitrary strength of mutual interactions and in presence of a scatterer of arbitrary strength are yet to be a part of the literature. Nevertheless, the attempts to reach this central goal of calculating the most general result has led to many peripheral results where the arbitrariness of one or more parameters had to be compromised. The most prevalent analytical tool to deal with these systems, which belong to the universal class of Luttinger liquids [5] is bosonization, where a fermion field operator is expressed as the exponential of a bosonic field [55, 56]. However the results of this method are exact only for the extreme cases of very weak impurity and very strong impurity (apart from some isolated examples such as the Bethe ansatz solutions [57, 58]) and to deal with the general result one has to rely on a perturbative approach in terms of the impurity strength (or inter-chain hopping in the other extreme) and renormalize the series to obtain a finite answer [37].
The study of the density correlations of a Luttinger liquid is important, which is well reflected in the literature. Iucci et al. obtained a closed-form analytical expression for the zero-temperature Fourier transform of the $2k_F$ component of the density-density correlation function in a spinful Luttinger liquid [59]. Schulz studied the density correlations in a one dimensional electron gas interacting with long ranged Coulomb forces and calculated the $4k_F$ component of the density which decays extremely slowly and represents a 1D Wigner crystal [60]. Parola presented an exact analytical evaluation of the asymptotic spin-spin correlations of the 1D Hubbard model with infinite on-site interaction ($U \to \infty$) and away from half filling [61]. Their results suggested that the renormalization-group scaling to the Tomonaga-Luttinger model is exact in the $U \to \infty$ Hubbard model. Stephan et al. calculated the dynamical density-density correlation function for the one-dimensional, half-filled Hubbard model extended with nearest-neighbor repulsion for large on-site repulsion compared to hopping amplitudes [62]. Caux et al. studied the dynamical density density correlations in a 1D Bose gas with a delta function interaction using a Bethe-ansatz-based numerical method [63]. Gambetta et al. have done a study of the correlation functions in a one-channel finite size Luttinger liquid quantum dot [64]. Protopopov et al. investigated the four-point correlations of a Luttinger liquid in a non-equilibrium setting [65]. In [66] Sen et al. performed a numerical study of the Luttinger liquid type behaviour of the density-density correlation functions in the lattice Calogero-Sutherland model. Aristov analyzed the modified density-density correlations when curvature in the fermionic dispersion is present [67].

In addition to this, numerical methods such as the density matrix renormalization group (DMRG) [68, 69] have been employed to study gapped fermionic and spin systems in 1D [70, 71, 72, 73]. However the system which we are interested in viz. Luttinger Liquids with impurities, are gapless and it is difficult to justify the application of DMRG to such systems. But notable attempts have been made in this

direction [74, 75, 76, 77].
A recently developed alternative to this was developed by our group. This technique which goes by the name 'Non chiral bosonization technique (NCBT)', does a better job of avoiding RG methods and tackling impurities of arbitrary strengths [46, 45]. But it can yield only the most singular part of the Green functions. The four-point Green functions in the context of Friedel oscillations in a Luttinger Liquid were calculated using NCBT and the most singular contribution to the slow part of the local density oscillations were obtained in the form of power laws [78]. Closed analytical expressions for the dynamical density of states exponents were also obtained. In a previous work, Matveev et al. [17] dealt with impurities of arbitrary strengths using fermionic renormalization only and without using bosonization methods, however their results are valid only for weak strengths of mutual interactions between the fermions.

The main achievement of the NCBT method is the careful redefinition of the meaning of the random phase approximation (RPA) in the context of strongly inhomogeneous Luttinger Liquids (so called due to the presence of a localised impurity backscattering of arbitrary strength that breaks the translation invariance in the system) which is by no means an obvious extension of the corresponding notion in homogeneous systems. This redefinition involves a systematic truncation and resummation of the perturbation series in powers of the fermion-fermion coupling with the impurities being arbitrary. However, DMRG methods (and conventional bosonization) apply it to the full system without any such truncation carried out. As a result we don't expect favourable comparisons between DMRG and NCBT except in limiting cases (half line and fully homogeneous system). The truncation and resummation method employed in our work is not a shortcoming since it is only when such a procedure is implemented, closed analytical expressions for the N-point functions of the system become feasible. There is no other limit in which it is possible to write down closed formulae in terms of elementary functions of positions and times for the N-point functions of a system as complicated as mutually interacting fermions in presence of impurities.

Moreover in DMRG, both forward scattering and backward scattering interactions between the fermions are taken into account. But the model we study using NCBT, includes only short-range forward scattering interactions between the fermions. So our results are bound to be different than what is obtained using techniques like DMRG. We make this distinction clear by taking the example of Oshikawa et.al [77]. Equations (1) and (3) of [77] involve terms such as $VS_i^zS_{i+1}^z$. According to Jordan Wigner transformation, $S_i^z = n_i - \frac{1}{2}$, and $n_i = c_i^\dagger c_i$ is the full density which includes both the slowly varying and rapidly varying parts. By contrast, our model described in Section 2 considers only the slow part of the density (Eq. 2.10 of the present work). Hence a comparison between such approaches and ours is not possible. The reason why we don't include backward scattering between fermions is - the correlation functions cannot be expressed in terms of elementary functions thereby limiting its use to theoretical physics. Also it is unlikely that such numerical methods are able to capture the most singular parts of the Green's functions that NCBT provides. The NCBT results have already been validated analytically using the Schwinger-Dyson equation [45] and a host of other ways as shown in our group's published works [78, 46, 79, 47, 49, 51].

The contents of this chapter is based on our published work [80]. In the following sections of this chapter, it is shown that the most singular contributions (precise meaning defined later) of the slow part of the density density correlation functions of a spinful Luttinger liquid (also spinless) with short-range forward scattering mutual interactions in presence of localized static scalar impurities is expressible in terms of elementary functions of positions and times which involve only second order poles and also involve only the bare reflection and transmission coefficients of a single particle in the presence of these impurities.

2.2 Model description

Consider a quantum system that comprises a 1D gas of electrons with forward scattering short-range mutual interactions and in the presence of a scalar potential $V(x)$ that is localized near an origin. The full generic-Hamiltonian of the system contains three parts and can be written as follows,

$$H = H_0 + H_{imp} + H_{fs} \tag{2.1}$$

where H_0 is the Hamiltonian of *free fermions* and H_{imp} is that of the *impurity* (or impurities). This has an asymptotic form in terms of its Green function which is given in [46]. H_{fs} is the Hamiltonian of *short-range forward scattering* mutual interactions between the fermions. Using the linear dispersion relations near the Fermi level ($E = E_F + kv_F$) one can write,

$$H_0 = -iv_F \int dx \sum_{\sigma=\uparrow,\downarrow} (: \psi^\dagger_R(x,\sigma)\partial_x\psi_R(x,\sigma) : - : \psi^\dagger_L(x,\sigma)\partial_x\psi_L(x,\sigma) :) \tag{2.2}$$

and the impurity Hamiltonian is given to be in Hermitian form as follows (the expression below taken at face value is ill-defined and a regularization procedure is implied as discussed below).

$$H_{imp} = V_0 \sum_{\sigma=\uparrow,\downarrow} (\psi^\dagger_R(0,\sigma)\psi_R(0,\sigma) + \psi^\dagger_L(0,\sigma)\psi_L(0,\sigma)) + V_1 \sum_{\sigma=\uparrow,\downarrow} \psi^\dagger_R(0,\sigma)\psi_L(0,\sigma) + V_1^* \sum_{\sigma=\uparrow,\downarrow} \psi^\dagger_L(0,\sigma)\psi_R(0,\sigma) \tag{2.3}$$

where the subscripts R ($\nu = +1$) and L ($\nu = -1$) are the usual right and left movers. The impurity Hamiltonian in equation (2.3) is ambiguous without proper regularisation. The point of view taken here is that the meaning of equation (2.3) is indirectly fixed by demanding that the Green function of $H_0 + H_{imp}$ be given by the following equation.

$$< T\ \psi_\nu(x,\sigma,t)\psi^\dagger_{\nu'}(x',\sigma',t') >_0\ \equiv \delta_{\sigma,\sigma'} \sum_{\gamma,\gamma'=\pm1} G^{\nu,\nu'}_{\gamma,\gamma'}(x,t;x',t')\ \theta(\gamma x)\theta(\gamma' x') \tag{2.4}$$

where $\theta(x > 0) = 1$, $\theta(x < 0) = 0$ and $\theta(0) = \frac{1}{2}$ is Heaviside's step function. It can be shown that

$$G^{\nu,\nu'}_{\gamma,\gamma'}(x,t;x',t') = \frac{g_{\gamma,\gamma'}(\nu,\nu')}{\nu x - \nu' x' - v_F(t-t')} \tag{2.5}$$

The complex numbers $g_{\gamma,\gamma'}(\nu,\nu')$ may be related to V_0 and V_1, alternatively, to the (bare) transmission (T) and reflection (R) amplitudes as shown in [46]. In terms of the reflection and transmission amplitudes, we have

$$g_{\gamma_1,\gamma_2}(\nu_1,\nu_2) = \frac{i}{2\pi}\,[\delta_{\nu_1,\nu_2}\delta_{\gamma_1,\gamma_2} + (T\delta_{\nu_1,\nu_2} + R\delta_{\nu_1,-\nu_2})\delta_{\gamma_1,\nu_1}\delta_{\gamma_2,-\nu_2} + (T^*\delta_{\nu_1,\nu_2} + R^*\delta_{\nu_1,-\nu_2})\delta_{\gamma_1,-\nu_1}\delta_{\gamma_2,\nu_2}] \tag{2.6}$$

These amplitudes may also be related to the details of the impurity potentials. The relation between V_0, V_1 in equation (2.3) to the bare transmission and reflection amplitudes is given by (see Appendix A for derivation),

$$\begin{aligned} V_0 &= \frac{2i\ v_F\ (T - T^*)}{2TT^* + T + T^*} \\ V_1 = V_1^* &= -\frac{4i\ v_F\ \ R^*T}{2TT^* + T + T^*} = \frac{4i\ v_F\ \ RT^*}{2TT^* + T + T^*} \end{aligned} \tag{2.7}$$

The slow part of the asymptotic density-density correlation may be written down using Wick's theorem as (after subtracting the uncorrelated average product: $\tilde{\rho}_s = \rho_s - <\rho_s>$),

$$
\begin{aligned}
&< T\ \tilde{\rho}_s(x,\sigma,t)\tilde{\rho}_s(x^{'},\sigma^{'},t^{'}) >_0 \;=\; -\,\delta_{\sigma,\sigma^{'}} \sum_{\gamma,\gamma^{'}=\pm 1}\ \sum_{\nu,\nu^{'}=\pm 1} \frac{|g_{\gamma,\gamma^{'}}(\nu,\nu^{'})|^2\ \theta(\gamma x)\theta(\gamma^{'}x^{'})}{(\nu x - \nu^{'}x^{'} - v_F(t-t^{'}))^2} \\
&= -\frac{\delta_{\sigma,\sigma^{'}}}{(2\pi)^2}\Big(\ \mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\frac{|R|^2}{(v_F(t_1-t_2)+|x_1|+|x_2|)^2} + \mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\frac{|R|^2}{(v_F(t_1-t_2)-|x_1|-|x_2|)^2} \\
&+ \frac{1}{(v_F(t_1-t_2)+x_1-x_2)^2} + \frac{1}{(v_F(t_1-t_2)-x_1+x_2)^2}\Big)
\end{aligned}
\tag{2.8}
$$

where,

$$
\rho_s(x,\sigma,t) = (:\psi_R^{\dagger}(x,\sigma,t)\psi_R(x,\sigma,t):+:\psi_L^{\dagger}(x,\sigma,t)\psi_L(x,\sigma,t):) \tag{2.9}
$$

In the presence of short-range forward scattering given by the additional piece,

$$
H_{fs} = \frac{1}{2}\sum_{\sigma,\sigma^{'}=\uparrow,\downarrow}\int dx \int dx^{'}\ v(x-x^{'})\ \rho_s(x,\sigma,t)\rho_s(x^{'},\sigma^{'},t) \tag{2.10}
$$

it is not possible to write down a simple formula such as equation (2.8) for the correlation function described there. The reason is because $|g_{\gamma,\gamma^{'}}(\nu,\nu^{'})|^2$ involves only the bare reflection and transmission coefficients - but in conventional chiral bosonization, they are renormalized to become scale-dependent. Also there is no guarantee that the function will continue to have simple second order poles as shown in equation (2.8). The *main claim of the non-chiral bosonization technique (NCBT)* is that if one is willing to be content at the most singular part of this correlation function then it is indeed possible to write down a simple formula very similar to equation (2.8) even when short-range forward scattering i.e. equation (2.10) is present. Furthermore, this most singular contribution will only involve the bare transmission and reflection coefficients as is the case in equation (2.8). This most singular part of the slowly varying asymptotic density-density correlation (ie. DDCF in presence of equation (2.10)) is given below and the proof is given in the next section.

$$
\langle T\ \rho_s(x_1,\sigma_1,t_1)\rho_s(x_2,\sigma_2,t_2)\rangle = \frac{1}{4}\left(\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle + \sigma_1\sigma_2\langle T\ \rho_n(x_1,t_1)\rho_n(x_2,t_2)\rangle\right) \tag{2.11}
$$

where,

$$
\begin{aligned}
\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle =& \frac{v_F}{2\pi^2 v_h}\sum_{\nu=\pm 1}\left(-\frac{1}{(x_1-x_2+\nu v_h(t_1-t_2))^2} - \frac{\frac{v_F}{v_h}\ \mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\ \frac{|R|^2}{\left(1-\frac{(v_h-v_F)}{v_h}|R|^2\right)}}{(|x_1|+|x_2|+\nu v_h(t_1-t_2))^2}\right) \\
\langle T\ \rho_n(x_1,t_1)\rho_n(x_2,t_2)\rangle =& \frac{1}{2\pi^2}\sum_{\nu=\pm 1}\left(-\frac{1}{(x_1-x_2+\nu v_F(t_1-t_2))^2} - \frac{\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\ |R|^2}{(|x_1|+|x_2|+\nu v_F(t_1-t_2))^2}\right)
\end{aligned}
\tag{2.12}
$$

with $v_h = \sqrt{v_F^2 + \frac{2v_0 v_F}{\pi}}$ and $\sigma = \uparrow (+1)$ and $\sigma = \downarrow (-1)$.

One of the main results of NCBT is the assertion that the most singular contribution to $< T\ \tilde{\rho}_s(x_1,\sigma_1,t_1)\tilde{\rho}_s(x_2,\sigma_2,t_2) >$ in the presence of short-range forward scattering between fermions viz. equation (2.10) is given by equation (2.11) and equation (2.12).

2.3 Density density correlation function results

The density density correlation functions (DDCF) in absence of mutual interactions is given by equation (2.8). This has to be systematically transformed to include mutual interactions. Firstly, the space time

DDCF is related to the momentum frequency DDCF as follows ($x_1 \neq x_2$ and $x_1 \neq 0$ and $x_2 \neq 0$).

$$< T\ \rho_s(x_1,t_1;\sigma_1)\rho_s(x_2,t_2;\sigma_2) >_0 = \frac{1}{L^2}\sum_{q,q^{'},n} e^{-iqx_1}e^{-iq^{'}x_2}e^{-w_n(t_1-t_2)} < T\ \rho_s(q,n;\sigma_1)\rho_s(q^{'},-n;\sigma_2) >_0 \tag{2.13}$$

From this we can obtain the DDCF in momentum and frequency space as follows (here β is the inverse temperature which comes into the calculation because of converting summation to integration which is allowed in the zero temperature limit: $\sum_n f(z_n) = \frac{\beta}{2\pi}\int f(z)dz$ where $z_n = \frac{\pi(2n+1)}{\beta}$).

$$\begin{aligned} < T\ \rho_s(q,n;\sigma_1)\rho_s(q^{'},-n;\sigma_2) >_0 \ &= \frac{\delta_{\sigma_1,\sigma_2}}{\beta}\frac{(2v_Fq^{'})(2v_Fq)|g_{1,1}(1,-1)|^2|w_n|(2\pi)}{((v_Fq)^2+w_n^2)((v_Fq^{'})^2+w_n^2)} \\ &+\frac{\delta_{\sigma_1,\sigma_2}}{\beta}\frac{2q^2v_F}{w_n^2+(qv_F)^2}\delta_{q+q^{'},0}\frac{L}{(2\pi)} \end{aligned} \tag{2.14}$$

In Appendix B we show how to recover equation (2.8) from equation (2.13) and equation (2.14). Now the generating function for an auxiliary field U in presence of mutual interactions between particles given by $v(x_1-x_2)$ can be written as

$$Z[U] = \int D[\rho]e^{iS_{eff,0}[\rho]}e^{\sum_{q,n,\sigma}\rho_{q,n,\sigma}U_{q,n,\sigma}}e^{-i\int_0^{-i\beta}dt\int dx_1\int dx_2\frac{1}{2}v(x_1-x_2)\rho(x_1,t_1;.)\rho(x_2,t_1;.)} \tag{2.15}$$

where S_0 is the action of free fermions and

$$\begin{aligned} \rho(x_1,t_1;.) &= \frac{1}{L}\sum_{q,n}e^{-iqx}e^{w_nt}\ \rho_{q,n;.} \\ v(x_1-x_2) &= \frac{1}{L}\sum_{Q}e^{-iQ(x_1-x_2)}v_Q \\ \rho(x_1,t_1;.) &= \rho(x_1,t_1;\uparrow)+\rho(x_1,t_1;\downarrow) \end{aligned}$$

Thus the generating function can be written as follows,

$$Z[U] = \int D[\rho]e^{iS_{eff,0}[\rho]}e^{\sum_{q,n,\sigma}\rho_{q,n,\sigma}U_{q,n,\sigma}}e^{-\sum_{q,n}\frac{\beta v_0}{2L}\rho_{q,n;.}\rho_{-q,-n;.}} \tag{2.16}$$

If one denotes the generating function in absence of interactions as Z_0, then

$$\begin{aligned} Z_0[U] &= \int D[\rho]e^{iS_{eff,0}[\rho]}e^{\sum_{q,n,\sigma}\rho_{q,n,\sigma}U_{q,n,\sigma}} \\ \Rightarrow e^{iS_{eff,0}[\rho]} &= \int D[U^{'}]e^{-\sum_{q,n,\sigma}\rho_{q,n,\sigma}U^{'}_{q,n,\sigma}}\ Z_0[U^{'}] \end{aligned}$$

Inserting in equation (2.16),

$$Z[U] = \int D[\rho]\ \int D[U^{'}]\ Z_0[U^{'}]\ e^{\sum_{q,n,\sigma}\rho_{q,n,\sigma}(U_{q,n,\sigma}-U^{'}_{q,n,\sigma})}e^{-\sum_{q,n}\frac{\beta v_0}{2L}\rho_{q,n;.}\rho_{-q,-n;.}} \tag{2.17}$$

Set,

$$\rho_{q,n,\sigma} = \frac{1}{2}\rho_{q,n;.}+\frac{1}{2}\sigma\ \sigma_{q,n}\ ;\ U_{q,n,\sigma} = \frac{1}{2}U_{q,n;.}+\frac{1}{2}\sigma\ W_{q,n}\ ;\ U^{'}_{q,n,\sigma} = \frac{1}{2}U^{'}_{q,n;.}+\frac{1}{2}\sigma\ W^{'}_{q,n} \tag{2.18}$$

Using these relations the generating function can be written as

$$Z[U] = \int D[\rho]\ \int D[U^{'}]\ Z_0[U^{'}]\ e^{\frac{1}{4}\sum_{q,n,\sigma}(\rho_{q,n;.}+\sigma\ \sigma_{q,n})((U_{q,n;.}-U^{'}_{q,n;.})+\sigma\ (W_{q,n}-W^{'}_{q,n}))}e^{-\sum_{q,n}\frac{\beta v_0}{2L}\rho_{q,n;.}\rho_{-q,-n;.}} \tag{2.19}$$

where in the RPA sense (for the homogeneous system, this choice corresponds to RPA, for the present steeplechase problem this choice corresponds to the most singular truncation of the RPA generating function)

$$Z_0[U^{'}] = e^{\frac{1}{2}\sum_{q,q^{'},n;\sigma} <\rho_{q,n}\rho_{q^{'},-n}>_0 \; U^{'}_{q,n;\sigma}U^{'}_{q^{'},-n;\sigma}} \tag{2.20}$$

where $<\rho_{q,n}\rho_{q^{'},-n}>_0$ is equation (2.14) with $\sigma_1 = \sigma_2$. It is to be noted that we have neglected the higher moments of ρ in $Z_0[U']$ beyond the quadratic as they are less singular than the second moment (see section 2.4.1). Now,

$$\begin{aligned}
Z[U] = &\int D[\rho] \int D[U^{'}] \; e^{\frac{1}{2}\sum_{q,q^{'},n;\sigma} <\rho_{q,n}\rho_{q^{'},-n}>_0 \; \frac{1}{4}(U^{'}_{q,n;.}+\sigma \; W^{'}_{q,n})(U^{'}_{q^{'},-n;.}+\sigma \; W^{'}_{q^{'},-n})} \\
&e^{\frac{1}{4}\sum_{q,n,\sigma}(\rho_{q,n;.}+\sigma \; \sigma_{q,n})((U_{q,n;.}-U^{'}_{q,n;.})+\sigma \; (W_{q,n}-W^{'}_{q,n}))} e^{-\sum_{q,n}\frac{\beta v_0}{2L}\rho_{q,n;.}\rho_{-q,-n;.}} \\
= &\int D[U^{'}] \; e^{\frac{1}{2}\sum_{q,q^{'},n} <\rho_{q,n}\rho_{q^{'},-n}>_0 \; \frac{1}{2}(U^{'}_{q,n;.}U^{'}_{q^{'},-n;.}+W^{'}_{q,n}W^{'}_{q^{'},-n})} \\
&\int D[\rho] \; e^{\frac{1}{2}\sum_{q,n}(\rho_{q,n;.}(U_{q,n;.}-U^{'}_{q,n;.})+\sigma_{q,n}(W_{q,n}-W^{'}_{q,n}))} e^{-\sum_{q,n}\frac{\beta v_0}{2L}\rho_{q,n;.}\rho_{-q,-n;.}} \\
= &\int D[U^{'}] \; e^{\frac{1}{4}\sum_{q,q^{'},n} <\rho_{q,n}\rho_{q^{'},-n}>_0 \; W_{q,n}W_{q^{'},-n}} \; e^{\frac{1}{4}\sum_{q,q^{'},n} <\rho_{q,n}\rho_{q^{'},-n}>_0 \; U^{'}_{q,n;.}U^{'}_{q^{'},-n;.}} \\
&\int D[\rho] \; e^{\frac{1}{2}\sum_{q,n}\rho_{q,n;.}(U_{q,n;.}-U^{'}_{q,n;.})} e^{-\sum_{q,n}\frac{\beta v_0}{2L}\rho_{q,n;.}\rho_{-q,-n;.}}
\end{aligned}$$

The last result follows from the extremum condition viz.

$$\rho_{-q,-n;.} = \frac{L}{2\beta v_0}(U_{q,n;.} - U^{'}_{q,n;.}) \tag{2.21}$$

Thus (including only the holon part),

$$Z[U] = \int D[U^{'}] \; e^{\frac{1}{4}\sum_{q,q^{'},n} <\rho_{q,n}\rho_{q^{'},-n}>_0 \; U^{'}_{q,n;.}U^{'}_{q^{'},-n;.}} \; e^{\frac{1}{4}\sum_{q,n}\frac{L}{2\beta v_0}(U_{-q,-n;.}-U^{'}_{-q,-n;.})(U_{q,n;.}-U^{'}_{q,n;.})} \tag{2.22}$$

The integration has to be done using the saddle point method which involves finding the extremum of the log of the integrand with respect to $U^{'}$ and that leads to the answer we are looking for. This means,

$$\sum_{q^{'}} <\rho_{q,n}\rho_{q^{'},-n}>_0 \;\; U^{'}_{q^{'},-n;.} - \frac{L}{2\beta v_0}(U_{-q,-n;.} - U^{'}_{-q,-n;.}) = 0 \tag{2.23}$$

From equation 2.14 we have,

$$\beta \;\; <\rho_{q,n}\rho_{q^{'},-n}>_0 = \frac{(2v_F q^{'})(2v_F q)}{((v_F q)^2 + w_n^2)((v_F q^{'})^2 + w_n^2)}|g_{1,1}(1,-1)|^2|w_n|(2\pi) + \frac{2q^2 v_F}{w_n^2 + (qv_F)^2}\delta_{q+q^{'},0}\frac{L}{(2\pi)} \tag{2.24}$$

This means,

$$\begin{aligned}
&\frac{(2v_F q)}{((v_F q)^2 + w_n^2)}\sum_{q^{'}}\frac{(2v_F q^{'})}{((v_F q^{'})^2 + w_n^2)} \; U^{'}_{q^{'},-n;.}|g_{1,1}(1,-1)|^2|w_n|(2\pi) \\
&\qquad + \frac{2q^2 v_F}{w_n^2 + (qv_F)^2} \; U^{'}_{-q,-n;.}\frac{L}{(2\pi)} - \frac{L}{2v_0}(U_{-q,-n;.} - U^{'}_{-q,-n;.}) = 0
\end{aligned} \tag{2.25}$$

Let us define the Hilbert transform,

$$\frac{1}{L}\sum_{q^{'}}\frac{(2v_F q^{'})}{((v_F q^{'})^2 + w_n^2)} \; U^{'}_{q^{'},-n;.} = u^{'}_n \tag{2.26}$$

then,

$$\frac{(2v_F q)}{((v_F q)^2 + w_n^2)}u^{'}_n L|g_{1,1}(1,-1)|^2|w_n|(2\pi) + \frac{2q^2 v_F}{w_n^2 + (qv_F)^2} \; U^{'}_{-q,-n;.}\frac{L}{(2\pi)} - \frac{L}{2v_0}(U_{-q,-n;.} - U^{'}_{-q,-n;.}) = 0$$

or,

$$\left(\frac{L}{2v_0}+\frac{L}{(2\pi)}\frac{2q^2v_F}{w_n^2+(qv_F)^2}\right)\ U^{'}_{-q,-n;.}=-\frac{(2v_Fq)}{((v_Fq)^2+w_n^2)}u^{'}_nL|g_{1,1}(1,-1)|^2|w_n|(2\pi)+\frac{L}{2v_0}U_{-q,-n;.}$$

This means,

$$U^{'}_{-q,-n;.}=-\frac{(2v_Fq)}{((v_Fq)^2+w_n^2)}\frac{u^{'}_nL|g_{1,1}(1,-1)|^2|w_n|(2\pi)}{\left(\frac{L}{2v_0}+\frac{L}{(2\pi)}\frac{2q^2v_F}{w_n^2+(qv_F)^2}\right)}+\frac{\frac{L}{2v_0}U_{-q,-n;.}}{\left(\frac{L}{2v_0}+\frac{L}{(2\pi)}\frac{2q^2v_F}{w_n^2+(qv_F)^2}\right)} \tag{2.27}$$

Set the holon velocity $v^2=v_F^2+\frac{2v_Fv_0}{\pi}$.

$$U^{'}_{-q,-n;.}=-(2v_Fq)\frac{u^{'}_n2v_0|g_{1,1}(1,-1)|^2|w_n|(2\pi)}{(w_n^2+(qv)^2)}+\frac{(w_n^2+(qv_F)^2)U_{-q,-n;.}}{(w_n^2+(qv)^2)} \tag{2.28}$$

Hence,

$$u^{'}_n\frac{1}{L}\sum_{q^{'}}\frac{(2v_Fq^{'})^2}{((v_Fq^{'})^2+w_n^2)(w_n^2+(q^{'}v)^2)}\ 2v_0|g_{1,1}(1,-1)|^2|w_n|(2\pi)+\frac{1}{L}\sum_{q^{'}}\frac{(2v_Fq^{'})U_{q^{'},-n;.}}{(w_n^2+(q^{'}v)^2)}=u^{'}_n \tag{2.29}$$

and we obtain the expression,

$$u^{'}_n=\frac{1}{L}\sum_{q^{'}}\frac{(2v_Fq^{'})U_{q^{'},-n;.}}{\left(1-\frac{4v_0v_F}{(v+v_F)v}\ |g_{1,1}(1,-1)|^2(2\pi)\right)(w_n^2+(q^{'}v)^2)} \tag{2.30}$$

We have,

$$\sum_{q^{'}}<\rho_{q,n}\rho_{q^{'},-n}>_0\ U^{'}_{q^{'},-n;.}-\frac{L}{2\beta v_0}(U_{-q,-n;.}-U^{'}_{-q,-n;.})=0 \tag{2.31}$$

Hence,

$$Z[U]=\ e^{\frac{1}{4}\sum_{q,n}\frac{L}{2\beta v_0}(U_{-q,-n;.}-U^{'}_{-q,-n;.})U_{q,n;.}} \tag{2.32}$$

and

$$U_{-q,-n;.}-U^{'}_{-q,-n;.}=\frac{1}{L}\sum_{q^{'}}\frac{(2v_Fq^{'})U_{q^{'},-n;.}}{\left(1-\frac{4v_0v_F}{(v+v_F)v}\ |g_{1,1}(1,-1)|^2(2\pi)\right)(w_n^2+(q^{'}v)^2)}(2v_Fq)\frac{2v_0|g_{1,1}(1,-1)|^2|w_n|(2\pi)}{(w_n^2+(qv)^2)}$$
$$+U_{-q,-n;.}\frac{(qv)^2-(qv_F)^2}{w_n^2+(qv)^2} \tag{2.33}$$

But we had

$$Z[U]=\int D[U^{'}]\ e^{\frac{1}{4}\sum_{q,q^{'},n}<\rho_{q,n}\rho_{q^{'},-n}>_0\ U^{'}_{q,n;.}U^{'}_{q^{'},-n;.}}\ e^{\frac{1}{4}\sum_{q,n}\frac{L}{2\beta v_0}(U_{-q,-n;.}-U^{'}_{-q,-n;.})(U_{q,n;.}-U^{'}_{q,n;.})}$$

This means

$$Z[U]=e^{\sum_{q,n}\frac{L}{8\beta v_0}U_{q,n;.}U_{-q,-n;.}\frac{(qv_h)^2-(qv_F)^2}{w_n^2+(qv_h)^2}}$$
$$e^{\sum_{q,n}\frac{L}{8\beta v_0}\frac{1}{L}\sum_{q^{'}}\frac{(2v_Fq^{'})U_{q,n;.}U_{q^{'},-n;.}}{\left(1-\frac{4v_0v_F}{(v_h+v_F)v_h}\ |g_{1,1}(1,-1)|^2(2\pi)\right)\left(w_n^2+(q^{'}v_h)^2\right)}(2v_Fq)\frac{2v_0|g_{1,1}(1,-1)|^2|w_n|(2\pi)}{(w_n^2+(qv_h)^2)}} \tag{2.34}$$

Here,

$$v_h=\sqrt{v_F^2+\frac{2v_Fv_0}{\pi}} \tag{2.35}$$

The two-point density correlations are obtained from the generating functional by differenting $Z[U]$ twice w.r.t the auxiliary field U which is taken to zero,

$$\frac{\partial^2 Z[U]}{\partial U_{q,n}\partial U_{q',-n}}|_{U=0} \quad = \quad \langle \rho_{q,n;.}\rho_{q',-n;.}\rangle \tag{2.36}$$

Since we have,

$$\rho_{q,n;.} = \rho_{q,n;\uparrow} + \rho_{q,n;\downarrow} \; ; \; \sigma_{q,n} = \rho_{q,n;\uparrow} - \rho_{q,n;\downarrow} \tag{2.37}$$

Thus the holon part of the density correlations is obtained as,

$$\begin{aligned} \frac{1}{4} < \rho_{q,n;.}\rho_{q',-n;.} > &= \delta_{q+q',0}\, \frac{L}{4\beta}\frac{2v_F q^2}{\pi(w_n^2+(qv_h)^2)} \\ &+\frac{1}{2\beta}\frac{(2v_F q)(2v_F q')|g_{1,1}(1,-1)|^2|w_n|(2\pi)}{\left(1-\frac{(v_h-v_F)}{v_h}\;|g_{1,1}(1,-1)|^2(2\pi)^2\right)(w_n^2+(q'v_h)^2)\,(w_n^2+(qv_h)^2)} \end{aligned} \tag{2.38}$$

and the spinon density correlation is

$$\begin{aligned} \frac{1}{4} < \sigma_{q,n}\sigma_{q',-n} > &= \frac{(2v_F q')(2v_F q)}{2\beta((v_F q)^2+w_n^2)((v_F q')^2+w_n^2)}|g_{1,1}(1,-1)|^2|w_n|(2\pi) + \frac{q^2 v_F L}{2\pi\beta(w_n^2+(qv_F)^2)}\delta_{q+q',0} \\ < \sigma_{q,n}\rho_{q',-n;.} > &= 0 \end{aligned} \tag{2.39}$$

Finally, the full density density correlation functions in momentum frequency space can be written as,

$$< \rho_{q,n,\sigma}\rho_{q',-n,\sigma'} >= \frac{1}{4} < \rho_{q,n;.}\rho_{q',-n;.} > + \frac{1}{4}\sigma\sigma' < \sigma_{q,n}\sigma_{q',-n} > \tag{2.40}$$

Doing an inverse Fourier transform of the above we get the DDCF in real space time.

$$\begin{aligned} &<T\; \rho_s(x_1,t_1;\sigma)\rho_s(x_2,t_2;\sigma') >= \\ &-\frac{v_F^2}{v_h^4}\,\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\left(\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)+\frac{|x_1|+|x_2|}{v_h}]^2}+\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)-\frac{|x_1|+|x_2|}{v_h}]^2}\right)\frac{1}{2\beta}\frac{|g_{1,1}(1,-1)|^2(2\pi)}{\left(1-\frac{4v_0v_F}{(v_h+v_F)v_h}\;|g_{1,1}(1,-1)|^2(2\pi)\right)} \\ &-\frac{\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)}{v_F^2}\left(\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)+\frac{|x_1|+|x_2|}{v_F}]^2}+\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)-\frac{|x_1|+|x_2|}{v_F}]^2}\right)\frac{1}{2\beta}|g_{1,1}(1,-1)|^2(2\pi\sigma\sigma') \\ &-\frac{v_F}{4\pi\beta v_h^3}\left(\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)+\frac{|x_1-x_2|}{v_h}]^2}+\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)-\frac{|x_1-x_2|}{v_h}]^2}\right) \\ &-\frac{\sigma\sigma'}{4\pi\beta v_F^2}\left(\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)+\frac{|x_1-x_2|}{v_F}]^2}+\frac{\beta}{2\pi}\,\frac{1}{[(t_1-t_2)-\frac{|x_1-x_2|}{v_F}]^2}\right) \end{aligned} \tag{2.41}$$

One can separate the holon and spinon parts of the DDCF and write as follows.

$$\begin{aligned} \langle T\; \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle &= \frac{v_F}{2\pi^2 v_h}\sum_{\nu=\pm1}\left(-\frac{1}{(x_1-x_2+\nu v_h(t_1-t_2))^2}-\frac{\frac{v_F}{v_h}\;\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\;\frac{|R|^2}{\left(1-\frac{(v_h-v_F)}{v_h}|R|^2\right)}}{(|x_1|+|x_2|+\nu v_h(t_1-t_2))^2}\right) \\ \langle T\; \rho_n(x_1,t_1)\rho_n(x_2,t_2)\rangle &= \frac{1}{2\pi^2}\sum_{\nu=\pm1}\left(-\frac{1}{(x_1-x_2+\nu v_F(t_1-t_2))^2}-\frac{\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\;|R|^2}{(|x_1|+|x_2|+\nu v_F(t_1-t_2))^2}\right) \end{aligned} \tag{2.42}$$

The full density density correlation functions can thus be written in a compact form as follows.

$$\langle T\; \rho_s(x_1,\sigma_1,t_1)\rho_s(x_2,\sigma_2,t_2)\rangle = \frac{1}{4}\left(\langle T\; \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle + \sigma_1\sigma_2\langle T\; \rho_n(x_1,t_1)\rho_n(x_2,t_2)\rangle\right) \tag{2.43}$$

2.4 Key aspects of our result

In this section we highlight and discuss certain key aspects of our result for the DDCF. We justify the use of the Gaussian approximation in the generating functional $Z_0[U']$ allowing us to obtain the most singular part of the correlations. The relation between the fast and slow parts of the density through the non-standard harmonic analysis is an important aspect of our work and it is elaborated. We also show how to obtain the DDCF for the case of spinless (spin polarized) fermions using our general result.

2.4.1 Gaussian approximation

In equation (2.20), it is seen that only the quadratic moment of ρ in the $Z_0[U']$ is included, neglecting all the higher moments. The reason for this is the following. The connected parts of the odd moments ρ vanish identically (in the RPA limit) and that of the even moments are less singular than the second moment, etc. For example, the connected 4-density function,

$$\begin{aligned} < T\rho(x_1)\rho(x_2)\rho(x_3)\rho(x_4) >_c \quad \sim \quad & < T\psi(x_1)\psi^*(x_2) >< T\psi(x_2)\psi^*(x_3) >< T\psi(x_3)\psi^*(x_4) >< T\psi(x_4)\psi^*(x_1) > \\ & + \text{ permutations} \end{aligned} \tag{2.44}$$

has only first order poles and no second order poles. The NCBT does not claim to provide the asymptotic Green functions of strongly inhomogeneous interacting Fermi systems exactly but it does claim to provide the *most singular part of* the asymptotic Green functions of strongly inhomogeneous interacting Fermi systems exactly. However this is not the case for the homogeneous systems ($|R| = 0$) and half-lines ($|R| = 1$) where both the even and the odd moments higher than the quadratic moment of density functions vanishes. Hence for these extreme cases, the density-density correlation functions are not just the most singular part but the full story. To prove this, a quantity 'Δ' is defined as follows.

$$\begin{aligned} \Delta \equiv & < T \ \rho_s(x_1,\sigma_1,t_1)\rho_s(x_2,\sigma_2,t_2)\rho_s(x_3,\sigma_3,\sigma_3,t_3)\rho_s(x_4,\sigma_4,t_4) >_0 \\ & - < T \ \rho_s(x_1,\sigma_1,t_1)\rho_s(x_2,\sigma_2,t_2) >_0 \quad < T \ \rho_s(x_3,\sigma_3,t_3)\rho_s(x_4,\sigma_4,t_4) >_0 \\ & - < T \ \rho_s(x_1,\sigma_1,t_1)\rho_s(x_3,\sigma_3,t_3) >_0 \quad < T \ \rho_s(x_2,\sigma_2,t_2)\rho_s(x_4,\sigma_4,t_4) >_0 \\ & - < T \ \rho_s(x_1,\sigma_1,t_1)\rho_s(x_4,\sigma_4,t_4) >_0 \quad < T \ \rho_s(x_2,\sigma_2,t_2)\rho_s(x_3,\sigma_3,t_3) >_0 \end{aligned} \tag{2.45}$$

If $\Delta = 0$ it means Wick's theorem is applicable at the level of pairs of fermions which makes the Gaussian theory valid. So now, the question is whether Δ is zero or not. Expanding the above expression using conventional Wick's theorem and using the form of the Green function shown in equation (2.4) we have the following results for various cases.

Case I: All four points on same side $(x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0)$

We have $\rho_s(x,\sigma,t) =: \psi^\dagger_R(x,\sigma,t)\psi_R(x,\sigma,t) + \psi^\dagger_L(x,\sigma,t)\psi_L(x,\sigma,t) :$ where :: represents normal ordering. For this case we obtain,

$$\begin{aligned} < T \ \rho_s(x_1,\sigma,t_1)\rho_s(x_2,\sigma,t_2)\rho_s(x_3,\sigma,t_3)\rho_s(x_4,\sigma,t_4) >_c \ = \ \frac{|R|^2|T|^2}{2\pi^4}\times \\ \Big(\frac{1}{((t_3-t_1)v_F+x_1+x_3)((t_3-t_2)v_F+x_2+x_3)((t_4-t_1)v_F+x_1+x_4)((t_4-t_2)v_F+x_2+x_4)} \\ + \frac{1}{((t_1-t_2)v_F+x_1+x_2)((t_3-t_2)v_F+x_2+x_3)((t_1-t_4)v_F+x_1+x_4)((t_3-t_4)v_F+x_3+x_4)} \\ + \frac{1}{((t_2-t_1)v_F+x_1+x_2)((t_3-t_1)v_F+x_1+x_3)((t_2-t_4)v_F+x_2+x_4)((t_3-t_4)v_F+x_3+x_4)} \\ + \frac{1}{((t_2-t_1)v_F+x_1+x_2)((t_2-t_3)v_F+x_2+x_3)((t_4-t_1)v_F+x_1+x_4)((t_4-t_3)v_F+x_3+x_4)} \\ + \frac{1}{((t_1-t_2)v_F+x_1+x_2)((t_1-t_3)v_F+x_1+x_3)((t_4-t_2)v_F+x_2+x_4)((t_4-t_3)v_F+x_3+x_4)} \\ + \frac{1}{((t_1-t_3)v_F+x_1+x_3)((t_2-t_3)v_F+x_2+x_3)((t_1-t_4)v_F+x_1+x_4)((t_2-t_4)v_F+x_2+x_4)} \Big) \end{aligned} \tag{2.46}$$

Case II: Three points on same side ($x_1 < 0, x_2 > 0, x_3 > 0, x_4 > 0$)

$$
\begin{aligned}
< T\ \rho_s(x_1,\sigma,t_1)\rho_s(x_2,\sigma,t_2)\rho_s(x_3,\sigma,t_3)\rho_s(x_4,\sigma,t_4) >_c \; &= \frac{|R|^2|T|^2}{2\pi^4}\times \\
\Bigg(-\frac{1}{((t_1-t_3)v_F + x_1 - x_3)((t_3-t_2)v_F + x_2 + x_3)((t_1-t_4)v_F + x_1 - x_4)((t_4-t_2)v_F + x_2 + x_4)} \\
-\frac{1}{((t_1-t_2)v_F - x_1 + x_2)((t_3-t_2)v_F + x_2 + x_3)((t_1-t_4)v_F - x_1 + x_4)((t_3-t_4)v_F + x_3 + x_4)} \\
-\frac{1}{((t_1-t_2)v_F + x_1 - x_2)((t_1-t_3)v_F + x_1 - x_3)((t_2-t_4)v_F + x_2 + x_4)((t_3-t_4)v_F + x_3 + x_4)} \\
-\frac{1}{((t_1-t_2)v_F + x_1 - x_2)((t_2-t_3)v_F + x_2 + x_3)((t_1-t_4)v_F + x_1 - x_4)((t_4-t_3)v_F + x_3 + x_4)} \\
-\frac{1}{((t_1-t_2)v_F - x_1 + x_2)((t_1-t_3)v_F - x_1 + x_3)((t_4-t_2)v_F + x_2 + x_4)((t_4-t_3)v_F + x_3 + x_4)} \\
+\frac{1}{((t_3-t_1)v_F + x_1 - x_3)((t_2-t_3)v_F + x_2 + x_3)((t_1-t_4)v_F - x_1 + x_4)((t_2-t_4)v_F + x_2 + x_4)}\Bigg)
\end{aligned}
\tag{2.47}
$$

Case III: Two points on same side ($x_1 < 0, x_2 < 0, x_3 > 0, x_4 > 0$)

$$
\begin{aligned}
< T\ \rho_s(x_1,\sigma,t_1)\rho_s(x_2,\sigma,t_2)\rho_s(x_3,\sigma,t_3)\rho_s(x_4,\sigma,t_4) >_c \; &= \frac{|R|^2|T|^2}{4\pi^4} \\
\Bigg(-\frac{1}{((t_1-t_3)v_F + x_1 - x_3)((t_3-t_2)v_F - x_2 + x_3)((t_1-t_4)v_F + x_1 - x_4)((t_2-t_4)v_F + x_2 - x_4)} \\
-\frac{2}{((t_3-t_2)v_F + x_2 - x_3)((t_1-t_3)v_F - x_1 + x_3)((t_1-t_4)v_F - x_1 + x_4)((t_2-t_4)v_F - x_2 + x_4)} \\
-\frac{1}{((t_2-t_3)v_F + x_2 - x_3)((t_3-t_1)v_F - x_1 + x_3)((t_1-t_4)v_F + x_1 - x_4)((t_2-t_4)v_F + x_2 - x_4)}\Bigg) \\
+\frac{R^{*2}T^2}{4\pi^4}\Bigg(-\frac{1}{((t_1-t_2)v_F + x_1 + x_2)((t_1-t_3)v_F + x_1 - x_3)((t_2-t_4)v_F - x_2 + x_4)((t_3-t_4)v_F + x_3 + x_4)} \\
-\frac{1}{((t_1-t_2)v_F + x_1 + x_2)((t_2-t_3)v_F - x_2 + x_3)((t_1-t_4)v_F + x_1 - x_4)((t_4-t_3)v_F + x_3 + x_4)} \\
-\frac{1}{((t_2-t_1)v_F + x_1 + x_2)((t_1-t_3)v_F - x_1 + x_3)((t_2-t_4)v_F + x_2 - x_4)((t_4-t_3)v_F + x_3 + x_4)} \\
-\frac{1}{((t_2-t_1)v_F + x_1 + x_2)((t_2-t_3)v_F + x_2 - x_3)((t_1-t_4)v_F - x_1 + x_4)((t_3-t_4)v_F + x_3 + x_4)}\Bigg) + \\
\frac{R^2T^{*2}}{4\pi^4}\Bigg(-\frac{1}{((t_1-t_2)v_F + x_1 + x_2)((t_1-t_3)v_F + x_1 - x_3)((t_2-t_4)v_F - x_2 + x_4)((t_3-t_4)v_F + x_3 + x_4)} \\
-\frac{1}{((t_1-t_2)v_F + x_1 + x_2)((t_2-t_3)v_F - x_2 + x_3)((t_1-t_4)v_F + x_1 - x_4)((t_4-t_3)v_F + x_3 + x_4)} \\
-\frac{1}{((t_2-t_1)v_F + x_1 + x_2)((t_1-t_3)v_F - x_1 + x_3)((t_2-t_4)v_F + x_2 - x_4)((t_4-t_3)v_F + x_3 + x_4)} \\
-\frac{1}{((t_2-t_1)v_F + x_1 + x_2)((t_2-t_3)v_F + x_2 - x_3)((t_1-t_4)v_F - x_1 + x_4)((t_3-t_4)v_F + x_3 + x_4)}\Bigg)
\end{aligned}
\tag{2.48}
$$

Now it is easy to see that for all the three possible cases above, the fourth moment of density will vanish when either of $|R|$ or $|T|$ vanishes. Hence the Gaussian theory is exact for a homogeneous system and a half line. For all intermediate cases ($0 < |R| < 1$), the fourth moment (and similarly the higher even moments) are less singular with first order poles as compared to the second moment which contains second order poles. Also note that all the above cases were done for all the four spins to be identical. If any one of them is different, then all the results on the RHSs of the above cases will also vanish.

2.4.2 Relation between fast and slow parts of DDCF

In the RPA sense, the density $\rho(x,\sigma,t)$ may be "harmonically analysed" as follows.

$$\rho(x,\sigma,t) = \rho_s(x,\sigma,t) + e^{2ik_Fx}\ \rho_f(x,\sigma,t) + e^{-2ik_Fx}\ \rho_f^*(x,\sigma,t) \tag{2.49}$$

Here ρ_s and ρ_f are the slowly varying and the rapidly varying parts respectively. The most singular parts of the rapidly varying components of the DDCF may be obtained by the following "non-standard harmonic analysis"

$$\rho_f(x,\sigma,t) \sim e^{2\pi i\int^x dy(\rho_s(y,t)+\lambda\ \rho_s(-y,t))} \tag{2.50}$$

where $\lambda = 0$ or $\lambda = 1$ depending upon which correlation function we want to reproduce. The presence of $\rho_s(-y,t)$ automatically incorporates (at the level of correlation functions), the presence of localised impurities in the system. This is unlike the conventional chiral bosonization method where the impurities lead to non-local hamiltonians whereas the Fermi-Bose correspondence is left untouched. The conventional method, though correct in principle, is unwieldy and unnatural since the impurities which lead to strong qualitative changes in the ground state of the system are introduced as an afterthought whereas the short-range forward scattering which leads only to subtle (but qualitative) changes are treated exactly. NCBT does a good job of including both but it is still not exact as it can only provide the most singular parts of the correlation functions in terms of elementary functions of positions and times.

Furthermore, in NCBT, the field "operator" (only a mnemonic for the correlation functions it generates) has a modified form which automatically takes into account the presence of impurities.

$$\begin{aligned} &\psi_\nu(x,\sigma,t) \sim e^{i\Xi_\nu(x,\sigma,t)}; \\ &\Xi_\nu(x,\sigma,t) \equiv \theta_\nu(x,\sigma,t) + 2\pi\lambda\nu\int^x dy\ \rho_s(-y,\sigma,t) \end{aligned} \tag{2.51}$$

where $\nu = R, L$ and $\lambda = 0, 1$ depending upon which 2-point correlation function we are looking for (right-right, right-left, left-right, left-left movers and both points on same (opposite) sides of the origin) . Also $\theta_\nu(x,\sigma,t)$ is the same as what is found in chiral bosonization. It may be related to currents and densities and through the continuity equation, finally only to the slow parts of the density.

$$\theta_\nu(x,\sigma,t) = \pi\int^x dy\left(\nu\rho_s(y,\sigma,t) - \int^y dy'\partial_{v_Ft}\rho_s(y',\sigma,t)\right) \tag{2.52}$$

As usual $\rho_s(y,\sigma,t) \equiv: \psi^\dagger_R(y,\sigma,t)\psi_R(y,\sigma,t) + \psi^\dagger_L(y,\sigma,t)\psi_L(y,\sigma,t)$: Note that unlike in chiral bosonization, equation (2.51) is *not* an operator identity. It is only meant to capture the most singular parts of the two point functions by employing the following device. We assert that the most singular parts of the 2-point functions are given by retaining only the leading terms in the cumulant expansion.

$$< \psi_\nu(x,\sigma,t)\psi^\dagger_{\nu'}(x',\sigma,t') > \ \sim\ < e^{i\Xi_\nu(x,\sigma,t)}e^{-i\Xi_{\nu'}(x',\sigma,t')} > \sim e^{-\frac{1}{2}<\Xi^2_\nu(x,\sigma,t)>}e^{-\frac{1}{2}<\Xi^2_{\nu'}(x',\sigma,t')>}e^{<\Xi_\nu(x,\sigma,t)\Xi_{\nu'}(x',\sigma,t')>} \tag{2.53}$$

As long as the symbol $< ... >$ on both sides of the equation (2.53) is read as "most singular part of the expectation value", equation (2.53) is in fact the exact answer for the (most singular part of the) two-point functions of a strongly inhomogeneous Luttinger liquid. The higher order terms in the cumulant expansion are purposely dropped as they can be shown to be less singular (see subsection 2.4.1).

The other important point worth mentioning is that in chiral bosonization, the 2-point Green functions in presence of impurities are discontinuous functions of the short-range forward scattering strength. This means in presence of impurities, the full asymptotic Green functions for weak short-range forward scattering are qualitatively (discontinuously) different from the corresponding quantities for no short-range forward scattering. This is the origin of the "cutting the chain" and "healing the chain" metaphors applicable for repulsive and attractive short-range forward scattering respectively.

However NCBT only yields the *most singular* part of the asymptotic Green functions. This attribute exhibits a behaviour complementary to what is seen in the full Green function. The *most singular* part of the 2-point Green functions are discontinuous functions of the *impurity strength* in presence of short-range forward scattering between fermions. This means the NCBT Green functions with weak impurities are qualitatively (ie. discontinuously) different from the corresponding quantity for no impurities.

This is the main reason why the results of chiral bosonization and NCBT cannot be easily compared with one another as they are fully complementary. The only exception is for a fully homogeneous system or its antithesis viz. the half line where the two results coincide.

2.4.3 Results for the spinless case

It is easy to transform equation (2.42) and equation (2.43) to show the results for the spinless fermions. For doing so, the density density correlation functions of the holons $\langle \rho_h \rho_h \rangle$ needs to be doubled and that of the spinons $\langle \rho_n \rho_n \rangle$ are allowed to vanish. The holon velocity in this case will be related to the Fermi velocity as $v_h = \sqrt{v_F^2 + v_0 v_F/\pi}$. Hence the density density correlation function for the spinless case is the following.

$$\langle T\ \rho_s(x_1,t_1)\rho_s(x_2,t_2)\rangle = \frac{v_F}{4\pi^2 v_h} \sum_{\nu=\pm1} \left(-\frac{1}{(x_1-x_2+\nu v_h(t_1-t_2))^2} - \frac{|R|^2}{\left(1-\frac{(v_h-v_F)}{v_h}|R|^2\right)} \frac{\frac{v_F}{v_h}\,\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)}{(|x_1|+|x_2|+\nu v_h(t_1-t_2))^2} \right) \tag{2.54}$$

2.5 Perturbative comparison of our results

The density density correlation functions of a strongly inhomogeneous Luttinger liquid given by equation (2.42) can be verified by a comparison with those obtained using standard fermionic perturbation (the comparison for the limiting cases, viz., $|R| = 0$ and $|R| = 1$ are given in Appendix C and Appendix D respectively). The general case viz. $0 < |R| < 1$ is given in Appendix E. Since the spinon DDCF is the same as that of the non-interacting DDCF (it is only the total density that couples with the interaction), hence it will suffice only to compare the holon DDCF to perturbative results. For this, the holon DDCF is expanded in powers of the interaction parameter v_0. Note that the holon velocity v_h is related to the Fermi velocity and the interaction parameter v_0 by the relation $v_h = v_F\sqrt{1+2v_0/(\pi v_F)}$. The holon DDCF is given as follows.

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle = \frac{v_F}{2\pi^2 v_h} \sum_{\nu=\pm1} \left(-\frac{1}{(x_1-x_2+\nu v_h(t_1-t_2))^2} - \frac{\frac{v_F}{v_h}\,\mathrm{sgn}(x_1x_2)\ \frac{|R|^2}{\left(1-\frac{(v_h-v_F)}{v_h}|R|^2\right)}}{(|x_1|+|x_2|+\nu v_h(t_1-t_2))^2} \right) \tag{2.55}$$

Expanding in powers of interaction parameter v_0 and retaining up to the first order, the following is obtained.

$$\begin{aligned} &\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle \\ &= \langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^0 + v_0\ \langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 + O[v_0^2] \end{aligned} \tag{2.56}$$

Now

$$\begin{aligned} \langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^0 = -\frac{1}{2\pi^2}\Bigg(&\frac{\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)|R|^2}{(|x_1|+|x_2|+v_F(t_1-t_2))^2} + \frac{\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)|R|^2}{(|x_1|+|x_2|-v_F(t_1-t_2))^2} \\ &+ \frac{1}{(x_1-x_2+v_F(t_1-t_2))^2} + \frac{1}{(x_1-x_2-v_F(t_1-t_2))^2}\Bigg) \end{aligned} \tag{2.57}$$

and

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 =$$
$$\frac{1}{4\pi^3 v_F}\Big(\frac{|R|^2\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)(-(|R|^2-1)\,|x_1|-(|R|^2-1)\,|x_2|+(|R|^2-3)v_F(t_1-t_2))}{(|x_1|+|x_2|+v_F(t_2-t_1))^3}$$
$$+\frac{|R|^2\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)}{(|x_1|+|x_2|+v_F(t_1-t_2))^2}+\frac{|R|^2\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)}{(|x_1|+|x_2|+v_F(t_2-t_1))^2}$$
$$-\frac{|R|^2\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)((|R|^2-1)\,|x_1|+(|R|^2-1)\,|x_2|+(|R|^2-3)v_F(t_1-t_2))}{(|x_1|+|x_2|+v_F(t_1-t_2))^3}$$
$$+\frac{2v_F(t_1-t_2)}{(t_1v_F-t_2v_F+x_1-x_2)^3}+\frac{2v_F(t_1-t_2)}{(t_1v_F-t_2v_F-x_1+x_2)^3}+\frac{1}{(t_1v_F-t_2v_F+x_1-x_2)^2}+\frac{1}{(t_1v_F-t_2v_F-x_1+x_2)^2}\Big) \tag{2.58}$$

The first order term viz. equation (2.58) has to be compared with that obtained using standard fermionic perturbation theory. Using the perturbative approach, the density density correlation functions in presence of interactions can be written in terms of the non-interacting ones as follows.

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle = \frac{\langle TS\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle_0}{\langle TS\rangle_0} \tag{2.59}$$

Here T represents the time ordering and the action S can be written as follows.

$$S = e^{-i\int H_{fs}(t)dt} = 1 - i\int H_{fs}(t)dt + ... \tag{2.60}$$

Hence the zeroth order term is simply $\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle_0$ and the first order perturbation term can be written as follows.

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = -i\int \langle T\ H_I(t)\rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle_0 dt$$

From equation (2.10) the interacting part of the Hamiltonian can be written as

$$H_{fs}(t) \;=\; \frac{1}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\ v(x-x')\ \rho_h(x,t)\rho_h(x',t)$$

Hence the first order term in the perturbation series can be written as follows.

$$\langle T\rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = -\frac{i}{2}\int d\tau\ \int dy\int dy'\ v(y-y')\langle T\ \rho_s(y,\tau_+)\rho_s(y',\tau)\rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle_0 \tag{2.61}$$

Here $v(y-y') = v_0\delta(y-y')$ is the short ranged mutual interaction term. The symbol $\langle ..\rangle_0$ on the RHS indicates single particle functions. The next step is crucial. We wish to only include the most singular contributions to the exact first order term in Eq.(2.61). This involves simply pairing up the densities (as explained in section 2.4.1). Using equation (2.61), the most singular contribution up to the first order in interaction parameter can be obtained as follows:

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = -iv_0\ \int_C d\tau\ \int_{-\infty}^{\infty}dy\ \langle T\ \rho_h(y,\tau)\rho_h(x_1,t_1)\rangle_0\ \langle T\ \rho_h(y,\tau)\rho_h(x_2,t_2)\rangle_0 \tag{2.62}$$

This has been worked out separately for $|R|=0$, $|R|=1$ and $0<|R|<1$ in Appendix C, Appendix D and Appendix E respectively. In Appendix E, the perturbation series to all orders is exhibited in momentum and frequency space and is explicitly evaluated up to second order in v_0.

2.6 Summary

The most singular contributions of the slow part of the density density correlation functions of a Luttinger liquid with short-range forward scattering mutual interactions in presence of static scalar impurities has been rigorously shown to be expressible in terms of elementary functions of positions and times. This formula has simple second order poles (when the 2-point functions are seen as functions of the complex variable $\tau = t - t'$). Furthermore, it only involves the bare reflection and transmission coefficients of a single fermion in presence of the localised impurities.

In chiral bosonization, the 2-point Green functions in presence of impurities are discontinuous functions of the short-range forward scattering strength. This means in presence of impurities, the full asymptotic Green functions for weak short-range forward scattering are qualitatively (discontinuously) different from the corresponding quantities for no short-range forward scattering. This is the origin of the "cutting the chain" and "healing the chain" metaphors applicable for repulsive and attractive short-range forward scattering respectively.

However NCBT only yields the *most singular* part of the asymptotic Green functions. This attribute exhibits a behaviour complementary to what is seen in the full Green function. The *most singular* part of the 2-point Green functions are discontinuous functions of the *impurity strength* in presence of short-range forward scattering between fermions. This means the NCBT Green functions with weak impurities are qualitatively (ie. discontinuously) different from the corresponding quantity for no impurities.

This is the main reason why the results of chiral bosonization and NCBT cannot be easily compared with one another as they are fully complementary. The only exception is for a fully homogeneous system or its antithesis viz. the half line where the two results coincide.

Chapter 3

Non-equilibrium Green functions of chiral quantum wires coupled through a point-contact

3.1 Introduction

This chapter is based on our published work [81], in which we study non-equilibrium transport in a one-dimensional system using the exact non-equilibrium Green function. Even when inter-particle interactions are absent, investigation of non-equilibrium dynamics is a non-trivial task. Experimental techniques have advanced in recent years to observe ultrafast transient responses in nanoscale electronic devices. Also experimental methods using tunneling spectroscopy have assisted researchers in understanding the non-equilibrium dynamics of low-dimensional systems better [82, 83]. Computational Wigner-function approach has been used in the past to study steady state as well as transient regime in a resonant tunneling diode [84]. But the theoretical tool of choice to study the subtle physics of non-equilibrium systems is the non-equilibrium Green function (NEGF) method (also known as Keldysh formalism). Although there are several techniques such as reduced density matrix methods [85], time-dependent DFT [86, 87], quantum master equation [88], density matrix renormalisation group [89, 90, 91], quantum Langevin equation [92, 93], Bohm trajectories [94, 95], Wigner transport equation [96, 97, 98], none are well suited to naturally include interactions in the system. The Keldysh formalism to handle the non-equilibrium Green function (NEGF) is a popular choice [99, 100]. This method has been extensively used to study tunneling conductance in generic tight binding junctions [101], nonequilibrium currents in quantum dots [102] as well as time dependent transport in double barrier resonant tunneling systems [103].

In this work, we explore the problem of non-equilibrium transport across a point-like contact between two chiral (i.e. unidirectional) non-interacting one-dimensional quantum wires (see Fig.3.1). In [104] the authors consider tunneling between interacting chiral quantum wires albeit treating the point contact perturbatively. Since we treat the point contact exactly, our work complements this earlier work [104] and contributes to a comprehensive understanding of this model. We exactly solve for the non-equilibrium Green functions (NEGF) in real space and time from the equation of motion of the Fermi fields as opposed to Fourier transformed energy domain. We extend our analysis beyond the infinite bandwidth limit and consider the case of a finite bandwidth in the point-contact. We obtain a transient in the tunneling current and this term involves an integral over the past history of the system and hence exhibits non-Markovian dynamics. Non-Markovian effects in quantum transport in nanodevices has attracted much attention in recent years over potential applications in quantum computing and nanotechnology. The non-Markovian master equation approach [105], analyses based on Keldysh non-equilibrium Green function formalism [103, 106, 107, 108] and theories based on the Feynman-Vernon influence functional [109] have proven to be very useful in studying transient transport beyond the infinite bandwidth limit. In this work we treat the infinite bandwidth case to be the zeroth order result of a systematic perturbation approach and consider the effects of a finite bandwidth in the point-contact upto first order.

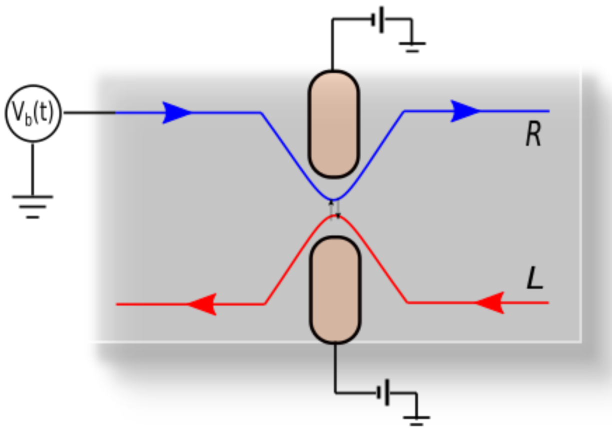

Figure 3.1: Schematic diagram of a tunneling point-contact between two chiral (unidirectional) quantum wires labelled R and L. An arbitrary time dependent potential $V_b(t)$ is applied on the R branch. A point contact is formed by applying an electrostatic gate voltage.

Exact analyses on time-dependent transport properties in out of equilibrium systems have been done by other authors that go beyond the infinite bandwidth limit [106]. However in our approach, we analytically obtain the exact non-equilibrium Green functions in space and time domain (two space-time points) as opposed to these other approaches that only consider constant bias and obtain the Green functions in a Fourier transformed frequency domain. When the point-contact has a finite bandwidth, closed formulas for the full NEGF are possible only as a perturbative series in the inverse of the bandwidth. It is remarkable indeed that a closed formula for the out-of-equilibrium space-time Green function with a general bias of a problem as important as the present one has not been written down till now. This formula provides a convincing derivation of the I-V characteristics that is able to account for a variety of situations such as, time-dependent bias and nonlinearities in the I-V characteristics and so on. Given these observations, it is hard to overstate the importance of the work discussed in this chapter.

3.2 Model and formalism

The tunneling through a point-contact is described by the addition of a tunneling Hamiltonian [110]. The full Hamiltonian for the system under consideration is,

$$H = \sum_p (v_F p + eV_b(t))c^\dagger_{p,R}c_{p,R} + \sum_p (-v_F p)c^\dagger_{p,L}c_{p,L} + \frac{\Gamma}{L}(c^\dagger_{.,R}c_{.,L} + c^\dagger_{.,L}c_{.,R}) \tag{3.1}$$

where R and L label the right and left moving chiral spinless modes and $V_b(t)$ is the generic time dependent bias voltage applied to one of the contacts , the right(R) moving one in this case. $c^\dagger_p$ and c_p are the spinless fermion creation and annihilation operators in momentum space and we use the notation $c^\dagger_{.,R} = \sum_p c^\dagger_{p,R}$ and $\Gamma = \Gamma^*$ is the tunneling amplitude of the symmetric point-contact junction and L that does not appear in the subscript is the system size. *Note that a regularisation scheme is implicit when writing down Eq.3.1. We call this Dirichlet's regularisation. Here summing over all momenta implicitly means summing over an interval* $p \in \{-\Lambda, \Lambda\}$ *and then setting* $\Lambda \to \infty$. *This leads us to conclude, among other things, that the values of discontinuous functions are always the average of left and right hand limits. Specifically in the Dirichlet regularisation, the step function that appears repeatedly in the rest of this chapter is defined as* $\theta(x > 0) = 1, \theta(x < 0) = 0, \theta(x = 0) = \frac{1}{2}$. We postulate that the right mover experiences a potential $V_b(t)$ more than the left mover at all spatial locations. Note that we have considered a generic time-dependent bias and hence our method is not restricted to the case of a dc-bias alone.

3.2.1 Equations of Motion

We may now go ahead and write down the equations of motion for the Fermi fields for the Hamiltonian in Eq.3.1 and systematically solve them to obtain the position-time non-equilibrium Green's function. The equations of motion are

$$\begin{aligned} i\partial_t c_{p,R}(t) &= (v_F p + eV_b(t))\ c_{p,R}(t) + \frac{\Gamma}{L}\ c_{.,L}(t) \\ i\partial_t c_{p,L}(t) &= -v_F p\ c_{p,L}(t) + \frac{\Gamma}{L}\ c_{.,R}(t) \end{aligned} \tag{3.2}$$

On solving the above coupled differential equations we write down the solutions explicitly in real space and time (where $\psi_{R/L}(x) = \frac{1}{\sqrt{L}}\sum_p e^{ipx}\ c_{p,R/L}$).

$$\begin{aligned} \psi_R(x,t) \ &= \ U(t_0,t)\left[1 + 2v_F\Gamma\ sgn(t_0 - t + \frac{x}{v_F})\xi(x,t)\right]\ \psi_R(x - v_F(t-t_0),t_0) \\ &\quad -i(2v_F)^2 U(t - \frac{x}{v_F},t)\ \xi(x,t)\ \psi_L(-x + v_F(t-t_0),t_0) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \psi_L(-x,t) \ &= \ \left[1 + 2v_F\Gamma\ sgn(t_0 - t + \frac{x}{v_F})\ \xi(x,t)\right]\ \psi_L(-x + v_F(t-t_0),t_0) \\ &\quad -i(2v_F)^2\ U(t_0, t - \frac{x}{v_F})\ \xi(x,t)\ \psi_R(x - v_F(t-t_0),t_0) \end{aligned} \tag{3.4}$$

Where we have set,

$$\xi(x,t) \equiv \frac{\Gamma}{v_F}\frac{[\theta(x) - \theta(v_F t_0 + x - v_F t)]}{\Gamma^2 - 4\Gamma^2\theta(t - \frac{x}{v_F} - t_0)\theta(t_0 - t + \frac{x}{v_F}) + 4v_F^2} \tag{3.5}$$

and we have defined $U(\tau,t) \equiv e^{-i\int_\tau^t eV_b(s)ds}$. The time t_0 specifies, at this stage, some (arbitrary) initial time. Here $\theta(x)$ is the Heaviside step function but regularised using the Dirichlet criterion (i.e. $\theta(0) = \frac{1}{2}$). Also $sgn(x) \equiv \theta(x) - \theta(-x)$. It is easy to verify that the above expressions for the fields do satisfy the equations of motion in real space and time which are

$$\begin{aligned} i\partial_t\psi_R(x,t) \ &= \ (-iv_F\partial_x + eV_b(t))\psi_R(x,t) + \Gamma\ \delta(x)\ \psi_L(0,t) \\ i\partial_t\psi_L(x,t) \ &= \ iv_F\partial_x\psi_L(x,t) + \Gamma\ \delta(x)\ \psi_R(0,t) \end{aligned} \tag{3.6}$$

3.3 The non-equilibrium Green functions

We assume that the system is in thermal equilibrium with a reservoir at temperature $T_{temp} = \frac{1}{\beta}$ in the remote past since the voltage bias is zero at these early times. Now setting the time $t_0 = -\infty$ means the same as the system being in the equilibrium state, i.e. before the bias voltage is applied. Using this condition in Eqs.3.3 and 3.4 we may write,

$$\begin{aligned} \psi_R(x,t) \ &= \ U(-\infty,t)\left[1 - \theta(x)\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\psi_{R,0}(x - v_F(t-t_0) < 0, t_0) \\ &- i\frac{\Gamma}{v_F}\theta(x)\ U(t - \frac{x}{v_F},t)\ \frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\ \psi_{L,0}(-x + v_F(t-t_0) > 0, t_0) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \psi_L(-x,t) \ &= \ \left[1 - \theta(x)\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\psi_{L,0}(-x + v_F(t-t_0) > 0, t_0) \\ &- i\frac{\Gamma}{v_F}\theta(x)\ \frac{(2v_F)^2\ U(-\infty, t - \frac{x}{v_F})}{\Gamma^2 + 4v_F^2}\ \psi_{R,0}(x - v_F(t-t_0) < 0, t_0) \end{aligned} \tag{3.8}$$

Before calculating the two-point Green's function it is crucial to note how the point-contact tunneling amplitude is related to the reflection (R) and transmission (T) amplitudes ($|T|^2 + |R|^2 = 1$) when modelled as an isolated impurity as shown in the reference [80],

$$\Gamma = \Gamma^* = \frac{4iv_F RT}{2T^2 + 2T} \tag{3.9}$$

This means,

$$R \;=\; -i\frac{4\Gamma v_F}{\Gamma^2 + 4v_F^2} \;;\; T \;=\; \frac{4v_F^2 - \Gamma^2}{\Gamma^2 + 4v_F^2} \tag{3.10}$$

Note that since t_0 is in the equilibrium region ($t_0 = -\infty$) the quantities $x - v_F(t - t_0) < 0$ and $-x + v_F(t - t_0) > 0$ for any fixed x, t. The correlation functions at time $t_0 = -\infty$ then are the well-known finite temperature equilibrium Green functions [46]. Set $z \equiv x - v_F(t - t_0)$, $z' \equiv x' - v_F(t' - t_0)$,

$$\begin{aligned} &< \psi^\dagger_{R,0}(z' < 0, t_0)\psi_{R,0}(z < 0, t_0) > \;=\; < \psi^\dagger_{L,0}(-z' > 0, t_0)\psi_{L,0}(-z > 0, t_0) > \;= \\ &-\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x - x' - v_F(t - t')))} \end{aligned} \tag{3.11}$$

where β is the inverse temperature. The R, L and L, R correlations with the position coordinates at the opposite sides of the point-contact are zero.

$$\begin{aligned} &< \psi^\dagger_{R,0}(z' < 0, t_0)\psi_{L,0}(-z > 0, t_0) > \\ &= \; < \psi^\dagger_{L,0}(-z' > 0, t_0)\psi_{R,0}(z < 0, t_0) > \;=\; 0 \end{aligned} \tag{3.12}$$

Using Eqs.3.11-3.12 in Eqs.3.7-3.8 and the corresponding conjugates, we get the full non-equilibrium Green functions (NEGF),

$$< \psi^\dagger_{\nu'}(x', t')\psi_\nu(x, t) > \;=\; -\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(\nu x - \nu' x' - v_F(t - t')))}\kappa_{\nu,\nu'} \tag{3.13}$$

where ν,$\nu' = \pm 1$ with $R = 1$ and $L = -1$ and

$$\begin{aligned} \kappa_{1,1} \;=\; &\Bigg(U(t', t)\left[1 - \theta(x')\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\left[1 - \theta(x)\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] \\ &+ \left(\frac{\Gamma}{v_F}\,\frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2\,\theta(x)\theta(x')\,U(t', t' - \frac{x'}{v_F})\,U(t - \frac{x}{v_F}, t)\Bigg) \end{aligned} \tag{3.14}$$

$$\begin{aligned} \kappa_{-1,-1} \;=\; &\Bigg(\left[1 - \theta(-x')\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\left[1 - \theta(-x)\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] \\ &+ \left(\frac{\Gamma}{v_F}\,\frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2\,\theta(-x)\theta(-x')\,U(t' + \frac{x'}{v_F}, t + \frac{x}{v_F})\Bigg) \end{aligned} \tag{3.15}$$

$$\begin{aligned} \kappa_{1,-1} \;=\; &\Bigg(-U(t - \frac{x}{v_F}, t)\,\left[1 - \theta(-x')\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\theta(x) \\ &+ U(t' + \frac{x'}{v_F}, t)\,\left[1 - \theta(x)\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\theta(-x')\Bigg) i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2 + 4v_F^2} \end{aligned} \tag{3.16}$$

$$\begin{aligned} \kappa_{-1,1} \;=\; &\Bigg(-U(t', t + \frac{x}{v_F})\left[1 - \theta(x')\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\theta(-x) \\ &+ U(t', t' - \frac{x'}{v_F})\,\left[1 - \theta(-x)\,\frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\theta(x')\Bigg)\; i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2 + 4v_F^2} \end{aligned} \tag{3.17}$$

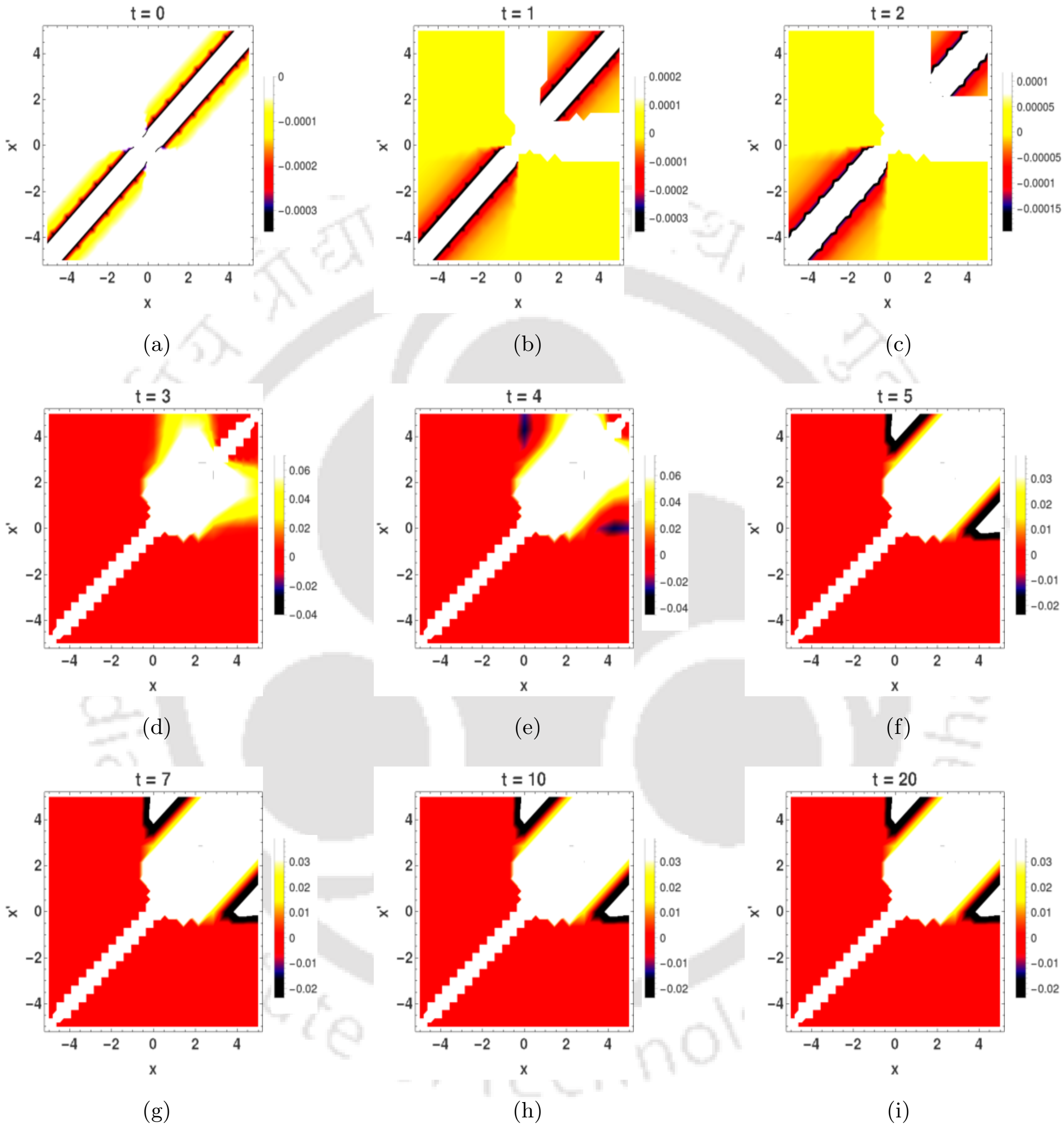

Figure 3.2: Transient in NEGF: This figure shows density plots of the real part of the equal-time RR Green function $Re[< \psi_R^{\dagger}(x^{'}, t)\psi_R(x, t) >]$ vs time (t) in presence of a step bias that is switched on at $t = 0$. The real part of the non-equilibrium Green functions shows transient dynamics (**a**)-(**e**) before reaching a steady state (**f**)-(**i**). The other parameters are chosen to be $\Gamma = 2$ and $v_F = 1$.

For illustrative purposes, let us consider a step bias that is initially zero but is turned on suddenly at $t = 0$ and remains constant for all positive times. We can show by plotting the equal-time NEGF versus the spatial coordinates for different times that at small times the NEGF captures the transient behaviour of the system and reaches a steady state at large positive times. In Fig.3.2 the zero temperature equal-time RR non-equilibrium Green function for various times is plotted, clearly showing the initial transient regime followed by the appearance of a steady state at sufficiently long times after the switching on of a constant bias. We show below how the space-time non-equilibrium Green function we have written in Eq.3.13 can be interpreted as the Keldysh contour ordered NEGF.

The meaning of the average of an observable $< Q >$ has the following interpretation. Let $|\Psi >_{0,i}$ be the eigenstate of a time independent Hamiltonian with eigenvalue $E_{0,i}$. In the remote past, the system was in such a state. Let us consider an adiabatic switch on of a bias. This means that the state now obeys a time-dependent Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)> \; = \; H_0|\Psi(t)> + H_{bias}(t)|\Psi(t)> \tag{3.18}$$

This has a unique solution with the initial condition $|\Psi(t = -\infty) >= |\Psi >_{0,i}$. Let us call this solution: $|\Phi_i(t) >$. The average of the observable $< Q >$ is

$$< Q > \; \equiv \; \frac{\sum_i e^{-\beta E_{0,i}} < \Phi_i(t)|Q|\Phi_i(t) >}{\sum_i e^{-\beta E_{0,i}}} \tag{3.19}$$

For Green functions $Q = c^\dagger(t_1)c(t_2)$ we need to solve (with H_0 and $H_{bias}(t)$ written in second quantised notation):

$$i\hbar\frac{\partial}{\partial \tau}c(\tau) \; = \; [c(\tau), H_0 + H_{bias}(\tau)] \tag{3.20}$$

subject to the condition that $c(\tau = -\infty) = c_{eq}$ where $< c^\dagger_{eq}c_{eq} >= 1- < c_{eq}c^\dagger_{eq} >$ is the Fermi-Dirac distribution at temperature β^{-1}.

3.3.1 Time ordering on the Keldysh contour

It is easy to reinterpret the space-time dependent Green functions as Keldysh Green functions where the times are ordered on the Keldysh contour as shown in Fig.3.3. For this we reinterpret U as follows,

$$U(t,t^{'}) \; = \; e^{-i\int_C d\tau \;\theta_C(t-\tau)\theta_C(\tau-t^{'})\; eV_b(\tau)} \tag{3.21}$$

The standard meaning of the contour ordering $\theta_C(t-t^{'})$ is that $\theta_C(t-t^{'}) = 1$ if t is to the right of $t^{'}$ on the contour C and $\theta_C(t-t^{'}) = 0$ if t is to the left of $t^{'}$ on this contour and $\theta_C(0) = \frac{1}{2}$ consistent with Dirichlet regularisation. The claim is that the Keldysh contour ordered Green function of the system may simply be written down as,

$$< T_C \;\psi_\nu(x,t)\psi^\dagger_{\nu^{'}}(x^{'},t^{'}) > \; = \; \frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(\nu x - \nu^{'}x^{'} - v_F(t-t^{'})))}\kappa_{\nu,\nu^{'}} \tag{3.22}$$

where ν,$\nu^{'} = \pm 1$. We may recall that the quantities κ contains evolution functions such as $U(t+\frac{x}{v_F},t^{'}+\frac{x^{'}}{v_F})$. For example if t is on the lower branch and $t^{'}$ is on the upper branch of the contour and $V_b(\tau) = 0$ when τ is purely imaginary then,

$$U(t+\frac{x}{v_F},t^{'}+\frac{x^{'}}{v_F}) \; = \; e^{-i\int_C d\tau \;\theta_C(t+\frac{x}{v_F}-\tau)\theta_C(\tau-t^{'}-\frac{x^{'}}{v_F})\; eV_b(\tau)} \tag{3.23}$$

But since,

$$\int_C d\tau \;\theta_C(t+\frac{x}{v_F}-\tau)\theta_C(\tau-t^{'}-\frac{x^{'}}{v_F})\; eV_b(\tau) \; = \; \int_{t^{'}+\frac{x^{'}}{v_F}}^{t+\frac{x}{v_F}} d\tau \; eV_b(\tau) \tag{3.24}$$

as before, we see that the reinterpretation has no effect on the results. Similarly, if $t, t^{'}$ are on the upper branch,

$$\int_C d\tau\ \theta_C(t + \frac{x}{v_F} - \tau)\theta_C(\tau - t^{'} - \frac{x^{'}}{v_F})\ eV_b(\tau) \ = \ \theta(t + \frac{x}{v_F} - t^{'} - \frac{x^{'}}{v_F}) \int_{t^{'}+\frac{x^{'}}{v_F}}^{t+\frac{x}{v_F}} d\tau\ eV_b(\tau) \tag{3.25}$$

where θ is now the ordinary Heaviside step function with a real argument.

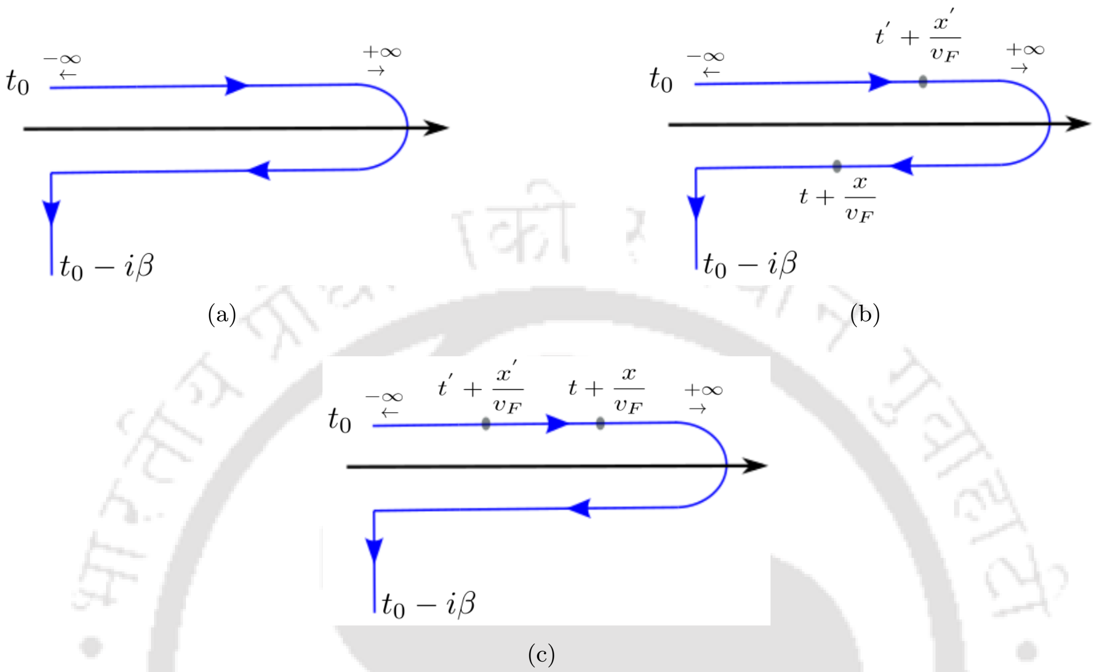

Figure 3.3: (**a**) The extended complex-time Keldysh contour on which Keldysh Green function theory is constructed. Times on the lower branch are greater than the upper branch. The contour is extended along the imaginary axis in a third branch to include the possibility of finite temperature Green functions. (**b**) t is on the lower branch and $t^{'}$ is on the upper branch, so $t > t^{'}$ although on the real time axis it appears that $t^{'} > t$. (**c**) Both t and $t^{'}$ are on the same branch of the contour.

Therefore reinterpreting the evolution function in the manner shown is sufficient to allow a recasting of the Green function in the Keldysh language where the time ordering is on the Keldysh contour.

3.3.2 Consistency check in the equilibrium limit

In a steady-state, the NEGF is a function of the time-difference $t-t^{'}$. The equilibrium Green's function also depends only on the time difference but the non-equilibrium nature of the problem is evident in the fact that the NEGF does not obey the KMS boundary conditions in imaginary time [111, 112] even when the system has reached a steady state. It is an important consistency check to make sure that the NEGF reduces to the equilibrium Green's function when the times t and $t^{'}$ are set to be in the equilibrium region (remote past). This is equivalent to setting $U(t^{'}, t) = 1$ since there is no voltage bias when the system is in equilibrium. Note that when $xx^{'} > 0$,

$$\left(\left[1 - \theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\ \left[1 - \theta(x)\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] + \theta(x)\theta(x^{'})\ \left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2\ \right) = 1 \tag{3.26}$$

whereas from Eq.3.10 it is clear that when $xx^{'} < 0$,

$$\begin{aligned} &\left(\left[1 - \theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\ \left[1 - \theta(x)\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] + \theta(x)\theta(x^{'})\ \left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2\ \right) \\ &= \left[1 - \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] = T = T^* \end{aligned} \tag{3.27}$$

The NEGF in this case, reduces to the well-known [46] equilibrium Green functions given in Eq.3.28.

$$< T\psi^{\dagger}_{\nu'}(x',t')\psi_{\nu}(x,t) >_{equil} \;=\; -\sum_{\gamma,\gamma'=\pm1}\frac{\pi}{\beta v_F}\frac{\theta(\gamma x)\theta(\gamma' x')g_{\gamma,\gamma'}(\nu,\nu')}{\sinh(\frac{\pi}{\beta v_F}(\nu x-\nu' x'-v_F(t-t')))} \tag{3.28}$$

where in terms of the reflection and transmission amplitudes

$$g_{\gamma,\gamma'}(\nu,\nu') \;=\; \frac{i}{2\pi}\Big(\delta_{\nu,\nu'}\delta_{\gamma,\gamma'}+(T\delta_{\nu,\nu'}+R\delta_{\nu,-\nu'})\delta_{\gamma,\nu}\delta_{\gamma',-\nu'}+(T^{*}\delta_{\nu,\nu'}+R^{*}\delta_{\nu,-\nu'})\delta_{\gamma,-\nu}\delta_{\gamma',\nu'}\Big) \tag{3.29}$$

It is easy to show that the non-equilibrium Green functions satisfy the equation of motion for the two-point functions. Now that we have obtained the NEGF we can use it to study the transport properties of the system particularly the tunneling current and differential tunneling conductance with general time-dependent bias voltage of which the well studied problem of a dc-bias [101, 113, 114] is a special case. In the infinite bandwidth case only steady state behaviour in the $I-V$ characteristics is observed. When we consider the case of a finite bandwidth in the point-contact we observe non-Markovian transient transport dynamics in presence of an arbitrary time-dependent bias voltage.

3.4 Tunneling current and conductance in the infinite bandwidth case

In this section, we evaluate the tunneling current and differential tunneling conductance. The tunneling current is defined usually as the rate of change of the difference in the number of right and left movers,

$$I_{tun}(t) = e\;\partial_t\frac{\Delta N}{2} = e\frac{i}{2}\,[H,\Delta N] = e\frac{i}{2}\,[H,N_R-N_L] \tag{3.30}$$

We use the convention as in [115], where they have considered $\mu_L-\mu_R = eV$. In our case the bias is applied to the right movers so that, $\mu_R = eV_b(t)$ and $\mu_L = 0$. Hence in our case we use $\mu_L-\mu_R = -eV_b(t) = eV(t)$. The current becomes,

$$I_{tun}(t) \;=\; -ie\Gamma\lim_{t'\to t}\Big(< \psi^{\dagger}_R(0,t')\psi_L(0,t) > - < \psi^{\dagger}_L(0,t)\psi_R(0,t') >\Big) \tag{3.31}$$

From the expression for the Green functions Eqs.3.13-4.6 and using the Dirichlet criterion $\theta(0) = \frac{1}{2}$ we can write

$$< \psi^{\dagger}_R(0,t')\psi_L(0,t) > \;=\; -\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(-v_F(t-t')))}\Big(-U(t',t)+1\Big)\;i\frac{\Gamma}{v_F}\frac{2v_F^2}{\Gamma^2+4v_F^2}\left[1-\frac{\Gamma^2}{\Gamma^2+4v_F^2}\right] \tag{3.32}$$

then we get,

$$I_{tun}(t) \;=\; ie\Gamma\lim_{t'\to t}\left(\frac{1}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(v_F(t-t')))}\Big(U(t,t')-U(t',t)\Big)\;\frac{\Gamma}{v_F}\frac{2v_F^2}{\Gamma^2+4v_F^2}\left[1-\frac{\Gamma^2}{\Gamma^2+4v_F^2}\right]\right) \tag{3.33}$$

Note that we have defined $U(t,t') \;=\; e^{-i\int_t^{t'}d\tau\;eV_b(\tau)}$. Evaluating the limit using L'Hospital's rule,

$$I_{tun}(t) \;=\; -\Gamma^2\;\frac{4}{\Gamma^2+4v_F^2}\left[1-\frac{\Gamma^2}{\Gamma^2+4v_F^2}\right]\;\frac{e^2}{2\pi}V_b(t) \tag{3.34}$$

Expressing the tunneling amplitude in terms of a tunneling parameter t_p defined as $\Gamma = 2v_Ft_p$ [116], so that the tunneling current upon restoring dimensional units and using $V_b(t) = -V(t)$ is obtained as

$$I_{tun}(t) \;=\; \frac{4t_p^2}{(t_p^2+1)^2}\;\frac{e^2}{h}V(t) \tag{3.35}$$

and the differential tunneling conductance

$$G = \frac{dI_{tun}}{dV(\tau)} \;=\; G_{tun} \;=\; \frac{4t_p^2}{(t_p^2+1)^2}\,\frac{e^2}{h} \tag{3.36}$$

is obtained as a function of the tunneling parameter and agrees with the predictions of standard scattering theory [117, 118]. The tunneling conductance shows $t_p \rightarrow \frac{1}{t_p}$ duality between the strong and weak tunneling regimes as predicted [116, 115]. Although our calculations are done when the system was at a finite temperature in the distant past, the temperature dependence naturally drops out of the expression for tunneling current when there are no interparticle interactions. Also it is worth mentioning that the current is linearly dependent on the time-dependent voltage bias (linear response) and the nonlinear dependence arises only when interactions between the fermions are taken into account.
For a step bias it is clear from Eq.3.35 that for the case of an infinite bandwidth no transients appear in the tunneling current even though the NEGF (Eq.3.13) shows transients. But these transients drop out when one evaluates the equal space-equal time two-point functions at the origin such as $< \psi_R^\dagger(0,t)\psi_L(0,t) >$ and $< \psi_L^\dagger(0,t)\psi_R(0,t) >$ that are present in the expression for the tunneling current (Eq.3.31). This could be explained by thinking of the variable $|x-x^{'}|$ as serving as a length scale or inverse bandwidth (for $< \psi_R^\dagger(x,t)\psi_R(x^{'},t) >$ and $< \psi_L^\dagger(x,t)\psi_L(x^{'},t) >$). This proxy for a inverse bandwidth is present in the full NEGF but is zero when one evaluates the tunneling current (because in this case $x = x^{'}$). So in order to investigate transients in the tunneling current one has to introduce a finite bandwidth in the problem description explicitly which is what we do in Sec.3.9. In that case one has to forego the idea of an exact solution and resort to a systematic perturbative approach.

3.5 Bias-induced anomalies

In this section, we point out two interesting features of the problem we are studying. They are caused by the presence of a bias in the system which leads to physical quantities that normally vanish identically to be non-zero.

Specifically, we contrast the behavior of a) the time-rate of change of the density difference between the local right and left mover densities and b) the time-rate of change of the difference between the total number of right movers and left movers.

The quantity in a) is,

$$\partial_t \Delta\rho(x,t) \;=\; \frac{d}{dt}(\rho_R(x,t) - \rho_L(x,t)) \tag{3.37}$$

where $\rho_R(x,t)$ and $\rho_L(x,t)$ are the (normal-ordered) right and left moving particle densities respectively, whereas the quantity in b) is,

$$\partial_t \Delta N(t) \;=\; \frac{d}{dt}(N_R(t) - N_L(t)) \tag{3.38}$$

where $N_{R/L}(t) = \int dx\ \rho_{R/L}(x,t)$. Naively, $\Delta N(t) \;=\; \int dx\ \Delta\rho(x,t)$

The striking result is this: in the limit when the bias becomes time independent, the answer for a) is zero but the answer for b) is non-zero. This is what we mean by bias-induced anomaly. Normally we expect that both should be zero since b) is obtained by spatially integrating a). The reason for this anomaly is that integrating over an infinite domain of x-values effectively multiplies the quantity by this infinity. Thus even if this quantity was tending to zero, when multiplied by a quantity tending to infinity, leads to a value that could be and in this case, is – finite.

To see this mathematically we write,

$$\partial_t \Delta\rho(x,t) = \lim_{x^{'} \to x} v_F \frac{d}{dt}\Big(< \psi_R^{\dagger}(x^{'},t)\psi_R(x,t) > - < \psi_L^{\dagger}(x^{'},t)\psi_L(x,t) > \Big)$$

$$= -\frac{v_F}{\pi}\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2 \frac{1}{2v_F}\, eV_b^{'}(t-\frac{|x|}{v_F}) \tag{3.39}$$

whereas,

$$\partial_t \Delta N(t) = -\frac{v_F}{\pi}\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2 \; e\,[V_b(t) - V_b(-\infty)] \tag{3.40}$$

Let us choose $V_b(t) = e^{\alpha t}V_b(0)$ with $\alpha \to 0^+$. This means the bias is slowly switched on from a zero value in the remote past and remains turned on at least until time t. In the limit $\alpha \to 0^+$, the bias is always on but the endpoints are mathematically well defined. In this case,

$$\partial_t \Delta\rho(x,t) = -Lim_{\alpha\to 0^+}\frac{v_F}{\pi}\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2 \frac{1}{2v_F}\, eV_b^{'}(t-\frac{|x|}{v_F}) \approx 0 \tag{3.41}$$

whereas,

$$\partial_t \Delta N(t) = -Lim_{\alpha\to 0^+}\frac{v_F}{\pi}\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2 \; e\,[V_b(t) - V_b(-\infty)]$$

$$= -Lim_{\alpha\to 0^+}\frac{v_F}{\pi}\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2 \; e\,V_b(0) \neq 0 \tag{3.42}$$

This is the first of the “bias-induced anomalies”.

The second example is the one-particle Green function itself - specifically the one which involves turning a right mover to a left mover i.e. $< \psi_L^{\dagger}(x,t)\psi_R(x^{'},t^{'}) >$. In the absence of a bias, this quantity vanishes identically when x and $x^{'}$ are on opposite sides of the origin. However in the present case we obtain something interesting,

$$< \psi_L^{\dagger}(x^{'}<0,t^{'})\psi_R(x>0,t) >= -\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}\, q_1}{\sinh(\frac{\pi}{\beta v_F}(x+x^{'}-v_F(t-t^{'})))}\left(-U(t-\frac{x}{v_F},t)+U(t^{'}+\frac{x^{'}}{v_F},t)\right) \tag{3.43}$$

where $q_1 = \left[1 - \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\; i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}$. This quantity is identically zero when there is no bias since the point contact causes reflection and turns a right mover to a left mover on the same side of the origin but not on opposite sides. But the presence of a bias leads to a non-equilibrium situation where what is opposite sides of the origin at one time is effectively the same side of the origin at other times as seen by the presence of the evolution factor $(-U(t-\frac{x}{v_F},t)+U(t^{'}+\frac{x^{'}}{v_F},t))$.

3.6 Dynamical density of states

In this section, we present the formula for the dynamical density of states $D(\omega;x,T)$ at position x. $D(\omega;x,T)\ d\omega$ is the number of fermionic states per unit length with energy between $\hbar\omega$ and $\hbar(\omega+d\omega)$ relative to the Fermi energy. The DDOS at $x^{'}=x$ is given by the formula

$$D(\omega;x,T) = \int d\tau\; e^{-i\omega\tau} < \{\psi(x,T+\frac{\tau}{2}),\psi^{\dagger}(x,T-\frac{\tau}{2})\} > \tag{3.44}$$

A closed expression for $D(\omega;x,T)$ can be obtained by using the formulas for the non-equilibrium Green functions obtained earlier. We evaluate the DDOS for the right movers in Appendix F and it can be evaluated in a similar manner for the left movers as well. The DDOS for the right movers is obtained as

$$D(\omega;x,T) \;=\; D_{equil}(\omega;x,T) = \frac{1}{2v_F} \tag{3.45}$$

In other words, it is the same as what one would expect when there is no bias, no mutual interactions between particles and the system is in equilibrium. The reason for this simple result is that in absence of mutual interactions between fermions, the anticommutator in eq.(3.44) is a Dirac delta-function at $\tau = 0$ (since the space dependence and time dependence appear additively for chiral fermions).

3.7 Time-dependent tunneling parameter

In this section we consider the case of a time-dependent tunneling parameter t_p, which means Γ is now time-dependent $\Gamma(t)$. We show that when the time-dependence of Γ is present only as a phase factor $\Gamma_{TD}(t) = \Gamma e^{i\theta_{TD}(t)}$ the results are same as in the previous sections. But when the magnitude of Γ itself is time-dependent i.e. $\Gamma_{TD}(t) = \Gamma(t)e^{i\theta_{TD}(t)}$ then these two approaches are not equivalent and we get different results. In general we can write a modified Hamiltonian for the system with a complex $\Gamma_{TD}(t)$,

$$H = \sum_p v_F p\; b^{\dagger}_{p,R} b_{p,R} + \sum_p (-v_F p) b^{\dagger}_{p,L} b_{p,L} + \frac{\Gamma_{TD}(t)}{L} b^{\dagger}_{.,R} b_{.,L} + \frac{\Gamma^{*}_{TD}(t)}{L} b^{\dagger}_{.,L} b_{.,R} \tag{3.46}$$

Here we have used the same notations as in section 3.2 and the Fermi fields are modified with the addition of a time-dependent phase $b_{p,R}(t) \;=\; c_{p,R}(t)\; e^{i\theta_{TD}(t)}$ and $b_{p,L}(t) \;=\; c_{p,L}(t)$. The equations of motion for the modified fields then become,

$$\begin{aligned} i\partial_t b_{p,R}(t) \;&=\; v_F p\; b_{p,R}(t) + \frac{\Gamma_{TD}(t)}{L} b_{.,L}(t) \\ i\partial_t b_{p,L}(t) \;&=\; -\,v_F p\; b_{p,L}(t) + \frac{\Gamma^{*}_{TD}(t)}{L} b_{.,R}(t) \end{aligned} \tag{3.47}$$

3.7.1 $|\Gamma_{TD}(t)|$ is independent of time

When the magnitude of the tunneling amplitude $|\Gamma_{TD}(t)|$ is independent of time we can write

$$\Gamma_{TD}(t) = \Gamma\; e^{i\theta_{TD}(t)} \tag{3.48}$$

On comparing Eqs.3.47 and 3.2 so that,

$$\begin{aligned} i\partial_t c_{p,R}(t) \;&=\; (v_F p + \partial_t\theta_{TD}(t))\; c_{p,R}(t) + \frac{|\Gamma|}{L} c_{p,L}(t) \\ i\partial_t c_{p,L}(t) \;&=\; -\,v_F p\; c_{p,L}(t) + \frac{|\Gamma|}{L} c_{p,R}(t) \end{aligned} \tag{3.49}$$

we can see that the case of $|\Gamma_{TD}(t)|$ independent of time is equivalent to the case of a time-independent real Γ considered in the previous sections, provided one identifies $\partial_t\theta_{TD}(t) = eV_b(t)$. However when the time-dependence of the tunneling amplitude is such that the magnitude $|\Gamma_{TD}(t)|$ is dependent on time the two approaches are not equivalent.

3.7.2 $|\Gamma_{TD}(t)|$ is time-dependent

When the magnitude of the tunneling amplitude is itself time-dependent we may write,

$$\Gamma_{TD}(t) = \Gamma(t)\; e^{i\theta_{TD}(t)} \tag{3.50}$$

Following a similar procedure as in Section 3.2 we write down the finite temperature non-equilibrium Green functions for the case of a time-dependent tunneling parameter,

$$< \psi^{\dagger}_{\nu'}(x',t')\psi_{\nu}(x,t) > \;=\; -\,\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(\nu x - \nu' x' - v_F(t-t')))}\zeta_{\nu,\nu'} \tag{3.51}$$

where $\nu,\nu^{'} = \pm 1$ with $R = 1$ and $L = -1$ and

$$\zeta_{1,1} = \left(1 + \frac{i}{2v_F}\ \Gamma^{*}_{TD}(t - \frac{x}{v_F})\ \Xi_R(x,t)\right)\left(1 - \frac{i}{2v_F}\ \Gamma_{TD}(t^{'} - \frac{x^{'}}{v_F})\ \Xi^{*}_R(x^{'},t^{'})\right) + \Xi_R(x,t)\ \Xi^{*}_R(x^{'},t^{'}) \tag{3.52}$$

$$\zeta_{1,-1} = \Xi^{*}_L(x^{'},t^{'})\ \left(1 + \frac{i}{2v_F}\ \Gamma^{*}_{TD}(t - \frac{x}{v_F})\ \Xi_R(x,t)\right) - \Xi_R(x,t)\ \left(1 + \frac{i}{2v_F}\ \Gamma^{*}_{TD}(t^{'} + \frac{x^{'}}{v_F})\ \Xi^{*}_L(x^{'},t^{'})\right) \tag{3.53}$$

$$\zeta_{-1,1} = \Xi_L(x,t)\left(1 - \frac{i}{2v_F}\ \Gamma_{TD}(t^{'} - \frac{x^{'}}{v_F})\ \Xi^{*}_R(x^{'},t^{'})\right) - \Xi^{*}_R(x^{'},t^{'})\ \left(1 - \frac{i}{2v_F}\ \Gamma_{TD}(t + \frac{x}{v_F})\ \Xi_L(x,t)\right) \tag{3.54}$$

$$\zeta_{-1,-1} = \left(1 - \frac{i}{2v_F}\ \Gamma_{TD}(t + \frac{x}{v_F})\ \Xi_L(x,t)\right)\ \left(1 + \frac{i}{2v_F}\ \Gamma^{*}_{TD}(t^{'} + \frac{x^{'}}{v_F})\ \Xi^{*}_L(x^{'},t^{'})\right) + \Xi_L(x,t)\ \Xi^{*}_L(x^{'},t^{'}) \tag{3.55}$$

where

$$\Xi_R(x,t)\ =\ \frac{i}{v_F}\frac{\theta(x)\ \Gamma_{TD}(t-\frac{x}{v_F})}{1+\frac{|\Gamma_{TD}(t-\frac{x}{v_F})|^2}{(2v_F)^2}};\ \ \Xi_L(x,t)\ =\ -\frac{i}{v_F}\frac{\theta(-x)\ \Gamma^{*}_{TD}(t+\frac{x}{v_F})}{1+\frac{|\Gamma_{TD}(t+\frac{x}{v_F})|^2}{(2v_F)^2}} \tag{3.56}$$

The tunneling current can be computed using the same definition as in Eq.3.30 but with the Hamiltonian given in Eq.3.46. This gives,

$$I_{tun}(t)\ =\ -\,i\ e\lim_{t^{'}\to t}\left(\Gamma_{TD}(t)\ \ < \psi^{\dagger}_R(0,t^{'})\psi_L(0,t) > -\Gamma^{*}_{TD}(t)\ \ < \psi^{\dagger}_L(0,t)\psi_R(0,t^{'}) > \right) \tag{3.57}$$

After some effort we conclude,

$$I_{tun}(t)\ =\ \frac{4iev_F^2\left(\Gamma^{*}_{TD}(t)\Gamma^{'}_{TD}(t) - \Gamma_{TD}(t)\Gamma^{*'}_{TD}(t)\right)}{\pi\left(|\Gamma_{TD}(t)|^2 + 4v_F^2\right)^2} \tag{3.58}$$

As the magnitude of $\Gamma_{TD}(t)$ is time-dependent in this case we set $\Gamma_{TD}(t)\ =\ \Gamma(t)\ e^{i\theta_{TD}(t)}$ and using the relation $\theta^{\prime}_{TD}(t) = eV_b(t)$ we get

$$I_{tun}(t)\ =\ -\frac{8e^2v_F^2\Gamma(t)^2}{\pi\left(\Gamma(t)^2 + 4v_F^2\right)^2}\ \ V_b(t) \tag{3.59}$$

The time-dependent tunneling amplitude in terms of the tunneling parameter $t_p(t)$ can be written as $\Gamma(t) = 2v_F t_p(t)$. Also using the convention as in [115] we pick up a minus sign $V_b(t) = -V(t)$. Finally the following expression for the time-dependent tunneling current is obtained after restoring dimensional units

$$I_{tun}(t)\ =\ \frac{e^2}{h}\ \frac{4\ t_p(t)^2}{(1 + t_p(t)^2)^2}\ V(t) \tag{3.60}$$

The above expression is similar in form to Eq. 3.35 except that t_p is now time-dependent and this tells us that so long as $t_p(t)$ is switched on and held at a constant value, the current is proportional to voltage and there are no transients in the current. However if $t_p(t)$ itself depends in some complicated way on the bias, this may not be the case. In such situations, this dependence has to be specified separately.

3.8 Double barrier resonant tunneling

The type of impurity present at the point contact is given by the form of $\Gamma_{TD}(t)$. Here we show that the results from the previous section can be used to easily study resonant tunneling by treating the point contact impurity as a double barrier structure such as a lead-device-lead junction at the origin. It can be pictured as a symmetric double delta potential with an inter-barrier separation a. We neglect

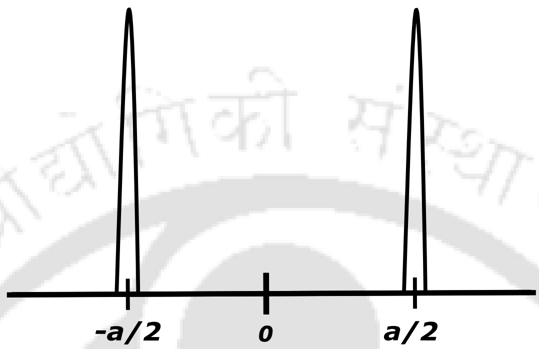

Figure 3.4: Schematic of a double delta potential with inter-barrier separation a

Coulomb interactions and dynamics in the device and only focus on point-contact coupling between the left-lead and the device and the right-lead and the device. In this case the tunneling amplitude takes the form

$$\Gamma_{TD}(t) = \Gamma(t)e^{i\Theta_{TD}} = (W_L e^{2i\xi_0} + W_R e^{-2i\xi_0})e^{i\Theta_{TD}(t)} \tag{3.61}$$

where $\Theta'_{TD}(t) = eV_b(t) = -eV(t)$ and the coupling between the double barrier at the origin and the left and right chiral quantum wires is W_L and W_R respectively which can be time-dependent in general but we assume them to be independent of time for simplicity as it doesn't change the form of the I-V characteristics. Eq.3.61 is valid in the limit $k_F \to \infty$ and inter-barrier separation $a \to 0$ such that $0 < k_F a = \xi_0 < \infty$ is fixed. For the case of a symmetric double barrier $W_L = W_R = W$ and substituting Eq.3.61 in Eq.3.58 we obtain

$$I_{tun}(t) = \frac{8e^2 v_F^2 W^2 (e^{2i\xi_0} + e^{-2i\xi_0})^2}{\pi(4v_F^2 + W^2(e^{2i\xi_0} + e^{-2i\xi_0})^2)^2} V(t) \tag{3.62}$$

The tunneling conductance is given by

$$G = \frac{8e^2 v_F^2 W^2 (e^{2i\xi_0} + e^{-2i\xi_0})^2}{\pi(4v_F^2 + W^2(e^{2i\xi_0} + e^{-2i\xi_0})^2)^2} \tag{3.63}$$

This is of the form $G = \frac{8e^2 v_F^2 |\Gamma|^2}{\pi(4v_F^2 + |\Gamma|^2)^2}$, which for $|\Gamma|^2 = 4v_F^2$ peaks at $G = \frac{e^2}{2\pi} = \frac{e^2}{h}$ (since $\hbar = 1$). This is used to determine the condition for resonant tunneling. One can tune the inter-barrier separation in the device region by external means in order to achieve resonance. When the resonance condition is satisfied the conductance attains its peak value $G = G_0 = \frac{e^2}{h}$ (see Fig.3.5). The condition for resonance in the symmetric double barrier case is determined to be

$$W^2(2\cos(4\xi_0) + 2) - 4v_F^2 = 0 \tag{3.64}$$

For the more general case of an asymmetrical double barrier the expression for current is obtained as

$$I_{tun}(t) = \frac{8e^2 v_F^2 \left(e^{2i\xi_0} W_L + e^{-2i\xi_0} W_R\right)\left(e^{-2i\xi_0} W_L + e^{2i\xi_0} W_R\right)}{\pi(4v_F^2 + \left(e^{2i\xi_0} W_L + e^{-2i\xi_0} W_R\right)\left(e^{-2i\xi_0} W_L + e^{2i\xi_0} W_R\right))^2} V(t) \tag{3.65}$$

The tunneling conductance attains its maximum value $G_0 = \frac{e^2}{h}$ when the resonance condition for this case is satisfied, which is

$$W_L^2 + 2\cos(4\xi_0) W_L W_R + W_R^2 - 4v_F^2 = 0 \tag{3.66}$$

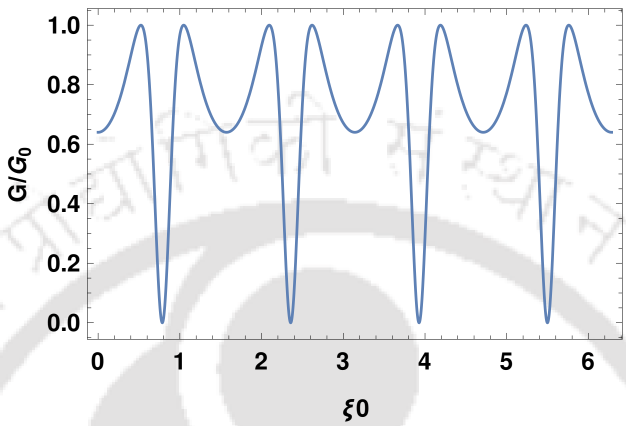

Figure 3.5: This figure shows G/G_0 vs ξ_0 for a symmetric double barrier. The conductance peaks when the resonance condition (Eq.3.64) is satisfied. The figure is plotted choosing $W = 2$ and $v_F = 1$.

3.9 Transient quantum transport in the case of finite bandwidth

So far what we have considered is the simplest possible scenario of non-interacting fermions with a linear dispersion and no momentum cutoffs (infinite bandwidth). In such a circumstance the I-V characteristic (Eq.3.35) for point-contact tunneling is linear and only shows steady state behaviour when a constant voltage is turned on suddenly. We expect deviations from these properties when we take into account the effects of a momentum cutoff (finite bandwidth) Λ in the point contact such that $|p| < \Lambda$. This implicitly means that there is a short distance cutoff ($\frac{1}{\Lambda}$) in the problem, which makes the quantum point contact (QPC) region to occupy a non-zero size. Here we calculate the leading correction to the tunneling current due to a finite bandwidth and show the appearance of a transient in the current in presence of a (subsequently constant) bias that is suddenly switched on. The correction to the current involves integration over the past history of the system and thus encodes memory effects ie. it displays non-Markovian behaviour.

In presence of a finite bandwidth, the Hamiltonian is written as

$$H = \sum_p (v_F p + eV_b(t)) c_{p,R}^\dagger c_{p,R} + \sum_p (-v_F p) c_{p,L}^\dagger c_{p,L} + \frac{\Gamma}{L}(c_{.,R}^\dagger c_{.,L} + c_{.,L}^\dagger c_{.,R}) \tag{3.67}$$

where now we have defined $c_{.,\nu} = \sum_{|p|<\Lambda} c_{p,\nu}$ with the momentum restricted to a finite bandwidth Λ. We set $c_{.,\nu} = c_{.,\nu}^\infty + \delta c_{.,\nu}$ where $c_{.,\nu}^\infty = \sum_p c_{p,\nu}$ and $\delta c_{.,\nu}$ is the deviation from the infinite bandwidth case. The other quantities have their usual meaning as in the previous sections. The equations of motion for the right and left moving Fermi fields in this case are

$$\begin{aligned} i\partial_t c_{p,R}(t) &= (v_F p + eV_b(t)) c_{p,R}(t) + \theta(\Lambda - |p|)\frac{\Gamma}{L} c_{.,L}(t) \\ i\partial_t c_{p,L}(t) &= -v_F p c_{p,L}(t) + \theta(\Lambda - |p|)\frac{\Gamma}{L} c_{.,R}(t) \end{aligned} \tag{3.68}$$

with $\theta(\Lambda - |p|)$ taken to be the Dirichlet regularized step function. Exact solution to the problem is possible only in the simplest case of infinite bandwidth. If we wish to investigate the case of a finite bandwidth analytically, we shall have to settle for perturbative corrections in powers of $\frac{1}{\Lambda}$. In this section, we are interested in the finite bandwidth correction to the current $\delta I_{tun}(t)$ and we shall evaluate it upto $O(\frac{1}{\Lambda})$ i.e. we assume a large but finite bandwidth. In this case we may write,

$$\begin{aligned}\delta I_{tun}(t) = &-\frac{ie\Gamma}{L}Lim_{t'\to t}(< \delta c^{\dagger}_{.,R}(t')c^{\infty}_{.,L}(t) > - < \delta c^{\dagger}_{.,L}(t)c^{\infty}_{.,R}(t') >)\\ &-\frac{ie\Gamma}{L}Lim_{t'\to t}(< c^{\dagger\infty}_{.,R}(t')\delta c_{.,L}(t) > - < c^{\dagger\infty}_{.,L}(t)\delta c_{.,R}(t') >)\end{aligned} \quad (3.69)$$

Upon solving the equations in Eq.3.68 we may write,

$$c_{.,R}(t) = \left(\sum_{|p|<\Lambda} c_{p,R}(t_0)e^{-i(t-t_0)pv_F}\right) U(t_0,t) - i\Gamma\int_{t_0}^{t} c_{.,L}(t_2)\delta_\Lambda(v_F(t-t_2))\,U(t_2,t)\,dt_2 \quad (3.70)$$

and

$$c_{.,L}(t) = \left(\sum_{|p|<\Lambda} c_{p,L}(t_0)e^{i(t-t_0)pv_F}\right) - i\Gamma\int_{t_0}^{t} c_{.,R}(t_2)\delta_\Lambda(v_F(t-t_2))\,dt_2 \quad (3.71)$$

where we have defined the broadened Dirac delta as $\delta_\Lambda(x) = \frac{1}{L}\sum_{|p|<\Lambda} e^{ipx}$. We find the need to introduce a quantity $\Delta(x)$ that corresponds to the difference between this broadened and actual Dirac delta function. In terms of the usual Dirac delta function $\delta(X) = \frac{1}{L}\sum_p e^{ipX}$ this can be written as

$$\delta_\Lambda(X) = \delta(X) - \Delta(X) \quad (3.72)$$

Computing the finite bandwidth correction to leading order in $1/\Lambda$ is itself not a straightforward task and involves a tedious calculation. In order to calculate $\delta I_{tun}(t)$ we have to evaluate correlations of the type $< \delta c^{\dagger}_{.,\nu'}(t')c^{\infty}_{.,\nu}(t) >$ and $< c^{\dagger\infty}_{.,\nu'}(t')\delta c_{.,\nu}(t) >$. The procedure to calculate these correlations is as follows. We already know from the derivation for the infinite bandwidth case the following expressions,

$$c^{\infty}_{.,R}(t) = \left(\sum_{p} c^{\infty}_{p,R}(t_0)e^{-i(t-t_0)pv_F}\right) U(t_0,t) - i\Gamma\int_{t_0}^{t} c^{\infty}_{.,L}(t_2)\delta(v_F(t-t_2))\,U(t_2,t)\,dt_2 \quad (3.73)$$

and

$$c^{\infty}_{.,L}(t) = \left(\sum_{p} c^{\infty}_{p,L}(t_0)e^{i(t-t_0)pv_F}\right) - i\Gamma\int_{t_0}^{t} c^{\infty}_{.,R}(t_2)\delta(v_F(t-t_2))\,dt_2 \quad (3.74)$$

Making use of the relation $c_{.,\nu}(t) = c^{\infty}_{.,\nu}(t) + \delta c_{.,\nu}(t)$ and Eqs.3.73, 3.74, 3.70 and 3.71 we obtain the following coupled equations

$$\begin{aligned}\delta c_{.,R}(t) \;=\; &(-\sum_{|p|>\Lambda} c^{\infty}_{p,R}(t_0) + \sum_{p}\delta c_{p,R}(t_0))e^{-i(t-t_0)pv_F}\;U(t_0,t)\\ &- i\Gamma\int_{t_0}^{t}(\delta(v_F(t-t_2))\delta c_{.,L}(t) - c^{\infty}_{.,L}(t_2)\Delta(v_F(t-t_2)))\;U(t_2,t)\,dt_2\end{aligned} \quad (3.75)$$

and

$$\begin{aligned}\delta c_{.,L}(t) \;=\; &(-\sum_{|p|>\Lambda} c^{\infty}_{p,L}(t_0) + \sum_{p}\delta c_{p,L}(t_0))e^{i(t-t_0)pv_F}\\ &- i\Gamma\int_{t_0}^{t}(\delta(v_F(t-t_2))\delta c_{.,R}(t_2) - \Delta(v_F(t-t_2))c^{\infty}_{.,R}(t_2))\,dt_2\end{aligned} \quad (3.76)$$

We solve the above two equations and write separate expressions for $\delta c_{.,R}(t)$ and $\delta c_{.,L}(t)$.

$$\begin{aligned}\delta c_{.,R}(t) &= \frac{2v_F}{\Gamma^2+4v_F^2}(2v_F((-\sum_{|p|>\Lambda} c^{\infty}_{p,R}(t_0)+\sum_p \delta c_{p,R}(t_0))e^{-i(t-t_0)pv_F}\ U(t_0,t)\\ &+ i\Gamma\int_{t_0}^{t} c^{\infty}_{.,L}(t_2)\Delta(v_F(t-t_2))\ U(t_2,t)\,dt_2) - i\Gamma((-\sum_{|p|>\Lambda} c^{\infty}_{p,L}(t_0)+\sum_p \delta c_{p,L}(t_0))e^{i(t-t_0)pv_F}\\ &+ i\Gamma\int_{t_0}^{t}\Delta(v_F(t-t_2))c^{\infty}_{.,R}(t_2)\,dt_2))\end{aligned} \tag{3.77}$$

and

$$\begin{aligned}\delta c_{.,L}(t) &= \frac{2v_F}{\Gamma^2+4v_F^2}(2v_F((-\sum_{|p|>\Lambda} c^{\infty}_{p,L}(t_0)+\sum_p \delta c_{p,L}(t_0))e^{i(t-t_0)pv_F}\\ &+ i\Gamma\int_{t_0}^{t}\Delta(v_F(t-t_2))c^{\infty}_{.,R}(t_2)\,dt_2) - i\Gamma((-\sum_{|p|>\Lambda} c^{\infty}_{p,R}(t_0)+\sum_p \delta c_{p,R}(t_0))e^{-i(t-t_0)pv_F}\ U(t_0,t)\\ &+ i\Gamma\int_{t_0}^{t} c^{\infty}_{.,L}(t_2)\Delta(v_F(t-t_2))\ U(t_2,t)\,dt_2))\end{aligned} \tag{3.78}$$

Also Eqs.3.73 and 3.74 reduce to

$$c^{\infty}_{.,R}(t) = \frac{2v_F(2v_F(\sum_p c^{\infty}_{p,R}(t_0)e^{-i(t-t_0)pv_F}U(t_0,t)) - i\Gamma(\sum_p c^{\infty}_{p,L}(t_0)e^{i(t-t_0)pv_F}))}{\Gamma^2+4v_F^2} \tag{3.79}$$

and

$$c^{\infty}_{.,L}(t) = \frac{2\left(2v_F^2(\sum_p c^{\infty}_{p,L}(t_0)e^{i(t-t_0)pv_F}) - i\Gamma v_F(\sum_p c^{\infty}_{p,R}(t_0)e^{-i(t-t_0)pv_F}U(t_0,t))\right)}{\Gamma^2+4v_F^2} \tag{3.80}$$

Using Eqs. 3.77, 3.78, 3.79 and 3.80 and the corresponding complex conjugates we can write down expressions for the correlations of the type $<\delta c^{\dagger}_{.,\nu'}(t')c^{\infty}_{.,\nu}(t)>$ and $<c^{\dagger\infty}_{.,\nu'}(t')\delta c_{.,\nu}(t)>$. After some simplification these correlation functions are obtained as some non-trivial combinations of the bias $V_b(t)$ and the equal-time equilibrium infinite bandwidth Green functions which we already know but for $<T\ \delta\psi_R(x,t_0)\psi^{\dagger,\infty}_{\nu'}(x',t_0)>_{eq}$ and $<T\ \delta\psi_L(x,t_0)\psi^{\dagger,\infty}_{\nu'}(x',t_0)>_{eq}$ (where $\nu'=R,L$) which we are required to explicitly calculate. Note that the Green functions at equal-time t_0 implies equilibrium Green functions as we take $t_0\to-\infty$ i.e. long before the bias is switched on. In equilibrium we have

$$\begin{aligned} i\partial_t\delta c_{p,R}(t) &= v_F p\ \delta c_{p,R}(t) + \frac{\Gamma}{L}\delta c_{.,L}(t) - \theta(|p|-\Lambda)\frac{\Gamma}{L}c^{\infty}_{.,L}(t)\\ i\partial_t\delta c_{p,L}(t) &= -v_F p\ \delta c_{p,L}(t) + \frac{\Gamma}{L}\delta c_{.,R}(t) - \theta(|p|-\Lambda)\frac{\Gamma}{L}c^{\infty}_{.,R}(t)\end{aligned} \tag{3.81}$$

We transform from time to discrete Matsubara frequency ($z_n=\frac{(2n+1)\pi}{\beta}$) and write down the correlations,

$$<T\ \delta c_{p,R}(n)c^{\dagger,\infty}_{p',\nu'}(n)> = -\theta(|p|-\Lambda)\frac{\Gamma}{L}\frac{1}{(iz_n-v_Fp)}<T\ c^{\infty}_{.,L}(n)c^{\dagger,\infty}_{p',\nu'}(n)> + \frac{\Gamma}{L}\frac{1}{(iz_n-v_Fp)}<T\ \delta c_{.,L}(n)c^{\dagger,\infty}_{p',\nu'}(n)> \tag{3.82}$$

and

$$<T\ \delta c_{p,L}(n)c^{\dagger,\infty}_{p',\nu'}(n)> = -\theta(|p|-\Lambda)\frac{\Gamma}{L}\frac{1}{(iz_n+v_Fp)}<T\ c^{\infty}_{.,R}(n)c^{\dagger,\infty}_{p',\nu'}(n)> + \frac{\Gamma}{L}\frac{1}{(iz_n+v_Fp)}<T\ \delta c_{.,R}(n)c^{\dagger,\infty}_{p',\nu'}(n)> \tag{3.83}$$

After some algebra we Fourier transform to real space which allows for further simplification. We then transform the discrete frequencies back to time taking the time interval $t-t'$ to be small. Finally we obtain the following correlations for small $t-t'$ and large finite bandwidth

$$< T\ \delta\psi_R(x,t)\psi^{\dagger;\infty}_{\nu'}(x',t') >_{eq} \ \approx\ 4i\Gamma v_F\Gamma\frac{(2v_F)^2}{(\Gamma^2+4v_F^2)^2}\frac{i}{-i\beta}\delta_{\nu',-1}\ \frac{i\Gamma\theta(xx')\coth\left(\frac{\pi(x+x')}{\beta v_F}\right)\mathrm{csch}\left(\frac{\pi(x+x')}{\beta v_F}\right)}{2\beta\Lambda v_F^4}$$
$$+(\Gamma^2-(2v_F)^2)\ \Gamma\frac{(2v_F)^2}{(\Gamma^2+4v_F^2)^2}\frac{i}{-i\beta}\delta_{\nu',1}\ \frac{1}{L^2}\ \frac{i\Gamma L^2(\mathrm{sgn}(x)\theta(-xx'))\coth\left(\frac{\pi(x'-x)}{\beta v_F}\right)\mathrm{csch}\left(\frac{\pi(x'-x)}{\beta v_F}\right)}{2\beta\Lambda v_F^4} \tag{3.84}$$

and

$$<T\ \delta\psi_L(x,t)\psi^{\dagger;\infty}_{\nu'}(x',t') >_{eq} \ \approx$$
$$-\frac{\Gamma}{L^2}\ \frac{iL}{v_F}\ \frac{(2v_F)^2}{(\Gamma^2+4v_F^2)^2}\frac{i\Gamma}{\pi\Lambda v_F^2}\frac{i}{-i\beta}4iv_F\Gamma\delta_{\nu',1}\ \theta(xx')\ \frac{iL}{v_F}\left(\frac{\pi\coth\left(\frac{\pi(x+x')}{\beta v_F}\right)\mathrm{csch}\left(\frac{\pi(x+x')}{\beta v_F}\right)}{2\beta}\right)$$
$$-\frac{\Gamma}{L^2}\ \frac{iL}{v_F}\ \frac{(2v_F)^2}{(\Gamma^2+4v_F^2)^2}\frac{i\Gamma}{\pi\Lambda v_F^2}\frac{i}{-i\beta}(\Gamma^2-(2v_F)^2)\delta_{\nu',-1}\ \ (\mathrm{sgn}(x')\theta(-xx'))$$
$$\frac{iL}{v_F}\left(\frac{\pi\coth\left(\frac{\pi(x-x')}{\beta v_F}\right)\mathrm{csch}\left(\frac{\pi(x-x')}{\beta v_F}\right)}{2\beta}\right) \tag{3.85}$$

Now we have all the ingredients to calculate $< \delta c^{\dagger}_{.,\nu'}(t')c^{\infty}_{.,\nu}(t) >$ and $< c^{\dagger\infty}_{.,\nu'}(t')\delta c_{.,\nu}(t) >$. Substituting in Eq.3.69 and simplifying we obtain the following expression

$$\delta I_{tun}(t) =$$
$$-\frac{ie\Gamma}{L}\Bigg(-\frac{4Lv_F^2\left(4v_F^2-\Gamma^2\right)\left(U(t,t_0)-U(t_0,t)\right)}{(\Gamma^2+4v_F^2)^2}\int dx\frac{\Delta(x-v_F(t-t_0))}{2\pi i}\frac{\frac{\pi}{\beta v_F}\ \theta(-x)(R-R^*)}{\sinh(\frac{\pi}{\beta v_F}(-x+v_F(t-t_0)))}$$
$$+\frac{32i\Gamma^3Lv_F^4}{(\Gamma^2+4v_F^2)^3}\int_{t_0}^{t}\Delta(v_F(t-t_2))\ \frac{1}{2\pi i}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta}(t-t_2))}\ (U(t,t_2)U(t,t_2)-U(t_2,t)\ U(t_2,t))\,dt_2\Bigg) \tag{3.86}$$

Here R is the reflection amplitude (for the infinite band-width point contact), β is inverse temperature, x is position and t_0 is an arbitrary initial time when the system is in equilibrium. Since we are going to set $t_0 \to -\infty$ as we did in the case of infinite bandwidth, the first term in Eq.3.86 drops out and we are left with only the second term which involves an integral over the past history of the system.

$$\delta I_{tun}(t) = -\frac{ie\Gamma}{L}\frac{32i\Gamma^3Lv_F^4}{(\Gamma^2+4v_F^2)^3}\ \int_{-\infty}^{t}\frac{\Delta(v_F(t-t_2))}{2\pi i}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta}(t-t_2))}(U(t,t_2)U(t,t_2)-U(t_2,t)\ U(t_2,t))\,dt_2 \tag{3.87}$$

Note that the function Δ may be written as $\Delta(v_F(t-t_2)) = \delta(v_F(t-t_2)) - \frac{\sin(\Lambda v_F(t-t_2))}{\pi v_F(t-t_2)}$. The expression for $\delta I_{tun}(t)$ in Eq.3.87 exhibits non-Markovian behaviour as the current at a given time is depends not only on the voltage applied at that time but also on the voltage at all previous times. When the bias is switched on suddenly and remains constant thereafter, the tunneling current shows a transient (gradual) build up before settling into its steady state constant value. Even when a constant voltage is present eternally, there is a non-linear dependence of the current on this bias voltage which is seen in terms such as $U(t,t_2)$, where $U(t,t_2) \equiv e^{-i\int_t^{t_2} eV_b(s)ds}$. Note that according to our convention $V_b(t) = -V(t)$ and hence $U(t,t_2) \equiv e^{i\int_t^{t_2} eV(s)ds}$ (where $V(t)$ is the convention of [115]). In Fig.3.6 we plot $\delta I_{tun}(t)$ vs t for the case of a step bias (sudden switching on) $V(t) = V_0\ \theta(t)$ showing the

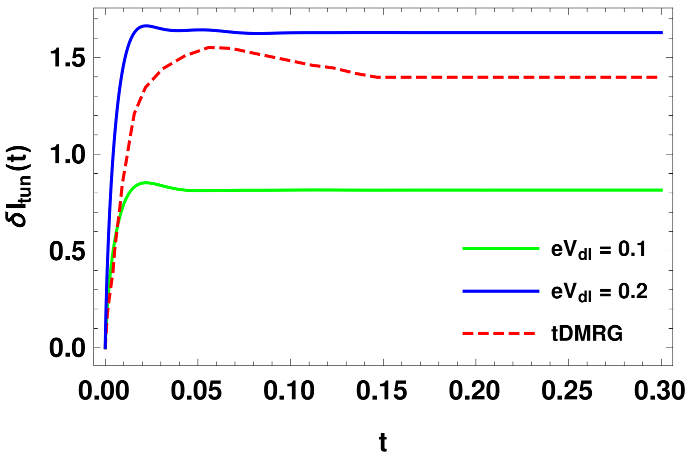

Figure 3.6: Current transients: This figure shows $\delta I_{tun}(t)$ vs t for different values of dimensionless bias eV_{dl} defined as $eV_{dl} = \frac{eV_0}{v_F\Lambda}$ where we have considered a step bias $V(t) = \theta(t)V_0$ and for dimensionless temperature $T_{dl} = \frac{T}{v_F\Lambda} = 0.1$. The other parameters are $\Gamma\Lambda = 100$ and $v_F\Lambda = 100$ in appropriate units. The red-dashes indicate the $tDMRG$ result for the time-evolution of current in an Anderson dot model (see Fig.1 in [1]), which we have rescaled and overlayed on our plot to show the qualitative similarity between the current transients even in a completely different model to ours.

transients in the current before reaching a steady state. The transients are qualitatively similar to that observed in the experimental investigation of split-gate quantum point-contacts [119] and also in numerical simulations of non-equilibrium transport through an Anderson dot using methods like time-dependent density matrix renormalization group ($tDMRG$) [1] and the iterative summation of path integrals ($ISPI$) [120] approach.
Our analysis shows that even in non-equilibrium transport through a simple quantum point-contact, transient dynamics appear in the tunneling current when a short distance cutoff is introduced in the problem. This is the reason why numerical methods like $tDMRG$, that work on a lattice, predict a transient in the current before a steady state is reached. Fig.3.7 shows the steady state current δI_{tun} as a function of constant time-independent bias ($eV_{dl} = \frac{eV}{v_F\Lambda}$) for different temperatures $T_{dl} = \frac{T}{v_F\Lambda}$. The three energy scales in the problem are temperature (T), bias potential (V) and the energy scale associated with the bandwidth ($v_F\Lambda$). The energy scale due to the bandwidth is the dominant one as we assume the bandwidth to be large but finite in our calculation.

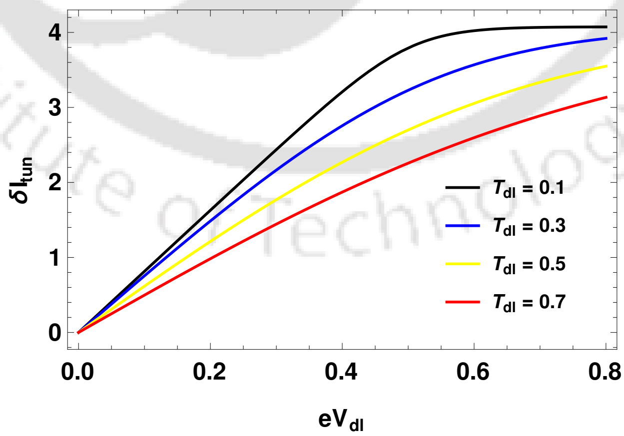

Figure 3.7: Nonlinear I-V characteristics: The temperature dependence of δI_{tun} in steady state is shown when plotted vs $eV_{dl} = \frac{eV}{v_F\Lambda}$. The dimensionless temperature is defined as $T_{dl} = \frac{T}{v_F\Lambda}$. The other parameters are $\Gamma\Lambda = 100$ and $v_F\Lambda = 100$ in appropriate units.

3.10 Nonequilibrium Green functions of a 1D Fermi gas with a mobile impurity

Let us consider a Fermi gas in one dimension with a mobile impurity injected into the system, described by the Hamiltonian

$$H = \sum_p \frac{p^2}{2m} c_p^\dagger c_p + V_0 \ \psi^\dagger(X(t))\psi(X(t)) \tag{3.88}$$

where V_0 is the impurity potential at the position $X(t)$. We consider here a classical impurity injected into the quantum gas of free fermions. Here we consider the case of the impurity moving with a constant drift velocity proportional to some external applied force i.e. a nonequilibrium steady state. The fermion operator $\psi(x)$ can be expressed as a summation over the momentum states as follows

$$\psi(X) = \frac{1}{\sqrt{L}} \sum_p e^{ipX} \ c_p \tag{3.89}$$

where L is the system size. The equation of motion for the momentum states is

$$i\partial_t c_p(t) = \frac{p^2}{2m} c_p(t) + V_0 \frac{1}{\sqrt{L}} e^{-ipX} \ \psi(X,t) \tag{3.90}$$

Linearizing the energy momentum dispersion near the Fermi surface and confining to the RPA limit we can analytically solve the equation of motion for the Fermi field

$$\begin{aligned}
&e^{ik_F(x-X(t))} \ \tilde{\psi}_R(x,t) + e^{-ik_F(x-X(t))} \ \tilde{\psi}_L(x,t) \\
&= -iV_0 \int_{t_1}^{t} d\tau \ e^{ik_F(x-X(\tau))} \ \delta\left((x-X(\tau)) - v_F(t-\tau)\right) \ [\tilde{\psi}_R(X(\tau),\tau) + \tilde{\psi}_L(X(\tau),\tau)] \\
&\quad - iV_0 \int_{t_1}^{t} d\tau \ \ e^{-ik_F(x-X(\tau))} \ \delta\left((x-X(\tau)) + v_F(t-\tau)\right) \ [\tilde{\psi}_R(X(\tau),\tau) + \tilde{\psi}_L(X(\tau),\tau)] \\
&\quad + e^{ik_F(x-X(t_1))} \ \tilde{\psi}_R(x - v_F(t-t_1), t_1) + e^{-ik_F(x-X(t_1))} \ \tilde{\psi}_L(x + v_F(t-t_1), t_1)
\end{aligned} \tag{3.91}$$

where we have decomposed the field in terms of the slowly varying parts in the following manner

$$\psi(x,t) = e^{ik_F(x-X(t))} e^{-\frac{i}{2}mv_F^2 t} \ \tilde{\psi}_R(x,t) + e^{-ik_F(x-X(t))} e^{-\frac{i}{2}mv_F^2 t} \ \tilde{\psi}_L(x,t) \tag{3.92}$$

The solution is obtained as

$$\begin{aligned}
\tilde{\psi}_R(x,t) = -iV_0 \ \int_{t_1}^{t} d\tau \ e^{ik_F(X(t)-X(\tau))} \ \delta\left((x-X(\tau)) - v_F(t-\tau)\right) \ (\tilde{\psi}_R(X(\tau),\tau) + \tilde{\psi}_L(X(\tau),\tau)) \\
+ e^{ik_F(X(t)-X(t_1))} \ \tilde{\psi}_R(x - v_F(t-t_1), t_1)
\end{aligned} \tag{3.93}$$

and

$$\begin{aligned}
\tilde{\psi}_L(x,t) = -iV_0 \ \int_{t_1}^{t} d\tau \ \ e^{-ik_F(X(t)-X(\tau))} \ \delta\left((x-X(\tau)) + v_F(t-\tau)\right) \ (\tilde{\psi}_R(X(\tau),\tau) + \tilde{\psi}_L(X(\tau),\tau)) \\
+ e^{-ik_F(X(t)-X(t_1))} \ \tilde{\psi}_L(x + v_F(t-t_1), t_1)
\end{aligned} \tag{3.94}$$

In the remote past the system is assumed to be in equilibrium with a temperature β^{-1} with a stationary impurity such that $X(\tau < t_1) = X$ where t_1 is an arbitrary initial time. But for $\tau \gg t_1$ the impurity moves with a drift velocity $X(\tau) = v_D\tau$ where $v_D > 0$ is proportional to some applied force on the impurity.

3.10.1 Consistency in the equilibrium limit

We can check that this solution is consistent with the equilibrium limit. At equilibrium the impurity is localised at a position X. Since the system is in equilibrium in the remote past we may set $t_1 = -\infty$ and in this limit we have $x - v_F(t-t_1) < 0$, $x + v_F(t-t_1) > 0$, $2X - x - v_F(t-t_1) < 0$ and $2X - x + v_F(t-t_1) > 0$ and equal-time correlations at t_1 such as $< T\ \tilde{\psi}_{\nu}(z,t_1)\tilde{\psi}^{\dagger}_{\nu'}(z',t_1) >_0$ are the known equilibrium Green functions. Taking the equilibrium limit of the above solutions for the fields we obtain the known expressions for the Green functions with a localised impurity at X,

$$
\begin{aligned}
< T\ \tilde{\psi}_R(x,t)\tilde{\psi}^{\dagger}_R(x^{'},t^{'}) >_0 \ &= \ \frac{\theta(X-x)\theta(x^{'}-X)\left(csch\left(\frac{\pi((t-t^{'})v_F-x+x^{'})}{\beta v_F}\right)\right)}{2\beta(V_0+iv_F)} \\
&- \frac{\theta(x-X)\theta(X-x^{'})\left(csch\left(\frac{\pi((t-t^{'})v_F-x+x^{'})}{\beta v_F}\right)\right)}{2\beta(V_0-iv_F)} \\
&- \frac{\theta((x-X)(x^{'}-X))\left(icsch\left(\frac{\pi((t-t^{'})v_F-x+x^{'})}{\beta v_F}\right)\right)}{2\beta v_F}
\end{aligned} \tag{3.95}
$$

and

$$
\begin{aligned}
< T\ \tilde{\psi}_L(x,t)\tilde{\psi}^{\dagger}_L(x^{'},t^{'}) >_0 \ &= \ -\frac{\theta(X-x)\theta(x^{'}-X)\left(csch\left(\frac{\pi((t-t^{'})v_F+x-x^{'})}{\beta v_F}\right)\right)}{2\beta(V_0-iv_F)} \\
&+ \frac{\theta(x-X)\theta(X-x^{'})\left(csch\left(\frac{\pi((t-t^{'})v_F+x-x^{'})}{\beta v_F}\right)\right)}{2\beta(V_0+iv_F)} \\
&- \frac{\theta((x-X)(x^{'}-X))\left(icsch\left(\frac{\pi((t-t^{'})v_F+x-x^{'})}{\beta v_F}\right)\right)}{2\beta v_F}
\end{aligned} \tag{3.96}
$$

and

$$
\begin{aligned}
< T\ \tilde{\psi}_R(x,t)\tilde{\psi}^{\dagger}_L(x^{'},t^{'}) >_0 \ &= \ \frac{\theta(x-X)\theta(x^{'}-X)\left(V_0\ csch\left(\frac{\pi(-(t-t^{'})v_F+x-2X+x^{'})}{\beta v_F}\right)\right)}{2\beta v_F(v_F+iV_0)} \\
&- \frac{\theta(X-x)\theta(X-x^{'})\left(V_0\ csch\left(\frac{\pi(-(t-t^{'})v_F+x-2X+x^{'})}{\beta v_F}\right)\right)}{2\beta v_F(v_F-iV_0)}
\end{aligned} \tag{3.97}
$$

and

$$
\begin{aligned}
< T\ \tilde{\psi}_L(x,t)\tilde{\psi}^{\dagger}_R(x^{'},t^{'}) >_0 \ &= \ \frac{\theta(x-X)\theta(x^{'}-X)\left(V_0\ csch\left(\frac{\pi((t-t^{'})v_F+x-2X+x^{'})}{\beta v_F}\right)\right)}{2\beta v_F(v_F-iV_0)} \\
&- \frac{\theta(X-x)\theta(X-x^{'})\left(V_0\ csch\left(\frac{\pi((t-t^{'})v_F+x-2X+x^{'})}{\beta v_F}\right)\right)}{2\beta v_F(v_F+iV_0)}
\end{aligned} \tag{3.98}
$$

Here the step functions are the Dirichlet regularized step functions defined as $\theta(x) = \frac{1}{2}(1+sgn(x))$.

3.10.2 Nonequilibrium Green functions with a mobile impurity

Given that our solution is consistent with the equilibrium answer we can compute the full nonequilibrium Green function with a mobile impurity. Let us consider $X(t) = v_D t$ where $0 < v_D \ll v_F$ is the drift velocity. The nonequilibrium Green functions (NEGF) are obtained from the solutions to the equation of motion,

$$< \tilde{\psi}_R^{\dagger}(x',t')\tilde{\psi}_R(x,t) > \quad \approx \tag{3.99}$$

$$\frac{\theta(x)\theta(x')\left(i\text{csch}\left(\frac{\pi(tv_F-t'v_F-x+x')}{\beta v_F}\right)\left(V_0^2 e^{ik_F v_D\left(-t+t'+\frac{2(x-x')}{v_F}\right)}+v_F^2 e^{ik_F v_D(t-t')}\right)\right)}{2\beta v_F\left(V_0^2+v_F^2\right)} \tag{3.100}$$

$$+\frac{\theta(x)\theta(-x')\left(e^{ik_F v_D(t-t')}\text{csch}\left(\frac{\pi(tv_F-t'v_F-x+x')}{\beta v_F}\right)\right)}{2\beta V_0-2i\beta v_F} \tag{3.101}$$

$$-\frac{\theta(-x)\theta(x')\left(e^{ik_F v_D(t-t')}\text{csch}\left(\frac{\pi(tv_F-t'v_F-x+x')}{\beta v_F}\right)\right)}{2\beta V_0+2i\beta v_F} \tag{3.102}$$

$$+\frac{\theta(-x)\theta(-x')\left(ie^{ik_F v_D(t-t')}\text{csch}\left(\frac{\pi(tv_F-t'v_F-x+x')}{\beta v_F}\right)\right)}{2\beta v_F} \tag{3.103}$$

and

$$< \tilde{\psi}_R^{\dagger}(x',t')\tilde{\psi}_L(x,t) > \quad \approx \tag{3.104}$$

$$\frac{V_0\theta(-x)\theta(x')\left(e^{ik_F v_D\left(t-t'+\frac{2x}{v_F}\right)}-e^{ik_F v_D\left(-t+t'-\frac{2x'}{v_F}\right)}\right)\text{csch}\left(\frac{\pi(tv_F-t'v_F+x+x')}{\beta v_F}\right)}{2\beta\left(V_0^2+v_F^2\right)} \tag{3.105}$$

$$+\frac{V_0\theta(-x)\theta(-x')e^{ik_F v_D\left(t-t'+\frac{2x}{v_F}\right)}\text{csch}\left(\frac{\pi(tv_F-t'v_F+x+x')}{\beta v_F}\right)}{2\beta v_F^2+2i\beta V_0 v_F} \tag{3.106}$$

$$-\frac{V_0\theta(x)\theta(x')e^{ik_F v_D\left(-t+t'-\frac{2x'}{v_F}\right)}\text{csch}\left(\frac{\pi(tv_F-t'v_F+x+x')}{\beta v_F}\right)}{2\beta v_F^2-2i\beta V_0 v_F} \tag{3.107}$$

and

$$< \tilde{\psi}_L^{\dagger}(x',t')\tilde{\psi}_L(x,t) > \quad \approx \tag{3.108}$$

$$\frac{i\theta(-x)\theta(-x')\text{csch}\left(\frac{\pi(tv_F-t'v_F+x-x')}{\beta v_F}\right)\left(2iV_0^2 e^{\frac{ik_F v_D(x-x')}{v_F}}\sin\left(\frac{k_F v_D(tv_F-t'v_F+x-x')}{v_F}\right)+V_0^2+v_F^2\right)}{2\beta v_F\left(V_0^2+v_F^2\right)} \tag{3.109}$$

$$+\frac{\theta(x)\theta(-x')\left(ie^{-ik_F t v_D}\left((V_0+iv_F)e^{ik_F t v_D}-V_0 e^{ik_F t' v_D}\right)\text{csch}\left(\frac{\pi(tv_F-t'v_F+x-x')}{\beta v_F}\right)\right)}{2\beta v_F(V_0+iv_F)} \tag{3.110}$$

$$+\frac{\theta(-x)\theta(x')\left(e^{-ik_F t v_D}\left(V_0 e^{ik_F t' v_D}-(V_0-iv_F)e^{ik_F t v_D}\right)\text{csch}\left(\frac{\pi(tv_F-t'v_F+x-x')}{\beta v_F}\right)\right)}{2\beta v_F(v_F+iV_0)} \tag{3.111}$$

$$+\frac{\theta(x)\theta(x')\left(i\text{csch}\left(\frac{\pi(tv_F-t'v_F+x-x')}{\beta v_F}\right)\right)}{2\beta v_F} \tag{3.112}$$

and

$$< \tilde{\psi}_L^{\dagger}(x^{'},t^{'})\tilde{\psi}_R(x,t) > \;\; \approx \tag{3.113}$$

$$\frac{\theta(x)\theta(-x^{'})\left(V_0\left(e^{\frac{ik_F v_D(-tv_F+t^{'}v_F+2x)}{v_F}}-e^{\frac{ik_F v_D(tv_F-t^{'}v_F-2x^{'})}{v_F}}\right)\text{csch}\left(\frac{\pi(tv_F-t^{'}v_F-x-x^{'})}{\beta v_F}\right)\right)}{2\beta\left(V_0^2+v_F^2\right)} \tag{3.114}$$

$$-\frac{\theta(-x)\theta(-x^{'})\left(V_0 e^{\frac{ik_F v_D(tv_F-t^{'}v_F-2x^{'})}{v_F}}\text{csch}\left(\frac{\pi(tv_F-t^{'}v_F-x-x^{'})}{\beta v_F}\right)\right)}{2\beta v_F^2-2i\beta V_0 v_F} \tag{3.115}$$

$$+\frac{\theta(x)\theta(x^{'})\left(V_0 e^{-\frac{ik_F v_D(tv_F-t^{'}v_F-2x)}{v_F}}\text{csch}\left(\frac{\pi(tv_F-t^{'}v_F-x-x^{'})}{\beta v_F}\right)\right)}{2\beta v_F^2+2i\beta V_0 v_F} \tag{3.116}$$

3.10.3 Mobility of the impurity

The current operator at a point x and time t is given by the following expression.

$$j(x,t) = v_F(< \tilde{\psi}_R^{\dagger}(x,t)\tilde{\psi}_R(x,t) > - < \tilde{\psi}_L^{\dagger}(x,t)\tilde{\psi}_L(x,t) >) \tag{3.117}$$

The total momentum of the fermions in the system, assuming the mass of each fermion to be m, is given by

$$P(t) = m\int_{-\infty}^{\infty} dx\; j(x,t) \tag{3.118}$$

We may reinterpret $j(x,t)$ as

$$j(x,t) = v_F\; Lim_{x^{'}\to x}\; Re(< \tilde{\psi}_R^{\dagger}(x^{'},t)\tilde{\psi}_R(x,t) > - < \tilde{\psi}_L^{\dagger}(x^{'},t)\tilde{\psi}_L(x,t) >) \tag{3.119}$$

This is evaluated using the NEGF

$$j(x,t) = \frac{k_F V_0^2 v_D}{\pi(V_0^2+v_F^2)}\;\theta(x-v_F t_1+v_F t)\theta(tv_F-t_1 v_F-x) \tag{3.120}$$

The total momentum of the fermions is

$$P(t) = m\int_{-\infty}^{\infty} dx\; j(x,t) = \frac{2k_F m V_0^2 v_D v_F(t-t_1)\theta((t-t_1)v_F)}{\pi\left(V_0^2+v_F^2\right)} \tag{3.121}$$

Since we set the initial time to be $t_1 \to -\infty$, we always have $t > t_1$ and the force acting on the Fermi gas due to the mobile impurity (which is equal to the force on the mobile impurity by the fermions in the system) is given by

$$F = \frac{dP(t)}{dt} = \frac{2k_F m V_0^2 v_D v_F}{\pi\left(V_0^2+v_F^2\right)} \tag{3.122}$$

Here v_D is the drift velocity of the impurity and V_0 is the strength of the coupling between the heavy particle and the fermions. Mobility μ is defined as the ratio between the terminal velocity of the impurity and the force acting on it so that,

$$\mu = \frac{\pi\left(V_0^2+v_F^2\right)}{2V_0^2 k_F^2} \tag{3.123}$$

The mobility diverges when the coupling between the impurity and fermions tends to zero, indicating that the impurity accelerates in reaction to an external force instead of attaining a fixed velocity. Conversely, when the coupling diverges - which means no tunneling of fermions through the impurity is permitted, the mobility saturates to its minimum value of $\mu_0 = \frac{\pi}{2k_F^2}$ [121].

3.11 Summary

In this work, we have investigated the non-equilibrium transport between two non-interacting chiral quantum wires by computing exactly the non-equilibrium Green function (NEGF) expressed in terms of simple functions of positions and times unlike previous analytical approaches that deal only with the steady state response to a constant bias. We have calculated the time-dependent tunneling current and differential conductance across an infinite bandwidth point contact for an arbitrary time-dependent bias. In this case no transients in the tunneling current are seen when a step voltage bias is considered $V_b(t) = \theta(t)\ V_0$. In this case, the steady state in the current is reached instantaneously which may be attributed to the extreme ideal situation of absence of interactions between fermions and infinite bandwidth in the point contact. The NEGF method allows us to study transient phenomena as well as steady state properties. The full space-time NEGF that we have obtained exhibits transient behavior upon sudden switch on of a bias even when the bandwidth of the point-contact is infinite. We also examine the situation where in addition to the bias voltage, the tunneling amplitude also becomes time-dependent. We calculate the NEGF and the time-dependent transport properties in such a scenario. We also demonstrate how resonant tunneling through a simple double barrier structure can be easily studied using our method.

We go beyond the infinite bandwidth limit and consider the situation when a finite bandwidth (Λ) in the point-contact is introduced in the problem. Although exact treatment of time-dependent non-equilibrium transport in a similar system has been studied before using Keldysh NEGF formalism, our method is different as we work in the position and time domain and our method applies for an arbitrary time-dependent bias unlike previous works [106] that are restricted to special cases like a sharp step bias or a square pulse voltage bias. When a finite bandwidth is introduced in the point-contact, a short distance cutoff ($\frac{1}{\Lambda}$) becomes implicit. A systematic perturbative treatment in this parameter allows the calculation of the correction to the tunneling current upto $O(\frac{1}{\Lambda})$. We have shown that the transport properties are non-Markovian in this case. The tunneling current now shows transient behaviour before reaching a steady state (for a bias that is suddenly switched on) which is merely a consequence of the presence of a short-distance cutoff in the problem description and not on other details.

In addition to the non-equilibrium two-point function, it is also possible to write down (using Wick's theorem) the four-point functions. Using these correlations in conjunction with powerful novel bosonization techniques [46] it is possible to extend our NEGF approach to study non-equilibrium transport between chiral fermionic edges with mutually interacting particles like in the case of Fractional Quantum Hall edge states [122]. This approach could shed more light on the universality of power-law exponents, or more generally, the scaling functions [123], in systems with chiral Luttinger liquid character.

Chapter 4

Unconventional bosonization of nonequilibrium chiral fermions

4.1 Introduction

The quantum physics of low-dimensional systems has been a source of rich theoretical results in condensed matter physics for the past few decades. The advent of modern experimental techniques has made it possible to investigate such systems in practice in the context of nanostructures, carbon nanotubes, quantum Hall edges and confined ultracold atoms. In one-dimension, interacting fermionic systems are described by the Luttinger liquid paradigm [33, 32, 15, 3]. Since the early works of Tomonaga [124] and Luttinger [125] it has been evident that even at low energies the interparticle interactions cannot be treated perturbatively in Luttinger liquids in contrast to the case of Fermi liquids in higher dimensions. The technique of bosonization was developed to deal with interactions in one-dimensional systems and it has been successful to a large extent. Bosonization involves replacing fermionic excitations with bosonic degrees of freedom. This technique is widely used and is well understood in the context of one-dimensional systems in equilibrium. Recently there is a growing interest in nonequilibrium phenomena in nanostructures and quantum wires [82, 126, 83]. Bosonization is a very versatile technique and a major fraction of the theoretical research on one-dimensional systems relies on this technique which could, in principle be extended to out-of-equilibrium situations as well. It has been well demonstrated in the literature that a consistent bosonization scheme could prove to be a powerful tool in analysing nonequilibrium transport problems in one-dimension. Using the framework of the Keldysh action formalism Gutman et.al [127, 128] have developed a bosonization technique for one-dimensional fermions in nonequilibrium and subsequently used this method to study an interacting quantum wire attached to two electrodes with arbitrary energy distributions but their methods work with the crucial assumption of no electron backscattering due to impurities. Conventional or standard bosonization is ill-suited to study systems where there is backscattering from impurities. In this work, we restrict to leads with noninteracting chiral fermions in contact with reservoirs at different chemical potentials and with a point contact tunneling junction at the origin [129, 130]. This is similar to the situation considered in [131], where it was shown that the conventional bosonization-debosonization procedure fails to give results that are consistent with the exact solution to this noninteracting problem. In this paper we set up a bosonization formalism for this problem in the spirit of the *Non-chiral bosonization technique (NCBT)* [132] which involves a radical new way of looking at the Fermi-Bose correspondence as a mnemonic to obtain the correct correlation functions rather than an operator identity and has been proven to be more successful than conventional bosonization in studying Luttinger liquids with impurities [45, 132, 133, 134, 49, 135, 136, 52]. The present chapter which is based on our preprint [137], extends this idea to systems that are driven out of equilibrium by the application of a bias between the left and right movers. We show that an appropriate version of NCBT reproduces the exact single-particle Green functions of this system under bias [137].

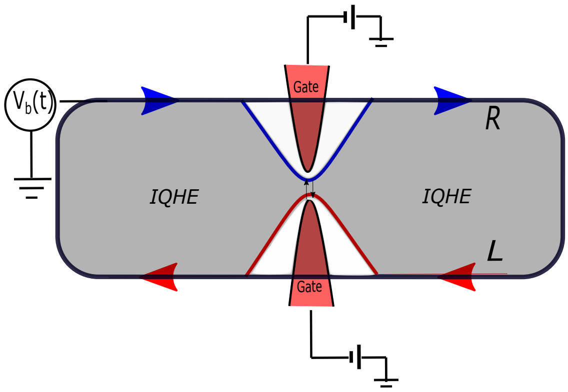

Figure 4.1: Schematic diagram of a tunneling point-contact between two chiral (unidirectional) quantum wires labelled R and L. This can be realized as opposite edge states of an integer quantum Hall effect (IQHE) fluid brought into close proximity with each other using an electrostatic gate voltage. An arbitrary bias potential $V_b(t)$ is applied on the R branch.

4.2 Model Description

We consider the same model as we did in our previous work [81] wherein we have computed the exact space-time nonequilibrium Green function (NEGF) for the problem. The model Hamiltonian is

$$H = \sum_p (v_F p + eV_b(t)) c^\dagger_{p,R} c_{p,R} + \sum_p (-v_F p) c^\dagger_{p,L} c_{p,L} + \frac{\Gamma}{L}(c^\dagger_{.,R} c_{.,L} + c^\dagger_{.,L} c_{.,R}) \tag{4.1}$$

where R and L label the right and left moving chiral spinless modes and $V_b(t)$ is the generic time dependent bias voltage applied to one of the contacts , the right(R) moving one in this case. $c^\dagger_p$ and c_p are the spinless fermion creation and annihilation operators in momentum space and we use the notation $c^\dagger_{.,R} = \sum_p c^\dagger_{p,R}$ and $\Gamma = \Gamma^*$ is the tunneling amplitude of the symmetric point-contact junction and L that does not appear in the subscript is the system size. We have considered a generic time-dependent bias potential defined as $\mu_L - \mu_R = -\mu_R = -eV_b(t) = eV(t)$ following the same convention as in [131]. Note that we assume the bias to be applied only to the right movers, hence $\mu_L = 0$ and $\mu_R = eV_b(t)$. A schematic of the model is shown in Fig.4.1. In [81] we obtain the exact dynamical nonequilibrium Green function (NEGF) by solving Dyson's equation analytically and this allows us to write down the two-point and four-point functions for this system in a closed form in terms of simple functions of position and time. The full NEGF as obtained in [81] is,

$$< T\ \psi^\dagger_{\nu'}(x',t')\psi_\nu(x,t) > \;=\; -\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(\nu x - \nu' x' - v_F(t-t')))} \kappa_{\nu,\nu'} \tag{4.2}$$

$$\kappa_{1,1} = \Bigg(U(t',t) \left[1 - \theta(x') \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] \left[1 - \theta(x) \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] + \left(\frac{\Gamma}{v_F} \frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2 \theta(x)\theta(x')\ U(t', t' - \frac{x'}{v_F})\ U(t - \frac{x}{v_F}, t)\Bigg) \tag{4.3}$$

$$\kappa_{-1,-1} = \Bigg(\left[1 - \theta(-x') \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] \left[1 - \theta(-x) \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] + \left(\frac{\Gamma}{v_F} \frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2 \theta(-x)\theta(-x')\ U(t' + \frac{x'}{v_F}, t + \frac{x}{v_F})\Bigg) \tag{4.4}$$

$$\kappa_{1,-1} = \Bigg(- U(t - \frac{x}{v_F}, t) \left[1 - \theta(-x') \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] \theta(x) + U(t' + \frac{x'}{v_F}, t) \left[1 - \theta(x) \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right] \theta(-x') \Bigg) i \frac{\Gamma}{v_F} \frac{(2v_F)^2}{\Gamma^2 + 4v_F^2} \tag{4.5}$$

$$\kappa_{-1,1} = \Big(- U(t^{'}, t + \frac{x}{v_F}) \left[1 - \theta(x^{'}) \ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2} \right] \theta(-x)$$
$$+ U(t^{'}, t^{'} - \frac{x^{'}}{v_F}) \ \left[1 - \theta(-x) \ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2} \right] \theta(x^{'}) \Big) \ i \frac{\Gamma}{v_F} \frac{(2v_F)^2}{\Gamma^2 + 4v_F^2} \quad (4.6)$$

where $\nu, \nu^{'} = \pm 1$ with $R = 1$ and $L = -1$. Here $U(\tau, t) \equiv e^{-i \int_{\tau}^{t} eV_b(s) ds}$ and $\theta(x)$ is the Dirichlet regularized step function (this means $\theta(x > 0) = 1, \theta(x < 0) = 0, \theta(0) = \frac{1}{2}$) and the time ordering is along the Keldysh contour [138]. In the equilibrium limit (limit of zero bias) i.e. $U(\tau, t) \equiv 1$, the NEGF in the above equation reduces to a similar form as the equilibrium noninteracting Green functions in [132] but they are not fully the same since in [132] the impurity terms in the Hamiltonian include both forward scattering and backward scattering terms, however in our present case only backward scattering (tunneling term) from the impurity (point-contact) is considered. Upon sudden switch on of a bias, the NEGF shows an initial transient regime before the appearance of a steady state. However, the transport characteristics show only steady state behaviour (at least in the case of a point contact with infinite bandwidth in the momentum to which we will be restricting ourselves in the rest of the paper). To develop a bosonization scheme that can reproduce the nonequilibrium Green functions is a challenging task even in the absence of interparticle interactions. The Fermi-Bose correspondence of chiral fermions is an exact operator identity for chiral fermions. But the presence of a point-contact in the model breaks translation invariance and hence conventional bosonization formulas are unwieldy in this case since the framework has been developed keeping the free particle in mind (e.g. all momentum states upto some level are filled for each chirality suggesting free particles and the number of each chirality is conserved). But in case of a half-line, it is possible to adapt this approach to even account for mutual interactions between fermions. Such situations have been dealt with in [139] by bosonizing a translation invariant reference system and imposing appropriate boundary conditions.
But in order to obtain the full space-time correlation functions in a closed form for an impurity of a finite strength (not vanishingly small or infinitely large), the use of radical new ideas like the non-chiral bosonization technique (NCBT) becomes important. In the NCBT approach, the notion that the Fermi-Bose correspondence is a strict operator identity is eased and the point of view taken is that the Fermi-Bose correspondence is a mnemonic valid at the level of correlation functions. This technique has been successfully used to obtain the most singular parts of the correlation functions in strongly inhomogeneous Luttinger liquids in equilibrium [132]. The effect of the backward scattering of the fermions due to an impurity is encoded in the Fermi-Bose correspondence and the Hamiltonian remains local in this method (unlike conventional approaches where the Fermi-Bose correspondence is sacred, as a result the Hamiltonian is nonlinear and nonlocal when impurities are present).

4.3 Density-density correlation functions with a constant bias

The density-density correlation functions (DDCFs) are certain four-point functions that are crucial to this approach. In the absence of interparticle interactions and assuming a constant bias one can use Wick's theorem to write the DDCF as follows,

$$< T \ \rho_{\nu}(x,t) \rho_{\nu^{'}}(x^{'},t^{'}) >_0 \ - \ < \rho_{\nu}(x,t) >_0 < \rho_{\nu^{'}}(x^{'},t^{'}) >_0$$
$$= \ - < T \ \psi_{\nu}(x,t) \psi^{\dagger}_{\nu^{'}}(x^{'},t^{'}) >_0 \ \ < T \ \psi_{\nu^{'}}(x^{'},t^{'}) \psi^{\dagger}_{\nu}(x,t) >_0 \quad (4.7)$$

Here the subscript $< \;\; >_0$ denotes the absence of mutual interactions. Assuming a constant bias we get the following expressions for the DDCF:

$$
\begin{aligned}
&<T\ \rho_R(x,t)\rho_R(x^{'},t^{'})>_0 - <\rho_R(x,t)>_0<\rho_R(x^{'},t^{'})>_0 \\
&= \ <T\ \rho_L(-x,t)\rho_L(-x^{'},t^{'})>_0 - <\rho_L(-x,t)>_0<\rho_L(-x^{'},t^{'})>_0 \\
&= \left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'})))}\right]^2 \\
&\Bigg(\left[1-\theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]^2\ \left[1-\theta(x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]^2 + \left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^4\ \theta(x)\theta(x^{'}) \\
&+\ \left[1-\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]^2\ \left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2\ \theta(x)\theta(x^{'})\ \left(e^{-ieV_b(t-t^{'}-\frac{x}{v_F}+\frac{x^{'}}{v_F})}+e^{-ieV_b(t^{'}-t-\frac{x^{'}}{v_F}+\frac{x}{v_F})}\right)\Bigg)
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&<T\ \rho_R(x,t)\rho_L(-x^{'},t^{'})>_0 - <\rho_R(x,t)>_0<\rho_L(-x^{'},t^{'})>_0 \\
&= \ <T\ \rho_L(-x,t)\rho_R(x^{'},t^{'})>_0 - <\rho_L(-x,t)>_0<\rho_R(x^{'},t^{'})>_0 \\
&= \left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'})))}\right]^2 \\
&\Bigg(\left(i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2\ \left(-\left[1-\theta(x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]^2\theta(x^{'})-\left[1-\theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]^2\theta(x)\right) \\
&+\left(i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2\ \left(e^{-ieV_b(t^{'}-t-\frac{x^{'}}{v_F}+\frac{x}{v_F})}+e^{-ieV_b(t-t^{'}-\frac{x}{v_F}+\frac{x^{'}}{v_F})}\right)\ \left[1-\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]^2\theta(x)\theta(x^{'})\Bigg)
\end{aligned}
\tag{4.9}
$$

It is easy to check using the above equations that the density-density correlation functions satisfy the following identities,

$$
\begin{aligned}
&<T\ \rho_\nu(\nu\ x,t)(\rho_\nu(\nu\ x^{'},t^{'})+\rho_{-\nu}(-\nu\ x^{'},t^{'}))>_0 - <\rho_\nu(x,t)>_0<(\rho_\nu(\nu\ x^{'},t^{'})+\rho_{-\nu}(-\nu\ x^{'},t^{'}))>_0 \\
&= \ <T\ (\rho_\nu(\nu\ x,t)+\rho_{-\nu}(-\nu\ x,t))\rho_\nu(\nu\ x^{'},t^{'})>_0 - <(\rho_\nu(\nu\ x,t)+\rho_{-\nu}(-\nu\ x,t))>_0<\rho_\nu(\nu\ x^{'},t^{'})>_0 \\
&= \left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'})))}\right]^2
\end{aligned}
\tag{4.10}
$$

where ν take values ± 1 with 1 denoting R (right movers) and -1 denoting L (left movers).

4.4 Bosonization scheme

In this section we introduce an unconventional bosonization scheme that reproduces the noninteracting NEGF for the inhomogeneous system under consideration.

4.4.1 General formalism

Because the fermions have a linear dispersion, it is expected that one should be able to express the low-energy degrees of freedom in terms of bosons. The Fourier transform of the right mover density is,

$$
\rho_R(x,t) = \frac{1}{L}\sum_q \rho_q^R e^{-iqx} \tag{4.11}
$$

where $\rho_q^R = \sum_k : c_{k,R}^{\dagger} c_{k+q,R} :$. The commutator of the densities is

$$
[\rho_q^R, \rho_{q^{'}}^R] = \frac{qL}{2\pi}\delta_{q+q^{'},0} \tag{4.12}
$$

A similar calculation gives $[\rho^L_q,\rho^L_{q'}] = -\frac{qL}{2\pi}\delta_{q+q',0}$, hence apart from a numerical factor the density operators are bosons. In conventional bosonization, the chiral Fermi fields are expressed in terms of the density as,

$$\psi_\nu(x,t) = e^{2\pi i\ \nu\int^x \rho_\nu(y,t)dy} \tag{4.13}$$

where $\nu = \pm$ for R or L respectively. The density is related to bosonic fields $\phi_\nu(x,t)$ as

$$2\pi\int^x dy\ \rho_\nu(y,t) = \phi_\nu(x,t) \tag{4.14}$$

and they obey the commutation relations

$$[\phi_R(x,t),\phi_R(x^{'},t)] = -[\phi_L(x,t),\phi_L(x^{'},t)] = -i\pi\ sgn(x-x^{'}) \tag{4.15}$$

The fermionic anticommutation relations hold in this boson representation [140]. The Fermi-Bose correspondence in Eq. 4.13 is proved by resorting to the basis states of homogeneous (translation-invariant) systems. When we have an impurity in the system like the point-contact tunnel junction present in the model under consideration, the number of right movers and left movers are not separately conserved, hence bosonization using the conventional approach is very inconvenient.

4.4.2 Unconventional bosonization

We handle this situation by modifying the standard Fermi-Bose correspondence to include the effects of backscattering of fermions from the impurity. The modified Fermi-Bose correspondence takes the form,

$$\psi_\nu(x,t) = e^{2\pi i\nu\int^x(\rho_\nu(y,t)+\lambda\ \rho_{-\nu}(-y,t))dy} \tag{4.16}$$

where the value of λ which is either 0 or 1 dictates the presence of the additional $\rho_{-\nu}(-y,t)$ term. The presence of a bias completely spoils the partial symmetry that previously existed between left and right movers. This necessitates a completely new ansatz to replace the original ansatz found in previous works published on the non-chiral bosonization technique, although superficially they look similar. We show that the correlation functions calculated using the correspondence in Eq. 4.16 reproduce the exact Green functions in Eq. 4.2.

Since $[\rho_R(x),\rho_L(x^{'})] = 0$ and $[\rho_\nu(x),\rho_\nu(x^{'})] = \nu\frac{i}{2\pi}\delta^{'}(x-x^{'})$, fermion anticommutation relations are satisfied when $x \neq x^{'}$ at the level of operators even for this modified Fermi-Bose correspondence seen in Eq.(4.16). For $x = x^{'}$, however, the correct anticommutation relations are only recovered at the level of correlation functions. One should note that the ansatz in Eq.4.16 is not a strict operator identity but merely a mnemonic to get the correlation functions. It has been shown in previous works that the non-chiral bosonization technique (NCBT) can be used to obtain the most singular parts of the Green's functions of interacting Luttinger liquids with impurities exactly [45]. The series expansion of the NCBT Green's functions in powers of fermion-fermion interaction strength matches term by term with standard fermionic perturbation theory (most singular terms). It has also been shown that the NCBT Green's functions with forward scattering between fermions satisfy the (most singular parts of the) exact Schwinger-Dyson equations [45]. These results settle any suspicion the reader may have that this formalism is mere phenomenology.

There are several terms that can possibly contribute to the evaluation of the two-point function. These are listed below:

$$<T\ \psi_\nu(x,t)\psi^\dagger_{\nu'}(x^{'},t^{'})> \;=\; <e^{2\pi i\nu\int^x\rho_\nu(y,t)\ dy}\ e^{-2\pi i\nu^{'}\int^{x^{'}}\rho_{\nu^{'}}(y^{'},t^{'})\ dy^{'}}> \tag{4.17}$$

$$<T\ \psi_\nu(x,t)\psi^\dagger_{\nu'}(x^{'},t^{'})> \;=\; <e^{2\pi i\nu\int^x(\rho_\nu(y,t)+\rho_{-\nu}(-y,t))\ dy}\ e^{-2\pi i\nu^{'}\int^{x^{'}}\rho_{\nu^{'}}(y^{'},t^{'})\ dy^{'}}> \tag{4.18}$$

$$<T\ \psi_\nu(x,t)\psi^\dagger_{\nu'}(x^{'},t^{'})> \;=\; <e^{2\pi i\nu\int^x\rho_\nu(y,t)\ dy}\ e^{-2\pi i\nu^{'}\int^{x^{'}}(\rho_{\nu^{'}}(y^{'},t^{'})+\rho_{-\nu^{'}}(-y^{'},t^{'}))\ dy^{'}}> \tag{4.19}$$

$$<T\ \psi_\nu(x,t)\psi^\dagger_{\nu'}(x^{'},t^{'})> \;=\; <e^{2\pi i\nu\int^x(\rho_\nu(y,t)+\rho_{-\nu}(-y,t))\ dy}\ e^{-2\pi i\nu^{'}\int^{x^{'}}(\rho_{\nu^{'}}(y^{'},t^{'})+\rho_{-\nu^{'}}(-y^{'},t^{'}))dy^{'}}> \tag{4.20}$$

These terms are evaluated using a version of the cumulant expansion (Baker-Campbell-Haussdorff formula).

$$< e^{A}e^{B} >\sim e^{\frac{1}{2}<A^2>}e^{\frac{1}{2}<B^2>}e^{<AB>} \tag{4.21}$$

For a homogeneous system this identity is exact since all higher order moments of the density are zero. But for an inhomogeneous system all odd moments of the density vanish but the even moments do not. We make the crucial observation that all higher order even moments of the density are less singular than the second moment (refer to [52] for the proof). So if only the most singular parts of the correlations are desired then all higher order even moments can be ignored. In other words using Eq.4.21 yields the most singular parts of the correlation functions exactly. The identity in Eq.4.21 is the most singular truncation of the cumulant expansion and this what we employ to calculate the most singular parts of the Green's function exactly.
The terms of the type in Eq.4.20 can be discarded straightaway as they do not even correctly reproduce the correct Green's function, and more importantly violate point-splitting constraints (shown in [45]). A linear combination of the other three terms reproduce the exact Green functions apart from the bias dependent exponential prefactors $U(\tau,t) = e^{-i\int_{\tau}^{t} eV_b dt}$. However we note that terms of the standard bosonization form $< e^{2\pi i\nu\int^{x}\rho_{\nu}(y,t)\ dy}\ e^{-2\pi i\nu^{'}\int^{x^{'}}\rho_{\nu^{'}}(y^{'},t^{'})\ dy^{'}} >$ do not always reproduce the correct form of correlation functions. For example let us evaluate for $x > 0, x^{'} > 0$,

$$\begin{aligned} < e^{2\pi i\int^{x}\rho_R(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}}\rho_R(y^{'},t^{'})\ dy^{'}} > \ &= e^{\frac{1}{2}(2\pi i)^2\int^{x}\ dy\ \int^{x}\ dy^{'}\ <\rho_R(y,t)\rho_R(y^{'},t)>} \\ & e^{\frac{1}{2}(2\pi i)^2\int^{x^{'}}\ dy\int^{x^{'}}\ dy^{'}\ <\rho_R(y,t^{'})\rho_R(y^{'},t^{'})>} \\ & e^{-(2\pi i)^2\int^{x}\ dy\int^{x^{'}}\ dy^{'}\ <\rho_R(y,t)\rho_R(y^{'},t^{'})>} \end{aligned} \tag{4.22}$$

The rhs of the above equation upon evaluation for $x > 0, x^{'} > 0$ using Eq.4.8 doesn't reproduce the dynamical part of the exact RR Green function. The conventional bosonization choice in Eq.4.17 only works for the cases: $< \psi_R(x < 0,t)\psi_R^{\dagger}(x^{'} < 0,t^{'}) >$ and $< \psi_L(x > 0,t)\psi_L^{\dagger}(x^{'} > 0,t^{'}) >$. We show in Section.4.5 where we evaluate the four-point functions that for these two cases the conventional choice that amounts to setting $\lambda = 0$ in Eq.4.16 is the one that works. As for all other cases it is necessary to use the modified Fermi-Bose correspondence i.e. $\lambda = 1$ in Eq.4.16, this means that in order to obtain the two-point functions in these cases the Eqs.4.18 and 4.19 must be used. This makes sense because the Green functions of the inhomogeneous system are in fact piecewise discontinuous.
In order to obtain the correct bias dependent $U(\tau,t)$ terms as in Eq.4.2 using this bosonization scheme, we make the following unitary field transformations

$$\tilde{\psi}_R(x,t) = U(-\infty,t)\psi_R(x,t) \tag{4.23}$$
$$\hat{\psi}_R(x,t) = U(t-\frac{x}{v_F},t)\psi_R(x,t) \tag{4.24}$$
$$\tilde{\psi}_L(x,t) = \psi_L(x,t) \tag{4.25}$$
$$\hat{\psi}_L(x,t) = U(-\infty,t+\frac{x}{v_F})\psi_L(x,t) \tag{4.26}$$

This asymmetrical transformations are invoked because the bias is only applied to one of the chiralities. We construct a bosonization prescription using a combination of the above mentioned field transformations along with the bosonized version of the two-point functions in Eqs.4.17,4.18 and 4.19 that reproduces the full non-interacting Green functions apart from constant prefactors. Since bosonization procedures don't give the prefactors, they are fixed on comparison with the known result for the non-interacting inhomogeneous Green functions. The correlations of the transformed fields can be written in a bosonized form as follows.

For the RR case,

$$< T\ \hat{\psi}_R(x,t)\hat{\psi}^\dagger_R(x^{'},t^{'}) >_{\lambda=1} = U(t^{'},t^{'}-\frac{x^{'}}{v_F})U(t-\frac{x}{v_F},t) \frac{1}{2}\Big(< e^{2\pi i\int^x (\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{-2\pi i\int^{x^{'}} \rho_R(y^{'},t^{'})dy^{'}} > + < e^{2\pi i\int^x \rho_R(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}} (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))\ dy^{'}} > \Big) \tag{4.27}$$

and for

$$< T\ \tilde{\psi}_R(x,t)\tilde{\psi}^\dagger_R(x^{'},t^{'}) >_{\lambda=1} = U(t^{'},t)\ \frac{1}{2}\Big(< e^{2\pi i\int^x (\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{-2\pi i\int^{x^{'}} \rho_R(y^{'},t^{'})dy^{'}} > + < e^{2\pi i\int^x \rho_R(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}} (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))\ dy^{'}} > \Big) \tag{4.28}$$

and we will also use the conventional choice in this case,

$$< T\ \tilde{\psi}_R(x,t)\tilde{\psi}^\dagger_R(x^{'},t^{'}) >_{\lambda=0} = U(t^{'},t)\ < e^{2\pi i\int^x \rho_R(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}} \rho_R(y^{'},t^{'})\ dy^{'}} > \tag{4.29}$$

The LR correlations of the transformed fields are written as,

$$< T\ \tilde{\psi}_L(x,t)\hat{\psi}^\dagger_R(x^{'},t^{'}) >_{\lambda=1} = \frac{1}{2}\Big(< e^{-2\pi i\int^x \rho_L(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}} (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))\ dy^{'}} > + < e^{-2\pi i\int^x (\rho_L(y,t)+\rho_R(-y,t))\ dy}\ e^{-2\pi i\int^{x^{'}} \rho_R(y^{'},t^{'})\ dy^{'}} > \Big)U(t^{'},t^{'}-\frac{x^{'}}{v_F}) \tag{4.30}$$

and

$$< T\ \hat{\psi}_L(x,t)\tilde{\psi}^\dagger_R(x^{'},t^{'}) >_{\lambda=1} = U(t^{'},t+\frac{x}{v_F})\ \frac{1}{2}\Big(< e^{-2\pi i\int^x \rho_L(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}} (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))\ dy^{'}} > + < e^{-2\pi i\int^x (\rho_L(y,t)+\rho_R(-y,t))\ dy}\ e^{-2\pi i\int^{x^{'}} \rho_R(y^{'},t^{'})\ dy^{'}} > \Big) \tag{4.31}$$

The choice of $\lambda = 0$ is not valid in this case.

(4.32)

The RL correlations of the transformed fields can be written as,

$$< T\ \hat{\psi}_R(x,t)\tilde{\psi}^\dagger_L(x^{'},t^{'}) >_{\lambda=1} = U(t-\frac{x}{v_F},t)\ \frac{1}{2}\Big(< e^{2\pi i\int^x (\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{2\pi i\int^{x^{'}} \rho_L(y^{'},t^{'})\ dy^{'}} > + < e^{2\pi i\int^x \rho_R(y,t)\ dy}\ e^{2\pi i\int^{x^{'}} (\rho_L(y^{'},t^{'})+\rho_R(-y^{'},t^{'}))\ dy^{'}} > \Big) \tag{4.33}$$

and

$$< T\ \tilde{\psi}_R(x,t)\hat{\psi}^\dagger_L(x^{'},t^{'}) >_{\lambda=1} = U(t^{'}+\frac{x^{'}}{v_F},t)\ \frac{1}{2}\Big(< e^{2\pi i\int^x (\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{2\pi i\int^{x^{'}} \rho_L(y^{'},t^{'})\ dy^{'}} > + < e^{2\pi i\int^x \rho_R(y,t)\ dy}\ e^{2\pi i\int^{x^{'}} (\rho_L(y^{'},t^{'})+\rho_R(-y^{'},t^{'}))\ dy^{'}} > \Big) \tag{4.34}$$

Again in this case the choice of $\lambda = 0$ is invalid. The LL correlations of the transformed fields are written as

$$< T\ \hat{\psi}_L(x,t)\hat{\psi}^\dagger_L(x^{'},t^{'}) >_{\lambda=1} = U(t^{'}+\frac{x^{'}}{v_F},t+\frac{x}{v_F})\ \frac{1}{2}\Big(< e^{-2\pi i\int^x \rho_L(y,t)\ dy}\ e^{2\pi i\int^{x^{'}} (\rho_L(y^{'},t^{'})+\rho_R(-y^{'},t^{'}))\ dy^{'}} > + < e^{-2\pi i\int^x (\rho_L(y,t)+\rho_R(-y,t))\ dy}\ e^{2\pi i\int^{x^{'}} \rho_L(y^{'},t^{'})\ dy^{'}} > \Big) \tag{4.35}$$

and

$$< T\ \tilde{\psi}_L(x,t)\tilde{\psi}_L^\dagger(x^{'},t^{'}) >_{\lambda=1} = \frac{1}{2}\Bigg(< e^{-2\pi i\int^{x} \rho_L(y,t)\ dy}\ e^{2\pi i\int^{x^{'}} (\rho_L(y^{'},t^{'})+\rho_R(-y^{'},t^{'}))\ dy^{'}} > \\ + < e^{-2\pi i\int^{x} (\rho_L(y,t)+\rho_R(-y,t))\ dy}\ e^{2\pi i\int^{x^{'}} \rho_L(y^{'},t^{'})\ dy^{'}} > \Bigg) \tag{4.36}$$

and we also use the conventional choice

$$< T\tilde{\psi}_L(x,t)\tilde{\psi}_L^\dagger(x^{'},t^{'}) >_{\lambda=0} =< e^{-2\pi i\int^{x} \rho_L(y,t)\ dy}\ e^{2\pi i\int^{x^{'}} \rho_L(y^{'},t^{'})\ dy^{'}} > \tag{4.37}$$

4.4.3 RR Green function:

The NCBT ansatz for the nonequilibrium RR Green function is

$$\begin{aligned} < T\ \Psi_R(x,t)\Psi_R^\dagger(x^{'},t^{'}) > \ &= \ \theta(-x)\theta(-x^{'}) < T\ \tilde{\psi}_R(x,t)\tilde{\psi}_R^\dagger(x^{'},t^{'}) >_{\lambda=0} \\ &+\ C_1 < T\ \hat{\psi}_R(x,t)\hat{\psi}_R^\dagger(x^{'},t^{'}) >_{\lambda=1} \\ &+\ C_2\ < T\ \tilde{\psi}_R(x,t)\tilde{\psi}_R^\dagger(x^{'},t^{'}) >_{\lambda=1} \end{aligned} \tag{4.38}$$

Bosonization doesn't give us the prefactors so by comparing with the exact results we fix the constant prefactors (see Appendix G for details),

$$C_1 = \frac{i}{2\beta v_F}\frac{\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2}{\sinh(\frac{\pi\epsilon}{\beta v_F})\ \mathscr{C}_1}\theta(x)\theta(x^{'}) \tag{4.39}$$

where

$$e^{\frac{1}{2}(2\pi i)^2\int^{x}\ dy\int^{x}\ dy^{'}\ <(\rho_R(y,t)+\rho_L(-y,t))(\rho_R(y^{'},t)+\rho_L(-y^{'},t))>}\ =\ \sinh(\frac{\pi\epsilon}{\beta v_F})$$

where ϵ is a regularization factor that is eventually taken to zero and we define $\mathscr{C}_1 = e^{\frac{1}{2}(2\pi i)^2\int^{x^{'}}\ dy^{'}\int^{x^{'}}\ dy\ <\rho_R(y^{'},t^{'})\rho_R(y,t^{'})>}$ with $x^{'} > 0$ which is ultimately independent of $x^{'}$ and $t^{'}$, and

$$C_2 = \frac{i}{2\beta v_F}\frac{\left(1-\ \theta(x^{'})\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right)\left(1-\ \theta(x)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right) - \theta(-x)\theta(-x^{'})}{\sinh(\frac{\pi\epsilon}{\beta v_F})\ \frac{1}{2}((\theta(x)+\theta(x^{'}))\mathscr{C}_1 + (\theta(-x)+\theta(-x^{'}))\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})} \tag{4.40}$$

The term $\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}}$ comes from evaluating $e^{\frac{1}{2}(2\pi i)^2\int^{x}\ dy^{'}\int^{x}\ dy\ <\rho_R(y,t^{'})\rho_R(y^{'},t^{'})>}$ with $x < 0$. This form of C_1 and C_2 in Eq.4.38 conspires to give us the correct constant prefactors as in the exact Green functions.

4.4.4 LR Green function:

The LR Green function can be written as

$$< T\ \Psi_L(x,t)\Psi_R^\dagger(x^{'},t^{'}) > \ = \ C_3\ < T\ \tilde{\psi}_L(x,t)\hat{\psi}_R^\dagger(x^{'},t^{'}) >_{\lambda=1}\ -\ C_4\ < T\ \hat{\psi}_L(x,t)\tilde{\psi}_R^\dagger(x^{'},t^{'}) >_{\lambda=1} \tag{4.41}$$

On comparison with the exact results we fix the constant prefactors to be (see Appendix H for details),

$$C_3 = -\frac{i}{2\beta v_F}\frac{\left[1-\theta(-x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}}{\frac{1}{2}\sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(-x)\mathscr{C}_2+\mathscr{C}_1+\theta(x)\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})}\theta(x^{'}) \tag{4.42}$$

and

$$C_4 = -\frac{i}{2\beta v_F}\frac{\left[1-\theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}}{\frac{1}{2}\sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(x^{'})\mathscr{C}_1+\mathscr{C}_2+\theta(-x^{'})\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})}\theta(-x) \tag{4.43}$$

where $\mathscr{C}_2 = e^{\frac{1}{2}(2\pi i)^2\int^{x^{'}}\ dy^{'}\int^{x^{'}}\ dy\ <\rho_L(y,t^{'})\rho_L(y^{'},t^{'})>}$ with $x^{'} < 0$ which is ultimately a constant independent of $x^{'}$ and $t^{'}$.

4.4.5 RL Green function:

The RL Green function can be written as

$$< T\ \Psi_R(x,t)\Psi_L^\dagger(x',t') > \ = \ -C_5\ < T\ \hat{\psi}_R(x,t)\tilde{\psi}_L^\dagger(x',t') >_{\lambda=1}\ +\ C_6\ < T\ \tilde{\psi}_R(x,t)\hat{\psi}_L^\dagger(x',t') >_{\lambda=1} \tag{4.44}$$

Comparing with the exact results we can express,

$$C_5 = \frac{i}{2\beta v_F}\frac{\left[1-\theta(-x')\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}}{\frac{1}{2}\sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(-x')\mathscr{C}_2+\mathscr{C}_1+\theta(x')\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})}\theta(x) \tag{4.45}$$

and

$$C_6 = \frac{i}{2\beta v_F}\frac{\left[1-\theta(x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}}{\frac{1}{2}\sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(x)\mathscr{C}_1+\mathscr{C}_2+\theta(-x)\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})}\theta(-x') \tag{4.46}$$

4.4.6 LL Green function:

Finally the LL Green function is expressed as,

$$\begin{aligned} < T\ \Psi_L(x,t)\Psi_L^\dagger(x',t') > \ &= \ \theta(x)\theta(x') < T\tilde{\psi}_L(x,t)\tilde{\psi}_L^\dagger(x',t') >_{\lambda=0}\ +\ C_7\ < T\ \hat{\psi}_L(x,t)\hat{\psi}_L^\dagger(x',t') >_{\lambda=1} \\ &+\ C_8 < T\ \tilde{\psi}_L(x,t)\tilde{\psi}_L^\dagger(x',t') >_{\lambda=1} \end{aligned} \tag{4.47}$$

Again on comparing with exact results we can express,

$$C_7 = -\frac{i}{2\beta v_F}\frac{\left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2}{\sinh(\frac{\pi\epsilon}{\beta v_F})\mathscr{C}_2}\theta(-x)\theta(-x') \tag{4.48}$$

and

$$C_8 = -\frac{i}{2\beta v_F}\frac{\left[1-\theta(-x')\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\left[1-\theta(-x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]-\theta(x)\theta(x')}{\frac{1}{2}\sinh(\frac{\pi\epsilon}{\beta v_F})((\theta(-x)+\theta(-x'))\mathscr{C}_2+(\theta(x)+\theta(x'))\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})} \tag{4.49}$$

Although this bosonization procedure seems like a roundabout and complicated way to study an exactly solvable system (i.e. free fermions plus infinite bandwidth impurity and time dependent bias) it should prove to be very useful for future prospective research on systems with interparticle interactions like fractional quantum Hall edges out of equilibrium.

4.5 Four-point functions

In this section we evaluate four point-functions using our unconventional bosonization scheme. We still have not made clear why we have chosen only the conventional bosonization form (with $\lambda = 0$) for the cases $< T\ \Psi_R(x < 0,t)\Psi_R^\dagger(x' < 0,t') >$ and $< T\ \Psi_L(x > 0,t)\Psi_L^\dagger(x' > 0,t') >$ whereas the unconventional choice ($\lambda = 1$) superficially seems to work for all cases. Evaluating the four-point functions will provide clarity on this issue by means of some consistency checks. Since we are not considering interparticle interactions we can use Wick's theorem to write down the four-point functions. Let us first consider

$$< T\ \rho_R(x_1,t_1)\psi_R^\dagger(x,t)\psi_R(x',t') > \ = \ -< T\ \psi_R(x',t')\psi_R^\dagger(x_1,t_1) > \ < T\ \psi_R(x_1,t_1)\psi_R^\dagger(x,t) > \tag{4.50}$$

We shall use the proposed bosonization scheme to reproduce the RHS of Eq.4.50. Expressing $\psi^{\dagger}_R(x,t)$ and $\psi_R(x^{'},t^{'})$ in bosonized form we can write this as

$$\begin{aligned}
&- < T\ \psi_R(x^{'},t^{'})\psi^{\dagger}_R(x_1,t_1) > \ \ < T\ \psi_R(x_1,t_1)\psi^{\dagger}_R(x,t) > \\
&\quad = \ w_0 < \rho_R(x_1,t_1)e^{-2\pi i\int^x dy\ \rho_R(y,t)}\ e^{2\pi i\int^{x^{'}} dy^{'}\ \rho_R(y^{'},t^{'})} > \\
&\quad + w_{11} < \rho_R(x_1,t_1)e^{-2\pi i\int^x dy\ (\rho_R(y,t)+\rho_L(-y,t))}\ e^{2\pi i\int^{x^{'}} dy^{'}\ \rho_R(y^{'},t^{'})} > \\
&\quad + w_{21} < \rho_R(x_1,t_1)e^{-2\pi i\int^x dy\ \rho_R(y,t)}\ e^{2\pi i\int^{x^{'}} dy^{'}\ (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))} >
\end{aligned} \tag{4.51}$$

where w_0 is the prefactor for the term with $\lambda = 0$ in the exponent (conventional choice) and w_{11} and w_{12} are the prefactors for the terms with $\lambda = 1$ in the exponent of $\psi^{\dagger}_R(x,t)$ and $\psi_R(x^{'},t^{'})$ respectively. Without loss of generality we can write $\rho_R(x_1,t_1) = \lim_{a\to 0}\frac{d}{da}e^{a\ \rho_R(x_1,t_1)}$ and we consider only the most singular parts of the RHS of Eq.4.51. In order to reproduce the exact result for the four-point function obtained using Wick's theorem we come to the conclusion that the terms on the RHS with $\lambda = 0$ are only valid for the case with $x < 0, x^{'} < 0$ (see Appendix I for details) while for all remaining cases only the unconventional bosonization terms with $\lambda = 1$ will give us the correct results with the prefactors appropriately fixed as shown in Appendix I. For the case $x < 0, x^{'} < 0$ it appears at the outset that both the conventional as well as the unconventional bosonization choices should work, but evaluating the four-point function of the type $< T\ \rho_L(-x_1,t_1)\psi^{\dagger}_R(x,t)\psi_R(x^{'},t^{'}) >$ gives additional constraints that force the condition that the prefactors for the $\lambda = 1$ terms be zero for the case $x < 0, x^{'} < 0$. Using Wick's theorem we write

$$< T\ \rho_L(-x_1,t_1)\psi^{\dagger}_R(x,t)\psi_R(x^{'},t^{'}) > \ = \ - < T\ \psi_R(x^{'},t^{'})\psi^{\dagger}_L(-x_1,t_1) > \ \ < T\ \psi_L(-x_1,t_1)\psi^{\dagger}_R(x,t) > \tag{4.52}$$

Similar to Eq.4.51 we can write this as,

$$\begin{aligned}
&- < T\ \psi_R(x^{'},t^{'})\psi^{\dagger}_L(-x_1,t_1) > \ \ < T\ \psi_L(-x_1,t_1)\psi^{\dagger}_R(x,t) > \\
&\quad = \ q_0\ \ < \rho_L(-x_1,t_1)e^{-2\pi i\int^x dy\ \rho_R(y,t)}\ e^{2\pi i\int^{x^{'}} dy^{'}\ \rho_R(y^{'},t^{'})} > \\
&\quad + \ q_{11} < \rho_L(-x_1,t_1)e^{-2\pi i\int^x dy\ (\rho_R(y,t)+\rho_L(-y,t))}\ e^{2\pi i\int^{x^{'}} dy^{'}\ \rho_R(y^{'},t^{'})} > \\
&\quad + \ q_{21}\ \ < \rho_L(-x_1,t_1)e^{-2\pi i\int^x dy\ \rho_R(y,t)}\ e^{2\pi i\int^{x^{'}} dy^{'}\ (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))} >
\end{aligned} \tag{4.53}$$

In Appendix I we show that in order to reproduce the correct Wick's theorem result for the case $x < 0, x^{'} < 0$, the prefactors q_{11} and q_{12} should be identically zero. This means that the unconventional bosonization terms with $\lambda = 1$ in the exponent do not play any role in this particular calculation. So we can argue on grounds of self consistency that when evaluating the two-point functions the conventional bosonization procedure is the only valid choice for the $< T\ \Psi_R(x < 0,t)\Psi^{\dagger}_R(x^{'} < 0,t^{'}) >$ correlations. A similar procedure evaluating the four-point functions $< T\ \rho_R(-x_1,t_1)\psi^{\dagger}_L(x,t)\psi_L(x^{'},t^{'}) >$ and $< T\ \rho_L(x_1,t_1)\psi^{\dagger}_L(x,t)\psi_L(x^{'},t^{'}) >$ can be used to show that for writing the bosonized form of $< T\ \Psi_L(x > 0,t)\Psi^{\dagger}_L(x^{'} > 0,t^{'}) >$ the conventional bosonization formula is the only valid choice. This makes sense because the right mover fermions in the spatial region $x < 0$ and the left mover fermions in the spatial region $x > 0$ have not scattered across the localised point-contact impurity at the origin and so the conventional bosonization method (which works for the homogeneous system) gives the correlation functions of these fermions. But for all other cases it is necessary to use the unconventional bosonization scheme to obtain the correct form of the two-point correlations in presence of impurity.

4.6 Tunneling current and conductance

The tunneling current is defined in the usual sense as the rate of change of the difference in the number of right and left movers

$$I_{tun} = e\ \partial_t\frac{\Delta N}{2} = e\frac{i}{2}\left[H,\Delta N\right] = e\frac{i}{2}\left[H,N_R - N_L\right] \tag{4.54}$$

From the Green functions obtained using our bosonization ansatz we obtain the expected results for the tunneling current [81, 141, 142],

$$I_{tun} \;=\; -ie\Gamma \lim_{t' \to t}\left(< \Psi_R^{\dagger}(0,t^{'})\Psi_L(0,t) > - < \Psi_L^{\dagger}(0,t)\Psi_R(0,t^{'}) > \right) \tag{4.55}$$

Using Eqs.4.44 and 4.41 along with the bosonized correlations in Eqs.4.30-4.34 we get,

$$< \Psi_R^{\dagger}(0,t^{'})\Psi_L(0,t) > \;=\; -\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(-v_F(t-t^{'})))}\left(-U(t^{'},t)+1\right)\, i\frac{\Gamma}{v_F}\frac{2v_F^2}{\Gamma^2+4v_F^2}\left[1-\frac{\Gamma^2}{\Gamma^2+4v_F^2}\right] \tag{4.56}$$

Note that we use the Dirichlet regularized step function ($\theta(0) = \frac{1}{2}$). This gives us

$$\begin{aligned} I_{tun} &= ie\Gamma \lim_{t' \to t}\left(\frac{1}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(v_F(t-t^{'})))}\left(U(t,t^{'})-U(t^{'},t)\right)\frac{\Gamma}{v_F}\frac{2v_F^2}{\Gamma^2+4v_F^2}\left[1-\frac{\Gamma^2}{\Gamma^2+4v_F^2}\right]\right) \\ &= -\Gamma^2\;\frac{4}{\Gamma^2+4v_F^2}\left[1-\frac{\Gamma^2}{\Gamma^2+4v_F^2}\right]\frac{e^2}{2\pi}V_b \end{aligned} \tag{4.57}$$

We define the tunneling amplitude in terms of a tunneling parameter t_p as $\Gamma = 2v_F t_p$ [143] and since we adopt the same convention as in our previous work we replace $V_b = -V$ and we get

$$I_{tun} \;=\; \frac{4t_p^2}{(t_p^2+1)^2}\,\frac{e^2}{h}V \tag{4.58}$$

and the differential tunneling conductance is obtained as

$$G = \frac{dI_{tun}}{dV} \;=\; G_{tun} \;=\; \frac{4t_p^2}{(t_p^2+1)^2}\,\frac{e^2}{h} \tag{4.59}$$

These results agree with standard scattering theory [117, 118] and also it is apparent that there is a duality between the strong and weak tunneling regimes ($t_p \to \frac{1}{t_p}$) in the tunneling conductance. The $I(V)$ characteristics is linear and nonlinearities in the transport are expected only when mutual interactions between the fermions are present or when a finite bandwidth point-contact is considered [81].

4.7 Summary

In summary, we use a modified version of the Fermi-Bose correspondence (viz. Eq.4.16) and construct an ansatz to obtain the bosonized version of the nonequilibrium Green functions (NEGF) for two non-interacting chiral quantum wires coupled through a point contact at the origin with a bias between the right and left movers. The standard bosonization formalism cannot be employed easily as it does not yield the exact Green functions in a closed form for models with impurity backscattering with or without bias while treating the impurity non perturbatively. The main result of this paper viz. the non-interacting NEGF using bosonization is remarkable as there is no guarantee (and no precedent either) that the full Green functions of such a complicated nonequilibrium system would come out correctly when described using commuting variables rather than anticommuting variables. We also show that the modified unconventional bosonization ansatz can be used to evaluate the relevant four-point functions consistent with Wick's theorem in the absence of interparticle interactions. This is a crucial cross-check that validates our method. The purpose of this article is to lay the foundation for our future work which is to show that the non-chiral bosonization technique (NCBT) can be used to write down the most singular parts of the full NEGF for the above mentioned system (considered in [123]) in terms of simple functions of position and time, when mutual interactions between fermions are included. Evaluating the full nonequilibrium Green function (NEGF) in presence of interparticle interactions is a highly nontrivial task. It is known that the edge states of fractional quantum Hall fluids can be described as chiral Luttinger liquids [144, 145, 146]. Due to the nature of the chiral Luttinger liquid a power-law dependence in the tunneling $I(V)$ characteristics is expected (at zero temperature). More generally, the tunneling behaviour as a function of bias voltage V or temperature T follows a universal scaling form. Fendley, Ludwig and Saleur [123] have shown that the problem of tunneling between chiral Luttinger edges is integrable for certain values of the filling fractions (between 1/4 and 1) using thermodynamic Bethe ansatz and they obtain the universal scaling functions for the nonequilibrium tunneling current and conductance.

In bosonization (both NCBT and conventional) inclusion of forward scattering is as easy (or difficult) as solving the theory without forward scattering. This is because bosonization is essentially describing fermions using commuting variables. Ultimately forward scattering mutual interaction between the fermions results in just a Gaussian deformation of the non-interacting theory (Hubbard Stratanovich transformation). But including the effects of impurity backscattering (the point-contact in this case) even without interparticle interactions is tremendously nontrivial. Being able to solve for the properties of free fermions but in presence of backward scattering from an impurity requires radically new approaches such as NCBT if one desires to treat the impurity properly (non-perturbatively). In our formalism, interparticle interactions can be treated non-perturbatively by calculating the density-density correlation functions (DDCF) using a generating functional with an auxiliary field in a manner similar to the procedure in [52]. The interacting DDCF can be used in our bosonization ansatz for the nonequilibrium correlation functions. The previously obtained NCBT Green's functions (most singular part) [132] for an interacting Luttinger liquid with impurities in equilibrium does indeed show universal power-law scaling behaviour in the equal space-equal time limit (see Appendix J). However, in our earlier works, the equal space but unequal time NCBT Green functions (most singular parts) which are important for extracting the density of states near the impurity have exponents that are not universal [132]. At first glance this may seem to contradict previous established results that predict the exponent to be independent of (bare) impurity strength [147, 54, 44, 148], but in fact these works study the full Green functions including non-singular parts but only in the vicinity of the half-line limit and the homogeneous limit. On the other hand, in our previous works and also in the present one, we only focus on the most singular parts of the Green functions for arbitrary impurity strength as they are given exactly [45] by the NCBT formalism even when mutual interaction between fermions are included. Hence it is a mistake to naively compare the NCBT results directly with previous works of other authors.
This unconventional approach to bosonization enables the calculation of the most singular parts of the correlation functions of fermions with localized scattering centers where backscattering takes place and in presence of mutual fermion-fermion forward scattering interactions (in the case of a Luttinger liquid [132]) to be expressed in terms of elementary functions of position and time.
The conventional bosonization scheme of the above mentioned model leads to a Lagrangian density that is nonlinear (Sine-Gordon) [149] but has a spatially local expression. Because of the Sine-Gordon term, it is not soluble. The most singular parts of the mutual (auto) correlations of density fluctuations involves realising (with proof [52]) that all the odd moments of the density fluctuations of the above system vanish identically whereas all the higher order even moments although non-zero are less singular than the leading second order moment and therefore ignorable. The most singular part of the Lagrangian density then becomes spatially non-local but purely quadratic in the bosons and therefore soluble (refer to Section 6.7). The conventional bosonization scheme has a Fermi-Bose correspondence that is an operator identity that is tied to the translationally invariant fermion basis.

The main message is that it is possible to study backward scattering from a stationary impurity in such a way that **(only) the most singular parts** of the correlation functions can be written down exactly using a modified bosonization framework that transfers the burden of shouldering the Sine Gordon character of the conventional approach to the modified Fermi Bose correspondence while retaining the action of the theory quadratic (albeit spatially non-local) in the bosons and therefore, soluble. The central achievement of this paper is being able to describe fermions backward scattering from an impurity in presence of a bias that drives the system out of equilibrium exactly using commuting variables as opposed to anticommuting variables. In the former approach, inclusion of mutual interaction between fermions is a (relatively) trivial Gaussian deformation of these results (formidable practically). In the latter approach, inclusion of even forward scattering between fermions is intractable. Technically, being able to solve the free fermion theory using bosonization is the crux of the problem, even though nontrivial physical phenomena are seen upon inclusion of forward scattering between fermions.

Chapter 5

Density-density correlation function of chiral Luttinger liquids with a point-contact

In this chapter based on our work [150], we evaluate the most singular parts of the density-density correlation functions (DDCF) of chiral Luttinger liquids (assumed spinless) with a point-contact junction at the origin. This is isomorphic to a system of fractional quantum Hall effect (FQHE) edge states with a point-contact constriction. The right (R) and left (L) moving edges are spatially separated by the bulk and the densities of the right and left moving edges are coupled through an interaction mediated by the bulk. The most singular part of the DDCF is obtained using a generating functional method and the results are compared with standard perturbation theory.

5.1 Model Hamiltonian

We consider a system of two chiral Luttinger liquids with a point-contact tunnel junction at the origin. This can be realized as edge states of a fractional quantum Hall effect (FQHE) fluid with a narrow constriction in the middle that forms a point-contact between the opposite edges. The FQHE systems belonging to the Laughlin series [151] with filling fraction $\nu = 1/m$ where m is an odd integer have only single edge modes that can be treated as chiral Luttinger liquids [144]. The non-interacting part of the Hamiltonian is

$$H_0 = \sum_p (v_F p) c^\dagger_{p,R} c_{p,R} + \sum_p (-v_F p) c^\dagger_{p,L} c_{p,L} + \frac{\Gamma}{L}(c^\dagger_{.,R} c_{.,L} + c^\dagger_{.,L} c_{.,R}) \tag{5.1}$$

where R and L label the right and left moving chiral spinless modes. The fermion creation and annihilation operators in momentum space are $c^\dagger_p$ and c_p respectively and we use the notation $c^\dagger_{.,R} = \sum_p c^\dagger_{p,R}$. The tunneling amplitude of the symmetric point-contact junction is Γ and L is the system size. The interaction part of the Hamiltonian is

$$H_{int} = \int dx \; v_0 \; \rho_R(x,t) \rho_L(x,t) \tag{5.2}$$

The right mover and left mover densities are coupled through an interaction parametrized by v_0. The full Hamiltonian of interest is

$$H \;=\; H_0 + H_{int} \tag{5.3}$$

The exact Green functions in space-time domain for the noninteracting problem was obtained using analytic methods in [81]. The noninteracting density-density correlation functions (DDCF) are obtained from the two-point Green's functions using Wick's theorem and are written as,

$$
\begin{aligned}
&< T\ \rho_R(x,t)\rho_R(x',t') >_0 - < \rho_R(x,t) >_0 < \rho_R(x',t') >_0 \\
&= \ < T\ \rho_L(-x,t)\rho_L(-x',t') >_0 - < \rho_L(-x,t) >_0 < \rho_L(-x',t') >_0 \\
&= \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F(t-t')))} \right]^2 \\
&\left(\left[1-\theta(x')\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2} \right]^2 \ \left[1-\theta(x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2} \right]^2 + \left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2+4v_F^2} \right)^4 \ \theta(x)\theta(x') \right. \\
&\left. +\ 2 \left[1-\frac{2\Gamma^2}{\Gamma^2+4v_F^2} \right]^2 \ \left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2+4v_F^2} \right)^2 \ \theta(x)\theta(x') \right)
\end{aligned} \tag{5.4}
$$

and

$$
\begin{aligned}
&< T\ \rho_R(x,t)\rho_L(-x',t') >_0 - < \rho_R(x,t) >_0 < \rho_L(-x',t') >_0 \\
&= \ < T\ \rho_L(-x,t)\rho_R(x',t') >_0 - < \rho_L(-x,t) >_0 < \rho_R(x',t') >_0 \\
&= \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F(t-t')))} \right]^2 \\
&\left(\left(i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2} \right)^2 \ \left(-\left[1-\theta(x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2} \right]^2 \theta(x') - \left[1-\theta(x')\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2} \right]^2 \theta(x) \right) \right. \\
&\left. + \left(i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2} \right)^2 \ \left(e^{-ieV_b(t'-t-\frac{x'}{v_F}+\frac{x}{v_F})} + e^{-ieV_b(t-t'-\frac{x}{v_F}+\frac{x'}{v_F})} \right) \ \left[1-\frac{2\Gamma^2}{\Gamma^2+4v_F^2} \right]^2 \theta(x)\theta(x') \right)
\end{aligned} \tag{5.5}
$$

The density-density correlations satisfy the following identities,

$$
\begin{aligned}
&< T\ \rho_\chi(\chi\ x,t)(\rho_\chi(\chi\ x',t') + \rho_{-\chi}(-\chi\ x',t')) >_0 - < \rho_\chi(x,t) >_0 < (\rho_\chi(\chi\ x',t') + \rho_{-\chi}(-\chi\ x',t')) >_0 \ = \\
&< T\ (\rho_\chi(\chi\ x,t) + \rho_{-\chi}(-\chi\ x,t))\rho_\chi(\chi\ x',t') >_0 - < (\rho_\chi(\chi\ x,t) + \rho_{-\chi}(-\chi\ x,t)) >_0 < \rho_\chi(\chi\ x',t') >_0 \\
&= \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F(t-t')))} \right]^2
\end{aligned} \tag{5.6}
$$

where χ takes values ± 1 with 1 denoting R (right movers) and -1 denoting L (left movers). The regularization scheme implied in writing Eq.5.1 is that summimg over all momenta implicitly means summing over momenta in the interval $p \in \{-\Lambda, \Lambda\}$ and taking $\Lambda \to \infty$. The implication of this Dirichlet regularization scheme is that the values of discontinuous functions are always the average of the left and right hand limits. The step function appearing throughout this paper is the Dirichlet regularized step function defined as: $\theta(x>0) = 1,\ \theta(x<0) = 0,\ \theta(x=0) = \frac{1}{2}$. In presence of the interactions it is in general not possible to write down a simple formula such as in Eqs.5.4 and 5.5 for the correlation functions when the point-contact is present. There is no guarantee that the correlation function will have simple second order poles. However it is possible to obtain the most singular part of the density-density correlation function even in the presence of forward scattering interactions, that is when Eq.5.2 is present in the Hamiltonian. By most singular we mean terms with higher order poles (second order in this case). In the following sections we derive the most singular contribution to the interacting DDCF using a generating functional. Before proceeding let us make the following definition of the symmetric and antisymmetric density (fluctuation) operators

$$
\begin{aligned}
\rho_{sym}(x,t) &\equiv \rho_R(x,t) + \rho_L(-x,t)\ ; \\
\rho_{asy}(x,t) &\equiv \rho_R(x,t) - \rho_L(-x,t)
\end{aligned} \tag{5.7}
$$

This means we can write,

$$
\rho_R(x,t) = \frac{\rho_{sym}(x,t) + \rho_{asy}(x,t)}{2} \tag{5.8}
$$

and

$$\rho_L(-x,t) = \frac{\rho_{sym}(x,t) - \rho_{asy}(x,t)}{2} \tag{5.9}$$

Using Eqs.5.4,5.5 and 5.6 we express the noninteracting correlations of the symmetric and antisymmetric densities as,

$$< \rho_{sym}(x,t)\rho_{sym}(x',t') >_0 \;=\; 2\left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F(t-t')))}\right]^2 \tag{5.10}$$

$$< \rho_{sym}(x,t)\rho_{asy}(x',t') >_0 \;=\; 0 \tag{5.11}$$

$$< \rho_{asy}(x,t)\rho_{sym}(x',t') >_0 \;=\; 0 \tag{5.12}$$

$$< \rho_{asy}(x,t)\rho_{asy}(x',t') >_0 \;=\; (\theta(xx') + r_1\,\theta(-xx'))\;\; < T\rho_{sym}(x,t)\rho_{sym}(x',t') >_0 \tag{5.13}$$

where $r_1 = \left(1 - \frac{32\Gamma^2 v_F^2}{(\Gamma^2+4v_F^2)^2}\right)$. We choose to work with the ρ_{sym} and ρ_{asy} fields as they prove to be convenient in evaluating the path integrals in the generating functional. The interaction part of the Hamiltonian is expressed in terms of these new fields as

$$H_{int} = \int dx\; v_0\; \rho_R(x,t)\rho_L(x,t) = \frac{v_0}{4}\int dx\; (\rho_{sym}(x,t) + \rho_{asy}(x,t))(\rho_{sym}(-x,t) - \rho_{asy}(-x,t)) \tag{5.14}$$

5.2 Generating functional for the density-density correlations

Interparticle interactions are systematically introduced non-perturbatively in the density correlations by means of a generating functional approach. The generating functional we employ to calculate the $< \rho_{sym}\rho_{sym} >$ and $< \rho_{asy}\rho_{asy} >$ correlations with auxiliary fields U_{sym} and U_{asy} in presence of interactions and the point-contact impurity can be written as

$$Z[U] = \int D[\rho_{sym}]\int D[\rho_{asy}]\;\; e^{iS_0}e^{iS_{int}}e^{\int \rho_{sym}U_{sym} + \int \rho_{asy}U_{asy}} \tag{5.15}$$

where S_0 and S_{int} are the free and interacting actions respectively. The generating functional in the absence of interactions is

$$\begin{aligned} Z_0[U] &= \int D[\rho_{sym}]\int D[\rho_{asy}]\;\; e^{iS_0}e^{\int \rho_{sym}U_{sym} + \int \rho_{asy}U_{asy}} \\ \Rightarrow e^{iS_0} &= \int D[U^{'}_{sym}]\int D[U^{'}_{asy}]\;\; e^{-\int \rho_{sym}U^{'}_{sym} - \int \rho_{asy}U^{'}_{asy}}\;\; Z_0[U^{'}] \end{aligned} \tag{5.16}$$

Hence we can write

$$\begin{aligned} Z[U] = &\int D[U^{'}_{sym}]\int D[U^{'}_{asy}]\; Z_0[U^{'}]\;\;\int D[\rho_{sym}]\int D[\rho_{asy}] \\ &e^{-i\int_C dt\;\int dx\;\frac{v_0}{4}\;(\rho_{sym}(x,t)+\rho_{asy}(x,t))(\rho_{sym}(-x,t)-\rho_{asy}(-x,t))} \\ &e^{\int_C dt\;\int dx\;\rho_{sym}(x,t)(U_{sym}(x,t)-U^{'}_{sym}(x,t)) + \int_C dt\;\int dx\;\rho_{asy}(x,t)(U_{asy}(x,t)-U^{'}_{asy}(x,t))} \end{aligned} \tag{5.17}$$

since $e^{iS_{int}} = e^{-i\int_C dt\; H_{int}}$. The path integrals are evaluated using the saddle point method and after integrating over the ρ_{sym} and ρ_{asy} fields we get

$$Z[U] = \int D[U^{'}_{sym}]\int D[U^{'}_{asy}]\; Z_0[U^{'}] \tag{5.18}$$

$$\times\, e^{-\frac{1}{iv_0}\int_C dt\ \int dx\ (U_{asy}(-x,t)-U^{'}_{asy}(-x,t)-U^{'}_{sym}(-x,t)+U_{sym}(-x,t))(U_{asy}(x,t)-U^{'}_{asy}(x,t)+U^{'}_{sym}(x,t)-U_{sym}(x,t))}$$

We perform the most singular truncation of Z_0 by making the Gaussian approximation

$$\begin{aligned} Z_0[U^{'}] \;=\; & e^{\frac{1}{2}\int_C dt\int dx\int_C dt'\int dx'\ <T\rho_{sym}(x,t)\rho_{sym}(x',t')>_0\ U^{'}_{sym}(x,t)U^{'}_{sym}(x',t')} \\ & e^{\frac{1}{2}\int_C dt\int dx\int_C dt'\int dx'\ <T\rho_{asy}(x,t)\rho_{asy}(x',t')>_0\ U^{'}_{asy}(x,t)U^{'}_{asy}(x',t')} \end{aligned} \tag{5.19}$$

The idea here is that we ignore the higher order moments of ρ in $Z_0[U^{'}]$ as they are less singular than the second moment. The connected parts of all the odd moments of ρ vanish and all the higher order even moments are less singular than the second moment as shown in [52]. The higher order even moments only involve first order simple poles whereas the second moment of the density involves second order poles hence is more singular. This means that this procedure will yield only the most singular part of the DDCF. The most singular part captures essentially the important physics and when used in conjunction with novel bosonization techniques like the non-chiral bosonization technique (NCBT) [45, 132, 137] give the most singular part of the two-point Green functions of the interacting inhomogeneous system. Solving for the saddle point of the $U^{'}$ fields we obtain the following equations

$$\int dt'\int dx'\ <T\rho_{sym}(x,t)\rho_{sym}(x',t')>_0\ U^{'}_{sym}(x',t')+\frac{2i}{v_0}\ (U_{sym}(-x,t)-U^{'}_{sym}(-x,t))=0 \tag{5.20}$$

and

$$\int_C dt'\int dx'\ <T\rho_{asy}(x,t)\rho_{asy}(x',t')>_0\ U^{'}_{asy}(x',t')-\frac{2i}{v_0}(U_{asy}(-x,t)-U^{'}_{asy}(-x,t))=0 \tag{5.21}$$

The generating functional now in terms of these saddle points is

$$Z[U]=e^{-\frac{1}{iv_0}\int_C dt\int dx\ (U^{'}_{sym}(-x,t)-U_{sym}(-x,t))U_{sym}(x,t)-\frac{1}{iv_0}\int_C dt\int dx\ (U_{asy}(-x,t)-U^{'}_{asy}(-x,t))U_{asy}(x,t)} \tag{5.22}$$

We see that the auxiliary fields of the symmetric and antisymmetric densities decouple and hence the $<\rho_{sym}\rho_{sym}>$ and $<\rho_{asy}\rho_{asy}>$ correlations can be separately obtained whereas the $<\rho_{sym}\rho_{asy}>$ correlations are zero.

5.3 Correlation function of the symmetric density fields

In this section we obtain the correlations of the symmetric densities from the generating functional. In order to proceed we have to obtain the saddle point by solving Eq.5.20 for $U^{'}_{sym}$. It helps to work in the Fourier space. Transforming Eq.5.20 to momentum and Matsubara frequency space we get

$$-i\beta\ <\rho_{sym}(q,n)\rho_{sym}(-q,-n)>_0\ U^{'}_{sym}(q,n)+\frac{2i}{v_0}\ (U_{sym}(-q,n)-U^{'}_{sym}(-q,n))=0 \tag{5.23}$$

where we have used the transformations

$$<T\rho_{sym}(x,t)\rho_{sym}(x',t')>_0\ =\ \frac{1}{L}\sum_{q,n}e^{-iq(x-x')}\ e^{w_n(t-t')}\ <\rho_{sym}(q,n)\rho_{sym}(-q,-n)>$$

and $U_{sym}(-x,t)\ =\ \frac{1}{-i\beta L}\sum_{q,n}e^{iqx}e^{w_nt}\ \ U_{sym}(q,n)$ and $w_n=\frac{2\pi n}{\beta}$ are bosonic Matsubara frequencies with $n\in\mathbb{Z}$. The saddle point for $U^{'}_{sym}$ is obtained from Eq.5.23.

$$U^{'}_{sym}(q,n)=\frac{2(v_0\beta R_{sym}(-q,n)U_{sym}(-q,n)-2U_{sym}(q,n))}{-4+\beta^2v_0^2R_{sym}(-q,n)R_{sym}(q,n)} \tag{5.24}$$

where $R_{sym}(q,n)\ =\ <\rho_{sym}(q,n)\rho_{sym}(-q,-n)>_0\ =\ -\frac{1}{\pi\beta v_F}\frac{iv_Fq}{(w_n-iv_Fq)}$ is the noninteracting correlation in momentum-Matsubara frequency space. The generating functional for the $<\rho_{sym}\rho_{sym}>$ correlations is

$$\begin{aligned} Z[U_{sym}] &= e^{-\sum_{q,n}\ \frac{1}{v_0\beta L}(U^{'}_{sym}(q,-n)-U_{sym}(q,-n))U_{sym}(q,n)} \\ &= e^{\frac{1}{L}\sum_{q,n}\frac{R_{sym}(-q,n)(2U_{sym}(-q,n)-\beta v_0R_{sym}(q,n)U_{sym}(q,n))}{4-\beta^2v_0^2R_{sym}(-q,n)R_{sym}(q,n)}U_{sym}(q,-n)} \end{aligned} \tag{5.25}$$

Now the correlation functions can be evaluated using the relation

$$\frac{1}{L}\sum_{q,n}\frac{R_{sym}(-q,n)(2U_{sym}(-q,n)-\beta v_0 R_{sym}(q,n)U_{sym}(q,n))}{4-\beta^2 v_0^2 R_{sym}(-q,n)R_{sym}(q,n)}U_{sym}(q,-n)$$
$$= \frac{1}{2L}\sum_{q,q',n} < \rho_{sym}(q,n)\rho_{sym}(q',-n) > \ U_{sym}(q,n)U_{sym}(q',-n) \tag{5.26}$$

This allows us to write down the interacting density-density correlations in momentum space

$$< \rho_{sym}(q,n)\rho_{sym}(-q,-n) > \ = \ \frac{4\pi q(qv_F - iw_n)}{\beta\left(4\pi^2\left(q^2v_F^2+w_n^2\right)-q^2v_0^2\right)} \tag{5.27}$$

and

$$< \rho_{sym}(q,n)\rho_{sym}(q,-n) > \ = \ -\frac{2q^2 v_0}{\beta\left(4\pi^2\left(q^2v_F^2+w_n^2\right)-q^2v_0^2\right)} \tag{5.28}$$

The correlations in space-time are calculated using

$$\begin{aligned} < T\rho_{sym}(x,t)\rho_{sym}(x',t') > \ &= \ \frac{1}{L}\sum_{q,n} e^{-iq(x-x')}\ e^{w_n(t-t')} \ < \rho_{sym}(q,n)\rho_{sym}(-q,-n) > \\ &+ \frac{1}{L}\sum_{q,n} e^{-iq(x+x')}\ e^{w_n(t-t')} \ < \rho_{sym}(q,n)\rho_{sym}(q,-n) > \end{aligned} \tag{5.29}$$

and we obtain the most singular part of the correlation function of the symmetric density fields

$$< T\rho_{sym}(x,t)\rho_{sym}(x',t') > \ = \ \frac{\begin{array}{c} v_0\left(csch^2\left(\frac{\pi((t-t')v_h+x+x')}{\beta v_h}\right)+csch^2\left(\frac{\pi(-(t-t')v_h+x+x')}{\beta v_h}\right)\right) \\ +\ 2\pi\left((v_h-v_F)csch^2\left(\frac{\pi((t-t')v_h+x-x')}{\beta v_h}\right)-(v_F+v_h)csch^2\left(\frac{\pi((t-t')v_h-x+x')}{\beta v_h}\right)\right)\end{array}}{8\pi\beta^2 v_h^3} \tag{5.30}$$

where the holon velocity v_h is defined by the relation $v_h^2 = v_F^2 - \frac{v_0^2}{4\pi^2}$. In the noninteracting limit ($v_0 \to 0$) Eq.5.30 reduces to Eq.5.10.

5.4 Correlation function of the antisymmetric density fields

In this section we calculate the correlation function of the antisymmetric densities from the generating functional. The most singular part of the correlation function of the ρ_{asy} fields will involve the bare reflection and transmission coefficients of the point-contact impurity. We obtain the saddle point of the U'_{asy} field by solving Eq.5.21. In momentum and frequency space we have

$$-i\beta\sum_{q'} < \rho_{asy}(q,n)\rho_{asy}(q',-n) >_0 U'_{asy}(-q',n) - \frac{2i}{v_0}(U_{asy}(-q,n)-U'_{asy}(-q,n)) = 0 \tag{5.31}$$

The Fourier transform to momentum-Matsubara frequency space of the noninteracting correlations is

$$< \rho_{asy}(q,n)\rho_{asy}(q',-n) >_0 \ = \ -\frac{(iqv_F)\delta_{q+q',0}}{(\pi\beta v_F)(w_n - iqv_F)} - \frac{(r_1-1)w_n sgn(w_n)}{\pi\beta L(qv_F+iw_n)(q'v_F - iw_n)} \tag{5.32}$$

Therefore we may write Eq.5.31 in the form

$$\frac{-i(-1+r_1)w_n sgn(w_n)}{\pi(qv_F+iw_n)}U'_{H,asy}(n) + \frac{i}{\pi v_F}\frac{iv_F q}{(w_n - iv_F q)}U'_{asy}(q,n) - \frac{2i}{v_0}(U_{asy}(-q,n)-U'_{asy}(-q,n)) = 0 \tag{5.33}$$

where we have introduced the Hilbert transform $U'_{H,asy}(n)$ which is defined as

$$U'_{H,asy}(n) = \frac{1}{L}\sum_{q'} \frac{U'_{asy}(q',n)}{(q'v_F + iw_n)} \tag{5.34}$$

This allows us to obtain the saddle point in terms of the Hilbert transform

$$U'_{asy}(q,n) = \frac{\begin{pmatrix} (r_1-1)v_0 w_n U'_{H,asy}(n) sgn(w_n)(-qv_0+2\pi q v_F+2i\pi w_n) \\ -2\pi(qv_F+iw_n)(qv_0 U_{asy}(-q,n)+2\pi(qv_F-iw_n)U_{asy}(q,n)) \end{pmatrix}}{q^2\left(v_0^2-4\pi^2 v_F^2\right)-4\pi^2 w_n^2} \tag{5.35}$$

The generating functional for the U_{asy} fields is

$$Z[U_{asy}] = e^{-\frac{1}{iv_0}\int_C dt\int dx\ (U_{asy}(-x,t)-U'_{asy}(-x,t))U_{asy}(x,t)} \tag{5.36}$$

which in Fourier space is

$$Z_{asy}[U] = e^{-\frac{1}{v_0\beta L}\sum_{q,n}(U_{asy}(q,n)-U'_{asy}(q,n))U_{asy}(q,-n)} \tag{5.37}$$

Using the definition of the Hilbert transform we may write,

$$U'_{H,asy}(n) = -\sum_q \frac{2\pi(qv_0U_{asy}(-q,n)+2\pi(qv_F-iw_n)U_{asy}(q,n))}{L\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)\left(\frac{(r_1-1)(v_0+2\pi(v_F-v_h))}{2(v_0+2\pi v_F)}+1\right)} \tag{5.38}$$

Hence we have,

$$\begin{aligned} &\ln(Z[U_{asy}]) = \\ &-\Sigma_{q,n}\frac{v_0(2\pi q(qv_F+iw_n)U_{asy}(-q,n)U_{asy}(q,-n))}{(\beta L v_0)\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)}-\Sigma_{q,n}\frac{q^2v_0^2U_{asy}(q,n)U_{asy}(q,-n)}{(\beta Lv_0)\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)} \\ &+\Sigma_{p,q,n}\frac{(r_1-1)v_0w_nsgn(w_n)(qv_0-2\pi qv_F-2i\pi w_n)2\pi(-pv_0+2\pi(pv_F-iw_n))U_{asy}(p,n)U_{asy}(q,-n)}{(\beta L^2v_0)\left(\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)\left(\left(p^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)\left(\frac{(r_1-1)(v_0+2\pi(v_F-v_h))}{2(v_0+2\pi v_F)}+1\right)\right)\right)} \end{aligned} \tag{5.39}$$

This allows us to evaluate the most singular part of the correlation from the relation

$$\begin{aligned} &-\Sigma_{q,n}\frac{v_0(2\pi q(qv_F+iw_n)U_{asy}(-q,n)U_{asy}(q,-n))}{(\beta L v_0)\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)}-\Sigma_{q,n}\frac{q^2v_0^2U_{asy}(q,n)U_{asy}(q,-n)}{(\beta Lv_0)\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)} \\ &+\Sigma_{p,q,n}\frac{(r_1-1)v_0w_nsgn(w_n)(qv_0-2\pi qv_F-2i\pi w_n)2\pi(-pv_0+2\pi(pv_F-iw_n))U_{asy}(p,n)U_{asy}(q,-n)}{(\beta L^2v_0)\left(\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)\left(\left(p^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)\left(\frac{(r_1-1)(v_0+2\pi(v_F-v_h))}{2(v_0+2\pi v_F)}+1\right)\right)\right)} \\ &= \frac{1}{2L}\sum_{p,q,n} < \rho_{asy}(p,n)\rho_{asy}(q,-n) > \ U_{asy}(p,n)U_{asy}(q,-n) \end{aligned} \tag{5.40}$$

We obtain

$$\begin{aligned} < T\ \rho_{asy}(x,t)\rho_{asy}(x',t') > \ &= \ -\frac{2}{v_0\beta L}\Sigma_{q,n}e^{-iq(-x+x')}e^{w_n(t-t')}\frac{v_0(2\pi q(qv_F+iw_n))}{\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)} \\ &-\frac{2}{v_0\beta L}\Sigma_{q,n}e^{-iq(x+x')}e^{w_n(t-t')}\frac{q^2v_0^2}{\left(q^2\left(v_0^2-4\pi^2v_F^2\right)-4\pi^2w_n^2\right)} \\ &+\Sigma_{q,q',n}e^{-iqx}e^{-iq'x'}e^{w_n(t-t')}\frac{2(r_1-1)v_0w_nsgn(w_n)(2\pi(-qv_0+2\pi(qv_F-iw_n)))(q'v_0-2\pi q'v_F-2i\pi w_n)}{(\beta L^2v_0)\left(q'^2\left(-4\pi^2v_h^2\right)-4\pi^2w_n^2\right)\left(\left(q^2\left(-4\pi^2v_h^2\right)-4\pi^2w_n^2\right)\left(\frac{(r_1-1)(v_0+2\pi(v_F-v_h))}{2(v_0+2\pi v_F)}+1\right)\right)} \end{aligned} \tag{5.41}$$

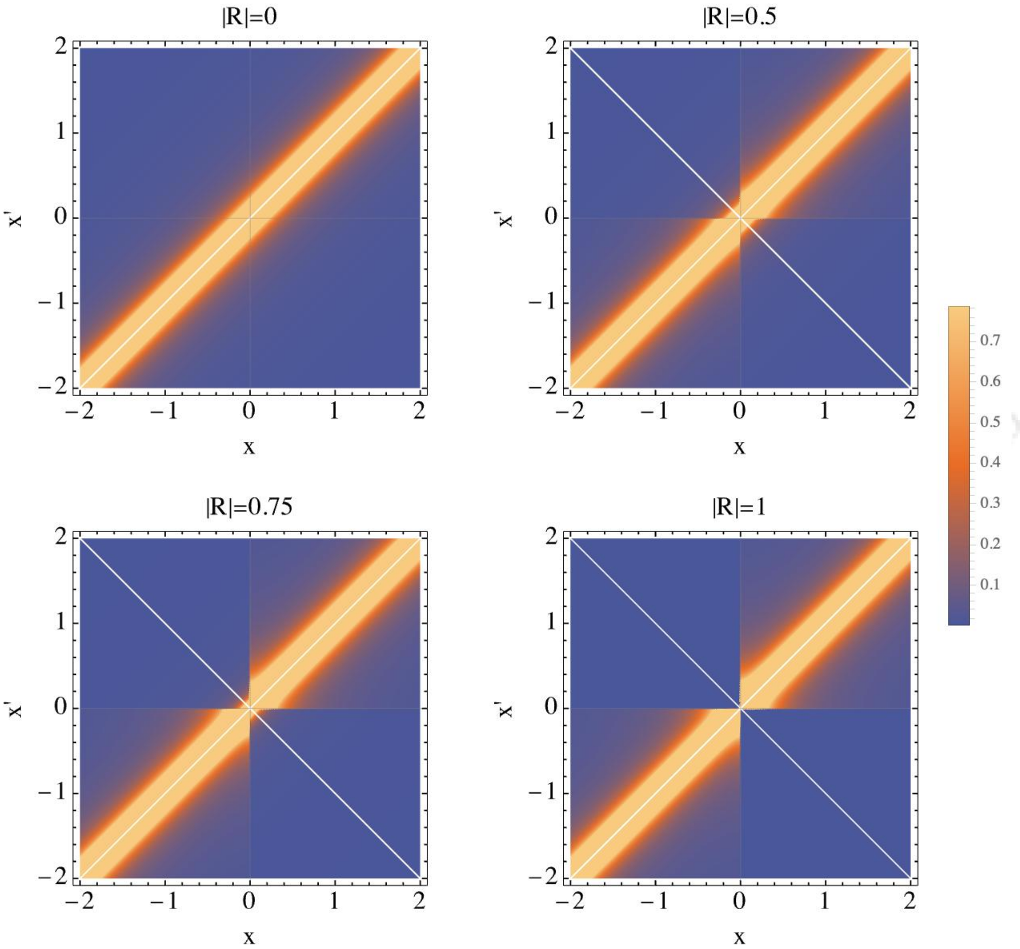

Figure 5.1: Equal time density density correlation function $| < \rho_R(x,t)\rho_R(x',t) > |$ as a function of spatial coordinates x and x' for different values of the reflection amplitude $|R|$ of the point contact impurity. We have taken $v_F\beta = 100$ for the plots.

The correlations in space-time are obtained as

$$
\begin{aligned}
< T\ \rho_{asy}(x,t)\rho_{asy}(x',t') > \ &= \\
&- \frac{v_0\left(csch^2\left(\frac{\pi(\tau v_h+x+x')}{\beta v_h}\right)+csch^2\left(\frac{\pi(-\tau v_h+x+x')}{\beta v_h}\right)\right)}{8\pi\beta^2 v_h^3} - \frac{(v_F-v_h)csch^2\left(\frac{\pi(\tau v_h+x-x')}{\beta v_h}\right)+(v_F+v_h)csch^2\left(\frac{\pi(\tau v_h-x+x')}{\beta v_h}\right)}{4\beta^2 v_h^3} \\
&- \frac{\theta(xx')\left((r_1-1)(v_0+2\pi v_F)(v_0+2\pi(v_h-v_F))(2\pi(v_F+v_h)-v_0)\left(csch^2\left(\frac{\pi(\tau v_h+x+x')}{\beta v_h}\right)+csch^2\left(\frac{\pi(-\tau v_h+x+x')}{\beta v_h}\right)\right)\right)}{16\pi^2\beta^2 v_h^4((r_1+1)v_0+2\pi(r_1 v_F-r_1 v_h+v_F+v_h))} \\
&+ \theta(-xx')\frac{(r_1-1)(v_0+2\pi v_F)\left(-(v_0-2\pi(v_F+v_h))^2 csch^2\left(\frac{\pi(-\tau v_h+x-x')}{\beta v_h}\right)-(v_0+2\pi(v_h-v_F))^2 csch^2\left(\frac{\pi(-\tau v_h-x+x')}{\beta v_h}\right)\right)}{16\pi^2\beta^2 v_h^4((r_1+1)v_0+2\pi(r_1 v_F-r_1 v_h+v_F+v_h))}
\end{aligned}
\tag{5.42}
$$

where $\tau = t - t'$ and $v_h^2 = v_F^2 - \frac{v_0^2}{4\pi^2}$. We have defined $r_1 = \left(1 - \frac{32\Gamma^2 v_F^2}{(\Gamma^2+4v_F^2)^2}\right)$. The point-contact tunneling amplitude is related to the bare reflection and transmission amplitudes of the point-contact impurity [81].

5.5 Compact expression for the density density correlation function

The density-density correlation functions of the right and left moving fermions can be written compactly in terms of the correlations of the symmetric and antisymmetric densities in the following manner,

$$
< T\ \rho_{\chi}(x,t)\rho_{\chi'}(x',t') > \ = \ \frac{1}{4}(< T\ \rho_{sym}(\chi\ x,t)\rho_{sym}(\chi'\ x',t') > +\chi\chi' < T\ \rho_{asy}(\chi\ x,t)\rho_{asy}(\chi'\ x',t') >)
\tag{5.43}
$$

where χ and χ' can take values ± 1 for R and L movers respectively. In Fig.5.1 the equal time density correlations are plotted for various values of the impurity reflection amplitude.

5.6 Comparison with perturbation theory

In this section we compare the results we have obtained using the generating functional method with those obtained using standard fermionic perturbation theory. We expand the most singular parts of the DDCF we have obtained in powers of the interaction parameter v_0 and perform a term by term comparison with S-matrix perturbation expansion of the DDCF retaining only the most singular terms. We explicitly show upto $O(v_0^2)$ that each term in the perturbation expansion matches with the S-matrix expansion and this implicitly means that perturbation expansion of the most singular terms matches upto all orders with standard perturbation theory.

5.6.1 Perturbative comparison of $< \rho_{sym}\rho_{sym} >$

The density-density correlations in presence of interactions is written in terms of the noninteracting ones in the following manner,

$$
< T\ \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) > \ = \ \frac{< T\ S\ \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >_0}{< T\ S >_0}
\tag{5.44}
$$

where $S = e^{-i\int_c dt H_{int}}$ and H_{int} is given by Eq.5.2. The perturbaton expansion in powers of v_0 gives

$$
\begin{aligned}
< T\ \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) > \ &= \ \frac{< T\ S\ \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >_0}{< T\ S >_0} \\
&= \ < T\ \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >_0 -i\int_C dt < T\ H_{int}(t)\rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >_{0,c} \\
&\quad + \frac{(-i)^2}{2}\int_C dt\int_C dt' < T\ H_{int}(t)H_{int}(t')\rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >_{0,c} + ...
\end{aligned}
\tag{5.45}
$$

where $< \;\; >_{0,c}$ denotes the noninteracting connected correlation functions. We retain only the most singular terms and use Eqs.5.11, 5.12 and 5.14 and express the first order term in the perturbation series in the form (note that $< \rho_{sym}\rho_{asy} >_0$ = zero),

$$
\begin{aligned}
&< T\ \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >^{(1)} \\
&= -i\frac{v_0}{4}\int_C dt\int dx\Big(< T\ \rho_{sym}(x,t)\rho_{sym}(x_1,t_1) >_0 < T\ \rho_{sym}(-x,t)\rho_{sym}(x_2,t_2) >_0 \\
&\quad + < T\ \rho_{sym}(x,t)\rho_{sym}(x_2,t_2) >_0 < T\ \rho_{sym}(-x,t)\rho_{sym}(x_1,t_1) >_0 \Big)
\end{aligned} \tag{5.46}
$$

It is more convenient to work in momentum and frequency space,

$$
\begin{aligned}
&<\rho_{sym}(q',n)\rho_{sym}(q'',-n) >^{(1)} \\
&= -\frac{v_0\beta}{4}\Big(\sum_q < \rho_{sym}(q,-n)\rho_{sym}(q',n) >_0 < \rho_{sym}(q,n)\rho_{sym}(q'',-n) >_0 \\
&\quad + \sum_q < \rho_{sym}(q,n)\rho_{sym}(q'',-n) >_0 < \rho_{sym}(q,-n)\rho_{sym}(q',n) >_0 \Big)
\end{aligned} \tag{5.47}
$$

Making use of $< \rho_{sym}(q,n)\rho_{sym}(q',-n) >_0 = -\frac{1}{\pi\beta v_F}\frac{iv_F q}{(w_n - iv_F q)}\delta_{q+q',0}$ we get,

$$
< \rho_{sym}(q',n)\rho_{sym}(q'',-n) >^{(1)} = -\frac{{q'}^2 v_0}{2\left(\pi^2\beta\left({q'}^2 v_F^2 + w_n^2\right)\right)}\ \delta_{q',q''} \tag{5.48}
$$

It is easy to see that the first order term in the expansion of Eq.5.28 in powers of v_0 is equal to the expression in Eq.5.48. The second order term in the expansion in Eq.5.45 is written as

$$
\begin{aligned}
&< T\ \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >^{(2)} \\
&= \tfrac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx' < T\ \rho_{sym}(x,t)\rho_{sym}(-x,t)\rho_{sym}(x',t')\rho_{sym}(-x',t')\rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >_{0,c}
\end{aligned} \tag{5.49}
$$

Evaluating this term in the momentum and frequency space (refer to Appendix K) while retaining only the most singular parts we get,

$$
< \rho_{sym}(q',n)\rho_{sym}(q'',-n) >^{(2)} \ = \ \frac{{q'}^3 v_0^2}{4\pi^3\beta(q'v_F - iw_n)(q'v_F + iw_n)^2}\ \delta_{q'+q'',0} \tag{5.50}
$$

This is precisely equal to the second order term we get on expanding Eq.5.27 in powers of v_0.

5.6.2 Perturbative comparison of $< \rho_{asy}\rho_{asy} >$

Similar to the previous section the standard perturbation expansion of the $< \rho_{asy}\rho_{asy} >$ correlation function in powers of v_0 is,

$$
\begin{aligned}
< T\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2) > &= \frac{< T\ S\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2) >_0}{< T\ S >_0} \\
&= < T\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2) >_0 - i\int_C dt < T\ H_{int}(t)\rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2) >_{0,c} \\
&\quad + \frac{(-i)^2}{2}\int_C dt\int_C dt' < T\ H_{int}(t)H_{int}(t')\rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2) >_{0,c} + \ldots\ldots\ldots
\end{aligned} \tag{5.51}
$$

Retaining only the most singular parts the first order term can be written as (note that $< \rho_{sym}\rho_{asy} >_0$ = zero),

$$
\begin{aligned}
&< T\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2) >^{(1)} \\
&= \frac{iv_0}{4}\int_c dt\int dx\Big(< T\ \rho_{asy}(x,t)\rho_{asy}(x_1,t_1) >_0 < T\ \rho_{asy}(-x,t)\rho_{asy}(x_2,t_2) >_0 \\
&\quad + < T\ \rho_{asy}(x,t)\rho_{asy}(x_2,t_2) >_0 < T\ \rho_{asy}(-x,t)\rho_{asy}(x_1,t_1) >_0 \Big)
\end{aligned} \tag{5.52}
$$

In frequency and momentum space this is

$$
\begin{aligned}
&<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>^{(1)} \\
&= \frac{iv_0}{4}(-i\beta)\Bigg(\sum_q <\rho_{asy}(q,-n)\rho_{asy}(q',n)>_0<\rho_{asy}(q,n)\rho_{asy}(q'',-n)>_0 \\
&\quad + \sum_q <\rho_{asy}(q,n)\rho_{asy}(q'',-n)>_0<\rho_{asy}(q,-n)\rho_{asy}(q',n)>_0\Bigg)
\end{aligned}
\tag{5.53}
$$

Using Eq.5.32 we evaluate this term and obtain

$$
\begin{aligned}
&<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>^{(1)} \\
&= v_0\Bigg(\frac{(r_1-1)w_n sgn(w_n)\big(qv_F^2(q'(2(r_1+1)v_F-r_1v_h+v_h)+i(r_1+1)w_n)+w_n\big((r_1-1)v_hw_n-iq'(r_1+1)v_F^2\big)\big)}{\pi^2\beta L\left(q^2v_h^2+w_n^2\right)\left(q'^2v_h^2+w_n^2\right)(r_1v_F-r_1v_h+v_F+v_h)^2} \\
&\quad + \frac{q^2\delta_{q,q'}}{2\pi^2\beta\left(q^2v_F^2+w_n^2\right)}\Bigg)
\end{aligned}
\tag{5.54}
$$

This is equal to the first order term obtained upon expanding the $<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>$ correlation function that can be inferred from the rhs of Eq.5.41. The second order term in the expansion is

$$
\begin{aligned}
&<T\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2)>^{(2)} \\
&= \tfrac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx' <T\ \rho_{asy}(x,t)\rho_{asy}(-x,t)\rho_{asy}(x',t')\rho_{asy}(-x',t')\rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2)>_{0,c}
\end{aligned}
\tag{5.55}
$$

Retaining the most singular parts and evaluating this term (refer to Appendix L) in momentum and frequency space we get

$$
\begin{aligned}
&<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>^{(2)} \\
&= v_0^2\Bigg(-\frac{(r_1-1)w_n sgn(w_n)\left(\begin{aligned}&q'\big(q''v_F\big(\big(r_1^2-1\big)v_Fv_h+(r_1+1)^2v_F^2-(r_1-1)^2v_h^2\big)+i(r_1-1)v_hw_n(2(r_1+1)v_F-r_1v_h+v_h)\big)\\&\quad+(r_1-1)v_hw_n((r_1+1)w_n-iq''(2(r_1+1)v_F-r_1v_h+v_h))\end{aligned}\right)}{2\pi^3\beta L\big(q'^2v_h^2+w_n^2\big)\big(q''^2v_h^2+w_n^2\big)(r_1v_F-r_1v_h+v_F+v_h)^3} \\
&\quad + \frac{q'^3\delta_{q',-q''}}{4\pi^3\beta(q'v_F-iw_n)(q'v_F+iw_n)^2}\Bigg)
\end{aligned}
\tag{5.56}
$$

This is exactly equal to the second order term obtained upon perturbative expansion of the $<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>$ correlation function (present in the rhs of Eq.5.41) in powers of v_0.

5.7 Two-terminal current in response to a difference in potential between the edges

We use the density correlations we have obtained in the previous sections to calculate the two-terminal conductance (source-drain conductance) both in the absence and presence of the point-contact. The right movers and left movers are separated with a different potential for the right and left moving edges. We consider that the difference in chemical potentials is $\mu_L - \mu_R = -\mu_R = eV$, that is a bias is applied to the right movers and the chemical potential of the left movers is taken to be zero. This means we have

$$
H_{bias} = -eV\int dy\rho_R(y,t) \tag{5.57}
$$

The current is calculated using the definition [3],

$$
j_e(x,t) \;=\; ev_h\nu(\Pi_R(x,t)-\Pi_L(x,t)) \;=\; ev_h(\rho_R(x,t)-\rho_L(x,t)) \tag{5.58}
$$

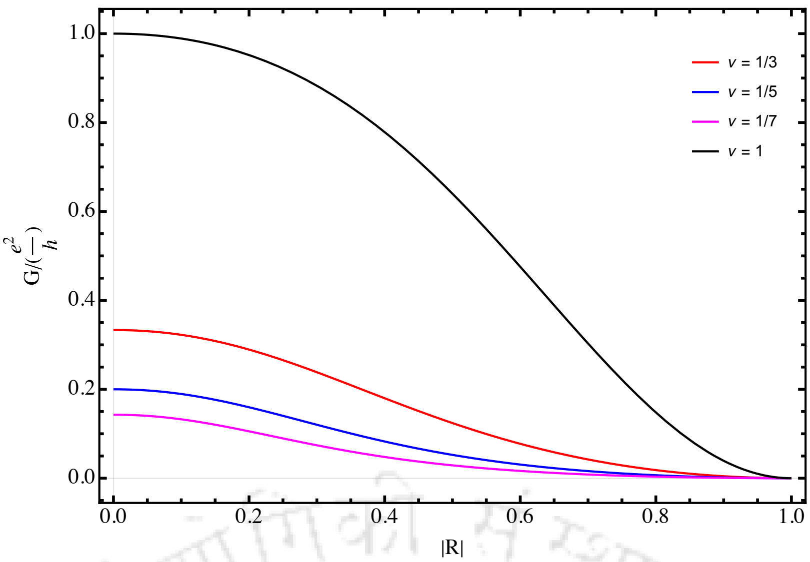

Figure 5.2: Conductance in units of $\frac{e^2}{h}$ for different values of ν in the Laughlin series as a function of the reflection amplitude $|R|$. The black line indicates the noninteracting case $\nu = 1$.

where $\Pi(x,t) = \rho(x,t)/\nu$ is the conjugate momentum of the ϕ fields for fractional quantum Hall edge states with $\nu = 1/m$ where m is an odd integer [152]. The average current is evaluated using

$$< j_e(x,t) > \;=\; ev_h \frac{< T\ S(\rho_R(x,t) - \rho_L(x,t)) >_{V_b=0}}{< TS >_{V_b=0}} \tag{5.59}$$

where $S = e^{-i\int_C dt H_{bias}}$. The current in linear response to the bias maybe computed directly

$$< j_e(x,t) > \;=\; ev_h(ieV)\int_{-\infty}^{\infty} dy \int_0^{-i\beta} dt_1\, (< \rho_R(y,t_1)\rho_R(x,t) > - < \rho_R(y,t_1)\rho_L(x,t) >) \tag{5.60}$$

In the absence of the point-contact ($\Gamma = 0$) we use Eq.5.43 with $r_1 = 1$ and evaluate the current to obtain

$$< j_e > \;=\; \frac{e^2}{h}\frac{v_0 + 2\pi v_F}{2\pi v_h} V \tag{5.61}$$

The interaction parameter v_0 of the Luttinger liquid is related to the bulk filling factor through the equation $v_0 = \frac{\pi(2(\nu^2-1))v_F}{\nu^2+1}$. This leads to the correct formula for the Hall conductance

$$G = \nu\frac{e^2}{h} \tag{5.62}$$

In the presence of a point-contact, there appears an additional term quantifying the loss of current due to backscattering at the point-contact. The linear response conductance in this case is given by

$$G \;=\; \frac{e^2\nu\left(\frac{r_1-1}{\nu+(\nu-1)r_1+1}+1\right)}{h} \tag{5.63}$$

This expression is valid only at zero temperature ($T = 0$), but the impurity is treated exactly to all orders in the tunneling amplitude.

5.7.1 Limiting case checks

No interactions: In the absence of interactions $\nu = 1$ the conductance reduces to

$$G = \frac{e^2}{h}|T|^2 \tag{5.64}$$

which is the Landauer's formula [153] for conductance. Here we have used $r_1 = \left(1 - \frac{32\Gamma^2 v_F^2}{\left(\Gamma^2+4v_F^2\right)^2}\right)$ and the relation between the tunneling amplitude Γ and the bare reflection and transmission amplitudes, $|R|$ and $|T|$ respectively [81],

$$|\frac{\Gamma}{v_F}| = \frac{2|R|}{1+|T|} \tag{5.65}$$

No impurity: In the absence of the point-contact we set $r_1 = 1$ and we get back $G = \nu \frac{e^2}{h}$

Half-line limit: In the case of a half-line $|R| = 1$ and we get

$$G = 0 \tag{5.66}$$

irrespective of the interactions. In Fig.5.2 the linear conductance in units of the conductance quantum is plotted as a function of the reflection amplitude of the point-contact impurity.

5.8 Summary

In this chapter, the most singular part of the density density correlation functions (DDCF) of chiral Luttinger liquids at the edges of fractional quantum Hall systems coupled through an interaction mediated by the bulk in presence of a point-contact impurity has been rigorously derived and shown to be expressible in terms of simple functions of position and time. The expressions for the DDCF have simple second order poles and involve the scale independent bare reflection and transmission amplitudes of a single fermion in the presence of a localised impurity potential. Our results are compared with standard perturbation theory and are shown to match term by term. The standard linear response Hall conductance is also obtained from the density correlations and is a crucial validation of our results. For the inhomogeneous system under consideration, the idea is to neglect all higher order (connected) moments of the density operator beyond the second order. The reason being that all odd moments of the density vanish identically whereas all the higher order even moments are less singular than the second moment as shown in [52]. This result is a crucial input to unconventional bosonization techniques that retrieve the most singular part of the two-point correlations for arbitrary impurity strengths and interactions. On the other hand conventional bosonization methods are only able to obtain analytical expressions for the Green's functions in the homogeneous limit ($|R| = 0$) or the half-line limit ($|R| = 1$) where higher order moments of the density beyond the second moment vanish identically. In order to bosonize the interacting nonequilibrium problem we need to obtain themost singular part of the nonequilibrium interacting DDCF with an arbitrary bias. This is necessary to recover the scaling function of the tunneling current. It is an immensely arduous task to obtain the full nonequilibrium Green's functions in presence of interactions and arbitrary impurities and is beyond the scope of the current thesis. But the equilibrium DDCF we have obtained can be used in conjunction with the unconventional bosonization method to extract the tunneling density of states (TDOS) at the point-contact. Formally this can be used to study the tunneling transport properties. In Fig.5.3 a flowchart diagram giving an overview of the steps involved in calculating the correlation functions with interactions using the proposed unconventional bosonization method is shown. In the next chapter we obtain the correct TDOS power-law exponents using the unconventional bosonization method.

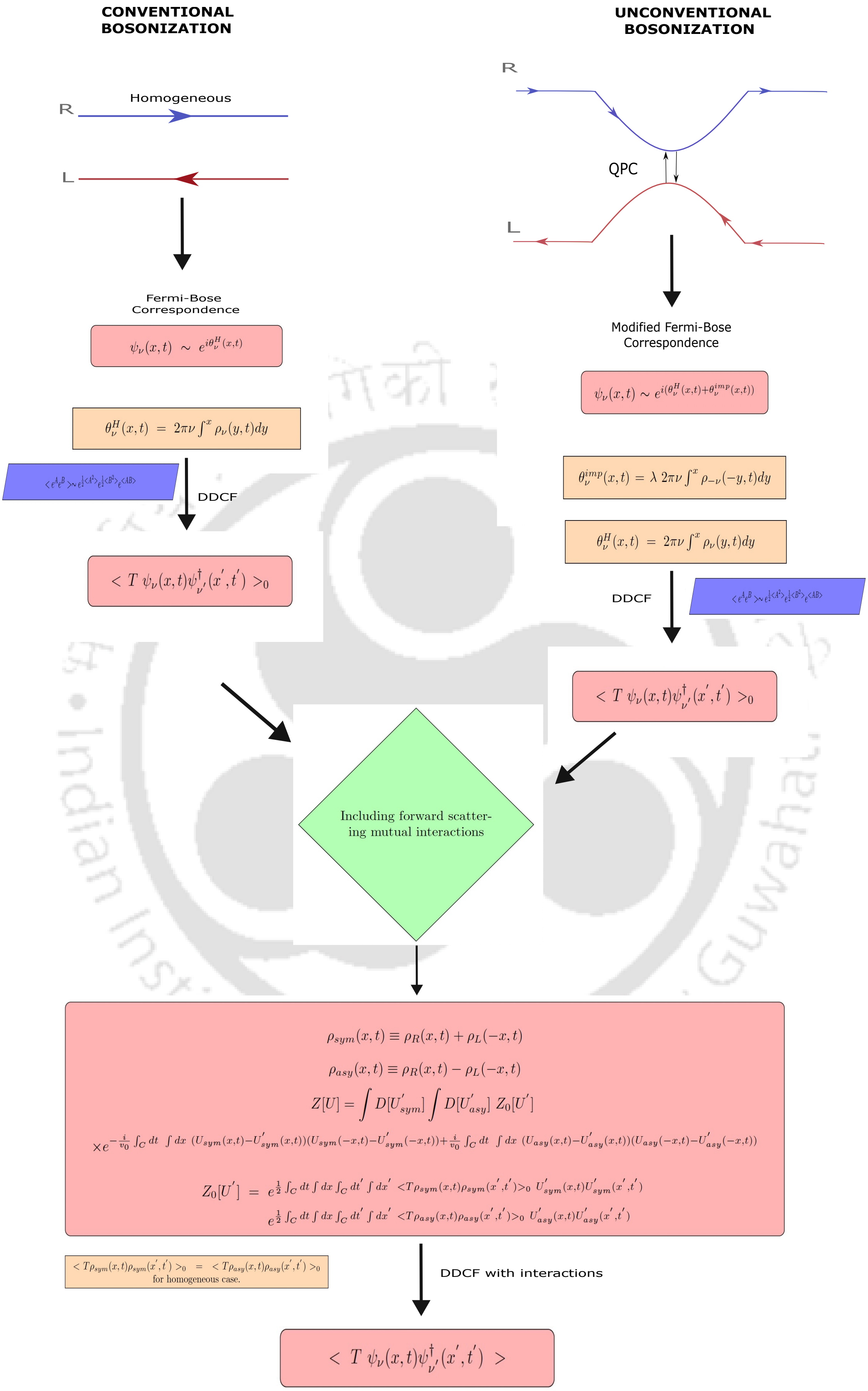

Figure 5.3: Flowchart describing the steps involved in obtaining the Green functions using the proposed unconventional bosonization method.

Chapter 6

Tunneling density of states of fractional quantum Hall edges: an unconventional bosonization approach

6.1 Introduction

In one dimension, electron interactions have a drastic effect and a Fermi liquid description of such a system fails [154, 13, 155]. Instead it was found that the 1D interacting system exhibits a Luttinger liquid phase [36]. The Luttinger liquid exhibits only collective low energy excitations moving to the right and left. It was shown by Wen [156] that the fractional quantum Hall effect (FQHE) [157] edge excitations could be described in terms of 1D interacting electrons. In fact it was shown that the edge states of a FQHE bulk with filling factor $\nu = 1/m$ with m an odd integer (termed the Laughlin series [158]) can be described as chiral Luttinger liquids. The Laughlin FQHE states have only a single edge excitation and the top edge or bottom edge can be considered equivalent to the right moving or left moving half of a conventional Luttinger liquid. The FQHE edge states are the most experimentally accessible systems where one encounters Luttinger liquid physics. The FQHE edge states are immune to impurity backscattering and the low energy properties cannot be inferred by probing the bulk. But inter edge tunneling can be made possible by bringing the opposite edges of the sample together by forming a point contact. This is achieved by forming a constriction in the bulk through the application of a gate voltage. The low energy physics of the edge states can be probed by studying the tunneling transport properties through the point contact. The point contact in a $\nu = 1/m$ Hall fluid is isomorphic to a point impurity in a conventional one-channel Luttinger liquid. The transport properties through the point contact depends on the tunneling density of states (TDOS) of each edge. Power law suppression of TDOS is characteristic of a Luttinger liquid.
The proper way to deal with interactions in one dimensional systems is using bosonization. This involves replacing fermions with bosonic degrees of freedom. But conventional or standard bosonization is ill suited to be applied to systems with impurity backscattering. Therefore the common method to study transport through the point contact is by a perturbative treatment of the impurity [152]. As an alternative, we can treat the impurity exactly by modifying the standard Fermi-Bose correspondence itself to take into account the impurity backscattering. This radical new method involves treating the Fermi-Bose correspondence as a mnemonic to obtain the correct correlation functions rather than a strict operator identity. This allows one to compute the most singular parts of the interacting Green's functions in presence of an impurity in one dimension and this technique has proven to be very successful in studying Luttinger liquids with impurities [52, 132, 45, 133, 134, 135, 136]. In the present chapter based on our preprint [159] we extend this idea to FQHE edge states with a point contact constriction at the origin. We show that the universal power law in the TDOS is recovered using the unconventional bosonization procedure in the presence of a point contact in both the cases of electron tunneling as well as quasiparticle tunneling.

6.2 Model Hamiltonian

We consider a system of two chiral Luttinger liquids with a point-contact tunnel junction at the origin. The non-interacting part of the Hamiltonian is

$$H_0 = \sum_p (v_F p + eV_b) c^\dagger_{p,R} c_{p,R} + \sum_p (-v_F p) c^\dagger_{p,L} c_{p,L} + \frac{\Gamma}{L}(c^\dagger_{.,R} c_{.,L} + c^\dagger_{.,L} c_{.,R}) \tag{6.1}$$

where R and L label the right and left moving chiral spinless modes and V_b is the bias voltage applied to the right movers. The fermion creation and annihilation operators in momentum space are $c^\dagger_p$ and c_p respectively and we use the notation $c^\dagger_{.,R} = \sum_p c^\dagger_{p,R}$. We consider a symmetric point-contact junction with tunneling amplitude Γ and the L that does not appear in the subscript is the system size. We define the bias potential as $\mu_L - \mu_R = -\mu_R = -eV_b = eV$ following the same convention as in [131]. The coupling at the point contact is not treated as a weak perturbation in our approach but is treated exactly as an impurity of arbitrary strength. The right mover and left mover fermions are coupled through a density-density interaction governed by v_0,

$$H_{int} = \int dx \; v_0 \; \rho_R(x,t) \rho_L(x,t) \tag{6.2}$$

This system of chiral Luttinger liquids is a good model for the edge states of fractional quantum Hall systems, at least the ones with a single edge mode as in the Laughlin series with filling fraction $\nu = 1/m$ with m an odd integer. We will show how this connection is made in the subsequent sections. The full Hamiltonian of interest is

$$H \;=\; H_0 + H_{int} \tag{6.3}$$

The non-equilibrium Green functions for H_0 were obtained in [81] and are discussed in Chapter 2 of this thesis.

6.3 Bosonization of the noninteracting edge states with interedge tunneling

In this section we show that the Integer Quantum Hall Effect (IQHE) edge can be bosonized in the presence of backscattering due to a point contact that brings the opposite edges into close proximity. Although the bosonized description is not necessary for the IQHE edges, it is of use since it can be generalized to the case of Fractional Quantum Hall Effect (FQHE) edges where fermion interactions need to be taken into account. So in this section our Hamiltonian is $H = H_0$. Let us consider an IQHE edge with only one edge mode i.e. one filled Landau level. One can linearize the low energy states near the Fermi momentum and it is possible to express the low energy degrees of freedom in terms of bosons. The Fermi surface consists of two Fermi points at $+k_F$ and $-k_F$. We are dealing with chiral fermions here i.e. right movers (k_F) and left movers ($-k_F$). Let us consider the edge with right mover fermions (i.e. with momentum k_F). We may define the edge density (fluctuation) operator

$$\rho(x) =: \psi^\dagger(x)\psi(x) : \tag{6.4}$$

It is normal ordered with respect to the filled Fermi sea. The Fourier transform of the density is

$$\rho_R(x) = \frac{1}{L}\sum_q \rho^R_q e^{-iqx} \tag{6.5}$$

where $\rho^R_q = \sum_k : c^\dagger_{k,R} c_{k+q,R} :$. The commutator of the densities can be shown to be

$$[\rho^R_q, \rho^R_{q'}] = \frac{qL}{2\pi}\delta_{q+q',0} \tag{6.6}$$

Simlarly for the edge with left movers it is $[\rho^L_q, \rho^L_{q'}] = -\frac{qL}{2\pi}\delta_{q+q',0}$. The commutator is zero for densities of opposite chiralities. Hence the commutation relation between the density operators is bosonic like. Thus we can construct bosonic annihilation and creation operators of the form

$$b_q = \sqrt{\frac{2\pi}{L|q|}} \sum_{\chi} \theta(\chi q)\rho_{\chi}(q) \tag{6.7}$$

and

$$b_q^{\dagger} = \sqrt{\frac{2\pi}{L|q|}} \sum_{\chi} \theta(\chi q)\rho_{\chi}(-q) \tag{6.8}$$

where χ can take values R $(+1)$ and L (-1). These operators satisfy the usual bosonic commutation relations $[b_q, b_{q'}] = 0$, $[b^{\dagger}_q, b^{\dagger}_{q'}] = 0$ and $[b_q, b^{\dagger}_{q'}] = \delta_{q,q'}$. So we may write the right mover density operator in terms of these bosonic operators as

$$\rho_R(q) = \sqrt{\frac{Lq}{2\pi}} b_q \tag{6.9}$$

or

$$\rho_R(-q) = \sqrt{\frac{Lq}{2\pi}} b^{\dagger}_q \tag{6.10}$$

The commutator of the densities in real space is

$$[\rho_R(x), \rho_R(x')] = \sum_q \frac{q}{2\pi L} e^{-iq(x-x')} = \frac{i}{2\pi}\delta'(x-x') \tag{6.11}$$

It is convenient to introduce a new field ϕ defined by

$$\phi_{\chi}(x,t) = 2\pi \int^x dy \quad \rho_{\chi}(y,t) \tag{6.12}$$

where χ can be R or L. It can be shown that the ϕ operators obey a Kac Moody commutation relation

$$\begin{aligned}[\phi_R(x,t), \phi_R(x',t)] &= -[\phi_L(x,t), \phi_L(x',t)] \\ &= -i\pi \; sgn(x-x')\end{aligned} \tag{6.13}$$

In conventional bosonization the chiral Fermi fields are expressed as [152]

$$\psi_{\chi}(x,t) = e^{2\pi i \; \chi \int^x \rho_{\chi}(y,t)dy} \tag{6.14}$$

where $\chi = \pm 1$ for R or L respectively. The Fermi-Bose correspondence in Eq.6.14 is proved by resorting to the basis states of homogeneous (translation-invariant) systems. Edge states are generally insensitive to disorder. Backscattering becomes possible only when the edges of opposite chirality are brought close together as in a point contact and inter-edge tunneling can take place. When we have an impurity in the system like the point-contact tunnel junction present in the model under consideration, the number of right movers and left movers are not separately conserved, hence bosonization using the conventional approach is not suitable.

We propose a method to handle this issue by modifying the standard Fermi-Bose correspondence in Eq.6.14 to include the effects of backscattering of fermions from the point contact impurity. The modified Fermi-Bose correspondence takes the form,

$$\psi_{\chi}(x,t) = e^{2\pi i\chi \int^x (\rho_{\chi}(y,t) + \lambda \; \rho_{-\chi}(-y,t))dy} \tag{6.15}$$

where the value of λ which is either 0 or 1 dictates the absence or presence of the additional $\rho_{-\nu}(-y,t)$ term respectively. This is similar to the non-chiral bosonization technique (NCBT) that has been used to obtain the most singular parts of the Green's functions of interacting Luttinger liquids with impurities [52, 132]. It has been shown that the series expansion of the NCBT Green's functions in powers of fermion-fermion interaction strength matches term by term with standard fermionic perturbation theory (most singular terms). It has also been shown that the NCBT Green's functions with forward scattering between fermions satisfy the (most singular parts of the) exact Schwinger-Dyson equations [45]. These results are a clear indication that this formalism is not mere phenomenology. In [137] it is shown how one can construct a bosonization ansatz using Eq.4.16 and reproduce the nonequilibrium Green functions for this noninteracting problem. The two point correlations are recovered using

$$< T\ \psi^{\dagger}_{\chi}(x,t)\psi_{\chi}(x^{'},t^{'}) > \ \sim \ < e^{-2\pi i\chi\int^{x}(\rho_{\chi}(y,t)+\rho_{-\chi}(-y,t))dy}\ e^{2\pi i\chi\int^{x^{'}}\rho_{\chi}(y^{'},t^{'})dy^{'}} > \tag{6.16}$$

or equivalently

$$< T\ \psi^{\dagger}_{\chi}(x,t)\psi_{\chi}(x^{'},t^{'}) > \ \sim \ < e^{-2\pi i\chi\int^{x}\rho_{\chi}(y,t)dy}\ e^{2\pi i\chi\int^{x^{'}}(\rho_{\chi}(y^{'},t^{'})+\rho_{-\chi}(-y^{'},t^{'}))dy^{'}} > \tag{6.17}$$

The choice of taking $\lambda = 1$ for both the Fermi operators while evaluating the two-point correlations has been shown to be invalid as it doesn't obey the point-splitting constraints (as shown in Eq.23 of [45]). In a homogeneous system all but the lowest nontrivial moment (second moment) of the density vanish identically. In a system with impurity all odd moments of the density vanish identically but none of the even moments do. We make the crucial assertion that dropping all but the second moment in an inhomogeneous system amounts to studying the most singular parts of the Green functions (the proof of which is shown in [52]). Hence the Green functions are evaluated using the truncated version of the cumulant expansion

$$< e^{A}e^{B} > \sim e^{\frac{1}{2}<A^2>}e^{\frac{1}{2}<B^2>}e^{<AB>} \tag{6.18}$$

The piecewise constant prefactors are fixed by comparing with the exact solution obtained in [81].

6.4 Bosonizing the FQHE edge states with interedge tunneling

A two dimensional electron system at low temperatures subject to a perpendicular magnetic field gives rise to some of the most important phenomena in quantum condensed matter physics namely the integer quantum Hall effect (IQHE) and the fractional quantum Hall effect (FQHE). The FQHE is an example of a collective quantum state of matter that can be attributed to strong electronic correlations in the system. A perpendicular magnetic field applied to the 2D electron fluid leads to the creation of degenerate discrete quantum states known as Landau levels. The filling factor is an important characteristic parameter of the system and is defined as the ratio of the number of electrons to the degeneracy (number of available states) of each Landau level. At low temperatures and low amount of disorder, the FQHE leads to the condensation of electrons into an incompressible quantum fluid state formed at specific filling factors ν. Typically the FQHE states are more pronounced at very high perpendicular magnetic fields when the lowest Landau level is fractionally filled with electrons. The striking experimental signature of this collective state of matter is the observation of quantized plateaus of the Hall resistance,

$$R_H = \frac{h}{\nu e^2} \tag{6.19}$$

where h is Planck's constant, e is the magnitude of the electron charge and ν is the filling factor which is a fraction. These phases are generally stable for filling fractions with an odd denominator. The most robust FQHE states occur when the lowest Landau level is fractionally filled with filling factors $\nu = \frac{1}{3}$ and $\nu = \frac{1}{5}$. These states are well described by Laughlin's [151] theory in terms of trial wavefunctions that belong to the Hilbert space of the lowest Landau level. The bulk quantum Hall states exhibit gapped excitations but the edge states that live on the boundary of the two dimensional system behave differently and exhibit gapless modes that can be effectively described by a 1D chiral Luttinger liquid model. The nontrivial transport properties of quantum Hall states come from these gapless edge excitations. A FQHE fluid with filling factor $\nu = 1/m$ with m an odd integer (Laughlin series) has only a single edge mode. The right (left) edge state is equivalent to the right (left) half of a conventional Luttinger liquid. A point contact constriction in the fluid is isomorphic to an impurity barrier in a 1D Luttinger liquid. It follows from Wen's [156, 160] hydrodynamic approach to describe the low-energy physics of the fractional quantum Hall edge states, that the edge density operators have the following commutation relation

$$[\rho_q^{\chi}, \rho_{q'}^{\chi}] = \chi\nu \frac{Lq}{2\pi}\delta_{q+q',0} \tag{6.20}$$

where ν is the filling factor. This is pretty much same as the corresponding relation for the ordinary Luttinger liquid except for the additional factor of ν. Hence it follows that the Kac Moody algebra of the ϕ fields picks up an extra ν factor.

$$\begin{aligned}[\phi_R(x,t), \phi_R(x^{'},t)] &= -[\phi_L(x,t), \phi_L(x^{'},t)] \\ &= -i\ \pi\ \nu\ sgn(x - x^{'})\end{aligned} \tag{6.21}$$

So the operator for an edge excitation is written down with the following Fermi-Bose correspondence

$$\psi_{\chi}(x,t) = e^{i\chi\Phi/\nu} = e^{2\pi i\ \chi\frac{1}{\nu}\int^{x}\rho_{\chi}(y,t)dy} \tag{6.22}$$

This operator generally has fractional statistics and is not exactly fermionic but for the special case of $\nu = 1/m$ with m odd this excitation is fermionic with charge e and it represents an edge electron. In order to consider the effect of backscattering from a point contact between the edges we modify the Fermi-Bose correspondence as

$$\psi_{\chi}(x,t) = e^{2\pi i\chi\frac{1}{\nu}\int^{x}(\rho_{\chi}(y,t)+\lambda\ \rho_{-\chi}(-y,t))dy} \tag{6.23}$$

The idea is that Eq.6.23 should be seen as a mnemonic that gives the most singular parts of the Green's functions for FQHE edge states with a point contact at the origin. We need to determine the density-density correlation functions in presence of interactions before we can evaluate the two-point functions. This calculation was done in [150] and is discussed in chapter 5 of this thesis.

6.5 Tunneling density of states of FQHE edge states

Power-law suppression of the tunneling density of states (TDOS) at zero energy is characteristic of Luttinger liquid physics. In the case of FQHE edge states one can consider a case of electron tunneling through the point-contact or there can also be a situation where the fractionally charged quasiparticles tunnel through the point-contact as explained below. The TDOS at the point-contact shows enhancement at low energies for quasiparticle tunneling rather than being suppressed. We recover the known results for the TDOS exponents using the proposed unconventional bosonization method.

6.5.1 Electron tunneling

A gate voltage can be used to control the point contact electrostatically. Let us first consider the case where the point contact is completely pinched off such that now the tunneling between the edge states does not happen through the bulk quantum Hall fluid and the particles that are transferred between the edges during the tunneling process are actual electrons of charge e (see Fig.6.1). We use the unconventional bosonization ansatz discussed in chapter 4 in conjunction with the interacting density density correlations obtained in chapter 5 to calculate the tunneling density of states for the right-moving edge. For the case of electron tunneling the Fermi-Bose correspondence to be used is that given in Eq.6.23, modified to include impurity backscattering. The RR Green function written using the unconventional bosonization ansatz, apart from prefactors will involve terms like

$$< T\psi_R^{\dagger}(x,t)\psi_R(x',t') > \ \sim\ < e^{-2\pi i\frac{1}{\nu}\int^{x}\rho_R(y,t)dy} e^{2\pi i\frac{1}{\nu}\int^{x'}(\rho_R(y',t')+\rho_L(-y',t'))dy'} > \tag{6.24}$$

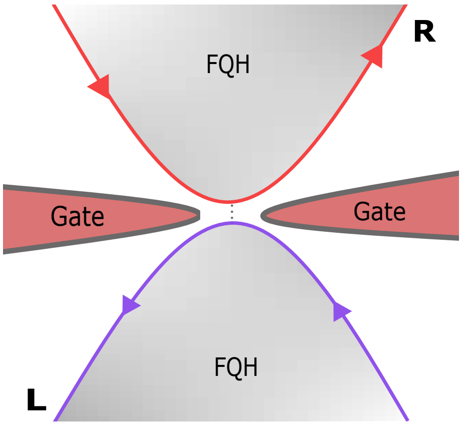

Figure 6.1: Schematic diagram of a completely pinched off point-contact geometry for electron tunneling. The point contact is controlled by electrostatic gate voltage.

The expectation value on the RHS is evaluated using a variant of the Baker-Campbell-Hausdorff formula,

$$
\begin{aligned}
< e^{-2\pi i \int^{x} \rho_R(y,t)dy} e^{2\pi i \int^{x'} (\rho_R(y',t')+\rho_L(-y',t'))dy'} > \\
= e^{\frac{1}{2}(2\pi i)^2 \frac{1}{\nu^2} \int^{x} dy \int^{x} dy' <\rho_R(y,t)\rho_R(y',t)>} \\
e^{\frac{1}{2}(2\pi i)^2 \frac{1}{\nu^2} \int^{x'} dy \int^{x'} dy' <(\rho_R(y,t')+\rho_L(-y,t'))(\rho_R(y',t')+\rho_L(-y',t')>} \\
e^{-(2\pi i)^2 \frac{1}{\nu^2} \int^{x} dy \int^{x'} dy' <\rho_R(y,t)(\rho_R(y',t')+\rho_L(-y',t')>}
\end{aligned}
\tag{6.25}
$$

Using the expression for the density-density correlations in Eq. 5.43 of chapter 5 we calculate the tunneling density of states (TDOS) at the point contact for the right-moving edge at zero temperature and obtain

$$
< T\psi_R^{\dagger}(0,t)\psi_R(0,t') > \;\sim\; \frac{1}{(t-t')^{\frac{1}{\nu^2}(\frac{v_0+2\pi v_F}{2\pi v_h})}}
\tag{6.26}
$$

The filling factor ν is related to the interaction parameter in order to make a connection between the Luttinger liquid paradigm and FQHE edge states [161]. So we have

$$
\nu = \sqrt{\frac{\frac{v_0}{2\pi v_F}+1}{1-\frac{v_0}{2\pi v_F}}}
\tag{6.27}
$$

From this we obtain the relation $\frac{v_0}{\pi v_F} = \frac{2(\nu^2-1)}{\nu^2+1}$. It is easy to check that the exponent turns out to be

$$
< T\psi_R^{\dagger}(0,t)\psi_R(0,t') > \;\sim\; \frac{1}{(t-t')^{\frac{1}{\nu}}}
\tag{6.28}
$$

After a Fourier transform the result for the TDOS at zero temperature may be written in general as

$$
D_R(\omega) \sim |\omega|^{m-1}
\tag{6.29}
$$

where $\nu = 1/m$ with m an odd integer. This is a power-law with universal exponent as expected for Luttinger liquid behaviour [162, 152]. For $\nu = 1$ that is $m = 1$ (IQHE edge state) the TDOS is a constant at the Fermi energy. For the most robust fractional quantum Hall state with $\nu = 1/3$ ($m = 3$) [158] the electron tunneling density of states exponent has a value of 2, in agreement with previous results [162]. A similar calculation will give the same exponent for the TDOS of the left-moving edge as well.

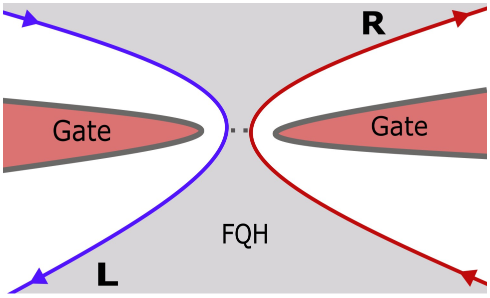

Figure 6.2: Schematic diagram of a point-contact geometry for quasiparticle tunneling through the bulk quantum Hall fluid. The point contact is controlled by electrostatic gate voltage.

6.5.2 Quasiparticle tunneling

Let us consider the case where the point-contact is not pinched off. In this case the particles can tunnel between the top edge and the bottom edge through the bulk quantum Hall fluid (see Fig.6.2). Therefore, it is the Laughlin quasiparticles with charge e/m that tunnel between edges. In this case the operators we consider are for charge e/m edge excitations and the Fermi-Bose correspondence to be considered is

$$\psi_\chi(x,t) = e^{i\chi\Phi} = e^{2\pi i\ \chi \int^x \rho_\chi(y,t)dy} \tag{6.30}$$

These particles show fractional statistics, acquiring a phase factor $e^{\pm i\nu\pi}$ under exchange. The modified Fermi-Bose correspondence we consider due to the presence of the point contact is

$$\psi_\chi(x,t) = e^{2\pi i\chi \int^x (\rho_\chi(y,t)+\lambda\ \rho_{-\chi}(-y,t))dy} \tag{6.31}$$

The two-point functions are evaluated in the same manner but without the $\frac{1}{\nu^2}$ factor appearing in the exponent. The TDOS at the point-contact for the right-moving edge is obtained as

$$< T\psi_R^\dagger(0,t)\psi_R(0,t') > \ \sim \ \frac{1}{(t-t')^{\left(\frac{v_0+2\pi v_F}{2\pi v_h}\right)}} \ = \ \frac{1}{(t-t')^\nu} \tag{6.32}$$

Upon a Fourier transform, the zero temperature TDOS takes the form

$$D_R(\omega) \sim |\omega|^{\frac{1}{m}-1} \tag{6.33}$$

Unlike the case of electron tunneling the TDOS at zero energy is enhanced for the case of quasiparticle tunneling [152]. The TDOS for the left-moving edge can be calculated in a similar manner to obtain the same exponent. The tunneling density of states is a constant at zero energy (Fermi energy) for the integer quantum Hall effect (IQHE) at $\nu = 1$.

6.6 Tunneling through a point-contact

Once we know the low-energy behaviour of the tunneling density of states (TDOS), this can be used to calculate the well known standard result of the two-terminal conductance through the point contact in the perturbative regime for small tunneling amplitude. For a voltage V across the junction, the tunneling rate to leading order can be obtained using Fermi's Golden rule and is expressed in terms of the TDOS as

$$I_{tun} = \frac{e\Gamma^2}{2\pi}\int dE\ \left(D_L^>(E)D_R^<(E-eV) - D_L^<(E-eV)D_R^>(E)\right) \tag{6.34}$$

where $D_a^>$ and $D_a^<$ are local tunneling in and tunneling out density of states for the edge modes. They are related to each other by $D^>(E) = D^<(-E)$. The electron tunneling density of states vanishes at zero energy and this gives rise to a non-Ohmic I-V characteristic of the form

$$I_{tun} \sim \Gamma^2 V^{2m-1} \tag{6.35}$$

At non-zero temperature the tunneling conductance vanishes as power-law for low temperatures

$$G \propto \Gamma^2 T^{2m-2} \tag{6.36}$$

This is a consequence of the electrons at the edge of a Laughlin FQHE fluid forming a Luttinger liquid phase. For the case of quasiparticle tunneling, the tunneling current and the conductance is of the form,

$$I_{tun} \sim \Gamma^2 V^{\frac{2}{m}-1} \tag{6.37}$$

and

$$G \propto \Gamma^2 T^{\frac{2}{m}-2} \tag{6.38}$$

6.7 Most singular part of the action

The densities ρ_{sym}, ρ_{asy} are defined such that they are decoupled, $< \rho_{sym}(x,t)\rho_{asy}(x^{'},t^{'}) >= 0$ even when there is backscattering at the impurity. The full action can be written as,

$$S = S_{sym}[\rho_{sym}] + S_{asy}[\rho_{asy}]$$

The crucial claim (proved in [52]) is that the generating function of the current and density include only the most singular quadratic terms. We define,

$$Z_0[U_{sym}, U_{asy}] = \int D[\rho_{sym}]D[\rho_{asy}] \;\; e^{iS[\rho_{sym},\rho_{asy}]} \; e^{\int dx \int_c dt \;\; \rho_{asy}(x,t)U_{asy}(x,t) + \int dx \int_c dt \;\; \rho_{sym}(x,t)U_{sym}(x,t)}$$

Given that we are only including the most singular parts of the correlation functions (all higher odd moments anyway vanish identically, but the even ones are less singular than the leading quadratic terms included), the form of Z_0 is the exponential of a quadratic form in ρ_{sym}, ρ_{asy}.

$$Z_{0,most-sing}[U_{sym}, U_{asy}] \;\; = \;\; e^{\frac{1}{2}\int dx \int dx^{'} \int dt \int dt^{'} \;\; \sum_{a=asy,sym} \;\; <T \;\; \rho_a(x,t)\rho_a(x^{'},t^{'})>_0 \;\; U_a(x,t)U_a(x^{'},t^{'})}$$

These non-interacting density correlations viz. $< T \;\rho_a(x,t)\rho_a(x^{'},t^{'}) >_0$ maybe derived explicitly using elementary Fermi algebra and are written down below:

$$< \rho_{sym}(x,t)\rho_{sym}(x^{'},t^{'}) >_0 \;\; = \;\; 2\left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x - x^{'} - v_F(t - t^{'})))}\right]^2 \tag{6.39}$$

$$< \rho_{sym}(x,t)\rho_{asy}(x^{'},t^{'}) >_0 \;\; = \;\; 0 \tag{6.40}$$

$$\begin{aligned} < \rho_{asy}(x,t)\rho_{asy}(x^{'},t^{'}) >_0 \;\; = \;\; & \frac{-((4v_F^2 + \Gamma^2)^2 - 32v_F^2\Gamma^2\theta(x))((4v_F^2 + \Gamma^2)^2 - 32v_F^2\Gamma^2\theta(x^{'}))}{2v_F^2\beta^2(4v_F^2 + \Gamma^2)^4 \sinh(\frac{\pi}{\beta v_F}(x - x^{'} - v_F(t - t^{'})))^2} \\ & - \frac{64(-4v_F^3\Gamma + v_F\Gamma^3)^2 \cos(\frac{-eV_b(x - x^{'} - v_F(t - t^{'}))}{v_F}) \;\theta(x)\theta(x^{'})}{2v_F^2\beta^2(4v_F^2 + \Gamma^2)^4 \sinh(\frac{\pi}{\beta v_F}(x - x^{'} - v_F(t - t^{'})))^2} \end{aligned} \tag{6.41}$$

This means the most singular part of the action is spatially non-local but quadratic in the boson terms which we may formally write,

$$\begin{aligned} & e^{iS_{most-sing}[\rho_{sym},\rho_{asy}]} \;\; = \\ \int D[U_{sym}] \int D[U_{asy}] \;\; & e^{\frac{1}{2}\int dx \int dx^{'} \int dt \int dt^{'} \;\; \sum_{a=asy,sym} \;\; <T \;\; \rho_a(x,t)\rho_a(x^{'},t^{'})>_0 \;\; U_a(x,t)U_a(x^{'},t^{'})} \\ & \times \; e^{-\int dx \int_c dt \;\; \rho_{asy}(x,t)U_{asy}(x,t) - \int dx \int_c dt \;\; \rho_{sym}(x,t)U_{sym}(x,t)} \end{aligned}$$

It is this action we are (indirectly) using in lieu of the full Sine-Gordon action. While the full action correctly captures all aspects of the model, it is not soluble. The effective quadratic (albeit spatially non-local) action above only gives the most singular parts of the correlation functions of the theory provided in addition we also modify the Fermi-Bose correspondence to shoulder part of the burden of capturing the effect of back-scattering at the impurity in the manner we have described in the manuscript. Thus the claim is

$$[S_{full=Sine-Gordon}]_{most-singular-part} = S_{spatially-nonlocal-but-quadratic-and-soluble}$$

and

$$[\psi_R(x,t)]_{most-singular-part} = [e^{2\pi i \int^x dy\ \rho_R(y,t)}]_{most-singular-part} = [e^{2\pi i \int^x dy\ (\rho_R(y,t)+\lambda\ \rho_L(-y,t))}]_{\lambda=0,1}$$

The modified Fermi-Bose correspondence rather than being tied to the inappropriate spatially homogeneous basis now encodes the effect of back-scattering at the impurity through the additional term in the exponent (while calculating the two point function $\lambda = 1$ should be used at most once, the reason is due to point splitting considerations [45]).

6.8 Summary

We have used the modified Fermi-Bose correspondence in Eq.6.23 to recover the correct tunneling density of states of FQHE edge states with a point contact. A major advantage of this approach is that the action remains quadratic in the bosonic operators even when backscattering due to the point contact impurity is included. However this method only gives the most singular parts of the correlation functions correctly. This quadratic (bosonic) action is used to derive the correlation functions in presence of mutual interactions using the generating functional method in Chapter 5. In conventional bosonization, the backscattering between the right and left movers due to the impurity produces a boundary sine-Gordon model that is not quadratic in the bosonic fields [149]. But in this alternative approach, the modified Fermi-Bose correspondence encodes the backscattering while leaving the action quadratic even with impurity and mutual (forward scattering) interaction between the fermions present. This method gives the most singular parts of the correlation functions of inhomogeneous Luttinger liquids exactly [52]. Our goal is to obtain the most singular parts of the non-equilibrium Green's function for this system exactly with a non-perturbative treatment of the point contact impurity. This is crucial in order to obtain the non-Ohmic tunneling I-V characteristics and express the conductance as a universal scaling function for a generic impurity strength. It involves calculating the $< \rho_{asy}\rho_{asy} >$ correlations out of equilibrium (with bias) with interactions and in presence of a point contact, which is a challenging task and we do not have the final results yet.
To summarize, this unconventional approach to bosonization enables the calculation of the most singular parts of the correlation functions of fermions with localized scattering centers where backscattering takes place and in presence of mutual fermion-fermion forward scattering interactions to be expressed in terms of elementary functions of position and time. This is accomplished by verifying, a-posteriori, the following claims:

(i) *Fact I:* The conventional bosonization scheme of the above mentioned model leads to a Lagrangian density that is nonlinear (Sine-Gordon) [149] but has a spatially local expression. Because of the Sine-Gordon term, it is not soluble. The most singular parts of the mutual (auto) correlations of density fluctuations involves realising (with proof [52]) that all the odd moments of the density fluctuations of the above system vanish identically whereas all the higher order even moments although non-zero are less singular than the leading second order moment and therefore ignorable.

(ii) *Claim I:* The most singular part of the above Lagrangian density in *Fact I* that recovers the density density correlation correctly and yields the result that all higher moments vanish (they are less singular so we feel entitled to ignore them) is spatially non-local but purely quadratic in the bosons and therefore soluble.

(iii) *Fact II:* The conventional bosonization scheme has a Fermi-Bose correspondence that is an operator identity that is tied to the translationally invariant fermion basis.

(iv) *Claim II:* The main message is that it is possible to study backward scattering from a stationary impurity in such a way that **(only) the most singular parts** of the correlation functions can be written down exactly using a modified bosonization framework that transfers part of the burden of shouldering the Sine Gordon character of the conventional approach to the modified Fermi Bose correspondence while retaining the action of the theory quadratic (albeit spatially non-local) in the bosons and therefore, soluble.

The main contribution of this work is to show that important physical attributes such as tunneling density of states exhibit universal power law behavior consistent with other well-established approaches found in the literature, even as the (most singular parts of the) full space-time Green functions - which only the present approach can calculate - will involve non-universal exponents.

Chapter 7

Conclusions

The Luttinger liquid model represents a fascinating paradigm in condensed matter physics, specifically for the description of one-dimensional interacting electron systems. Unlike conventional Fermi liquids prevalent in higher dimensions, Luttinger liquids emerge due to the unique nature of electron-electron interactions in 1D and such systems could be experimentally realised in quasi-one-dimensional materials like carbon nanotubes or quantum wires and also in edge states of quantum Hall systems. The collective behavior of the electrons dominates the physics of Luttinger liquids and well defined quasi-particles characteristic of Landau's Fermi liquid theory are absent. The interactions in these systems lead to a breakdown of the conventional Fermi liquid theory, manifesting as power-law correlations and fractionalized excitations. These novel properties make Luttinger liquids a unique and intriguing field of study with implications for understanding quantum transport in low-dimensional systems. In one-dimension there is the absence of a Fermi surface in the traditional sense. The Fermi surface is constituted by just two points at $+k_F$ and $-k_F$. Due to the totally nested nature of the 1D Fermi surface a perturbative treatment of the interactions is plagued by divergences. Bosonization is a powerful theoretical technique used to describe interacting fermion systems nonperturbatively in 1D. The complexity in handling the interactions directly arises due to the quartic fermion-fermion interaction term and hence the Hamiltonian is non diagonalizable. Bosonization involves mapping the fermionic degrees of freedom onto equivalent bosonic degrees of freedom and it turns out that the interaction term becomes quadratic in the boson operators thereby making the Hamiltonian diagonalizable, facilitating the study of the low-energy properties of the Luttinger liquid.
The conventional field theoretic bosonization (g-ology) provides the n-point correlations of a clean Luttinger liquid in presence of mutual forward scattering interactions [36]. However, when the translation invariance is broken by an impurity, one has to rely on the renormalization group (RG) treatment, yet these methods can't provide closed analytical expressions of the Green functions of the inhomogeneous system. Impurities in one-dimensional quantum many body systems have a significant effect on the low-energy physics [37]. The work in this thesis promulgates a novel method of bosonization, which is ideally suited to study inhomogeneous systems. The non chiral bosonization technique (NCBT) has been successful in obtaining the most singular part of the Green functions of a Luttinger liquid with an impurity at the origin [46], for a one step fermionic ladder [79] and for mobile impurities in a Luttinger liquid [47].

In the second chapter of this thesis we show in pedagogical detail that the most singular part of the asymptotic density-density correlation function (DDCF) of Luttinger liquids with localised scalar static impurities and short-range forward scattering mutual interactions, has a compact analytical expression with second order poles and involves only the scale-independent bare reflection and transmission coefficients. We also show that for such systems the (connected) moments of the density operator can be ignored beyond second order since the odd ones vanish identically and the even moments are less singular than the second order moment and can be neglected as long as only the most singular parts are desired. We show that our results match with conventional fermionic perturbation theory. The DDCF is a crucial input to NCBT and this work [52] clarifies the technical details behind the applicability of NCBT. For a homogeneous system or its opposite extreme viz. the half-line, all higher order connected moments of the density vanish identically which means the result of chiral bosonization and NCBT are

indeed the same in these two limiting cases.
The third chapter deals with noninteracting one-dimensional chiral quantum wires coupled through a point-contact and driven out of equilibrium [81]. We systematically solve Dyson's equation analytically and obtain the exact dynamical non-equilibrium Green function (NEGF). We also obtain the DDCF in a closed form in terms of simple functions of position and time. The tunneling I-V characteristics are obtained for an arbitrary time dependent bias. By considering a finite bandwidth in the point-contact we account for transient phenomena in the transport properties which exhibit non-Markovian behavior. The transients in the I-V are consistent with numerical simulations (tDMRG) of lattice systems which suggests that the transient behavior is merely due to the presence of a short-distance cutoff in the problem. Our aim in studying this problem was to construct a bosonization technique in the spirit of NCBT that reproduces exactly the nonequilibrium Green functions. This is precisely what we have shown in our next work.
In the fourth chapter we develop an unconventional bosonization method that provides the NEGF for chiral quantum wires coupled through a point-contact driven out of equilibrium [137]. The proposed unconventional bosonization scheme is shown to be internally consistent as the four-point functions obtained through bosonization are related to the two-point functions through Wick's theorem as it should be. This system of noninteracting chiral quantum wires is isomorphic to integer quantum Hall effect (IQHE) edge states with filling factor $\nu = 1$, with a point-contact constriction across which tunneling can take place upon application of a bias. For a similar setup involving fractional quantum Hall effect (FQHE) edge states, interparticle interactions need to be taken into account and this is where bosonization is very useful. In order to apply the unconventional bosonization method to FQHE edge states it becomes necessary to compute the interacting density-density correlations of the edge states in presence of a point-contact tunnel junction. This is precisely what we compute in the next chapter. We systematically derive the DDCF (most singular parts) of chiral Luttinger liquids forming the FQHE edge in presence of a point-contact constriction acting as a localised scalar impurity, using a generating functional. The results are validated on comparison with standard fermionic perturbation theory. The two-terminal current in linear response to a potential difference between the right moving and left moving edges is calculated using the DDCF at zero temperature. Obtaining the nonequilibrium DDCF for this problem analytically is very difficult but important transport properties can be gleaned by using the equilibrium DDCF in the unconventional bosonization method to extract the power-law exponent of the tunneling density of states (TDOS). The TDOS for both the electron tunneling and Laughlin quasiparticle tunneling is obtained in the sixth chapter and is found to agree with previous results in the literature. This is a convincing demonstration of the utility of the novel bosonization technique discussed in this thesis, that treats impurity backscattering in a Luttinger liquid exactly yet remains solvable for the most singular parts of the correlations.

Future directions

An immediate extension of this work is the challenging task of obtaining the full nonequilibrium Green functions for FQHE edge states coupled through a point-contact by solving for the nonequilibrium DDCF first. Also, the unconventional bosonization idea may be extended to systems with impurities with quantum degrees of freedom like a Kondo impurity. Another promising endeavour is the generalization of bosonization to higher dimensions.

Appendices

APPENDIX A: Derivation of formulas for the parameters of the generalized Hamiltonian (V_R, V_L, V_1, V_1^*) in terms of T, R

$$
\begin{aligned}
H_0 = & -i\ v_F \int dx\ (\psi_R^\dagger(x)\partial_x\psi_R(x) - \psi_L^\dagger(x)\partial_x\psi_L(x)) + V_R\psi_R^\dagger(0)\psi_R(0) + V_L\psi_L^\dagger(0)\psi_L(0) \\
& + V_1\ \psi_R^\dagger(0)\psi_L(0) + V_1^*\ \psi_L^\dagger(0)\psi_R(0)
\end{aligned}
\tag{A.1}
$$

Note that we may write $V_1 = |V_1|\ e^{i\delta}$. This phase may be absorbed by a redefinition of the fields $\psi_R(x) \to e^{i\delta}\psi_R(x)$ for example. This means V_1 can be chosen to be real without loss of generality.

$$
< T\ \psi_\nu(x,t)\psi_{\nu'}^\dagger(x',t') > \ = \ \frac{\Gamma^{\nu,\nu'}(x,x')}{\nu x - \nu' x' -\ v_F\ (t-t')}
\tag{A.2}
$$

and

$$
\Gamma^{\nu,\nu'}(x,x') = \sum_{\gamma,\gamma'=\pm1} \theta(\gamma x)\theta(\gamma' x')\ g_{\gamma,\gamma'}(\nu,\nu')
\tag{A.3}
$$

$$
i\partial_t < \psi_\nu(x,t)\psi_{\nu'}^\dagger(x',t') > \ = i\ v_F\ \frac{\Gamma^{\nu,\nu'}(x,x')}{(\nu x - \nu' x' -\ v_F\ (t-t'))^2}
\tag{A.4}
$$

$$
i\partial_t < \psi_\nu(x,t)\psi_{\nu'}^\dagger(x',t') > \ = \ < [\psi_\nu(x,t), H_0]\psi_{\nu'}^\dagger(x',t') >
\tag{A.5}
$$

or

$$
\begin{aligned}
i\partial_t < \psi_\nu(x,t)\psi_{\nu'}^\dagger(x',t') > \ = & \ -i\ v_F\ \nu\ < \partial_x\psi_\nu(x,t)\psi_{\nu'}^\dagger(x',t') > \\
& + < [\psi_\nu(x,t), V_R\psi_R^\dagger(0)\psi_R(0)]\psi_{\nu'}^\dagger(x',t') > + < [\psi_\nu(x,t), V_L\psi_L^\dagger(0)\psi_L(0)]\psi_{\nu'}^\dagger(x',t') > \\
& + < [\psi_\nu(x,t), V_1\ \psi_R^\dagger(0)\psi_L(0)]\psi_{\nu'}^\dagger(x',t') > + < [\psi_\nu(x,t), V_1^*\ \psi_L^\dagger(0)\psi_R(0)]\psi_{\nu'}^\dagger(x',t') >
\end{aligned}
$$

or

$$
\begin{aligned}
i\ v_F\ \frac{\Gamma^{\nu,\nu'}(x,x')}{(\nu x - \nu' x' -\ v_F\ (t-t'))^2} = & \ i\ v_F\ \frac{\Gamma^{\nu,\nu'}(x,x')}{(\nu x - \nu' x' -\ v_F\ (t-t'))^2} - i\ v_F\ \nu\ \frac{[\partial_x\Gamma^{\nu,\nu'}(x,x')]}{\nu x - \nu' x' -\ v_F\ (t-t')} \\
& + \delta_{\nu,1}\delta(x)\ V_R < \psi_R(0,t)\psi_{\nu'}^\dagger(x',t') > + V_L\delta_{\nu,-1}\delta(x) < \psi_L(0,t)\psi_{\nu'}^\dagger(x',t') > \\
& + V_1\delta(x)\delta_{\nu,1} < \psi_L(0,t)\psi_{\nu'}^\dagger(x',t') > + V_1^*\delta_{\nu,-1}\delta(x)\ < \psi_R(0,t)\psi_{\nu'}^\dagger(x',t') >
\end{aligned}
$$

or

$$
\begin{aligned}
0 = & -i\ v_F\ \nu\ \frac{\sum_{\gamma,\gamma'=\pm1}\delta(x)\gamma\theta(\gamma' x')\ g_{\gamma,\gamma'}(\nu,\nu')}{\nu x - \nu' x' -\ v_F\ (t-t')} + \delta_{\nu,1}\delta(x)\ V_R\frac{\Gamma^{1,\nu'}(0,x')}{-\nu' x' -\ v_F\ (t-t')} + V_L\delta_{\nu,-1}\delta(x)\frac{\Gamma^{-1,\nu'}(0,x')}{-\nu' x' -\ v_F\ (t-t')} \\
& + V_1\delta(x)\delta_{\nu,1}\frac{\Gamma^{-1,\nu'}(0,x')}{-\nu' x' -\ v_F\ (t-t')} + V_1^*\delta_{\nu,-1}\delta(x)\ \frac{\Gamma^{1,\nu'}(0,x')}{-\nu' x' -\ v_F\ (t-t')}
\end{aligned}
$$

or

$$
\begin{aligned}
0 = & \ -i\ v_F\ \nu\ \frac{\sum_{\gamma,\gamma'=\pm1}\delta(x)\gamma\theta(\gamma' x')\ g_{\gamma,\gamma'}(\nu,\nu')}{-\nu' x' -\ v_F\ (t-t')} + \delta_{\nu,1}\delta(x)\ V_R\frac{\sum_{\gamma,\gamma'=\pm1}\frac{1}{2}\theta(\gamma' x')\ g_{\gamma,\gamma'}(1,\nu')}{-\nu' x' -\ v_F\ (t-t')} \\
& + V_L\delta_{\nu,-1}\delta(x)\frac{\sum_{\gamma,\gamma'=\pm1}\frac{1}{2}\theta(\gamma' x')\ g_{\gamma,\gamma'}(-1,\nu')}{-\nu' x' -\ v_F\ (t-t')} + V_1\delta(x)\delta_{\nu,1}\frac{\sum_{\gamma,\gamma'=\pm1}\frac{1}{2}\theta(\gamma' x')\ g_{\gamma,\gamma'}(-1,\nu')}{-\nu' x' -\ v_F\ (t-t')} \\
& + V_1^*\delta_{\nu,-1}\delta(x)\ \frac{\sum_{\gamma,\gamma'=\pm1}\frac{1}{2}\theta(\gamma' x')\ g_{\gamma,\gamma'}(1,\nu')}{-\nu' x' -\ v_F\ (t-t')}
\end{aligned}
$$

or

$$0 = \sum_{\gamma=\pm1} (-i\ v_F\ \nu\ \gamma\ g_{\gamma,\gamma'}(\nu,\nu') + \delta_{\nu,1}\ V_R \frac{1}{2}\ g_{\gamma,\gamma'}(1,\nu') + V_L \delta_{\nu,-1} \frac{1}{2}\ g_{\gamma,\gamma'}(-1,\nu')$$
$$+ V_1 \delta_{\nu,1} \frac{1}{2}\ g_{\gamma,\gamma'}(-1,\nu') + V_1^* \delta_{\nu,-1}\ \frac{1}{2}\ g_{\gamma,\gamma'}(1,\nu'))$$

The above equation gives,

$$\begin{aligned} V_R = V_L &= \frac{2i\ v_F\ (T-T^*)}{2TT^* + T + T^*} \\ V_1 = V_1^* &= -\frac{4i\ v_F\ \ R^*T}{2TT^* + T + T^*} = \frac{4i\ v_F\ \ RT^*}{2TT^* + T + T^*} \end{aligned} \tag{A.6}$$

This automatically means

$$-R^*T = RT^* \tag{A.7}$$

APPENDIX B: Fourier transform of the DDCF

In the main text, we are required to show that equation (2.8) may be recovered from equation (2.13) and equation (2.14). For this we are required to make sense of integrals such as ($x \neq x'$ and $x, x' \neq 0$),

$$I_0(x-x') = \sum_q e^{-iq(x-x')}\ \frac{2q^2 v_F}{w_n^2 + (qv_F)^2} \tag{B.1}$$

and

$$I_1(x) = \sum_q e^{-iqx}\ \frac{2v_F q}{w_n^2 + (qv_F)^2} \tag{B.2}$$

We write,

$$I_0(x-x') = \sum_q e^{-iq(x-x')}\ \ (\frac{q}{iw_n + (qv_F)} + \frac{q}{-iw_n + (qv_F)}) \tag{B.3}$$

Set $X = x - x'$. Then,

$$I_0(X) = i\frac{d}{dX} \sum_q e^{-iqX}\ \ (\frac{1}{iw_n + (qv_F)} + \frac{1}{-iw_n + (qv_F)}) = i\frac{d}{dX} I_1(X) \tag{B.4}$$

and

$$I_1(x) = \sum_q e^{-iqx}\ \ (\frac{1}{iw_n + (qv_F)} + \frac{1}{-iw_n + (qv_F)}) = -\frac{iL\ \mathrm{sgn}(x)\ e^{-\frac{|w_n|\ |x|}{v_F}}}{v_F} \tag{B.5}$$

Now,

$$I_1(X>0) = -\frac{iL\ e^{-\frac{|w_n|\ X}{v_F}}}{v_F}\ \ ;\quad I_1(X<0) = \frac{iL\ e^{\frac{|w_n|\ X}{v_F}}}{v_F} \tag{B.6}$$

and

$$I_0(X>0) = -\frac{|w_n|}{v_F} \frac{L\ e^{-\frac{|w_n|\ X}{v_F}}}{v_F}\ \ ;\quad I_0(X<0) = -\frac{|w_n|}{v_F} \frac{L\ e^{\frac{|w_n|\ X}{v_F}}}{v_F} \tag{B.7}$$

or

$$I_0(X) = -\frac{L\,|w_n|}{v_F^2} e^{-\frac{|w_n|\ |X|}{v_F}} \tag{B.8}$$

Finally we are called upon to transform the Matsubara frequency to (imaginary) time.

$$J_0(x) = \sum_n e^{-w_n\tau}\ I_0(x)\ ;\quad J_1(x) = \sum_n e^{-w_n\tau}\ I_1(x) \tag{B.9}$$

or,

$$J_0(X) = -\sum_n e^{-w_n \tau} \frac{L\,|w_n|}{v_F^2} e^{-\frac{|w_n|\ |X|}{v_F}} \quad ; \quad J_1(x) = -\sum_n e^{-w_n \tau} \frac{iL\ \mathrm{sgn}(x)\ e^{-\frac{|w_n|\ |x|}{v_F}}}{v_F}$$

or,

$$J_0(X) = \frac{\beta}{2\pi}\int_{-\infty}^{0} dw_n\ e^{-w_n\tau}\ \frac{Lw_n}{v_F^2} e^{\frac{w_n\ |X|}{v_F}} - \frac{\beta}{2\pi}\int_{0}^{\infty} dw_n\ e^{-w_n\tau}\ \frac{Lw_n}{v_F^2} e^{-\frac{w_n\ |X|}{v_F}}$$

$$J_1(x) = -\frac{\beta}{2\pi}\int_{-\infty}^{0} dw_n\ e^{-w_n\tau}\ \frac{iL\ \mathrm{sgn}(x)\ e^{\frac{w_n\ |x|}{v_F}}}{v_F} - \frac{\beta}{2\pi}\int_{0}^{\infty} dw_n\ e^{-w_n\tau}\ \frac{iL\ \mathrm{sgn}(x)\ e^{-\frac{w_n\ |x|}{v_F}}}{v_F}$$

or,

$$J_0(X) = \frac{\beta}{2\pi}\int_{-\infty}^{0} dw_n\ e^{-w_n\tau}\ \frac{Lw_n}{v_F^2} e^{\frac{w_n\ |X|}{v_F}} - \frac{\beta}{2\pi}\int_{0}^{\infty} dw_n\ e^{-w_n\tau}\ \frac{Lw_n}{v_F^2} e^{-\frac{w_n\ |X|}{v_F}}$$

$$J_1(x) = -\frac{\beta}{2\pi}\int_{-\infty}^{0} dw_n\ e^{-w_n\tau}\ \frac{iL\ \mathrm{sgn}(x)\ e^{\frac{w_n\ |x|}{v_F}}}{v_F} - \frac{\beta}{2\pi}\int_{0}^{\infty} dw_n\ e^{-w_n\tau}\ \frac{iL\ \mathrm{sgn}(x)\ e^{-\frac{w_n\ |x|}{v_F}}}{v_F}$$

or,

$$J_0(X) = -\frac{\beta L}{2\pi(|X|-\tau v_F)^2} - \frac{\beta L}{2\pi(|X|+\tau v_F)^2} \tag{B.10}$$

$$J_1(x) = \frac{\beta iL\ \mathrm{sgn}(x)}{2\pi(\tau v_F - |x|)} - \frac{\beta iL\ \mathrm{sgn}(x)}{2\pi(|x|+\tau v_F)} \tag{B.11}$$

Thus equation (2.41) is just some combination of these functions J_0, J_1.

APPENDIX C: Perturbative comparison of DDCF for $|R| = 0$ case

$$\langle T\ \rho_a(x_1,t_1)\rho_a(x_2,t_2)\rangle = \frac{v_F}{2\pi^2 v_a}\sum_{\nu=\pm1}\left(-\frac{1}{(x_1-x_2+\nu v_a(t_1-t_2))^2}\right) \tag{C.1}$$

where $a = n\ or\ h$,

$$v_h = \sqrt{v_F^2 + \frac{2v_F v_0}{\pi}} \tag{C.2}$$

$$v_n = v_F$$

$$\langle T\ \rho_n(x_1,t_1)\rho_n(x_2,t_2)\rangle = \frac{1}{2\pi^2}\sum_{\nu=\pm1}\left(-\frac{1}{(x_1-x_2+\nu v_F(t_1-t_2))^2}\right) \tag{C.3}$$

and $\langle T\ \rho_n(x_1,t_1)\rho_h(x_2,t_2)\rangle = 0$. This means $\langle T\ \rho_n(x_1,t_1)\rho_n(x_2,t_2)\rangle^1 = 0$.

On the one hand

Simply expanding the full final answer equation (C.1) to first power in v_0 we get,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \frac{v_0}{2\pi^3 v_F}$$
$$\left(\frac{2v_F(t_1-t_2)}{((t_1-t_2)v_F+x_1-x_2)^3} + \frac{2v_F(t_1-t_2)}{((t_1-t_2)v_F-x_1+x_2)^3} + \frac{1}{((t_1-t_2)v_F+x_1-x_2)^2} + \frac{1}{((t_1-t_2)v_F-x_1+x_2)^2}\right) \tag{C.4}$$

On the other hand

Using standard perturbation and retaining the most singular terms we get,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\int_C d\tau\int_{-\infty}^{\infty} dy\ \langle T\ \rho_h(y,\tau)\rho_h(x_1,t_1)\rangle_0\ \langle T\ \rho_h(y,\tau)\rho_h(x_2,t_2)\rangle_0 \tag{C.5}$$

But,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle_0 = \frac{1}{2\pi^2}\sum_{\nu=\pm1}\left(-\frac{1}{(x_1-x_2+\nu v_F(t_1-t_2))^2}\right) \tag{C.6}$$

Hence,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\ \int_C d\tau\ \int_{-\infty}^{\infty} dy\ \frac{1}{4\pi^4(-v_F(\tau-t_1)-x_1+y)^2(v_F(\tau-t_2)-x_2+y)^2}$$
$$+ (-i)v_0\ \int_C d\tau\ \int_{-\infty}^{\infty} dy\ \frac{1}{4\pi^4(v_F(\tau-t_1)-x_1+y)^2(-v_F(\tau-t_2)-x_2+y)^2}$$
$$+ (-i)v_0\ \int_C d\tau\ \int_{-\infty}^{\infty} dy\ \frac{1}{4\pi^4(-v_F(\tau-t_1)-x_1+y)^2(-v_F(\tau-t_2)-x_2+y)^2}$$
$$+ (-i)v_0\ \int_C d\tau\ \int_{-\infty}^{\infty} dy\ \frac{1}{4\pi^4(v_F(\tau-t_1)-x_1+y)^2(v_F(\tau-t_2)-x_2+y)^2}$$

Hence,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1$$
$$= (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\frac{4\pi i}{(-v_F(\tau-t_1)-x_1-v_F(\tau-t_2)+x_2)^3}(\theta(-(\tau-t_1))\theta(-(\tau-t_2))-\theta(\tau-t_1)\theta(\tau-t_2))$$
$$+ (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\frac{4\pi i}{(v_F(\tau-t_1)-x_1+v_F(\tau-t_2)+x_2)^3}(\theta(\tau-t_1)\theta(\tau-t_2)-\theta(-(\tau-t_1))\theta(-(\tau-t_2))$$
$$+ (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-v_F(\tau-t_1)-x_1+v_F(\tau-t_2)+x_2)^3}(\theta(-(\tau-t_1))\theta(\tau-t_2)-\theta(\tau-t_1)\theta(-(\tau-t_2)))$$
$$+ (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(v_F(\tau-t_1)-x_1-v_F(\tau-t_2)+x_2)^3}(\theta(\tau-t_1)\theta(-(\tau-t_2))-\theta(-(\tau-t_1))\theta(\tau-t_2))$$

Specifically consider $t_1 > t_2$ (t_1 is on the lower contour and t_2 is on the upper contour). In this case,

$$\theta(-(\tau-t_1))\theta(-(\tau-t_2))-\theta(\tau-t_1)\theta(\tau-t_2) = \theta(-(\tau-t_2))-\theta(\tau-t_1)$$
$$\theta(\tau-t_1)\theta(\tau-t_2)-\theta(-(\tau-t_1))\theta(-(\tau-t_2)) = \theta(\tau-t_1)-\theta(-(\tau-t_2))$$
$$\theta(-(\tau-t_1))\theta(\tau-t_2)-\theta(\tau-t_1)\theta(-(\tau-t_2)) = \theta(-(\tau-t_1))\theta(\tau-t_2)$$
$$\theta(\tau-t_1)\theta(-(\tau-t_2))-\theta(-(\tau-t_1))\theta(\tau-t_2) = -\theta(-(\tau-t_1))\theta(\tau-t_2)$$

or,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\frac{4\pi i}{(-v_F(\tau-t_1)-x_1-v_F(\tau-t_2)+x_2)^3}(\theta(-(\tau-t_2))-\theta(\tau-t_1))$$
$$+ (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\frac{4\pi i}{(v_F(\tau-t_1)-x_1+v_F(\tau-t_2)+x_2)^3}(\theta(\tau-t_1)-\theta(-(\tau-t_2)))$$
$$+ (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-v_F(\tau-t_1)-x_1+v_F(\tau-t_2)+x_2)^3}(\theta(-(\tau-t_1))\theta(\tau-t_2))$$
$$+ (-i)v_0\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(v_F(\tau-t_1)-x_1-v_F(\tau-t_2)+x_2)^3}(-\theta(-(\tau-t_1))\theta(\tau-t_2))$$

or,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\ \frac{1}{4\pi^4}(\int_{-\infty}^{t_2} d\tau - \int_{t_1}^{-\infty} d\tau)\frac{4\pi i}{(-v_F(\tau-t_1)-x_1-v_F(\tau-t_2)+x_2)^3}$$
$$+ (-i)v_0\ \frac{1}{4\pi^4}(\int_{t_1}^{-\infty} d\tau - \int_{-\infty}^{t_2} d\tau)\frac{4\pi i}{(v_F(\tau-t_1)-x_1+v_F(\tau-t_2)+x_2)^3}$$
$$+ (-i)v_0\ \frac{1}{4\pi^4}\int_{t_2}^{t_1} d\tau\ \frac{4\pi i}{(-v_F(\tau-t_1)-x_1+v_F(\tau-t_2)+x_2)^3}$$
$$+ (-i)v_0\ \frac{1}{4\pi^4}(-\int_{t_2}^{t_1} d\tau)\ \frac{4\pi i}{(v_F(\tau-t_1)-x_1-v_F(\tau-t_2)+x_2)^3}$$

Or,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\ \frac{1}{4\pi^4}(\frac{i\pi}{v_F(t_1v_F-t_2v_F+x_1-x_2)^2}+\frac{i\pi}{v_F(t_1v_F-t_2v_F-x_1+x_2)^2})$$
$$+(-i)v_0\ \frac{1}{4\pi^4}(\frac{i\pi}{v_F(t_1v_F-t_2v_F+x_1-x_2)^2}+\frac{i\pi}{v_F(t_1v_F-t_2v_F-x_1+x_2)^2})$$
$$+(-i)v_0\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(x_2-x_1+v_F(t_1-t_2))^3}(t_1-t_2)$$
$$+(-i)v_0\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(x_2-x_1-v_F(t_1-t_2))^3}(-(t_1-t_2))$$

Or,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \frac{v_0}{2\pi^3 v_F}$$
$$\left(\frac{2v_F(t_1-t_2)}{((t_1-t_2)v_F+x_1-x_2)^3}+\frac{2v_F(t_1-t_2)}{((t_1-t_2)v_F-x_1+x_2)^3}+\frac{1}{((t_1-t_2)v_F+x_1-x_2)^2}+\frac{1}{((t_1-t_2)v_F-x_1+x_2)^2}\right) \tag{C.7}$$

equation (C.7) matches with the earlier result equation (C.4).

APPENDIX D: Perturbative comparison of DDCF for $|R|=1$ case

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle = \frac{v_F}{2\pi^2 v_h}\sum_{\nu=\pm1}\left(-\frac{1}{(x_1-x_2+\nu v_h(t_1-t_2))^2}-\frac{\frac{v_F}{v_h}\ \mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\ Z_h}{(|x_1|+|x_2|+\nu v_h(t_1-t_2))^2}\right) \tag{D.1}$$

$$Z_h = \frac{1}{\left(1-\frac{4v_0v_F}{(v_h+v_F)v_h}\ \frac{1}{(2\pi)}\right)}\ ;\ v_h=\sqrt{v_F^2+\frac{2v_0v_F}{\pi}} \tag{D.2}$$

Expanding to first power of v_0 we get,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle \approx$$
$$-\frac{1}{2\pi^2}(\frac{\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)}{(|x_1|+|x_2|+v_F(t_1-t_2))^2}+\frac{\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)}{(|x_1|+|x_2|-v_F(t_1-t_2))^2}+\frac{1}{(x_1-x_2+v_F(t_1-t_2))^2}+\frac{1}{(x_1-x_2-v_F(t_1-t_2))^2})$$
$$+\frac{v_0}{2\pi^3v_F}\Bigg(\mathrm{sgn}(x_1)\mathrm{sgn}(x_2)\Big(\frac{1}{(|x_1|+|x_2|+v_F(t_1-t_2))^2}+\frac{1}{(|x_1|+|x_2|-v_F(t_1-t_2))^2}$$
$$+\frac{2v_F(t_1-t_2)}{(|x_1|+|x_2|+v_F(t_1-t_2))^3}-\frac{2v_F(t_1-t_2)}{(|x_1|+|x_2|-v_F(t_1-t_2))^3}\Big)$$
$$-\frac{2v_F(t_1-t_2)}{(x_1-x_2-v_F(t_1-t_2))^3}+\frac{2v_F(t_1-t_2)}{(x_1-x_2+v_F(t_1-t_2))^3}+\frac{1}{(x_1-x_2+v_F(t_1-t_2))^2}+\frac{1}{(x_1-x_2-v_F(t_1-t_2))^2}\Bigg)$$

Or,

$$\delta\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle \approx$$
$$\theta(x_1x_2)\frac{v_0}{2\pi^3v_F}(\frac{1}{(x_1+x_2+v_F(t_1-t_2))^2}+\frac{1}{(x_1+x_2-v_F(t_1-t_2))^2}+\frac{1}{(x_1-x_2+v_F(t_1-t_2))^2}+\frac{1}{(x_1-x_2-v_F(t_1-t_2))^2})$$
$$+\theta(x_1x_2)\frac{v_0}{2\pi^3v_F}(\frac{2v_F(t_1-t_2)}{(x_1+x_2+v_F(t_1-t_2))^3}-\frac{2v_F(t_1-t_2)}{(x_1+x_2-v_F(t_1-t_2))^3}-\frac{2v_F(t_1-t_2)}{(x_1-x_2-v_F(t_1-t_2))^3}+\frac{2v_F(t_1-t_2)}{(x_1-x_2+v_F(t_1-t_2))^3}) \tag{D.3}$$

On the other hand, using standard perturbation and retaining the most singular terms we get,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\ \int_C d\tau\ \int_{-\infty}^{\infty} dy\ \langle T\ \rho_h(y,\tau)\rho_h(x_1,t_1)\rangle_0\ \langle T\ \rho_h(y,\tau)\rho_h(x_2,t_2)\rangle_0 \tag{D.4}$$

This means,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0 \int_C d\tau \int_{-\infty}^{0} dy$$
$$\frac{1}{2\pi^2}\left(\frac{-\mathrm{sgn}(x_1)}{(-y+|x_1|+v_F(\tau-t_1))^2}+\frac{-\mathrm{sgn}(x_1)}{(-y+|x_1|-v_F(\tau-t_1))^2}+\frac{1}{(y-x_1+v_F(\tau-t_1))^2}+\frac{1}{(y-x_1-v_F(\tau-t_1))^2}\right)$$
$$\frac{1}{2\pi^2}\left(\frac{-\mathrm{sgn}(x_2)}{(-y+|x_2|+v_F(\tau-t_2))^2}+\frac{-\mathrm{sgn}(x_2)}{(-y+|x_2|-v_F(\tau-t_2))^2}+\frac{1}{(y-x_2+v_F(\tau-t_2))^2}+\frac{1}{(y-x_2-v_F(\tau-t_2))^2}\right)$$
$$+(-i)v_0 \int_C d\tau \int_{0}^{\infty} dy$$
$$\frac{1}{2\pi^2}\left(\frac{\mathrm{sgn}(x_1)}{(y+|x_1|+v_F(\tau-t_1))^2}+\frac{\mathrm{sgn}(x_1)}{(y+|x_1|-v_F(\tau-t_1))^2}+\frac{1}{(y-x_1+v_F(\tau-t_1))^2}+\frac{1}{(y-x_1-v_F(\tau-t_1))^2}\right)$$
$$\frac{1}{2\pi^2}\left(\frac{\mathrm{sgn}(x_2)}{(y+|x_2|+v_F(\tau-t_2))^2}+\frac{\mathrm{sgn}(x_2)}{(y+|x_2|-v_F(\tau-t_2))^2}+\frac{1}{(y-x_2+v_F(\tau-t_2))^2}+\frac{1}{(y-x_2-v_F(\tau-t_2))^2}\right)$$

This means,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\ \theta(-x_1)\theta(-x_2) \int_C d\tau \int_{-\infty}^{0} dy$$
$$\frac{1}{2\pi^2}\left(\frac{1}{(-y-x_1+v_F(\tau-t_1))^2}+\frac{1}{(-y-x_1-v_F(\tau-t_1))^2}+\frac{1}{(y-x_1+v_F(\tau-t_1))^2}+\frac{1}{(y-x_1-v_F(\tau-t_1))^2}\right)$$
$$\frac{1}{2\pi^2}\left(\frac{1}{(-y-x_2+v_F(\tau-t_2))^2}+\frac{1}{(-y-x_2-v_F(\tau-t_2))^2}+\frac{1}{(y-x_2+v_F(\tau-t_2))^2}+\frac{1}{(y-x_2-v_F(\tau-t_2))^2}\right)$$
$$+(-i)v_0\ \theta(x_1)\theta(x_2) \int_C d\tau \int_{0}^{\infty} dy$$
$$\frac{1}{2\pi^2}\left(\frac{1}{(y+x_1+v_F(\tau-t_1))^2}+\frac{1}{(y+x_1-v_F(\tau-t_1))^2}+\frac{1}{(y-x_1+v_F(\tau-t_1))^2}+\frac{1}{(y-x_1-v_F(\tau-t_1))^2}\right)$$
$$\frac{1}{2\pi^2}\left(\frac{1}{(y+x_2+v_F(\tau-t_2))^2}+\frac{1}{(y+x_2-v_F(\tau-t_2))^2}+\frac{1}{(y-x_2+v_F(\tau-t_2))^2}+\frac{1}{(y-x_2-v_F(\tau-t_2))^2}\right)$$

This means,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = (-i)v_0\ \theta(x_1x_2)\ \partial_{x_1}\partial_{x_2} \int_C d\tau \int_{-\infty}^{0} dy\ \frac{1}{4\pi^4}$$
$$\times\left(\frac{1}{(y-x_1-v_F(\tau-t_1))}+\frac{1}{(y-x_1+v_F(\tau-t_1))}-\frac{1}{(y+x_1-v_F(\tau-t_1))}-\frac{1}{(y+x_1+v_F(\tau-t_1))}\right)$$
$$\times\left(\frac{1}{(y-x_2-v_F(\tau-t_2))}+\frac{1}{(y-x_2+v_F(\tau-t_2))}-\frac{1}{(y+x_2-v_F(\tau-t_2))}-\frac{1}{(y+x_2+v_F(\tau-t_2))}\right)$$

This means,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1$$
$$= \sum_{a_1,\nu_1,a_2,\nu_2=\pm1} (-i)v_0\ \theta(x_1x_2)\ \partial_{x_1}\partial_{x_2} \int_C d\tau \int_{-\infty}^{0} dy\ \frac{1}{4\pi^4}\ \frac{a_1}{(y-a_1x_1-\nu_1v_F(\tau-t_1))}\frac{a_2}{(y-a_2x_2-\nu_2v_F(\tau-t_2))}$$

This also means,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1$$
$$= \sum_{a_1,\nu_1,a_2,\nu_2=\pm1} (-i)v_0\ \theta(x_1x_2)\ \partial_{x_1}\partial_{x_2} \int_C d\tau \int_{0}^{\infty} dy\ \frac{1}{4\pi^4}\ \frac{a_1}{(y-a_1x_1-\nu_1v_F(\tau-t_1))}\frac{a_2}{(y-a_2x_2-\nu_2v_F(\tau-t_2))}$$

This means,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1$$
$$= \sum_{a_1,\nu_1,a_2,\nu_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2) \int_C d\tau \int_{-\infty}^{\infty} dy\ \frac{1}{4\pi^4}\ \frac{1}{(y-a_1x_1-\nu_1v_F(\tau-t_1))^2(y-a_2x_2-\nu_2v_F(\tau-t_2))^2}$$

This means,

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \sum_{a_1,\nu_1,a_2,\nu_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \int_C d\tau\ \int_{-\infty}^{\infty} dy\ \frac{1}{4\pi^4}\ \frac{1}{(y-a_1x_1-\nu_1v_F(\tau-t_1))^2(y-a_2x_2-\nu_2v_F(\tau-t_2))^2} \tag{D.5}$$

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \sum_{a_1,\nu_1,a_2,\nu_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1-\nu_1v_F(\tau-t_1)-(-a_2x_2-\nu_2v_F(\tau-t_2)))^3} \times(\theta(-Im[-\nu_1(\tau-t_1)])\theta(Im[-\nu_2(\tau-t_2)])-\theta(Im[-\nu_1(\tau-t_1)])\theta(-Im[-\nu_2(\tau-t_2)])) \tag{D.6}$$

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \sum_{a_1,\nu_1,a_2,\nu_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1-\nu_1v_F(\tau-t_1)-(-a_2x_2-\nu_2v_F(\tau-t_2)))^3} (\theta_C(-\nu_1(\tau-t_1))\theta_C(\nu_2(\tau-t_2))-\theta_C(\nu_1(\tau-t_1))\theta_C(-\nu_2(\tau-t_2)))$$

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \sum_{a_1,a_2,\nu=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2+\nu v_F(t_1-t_2))^3} (\theta_C(-\nu(\tau-t_1))\theta_C(\nu(\tau-t_2))-\theta_C(\nu(\tau-t_1))\theta_C(-\nu(\tau-t_2))) + \sum_{a_1,a_2,\nu=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2+\nu v_F(t_1+t_2)-2\nu v_F\tau)^3} (\theta_C(-\nu(\tau-t_1))\theta_C(-\nu(\tau-t_2))-\theta_C(\nu(\tau-t_1))\theta_C(\nu(\tau-t_2)))$$

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)$$
$$\Bigg(\int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2+v_F(t_1-t_2))^3}\ (\theta_C(-(\tau-t_1))\theta_C(\tau-t_2)-\theta_C(\tau-t_1)\theta_C(-(\tau-t_2)))$$
$$+\int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2-v_F(t_1-t_2))^3}\ (\theta_C(\tau-t_1)\theta_C(-(\tau-t_2))-\theta_C(-(\tau-t_1))\theta_C(\tau-t_2))$$
$$+\int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2-v_F(t_1+t_2)+2v_F\tau)^3}\ (\theta_C(\tau-t_1)\theta_C(\tau-t_2)-\theta_C(-(\tau-t_1))\theta_C(-(\tau-t_2)))$$
$$+\int_C d\tau\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2+v_F(t_1+t_2)-2v_F\tau)^3}\ (\theta_C(-(\tau-t_1))\theta_C(-(\tau-t_2))-\theta_C(\tau-t_1)\theta_C(\tau-t_2)\Bigg)$$

If $t_1 > t_2$,

$$\begin{aligned}
&\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 \\
&= \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2+v_F(t_1-t_2))^3}\ \left(\int_{t_2}^{t_1} d\tau\right) \\
&+ \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2-v_F(t_1-t_2))^3}\ \left(-\int_{t_2}^{t_1} d\tau\right) \\
&+ \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \left(-\int_{-\infty}^{t_1} d\tau - \int_{-\infty}^{t_2} d\tau\right)\frac{4\pi i}{(-a_1x_1+a_2x_2-v_F(t_1+t_2)+2v_F\tau)^3} \\
&+ \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \left(\int_{-\infty}^{t_2} d\tau + \int_{-\infty}^{t_1} d\tau\right)\frac{4\pi i}{(-a_1x_1+a_2x_2+v_F(t_1+t_2)-2v_F\tau)^3}
\end{aligned} \tag{D.7}$$

$$\begin{aligned}
&\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 \\
&= \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2+v_F(t_1-t_2))^3}\ (t_1-t_2) \\
&+ \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2-v_F(t_1-t_2))^3}\ (t_2-t_1) \\
&+ \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \left(-\int_{-\infty}^{t_1} d\tau - \int_{-\infty}^{t_2} d\tau\right)\frac{4\pi i}{(-a_1x_1+a_2x_2-v_F(t_1+t_2)+2v_F\tau)^3} \\
&+ \sum_{a_1,a_2=\pm1} (-i)\frac{v_0}{2}\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \left(\int_{-\infty}^{t_2} d\tau + \int_{-\infty}^{t_1} d\tau\right)\frac{4\pi i}{(-a_1x_1+a_2x_2+v_F(t_1+t_2)-2v_F\tau)^3}
\end{aligned} \tag{D.8}$$

$$\begin{aligned}
\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 &= \sum_{a_1,a_2=\pm1} (-i)v_0\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \frac{4\pi i}{(-a_1x_1+a_2x_2+v_F(t_1-t_2))^3}\ (t_1-t_2) \\
&+ \sum_{a_1,a_2=\pm1} (-i)v_0\ \theta(x_1x_2)\ \frac{1}{4\pi^4}\ \left(\frac{i\pi}{v_F(a_1x_1-a_2x_2+t_1v_F-t_2v_F)^2} + \frac{i\pi}{v_F(-a_1x_1+a_2x_2+t_1v_F-t_2v_F)^2}\right)
\end{aligned} \tag{D.9}$$

$$\langle T\ \rho_h(x_1,t_1)\rho_h(x_2,t_2)\rangle^1 = \sum_{a_1,a_2=\pm1} \frac{v_0}{2\pi^3 v_F}\ \theta(x_1x_2)\ \left(\frac{2v_F(t_1-t_2)}{(-a_1x_1+a_2x_2+v_F(t_1-t_2))^3} + \frac{1}{(a_1x_1-a_2x_2+v_F(t_1-t_2))^2}\right) \tag{D.10}$$

Hence equation (D.3) matches with equation (D.10)

APPENDIX E: Perturbative comparison of DDCF for $0 < |R| < 1$ case

The conventional method of performing perturbation expansion is the S-matrix method viz. $\rho_s(x_1,t_1) = \rho_s(x_1,\uparrow,t_1) + \rho_s(x_1,\downarrow,t_1)$,

$$\begin{aligned}
< T\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) > &= \frac{< T\ S\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) >_0}{< T\ S >_0} \\
&= < T\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) >_0 - i\int_C dt\ < T\ H_{fs}(t)\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) >_{0,c} \\
&+ \frac{(-i)^2}{2}\int_C dt \int_C dt'\ < T\ H_{fs}(t)H_{fs}(t')\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) >_{0,c}
\end{aligned} \tag{E.1}$$

where $S = e^{-i\int_C dt\ H_{fs}(t)}$ where H_{fs} is given by equation (2.10). Retaining only the most singular terms,

$$< T\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) >=< T\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) >_0 -iv_0\int_C dt\ \int dy < T\ \rho_s(x_1,t_1)\rho_s(y,t) >_0< T\ \rho_s(y,t)\rho_s(x_2,t_2) >_0$$
$$+ v_0^2(-i)^2\int_C dt\ \int_C dt^{'}\ \int dy\ \int dz\ < T\ \rho_s(x_1,t_1)\rho_s(z,t) >_0\ \ < T\ \rho_s(z,t)\rho_s(y,t^{'}) >_0\ \ < T\ \rho_s(y,t^{'})\rho_s(x_2,t_2) >_0 +.... \tag{E.2}$$

Since we are going beyond leading order, it is more convenient to work in momentum and frequency space.

$$< T\ \rho_s(x_1,t_1)\rho_s(x_2,t_2) >= \frac{1}{L^2}\sum_{q,q^{'},n} e^{-iqx_1}e^{-iq^{'}x_2}e^{-w_n(t_1-t_2)} < \rho_{q,n;.}\rho_{q^{'},-n;.} > \tag{E.3}$$

Thus the most singular parts are captured by the following perturbation series,

$$< \rho_{q,n;.}\rho_{q^{'},-n;.} > = \ < \rho_{q,n;.}\rho_{q^{'},-n;.} >_0 - \frac{v_0\beta}{L}\sum_{Q^{'}} < \rho_{q,n;.}\rho_{Q^{'},-n;.} >_0< \rho_{-Q^{'},n;.}\rho_{q^{'},-n;.} >_0$$
$$+ \frac{v_0^2\beta^2}{L^2}\sum_{Q^{'},Q^{''}} < \rho_{q,n;.}\rho_{Q^{'},-n;.} >_0< \rho_{-Q^{'},n;.}\rho_{Q^{''},-n;.} >_0< \rho_{-Q^{''},n;.}\rho_{q^{'},-n;.} >_0 +.... \tag{E.4}$$

NCBT says that when the above series is summed to all orders we get,

$$< \rho_{q,n;.}\rho_{q^{'},-n;.} >= \delta_{q+q^{'},0}\ \frac{L}{\beta}\frac{2v_Fq^2}{\pi(w_n^2+(qv_h)^2)} + \frac{|w_n|}{\pi\beta}\frac{|R|^2}{\left(1-\frac{(v_h-v_F)}{v_h}\ |R|^2\right)}\ \frac{(2v_Fq)(2v_Fq^{'})}{(w_n^2+(q^{'}v_h)^2)\,(w_n^2+(qv_h)^2)} \tag{E.5}$$

where $v_h = \sqrt{v_F^2 + \frac{2v_0v_F}{\pi}}$. However when $v_h = v_F$ (no short-range forward scattering between fermions),

$$< \rho_{q,n;.}\rho_{q^{'},-n;.} >_0 \ \ = \ \delta_{q+q^{'},0}\ \frac{L}{\beta}\frac{2v_Fq^2}{\pi(w_n^2+(qv_F)^2)} + \frac{|w_n|}{\pi\beta}|R|^2\ \frac{(2v_Fq)(2v_Fq^{'})}{(w_n^2+(q^{'}v_F)^2)\,(w_n^2+(qv_F)^2)} \tag{E.6}$$

Here we want to verify that this is consistent upto second order. The resummation to all orders has already been done using the generating function method in the main text. The series equation (E.4) may be evaluated as follows:

The term proportional to v_0 *in* $< \rho_{q,n;.}\rho_{q^{'},-n;.} > - < \rho_{q,n;.}\rho_{q^{'},-n;.} >_0$ *in the series equation (E.4) is:*

$$-\frac{4Lq^2q^{'2}v_0v_F^2\delta_{0,q+q^{'}}}{\pi^2\beta\left(q^2v_F^2+w_n^2\right)\left(q^{'2}v_F^2+w_n^2\right)} + \frac{L\int_{-\infty}^{\infty}\frac{16qq_1^2q^{'}|R|^4v_0v_F^4|w_n|^2}{\pi^2\beta L\left(q^2v_F^2+w_n^2\right)\left(q_1^2v_F^2+w_n^2\right)^2\left(q^{'2}v_F^2+w_n^2\right)}dq_1}{2\pi}$$
$$-\frac{8qq^{'3}|R|^2v_0v_F^3\,|w_n|}{\pi^2\beta\left(q^2v_F^2+w_n^2\right)\left(q^{'2}v_F^2+w_n^2\right)^2} - \frac{8q^3q^{'}|R|^2v_0v_F^3\,|w_n|}{\pi^2\beta\left(q^2v_F^2+w_n^2\right)^2\left(q^{'2}v_F^2+w_n^2\right)} \tag{E.7}$$

This simplifies to,

$$T_1 = -\frac{4qq^{'}v_0v_F\left(Lqq^{'}v_F\left(q^2v_F^2+w_n^2\right)\left(q^{'2}v_F^2+w_n^2\right)\delta_{0,q+q^{'}} - |R|^2|w_n|\left(q^2q^{'2}(|R|^2-4)v_F^4+(|R|^2-2)v_F^2w_n^2\left(q^2+q^{'2}\right)+|R|^2w_n^4\right)\right)}{\pi^2\beta\left(q^2v_F^2+w_n^2\right)^2\left(q^{'2}v_F^2+w_n^2\right)^2} \tag{E.8}$$

The term proportional to v_0^2 *in* $< \rho_{q,n;.}\rho_{q',-n;.} > - < \rho_{q,n;.}\rho_{q',-n;.} >_0$ *in the series equation (E.4) is:*

$$\begin{aligned}
&\frac{8Lq^2q^4v_0^2v_F^3\delta_{0,q+q'}}{\pi^3\beta\left(q^2v_F^2+w_n^2\right)\left(q^2v_F^2+w_n^2\right)\left(q^2v_F^2+w_n^2\right)} \\
&+\left(\frac{L}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{64qq'Q'^2Q''^2|R|^6v_0^2v_F^6\,|w_n|^3}{\pi^3\beta L^2\left(q^2v_F^2+w_n^2\right)\left(q'^2v_F^2+w_n^2\right)\left(Q'^2v_F^2+w_n^2\right)^2\left(Q''^2v_F^2+w_n^2\right)^2}dQ''dQ' \\
&-\frac{L}{2\pi}\int_{-\infty}^{\infty}\frac{32qq'Q'^3Q'|R|^4v_0^2v_F^5\,|w_n|^2}{\pi^3\beta L\left(q^2v_F^2+w_n^2\right)\left(q'^2v_F^2+w_n^2\right)\left(Q'^2v_F^2+w_n^2\right)^2\left(Q'^2v_F^2+w_n^2\right)}dQ' \\
&+\frac{L}{2\pi}\int_{-\infty}^{\infty}\frac{32q^2(-q)q'Q''^2|R|^4v_0^2v_F^5\,|w_n|^2}{\pi^3\beta L\left(q^2v_F^2+w_n^2\right)\left(q^2v_F^2+w_n^2\right)\left(q'^2v_F^2+w_n^2\right)\left(Q''^2v_F^2+w_n^2\right)^2}dQ'' \\
&-\frac{L}{2\pi}\int_{-\infty}^{\infty}\frac{32qq'^3Q'^2|R|^4v_0^2v_F^5\,|w_n|^2}{\pi^3\beta L\left(q^2v_F^2+w_n^2\right)\left(q'^2v_F^2+w_n^2\right)^2\left(Q'^2v_F^2+w_n^2\right)^2}dQ' \\
&-\frac{16q^2(-q)^3q'|R|^2v_0^2v_F^4\,|w_n|}{\pi^3\beta\left(q^2v_F^2+w_n^2\right)\left(q^2v_F^2+w_n^2\right)\left(q^2v_F^2+w_n^2\right)\left(q'^2v_F^2+w_n^2\right)} \\
&+\frac{16qq'^3q'^2|R|^2v_0^2v_F^4\,|w_n|}{\pi^3\beta\left(q^2v_F^2+w_n^2\right)\left(q'^2v_F^2+w_n^2\right)^2\left(q'^2v_F^2+w_n^2\right)}-\frac{16q^2(-q)q'^3|R|^2v_0^2v_F^4\,|w_n|}{\pi^3\beta\left(q^2v_F^2+w_n^2\right)\left(q^2v_F^2+w_n^2\right)\left(q'^2v_F^2+w_n^2\right)^2}
\end{aligned} \tag{E.9}$$

This simplifies to,

$$\begin{aligned}
T_2 = &\frac{2qq'v_0^2}{\pi^3\beta(q^2v_F^2+w_n^2)^3(q'^2v_F^2+w_n^2)^3}\Big(4Lq^2q'^2v_F^3(q^2v_F^2+w_n^2)(q'^2v_F^2+w_n^2)(qq'v_F^2-w_n^2)\delta_{0,q+q'} \\
&+|R|^2\,|w_n|\,(q^4q'^4(|R|^2(2|R|^2-11)+24)v_F^8+2q^2q'^2(|R|^2(2|R|^2-9)+12)v_F^6w_n^2(q^2+q'^2) \\
&+2|R|^2(2|R|^2-5)v_F^2w_n^6(q^2+q'^2)+v_F^4w_n^4((|R|^2(2|R|^2-7)+8)(q^4+q'^4) \\
&+4q^2q'^2(|R|^2(2|R|^2-7)+2))+|R|^2(2|R|^2-3)w_n^8)\Big)
\end{aligned} \tag{E.10}$$

It is easy to verify that both T_1 and T_2 may also be obtained by simply expanding equation (E.5) in powers of v_0 and retaining upto order v_0^2.

APPENDIX F: DDOS for right movers

The dynamical density of states for the right movers is given by the equation

$$D(\omega;x,T) \;=\; \int d\tau\; e^{-i\omega\tau} < \{\psi(x,T+\frac{\tau}{2}),\psi^\dagger(x,T-\frac{\tau}{2})\} > \tag{F.1}$$

Using Eq.3.13 we can write,

$$\begin{aligned}
&< \{\psi_R(x,T+\frac{\tau}{2}),\psi_R^\dagger(x',T-\frac{\tau}{2})\} > \\
&\quad= \frac{i}{2\pi}\left(\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F\tau+iv_F\epsilon))}-\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F\tau-iv_F\epsilon))}\right) \\
&\quad\left(U(T-\frac{\tau}{2},T+\frac{\tau}{2})\left[1-\theta(x')\;\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\;\left[1-\theta(x)\;\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\right. \\
&\quad\left.+\left(\frac{\Gamma}{v_F}\;\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2\;\theta(x)\theta(x')\;U(T-\frac{\tau}{2},T-\frac{\tau}{2}-\frac{x'}{v_F})\;U(T+\frac{\tau}{2}-\frac{x}{v_F},T+\frac{\tau}{2})\right)
\end{aligned} \tag{F.2}$$

where we have taken $t = T + \frac{\tau}{2}$ and $t^{'} = T - \frac{\tau}{2}$ and $\epsilon > 0$. In the zero temperature limit $\beta \to \infty$ doing an expansion in powers of ϵ and finally taking the limit $\epsilon \to 0$ we can write

$$\frac{i}{2\pi}\left(\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x - x^{'} - v_F\tau + iv_F\epsilon))} - \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x - x^{'} - v_F\tau - iv_F\epsilon))}\right) = \tag{F.3}$$

$$\frac{\epsilon/\pi}{2v_F\left((\tau + \frac{-x+x^{'}}{v_F})^2 + \epsilon^2\right)} = \frac{1}{2v_F}\delta(\tau + \frac{-x + x^{'}}{v_F}) \tag{F.4}$$

This means,

$$< \{\psi_R(x, T + \frac{\tau}{2}), \psi_R^{\dagger}(x^{'}, T - \frac{\tau}{2})\} >$$
$$= \frac{1}{2v_F}\delta(\tau + \frac{-x + x^{'}}{v_F})U(T - \frac{x - x^{'}}{2v_F}, T + \frac{x - x^{'}}{2v_F})\ \left(\left[1 - \theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\ \left[1 - \theta(x)\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]\right.$$
$$\left. + \left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2\ \theta(x)\theta(x^{'})\right) \tag{F.5}$$

At $x^{'} = x$, the DDOS for the right movers is,

$$D(\omega; x, T)\ =\ \int d\tau\ e^{-i\omega\tau} < \{\psi_R(x, T + \frac{\tau}{2}), \psi_R^{\dagger}(x, T - \frac{\tau}{2})\} >$$
$$= \frac{1}{2v_F}\ \left(\left[1 - \theta(x)\ \frac{2\Gamma^2}{\Gamma^2 + 4v_F^2}\right]^2 + \left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2 + 4v_F^2}\right)^2\ \theta(x)\right)\ =\ \frac{1}{2v_F} \tag{F.6}$$

Appendix G: Calculation of the RR Green function from the unconventional bosonization ansatz

In this appendix we show in detail the calculation of the RR Green function from the unconventional bosonization (NCBT) ansatz. The LL Green function is also obtained in a similar manner and hence is not shown separately. We have to begin with,

$$< T\ \Psi_R(x,t)\Psi_R^{\dagger}(x^{'},t^{'}) >\ =\ \theta(-x)\theta(-x^{'}) < T\ \tilde{\psi}_R(x,t)\tilde{\psi}_R^{\dagger}(x^{'},t^{'}) >_{\lambda=0}$$
$$+\ C_1 < T\ \hat{\psi}_R(x,t)\hat{\psi}_R^{\dagger}(x^{'},t^{'}) >_{\lambda=1} + C_2\ < T\ \tilde{\psi}_R(x,t)\tilde{\psi}_R^{\dagger}(x^{'},t^{'}) >_{\lambda=1} \tag{G.1}$$

Using Eqs.4.27 and 4.28 we can write this as,

$$< T\ \Psi_R(x,t)\Psi_R^{\dagger}(x^{'},t^{'}) >$$
$$=\ \theta(-x)\theta(-x^{'})U(t^{'},t)\ < e^{2\pi i\int^{x}\rho_R(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}}\ \rho_R(y^{'},t^{'})\ dy^{'}}\ >$$
$$+ C_1\ U(t^{'},t^{'} - \frac{x^{'}}{v_F})U(t - \frac{x}{v_F},t)\ \frac{1}{2}\left(< e^{2\pi i\int^{x}(\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{-2\pi i\int^{x^{'}}\ \rho_R(y^{'},t^{'})dy^{'}}\ >\right.$$
$$\left. + < e^{2\pi i\int^{x}\rho_R(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}}(\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))\ dy^{'}}\ >\right)$$
$$+\ C_2\ U(t^{'},t)\ \frac{1}{2}\left(< e^{2\pi i\int^{x}(\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{-2\pi i\int^{x^{'}}\ \rho_R(y^{'},t^{'})dy^{'}}\ >\right.$$
$$\left. + < e^{2\pi i\int^{x}\rho_R(y,t)\ dy}\ e^{-2\pi i\int^{x^{'}}(\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))\ dy^{'}}\ >\right) \tag{G.2}$$

Let us now evaluate the expectation using a version of the Baker-Campbell-Haussdorff formula (Eq. 4.21),

$$< e^{2\pi i\int^{x}(\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{-2\pi i\int^{x^{'}}\ \rho_R(y^{'},t^{'})dy^{'}}\ > = e^{\frac{1}{2}(2\pi i)^2\int^{x}dy\int^{x}dy^{'}<(\rho_R(y,t)+\rho_L(-y,t))(\rho_R(y^{'},t)+\rho_L(-y^{'},t))>}$$
$$e^{\frac{1}{2}(2\pi i)^2\int^{x^{'}}\ dy\int^{x^{'}}\ dy^{'}<\rho_R(y,t^{'})\rho_R(y^{'},t^{'})>}$$
$$e^{-(2\pi i)^2\int^{x}dy\int^{x^{'}}\ dy^{'}<(\rho_R(y,t)+\rho_L(-y,t))\rho_R(y^{'},t^{'})>} \tag{G.3}$$

Using the form of the density-density correlation functions in Sec.4.3 we evaluate the integrals in the exponents,

$$< e^{2\pi i \int^{x}(\rho_R(y,t)+\rho_L(-y,t))\ dy}\ e^{-2\pi i \int^{x'} \rho_R(y',t')dy'} > \ = \ \sinh(\frac{\pi\epsilon}{\beta v_F})\ (\theta(x')\mathscr{C}_1 + \theta(-x')\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}}) \frac{1}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F(t-t')))} \tag{G.4}$$

Here ϵ is a regularization factor that will eventually be taken to zero ($\epsilon \to 0$) and $\mathscr{C}_1 = e^{\frac{1}{2}(2\pi i)^2 \int^{x'} dy' \int^{x'} dy\ <\rho_R(y',t')\rho_R(y,t')>}$ evaluated with $x' > 0$ is ultimately a constant term. Similarly we get,

$$< e^{2\pi i \int^{x} \rho_R(y,t)\ dy}\ \ e^{-2\pi i \int^{x'} (\rho_R(y',t')+\rho_L(-y',t'))\ dy'} >= \sinh(\frac{\pi\epsilon}{\beta v_F})\ (\theta(x)\mathscr{C}_1 + \theta(-x)\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}}) \frac{1}{\sinh(\frac{\pi}{\beta v_F}(x-x'-v_F(t-t')))} \tag{G.5}$$

On comparing with the exact Green functions in Eq.4.2 we see that for the first term the constant prefactors are of the form $\frac{i}{2\beta v_F}\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2 \theta(x)\theta(x')$ and since bosonization doesn't give the prefactors we fix C_1 such that

$$C_1 = \frac{i}{2\beta v_F} \frac{\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2}{\sinh(\frac{\pi\epsilon}{\beta v_F})\ \mathscr{C}_1}\theta(x)\theta(x') \tag{G.6}$$

so that the term in the denominator cancels with a corresponding term that comes from evaluating the expectation values of the bosonized fields (Eqs.G.4 and G.5) and substituting in Eq.G.2. Note that the $\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}}$ term drops out since C_1 is non-zero only for $x, x' > 0$.

Now in the second term the prefactor is $\frac{i}{2\beta v_F}\left(1-\ \theta(x')\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right)\left(1-\ \theta(x)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right)$, hence we fix C_2 to be

$$C_2 = \frac{i}{2\beta v_F} \frac{\left(1-\ \theta(x')\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right)\left(1-\ \theta(x)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right) - \theta(-x)\theta(-x')}{\sinh(\frac{\pi\epsilon}{\beta v_F})\ \frac{1}{2}((\theta(x)+\theta(x'))\mathscr{C}_1 + (\theta(-x)+\theta(-x'))\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})} \tag{G.7}$$

The term in the denominator cancels with the same term appears from substituting Eqs.G.4 and G.5 in Eq.G.2. So evaluating Eq.G.2 reproduces the correct exact RR Green function as in Eq.4.2.

$$\begin{aligned} < T\ \Psi_R(x,t)\Psi_R^{\dagger}(x',t') > \ &= \ \left(\frac{i}{2\beta v_F}\left(\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2 \theta(x)\theta(x')\ U(t',t'-\frac{x'}{v_F})U(t-\frac{x}{v_F},t)\right. \\ &+ \frac{i}{2\beta v_F}\left(1-\ \theta(x')\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right) \\ &\left.\left(1-\ \theta(x)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right)\ U(t',t)\right)\ csch(\frac{\pi}{\beta v_F}(x-x'-v_F(t-t'))) \end{aligned} \tag{G.8}$$

A similar calculation gives us the correct LL Green function as well.

Appendix H: Calculation of the LR Green function from the unconventional bosonization ansatz

Here we show the calculation of the LR Green function from the unconventional bosonization ansatz. The RL Green function is also obtained in a similar manner and hence is not shown separately. We have

$$< T\ \Psi_L(x,t)\Psi_R^{\dagger}(x',t') > \ = \ C_3\ < T\ \tilde{\psi}_L(x,t)\hat{\psi}_R^{\dagger}(x',t') >_{\lambda=1}\ -\ C_4\ < T\ \hat{\psi}_L(x,t)\tilde{\psi}_R^{\dagger}(x',t') >_{\lambda=1} \tag{H.1}$$

Using Eqns.4.30 and 4.32 we write,

$$
\begin{aligned}
&< T\ \Psi_L(x,t)\Psi_R^\dagger(x',t') > \ = \\
&C_3\ U(t',t-\frac{x'}{v_F})\ \frac{1}{2}\Big(< e^{-2\pi i\int^x \rho_L(y,t)\ dy}\ e^{-2\pi i\int^{x'}(\rho_R(y',t')+\rho_L(-y',t'))\ dy'} > \\
&+ < e^{-2\pi i\int^x(\rho_L(y,t)+\rho_R(-y,t))\ dy}\ e^{-2\pi i\int^{x'}\rho_R(y',t')\ dy'} > \Big) \\
&- C_4\ U(t',t+\frac{x}{v_F})\ \frac{1}{2}\Big(< e^{-2\pi i\int^x \rho_L(y,t)\ dy}\ e^{-2\pi i\int^{x'}(\rho_R(y',t')+\rho_L(-y',t'))\ dy'} > \\
&+ \ < e^{-2\pi i\int^x(\rho_L(y,t)+\rho_R(-y,t))\ dy}\ e^{-2\pi i\int^{x'}\rho_R(y',t')\ dy'} > \Big)
\end{aligned} \tag{H.2}
$$

Let us evaluate

$$
\begin{aligned}
&< e^{-2\pi i\int^x \rho_L(y,t)\ dy}\quad e^{-2\pi i\int^{x'}(\rho_R(y',t')+\rho_L(-y',t'))\ dy'} > \\
&= \ e^{\frac{1}{2}(2\pi i)^2\int^x dy\int^x dy' <\rho_L(y,t)\rho_L(y',t)>}\ e^{\frac{1}{2}(2\pi i)^2\int^{x'} dy\int^{x'} dy' <(\rho_R(y,t')+\rho_L(-y,t'))(\rho_R(y',t')+\rho_L(-y',t'))>} \\
&\ e^{(2\pi i)^2\int^x dy\int^{x'} dy' <\rho_L(y,t)(\rho_R(y',t')+\rho_L(-y',t'))>}
\end{aligned} \tag{H.3}
$$

Using the form of the density-density correlation functions in Sec.4.3 we evaluate the integrals in the exponents,

$$
\begin{aligned}
< e^{-2\pi i\int^x \rho_L(y,t)\ dy}\quad e^{-2\pi i\int^{x'}(\rho_R(y',t')+\rho_L(-y',t'))\ dy'} > \ &= \ \sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(-x)\mathscr{C}_2+\theta(x)\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}}) \\
&\frac{1}{\sinh(\frac{\pi}{\beta v_F}(x+x'+v_F(t-t')))}
\end{aligned} \tag{H.4}
$$

Here ϵ is a regularization factor that is eventually made zero and $\mathscr{C}_2 = e^{\frac{1}{2}(2\pi i)^2\int^{x'} dy'\int^{x'} dy\ <\rho_L(y,t')\rho_L(y',t')>}$ with $x' < 0$ is a constant term. Similarly we get

$$
\begin{aligned}
< e^{-2\pi i\int^x(\rho_L(y,t)+\rho_R(-y,t))\ dy}\ e^{-2\pi i\int^{x'}\rho_R(y',t')\ dy'} > \ &= \ \sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(x')\mathscr{C}_1+\theta(-x')\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}}) \\
&\frac{1}{\sinh(\frac{\pi}{\beta v_F}(x+x'+v_F(t-t')))}
\end{aligned} \tag{H.5}
$$

On comparing with the exact Green functions in Eq.4.2 we fix

$$
C_3 = -\frac{i}{2\beta v_F}\frac{\left[1-\theta(-x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}}{\frac{1}{2}\sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(-x)\mathscr{C}_2+\mathscr{C}_1+\theta(x)\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})}\theta(x') \tag{H.6}
$$

such that the terms in the denominator cancel with corresponding terms that appear from substituting Eqs.H.4 and H.5 in Eq.H.2 and we get the correct prefactors. For the other prefactor we fix it to be

$$
C_4 = -\frac{i}{2\beta v_F}\frac{\left[1-\theta(x')\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}}{\frac{1}{2}\sinh(\frac{\pi\epsilon}{\beta v_F})(\theta(x')\mathscr{C}_1+\mathscr{C}_2+\theta(-x')\sinh(\frac{\pi\epsilon}{\beta v_F})^{\frac{1}{2}})}\theta(-x) \tag{H.7}
$$

so that the terms in the denominator cancel with corresponding terms that appear from substituting Eqs.H.4 and H.5 in Eq.H.2 and we get the correct prefactors. So upon evaluating Eq.H.2 we obtain the exact LR Green functions as in Eq.4.2,

$$< T \ \Psi_L(x,t)\Psi_R^\dagger(x^{'},t^{'}) > \ = \Bigg(\frac{i}{2\beta v_F}\left[1-\theta(-x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\theta(x^{'}) \\ U(t^{'},t^{'}-\frac{x^{'}}{v_F}) - \frac{i}{2\beta v_F}\left[1-\theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] \\ i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\theta(-x)\ U(t^{'},t+\frac{x}{v_F})\Bigg) \\ csch(\frac{\pi}{\beta v_F}(-x-x^{'}-v_F(t-t^{'}))) \tag{H.8}$$

Appendix I: Evaluating the four-point functions using unconventional bosonization ansatz

First let us consider the general form of following four-point function,

$$< T \ \rho_R(x_1,t_1)\psi_R^\dagger(x,t)\psi_R(x^{'},t^{'}) > \ = \ - < T \ \psi_R(x^{'},t^{'})\psi_R^\dagger(x_1,t_{1+}) > \ < T \ \psi_R(x_1,t_1)\psi_R^\dagger(x,t) > \\ = \ w_0 \ < \rho_R(x_1,t_1) e^{-2\pi i \int^x dy \ \rho_R(y,t)} \\ e^{2\pi i \int^{x^{'}} dy^{'} \ \rho_R(y^{'},t^{'})} > \\ + w_{11} \ < \rho_R(x_1,t_1) e^{-2\pi i \int^x dy \ (\rho_R(y,t)+\rho_L(-y,t))} \\ e^{2\pi i \int^{x^{'}} dy^{'} \ \rho_R(y^{'},t^{'})} > \\ + w_{21} \ < \rho_R(x_1,t_1) e^{-2\pi i \int^x dy \ \rho_R(y,t)} \\ e^{2\pi i \int^{x^{'}} dy^{'} \ (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))} > \tag{I.1}$$

where w_0, w_{11} and w_{21} are prefactors. We can write $\rho_R(x_1,t_1) = \lim_{a\to 0}\frac{d}{da}e^{a\ \rho_R(x_1,t_1)}$ hence we get,

$$- < T \ \psi_R(x^{'},t^{'})\psi_R^\dagger(x_1,t_1) >< T \ \psi_R(x_1,t_1)\psi_R^\dagger(x,t) > \\ = \ w_0\Big(-2\pi i \int^x dy \ \ < T \ \rho_R(x_1,t_1)\rho_R(y,t) > +2\pi i \int^{x^{'}} dy^{'} \ \ < T \ \rho_R(x_1,t_1)\rho_R(y^{'},t^{'}) > \Big) \\ e^{\frac{1}{2}(2\pi i)^2 \int^x dy \ \int^x dy^{'} \ <T \ \rho_R(y,t)\rho_R(y^{'},t)>} \\ e^{\frac{1}{2}(2\pi i)^2 \int^{x^{'}} dy^{'} \ \int^{x^{'}} dy \ <T \ \rho_R(y^{'},t^{'})\rho_R(y,t^{'})>} \ e^{-(2\pi i)^2 \int^x dy \ \int^{x^{'}} dy^{'} \ <T\rho_R(y,t)\rho_R(y^{'},t^{'})>} \\ + \ w_{11}\Big(-2\pi i \int^x dy \ \ < T \ \rho_R(x_1,t_1)(\rho_R(y,t)+\rho_L(-y,t)) > +2\pi i \int^{x^{'}} dy^{'} \ \ < T \ \rho_R(x_1,t_1)\rho_R(y^{'},t^{'}) > \Big) \\ e^{\frac{1}{2}(2\pi i)^2 \int^x dy \ \int^x dy^{'} \ <T \ (\rho_R(y,t)+\rho_L(-y,t))(\rho_R(y^{'},t)+\rho_L(-y^{'},t))>} \ e^{\frac{1}{2}(2\pi i)^2 \int^{x^{'}} dy^{'} \ \int^{x^{'}} dy \ <T \ \rho_R(y^{'},t^{'})\rho_R(y,t^{'})>} \\ e^{-(2\pi i)^2 \int^x dy \ \int^{x^{'}} dy^{'} \ <T \ (\rho_R(y,t)+\rho_L(-y,t))\rho_R(y^{'},t^{'})>} + w_{21}\Big(-2\pi i \int^x dy \ \ < T \ \rho_R(x_1,t_1)\rho_R(y,t) > \\ + 2\pi i \int^{x^{'}} dy^{'} \ \ < T \ \rho_R(x_1,t_1)(\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'})) > \Big) \\ e^{\frac{1}{2}(2\pi i)^2 \int^x dy \ \int^x dy^{'} \ <T \ \rho_R(y,t)\rho_R(y^{'},t)>} \ e^{\frac{1}{2}(2\pi i)^2 \int^{x^{'}} dy^{'} \ \int^{x^{'}} dy \ <T \ (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))(\rho_R(y,t^{'})+\rho_L(-y,t^{'}))>} \\ e^{-(2\pi i)^2 \int^x dy \ \int^{x^{'}} dy^{'} \ <T \ \rho_R(y,t)(\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))>} \tag{I.2}$$

We define the symmetric and antisymmetric density fields as follows

$$\rho_{sym}(x,t) \equiv \rho_R(x,t) + \rho_L(-x,t) \\ \rho_{asy}(x,t) \equiv \rho_R(x,t) - \rho_L(-x,t) \tag{I.3}$$

This means that we can write

$$\begin{aligned} \rho_R(x,t) &= \frac{\rho_{sym}(x,t)+\rho_{asy}(x,t)}{2} \\ \rho_L(-x,t) &= \frac{\rho_{sym}(x,t)-\rho_{asy}(x,t)}{2} \end{aligned} \tag{I.4}$$

Using this in Eq.I.2 we get,

$$\begin{aligned}
&- <T\ \psi_R(x^{'},t^{'})\psi_R^{\dagger}(x_1,t_1)><T\ \psi_R(x_1,t_1)\psi_R^{\dagger}(x,t)> \\
&= \ w_0\Big(-2\pi i\int^x dy\ \frac{1}{4}(<T\rho_{sym}(x_1,t_1)\rho_{sym}(y,t)> + <T\rho_{asy}(x_1,t_1)\rho_{asy}(y,t)> \\
&\quad + 2\pi i\int^{x^{'}} dy^{'}\ \frac{1}{4}(<T\rho_{sym}(x_1,t_1)\rho_{sym}(y^{'},t^{'})> + <T\rho_{asy}(x_1,t_1)\rho_{asy}(y^{'},t^{'})>)\Big) \\
&\quad e^{\frac{1}{2}(2\pi i)^2\frac{1}{4}\int^x dy\ \int^x dy^{'}\ (<T\rho_{sym}(y,t)\rho_{sym}(y^{'})>+<T\rho_{asy}(y,t)\rho_{asy}(y^{'},t)>)} \\
&\quad e^{\frac{1}{2}(2\pi i)^2\frac{1}{4}\int^{x^{'}} dy^{'}\ \int^{x^{'}} dy\ (<T\rho_{sym}(y^{'},t^{'})\rho_{sym}(y,t^{'})>+<T\rho_{asy}(y^{'},t^{'})\rho_{asy}(y,t^{'})>)} \\
&\quad e^{-(2\pi i)^2\frac{1}{4}\int^x dy\ \int^{x^{'}} dy^{'}\ (<T\rho_{sym}(y,t)\rho_{sym}(y^{'},t^{'})>+<T\rho_{asy}(y,t)\rho_{asy}(y^{'},t^{'})>)} \\
&\quad + w_{11}\ (-2\pi i\int^x dy\ \frac{1}{2} <T\rho_{sym}(x_1,t_1)\rho_{sym}(y,t)> \\
&\quad + 2\pi i\frac{1}{4}\int^{x^{'}} dy^{'}\ (<T\rho_{sym}(x_1,t_1)\rho_{sym}(y^{'},t^{'})> + <T\rho_{asy}(x_1,t_1)\rho_{asy}(y^{'},t^{'})>)) \\
&\quad e^{\frac{1}{2}(2\pi i)^2\int^x dy\ \int^x dy^{'}\ <T\rho_{sym}(y,t)\rho_{sym}(y^{'},t)>} e^{\frac{1}{2}(2\pi i)^2\frac{1}{4}\int^{x^{'}} dy^{'}\ \int^{x^{'}} dy\ (<T\rho_{sym}(y^{'},t^{'})\rho_{sym}(y,t^{'})>+<T\rho_{asy}(y^{'},t^{'})\rho_{asy}(y,t^{'})>)} \\
&\quad e^{-(2\pi i)^2\frac{1}{2}\int^x dy\ \int^{x^{'}} dy^{'}\ <T\rho_{sym}(y,t)\rho_{sym}(y^{'},t^{'})>} \\
&\quad + \ w_{21}\Big(-2\pi i\frac{1}{4}\int^x dy\ (<T\rho_{sym}(x_1,t_1)\rho_{sym}(y,t)> + <T\rho_{asy}(x_1,t_1)\rho_{asy}(y,t)> \\
&\quad + 2\pi i\frac{1}{2}\int^{x^{'}} dy^{'}\ <T\rho_{sym}(x_1,t_1)\rho_{sym}(y^{'},t^{'})>\Big) \\
&\quad e^{\frac{1}{2}(2\pi i)^2\frac{1}{4}\int^x dy\ \int^x dy^{'}\ (<T\rho_{sym}(y,t)\rho_{sym}(y^{'},t)>+<T\rho_{asy}(y,t)\rho_{asy}(y^{'},t)>)} e^{\frac{1}{2}(2\pi i)^2\int^{x^{'}} dy^{'}\ \int^{x^{'}} dy\ <T\rho_{sym}(y^{'},t^{'})\rho_{sym}(y,t^{'})>} \\
&\quad e^{-(2\pi i)^2\frac{1}{2}\int^x dy\ \int^{x^{'}} dy^{'}\ <T\rho_{sym}(y,t)\rho_{sym}(y^{'},t^{'})>}
\end{aligned} \tag{I.5}$$

The correlations of the symmetric and antisymmetric fields are,

$$<T\ \rho_{sym}(x,t)\rho_{sym}(x^{'},t^{'})>_0 \ = \ 2\left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'})))}\right]^2 \tag{I.6}$$

and

$$\begin{aligned} &<T\ \rho_{asy}(x,t)\rho_{asy}(x^{'},t^{'})>_0 \ = \\ &\frac{-((4v_F^2+\Gamma^2)^2-32v_F^2\Gamma^2\theta(x))((4v_F^2+\Gamma^2)^2-32v_F^2\Gamma^2\theta(x^{'}))-64(-4v_F^3\Gamma+v_F\Gamma^3)^2\cos(\frac{-eV_b(x-x^{'}-v_F(t-t^{'}))}{v_F})\ \theta(x)\theta(x^{'})}{2v_F^2\beta^2(4v_F^2+\Gamma^2)^4\sinh(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'})))^2} \end{aligned} \tag{I.7}$$

Let us consider the cases $x>0, x^{'}>0$ or $x>0, x^{'}<0$ or $x<0, x^{'}>0$, the $\lambda=0$ term does not reproduce the correct terms for these cases as is evident from the presence of terms of the type

$$e^{-(2\pi i)^2\frac{1}{4}\int^x dy\ \int^{x^{'}} dy^{'}\ (<T\rho_{sym}(y,t)\rho_{sym}(y^{'},t^{'})>+<T\rho_{asy}(y,t)\rho_{asy}(y^{'},t^{'})>)}$$

. So for these cases $w_0 = 0$ and we can write Eq.I.5 as,

$$
\begin{aligned}
&- < T\ \psi_R(x^{'}, t^{'}) \psi_R^{\dagger}(x_1, t_{1+}) >< T\ \psi_R(x_1, t_1) \psi_R^{\dagger}(x, t) > \;\; = \\
&\quad w_{11}\Big(- 2\pi i \int^{x} dy\ \frac{1}{2} 2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x_1 - y - v_F(t_1 - t)))} \right]^2 \\
&\quad + 2\pi i \frac{1}{4} \int^{x^{'}} dy^{'}\ (2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x_1 - y^{'} - v_F(t_1 - t^{'})))} \right]^2 + < \rho_{asy}(x_1, t_1) \rho_{asy}(y^{'}, t^{'}) >) \Big) \\
&\quad e^{\frac{1}{2}(2\pi i)^2 \int^{x} dy\ \int^{x} dy^{'}\ 2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y - y^{'}))} \right]^2} \\
&\quad e^{\frac{1}{2}(2\pi i)^2 \frac{1}{4} \int^{x^{'}} dy^{'}\ \int^{x^{'}} dy\ (2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y^{'} - y))} \right]^2 + <\rho_{asy}(y^{'}, t^{'}) \rho_{asy}(y, t^{'})>)} \\
&\quad e^{-(2\pi i)^2 \frac{1}{2} \int^{x} dy\ \int^{x^{'}} dy^{'}\ 2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y - y^{'} - v_F(t - t^{'})))} \right]^2} \\
&\quad + \ w_{21}\Big(- 2\pi i \frac{1}{4} \int^{x} dy\ (2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x_1 - y - v_F(t_1 - t)))} \right]^2 + < \rho_{asy}(x_1, t_1) \rho_{asy}(y, t) > \\
&\quad + 2\pi i \frac{1}{2} \int^{x^{'}} dy^{'} 2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x_1 - y^{'} - v_F(t_1 - t^{'})))} \right]^2 \Big) \\
&\quad e^{\frac{1}{2}(2\pi i)^2 \frac{1}{4} \int^{x} dy\ \int^{x} dy^{'}\ (2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y - y^{'}))} \right]^2 + <\rho_{asy}(y, t) \rho_{asy}(y^{'}, t)>)} \\
&\quad e^{\frac{1}{2}(2\pi i)^2 \int^{x^{'}} dy^{'}\ \int^{x^{'}} dy\ 2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y^{'} - y))} \right]^2} \\
&\quad e^{-(2\pi i)^2 \frac{1}{2} \int^{x} dy\ \int^{x^{'}} dy^{'}\ 2 \left[\frac{i}{2\pi} \frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y - y^{'} - v_F(t - t^{'})))} \right]^2}
\end{aligned}
\tag{I.8}
$$

We include only the most singular terms of the expression in the RHS of the above equation. Then Eq.I.8 reduces to,

$$
\begin{aligned}
-<T\ \psi_R(x^{'},t^{'})\psi_R^{\dagger}(x_1,t_{1+})><T\ \psi_R(x_1,t_1)\psi_R^{\dagger}(x,t)> \ &= \\
& -\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x^{'}-x_1-v_F(t^{'}-t_1)))}\Bigg(U(t_1,t^{'})\left[1-\theta(x_1)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\ \left[1-\theta(x^{'})\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] \\
& +\left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2\ \theta(x_1)\theta(x^{'})\ U(t_1,t_1-\frac{x_1}{v_F})\ U(t^{'}-\frac{x^{'}}{v_F},t^{'})\Bigg) \\
& \frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x_1-x-v_F(t_1-t)))} \\
& \Bigg(U(t,t_1)\left[1-\theta(x)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\ \left[1-\theta(x_1)\ \frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right] \\
& +\left(\frac{\Gamma}{v_F}\ \frac{(2v_F)^2}{\Gamma^2+4v_F^2}\right)^2\ \theta(x)\theta(x_1)\ U(t,t-\frac{x}{v_F})\ U(t_1-\frac{x_1}{v_F},t_1)\Bigg) \\
& =\ w_{11}\ \frac{i}{4\beta v_F} \\
& \Bigg(2\coth\left(\frac{\pi(v_F(t-t_1)-x+x_1)}{\beta v_F}\right)-\frac{2\left(\left(\Gamma^2+4v_F^2\right)^2-16\Gamma^2v_F^2(-2\theta(x_1,x^{'})+\theta(x_1)+\theta(x^{'}))\right)}{\left(\Gamma^2+4v_F^2\right)^2} \\
& \coth\left(\frac{\pi(x_1-x^{'}-v_F(t_1-t^{'}))}{\beta v_F}\right)\Bigg)\ e^{\Delta}e^{\zeta(x^{'})}\ \mathrm{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right) \\
& +w_{21}\Bigg((-2\pi i)\frac{1}{4}\ \frac{\left(\frac{32\Gamma^2v_F^2(-2\theta(x,x_1)+\theta(x)+\theta(x_1))}{\left(\Gamma^2+4v_F^2\right)^2}-2\right)}{2\pi\beta v_F}\ \coth\left(\frac{\pi(v_F(t-t_1)-x+x_1)}{\beta v_F}\right) \\
& -\frac{i}{2v_F\beta}\coth\left(\frac{\pi(v_F(t^{'}-t_1)-x^{'}+x_1)}{\beta v_F}\right)\Bigg)\ e^{\zeta(x)}e^{\Delta}\ \mathrm{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right) \qquad \text{(I.9)}
\end{aligned}
$$

where

$$
\zeta(x)\ =\ \frac{1}{2}(2\pi i)^2\frac{1}{4}\int^{x}dy\int^{x}dy^{'}\ (2\left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y-y^{'}))}\right]^2+<\rho_{asy}(y,t)\rho_{asy}(y^{'},t)>)
$$

and

$$
\Delta\ =\ \frac{1}{2}(2\pi i)^2\int^{x^{'}}dy^{'}\int^{x^{'}}dy\ 2\left[\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(y^{'}-y))}\right]^2
$$

It is easy to appropriately fix the prefactors w_{11} and w_{21} such that the correct Wick's theorem expression for the four-point function is obtained. For the case $x, x^{'}, x_1 < 0$ Eq.I.5 reduces to

$$
\begin{aligned}
&-\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x^{'}-x_1-v_F(t^{'}-t_1)))}\ U(t,t^{'})\ \frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x_1-x-v_F(t_1-t)))} \\
&= \frac{2\pi i\ w_0}{4\pi\beta v_F}\ (\coth\left(\frac{\pi(x_1-x+v_F(t-t_1))}{\beta v_F}\right) \\
&\quad -\coth\left(\frac{\pi(x_1-x^{'}-v_F(t_1-t^{'}))}{\beta v_F}\right))\ \ \sinh\left(\frac{\pi\delta}{\beta v_F}\right)\ \operatorname{csch}\left(\frac{\pi(x-x^{'}-v_F(t-t^{'}))}{\beta v_F}\right) \\
&\quad + w_{11}\ \frac{i}{4\beta v_F}\ \left(2\coth\left(\frac{\pi(v_F(t-t_1)-x+x_1)}{\beta v_F}\right)-2\coth\left(\frac{\pi(x_1-x^{'}-v_F(t_1-t^{'}))}{\beta v_F}\right)\right) \\
&\quad e^{\Delta}e^{\zeta_<}\ \operatorname{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right)+w_{21}\ \left(\frac{i}{2v_F\beta}\ \coth\left(\frac{\pi(v_F(t-t_1)-x+x_1)}{\beta v_F}\right)\right. \\
&\quad \left.-\frac{i}{2v_F\beta}\coth\left(\frac{\pi(v_F(t^{'}-t_1)-x^{'}+x_1)}{\beta v_F}\right)\right)\ e^{\zeta_<}e^{\Delta}\ \operatorname{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right) \qquad \text{(I.10)}
\end{aligned}
$$

and for $x, x^{'} < 0,\ x_1 > 0$ Eq.I.5 becomes

$$
\begin{aligned}
&-U(t,t^{'})\gamma_1\ \frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x^{'}-x_1-v_F(t^{'}-t_1)))}\ \ \frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x_1-x-v_F(t_1-t)))} \\
&= w_0\ 2\pi i\frac{1}{4}\frac{\gamma_1}{\beta^2 v_F^2}\frac{\beta v_F}{\pi}(\coth\left(\frac{\pi}{\beta v_F}(x_1-x+v_F(t-t_1))\right)-\coth\left(\frac{\pi}{\beta v_F}(x_1-x^{'}+v_F(t^{'}-t_1))\right)) \\
&\quad \sinh\left(\frac{\pi\delta}{\beta v_F}\right)\ \operatorname{csch}\left(\frac{\pi(x-x^{'}-v_F(t-t^{'}))}{\beta v_F}\right)+\frac{i}{2\beta v_F}\left(\begin{matrix}(w_{11}+w_{21}\ \gamma_1) \\ \coth\left(\frac{\pi}{\beta v_F}(v_F(t-t_1)-x+x_1)\right) \\ -(w_{11}\ \gamma_1+w_{21}) \\ \coth\left(\frac{\pi}{\beta v_F}(x_1-x^{'}-v_F(t_1-t^{'}))\right)\end{matrix}\right) \\
&\quad e^{\Delta+\zeta_<}\ \operatorname{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right) \qquad \text{(I.11)}
\end{aligned}
$$

where $\gamma_1 = \frac{(\Gamma^2-4v_F^2)^2}{(\Gamma^2+4v_F^2)^2}$. Hence for $x, x^{'} < 0$ it appears that both $\lambda = 0$ and $\lambda = 1$ terms can be used to obtain the four-point functions with the only constraint on the prefactors being $w_{11} = w_{21}$ which is evident from Eq.I.11. But by evaluating $< T\ \rho_L(-x_1,t_1)\psi_R^{\dagger}(x,t)\psi_R(x^{'},t^{'}) >$ we'll see that only the conventional ($\lambda = 0$) choice is the consistent one for $x, x^{'} < 0$. We can write,

$$
\begin{aligned}
< T\ \rho_L(-x_1,t_1)\psi_R^{\dagger}(x,t)\psi_R(x^{'},t^{'}) >= &-< T\ \psi_R(x^{'},t^{'})\psi_L^{\dagger}(-x_1,t_1) >\ \ < T\ \psi_L(-x_1,t_1)\psi_R^{\dagger}(x,t) > \\
&= q_0\ \ < \rho_L(-x_1,t_1)e^{-2\pi i\int^{x}dy\ \rho_R(y,t)}\ e^{2\pi i\int^{x^{'}}dy^{'}\ \rho_R(y^{'},t^{'})} > \\
&+ q_{11} < \rho_L(-x_1,t_1)\ e^{-2\pi i\int^{x}dy\ (\rho_R(y,t)+\rho_L(-y,t))}\ e^{2\pi i\int^{x^{'}}dy^{'}\ \rho_R(y^{'},t^{'})} > \\
&+ q_{21}\ \ < \rho_L(-x_1,t_1)e^{-2\pi i\int^{x}dy\ \rho_R(y,t)}\ e^{2\pi i\int^{x^{'}}dy^{'}\ (\rho_R(y^{'},t^{'})+\rho_L(-y^{'},t^{'}))} > \\
&\qquad \text{(I.12)}
\end{aligned}
$$

Following a similar procedure as shown above we can conclude that for the cases $x > 0, x^{'} > 0$ or $x > 0, x^{'} < 0$ or $x < 0, x^{'} > 0$, the $\lambda = 0$ term will not reproduce the correct form of the concerned four-point function. Considering only the most singular terms in Eq.I.12 we get,

$$
\begin{aligned}
&-\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x^{'}-x_1-v_F(t^{'}-t_1)))}\left(-U(t^{'}-\frac{x^{'}}{v_F},t^{'})\left[1-\theta(x_1)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x^{'})\right.\\
&\left.+U(t_1-\frac{x_1}{v_F},t^{'})\left[1-\theta(x^{'})\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x_1)\right)i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x_1-v_F(t-t_1)))}\\
&\left(-U(t,t-\frac{x}{v_F})\left[1-\theta(x_1)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x)+U(t,t_1-\frac{x_1}{v_F})\left[1-\theta(x)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x_1)\right)i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\\
&=\ q_{11}\ (\frac{i}{2\beta v_F}\coth\left(\frac{\pi}{\beta v_F}(x_1-x+v_F(t-t_1))\right)\\
&-\frac{8i\pi\Gamma^2}{\beta^2\left(\Gamma^2+4v_F^2\right)^2}\ \theta(-x_1x^{'})\ \frac{\beta v_F}{\pi}\coth\left(\frac{\pi}{\beta v_F}(x_1-x^{'}-v_F(t_1-t^{'}))\right))\ e^{\Delta}e^{\zeta(x^{'})}\\
&\text{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right)+q_{21}\ (-\frac{i}{2\beta v_F}\coth\left(\frac{\pi}{\beta v_F}(x_1-x^{'}+v_F(t^{'}-t_1))\right)\\
&+\frac{8i\pi\Gamma^2}{\beta^2\left(\Gamma^2+4v_F^2\right)^2}\ \theta(-x_1x)\ \frac{\beta v_F}{\pi}\coth\left(\frac{\pi}{\beta v_F}(x_1-x-v_F(t_1-t))\right))\\
&e^{\Delta}e^{\zeta(x)}\ \text{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right)
\end{aligned}
\tag{I.13}
$$

Solving for the prefactors so that the above equation holds gives us,

$$
q_{11}=-\frac{i\eta e^{-\Delta-\zeta(x^{'})}\left(\Gamma^2+4v_F^2\right)^2\left(\left(\Gamma^2+4v_F^2\right)^2-16\Gamma^2v_F^2\theta(-xx_1)\right)}{2\beta v_F(\left(\Gamma^2+4v_F^2\right)^4-256\Gamma^4v_F^4\theta(-xx_1)\theta(-x_1x'))}
\tag{I.14}
$$

$$
q_{21}=-\frac{i\eta e^{-\Delta-\zeta(x)}\left(\Gamma^2+4v_F^2\right)^2\left(\left(\Gamma^2+4v_F^2\right)^2-16\Gamma^2v_F^2\theta(-x_1x^{'})\right)}{2\beta v_F\left(\left(\Gamma^2+4v_F^2\right)^4-256\Gamma^4v_F^4\theta(-xx_1,-x_1x^{'})\right)}
\tag{I.15}
$$

where,

$$
\begin{aligned}
\eta=-\frac{16\Gamma^2v_F^2}{(\Gamma^2+4v_F^2)^2}&\left(-U(t^{'}-\frac{x^{'}}{v_F},t^{'})\left[1-\theta(x_1)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x^{'})+U(t_1-\frac{x_1}{v_F},t^{'})\left[1-\theta(x^{'})\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x_1)\right)\\
&\left(-U(t,t-\frac{x}{v_F})\left[1-\theta(x_1)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x)+U(t,t_1-\frac{x_1}{v_F})\left[1-\theta(x)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x_1)\right)
\end{aligned}
\tag{I.16}
$$

So using only the terms with $\lambda=1$ in the exponent we obtain the correct four-point function as well as the correct two-point functions for these cases. Now let us consider the case $x<0,x^{'}<0$ in Eq.I.12

and including only the most singular terms we get,

$$
\begin{aligned}
&-\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x^{'}-x_1-v_F(t^{'}-t_1)))}\left(-U(t^{'}-\frac{x^{'}}{v_F},t^{'})\left[1-\theta(x_1)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x^{'})\right.\\
&\left.+U(t_1-\frac{x_1}{v_F},t^{'})\left[1-\theta(x^{'})\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x_1)\right)i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\\
&\frac{i}{2\pi}\frac{\frac{\pi}{\beta v_F}}{\sinh(\frac{\pi}{\beta v_F}(x-x_1-v_F(t-t_1)))}\left(-U(t,t-\frac{x}{v_F})\left[1-\theta(x_1)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x)\right.\\
&\left.+U(t,t_1-\frac{x_1}{v_F})\left[1-\theta(x)\frac{2\Gamma^2}{\Gamma^2+4v_F^2}\right]\theta(x_1)\right)i\frac{\Gamma}{v_F}\frac{(2v_F)^2}{\Gamma^2+4v_F^2}\\
&= q_{11}\ (\frac{i}{2\beta v_F}\coth\left(\frac{\pi}{\beta v_F}(x_1-x+v_F(t-t_1))\right)\\
&-\frac{8i\pi\Gamma^2}{\beta^2\left(\Gamma^2+4v_F^2\right)^2}\ \theta(-x_1x^{'})\ \frac{\beta v_F}{\pi}\coth\left(\frac{\pi}{\beta v_F}(x_1-x^{'}-v_F(t_1-t^{'}))\right))\\
&e^{\Delta}e^{\zeta(x^{'})}\ \mathrm{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right)\\
&+q_{21}\ (-\frac{i}{2\beta v_F}\coth\left(\frac{\pi}{\beta v_F}(x_1-x^{'}+v_F(t^{'}-t_1))\right)\\
&+\frac{8i\pi\Gamma^2}{\beta^2\left(\Gamma^2+4v_F^2\right)^2}\ \theta(-x_1x)\ \frac{\beta v_F}{\pi}\coth\left(\frac{\pi}{\beta v_F}(x_1-x-v_F(t_1-t))\right))\\
&e^{\Delta}e^{\zeta(x)}\ \mathrm{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right)\\
&+q_0(\frac{8i\Gamma^2v_F\theta(x_1)\left(\coth\left(\frac{\pi}{\beta v_F}(x_1-x-v_F(t_1-t))\right)-\coth\left(\frac{\pi}{\beta v_F}(x_1-x^{'}-v_F(t_1-t^{'}))\right)\right)}{\beta\left(\Gamma^2+4v_F^2\right)^2})\\
&\sinh\left(\frac{\pi\delta}{\beta v_F}\right)\ \mathrm{csch}\left(\frac{\pi}{\beta v_F}(x-x^{'}-v_F(t-t^{'}))\right)
\end{aligned} \tag{I.17}
$$

The only relevant case here is $x<0,x^{'}<0,x_1>0$ as the Wick's theorem result is zero for $x<0,x^{'}<0,x_1>0$. A subtle point to note is that in both the four-point functions of interest $< T\ \rho_L(-x_1,t_1)\psi_R^{\dagger}(x,t)\psi_R(x^{'},t^{'}) >$ and $< T\ \rho_R(x_1,t_1)\psi_R^{\dagger}(x,t)\psi_R(x^{'},t^{'}) >$, we write the term $\psi_R^{\dagger}(x,t)\psi_R(x^{'},t^{'})$ in bosonized form which is the same in both cases with the only difference being the presence of the $\rho_L(-x_1,t_1)$ or $\rho_R(x_1,t_1)$ term when evaluating the expectation value. This means that the prefactors in both the cases i.e. the w's and the q's should have the same qualitative properties in order to match the Wick's theorem result for the four-point functions. We have already obtained the constraint $w_{11}=w_{21}$ for $x,x^{'}<0$. So it follows that $q_{11}=q_{21}$ for $x<0,x^{'}<0,x_1>0$ and solving for the prefactors to obtain the correct Wick's theorem result we get $q_{11}=q_{21}=0$ and $q_0=-\frac{i\eta\left(\Gamma^2+4v_F^2\right)^2\mathrm{csch}\left(\frac{\pi\delta}{\beta v_F}\right)}{32\beta\Gamma^2v_F^3}$, where η is as defined in Eq.I.16. This means that the unconventional bosonization choice does not play any role this case as its prefactors are identically zero and it entirely drops out from the calculation. The implication here is that the conventional bosonization procedure (ie. without the anomalous term in the exponent) is the only valid choice for writing the bosonized version of the $< T\ \Psi_R(x<0,t)\Psi_R^{\dagger}(x^{'}<0,t^{'}) >$ correlation function. A similar analysis shows that for $< T\ \Psi_L(x>0,t)\Psi_L^{\dagger}(x^{'}>0,t^{'}) >$ also only the conventional method of bosonization is valid. But for all the other cases it is necessary to include the unconventional anomalous term in the exponent to obtain the correct form of the Green functions.

Appendix J: Universal power-law behaviour in NCBT Green's functions of strongly inhomogeneous Luttinger liquids in equilibrium

The most singular parts of the full interacting Green's functions of a strongly inhomogeneous Luttinger liquid in equilibrium obtained using NCBT are shown in Eqs.14 and 15 of [132]. Substituting $x_1 = x$, $x_2 = x + \epsilon$ (to avoid infinities/zeroes) and $t_1 = t_2 = t$ in the RL same side (x_1 and x_2 on same side of the origin) Green's function, we obtain

$$< \psi_R(x)\psi_L^\dagger(x) > \;\sim\; x^X(-2x-\epsilon)^{C-X}(-\epsilon)^{2Q-\frac{Q}{C}}(x+\epsilon)^X \;(2x+\epsilon)^{-1+C-X}(x^{2C}+(x+\epsilon)^{2C}) \sim\; x^{-g} \tag{J.1}$$

where $g = \frac{v_F}{v_h}$ is just the Luttinger liquid interaction parameter. The expressions for the anomalous exponents that appear in the NCBT Green functions are as follows,

$$Q = \frac{(v_h - v_F)^2}{8v_h v_F} \;;\; X = \frac{|R|^2(v_h - v_F)(v_h + v_F)}{8v_h(v_h - |R|^2(v_h - v_F))} \;;\; C = \frac{v_h - v_F}{4v_h}$$

where v_F is the Fermi velocity, v_h is the holon velocity and $|R|$ is the reflection amplitude. Hence in the limit $x_1 \to x_2$ the power law exponents turn out to be universal (i.e. independent of impurity strength) although the exponents in the general expresssion for the Green functions do depend on the impurity strength. For the RR same side Green's function we obtain,

$$<\psi_R(x)\psi_R^\dagger(x)> \;\sim\; 4^X(-2x-\epsilon)^{-X}(-\epsilon)^{-1-Q}(\epsilon)^{-Q}\;(x(x+\epsilon))^X(2x+\epsilon)^{-X} \sim\; x^0 \tag{J.2}$$

The x-dependence drops out in the RR and LL cases. The impurity dependence in the exponent drops out in one case and the final power-law exponent is universal as it depends only on the interaction parameter and in the other case the overall exponent adds up to zero making it trivially universal. On the basis of these observations, it is reasonable to suspect that similar universal behaviour (i.e. impurity strength independence) may be expected even in a system driven out of equilibrium by the application of a bias. In the present problem we expect to obtain the correct universal exponent for the tunneling density of states (TDOS) at the point-contact for fractional quantum Hall edge (FQHE) states.

APPENDIX K: Perturbative comparison of second order ($O(v_0^2)$) term for $< \rho_{sym}\rho_{sym} >$

The second order term in the expansion in Eq.5.45 is

$$\begin{aligned} &< T\; \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >^{(2)} \\ &\quad = \frac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx' <T\;\rho_{sym}(x,t)\rho_{sym}(-x,t)\rho_{sym}(x',t')\rho_{sym}(-x',t')\rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2)>_{0,c} \end{aligned} \tag{K.1}$$

Retaining only the most singular parts involves making the Gaussian approximation [52] which means that Wick's theorem is applicable at the level of pairs of fermions. This means we can write

$$\begin{aligned} &< T\; \rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2) >^{(2)} \\ &= \frac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx' <T\;\rho_{sym}(x,t)\rho_{sym}(-x,t)\rho_{sym}(x',t')\rho_{sym}(-x',t')\rho_{sym}(x_1,t_1)\rho_{sym}(x_2,t_2)>_{0,c} \\ &= \frac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx' \Big(<T\;\rho_{sym}(x,t)\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(x',t')\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(-x,t)\rho_{sym}(-x',t')>_0 \\ &+ <T\;\rho_{sym}(x,t)\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(x',t')\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(-x,t)\rho_{sym}(-x',t')>_0 \\ &+ <T\;\rho_{sym}(-x,t)\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(x',t')\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(x,t)\rho_{sym}(-x',t')>_0 \\ &+ <T\;\rho_{sym}(-x,t)\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(x',t')\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(x,t)\rho_{sym}(-x',t')>_0 \\ &+ <T\;\rho_{sym}(-x',t')\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(x,t)\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(-x,t)\rho_{sym}(x',t')>_0 \\ &+ <T\;\rho_{sym}(-x',t')\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(x,t)\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(-x,t)\rho_{sym}(x',t')>_0 \\ &+ <T\;\rho_{sym}(-x',t')\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(-x,t)\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(x,t)\rho_{sym}(x',t')>_0 \\ &+ <T\;\rho_{sym}(-x',t')\rho_{sym}(x_2,t_2)>_0<T\;\rho_{sym}(-x,t)\rho_{sym}(x_1,t_1)>_0<T\;\rho_{sym}(x,t)\rho_{sym}(x',t')>_0 \Big) \end{aligned} \tag{K.2}$$

In momentum and Matsubara frequency space the second order term takes the form,

$$\begin{aligned}
&<\rho_{sym}(q',n)\rho_{sym}(q'',-n)>^{(2)} \\
&= \frac{(-iv_0)^2}{32}(-i\beta)^2 \sum_{q,Q}\Big(<\rho_{sym}(q,-n)\rho_{sym}(q',n)>_0 \quad <\rho_{sym}(Q,n)\rho_{sym}(q'',-n)>_0 \\
&\quad <\rho_{sym}(q,n)\rho_{sym}(Q,-n)>_0 \quad + ...\Big)
\end{aligned} \tag{K.3}$$

Using the expression for the noninteracting correlation in Fourier space

$$<\rho_{sym}(q,n)\rho_{sym}(q',-n)>_0 = -\frac{1}{\pi\beta v_F}\frac{iv_F q}{(w_n - iv_F q)}\delta_{q+q',0}$$

we evaluate the second order term and obtain

$$<\rho_{sym}(q',n)\rho_{sym}(q'',-n)>^{(2)} \quad = \frac{q'^3 v_0^2}{4\pi^3\beta(q'v_F - iw_n)(q'v_F + iw_n)^2}\ \delta_{q'+q'',0} \tag{K.4}$$

Upon expanding Eq.5.27 in powers of v_0, the $O(v_0^2)$ term is equal to Eq.K.4.

APPENDIX L: Perturbative comparison of second order ($O(v_0^2)$) term for $<\rho_{asy}\rho_{asy}>$

The second order term in the expansion $<T\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2)>$ is

$$\begin{aligned}
&<T\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2)>^{(2)} \\
&= \tfrac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx' <T\ \rho_{asy}(x,t)\rho_{asy}(-x,t)\rho_{asy}(x',t')\rho_{asy}(-x',t')\rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2)>_{0,c}
\end{aligned} \tag{L.1}$$

Retaining the most singular parts alone we may write,

$$\begin{aligned}
&<T\ \rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2)>^{(2)} \\
&= \tfrac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx' <T\ \rho_{asy}(x,t)\rho_{asy}(-x,t)\rho_{asy}(x',t')\rho_{asy}(-x',t')\rho_{asy}(x_1,t_1)\rho_{asy}(x_2,t_2)>_{0,c} \\
&= \tfrac{(-iv_0)^2}{32}\int_C dt\int_C dt'\int dx\int dx'\Big(<T\ \rho_{asy}(x,t)\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(x',t')\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(-x,t)\rho_{asy}(-x',t')>_0 \\
&+ <T\ \rho_{asy}(x,t)\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(x',t')\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(-x,t)\rho_{asy}(-x',t')>_0 \\
&+ <T\ \rho_{asy}(-x,t)\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(x',t')\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(x,t)\rho_{asy}(-x',t')>_0 \\
&+ <T\ \rho_{asy}(-x,t)\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(x',t')\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(x,t)\rho_{asy}(-x',t')>_0 \\
&+ <T\ \rho_{asy}(-x',t')\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(x,t)\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(-x,t)\rho_{asy}(x',t')>_0 \\
&+ <T\ \rho_{asy}(-x',t')\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(x,t)\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(-x,t)\rho_{asy}(x',t')>_0 \\
&+ <T\ \rho_{asy}(-x',t')\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(-x,t)\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(x,t)\rho_{asy}(x',t')>_0 \\
&+ <T\ \rho_{asy}(-x',t')\rho_{asy}(x_2,t_2)>_0 <T\ \rho_{asy}(-x,t)\rho_{asy}(x_1,t_1)>_0 <T\ \rho_{asy}(x,t)\rho_{asy}(x',t')>_0 \Big)
\end{aligned} \tag{L.2}$$

This is written in Fourier space as

$$\begin{aligned}
&<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>^{(2)} \\
&= \frac{(-iv_0)^2}{32}(-i\beta)^2 \sum_{q,Q}\Big(<\rho_{asy}(q,-n)\rho_{asy}(q',n)>_0 \quad <\rho_{asy}(Q,n)\rho_{asy}(q'',-n)>_0 \\
&\quad <\rho_{asy}(q,n)\rho_{asy}(Q,-n)>_0 \quad + ...\Big)
\end{aligned} \tag{L.3}$$

Making use of

$$<\rho_{asy}(q,n)\rho_{asy}(q',-n)>_0 \quad = \quad -\frac{(iqv_F)\delta_{q+q',0}}{(\pi\beta v_F)(w_n - iqv_F)} - \frac{(r_1-1)w_n sgn(w_n)}{\pi\beta L(qv_F + iw_n)(q'v_F - iw_n)} \tag{L.4}$$

we evaluate the second order term and obtain the following expression,

$$\begin{aligned}
&<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>^{(2)} \\
&= v_0^2\Bigg(-\frac{(r_1-1)w_n sgn(w_n)\begin{pmatrix} q'\big(q''v_F\big(\big(r_1^2-1\big)v_F v_h+(r_1+1)^2v_F^2-(r_1-1)^2v_h^2\big)+i(r_1-1)v_h w_n(2(r_1+1)v_F-r_1v_h+v_h)\big) \\ +(r_1-1)v_h w_n((r_1+1)w_n-iq''(2(r_1+1)v_F-r_1v_h+v_h))\end{pmatrix}}{2\pi^3\beta L\big(q'^2v_h^2+w_n^2\big)\big(q''^2v_h^2+w_n^2\big)(r_1v_F-r_1v_h+v_F+v_h)^3} \\
&\quad+\frac{q'^3\delta_{q',-q''}}{4\pi^3\beta(q'v_F-iw_n)(q'v_F+iw_n)^2}\Bigg) \qquad \text{(L.5)}
\end{aligned}$$

This is exactly equal to the second order term obtained upon perturbative expansion of the $<\rho_{asy}(q',n)\rho_{asy}(q'',-n)>$ correlation function (present in the rhs of Eq.5.41) in powers of v_0.

List of Publications

1. *Density density correlation function of strongly inhomogeneous Luttinger liquids*, **Babu N D**, Das J P and Setlur G S 2020, Physica Scripta 95 115206, URL: https://dx.doi.org/ 10.1088/1402-4896/abbc2f

2. *Non Markovian transients in transport across chiral quantum wires using space-time non equilibrium Green functions*, **Babu N D** and Setlur G S 2022, Journal of Physics: Condensed Matter 34 125602, URL: https://doi.org/10.1088/1361-648x/ac45b6

3. *Density density correlation functions of chiral Luttinger liquids with a point contact impurity*, **Babu N D** and Setlur G S 2024 (Physica Scripta 99(10), Physica Scripta 99(10), 105944 URL https://iopscience.iop.org/article/10.1088/1402-4896/ad7363).

4. *Unconventional bosonization of chiral quantum wires coupled through a point-contact driven out of equilibrium*, **Babu N D** and Setlur G S 2023 (arXiv preprint 2204.13517).

5. *Tunneling density of states of fractional quantum hall edges: an unconventional bosonization approach*, **Babu N D** and Setlur G S 2023 (arXiv preprint 2310.10319).

Book Chapters

1. *Artificial Intelligence in Biological Sciences: A brief overview*, Uma Dutta, **Nikhil Danny Babu** and Girish S. Setlur (Information Retrieval in Bioinformatics: A Practical Approach, Palgrave Macmillan, Singapore. Print ISBN 978-981-19-6505-0).

Conference Presentations

1. Gave a talk titled "Non-Markovian transients in nonequilibrium transport between chiral quantum wires coupled through a point-contact.", at the *CMD 30 - FisMat 2023 conference on condensed matter physics* held at Politecnico di Milano, Milan, Italy.

2. Gave a talk titled "Non-equilibrium transport in chiral quantum wires coupled through a point-contact", at the *QMAT 2022 annual national conference on Quantum Condensed Matter* in IIT Kanpur, Kanpur, India.

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