# High-Order Weak Galerkin Finite Element Methods for Some Hyperbolic PDEs

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*This thesis*

*is dedicated*

*to*
*my family*

# CERTIFICATE

It is certified that the work contained in this thesis entitled **"High-Order Weak Galerkin Finite Element Methods for Some Hyperbolic PDEs"** by **PUSPENDU JANA**, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

August, 2024

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# ABSTRACT

The main objective of this thesis is to develop high-order numerical schemes for various hyperbolic problems such as linear hyperbolic equations, semilinear hyperbolic equations and quasi-linear hyperbolic equations. Numerous numerical techniques are available in the literature that are intended for the numerical approximation of this type of hyperbolic problem. The collection of the existing literature has suggested that finite element methods (FEMs) are one of the most accurate, efficient and reliable approximation schemes in scientific computing due to their significant applications for real-world physical phenomena. However, there are still challenges in designing high-order accurate and computationally efficient methods for PDEs defined in the complicated domain, particularly for hyperbolic problems. The consistency of classical FEMs is compromised by limited options for the approximation spaces, underlying finite element partitions, low global regularity of the exact solutions and interfaces with geometric singularities. In reality, finite element partition using arbitrary shapes provides more flexibility in both numerical approximation and general mesh generation such as hybrid meshes, polygonal/polyhedral meshes, and mesh with hanging nodes.

In recent years, newly developed weak Galerkin finite element methods (skeletal DG methods) offer a versatile numerical approach for solving partial differential equations on polytopal partitions. It utilizes discontinuous functions to approximate the exact solution and its gradient, a characteristic feature of the WG-FEMs. Moreover, constructing high-order WG spaces is generally more straightforward compared to conforming FEM spaces due to the absence of continuity requirements on the approximation spaces. The approach involves polynomial approximation within each polytopal element's interior and boundary, complemented by a straightforward stabilization technique to ensure the necessary continuity of weak approximation functions. The WG formulation serves as a natural extension of the conforming finite element formulation when nonconforming elements are utilized.

In this thesis, we design and analyze high-order convergence of weak Galerkin finite element approximations to the true solutions for hyperbolic problems.

First, we describe a systematic numerical study for second order linear wave equa-

tions using weak Galerkin finite element methods (WG-FEMs) for semidiscrete cases. Various degrees of polynomials are used to construct weak Galerkin finite spaces. Error estimates in $L^\infty(L^2)$ norm as well as in $L^\infty(H^1)$ norms have been established for general weak Galerkin space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$, where $k, j$ & $l$ are non-negative integers with $k \geq 1$. The fully discrete Scheme is based on the Crank-Nicolson time discretization.

Next, we propose weak Galerkin finite element methods (WG-FEMs) for the semilinear Klein-Gordon equation. Optimal order error estimate in both $L^\infty(L^2)$ norm and $L^\infty(H^1)$ norm have been executed for WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$ is an integer. For a fully discrete scheme, we employ the Newmark scheme for temporal discretization.

Further, we propose weak Galerkin finite element methods for Westervelt's model of ultrasound waves. Specifically, we investigate the spatial discretization of Westervelt's quasi-linear, strongly damped wave equation using high-order weak Galerkin discretization. Convergence analysis in $L^2$-based spatial norms for linearized Westervelt's equation has been performed with variable coefficients. Then the results have been extended for Westervelt's quasi-linear acoustic wave equation relying on the Banach fixed-point theorem for sufficiently small data and mesh size, given an appropriate choice of initial data.

In the last problem, we describe WG-FEMs for general linear hyperbolic interface problems. Convergence analysis is carried out for both semidiscrete and fully discrete schemes. For locally smooth solutions, optimal error estimates in $L^\infty(L^2)$ norm as well as $L^\infty(H^1)$ norm have been obtained for the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, $k \geq 1$. For a fully discrete scheme, we employ the Crank-Nicolson scheme for temporal discretization.

Finally, several numerical experiments have been performed to justify the accuracy, efficiency, flexibility and robustness of each proposed algorithm. Our numerical results cover a wide range of applied problems such as discontinuous coefficients, variable coefficients and interface problems aiming to fill the gap in the existing literature.

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# 1

# Introduction

This thesis proposes to study high-order weak Galerkin finite element methods (WG-FEMs) for some linear and nonlinear hyperbolic problems on polygonal meshes. Numerical solutions of hyperbolic problems play a significant role in numerous fields of applied sciences, medical sciences, engineering and industrial applications such as acoustics, fluid dynamics, astrophysics, aerodynamics, shock wave lithotripsy or diagnostic ultrasound imaging, ultrasound cleaning, welding and sonochemistry. This is an introductory chapter that includes a description of problems, some common notations used in this thesis and some preliminary material. A brief overview of relevant literature and the motivation behind the current study are included. Finally, the last section of this chapter presents a chapter-by-chapter summary of the thesis.

## 1.1 Problem Description

In this section, we introduce some linear and nonlinear hyperbolic problems that are to be studied in this thesis. It also includes a brief description of these problems and their applications in various fields of science and engineering.

**Second Order Linear Wave Equation:** We consider following second order linear wave equation

$$u_{tt} - \nabla \cdot (\boldsymbol{\mu} \nabla u) = f \ \text{ in } \Omega \times (0, T],\ T < \infty, \tag{1.1.1}$$

with the initial conditions

$$u(0) = u^0, \ \ u_t(0) = v^0 \ \text{ in } \Omega \tag{1.1.2}$$

and boundary condition

$$u = 0 \ \text{ on } \partial\Omega \times (0, T]. \tag{1.1.3}$$

Here $\Omega$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $J = (0, T]$ is the observation time period. The coefficient matrix $\boldsymbol{\mu} = (\mu_{i,j}(\mathbf{x}))_{2\times 2}$ is considered to be symmetric and positive definite in the domain $\Omega$. The initial functions $\{u^0, v^0\}$, coefficient $\boldsymbol{\mu}$ and the source function $f$ are considered to be sufficiently smooth.

The wave equation is a fundamental example of a hyperbolic partial differential equation. Several forms of waves exist in the areas of acoustics, electromagnetics and fluid dynamics, namely sound waves, light waves, water waves and seismic waves. The wave equation models the propagation of numerous types of waves such as elastic waves [70], sound waves in fluid and gas [73], electromagnetic wave [81]. Also, modeling of wave equations is important in acoustics [65] and cosmology [123]. Moreover, the wave equation has uses in geophysics [95], earthquake prediction and other seismic activities [114].

**Semilinear Klein-Gordon Equation:** Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary $\partial\Omega$. Consider the semilinear Klein-Gordon (KG) equation

$$u_{tt} - \Delta u + f(u) = g \ \text{ in } \ \Omega \times (0, T], \ T < \infty, \tag{1.1.4}$$

with the initial conditions

$$u(0) = u^0, \ \ u_t(0) = v^0 \ \text{ in } \Omega \tag{1.1.5}$$

and homogeneous boundary condition

$$u = 0 \ \text{ on } \partial\Omega \times (0, T]. \tag{1.1.6}$$

Initial data $\{u^0, v^0\}$ and source function $g$ are assumed to be smooth functions in their respective domains of definition, and $J = (0, T]$ is the observation time period.

The nonlinear KG equation is used to model many nonlinear phenomena. In the well-known sine-Gordon equation, the nonlinear force is given by $f(u) = \sin u$. In the physical applications, the nonlinear force has also other forms. The cases $f(u) = \sin u + \sin 2u$ and $f(u) = \sinh u + \sinh 2u$ are called the double sine-Gordon equation and the double sinh-Gordon equation, respectively.

Equation (1.1.4) plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory [155]. Klein-Gordon equation is the relativistic version of Schrôdinger equation; however, the KG equation is second order in time derivatives. The Klein-Gordon equation governs the quantum evolution of wave functions for relativistic spinless particles [118], which is of great importance for the high energy physicists and is used to model many different phenomena, including the propagation of dislocations in crystals and the behaviour of elementary particles. The Klein-Gordon equation plays an important role in mathematical physics. It can be applied to model and predict the behavior of fields and particles under different scenarios such as quantum mechanics [157], particle physics [72] and cosmology [13].

The equation has attracted much attention in studying solitons and condensed matter physics [26], in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states and in examining the nonlinear wave equations [50].

**Westervelt's Quasi-linear Wave Equation:** The Westervelt's equation, modeling nonlinear ultrasound propagation through a homogeneous medium for the acoustic pressure $u$ in dimensionless form, can be expressed as

$$(1-2\sigma u)u_{tt} - c^2\Delta u - b\Delta u_t = 2\sigma u_t^2 \quad \text{in } \ \Omega \times (0,T], \tag{1.1.7}$$

with the following initial and boundary conditions

$$\begin{cases} u = 0 \ \text{ on } \partial\Omega \times (0,T], \\ u(0) = u_*, u_t(0) = u^* \ \text{ in } \Omega. \end{cases} \tag{1.1.8}$$

Here $\Omega$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $J = (0,T]$ is the observation time interval. The constant $c$ denotes the speed of sound, $b$ is the sound diffusivity and $\sigma = \beta_a/(\rho_m c^2)$, where $\rho_m$ is the mass density, and $\beta_a$ is the coefficient of non-linearity of the medium. Again, setting $\hat{\alpha} = 1 - 2\sigma u$ and $\hat{\beta} = -2\sigma u_t$, equation (1.1.7) can be rewritten as the following viscous wave equation

$$\hat{\alpha}(\mathbf{x},t)u_{tt} - c^2\Delta u - b\Delta u_t + \hat{\beta}(\mathbf{x},t)u_t = 0 \ \text{ in } \Omega \times J. \tag{1.1.9}$$

Analysis of the linearization (1.1.9) will allow us to study the Westervelt's equation with quadratic nonlinearities.

Westervelt's equation is a strongly damped quasi-linear wave equation. Wave-type problems cover a wide range of applications in science and engineering. Due to their simplicity, the consideration of linear wave-type problems based on linearized material models is very appealing. However, this is not reasonable if waves with high frequency or intensity occur. The modeling of nonlinear effects arising in the presence of high-intensity acoustic fields plays an increasingly important role in diagnostic and therapeutic medicine [77]. Thus, nonlinear material models have to be considered, which yield quasi-linear wave-type problems. Nonlinear acoustics plays a central role in numerous applications ranging from medical treatment like shock wave lithotripsy or diagnostic ultrasound imaging to industrial applications like ultrasound cleaning or welding and sonochemistry [22, 52, 130, 134].

**General Hyperbolic Equations with interface:** Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and $\Omega_1$ is an open domain with $C^2$ smooth interface $\Gamma = \partial\Omega_1$ such that $\bar{\Omega}_1 \subsetneq \Omega$, and $\Omega_2 = \Omega \setminus \Omega_1$(see, Figure 1.1.1). In $\Omega$, we

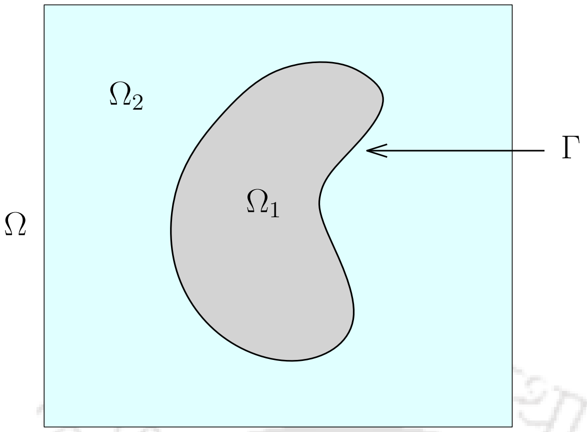

Figure 1.1.1: Domain $\Omega$ and its sub domains $\Omega_1$, $\Omega_2$ with interface $\Gamma$.

consider the following general linear second order hyperbolic equation

$$u_{tt} + \Theta u_t + \xi u - \nabla.(\eta\nabla u + \upsilon\nabla u_t) = f, \quad \text{in } \Omega \times (0, T], \tag{1.1.10}$$

with the initial and boundary conditions

$$\begin{cases} u(0) = u^0, \quad u_t(0) = v^0 \;\text{ in } \Omega, \\ u = 0 \;\text{ on } \partial\Omega \times (0, T], \end{cases} \tag{1.1.11}$$

where $J = (0, T]$ is the finite terminal period. Coefficients are assumed to be positive real valued functions defined in $\Omega$. Further, initial data $\{u^0, v^0\}$ and the source function $f$ are sufficiently smooth in their respective domain.

Equation (1.1.10) has received a great attention in the context of various engineering and medical applications. For example, dual-phase-lag bio heat model is described by a time-dependent equation

$$\begin{aligned} \tau_q\rho c\frac{\partial^2\hat{T}}{\partial t^2} &= k\nabla^2\hat{T} + \tau_{\hat{T}}k\nabla^2\frac{\partial\hat{T}}{\partial t} - \omega_b\rho_b c_b\hat{T} - (\tau_q\omega_b\rho_b c_b + \rho c)\frac{\partial\hat{T}}{\partial t} \\ &\quad + \big(\omega_b\rho_b c_b\hat{T}_a + q_{\text{met}} + q_{\text{ext}} + \tau_q\frac{\partial q_{\text{met}}}{\partial t} + \tau_q\frac{\partial q_{\text{ext}}}{\partial t}\big), \end{aligned}$$

where $\rho, c, k$ are the density, specific heat and thermal conductivity of skin tissue, respectively; $\rho_b, c_b$ are the density and specific heat of blood, $\omega_b$ is the blood perfusion rate; $\hat{T}_a$ and $\hat{T}$ are the temperatures of arterial blood and skin tissue respectively; $q_{\text{met}}$ is the metabolic heat generation in the skin tissue and $q_{\text{ext}}$ is the heat source due to external heating and $\tau_q$ is defined as the thermal relaxation time. Due to heterogeneity in the underlying media, the thermal properties of biological media vary between different layers. In $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ (see, Figure 1.1.1), we assume that the physical coefficients are discontinuous and piecewise constants. We write

$$(\Theta, \xi, \eta, \upsilon) = \begin{cases} (\Theta_1, \xi_1, \eta_1, \upsilon_1) \ \text{ in } \ \Omega_1, \\ (\Theta_2, \xi_2, \eta_2, \upsilon_2) \ \text{ in } \ \Omega_2. \end{cases}$$

Then the information between both the domains are transferred via following interface conditions

$$[u] = \Phi, \quad \left[\eta \frac{\partial u}{\partial \mathbf{n}} + \upsilon \frac{\partial u_t}{\partial \mathbf{n}}\right] = \Psi \ \text{ on } \ \Gamma \times [0, T], \tag{1.1.12}$$

where $[u] = u_1|_\Gamma - u_2|_\Gamma$ and $\left[\eta \frac{\partial u}{\partial \mathbf{n}} + \upsilon \frac{\partial u_t}{\partial \mathbf{n}}\right] = \eta_1 \frac{\partial u_1}{\partial \mathbf{n}_1} + \upsilon_1 \frac{\partial u_{1t}}{\partial \mathbf{n}_1} + \eta_2 \frac{\partial u_2}{\partial \mathbf{n}_2} + \upsilon_2 \frac{\partial u_{2t}}{\partial \mathbf{n}_2}$. Here $u_i$ is the restriction of $u$ in $\Omega_i$ and $\frac{\partial}{\partial \mathbf{n}_i}$ is the outer normal derivative with respect to $\Omega_i$ for $i = 1, 2$.

The diffusive-viscous theory is provoked by numerous applications of nonlinear hyperbolic problems in medicine and industrial fields like acoustics, fluid mechanics, astrophysics, aerodynamics and high-intensity focused ultrasound by appropriate selection of the damping coefficients. When $\Theta = 0$, Eq. (1.1.10) describes the wave propagation phenomena of actual vibration through a viscoelastic medium representing a viscoelastic wave equation [110]. The viscoelastic wave equation can describe the attenuation of seismic waves in fluid-saturated systems. For instance, during the heat conduction in memory materials [76], gas diffusion [116], propagation of sound through viscous media [137]. Upon $\Theta \neq 0$, as given by (1.1.10), represents a crucial viscous wave problem that appears in simulating microscale heat transfer and propagation of acoustic waves. The recent applications of deriving the viscous wave from physical principles are provided in [108].

## 1.2 Notations and Preliminaries

In this section, we shall introduce some standard notations, function spaces and preliminary materials, which are essential for the writing of this thesis. All functions considered in this thesis are real-valued. Now to introduce notations, we assume that $\Omega$ be a convex polygonal domain in $\mathbb{R}^d$ ($d-$dimensional Euclidean space) and $\partial\Omega$ denote its boundary. Now for $\mathbf{x} = (x_1, \cdots, x_d) \in \Omega$, set $d\mathbf{x} = dx_1 \cdots dx_d$. Further, let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \cdots \lambda_d)$ be an $d-$tuple with non negative integer components and the order of $\boldsymbol{\lambda}$ is defined as $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \cdots + \lambda_d$. Then $\boldsymbol{\lambda}$th derivative of $\varphi$ denoted by $D^{\boldsymbol{\lambda}}\varphi$ and defined as

$$D^{\boldsymbol{\lambda}}\varphi = \frac{\partial^{|\boldsymbol{\lambda}|}\varphi}{\partial {x_1}^{\lambda_1} \partial {x_2}^{\lambda_2} \cdots \partial {x_d}^{\lambda_d}}.$$

Next, for a function $\varphi : \Omega \to \mathbb{R}$, the support of it denoted by $\text{supp}(\varphi)$ and defined as closure of the set $\{\mathbf{x} \in \Omega : \varphi(\mathbf{x}) \neq 0\}$, i.e.,

$$\text{supp}(\varphi) = \overline{\{\mathbf{x} \in \Omega : \varphi(\mathbf{x}) \neq 0\}}.$$

Now for any integer $m \geq 0$, $C^m(\bar{\Omega})$ denotes the space of functions with continuous derivative upto order $m$ in $\bar{\Omega}$. Also, $C_0^m(\Omega)$ denote the space of functions with continuous derivative upto order $m$ with compact support in $\Omega$ and $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in $\Omega$.

Next, we recall some basic function spaces that we shall refer frequently. For any domain $\Omega \subset \mathbb{R}^d, d = 2$ and $\mathcal{K} \subseteq \Omega$ with $1 \leq p \leq \infty$, $L^p(\mathcal{K})$ be the normed linear space of equivalence classes of measurable functions $\varphi$ in $\mathcal{K}$ with $\|\varphi\|_{L^p(\mathcal{K})} < \infty$, where

$$\begin{aligned} \|\varphi\|_{L^p(\mathcal{K})} &= \left( \int_{\mathcal{K}} |\varphi(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \\ \|\varphi\|_{L^\infty(\mathcal{K})} &= \operatorname*{ess\,sup}_{\mathbf{x} \in \mathcal{K}} |\varphi(\mathbf{x})| < \infty. \end{aligned}$$

For simplicity, we write the norm $\| \cdot \|_{L^2(\mathcal{K})}$ on $L^2(\mathcal{K})$ by $\| \cdot \|_{\mathcal{K}}$ and when $\mathcal{K} = \Omega$, we remove the subscript $\mathcal{K}$.

Now we introduce the concept of Sobolev space. For any integer $m \geq 0$ and real number $p$ with $1 \leq p \leq \infty$, the standard Sobolev space, denoted by $W^{m,p}(\mathcal{K})$, is a space of all functions whose distributional derivatives of order upto $m$ are in $L^p(\mathcal{K})$. More precisely,

$$W^{m,p}(\mathcal{K}) = \{\varphi \in L^p(\mathcal{K}) : D^{\boldsymbol{\lambda}}\varphi \in L^p(\mathcal{K}) \text{ for } 0 \leq |\boldsymbol{\lambda}| \leq m\}.$$

The spaces $W^{m,p}(\mathcal{K})$ are Banach spaces associated with the norm

$$\begin{aligned} \|\varphi\|_{m,p,\mathcal{K}} &= \left( \sum_{0 \leq |\boldsymbol{\lambda}| \leq m} \|D^{\boldsymbol{\lambda}}\varphi\|^p_{L^p(\mathcal{K})} \right)^{\frac{1}{p}}, \ 1 \leq \theta < \infty, \\ \|\varphi\|_{m,\infty,\mathcal{K}} &= \max_{0 \leq |\boldsymbol{\lambda}| \leq m} \|D^{\boldsymbol{\lambda}}\varphi\|_{L^\infty(\mathcal{K})} \end{aligned}$$

and the semi-norm on $W^{m,p}(\mathcal{K})$ is defined as

$$|\varphi|_{m,p,\mathcal{K}} = \sum_{|\lambda| = m} \|D^{\boldsymbol{\lambda}}\varphi\|_{L^p(\mathcal{K})}.$$

When $p = 2$, we denote $W^{m,2}(\mathcal{K})$ by $H^m(\mathcal{K})$ with the norm $\| \cdot \|_{m,2,\mathcal{K}} = \| \cdot \|_{m,\mathcal{K}}$. For simplicity of notation, when $\mathcal{K} = \Omega$, we omit the subscript $\mathcal{K}$ from the norm and inner product.

The space $H^m(\mathcal{K})$ is a Hilbert space with the natural inner product given by

$$(\varphi_1, \varphi_2)_{m,\mathcal{K}} = \sum_{0\leq|\boldsymbol{\lambda}|\leq m} \int_{\mathcal{K}} D^{\boldsymbol{\lambda}}\varphi_1 D^{\boldsymbol{\lambda}}\varphi_2 d\mathbf{x} \ \forall \ \varphi_1, \varphi_2 \in H^m(\mathcal{K}).$$

The Sobolev space $H_0^m(\Omega)$ is the space of all functions in $H^m(\Omega)$ whose traces up to order $m-1$ vanish on $\partial\Omega$.

We shall also use the following space-time function spaces in this thesis. Let $\mathbf{B}$ be a Banach space with norm $\|\cdot\|_{\mathbf{B}}$. Then for any integer $p$ with $1 \leq p \leq \infty$, define the standard Bôchner space $L^p(J;\mathbf{B})$, where $J = (0,T]$, consisting of all the measurable functions $\varphi : J \to \mathbf{B}$ such that

$$\|\varphi\|_{L^p(J;\mathbf{B})} = \left(\int_0^T \|\varphi(t)\|_{\mathbf{B}}^p dt\right)^{\frac{1}{p}} < \infty \text{ for } 1 \leq p < \infty,$$

$$\|\varphi\|_{L^\infty(J;\mathbf{B})} = \text{ess} \sup_{t\in[0,T]} \|\varphi(t)\|_{\mathbf{B}} < \infty \quad \text{for } p = \infty.$$

Also, for $1 \leq k < \infty$, we denote by $H^k(J;\mathbf{B})$ as the space of all measurable functions $\varphi : J \to \mathbf{B}$ for which

$$\|\varphi\|_{H^k(J;\mathbf{B})} = \left(\sum_{i=0}^{k} \int_0^T \left\|\frac{\partial^i \varphi(t)}{\partial t^i}\right\|_{\mathbf{B}}^2 dt\right)^{\frac{1}{2}} < \infty.$$

Without any confusion, for our notational convenience, we use $H^k(\mathbf{B})$ instead of $H^k(J;\mathbf{B})$, $L^2(\mathbf{B})$ instead of $L^2(J;\mathbf{B})$ and $L^\infty(\mathbf{B})$ instead of $L^\infty(J;\mathbf{B})$. Further, $C(\bar{J};\mathbf{B})$ denote the space of continuous functions $\varphi : \bar{J} \to \mathbf{B}$ induced with the norm $\|\varphi\|_{C(\bar{J};\mathbf{B})} = \max_{t\in[0,T]} \|\varphi(t)\|_{\mathbf{B}} < \infty$. For a complete discussion on Sobolev spaces one may refer to Adams and Fournier [5], Dautray and Lions [38] and Evans [58]. Furthermore, $H^{-m}(\Omega)$ denotes the space of all bounded linear functionals on $H_0^m(\Omega)$. For a functional $f \in H^{-m}(\Omega)$, its action on a function $\varphi \in H_0^m(\Omega)$ is defined by $(f,\varphi)$, which represents the duality pairing between $H^{-m}(\Omega)$ and $H_0^m(\Omega)$. The negative Sobolev norm is define by

$$\|f\|_{-m} = \sup_{0\neq\varphi\in H_0^m(\Omega)} \frac{(f,\varphi)}{\|\varphi\|_m}.$$

Also, to deal with the interface problem (1.1.10)-(1.1.11), we consider the following Banach space

$$\mathcal{X}^* = \{\varphi \in L^2(\Omega) : \ \varphi|_{\Omega_i} \in H^2(\Omega_i), \quad i = 1, 2\}, \tag{1.2.1}$$

along with the norm

$$\|\varphi\|_{\mathcal{X}^*} = \|\varphi\|_{L^2(\Omega)} + \|\varphi\|_{H^2(\Omega_1)} + \|\varphi\|_{H^2(\Omega_2)}.$$

Next, we shall review a few important inequalities for our subsequent analysis [80].

**Young's inequality:** For any two positive real numbers $a_1, a_2$ and $\epsilon > 0$, the following inequality holds

$$a_1 a_2 \leq \frac{a_1^2}{4\epsilon} + \epsilon a_2^2.$$

An important consequence of Young's inequality is the Hölder's inequality. The discrete version of Hölder's inequality is stated below.

**Hölder's inequality:** Let $p > 1$ and $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any sequences of real numbers $a_i, b_i \in \mathbb{R},\ i = 1, 2, \cdots, n$,

$$\sum_{i=1}^{n} |a_i b_i| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}}.$$

In particular, when $p = q = 2$, the above inequality is known as the Cauchy-Schwarz inequality in $\mathbb{R}^n$.

The integral form of Hölder's inequality is as follows: Let $p > 1$ and $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any measurable function $\varphi, \upsilon : \Omega \to \mathbb{R}$ such that $\varphi \in L^p(\Omega)$ and $\upsilon \in L^q(\Omega)$ then following inequality holds

$$\|\varphi \upsilon\|_{L^1(\Omega)} \leq \|\varphi\|_{L^p(\Omega)} \|\upsilon\|_{L^q(\Omega)}.$$

For $p = q = 2$, the above inequality is known as the Cauchy-Schwarz inequality. Further, a general version of Hölder's inequality is as follows:

**General Hölder's inequality:** Let $p > 1$ and $q, r > 0$ are real numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Then for $\varphi \in L^p(\Omega),\ \upsilon \in L^q(\Omega),\ \psi \in L^r(\Omega)$, then following inequality holds

$$\|\varphi \upsilon \psi\|_{L^1(\Omega)} \leq \|\varphi\|_{L^p(\Omega)} \|\upsilon\|_{L^q(\Omega)} \|\psi\|_{L^r(\Omega)}.$$

**Poincaré inequality:** Let $\Omega$ be an open bounded domain in $\mathbb{R}^d$. Then there exists a positive constant $C = C(\Omega)$ such that

$$\|\varphi\| \leq C \|\nabla \varphi\| \quad \forall \varphi \in H_0^1(\Omega).$$

In view of the Poincaré inequality, semi-norm $\|\nabla \cdot\|$ defines a norm in $H_0^1(\Omega)$.

**Gronwall's inequality:** Let $\varphi(t)$ be a continuous function and $\upsilon(t)$ is a non negative continuous function in the interval $t_0 \leq t \leq t_0 + a$. If a continuous function $\psi(t)$ has the property that

$$\psi(t) \leq \varphi(t) + \int_{t_0}^{t} \psi(s) \upsilon(s) ds \quad \text{for } t \in [t_0, t_0 + a], \tag{1.2.2}$$

then

$$\psi(t) \leq \varphi(t) + \int_{t_0}^{t} \varphi(s)\upsilon(s) \exp\left[\int_{s}^{t} \upsilon(\tau)d\tau\right] ds \quad \text{for } t \in [t_0, t_0 + a]. \tag{1.2.3}$$

In particular, when $\varphi(t) = C$ a non-negative constant, then we have

$$\psi(t) \leq C \exp\left[\int_{t_0}^{t} \upsilon(s)ds\right] \quad \text{for } t \in [t_0, t_0 + a].$$

A modified version of Gronwall's inequality is required for our analysis (cf. Lemma 3.1 [63]).

**Lemma 1.2.1.** *For two non negative continuous functions $u$, $v$ and two positive constants $C_1, C_2 < \infty$ such that*

$$u(t) + v(t) \leq C_1 + C_2 \int_0^t u(s)ds \quad \forall\, t \in [0, T].$$

*Then the following modification of Gronwall's inequality holds*

$$u(t) + v(t) \leq C_1 e^{C_2 T} \quad \forall\, t \in [0, T].$$

**Remark 1.2.1.** *From Proposition 7.1 in [139], for any $\phi \in H^1(J; \mathbf{B})$, we have*

$$\sup_{t \in \bar{J}} \|\phi(t)\|_{\mathbf{B}} \leq C \|\phi\|_{H^1(J;\mathbf{B})}. \tag{1.2.4}$$

*More precisely, we have that $\phi \in C(\bar{J}; \mathbf{B})$ for $\phi \in H^1(J; \mathbf{B})$.*

For our notational convenience, we shall be using $\frac{\partial \varphi}{\partial t}$ or $\varphi'$ or $\varphi_t$ interchangeably to denote first order time derivative of $\varphi$. Similar notions are used for higher-order time derivatives.

## 1.3 Background and Motivation

The existing relevant literature, our contributions and the reasons for the current study are succinctly explained in this section.

The modeling of partial differential equations (PDEs) has become a prime topic in science and engineering. Its application ranges from simulating the aerodynamics of large aircraft to modeling of atoms on a quantum mechanic level. For instance, quantum physics problems are described by Schrödinger equation and Klein-Gordon equation; electromagnetic fields involve the Maxwell equations. The Westervelt's equation can be utilized to model the non-linear propagation of ultrasound waves in biological tissues. Also, Navier-Stokes equations describe the motion of viscous fluids such as liquids and gases. The analytic solutions of these PDEs are often hard to determine. Therefore, developing effective numerical techniques that are computationally economical is beneficial for such PDEs. This thesis focuses on some hyperbolic equations, such as wave equations, semilinear Klein-Gordon equations and quasi-linear Westervelt's acoustic wave equations. Further, we expand the analysis to second order general hyperbolic equations with interfaces. Numerous numerical techniques are available in the literature that are intended for the numerical approximations of these types of hyperbolic problems. For many years, the finite difference method (FDM) has been known as the most practical and dominant numerical algorithm for solving PDEs appeared in various scientific fields, which approximates the PDEs on a uniform grid and its implementation is very efficient on simple geometries. The hyperbolic equations, particularly for nonlinear conservation laws, the finite difference methods have been continuously playing a significant role till now initiated with work by several researchers, e.g., Friedrichs, Lax and Wendroff (cf. [8, 57, 103]). However, unstructured meshes offer more flexibility as they show alignment with many complex geometries and thus, can be used with finite element methods. In the last few decades, tremendous progress has been made in developing improved finite element methods for different classes of problems and it has become an active research area for applied mathematicians. The ability of this method to solve a wide class of problems with complicated structures systematically facilitates its use over other numerical procedures in different fields of science and engineering. In this thesis, we have analyzed newly developed efficient and accurate WG-FEMs for some hyperbolic problems.

FEMs to accurately simulate hyperbolic problems are constantly being developed and applied with increasing levels of sophistication. Numerical methods applied for hyperbolic problems based on finite element framework can be mainly grouped by standard conforming FEMs, Mixed FEMs and Discontinuous Galerkin (DG) methods. A substantial amount of research on *a priori* and a posteriori error estimates in the design of standard conforming finite element methods for the hyperbolic equations are available in literature (e.g., [14, 15, 16, 21, 53, 67, 68, 92, 136] to name a few). The standard Galerkin approximation of the second order wave equation can also be interpreted as mixed discretization of the first order system. This method is especially beneficial for problems where secondary variables, such as stresses or fluxes are significant. We refer to [35, 55, 69, 87] and references therein for broad analysis of mixed finite element methods for wave propagation problems. In practice, allowing arbitrary shape in a finite element partition provides a convenient flexibility in both numerical approximation and mesh generation. The classical finite element methods based on conforming finite element discretization, have limitations in practical computation. The conforming finite element space is restricted to piecewise polynomials with prescribed continuity that ensures conformity and stability of the corresponding weak formulation, which is often very difficult to implement, particularly for problems in high dimensions and/or on general polytopal partitions. Contrarily, discontinuous Galerkin (DG) methods can be used for different types and shapes of irregular non matching grids and for locally varying polynomial order. Since their inception DG-FEMs have become popular in solving hyperbolic equations among the research community. In the last two decades several discontinuous finite element methods for solving the wave equation have appeared in the literature, e.g. penalty DG-FEM [6, 74, 75], local DG-FEM [10, 30], the hybrid DG (HDG) [28, 34] and hybrid high-order (HHO) method [24, 25, 144].

Nonlinear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics, can be modeled by partial differential equations. A broad class of analytical solution methods and numerical solution methods were used to handle these problems. For the finite element approximations to semilinear hyperbolic problems, we refer to [2, 4, 135] and the references therein. The numerical solutions to the nonlinear Klein-Gordon equation have received considerable attention in the literature. Several numerical techniques have been applied to solve the KG equation, namely finite difference method [1, 18, 78], Galerkin finite element method [94], spectral methods [160, 161], among the others. Kirby et al. [94] explored the finite element method for more general nonlinear term $f$ satisfying the growth condition $|f'(u)| \leq C_f(1+|u|^p)$ for some number $p \geq 0$, but, they did not explore fully discrete schemes properly. For the two-dimensional sine-Gordon equation, the first local DG scheme was introduced in [12] and is based on the velocity-stress formulation. This reference also provides a complete list of numerical methods for approximating the sine-Gordon equation, including finite differences and finite element methods. The LDG method in [11, 12] uses alternating fluxes and it is also energy-conserving. Another development using the high-order local discontinuous Galerkin (LDG) technique for the KG equation can be found in [159]. Zhang et al. [164] approached with a linearized high-order Galerkin finite element method for two-dimensional nonlinear time fractional Klein-Gordon equations. In [84], a high-order numerical solution was obtained using the cubic B-spline Galerkin method for space discretization and the finite difference method of order four for time discretization. Recently, energy-preserving fully discrete finite element schemes were analyzed by Dautov et al. [37]. An energy-conserving finite element method for coupled nonlinear KG equations using the error splitting technique and postprocessing interpolation were proposed by Cui et al. [36]. Hamiltonian finite element methods, which combine finite element methods for space discretization with symplectic methods for time discretization, can be found in [33, 140, 141]. Finite element methods discretize the Hamiltonian system of PDEs in a way that results in a system of ODEs that is also Hamiltonian. As a result, when the system is discretized by a symplectic method, the conservation or non-drifting properties of the time integrators apply, making it suitable for long-term simulations.

The generalized wave equation (1.1.10) serves as a linear model problem for more challenging nonlinear models for wave propagation, which contain dissipative and attenuation parameters. There is plenty of literature available for the convergence analysis for the general linear second order hyperbolic equation via the classical finite element algorithm (cf. [62, 79, 91, 102, 133, 148] to name a few). In recent years, studies on finite element error analysis for the Westervelt's equation have gradually increased (cf. [7, 121, 129]). For the strongly damped Westervelt's equation, error estimates for space discretization using continuous and discontinuous Galerkin finite elements were derived by Nikolić & Wohlmuth [131] and Antonietti et al. [7], respectively. Both analyses rely on the Banach fixed-point theorem combined with the stability and convergence of a linearized Westervelt's equation with variable coefficients. Optimal convergence rates in $L^2$-based spatial norms were derived for the approximate solution using the linear conforming finite element method, which has been extended to the mixed finite element method in [121]. The standard finite element method has limitations due to the continuity requirement on the approximation spaces, which is often challenging to implement, particularly for problems in high dimensions or on general polytopal partitions. In contrast, the discontinuous Galerkin (DG) method offers flexibility in constructing approximation spaces on hybrid meshes. In [7], the authors proposed a high-order DG method for the Westervelt's equation and derived an *a priori* error estimate in a suitable energy norm, extending the results to nonlinear elasto-acoustic problems with discontinuous coefficients by Muhr & Wohlmuth [129]. An alternative approach based on semigroup theory was proposed in [82, 119] for a general class of first and second order quasi-linear wave-type problems. Using piecewise polynomials of degree at least $p \geq 2$, error estimates of $\mathcal{O}(h^p)$ under $L^\infty(H^1)$ norm and $H^1(L^2)$ norm were derived for the Westervelt's equation in inviscid media ($b = 0$) (see, Theorem 6.2, [82]) in a conforming setting, with results extended to an implicit fully discrete scheme in [119] (see, Theorem 7.2 therein). Although stronger convergence rates were derived in [7, 131] compared to Theorem 6.4 in [82], the results in [7, 131] deteriorate as $b \to 0^+$. However, the convergence analysis for the inviscid Westervelt's equation can be recovered in the standard energy norm as well as a higher-order norm as $b \to 0^+$ for small data (see, Theorem 5.1 in [88]). Recently, a hybridizable discontinuous Galerkin method for the Westervelt's model was discussed in [71], establishing optimal convergence for the gradient approximation ($\nabla u$), stable as $b \to 0^+$. Algorithmic aspects of finite element discretizations without any *a priori* analysis for the Westervelt's equation and in general, nonlinear sound propagation by the finite element method, can be found in [61, 90, 149]. Interface problems are frequently encountered in scientific computing and many applied sciences. Typical examples are the elliptic, parabolic and hyperbolic equations with discontinuous coefficients. Due to the practical relevance of interface problems in many engineering and industrial applications, numerical methods for interface problems have been investigated widely. Finite element method (FEM) is another class of important approaches for interface problems and a wide variety of FEM approaches have been proposed in the literature. Classical finite element methods for interface problems are mainly based on the interface-fitted discretization. The performance of such kind of interface-fitted FEMs depends on the quality of underlying finite element partition and how well the interface is resolved by the finite element mesh. A fitted linear finite element method was proposed and analyzed for the interface problem (1.1.10)-(1.1.12) in [54]. Optimal *a priori* error estimates for both semidiscrete and fully discrete schemes were proved in $L^\infty(L^2)$ norm. The fully discrete space-time finite element discretization is based on second order in time Newmark scheme.

Weak Galerkin finite element method (WG-FEMs) was proposed by Wang and Ye in [152] for standard meshes, and later extended for general polygonal meshes in [153]. The computing procedure of the WG method for elliptic equations have been explained in [125]. WG-FEMs refers to a numerical technique which permits domain to be discretized into arbitrary shape of polygons. As a result, mesh formation becomes much more versatile for complicated geometries common in real-world problems. The weak

Galerkin algorithm is based on some basic principles: the use of weak functions and weak derivatives instead of the classical derivative present in the variational form of the given problem, and a parameter-free stabilizer added to handle the discontinuity of function. The use of weak functions allows WG-FEMs to be more adaptable and it has been developed to handle a wide range of PDEs emerging from real-world engineering and technological applications. However, one disadvantage is that, we might have to evaluate integrals of rational functions rather than integrals of polynomials on the reference element [166]. There is a considerable collection of literature on such applications; for example elliptic problems [105, 109, 112, 113, 126, 127, 150, 151], parabolic problems [42, 43, 45, 107, 165, 168], hyperbolic equation [83, 124, 154, 163] etc. Hybrid high-order (HHO) method is reported to be very closely related to WG finite element method as the reconstruction operator in the HHO method corresponds to the weak gradient in WG methods [25, 51]. The only difference between the HHO and WG methods is in the selection of discrete unknowns and the stabilization strategy. However, the links between HHO and WG methods are not fully explored yet, nevertheless they share something in their roots (cf. [19, 32, 51]). It is noteworthy that WG, HHO and HDG use somewhat different analytical techniques and are based on distinct conceptual frameworks. The goal of this thesis is to analyze the weak Galerkin finite element method for linear wave equations, the semilinear and quasi-linear hyperbolic equations, and extend it to the general hyperbolic equation with the interface.

At first, we develop a systematic framework on weak Galerkin finite element methods for second order linear wave equation (1.1.1)-(1.1.2) by allowing different degree polynomial approximations for every local element. A general local WG finite space looks like $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$, where $k \geq 1$ is the degree of polynomials in the interior of the element $\mathcal{K}$, $j \geq 0$ is the degree of polynomials on the boundary of $\mathcal{K}$ and $l \geq 0$ is the degree of polynomials employed in the computation of weak gradients or weak partial derivatives. The choice of such polynomials has a considerable influence on the accuracy and computing difficulty of the associated WG method. Error estimates in $L^2$ norm as well as in discrete $H^1$ norm for semidiscrete and $L^2$ estimates for fully discrete schemes have been established in general weak Galerkin space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$ for sufficiently smooth solution. The time discretization for fully discrete scheme is based on second order in time Crank-Nicolson scheme. The results illustrate the robustness of the WG-FEMs with various polynomial combinations in the numerical scheme, addressing a gap in the existing literature.

Next, we focus to study weak Galerkin finite element method for semilinear Klein-Gordon equation (1.1.4)-(1.1.5). We have assumed that the nonlinear term $f$ is globally

Lipschitz i.e. there exists a constant $C_f$ such that $\|f(u)-f(v)\| \leq C_f\|u-v\|$ for $u, v$. Optimal *a priori* error estimate in both $L^2$ norm as well as discrete $H^1$ norm for semidiscrete case have been established for the weak Galerkin space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$ is an integer. Also, for fully discrete scheme, we employ implicit type time Newmark scheme for temporal discretization and the error estimate in $L^2$ norm demonstrates that the optimal convergence rate has been achieved. Some numerical examples are provided to validate theoretical results.

Our third problem focuses on studying the weak Galerkin finite element method for the Westervelt's model [156] of ultrasound waves (1.1.7)-(1.1.8) on polygonal meshes. More precisely, we study the spatial discretization of Westervelt's quasi-linear strongly damped wave equation by high-order weak Galerkin discretization in space. A significant challenge in analyzing the Westervelt's equation lies in avoiding the degeneracy of the coefficient $(1 - 2\sigma u)$ for the second time derivative $u_{tt}$ in (1.1.7). Typically, non-degeneracy can be attained under high regularity for the solution and the application of embedding results (e.g., $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$); however, this approach is not feasible for nonconforming finite element approximations. Instead, reliance on inverse estimates for finite element functions and the stability and approximation properties of the interpolation operator is necessary. Our work is inspired by a growing body of recent literature on finite element approximations for Westervelt's equation. Our contribution lies in analyzing the spatial discretization of Westervelt's quasi-linear, strongly damped wave equation within a weak Galerkin finite space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), \left[\mathbf{P}_{k-1}(\mathcal{K})\right]^2)$. We derive optimal convergence rates in $L^2$ based spatial norms for sufficiently small data and mesh size. Our approach hinges on the Banach fixed-point theorem in conjunction with stability and *a priori* error analysis of linear diffusive viscous wave equation (1.1.9) with time-dependent coefficients. Notably, the numerical approximation of the linear diffusive viscous wave equation is significant in its own right, as evidenced by existing literature. Wave equations with time-dependent coefficients have received relatively less attention, particularly for diffusive viscous wave models (1.1.9). Our results aim to enhance the numerical treatment of strongly damped quasi-linear wave equations, with numerical experiments in two-dimensional settings serving to illustrate the theoretical convergence results.

Next, we have made an attempt to extend the convergence analysis to the general linear second order hyperbolic equation (1.1.10) with interfaces. It is worth noting that rigorous finite element error analysis for the interface problem (1.1.10)-(1.1.12) with non-homogeneous jump conditions is mostly missing. Only linear classical finite element method was discussed in [54] for continuous solution. On the other hand, discontinuity of the solution along the interfaces adds more challenges than one would imagine. Further, higher order of convergence is always one of the major research goals, because high order methods are more accurate. Semidiscrete error estimate in $L^2$ norm as well as $H^1$ norm have been executed for the weak space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$ is an integer. For the fully discrete scheme, we employ implicit second order Crank-Nicolson scheme for temporal discretization. Optimal order error estimate in the $L^\infty(L^2)$ norm is derived as $\mathrm{O}(h^{k+1} + \tau^2)$, where $h$ is the mesh size and $\tau$ the time step. Few numerical experiments are presented to illustrate our theoretical convergence findings.

## 1.4 WG Discretization for Elliptic Problem

In this section, we shall briefly discuss WG-FEMs for elliptic problems. We also review the definition of weak gradient operator and its discrete counterpart.

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary $\partial\Omega$. We consider the following linear elliptic problem

$$-\nabla \cdot (\boldsymbol{\mu} \nabla u) = f \quad \text{in } \Omega, \tag{1.4.1}$$

with the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.4.2}$$

We consider that the coefficient matrix $\boldsymbol{\mu} = (\mu_{i,j}(\mathbf{x}))_{2\times 2} \in [L^\infty(\Omega)]^{2^2}$ is symmetric and positive definite in $\Omega$. The load function $f$ is considered to be sufficiently smooth in its domain of definition.

For some $h_0 > 0$ and $h \in (0, h_0]$, let $\mathcal{T}_h$ be a polygonal partition of the two-dimensional domain $\Omega$. Assume that it meets a set of conditions for shape regularity described in [151, 153]. Let $\mathcal{E}_h$ be the collection of all edges of the polygons in $\mathcal{T}_h$ and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$. For any $\mathcal{K} \in \mathcal{T}_h$, we denote by $|\mathcal{K}|$ the measure of $\mathcal{K}$ and its diameter by $h_{\mathcal{K}}$ and for $\mathcal{T}_h$, mesh size by $h$, defined as $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$.

Weak Galerkin algorithm is based on two fundamental principles, namely, the use of discrete weak functions defined on $\mathcal{T}_h$ and substituting the classical gradient with a weak gradient operator presented in the variational form of the given problems. The weak gradient operator and its discrete counterpart were discussed in [152, 153].

Let $\mathcal{K} \in \mathcal{T}_h$ be an element with boundary $\partial\mathcal{K}$. A weak function $\phi_h$ on $\mathcal{K}$ is a pair of scalar functions $\phi_h = \{\phi_0, \phi_b\}$ such that $\phi_0 \in L^2(\mathcal{K})$ and $\phi_b \in H^{\frac{1}{2}}(\partial\mathcal{K})$. Note that $\phi_b$ need not be necessarily related to the trace of $\phi_0$ on $\partial\mathcal{K}$. Let the space of all weak functions on $\mathcal{K}$ be represented by $\mathcal{W}(\mathcal{K})$; i.e.,

$$\mathcal{W}(\mathcal{K}) = \big\{\phi_h = \{\phi_0, \phi_b\} : \phi_0 \in L^2(\mathcal{K}),\ \phi_b \in H^{\frac{1}{2}}(\partial\mathcal{K})\big\}. \tag{1.4.3}$$

Define a space

$$H(\text{div}, \mathcal{K}) = \{\mathbf{q} : \mathbf{q} \in [L^2(\mathcal{K})]^2, \nabla \cdot \mathbf{q} \in L^2(\mathcal{K})\}.$$

For any weak function $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}(\mathcal{K})$, its weak gradient $\nabla_w \phi_h$ is defined (interpreted) as a linear functional on $H(\text{div}, \mathcal{K})$, which acts on each $\mathbf{q} \in H(\text{div}, \mathcal{K})$ is defined as

$$(\nabla_w \phi_h, \mathbf{q}) = -\int_{\mathcal{K}} \phi_0 (\nabla \cdot \mathbf{q}) d\mathcal{K} + \int_{\partial \mathcal{K}} \phi_b (\mathbf{q} \cdot \mathbf{n}) ds, \tag{1.4.4}$$

where $\mathbf{n}$ is the unit outword drawn normal to $\partial \mathcal{K}$.

For non-negative integer $k$, let $\mathbf{P}_k(\mathcal{K})$ be the space of all polynomials of degree not greater than $k$ on the polygon $\mathcal{K} \in \mathcal{T}_h$. Similarly, for a non-negative integer $j$, let $\mathbf{P}_j(e)$ be the set of all polynomials of degree not greater than $j$ on the edge $e \in \mathcal{E}_h$. A discrete weak function $\phi_h = \{\phi_0, \phi_b\}$ on $\mathcal{K}$ refers to the weak function $\phi_h = \{\phi_0, \phi_b\}$, where $\phi_0 \in \mathbf{P}_k(\mathcal{K})$ and $\phi_b|_e \in \mathbf{P}_j(e)$, $e \subset \partial \mathcal{K}$. Let $\mathcal{W}(k, j, \mathcal{K})$ be the space of all discrete weak functions on $\mathcal{K}$; i.e.

$$\mathcal{W}(k, j, \mathcal{K}) = \Big\{ \phi_h = \{\phi_0, \phi_b\} : \phi_0 \in \mathbf{P}_k(\mathcal{K}), \ \phi_b|_e \in \mathbf{P}_j(e), \ e \subset \partial \mathcal{K} \Big\}. \tag{1.4.5}$$

Then the global weak Galerkin finite space $\mathcal{W}_h$ is constructed from the local finite space $\mathcal{W}(k, j, \mathcal{K})$ as

$$\mathcal{W}_h = \{\phi_h = \{\phi_0, \phi_b\} : \phi|_{\mathcal{K}} \in \mathcal{W}(k, j, \mathcal{K}), [\phi_h]_e = 0, \forall e \in \mathcal{E}_h^0\}.$$

Here $[\phi_h]_e = [\phi_b]_e = \phi_b|_{\mathcal{K}_1 \cap e} - \phi_b|_{\mathcal{K}_2 \cap e}$ denotes the jump of $\phi_h$ across interior edge $e \in \mathcal{E}_h^0$, such that $e \subset \partial \mathcal{K}_1 \cap \partial \mathcal{K}_2$. Denote $\mathcal{W}_h^0$ the subspace of $\mathcal{W}_h$ containing all weak functions, which vanishes on boundary $\partial \Omega$; i.e.

$$\mathcal{W}_h^0 = \{\phi_h \in \mathcal{W}_h : \phi_b|_{\partial \Omega} = 0\}.$$

Next, for $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, we introduce discrete weak gradient operator, denoted by $\nabla_w$, is defined as the unique polynomial $\nabla_w \phi_h \in [\mathbf{P}_l(\mathcal{K})]^2$ that satisfies the following equation

$$(\nabla_w \phi_h, \mathbf{q}) = -\int_{\mathcal{K}} \phi_0 (\nabla \cdot \mathbf{q}) d\mathcal{K} + \int_{\partial \mathcal{K}} \phi_b (\mathbf{q} \cdot \mathbf{n}) ds \quad \forall \mathbf{q} \in [\mathbf{P}_l(\mathcal{K})]^2, \tag{1.4.6}$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial \mathcal{K}$ and $l \geq 0$ is the prescribed non-negative integer. By using divergence theorem to equation (1.4.6), we obtain

$$(\nabla_w \phi_h, \mathbf{q}) = (\nabla \phi_0, \mathbf{q})_{\mathcal{K}} + \langle \phi_b - \phi_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial \mathcal{K}} \ \ \forall \ \mathbf{q} \in [\mathbf{P}_l(\mathcal{K})]^2. \tag{1.4.7}$$

For the weak Galerkin approximation, using weak gradient $\nabla_w$, we define a bilinear map $\mathcal{A}_w : \mathcal{W}_h \times \mathcal{W}_h \to \mathbb{R}$ as

$$\mathcal{A}_w(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} (\boldsymbol{\mu}\nabla_w u_h, \nabla_w \phi_h)_{\mathcal{K}} + \mathbf{S}(u_h, \phi_h) \ \ \forall u_h, \phi_h \in \mathcal{W}_h. \tag{1.4.8}$$

Here $\mathbf{S}(\cdot,\cdot) : \mathcal{W}_h \times \mathcal{W}_h \to \mathbb{R}$ is a semi-positive bilinear map. It is referred to a stabilizer or smoother. We choose the stabilizer $\mathbf{S}(\cdot,\cdot)$ in such a manner that it is compatible with both the WG theory and the execution of WG numerical schemes. Here two examples are presented (cf. [104, 153]).

**Example 1.4.1.** *(Projected Element-Boundary Discrepancy) For $u_h = \{u_0, u_b\}$, $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, the continuity of $u_h$ and $\phi_h$ can be measured by the quantities $u_b - u_0|_{\partial\mathcal{K}}$, $\phi_b - \phi_0|_{\partial\mathcal{K}}$ respectively, for each element $\mathcal{K} \in \mathcal{T}_h$. The stabilizer based on projected-element-boundary-discrepancy is given by*

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle \mathcal{Q}_m(u_b - u_0|_{\partial\mathcal{K}}), \mathcal{Q}_m(\phi_b - \phi_0|_{\partial\mathcal{K}})\rangle_{\partial\mathcal{K}},$$

*where $m = \max\{j, l\}$ and $\mathcal{Q}_m : L^2(\partial\mathcal{K}) \to \mathbf{P}_m(\partial\mathcal{K})$ is the standard $L^2$-projection.*

**Example 1.4.2.** *(Element-Boundary Discrepancy) For $u_h, \phi_h \in \mathcal{W}_h$, the stabilizer based on element-boundary-discrepancy is given by: For $u_h = \{u_0, u_b\}$, $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$*

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle (u_b - u_0|_{\partial\mathcal{K}}), (\phi_b - \phi_0|_{\partial\mathcal{K}})\rangle_{\partial\mathcal{K}}.$$

The notation $\langle\cdot,\cdot\rangle_{\partial\mathcal{K}}$ denotes the $L^2$ inner product on $\partial\mathcal{K}$ and accordingly, we write

$$\langle\cdot,\cdot\rangle_{\partial\mathcal{K}} = \sum_{e\subset\partial\mathcal{K}} \langle\cdot,\cdot\rangle_e,$$

where $\langle\cdot,\cdot\rangle_e$ denotes $L^2$ inner product on $e \subset \partial\mathcal{K}$.

The polynomial degree and stabilizer must be selected in the WG techniques so that the bilinear map $\mathcal{A}_w(\cdot,\cdot)$ is coercive with regard to the mesh-dependent semi-norm $\|\cdot\|_{1,h}$ (cf. [151]) defined as

$$\|\phi_h\|_{1,h} = \left( \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla\phi_0\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^{-1}\|\phi_b - \phi_0\|_{\partial\mathcal{K}}^2) \right)^{\frac{1}{2}} \ \ \forall \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h. \tag{1.4.9}$$

In fact, the semi-norm $\|\cdot\|_{1,h}$ is a norm in $\mathcal{W}_h^0$.

The following lemma establishes the coercivity of the bilinear map $\mathcal{A}_w(\cdot,\cdot)$ on WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$. Details follow from [151].

**Lemma 1.4.1.** *Suppose that $l \geq k-1$ and $m = \max\{j, l\}$. Then there exist two positive constants $C_*$ & $C^*$ such that following inequality holds true for any $\phi_h \in \mathcal{W}_h$*

$$C_* \|\phi_h\|_{1,h}^2 \leq \mathcal{A}_w(\phi_h, \phi_h) \leq C^* \|\phi_h\|_{1,h}^2. \tag{1.4.10}$$

Under the mesh assumptions prescribed as in [153], the following trace inequality holds on $\mathcal{T}_h$.

**Lemma 1.4.2.** (Lemma A.3, [153]) *For each $e \in \mathcal{E}_h$ and $\varphi \in H^1(\mathcal{K})$, we have*

$$\|\varphi\|_e^2 \leq C(h_{\mathcal{K}}^{-1} \|\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}} \|\nabla \varphi\|_{\mathcal{K}}^2). \tag{1.4.11}$$

The usual $L^2$ inner product on $\Omega$ can be written locally on each element as follows

$$(\nabla_w u_h, \nabla_w \phi_h) = \sum_{\mathcal{K} \in \mathcal{T}_h} (\nabla_w u_h, \nabla_w \phi_h)_{\mathcal{K}} \quad \text{for } u_h, \phi_h \in \mathcal{W}_h.$$

**Remark 1.4.1.** *For a time dependent discrete weak function $\phi_h : J = (0, T] \to \mathcal{W}_h^0$ such that $\phi_h \in C^1(\bar{J}; \mathcal{W}_h^0)$, we write $\phi_h(t) = \{\phi_0(t), \phi_b(t)\}$ for all $t \in \bar{J}$ and subsequently we define $\phi_h'(t) = \{\phi_0'(t), \phi_b'(t)\}$ for all $t \in \bar{J}$, where "$\prime$" stands for time derivative. For simplicity, for all $t \in \bar{J}$, we use $\phi_h = \{\phi_0, \phi_b\}$ for $\phi_h(t)$ and $\phi_h' = \{\phi_0', \phi_b'\}$ for $\phi_h'(t)$. Further, from the definition (1.4.7), we note that $(\nabla_w \phi_h)' = \nabla_w \phi_h'$ and $(\nabla_w \phi_h)|_{t=0} = \nabla_w \phi_h(0)$ for all $\phi_h \in C^1(\bar{J}; \mathcal{W}_h^0)$ with $J = (0, T]$. We shall follow similar notations for other higher order time derivatives.*

Throughout the thesis, $C$ is a positive constant independent of the mesh parameters, whose value varies depending on the situation.

## 1.5 Organization of the Thesis

This thesis is composed of seven chapters, which are listed below.

**Chapter 1**, contains the description of the problems, notations and preliminary materials to be used in the thesis. It also provides a brief survey on the relevant literature concerning the problems and their numerical solutions. Further, motivations for the present study is discussed.

In **Chapter 2**, we describe a systematic numerical study on weak Galerkin finite element method for second order linear wave equation (1.1.1) with variable coefficients by allowing polynomial approximations with various degrees for each local element. Convergence of semidiscrete WG solutions are established in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms for different degree polynomials.

In **Chapter 3**, we extend the spatially discrete *a priori* error analysis to the fully discrete approximation for second order wave equation (1.1.1). The space-time discretization is based on a second order in time Crank-Nikolson scheme. Numerical experiments are reported to justify the robustness, reliability and accuracy of the WG finite element method. Results and findings of this Chapter are published in [85].

**Chapter 4** is concerned about the weak Galerkin finite element approximations of semilinear Klein-Gordon equation (1.1.4). Optimal order error estimates in $L^\infty(L^2)$ as well as in $L^2(H^1)$ norms are shown to hold for spatially discrete continuous time scheme. The fully discrete schemes based on implicit type Newmark scheme. Some numerical examples are provided to validate theoretical results. Results and findings of this Chapter are published in [86].

**Chapter 5** is devoted to the convergence analysis of weak Galerkin finite element method for Westervelt quasi-linear wave equation. Stability and optimal $L^2$ error estimates of linear diffusive viscous wave equation with time dependent coefficients are discussed for continuous weak Galerkin scheme. The error analysis for the Westervelt's equation has been carried out based on the Banach fixed-point theorem combined with results for linearized equation. Numerical experiments carried out to illustrate the theoretical convergence results.

In **Chapter 6**, we design and analyze a weak Galerkin finite element method to approximate the second order general linear hyperbolic equation with interface. Semidiscrete error estimate in $L^2$ norm as well as $H^1$ norm have been executed for the weak space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$ is an integer. For fully discrete scheme, optimal $L^2$ error estimate have been carried out by employing Crank-Nicolson scheme for temporal discretization. Also, some numerical results are provided to support our theoretical results.

Finally in **Chapter 7**, we discuss the critical evaluation of the results reported in this thesis. This chapter concludes with a brief discussion on the possible extensions and future scopes.

For clarity of presentation, we have repeatedly mentioned the problems and relevant preliminary materials at the beginning of each chapter.

# 2

# Semidiscrete WG-FEMs for Second Order Linear Wave Equation

In this chapter, we present a systematic numerical study for second order linear wave equation (1.1.1)-(1.1.3) using weak Galerkin finite element methods (WG-FEMs) for the semidiscrete case. Various degrees of polynomials are used to construct weak Galerkin finite spaces. Error estimates in $L^2$ norm as well as in discrete $H^1$ norm have been established for general weak Galerkin space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$, where $k, j$ & $l$ are non-negative integers with $k \geq 1$. Our results extend the numerical analysis of WG-FEMs for elliptic problems [151] and parabolic problems [43] to hyperbolic problems.

## 2.1 Introduction

To begin with, let us first consider the following second order linear wave equation

$$u_{tt} - \nabla \cdot (\boldsymbol{\mu} \nabla u) = f \quad \text{in } \Omega \times (0, T], \; T < \infty, \tag{2.1.1}$$

with the initial conditions

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = v^0(\mathbf{x}) \;\; \forall \mathbf{x} \in \Omega \tag{2.1.2}$$

and boundary condition

$$u(\mathbf{x}, t) = 0 \;\; \forall (\mathbf{x}, t) \in \partial\Omega \times (0, T]. \tag{2.1.3}$$

Here $\Omega$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $J = (0, T]$ is the observation time period. The coefficient matrix $\boldsymbol{\mu} = [\mu_{i,j}(\mathbf{x})]_{2\times 2}$ is considered to be symmetric and positive definite in the domain $\Omega$. The initial functions $\{u^0, v^0\}$, the coefficient matrix $\boldsymbol{\mu}$ and the source function $f$ are considered to be sufficiently smooth. Additional regularity assumptions were made throughout the chapter to carry out the convergence analysis. For the related regularity results, we refer to [58, 111].

The numerical solution of the wave equation is of fundamental importance to the simulation of time dependent acoustic, electromagnetic, or elastic waves. For such wave phenomena the second order wave equation often serves as a model problem. Numerous numerical techniques based on the finite element structure have been developed namely conforming finite element methods, Ritz-Galerkin finite element methods, Mixed-FEMs and discontinuous Galerkin methods for solving such PDEs. There have been several studies on error estimates using standard conforming FEMs for hyperbolic problems in literature [15, 67, 92, 136]. Also, several discontinuous finite element methods are developed to solve the wave equation such as penalty DG-FEM [6, 74, 75] and local DG-FEM [10, 30]. The use of discontinuous approximation functions makes WG-FEMs highly flexible in the construction of finite element spaces of any order with the price of more degrees of freedom and more complex formulations. Unlike classical FEMs, WG-FEMs refer to a numerical technique that permits domain to be discretized into arbitrary shapes of polygons. As a result, mesh formation becomes much more versatile for complicated geometries that are common in real-world problems. The present chapter is aimed to investigate a systematic structure for the weak Galerkin finite element methods for second order linear wave equation. For any $\mathcal{K} \subset \mathbb{R}^2$, a general local WG finite space is presented as $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$, where $k, j, l$ are positive integers with $k \geq 1$. In this study, we explore all possible combinations of polynomial functions in the reconstruction of underlying differential operators. Systematic analysis on weak Galerkin technique for elliptic problems was studied by Wang et al.[151], later extended to parabolic problems in [43]. Our study is intended to expand the error analysis in [43, 151] for the second order linear wave equation and it fills a gap in the existing literature. The error analysis reported in this chapter shows that the WG finite element solutions approximate the exact solutions with an optimal order in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms.

The rest of this chapter is organized as follows. In Sec. 2.2, we go through the weak Galerkin finite element discretization, the definition of the weak gradient with its discrete analogs with suitable polynomial spaces and derive an important error equation. Sec. 2.3 is devoted to optimal error analysis in both $L^\infty(L^2)$ and energy norm based on elliptic projection for the semidiscrete WG-FEMs technique.

## 2.2 Weak Galerkin Space and Semidiscrete Approximation

In this section, we deal with a spatially discrete scheme for the second order linear wave equation (2.1.1)-(2.1.3).

Let $\mathcal{T}_h$ be a polygonal partition of the two-dimensional domain $\Omega$ as described in

Chapter 1. Based on the discretization $\mathcal{T}_h$ for $k \geq 1$ and $j \geq 0$, we recall the following weak Galerkin finite space $\mathcal{W}_h$ defined as

$$\mathcal{W}_h = \big\{\phi_h = \{\phi_0, \phi_b\} : \phi|_{\mathcal{K}} \in \mathcal{W}(k, j, \mathcal{K}), [\phi_h]_e = 0, \forall e \in \mathcal{E}_h^0\big\}$$

and

$$\mathcal{W}_h^0 = \{\phi_h \in \mathcal{W}_h : \phi_b|_{\partial\Omega} = 0\}.$$

Here $[\phi_h]_e = [\phi_b]_e$ denotes the jump of $\phi_h \in \prod_{\mathcal{K}\in\mathcal{T}_h} \mathcal{W}(k, j, \mathcal{K})$ along interior edge $e \in \mathcal{E}_h^0$, where $\mathcal{W}(k, j, \mathcal{K})$ is the local weak Galerkin space as defined in (1.4.5).

Next, for $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, we recall the discrete weak gradient operator, denoted by $\nabla_w \phi_h$, is defined as the unique polynomial $\nabla_w \phi_h \in [\mathbf{P}_l(\mathcal{K})]^2$ that satisfies the following equation

$$(\nabla_w \phi_h, \mathbf{q}) = -\int_{\mathcal{K}} \phi_0 (\nabla \cdot \mathbf{q}) d\mathcal{K} + \int_{\partial\mathcal{K}} \phi_b (\mathbf{q} \cdot \mathbf{n}) ds \quad \forall \mathbf{q} \in [\mathbf{P}_l(\mathcal{K})]^2, \tag{2.2.1}$$

where $\mathbf{n}$ is the outword normal to the boundary $\partial\mathcal{K}$ and $l \geq 0$ is the prescribed non-negative integer. Applying divergence theorem to equation (2.2.1), we obtain

$$(\nabla_w \phi_h, \mathbf{q}) = (\nabla \phi_0, \mathbf{q})_{\mathcal{K}} + \langle \phi_b - \phi_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial\mathcal{K}} \;\; \forall \; \mathbf{q} \in [\mathbf{P}_l(\mathcal{K})]^2. \tag{2.2.2}$$

We recall some usual $L^2$ projections. On each $\mathcal{K} \in \mathcal{T}_h$, let the operator $\mathcal{Q}_k^0 : L^2(\mathcal{K}) \to \mathbf{P}_k(\mathcal{K})$ be the standard $L^2$ projection and for an edge $e \in \mathcal{E}_h$, let the operator $\mathcal{Q}_j^b : L^2(e) \to \mathbf{P}_j(e)$ be the standard $L^2$ projection. Similarly, let $\mathcal{Q}_h$ be the standard $L^2$ projection onto the WG finite space $\mathcal{W}_h$ such that $\mathcal{Q}_h|_{\mathcal{K}} = \{\mathcal{Q}_k^0, \mathcal{Q}_j^b\}$. Also, let $\mathbb{Q}_l : [L^2(\mathcal{K})]^2 \to [\mathbf{P}_l(\mathcal{K})]^2$ be another $L^2$ projection.

Let us discuss following important approximation properties of the projections $\mathcal{Q}_k^0$ and $\mathbb{Q}_l$ (cf. [153]).

**Lemma 2.2.1.** *Let $\mathcal{T}_h$ be a polygonal partition of $\Omega$ that meets the shape regularity conditions as described in [153]. Then for $\varphi \in H^{r+1}(\Omega)$, we have*

$$\begin{aligned} \sum_{\mathcal{K}\in\mathcal{T}_h} \Big( \|\varphi - \mathcal{Q}_k^0 \varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^2 \|\nabla(\varphi - \mathcal{Q}_k^0 \varphi)\|_{\mathcal{K}}^2 \Big) &\leq Ch^{2(r+1)} \|\varphi\|_{H^{r+1}(\Omega)}^2, \\ \sum_{\mathcal{K}\in\mathcal{T}_h} \Big( \|\nabla\varphi - \mathbb{Q}_l \nabla\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^2 \|\nabla(\nabla\varphi - \mathbb{Q}_l(\nabla\varphi))\|_{\mathcal{K}}^2 \Big) &\leq Ch^{2r} \|\varphi\|_{H^{r+1}(\Omega)}^2. \end{aligned}$$

The continuous-time WG approximation to the problem (2.1.1)-(2.1.3) can be described as follows: Find $u_h = \{u_0, u_b\} : (0, T] \to \mathcal{W}_h^0$, that solves the following equation

$$(u_{htt}, \phi_0) + \mathcal{A}_w(u_h, \phi_h) = (f, \phi_0) \;\; \forall \; \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0, \tag{2.2.3}$$

with $u_h(0) = \mathcal{Q}_h u^0$ and $u_{ht}(0) = \mathcal{Q}_h v^0$.

The above scheme (2.2.3) is well-posed because WG space $\mathcal{W}_h^0$ is a normed linear space with regard to the energy norm $|||\cdot|||$ associated with the bilinear map $\mathcal{A}_w(\cdot,\cdot)$ defined as

$$|||\phi_h||| = \sqrt{\mathcal{A}_w(\phi_h,\phi_h)} \ \text{ for } \ \phi_h \in \mathcal{W}_h^0. \tag{2.2.4}$$

Now from the coercive inequality (1.4.10), we can conclude that discrete $H^1$ norm $\|\cdot\|_{1,h}$ and triple norm $|||\cdot|||$ are equivalent.

Let $u$ be the exact solution of (2.1.1)-(2.1.3) and $u_h$ be its semidiscrete solution given by (2.2.3). Then due to Lemma 2.2.1, we only try to bound following projected error

$$e_h(t) = \{e_0(t), e_b(t)\} = u_h(t) - \mathcal{Q}_h u(t), \ t \in [0,T]. \tag{2.2.5}$$

For simplicity, we write $e_h = \{e_0, e_b\} = u_h - \mathcal{Q}_h u$.

Further, we derive the following error equation for $e_h$, which is very important for the subsequent error analysis.

**Lemma 2.2.2.** *Let $e_h(t) = u_h(t) - \mathcal{Q}_h u(t)$. Then for all $\phi_h = \{\phi_0,\phi_b\} \in \mathcal{W}_h^0$, we have*

$$(e_{htt},\phi_0) + \mathcal{A}_w(e_h,\phi_h) = -\left\{\ell_1(u,\phi_h) + \ell_2(u,\phi_h) + \ell_3(u,\phi_h) + \mathbf{S}(\mathcal{Q}_h u,\phi_h)\right\}, \tag{2.2.6}$$

*where $\ell_1(\cdot,\cdot)$, $\ell_2(\cdot,\cdot)$, $\ell_3(\cdot,\cdot)$ are bilinear maps given as*

$$\begin{cases} \ell_1(u,\phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l\nabla \mathcal{Q}_k^0 u) - \boldsymbol{\mu}\nabla u, \nabla\phi_0\right)_{\mathcal{K}}, \\ \ell_2(u,\phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} \left\langle(\mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l\nabla \mathcal{Q}_k^0 u) - \boldsymbol{\mu}\nabla u)\cdot\mathbf{n}, \phi_b - \phi_0\right\rangle_{\partial\mathcal{K}}, \\ \ell_3(u,\phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} \left\langle \mathcal{Q}_j^b u - \mathcal{Q}_k^0 u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi_h)\cdot\mathbf{n}\right\rangle_{\partial\mathcal{K}}. \end{cases} \tag{2.2.7}$$

*Proof.* For any $\phi_h = \{\phi_0,\phi_b\} \in \mathcal{W}_h^0$, we test the equation (2.1.1) against $\phi_0$ on each $\mathcal{K} \in \mathcal{T}_h$ to obtain

$$\begin{aligned} (f,\phi_0) &= (u_{tt},\phi_0) - \sum_{\mathcal{K}\in\mathcal{T}_h}(\nabla\cdot(\boldsymbol{\mu}\nabla u),\phi_0)_{\mathcal{K}} \\ &= (\mathcal{Q}_h u_{tt},\phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h}(\boldsymbol{\mu}\nabla u,\nabla\phi_0)_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h}\langle\boldsymbol{\mu}\nabla u\cdot\mathbf{n},\phi_0\rangle_{\partial\mathcal{K}} \\ &= ((\mathcal{Q}_h u)_{tt},\phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h}(\boldsymbol{\mu}\nabla u,\nabla\phi_0)_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h}\langle\boldsymbol{\mu}\nabla u\cdot\mathbf{n},\phi_0-\phi_b\rangle_{\partial\mathcal{K}}. \end{aligned} \tag{2.2.8}$$

Here we have used the Gauss divergence theorem as well as the fact that $\sum_{\mathcal{K}\in\mathcal{T}_h}\langle\boldsymbol{\mu}\nabla u\cdot\mathbf{n},\phi_b\rangle_{\partial\mathcal{K}} = 0$. Combining (2.2.3) and (2.2.8), we have

$$\begin{aligned} &(u_{htt},\phi_0) + \mathcal{A}_w(u_h,\phi_h) \\ &= ((\mathcal{Q}_h u)_{tt},\phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h}(\boldsymbol{\mu}\nabla u,\nabla\phi_0)_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h}\langle\phi\nabla u\cdot\mathbf{n},\phi_0-\phi_b\rangle_{\partial\mathcal{K}}. \end{aligned} \tag{2.2.9}$$

Now using projection operator $\mathbb{Q}_l$ and weak gradient $\nabla_w$, we have

$$\begin{aligned}(\boldsymbol{\mu}\nabla_w \mathcal{Q}_h u, \nabla_w\phi)_{\mathcal{K}} &= (\nabla_w \mathcal{Q}_h u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi))_{\mathcal{K}}\\ &= (\nabla \mathcal{Q}_k^0 u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi))_{\mathcal{K}} + \langle Q_j^b u - Q_k^0 u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &= \big(\mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l(\nabla \mathcal{Q}_k^0 u)), \nabla_w\phi\big)_{\mathcal{K}} + \langle Q_j^b u - Q_k^0 u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &= \big(\mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l(\nabla \mathcal{Q}_k^0 u)), \nabla\phi_0\big)_{\mathcal{K}} + \langle Q_j^b u - Q_k^0 u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &\quad + \langle \phi_b - \phi_0, \mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l(\nabla \mathcal{Q}_k^0 u)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{A}_w(\mathcal{Q}_h u, \phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h}(\boldsymbol{\mu}\nabla_w \mathcal{Q}_h u, \nabla_w\phi_h)_{\mathcal{K}} + \mathbf{S}(\mathcal{Q}_h u, \phi_h)\\ &= \sum_{\mathcal{K}\in\mathcal{T}_h}\big(\mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l(\nabla \mathcal{Q}_k^0 u)), \nabla\phi_0\big)_{\mathcal{K}} + \sum_{\mathcal{K}\in\mathcal{T}_h}\langle Q_j^b u - Q_k^0 u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &\quad + \sum_{\mathcal{K}\in\mathcal{T}_h}\langle \phi_b - \phi_0, \mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l(\nabla \mathcal{Q}_k^0 u)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} + \mathbf{S}(\mathcal{Q}_h u, \phi_h).\end{aligned}$$

Therefore,

$$\begin{aligned}&((\mathcal{Q}_h u)_{tt}, \phi_0) + \mathcal{A}_w(\mathcal{Q}_h u, \phi_h)\\ &= ((\mathcal{Q}_h u)_{tt}, \phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h}\big(\mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l(\nabla \mathcal{Q}_k^0 u)), \nabla\phi_0\big)_{\mathcal{K}} + \sum_{\mathcal{K}\in\mathcal{T}_h}\langle Q_j^b u - Q_k^0 u, \mathbb{Q}_l(\boldsymbol{\mu}\nabla_w\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &\quad + \sum_{\mathcal{K}\in\mathcal{T}_h}\langle \phi_b - \phi_0, \mathbb{Q}_l(\boldsymbol{\mu}\mathbb{Q}_l(\nabla \mathcal{Q}_k^0 u)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} + \mathbf{S}(\mathcal{Q}_h u, \phi_h). \qquad (2.2.10)\end{aligned}$$

Subtracting equation (2.2.10) from equation (2.2.9), we obtain the desired result. □

Next, for any shape regular polygonal discretization $\mathcal{T}_h$ of $\Omega$, we recall the following significant estimation results for bilinear maps $\ell_1(\cdot,\cdot)$, $\ell_2(\cdot,\cdot)$ and $\ell_3(\cdot,\cdot)$ . The following estimates are borrowed from Lemma 4.2 - Lemma 4.4 of [151].

**Lemma 2.2.3.** *Let* $\lambda = \min\{l+1, k\}$. *Assume that* $u \in H^{\lambda+1}(\Omega)\cap H_0^1(\Omega)$, *then there exists a constant* $C > 0$ *such that*

$$|\ell_1(u,\phi_h)| \le Ch^{\lambda}\|u\|_{H^{\lambda+1}(\Omega)}\|\phi_h\|_{1,h} \quad \forall \phi_h \in \mathcal{W}_h^0,$$

*where* $C$ *depends only on* $\|\boldsymbol{\mu}\|_{l+1,\infty}$, *the element-wise* $W^{l+1,\infty}$ *norm of the matrix* $\boldsymbol{\mu}$.

**Lemma 2.2.4.** *Under the same assumptions as in Lemma 2.2.3, the following estimate holds*

$$|\ell_2(u,\phi_h)| \le Ch^{\lambda}\|u\|_{H^{\lambda+1}(\Omega)}\|\phi_h\|_{1,h} \quad \forall \phi_h \in \mathcal{W}_h^0.$$

**Lemma 2.2.5.** *Suppose that the non-negative integers $k, j$ and $l$ are used to define the weak finite space $\mathcal{W}_h$. Then the following estimates hold:*

(a) *Suppose $j < l$ and define $s = \min\{k, j\} \geq 1$. Assume that $u \in H^{s+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$|\ell_3(u, \phi_h)| \leq Ch^s \|u\|_{H^{s+1}(\Omega)} \|\phi_h\|_{1,h} \ \ \forall \phi_h \in \mathcal{W}_h^0.$$

(b) *If $j \geq l$ and $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$|\ell_3(u, \phi_h)| \leq Ch^k \|u\|_{H^{k+1}(\Omega)} \|\phi_h\|_{1,h} \ \ \forall \phi_h \in \mathcal{W}_h^0.$$

## 2.3 Semidiscrete Error Analysis for Wave Equation

This section is devoted to error estimates for the semidiscrete scheme. The optimal order of convergence in both $L^\infty(H^1)$ and $L^\infty(L^2)$ norms are determined under suitable regularity assumptions of the true solution.

### 2.3.1 Discrete $H^1$ norm Error Estimate for Wave Equation

In this subsection, we derive error estimate in discrete energy norm for the semidiscrete scheme with stabilizer based on projected element-boundary-discrepancy. The following crucial estimates for the stabilizer followed from the existing literature (cf. Lemma 4.7 in [151]).

**Lemma 2.3.1.** *Suppose that the non-negative integers $k, j$ and $l$ are used to define the weak finite space $\mathcal{W}_h$ and let*

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K} \in \mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle \mathcal{Q}_m(u_b - u_0|_{\partial\mathcal{K}}), \mathcal{Q}_m(\phi_b - \phi_0|_{\partial\mathcal{K}}) \rangle_{\partial\mathcal{K}},$$

*where $m = max\{j, l\}$ and $\mathcal{Q}_m$ is the standard $L^2$ projection onto $\mathbf{P}_m(\partial\mathcal{K})$. Then following hold:*

(a) *If $j < l$ and define $s = \min\{k, j\} \geq 1$. Also, assume the exact solution $u$ of (2.1.1)-(2.1.3) is so regular such that $u \in H^{s+1}(\Omega) \cap H_0^1(\Omega)$. Then there exists a constant $C$ such that*

$$|\mathbf{S}(\mathcal{Q}_h u, \phi_h)| \leq Ch^{2s} \|u\|_{H^{s+1}(\Omega)} \|\phi_h\|_{1,h}.$$

(b) *If $j \geq l$ and let the exact solution $u$ of (2.1.1)-(2.1.3) is so regular such that $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$. Then there exists a constant $C$ such that*

$$|\mathbf{S}(\mathcal{Q}_h u, \phi_h)| \leq Ch^{2k} \|u\|_{H^{k+1}(\Omega)} \|\phi_h\|_{1,h}.$$

The following theorem establishes convergence results in discrete energy norm with stabilizer based on projected element-boundary-discrepancy.

**Theorem 2.3.1.** *Suppose that the non-negative integers $k, j$ and $l \geq k-1$ are used to define weak finite space $\mathcal{W}_h$. Let*

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^{-1}\langle \mathcal{Q}_m(u_b - u_0|_{\partial\mathcal{K}}), \mathcal{Q}_m(\phi_b - \phi_0|_{\partial\mathcal{K}})\rangle_{\partial\mathcal{K}},$$

*where $m = max\{j, l\}$ and $\mathcal{Q}_m$ is the standard $L^2$ projection onto $\mathbf{P}_m(\partial\mathcal{K})$. Then the following estimates hold:*

(a) *If $j < l$ and define $s = \min\{k, j\} \geq 1$. Also, assume that the exact solution $u$ of (2.1.1)-(2.1.3) is so regular such that $u_t \in H^{s+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|e_h\|_{1,h}^2 \leq Ch^{2s}\Big\{\|u\|^2_{H^{s+1}(\Omega)} + \|u_t\|^2_{L^2(H^{s+1})}\Big\}.$$

(b) *If $j \geq l$ and the exact solution $u$ of (2.1.1)-(2.1.3) is so regular such that $u_t \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|e_h\|_{1,h}^2 \leq Ch^{2k}\Big\{\|u\|^2_{H^{k+1}(\Omega)} + \|u_t\|^2_{L^2(H^{k+1})}\Big\}.$$

*Proof.* Setting $\phi_h = e_{ht}$ in the error equation (2.2.6), we have

$$(e_{htt}, e_{ht}) + \mathcal{A}_w(e_h, e_{ht}) = -\left\{\ell_1(u, e_{ht}) + \ell_2(u, e_{ht}) + \ell_3(u, e_{ht}) + \mathbf{S}(\mathcal{Q}_h u, e_{ht})\right\}.$$

We can rearrange above equation as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|e_{ht}\|^2 + \frac{1}{2}\frac{d}{dt}\mathcal{A}_w(e_h, e_h)\\ &= -\left\{\ell_1(u, e_{ht}) + \ell_2(u, e_{ht}) + \ell_3(u, e_{ht}) + \mathbf{S}(\mathcal{Q}_h u, e_{ht})\right\}.\end{aligned} \tag{2.3.1}$$

Now integrating both sides of the equation (2.3.1) from 0 to $t$, we have

$$\begin{aligned}&\int_0^t \frac{1}{2}\frac{d}{dt}\|e_{ht}\|^2 + \int_0^t \frac{1}{2}\frac{d}{dt}\mathcal{A}_w(e_h, e_h)\\ &= -\left\{\int_0^t \ell_1(u, e_{ht}) + \int_0^t \ell_2(u, e_{ht}) + \int_0^t \ell_3(u, e_{ht}) + \int_0^t \mathbf{S}(\mathcal{Q}_h u, e_{ht})\right\}\\ &:= H_1 + H_2 + H_3 + H_4.\end{aligned}$$

Further, using the facts $e_h(0) = 0$ and $e_{ht}(0) = 0$, we obtain

$$\frac{1}{2}\|e_{ht}(t)\|^2 + \frac{1}{2}\mathcal{A}_w(e_h(t), e_h(t)) = H_1 + H_2 + H_3 + H_4. \tag{2.3.2}$$

Term $H_1$ can be expressed as

$$H_1 = \int_0^t \ell_1(u_t, e_h)ds - \ell_1(u(t), e_h(t)).$$

Here we have used integration by parts and the fact that $e_h(0) = 0$. Now for $\lambda = \min\{l+1, k\} = k$, Lemma 2.2.3 leads to

$$|H_1| \leq Ch^k\bigg\{\|u\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h} + \int_0^t \|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\bigg\}. \tag{2.3.3}$$

Similarly, using Lemma 2.2.4, we have

$$|H_2| \leq Ch^k\bigg\{\|u\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h} + \int_0^t \|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\bigg\}. \tag{2.3.4}$$

If $j < l$ and $s = \min\{k, j\} \geq 1$. Then Lemma 2.2.5 and Lemma 2.3.1 yield

$$|H_3| \;\&\; |H_4| \leq Ch^s\bigg\{\|u\|_{H^{s+1}(\Omega)}\|e_h\|_{1,h} + \int_0^t \|u_t\|_{H^{s+1}(\Omega)}\|e_h\|_{1,h}ds\bigg\}. \tag{2.3.5}$$

Now using inequalities (2.3.3), (2.3.5) in (2.3.2), we have

$$\frac{1}{2}\|e_{ht}(t)\|^2 + \frac{1}{2}\mathcal{A}_w(e_h(t), e_h(t)) \leq Ch^s\bigg\{\|u\|_{H^{s+1}(\Omega)}\|e_h\|_{1,h} + \int_0^t \|u_t\|_{H^{s+1}(\Omega)}\|e_h\|_{1,h}ds\bigg\}.$$

Next, applying coercivity (1.4.10) and further using Young's inequality with appropriate $\epsilon > 0$, we obtain

$$C_*\|e_h(t)\|_{1,h}^2 \leq C\bigg\{\frac{h^{2s}}{2\epsilon}\|u\|_{H^{s+1}(\Omega)}^2 + \frac{\epsilon}{2}\|e_h\|_{1,h}^2 + \int_0^t \frac{h^{2s}}{2}\|u_t\|_{H^{s+1}(\Omega)}^2 ds + \frac{1}{2}\int_0^t \|e_h\|_{1,h}^2 ds\bigg\}.$$

Therefore, for $\epsilon = C_*$ and $t \in (0, T]$, we obtain

$$\|e_h(t)\|_{1,h}^2 \leq Ch^{2s}\bigg\{\|u\|_{H^{s+1}(\Omega)}^2 + \int_0^T \|u_t\|_{H^{s+1}(\Omega)}^2 ds\bigg\} + C\int_0^T \|e_h\|_{1,h}^2 ds.$$

Now using standard Gronwall's inequality, we have

$$\|e_h(t)\|_{1,h}^2 \leq Ch^{2s}\bigg\{\|u\|_{H^{s+1}(\Omega)}^2 + \int_0^T \|u_t\|_{H^{s+1}(\Omega)}^2 ds\bigg\}.$$

This completes the proof of part (a).

In the similar manner, when $j \geq l$, we can prove that

$$\|e_h(t)\|_{1,h}^2 \leq Ch^{2k}\bigg\{\|u\|_{H^{k+1}(\Omega)}^2 + \|u_t\|_{L^2(H^{k+1})}^2\bigg\},$$

which completes the rest of the proof. □

### 2.3.2 $L^2$ norm Error Estimate for Wave Equation

In this subsection, we deal with the optimal order of error estimate in $L^2$ norm. The basic concept is to employ elliptic projection as in [158].

For $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, we write

$$f_\varphi = -\nabla \cdot (\boldsymbol{\mu} \nabla \varphi) \ \text{ in } \ \Omega.$$

Then we define elliptic projection $\mathcal{R}_h : H^2(\Omega) \cap H_0^1(\Omega) \to \mathcal{W}_h^0$ by

$$\mathcal{A}_w(\mathcal{R}_h \varphi, \phi_h) = (f_\varphi, \phi_h) \ \ \forall \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0. \tag{2.3.6}$$

Equation (2.3.6) implies that $\mathcal{R}_h \varphi$ is the standard weak Galerkin solution of the following second order elliptic problem with exact solution $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$-\nabla \cdot (\boldsymbol{\mu} \nabla \varphi) = f_\varphi \text{ in } \Omega \text{ and } \varphi = 0 \text{ on } \partial\Omega.$$

Next, we assume the following important approximation result, which can be proved by a slight modification in existing result in [43].

**Lemma 2.3.2.** *Consider the non-negative integers $k, j$ and $l \geq k-1$ that define the weak finite space $\mathcal{W}_h$ and $s = min\{k, j\}$. Then for sufficiently smooth $\varphi$, there exists a constant $C$ such that*

$$\|\mathcal{Q}_h \varphi - \mathcal{R}_h \varphi\|^2 \leq \begin{cases} Ch^{2(s+1)} \|\varphi\|_{H^{s+1}(\Omega)}^2 & \textit{for } \ j < l, \\ Ch^{2(k+1)} \|\varphi\|_{H^{k+1}(\Omega)}^2 & \textit{for } \ j \geq l. \end{cases} \tag{2.3.7}$$

Hence, for any smooth function $\varphi$, using the fact that $\frac{\partial}{\partial t}(\mathcal{Q}_h \varphi - \mathcal{R}_h \varphi) = \mathcal{Q}_h \varphi_t - \mathcal{R}_h \varphi_t$ and then applying Lemma 2.3.2, we obtain

$$\|(\mathcal{Q}_h \varphi_t - \mathcal{R}_h \varphi_t)(t)\|^2 \leq \begin{cases} Ch^{2(s+1)} \|\varphi_t(t)\|_{H^{s+1}(\Omega)}^2 ds & \text{for } \ j < l, \\ Ch^{2(k+1)} \|\varphi_t(t)\|_{H^{k+1}(\Omega)}^2 ds & \text{for } \ j \geq l. \end{cases} \tag{2.3.8}$$

Next, we further split the error $e_h = u_h - \mathcal{Q}_h u$ with the help of elliptic projection as

$$e_h(t) = u_h(t) - \mathcal{Q}_h u(t) = u_h(t) - \mathcal{R}_h u(t) + \mathcal{R}_h u(t) - \mathcal{Q}_h u(t) = \theta(t) - \rho(t),$$

where $\theta = u_h - \mathcal{R}_h u$ and $\rho = \mathcal{Q}_h u - \mathcal{R}_h u$.

Then it is easy to verify that

$$(\theta_{tt}, \phi_h) + \mathcal{A}_w(\theta, \phi_h) = (\rho_{tt}, \phi_h) \ \ \forall \phi_h \in \mathcal{W}_h^0. \tag{2.3.9}$$

Next, for some $\zeta \in (0,T]$, we define $\hat{\theta}$ as

$$\hat{\theta}(\cdot,t) = \int_t^{\zeta} \theta(\cdot,s)ds \quad 0 \le t \le T.$$

Therefore,

$$\hat{\theta}(\zeta) = 0 \text{ and } \hat{\theta}_t = -\theta(\cdot,t) \ \ t \in [0,T]. \tag{2.3.10}$$

Setting $\phi_h = \hat{\theta}$ in (2.3.9) and integrating from 0 to $\zeta$ with respect to time $t$, we have

$$\begin{aligned} &-\int_0^{\zeta} (\theta_t, \hat{\theta}_t)ds + (\theta_t(\zeta), \hat{\theta}(\zeta)) - (\theta_t(0), \hat{\theta}(0)) + \int_0^{\zeta} \mathcal{A}_w(\theta, \hat{\theta})ds \\ &= -\int_0^{\zeta} (\rho_t, \hat{\theta}_t)ds + (\rho_t(\zeta), \hat{\theta}(\zeta)) - (\rho_t(0), \hat{\theta}(0)). \end{aligned}$$

Then (2.3.10) yields

$$\int_0^{\zeta} \frac{1}{2}\frac{d}{dt}\|\theta\|^2 dt - \int_0^{\zeta} \frac{1}{2}\frac{d}{dt}\mathcal{A}_w(\hat{\theta}, \hat{\theta})ds = \int_0^{\zeta} (\rho_t, \theta)ds + (\theta_t(0) - \rho_t(0), \hat{\theta}(0)). \tag{2.3.11}$$

Now we select $\zeta \in (0,T]$ such that $\|\theta(\zeta)\| = \max_{0\le t\le T} \|\theta(t)\|$. Further, using the fact that $\theta_t(0) - \rho_t(0) = e_{ht}(0) = 0$ in (2.3.11), we obtain

$$\frac{1}{2}\|\theta(\zeta)\|^2 - \frac{1}{2}\|\theta(0)\|^2 - \frac{1}{2}\mathcal{A}_w(\hat{\theta}(\zeta), \hat{\theta}(\zeta)) + \frac{1}{2}\mathcal{A}_w(\hat{\theta}(0), \hat{\theta}(0)) = \int_0^{\zeta} (\rho_t, \theta)ds.$$

Again, using (2.3.10), we obtain

$$\frac{1}{2}\|\theta(\zeta)\|^2 + \frac{1}{2}\mathcal{A}_w(\hat{\theta}(0), \hat{\theta}(0)) = \frac{1}{2}\|\theta(0)\|^2 + \int_0^{\zeta} (\rho_t, \theta)ds.$$

Then standard inequalities and positivity of $\mathcal{A}_w$ lead to

$$\|\theta(\zeta)\|^2 \le C\big(\|\theta(0)\|^2 + \|\rho_t\|^2_{L^2(L^2)}\big). \tag{2.3.12}$$

Again, as a consequence of Lemma 2.3.2, we obtain

$$\|\theta(0)\|^2 = \|\mathcal{Q}_h u^0 - \mathcal{R}_h u^0\|^2 \le \begin{cases} Ch^{2(s+1)}\|u^0\|^2_{H^{s+1}(\Omega)} & \text{for } j < l, \\ Ch^{2(k+1)}\|u^0\|^2_{H^{k+1}(\Omega)} & \text{for } j \ge l. \end{cases} \tag{2.3.13}$$

Therefore, using (2.3.8) and (2.3.13) in (2.3.12), we have

$$\begin{aligned} \|\theta(t)\|^2 &\le \|\theta(\zeta)\|^2 \\ &\le C \begin{cases} Ch^{2(s+1)}\big(\|u^0\|^2_{H^{s+1}(\Omega)} + \|u_t\|^2_{L^2(H^{s+1})}\big) & \text{for } j < l, \\ Ch^{2(k+1)}\big(\|u^0\|^2_{H^{k+1}(\Omega)} + \|u_t\|^2_{L^2(H^{k+1})}\big) & \text{for } j \ge l, \end{cases} \end{aligned} \tag{2.3.14}$$

for all $t \in [0,T]$, where $s = \min\{k,j\}$.

Finally, using Lemma 2.3.2 and inequality (2.3.14), we estimate error in $L^\infty(L^2)$ norm.

**Theorem 2.3.2.** *Consider the non-negative integers $k, j$ and $l \geq k-1$ that define the weak finite space $\mathcal{W}_h$. Let*

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle \mathcal{Q}_m(u_b - u_0|_{\partial\mathcal{K}}), \mathcal{Q}_m(\phi_b - \phi_0|_{\partial\mathcal{K}})\rangle_{\partial\mathcal{K}},$$

*where $m = max\{j, l\}$ and $\mathcal{Q}_m$ is the standard $L^2$ projection onto $\mathbf{P}_m(\partial\mathcal{K})$. Then we have the following error estimates:*

(a) *Suppose $j < l$ and define $s = min\{k, j\} \geq 1$. Also, assume that the exact solution $u$ of (2.1.1)-(2.1.3) is so regular such that $u_t \in H^{s+1}(\Omega) \cap H_0^1(\Omega)$. Then we have*

$$\|e_h(t)\| \leq Ch^{s+1}\big(\|u^0\|_{H^{s+1}(\Omega)} + \|u_t\|_{L^2(H^{s+1})}\big).$$

(b) *If $j \geq l$ and the exact solution $u$ of (2.1.1)-(2.1.3) is so regular such that $u_t \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$. Then we have*

$$\|e_h(t)\| \leq Ch^{k+1}\big(\|u^0\|_{H^{k+1}(\Omega)} + \|u_t\|_{L^2(H^{k+1})}\big).$$

*Proof.* We have

$$\begin{aligned} e_h(t) &= u_h(t) - \mathcal{Q}_h u(t) \\ &= u_h(t) - \mathcal{R}_h u(t) + \mathcal{R}_h(t) - \mathcal{Q}_h u(t) = \theta(t) - \rho(t). \end{aligned}$$

Now triangle inequality yields

$$\|e_h(t)\| \leq \|\theta(t)\| + \|\rho(t)\|.$$

Then Lemma 2.3.2 and estimate (2.3.14) lead to the desired result. □

# 3

# Fully Discrete WG-FEMs for Second Order Linear Wave Equation

The preceding chapter's spatially semidiscrete *a priori* error analysis is now expanded to a fully discrete weak Galerkin approximation. The fully discrete space-time finite element discretizations can be reviewed as the Crank-Nicolson discretization of the reformulation of the governing equation in the first-order system as in Baker [14]. The optimal error estimate in $L^\infty(L^2)$ norm is derived for the fully discrete solution. Finally, numerical experiments are reported for several cases to justify theoretical convergence results.

## 3.1 Introduction

We shall begin with recalling the second order linear wave equation

$$u_{tt} - \nabla \cdot (\boldsymbol{\mu} \nabla u) = f \ \text{ for } \ (\mathbf{x}, t) \in \Omega \times (0, T], \ T < \infty, \tag{3.1.1}$$

with the initial conditions

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \ \ u_t(\mathbf{x}, 0) = v^0(\mathbf{x}) \ \ \forall \ \mathbf{x} \in \Omega \tag{3.1.2}$$

and boundary condition

$$u(\mathbf{x}, t) = 0 \ \ \forall \ (\mathbf{x}, t) \in \partial\Omega \times (0, T], \tag{3.1.3}$$

where $\Omega$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $J = (0, T]$ is the observation time period. The coefficient matrix $\boldsymbol{\mu} = [\mu_{i,j}(\mathbf{x})]_{2\times 2}$ is considered to be symmetric and positive definite in the domain $\Omega$. The initial functions $\{u^0, v^0\}$, coefficient $\boldsymbol{\mu}$ and the source function $f$ are considered to be sufficiently smooth. Additional regularity assumptions were made throughout the chapter to carry out the convergence analysis. For the related regularity results, we refer to [58, 111].

The rest of this chapter is as follows Sec. 3.2 introduces time discretization and fully discrete approximation of the given problem using the Crank-Nicolson scheme. Also, convergence analysis of the fully discrete solution is discussed in Sec. 3.2. In Sec. 3.3 several numerical data are provided to verify the rate of convergence. Also, we present graphical representations of the errors for several combinations of $k$, $j$ and $l$ in Sec. 3.3.

## 3.2 Fully Discrete WG-FEMs for Wave Equation

First, to apply the Crank-Nicolson scheme, we convert second order problem (3.1.1) into a system of first order equations. Let us take $u_t = p$, where $p$ is an auxiliary unknown function. Then our model equation (3.1.1) is reduced to following first order system of equation in time

$$\begin{cases} u_t - p = 0 \text{ in } \Omega \times (0, T], \\ p_t - \nabla \cdot (\boldsymbol{\mu} \nabla u) = f \quad \text{ in } \Omega \times (0, T]. \end{cases} \tag{3.2.1}$$

We now turn our attention to some discrete time weak Galerkin methods. First, let us divide the time period $(0, T]$ into $M$ uniformly distributed sub-intervals $J_n = (t_{n-1}, t_n]$ for $n = 1, 2, \cdots M$ with $t_0 = 0$, $t_n = n\tau$ and $t_M = T$, where $\tau = \frac{T}{M}$ is the time step.

For a continuous map $\varphi : [0, T] \to L^2(\Omega)$, define $\varphi^n = \varphi(\cdot, t_n)$, $n = 0, 1, \cdots M$. Also, for a sequence $\{\varphi^n\}_0^M \subset L^2(\Omega)$, we define

$$\partial_\tau \varphi^n = \frac{\varphi^{n+1} - \varphi^n}{\tau} \quad \text{and} \quad \varphi^{n+\frac{1}{2}} = \frac{1}{2}(\varphi^{n+1} + \varphi^n). \tag{3.2.2}$$

Now by applying the Crank-Nicolson scheme to the system of equations in (3.2.1), the fully discrete weak Galerkin finite element estimate to the equation (3.1.1)-(3.1.3) is given by: Find $U^n \in \mathcal{W}_h^0$ such that

$$\partial_\tau U^n = p^{n+\frac{1}{2}} \text{ for } n = 0,\ 1,\ \cdots, M-1 \tag{3.2.3}$$

and

$$(\partial_\tau p^n, \phi_0) + \mathcal{A}_w(U^{n+\frac{1}{2}}, \phi_h) = (f^{n+\frac{1}{2}}, \phi_0) \ \ \forall \phi_h \in \mathcal{W}_h^0, \tag{3.2.4}$$

with $U^0 = \mathcal{Q}_h u^0$ and $p^0 = \mathcal{Q}_h v^0$.

The following important result follows from the Taylor's series expansion.

**Lemma 3.2.1.** *For any $\varphi \in H^3(J; L^2(\Omega))$, we have*

$$\left\| \partial_\tau \varphi^n - \varphi_t^{n+\frac{1}{2}} \right\|^2 \leq C\tau^3 \int_{t_n}^{t_{n+1}} \|\varphi_{ttt}\|^2 dt. \tag{3.2.5}$$

Now the error $e_h^n = U^n - \mathcal{Q}_h u^n$ at time level $t = t_n$ is divided as

$$e_h^n = U^n - \mathcal{Q}_h u^n = U^n - \mathcal{R}_h u^n + \mathcal{R}_h u^n - \mathcal{Q}_h u^n,$$

where $\mathcal{Q}_h$ and $\mathcal{R}_h$ are the standard $L^2$ projection and elliptic projection respectively.

For $1 \leq n \leq M$, we write $\gamma^n = U^n - \mathcal{R}_h u^n$. Also, for our convenience, we introduce

$$q^n = p^n - \mathcal{R}_h u_t^n \quad \text{and} \quad \chi^n = u^n - \mathcal{R}_h u^n.$$

The following estimate holds for $\gamma^n$, which is decisive for our $L^2$ norm estimate.

**Lemma 3.2.2.** *Let $u$ be exact solution of (3.1.1)-(3.1.3) and $U^n$ be fully discrete WG solution of (3.2.3)-(3.2.4). Then following estimates hold:*

(a) *Suppose $j < l$ and define $s = \min\{k, j\} \geq 1$. Also, assume that $u$ is so regular such that $u \in H^1(J; H^{s+1}(\Omega)) \cap H^4(J; L^2(\Omega))$. Then we have*

$$\max_{0 \leq n \leq M} \|\gamma^n\|^2 \leq Ch^{2(s+1)}\Big(\|u^0\|^2_{H^{s+1}(\Omega)} + \|u_t\|^2_{L^2(H^{s+1})}\Big) + C\tau^4\|u\|^2_{H^4(L^2)}.$$

(b) *If $j \geq l$ and the solution $u \in H^1(J; H^{k+1}(\Omega)) \cap H^4(J; L^2(\Omega))$. Then we have*

$$\max_{0 \leq n \leq M} \|\gamma^n\|^2 \leq Ch^{2(k+1)}\Big(\|u^0\|^2_{H^{k+1}(\Omega)} + \|u_t\|^2_{L^2(H^{k+1})}\Big) + C\tau^4\|u\|^2_{H^4(L^2)},$$

*where $C > 0$ is a constant independent of $h$ and $\tau$.*

*Proof.* From the equation (3.2.4), for each $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, we have

$$\begin{aligned}
(\partial_\tau q^n, \phi_0) + \mathcal{A}_w(\gamma^{n+\frac{1}{2}}, \phi_h) &= (\partial_\tau p^n, \phi_0) + \mathcal{A}_w(U^{n+\frac{1}{2}}, \phi_h) - (\partial_\tau \mathcal{R}_h u_t^n, \phi_0) \\
&\quad + \big(\nabla \cdot (\boldsymbol{\mu} \nabla u^{n+\frac{1}{2}}), \phi_0\big) \\
&= \{(\partial_\tau p^n, \phi_0) + \mathcal{A}_w(U^{n+\frac{1}{2}}, \phi_h) - (f^{n+\frac{1}{2}}, \phi_0)\} \\
&\quad + (u_{tt}^{n+\frac{1}{2}}, \phi_0) - (\partial_\tau \mathcal{R}_h u_t^n, \phi_0) \\
&= (u_{tt}^{n+\frac{1}{2}}, \phi_0) - (\partial_\tau \mathcal{R}_h u_t^n, \phi_0) \\
&= (u_{tt}^{n+\frac{1}{2}} - \partial_\tau u_t^n + \partial_\tau u_t^n - \partial_\tau \mathcal{R}_h u_t^n, \phi_0).
\end{aligned}$$

Setting $w^n = \partial_\tau u_t^n - u_{tt}^{n+\frac{1}{2}}$ in the above equation, we obtain

$$(\partial_\tau q^n, \phi_0) + \mathcal{A}_w(\gamma^{n+\frac{1}{2}}, \phi_h) = (\partial_\tau \chi_t^n - w^n, \phi_0) \;\; \forall \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0. \tag{3.2.6}$$

It is easy to derive that

$$\partial_\tau \gamma^n = q^{n+\frac{1}{2}} + \partial_\tau \chi^n - \chi_t^{n+\frac{1}{2}} - \zeta^n, \tag{3.2.7}$$

where $\zeta^n = \partial_\tau u^n - u_t^{n+\frac{1}{2}}$.

Hence, we have

$$\partial_\tau \gamma^n = \begin{cases} q^0 + \frac{\tau}{2}\partial_\tau q^0 + \partial_\tau \chi^0 - \chi_t^{\frac{1}{2}} - \zeta^0 & \text{for} \quad n = 0, \\ q^0 + \frac{\tau}{2}\sum_{i=0}^{n}\partial_\tau q^i + \frac{\tau}{2}\sum_{i=0}^{n-1}\partial_\tau q^i + \partial_\tau \chi^n - \chi_t^{n+\frac{1}{2}} - \zeta^n & \text{for } 1 \le n \le M-1. \end{cases} \tag{3.2.8}$$

Now consider a sequence $\{s^n\}_{n=0}^{M}$ defined as: $s^0 = 0$ and $s^n = \tau \sum_{i=0}^{n-1} \gamma^{i+\frac{1}{2}}$ for $1 \le n \le M$. Then we have

$$s^{n+\frac{1}{2}} = \begin{cases} \frac{\tau}{2}\gamma^{\frac{1}{2}} & \text{for} \quad n = 0, \\ \frac{\tau}{2}\left\{ \sum_{i=0}^{n}\gamma^{i+\frac{1}{2}} + \sum_{i=0}^{n-1}\gamma^{i+\frac{1}{2}} \right\}, & \text{for} \quad 1 \le n \le M-1. \end{cases} \tag{3.2.9}$$

Using (3.2.8)-(3.2.9) in (3.2.6), we obtain

$$(\partial_\tau \gamma^n, \phi_0) + \mathcal{A}_w(s^{n+\frac{1}{2}}, \phi_h) = (\epsilon^n, \phi_0) \quad \text{for} \quad 0 \le n \le M-1, \quad \forall \phi_h \in \mathcal{W}_h^0, \tag{3.2.10}$$

where $\epsilon^n$ is given by

$$\epsilon^n = \begin{cases} \partial_\tau \chi^0 - \zeta^0 - \frac{\tau}{2} w^0 & \text{for} \quad n = 0, \\ \partial_\tau \chi^n - \zeta^n - \frac{\tau}{2} w^n - \tau \sum_{i=0}^{n-1} w^{i+\frac{1}{2}} & \text{for} \quad 1 \le n \le M-1. \end{cases} \tag{3.2.11}$$

Now setting $\phi_h = \partial_\tau s^n = \dfrac{\gamma^{n+1} + \gamma^n}{2}$ in (3.2.10) and then summing the resultant equation over $n = 0$ to $r-1$ for $1 \le r \le M$, we arrive at

$$\|\gamma^r\|^2 - \|\gamma^0\|^2 + |\!|\!| s^r |\!|\!|^2 - |\!|\!| s^0 |\!|\!|^2 = 2\tau \sum_{n=0}^{r-1} \left(\epsilon^n, \gamma^{n+\frac{1}{2}}\right). \tag{3.2.12}$$

Then using the facts that $s^0 = 0$ and $|\!|\!| s^r |\!|\!|^2 \ge 0$, we obtain

$$\begin{aligned} \|\gamma^r\|^2 &\le \|\gamma^0\|^2 + 4T\tau \sum_{n=0}^{r-1} \|\epsilon^n\|^2 + \frac{\tau}{4T}\sum_{n=0}^{r-1} \|\gamma^{n+\frac{1}{2}}\|^2 \\ &\le \|\gamma^0\|^2 + 4T\tau \sum_{n=0}^{r-1} \|\epsilon^n\|^2 + \frac{1}{2}\max_{0 \le n \le M} \|\gamma^n\|^2. \end{aligned}$$

Therefore, we have

$$\max_{0 \le n \le M} \|\gamma^n\|^2 \le 2\|\gamma^0\|^2 + 8T\tau \sum_{n=0}^{M-1} \|\epsilon^n\|^2. \tag{3.2.13}$$

Again, for the term $\epsilon^n$ $(1 \leq n \leq M-1)$, we have

$$\begin{aligned}
\epsilon^n &= \partial_\tau \chi^n - \zeta^n - \frac{\tau}{2} w^n - \tau \sum_{i=0}^{n-1} w^{i+\frac{1}{2}} \\
&= \partial_\tau \chi^n - \zeta^n - \frac{\tau}{2} w^n - \frac{\tau}{2} \sum_{i=0}^{n-1} w^{i+1} - \frac{\tau}{2} \sum_{i=0}^{n-1} w^i \\
&= \partial_\tau \chi^n - \zeta^n - \frac{\tau}{2} \sum_{i=1}^{n} w^i - \frac{\tau}{2} \sum_{i=0}^{n} w^i.
\end{aligned}$$

Now for $0 \leq n \leq M-1$, applying triangle inequality and the Cauchy-Schwarz inequality to have

$$\begin{aligned}
\|\epsilon^n\|^2 &\leq \|\partial_\tau \chi^n\|^2 + \|\zeta^n\|^2 + \frac{\tau^2}{4} \|\sum_{i=1}^{n} w^i\|^2 + \frac{\tau^2}{4} \|\sum_{i=0}^{n} w^i\|^2 \\
&\leq \|\partial_\tau \chi^n\|^2 + \|\zeta^n\|^2 + \frac{\tau^2}{2} M \sum_{i=0}^{M-1} \|w^i\|^2 \\
&\leq \|\partial_\tau \chi^n\|^2 + \|\zeta^n\|^2 + \frac{T}{2} (\tau \sum_{i=0}^{M-1} \|w^i\|^2).
\end{aligned}$$

Summing over $n = 0$ to $n = M-1$ and then multiplying both sides by $\tau$, we obtain

$$\tau \sum_{n=0}^{M-1} \|\epsilon^n\|^2 \leq \tau \sum_{n=0}^{M-1} \|\partial_\tau \chi^n\|^2 + \tau \sum_{n=0}^{M-1} \|\zeta^n\|^2 + \tau \sum_{n=0}^{M-1} \frac{T}{2} \left( \tau \| \sum_{i=0}^{M-1} w^i \|^2 \right). \tag{3.2.14}$$

Using Lemma 3.2.1, we have

$$\tau \sum_{i=0}^{M-1} \|w^i\|^2 \leq C \tau^4 \|u_{tttt}\|^2_{L^2(L^2)}. \tag{3.2.15}$$

Similarly, we have

$$\tau \sum_{n=0}^{M-1} \|\zeta^n\|^2 \leq C \tau^4 \|u_{ttt}\|^2_{L^2(L^2)}. \tag{3.2.16}$$

Further, Lemma 2.2.1 and Lemma 2.3.2 yield

$$\tau \sum_{n=0}^{M-1} \|\partial_\tau \chi^n\|^2 \leq C \|\chi_t\|^2_{L^2(L^2)} \leq \begin{cases} C h^{2(s+1)} \|u_t\|_{L^2(H^{s+1})} & \text{if } \ j < l, \\ C h^{2(k+1)} \|u_t\|_{L^2(H^{k+1})} & \text{if } \ j \geq l. \end{cases} \tag{3.2.17}$$

Using the estimates (3.2.15)-(3.2.17) in (3.2.14), we obtain

$$\max_{0\le n\le M}\|\gamma^n\|^2 \le \begin{cases} 2\|\gamma^0\|^2 + C\tau^4\left(\|u_{tttt}\|^2_{L^2(L^2)} + \|u_{ttt}\|^2_{L^2(L^2)}\right) \\ \qquad\qquad + Ch^{2(s+1)}\|u_t\|^2_{L^2(H^{s+1})} & \text{if } j<l, \\ 2\|\gamma^0\|^2 + C\tau^4\left(\|u_{tttt}\|^2_{L^2(L^2)} + \tau^4\|u_{ttt}\|^2_{L^2(L^2)}\right) \\ \qquad\qquad + Ch^{2(k+1)}\|u_t\|^2_{L^2(H^{k+1})} & \text{if } j\ge l. \end{cases} \tag{3.2.18}$$

Also, as a consequence of Lemma 2.3.2, we have

$$\|\gamma^0\|^2 = \|\mathcal{Q}_h u^0 - \mathcal{R}_h u^0\|^2 \le \begin{cases} Ch^{2(s+1)}\|u^0\|^2_{H^{s+1}(\Omega)} & \text{if } j<l, \\ Ch^{2(k+1)}\|u^0\|^2_{H^{k+1}(\Omega)} & \text{if } j\ge l. \end{cases}$$

Finally, the above estimate together with (3.2.18) lead to desired result. □

**Theorem 3.2.1.** *Let $u$ be the exact solution of (3.1.1)-(3.1.3) and $U^n$ be its fully discrete approximation at $t=t_n$ given by (3.2.3)-(3.2.4). Then following error estimates hold:*

(a) *For $j<l$ and define $s=\min\{k,j\}\ge 1$. Also, assume that $u\in H^1(J;H^{s+1}(\Omega))\cap H^4(J;L^2(\Omega))$. Then we have*

$$\max_{0\le n\le M}\|e_h^n\| \le \tilde{C}(u^0,u)(h^{s+1}+\tau^2),$$

*where*

$$\tilde{C}(u^0,u) = \|u^0\|_{H^{s+1}(\Omega)} + \|u\|_{L^\infty(H^{s+1})} + \|u_t\|_{L^2(H^{s+1})} + \|u\|_{H^4(L^2)}.$$

*(b) Suppose $j\ge l$ and $u\in H^1(J;H^{k+1}(\Omega))\cap H^4(J;L^2(\Omega))$. Then we have*

$$\max_{0\le n\le M}\|e_h^n\| \le \tilde{C}(u^0,u)(h^{k+1}+\tau^2),$$

*where*

$$\tilde{C}(u^0,u) = \|u^0\|_{H^{k+1}(\Omega)} + \|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^2(H^{k+1})} + \|u\|_{H^4(L^2)}.$$

*Proof.* We split the error $e_h^n$ as

$$e_h^n = U^n - \mathcal{Q}_h u^n = U^n - \mathcal{R}_h u^n + \mathcal{R}_h u^n - \mathcal{Q}_h u^n.$$

Then using triangle inequality, Lemma 2.3.2 and Lemma 3.2.2, we obtain

$$\|e_h^n\|^2 \;\le\; Ch^{2(s+1)}\left(\|u^0\|^2_{H^{s+1}(\Omega)} + \|u\|^2_{L^\infty(H^{s+1})} + \|u_t\|^2_{L^2(H^{s+1})}\right) + C\tau^4\|u\|^2_{H^4(L^2)},$$

when $j<l$ and $s=\min\{k,j\}\ge 1$. Further, for $j\ge l$, we have

$$\|e_h^n\|^2 \;\le\; Ch^{2(k+1)}\left(\|u^0\|^2_{H^{k+1}(\Omega)} + \|u\|^2_{L^\infty(JH^{k+1}))} + \|u_t\|^2_{L^2(H^{k+1})}\right) + C\tau^4\|u\|^2_{H^4(L^2)}.$$

This completes the proof. □

## 3.3 Numerical Experiments

In this section, we have examined the results of computations for the wave equation (3.1.1)-(3.1.3) in $\Omega \times J$.

**Example 3.3.1.** Consider following second order linear wave equation

$$u_{tt} - \nabla \cdot (\boldsymbol{\mu} \nabla u) = f, \text{ in } \Omega \times (0, T], \; T < \infty, \tag{3.3.1}$$

with the initial and boundary conditions are given by

$$\begin{cases} u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = v^0(\mathbf{x}) \;\; \forall \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) = 0 \;\; \forall (\mathbf{x}, t) \in \partial\Omega \times (0, T], \end{cases} \tag{3.3.2}$$

where $\Omega = (0,1) \times (0,1)$ and $(0, T] = (0, 1]$. All the initial and boundary data appeared in (3.3.1)-(3.3.2) are selected by setting $\boldsymbol{\mu} = I$(identity matrix), and

$$u = t^2 \exp(-t) \sin(\pi x) \sin(\pi y) \;\; \forall (x, y, t) \in \Omega \times (0, T].$$

Here we have used rectangular meshes with various values on the degree of polynomials in the weak Galerkin finite spaces. We have done uniform partitioning of the domain into $m \times m$ sub rectangles with mesh size $h = 1/m$, where $m$ is any non-negative integer. We have performed various experiments using different combinations of polynomial spaces to justify our theoretical findings.

Let $U^n$ be the fully discrete weak Galerkin solution defined by (3.2.3)-(3.2.4). Then we have calculated the error

$$e_h^n = U^n - \mathcal{Q}_h u^n = \{e_0^n, e_b^n\},$$

with respect to the $H^1$ norm and $L^2$ norm at final time $T = 1$. We have carried out the WG scheme (3.2.3)-(3.2.4) for the problem (3.3.1)-(3.3.2) with the stabilizer $\mathbf{S}(\cdot, \cdot)$ which is given by

$$\mathbf{S}(u_h, v_h) = \sum_{\mathcal{K} \in \mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle \mathcal{Q}_m(u_b - u_0|_{\partial\mathcal{K}}), \mathcal{Q}_m(v_b - v_0|_{\partial\mathcal{K}}) \rangle_{\partial\mathcal{K}},$$

where $m = \min\{j, l\}$. Also, we have applied the WG finite element scheme (3.2.3)-(3.2.4) with the stabilizer term based on element-boundary-discrepancy given by

$$\mathbf{S}(u_h, v_h) = \sum_{\mathcal{K} \in \mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle u_b - u_0|_{\partial\mathcal{K}}, v_b - v_0|_{\partial\mathcal{K}} \rangle_{\partial\mathcal{K}}.$$

We have reported the order of accuracy with distinct values of $k$ $(1 \leq k \leq 4)$, $j$ $(0 \leq j \leq 4)$ and $l$ $(0 \leq l \leq 4)$ in Table 3.3.1, Table 3.3.2, Table 3.3.3 with the time step as

$$\tau = \begin{cases} h^s & \text{for } j < l, \quad \text{where } \; s = \min\{k, j\}, \\ h^k & \text{for } j \geq l. \end{cases}$$

Table 3.3.1: Order of convergence for Example 3.3.1 with stabilizer based on projected-boundary-discrepancy

| | **k=1** | | | | | **k=2** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | j=0 | j=1 | j=2 | j=3 | j=4 | j=0 | j=1 | j=2 | j=3 | j=4 |
| l=0 | 1/2 | 1/2 | 1/2 | 1/2 | 1/2 | NI | NI | NI | NI | NI |
| l=1 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 2/3 | 2/3 | 2/3 | 2/3 |
| l=2 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | 2/3 | 2/3 | 2/3 |
| l=3 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | 2/3 | 2/3 | 2/3 |
| l=4 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | 2/3 | 2/3 | 2/3 |

Here "NI" means the corresponding WG scheme is unstable or not consistent. The convergence order for each particular combination is indicated in the form $n/m$, where $n$ stands for the order of convergence in the triple bar norm and $m$ for the order of convergence in the $L^2$ norm. For example, $2/3$ would mean that the method is convergent at the rate of $h^2$ in the triple bar norm and $h^3$ in the $L^2$ norm. It is observed from computation that for $l < k - 1$, the WG scheme works imperfectly. For numerical justification, we refer to Figure 3.3.1.

From numerical observation, we have noticed that the WG scheme is solvable but not consistent for $l = k - 2$ with $j \geq k$ and $l < k - 2$, which is illustrated in Figure 3.3.2 - Figure 3.3.3. We do not have a mathematical justification for this curious development and we leave it to interested readers.

Table 3.3.2: Order of convergence for Example 3.3.1 with stabilizer based on projected-boundary-discrepancy

| | **k=3** | | | | | **k=4** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | j=0 | j=1 | j=2 | j=3 | j=4 | j=0 | j=1 | j=2 | j=3 | j=4 |
| l=0 | NI | NI | NI | NI | NI | NI | NI | NI | NI | NI |
| l=1 | NI | NI | NI | NI | NI | NI | NI | NI | NI | NI |
| l=2 | 0/0 | 1/2 | 3/4 | 3/4 | 3/4 | NI | NI | NI | NI | NI |
| l=3 | 0/0 | 1/2 | 2/3 | 3/4 | 3/4 | 0/0 | 1/2 | 2/3 | 4/5 | 4/5 |
| l=4 | 0/0 | 1/2 | 2/3 | 3/4 | 3/4 | 0/0 | 1/2 | 2/3 | 3/4 | 4/5 |

**Remark 3.3.1.** By comparing Table 3.3.2 with Table 3.3.3, we may observe some differences in the performance of the two stabilizers discussed in this article. The method of element-boundary-discrepancy seems to be more stable than the method of projected element boundary-discrepancy. For example, the former one provides WG schemes that are stable and convergent for $j = 1, 2, 3$ and $l = 2$ with fix $k = 4$ while the latter one is unsolvable for such combinations. From Figure 3.3.1, we can see that the WG solution is stable but not convergent with the projected based stabilizer term. On the other hand, the latter one is more accurate than the former one for the case of $j = k-1$ and $l = k-1$. We can observed from Figure 3.3.4 (bottom) that the order of convergence is O$(h^5)$ and O$(h^4)$ in the $L^2$ norm and discrete $H^1$ norm respectively. Again, for such combinations, we have obtained less order of accuracy using element-boundary-discrepancy stabilizer term, see Figure 3.3.5 (bottom).

**Log-log plots for Computational Results:** We have provided numerical results through Figures 3.3.4-3.3.5 for some combinations of weak Galerkin spaces.

Table 3.3.3: Order of convergence for Example 3.3.1 with stabilizer based on element-boundary-discrepancy

| | **k=1** | | | | | **k=2** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | j=0 | j=1 | j=2 | j=3 | j=4 | j=0 | j=1 | j=2 | j=3 | j=4 |
| l=0 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | NI | NI | NI |
| l=1 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | 2/3 | 2/3 | 2/3 |
| l=2 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | 2/3 | 2/3 | 2/3 |
| l=3 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | 2/3 | 2/3 | 2/3 |
| l=4 | 0/0 | 1/2 | 1/2 | 1/2 | 1/2 | 0/0 | 1/2 | 2/3 | 2/3 | 2/3 |
| | **k=3** | | | | | **k=4** | | | | |
| | j=0 | j=1 | j=2 | j=3 | j=4 | j=0 | j=1 | j=2 | j=3 | j=4 |
| l=0 | NI | NI | NI | NI | NI | NI | NI | NI | NI | NI |
| l=1 | 0/0 | 1/2 | 2/3 | NI | NI | NI | NI | NI | NI | NI |
| l=2 | 0/0 | 1/2 | 2/3 | 3/4 | 3/4 | 0/0 | 1/2 | 2/3 | 3/4 | NI |
| l=3 | 0/0 | 1/2 | 2/3 | 3/4 | 3/4 | 0/0 | 1/2 | 2/3 | 3/4 | 4/5 |
| l=4 | 0/0 | 1/2 | 2/3 | 3/4 | 3/4 | 0/0 | 1/2 | 2/3 | 3/4 | 4/5 |

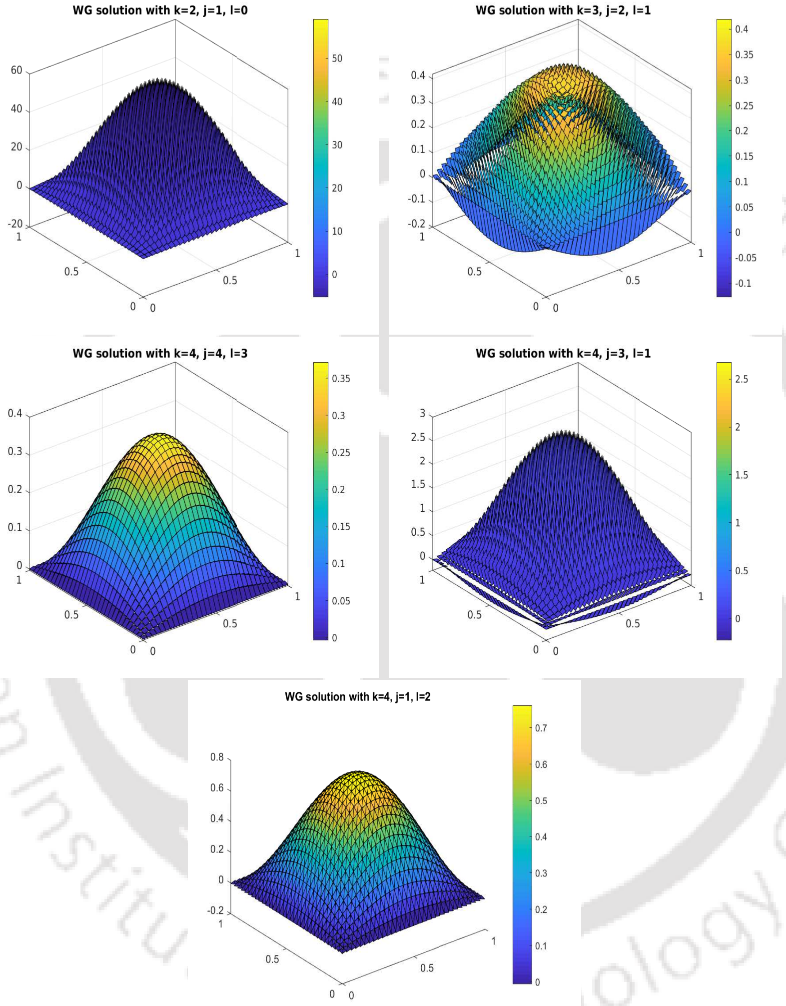

Figure 3.3.1: Plots of the WG approximations at time $t = 1$ for Example 3.3.1 with stabilizer based on projected element-boundary-discrepancy for $h = 1/32$.

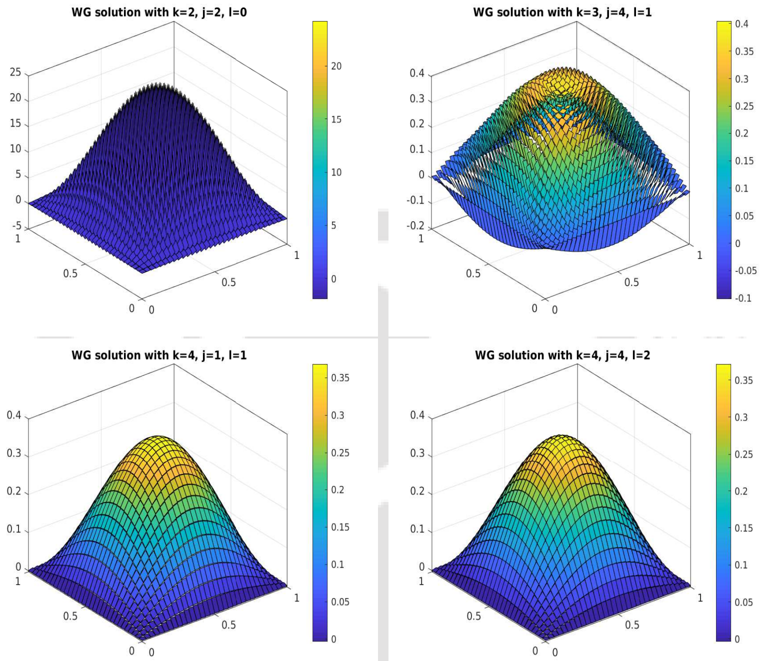

Figure 3.3.2: Plots of the WG approximations at time $t = 1$ for Example 3.3.1 with stabilizer based on element-boundary-discrepancy with $h = 1/32$.

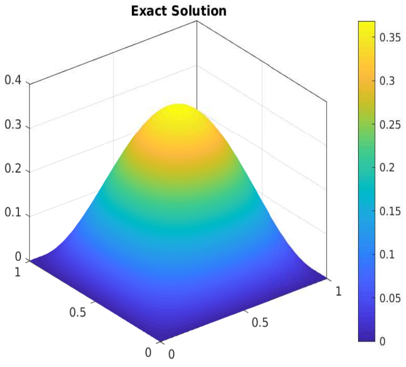

Figure 3.3.3: Exact solution for Example 3.3.1 at $t = 1$.

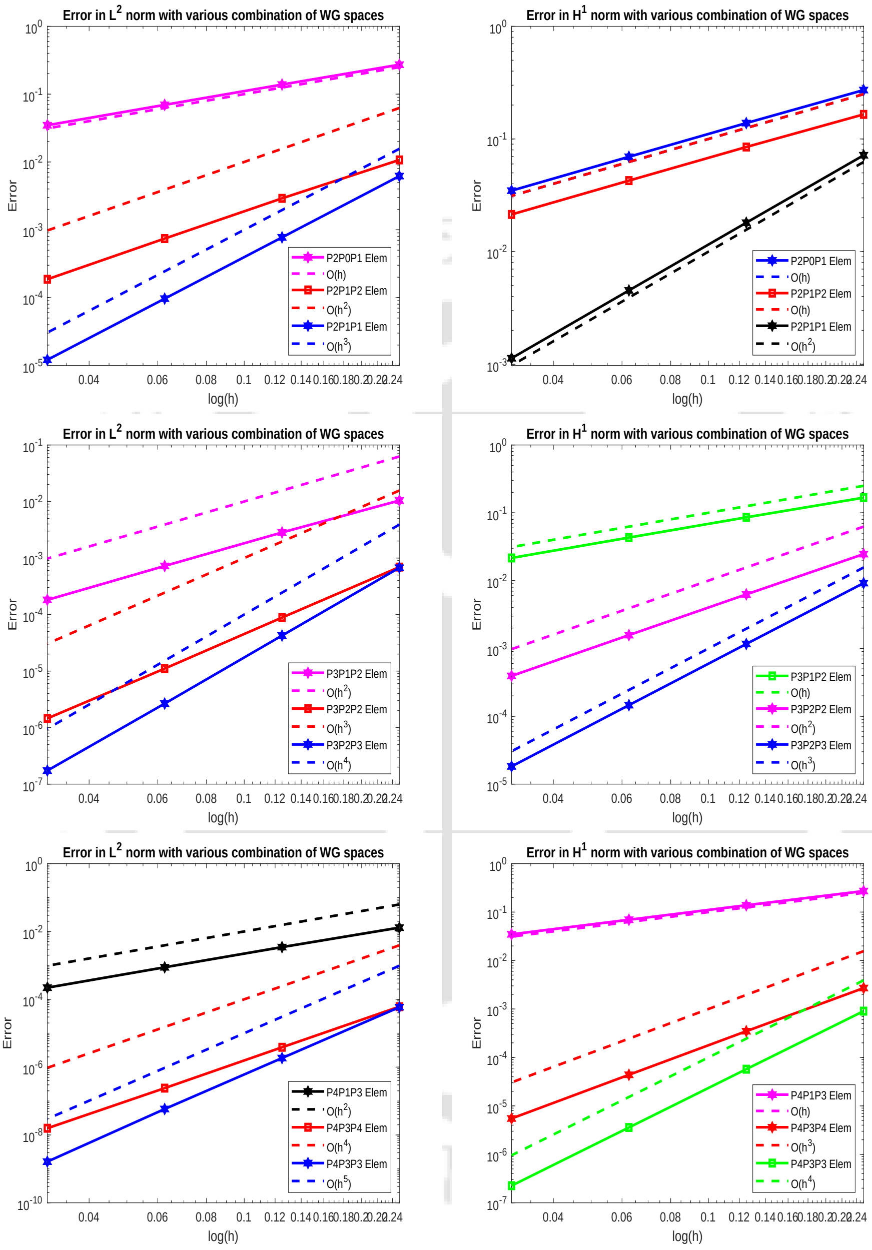

Figure 3.3.4: Log-log plot of the $L^2$ norm (left) and $H^1$ norm (right) errors versus mesh size at time $t = 1$ for Example 3.3.1 with projected element-boundary-discrepancy stabilizer term.

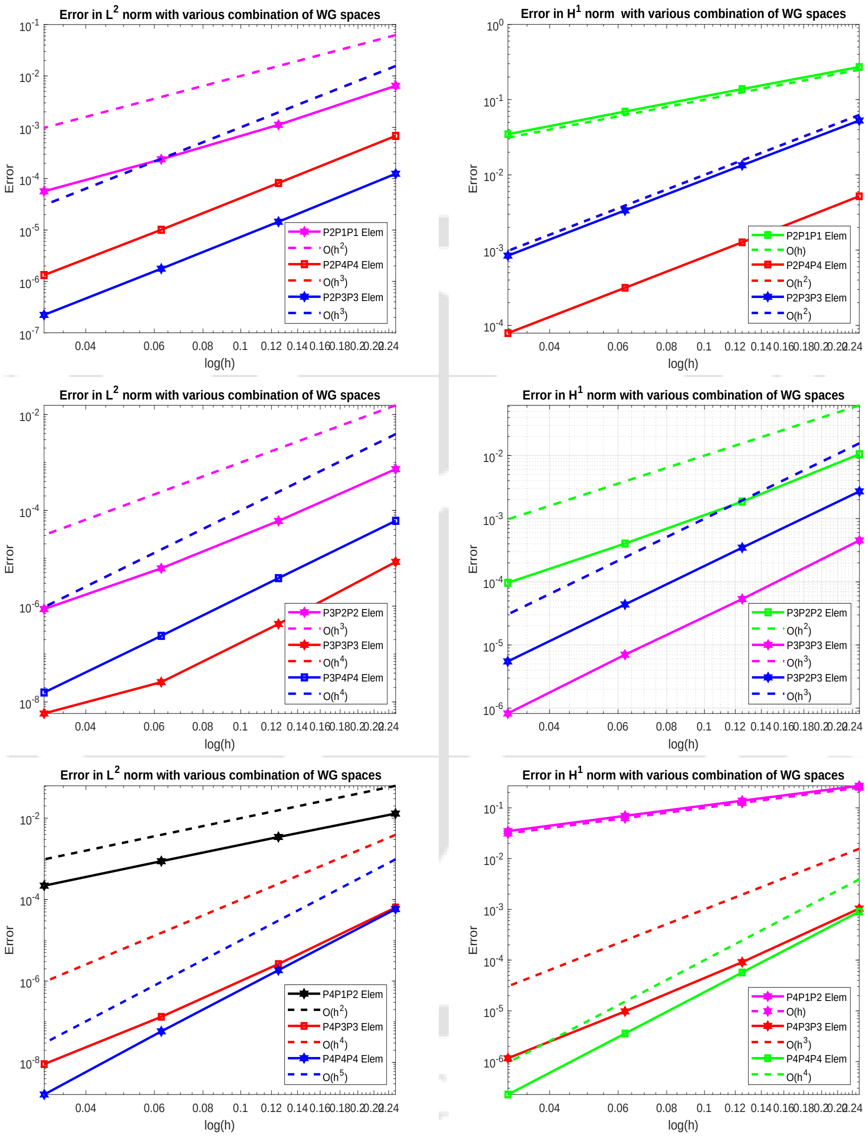

Figure 3.3.5: Log-log plot of the $L^2$ norm (left) and $H^1$ norm (right) errors versus mesh size at time $t = 1$ for Example 3.3.1 with element-boundary-discrepancy stabilizer term.

# 4

# WG-FEMs for Semilinear Klein-Gordon Equation

This chapter presents the development of the weak Galerkin finite element methods (WG-FEMs) for semilinear hyperbolic problems. Semidiscrete error estimate in $L^2$ norm as well as $H^1$ norm have been executed for the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$ is an integer. For a fully discrete scheme, we employ the Newmark scheme for temporal discretization. Finally, numerical results are provided to validate theoretical results.

## 4.1 Introduction

We consider the following semilinear Klein-Gordon equation

$$u_{tt} - \Delta u + f(u) = g, \quad \text{in } \ \Omega \times (0, T], \tag{4.1.1}$$

with the initial conditions

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = v^0(\mathbf{x}) \ \ \forall\, \mathbf{x} \in \Omega \tag{4.1.2}$$

and homogeneous boundary condition

$$u(\mathbf{x}, t) = 0 \ \ \forall (\mathbf{x}, t) \in \partial\Omega \times (0, T]. \tag{4.1.3}$$

Here $\Omega$ be a polygonal convex domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and $J = (0, T]$ with $T > 0$ be the final observation time. The initial functions $\{u^0, v^0\}$ and source function $g$ are smooth enough in their respective domain. Further, we assume that the nonlinear function $f$ is globally Lipschitz continuous with respect to $u$, i.e., there exists a constant $C_f > 0$ such that

$$\|f(u) - f(v)\| \leq C_f \|u - v\|, \tag{4.1.4}$$

for all $u, v$.

Semilinear hyperbolic equations play a crucial role in geometry and many areas of physics, namely relativity, fluid dynamics, acoustic, electromagnetic, mathematical physics and many more; due to such variety of applications, semilinear hyperbolic equations are considered to be an essential research topic in the field of numerical analysis. In particular, the existence and uniqueness of the semilinear degenerate hyperbolic Goursat problem have been examined in [117]. The unique solvability of local and nonlocal boundary problems for the semilinear Schrödinger equation is investigated in [9] in a Hilbert space. FEMs for semilinear elliptic problems with discontinuous coefficients are proposed in [60, 162]. Also, FEMs for semilinear elliptic and parabolic interface problems are discussed in [143] for the polygonal domain. Error estimates using the semidiscrete finite element technique for linear and semilinear parabolic problems are discussed in [31] and for nonlinear heat equation [27, 169].

Equation of this model (4.1.1) often arises in relativistic quantum physics. It is used to illustrate the free particle wave function, such as the Higgs boson. Several mathematical techniques have been applied to solve the Klein-Gondon equation, namely finite difference method [1, 78], Galerkin finite element method [94], spectral methods [160, 161]. Also, very recently, virtual element methods have been proposed for semilinear hyperbolic problems on polygonal meshes in [4]. Another development using the high-order local discontinuous Galerkin (LDG) technique for the Klein-Gordon equation can be found in [159]. In this chapter, we are going to employ the weak Galerkin finite element methods (WG-FEMs) to solve the Klein-Gordon equation (4.1.1). WG-FEMs have been studied extensively for the approximation of linear problems; however, there are only very few contributions dealing with nonlinear equations, for instance, Sun et al. [145] studied the WG-FEMs for a class of quasi-linear elliptic equations. Then the extension to semilinear parabolic problems using a linearized backward Euler scheme can be found in [115]. For other related works, we refer to [97, 98, 167].

The weak formulation for the model problem (4.1.1)-(4.1.3) can be defined as follows: Find $u : (0,T] \to H_0^1(\Omega)$ satisfying

$$\begin{cases} (u_{tt}, v) + \mathcal{A}(u,v) + (f(u), v) = (g, v) \ \ \forall \ \ v \in H_0^1(\Omega) \ \text{ and } \ t \in (0,T], \\ u(\mathbf{x},0) = u^0(\mathbf{x}), \ \ u_t(\mathbf{x},0) = v^0(\mathbf{x}) \ \ \forall \, \mathbf{x} \in \Omega. \end{cases} \tag{4.1.5}$$

Here $(.,.)$ is $L^2$ inner product and $\mathcal{A}(u,w) = (\nabla u, \nabla w)$. The global Lipschitz continuity of $f$ ensures that equation (4.1.5) has unique solution.

The rest of the chapter is as follows: In Sec. 4.2, we have described the WG finite element discretization, existence and uniqueness of the semidiscrete WG solution. Additionally, we have discussed the error equation. Sec. 4.3, devoted to optimal error estimates in both $L^\infty(L^2)$ and discrete $H^1$ norms for the semidiscrete weak Galerkin approximation. We deal with error estimates for fully discrete cases using the Newmark scheme in Sec. 4.4. At last, in Sec. 4.5, some numerical experiments are presented to validate theoretical results.

## 4.2 Weak Galerkin Space and Semidiscrete Approximation

In this section, WG finite element discretization is described and we recall the discrete weak gradient operator.

Let $\mathcal{T}_h$ be a partition of $\Omega \subset \mathbb{R}^2$ consisting of polygons as discussed in Chapter 1. Suppose $\mathcal{K} \in \mathcal{T}_h$ be a polygon with boundary $\partial\mathcal{K}$. For an integer $k \geq 1$, let $\mathbf{P}_k(\mathcal{K})$ be the set of all polynomials of degree $k$ in $\mathcal{K} \in \mathcal{T}_h$. Analogously, $\mathbf{P}_k(e)$ be the collection of all polynomials of degree $k$ on $e \in \mathcal{E}_h$. Therefore, a discrete weak function $\phi_h = \{\phi_0, \phi_b\}$ on $\mathcal{K}$ is the weak function $\phi_h = \{\phi_0, \phi_b\}$, where $\phi_0 \in \mathcal{P}_k(\mathcal{K})$ and $\phi_b \in \mathbf{P}_k(e), e \subset \partial\mathcal{K}$. Suppose $\mathcal{W}(k,k;\mathcal{K})$ be the local discrete weak finite space corresponding to $\mathcal{K}$; i.e.

$$\mathcal{W}(k,k;\mathcal{K}) = \Big\{\phi_h = \{\phi_0, \phi_b\} : \phi_0 \in \mathbf{P}_k(\mathcal{K}), \phi_b \in \mathbf{P}_k(e), e \subset \partial\mathcal{K}\Big\}.$$

Therefore, the global weak finite space $\mathcal{W}_h$ corresponding to $\mathcal{T}_h$ is constructed by patching local weak finite spaces $\mathcal{W}(k,k;\mathcal{K})$ as

$$\mathcal{W}_h = \{\phi_h = \{\phi_0, \phi_b\} : \phi_h|_{\mathcal{K}} \in \mathcal{W}(k,k;\mathcal{K}), [\phi_h]_e = 0, \forall e \in \mathcal{E}_h^0\}$$

and

$$\mathcal{W}_h^0 = \{\phi_h \in \mathcal{W}_h : \phi_b|_{\partial\Omega} = 0\}.$$

where $[\phi_h]_e = [\phi_b]_e$ is the jump of $\phi_h$ across interior edge $e \in \mathcal{E}_h^0$.

Now for any $\phi_h \in \mathcal{W}_h$, the discrete weak gradient of it, denoted by $\nabla_w\phi_h \in [\mathbf{P}_{k-1}(\mathcal{K})]^2$ that satisfies the following equation

$$(\nabla_w\phi_h, \mathbf{q}) = -\int_{\mathcal{K}} \phi_0(\nabla\cdot\mathbf{q})d\mathcal{K} + \int_{\partial\mathcal{K}} \phi_b(\mathbf{q}\cdot\mathbf{n})ds \quad \forall\mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2. \tag{4.2.1}$$

Here $\mathbf{n}$ is outward normal to $\partial\mathcal{K}$.

Hence, applying divergence theorem to equation (4.2.1), we have

$$(\nabla_w\phi_h, \mathbf{q}) = (\nabla\phi_0, \mathbf{q})_{\mathcal{K}} + \langle\phi_b - \phi_0, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} \quad \forall\mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2. \tag{4.2.2}$$

Now we recall some standard $L^2$ projections. For $\mathcal{K} \in \mathcal{T}_h$, let $\mathcal{Q}_k^0 : L^2(\mathcal{K}) \to \mathbf{P}_k(\mathcal{K})$ be the usual $L^2$ projection and for $e \in \mathcal{E}_h$, let $\mathcal{Q}_k^b : L^2(e) \to \mathbf{P}_k(e)$ be another $L^2$ projection. Further, we define the $L^2$ projection $\mathcal{Q}_h$ onto the WG space $\mathcal{W}_h$ which satisfies $\mathcal{Q}_h|_{\mathcal{K}} = \{\mathcal{Q}_k^0, \mathcal{Q}_k^b\}$. Also, define another $L^2$ projection $\mathbb{Q}_{k-1} : [L^2(\mathcal{K})]^2 \to [\mathbf{P}_{k-1}(\mathcal{K})]^2$. For the $L^2$ projections, we have the following results (cf. [153]).

**Lemma 4.2.1.** *Let $\mathcal{T}_h$ be a polygonal partition of $\Omega$ that meets the shape regularity conditions as described in [153]. Then for $\varphi \in H^{r+1}(\Omega)$, we have*

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\left(\|\varphi-\mathcal{Q}_k^0\varphi\|_{\mathcal{K}}^2+h_{\mathcal{K}}^2\|\nabla(\varphi-\mathcal{Q}_k^0\varphi)\|_{\mathcal{K}}^2\right)\le Ch^{2(r+1)}\|\varphi\|_{H^{r+1}(\Omega)}^2,$$

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\left(\|\nabla\varphi-\mathbb{Q}_{k-1}\nabla\varphi\|_{\mathcal{K}}^2+h_{\mathcal{K}}^2\|\nabla(\nabla\varphi-\mathbb{Q}_{k-1}(\nabla\varphi))\|_{\mathcal{K}}^2\right)\le Ch^{2r}\|v\|_{H^{r+1}(\Omega)}^2.$$

Now for weak Galerkin approximation, define the bilinear map $\mathcal{A}_w$ from $\mathcal{W}_h^0\times\mathcal{W}_h^0$ to $\mathbb{R}$ as

$$\mathcal{A}_w(u_h,\phi_h)=\sum_{\mathcal{K}\in\mathcal{T}_h}(\nabla_w u_h,\nabla_w\phi_h)_{\mathcal{K}}+\mathbf{S}(u_h,\phi_h)\ \ \forall u_h,\phi_h\in\mathcal{W}_h^0. \tag{4.2.3}$$

Here $\mathbf{S}(\cdot,\cdot)$ is the stabilizer based on element-boundary-discrepancy as defined in Example 1.4.2 in Chapter 1. So for $u_h=\{u_0,u_b\}$, $\phi_h=\{\phi_0,\phi_b\}\in\mathcal{W}_h^0$, the stabilizer $\mathbf{S}(\cdot,\cdot)$ (cf. [104]) is given by

$$\mathbf{S}(u_h,\phi_h)=\sum_{\mathcal{K}\in\mathcal{T}_h}h_{\mathcal{K}}^{-1}\langle(u_b-u_0|_{\partial\mathcal{K}}),(\phi_b-\phi_0|_{\partial\mathcal{K}})\rangle_{\partial\mathcal{K}}. \tag{4.2.4}$$

Following Lemma illustrates the coercive property of $\mathcal{A}_w(\cdot,\cdot)$ with respect to $\|\cdot\|_{1,h}$ norm on WG space $\mathcal{W}_h$. Details follow from [151].

**Lemma 4.2.2.** *Let $\phi_h\in\mathcal{W}_h$, then there exit constants $C_*>0$ & $C^*>0$, such that following inequality holds*

$$C_*\|\phi_h\|_{1,h}^2\le\mathcal{A}_w(\phi_h,\phi_h)\le C^*\|\phi_h\|_{1,h}^2. \tag{4.2.5}$$

Now we define energy norm(triple bar norm) $|||\cdot|||$ related to the bilinear map $\mathcal{A}_w(\cdot,\cdot)$ defined by

$$|||\phi_h|||=\sqrt{\mathcal{A}_w(\phi_h,\phi_h)}\ \ \text{for}\ \ \phi_h\in\mathcal{W}_h^0. \tag{4.2.6}$$

Hence, from above inequality (4.2.5), it can be conclude that $\|\cdot\|_{1,h}$ norm and triple bar norm $|||\cdot|||$ are equivalent in $\mathcal{W}_h^0$.

The continuous-time WG approximation to the problem (4.1.1)-(4.1.3) can be defined as follows: Find $u_h=\{u_0,u_b\}:(0,T]\to\mathcal{W}_h^0$ that satisfies following equation

$$(u_{htt},\phi_h)+\mathcal{A}_w(u_h,\phi_h)+(f(u_h),\phi_h)=(g,\phi_0)\ \ \forall\ \phi_h\in\mathcal{W}_h^0, \tag{4.2.7}$$

with $u_h(0)=\mathcal{Q}_hu^0$ and $u_{ht}(0)=\mathcal{Q}_hv^0$. Here $(f(u_h),\phi_h)$ is defined as

$$(f(u_h),\phi_h)=\sum_{\mathcal{K}\in\mathcal{T}_h}(f(u_0),\phi_0)_{\mathcal{K}}. \tag{4.2.8}$$

**Theorem 4.2.1.** *For global Lipschitz continuous function $f(u)$, semidiscrete approximation (4.2.7) has a unique solution.*

*Proof.* For a given $\mathcal{K} \in \mathcal{T}_h$, let $\{\phi_{0,i} : i = 1 \cdots N_0\}$ be the set of basis functions for $\mathcal{P}_k(\mathcal{K})$ and $\{\phi_{b,j} : j = 1 \cdots N_b\}$ be the set of basis functions for $\sum_{e \in \partial\mathcal{K} \cap \mathcal{E}_h} \mathcal{P}_k(e)$. Then for any $u_h = \{u_0, u_b\} \in \mathcal{W}_h^0$ can be written as

$$u_h|_{\mathcal{K}} = \left\{ \sum_{i=1}^{N_0} d_{0,i}(t)\phi_{0,i}, \sum_{j=1}^{N_b} d_{b,j}(t)\phi_{b,j} \right\},$$

where $d_{0,i}, d_{b,j} : [0,T] \to \mathbb{R}$ are the coefficients functions for $1 \leq i \leq N_0$ and $1 \leq j \leq N_b$.

Now for $1 \leq i \leq N_0 + N_b$, we write $\phi_{i,h} = \{\bar{\phi}_{0,i}, \bar{\phi}_{b,i}\}$ with $\bar{\phi}_{0,i} = \phi_{0,i}$ for $i = 1, \cdots, N_0$, $\bar{\phi}_{0,i} = 0$ for $i = N_0+1, \cdots, N_0+N_b$ and $\bar{\phi}_{b,i} = 0$ for $i = 1, \cdots, N_0$, $\bar{\phi}_{b,i} = \phi_{b,i-N_0}$ for $i = N_0 + 1, \cdots, N_0 + N_b$.

Similarly, we define $d_{i,h} = d_{0,i}$ for $i = 1 \cdots N_0$ and $d_{i,h} = d_{b,i-N_0}$ for $i = N_0+1, \cdots, N_0+N_b$.

Hence,

$$u_h|_{\mathcal{K}} = \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\phi_{i,h} = \left\{ \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\bar{\phi}_{0,i}, \sum_{j=1}^{N_0+N_b} d_{j,h}(t)\bar{\phi}_{b,j} \right\}, \quad \mathcal{K} \in \mathcal{T}_h.$$

Next, set $\phi_h = \phi_{j,h}$ for $j = 1, 2, \cdots, N_0 + N_b$ in (4.2.7) to obtain

$$\left( \sum_{i=1}^{N_0+N_b} d''_{i,h}(t)\phi_{i,h}, \phi_{j,h} \right) + \mathcal{A}_w\left( \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\phi_{i,h}, \phi_{j,h} \right)$$
$$+ \left( f\left( \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\phi_{i,h} \right), \phi_{j,h} \right) = (g, \bar{\phi}_{0,j}) \quad \text{for } \; j = 1, 2, \cdots, N_0 + N_b.$$

After rearranging, we obtain

$$\sum_{i=1}^{N_0+N_b} d''_{i,h}(t)(\bar{\phi}_{0,i}, \bar{\phi}_{b,j}) + \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\mathcal{A}_w(\phi_{i,h}, \phi_{j,h}) \left( f\left( \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\bar{\phi}_{0,i} \right), \bar{\phi}_{0,j} \right)$$
$$+ = (g, \bar{\phi}_{0,j}) \quad \text{for } \; j = 1, 2, \cdots, N_0 + N_b.$$

Therefore, in matrix form, we need to find

$$D_h(t) = \begin{bmatrix} D_{0h} \\ D_{bh} \end{bmatrix}$$

with

$$D_{0h} = [d_{1,h}, d_{2,h}, \cdots, d_{N_0,h}]^T, \quad D_{bh} = [d_{N_0+1,h}, d_{N_0+2,h}, \cdots, d_{N_0+N_b,h}]^T$$

such that

$$\begin{cases} \mathcal{M}_{h\mathcal{K}} D_h''(t) + \mathcal{A}_{h\mathcal{K}} D_h(t) = \mathcal{F}_{h\mathcal{K}}(D_h(t)), \\ D_h(0) = [d_{1,h,0}, d_{2,h,0}, \cdots, d_{N_0+N_b,h,0}]^T, \\ D_{ht}(0) = [d_{1,h,1}, d_{2,h,1}, \cdots, d_{N_0+N_b,h,1}]^T, \end{cases} \tag{4.2.9}$$

where $D_h(0)$ and $D_{ht}(0)$ are the coefficient vectors of $\mathcal{Q}_h u^0$ and $\mathcal{Q}_h v^0$ when expressed as a linear combination of the basis functions, respectively. The matrix $\mathcal{F}_{h\mathcal{K}}$ is given by

$$\mathcal{F}_{h\mathcal{K}}(t) = [F_1(t), F_2(t), \cdots, F_{N_0+N_b}(t)]^T = \begin{bmatrix} \mathcal{F}_0 \\ \mathcal{F}_b \end{bmatrix},$$

where $\mathcal{F}_0 = [F_1, F_2, \cdots, F_{N_0}]^T$, $\mathcal{F}_b = [F_{N_0+1}, F_{N_0+2}, \cdots, F_{N_0+N_b}]^T = \mathbb{O}_{N_b \times 1}$ and

$$F_j = (g, \bar{\phi}_{0,j}) - \left( f\left( \sum_{i=1}^{N_0} d_{i,h}(t) \bar{\phi}_{0,i} \right), \bar{\phi}_{0,j} \right) \text{ for } 1 \leq j \leq N_0,$$

and $F_j = 0$for $N_0 + 1 \leq j \leq N_0 + N_b$.

Also, on each element $\mathcal{K}$, the local stiffness matrix $\mathcal{A}_{h\mathcal{K}}$ associated with the bilinear map $\mathcal{A}_w$ can be written as a block matrix defined as

$$\mathcal{A}_{h\mathcal{K}} = \begin{bmatrix} \mathcal{A}_{00} & \mathcal{A}_{0b} \\ \mathcal{A}_{b0} & \mathcal{A}_{bb} \end{bmatrix},$$

where $\mathcal{A}_{00}, \mathcal{A}_{0b}, \mathcal{A}_{b0}$ and $\mathcal{A}_{bb}$ are given by

$$\mathcal{A}_{00} = [\mathcal{A}_w(\phi_{j,h}, \phi_{i,h})]_{1 \leq i,j \leq N_0}, \qquad \mathcal{A}_{0b} = [\mathcal{A}_w(\phi_{N_0+j,h}, \phi_{i,h})]_{1 \leq i \leq N_0,\, 1 \leq j \leq N_b},$$
$$\mathcal{A}_{b0} = [\mathcal{A}_w(\phi_{j,h}, \phi_{N_0+i,h})]_{1 \leq i \leq N_b,\, 1 \leq j \leq N_0}, \qquad \mathcal{A}_{bb} = [\mathcal{A}_w(\phi_{N_0+j,h}, \phi_{N_0+i,h})]_{1 \leq i,j \leq N_b}.$$

Further, mass matrix $\mathcal{M}_{h\mathcal{K}}$ can be written as a block matrix defined as below

$$\mathcal{M}_{h\mathcal{K}} = \begin{bmatrix} \mathcal{M}_{00} & \mathcal{M}_{0b} \\ \mathcal{M}_{b0} & \mathcal{M}_{bb} \end{bmatrix},$$

where $\mathcal{M}_{00}, \mathcal{M}_{0b}, \mathcal{M}_{b0}$ and $\mathcal{M}_{bb}$ are given by

$$\mathcal{M}_{00} = [(\bar{\phi}_{0,j}, \bar{\phi}_{0,i})]_{1 \leq i,j \leq N_0}, \quad \mathcal{M}_{0b} = \mathbb{O}_{N_0 \times N_b}, \quad \mathcal{M}_{b0} = \mathbb{O}_{N_b \times N_0}, \quad \mathcal{M}_{bb} = \mathbb{O}_{N_b \times N_b}.$$

Now for any $\mathbf{v} = \{v_1, v_2, \cdots, v_{N_0}\}^T \in \mathbb{R}^{N_0} \setminus \{0\}$, we have

$$\mathbf{v}^T \mathcal{M}_{00} \mathbf{v} = \int_{\mathcal{K}} \left| \sum_{i=1}^{N_0} v_i \bar{\phi}_{0,i} \right|^2 dx = \left\| \sum_{i=1}^{N_0} v_i \bar{\phi}_{0,i} \right\|_{L^2(\mathcal{K})}^2 > 0$$

and for any $\mathbf{w} = \{w_1, w_2, \cdots, w_{N_b}\}^T \in \mathbb{R}^{N_b} \setminus \{0\}$, using coercivity property of $\mathcal{A}_w$ we have

$$\mathbf{w}^T \mathcal{A}_{bb} \mathbf{w} = \mathcal{A}_w\left(\sum_{j=1}^{N_b} w_j \phi_{N_0+j,h}, \sum_{j=1}^{N_b} w_j \phi_{N_0+j,h}\right) > 0.$$

Hence, matrices $\mathcal{M}_{00}$ and $\mathcal{A}_{bb}$ are positive definite.

Therefore, the matrix equation (4.2.9) can be written as

$$\begin{cases} D''_{0h}(t) + \mathcal{M}_{00}^{-1}\mathcal{A}_{00}D_{0h}(t) + \mathcal{M}_{00}^{-1}\mathcal{A}_{0b}D_{bh}(t) = \mathcal{M}_{00}^{-1}\mathcal{F}_0(t), \\ D_{bh}(t) = -\mathcal{A}_{bb}^{-1}\mathcal{A}_{b0}D_{0h}(t) \\ D_h(0) = [d_{1,h,0}, d_{2,h,0}, \cdots, d_{N_0+N_b,h,0}]^T, \\ D_{ht}(0) = [d_{1,h,1}, d_{2,h,1}, \cdots, d_{N_0+N_b,h,1}]^T. \end{cases} \tag{4.2.10}$$

Existence of the solution $u_h \in H^2(0,T;\mathcal{W}_h^0)$ follows from standard ODE theory using the fact $f$ is globally Lipschitz (see, Chap. 13 in [147]). □

Let $u$ be the exact solution of (4.1.5) and $u_h$ be the solution of semidiscrete problem (4.2.7). Then the projected error can be presented as:

$$e_h(t) = \{e_0(t), e_b(t)\} = u_h(t) - \mathcal{Q}_h u(t), \ t \in [0,T]. \tag{4.2.11}$$

Simply, we write $e_h = \{e_0, e_b\} = u_h - \mathcal{Q}_h u$.

Now we establish the following error equation for $e_h$, which is very crucial for further analysis.

**Lemma 4.2.3.** *Let $e_h(t) = u_h(t) - \mathcal{Q}_h u(t)$. Then for all $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, we have*

$$\begin{aligned} &(e_{htt}, \phi_0) + \mathcal{A}_w(e_h, \phi_h) \\ &= \ell_1(u,\phi_h) + \ell_2(u,\phi_h) + \ell_3(u,\phi_h) - \mathbf{S}(\mathcal{Q}_h u, \phi_h) + (f(u), \phi_0) - \sum_{\mathcal{K}\in\mathcal{T}_h} (f(u_0), \phi_0)_{\mathcal{K}}, \end{aligned} \tag{4.2.12}$$

*where $\ell_1(\cdot,\cdot)$, $\ell_2(\cdot,\cdot)$, $\ell_3(\cdot,\cdot)$ are bilinear maps given as*

$$\begin{cases} \ell_1(u,\phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\nabla u - \nabla \mathcal{Q}_k^0 u, \nabla \phi_0\right)_{\mathcal{K}}, \\ \ell_2(u,\phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h} \left\langle (\nabla u - \nabla \mathcal{Q}_k^0 u) \cdot \mathbf{n}, \phi_b - \phi_0 \right\rangle_{\partial\mathcal{K}}, \\ \ell_3(u,\phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h} \left\langle \mathcal{Q}_k^0 u - \mathcal{Q}_k^b u, \nabla_w \phi_h \cdot \mathbf{n} \right\rangle_{\partial\mathcal{K}}. \end{cases} \tag{4.2.13}$$

*Proof.* For any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, we test the equation (4.1.1) against $\phi_0$ on each $\mathcal{K} \in \mathcal{T}_h$ to obtain

$$\begin{aligned}(g, \phi_0) &= (u_{tt}, \phi_0) - \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla\cdot(\nabla u), \phi_0)_{\mathcal{K}} + (f(u), \phi_0)\\ &= (\mathcal{Q}_h u_{tt}, \phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla u, \nabla\phi_0)_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \nabla u\cdot\mathbf{n}, \phi_0\rangle_{\partial\mathcal{K}} + (f(u), \phi_0)\\ &= ((\mathcal{Q}_h u)_{tt}, \phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla u, \nabla\phi_0)_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \nabla u\cdot\mathbf{n}, \phi_0 - \phi_b\rangle_{\partial\mathcal{K}} + (f(u), \phi_0).\end{aligned} \tag{4.2.14}$$

Here we have applied the Gauss divergence theorem and the fact that $\sum_{\mathcal{K}\in\mathcal{T}_h}\langle\nabla u\cdot\mathbf{n}, \phi_b\rangle_{\partial\mathcal{K}} = 0$. Combining (4.2.7) and (4.2.14), we obtain

$$\begin{aligned}&(u_{htt}, \phi_0) + \mathcal{A}_w(u_h, \phi_h)\\ &\quad= ((\mathcal{Q}_h u)_{tt}, \phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla u, \nabla\phi_0)_{\mathcal{K}} + \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \nabla u\cdot\mathbf{n}, \phi_b - \phi_0\rangle_{\partial\mathcal{K}}\\ &\qquad + (f(u), \phi_0) - \sum_{\mathcal{K}\in\mathcal{T}_h} (f(u_0), \phi_0)_{\mathcal{K}}.\end{aligned} \tag{4.2.15}$$

Now employing definition of weak gradient $\nabla_w$, we obtain

$$\begin{aligned}&(\nabla_w \mathcal{Q}_h u, \nabla_w\phi_h)_{\mathcal{K}}\\ &= (\nabla \mathcal{Q}_k^0 u, \nabla_w\phi_h)_{\mathcal{K}} + \langle Q_k^b u - Q_k^0 u, \nabla_w\phi_h\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &= \left(\nabla \mathcal{Q}_k^0 u, \nabla\phi_0\right)_{\mathcal{K}} + \langle Q_k^b u - Q_k^0 u, \nabla_w\phi_h\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} + \langle \phi_b - \phi_0, \nabla \mathcal{Q}_k^0 u\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{A}_w(\mathcal{Q}_h u, \phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla_w \mathcal{Q}_h u, \nabla_w\phi_h)_{\mathcal{K}} + \mathbf{S}(\mathcal{Q}_h u, \phi_h)\\ &= \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\nabla \mathcal{Q}_k^0 u, \nabla\phi_0\right)_{\mathcal{K}} + \sum_{\mathcal{K}\in\mathcal{T}_h} \langle Q_k^b u - Q_k^0 u, \nabla_w\phi_h\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &\quad + \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \phi_b - \phi_0, \nabla \mathcal{Q}_k^0 u\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} + \mathbf{S}(\mathcal{Q}_h u, \phi_h).\end{aligned}$$

Therefore,

$$\begin{aligned}&((\mathcal{Q}_h u)_{tt}, \phi_0) + \mathcal{A}_w(\mathcal{Q}_h u, \phi_h)\\ &= ((\mathcal{Q}_h u)_{tt}, \phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\nabla \mathcal{Q}_k^0 u, \nabla\phi_0\right)_{\mathcal{K}} + \sum_{\mathcal{K}\in\mathcal{T}_h} \langle Q_k^b u - Q_k^0 u, \nabla_w\phi_h\cdot\mathbf{n}\rangle_{\partial\mathcal{K}}\\ &\quad + \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \phi_b - \phi_0, \nabla \mathcal{Q}_k^0 u\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} + \mathbf{S}(\mathcal{Q}_h u, \phi_h).\end{aligned} \tag{4.2.16}$$

Subtracting equation (4.2.16) from equation (4.2.15), we have the desired result. ☐

We now require following important estimates for bilinear maps $\ell_1(\cdot,\cdot)$, $\ell_2(\cdot,\cdot)$ and $\ell_3(\cdot,\cdot)$ for polygonal discretization $\mathcal{T}_h$ of $\Omega$, which is shape regular. Following results followed directly from Lemma 4.2-Lemma 4.4 of the literature [151].

**Lemma 4.2.4.** *Suppose that $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, then there exists a constant $C > 0$ such that*

$$|\ell_1(u, \phi_h)| \leq Ch^k \|u\|_{H^{k+1}(\Omega)} \|\phi_h\|_{1,h} \ \ \forall \phi_h \in \mathcal{W}_h^0.$$

**Lemma 4.2.5.** *Under the similar assumptions as in above Lemma 4.2.4 the following result holds*

$$|\ell_2(u, \phi_h)| \leq Ch^k \|u\|_{H^{k+1}(\Omega)} \|\phi_h\|_{1,h} \ \ \forall \phi_h \in \mathcal{W}_h^0,$$

*where $C > 0$ is a constant.*

**Lemma 4.2.6.** *Under the similar assumptions as in above Lemma 4.2.4, the following result holds*

$$|\ell_3(u, \phi_h)| \leq Ch^k \|u\|_{H^{k+1}(\Omega)} \|\phi_h\|_{1,h} \ \ \forall \phi_h \in \mathcal{W}_h^0,$$

*where $C > 0$ is a constant.*

## 4.3 Semidiscrete Error Analysis for Klein-Gordon Equation

This section deals with the error analysis for the spatially discrete scheme (4.2.7). Optimal order of convergence in both $L^2$ and discrete $H^1$ norms are established for the semidiscrete scheme with stabilizer based on element-boundary discrepancy.

### 4.3.1 $L^2$ norm Error Estimate for Klein-Gordon Equation

In this subsection, we discuss the optimal order of error estimates in $L^2$ norm. The basic idea is to use elliptic projection as in [158].

For $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, we write

$$f_\varphi = -\nabla \cdot (\nabla \varphi) \ \text{ in } \ \Omega.$$

Define the elliptic projection $\mathcal{R}_h : H^2(\Omega) \cap H_0^1(\Omega) \to \mathcal{W}_h^0$ by

$$\mathcal{A}_w(\mathcal{R}_h \varphi, \phi_h) = (f_\varphi, \phi_h) \ \ \forall \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0. \tag{4.3.1}$$

Equation (4.3.1) suggests that $\mathcal{R}_h \varphi$ is weak Galerkin solution to the following second order elliptic problem with true solution $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$

$$-\nabla \cdot (\nabla \varphi) = f_\varphi \text{ in } \Omega \text{ and } \varphi = 0 \text{ on } \partial\Omega.$$

Next, the following approximation result can be followed from [43].

**Lemma 4.3.1.** *For sufficiently smooth function $v$, we have*

$$\|\mathcal{Q}_h v(t) - \mathcal{R}_h v(t)\|^2 \le Ch^{2(k+1)}\|v(t)\|^2_{H^{k+1}(\Omega)}.$$

Therefore, employing elliptic projection $\mathcal{R}_h$ we further divide the projected error $e_h = u_h - \mathcal{Q}_h u$ as

$$e_h(t) = u_h(t) - \mathcal{Q}_h u(t) = u_h(t) - \mathcal{R}_h u(t) + \mathcal{R}_h u(t) - \mathcal{Q}_h u(t) = \theta(t) - \rho(t),$$

where $\theta = u_h - \mathcal{R}_h u$ and $\rho = \mathcal{Q}_h u - \mathcal{R}_h u$.

Then it is easy to verify that for $\phi_h \in \mathcal{W}_h^0$,

$$(\theta_{tt}, \phi_h) + \mathcal{A}_w(\theta, \phi_h) = (\rho_{tt}, \phi_h) - \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{d}{dt}(G, \phi_0)_{\mathcal{K}} + \sum_{\mathcal{K}\in\mathcal{T}_h}(G, \phi_{0t})_{\mathcal{K}}, \tag{4.3.2}$$

where $G$ is defined as

$$G(t) = \int_0^t (f(u_h) - f(u))(s)ds \quad \text{for } 0 \le t \le T.$$

Next, for some $\zeta \in (0, T]$, we define $\hat{\theta}$ as

$$\hat{\theta}(\cdot, t) = \int_t^{\zeta} \theta(\cdot, s)ds \quad 0 \le t \le T.$$

Therefore,

$$\hat{\theta}(\zeta) = 0 \ \text{ and } \ \hat{\theta}_t = -\theta(\cdot, t) \ \text{ for } t \in [0, T]. \tag{4.3.3}$$

Setting $\phi_h = \hat{\theta}$ in (4.3.2) and integrating with respect to time $t$ from 0 to $\zeta$, and using the fact $\hat{\theta}(\zeta) = 0$, we have

$$\begin{aligned} &-\int_0^{\zeta}(\theta_t, \hat{\theta}_t)ds - (\theta_t(0), \hat{\theta}(0)) + \int_0^{\zeta}\mathcal{A}_w(\theta, \hat{\theta})ds \\ &= -\int_0^{\zeta}(\rho_t, \hat{\theta}_t)ds - (\rho_t(0), \hat{\theta}(0)) - \int_0^{\zeta}\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{d}{dt}(G, \hat{\theta})_{\mathcal{K}} + \int_0^{\zeta}\sum_{\mathcal{K}\in\mathcal{T}_h}(G, \hat{\theta}_t)_{\mathcal{K}}. \end{aligned}$$

Again, (4.3.3) and the fact that $\theta_t(0) - \rho_t(0) = e_{ht}(0) = 0$ yield

$$\int_0^{\zeta}\frac{1}{2}\frac{d}{dt}\|\theta\|^2 dt - \int_0^{\zeta}\frac{1}{2}\frac{d}{dt}\mathcal{A}_w(\hat{\theta}, \hat{\theta})ds = \int_0^{\zeta}(\rho_t, \theta)ds - \int_0^{\zeta}\sum_{\mathcal{K}\in\mathcal{T}_h}(G, \theta)_{\mathcal{K}}. \tag{4.3.4}$$

Now using Lipschitz condition (4.1.4) of $f$, Lemma 4.2.1 and Lemma 4.3.1, we have

$$\begin{aligned} \|f(u_h) - f(u)\|^2 &\le C_f^2\|u_h - u\|^2 \\ &\le C_f^2\|u_h - \mathcal{Q}_h u\|^2 + C_f^2\|\mathcal{Q}_h u - u\|^2 \\ &\le C_f^2\|\theta\|^2 + C_f^2\|\rho\|^2 + C_f^2\|\mathcal{Q}_h u - u\|^2 \\ &\le C\|\theta\|^2 + Ch^{2(k+1)}\|u\|^2_{H^{k+1}(\Omega)}. \end{aligned} \tag{4.3.5}$$

Therefore,

$$\int_0^\zeta (G,\theta) \le C \int_0^\zeta \left( \int_0^t \|f(u_h)-f(u)\| \right) \|\theta(t)\| dt$$
$$\le Ch^{2(k+1)} \|u\|^2_{L^2(H^{k+1})} + C\int_0^\zeta \|\theta(t)\|^2 dt. \tag{4.3.6}$$

Now we consider $\zeta \in (0,T]$ such that $\|\theta(\zeta)\| = \max_{0\le t\le T} \|\theta(t)\|$. Hence, from equation (4.3.4), we obtain

$$\frac{1}{2}\|\theta(\zeta)\|^2 - \frac{1}{2}\|\theta(0)\|^2 - \frac{1}{2}\mathcal{A}_w(\hat{\theta}(\zeta),\hat{\theta}(\zeta)) + \frac{1}{2}\mathcal{A}_w(\hat{\theta}(0),\hat{\theta}(0)) = \int_0^\zeta (\rho_t,\theta)ds - \int_0^\zeta (G,\theta).$$

Again, using (4.3.3), we have

$$\frac{1}{2}\|\theta(\zeta)\|^2 + \frac{1}{2}\mathcal{A}_w(\hat{\theta}(0),\hat{\theta}(0)) = \frac{1}{2}\|\theta(0)\|^2 + \int_0^\zeta (\rho_t,\theta)ds - \int_0^\zeta (G,\theta).$$

Then inequality (4.3.5) together with positivity of $\mathcal{A}_w$ and some standard inequalities lead to

$$\|\theta(\zeta)\|^2 \le C\left\{ h^{2(k+1)}\|u\|^2_{L^2(H^{k+1})} + \|\theta(0)\|^2 + \|\rho_t\|^2_{L^2(L^2)} \right\}. \tag{4.3.7}$$

Consequently, for $\rho_t = \mathcal{Q}_h u_t - \mathcal{R}_h u_t$, using Lemma 4.3.1, we obtain

$$\|\rho_t(t)\|^2 \le Ch^{2(k+1)}\|u_t(t)\|^2_{H^{k+1}(\Omega)} ds. \tag{4.3.8}$$

Once again, Lemma 4.3.1, yields

$$\|\theta(0)\|^2 = \|\mathcal{Q}_h u^0 - \mathcal{R}_h u^0\|^2 \le Ch^{2(k+1)}\|u^0\|^2_{H^{k+1}(\Omega)}. \tag{4.3.9}$$

Hence, using (4.3.8), (4.3.9) in (4.3.7), for $t \in (0,T]$, we have

$$\|\theta(t)\|^2 \le \|\theta(\zeta)\|^2$$
$$\le Ch^{2(k+1)}\left( \|u\|^2_{L^2(H^{k+1})} + \|u^0\|^2_{H^{k+1}(\Omega)} + \|u_t\|^2_{L^2(H^{k+1})} \right). \tag{4.3.10}$$

Finally, above inequality (4.3.10) and Lemma 4.3.1 lead to following error estimate in $L^\infty(L^2)$ norm.

**Theorem 4.3.1.** *Suppose that* $u \in L^2(J; H^{k+1}(\Omega))$ *and* $u_t \in L^2(J; H^{k+1}(\Omega))$. *Then there exists a constant* $C > 0$, *such that the following estimate holds*

$$\|e_h\|_{L^\infty(L^2)} \le Ch^{k+1}\left( \|u^0\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(H^{k+1})} + \|u_t\|_{L^2(H^{k+1})} \right).$$

*Proof.* We have

$$\begin{aligned}e_h(t) &= u_h(t) - \mathcal{Q}_h u(t)\\ &= u_h(t) - \mathcal{R}_h u(t) + \mathcal{R}_h u(t) - \mathcal{Q}_h u(t) = \theta(t) - \rho(t).\end{aligned}$$

Hence, using triangle inequality, we have

$$\|e_h(t)\| \leq \|\theta(t)\| + \|\rho(t)\|.$$

Therefore, Lemma 4.3.1 and inequality (4.3.10) lead to desired result. $\square$

### 4.3.2 Discrete $H^1$ norm Error Estimate for Klein-Gordon Equation

The following estimate result for the stabilizer follows from the existing results (cf. Lemma 4.7 in [151]).

**Lemma 4.3.2.** *Suppose that* $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ *and*

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle (u_b - u_0|_{\partial\mathcal{K}}), (\phi_b - \phi_0|_{\partial\mathcal{K}}) \rangle_{\partial\mathcal{K}}.$$

*Then there exists a constant* $C > 0$ *such that the following result holds*

$$|\mathbf{S}(\mathcal{Q}_h u, \phi_h)| \leq C h^{2k} \|u\|_{H^{k+1}(\Omega)} \|\phi_h\|_{1,h}.$$

Now we shall present the main result of this section.

**Theorem 4.3.2.** *Suppose that* $u \in L^\infty(J; H^{k+1}(\Omega))$ *and* $u_t \in L^2(J; H^{k+1}(\Omega))$*. Then the following error estimate holds*

$$\begin{aligned}|\!|\!| e_h |\!|\!| \leq & C h^k \Big( \|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^2(H^{k+1})} \Big)\\ & + C h^{k+1} \Big( \|u^0\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(H^{k+1})} + \|u_t\|_{L^2(H^{k+1})} \Big),\end{aligned}$$

*where* $C > 0$ *is a constant independent of* $h$*.*

*Proof.* Taking $\phi_h = e_{ht} = \{e_{0t}, e_{bt}\}$ in the error equation (4.2.12), we obtain

$$\begin{aligned}&(e_{htt}, e_{ht}) + \mathcal{A}_w(e_h, e_{ht})\\ &= \ell_1(u, e_{ht}) + \ell_2(u, e_{ht}) + \ell_3(u, e_{ht}) - \mathbf{S}(\mathcal{Q}_h u, e_{ht}) + (f(u), e_{ht}) - \sum_{\mathcal{K}\in\mathcal{T}_h} (f(u_0), e_{0t})_{\mathcal{K}}.\end{aligned}$$

We can rearrange the above equation as

$$\frac{1}{2}\frac{d}{dt}\|e_{ht}\|^2 + \frac{1}{2}\frac{d}{dt}\mathcal{A}_w(e_h, e_h)$$
$$= \ell_1(u, e_{ht}) + \ell_2(u, e_{ht}) + \ell_3(u, e_{ht}) - \mathbf{S}(\mathcal{Q}_h u, e_{ht}) + \sum_{\mathcal{K}\in\mathcal{T}_h}(f(u), e_{0t})_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h}(f(u_0), e_{0t})_{\mathcal{K}}. \tag{4.3.11}$$

Now integrating both sides of the equation (4.3.11) from 0 to $t$ for some $t \in (0, T]$, we have

$$\begin{aligned}
&\int_0^t \frac{1}{2}\frac{d}{dt}\|e_{ht}\|^2 + \int_0^t \frac{1}{2}\frac{d}{dt}\mathcal{A}_w(e_h, e_h) \\
&= \int_0^t \ell_1(u, e_{ht}) + \int_0^t \ell_2(u, e_{ht}) + \int_0^t \ell_3(u, e_{ht}) \\
&\quad - \int_0^t \mathbf{S}(\mathcal{Q}_h u, e_{ht}) + \int_0^t \sum_{\mathcal{K}\in\mathcal{T}_h}((f(u) - f(u_0)), e_{0t})_{\mathcal{K}}\,.
\end{aligned}$$

Further, using the fact $e_h(0) = 0$ and $e_{ht}(0) = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|e_{ht}(t)\|^2 + \frac{1}{2}\mathcal{A}_w(e_h(t), e_h(t)) \\
&= \int_0^t \ell_1(u, e_{ht}) + \int_0^t \ell_2(u, e_{ht}) + \int_0^t \ell_3(u, e_{ht}) \\
&\quad - \int_0^t \mathbf{S}(\mathcal{Q}_h u, e_{ht}) + \int_0^t ((f(u) - f(u_h)), e_{ht})\,.
\end{aligned} \tag{4.3.12}$$

Now using integration by parts and the fact that $e_h(0) = 0$, we have

$$\int_0^t \ell_1(u, e_{ht}) = \ell_1(u(t), e_h(t)) - \int_0^t \ell_1(u_t, e_h)ds.$$

Again, Lemma 4.2.4 yields

$$\left|\int_0^t \ell_1(u, e_{ht})\right| \leq Ch^k\left\{\|u(t)\|_{H^{k+1}(\Omega)}\|e_h(t)\|_{1,h} + \int_0^t \|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\right\}. \tag{4.3.13}$$

Similarly, using Lemma 4.2.5-Lemma 4.3.2, we obtain

$$\left|\int_0^t \ell_2(u, e_{ht})\right| \leq Ch^k\left\{\|u(t)\|_{H^{k+1}(\Omega)}\|e_h(t)\|_{1,h} + \int_0^t \|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\right\}, \tag{4.3.14}$$

$$\left|\int_0^t \ell_3(u, e_{ht})\right| \leq Ch^k\left\{\|u(t)\|_{H^{k+1}(\Omega)}\|e_h(t)\|_{1,h} + \int_0^t \|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\right\}, \tag{4.3.15}$$

$$\left|\int_0^t \mathbf{S}(\mathcal{Q}_h u, e_{ht})\right| \leq Ch^{2k}\left\{\|u(t)\|_{H^{k+1}(\Omega)}\|e_h(t)\|_{1,h} + \int_0^t \|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\right\}. \tag{4.3.16}$$

Therefore, using inequalities (4.3.13)-(4.3.16) in (4.3.12) and Cauchy-Schwarz inequality we have

$$\begin{aligned}&\frac{1}{2}\|e_{ht}(t)\|^2+\frac{1}{2}\mathcal{A}_w(e_h(t),e_h(t))\\&\leq Ch^k\Big\{\|u(t)\|_{H^{k+1}(\Omega)}\|e_h(t)\|_{1,h}+\int_0^t\|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\Big\}\\&\quad+\frac{1}{2}\int_0^t\|f(u)-f(u_h)\|^2+\frac{1}{2}\int_0^t\|e_{ht}\|^2.\end{aligned}\tag{4.3.17}$$

Again, from inequality (4.3.5), we have

$$\|f(u)-f(u_h)\|^2\leq C\|\theta\|^2+Ch^{2(k+1)}\|u\|^2_{H^{k+1}(\Omega)}.\tag{4.3.18}$$

Hence, using (4.3.10) in above inequality (4.3.18), we obtain

$$\int_0^t\|f(u)-f(u_h)\|^2\leq Ch^{2(k+1)}\left(\|u^0\|^2_{H^{k+1}(\Omega)}+\|u\|^2_{L^2(H^{k+1})}+\|u_t\|^2_{L^2(H^{k+1})}\right).\tag{4.3.19}$$

Therefore, using (4.3.19) in (4.3.17) and coercive inequality (4.2.5), we have

$$\begin{aligned}&\frac{1}{2}\|e_{ht}(t)\|^2+\frac{C_*}{2}\|e_h(t)\|^2_{1,h}\\&\leq Ch^k\left\{\|u(t)\|_{H^{k+1}(\Omega)}\|e_h(t)\|_{1,h}+\int_0^t\|u_t\|_{H^{k+1}(\Omega)}\|e_h\|_{1,h}ds\right\}\\&\quad+Ch^{2(k+1)}\left(\|u^0\|^2_{H^{k+1}(\Omega)}+\|u\|^2_{L^2(H^{k+1})}+\|u_t\|^2_{L^2(H^{k+1})}\right)+\frac{1}{2}\int_0^t\|e_{ht}\|^2.\end{aligned}\tag{4.3.20}$$

Now applying Young's inequality with suitable $\epsilon>0$, we have

$$\begin{aligned}&\frac{1}{2}\|e_{ht}(t)\|^2+\frac{C_*}{2}\|e_h(t)\|^2_{1,h}\\&\leq\frac{Ch^{2k}}{2\epsilon}\|u(t)\|^2_{H^{k+1}(\Omega)}+\frac{\epsilon}{2}\|e_h(t)\|^2_{1,h}+\frac{1}{2}\int_0^t\|u_t\|^2_{H^{k+1}(\Omega)}+\frac{1}{2}\int_0^t\|e_h\|^2_{1,h}ds\\&\quad+\frac{1}{2}\int_0^t\|e_{ht}\|^2+Ch^{2(k+1)}\left(\|u^0\|^2_{H^{k+1}(\Omega)}+\|u\|^2_{L^2(H^{k+1})}+\|u_t\|^2_{L^2(H^{k+1})}\right).\end{aligned}\tag{4.3.21}$$

Finally, setting $\epsilon=\frac{C_*}{2}$ and applying Gronwall's inequality, we obtain

$$\begin{aligned}&\|e_{ht}(t)\|^2+\|e_h(t)\|^2_{1,h}\\&\leq Ch^{2k}\Big\{\|u\|^2_{L^\infty(H^{k+1})}+\|u_t\|^2_{L^2(H^{k+1})}\Big\}\\&\quad+Ch^{2(k+1)}\left(\|u^0\|^2_{H^{k+1}(\Omega)}+\|u\|^2_{L^2(H^{k+1})}+\|u_t\|^2_{L^2(H^{k+1})}\right).\end{aligned}$$

Now the Theorem 4.3.2 follows form the fact that $|||\cdot|||$ and $\|\cdot\|_{1,h}$ are equivalent. □

## 4.4 Fully Discrete WG-FEMs for Semilinear Klein-Gordon Equation

We shall expand analysis of previous section to a fully discrete weak Galerkin finite element scheme. We employ the Newmark scheme for the temporal variable. Let $M$ be a natural number. First we split the time period $(0,T]$ into $M$ uniform sub-intervals $J_n = (t_{n-1}, t_n]$ for $n = 1, 2, \cdots M$ with $t_0 = 0$, $t_n = n\tau$ and $t_M = T$, where $\tau = \frac{T}{M}$ is time step.

For any continuous function $\varphi : [0,T] \to L^2(\Omega)$, define $\varphi^n = \varphi(\cdot, t_n)$ for $n = 0, 1, \cdots M$. Also, for any sequence $\{\varphi^n\}_0^M \subset L^2(\Omega)$, define

$$\begin{cases} \varphi^{n+\frac{1}{2}} = \frac{1}{2}(\varphi^{n+1} + \varphi^n), & \partial_\tau \varphi^{n+\frac{1}{2}} = \frac{\varphi^{n+1} - \varphi^n}{\tau}; \\ \partial_\tau^2 \varphi^n = \frac{\varphi^{n+2} - 2\varphi^{n+1} + \varphi^n}{\tau^2}, & \delta_\tau^2 \varphi^n = \frac{\varphi^{n+2} + 2\varphi^{n+1} + \varphi^n}{4}. \end{cases} \tag{4.4.1}$$

Now applying the above scheme, the fully discrete weak-Galerkin finite element estimate to (4.1.1)-(4.1.3) is given by: For $0 \le n \le M-2$, find $U^{n+2} \in \mathcal{W}_h^0$ such that

$$(\partial_\tau^2 U^n, \phi_h) + \mathcal{A}_w(\delta_\tau^2 U^n, \phi_h) + (\delta_\tau^2 F^n, \phi_h) = (\delta_\tau^2 g^n, \phi_h) \ \forall \ \phi_h \in \mathcal{W}_h^0, \tag{4.4.2}$$

with $U^0 = \mathcal{Q}_h u^0$ and

$$\begin{aligned} &(U^1, \phi_h) + \frac{\tau^2}{4}\mathcal{A}_w(U^1, \phi_h) \\ &= (U^0, \phi_h) - \frac{\tau^2}{4}\mathcal{A}_w(U^0, \phi_h) + \tau(w_{h,o}, \phi_h) + \frac{\tau^2}{4}(g^1 + g^0 - F^1 - F^0, \phi_h) \ \forall \ \phi_h \in \mathcal{W}_h. \end{aligned}$$

Here $F^n = f(U(t_n))$ and $g^n = g(t_n)$ for $0 \le n \le M$, and $w_{h,0}$ is the suitable approximation of $v^0$. The above scheme is unconditionally stable [106] as the CFL condition automatically holds.

Now the projected error at the time level $t = t_n$ is divided as

$$e_h^n = U^n - \mathcal{Q}_h u^n = U^n - \mathcal{R}_h u^n + \mathcal{R}_h u^n - \mathcal{Q} u^n = \gamma^n - \rho^n,$$

where $\gamma^n = U^n - \mathcal{R}_h u^n$ and $\rho^n = \mathcal{Q} u^n - \mathcal{R}_h u^n$.

**Lemma 4.4.1.** *Let $u$ be the exact solution of the problem (4.1.5) and $U^n$ be the fully discrete solution. Assume that $u_{tttt} \in L^1(J; L^2(\Omega)), u \in L^\infty(J; H^{k+1}(\Omega))$ and $u_{tt} \in L^\infty(J; H^{k+1}(\Omega))$. Then there exists a constant $C > 0$, independent of $\tau$ and $h$ such that following estimate holds*

$$\begin{aligned} &\max_{0 \le n \le M} \|\gamma^n\| \\ &\le C\left(\|\partial_\tau \gamma^{\frac{1}{2}}\| + \|\gamma^{\frac{1}{2}}\|_{1,h}\right) + C\tau^2 \|u_{tttt}\|_{L^1(L^2)} + Ch^{k+1}\left\{\|u\|_{L^\infty(H^{k+1})} + \|u_{tt}\|_{L^\infty(H^{k+1})}\right\}. \end{aligned}$$

*Proof.* For $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, we have

$$\begin{aligned}
&(\partial_\tau^2 \gamma^n, \phi_h) + \mathcal{A}_w(\delta_\tau^2 \gamma^n, \phi_h) \\
&= (\partial_\tau^2 U^n, \phi_h) + \mathcal{A}_w(\delta_\tau^2 U^n, \phi_h) - (\partial_\tau^2 \mathcal{R}_h u^n, \phi_h) - \mathcal{A}_w(\delta_\tau^2 \mathcal{R}_h u^n, \phi_h) \\
&= (\delta_\tau^2 g^n, \phi_h) - (\delta_\tau^2 F^n, \phi_h) - (\partial_\tau^2 \mathcal{R}_h u^n, \phi_h) - \mathcal{A}_w(\delta_\tau^2 \mathcal{R}_h u^n, \phi_h) \\
&= (\delta_\tau^2 g^n, \phi_h) - (\delta_\tau^2 F^n, \phi_h) - (\partial_\tau^2 \mathcal{R}_h u^n, \phi_h) + (\delta_\tau^2 (\nabla \cdot \nabla u^n), \phi_h) \\
&= (\delta_\tau^2 g^n, \phi_h) - (\delta_\tau^2 F^n, \phi_h) - (\partial_\tau^2 \mathcal{R}_h u^n, \phi_h) + (\delta_\tau^2 (u_{tt}^n + f(u^n) - g^n), \phi_h) \\
&= (\delta_\tau^2 u_{tt}^n, \phi_h) - (\delta_\tau^2 (F^n - f(u^n)), \phi_h) + (\partial_\tau^2 \rho^n, \phi_h) \\
&\quad - (\partial_\tau^2 (\mathcal{Q}_h u^n - u^n), \phi_h) - (\partial_\tau^2 u^n, \phi_h).
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\partial_\tau^2 \gamma^n, \phi_h) + \mathcal{A}_w(\delta_\tau^2 \gamma^n, \phi_h) \\
&= (\delta_\tau^2 u_{tt}^n, \phi_h) - (\delta_\tau^2 (F^n - f(u^n)), \phi_h) + (\partial_\tau^2 \rho^n, \phi_h) - (\partial_\tau^2 (\mathcal{Q}_h u^n - u^n), \phi_h) - (\partial_\tau^2 u^n, \phi_h).
\end{aligned} \tag{4.4.3}$$

Next, we estimate the nonlinear term. Using Lipschitz condition (4.1.4) of $f$, we have

$$|(f(u_h) - f(u), \phi_h)| \leq C_f \|u_h - u\| \|\phi_h\| \leq C_f \|u_h - \mathcal{Q}_h u\| \|\phi_h\| + C_f \|\mathcal{Q}_h u - u\| \|\phi_h\|.$$

Now using Lemma 4.2.1, we obtain

$$|(f_h(u_h) - f(u), \phi_h)| \leq C \|u_h - \mathcal{Q}_h u\| \|\phi_h\| + C h^{k+1} \|u\|_{H^{k+1}(\Omega)} \|\phi_h\|. \tag{4.4.4}$$

Therefore,

$$|(\delta_\tau^2 (F_h^n - f(u^n)), \phi_h)| \leq C h^{k+1} \|\delta_\tau^2 u^n\|_{H^{k+1}(\Omega)} \|\phi_h\| + C \|\delta_\tau^2 \gamma^n\| \|\phi_h\|. \tag{4.4.5}$$

Further,

$$\partial_\tau^2 u^n = \frac{1}{(\tau)^2} \int_{-\tau}^{\tau} (\tau - |s|) u_{tt}(\cdot, t_{n+1} + s) ds.$$

Therefore, applying integration by parts twice, we have

$$\partial_\tau^2 u^n = u_{tt}^{n+1} + \frac{1}{6\tau^2} \int_{-\tau}^{\tau} (\tau - |s|)^3 u_{tttt}(\cdot, t_{n+1} + s) ds. \tag{4.4.6}$$

Also, using Taylor's theorem

$$\begin{aligned}
u_{tt}^{n+2} &= u_{tt}^{n+1} + \tau u_{ttt}^{n+1} + \int_0^{\tau} (\tau - |s|) u_{tttt}(\cdot, t_{n+1} + s) ds, \\
u_{tt}^{n} &= u_{tt}^{n+1} - \tau u_{ttt}^{n+1} + \int_{-\tau}^{0} (\tau - |s|) u_{tttt}(\cdot, t_{n+1} + s) ds,
\end{aligned}$$

which implies

$$\delta_\tau^2(u_{tt}^n) = u_{tt}^{n+1} + \frac{1}{4}\int_{-\tau}^{\tau} (\tau - |s|) u_{tttt}(\cdot, t_{n+1} + s) ds. \tag{4.4.7}$$

Again, using (4.4.6)-(4.4.7), we have

$$\|\partial_\tau^2 u^n - \delta_\tau^2(u_{tt}^n)\| \leq C(\tau) \int_{-\tau}^{\tau} \|u_{tttt}(\cdot, t_{n+1} + s)\| ds. \tag{4.4.8}$$

Hence, using (4.4.5) and (4.4.8) in (4.4.3), we have

$$\begin{aligned}
&(\partial_\tau^2 \gamma^n, \phi_h) + \mathcal{A}_w(\delta_\tau^2 \gamma^n, \phi_h) \\
&\leq C\|\delta_\tau^2 \gamma^n\| \|\phi_h\| + C\|\partial_\tau^2 \rho^n\| \|\phi_h\| + Ch^{k+1} \|\delta_\tau^2 u^n\|_{H^{k+1}(\Omega)} \|\phi_h\| \\
&\quad + Ch^{k+1}\|\partial_\tau^2 u^n\| \|\phi_h\| + C\tau \int_{-\tau}^{\tau} \|u_{tttt}(\cdot, t_{n+1} + s)\| ds \|\phi_h\|.
\end{aligned} \tag{4.4.9}$$

Next, setting $\phi_h = \frac{\gamma^{n+2} - \gamma^n}{2\tau}$ in (4.4.9) and applying Lemma 4.3.1, we obtain

$$\begin{aligned}
&\frac{1}{2\tau}\left[\frac{\|\gamma^{n+2} - \gamma^{n+1}\|^2}{\tau^2} - \frac{\|\gamma^{n+1} - \gamma^n\|^2}{\tau^2} + \frac{\|\gamma^{n+2} - \gamma^{n+1}\|_{1,h}^2}{4} - \frac{\|\gamma^{n+1} - \gamma^n\|_{1,h}^2}{4}\right] \\
&\leq Ch^{k+1}\left(\|\delta_\tau^2 u^n\|_{H^{k+1}(\Omega)} + \|\partial_\tau^2 u^n\|_{H^{k+1}(\Omega)}\right) \left\|\frac{\gamma^{n+2} - \gamma^n}{2\tau}\right\| \\
&\quad + C\left\{\tau \int_{-\tau}^{\tau} \|u_{tttt}(\cdot, t_{n+1} + s)\| ds + \|\delta_\tau^2 \gamma^n\|\right\} \left\|\frac{\gamma^{n+2} - \gamma^n}{2\tau}\right\|.
\end{aligned}$$

Therefore, by multiplying $\tau$ both sides of the above and applying Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\left[\frac{\|\gamma^{n+2} - \gamma^{n+1}\|^2}{\tau^2} - \frac{\|\gamma^{n+1} - \gamma^n\|^2}{\tau^2} + \frac{\|\gamma^{n+2} - \gamma^{n+1}\|_{1,h}^2}{4} - \frac{\|\gamma^{n+1} - \gamma^n\|_{1,h}^2}{4}\right] \\
&\leq Ch^{2(k+1)}\left\{\|\delta_\tau^2 u^n\|_{H^{k+1}(\Omega)}^2 + \|\partial_\tau^2 u^n\|_{H^{k+1}(\Omega)}^2\right\} + C\tau^4 \left(\int_{-\tau}^{\tau} \|u_{tttt}(\cdot, t_{n+1} + s)\| ds\right)^2 \\
&\quad + C\tau^2 \|\delta_\tau^2 \gamma^n\|^2 + \left\|\frac{\gamma^{n+2} - \gamma^n}{2\tau}\right\|^2,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\left[\frac{\|\gamma^{n+2} - \gamma^{n+1}\|^2}{\tau^2} - \frac{\|\gamma^{n+1} - \gamma^n\|^2}{\tau^2} + \frac{\|\gamma^{n+2} - \gamma^{n+1}\|_{1,h}^2}{4} - \frac{\|\gamma^{n+1} - \gamma^n\|_{1,h}^2}{4}\right] \\
&\leq Ch^{2(k+1)}\left\{\|\delta_\tau^2 u^n\|_{H^{k+1}(\Omega)}^2 + \|\partial_\tau^2 u^n\|_{H^{k+1}(\Omega)}^2\right\} + C\tau^4 \left(\int_{-\tau}^{\tau} \|u_{tttt}(\cdot, t_{n+1} + s)\| ds\right)^2 \\
&\quad + C\tau^2\left(\|\gamma^{n+2}\|^2 + \|\gamma^{n+1}\|^2 + \|\gamma^n\|^2\right) + C\left(\left\|\frac{\gamma^{n+2} - \gamma^{n+1}}{\tau}\right\|^2 + \left\|\frac{\gamma^{n+1} - \gamma^n}{\tau}\right\|^2\right).
\end{aligned}$$

Now applying discrete Gronwall's inequality and summing over 0 to $n-2$, we obtain

$$\begin{aligned}&\left(\|\partial_\tau \gamma^{n-\frac{1}{2}}\|^2 + \|\gamma^{n-\frac{1}{2}}\|_{1,h}^2\right)\\ &\le \left(\|\partial_\tau\gamma^{\frac{1}{2}}\|^2+\|\gamma^{\frac{1}{2}}\|_{1,h}^2\right) + C\tau^4\|u_{tttt}\|^2_{L^1(L^2)} + C\tau^2\sum_{r=0}^{n} \|\gamma(t_r)\|^2\\ &\quad + Ch^{2(k+1)}\left\{\|u\|^2_{L^\infty(H^{k+1})}+\|u_{tt}\|^2_{L^\infty(H^{k+1})}\right\}.\end{aligned}$$

Since, $\gamma^n = \gamma^{n-\frac{1}{2}}+\frac{\tau}{2}\partial_t\gamma^{n-\frac{1}{2}}$, so we have

$$\begin{aligned}\|\gamma^n\|^2 &\le C\left(\|\partial_\tau\gamma^{\frac{1}{2}}\|^2+\|\gamma^{\frac{1}{2}}\|_{1,h}^2\right) + C\tau^4\|u_{tttt}\|^2_{L^1(L^2)}\\ &\quad + C\tau^2\sum_{r=0}^{n} \|\gamma(t_r)\|^2 + Ch^{2(k+1)}\left\{\|u\|^2_{L^\infty(H^{k+1})}+\|u_{tt}\|^2_{L^\infty(H^{k+1})}\right\}.\end{aligned}$$

Again, applying discrete Gronwall's inequality, we obtain

$$\begin{aligned}\|\gamma^n\|^2 &\le C\left(\|\partial_\tau\gamma^{\frac{1}{2}}\|^2+\|\gamma^{\frac{1}{2}}\|_{1,h}^2\right) + C\tau^4\|u_{tttt}\|^2_{L^1(L^2)}\\ &\quad + Ch^{2(k+1)}\left\{\|u\|^2_{L^\infty(H^{k+1})}+\|u_{tt}\|^2_{L^\infty(H^{k+1})}\right\}.\end{aligned}$$

□

Further, using Lemma 4.3.1, we have

$$\begin{aligned}\|\rho^n\| = \|\mathcal{Q}_h u^n - \mathcal{R}_h u^n\| &\le Ch^{k+1}\|u^n\|_{H^{k+1}(\Omega)}\\ &\le Ch^{k+1}\left\{\|u^0\|_{H^{k+1}(\Omega)} + \|u_t\|_{L^1(H^{k+1})}\right\}. \qquad (4.4.10)\end{aligned}$$

**Theorem 4.4.1.** *Let $u$ be solution of the problem (4.1.5) and $U^n$ be the fully discrete solution. Further, assume that $\|\partial_\tau\gamma^{\frac{1}{2}}\| + \|\gamma^{\frac{1}{2}}\|_{1,h} = \mathcal{O}(\tau^2+h^{k+1})$ and all the conditions of Lemma 4.4.1 are satisfied. Then we have*

$$\begin{aligned}\max_{0\le n\le M}\|e_h^n\| &\le C\left(\|\partial_\tau\gamma^{\frac{1}{2}}\|+\|\gamma^{\frac{1}{2}}\|_{1,h}\right) + C\tau^2\|u_{tttt}\|_{L^1(L^2)}\\ &\quad + Ch^{k+1}\left\{\|u_0\|_{H^{k+1}(\Omega)} + \|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^1(H^{k+1})} + \|u_{tt}\|_{L^\infty(H^{k+1})}\right\},\end{aligned}$$

*where $C>0$ is a constant independent of $\tau$ and $h$.*

*Proof.* Applying triangle inequality, we obtain

$$\|e_h^n\| \le \|\gamma^n\| + \|\rho^n\|.$$

Hence, Lemma 4.4.1 and inequality (4.4.10) lead to the above Theorem 4.4.1. □

## 4.5 Numerical Experiments

In order to examine the effectiveness of the proposed WG-FEMs for solving semilinear initial boundary value problems, we have taken different test problems in $\mathbb{R}^d(d = 1.2)$. The time interval is fixed as $(0, T]$ to verify the theoretical results provided in the preceding section. Let $U^n$ be the fully discrete weak Galerkin solution obtained from equation (4.4.2). To demonstrate the convergence history of the WG technique in terms of the discretization error, we calculate the projected error $e_h^n = U^n - \mathcal{Q}_h u(\mathbf{x}_r, t_n)$ at the final time $t_n = T$. The errors $e_h^n$ are evaluated with respect to the $|||\cdot|||$ norm, defined in equation (4.2.6). These errors are reported in tables corresponding to the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(e), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $h$ is the spatial step size and $\tau = \mathcal{O}(h^k)$ is the time step size. Additionally, we examine the order of convergence of the error $e_h^n$ measured in the $L^2$ norm

$$\|e_h^n\|_{L^2(\Omega)} = \Big( \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} |e_0^n|^2 d\mathcal{K} \Big)^{1/2}.$$

Experimental order of convergence (EOC) is defined as: $\text{EOC}(i+1) = \frac{\log(e_h(i+1)/e_h(i))}{\log(h(i+1)/h(i))}$. By analyzing the EOC, we can determine the convergence rate of the error $e_h^n$ as the mesh is refined.

**Example 4.5.1.** Consider the following sine-Gordon equation in a one dimensional domain

$$u_{tt} - u_{xx} + \sin u = 0 \ \text{ in } \ \Omega \times (0, T], \tag{4.5.1}$$

where the domain $\Omega = [-20, 20]$ and the analytical solution is given by

$$u(x,t) = 4tan^{-1}\left( \frac{\sqrt{1-\mu^2}\, cos(\mu t)}{\mu \cosh(\sqrt{1-\mu^2}\ x)} \right), \quad 0 < \mu < 1.$$

Thus, the initial conditions are defined as follows

$$\begin{cases} u(x,0) = 4tan^{-1}\left( \frac{\sqrt{1-\mu^2}}{\mu \cosh(\sqrt{1-\mu^2}\ x)} \right) \text{ in } [-20, 20], \\ u_t(x,0) = 0 \ \text{ in } \ [-20, 20]. \end{cases}$$

We determine spatial and temporal convergence rates. The $L^2$ norm and triple-bar $|||\cdot|||$ norm errors along with corresponding convergence orders of the schemes are shown in Table 4.5.1 and Table 4.5.2. With triple-bar and $L^2$ norms, respectively, we achieved order of $\mathcal{O}(h^k)$ and $\mathcal{O}(h^{k+1})$ accuracy in space, and second order in time. This agrees with our theoretical investigation.

Table 4.5.1: The history of convergence with linear WG space under the $|||\cdot|||$ norm and $L^2$ norm at final time $t = 1$ with $\mu = 0.7$ for Example 4.5.1.

| $h$ | $\|\|\|e_h\|\|\|$ | EOC | $\|\|e_h\|\|$ | EOC |
|---|---|---|---|---|
| 1/8 | 9.704e-01 | - | 9.4167e-01 | - |
| 1/16 | 7.2022e-01 | 0.43 | 5.1871e-01 | 0.86 |
| 1/32 | 6.6421e-01 | 0.11 | 4.4117e-01 | 0.23 |
| 1/64 | 3.3125e-01 | 1.00 | 1.0973e-01 | 2.00 |
| 1/128 | 1.4815e-01 | 1.16 | 2.1949e-02 | 2.32 |
| 1/256 | 7.5431e-02 | 0.97 | 5.6899e-03 | 1.95 |

**Example 4.5.2.** Next, consider another example with a nonlinear source term

$$\begin{cases} u_{tt} - \Delta u = f(u) + g(x,y,t) \ \text{ in } \ \Omega \times (0,T], \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{4.5.2}$$

where $\Omega = [0,1] \times [0,1]$, $T = 1$ be the final time and $f(u) = u^2$. We choose the source term $g(x,y,t)$ such a way that $u(x,y,t) = e^{-t}xy(1-x)(1-y)$ becomes the exact solution of (4.5.2).

We discretized the domain into Voronoi meshes. The results in Table 4.5.3 are obtained for different refinement mesh sizes, denoted as $h$ and time steps denoted as $\tau$. The final time of the simulation is $T = 1$ and the time step $\tau$ is chosen to be equal to $h$. This choice ensures that the time step is proportional to the spatial mesh size, which can affect the accuracy of the numerical solution. It is inferred that the current approach needs fewer Newton iterations, which impacts the CPU time necessary for a similar order of accuracy. We have observed that a WG scheme has high order accuracy in space, which guarantees the theoretical results.

Table 4.5.2: The history of convergence with linear WG space under the $|||\cdot|||$ norm and $L^2$ norm at final time $t = 1$ with $\mu = 0.5$ for Example 4.5.1.

| $h$ | $\|\|\|e_h\|\|\|$ | EOC | $\|\|e_h\|\|$ | EOC |
|---|---|---|---|---|
| 1/8 | 1.2281e+00 | - | 1.5081e+00 | - |
| 1/16 | 8.4897e-01 | 0.53 | 7.2074e-01 | 1.06 |
| 1/32 | 8.3467e-01 | 0.02 | 6.9667e-01 | 0.04 |
| 1/64 | 4.2126e-01 | 0.98 | 1.7746e-01 | 1.97 |
| 1/128 | 1.8657e-01 | 1.17 | 3.4808e-02 | 2.35 |
| 1/256 | 9.5436e-02 | 0.96 | 9.1081e-03 | 1.93 |
| 1/512 | 4.7243e-02 | 1.01 | 2.2319e-03 | 2.02 |

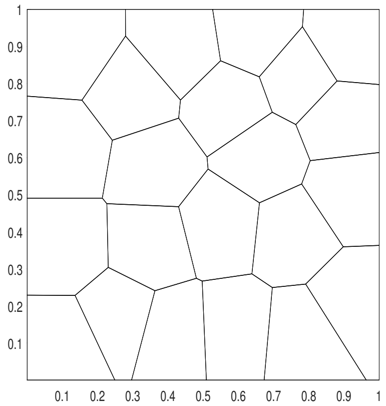

Figure 4.5.1: Sample of polygonal mesh for unit square domain

**Example 4.5.3.** Next, we consider 2-dimensional sine-Gordon equation given by

$$u_{tt} - \Delta u + \sin u = 0 \ \text{ in } \ \Omega \times J, \tag{4.5.3}$$

Table 4.5.3: The history of convergence with linear WG space under the $|||\cdot|||$ norm and $L^2$ norm at final time $t = 1$ for Example 4.5.2.

| $h$ | $\|\|\|e_h\|\|\|$ | EOC | $\|e_h\|$ | EOC |
|---|---|---|---|---|
| 1/8 | 2.2737e-01 | - | 7.0406e-02 | - |
| 1/16 | 1.0402e-01 | 1.12 | 1.1948e-02 | 2.55 |
| 1/32 | 3.3064e-02 | 1.65 | 1.8622e-03 | 2.68 |
| 1/64 | 1.5376e-02 | 1.10 | 3.9793e-04 | 2.22 |
| 1/128 | 7.1021e-03 | 1.11 | 7.9127e-05 | 2.33 |

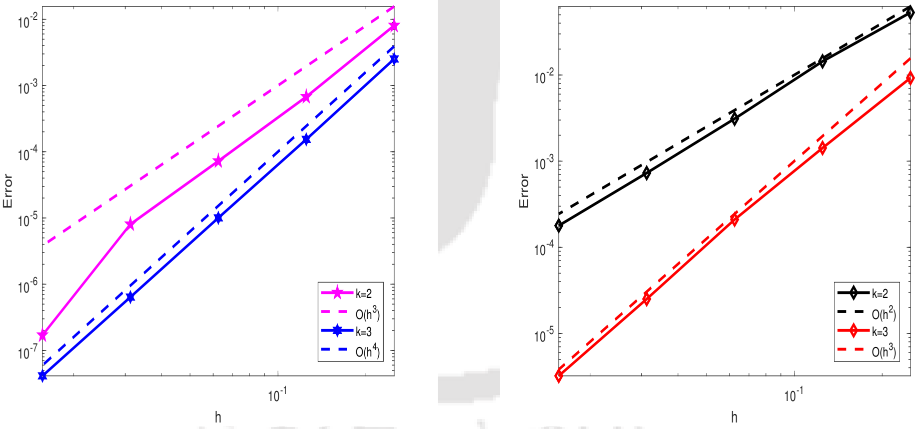

Figure 4.5.2: Log-log plot of the $L^2$ norm (left) and energy norm (right) errors versus mesh size at time $t = 1$ for WG spaces with $k = 2$ & $k = 3$ for Example 4.5.2.

where the computational domain $\Omega = [-7, 7] \times [-7, 7]$ and time interval $J = (0, 1]$. We have assumed the initial conditions as

$$\begin{cases} u(x, y, 0) = 4 \ \ \tan^{-1}(\exp(3 - \sqrt{x^2 + y^2})) \ \text{ in } \ [-7, 7] \times [-7, 7], \\ u_t(x, y, 0) = 0 \ \text{ in } \ [-7, 7] \times [-7, 7]. \end{cases}$$

We discretize the domain into Voronoi mesh with mesh size $h = 0.25$. The solutions that appear in Figure 4.5.3 are homocentric. Figure 4.5.3 shows the numerical solution includes shrinking and expanding phases.

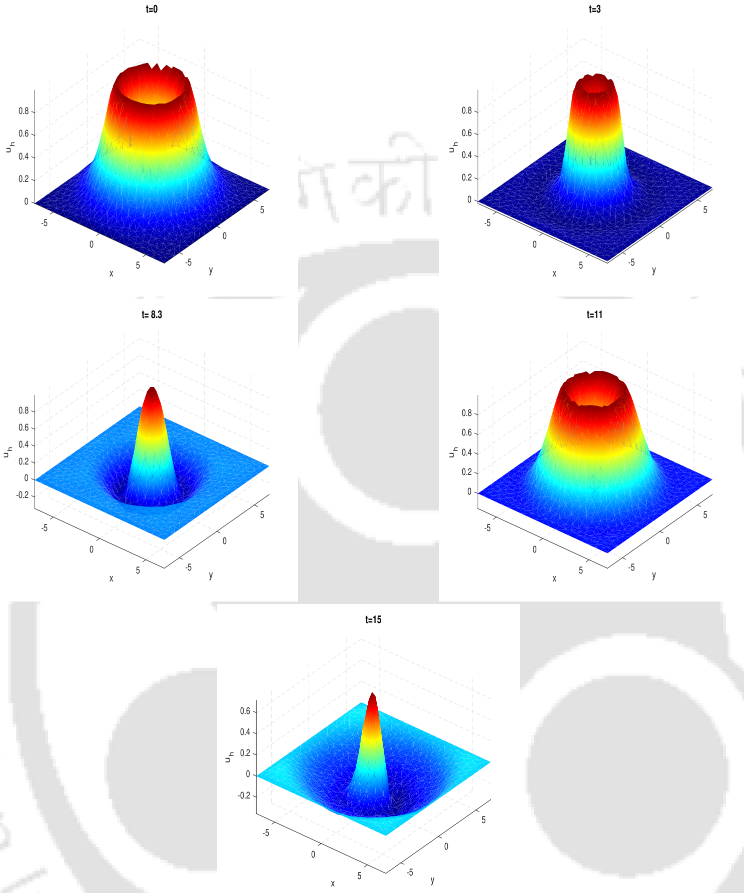

Figure 4.5.3: Numerical solutions at different time levels for Example 4.5.3.

# 5

# WG-FEMs for Analysis and Simulation of Nonlinear Wave Propagation in Homogeneous Media

In this chapter, we propose a weak Galerkin finite element method for the Westervelt's model of ultrasound waves on polygonal meshes. Specifically, we investigate the spatial discretization of Westervelt's quasi-linear, strongly damped wave equation using high-order weak Galerkin discretization. The primary challenges in the numerical analysis include managing the nonlinear terms in the model and preventing the equation from degenerating. We avoid the degeneracy of the semi-discrete Westervelt's equation by employing inverse estimates, the stability and approximation properties of the $L^2$ projection. Our convergence analysis relies on the Banach fixed-point theorem, along with a stability and convergence analysis of a linear diffusive viscous wave equation with variable coefficients for the first and second time derivatives. This approach yields optimal convergence rates in $L^2$-based spatial norms for sufficiently small data and mesh size, given an appropriate choice of initial data. Numerical experiments conducted in two-dimensional settings illustrate the theoretical convergence results.

## 5.1 Introduction

The modeling of nonlinear effects arising in the presence of high-intensity acoustic fields plays an increasingly important role in diagnostic and therapeutic medicine [77]. This chapter considers the Westervelt's equation [156], which is a fundamental mathematical model of nonlinear acoustics and addresses its numerical solution using the weak Galerkin finite element methods (WG-FEMs). The Westervelt's equation, modeling nonlinear ultrasound propagation through a homogeneous medium for the acoustic pressure $u$ in dimensionless form, can be expressed as

$$(1-2\sigma u)u_{tt} - c^2\Delta u - b\Delta u_t = 2\sigma u_t^2. \tag{5.1.1}$$

Here $c$ represents the speed of sound, $b$ is the sound diffusivity and $\sigma = \beta_a/(\rho_m c^2)$, where $\rho_m$ is the mass density and $\beta_a$ is the coefficient of nonlinearity of the medium (cf. [56, 142]). The acoustic velocity potential $\psi$ can be obtained via the following equation (cf. [7])

$$(1 - 2\kappa\psi_t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0, \tag{5.1.2}$$

where the constant $\kappa$ is given by $\kappa = \beta_a/c^2$.

Wave-type problems encompass a wide range of applications in science and engineering. Linear wave-type problems based on linearized material models are often favored for their simplicity. However, this approach is inadequate for high-frequency or high-intensity waves. In certain parts of the domain, material properties necessitate the use of the full nonlinear model, leading to nonlinear wave-type problems. Nonlinear acoustics plays a crucial role in many applications, from medical treatments like shock wave lithotripsy and diagnostic ultrasound imaging to industrial processes such as ultrasound cleaning, welding and sonochemistry [22, 52, 130, 134]. Notable analytical studies have been conducted on the Westervelt's equation [3, 88, 122], yet numerical methods are often required to further investigate these models. Despite advancements in finite element approximations for linear wave equations, studies on quasi-linear acoustic wave equations remain limited. For space discretizations of quasi-linear elastic wave equations, we refer to the work of Makridakis [120] and Ortner & Süli [132]. Makridakis [120] proved an error estimate using Banach's fixed-point theorem and inverse estimates. Ortner & Süli [132] extended these results to nonconforming discretizations using discontinuous Galerkin finite elements. Additionally, there are some results considering the full discretization of quasi-linear second order wave-type problems, see [17, 59, 64, 89]. Due to several applications of Westervelt's equation in many field of medicine and industrial applications, several numerical methods such as classical FEM, mixed FEM, discontinuous Galerkin method have been investigated widely (for instance, see Section 1.3). There are plenty of literature available on the numerical study of the Westervelt's quasi-linear wave equation (5.1.1). One may refer to [7, 119, 121, 129, 131] and reference therein. It is worthwhile to note that only the semidiscrete scheme (space discretization) has been discussed in [7, 131]. Recently error estimates of $\mathcal{O}(h^k)$ under $L^\infty(H^1)$ norm and $H^1(L^2)$ norm using piecewise polynomials of degree at least $k \geq 2$ were derived for the Westervelt's equation in inviscid media ($b = 0$) in conforming settings [82, 119]. The approximation results have been extended to an implicit fully discrete scheme in [119]). The fully discrete scheme (space-time discretization) error analysis is still unexplored for $b \neq 0$. Although, Antonietti et al. [7] implemented discontinuous Galerkin FEM for the Westervelt's equation, but they have analyzed error estimate in suitable energy norm only.

The numerical method employed in this study relies on weak Galerkin (WG) space discretization for the strongly damped Westervelt's equation, assuming sufficiently smooth and small data. Challenges in the numerical analysis lie in handling the nonlinearity in the model and in preventing the equation from degenerating. We provide an analysis of the spatial discretization of the strongly damped Westervelt's quasi-linear wave equation in the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), \left[\mathbf{P}_{k-1}(\mathcal{K})\right]^2)$. A priori error analysis and stability analysis of linearized Westervelt's equation, together with the Banach fixed point theorem, have been used to explore optimal order convergence analysis in the $L^2$ norm under suitably small data and mesh size. Finally, numerical experiments in two-dimensional settings illustrate theoretical results.

The rest of the chapter is organized as follows: In Sec. 5.2 we have described Model problem with its weak form and regularity of the weak solution are presented. We have discussed Linearized Westervelt's equation with variable coefficients, its weak Galerkin approximation along with wellposedness and *a priori* bound in Sec. 5.3 . Sec. 5.4 devoted to analyzed error of linearized Westervelt's equation in $L^2$ norm. We extended the results to quasi-linear Westervelt's equation in Sec. 5.5. Sec.5.6 is concerned to fully discrete approximation to equation (5.1.1) using backward Euler method. Also, some numerical examples are provided in Sec. 5.6 to validate theoretical findings.

## 5.2 Model Problem and Weak Galerkin Discretization

### 5.2.1 Model Problem

We study a discretization of Westervelt's wave equation for the acoustic pressure $u$

$$(1-2\sigma u)u_{tt} - c^2\Delta u - b\Delta u_t = 2\sigma u_t^2 \ \text{ in } \Omega\times(0,T],\ T<\infty, \tag{5.2.1}$$

with the following initial-boundary conditions

$$\begin{cases} u = 0 \ \text{ on } \partial\Omega\times(0,T], \\ (u,u_t) = (u_*, u^*) \ \text{ in } \Omega\times\{t=0\}. \end{cases} \tag{5.2.2}$$

Here $\Omega$ is bounded domain in $\mathbb{R}^n$ $(n=2,3)$ with boundary $\partial\Omega$ and $J=(0,T]$ is the observation time interval.

The weak form of the problem (5.2.1)-(5.2.2) is given by

$$\begin{cases} ((1-2\sigma u)u_{tt},\phi)+c^2(\nabla u,\nabla\phi)+b(\nabla u_t,\nabla\phi)-2\sigma(u_t^2,\phi)=0, \\ \qquad\qquad \text{for all} \;\; \phi\in H_0^1(\Omega) \;\; \text{a.e. in} \;\; (0,T], \\ u=u_*, \quad u_t=u^* \;\; \text{on } \Omega\times\{t=0\}. \end{cases} \tag{5.2.3}$$

The wellposedness of this problem holds for sufficiently small data, which by continuity implies that the solution $u$ of (5.2.3) is so small in appropriate norms.

**Theorem 5.2.1.** *Let* $b,\ \sigma,\ c^2>0$ *and* $T>0$ *be arbitrary. Also, let* $0<\hat{m}<\frac{1}{4\sigma}$ *and* $\hat{M}>0$ *be arbitrary. Assume that*

$$\|u_*\|^2_{H^2(\Omega)}+\|u^*\|^2_{H^2(\Omega)}\leq\tilde{\rho}_T$$

*with* $\tilde{\rho_T}$ *sufficiently small ( but depends on* $T$*). Then there exists a unique solution* $u$ *of (5.2.3) such that*

$$u\in\mathbf{D}=\Big\{u\in L^\infty(J;L^\infty(\Omega)):\|u\|_{L^\infty(L^\infty)}\leq\hat{m},\ \|u_{tt}\|_{L^2(H^1)}\leq\hat{M}, \\ \|u_t\|_{L^\infty(H^1)}\leq\hat{M} \text{ and } u(0)=u_*,\ u_t(0)=u^*\Big\}$$

*and satisfies* $\Delta u, u_{tt}, \nabla u_t\in L^\infty(J;L^2(\Omega)),\ \nabla u_{tt}\in L^2(J;L^2(\Omega))$.

**Remark 5.2.1.** *Let* $R>0$ *and* $T$ *be chosen such that*

$$\mathcal{C}_A\hat{\mathcal{C}}_1\exp{(\hat{\mathcal{C}}_2RT)}(\|u^*\|^2_{L^2(\Omega)}+\|\nabla u_*\|^2_{L^2(\Omega)})^{\frac{1}{4}}R^{\frac{1}{2}}\leq\frac{1}{4|\sigma|} \tag{5.2.4}$$

*and*

$$\mathcal{C}_1\exp{(\mathcal{C}_2|\sigma|(R+R^2+R^3)T)}(\|\nabla\Delta u_*\|^2_{L^2(\Omega)}+\|\Delta u_*\|^2_{L^2(\Omega)})\leq R^2, \tag{5.2.5}$$

*where the positive constants* $\mathcal{C}_A,\ \hat{\mathcal{C}}_1,\ \hat{\mathcal{C}}_2,\ \mathcal{C}_1$ *and* $\mathcal{C}_2$ *are as defined in Proposition 3.1 in [88]. Under the following higher regularity assumptions*

$$(u_*,u^*)\in\{v\in H^3(\Omega):\ v|_{\partial\Omega}=0,\ \ \Delta v|_{\partial\Omega}=0\}\times(H_0^1(\Omega)\cap H^2(\Omega))$$

*on the initial data, the solution* $u$ *of (5.2.3) satisfies the energy estimate*

$$\|\nabla u_{tt}\|^2_{L^2(L^2)}+\operatorname*{ess\,sup}_{t\in[0,T]}\|\Delta u_t(t)\|^2_{L^2}+\operatorname*{ess\,sup}_{t\in[0,T]}\|\nabla\Delta u(t)\|^2_{L^2}+b\|\nabla\Delta u_t\|^2_{L^2(L^2)}\leq R^2.$$

*Both smallness conditions (5.2.4) and (5.2.5) are mitigated by the fact that the nonlinearity parameter* $\sigma$ *is typically small in magnitude in ultrasonic applications. As we going forward, we shall impose additional regularity condition on the solution* $u$ *for convergence analysis.*

### 5.2.2 Weak Galerkin Discretization

For $h \in (0, h_0]$, let $\mathcal{T}_h$ be a polygonal partition of a two-dimensional domain $\Omega$ satisfying certain shape regularity assumptions (cf. [153]). Suppose $\mathcal{E}_h$ denotes the collection of all edges in $\mathcal{T}_h$ and $\mathcal{E}_h^0$ represents the set of all interior edges. The mesh parameter $h$ is defined as $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$, where $h_{\mathcal{K}}$ represents the diameter of an element $\mathcal{K} \in \mathcal{T}_h$.

Let $\mathcal{K} \in \mathcal{T}_h$ be a polygon with boundary $\partial\mathcal{K}$. For an integer $k \geq 1$, a local discrete weak functions space on $\mathcal{K}$ is denoted by $\mathcal{W}(k, k; \mathcal{K})$ and it is defined as

$$\mathcal{W}(k, k; \mathcal{K}) = \Big\{ \phi_h = \{\phi_0, \phi_b\} : \phi_0 \in \mathbf{P}_k(\mathcal{K}), \phi_b \in \mathbf{P}_k(e), e \subset \partial\mathcal{K} \Big\}.$$

Therefore, the global weak finite space $\mathcal{W}_h$ corresponding to $\mathcal{T}_h$ is constructed by patching local weak finite spaces $\mathcal{W}(k, k; \mathcal{K})$ as

$$\mathcal{W}_h = \{\phi_h = \{\phi_0, \phi_b\} : \phi|_{\mathcal{K}} \in \mathcal{W}(k, k; \mathcal{K}), [\phi_h]_e = 0, \forall e \in \mathcal{E}_h^0\},$$

where $[\phi_h]_e = [\phi_b]_e$ is the jump of $\phi_h$ across interior edge $e \in \mathcal{E}_h^0$. Recall $\mathcal{W}_h^0$, subspace of $\mathcal{W}_h$, containing all weak functions that vanishes on $\partial\Omega$; i.e.

$$\mathcal{W}_h^0 = \{\phi_h \in \mathcal{W}_h : \phi_b|_{\partial\Omega} = 0\}.$$

For each $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, the discrete weak gradient $\nabla_w \phi_h \in [\mathbf{P}_{k-1}(\mathcal{K})]^2$ is defined as follows

$$(\nabla_w \phi_h, \mathbf{q}) = -\int_{\mathcal{K}} \phi_0 (\nabla \cdot \mathbf{q}) d\mathcal{K} + \int_{\partial\mathcal{K}} \phi_b (\mathbf{q} \cdot \mathbf{n}) ds \quad \forall \mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2, \tag{5.2.6}$$

where $\mathbf{n}$ is the outword normal to the boundary $\partial\mathcal{K}$. By using divergence theorem to equation (5.2.6), we obtain

$$(\nabla_w \phi_h, \mathbf{q}) = (\nabla \phi_0, \mathbf{q})_{\mathcal{K}} + \langle \phi_b - \phi_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial\mathcal{K}} \;\; \forall \;\; \mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2. \tag{5.2.7}$$

Next, for the purpose of weak Galerkin approximation, we introduce a bilinear map $\mathcal{A}_w : \mathcal{W}_h \times \mathcal{W}_h \to \mathbb{R}$, which is being considered in this work as

$$\mathcal{A}_w(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla_w u_h, \nabla_w \phi_h)_{\mathcal{K}} + \mathbf{S}(u_h, \phi_h). \tag{5.2.8}$$

where the stabilizer or smoother $\mathbf{S}(\cdot, \cdot)$ is based on element-boundary-discrepancy as defined in Example 1.4.2 of Chapter 1. More precisely, for $u_h = \{u_0, u_b\}$, $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, the stabilizer $\mathbf{S}(\cdot, \cdot)$ (cf. [104]) is given by

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle (u_b - u_0|_{\partial\mathcal{K}}), (\phi_b - \phi_0|_{\partial\mathcal{K}}) \rangle_{\partial\mathcal{K}}. \tag{5.2.9}$$

For any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, we define a mesh-dependent semi-norm $\|\cdot\|_{1,h}$ as in Chapter 1 by

$$\|\phi_h\|_{1,h}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} \left( \|\nabla\phi_0\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^{-1}\|\phi_b - \phi_0\|_{\partial\mathcal{K}}^2 \right). \tag{5.2.10}$$

In fact, the semi-norm $\|\cdot\|_{1,h}$ is a norm in $\mathcal{W}_h^0$ and following inequalities hold (cf. [151])

$$C_*\|\phi_h\|_{1,h}^2 \leq \mathcal{A}_w(\phi_h, \phi_h) \leq C^*\|\phi_h\|_{1,h}^2 \tag{5.2.11}$$

on $\mathcal{W}_h$ for some constants $C_*$, $C^* > 0$.

Further, we have the following Poincaré-type inequality (cf. [126])

$$\|\phi_0\|^2 \leq C\mathcal{A}_w(\phi_h, \phi_h), \ \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0. \tag{5.2.12}$$

Also, define following energy norm in $\mathcal{W}_h^0$ as

$$|||\phi_h|||^2 = \mathcal{A}_w(\phi_h, \phi_h), \ \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0. \tag{5.2.13}$$

As a consequence to (5.2.11), triple-bar norm $|||\cdot|||$ is equivalent to norm $\|\cdot\|_{1,h}$ in $\mathcal{W}_h^0$. Finally, we define a mesh dependent discrete supremum norm $\|\cdot\|_{\infty,h}$ on $\mathcal{W}_h$ as below

$$\|\phi_h\|_{\infty,h} = \sup_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|_{L^\infty(\mathcal{K})} \quad \forall\ \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h. \tag{5.2.14}$$

For the purpose of error analysis, we introduce following standard $L^2$ projections on each $\mathcal{K} \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$:

- $\mathcal{Q}_k^0 : L^2(\mathcal{K}) \to \mathbf{P}_k(\mathcal{K})$,
- $\mathcal{Q}_k^b : L^2(e) \to \mathbf{P}_k(e)$
- $\mathbb{Q}_{k-1} : [L^2(\mathcal{K})]^2 \to [\mathbf{P}_{k-1}(\mathcal{K})]^2$.

By setting $\mathcal{Q}_h\phi = \{\mathcal{Q}_k^0\phi, \mathcal{Q}_k^b\phi\} \in \mathcal{W}_h$, we shall combine both the local $L^2$ projections $\mathcal{Q}_k^0$ and $\mathcal{Q}_k^b$ for any $\phi \in H^1(\Omega)$.

We borrow following important approximation results for the local $L^2$ projections from [153] (see, Lemma 4.1).

**Lemma 5.2.1.** *Let $\mathcal{T}_h$ be a polygonal partition of $\Omega$ that meets the shape regularity conditions as described in [153]. Then for $v \in H^{r+1}(\Omega)$ with $0 \leq r \leq k$, we have*

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \left( \|v - \mathcal{Q}_k^0 v\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^2\|\nabla(v - \mathcal{Q}_k^0 v)\|_{\mathcal{K}}^2 \right) \leq Ch^{2(r+1)}\|v\|_{H^{r+1}(\Omega)}^2,$$

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \left( \|\nabla v - \mathbb{Q}_{k-1}\nabla v\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^2\|\nabla(\nabla v - \mathbb{Q}_{k-1}(\nabla v))\|_{\mathcal{K}}^2 \right) \leq Ch^{2r}\|v\|_{H^{r+1}(\Omega)}^2.$$

The following identity will be used in our subsequent analysis (cf. [126])

$$\mathbb{Q}_{k-1}(\nabla\varphi) = \nabla_w\left(\mathcal{Q}_h\varphi\right) \quad \forall\, \varphi \in H^1(\mathcal{K}). \tag{5.2.15}$$

Under the mesh assumptions prescribed as in [153], the following trace and domain inverse inequalities hold on $\mathcal{T}_h$.

**Lemma 5.2.2.** (Lemma A.3, [153]) *For each $e \in \mathcal{E}_h$ and $w \in H^1(\mathcal{K})$, we have*

$$\|w\|^2_{L^2(e)} \leq C(h_{\mathcal{K}}^{-1}\|w\|^2_{L^2(\mathcal{K})} + h_{\mathcal{K}}\|\nabla w\|^2_{L^2(\mathcal{K})}). \tag{5.2.16}$$

**Lemma 5.2.3.** (Lemma A.5, [153]) *Suppose $\mathbf{K}$ be a shape regular 2-simplex in $\mathbb{R}^2$ with diameter $h_{\mathbf{K}}$. For any ball $B \subset \mathbf{K}$ which has diameter $h_B$ such that $h_B \geq \zeta_* h_{\mathbf{K}}$ for some fixed $\zeta_* > 0$. Then following inequality holds true*

$$\|\psi\|_{\mathbf{K}} \leq C(\zeta_*, k)\|\psi\|_B$$

*for any polynomial $\psi$ of degree not more than $k$, $C$ depends on $\zeta_*$ and $k$.*

**Remark 5.2.2.** *Under the conditions of Lemma 5.2.3 and relying on the $L^\infty$ inverse estimates on shape regular simplex $S(\mathcal{K})$ containing $\mathcal{K} \in \mathcal{T}_h$, we have (see, equation (3.8) in [66], Lemma 1 in [7])*

$$\|\psi\|_{L^\infty(\mathcal{K})} \leq Ch_{\mathcal{K}}^{-1}\|\psi\|_{L^2(\mathcal{K})} \tag{5.2.17}$$

*for any polynomial $\psi$ of degree not more than $k$. In fact, under the slightly more restrictive mesh assumption*

$$\max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}} \leq C \min_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}, \tag{5.2.18}$$

*one might substitute $h_{\mathcal{K}}$ by the mesh size $h$. More precisely, for quasi-uniform partitions; i.e., partitions for which $h_{\mathcal{K}}/h$ is bounded from below and above uniformly in $\{\mathcal{K} \in \mathcal{T}_h\}$, we have*

$$\|\psi\|_{L^\infty(\mathcal{K})} \leq Ch^{-1}\|\psi\|_{L^2(\mathcal{K})} \tag{5.2.19}$$

*for any polynomial $\psi$ of degree not more than $k$.*

**Corollary 5.2.1.** *Let $\mathcal{T}_h$ be quasi-uniform polygonal partition of $\Omega$ satisfying all the shape regularity conditions A1-A4 in [153]. Then for $u \in L^\infty(J; L^\infty(\mathcal{K}))$, we have*

$$\|\mathcal{Q}_k^0 u\|_{L^\infty(\mathcal{K})} \leq C\|u\|_{L^\infty(\mathcal{K})} \quad \forall\mathcal{K} \in \mathcal{T}_h.$$

*Proof.* For a quasi-uniform polygonal partition $\mathcal{T}_h$ of $\Omega$ and for a shape regular simplex $S(\mathcal{K})$ containing $\mathcal{K} \in \mathcal{T}_h$, and $u \in L^\infty(J; L^\infty(\Omega))$, we have

$$\begin{aligned}
\|\mathcal{Q}_k^0 u\|_{L^\infty(\mathcal{K})} &\le Ch^{-1}\|\mathcal{Q}_k^0 u\|_{L^2(S(\mathcal{K}))} \\
&\le Ch^{-1}\|\mathcal{Q}_k^0 u\|_{L^2(\mathcal{K})} \\
&\le Ch^{-1}\|u\|_{L^2(\mathcal{K})} \\
&\le Ch^{-1}\|u\|_{L^\infty(\mathcal{K})}|\mathcal{K}|^{\frac{1}{2}} \\
&\le Ch^{-1}\|u\|_{L^\infty(\mathcal{K})}|S(\mathcal{K})|^{\frac{1}{2}} \\
&\le Ch^{-1}\|u\|_{L^\infty(\mathcal{K})}h_{S(\mathcal{K})}.
\end{aligned}$$

Since, $\mathcal{T}_h$ satisfies the shape regularity conditions A1-A4 as in [153], there exists $\gamma_* > 0$ such that $h_{S(\mathcal{K})} \le \gamma_* h_{\mathcal{K}}$ leading to the desired estimate. □

**Remark 5.2.3.** *The following result is related to the stability of the projection operator* $\mathcal{Q}_h$*. For any* $v \in H^1(\Omega)$*, we have*

$$\begin{aligned}
|||\mathcal{Q}_h v|||^2 &\le C \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla \mathcal{Q}_0^k v\|^2_{L^2(\mathcal{K})} + h_{\mathcal{K}}^{-1}\|\mathcal{Q}_0^k v - \mathcal{Q}_b^k v\|^2_{L^2(\partial\mathcal{K})}) \\
&\le C \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla \mathcal{Q}_0^k v\|^2_{L^2(\mathcal{K})} + h_{\mathcal{K}}^{-1}\|\mathcal{Q}_0^k v - v\|^2_{L^2(\partial\mathcal{K})}) \\
&\le C \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla \mathcal{Q}_0^k v\|^2_{L^2(\mathcal{K})} + h_{\mathcal{K}}^{-2}\|\mathcal{Q}_0^k v - v\|^2_{L^2(\mathcal{K})} + \|\nabla(\mathcal{Q}_0^k v - v)\|^2_{L^2(\mathcal{K})}) \\
&\le C\|v\|^2_{H^1(\Omega)}. \qquad (5.2.20)
\end{aligned}$$

*Here we have used the fact* $\|\mathcal{Q}_0^k v - \mathcal{Q}_b^k v\|_{L^2(\partial\mathcal{K})} \le \|\mathcal{Q}_0^k v - v\|_{L^2(\partial\mathcal{K})}$*, Lemma 5.2.2 and Lemma 5.2.1.*

We end this section with the following important Lemma.

**Lemma 5.2.4.** *For any* $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$ *and given integer* $2 < p \le 6$*, there exists a constant* $\mathcal{C}(p) > 0$*, depending on p, such that*

$$\|\phi_h\|_{L^p(\Omega)} \le \mathcal{C}(p)|||\phi_h|||. \qquad (5.2.21)$$

*Proof.* For $p = 2m$ $(m > 1)$, we obtain

$$\|\phi_h\|^p_{L^p(\Omega)} = \|\phi_0\|^p_{L^p(\Omega)} = \sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^p_{L^p(\mathcal{K})} \le \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(\mathcal{K})}\right)^m. \qquad (5.2.22)$$

Again, for a shape regular simplex $S(\mathcal{K})$ containing $\mathcal{K}$, as described in **A4** of [153], we obtain

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(\mathcal{K})} \leq \sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(S(\mathcal{K}))} \leq \sum_{\mathcal{K}\in\mathcal{T}_h} C_{S(\mathcal{K})} p \|\phi_0\|^2_{H^1(S(\mathcal{K}))}. \tag{5.2.23}$$

In the last inequality, we have used the Sobolev embedding inequality for two dimensions (cf. Ren & Wei [138], Theorem 4.12 in [5]).

Since, $\mathcal{T}_h$ is quasi-uniform, there exists $\mu^* > 0$ ensuring $\mu^* h \leq h_{\mathcal{K}}$ for all $\mathcal{K} \in \mathcal{T}_h$. Consequently, $\{S(\mathcal{K}) : \mathcal{K} \in \mathcal{T}_h\}$ is quasi-uniform and non-degenerate. Employing similar arguments as in Proposition 4.4.11 of [23], we have:

$$C_{S(\mathcal{K})} \leq \zeta^*(a_{S(\mathcal{K})}) C(\mu^*),$$

where $A\mathbf{x} = a_{S(\mathcal{K})}\mathbf{x} + b_{S(\mathcal{K})}$ represents the affine transformation and $\zeta^*$ is a positive function dependent continuously on $a_{S(\mathcal{K})} \in GL(\mathbb{R}^2)$. Since, $\{S(\mathcal{K}) : \mathcal{K} \in \mathcal{T}_h\}$ is non-degenerate, $\{a_{S(\mathcal{K})} : \mathcal{K} \in \mathcal{T}_h\}$ forms a compact set of $GL(\mathbb{R}^2)$.

Now using domain inverse inequality Lemma 5.2.3, we have

$$\begin{aligned}\|\phi_0\|^2_{H^1(S(\mathcal{K}))} &= \|\nabla\phi_0\|^2_{L^2(S(\mathcal{K}))} + \|\phi_0\|^2_{L^2(S(\mathcal{K}))} \\ &\leq C\left(\|\nabla\phi_0\|^2_{L^2(B)} + \|\phi_0\|^2_{L^2(B)}\right) \leq C\left(\|\nabla\phi_0\|^2_{L^2(\mathcal{K})} + \|\phi_0\|^2_{L^2(\mathcal{K})}\right),\end{aligned}$$

where $B$ is a ball inside of $\mathcal{K}$ with diameter proportional to $h_{\mathcal{K}}$. Using above inequality in (5.2.23) and estimates (5.2.11)-(5.2.12), we obtain

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(\mathcal{K})} \leq Cp \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla\phi_0\|^2_{L^2(\mathcal{K})} + \|\phi_0\|^2_{L^2(\mathcal{K})}) \leq Cp |||\phi_h|||^2.$$

Finally,

$$\|\phi_0\|_{L^p(\Omega)} \leq \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(\mathcal{K})}\right)^{\frac{m}{p}} \leq \mathcal{C}(p) |||\phi_h|||.$$

For $p = 2m+1$ $(m \geq 1)$, we have

$$\begin{aligned}\|\phi_0\|^p_{L^p(\Omega)} &= \sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^{2m}_{L^p(\mathcal{K})} \|\phi_0\|_{L^p(\mathcal{K})} \leq \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^{4m}_{L^p(\mathcal{K})}\right)^{\frac{1}{2}} \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(\mathcal{K})}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(\mathcal{K})}\right)^{m} \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\phi_0\|^2_{L^p(\mathcal{K})}\right)^{\frac{1}{2}} \leq (Cp)^{m+\frac{1}{2}} |||\phi_h|||^{2(m+\frac{1}{2})}.\end{aligned}$$

This completes the proof of Lemma 5.2.4. □

## 5.3 WG-FEMs for the Linearized Westervelt's Equation

We first consider a general linear second order wave-type equation with variable coefficients which can be interpreted as a linearization of the Westervelt's equation. We analyze the following initial-boundary value problem for a non-degenerate equation

$$\begin{cases} \alpha(\mathbf{x},t)u_{tt} - c^2\Delta u - b\Delta u_t + \beta(\mathbf{x},t)u_t = f(\mathbf{x},t) \quad \text{in } \Omega\times(0,T], \\ u = 0 \quad \text{on } \partial\Omega\times(0,T]. \\ u = u_*, \quad u_t = u^* \quad \text{on } \Omega\times\{t=0\}. \end{cases} \tag{5.3.1}$$

We assume that there exists $\alpha_*, \alpha^* > 0$ such that

$$0 < \alpha_* \le \alpha(\mathbf{x},t) \le \alpha^* \ \text{ a.e. } \ \text{in } \ \Omega\times[0,T]. \tag{5.3.2}$$

Existence and uniqueness of a solution to the problem (5.3.1) is proved in [89] (Proposition 7.2) and [88] (Proposition 3.1). For suitable coefficients $\alpha$, $\beta$, initial data $(u_*, u^*)$ and forcing function $f$, the problem (5.3.1) has a unique solution such that $(u, u_t) \in C([0,T]; H^j(\Omega)\times H^{j-1}(\Omega))$, where $j \in \{2,4\}$. If $b > 0$, then additionally

$$u \in \mathcal{W}^{1,\infty}(0,T; H^3_\diamond(\Omega)) \cap H^2(0,T; H^2(\Omega)),$$

where $H^3_\diamond(\Omega) = \{u \in H^3(\Omega) : \ u|_{\partial\Omega} = 0 \ \ \& \ \ \Delta u|_{\partial\Omega} = 0\}$.

Our aim is to provide convergence analysis of the linearization (5.3.1) and this will allow us to study the Westervelt's equation with quadratic nonlinearities. The case of a damped wave equation and with $\alpha = 1, \ \beta = 1$ is analyzed in [96]. For $\alpha \in \mathcal{X} = L^\infty(0,T; H^3(\Omega)) \cap \mathcal{W}^{1,\infty}(0,T; H^2(\Omega))$ and $\beta \in L^\infty(0,T; H^2(\Omega))$ (cf. [88]), let

$$\alpha_h(t) = \{\alpha_0(t), \alpha_b(t)\} \ \ \& \ \ \beta_h(t) = \{\beta_0(t), \beta_b(t)\}$$

be suitable approximations of $\alpha(t)$ and $\beta(t)$, respectively. We assume $\alpha_h(t), \ \beta_h(t) \in \mathcal{W}_h^0$ for a.e. $t \in [0,T]$.

Now we are in a position to define the continuous time weak Galerkin finite element approximation to (5.3.1), which is defined as follows: Find $u_h = \{u_0, u_b\} \in H^2(0,T; \mathcal{W}_h^0)$ satisfying

$$(\alpha_h u_{htt}, \phi_h) + c^2\mathcal{A}_w(u_h, \phi_h) + b\mathcal{A}_w(u_{ht}, \phi_h) + (\beta_h u_{ht}, \phi_h) = (f_h, \phi_h) \tag{5.3.3}$$

for all $\phi_h(t) = \{\phi_0(t), \phi_b(t)\} \in \mathcal{W}_h^0$, a.e. in time and

$$u_h = u_{h,0} \qquad u_{ht} = u_{h,1} \quad \text{in } \Omega\times\{t=0\}. \tag{5.3.4}$$

Here $u_{h,0}, \ u_{h,1}$ and $f_h$ are suitable approximations in $\mathcal{W}_h$ of $u_*, \ u^*$ and $f$, respectively.

**Assumption 5.3.1.** *We assume that the approximate coefficients* $\alpha_h = \{\alpha_0, \alpha_b\}, \beta_h = \{\beta_0, \beta_b\}$ *and the source term* $f_h$ *satisfy the following conditions*

$$\begin{cases} \operatorname{ess\,sup}_{t\in[0,T]} \|\alpha_h(t)\|_{\infty,h} < \infty \ \ \textit{and} \ \ \exists\ \alpha_* : \alpha_h(\mathbf{x},t) \geq \alpha_* > 0 \ \ \textit{a.e. in} \ \ \Omega \times [0,T], \\ \alpha_h, \beta_h \in L^\infty(J; L^3(\Omega)) \ \ \textit{and} \ \ f_h \in L^2(J; L^2(\Omega)). \end{cases}$$

**Remark 5.3.1.** *Under above Assumption 5.3.1 and Lemma 5.2.4, together with standard Hölder's inequality, for any* $w_h = \{w_0, w_b\}$, $v_h = \{v_0, v_b\} \in \mathcal{W}_h^0$, *we obtain*

$$|(\alpha_h w_h, v_h)| \leq \|\alpha_h\|_{\infty,h} \|w_h\| \|v_h\|, \ \ |(\alpha_h w_h, v_h)| \leq \mathcal{C}(6) \|\alpha_h\|_{L^3(\Omega)} |||w_h||| \|v_h\|.$$

*Similarity, we obtain*

$$|(\beta_h w_h, v_h)| \leq \|\beta_h\|_{L^3(\Omega)} \|w_h\|_{L^3(\Omega)} \|v_h\|_{L^3(\Omega)} \leq (\mathcal{C}(3))^2 \|\beta_h\|_{L^3(\Omega)} |||w_h||| |||v_h|||.$$

The following result deals with the existence and uniqueness of the WG solution $u_h$. The basic technique is borrowed from [131].

**Theorem 5.3.1.** *Let* $c^2, b > 0$ *and let Assumption 5.3.1 holds. For each* $h \in (0, h_0]$, *there exists a unique discrete weak function* $u_h \in H^2(0,T; \mathcal{W}_h^0)$ *satisfying (5.3.3)-(5.3.4).*

*Proof.* For a given $\mathcal{K} \in \mathcal{T}_h$, let $\{\phi_{0,i} : i = 1 \cdots N_0\}$ be the set of basis functions for $\mathcal{P}_k(\mathcal{K})$ and $\{\phi_{b,j} : j = 1 \cdots N_b\}$ be the set of basis functions for $\sum_{e\in\partial\mathcal{K}\cap\mathcal{E}_h} \mathcal{P}_k(e)$. Then for any $u_h = \{u_0, u_b\} \in \mathcal{W}_h^0$ can be written as

$$u_h|_{\mathcal{K}} = \left\{ \sum_{i=1}^{N_0} d_{0,i}(t)\phi_{0,i}, \sum_{j=1}^{N_b} d_{b,j}(t)\phi_{b,j} \right\},$$

where $d_{0,i}, d_{b,j} : [0,T] \to \mathbb{R}$ are the coefficient functions for $1 \leq i \leq N_0$ and $1 \leq j \leq N_b$.

Now for $1 \leq i \leq N_0 + N_b$, we write

$$\phi_{i,h} = \{\bar{\phi}_{0,i}, \bar{\phi}_{b,i}\},$$

where $\bar{\phi}_{0,i} = \phi_{0,i}$ for $1 \leq i \leq N_0$, $\bar{\phi}_{0,i} = 0$ for $N_0 + 1 \leq i \leq N_0 + N_b$ and $\bar{\phi}_{b,i} = 0$ for $1 \leq i \leq N_0$, $\bar{\phi}_{b,i} = \phi_{b,i-N_0}$ for $N_0 + 1 \leq i \leq N_0 + N_b$. Similarly, we define $d_{i,h} = d_{0,i}$ for $1 \leq i \leq N_0$ and $d_{i,h} = d_{b,i-N_0}$ for $N_0 + 1 \leq i \leq N_0 + N_b$. Therefore,

$$u_h|_{\mathcal{K}} = \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\phi_{i,h} = \left\{ \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\bar{\phi}_{0,i}, \sum_{j=1}^{N_0+N_b} d_{j,h}(t)\bar{\phi}_{b,j} \right\}, \ \ \mathcal{K} \in \mathcal{T}_h.$$

Now setting $\phi_h = \phi_{j,h}$ for $j = 1, 2, \cdots, N_0 + N_b$ in (5.3.3), we obtain

$$\left(\alpha_h(t) \sum_{i=1}^{N_0+N_b} d''_{i,h}(t)\bar{\phi}_{0,i}, \bar{\phi}_{0,j}\right) + c^2\mathcal{A}_w\left(\sum_{i=1}^{N_0+N_b} d_{i,h}(t)\phi_{i,h}, \phi_{j,h}\right)$$
$$+ b\mathcal{A}_w\left(\sum_{i=1}^{N_0+N_b} d'_{i,h}(t)\phi_{i,h}, \phi_{j,h}\right) + \left(\beta_h(t) \sum_{i=1}^{N_0+N_b} d'_{i,h}(t)\bar{\phi}_{0,i}, \bar{\phi}_{0,j}\right) = (f_h, \bar{\phi}_{0,j}).$$

Equivalently, in matrix form, we need to evaluate unknown vector $D_h(t) = \begin{bmatrix} D_{0h} \\ D_{bh} \end{bmatrix}$, with $D_{0h} = [d_{1,h}\ d_{2,h}\ \cdots\ d_{N_0,h}]^T$ & $D_{bh} = [d_{N_0+1,h}\ d_{N_0+2,h}\ \cdots\ d_{N_0+N_b,h}]^T$, satisfying the following IVP

$$\begin{cases} M_{h\mathcal{K}}D''_h(t) + A^1_{h\mathcal{K}}D_h(t) + A^2_{h\mathcal{K}}D'_h(t) + K_{h\mathcal{K}}D'_h(t) = F_{h\mathcal{K}}(t), \\ D_h(0) = [d_{1,h,0}\ d_{2,h,0} \cdots d_{N_0+N_b,h,0}]^T, \\ D_{ht}(0) = [d_{1,h,1}\ d_{2,h,1} \cdots d_{N_0+N_b,h,1}]^T, \end{cases} \tag{5.3.5}$$

where $D_h(0)$ and $D_{ht}(0)$ are the coefficient vectors of $u_{h,0}$ and $u_{h,1}$, when expressed as the linear combinations of the basis functions, respectively. Further, matrices $M_{h\mathcal{K}}$ and $K_{h\mathcal{K}}$ are given by

$$M_{h\mathcal{K}} = \begin{bmatrix} \mathcal{M}_{00} & \mathcal{M}_{0b} \\ \mathcal{M}_{b0} & \mathcal{M}_{bb} \end{bmatrix} \ \&\ K_{h\mathcal{K}} = \begin{bmatrix} \mathcal{K}_{00} & \mathcal{K}_{0b} \\ \mathcal{K}_{b0} & \mathcal{K}_{bb} \end{bmatrix},$$

where

$$\begin{aligned} \mathcal{M}_{00} &= [(\alpha_h\bar{\phi}_{0,j}, \bar{\phi}_{0,i})]_{N_0\times N_0},\ \mathcal{M}_{0b} = [0]_{N_0\times N_b},\ \mathcal{M}_{b0} = [0]_{N_b\times N_0},\ \mathcal{M}_{bb} = [0]_{N_b\times N_b}, \\ \mathcal{K}_{00} &= [(\beta_h\bar{\phi}_{0,j}, \bar{\phi}_{0,i})]_{N_0\times N_0},\ \mathcal{K}_{0b} = [0]_{N_0\times N_b},\ \mathcal{K}_{b0} = [0]_{N_b\times N_0},\ \mathcal{K}_{bb} = [0]_{N_b\times N_b}. \end{aligned}$$

Coefficient matrices $A^i_{h\mathcal{K}}$ $(i = 1, 2)$ are given by

$$\begin{aligned} A^1_{h\mathcal{K}} &= [A_{i,j}],\ \ A_{i,j} = c^2\mathcal{A}_w(\phi_{j,h}, \phi_{i,h}) \text{ for } 1 \le i, j \le N_0 + N_b, \\ A^2_{h\mathcal{K}} &= [B_{i,j}],\ \ B_{i,j} = b\mathcal{A}_w(\phi_{j,h}, \phi_{i,h}) \text{ for } 1 \le i, j \le N_0 + N_b. \end{aligned}$$

In fact, matrices $A^i_{h\mathcal{K}}$ $(i = 1, 2)$ can be expressed as follows

$$A^1_{h\mathcal{K}} = \begin{bmatrix} A_{00} & A_{0b} \\ A_{b0} & A_{bb} \end{bmatrix} \ \&\ A^2_{h\mathcal{K}} = \begin{bmatrix} B_{00} & B_{0b} \\ B_{b0} & B_{bb} \end{bmatrix},$$

where block matrices with subscripts $00, 0b, b0$ and $bb$ are of orders $N_0 \times N_0$, $N_0 \times N_b$, $N_b \times N_0$ and $N_b \times N_b$, respectively. The source matrix $F_{h\mathcal{K}}$ is given by

$$F_{h\mathcal{K}} = \begin{bmatrix} F_{0h} \\ F_{bh} \end{bmatrix} \quad \text{with} \quad F_{0h} = [(f_h, \bar{\phi}_{0,j})]_{N_0 \times 1} \;\; \& \quad F_{bh} = [0]_{N_b \times 1}.$$

Due to Remark 5.3.1, note that the matrices and the right-hand-side vector are all well-defined a.e. in time. Indeed, for any $\mathbf{v} = [v_1 \; v_2 \cdots v_{N_0}]^T \in \mathbb{R}^{N_0} \setminus \{0\}$ and $\mathbf{w} = [w_1 \; w_2 \cdots w_{N_b}]^T \in \mathbb{R}^{N_b} \setminus \{0\}$, we have

$$\mathbf{v}^T \mathcal{M}_{00} \mathbf{v} \geq \alpha_* \int_{\mathcal{K}} \left| \sum_{i=1}^{N_0} v_i \bar{\phi}_{0,i} \right|^2 dx = \alpha_* \left\| \sum_{i=1}^{N_0} v_i \bar{\phi}_{0,i}(t) \right\|^2_{L^2(\mathcal{K})} > 0,$$

$$\mathbf{w}^T B_{bb} \mathbf{w} = b\mathcal{A}_w \left( \sum_{j=1}^{N_b} w_j \phi_{N_0+j,h}, \sum_{j=1}^{N_b} w_j \phi_{N_0+j,h} \right) > 0.$$

Therefore, $\mathcal{M}_{00}$ and $B_{bb}$ are both positive definite, so are invertible. Hence, the system (6.3.4) splits in two systems

$$\begin{cases} D''_{0h}(t) + \mathcal{M}_{00}^{-1} A_{00} D_{0h}(t) + \mathcal{M}_{00}^{-1} A_{0b} D_{bh}(t) + \mathcal{M}_{00}^{-1} B_{00} D'_{0h}(t) + \mathcal{M}_{00}^{-1} B_{0b} D'_{bh}(t) \\ \qquad\qquad + \mathcal{M}_{00}^{-1} \mathcal{K}_{00} D'_{0h}(t) = \mathcal{M}_{00}^{-1} F_{0h}(t), \\ D_{0h}(0) = [d_{1,h,0} \; d_{2,h,0} \cdots d_{N_0,h,0}]^T, \;\; D_{0ht}(0) = [d_{1,h,1} \; d_{2,h,1} \cdots d_{N_0,h,1}]^T \end{cases} \tag{5.3.6}$$

and

$$\begin{cases} D'_{bh}(t) + B_{bb}^{-1} A_{bb} D_{bh}(t) = -B_{bb}^{-1} B_{b0} D'_{0h}(t) - B_{bb}^{-1} A_{b0} D_{0h}(t), \\ D_{bh}(0) = [d_{N_0+1,h,0} \; d_{N_0+2,h,0} \cdots d_{N_0+N_b,h,0}]^T, \\ D_{bht}(0) = [d_{N_0+1,h,1} \; d_{N_0+2,h,1} \cdots d_{N_0+N_b,h,1}]^T. \end{cases} \tag{5.3.7}$$

Again, differentiating equation (5.3.7) with respect to time variable once, we obtain

$$\begin{cases} B_{bb}^{-1} B_{b0} D''_{0h}(t) + D''_{bh}(t) + B_{bb}^{-1} A_{b0} D'_{0h}(t) + B_{bb}^{-1} A_{bb} D'_{bh}(t) = \mathbf{0}, \\ D_{bh}(0) = [d_{N_0+1,h,0} \; d_{N_0+2,h,0} \cdots d_{N_0+N_b,h,0}]^T, \\ D_{bht}(0) = [d_{N_0+1,h,1} \; d_{N_0+2,h,1} \cdots d_{N_0+N_b,h,1}]^T. \end{cases} \tag{5.3.8}$$

Finally, we combine (5.3.6) and (5.3.8) to have

$$\begin{cases} \mathbf{M}_{h\mathcal{K}} D_h''(t) + \mathbf{A}_{h\mathcal{K}} D_h'(t) + \mathbf{G}_{h\mathcal{K}} D_h(t) = \mathbf{F}_{h\mathcal{K}}(t), \\ D_h(0) = [d_{1,h,0}\ d_{2,h,0} \cdots d_{N_0+N_b,h,0}]^T, \\ D_{ht}(0) = [d_{1,h,1}\ d_{2,h,1} \cdots d_{N_0+N_b,h,1}]^T, \end{cases} \tag{5.3.9}$$

where

$$\mathbf{M}_{h\mathcal{K}} = \begin{bmatrix} I & \mathbf{0} \\ B_{bb}^{-1} B_{b0} & I \end{bmatrix}, \quad \mathbf{A}_{h\mathcal{K}} = \begin{bmatrix} \mathcal{M}_{00}^{-1} B_{00} + \mathcal{M}_{00}^{-1} \mathcal{K}_{00} & \mathcal{M}_{00}^{-1} B_{0b} \\ B_{bb}^{-1} A_{b0} & B_{bb}^{-1} A_{bb} \end{bmatrix},$$

$$\mathbf{G}_{h\mathcal{K}} = \begin{bmatrix} \mathcal{M}_{00}^{-1} A_{00} & \mathcal{M}_{00}^{-1} A_{0b} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{F}_{h\mathcal{K}} = \begin{bmatrix} \mathcal{M}_{00}^{-1} F_{0h} \\ \mathbf{O} \end{bmatrix}.$$

Clearly, $\mathbf{M}_{h\mathcal{K}}$ is invertible. Hence, the existence of the solution $u_h \in H^2(0,T;\mathcal{W}_h^0)$ follows from the standard ODE theory. $\square$

**Remark 5.3.2.** *For $u_h \in H^2(0,T;\mathcal{W}_h^0)$, from (1.2.4), we obtain*

$$\sup_{t\in[0,T]} \|u_{ht}(t)\|_{1,h} \leq C\|u_h\|_{H^2(0,T;\mathcal{W}_h^0)}. \tag{5.3.10}$$

*Rigorous a priori bounds of $u_h$ on $[0,T]$ in the appropriate norms are presented in the next result.*

**Lemma 5.3.1.** *Let $u_h$ be the solution of the problem (5.3.3)-(5.3.4) and let Assumption 5.3.1 holds. Then for $0 < h < h_0$, we have*

$$\begin{aligned} &\|u_{htt}\|_{L^2(L^2)}^2 + \operatorname*{ess\,sup}_{t\in[0,T]} |||u_h(t)|||^2 + \operatorname*{ess\,sup}_{t\in[0,T]} |||u_{ht}(t)|||^2 + \int_0^T |||u_{ht}|||^2 ds \\ &\leq \tilde{C}(\alpha_h, \beta_h, T)\Big( |||u_{h,0}|||^2 + |||u_{h,1}|||^2 + \|f_h\|_{L^2(L^2)}^2 \Big), \end{aligned} \tag{5.3.11}$$

*where $\tilde{C}$ is given by*

$$\tilde{C}(\alpha_h, \beta_h, T) = C_3 \exp\Big( C_4\big(\|\alpha_h\|_{L^\infty(L^3)}^2 + \|\beta_h\|_{L^\infty(L^3)}^2 + 1\big)T \Big), \tag{5.3.12}$$

*for some positive constants $C_3$ and $C_4$.*

*Proof.* For suitable $\mu > 0$, we set $\phi_h = \mu u_{ht}$ in (5.3.3) and then integrating from 0 to $t \in (0,T]$, we obtain

$$\begin{aligned} &\mu \frac{c^2}{2} \mathcal{A}_w(u_h(t), u_h(t)) + \mu b \int_0^t \mathcal{A}_w(u_{ht}, u_{ht}) ds \\ &= \mu \frac{c^2}{2} \mathcal{A}_w(u_h(0), u_h(0)) - \int_0^t (\beta_h u_{ht}, \mu u_{ht}) ds - \int_0^t (\alpha_h u_{htt}, \mu u_{ht}) ds + \int_0^t (f_h, \mu u_{ht}) ds. \end{aligned}$$

Therefore, using Remark 5.3.1 and Poincaré type inequality, we have

$$
\begin{aligned}
&\mu\frac{c^2}{2}\mathcal{A}_w(u_h(t),u_h(t))+\mu b\int_0^t \mathcal{A}_w(u_{ht},u_{ht})ds\\
&\le \mu\frac{c^2}{2}|||u_h(0)|||^2+C\mu\int_0^t \|\beta_h\|_{L^3(\Omega)}|||u_{ht}|||^2ds\\
&\quad + C\mu\int_0^t \|\alpha_h\|_{L^3(\Omega)}|||u_{ht}|||\|u_{htt}\|ds + C\mu\int_0^t \|f_h\||||u_{ht}|||ds.
\end{aligned}
$$

Applying the Young's inequality with suitable $\epsilon > 0$, we obtain

$$
\begin{aligned}
&\mu\frac{c^2}{2}|||u_h(t)|||^2+\mu b\int_0^t |||u_{ht}(t)|||^2ds\\
&\le \mu\frac{c^2}{2}|||u_{h,0}|||^2+\epsilon\int_0^t \|u_{htt}\|^2ds + C\frac{\mu^2}{4\epsilon}\int_0^t \|f_h\|^2ds\\
&\quad + \left(C\frac{\mu^2}{4\epsilon}\left(\|\alpha_h\|^2_{L^\infty(L^3)}+\|\beta_h\|^2_{L^\infty(L^3)}\right)+2\epsilon\right)\int_0^t |||u_{ht}|||^2ds. \qquad (5.3.13)
\end{aligned}
$$

We now turn to estimate the term $\|u_{htt}\|_{L^2(L^2)}$ appears in the right hand side of (5.3.13). Setting $\phi_h = u_{htt}$ in (5.3.3), we have

$$
\begin{aligned}
&(\alpha_h u_{htt},u_{htt})+c^2\left(\frac{d}{dt}\mathcal{A}_w(u_h,u_{ht})-|||u_{ht}|||^2\right)+\frac{b}{2}\frac{d}{dt}\mathcal{A}_w(u_{ht},u_{ht})+(\beta_h u_{ht},u_{htt})\\
&=(f_h,u_{htt}).
\end{aligned}
$$

Next, integrating from 0 to $t \in (0,T]$ and using the fact $\alpha_h(\mathbf{x},t) \ge \alpha_* > 0$ a.e. in $\Omega\times[0,T]$, and Remark 5.3.1 we have

$$
\begin{aligned}
&\alpha_*\int_0^t \|u_{htt}\|^2ds+\frac{b}{2}|||u_{ht}(t)|||^2\\
&\le c^2|||u_h(0)||||||u_{ht}(0)|||+\frac{b}{2}|||u_{ht}(0)|||^2+c^2|||u_h(t)||||||u_{ht}(t)|||\\
&\quad + c^2\int_0^t |||u_{ht}|||^2ds+\int_0^t \|f_h\|\|u_{htt}\|ds + C\int_0^t \|\beta_h\|_{L^3(\Omega)}|||u_{ht}|||\|u_{htt}\|ds.
\end{aligned}
$$

Again, for suitable $\epsilon > 0$ (same $\epsilon$ as in (5.3.13)) and $\delta > 0$, we apply Young's inequality to obtain

$$
\begin{aligned}
&(\alpha_*-2\epsilon)\int_0^t \|u_{htt}\|^2ds+\left(\frac{b}{2}-\delta\right)|||u_{ht}(t)|||^2\\
&\le \frac{c^2}{2}|||u_h(0)|||^2+\frac{c^2+b}{2}|||u_{ht}(0)|||^2+\frac{c^4}{4\delta}|||u_h(t)|||^2+\frac{1}{4\epsilon}\int_0^t \|f_h\|^2ds\\
&\quad + \left(\frac{C}{4\epsilon}\|\beta_h\|^2_{L^\infty(L^3)}+c^2\right)\int_0^t |||u_{ht}|||^2ds.
\end{aligned}
$$

Setting $\delta = \frac{b}{4}$ and applying Gronwall's inequality, we have

$$\begin{aligned}
&(\alpha_* - 2\epsilon)\int_0^t \|u_{htt}\|^2 ds + \frac{b}{4}|||u_{ht}(t)|||^2 \\
&\le \frac{c^2}{2}|||u_{h,0}|||^2 + \frac{c^2+b}{2}|||u_{h,1}|||^2 + \frac{c^4}{b}|||u_h(t)|||^2 + \frac{1}{4\epsilon}\int_0^t \|f_h\|^2 ds \\
&\quad + \left(\frac{C}{4\epsilon}\|\beta_h\|^2_{L^\infty(L^3)} + c^2\right)\int_0^t |||u_{ht}|||^2 ds. \qquad (5.3.14)
\end{aligned}$$

Adding inequalities (5.3.13) and (5.3.14), we obtain

$$\begin{aligned}
&(\alpha_* - 3\epsilon)\int_0^t \|u_{htt}\|^2 ds + \left(\mu\frac{c^2}{2} - \frac{c^4}{b}\right)|||u_h(t)|||^2 + \frac{b}{4}|||u_{ht}(t)|||^2 + \mu b\int_0^t |||u_{ht}|||^2 ds \\
&\le (\mu+1)\frac{c^2}{2}|||u_{h,0}|||^2 + \frac{c^2+b}{2}|||u_{h,1}|||^2 + \frac{C\mu^2+1}{4\epsilon}\int_0^t \|f_h\|^2 ds \\
&\quad + \left(2\epsilon + c^2 + \frac{C}{4\epsilon}\left(\|\beta_h\|^2_{L^\infty(L^3)} + \mu^2\left(\|\alpha_h\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)}\right)\right)\right)\int_0^t |||u_{ht}|||^2.
\end{aligned}$$

Finally, we select $\mu = \frac{4c^2}{b}$, $\epsilon = \frac{\alpha_*}{6}$ and then applying Gronwall's inequality, and taking essential supremum over $[0,T]$, we have

$$\begin{aligned}
&\|u_{htt}\|^2_{L^2(L^2)} + \operatorname*{ess\,sup}_{t\in[0,T]} |||u_h(t)|||^2 + \operatorname*{ess\,sup}_{t\in[0,T]} |||u_{ht}(t)|||^2 + \int_0^T |||u_{ht}|||^2 ds \\
&\le C_1 \exp\left(C_2\left(\|\alpha_h\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + 1\right)T\right) \times \left(|||u_{h,0}|||^2 + |||u_{h,1}|||^2 + \|f_h\|^2_{L^2(L^2)}\right).
\end{aligned}$$

This completes the proof of Lemma 5.3.1. □

**Remark 5.3.3.** *Using Lemma 5.2.4, for all $t \in (0,T]$, we have*

$$\|u_h(t)\| = \|u_0(t)\| \le C|||u_h(t)||| \;\; \& \;\; \|u_{ht}(t)\| = \|u_0'(t)\| \le C|||u_{ht}(t)|||.$$

*Then using Lemma 5.3.1, we obtain*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u_h(t)\|^2 + \operatorname*{ess\,sup}_{t\in[0,T]} \|u_{ht}(t)\|^2 \le \tilde{C}(\alpha_h, \beta_h, T)\left(|||u_{h,0}|||^2 + |||u_{h,1}|||^2 + \|f_h\|^2_{L^2(L^2)}\right).$$

## 5.4 A Priori Error Analysis for Linearized Westervelt's Equation

In this section, we now focus on proving an optimal *a priori* error estimate under $L^\infty(L^2)$ norm for the linearized Westervelt's equation that also take into account approximation error of the coefficients and the source term. We employ standard elliptic projection (cf. [158]), which is crucial for $L^2$ norm error analysis

Let $u$ be the exact solution of (5.3.1) and $u_h$ be the weak Galerkin solution as defined in (5.3.3)-(5.3.4) respectively. We define the projected error $e_h$ as

$$e_h(t) = \{e_0(t), e_b(t)\} = u_h(t) - \mathcal{Q}_h u(t) \text{ for } t \in [0, T]. \tag{5.4.1}$$

To simplify the notation, we use $e_h = \{e_0, e_b\} = u_h - \mathcal{Q}_h u$ for $e_h(t) = \{e_0(t), e_b(t)\}$.

For $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, we define

$$f_\varphi^* = -\nabla \cdot \nabla\varphi \ \text{ in } \ \Omega.$$

Clearly $f_\varphi^* \in L^2(\Omega)$. Now we recall standard Ritz projection $\mathcal{R}_h : H^2(\Omega) \cap H_0^1(\Omega) \to \mathcal{W}_h^0$ defined as

$$\mathcal{A}_w(\mathcal{R}_h\varphi, \phi_h) = (f_\varphi^*, \phi_0) \ \forall \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0. \tag{5.4.2}$$

Next, we follow the standard approach in the finite element error analysis and split the projected error $e_h = \{e_0, e_b\} = u_h - \mathcal{Q}_h u$ into

$$e_h = u_h - \mathcal{R}_h u + \mathcal{R}_h u - \mathcal{Q}_h u = \theta - \rho,$$

where $\theta = u_h - \mathcal{R}_h u$ and $\rho = \mathcal{Q}_h u - \mathcal{R}_h u$.

We first recall the following useful result, which deals with the bounds for $\rho$ (cf. [43]).

**Lemma 5.4.1.** *Let $\varphi \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|\mathcal{Q}_h\varphi - \mathcal{R}_h\varphi\| + h|\!|\!|\mathcal{Q}_h\varphi - \mathcal{R}_h\varphi|\!|\!| \le Ch^{k+1}\|\varphi\|_{H^{k+1}(\Omega)}.$$

**Remark 5.4.1.** *It is noteworthy that $(\mathcal{Q}_h u)_t = \mathcal{Q}_h u_t$ & $(\mathcal{R}_h u)_t = \mathcal{R}_h u_t$. Thus, for $u \in H^2(J; H^{k+1}(\Omega))$, using Lemma 5.4.1 following bounds can be obtained for $\rho_t$ and $\rho_{tt}$*

$$\|\rho_t(t)\| + h|\!|\!|\rho_t(t)|\!|\!| \le Ch^{k+1}\|u_t(t)\|_{H^{k+1}(\Omega)} \ \ \&$$
$$\|\rho_{tt}(t)\| + h|\!|\!|\rho_{tt}(t)|\!|\!| \le Ch^{k+1}\|u_{tt}(t)\|_{H^{k+1}(\Omega)}$$

*for a.e. $t \in [0, T]$.*

Since, we are able to estimate $\rho$ and analogous bounds for $\rho_t$ and $\rho_{tt}$, we now interested on deriving *a priori* bounds for $\theta$. As a first step, for any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, recalling the definition of projection $\mathcal{R}_h$ and (5.3.1), (5.3.3), we note that

$$
\begin{aligned}
&(\alpha_h\theta_{tt},\phi_h)+c^2\mathcal{A}_w(\theta,\phi_h)+b\mathcal{A}_w(\theta_t,\phi_h)+(\beta_h\theta_t,\phi_h)\\
&=(\alpha_h u_{htt},\phi_h)+c^2\mathcal{A}_w(u_h,\phi_h)+b\mathcal{A}_w(u_{ht},\phi_h)+(\beta_h u_{ht},\phi_h)\\
&\quad-(\alpha_h(\mathcal{R}_h u)_{tt},\phi_h)-c^2\mathcal{A}_w(\mathcal{R}_h u,\phi_h)-b\mathcal{A}_w((\mathcal{R}_h u)_t,\phi_h)-(\beta_h(\mathcal{R}_h u)_t,\phi_h)\\
&=(f_h,\phi_h)-(\alpha_h\mathcal{R}_h u_{tt},\phi_h)-c^2\mathcal{A}_w(\mathcal{R}_h u,\phi_h)-b\mathcal{A}_w(\mathcal{R}_h u_t,\phi_h)-(\beta_h\mathcal{R}_h u_t,\phi_h)\\
&=(f_h,\phi_h)-(\alpha_h\mathcal{R}_h u_{tt},\phi_h)+(c^2\nabla\cdot\nabla u,\phi_h)+(b\nabla\cdot\nabla u_t,\phi_h)-(\beta_h\mathcal{R}_h u_t,\phi_h)\\
&=(f_h,\phi_h)-(\alpha_h\mathcal{R}_h u_{tt},\phi_h)-(f,\phi_h)+(\alpha u_{tt},\phi_h)+(\beta u_t,\phi_h)-(\beta_h\mathcal{R}_h u_t,\phi_h).
\end{aligned}
$$

Further, using $L^2$ projection $\mathcal{Q}_h$, we have

$$
\begin{aligned}
&(\alpha_h\theta_{tt},\phi_h)+c^2\mathcal{A}_w(\theta,\phi_h)+b\mathcal{A}_w(\theta_t,\phi_h)+(\beta_h\theta_t,\phi_h)\\
&=(f_h-f,\phi_h)+((\alpha-\alpha_h)u_{tt},\phi_h)+(\alpha_h\rho_{tt},\phi_h)+(\alpha_h(u_{tt}-\mathcal{Q}_h u_{tt}),\phi_h)\\
&\quad+((\beta-\beta_h)u_t,\phi_h)+(\beta_h\rho_t,\phi_h)+(\beta_h(u_t-\mathcal{Q}_h u_t),\phi_h).
\end{aligned}
\tag{5.4.3}
$$

**Proposition 5.4.1.** *Let $c^2, b > 0$ and let $u$ be the solution of (5.3.1) that satisfies*

$$u_t, u_{tt} \in L^2(J; H^{k+1}(\Omega)) \cap L^2(J; L^3(\Omega)).$$

*Also, assume that Assumption 5.3.1 hold and $\alpha_{ht} \in L^\infty(J; L^3(\Omega))$. Furthermore, let $u_{h,0} = \mathcal{R}_h u_*$ and $u_{h,1} = \mathcal{R}_h u^*$. Then there exists a positive constant $C = C(\alpha_h, \beta_h, T)$ such that*

$$
\begin{aligned}
&\|\theta_t\|^2_{L^\infty(L^2)}+\operatorname*{ess\,sup}_{t\in[0,T]}|||\theta(t)|||^2+\int_0^T|||\theta_t|||^2ds\\
&\le C\Big(h^{2(k+1)}\big(\|u_t\|^2_{L^2(H^{k+1})}+\|u_{tt}\|^2_{L^2(H^{k+1})}\big)+\|f_h-f\|^2_{L^2(L^2)}\\
&\quad+\|\alpha-\alpha_h\|^2_{L^\infty(L^2)}\|u_{tt}\|^2_{L^2(L^3)}+\|\beta-\beta_h\|^2_{L^\infty(L^2)}\|u_t\|^2_{L^2(L^3)}\Big).
\end{aligned}
\tag{5.4.4}
$$

*Proof.* Setting $\phi_h = \theta_t$ in (5.4.3) and then integrating from 0 to $t \in (0, T]$, we have

$$
\begin{aligned}
&\frac{1}{2}(\alpha_h(t)\theta_t(t),\theta_t(t))+\frac{c^2}{2}\mathcal{A}_w(\theta(t),\theta(t))+b\int_0^t\mathcal{A}_w(\theta_t,\theta_t)ds\\
&=\frac{1}{2}\int_0^t(\alpha_{ht}\theta_t,\theta_t)ds+\int_0^t(f_h-f,\theta_t)ds+\int_0^t((\alpha-\alpha_h)u_{tt},\theta_t)ds\\
&\quad+\int_0^t(\alpha_h\rho_{tt},\theta_t)ds+\int_0^t(\alpha_h(u_{tt}-\mathcal{Q}_h u_{tt}),\theta_t)ds+\int_0^t((\beta-\beta_h)u_t,\theta_t)ds\\
&\quad+\int_0^t(\beta_h\rho_t,\theta_t)ds+\int_0^t(\beta_h(u_t-\mathcal{Q}_h u_t),\theta_t)ds-\int_0^t(\beta_h\theta_t,\theta_t)ds.
\end{aligned}
$$

Here we have used the fact that $\theta_t(0) = u_{ht}(0) - \mathcal{R}_h u_t(0) = 0$ and the following identity

$$\int_0^t \int_\Omega \alpha_h \theta_{tt} \theta_t dx ds = \frac{1}{2}(\alpha_h(t)\theta_t(t), \theta_t(t)) - \frac{1}{2}\int_0^t (\alpha_{ht}\theta_t, \theta_t) ds.$$

Using Remark 5.3.1 and the fact $\alpha_h(\mathbf{x}, t) \geq \alpha_* > 0$ a.e. in $\Omega \times [0, T]$, we have

$$\begin{aligned}
&\frac{\alpha_*}{2}\|\theta_t(t)\|^2 + \frac{c^2}{2}|||\theta(t)|||^2 + b\int_0^t |||\theta_t|||^2 ds \\
&\leq \frac{C}{2}\int_0^t \|\alpha_{ht}\|_{L^3(\Omega)}\|\theta_t\||||\theta_t|||ds + \int_0^t \|f_h - f\|\|\theta_t\|ds \\
&\quad + C\int_0^t \|\alpha - \alpha_h\|\|u_{tt}\|_{L^3(\Omega)}|||\theta_t|||ds + C\int_0^t \|\alpha_h\|_{L^3(\Omega)}\|\rho_{tt}\||||\theta_t|||ds \\
&\quad + C\int_0^t \|\alpha_h\|_{L^3(\Omega)}\|u_{tt} - \mathcal{Q}_h u_{tt}\||||\theta_t|||ds + C\int_0^t \|\beta - \beta_h\|\|u_t\|_{L^3(\Omega)}|||\theta_t|||ds \\
&\quad + C\int_0^t \|\beta_h\|_{L^3(\Omega)}\|\rho_t\||||\theta_t|||ds + C\int_0^t \|\beta_h\|_{L^3(\Omega)}\|u_t - \mathcal{Q}_h u_t\||||\theta_t|||ds \\
&\quad + C\int_0^t \|\beta_h\|_{L^3(\Omega)}\|\theta_t\||||\theta_t|||ds.
\end{aligned}$$

Therefore, applying Young's inequality with suitable $\epsilon > 0$, results in

$$\begin{aligned}
&\frac{\alpha_*}{2}\|\theta_t(t)\|^2 + \frac{c^2}{2}|||\theta(t)|||^2 + b\int_0^t |||\theta_t|||^2 ds \\
&\leq \frac{C}{16\epsilon}\|\alpha_{ht}\|^2_{L^\infty(L^3)}\int_0^t \|\theta_t\|^2 ds + \frac{C}{4\epsilon}\|\alpha - \alpha_h\|^2_{L^\infty(L^2)}\int_0^t \|u_{tt}\|^2_{L^3(\Omega)} ds \\
&\quad + \frac{C}{4\epsilon}\|\alpha_h\|^2_{L^\infty(L^3)}\int_0^t \|\rho_{tt}\|^2 ds + \frac{C}{4\epsilon}\|\alpha_h\|^2_{L^\infty(L^3)}\int_0^t \|u_{tt} - \mathcal{Q}_h u_{tt}\|^2 ds \\
&\quad + \frac{C}{4\epsilon}\|\beta - \beta_h\|^2_{L^\infty(L^2)}\int_0^t \|u_t\|^2_{L^3(\Omega)} ds + \frac{C}{4\epsilon}\|\beta_h\|^2_{L^\infty(L^3)}\int_0^t \|\rho_t\|^2 ds \\
&\quad + \frac{C}{4\epsilon}\|\beta_h\|^2_{L^\infty(L^3)}\int_0^t \|u_t - \mathcal{Q}_h u_t\|^2 ds + \frac{C}{4\epsilon}\|\beta_h\|^2_{L^\infty(L^3)}\int_0^t \|\theta_t\|^2 ds \\
&\quad + 8\epsilon\int_0^t |||\theta_t|||^2 ds + \frac{1}{4}\int_0^t \|f_h - f\|^2 ds + \int_0^t \|\theta_t\|^2 ds.
\end{aligned}$$

Then $\epsilon = \frac{b}{16}$ yields

$$
\begin{aligned}
&\frac{\alpha_*}{2}\|\theta_t(t)\|^2 + \frac{c^2}{2}|||\theta(t)|||^2 + \frac{b}{2}\int_0^t |||\theta_t|||^2 ds \\
&\le \left(\frac{4C}{b}\left(\|\alpha_{ht}\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)}\right) + 1\right)\int_0^t \|\theta_t\|^2 ds + \frac{1}{4}\int_0^t \|f_h - f\|^2 ds \\
&\quad + \frac{4C}{b}\Bigg(\|\alpha - \alpha_h\|^2_{L^\infty(L^2)}\int_0^t \|u_{tt}\|^2_{L^3(\Omega)} ds + \|\alpha_h\|^2_{L^\infty(L^3)}\int_0^t \|\rho_{tt}\|^2 ds \\
&\quad + \|\alpha_h\|^2_{L^\infty(L^3)}\int_0^t \|u_{tt} - \mathcal{Q}_h u_{tt}\|^2 ds + \|\beta - \beta_h\|^2_{L^\infty(L^2)}\int_0^t \|u_t\|^2_{L^3(\Omega)} ds \\
&\quad + \|\beta_h\|^2_{L^\infty(L^3)}\int_0^t \|\rho_t\|^2 ds + \|\beta_h\|^2_{L^\infty(L^3)}\int_0^t \|u_t - \mathcal{Q}_h u_t\|^2 ds\Bigg).
\end{aligned}
\tag{5.4.5}
$$

Finally, applying Lemma 5.2.1 and results for the estimates $\|\rho_t\|_{L^2(L^2)}, \|\rho_{tt}\|_{L^2(L^2)}$ stated in Remark 5.4.1 to the above estimate (5.4.5), and then Gronwall's inequality lead to

$$
\begin{aligned}
&\|\theta_t(t)\|^2 + |||\theta(t)|||^2 + \int_0^t |||\theta_t|||^2 ds \\
&\le \hat{C}(\alpha_h, \beta_h, T)\Bigg(\|f_h - f\|^2_{L^2(L^2)} + \|\alpha - \alpha_h\|^2_{L^\infty(L^2)}\|u_{tt}\|^2_{L^2(L^3)} + \|\beta - \beta_h\|^2_{L^\infty(L^2)}\|u_t\|^2_{L^2(L^3)} \\
&\quad + Ch^{2(k+1)}\left(\|\alpha_h\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)}\right)\left(\|u_{tt}\|^2_{L^2(H^{k+1})} + \|u_t\|^2_{L^2(H^{k+1})}\right)\Bigg),
\end{aligned}
$$

for $t \in (0, T]$ and $\hat{C}(\alpha_h, \beta_h, T) = \hat{C}_5 \exp(C_6(\|\alpha_{ht}\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + 1)T)$.

Therefore, taking essential supremum over $[0, T]$ we have desired error bound (5.4.4) and the constant appeared is given by

$$
\begin{aligned}
&C(\alpha_h, \beta_h, T) \\
&= C_5(\|\alpha_h\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + 1) \times \exp(C_6(\|\alpha_{ht}\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + 1)T).
\end{aligned}
$$

This completes the proof of Proposition 5.4.1. □

**Proposition 5.4.2.** *Let $c^2$, $b > 0$ and let $u$ be the solution of (5.3.1) that satisfies*

$$
\begin{cases}
u \in L^\infty(J; H^{k+1}(\Omega)), \\
u_t \in L^2(J; L^\infty(\Omega)) \cap L^\infty(J; H^{k+1}(\Omega)), \\
u_{tt} \in L^2(J; H^{k+1}(\Omega) \cap L^\infty(\Omega)).
\end{cases}
\tag{5.4.6}
$$

*Also, assume that Assumption 5.3.1 holds and $\alpha_{ht} \in L^\infty(J; L^3(\Omega))$, and $\int_0^T \|\beta_h\|^2_{\infty,h} < \infty$. Furthermore, let $u_{h,0} = \mathcal{R}_h u_*$ and $u_{h,1} = \mathcal{R}_h u^*$. Then there exists a positive constant*

*$C = C(\alpha_h, \beta_h, T)$ such that*

$$\begin{aligned}
&\|\theta_{tt}\|^2_{L^2(L^2)} + \|\theta_t\|^2_{L^\infty(L^2)} + \operatorname*{ess\,sup}_{t\in[0,T]} |\!|\!|\theta(t)|\!|\!|^2 + \operatorname*{ess\,sup}_{t\in[0,T]} |\!|\!|\theta_t(t)|\!|\!|^2 + \int_0^T |\!|\!|\theta_t|\!|\!|^2 \\
&\leq C\Big( h^{2(k+1)} \Big( \|u_t\|^2_{L^\infty(H^{k+1})} + \|u_{tt}\|^2_{L^2(H^{k+1})} \Big) + \|f_h - f\|^2_{L^2(L^2)} \\
&\qquad + \|\alpha - \alpha_h\|^2_{L^\infty(L^2)} \|u_{tt}\|^2_{L^2(L^\infty)} + \|\beta - \beta_h\|^2_{L^\infty(L^2)} \|u_t\|^2_{L^2(L^\infty)} \Big). \qquad (5.4.7)
\end{aligned}$$

*Proof.* Setting $\phi_h = \theta_{tt}$ in (5.4.3) and using properties of projector $\mathcal{Q}_h$, we have

$$\begin{aligned}
&(\alpha_h\theta_{tt}, \theta_{tt}) + c^2 \frac{d}{dt}\mathcal{A}_w(\theta, \theta_t) - c^2 \mathcal{A}_w(\theta_t, \theta_t) + \frac{b}{2}\frac{d}{dt}\mathcal{A}_w(\theta_t, \theta_t) + (\beta_h\theta_t, \theta_{tt}) \\
&= (f_h - f, \theta_{tt}) + ((\alpha - \alpha_h)u_{tt}, \theta_{tt}) + (\alpha_h\rho_{tt}, \theta_{tt}) + (\alpha_h(u_{tt} - \mathcal{Q}_h u_{tt}), \theta_{tt}) \\
&\quad + ((\beta - \beta_h)u_t, \theta_{tt}) + (\beta_h\rho_t, \theta_{tt}) + (\beta_h(u_t - \mathcal{Q}_h u_t), \theta_{tt}).
\end{aligned}$$

Now integrating from 0 to $t \in (0, T]$ and using the facts $\theta(0) = 0$ and $\theta_t(0) = 0$, we have

$$\begin{aligned}
&\int_0^t (\alpha_h\theta_{tt}, \theta_{tt})ds + c^2\mathcal{A}_w(\theta(t), \theta_t(t)) - c^2\int_0^t \mathcal{A}_w(\theta_t, \theta_t)ds + \frac{b}{2}\mathcal{A}_w(\theta_t(t), \theta_t(t)) \\
&= \int_0^t (f_h - f, \theta_{tt})ds + \int_0^t ((\alpha - \alpha_h)u_{tt}, \theta_{tt})ds + \int_0^t (\alpha_h\rho_{tt}, \theta_{tt})ds \\
&\quad + \int_0^t (\alpha_h(u_{tt} - \mathcal{Q}_h u_{tt}), \theta_{tt})ds + \int_0^t ((\beta - \beta_h)u_t, \theta_{tt})ds \\
&\quad + \int_0^t (\beta_h\rho_t, \theta_{tt})ds + \int_0^t (\beta_h(u_t - \mathcal{Q}_h u_t), \theta_{tt})ds - \int_0^t (\beta_h\theta_t, \theta_{tt})ds.
\end{aligned}$$

Using the fact $\alpha_h(\mathbf{x}, t) \geq \alpha_* > 0$ a.e. in $\Omega \times [0, T]$ and using Hölder's inequality and Remark 5.3.1, we have

$$\begin{aligned}
&\alpha_* \int_0^t \|\theta_{tt}\|^2_{L^2(\Omega)} + \frac{b}{2}\mathcal{A}_w(\theta_t(t), \theta_t(t)) \\
&\leq c^2 |\!|\!|\theta(t)|\!|\!| |\!|\!|\theta_t(t)|\!|\!| + c^2 \int_0^t |\!|\!|\theta_t|\!|\!|^2 ds \\
&\quad + \int_0^t \|f_h - f\|\|\theta_{tt}\|ds + \int_0^t \|\alpha - \alpha_h\|\|u_{tt}\|_{L^\infty(\Omega)}\|\theta_{tt}\|ds \\
&\quad + \int_0^t \|\alpha_h\|_{\infty,h}\|\rho_{tt}\|\|\theta_{tt}\|ds + \int_0^t \|\alpha_h\|_{\infty,h}\|u_{tt} - \mathcal{Q}_h u_{tt}\|\|\theta_{tt}\|ds \\
&\quad + \int_0^t \|\beta - \beta_h\|\|u_t\|_{L^\infty(\Omega)}\|\theta_{tt}\|ds + \int_0^t \|\beta_h\|_{\infty,h}\|\rho_t\|\|\theta_{tt}\|ds \\
&\quad + \int_0^t \|\beta_h\|_{\infty,h}\|u_t - \mathcal{Q}_h u_t\|\|\theta_{tt}\|ds + C\int_0^t \|\beta_h\|_{L^3(\Omega)} |\!|\!|\theta_t|\!|\!| \|\theta_{tt}\|ds.
\end{aligned}$$

For suitable $\epsilon > 0$ and $\delta > 0$, apply Young's inequality to obtain

$$\begin{aligned}
&(\alpha_* - 8\epsilon)\int_0^t \|\theta_{tt}\|^2 ds + \frac{b}{2}|||\theta_t(t)|||^2 \\
&\leq \frac{c^4}{4\delta}|||\theta(t)|||^2 + \delta|||\theta_t(t)|||^2 + c^2\int_0^t |||\theta_t|||^2 ds + \frac{1}{4\epsilon}\int_0^t \|f_h - f\|^2 ds \\
&\quad + \frac{1}{4\epsilon}\|\alpha - \alpha_h\|^2_{L^\infty(L^2)}\int_0^t \|u_{tt}\|^2_{L^\infty(\Omega)} ds + \frac{1}{4\epsilon}\operatorname*{ess\,sup}_{t\in[0,T]}\|\alpha_h\|^2_{\infty,h}\int_0^t \|\rho_{tt}\|^2 ds \\
&\quad + \frac{1}{4\epsilon}\operatorname*{ess\,sup}_{t\in[0,T]}\|\alpha_h\|^2_{\infty,h}\int_0^t \|u_{tt} - \mathcal{Q}_h u_{tt}\|^2 ds \\
&\quad + \frac{1}{4\epsilon}\|\beta - \beta_h\|^2_{L^\infty(L^2)}\int_0^t \|u_t\|^2_{L^\infty(\Omega)} ds + \frac{1}{4\epsilon}\|\rho_t\|^2_{L^\infty(L^2)}\int_0^t \|\beta_h\|^2_{\infty,h} ds \\
&\quad + \frac{1}{4\epsilon}\|u_t - \mathcal{Q}_h u_t\|^2_{L^\infty(L^2)}\int_0^t \|\beta_h\|^2_{\infty,h} ds + \frac{C}{4\epsilon}\|\beta_h\|^2_{L^\infty(L^3(\Omega))}\int_0^t |||\theta_t|||^2 ds.
\end{aligned}$$

Setting $\epsilon = \frac{\alpha_*}{16}$ and $\delta = \frac{b}{4}$, we have

$$\begin{aligned}
&\frac{\alpha_*}{2}\int_0^t \|\theta_{tt}\|^2 ds + \frac{b}{4}|||\theta_t(t)|||^2 \\
&\leq \frac{c^4}{b}|||\theta(t)|||^2 + \left(\frac{4C}{\alpha_*}\|\beta_h\|^2_{L^\infty(L^3)} + c^2\right)\int_0^t |||\theta_t|||^2 ds \\
&\quad + \frac{4}{\alpha_*}\Bigg\{\|f_h - f\|^2_{L^2(L^2)} + \|\alpha - \alpha_h\|^2_{L^\infty(L^2)}\|u_{tt}\|^2_{L^2(L^\infty)} + \|\beta - \beta_h\|^2_{L^\infty(L^2)}\|u_t\|^2_{L^2(L^\infty)} \\
&\quad + Ch^{2(k+1)}\left(\operatorname*{ess\,sup}_{t\in[0,T]}\|\alpha_h\|^2_{\infty,h}\|u_{tt}\|^2_{L^2(H^{k+1})} + \|u_t\|^2_{L^\infty(H^{k+1})}\int_0^T \|\beta_h\|^2_{\infty,h}\right)\Bigg\}. \qquad (5.4.8)
\end{aligned}$$

Next, to handle the term $\frac{c^4}{b}|||\theta(t)|||$ present in the right hand side of above estimate (5.4.8), we first multiply (5.4.5) by some positive $\lambda$ such that $\lambda\frac{c^2}{2} > \frac{c^4}{b}$ and then add to (5.4.8). Finally, Gronwall's inequality and standard Sobolev embedding results, and the facts $\|u_t\|^2_{L^2(L^3)} \leq C(\Omega)\|u_t\|^2_{L^2(L^\infty)}$, $\|u_{tt}\|^2_{L^2(L^3)} \leq C(\Omega)\|u_{tt}\|^2_{L^2(L^\infty)}$ yield

$$\begin{aligned}
&\int_0^t \|\theta_{tt}\|^2 ds + \|\theta_t(t)\|^2 + |||\theta(t)|||^2 + |||\theta_t(t)|||^2 + \int_0^t |||\theta_t|||^2 ds \\
&\leq \hat{C}(\alpha_h, \beta_h, T)\Bigg\{\|f_h - f\|^2_{L^2(L^2)} + \|\alpha - \alpha_h\|^2_{L^\infty(L^2)}\|u_{tt}\|^2_{L^2(L^\infty)} + \|\beta - \beta_h\|^2_{L^\infty(L^2)}\|u_t\|^2_{L^2(L^\infty)} \\
&\quad + Ch^{2(k+1)}\left(\operatorname*{ess\,sup}_{t\in[0,T]}\|\alpha_h\|^2_{\infty,h}\|u_{tt}\|^2_{L^2(H^{k+1})} + \|u_t\|^2_{L^\infty(H^{k+1})}\int_0^T \|\beta_h\|^2_{\infty,h}\right) \\
&\quad + Ch^{2(k+1)}\left(\|\alpha_h\|^2_{L^\infty(L^3)}\|u_{tt}\|^2_{L^2(H^{k+1})} + \|\beta_h\|^2_{L^\infty(L^3)}\|u_t\|^2_{L^2(H^{k+1})}\right)\Bigg\},
\end{aligned}$$

for all $t \in (0, T]$ and $\hat{C}(\alpha_h, \beta_h, T) = \hat{C}_7\exp(C_8(\|\alpha_{ht}\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + 1)T)$.

Therefore, taking essential supremum over $[0,T]$ and the fact that $\|u_t\|^2_{L^2(H^{k+1})} \leq T\|u_t\|^2_{L^\infty(H^{k+1})}$ lead to the desired error bound (5.4.7), where the constant appeared in the desired error bound (5.4.7) is given by

$$C(\alpha_h,\beta_h,T) = C_7\left(\operatorname*{ess\,sup}_{t\in[0,T]}\|\alpha_h\|^2_{\infty,h} + \int_0^T \|\beta_h\|^2_{\infty,h} + \|\alpha_h\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + T + 1\right)$$
$$\times \exp\Big(C_8(\|\alpha_{ht}\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + 1)T\Big).$$

This completes the proof of Proposition 5.4.2. □

**Theorem 5.4.1.** *Let $c^2, b > 0$ and all the assumptions of Proposition 5.4.2 hold, and $u, u_h$ be the solution of (5.3.1) and (5.3.3). Then following a priori estimate holds*

$$\begin{aligned}
&\|e_{htt}\|^2_{L^2(L^2)} + \|e_h\|^2_{L^\infty(L^2)} + \|e_{ht}\|^2_{L^\infty(L^2)} + h^2\operatorname*{ess\,sup}_{t\in[0,T]}\|\!|e_h(t)|\!\|^2 + h^2\operatorname*{ess\,sup}_{t\in[0,T]}\|\!|e_{ht}(t)|\!\|^2 \\
&\leq C(\alpha_h,\beta_h,T)\bigg(h^{2(k+1)}\Big(\|u\|^2_{L^\infty(H^{k+1})} + \|u_t\|^2_{L^\infty(H^{k+1})} + \|u_{tt}\|^2_{L^2(H^{k+1})}\Big) \\
&\quad + \|f_h - f\|^2_{L^2(L^2)} + \|\alpha-\alpha_h\|^2_{L^\infty(L^2)}\|u_{tt}\|^2_{L^2(L^\infty)} + \|\beta-\beta_h\|^2_{L^\infty(L^2)}\|u_t\|^2_{L^2(L^\infty)}\bigg),
\end{aligned} \tag{5.4.9}$$

*where the constant $C$ is given by*

$$\begin{aligned}
C(\alpha_h,\beta_h,T) = C_9\bigg[&\Big(\operatorname*{ess\,sup}_{t\in[0,T]}\|\alpha_h\|^2_{\infty,h} + \int_0^T \|\beta_h\|^2_{\infty,h} + \|\alpha_h\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + T + 1\Big) \\
&\times \exp\Big(C_{10}(\|\alpha_{ht}\|^2_{L^\infty(L^3)} + \|\beta_h\|^2_{L^\infty(L^3)} + 1)T + 1\Big)\bigg].
\end{aligned} \tag{5.4.10}$$

*Proof.* Clearly by splitting $e_h = u_h - \mathcal{Q}_h u$ into $\theta$ and $\rho$, and applying triangle inequality, we have

$$\begin{aligned}
&\|e_{htt}\|^2_{L^2(L^2)} + \|e_h\|^2_{L^\infty(L^2)} + \|e_{ht}\|^2_{L^\infty(L^2)} + h^2\operatorname*{ess\,sup}_{t\in[0,T]}\|\!|e_h(t)|\!\|^2 + h^2\operatorname*{ess\,sup}_{t\in[0,T]}\|\!|e_{ht}(t)|\!\|^2 \\
&\leq \|\theta_{tt}\|^2_{L^2(L^2)} + \|\theta\|^2_{L^\infty(L^2)} + \|\theta_t\|^2_{L^\infty(L^2)} + h^2\operatorname*{ess\,sup}_{t\in[0,T]}\|\!|\theta(t)|\!\|^2 + h^2\operatorname*{ess\,sup}_{t\in[0,T]}\|\!|\theta_t(t)|\!\|^2 \\
&\quad + \|\rho_{tt}\|^2_{L^2(L^2)} + \operatorname*{ess\,sup}_{t\in[0,T]}\Big(\|\rho(t)\|^2 + h^2\|\!|\rho(t)|\!\|^2\Big) + \operatorname*{ess\,sup}_{t\in[0,T]}\Big(\|\rho_t(t)\|^2 + h^2\|\!|\rho_t(t)|\!\|^2\Big).
\end{aligned}$$

Hence, using the fact $\|\theta\|_{L^\infty(L^2)} \leq C\operatorname{ess\,sup}_{t\in[0,T]}\|\!|\theta(t)|\!\|$, Proposition 5.4.2 and Lemma 5.4.1 along with Remark 5.4.1, we have the desired result. □

**Remark 5.4.2.** *If we do not have to take the coefficient error into account, then $\|\alpha - \alpha_h\|_{L^\infty(L^2)} = 0$ and $\|\beta - \beta_h\|_{L^\infty(L^2)} = 0$. Therefore, regularity condition (5.4.6) can be relaxed to*

$$u \in L^\infty(J; H^{k+1}(\Omega)), \quad u_t \in L^\infty(J; H^{k+1}(\Omega)), \quad u_{tt} \in L^2(J; H^{k+1}(\Omega)).$$

## 5.5 WG Finite Element Approximations of Westervelt's Equation

In this section our aim is to study weak Galerkin discretization of the initial boundary value problem for the Westervelt's equation (5.2.1). To this extend, we shall employ Banach fixed-point theorem alongwith results for the linearized Westervelt's equation.

**Theorem 5.5.1.** *Let* $c^2, b, \sigma > 0$ *and* $T > 0$. *Assume that the initial-boundary value problem for the Westervelt's equation (5.2.1) has a unique solution which satisfies*

$$u \in L^\infty(J; L^\infty(\Omega) \cap H^{k+1}(\Omega)), \quad u_t \in L^2(J; L^\infty(\Omega)) \cap L^\infty(J; H^{k+1}(\Omega)),$$
$$u_{tt} \in L^2(J; L^\infty(\Omega) \cap H^{k+1}(\Omega))$$

*Then for sufficiently small* $\boldsymbol{m}$ *and* $\mathcal{M}$, *defined by*

$$\boldsymbol{m} = \|u\|_{L^\infty(L^\infty)}$$
$$\mathcal{M} = max \left\{ \|u\|_{L^\infty(H^{k+1})}, \|u_t\|_{L^\infty(H^{k+1})}, \|u_t\|_{L^2(L^\infty)}, \|u_{tt}\|_{L^2(H^{k+1})}, \|u_{tt}\|_{L^2(L^\infty)} \right\},$$

*and* $h_0$, *there exists a unique* $u_h = \{u_0, u_b\} \in H^2(J; \mathcal{W}_h^0)$ *in the neighborhood of* $\mathcal{Q}_h u$, *such that*

$$\begin{cases} ((1 - 2\sigma u_h)u_{htt}, \phi_h) + c^2 \mathcal{A}_w(u_h, \phi_h) + b\mathcal{A}_w(u_{ht}, \phi_h) = 2\sigma(u_{ht}^2, \phi_h), \ \forall \ \phi_h \in \mathcal{W}_h^0, \\ u_h(0) = \mathcal{R}_h u_*, \quad u_{ht}(0) = \mathcal{R}_h u^*. \end{cases} \tag{5.5.1}$$

*Furthermore, there exists a positive constant* $C$ *independent of* $h$, *such that*

$$\|\mathcal{Q}_h u_{tt} - u_{htt}\|_{L^2(L^2)} + \|\mathcal{Q}_h u - u_h\|_{L^\infty(L^2)} + \|\mathcal{Q}_h u_t - u_{ht}\|_{L^\infty(L^2)}$$
$$+ h \operatorname*{ess\,sup}_{t \in [0,T]} |||(\mathcal{Q}_h u - u_h)(t)||| + h \operatorname*{ess\,sup}_{t \in [0,T]} |||(\mathcal{Q}_h u_t - u_{ht})(t)||| \leq Ch^{k+1}.$$

*Proof.* We shall define an iterative map on which we employ Banach fixed-point theorem relaying on the results for linearized problem. To this end, we consider the set $\mathcal{B}_h \subset \mathcal{W}_h$ defined as

$$\mathcal{B}_h = \Big\{ w_h \in \mathfrak{B} : \ \|\mathcal{Q}_h u_{tt} - w_{htt}\|_{L^2(L^2)} + \|\mathcal{Q}_h u - w_h\|_{L^\infty(L^2)} + \|\mathcal{Q}_h u_t - w_{ht}\|_{L^\infty(L^2)}$$
$$+ h \operatorname*{ess\,sup}_{t \in [0,T]} |||(\mathcal{Q}_h u - w_h)(t)||| + h \operatorname*{ess\,sup}_{t \in [0,T]} |||(\mathcal{Q}_h u_t - w_{ht})(t)||| \leq Lh^{k+1}$$
$$\text{and} \ \ w_h(0) = \mathcal{R}_h u_*, w_{ht}(0) = \mathcal{R}_h u^* \Big\},$$

where $\mathfrak{B} = W^{1,\infty}(J; \mathcal{W}_h) \cap H^2(J; L^2(\Omega))$ and the constant $L > 0$ independent of $h \leq h_0$. Clearly, $\mathcal{B}_h$ is non empty as $\mathcal{Q}_h u \in \mathcal{B}_h$. For $w_h = \{w_0, w_b\} \in \mathcal{B}_h$, consider the following linearization problem: Find $u_h$ such that

$$\begin{cases} ((1 - 2\sigma w_h)u_{htt}, \phi_h) + c^2 \mathcal{A}_w(u_h, \phi_h) + b\mathcal{A}_w(u_{ht}, \phi_h) = 2\sigma(w_{ht}u_{ht}, \phi_h) \\ \qquad\qquad \forall\ \phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h, \text{ a.e. in } [0, T], \\ u_h(0) = \mathcal{R}_h u_*, u_{ht}(0) = \mathcal{R}_h u^*. \end{cases} \tag{5.5.2}$$

Approximation (5.5.2) can be treated as the weak Galerkin formulation for (5.3.1) with

$$\alpha = 1 - 2\sigma u, \quad \beta = -2\sigma u_t \quad \text{and} \quad f = 0.$$

We define an iterative map $\mathcal{F} : \mathcal{B}_h \to \mathcal{W}_h^0$ given by $\mathcal{F}(w_h) = u_h$, where $u_h$ is the solution of (5.5.2). $\mathcal{F}$ is well defined, since, (5.5.2) has unique solution thanks to Theorem 5.3.1. Clearly, a fixed point of $\mathcal{F}$ will solve (5.5.1). So we apply Banach fixed-point theorem using the result for the linearized Westervelt's problem.

Now we proceed to verify the conditions of Banach fixed-point theorem.

**Step 1:** $(\mathcal{B}_h, d)$ ***is a complete metric space:***

$\mathcal{B}_h$ is closed in the Banach space $\mathfrak{B} = W^{1,\infty}(J; \mathcal{W}_h^0) \cap H^2(J; L^2(\Omega))$ with respect to the metric induced by the norm

$$\|w_h\|_{\mathfrak{B}} = \max\{\|w_h\|_{H^2(L^2)}, \|w_h\|_{W^{1,\infty}(L^2)}, \|w_h\|_{W^{1,\infty}(\mathcal{W}_h^0)}\}.$$

Hence, $(\mathcal{B}_h, d)$ is a complete metric space.

**Step 2:** ***The self mapping property of*** $\mathcal{F}$***:***

For any $w_h \in \mathcal{B}_h$, we want to show $u_h = \mathcal{F}_h(w_h) \in \mathcal{B}_h$. Comparing (5.5.2) with the equation (5.3.3), we have

$$\alpha_h = 1 - 2\sigma w_h, \quad \beta_h = -2\sigma w_{ht} \quad \text{and} \quad f_h = 0.$$

First we verify the conditions of Theorem 5.3.1 and Theorem 5.4.1. Now for the degeneracy condition of $\alpha_h$, we shall use Remark 5.2.2 and Corollary 5.2.1.

For $\mathcal{K} \in \mathcal{T}_h$, we have

$$\begin{aligned} \|w_0(t)\|_{L^\infty(\mathcal{K})} &\leq \|(w_0 - \mathcal{Q}_k^0 u)(t)\|_{L^\infty(\mathcal{K})} + \|\mathcal{Q}_k^0 u(t)\|_{L^\infty(\mathcal{K})} \\ &\leq Ch^{-1}\|(w_h - \mathcal{Q}_h u)(t)\|_{L^2(\mathcal{K})} + C\|u(t)\|_{L^\infty(\Omega)} \\ &\leq Ch^{-1}\|(w_h - \mathcal{Q}_h u)(t)\|_{L^2(\Omega)} + C\|u(t)\|_{L^\infty(\Omega)}. \end{aligned}$$

Therefore, for a.e. $t \in [0,T]$, we obtain

$$\sup_{\mathcal{K}\in\mathcal{T}_h} \|w_0(t)\|_{L^\infty(\mathcal{K})} \le Ch^{-1}\|(w_h - \mathcal{Q}_h u)(t)\|_{L^2(\Omega)} + C\|u(t)\|_{L^\infty(\Omega)}. \tag{5.5.3}$$

Hence,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|w_h(t)\|_{\infty,h} \le CLh^k + C\boldsymbol{m} \le CLh_0^k + C\boldsymbol{m}. \tag{5.5.4}$$

Now we choose $h_0$ and $\boldsymbol{m}$ sufficiently small such that

$$\boldsymbol{m}_0 = CLh_0^k + C\boldsymbol{m} < \frac{1}{2\sigma},$$

which yields

$$0 < \alpha_* = 1 - 2\sigma\boldsymbol{m}_0 \le \alpha_h(\mathbf{x},t) \le 1 + 2\sigma\boldsymbol{m}_0 \quad \text{for a.e. } (\mathbf{x},t) \in \Omega\times[0,T].$$

Also, we need to bound $\|\alpha_h\|_{L^\infty(L^3)}$, $\|\beta_h\|_{L^\infty(L^3)}$ or $\|\alpha_{ht}\|_{L^\infty(L^3)}$ and $\int_0^T \|\beta_h\|_{\infty,h}^2$.

Using Lemma 5.2.4, Remark 5.2.3 and the fact $w_h \in \mathcal{B}_h$, we have

$$\begin{aligned}
\|\alpha_h\|_{L^\infty(L^3)} &\le T|\Omega| + 2\sigma \operatorname*{ess\,sup}_{t\in[0,T]} \|w_h\|_{L^3(\Omega)} \\
&\le T|\Omega| + 2\sigma C \operatorname*{ess\,sup}_{t\in[0,T]} |||w_h||| \\
&\le T|\Omega| + 2\sigma C \operatorname*{ess\,sup}_{t\in[0,T]} |||w_h - \mathcal{Q}_h u||| + 2\sigma C \operatorname*{ess\,sup}_{t\in[0,T]} |||\mathcal{Q}_h u||| \\
&\le T|\Omega| + 2\sigma CLh^k + 2\sigma C \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_{H^1(\Omega)} \\
&\le T|\Omega| + 2\sigma CLh_0^k + 2\sigma C\|u\|_{L^\infty(H^1)} \le T|\Omega| + 2\sigma CLh_0^k + 2\sigma C\mathcal{M},
\end{aligned}$$

where $|\Omega|$ is Lebesgue measure of $\Omega$.

Similar arguments yield

$$\|\beta_h\|_{L^\infty(L^3)} \le 2\sigma CLh_0^k + 2\sigma C\|u_t\|_{L^\infty(H^1)} \le 2\sigma CLh_0^k + 2\sigma C\mathcal{M}$$

and as a consequence, we obtain

$$\|\alpha_{ht}\|_{L^\infty(L^3)} = \|\beta_h\|_{L^\infty(L^3)} \le 2\sigma CLh_0^k + 2\sigma C\mathcal{M}.$$

Arguing as deriving (5.5.3), we can deduce

$$\begin{aligned}
\|\beta_h(t)\|_{\infty,h} = 2\sigma \sup_{\mathcal{K}\in\mathcal{T}_h} \|w_{0t}(t)\|_{L^\infty(\mathcal{K})} &\le Ch^{-1}\|(w_{ht} - \mathcal{Q}_h u_t)(t)\|_{L^2(\Omega)} + C\|u_t(t)\|_{L^\infty(\Omega)} \\
&\le Ch^{-1}\|w_{ht} - \mathcal{Q}_h u_t\|_{L^\infty(L^2)} + C\|u_t(t)\|_{L^\infty(\Omega)},
\end{aligned}$$

for a.e. $t \in [0,T]$. Therefore, using the fact $w_h \in \mathcal{B}_h$, we obtain

$$\begin{aligned}\left(\int_0^T \|\beta_h\|_{\infty,h}^2 dt\right)^{\frac{1}{2}} &\leq Ch^{-1}T\|w_{ht} - \mathcal{Q}_h u_t\|_{L^\infty(L^2)} + C\|u_t\|_{L^2(L^\infty)} \\ &\leq CTLh^k + C\mathcal{M} \leq CTLh_0^k + C\mathcal{M}. \qquad (5.5.5)\end{aligned}$$

Hence, all the conditions of Theorem 5.3.1 and Theorem 5.4.1 hold. Hence, according to Theorem 5.3.1, there exists a unique solution $u_h \in \mathcal{W}_h^0$ of (5.5.2). Further, by Theorem 5.4.1, *a priori* estimate for the linearized Westervelt's equation stated in, we obtain

$$\begin{aligned}&\|\mathcal{Q}_h u_{tt} - u_{htt}\|_{L^2(L^2)}^2 + \|\mathcal{Q}_h u - u_h\|_{L^\infty(L^2)}^2 + \|\mathcal{Q}_h u_t - u_{ht}\|_{L^\infty(L^2)}^2 \\ &+ h^2 \operatorname*{ess\,sup}_{t\in[0,T]} |||(\mathcal{Q}_h u - u_h)(t)|||^2 + h^2 \operatorname*{ess\,sup}_{t\in[0,T]} |||(\mathcal{Q}_h u_t - u_{ht})(t)|||^2 \\ &\leq \tilde{C}^* \Big\{ h^{2(k+1)}\{\|u\|_{L^\infty(H^{k+1})}^2 + \|u_t\|_{L^\infty(H^{k+1})}^2 + \|u_{tt}\|_{L^2(H^{k+1})}^2\} \\ &\qquad + 4\sigma^2\|u - w_h\|_{L^\infty(L^2)}^2 \|u_{tt}\|_{L^2(L^\infty)}^2 + 4\sigma^2\|u_t - w_{ht}\|_{L^\infty(L^2)}^2 \|u_t\|_{L^2(L^\infty)}^2 \Big\},\end{aligned}$$

where the constant $\tilde{C}^*$ is computed according to (5.4.10) and uniform bounds of $\alpha_h, \beta_h$,

$$\begin{aligned}\tilde{C}^* = C_{11}\Big\{ \Big( &(1 + 2\sigma L h_0^k + \boldsymbol{m})^2 + (TLh_0^k + \mathcal{M})^2 + (T|\Omega| + 2\sigma L h_0^k + 2\sigma\mathcal{M})^2 \\ &+ (2\sigma L h_0^k + 2\sigma\mathcal{M})^2 + T + 1 \Big) \times \exp\left(C_{12}(4(2\sigma L h_0^k + 2\sigma\mathcal{M})^2 + 1)T + 1\right) \Big\}.\end{aligned}$$

Further, using triangle inequality as $\|u - w_h\|_{L^\infty(L^2)} \leq \|u - \mathcal{Q}_h u\|_{L^\infty(L^2)} + \|\mathcal{Q}_h u - w_h\|_{L^\infty(L^2)}$, $\|u_t - w_{ht}\|_{L^\infty(L^2)} \leq \|u_t - \mathcal{Q}_h u_t\|_{L^\infty(L^2)} + \|\mathcal{Q}_h u_t - w_{ht}\|_{L^\infty(L^2)}$ and Lemma 5.2.1, and the fact $w_h \in \mathcal{B}_h$, we obtain

$$\begin{aligned}&\|\mathcal{Q}_h u_{tt} - u_{htt}\|_{L^2(L^2)} + \|\mathcal{Q}_h u - u_h\|_{L^\infty(L^2)} + \|\mathcal{Q}_h u_t - u_{ht}\|_{L^\infty(L^2)} \\ &\quad + h \operatorname*{ess\,sup}_{t\in[0,T]} |||(\mathcal{Q}_h u - u_h)(t)||| + h \operatorname*{ess\,sup}_{t\in[0,T]} |||(\mathcal{Q}_h u_t - u_{ht})(t)||| \\ &\leq \sqrt{\tilde{C}^*} h^{k+1} \left\{3\mathcal{M} + 2\sigma\left((C+L)(\|u\|_{L^\infty(H^{k+1})}\|u_{tt}\|_{L^2(L^\infty)} + \|u_t\|_{L^\infty(H^{k+1})}\|u_t\|_{L^2(L^\infty)})\right)\right\} \\ &\leq C\mathcal{M}h^{k+1}\left(3 + 2\sigma(C+L)\mathcal{M}\right) \leq Lh^{k+1},\end{aligned}$$

for sufficiently small $m, \mathcal{M}$ and $h_0$. So we can conclude that $\mathcal{F}(w_h) = u_h \in \mathcal{B}_h$, which implies $\mathcal{F}(\mathcal{B}_h) \subset \mathcal{B}_h$.

**Step 3: *$\mathcal{F}$ is a contraction map:***

Now to prove the operator $\mathcal{F}$ is strictly contractive, we take $w_h, \hat{w}_h \in \mathcal{B}_h$ and set

$$u_h = \{u_0, u_b\} = \mathcal{F}(w_h) \text{ and } \hat{u}_h = \{\hat{u}_0, \hat{u}_b\} = \mathcal{F}(\hat{w}_h).$$

Therefore, $u_h, \hat{u}_h \in \mathcal{B}_h$. So to prove $\mathcal{F}$ is strictly contractive, we need to show that

$$\|\mathcal{F}(w_h) - \mathcal{F}(\hat{w}_h)\|_{\mathfrak{B}} \leq q\|w_h - \hat{w}_h\|_{\mathfrak{B}} \quad \text{for some} \quad 0 < q < 1.$$

Then $\psi_h = u_h - \hat{u}_h$ satisfies the following equation

$$\begin{aligned}&\Big((1-2\sigma w_h)\psi_{htt}, \phi_h\Big) + c^2\mathcal{A}_w(\psi_h, \phi_h) + b\mathcal{A}_b(\psi_{ht}, \phi_h) - 2\sigma\Big(\hat{w}_{ht}\psi_{ht}, \phi_h\Big)\\ &= \Big(2\sigma(w_{ht} - \hat{w}_{ht})u_{ht} + 2\sigma(w_h - \hat{w}_h)\hat{u}_{htt}, \phi_h\Big) \ \forall \phi_h \in \mathcal{W}_h^0,\end{aligned}$$

with $\psi_h(0) = 0$ and $\psi_{ht}(0) = 0$.

Now comparing with (5.3.3), the above equation is particular case of (5.3.3) with coefficients and force function given by

$$\alpha_h = 1 - 2\sigma w_h, \quad \beta_h = -2\sigma\hat{w}_{ht} \quad \text{and} \quad f_h = 2\sigma(w_{ht} - \hat{w}_{ht})u_{ht} + 2\sigma(w_h - \hat{w}_h)\hat{u}_{htt}.$$

Similar to **Step 2**, we have

$$\begin{aligned}&\|\alpha_h\|_{L^\infty(L^3)} \leq T|\Omega| + 2\sigma CLh_0^k + 2\sigma C\mathcal{M}, \ \|\beta_h\|_{L^\infty(L^3)} \leq 2\sigma CLh_0^k + 2\sigma C\mathcal{M} \ \ \&\\ &0 < \alpha_* = 1 - 2\sigma\boldsymbol{m}_0 \leq \alpha_h(\mathbf{x}, t) \leq 1 + 2\sigma\boldsymbol{m}_0 \ \text{ for a.e. } \ (\mathbf{x}, t) \in \Omega \times [0, T],\end{aligned}$$

with

$$\boldsymbol{m}_0 = CLh_0^k + C\boldsymbol{m} < \frac{1}{2\sigma}.$$

Hence, all the conditions of Lemma 5.3.1 hold. Therefore, from Lemma 5.3.1 and Remark 5.3.3, we have the following *a priori* estimate

$$\begin{aligned}&\|\psi_{htt}\|^2_{L^2(L^2)} + \|\psi_h\|^2_{L^\infty(L^2)} + \|\psi_{ht}\|^2_{L^\infty(L^2)}\\ &\qquad\qquad + \operatorname*{ess\,sup}_{t\in[0,T]} |||\psi_h(t)|||^2 + \operatorname*{ess\,sup}_{t\in[0,T]} |||\psi_{ht}(t)|||^2 + \int_0^T |||\psi_{ht}|||^2 ds\\ &\leq \tilde{C}_*\|f_h\|^2_{L^2(L^2)}\\ &\leq \tilde{C}_*\|2\sigma(w_{ht} - \hat{w}_{ht})u_{ht} + 2\sigma(w_h - \hat{w}_h)\hat{u}_{htt}\|^2_{L^2(L^2)}\\ &\leq 4\sigma^2\tilde{C}_*\left\{\int_0^T \|(w_{ht} - \hat{w}_{ht})u_{ht}\|^2 dt + \int_0^T \|(w_h - \hat{w}_h)\hat{u}_{htt}\|^2 dt\right\}\\ &\leq 4\sigma^2\tilde{C}_*\left\{\|w_{ht} - \hat{w}_{ht}\|^2_{L^\infty(L^2)}\int_0^T \|u_{ht}\|^2_{\infty,h} dt + \|w_h - \hat{w}_h\|^2_{L^\infty(L^2)}\int_0^T \|\hat{u}_{htt}\|^2_{\infty,h} dt\right\},\end{aligned}$$

where the above constant $\tilde{C}_*$, according to (5.3.12), is

$$\tilde{C}_* = C_{13}\exp\left(C_{14}\left((T|\Omega| + 2\sigma Lh_0^k + 2\sigma\mathcal{M})^2 + (2\sigma Lh_0^k + 2\sigma\mathcal{M})^2 + 1\right)T\right).$$

Arguing as deriving (5.5.5), we obtain

$$\begin{aligned}\int_0^T \|u_{ht}\|^2_{\infty,h}dt &\le Ch^{-2}T^2\|u_{ht} - \mathcal{Q}_h u_t\|^2_{L^\infty(L^2)} + C\|u_t\|^2_{L^2(L^\infty)}\\ &\le C(TL)^2h^{2k} + C\mathcal{M}^2 \le C(TL)^2h_0^{2k} + C\mathcal{M}^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_0^T \|\hat{u}_{htt}\|^2_{\infty,h}dt &\le Ch^{-2}\|\hat{u}_{htt} - \mathcal{Q}_h u_{tt}\|^2_{L^2(L^2)} + C\|u_{tt}\|_{L^2(L^\infty)}\\ &\le Ch^{2k}L^2 + C\mathcal{M}^2 \le Ch_0^{2k}L^2 + C\mathcal{M}^2.\end{aligned}$$

Hence, for sufficiently small $\mathcal{M}$ and $h_0$, there exists a $0 < q < 1$ such that

$$\begin{aligned}&\|\psi_{htt}\|^2_{L^2(L^2)} + \|\psi_h\|^2_{L^\infty(L^2)} + \|\psi_{ht}\|^2_{L^\infty(L^2)}\\ &\qquad + \operatorname*{ess\,sup}_{t\in[0,T]} |||\psi_h(t)|||^2 + \operatorname*{ess\,sup}_{t\in[0,T]} |||\psi_{ht}(t)|||^2 + \int_0^T |||\psi_{ht}|||^2 ds\\ &\le q\left(\|w_{ht} - \hat{w}_{ht}\|^2_{L^\infty(L^2)} + \|w_h - \hat{w}_h\|^2_{L^\infty(L^2)}\right).\end{aligned}$$

Therefore,

$$\|\mathcal{F}w_h - \mathcal{F}\hat{w}_h\|^2_{\mathfrak{B}} \le q\|w_h - \hat{w}_h\|^2_{\mathfrak{B}}.$$

Finally, we can conclude that $\mathcal{F}$ is a contraction map with respect to the topology induced by $\|\cdot\|_{\mathfrak{B}}$ in the Banach space $\mathcal{B}_h$. Therefore, Theorem 5.5.1 follows by applying Banach fixed-point theorem. $\square$

## 5.6 Fully Discrete Weak Galerkin Scheme

We divide the time period $(0,T]$ into $M$ uniformly distributed sub-intervals $J_n = (t_{n-1}, t_n]$ for $n = 1, 2, \cdots M$ with $t_0 = 0$, $t_n = n\tau$ and $t_M = T$, where $\tau = \frac{T}{M}$ is the time step.

For any continuous function $\varphi : [0,T] - L^2(\Omega)$, define $\varphi^n = \varphi(t_n)$. We use backward Euler finite difference scheme for the temporal discretization. So for a sequence $\{\varphi^n\}_0^M \subset L^2(\Omega)$, we define

$$\delta_t^2\varphi^n = (\varphi^n - 2\varphi^{n-1} + \varphi^{n-2})/\tau^2 \quad \text{and} \quad \delta_t\varphi^n = (\varphi^n - \varphi^{n-1})/\tau.$$

---

**Algorithm 1** : Fully Discrete WG Method

The fully discrete weak Galerkin finite element scheme for solving problem (5.2.1) is defined as follows: Find $U^n = \{U_0^n, U_b^n\} \in \mathcal{W}_h$ such that $U_b^n = \mathcal{Q}_k^b \tilde{g}$ on $\partial\Omega$ and for any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, there holds

$$(\delta_t^2 U_0^n, \phi_0) + c^2 \mathcal{A}_w(U^n, \phi_h) + b\mathcal{A}_w(\delta_t U^n, \phi_h) = 2\sigma(U_0^n \delta_t^2 U_0^n + (\delta_t U_0^n)^2, \phi_0) + (f^n,\ \phi_0),\ n \geq 2, \tag{5.6.1}$$

with $U^0 = \mathcal{Q}_h u_*$ and $U^1 = U^0 + \tau \mathcal{Q}_h u^*$.

---

Again, for any continuous function $\varphi : [0, T] - L^2(\Omega)$, define

$$N_h(\varphi^n, \varphi^{n-1}, \varphi^{n-2}) = \varphi^n(\varphi^n - 2\varphi^{n-1} + \varphi^{n-2}) + (\varphi^n - \varphi^{n-1})^2 \text{ for } 2 \leq n \leq M.$$

Then the fully discrete WG scheme (5.6.1) can be rewritten as

$$\begin{aligned}&\frac{1}{\tau^2}(U_0^n - 2U_0^{n-1} + U_0^{n-2}, \phi_0) + c^2\mathcal{A}_w(U^n, \phi_h) + \frac{b}{\tau}\mathcal{A}_w(U^n - U^{n-1}, \phi_h)\\ &= \frac{2\sigma}{\tau^2}(N_h(U_0^n, U_0^{n-1}, U_0^{n-2}), \phi_0) + (f^n,\ \phi_0),\end{aligned}$$

or equivalently

$$\begin{aligned}&(U_0^n, \phi_0) + \tau^2 c^2 \mathcal{A}_w(U^n, \phi_h) + \tau b\mathcal{A}_w(U^n, \phi_h)\\ &= (2U_0^{n-1} - U_0^{n-2}, \phi_0) + \tau b\mathcal{A}_w(U^{n-1}, \phi_h) + 2\sigma(N_h(U_0^n, U_0^{n-1}, U_0^{n-2}), \phi_0) + \tau^2(f^n,\ \phi_0).\end{aligned}$$

To resolve the nonlinear system, we use a fixed-point iteration as follows:

---

**Algorithm 2** : Iterative Fully Discrete WG Method

---

The iterative fully discrete weak Galerkin finite element scheme for solving problem (5.2.1) is defined as follows: iterate over $i = 1, 2, \cdots, N_{iter}$ with $U^{n,0} = U^{n-1}$ and

1. Solve step: find $U^{n,i} = \{U_0^{n,i}, U_b^{n,i}\} \in \mathcal{W}_h$ such that $U_b^{n,i} = \mathcal{Q}_k^b \tilde{g}$ on $\partial\Omega$ and

$$\begin{aligned}
&(U_0^{n,i}, \phi_0) + \tau^2 c^2 \mathcal{A}_w(U^{n,i}, \phi_h) + \tau b \mathcal{A}_w(U^{n,i}, \phi_h)\\
&= (2U_0^{n-1} - U_0^{n-2}, \phi_0) + \tau b \mathcal{A}_w(U^{n-1}, \phi_h)\\
&\quad + 2\sigma(N_h(U_0^{n,i-1}, U_0^{n-1}, U_0^{n-2}), \phi_0) + \tau^2(f^n,\ \phi_0),
\end{aligned} \tag{5.6.2}$$

   for $n \geq 2$ with $U^0 = \mathcal{Q}_h u_*$ and $U^1 = U^0 + \tau \mathcal{Q}_h u^*$.

2. Check termination criterion: if

$$\frac{\|U^{n,i} - U^{n,i-1}\|}{\|U^{n,i}\|} < TOL,$$

   then set $U^n = U^{n,i}$.

---

### 5.6.1 Numerical Experiments

In this section, we conduct numerical experiments to illustrate our theoretical findings. The parameters in fixed-point iteration are chosen to be $TOL = 10^{-10}$ and the maximum number of iteration $N_{iter} = 100$ throughout all experiments. Figure 5.6.1 shows the first two level of polygonal grids used in our computation, which are generated by the MATLAB software package `PolyMesher` [146].

**Example 5.6.1.** (Test case 1: Exact solution known) Consider the Westervelt's equation (5.2.1) with the domain $\Omega = [0, 1] \times [0, \frac{2}{3}\sqrt{3}]$ and the final time $T = 0.8$. The parameters in the equation (5.2.1) are chosen to be

$$c = 1, b = 10^{-5}, \beta_a = 10^{-4}, \rho_m = 1.$$

Choosing the source term $f$, the boundary condition data $\tilde{g}$ and the initial conditions data $(u_*, u^*)$ such that the exact solution is

$$u(t, x, y) = \sin(4\pi t)\sin(4\pi x).$$

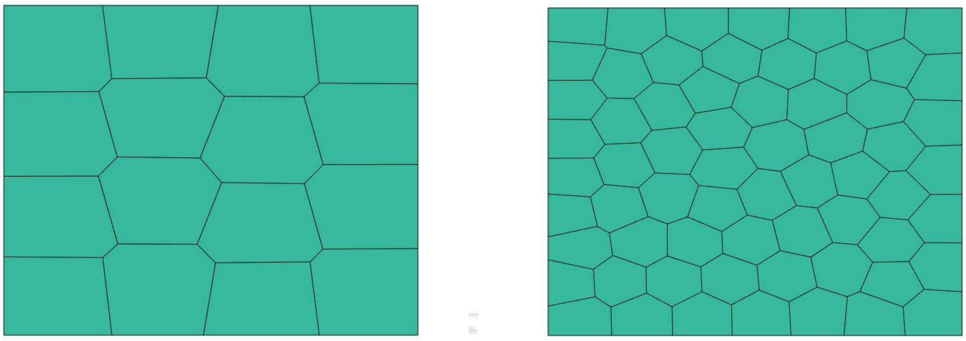

Figure 5.6.1: Sample of polygonal grids with $4 \times 4$ elements(left) and $8 \times 8$ elements(right).

We perform the numerical test on a polygonal mesh, as shown in Figure 5.6.1. The convergence histories of both the $L^2$ error and gradient error are illustrated in Figure 5.6.2 for polynomial degrees $k = 1, 2$. The obtained convergence rates align with the theoretical insights outlined in Theorem 5.5.1.

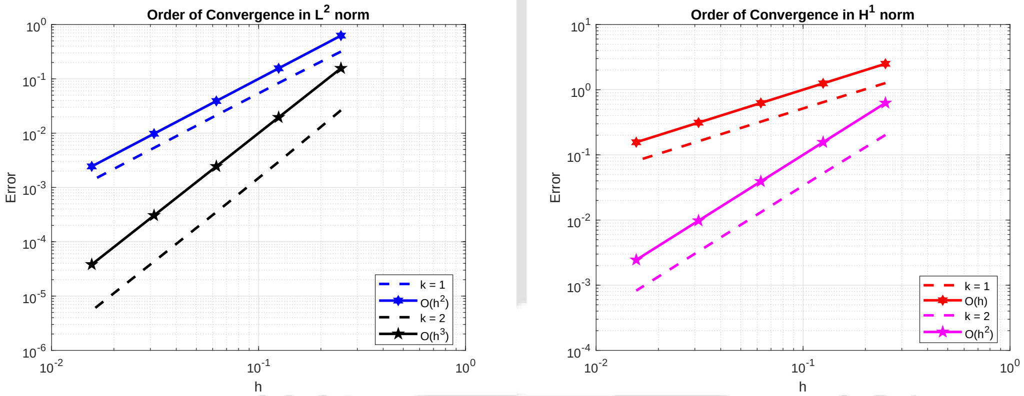

Figure 5.6.2: Log-log plot of the errors versus mesh size at time $T = 0.8$ for WG spaces $k = 1$ & $k = 2$ with $\tau = h^{k+1}$ for Example 5.6.1.

**Example 5.6.2.** (Test case 2: Exact solution unknown) Consider the Westervelt's equation (5.2.1) with a more realistic setting. The computational domain is $\Omega = [0, S] \times [0, H]$ with $S = 0.02\sqrt{3}$ m and $H = 0.02$ m. The physical parameters are now set to be

$$c = 1500\, m/s, b = 6 \times 10^{-9}\, m^2/s, \beta_a = 7, \rho_m = 1000\, kg/m^3.$$

The final time is chosen to be $T = 2.4 \times 10^{-5}\, s$, resolved by a step size of $\tau = 2 \times 10^{-9}\, s$. The source term $f$ is set to be zero. The excitation signal is given in the form $\psi(t, x, y) =$

$\psi_1(t)\psi_2(x,y)$. Here in the temporal part responsible for the initialization of the wave oscillations is given by

$$\psi_1(t) = \begin{cases} \frac{1}{4}(\nu t)^2 A\sin(\omega t), & t < 2/v, \\ A\sin(\omega t), & t \geq 2/v, \end{cases}$$

where the driving frequency $\nu = 210\ kHz$ and the amplitude $A = 0.01\,m^2/s^2$, the angular frequency $\omega = 2\pi\nu$. The computational WG solution is visually depicted by the polygonal grid as shown in Figure 5.6.3. Boundary parts enforce homogeneous Dirichlet data, except for the left boundary's excitation segment, which features non-zero Dirichlet data denoted by $\tilde{g}$. On the right side, the solution is assessed along the axis of symmetry, indicated by the blue line. The spatial part is given by a mollifier-type function in order to get a spatially smooth transition between the inhomogeneous excitation and the homogeneous remaining boundary data. In particular, we have

$$\psi_2(x,y) = \begin{cases} 0, & (x,y) = (0,0), \\ \exp\left(1 - \frac{1}{1-\left(\frac{y-0.005}{0.005}\right)^2}\right), & (x,y) \in \{0\} \times (0, 0.005), \\ 1, & (x,y) \in \{0\} \times [0.005, 0.015], \\ \exp\left(1 - \frac{1}{1-\left(\frac{y-0.015}{0.005}\right)^2}\right), & (x,y) \in \{0\} \times (0.015, 0.02), \\ 0, & (x,y) = (0, 0.02), \\ 0, & (x,y) \in D = \{(x,y) \in \Omega :\ x > 0\}. \end{cases}$$

The function $u$ restricted to $\partial\Omega$ is used as the boundary data $\tilde{g}$ in problem (5.2.1) and the initial data are taken to be $(u_*, u^*) = (\psi(0,x,y), \psi_t(0,x,y))$. For the details, we refer to numerical Example 8.2 in [7].

The exact solution is unknown in this more realistic setting. To analyze the convergence behavior of the WG solution $u_h = \{u_0, u_b\}$ with respect to $h$-refinement, we compute the measurement

$$\mathbf{M}(u_h) = \|u_0\|_{L^\infty(0,T;L^2(\Omega))}$$

on different discretization levels. Noting that (cf. [7])

$$|\mathbf{M}(u) - \mathbf{M}(u_h)| \leq \|u - u_0\|_{L^\infty(0,T;L^2(\Omega))},$$

so we expect that, for $k$ fixed, $\mathbf{M}(u_h)$ behaves asymptotically as $a_1 + a_2 h^{k+1}$ for some constants $a_1$ and $a_2$. The resulting data has been then fitted to a curve $a_1 + a_2 h^{\tilde{\gamma}}$ by employing the nonlinear least-squares solver `lsqcurvefit` in MATLAB with the starting point $(1, 1, 2)$. We obtain $\tilde{\gamma} = 2.0$ for the rate of convergence.

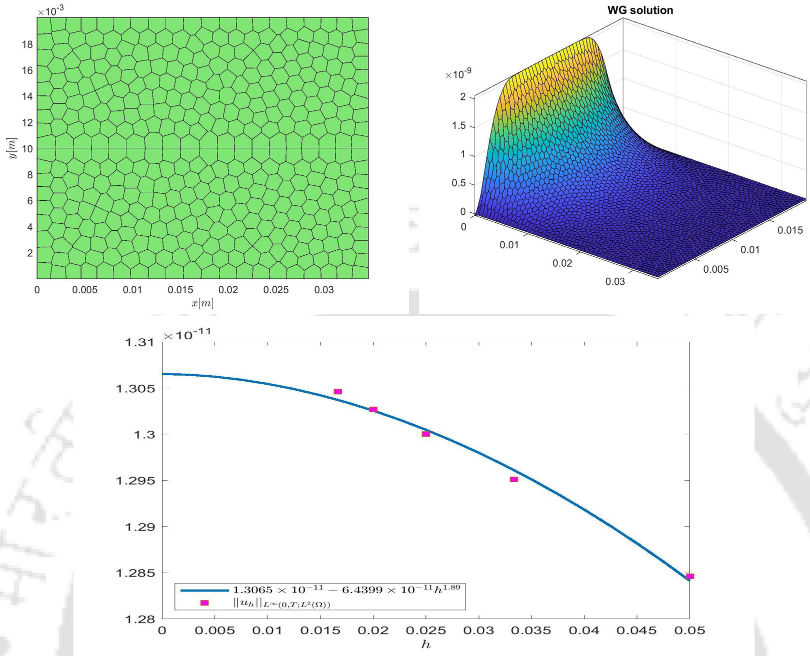

Figure 5.6.3: The WG surf plot and a least-square fitted extrapolation-curve for Example 5.6.2.

**Example 5.6.3.** (Test case 3: Linearized Westervelt's equation) Next, we have executed a numerical example of linearized Westervelt's equation to examine the theoretical results. Here we numerically verified the following linearized Westervelt's equation with variable coefficients.

$$\alpha(\mathbf{x}, t)u_{tt} - \nabla \cdot (c^2 \nabla u + b \nabla u_t) + \beta(\mathbf{x}, t)u_t = f(\mathbf{x}, t) \quad \text{in } \Omega \times [0, T], \tag{5.6.3}$$

where $\Omega = (0, 1) \times (0, 1)$. It is worth to note that only semi-discrete error analysis has been discussed in [131] and its provides a scope for the generalization of these works to higher order of accuracy methods. The source term appearing in the above problem is selected by setting the exact solution as

$$u = \exp(-t) \sin(\pi x) \sin(\pi y) \sin(\pi x + \pi y - t),$$

with coefficients choosing as $\beta = x^2 y + t, c^2 = 1500m/s, b = 1$ and $\alpha = xyt^2$. Here we have implemented the equation (5.6.3) with linear and quadratic weak Galerkin approximation space at final time $T = 1$. It is clear from Table 5.6.1 that we have obtained optimal order of convergence in both $L^2$ and discrete $H^1$ norms.

Table 5.6.1: The history of convergence under the $|||\cdot|||$ norm and $L^2$ norm at final time $t = 1$ for Example 5.6.3 with $\tau = h^{k+1}$.

| $k = 1$ | | | | | $k = 2$ | | | |
|---|---|---|---|---|---|---|---|---|
| $h$ | $\|e_h\|$ | EOC | $\|\|\|e_h\|\|\|$ | EOC | $\|e_h\|$ | EOC | $\|\|\|e_h\|\|\|$ | EOC |
| 1/2 | 7.786e-02 | - | 4.753e-01 | - | 8.081e-02 | - | 5.309e-01 | - |
| 1/4 | 4.349e-02 | 0.84 | 4.040e-01 | 0.23 | 2.198e-02 | 1.87 | 2.419e-01 | 1.13 |
| 1/8 | 1.609e-02 | 1.43 | 2.536e-01 | 0.67 | 3.971e-03 | 2.46 | 9.177e-02 | 1.39 |
| 1/16 | 4.186e-03 | 1.94 | 1.280e-01 | 0.98 | 5.099e-04 | 2.96 | 2.410e-02 | 1.92 |
| 1/32 | 1.052e-03 | 1.99 | 6.393e-02 | 1.00 | 6.386e-05 | 2.99 | 6.094e-03 | 1.98 |
| 1/64 | 2.634e-04 | 1.99 | 3.194e-02 | 1.00 | 7.982e-06 | 2.99 | 1.527e-03 | 1.99 |

# 6

# WG-FEMs for General Hyperbolic Equations with Interface

In this chapter, we are concerned with *a priori* error analysis of weak Galerkin (WG) finite element approximations to a general linear second order hyperbolic interface problem (1.1.10) with variable coefficients on polygonal meshes. Semidiscrete error analysis in $L^\infty(L^2)$ norm as well as discrete $H^1$ norm have been executed for the weak space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$ is an integer. For a fully discrete scheme, we employ the Crank-Nicolson scheme for temporal discretization after reformulating the governing equation as a first-order system. The optimal order of convergence in $L^\infty(L^2)$ norm is derived for the fully discrete scheme. Finally, some numerical experiments are performed in a two-dimensional setting to support theoretical results.

## 6.1 Introduction

Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and $\Omega_1$ is an open domain with $C^2$ smooth interface $\Gamma = \partial\Omega_1$, such that $\bar{\Omega}_1 \subsetneq \Omega$, and $\Omega_2 = \Omega \setminus \Omega_1$ (see, Figure 1.1.1). We consider the following general linear second order hyperbolic equation in multi-layered media

$$u_{tt} + \Theta u_t + \xi u - \nabla.(\eta \nabla u + \upsilon \nabla u_t) = f \quad \text{in} \quad \Omega \times (0, T], \quad T < \infty, \tag{6.1.1}$$

with the initial and boundary conditions

$$\begin{cases} u(0) = u^0, \quad u_t(0) = v^0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \times (0, T]. \end{cases} \tag{6.1.2}$$

The information between both the domains are transferred via following interface conditions

$$[u] = \Phi, \quad \& \quad \left[\eta \frac{\partial u}{\partial \mathbf{n}} + \upsilon \frac{\partial u_t}{\partial \mathbf{n}}\right] = \Psi \quad \text{on} \quad \Gamma \times [0, T], \tag{6.1.3}$$

where $J = (0, T]$ is the finite terminal period, $[u] = u_1|_\Gamma - u_2|_\Gamma$ and $\left[\eta\frac{\partial u}{\partial \mathbf{n}} + \upsilon\frac{\partial u_t}{\partial \mathbf{n}}\right] = \eta_1\frac{\partial u_1}{\partial \mathbf{n}_1} + \upsilon_1\frac{\partial u_{1t}}{\partial \mathbf{n}_1} + \eta_2\frac{\partial u_2}{\partial \mathbf{n}_2} + \upsilon_2\frac{\partial u_{2t}}{\partial \mathbf{n}_2}$. Here $u_i$ is the restriction of $u$ in $\Omega_i$ and $\frac{\partial}{\partial \mathbf{n}_i}$ is the outer normal derivative with respect to $\Omega_i$ for $i = 1, 2$.

Coefficients are assumed to be positive real valued functions defined in $\Omega$. Further, initial data $\{u^0, v^0\}$ and the source function $f$ are sufficiently smooth in their respective domain of definition. Due to heterogeneity in the underlying media, the thermal properties of biological media vary between different layers. In $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ (see, Figure 1.1.1), we assume that the physical coefficients are discontinuous and piecewise constants. So we write

$$(\Theta, \xi, \eta, \upsilon) = \begin{cases} (\Theta_1, \xi_1, \eta_1, \upsilon_1) \;\text{ in }\; \Omega_1, \\ (\Theta_2, \xi_2, \eta_2, \upsilon_2) \;\text{ in }\; \Omega_2. \end{cases}$$

Equation (6.1.1) is provoked by numerous applications of hyperbolic interface problems in medicine and industrial fields like viscous wave equation, network of linked beams, hybrid chimney, bio heat transfer etc. Various numerical methods have been attempted to handle interface problems as a consequence of practical achievements in engineering and industrial applications. Numerical techniques based on finite element framework can be grouped by conforming FEMs, Mixed FEMs, Discontinuous Galerkin(DG) and Immersed FEMs. Based on the domain discretization, there are two major classes of FEMs, namely, interface-fitted FEMs and unfitted FEMs. Convergence analysis without the interface for a general linear hyperbolic equation using FEMs has been studied in the literature (cf. [20, 91, 102, 133]). Classical FEMs for interface problems are mainly based on interface-fitted meshes. Although the flux discontinuity of the solutions can be captured in variation formulation, the discontinuity of the solution neither fits in the variation formulation nor is satisfied in classical FEM solution spaces. So the conforming FEMs for interface problems assume continuity of the solutions along the interfaces. Finite element analysis with the interface for linear hyperbolic equations has been extensively discussed in [39, 40, 41, 49] and references therein. Also, *a priori* error estimates of DPL-Bio heat problem in the heterogeneous medium using FEMs has been studied by J. Dutta et al. (cf. [54] ) with a homogeneous jump. Recently, weak Galerkin FEMs for interface problems are well studied in the literature (cf. [44, 46, 47, 48, 100, 101]). However, to our knowledge, weak Galerkin finite element analysis for general second order hyperbolic equations with a non-homogeneous interface condition has not been studied yet.

In this chapter, we focus on developing and analyzing a WG finite element approach to the general linear second order hyperbolic interface problem (6.1.1). The main aspect of our proof is to use a non standard elliptic type projection operator instead of the usual elliptic projection. We have achieved $\mathcal{O}(h^k)$ and $\mathcal{O}(h^{k+1})$ for the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$ in energy norm and $L^2$ norm, respectively for semidiscrete error estimates. Further, semidiscrete error analysis has been extended to fully discrete scheme. The fully discrete space-time discretizations can be revised as the Crank-Nicolson discretization of the reformulation of the governing equation in the first-order system as in Baker [14]. We have achieved $\mathcal{O}(\tau^2 + h^{k+1})$ convergence rate for fully discrete $L^2$ error estimate. Finally, several numerical experiments have been reported in order to establish the efficiency of the WG method in scientific computing.

The rest of this chapter is organized as follows: Sec. 6.2 concerns with weak Galerkin discretization for interface problems. Also, in Sec. 6.3 weak Galerkin approximation to IBVP (6.1.1)-(6.1.2) and the existence, uniqueness of approximate solutions is described. Further, error estimates in both $L^\infty(H^1)$ and $L^\infty(L^2)$ norms are analyzed in this section. Sec. 6.4 is devoted to fully discrete analysis by employing the Crank-Nicolson implicit scheme. Finally in Sec. 6.5 some numerical experiments are presented to validate theoretical results.

## 6.2 Weak Galerkin Discretization

Let $\mathcal{T}_h$ be a polygonal partition of the domain $\Omega$ with mesh size $h$. We require that the edges of the elements in $\mathcal{T}_h$ align with the interface $\Gamma$. A simple and efficient interface fitted mesh generation algorithm has been proposed in [29]. Elements in such interface fitted meshes are not restricted to simplices but can be polygons or polyhedra. A typical body fitted discretization is presented in Figure 6.2.1. Thus, the partition $\mathcal{T}_h$ can be grouped into two set of elements denoted by $\mathcal{T}_h^1 = \mathcal{T}_h \cap \Omega_1$ and $\mathcal{T}_h^2 = \mathcal{T}_h \cap \Omega_2$, respectively. Let $\mathcal{E}_h$ be the collection of all edges of the polygons in $\mathcal{T}_h$ and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the collection of all interior edges. Also, let $\Gamma_h$ denote the subset of $\mathcal{E}_h$ of all edges on $\Gamma$. Further, for any $\mathcal{K} \in \mathcal{T}_h$, denote its diameter by $h_{\mathcal{K}}$ and for $\mathcal{T}_h$, mesh size by $h$, defined as $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$. Note that,

$$\begin{aligned} \mathcal{T}_h &= \{\mathcal{K} \in \mathcal{T}_h : \mathcal{K} \not\subseteq \Omega_2 \text{ or } \partial\mathcal{K} \cap \Gamma = \emptyset\} \cup \{\mathcal{K} \in \mathcal{T}_h : \mathcal{K} \subseteq \Omega_2 \text{ and } \partial\mathcal{K} \cap \Gamma \neq \emptyset\} \\ &= \mathcal{T}_1 \cup \mathcal{T}_2. \end{aligned} \tag{6.2.1}$$

Clearly, $\mathcal{T}_1$ consist all elements in $\Omega_1$ as well as all non interface elements in $\Omega_2$ and $\mathcal{T}_2$ contains all interface elements in $\Omega_2$. Details follow in [128].

Let $\mathcal{K} \in \mathcal{T}_h$ be an element with boundary $\partial\mathcal{K}$. For an integer $k \geq 1$, let $\mathbf{P}_k(\mathcal{K})$ be the space of all polynomials of degree not greater than $k$ on the polygon $\mathcal{K} \in \mathcal{T}_h$. Similarly, $\mathbf{P}_k(e)$ be the set of all polynomials of degree not greater than $k$ on the edge $e \in \mathcal{E}_h$. A

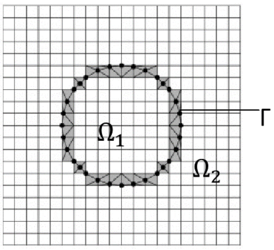

Figure 6.2.1: A typical interface fitted discretization

discrete weak function $\phi_h = \{\phi_0, \phi_b\}$ on $\mathcal{K}$ refers to the weak function $\phi_h = \{\phi_0, \phi_b\}$, where $\phi_0 \in \mathcal{P}_k(\mathcal{K})$ and $\phi_b \in \mathbf{P}_k(e), e \subset \partial\mathcal{K}$. Let $\mathcal{W}(k, k; \mathcal{K})$ be the space of all discrete weak functions on $\mathcal{K}$; i.e.

$$\mathcal{W}(k, k; \mathcal{K}) = \Big\{\phi_h = \{\phi_0, \phi_b\} : \phi_0 \in \mathbf{P}_k(\mathcal{K}), \phi_b \in \mathbf{P}_k(e), e \subset \partial\mathcal{K}\Big\}.$$

Then the global weak Galerkin finite space $\mathcal{W}_h$ is constructed from the local finite space $\mathcal{W}(k, k; \mathcal{K})$ as

$$\mathcal{W}_h = \big\{\phi_h = \{\phi_0, \phi_b\} : \phi_h|_{\mathcal{K}} \in \mathcal{W}(k, \mathcal{K}), [\phi_h]_e = 0, \forall e \in \mathcal{E}_h^0\big\}.$$

Here $[\phi_h]_e$ denotes the jump of $\phi_h$ along interior edges $e \in \mathcal{E}_h^0$.

Let $\mathcal{W}_h^0$ be the subspace of $\mathcal{W}_h$ containing all weak functions, which vanishes on boundary $\partial\Omega$; i.e.

$$\mathcal{W}_h^0 = \{\phi_h \in \mathcal{W}_h : \phi_b|_{\partial\Omega} = 0\}.$$

Now for $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, the weak gradient of $\phi_h$, denoted by $\nabla_w\phi_h$, is a linear functional from $[\mathbf{P}_{k-1}(\mathcal{K})]^2$ to $\mathbb{R}$, which acts on each $\mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2$ as

$$(\nabla_w\phi_h, \mathbf{q}) = -\int_{\mathcal{K}} \phi_0(\nabla \cdot \mathbf{q})d\mathcal{K} + \int_{\partial\mathcal{K}} \phi_b(\mathbf{q} \cdot \mathbf{n})ds \quad \forall \mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2, \tag{6.2.2}$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial\mathcal{K}$.

By using divergence theorem to equation (6.2.2), we obtain

$$(\nabla_w\phi_h, \mathbf{q}) = (\nabla\phi_0, \mathbf{q})_{\mathcal{K}} + \langle\phi_b - \phi_0, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial\mathcal{K}} \;\; \forall\, \mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2. \tag{6.2.3}$$

For the weak Galerkin approximation, we define bilinear maps $\mathcal{A}_1, \mathcal{A}_2$ as:

$$\mathcal{A}_1(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} (\eta\nabla_w u_h, \nabla_w\phi_h)_{\mathcal{K}} + \mathbf{S}(u_h, \phi_h),$$

$$\mathcal{A}_2(u_h, \phi_h) = \sum_{\mathcal{K}\in\mathcal{T}_h} (\upsilon\nabla_w u_h, \nabla_w\phi_h)_{\mathcal{K}} + \mathbf{S}(u_h, \phi_h),$$

for $u_h, \phi_h \in \mathcal{W}_h$. Here $\mathbf{S}(\cdot,\cdot) : \mathcal{W}_h \times \mathcal{W}_h \to \mathbb{R}$ is the stabilizer based on element-boundary-discrepancy as defined in Example 1.4.2 in Chapter 1. So for $u_h = \{u_0, u_b\}$, $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, the stabilizer $\mathbf{S}(\cdot,\cdot)$ (cf. [104]) is given by

$$\mathbf{S}(u_h, \phi_h) = \sum_{\mathcal{K} \in \mathcal{T}_h} h_{\mathcal{K}}^{-1} \langle (u_b - u_0|_{\partial\mathcal{K}}), (\phi_b - \phi_0|_{\partial\mathcal{K}}) \rangle_{\partial\mathcal{K}}. \tag{6.2.4}$$

For any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$, we define a mesh-dependent semi-norm $\|\cdot\|_{1,h}$ as in Chapter 1 by

$$\|\phi_h\|_{1,h}^2 = \sum_{\mathcal{K} \in \mathcal{T}_h} \left( \|\nabla \phi_0\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^{-1} \|\phi_b - \phi_0\|_{\partial\mathcal{K}}^2 \right). \tag{6.2.5}$$

In fact, the semi-norm $\|\cdot\|_{1,h}$ is a norm in $\mathcal{W}_h^0$.

Next, the following Lemma establishes the coercivity of the bilinear maps $\mathcal{A}_1(\cdot,\cdot)$ and $\mathcal{A}_2(\cdot,\cdot)$ on WG space $\mathcal{W}_h$. Details follows from [151].

**Lemma 6.2.1.** *Let $\phi_h \in \mathcal{W}_h$, then there are two positive constants $\mathbf{C}_*$ & $\mathbf{C}^* > 0$ such that following inequality holds true*

$$\mathbf{C}_* \|\phi_h\|_{1,h}^2 \leq \mathcal{A}_i(\phi_h, \phi_h) \leq \mathbf{C}^* \|\phi_h\|_{1,h}^2 \quad \textit{for} \ \ i = 1, 2. \tag{6.2.6}$$

Now we define the energy norm (triple bar norm) $|||\cdot|||$ on the space $\mathcal{W}_h^0$ as:

$$|||\phi_h|||^2 = (\nabla_w \phi_h, \nabla_w \phi_h) + \mathbf{S}(\phi_h, \phi_h) \ \ \forall \ \phi_h \in \mathcal{W}_h^0. \tag{6.2.7}$$

Therefore, it can be proved (cf. Lemma 7.2 [126]) that there exist constants $C_* > 0$ and $C^* > 0$ such that

$$C_* |||\phi_h|||^2 \leq \mathcal{A}_i(\phi_h, \phi_h) \leq C^* |||\phi_h|||^2 \ \ \text{for} \ \ i = 1, 2. \tag{6.2.8}$$

Hence, from the coercive inequality (6.2.6) and inequality (6.2.8) we can conclude that discrete $H^1$ norm $\|\cdot\|_{1,h}$ and triple norm $|||\cdot|||$ are equivalent. Further, following Poincaré type inequality holds true (cf. [126])

$$\|\phi_h\| = \|\phi_0\| \leq C |||\phi_h||| \ \forall \ \phi_h \in \mathcal{W}_h^0, \tag{6.2.9}$$

where $C$ is a positive constant.

Now we define some usual $L^2$ projections. On each $\mathcal{K} \in \mathcal{T}_h$, let the operator $\mathcal{Q}_k^0 : L^2(\mathcal{K}) \to \mathbf{P}_k(\mathcal{K})$ be the standard $L^2$ projection and for an edge $e \in \mathcal{E}_h$, let the operator $\mathcal{Q}_k^b : L^2(e) \to \mathbf{P}_k(e)$ be the standard $L^2$ projection. Also, let $\mathcal{Q}_h$ be the standard $L^2$ projection onto the WG finite space $\mathcal{W}_h$ such that $\mathcal{Q}_h v = \{\mathcal{Q}_k^0, \mathcal{Q}_k^b v\}$. To ensure that

$\mathcal{Q}_k^b v$ takes unique value on any $e \in \mathcal{E}_h$, $\mathcal{Q}_k^b v$ is defined as follows:

$$\mathcal{Q}_k^b v = \begin{cases} \mathcal{Q}_k^b(v|_{\mathcal{K}\cap e}) & \text{if } \ e \subseteq \Gamma \ \text{ and } \ \mathcal{K} \subset \Omega_1, \\ \mathcal{Q}_k^b(v|_{\mathcal{K}\cap e}) + \mathcal{Q}_k^b \Phi & \text{if } \ e \subseteq \Gamma \ \text{ and } \ \mathcal{K} \subset \Omega_2, \\ \mathcal{Q}_k^b(v|_{\mathcal{K}\cap e}) & \text{if } \ e \nsubseteq \Gamma \ \text{ and } \ \mathcal{K} \in \mathcal{T}_h. \end{cases}$$

Similarly, let $\mathbb{Q}_{k-1} : [L^2(\mathcal{K})]^2 \to [\mathbf{P}_{k-1}(\mathcal{K})]^2$ be standard $L^2$ projection.
Now we recall the following Lemma that connects $\mathcal{Q}_h$ and $\mathbb{Q}_{k-1}$ operators from the literature (cf. [128]).

**Lemma 6.2.2.** *Let $\mathcal{Q}_h$ and $\mathbb{Q}_{k-1}$ be $L^2$ projections as defined. Then for each element $\mathcal{K} \in \mathcal{T}_h$ and for $\mathbf{q} \in [\mathbf{P}_{k-1}(\mathcal{K})]^2$, we have*

$$\begin{aligned} (\nabla_w(\mathcal{Q}_h v), \mathbf{q})_{\mathcal{K}} &= (\mathbb{Q}_{k-1}(\nabla v), \mathbf{q})_{\mathcal{K}}, \quad \text{if } \ \mathcal{K} \in \mathcal{T}_1, \\ (\nabla_w(\mathcal{Q}_h v), \mathbf{q})_{\mathcal{K}} &= (\mathbb{Q}_{k-1}(\nabla v), \mathbf{q})_{\mathcal{K}} + \langle \Phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial\mathcal{K}\cap\Gamma}, \quad \text{if } \ \mathcal{K} \in \mathcal{T}_2, \end{aligned} \tag{6.2.10}$$

*where $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined as in (6.2.1).*

Also, recall another crucial Lemma for approximation properties of the projections $\mathcal{Q}_k^0$ and $\mathbb{Q}_{k-1}$. For details, we refer to [128] (Proof of Lemma 3.4 therein).

**Lemma 6.2.3.** *Let $\mathcal{T}_h$ be a polygonal partition of $\Omega$ that meets the shape regularity conditions as described in [153]. Then for $\varphi \in H^{k+1}(\Omega_i)$, $i = 1, 2$, we have for $0 \le r \le k$,*

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \left( \|\varphi - \mathcal{Q}_k^0\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^2 \|\nabla(\varphi - \mathcal{Q}_k^0\varphi)\|_{\mathcal{K}}^2 \right) \le C h^{2(r+1)} \sum_{i=1}^{2} \|\varphi\|_{H^{r+1}(\Omega_i)}^2,$$

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \left( \|\nabla\varphi - \mathbb{Q}_{k-1}\nabla\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^2 \|\nabla(\nabla\varphi - \mathbb{Q}_{k-1}(\nabla\varphi))\|_{\mathcal{K}}^2 \right) \le C h^{2r} \sum_{i=1}^{2} \|\varphi\|_{H^{r+1}(\Omega_i)}^2.$$

**Remark 6.2.1.** *Above Lemma 6.2.3 is also true element-wise for every $\mathcal{K} \in \mathcal{T}_h$, i.e.,*

$$\|\varphi - \mathcal{Q}_k^0\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^2 \|\nabla(\varphi - \mathcal{Q}_k^0\varphi)\|_{\mathcal{K}}^2 \le C h_{\mathcal{K}}^{2(r+1)} \|\varphi\|_{H^{r+1}(\mathcal{K})}^2 \quad \text{for } \ 0 \le r \le k.$$

Also, for any $e \subset \Gamma$ be a common edge of two elements $\mathcal{K}_1 \subset \Omega_1$ and $\mathcal{K}_2 \subset \Omega_2$, we define following forms

$$\begin{aligned} \langle \Phi, \eta\nabla_w\phi_h.\mathbf{n}\rangle_{\Gamma_h} &= \sum_{\mathcal{K}\in\mathcal{T}_2} \langle \Phi, \eta_2\nabla_w\phi_h.\mathbf{n}\rangle_{\partial\mathcal{K}\cap\Gamma_h} = \sum_{e\in\Gamma_h} \langle \Phi, \eta\nabla_w(\phi_h|_{\mathcal{K}_2}).\mathbf{n}\rangle_e, \\ \langle \Phi, \upsilon\nabla_w\phi_h.\mathbf{n}\rangle_{\Gamma_h} &= \sum_{\mathcal{K}\in\mathcal{T}_2} \langle \Phi, \upsilon_2\nabla_w\phi_h.\mathbf{n}\rangle_{\partial\mathcal{K}\cap\Gamma_h} = \sum_{e\in\Gamma_h} \langle \Phi, \upsilon\nabla_w(\phi_h|_{\mathcal{K}_2}).\mathbf{n}\rangle_e, \\ h^{-1}\langle \Phi, \phi_0 - \phi_b\rangle_{\Gamma_h} &= \sum_{\mathcal{K}\in\mathcal{T}_2} h^{-1}\langle \Phi, \phi_0 - \phi_b\rangle_{\partial\mathcal{K}\cap\Gamma_h} = \sum_{e\in\Gamma_h} h^{-1}\langle \Phi, \phi_0|_{\mathcal{K}_2} - \phi_b\rangle_e, \\ \langle \Psi, \phi_b\rangle_{\Gamma_h} &= \sum_{\mathcal{K}\in\mathcal{T}_2} \langle \Psi, \phi_b\rangle_{\partial\mathcal{K}\cap\Gamma_h} = \sum_{e\in\Gamma_h} \langle \Psi, \phi_b\rangle_e. \end{aligned}$$

Next, we consider another result related to the projection operator $\mathcal{Q}_h$, for every $\varphi \in H^1(\Omega_1) \cap H^1(\Omega_2)$, we have

$$\begin{aligned}
|\!|\!|\mathcal{Q}_h\varphi|\!|\!|^2 &\leq C \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla \mathcal{Q}_k^0\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^{-1}\|\mathcal{Q}_k^0\varphi - \mathcal{Q}_k^b\varphi\|_{\partial\mathcal{K}}^2) \\
&\leq C \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla \mathcal{Q}_k^0\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^{-1}\|\mathcal{Q}_k^0\varphi - \varphi\|_{\partial\mathcal{K}}^2) \\
&\leq C \sum_{\mathcal{K}\in\mathcal{T}_h} (\|\nabla \mathcal{Q}_k^0\varphi\|_{\mathcal{K}}^2 + h_{\mathcal{K}}^{-2}\|\mathcal{Q}_k^0\varphi - \varphi\|_{\mathcal{K}}^2 + \|\nabla(\mathcal{Q}_k^0\varphi - \varphi)\|_{\mathcal{K}}^2) \\
&\leq C \sum_{i=1}^{2} \|\varphi\|_{H^1(\Omega_i)}^2. \qquad (6.2.11)
\end{aligned}$$

Here we have used the fact $\|\mathcal{Q}_k^0\varphi - \mathcal{Q}_k^b\varphi\|_{\partial\mathcal{K}} \leq \|\mathcal{Q}_k^0\varphi - \varphi\|_{\partial\mathcal{K}}$, standard trace inequality and Remark 6.2.1.

## 6.3 Semidiscrete WG-FEMs for General Hyperbolic Interface Problem

In this section, we deal with error estimates for the semidiscrete scheme. The optimal order of estimates in both $L^\infty(H^1)$ and $L^\infty(L^2)$ norms are determined under suitable regularity assumptions of the true solution.

The continuous-time weak Galerkin finite element approximation to the problem (6.1.1)-(6.1.2) can be described as follows: Find $u_h = \{u_0, u_b\} : (0, T] \to \mathcal{W}_h^0$, such that

$$\begin{aligned}
(u_{htt}, \phi_h) + \mathcal{B}(u_{ht}, \phi_h) + \mathcal{D}(u_h, \phi_h) + \mathcal{A}_1(u_h, \phi_h) + \mathcal{A}_2(u_{ht}, \phi_h) = (f, \phi_0) + \langle \Psi, \phi_b\rangle_{\Gamma_h} \\
+ \langle \Phi, \eta\nabla_w\phi_h.\mathbf{n}\rangle_{\Gamma_h} + \langle \Phi_t, \upsilon\nabla_w\phi_h.\mathbf{n}\rangle_{\Gamma_h} - h^{-1}\langle \Phi + \Phi_t, \phi_0 - \phi_b\rangle_{\Gamma_h}, \ \forall\ \phi_h \in \mathcal{W}_h^0,
\end{aligned} \qquad (6.3.1)$$

with $u_h(0) = \mathcal{Q}_h u^0$, $u_{ht}(0) = \mathcal{Q}_h v^0$ and bilinear maps $\mathcal{B}$, $\mathcal{D}$ are defined as

$$\mathcal{B}(u_h, \phi_h) = (\Theta u_h, \phi_h), \quad \mathcal{D}(u_h, \phi_h) = (\xi u_h, \phi_h).$$

Now we define the bilinear map $\mathcal{H}$ as

$$\mathcal{H}(\Psi, \Phi)(\phi_h) = \langle \Psi, \phi_b\rangle_{\Gamma_h} + \langle \Phi, \eta\nabla_w\phi_h.\mathbf{n}\rangle_{\Gamma_h} + \langle \Phi_t, \upsilon\nabla_w\phi_h.\mathbf{n}\rangle_{\Gamma_h} - h^{-1}\langle \Phi + \Phi_t, \phi_0 - \phi_b\rangle_{\Gamma_h}. \qquad (6.3.2)$$

Hence, equation (6.3.1) becomes

$$(u_{htt}, \phi_h) + \mathcal{B}(u_{ht}, \phi_h) + \mathcal{D}(u_h, \phi_h) + \mathcal{A}_1(u_h, \phi_h) + \mathcal{A}_2(u_{ht}, \phi_h) = (f, \phi_0) + \mathcal{H}(\Psi, \Phi)(\phi_h). \qquad (6.3.3)$$

Existence and uniqueness of the solution of (6.3.1) follows from the following theorem.

**Theorem 6.3.1.** *For each $h > 0$ there exists a unique function $u_h \in H^2(J; \mathcal{W}_h^0)$ satisfying (6.3.1).*

*Proof.* For a given $\mathcal{K} \in \mathcal{T}_h$, let $\{\phi_{0,i} : i = 1 \cdots N_0\}$ be the set of basis functions for $\mathcal{P}_k(\mathcal{K})$ and $\{\phi_{b,j} : j = 1 \cdots N_b\}$ be the set of basis functions for $\sum_{e \in \partial\mathcal{K} \cap \mathcal{E}_h} \mathcal{P}_k(e)$. Then for any $u_h = \{u_0, u_b\} \in \mathcal{W}_h^0$ can be written as

$$u_h|_{\mathcal{K}} = \left\{ \sum_{i=1}^{N_0} d_{0,i}(t)\phi_{0,i}, \sum_{j=1}^{N_b} d_{b,j}(t)\phi_{b,j} \right\},$$

where $d_{0,i}, d_{b,j} : [0, T] \to \mathbb{R}$ are the coefficient functions for $1 \leq i \leq N_0$ and $1 \leq j \leq N_b$.

Now for $1 \leq i \leq N_0 + N_b$, we write

$$\phi_{i,h} = \{\bar{\phi}_{0,i}, \bar{\phi}_{b,i}\},$$

where $\bar{\phi}_{0,i} = \phi_{0,i}$ for $1 \leq i \leq N_0$, $\bar{\phi}_{0,i} = 0$ for $N_0 + 1 \leq i \leq N_0 + N_b$ and $\bar{\phi}_{b,i} = 0$ for $1 \leq i \leq N_0$, $\bar{\phi}_{b,i} = \phi_{b,i-N_0}$ for $N_0 + 1 \leq i \leq N_0 + N_b$.

Similarly, we define $d_{i,h} = d_{0,i}$ for $1 \leq i \leq N_0$ and $d_{i,h} = d_{b,i-N_0}$ for $N_0 + 1 \leq i \leq N_0 + N_b$. Therefore,

$$u_h|_{\mathcal{K}} = \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\phi_{i,h} = \left\{ \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\bar{\phi}_{0,i}, \sum_{j=1}^{N_0+N_b} d_{j,h}(t)\bar{\phi}_{b,j} \right\} \text{ for } \mathcal{K} \in \mathcal{T}_h.$$

Now setting $\phi_h = \phi_{j,h}$ for $j = 1, 2, \cdots, N_0 + N_b$ in (6.3.3), we obtain

$$\begin{aligned} &\left( \sum_{i=1}^{N_0+N_b} d''_{i,h}(t)\bar{\phi}_{0,i}, \bar{\phi}_{0,j} \right) + \mathcal{B}\left( \sum_{i=1}^{N_0+N_b} d'_{i,h}(t)\bar{\phi}_{0,i}, \bar{\phi}_{0,j} \right) + \mathcal{D}\left( \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\bar{\phi}_{0,i}, \bar{\phi}_{0,j} \right) \\ &+ \mathcal{A}_1\left( \sum_{i=1}^{N_0+N_b} d_{i,h}(t)\phi_{i,h}, \phi_{j,h} \right) + \mathcal{A}_2\left( \sum_{i=1}^{N_0+N_b} d'_{i,h}(t)\phi_{i,h}, \phi_{j,h} \right) = (f, \bar{\phi}_{0,j}) + \mathcal{H}(\Psi, \Phi)(\phi_{j,h}). \end{aligned}$$

Hence, in matrix form, we need to find $D_h(t) = \begin{bmatrix} D_{0h} \\ D_{bh} \end{bmatrix}$, where $D_{0h} = [d_{1,h}\ d_{2,h} \cdots d_{N_0,h}]^T$ and $D_{bh} = [d_{N_0+1,h}\ d_{N_0+2,h} \cdots d_{N_0+N_b,h}]^T$ such that

$$\begin{cases} M_{h\mathcal{K}}D''_h(t) + K_{h\mathcal{K}}D'_h(t) + L_{h\mathcal{K}}D_h(t) + A_{1h\mathcal{K}}D_h(t) + A_{2h\mathcal{K}}D'_h(t) = F_{h\mathcal{K}}(t) + H_{h\mathcal{K}}(t), \\ D_h(0) = [d_{1,h,0}\ d_{2,h,0} \cdots d_{N_0+N_b,h,0}]^T, \\ D_{ht}(0) = [d_{1,h,1}\ d_{2,h,1} \cdots d_{N_0+N_b,h,1}]^T, \end{cases} \tag{6.3.4}$$

where $D_h(0)$ and $D_{ht}(0)$ are the coefficient vectors of $\mathcal{Q}_h u^0$ and $\mathcal{Q}_h v^0$ when expressed as a linear combination of the basis functions respectively, and $M_{h\mathcal{K}}, K_{h\mathcal{K}}, L_{h\mathcal{K}}$ are given by

$$M_{h\mathcal{K}} = \begin{bmatrix} \mathcal{M}_{00} & \mathcal{M}_{0b} \\ \mathcal{M}_{b0} & \mathcal{M}_{bb} \end{bmatrix}, \quad K_{h\mathcal{K}} = \begin{bmatrix} \mathcal{K}_{00} & \mathcal{K}_{0b} \\ \mathcal{K}_{b0} & \mathcal{K}_{bb} \end{bmatrix}, \quad L_{h\mathcal{K}} = \begin{bmatrix} \mathcal{L}_{00} & \mathcal{L}_{0b} \\ \mathcal{L}_{b0} & \mathcal{L}_{bb} \end{bmatrix},$$

where

$$\mathcal{M}_{00} = [(\bar{\phi}_{0,j},\ \bar{\phi}_{0,i})]_{N_0\times N_0},\ \mathcal{M}_{0b} = O_{0b},\ \mathcal{M}_{b0} = O_{b0},\ \mathcal{M}_{bb} = O_{bb},$$

$$\mathcal{K}_{00} = [\mathcal{B}(\bar{\phi}_{0,j}, \bar{\phi}_{0,i})]_{N_0\times N_0},\quad \mathcal{K}_{0b} = O_{0b},\quad \mathcal{K}_{b0} = O_{b0},\quad \mathcal{K}_{bb} = O_{bb},$$

$$\mathcal{L}_{00} = [\mathcal{D}(\bar{\phi}_{0,j}, \bar{\phi}_{0,i})]_{N_0\times N_0},\quad \mathcal{L}_{0b} = O_{0b},\quad \mathcal{L}_{b0} = O_{b0},\quad \mathcal{L}_{bb} = O_{bb}.$$

Also, $A_{1h\mathcal{K}}$ and $A_{2h\mathcal{K}}$ are given by

$$A_{1h\mathcal{K}} = [A_{i,j}]_{N_0+N_b\times N_0+N_b} = \begin{bmatrix} A_{00} & A_{0b} \\ A_{b0} & A_{bb} \end{bmatrix}, \quad A_{2h\mathcal{K}} = [B_{i,j}]_{N_0+N_b\times N_0+N_b} = \begin{bmatrix} B_{00} & B_{0b} \\ B_{b0} & B_{bb} \end{bmatrix},$$

where $A_{i,j} = \mathcal{A}_1(\phi_{j,h}, \phi_{i,h})$ for $1 \le i, j \le N_0 + N_b$ and $B_{i,j} = \mathcal{A}_2(\phi_{j,h}, \phi_{i,h})$ for $1 \le i, j \le N_0 + N_b$.

Here $A_{00}, A_{0b}, A_{b0}, A_{bb}$ are $N_0 \times N_0, N_0 \times N_b, N_b \times N_0, N_b \times N_b$ matrices respectively. Similarly, $B_{00}, B_{0b}, B_{b0}, B_{bb}$ are $N_0 \times N_0, N_0 \times N_b, N_b \times N_0, N_b \times N_b$ matrices respectively.

Further, $F_{h\mathcal{K}}$ and $H_{h\mathcal{K}}$ are given by,

$$F_{h\mathcal{K}} = \begin{bmatrix} F_{0h} \\ F_{bh} \end{bmatrix}, \quad H_{h\mathcal{K}} = \begin{bmatrix} H_{0h} \\ H_{bh} \end{bmatrix},$$

where

$$F_{0h} = [F_j]_{N_0\times 1} \text{ with } F_j = (f, \bar{\phi}_{0,j}) \text{ and } F_{bh} = [F_{N_0+1},\ F_{N_0+2}, \cdots F_{N_0+N_b}]^T = O_{N_b\times 1},$$

$$H_{0h} = [H_j]_{N_0\times 1},\ H_{bh} = [H_{N_0+j}]_{N_b\times 1},\ \text{where } H_j = \mathcal{H}(\Psi, \Phi)(\phi_{jh}) \text{ for1} \le j \le N_0 + N_b.$$

Next, for any $\mathbf{v} = [v_1\ v_2 \cdots v_{N_0}]^T \in \mathbb{R}^{N_0} \setminus \{0\}$ and $\mathbf{w} = [w_1\ w_2 \cdots w_{N_b}]^T \in \mathbb{R}^{N_b} \setminus \{0\}$ we have,

$$\mathbf{v}^T \mathcal{M}_{00} \mathbf{v} = \int_{\mathcal{K}} \left| \sum_{i=1}^{N_0} v_i \bar{\phi}_{0,i} \right|^2 dx = \left\| \sum_{i=1}^{N_0} v_i \bar{\phi}_{0,i}(t) \right\|^2_{L^2(\mathcal{K})} > 0,$$

$$\mathbf{w}^T B_{bb} \mathbf{w} = \mathcal{A}_2\left( \sum_{j=1}^{N_b} w_j \phi_{N_0+j,h}, \sum_{j=1}^{N_b} w_j \phi_{N_0+j,h} \right) > 0.$$

Therefore, $\mathcal{M}_{00}$ and $B_{bb}$ are both positive definite, so are invertible. Hence, the matrix equation (6.3.4) can be written as

$$\begin{cases} D''_{0h}(t)+\mathcal{M}_{00}^{-1}\mathcal{K}_{00}D'_{0h}(t)+\mathcal{M}_{00}^{-1}\mathcal{L}_{00}D_{0h}(t)+\mathcal{M}_{00}^{-1}A_{00}D_{0h}(t)+\mathcal{M}_{00}^{-1}A_{0b}D_{bh}(t) \\ \qquad\qquad +\mathcal{M}_{00}^{-1}B_{00}D'_{0h}(t)+\mathcal{M}_{00}^{-1}B_{0b}D'_{bh}(t)=\mathcal{M}_{00}^{-1}F_{0h}(t)+\mathcal{M}_{00}^{-1}H_{0h}(t), \\ D_{0h}(0)=[d_{1,h,0}\ d_{2,h,0}\cdots d_{N_0,h,0}]^T, \\ D_{0ht}(0)=[d_{1,h,1}\ d_{2,h,1}\cdots d_{N_0,h,1}]^T. \end{cases} \tag{6.3.5}$$

$$\begin{cases} D'_{bh}(t)+B_{bb}^{-1}A_{bb}D_{bh}(t)=B_{bb}^{-1}H_{bh}-B_{bb}^{-1}A_{b0}D_{0h}(t)-B_{bb}^{-1}B_{b0}D'_{0h}(t), \\ D_{bh}(0)=[d_{N_0+1,h,0}\ d_{N_0+2,h,0}\cdots d_{N_0+N_b,h,0}]^T, \\ D_{bht}(0)=[d_{N_0+1,h,1}\ d_{N_0+2,h,1}\cdots d_{N_0+N_b,h,1}]^T. \end{cases} \tag{6.3.6}$$

Now differentiating (6.3.6) with respect to time once, we have

$$\begin{cases} B_{bb}^{-1}B_{b0}D''_{0h}(t)+D''_{bh}(t)+B_{bb}^{-1}A_{b0}D'_{0h}(t)+B_{bb}^{-1}A_{bb}D'_{bh}(t)=B_{bb}^{-1}H'_{bh}, \\ D_{bh}(0)=[d_{N_0+1,h,0}\ d_{N_0+2,h,0}\cdots d_{N_0+N_b,h,0}]^T, \\ D_{bht}(0)=[d_{N_0+1,h,1}\ d_{N_0+2,h,1}\cdots d_{N_0+N_b,h,1}]^T. \end{cases} \tag{6.3.7}$$

Therefore, combining (6.3.5) and (6.3.7), we obtain

$$\begin{cases} \mathbf{M}_{h\mathcal{K}}D''_h(t)+\mathbf{A}_{h\mathcal{K}}D'_h(t)+\mathbf{G}_{h\mathcal{K}}D_h(t)=\mathbf{F}_{h\mathcal{K}}(t)+\mathbf{E}_{h\mathcal{K}}(t), \\ D_h(0)=[d_{1,h,0}\ d_{2,h,0}\cdots d_{N_0+N_b,h,0}]^T, \\ D_{ht}(0)=[d_{1,h,1}\ d_{2,h,1}\cdots d_{N_0+N_b,h,1}]^T, \end{cases} \tag{6.3.8}$$

where

$$\mathbf{M}_{h\mathcal{K}}=\begin{bmatrix} I & O \\ B_{bb}^{-1}B_{b0} & I \end{bmatrix},\quad \mathbf{A}_{h\mathcal{K}}=\begin{bmatrix} \mathcal{M}_{00}^{-1}B_{00}+\mathcal{M}_{00}^{-1}\mathcal{K}_{00} & \mathcal{M}_{00}^{-1}B_{0b} \\ B_{bb}^{-1}A_{b0} & B_{bb}^{-1}A_{bb} \end{bmatrix},$$

$$\mathbf{G}_{h\mathcal{K}}=\begin{bmatrix} \mathcal{M}_{00}^{-1}L_{00}+\mathcal{M}_{00}^{-1}A_{00} & \mathcal{M}_{00}^{-1}A_{0b} \\ O & O \end{bmatrix},\quad \mathbf{F}_{h\mathcal{K}}=\begin{bmatrix} \mathcal{M}_{00}^{-1}F_{0h} \\ O \end{bmatrix},\quad \mathbf{E}_{h\mathcal{K}}=\begin{bmatrix} \mathcal{M}_{00}^{-1}H_{0h} \\ B_{bb}^{-1}H'_{bh} \end{bmatrix}.$$

Clearly, $\mathbf{M}_{h\mathcal{K}}$ is invertible. Hence, the existence of the solution $u_h \in H^2(J;\mathcal{W}_h^0)$ follows from the standard ODE theory. □

Let $u$ be the solution of (6.1.1)-(6.1.2) and $u_h$ be the solution of semi discrete problem (6.3.1). Then similar to the finite element method we split the error as:

$$u - u_h = (u - \mathcal{Q}_h u) + (\mathcal{Q}_h u - u_h).$$

First part of the above error can be estimated by Lemma 6.2.3. So we choose the projected error as

$$e_h(t) = \{e_0(t), e_b(t)\} = u_h(t) - \mathcal{Q}_h u(t),\ t \in (0, T]. \tag{6.3.9}$$

For simplicity, we write $e_h = \{e_0, e_b\} = u_h - \mathcal{Q}_h u$.

### 6.3.1 Discrete $H^1$ norm Error Estimate

Now we derive some error equation for $e_h$, which is crucial for the subsequent analysis.

**Lemma 6.3.1.** *Let $e_h(t)$ be the projected error defined in (6.3.9) and $u \in H^2(J; \mathcal{X}^*)$ (here $\mathcal{X}^*$ is the space as defined in (1.2.1) ) be the exact solution of (6.1.1)-(6.1.2). Then for any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, we have*

$$\begin{aligned}(e_{htt}, \phi_h) &+ \mathcal{B}(e_{ht}, \phi_h) + \mathcal{D}(e_h, \phi_h) + \mathcal{A}_1(e_h, \phi_h) + \mathcal{A}_2(e_h, \phi_h)\\ &= \ell_1(u, \phi_h) + \ell_2(u_t, \phi_h) + \ell_3(u, \phi_h) + \ell_3(u_t, \phi_h),\end{aligned} \tag{6.3.10}$$

*where $\ell_1(\cdot,\cdot)$, $\ell_2(\cdot,\cdot)$, $\ell_3(\cdot,\cdot)$ are bilinear maps given by*

$$\begin{cases} \ell_1(u, \phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \eta(\nabla u - \mathbb{Q}_{k-1}(\nabla u)).\mathbf{n}, \phi_b - \phi_0\rangle_{\partial\mathcal{K}},\\ \ell_2(u, \phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \upsilon(\nabla u - \mathbb{Q}_{k-1}(\nabla u)).\mathbf{n}, \phi_b - \phi_0\rangle_{\partial\mathcal{K}},\\ \ell_3(u, \phi_h) &= \sum_{\mathcal{K}\in\mathcal{T}_h} h^{-1}\langle \mathcal{Q}_k^0 u - \mathcal{Q}_k^b(u|_{\partial\mathcal{K}}), \phi_b - \phi_0\rangle_{\partial\mathcal{K}}. \end{cases} \tag{6.3.11}$$

*Proof.* For each element $\mathcal{K} \in \mathcal{T}_h$, either $\mathcal{K} \subset \overline{\Omega}_1$ or $\mathcal{K} \subset \overline{\Omega}_2$. So for $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, we test the equation (6.1.1) against $\phi_0$ to obtain

$$(f, \phi_0) = (u_{tt}, \phi_0) + \mathcal{B}(u_t, \phi_0) + \mathcal{D}(u, \phi_0) - \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla\cdot(\eta\nabla u), \phi_0)_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla\cdot(\upsilon\nabla u_t), \phi_0)_{\mathcal{K}}. \tag{6.3.12}$$

Now for $\mathcal{K} \subset \overline{\Omega}_1$ or $\mathcal{K} \subset \overline{\Omega}_2$, we have

$$-(\nabla\cdot(\eta\nabla u), \phi_0)_{\mathcal{K}} = -(\nabla\cdot(\eta_i\nabla u) = (\eta_i\nabla u, \nabla\phi_0)_{\mathcal{K}} - \langle \eta_i\nabla u\cdot\mathbf{n}, \phi_0\rangle_{\partial\mathcal{K}}.$$

Hence, summing over $\mathcal{K} \in \mathcal{T}_h$ we have

$$\begin{aligned}
&-\sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla\cdot(\eta\nabla u), \phi_0)_{\mathcal{K}} \\
&= \sum_{\mathcal{K}\in\mathcal{T}_h} (\eta\nabla u, \nabla\phi_0)_{\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \eta\nabla u\cdot\mathbf{n}, \phi_0-\phi_b\rangle_{\partial\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h} \langle \eta\nabla u\cdot\mathbf{n}, \phi_b\rangle_{\partial\mathcal{K}}. \qquad (6.3.13)
\end{aligned}$$

Here $\eta\nabla u\cdot\mathbf{n} = \eta_i\nabla u\cdot\mathbf{n}$ on $\partial\mathcal{K}$ if $\mathcal{K}\subset\overline{\Omega}_i$ for $i=1,2$.
Again, using definition of weak gradient and properties of $L^2$ projector $\mathbb{Q}_{k-1}$, we have

$$\begin{aligned}
(\eta\mathbb{Q}_{k-1}(\nabla u), \nabla_w\phi_h)_{\mathcal{K}} &= (\nabla\phi_0, \eta\mathbb{Q}_{k-1}(\nabla u))_{\mathcal{K}} + \langle\phi_b-\phi_0, (\eta\mathbb{Q}_{k-1}(\nabla u))\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} \\
&= (\eta\nabla u, \nabla\phi_0)_{\mathcal{K}} - \langle\phi_0-\phi_b, (\eta\mathbb{Q}_{k-1}(\nabla u))\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} \qquad (6.3.14)
\end{aligned}$$

Combining (6.3.13) and (6.3.14 ), we have

$$\begin{aligned}
-\sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla\cdot(\eta\nabla u), \phi_0)_{\mathcal{K}} &= \sum_{\mathcal{K}\in\mathcal{T}_h} (\eta\mathbb{Q}_{k-1}(\nabla u), \nabla_w\phi_h)_{\mathcal{K}} \\
&+ \sum_{\mathcal{K}\in\mathcal{T}_h} \langle\phi_0-\phi_b, (\eta\left(\mathbb{Q}_{k-1}(\nabla u)-\nabla u\right)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h} \langle\eta\nabla u\cdot\mathbf{n}, \phi_b\rangle_{\partial\mathcal{K}}. \qquad (6.3.15)
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\sum_{\mathcal{K}\in\mathcal{T}_h} (\nabla\cdot(\upsilon\nabla u_t), \phi_0)_{\mathcal{K}} &= \sum_{\mathcal{K}\in\mathcal{T}_h} (\upsilon\mathbb{Q}_{k-1}(\nabla u_t), \nabla_w\phi_h)_{\mathcal{K}} \\
&+ \sum_{\mathcal{K}\in\mathcal{T}_h} \langle\phi_0-\phi_b, (\upsilon\left(\mathbb{Q}_{k-1}(\nabla u_t)-\nabla u_t\right)\cdot\mathbf{n}\rangle_{\partial\mathcal{K}} - \sum_{\mathcal{K}\in\mathcal{T}_h} \langle\upsilon\nabla u_t\cdot\mathbf{n}, \phi_b\rangle_{\partial\mathcal{K}}. \\
&\qquad (6.3.16)
\end{aligned}$$

Hence, using (6.3.15) and (6.3.16) in (6.3.12), we obtain

$$\begin{aligned}
(f,\phi_0) &= (u_{tt},\phi_0) + \mathcal{B}(u_t,\phi_0) + \mathcal{D}(u,\phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h} (\eta\mathbb{Q}_{k-1}(\nabla u), \nabla_w\phi_h)_{\mathcal{K}} \\
&+ \sum_{\mathcal{K}\in\mathcal{T}_h} (\upsilon\mathbb{Q}_{k-1}(\nabla u_t), \nabla_w\phi_h)_{\mathcal{K}} + \ell_1(u,\phi_h) + \ell_2(u_t,\phi_h) - \langle\Psi,\phi_b\rangle_{\partial\mathcal{K}}. \qquad (6.3.17)
\end{aligned}$$

Again, using Lemma 6.2.2, we have

$$\begin{aligned}
&(u_{tt},\phi_0) + \mathcal{B}(u_t,\phi_0) + \mathcal{D}(u,\phi_0) + \sum_{\mathcal{K}\in\mathcal{T}_h} (\eta\nabla_w\left(\mathcal{Q}_h u\right), \nabla_w\phi_h)_{\mathcal{K}} + \sum_{\mathcal{K}\in\mathcal{T}_h} (\upsilon\nabla_w\left(\mathcal{Q}_h u_t\right), \nabla_w\phi_h)_{\mathcal{K}} \\
&= \langle\Phi, \eta\nabla_w\phi_h\cdot\mathbf{n}\rangle_{\Gamma_h} + \langle\Phi_t, \eta\nabla_w\phi_h\cdot\mathbf{n}\rangle_{\Gamma_h} - \ell_1(u,\phi_h) - \ell_2(u_t,\phi_h) + (f,\phi_0) + \langle\Psi,\phi_b\rangle_{\partial\mathcal{K}}. \\
&\qquad (6.3.18)
\end{aligned}$$

Now adding $\mathbf{S}(\mathcal{Q}_h u, \phi_h)$ and $\mathbf{S}(\mathcal{Q}_h u_t, \phi_h)$ to both side of (6.3.18) we have

$$\begin{aligned}
&(\mathcal{Q}_h u_{tt}, \phi_0) + \mathcal{B}(\mathcal{Q}_h u_t, \phi_0) + \mathcal{D}(\mathcal{Q}_h u, \phi_0) + \mathcal{A}_1(\mathcal{Q}_h u, \phi_h) + \mathcal{A}_2(\mathcal{Q}_h u_t, \phi_h) \\
&\quad = (f, \phi_0) - \ell_1(u, \phi_h) - \ell_2(u_t, \phi_h) - \ell_3(u, \phi_h) - \ell_3(u_t, \phi_h) + \mathcal{H}(\Psi, \Phi)(\phi_h). \qquad (6.3.19)
\end{aligned}$$

Therefore, subtracting equation (6.3.19) from equation (6.3.1), we have the desired error equation. □

Next, we recall following important Lemma for the estimate of bilinear maps $\ell_1, \ell_2$ and $\ell_3$ (cf. [128]).

**Lemma 6.3.2.** *Assume that $\mathcal{T}_h$ is shape regular discretization of $\Omega$. Then for $u \in H^{k+1}(\Omega_i)$, $i = 1, 2$ and $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$, we have*

$$\begin{aligned}
|\ell_1(u, \phi_h)| + |\ell_3(u, \phi_h)| &\le Ch^k(\|u\|_{H^{k+1}(\Omega_1)} + \|u\|_{H^{k+1}(\Omega_2)}) |||\phi_h|||, \\
|\ell_2(u_t, \phi_h)| + |\ell_3(u_t, \phi_h)| &\le Ch^k(\|u_t\|_{H^{k+1}(\Omega_1)} + \|u_t\|_{H^{k+1}(\Omega_2)}) |||\phi_h|||.
\end{aligned}$$

Now we are ready to present the main result of this subsection

**Theorem 6.3.2.** *Let $u_h \in \mathcal{W}_h^0$ be the solution of (6.3.1) and assume that the exact solution $u \in H^1(J; H^{k+1}(\Omega_i))$ for $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\|e_{ht}\|_{L^\infty(L^2)} + \operatorname*{ess\,sup}_{t\in[0,T]} |||e_h(t)||| + \int_0^T |||e_{ht}(s)||| ds \\
&\quad \le Ch^k(\|u\|_{H^1(J;H^{k+1}(\Omega_1))} + \|u\|_{H^1(J;H^{k+1}(\Omega_2))}). \qquad (6.3.20)
\end{aligned}$$

*Proof.* Setting, $\phi_h = e_{ht}$ in the error equation (6.3.10) and then integrating from 0 to $t$ for some $t \in (0, T]$, we have

$$\begin{aligned}
&\int_0^t \frac{1}{2}\frac{d}{dt}(e_{ht}, e_{ht}) + \int_0^t \mathcal{B}(e_{ht}, e_{ht}) + \int_0^t \frac{1}{2}\frac{d}{dt}\mathcal{D}(e_h, e_h) \\
&\quad + \int_0^t \frac{1}{2}\frac{d}{dt}\mathcal{A}_1(e_h, e_h) + \int_0^t \mathcal{A}_2(e_{ht}, e_{ht}) \\
&= \int_0^t \ell_1(u, e_{ht}) + \int_0^t \ell_2(u_t, e_{ht}) + \int_0^t \ell_3(u, e_{ht}) + \int_0^t \ell_3(u_t, e_{ht}).
\end{aligned}$$

Hence, using the facts $e_h(0) = 0$ and $e_{ht}(0) = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|e_{ht}(t)\|^2 + \int_0^t \mathcal{B}(e_{ht}, e_{ht}) + \frac{1}{2}\mathcal{D}(e_h(t), e_h(t)) + \frac{1}{2}\mathcal{A}_1(e_h(t), e_h(t)) + \int_0^t \mathcal{A}_2(e_{ht}, e_{ht}) \\
&\le \int_0^t |\ell_1(u, e_{ht}) + \ell_3(u, e_{ht})| + \int_0^t |\ell_2(u_t, e_{ht}) + \ell_3(u_t, e_{ht})|.
\end{aligned}$$

Therefore, using coercive inequality (6.2.8) and Lemma 6.3.2, we have

$$\begin{aligned}\frac{1}{2}\|e_{ht}(t)\|^2 + \int_0^t \mathcal{B}(e_{ht}, e_{ht}) + \frac{1}{2}\mathcal{D}(e_h(t), e_h(t)) + \frac{C_*}{2}|||e_h(t)|||^2 + C_* \int_0^t |||e_{ht}|||^2 \\ \leq \int_0^t Ch^k \left(\sum_{i=1}^{2} \|u\|_{H^{k+1}(\Omega_i)}\right) |||e_{ht}||| + \int_0^t Ch^k \left(\sum_{i=1}^{2} \|u_t\|_{H^{k+1}(\Omega_i)}\right) |||e_{ht}|||.\end{aligned}$$

Now applying Young's inequality for suitable $\mu > 0$ we have

$$\begin{aligned}\frac{1}{2}\|e_{ht}(t)\|^2 + \int_0^t \mathcal{B}(e_{ht}, e_{ht}) + \frac{1}{2}\mathcal{D}(e_h(t), e_h(t)) + \frac{C_*}{2}|||e_h(t)|||^2 + C_* \int_0^t |||e_{ht}|||^2 \\ \leq \int_0^t Ch^{2k} \left(\sum_{i=1}^{2} \|u\|^2_{H^{k+1}(\Omega_i)} + \sum_{i=1}^{2} \|u_t\|^2_{H^{k+1}(\Omega_i)}\right) + 2\mu \int_0^t |||e_{ht}|||^2.\end{aligned}$$

Choosing $\mu = \frac{C_*}{4}$ to obtain

$$\begin{aligned}\|e_{ht}(t)\|^2 + \int_0^t \mathcal{B}(e_{ht}, e_{ht}) + \mathcal{D}(e_h(t), e_h(t)) + \frac{C_*}{2}|||e_h(t)|||^2 + \frac{C_*}{2} \int_0^t |||e_{ht}|||^2 \\ \leq \int_0^t Ch^{2k} \left(\sum_{i=1}^{2} \|u\|^2_{H^{k+1}(\Omega_i)} + \sum_{i=1}^{2} \|u_t\|^2_{H^{k+1}(\Omega_i)}\right).\end{aligned}$$

Finally, taking essential supremum over $[0, T]$ we have

$$\begin{aligned}\|e_{ht}\|^2_{L^\infty(L^2)} + \int_0^T \mathcal{B}(e_{ht}, e_{ht}) + \operatorname*{ess\,sup}_{t\in[0,T]} \mathcal{D}(e_h(t), e_h(t)) + \operatorname*{ess\,sup}_{t\in[0,T]} |||e_h(t)|||^2 + \int_0^T |||e_{ht}|||^2 \\ \leq Ch^{2k} \left(\|u\|^2_{H^1(J;H^{k+1}(\Omega_1))} + \|u\|^2_{H^1(J;H^{k+1}(\Omega_2))}\right).\end{aligned}$$

Hence, the Theorem follows. □

### 6.3.2 $L^2$ norm Error Estimate

Now for optimal order of error estimate in $L^2$ norm we employ a non standard projection operator. Let $\varphi \in H^1(J; \mathbb{X}^*)$, where $\mathbb{X}^* = L^2(\Omega) \cap H^{k+1}(\Omega_1) \cap H^{k+1}(\Omega_2)$, with $[\varphi] = \Phi_\varphi$ and $[\eta \frac{\partial \varphi}{\partial \mathbf{n}} + \upsilon \frac{\partial \varphi_t}{\partial \mathbf{n}}] = \Psi_\varphi$, we define

$$f_\varphi = \begin{cases} -\nabla \cdot (\eta_1 \nabla \varphi + \upsilon_1 \nabla \varphi_t) & \text{in } \Omega_1 \\ -\nabla \cdot (\eta_2 \nabla \varphi + \upsilon_2 \nabla \varphi_t) & \text{in } \Omega_2. \end{cases}$$

Clearly $f_\varphi \in L^2(\Omega)$. Define a non standard projection $\mathbf{R}_h : H^2(J; \mathbb{X}^*) \to \mathcal{W}_h^0$ by

$$\begin{aligned}\mathcal{A}_1(\mathbf{R}_h\varphi, \phi_h) + \mathcal{A}_2(\mathbf{R}_h\varphi_t, \phi_h) =& (f_\varphi, \phi_h) + \langle \Psi_\varphi, \phi_b \rangle_{\Gamma_h} + \langle \Phi_\varphi, \eta \nabla_w \phi_h.\mathbf{n} \rangle_{\Gamma_h} \\ &+ \langle \Phi_{\varphi t}, \upsilon \nabla_w \phi_h.\mathbf{n} \rangle_{\Gamma_h} - h^{-1} \langle \Phi_\varphi + \Phi_{\varphi t}, \phi_0 - \phi_b \rangle_{\Gamma_h},\end{aligned} \tag{6.3.21}$$

with $\mathbf{R}_h\varphi(0) = \mathcal{Q}_h\varphi(0)$ for all $\phi_h \in \mathcal{W}_h$ and $[\varphi] = \Phi_\varphi$, $[\eta\frac{\partial\varphi}{\partial\mathbf{n}} + \upsilon\frac{\partial\varphi_t}{\partial\mathbf{n}}] = \Psi_\varphi$ along $\Gamma_h$.

In view of equation (6.3.21) it can be observed that $\mathbf{R}_h\varphi$ is the standard weak Galerkin solution of the following interface problem with exact solution $\varphi$

$$-\nabla\cdot(\eta\nabla\varphi + \upsilon\nabla\varphi_t) = f_\varphi \text{ in } \Omega\times(0,T],$$

$$\varphi = 0 \text{ on } \partial\Omega\times(0,T],$$

with $[\varphi] = \Phi_\varphi$, $[\eta\frac{\partial\varphi}{\partial\mathbf{n}} + \upsilon\frac{\partial\varphi_t}{\partial\mathbf{n}}] = \Psi_\varphi$ along $\Gamma_h$.

Hence, error estimate between $\mathcal{Q}_h\varphi$ and $\mathbf{R}_h\varphi$ can be imported from the existing result (Theorem 3.1 and Theorem 3.2 in [46]) as in the following Lemma.

**Lemma 6.3.3.** *Let* $\mathbf{R}_h$ *be defines as in (6.3.21). Assume that* $\varphi \in H^1(J;H^{k+1}(\Omega_i))$ *for* $i = 1,2$. *Then there exists a constant* $C > 0$ *such that*

$$|||\mathcal{Q}_h\varphi(t) - \mathbf{R}_h\varphi(t)||| \le Ch^k\sum_{i=1}^{2}\|\varphi\|_{H^1(J;H^{k+1}(\Omega_i))},$$

$$\|\mathcal{Q}_h\varphi(t) - \mathbf{R}_h\varphi(t)\| \le Ch^{k+1}\sum_{i=1}^{2}\|\varphi\|_{H^1(J;H^{k+1}(\Omega_i))}, \quad \forall\, t \in J.$$

Next, we further split the projected error $e_h = u_h - \mathcal{Q}_h u$ with the help of non standard projection $\mathbf{R}_h$ as

$$e_h(t) = u_h(t) - \mathcal{Q}_h u(t) = u_h(t) - \mathbf{R}_h u(t) + \mathbf{R}_h u(t) - \mathcal{Q}_h u(t) = \theta(t) - \rho(t),$$

where $\theta = u_h - \mathbf{R}_h u$ and $\rho = \mathcal{Q}_h u - \mathbf{R}_h u$. Clearly $\rho$ can be approximated by Lemma 6.3.3. So we try to bound $\theta$ in $L^2$ norm.

Now for $\phi_h \in \mathcal{W}_h^0$, using the definition of $\mathbf{R}_h$ we have

$$\begin{aligned}
&(\theta_{tt},\phi_h) + \mathcal{B}(\theta_t,\phi_h) + \mathcal{D}(\theta,\phi_h) + \mathcal{A}_1(\theta,\phi_h) + \mathcal{A}_2(\theta_t,\phi_h)\\
&= (f,\phi_0) + \mathcal{H}(\Phi,\Psi)(\phi_h) - (\mathbf{R}_h u_{tt},\phi_h) - \mathcal{B}(\mathbf{R}_h u_t,\phi_h) - \mathcal{D}(\mathbf{R}_h u,\phi_h)\\
&\quad - \mathcal{A}_1(\mathbf{R}_h u,\phi_h) - \mathcal{A}_2(\mathbf{R}_h u_t,\phi_h)\\
&= (f,\phi_0) - (f_u^*,\phi_0) - (\mathbf{R}_h u_{tt},\phi_h) - \mathcal{B}(\mathbf{R}_h u_t,\phi_h) - \mathcal{D}(\mathbf{R}_h u,\phi_h)\\
&= (\mathcal{Q}_h u_{tt},\phi_h) + \mathcal{B}(\mathcal{Q}_h u_t,\phi_h) + \mathcal{D}(\mathcal{Q}_h u,\phi_h) - (\mathbf{R}_h u_{tt},\phi_h) - \mathcal{B}(\mathbf{R}_h u_t,\phi_h) - \mathcal{D}(\mathbf{R}_h u,\phi_h)\\
&= (\rho_{tt},\phi_h) + \mathcal{B}(\rho_t,\phi_h) + \mathcal{D}(\rho,\phi_h).
\end{aligned}$$

Hence,

$$(\theta_{tt},\phi_h) + \mathcal{B}(\theta_t,\phi_h) + \mathcal{D}(\theta,\phi_h) + \mathcal{A}_1(\theta,\phi_h) + \mathcal{A}_2(\theta_t,\phi_h) = (\rho_{tt},\phi_h) + \mathcal{B}(\rho_t,\phi_h) + \mathcal{D}(\rho,\phi_h). \tag{6.3.22}$$

Next, for any $\zeta \in (0,T]$, we define $\hat{\theta}$ as

$$\hat{\theta}(\cdot,t) = \int_t^{\zeta} \theta(\cdot,s)ds \quad 0 \le t \le T.$$

Therefore,

$$\hat{\theta}(\zeta) = 0 \ \text{ and } \ \hat{\theta}_t(\cdot,t) = -\theta(\cdot,t) \ \ t \in [0,T]. \tag{6.3.23}$$

Setting $\phi_h = \hat{\theta}$ in (6.3.22) and then integrating from 0 to $\zeta$ with respect to time $t$, and using (6.3.23), we have

$$\begin{aligned}
&\int_0^{\zeta} (\theta_t,\theta) - (\theta_t(0),\hat{\theta}(0)) + \int_0^{\zeta} \mathcal{B}(\theta,\theta) - \mathcal{B}(\theta(0),\hat{\theta}(0)) + \frac{1}{2}\mathcal{D}(\hat{\theta}(0),\hat{\theta}(0)) \\
&\quad + \frac{1}{2}\mathcal{A}_1(\hat{\theta}(0),\hat{\theta}(0)) + \int_0^{\zeta} \mathcal{A}_2(\theta,\theta)ds - \mathcal{A}_2(\theta(0),\hat{\theta}(0)) \\
&= -\int_0^{\zeta} (\rho_t,\hat{\theta}_t) - (\rho_t(0),\hat{\theta}(0)) + \int_0^{\zeta} \mathcal{B}(\rho,\theta) - \mathcal{B}(\rho(0),\hat{\theta}(0)) + \int_0^{\zeta} \mathcal{D}(\rho,\hat{\theta}).
\end{aligned}$$

Again, using (6.3.23) and the facts that $\rho_t(0) = \theta_t(0), \ \ \rho(0) = \theta(0) = 0$, we obtain

$$\begin{aligned}
&\int_0^{\zeta} \frac{1}{2}\frac{d}{dt}\|\theta\|^2 dt + \int_0^{\zeta} \mathcal{B}(\theta,\theta) + \frac{1}{2}\mathcal{D}(\hat{\theta}(0),\hat{\theta}(0)) + \frac{1}{2}\mathcal{A}_1(\hat{\theta}(0),\hat{\theta}(0)) + \int_0^{\zeta} \mathcal{A}_2(\theta,\theta) \\
&= -\int_0^{\zeta} (\rho_t,\hat{\theta}_t) + \int_0^{\zeta} \mathcal{B}(\rho,\theta) + \int_0^{\zeta} \mathcal{D}(\rho,\hat{\theta}).
\end{aligned} \tag{6.3.24}$$

Further, the fact $\theta(0) = 0$ leads to

$$\begin{aligned}
&\frac{1}{2}\|\theta(\zeta)\|^2 + \int_0^{\zeta} \mathcal{B}(\theta,\theta) + \frac{1}{2}\mathcal{D}(\hat{\theta}(0),\hat{\theta}(0)) + \frac{1}{2}\mathcal{A}_1(\hat{\theta}(0),\hat{\theta}(0)) + \int_0^{\zeta} \mathcal{A}_2(\theta,\theta) \\
&= \int_0^{\zeta} (\rho_t,\theta) + \int_0^{\zeta} \mathcal{B}(\rho,\theta) + \int_0^{\zeta} \mathcal{D}(\rho,\hat{\theta}).
\end{aligned} \tag{6.3.25}$$

Since, $\theta$ is continuous in temporal direction, we select $\zeta \in (0,T]$ such that $\|\theta(\zeta)\| = \max_{0 \le t \le T} \|\theta(t)\|$. Also,

$$\|\hat{\theta}(0)\| \le \int_0^{\zeta} \|\theta(s)\|ds \le \zeta\|\theta(\zeta)\| \ \text{ and } \ \|\hat{\theta}(t)\| \le \sqrt{T}\|\theta(\zeta)\|.$$

Therefore, using Holder's inequality and then Young's inequalities with suitable $\mu > 0$, we have

$$\begin{aligned}
&\|\theta(\zeta)\|^2 + \int_0^{\zeta} \|\theta\|^2 + \|\hat{\theta}(0)\|^2 + \left|\left|\left|\hat{\theta}(0)\right|\right|\right|^2 + \int_0^{\zeta} |||\theta|||^2 \\
&\le \frac{C}{2}\int_0^{\zeta} \|\rho_t\|^2 + \frac{C}{2}\int_0^{\zeta} \|\rho\|^2 + \int_0^{\zeta} \|\theta\|^2 + \mu\int_0^{\zeta} \|\theta(\zeta)\|^2 + \frac{C^2T}{4\mu}\int_0^{\zeta} \|\rho\|^2.
\end{aligned}$$

Now choose $\mu = \frac{1}{2T}$ to obtain

$$\begin{aligned} &\frac{1}{2}\|\theta(\zeta)\|^2 + \int_0^\zeta \|\theta\|^2 + \left|\left|\left|\hat{\theta}(0)\right|\right|\right|^2 + \int_0^\zeta |||\theta|||^2 \\ &\leq C\left(\int_0^\zeta \|\rho_t\|^2 + \int_0^\zeta \|\rho\|^2\right) + \int_0^\zeta \|\theta(s)\|^2 ds + \frac{C^2T^2}{2}\int_0^\zeta \|\rho\|^2. \end{aligned}$$

Hence, using Gronwall's inequality, we have

$$\|\theta(\zeta)\|^2 + \int_0^\zeta \|\theta\|^2 + \left|\left|\left|\hat{\theta}(0)\right|\right|\right|^2 + \int_0^\zeta |||\theta|||^2 \leq C\Big(\int_0^\zeta \|\rho_t\|^2 + \int_0^\zeta \|\rho\|^2\Big). \tag{6.3.26}$$

Again, using the fact that $\rho_t = \mathcal{Q}_h u_t - \mathbf{R}_h u_t$ and then applying Lemma 6.3.3, we obtain

$$\|\rho_t(t)\|^2 \leq Ch^{2(k+1)} \sum_{i=1}^{2} \|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))} \quad \forall t \in [0,T]. \tag{6.3.27}$$

Therefore, using (6.3.27) and in (6.3.26), we have for $t \in [0,T]$,

$$\|\theta(t)\|^2 \leq \|\theta(\zeta)\|^2 \leq Ch^{2(k+1)}\left(\sum_{i=1}^{2} \|u\|^2_{H^1(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2} \|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}\right). \tag{6.3.28}$$

Finally, using Lemma 6.3.3 and inequality (6.3.28), we estimate error in $L^\infty(L^2)$ norm.

**Theorem 6.3.3.** *Suppose that the exact solution* $u \in H^1(J; H^{k+1}(\Omega_i)$ *for* $i = 1, 2$ *and* $u_t \in H^1(J; H^{k+1}(\Omega_i)$ *for* $i = 1, 2$*. Then there exists a positive constant* $C > 0$ *such that following holds*

$$\|e_h\|_{L^\infty(J;L^2(\Omega))} \leq Ch^{k+1}\left(\sum_{i=1}^{2} \|u\|_{H^1(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2} \|u_t\|_{H^1(J;H^{k+1}(\Omega_i))}\right).$$

*Proof.* We have

$$e_h(t) = u_h(t) - \mathcal{Q}_h u(t) = \theta(t) - \rho(t).$$

Hence, triangle inequality yields

$$\|e_h(t)\| \leq \|\theta(t)\| + \|\rho(t)\| \text{ for a.e. } t \in [0,T].$$

Then Lemma 6.3.3 and (6.3.28) lead to the desired result. □

## 6.4 Fully Discrete WG-FEMs for General Hyperbolic Interface Problem

In this section, we derive $L^2$ error estimate for a fully discrete weak Galerkin finite element scheme. We employ the Crank-Nicolson scheme for the temporal variable. First, we convert second order problem (6.1.1) into a system of first order equations. Let us take $u_t = p$, where $p$ is an auxiliary unknown function. Then our model equation (6.1.1) is reduced to following first order system of equation in time

$$\begin{cases} u_t - p = 0 \text{ in } \Omega \times (0,T], \\ p_t + \Theta p + \xi u - \nabla.(\eta \nabla u + \upsilon \nabla p) = f \quad \text{in} \quad \Omega \times (0,T]. \end{cases} \tag{6.4.1}$$

Let $M$ be a natural number. First divide the time period $(0,T]$ into $M$ uniform sub-intervals $J_n = (t_{n-1}, t_n]$ for $n = 1, 2, \cdots, M$ with $t_0 = 0$, $t_n = n\tau$ and $t_M = T$, where $\tau = \frac{T}{M}$ is the time step.

For a continuous function $\varphi : [0,T] \to L^2(\Omega)$, define $\varphi^n = \varphi(\cdot, t_n)$, $n = 0, 1, \cdots, M$. Also, for any sequence $\{\varphi^n\}_0^M \subset L^2(\Omega)$, define

$$\varphi^{n+\frac{1}{2}} = \frac{1}{2}(\varphi^{n+1} + \varphi^n) \text{ and } \partial_\tau \varphi^n = \frac{\varphi^{n+1} - \varphi^n}{\tau} \quad \text{for } n = 0, 1, \cdots, M-1. \tag{6.4.2}$$

Then the fully discrete weak Galerkin finite element estimate to (6.1.1)-(6.1.2) is defined as follows: For $n = 0, 1, \cdots M-1$, find $U^n \in \mathcal{W}_h^0$ such that

$$\partial_\tau U^n = p^{n+\frac{1}{2}}, \tag{6.4.3}$$

$$\begin{aligned} &(\partial_\tau p^n, \phi_h) + \mathcal{B}(p^{n+\frac{1}{2}}, \phi_h) + \mathcal{D}(U^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_1(U^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_2(p^{n+\frac{1}{2}}, \phi_h) \\ &= (f^{n+\frac{1}{2}}, \phi_0) + \mathcal{H}(\Psi^{n+\frac{1}{2}}, \Phi^{n+\frac{1}{2}})(\phi_h) \quad \forall \phi_h \in \mathcal{W}_h^0, \end{aligned} \tag{6.4.4}$$

with $U^0 = \mathbf{R}_h u^0$, $p^0 = \mathbf{R}_h v^0$ and $\mathcal{H}$ is defined as in (6.3.2).

**Remark 6.4.1.** *We have incorporated jump functions $\Phi$ and $\Psi$ in the weak Galerkin algorithm to avoid residue in the error estimate.*

The existence and uniqueness of the fully discrete solution $U^n$ in terms of auxiliary variable $p^n$ follows from the following Lemma.

**Lemma 6.4.1.** *There exists a unique sequence $\{U^n\}_{n=0}^M \subset \mathcal{W}_h^0$ satisfying (6.4.4) and a corresponding unique sequence $\{p^n\}_{n=0}^M \subset \mathcal{W}_h^0$ satisfying (6.4.3).*

*Proof.* Clearly,

$$U^{n+\frac{1}{2}} = \frac{\tau}{2}(p^{n+1} + p^n) + U^n. \tag{6.4.5}$$

Therefore, using above equation in (6.4.4), we have

$$\mathcal{A}_\tau(p^{n+1}, \phi_h) = \mathcal{F}^n(\phi_h), \tag{6.4.6}$$

where $\mathcal{A}_\tau$ is the bilinear form defined as

$$\mathcal{A}_\tau(\Psi_h, \phi_h) = (\Psi_h, \phi_h) + \frac{\tau}{2}\mathcal{B}(\Psi_h, \phi_h) + \frac{\tau^2}{4}\mathcal{D}(\Psi_h, \phi_h) + \frac{\tau^2}{4}\mathcal{A}_1(\Psi_h, \phi_h) + \frac{\tau}{2}\mathcal{A}_2(\Psi_h, \phi_h),$$

for $\Psi_h, \phi_h \in \mathcal{W}_h$ and the linear functional $\mathcal{F}^n$ is given by

$$\begin{aligned}\mathcal{F}^n(\phi_h) = (f^{n+\frac{1}{2}}, \phi_0) + (p^n, \phi_h) - \frac{\tau}{2}\mathcal{B}(p^n, \phi_h) - \frac{\tau^2}{4}\mathcal{D}(p^n, \phi_h)) - \tau\mathcal{D}(U^n, \phi_h)) \\ - \frac{\tau^2}{4}\mathcal{A}_1(p^n, \phi_h) - \tau\mathcal{A}_1(U^n, \phi_h)) - \frac{\tau}{2}\mathcal{A}_2(p^n, \phi_h) + \mathcal{H}(\Psi^{n+\frac{1}{2}}, \Phi^{n+\frac{1}{2}})(\phi_h).\end{aligned}$$

Clearly, $\mathcal{A}_\tau$ is a bounded as well as positive bilinear map and $\mathcal{F}^n$ is bounded. Therefore, there exists unique $p^{n+1} \in \mathcal{W}_h^0$ satisfying (6.4.6), so is $U^{n+1}$ for $0 \le n \le M-1$. □

Now we discuss following important Lemma, which can be proved using Taylor's series expansion.

**Lemma 6.4.2.** *For any Banach space* $\mathbf{B}$ *and* $\varphi \in H^3(J; \mathbf{B})$*, the following inequality holds*

$$\|\partial_\tau \varphi^n - \varphi_t^{n+\frac{1}{2}}\|_{\mathbf{B}}^2 \le C\tau^3 \int_{t_n}^{t_{n+1}} \|\varphi_{ttt}\|_{\mathbf{B}}^2 dt,$$

*where* $C > 0$ *is a constant independent of* $\tau$.

Next, to compute projected error $e_h^n = U^n - \mathcal{Q}_h u^n$, we first establish the error

$$\gamma^n = U^n - \mathbf{R}_h u^n \text{ for } 1 \le n \le M.$$

Now we define the following sequences

$$q^n = p^n - \mathbf{R}_h u_t^n \quad \text{and} \quad \chi^n = u^n - \mathbf{R}_h u^n \quad \text{for } \ 0 \le n \le M.$$

The following Lemma is crucial for the fully discrete error estimate.

**Lemma 6.4.3.** *Let* $u$ *be the exact solution of (6.1.1) and* $U^n$ *be solution of fully discrete approximation (6.4.4). Assume that* $u, u_t \in L^\infty(J; H^{k+1}(\Omega_i))$ *for* $i = 1, 2$ *and* $u, u_t \in H^1(J; H^{k+1}(\Omega_i))$ *for* $i = 1, 2$*. Further, assume that* $u_{ttt} \in H^1(J; H^2(\Omega_i))$ *for* $i = 1, 2$*.*

*Then there exists a constant $C > 0$, independent of $\tau$ and $h$ such that following estimate holds*

$$
\begin{aligned}
\max_{1\le n\le M}\|\gamma^n\| \le Ch^{k+1}\Big\{ &\sum_{i=1}^{2}\|u\|_{L^\infty(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u_t\|_{L^\infty(J;H^{k+1}(\Omega_i))} \\
&+ \sum_{i=1}^{2}\|u\|_{H^1(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u_t\|_{H^1(J;H^{k+1}(\Omega_i))}\Big\} \\
+ C\tau^2\bigg( \sum_{i=1}^{2}\|u_{ttt}\|_{L^2(J;H^1(\Omega_i))} &+ \sum_{i=1}^{2}\|u_{ttt}\|_{H^1(J;H^2(\Omega_i))} + \int_0^T \|u_{ttt}\|dt + \int_0^T \|u_{tttt}\|dt\bigg).
\end{aligned}
$$

*Proof.* For each $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h^0$ and for $1 \le n \le M-1$, we have

$$
\begin{aligned}
&(\partial_\tau q^n, \phi_h) + \mathcal{B}(q^{n+\frac{1}{2}}, \phi_h) + \mathcal{D}(\gamma^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_1(\gamma^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_2(q^{n+\frac{1}{2}}, \phi_h) \\
&= (\partial_\tau p^n, \phi_h) + \mathcal{B}(p^{n+\frac{1}{2}}, \phi_h) + \mathcal{D}(U^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_1(U^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_2(p^{n+\frac{1}{2}}, \phi_h) \\
&\quad - (\partial_\tau \mathbf{R}_h u_t^n, \phi_h) - \mathcal{B}(\mathbf{R}_h u_t^{n+\frac{1}{2}}, \phi_h) - \mathcal{D}(\mathbf{R}_h u^{n+\frac{1}{2}}, \phi_h) \\
&\quad - \mathcal{A}_1(\mathbf{R}_h u^{n+\frac{1}{2}}, \phi_h) - \mathcal{A}_2(\mathbf{R}_h u_t^{n+\frac{1}{2}}, \phi_h) \\
&= (f^{n+\frac{1}{2}}, \phi_0) + \mathcal{H}(\Psi^{n+\frac{1}{2}}, \Phi^{n+\frac{1}{2}})(\phi_h) - (\partial_\tau \mathbf{R}_h u_t^n, \phi_h) - \mathcal{B}(\mathbf{R}_h u_t^{n+\frac{1}{2}}, \phi_h) \\
&\quad - \mathcal{D}(\mathbf{R}_h u^{n+\frac{1}{2}}, \phi_h) - (f_u^{n+\frac{1}{2}}, \phi_0) - \mathcal{H}(\Psi^{n+\frac{1}{2}}, \Phi^{n+\frac{1}{2}})(\phi_h) \\
&= (f^{n+\frac{1}{2}}, \phi_0) - (\partial_\tau \mathbf{R}_h u_t^n, \phi_h) - \mathcal{B}(\mathbf{R}_h u_t^{n+\frac{1}{2}}, \phi_h) - \mathcal{D}(\mathbf{R}_h u^{n+\frac{1}{2}}, \phi_h) \\
&\quad + (\nabla\cdot(\eta\nabla u^{n+\frac{1}{2}} + \upsilon\nabla u_t^{n+\frac{1}{2}}), \phi_0) \\
&= (f^{n+\frac{1}{2}}, \phi_0) - (\partial_\tau \mathbf{R}_h u_t^n, \phi_h) - \mathcal{B}(\mathbf{R}_h u_t^{n+\frac{1}{2}}, \phi_h) - \mathcal{D}(\mathbf{R}_h u^{n+\frac{1}{2}}, \phi_h) \\
&\quad + (u_{tt}^{n+\frac{1}{2}}, \phi_0) + \mathcal{B}(u_t^{n+\frac{1}{2}}, \phi_0) + \mathcal{D}(u^{n+\frac{1}{2}}, \phi_0) - (f^{n+\frac{1}{2}}, \phi_0) \\
&= (u_{tt}^{n+\frac{1}{2}} - \partial_\tau \mathbf{R}_h u_t^n, \phi_0) + \mathcal{B}(u_t^{n+\frac{1}{2}} - \mathbf{R}_h u_t^{n+\frac{1}{2}}, \phi_0) + \mathcal{D}(u^{n+\frac{1}{2}} - \mathbf{R}_h u^{n+\frac{1}{2}}, \phi_0).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&(\partial_\tau q^n, \phi_h) + \mathcal{B}(q^{n+\frac{1}{2}}, \phi_h) + \mathcal{D}(\gamma^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_1(\gamma^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_2(q^{n+\frac{1}{2}}, \phi_h) \\
&= (\partial_\tau \chi_t^n - w^n, \phi_0) + \mathcal{B}(\chi_t^{n+\frac{1}{2}}, \phi_0) + \mathcal{D}(\chi^{n+\frac{1}{2}}, \phi_0),
\end{aligned}
\tag{6.4.7}
$$

where $w^n = \partial_\tau u_t^n - u_{tt}^{n+\frac{1}{2}}$.

Now

$$
\begin{aligned}
\partial_\tau \gamma^n = \partial_\tau U^n - \partial_\tau \mathbf{R}_h u^n &= p^{n+\frac{1}{2}} - \partial_\tau \mathbf{R}_h u^n \\
&= q^{n+\frac{1}{2}} + \mathbf{R}_h u_t^{n+\frac{1}{2}} - \partial_\tau \mathbf{R}_h u^n \\
&= q^{n+\frac{1}{2}} + \alpha^n,
\end{aligned}
$$

where $\alpha^n = \mathbf{R}_h u_t^{n+\frac{1}{2}} - \partial_\tau \mathbf{R}_h u^n$.

Therefore, using the fact $\gamma^0 = U^0 - \mathbf{R}_h u(0) = 0$ and $q^0 = p^0 - \mathbf{R}_h u_t(0) = \mathbf{R}_h v^0 - \mathbf{R}_h v^0 = 0$, we have

$$\gamma^n = \tau \sum_{j=0}^{n-1} \partial_\tau \gamma^j = \tau \sum_{j=0}^{n-1} q^{j+\frac{1}{2}} + \tau \sum_{j=0}^{n-1} \alpha^j.$$

Also, $q^n = \tau \sum_{j=0}^{n-1} \partial_\tau q^j$. Hence, we have

$$\begin{cases} \partial_\tau \gamma^n = \frac{\tau}{2}\Big( \sum_{j=0}^{n} \partial_\tau q^j + \sum_{j=0}^{n-1} \partial_\tau q^j \Big) + \alpha^n, \\ \gamma^{n+\frac{1}{2}} = \frac{\tau}{2}\Big( \sum_{j=0}^{n} q^{j+\frac{1}{2}} + \sum_{j=0}^{n-1} q^{j+\frac{1}{2}} \Big) + \frac{\tau}{2}\Big( \sum_{j=0}^{n} \alpha^j + \sum_{j=0}^{n-1} \alpha^j \Big). \end{cases} \tag{6.4.8}$$

Now we define a sequence $\{s^n\}_{n=0}^{M}$ such that $s^0 = 0$ and

$$s^n = \tau \sum_{j=0}^{n-1} \gamma^{j+\frac{1}{2}} \quad \text{for} \ \ n = 1, 2, \cdots, M.$$

So

$$s^{n+\frac{1}{2}} = \frac{\tau}{2}\Big( \sum_{j=0}^{n} \gamma^{j+\frac{1}{2}} + \sum_{j=0}^{n-1} \gamma^{j+\frac{1}{2}} \Big). \tag{6.4.9}$$

Therefore, for any $\phi_h = \{\phi_0, \phi_b\} \in \mathcal{W}_h$ using equation (6.4.7), we have for $0 \leq n \leq M-1$

$$\begin{aligned} &(\partial_\tau \gamma^n, \phi_h) + \mathcal{B}(\gamma^{n+\frac{1}{2}}, \phi_h) + \mathcal{D}(s^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_1(s^{n+\frac{1}{2}}, \phi_h) + \mathcal{A}_2(\gamma^{n+\frac{1}{2}}, \phi_h) \\ &= \frac{\tau}{2}\Big( \sum_{j=0}^{n} (\partial_\tau q^j, \phi_h) + \mathcal{B}(q^{j+\frac{1}{2}}, \phi_h) + \mathcal{D}(\gamma^{j+\frac{1}{2}}, \phi_h) + \mathcal{A}_1(\gamma^{j+\frac{1}{2}}, \phi_h) + \mathcal{A}_2(q^{j+\frac{1}{2}}, \phi_h) \Big) \\ &\quad + \frac{\tau}{2}\Big( \sum_{j=0}^{n-1} (\partial_\tau q^j, \phi_h) + \mathcal{B}(q^{j+\frac{1}{2}}, \phi_h) + \mathcal{D}(\gamma^{j+\frac{1}{2}}, \phi_h) + \mathcal{A}_1(\gamma^{j+\frac{1}{2}}, \phi_h) + \mathcal{A}_2(q^{j+\frac{1}{2}}, \phi_h) \Big) \\ &\quad + (\alpha^n, \phi_h) + \frac{\tau}{2}\mathcal{B}\Big( \sum_{j=0}^{n} \alpha^j + \sum_{j=0}^{n-1} \alpha^j, \phi_h \Big) + \frac{\tau}{2}\mathcal{A}_2\Big( \sum_{j=0}^{n} \alpha^j + \sum_{j=0}^{n-1} \alpha^j, \phi_h \Big) \\ &= \frac{\tau}{2} \sum_{j=0}^{n} \Big\{ (\partial_\tau \chi_t^j - w^j, \phi_h) + \mathcal{B}(\chi_t^{j+\frac{1}{2}}, \phi_h) + \mathcal{D}(\chi^{j+\frac{1}{2}}, \phi_h) \Big\} \\ &\quad + \frac{\tau}{2} \sum_{j=0}^{n-1} \Big\{ (\partial_\tau \chi_t^j - w^j, \phi_h) + \mathcal{B}(\chi_t^{j+\frac{1}{2}}, \phi_h) + \mathcal{D}(\chi^{j+\frac{1}{2}}, \phi_h) \Big\} + (\alpha^n, \phi_h) \\ &\quad + \frac{\tau}{2}\mathcal{B}\Big( \sum_{j=0}^{n} \alpha^j + \sum_{j=0}^{n-1} \alpha^j, \phi_h \Big) + \frac{\tau}{2}\mathcal{A}_2\Big( \sum_{j=0}^{n} \alpha^j + \sum_{j=0}^{n-1} \alpha^j, \phi_h \Big). \end{aligned}$$

Hence, we have

$$\begin{aligned}&(\partial_\tau\gamma^n,\phi_h)+\mathcal{B}(\gamma^{n+\frac{1}{2}},\phi_h)+\mathcal{D}(s^{n+\frac{1}{2}},\phi_h)+\mathcal{A}_1(s^{n+\frac{1}{2}},\phi_h)+\mathcal{A}_2(\gamma^{n+\frac{1}{2}},\phi_h)\\&=(G_1^n,\phi_h)+\mathcal{B}(G_2^n+G_3^n,\phi_h)+\mathcal{D}(G_4^n,\phi_h)+\mathcal{A}_2(G_3^n,\phi_h),\end{aligned}\tag{6.4.10}$$

where $G_1^n, G_2^n, G_3^n$ and $G_4^n$ are given by

$$\begin{cases}G_1^n=\frac{\tau}{2}\Big\{\sum_{j=0}^{n}\partial_\tau\chi_t^j+\sum_{j=0}^{n-1}\partial_\tau\chi_t^j-\sum_{j=0}^{n}w^j-\sum_{j=0}^{n-1}w^j\Big\}+\alpha^n,\\ G_2^n=\frac{\tau}{2}\Big(\sum_{j=0}^{n}\chi_t^{j+\frac{1}{2}}+\sum_{j=0}^{n-1}\chi_t^{j+\frac{1}{2}}\Big),\\ G_3^n=\frac{\tau}{2}\Big(\sum_{j=0}^{n}\alpha^j+\sum_{j=0}^{n-1}\alpha^j\Big),\\ G_4^n=\frac{\tau}{2}\Big(\sum_{j=0}^{n}\chi^{j+\frac{1}{2}}+\sum_{j=0}^{n-1}\chi^{j+\frac{1}{2}}\Big).\end{cases}$$

Now setting $\phi_h=\partial_\tau s^n=\gamma^{n+\frac{1}{2}}$ in above equation (6.4.10), we obtain

$$\begin{aligned}&(\gamma^{n+1},\gamma^{n+1})+2\tau\mathcal{B}(\gamma^{n+\frac{1}{2}},\gamma^{n+\frac{1}{2}})+\mathcal{D}(s^{n+1},s^{n+1})+\mathcal{A}_1(s^{n+1},s^{n+1})+2\tau\mathcal{A}_2(\gamma^{n+\frac{1}{2}},\gamma^{n+\frac{1}{2}})\\&=(\gamma^n,\gamma^n)+\mathcal{D}(s^n,s^n)+\mathcal{A}_1(s^n,s^n)+2\tau(G_1^n,\gamma^{n+\frac{1}{2}})+2\tau\mathcal{B}(G_2^n+G_3^n,\gamma^{n+\frac{1}{2}})\\&\quad+2\tau\mathcal{D}(G_4^n,\gamma^{n+\frac{1}{2}})+2\tau\mathcal{A}_2(G_3^n,\gamma^{n+\frac{1}{2}}).\end{aligned}$$

Next, using Cauchy-Schwartz inequality and inequality (6.2.8), we have

$$\begin{aligned}&(\gamma^{n+1},\gamma^{n+1})+2\tau\mathcal{B}(\gamma^{n+\frac{1}{2}},\gamma^{n+\frac{1}{2}})+\mathcal{D}(s^{n+1},s^{n+1})+\mathcal{A}_1(s^{n+1},s^{n+1})+2\tau C_*\Big|\Big|\Big|\gamma^{n+\frac{1}{2}}\Big|\Big|\Big|^2\\&\le\|\gamma^n\|^2+\mathcal{D}(s^n,s^n)+\mathcal{A}_1(s^n,s^n)+2\tau\|G_1^n\|\|\gamma^{n+\frac{1}{2}}\|+2\tau\|G_2^n+G_3^n\|\|\gamma^{n+\frac{1}{2}}\|\\&\quad+2\tau\|G_4^n\|\|\gamma^{n+\frac{1}{2}}\|+2\tau|||G_3^n|||\Big|\Big|\Big|\gamma^{n+\frac{1}{2}}\Big|\Big|\Big|.\end{aligned}$$

Hence, using Poincaré type inequality (6.2.9) and then applying Young's inequality with suitable $\mu>0$, we obtain

$$\begin{aligned}&(\gamma^{n+1},\gamma^{n+1})+2\tau\mathcal{B}(\gamma^{n+\frac{1}{2}},\gamma^{n+\frac{1}{2}})+\mathcal{D}(s^{n+1},s^{n+1})+\mathcal{A}_1(s^{n+1},s^{n+1})+2C_*\tau\Big|\Big|\Big|\gamma^{n+\frac{1}{2}}\Big|\Big|\Big|^2\\&\le\|\gamma^n\|^2+\mathcal{D}(s^n,s^n)+\mathcal{A}_1(s^n,s^n)+\frac{C^2\tau^2}{\mu}\left(\|G_1^n\|^2+\|G_2^n+G_3^n\|^2+\|G_4^n\|^2\right)\\&\quad+\frac{\tau^2}{\mu}|||G_3^n|||^2+4\mu\Big|\Big|\Big|\gamma^{n+\frac{1}{2}}\Big|\Big|\Big|^2.\end{aligned}$$

Setting $\mu=\frac{C_*\tau}{4}$ leads to

$$\begin{aligned}&(\gamma^{n+1},\gamma^{n+1})+2\tau\mathcal{B}(\gamma^{n+\frac{1}{2}},\gamma^{n+\frac{1}{2}})+\mathcal{D}(s^{n+1},s^{n+1})+\mathcal{A}_1(s^{n+1},s^{n+1})+C_*\tau\Big|\Big|\Big|\gamma^{n+\frac{1}{2}}\Big|\Big|\Big|^2\\&\le\|\gamma^n\|^2+\mathcal{D}(s^n,s^n)+\mathcal{A}_1(s^n,s^n)+\frac{4\tau C^2}{C_*}\left(\|G_1^n\|^2+\|G_2^n+G_3^n\|^2+\|G_4^n\|^2\right)+\frac{4\tau}{C_*}|||G_3^n|||^2.\end{aligned}$$

Therefore, using the positivity of $\mathcal{B}$, we have

$$
\begin{aligned}
&\|\gamma^{n+1}\|^2 + \mathcal{D}(s^{n+1}, s^{n+1}) + \mathcal{A}_1(s^{n+1}, s^{n+1})\\
&\leq \|\gamma^n\|^2 + \mathcal{D}(s^n, s^n) + \mathcal{A}_1(s^n, s^n) + C\tau\Big( \|G_1^n\|^2 + \|G_2^n + G_3^n\|^2 + \|G_4^n\|^2 + |||G_3^n|||^2 \Big).
\end{aligned}
$$

Now summing above inequality over $n = 1$ to $n = r - 1$ for $2 \leq r \leq M$, we have

$$
\begin{aligned}
\max_{2\leq n\leq r} \|\gamma^n\|^2 \leq & \|\gamma^1\|^2 + \|s^1\|^2 + |||s^1|||^2\\
& + C\tau \sum_{n=0}^{r-1} \Big( \|G_1^n\|^2 + \|G_2^n + G_3^n\|^2 + \|G_4^n\|^2 + |||G_3^n|||^2 \Big). \qquad (6.4.11)
\end{aligned}
$$

Next, for the estimate of $\|\gamma^1\|, \|s^1\|$ and $|||s^1|||$, we set $n = 0$ in (6.4.7), to obtain

$$
\begin{aligned}
&(\partial_\tau q^0, \phi_h) + \mathcal{B}(q^{\frac{1}{2}}, \phi_h) + \mathcal{D}(\gamma^{\frac{1}{2}}, \phi_h) + \mathcal{A}_1(\gamma^{\frac{1}{2}}, \phi_h) + \mathcal{A}_2(q^{\frac{1}{2}}, \phi_h)\\
&= (\partial_\tau \chi_t^0 - w^0, \phi_h) + \mathcal{B}(\chi_t^{\frac{1}{2}}, \phi_h) + \mathcal{D}(\chi^{\frac{1}{2}}, \phi_h).
\end{aligned}
$$

Therefore, using the identities $s^1 = \frac{\tau}{2}\gamma^1$, $q^{\frac{1}{2}} = \partial_\tau \gamma^0 - \alpha^0 = \frac{\gamma^1}{\tau}$ and $\partial_\tau q^0 = \frac{2}{\tau^2}\gamma^1 - \frac{2}{\tau}\alpha^0$, we have

$$
\begin{aligned}
&\frac{2}{\tau^2}(\gamma^1, \phi_h) + \frac{1}{\tau}\mathcal{B}(\gamma^1, \phi_h) + \frac{1}{2}\mathcal{D}(\gamma^1, \phi_h) + \frac{1}{\tau}\mathcal{A}_1(s^1, \phi_h) + \frac{1}{\tau}\mathcal{A}_2(\gamma^1, \phi_h)\\
&= \frac{2}{\tau}(\alpha^0, \phi_h) + \mathcal{B}(\alpha^0, \phi_h) + \mathcal{A}_2(\alpha^0, \phi_h) + (\partial_\tau \chi_t^0 - w^0, \phi_h) + \mathcal{B}(\chi_t^{\frac{1}{2}}, \phi_h) + \mathcal{D}(\chi^{\frac{1}{2}}, \phi_h).
\end{aligned}
$$

Hence, setting $\phi_h = \gamma^1 = \frac{2}{\tau}s^1$ and using inequality (6.2.8) and Poincaré type inequality (6.2.9), we have

$$
\begin{aligned}
&\|\gamma^1\|^2 + \frac{\tau}{2}\mathcal{B}(\gamma^1, \gamma^1) + \frac{\tau^2}{4}\mathcal{D}(\gamma^1, \gamma^1) + C_*|||s^1|||^2 + \frac{C_*\tau}{2}|||\gamma^1|||^2\\
&\leq \tau\|\alpha^0\||||\gamma^1||| + C\frac{\tau^2}{2}\left( \left\{ \|\chi_t^0 - w^0\| + \|\alpha^0\| + \|\chi_t^{\frac{1}{2}}\| + \|\chi^{\frac{1}{2}}\| \right\} |||\gamma^1||| + |||\alpha^0||||||s^1||| \right).
\end{aligned}
$$

Therefore, applying Young's inequality with suitable $\mu_1, \mu_2 > 0$, we have

$$
\begin{aligned}
&\|\gamma^1\|^2 + \frac{\tau}{2}\mathcal{B}(\gamma^1, \gamma^1) + \frac{\tau^2}{4}\mathcal{D}(\gamma^1, \gamma^1) + C_*|||s^1|||^2 + \frac{C_*\tau}{2}|||\gamma^1|||^2\\
&\leq \frac{\tau^2}{4\mu_1}\|\alpha^0\|^2 + C\frac{\tau^4}{8\mu_1}\left\{ \|\chi_t^0 - w^0\|^2 + \|\alpha^0\|^2 + \|\chi_t^{\frac{1}{2}}\|^2 + \|\chi^{\frac{1}{2}}\|^2 \right\}\\
&\quad + 2\mu_1|||\gamma^1|||^2 + C\frac{\tau^4}{8\mu_2}|||\alpha^0|||^2 + \mu_2|||s^1|||^2.
\end{aligned}
$$

Setting $\mu_1 = \frac{C_*\tau}{8}$, $\mu_2 = \frac{C_*}{2}$ and Poincaré type inequality (6.2.9) lead to

$$
\begin{aligned}
&\|\gamma^1\|^2 + \|s^1\|^2 + |||s^1|||^2\\
&\leq C\tau^2\|\alpha^0\|^2 + C\tau^4\Big\{ \|\chi_t^0 - w^0\|^2 + \|\alpha^0\|^2 + |||\alpha^0|||^2 + \|\chi_t^{\frac{1}{2}}\|^2 + \|\chi^{\frac{1}{2}}\|^2 \Big\}. \qquad (6.4.12)
\end{aligned}
$$

Therefore, using above equation (6.4.12) in (6.4.11), we have

$$\max_{2\le n\le r}\|\gamma^n\|^2 \le C\Big\{\tau^4\|\chi_t^0 - w^0\|^2 + \tau^2\|\alpha^0\|^2 + \tau^4|||\alpha^0|||^2\Big\}$$
$$+ C\tau\sum_{n=0}^{r-1}\Big(\|G_1^n\|^2 + \|G_2^n + G_3^n\|^2 + \|G_4^n\|^2 + |||G_3^n|||^2\Big),$$

i.e.,

$$\max_{2\le n\le r}\|\gamma^n\|^2 \le C\tau\sum_{n=0}^{r-1}\Big(\|G_1^n\|^2 + \|G_2^n + G_3^n\|^2 + \|G_4^n\|^2 + |||G_3^n|||^2\Big) \quad \text{for} \quad 2\le r\le M. \tag{6.4.13}$$

Now

$$\|G_1^n\|^2 \le \|\frac{\tau}{2}\sum_{j=0}^{n}\partial_\tau\chi_t^j + \frac{\tau}{2}\sum_{j=0}^{n-1}\partial_\tau\chi_t^j\|^2 + \|\frac{\tau}{2}\sum_{j=0}^{n}w^j + \frac{\tau}{2}\sum_{j=0}^{n-1}w^j\|^2 + \|\alpha^n$$
$$= \frac{1}{2}\|\chi_t^{n+1} - 2\chi_t^0 + \chi_t^n\|^2 + \|\frac{\tau}{2}w^n + \tau\sum_{j=0}^{n-1}w^j\|^2 + \|\alpha^n\|^2.$$

Again,

$$\|\chi_t^0\|^2 = \|u_t^0 - \mathcal{Q}_h u_t^0\|^2 + \|\mathcal{Q}_h u_t^0 - \mathbf{R}_h u_t^0\|^2$$
$$\le Ch^{2(k+1)}\Big\{\sum_{i=1}^{2}\|v^0\|^2_{H^{k+1}(\Omega_i)} + \sum_{i=1}^{2}\|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}\Big\}.$$

Also, for $1\le n\le M$, we have

$$\|\chi_t^n\|^2 \le \|u_t^n - \mathcal{Q}_h u_t^n\|^2 + \|\mathcal{Q}_h u_t^n - \mathbf{R}_h u_t^n\|^2$$
$$\le Ch^{2(k+1)}\Big\{\sum_{i=1}^{2}\|u_t^n\|^2_{H^{k+1}(\Omega_i)} + \sum_{i=1}^{2}\|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}\Big\}.$$

Hence,

$$\|G_1^n\|^2 \le Ch^{2(k+1)}\sum_{i=1}^{2}\Big(\|v^0\|^2_{H^{k+1}(\Omega_i)} + \|u_t^n\|^2_{H^{k+1}(\Omega_i)} + \|u_t^{n+1}\|^2_{H^{k+1}(\Omega_i)}\Big)$$
$$+ Ch^{2(k+1)}\sum_{i=1}^{2}\|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}$$
$$+ C\Big\{\tau^5\int_{t_n}^{t_{n+1}}\|u_{tttt}\|^2 dt + \tau^5\int_0^T\|u_{tttt}\|^2 dt + \tau^3\int_{t_n}^{t_{n+1}}\|\mathbf{R}_h u_{ttt}\|^2 dt\Big\}.$$

Further, using $\|\mathbf{R}_h u_{ttt}\|^2 \le \|\mathcal{Q}_h u_{ttt}\|^2 + \|\mathbf{R}_h u_{ttt} - \mathcal{Q}_h u_{ttt}\|^2$ and then Lemma 6.3.3, we have

$$\begin{aligned}\|G_1^n\|^2 \le Ch^{2(k+1)}\Big\{\sum_{i=1}^{2}\|v^0\|^2_{H^{k+1}(\Omega_i)} + \sum_{i=1}^{2}\|u_t\|^2_{L^\infty(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}\Big\}\\ + C\Big\{\tau^5\int_{t_n}^{t_{n+1}}\|u_{tttt}\|^2dt + \tau^5\int_0^T\|u_{tttt}\|^2dt\\ + \tau^3\int_{t_n}^{t_{n+1}}\|u_{ttt}\|^2dt + \tau^3h^4\sum_{i=1}^{2}\|u_{ttt}\|^2_{H^1(J;H^2(\Omega_i))}\Big\}. \qquad (6.4.14)\end{aligned}$$

Also,

$$\begin{aligned}\|G_2^n\|^2 &= \|\frac{\tau}{2}\Big(\sum_{j=0}^{n}\chi_t^{j+\frac{1}{2}} + \sum_{j=0}^{n-1}\chi_t^{j+\frac{1}{2}}\Big)\|^2 \le \|\tau\sum_{j=0}^{n+1}\chi_t^j\|^2\\ &\le Ch^{2(k+1)}\tau\Big\{\sum_{j=0}^{n+1}\sum_{i=1}^{2}\|u_t^j\|^2_{H^{k+1}(\Omega_i)} + \sum_{j=0}^{n+1}\sum_{i=1}^{2}\|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}\Big\}\\ &\le Ch^{2(k+1)}\Big\{\sum_{i=1}^{2}\|u_t\|^2_{L^\infty(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}\Big\}. \qquad (6.4.15)\end{aligned}$$

Similarly,

$$\|G_4^n\|^2 \le Ch^{2(k+1)}\Big\{\sum_{i=1}^{2}\|u\|^2_{L^\infty(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u\|^2_{H^1(J;H^{k+1}(\Omega_i))}\Big\}. \qquad (6.4.16)$$

Again, using Lemma 6.4.2, we have

$$|||G_3^n|||^2 = \Big|\Big|\Big|\frac{\tau}{2}\alpha^n + \tau\sum_{j=0}^{n-1}\alpha^j\Big|\Big|\Big|^2 \le C\Big(\tau^5\int_{t_n}^{t_{n+1}}|||\mathbf{R}_h u_{ttt}|||^2dt + \tau^5\int_0^T|||\mathbf{R}_h u_{ttt}|||^2dt\Big).$$

Also, using Lemma 6.3.3, we obtain

$$\begin{aligned}|||\mathbf{R}_h u_{ttt}|||^2 &\le |||\mathcal{Q}_h u_{ttt}|||^2 + |||\mathbf{R}_h u_{ttt} - \mathcal{Q}_h u_{ttt}|||^2\\ &\le C\sum_{i=1}^{2}\|u_{ttt}\|^2_{H^1(\Omega_i)} + Ch^2\sum_{i=1}^{2}\|u_{ttt}\|^2_{H^1(J;H^2(\Omega_i))}.\end{aligned}$$

Therefore,

$$|||G_3^n|||^2 \le C\tau^5\Big(\sum_{i=1}^{2}\|u_{ttt}\|^2_{L^2(J;H^1(\Omega_i))} + Th^2\sum_{i=1}^{2}\|u_{ttt}\|^2_{H^1(J;H^2(\Omega_i))}\Big). \qquad (6.4.17)$$

Similarly, using Lemma 6.3.3

$$\|G_3^n\|^2 \leq C\tau^5\left(\sum_{i=1}^{2}\|u_{ttt}\|^2_{L^2(J;L^2(\Omega_i))} + Th^4\sum_{i=1}^{2}\|u_{ttt}\|^2_{H^1(J;H^2(\Omega_i))}\right). \tag{6.4.18}$$

Hence, using above two inequalities (6.4.14), (6.4.18) in (6.4.13), we have

$$\begin{aligned}\max_{1\leq n\leq M}\|\gamma^n\|^2 \leq Ch^{2(k+1)}&\Bigg\{\sum_{i=1}^{2}\|v^0\|^2_{H^{k+1}(\Omega_i)} + \sum_{i=1}^{2}\|u\|^2_{L^\infty(J;H^{k+1}(\Omega_i))}\\ &+ \sum_{i=1}^{2}\|u_t\|^2_{L^\infty(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u\|^2_{H^1(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u_t\|^2_{H^1(J;H^{k+1}(\Omega_i))}\Bigg\}\\ &+ C\Bigg(\tau^6\sum_{i=1}^{2}\|u_{ttt}\|^2_{L^2(J;H^1(\Omega_i))} + \tau^4\sum_{i=1}^{2}\|u_{ttt}\|^2_{H^1(J;H^2(\Omega_i))}\\ &\qquad + \tau^4\int_0^T\|u_{ttt}\|^2dt + \tau^6\int_0^T\|u_{tttt}\|^2dt\Bigg).\end{aligned} \tag{6.4.19}$$

□

Finally, the following main theorem of this section states the optimal $L^2$ error estimate for the fully discrete case.

**Theorem 6.4.1.** *Let $u$ be the exact solution of the interface problem (6.1.1)- (6.1.2) and $U^n$ be solution of fully discrete approximation (6.4.4), respectively. Assume that $u, u_t \in L^\infty(J;H^{k+1}(\Omega_i))$ for $i = 1,2$ and $u, u_t \in H^1(J;H^{k+1}(\Omega_i))$ for $i = 1,2$. Further, assume that $u_{ttt} \in H^1(J;H^2(\Omega_i))$ for $i = 1,2$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}\max_{0\leq n\leq M}\|e_h^n\| \leq Ch^{k+1}&\Bigg\{\sum_{i=1}^{2}\|u\|_{L^\infty(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u_t\|_{L^\infty(J;H^{k+1}(\Omega_i))}\\ &+ \sum_{i=1}^{2}\|u\|_{H^1(J;H^{k+1}(\Omega_i))} + \sum_{i=1}^{2}\|u_t\|_{H^1(J;H^{k+1}(\Omega_i))}\Bigg\}\\ + C\tau^2\Bigg(\sum_{i=1}^{2}\|u_{ttt}\|_{L^2(J;H^1(\Omega_i))} &+ \sum_{i=1}^{2}\|u_{ttt}\|_{H^1(J;H^2(\Omega_i))} + \int_0^T\|u_{ttt}\|^2dt + \int_0^T\|u_{tttt}\|^2dt\Bigg).\end{aligned}$$

*Proof.* Applying triangle inequality, we have

$$\|U^n - \mathcal{Q}_hu^n\| \leq \|U^n - \mathbf{R}_hu^n\| + \|\mathbf{R}_hu^n - \mathcal{Q}_hu^n\|.$$

Using Lemma 6.3.3 and Lemma 6.4.3 in the above inequality we have the required results. □

## 6.5 Numerical Experiments

In this section, we conduct numerical experiments to illustrate our theoretical findings. To demonstrate high-order convergence, we consider different kinds of Lipschitz continuous interfaces with complex structures in 2D model problems with sufficiently smooth exact solutions. The non-uniform triangular meshes aligned with interface $\Gamma$ (see, Figure 6.5.1 (right)) have been taken for the WG algorithm. The errors $e_h^n$ are evaluated with respect to the $|||\cdot|||$ and $L^2$ norms, defined in equation (6.2.7). These errors are reported in tables corresponding to the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(e), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $h$ is the spatial step size and $\tau$ is the time step size.

**Example 6.5.1.** Consider unit square domain $\Omega = (0,1)^2$ with the circular interface with center $\left(\frac{1}{2}, \frac{1}{2}\right)$ with radius $r = \frac{1}{4}$ (see, Figure 6.5.1(left)). Here $\Omega_1$ is the interior portion of the circular interface and $\Omega_2 = \Omega \setminus \Omega_1$. The time interval $J = (0,1]$.

The exact solution $u$ and physical coefficients are given by

$$u = \begin{cases} \exp(-t)\cos(\pi(x+3y))\sin(\pi y) & \text{in } \Omega_1, \\ t(xy + cos(\pi y)) & \text{in } \Omega_2, \end{cases}$$

$$(\Theta, \xi, \eta, \upsilon) = \begin{cases} \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{10}, \frac{1}{3}\right) \text{ in } \Omega_1, \\ \left(\frac{1}{10}, \frac{1}{50}, \frac{1}{10^2}, \frac{1}{2}\right) \text{ in } \Omega_2. \end{cases}$$

Figure 6.5.1: The domain $\Omega$ with interface $\Gamma$ (left) and a refined mesh (right) at level 6.

Table 6.5.1: The history of convergence under $|||\cdot|||$ and $L^2$ norms at time $t = 1$ for Example 6.5.1.

| $k$ | Level | $h$ | $\|e_h\|$ | EOC | $\|\|\|e_h\|\|\|$ | EOC |
|---|---|---|---|---|---|---|
| 1 | 1 | 2.78e-01 | 9.35e-02 | — | 3.96e-01 | — |
| | 2 | 1.39e-01 | 2.39e-02 | 1.96 | 1.99e-01 | 0.99 |
| | 3 | 6.96e-02 | 6.00e-03 | 1.99 | 9.98e-02 | 1.00 |
| | 4 | 3.48e-02 | 1.50e-03 | 2.00 | 4.99e-02 | 1.00 |
| | 5 | 1.74e-02 | 3.75e-04 | 2.00 | 2.49e-02 | 1.00 |
| | 6 | 8.70e-03 | 9.39e-05 | 2.00 | 1.24e-02 | 1.00 |

Table 6.5.2: The history of convergence under $|||\cdot|||$ and $L^2$ norms at time $t = 1$ for Example 6.5.1.

| $k$ | Level | $h$ | $\|e_h\|$ | EOC | $\|\|\|e_h\|\|\|$ | EOC |
|---|---|---|---|---|---|---|
| 2 | 1 | 2.78e-01 | 1.94e-02 | — | 1.13e-01 | — |
| | 2 | 1.39e-01 | 2.63e-03 | 2.88 | 3.04e-02 | 1.89 |
| | 3 | 6.96e-02 | 3.35e-04 | 2.97 | 7.76e-03 | 2.00 |
| | 4 | 3.48e-02 | 4.20e-05 | 3.00 | 1.95e-03 | 2.00 |
| | 5 | 1.74e-02 | 5.26e-06 | 3.00 | 4.88e-04 | 2.00 |
| | 6 | 8.70e-03 | 6.57e-07 | 3.00 | 1.22e-04 | 2.00 |

**Example 6.5.2.** Consider the square domain $\Omega = (-1, 1)^2$ and the time period $J = (0, 1]$. This example explores the efficacy of the WG algorithm in handling a problem characterized by a five-leaf shape interface embedded in $\Omega = (-1, 1)^2$ (see, Figure 6.5.2 (left) for interface and subdomains $\Omega_1$, $\Omega_2$), described by its parametric equation as

Table 6.5.3: The history of convergence under $|||\cdot|||$ and $L^2$ norms at time $t = 1$ for Example 6.5.1.

| $k$ | Level | $h$ | $\|e_h\|$ | EOC | $\|\|\|e_h\|\|\|$ | EOC |
|---|---|---|---|---|---|---|
| 3 | 1 | 2.78e-01 | 3.59e-03 | — | 2.46e-02 | — |
| | 2 | 1.39e-01 | 2.34e-04 | 3.94 | 3.18e-03 | 2.95 |
| | 3 | 6.96e-02 | 1.47e-05 | 3.99 | 4.01e-04 | 2.99 |
| | 4 | 3.48e-02 | 9.23e-07 | 4.00 | 5.03e-05 | 2.99 |
| | 5 | 1.74e-02 | 5.77e-08 | 4.00 | 6.29e-06 | 3.00 |
| | 6 | 8.70e-03 | 3.60e-09 | 4.00 | 7.86e-07 | 3.00 |

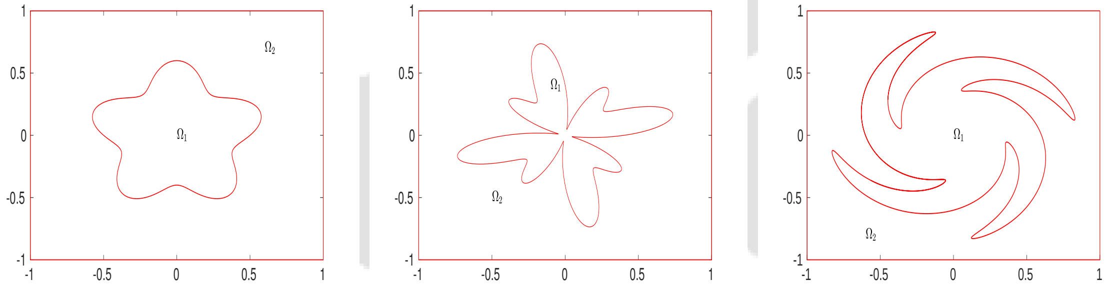

Figure 6.5.2: The shape of domain, subdomains and interface.

follows [99]

$$\begin{cases} x(\omega) = r(1 + (0.2)\sin(5\omega))\cos(\omega), \\ y(\omega) = r(1 + (0.2)\sin(5\omega))\sin(\omega), \end{cases}$$

where $r = 1/2$ and $\omega \in [0, 2\pi]$.

The exact solution and physical coefficient are given by

$$u = \begin{cases} (t+3)\exp(x)\cos(x+y) & \text{in } \Omega_1, \\ (1+\sin(t))\cos(9x)\sin(9y) & \text{in } \Omega_2, \end{cases}$$

$$(\Theta,\xi,\eta,\upsilon)=\begin{cases}(\frac{1}{10},\frac{1}{5},1,\frac{1}{10^3}) & \text{in } \Omega_1,\\ (\frac{1}{10^2},\frac{1}{10},\frac{1}{2},\frac{1}{10^5}) & \text{in } \Omega_2.\end{cases}$$

Table 6.5.4: The history of convergence under $|||\cdot|||$ and $L^2$ norms at time $t = 1$ for Example 6.5.2.

| $k$ | Level | $h$ | $\|e_h\|$ | EOC | $\|\|\|e_h\|\|\|$ | EOC |
|---|---|---|---|---|---|---|
| 1 | 1 | 2.75e-01 | 2.05e-02 | — | 1.00e-01 | — |
|  | 2 | 1.37e-01 | 5.31e-03 | 1.94 | 5.21e-02 | 0.94 |
|  | 3 | 6.89e-02 | 1.33e-03 | 1.99 | 2.64e-02 | 0.97 |
|  | 4 | 3.44e-02 | 3.34e-04 | 2.00 | 1.33e-02 | 0.99 |
|  | 5 | 1.72e-02 | 8.35e-05 | 2.00 | 6.67e-03 | 1.00 |
|  | 6 | 8.61e-03 | 2.08e-05 | 2.00 | 3.33e-03 | 1.00 |

**Example 6.5.3.** Consider the domain $\Omega = (-1,1)^2$ and the time period $J = (0,1]$. This example concerns with the arbitrary shape interface $\Gamma$, whose parametric equation in polar coordinate $\omega$ is given by [128]

$$\begin{cases} x(\omega) = (0.40178 + (0.40178)\cos(2\omega)\sin(6\omega))\cos(\omega),\\ y(\omega) = (0.40178 + (0.40178)\cos(2\omega)\sin(6\omega))\sin(\omega),\end{cases}$$

where $\omega = [0, 2\pi]$. The subdomain $\Omega_1$ is the interior portion of interface $\Gamma$ and $\Omega_2 = \Omega \setminus \Omega_1$. For more details, see Figure 6.5.2 (middle).

The exact solution and physical coefficients are given by

$$u=\begin{cases}(\cos(t+1)+\exp(t))\exp(x+2y) & \text{in } \Omega_1,\\ (t+3)(\sin(x)+y+1) & \text{in } \Omega_2,\end{cases}$$

$$(\Theta,\xi,\eta,\upsilon)=\begin{cases}(\frac{1}{10^2},\frac{1}{2},\frac{1}{10},\frac{1}{10^3}) & \text{in } \Omega_1,\\ (1,\frac{1}{10},\frac{1}{10^4},\frac{1}{10^6}) & \text{in } \Omega_2.\end{cases}$$

Table 6.5.5: The history of convergence under $|||\cdot|||$ and $L^2$ norms at time $t = 1$ for Example 6.5.3.

| $k$ | Level | $h$ | $\|e_h\|$ | EOC | $|||e_h|||$ | EOC |
|---|---|---|---|---|---|---|
| 1 | 1 | 2.05e-01 | 4.62e-01 | — | 2.03e+00 | — |
| | 2 | 1.02e-01 | 1.50e-01 | 1.61 | 1.22e+00 | 0.72 |
| | 3 | 5.14e-02 | 3.95e-02 | 1.93 | 6.35e-01 | 0.95 |
| | 4 | 2.57e-02 | 9.94e-03 | 1.99 | 3.19e-01 | 0.98 |
| | 5 | 1.28e-02 | 2.48e-03 | 2.00 | 1.60e-01 | 1.00 |
| | 6 | 6.42e-03 | 6.22e-04 | 2.00 | 8.01e-02 | 1.00 |

**Example 6.5.4.** Let $\Omega = (-1,1)^2$ be square domain and time period $J = (0,1]$. Consider an arbitrary shape complex interface (see Figure 6.5.2 (right)). The parametric equation of the interface $\Gamma$ embedded in $\Omega$ is set to be [128]

$$\begin{cases} y(\omega) = r(\omega)\sin(g(\omega)), \\ x(\omega) = r(\omega)\cos(g(\omega)), \end{cases}$$

where $r(\omega) = 0.2401256\cos(4\omega + \pi/2) + 0.6012563,\ g = \sin(4\omega) + \omega.$

The exact solution and physical coefficients are given by

$$u = \begin{cases} (t^2+2)(\cos(xy) + x^2 + 7) & \text{in } \Omega_1, \\ t(x+y+3) & \text{in } \Omega_2, \end{cases}$$

$$(\Theta, \xi, \eta, \upsilon) = \begin{cases} (1, \frac{1}{10^2}, \frac{1}{5}, \frac{1}{10^4}) \ \text{ in } \ \Omega_1, \\ (\frac{1}{2}, \frac{1}{10}, 1, \frac{1}{10}) \ \text{ in } \ \Omega_2. \end{cases}$$

Table 6.5.6: The history of convergence under $|||\cdot|||$ and $L^2$ norms at time $t = 1$ for Example 6.5.4.

| $k$ | Level | $h$ | $\|e_h\|$ | EOC | $|||e_h|||$ | EOC |
|---|---|---|---|---|---|---|
| 1 | 1 | 1.82e-01 | 6.37e-01 | — | 2.57e+00 | — |
|  | 2 | 9.13e-02 | 1.85e-01 | 1.78 | 1.40e+00 | 0.88 |
|  | 3 | 4.56e-02 | 4.62e-02 | 2.00 | 6.93e-01 | 1.01 |
|  | 4 | 2.28e-02 | 1.15e-02 | 2.00 | 3.44e-01 | 1.00 |
|  | 5 | 1.14e-02 | 2.87e-03 | 2.00 | 1.72e-01 | 1.00 |
|  | 6 | 5.70e-03 | 7.19e-04 | 2.00 | 8.60e-02 | 1.00 |

7

# Conclusions and Future Scopes

In this chapter, we highlight the significance of current thesis work, the corresponding results and techniques to derive them. We also provide information for the scope of possible extensions and future investigations.

## 7.1 Summary of results

This thesis has considered recently developed high-order weak Galerkin finite element methods for different hyperbolic problems on polygonal meshes. The work in this thesis could be a crucial step for error analysis of the WG-FEMs for linear and nonlinear hyperbolic problems. We aim to derive *a priori* error estimates in suitable norms for the WG approximation solution. Further, theoretical results are supported by numerical experiments. In the following, we highlight the critical review of the results obtained in each chapter.

In Chapter 2, we have presented *a priori* error estimates for the spatially semidiscrete scheme for second order linear wave equation (2.1.1)-(2.1.3) by allowing polynomial approximations with various degrees for each local WG element. Our results are intended to extend the work of [43, 151] to second order wave equation based on WG finite element space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$, with arbitrary non negative integers$\{k, j, l\}$, where $k \geq 1$. Optimal order error estimates in both $L^\infty(H^1)$ (see, Theorem 2.3.1) as well as $L^\infty(L^2)$ (see, Theorem 2.3.2) norms are established. Error equation (2.2.6) played a crucial role in the derivation of optimal order error with respect to point-wise in time discrete $H^1$ norm. Further, elliptic type projection operator $\mathcal{R}_h$ has been introduced in (2.3.6) to obtain optimal order of convergence for semidiscrete solution with respect to $L^\infty(L^2)$ norm.

In Chapter 3, we have extended the spatially discrete *a priori* error analysis to the fully discrete approximation for second order linear wave equation (3.1.1)-(3.1.3) in local WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_j(\partial\mathcal{K}), [\mathbf{P}_l(\mathcal{K})]^2)$. The Crank-Nicolson scheme is applied for the time discretization. We have obtained the optimal order of convergence in $L^\infty(L^2)$ (see, Theorem 3.2.1) norm for the fully discrete scheme. Finally, we have presented some numerical experiments to justify the robustness, reliability and accuracy of WG-FEMs. We observed that the method of element-boundary-discrepancy seems to be more stable than the method of projected element boundary-discrepancy.

In Chapter 4, we have developed WG-FEMs for solving semilinear Klein-Gordon equation (4.1.1)-(4.1.3) on polygonal meshes. The optimal order of convergence in $L^\infty(L^2)$ norm is proved in both semidiscrete and fully-discrete schemes (see, Theorem 4.3.1 and Theorem 4.4.1) in the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$. Also, semidiscrete error estimate in discrete $H^1$ norm (see, Theorem 4.3.2) has been established. For the fully-discrete case we have employed time Newmark scheme for temporal discretization. At the end of the chapter, we have performed various numerical experiments that justify the theoretical results.

In Chapter 5, we have analyzed a weak Galerkin finite element method for the quasi-linear Westervelt's model (5.2.1)-(5.2.2) of ultrasound waves on polygonal meshes. Specifically, we investigate the spatial discretization of Westervelt's quasi-linear, strongly damped wave equation using high-order weak Galerkin discretization. Our convergence analysis relies on the Banach fixed-point theorem, along with stability (see, Lemma 5.3.1) and convergence analysis of a linear diffusive viscous wave equation (5.3.1) with variable coefficients for the first and second time derivatives. This approach yields optimal convergence rates in $L^2$-based spatial norm (see, Theorem 5.5.1) for sufficiently small data and mesh size, given an appropriate choice of initial data and error estimate in $L^\infty(L^2)$ norm (see, Theorem 5.4.1) for linearized Westervelt's equation in the WG space $(\mathbf{P}_k(\mathcal{K}), \mathbf{P}_k(\partial\mathcal{K}), [\mathbf{P}_{k-1}(\mathcal{K})]^2)$, where $k \geq 1$. Numerical experiments in two-dimensional settings based on backward Euler time discretization illustrate the theoretical convergence results.

Finally, in Chapter 6, we have studied *a priori* error analysis of weak Galerkin finite element approximations to a general linear second order hyperbolic interface problem (6.1.1) with variable coefficients on polygonal meshes. Apart from standard projection operators, a new non-standard elliptic type projection operator $\mathbf{R}_h$ has been introduced. The derivation of error bound heavily depends on this new non-standard elliptic type projection operator $\mathbf{R}_h$. Optimal order of convergence in both $L^\infty(H^1)$ norm and $L^\infty(L^2)$ norm are established (see, Theorem 6.3.2 and Theorem 6.3.3). We have extended the spatially discrete *a priori* error analysis to a fully discrete approximation for second order hyperbolic interface problem (1.1.10). The fully discrete scheme is the well known Crank-Nicolson scheme. Optimal order of convergence in $L^\infty(L^2)$ norm is derived (see, Theorem 6.4.1). Numerical results for two dimensional test problems are presented in support of theoretical convergence results. In each of these numerical experiments, we have used different forms of interfaces indicating the adaptability, reliability and accuracy of the method.

## 7.2 Future Scopes and Remarks

In this section, we have described some possible extensions of our thesis work. Here we are briefly presented some uninvestigated models that can be demonstrated as the future scopes of our findings.

**Weak Galerkin finite element general Newmark scheme for acoustic wave equation:** In this thesis, we have employed the Crank-Nicolson scheme for temporal discretization for wave equation (see, Chapter 3) and general second order linear hyperbolic equation (see, Chapter 6) and particular Newmark scheme for semilinear Klein-Gordon equation (see, chapter 4). Although, the above approaches are implicit, mesh parameters have no stability restriction. In Kim and Lim [93], it is found that the implicit technique yields solutions that are less dispersive than the explicit one, which is helpful for the numerical solution in highly oscillatory media. It could be interesting to apply a more general Newmark scheme for the acoustic wave equation, defined as

$$\left(\partial_\tau^2 U^n, \phi_h\right) + \mathcal{A}_w(U^{n,z_1,z_2}, \phi_h) = (f^{n,z_1,z_2}, \phi_h)\ \forall \phi_h \in \mathcal{W}_h^0, \tag{7.2.1}$$

where $\partial_\tau^2 U^n$ is defines by (3.2.2) and

$$U^{n,z_1,z_2} = z_1 U^{n+1} + \left(\frac{1}{2} - 2z_1 + z_2\right) U^n + \left(\frac{1}{2} + z_1 - z_2\right) U^{n-1}, \tag{7.2.2}$$

and $f^{n,z_1,z_2}$ is defined accordingly. Here $z_1,\ z_2 \geq 0$ are free parameters. Above scheme (7.2.2) reduces to explicit scheme when $z_1 = 0$ and $z_2 = \frac{1}{2}$.

**Fully discrete a priori error analysis for semilinear Westervelt's wave equation:**

In Chapter 5, we have developed and examined efficient computational methods for accurately approximating nonlinear wave propagation in uniform media. Our focus was on investigating the Westervelt's equation to address nonlinearity. Our convergence analysis relies on the Banach fixed-point theorem, supplemented by a stability and convergence evaluation of a linear diffusive viscous wave equation with variable coefficients. Subsequently, we conducted several numerical experiments to validate the efficacy of the WG method in scientific computing. However, a fully discrete convergent analysis for the semilinear Westervelt's equation is still unexplored. We are interested in extending spatially semidiscrete *a priori* error analysis to a fully discrete weak Galerkin approximation for the semilinear Westervelt's equation.

**A priori $L^\infty(L^2)$ error estimates of WG-FEMs for nonlinear sound waves:** Consider the Westervelt's equation (5.1.1) to model nonlinear sound propagation through a homogeneous media given in terms of the acoustic velocity potential $\psi$ can be obtained via the following equation (cf. [156]):

$$(1 - 2\kappa\psi_t)\psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0. \tag{7.2.3}$$

Here the constants $c$ represents the speed of sound, $b$ is the so-called sound diffusivity, $\kappa$ is given by $\kappa = \beta_a/\rho_m c^2$, where $\rho_m$ is the mass density and $\beta_a$ is the coefficient of nonlinearity of the medium (cf. [56, 142]). Challenges in the numerical analysis lie in handling the nonlinearity in the model (7.2.3), which involves the derivatives in time of the acoustic velocity potential $\psi$ and preventing the equation from degenerating. Relying on Banach fixed point theorem combined with a stability and convergent analysis of the linear wave equation Antonietti et al. [7] implemented discontinuous Galerkin methods to derive a priori error estimate in a suitable energy norm. Our aim is to analyze *a priori* $L^\infty(L^2)$ error estimates for the model problem (7.2.3).

**Weak Galerkin finite element approximation for linearized Westervelt's equation in heterogeneous media:** In Chapter 5, we have developed and examined efficient computational methods for accurately approximating nonlinear wave propagation in uniform media. We focused on investigating the linearized Westervelt equation and extending the results to quasi-linear Westervelt's acoustic wave equation based on Banach fixed point theorem. We aim to extend these findings to linearized Westervelt's equation, which incorporates heterogeneity in the form of varying material parameters.

Let $\Omega$ is a convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ with $C^2$ smooth interface $\Gamma = \partial\Omega_1$ and $\Omega_2 = \Omega \setminus \Omega_1$(see, Figure 1.1.1). In $\Omega$, consider linearized Westervelt's equation

$$\alpha(\mathbf{x}, t)u_{tt} - c^2\Delta u - b\Delta u_t + \beta(\mathbf{x}, t)u_t = f(\mathbf{x}, t) \quad \text{in } \Omega \times (0, T], \tag{7.2.4}$$

We assume that the physical coefficients $\alpha, \beta$ are discontinuous and piecewise regular in $\Omega$. Specifically,

$$(\alpha(\mathbf{x}, t), \beta(\mathbf{x}, t)) = \begin{cases} (\alpha_1(\mathbf{x}, t), \beta_1(\mathbf{x}, t)) & \text{in } \Omega_1, \\ (\alpha_2(\mathbf{x}, t), \beta_2(\mathbf{x}, t)) & \text{in } \Omega_2. \end{cases}$$

Further, assume that physical coefficients $c^2, b$ are piecewise constants, i.e.

$$(c^2, b) = \begin{cases} (c_1^2, b_1) & \text{in } \Omega_1, \\ (c_2^2, b_2) & \text{in } \Omega_2. \end{cases}$$

The problem (7.2.4) is supplemented with the following physical interface conditions:

$$[u] = \Phi, \quad \left[\alpha(\mathbf{x})\frac{\partial u'}{\partial \mathbf{n}} + b(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}\right] = \Psi \quad \text{along } \Gamma \times (0, T], \tag{7.2.5}$$

where $[u]$ denotes the jump of a quantity across the interface $\Gamma$, i.e., $[u](\mathbf{x}) = u_1|_\Gamma - u_2|_\Gamma$ with $u_i(\mathbf{x}) = u(\mathbf{x})|_{\Omega_i}$, $i = 1, 2$ and $\left[\alpha \partial u'/\partial \mathbf{n} + b\partial u/\partial \mathbf{n}\right] = \alpha_1 \partial u_1'/\partial \mathbf{n}_1 + \alpha_2 \partial u_2'/\partial \mathbf{n}_2 + b_1 \partial u_1/\partial \mathbf{n}_1 + b_2 \partial u_2/\partial \mathbf{n}_2$ with $\partial/\partial \mathbf{n}_i$ denoting the outer normal derivative with respect to $\Omega_i$, $i = 1, 2$. We shall subsequently employ these results to establish coupling between the linear acoustic equation and the Westervelt's equation in heterogeneous media.

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