Abstract

Graphs with positive or negative edges are called *signed graphs*. We denote a signed graph Σ by (G, ϕ), where G is called the underlying graph of Σ and ϕ is a function that assigns $+1$ or -1 to the edges of G. The set of negative edges in Σ is known as the *signature* of Σ. An unsigned graph can be realized as a signed graph in which all edges are positive. *Switching* Σ by a vertex v is to change the sign of each edge incident to v. Switching is a way of turning one signed graph into another. Two signed graphs are called *switching equivalent* if one can be obtained from the other by a sequence of switchings. Further, two signed graphs are said to be *switching isomorphic* to each other if one is isomorphic to a switching of the other. In Chapter 2 of the thesis, we classify the switching non-isomorphic signed graphs arising from K_6, $P_{3,1}$, $P_{5,1}$, $P_{7,1}$, and $B(m,n)$ for $m \geq 3$, $n \geq 1$, where K_6 is the complete graph on six vertices, $P_{n,k}$ denotes the generalized Petersen graph and $B(m,n)$ denotes the book graph consisting of n copies of the cycle $C_m$with exactly one common edge. We also count the switching non-isomorphic signatures of size two in $P_{2n+1,1}$ for $n \geq 1$. We prove that the size of a minimum signature of $P_{2n+1,1}$, up to switching, is at most $n+1$.

A signed cycle is *positive* if the product of signs of the edges in the cycle is positive, and *negative*, otherwise. A signed graph is said to be *balanced* if every cycle in the graph is positive. A signed graph is called *unbalanced* if it is not balanced, that is, it has at least one negative cycle. The minimum number of edges required to delete from a signed graph to make it balanced is called the *frustration index* of the signed graph. The frustration index of a signed graph Σ is denoted by $l(\Sigma)$. The frustration index is a significant way to measure how unbalanced a signed graph is. V. Sivaraman [49] proved that if Σ is a signed graph whose underlying graph is simple, triangle-free and cubic, then $l(\Sigma) \leq \frac{3}{8}|V(\Sigma)|$. The *maximum frustration* of G, denoted $D(G)$, is the maximum frustration index over all possible signatures of G. Let $P_{n,k}$ be the generalized Petersen graph and $d \coloneqq \gcd(n,k)$. In Chapter 3, we prove that $D(P_{n,k}) \leq 1 + \lfloor \frac{n}{2} \rfloor$ for $d = 1$, and $D(P_{n,k}) \leq 1 + d(1 + \lfloor \frac{n}{2d} \rfloor)$ for $d > 1$. These upper bounds on $D(P_{n,k})$ improve the Sivaraman's [49] bound for these graphs. We also determine the frustration indices of switching non-isomorphic signed complete graphs on six vertices. Finally, we show that if $B(m,n)$ is the book graph with $m \geq 3$ and $n \geq 1$, then $D(B(m,n)) = \lceil \frac{n}{2} \rceil$.

Ashraf and Germina [9] defined double domination in signed graphs as follows. Given a signed graph Σ with vertex set V, a subset D of V is said to be a *double dominating set* (DDS) of Σ if it satisfies the following conditions: (i) $|N[v] \cap D| \geq 2$ for each $v \in V$, and (ii) $\Sigma[D, D^c]$ is balanced, where $\Sigma[D, D^c]$ is the signed graph induced by the edges of Σ with one end vertex in D and the other end vertex in D^c. The size of a minimum DDS of Σ is called the *double domination number* (DDN) of Σ, and it is denoted by $\gamma_{\times 2}(\Sigma)$. In Chapter 4, we prove that if D is a DDS of a cubic unsigned graph G such that $|D| = \frac{|V(G)|}{2}$, then $G[D, D^c]$ admits a cycle decomposition. We further give an example which shows that if D, with $|D| = \frac{|V(G)|}{2}$, is not a DDS of a cubic graph G, then it is not necessarily true that $G[D, D^c]$ admits a cycle decomposition. A lower bound on the DDN of signed cubic graphs are also obtained in Chapter 4. We also obtain some bounds on the DDN of signed generalized Petersen graphs and signed I-graphs. We prove that the DDN of a signed complete graph on n vertices is $n-1$ for $n \geq 5$. However, this number is known to be exactly 2 for unsigned complete graph on n vertices, when $n \geq 2$. Also, we compute the DDN of all switching non-isomorphic signed Petersen graphs and of all switching non-isomorphic signed complete graphs over K_6.

For $n \geq 1$, let $M_n = \{0, \pm 1, \ldots, \pm k\}$ if $n = 2k+1$ and $M_n = \{\pm 1, \ldots, \pm k\}$ if $n = 2k$. A *proper n-coloring* of a signed graph Σ is defined to be a mapping $c : V(G) \longrightarrow M_n$ that satisfies $c(x) \neq \phi(e)c(y)$ for each edge e, where $\Sigma = (G, \phi)$, $e = xy$ and $\phi(e)$ is the sign of e. Such a coloring is said to be *zero-free* if it never takes the value zero. The *chromatic number* $\chi(G, \phi)$ of (G, ϕ) is the smallest number n such that (G, ϕ) admits a proper n-coloring. The *chromatic polynomial* $\chi_\Sigma(2k+1)$ of Σ is defined to be the function whose value is equal to the number of proper colorings of Σ in $2k+1$ colors. The *zero-free chromatic polynomial* $\chi^b_\Sigma(\lambda)$ is the function such that $\chi^b_\Sigma(2k)$ counts the zero-free proper colorings in $2k$ colors. In Chapter 5, we first prove that the chromatic number of every signed book graph is either 2 or 3. We also obtain explicit formulas for the chromatic polynomials and the zero-free chromatic polynomials of switching non-isomorphic signed book graphs.

A signed graph Σ is said to be *parity signed* if there exists a bijective map $f : V(G) \longrightarrow \{1, \ldots, |V(G)|\}$ such that $\phi(uv) = +1$ if and only if $f(u)$ and $f(v)$ are of the same parity for each $uv \in E(G)$, where $\Sigma = (G, \phi)$. The rna number of a graph G, denoted $\sigma^-(G)$, is the least number of negative edges among all possible parity signed graphs over G. The rna number is also equal to the least size of a cut that has nearly equal sides. In Chapter 6, we prove that $3 \leq \sigma^-(P_{n,k}) \leq n$ and that these bounds are sharp. The values of $\sigma^-(P_{n,k})$ are also determined for $k \in \{1, 2\}$. The rna numbers of some famous generalized Petersen graphs are also determined. We show that the smallest order of $(4n-1)$-regular graphs having rna number 1 is bounded above by $12n-2$. The sharpness of this upper bound is established for $n = 1$ by showing that there is a unique cubic graph of order 10 whose rna number is 1. We also show that the smallest order of a $(4n+1)$-regular graph having rna number 1 is $8n+6$.