

DYNAMIC ANALYSES OF PZT BASED ACTIVE NONLINEAR VIBRATION ABSORBERS

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by

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Dedicated to my Mother



Declaration

I declare that this thesis entitled “Dynamic analyses of PZT based active nonlinear vibration absorbers”, submitted by me is a presentation of my original research work done under the guidance of Dr. Santosha K. Dwivedy, Professor, Department of Mechanical Engineering, Indian Institute of Technology Guwahati. This work has not been submitted elsewhere for the award of any degree, diploma, associateship, fellowship, titles in this institute or any other university or institution of higher learning.

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Certificate

This is to certify that this thesis entitled “**Dynamic Analyses of PZT based Active Nonlinear Vibration Absorbers**” submitted by Sibananda Mohanty to Indian Institute of Technology Guwahati, Assam, India, for the award of the degree of Doctor of Philosophy in the Department of Mechanical Engineering is a *bona fide* record of research work carried out by the student himself under my guidance and supervision. To the best of my knowledge, no part of the work reported in this thesis has been presented for the award of any degree, diploma, associateship, fellowship, titles in this institute or any other university or institution of higher learning.

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ABSTRACT

There are many physical systems such as engineering structures or machines which undergo severe vibration and need a reduction in vibration by using passive or active vibration absorbers. In the passive vibration absorber, structural modification is carried out by attaching an additional spring-mass-damper system to the vibrating main or primary system to reduce its vibration. Whereas in the active vibration absorber (AVA) various sensors and actuators are integrated with the passive vibration absorber to suppress the vibration of the primary system. The vibration absorbers are generally designed based on the assumption that the absorber structure possesses linear characteristics. However, an effective vibration absorber vibrates with a large amplitude leading to the dominance of structural nonlinearity. Thus, the linearity assumption regarding the elastic properties of the absorber substructure does not hold good in reality. Also, nonlinearity is inherently present in the main vibrating primary systems due to prolonged use, certain applications, or being subjected to various forms of external excitations. The available passive and active vibration absorbers also come up short to suppress the vibration in nonlinear primary systems under varied resonance conditions. Since the existence of nonlinearity in the primary systems and the absorber system are inevitable. It is observed from the literature that with the available passive or active vibration absorbers the vibration reduction of the primary system under varied resonance conditions is still high or limited to a very narrow range of operating frequencies. Also, complete vibration suppression of the nonlinear vibrating primary system is not achieved under various resonance conditions. The investigations of AVA with acceleration or combination feedbacks (displacement, velocity and acceleration) and its effectiveness in vibration suppression are also not explored. So, in the present thesis, a modified designed piezoelectric stack actuator based active nonlinear vibration absorber (ANVA) with various feedbacks and time delay is considered, where the frequency of the absorber can be actively changed to suppress the vibration of the primary system. In the modified designed ANVA the PZT (lead zirconium titanate) stack actuator is connected in series connection with a spring in the absorber configuration for a fail-safe design.

In this work, five problems related to the ANVA have been investigated considering the linear and nonlinear primary systems under various resonance conditions. In the first

problem, the linear single degree of freedom (SDOF) spring-mass-damper primary system is subjected to an external harmonic excitation and its vibration suppression is carried out considering linear active vibration absorber (AVA) with acceleration feedback. Optimized parameters such as tuning ratio and damping ratio for the proposed AVA are obtained from the fixed-point theory (FPT) and genetic algorithm (GA). The optimal feedback gain is also evaluated maintaining Den Hartog's equal peaks in the frequency response and it is shown that with the active force the response amplitude of the primary system reduces close to its static deflection.

In the second problem, a similar system to the first problem is analyzed but here cubic nonlinear springs are considered in both the primary system and the absorber. Also in this problem, the primary system is subjected to both external harmonic and base excitations. The active force by the ANVA is produced considering the time delay in the acceleration feedback. The modified harmonic balancing method (HBM), and Newton's method are used to study the system responses and stability. Fundamentally based on nonlinearity, presence and absence of PZT stack actuator (active force), delay and base excitation twenty different cases have been investigated. The system steady-state responses are investigated through the frequency responses, time responses, phase portraits, Poincare' sections, basin of attractions, force responses and different time delay effects in the feedback. The bifurcation and stability of the system are studied from various system parameters. It is shown that with the proposed ANVA significant vibration suppression for the nonlinear system is achieved and Den Hartog's equal peaks in the frequency response of the system are obtained when the primary system is subjected to both harmonic and base excitations.

In the third problem, a few modifications to the second problem are carried out by considering the undamped primary system subjected to external harmonic and parametric excitation. Here two types of ANVA are studied viz., traditional and non-traditional ANVA. In the traditional ANVA the damper is attached to the primary system whereas in the non-traditional ANVA damper is connected to the base of the primary system. The active feedback force by the ANVA is modified considering time delay in displacement, velocity and acceleration feedback. The method of multiple scales (MMS) is used to obtain the solutions which are found to be in good agreement with the linear, HBM and numerical methods. It is shown that with the proposed active forces by the ANVA, 100% vibration reduction in the primary system is achieved for the primary, principal parametric and simultaneous primary and principal parametric resonance conditions. Also, for a wide range

of operating frequencies, the vibration of the primary system is reduced by changing the frequency of the absorber actively.

In the fourth problem, the previous problem (third problem) is modified considering the nonlinear primary system subjected to external hard harmonic excitation, parametric excitation and base excitations. The ANVA is also modified considering time delay in the damper along with combinations of time-delayed displacement, velocity and acceleration feedback. The MMS is used to obtain the solution of the system for primary, principal parametric, sub and superharmonic resonance conditions considering 1:1, 1:2, 1:3 and 3:1 internal resonance conditions. From the investigations on various active forces by different feedbacks and time delays, 100% vibration suppression of the nonlinear system is analyzed.

In the fifth problem, three different systems are studied. In the first system vibration suppression of the forearm tremor of the human hand with ANVA is carried out. This problem is similar to the second problem but here the primary system parameters are considered from the human forearm. In the second problem, a continuous beam system with various boundary conditions is considered as the primary system. The ANVA is attached to the beam which is subjected to a point load. The analyses are carried out to suppress the vibration of the beam at the first three modal frequencies with the displacement and velocity feedbacks by the ANVA. In the third problem, the ANVA is modified by a nonlinear rotational inertial double-tuned mass damper (RIDTMD) and the vibration suppression of the nonlinear primary system is carried out through various nonlinear stiffness coefficients in the system. In all these systems HBM is used to obtain the solution of the governing equations of motion.

These studies will give a clear idea to the designer to know the safe operating zone and the system responses for various system parameters. From this research work, it is shown that one can completely suppress the vibration of the linear and nonlinear primary system for various resonance conditions. Also, the proposed ANVA is a fail-safe design i.e., when the active part fails due to electronic malfunction, then the passive optimal absorber can still protect the primary system from severe vibration.



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Abbreviations

ANVA	Active nonlinear Vibration absorber
AVA	Active vibration absorber
DC	Direct current
DOF	Degree of freedom
DVA	Dynamic vibration absorber
EOM	Equations of motion
FPT	Fixed point theory
GA	Genetic algorithm
HBM	Harmonic balance method
LTVA	Linear tuned vibration absorber
MMS	Method of multiple scales
MR	Magneto rheological
MTMD	Multiple tuned mass damper
NVA	Non-traditional vibration absorber
PVA	Passive vibration absorber
PZT	Lead zirconate titanate
RIDTMD	Rotational inertial double tuned mass damper
SADVA	Semi-active dynamic vibration absorbers
SDOF	Single degree of freedom
SMA	Shape memory alloys
TMD	Tuned mass damper
TVA	Traditional vibration absorber
VA	Vibration absorber



Notations

A_T	Response amplitude of the primary system using traditional vibration absorber
A_N	Response amplitude of the secondary system using non-traditional vibration absorber
B_T	Response amplitude of the primary system using traditional vibration absorber
B_N	Response amplitude of the secondary system using non-traditional vibration absorber
c_1	Damping coefficient of the primary system
c_2	Damping coefficient of the secondary system
d_{33}	Dielectric charge constant
E	Electric field
F_1	Amplitude of external harmonic force of excitation
F_2	Amplitude of parametric excitation
f_a	Acceleration feedback control active force
f_d	Displacement feedback control active force
f_c	Active control force by PZT stack actuator and spring in series
f_v	Velocity feedback control active force
J	Jacobian matrix
k_c	Control gain
k_1	Spring stiffness of the primary system
k_2	Spring stiffness of the secondary system
k_3	Stiffness of the spring attached to the PZT stack actuator
k_r	Equivalent stiffness of the PZT stack actuator and spring in series
k_{13}	Cubic nonlinear stiffness of the primary system
k_{23}	Cubic nonlinear stiffness of the secondary system
k_p^E	Stiffness of the PZT stack actuator

m_1	Mass of the primary system
m_2	Mass of the secondary system
n	Number of wafers used in the PZT stack actuator
$[\tilde{P}]$	Weighted modal matrix
s^E	Compliance of the piezoelectric material under constant electric field
t	Time
Y	Amplitude of base excitation
V	Voltage
x_1	Displacements of the primary system
x_2	Displacements of the secondary system

Greek Symbols

ε	Bookkeeping parameter
δ_0	Nominal displacement of the PZT stack actuator
Ω_i	Frequency of external harmonic excitation
μ	Mass ratio between the mass of the absorber and the primary system
ζ_1	Damping ratio of the primary system
ζ_2	Damping ratio of the secondary system

CHAPTER 1

Introduction and Literature Review

1.1 Introduction

An engineering structure or a machine undergoes multiple sources of vibrations throughout its life. These vibrations often lead to material fatigue, structural damage and failure. A machine part on which a steady alternating force of constant frequency is acting may take up unpleasant vibration when it is close to resonance. So, the control of vibration in mechanical systems is still a flourishing research field [1-3]. There are fundamentally two different primary methods of vibration reduction namely, vibration isolation and vibration absorption [3]. The isolators are positioned between the source of disturbance and the object to be protected using resilient elements i.e., springs, dampers, or some other vibration-isolating materials. Isolators reduce the magnitude of force transmitted from the vibrating object to its foundation or the transmitted motion from the base to the object. In the vibration absorber system, a secondary spring-mass system is added to the primary system. The added mass and spring are designed to produce a two-degree-of-freedom system whose only purpose is to absorb the vibration level of the primary system. The vibration absorber reduces vibrations of the primary system by channelling energy to the absorber itself [1-6]. In linear vibration absorbers, this channelled energy is manifested as a deflection in the absorber's mass [1-2].

The undamped vibration absorber attached to the main vibrating primary system attenuates the vibration of the primary system only at the resonating frequency i.e., when the external excitation frequency matches with the natural frequency of the primary system and that of the absorber. This is also known as a tuned vibration absorber, but its suppression is only at the resonant frequency and outside this frequency high response amplitude of the primary system takes place [1-4]. To suppress this high response amplitude a damper is added to the mass and spring configuration of the vibration absorber which increases the band of operating frequency for which the vibration of the primary system is minimum [1-4]. These mass, spring and damper type of vibration absorber which is attached to the primary system

is often known as tuned mass damper (TMD) or tuned dynamic vibration absorber (DVA). However, it is observed that TMD cannot suppress completely the steady-state response of the primary system under external excitation. The efficacy of a TMD largely depends upon the optimised value of its spring stiffness, damping ratio and mass [1-5]. Due to this, many optimization techniques such as fixed point, H_∞ , H_2 optimization etc. are developed to reduce the amplitude of the primary system for a wide band of operating frequencies [3-9]. There are basically two types of DVAs in terms of the attached damper. One is the traditional DVA where the damper of the absorber is attached to the primary system and the other one is a variant/non-traditional DVA where the damper of the absorber is connected to the base of the primary system [6]. DVAs are commonly designed and tuned to suppress vibrations at one vibration mode or at the resonant frequency of the primary system. But if the primary system is subjected to multiple excitations with various resonance conditions, then the resonance may still occur at other vibration modes of the structure [5-9]. Thus using smart materials such as PZT (lead zirconate titanate) actuator, shape memory alloy, magnetostrictive materials etc. one can tune the frequency of the absorber to various resonance conditions actively in real-time through various feedbacks and control laws. The use of smart materials in the DVA is often known as adaptive dynamic vibration absorber (ADVA) or active vibration absorber (AVA) [5, 9-10].

The nonlinear analysis of the passive and AVA considering nonlinearity in the main system is equally important [10-21]. The nonlinearity in the system arises as the vibration absorber undergoes a high amplitude of excitations and prolonged use. It may occur in certain applications, or it may be inherently present in the system [10-15, 21]. The nonlinear analysis of DVA is more practical and can be used in most applications [10-20]. The classical theory of spring and viscous damper-based vibration absorbers can be classified into three distinct categories in terms of smart materials [1-9, 20] used in the absorber/secondary system. They are (i) passive vibration absorber (PVA)/tuned mass damper, (ii) semi-active or adaptive vibration absorber and (iii) active/hybrid vibration.

1.1.1 Passive vibration absorber

In Fig. 1.1, a linear undamped vibration absorber or neutralizer [1, 16] is shown which consists of a primary and a secondary system. The primary system has a mass m_1 and

stiffness k_1 which is subjected to an external excitation force, $F_0 \cos \omega t$ having a displacement, x_1 . The secondary system which is attached to the primary system has a mass m_2 , spring stiffness k_2 and displacement x_2 . The addition of the secondary system adds one degree of freedom to the primary system and shifts the natural frequencies away from the excitation frequency. The lower frequency is less than the natural frequency of the primary system while the higher frequency is greater than the natural frequency of the primary system. The steady-state amplitude of the primary and the secondary system can be given by the following expressions.

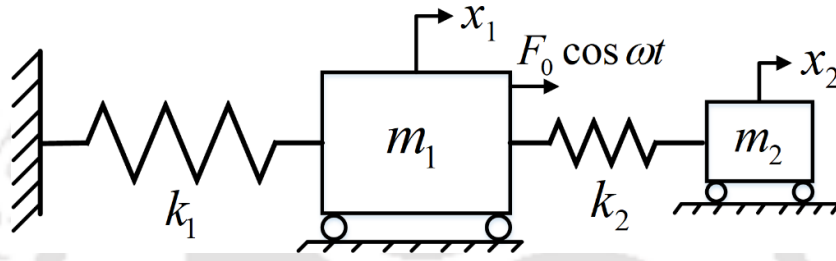


Fig. 1.1: Linear undamped vibration absorber (Neutraliser).

$$x_1 = \frac{\left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] x_{st}}{\left[1 + \mu \left(\frac{\omega}{\omega_n}\right)^2 - \left(\frac{\omega}{\omega_n}\right)^2\right] \left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] - \mu \left(\frac{\omega}{\omega_n}\right)^2} \quad (1.1)$$

$$x_2 = \frac{x_{st}}{\left[1 + \mu \left(\frac{\omega}{\omega_n}\right)^2 - \left(\frac{\omega}{\omega_n}\right)^2\right] \left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] - \mu \left(\frac{\omega}{\omega_n}\right)^2} \quad (1.2)$$

where $\omega_n = \sqrt{k_1/m_1}$, $\omega_a = \sqrt{k_2/m_2}$, $x_{st} = F_0/k_1$, $\mu = m_2/m_1$

From Eq. (1.1), one can conclude that for $\omega = \omega_a$, the response amplitude x_1 of the primary system is zero. Hence, the absorber can eliminate the vibration of the main mass, provided the natural frequency of the absorber is the same as the operating frequency of the primary system i.e., $\omega = \omega_a = \sqrt{k_2/m_2}$. When a vibration absorber is added to a system with a harmonic excitation and if the absorber is tuned to the excitation frequency, then the system

has zero steady-state amplitude. This concept of tuned vibration absorber was first patented by Frahm in the year 1911 [22]. So, the values of the absorber parameters are chosen by considering the constant exciting frequency and the maximum permissible amplitude of the auxiliary mass. Hence Eqs. (1.1) and (1.2) can be used to design a linear undamped vibration absorber. Fig. 1.2(a), shows the response of the primary system from Eq. (1.1) and Eq. (1.2). From this figure, one can study the effect of a tuned vibration absorber on the response in a non-synchronous operation (when the excitation frequency varies over a wide range). Although the vibration absorber is designed for a given operating frequency ω , the absorber can perform satisfactorily for operating frequencies that vary slightly away from ω . In this case, the motion of m_1 is not zero, but its amplitude is very small. However, the drawback of using an undamped linear tuned vibration absorber (LTVA) is that the system, now being a 2-DOF system, has two undamped Eigen frequencies, with infinitely high resonance peaks. This is problematic as a small variation

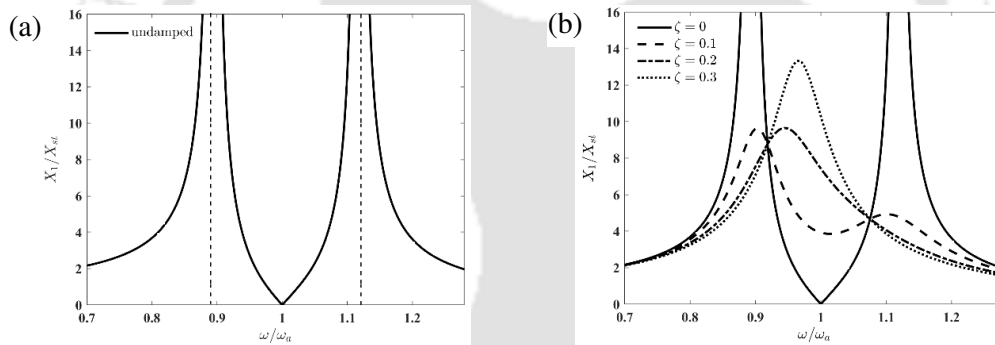


Fig. 1.2: Response amplitude of the primary system for $\mu = 0.05$ at various frequencies ω . (a) with undamped LTVA (b) with various damping ratios in the damped LTVA.

can move the excitation frequency close to one of the two new resonance peaks and cause vibration in the main system to be amplified. Therefore, the use of undamped LTVAs is limited to very well-controlled systems [1-6]. When damping is introduced in the absorber considering the damping ratio ζ the system response amplitude becomes lower and the two resonance peaks show finite values, as shown in Fig. 1.2(b). The response amplitude at the peaks can be further reduced by incorporating optimized damping and tuning ratio (ω_a/ω). The theory of the LTVA or tuned mass damper was first published in 1928 by Ormondroyd, and Den Hartog [23], where the fixed-point theory of optimisation is used to

obtain the optimal tuning ratio $(\omega/\omega_a)_{opt}$ and damping ratio (ζ_{opt}) to obtain the Den Hartog's equal peaks in the response (Fig. 1.3). These concepts were used as passive vibration control since the 1970s and are found in many structural systems to suppress

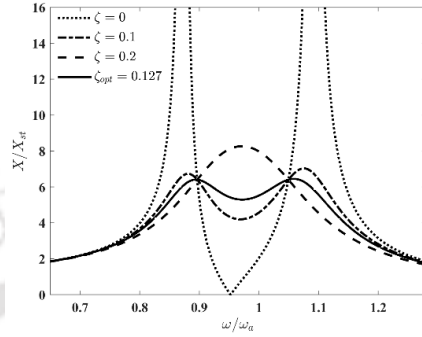


Fig. 1.3: Comparison of the response of the primary system with the optimal damped LTVA for $\mu=0.05$ at various distributing frequencies ω .

vibration. However, with the optimal tuning and damping ratio also, the vibration of the primary system is still high at the peaks and no longer completely suppressed at the resonant frequency [1-2]. Because of this semi-active and active/hybrid vibration absorber model are developed where new modification in the absorber design and the use of smart materials further minimizes the response amplitudes of the primary system for a broader range of operating frequencies with lesser absorber mass than the PVA, so, that the overall structural weight of the system is minimum [4-9]. In the following section, the semi-active vibration absorber is discussed.

1.1.2 Semi-active or adaptive vibration absorber

Semi-active dynamic vibration absorbers (SADVA) are often used as intelligent structures that can change or adapt to the environment dynamically. MR (Magneto rheological) fluid damper is an example of these types of intelligent structures. SADVA also can provide actuation by the absorber with the modified design in the absorber configuration without any smart or intelligent structures [16, 20, 24]. In MR damper based SADVA, the magnetic particles embedded in the fluid produce a large damping force by varying the magnetic field in real-time to actively control the vibration which is shown in Fig. 1.4. Also, one may use electro-rheological (ER) fluids where the damping properties can be changed actively by applying an electric field. Another type of intelligent structure used in SADVA



Fig. 1.4: Magnetorheological fluid dampers working principle.

is the shape memory alloys (SMA's) where the variation of temperature provides actuation. When an SMA is cold, or below its transformation temperature has a very low yield strength and can be deformed quite easily into any new shape, which it will retain. However, when the material is heated above its transformation temperature its crystal structure changes which cause it to return to its original shape. If the SMA encounters any resistance during this transformation, it can generate forces, which provides a unique mechanism for remote actuation. But these type of materials produces very small actuation and are slow to react. The most common shape memory material is an alloy of nickel and titanium called Nitinol. Shape memory alloys, however, are not for all applications. One must take into account the forces, displacements, temperature conditions, time for actuation and cycle rates required of a particular actuator. The SADVA is limited primarily by the maximum displacement of the absorber and the accuracy of the modal state measurements of the structure. Due to these limitations further, smart materials are developed which can produce large actuating forces such as piezoelectric based smart materials. In the next subsection, piezoelectric based smart materials used in AVA are discussed to actively control the vibration of the structures.

1.1.3 Active vibration absorber

In an active vibration absorber (AVA), an actuator and sensor are used in the secondary system by various control feedbacks to minimize the steady-state response of the primary system for a wider band of operating frequencies [8, 9, 16, 19, 20, 24, 25]. The AVA uses various types of actuators such as electromagnetic actuators and voice coil motors. However, its successful attainment is sometimes found to be difficult due to hardware limitations such as saturation and response speed under certain conditions. This difficulty can be resolved by using smart material actuators. Among many smart materials, piezoelectric materials provide a high actuation force compared to others and can be used simultaneously as a sensor and actuator. The piezoelectric material can also be useful to tune the frequency of the absorber for various modes/resonance conditions. Initially, the

AVA is used with only an actuator or actuator and mass in the secondary system [20] without considering spring or damper, this arrangement can be called pure AVA which is shown in Fig. 1.5(a). However, this arrangement requires more voltage supply to the

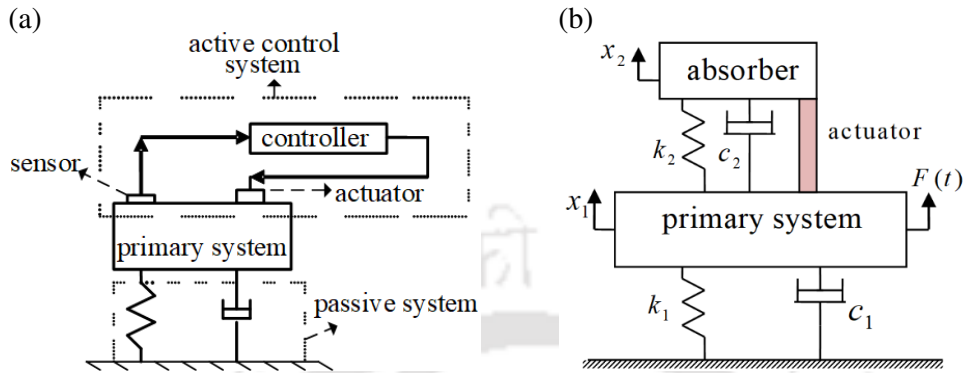


Fig 1.5: Schematic diagram of (a) pure AVA (b) active/hybrid vibration absorber.

actuator which is not economical. Also, if the primary system is large then more actuators may be needed. To overcome these difficulties AVA are often integrated with PVA, this combination is also known as a hybrid vibration absorber or sometimes simply AVA [20] to suppress vibration of the primary system effectively with less absorber mass and control force, which is shown in Fig. 1.6(b). These types of AVA are economical and also a failsafe design as when the control is on it performs better than a PVA and in the event of failure of the active part, the PVA can still protect the system from being damaged due to severe vibration. Also, since energy is almost always being dissipated/ redistributed by the passive components, they are much more robust and stable than the purely active approach [3-7]. In the next section, a brief description of how the actuation developed in the AVA by the piezoelectric materials is given.

1.1.3.1 Piezoelectric materials

Piezoelectric ceramics are prepared of metal oxides and an organic binder [20, 24, 25] to form a dense crystalline structure. Actuators are configured by giving the sintered ceramic desired shapes and bonding electrodes at appropriate locations. The crystal exhibits dipole moment but due to the random distribution of these dipoles has zero overall polarization. However, a permanent polarization is achieved by heating the material above its Curie temperature under a strong direct current (DC) electric field as shown in Fig. 1.6. As a result, the dipoles align in the direction of the applied electric field called the polarization

axis. At this stage, the material is said to be poled and even after the removal of the poling electric field, the polarization is still retained to a large extent. In the present work PZT based stack actuator is used due to its high actuation capability and its constitutive relations are detailed below.

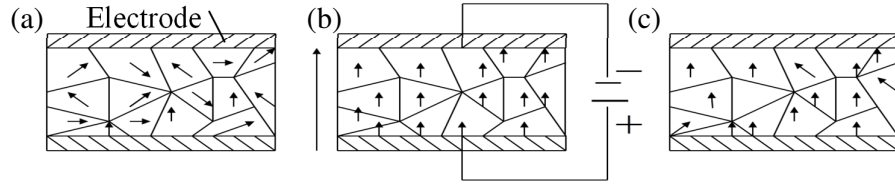


Fig 1.6: Steps of poling process of piezoceramic materials. (a) unpoled material with randomly distributed dipoles, (b) dipoles are aligned due to external DC electric field, (c) permanently polarised material after withdrawal of the DC field [20].

1.1.3.2 Displacement modes and force of the piezoelectric actuator

The piezoelectric materials exhibit direct or converse effects. In converse effect, the piezoelectric rod elongates when subjected to an electric field E in the poling direction. In this case, considering linear behaviour, the strain is calculated as dE , where d is the piezoelectric charge constant. Now by keeping E fixed if the rod is stressed by an external mechanical load in the same direction as the electric field, the additional strain-induced is $s^E \sigma$, where s^E is the compliance of the material under a constant electric field. Thus, the total strain induced in the material due to the combined effects of electrical and mechanical action is given below.

$$\epsilon_T = s^E \sigma + dE \quad (1.3)$$

Similarly when the rod is first mechanically stressed (σ) to induce a charge per unit area of the amount $d\sigma$ then keeping the mechanical load fixed, E is applied in the poling direction to produce an additional charge of the amount $e^\sigma E$, where e^σ denotes the dielectric constant under constant stress. Thus, the total electric displacement D is given below.

$$D = d\sigma + e^\sigma E, \quad (1.4)$$

Considering one-dimensional piezoelectric material of constitutive equations [20] (using the notations of the IEEE Standard on Piezoelectricity)

$$D_3 = d_{33}\sigma_3 + e_{33}^\sigma E_3, \quad (1.5)$$

$$\epsilon_3 = s_{33}^E \sigma_3 + d_{33} E_3 \quad (1.6)$$

The subscript ‘33’ is used in the piezoelectric charge constant since, by convention, index 3 is always aligned to the poling direction of the material, and it is assumed that the electric field is parallel to the poling direction. Piezoelectric actuators are generally classified as linear actuators (stack actuators), laminar actuators (patch actuators) and shear actuators. In the next section, only the linear stack actuator is discussed, which is used in the present work.

1.1.3.3 Stack actuators

Stack actuators are made of stack thin wafers of piezoelectric materials sandwiched between electrodes. The piezoceramic wafers are arranged in such a way that any two consecutive wafers have opposite polarization directions as shown in Fig.1.8. For applying an electric field along the direction of polarization of each wafer, the electrodes are alternately connected [20] to the positive and negative terminals of a voltage source. Consequently, each piezoceramic layer expands along the direction normal to the wafer surface and the total expansion of the stack is the cumulative effect of the extensions of all individual wafers. Though stack actuators can generate a very large actuation force, the stroke length is limited to only a few micrometres. As the piezoelectric coefficient d_{33} determines the extension of each wafer of a stack. So, using Eq. (1.6) the constitutive equations, for the maximum displacement of the stack actuator in the absence of mechanical load or stress ($\sigma_3 = 0$) can be written as follows.

$$\Delta L = \delta_0 = d_{33} V L / t = n d_{33} V \quad (1.7)$$

where $n = L/t$, is the number of wafers used in the stack. If the displacement of the stack is completely blocked, ($\varepsilon_3 = 0$), then the compressive force generated by the actuator, called blocking force, can be written as follows.

$$F_b = \sigma_3 A = k_p^E \delta_0 = k_p^E n d_{33} V, \quad (1.8)$$

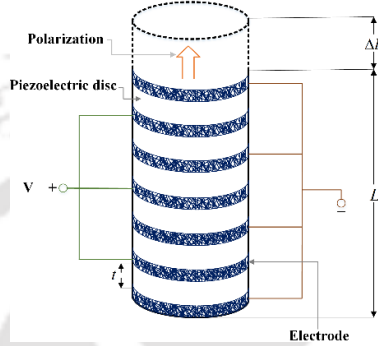


Fig 1.7: Schematic diagram of a stacked piezoelectric actuator.

where $k_p^E \approx A / (s_{33}^E L)$ and “A” is the cross-sectional area of the piezo-stack actuator. PZT stack actuator is one of the world’s most widely used smart materials which has been widely used in many fields, such as active vibration control, precision positioning system and MEMS. The PZT stack actuator has a very small induced displacement, of $1000 \mu\text{m/m}$ but can provide very a high blocking force. The blocking forces of the PZT stack actuator are measured to be 230 N, 369 N, and 478 N for 100 V, 200 V, and 300 V, respectively [26]. The PZT materials are first developed in the year 1952 by scientists at the Tokyo Institute of Technology. In the next section, the nonlinear vibration absorber is briefly discussed.

1.1.4 Nonlinear vibration absorber

From the previous discussion it can be observed that in LTVA, the vibration suppression of the primary system is limited to a very narrow frequency range around their eigenfrequency. Also, from Eq. (1.2), it is evident that the deflection of the secondary system (x_2) is inversely proportional to the mass ratio (μ). So, if m_2 is small then it produces excessive vibration in the secondary system which leads to nonlinear analysis. Due to this, apart from using smart materials in the LTVA researchers also investigated the use of non-linear tuned vibration absorbers (VA) for over 65 years to broaden the suppression frequency band for

non-linear structures or mechanical systems with varying excitation frequencies [5, 6, 11, 21]. Roberson [27] in the year 1952 first showed theoretically that a tuned VA with a cubic softening spring has a much broader suppressing frequency band than an LTVA. Pipes [28] showed similar results in 1953 and Arnold [29] in 1955. Arnold [29] concluded from their research finding that a softening spring in the absorber would increase the vibration suppression bandwidth, while the hardening spring would do contrarily. The effectiveness of the nonlinear absorber to increase absorber performance can be observed in many research works [11-21, 27-29]. It is therefore unsuitable for analysing the linear PVA or AVA only, so investigations of active nonlinear vibration absorber (ANVA) for vibration suppression is a crucial area to study [18-19]. The practical applications of vibration absorbers are briefly discussed in the next section.

1.1.5 Applications of passive and active vibration absorbers

PVAs and AVAs are widely used in many engineering systems [5-7, 30, 31], structures, bridges, automobiles [30-35], railways [35], ships [36], space crafts [30, 31, 36-39], machining systems like turning, milling, boring or drilling system [39-41], noise reductions, tremor suppression due to Parkinson disease [42-46] and in consumer's goods which undergo severe vibration. The initial application of PVA is in civil structure and thereafter its application is extended to suppress vibration of many kinds of vibrating systems starting from small to high procession machinery to big industrial structures [7, 47-49]. Mostly PVAs and AVAs are used in civil engineering, mechanical engineering, aerospace engineering and currently bio-medical engineering [5-7, 42]. Some of these applications are discussed in the next section.

1.1.5.1 Civil Engineering

Passive vibration absorbers or tuned mass dampers (TMDs), active TMDs, and pendulum TMDs are widely used in many buildings, towers and bridges to attenuate vibration [7]. Some of these structures are shown in Fig. 1.8. One of the first applications of TMD was used in the John Hancock building (241 m) in Boston, USA in 1976 to reduce response to wind vibrations. TMD has deployed in over 60 structures in several countries [7] including the U.S.A, Japan, the U.K, China, Korea and recently in India. There are several practical examples where DVAs/AVAs are used in many high-rise structures, such as Shinjuku Park

Tower (227 m) in Tokyo, Japan, Incheon International Airport Control Tower (100.4 m) in Incheon, Korea, Air Traffic Control Tower (57 m) in Edinburgh, UK, Shanghai World Financial Center (492 m) in Shanghai, China [7, 21], and Taipei 101 with a sphere-shaped



Fig. 1.8: (a) 634 m tall Tokyo sky tree, Japan (b) closed view (two TMDs) [7]. (c) Taipei 101 with a sphere-shaped TMD [48]. (d) 182 m Statue of unity in India, where two 250 tonne TMDs are used for stability against seismic and wind excitations [49].

TMD. Passive and active TMDs have been also employed in many bridges to suppress vibration of the primary system since 1994. Some recent applications of active and passive TMDs [6, 7, 21, 30, 31] are in 600 m canton tower in Guangzhou, China in 2010, 104m Estela de la Luz tower in Mexico in the year 2010, 634 m tall Tokyo Skytree tower in Tokyo, Japan in the year 2012 and statue of unity (182 m) in India, in the year 2013.

1.1.5.2 Mechanical Engineering

In mechanical engineering fields, PVA and AVA are mainly used in automobile sectors to improve stability and reduce vibration in suspension systems. Vibration absorbers (VAs) are used to control rotary vibrations in power trains and rotor hubs, noise radiation, wind-induced oscillations of pipelines and vibrations of tension stabilized structures using smart

AVA [6, 7, 21, 30, 31, 36-38, 50-52]. The VAs are also used to reduce vibration in chattering and improve the stability of the system in machining processes such as turning [53], milling and boring operations [30-40]. These applications with various models of passive and active TMD in a quarter car model are shown in Fig. 1.9. TMD is first used by

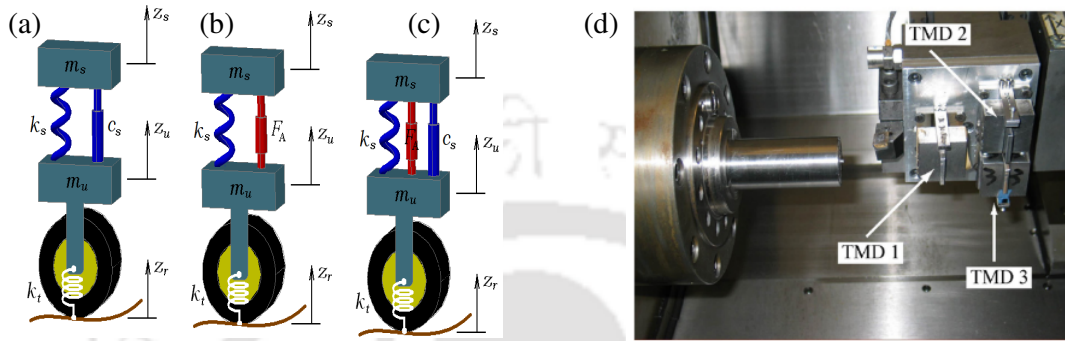


Fig. 1.1.19: Quarter-car suspension systems: (a) passive suspension, (b) active electromagnetic suspension, (c) active hydraulic suspension, (d) TMD used on a lathe turret [53].

Renault in F1 car racing to improve the speed of the vehicle, especially in turning and bumps, which won them the world championship race in the year 2006. Later many automobile companies like Ferrari and McLaren also tried the concept of TMD to improve vehicle performance but were banned due to safety reasons. The TMDs are also used in suspension systems of railways, motor vehicles, robots, boring bars, gun barrels, transmission lines and energy harvesting to reduce both vibrations, and noise and improve the stability of the system. AVAs are more suited to satisfy serviceability criteria in changing environments supporting structures with active struts under vibratory load and also have the potential to reduce environmental vibration originating from machinery and vehicles in space where human sensitivity is a concern.

1.1.5.3 Aerospace Engineering

Active vibration absorbers find applications in vibration suppression of individual blades in the helicopter on several turboprop aircraft [50, 51]. Some of the applications of VAs used in the helicopter blade to reduce noise and vibration are shown in Fig. 1.10(a-c). In the aeroplane wings also AVAs are used, to reduce the flutter vibrations [52] by using piezoelectric patch actuators as shown in Fig. 1.10(d).

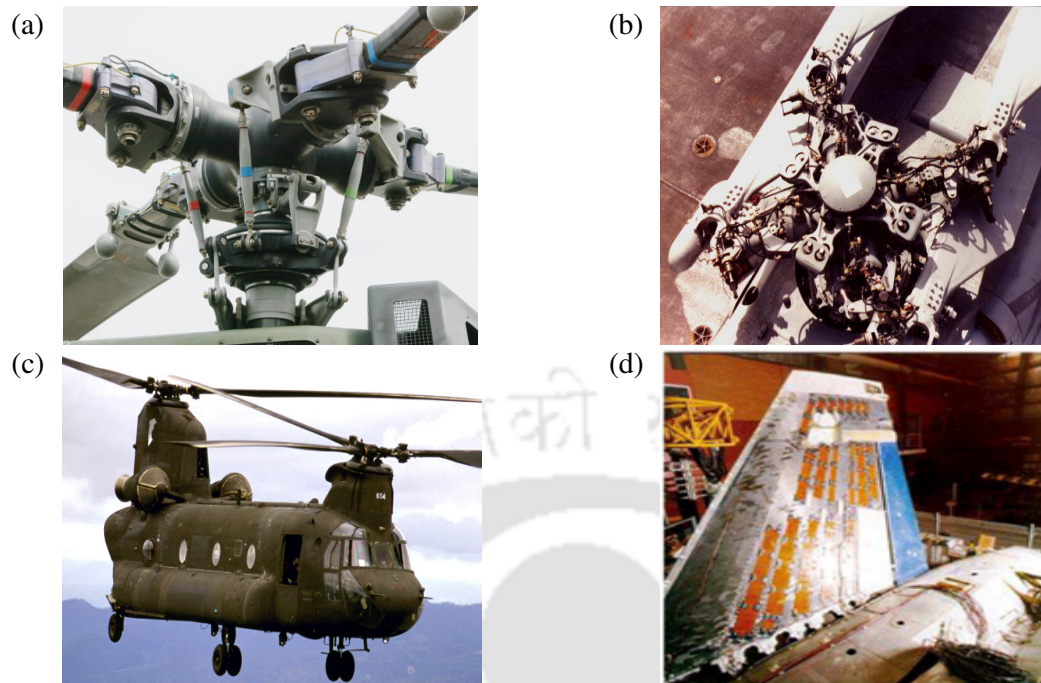


Fig. 1.10: (a) Helicopter blade of MBB Bo105 Helicopter (b) closed view showing active pendulum-type vibration absorber. (c) Boeing vertol ch-47 chinook where five AVAs are used one in the nose, two under the cockpit floor and two inside the aft pylon, (d) F-18 fighter plane wing with piezoelectric actuators for buffeting (shock wave oscillations) control [52].

1.1.5.4 Bio medical Engineering

The TMDs are used to suppress tremors of the patient suffering from Parkinson's disease by adding an auxiliary/absorber mass on the oscillating human hand or in any other body parts [42-46]. Two such devices are shown in Fig. 1.11. The PVA showed in Fig. 1.11(a) suppress around 80% of oscillations in the forearm but for a limited range of operating frequencies [42]. For complete suppression of oscillation of tremor in the forearm, one may use AVAs for a wide frequency range with less absorber mass. In the following section, an extensive literature on vibration absorbers is presented and the current work is also outlined.

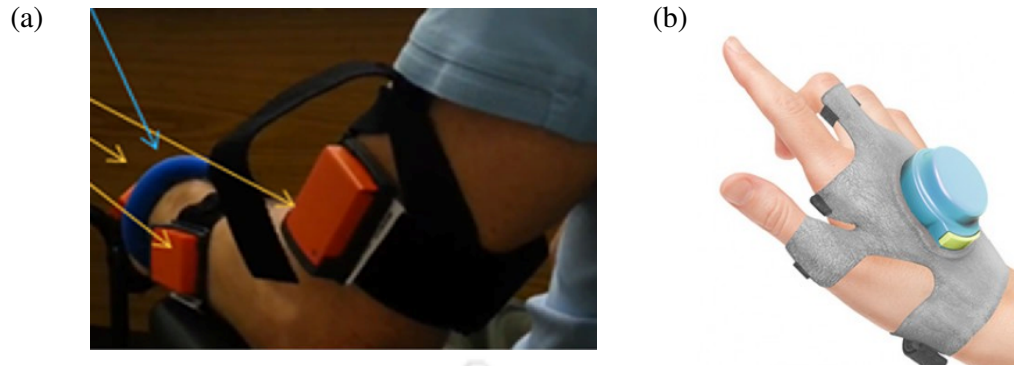


Fig. 1.11: (a) Vib-bracelet based PVA for attenuating forearm tremor [42] (b) A wearable device promises to help steady hand tremors [43].

around 80% of oscillations in the forearm but for a limited range of operating frequencies [42]. For complete suppression of oscillation of tremor in the forearm, one may use AVAs for a wide frequency range with less absorber mass. In the following section, an extensive literature on vibration absorbers is presented and the current work is also outlined.

1.2 Literature Review

The objective of this literature survey is to review the available articles on vibration suppression of primary systems by the use of passive and active vibration absorbers considering both linear and nonlinear analysis. The main focus is on the passive and PZT stack actuator based active nonlinear vibration absorber for the nonlinear primary system. As observed from the introduction section vibration absorber using linear vibration theory is well established which includes small amplitude vibrations of long, slender objects like long bridges, aeroplane wings, and helicopter blades; small rocking motions of ships in calm waters; the simplest whirling motions of flexible shafts, and so on. However, interactions between bridges and foundations, between wings/blades and air, between ships and waves, between shafts and bearings, and so on, are all nonlinear. These nonlinear interactions make the system more practical in nature and its investigations using nonlinear absorbers from the previous literature are more research worthy [54]. It will also be of interest to know the solution approach for the nonlinear governing equation of motion (EOM) for various nonlinear mathematical models on passive/active vibration absorbers to analyse the dynamic behaviour of the system. The comparisons of the responses of the primary system by the use of active/passive linear and nonlinear vibration absorbers are

also an important area to learn from the previous articles. Finally, from these literature studies, potential gaps will be identified to set the objectives for the current research work. The stability analyses of the nonlinear system with various critical system parameters are also considered which contribute to the dynamic behaviour and its stability. Hence, research works related to vibration suppression of the primary nonlinear system under various external excitations, use of various combinations of active control forces using PZT stack actuators and stability prediction of the system should be investigated. In the following section, literature studies on linear passive vibration absorbers are discussed.

1.2.1 Linear passive vibration absorbers

In this section, literature on the vibration absorption of the linear primary system with linear absorbers has been studied. As discussed in the introduction sections, it can be observed that the undamped tuned vibration absorber completely suppresses the vibration of the primary system only at the resonating frequency and produces two resonating peaks with high amplitudes near the original resonant frequency [1-6]. To reduce these unwanted two high amplitude peaks, initially, dampers are used in the absorber configuration. Ormondroyd and Den Hartog used an additional damper in the absorber configuration [23] connected to the primary system. Though with this configuration, the amplitude at the peaks reduces to a great extent at resonance, however, the vibration of the structure is no longer zero as with the undamped vibration absorber [1, 23]. They obtained the optimal parameters (tuning ratio and damping ratio) using fixed point theory for the absorber which shows minimum response amplitude with two equal peaks in the frequency response of the primary system. Later, many optimization techniques such as H_2 , H_∞ [55, 56] linear quadratic regulator (LQR), genetic algorithm etc. are developed to obtain the optimized values for the stiffness, damping ratio for the absorber to minimize the vibration of the structure at both peaks and also at the resonating frequency of operation [5, 6, 16, 30, 31, 55]. These optimization procedures are mainly focused to obtain the two equal peaks in the frequency response amplitudes often known as Den Hartog's equal peaks [1, 16, 55] and minimizing the total vibrational energy of the primary system, so, that the maximum vibration of the primary system reduces. These types of vibration absorbers, often known as passive traditional dynamic vibration absorbers (DVA) and are ideal for large absorber masses, which are typically 10 -15% of the primary system's mass, as many experimental

studies have shown [5-7, 42]. Though the absorber mass is assumed to be 5% of the primary system mass in theoretical research [5, 6, 11, 16, 19, 30, 31, 55]. From the optimal studies, it is observed that there is a limitation in the vibration suppression of the primary system with these DVAs. Due to this, various modifications in the design of absorber models are developed and one of them is a non-traditional vibration absorber (NVA) [6, 16]. In these types of vibration models, the damper of the absorber is connected to the base of the primary structure. There are many studies carried out with new optimization procedures, and various designs in the absorber model to reduce the vibration of the primary system with a passive approach. Few of the important initial review articles where passive linear vibration absorbers are extensively studied are from the work of Sun et al. [5] and Housner et al. [55]. Sun et al. [5] in the year 1995 first reviewed the use of PVA in reducing vibration and noise of the primary system through various design models in the absorber configuration and its applications in various fields. They also extended a brief discussion on the recent progress of vibration suppression capabilities of semi-active and active vibration absorbers. Housner et al. [55] in the year 1997 also, reviewed the literature on structural vibration control by the use of passive TMDs, AVA and many control methods to suppress vibration of the primary system. In the next section, brief discussions on the various optimization techniques used by the researchers to obtain the optimal tuning ratio and damping ratio for the linear vibration absorbers are given.

1.2.1.1 Optimization techniques used in the linear vibration absorber

Cheung and Wong [56-58] contributed a significant amount of work on vibration absorbers both by passive and active methods to suppress vibration of the spring-mass-damper SDOF primary system and continuous systems such as beams and plates effectively for a wider range of operating frequencies. They studied various types of passive absorber models and obtained respective optimal parameters for the tuning ratios and damping ratios using optimization techniques such as H_∞ and H_2 optimal theories to obtain Den Hartog's equal peaks in the frequency response of the primary system. Zilletti et al. [59] demonstrated through the optimal parameters that the maximum power dissipation by the absorber is the same as those that minimize the kinetic energy of the damped SDOF primary system under random excitation, which provides a method for self-tuning DVA. They carried out extensive work both theoretically and experimentally on the self-tuning of the absorber [60]. Brennan and Dayou [61] studied the global vibration control of the primary system

using VAs. Brennan [62-64] also studied many aspects of the optimal parameters and design configuration of the absorber for widening the vibration suppression band of the primary system. Anh and Nguyen [65] studied vibration suppression of the SDOF damped primary system under external force excitation and ground motion by the use of TMD. They obtained optimum parameters for the absorber by replacing the original damped SDOF primary system with an undamped one by equivalent linearization and dual criterion method. The results are compared with previous literature showing its effectiveness in suppressing vibrations. Issa [66] studied to suppress the vibration of an undamped SDOF primary system subjected to harmonic excitation by the use of PVA. They obtained optimal parameters for the absorber and figured out the optimal mass ratio required to get the utmost performance in suppressing the vibration of the system. Brennan et al. [67] investigated the use of PVA simultaneously as an absorber and an energy harvester when the SDOF primary system is subjected to single harmonic and random excitations. They compared various optimization criteria with each other and found that the Den Hartog method of optimization procedure flattened the amplitude response curve of the primary system. Also, they concluded that with single harmonic excitation, the absorber cannot be optimally tuned and harvest maximum energy simultaneously. It is observed from the literature that the correct damping ratio and tuning ratio for the damped DVA play a significant role in vibration suppression of the primary system for a wider band of operating frequencies. The optimal parameters of the passive vibration absorbers (PVA) model are still an active area of research as observed from many of the recently published articles [6, 68-87] incorporating recent techniques such as genetic algorithms, artificial intelligence and machine learning. The optimum parameters of the PVA not only depend upon the tuning ratio or damping ratio but also its mass, location of placement with various excitations acting on the primary system and also DOF of the primary system. The vibration suppression of the primary system under various excitations can also be minimized by the use of various modifications in the traditional passive vibration model which are briefly discussed in the next section.

1.2.1.2 New modification in the design of the absorber model

In this section, a brief overview of the literature is presented where various modifications in the conventional PVA are considered to increase the efficacy of the vibration absorber. Krenk and Høgsberg [88] modified the conventional TMDs design by considering

additional flexibility between the TMD and the primary system. This flexibility is represented by connecting the support of the TMD with an additional spring stiffness. They obtained the optimum parameters for this newly designed absorber by fixed point theory and showed its effectiveness in vibration suppression of 10 storey buildings, taut cable and pedestrian bridges showing a fully balanced frequency response curve. Jang et al. [89] proposed a new design in the absorber configuration by considering two degrees of freedom (DOF) vibration absorber with translational and rotational DOF to suppress the vibration of the damped SDOF primary system subjected to base excitation. They obtained the optimal parameters for the proposed absorber by a similar approach as considered by Den Hartog and showed improved performance than conventional VAs. Asami [90] compared the effectiveness of vibration suppression of the undamped primary system by using three different types of absorber configuration i.e., conventional DVA, double mass DVA in series and double mass DVA in parallel. They obtained optimal parameters of the proposed absorbers by three different optimization criteria viz., the H_∞ optimization, H_2 optimization, and stability maximization criteria. From the analyses, they concluded that series-type double mass DVA is the best among the three DVAs considered and the conventional single-mass DVA gives the worst vibration suppression of the primary system. Viana et al. [91] designed and compared vibration suppression of the primary system by two special types of DVAs viz., the vibrating blade dynamic vibration absorber and the multi-mode dynamic vibration absorber. They obtained the optimal parameters of these DVAs using the ant colony theory of optimization, which reduces the vibration of the primary system more than without DVA. Modification in the TMD design by using multiple absorber mass where the primary system is subjected to various excitation under different applications such as in wind turbines and ship models etc are still a flourishing research area as studied from many current literature [92-98]. In the next section, one major type of modification in the traditional TMDs where the damper of the absorber is connected between the base of the primary system and the absorber mass, known as a non-traditional vibration absorber is studied considering previous literature.

1.2.1.3 Non-traditional vibration absorber

In this section, previous works on non-traditional/ variant vibration absorbers are discussed. Ren et al. [99] in the year 2001 first obtained the formulae for the optimized tuning ratio and damping ratio for the non-traditional vibration absorber by using the fixed-point theory

of optimization. They showed that the two-frequency response amplitude of the SDOF primary system under external harmonic excitation at the peaks reduces slightly than the traditional passive vibration absorber. Cheung et al. [100-102] also did a significant amount of work on the non-traditional vibration models to obtain the optimal parameters of the absorber using fixed point theory and H_2 optimization procedure, when the primary subjected to harmonic and random excitations. They also compared the responses of the primary system with various modifications in the VAs design models with the passive approach. Chun et.al. [103] designed a non-traditional PVA for suppressing vibration of a damped primary system subjected to high amplitude excitation. They obtained the optimal parameters for the absorber by proposing a diversified constraint particle swarm optimization method, which reduces the significant response amplitude of the primary system compared to the optimal parameters obtained by H_∞ the optimization technique. Xiang and Nishitani [104] studied vibration absorption of the damped primary system under seismic induced base excitation by non-traditional TMDs. They obtained optimum parameters for the absorber by quasi-fixed-point theory of optimization. The proposed absorber showed less space and stroke length than the traditional TMDs to suppress the vibration of the primary system for a wide range of operating frequencies. Anh et al. [105] designed a non-traditional vibration absorber (NVA), to suppress vibration of SDOF damped primary system under force excitation. They obtained optimal parameters for the NVA by H_∞ optimization procedure and use the dual criterion method to replace the original damped primary system with an equivalent undamped primary system. Noori and Farshidianfar [106] used H_∞ and H_2 techniques to obtain the optimal parameters of the non-traditional DVA to suppress the vibration of a beam under random force excitation. They showed that the non-traditional DVA outperforms the traditional DVA in suppressing the vibration of the primary system. Araz [107] in the year 2020 studied the effect of frequency detuning on the performance of the non-traditional TMD. They obtained the optimum parameters of the non-traditional TMD system and studied the corresponding effectiveness for different mass ratios. Through their numerical results, it is also indicated that the non-traditional TMD with a high mass ratio provides better robustness to the changes in the target frequency ratio than the traditional TMD. It is observed from the literature that very few works were carried out using non-traditional TMDs due to their damper configuration, whereas its vibration suppression capability is better than traditional TMDs at the same mass ratio. Now researchers [108] are applying the same methodology

of various optimization techniques and model design of traditional TMDs to the non-traditional TMDs which has a significant literature gap. But both in traditional and non-traditional VA models the frequency response amplitude at the peaks remains to a greater extent and there is a limit to suppressing the vibration of the main structure unless the absorber mass is increased. Also, non-traditional vibration absorber (NVA) models are not suitable in many applications due to their damper configuration so, further studies are carried out considering active force by the vibration absorber on the primary system. These types of vibration absorber are often known as active/ hybrid vibration absorber which utilizes various smart materials and control algorithm to suppress vibration of the primary system for a wider range of operating frequencies. In the next section, brief literature studies on linear active vibration absorbers are studied focusing mainly on PZT stack actuators, as it has high actuation capability and can be tuned more actively than other types of smart materials.

1.2.2 Linear active vibration absorber

It can be observed from the literature review section 1.2.1 that the mass ratio (μ) between the absorber and the primary system plays a significant role in the suppression of the vibration and most of the researchers have taken μ as 1:10 experimentally and 1:20 theoretically [5-7, 16, 30, 31, 34, 40, 42, 43, 46, 55, 56, 59, 62, 66, 67, 69, 72, 79, 82, 83, 89, 96, 99, 101-104, 106, 108]. This high mass ratio increases the overall structural weight of the system. But with the use of various optimization techniques [56-87], different model designs [88-108] along with the advanced materials [5, 6, 8, 9, 19, 20, 30, 31, 55] such as piezoelectric actuator and sensor, shape memory alloy [109-111], voice coil motor [112, 113], electrodynamic vibration absorber [114-116], magnetostrictive elastomers and dampers [117, 118], and other smart materials one may go for the lower mass ratio to absorb the vibration of the system for a larger band of operating frequencies [30, 31]. Therefore, further various modifications in the design of absorber models are extensively studied in the recent literature where the active elements are integrated with the PVA to enhance its performance. The use of smart materials in the traditional DVAs are commonly known as active or semi-active vibration absorbers or smart TMDs [119] by which one can tune the frequency of the absorber actively in real-time. The smart materials used in the active vibration absorber (AVA) use various feedbacks such as displacement, velocity and

acceleration feedback to provide the necessary actuation force to suppress the vibration of the main system [6-9, 120-134]. There are many review articles on active/hybrid vibration absorbers with various actuators and feedback which are given below.

Fisco and Adeli [30, 31] presented a state-of-the-art review of journal articles from 1997 to 2010 on different types of active and semi-active vibration absorbers that are used to control the structural vibration of the primary system and also reviewed many applications and control algorithms used in the active TMDs. El-Khoury and Adeli [120] reviewed recent journal articles on passive, active, semi-active, and hybrid vibration control of structures subjected to dynamic loading and suggested that the hybrid and semi-active control approaches suppress the vibration of the structure is an integral way to look forward in future research. Kela and Vähäoja [16] gathered recent articles on adaptively tuned vibration absorbers using various smart structures which are used to damp out vibrations of the primary system for a wide frequency band. There are many review articles [5, 6, 7, 11, 15, 16, 19, 30, 31, 36, 55, 62, 119-121] published where the importance of active TMDs is discussed to suppress the vibration of the host or the main structure effectively for a broad range of operating frequencies. The active TMDs utilize various types of feedback signals either from the primary system or the absorber to provide the necessary actuation to suppress the vibration of the primary system. In the next section, a brief overview of the various feedbacks used in the AVA is discussed.

1.2.2.1 Displacement, velocity and acceleration feedback

In the investigation of active linear vibration absorber, Cheung et al. [122-126] have done a considerable amount of work to analyze the vibration suppression of SDOF primary system and a simply supported beam by using AVA using various combinations of displacement and velocity feedbacks from the primary system and the absorber. They used H_{∞} optimization criteria to obtain the optimal parameters for the AVA to suppress vibration of the primary system with minimal absorber mass. Chatterjee [127] studied the efficacy of an AVA by considering the feedback from the absorber mass. The optimal parameters for the absorber are obtained by using H_{∞} optimization technique considering displacement and velocity feedback from the absorber. The effectiveness of the AVA with optimal parameters is studied for both SDOF primary system and a fixed-fixed Euler Bernoulli beam under external excitations. Nagashima [128] studied active tuned mass

damper (TMD) with optimal displacement feedback control law and LQR feedback gains to suppress the vibration of an undamped SDOF primary system. However, in the above literature, the types of actuators used for the theoretical analysis are not mentioned. Alujević et al. [129] considered velocity feedback for the inertial actuator to suppress the vibration of an undamped SDOF primary system subjected to random excitation. They obtained optimal parameters for the active absorber and showed that the AVA outperforms optimal passive TMD. Guo et.al. [130] presented an improved force feedback algorithm, with velocity feedback to enhance the damping ratio of the large space structure (free-free truss) using an active tendon/spring. Collette and Chesne [131] compared the efficacy of vibration suppression by pure AVA, a hybrid vibration absorber with velocity feedback to that of the PVA. They considered a three DOF model as the primary system and demonstrated that the vibration suppression is better with the proposed hybrid vibration absorber than that with the pure AVA which requires more active forces. The proposed absorber also outperforms the PVA. From the literature, it is observed that many researchers used displacement and velocity feedbacks in their analysis [5, 6, 7, 11, 15, 16, 19, 30, 31, 36, 55, 62, 119-121] while analysis on acceleration feedback and combination of these feedbacks are limited. The application of the AVA initially started in civil structures but its application with various control algorithms is now found in many fields [119, 120, 132-134]. The smart materials used in the AVA to provide actuation often accounted for a time delay in the feedback [135, 136], which occurs in the process of signal acquiring from the vibrating main system, processing, computing and then actuating. There is also a time delay produced intentionally by the controller or the user to enhance the vibration suppression and increase the stability of the system. In the next section, brief discussions on time delay in the feedback/ control algorithms of the AVAs are briefly discussed.

1.2.2.2 Time delay feedback

Time delay in the feedback or actuation process is a typical occurrence and useful in AVA applications. Andrade et al. [137] in the year 1995 studied the importance and influence of time delay in the active vibration absorber for vibration suppression. They showed that the time-delayed AVA excel in more vibration suppression than PVA. However, the time delay in AVA may deteriorate the performance if they are not estimated properly. Agarwal and Yang [138] presented a state-of-the-art review of the fixed time delay of actively controlled

vibration absorbers for civil engineering structures. They studied various literature on the identification, effect on the stability and performance of the AVA/active TMDs with the critical time delay in controlling vibration of the primary structures under earthquake excitations. Further, they presented a method of stability analysis for the critical time delay of the multi DOF systems equipped with single or multiple actuators. Vyhlídal et al. [139] analysed active vibration suppression of the SDOF primary system using a time-delayed absorber with acceleration feedback. They determined the stability of the system using the spectral element method and showed desirable vibration suppression of the primary system with various case studies. Di Egidio et al. [140] in the year 2020 used time-delayed AVA where they proposed an active control method for symmetric rigid blocks under harmonic and seismic excitation based on the LQR approach. They showed that the control law has good robustness with respect to the sampling time and the time delay. Also, Kayabekir et al. [141] recently, obtained the optimized values of the AVA using a modified harmony search method for reduction of responses of the seismic structures for different values in the time delay. A brief review of the previous literature is presented in the following section, where the actuation by the AVA is achieved using piezoelectric materials in the stack and patch forms.

1.2.2.3 Piezoelectric smart actuator

Tewani et al. [142] suppressed the machine tool chatter in a boring bar by the use of an AVA with a piezoelectric actuator. They compared their findings on AVA to PVA and found that using the proposed AVA, they were able to achieve a significant increase in the maximum permissible cut width in a stable range for a wide band of cutting speeds. Zhao et al. [143] analysed the dynamics of piezoelectric based VAs where the piezoelectric stack actuator is connected to an RLC circuit to suppress vibration of the SDOF primary system. They compared vibration suppression of the primary system using various shunt circuits and H_2 optimization techniques by considering piezoelectric stack actuator and RLC circuit as an additional DOF. Fallah and Ebrahimnejad [144] studied the performance of PZT stack actuators which is capable of providing 80 kN blocking forces, for vibration suppression of 10 storey tall building due to seismic response. They used three control schemes namely LQR, supervisory and fuzzy concept to command piezoelectric actuators and obtained the optimum arrangements of actuators by genetic algorithm. Chatterjee [145]

demonstrated new control laws for the active vibration control using the PZT stack actuator as an impact damper to suppress the vibration of the lightly damped SDOF primary system under free, forced and self-excited excitations. Chen et al. [146] conducted a review of literature on the application of piezoelectric materials for sustainable building development and energy harvesting via host structure vibration mitigation. There are also extensive studies [147] on vibration control of various structures using piezoelectric materials in the AVA.

The piezoelectric materials in patches form are also widely used in the AVAs, mostly to suppress vibration in continuous systems; some of the literature on this topic is reviewed here. Fallah and Ebrahimirjad [148] use PZT actuators and sensors as patches over thick and thin beam systems to actively control the vibration. They used the finite volume approach and negative velocity feedback control to study the system responses for various applications considering cantilevers and simply supported beams. Lee et al. [149] studied the effects of surface-bonded PZT layers on a laminated Timoshenko beam model by adopting first-order shear deformation theory. Nikkhoo [150] parametrically studied the performance of piezoelectric patches with displacement, velocity and acceleration feedback to suppress vibration of a single span and multi-span thin beam by bonding the number of piezoelectric patches. Høgsberg et al. [151] showed a reduction in response amplitude of the primary system in a series of the hybrid unit of passive and active vibration absorbers. They used an integrated feedback control law and demonstrated good damping performance by the smart actuator which can be made of piezoelectric material. Shin et al. [152] studied an active mass damper using acceleration feedback to suppress vibrations of a continuous system such as a plate. It is observed from the above literature that many researchers used displacement and velocity feedback [6, 16, 19, 30, 31, 55, 62, 119, 120, 122-132, 138, 142-145, 148, 153] in the active vibration absorber to suppress the vibration of the main system with very few literature on acceleration feedback [135, 136, 139, 140, 152] or combination of these feedbacks [150]. Among the various smart/active materials, piezoelectric based materials or PZT (lead zirconium titanate) in patch form are widely used in the absorber [143, 145, 154] compared to its stack form as these patches are more flexible to withstand failures. But the PZT actuator in stack form provides a much higher actuation force which will be much more useful to suppress the vibration of the host primary structure. The nonlinear analysis of the passive and active vibration absorbers

considering nonlinearity in the primary system and the absorber are equally important as cited in many works [2, 4, 7, 43-63]. In the next section, nonlinear passive vibration absorbers are briefly reviewed.

1.2.3 Nonlinear passive vibration absorber

The nonlinearities in the structure of TMDs be it passive or active is an important aspect of the analysis which are discussed in this section considering works from the previous literature. As discussed in the introduction section, the nonlinearity in the main system arises, due to the high amplitude of excitations of the structures, prolonged use of the system, and irregularity in the cross-sections. They are also inherently present in many systems in different applications [5-8, 10-21, 27-32, 35, 37, 39-41, 43, 54, 55]. Also, when the primary system is subjected to multiple excitations, parametric excitations or combinations of multi-frequency excitations then the nonlinear phenomena arise in the response of the system. Further, the nonlinear phenomena develop as the absorber vibrates very rapidly due to the energy transfer from the main system to the absorber. The nonlinear analysis of DVA is more practical and can be used in most applications [5-8, 10-21, 55].

To obtain the solutions to the nonlinear governing equation of motion many analytical methods are used but the method of multiple scales (MMS) [29, 31, 34, 46-55] is mostly used due to its easier solving procedures to obtain the reduced equations for various resonance conditions with good accuracy, especially for low nonlinearity in the system. The other methods such as the harmonic balance method (HBM) [39, 44, 45, 56-62] and the method of averaging [63], are also used in many literature for high and low nonlinear systems. The nonlinear studies of DVA are often accounted for nonlinearity in the stiffness of the absorber or the main system [4, 8]. The optimization techniques for vibration suppression of the nonlinear primary system by nonlinear passive or active DVAs are still a new area of research [6, 155]. In the next section, HBM and MMS solution methodology and optimization parameters for the nonlinear vibration absorber are briefly studied from the previous literature.

1.2.3.1 Harmonic balance method

The nonlinear vibration absorber and its application in various fields is still a broad area for research as mentioned in the books of Nayfeh and Balachandran [156] and Lacarbonara et al. [157]. Balaji and SelvaKumar [155] recently reviewed articles on the application of nonlinearity in the passive vibration absorber. The methodology to solve the nonlinear governing EOM of PVA by HBM is extensively carried out in the work of Kerschen et al. [158-164] and their group. Kerschen et al. [158-160] made a significant contribution to the passive nonlinear vibration absorbers by studying various nonlinear phenomena and optimization parameters. Habib et al. [161] used HBM to obtain the solution of the EOM and derived the optimal formulae for the stiffness and damping ratio of the nonlinear passive DVA which shows two equal Den Hartog's peaks in the frequency response of the SDOF nonlinear primary system. Habib et al. [161-166], also performed extended analyses on the vibration suppression of the nonlinear primary systems and study the nonlinear dynamics using nonlinear tuned vibration absorbers for many applications. Habib et al. [166] in the year 2019 studied the vibration suppression of the undamped SDOF primary system by using the Duffing type of oscillator in the absorber. They combine HBM with straightforward expansion to obtain the solution of the governing EOM and illustrated that two different dynamical paths exist out of which in one path vibration mitigation is faster. Viguié and Kerschen [167] developed a tuning methodology to mitigate the vibration of the nonlinear SDOF primary system with the nonlinear DVA. Detroux et al. [168] identified the safe, unsafe and unacceptable regions of operations for the nonlinear VAs when the SDOF primary system is subjected to harmonic excitation. They used the harmonic balance method (HBM) to obtain the solution of the governing equation of motion and analysed bifurcation, detached resonance curves and quasiperiodic responses of the system. Cirillo et al. [169] explored the qualitative and structural behaviour of the SDOF nonlinear primary system under harmonic excitation with an attached SDOF nonlinear absorber under harmonic excitation by singularity theory. They used a single harmonic term in HBM to obtain the solution and bifurcation analysis has been carried out by studying detached resonance curves in the frequency response of the primary system. Raze and Kerschen [170] studied multiple nonlinear vibration absorbers that mitigate several nonlinear resonances simultaneously. They used HBM and maintain equal peaks in the frequency response for the nonlinear system over a broader range of forcing amplitudes than the linear absorber. They studied many nonlinear phenomena such as adverse attractors, detached

resonance curves, quasiperiodic and superharmonic resonances, highlighting the performance of the proposed nonlinear absorbers superior to that of linear absorbers. Gatti [171] studied the efficacy of cubic nonlinear stiffness in the tuned mass damper (TMD) to suppress the vibration of the SDOF primary system under the harmonic excitation. They used HBM and Floquet's theory to obtain the response of the system and study the effect of hardening and softening nonlinear stiffness on the frequency response and stability of the system. Renault et al. [172] investigated the enhanced performance of nonlinear DVA by tracking anti-resonances in frequency response curve using coupled numerical technique and HBM when the SDOF primary system is subjected to external harmonic excitation. They studied both geometric and/or inertial nonlinearities and showed hardening and softening effects in the response. Li and Cui [173] used HBM to obtain the solution of the equation of motion (EOM) for the single degree of freedom (SDOF) primary system with attached nonlinear TMD under primary resonance conditions. They provided a new design method for the nonlinear TMD which sufficiently eliminates the adverse effects of nonlinear systems. Li and Du [174] in the year 2020 used HBM to determine the steady-state responses of the SDOF primary system under harmonic excitation by a nonlinear TMD. They obtained the optimal parameters of the nonlinear TMD and studied hardening and softening effects in the responses. Kong et al. [175] studied the nonlinear energy sink with nonlinear cubic damping attached to the SDOF nonlinear primary system by using incremental HBM and Floquet theory. The nonlinear energy sinks refer to the system where the secondary/ absorber system consists of only nonlinear spring stiffness with no linear stiffness. They demonstrated the proposed vibration absorber is more effective than previous studies. Ding and Chen [176] reviewed 383 articles in the year 2020 on nonlinear energy sinks which can suppress vibrations over a wide frequency range. They suggested promising results for multidirectional vibration reduction where the absorber can be cubic nonlinear stiffness. There are other nonlinear methods such as MMS which has been extensively used to study vibration absorbers for various resonance conditions under various external excitations on the primary system. In the next section, the solution of the nonlinear passive vibration absorber has been studied considering MMS.

1.2.3.2 Method of multiple scales

In this section, the nonlinear PVA is studied from the previous literature, where MMS is used to obtain the reduced equations from the nonlinear governing EOM. Ji [177, 178]

studied a nonlinear vibration absorber tuned under 3:1 internal resonance condition to suppress the primary resonance of an SDOF nonlinear primary system under weak harmonic excitation. They used the method of multiple scales (MMS) to obtain reduced equations and parametric studies are undertaken. Li and Zhang [179] recently showed that neglecting the nonlinearity in the design process of TMD can produce adverse effects on the control performance. They considered nonlinearity in the TMD and deduces an optimum formula by using the complexification averaging MMS. They manifested that the nonlinear TMD performs better than the linear TMD. Luongo and Zulli [180] addressed the passive control of aeroelastic instability in multi DOF systems due to Hopf bifurcation by using a nonlinear energy sink type of absorber in the system. They used combined MMS and HBM techniques to obtain the reduced amplitude modulation equations. They showed that a nonlinear energy sink, can shift forward the bifurcation point and reduce the amplitude of the limit cycles. There are many recent studies [180-182] on the passive nonlinear energy sinks i.e. where essential nonlinear stiffness is considered in the absorber configuration to suppress the vibration of the linear/ nonlinear main system under various applications and excitations. Sayed and Kamel [183] studied the response of the two DOF systems by considering coupled nonlinear equations applicable to rotor blade flapping motion. They used MMS to study the response of the system for various values of linear and nonlinear stiffness coefficients. Sayed and Hamed [184] carried out a parametric study on the response of two DOF coupled nonlinear equations under primary and principal parametric resonance conditions. They used the MMS perturbation method to obtain the reduced equations. EL-Sayed [185, 186] proposed a linear vibration absorber to suppress the vibration of the nonlinear dynamical system at simultaneous primary resonance and the presence of a 1:1 internal resonance condition. They used the MMS technique for the analytical study and compared their results with the related published literature showing the effectiveness of the proposed absorber. From their analysis, it is found that primary resonance is one of the worst resonance cases and with the proposed absorber the efficiency of controlling the vibrating system is increased. El-Ghareeb et al. [187] studied the vibration suppression of a nonlinear system by a nonlinear absorber by considering different resonance conditions using multiple scale perturbation techniques. They concluded different characteristics of frequency response curves by taking various values of forces and nonlinearity in the stiffness and showed that by increasing the nonlinear stiffness value the response amplitude of the primary system reduces significantly.

Karimpour and Eftekhari [188] studied the flexural vibration of a symmetrically laminated composite cantilever beam carrying a sliding mass under harmonic base excitations. The composite beam carrying a moving mass, which reciprocates inside a wire coil, is considered to act partially as a vibration absorber and also as an electric generator. Equation of motions of the system is derived according to extended Hamilton's principle and they employed the method of multiple scales for the analysis. Their proposed study reveals that such absorbers can be successfully implemented to attenuate the large amplitude of the beam vibrations and also harvest energy. Wang et al. [189] analysed vibration suppression of a hinged-hinged nonlinear Euler-Bernoulli beam using a linear/nonlinear TMD. They used MMS and examined the effectiveness of the linear and nonlinear TMDs by obtaining the optimal damping ratio. Bellizzi et al. [190] in the year 2019 combined complexification, averaging, and multiple-scale approaches to obtain the solution of the linear SDOF primary system under the periodic excitation with the attached essential nonlinear SDOF absorber/nonlinear energy sink. They showed that a relatively small-time delay causes significant suppression of quasiperiodic response regimes and reduction of the response of the primary system. Very recent work (2021) includes the work of Savadkoobi et al. [191] where they modelled a nonlinear vibration absorber for passive vibration control of the human arm under essential tremor. They used MMS to obtain the solution. There are various other methods to solve the nonlinear equations of motion of the passive nonlinear vibration absorber which are discussed in the next section.

1.2.3.3 Other nonlinear methods

In this section, various other nonlinear methods by which the reduced equations are obtained to study the response of the primary and the secondary/absorber system are studied. Esen and Koç [192] suppressed the nonlinear vibration of the gun barrel tip modelled as the primary system by using a PVA. They obtained the optimized parameters of the PVA by the combination of the genetic algorithm and finite element method. Yang et al. [193] studied the performance of the nonlinear vibration absorber attached to a nonlinear primary system under harmonic excitation in terms of power flow characteristics. They considered cubic nonlinearity in both spring and damper and used the method of averaging to obtain the solution of EOM. The efficacy of nonlinear damper, hardening and softening stiffness in the absorber over linear absorber are highlighted. Lacarbonara et al. [194-196] and their group Carpineto et al. [197-199] and Vestroni et al. [200] exploit the

nonlinearities and the hysteresis in the vibration absorber to suppress the flutter vibration using linearization technique and extended their analyses to many other applications [201]. Ebrahimzade et al. [202] studied the performance of linear PVAs, nonlinear PVAs and nonlinear energy sink (NES) on the stability properties and nonlinear behaviours of an aeroelastic model. They extended their analysis to a two DOF linear DVA, linear spring parallel to Maxwell DVA and Maxwell model DVA. Franzini [203] recently (2021) studied an elastic rotative nonlinear vibration absorber to passively suppress the vortex-induced vibration of a rigid cylinder immersed in a fluid as the primary system. The nonlinear TMDs systems that may have better control performance than the linear TMDs remain still in the developmental stage. These passive nonlinear TMDs can be widely applicable to many vibrating systems to suppress vibration in a wide frequency range and there are many new techniques such as the use of inerter as passive vibration control and various new methods to solve the nonlinear equations [204-210]. In the next section, the studies on the nonlinear vibration absorber from the previous works are extended by considering active forces by actuator mainly focusing on PZT nonlinear AVA.

1.2.4 Nonlinear active vibration absorber

It is observed from the previous literature review sections the importance of considering nonlinearity [5-8, 10-21, 37, 39-41, 43, 54, 55, 155-210] in the system while designing/analysing the vibration suppression capability of the absorber. To attenuate the vibration of the host primary structure using active control methods is well established, which imparts not only a significant reduction in the response amplitude but also makes the system stable for a wide range of frequencies [3-9, 16-20, 24-26, 30, 31, 35-38, 50-55, 60-62, 109-154]. In the next section, the nonlinear vibration absorber where active components such as actuators and sensors are integrated into the passive nonlinear vibration absorber is discussed. Various types of actuators are integrated with the nonlinear vibration absorbers among them PZT based actuator is found in many studies due to their high actuation capability and through various feedbacks [20, 142-154] absorber frequency can also be actively controlled for various resonance conditions. In the previous literature, MMS techniques are widely used for the analyses of nonlinear AVA with PZT based actuators and other forms due to their simplicity in the formulation to study various resonance cases simultaneously. These studies from the literature are briefly studied in the next section.

1.2.4.1 MMS for primary and internal resonance cases

In the present section, works from the previous literature have been considered on the nonlinear analyses of AVA using the MMS solution approach to study the response of the system to the primary and internal resonance conditions. Oueini et al. [19] first studied the active nonlinear vibration absorber using various actuators such as PZT ceramics, Terfenol-D, motor and a potentiometer to successfully suppress the response amplitude of the SDOF, multi DOF and flexible beam. They used MMS to obtain an approximate solution of the EOM considering primary and 2:1 internal resonance condition. They further extend their analysis [211, 212] using the MMS considering quadratic and cubic nonlinearities in the stiffness of the system for primary, 1:1 internal resonance and 2:1 auto parametric resonance condition. They demonstrated the AVA exhibits various complicated dynamics, such as Hopf bifurcations, intermittency, and chaotic responses. Jun et al [213] investigated nonlinear AVA with quadratic nonlinearities and saturation phenomena using MMS for primary and 2:1 internal resonance condition. They performed the bifurcation analysis and examine the stability of the system for various feedback gains. Jun et al. [214] extended their analysis on the saturation-based control method by the nonlinear AVA in reducing the vibration response of the primary system. They used MMS and showed a reduction in the self-excited oscillation of the primary system when the frequency of the absorber is exactly tuned to half of the natural frequency of the primary system.

Carbajal and Navarro [215] analysed linear AVA using a differential flatness scheme to suppress the multi-frequency harmonic vibrations of the SDOF nonlinear primary system. They proposed a robust active vibration controller and used the MMS to obtain the approximate solution for the primary resonance condition. The nonlinear system exhibits various forms of resonances either from various nonlinearities in the stiffness, damper or the amplitude and type of external excitations. In the next section literature related to the use of MMS and HBM for various resonance conditions (besides primary resonance) are presented.

1.2.4.2 MMS and HBM for various other resonance cases.

In this section, various other resonance conditions are considered from the works of previous literature such as super-harmonic, sub-harmonic, and principal parametric with

the combination of internal resonance and primary resonance for the nonlinear system. Bauomy [216] studied the suppression of the vibration of the SDOF nonlinear primary system under multiple parametric excitations by using a negative velocity feedback controller. They obtained the reduced equations using MMS for simultaneous superharmonic and 1:1 internal resonance condition. Sayed and Kamel [217] considered two coupled governing nonlinear EOM, where the primary system is subjected to harmonic and parametric excitations. They used MMS to obtain the solution of the EOM and suppress vibration of the SDOF primary system for simultaneous primary, principal parametric, 1:2, 1:3 internal resonance conditions, using active nonlinear control force. Til et al. [218] studied nonsmooth dynamics of TMD with lateral stops using an alternating frequency/time-harmonic balancing method. They considered single harmonic- and a wide-spectrum excitation and provided insights into different system responses. Raj and Santosh [219] used the multi HBM and arc continuation technique to obtain the responses of a two DOF nonlinear system where the SDOF primary system is subjected to harmonic excitation. They considered a piezoelectric energy harvesting circuit in the nonlinear absorber configuration for vibration suppression and energy harvesting through optimal parameters from the genetic algorithm. These nonlinear TMDs can be widely applicable to many vibrating systems to suppress vibration by integrating actuators and sensors with numerous control feedbacks. However, during sensing, processing and actuating often a time delay is encountered in the feedback. In the next section works from the previous literature are considered to study its effect on vibration suppression of the nonlinear primary and secondary/absorber system for various resonance conditions.

1.2.4.3 Time-delayed system for various other resonance cases.

The active control of structures through linear and nonlinear analyses is a widely researched area but there are still several limitations to preventing unexpected high response amplitude of the primary system. In the nonlinear AVA, the time delay in the feedback or in the process of actuating is a common phenomenon, but its effect on the system response can vary differently. These aspects are considered in this section from the previous articles. The time delay aspect of study in both active linear and nonlinear systems is important in terms of practical implementation of the device as studied in the work of Balachandran et al. [220]. Zhao and Xu [221] studied vibration suppression and improved the stability of the SDOF primary system under harmonic excitation by using time-delayed position feedback

in the autoparametric pendulum-type vibration absorber. They used MMS considering 1:2 internal resonances and showed that the delayed feedback control can be used to suppress the vibration or stabilize the system. They [222] also, studied the effects of time delay with positive and negative position feedback in the absorber to suppress vibration of SDOF nonlinear primary system for primary and 1:1 internal resonance condition. They used MMS to obtain the solutions and studied various bifurcation points with variations of control gain. Rabelo et al. [223] studied time-delayed linear and nonlinear damping effects in the nonlinear vibration absorber to suppress vibration of the SDOF primary system for simultaneous primary, principal parametric and 1:3 internal resonance cases. They used MMS and modified Routh-Hurwitz criteria to study the system responses and stability. Sun and Xu [224] proposed an absorber-isolator-combined structure with timed delay displacement feedback for vibration control and isolation improvement of the SDOF primary system. They used MMS to study the vibration suppression effectiveness and stability of the steady states for the primary resonance, 1:2 and 1:3 internal resonances. El-Gohary and El-Ganaini [225] used a piezoelectric actuator with time delay in displacement and velocity feedback to suppress the vibration of a hinge-hinge beam under simultaneous primary and sub-harmonic resonance conditions. The governing equation of motion of the system is obtained by considering two DOF system models. They used MMS up to second-order to obtain the system responses and showed time delay is an effective parameter for vibration suppression. El-Ganaini and El-Gohary [226] studied time delayed absorbers for vibration suppression of two DOF nonlinear systems. They modelled the primary system as a cantilever beam, which is subjected to multi tuned excitation force and used MMS to study responses for simultaneous combined and 1:1 internal resonance condition. Rabelo et al. [227] analysed that time delay in the damping of the absorber can act as a stability factor for vibration control of SDOF nonlinear primary system under harmonic excitation. They used MMS for obtaining the solution of the EOM, considering primary and internal resonance conditions. Shao et al. [228] proposed an H_∞ controller in the DVA to improve the electric vehicle suspension and ride quality. They used the particle swarm optimization technique to obtain the optimal parameters of the DVA. In their analysis, time-delayed feedback control is considered and they demonstrated the effectiveness of the proposed DVA on the quarter vehicle under bumps and random road excitations. Yao et al. [229] investigated the active feedback control by using linear and nonlinear DVA to increase the hunting stability of the motor flexible suspension train bogie at high speed. They used the

multi-objective optimal method and evaluated the effect of time delay on the system stability. Casciati et al. [230] reviewed articles from 2009 to 2011, emphasizing the developments of theoretical, numerical, control algorithms and experimental studies, as well as actual implementations of the devices to suppress vibration. These actual implementations or the experimental works of the passive/active linear/ nonlinear vibration absorber are studied in the next section.

1.2.5 Experimental analysis of vibration absorber

The vibration absorbers are used in many applications as discussed in the introduction section. Some of the initial experimental verification of TMD to reduce the vibration include the works of Jones et al. [231] in the year 1967 and Abu-Akeel [232] in the year 1969. The experimental analyses for vibration suppression of the host or primary system by using the auxiliary spring–mass–damper absorber systems can be found in the review articles [5-7, 11, 15, 16, 19, 55, 73, 121, 146, 155, 230], where a large number of realistic applications have been considered. In the next section, literature related to experimental verification of PVA and AVA are discussed.

1.2.5.1 Passive vibration absorber

In this section, experimental works on the implementation of linear and nonlinear PVA are studied. Brennan [233] theoretically and experimentally designed a tunable vibration absorber/ neutralizer consisting of a mass supported by an air mount, whose natural frequency can be adjusted actively by air pressure to tune the absorber frequencies for a wide range of frequencies. Zárate et al. [234] experimentally showed the effective performance of PVA to dissipate the overall energy of the three-level building primary system, which is harmonically excited by an electromagnetic actuator at the base. Dayou and Brennan [235] extended their theoretical analysis of optimal VAs to experimental verifications for global vibration control using the passive approach. They show the effectiveness of the proposed vibration absorber is comparable to that of the AVA. Yang et al. [236] experimentally designed and developed an optimization strategy using a minimax numerical method for multiple tuned mass dampers (MTMD) to suppress machine tool chattering and to increase chatter-free depth cuts. They showed that MTMD is more robust to uncertainties in damping and input dynamic parameters in comparison

with a single TMD. Yang et. al. [237] also, designed and experimented with the use of two DOF based TMD possessing translation and rotation motion for mitigation of vibration of milling machine, which is considered as the SDOF damped primary system. They obtained optimum parameters for the absorber using H_{∞} optimization technique which resulted in improved machining stability and depth of cut, as well as an 80.8 % reduction in the vibration.

Kandaswamy et al. [238] reviewed articles on vibration control of offshore structures by the use of passive, active, semi-active, and hybrid designs considering both the linear and nonlinear vibration absorbers. The recent applications of the passive dynamic vibration absorber include in the area of biomedical engineering to suppress tremors of the forearm, wrist, neck or in any other body parts of a human due to Parkinson's disease [42-46, 239]. In these studies, PVA is designed as a bracelet to attenuate 85% of tremors [42] in the human forearm. Vibration control using vibration absorbers is widely used in many structures and devices around the world. Some of this use of passive and active vibration absorbers used in realistic applications can be found in the highly cited review articles of Spencer and Nagarajaiah [240] and Soong and Spencer [241]. The nonlinear analyses of these vibration absorbers can be applied to many practical systems as nonlinearity is inherently present, particularly in the joints, bearings and other moving parts. Also, nonlinearities arise due to various operations in the machining process [41, 39], automobiles [32], ships [35], aircraft, tall and large structures such as buildings, bridges, beams, flexible structures, off-shore structures, and railroad wheelsets, etc. as studied in many review articles [5-7, 11, 15, 16, 19, 55, 155, 230]. Nayfeh and Balachandran [242] analyse the applications of nonlinearities in various experimental works and their importance to study in various application fields. Jo and Yabuno [243] designed and experimented with a new type of nonlinear vibration absorber for the primary system under primary resonance conditions. They validated the theoretical results by using MMS with that of the experimental results. Jo and Yabuno [244] also analyzed vibration reduction of the primary system under parametric resonance conditions by using the vibration absorber with quadratic nonlinearity. They tuned the natural frequency of the vibration absorber to twice that of the primary system and used MMS to obtain the solution of the EOM. They also validated the analytical results from MMS with that of the experimental results. Kachapi and Dukkipati [245] developed a new method to transform the two nonlinear

differential equations of motion of a two DOF system into a single nonlinear differential equation of motion and then linearize the EOM to study the response of the system. Feng et al. [246] theoretically and experimentally demonstrated the benefit of nonlinearity in muscle characteristics in improving vibration isolation at low frequencies in the human body. They demonstrated that the proposed anti-vibration system has a wide range of engineering applications. Zhao et al. [247] investigated experimentally the effectiveness of the TMD on a transmission tower under seismic excitations through various parametric studies. From the previous literature, it can be inferred that the use of active approaches in these linear or nonlinear vibration absorbers significantly improves the vibration suppression for a wide range of operating frequencies. So, in the next section, experimental works on AVA are studied considering mainly the piezoelectric based sensor and actuator in the absorber configuration.

1.2.5.2 Active vibration absorber

In this section, experimental analyses of the AVA to suppress the vibration of both discrete and continuous primary systems are studied from the previously published literature. The AVA with various types of actuators and sensors are synthesized in conjunction with the PVA such that their combined effect is superior to the PVAs. These types of vibration absorbers are used in various applications which are discussed in the next section, considering discrete primary systems.

Linear systems

Here, the use of AVAs on discrete SDOF primary systems is studied from the previously published works. Jang and Tarng [248] experimentally showed that the piezoelectric based AVA can suppress a maximum of 90% of undesired vibration of the cutting tool excited by a harmonic disturbance at the resonance frequency. The vibration suppression by their proposed AVA suppresses both free and force vibration for a wide range of operating frequencies by changing the inertial mass of the AVA. Kim et al. [249] through simulations and experimental studies obtained the optimal parameters of the passive and active piezoelectric based AVA to suppress vibration of the SDOF primary system. They used acceleration feedback for AVA and validated the theoretical results through simulations and experiments.

Kim and Brennan [250] used delayed feedback control in an experimental analysis to suppress multiple vibration modes of the beam system. They showed that the primary beam system vibration is suppressed at each mode by an attached DVA. Yong et al. [251] both theoretically and experimentally showed that the vibration of the helicopter's individual blade can be reduced by using AVA. They used a piezoelectric based smart spring to provide actuation that allows them to actively change the structural response. Further, they [252] also used a piezoelectric based stack actuator as a smart spring for vibration reduction of the individual blade control of the helicopter flight. From this study, they demonstrated the potency of the piezoelectric stack actuator which can produce greater than 3.3 kN of blocked force at the maximum input voltage of 100 V to suppress single harmonic and multiple harmonic excitations acting on the primary system. Nambu et al. [253] studied and experimented using a smart self-sensing and actuating piezoelectric based AVA to suppress the vibration of a string supported by flexible beams. They showed robustness, effectiveness and compactness of the AVA and suggested using several smart AVAs for suppression of spatial and multimodal vibrations. Lu et al. [254] analysed both theoretically and experimentally to suppress wind-induced earthquake vibration of a 492-m height Shanghai World Financial Centre by the use of two active TMDs. They developed the active control force by considering displacement and velocity control gains while the nonlinear control force is neglected. From their study, it is observed that significant vibration is reduced by the active TMDs under wind loading while under earthquake conditions active TMDs are disabled and PVA is used to suppress the vibration of the primary system. Casado et al. [255] designed and experimented on the passive TMD and active mass damper to mitigate human-induced vibrations of the footbridge in Valladolid, Spain. They used a proof mass actuator for the active mass damper with acceleration feedback and H_{∞} control strategy is used to obtain optimal parameters.

Peng and Gong [256] designed and experimented with vibration suppression of the SDOF primary system by an active adaptive vibration absorber and showed its effectiveness compared to active and passive vibration absorbers. Tso et al. [257] analytically and experimentally showed global vibration control of flexible structures under random stationary excitations by using an AVA/hybrid vibration absorber. They showed the proposed AVA is economical and provide better performance than the conventional TMD. Yang et al. [258] and Lin et al. [259] experimented with a novel resettable semi-active stiffness damper, where the stiffness of the TMD can be changed through the controller and

suppress vibration of civil structures under seismic excitations. Zárate et al. [260] theoretically and experimentally suppress the vibration of a three-story building structure for a wide range of excitation frequencies using PZT stack actuator-based AVA with acceleration feedback. They validated the theoretical results of FEM with the experimental observations.

Herold and Mayer [261] investigated experimentally broadband vibration control of the primary system by using a tunable piezoelectric stack actuator based AVA. They tested the AVA on a basic truss structure and actively controlled the properties of the absorber. Groots et al. [262] experimented on a deployable active control device that consists of an electromechanical mass damper mounted on an uncrewed ground vehicle for rapid deployment. They validated the effectiveness of the control through the experimental studies and showed its application in lightweight bridge structures. Yang et al. [263] experimentally showed to suppress vibration of multi DOF system by using AVA with negative acceleration feedback. Talib et al. [264] designed and experimented on multiple AVA with multi-input and multi-output acceleration feedback for vibration suppression of a two-story building-like structure. Soleymani et al. [265] proposed a new robust control system for the AVA to suppress the vibration of a high rise building under seismic excitation. They showed the effectiveness of the proposed controller through experiments at tackling the simulated disturbances in the presence of model uncertainties and input delay. Kopylov et al. [266] investigated the practical implementation of controlled TMDs by manufacturing electromagnetic regenerative TMD with acceleration-based ground-hook control. They showed with the proposed TMD vibration suppression is achieved for a protected mass test bench similar to a typical vehicle unsprung mass under random excitations. Sethi and Song [267] demonstrated the effectiveness of vibration suppression through experimental studies by using surface-bonded PZT actuators on a three-story model structure. Tang and Zuo [268] experimentally presented, their proposed TMD for vibration reduction and energy harvesting of a three-story building prototype.

Buelga et al. [269] also showed results both analytically and experimentally to suppress vibration and harvest energy simultaneously by using active TMD on the SDOF vibrating primary system under external excitation. Kučera et al. [270] modified the delayed resonator, to widen the operable frequency, range by using both delayed and non-delayed acceleration feedback control laws together. They showed with experimental studies the easier implementation of the AVA with delay and better vibration suppression of the SDOF

primary system under harmonic excitation. Eris and Ergenc [271] experimentally showed the effectiveness of a multiple-delay delayed resonator with the combination of speed and position feedback to suppress car body vibrations. Kassem et al. [272] experimentally studied active DVA consisting of a cantilever beam with a bonded macro-fibre composite and a lumped mass for flutter vibration suppression. They showed through experiments with the proposed active DVA up to 93.3% suppression in the response amplitude is achieved.

In most of the previous literature studied till now emphasis is put on the vibration suppression of discrete systems (SDOF and MDOF), while active or passive TMDs are used both theoretically and experimentally in the continuous systems such as beams, and plates, rods etc. An et al. [273] through simulation and experiments on a cantilever beam showed that their proposed time delayed acceleration feedback controller using piezoelectric patches can change the mass properties and damping of the controlled system simultaneously and reduce the vibration of the beam. Yamada et al. [274] experimentally investigated vibration suppression of a cantilever beam using a piezoelectric element and LR circuit where voltage is applied by displacement or acceleration of the primary system. They also obtained optimum values of inductance and resistance using fixed point theory and showed vibration suppression with minimal applied voltage with more stability. Yamada et al. [275] also, investigated both theoretically and experimentally to improve the bonding and location of piezoelectric elements used for vibration suppression of a piezoelectric based beam. They considered the mechanical impedance matching method and used spacers for the piezoelectric element. Harne [276] designed and experimented with an energy harvesting vibration absorber using a circularly corrugated piezoelectric film spring and distributed mass layer for simultaneous vibration attenuation and energy harvesting. Matsubara et al. [277] suppress the chatter vibration of a boring bar experimentally by the piezoelectric actuator and LR circuit with acceleration feedback. They modelled the cantilever beam based boring bar as the SDOF primary system and used the fixed-point theory of optimization to obtain the optimal parameters for the absorber. Jovanovic et al. [278] experimentally investigated a composite cantilever beam with fibre-reinforced piezoelectric actuators subjected to a periodic excitation. They compared the control performances of the proposed modified PID controller with the PD, PI and P controllers. Xu and Sun [279] used time delayed acceleration feedback-based AVA with a servo motor as an actuator to suppress vibration of the SDOF primary system under base

excitation. They showed the effectiveness of the time-delayed absorber both by analytical and experimental methods to suppress the vibration of the primary system in a wide frequency range. Kammmer and Olgac [280] controlled the vibration of the SDOF system and a cantilever beam by time-delayed acceleration feedback in the piezoelectric networks by both theoretical and experimental results. Some of the experimental works on nonlinear active vibration absorbers are studied in the next section.

Nonlinear vibration absorber systems

Bronkhorst et al. [281] designed an optimal viscoelastic DVA to suppress vibration of SDOF nonlinear primary system with cubic nonlinearity. They used HBM for the theoretical analyses which are validated with the experimental work. They also developed methodologies to identify the vibration parameters of the SDOF nonlinear system. Chen and Yang [282] experimentally controlled the vibration of the nonlinear building like structure under free, forced and earthquake excitation using active TMD with the modified Newton's method. Brennan and Gatti [283] investigated experimentally the use of cubic nonlinearity in the vibration absorber which shift the resonating peak to a higher frequency range, in contrast to linear vibration absorber where additional mass is required for this purpose. They used HBM for the theoretical analysis which is validated by experimental studies. Abundis-Fong et al. [284] experimentally designed an optimal autoparametric cantilever beam as a PZT based nonlinear AVA for a linear spring-mass-damper SDOF primary system subject to external harmonic excitation. They used the MMS perturbation method and obtain the solution and attenuated the response amplitude of the primary system under the principal parametric resonant frequency. Rizos et al. [285] designed and experimented with pre-stressed leaf-springs based TMD with four PZT stack actuators to mitigate vibration of the primary system which undergoes free, force and base excitation for a wide band of operating frequencies. They also identified and characterize various nonlinear stiffness developed in the system in suppressing the vibration. Bian and Gao [286] theoretically and experimentally studied vibration absorption of a flexible arm using a virtual vibration absorber considering internal resonance. The virtual vibration absorber consists of a servomotor, by which its stiffness and damping coefficients can be readily adjusted by position feedback gain and variable velocity feedback gain, respectively. They used MMS to obtain the system response theoretically which is validated by the experimental results. Zhao et al. [287] investigated both theoretically and experimentally

vibration mitigation of a Duffing oscillator by a nonlinear positive position feedback controller. They used HBM to obtain the solutions and H_{∞} optimal control parameters to minimise the response amplitude of the main system. Rasid et al. [288] developed experimentally an active vibration isolation system where the active DVA acts as an accelerometer in a low-frequency range and as a vibration control device in a high-frequency range. They used PID controller in their analyses and through experiments and simulation results demonstrated an almost 50% reduction in the vibration by the acceleration and velocity feedback force. Wang et.al. [289] numerically and experimentally explored vibration attenuation of a primary system subjected to single and multi-harmonic excitation by using an undamped voice coil based active dynamic undamped dynamic vibration absorber. Wang and Yang [290] used a voice coil motor in place of a spring and damper in the absorber configuration as a nonlinear active dynamic vibration absorber to control the transient vibration caused by the impulse excitation on the SDOF primary system. They showed through theoretical and experimental analysis that the nonlinear active DVA is better than the linear active DVA for vibration suppression. Further Wang et.al. [291] showed the efficacy of active DVA for vibration suppression of SDOF primary system under single and multiple frequency excitations. Thenozhi and Yu [292] reviewed 189 articles on passive, active, semi-active, hybrid, base isolators with various types of control strategies for vibration control of building structures.

The nonlinear vibration absorbers are long been studied both theoretically and experimentally in many realistic applications such as long bridges [197, 293], tall towers [254], skyscrapers [7], aircraft engines [294, 295], helicopter blades, production machines, structures subjected to human-induced vibrations [296-298], with various modification in the design of the absorbers [299-310]. It is observed that the consideration of nonlinearities in the analysis provides robust performance in the vibration absorbers [311-318].

1.3 Gaps in the Literature and Motivation for the Present Study

It is observed from the literature discussed in the above sections that the linear and nonlinear analyses of the vibration absorbers considering active force are predominantly more effective in vibration suppression of most of the SDOF, MDOF and continuous

systems, which are subjected to various forms of excitations. The active part in these vibration absorbers which produces the actuation to suppress the vibration of the primary system is often considered the main integral part of the active linear or nonlinear vibration absorber. From the literature, it is observed that in very few literature, acceleration feedback is considered to provide the actuation by the AVA. Also, in no literature, the comparison between various feedbacks such as displacement, velocity, acceleration feedback and a combination of these feedbacks are studied to suppress the vibration of the primary system. The acceleration feedback is considered, easy to implement in the vibration absorber configuration just by using an accelerometer. The comparisons between the sensor position which can be either on the primary system or on the absorber are also not explored in the previously published literature. These changes in the feedback process can be significantly useful in suppressing the vibration of the primary system under various forms of excitations. The optimal value of control gains used in the displacement, velocity, acceleration and combination of these feedbacks are also not reported in the earlier works.

In few literature more than 60% of vibration attenuation of the primary system is reported and in no literature complete vibration suppression of an undamped primary system under various excitations for a wide range of operating frequencies is attempted. The nonlinear studies of AVA are often accounted for nonlinearity in the stiffness of the absorber or the primary system [4, 8] but considering nonlinearity in the active force by the AVA is also not reported in earlier literature. In previous literature, the actuators are considered only to provide a counteracting force with different feedback but the property and type of the actuators and how the active force is generated are not studied much. From the introduction and literature review sections, one can observe that piezoelectric materials can produce very high actuation force with very limited voltage supply, but their use in AVA is not explored more. In very few literature types of actuators used in the AVAs are mentioned and in those articles, the researchers considered PZT in patch forms but the difficulty in the PZT patches is that the suppression of the vibration takes place in the shear direction and also the actuating force is very less compared to stack form [5-10, 41, 42]. However, PZT patches in stack form can produce a high actuation force [6, 7, 41, 42] of more than 300N along the direction of vibration and can also change its stiffness actively. But, the use of the PZT actuator in stack form is less investigated in earlier literature [2-11] for vibration suppression of undamped primary systems under external excitations. The time delay in

the feedbacks, comparisons among different feedbacks gains and the active nonlinear force used to suppress vibration of the main system are also not studied much.

The parametric comparisons between the effectiveness of the linear and nonlinear vibration absorbers for various nonlinear stiffness values attached to the linear/nonlinear vibrating undamped primary system are also not investigated. The optimal range of nonlinear stiffness coefficient value of the primary system and the absorber are also not reported at which the primary system response amplitude will be minimum than the linear primary system considering the linear vibration absorber. From the literature review section, it can be observed that basically the vibration suppression of the primary system is mainly achieved by the transfer of energy at the resonance to the absorber. During this process, the absorber vibrates at a much larger amplitude which may fail the total system or in some cases there is some limitation of the stroke length of the absorber. But in no literature, the response amplitude of the absorber is minimized or its response amplitude is compared with the primary vibrating system. The simultaneous vibration suppression of the primary system and the absorber are also not carried out in the previous literature. In most of the previous literature a damper is attached to the primary system, which is subjected to various excitations and vibration suppression of this primary system under various absorber designs are carried out. But in very few literature undamped primary system is considered where the response amplitude of the primary system at the resonance becomes infinite for the linear system. The vibration attenuation of the undamped primary system under various external excitations is not explored in earlier literature.

The Den Hartogs' equal peaks in the frequency response amplitude of the linear primary system under external harmonic excitations with the attached linear absorber are long been studied. But for multiple excitations on the primary system, Den Hartogs' equal peaks in the frequency response amplitude of the system are not carried out. The optimization techniques for vibration suppression of nonlinear main systems by nonlinear passive or active DVAs are still a new area of research as very limited literature on passive nonlinear systems is considered. The optimal nonlinear stiffness of the nonlinear active absorber to obtain Den Hartog's equal peaks in the frequency response amplitude when the primary system is subjected to multiple external excitations is not carried out in previous literature.

The nonlinear phenomena in the nonlinear vibration absorbers are not only dependent on the nonlinearity in the stiffness but also develop when the primary system is subjected to

parametric excitations, combinations of harmonic and parametric and combinations of harmonic, parametric and base excitations or due to the applied nonlinear active force by the AVA but there are very few works available in this field. The nonlinear studies considering the delay in the active force with various feedbacks and gains are also not studied in the previous literature. The consideration of quadratic and cubic nonlinearities in the stiffness and time-delayed damper of the primary system and the absorber are not investigated for the nonlinear AVA. Vibration suppression of the nonlinear primary system under simultaneous primary and principal parametric resonance conditions, and simultaneous primary, principal parametric, sub-harmonic and superharmonic resonance conditions are not carried out in the previously published works. The effects of active forces by different gains and time delay in the feedback on the trivial state instability regions and the critical value of excitations are also not studied in the previous literature.

The nonlinear study of AVA used various methods to obtain the reduced amplitude-phase relations for studying the response of the system. Out of the many nonlinear methods, the method of multiple scales (MMS) [29, 31, 34, 46-55] is mostly used due to its easier solving procedures to obtain the reduced equations for various resonance conditions with good accuracy, especially for low nonlinearity under weak external excitations in the system. The other methods such as the harmonic balance method (HBM) [39, 44, 45, 56-62] and the method of averaging [63], are also used in several works for high and low nonlinear systems. However, there are very few works in which the results obtained using these methods are compared with each other. It is observed that using MMS reduced equations the frequency responses of the system show only one peak instead of two Den Hartog peaks and its frequency of operation is also difficult to predict as many researchers used non-dimensional parameters [29, 31, 34, 46-55], whereas in HBM [39, 44, 45, 56-60], method of averaging [63], various linear methods such as Laplace transformation etc [12-25, 27, 28] shows two peaks in the frequency response curves signifying Den Hartog's equal peaks response. These peaks in the response signify the vibrational energy transfer at the resonance from the primary system to the absorber by the two modal frequencies through tuning the absorber frequency.

In the literature, the traditional vibration absorber is well investigated but the comparisons of vibration suppression of the primary system between the traditional nonlinear AVA and non-traditional AVA are not explored in previous studies. The parametric studies on

various nonlinear stiffness, the amplitude of external excitations, various feedback gains, and time delay for the traditional and non-traditional nonlinear AVA are not studied in the previous literature. The mass of the absorber used in most of the literature remains high usually 0.1 to a minimum value of 0.05 times the main system mass, which increases the overall weight of the system. The studies on considering a mass ratio of 0.01 while suppressing the vibration of the primary system under various external excitations are also not considered in the previous literature. Modification in the AVA models so that the PZT actuator used in the AVA provides more blocking force with less voltage supply was also not considered in the previous analyses. The applications of the nonlinear PVA are well studied but its active approaches to suppress vibration of the continuous system such as beam under various boundary conditions and to suppress vibrations due to tremors in the human forearm, wrist and shoulders are also not explored in the previous literature.

Based on the above discussion addressing many gaps in the literature if implemented can certainly enhance the effectiveness of the vibration absorber for suppressing vibration of the primary system more effectively for a wider band of operating frequencies under various excitations with less absorber mass. This will increase the applicability of the vibration absorber in many fields which will be cheap and safe to operate for a prolonged duration of time. The lower mass ratio in the absorber, economic voltage supply, considering nonlinearity in all the system parameters and suppressing vibration of the primary system for a wide band of operating frequencies are not only very significant from the academic point of view but also for many industrial applications which will have tremendous commercial value. These technical gaps motivate the author to carry out research on the active nonlinear vibration absorbers with the following objectives.

1.4 Objectives of the Present Work

The main objective of the present work is to develop a novel active nonlinear vibration absorber and to study the dynamics of the systems under various resonance conditions. Efforts will be made to achieve 100% vibration reduction in the primary system. The proposed novel vibration absorber consists of a PZT stack actuator along with a spring in series which is connected to the primary system. The primary system may be subjected to external harmonic, parametric, and base excitations. Several combinations of linear and nonlinear springs, damping with and without time delay will be used in both the primary

and secondary systems. Time delay in the active force by various combinations of displacement, velocity and acceleration feedback may be considered under primary, principal parametric, sub-superharmonic resonance conditions along with or without internal resonance conditions. To achieve this main objective following five objectives are considered for the present work.

- (i) Dynamic analysis of linear PZT based AVA with acceleration feedback under external harmonic excitation.
- (ii) Dynamics of ANVA using time delay acceleration feedback under both base and harmonic excitations.
- (iii) Traditional and non-traditional ANVA with time delay combination feedback for primary and principal parametric resonance conditions.
- (iv) Traditional and non-traditional ANVA with time delay combination feedback for hard excitations.
- (v) Application of ANVA for vibration suppression of tremors in the forearm, beams and RIDTMD.

1.5 Thesis Organization

The thesis is organized into seven chapters, which are briefly discussed as follows.

In Chapter 1, the vibration absorber is briefly introduced and an extensive literature review is carried out. The technical gaps have been identified and the objectives of the present works are highlighted.

Chapter 2 contains the linear analysis of the PZT stack actuator-based AVA with acceleration feedback to suppress the response amplitude of a single degree of freedom (SDOF) spring-mass-damper primary system under external harmonic excitation. Here the optimized parameters of the proposed AVA are evaluated.

Chapter 3 presents the dynamics of PZT stack actuator-based ANVA with time delay in acceleration feedback to suppress the vibration of the SDOF spring, mass, and damper primary system under external harmonic and base excitations. Here Den Hartog's equal peaks in the frequency response are obtained using a modified harmonic balance method (HBM). Analyses are carried out to effectively suppress the vibration of the primary system

by studying the frequency response, time responses, phase portraits, Poincare' sections, the basin of attractions and force responses, for various system parameters and active forces.

Chapter 4 investigates the dynamics of the traditional and nontraditional PZT stack actuator-based ANVA with the time-delayed combination of displacement, velocity, and acceleration feedback to completely absorb (100% suppression) the response amplitude of the SDOF spring-mass primary system under external harmonic and parametric excitations. The reduced equations are obtained using the method of multiple scales (MMS) which are then compared with HBM and classical methods. Further, stability and bifurcation analyses are carried out.

In Chapter 5, the dynamics of the traditional and nontraditional PZT stack actuator-based ANVA are investigated considering time delay in both the damper and feedback where the primary system is under hard harmonic, parametric, and base excitations. Here the feedback is a combination of displacement, velocity, and acceleration feedback.

In Chapter 6, the applications of the ANVA with and without active forces are studied for three cases viz., (i) suppress tremors in the forearm of Parkinson's disease patients, (ii) continuous beam model with various boundary conditions, and (iii) three degrees of freedom system with rotational inertial double-tuned mass damper (RIDTMD).

Finally, in Chapter 7, the general and specific conclusions from this work are presented, and the scope of the future works have been highlighted.

CHAPTER 2

Dynamics of linear PZT based active vibration absorber with acceleration feedback under external harmonic excitation.

2.1 Introduction

In this chapter vibration suppression of a single degree of freedom (SDOF) spring-mass-damper primary system is carried out using an active vibration absorber (AVA). The secondary system consists of an active PZT stack actuator along with spring-mass-damper arrangements with acceleration feedback of the primary system. The governing equations of motion of the system are developed in section 2.2 and the Laplace transform is used to obtain the solution of the governing equation as described in section 2.3. Two different optimization techniques viz., fixed point theory (FPT) or H_∞ optimization and genetic algorithm (GA) have been used for finding the optimum tuning ratio and damping ratio for the AVA. This is explained in section 2.4. Numerical results have been discussed in section 2.5 by studying the responses of the system and comparing different optimized damping ratios obtained by the FPT and GA techniques. The optimal feedback gain is also evaluated at which the total vibrational energy of the primary system is minimum. In the next section modeling of the linear vibration absorber system has been described.

2.2 Derivation of the dynamic equation of motion of the AVA

In this section, the governing equation of motion of the active vibration absorber (AVA) system is derived. Fig. 2.1 shows both the primary and secondary systems where, m_i , k_i , c_i and x_i , denotes the mass, spring stiffness, damping coefficient and displacements, respectively of the primary system ($i = 1$) and the secondary system ($i = 2$). The primary system is subjected to an external excitation $F(t)$ where t represents the time and the feedback gain f_c is produced by the combination of a spring having stiffness k_3 and a PZT

stack actuator in a series connection. The stiffness of the PZT stack actuator is denoted as k_p^E . The negative acceleration feedback of the primary system is used for the actuation by the AVA using control gain k_c . Using Newton's second law, the governing equation of motion of the system can be written as follows.

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) + c_1 \dot{x}_1 + c_2 (\dot{x}_1 - \dot{x}_2) = F(t) - f_c \quad (2.1)$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) + c_2 (\dot{x}_2 - \dot{x}_1) = f_c \quad (2.2)$$

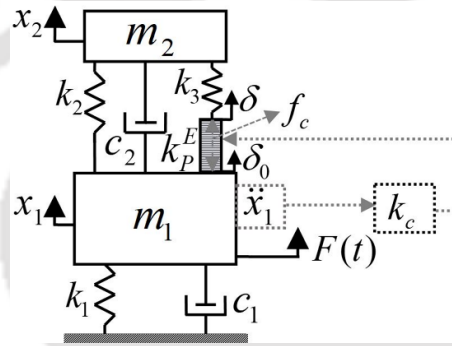


Fig. 2.1: PZT-based active vibration absorber with acceleration feedback.

It may be noted that the stack actuator has a displacement of $(x_1 + \delta_0 - \delta)$ where δ_0 is the nominal displacement of the actuator and δ is the displacement of the spring at the junction of the PZT stack actuator and secondary absorber mass m_2 . Hence the feedback gain produced by the AVA with the PZT stack actuator can be expressed as $f_c = k_p^E (x_1 + \delta_0 - \delta)$ which is equal to $k_3 (\delta - x_2)$, due to the presence of the spring with stiffness k_3 in series.

From this relation, the expression of δ can be obtained as $\delta = \frac{k_3 x_2 + k_p^E (x_1 + \delta_0)}{k_3 + k_p^E}$.

Substituting this δ value in the above f_c expression one can express $f_c = k_r (x_1 + \delta_0 - x_2)$, where $k_r = k_p^E k_3 / (k_p^E + k_3)$. The δ_0 for the PZT stack actuator can also be expressed as $\delta_0 = n d_{33} v$, where n is the number of wafers used in the PZT stack actuator and v is the applied voltage and d_{33} is the dielectric charge constant [20]. Considering the non-

dimensional time $\tau = \omega t$, x_0 and v_0 as reference displacement, and voltage, respectively the governing Eqs. (2.1) and (2.2) can be rewritten as given below.

$$\ddot{Z}_1 + Z_1 + 2\zeta_1 \dot{Z}_1 - \mu \omega_r^2 Z - 2\zeta_2 \mu \omega_r \dot{Z} = F_1(\tau) - f_a V \quad (2.3)$$

$$\ddot{Z} + \omega_r^2 Z + 2\zeta_2 \omega_r \dot{Z} = -\ddot{Z}_1 + \frac{f_a V}{\mu} \quad (2.4)$$

The non-dimensional parameters used in Eqs. (2.3) and (2.4) are defined as follows.

$$Z_1 = \frac{x_1}{x_0}, Z_2 = \frac{x_2}{x_0}, Z = Z_2 - Z_1, \mu = \frac{m_2}{m_1}, \omega_r = \frac{\omega_2}{\omega_1}, \zeta_1 = \frac{c_1}{2m_1\omega_1}, \zeta_2 = \frac{c_2}{2m_2\omega_2},$$

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}, \omega_2 = \sqrt{\frac{k_2 + k_r}{m_2}}, \lambda = \frac{nd_{33}v_0}{x_0}, V = \frac{v}{v_0}, f_a = \frac{k_r \lambda}{k_1}, F_1 = \frac{F}{m_1 \omega_1^2 x_0}$$

In the next section, the transfer function of the system is obtained from the non-dimensional governing equation of motion (Eq. (2.3) and Eq. (2.4)) and the stability analysis is carried out.

2.3 Stability analysis using Routh Criteria

In this section, the stability analysis of the system is carried out by obtaining the transfer function of the system. The acceleration feedback of the primary system is considered as shown in Fig. 2.2, with control gain (k_c) and accordingly the control law is written as,

$V = -k_c \ddot{Z}_1$. The block diagram of the primary system from Eq. 2.3 is shown in Fig. 2.2. The

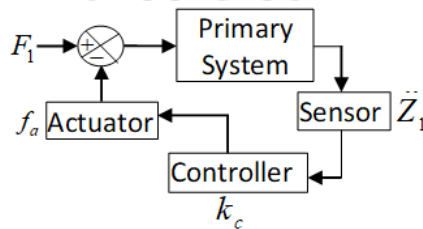


Fig.2.2: Block diagram for acceleration feedback of the primary system.

transfer function of the primary system and the absorber are obtained by taking Laplace transformations of Eqs. (2.3) and (2.4), which yield the following equations.

$$(s^2 + 2\zeta_1 s + 1)Z_1(s) - (\mu\omega_r^2 + 2\zeta_2\mu\omega_r s)Z(s) = F_1(s) + F_a(s)s^2Z_1(s) \quad (2.5)$$

$$(s^2 + 2\zeta_2\omega_r s + \omega_r^2)Z(s) + s^2Z_1(s) = -\frac{F_a(s)s^2Z_1(s)}{\mu} \quad (2.6)$$

where $F_a = k_c f_a$

Solving Eqs. (2.5) and (2.6), the transfer function of the primary system and the AVA are obtained as follows.

$$G_1(s) = \frac{Z_1(s)}{F_1(s)} = \frac{s^2 + 2\zeta_2\omega_r s + \omega_r^2}{N_{14}s^4 + N_{13}s^3 + N_{12}s^2 + N_{11}s + N_{10}} \quad (2.7)$$

$$G_2(s) = \frac{Z(s)}{F_1(s)} = -\frac{\left(1 + \frac{F_a}{\mu}\right)s^2}{N_{14}s^4 + N_{13}s^3 + N_{12}s^2 + N_{11}s + N_{10}} \quad (2.8)$$

where the coefficients N_{10} , N_{11} , N_{12} , N_{13} and N_{14} are expressed as follows.

$$N_{10} = \omega_r^2, N_{11} = 2\zeta_1\omega_r^2 + 2\zeta_2\omega_r, N_{12} = \omega_r^2 + 4\zeta_1\zeta_2\omega_r + \mu\omega_r^2 + 1, \\ N_{13} = 2\zeta_2\omega_r + 2\zeta_1 + 2\zeta_2\omega_r\mu, N_{14} = 1 - F_a$$

Following Routh's stability criterion to obtain the range of the feedback gain (F_a) by the absorber at which the system responses are stable, the following array of the coefficient are obtained from Eq. 2.7 for the primary system.

$$\begin{array}{c|ccc} s^4 & N_{14} & N_{12} & N_{10} \\ s^3 & N_{13} & N_{11} & \\ s^2 & \frac{N_{13}N_{12} - N_{11}N_{14}}{N_{13}} & N_{10} & \\ s^1 & \frac{N_{13}N_{12}N_{11} - N_{14}N_{11}^2 - N_{13}^2N_{10}}{N_{13}N_{12} - N_{11}N_{14}} & & \\ s^0 & N_{10} & & \end{array} \quad (2.9)$$

The stable responses are ensured when the real parts of all poles are negative. From Eq. (2.9) it can be derived from the first column of the Routh table that all the coefficients of

the array are positive for $-\mu < F_a < 1$. Which means the proposed system is stable within the above range of feedback gain (F_a). In the next section two optimization procedures viz., FPT and GA, techniques are undertaken to obtain the optimal tuning ratio, damping ratio required by the AVA to attenuates the response amplitude of the primary system.

2.4 Theory of optimization of the AVA

The tuning ratio (ω_r) which is the ratio between the natural frequency of the absorber (ω_2) and the primary system (ω_1) and the damping ratio (ζ_2) are important parameters which are usually varied to suppress the vibration amplitude of the primary system. In this section, the theory of FPT optimization is detailed and correspondingly the optimum (ω_r) and (ζ_2) values of the AVA are obtained following the fixed point theory (FPT) and the genetic algorithm (GA) optimization techniques in terms of mass ratio μ . To obtain the optimal parameters of the proposed AVA, initially, the response of the system is studied with various ω_r and ζ_2 values, which are shown in Fig. 2.3. It may be noted that the frequency response of the system is obtained by solving Eqs. (2.7) and (2.8) by considering $s = j\omega$, where $\omega = \frac{\omega_{11}}{\omega_1}$, $j = \sqrt{-1}$. Here ω_{11} is the frequency of external harmonic excitation acting on the primary system. The reference displacement (x_0) and reference voltage (v_0) are considered equal to 1, and the damping ratio of the primary system (ζ_1) is considered equal to 0 throughout the analysis. In Fig. 2.3 the frequency response of the primary system is shown for four different values of ζ_2 considering zero feedback gain ($F_a = 0$) in the primary system with $\mu = 0.05$. In Fig. 2.3(a) and Fig. 2.3(b) the ω_r is considered equal to 1, which is the same as undamped TVA. From Fig. 2.3(a) it can be observed that at ω_a and ω_b the response amplitude of the primary system remains the same irrespective of the various values of damping ratio ζ_2 . These two points (A and B) at ω_a and ω_b in the frequency responses are known as fixed points [1, 122-126]. There are many studies carried out on these fixed points starting with the first work from Den Hartog in the year 1942 to the many recent works [56-59, 68-74, 76-87] to obtain the optimum tuning and damping ratio of the absorber so that these peaks are minimum. One of the widely used methods is

the fixed point theory of optimization technique which was first proposed by Den Hartog [1, 5-7, 55-58, 100-103, 122-126] and famously known as Den Hartog equal peaks in the frequency response. The response of the absorber is shown in Fig. 2(b), from which one can observe that the vibrational energy of the primary system is transferred to the absorber due to which its response is high. Also, from Fig. 2.3(a) one can observe that with an increase in the ζ_2 values the response amplitude of the primary system increases instead of decreasing. Further, Fig. 2.3(b) shows the common phenomenon that is the response amplitude of the absorber decreases with increases in ζ_2 values, which means the vibrational energy from the primary system is not effectively transferred to the absorber.

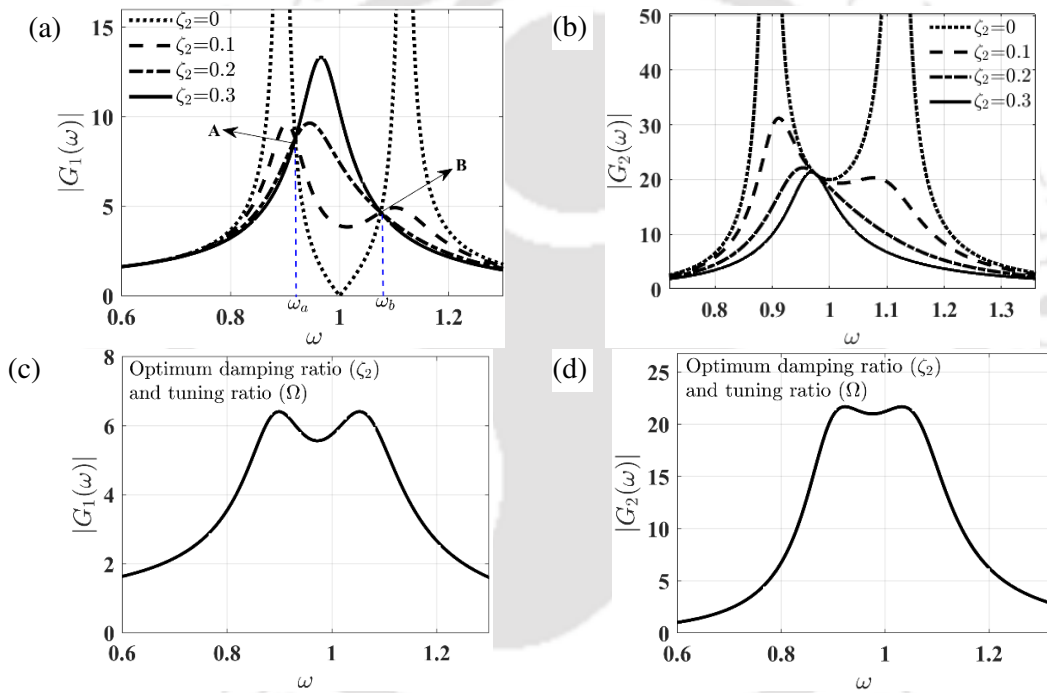


Fig. 2.3: Responses of the system with $\mu = 0.05$ (a) undamped primary system (b) passive vibration absorber (PVA) (c) primary system with optimized PVA (d) PVA with optimum parameters.

From these figures, one can notice that there should be some optimal parameters for the tuning ratio and damping ratio (ζ_2) at which the response of the primary system is suppressed effectively by the passive vibration absorber (PVA). From the previous literature [100-103, 122-126] it is found that for the passive TMD the optimum parameters for damping ratio (ζ_2) and tuning ratio (ω_r) in terms of the mass ratio μ are given as

$\omega_r = 1/(1+\mu)$ and $\zeta_2 = \sqrt{3\mu/(8(1+\mu))}$. Using these optimal parameters the response of the system with the passive absorber are shown in Fig. 2.3(c) and Fig. 2.3(d). From these figures, one can notice that the primary system response is considerably decreased and the peaks are at the same amplitude which is known as Den Hartog equal peaks in the response. However, it can be observed that with these optimal parameters, the response of the primary system is still very high. Hence one may consider an AVA to reduce the primary system response. The optimal parameters for the proposed AVA are obtained in the next section.

2.4.1 Fixed point theory and GA techniques for the proposed AVA

It may be noted that in the previous literature, the optimized damping ratio and tuning ratio for the PZT-based AVA with acceleration feedback was not found. So the proposed AVA uses acceleration feedback which is easier to implement in many applications in the form of an accelerometer compared to other feedbacks. To obtain the optimized ω_r and ζ_2 , the frequency response function of the primary system and the absorber which are obtained in Eqs. (2.7) and (2.8) are modified as given below by considering $s = j\omega$ where $j = \sqrt{-1}$.

$$G_1(j\omega) = \frac{P + jQ}{R + jS} \quad (2.10)$$

$$G_2(j\omega) = \frac{\left(1 + \frac{F_a}{\mu}\right)\omega^2}{R + jS} \quad (2.11)$$

where

$$P = \omega_r^2 - \omega^2, \quad Q = 2\zeta_2\omega_r\omega, \quad R = N_{14}\omega^4 - N_{12}\omega^2 + N_{10}, \quad S = N_{11}\omega - N_{13}\omega^3$$

Now considering the FPT optimization techniques, one can observe from Fig. 2.3(a) that the response amplitude of the primary system at the fixed points (A and B) remains the same whether $\zeta_2 = 0$ or $\zeta_2 = \infty$. Substituting these two values of ζ_2 in Eq. (2.10), and equating the resulting equations one may obtain the following expression.

$$\omega^4 - \frac{2\omega_r^2(1+\mu)+2}{2+\mu-F_a}\omega^2 + \frac{2\omega_r^2}{2+\mu-F_a} = 0 \quad (2.12)$$

The roots of the Eq. (2.12) and the corresponding response amplitude of the primary system at A and B with the proposed AVA are obtained as follows.

$$\omega_a^2, \omega_b^2 = \frac{1+(1+\mu)\omega_r^2}{2+\mu-F_a} \pm \sqrt{\left(\left[\frac{1+(1+\mu)\omega_r^2}{2+\mu-F_a}\right]^2 - \frac{2\omega_r^2}{2+\mu-F_a}\right)} \quad (2.13)$$

$$|G_1(\omega_a)| = \left| \frac{1}{1-(1+\mu)\omega_a^2} \right| \quad (2.14)$$

$$|G_1(\omega_b)| = \left| \frac{1}{1-(1+\mu)\omega_b^2} \right| \quad (2.15)$$

Following the observation in Fig. 2.3, one can notice that the optimum damping ratio of the AVA can be evaluated, when the response amplitude of the primary system at ω_a and ω_b are same and minimum, which can be mathematically expressed as $|G_1(\omega_a)| = |G_1(\omega_b)|$. From this condition the optimum tuning ratio (ω_r) of the system for the proposed AVA with feedback gain F_a , by the acceleration feedback can be written as given below.

$$\omega_{opt} = \sqrt{(1-F_a)/(1+\mu)} \quad (2.16)$$

This optimum value of the tuning ratio (ω_{opt}) gives rise to the condition that the feedback gain by the actuator must be $F_a \leq 1$. The obtained optimal tuning ratio (ω_{opt}) of the AVA is also verified in Fig. 2.4 by substituting Eq. (2.13) into Eqs. (2.14) and (2.15) and plotting the curves for the PVA and AVA with respect to variation in ω_r for $\mu = 0.05$. From Fig. 2.4(a) one can observe that the optimal tuning ratio (ω_{opt}) is the intersection point of curves obtained from Eq. (2.14) and Eq. (2.15). These points are marked as C and D for the PVA and AVA, respectively. The optimal tuning ratio (ω_{opt}) obtained in Eq. (2.16) satisfied with Fig. 2.4 for the PVA ($F_a = 0$) and AVA ($F_a = 0.5$) for $\mu = 0.05$. Substituting ω_{opt} value from Eq. (2.16) into Eq. (2.10) one may obtain the following expression for ω_a and ω_b at the fixed points.

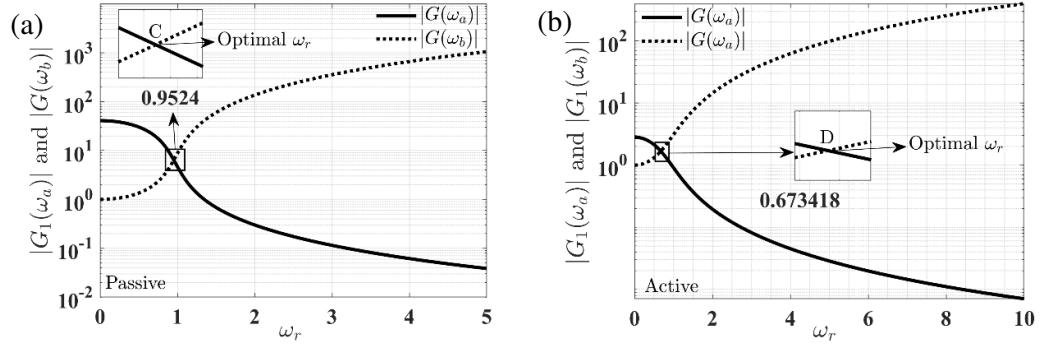


Fig. 2.4: Response amplitude of the primary system with AVA at the fixed points (a) $F_a = 0$ (passive) (b) $F_a = 0.5$ (AVA).

$$\omega_a^2, \omega_b^2 = \frac{1}{(1+\mu)} \left(1 \pm \sqrt{\left(\frac{\mu + F_a}{(2+\mu - F_a)} \right)} \right) \quad (2.17)$$

Substituting Eq. (2.17) into Eqs. (2.15) and (2.14) the corresponding response amplitude at the fixed points ω_a and ω_b are obtained as $\sqrt{(2+\mu - F_a)/(F_a + \mu)}$. To obtain the optimal damping ratio ζ_2 for the AVA, $|G_1(j\omega)|$ is obtained from Eq. 2.7, which is given below.

$$|G_1(j\omega)| = \sqrt{\frac{P^2 + Q^2}{R^2 + S^2}} \quad (2.18)$$

The optimum damping ratio of the AVA can be obtained at which the fixed points become the maximum response amplitude of the primary system, which gives rise to the following expression [5-7, 16, 55-58, 100-103, 122-126].

$$\frac{\partial}{\partial \omega^2} |G_1(\omega)|^2 \Big|_{\omega=\omega_a} = \frac{\partial}{\partial \omega^2} |G_1(\omega)|^2 \Big|_{\omega=\omega_b} = 0 \quad (2.19)$$

The optimal damping ratio of the AVA leads to the maximum response amplitude of the primary system at the fixed points which are obtained by solving Eq. (2.16), Eq. (2.17), Eq. (2.18) and Eq. (2.19). The resulting optimal damping ratio (ζ_2) for the proposed AVA at the fixed point ω_a is obtained as given by the following expression.

$$\zeta_2^2(\omega_a) = \frac{-C_{12} \pm \sqrt{C_{12}^2 - 4C_{11}C_{13}}}{2C_{11}} \quad (2.20)$$

Similarly, the optimal damping ratio ζ_2 at the fixed point ω_b can be written as follows.

$$\zeta_2^2(\omega_b) = \frac{-\tilde{C}_{12} \pm \sqrt{\tilde{C}_{12}^2 - 4\tilde{C}_{11}\tilde{C}_{13}}}{2\tilde{C}_{11}} \quad (2.21)$$

The $\zeta_2(\omega_a)$ and $\zeta_2(\omega_b)$ values provide the peak value for $|G_1(\omega_a)|$ and $|G_1(\omega_b)|$, respectively [126]. It may be noted that the optimized $\zeta_2(\omega_a)$ and $\zeta_2(\omega_b)$ value are very close to each other. The optimum damping ratio is when the fixed points are equal, so the convenient approximate of the optimal damping ratio (ζ_{opt}) for the proposed AVA with the acceleration feedback can be given by the following equation in terms of mass ratio μ .

$$\zeta_{opt} = \sqrt{\frac{\zeta_2^2(\omega_a) + \zeta_2^2(\omega_b)}{2}} \quad (2.22)$$

where

$$\begin{aligned} C_{11} &= C_2C_4 - C_6C_9, \quad C_{12} = C_2C_3 + C_1C_4 - C_6C_7C_8 - C_5C_9, \quad C_{13} = C_1C_3 - C_5C_7C_8, \\ \tilde{C}_{11} &= \tilde{C}_2\tilde{C}_4 - \tilde{C}_6\tilde{C}_9, \quad \tilde{C}_{12} = \tilde{C}_2\tilde{C}_3 + \tilde{C}_1\tilde{C}_4 - \tilde{C}_6\tilde{C}_7\tilde{C}_8 - \tilde{C}_5\tilde{C}_9, \quad \tilde{C}_{13} = \tilde{C}_1\tilde{C}_3 - \tilde{C}_5\tilde{C}_7\tilde{C}_8, \\ C_1 &= ((1-F_a)\omega_a^4 - (1+(1+\mu)\omega_{opt}^2)\omega_a^2 + \omega_{opt}^2)^2, \\ C_2 &= 4\omega_{opt}^2\omega_a^2 + 4\omega_{opt}^2(1+\mu)^2\omega_a^6 - 8\omega_{opt}^2(1+\mu)\omega_a^4, \\ C_3 &= -2(\omega_{opt}^2 - \omega_a^2), \quad C_4 = 4\omega_{opt}^2, \quad C_5 = (\omega_{opt}^2 - \omega_a^2)^2, \quad C_6 = 4\omega_a^2\omega_{opt}^2, \\ C_7 &= 2((1-F_a)\omega_a^4 - (1+(1+\mu)\omega_r^2)\omega_a^2 + \omega_{opt}^2), \quad C_8 = 2(1-F_a)\omega_a^2 - (1+(1+\mu)\omega_{opt}^2), \\ C_9 &= 4\omega_{opt}^2 + 12\omega_{opt}^2(1+\mu)^2\omega_a^4 - 16\omega_{opt}^2(1+\mu)\omega_a^2, \\ \tilde{C}_1 &= ((1-F_a)\omega_b^4 - (1+(1+\mu)\omega_{opt}^2)\omega_b^2 + \omega_{opt}^2)^2, \\ \tilde{C}_2 &= 4\omega_{opt}^2\omega_b^2 + 4\omega_{opt}^2(1+\mu)^2\omega_b^6 - 8\omega_{opt}^2(1+\mu)\omega_b^4, \\ \tilde{C}_3 &= -2(\omega_{opt}^2 - \omega_b^2), \quad \tilde{C}_4 = 4\omega_r^2, \quad \tilde{C}_5 = (\omega_{opt}^2 - \omega_b^2)^2, \quad \tilde{C}_6 = 4\omega_b^2\omega_{opt}^2, \\ \tilde{C}_7 &= 2((1-F_a)\omega_b^4 - (1+(1+\mu)\omega_{opt}^2)\omega_b^2 + \omega_{opt}^2), \quad \tilde{C}_8 = 2(1-F_a)\omega_b^2 - (1+(1+\mu)\omega_{opt}^2), \\ \tilde{C}_9 &= 4\omega_{opt}^2 + 12\omega_{opt}^2(1+\mu)^2\omega_b^4 - 16\omega_{opt}^2(1+\mu)\omega_b^2, \quad \omega_{opt} = \sqrt{(1-F_a)/(1+\mu)}, \end{aligned}$$

$$\omega_a^2 = \frac{1}{(1+\mu)} \left(1 + \sqrt{\left(\frac{\mu + F_a}{(2+\mu - F_a)} \right)} \right), \quad \omega_b^2 = \frac{1}{(1+\mu)} \left(1 - \sqrt{\left(\frac{\mu + F_a}{(2+\mu - F_a)} \right)} \right)$$

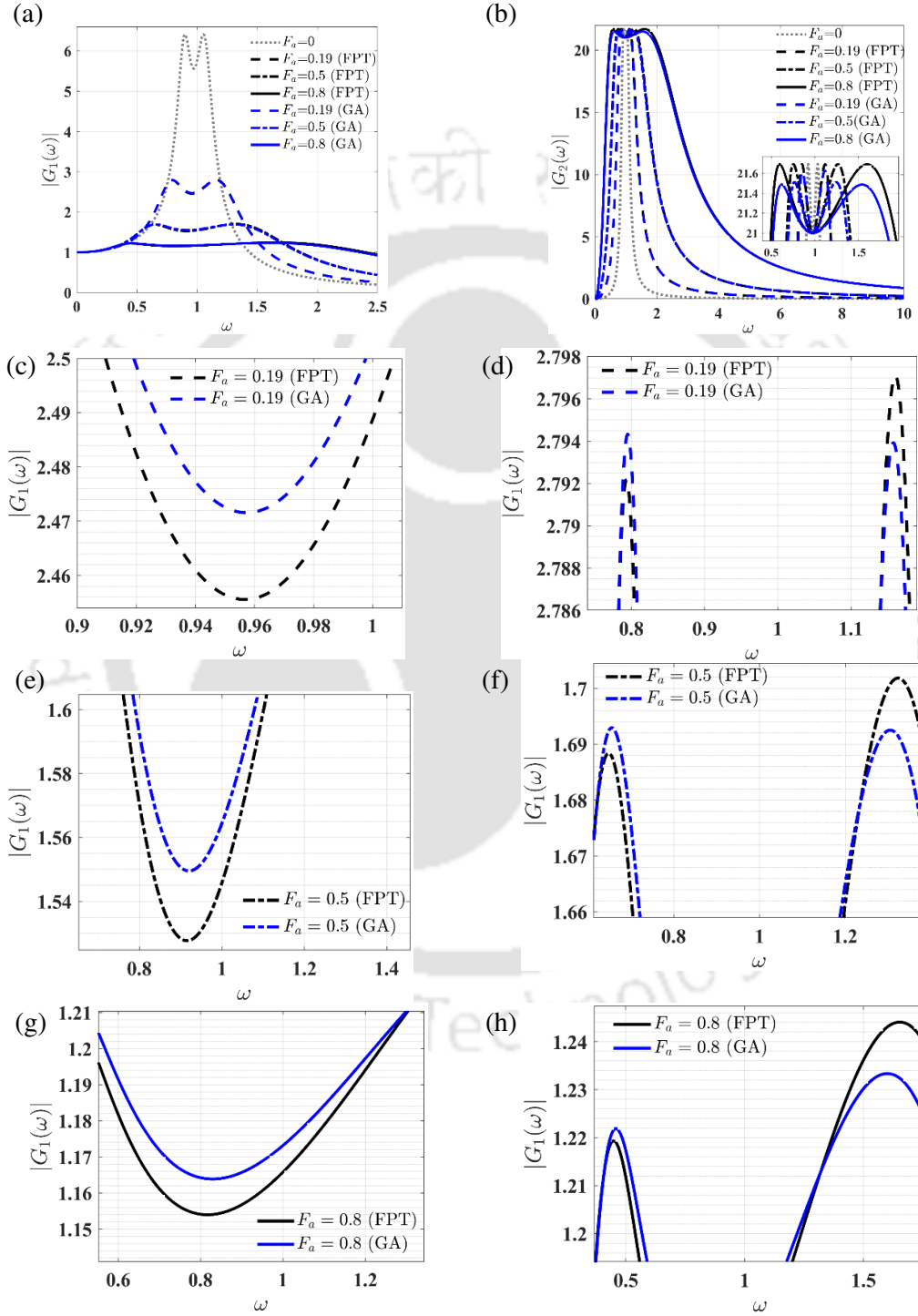
The damping ratio of the proposed AVA is also optimized by using the GA toolbox in MATLAB. The optimized values of the proposed AVA by GA are obtained by the following steps. First, the maximum peak value of the frequency response curve is obtained from Eq. (2.18) and Eq. (2.16) by considering $|G_1(j\omega)|$ as the fitness function and selecting the upper and lower bounds of ζ_2 between 0 and 1. Then in the next step, the GA optimization toolbox in MATLAB is used to obtain the optimal damping ratio of the AVA from the above range of damping ratios. In the next section, the performance of the proposed AVA is studied with the obtained optimal tuning ratio (ω_{opt}) and damping ratio (ζ_{opt}) by the FPT and GA for various values of feedback gain F_a .

2.5 Results and discussion

In this section, the responses of the system are studied with the obtained optimal parameters viz., ω_{opt} and ζ_{opt} for the AVA by acceleration feedback considering $\mu = 0.05$ in the entire analysis. In Fig. 2.5 the frequency response of the system is shown for various values of F_a with the optimized parameters obtained from Eq. (2.16) and Eq. (2.22) by using FPT and GA techniques and substituting these optimal parameters in Eqs. (2.7) and (2.8). From Fig. 2.5(a) it can be observed that with increasing value of F_a there is a corresponding decrease in the response amplitude of the primary system. For $F_a = 0.19$ there is an almost 60% decrease in the response amplitude of the primary system than the PVA in between the peaks. Similarly, for $F_a = 0.5$ and 0.8 , the response amplitude of the primary system reduces significantly close to the static deflection of the primary system. It may be also noted from Fig. 2.5(a) that the response amplitude of the primary system shows two equal Den Hartog's peaks with the changes in F_a values, which justifies the correctness of the ω_{opt} and ζ_{opt} relations by the FPT and GA optimization techniques. Fig. 2.5(b) shows that with increasing F_a values, the response amplitude of the secondary system increases. The

responses obtained by using the ζ_{opt} by GA also show a similar response as with FPT, and they are shown in Fig. 2.5(a) and Fig. 2.5(b) for three values of F_a . However, the comparisons of the responses at the peaks and valleys obtained by these two optimization techniques are highlighted in Fig. 2.5(c) to Fig. 2.5(h). From Fig. 2.5(c) and Fig. 2.5(d) with $F_a = 0.19$, one can observe that the Den Hartog's equal peaks with GA ζ_{opt} value, are slightly less than those obtained by using FPT. This difference is found to be approximately 0.14%. But the response amplitude of the primary system at the valley (close to the resonance), with $F_a = 0.19$ by FPT ζ_{opt} , is slightly less compared to the GA. Similar observations are also noticed in Fig. 2.5(e) to Fig. 2.5(h). In Fig. 2.5(i) to Fig. 2.5(k), the difference between the response amplitude of the absorber at the peaks and valleys are studied with the ζ_{opt} values by the FPT and GA techniques considering Fig. 2.5(b). From Fig. 2.5(i) to Fig. 2.5(k) it is observed that at the peaks with FPT the response amplitude of the absorber is slightly more than the GA. The damping ratios obtained for various values of the F_a are given in Table 2.1. From this table, one can observe that the ζ_{opt} by FPT technique is slightly less than that of the GA. Due to this, it can be observed that with GA the response amplitude of the absorber is comparatively less than with FPT. From Fig. 2.5 it can be noticed that with the proposed AVA, both the FPT and GA optimization technique gives almost similar responses. But the with GA the time required to obtain the optimal damping ratio by the AVA is more. So one can use the close form optimal value of damping ratio (Eq.22) obtained by the FPT technique to obtain the frequency response of the system for various F_a values quickly. The optimized damping ratios (ζ_{opt}) obtained for various values of the F_a are given in Table 2.1 from the two optimization techniques. From this table, one can observe that the ζ_{opt} by FPT technique is slightly less than that of the GA. Due to this, it can be observed that with GA the response amplitude of the absorber is comparatively less than with FPT. From Fig. 2.5 it can be noticed that with proposed AVA, if slight improvement in the simultaneous vibration attenuation of the primary system and the absorber is required, then one can use GA optimization or if the vibration attenuation of the primary system is the sole purpose then one can use FPT technique. But, with the GA optimization technique, the time consumption is more to obtain the response. The response amplitude shown in Fig. 2.5(a) for the passive vibration absorber (PVA, $F_a = 0$)

are also in good agreement with the work of Batou and Adhikari [319] (Fig. 8 classical TMD). Further, from Table 2.1, one can observe that ζ_{opt} for the PVA by FPT methods are also in good agreement with the published work [20, 56-58, 100-103, 122-126]. From Fig.



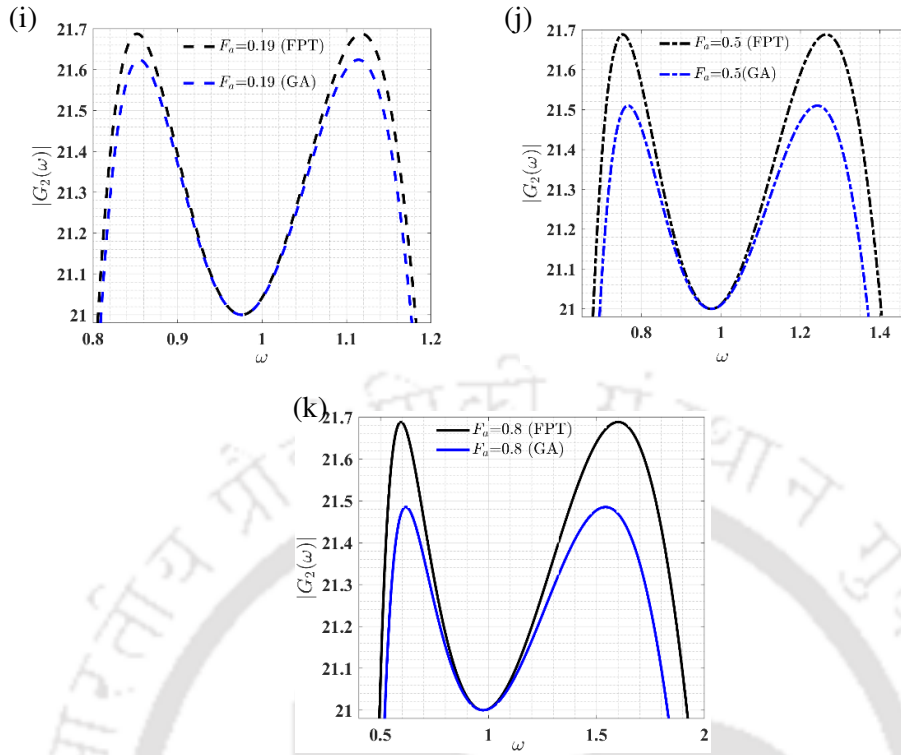


Fig. 2.5: Frequency response of the system with the optimized AVA (a) primary system (b) AVA (c-h) comparison of responses at the peaks and valleys of the primary system by FPT and GA techniques for different F_a values, (i-k) comparison of responses at the peaks and valleys of the absorber by FPT and GA techniques for different F_a values.

2.5(a) one can also observe that the response amplitude of the primary system significantly reduces from $F_a = 0$ to $F_a = 0.19$ but this large reduction in the response amplitude is not

Table 2.1: Optimized damping ratio (ζ_{opt}) values for different feedback gains (F_a) by using FPT and GA optimization.

Active control force (F_a)	ζ_{opt} by FPT	ζ_{opt} by GA
0	0.1336	0.133
0.19	0.2928	0.295
0.5	0.4432	0.453
0.8	0.5510	0.565

observed when F_a is increased from 0.5 to 0.8. These observations suggest that one can obtain the optimal feedback gain for which the total vibrational energy of the primary system is minimum. The optimal feedback gain F_a at which the total vibration amplitude of the primary system is minimum is shown in Fig. 2.6(a) and Fig. 2.6(b). In Fig. 2.6(a)

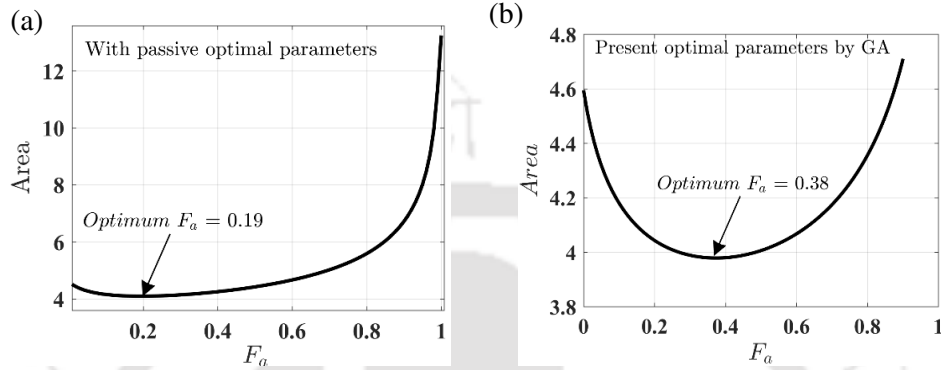


Fig. 2.6: The vibrational energy of the primary system with (a) passive optimal parameters (b) present optimal parameters.

the optimal feedback gain F_a is obtained by using passive optimal parameters $\omega_r = 1/(1+\mu)$ and $\zeta_2 = \sqrt{3\mu/8(1+\mu)}$. The total vibrational energy shown in Fig. 2.6(a) is calculated from Eq. 2.7, which is nothing but the area under the frequency response curve. From this figure, it is found that for $F_a = 0.19$ the area under the frequency response curve is minimum, which corresponds to the optimal feedback gain. In the same way for the proposed AVA using the obtained optimal tuning ratio and damping ratio obtained from Eq. (2.16) and Eq. (2.22), the corresponding optimal feedback gain F_a is obtained and shown in Fig. 5(b). From this figure, the optimal F_a at which the vibrational energy of the primary system is minimum is found to be 0.38. Figures 2.7(a) and 2.7(b) provide a comparison of the PVA and AVA with GA optimized parameters, taking into account how quickly the mean fitness value converges to optimal fitness and the corresponding fitness value. From Fig. 2.7(a) the best fitness value by the PVA is found to be around 6.46 and the mean fitness value starts from 28 which requires more time to converge to the best fitness value than the proposed AVA, which is shown in Fig. 2.7(b). From Fig. 2.7(b) the best fitness value is around 1.66 and the mean fitness value starts from 2.9 which is

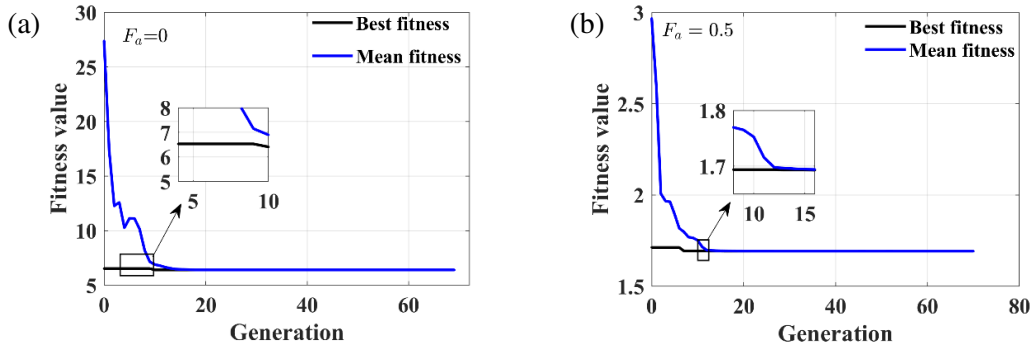


Fig. 2.7: Fitness value of the primary system with optimized (a) PVA (b) AVA.

converged faster than PVA (Fig.2.7(a)). The fitness value shown in Fig 2.7 can be termed as the maximum response amplitude of the primary system. From Fig. 2.7(b) it can be noticed that with the AVA, the response of the primary system is nearly one-fourth of that with PVA (Fig. 2.7(a)). In Fig. 2.8(a) and Fig. 2.8(b), the step response of the system is shown with PVA and AVA, respectively. From these figures, one can observe the settling time required to reach the steady-state response is more in the case of PVA (Fig. 2.8(a)) than with AVA (Fig. 2.8(b)). The response of the primary system improves tellingly in all aspects with feedback gain as illustrated in Fig. 2.8(b) than with PVA (Fig. 2.8(a)). From Fig. 2.8(b) it can be observed that with GA (black color) the primary system response is marginally better than the FPT optimization technique (blue color). In Fig 2.9 the root locus plot of the system is shown for PVA and AVA. All four poles of the system have negative real parts, as shown in Fig. 2.9(a), but the roots are marginally stable with the PVA because the solution approaches zero much more closely than with the AVA (Fig. 2.9(b)), where

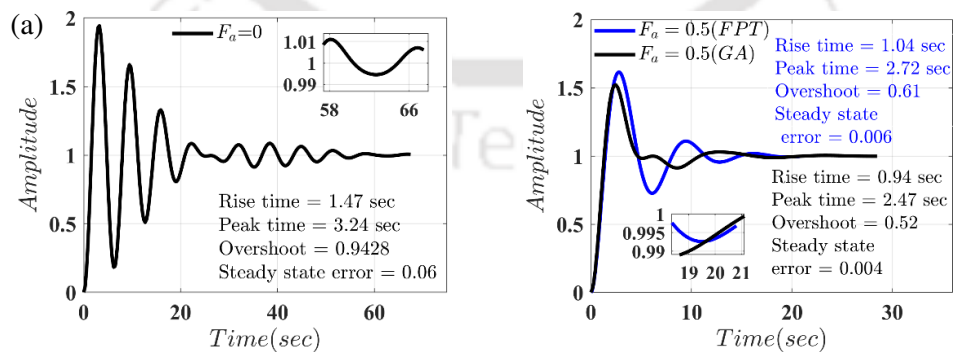


Fig. 2.8: Step response of the primary system with the optimal parameter using (a) PVA (b) AVA.

for $F_a = 0.5$ is considered. The two complex conjugate poles of the primary system transfer function (Eq. 2.7) with PVA are crossing the real axis at -0.02 whereas with AVA these poles are at -0.36. From these figures, one can observe that with feedback gain by the proposed AVA the primary system is more stable than with PVA, and it reduces the response amplitude of the primary system significantly.

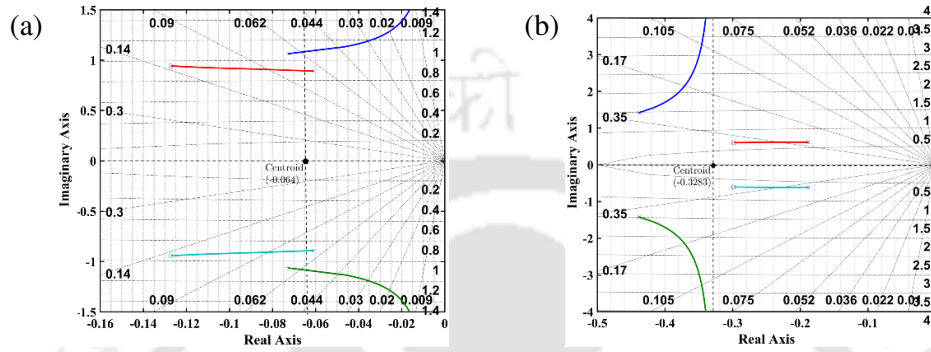


Fig. 2.9: Root locus plot of the primary system (a) PVA (b) AVA.

2.6 Summary

In this work, the fixed point theory (FPT) and genetic algorithm (GA) optimization techniques are used to obtain the optimum tuning and damping ratio of a PZT stack actuator-based SDOF spring-mass-damper active vibration absorber (AVA) with acceleration feedback to suppress resonant vibration of a SDOF spring-mass-damper primary system. The primary system is subjected to external harmonic force excitation. The closed-form solutions are obtained by using the FPT technique for the optimized tuning ratio and damping ratio in terms of the mass ratio (μ) between the absorber and the primary system for the proposed AVA with acceleration feedback. The optimum feedback gain is also evaluated and significant vibration suppression in terms of total energy is achieved. From the previous literature, the optimized damping ratio and tuning ratio for the PZT based AVA with acceleration feedback was not found. So the proposed AVA uses acceleration feedback which is easier to implement in many applications in the form of an accelerometer compared to other feedbacks. The optimum damping ratio by the GA takes more time to converge for different feedback gains by acceleration feedback, so one can use the obtained close form solution for the optimal damping ratio by the FPT optimization technique.



CHAPTER 3

Dynamics of ANVA using Time Delay Acceleration Feedback

3.1 Introduction

In this chapter, dynamic analysis of an active nonlinear vibration absorber (ANVA) using a lead zirconate titanate (PZT) stack actuator has been carried out considering time delay in the acceleration feedback. Here the primary system is modeled as a harmonically base excited system with a nonlinear spring, damper and mass which is subjected to an external harmonic force. The smart absorber consists of a nonlinear spring, mass and damper system along with a linear spring connected in series with the PZT stack actuator. In the proposed model the feedback gain is produced by the combination of spring and the PZT stack actuator which requires less voltage compared to that of the conventional system where the actuator is directly connected to the primary system and one can tune the frequency ratio of the absorber actively. The nonlinear governing equation of motion of the system is derived and solved by using a modified harmonic balance method (HBM) and presented in section 3.2. The steady-state response is obtained by using Newton's method and the stability of the system is studied using the reduced equations. It has been shown that the proposed novel vibration absorber which has an inbuilt fail-safe design can absorb the vibration of the system more effectively with negligible damping than the available passive and active vibration absorbers. Also, Den Hartog's equal peaks are achieved when the primary system is subjected to both harmonic force and base excitation. These studies are given in the results and discussion section (section 3.3).

It is observed that most of the works on the DVAs are based on linear primary and secondary (absorber) systems and very few researchers have studied the nonlinear vibration absorber to absorb the vibration of the nonlinear primary system. Though several studies have been made on PZT patch actuators for vibration suppression, in very few literature PZT stack actuators are used [144, 145, 154, 281, 285]. Though accelerometers are commonly used for vibration measurement [38] most of the studies considered displacement or velocity feedback [5, 20, 30,

31, 122-131, 144, 145, 154, 215, 216, 221, 251, 290, 320] instead of acceleration feedback of the system which shows better vibration suppression [321-324]. Hence, few studies have been made on acceleration feedback. Further, in most cases, the time delay in the feedback between sensing and actuation is ignored. In a few works, both the external forcing and base excitation are considered to obtain the Den Hartog's equal peaks for the nonlinear primary and secondary systems. Hence, in the present chapter, an attempt has been made to develop a novel vibration absorber for suppressing the vibration of the nonlinear primary system with a nonlinear absorber containing a linear spring with the PZT stack actuator and time delay acceleration feedback system. In the next section, the mathematical modeling of the proposed ANVA is presented.

3.2 Mathematical Modeling of the System

Figure 3.1 shows the PZT stack actuator-based active nonlinear vibration absorber (ANVA) connected to a harmonically base excited single degree of freedom primary system with a nonlinear spring, mass and a damper. In this figure m_i , c_i , k_i , and k_{i3} denote mass, damping, linear and nonlinear stiffness of the primary system ($i = 1$) and the absorber ($i = 2$). The primary system is subjected to an external harmonic force $F_{11} \cos(\Omega_1 t)$ and a base excitation $Y_0 \cos(\Omega_2 t)$. The feedback gain f_c is produced by the combination of a PZT stack actuator with stiffness k_p^E and a linear spring with stiffness k_3 . The governing equations of motion of the system in terms of the displacements of the primary system (x_1) and the secondary system (x_2) are described by two ordinary coupled nonlinear differential equations of motion using Newton's second law or d'Alembert's principle which are given below.

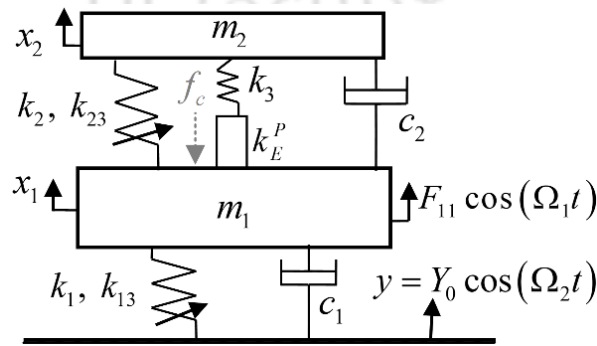


Fig. 3.1: PZT stack actuator based active nonlinear vibration absorber.

$$m_1 \ddot{x}_1 + c_1 (\dot{x}_1 - \dot{y}) + c_2 (\dot{x}_1 - \dot{x}_2) + k_1 (x_1 - y) + k_{13} (x_1 - y)^3 + k_2 (x_1 - x_2) + k_{23} (x_1 - x_2)^3 = F_{11} \cos(\Omega_1 t) - f_c \quad (3.1)$$

$$m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) + k_{23} (x_2 - x_1)^3 = f_c \quad (3.2)$$

In the absence of feedback gain f_c , spring stiffness k_3 and base excitation, Eqs. (3.1) and (3.2) can be reduced to that of Cirillo et al. [169] and Habib et al. [162]. The feedback gain produced by the PZT stack actuator and the spring stiffness k_3 in series may be written as $f_c = k_r (x_1 + \delta_0 - x_2)$ where the equivalent stiffness $k_r = (k_3 k_p^E) / (k_3 + k_p^E)$, and δ_0 is the nominal displacement of the PZT stack actuator at the juncture of the primary system. The nominal displacement of the actuator can be written as $\delta_0 = n d_{33} v$ [20], where, n , d_{33} and v denote the number of wafers in the PZT actuator, dielectric charge constant and the voltage applied to the actuator, respectively. The acceleration of the primary system can be measured by a separate non-contact type accelerometer and by considering a time delay of τ in the acceleration feedback the actuation is provided by the PZT actuator. The voltage applied to the PZT actuator by the time delay τ in the acceleration feedback can be written as $v = k_c (\ddot{x}_1 (t - \tau))$, where, k_c denote controller gain in the acceleration feedback. Substituting the value of v in f_c , the modified equation can be written as follows.

$$f_c = k_r (x_1 - x_2 + n d_{33} k_c \ddot{x}_1 (t - \tau)) \quad (3.3)$$

Equations (3.1) and (3.2) are rewritten below in the non-dimensional form by assuming non-dimensional time $\tau_1 = \omega_1 t$, where, $\omega_1 = \sqrt{k_1 / m_1}$.

$$\begin{aligned} \ddot{u}_1 + 2\zeta_1 \dot{u}_1 + 2\zeta_1 \Omega Y \sin(\Omega \tau_1 - \gamma) - 2\zeta_2 \dot{u}_2 + u_1 + \alpha_{13} u_1^3 - (\alpha + \alpha_r) u_2 - \beta u_2^3 = F_1 \cos \Omega \tau_1 \\ + Y \cos(\Omega \tau_1 - \gamma) + \alpha_{13} (Y \cos(\Omega \tau_1 - \gamma))^3 + 3\alpha_{13} \left(\frac{u_1^2 Y \cos(\Omega \tau_1 - \gamma) -}{u_1 (Y \cos(\Omega \tau_1 - \gamma))^2} \right) - F_{c1} \ddot{u}_1 (\tau_1 - \tau) \end{aligned} \quad (3.4)$$

$$\mu \ddot{u}_2 + 2\zeta_2 \dot{u}_2 + (\alpha + \alpha_r) u_2 + \beta u_2^3 = F_{c1} \ddot{u}_1 (\tau_1 - \tau) - \mu \ddot{u}_1 \quad (3.5)$$

where

$$u_1 = x_1 / x_0, u_2 = (x_2 - x_1) / x_0, \mu = \frac{m_2}{m_1}, \zeta_1 = \frac{c_1}{2m_1\omega_1}, \zeta_2 = \frac{c_2}{2m_1\omega_1}, \alpha = \frac{k_2}{k_1}, \alpha_r = \frac{k_r}{k_1}, Y = Y_0 / x_0,$$

$$\alpha_{13} = \frac{k_{13}x_0^2}{k_1}, \beta = \frac{k_{23}x_0^2}{k_1}, F_1 = \frac{F_{11}}{m_1\omega_1^2 x_0}, F_{c1} = \alpha_r k_c n d_{33}, \Omega = \frac{\Omega_1}{\omega_1}, \Omega_2 t = \frac{\Omega_2 \tau_1}{\omega_1} = \Omega \tau_1 - \gamma$$

It may kindly be noted that the term $\Omega_2 t = \frac{\Omega_2 \tau_1}{\omega_1}$ is assumed to be $\Omega \tau_1 - \gamma$ and $x_0 =$ reference length. Here the frequency of the base excitation term is chosen in such a way that there is a phase difference γ between F_1 and Y . It may be noted that Eqs. (3.4) and (3.5) are written by considering the relative displacement of the absorber i.e, $u_2 = (x_2 - x_1) / x_0$, by which the complexity associated with the expansion of the nonlinear term $(x_2 - x_1)^3$ reduces and one can easily obtain the solution. The approximate solution of Eqs. (3.4) and (3.5) are discussed in the following section.

3.2.1 Approximate solution by HBM

In this section, the steady-state solution of the Eqs. (3.4) and (3.5) are obtained by using the harmonic balance method (HBM). Here the solution of u_1 and u_2 are written using the slowly varying function of time A , B , φ_1 and φ_2 as given below.

$$u_1 = A(\tau_1) \cos(\Omega \tau_1 - \varphi_1(\tau_1)) \quad (3.6)$$

$$u_1(\tau_1 - \tau) = A(\tau_1) \cos(\Omega(\tau_1 - \tau) - \varphi_1(\tau_1 - \tau)) \quad (3.7)$$

$$u_2 = B(\tau_1) \cos(\Omega \tau_1 - \varphi_2(\tau_1)) \quad (3.8)$$

It may be noted that in the conventional HBM [12] generally the coefficients are taken to be constants. But here, both the amplitudes ($A(\tau_1)$, $B(\tau_1)$) and the phases ($\varphi_1(\tau_1)$, $\varphi_2(\tau_1)$) are considered to be slowly varying functions of time τ_1 such that one can neglect the following terms: $\ddot{A}$, $\ddot{\varphi}_1$, $\ddot{B}$, $\ddot{\varphi}_2$, $\dot{A}\dot{\varphi}_1$, $\dot{B}\dot{\varphi}_2$, $\dot{\varphi}_1^2$, $\dot{\varphi}_2^2$. Substituting Eqs. (3.6) to (3.8) into

Eqs. (3.4) and (3.5) and equating coefficient of $\sin \Omega \tau_1$ and $\cos \Omega \tau_1$ terms, Eqs. (3.4) and (3.5) can be written in the following compact matrix form:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \begin{Bmatrix} \dot{A} \\ \dot{\phi}_1 \\ \dot{B} \\ \dot{\phi}_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix} \quad (3.9)$$

where a_1 to a_{16} and b_1 to b_4 are given as

$$\begin{aligned} a_1 &= -2\Omega \cos \phi_1 + 2\zeta_1 \sin \phi_1 + 2F_{c1}\Omega \sin \phi_1 \sin \Omega \tau - 2F_{c1}\Omega \cos \phi_1 \cos \Omega \tau, \\ a_2 &= 2\zeta_1 A \cos \phi_1 + 2\Omega A \sin \phi_1 + 2F_{c1}A\Omega \sin \phi_1 \cos \Omega_d + 2F_{c1}A\Omega \cos \phi_1 \sin \Omega \tau, \\ a_3 &= -2\zeta_2 \sin \phi_2, \quad a_4 = -2\zeta_2 B \cos \phi_2, \\ a_5 &= 2\Omega \sin(\phi_1) + 2\zeta_1 \cos(\phi_1) + 2F_{c1}\Omega \sin \phi_1 \cos \Omega \tau + 2F_{c1}\Omega \cos \phi_1 \sin \Omega \tau, \\ a_6 &= -2\zeta_1 A \sin \phi_1 + 2\Omega A \cos \phi_1 - 2F_{c1}A\Omega \sin \phi_1 \sin \Omega \tau + 2F_{c1}A\Omega \cos \phi_1 \cos \Omega \tau, \\ a_7 &= -2\zeta_2 \cos \phi_2, \quad a_8 = 2\zeta_2 B \sin \phi_2, \\ a_9 &= -2\mu\Omega \cos \phi_1 - 2F_{c1}\Omega \sin \phi_1 \sin \Omega \tau + 2F_{c1}\Omega \cos \phi_1 \cos \Omega \tau, \\ a_{10} &= 2\mu\Omega A \sin \phi_1 - 2F_{c1}A\Omega \sin \phi_1 \cos \Omega \tau - 2F_{c1}A\Omega \cos \phi_1 \sin \Omega \tau, \\ a_{11} &= -2\mu\Omega \cos \phi_2 + 2\zeta_2 \sin \phi_2, \\ a_{12} &= 2\zeta_2 B \cos \phi_2 + 2\mu\Omega B \sin \phi_2, \\ a_{13} &= 2\mu\Omega \sin \phi_1 - 2F_{c1}\Omega \sin \phi_1 \cos \Omega \tau - 2F_{c1}\Omega \cos \phi_1 \sin \Omega \tau, \\ a_{14} &= 2\Omega \mu A \cos \phi_1 + 2F_{c1}A\Omega \sin \phi_1 \sin \Omega \tau - 2F_{c1}A\Omega \cos \phi_1 \cos \Omega \tau \\ a_{15} &= 2\mu\Omega \sin \phi_2 + 2\zeta_2 \cos \phi_2, \\ a_{16} &= -2\zeta_2 B \sin \phi_2 + 2\mu\Omega B \cos \phi_2, \\ b_1 &= N_1 \cos \phi_1 + N_2 \sin \phi_1 + N_3 \cos \phi_2 + N_4 \sin \phi_2 + N_{b1}, \\ b_2 &= -N_1 \sin \phi_1 + N_2 \cos \phi_1 - N_3 \sin \phi_2 + N_4 \cos \phi_2 - F_1 + N_{b2}, \\ b_3 &= N_5 \cos \phi_2 + N_6 \sin \phi_2 + N_7 \sin \phi_1 + N_8, \\ b_4 &= -N_5 \sin \phi_2 + N_6 \cos \phi_2 + N_7 \cos \phi_1 + N_9. \end{aligned}$$

From Eq. (3.9) the following amplitude and phase equations are obtained.

$$\dot{A} = f_1(A, B, \phi_1, \phi_2), \quad (3.10)$$

$$\dot{\phi}_1 = f_2(A, B, \phi_1, \phi_2), \quad (3.11)$$

$$\dot{B} = f_3(A, B, \varphi_1, \varphi_2), \quad (3.12)$$

$$\dot{\varphi}_2 = f_4(A, B, \varphi_1, \varphi_2). \quad (3.13)$$

The steady-state solutions of the slow-flow equations are obtained from Eq. (3.9) by taking $\dot{A} = \dot{\varphi}_1 = \dot{B} = \dot{\varphi}_2 = 0$ and the corresponding equations are given below.

$$N_1 \cos \varphi_1 + N_2 \sin \varphi_1 + N_3 \cos \varphi_2 + N_4 \sin \varphi_2 + N_{b1} = 0 \quad (3.14)$$

$$-N_1 \sin \varphi_1 + N_2 \cos \varphi_1 - N_3 \sin \varphi_2 + N_4 \cos \varphi_2 - F_1 + N_{b2} = 0 \quad (3.15)$$

$$N_5 \cos \varphi_2 + N_6 \sin \varphi_2 + N_7 \sin \varphi_1 + N_8 = 0 \quad (3.16)$$

$$-N_5 \sin \varphi_2 + N_6 \cos \varphi_2 + N_7 \cos \varphi_1 + N_9 = 0 \quad (3.17)$$

where, N_1 to N_9 , N_{b1} and N_{b2} are given as

$$\begin{aligned} N_1 &= -2\zeta_1 A \Omega, \quad N_2 = (-\Omega^2 + 1)A, \quad N_3 = 2\zeta_2 B \Omega, \quad N_4 = -(\alpha B + 0.75\beta B^3), \\ N_{b1} &= -\alpha_r B \sin \varphi_2 + 0.75\alpha_{13} A^3 \sin \varphi_1 - F_{c1} A \Omega^2 \sin(\varphi_1 + \Omega \tau) \\ &\quad - 0.75 A^2 Y \sin(2\varphi_1) \cos(\gamma) + 1.5\alpha_{13} A Y^2 \sin \varphi_1 - 0.75\alpha_{13} A Y^2 \sin \varphi_1 \cos(2\gamma) \\ &\quad - Y \sin(\gamma) - 0.75\alpha_{13} Y^3 \sin(\gamma) - 1.5\alpha_{13} A^2 Y \sin(\gamma) + 2\zeta_1 \Omega Y \cos(\gamma) \\ &\quad + 0.75\alpha_{13} A^2 Y \cos(2\varphi_1) \sin(\gamma) - 0.75\alpha_{13} A Y^2 \sin(2\gamma) \cos(\varphi_1), \\ N_5 &= -2\zeta_2 B \Omega, \quad N_6 = -\mu \Omega^2 B + \alpha B + 0.75\beta B^3, \quad N_7 = -\mu \Omega^2 A, \\ N_{b2} &= \alpha_r B \cos \varphi_2 + 0.75\alpha_{13} A^3 \cos \varphi_1 - F_{c1} A \Omega^2 \cos(\varphi_1 + \Omega \tau) - Y \cos(\gamma) \\ &\quad - 0.75\alpha_{13} Y^3 \cos(\gamma) - 1.5\alpha_{13} A^2 Y \cos(\gamma) - 0.75\alpha_{13} A^2 Y \cos(\gamma) \cos(2\varphi_1) \\ &\quad + 1.5\alpha_{13} A Y^2 \cos(\varphi_1) + 0.75\alpha_{13} A Y^2 \cos(\varphi_1) \cos(2\gamma) - 2\zeta_1 \Omega Y \sin(\gamma) \\ &\quad + 0.75\alpha_{13} A^2 Y \sin(2\varphi_1) \sin(\gamma) + 0.75\alpha_{13} A Y^2 \sin(2\gamma) \sin(\varphi_1), \\ N_8 &= -\alpha_r B \sin \varphi_2 + F_{c1} A \Omega^2 \sin(\varphi_1 + \Omega \tau), \quad N_9 = -\alpha_r B \cos \varphi_2 + F_{c1} A \Omega^2 \cos(\varphi_1 + \Omega \tau). \end{aligned}$$

It may be noted that in the conventional HBM one has to do the stability analysis using the original equation after getting the response [12] but in the present case, both the response

and stability analysis can be performed simultaneously by using the reduced Eqs. (3.10) to (3.13). The stability of the steady-state equations of the slow-flow equation thus obtained is ensured by the negative real part of the eigenvalues of the Jacobian matrix which is given below.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial A} & \frac{\partial f_1}{\partial \phi_1} & \frac{\partial f_1}{\partial B} & \frac{\partial f_1}{\partial \phi_2} \\ \frac{\partial f_2}{\partial A} & \frac{\partial f_2}{\partial \phi_1} & \frac{\partial f_2}{\partial B} & \frac{\partial f_2}{\partial \phi_2} \\ \frac{\partial f_3}{\partial A} & \frac{\partial f_3}{\partial \phi_1} & \frac{\partial f_3}{\partial B} & \frac{\partial f_3}{\partial \phi_2} \\ \frac{\partial f_4}{\partial A} & \frac{\partial f_4}{\partial \phi_1} & \frac{\partial f_4}{\partial B} & \frac{\partial f_4}{\partial \phi_2} \end{bmatrix} \quad (3.18)$$

The steady-state solutions of the slow-flow Eqs. (3.14) to (3.17) are solved by using Newton's method and are discussed in the following section.

3.3 Results and Discussions

In this section, the performance of the active nonlinear vibration absorber (ANVA) is studied for a mass ratio of 1:20 (i.e., $\mu = 0.05$) by using the frequency response curves (FRC), time responses and phase portraits for various system parameters such as cubic nonlinear stiffness in the primary system α_{13} , cubic nonlinear stiffness of the absorber β , feedback gain F_{c1} , time delay τ , amplitude of base excitation Y , the amplitude of harmonic force excitation F_1 and stiffness of the spring k_3 between the PZT stack actuator and the absorber. The FRC for the primary system and the ANVA are found by numerically solving the four equations (Eqs. (3.14 - 3.17)) by using Newton's method. It may be noted that these equations are obtained by using HBM. These results are initially compared with the developed SIMULINK model (Appendix-A) by solving Eqs. (3.4) and (3.5). Further, it is compared by studying Eqs. (3.4) and (3.5) using the fourth-order Runge Kutta method. The MATLAB command *ode45* is used for the system without delay and *dde23* is used for the system with delay. The stiffness and damping parameters for the ANVA are taken from the work of Habib et al. [161], and Mallik and Chatterjee [20] have been referred for the

properties of PZT stack actuator. With reference to Eq. (3.4) and (3.5), the nondimensional natural frequency of the primary system is $\omega_1 = 1$, the damping ratio of the primary system $\zeta_1 = 0.001$ and for the absorber is $\zeta_2 = 0.0064$, the linear stiffness ratio between the absorber and the primary system is $(\alpha + \alpha_r) = 0.0454$, external forcing amplitude $F_1 = 0.1$ and $x_0 = 1$. The amplitude of the external forcing F_1 is considered to be equal to 0.1 in the sub-section 3.1 - 3.5, 3.7 and 3.8 and in other sub-sections F_1 values are varied from 0.0005 to 1 while all other system parameters are kept the same in all sections as mentioned above.

In sub-section 3.3.1 - 3.3.6, 3.3.8 and 3.3.9 frequency responses, time responses phase portraits, Poincare's section and basin of attractions of the system are studied by considering only the external harmonic excitation $F_1 \cos \Omega \tau_1$ and hence the amplitude of harmonic base excitation Y is considered to be zero. In other sub-sections, 3.3.7, 3.3.8 and 3.3.10 FRC of the system are studied for both external harmonic and base excitation. In the following subsection, the frequency response of the passive linear system is studied.

3.3.1 Passive linear system

The FRC of the primary system and the ANVA in the absence of any nonlinearity are shown in Fig. 3.2 with the optimal system parameters taken from Habib et al. [161]. The results obtained in the frequency responses are compared with the numerical simulations using the Simulink model and they are found to be in good agreement as shown in Fig. 3.2(a) and 3.2(b). From Fig. 3.2(a) it is observed that the amplitudes of the two resonating peaks are the same and equal to 0.62 at a frequency $\Omega = 0.89$ and 1.05. The response amplitude of the primary system at resonance (i.e., $\Omega = 1$) is 0.54. The absorber response is shown in Fig. 3.2(b) where the response amplitude at the peaks and the valley are 2.102 and 2.05, respectively. These results are similar to earlier theoretical analysis [216, 161, 12] and experimental studies [65, 325]. In the literature [38, Fig. 3.2(a)] the amplitude of external excitation is considered as $F_1 = 0.001$ and they observed linear behavior with a maximum amplitude of 0.0062. In the present work, the forcing amplitude $F_1 = 0.1$, (100 times more) and the corresponding response amplitude is found to be 0.62, which is also 100 times more than the response amplitude in [161]. Hence both results are consistent as per the

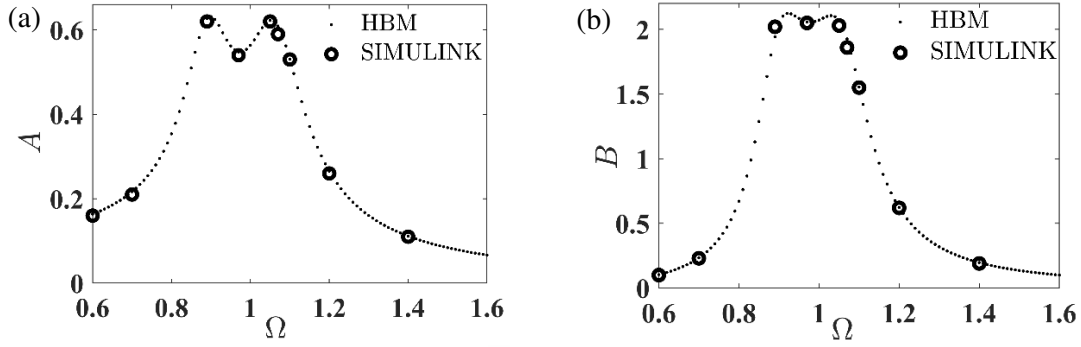


Fig. 3.2: Frequency response of the (a) primary system and the (b) absorber.

superposition rule for the linear system. This validates the use of the harmonic balance method (HBM) formulation for the linear case and validation for the nonlinear cases is given in subsequent sections. In the next sub-section, the effect of cubic nonlinear stiffness in the absorber on the FRC is studied.

3.3.2 Effects of cubic nonlinear stiffness in PVA

The FRC of the primary system and the passive absorber are shown in Fig. 3.3 for two values of cubic nonlinear stiffness β equal to 2.2% (0.001) and 22% (0.01) of the linear stiffness of the absorber. It is observed from Fig. 3.3(a) that for $\beta = 0.001$ the first resonating peak is at 0.86 for $\Omega = 0.92$ and the second peak is at 0.53 for $\Omega = 1.1$. Similarly, the response amplitudes of 2.49 and 1.73 are observed for the absorber (Fig. 3.3(b)) at the same frequencies. From Fig. 3.2 and Fig. 3.3(a, b) it is observed that when the operating frequency Ω is greater than 1, the response of the system is better in Fig. 3.3

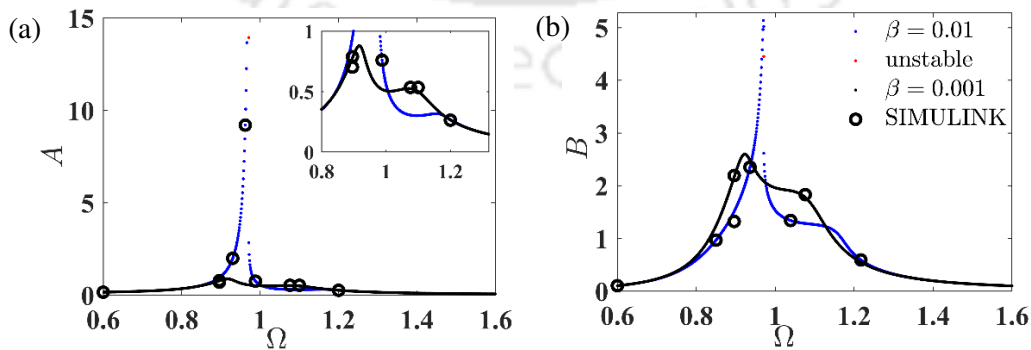


Fig. 3.3: Frequency response curve of the system for $\beta = 0.001$ and $\beta = 0.01$.

which reduced the second peak response 15% more than the linear system. For $\beta = 0.01$ it is observed from Fig. 3.3(a) that the response amplitude of the primary system increased sharply from 0.9 to 1.5 for Ω in the range of 0.9 to 0.972. Similar frequency response for the absorber is observed in Fig. 3.3(b). The second peak amplitude of the primary system and the absorber are observed at $\Omega = 1.15$ with amplitudes of 0.31 and 1.12, respectively. The unstable solution in the frequency response is observed for $\Omega = 0.971$. It is inferred from Fig. 3.3 that for β equals to 2.2% or 0.001 the system amplitude reduces by 15% in the second peak, however, an increase in β value to 22% or 0.01 makes the equilibrium points unstable and creates a larger response in the system. Similar responses are also observed for β equals 10% and 15% of linear stiffness. Hence the cubic nonlinear stiffness value in the absorber must be less than 2.2% of the absorber's linear stiffness. So one may investigate the use of an active vibration absorber (AVA) if the nonlinearity is more than 2.2%. As shown in this figure, the results obtained from HBM and Newton's method are also compared with the numerical simulations and are found to be in good agreement.

3.3.3 Effects of cubic nonlinear stiffness in the primary system

The FRC of the primary system and the passive absorber are shown in Fig. 3.4 for cubic nonlinear stiffness in the primary system α_{13} equal to 10% i.e., 0.1 and 100% i.e., 1 of the linear stiffness of the primary system. The FRC of the system obtained by Newton's method is also compared numerically by the fourth-order Runge Kutta method by solving Eq. (3.4) and (3.5) showing a good match with each other. For $\alpha_{13} = 0.1$, it is observed from Fig. 3.4(a) and Fig. 3.2(a), that the response amplitude in the first resonant peak decreases from 0.62 to 0.57 and at the second peak it increases from 0.62 to 0.66 than the linear primary system. The absorber frequency response is shown in Fig. 3.4(b) which showed similar characteristics as the primary system. For $\alpha_{13} = 1$, in Fig. 3.4(a) one can observe two saddle points at $\Omega = 1.18$ and $\Omega = 2.02$ showing jump up and jump down phenomena respectively. The second peak in the frequency response is at 1.7 due to the hardening effect in the frequency response while the first peak amplitude is 0.39 at $\Omega = 0.89$. The FRC of the passive absorber is shown in Fig. 3.4(b) where the same nonlinear behavior is observed as the primary system and the maximum amplitude response is found to be 2.32. The frequency response for Ω in the range of 1.18 to 2.02 shows two stable and one unstable

branch. Hence for nonlinear stiffness in the primary system below 10%, the system does not show any appreciable changes in the FRC than the linear system but when the primary system is highly nonlinear then the response amplitude and unstable region increase as evident from Fig. 3.4. It may be noted that the computational time

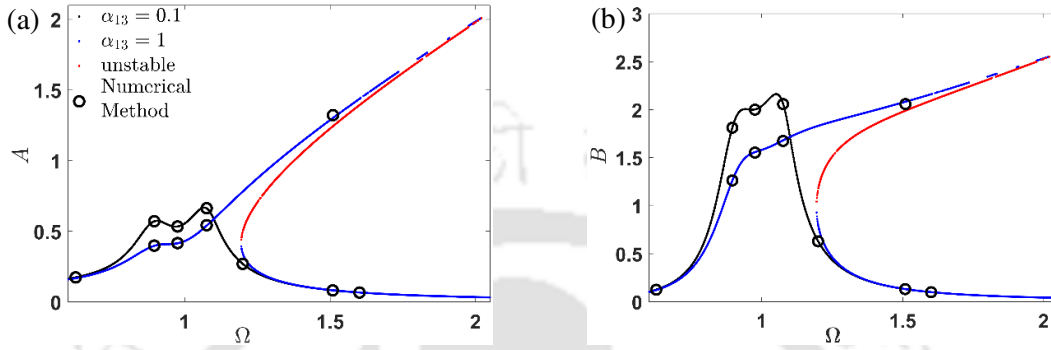


Fig. 3.4: Frequency response of the system for $\alpha_{13} = 0.1$ and $\alpha_{13} = 1$.

required to obtain the FRC by SIMULINK (Fig. 3.2 and 3.3) or the fourth-order Runge Kutta method (Fig. 3.4) is more as one has to get the response at each frequency point. Time taken to obtain a single point on the response curve takes 1 minute (intel 7 (3rd Gen), 3.12 GHz, 8 GB RAM, Windows 10), so to obtain complete FRC which consists of more than 100 points it will take more than 1 hour and 40 minutes, and also unstable points are difficult to obtain. Using the HBM method the computational time required to obtain the plot is only 6 minutes so, one can use the proposed HBM method to reduce the computational time and study the stability of the response. Unlike other perturbation methods where the small divisor terms in the equation motion are required to obtain the solution but in HBM one can use the procedure without ordering the equation. Further, in this modified HBM one can obtain the stability of the system by using the Jacobian matrix of the reduced equations. While the above-mentioned FRC shows the response of the system with the PVA, in the next section the study has been carried out for AVA by using the PZT stack actuator with acceleration feedback.

3.3.4 Active vibration absorber

In this section, the FRC of the proposed ANVA are found for four different cases namely, linear, nonlinear spring in the primary system, nonlinear spring in the absorber and

nonlinear spring in both the primary system and the absorber, with negative feedback gain $F_{c1} = 0.1$ and they are shown in Fig. 3.5. For the linear case, it is observed from Fig. 3.5(a) that with the active feedback gain (blue) the response amplitude decreases from 0.62 (Fig. 3.2(a)) to 0.49 in the first peak (21%) at $\Omega = 0.8$ and for second peak it decreases from 0.62

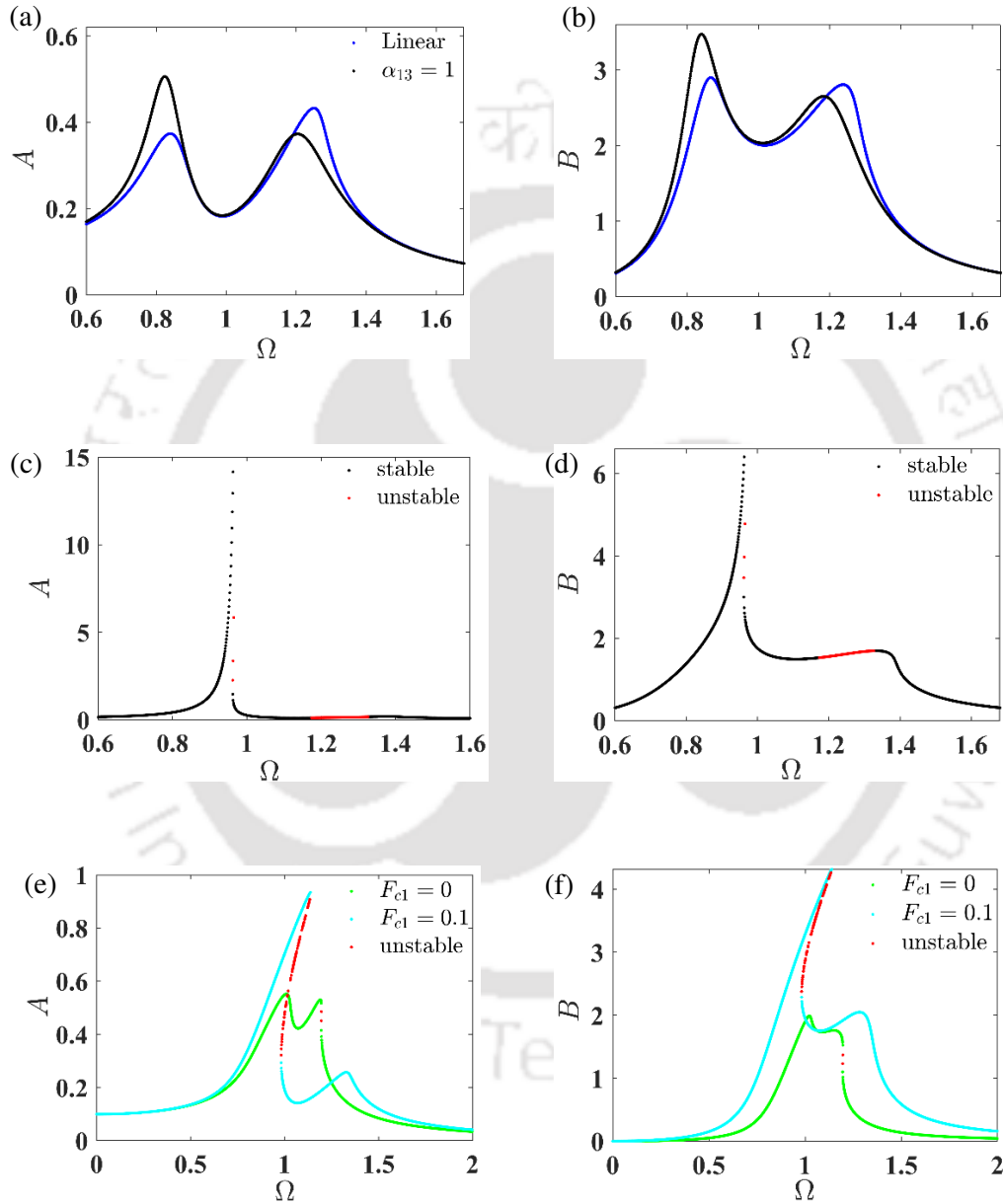


Fig. 3.5: Frequency response of the system (a, b) linear and nonlinear stiffness ($\alpha_{13} = 1$) in the primary system. (c, d) nonlinear stiffness ($\beta = 0.01$) in the absorber (e, f) nonlinear stiffness in both the primary system ($\alpha_{13} = 1$) and the absorber ($\beta = 0.0042$).

(Fig. 3.2(a)) to 0.34 (45%) at $\Omega = 1.2$. It may be observed that these peaks are occurring at different frequencies due to the presence of feedback gain in k_r . The response amplitude at the resonating frequency ($\Omega = 1$) decreases from 0.54 (i.e., in the passive case, Fig 3.2(a)) to 0.15 in the present active case which is a 72% reduction. But the response amplitude of the absorber increases from 2.4 to 3.6 (50%) for the first peak and from the same value to 2.6 (8%) in the second peak as shown in Fig. 3.5(b) and Fig. 3.2(b). Further considering a realistic system where the primary system is considered to be highly nonlinear ($\alpha_{13} = 1$), the FRC is shown in Fig. 3.5(a) (black). Unlike in Fig. 3.4(a) here, the frequency response of the primary system shows no unstable region and the amplitude of vibration reduces from 0.39 to 0.36 (7.7%) in the first peak and from 1.8 to 0.42 (77%) in the second peak. The FRC for the absorber (Fig. 3.5(b)) seems to be similar to the linear system and no unstable region is found unlike in the case of Fig. 3.4(b) but the amplitude of the response increases from 1.5 to 2.7 in the first peak and 2.3 to 2.8 in the second peak. Comparing the linear and the nonlinear responses it is observed that the response amplitude for the linear system decreases 34% in the first peak but at the second peak it increases 14% while, at the resonating frequency, the response amplitude remains the same as the nonlinear primary system. In Fig. 3.5(c, d) the FRC of the system is studied for $\beta = 0.01$. From Fig. 3.5(c) it is observed that by the applied feedback gain a small range of unstable zone is found when $0.962 < \Omega < 0.965$ with jump up and jump down phenomena. In addition, the equilibrium points show an unstable region when $1.195 < \Omega < 1.373$ showing Hopf bifurcation points at $\Omega = 1.195$ and 1.373 . The response amplitude of the nonlinear absorber is shown in Fig. 3.5(d) which increases as compared to Fig. 3.3(d). In Fig. 3.5(e, f) nonlinear stiffness in both the primary system ($\alpha_{13} = 1$) and the absorber ($\beta = 0.0042$) are considered. It may be noted that here, optimum nonlinear stiffness coefficients are considered from the work of Habib et al. [161]. It is observed from Fig. 3.5(e) that with the feedback gain $F_{c1} = 0.1$, the response amplitude of the primary system reduces for $1.136 < \Omega < 1.32$ and shows saddle-node bifurcation points at $\Omega = 0.982$ and 1.136 . Here, also it may be noted that at the resonant frequency i.e. $\Omega = 1$ the nonlinear system has two stable and one unstable solution out of which the lower stable solution has a value that is less than that found in [161], where the $F_{c1} = 0$. Also, $F_{c1} = 0$ gives less response amplitude than $F_{c1} = 0.1$ above $\Omega > 1.32$ and for $0.62 < \Omega < 0.98$. From Fig. 3.5(f) it is observed that the response amplitude of the absorber is more in the case of AVA than the PVA at the peaks.

From Fig. 3.5 it is observed that with the applied feedback gain considering only high nonlinear stiffness in the primary system there is much decrease in the unstable region of operating frequency and also the vibration of the primary system, however, the amplitude of the absorber increases at the peaks. In the following section nonlinear stiffness is considered in both the primary system and the absorber and the variation of feedback gain on the frequency response is studied.

3.3.5 Variation of feedback gain on the frequency response

The effect of four values of feedback gains F_{c1} viz 0 (green), 0.1 (cyan), 0.2 (black) and 0.5 (blue) on the FRC of the system is studied in this section.

3.3.5.1 Nonlinear stiffness in both the primary system and the absorber

In Fig. 3.6 (a, b) and (e, f) the FRC of the system is shown by considering two values of nonlinear spring stiffness in both the primary system and the absorber. In Fig. 3.6(a, b) the nonlinear stiffness α_{13} and β are considered equal to 0.1 and 0.001, respectively. From Fig. 3.6(a) it is observed that with $F_{c1} = 0$ the primary system shows two peaks with amplitude 0.69 and 0.54 at Ω equals to 0.92 and 1.08, respectively with no unstable zone. But by increasing F_{c1} values from 0 to 0.1, the amplitude of the two peaks shifts and occurs at Ω equals to 0.908 and 1.24. A minimum of 55% reduction in the vibration of the primary system is observed with $F_{c1} = 0.1$ than $F_{c1} = 0$ when $0.91 < \Omega < 2$. However, the system shows an unstable frequency zone with $F_{c1} = 0.1$, for a small range when $0.906 < \Omega < 0.91$ with two stable and one unstable branch. The jump up and jump down phenomena are observed with saddle-node bifurcation points at $\Omega = 0.906$ and 0.911. Further increasing F_{c1} values from 0.1 to 0.2 the primary system amplitude reduces to 0.003 (96%) than for $F_{c1} = 0$ when $\Omega > 0.96$, but at the same time more hardening effect in the FRC is observed with instability zone when $0.888 < \Omega < 0.956$. Two saddle points are observed at $\Omega = 0.888$ with the jump-up amplitude height 0.7 and at $\Omega = 0.956$ with the jump-down amplitude height 1.44 for $F_{c1} = 0.2$. It is observed from the FRC that with an increase in the F_{c1} at first peak the system amplitude increases and for a very small range of operating frequency the system shows an

unstable fixed-point solution but for a larger band the amplitude of the response reduces significantly. In Fig. 3.6(c, d) basin of attraction for the primary system is plotted at the points D ($\Omega = 0.82$) and E ($\Omega = 0.889$) of Fig. 3.6(a). From Fig. 3.6(c) at point D, one can observe a single stable point with a response amplitude of 0.64. Also in Fig. 3.6(d) at point E two stable points with response amplitude of 1.2 and 0.4, with a separatrix at response amplitude of 0.62 can be observed which are evident from the FRC curve shown in Fig. 3.6(a). In Fig. 3.6(b) the amplitude of the absorber for $F_{c1} = 0.1$ shows marginal higher

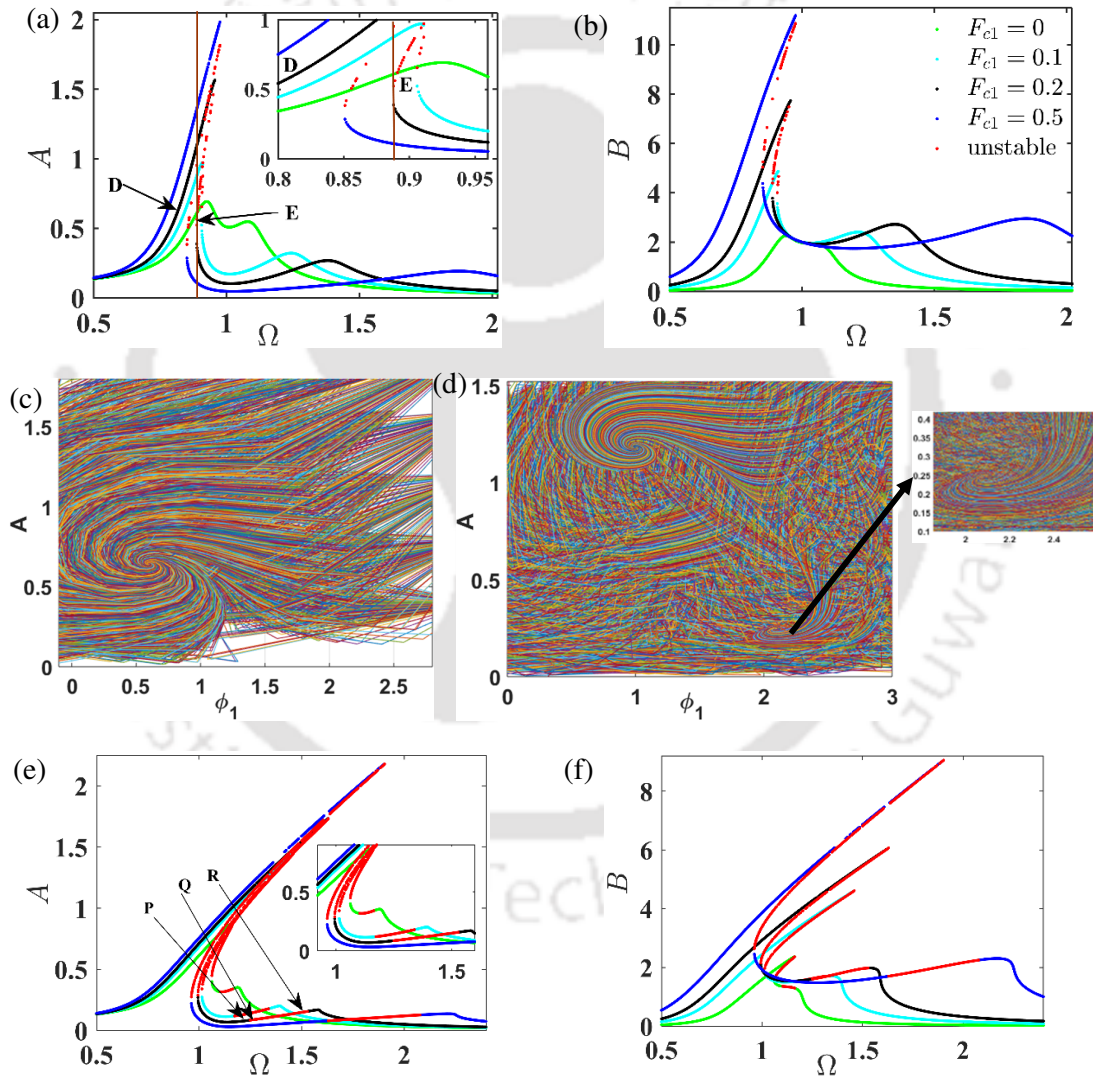


Fig. 3.6: Frequency response of the system for variation in feedback gain F_{c1} for (a, b) $\alpha_{13} = 0.1$ and $\beta = 0.001$. (c, d) Basin of attraction of the primary system at $\Omega = 0.82$ and 0.889 . Frequency response of the system with different F_{c1} for (e-f) $\alpha_{13} = 1$ and $\beta = 0.01$.

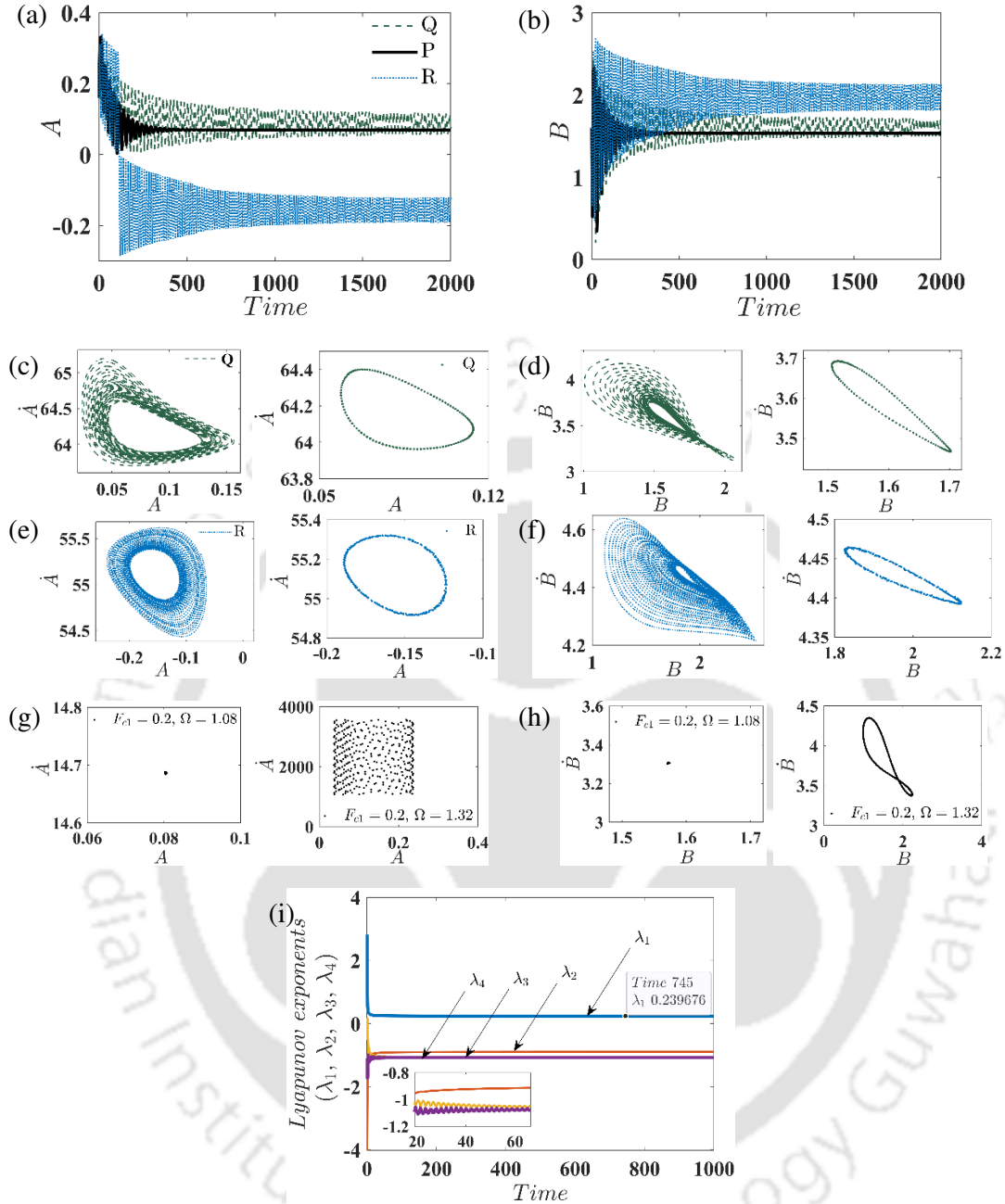


Fig. 3.7: Time responses (a, b), phase portraits and Poincaré's section (c, f) at the frequency points P ($\Omega=1.15$), Q ($\Omega=1.25$), and R ($\Omega=1.52$) for $F_{cl} = 0.2$. (g, h) phase portraits and Poincaré's section for $F_{cl} = 0.2$ at $\Omega=1.08$ and $\Omega = 1.32$ (i) Lyapunov exponents for $F_{cl} = 0.2$ at $\Omega = 1.08$ and $\Omega = 1.32$ for initial condition (1, 0, 1, 0)

amplitude for $\Omega > 1$ but for $\Omega < 0.9$ the absorber amplitude increases with the same unstable zone as discussed above. In Fig. 3.6(e, f) highly nonlinear system is considered with α_{13} and β equals to 1 and 0.01, respectively. In Fig. 3.6(e) for $F_{cl} = 0$ two high peaks with

amplitude equals to 0.87 and 0.35 at $\Omega = 1.16$ and 1.18, respectively with the stable and unstable solutions appear when $1.062 < \Omega < 1.162$ are shown. The jump-up amplitude of 0.32 and jump-down amplitude of 0.54 are observed at $\Omega = 1.062$ and 1.162, respectively. The Hopf bifurcation points are observed at $\Omega = 1.103$ and at 1.163. The response amplitude at the first peak of the primary system reduces from 0.87 to 0.56 with $F_{c1} = 0.1$ compares to when $F_{c1} = 0$ but creates a wider range of unstable zone with saddle-node bifurcation points at $\Omega = 1.014$ and 1.146 and Hopf bifurcation points at $\Omega = 1.172$ and 1.338. The same effects are also observed for $F_{c1} = 0.2$ and 0.5 with more hardening effect and unstable region. The amplitude of the primary system at the first peak ($\Omega = 1.016$) decreases 80% with an increase in F_{c1} from 0 to 0.5 but shows both stable and unstable fixed-point solutions. However, the jump up and jump down amplitude height increases with a more hardening effect. In Fig. 3.7(a, f) for $F_{c1} = 0.2$ (black line) the time responses, phase portraits and Poincare's section at point P ($\Omega = 1.15$, stable point), Q ($\Omega = 1.25$, unstable point) and R ($\Omega = 1.52$, unstable point) of Fig. 3.6 (e) are shown. From Fig. 3.7(a, b) time response, fixed point response is observed at point P, quasi-periodic responses are observed at points Q and R. From the phases portraits Poincare's section and Lyapunov exponents in Fig. 3.7(a-i) existence of periodic, quasiperiodic and chaotic responses in addition to fixed point responses are clearly observed for different system parameters. Hence one must choose the system parameters carefully to suppress the excessive vibration in the system.

3.3.5.2 Linear system

The FRC of the linear system with a variation of F_{c1} is shown in Fig. 3.8(a, b). It is observed from Fig. 3.8(a) that with higher F_{c1} the amplitude of the primary system reduces for all frequency of operation with no unstable zone and shows a significant decrease in the amplitude response at the second peak. The system with $F_{c1} = 0.5$ shows around 90 % reduction in vibration than $F_{c1} = 0$ for $\Omega > 0.85$ but at the same time the absorber amplitude (Fig. 3.8(b)) at the peaks increases. Hence from Fig. 3.8, it is observed that large vibration reduction in the primary system is achieved with higher values of F_{c1} when the system is linear. But when both the primary system and absorber are nonlinear (Fig. 3.6 and 3.7), then for a particular operating frequency higher feedback gain may not be effective. It is also

observed that for $\alpha_{13} > 10\%$ and $\beta > 2.2\%$ of the linear stiffness the system shows a wider unstable range of frequency. So, one has to choose F_{c1} values judiciously to suppress the vibration of the system for a broader range of operating frequencies. In the following section, the effects of variation in the amplitude of external force excitation on the FRC is studied.

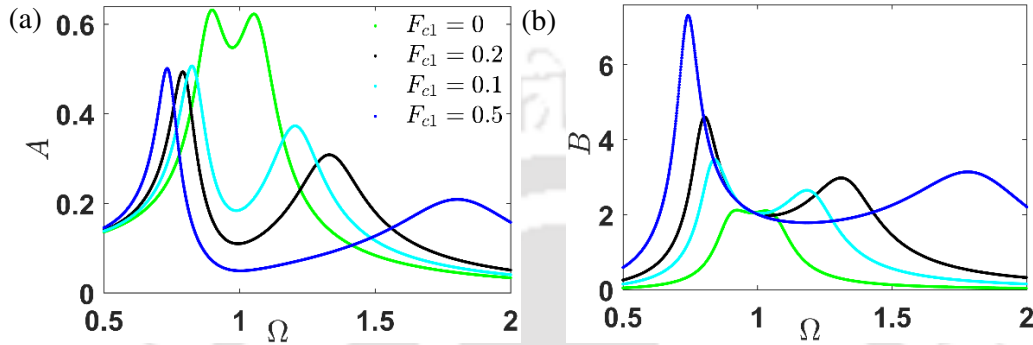


Fig. 3.8: Frequency response of the linear system for variation in feedback gain F_{c1} .

3.3.6 Effects of variation in the amplitude of external excitation

In this section, the effect of three values of the amplitude of external excitation F_1 equal to 0.1 (cyan), 0.5 (black) and 1 (green) on the FRC of the system are studied without any applied feedback gain and the results are compared with those obtained by numerically solving Eqs. 3.4 and 3.5. In Fig. 3.9 (a-b) the FRC of the system is shown for α_{13} and β equal to 0.1 and 0.01, respectively. From these figures, it is observed that with an increase in the amplitude of external excitation the peak response amplitude of the system increases. For $F_1 = 0.1$ the jump up and jump down phenomena are observed with saddle-node bifurcation points at Ω equals to 1.005 and 1.087, respectively. An isolated stable and unstable region is also observed when $1.25 < \Omega < 1.356$. Further, increase in the F_1 value from 0.1 to 1, the system response amplitude increases with a large increase in the jump up and jump down height as shown in Fig. 3.9 (a, b). The Primary system shows only a single peak (Fig. 3.8(a)) but the absorber shows two peaks (Fig. 3.9(b)). In the case of point Hopf bifurcation point S ($\Omega = 1.4$), the time response of the primary system is found to be periodic with many

harmonics and for the absorber, it is found to be quasiperiodic. For other points near Hopf bifurcation point R ($\Omega=1.552$) both the primary system and the absorber have a quasiperiodic response which is clearly evident from Poincare's section (Fig. 3.9(c, d)). In the following section, the same system is studied where the primary system is also subjected to base excitation.

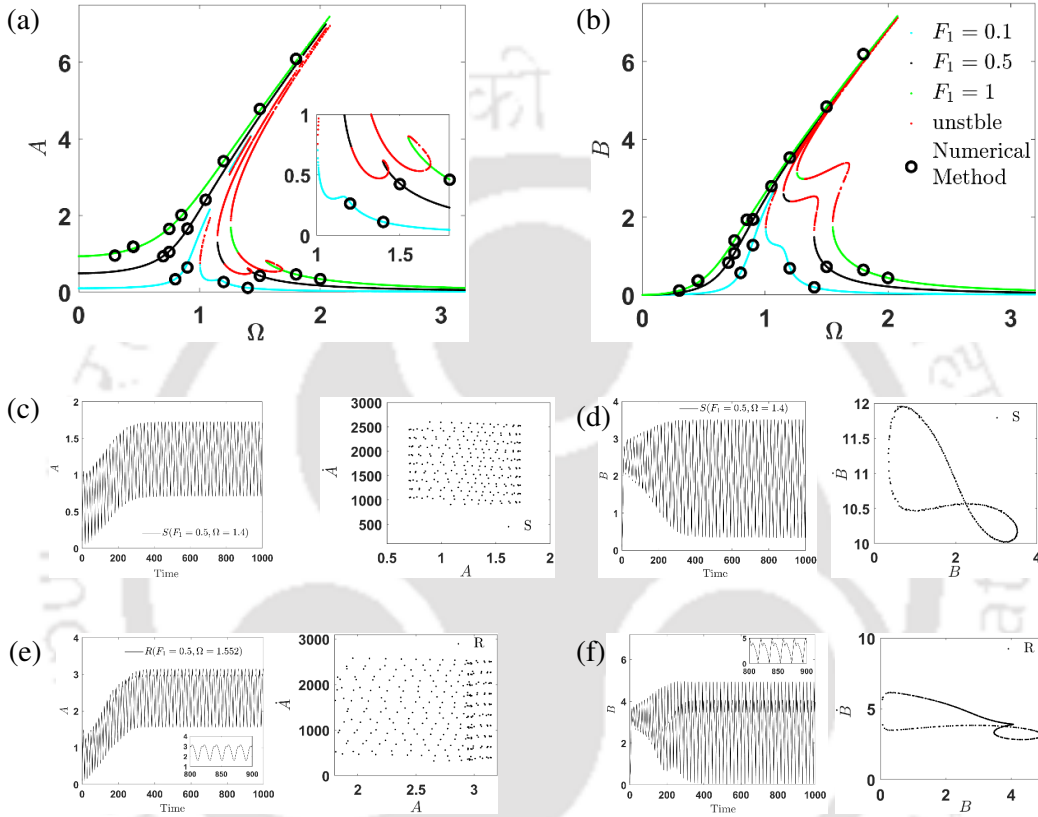


Fig. 3.9: Frequency response of the system for variation in the amplitude of external force F_1 (a, b) $\alpha_{13} = 0.1$ and $\beta = 0.01$. Time response and Poincare's section at S ($\Omega = 1.4$) and R ($\Omega = 1.552$) for $F_1 = 0.5$.

In Fig. 3.10 the response of the highly nonlinear system is analyzed for the variable amplitude of external excitation for three different frequencies i.e., $\Omega = 1.2, 1$ and 0.8 . From Fig. 3.10 (a) and Fig. 3.10 (c) it is observed that with $F_{c1} = 0.1$ the response amplitude of the highly nonlinear primary system ($\alpha_{13} = 1$) decreases and no unstable region is observed for different frequencies. The Hopf bifurcation points without feedback gain are observed at $F_1 = 0.062$ and $F_1 = 0.104$ for $\Omega = 1.2$. The effectiveness of the feedback gain in

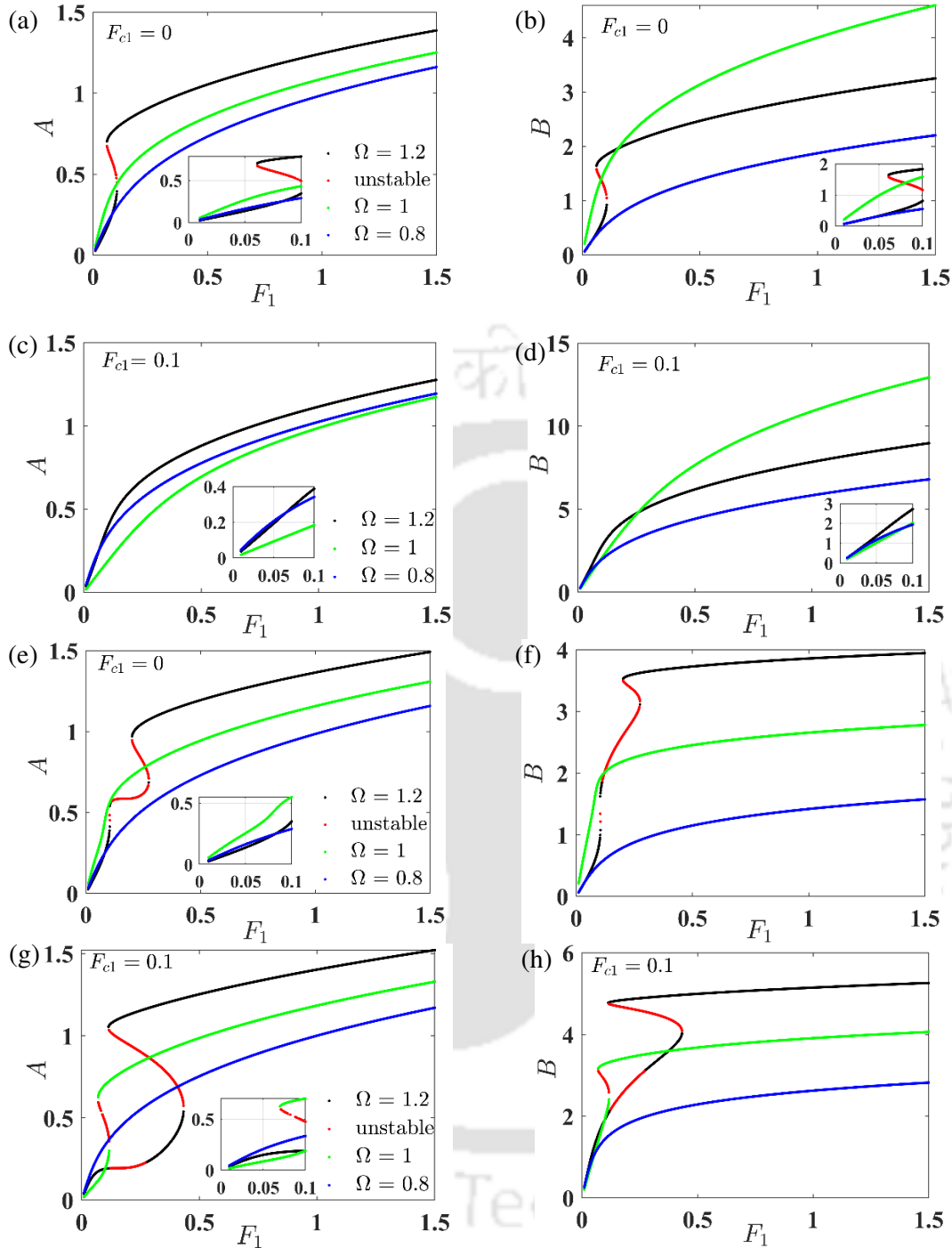


Fig. 3.10: Response of the system for variation in the amplitude of external force F_1 (a, b) $\alpha_{13} = 1$ and $F_{cl} = 0$. (c, d) $\alpha_{13} = 1$ and $F_{cl} = 0.1$. (e, f) $\alpha_{13} = 1$, $\beta = 0.0042$ and $F_{cl} = 0$. (g, h) $\alpha_{13} = 1$, $\beta = 0.0042$ and $F_{cl} = 0.1$.

suppressing the vibration is more observed for external amplitude excitation in the range of 0 to 0.2. In Fig. 3.10 (e-h) optimal nonlinear stiffness parameters ($\alpha_{13} = 1$, $\beta = 0.0042$)

from the work of Habib et al. [161] are considered. From Fig. 3.10 (e) and (g) it is observed that with $F_{c1} = 0.1$ the response amplitude of the primary system decreases for F_1 in the range of 0 to 0.1. The Hopf bifurcation points is observed at $F_1 = 0.107, 0.2$ and $\Omega = 0.274$ for $\Omega = 1.2$ without feedback gain. But with $F_{c1} = 0.1$ the Hopf bifurcation points are observed at $F_1 = 0.116, 0.124, 0.276$ and 0.432 for $\Omega = 1.2$ and $F_1 = 0.071$ and 0.115 for $\Omega = 1$. The response amplitude of the absorber increases with $F_{c1} = 0.1$ compared to $F_{c1} = 0$. From Fig. 3.10 it is observed that with $F_{c1} = 0.1$ the response amplitude of the highly nonlinear primary system decreases with feedback gain but for the higher amplitude of excitation, the effectiveness of feedback gain is not observed. However, one can suppress the vibration for a higher amplitude of external excitation by increasing the feedback gain.

3.3.7 AVA with both harmonic force and base excitation

Considering both external harmonic and base excitations, the FRC of the system is shown in Fig. 3.11 considering linear and nonlinear stiffness in the system. The amplitude of force F_1 and base excitation Y in Fig. 3.11(a-d) are considered to be equal to 0.1 and 0.01, respectively in the same phase. In Fig. 3.11(e, f) F_1 remains the same but the base excitation amplitude is taken as $Y = 0.1$ with phase angle $\gamma = \pi/2$.

3.3.7.1 Linear system

Initially, the response for the primary system and the passive absorber is shown in Fig. 3.11(a, b), where the response amplitude of the primary system at the two peaks is the same and equals 0.69 and for the absorber, it is 2.33 and 2.29, respectively. The response amplitude of the primary system decreases from 0.69 to 0.55 (21%) in the first peak and from 0.69 to 0.41 (40%) in the second peak with the negative feedback gain F_{c1} equal to 0.1. At the resonating frequency ($\Omega = 1$), the amplitude of the primary system also decreases from 0.6 to 0.2 (67%). However, the response amplitude of the absorber (Fig. 3.11(b)) at the peaks is more in the case of AVA than PVA.

3.3.7.2 Nonlinear stiffness in the primary system

Now considering $\alpha_{13} = 0.1$ in Fig. 3.11(c), the response amplitudes of the two peaks are found to be 0.52 and 0.41 at Ω equals 0.82 and 1.21, respectively. From Fig. 3.11(c), it can be observed that a reduction of 16% is achieved in the first peak and 41% in the second peak with the AVA than the PVA (Fig. 3.4(a)). It may be noted that in Fig. 3.4(a) the primary system is subjected to only harmonic excitation force. For the absorber (Fig. 3.11(d)) the two peaks are found to be 3.6 and 2.8 respectively. Now, considering the highly

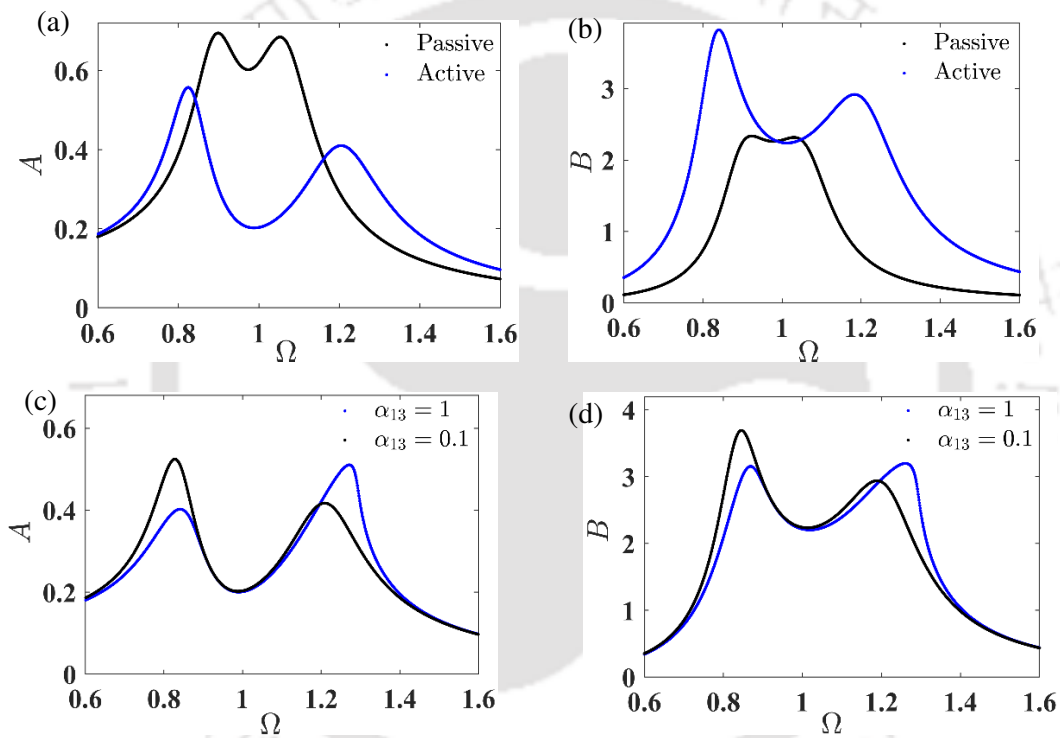


Fig. 3.11: Frequency response curve for $Y = 0.01$ and $F_1 = 0.1$. Linear system (a, b), Nonlinear primary system (c, d) with $\alpha_{13} = 0.1$ and $\alpha_{13} = 1$.

nonlinear primary system ($\alpha_{13} = 1$) shown in Fig. 3.11(c), one can observe that the two resonating peaks are 0.42 and 0.52. It may be noted that an 80% reduction in response amplitude in the second peak is observed in the active case in comparison to the passive case (Fig. 3.4(a)). However, for the absorber (Fig. 3.11(d)) the response amplitudes at the peaks are found to be 3.14 and 3.16. From Fig. 3.11 it is clearly observed that the suppression of vibration of the primary system under harmonic force and base excitation can be achieved by applying the appropriate feedback gain using the developed equations

for the proposed absorber. Here the nonlinear system behaves much like a linear system by using the acceleration feedback-based gain.

3.3.7.3 Nonlinear stiffness in both the primary system and the absorber with different forcing frequencies.

Considering nonlinear stiffness in both the primary system and absorber with different phases between the external forcing and base excitation is shown in Fig. 3.12. From this figure, it is observed that the system response amplitude is more when both the frequency of external forcing and the base excitations are the same. The higher amplitude of base excitation force produces a higher response amplitude. The applied feedback gain reduces the response

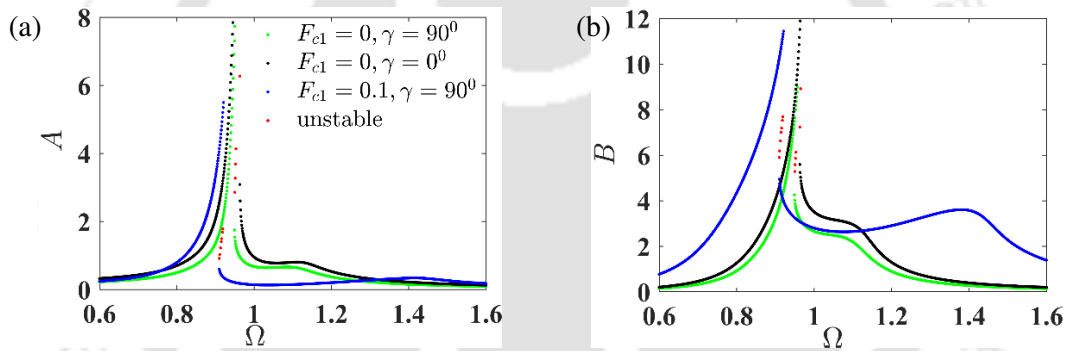


Fig. 3.12: Frequency response curve with $Y = 0.01$ and $F_1 = 0.1$. (a, b) $\alpha_{13} = 0.1$ and $\beta = 0.001$.

amplitude when the operating frequency is away from the resonating frequency. Here also similar nonlinear effects as stated in the previous sections are observed. In the following section, the effect of time delay with the AVA is investigated.

3.3.8 Effects of time delay in Active vibration absorber

The effects of four values of time delay $\tau = 0, 0.1, 1$ and 10 on the FRC of the system are studied in this section for various parameters of nonlinear stiffness. In the sections, 3.3.8.1 to 3.3.8.4 only external force excitation is considered while in section 3.3.8.5 both external force and base excitations are considered.

3.3.8.1 Linear System

In Fig. 3.13(a, b) the FRC of the system is shown by considering a linear system when the primary system is subjected to only force excitation. From Fig. 3.13(a) it is observed that for τ equals to 0 and 0.1, the system does not show any unstable region but for τ equals to 1 and 10, a broader range of unstable regions appears. The response amplitude at the first peak is 7% more at $\tau = 0$ than at $\tau = 0.1$, while at the second peak it is 1% less. For

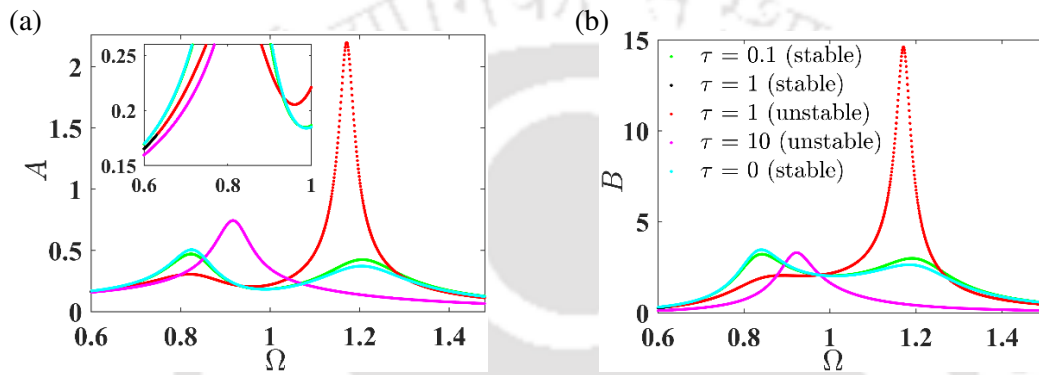


Fig. 3.13: Frequency response of the active linear system with variation in the time delays. $\tau = 0$ cyan (stable), $\tau = 0.1$ green (stable), $\tau = 1$ black (stable) red (unstable), $\tau = 10$ magenta (unstable).

$\tau = 1$ the stable region is observed for $\Omega < 0.63$ and $\Omega > 2.16$ and for $\tau = 10$ the equilibrium points are completely unstable. Fig. 3.13(b) shows similar characteristics as Fig. 3.13(a) but with higher response amplitude. Further, from Fig. 3.13(b) it is observed that at $\Omega = 0.977$ all the curves pass through a fixed point at $B = 2.08$ irrespective of the different time delays which will be further investigated in the future. From Fig. 3.13(a, b) it can be inferred that the vibration suppression of the system is better for $\tau = 0$ and 0.1 than for higher time delays τ . It is also deduced that for $\tau = 0$ the response amplitude of the primary system is better than $\tau = 0.1$ for $\Omega > 0.96$ and that for $\Omega < 0.96$, $\tau = 0.1$ is better than $\tau = 0$.

3.3.8.2 Nonlinear stiffness in the absorber

In Fig. 3.14(a, b) only the nonlinear stiffness in the absorber is considered for $\beta = 0.001$. From Fig. 3.14(a) it is observed that for $\tau = 0$ the primary system shows high response amplitude at the first peak with jump up and jump down phenomena at saddle-node bifurcation points $\Omega = 0.902$ and 0.916 , respectively, but for $\tau = 0.1$ the response amplitude at the first peak reduces

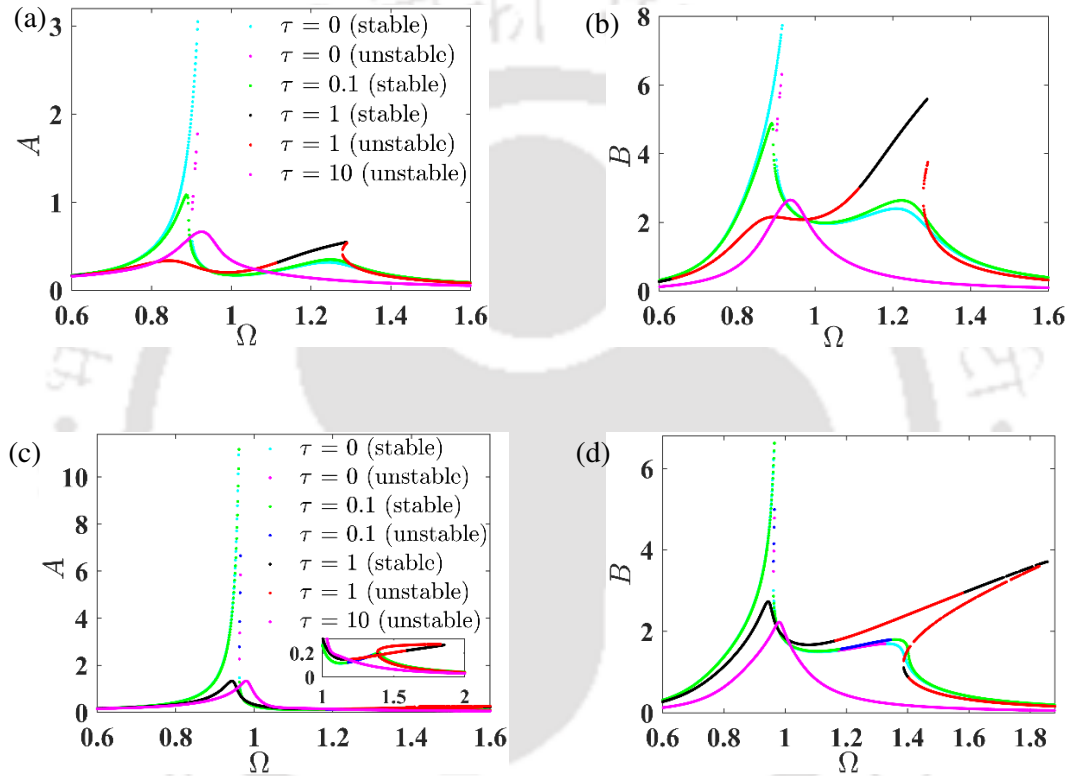


Fig. 3.14: Frequency response of the system for different time delays (a, b) $\beta = 0.001$ (c, d) $\beta = 0.01$. $\tau = 0$ cyan (stable) magenta (unstable), $\tau = 0.1$ green (stable) blue (unstable), $\tau = 1$ black (stable) red (unstable), $\tau = 10$ magenta (unstable).

to 65% with no unstable region. Further, increasing τ to 1 it is observed that the equilibrium points become unstable with a stable region is found for a narrow band of $1.112 < \Omega < 1.228$, while for $\tau = 10$ no stable region is observed. From Fig. 3.14(a, b) it is inferred that for $\beta = 0.001$ a time delay of $\tau = 0.1$ shows better vibration suppression and stability than other time delays. In Fig. 3.14(c, d) FRC of the system is shown by increasing

the nonlinear stiffness i.e. $\beta = 0.01$. From Fig. 3.14(c) it is observed that for $\tau = 1$ the response amplitude near the first peak reduces to 90% than for $\tau = 0$ or 0.1 and shows a stable region for $\Omega < 1.164$ but for $\tau = 10$ the solutions are completely unstable. The Hopf bifurcation points are observed for $\tau = 1$ at $\Omega = 1.165$ and 1.515, where the stability changes from stable to unstable and vice versa respectively, which is more clearly shown in Fig. 3.14(d). From Fig. 3.14(a-d) it is concluded that for lower nonlinear stiffness in the absorber (2.2% or 0.001) the time delay of 0.1 shows better vibration suppression with no unstable region. While for higher nonlinear stiffness in the absorber (22% or 0.01) the time delay of 1 shows better vibration suppression with no unstable operating frequency for $\Omega < 1.164$.

3.3.8.3 Nonlinear stiffness in the primary system

In Fig. 3.15(a, b) only the nonlinearity in the primary system is considered i.e. $\alpha_{13} = 0.1$. From these figures, it is observed that for $\tau = 0$ and 0.1 the response amplitude is stable. The response amplitude of the primary system for $\tau = 0$ is 6% more at the first peak and 12% less at the second peak than for $\tau = 0.1$. For $\tau = 1$ the system shows an unstable region for $0.62 < \Omega < 2.162$ and for $\tau = 10$ the system shows a broader unstable region and an isolated stable and unstable branch for $0.996 < \Omega < 1.071$. From these figures, it can be inferred that for $\alpha_{13} = 0.1$ the delay in the range of 0 to 0.1 shows stable behavior and less peak response. In Fig. 3.15(c, d) the nonlinear stiffness is increased by considering $\alpha_{13} = 0.2$. Here also the same analogy is observed as in Fig. 3.15(a, b) but for $\tau = 10$ the isolated branch is closed with stable and unstable solutions. Further increasing the nonlinear stiffness i.e. $\alpha_{13} = 1$ in Fig. 3.15(e, f) the close orbit opens up for $\tau = 10$ and shows stable region for $0.931 < \Omega < 1.04$. The Hopf bifurcation points are observed at frequency points $\Omega = 0.93$ and at 1.04. A comparison of time responses at frequency K ($\Omega = 0.92$, unstable) and L ($\Omega = 0.94$, stable of Fig. 3.15(e, f)) are shown in Fig. 3.15(g, h). From the time responses in Fig. 3.15(g, h) one can observe a fixed-point response at the L point and a higher periodic response amplitude at the K point which is evident from the FRC of Fig. 3.15(e, f). From Fig. 3.15(a-h) it can be inferred that for time delays equal to 0 and 0.1 the system amplitude is less than that with higher delay and no unstable region is observed for α_{13} equals to 0.1 to 1. Also, the system amplitude at both peaks decreases with increasing

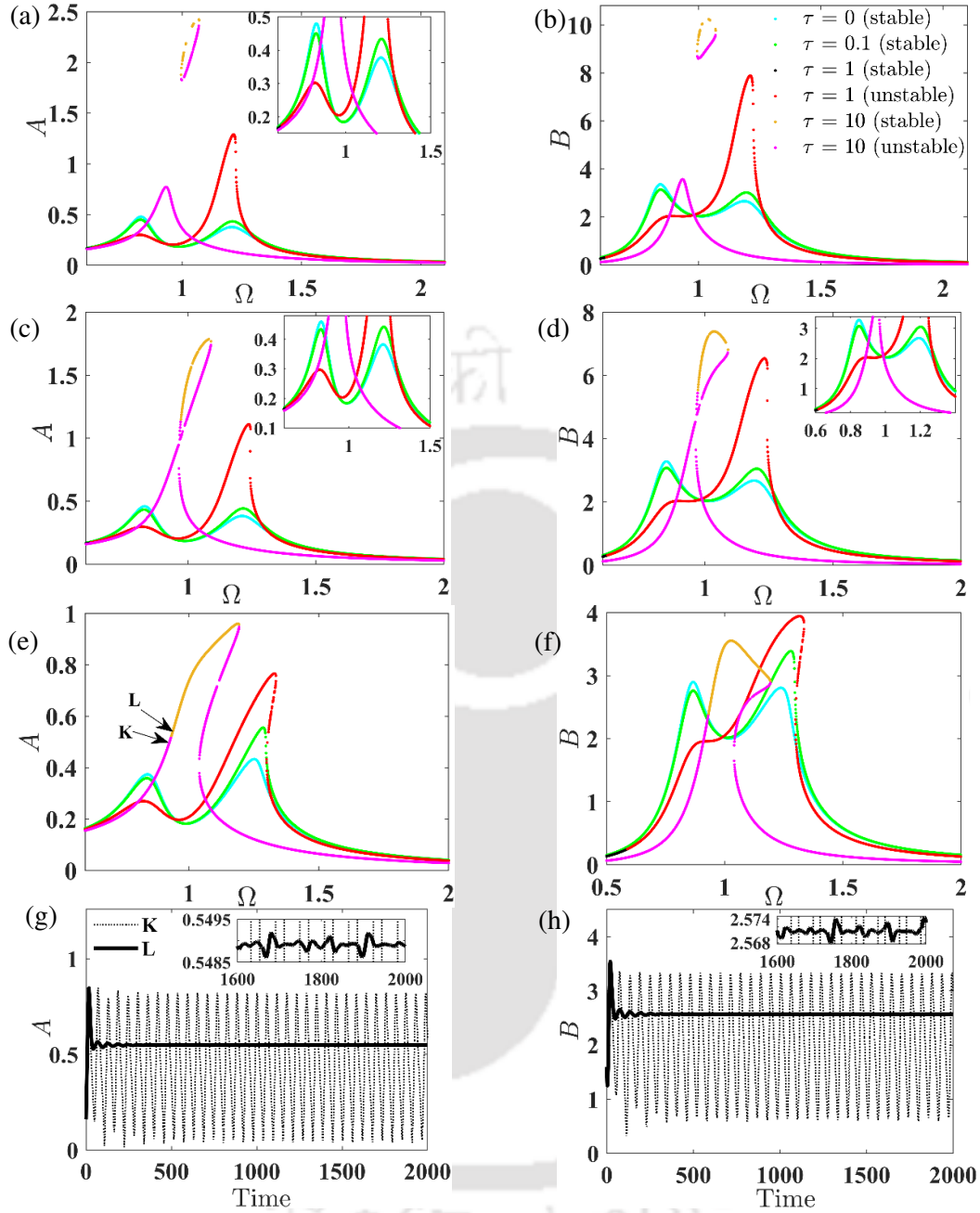


Fig. 3.15: Frequency response of the system for different time delays (a, b) $\alpha_{13} = 0.1$ (c, d) $\alpha_{13} = 0.2$ (e, f) $\alpha_{13} = 1$. (g, h) Time responses at frequency K ($\Omega = 0.92$) and L ($\Omega = 0.94$). $\tau = 0$ cyan (stable), $\tau = 0.1$ green (stable), $\tau = 1$ black (stable) red (unstable), $\tau = 10$ yellow (stable) magenta (unstable).

α_{13} from 0.1 to 1. For $\tau = 1$ and 10, the response amplitude and unstable region are more pronounced and only for a limited range the solutions are stable.

3.3.8.4 Nonlinear stiffness in both the absorber and the primary system

In Fig. 3.16(a, b) α_{13} and β are considered equals to 0.1 and 0.001 respectively and the color codes are the same as in Fig. 3.14. From Fig. 3.16(a) it is observed that for $\tau = 0$ the response amplitude at the first peak increases by 18% while at the second peak it is decreased by 8% than for $\tau = 0.1$ and shows an unstable solution at the first peak with two

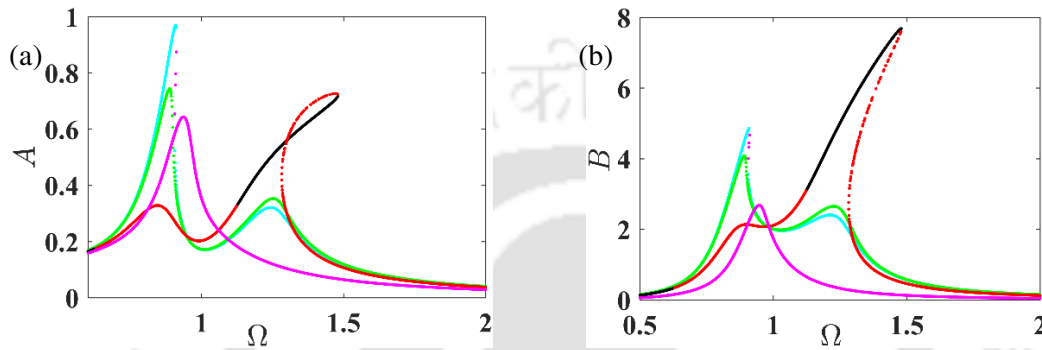


Fig. 3.16: Frequency response of the system with different time delay for $\alpha_{13} = 0.1$ and $\beta = 0.001$. $\tau = 0$ cyan (stable) magenta (unstable), $\tau = 0.1$ green (stable), $\tau = 1$ black (stable) $\tau = 1$ red (unstable), $\tau = 10$ magenta (unstable).

saddle-node bifurcation points at $\Omega = 0.906$ and 0.911 . For $\tau = 1$ the unstable region broadens with Hopf bifurcation points at $\Omega = 0.628, 1.123$ and 1.479 . The system becomes completely unstable for $\tau = 10$ and shows a single peak. It is inferred from these figures that for $\tau = 0.1$ the system response amplitude shows better vibration suppression for a broader range of operating frequency when $\alpha_{13} = 0.1$ and $\beta = 0.001$ with no unstable region.

3.3.8.5 Nonlinear primary system with both external force and base excitation

Considering both external harmonic force $F_1 = 0.1$ and base excitation $y = 0.1$ the FRC of the system for the above four-time delays τ is shown in Fig. 3.17(a, b). From Fig. 3.17(a, b) one can observe for $\tau = 0$ and 0.1 the response amplitude of the system is less and shows no unstable range of operating frequency than for higher time delays $\tau = 1$ or 10 . For $\tau = 1$, the Hopf bifurcation points are observed at $\Omega = 0.62$ where the system stability changes from stable to unstable as shown in red color. For $\tau = 10$ the system shows Hopf bifurcation

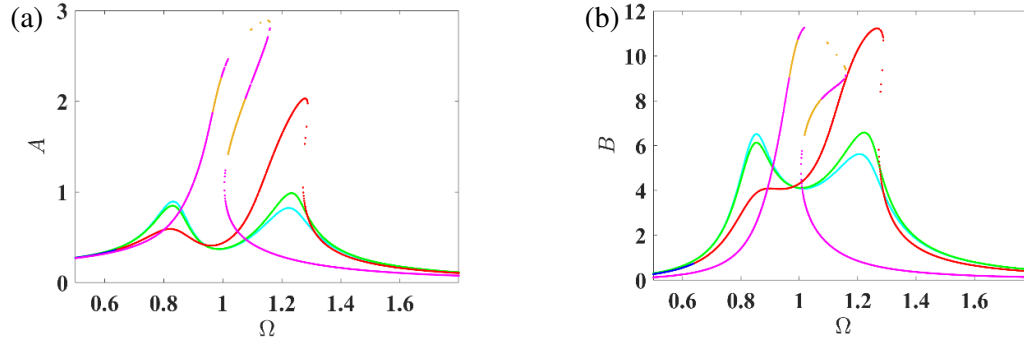


Fig. 3.17: Frequency response of the system with different time delay with $Y = 0.1$ and $\alpha_{13} = 0.1$. $\tau = 0$ cyan (stable) magenta (unstable), $\tau = 0.1$ green (stable), $\tau = 1$ black (stable) $\tau = 1$ red (unstable), $\tau = 10$ yellow (stable) $\tau = 10$ magenta (unstable).

points at $\Omega = 0.964, 1.017, 1.018$ and 1.073 . It is observed from these figures, that for time delays $\tau = 0$ and 0.1 the system is stable and response amplitude is less than higher time delays 1 or 10 . In the following section effects of the spring stiffness k_3 attached to the PZT stack actuator in the active vibration, the absorber is investigated.

3.3.9 Effects of the spring stiffness attached to PZT stack actuator

The displacement of the nonlinear primary system and the active nonlinear absorber are shown in Fig. 3.18 when the applied feedback gain $F_{c1} = 0.005$ and stiffness $k_3 = 0.05$ N/m. The parameters are kept the same as in the work of Habib et al. [38] which are $m_1 = 1$ kg, $c_1 = 0.002$ Ns/m, $k_1 = 1$ N/m, $k_{13} = 1$ N/m³, $m_2 = 0.05$ kg, $c_2 = 0.0128$ Ns/m, $k_2 = 0.0454$ N/m and $k_{23} = 0.0042$ N/m³. In Fig. 3.18 a comparison of the displacements of the primary system and the secondary systems for the present case of ANVA and that of the Habib et al. [161] are shown. The responses in both cases are quasi-periodic for external excitation $F_1 = 0.12$. It shows clearly that in the present case, the response of the primary system decreases by 31% i.e. from 0.75 to 0.52 than that of Habib et.al. [161]. Similarly, the vibration suppression of the nonlinear absorber is more noticeable as shown in Fig. 3.18(b). Here the amplitude in the present case is 0.32 and with PVA [161] it is 2.4 . Hence a reduction of 87% is achieved in the present case due to the feedback gain.

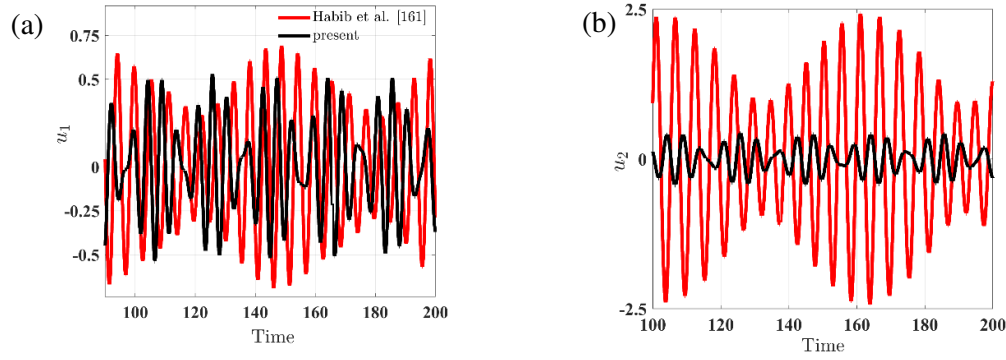


Fig. 3.18: Comparison of time responses with passive and active absorber for the (a) primary system and the (b) absorber with $F_1 = 1.2$.

In all the previous analyses it is observed that with the applied feedback gain F_{c1} , the response amplitude of the absorber at the peaks increases but here it is shown that with proper stiffness value of k_3 one can reduce the vibration of both the absorber and the primary system. Also, it is observed that with the applied feedback gain the system settles down more quickly than the passive vibration absorber. The stiffness of the spring k_3 can be increased to provide more blocking force to the PZT stack actuator and at the same time, spring stiffness k_2 can be decreased to maintain the optimum tuning ratio required to suppress the vibration of the primary system actively. The increase in spring stiffness k_3 suppresses the vibration of the absorber and by the combination of both feedback gain and spring k_3 , the required vibration suppression of the primary system and the absorber can be achieved. In the following section, the effect of spring stiffness k_3 is studied on FRC to obtain Den Hartog's equal peaks when the primary system is subjected to both external harmonic and base excitation.

3.3.10 Effects of attached spring in the PZT stack actuator

While in the previous sections attempts were made to reduce the amplitude of the primary system, in this section it is tried to achieve the Den Hartog's equal peaks in both the nonlinear primary system and the active nonlinear absorber, when the primary system is subjected to both external force and base excitation. While it is absolutely required to have equal peaks in the primary system to minimize its maximum response amplitude, sometimes due to space/kinematic constraints, it is also required to have equal peaks of the

absorber to minimize its response amplitude. The equal peaks in the FRC are tried to achieve by variation of spring stiffness k_3 and F_{c1} . It may be noted that the taken system parameters are the same as those in Habib et al. [161], where they obtained equal peaks in the FRC of the primary nonlinear system which is subjected to only external excitation force by considering nonlinear passive vibration absorber.

3.3.10.1 External force excitation

In Fig. 3.19(a, b) the spring stiffness $k_3 = 0.0005$ N/m, feedback gain $F_{c1} = 0.0005$ and external harmonic force $F_1 = 0.001$ are considered. From Fig. 3.18(a) the two resonating peaks in the FRC of the primary system are observed to be 0.0042 and 0.0082 at $\Omega = 0.88$ and 1.08, respectively. The resonating peak amplitude of the absorber (Fig. 3.19(b)) are at

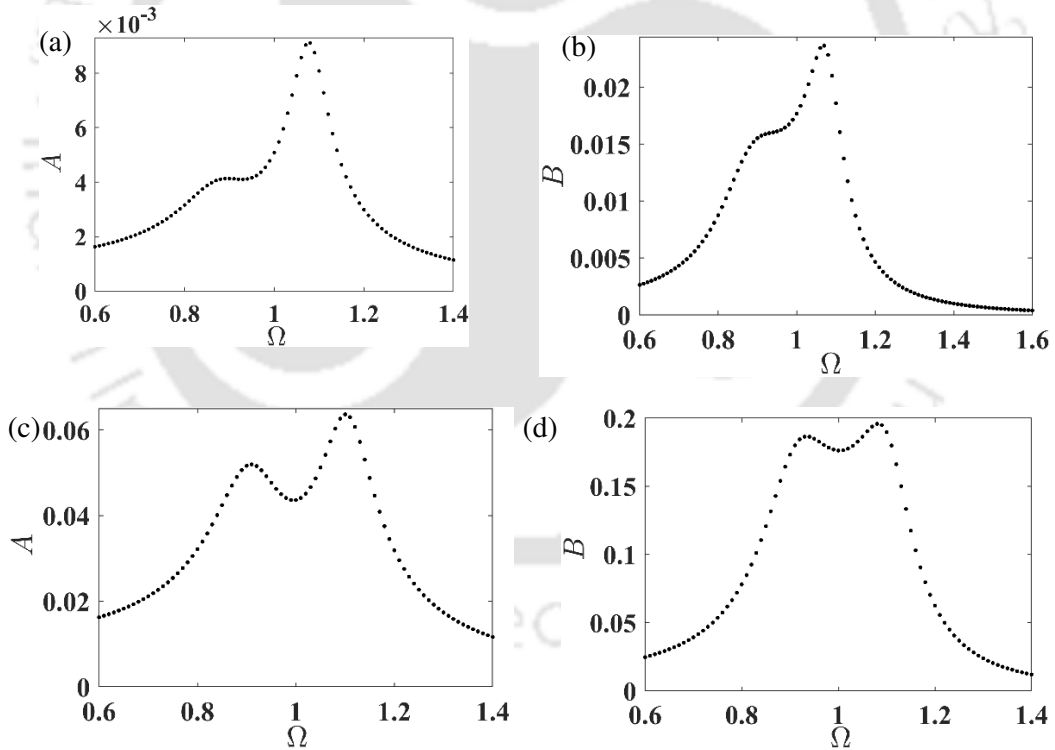


Fig. 3.19: Frequency response of the system with the external harmonic force for (a, b) $k_3 = 0.0005$ N/m, $F_{c1} = 0.0005$ and $F_1 = 0.001$ (c, d) $k_3 = 0.0005$ N/m, $F_{c1} = 0.005$ and $F_1 = 0.01$.

different amplitudes of 0.016 and 0.022 at $\Omega = 0.9$ and 1.07, respectively. It may be noted from Fig. 3.19(a) that with the applied feedback gain the primary system amplitude is lesser at the first resonating peak than the passive nonlinear vibration absorber by Habib et al. [161]. In Fig. 3.19(c, d) F_1 and F_{c1} are increased to 0.01 and 0.005, respectively, keeping all other system parameters the same as in Fig. 3.19(a, b). The difference between the resonating peak amplitudes and valley amplitude of both the primary system and the absorber reduces as shown in Fig. 3.19(c, d). The maximum amplitude for the primary system is observed at the second peak with a value of 0.062 at $\Omega = 1.08$. The maximum amplitude for the absorber is found to be 1.194 (Fig. 3.19 (d)) at $\Omega = 1.08$. The resonating peak and the valley amplitude in the FRC for both the primary system and the absorber are minimized by increasing the active feedback gain as observed from Fig. 3.19(a-d) and also the Den Hartog's equal peaks in the FRC of the nonlinear primary system and the active nonlinear vibration absorber can be found.

3.3.10.2 Both harmonic external force and base excitation

Now in Fig. 3.20(a, b) to obtain equal peaks in the FRC for base excitation $Y = 0.0012$ and $F_1 = 0.001$, F_{c1} and k_3 are increased to 0.05 while keeping all other system parameters the same as in Fig. 3.19(c, d). But it is found that with the increase F_{c1} and k_3 the primary system (Fig. 3.19(a)) shows a single peak of amplitude = 0.13 at $\Omega = 0.88$ and for the absorber (Fig. 20(b)) the first resonant peak amplitude is 0.25 and a small resonating peak is observed with an amplitude of 0.34 at $\Omega = 1.4$. So, in Fig. 3.20(c, d) the stiffness of k_3 the spring is decreased to 0.005 from 0.05 while all other system parameters are kept the same as in Fig. 3.20(a, b). It is observed from Fig. 3.20(c, d) that vibration suppression by the absorber is significantly better than in Fig. 3.19(a, b) in reducing the maximum amplitude for the primary system to 0.037 and 0.052 at $\Omega = 0.93$ and 1.09 respectively. But the two resonating peaks are found at different response amplitudes. In Fig. 3.20(e, f) the comparisons between the responses of the system are shown when the primary system is subjected to external harmonic excitation $F_1 = 0.0012$ and base excitation $Y = 0.0012$ with and without using the attached spring k_3 to the PZT stack actuator. It is observed from Fig. 3.20(e, f) that for the same spring stiffness k_3 and F_{c1} equal to 0.005 the vibration suppression is better for both the primary system and the absorber and also the response

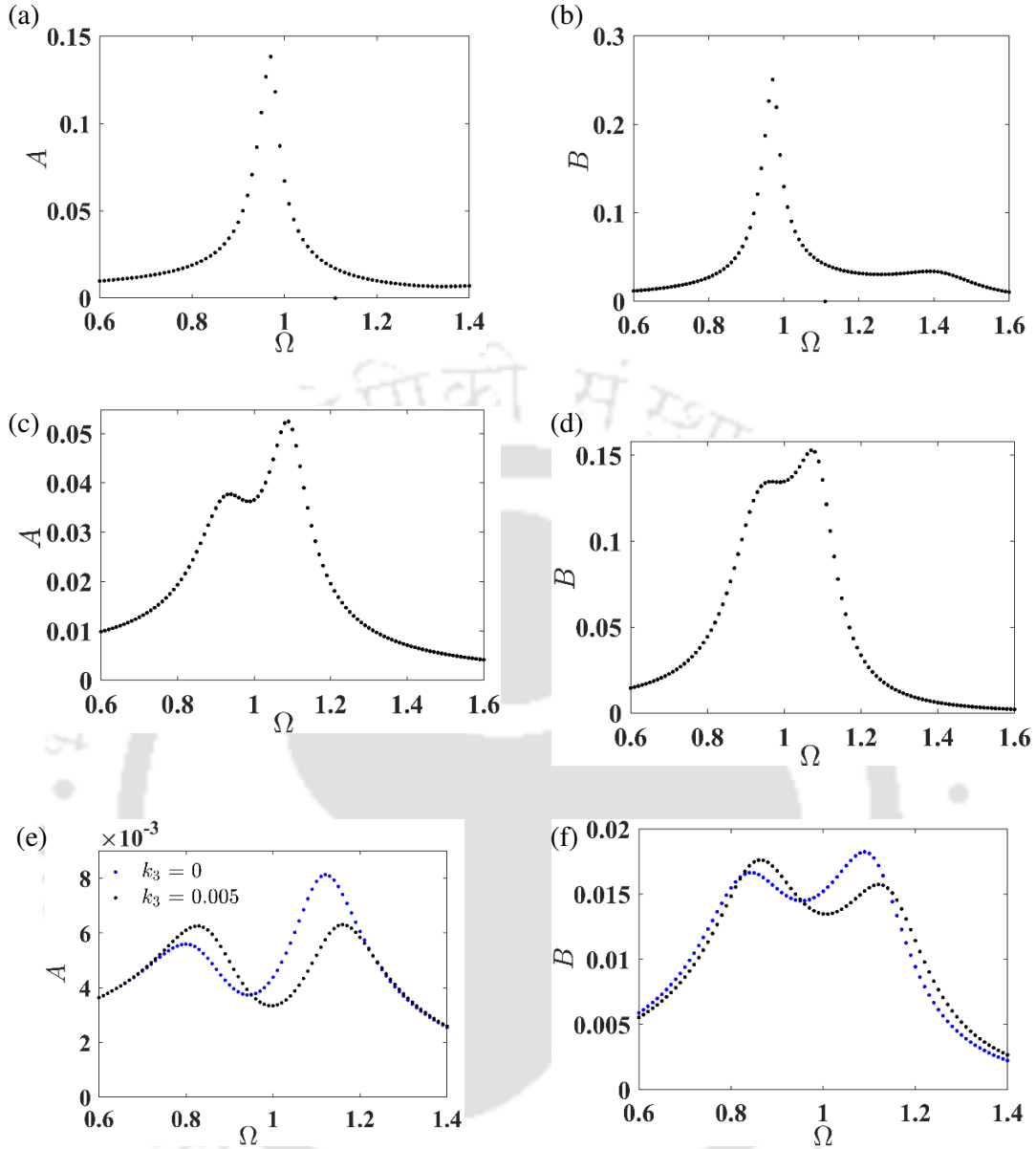


Fig. 3.20: Frequency response of the system for $F_{cl} = 0.05$, $F_1 = 0.01$ and $Y = 0.0012$, (a, b) $k_3 = 0.05$ N/m (c, d) $k_3 = 0.005$ N/m. Frequency response of the system for $F_{cl} = 0.005$ and $F_1 = Y = 0.0012$ (e, f) (black) for $k_3 = 0.005$ N/m and (blue) for $k_3 = 0$.

amplitudes of the resonating peaks are the same. From Fig. 3.20 it is observed that with applied feedback gain the vibration suppression is better than the passive vibration absorber and also sensitiveness of the vibration for the system reduces which shows two equal peaks in the response curves when both external harmonic force and base excitation are acting on the primary system. With the proposed active nonlinear vibration absorber, the two

resonating peak amplitudes of the primary system are found to be 0.0062, which shows better vibration suppression than the passive nonlinear vibration absorber. In Fig. 3.20(f) the FRC of the absorber shows the two resonating peaks' amplitudes are slightly at different amplitudes of 0.015 and 0.017, respectively. From these curves, it is found that with the attached spring and PZT stack actuator combination the vibration suppression of both systems is reduced for a wider band of operating frequency when the primary system undergoes both harmonic force and base excitation. From Fig. 3.20 and section 3.3.7 it is observed that the vibration suppression of the primary system is better than the work reported in [215, 177] where the mass ratio is considered to be 1:10.

3.3.11 Reliability and cost of the proposed ANVA

In the proposed model, as both spring and actuator provide the feedback gain, one can obtain higher feedback gain by the PZT stack actuator due to the blocking force produced by the spring which is in series connection with the PZT stack actuator. Also, when the actuator i.e., the active part fails due to electronic malfunction, then the passive optimal absorber can still protect the primary system with a very low value of damping as considered in the present analysis from severe vibration. Thus the proposed vibration absorber is a more economical and fail-safe design. Also, it can be observed that the natural frequency of the primary system ($\omega_{n1} = \sqrt{(k_1 + k_2 + k_r)/m_1}$, $k_r = k_3 k_p^E / (k_3 + k_p^E)$) does not change much with the variation of spring k_3 attached to the PZT stack actuator but the natural frequency of the absorber ($\omega_{n2} = \sqrt{(k_2 + k_r)/(\mu m_1)}$, $\mu = m_2/m_1$) changes significantly as the absorber mass is 20 times smaller than the mass of the primary system. So the designer can tune the frequency of the absorber actively in accordance with the forcing frequency of the primary system by the spring and actuator combination for various other resonance conditions. The proposed active nonlinear vibration absorber shows better vibration suppression than Fig. 3.16 and Fig. 3.17 from the work of Kučera et al. [270] where linear analysis is carried out experimentally considering the delayed resonator. The vibration suppression by the proposed absorber is almost the same (90%) as the vibration for the highly nonlinear primary system as in the experimental work of Bronkhorst et al. [281]. But in [281], the absorber mass is 20% of the primary mass, whereas in the present analysis the absorber mass is only 5% of the primary system. Hence the proposed absorber is very much lighter than in [281]. The proposed absorber also gives better performance

than the work of Xiang et al. [65] where absorber mass is similar to that in [281] i.e., 20% of the primary system where a higher damping ratio for the absorber is used. It may be noted that in the present case, a very low damping ratio for the primary system ($\zeta_1 = 0.001$) and the absorber ($\zeta_2 = 0.0064$) are used. Hence the present model is more reliable and cost-effective as a lighter weight absorber is used.

3.4 Summary

In this work, the performance of the active nonlinear vibration absorber (ANVA) is investigated for the primary resonance condition by considering a new design in the active part where the PZT stack actuator is connected in series with a spring in the absorber configuration. The analysis is carried out by considering time delay in the acceleration feedback of the primary system when it undergoes both external harmonic force and base excitation. The steady-state response and its stability are obtained by using a modified harmonic balance method, where the amplitude and the phase are considered as a slowly varying function of time. Time responses, phase portraits, Poincare's sections, the basin of attractions, frequency responses with stable and unstable branches are plotted to investigate the effect of cubic nonlinear stiffness, feedback gain, time delay, attached spring stiffness, the amplitude and frequency of external harmonic force and base excitation on the suppression of vibration of the system. The frequency response curves obtained by harmonic balance and Newton's method are compared with those obtained by the fourth-order Runge Kutta method and are found to be in good agreement. Fundamentally based on nonlinearity the considered systems are divided into four cases viz., case-1: both primary and absorber are linear, case-2: primary system linear and absorber nonlinear, case-3: primary nonlinear and absorber linear, case-4: both primary and absorber nonlinear. Further considering the presence and absence of PZT stack actuator (feedback gain), delay; and base excitation one may obtain twenty different cases which have been investigated.

From the analysis, it can be observed that by suitably applying the feedback gain by the spring and PZT stack actuator combination (30% - 90%) vibration suppression is achieved in the primary system for various system parameters. Further, it is also observed that with the variation of both k_3 and F_{c1} , the vibration of the primary system and the absorber are reduced up to 30% and 80%, respectively. Also, the Den Hartog's equal peaks in the

frequency response are achieved when the nonlinear primary system undergoes both harmonic and base excitation. and the operating frequency range to minimize the vibration of the system.



CHAPTER 4

Traditional and Non-Traditional ANVA with Time Delay Combination Feedback for Primary and Principal Parametric Resonance Conditions

4.1 Introduction

In the present work, vibration suppression of a single degree of freedom nonlinear spring-mass primary (main) system subjected to simultaneous external harmonic and parametric excitation by using traditional and non-traditional active nonlinear vibration absorber (ANVA) is carried out. The traditional and non-traditional ANVA consists of a PZT (lead zirconate titanate) stack actuator in series with a nonlinear spring and a damper. In the traditional ANVA, the damper is connected to the main system whereas, in non-traditional ANVA, the damper is connected to the base of the main system. The governing nonlinear equations of motion for the system have been derived in section 4.2, using the weighted modal matrix approach considering novel actuation force by time delay in displacement, velocity and acceleration feedback of the main system by the PZT stack actuator and the nonlinear spring in series combination. In section 4.3, the method of multiple scales (MMS) is used to obtain the reduced equations which are solved using Newton's method. In section 4.4, the responses of the system for the primary, principal parametric and simultaneous primary and principal parametric resonance conditions have been studied. The results are compared with those obtained by using the harmonic balance method (HBM), numerical method and with previous literature. From these results, two resonating peaks in the frequency response of the main system are shown by using the reduced equation of the MMS which was not reported in earlier literature. It is shown that for the active case the vibration of the undamped main system is completely reduced at the specified operating frequency for both the linear and nonlinear systems. Also, for a wider range of operating frequencies, the vibration of the main system is reduced by actively tuning the absorber for various mass ratios. Further, the efficacy of the feedback gain is observed in reducing the instability regions and improving the critical value of excitations in the trivial state.

4.2 Mathematical Modeling of the System

The studies on parametric excitations acting on the nonlinear SDOF primary system can be found in many applications a few of which includes [216, 217, 223, 244, 284, 184]. Figures 4.1(a) and 4.1(b) shows the schematic diagram of an SDOF nonlinear main system, attached to a PZT-based active nonlinear traditional and non-traditional vibration absorber, respectively. The parameters m_i , c_i , k_i denote mass, damping coefficient, linear stiffness of the main system ($i = 1$) and the absorber ($i = 2$) and k_{13} is the cubic nonlinear stiffness of the main system. In the presence of a nonlinear spring in the absorber, as the displacement at the junction between the PZT stack actuator and the spring k_2 is difficult to calculate, so, here k_{23} is assumed to be the equivalent cubic nonlinear stiffness of having a displacement of $x_1 + \delta_0 - x_2$. The main system is subjected to an external harmonic force $F_1 \cos(\Omega_1 t)$ and a parametric force $x_1 F_2 \cos(\Omega_2 t)$. The feedback gain by the absorber is produced with the combination of the PZT stack actuator having stiffness k_p^E and a nonlinear spring considering displacement, velocity and acceleration feedback of the main system. The equations of motion of the system shown in Fig. 4.1 (a) in terms of the displacements of the main system (x_1) and the absorber (x_2) are described by two ordinary coupled nonlinear differential equations of motion which are given below.

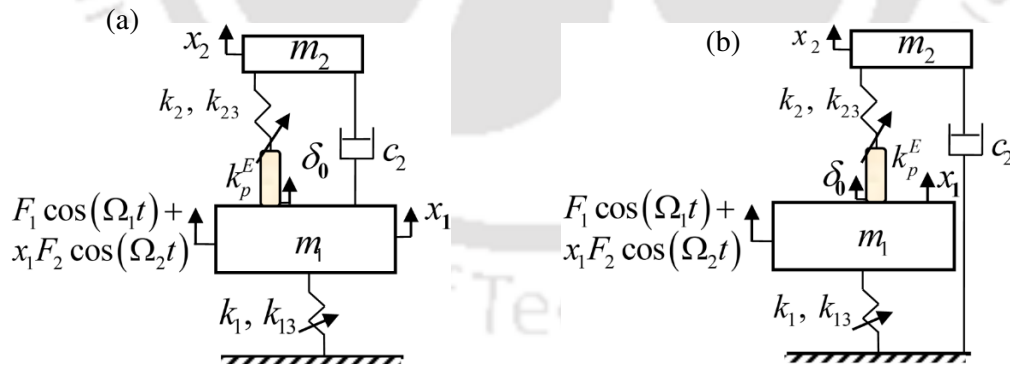


Fig. 4.1: Schematic diagram of a PZT based active nonlinear vibration absorber (a) traditional (b) non-traditional.

$$\begin{aligned}
 m_1 \ddot{x}_1 + c_2 (\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_{13} x_1^3 + (k_2 k_p^E (x_1 - x_2 + \delta_0)) / (k_2 + k_p^E) + \\
 k_{23} (x_1 - x_2 + \delta_0)^3 = F_1 \cos(\Omega_1 t) + x_1 F_2 \cos(\Omega_2 t)
 \end{aligned} \tag{4.1a}$$

$$m_2\ddot{x}_2 - c_2(\dot{x}_1 - \dot{x}_2) - (k_2 k_p^E (x_1 - x_2 + \delta_0)) / (k_2 + k_p^E) - k_{23}(x_1 - x_2 + \delta_0)^3 = 0 \quad (4.1b)$$

Similarly, the first equation of the system shown in Fig. 4.1(b) is obtained by considering the term $c_2(\dot{x}_1 - \dot{x}_2)$ in Eq. 4.1(a) equal to zero. The equation of motion for the absorber is given below.

$$m_2\ddot{x}_2 + c_2\dot{x}_2 - (k_2 k_p^E (x_1 - x_2 + \delta_0)) / (k_2 + k_p^E) - k_{23}(x_1 - x_2 + \delta_0)^3 = 0 \quad (4.2)$$

The nominal displacement of the PZT stack actuator δ_0 is expressed as $\delta_0 = nd_{33}V$ [20]. Here the terms n , d_{33} and V are the number of wafers, dielectric charge constant and the voltage developed in the PZT stack actuator, respectively. It may also be noted that Eq. (4.1a) and Eq. (4.1b) can be reduced to the work of Habib et. al [161] and Detroux et. al [168] in the absence of nonlinear stiffness in the absorber (k_{23}), parametric excitation ($x_1 F_2 \cos(\Omega_2 t)$), PZT stack actuator stiffness (k_p^E) and displacement of PZT stack actuator (δ_0). Also in [161, 168], they have used a damper in the main system which is not considered in the present analysis. Considering the effect of time delay t_d in the displacement, velocity and acceleration feedback of the main system, the voltage developed in the PZT stack actuator can be expressed as $V = \kappa_d(x_1(t-t_d)) + \kappa_v(\dot{x}_1(t-t_d)) + \kappa_a(\ddot{x}_1(t-t_d))$, where κ_d , κ_v and κ_a are the controller gain for displacement, velocity and acceleration feedback respectively. It may be noted that here the time delay is produced during sensing the vibration of the main system, processing, computing and actuating by the PZT stack actuator through the controller [135-141]. Substituting V in the displacement of δ_0 and then in Eq. (4.1a) and Eq. (4.1b) the governing equations of motion are modified. Now applying the weighted modal matrix method by considering $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\tilde{P}] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ where $[\tilde{P}]$ is the weighted modal matrix, the governing equations of motion are rewritten using a small bookkeeping parameter ε as follows.

$$\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \varepsilon \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} + \varepsilon [\tilde{P}]^T [R] = \varepsilon [\tilde{P}]^T [R_f] \quad (4.3)$$

where

$$\begin{aligned} [\tilde{P}] &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, k_a = \frac{k_2 k_p^E}{(k_2 + k_p^E)}, \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = [\tilde{P}]^T \begin{bmatrix} k_1 + k_a & -k_a \\ -k_a & k_a \end{bmatrix} [\tilde{P}], \\ [Z] &= \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = [\tilde{P}]^T \frac{1}{\varepsilon} \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} [P], F_{11} = \frac{F_1}{\varepsilon}, F_{12} = \frac{F_2}{\varepsilon}, \\ [R_f] &= \begin{bmatrix} F_{11} \cos \Omega_1 t + x_1 F_{12} \cos \Omega_2 t \\ 0 \end{bmatrix}, [R] = \begin{bmatrix} R_{n1} \\ R_{n2} \end{bmatrix}, \bar{k}_{13} = \frac{k_{13}}{\varepsilon}, \bar{k}_{23} = \frac{k_{23}}{\varepsilon}, \delta = \frac{\delta_0}{\varepsilon}, \\ R_{n1} &= \bar{k}_{13} x_1^3 + k_a (\delta) + \bar{k}_{23} (x_1 - x_2 + \delta)^3, R_{n2} = -R_{n1} + \bar{k}_{13} x_1^3, \\ \delta_0 &= f_d (x_1 (t - t_d)) + f_v (\dot{x}_1 (t - t_d)) + f_a (\ddot{x}_1 (t - t_d)), \\ f_d &= n d_{33} \kappa_d, f_v = n d_{33} \kappa_v, f_a = n d_{33} \kappa_a \end{aligned}$$

It may be noted that the governing equation of motion for the model shown in Fig. 4.1(b) is the same as in Eq. 4.3 except it has different values of the damping matrix $[Z]$. Also, here for the first-time weighted modal matrix approach for the traditional and non-traditional time delayed PZT stack actuator based ANVA are used to obtain the governing equation of motion (Eq. 4.3). The method of multiple scales is used to solve Eq. 4.3 which is discussed in the following section.

4.3 Approximate Solution using Perturbation Method

In this subsection, the standard method of multiple scales is employed to obtain an approximate analytical solution of the governing equation of motion Eq. 4.3. The perturbed solutions in the new time scale are expanded by considering the system response at the past state $X_i(t - t_d)$ which is approximated as a summation of the responses of different time scales ($T_n = \varepsilon^n t$ for $n = 0, 1, \dots$) in the following form:

$$X_i(t - t_d) = X_{i0}(T_0 - t_d, T_1 - \varepsilon t_d, \dots) + \varepsilon X_{i1}(T_0 - t_d, T_1 - \varepsilon t_d, \dots) \quad (4.4)$$

As indicated previously here $i = 1$ for the main system, $i = 2$ for the absorber and t_d is the time delay in the feedback. If the system has no delay, then the delay term t_d can be equated to zero and the corresponding system response at the time t (i.e., $X_i(t)$) can be expressed. Substituting the above-assumed responses for the main system and the absorber from Eq.

4.4 into Eq. 4.3 and collecting the coefficients of ε the following expression can be obtained.

$$\varepsilon^0 : D_0^2 X_{10} + \omega_1^2 X_{10} = 0 \quad (4.5a)$$

$$D_0^2 X_{20} + \omega_2^2 X_{20} = 0 \quad (4.5b)$$

$$\begin{aligned} \varepsilon^1 : D_0^2 X_{11} + \omega_1^2 X_{11} = & -2D_0 D_1 X_{10} - z_{11} D_0 X_{10} - z_{12} D_0 X_{20} \\ & - p_{11} \bar{k}_{13} (p_{11} X_{10} + p_{12} X_{20})^3 - (p_{21} - p_{11}) \bar{k}_{23} \left(\frac{(p_{11} - p_{21}) X_{10} + (p_{12} - p_{22}) X_{20}}{(p_{12} - p_{22}) X_{20}} \right)^3 \\ & - (p_{21} - p_{11}) \left(k_2 F_c + \bar{k}_{23} F_c^3 + 3\bar{k}_{23} F_c \left((p_{11} - p_{21}) X_{10} + (p_{12} - p_{22}) X_{20} \right)^2 \right) \\ & - 3(p_{21} - p_{11}) \bar{k}_{23} \left((p_{11} - p_{21}) X_{10} + (p_{12} - p_{22}) X_{20} \right) F_c^2 \\ & + p_{11} F_{11} \cos(\Omega_1 T_0) + p_{11} (p_{11} X_{10} + p_{12} X_{20}) F_{12} \cos(\Omega_2 T_0) \end{aligned} \quad (4.6a)$$

$$\begin{aligned} D_0^2 X_{21} + \omega_2^2 X_{21} = & -2D_0 D_1 X_{20} - z_{21} D_0 X_{10} - z_{22} D_0 X_{20} \\ & - p_{12} \bar{k}_{13} (p_{11} X_{10} + p_{12} X_{20})^3 - (p_{22} - p_{12}) \bar{k}_{23} \left(\frac{(p_{11} - p_{21}) X_{10} + (p_{12} - p_{22}) X_{20}}{(p_{12} - p_{22}) X_{20}} \right)^3 \\ & - (p_{22} - p_{12}) \left(k_2 F_c - \bar{k}_{23} F_c^3 + 3\bar{k}_{23} F_c \left((p_{11} - p_{21}) X_{10} + (p_{12} - p_{22}) X_{20} \right)^2 \right) \\ & - 3(p_{22} - p_{12}) \bar{k}_{23} \left((p_{11} - p_{21}) X_{10} + (p_{12} - p_{22}) X_{20} \right) F_c^2 + p_{12} F_{11} \cos(\Omega_1 T_0) \\ & + p_{12} (p_{11} X_{10} + p_{12} X_{20}) F_{12} \cos(\Omega_2 T_0) \end{aligned} \quad (4.6b)$$

where

$$F_c = \left(p_{11} (f_d X_{10d} + f_v D_0 X_{10d} + f_a D_0^2 X_{10d}) + p_{12} (f_d X_{20d} + f_v D_0 X_{20d} + f_a D_0^2 X_{20d}) \right)$$

The solution of the Eqs. (4.6a) and (4.6b) can be written as

$$X_{10} = A_1(T_1) \exp(i\omega_1 T_0) + \bar{A}_1(T_1) \exp(-i\omega_1 T_0) \quad (4.7a)$$

$$X_{20} = A_2(T_1) \exp(i\omega_2 T_0) + \bar{A}_2(T_1) \exp(-i\omega_2 T_0) \quad (4.7b)$$

where A_1 and A_2 are an unknown complex function of amplitude to be determined later.

$\bar{A}_1$ and $\bar{A}_2$ are the complex conjugate terms of A_1 and A_2 , respectively. Similarly, the solutions for the delay terms for the main system (X_{10d}) and the absorber (X_{20d}) can be written as

$$X_{10d} = A_{1d}(T_1 - \varepsilon t_d) \exp(i\omega_1(T_0 - t_d)) + \bar{A}_{1d}(T_1 - \varepsilon t_d) \exp(-i\omega_1(T_0 - t_d)) \quad (4.7c)$$

$$X_{20d} = A_{2d}(T_1 - \varepsilon t_d) \exp(i\omega_2(T_0 - t_d)) + \bar{A}_{2d}(T_1 - \varepsilon t_d) \exp(-i\omega_2(T_0 - t_d)) \quad (4.7d)$$

where $A_{1d} = A_1(T_1 - \varepsilon t_d)$ and $A_{2d} = A_2(T_1 - \varepsilon t_d)$. $\bar{A}_{1d}$ and $\bar{A}_{2d}$ are the complex conjugate terms of A_{1d} and A_{2d} respectively. It may be noted that Eq. 4.6(a) and Eq. 4.6(b) include nonlinearity in both the main system and absorber, nonlinearity in feedback gain by displacement, velocity, acceleration and combination of all these feedbacks with and without time delay for both passive, active, traditional and non-traditional vibration absorbers. Substituting Eqs. (4.7a) to (4.7d) into Eqs. (4.6a) and (4.6b) yields the following set of equations.

$$\begin{aligned} D_0^2 X_{11} + \omega_1^2 X_{11} = & -2i\omega_1 D_1 A_1 \exp(i\omega_1 T_0) - z_{11} i\omega_1 A_1 \exp(i\omega_1 T_0) \\ & - z_{12} i\omega_2 A_2 \exp(i\omega_2 T_0) p_{11} \bar{k}_{13} (p_{11} A_1 \exp(i\omega_1 T_0) + p_{12} A_2 \exp(i\omega_2 T_0))^3 \\ & - (p_{21} - p_{11}) \bar{k}_{23} S_1^3 - (p_{21} - p_{11}) k_2 S_2 - (p_{21} - p_{11}) \bar{k}_{23} S_2^3 \\ & - 3(p_{21} - p_{11}) \bar{k}_{23} S_1^2 S_2 - 3(p_{21} - p_{11}) \bar{k}_{23} S_1 S_2^2 + 0.5 p_{11} F_{11} \exp(\Omega_1 T_0) \\ & + 0.5 p_{11} F_{12} \exp(\Omega_2 T_0) (p_{11} \bar{A}_1 \exp(-i\omega_1 T_0) + p_{12} \bar{A}_2 \exp(-i\omega_2 T_0)) + cc \end{aligned} \quad (4.8a)$$

$$\begin{aligned} D_0^2 X_{21} + \omega_2^2 X_{21} = & -2i\omega_2 D_1 A_2 \exp(i\omega_2 T_0) - z_{21} i\omega_2 A_2 \exp(i\omega_2 T_0) \\ & - z_{22} i\omega_1 A_1 \exp(i\omega_1 T_0) p_{12} \bar{k}_{13} (p_{11} A_1 \exp(i\omega_1 T_0) + p_{12} A_2 \exp(i\omega_2 T_0))^3 \\ & - (p_{22} - p_{12}) \bar{k}_{23} S_1^3 - (p_{22} - p_{12}) k_2 S_2 - (p_{21} - p_{11}) \bar{k}_{23} S_2^3 - 3(p_{22} - p_{12}) \bar{k}_{23} S_1^2 S_2 \\ & - 3(p_{22} - p_{12}) \bar{k}_{23} S_1 S_2^2 + 0.5 p_{12} F_{11} \exp(\Omega_1 T_0) \\ & + 0.5 p_{12} F_{12} \exp(\Omega_2 T_0) (p_{11} \bar{A}_1 \exp(-i\omega_1 T_0) + p_{12} \bar{A}_2 \exp(-i\omega_2 T_0)) + cc \end{aligned} \quad (4.8b)$$

where

$$S_1 = ((p_{11} - p_{21}) A_1 \exp(i\omega_1 T_0) + (p_{12} - p_{22}) A_2 \exp(i\omega_2 T_0)),$$

$$S_2 = \begin{pmatrix} p_{11} \left(A_{1d} \exp(i\omega_1(T_0 - t_d)) (f_d + i\omega_1 f_v - \omega_1^2 f_a) \right) + \\ p_{12} \left(A_{2d} \exp(i\omega_2(T_0 - t_d)) (f_d + i\omega_2 f_v - \omega_2^2 f_a) \right) \end{pmatrix}$$

It may be noted that the term S_2 contains the time delay feedback gains by displacement, velocity and acceleration feedback and its higher-order terms such as S_2^2 and S_2^3 are neglected as they are of the order of ε^2 or higher-order terms of ε . The expansion of Eqs. (4.8a) and (4.8b) contains secular or nearly secular terms when $\Omega_1 \cong \omega_1$ or ω_2 (primary resonance) and $\Omega_2 \cong 2\omega_1$ or $2\omega_2$ (principal parametric resonance), considering 1:1 internal resonance. From the literature survey also, it is observed that the vibration absorber performs best when the absorber frequency (ω_2) is nearer to the main system frequency (ω_1), which gives rise to the 1:1 internal resonance condition. Considering the nearness for primary resonance, principal parametric resonance and 1:1 internal resonance four detuning parameters σ_1 , σ_2 , σ_3 and σ_4 are introduced as $\Omega_1 = \omega_1 + \varepsilon\sigma_1 = \omega_2 + \varepsilon\sigma_2$ and $\Omega_2 = 2\omega_1 + \varepsilon\sigma_3 = 2\omega_2 + \varepsilon\sigma_4$. Substituting the above primary, principal parametric and 1:1 internal resonance ($\omega_2 = \omega_1 + \varepsilon(\sigma_1 - \sigma_2)$) conditions and eliminating the secular terms from Eqs. (4.8a) and (4.8b), a set of bounded equations are obtained, which are given in **Appendix B** (B. 1, B. 2).

The amplitude functions A_1 and A_2 can be expressed in polar form as $A_1 = \frac{1}{2}a_1 e^{i\beta_1}$ and $A_2 = \frac{1}{2}a_2 e^{i\beta_2}$. Similarly $A_{1d} = \frac{1}{2}a_{1d} e^{i\beta_1(1-\tau_d)}$ and $A_{2d} = \frac{1}{2}a_{2d} e^{i\beta_2(1-\tau_d)}$. Here a_1 , a_2 , β_1 and β_2 represents amplitude and phase responses of the main system and the absorber and τ_d represents the non-dimensional time delay. In the same way, the complex conjugate terms $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_{1d}$ and $\bar{A}_{2d}$ are expressed. Substituting these expressions in the secular equations (**Appendix B** (B. 1, B. 2)) and separating real and imaginary terms one may obtain the following reduced autonomous equations.

$$\begin{aligned}
a_1' = & -0.5z_{11}a_1 - 0.5z_{12}a_2 \frac{\omega_2}{\omega_1} \cos(\gamma_1 - \gamma_2) - \frac{3p_{11}p_{12}\bar{k}_{13}a_2}{8\omega_1} G_1 \\
& + \frac{(p_{21} - p_{11})\bar{k}_{23}a_2}{8\omega_1} G_2 + \frac{(p_{21} - p_{11})k_2}{2\omega_1} (-p_{11}a_{1d}Q_{11} + p_{12}a_{2d}Q_{12}) \\
& + \frac{(p_{21} - p_{11})\bar{k}_{23}}{8\omega_1} G_3 + 0.5 \frac{p_{11}F_{11}}{\omega_1} \sin \gamma_1 + \\
& 0.25 \frac{p_{11}F_{12}}{\omega_1} (p_{11}a_1 \sin 2\gamma_1 + p_{12}a_2 \sin(\gamma_1 + \gamma_2))
\end{aligned} \tag{4.9a}$$

$$\begin{aligned}
a_1\gamma_1' = & \sigma_1 a_1 + 0.5z_{12} \frac{a_2\omega_2}{\omega_1} \sin(\gamma_1 - \gamma_2) - \frac{3p_{11}\bar{k}_{13}}{8\omega_1} \tilde{G}_1 + \frac{(p_{21} - p_{11})\bar{k}_{23}}{8\omega_1} \tilde{G}_2 + \\
& \frac{(p_{21} - p_{11})k_2}{2\omega_1} (p_{11}a_{1d}\tilde{Q}_{11} + p_{12}a_{2d}\tilde{Q}_{12}) + \frac{(p_{21} - p_{11})\bar{k}_{23}}{8\omega_1} \tilde{G}_3 + 0.5 \frac{p_{11}F_{11}}{\omega_1} \cos \gamma_1 \\
& + 0.25 \frac{p_{11}F_{12}}{\omega_1} (p_{11}a_1 \cos 2\gamma_1 + p_{12}a_2 \cos(\gamma_1 + \gamma_2))
\end{aligned} \tag{4.9b}$$

$$\begin{aligned}
a_2' = & -0.5z_{21} \frac{a_1\omega_1}{\omega_2} \cos(\gamma_2 - \gamma_1) - 0.5z_{22}a_2 - \frac{3p_{11}p_{12}\bar{k}_{13}a_1}{8\omega_2} G_4 \\
& + \frac{(p_{22} - p_{12})\bar{k}_{23}a_1}{8\omega_2} G_5 + \frac{(p_{22} - p_{12})k_2}{2\omega_2} (-p_{11}a_{1d}q_{11} + p_{12}a_{2d}q_{12}) \\
& + \frac{(p_{22} - p_{12})\bar{k}_{23}}{8\omega_2} G_6 + 0.5 \frac{p_{12}F_{11}}{\omega_1} \sin \gamma_2 + \\
& 0.25 \frac{p_{12}F_{12}}{\omega_1} (p_{12}a_2 \sin 2\gamma_1 + p_{11}a_1 \sin(\gamma_1 + \gamma_2))
\end{aligned} \tag{4.9c}$$

$$\begin{aligned}
a_2\gamma_2' = & \sigma_2 a_2 + 0.5z_{21} \frac{a_1\omega_1}{\omega_2} \sin(\gamma_2 - \gamma_1) - \frac{3p_{12}\bar{k}_{13}}{8\omega_2} \tilde{G}_4 + \frac{(p_{22} - p_{12})\bar{k}_{23}}{8\omega_2} \tilde{G}_5 \\
& + \frac{(p_{22} - p_{12})k_2}{2\omega_2} (p_{11}a_{1d}\tilde{q}_{11} + p_{12}a_{2d}\tilde{q}_{12}) + \frac{(p_{22} - p_{12})\bar{k}_{23}}{8\omega_2} \tilde{G}_6 + \\
& 0.5 \frac{p_{12}F_{11}}{\omega_2} \cos \gamma_2 + 0.25 \frac{p_{12}F_{12}}{\omega_2} (p_{12}a_2 \sin 2\gamma_1 + p_{11}a_1 \cos(\gamma_1 + \gamma_2))
\end{aligned} \tag{4.9d}$$

where $\gamma_1 = \sigma_1 T_1 - \beta_1$, $\gamma_2 = \sigma_2 T_1 - \beta_2$, $\sigma_3 = 2\sigma_1$, $\sigma_4 = 2\sigma_2$.

The terms Q_{11} , Q_{12} , $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, q_{11} , q_{12} , $\tilde{q}_{11}$, $\tilde{q}_{12}$, G_1 to G_6 and $\tilde{G}_1$ to $\tilde{G}_6$ are given in **Appendix-B**. These four reduced equations are used to study the steady-state response of

the system by equating $a_1' = \gamma_1' = a_2' = \gamma_2' = 0$ for the 1:1 internal, primary and principal parametric resonance conditions. These steady-state equations are solved by using the Newton's method and the stability of the system is ensured by finding the eigenvalue of the Jacobian matrix. It may be noted that a_1 , γ_1 , a_2 and γ_2 are the modal amplitudes and phase of the main system and absorber, respectively. The original amplitude of the system

is obtained as $\begin{bmatrix} A_T \\ B_T \end{bmatrix} = [\tilde{P}] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, where A_T and B_T represents the amplitudes of the main

system and the absorber for the traditional vibration absorber. Similarly, for the non-traditional vibration absorber, the original amplitude of the main system and the absorber are represented as A_N and B_N , respectively. One can observe from the reduced Eqs. (4.10a) to (4.10d) have both trivial and nontrivial solutions. However, the stability of the trivial state solutions cannot be studied due to the presence of $a_1\gamma_1'$ and $a_2\gamma_2'$. So, the normalization method is adopted by introducing the transformation as $p_1 = a_1 \cos \gamma_1$, $q_1 = a_1 \sin \gamma_1$, $p_2 = a_2 \cos \gamma_2$ and $q_2 = a_2 \sin \gamma_2$. Substituting the above transformation one can obtain the following Cartesian form of normalized reduced equations.

$$\begin{aligned} p_1' = & -0.5z_{11}p_1 - 0.5z_{12}p_2 \frac{\omega_2}{\omega_1} - \sigma_1 q_1 + \frac{p_{11}^2 F_{12}}{4\omega_1} q_1 + \frac{p_{11}p_{12}F_{12}}{4\omega_1} q_2 + \\ & \frac{(p_{21} - p_{11})k_2}{4\omega_1} \left(\begin{aligned} & -p_{11}(f_d - f_a \omega_1^2)h_{11} + p_{11}f_v \omega_1 h_{12} - \\ & p_{12}(f_d - f_a \omega_2^2)h_{13} + p_{12}f_v \omega_2 h_{14} \end{aligned} \right) \end{aligned} \quad (4.10a)$$

$$\begin{aligned} q_1' = & -0.5z_{11}q_1 - 0.5z_{12} \frac{\omega_2}{\omega_1} q_2 + \sigma_1 p_1 + \frac{p_{11}F_{11}}{2\omega_1} + \frac{p_{11}^2 F_{12}}{4\omega_1} p_1 + \frac{p_{11}p_{12}F_{12}}{4\omega_1} p_2 + \\ & \frac{(p_{21} - p_{11})k_2}{4\omega_1} \left(\begin{aligned} & p_{11}(f_d - f_a \omega_1^2)h_{12} + p_{11}f_v \omega_1 h_{11} + \\ & p_{12}(f_d - f_a \omega_2^2)h_{14} + p_{12}f_v \omega_2 h_{13} \end{aligned} \right) \end{aligned} \quad (4.10b)$$

$$\begin{aligned} p_2' = & -0.5z_{21} \frac{\omega_1}{\omega_2} p_1 - 0.5z_{22}p_2 - \sigma_2 q_2 + \frac{p_{12}^2 F_{12}}{4\omega_2} q_2 + \frac{p_{11}p_{12}F_{12}}{4\omega_2} q_1 + \\ & \frac{(p_{22} - p_{12})k_2}{4\omega_2} \left(\begin{aligned} & -p_{11}(f_d - f_a \omega_1^2)h_{11} + p_{11}f_v \omega_1 h_{12} - \\ & p_{12}(f_d - f_a \omega_2^2)h_{13} + p_{12}f_v \omega_2 h_{14} \end{aligned} \right) \end{aligned} \quad (4.10c)$$

$$\begin{aligned}
q_2' = & -0.5z_{21} \frac{\omega_1}{\omega_2} q_1 - 0.5z_{22} q_2 + \sigma_2 p_2 + \frac{p_{12} F_{11}}{2\omega_2} + \frac{p_{12}^2 F_{12}}{4\omega_2} p_2 + \frac{p_{11} p_{12} F_{12}}{4\omega_2} p_1 + \\
& \frac{(p_{22} - p_{12}) k_2}{4\omega_2} \left(p_{11} (f_d - f_a \omega_1^2) h_{12} + p_{11} f_v \omega_1 h_{11} + \right. \\
& \left. p_{12} (f_d - f_a \omega_2^2) h_{14} + p_{12} f_v \omega_2 h_{13} \right)
\end{aligned} \tag{4.10d}$$

where h_{11} to h_{14} are given in **Appendix B**.

The steady-state response and stability of both trivial and non-trivial solutions are studied in the next section by considering physical system parameters.

4.4 Results and Discussions

In this section efficacy of the feedback gains by displacement, velocity and acceleration feedback by the spring and PZT stack actuator combinations are studied considering physical system parameters to suppress the vibration of the main system. The analysis is carried out by studying the trivial instability regions, frequency responses, time responses, phase portraits, Poincare' sections and basin of attractions for different feedbacks, feedback gains (f_d, f_v, f_a) , time delays (t_d) , nonlinear stiffness (k_{13}, k_{23}) , the amplitude of external force and parametric excitations (F_{11}, F_{12}) and mass ratios (m_2/m_1) for both traditional and non-traditional active nonlinear vibration absorber. The following physical parameters for the traditional vibration absorber (TVA) are considered from the work of Habib et al. [161], which have been cited in more than 99 recent published works. It may be noted that these parameters are also the same as in the work of Detroux et al. [168]. The system parameters considered for the TVA in all the analysis are $m_1 = 1$ kg, $k_1 = 1$ N/m, $m_2 = 0.05$ kg, $k_2 = 0.0454$ N/m and $c_2 = 0.0128$ N-s/m. These parameters are taken in such a way that the natural frequency (ω_n) of the main system becomes unity. It may be noted that by taking any real system parameters one can reduce the governing equation to that of a system with unit mass and stiffness by taking the non-dimensional time $\tau = \omega_n t$. Here the mass ratio (m_2/m_1) is considered to be 1:20 which is taken in most of the literature and it is further varied up to 1:100 so that one can have a very lightweight absorber. The nonlinear stiffness of the absorber (k_{23}) is calculated from the formula given in [161] as $k_{23} = \frac{2(m_2/m_1)^2 k_{13}}{1 + 4(m_2/m_1)}$.

It may be noted that in [161] these system parameters for the traditional passive nonlinear

vibration absorber are the optimized values to suppress maximum vibration of the nonlinear main system and shows Den Hartog's equal peaks in the frequency response. Also, in [161] they have used a damper in the main system which is not included in the present system. Further, for different mass ratios (m_2 / m_1) , the optimized spring stiffness (k_2) , cubic nonlinear stiffness (k_{23}) and damping coefficient (c_2) values for the TVA are calculated from the formulae provided in Habib et al. [161].

For the non-traditional vibration absorber (NVA), the optimal spring stiffness (k_2) and damping coefficient (c_2) are calculated from the formulae provided in the work of Ren [99] for a mass ratio of 0.05. These optimal values are used to suppress the maximum vibration of the main system and show Den Hartog's equal peaks in the frequency response. The physical system parameters for the non-traditional vibration absorber are considered as $m_2 = 0.05$ kg, $k_2 = 0.0556$ N/m and $c_2 = 0.1446$ N-s/m. The main system parameters are the same as mentioned before. The bookkeeping parameter ε is considered equal to 0.1 throughout the analysis and PZT actuator properties are considered from Mallik and Chatterjee [20]. The amplitude of excitations F_{11} and F_{12} values for both TVA and NVA are varied from 0.05 N and 0.5 N/m to 5 N and 5 N/m, respectively. These variations are also mentioned in the corresponding sections. In the next sub-sections, the efficacy of the feedback gain on the trivial state instability regions of the system is studied for the traditional and non-traditional vibration absorber.

4.4.1 Trivial state instability regions

In this sub-section, the trivial state instability regions of the system are studied for the different amplitudes of parametric excitation, harmonic excitation, feedback gain and time delay in the feedback using traditional and non-traditional vibration absorbers. The trivial state instability regions are plotted by obtaining the Jacobian matrix from Eq. (4.10a) to Eq. (4.10d) and these regions are compared by solving the governing Eq. 3 by the fourth-order Runge Kutta method. The stability of the system is ensured by the negative real part of the Jacobian matrix obtained from Eq. (4.10a) to Eq. (4.10d). It may be noted that studies on the trivial instability regions of the system by the time-delayed traditional and non-traditional ANVA are not explored in previous literature.

4.4.1.1 Traditional vibration absorber

The trivial state boundary of the system with a variation of parametric excitation F_{12} and frequency Ω_2 is studied in Fig 4.2 for the traditional vibration absorber (TVA). In Fig. 4.2 the effectiveness of the feedback gain is studied by comparing passive and active feedback gain (Fig. 4.2(a)), with different time delays (Fig. 4.2(b)), passive TVA for simultaneous excitations (Fig. 4.2(c)) and active TVA for simultaneous excitations (Fig. 4.2(d)). Fig. 4.2(a) shows the trivial state instability region of the system for the passive and two active cases without time delay and considering F_{11} equal to zero. The *active1* case produced by the PZT stack actuator and the nonlinear spring in combinations with control gains $f_d =$

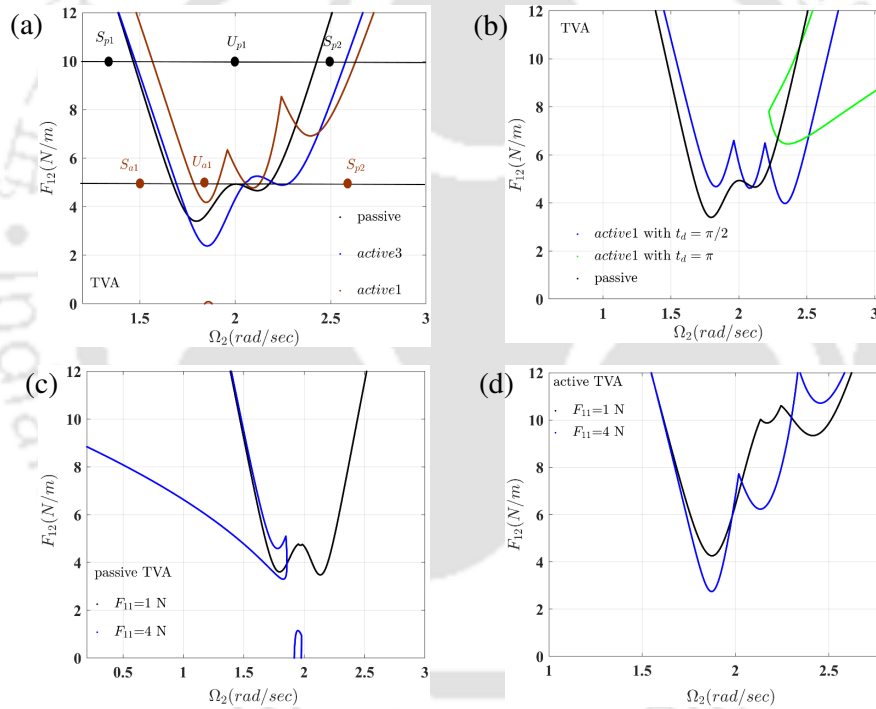


Fig. 4.2: Trivial state instability region with TVA (a) passive and active (b) active with time delay (c) passive with F_{11} (d) active with F_{11} .

5, $f_v = 0$, $f_a = -10$ and $f_a = -10$ and $k_a = 0.0544$ N/m and for the *active3* case the control gains are $f_d = 0$, $f_v = -5$, $f_a = 0$ and $k_a = 0.0544$ N/m. These combinations of feedback gains are selected after numerous variations and are found to suppress the vibration of the main system more effectively than other combinations. From Fig. 4.2(a) one can observe with the *active1* case the critical value of F_{12} (i.e., below which the system

remains stable for all operating frequency Ω_2) is found to be 4.2 N/m whereas for the passive it is 3.4 N/m. This suggests that with the feedback gain the system remains stable for a higher amplitude of F_{12} . However, the *active3* case produces a broader instability region with a lower critical value of F_{12} . To verify the system instability region for the TVA, time responses and phase portraits for the passive and active cases are shown in Fig.3 for three values of Ω_2 marked as S_{p1} , U_{p1} and S_{p2} for the passive case and S_{a1} , U_{a1} and S_{a2} for the active case, respectively. Clearly from Fig. 4.3(a) to (h) for the points S_{p1} and S_{p2} the system shows responses are bounded and hence stable and for U_{p1} the system is

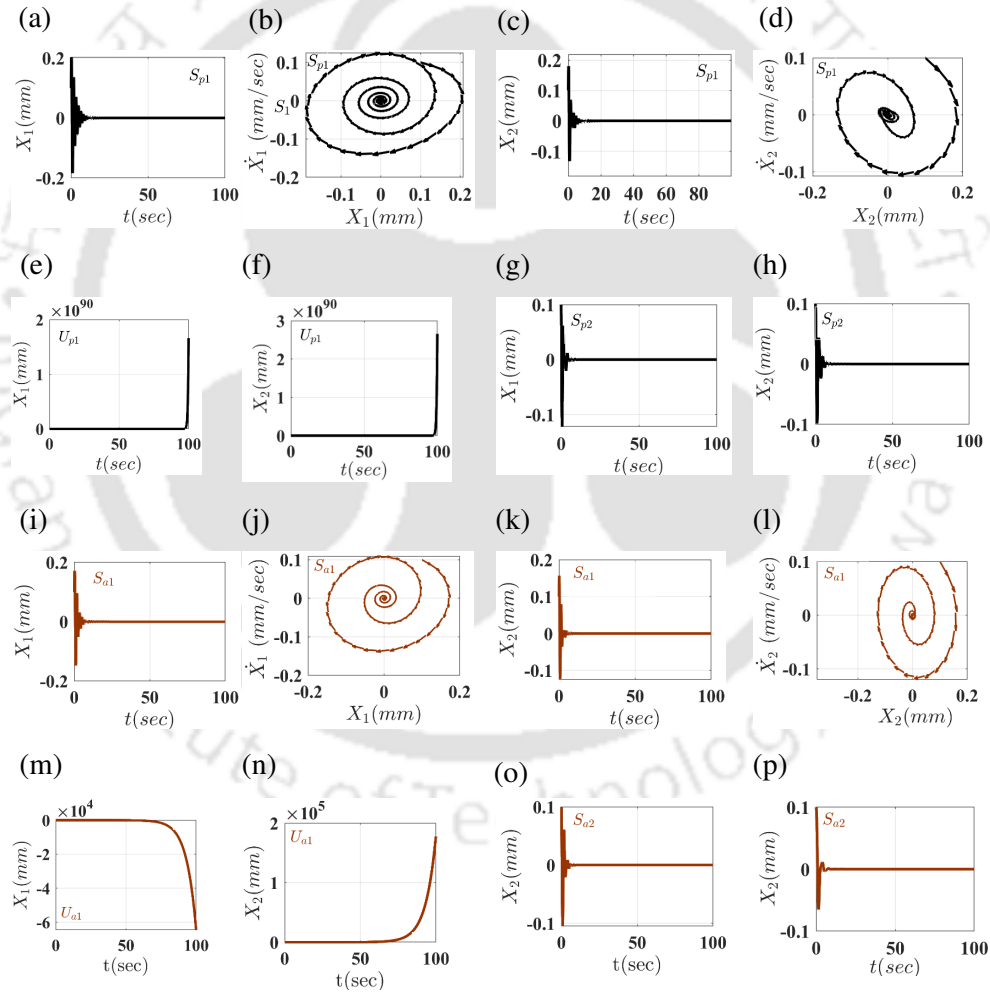


Fig. 4.3: Time responses (a, c, e-h, i, k, m-p) and phase portraits (b, d, j, l) with the passive and active TVA at the marked frequency points in Fig. 2(a).

trivially unstable. Similar observations are also perceived for the active case for the frequency points S_{a1} , U_{a1} and S_{a2} from Fig. 4.3(i) to 4.3(p). Also, from Fig. 4.2(a) one can observe for F_{12} equal to 5 N/m the instability region with passive TVA is in between 1.68 rad/sec to 2.19 rad/sec whereas for the active TVA this region is from 1.78 rad/sec to 1.91 rad/sec and again from 2.04 to 2.12 rad/sec. Since the *active1* case gives a higher critical value of forcing amplitude F_{12} therefore further analysis is carried out considering only the *active1* case. The various time delay effect in the *active1* case on the trivial instability regions are shown in Fig. 4.2(b). From this figure, one can observe that the critical value F_{12} significantly improves from 3.4 N/m in the passive case to 6 N/m in the active case (76% increase) with a time delay of π sec and the system shows a narrow unstable band of Ω_2 . The efficacy of the feedback gain is also noticed in Fig. 4.2(c) and Fig. 4.2(d) where simultaneous excitations F_{11} and F_{12} are considered. From Fig 4.2(c) it can be observed that with an increase in F_{11} values from 1 N to 4 N the stability boundary shifts to the lower values of Ω_2 and shows the unstable region from 0 to 2 rad/sec for the passive case. Also, a small unstable region is observed from $\Omega_2 = 1.91$ rad/sec to 1.97 rad/sec for F_{12} values from 0 N/m to 1 N/m, this unstable region grows for the passive case with the increased value of F_{11} , which is noticeable in Fig. 4.4(c) for the NVA. The unstable region of the system under F_{11} and F_{12} with passive TVA is significantly reduced once the feedback gain is applied which is shown in Fig. 4.2(d). Further, with the gain, the critical value of F_{12} increases when $F_{11} = 1$ N and for $F_{11} = 4$ N/m, no isolated instability region is observed. In all the instability regions shown in Fig. 4.2 kinks can be noted which are due to the 1:1 internal resonance considered in the present analysis.

4.4.1.2 Non-traditional vibration absorber

The similar analogy as in subsection 4.4.1.1 is carried out for the non-traditional vibration absorber (NVA) and the corresponding trivial state instability regions are shown in Fig. 4.4. From Fig. 4.4(a) one can observe with the passive NVA the critical value of F_{12} is 3.16 N/m. The corresponding value with the *active1* case is 6.24 N/m. Hence it is observed that with the *active1* case around double forcing amplitude F_{12} is required to make the system trivially unstable and also the instability region is reduced. The effects of time

delays on the trivial state instability region for the *activeI* case are shown in Fig. 4.4(b). From this figure, it is observed that with the $t_d = \pi$ sec in the *activeI* case the critical value does not reduce unlike with TVA (Fig. 4.2(b)) and also the frequency range of the instability region increases. In Fig. 4.4(c) and Fig. 4.4(d) the trivial state instability region are shown for simultaneous excitations F_{11} and F_{12} with the passive NVA and active NVA, respectively. From these figures, it is observed that for the *activeI* case not only the critical value of F_{12} increases but also the instability region is significantly reduced than the passive NVA. Comparing the effects of the feedback gain on the trivial state instability regions between TVA and NVA, it is observed that for TVA with a time delay equal to π sec in the *activeI* case, a significant reduction in the trivial state instability the region is observed and the critical value of forcing amplitude F_{12} increases to 1.76 times than the passive TVA. But for the NVA with only feedback gain without delay, a

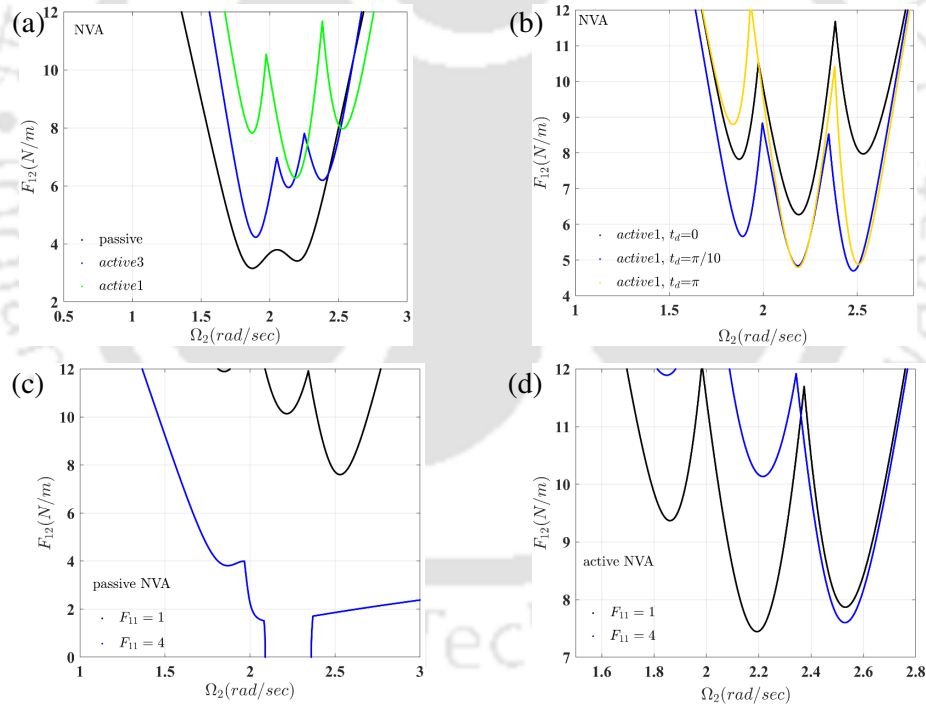


Fig. 4.4: Trivial state instability region with NVA (a) passive and active (b) active with time delay (c) passive with F_{11} (d) active with F_{11} .

significant improvement (200%) of the critical value of F_{12} and reduction in instability region is observed. The effectiveness of the feedback gain is also, noticeable when the main

system is subjected to simultaneous F_{11} and F_{12} excitations. In the next sub-section frequency responses and time responses of the linear system are studied.

4.4.2 Response of the linear system

In this sub-section, the frequency responses of the linear main system and the linear absorber are studied for both TVA and NVA by solving the reduced equations by the first order MMS from Eq. (4.9a) to (4.9d) using Newton's method. The analysis is carried out by studying the effects of various feedback gains (f_d, f_v, f_a) by different feedbacks, active tuning of the absorber (k_a) and time delay (t_d) in the various feedbacks to suppress the vibration of the main system. In all the frequency response plots the 'dot' or line signifies the stable solution and the 'circle' signifies the unstable solution. In the following subsection frequency responses of the linear main system are studied with only harmonic excitation.

4.4.2.1 Passive TVA under harmonic excitation

In this sub-section, the frequency responses obtained from the reduced first order MMS Eq. (4.9a) to Eq. (4.9d) are compared with various methods such as 2nd order MMS, HBM and fourth-order Runge Kutta method for the primary resonance condition and shown in Fig. 4.5. One can obtain the reduced equations by the 2nd order MMS for the passive absorber by following the standard procedure very easily, which is given in **Appendix B** (B. 3 to B. 6). In Fig. 4.5 the amplitude of forcing $F_1 = 0.005$ N is considered, which is the same as in the work of Detroux et al. [168] (Fig. 4.3(a)). It may be noted that in [168], HBM is used to obtain the frequency response of the main system with the same linear physical parameters as considered in the present work. In the present analysis, the frequency response obtained from the reduced equations by using the MMS is first time compared with the HBM. In previous literature, the frequency response obtained by reduced equations of MMS shows only a single peak [177-186, 215-217, 222-225, 286] with no Den Hartog's equal peaks in the frequency response. Most of the analyses in previous literature are non-dimensional and there is a clear deviation from the results by HBM [161-174, 219, 287] and various linear methods [122-152] in terms of the trend of the frequency response curves and the amplitudes of the main system at the operating frequency. Hence in the present

work, the governing Eq. 4.3 is obtained using the weighted modal matrix and then first order MMS is used to obtain the reduced equations. Fig. 4.5(a) shows the original response amplitude (A_T) and Fig. 4.5(b) shows the modal amplitude (a_1) response. From Fig. 4.5(a) one can observe the same trend of frequency response curves as other linear [1, 23, 48, 56, 99-107, 122-128, 192, 281, 325-329] and nonlinear methods [160-176, 193, 219, 287] with similar operating frequency range where the main system shows Den Hartog equal peaks. From Fig. 4.5(a) one can observe that there is a small deviation (0.006 m) of the amplitude of response between the 1st order MMS and the HBM results [168] for Ω_1 in the range of 0.88 rad/sec to 1.1 rad/sec. The amplitude response at $\Omega_1 = 1$ rad/sec by first-order MMS is 0.036 m while with HBM [168] it is 0.02851 m. This small deviation is also observed in the time response shown in Fig. 4.5(c) between the numerical method (NM) and the first order MMS. In Fig. 4.5(c) and Fig. 4.5(d), time responses of the original displacement (x_1) and modal displacement (X_1) of the main system is shown, respectively. The time response of the main system in Fig. 4.5(c) and Fig. 4.5(d) are obtained by NM from Eq. 3. The modal response amplitudes (a_1) of the main system are shown in Fig. 4.5(b) and the

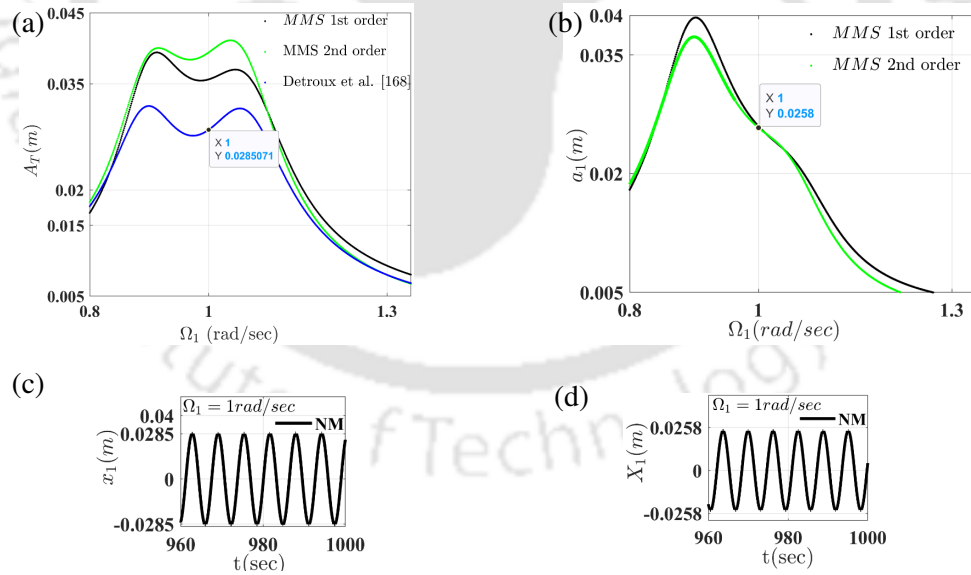


Fig. 4.5: Comparison of frequency responses by different methods for the (a) original and (b) modal response amplitude. Comparison of time responses at $\Omega_1 = 1$ rad/sec for (c) original and (d) modal displacement.

time responses shown in Fig 4.5(d) are exactly matching with each other at the corresponding frequency points ($\Omega_1 = 1$ rad/sec). The modal response amplitude in Fig. 4.5(b) shows a single large peak, unlike Fig. 4.5(a). The small deviation of 0.006 m in the original amplitude response (A_T) of the main system (Fig. 4.5(c)) between the MMS and NM may be due to the multiplication of the weighted modal matrix $[\tilde{P}]$ which increases the error. Also, From Fig 4.5(a) and Fig. 4.5(b) it can be observed that 2nd order MMS gives a larger deviation than the 1st order MMS, so further analysis is carried out by considering 1st order MMS reduced equations. It is inferred from Fig. 4.5 that the 1st order MMS gives a conservative response for the original amplitude response (A_T) and exactly matches with the modal amplitude response (a_1). In the next sub-sections analysis is carried out for the active cases.

4.4.2.2 Active vibration absorber for harmonic excitation

In this subsection, the vibration suppression of the main system under F_{11} equal to 1 N ($F_1 = 0.1$ N) for the primary resonance condition is studied by various feedbacks, gains, time delays and actively changing the stiffness of the absorber. In Fig. 4.6 frequency response of the main system is compared with the passive vibration absorber by considering only displacement feedback (f_d), velocity feedback (f_v), acceleration feedback (f_a) and a combination of these feedbacks with time delays (t_d). Figs. 4.6(a-d) and Figs. 4.6(e, f) shows the frequency response of the main system by using TVA and NVA, respectively. In Fig. 4.6(a) the response amplitude of the main system is plotted for different f_d values. From this figure, it can be observed that the response amplitude of the main system decreases for Ω_1 in the range of 0.82 rad/sec to 1.15 rad/sec with the increasing value of f_d . This may happen as f_d increases the stiffness of the absorber increases and tunes the absorber frequency to the main system frequency. But this vibration suppression is limited close to the resonance condition ($\Omega_1 = 1$ rad/sec) and also the response amplitude increases at the peaks with higher feedback force. It is observed that the peak amplitude increases for f_d value greater than 5. The frequency response with the passive TVA shown in Fig. 4.6(a) is well-matched with Fig. 2(a) from the work of Habib et al. [161], where the amplitude of forcing (F_1) is considered 100 times higher than the present work. The modal amplitude-

frequency response is shown in Fig. 4.6(b) where similar observations are noticed but the trend of the curves are different. From Fig. 4.6(c) one can observe that the response amplitude increases with $f_v = 5$ for Ω_1 in the range of 0.82 rad/s to 0.999 rad/s but decreases for Ω_1 in the range of 1 rad/s to 1.15 rad/s. One can notice that the negative and positive f_v forces are mirror images of each other. In Fig. 4.6(d) frequency response is plotted for different values of f_a and it is observed that more reduction in vibration for Ω_1 in the range of 0.88 rad/sec to 1.08 rad/sec than with f_d (Fig. 4.6(a)) and f_v (Fig. 4.6(c)). However, here also a higher response amplitude near the first peak is observed. In Fig. 4.6(e) the frequency response of the system is plotted when the main system is subjected to a combination of f_d, f_v, f_a forces and actively changes the stiffness of the absorber (k_a). In Fig. 4.6(e) the feedback gains by the TVA are represented by four combinations of control gains namely *active1*, *active2*, *active3*, and *active4*. The *active1* and *active3* are the same as mentioned in the sub-section 4.1.1 while the *active2* case is obtained for $k_a = 0.0544$ N/m. In *active4* case $k_a = 0.0544$ N/m, $f_d = 2$, $f_v = -5$ and $f_a = -10$ is considered. It is observed that out of many combinations considered above with the *active3* case the vibration suppression is better than the other combinations which reduce 50% of the amplitude response at the peaks and also completely suppress the vibration at $\Omega_1 = 1.042$ rad/sec. Moreover, for a wider band of Ω_1 the vibration suppression of the main system is achieved. The two-time delays effect in the *active1* case and *active3* case are studied in Fig. 4.6(f) using the TVA. From Fig. 4.6(f) one can observe for lower time delay i.e., $t_d = \pi/10$ sec for both *active1* and *active3* the response amplitudes are less in comparison to the active system without delay (Fig. 4.6(e)). From Fig. 4.6(f) the maximum response amplitude by *active1* at the first peak with $t_d = \pi/10$ sec is found to be 0.32 m whereas without delay it is 0.76 m (Fig. 4.6(e)) which is a reduction of 58% of amplitude. Also, with *active1* and $t_d = \pi/10$ sec the vibration reduction is for a wider range of Ω_1 than without delay. However, with a higher delay of $t_d = \pi/2$ sec, the system becomes completely unstable as shown in Fig. 6(f). Similar observations are also noticed in Fig. 4.6(f) for *active3* with lower and higher time delays. Here with *active3* and $t_d = \pi/10$ sec the variation of response amplitude in comparison to the active system with $t_d = 0$ (Fig. 4.6(e)) is found to be negligible and the system response is found to be stable with very high response amplitude. In Fig. 4.6(g) and Fig. 4.6(h), the response amplitude of the main

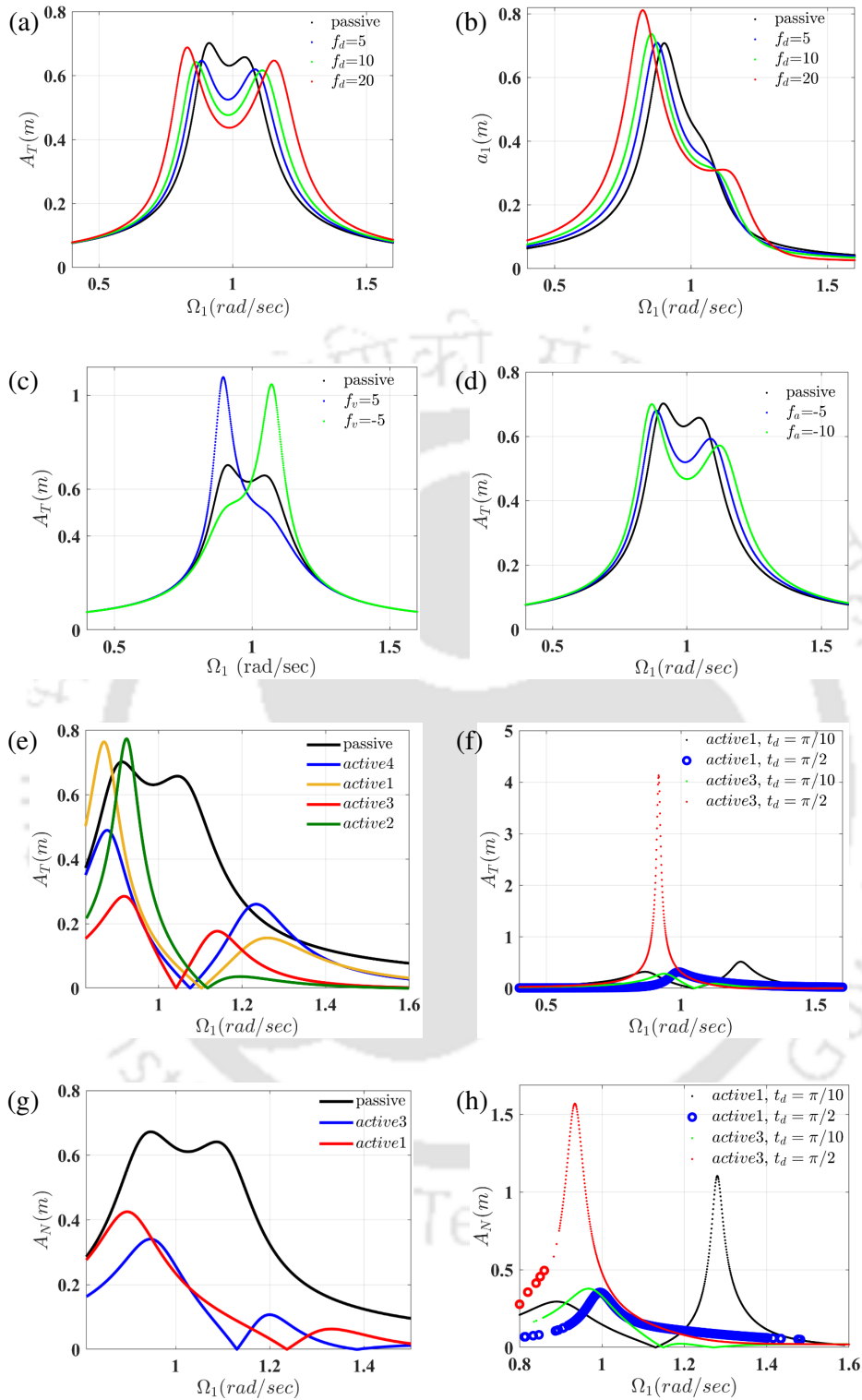


Fig. 4.6: Frequency responses with different feedbacks, feedback gains and time delays with the TVA (a-f) and with NVA (g, h).

system with NVA is shown. In Fig. 6(g) the effectiveness of the *active1* and *active2* cases can be observed for completely suppressing the vibration of the main system at Ω_1 equal to 1.22 rad/sec and also for a wider range of operating Ω_1 . It may be noted from Fig. 4.6(g) that the response amplitude of the main system with the passive NVA is closely matching with Fig. 4.4(b) of Shen et al. work [329], where they compared their results with the passive NVA [24]. Also, from Fig. 4.6(a) and Fig. 4.6(g) one can observe the maximum response amplitude of the main system at peak using passive TVA is 0.7 m whereas with passive NVA the maximum amplitude is 0.67 m. These comparative results also matched with the work of Ren [99] where they showed that with the NVA the vibration of the main system can be little more effectively suppressed than the TVA. Hence from the present analysis, it is shown that with active NVA the reduction in response amplitude is more effective than those in the published literature [99, 329]. In the next sub-section, the analysis is carried out when the main system is subjected to simultaneous excitations.

4.4.2.3 Active vibration absorber for simultaneous excitation

In this subsection, a similar analogy as in subsection 4.2.2 is carried out but here the main system is subjected to simultaneous forcing amplitude $F_{11} = 1$ N and $F_{12} = 1$ N/m for the primary and principal parametric resonance conditions. It may be noted Ω_2 is equal to $2\Omega_1$ from the resonance conditions. Fig. 4.7(a) shows that an increase in f_d the response amplitude of the main system decreases for Ω_1 in the range of 0.85 rad/sec to 1.05 rad/sec but thereafter increases compared to the passive case. Fig. 4.7(b) shows for $f_v = -5$ the vibration suppression is in narrower range of 0.8 rad/sec to 0.99 rad/sec compared to f_d (Fig. 4.7(a)). It is observed that a further increase in the magnitude of f_v induces a high amplitude response in the second peak with unstable solutions in the frequency response plot. Fig. 4.7(c) shows the frequency response plot for three different values of f_d . This figure is similar to Fig. 4.7(a) but in this case, the response amplitude is slightly more. In Fig. 4.7(d) the response amplitude of the main system is compared between the passive case and the combination of feedbacks. Fig. 4.7(d) shows that for both *active1* and *active3* cases there is a reduction in the response amplitude of the main system for the whole range of Ω_1 compared to the passive case. The response amplitude of the main system is completely suppressed for the *active1* and *active3* cases at $\Omega_1 = 1.075$ rad/sec and at Ω_1

=1.144 rad/sec, respectively. Though in the *active3* case the reduction in the response amplitude is better than in the *active1* case. For the *active1* case with a time delay of $\pi/10$ sec reduces the first peak amplitude, but produces a higher amplitude response in the second peak and also the system becomes unstable as shown in Fig. 4.7(d). In Fig. 4.7(e) and Fig. 4.7(f) frequency responses of the main system with NVA are studied for the passive, *active1* and *active3* cases. From Fig. 4.7(e) the effectiveness of *active1* and *active3* cases can also be observed as in TVA for completely suppressing the vibration at the indicated

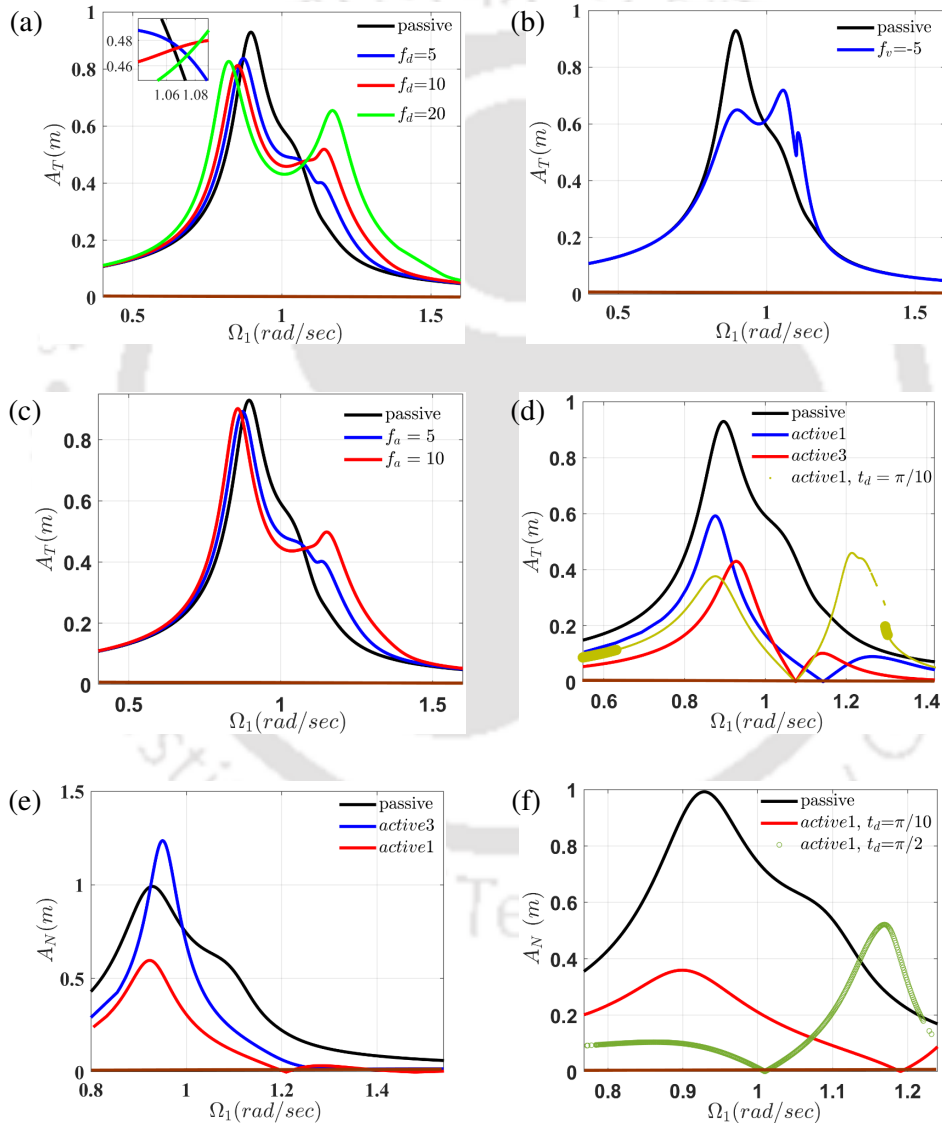


Fig. 4.7: Frequency responses with F_{11} and F_{12} for different feedbacks, feedback gains and time delays with the (a-d) TVA and with the (e, f) NVA.

Ω_1 and also suppressing the vibration for a wider range of Ω_1 . In Fig. 4.7(f) time delay in the *active1* case is studied and compared with the passive case. From Fig. 4.7(f) one can observe that for $t_d = \pi/2$ sec the systems become unstable for a large range of Ω_1 . But for $t_d = \pi/10$ sec the system shows no unstable solutions with a better reduction in the response amplitude than without delay (Fig. 4.7(e)). All the trivial state responses show stable solutions in Fig. 4.7 as evident from the trivial state instability regions curves are shown in Fig. 4.2 and Fig. 4.4 for $F_{11} = 1$ N and $F_{12} = 1$ N/m. The efficacy of the proper time-delayed feedback gain by the absorber in suppressing the vibration of the main system is also noticed in the experimental work of Xu and Sun [279]. They showed experimentally that with proper time delay in the feedback gain by acceleration feedback significant reduction in the vibration of the linear main system under base excitation is achieved for a certain range of operating frequencies.

4.4.3 Response of the nonlinear system

In this sub-section, the frequency responses of the nonlinear main system and the nonlinear absorber are studied for both TVA and NVA by solving the reduced equations by the MMS from Eq. (4.9a) to (4.9d) using Newton's method. The similar analyses are carried out as in the linear system but here different cubic nonlinear stiffnesses in the springs are considered and vibration suppression by *active1* and *active2* cases are discussed for simultaneous primary and principal parametric resonance conditions. The nonlinear studies carried out in the next section can be applied to many practical systems as nonlinearity is inherently present, particularly in the joints, bearings and other moving parts. Also, nonlinearities arise due to various operations in the machining process [39, 41], automobiles [32], ships [184], aircraft, tall and large structures such as buildings, bridges, beams, flexible structures, off-shore structures, and railroad wheelsets, etc. [5-8, 10-21, 27-32, 35, 37, 39-41, 43, 54-62, 156-230, 286, 287, 330] due to various form of external excitations on the primary systems. So, the present nonlinear analyses are more practical in nature. In the following subsection frequency responses, time responses, and phase portraits of the main system are studied for $F_{11} = 1$ N and $F_{12} = 1$ N/m.

4.4.3.1 Nonlinear system for simultaneous excitation

In this subsection frequency response of the system is studied in Fig. 8 for two values of cubic nonlinear stiffness in the main system ($\bar{k}_{13}$) as 1 N/m³ and 5 N/m³ and the corresponding optimal $\bar{k}_{23}$ values are calculated from the work of Habib et al. [161] with $F_{11} = F_{12} = 1$ N. In Fig. 8(a) $\bar{k}_{13} = 1$ N/m³ and $\bar{k}_{23} = 4.2 \times 10^{-3}$ N/m³. Fig. 4.8(a) shows that for the passive TVA the main system vibrates with a maximum response amplitude of 0.8m and an unstable isolated region of higher amplitude is also observed for Ω_1 in the range of 1.2 rad/sec to 1.5 rad/sec with Hopf bifurcation point at $\Omega_1 = 1.32$. The response amplitude of the main system reduces once the feedback gain is applied by the TVA as shown in Fig. 4.8(a). For the *active1* case, a minimum of 80% of vibration reduction is achieved for Ω_1 in the range of 0.78 rad/sec to 1.1 rad/sec. Further, 100% vibration suppression of the main system is observed for $0.88 \text{ rad/sec} \leq \Omega_1 \leq 1 \text{ rad/sec}$ compared to the passive case. For the *active2* case the response amplitude is not decreased as compared to the *active1* case and also an isolated unstable region is observed with a lesser amplitude response than the passive TVA. Fig. 4.8(b) shows the modal amplitude response of the main system with the TVA for the same parameters as in Fig. 4.8(a). Here a higher response amplitude is observed than in Fig. 8(a). The Hopf bifurcation points (H_1, H_2) and saddle-node bifurcation points (S) for the *active1* case are marked in Fig. 4.8(a) and Fig. 4.8(b), which are verified by the time responses and phase portraits curves and shown in Fig. 4.9. From Fig. 4.9 time responses and phase portraits corresponding to marked points in Fig. 4.8(a) and 4.8(b) one can observe periodic responses at H_1 and H_2 points and fixed point responses at S point, which confirms the Hopf bifurcation points (H_1 and H_2). In Fig. 4.8(c) response of the main system is shown with TVA for $\bar{k}_{13} = 5$ N/m³ and $\bar{k}_{23} = 4.2 \times 10^{-3}$ N/m³. It may be noted that here $\bar{k}_{23}$ value is not calculated from the Habib et al. [161] work and kept the same as in Fig. 4.8(a). From Fig. 4.8(c) with the passive TVA two peaks in the frequency response can be observed for the passive, which is 28% less than in Fig. 4.8(a) for the passive case. The response amplitude is considerably reduced for the *active1* and *active2* cases as shown in Fig. 4.8(c). The passive TVA shows a maximum amplitude response of 0.736 m with an isolated unstable region for $\Omega_1 \geq 1.16$ rad/sec and Hopf bifurcation point at $\Omega_1 = 1.209$ rad/sec. The 100% vibration suppression for *active1*

case is achieved at $\Omega_1 = 0.95$ rad/sec. It can be observed that for the *active1* case though significant amplitude reduction (minimum of 75%) is achieved from $0.625 \text{ rad/sec} \leq \Omega_1 \leq 1.12 \text{ rad/sec}$ but produces unstable response with saddle-node bifurcation point at $\Omega_1 =$

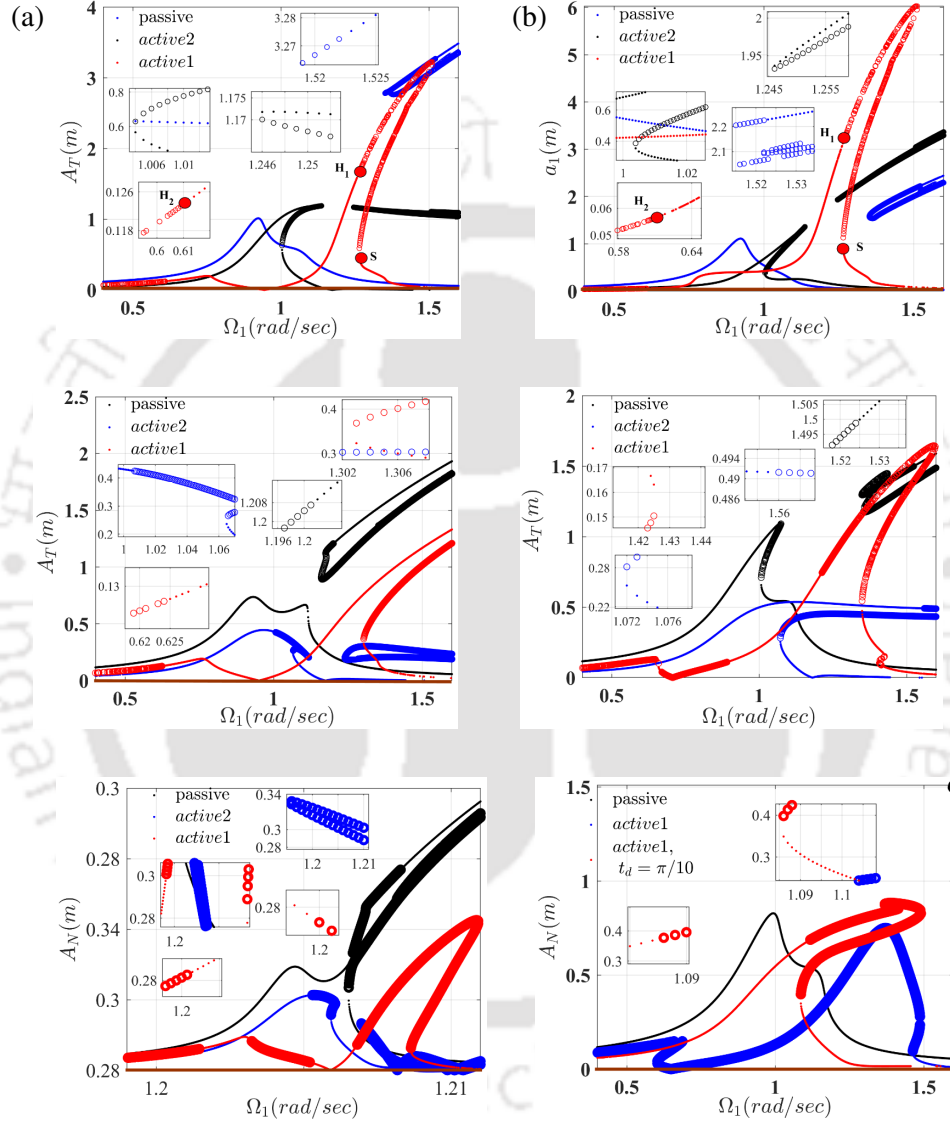


Fig. 4.8: Frequency responses with F_{11} and F_{12} for different $\bar{k}_{13}$ and $\bar{k}_{23}$ values using the TVA (a-d) and NVA (e, f). (a, b) $\bar{k}_{13} = 1 \text{ N/m}^3$ and $\bar{k}_{23} = 4.2 \times 10^{-3} \text{ N/m}^3$ (c) $\bar{k}_{13} = 5 \text{ N/m}^3$ and $\bar{k}_{23} = 4.2 \times 10^{-3} \text{ N/m}^3$ (d) $\bar{k}_{13} = 5 \text{ N/m}^3$ and $2.1 \times 10^{-2} \text{ N/m}^3$ (e) $\bar{k}_{13} = 5 \text{ N/m}^3$ and $\bar{k}_{23} = 4.2 \times 10^{-3} \text{ N/m}^3$ (f) $\bar{k}_{13} = 5 \text{ N/m}^3$ and $\bar{k}_{23} = 2.1 \times 10^{-2} \text{ N/m}^3$.

1.302 rad/sec. Also, for the *active1* case, no isolated unstable region is observed, unlike the passive case. The 100% vibration suppression for *active1* case is achieved at $\Omega_1 = 0.95$ rad/sec. The unstable response with *active1* case is observed for $\Omega_1 \leq 0.625$ rad/sec with Hopf bifurcation point at $\Omega_1 = 0.625$ rad/sec. So, it is suggested that the *active1* case can be utilized for $0.625 \text{ rad/sec} \leq \Omega_1 \leq 1.12 \text{ rad/sec}$ and outside this region, the *active2* case can be utilized for maximum vibration suppression of the main system for a wider band of Ω_1 . In Fig. 8(d) $\bar{k}_{13} = 5 \text{ N/m}^3$ is considered and the corresponding $\bar{k}_{23}$ value is calculated from [161], which is equal to 0.021 N/m^3 . Here the passive TVA shows jump up and jump down phenomena at saddle-node bifurcation points 1.007 rad/sec and 1.19 rad/sec, respectively. The response amplitude of the main system decreases for *active1* case from 0.4 rad/sec to 1.072 rad/sec and shows saddle-node bifurcation point at $\Omega_1 = 1.072$ rad/sec. In Fig. 4.8(d) the effectiveness of the *active1* case is observed for a very small range of Ω_1 i.e from 0.89 rad/sec to 1.21 rad/sec and produces a more unstable region. From Fig. 4.8(d) one can observe that the *active2* case is more effective than the *active1* case as it produces

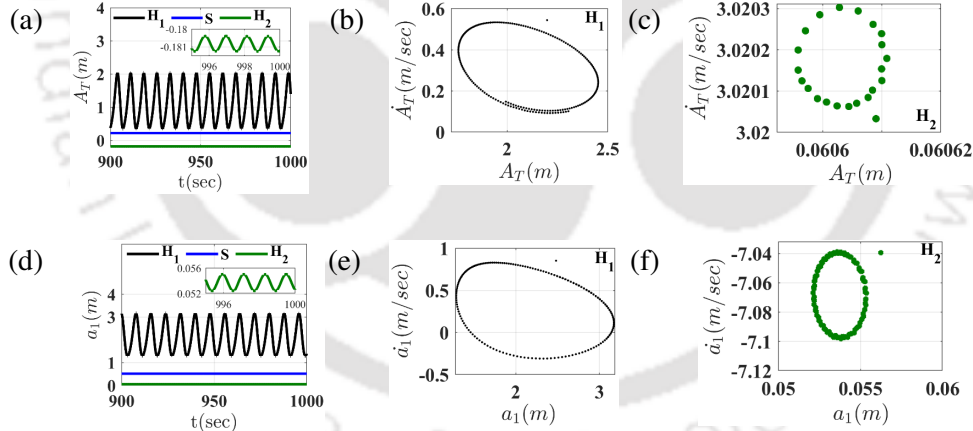


Fig. 4.9: Time responses (a, d) phase portraits (b, c, e, f) at the marked H_1 , H_2 and S frequency points in Fig. 4.8(a) and Fig. 4.8(b).

a more stable range of Ω_1 . All the trivial state solutions in Fig. 4.8 shows stable solutions as evident from Fig. 4.2 for $F_{11} = 1 \text{ N}$ and $F_{12} = 1 \text{ N/m}$. From the various $\bar{k}_{13}$ and $\bar{k}_{23}$ values, it is inferred that the nonlinear cubic stiffness in the absorber ($\bar{k}_{23}$) makes the system more unstable than $\bar{k}_{13}$. So one can use less k_{23} value in the range of 1 to 2% of the

linear stiffness (k_2) and higher k_{13} in the main system in the range of 10% to 50% of the linear stiffness (k_1) for better vibration reduction in a wider range of operating frequency for the *active1* case. In Fig. 4.8(e) and Fig. 4.8(f) frequency response of the nonlinear main system is studied for the passive and active cases with the nonlinear NVA. The nonlinear stiffness in Fig. 4.8(e) and 4.8(f) are taken the same as in Fig. 4.8(c) and 4.8(d), respectively. From Fig. 4.8(e) one may observe that in the *active1* case the system is unstable for a larger range of Ω_1 than the *active2* case. So, here for $\bar{k}_{13} = 5 \text{ N/m}^3$ and $\bar{k}_{23} = 4.2 \times 10^{-3} \text{ N/m}^3$ the control parameter in the *active1* case can be utilised for $0.646 \text{ rad/sec} \leq \Omega_1 \leq 0.813 \text{ rad/sec}$ and $1.02 \text{ rad/sec} \leq \Omega_1 \leq 1.18 \text{ rad/sec}$ and outside this region *active2* case may be utilised. In Fig. 4.8(f) for $\bar{k}_{13} = 0.5 \text{ N/m}^3$ and $\bar{k}_{23} = 0.021 \text{ N/m}^3$, it is shown that with $t_d = \pi/10$ sec in the *active1* case the unstable region is reduced than without delay and also decreases the response amplitude than the passive case for $\Omega_1 \leq 1.08 \text{ rad/sec}$. From Fig. 4.8 it is observed that with *active1* and *active2* cases the response amplitude of the main system is significantly reduced for different nonlinear stiffness in the system. The unstable region of frequency response by the *active1* case can be eliminated by using the *active2* case. It may be noted that in the *active2* case all control gains f_d , f_v and f_a are equal to zero. So, one can switch from *active2* case to *active1* case easily by providing the mentioned f_d and f_a forces. These results are also in good agreement qualitatively with [171, 193] where the efficacy of the feedback gains are shown for the nonlinear absorber. Wang et al. [289, 290] showed that with the active dynamic vibration absorber (ADVA), vibration suppression of the main system is better than the passive case and also the nonlinear ADVA further reduces the convergence time of oscillations than the linear ADVA. The nonlinearity in the absorber is more practical and also suppresses the vibration better than the linear vibration absorber as also evident from the present studies. The feedback gain produced in the present work is also nonlinear due to the attached nonlinear spring with PZT stack actuator, which provides more blocking force and requires less voltage supply. The effectiveness of the feedback gain in the present work is also in good agreement with the experimental works [153, 154, 331, 287, 260, 279, 280] where nonlinear analyses are considered. The time-delayed effects in the feedback gain by the piezoelectric-based active vibrations and its importance are also in good agreement with the experimental works [279, 280]. However, unlike in the present work the complete vibration suppression of the undamped main system by harmonic and parametric

excitations is not achieved by the previous literature. In the next sub-section frequency response of the nonlinear system under parametric excitation $F_{12} = 5 \text{ N/m}$ is studied.

4.4.3.2 Nonlinear system with parametric excitation

In this sub-section, the response of the main system is studied when it is subjected to only parametric excitation with $F_{12} = 5 \text{ N/m}$. The frequency response, time response, Poincare' sections and basin of attractions with TVA are plotted considering nonlinear stiffness $\bar{k}_{13} = 5 \text{ N/m}^3$ and $\bar{k}_{23} = 0.021 \text{ N/m}^3$, which are the same as those in Fig. 4.8(d). It may be noted that here only the principal parametric resonance condition is studied for the passive case (Fig. 4.10(a)) and *active1* case (Fig. 4.10(b)). Fig. 4.10(a) shows the frequency response for the passive case where the trivial state becomes unstable for $1.7 \text{ rad/sec} \leq \Omega_2 \leq 2.2 \text{ rad/sec}$ and this trivial unstable region are well matches with the trivial state instability region shown in Fig. 4.2(a) for the passive case. A special type of pitchfork bifurcation is observed at $\Omega_2 = 2.32 \text{ rad/sec}$ and 2.6 rad/sec where the stable trivial state leads to an unstable nontrivial and a stable trivial state. Further, at B i.e., $\Omega_2 = 1.38 \text{ rad/sec}$, another special type of bifurcation occurs where the unstable nontrivial branch bifurcates into a stable and an unstable branch. The time responses and Poincare' sections (Fig. 4.11) clearly show the chaotic response of the system at point B . One can observe a crisis phenomenon with a slight increase in the frequency Ω_2 as the chaotic response vanishes leading to a stable nontrivial response. In Fig. 4.10(b) the trivial state instability region is observed for 1.8

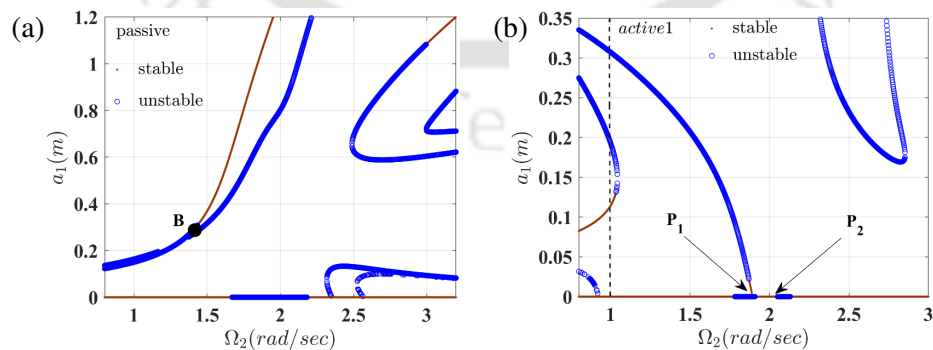


Fig. 4.10: Frequency responses with F_{12} using the (a) passive TVA and the (b) active TVA.

$\text{rad/sec} \leq \Omega_2 \leq 1.9 \text{ rad/sec}$ and $2.04 \text{ rad/sec} \leq \Omega_2 \leq 2.16 \text{ rad/sec}$ which are in good agreement with Fig. 4.2(a) for the active TVA. Clearly, five bifurcation points are observed in the trivial states which are subcritical pitchfork, Hopf, supercritical pitchfork (P_1), supercritical Hopf (P_2) and another Hopf bifurcation. At P_1 ($\Omega_2 = 1.88 \text{ rad/sec}$) the time response and Poincare' section of the response of the system are shown in Fig. 4.12(a) and Fig. 4.12(b), respectively. Similarly, at P_2 ($\Omega_2 = 2.05 \text{ rad/sec}$) time response and Poincare'

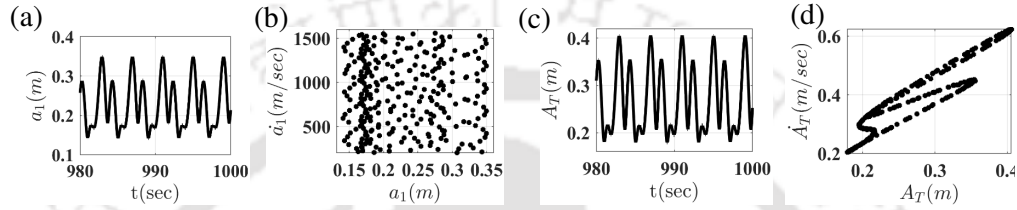


Fig. 4.11: Time responses (a, c) and Poincare's sections (b, d) at the marked point B in Fig. 4.10(a).

sections are shown in Fig. 4.12(c) and Fig. 4.12(d), respectively. From Fig. 4.12(c) and Fig. 4.12(d) the response of the system is periodic which matches with Fig. 4.10(b) Hopf bifurcation point at P_2 . The time response and Poincare' section in Fig. 4.12(a) and Fig. 4.12(b) show chaotic responses which are also in good agreement with Fig. 4.10(b). Fig.

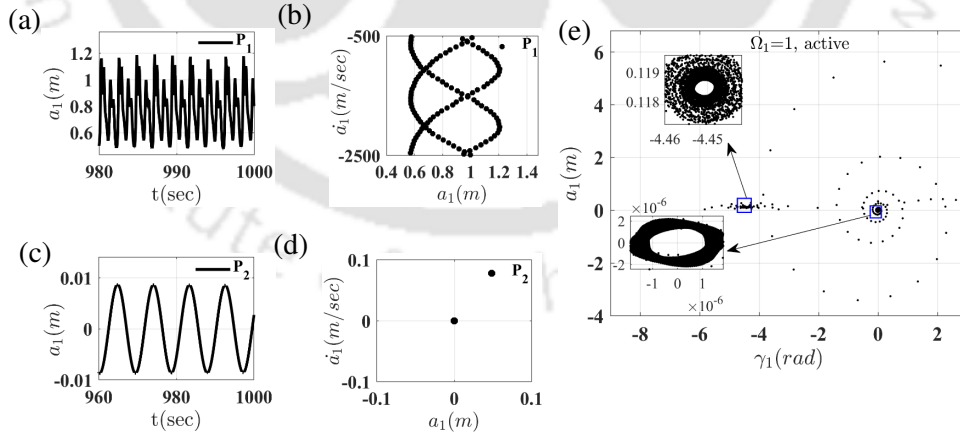


Fig. 4.12: Time responses, Poincare's sections and basin of attraction at the marked points in Fig. 4.10(b). (a, b) P_1 point (c, d) P_2 point and (e) basin of attraction at $\Omega_2 = 1 \text{ rad/sec}$.

4.12(e) shows a part of the basin of attraction with two stable solutions corresponding to the frequency response shown in Fig. 10(b) at $\Omega_2 = 1$ rad/sec. Here one can observe the stable trivial state ($a_1 = 0$) and stable nontrivial state at ($a_1 = 0.1185$ m), which are closely matched with the solutions shown in Fig. 4.10(b). It may be noted that as in the present work four state variables (a_1 , γ_1 , a_2 and γ_2) are existing, so it is difficult to show a clear basin of attraction. In the next sub-section response amplitudes of the highly nonlinear system for the passive and active cases are studied with variation in the amplitude of external excitations.

4.4.4 Force response analysis

In this sub-section, the response amplitude of the main system with a simultaneous variation of F_{11} and F_{12} values for the simultaneous primary and principal parametric resonance conditions are discussed. Fig. 4.13 shows three-dimensional plots for the variation of response amplitude (A_r) with F_{11} and F_{12} at $\Omega_1 = 0.9$ rad/sec, 1 rad/sec and 1.1 rad/sec for the passive and active TVAs. The nonlinear stiffness value $\bar{k}_{13}$ and $\bar{k}_{23}$ are equal to 5 N/m³ and 0.021 N/m³, respectively which are the same as in Fig. 4.8(d) and Fig 4.10. For the passive TVA in Fig. 4.13 (a), one can observe that for $\Omega_1 = 0.9$ rad/sec, the system is completely stable but for $\Omega_1 = 1$ rad/sec the system has saddle-node bifurcation points ($F_{11} = 0.82$ N, 0.91 N and $F_{12} = 0.82$ N/m, 0.91 N/m) and for $\Omega_1 = 1.1$ rad/sec the system has multiple bifurcation points viz. Hopf bifurcations ($F_{11} = 1.24$ N, 2.19 N and $F_{12} = 1.24$ N/m, 2.19 N/m) and saddle node bifurcations ($F_{11} = 2.29$ N and 1.06 N and $F_{12} = 2.29$ N/m and 1.06 N/m) leading to jump up and jump down phenomena. For $\Omega_1 = 1.1$ rad/sec the jump up and jump down phenomena are more prominent than those for $\Omega_1 = 1$ rad/sec. The jump down length at $\Omega_1 = 1.1$ rad/sec for a slight decrease in $F_{11} = 1.06$ N and $F_{12} = 1.06$ N/m is found to be 0.56 m and for the slight increase in $F_{11} = 2.29$ N and $F_{12} = 2.29$ N/m the jump up length is 0.38 m. This sudden decrease or increase in the response amplitude of the main system may fail the system, so the active TVA is more useful in these conditions. In the active TVA, shown in Fig. 4.13(b), there are no jump up or jump down phenomena are observed. It is observed that for the *activeI* case (active

TVA) the response of the main system is reduced to a minimum of 48% to a maximum of 99% in comparison to the passive case (Fig. 4.13(a)) for F_{11} and F_{12} values in the range 0

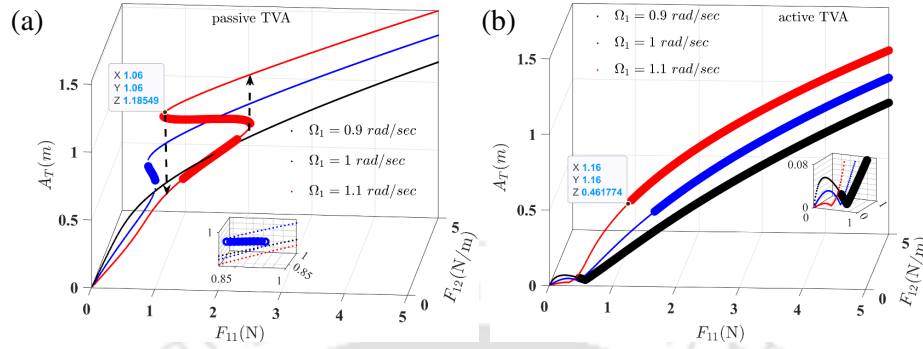


Fig. 4.13: Response amplitudes with variation in F_{11} and F_{12} using the (a) passive TVA and the (b) active TVA.

to 1 for $\Omega_1 = 1$ rad/sec. Similarly, $\Omega_1 = 1.1$ rad/sec the reduction is found to be a minimum of 80% to a maximum of 99% than the passive case. For $\Omega_1 = 0.9$ rad/sec, from Fig. 4.13(b) it is observed that in the *activeI* case the system becomes unstable after $F_{11} = 0.42$ N and $F_{12} = 0.42$ N/m. In the active TVA (Fig. 4.13(b)) the Hopf bifurcation points for $\Omega_1 = 1$ rad/sec and 1.1 rad/sec are found to be for $F_{11} = 1.48$ N and 1.16 N and $F_{12} = 1.48$ N/m and 1.16 N/m, respectively. The response of the main system with NVA is not shown here as the results are of similar characteristics. From the above figures, it is observed that the active TVA reduces significant response amplitude of the main system for F_{11} and F_{12} values in the range of 0 to 1 and for higher excitation amplitudes one may increase the feedback gain parameters f_d , f_v and f_a values. Further, with the active TVA no jump up or jump down phenomena are observed. In the next sub-section responses of the nonlinear system for the passive and active cases are studied with variation in the nonlinear stiffness of the system.

4.4.5 Variation of response amplitude with the nonlinear stiffness

Fig 4.14(a, b) shows the variation of response amplitude of the main system with $\bar{k}_{13}$ and $\bar{k}_{23}$ for three values of Ω_1 i.e. 0.9 rad/sec, 1 rad/sec and 1.1 rad/sec for the simultaneous primary and principal parametric resonance conditions considering $F_{11} = 1$ N and $F_{12} = 1$

N/m with the passive and active TVA. The optimized $\bar{k}_{23}$ values for different $\bar{k}_{13}$ values (0 to 6 N/m³) are calculated from [161]. From Fig 4.14(a) with the passive TVA the response remains stable for $\Omega_1 = 0.9$ rad/sec but for 1 rad/sec and 1.1 rad/sec saddle-node bifurcations points lead to jump up and jump down phenomena can be observed. For $\Omega_1 = 1$ rad/sec at the saddle-node bifurcation point ($\bar{k}_{13} = 4.28$ N/m³ and $\bar{k}_{23} = 0.01798$ N/m³) the jump up length is 0.269 m and at the second saddle-node bifurcation point ($\bar{k}_{13} = 3.16$ N/m³ and $\bar{k}_{23} = 0.0132$ N/m³) the jump down length is found to be 0.4342 m. Similar bifurcation points are also observed in Fig. 4.8(d) where higher nonlinear cubic stiffness is considered which produces a more hardening effect on the frequency response for $\Omega_1 \geq 1$ rad/sec. These phenomena are not observed with the *activeI* case (active TVA) as shown in Fig. 4.14(b). But here unstable responses are noticed for $\Omega_1 = 0.9$ rad/sec with Hopf bifurcation point ($\bar{k}_{13} = 0.73$ N/m³ and $\bar{k}_{23} = 0.003$ N/m³). For $\Omega_1 = 0.9$ rad/sec a minimum of 50% reduction in the response amplitude is observed compared to the passive case for $\bar{k}_{13}$ values in the range of 0 to 0.73 N/m³. Similarly, for $\Omega_1 = 1$ rad/sec a minimum 80% reduction in the response of the main system is observed with the *activeI* case than the passive case for the whole range of $\bar{k}_{13}$ values. Also, the effectiveness of the *activeI* case can be noticeable for $\Omega_1 = 1.1$ rad/sec with the variation in $\bar{k}_{13}$ and $\bar{k}_{23}$ stiffness

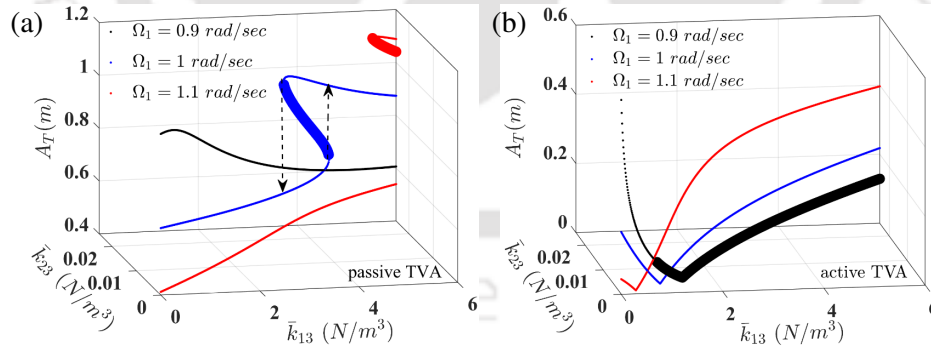


Fig. 4.14: Response amplitudes with variation in $\bar{k}_{13}$ and $\bar{k}_{23}$ using the (a) passive TVA and the (b) active TVA.

values, with no jump up and jump down phenomena. In the next sub-section, the system responses with the variation of feedback forces and the time delays are studied.

4.4.6 Variation of response amplitude with feedbacks gains and time delays

In Fig. 4.15 and Fig. 4.16 the variation of response amplitudes of the main system with time delays (t_d) and feedback forces by using the traditional vibration absorber are shown considering the same parameters as in Fig. 4.8(a, b). Fig. 4.15(a) shows the response amplitude of the main system with the variation in f_d and t_d . From this figure, the unstable response is observed for t_d in the range of 1.22π sec to 1.59π sec for f_d in the range of 3.67 to 4.78. Also, the response amplitude of the main system is found to be stable with almost the same amplitude for $t_d = 0$ to 1.22π sec. The sharp increase in the response amplitude of the main system is observed for $t_d = 0.82\pi$ sec to 1.22π sec and $f_d = 2.52$ to 3.67. The response is found to be minimum at $f_d = 5.61$ and $t_d = 1.86\pi$ sec. The same analogy as in Fig. 15(a) is carried out in Fig. 4.15(b) and 4.15(c) considering variation in f_v and f_a , respectively. In Fig. 4.15(b), the unstable solutions appear for less time delay i.e., t_d in the range of 0 to 0.24π sec and 0.87π sec to 1.22π sec unlike Fig. 15(a). The response amplitude of the main system also increases sharply for small changes in the delay from 0.24π sec and 0.43π sec. In Fig. 4.16(c) the response amplitude of the main system is shown with variation in f_a and t_d . Here a broader range of unstable responses of the system is observed compared to Fig. 4.15(a) and Fig. 4.15(b) for t_d in the range of 0.24π sec and 0.43π sec. However, with variation, f_a a comparative minimum response amplitude is observed at 0.82π sec than with f_d and f_v (Fig. 4.15(a) and Fig. 4.15(b)). Also, for a wider range of t_d i.e., from 0.66π sec to 0.97π sec the system response amplitude is less than with f_d or f_v feedback forces (Fig. 4.15(a) and Fig. 4.15(b)). In Fig. 4.15(d) the response amplitude of the system is studied by varying in f_d , f_v and f_a simultaneously with t_d . From Fig. 4.15(d) it can be observed that the unstable responses are comparatively in a wider range of t_d than with other individual feedbacks shown in Fig. 4.15(a) to Fig. 4.15(c). However, with this combination of feedbacks vibration suppression effect is greatly improved than the individual feedback at some particular value of t_d . The maximum suppression in the response amplitude of the main system is observed at $t_d =$

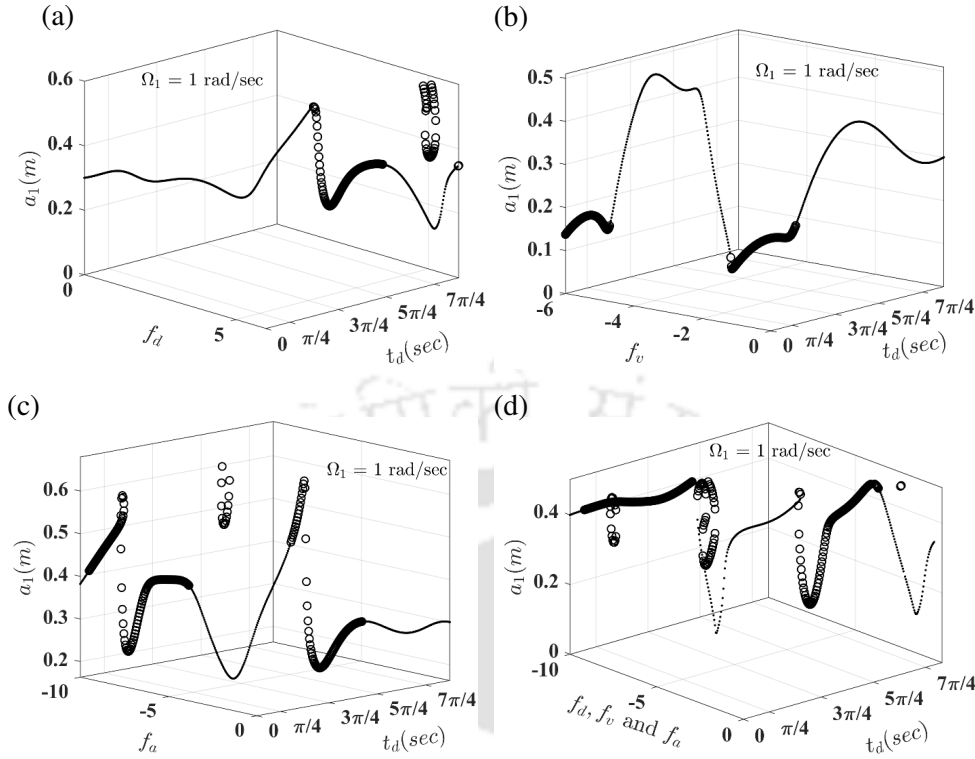
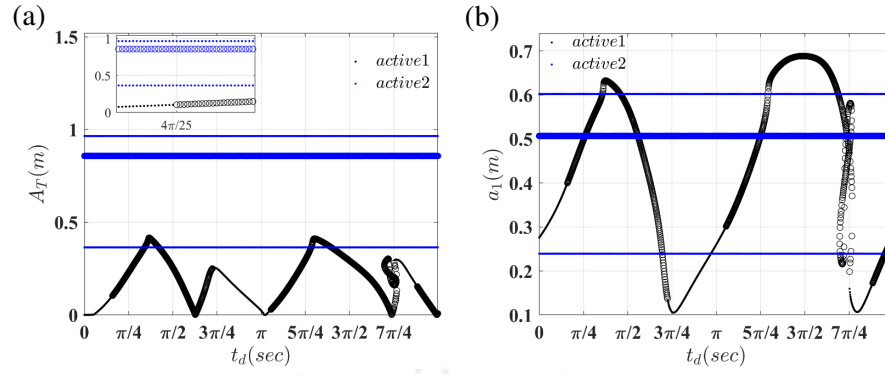


Fig. 4.15: Response amplitude of the main system at $\Omega_1 = 1 \text{ rad/sec}$ with variation in delay and different feedbacks gains (a) f_d (b) f_v (c) f_a and (d)

$$f_d, f_v, f_a.$$

0.76 π sec. The unstable regions in Fig. 4.15(d) are observed for t_d in the range of 0.076 π sec to 0.66 π sec and 1.2 π sec to 1.6 π sec. From Fig. 4.15 it can also be observed that the stability of the system largely depends upon time delay t_d . The time delay t_d in the active vibration absorber is a common occurrence as during sensing, processing and actuation some delay [221- 230, 279, 280] in the feedback takes place. In Fig. 4.16 the variation of response amplitudes A_r and the modal response a_1 of the main system with t_d for *active1* and *active2* cases are shown at $\Omega_1 = 1 \text{ rad/sec}$, with the same parameters that are used in Fig. 4.15 and in Fig. 4.8(a, b). From these figures for the *active1* case the response amplitudes of the main system is found to be stable for t_d in the range of 0 to 0.16 π sec, 0.72 π sec to 1.06 π sec and 1.75 π sec to 1.88 π sec and outside this range the system response is unstable. The Hopf bifurcation points are observed for t_d equals to 0.16



ig. 4.16: Response amplitude of the main system at $\Omega_1 = 1$ rad/sec with variation in delay with *active1* and *active2* cases (a) original response (b) modal response.

π sec, 0.72π sec, 1.06π sec and 1.75π sec. The 100% vibration suppression of the original response amplitude (A_T) of the main system is observed with *active1* case for t_d in the range of 0 to 0.04π sec and from 1.01π sec and 1.03π sec, as shown in Fig. 4.16(a). The result is shown in Fig. 4.16(a) for the *active1* case is matching well with Fig. 4.8(a) for zero t_d at $\Omega_1 = 1$ rad/sec. From Fig. 4.16(a) and Fig. 4.106(b) it is also observed that the response amplitude of the main system with the *active2* case shows two stable responses and one unstable response and remains constant irrespective of t_d values. These results for the *active2* case are also well matched with Fig. 4.8(a) and Fig. 4.8(b), respectively. One can observe from Fig. 4.8(a) and Fig. 4.8(b) that for $t_d = 0$ at $\Omega_1 = 1$ rad/sec, the main system shows two stable and one unstable response. The response amplitude remains constant with the *active2* case as in this case the control gains parameters (f_d , f_v and f_a) are considered equal to zero and only the frequency of the absorber is actively changed, as mentioned in sections 4.2.2 and 4.3.1. The results shown in Fig. 4.16 are also, well-matched with Fig. 4.8(a) and Fig. 4.8(b) for $t_d = 0$ at $\Omega_1 = 1$ rad/sec. From these analyses, it is observed that the introduction of feedback gains induces large suppression of the response amplitude of the main system at some particular value of t_d . Hence the ANVA can be operated within the mentioned t_d range and feedback gains for suppressing maximum vibration of the main system for the simultaneous primary and principal parametric resonance conditions.

The three control gains considered here can be used to vary stiffness, damping, and inertia of the absorber actively by changing the corresponding displacement, velocity and

acceleration feedback gains as studied in section 4. The consideration of the time delay in these feedbacks will also help to change the frequency of the absorber actively which can be observed from the parameter S_2 used in Eq. 4.8(a) and 4.8 (b). Thus with the proposed feedback, the active absorber can be useful to actively change its frequency and simultaneously impart a counteracting (blocking) force on the main system to reduce its vibration. The present analyses can be applied using either a three-vibration sensor or a single sensor (accelerometer) by varying the delay parameter. In the next sub-section for various mass ratios, the response of the linear and nonlinear systems for the passive and active cases are studied.

4.4.7 Amplitude response with the variation in the mass ratio

In this sub-section mass of the absorber (m_2) used in the TVA, F_{11} and F_{12} are varied for the linear and nonlinear system and the corresponding response amplitude of the main system is obtained for $\Omega_1 = 1$ rad/sec. It may be noted that since the mass of the main system is 1 kg hence the mass ratio (m_2/m_1) is the same as the mass of the absorber. Also, the optimized k_2 , c_2 and $\bar{k}_{23}$ values for different mass ratios and $\bar{k}_{13}$ values are calculated from the formulae given in [161]. The response amplitude of the main system is obtained and shown in Figs. 4.16 and 4.17 for five mass ratios i.e., 0.01, 0.025, 0.05, 0.075 and 0.1 with a variation in F_{11} values from 0.2 N to 2 N and F_{12} values from 0.2 N/m to 2 N/m. In the following sub-section, the response amplitude of the linear main system with linear TVA is obtained for simultaneous primary and principal parametric resonance conditions with different mass ratios and amplitude of excitations F_{11} and F_{12} .

4.4.7.1 Linear system

In Fig. 4.17 the response amplitude of the linear main system with linear TVA is shown for the passive and active cases. In the linear passive TVA shown in Fig. 4.17(a) the main system response amplitude gradually reduces with an increase in the mass ratios for $\Omega_1 = 1$ rad/sec which is generic from the previous literature. The effect of active linear TVA considering the *active1* case is shown in Fig. 4.17(b). From this figure, one can observe a reduction of 72% in the vibration of the main system than the passive TVA (Fig. 4.17(a)) for the mass ratio of 0.05 with $F_{11} = 2$ N and $F_{12} = 2$ N/m at $\Omega_1 = 1$ rad/sec. The response

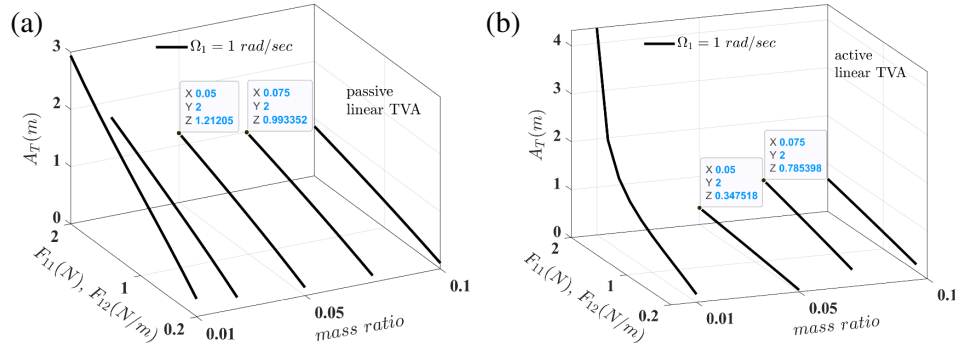


Fig. 4.17: Response amplitudes with variation in mass ratios using the (a) passive TVA and the (b) active TVA for the linear system.

amplitude of the main system is found to be more than the passive case only for mass ratio equal to 0.01 and $F_{11} = 2$ N and $F_{12} = 2$ N/m. Though vibration reduction by the active TVA is achieved for all mass ratios but more noticeable for the 0.05 mass ratio. For other mass ratios, one can use different gains in feedback to maximize the reduction in the amplitude response. In the next subsection vibration reduction by the *active1* case is compared with the passive case considering nonlinear stiffness in the system.

4.4.7.2 Nonlinear system

The amplitude response of the main system is compared for the passive and active TVA for different mass ratios and shown in Fig. 18 by considering higher nonlinear stiffness ($\bar{k}_{13} = 5$ N/m³ and $\bar{k}_{23} = 0.021$ N/m³), which are the same as in Fig. 4.8(d) and Fig. 4.10. From Fig. 4.18(a) it is observed that for a lower mass ratio of 0.01 the response amplitude becomes unstable with the passive TVA for $F_{11} \geq 1$ N and $F_{12} \geq 1$ N/m at $\Omega_1 = 1$ rad/sec. For $F_{11} = 0.8$ N and $F_{12} = 0.8$ N/m the response amplitude of the main system is found to be 0.5134 m for the mass ratio of 0.05. This response amplitude increases sharply from 0.5134 m to 0.9375 m with an increase in F_{11} value from 0.8 N to 1 N and F_{12} value from 0.8 N/m to 1 N/m as shown in Fig. 4.18(a) and Table 4.1. However, this sharp increase in the response amplitude is not observed for the mass ratio of 0.075. Here, one can observe that for a mass ratio equal to 0.075, the response amplitude for $F_{11} = 0.8$ N and $F_{12} = 0.8$ N/m is 0.4242 m and for 1 N it is 0.516 m, which increases gradually unlike for the mass ratio of 0.05. Also, from the above results, one can notice that there is not much decrease

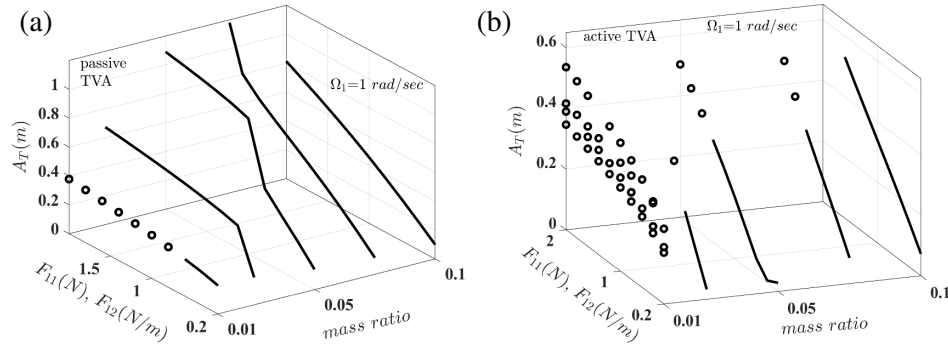


Fig. 4.18: Response amplitudes with variation in mass ratios for $\bar{k}_{13} = 5 \text{ N/m}^3$ and $\bar{k}_{23} = 0.021 \text{ N/m}^3$ using the (a) passive TVA and the (b) active TVA.

in the response amplitude with an increase in the mass ratio from 0.05 to 0.075 for $F_{11} = 0.8 \text{ N}$ and $F_{12} = 0.8 \text{ N/m}$. However, there is a significant reduction in the response amplitude when $F_{11} = 1 \text{ N}$ and $F_{12} = 1 \text{ N/m}$. In Fig. 4.18(b) the response amplitude with active TVA (*activeI* case) is shown. From this figure, one can observe there is a reduction in the response amplitudes of the main system for different mass ratios and forcing amplitudes. However, with active TVA more unstable regions are emerged with an increase in F_{11} and F_{12} values. In Table 4.1 few amplitude responses with different mass ratios, F_{11} and F_{12} values for the passive and active TVA are compared from the result shown in Fig. 4.18(a) and 4.18(b). From Table 4.1 one can observe the vibration reduction by the active TVA is more effective for the mass ratio of 0.05 and for other mass ratios one can use different active forces to suppress the vibration of the main system more effectively. The effectiveness in vibration suppression by the active TVA than the passive TVA can be noticeable for lower nonlinear stiffness in the absorber as discussed in Fig. 4.8(a-c) and Fig. 4.14, even so at higher nonlinear stiffness its efficacy can be observed from Fig. 4.8(d-f), Fig. 4.13, Fig. 4.14 and Fig. 4.18.

The nonlinear spring and the PZT stack actuator considered in this work provide not only the active nonlinear force but also provide more blocking force to generate more actuation with less voltage supply [30, 31, 20, 26]. The nonlinearity in spring is also important as, during the energy transfer from the primary system at the resonance, the absorber vibrates at a much larger amplitude. These nonlinearities can act both as a design parameter and a natural parameter depending upon the applications. When one changes the stiffness of the spring (actively or passively) then it can be taken as a design parameter. But in many

Table 4.1: Response of the main system for different mass ratios and amplitude of forcing excitations for $\bar{k}_{13} = 5 \text{ N/m}^3$ and $\bar{k}_{23} = 0.021 \text{ N/m}^3$.

Mass ratio	F_{11} (N) and F_{12} (N/m)	Passive TVA Amplitude (m)	Active TVA Amplitude (m)	Percentage of reduction
0.01	0.2	0.197	unstable	-
0.025	0.2	0.1895	0.032	
	0.4	0.488	0.1339	72
	0.6	0.523	0.2364	55
	0.8	0.5516	unstable	-
0.05	0.2	0.1392	0.028	80
	0.4	0.2684	0.008	97
	0.8	0.5134	0.1295	74
	1	0.9375	0.2039	78
	1.4	1.012	0.3373	66
	1.6	1.041	unstable	-
0.075	0.2	0.1141	0.084	26
	0.8	0.4242	0.3205	24
	1	0.516	0.3983	22
	1.2	0.6038	unstable	-
0.1	1.6	0.68	0.53	22
	1.8	0.7478	unstable	-

applications due to wear and tear the stiffness of the spring becomes nonlinear [15, 11, 12]. So, in those cases, it can be considered a natural parameter. The nonlinear studies will help the designer to operate the system according to the stability properties, bifurcations, jump

phenomena, etc. in the system. Also, in the present work main system is considered undamped, to study the efficacy of the proposed feedback gains by traditional/non-traditional vibration absorbers.

4.4.8 Application of the present work

In this section, the applicability of the design and control parameters considered in the present analyses are applied an undamped main system of mass $(m_1) = 240$ kg, stiffness $(k_1) = 160$ kN/m [330], nonlinear stiffness $(k_{13}) = 16$ kN/m³. It may be noted that in [330] the system considered is a quarter car model. The natural frequency (ω_n) of this main system is found to be 25.82 rad/sec. Now if this main system is subjected to excitations $F_1 \cos(\omega t)$ and $F_2 x_1 \cos(2\omega t)$, where F_1 (16 kN) and F_2 (16 kN/m) are the amplitude of excitations and forcing frequency $\omega \cong 25.82$ rad/sec, then one can observe simultaneous primary and principal parametric resonance conditions. Now to suppress the vibration of this main system under the above-mentioned resonance conditions by the present study and [161] the following absorber parameters are obtained for a mass ratio of 0.05, i.e., mass $(m_2) = 12$ kg, stiffness $(k_2) = 7.2561$ kN/m, nonlinear stiffness $(k_{23}) = 66.6667$ N/m³, and damping $c_2 = 79.0451$ N-sec/m. The equation of motion of this system can be non-dimensionalized by considering non-dimensionalized time equal to $\omega_n t$. So, here the system governing equation of motion is reduced to the same as the present analysis where coefficients of the main system mass and stiffness are equal to unity. Now to completely absorb the vibration of this main system for the worst possible resonance case i.e., at the simultaneous primary and principal parametric resonance conditions, one can follow the same procedure as discussed in Fig. 4.8(a) and Fig. 4.8(b). The system parameters considered in this example are the same as in Fig. 4.8(a) and Fig. 4.8(b) and the corresponding response amplitude of this main system will be the same as shown in Fig. 8(a) and Fig. 4.8(b). It can be observed that with the proposed *activeI* force by the traditional ANVA the main system response amplitude is 100% suppressed for forcing frequency (ω) in the range (22.72 rad/sec to 25.29 rad/sec) and (45.44 rad/sec to 50.58 rad/sec). Also, for a broad range of operating frequency (ω) , a minimum of 80% of vibration reduction is achieved i.e., for ω in the range of 19.72 rad/sec to 27.82 rad/sec and

39.44 rad/sec to 55.64 rad/sec compared to the optimal nonlinear passive tuned vibration absorber [161]. Further, at the other operating frequency of operation (ω) i.e. far away from the resonating frequencies (where the response amplitude is minimum) one can change the frequency of the absorber actively or provide zero feedback gain, the vibration suppression of the main system can be achieved for the whole range of operating frequencies. Hence the proposed analyses can be used for any real application to suppress the vibration of the main system more effectively under primary, principal parametric, and simultaneous primary and principal parametric resonance conditions. In most of the previously published works, only one or two effects have been taken considering damping in the main (primary) system, with various types of actuators and feedback systems [5, 6, 15, 16, 32, 39, 41, 161-168, 170-178, 202-227, 184-190, 286, 287, 330]. Further, it may be noted that the superposition rule does not apply to the nonlinear systems and hence, one will never get the actual response of the system, by combining all the previously published results which are only possible for a linear system.

4.5 Summary

In this work, the efficacy of the active traditional and non-traditional vibration absorber is studied to suppress the vibration of the undamped main system under primary, principal parametric and simultaneous primary and principal parametric resonance conditions. The feedback gain by the absorber is produced by the PZT stack actuator and a nonlinear spring in series combination considering time delay in the displacement, velocity and acceleration feedback. With this new design, the PZT stack actuator can produce both linear and nonlinear feedback gains while changing the natural frequency of the absorber actively.

The nonlinear governing equations of motion of the system are obtained by Newton's second law and modified by the weighted modal matrix, which is then solved by the first-order method of multiple scales (MMS) to obtain the reduced equations. These reduced equations are solved by Newton's method and both the linear and nonlinear results are compared with the second-order MMS, HBM, numerical method and with the published literature and they are found to be in good agreement. It is observed that for the linear system the first order MMS gives conservative results with HBM, numerical method and other linear methods, for the original response amplitude of the main system while for the modal response amplitude, the results are found to be in good agreement. From these results

for the first time, Den Hartog's equal peaks in the frequency responses are obtained by using MMS which is not reported in earlier literature. Linear and nonlinear dynamics of the system are studied by the trivial state instability regions, frequency responses, time responses, phase portraits, Poincaré sections and basin of attractions for various feedbacks, feedback gains, time delays, the amplitude of external excitations, nonlinear stiffnesses and mass ratios showing the potency of the proposed feedback gain in reducing vibration of the main system

The efficacy of the feedback gain is studied by comparing various feedbacks, feedback gains and time delays. It is shown that with the displacement, acceleration feedback and actively tuning of the absorber frequency, 100% vibration suppression of the linear and nonlinear undamped main system is achieved for the above-mentioned resonance conditions at the mentioned frequency. Moreover, with this feedback gain, significant vibration suppression of the undamped main system is accomplished for a wider range of operating frequencies for various mass ratios, especially for 0.05. Also, with the feedback gain the trivial state instability region reduces and the critical value of amplitude parametric excitation increases. The various nonlinear phenomena such as jump up, jump down, chaotic response, and Hopf bifurcation points are observed at the mentioned operating frequencies and amplitude of excitations. The feedback gain by the proposed vibration absorber eliminates jump up and jump down phenomena at the resonant operating frequency. The time delay of more than $\pi/10$ sec with the proposed *active1* feedback force makes the system unstable for linear and nonlinear systems. However, within the mentioned time delay range the system stability and response amplitude improve, especially for the nonlinear system. The non-traditional vibration absorber shows less amplitude response than the traditional vibration absorber in all cases, though its application is limited due to its damper configuration. The response amplitude of the main system and the unstable range of operating frequencies are specified for both the passive and active case for a wide range of mass ratios (0.01 to 0.1), external forcing amplitudes of harmonic (0.1 N to 0.5 N), and parametric excitations (0.1 N/m to 0.5 N/m), nonlinear stiffness in the main system (0.1 N/m³ to 0.6 N/m³) and absorber (4.2×10^{-4} N/m³ to 2.1×10^{-3} N/m³), which will provide the designer a clear idea for selecting proper system parameters for different operating conditions based on required application to suppress maximum vibration of the main system under primary, principal parametric and simultaneous primary and principal parametric resonance conditions.

CHAPTER 5

Traditional and Non-Traditional ANVA with Time Delay

Combination Feedback for Hard Excitation

5.1 Introduction

In this chapter, vibration absorption of a base excited nonlinear spring-mass based single degree of freedom (SDOF) subjected to simultaneous external hard harmonic and parametric excitation is studied by using a modified designed traditional and non-traditional SDOF active nonlinear vibration absorber (ANVA). The active part in the traditional and non-traditional absorber consists of a PZT stack actuator along with a linear and nonlinear spring in series combination and a time-delayed damper. This novel design is used to produce nonlinear feedback gain on the primary system by considering time delay in the displacement, velocity and acceleration feedback of the primary system. The time delay effect in the nonlinear damper is also studied for vibration suppression of the primary system. The governing coupled nonlinear equations of motion for the system with the weighted modal matrix approach have been derived. The first-order method of multiple scales (MMS) is used to obtain the reduced equations for the simultaneous superharmonic, subharmonic, primary and principal parametric resonance conditions considering 1:1, 1:2, 1:3 and 3:1 internal resonance cases. These reduced equations are used to obtain the steady-state responses of the nonlinear systems using Newton's method, which are then compared with the numerical methods, showing good agreement. The parametric analysis for various system parameters is carried out showing the effectiveness of the proposed time delayed active vibration absorber for the nonlinear primary system under simultaneous hard excitations for a wider band of operating frequencies. From these parametric studies, stable and unstable ranges of operating frequencies are evaluated at which the system response amplitude is minimum and stable. It is observed that with the proposed feedback gains the complete vibration absorption can be achieved at a certain range of operating frequencies for the above-mentioned resonance conditions for various system parameters.

5.2 Mathematical Modeling of the System

The schematic diagram of the proposed single degree of freedom (SDOF) traditional and nontraditional nonlinear AVA attached to a nonlinear SDOF primary system is shown in Fig. 5.1(a) and Fig. 5.1(b), respectively. In Fig. 5.1 the parameters of the systems m_i and k_i denote the mass, linear stiffness, of the primary system ($i = 1$) and the absorber ($i = 2$). k_{13} denotes cubic nonlinear stiffness of the primary system. The time-delayed (τ_1) damping coefficient of the absorber is denoted as c_2 . The gain produced by the PZT stack actuator having stiffness k_E^P and a spring in series combinations by time delay in displacement, velocity and acceleration feedback of the main system. In the presence of a nonlinear spring, as the displacement at the junction between the PZT stack actuator and the spring k_2 is difficult to calculate, so, here k_{23} is assumed to be the equivalent cubic nonlinear stiffness having a displacement of $x_1 + \delta_0 - x_2$. The primary system is subjected to external harmonic ($F_1 \cos(\Omega_1 t)$), parametric ($x_1 F_2 \cos(\Omega_2 t)$) and base ($Y_0 \cos(\Omega_3 t)$) excitations. Here F_1 , F_2 and Y_0 denotes the amplitude of force and base excitations and Ω_1 , Ω_2 and Ω_3 represents the corresponding external forcing frequencies. Considering x_1 as the displacements of the primary system, x_2 as the displacement of the absorber and δ_0 as the nominal displacement of the PZT stack actuator [20]. The coupled nonlinear differential equations of motion of the system are given below considering the system shown in Fig. 5.1(a).

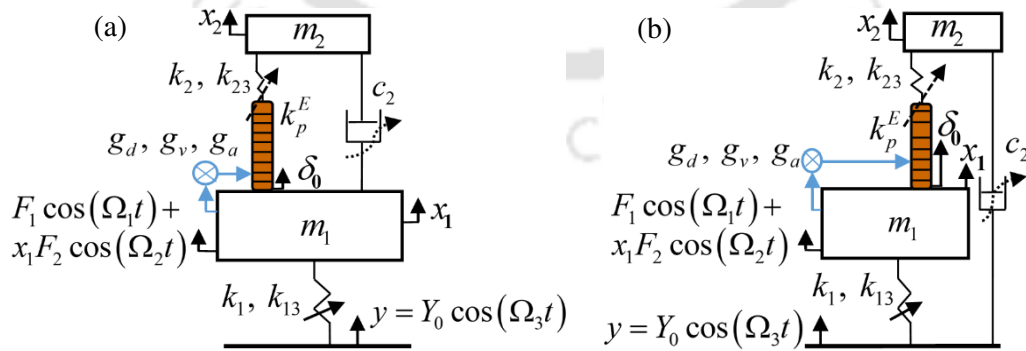


Fig 5.1: Schematic diagram of the nonlinear active dynamic vibration absorber (a) traditional (b) non-traditional.

$$m_1\ddot{x}_1 + k_1(x_1 - y) + k_{13}(x_1 - y)^3 + k_e(x_1 + \delta_0 - x_2) + k_{23}(x_1 + \delta_0 - x_2)^3 + c_2(\dot{x}_1(t - \tau_1) - \dot{x}_2(t - \tau_1)) = F_{11}\cos(\Omega_1 t) + x_1 F_2 \cos(\Omega_2 t) \quad (5.1)$$

$$m_2\ddot{x}_2 - k_e(x_1 - x_2 + \delta_0) - k_{23}(x_1 - x_2 + \delta_0)^3 - c_2(\dot{x}_1(t - \tau_1) - \dot{x}_2(t - \tau_1)) = 0 \quad (5.2)$$

where $k_e = k_2 k_p^E / (k_2 + k_p^E)$ and τ_1 is a time delay in the damper.

The first equation of motion for the nontraditional AVA, which is shown in Fig. 5.1(b) is also the same as Eq. 5.1 by considering the term $c_2(\dot{x}_1(t - \tau_1) - \dot{x}_2(t - \tau_1))$ equal to zero.

The second equation of motion of the system shown in Fig. 5.1(b) is given below.

$$m_2\ddot{x}_2 - k_e(x_1 - x_2 + \delta_0) - k_{23}(x_1 - x_2 + \delta_0)^3 + c_2(\dot{x}_2(t - \tau_1) - \dot{y}(t - \tau_1)) = 0 \quad (5.3)$$

where $\delta_0 = nd_{33}V$ [20], the terms n , d_{33} and V are the number of wafers, dielectric charge constant and the voltage developed in the PZT stack actuator, respectively. One can observe from the equations of motion for the two systems shown in Fig. 5.1(a) and Fig. 5.1(b) that only the damping terms are different, therefore in the present work analyses of both systems are considered. It may be noted that Eqs. (5.1) and (5.2) can be reduced to the work of Habib et. al [161] and Detroux et. al [168] in the absence of nonlinear stiffness in the absorber (k_{23}), time delay (τ_1) in the damper (c_2), base excitation $y = Y_0 \cos(\Omega_3 t)$, parametric excitation $x_1 F_2 \cos(\Omega_2 t)$, PZT stack actuator stiffness (k_p^E), and displacement (δ_0). Also in [161, 168], they have used a damper in the main system which is not considered in the present analysis. Further, in the present analyses, external hard harmonic excitation acting on the primary system is considered whereas in prior studies [161, 168] soft excitation was considered. Eqs. (1) and (2) can also be reduced to those of Rabelo et al. [227] by considering y , δ_0 and F_2 equal to zero. It may be noted that in their work base excitation, parametric excitation and actuator displacement were not considered and damping in the primary system was considered. Considering combination feedback of displacement, velocity and acceleration of the main system with a time delay τ_2 , the voltage supplied/developed in the PZT stack actuator are expressed as $V = g_d(x_1(t - \tau_2)) + g_v(\dot{x}_1(t - \tau_2)) + g_a(\ddot{x}_1(t - \tau_2))$, where g_d , g_v and g_a are the

feedback gains for displacement, velocity and acceleration feedbacks, respectively. The above feedbacks are considered as one can have a comparative study among the efficacy of individual and various combinations of these feedbacks. Substituting V in the nominal displacement of δ_0 , Eqs. (5.1) and Eq. (5.2) are reconstructed. Now the governing equations of motion are rewritten by the weighted modal matrix approach by considering $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\tilde{P}] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, where $[\tilde{P}]$ is the weighted modal matrix. These reformed governing equations of motion are ordered by considering a small bookkeeping parameter ε as follows.

$$\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \varepsilon [C] \begin{bmatrix} \dot{X}_1(\tau - \tau_1) \\ \dot{X}_2(\tau - \tau_1) \end{bmatrix} + \varepsilon [\tilde{P}]^T [K_n] = [\tilde{P}]^T [F] \quad (5.4)$$

where

$$\begin{aligned} [\tilde{P}] &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = [\tilde{P}]^T \begin{bmatrix} k_1 + k_e & -k_e \\ -k_e & k_e \end{bmatrix} [\tilde{P}], \\ [C] &= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = [\tilde{P}]^T \frac{1}{\varepsilon} \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} [\tilde{P}], \\ [K_n] &= \begin{bmatrix} K_{n1} \\ K_{n2} \end{bmatrix}, \bar{k}_{13} = \frac{k_{13}}{\varepsilon}, \bar{k}_{23} = \frac{k_{23}}{\varepsilon}, K_{n1} = \bar{k}_{13} x_1^3 + k_e (\delta_0) + \bar{k}_{23} (x_1 - x_2 + \delta_0)^3, \\ K_{n2} &= -K_{n1} + \bar{k}_{13} x_1^3, \delta_0 = f_d (x_1(t - \tau_2)) + f_v (\dot{x}_1(t - \tau_2)) + f_a (\ddot{x}_1(t - \tau_2)), \\ f_d &= \frac{nd_{33}g_d}{\varepsilon}, f_v = \frac{nd_{33}g_v}{\varepsilon}, f_a = \frac{nd_{33}g_a}{\varepsilon}, F_{12} = \frac{F_2}{\varepsilon}, Y = \frac{Y_0}{\varepsilon}, [F] = \begin{bmatrix} \bar{F} \\ 0 \end{bmatrix}, \\ \bar{F} &= F_1 \cos \Omega_1 t + \varepsilon x_1 F_{12} \cos \Omega_2 t + \varepsilon k_1 Y \cos \Omega_3 t + \varepsilon^4 \bar{k}_{13} Y^3 \cos^3 \Omega_3 t + \\ & 3\varepsilon^2 \bar{k}_{13} (x_1^2 Y \cos \Omega_3 t - \varepsilon x_1 Y^2 \cos^2 \Omega_3 t) \end{aligned}$$

Similarly, the governing equation of motion for the mathematical model shown in Fig. 5.1(b) is obtained which is the same as Eq. (5.4) except in this case the damping matrix $[C]$ has different values and an additional term of $-\varepsilon c_2 \Omega_3 Y \sin(\Omega_3(t - \tau_1))$ is added in $\bar{F}$. In the next section, the method of multiple scales (MMS) is used to obtain the approximate solution of Eq. 5.4.

5.3 Approximate Solution using MMS

In this section, the governing equation of motion Eq. 5.4 is solved using the MMS to obtain an approximate analytical solution of the system. The perturbed solutions in the new time scale are expanded by considering the system response at the past state $X_i(t - \tau_j)$ with time delay in the damper for $j = 1$ and feedback for $j = 2$. The responses of the system at different time scales ($T_n = \varepsilon^n t$ for $n = 0, 1, \dots$) can be written by following the standard MMS procedure as follows.

$$X_i(t - \tau_j) = X_{i0}(T_0 - \tau_j, T_1 - \varepsilon\tau_j, \dots) + \varepsilon X_{i1}(T_0 - \tau_j, T_1 - \varepsilon\tau_j, \dots) \quad (5.5)$$

It may be noted that if the system has no time delay in the feedback or the damping then the delay terms τ_j can be equated to zero and the corresponding system response at the t^{th} time i.e. $X_i(t)$ can be expressed. Substituting the above-assumed responses from Eq. 5.5 into Eq. 5.4 and separating the terms with different order of ε the following expression can be obtained.

$$\varepsilon^0 : D_0^2 X_{10} + \omega_1^2 X_{10} = p_{11} F_1 \cos \Omega_1 T_0 \quad (5.6a)$$

$$D_0^2 X_{20} + \omega_2^2 X_{20} = p_{12} F_1 \cos \Omega_1 T_0 \quad (5.6b)$$

$$\begin{aligned} \varepsilon^1 : D_0^2 X_{11} + \omega_1^2 X_{11} = & -2D_0 D_1 X_{10} - c_{11} D_0 X_{10d} - c_{12} D_0 X_{20d} - p_{11} \bar{k}_{13} (p_{11} X_{10} + p_{12} X_{20})^3 \\ & - p_1 k_e \left(f_d (p_{11} X_{10d} + p_{12} X_{20d}) + f_v (p_{11} D_0 X_{10d} + p_{12} D_0 X_{20d}) \right) \\ & + f_a (p_{11} D_0^2 X_{10d} + p_{12} D_0^2 X_{20d}) \\ & - p_1 \bar{k}_{23} (p_1 X_{10} + p_2 X_{20})^3 + p_{11} (p_{11} X_{10} + p_{12} X_{20}) F_{12} \cos(\Omega_2 T_0) + p_{11} k_1 Y \cos \Omega_3 T_0 \end{aligned} \quad (5.7a)$$

$$\begin{aligned} D_0^2 X_{21} + \omega_2^2 X_{21} = & -2D_0 D_1 X_{20} - c_{21} D_0 X_{10d} - c_{22} D_0 X_{20d} - p_{12} \bar{k}_{13} (p_{11} X_{10} + p_{12} X_{20})^3 \\ & - p_2 k_e \left(f_d (p_{11} X_{10d} + p_{12} X_{20d}) + f_v (p_{11} D_0 X_{10d} + p_{12} D_0 X_{20d}) \right) \\ & + f_a (p_{11} D_0^2 X_{10d} + p_{12} D_0^2 X_{20d}) \\ & - p_2 \bar{k}_{23} (p_1 X_{10} + p_2 X_{20})^3 + p_{12} (p_{11} X_{10} + p_{12} X_{20}) F_{12} \cos \Omega_2 T_0 + p_{12} k_1 Y \cos \Omega_3 T_0 \end{aligned} \quad (5.7b)$$

Similarly, for the non-traditional ANVA an additional term of $-p_{11}c_2\Omega_3Y\sin(\Omega_3(T_0-\tau_1))$ and $-p_{12}c_2\Omega_3Y\sin(\Omega_3(T_0-\tau_1))$ are added on the right-hand side of Eq. (5.7a) and Eq. (5.7b), respectively. The solution of the Eqs. (5.6a) and (5.6b) can be written as

$$X_{10} = A_1(T_1)\exp(i\omega_1T_0) + \Lambda_1(T_1)\exp(i\Omega_1T_0) + cc \quad (5.8a)$$

$$X_{20} = A_2(T_1)\exp(i\omega_2T_0) + \Lambda_2(T_1)\exp(i\Omega_1T_0) + cc \quad (5.8b)$$

where

$$\Lambda_1 = \frac{p_{11}F_1}{2(\omega_1^2 - \Omega_1^2)}, \quad \Lambda_2 = \frac{p_{12}F_1}{2(\omega_2^2 - \Omega_1^2)}, \quad p_1 = p_{11} - p_{21}, \quad p_2 = p_{12} - p_{22}$$

where the complex function of amplitudes A_1 , A_2 and its complex conjugate terms ($\bar{A}_1$ and $\bar{A}_2$) are to be determined later. The solutions for the delay terms (X_{10d} and X_{20d}) for the system response can also be written in similar forms as given below.

$$X_{10d} = A_{1d}(T_1 - \varepsilon\tau_j)\exp(i\omega_1(T_0 - \tau_j)) + \Lambda_{1d}(T_1 - \varepsilon\tau_j)\exp(i\Omega_1(T_0 - \tau_j)) \quad (5.8c)$$

$$X_{20d} = A_{2d}(T_1 - \varepsilon\tau_j)\exp(i\omega_2(T_0 - \tau_j)) + \Lambda_{2d}(T_1 - \varepsilon\tau_j)\exp(i\Omega_1(T_0 - \tau_j)) \quad (5.8d)$$

where $A_{1d} = A_1(T_1 - \varepsilon t_d)$ and $A_{2d} = A_2(T_1 - \varepsilon t_d)$. $\bar{A}_{1d}$ and $\bar{A}_{2d}$ are the complex conjugate terms of A_{1d} and A_{2d} respectively. The solutions obtained in Eqs. (5.8a) to (5.8d) are substituted into Eqs. (5.7a) and (5.7b). The expansion of Eqs. (5.7a) and (5.7b) contains secular or nearly secular terms for many resonance conditions which are discussed in the next section.

5.3.1 Resonance cases

In this section, all the possible external and internal resonance cases are given below which are obtained from the expansion of Eqs. (5.7a) and (5.7b).

- (i) Primary resonance ($\Omega_3 \cong \omega_1$ or ω_2),
- (ii) Principal parametric resonance ($\Omega_2 \cong 2\omega_1$ or $2\omega_2$),
- (iii) Super-harmonic resonance ($\Omega_1 \cong \omega_1/3$ or $\omega_2/3$),
- (iv) Sub-harmonic resonance ($\Omega_1 \cong 3\omega_1$ or $3\omega_2$),
- (v) **Simultaneous (i), (ii) and (iii),**
- (vi) **Simultaneous (i), (ii) and (iv),**
- (vii) Combination resonance $\Omega_1 \cong 2\omega_1 + \omega_2$ or $\Omega_1 \cong 2\omega_1 - \omega_2$
and $\Omega_1 + \Omega_2 \cong \omega_1$ or $\Omega_1 - \Omega_2 \cong \omega_1$.

In the present chapter all these resonance conditions are considered especially case(v) and case (vi), where one can get all resonance conditions simultaneously. From these resonance conditions, an attempt has been made to suppress vibration of the primary system effectively through the proposed time delayed gains through various combinations of feedbacks and time delay in the damping by the novel nonlinear AVA. From (5.7a) and (5.7b), one can also observe internal resonance conditions when ω_2 is an integer multiple of ω_1 , which leads to the following internal resonance conditions. The internal resonance cases are observed for 1:1 ($\omega_2 \cong \omega_1$), 1:2 ($\omega_2 \cong 2\omega_1$), 3:1 ($\omega_2 \cong \omega_1/3$) and 1:3 ($\omega_2 \cong 3\omega_1$) internal resonance conditions. In the next section, the reduced amplitude and phase equations are obtained by considering case (v) with 1:1, 2:1 and 3:1 internal resonance conditions.

5.3.1.1 Simultaneous primary, principal parametric and super-harmonic resonance conditions

In this section simultaneous (i), (ii) and (iii) or case (v) resonance conditions are considered i.e. for simultaneous primary, principal parametric and super-harmonic resonances with 1:1, 2:1 and 3:1 internal resonance conditions. It may be noted that here as the PZT stack actuator is integrated with the spring in the absorber configuration, so one can actively change the frequency of the absorber in accordance with various internal resonance cases as mentioned above. In the next section, 1:1 internal resonance conditions to obtain the secular terms.

1:1 internal resonance condition

In this section, ω_1 is taken to be nearly equal to ω_2 which gives rise to 1:1 internal resonance condition. The secular terms are obtained by substituting the 1:1 internal resonance condition ($\Omega_3 = \omega_1 + \varepsilon\sigma_m = \omega_2 + \varepsilon\sigma_{mm}$) for $m = 3$, with simultaneous primary, principal parametric and super-harmonic resonances. Here σ_m and σ_{mm} (for $m = 1, 2$ and 3) denotes the detuning parameters. It may be noted that here 1:1 internal resonance condition ($\omega_2 = \omega_1 + \varepsilon(\sigma_3 - \sigma_{33})$), is considered to suppress response amplitude for the primary resonance condition. Substituting $\omega_2 = \omega_1 + \varepsilon(\sigma_3 - \sigma_{33})$, in the expansion of Eqs. (5.7a) and (5.7b) for case (v), the following sets of secular terms are obtained which are equated to zero to get bounded solutions for X_{11} and X_{12} . It may be noted that the secular terms are obtained from the expansion of Eqs. (5.7a) and (5.7b) consist of coefficients $\exp(i\omega_1 T_0)$, $\exp(i\omega_2 T_0)$, $\exp(i(2\omega_1 - \omega_2)T_0)$, $\exp(3i\Omega_1 T_0)$, $\exp(i(2\omega_2 - \omega_1)T_0)$, $\exp(i(\Omega_2 - \omega_1)T_0)$ and $\exp(i\Omega_3 T_0)$. These secular terms are giving rise to the resonance conditions as $\Omega_1 \cong \omega_1/3$ or $\omega_2/3$, $\Omega_2 \cong 2\omega_1$ or $2\omega_2$ and $\Omega_3 \cong \omega_1$ or ω_2 .

$$\begin{aligned}
2i\omega_1 D_1 A_1 = & -ic_{11} A_{1d} \omega_1 \exp(-i\omega_1 \tau_1) - ic_{12} A_2 \omega_2 \exp(i(\sigma_3 - \sigma_{33})T_1) \exp(-i\omega_2 \tau_1) \\
& -k_{n1} (\Lambda_1^3 \exp(3i\sigma_1 T_1) + 3A_1^2 \bar{A}_1 + 6A_1 \Lambda_1^2) \\
& -3k_{n2} \left(A_1^2 \bar{A}_2 \exp(-i(\sigma_3 - \sigma_{33})T_1) + \Lambda_1^2 \Lambda_2 \exp(3i\sigma_1 T_1) + 4A_1 \Lambda_1 \Lambda_2 \right) \\
& + (2A_1 \bar{A}_1 A_2 + 2\Lambda_1^2 A_2) \exp(i(\sigma_3 - \sigma_{33})T_1) \\
& -3k_{n3} \left(A_2^2 \bar{A}_1 \exp(2i(\sigma_3 - \sigma_{33})T_1) + \Lambda_2^2 \Lambda_1 \exp(3i\sigma_1 T_1) \right) \\
& + 4A_2 \Lambda_1 \Lambda_2 \exp(i(\sigma_3 - \sigma_{33})T_1) + 2A_2 \bar{A}_2 A_1 + 2\Lambda_2^2 A_1 \\
& -k_{n4} (\Lambda_2^3 \exp(3i\sigma_1 T_1) + 3A_2^2 \bar{A}_2 + 6A_2 \Lambda_2^2) \exp(i(\sigma_3 - \sigma_{33})T_1) - g_{11} A_{1d} \exp(-i\omega_1 \tau_2) \\
& -g_{12} A_{2d} \exp(-i\omega_2 \tau_2) \exp(i(\sigma_3 - \sigma_{33})T_1) + 0.5 p_{11}^2 F_{12} \bar{A}_1 \exp(i\sigma_2 T_1) \\
& + 0.5 p_{11} p_{12} F_{12} \bar{A}_2 \exp(i(\sigma_2 - (\sigma_3 - \sigma_{33}))) T_1 + 0.5 p_{11} k_1 Y \exp(i\sigma_3 T_1)
\end{aligned} \tag{5.9a}$$

$$\begin{aligned}
2i\omega_2 D_1 A_2 = & -ic_{21} A_1 \omega_1 \exp(-i(\sigma_3 - \sigma_{33})T_1) \exp(-i\omega_1 \tau_1) - ic_{22} A_2 \omega_2 \exp(-i\omega_2 \tau_1) \\
& -\tilde{k}_{n1} \exp(-i(\sigma_3 - \sigma_{33})T_1) (\Lambda_1^3 \exp(3i\sigma_1 T_1) + 3A_1^2 \bar{A}_1 + 6A_1 \Lambda_1^2) \\
& -3\tilde{k}_{n2} \exp(-i(\sigma_3 - \sigma_{33})T_1) \left(A_1^2 \bar{A}_2 \exp(-i(\sigma_3 - \sigma_{33})T_1) + \Lambda_1^2 \Lambda_2 \exp(3i\sigma_1 T_1) \right) \\
& + 4A_1 \Lambda_1 \Lambda_2 + 2A_1 \bar{A}_1 A_2 + 2\Lambda_1^2 A_2
\end{aligned}$$

$$\begin{aligned}
 & 3\tilde{k}_{n3} \left(A_2^2 \bar{A}_1 \exp(i(\sigma_3 - \sigma_{33})T_1) + \Lambda_2^2 \Lambda_1 \exp(i(3\sigma_1 - (\sigma_3 - \sigma_{33}))T_1) \right) \\
 & + 4A_2 \Lambda_1 \Lambda_2 + (2A_2 \bar{A}_2 A_1 + 2\Lambda_2^2 A_1) \exp(-i(\sigma_3 - \sigma_{33})T_1) \\
 & - \tilde{k}_{n4} (\Lambda_2^3 \exp(i(3\sigma_1 - (\sigma_3 - \sigma_{33}))T_1) + 3A_2^2 \bar{A}_2 + 6A_2 \Lambda_2^2) \\
 & - g_{21} A_{1d} \exp(-i(\sigma_3 - \sigma_{33})T_1 - \omega_1 \tau_2) - g_{22} A_{2d} \exp(-i\omega_2 \tau_2) + 0.5 p_{12} k_1 Y \exp(i\sigma_{33} T_1) \\
 & + 0.5 p_{11} p_{12} F_{12} \bar{A}_1 \exp(i(\sigma_2 - (\sigma_3 - \sigma_{33}))T_1) + 0.5 p_{12}^2 F_{12} \bar{A}_2 \exp(-2i(\sigma_3 - \sigma_{33})T_1)
 \end{aligned} \tag{5.9b}$$

where

$$\begin{aligned}
 k_{n1} &= \bar{k}_{13} p_{11}^4 + \bar{k}_{23} p_1^4, \quad k_{n2} = \bar{k}_{13} p_{11}^3 p_{12} + \bar{k}_{23} p_1^3 p_2, \quad k_{n3} = \bar{k}_{13} p_{11}^2 p_{12}^2 + k_{23} p_1^2 p_2^2, \\
 k_{n4} &= \bar{k}_{13} p_{11} p_{12}^3 + \bar{k}_{23} p_1 p_2^3, \quad \tilde{k}_{n1} = \bar{k}_{n2}, \quad \tilde{k}_{n2} = k_{n3}, \quad \tilde{k}_{n3} = k_{n4}, \quad \tilde{k}_{n4} = \bar{k}_{13} p_{12}^4 + \bar{k}_{23} p_2^4, \\
 g_{11} &= p_1 p_{11} k_e (f_d + i\omega_1 f_v - \omega_1^2 f_a), \quad g_{12} = p_1 p_{12} k_e (f_d + i\omega_2 f_v - \omega_2^2 f_a), \\
 g_{21} &= p_2 p_{11} k_e (f_d + i\omega_1 f_v - \omega_1^2 f_a), \quad g_{22} = p_2 p_{12} k_e (f_d + i\omega_2 f_v - \omega_2^2 f_a)
 \end{aligned}$$

Further, for the non-traditional ANVA the expansion of Eqs. (5.7a) and (5.7b) contains additional secular terms $-0.5 p_{11} c_2 \Omega_3 Y \exp(-i(0.5\pi - (\Omega_3(T_0 - \tau_1))))$ and $-0.5 p_{12} c_2 \Omega_3 Y \exp(-i(0.5\pi - (\Omega_3(T_0 - \tau_1))))$ on the right-hand side of Eqs. (5.9a) and (5.9b), respectively. Eliminating secular terms, the solution of the Eqs. (5.7a) and (5.7b) are expressed as follows.

$$X_{11} = (c_{11} \Lambda_1 + c_{12} \Lambda_2) D_1 - k_{n1} D_2 - 3k_{n2} D_3 - 3k_{n3} D_4 - k_{n4} D_5 - D_6 + p_{11} F_{12} D_7 \tag{5.10a}$$

$$X_{21} = (c_{21} \Lambda_1 + c_{22} \Lambda_2) \tilde{D}_1 - \tilde{k}_{n1} \tilde{D}_2 - 3\tilde{k}_{n2} \tilde{D}_3 - 3\tilde{k}_{n3} \tilde{D}_4 - \tilde{k}_{n4} \tilde{D}_5 - \tilde{D}_6 + p_{12} F_{12} \tilde{D}_7 \tag{5.10b}$$

The terms D_1 to D_7 and $\tilde{D}_1$ to $\tilde{D}_7$ are given in **Appendix C**. Following the standard procedure of MMS by considering $A_1 = \frac{1}{2} a_1 e^{i\beta_1}$ and $A_2 = \frac{1}{2} a_2 e^{i\beta_2}$, the following steady-state autonomous reduced equations are obtained for the system.

$$\begin{aligned}
 \dot{a}_1 &= -0.5 c_{11} a_1 \cos(\omega_1 \tau_1) - 0.5 c_{12} a_2 \frac{\omega_2}{\omega_1} \cos(\gamma_1 - \gamma_2 - \omega_2 \tau_1) - \frac{k_{n1}}{\omega_1} \Lambda_1^3 \sin \gamma_1 \\
 &- \frac{3k_{n2}}{\omega_1} \left(\frac{a_1^2 a_2}{8} \sin(\gamma_2 - \gamma_1) + \Lambda_1^2 \Lambda_2 \sin \gamma_1 + \left(\frac{a_1^2 a_2}{4} + \Lambda_1^2 a_2 \right) \sin(\gamma_1 - \gamma_2) \right) -
 \end{aligned}$$

$$\begin{aligned}
& \frac{3k_{n3}}{\omega_1} \left(\frac{a_1 a_2^2}{8} \sin(2(\gamma_1 - \gamma_2)) + \Lambda_1 \Lambda_2^2 \sin \gamma_1 + 2a_2 \Lambda_1 \Lambda_2 \sin(\gamma_1 - \gamma_2) \right) \\
& - \frac{k_{n4}}{\omega_1} \left(\Lambda_2^3 \sin \gamma_1 + \left(\frac{3a_2^3}{8} + 3a_2 \Lambda_2^2 \right) \sin(\gamma_1 - \gamma_2) \right) - 0.5 \frac{a_{1d}}{\omega_1} \tilde{g}_2 \cos(\omega_1 \tau_2) \\
& + 0.5 \frac{a_{1d}}{\omega_1} \tilde{g}_1 \sin(\omega_1 \tau_2) - 0.5 \frac{a_{2d}}{\omega_1} (\tilde{g}_4 \cos(\gamma_1 - \gamma_2 - \omega_2 \tau_2) + \tilde{g}_3 \sin(\gamma_1 - \gamma_2 - \omega_2 \tau_2)) \\
& + \frac{p_{11}^2 F_{12}}{4\omega_1} a_1 \sin 2\gamma_1 + \frac{p_{11} p_{12} F_{12}}{4\omega_1} a_2 \sin(\gamma_1 + \gamma_2) + \frac{p_{11} k_1 Y}{2\omega_1} \sin \gamma_1
\end{aligned} \tag{5.11a}$$

$$\begin{aligned}
a_1 \dot{\gamma}_1 &= 3\sigma_1 a_1 - 0.5c_{11} a_1 \sin(\omega_1 \tau_1) + 0.5c_{12} a_2 \frac{\omega_2}{\omega_1} \sin(\gamma_1 - \gamma_2 - \omega_2 \tau_1) \\
& - \frac{k_{n1}}{\omega_1} \left(\Lambda_1^3 \cos \gamma_1 + \frac{3a_1^3}{8} + 3a_1 \Lambda_1^2 \right) - \frac{3k_{n2}}{\omega_1} \frac{a_1^2 a_2}{8} \cos(\gamma_2 - \gamma_1) - \\
& \frac{3k_{n2}}{\omega_1} \left(\Lambda_1^2 \Lambda_2 \cos \gamma_1 + 2a_1 \Lambda_1 \Lambda_2 + \left(\frac{a_1^2 a_2}{4} + \Lambda_1^2 a_2 \right) \cos(\gamma_1 - \gamma_2) \right) - \frac{3k_{n3} \Lambda_2^2 a_1}{\omega_1} \\
& - \frac{3k_{n3}}{\omega_1} \left(\frac{a_1 a_2^2}{8} \cos(2(\gamma_1 - \gamma_2)) + \Lambda_1 \Lambda_2^2 \cos \gamma_1 + 2a_2 \Lambda_1 \Lambda_2 \cos(\gamma_1 - \gamma_2) + \frac{a_2^2 a_1}{4} \right) \\
& - \frac{k_{n4}}{\omega_1} \left(\Lambda_2^3 \cos \gamma_1 + \left(\frac{3a_2^3}{8} + 3a_2 \Lambda_2^2 \right) \cos(\gamma_1 - \gamma_2) \right) - 0.5 \frac{a_{1d}}{\omega_1} \tilde{g}_1 \cos(\omega_1 \tau_2) \\
& - \frac{a_{1d}}{2\omega_1} \tilde{g}_2 \sin(\omega_1 \tau_2) - \frac{a_{2d}}{2\omega_1} (\tilde{g}_3 \cos(\gamma_1 - \gamma_2 - \omega_2 \tau_2) - \tilde{g}_4 \sin(\gamma_1 - \gamma_2 - \omega_2 \tau_2)) \\
& + \frac{p_{11}^2 F_{12}}{4\omega_1} a_1 \cos 2\gamma_1 + \frac{p_{11} p_{12} F_{12}}{4\omega_1} a_2 \cos(\gamma_1 + \gamma_2) + \frac{p_{11} k_1 Y}{2\omega_1} \cos \gamma_1
\end{aligned} \tag{5.11b}$$

$$\begin{aligned}
\dot{a}_2 &= -0.5c_{21} \frac{\omega_1}{\omega_2} a_1 \cos(\gamma_2 - \gamma_1 - \omega_1 \tau_1) - 0.5c_{22} a_2 \cos(\omega_2 \tau_1) \\
& - \frac{\tilde{k}_{n1}}{\omega_2} \left(\Lambda_1^3 \sin \gamma_2 + \left(\frac{3a_1^3}{8} + 3a_1 \Lambda_1^2 \right) \sin(\gamma_2 - \gamma_1) \right) + \frac{p_{12}^2 F_{12}}{4\omega_2} a_2 \sin 2\gamma_2 \\
& - \frac{3\tilde{k}_{n2}}{\omega_2} \left(\frac{a_1^2 a_2}{8} \sin(2(\gamma_2 - \gamma_1)) + \Lambda_1^2 \Lambda_2 \sin \gamma_2 + 2a_1 \Lambda_1 \Lambda_2 \sin(\gamma_2 - \gamma_1) \right) \\
& - \frac{3\tilde{k}_{n3}}{\omega_2} \left(\frac{a_1 a_2^2}{8} \sin(\gamma_1 - \gamma_2) + \Lambda_1 \Lambda_2^2 \sin \gamma_2 + \left(\frac{a_1 a_2^2}{4} + \Lambda_2^2 a_1 \right) \sin(\gamma_2 - \gamma_1) \right) \\
& - \frac{\tilde{k}_{n4}}{\omega_2} \Lambda_2^3 \sin \gamma_2 - \frac{a_{1d}}{2\omega_2} (\tilde{g}_5 \sin(\gamma_2 - \gamma_1 - \omega_1 \tau_2) + \tilde{g}_6 \cos(\gamma_2 - \gamma_1 - \omega_1 \tau_2)) \\
& - \frac{a_{2d}}{2\omega_2} (\tilde{g}_8 \cos(\omega_2 \tau_2) - \tilde{g}_7 \sin(\omega_2 \tau_2)) + \frac{p_{11} p_{12} F_{12}}{4\omega_2} a_1 \sin(\gamma_1 + \gamma_2) + \frac{p_{12} k_1 Y}{2\omega_2} \sin \gamma_2
\end{aligned} \tag{5.11c}$$

$$\begin{aligned}
 a_2 \dot{\gamma}_2 = & \sigma_{33} a_2 + 0.5 c_{21} \frac{\omega_1}{\omega_2} a_1 \sin(\gamma_2 - \gamma_1 - \omega_1 \tau_1) - 0.5 c_{22} a_2 \sin(\omega_2 \tau_1) \\
 & - \frac{\tilde{k}_{n1}}{\omega_2} \left(\Lambda_1^3 \cos \gamma_2 + \left(\frac{3a_1^3}{8} + 3a_1 \Lambda_1^2 \right) \cos(\gamma_2 - \gamma_1) \right) - \frac{3\tilde{k}_{n2}}{\omega_2} \left(\frac{a_1^2 a_2}{4} + a_2 \Lambda_1^2 \right) \\
 & - \frac{3\tilde{k}_{n2}}{\omega_2} \left(\frac{a_1^2 a_2}{8} \cos(2(\gamma_2 - \gamma_1)) + \Lambda_1^2 \Lambda_2 \cos \gamma_2 + 2a_1 \Lambda_1 \Lambda_2 \cos(\gamma_2 - \gamma_1) \right) \\
 & - \frac{3\tilde{k}_{n3}}{\omega_2} \left(\frac{a_1 a_2^2}{8} \cos(\gamma_1 - \gamma_2) + \Lambda_1 \Lambda_2^2 \cos \gamma_2 + \left(\frac{a_1 a_2^2}{4} + \Lambda_2^2 a_1 \right) \cos(\gamma_2 - \gamma_1) \right) \\
 & - \frac{6\tilde{k}_{n3} a_2 \Lambda_1 \Lambda_2}{\omega_2} - \frac{\tilde{k}_{n4}}{\omega_2} \left(\Lambda_2^3 \cos \gamma_2 + \frac{3a_2^3}{8} + 3a_2 \Lambda_2^2 \right) - \frac{a_{1d}}{2\omega_2} (\tilde{g}_5 \cos(\gamma_2 - \gamma_1 - \omega_1 \tau_2)) \\
 & + \frac{a_{1d}}{2\omega_2} \tilde{g}_6 \sin(\gamma_2 - \gamma_1 - \omega_1 \tau_2) - \frac{a_{2d}}{2\omega_2} (\tilde{g}_7 \cos(\omega_2 \tau_2) + \tilde{g}_8 \sin(\omega_2 \tau_2)) \\
 & + \frac{p_{11} p_{12} F_{12}}{4\omega_2} a_1 \cos(\gamma_1 + \gamma_2) + \frac{p_{12}^2 F_{12}}{4\omega_2} a_2 \cos 2\gamma_2 + \frac{p_{12} k_1 Y}{2\omega_2} \cos \gamma_2
 \end{aligned} \tag{5.11d}$$

where

$$\begin{aligned}
 \gamma_1 = & 3\sigma_1 T_1 - \beta_1, \quad \gamma_2 = 3\sigma_1 T_1 - (\sigma_3 - \sigma_{33}) T_1 - \beta_2, \quad \sigma_2 = 6\sigma_1, \quad \sigma_3 = 3\sigma_1, \\
 \tilde{g}_1 = & p_1 p_{11} k_e (f_d - \omega_1^2 f_a), \quad \tilde{g}_2 = p_1 p_{11} k_e \omega_1 f_v, \quad \tilde{g}_3 = p_1 p_{12} k_e (f_d - \omega_2^2 f_a), \quad \tilde{g}_4 = p_1 p_{12} k_e \omega_2 f_v \\
 \tilde{g}_5 = & p_2 p_{11} k_e (f_d - \omega_1^2 f_a), \quad \tilde{g}_6 = p_2 p_{11} k_e \omega_1 f_v, \quad \tilde{g}_7 = p_2 p_{12} k_e (f_d - \omega_2^2 f_a), \quad \tilde{g}_8 = p_2 p_{12} k_e \omega_2 f_v \\
 g_{11} = & \tilde{g}_1 + i\tilde{g}_2, \quad g_{12} = \tilde{g}_3 + i\tilde{g}_4, \quad g_{21} = \tilde{g}_5 + i\tilde{g}_6, \quad g_{22} = \tilde{g}_7 + i\tilde{g}_8
 \end{aligned}$$

Similarly, for the non-traditional ANVA the following additional terms $0.5 p_{11} c_2 \Omega_3 Y \cos(\gamma_1 - \Omega_3 \tau_1)$, $-0.5 p_{11} c_2 \Omega_3 Y \sin(\gamma_1 - \Omega_3 \tau_1)$, $0.5 p_{12} c_2 \Omega_3 Y \cos(\gamma_2 - \Omega_3 \tau_1)$

and $-0.5 p_{12} c_2 \Omega_3 Y \sin(\gamma_2 - \Omega_3 \tau_1)$ are added on the right-hand side of the existing steady-state autonomous Eqs. (5.11a), (5.11b), (5.11c), and (5.11d), respectively. The steady-state solutions of the reduced MMS equations (Eq. (5.11a) to Eq. (5.11d)) are solved using Newton's method by equating $\dot{a}_1 = \dot{\gamma}_1 = \dot{a}_2 = \dot{\gamma}_2 = 0$. The stability of the system is ensured by finding the eigenvalue of the Jacobian matrix. The solutions obtained from Newton's method (a_1 , γ_1 , a_2 and γ_2) are substituted in Eq. (5.8a), Eq. (5.8b), Eq. (5.10a) and Eq. (5.10b) to obtain the approximate solution by MMS for X_1 and X_2 given in Eq. (5.5).

These solutions for X_1 and X_2 are given in **Appendix C**. By directly perturbing the reduced equations, one can study the stability of the non-trivial steady-state solution. But, as the

reduced Eqs. (5.11a) to (5.11d) have the coupled terms $a_1\dot{\gamma}_1$ and $a_2\dot{\gamma}_2$, the perturbed equations will not contain the perturbations $\Delta\dot{\gamma}_1$ and $\Delta\dot{\gamma}_2$ for trivial solutions and hence the stability of the trivial state cannot be studied directly perturbing Eqs. (5.11a) to (5.11d). To circumvent this difficulty, the normalization method is adopted by introducing the transformation $P_i = a_i \cos \gamma_i$ and $Q_i = a_i \sin \gamma_i$ for $i = 1, 2$. into Eqs. (5.11a) to (5.11d). Carrying out trigonometric manipulations, the following normalized reduced equations are obtained:

$$\begin{aligned} \dot{P}_1 = & -3\sigma_1 Q_1 - 0.5c_{11} (P_1 \cos(\omega_1 \tau_1) - Q_1 \sin(\omega_1 \tau_1)) - 0.5c_{12} \frac{\omega_2}{\omega_1} (P_2 \cos(\omega_2 \tau_1) - Q_2 \sin(\omega_2 \tau_1)) \\ & + \frac{\tilde{g}_1}{2\omega_1} (Q_1 \cos(\omega_1 \tau_2) + P_1 \sin(\omega_1 \tau_2)) - \frac{\tilde{g}_2}{2\omega_1} (P_1 \cos(\omega_1 \tau_2) - Q_1 \sin(\omega_1 \tau_2)) + \frac{p_{11}^2 F_{12}}{4\omega_1} Q_1 \\ & - \frac{\tilde{g}_4}{2\omega_1} (P_2 \cos(\omega_2 \tau_2) - Q_2 \sin(\omega_2 \tau_2)) + \frac{\tilde{g}_3}{2\omega_1} (Q_2 \cos(\omega_2 \tau_2) + P_2 \sin(\omega_2 \tau_2)) + \frac{p_{11} p_{12} F_{12}}{4\omega_1} Q_2 \end{aligned} \quad (5.12a)$$

$$\begin{aligned} \dot{Q}_1 = & 3\sigma_1 P_1 - 0.5c_{11} Q_1 \cos(\omega_1 \tau_1) - 0.5c_{11} P_1 \sin(\omega_1 \tau_1) - \frac{c_{12}\omega_2}{2\omega_1} (Q_2 \cos \omega_2 \tau_1 + P_2 \sin \omega_2 \tau_1) \\ & - \frac{Q_1}{2\omega_1} (\tilde{g}_2 \cos(\omega_1 \tau_2) - \tilde{g}_1 \sin(\omega_1 \tau_2)) - \frac{P_1}{2\omega_1} \tilde{g}_1 \cos(\omega_1 \tau_2) - \frac{P_1}{2\omega_1} \tilde{g}_2 \sin(\omega_1 \tau_2) \\ & - \frac{\tilde{g}_4}{2\omega_1} (Q_2 \cos(\omega_2 \tau_2) + P_2 \sin(\omega_2 \tau_1)) - \frac{\tilde{g}_3}{2\omega_1} (P_2 \cos(\omega_2 \tau_2) - Q_2 \sin(\omega_2 \tau_2)) \\ & + \frac{p_{11}^2 F_{12}}{4\omega_1} P_1 + \frac{p_{11} p_{12} F_{12}}{4\omega_1} P_2 + \frac{p_{11} k_1 Y}{2\omega_1} \end{aligned} \quad (5.12b)$$

$$\begin{aligned} \dot{P}_2 = & -\sigma_{33} Q_2 - 0.5c_{21} \frac{\omega_1}{\omega_2} (P_1 \cos(\omega_1 \tau_1) - Q_1 \sin(\omega_1 \tau_1)) - 0.5c_{22} P_2 \cos(\omega_2 \tau_1) \\ & + 0.5c_{22} Q_2 \sin(\omega_2 \tau_1) + \frac{\tilde{g}_5}{2\omega_2} (Q_1 \cos(\omega_1 \tau_2) + P_1 \sin(\omega_1 \tau_2)) - \frac{\tilde{g}_6}{2\omega_2} P_1 \cos(\omega_1 \tau_2) \\ & + \frac{\tilde{g}_6}{2\omega_2} Q_1 \sin(\omega_1 \tau_2) + \frac{\tilde{g}_8}{2\omega_2} (Q_2 \sin(\omega_2 \tau_2) - P_2 \cos(\omega_2 \tau_2)) + \frac{P_2}{2\omega_2} \tilde{g}_7 \sin(\omega_2 \tau_2) \\ & + \frac{Q_2}{2\omega_2} \tilde{g}_7 \cos(\omega_2 \tau_2) + \frac{p_{11} p_{12} F_{12}}{4\omega_2} Q_1 + \frac{p_{12}^2 F_{12}}{4\omega_2} Q_2 \end{aligned} \quad (5.12c)$$

$$\begin{aligned}
 \dot{Q}_2 = & \sigma_{33}P_2 - 0.5c_{21}\frac{\omega_1}{\omega_2}(Q_1 \cos(\omega_1\tau_1) + P_1 \sin(\omega_1\tau_1)) - 0.5c_{22}Q_2 \cos(\omega_2\tau_1) \\
 & - 0.5c_{22}P_2 \sin(\omega_2\tau_1) - \frac{\tilde{g}_5}{2\omega_2}(P_1 \cos(\omega_1\tau_1) - Q_1 \sin(\omega_1\tau_1)) - \frac{\tilde{g}_6}{2\omega_2}Q_1 \cos(\omega_1\tau_1) \\
 & - \frac{\tilde{g}_6}{2\omega_2}P_1 \sin(\omega_1\tau_1) - \frac{Q_2}{2\omega_2}(\tilde{g}_8 \cos(\omega_2\tau_2) - \tilde{g}_7 \sin(\omega_2\tau_2)) - \frac{P_2}{2\omega_2}\tilde{g}_7 \cos(\omega_2\tau_2) \\
 & - \frac{P_2}{2\omega_2}\tilde{g}_8 \sin(\omega_2\tau_2) + \frac{p_{11}p_{12}F_{12}}{4\omega_2}P_1 + \frac{p_{12}^2F_{12}}{4\omega_2}P_2 + \frac{p_{12}k_1Y}{2\omega_2}
 \end{aligned} \tag{5.12d}$$

Now perturbing the above equations, one can obtain the Jacobian matrix whose eigenvalues will determine the stability and bifurcation of the system. In the next sub-section, the 1:2 internal resonance condition ($\Omega_2 = 2\omega_1 + \varepsilon\sigma_2 = \omega_2 + \varepsilon\sigma_{22}$), for case(v) is considered to obtain the reduced equations for the system.

1: 2 internal resonance condition

In this section, ω_1 is taken to be nearly equal to 0.5 times of ω_2 which gives rise to 1:2 internal resonance condition. Substituting this 1:2 internal resonance condition the secular terms are obtained from the expansion of Eqs. (5.7a) which consists of coefficients of terms $\exp(i\omega_1T_0)$, $\exp(3i\Omega_1T_0)$, $\exp(i(\Omega_2 - \omega_1)T_0)$, $\exp(i\Omega_3T_0)$. Similarly, the secular terms are obtained from the expansion of Eqs. (6.7b) are from the resonance conditions of $\exp(i\omega_2T_0)$. The secular terms for 1:2 internal resonance condition ($\omega_2 = 2\omega_1 + \varepsilon(\sigma_2 - \sigma_{22})$) with simultaneous primary, principal parametric and super-harmonic resonances are given below to obtain the bounded solution for the system.

$$\begin{aligned}
 2i\omega_1 D_1 A_1 = & -ic_{11}A_1\omega_1 \exp(-i\omega_1\tau_1) - k_{n1}(\Lambda_1^3 \exp(3i\sigma_1T_1) + 3A_1^2\bar{A}_1 + 6A_1\Lambda_1^2) \\
 & - 3k_{n2}\Lambda_2(\Lambda_1^2 \exp(3i\sigma_1T_1) + 4A_1\Lambda_1) - 3k_{n3}(\Lambda_2^2\Lambda_1 \exp(3i\sigma_1T_1)) \\
 & - 6k_{n3}A_1(A_2\bar{A}_2 + \Lambda_2^2) - k_{n4}\Lambda_2^3 \exp(3i\sigma_1T_1) - g_{11}A_{1d} \exp(-i\omega_1\tau_2) \\
 & + 0.5p_{11}(p_{11}F_{12}\bar{A}_1 \exp(i\sigma_2T_1) + k_1Y \exp(i\sigma_3T_1))
 \end{aligned} \tag{5.13a}$$

$$\begin{aligned}
 2i\omega_2 D_1 A_2 = & -ic_{22}A_2\omega_2 \exp(-i\omega_2\tau_1) - 3\tilde{k}_{n2}(2A_1\bar{A}_1A_2 + 2\Lambda_1^2A_2) - 12\tilde{k}_{n3}A_2\Lambda_1\Lambda_2 \\
 & - \tilde{k}_{n4}(3A_2^2\bar{A}_2 + 6A_2\Lambda_2^2) - g_{22}A_{2d} \exp(-i\omega_2\tau_2)
 \end{aligned} \tag{5.13b}$$

It may be noted that here 1:2 internal resonance condition ($\Omega_2 = 2\omega_1 + \varepsilon\sigma_2 = \omega_2 + \varepsilon\sigma_{22}$), is considered to suppress the response amplitude for the principal parametric resonance condition. Following the same procedure as mentioned previously the following steady-state autonomous reduced equations are obtained for the system.

$$\begin{aligned} \dot{a}_1 = & -0.5c_{11}a_1 \cos(\omega_1\tau_1) - \left(\frac{k_{n1}}{\omega_1}\Lambda_1^3 + \frac{3k_{n2}}{\omega_1}\Lambda_1^2\Lambda_2 + \frac{3k_{n3}}{\omega_1}\Lambda_2^2\Lambda_1 + \frac{k_{n4}}{\omega_1}\Lambda_2^3 \right) \sin\gamma_1 \\ & -0.5\frac{a_{1d}}{\omega_1}(\tilde{g}_2 \cos(\omega_1\tau_2) - \tilde{g}_1 \sin(\omega_1\tau_2)) + 0.25\frac{p_{11}^2 F_{12}a_1}{\omega_1} \sin 2\gamma_1 + 0.5\frac{p_{11}k_1 Y}{\omega_1} \sin\gamma_1 \end{aligned} \quad (5.14a)$$

$$\begin{aligned} a_1\dot{\gamma}_1 = & 3\sigma_1 a_1 - 0.5c_{11}a_1 \sin(\omega_1\tau_1) - \frac{k_{n1}}{\omega_1} \left(\Lambda_1^3 \cos\gamma_1 + \frac{3a_1^3}{8} + 3a_1\Lambda_1^2 \right) - \frac{3k_{n2}\Lambda_1^2\Lambda_2}{\omega_1} \cos\gamma_1 \\ & -6\frac{k_{n2}}{\omega_1}a_1\Lambda_1\Lambda_2 - 3\frac{k_{n3}}{\omega_1}\Lambda_2^2\Lambda_1 \cos\gamma_1 - 3\frac{k_{n3}}{\omega_1}(0.25a_2^2a_1 + \Lambda_2^2a_1) - \frac{k_{n4}}{\omega_1}\Lambda_2^3 \cos\gamma_1 \\ & -0.5a_{1d}(\tilde{g}_1 \cos(\omega_1\tau_2) + \tilde{g}_2 \sin(\omega_1\tau_2)) + 0.25p_{11}^2 F_{12}a_1 \cos 2\gamma_1 + 0.5p_{11}k_1 Y \cos\gamma_1 \end{aligned} \quad (5.14b)$$

$$\dot{a}_2 = -0.5c_{22}a_2 \cos(\omega_2\tau_1) - 0.5\frac{a_{2d}}{\omega_2}(\tilde{g}_8 \cos(\omega_2\tau_2) - \tilde{g}_7 \sin(\omega_2\tau_2)) \quad (5.14c)$$

$$\begin{aligned} a_2\dot{\beta}_2 = & -0.5c_{22}a_2 \sin(\omega_2\tau_1) - 3\frac{\tilde{k}_{n2}}{\omega_2} \left(\frac{a_1^2 a_2}{4} + \Lambda_1^2 a_2 \right) - \frac{6\tilde{k}_{n3}a_2\Lambda_1\Lambda_2}{\omega_2} \\ & + \frac{\tilde{k}_{n4}}{\omega_2} \left(\frac{3a_2^3}{8} + 3a_2\Lambda_2^2 \right) - 0.5\frac{a_{2d}}{\omega_2}(\tilde{g}_7 \cos(\omega_2\tau_2) + \tilde{g}_8 \sin(\omega_2\tau_2)) \end{aligned} \quad (5.14d)$$

where

$$\gamma_1 = 3\sigma_1 T_1 - \beta_1, \quad \sigma_2 = 6\sigma_1, \quad \sigma_3 = 3\sigma_1$$

In the next sub-section, the 3:1 internal resonance condition ($\Omega_1 = \omega_1/3 + \varepsilon\sigma_1 = \omega_2 + \varepsilon\sigma_{11}$), for case(v) is considered to obtain the reduced equations for the system.

3: 1 internal resonance condition

In this section, ω_1 is taken to be nearly equal to 3 times the ω_2 which gives rise to 3:1 internal resonance condition. Here the secular terms are obtained from the expansion of

Eqs. (5.7a) which consists of the coefficients of the terms $\exp(i\omega_1 T_0)$, $\exp(3i\Omega_1 T_0)$, $\exp(i(\omega_1 - \omega_2 + \Omega_1)T_0)$, $\exp(i(\omega_1 + \omega_2 - \Omega_1)T_0)$, $\exp(i(2\omega_2 + \Omega_1)T_0)$, $\exp(i(2\Omega_1 + \omega_2)T_0)$, $\exp(3i\omega_2 T_0)$, $\exp(i(\Omega_2 - \omega_1)T_0)$, $\exp(i\Omega_3 T_0)$. Similarly, the secular terms are obtained from the expansion of Eqs. (5.7b) are from the resonance conditions of $\exp(i\omega_2 T_0)$, $\exp(i\Omega_1 T_0)$, $\exp(i(\omega_1 - 2\Omega_1)T_0)$, $\exp(i(2\Omega_1 - \omega_2)T_0)$, $\exp(i(\omega_1 - \Omega_1 - \omega_2)T_0)$, $\exp(i(\omega_1 - 2\omega_2)T_0)$, $\exp(i(2\omega_2 - \Omega_1)T_0)$. From these conditions, the secular terms for the 3:1 internal resonance condition $\left(\omega_2 = \frac{\omega_1}{3} + \varepsilon(\sigma_1 - \sigma_{11})\right)$ with simultaneous primary, principal parametric and super-harmonic resonances (case(v)) are given below.

$$\begin{aligned}
 2i\omega_1 D_1 A_1 = & -ic_{11} A_1 \omega_1 \exp(-i\omega_1 \tau_1) - k_{n1} \left(\Lambda_1^3 \exp(3i\sigma_1 T_1) + 3A_1^2 \bar{A}_1 + 6A_1 \Lambda_1^2 \right) \\
 & - 3k_{n2} \left(A_2 \Lambda_1^2 \exp(i(2\sigma_1 + (\sigma_1 - \sigma_{11}))T_1) + \Lambda_1^2 \Lambda_2 \exp(3i\sigma_1 T_1) + 4A_1 \Lambda_1 \Lambda_2 \right. \\
 & \left. + 2A_1 \Lambda_1 \bar{A}_2 \exp(i(\sigma_1 - (\sigma_1 - \sigma_{11}))T_1) + 2A_1 A_2 \Lambda_1 \exp(i((\sigma_1 - \sigma_{11}) - \sigma_1)T_1) \right) \\
 & - 3k_{n3} \left(A_2^2 \Lambda_1 \exp(i(3\sigma_1 - 2\sigma_{11})T_1) + 2A_2 \Lambda_1 \Lambda_2 \exp(i(2\sigma_1 + (\sigma_1 - \sigma_{11}))T_1) \right. \\
 & \left. + \Lambda_2^2 \Lambda_1 \exp(3i\sigma_1 T_1) + 2\bar{A}_2 A_1 \Lambda_2 \exp(i(\sigma_1 - (\sigma_1 - \sigma_{11}))T_1) \right. \\
 & \left. + 2A_1 A_2 \Lambda_2 \exp(i((\sigma_1 - \sigma_{11}) - \sigma_1)T_1) + 2A_2 \bar{A}_2 A_1 + 2\Lambda_2^2 A_1 \right) \\
 & - k_{n4} \left(A_2^3 \exp(3i(\sigma_1 - \sigma_{11})T_1) + \Lambda_2^3 \exp(3i\sigma_1 T_1) \right. \\
 & \left. + 3A_2 \Lambda_2^2 \exp(i(2\sigma_1 + (\sigma_1 - \sigma_{11}))T_1) + 3A_2^2 \Lambda_2 \exp(i(2(\sigma_1 - \sigma_{11}) + \sigma_1)T_1) \right) \\
 & - g_{11} A_{1d} \exp(-i\omega_1 \tau_2) + 0.5 p_{11}^2 F_{12} \bar{A}_1 \exp(i\sigma_2 T_1) + 0.5 p_{11} k_1 Y \exp(i\sigma_3 T_1) \\
 2i\omega_2 D_1 A_2 = & -ic_{22} \left(A_2 \omega_2 \exp(-i\omega_2 \tau_1) + \Lambda_2 \Omega_1 \exp(i(\sigma_{11} T_1 - \Omega_1 \tau_1)) \right) \\
 & - ic_{21} \Lambda_1 \Omega_1 \exp(i(\sigma_{11} T_1 - \Omega_1 \tau_1)) - \tilde{k}_{n1} \left(3\Lambda_1^3 \exp(i\sigma_{11} T_1) + 6A_1 \bar{A}_1 \Lambda_1 \exp(i\sigma_{11} T_1) \right. \\
 & \left. + 3A_1 \Lambda_1^2 \exp(i(-2\sigma_1 - (\sigma_1 - \sigma_{11}))T_1) \right) \\
 & - 3\tilde{k}_{n2} \left(\Lambda_1^2 \bar{A}_2 \exp(i(2\sigma_1 - 2(\sigma_1 - \sigma_{11}))T_1) + 2A_1 \bar{A}_2 \Lambda_1 \exp(i(-\sigma_1 - 2(\sigma_1 - \sigma_{11}))T_1) \right. \\
 & \left. + 2A_1 \Lambda_1 \Lambda_2 \exp(i(-2\sigma_1 - (\sigma_1 - \sigma_{11}))T_1) + 2A_1 \bar{A}_1 A_2 + 2\Lambda_1^2 A_2 \right) \\
 & - 3\tilde{k}_{n3} \left(\bar{A}_2^2 A_1 \exp(i(-3(\sigma_1 - \sigma_{11}))T_1) + A_2^2 \Lambda_1 \exp(i((\sigma_1 - \sigma_{11}) - \sigma_1)T_1) \right. \\
 & \left. + A_1 \Lambda_2^2 \exp(i(-2\sigma_1 - (\sigma_1 - \sigma_{11}))T_1) + 2\bar{A}_2 A_1 \Lambda_2 \exp(i(-\sigma_1 - 2(\sigma_1 - \sigma_{11}))T_1) \right. \\
 & \left. + 2\bar{A}_2 \Lambda_2 \Lambda_1 \exp(i(2\sigma_1 - 2(\sigma_1 - \sigma_{11}))T_1) + 4A_2 \Lambda_1 \Lambda_2 \right) \\
 & - k_{n4} \left(3\Lambda_2^3 \exp(i\sigma_{11} T_1) + 3A_2^2 \bar{A}_2 + 6A_2 \Lambda_2^2 + 6A_2 \bar{A}_2 \Lambda_2 \exp(i\sigma_{11} T_1) \right) -
 \end{aligned} \tag{5.15a}$$

$$\begin{aligned}
& k_{n4} \left(3\bar{A}_2 \Lambda_2^2 \exp(i(2\sigma_1 - 2(\sigma_1 - \sigma_{11}))T_1) + 3A_2^2 \Lambda_2 \exp(-i\sigma_{11}T_1) \right) \\
& - g_{22} A_{2d} \exp(-i\omega_2 \tau_2) - (g_{21} \Lambda_1 - g_{22} \Lambda_2) \exp(-i\Omega_1 \tau_2) \exp(i\sigma_{11}T_1)
\end{aligned} \quad (5.15b)$$

It may be noted that here 3:1 internal resonance condition $\left(\Omega_1 = \frac{\omega_1}{3} + \varepsilon \sigma_1 = \omega_2 + \varepsilon \sigma_{11} \right)$, is considered to suppress response amplitude for the superharmonic resonance condition. Similarly, the following steady-state autonomous reduced equations are obtained for the system as given below.

$$\begin{aligned}
\dot{a}_1 = & -0.5c_{11}a_1 \cos(\omega_1 \tau_1) - \frac{k_{n1}}{\omega_1} \Lambda_1^3 \sin \gamma_1 - \frac{3k_{n2}}{2\omega_1} a_2 \Lambda_1^2 \sin(\gamma_1 - \gamma_2) \\
& - 3 \frac{k_{n2}}{\omega_1} \left(\Lambda_1^2 \Lambda_2 \sin \gamma_1 + 0.5a_1 a_2 \Lambda_1 \sin \gamma_2 - 0.5a_1 a_2 \Lambda_1 \sin \gamma_2 \right) \\
& - 3 \frac{k_{n3}}{\omega_1} \left(0.25a_2^2 \Lambda_1 \sin(\gamma_1 - 2\gamma_2) + a_2 \Lambda_1 \Lambda_2 \sin(\gamma_1 - \gamma_2) \right. \\
& \left. + \Lambda_2^2 \Lambda_1 \sin \gamma_1 + 0.5a_2 a_1 \Lambda_2 \sin \gamma_2 - 0.5a_1 a_2 \Lambda_2 \sin \gamma_2 \right) \\
& - \frac{k_{n4}}{\omega_1} \left(\frac{a_2^3}{8} \sin(\gamma_1 - 3\gamma_2) + \Lambda_2^3 \sin \gamma_1 + 1.5a_2 \Lambda_2^2 \sin(\gamma_1 - \gamma_2) \right) \\
& - \frac{3k_{n4}}{4\omega_1} a_2^2 \Lambda_2 \sin(\gamma_1 - 2\gamma_2) - 0.5 \frac{a_{1d}}{\omega_1} (\tilde{g}_2 \cos(\omega_1 \tau_2) - \tilde{g}_1 \sin(\omega_1 \tau_2)) \\
& + 0.25 \frac{p_{11}^2 F_{12} a_1}{\omega_1} \sin 2\gamma_1 + 0.5 \frac{p_{11} k_1 Y}{\omega_1} \sin \gamma_1
\end{aligned} \quad (5.16a)$$

$$\begin{aligned}
a_1 \dot{\gamma}_1 = & 3a_1 \sigma_1 - 0.5c_{11}a_1 \sin(\omega_1 \tau_1) - \frac{k_{n1}}{\omega_1} \left(\Lambda_1^3 \cos \gamma_1 + \frac{3a_1^3}{8} + 3a_1 \Lambda_1^2 \right) \\
& - 3 \frac{k_{n2}}{\omega_1} \left(0.5a_2 \Lambda_1^2 \cos(\gamma_1 - \gamma_2) + 0.5a_1 a_2 \Lambda_1 \cos \gamma_2 \right. \\
& \left. + 0.5a_1 a_2 \Lambda_1 \cos \gamma_2 + 2a_1 \Lambda_1 \Lambda_2 + \Lambda_1^2 \Lambda_2 \cos \gamma_1 \right) - \frac{3k_{n3}}{4\omega_1} a_2^2 a_1 \\
& - 3 \frac{k_{n3}}{\omega_1} \left(0.25a_2^2 \Lambda_1 \cos(\gamma_1 - 2\gamma_2) + a_2 \Lambda_1 \Lambda_2 \cos(\gamma_1 - \gamma_2) + \Lambda_2^2 a_1 \right. \\
& \left. + \Lambda_2^2 \Lambda_1 \cos \gamma_1 + 0.5a_2 a_1 \Lambda_2 \cos \gamma_2 + 0.5a_1 a_2 \Lambda_2 \cos \gamma_2 \right) \\
& - \frac{k_{n4}}{\omega_1} \left(\frac{a_2^3}{8} \cos(\gamma_1 - 3\gamma_2) + 1.5a_2 \Lambda_2^2 \cos(\gamma_1 - \gamma_2) \right. \\
& \left. + 0.75a_2^2 \Lambda_2 \cos(\gamma_1 - 2\gamma_2) + \Lambda_2^3 \cos \gamma_1 \right) + 0.25 \frac{p_{11}^2 F_{12} a_1}{\omega_1} \cos 2\gamma_1 \\
& - 0.5 \frac{a_{1d}}{\omega_1} (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)) + 0.5 \frac{p_{11} k_1 Y}{\omega_1} \cos \gamma_1
\end{aligned} \quad (5.16b)$$

$$\begin{aligned}
 \dot{a}_2 = & -c_{22} \left(0.5a_2 \cos(\omega_2 \tau_1) + \Lambda_2 \frac{\Omega_1}{\omega_2} \cos(\gamma_2 - \Omega_1 \tau_1) \right) - c_{21} \Lambda_1 \frac{\Omega_1}{\omega_2} \cos(\gamma_2 - \Omega_1 \tau_1) \\
 & - \left(\tilde{k}_{n1} (3\Lambda_1^3 \sin \gamma_2 + 1.5a_1^2 \Lambda_1 \sin \gamma_2 + 1.5a_1 \Lambda_1^2 \sin(\gamma_2 - \gamma_1)) \right) / \omega_2 \\
 & - 3\tilde{k}_{n2} (0.5\Lambda_1^2 a_2 \sin 2\gamma_2 + 0.5a_1 a_2 \Lambda_1 \sin(2\gamma_2 - \gamma_1) + a_1 \Lambda_1 \Lambda_2 \sin(\gamma_2 - \gamma_1)) / \omega_2 \\
 & - 3\frac{\tilde{k}_{n3}}{\omega_2} \left(\frac{a_2^2 a_1}{8} \sin(3\gamma_2 - \gamma_1) - \frac{a_2^2}{4} \Lambda_1 \sin \gamma_2 + a_2 \Lambda_2 \Lambda_1 \sin(2\gamma_2) \right. \\
 & \quad \left. + 0.5a_1 \Lambda_2^2 \sin(\gamma_2 - \gamma_1) + 0.5a_2 a_1 \Lambda_2 \sin(2\gamma_2 - \gamma_1) \right) - \frac{a_{2d}}{2\omega_2} \tilde{g}_8 \cos(\omega_2 \tau_2) \quad (5.16c) \\
 & - \frac{\tilde{k}_{n4}}{\omega_2} \left(\left(3\Lambda_2^3 + 1.5a_2^2 \Lambda_2 - \frac{3a_2^2 \Lambda_2}{4} \right) \sin \gamma_2 + 1.5a_2 \Lambda_2^2 \sin 2\gamma_2 \right) + \frac{a_{2d}}{2\omega_2} \tilde{g}_7 \sin(\omega_2 \tau_2) \\
 & - \frac{(\tilde{g}_5 \Lambda_1 + \tilde{g}_7 \Lambda_2)}{\omega_2} \sin(\gamma_2 - \Omega_1 \tau_2) - \frac{(\tilde{g}_6 \Lambda_1 + \tilde{g}_8 \Lambda_2)}{\omega_2} \cos(\gamma_2 - \Omega_1 \tau_2)
 \end{aligned}$$

$$\begin{aligned}
 a_2 \dot{\gamma}_2 = & a_2 \sigma_{11} - c_{22} \left(0.5a_2 \sin(\omega_2 \tau_1) - \frac{\Lambda_2 \Omega_1}{\omega_2} \sin(\gamma_2 - \Omega_1 \tau_1) \right) \\
 & + \frac{c_{21} \Lambda_1 \Omega_1}{\omega_2} \sin(\gamma_2 - \Omega_1 \tau_1) - \frac{3\tilde{k}_{n1}}{\omega_2} \left((\Lambda_1^3 + 0.5a_1^2 \Lambda_1) \cos \gamma_2 + 0.5a_1 \Lambda_1^2 \cos(\gamma_2 - \gamma_1) \right) \\
 & - \frac{3\tilde{k}_{n2}}{2\omega_2} \left(\Lambda_1^2 a_2 \cos 2\gamma_2 + a_1 a_2 \Lambda_1 \cos(2\gamma_2 - \gamma_1) \right. \\
 & \quad \left. + 2a_1 \Lambda_1 \Lambda_2 \cos(\gamma_2 - \gamma_1) + 0.5a_1^2 a_2 + 2\Lambda_1^2 a_2 \right) \\
 & - \frac{3\tilde{k}_{n3}}{\omega_2} \left(\frac{a_2^2 a_1}{8} \cos(3\gamma_2 - \gamma_1) + \frac{a_2^2}{4} \Lambda_1 \cos \gamma_2 + 0.5a_1 \Lambda_2^2 \cos(\gamma_2 - \gamma_1) \right. \\
 & \quad \left. + 0.5a_2 a_1 \Lambda_2 \cos(2\gamma_2 - \gamma_1) + a_2 \Lambda_2 \Lambda_1 \cos(2\gamma_2) + 2a_2 \Lambda_1 \Lambda_2 \right) \\
 & - \frac{3\tilde{k}_{n4}}{\omega_2} \left(\left(\Lambda_2^3 + 0.5a_2^2 \Lambda_2 + \frac{a_2^2 \Lambda_2}{4} \right) \cos \gamma_2 + \frac{a_2^3}{8} + a_2 \Lambda_2^2 + 0.5a_2 \Lambda_2^2 \cos 2\gamma_2 \right) \\
 & - 0.5 \frac{a_{2d}}{\omega_2} (\tilde{g}_7 \cos(\omega_2 \tau_2) + \tilde{g}_8 \sin(\omega_2 \tau_2)) \\
 & - \left(\frac{(\tilde{g}_5 \Lambda_1 + \tilde{g}_7 \Lambda_2)}{\omega_2} \cos(\gamma_2 - \Omega_1 \tau_2) - \frac{(\tilde{g}_6 \Lambda_1 + \tilde{g}_8 \Lambda_2)}{\omega_2} \sin(\gamma_2 - \Omega_1 \tau_2) \right) \quad (5.16d)
 \end{aligned}$$

where

$$\gamma_1 = 3\sigma_1 T_1 - \beta_1, \quad \gamma_2 = \sigma_{11} T_1 - \beta_2, \quad \sigma_2 = 6\sigma_1, \quad \sigma_1 = \sigma_{11}$$

Similarly, for the non-traditional ANVA the following additional terms $0.5p_{11}c_2\Omega_3Y\cos(\gamma_1-\Omega_3\tau_1)$ and $-0.5p_{11}c_2\Omega_3Y\sin(\gamma_1-\Omega_3\tau_1)$ are added on the right-hand side of the steady-state autonomous Eqs. (5.16a) and (5.16b), respectively. As explained in

the previous section, the trivial state solutions of the above-reduced equations in Cartesian forms are given below by assuming $P_i = a_i \cos \gamma_i$, $Q_i = a_i \sin \gamma_i$ for $i = 1$ and 2 .

$$\begin{aligned} \dot{P}_1 = & -3\sigma_1 Q_1 + 0.5c_{11} Q_1 \sin(\omega_1 \tau_1) - 0.5c_{11} P_1 \cos(\omega_1 \tau_1) - \frac{P_1}{2\omega_1} \tilde{g}_2 \cos(\omega_1 \tau_2) \\ & + \frac{P_1}{2\omega_1} \tilde{g}_1 \sin(\omega_1 \tau_2) + \frac{Q_1}{2\omega_1} (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)) + \frac{p_{11}^2 F_{12} Q_1}{4\omega_1} \end{aligned} \quad (5.17a)$$

$$\begin{aligned} \dot{Q}_1 = & 3\sigma_1 P_1 - 0.5c_{11} P_1 \sin(\omega_1 \tau_1) - 0.5c_{11} Q_1 \cos(\omega_1 \tau_1) + \frac{p_{11} k_1 Y}{2\omega_1} - \frac{Q_1}{2\omega_1} \tilde{g}_2 \cos(\omega_1 \tau_2) \\ & + \frac{Q_1}{2\omega_1} \tilde{g}_1 \sin(\omega_1 \tau_2) - \frac{P_1}{2\omega_1} (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)) + \frac{p_{11}^2 F_{12} P_1}{4\omega_1} \end{aligned} \quad (5.17b)$$

$$\begin{aligned} \dot{P}_2 = & -\sigma_{11} Q_2 - 0.5c_{22} P_2 \cos(\omega_2 \tau_1) + 0.5c_{22} Q_2 \sin(\omega_2 \tau_1) - c_{22} \frac{\Lambda_2 \Omega_1}{\omega_2} \cos(\Omega_1 \tau_1) \\ & - c_{21} \frac{\Lambda_1 \Omega_1}{\omega_2} \cos(\Omega_1 \tau_1) - \frac{P_2}{2\omega_2} (\tilde{g}_8 \cos(\omega_2 \tau_2) - \tilde{g}_7 \sin(\omega_2 \tau_2)) + \frac{Q_2}{2\omega_2} \tilde{g}_7 \cos(\omega_2 \tau_2) \\ & + \frac{Q_2}{2\omega_2} \tilde{g}_8 \sin(\omega_2 \tau_2) + \frac{(\tilde{g}_5 \Lambda_1 + \tilde{g}_7 \Lambda_2)}{\omega_2} \sin(\Omega_1 \tau_2) - \frac{(\tilde{g}_6 \Lambda_1 + \tilde{g}_8 \Lambda_2)}{\omega_2} \cos(\Omega_1 \tau_2) \end{aligned} \quad (5.17c)$$

$$\begin{aligned} \dot{Q}_2 = & \sigma_{11} P_2 - 0.5c_{22} Q_2 \cos(\omega_2 \tau_1) - 0.5c_{22} P_2 \sin(\omega_2 \tau_1) - \frac{c_{22} \Lambda_2 \Omega_1}{\omega_2} \sin(\Omega_1 \tau_1) \\ & - \frac{c_{21} \Lambda_1 \Omega_1}{\omega_2} \sin(\Omega_1 \tau_1) - \frac{Q_2}{2\omega_2} (\tilde{g}_8 \cos(\omega_2 \tau_2) - \tilde{g}_7 \sin(\omega_2 \tau_2)) - \frac{P_2}{2\omega_2} \tilde{g}_7 \cos(\omega_2 \tau_2) \\ & - \frac{P_2}{2\omega_2} \tilde{g}_8 \sin(\omega_2 \tau_2) - \frac{(\tilde{g}_5 \Lambda_1 + \tilde{g}_7 \Lambda_2)}{\omega_2} \cos(\Omega_1 \tau_2) - \frac{(\tilde{g}_6 \Lambda_1 + \tilde{g}_8 \Lambda_2)}{\omega_2} \sin(\Omega_1 \tau_2) \end{aligned} \quad (5.17d)$$

In the next section, the resonance condition mentioned in case (vi) is considered i.e. for simultaneous primary, principal parametric and sub-harmonic resonances with three internal resonance conditions to obtain the reduced equation for studying responses of the system.

5.3.1.2 Simultaneous primary, principal parametric and sub-harmonic resonance conditions

In this section, the resonance conditions mentioned in case (vi) are considered i.e. for simultaneous primary, principal parametric and sub-harmonic resonances with 1:1, 1:2 and

1:3 internal resonance conditions. It may be noted that here as the PZT stack actuator is integrated with the spring in the absorber configuration, so one can actively change the frequency of the absorber in accordance with various internal resonance cases as mentioned above. In the next sub-section 1:1 internal resonance condition is considered to obtain the secular terms for the above-mentioned simultaneous resonance conditions by solving Eq. (5.7a) and Eq. (5.7b). These secular terms are equated to zero to obtain the solutions for X_1 and X_2 following the standard procedure of MMS.

1:1 internal resonance condition

In this section, the secular terms are obtained for the simultaneous primary, principal parametric and sub-harmonic resonances with 1:1 internal resonance condition ($\omega_2 = \omega_1 + \varepsilon(\sigma_3 - \sigma_{33})$). It may be noted that the secular terms are obtained from the expansion of Eqs. (5.7a) and (5.7b) consists of coefficients of terms $\exp(i\omega_1 T_0)$, $\exp(i\omega_2 T_0)$, $\exp(i(\Omega_1 - 2\omega_1)T_0)$, $\exp(i(2\omega_1 - \omega_2)T_0)$, $\exp(i(2\omega_2 - \omega_1)T_0)$, $\exp(i(\Omega_1 - \omega_1 - \omega_2)T_0)$, $\exp(i\Omega_3 T_0)$, $\exp(i(\Omega_1 - 2\omega_2)T_0)$, $\exp(i(\Omega_2 - \omega_1)T_0)$ and $\exp(i(\Omega_1 - \Omega_2)T_0)$.

$$\begin{aligned}
2i\omega_1 D_1 A_1 = & -ic_{11} A_1 \omega_1 \exp(-i\omega_1 \tau_1) - ic_{12} A_2 \omega_2 \exp(i(\sigma_3 - \sigma_{33})T_1) \exp(-i\omega_2 \tau_1) \\
& -k_{n1} (3A_1^2 \bar{A}_1 + 6A_1 \Lambda_1^2 + 3\bar{A}_1^2 \Lambda_1 \exp(i\sigma_1 T_1)) \\
& -3k_{n2} \left(A_1^2 \bar{A}_2 \exp(-i(\sigma_3 - \sigma_{33})T_1) + \bar{A}_1^2 \Lambda_2 \exp(i\sigma_1 T_1) + \right. \\
& \left. 4A_2 \Lambda_1 \Lambda_2 + 2\bar{A}_1 \bar{A}_2 \Lambda_1 \exp(i(\sigma_1 - (\sigma_3 - \sigma_{33}))T_1) \right. \\
& \left. + 2A_1 \bar{A}_1 A_2 \exp(i\omega_2 T_0) + 2\Lambda_1^2 A_2 \exp(i\omega_2 T_0) \right) - g_{11} A_{1d} \exp(-i\omega_1 \tau_2) \\
& -3k_{n3} \left(A_2^2 \bar{A}_1 \exp(2i(\sigma_3 - \sigma_{33})T_1) + 4A_2 \Lambda_1 \Lambda_2 \exp(i(\sigma_3 - \sigma_{33})T_1) \right. \\
& \left. + \bar{A}_2^2 \Lambda_1 \exp(i(\sigma_1 - 2(\sigma_3 - \sigma_{33}))T_1) + 2\Lambda_2^2 A_1 \exp(i\omega_1 T_0) \right. \\
& \left. + 2\bar{A}_1 \bar{A}_2 \Lambda_2 \exp(i(\sigma_1 - (\sigma_3 - \sigma_{33}))T_1) + 2A_2 \bar{A}_2 A_1 \exp(i\omega_1 T_0) \right) - 3k_{n4} A_2^2 \bar{A}_2 \\
& -k_{n4} (6A_2 \Lambda_2^2 + 3\bar{A}_2^2 \Lambda_2 \exp(3i(\sigma_{33} - \sigma_3)T_1)) \exp(i(\sigma_3 - \sigma_{33})T_1) \\
& -g_{12} A_{2d} \exp(-i\omega_2 \tau_2) \exp(i(\sigma_3 - \sigma_{33})T_1) + 0.5 p_{11}^2 F_{12} \bar{A}_1 \exp(i\sigma_2 T_1) \\
& + 0.5 p_{11} p_{12} F_{12} \bar{A}_2 \exp(i(\sigma_2 - (\sigma_3 - \sigma_{33}))T_1) + 0.5 p_{11} k_1 Y \exp(i\sigma_3 T_1) \\
& + 0.5 p_{11}^2 F_{12} \Lambda_1 \exp(i(\sigma_1 - \sigma_2)T_1) + 0.5 p_{11} p_{12} F_{12} \Lambda_2 \exp(i(\sigma_1 - \sigma_2)T_1)
\end{aligned} \tag{5.18a}$$

$$\begin{aligned}
& 2i\omega_2 D_1 A_2 = ic_{21} A_1 \omega_1 \exp(-i(\sigma_3 - \sigma_{33})T_1) \exp(-i\omega_1 \tau_1) - ic_{22} A_2 \omega_2 \exp(-i\omega_2 \tau_1) \\
& -\tilde{k}_{n1} \exp(-i(\sigma_3 - \sigma_{33})T_1) (\Lambda_1^3 \exp(3i\sigma_1 T_1) + 3A_1^2 \bar{A}_1 + 6A_1 \Lambda_1^2) \\
& -3\tilde{k}_{n2} \exp(-i(\sigma_3 - \sigma_{33})T_1) (A_1^2 \bar{A}_2 \exp(-i(\sigma_3 - \sigma_{33})T_1) + \Lambda_1^2 \Lambda_2 \exp(3i\sigma_1 T_1)) \\
& -12\tilde{k}_{n2} A_1 \Lambda_1 \Lambda_2 \exp(-i(\sigma_3 - \sigma_{33})T_1) - 6\tilde{k}_{n2} A_2 \exp(i\omega_2 T_0) (A_1 \bar{A}_1 + \Lambda_1^2) \\
& -3\tilde{k}_{n3} \left(A_2^2 \bar{A}_1 \exp(i(\sigma_3 - \sigma_{33})T_1) + A_2^2 \Lambda_1 \exp(i(\sigma_1 - 3(\sigma_3 - \sigma_{33}))T_1) + 4A_2 \Lambda_1 \Lambda_2 \right. \\
& \quad \left. + 2\bar{A}_1 \bar{A}_2 \Lambda_2 \sin((2\gamma_2 + \gamma_1)/3) + 2A_2 \bar{A}_2 A_1 \exp(i\omega_1 T_0) + 2\Lambda_2^2 A_1 \exp(i\omega_1 T_0) \right) \\
& -\tilde{k}_{n4} (3A_2^2 \Lambda_2 \exp(i(\sigma_1 - 3(\sigma_3 - \sigma_{33}))T_1) + 3A_2^2 \bar{A}_2 + 6A_2 \Lambda_2^2) \\
& -g_{21} A_{1d} \exp(-i\omega_1 \tau_2) \exp(-i(\sigma_3 - \sigma_{33})T_1) - g_{22} A_{2d} \exp(-i\omega_2 \tau_2) \\
& +0.5p_{11} p_{12} F_{12} \bar{A}_1 \exp(i(\sigma_2 - (\sigma_3 - \sigma_{33}))T_1) + 0.5p_{12}^2 F_{12} \bar{A}_2 \exp(-2i(\sigma_3 - \sigma_{33})T_1) \\
& +0.5p_{11} k_1 Y \exp(i\sigma_{33} T_1) + 0.5p_{11} p_{12} F_{12} \Lambda_1 \exp(i((\sigma_1 - \sigma_2) - (\sigma_3 - \sigma_{33}))T_1) \\
& +0.5p_{12}^2 F_{12} \Lambda_2 \exp(i((\sigma_1 - \sigma_2) - (\sigma_3 - \sigma_{33}))T_1)
\end{aligned} \tag{5.18b}$$

It may be noted that here 1:1 internal resonance condition ($\Omega_3 = \omega_1 + \varepsilon\sigma_3 = \omega_2 + \varepsilon\sigma_{33}$), is considered to suppress response amplitude for the primary resonance condition of the case (vi). Following the same standard procedure of MMS, the following steady-state autonomous reduced equations are obtained by separating real and imaginary parts from the corresponding equations.

$$\begin{aligned}
\dot{a}_1 = & -0.5c_{11} a_1 \cos(\omega_1 \tau_1) - \frac{0.5}{\omega_1} c_{12} \omega_2 a_2 \cos((\gamma_1 - \gamma_2)/3 - \omega_2 \tau_1) - \frac{3k_{n1} a_1^2 \Lambda_1 \sin \gamma_1}{4\omega_1} \\
& - \frac{3k_{n2}}{\omega_1} \left(0.125a_1^2 a_2 \sin((\gamma_2 - \gamma_1)/3) + 0.25a_1^2 \Lambda_2 \sin \gamma_1 + \Lambda_1^2 a_2 \sin((\gamma_1 - \gamma_2)/3) \right) \\
& - \frac{3k_{n3}}{\omega_1} \left(0.125a_1 a_2^2 \sin(2(\gamma_1 - \gamma_2)/3) + 2a_2 \Lambda_1 \Lambda_2 \sin((\gamma_1 - \gamma_2)/3) \right. \\
& \quad \left. + 0.25a_2^2 \Lambda_1 \sin((2\gamma_2 + \gamma_1)/3) + 0.5a_1 a_2 \sin((\gamma_2 + 2\gamma_1)/3) \right) \\
& - \frac{k_{n4}}{\omega_1} \left(0.375a_2^3 \sin\left(\frac{(\gamma_1 - \gamma_2)}{3}\right) + 3a_2 \Lambda_2^2 \sin\left(\frac{(\gamma_1 - \gamma_2)}{3}\right) + 3a_2^2 \Lambda_2 \sin\left(\frac{(2\gamma_2 + \gamma_1)}{3}\right) \right) \\
& - \frac{a_{1d}}{2\omega_1} (\tilde{g}_2 \cos(\omega_1 \tau_2) - \tilde{g}_1 \sin(\omega_1 \tau_2)) - \frac{a_{2d}}{2\omega_1} \tilde{g}_3 \sin\left(\frac{\gamma_1 - \gamma_2}{3} - \omega_2 \tau_2\right) + \frac{p_{11}}{2\omega_1} k_1 Y \sin\left(\frac{\gamma_1}{3}\right) \\
& - \frac{0.5a_{2d}}{\omega_1} \tilde{g}_4 \cos\left(\frac{\gamma_1 - \gamma_2}{3} - \omega_2 \tau_2\right) + \frac{p_{11}^2}{4\omega_1} F_{12} a_1 \sin\left(\frac{2\gamma_1}{3}\right) + \frac{p_{11} p_{12}}{4\omega_1} F_{12} a_2 \sin\left(\frac{\gamma_1 + \gamma_2}{3}\right)
\end{aligned} \tag{5.19a}$$

$$\begin{aligned}
 a_1 \dot{\gamma}_1 = & \sigma_1 a_1 - 1.5 c_{11} a_1 \sin(\omega_1 \tau_1) + \frac{1.5}{\omega_1} c_{12} \omega_2 a_2 \sin\left(\frac{(\gamma_1 - \gamma_2)}{3} - \omega_2 \tau_1\right) \\
 & - \frac{9k_{n1}}{\omega_1} \left(\frac{a_1^3}{8} + a_1 \Lambda_1^2 + 0.25 a_1^2 \Lambda_1 \cos \gamma_1 \right) \\
 & - \frac{9k_{n2}}{\omega_1} \left(\frac{a_1^2 a_2}{8} \cos\left(\frac{(\gamma_2 - \gamma_1)}{3}\right) + 0.25 a_1^2 \Lambda_2 \cos \gamma_1 + 0.5 a_1 a_2 \cos((\gamma_2 + 2\gamma_1)/3) \right. \\
 & \left. + 2a_1 \Lambda_1 \Lambda_2 + \frac{a_1^2 a_2}{4} \cos((\gamma_1 - \gamma_2)/3) + \Lambda_1^2 a_2 \cos((\gamma_1 - \gamma_2)/3) \right) \\
 & - \frac{9k_{n3}}{\omega_1} \left(\frac{a_1 a_2^2}{8} \cos(2(\gamma_1 - \gamma_2)/3) + 2a_2 \Lambda_1 \Lambda_2 \cos((\gamma_1 - \gamma_2)/3) + \frac{a_2^2 a_1}{4} \right. \\
 & \left. + 0.25 a_2^2 \Lambda_1 \cos((2\gamma_2 + \gamma_1)/3) + 0.5 a_1 a_2 \Lambda_2 \cos((\gamma_2 + 2\gamma_1)/3) + \Lambda_2^2 a_1 \right) \\
 & - \frac{9k_{n4}}{\omega_1} \left(\frac{a_2^3}{8} \cos((\gamma_1 - \gamma_2)/3) + a_2 \Lambda_2^2 \cos((\gamma_1 - \gamma_2)/3) + a_2^2 \Lambda_2 \cos((2\gamma_2 + \gamma_1)/3) \right) \\
 & - \frac{1.5 a_{1d}}{\omega_1} (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)) - \frac{1.5 a_{2d}}{\omega_1} \tilde{g}_3 \cos\left(\frac{\gamma_1 - \gamma_2}{3} - \omega_2 \tau_2\right) \\
 & + \frac{1.5 a_{2d}}{\omega_1} \tilde{g}_4 \sin\left(\frac{\gamma_1 - \gamma_2}{3} - \omega_2 \tau_2\right) + \frac{3 p_{11}^2 F_{12}}{4 \omega_1} a_1 \cos\left(\frac{2\gamma_1}{3}\right) \\
 & + \frac{3 p_{11} p_{12}}{4 \omega_1} F_{12} a_2 \cos\left(\frac{\gamma_1 + \gamma_2}{3}\right) + \frac{3 p_{11}}{2 \omega_1} k_1 Y \cos\left(\frac{\gamma_1}{3}\right)
 \end{aligned} \tag{5.19b}$$

$$\begin{aligned}
 \dot{a}_2 = & -\frac{c_{21} \omega_1}{2 \omega_2} a_1 \cos\left(\frac{(\gamma_2 - \gamma_1)}{3} - \omega_1 \tau_1\right) - 0.5 c_{22} a_2 \cos(\omega_2 \tau_1) \\
 & - \frac{3 \tilde{k}_{n1}}{\omega_2} \left(\frac{a_1^3}{8} \sin((\gamma_2 - \gamma_1)/3) + a_1 \Lambda_1^2 \sin((\gamma_2 - \gamma_1)/3) + 0.25 a_1^2 \Lambda_1 \sin((\gamma_2 + \gamma_1)/3) \right) \\
 & - \frac{3 \tilde{k}_{n2}}{\omega_2} \left(\frac{0.125 a_1^2 a_2 \sin(2(\gamma_2 - \gamma_1)/3) + 0.25 a_1^2 \Lambda_2 \sin((\gamma_2 + 2\gamma_1)/3)}{+ 2a_1 \Lambda_1 \Lambda_2 \sin((\gamma_2 - \gamma_1)/3) + 0.5 a_1 a_2 \Lambda_1 \sin((2\gamma_2 + \gamma_1)/3)} \right) \\
 & - \frac{3 \tilde{k}_{n3}}{\omega_2} \left(\frac{0.125 a_2^2 a_1 \sin((\gamma_1 - \gamma_2)/3) + 0.5 a_1 a_2 \Lambda_2 \sin((2\gamma_2 + \gamma_1)/3)}{+ 0.25 a_2^2 a_1 \sin((\gamma_2 - \gamma_1)/3) + \Lambda_2^2 a_1 \sin((\gamma_2 - \gamma_1)/3) + 0.25 a_2^2 \Lambda_1 \sin(\gamma_2)} \right) \\
 & - \frac{3 \tilde{k}_{n4}}{4 \omega_2} a_2^2 \Lambda_2 \sin \gamma_2 - \frac{a_{1d}}{2 \omega_2} \left(\tilde{g}_5 \sin\left(\frac{\gamma_2 - \gamma_1}{3} - \omega_1 \tau_2\right) + \tilde{g}_6 \cos\left(\frac{\gamma_2 - \gamma_1}{3} - \omega_1 \tau_2\right) \right) \\
 & - \frac{a_{2d}}{2 \omega_2} (\tilde{g}_8 \cos(\omega_2 \tau_2) - \tilde{g}_7 \sin(\omega_2 \tau_2)) + \frac{p_{11} p_{12}}{4 \omega_2} F_{12} a_2 \sin\left(\frac{\gamma_1 + \gamma_2}{3}\right) \\
 & + \frac{p_{12}^2}{4 \omega_2} F_{12} a_2 \sin\left(\frac{2\gamma_2}{3}\right) + \frac{p_{12}}{2 \omega_2} k_1 Y \sin\left(\frac{\gamma_2}{3}\right)
 \end{aligned} \tag{5.19c}$$

$$\begin{aligned}
a_2 \dot{\gamma}_2 = & \sigma_{33} a_2 + \frac{1.5}{\omega_2} c_{21} \omega_1 a_1 \sin \left(\frac{(\gamma_2 - \gamma_1)}{3} - \omega_1 \tau_1 \right) - 1.5 c_{22} a_2 \sin(\omega_2 \tau_1) \\
& - \frac{3 \tilde{k}_{n1}}{\omega_2} \left(3 \left(\frac{a_1^3}{8} + a_1 \Lambda_1^2 \right) \cos((\gamma_2 - \gamma_1)/3) + 0.75 a_1^2 \Lambda_1 \cos((\gamma_2 + \gamma_1)/3) \right) \\
& - \frac{9 \tilde{k}_{n2}}{\omega_2} \left(\frac{a_1^2 a_2}{8} \cos(2(\gamma_2 - \gamma_1)/3) + 0.25 a_1^2 \Lambda_2 \cos((\gamma_2 + 2\gamma_1)/3) \right. \\
& \left. + 2 a_1 \Lambda_1 \Lambda_2 \cos((\gamma_2 - \gamma_1)/3) + 0.5 a_1 a_2 \Lambda_1 \cos((2\gamma_2 + \gamma_1)/3) + \frac{a_1^2 a_2}{4} + \Lambda_1^2 a_2 \right) \\
& - \frac{9 \tilde{k}_{n3}}{\omega_2} \left(\frac{a_2^2 a_1}{8} \cos((\gamma_1 - \gamma_2)/3) + 0.25 a_2^2 \Lambda_1 \cos(\gamma_2) + 0.5 a_1 a_2 \Lambda_2 \cos((2\gamma_2 + \gamma_1)/3) \right. \\
& \left. + \frac{a_2^2 a_1}{4} \cos((\gamma_2 - \gamma_1)/3) + \Lambda_2^2 a_1 \cos((\gamma_2 - \gamma_1)/3) \right) \\
& - \frac{3 \tilde{k}_{n4}}{\omega_2} \left(\frac{3 a_2^2}{4} \Lambda_2 \cos \gamma_2 + \frac{3 a_2^3}{8} + 3 a_2 \Lambda_2^2 \right) - \frac{1.5 a_{1d}}{\omega_2} \tilde{g}_5 \cos \left(\frac{\gamma_2 - \gamma_1}{3} - \omega_1 \tau_2 \right) + \\
& \frac{1.5 a_{1d}}{\omega_2} \tilde{g}_6 \sin \left(\frac{\gamma_2 - \gamma_1}{3} - \omega_1 \tau_2 \right) - \frac{1.5 a_{2d}}{\omega_2} (\tilde{g}_7 \cos(\omega_2 \tau_2) + \tilde{g}_8 \sin(\omega_2 \tau_2)) \\
& + \frac{3 p_{11} p_{12}}{4 \omega_2} F_{12} a_1 \cos \left(\frac{\gamma_1 + \gamma_2}{3} \right) + \frac{3 p_{12}^2}{4 \omega_2} F_{12} a_2 \cos \left(\frac{2 \gamma_2}{3} \right) + \frac{3 p_{12}}{2 \omega_2} k_1 Y \cos \left(\frac{\gamma_2}{3} \right) \quad (5.19d)
\end{aligned}$$

where

$$\gamma_1 = \sigma_1 T_1 - 3\beta_1, \quad \gamma_2 = \sigma_1 T_1 - 3(\sigma_3 - \sigma_{33}) - 3\beta_2, \quad 3\sigma_2 = 2\sigma_1, \quad 3\sigma_3 = \sigma_1.$$

By directly perturbing the reduced equations, one can study the stability of the non-trivial steady-state solution. But, as the reduced Eqs. (5.19a) to (5.19d) have the coupled terms $a_1 \dot{\gamma}_1$ and $a_2 \dot{\gamma}_2$, the perturbed equations will not contain the perturbations $\Delta \dot{\gamma}_1$ and $\Delta \dot{\gamma}_2$ for trivial solutions and hence the stability of the trivial state cannot be studied by directly perturbing Eqs. (5.19a) to (5.19d). To circumvent this difficulty, the normalization method is adopted by introducing the transformation $P_i = a_i \cos \frac{\gamma_i}{3}$ and $Q_i = a_i \sin \frac{\gamma_i}{3}$ for $i = 1, 2$. into Eqs. (5.19a) to (5.19d). Carrying out trigonometric manipulations, one arrives at the following normalized reduced equations or the Cartesian form of modulation equations:

$$\begin{aligned}
\dot{P}_1 = & -\frac{1}{3}\sigma_1 Q_1 - 0.5c_{11}P_1 \cos(\omega_1 \tau_1) + 0.5c_{11}Q_1 \sin(\omega_1 \tau_1) - \frac{c_{12}\omega_2}{2\omega_1}P_2 \cos(\omega_2 \tau_1) \\
& + \frac{c_{12}\omega_2}{2\omega_1}Q_2 \sin(\omega_2 \tau_1) - \frac{P_1}{2\omega_1}(\tilde{g}_2 \cos(\omega_1 \tau_2) - \tilde{g}_1 \sin(\omega_1 \tau_2)) \\
& + \frac{Q_1}{2\omega_1}(\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)) + \frac{\tilde{g}_3}{2\omega_1}(Q_2 \cos(\omega_2 \tau_2) + P_2 \sin(\omega_2 \tau_2)) \\
& - \frac{\tilde{g}_4}{2\omega_1}(P_2 \cos(\omega_2 \tau_2) - Q_2 \sin(\omega_2 \tau_2)) + \frac{P_{11}^2}{4\omega_1}F_{12}Q_1 + \frac{P_{11}P_{12}}{4\omega_1}F_{12}Q_2
\end{aligned} \tag{5.20a}$$

$$\begin{aligned}
\dot{Q}_1 = & \frac{1}{3}\sigma_1 P_1 - 0.5c_{11}Q_1 \cos(\omega_1 \tau_1) - 0.5c_{11}P_1 \sin(\omega_1 \tau_1) - \frac{0.5}{2\omega_1}c_{12}\omega_2 Q_2 \cos(\omega_2 \tau_1) \\
& - \frac{0.5}{2\omega_1}c_{12}\omega_2 P_2 \sin(\omega_2 \tau_1) - \frac{\tilde{g}_2}{2\omega_1}Q_1 \cos(\omega_1 \tau_2) - \frac{\tilde{g}_2}{2\omega_1}P_1 \sin(\omega_1 \tau_2) + \frac{\tilde{g}_1}{2\omega_1}Q_1 \sin(\omega_1 \tau_2) \\
& - \frac{\tilde{g}_1}{2\omega_1}P_1 \cos(\omega_1 \tau_2) - \frac{\tilde{g}_3}{2\omega_1}(P_2 \cos(\omega_2 \tau_2) - Q_2 \sin(\omega_2 \tau_2)) - \frac{\tilde{g}_4}{2\omega_1}Q_2 \cos(\omega_2 \tau_2) \\
& - \frac{\tilde{g}_4}{2\omega_1}P_2 \sin(\omega_2 \tau_2) + \frac{P_{11}^2}{4\omega_1}F_{12}P_1 + \frac{P_{11}P_{12}}{4\omega_1}F_{12}P_2 + \frac{P_{11}}{2\omega_1}k_1 Y
\end{aligned} \tag{5.20b}$$

$$\begin{aligned}
\dot{P}_2 = & -\frac{1}{3}\sigma_{33}Q_2 + 0.5c_{22}Q_2 \sin(\omega_2 \tau_1) - 0.5c_{22}P_2 \cos(\omega_2 \tau_1) \\
& - \frac{\omega_1}{2\omega_2}c_{21}(P_1 \cos(\omega_1 \tau_1) - Q_1 \sin(\omega_1 \tau_1)) + \frac{0.5}{\omega_2}\tilde{g}_5(Q_1 \cos(\omega_1 \tau_2) + P_1 \sin(\omega_1 \tau_2)) \\
& - \frac{\tilde{g}_6}{2\omega_2}(P_1 \cos(\omega_1 \tau_2) - Q_1 \sin(\omega_1 \tau_2)) - \frac{\tilde{g}_8 P_2}{2\omega_2} \cos(\omega_2 \tau_2) + \frac{\tilde{g}_8 Q_2}{2\omega_2} \sin(\omega_2 \tau_2) \\
& + \frac{\tilde{g}_7 P_2}{2\omega_2} \sin(\omega_2 \tau_2) + \frac{\tilde{g}_7 Q_2}{2\omega_2} \cos(\omega_2 \tau_2) + \frac{P_{11}P_{12}}{4\omega_2}F_{12}Q_2 + \frac{P_{12}^2}{4\omega_2}F_{12}Q_2
\end{aligned} \tag{5.20c}$$

$$\begin{aligned}
\dot{Q}_2 = & \frac{1}{3}\sigma_{33}P_2 - 0.5c_{22}Q_2 \cos(\omega_2 \tau_1) - 0.5c_{22}P_2 \sin(\omega_2 \tau_1) + \frac{P_{12}}{2\omega_2}k_1 Y \\
& - \frac{\omega_1}{2\omega_2}c_{21}(Q_1 \cos(\omega_1 \tau_1) + P_1 \sin(\omega_1 \tau_1)) - \frac{\tilde{g}_5}{\omega_2}(P_1 \cos(\omega_1 \tau_2) - Q_1 \sin(\omega_1 \tau_2)) \\
& - \frac{\tilde{g}_6}{\omega_2}(Q_1 \cos(\omega_1 \tau_2) + P_1 \sin(\omega_1 \tau_2)) - \frac{\tilde{g}_8}{2\omega_2} \cos(\omega_2 \tau_2)Q_2 - \frac{\tilde{g}_8}{2\omega_2} \sin(\omega_2 \tau_2)P_2 \\
& + \frac{\tilde{g}_7}{2\omega_2} \sin(\omega_2 \tau_2)Q_2 - \frac{\tilde{g}_7}{2\omega_2} \cos(\omega_2 \tau_2)P_2 + \frac{P_{11}P_{12}}{4\omega_2}F_{12}P_1 \cos\left(\frac{\gamma_1}{3}\right) + \frac{P_{12}^2}{4\omega_2}F_{12}P_2
\end{aligned} \tag{5.20d}$$

In the next sub-section, the resonance condition mentioned in case (vi) is extended for 1:2 internal resonance condition $(\omega_2 = 2\omega_1 + \varepsilon(\sigma_2 - \sigma_{22}))$.

1: 2 internal resonance condition

In this case, the natural frequency of the secondary system is considered to be two times that of the primary system, giving rise to the 1:2 internal resonance condition. In this sub-section, the secular terms are obtained for case (vi) i.e. for simultaneous primary, principal parametric and subharmonic resonances with 1:2 internal resonance conditions $(\omega_2 = 2\omega_1 + \varepsilon(\sigma_2 - \sigma_{22}))$ and the corresponding equations are given below.

$$\begin{aligned}
 2i\omega_1 D_1 A_1 = & -ic_{11} A_1 \omega_1 \exp(-i\omega_1 \tau_1) - k_{n1} (3A_1^2 \bar{A}_1 + 6A_1 \Lambda_1^2) - 3k_{n2} \bar{A}_1^2 \Lambda_2 \exp(i\sigma_1 T_1) \\
 & - 12k_{n2} A_1 \Lambda_1 \Lambda_2 - 3k_{n3} (A_2^2 \Lambda_1 \exp(i(2(\sigma_2 - \sigma_{22}) - \sigma_1) T_1) + 2A_2 \bar{A}_2 A_1 + 2\Lambda_2^2 A_1) \\
 & - k_{n4} A_2^2 \Lambda_2 \exp(i(2(\sigma_2 - \sigma_{22}) - \sigma_1) T_1) - g_{11} A_{1d} \exp(-i\omega_1 \tau_2) \\
 & + 0.5 p_{11}^2 F_{12} \bar{A}_1 \exp(i\sigma_2 T_1) + 0.5 p_{11}^2 F_{12} \Lambda_1 \exp(i(\sigma_1 - \sigma_2) T_1) + 0.5 p_{11} k_1 Y \exp(i\sigma_3 T_1)
 \end{aligned}
 \tag{5.21a}$$

$$\begin{aligned}
 2i\omega_2 D_1 A_2 = & -ic_{22} A_2 \omega_2 \exp(-i\omega_2 \tau_1) - 3\tilde{k}_{n2} (2A_1 \Lambda_1 \bar{A}_2 \exp(i(\sigma_1 - 2(\sigma_2 - \sigma_{22})) T_1)) \\
 & - 6\tilde{k}_{n2} A_2 (A_1 \bar{A}_1 + \Lambda_1^2) - 12\tilde{k}_{n3} A_2 \Lambda_1 \Lambda_2 - 3\tilde{k}_{n4} A_2 (A_2 \bar{A}_2 + 2\Lambda_2^2) - g_{22} A_{2d} \exp(-i\omega_2 \tau_2)
 \end{aligned}
 \tag{5.21b}$$

It may be noted that here 1:2 internal resonance condition $(\Omega_2 = 2\omega_1 + \varepsilon\sigma_2 = \omega_2 + \varepsilon\sigma_{22})$, is considered to suppress response amplitude for the principal parametric resonance condition of the case (vi). The secular terms obtained from the expansion of Eq. (5.7a) are from the resonance conditions of $\exp(i\omega_1 T_0)$, $\exp(i(\Omega_1 - 2\omega_1) T_0)$, $\exp(i(2\omega_2 - \Omega_1) T_0)$, $\exp(i(\Omega_1 - \Omega_2) T_0)$, $\exp(i(\Omega_2 - \omega_1) T_0)$, $\exp(i\Omega_3 T_0)$. Similarly, the secular terms obtained from the expansion of Eq. (5.7b) are from the resonance conditions of $\exp(i\omega_2 T_0)$ and $\exp(i(\omega_1 - \omega_2 + \Omega_1) T_0)$. Following the standard procedure of MMS, the complex amplitude functions A_1 and A_2 are written in polar form and substituted in Eq. (5.21a) and Eq. (5.21b). Then from the corresponding equations, the real and imaginary parts are separated from which one may obtain the following reduced steady-state autonomous equations.

$$\begin{aligned} \dot{a}_1 = & -0.5c_{11}a_1 \cos(\omega_1\tau_1) - \frac{3k_{n2}a_1^2\Lambda_2}{4\omega_1} \sin\gamma_1 - \frac{3k_{n3}a_2^2}{4\omega_1} \Lambda_1 \sin\left(\frac{\gamma_1}{3} - 2\gamma_2\right) \\ & - k_{n4} \frac{a_2^2\Lambda_2}{4\omega_1} \sin\left(\frac{\gamma_1}{3} - 2\gamma_2\right) - 0.5a_{1d}(\tilde{g}_2 \cos(\omega_1\tau_2) - \tilde{g}_1 \sin(\omega_1\tau_2)) \\ & + 0.25 \frac{p_{11}^2}{\omega_1} F_{12}a_1 \sin\frac{2\gamma_1}{3} + 0.5 \frac{p_{11}^2}{\omega_1} F_{12}\Lambda_1 \sin\frac{\gamma_1}{3} + 0.5 \frac{p_{11}}{\omega_1} k_1 Y \sin\frac{\gamma_1}{3} \end{aligned} \quad (5.22a)$$

$$\begin{aligned} a_1\dot{\gamma}_1 = & a_1\sigma_1 - 1.5c_{11}a_1 \sin(\omega_1\tau_1) - 9 \frac{k_{n1}}{\omega_1} \left(\frac{a_1^3}{8} + a_1\Lambda_1^2 \right) - 9 \frac{k_{n2}}{\omega_1} \left(\frac{a_1^2\Lambda_2}{4} \cos\gamma_1 + 2a_1\Lambda_1\Lambda_2 \right) \\ & - 9 \frac{k_{n3}}{\omega_1} \left(\frac{a_2^2}{4} \Lambda_1 \cos\left(\frac{\gamma_1}{3} - 2\gamma_2\right) + \frac{a_2^2a_1}{4} + \Lambda_2^2a_1 \right) - \frac{3k_{n4}}{\omega_1} \frac{a_2^2\Lambda_2}{4} \cos\left(\frac{\gamma_1}{3} - 2\gamma_2\right) \end{aligned} \quad (5.22b)$$

$$\begin{aligned} & - 1.5 \frac{a_{1d}}{\omega_1} (\tilde{g}_1 \cos(\omega_1\tau_2) + \tilde{g}_2 \sin(\omega_1\tau_2)) \\ & + 0.75 \frac{p_{11}^2}{\omega_1} F_{12}a_1 \cos\frac{2\gamma_1}{3} + 1.5 \frac{p_{11}^2}{\omega_1} F_{12}\Lambda_1 \cos\frac{\gamma_1}{3} + 1.5 p_{11} \frac{k_1}{\omega_1} Y \cos\frac{\gamma_1}{3} \\ \dot{a}_2 = & -0.5c_{22}a_2 \cos(\omega_2\tau_1) - 3 \frac{\tilde{k}_{n2}}{\omega_2} a_1a_2\Lambda_1 \sin\left(2\gamma_2 - \frac{\gamma_1}{3}\right) \end{aligned} \quad (5.22c)$$

$$\begin{aligned} & - \frac{a_{2d}}{2\omega_2} (\tilde{g}_8 \cos(\omega_2\tau_2) - \tilde{g}_7 \sin(\omega_2\tau_2)) \\ a_2\dot{\gamma}_2 = & a_2\sigma_2 - 0.5c_{22}a_2 \sin(\omega_2\tau_1) - 3 \frac{\tilde{k}_{n2}}{\omega_2} \left(a_1a_2\Lambda_1 \cos\left(2\gamma_2 - \frac{\gamma_1}{3}\right) + \frac{a_1^2a_2}{4} + a_2\Lambda_1^2 \right) \\ & - 6 \frac{\tilde{k}_{n3}}{\omega_2} a_2\Lambda_1\Lambda_2 - \frac{\tilde{k}_{n4}}{\omega_2} \left(\frac{3a_2^3}{8} + 3a_2\Lambda_2^2 \right) - 0.5 \frac{a_{2d}}{\omega_2} (\tilde{g}_7 \cos(\omega_2\tau_2) + \tilde{g}_8 \sin(\omega_2\tau_2)) \end{aligned} \quad (5.22d)$$

where

$$\gamma_1 = \sigma_1 T_1 - 3\beta_1, \quad \gamma_2 = \sigma_{22} T_1 - \beta_2, \quad 3\sigma_2 = 2\sigma_1, \quad 3\sigma_3 = \sigma_1.$$

The steady-state solutions of the above-reduced equations (Eq. (5.22a) to Eq. (5.22d)) are solved by considering $\dot{a}_1 = \dot{\gamma}_1 = \dot{a}_2 = \dot{\gamma}_2 = 0$. From this condition and Eqs. (5.22a) to (5.22d) one can observe there are two solutions for a_2 , i.e, either $a_2 = 0$ or $a_2 \neq 0$. Considering $a_2 = 0$, the following equations are obtained for the steady-state response of the system.

$$W_1 a_1^2 \sin\gamma_1 + W_2 a_1 - W_3 a_1 \sin(2\gamma_1/3) - W_4 \sin(\gamma_1/3) = 0 \quad (5.23a)$$

$$W_5 a_1^2 + W_6 a_1 \cos(2\gamma_2 - \gamma_1/3) + W_7 = 0 \quad (5.23b)$$

$$W_8 a_1^3 - W_9 a_1^2 \cos \gamma_1 + W_{10} a_1 \cos(2\gamma_1/3) + W_{11} a_1 + W_{12} \cos(\gamma_1/3) = 0 \quad (5.23c)$$

where

$$\begin{aligned} W_1 &= 0.75 k_{n2} \Lambda_2 / \omega_1, \quad W_2 = 0.5 c_{11} \cos(\omega_1 \tau_1) + 0.5 (\tilde{g}_2 \cos(\omega_1 \tau_2) - \tilde{g}_1 \sin(\omega_1 \tau_2)), \\ W_3 &= 0.25 p_{11}^2 F_{12} / \omega_1, \quad W_4 = 0.5 (p_{11}^2 F_{12} \Lambda_1 + p_{11} k_1 Y) / \omega_1, \quad W_5 = 0.75 \tilde{k}_{n2} / \omega_2, \quad W_6 = 3 \tilde{k}_{n2} \Lambda_1 / \omega_2, \\ W_7 &= \frac{3 \tilde{k}_{n2} \Lambda_1^2}{\omega_2} - \sigma_2 + 0.5 c_{22} \sin(\omega_2 \tau_1) + \frac{6 \tilde{k}_{n3} \Lambda_1 \Lambda_2}{\omega_2} + \frac{3 \tilde{k}_{n4} \Lambda_2^2}{\omega_2} + \frac{0.5}{\omega_2} (\tilde{g}_7 \cos(\omega_2 \tau_2) + \tilde{g}_8 \sin(\omega_2 \tau_2)) \\ W_8 &= -9 k_{n1}, \quad W_9 = 18 k_{n2} \Lambda_2, \quad W_{10} = 6 p_{11}^2 F_{12}, \\ W_{11} &= 8 \omega_1 \sigma_1 - 72 k_{n1} \Lambda_1^2 - 12 c_{11} \omega_1 \sin(\omega_1 \tau_1) - 144 k_{n2} \Lambda_1 \Lambda_2 - 72 k_{n3} \Lambda_2^2 \\ &\quad - 12 (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)), \\ W_{12} &= 12 p_{11}^2 F_{12} \Lambda_1 + 12 k_1 p_{11} Y \end{aligned}$$

Similarly, considering $a_2 \neq 0$, from the Eqs. (5.22a) to (5.22d) the following steady state autonomous equations are obtained.

$$6 \tilde{k}_{n2} \Lambda_1 a_1 \sin\left(2\gamma_2 - \frac{\gamma_1}{3}\right) + c_{22} \omega_2 \cos(\omega_2 \tau_1) + \tilde{g}_8 \cos(\omega_2 \tau_2) - \tilde{g}_7 \sin(\omega_2 \tau_2) = 0 \quad (5.24a)$$

$$\begin{aligned} W_{11} a_1^3 + W_{12} a_1^2 \cos \gamma_1 - W_{13} a_1 - W_{14} a_1 \cos \frac{2\gamma_1}{3} + W_{15} a_2^2 \cos\left(\frac{\gamma_1}{3} - 2\gamma_2\right) \\ + W_{16} a_2^2 a_1 - W_{17} \cos \frac{\gamma_1}{3} = 0 \end{aligned} \quad (5.24b)$$

$$W_{18} a_1^2 \sin \gamma_1 + W_{19} a_1 - W_{20} a_1 \sin \frac{2\gamma_1}{3} + W_{21} a_2^2 \sin\left(\frac{\gamma_1}{3} - 2\gamma_2\right) - W_{22} \sin \frac{\gamma_1}{3} = 0 \quad (5.24c)$$

$$W_{23} a_2^2 + W_{24} a_1^2 + W_{25} a_1 \cos\left(2\gamma_2 - \frac{\gamma_1}{3}\right) - W_{26} = 0 \quad (5.24d)$$

where

$$\begin{aligned} W_{11} &= 9 k_{n1}, \quad W_{12} = 18 k_{n2} \Lambda_2, \\ W_{13} &= 8 \omega_1 \sigma_1 - 12 c_{11} \omega_1 \sin(\omega_1 \tau_1) - 72 k_{n1} \Lambda_1^2 - 144 k_{n2} \Lambda_1 \Lambda_2 - 72 k_{n3} \Lambda_2^2 \\ &\quad - 12 (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)), \end{aligned}$$

$$\begin{aligned}
 W_{14} &= 6p_{11}^2 F_{12}, \quad W_{15} = 6k_{n4}\Lambda_2 - 18k_{n3}\Lambda_1, \quad W_{16} = 18k_{n3}, \quad W_{17} = 12p_{11}^2 F_{12}\Lambda_1 + 12p_{11}k_1Y, \\
 W_{18} &= \frac{3k_{n2}\Lambda_2}{4\omega_1}, \quad W_{19} = 0.5c_{11} \cos(\omega_1\tau_1) + 0.5(\tilde{g}_2 \cos(\omega_1\tau_2) - \tilde{g}_1 \sin(\omega_1\tau_2)), \\
 W_{20} &= \frac{p_{11}^2 F_{12}}{4\omega_1}, \quad W_{21} = \frac{3k_{n3}\Lambda_1 + k_{n4}\Lambda_2}{4\omega_1}, \quad W_{22} = \frac{p_{11}^2 F_{12}\Lambda_1 + p_{11}k_1Y}{2\omega_1}, \\
 W_{23} &= \frac{3k_{n4}}{8\omega_2}, \quad W_{24} = \frac{3\tilde{k}_{n2}}{4\omega_2}, \quad W_{25} = \frac{3\tilde{k}_{n2}\Lambda_1}{\omega_2}, \\
 W_{26} &= \sigma_2 - \frac{3\tilde{k}_{n2}\Lambda_1^2}{\omega_2} - \frac{c_{22} \sin(\omega_2\tau_1)}{2\omega_2} - \frac{6\tilde{k}_{n3}\Lambda_1\Lambda_2}{\omega_2} - \frac{3\tilde{k}_{n4}\Lambda_2^2}{\omega_2} - \frac{0.5}{\omega_2}(\tilde{g}_7 \cos(\omega_2\tau_2) + \tilde{g}_8 \sin(\omega_2\tau_2))
 \end{aligned}$$

In the next sub-section, the resonance condition mentioned in case (vi) is extended for 1:3 internal resonance condition $(\omega_2 = 3\omega_1 + \varepsilon(\sigma_1 - \sigma_{11}))$ to obtain the reduced equation for studying responses of the system.

1: 3 internal resonance conditions for sub, principal & primary resonances

In this section, considering ω_2 nearly equal to 3 times ω_1 , one may obtain 1:3 internal resonance conditions. Here the secular terms are obtained from the expansion of Eq. (5.7a) from the resonance conditions of $\exp(i\omega_1 T_0)$, $\exp(i(\omega_2 - 2\omega_1)T_0)$, $\exp(i(\Omega_1 - 2\omega_1)T_0)$, $\exp(i(\omega_1 + \omega_2 - \Omega_1)T_0)$, $\exp(i(\Omega_1 - \omega_2 + \omega_1)T_0)$, $\exp(i(\Omega_1 - \Omega_2)T_0)$, $\exp(i(\Omega_2 - \omega_1)T_0)$, $\exp(i(\omega_2 - \Omega_2)T_0)$, $\exp(i\Omega_3 T_0)$. Similarly, the secular terms obtained from the expansion of Eq. (5.7b) are from the resonance conditions of $\exp(i\omega_2 T_0)$, $\exp(i(2\Omega_1 - \omega_2)T_0)$, $\exp(i\Omega_1 T_0)$, $\exp(i(2\omega_2 - \Omega_1)T_0)$ and $\exp(i(\Omega_2 + \omega_1)T_0)$. From these conditions, the secular terms obtained for the simultaneous primary, principal parametric and subharmonic resonances with 1:3 internal resonance conditions $(\omega_2 = 3\omega_1 + \varepsilon(\sigma_1 - \sigma_{11}))$ are given as follows.

$$\begin{aligned}
 2i\omega_1 D_1 A_1 &= -ic_{11}A_1\omega_1 \exp(-i\omega_1\tau_1) - k_{n1}(3A_1^2\bar{A}_1 + 6A_1\Lambda_1^2 + 3\bar{A}_1^2\Lambda_1 \exp(i\sigma_1 T_1)) \\
 &- 3k_{n2}(\bar{A}_1^2 A_2 \exp(i(\sigma_1 - \sigma_{11})T_1) + \bar{A}_1^2 \Lambda_2 \exp(i\sigma_1 T_1) + 4A_1\Lambda_1\Lambda_2 + 2A_1A_2\Lambda_1 \exp(-i\sigma_{11}T_1)) \\
 &- 6k_{n3}(\bar{A}_2 A_1\Lambda_2 \exp(i\sigma_{11}T_1) + A_2\Lambda_2 A_1 \exp(-i\sigma_{11}T_1) + A_2\bar{A}_2 A_1\Lambda_2 + \Lambda_2^2 A_1) \\
 &- g_{11}A_{1d} \exp(-i\omega_1\tau_2) + 0.5p_{11}^2 F_{12}\bar{A}_1 \exp(i\sigma_2 T_1) \\
 &+ 0.5p_{11}p_{12}F_{12}A_2 \exp(i(\sigma_1 - \sigma_{11} - \sigma_2)T_1) + 0.5p_{11}k_1Y \exp(i\sigma_3 T_1) +
 \end{aligned}$$

$$0.5 p_{11}^2 F_{12} \Lambda_1 \exp(i(\sigma_1 - \sigma_2) T_1) + 0.5 p_{11} p_{12} F_{12} \Lambda_2 \exp(i(\sigma_1 - \sigma_2) T_1) \quad (5.25a)$$

$$\begin{aligned} 2i\omega_2 D_1 A_2 = & -ic_{22} (A_2 \omega_2 \exp(-i\omega_2 \tau_1) + \Lambda_2 \Omega_1 \exp(i(\sigma_{11} T_1 - \Omega_1 \tau_1))) \\ & -ic_{21} \Lambda_1 \Omega_1 \exp(i(\sigma_{11} T_1 - \Omega_1 \tau_1)) - \tilde{k}_{n1} \left(A_1^3 \exp(i\omega_2 T_0 - i(\sigma_{1s} - \sigma_{11s}) T_1) \right. \\ & \left. + 3\Lambda_1^3 \exp(i\sigma_{11} T_1) + 6A_1 \bar{A}_1 \Lambda_1 \exp(i\sigma_{11s} T_1) \right) \\ & -3\tilde{k}_{n2} \left(\Lambda_1^2 \bar{A}_2 \exp(2i\sigma_{11} T_1) + \Lambda_1^2 \Lambda_2 \exp(i\sigma_{11} T_1) \right. \\ & \left. + 2A_1 \bar{A}_1 A_2 + 2\Lambda_1^2 A_2 + 2(A_1 \bar{A}_1 \Lambda_2 + \Lambda_1^2 \Lambda_2) \exp(i\sigma_{11} T_1) \right) \\ & -3\tilde{k}_{n3} \left(A_2^2 \Lambda_1 \exp(-i\sigma_{11} T_1) + \Lambda_2^2 \Lambda_1 \exp(i\sigma_{11} T_1) + 4A_2 \Lambda_1 \Lambda_2 \right. \\ & \left. + 2(A_2 \bar{A}_2 \Lambda_1 \exp(i\Omega_1 T_0) + \Lambda_2^2 \Lambda_1) \exp(i\sigma_{11} T_1) \right) \\ & -\tilde{k}_{n4} \left(3\Lambda_2^3 \exp(i\sigma_{11s} T_1) + 3A_2^2 \bar{A}_2 + 6A_2 \Lambda_2^2 + 6A_2 \bar{A}_2 \Lambda_2 \exp(i\sigma_{11s} T_1) \right. \\ & \left. + 3\Lambda_2^2 \bar{A}_2 \exp(2i\sigma_{11s} T_1) + 3A_2^2 \Lambda_2 \exp(-i\sigma_{11s} T_1) \right) \\ & -g_{22} A_{2d} \exp(-i\omega_2 \tau_2) - (g_{21} \Lambda_1 + g_{22} \Lambda_2) \exp(i\sigma_{11} T_1 - \Omega_1 \tau_2) \\ & + 0.5 p_{12} p_{11} F_{12} A_1 \exp(i(\sigma_2 - (\sigma_1 - \sigma_{11}) T_1)) \end{aligned} \quad (5.25b)$$

It may be noted that here 1:3 internal resonance condition is considered to suppress response amplitude for the subharmonic resonance condition of the case (vi). Following the standard MMS procedure, the steady-state autonomous reduced equations are given below.

$$\begin{aligned} \dot{a}_1 = & -0.5c_{11} a_1 \cos(\omega_1 \tau_1) - \frac{k_{n1}}{\omega_1} (0.75a_1^2 \Lambda_1 \sin \gamma_1) - \frac{3k_{n2}}{8\omega_1} a_1^2 a_2 \sin(\gamma_1 - \gamma_2) \\ & - \frac{3k_{n2}}{\omega_1} (0.25a_1^2 \Lambda_2 \sin \gamma_1 - 0.5a_1 a_2 \Lambda_1 \sin \gamma_2) - \frac{0.5a_{1d}}{\omega_1} (\tilde{g}_2 \cos(\omega_1 \tau_2) - \tilde{g}_1 \sin(\omega_1 \tau_2)) \\ & + 0.25 \frac{p_{11}^2}{\omega_1} F_{12} a_1 \sin \frac{2\gamma_1}{3} + 0.5 \frac{p_{11}}{\omega_1} k_1 Y \sin \frac{\gamma_1}{3} + 0.25 \frac{p_{11}}{\omega_1} p_{12} F_{12} a_2 \sin \left(\frac{\gamma_1}{3} - \gamma_2 \right) \\ & + 0.5 \frac{p_{11}^2}{\omega_1} F_{12} \Lambda_1 \sin \frac{\gamma_1}{3} + 0.5 \frac{p_{11}}{\omega_1} p_{12} F_{12} \Lambda_2 \sin \frac{\gamma_1}{3} \end{aligned} \quad (5.26a)$$

$$\begin{aligned} a_1 \dot{\gamma}_1 = & a_1 \sigma_1 - 1.5c_{11} a_1 \sin(\omega_1 \tau_1) - \left(3k_{n1} (0.375a_1^3 + 3a_1 \Lambda_1^2 + 0.75a_1^2 \Lambda_1 \cos \gamma_1) \right) / \omega_1 \\ & - \left(9k_{n2} (0.125a_1^2 a_2 \cos(\gamma_1 - \gamma_2) + 0.25a_1^2 \Lambda_2 \cos \gamma_1 + 2a_1 \Lambda_1 \Lambda_2 + 0.5a_1 a_2 \Lambda_1 \cos \gamma_2) \right) / \omega_1 \\ & - \frac{9k_{n3}}{\omega_1} \left(0.5a_2 a_1 \Lambda_2 \cos \gamma_2 + 0.5a_1 a_2 \Lambda_2 \cos \gamma_2 + \frac{a_2^2 a_1}{4} \Lambda_2 + \Lambda_2^2 a_1 \right) - 1.5 \frac{a_{1d}}{\omega_1} \tilde{g}_1 \cos(\omega_1 \tau_2) \\ & - 1.5 \frac{a_{1d}}{\omega_1} \tilde{g}_2 \sin(\omega_1 \tau_2) + 0.75 \frac{p_{11}^2}{\omega_1} F_{12} a_1 \cos \frac{2\gamma_1}{3} + 0.75 \frac{p_{11}}{\omega_1} p_{12} F_{12} a_2 \cos \left(\frac{\gamma_1}{3} - \gamma_2 \right) + \end{aligned}$$

$$1.5p_{11}\frac{k_1}{\omega_1}Y\cos\frac{\gamma_1}{3}+1.5\frac{p_{11}^2}{\omega_1}F_{12}\Lambda_1\cos\frac{\gamma_1}{3}+1.5\frac{p_{11}}{\omega_1}p_{12}F_{12}\Lambda_2\cos\frac{\gamma_1}{3} \quad (5.26b)$$

$$\begin{aligned} \dot{a}_2 = & -c_{22}\left(0.5a_2\cos(\omega_2\tau_1)+\Lambda_2\frac{\Omega_1}{\omega_2}\cos(\gamma_2-\Omega_1\tau_1)\right)-c_{21}\Lambda_1\frac{\Omega_1}{\omega_2}\cos(\gamma_2-\Omega_1\tau_1) \\ & -\frac{\tilde{k}_{n1}}{\omega_2}\left(\frac{a_1^3}{8}\sin(\gamma_2-\gamma_1)+3\Lambda_1^3\sin\gamma_2+1.5a_1^2\Lambda_1\sin\gamma_2\right) \\ & -\frac{3\tilde{k}_{n2}}{\omega_2}\left(0.5\Lambda_1^2a_2\sin(2\gamma_2)+\left(\Lambda_1^2+\frac{a_1^2}{2}+2\Lambda_1^2\right)\Lambda_2\sin\gamma_2\right) \\ & -\frac{3\tilde{k}_{n3}}{\omega_2}\left(-0.25a_2^2+\Lambda_2^2+0.5a_2^2+2\Lambda_2^2\right)\Lambda_1\sin\gamma_2 \\ & -\frac{\tilde{k}_{n4}}{\omega_2}\left(3\Lambda_2^3\sin\gamma_2+1.5a_2^2\Lambda_2\sin\gamma_2+1.5\Lambda_2^2a_2\sin(2\gamma_2)-0.75a_2^2\Lambda_2\sin\gamma_2\right) \\ & -0.5\frac{a_{2d}}{\omega_2}\left(\tilde{g}_8\cos(\omega_2\tau_2)-\tilde{g}_7\sin(\omega_2\tau_2)\right)+0.25\frac{p_{11}}{\omega_2}p_{12}F_{12}a_1\sin\left(\gamma_2-\frac{\gamma_1}{3}\right) \\ & -\left(\frac{(\tilde{g}_5\Lambda_1+\tilde{g}_7\Lambda_2)}{\omega_2}\sin(\gamma_2-\Omega_1\tau_2)+\frac{(\tilde{g}_6\Lambda_1+\tilde{g}_8\Lambda_2)}{\omega_2}\cos(\gamma_2-\Omega_1\tau_2)\right) \end{aligned} \quad (5.26c)$$

$$\begin{aligned} a_2\dot{\gamma}_2 = & \sigma_{11}a_2-c_{22}\left(\frac{a_2}{2}\sin(\omega_2\tau_1)-\frac{\Lambda_2\Omega_1}{\omega_2}\sin(\gamma_2-\Omega_1\tau_1)\right)+\frac{c_{21}\Lambda_1\Omega_1}{\omega_2}\sin(\gamma_2-\Omega_1\tau_1) \\ & -\frac{\tilde{k}_{n1}}{\omega_2}\left(0.125a_1^3\cos(\gamma_2-\gamma_1)+\left(3\Lambda_1^3+1.5a_1^2\Lambda_1\right)\cos\gamma_2\right) \\ & -\frac{3\tilde{k}_{n2}}{\omega_2}\left(0.5\Lambda_1^2a_2\cos2\gamma_2+0.25a_1^2a_2+\Lambda_1^2a_2+\left(\Lambda_1^2+0.5a_1^2+2\Lambda_1^2\right)\Lambda_2\cos\gamma_2\right) \\ & -\frac{3\tilde{k}_{n3}}{\omega_2}\left(\left(0.25a_2^2+\Lambda_2^2+0.5a_2^2+2\Lambda_2^2\right)\Lambda_1\cos\gamma_2+2a_2\Lambda_1\Lambda_2\right) \\ & -\frac{\tilde{k}_{n4}}{\omega_2}\left(0.375a_2^3+3a_2\Lambda_2^2+1.5\Lambda_2^2a_2\cos2\gamma_2+\left(3\Lambda_2^2+1.5a_2^2+0.75a_2^2\right)\Lambda_2\cos\gamma_2\right) \\ & -0.5\frac{a_{2d}}{\omega_2}\left(\tilde{g}_7\cos(\omega_2\tau_2)+\tilde{g}_8\sin(\omega_2\tau_2)\right)+0.25\frac{p_{11}}{\omega_2}p_{12}F_{12}a_1\cos\left(\gamma_2-\frac{\gamma_1}{3}\right) \\ & -\left(\frac{(\tilde{g}_5\Lambda_1+\tilde{g}_7\Lambda_2)}{\omega_2}\cos(\gamma_2-\Omega_1\tau_2)-\frac{(\tilde{g}_6\Lambda_1+\tilde{g}_8\Lambda_2)}{\omega_2}\sin(\gamma_2-\Omega_1\tau_2)\right) \end{aligned} \quad (5.26d)$$

where

$$\gamma_1 = \sigma_1 T_1 - 3\beta_1, \quad \gamma_2 = \sigma_{11} T_1 - \beta_2, \quad 3\sigma_2 = 2\sigma_1, \quad 3\sigma_3 = \sigma_1.$$

Similarly, as discussed in sub-sections 5.3.1.1 the trivial state equations of the system are expressed as $P_1 = a_1 \cos(\gamma_1/3)$, $Q_1 = a_1 \sin(\gamma_1/3)$, $P_2 = a_2 \cos \gamma_2$ and $Q_2 = a_2 \sin \gamma_2$.

$$\begin{aligned} \dot{P}_1 = & -\frac{\sigma_1}{3} Q_1 - 0.5c_{11} P_1 \cos(\omega_1 \tau_1) + 0.5c_{11} Q_1 \sin(\omega_1 \tau_1) - \frac{P_1}{2\omega_1} \tilde{g}_2 \cos(\omega_1 \tau_2) \\ & + \frac{P_1}{2\omega_1} \tilde{g}_1 \sin(\omega_1 \tau_2) + \frac{Q_1}{2\omega_1} (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)) + \frac{P_{11}^2}{4\omega_1} F_{12} Q_1 - \frac{P_{11}}{4\omega_1} p_{12} F_{12} Q_2 \end{aligned} \quad (5.27a)$$

$$\begin{aligned} \dot{Q}_1 = & \frac{\sigma_1}{3} Q_1 - 0.5c_{11} P_1 \sin(\omega_1 \tau_1) - 0.5c_{11} Q_1 \cos(\omega_1 \tau_1) \\ & - \frac{Q_1}{2\omega_1} (\tilde{g}_2 \cos(\omega_1 \tau_2) - \tilde{g}_1 \sin(\omega_1 \tau_2)) - \frac{P_1}{2\omega_1} (\tilde{g}_1 \cos(\omega_1 \tau_2) + \tilde{g}_2 \sin(\omega_1 \tau_2)) \\ & + 0.25 \frac{P_{11}^2}{\omega_1} F_{12} P_1 + 0.25 \frac{P_{11}}{\omega_1} p_{12} F_{12} P_2 + 0.5 \frac{P_{11}}{\omega_1} k_1 Y + 0.5 \frac{P_{11}^2}{\omega_1} F_{12} \Lambda_1 + 0.5 \frac{P_{11}}{\omega_1} p_{12} F_{12} \Lambda_2 \end{aligned} \quad (5.27b)$$

$$\begin{aligned} \dot{P}_2 = & -\sigma_{11} Q_2 - c_{22} \frac{P_2}{2} \cos(\omega_2 \tau_1) + c_{22} \frac{Q_2}{2} \sin(\omega_2 \tau_1) - \frac{\Lambda_2 \Omega_1 c_{22}}{\omega_2} \cos(\Omega_1 \tau_1) - \\ & - \frac{\Lambda_1 \Omega_1 c_{21}}{\omega_2} \cos(\Omega_1 \tau_1) - \frac{P_2}{2\omega_2} (\tilde{g}_8 \cos(\omega_2 \tau_2) - \tilde{g}_7 \sin(\omega_2 \tau_2)) + \frac{Q_2 \tilde{g}_7}{2\omega_2} \cos(\omega_2 \tau_2) + \\ & \frac{Q_2 \tilde{g}_8 \sin \omega_2 \tau_2}{2\omega_2} + \frac{p_{11} p_{12} F_{12} Q_1}{4\omega_2} + \frac{(\tilde{g}_5 \Lambda_1 + \tilde{g}_7 \Lambda_2) \sin(\Omega_1 \tau_2)}{\omega_2} - \frac{(\tilde{g}_6 \Lambda_1 + \tilde{g}_8 \Lambda_2) \cos(\Omega_1 \tau_2)}{\omega_2} \end{aligned} \quad (5.27c)$$

$$\begin{aligned} \dot{Q}_2 = & \sigma_{11} P_2 - 0.5c_{22} Q_2 \cos(\omega_2 \tau_1) - 0.5c_{22} P_2 \sin(\omega_2 \tau_1) - \frac{\Omega_1}{\omega_2} \sin(\Omega_1 \tau_1) (\Lambda_2 c_{22} + \Lambda_1 c_{21}) \\ & - \frac{Q_2}{2\omega_2} (\tilde{g}_8 \cos(\omega_2 \tau_2) - \tilde{g}_7 \sin(\omega_2 \tau_2)) - \frac{P_2}{2\omega_2} (\tilde{g}_7 \cos(\omega_2 \tau_2) - \tilde{g}_8 \sin(\omega_2 \tau_2)) \\ & + \frac{p_{11} p_{12}}{4\omega_2} F_{12} a_1 \cos\left(\frac{\gamma_1}{3}\right) - \frac{(\tilde{g}_5 \Lambda_1 + \tilde{g}_7 \Lambda_2) \cos(\Omega_1 \tau_2)}{\omega_2} - \frac{(\tilde{g}_6 \Lambda_1 + \tilde{g}_8 \Lambda_2) \sin(\Omega_1 \tau_2)}{\omega_2} \end{aligned} \quad (5.27d)$$

In the next section, the resonance condition mentioned in case (vii) is discussed i.e. for combination resonances with various internal resonance conditions.

5.3.1.3 Combination resonance conditions

In this section, the combination resonance conditions mentioned in case (vii) are considered. This resonance condition is produced due to the many combinations of external harmonic and principal parametric excitations which are given as $\Omega_1 \cong 2\omega_1 + \omega_2$,

$$\Omega_1 \cong 2\omega_1 - \omega_2, \Omega_1 \cong 0.5(\omega_1 + \omega_2), \Omega_1 \cong 0.5(\omega_1 - \omega_2), \Omega_1 + \Omega_2 \cong \omega_1, \text{ and } \Omega_1 - \Omega_2 \cong \omega_1.$$

These conditions of combination resonances are observed for the following cases considering 1:1 internal resonance condition. The combination resonance $\Omega_1 \cong 2\omega_1 + \omega_2$ with 1:1 internal resonance condition provides the same resonance case as in case (iii) i.e., for the subharmonic resonance condition. Now if the combination resonance condition is considered as $\Omega_1 \cong 2\omega_1 - \omega_2$ or $\Omega_1 \cong 0.5(\omega_1 + \omega_2)$, with the same 1:1 internal resonance conditions then this case is the same as the primary resonance condition for external harmonic excitation $F_1 \cos \Omega_1 t$, which is not carried out in the present analysis as the amplitude of external harmonic excitation (F_1) is considered as hard. Also, for resonance conditions when $\Omega_1 + \Omega_2 \cong \omega_1$ or $\Omega_1 - \Omega_2 \cong \omega_1$ then these resonance conditions are also the same as primary resonance as $\Omega_2 = 2\omega_1 + \varepsilon\sigma_2$, which is not considered in the present analyses for the external hard harmonic excitation.

The reduced equations obtained in the subsequent subsections of 5.3.1.1 and 5.3.1.2 for various internal resonance conditions viz., 1:1, 1:2, 1:3 and 3:1 are used to study the steady-state response of the system by equating $\dot{a}_1 = \dot{\gamma}_1 = \dot{a}_2 = \dot{\gamma}_2 = 0$. These steady-state equations are solved by using Newton's method and the stability is ensured by finding the eigenvalue of the Jacobian matrix. It may be noted that a_1 , γ_1 , a_2 and γ_2 are the modal amplitudes and phase of the primary system and the absorber respectively. The original amplitude of the system is obtained as $\begin{bmatrix} A_T \\ B_T \end{bmatrix} = [\tilde{P}] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, where A_T and B_T represents the amplitudes of the main system and the absorber for the traditional vibration absorber. Similarly, for the non-traditional vibration absorber, the original amplitude of the main system and the absorber are represented as A_N and B_N respectively. The steady-state responses of the system for various system parameters are studied and discussed in the next section.

5.4 Results and Discussions

In this section, the system responses and stability analyses are carried out considering both the active traditional and the active non-traditional vibration absorber with different combinations of feedbacks, gains and time delays for various external and internal

resonance conditions. The physical parameters for the traditional vibration absorber (TVA) are considered from the work Detroux et al. [168], as follows. $m_1 = 1$ kg, $k_1 = 1$ N/m, $m_2 = 0.05$ kg, $k_2 = 0.0454$ N/m and $c_2 = 0.0128$ N-s/m. The nonlinear stiffness of the absorber (k_{23}) is calculated from the formula given in [161] as $k_{23} = \frac{2(m_2/m_1)^2 k_{13}}{1 + 4(m_2/m_1)}$, here k_{13} values are varied from 0 to 1 N/m³. Further for different mass ratios (m_2/m_1) and internal resonance conditions, the optimized spring stiffness (k_2), cubic nonlinear stiffness (k_{23}) and damping coefficient (c_2) values for the TVA are calculated from the work of Habib et al. [161]. The analyses are carried out by studying the trivial state instability regions, frequency responses, time responses, phase portraits, Poincare' sections, Lyapunov exponents, for different feedbacks, gains (f_d, f_v, f_a), time delays (τ_1, τ_2), nonlinear stiffness (k_{13}, k_{23}), the amplitude of force excitations (F_{11}, F_{12}), and amplitude of base excitation (Y) for both traditional and non-traditional ANVA.

For the non-traditional vibration absorber (NVA), the optimal the optimal spring stiffness (k_2) and damping coefficient (c_2) are calculated from the formulae provided in the work of Ren et al [99] for a mass ratio of 0.05. It may be noted that these optimal values are used to suppress maximum vibration of the main system and shows Den Hartog equal peaks in the frequency response. The physical system parameters for the NVA are considered as $m_2 = 0.05$ kg, $k_2 = 0.0556$ N/mm and $c_2 = 0.1446$ N-s/m. The main system parameters are the same as mentioned before. The bookkeeping parameter $\mathcal{E}$ is considered equal to 0.1 throughout the analysis and PZT actuator properties are considered from Malik and Chatterjee [20]. These properties are given as $n = 100$, $d_{33} = 3.5 \times 10^{-10}$ m/volt and elastic compliance $S_{33}^E = 16 \times 10^{-12}$ m²/N and piezoelectric stiffness $k_p^E = 1.2 \times 10^8$ N/m. From these parameters, one can change the stiffness of the present absorber actively with the applied voltage V and feedback gains. The amplitude of excitations $F_1 = 1$ N, $F_{12} = 1$ N/m and amplitude of base of excitation $Y = 1$ m. The actual amplitude of parametric excitation $F_2 = 0.1$ N/m and base excitation $Y_0 = 0.1$ m. It may be noted that the feedback gains produced by the ANVA by displacement (f_d), negative velocity (f_v) negative acceleration feedback

(f_a) and a combination of these feedbacks (f_{c1}) are considered equal to 10 in most of the analyses. In a few cases, different feedback gains are considered those are mentioned in the corresponding sections. Further, the combination of f_d and f_a , are denoted as f_{da} , similarly a combination of f_v and f_a are denoted as f_{va} . The positive combination of f_a and f_v are denoted as f_{av} . The combination of f_d , f_v and f_a feedbacks are denoted as f_{c1} and the positive combinations of these (f_d, f_v, f_a) feedbacks are denoted as f_{c2} . Also, the time delay terms τ_1 and τ_2 denotes delay in the damping and delay in the feedback, respectively. In the next sub-section, the trivial state instability regions of the system are studied with various feedback gains and actively changing the frequency of the absorber with 1:1 internal resonance conditions.

5.4.1 Trivial state instability region for simultaneous superharmonic, principal parametric and primary resonance conditions

In this section, the trivial state instability regions of the system are studied by the TVA and the NVA by obtaining the eigenvalues of the Jacobian matrix from the Eq. (5.12a) and Eq. (5.12d) by considering $\dot{P}_1 = \dot{Q}_1 = \dot{P}_2 = \dot{Q}_2 = 0$. The analyses are carried out to study the responses of the system for 1:1 internal resonance condition with the various combination of feedbacks gains and time delays.

5.4.1.1 Traditional vibration absorber

Fig. 5.2(a, b) shows the trivial state boundary of the system with a variation of the amplitude of parametric excitation F_{12} and frequency Ω_2 considering various combination feedback gains and actively changing the stiffness of the absorber. The amplitude of external excitations F_1 and Y values are considered equal to zero in Fig. 5.2(a). Also, the amplitude of F_1 does not affect the instability regions as F_1 value is coupled with the nonlinear terms (k_{n1} to k_{n4} , $\tilde{k}_{n1}$ to $\tilde{k}_{n4}$) as Λ_1 and Λ_2 . So F_1 the parameter is not appeared in Eq. (5.12a) to Eq.(5.12d). In Fig. 5.2 the active changes in the stiffness of the absorber without any feedback gains are referred to as *active₁*, *active₂* and *active₄* TVA. In *active₁*, *active₂* and *active₄* TVA the k_2 values are taken as 0.05 N/m [1-6, 16], 0.0544 N/m, and 0.0635 N/m,

respectively. It may be noted that these values of k_2 are chosen after numerous analyses for the proposed ANVA with different feedback gains. From Fig. 5.2(a), it can be observed that with *active*₁, *active*₂ and *active*₄ TVA the instability region shifts to the higher side of Ω_2 values than the passive TVA. So, by using these TVAs one can avoid instability regions effectively in a passive way (PVA) without applying any feedback gains by operating the system at a higher frequency

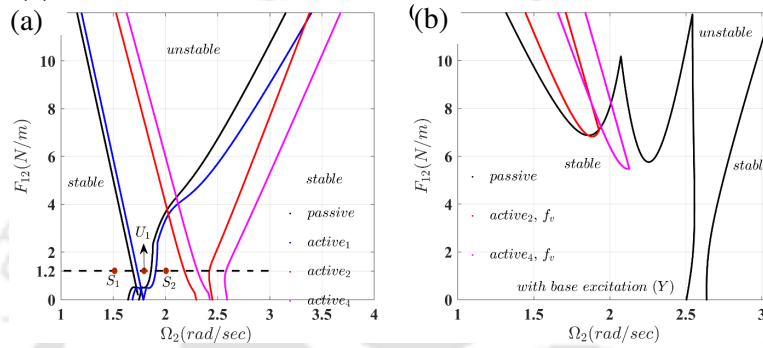


Fig. 5.2: Trivial state instability region with passive and active TVA (a) zero feedback gain (b) various feedback gains and with base excitation (Y).

range. In this figure also one can notice that the trivial state becomes unstable even at $F_{12} = 0$ N/m, for the passive as well as different active TVAs (*active*₁, *active*₂ and *active*₄). Fig. 5.2(b) shows the trivial instability region of the system considering $Y = 0.1$ m. From Fig. 5.2(b) it can be observed that with the passive TVA the range of the instability region is larger than with *active*₂ or *active*₄ TVA along with f_v feedback gain. The critical value of F_{12} below which the trivial system becomes stable is also higher in the case of *active*₂ or *active*₄ TVA than the passive TVA. In passive TVA the trivial state becomes unstable for $F_{12} = 0$ N/m but with *active*₂ and *active*₄ TVA along with f_v feedback gain the critical value of F_{12} is at 5.4 N/m and 6.8 N/m, respectively. From Fig. 5(a, b) one can observe the effectiveness of the *active*₂ or *active*₄ TVA along with or without feedback gain in increasing the stable range of operating frequency as well as increasing the critical value of F_{12} .

The instability region shown in Fig. 5.2(a) is verified at the three marked points S_1 , U_1 and S_2 for $F_{12} = 1.2$ N/m, by solving the governing Eq. (5.4) using the numerical method, and these results are shown in Fig. 5.3. From Fig. 5.3 (a, b, e, f) one can observe the stable responses at the marked points S_1 and S_2 which are in good agreement with Fig. 5.2(a). Similarly in Fig. 5.2(a) at point U_1 the system response is unstable which are also in good agreement with Fig. 5.3(c, d) at the corresponding point. In the next section, the trivial state instability region of the system is studied with NVA by various combinations of feedback gains.

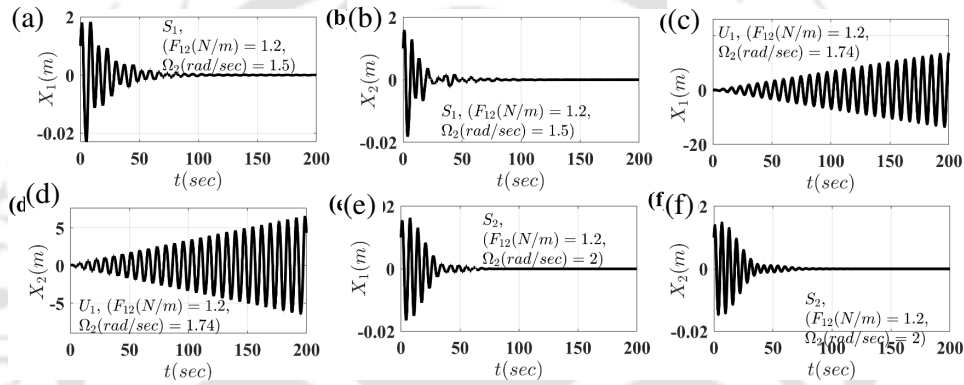


Fig. 5.3: Time responses of the system at the marked points S_1 , U_1 and S_2 of Fig. 5.2(a).

5.4.1.2 Non-traditional vibration absorber

The trivial state instability region of the system with a non-traditional vibration absorber (NVA) is shown in Fig. 5.4(a, b). Fig. 5.4(a) shows a comparison between the instability region with passive NVA and with *active*₁, *active*₂ and *active*₄ NVA. From this figure, one can observe that with the *active*₂ and *active*₄ NVA the instability region shifts to the higher side of frequencies, which is similar to the observation noticed in Fig. 5.2(a) with the TVA. However, with *active*₂ and *active*₄ NVA the instability region shows multiple stable and unstable regions than with TVA (Fig. 5.2(a)). In Fig. 5.4(b) f_v gain is used keeping all other parameters the same as in Fig. 5.4(a). Fig. 5.4(b) shows that with the f_v gain in the passive, *active*₂ and *active*₄ NVA, the instability region significantly reduces than Fig. 5.4(a). Considering only f_v gain in the passive NVA the critical value of F_{12} increases

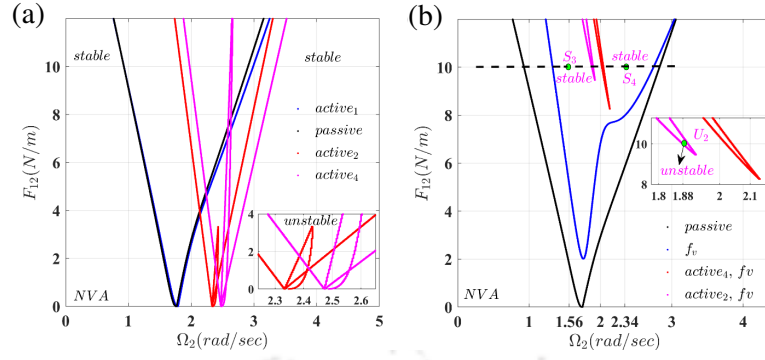


Fig. 5.4: Trivial state instability region with passive and active NVA (a) zero feedback gain (b) various feedback gains and with base excitation (Y).

from 0 N/m to 2 N/m and also the instability region is narrower than the passive NVA. It is shown in Fig. 5.4(b) that the critical value of the F_{12} remarkably increases from 0 N/m with passive TVA to 8.2 N/m and 9.4 N/m, respectively with *active₂* and *active₄* NVA along with the f_v active force. The results shown in Fig. 5.4(b) with *active₂* NVA along with the f_v gain is verified by obtaining the time responses of the system from the governing Eq. (5.4) and shown in Fig. 5.5 at the three marked points (S_3 , U_2 and S_4) in Fig. 5.4(b) for $F_{12} = 10$ N/m. From Fig. 5.5(a, b, e, f) it can be observed that at S_3 ($\Omega_2 = 1.56$ rad/sec)

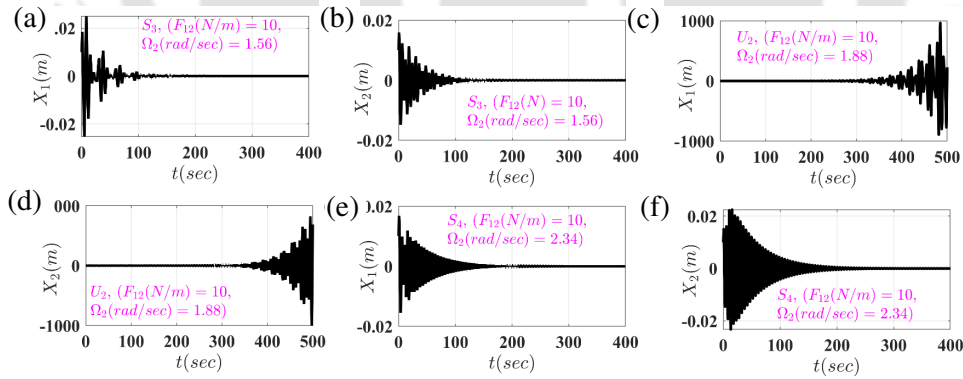


Fig. 5.5: Time responses of the system at the marked points S_3 , U_2 and S_4 of Fig. 5.4(b).

and S_4 ($\Omega_2 = 2.34$ rad/sec) the responses are stable which in good agreement with Fig. 5.4(b). Also, at U_2 ($\Omega_2 = 1.88$ rad/sec) and $F_{12} = 10$ N/m, the response is unstable as shown in Fig. 5.5(c, d) which are in good agreement with Fig. 5.4(b). Hence from Fig. 5.2 and Fig. 5.4 the efficacy of the proposed active TVA and the active NVA can be observed in

reducing the instability regions remarkably and significantly enhancing the critical value of F_{12} . In the next sub-section, the performance of the ANVA is studied with various system parameters through the frequency responses, time responses, phase portraits, Poincare' sections and basin of attractions by the ANVA for simultaneous superharmonic, principal parametric and primary resonance conditions.

5.4.2 Response of both TVA and NVA for simultaneous superharmonic, principal parametric and primary resonance conditions

In this sub-section, the frequency responses of the primary and the secondary system are studied for both TVA and NVA by solving the reduced equations derived in sections 5.3.1.1 by the MMS using Newton's method. The analyses are carried out to study the responses of the system for 1:1, 1:2 and 3:1 internal resonance conditions by considering various linear and nonlinear stiffness, time delays in the damping and various combinations of feedback gains. In the following subsection considering case (v) of simultaneous resonance with the 1:1 internal resonance condition, the responses of the system are investigated. It may be noted that the modal and original response amplitude of the primary system are denoted as a_1 and A_T , respectively. Similarly, for the secondary system, the response amplitudes are denoted as a_2 and B_T . Further, it may be also noted that in all the frequency response plots the 'dot' or line signifies the stable solutions and 'circle' signifies the unstable solutions.

5.4.2.1 1:1 internal resonance condition

In this subsection responses of the system are studied in three sub-sections based on the consideration of nonlinear stiffness in the system. The frequency responses of the system are obtained by solving the MMS reduced equations (Eq. (5.11a) to Eq. (5.11d)) using Newton's method. In the first sub-section linear system is studied and in the next two subsections, nonlinear stiffness in the primary system and nonlinear stiffness in both the primary system as well as in the absorber are considered. In the next subsections the effect of various feedback gains, time delays in the feedback and time delays in the damping on the frequency responses and time responses of the linear and nonlinear system by the AVA are analyzed.

Linear system

In this subsection, the frequency responses are studied considering linear stiffness in the systems with the variations in the feedback gains and time delays. Fig. 5.6(a, b) and Fig. 5.6(c, d) show the response amplitude of the primary system and the active TVA, respectively. The response amplitude of the system is studied for four cases. These four cases are given as passive TVA, passive TVA with a time delay (τ_1) of $\pi/10$ sec in the damping, active TVA with only negative acceleration feedback gain of $f_a = 10$ and the last case, active TVA with only negative velocity feedback gain ($f_v = 5$). From Fig. 5.6(b) it is observed that with $f_v = 5$ the response amplitude of the primary system is 20% less than that with the passive TVA. The time delay (τ_1) of $\pi/10$ sec produces a higher response amplitude than with acceleration or velocity feedback gains. It may be noted that here $\omega_1 = 0.87$ rad/sec and $\omega_2 = 1.09$ rad/sec which is obtained from the parameters given in section 5.4. The frequency responses in Fig. 5.6(a-d) show peaks for $\Omega_1 = \omega_1/3$ and $\Omega_3 = \omega_1$ which correspond to primary and superharmonic resonance conditions, respectively. The frequency responses of the secondary system are shown in Fig. 5.6(c) and Fig. 5.6(d).

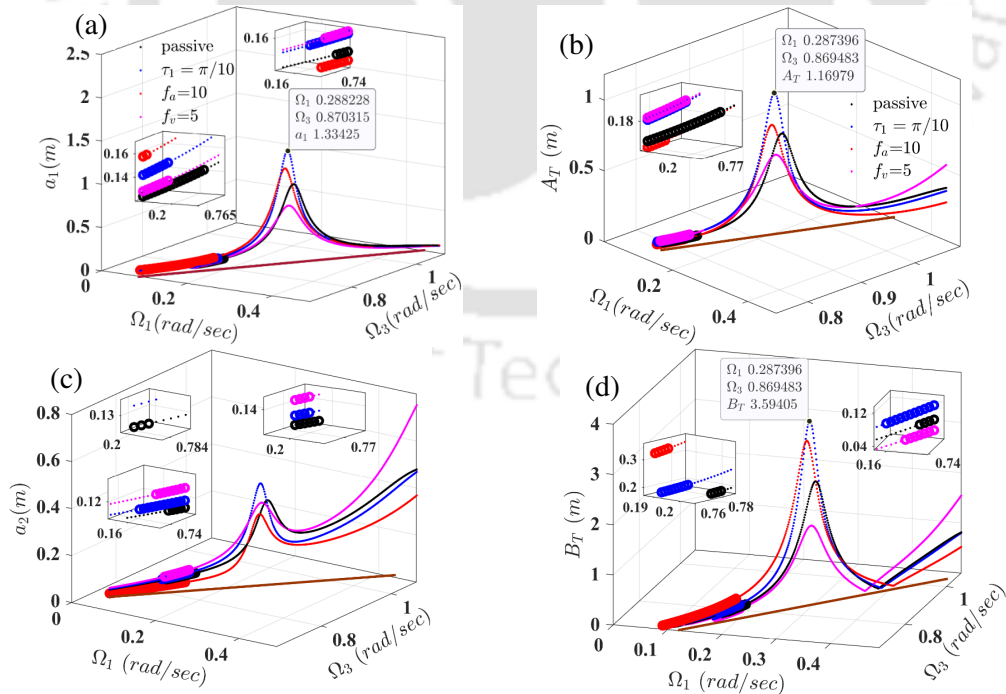


Fig. 5.6: (a-d) Frequency response of the system with passive and active TVA.

Comparing Fig. 5.6(d) with Fig. 5.6(b), it can be observed that the response amplitude of the TVA is almost 3 times the response amplitude of that of the primary system, which also matches the published literature [20] and also with the results shown in Chapter 2 and Chapter 3. The modal response amplitude (a_1) of the TVA (Fig. 5.6(c)) shows a comparatively low response amplitude than the original response amplitude (A_r) as shown in Fig. 5.6(d). The Hopf bifurcation points in the frequency responses where the stability of the system is changing are shown in each corresponding figure. The frequency response amplitude of the system shown in Fig. 5.6(a-d) for $\tau_1 = \pi/10$ sec is compared with the numerical method and shown in Fig. 5.7(a-h). From Fig. 5.7(a, b) one can observe that the

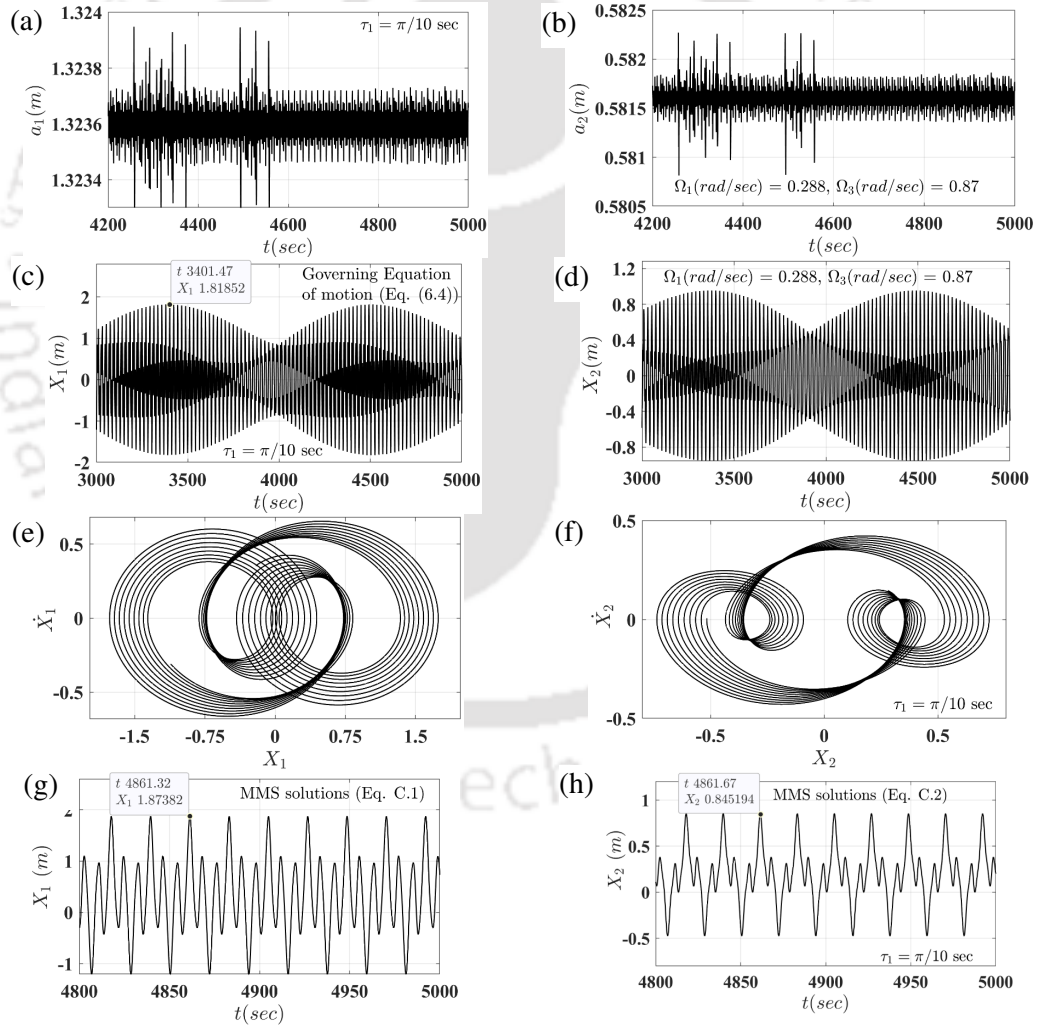


Fig. 5.7: (a-d, g, h) Time responses of the system at $\Omega_1 = 0.288$ rad/sec and $\Omega_3 = 0.87$ rad/sec, (e, f) phase portrait of the system at $\Omega_1 = 0.288$ rad/sec and $\Omega_3 = 0.87$ rad/sec.

time responses and the frequency responses are in good agreement with each other. Further, the frequency response of the system for $\tau_1 = \pi/10$ sec which is marked in Fig. 5.6(a) is compared with the numerical method by solving the governing Eq. (5.4) and the MMS solution (Appendix- C, (Eq. C.1 and Eq. C.2)). These comparisons are shown by the time responses and phase portraits in Fig. 5.7(c-f), respectively. From Fig. 5.7(c-f) one can observe three periodic responses as the primary system is subjected to three external excitations with frequencies Ω_1 , Ω_2 and Ω_3 as given in the governing Eq. 5.4. The time responses and phase portraits obtained from Eq. 5.4 (Fig. 5.7(c-f)) are compared with the MMS solution given in Appendix C (Eqs. C.1 and C.2) which are shown in Figs. 5.7(g) and 5.7(h). From Fig. 5.7(c-f) and Fig. 5.7(g, h) one can observe good agreement between the MMS solutions and the governing Eq. (5.4) by the fourth ordered Runge Kutta numerical method. In Fig. 5.8(a-d) the frequency responses of the system are studied for various combinations of feedback gains (f_d , f_{da} and f_{c1}) with time delays of $\tau_1 = \tau_2 = 0.1 \pi$ sec in the damping and also in the feedback which is more feasible in practical applications. From Fig. 5.8(b) it can be observed that the response of the primary system is minimum with f_{c1} gain at the super-harmonic ($\Omega_1 = \omega_1/3 = 0.29$ rad/sec) and primary resonance ($\Omega_3 = \omega_1 = 0.8731$ rad/sec) conditions than with other feedback gain

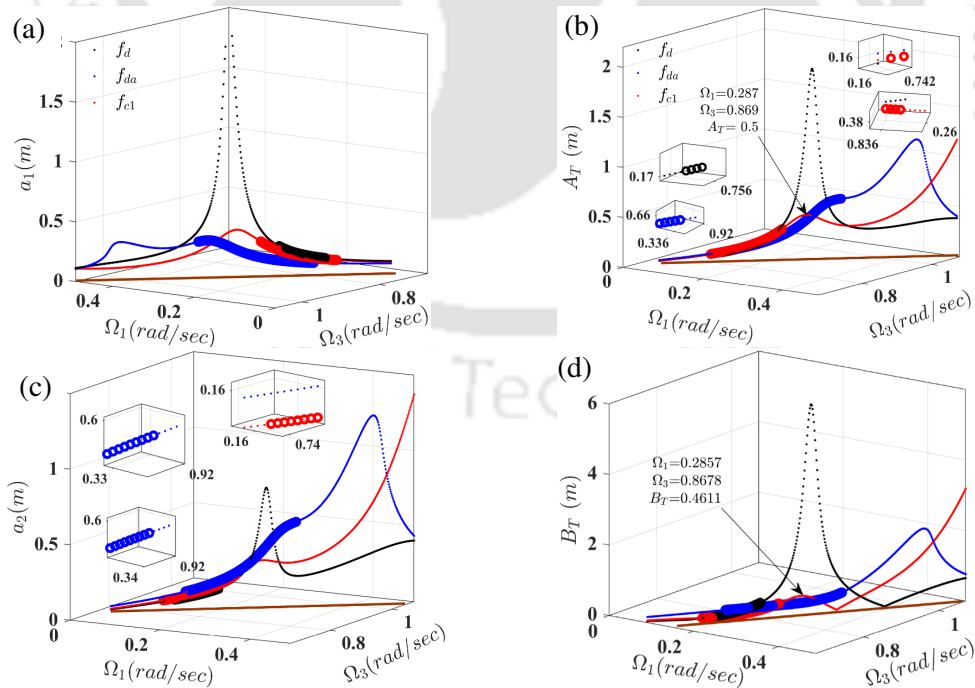


Fig. 5.8: (a-d) Frequency response of the system with active TVA.

(f_d, f_{da}) . The f_{da} gain produces a broader range of unstable regions at the simultaneous super-harmonic ($\Omega_1 \cong \omega_1/3$) and primary resonance conditions ($\Omega_3 \cong \omega_1$) than with the other combinations as shown in Fig. 5.8(a-d). The response amplitude of the primary system with only f_d gain is found to be more compared to other combinations which can be observed in Fig. 5.8(a-d) and Fig. 5.6(a-d). With the f_{c1} feedback gain (Fig. 5.8(b)) the response amplitude of the primary system reduces to almost 47% compared with the passive TVA (Fig. 5.6(b)). The TVA responses are shown in Fig. 5.8(c-d) which show similar characteristics as in Fig. 5.8(a, b). The trend of the response amplitude of the primary system (Fig. 5.8(b)) and the modal response amplitude of the absorber (TVA), which is shown in Fig. 5.8(c) are almost the same while their response amplitudes are at different magnitudes. In Fig. 5.9(a-d) the response amplitude of the system is studied with a fixed time delay of $\tau_2 = 0.1 \pi$ sec and with the various combinations of feedback gain and time delay (τ_1). From these figures, it can be observed that with f_{c1} feedback gain and with $\tau_1 = 0.2 \pi$ sec the stability of the system improves near the superharmonic and primary resonance condition than with other τ_1 values and with f_v feedback gain.

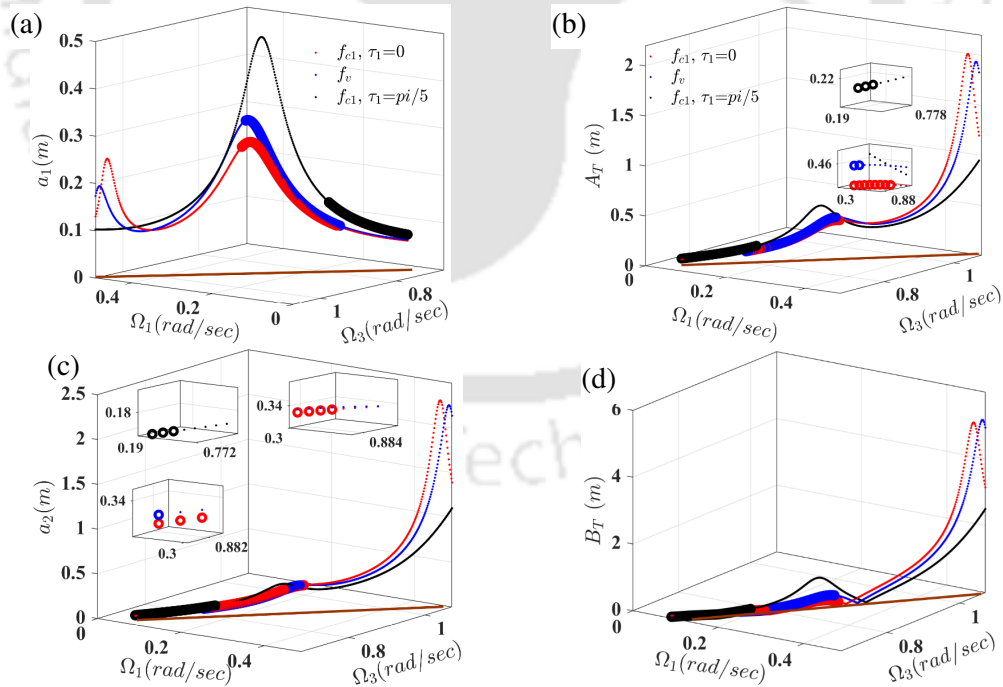


Fig. 5.9: (a-d) Frequency response of the system with the active TVA for fixed delay of

$$\tau_2 = 0.1 \pi \text{ sec.}$$

Comparing Fig. 5.9(b) and Fig. 5.8(b) for f_{c1} feedback gain, it can be observed that the response amplitude of the primary system is 15% less with a lesser time delay of $\tau_1 = 0.1 \pi$ sec, (Fig. 5.8(b)) than with a higher time delay of $\tau_1 = 0.2 \pi$ sec (Fig. 5.9(b)). The trivial state responses of the system are found to be stable for all the mentioned feedback gains and time delays which are shown in Fig. 5.6 to Fig. 5.9. The response of the system is also studied by actively changing the frequency of the absorber (*active*₂) with various combinations of feedback gains, and the corresponding results are shown in Fig. 5.10. It may be noted that the *active*₂ case is the same as mentioned in Chapter 4 (*active*₂), which also follows the 1:1 internal resonance condition. From Fig. 5.10, one can observe that the response amplitude of the primary system significantly reduces for $\Omega_1 > 0.32$ rad/sec and $\Omega_3 > 1.08$ rad/sec with only *active*₂ case (zero feedback gains) than that with passive TVA or with the consideration of various feedback gains in the active TVA which are shown in previous figures (Fig. 5.6, Fig. 5.8 and Fig. 5.9). From Fig. 5.10(b) it can be observed that with the *active*₂ case the response amplitude of the primary system is almost completely suppressed (99.9%) at $\Omega_1 = 0.37$ rad/sec and 0.42 rad/sec and for $\Omega_3 = 1.14$ rad/sec and 1.19 rad/sec unlike previous results shown in Fig. 5.6, Fig. 5.8 and Fig. 5.9. The high

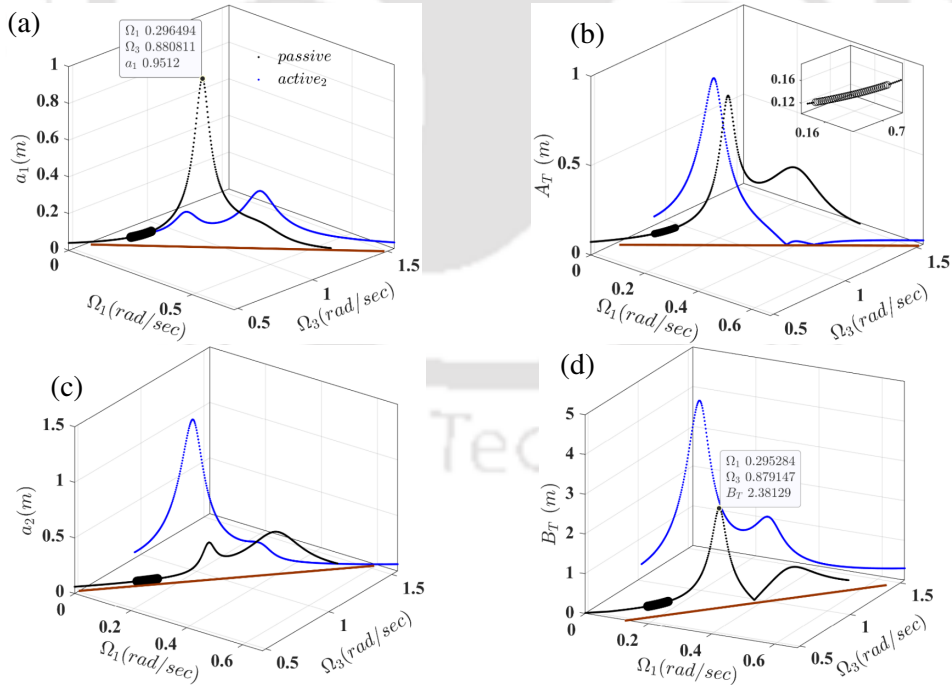


Fig. 5.10: (a-d) Comparison of frequency responses of the system with the *active*₂ and passive TVA.

response amplitude of the system is observed in Fig. 5.10(a-d) with only *active*₂ cases for Ω_1 in the range of 0.18 rad/sec to 0.32 rad/sec and Ω_3 in the range of 0.94 rad/sec to 1.08 rad/sec. However, these high response amplitudes at the above-mentioned frequencies for the primary system can be significantly reduced by providing the feedback gains f_v and f_{cl} , which are shown in blue and magenta colors in Fig. 5.11(a, b). The unstable regions in the responses are only observed for the f_{cl} feedback gain and $\tau_2 = 0.1 \pi$ sec, and the Hopf bifurcation frequency points are also shown in the corresponding figures. It is also observed from Fig. 5.11(a-d) that though the response amplitude significantly reduces near the superharmonic ($\Omega_1 \cong \omega_1 / 3 \cong 0.29$ rad/sec) and primary resonance conditions ($\Omega_3 \cong \omega_1 \cong 0.87$ rad/sec) by the feedback gains (f_v and f_{cl}) but with these gains, high response amplitude in the system is observed at $\Omega_1 = 0.36$ rad/sec and $\Omega_3 = 1.11$ rad/sec for f_v gain, and at Ω_3 equals to 0.37 rad/sec and 1.15 rad/sec for f_{cl} feedback gain than considering only the *active*₂ case (zero feedback gains). In the next sub-section, the responses of the system are studied for various feedback gains and time delays by considering only nonlinear stiffness in the primary system.

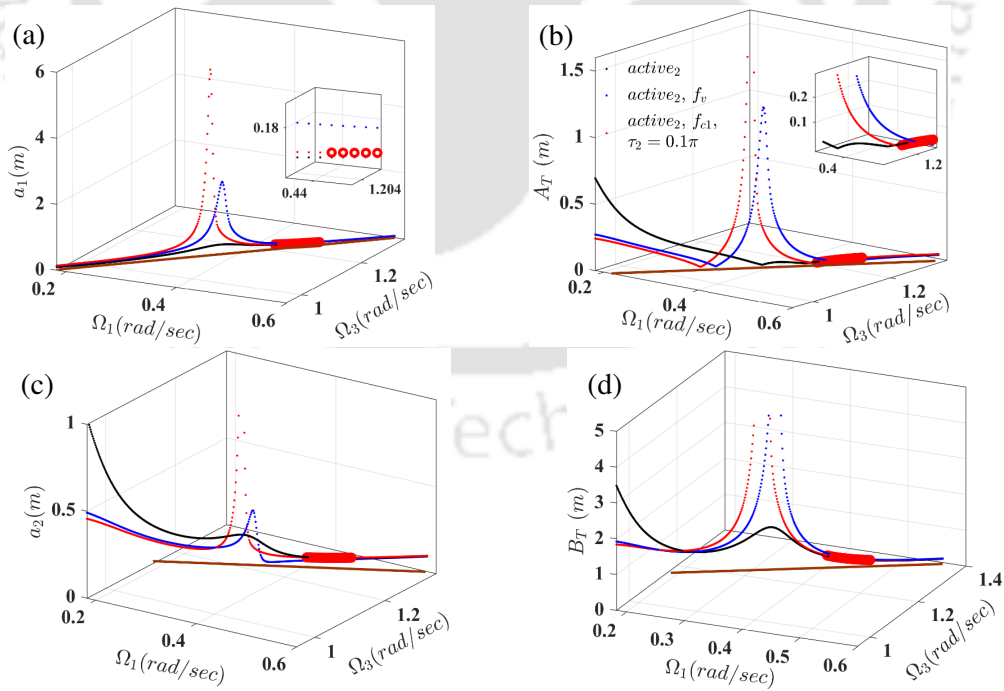


Fig. 5.11: (a-d) Frequency responses of the system with the *active*₂ TVA and control gains.

Nonlinear primary system

In this sub-section, the system responses are studied by considering only nonlinear stiffness in the primary system i.e., $\bar{k}_{13} = 1 \text{ N/m}^3$ and $\bar{k}_{23} = 0 \text{ N/m}^3$, for various feedback gains and time delays in the damping as well as in the feedbacks. Fig. 5.12(a-d) shows the comparison of responses between the passive and *active*₂ TVA. It may be noted that in *active*₂ TVA, only the stiffness of the absorber is actively changed without any feedback gains. The modal response amplitudes of the system are shown in Fig. 5.12(a) and Fig. 5.12(c). From these figures, two peaks at different response amplitudes are observed with *active*₂ TVA, whereas with passive TVA only a single peak is observed. Fig. 5.12(a-d) shows the unstable range of responses with *active*₂ TVA for Ω_1 and Ω_3 greater than 0.4676 rad/sec and 1.2648 rad/sec, respectively. To analyze the type of responses close to these unstable regions three points are marked (N_2, N_3, N_4) in Fig. 5.12(a) and corresponding time responses are shown in Fig. 5.13. From Fig. 5.12(a) also one can observe that with *active*₂ TVA the a_1 value is minimum at the superharmonic (0.291 rad/sec) and primary resonance

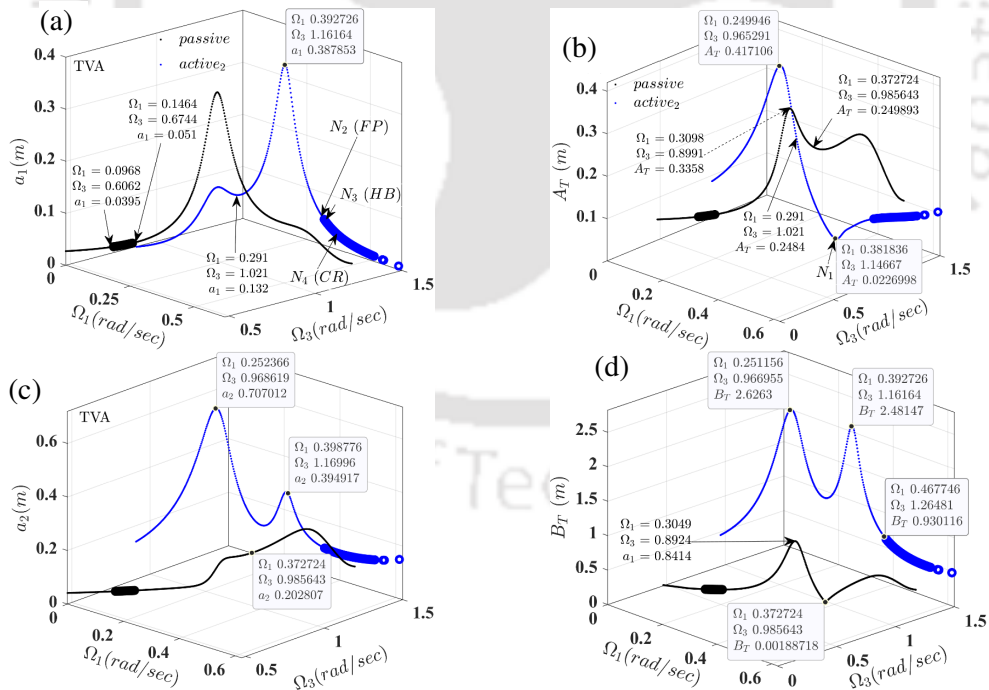


Fig. 5.12: (a-d) Comparison of frequency responses of the system by the *active*₂ and passive TVA, considering nonlinear stiffness in the primary system.

(0.8731 rad/sec) conditions whereas a_2 value shown in Fig. 5.12(c) remains at 3.5 times more than a_1 value at these frequencies. The unstable responses with the passive TVA appear for $0.0096 \text{ rad/sec} \leq \Omega_1 \leq 0.1464 \text{ rad/sec}$ and $0.6062 \text{ rad/sec} \leq \Omega_3 \leq 0.6744 \text{ rad/sec}$. In Fig. 5.12(b) the frequency response of the primary system is shown. From Fig. 5.12(b) one can observe that the response amplitude of the primary system almost reduces to zero amplitude ($A_r = 0.022 \text{ m}$) with the *active*₂ TVA at $\Omega_1 = 0.3818 \text{ rad/sec}$ and $\Omega_3 = 1.1466 \text{ rad/sec}$ but with the passive TVA A_r equal to 0.248 m close to this frequencies. The response amplitude (A_r) with the passive TVA also remains comparatively higher than that with the *active*₂ TVA for $0.3098 \text{ rad/sec} \leq \Omega_1 \leq 0.4676 \text{ rad/sec}$ and $0.8991 \text{ rad/sec} \leq \Omega_3 \leq 1.2648 \text{ rad/sec}$. Moreover, with *active*₂ TVA for a broad range of Ω_1 and Ω_3 values close to the resonance conditions the A_r value is lesser than that with passive TVA. Hence significant vibration absorption of the response amplitude of the nonlinear primary system can be achieved for a wide range of operating frequencies by first using passive TVA for the lower range of operating frequencies and then *active*₂ TVA can be used. The minimum response amplitude of the primary system with *active*₂ TVA (Fig. 5.12(b)) is marked as N_1 which is then verified by the time responses shown in Fig. 5.13. From Fig. 5.12(c, d) it is also observed that with the passive TVA the response amplitude of the absorber is smaller than that with *active*₂ TVA at close to the N_1 frequency point and also for a wide range of Ω_1 and Ω_3 . Due to this the response amplitude of the primary system is more with the passive TVA as the energy transfer from the primary system to the absorber is not taking place effectively. It is also observed from Fig. 5.12(d) that at $\Omega_1 = 0.3727 \text{ rad/sec}$ and at $\Omega_3 = 0.9856 \text{ rad/sec}$, the response amplitude of the absorber is minimum ($B_r = 0.00188 \text{ m}$) with the passive TVA. Hence from Fig. 5.12, it can be inferred that the response amplitude of both the nonlinear primary system and the linear secondary system can be minimized by the TVA without any feedback gains. In Fig. 5.13 the time response of the system is studied at three marked points (N_2, N_3, N_4) in Fig. 5.12(a) where the stability of the system changes from stable to unstable solutions. Figure 5.13(a-f) demonstrates that the system exhibits fixed point response (FP) at N_2 frequency point ($\Omega_1 = 0.4676 \text{ rad/sec}$ and $\Omega_3 = 1.2648 \text{ rad/sec}$), which is consistent with Fig. 5.12. Figure 5.13(g, i) and Fig.

5.13(h, j) show time responses and Poincare' sections of the system at N_3 frequency point ($\Omega_1 = 0.472$ rad/sec and $\Omega_3 = 1.28$ rad/sec), respectively which indicate periodic response. As a result, Hopf bifurcation (HB) occurs at N_3 frequency point as shown in Fig. 5.12. Time responses (Fig. 5.13(g, l)) and Poincare' sections (Fig. 5.13(h, j)), of the system at frequency point N_4 ($\Omega_1 = 0.52$ rad/sec and $\Omega_3 = 1.34$ rad/sec) reveal chaotic and two periodic types of responses in the primary system and the secondary system. From Fig. 5.13 it can be inferred that at the bifurcation point N_2 ($\Omega_1 = 0.4676$ rad/sec and $\Omega_1 = 1.2648$ rad/sec), shown in Fig. 5.12 the system shows fixed point response, periodic response (N_3), two periodic response and chaotic response (N_4) with a slight increase in the operating frequencies Ω_1 and Ω_3 . The N_3 and N_4 frequency points are further verified in Fig. 5.14 by the Lyapunov exponents which is used to quantitatively characterize any chaotic system.

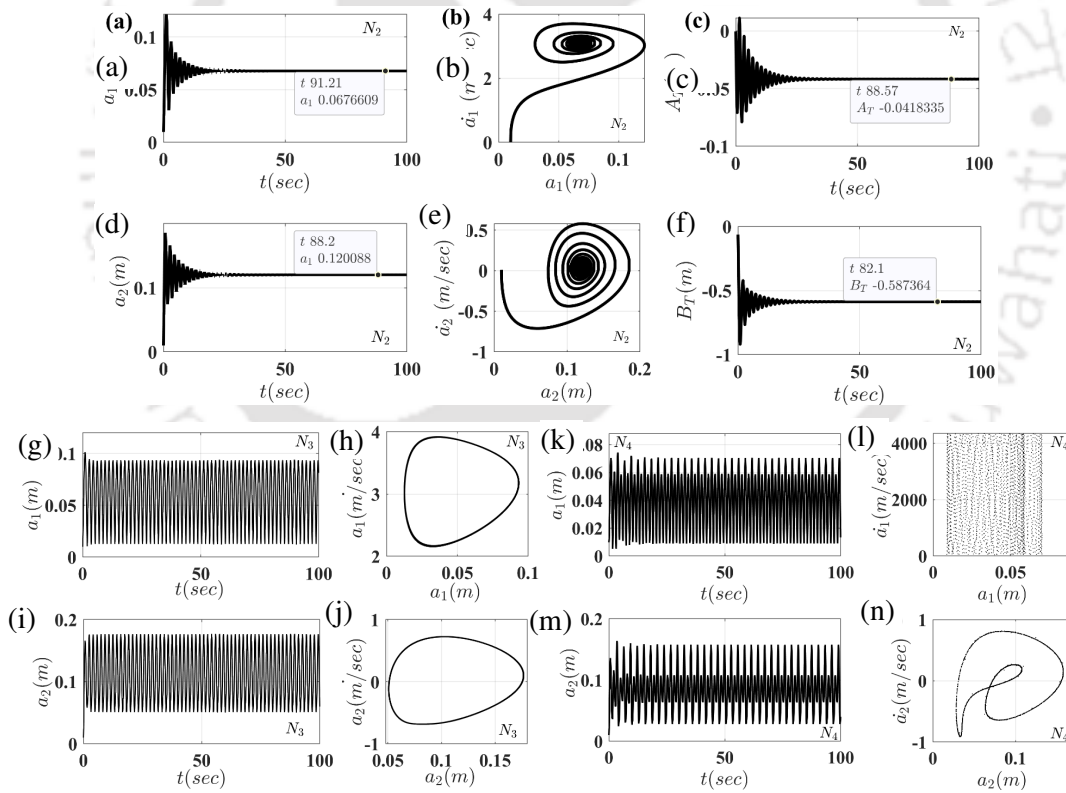


Fig. 5.13: Time responses (a, c, d, f, g, k, i, m), phase portraits (b, e) and Poincare' sections (h, j, l, n) of the primary system and the secondary system at the marked points N_2 , N_3 , N_4 of Fig. 5.12(a).

The Lyapunov exponents of the time series generated by the numerical method show periodic and chaotic responses at N_3 (Fig. 5.14(a)) and at N_4 (Fig. 5.14(b)) frequency points. Fig. 5.15 shows the comparisons of time responses between the governing Eq. (5.4) and the MMS solutions (Eqs. (5.11a) and (5.11b)) by the numerical method at the marked N_1 point in Fig. 5.12(b). From Figs. 5.15(a, b) one can observe the displacements of the primary system (X_1) and the secondary system (X_2) by solving Eq. (5.4) and Eqs. (5.11a) and (5.11b) are in good agreement with each other, showing multi-harmonic responses. It may be noted that the displacements X_1 and X_2 of the system are obtained from Appendix C, Eqs. (C.1) and (C.2), by substituting a_1 , γ_1 , a_2 , γ_2 values at the N_1 frequency point. It is noticed that at the N_1 frequency point shown in Fig. 5.12(b), the response amplitude of the primary system $a_1 = 0.3468$ m and $A_r = 0.022$ m, It is noticed that at the N_1 frequency point shown in Fig. 5.12(b), the response amplitude of the primary system $a_1 = 0.3468$ m and $A_r = 0.022$ m, whereas the displacement of the primary system (X_1) at the N_1 point is oscillating from -1.032 m to 1.032 m (Fig. 5.15(a)). This deviation between the solutions of the frequency responses amplitudes (a_1) and the time responses (X_1) are due to the additional terms $(2\Lambda_1 \cos(\Omega_1 T_0) + \varepsilon(\dots))$ which are given in Appendix C, Eqs. (C.1) and (C.2) where X_1 equals to $(a_1 \cos(3\Omega_1 T_0 - \gamma_1) + 2\Lambda_1 \cos(\Omega_1 T_0) + \varepsilon(\dots))$ and X_2 equal to $(a_2 \cos(3\Omega_1 T_0 - \gamma_2) + 2\Lambda_2 \cos(\Omega_1 T_0) + \varepsilon(\dots))$. It may be also noted these additional terms are developed owing to the hard excitation $\left(\Lambda_1 = \frac{p_{11} F_1}{2(\omega_1^2 - \Omega_1^2)}, \Lambda_2 = \frac{p_{12} F_1}{2(\omega_2^2 - \Omega_1^2)} \right)$ acting

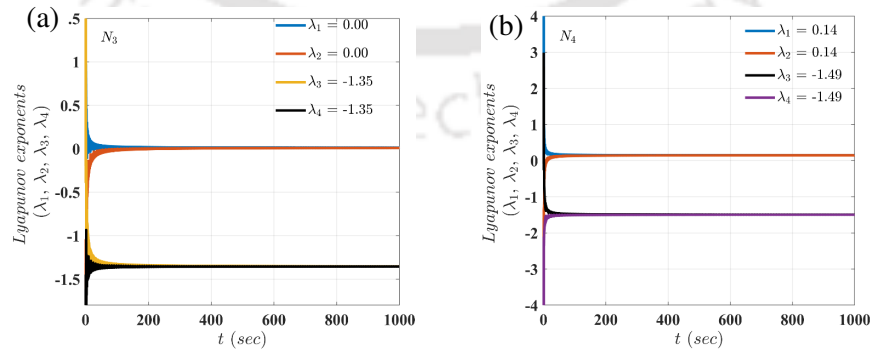


Fig. 5.14: Lyapunov exponents for initial condition (1, 0, 1, 0) at the marked points N_3 and N_4 of Fig. 5.12(a).

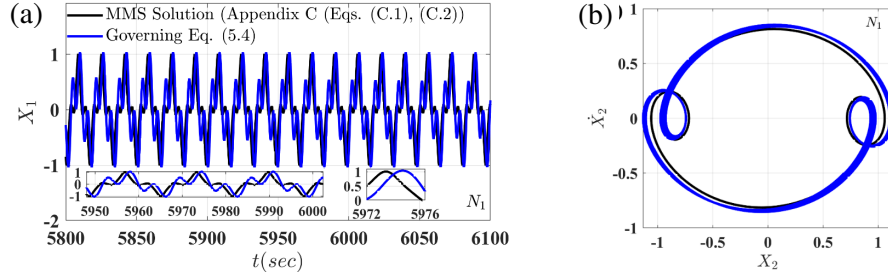


Fig. 5.15: Comparisons of time responses of the system between governing equations and MMS solutions at the marked points N_1 of Fig. 5.12(b).

on the primary system. The frequency response of the system with NVA is studied in Fig. 5.16 considering passive and *active2* NVA. From Fig. 5.16(a, b) it can be observed that the minimum and maximum response amplitude of the primary system with *active2* NVA and passive NVA are comparatively higher than that with the *active2* TVA and passive TVA (Fig. 5.12(a, b)). As observed from Fig. 5.16(b) the *active2* NVA performs better than passive NVA for $0.3236 \text{ rad/sec} \leq \Omega_1 \leq 0.4676 \text{ rad/sec}$ and $1.11 \text{ rad/sec} \leq \Omega_3 \leq 1.2648 \text{ rad/sec}$. However, for $0.11 \text{ rad/sec} \leq \Omega_1 \leq 0.3236 \text{ rad/sec}$ and $0.88 \text{ rad/sec} \leq \Omega_3 \leq 1.11 \text{ rad/sec}$ the passive NVA performs better in suppressing the response amplitude of the

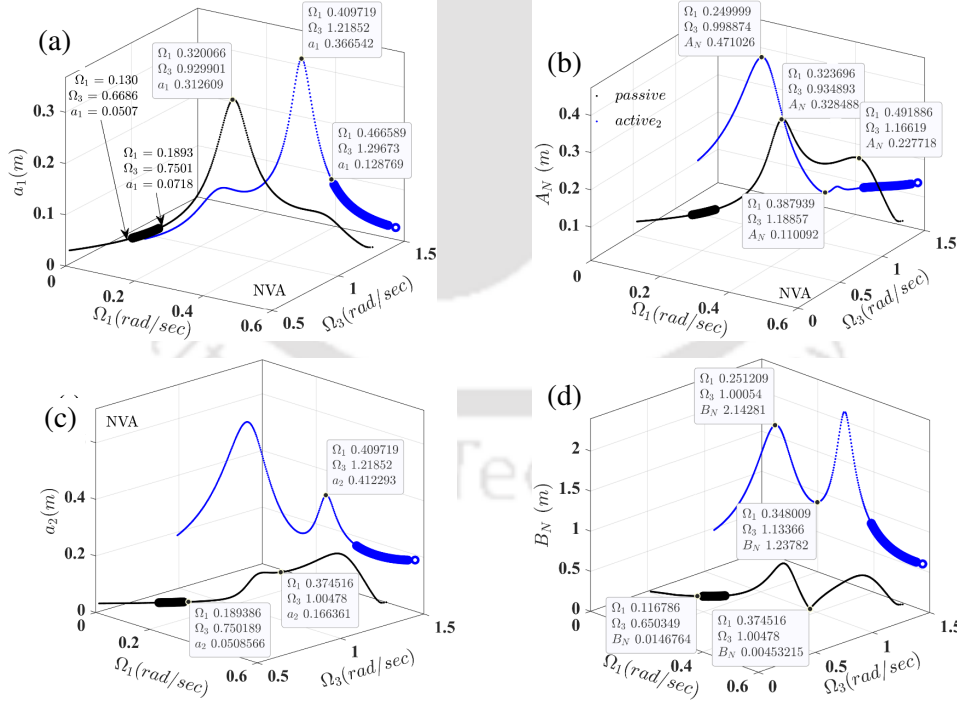


Fig. 5.16: (a-d) Comparison of frequency responses of the system by the *active2* and passive NVA, considering nonlinear stiffness in the primary system.

primary system than that of the *active2* NVA. The unstable responses in the system with the passive and *active2* NVA are observed similar to Fig. 5.12(a-d). The response amplitude of the absorber almost reduces to zero amplitude with the passive NVA at $\Omega_1 = 0.3745$ rad/sec and $\Omega_3 = 1$ rad/sec. From Fig. 5.16 it can be observed that the with *active2* TVA (Fig. 5.12) the vibration suppression of the primary system is less in comparison to the *active2* NVA (Fig. 5.16). However, the passive NVA (Fig. 5.16) performs better in reducing the maximum peaks in the response amplitude of the system than the passive TVA (Fig. 5.12).

The effect of gains on the frequency responses by the various combinations of feedbacks such as f_v , f_{va} and f_{c1} by with and without *active2* TVA are discussed and corresponding plots are shown in Fig. 5.17, Fig. 5.18 and Fig. 5.19, respectively. In Fig. 5.17(a-d) the response amplitude of the system is compared considering velocity feedback gain f_v in active TVA and with *active2* TVA. It may be noted that f_v in the active TVA refers to gain by velocity feedback in the passive TVA. The f_v with the *active2* TVA refers to active

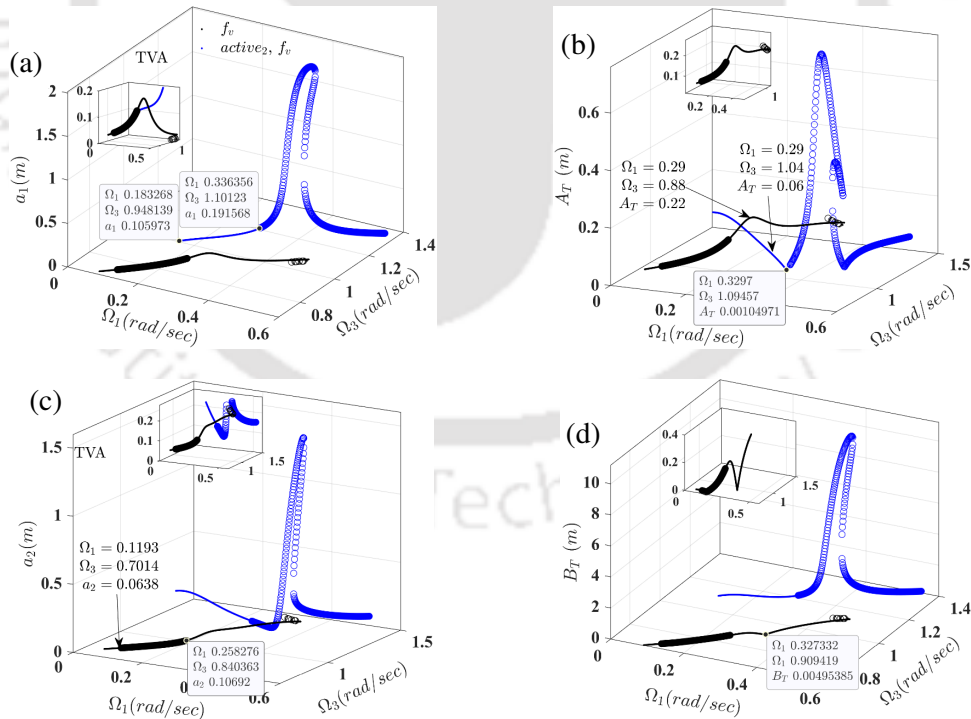


Fig. 5.17: (a-d) Comparison of frequency responses of the nonlinear primary system by f_v gain and with the *active2* TVA and f_v .

gain by velocity feedback and actively changing the frequency of the absorber as mentioned in section 5.4.1. From these figures, one can notice that the response shows a broader region of unstable response with the *active*₂ TVA and f_v gain in comparison with the only f_v gain. However, the response amplitude of the primary system with the *active*₂ TVA and f_v gain shows 72% less amplitude close to the super-harmonic ($\Omega_1 = 0.29$ rad/sec) and primary resonance conditions ($\Omega_3 = 0.87$ rad/sec) than with the only f_v gain. The response amplitude of the absorber with only f_v gain shows almost zero response amplitude at the above-mentioned resonance conditions, which is shown in Fig. 5.17(d). However, in this figure with *active*₂ TVA and f_v gain the response amplitude of the absorber at the above resonance condition is more compared to with only f_v gain. Because of this, the response amplitude of the primary system is more with only f_v gain than with *active*₂ TVA and f_v gain.

From Fig. 5.17 and Fig. 5.12 it can be observed that with f_v gain in *active*₂ TVA, there is a substantial decrease (52%) in the maximum peak value A_r and also for a broader range i.e. for $0.158 \text{ rad/sec} \leq \Omega_1 \leq 0.3297 \text{ rad/sec}$ and $0.838 \text{ rad/sec} \leq \Omega_3 \leq 0.952 \text{ rad/sec}$, the A_r value is smaller than the passive or *active*₂ TVA (Fig. 5.12). From these figures, it can be also observed that with the passive TVA (Fig. 5.12) and passive TVA with f_v gain (Fig. 5.17) the unstable region of responses are appeared in the lower range of frequencies whereas with *active*₂ TVA the unstable range of frequencies appear in the higher frequency range. Fig. 5.17 shows the range of unstable responses ($0.119 \text{ rad/sec} \leq \Omega_1 \leq 0.258 \text{ rad/sec}$ and $0.701 \text{ rad/sec} \leq \Omega_3 \leq 0.84 \text{ rad/sec}$) with passive TVA and f_v gain is comparatively more than with only passive TVA (Fig. 5.12), where the range is $0.0096 \text{ rad/sec} \leq \Omega_1 \leq 0.1464 \text{ rad/sec}$ and $0.6062 \text{ rad/sec} \leq \Omega_3 \leq 0.6744 \text{ rad/sec}$. From Fig. 5.17 it can be noted that one can effectively suppress the vibration of the primary system with the f_v gain in the *active*₂ TVA for $0.158 \text{ rad/sec} \leq \Omega_1 \leq 0.33 \text{ rad/sec}$ and $0.838 \text{ rad/sec} \leq \Omega_3 \leq 0.952 \text{ rad/sec}$ and outside this frequency range one can provide zero control gain in the *active*₂ TVA to suppress the vibration of the primary system for $\Omega_1 \geq 0.33 \text{ rad/sec}$ $\Omega_3 \geq 0.952 \text{ rad/sec}$. It is also observed that the A_r value with the passive or *active*₂ TVA (Fig. 5.12) is

slightly smaller (10-20%) than that with f_v gain in the *active₂* TVA (Fig. 5.17) for $0 \text{ rad/sec} \leq \Omega_1 \leq 0.158 \text{ rad/sec}$ and $0.47 \text{ rad/sec} \leq \Omega_3 \leq 0.838 \text{ rad/sec}$. It is observed that the Hopf bifurcation (*HB*) points with f_v gain in the passive TVA occur at $\Omega_1 = 0.1193 \text{ rad/sec}$ and $\Omega_3 = 0.7014 \text{ rad/sec}$. Similarly, for the *active₂* TVA and f_v gain, the *HB* points occurs at $\Omega_1 = 0.3297 \text{ rad/sec}$ and $\Omega_3 = 1.0945 \text{ rad/sec}$. It may be noted that the time responses and phase portraits at these bifurcations' points with f_v gain in the passive and *active₂* TVA are similar to Fig. 5.14. Also, the frequency response of the system with TVA is similar to the NVA, so in further studies, only TVA is considered for analyzing the frequency response of the system.

In Fig. 5.18(a-d) the response amplitude of the system is compared with the combination of velocity and acceleration feedback gains (f_{va}) in the active TVA and with *active₂* TVA. From Fig. 5.18(a, b) one can observe an 82% reduction in the A_T value with *active₂* TVA

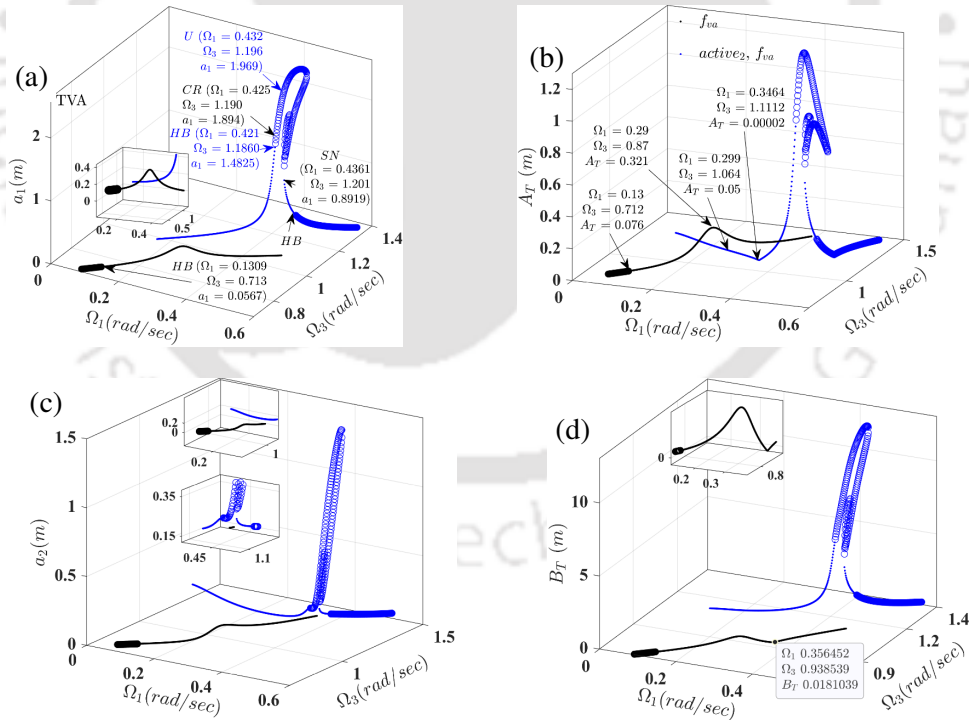


Fig. 5.18: (a-d) Comparison of frequency responses of the system considering nonlinear stiffness in the primary system by the combination of velocity and acceleration feedback gains (f_{va}).

and f_{va} gain, in compared to with only f_{va} at the super-harmonic ($\Omega_1 = 0.29$ rad/sec) and primary resonance ($\Omega_3 = 0.87$ rad/sec) conditions. Fig. 5.18(a) shows similar, observation as Fig. 5.18(b). The response amplitude of the primary system (A_r) is 100% suppressed at $\Omega_1 = 0.3464$ rad/sec and at $\Omega_3 = 1.1112$ rad/sec with f_{va} gain in the *active₂* TVA. It can be observed from these figures that the A_r value with f_{va} gain in the *active₂* TVA is comparatively very less than the with only f_{va} gain for Ω_1 in the range of 0.18 rad/sec to 0.38 rad/sec and for Ω_3 in the range of 0.94 rad/sec to 1.14 rad/sec. It may be noted that the above frequency ranges are within the superharmonic and primary resonance conditions. It is also observed that the A_r value with f_{va} gain in the *active₂* TVA at $\Omega_1 = 0.09$ rad/sec, $\Omega_3 = 0.7473$ rad/sec is 0.038 m and for a broad range i.e. from $0 \text{ rad/sec} \leq \Omega_1 \leq 0.38 \text{ rad/sec}$ and $0.7473 \text{ rad/sec} \leq \Omega_3 \leq 1.14 \text{ rad/sec}$, the A_r value is lesser (10-35%) than that shown in Figs. 5.12, 5.17 and 5.18. However, for $\Omega_1 \geq 0.38$ rad/sec and $\Omega_3 \geq 1.14$ rad/sec f_{va} gain in the *active₂* TVA produces a high response amplitude. It may be noted that as the frequency of the TVA can be changed actively so outside the above-mentioned frequency range one can use only f_{va} gain for effectively suppressing the vibration of the system for a wider range of operating frequencies. From these figures the Hopf bifurcations points (*HB* ($\Omega_1 = 0.421$ rad/sec, $\Omega_3 = 1.186$ rad/sec and $\Omega_1 = 0.458$ rad/sec, $\Omega_3 = 1.223$ rad/sec)), saddle node bifurcation (*SN* ($\Omega_1 = 0.4361$ rad/sec, $\Omega_3 = 1.201$ rad/sec)) frequency points, chaotic responses (*CR* ($\Omega_1 = 0.425$ rad/sec, $\Omega_3 = 1.19$ rad/sec)), and unstable responses (*U* ($\Omega_1 = 0.432$ rad/sec, $\Omega_3 = 1.196$ rad/sec)), are observed with f_{va} gain in the *active₂* TVA. The Hopf bifurcation point with the passive TVA and f_{va} gain is observed at (*HB* ($\Omega_1 = 0.1301$ rad/sec, $\Omega_3 = 0.713$ rad/sec)). These bifurcation points are verified in Fig. 5.19 by the time responses (Figs. 5.19(a), 5.19(b) and 5.19(c)) and Poincare' section (Fig. 5.19(d)) at the corresponding frequencies points. Fig. 5.18(a) shows with at $\Omega_1 = 0.4361$ rad/sec, $\Omega_3 = 1.201$ rad/sec the frequency response shows *SN* bifurcation and by sweeping down the Ω_1 value slightly from 0.4361 rad/sec to 0.420 rad/sec there is jump up in the fixed point response as observed from Fig. 5.19(a).

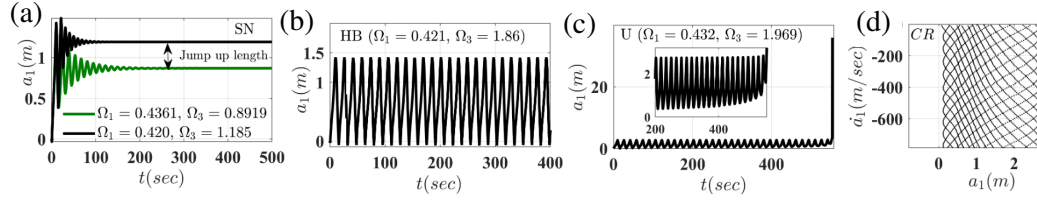


Fig. 5.19: (a-c) Time responses of the primary system at the marked points *SN*, *HB*, *U* of Fig. 5.18(a), (d) Poincare' section of at the marked points *CR* of Fig. 5.18(a).

Fig. 5.19(b) shows periodic response at $\Omega_1 = 0.421$ rad/sec, $\Omega_3 = 1.186$ rad/sec which is well matched with Fig. 5.18(a). The unstable response (*U*) and chaotic responses (*CR*) which are marked in Fig. 5.18(a) are also well matched with Fig. 5.19(c, d).

Fig. 5.20 shows the frequency response of the system with displacement feedback (f_d) gain in the passive and *active*₂ TVA. From Fig. 5.20(a-d), with the passive TVA and f_d gain the unstable responses appear at lower Ω_1 (0.0387 rad/sec to 0.0835 rad/sec) and Ω_3 (0.5263 rad/sec to 0.5879 rad/sec) ranges which are in the same trend as in figures (Figs. 5.12, 5.16, 5.17, 5.18). The f_d gain in the passive TVA shows comparatively higher responses than

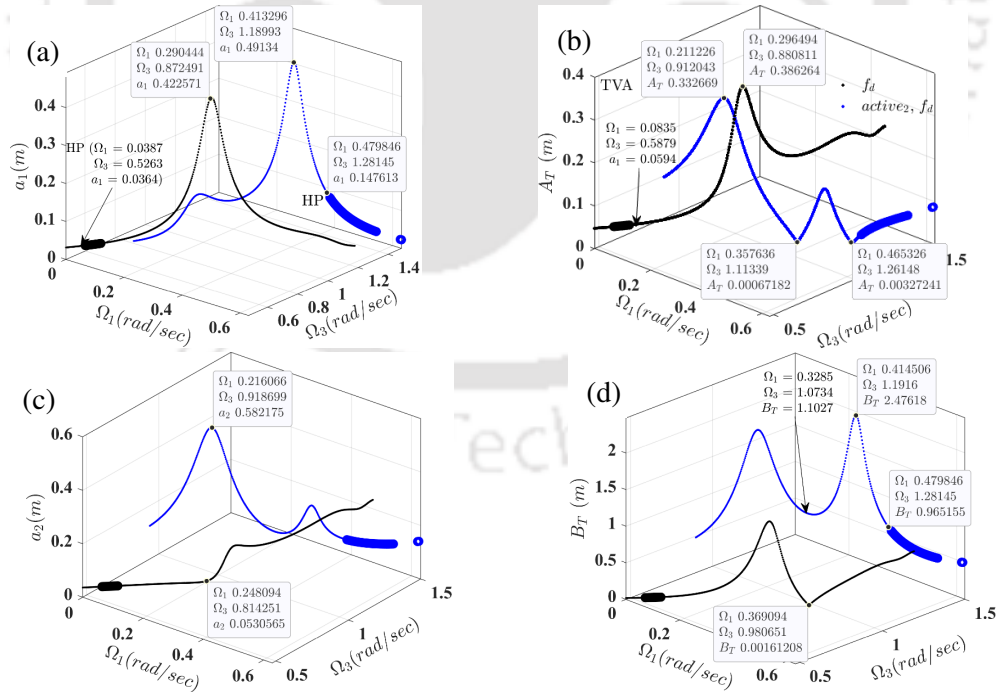


Fig. 5.20: (a-d) Frequency responses of the system considering nonlinear stiffness in the primary system by the displacement feedback gain (f_d).

the f_d gain in the *active*₂ TVA for a broader range of Ω_1 and Ω_3 values. The f_d gain in the passive TVA is found to be more effective than f_d gain in the *active*₂ TVA for lower frequencies range ($0.0835 \text{ rad/sec} \leq \Omega_1 \leq 0.2614 \text{ rad/sec}$ and $0.08325 \text{ rad/sec} \leq \Omega_3 \leq 0.5879 \text{ rad/sec}$). Fig. 5.20 (b) shows at $\Omega_1 = 0.3576 \text{ rad/sec}$ and $\Omega_3 = 1.1133 \text{ rad/sec}$ and at $\Omega_1 = 0.4653 \text{ rad/sec}$ and $\Omega_3 = 1.2614 \text{ rad/sec}$ the A_r value is almost zero, which suggest very low response in the primary system. The A_r value is found to be smaller with the f_d gain in the *active*₂ TVA in comparison with the f_d gain in the passive TVA for a broader range of Ω_1 and Ω_3 values ($0.2614 \text{ rad/sec} \leq \Omega_1 \leq 0.4798 \text{ rad/sec}$ and $0.5879 \text{ rad/sec} \leq \Omega_3 \leq 1.2814 \text{ rad/sec}$). It may be noted that these frequencies range are within the superharmonic ($\Omega_1 = 0.291 \text{ rad/sec}$) and primary resonance ($\Omega_3 = 0.873 \text{ rad/sec}$) conditions. However, the A_r value with the f_d gain in the *active*₂ TVA shows comparatively high peaks and comparatively larger responses for a wider range of Ω_1 and Ω_3 values ($0 \text{ rad/sec} \leq \Omega_1 \leq 0.38 \text{ rad/sec}$ and $0.7473 \text{ rad/sec} \leq \Omega_3 \leq 1.14 \text{ rad/sec}$ (Fig. 5.18), $0.158 \text{ rad/sec} \leq \Omega_1 \leq 0.3297 \text{ rad/sec}$ and $0.838 \text{ rad/sec} \leq \Omega_3 \leq 0.952 \text{ rad/sec}$ (Fig. 5.17)) than with f_v and f_{va} gains shown in Figs. 5.18, and 5.17. The frequency responses of the system with f_d gain in the *active*₂ TVA show similar responses as in Fig. 5.12 and Fig. 5.16, with a smaller range of Ω_1 and Ω_3 at which the system shows unstable responses than in Figs 5.17 and 5.18. Figs. 5.20(c) and 5.20(d) show that with f_d gain in the *active*₂ TVA the a_2 and B_r values are higher than that with f_d gain in the passive TVA. The B_r value is found to be almost zero at Ω_1 and Ω_3 equals to 0.369 rad/sec and 0.98 rad/sec , respectively. In the next analysis, the combination feedbacks (f_{c1}) in the passive *active*₂ TVA are used to study the frequency response of the system.

In Fig. 5.21(a-d) the effect of combination feedbacks (f_{c1}) by the passive and the *active*₂ TVA on the system response is studied. From Fig. 5.21(a-d) the response amplitudes of the system are found to be similar to Fig. 5.18(a-d), but here 15% to 30% lesser A_r values are observed for Ω_1 in the range of 0.18 rad/sec to 0.32 rad/sec and for Ω_3 in the range of 0.94 rad/sec to 1.08 rad/sec . The response amplitudes of the primary system (A_r) with f_{c1} and *active*₂ TVA are found to be better than the other combination of feedback as discussed in

Figs. 5.20, 5.18, 5.17 and 5.12, within the above-mentioned frequencies range. However, with f_{c1} feedback and *active*₂ TVA, the response amplitude of the system becomes unstable outside the above range of Ω_1 and Ω_3 values, which can be minimized by applying f_{c1} gain in the passive TVA to suppress the responses for a wider range of operating frequencies. From Fig. 5.21, the *HB*, and *SN* bifurcation points are observed with the f_{c1} gain in the *active*₂ TVA which are marked in Fig. 5.21(b). The *HB* point with the f_{c1} gain in the *active*₂ TVA is observed at $\Omega_1 = 0.4428$ rad/sec and $\Omega_3 = 0.1207$ rad/sec and also at $\Omega_1 = 0.4553$ rad/sec and $\Omega_3 = 0.122$ rad/sec. The *SN* bifurcation point is observed at $\Omega_1 = 0.4453$ rad/sec and $\Omega_3 = 0.1210$ rad/sec. The *HB* point with the f_{c1} gain in the passive TVA is observed at $\Omega_1 = 0.128$ rad/sec and $\Omega_3 = 0.711$ rad/sec. From Fig. 5.21(d) the minimum response amplitude of the absorber (B_T) is observed at $\Omega_1 = 0.3681$ rad/sec and $\Omega_3 = 0.9501$ rad/sec. From Fig. 5.21(c, d) one can observe that the B_T value with *active*₂ TVA and f_{c1} gain are found to be more than that with f_{c1} gain in the passive TVA. In the

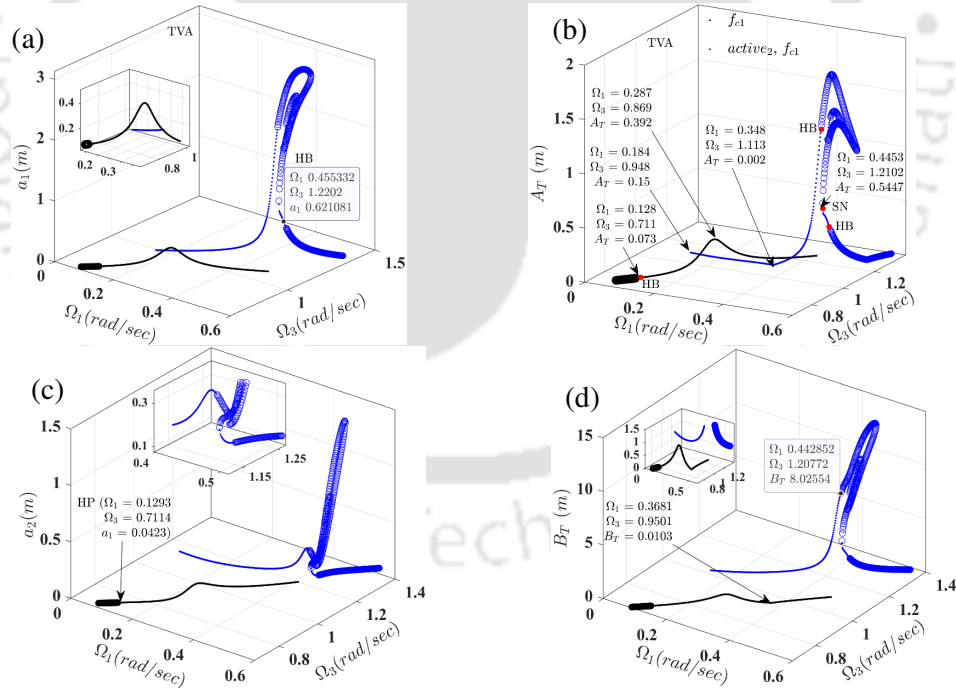


Fig. 5.21: (a-d) Comparison of frequency responses of the system considering nonlinear stiffness in the primary system by the combination of active displacement, velocity and acceleration feedback gains (f_{c1}).

next sub-section time delay effect in the damper and also in the feedback gains on the frequency response of the system is studied.

Nonlinear spring in the primary system with time delay effect

The time delay effect in the damping (τ_1) of the absorber is also an important aspect, of studying many structural systems [221-230]. In Fig. 5.22(a-d) effects of τ_1 on the system response amplitude are studied considering passive and the *active*₂ TVA. From Fig. 5.22(a-d) two peaks in the response amplitude can be observed with the *active*₂ TVA than the passive TVA. Also, 100% suppression in the response amplitude of the primary system is observed with *active*₂ TVA for different τ_1 values than that with the passive TVA. From the two values of τ_1 i.e. 0.01π sec and 0.1π sec considered in these figures, it can be observed that the higher τ_1 values in the damping produce higher response amplitude in the system for both the active and passive TVA. The HB points with *active*₂ TVA for $\tau_1 = 0.01\pi$ sec and 0.1π sec are observed at $\Omega_1 \cong 0.4568$ rad/sec and $\Omega_3 \cong 1.2217$ rad/sec.

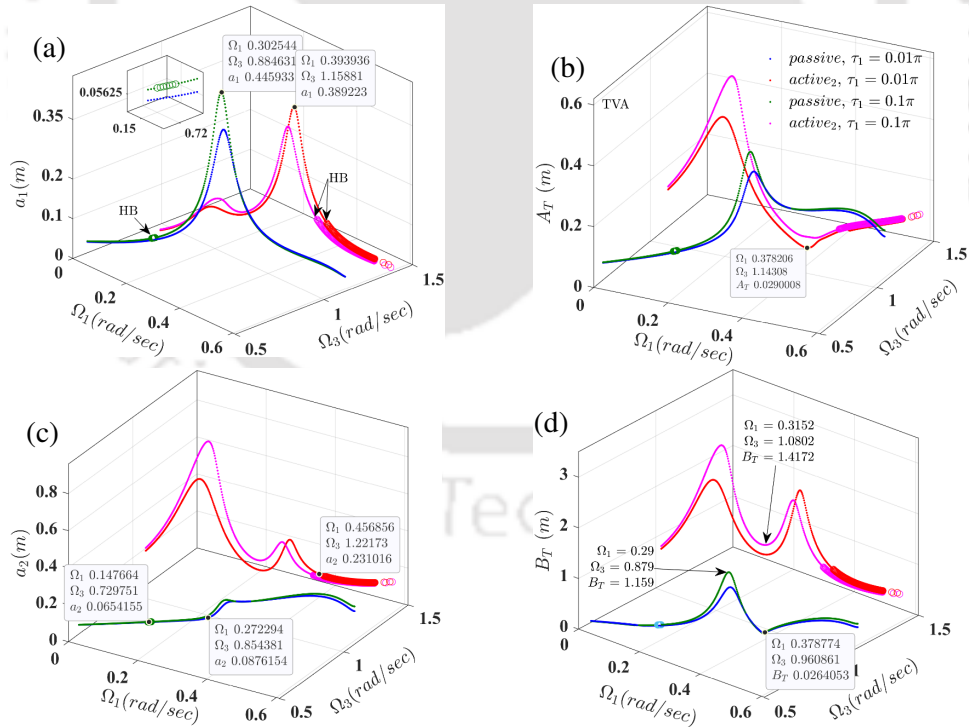


Fig. 5.22: (a-d) Time delay effect (τ_1) on the frequency responses of the system considering nonlinear stiffness in the primary system by the passive and the *active*₂ TVA.

The HB points with passive TVA for $\tau_1 = 0.1\pi$ sec are observed at $\Omega_1 \cong 0.1476$ rad/sec and $\Omega_3 \cong 0.7297$ rad/sec, while the system shows no unstable response for $\tau_1 = 0.1\pi$ sec with the passive TVA.

The time delays in the feedbacks (τ_2) are also a common phenomenon during the time lag between the sensing and actuating by the actuator through the feedbacks [138-1141, 221-230]. In Fig. 5.23(a-d) considering four cases, the effects of f_v and f_a feedback gains with $\tau_2 = 0.1\pi$ sec on the frequency response of the system are studied by using active and $active_2$ TVA. From these figures, it can be observed that with f_a feedback gains and $active_2$ TVA the response amplitude of the primary system is completely suppressed at $\Omega_1 = 0.36$ rad/sec and $\Omega_3 = 1.11$ rad/sec, which is not observed for other feedback gains with $\tau_2 = 0.1\pi$ sec. Also, the response amplitude of the primary system is minimum at the superharmonic and primary resonance conditions with the f_a feedback gain and $active_2$ TVA than with only f_v feedback gain. The time-delayed f_v gain by the TVA produces

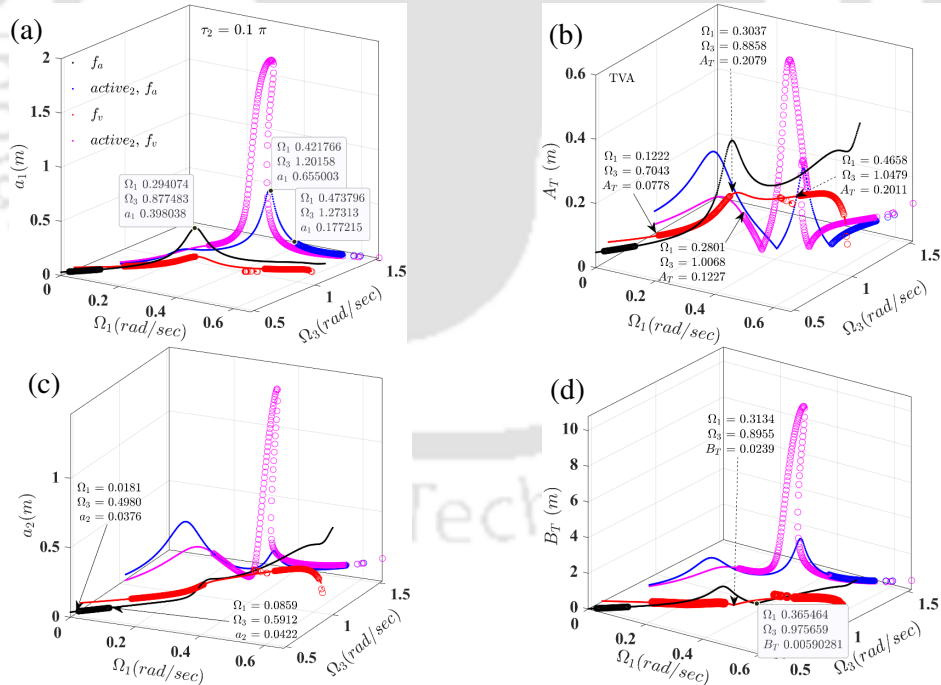


Fig. 5.23: (a-d) Comparison of frequency responses of the system considering nonlinear stiffness in the primary system with f_v and f_a with time delay (τ_2) in the feedback.

more range of unstable responses in the system compared to the time-delayed f_a feedback gain. Hence, from this figure, it can be inferred that for $\tau_2 = 0.1 \pi$ sec in the feedback the response amplitude shows better vibration suppression with f_a feedback gain and $active_2$ TVA than with f_v gain.

In Fig. 5.24 (a-d) keeping τ_2 value same as in Fig. 5.23 the combination of f_v and f_a feedback gains i.e. f_{va} or f_{av} feedback gains on the frequency response of the system is studied with active and $active_2$ TVA. The f_{va} or f_{av} feedback gains stand for negative and positive acceleration feedback gains. From Fig. 5.24 it can be observed that the response amplitude of the system is unstable with these combinations of feedback forces for a broader range of frequencies. The f_{va} feedback force by the active TVA shows comparatively less response amplitude and a more range of stable zone of operation than f_{av} with active or $active_2$ TVA. Comparing the frequency responses from Fig. 5.22 to Fig. 5.24 where variation in time delays τ_1 and τ_2 are considered, it can be observed the responses of the primary system are minimum for a wide range of operating frequencies

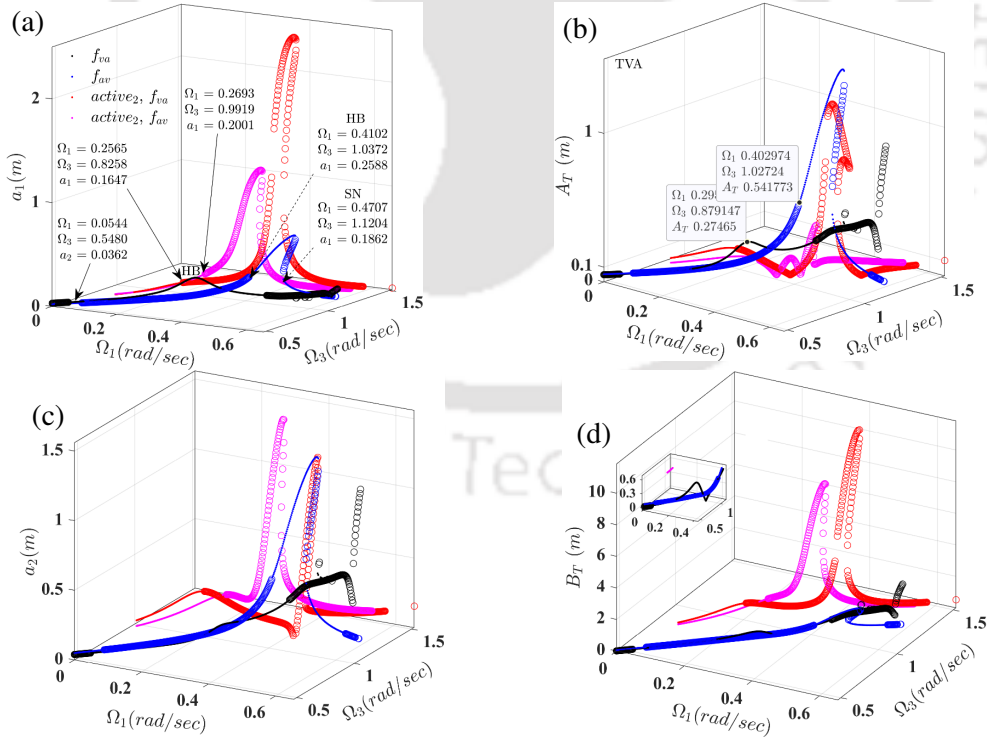


Fig. 5.24: (a-d) Comparison of frequency responses of the system considering nonlinear stiffness in the primary system with time delay (τ_2), f_{va} and f_{av} gains.

with f_a gain and *active*₂ TVA than with other combinations.

The time delays both in damping (τ_1) and in the feedback (τ_2) are a very common phenomenon in practical applications [6, 11, 15, 221-225] and its effect on the frequency response of the system is studied in Fig. 5.25(a-d) for f_{va} feedback gain. From Fig. 5.25 it can be observed that consideration of time delays τ_1 and τ_2 in the damping as well as in the feedbacks increase the unstable region than without both delays as shown in previous figures. From Fig. 5.25(a, b) it can be observed that the response amplitude of the system shows Hopf bifurcation points at $\Omega_1 = 0.06$ rad/sec and $\Omega_3 = 0.38$ rad/sec with f_{va} feedback gain. The system response is unstable within this frequency range of operation. However, considering f_{va} and changing the stiffness of the absorber actively with *active*₂ TVA, one can observe that the response amplitude of the primary system is minimum and stable. Hence one has to change the frequency of the absorber as well as with the application of f_{va} feedback gain one can suppress the response amplitude of the primary system when the time delay is present both in the damping and feedback. The response amplitude of the

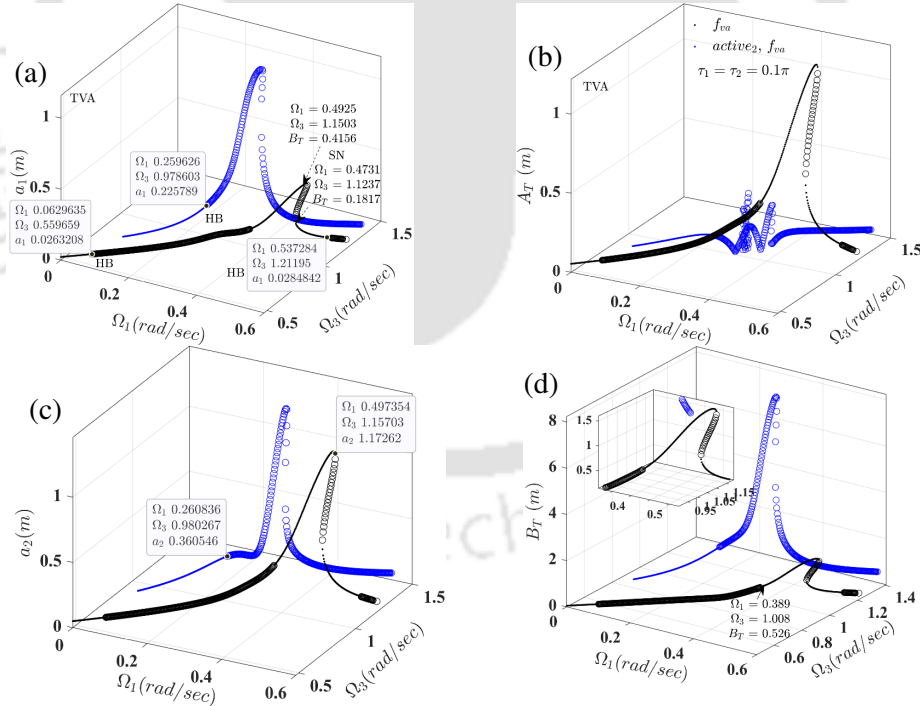


Fig. 5.25: (a-d) Frequency responses of the system considering nonlinear stiffness in the primary system with time delay (τ_1) in the damping and in the (τ_2) f_{va} feedback gain.

absorber is shown in Fig. 5.25(c, d), from which it can be observed that the response amplitude of the absorber is around five times higher than the response amplitude of the primary system within the above-mentioned frequency range with f_{va} feedback gain and with *active*₂ TVA. In the next section frequency response of the system is studied considering nonlinear spring stiffness in both the primary system and the absorber.

Nonlinear spring in both the primary and secondary system

In this section, nonlinear stiffness in both the primary system and the absorber are considered i.e., $\bar{k}_{13} = 1 \text{ N/m}^3$ and $\bar{k}_{23} = 0.0042 \text{ N/m}^3$ and the corresponding frequency responses are shown in Fig. 5.26. From Fig. 5.26(a-d) it can be observed that the delay in damping (τ_1) produces a higher response amplitude for the passive TVA while with *active*₂ TVA the response amplitude of the primary system (Fig. 5.26b) is zero at $\Omega_1 = 0.4 \text{ rad/sec}$ and $\Omega_3 = 1.17 \text{ rad/sec}$. Hence complete absorption of the response of the system is observed even by considering nonlinear spring in both the primary and secondary systems. However, with the *active*₂ TVA high peaks in the response amplitudes of the primary system are observed for Ω_1 in the range of 0.22 rad/sec to 0.29 rad/sec and Ω_3 in the range of 0.94

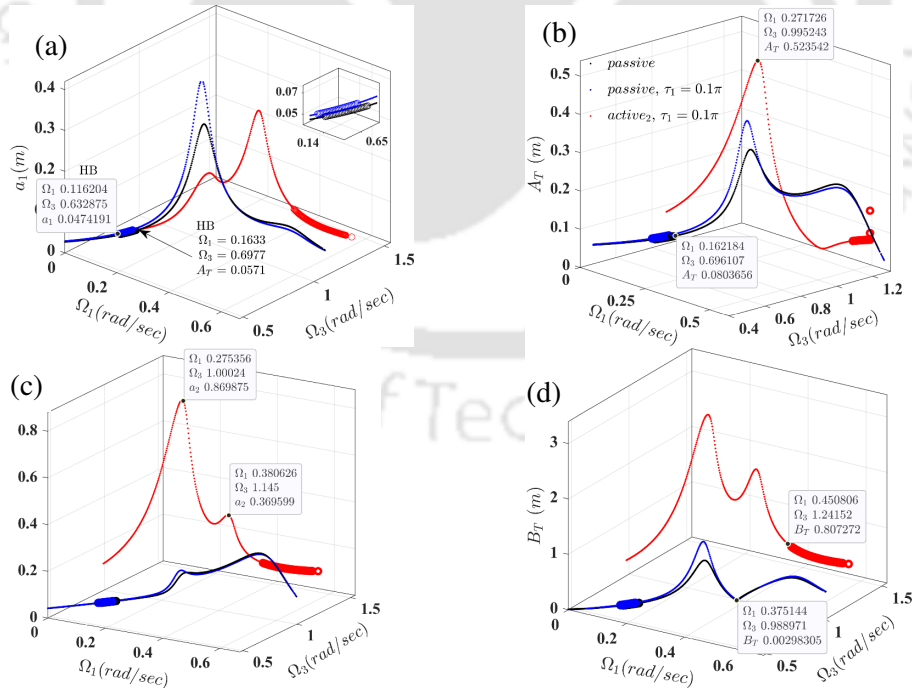


Fig. 5.26: (a-d) Frequency responses of the nonlinear system with time delay (τ_1) in the damping considering the passive and *active*₂ TVA.

rad/sec to 1.02 rad/sec than that with the passive TVA. The response amplitude of the passive TVA shown in Fig. 5.26(c, d) is found to be less than the *active*₂ TVA for all operating frequencies and zero response amplitude is noticed at $\Omega_1 = 0.37$ rad/sec and $\Omega_3 = 0.98$ rad/sec. In Fig. 5.27(a-d) the comparison of various τ_1 values on the frequency response of the system are studied considering *active*₂ TVA. From these figures, it can be observed that with increasing τ_1 values the response amplitude of the system also increases. The increasing τ_1 values also produces more hardening behaviour in the frequency response with jump up ($\Omega_1 = 0.31$ rad/sec and $\Omega_3 = 0.92$ rad/sec) and jump down ($\Omega_1 = 0.32$ rad/sec and $\Omega_3 = 0.93$ rad/sec) phenomena at saddle-node bifurcation points. The response of the absorber (Fig. 5.27(c, d)) shows more prominent two peaks

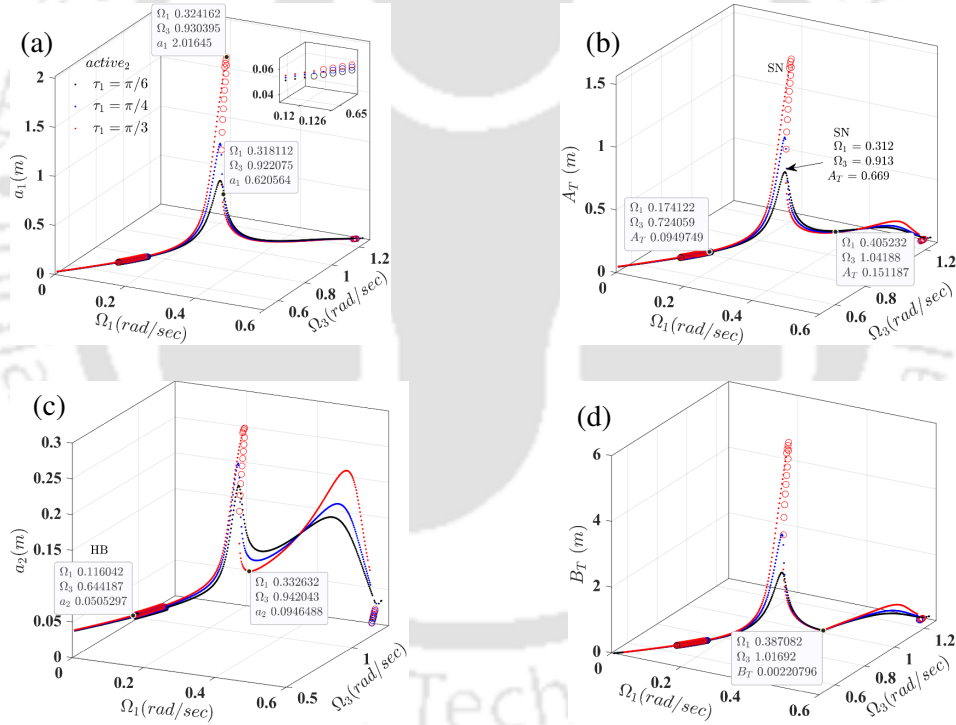


Fig. 5.27: (a-d) Frequency responses of the nonlinear system with various time delays (τ_1) in the damping with the *active*₂ TVA.

unlike primary system (Fig. 5.27(a, b)) and zero response amplitude at $\Omega_1 = 0.38$ rad/sec and $\Omega_3 = 1.01$ rad/sec. In Fig. 5.28(a-d) the frequency response of the system is studied using combination feedbacks (f_{c1} and f_{c2}) considering $\tau_2 = 0.05 \pi$ sec. It may be noted

that in f_{c1} and f_{c2} feedback gains are developed considering the negative and positive acceleration feedback (f_a), respectively. In these figures, the frequency of the absorber is actively changed to match exactly the natural frequency of the primary system, which is denoted as *active₁* TVA (zero feedback gains). From these figures, it can be observed that with f_{c2} gain the frequency response shows more hardening behavior and instability zone than with other combinations. The frequency response of the system is stable for a wider band of operating frequencies with *active₂* TVA than with other combinations. However, with *active₁* TVA and f_{c1} gain the system response is less than with *active₁* TVA for Ω_1 in the range of 0.31 rad/sec to 0.51 rad/sec and Ω_3 in the range of 0.91 rad/sec to 1.19 rad/sec, which is close to the resonance conditions considered in this section.

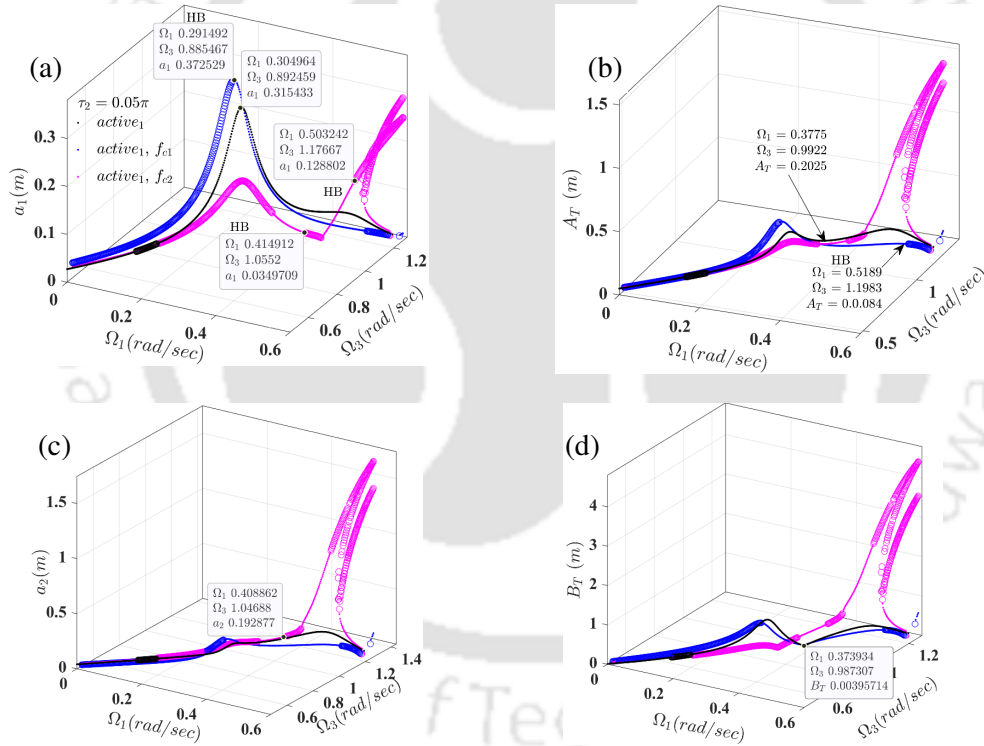


Fig. 5.28: (a-d) Frequency responses of the nonlinear system with various time delays (τ_2) in the combination feedbacks (f_{c1} and f_{c2}) and the *active₁* TVA.

In Fig. 5.29(a-d) the frequency response of the system is studied for a combination of f_v and f_a feedback and f_{c1} gain with $\tau_2 = 0.05\pi$ sec for the *active₂* TVA. Comparing Fig. 5.29(a-d) with Fig. 5.28(a-d) it can be observed that the instability region of operating

frequencies with f_{c1} gain and *active*₂ TVA is considerably reduced than *active*₁ TVA (Fig. 5.28(a-d)). Also, the response amplitude of the primary system is almost zero at $\Omega_1 = 0.34$ rad/sec and $\Omega_3 = 1.09$ rad/sec. Further, at the simultaneous superharmonic and primary resonance conditions, the response amplitude of the primary system is less than in other combinations as shown in Fig. 5.28(a-d) and Fig. 5.29(a-d). The saddle-node bifurcation points in frequency responses are marked in Fig. 5.29(d) with f_{va} gain. These bifurcation points with f_{va} gain are at $\Omega_1 = 0.4471$ rad/sec and $\Omega_3 = 1.2365$ rad/sec and at $\Omega_1 = 0.4544$ rad/sec and $\Omega_3 = 1.2465$ rad/sec. The A_r value with f_{c1} gain shows a comparatively lower amplitude response than f_{va} gain.

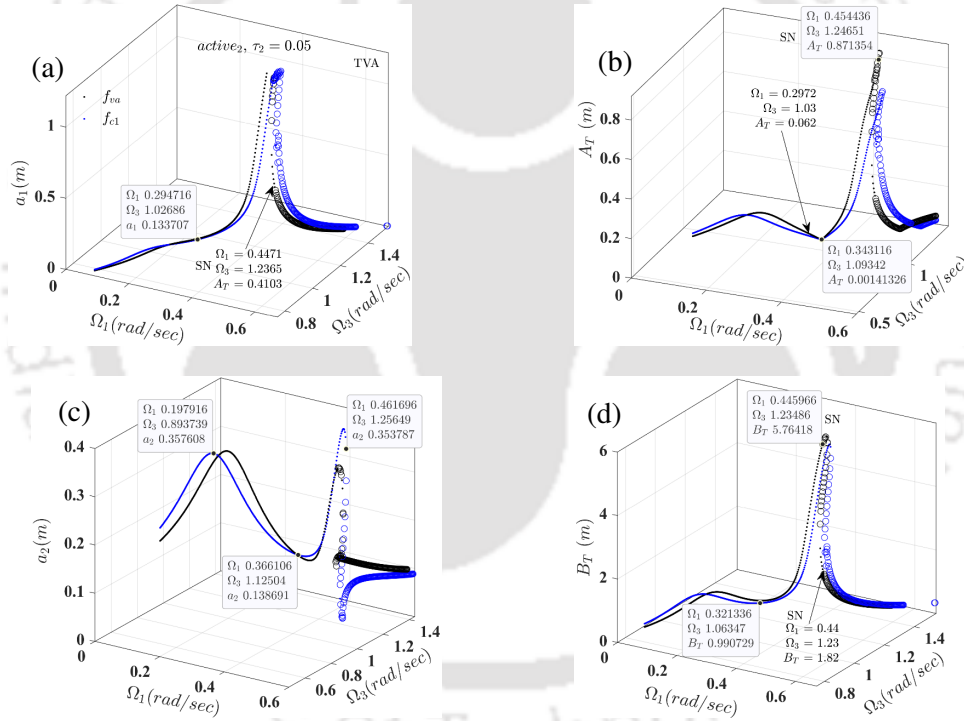


Fig. 5.29: (a-d) Frequency responses of the nonlinear system with various time delays (τ_2) in the combination feedbacks (f_{va} and f_{c1}) and the *active*₂ TVA.

In Fig. 5.30(a-d) the frequency responses of the system are studied by considering the time delay both in the damping as well as in the feedback i.e. $\tau_1 = \tau_2 = 0.1 \pi$ sec with various combinations of feedback gains viz., f_{av} , f_{va} and f_{c1} with the *active*₂ TVA. From Fig. 5.30(a-d) it can be observed that with f_{va} or f_{av} a wider range of unstable responses in the

system are observed as compared to the f_{c1} . The f_{va} feedback gain performs better in suppressing the response amplitude of the primary system for a very small range of Ω_1 (0.08 rad/sec to 0.26 rad/sec) and Ω_3 (0.72 rad/sec to 0.99 rad/sec) as observed from these figures. The absorber response f_{c1} feedback gain shown in Fig. 5.30(c, d) shows saddle-node bifurcation frequency points at Ω_1 equals to 0.49 rad/sec and 0.58 rad/sec, which leads to jump up and jump down phenomena. The time delay in the system introduces more range of instability regions in the frequency response as observed from these figures. The effect of f_{c1} feedback gain on the frequency responses of the system for various other time delays in the damping and the feedbacks are studied in Fig. 5.31. From Fig. 5.31(a-d) it can be observed that with an increase

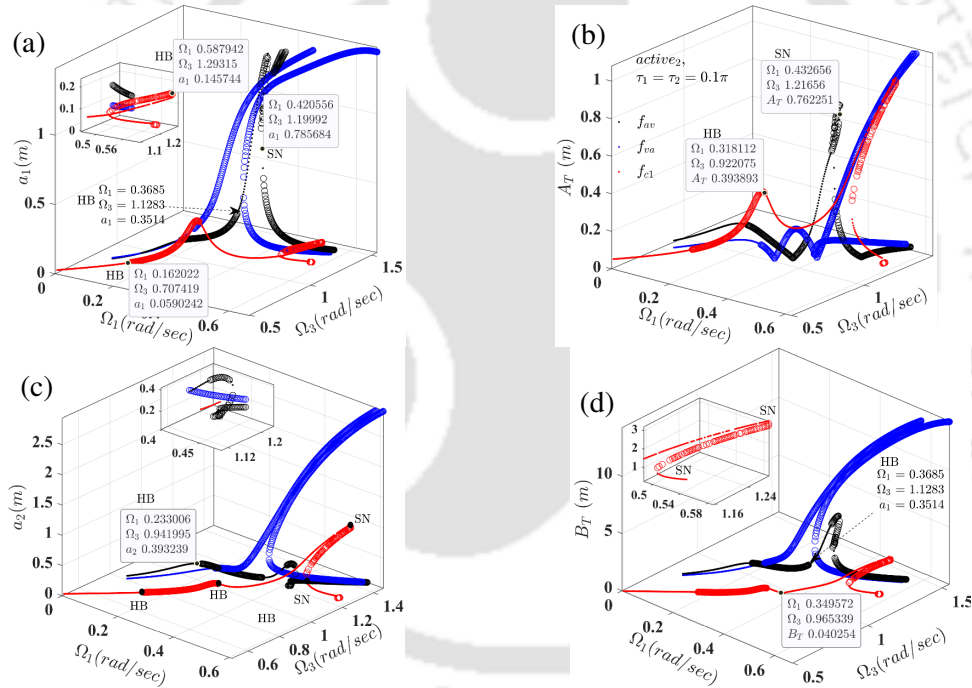


Fig. 5.30: (a-d) Frequency responses of the nonlinear system with time delays (τ_1 and τ_2) in the combination feedbacks (f_{va} , f_{va} and f_{c1}) with the $active_2$ TVA.

in τ_1 and τ_2 values the response amplitude of the system increases at the first peak. The frequency response of the system remains almost the same with different τ_1 and τ_2 values for Ω_1 in the range of 0.35 rad/sec to 0.54 rad/sec and Ω_3 in the range of 0.96 rad/sec to 1.22 rad/sec. However, in these figures (Fig. 5.31) f_{c1} feedback gain hardening effect in

the frequency response is observed with stable and unstable branches. From the above parametric study on the frequency responses of the system (Fig. 5.6 to Fig. 5.31) for various nonlinear stiffness in the system, various combinations of feedbacks, feedback gains, time delays in the damping and feedbacks, one can get a clear idea to use the different combination of feedback forces, time delay in the damping and time delay in the feedback, at which the response of the system is minimum and stable. It can be observed that with the feedback gains the system shows a broader range of unstable zone which can be minimized by providing zero feedback gains and only changing the frequency of the absorber actively (*active₂* TVA). It may be noted that through the active use of the vibration absorber by the displacement, velocity and acceleration feedbacks the dynamics of the system are changing actively in terms of stiffness, damping and inertia. So, with the proper use of these feedback gains as mentioned in (Fig. 5.6

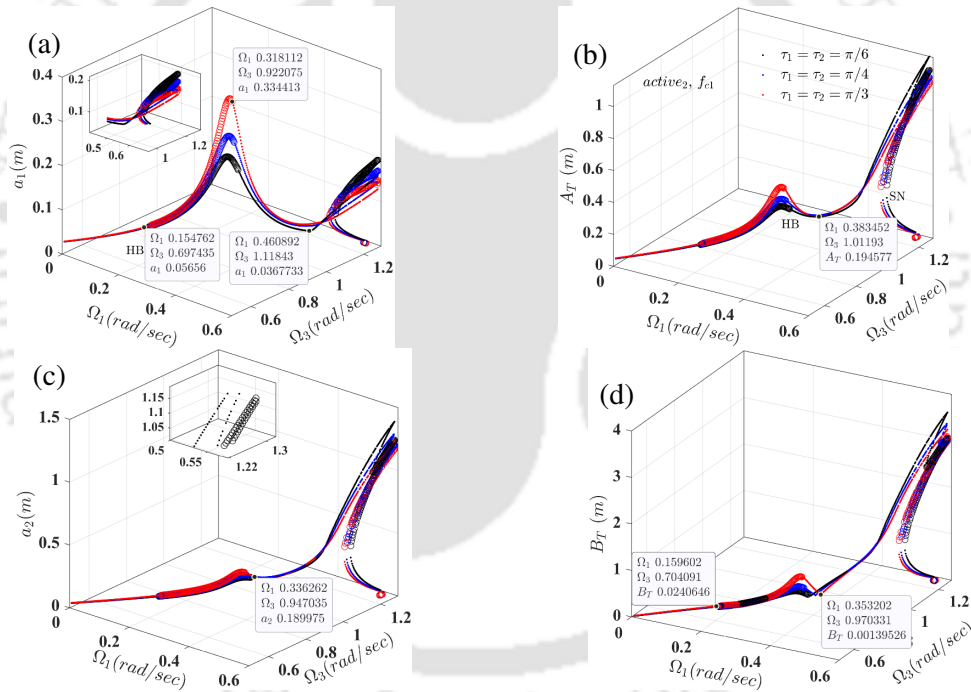


Fig. 5.31: (a-d) Frequency responses of the nonlinear system with variation in time delays (τ_1 and τ_2) in the combination feedbacks (f_{c1}) with the *active₂* TVA.

to Fig. 5.31) one can reduce the response of both the primary system and the secondary system for a wide range of operating frequencies at the simultaneous resonance conditions. In the next analysis frequency of the system with the non-traditional vibration absorber is studied, where the damper of the absorber is connected to the base of the primary system. In Fig. 5.32(a-d) the frequency response of the system is shown by using the non-traditional

vibration absorber (NVA) considering both linear and nonlinear stiffness in the system. From Fig. 5.32(a-d) it can be observed that for the linear system the f_{cl} gain by the active NVA produces more unstable responses in the system than the passive NVA. However, at the frequency range $(0.29 \text{ rad/sec} \leq \Omega_1 \leq 0.42 \text{ rad/sec}$ and $0.86 \text{ rad/sec} \leq \Omega_3 \leq 1.06 \text{ rad/sec})$ as shown in Fig. 26(a, b) the response amplitude of the primary system is 19% to 46% less in comparison with the passive NVA. The maximum peaks in the primary system response by the passive NVA (Fig. 5.32(a, b)) also produce 35% less response amplitude than the passive TVA shown in Fig. 5.10(a, b). The consideration of nonlinear stiffness in the primary system and the absorber produces less response amplitude in the system by the nonlinear passive NVA than the linear passive NVA. However, the nonlinearity in the

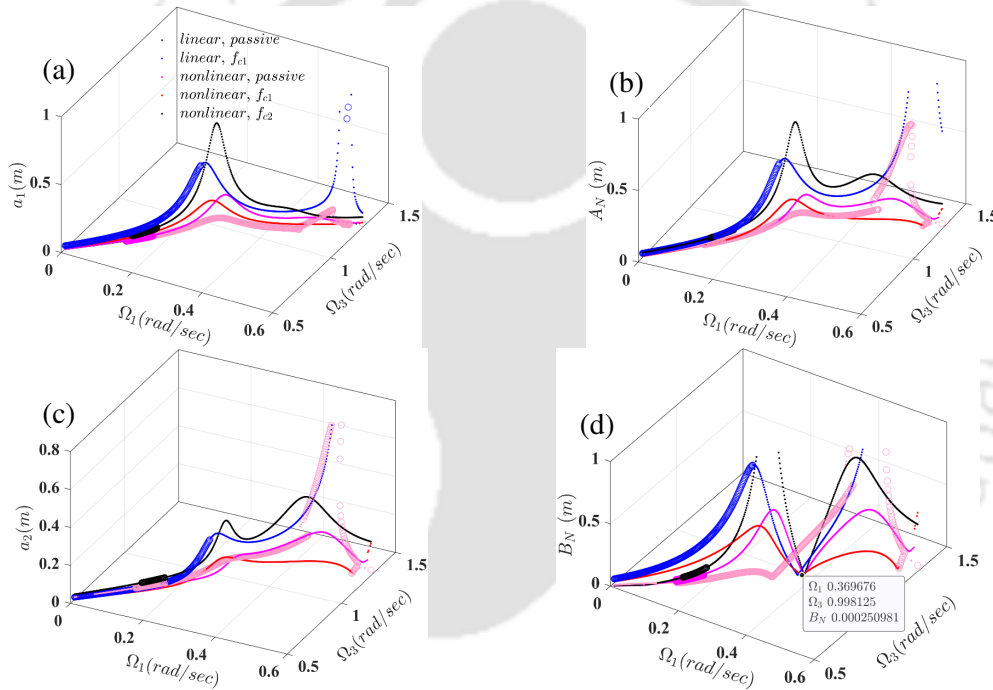


Fig. 5.32. (a-d) Frequency responses of the linear and nonlinear system by the passive and combination feedback (f_{cl}) gain using NVA.

system produces unstable responses as shown in Fig. 5.32(a-d). A reduction of 54% in the response amplitude of the primary system is observed at the peak by the passive nonlinear NVA than the passive linear NVA. The f_{cl} force in the nonlinear NVA further reduces the response amplitude to 62% at the peak than passive linear NVA. The zero response amplitude of the NVA is observed at Ω_1 and Ω_3 close to 0.37 and 0.99 rad/sec, respectively. From the above study, the designer can know the responses of the system

according to various parametric analyses (variation in nonlinear stiffness, various combinations of feedback gains, time delays in damping and feedbacks) for both linear and nonlinear systems to effectively suppress the vibration of the system for a wider band of operating frequencies for the simultaneous superharmonic, principal parametric and primary resonance conditions with 1:1 internal resonance. In the next section frequency response of the system is studied considering 3:1 internal resonance conditions for the simultaneous superharmonic, principal parametric and primary resonance conditions.

5.4.2.2 3:1 internal resonance condition

In this section, the frequency response of the system is studied considering 3:1 internal resonance conditions by solving Eqs. (5.16a) to (5.16d) by using Newton's method. It may be noted that here $\omega_1 \cong 3\omega_2$. The 3:1 internal resonance condition in the system is achieved by taking the stiffness of the absorber k_2 to 0.4032N/m, which can be actively changed. The corresponding optimal parameters of the nonlinear stiffness and damping in the absorber are also calculated from the work of Habib et al. [161] which are given as $\bar{k}_{23} = 0.037 \text{ N/m}^3$ and $c_2 = 0.0374 \text{ N-sec/m}$. The corresponding ω_1 and ω_2 values with 3:1 internal resonance condition are found to be 2.9189 rad/sec and 0.9728 rad/sec, respectively. Considering the above k_2 and k_{23} values the frequency response of the nonlinear system is studied keeping all other parameters the same as before. The active changes of the k_2 value is represented as *active₃* TVA. From Fig. 5.33(a-d) it can be observed that the system responses show high response amplitude for $0.94 \text{ rad/sec} \leq \Omega_1 \leq 1.04 \text{ rad/sec}$ and $2.86 \text{ rad/sec} \leq \Omega_3 \leq 3.12 \text{ rad/sec}$, i.e. when $\Omega_1 \cong \omega_2$, as the term Λ_2 is given in Eq. (5.8a) and Eq. (5.8b), $\left(\Lambda_2 = \frac{p_{12}F_1}{2(\omega_2^2 - \Omega_1^2)} \right)$, tends to infinity. It may be noted that here Ω_1 varies from $(\omega_1/3) - 0.1\omega_1$ to $(\omega_1/3) + 0.1\omega_1$ (i.e., 0.67 rad/sec to 1.27 rad/sec), considering superharmonic resonance conditions. So, when Ω_1 is close to ω_2 i.e. 0.9728 rad/sec, then a very high response amplitude in the system is observed due to the large value of Λ_2 . From Fig. 5.33(a-d) it can also be observed that with zero control gain and with *active₃* TVA the response amplitude of the system remains stable than with other combination of feedback gains and time delays. The effects of control feedback gains such

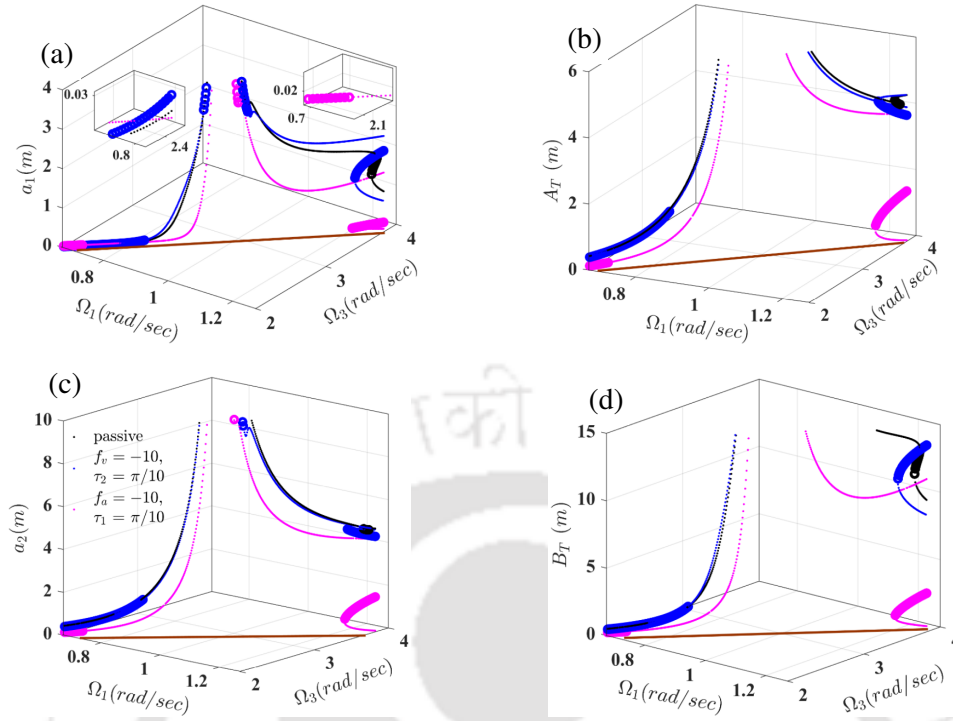


Fig. 5.33: (a-d) Frequency responses of the nonlinear system for 3:1 internal resonance condition by the passive and active TVA with time delays (τ_1 and τ_2).

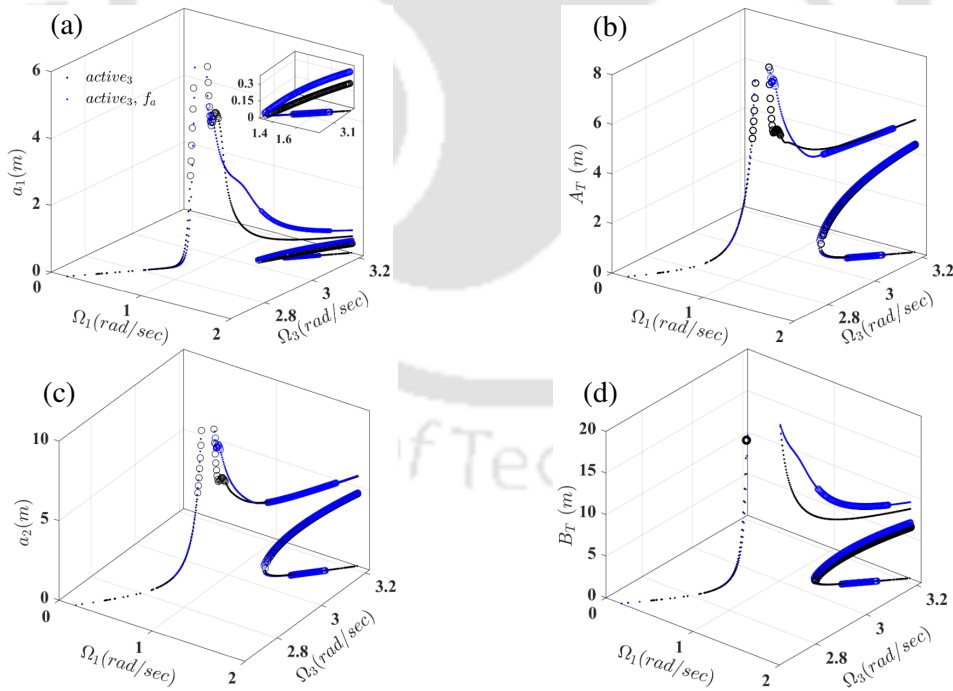


Fig. 5.34: (a-d) Frequency responses of the nonlinear system for 3:1 internal resonance condition by the *active₃* TVA and f_a gain.

as f_v or f_a and time delay in the damping or in the feedback induce a more unstable response in the system as observed from Fig. 5.33(a-d). However, the f_a gain with τ_1 by the active TVA produces less response amplitude than the passive TVA for $0.71 \text{ rad/sec} \leq \Omega_1 \leq 0.96 \text{ rad/sec}$ and $0.99 \text{ rad/sec} \leq \Omega_3 \leq 1.24 \text{ rad/sec}$. In Fig. 5.34 the response amplitude of the system is shown considering *active3* TVA and lower nonlinear stiffness in the absorber compared to Fig. 5.33, which is the same as in the 1:1 internal resonance condition i.e., $\bar{k}_{23} = 0.0042 \text{ N/m}^3$. From Fig. 5.34(a-d) the zero-response amplitude of the system is found for Ω_1 in the range of 0.18 rad/sec to 0.88 rad/sec for both *active3* TVA and *active3* TVA with f_a the feedback gain. The response amplitude of the system is

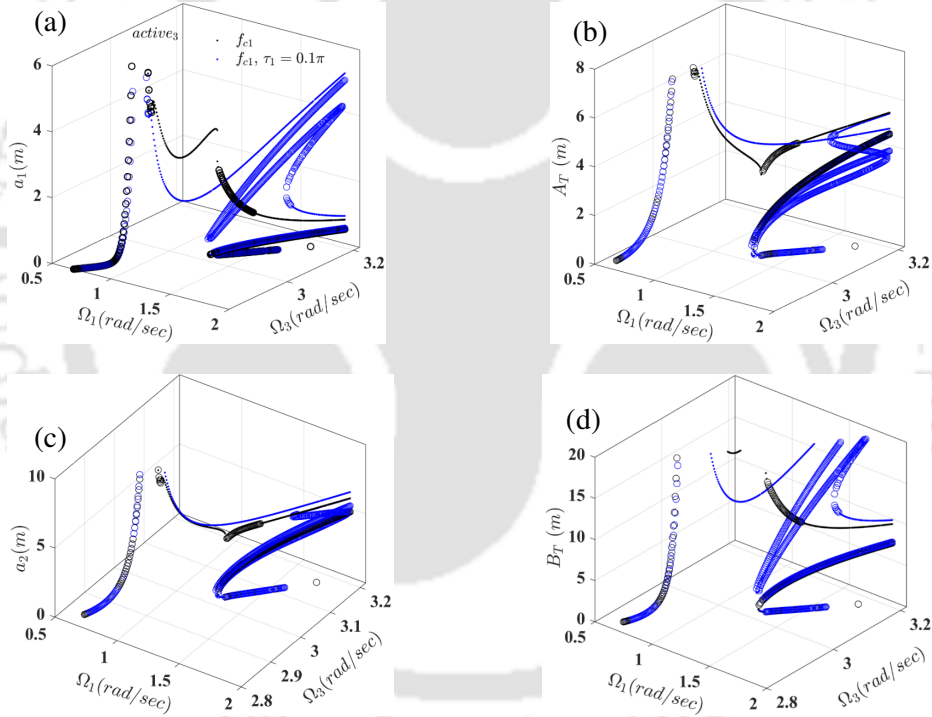


Fig. 5.35: (a-d) Frequency responses of the nonlinear system for 3:1 internal resonance condition by the *active3* TVA and f_{cl} gain with τ_1 time delay.

comparatively less than that of Fig. 5.33(a-d), where higher nonlinear stiffness in the absorber is considered. In Fig. 5.35(a-d) the response amplitude of the system is studied considering *active3* TVA and f_{cl} force with time delay in the damping. From these figures, it can be observed that with f_{cl} gain and τ_1 the response amplitude and the range of

unstable the zone is more than with only f_{cl} gain. Also, the f_{cl} gain produces a wider range of unstable responses than that with *active3* TVA or *active3* TVA with f_a feedback gain (Fig. 5.35(a-d)). In Fig. 5.36(a-d) the frequency response of the system is studied by using the passive and active NVA with f_{cl} and f_a feedback gains, keeping all other parameters the same as in Fig. 5.35. From these figures (5.36(a-d)) the two peaks in the frequency response of the system are observed with the passive NVA, but also a broader range of unstable responses are observed in comparison with the active NVA. The f_a feedback gain by the NVA produces more range of stable responses than the f_{cl} gain and comparatively lesser response amplitude in the primary system than the TVA. From Fig. 5.36(a-d) the high response amplitude in the system is observed for Ω_1 in the range of 0.88 rad/sec to 1.02 rad/sec and Ω_3 in the range of 2.88 rad/sec and 2.92 rad/sec. Hence, one has

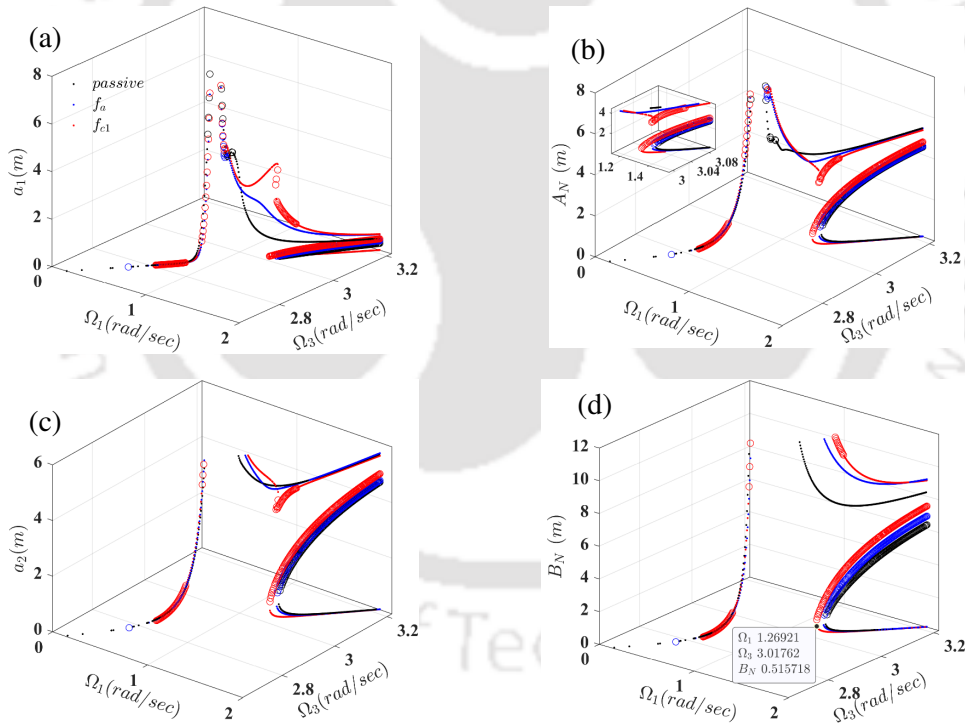


Fig. 5.36: (a-d) Frequency responses of the nonlinear system for 3:1 internal resonance condition by the passive and active NVA with f_a and f_{cl} gains.

to operate the system in a lower range ($0.32 \text{ rad/sec} < \Omega_1 < 0.64 \text{ rad/sec}$ and $2.72 \text{ rad/sec} < \Omega_3 < 2.82 \text{ rad/sec}$) of operating frequency to have minimum response amplitude in the system. In the same way, the system responses for various parameters of the nonlinear

stiffness and control feedback gains considering 1:2 internal resonance conditions can be studied from the MMS reduced equations (Eq. (5.14a) to Eq. (5.14d)). In the next section, the system time responses are studied considering 1:2 internal resonance conditions for the simultaneous superharmonic, principal parametric and primary resonance conditions.

5.4.2.3 1:2 internal resonance condition

In this section the time response of the system is obtained by solving the MMS reduced Eq. (5.14a) to Eq. (5.14b) using the numerical method. Fig. 5.37(a-d) shows time responses of the system using the passive TVA at the two neighboring operating frequencies (± 0.1 rad/sec) that lead to simultaneous superharmonic, principal parametric and primary resonance conditions. From these figures, one can observe that with the passive TVA at exact resonating frequencies the response of the system is more than slightly away (± 0.1 rad/sec) from the resonating frequencies.

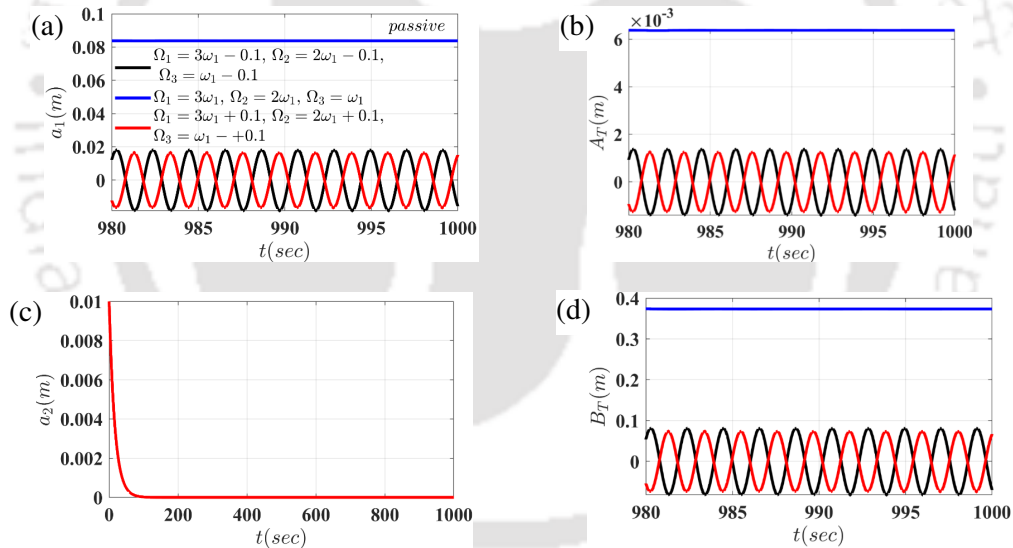


Fig. 5.37: (a-d) Time responses of the nonlinear system considering 1:2 internal resonance condition by the TVA with zero control gains.

In Fig. 5.38 the time response of the system is shown with f_{c1} force and time delays τ_1 and τ_2 . From Fig. 5.38(a-d) one can observe that the response of the system with the f_{c1} gain produces very high responses slightly away from the simultaneous superharmonic, principal parametric and primary resonance conditions. The feedback gains by the TVA make the system unstable so one has to only tune the frequencies of the TVA close to the

frequency of the primary system considering the 1:2 internal resonance condition for effectively suppressing the response of the system. It may be noted that to suppress the response of the system at the 1:2 internal resonance condition one has to actively tune the

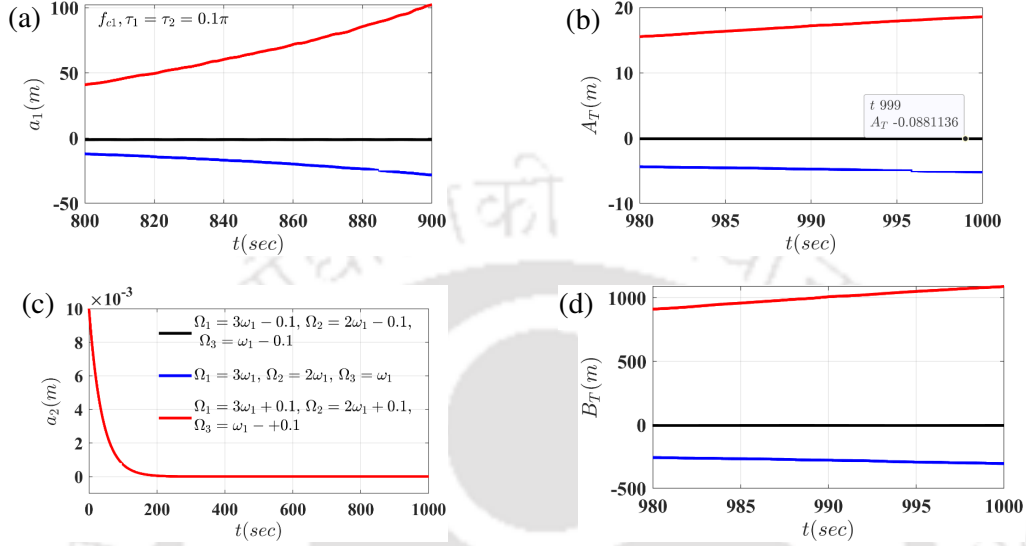


Fig. 5.38: (a-d) Time responses of the nonlinear system considering 1:2 internal resonance condition by the active TVA with f_{cl} gains.

frequency of the TVA for which the purposed design will be useful as the spring used in the TVA is connected in series with the PZT stack actuator. The spring stiffness of the absorber for the 1:2 internal resonance condition is found to be 0.0129 N/m. In the next subsection, the frequency and time responses of the system are studied considering simultaneous subharmonic, principal parametric and primary resonance conditions with 1:1, 1:3 and 1:2 internal resonance conditions from the reduced MMS equations.

5.4.3 Response of the system with TVA for simultaneous subharmonic, principal parametric and primary resonance conditions

In this sub-section, the frequency responses of the primary system and the absorber are studied for both TVA and NVA by solving the reduced equations derived in sections 5.3.1.2 by the MMS using Newton's method. The analyses are carried out to study the responses of the system for 1:1, 3:1 and 1:2 internal resonance conditions by considering various feedback gains and actively changing the stiffness of the absorber. In the following

subsection considering case (vi) with the 1:1 internal resonance condition, the responses of the system are studied.

5.4.3.1 1:1 internal resonance condition

In this subsection frequency responses of the system are studied by solving the reduced MMS Eq. (6.19a) to Eq. (6.19d) using Newton's method considering nonlinear stiffness same as in section 5.4.2.1, i.e. $\bar{k}_{13} = 1 \text{ N/m}^3$ and $\bar{k}_{23} = 0.0042 \text{ N/m}^3$. Fig. 5.39(a-d) shows the frequency response of the system for both the trivial and nontrivial solution with the passive TVA. From Fig. 5.39(a-d) it can be observed that the trivial state solution becomes unstable for Ω_1 in the range of 1.15 rad/sec to 1.65 rad/sec and the non-trivial solution shows a saddle-node bifurcation point at $\Omega_1 = 1.36 \text{ rad./sec}$. The response amplitude of the primary system shown in Fig. 5.39(b) shows Hopf bifurcation points at $\Omega_1 = 1.46 \text{ rad/sec}$. The supercritical and sub-critical pitchfork bifurcation points are observed in Fig. 5.30(a)

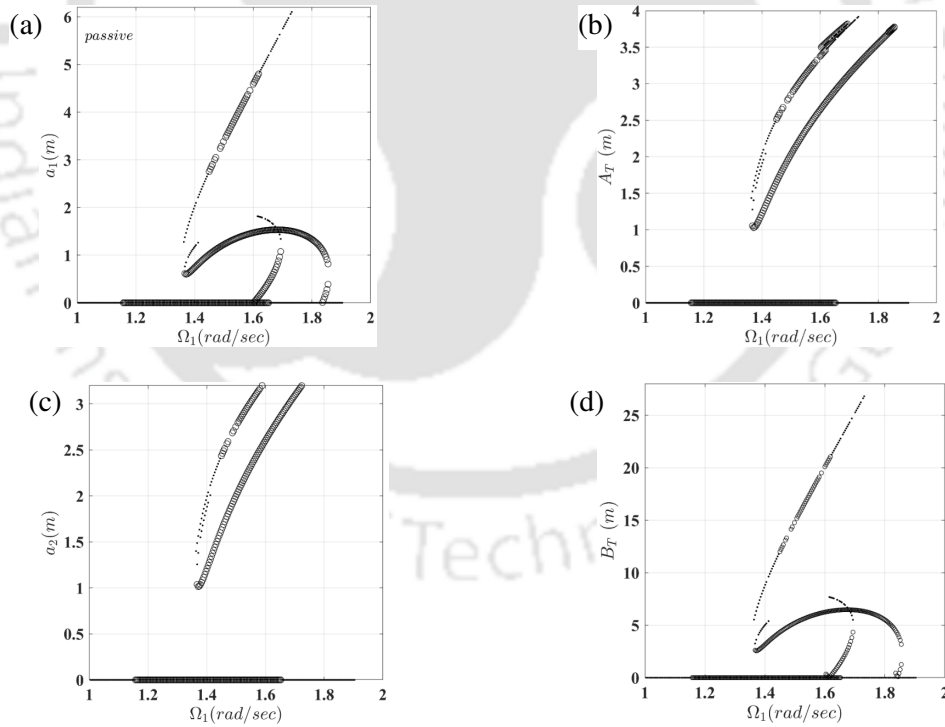


Fig. 5.39: (a-d) Frequency responses of the nonlinear system for 1:1 internal resonance condition by the TVA with zero control gains.

and Fig. 5.39(d) at $\Omega_1 = 1.82$ rad/sec during sweeping up and at $\Omega_1 = 1.65$ rad/sec during sweeping down the frequency, respectively. Fig. 5.40(a-d) shows the frequency response of the system with f_{cl} gain with all the same parameters as in Fig. 5.39. From these figures, it can be observed that unlike Fig. 5.39 in Fig. 5.40 the unstable responses reduces both in the trivial and non-trivial responses of the system. However, the magnitude of the response amplitude of the system in Fig. 5.40 remains the same as in Fig. 5.39. From Fig. 5.39 and Fig. 5.40 it can be observed that the response amplitude of the system improves with the

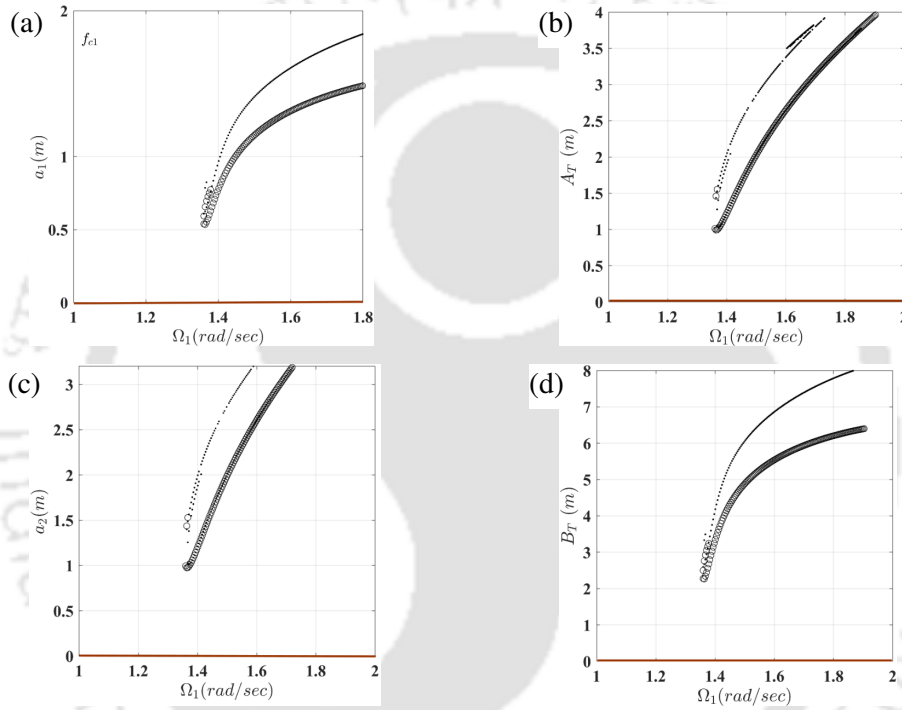


Fig. 5.40: (a-d) Frequency responses of the nonlinear system for 1:1 internal resonance condition by the TVA with f_{cl} .

application of combination feedback gains by the TVA than the zero control gain. In the next subsection time responses of the system are studied considering 1:3 and 1:2 internal resonance conditions for the simultaneous subharmonic, principal parametric and primary resonance conditions.

5.4.3.2 1:3 internal resonance condition

In this subsection the system time responses are obtained by solving the MMS reduced Eq. (5.26a) to Eq. (5.26d) by using the numerical method. In Fig. 5.41(a-d) the time responses

of the system are shown considering three frequencies close to the simultaneous subharmonic, principal parametric and primary resonance conditions by changing the stiffness of the absorber to maintain the 1:3 internal resonance condition. The stiffness of

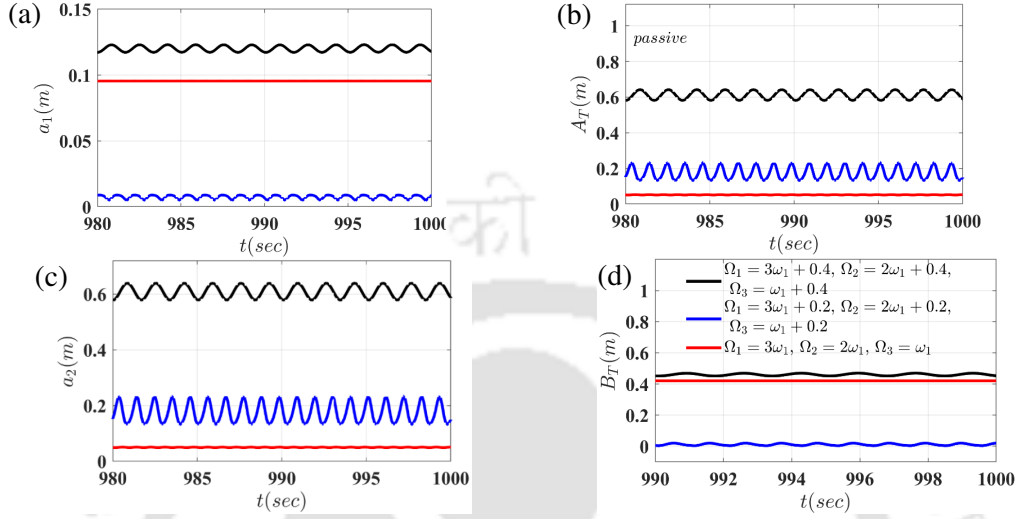


Fig. 5.41: (a-d) Time responses of the nonlinear system considering 1:3 internal resonance condition by the TVA with zero control gains.

the absorber (k_2) to maintain the 1:3 internal resonance condition is considered as 0.0056 N/m. From Fig. 5.41(a-d) and Fig. 5.42(a-d), it can be observed that with the application of feedback gain f_{c1} in the feedback the response of the system reduces than the passive

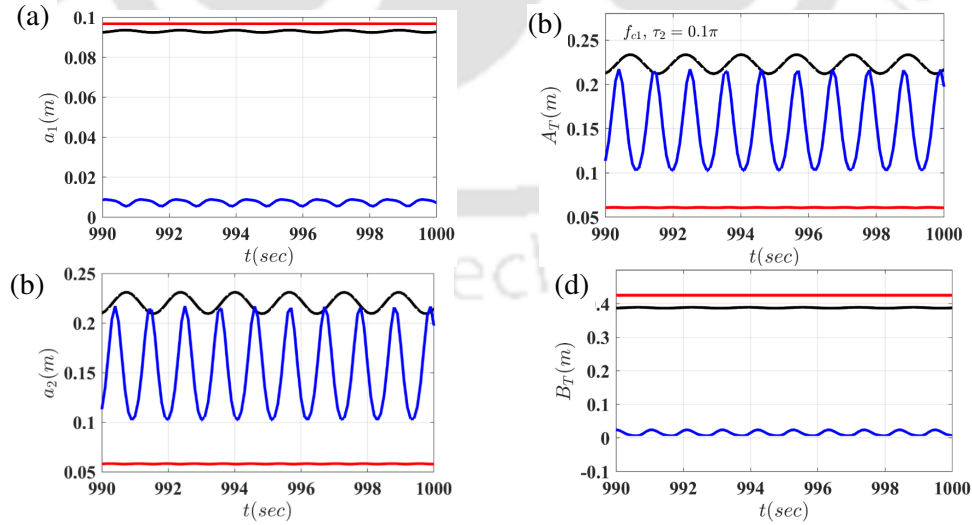


Fig. 5.42: (a-d) Time responses of the nonlinear system considering 1:3 internal resonance condition by the active TVA with f_{c1} gains.

system for the higher side of the above resonance conditions ($\Omega_1 = 3\omega_1 + 0.4$, $\Omega_2 = 2\omega_1 + 0.4$ and $\Omega_3 = \omega_1 + 0.4$). It is also observed that the response of the system remains almost the same with the passive and active TVA at other neighboring operating frequencies as shown in Fig. 5.41 and Fig. 5.42. The response amplitude of the system is comparatively smaller than in other internal resonance conditions which are explored in the previous subsections. The f_{c1} feedback gain with the time delay τ_2 suppresses around 60% of primary responses when the forcing frequencies are operated at a higher side (+0.4 rad/sec) of simultaneous subharmonic, principal parametric and primary resonance conditions which are shown in black color in Fig. 5.41 and Fig. 5.42. In 1:2 internal resonance conditions, the efficacy of the feedback gains in the feedback is not observed to reduce the vibration of the primary system compared to the zero feedback gain. However, the 1:2 internal resonance conditions are achieved by changing the stiffness of the absorber actively. Hence the proposed ANVA can be used to suppress the vibration of the system for these internal resonance conditions. In the next subsection, the frequency and time responses of the system are studied considering 1:2 internal resonance conditions from the reduced MMS equations.

5.4.3.3 1:2 internal resonance condition

In this subsection the time responses of the system are studied by solving the MMS reduced Eq. (5.23a) to Eq. (5.23c) by using the numerical method for the simultaneous subharmonic, principal parametric and primary resonance conditions considering 1:2 internal resonance condition. This internal resonance condition is obtained by considering the stiffness of the absorber equal to 0.0129 N/m. Fig. 5.43(a-b) shows the time response of the primary system

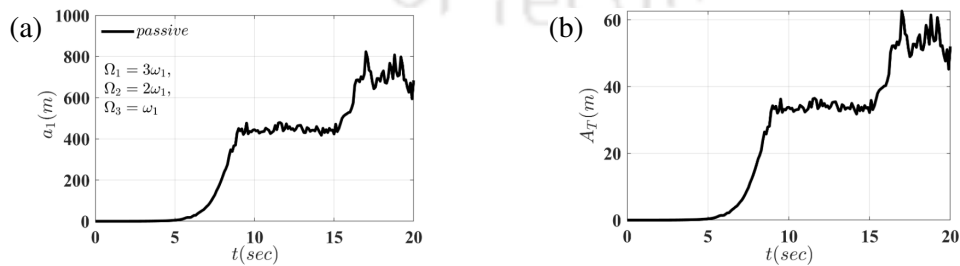


Fig. 5.43: (a-b) Time responses of the nonlinear system considering 1:2 internal resonance condition by the TVA with zero control gains.

for the passive TVA and Fig. 5.44(a-d) shows with f_{c1} feedback gain and with various time delays (τ_1) in the damping. From Fig. 5.43(a, b) high response in the system is observed when the external frequencies are exactly at the simultaneous subharmonic, principal parametric and primary resonance conditions considering only tuning the frequency of the absorber at 1:2 internal resonance condition with zero feedback gains. These high response amplitudes lead to an unstable response as observed in Fig. 5.43. However, when the time

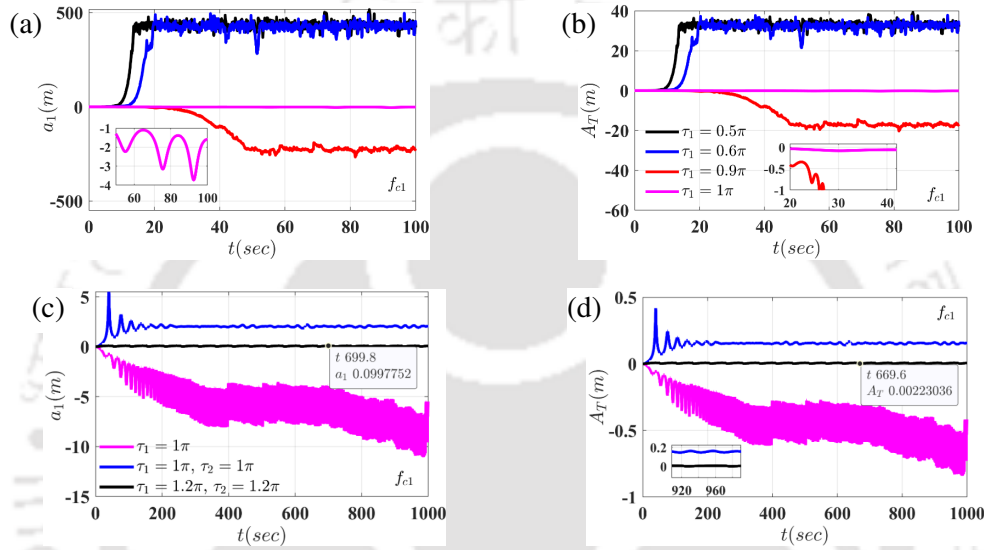


Fig. 5.44: (a-d) Time responses of the nonlinear system considering 1:2 internal resonance condition by the TVA with various time delays (τ_1 , τ_2) and f_{c1} feedback gain.

delay (τ_1) in the damping is considered with the f_{c1} feedback gain then the time response of the system is remarkably improved, which is shown in Fig. 5.44(a, b). The increase in the time delay ($\tau_1 = \pi$) reduces the response amplitude for a larger range of time (100 sec) but the steady response of the system is still unstable for $\tau_1 = \pi$ sec as observed in Fig. 5.44(c, d). Further, considering time delay in the feedback (τ_2) along with the time delay (τ_1) and f_{c1} active gain one can observe a significant reduction in response amplitude of the system at the steady state. Considering a higher time delay for both in the damping and feedback i.e. $\tau_1 = \tau_2 = 1.2\pi$ sec, along with the f_{c1} feedback gain the primary system response is almost zero at the steady-state as shown in Fig. 5.44(b) for the simultaneous subharmonic, principal parametric and primary resonance conditions considering 1:2

internal resonance condition. The conclusions of the present chapter are discussed in the next section.

From the above study, the designer can know the response amplitudes of both the primary and secondary system considering both modal and actual responses according to various parametric analyses (variation in nonlinear stiffness, various combinations of feedback gains, time delays in damping and feedbacks) for both linear and nonlinear systems to effectively suppress the vibration of the system for a wider band of operating frequencies for the simultaneous superharmonic, principal parametric and primary resonance conditions with 1:1 internal resonance. The response amplitude of the system mostly shows unstable responses with the ANVA for higher excitations amplitudes for various forcing frequencies considered in these analyses. However, these high response amplitudes of the system can also be minimized by proving different gains values or using different combination feedbacks on the primary system. From the various analyses carried out in this section, the efficacy of the ANVA with the f_{c1} feedback gain can be observed by reducing the response amplitudes of both the primary system and the secondary system for a wide band of forcing amplitudes for the simultaneous resonance conditions.

5.5 Summary

In this work the performances of the passive and active traditional and non-traditional vibration absorbers are studied with various combinations of feedback gains, time delays in the damping as well as in the feedback, to suppress the vibration of the primary system when it is subjected to external hard harmonic, soft parametric and soft base excitations. The cubic nonlinear stiffness in both the primary system and the absorber are considered and parametric studies have been carried out to study the response of the system. The resonance conditions considered in the analysis are simultaneous superharmonic, principal parametric and primary resonance conditions with 1:1, 1:2, and 3:1 internal resonance conditions. The analyses are further extended for the simultaneous sub-harmonic, principal parametric and primary resonance conditions with 1:1, 1:2, and 1:3 internal resonance conditions. The nonlinear governing equations of motion of the system are obtained by Newton's law and the weighted modal matrix approach, which are then solved by the method of multiple scales (MMS) to obtain the reduced equations. These reduced equations are solved by Newton's method and the obtained results are in good agreement with the

results from the numerical method. In the analyses, linear and nonlinear dynamics of the system are studied through the frequency responses, time responses, phase portraits and Poincare sections for various combinations of feedbacks, gains, nonlinear stiffness, time delays in the damping and the feedbacks.

From the analyses, it is observed that with simultaneous superharmonic, principal parametric, primary resonance and 1:1 internal resonance conditions the response amplitude of the linear and the nonlinear system are completely zero with the combination feedback for a certain range of operating frequencies for various nonlinear stiffness in the system. It is also shown that the primary system response amplitude can be effectively reduced by close to the 100% absorption by the proposed active vibration absorber for a wider range of operating frequencies by suitably changing the frequency of the absorber ($active_1$, $active_2$ and $active_4$ TVA) with various feedback gains (f_d , f_v , f_a). The combination feedback (f_{c1}) shows comparatively better vibration suppression (100%) than other feedback considering various nonlinear stiffness in the system. However, with these feedback gains at a certain range of operating frequencies, high response amplitudes in the system are observed than in the passive vibration absorber, which produces unstable responses. So one has to follow the safe range of operating frequencies as mentioned to effectively minimize the response amplitude of the system by using $active_2$ TVA. The trivial state instability region of the system with the $active_2$ TVA and f_v gain increases the critical value of F_{12} excitation from 0 N to 6.8 N and also reduces the region of the unstable range of operating frequencies. From, most of the parametric studies, it is observed that the time delay in the damping produces high response amplitude which produces unstable responses for the wider zone of operating frequencies. Considering 3:1 internal resonance conditions with the above-mentioned simultaneous resonances, an almost 100% reduction in the response amplitude of the primary system is observed for Ω_1 in the range of 0.18 rad/sec to 0.88 rad/sec without any feedback gain. However, the use of various combinations of feedback gains in the TVA produces comparatively higher response amplitude. Therefore, one has to actively tune the frequency of the absorber to obtain the 3:1 internal resonance condition and thereby effectively suppress the response amplitude of the primary system without any feedback gains. The same sorts of observations are made for 1:2 internal resonance conditions. So, for both 1:2 and 3:1 internal resonance conditions only by actively tuning the frequency of the absorber the primary system responses can be minimized.

In the case of simultaneous subharmonic, principal parametric, primary resonance and 1:1 internal resonance conditions, the trivial state of the system shows stable solutions with the feedback gains by the combination feedback than without feedback gains. The nontrivial state undergoes the transition from stable to unstable region by saddle-node bifurcation for both with and without feedback gains. For 1:3 internal resonance conditions, time-delayed combination feedback gains a 60% reduction in the response of the primary system is observed. For the 1:2 internal resonance condition, it is observed that with passive TVA the system response is unstable. However, with the active TVA along with delay in the damping and feedback, the system response is completely suppressed at the above-mentioned resonance conditions.



CHAPTER 6

Applications of ANVA for Vibration Suppression of Tremors in the Forearm, Beam and in RIDTMD

6.1 Introduction

In this chapter, the proposed active nonlinear vibration absorber (ANVA) model is used to suppress the vibration of three different physical primary systems. In the first study ANVA is used to suppress forearm tremors due to Parkinson's disease (PD), in the second study a beam system is considered with various boundary conditions and in the last study, a three degree of freedom model is undertaken. To suppress the tremor in the forearm, ANVA as a wearable bracelet is used whose mathematical analysis is undertaken in section 6.2.

In section 6.3, the ANVA is extended to a continuous beam system by considering the Euler-Bernoulli beam as the primary system. The ANVA uses displacement and velocity feedback from the absorber mass to reduce the vibration of the primary system subjected to a harmonically excited point force. In the next section, mathematical modelling of the ANVA used to suppress the response amplitude of the forearm is presented.

6.2 ANVA for Forearm Tremor

Figure 6.1 shows the active nonlinear vibration absorber (ANVA) attached to the nonlinear primary system. Here the forearm is considered as the nonlinear primary system with a mass (M), linear spring (K), nonlinear spring (K_{13}), and a damper (C). The primary system is subjected to external harmonic forcing excitation of $F_1 \cos(\Omega t)$ due to tremor or other disturbances. This model is similar to the theoretical and experimental work of Buki et al. [42], where the forearm is modelled as a linear spring-mass-damper primary system and a linear passive vibration absorber was used. But, in the present work, an active nonlinear vibration absorber is used to suppress the vibration of the nonlinear primary

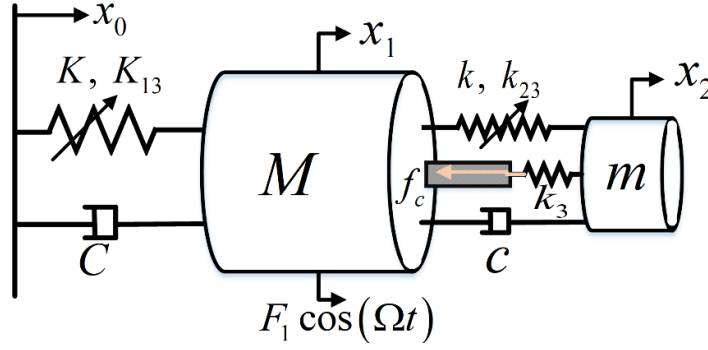


Fig. 6.1: Active nonlinear vibration absorber attached to the nonlinear primary system.

system for a wider range of frequencies. It may be noted that the model used in [42] is applicable for only a narrow band of frequencies. The use of vibration absorbers for tremor suppression is extensively studied in many literature [42-46, 332, 333] but the active approach and nonlinear analyses are limited. Hence this gap is attempted to be filled in the present work. Here the secondary system consists of a mass (m), linear spring (k), nonlinear spring (k_{23}), damper (c), and an actuator with active feedback gain (f_c) produced by the actuator and spring (k_3) in series connection by displacement, velocity and acceleration feedback from the primary system. It may be noted that the active force (f_c) used in the present work is similar to that used in Chapters 2 and 3. Here, x_0 , x_1 and x_2 , respectively indicate the displacement of the base, the primary system, and the vibration absorber (Fig. 6.1). Using Newton's second law one may write the governing equation of motion of the system by two coupled nonlinear differential equations which are given below.

$$M\ddot{x}_1 = K(x_0 - x_1) + C(\dot{x}_0 - \dot{x}_1) + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) - K_{13}(x_0 - x_1)^3 + k_{23}(x_2 - x_1)^3 + F_1 \cos(\Omega t) - f_c \quad (6.1)$$

$$m\ddot{x}_2 = c(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) + k_{23}(x_1 - x_2)^3 + f_c \quad (6.2)$$

It may be noted that the exclusion of the terms f_c , K_{13} , k_{23} , and $F_1 \cos(\Omega t)$, will reduce Eqs. (6.1) and (6.2) to that of Buki et al. [42]. The governing Eqs. (6.1) and (6.2) are modified by assuming $\omega_p = \sqrt{K/M}$.

$$\ddot{x}_1 = \omega_p^2 (x_0 - x_1) + 2\zeta_1 \omega_p (\dot{x}_0 - \dot{x}_1) + \mu \omega_a^2 (x_2 - x_1) + 2\mu \zeta_2 \omega_a (\dot{x}_2 - \dot{x}_1) + \alpha (x_0 - x_1)^3 + \mu \beta (x_2 - x_1)^3 + F \cos \Omega t - F_{act} \quad (6.3)$$

$$\ddot{x}_2 = 2\zeta_2 \omega_a (\dot{x}_1 - \dot{x}_2) + \omega_a^2 (x_1 - x_2) + \beta (x_1 - x_2)^3 + F_{act} / \mu \quad (6.4)$$

where

$$\omega_a = \sqrt{\frac{k}{m}}, \mu = \frac{m}{M}, \zeta_1 = \frac{C}{2\sqrt{KM}}, \zeta_2 = \frac{c}{2\sqrt{km}}, \alpha = \frac{K_{13}}{M}, \beta = \frac{k_{23}}{m}, F = \frac{F_1}{M}, F_{act} = \frac{f_c}{M}$$

The approximate solution of the Eqs. (6.3) and (6.4) are solved using the HBM and described in the next section.

6.2.1 Solutions obtained by HBM

In this section, the approximate steady-state solution of Eqs. (6.3) and (6.4) are solved by HBM by assuming the solution as follows.

$$x_1 = A(t) \cos(\Omega t - \theta_1(t)) \quad (6.5)$$

$$x_2 - x_1 = B(t) \cos(\Omega t - \theta_2(t)) \quad (6.6)$$

Here $A(t)$, $\theta_1(t)$, $B(t)$ and $\theta_2(t)$ are slowly varying functions of time t in a way that the following higher-order terms: $\ddot{A}_1$, $\ddot{\theta}_1$, $\ddot{B}$, $\ddot{\theta}_2$, $\dot{A}\dot{\theta}_1$, $\dot{B}\dot{\theta}_2$, $\dot{\theta}_1^2$ and $\dot{\theta}_2^2$ can be neglected. The base excitation x_0 is considered equal to $X_0 \cos \Omega t$. Substituting Eqs. (6.5) and (6.6) into Eqs. (6.3) and (6.4) one can obtain the following coefficients $\sin \Omega t$ and $\cos \Omega t$ which are represented in a matrix form as shown below.

$$\underbrace{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}}_P \underbrace{\begin{Bmatrix} \dot{A} \\ \dot{\theta}_1 \\ \dot{B} \\ \dot{\theta}_2 \end{Bmatrix}}_{\dot{Q}} = - \underbrace{\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}}_R \quad (6.7)$$

The terms $a_1, a_2 \dots a_{15}, a_{16}$ and b_1, b_2, b_3, b_4 are given in Appendix D. From Eq. (6.7) the steady-state amplitude and phase equations are obtained as follows.

$$[\dot{Q}] = [P]^{-1} [R] \quad (6.8)$$

The steady-state solutions are obtained from the reduced Eq. (6.8) by taking $\dot{A} = \dot{\theta}_1 = \dot{B} = \dot{\theta}_2 = 0$. These steady-state solutions are then solved by using Newton's method to study the responses of the system which are discussed in the next section. It may be noted that the above method is similar to the method described in detail in Chapter 3.

6.2.2 Results and Discussions

In the present section, the potential of the ANVA in vibration suppression of the base excited nonlinear SDOF primary system/forearm is studied by the frequency responses, time responses, phase portraits and Poincare' sections for various system parameters such as nonlinear stiffness, various feedback gains and external harmonic excitations on the primary system. The frequency response curves of the primary and secondary systems are found by numerically solving the steady-state equations (Eq. 6.8) by using Newton's method. For numerical analyses, the system parameters viz., the mass, linear stiffness, and damping parameters of the primary system (forearm) and the secondary system (bracelet) are considered from the work of Buki et al. [42]. In the next sub-sections, frequency responses are studied for the linear and nonlinear systems with and without feedback gains by considering various physical system parameters.

6.2.2.1 Frequency responses with variation in cubic nonlinear stiffness

The frequency response curves of the forearm and the bracelet are shown in Fig. 6.2 with and without the use of cubic nonlinear stiffness coefficient α and β in the bracelet which is taken equal to 0.1 and 0.01, respectively. Here feedback gain (F_{act}) is considered equal to zero. The stable responses are shown by lines and dots while unstable responses are shown by circles. From Fig. 6.2(a) it can be observed that the result of the present work which is obtained using HBM is matching exactly with the work of Buki et al. [42] considering linear stiffness. In this figure for $\alpha = 0.1$, the response amplitude of the

forearm (primary system) is comparatively less than that with linear stiffness for a wider range of operating frequencies. However, for $\beta = 0.01$, unstable responses have been observed. The bracelet (absorber) responses have been shown in Fig. 6.2(b) considering

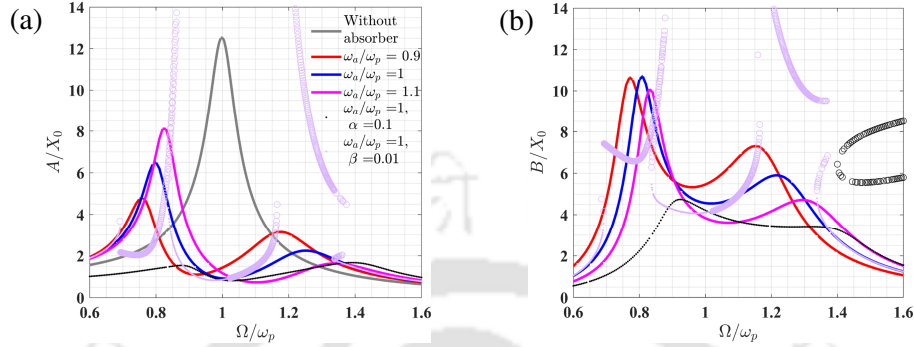


Fig. 6.2: Frequency response of the linear [42] and nonlinear system (a) forearm
(b) bracelet for $\mu = 0.2$, $\zeta_1 = 0.04$ and $\zeta_2 = 0.1$.

linear and nonlinear stiffness, where similar observations as that of Fig. 6.2(a) is noticed. It is inferred from Fig. 6.2 that the consideration of the nonlinear stiffness coefficient in the primary system reduces the response amplitude more than that with the linear stiffness. In the next section, the effect of active forces on the frequency response of the system is studied.

6.2.2.2 Effects of various feedback gains on the frequency responses

In this section, the frequency responses of the primary system with unit base excitation shown in Fig. 6.3 are studied with various feedback gains, and considering nonlinear stiffness in the system. Fig. 6.3 (a) shows the frequency responses of the primary system with the AVA by various feedback gains such as displacement (F_d), velocity (F_v), accelerations (F_{ac}) and combination of these feedbacks (F_d , F_v and F_{ac}) considering $\omega_a/\omega_p = 1$. In Fig. 6.3, the value of F_d , F_v and F_{ac} gains are considered equals to -0.9, 0.1 and -0.9, respectively which are obtained with several trials and error computations. From Fig. 6.3 (a) it can be observed with F_{ac} and combination feedbacks (F_d , F_v and F_{ac}) gains the response amplitude (A) of the primary system decreases for a broader range of Ω/ω_p values than with other combinations. From this figure, one can notice that with F_{ac}

gain the A/X_0 value is less than that with PVA for $\Omega/\omega_p > 0.77$, while with combination feedback gains A/X_0 value is less than that with PVA for $\Omega/\omega_p > 0.7$. The F_d gain by the AVA (bracelet) performs better than PVA for $0.735 < \Omega/\omega_p < 1.63$. However, with F_v gain, the response amplitude of the primary is reduced for a narrow band of Ω/ω_p values i.e., in the range of 1.07 to 1.37 and 0.53 to 0.82. Also from this figure, one can notice that the response amplitude of the primary system with F_v gain shows two peaks similar to the PVA. It may be noted that the F_v gain by the AVA increases the damping in the secondary system, which corresponds to the two peaks in the system. Also, by choosing the optimal F_v gain one can obtain the Den Hartog's equal peak in the frequency response curve shown in Fig. 6.3(a). The F_{ac} gain by the AVA reduces 52% and 96% of the response amplitude of the primary system at the first and second peaks, respectively than the PVA. However, with F_d , F_{ac} and combination feedback gains the first peak response amplitude of the

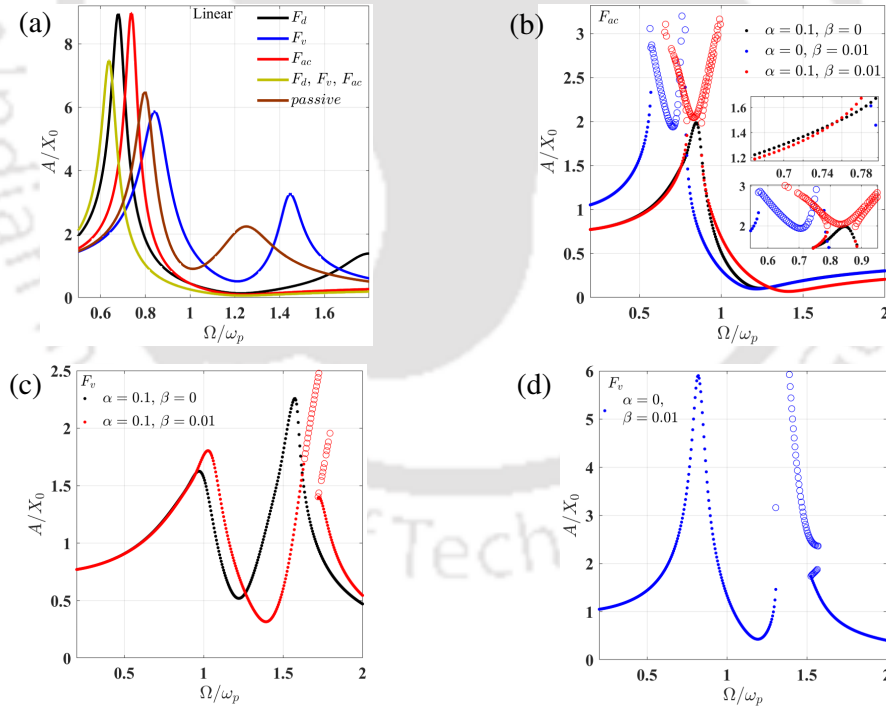


Fig. 6.3: Frequency response of the primary (forearm) system with various feedback gains by the AVA (bracelet) with linear and nonlinear stiffness for $\mu = 0.2$, $\zeta_1 = 0.04$, $\zeta_2 = 0.1$.

primary system is more than the PVA. The first peak in the frequency response by F_d , F_{ac} and combination feedback gains occur at a slightly lower frequency than the first peak of frequency response with PVA. From Fig. 6.3(a), it can be inferred that with F_{ac} and combination of feedback gains the primary system response can be reduced for a broader range of operating frequencies, especially at the close proximities of the resonating frequency of operation. In Fig. 6.3(b) the frequency response of the primary system is shown with F_{ac} gain and cubic nonlinear stiffness in the system. Considering $\alpha = 0.1$ and $\beta = 0$, it is observed that the high peak response amplitude of the primary system with F_{ac} gain reduces than the corresponding linear system (Fig.6.3(a)). The nonlinear primary system response amplitude with $\alpha = 0.1$ and $\beta = 0$ and F_{ac} gain remains very less than the linear primary system with PVA and AVA with F_{ac} gain (Fig.6.3(a)) for $0 < \Omega/\omega_p < 0.77$. It is noticed that with F_{ac} gain and considering $\alpha = 0.1$ and $\beta = 0$ (Fig. 6.3(b)), there is a minimum 48% and a maximum 68% reduction in the response amplitude of the primary system than the linear primary system with PVA (Fig. 6.3(a)) for $0 < \Omega/\omega_p < 0.8$. However, it is observed that with the consideration of $\alpha = 0.1$ and $\beta = 0$, the unstable solutions are appearing at the higher operating frequencies range. Considering nonlinear stiffness only in the AVA i.e. $\alpha = 0$ and $\beta = 0.01$ (Fig. 6.3(b)), one can notice that the response amplitude of the primary system shows unstable responses for $0.57 < \Omega/\omega_p < 0.79$. However, outside for $\Omega/\omega_p > 0.79$, the response amplitude of the primary system with F_{ac} gain is less than PVA. Similar, response amplitude is also observed considering $\alpha = 0.1$ and $\beta = 0.01$, with F_{ac} gain but here for $\Omega/\omega_p < 0.79$ the response amplitude of primary system is less than considering $\alpha = 0$ and $\beta = 0.01$. The unstable solutions appeared considering $\alpha = 0.1$ and $\beta = 0.01$ is in the range of $0.79 < \Omega/\omega_p < 0.88$. In Figs. 6.3(c) and 6.3(d), similar analyses as in Fig. 6.3(b) is carried out, but here F_v gain in the feedback is considered. From these figures, one can observe two peaks in the frequency response curves considering α and β coefficients in the stiffness of the primary system and the absorber. The response amplitude of the primary system shows comparatively better response with nonlinear stiffness present in the primary system (Fig. 6.3(c), $\alpha = 0.1$ and $\beta = 0$ and $\alpha = 0.1$ and $\beta = 0.01$) than considering only nonlinear stiffness in the absorber (Fig.6.3(d)).

From Fig. 6.3 one can observe the efficacy of the F_{ac} and combination feedback gains in reducing the response amplitude of the primary system for a broad range of operating frequencies than with PVA or other combinations of feedbacks. The PVA shows lesser responses in the lower operating frequencies region. Hence one can use the AVA with zero gain for the lower operating frequency region ($\Omega/\omega_p < 0.77$) and for the higher operating frequency region ($\Omega/\omega_p > 0.77$), one can use F_{ac} gain by the AVA to broadly suppress the vibration of the primary system. The consideration of cubic nonlinear stiffness in the system shows unstable responses for a particular region of operating frequencies. So, one can avoid operating the AVA in these regions for the nonlinear system.

Figure 6.4 illustrates the unit base excited primary system response amplitude with the F_{ac} gain by the AVA, taking into account physical system parameters of the forearm from the work of Buki et. al. [42] viz., $\omega_p = 28.65$ rad/sec, $\omega_a = 36.75$ rad/sec, $\zeta_1 = 0.079$ and $\zeta_2 = 0.0263$, and $\mu = 0.2248$. Fig. 6.4(a) shows the frequency response amplitude of the primary system with PVA, F_{ac} gain by the AVA and F_{ac} gain with optimal AVA parameters. From this figure, it can be observed that with PVA (bracelet) the two peaks are at 3.83 Hz and 6.95 Hz with response amplitudes of 7.165 and 1.99, respectively. It may be noted that in Fig. 6.4 in addition to the base excitation [42], a harmonic excitation $F \cos \Omega t$, where $F = 0.1$ N, is also considered, for the active case (AVA with F_{ac} gain), which may arise due to many other disturbance or feedback gains. The F_{ac} gain by the AVA reduces the response amplitude by 72.5% in the first peak (3.83 Hz) and 92.6% in the second peak (6.95 Hz) than the PVA. The response amplitude of the primary system shows a comparatively lesser amplitude with F_{ac} gain by the AVA for $\Omega > 3.5$ Hz than the PVA. However, with F_{ac} gain the first peak occurs at 3.36 Hz with a response amplitude equal to 7.5. Thus, one can use the AVA with zero feedback gain for $0 < \Omega < 3.5$ Hz and outside this range, F_{ac} gain in the feedback by the AVA can be used to suppress the forearm (primary system) tremor for a broad range of operating frequencies. It is also shown in Fig. 6.4(a) that with the optimal parameters for the frequency ($\omega_a(opt) = 27$ rad/sec) and damping ratio ($\zeta_2(opt) = 0.25$) in the AVA there is a significant reduction in the response amplitude of the primary system for a broad range of operating frequencies. With the

optimal AVA and F_{ac} gain the maximum peaks in the response amplitude is 2.86 at 2.95 Hz, which is 60% less than the maximum peak by the PVA. The response amplitude of the primary system with optimal AVA and F_{ac} gain is slightly more than PVA for a narrow band of operating frequency Ω in the range of 5.5 Hz to 6.12 Hz and 1.9 to 3.15 Hz.

One can obtain the mass ratio, mass of the primary system, stiffness of the primary system and the absorber from the data provided in the work of Buki et al. [42] as $m = 0.28$ kg, $M = 1.2456$ kg, $\mu = 0.2248$, $K = 1.0224$ kN/m, $k = 208.68$ N/m (for $\omega_a(opt)$) and $k = 378.1575$ N/m (for ω_a). Considering α and β equal to 10% and 1% of the corresponding linear stiffness the response amplitude of the primary system is shown in Fig. 6.4(b-c). In Figs. 6.4(b) and 6.4(c) the response amplitude of the primary system with PVA is shown. From Fig. 6.4(b) it can be observed that considering $\alpha = 10\%$ of K and $\beta = 1\%$ of k , the response amplitude of the primary system with PVA is less than that of the linear

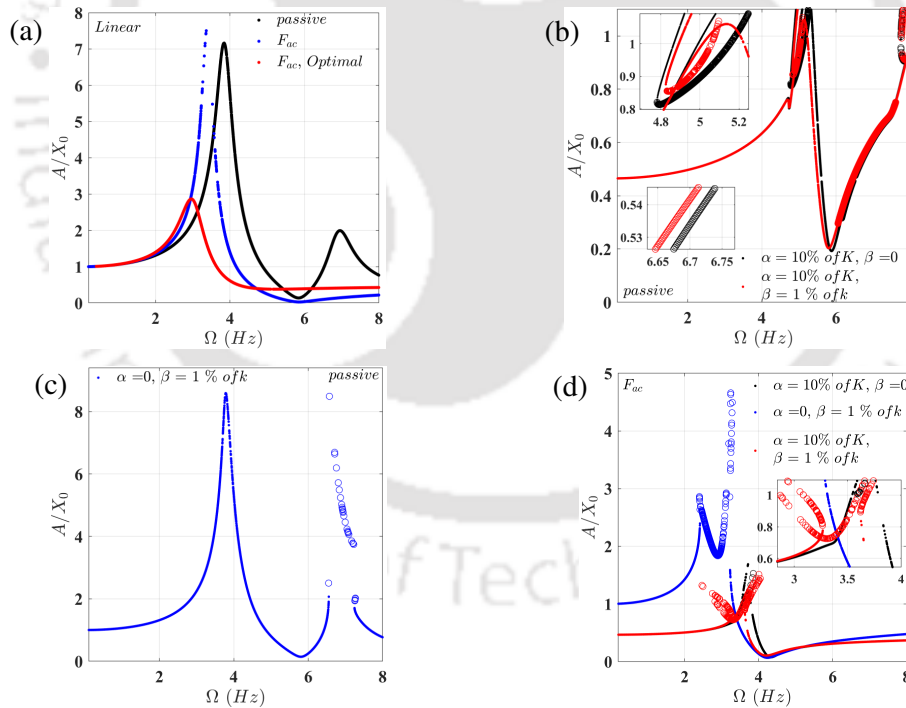


Fig. 6.4: Frequency response of the primary (forearm) system with F_{ac} gains by the AVA (active bracelet) considering physical system parameters of the forearm [42]. (a) linear system (b) nonlinear primary system with PVA (c) nonlinear PVA (d) nonlinear AVA.

passive system (Fig. 6.4(a)) for $\Omega < 4.78$ Hz. However, with the nonlinear stiffness, the responses of the primary system show unstable solutions for Ω in the range of 4.78 Hz to 5.36 Hz and 6.5 Hz to 7.66 Hz. Considering only $\beta = 1\%$ of k and $\alpha = 0$, high first peaks in the response amplitude of the primary system are observed, which is similar to the linear system with PVA shown in Fig. 6.4(a). In Fig. 6.4(d) the frequency response of the primary system with F_{ac} gain by the AVA (active bracelet) is shown considering nonlinear stiffness in the system. From this figure, one can observe a significant reduction (44% to 98%) in the response amplitude of the primary system with the F_{ac} gain by the active bracelet, considering nonlinear stiffness only in the absorber ($\alpha = 0$ and $\beta = 1\%$ of k), for Ω in the range of 3.2 Hz to 4.2 Hz. The maximum peak of the primary system with a nonlinear passive bracelet is 8.5354 at 3.799 Hz (Fig. 6.4(c)), whereas with an active bracelet the response amplitude is 0.259 at the same frequency (Fig. 6.4(d)), which is 97% reduction. However, with F_{ac} gain by the active bracelet unstable responses are appearing for Ω in the range of 2.41 Hz to 3.22 Hz. Also, with the passive bracelet, the unstable responses are observed for Ω in the range of 6.55 Hz to 7.24 Hz, whereas with F_{ac} gain by the active bracelet the primary system responses are stable and minimum in these regions of Ω . The efficacy of the F_{ac} gain by the active bracelet is also observed showing stable responses for $\alpha = 10\%$ of K and $\beta = 1\%$ of k and $\alpha = 10\%$ of K and $\beta = 0$ near the second peak of the frequency response plot. From Fig. 6.4 it can be inferred that the F_{ac} gain by the AVA (active bracelet) significantly suppresses the tremor suppression of the forearm of the patient both for the linear and nonlinear system. The nonlinear stiffness in the system produces unstable responses as shown in Fig. 6.4(b-d), so, one has to avoid these regions of operating frequencies to have less response amplitude in the forearm. In the next subsection time responses, phase portraits and Poincare sections of the forearm with the passive and active bracelet are studied considering various external excitations acting on the forearm.

6.2.2.3 Responses of the system by acceleration feedback

In this section, the time responses, phase portraits and Poincare sections of the unit base excited primary system (forearm) is shown in Fig. 6.5 considering passive and active bracelet with F_{ac} gain and nonlinear stiffness in the active bracelet. For the passive bracelet

(PVA) only base excitation is considered while for the active bracelet (AVA) both unit base and harmonic excitations ($F = 0.1$ N) are considered. In Fig. 6.5, effects of linear and only cubic nonlinear stiffness in the responses of the system are studied at the resonance i.e., when $\Omega = \omega_p$. Figure 6.5(a-c) shows with the linear passive bracelet the response of the primary system is periodic and its value is 1.682. From Fig. 6.5(a-c) and Fig. 6.4(a) responses for the linear system with PVA, one can observe good agreement between the HBM and numerical method results. It may be noted that in Fig. 6.4(a) the response amplitude of the linear primary system for the linear passive case is 1.683 at 4.5597 Hz ($\Omega = \omega_p$). The F_{ac} gain by the active bracelet reduces the response of the primary system from 1.682 to 0.514, which is a 69.4% reduction. The time response obtained with F_{ac} gain for the primary system at $\Omega = \omega_p$ with the linear bracelet is also in good agreement with the HBM results shown in Fig. 6.4(a), where the response amplitude is 0.512 at 4.5597 Hz. Considering nonlinear stiffness only in the absorber ($\alpha = 0$ and $\beta = 1\%$ of k (378.1575 N/m)) with the passive and active bracelet the responses are also closely matched with the corresponding Figs. 6.4(c) and 6.4(d). From Fig. 6.4(b-c) it is observed that the responses are periodic for the linear and nonlinear passive or active bracelet.

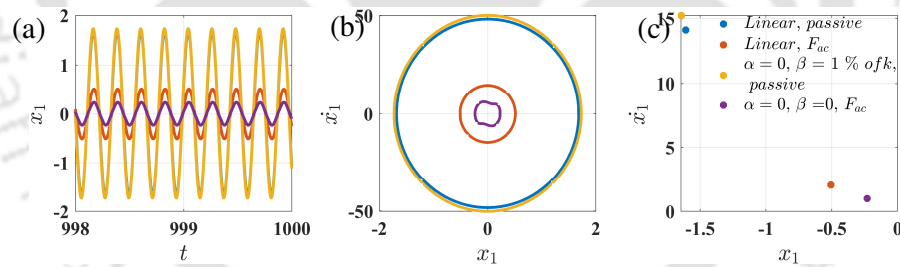


Fig. 6.5: (a) Time responses, (b) phase portraits and (c) Poincare sections with passive and active bracelets at $\Omega = \omega_p$.

6.2.2.4 Responses of the primary system for multi base excitations

In this sub-section, the performance of the F_{ac} gain by the active bracelet is studied in vibration reduction of the forearm (primary system) considering multi-frequency base excitations in the form $X_0 \cos(\Omega t) + X_0 \cos(2\sqrt{2}\Omega t)$. The corresponding time responses, phase portraits and Poincare sections are shown in Fig. 6.6. It may be noted as in the

previous figures (Figs. 6.4 and 6.5) for the active case in addition to the above base excitations, a harmonic excitation $F \cos \Omega t$, where $F = 0.1$ N, is also considered. From Fig. 6.6, one can observe around a 50% reduction in the response by the F_{ac} gain with the active bracelet. From the phase portraits (Fig. 6.6(b)) and Poincare sections (Fig. 6.6(c)), one can observe quasi-periodic responses.

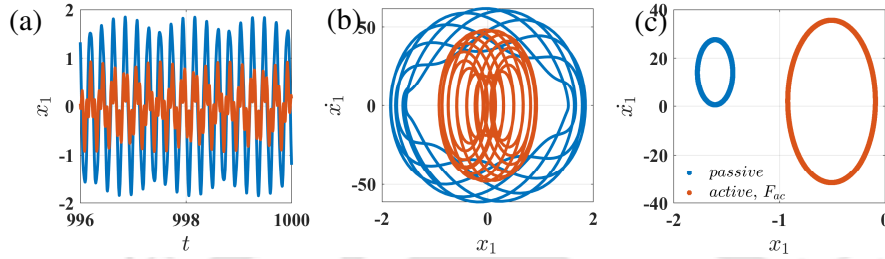


Fig. 6.6: (a) Time responses, (b) phase portraits and (c) Poincare sections with passive and active bracelet at $\Omega = \omega_p$ for multi base excitations.

6.2.2.5 Responses of the primary system for multi excitations

In this section, the efficacy of the F_{ac} feedback gain is studied in vibration suppression of the primary system when it is subjected to multi-base and harmonic excitations. Figures 6.7 show the responses when the primary/forearm undergoes multi-base excitations of the form $X_0 \cos(\Omega t) = \sum_{n=0}^7 X_0 \cos(2^n \Omega t)$. For the active case along with these base excitations

an additional harmonic excitation of the form $F \cos(\Omega t) = \sum_{n=0}^7 F \cos(2^n \Omega t)$, where $F =$

0.1 N is considered. Figure 6.7(a-c) shows the responses of the primary (forearm) system with the passive and active bracelets. From this figure, one can observe a slight improvement in suppressing the responses by the active bracelet in comparison to the passive. From the phase portraits (Fig. 6.7(b)) and Poincare sections (Fig. 6.7(c)), one can observe periodic responses. The performance of the active acceleration feedback gain F_{ac} by the bracelet is also studied considering the optimal frequency ($\omega_a(opt)$) and optimal frequency and damping ratio by the active bracelet and shown in Fig. 6.7(c-f). From Fig. 6.7(c-f), one can observe a substantial decrease in the response of the primary system by

the F_{ac} gain with the optimal parameters in comparison to the passive bracelet (Fig. 6.7(a-c)). For multiple excitations with higher amplitude of excitations, one can increase the gain

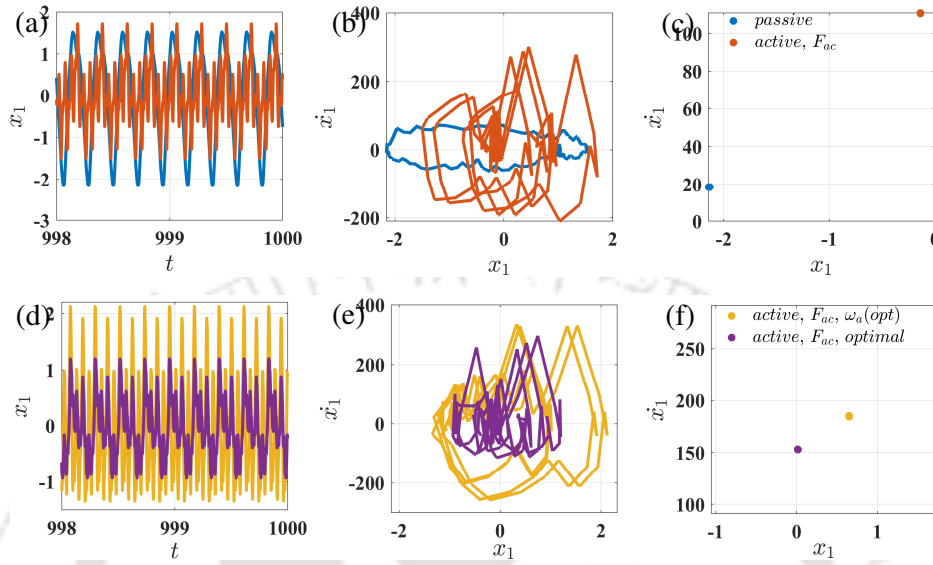


Fig. 6.7: Time responses (a, d), phase portraits (b, e) and Poincare sections (c, f) of the primary system under multiple excitations at $\Omega = \omega_p$.

used by the active bracelet to effectively suppress the responses of the primary system considering both linear and nonlinear stiffness in the system. Furthermore, from the results shown in section 6.2.2 one can observe the efficacy of F_{ac} gain in tremor suppression for various excitations and nonlinear stiffness in the system. The application of vibration absorbers with modified design is also employed to suppress the vibration of the beam system under external harmonic excitations [57, 106, 127, 189, 243, 253, 315, 326, 334, 335]. In the next section, one such example of the application of ANVA is extended to a beam system.

6.3 ANVA for Harmonically Excited Beam System

In this section, vibration suppression by ANVA of a continuous beam system is analysed. The secondary system consists of a nonlinear spring, mass and an actuator, which is attached to the primary beam to suppress its vibration. The beam is modelled as Euler-Bernoulli beam, which is excited by an external harmonic point force. Three different feedbacks namely proportional displacement or velocity combination of these are used by

the actuator to suppress the first three modes of vibrations of the beam for various boundary conditions. It may be noted that this work is an extension work of Chatterjee [127] where linear vibration absorbers with displacement and velocity feedback were studied.

The schematic diagram of a fixed-fixed beam subjected to harmonic excitation $F \cos \omega t$ at distance b from the left end is shown in Fig. 6.8. The parameters ρ , A , L , and EI denotes density, cross-sectional area, length, elastic modulus and, the flexural rigidity of the beam, respectively. The mass, stiffness of the spring and cubic nonlinear stiffness of the secondary

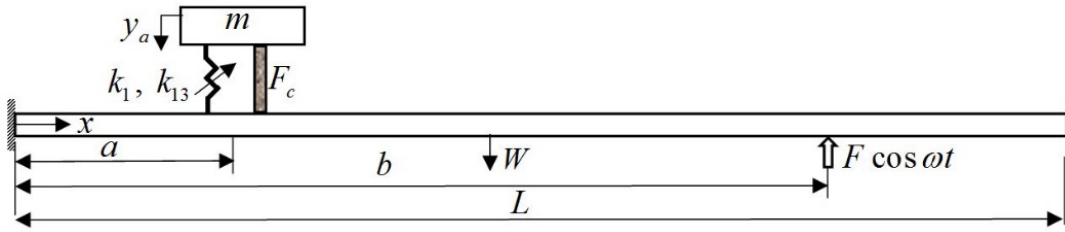


Fig. 6.8: Vibration control of a fixed-fixed beam subjected to harmonically excited point load by using an active nonlinear vibration absorber.

system is denoted by m , k_1 and k_{13} , respectively. The secondary system is attached at a distance a from the left end of the beam. The active control force F_c by the actuator is considered to be proportional to displacement or velocity feedback or a combination of these taken from the absorber mass. The transverse displacement of the beam about the neutral axis at any point x and time t is denoted by $W(x, t)$. The governing differential equation of motion of the system is obtained by considering the equilibrium of the forces and moments, which are given below.

$$\rho A \frac{\partial^2 W(x, t)}{\partial t^2} + EI \frac{\partial^4 W(x, t)}{\partial x^4} + k_1 (W(x(a), t) - y_a) \delta(x-a) + k_{13} (W(x(a), t) - y_a)^3 \delta(x-a) = F \delta(x-a) \cos \omega t - F_c \delta(x-a) \quad (6.9)$$

$$m \ddot{y}_a + k_1 (y_a - W(a, t)) \delta(x-a) + k_{13} (y_a - W(a, t))^3 \delta(x-a) = F_c \delta(x-a) \quad (6.10)$$

$$F_c = K_p y_a - K_v \dot{y}_a \quad (6.11)$$

where K_p and K_v are the control gains of the absorber mass, for displacement and velocity feedback, respectively. The term y_a is the displacement of the absorber mass. It may be noted that to make the design conservative, damping is not considered in the primary system while studying the ANVA. If the system vibration is controlled without damping, definitely it will also absorb the vibration of the primary system with damping having less control gain. Hence in the present analyses, no material damping is considered for the beam system. Taking $W(x, t) = \sum_{i=1}^3 \phi_i(x) q_i(t)$ in Eqs. (6.9) and (6.10), where $\phi_i(x)$ is the modal shape function and $q_i(t)$ is the time modulation of the i^{th} mode of the beam vibration, the temporal equations of the system is obtained by using Galerkin's method. Here normalization criteria is taken as $\int_0^L (\phi_i^2(x)) dx = 1$. Following non-dimensional parameters are used for further analysis. Modal displacement of the beam for i^{th} mode: $w_i = q_i / \delta_{st}$, displacement of the absorber: $y = y_a / \delta_{st}$, mass ratio ($\mu = m / (\rho AL)$), $\omega_a = \sqrt{k_1 / m} / \omega_0$, δ_{st} is the static deflection of the beam, ω_0 is the reference frequency, the normalized natural frequency of i^{th} mode the beam is equal to ω_{ni} , cubic nonlinear stiffness coefficient: $\alpha = k_{13} / k_1$, excitation force: $f = F / (\rho AL \omega_0^2 \delta_{st})$, control gains: $k_p = K_p / (\rho AL \omega_0^2)$, $k_v = K_v / (\rho AL \omega_0^2)$, $\Omega = \omega / \omega_0$ and non-dimensional time $\tau = \omega_0 t$. The obtained non-dimensional equations of motion are given below.

$$\ddot{w}_i(\tau) + \omega_{ni}^2 w_i(\tau) + \mu \omega_a^2 \phi_i(a) \left\{ \sum_{i=1}^3 \phi_i(a) w_i(\tau) - y(\tau) \right\} + \phi_i(a) \{k_p y(\tau) - k_v \dot{y}(\tau)\} + \alpha \mu \omega_a^2 \phi_i(a) \left\{ \sum_{i=1}^3 \phi_i(a) w_i(\tau) - y(\tau) \right\}^3 = \phi_i(b) f \cos(\Omega \tau) \quad (6.12)$$

$$\mu \ddot{y}(\tau) + \mu \omega_a^2 \left\{ y(\tau) - \sum_{i=1}^3 \phi_i(a) w_i(\tau) \right\} + \alpha \mu \omega_a^2 \left\{ y(\tau) - \sum_{i=1}^3 \phi_i(a) w_i(\tau) \right\}^3 - k_p y(\tau) + k_v \dot{y}(\tau) = 0 \quad (6.13)$$

It may be noted that Eqs. (6.12) and (6.13) are similar to Chatterjee [127] but here the cubic nonlinear stiffness in the absorber and different feedback forces are considered for various boundary conditions of the beam. The Eqs. (6.12) and (6.13) are solved for the first three

modal displacements of the beam i.e., for $i = 1, 2$ and 3 using HBM which is discussed in the following section.

6.3.1 Approximate solution by harmonic balance method

In this section, the harmonic balance method with slowly varying parameters is employed to Eqs. (6.12) and (6.13) to analyse the steady-state response of the system. The assumed solutions are stated as follows:

$$w_n(\tau) = A_n(\tau) \cos(\Omega\tau + \varphi_n(\tau)) \text{ for } n = 1, 2, 3 \quad (6.14)$$

$$y(\tau) = B(\tau) \cos(\Omega\tau + \varphi_4(\tau)) \quad (6.15)$$

where $A_1(\tau)$, $A_2(\tau)$, $A_3(\tau)$, $B(\tau)$, $\varphi_1(\tau)$, $\varphi_2(\tau)$, $\varphi_3(\tau)$ and $\varphi_4(\tau)$ are slowly-varying functions of time τ such that one can neglect the higher-order or multiplication of derivatives following the similar procedure as in the previous section. Substituting Eqs. (6.14) and (6.15) into Eqs. (6.12) and (6.13) and equating the coefficient of $\sin \Omega\tau$ and $\cos \Omega\tau$, the following set of algebraic equations in the matrix form are obtained.

$$\begin{bmatrix} a_{11} & \cdots & a_{18} \\ \vdots & \ddots & \vdots \\ a_{81} & \cdots & a_{88} \end{bmatrix} \begin{Bmatrix} \dot{A}_1 & \dot{\varphi}_1 & \dot{A}_2 & \dot{\varphi}_2 & \dot{A}_3 & \dot{\varphi}_3 & \dot{B} & \dot{\varphi}_4 \end{Bmatrix}^T = \begin{Bmatrix} b_1 & \cdots & b_8 \end{Bmatrix}^T \quad (6.16)$$

The terms a_{11} , $a_{12} \dots a_{87}$, a_{88} and b_1 , $b_2 \dots b_7$, b_8 are given in Appendix D. From Eq. (6.16) the eight sets of amplitudes (A_1, A_2, A_3, B) and phase ($\varphi_1, \varphi_2, \varphi_3, \varphi_4$) equations are obtained as given below.

$$\dot{A}_n = f_n(A_1, A_2, A_3, B, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \text{ for } n = 1, 2, 3 \quad (6.17)$$

$$\dot{\varphi}_n = f_n(A_1, A_2, A_3, B, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \text{ for } n = 4, 5, 6 \quad (6.18)$$

$$\dot{B} = f_7(A_1, A_2, A_3, B, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \quad (6.19)$$

$$\dot{\phi}_4 = f_8(A_1, A_2, A_3, B, \phi_1, \phi_2, \phi_3, \phi_4) \quad (6.20)$$

The steady-state solutions of the equations are obtained by setting the first derivatives of the slowly varying amplitudes and phases equal to zero, i.e., $\dot{A}_1 = \dot{A}_2 = \dot{A}_3 = \dot{B} = \dot{\phi}_1 = \dot{\phi}_2 = \dot{\phi}_3 = \dot{\phi}_4 = 0$ in the Eqs. (6.17 to 6.20) and the stability is studied by obtaining the eigenvalues of the Jacobian matrix from Eq. (6.16). In the following section, the performance of the active nonlinear vibration absorber is discussed for various system parameters and control gains to suppress the vibration of the beam for the first three modal frequencies.

6.3.2 Results and discussions

In this section, the performance of the active nonlinear vibration absorber on the beam is studied by the time responses, phase portraits and frequency responses of the system for the first three modal frequencies of the beam and also compared with those of the linear passive vibration absorber (PVA). The optimum absorber parameters and the active forces for displacement (k_p) and velocity feedback (k_v) are considered the same as that in the work of Chatterjee [127]. The system parameters are considered as follows: mass ratio $\mu = 0.2$, the amplitude of external excitation $f = 1$, absorber frequency $\omega_a = 2$, first three modal frequencies of the beam are $\omega_{n1} = 1, 4$ and 9 . The parametric study is carried out by varying cubic nonlinear stiffness coefficient α and active forces k_p and k_v for various boundary conditions of the beam. The time responses and phase portraits of the system are obtained by solving Eqs. (6.12) and (6.13) using the fourth-order Runge Kutta method and the frequency responses of the system are obtained by solving the steady Eqs. (6.17 to 6.20) using Newton's method. The solution is considered to be stable when the real part of the eigenvalues of the Jacobian matrix is negative.

6.3.2.1 Time responses of the system

The time responses and phase portraits of the fixed-fixed beam and with the passive absorber are shown in Fig. 6.9 for studying the effects of α and active forces in the vibration suppression of the beam. In Fig. 6.9(a-d) comparison of time responses at the

first modal frequency ($\omega_{n1} = \Omega = 1$) of the beam for linear (red colour) and nonlinear (black colour) PVA has been carried out. From Fig. 6.9(a) one can observe that with cubic

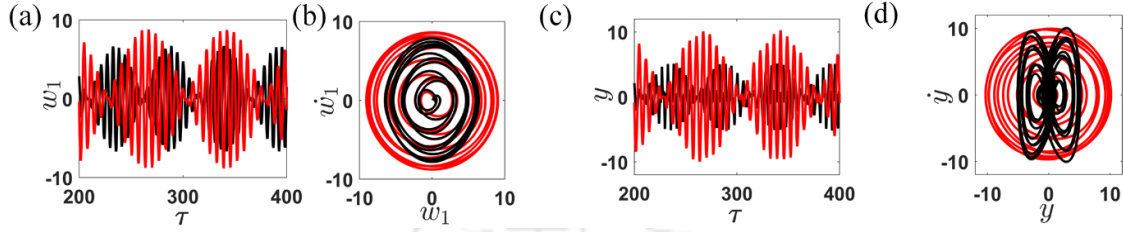


Fig. 6.9: Comparison of time responses (a, c) and phase portraits (b, d) of the passive fixed-fixed beam system at ($\omega_{n1} = \Omega = 1$) for linear ($\alpha = 0$) (red colour) and nonlinear absorber ($\alpha = 0.1$) (black colour).

nonlinear stiffness, the response amplitude of the beam and the absorber decreases by 24% and 50% respectively, compared to the linear vibration absorber. The corresponding phase portraits of the system are shown in Fig. 6.9(b, d) where the beam and absorber show quasi-periodic responses. In Fig. 6.10 responses of the fixed-fixed beam are studied for $\alpha = 0.1$ and 0.2 with different control gains. In Fig. 6.10(a-d) the black line shows the system response with only control gain k_v and the red lines shows zero control gain for the first

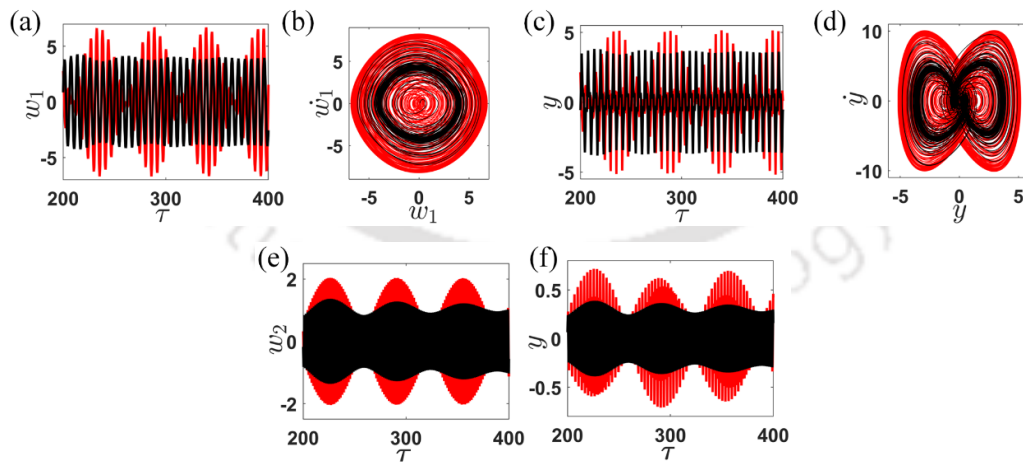


Fig. 6.10: Time responses and phase portraits of the system with fixed-fixed boundary conditions for different feedbacks and nonlinear stiffness in the absorber, where passive (red colour) and active (black colour). (a-d) $k_v = 0.3578$ and $\alpha = 0.1$ at $\omega_{n1} = \Omega = 1$, (e, f) $k_v = 1$ and $\alpha = 0.1$ at $\omega_{n1} = \Omega = 4$.

modal frequency ($\omega_{n1} = \Omega = 1$) of operation. It is observed from Fig. 6.10(a, c) that with $k_v = 0.3578$ the response of the beam and the absorber decreases by 52% and 28% respectively, compared to the nonlinear PVA. In Fig. 6.10(b) and Fig. 6.10(d) phase portraits of the beam and the absorber show quasiperiodic responses. In Fig. 6.10(e-f) time responses are shown to control the second modal frequency ($\omega_{n1} = \Omega = 4$) of the beam with $k_v = 1$, while all other system parameters are considered the same as in Fig. 6.10(a-d). Here one can observe that the system amplitude at the first modal frequency is more than that with the second modal frequency. From Fig. 6.10 it is observed that for the nonlinear absorber active forces with velocity feedback reduce the amplitude of the fixed-fixed beam system. In Fig. 6.11 the response of the system is studied for the simply supported beam (Fig. 6.11(a-b)) and the cantilevered beam (Fig. 6.11(c-d)) with the same α and control gains as in Fig. 6.10(a-d). From Fig. 6.11(a, b) the same beating time responses are observed as in Fig. 6.10 but with higher response amplitude. However, with the active forces, the amplitude is less which are shown in black colour. For the cantilever beam shown in Fig. 6.11(c, d) the vibration suppression is better than the simply supported beam with the same applied active forces.

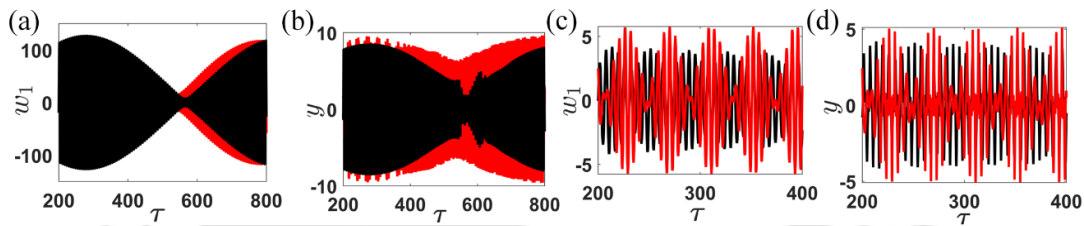


Fig. 6.11: Time responses of the (a, b) cantilevered beam and (c, d) simply supported beam at $\omega_{n1} = \Omega = 1$ for $\alpha = 0.1$, where passive (red colour) and active ($k_v = 0.3578$) (black colour).

6.3.2.2 Frequency responses curves of the system

In this section, the parametric study is undertaken to analyse the frequency response of the system with different values of α , boundary conditions of the beam viz., fixed-fixed, simply supported and cantilever beam and active forces for displacement (k_p) and velocity feedback (k_v). In Fig. 6.12(a, b) a comparison of frequency responses between the linear

(black colour) and nonlinear active vibration absorber (blue colour) with k_p and k_v are shown to suppress the first modal frequency ($\omega_{n1} = \Omega = 1$) of vibration in the fixed-fixed beam. From these figures, it is observed that for $\alpha = 0.01$ the amplitude of the beam and the absorber decreases for the first three modal frequencies than the linear absorber. In Fig. 6.12(c, d) only velocity feedback is used to suppress the second modal vibration ($\omega_{n1} = \Omega = 4$) of the beam. It is observed from these figures, that only k_v the second mode of vibration of the beam is suppressed and also the same analogy can be interpreted for the absorber. It may be noted that Figs. 6.12(a) and 6.12(c) are similar to the results obtained by Chatterjee [127] where only linear analysis has been carried out. In Fig. 6.12(e, f) the same k_v and negative k_p are considered to suppress the vibration of the beam at the third mode ($\omega_{n1} = \Omega = 9$). It is observed from these figures that with the control gains both the third mode and the second mode of vibration of the beam decrease. In Fig. 6.12(g) for α

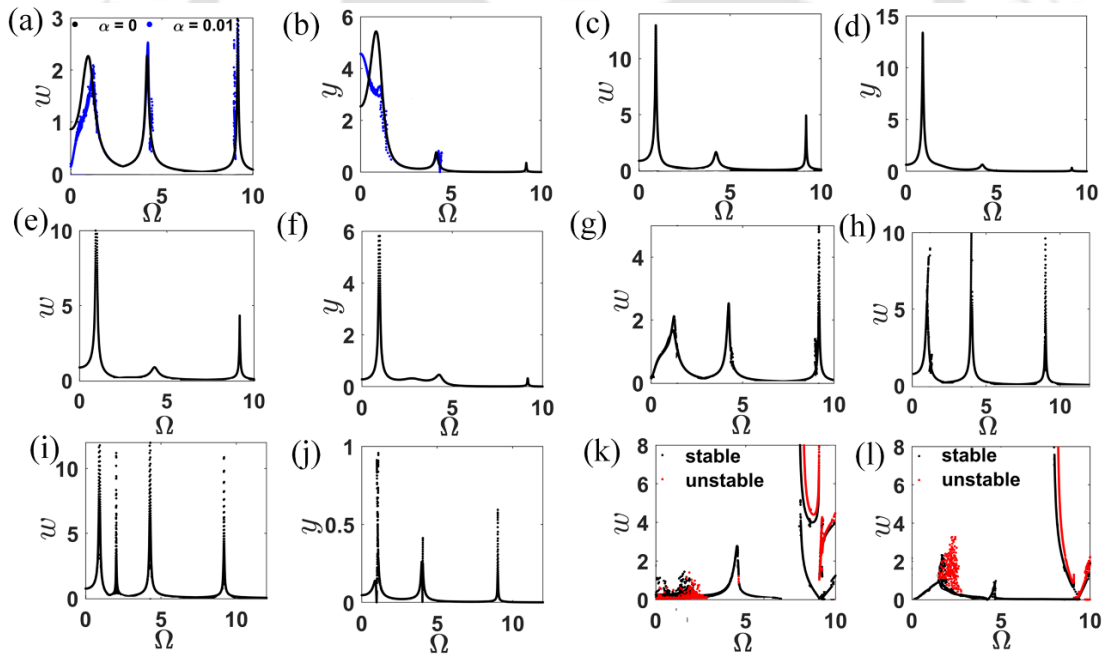


Fig. 6.12: Frequency responses of the fixed-fixed beam at the first three modal frequencies with $k_v = 0.3578$ and different k_p and α (a, b) $k_p = 0.6$, at $\omega_{n1} = \Omega = 1$, (c, d) $k_p = 0$, $\alpha = 0$ at $\omega_{n1} = \Omega = 4$, (e, f) $k_p = -1$, $\alpha = 0.05$ at $\omega_{n1} = \Omega = 9$, (g) $k_p = 0.6$, $\alpha = 0.05$ at $\omega_{n1} = \Omega = 1$, (h) $k_p = -0.6$, $\alpha = 0.1$ at $\omega_{n1} = \Omega = 1$, (i, j) $k_p = -1$, $\alpha = 0.01$ at $\omega_{n1} = \Omega = 1$ and (k, l) $k_p = -1$, $\alpha = 1$ at $\omega_{n1} = \Omega = 1$.

$= 0.05$, it is observed that the vibration of the beam further reduces in the first two modal frequencies than Fig 6.12(a). But for $\alpha = 0.1$ the beam vibration increases which are shown in Fig. 6.12(h). In Fig. 6.12(i, j) the vibration suppression of the third mode is analysed for $\alpha = 0.01$ with k_v and negative k_p . It is observed that the vibration suppression of the beam in the third mode is not achieved with the applied control gains. In Fig. 6.12(k, l), $\alpha = 1$ is considered to suppress the first mode of vibration ($\omega_{n1} = \Omega = 1$) of the beam. From Fig. 6.12(k, l), high response amplitude and unstable regions are observed in the system. From Fig. 6.12 it is observed that for α in the range of 0 to 0.1 with the applied control gains the response amplitude of the system reduces for the first three modal frequencies but higher values of α i.e. for $\alpha = 1$, make the system unstable with high response amplitude. In Fig. 6.13(a, d) the simply supported beam and cantilever beam are considered with k_p , k_v and α while all other parameters are the same as in Fig. 6.10(a-d). In simply supported beam (Fig. 6.13(a, b)) the unstable region is observed for $0.85 < \Omega < 0.95$ with jump up and jump down phenomena. For the cantilever beam (Fig. 6.13(c, d)) the response amplitude is higher than in the fixed-fixed beam with the same control gains.

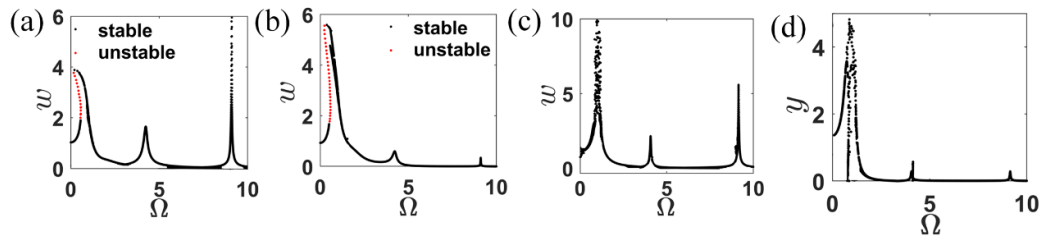


Fig. 6.13: Frequency responses of the (a, b) simply supported beam and (c, d) cantilever beam for $k_p = 0.6$, $k_v = 0.3578$ and $\alpha = 0.01$ at $\omega_{n1} = \Omega = 1$.

The vibration suppression technique by the PVA is not only limited to the use of active/smart materials (AVA) but also modifying the design PVA and use of various optimization techniques [334-340]. In the next section, a modified design in the vibration absorber is studied where in place of the PZT stack actuator a flywheel is used to provide an active force in the system known as a rotational inertial double-tuned mass damper (RIDTMD). The analyses are carried out to study the effects of nonlinear cubic spring stiffness on the response of the system.

6.4 Nonlinear Rotational Inertial Double Tuned Mass Damper

A single degree of freedom spring-mass primary system to which an active rotational inertial doubled tuned mass damper (RIDTMD) is attached is shown in Fig. 6.14. The proposed model is similar to Garrido et al. [340] but here cubic nonlinear stiffness is considered for both primary mass and the absorber configuration. RIDTMD consists of TMD using rotational inertial viscous damper (RIVD). Here m_1 and k_1 denote the mass and the stiffness of the primary mass. The RIDTMD consists of a moving block mass, the primary and secondary spring stiffness and the equivalent damping factor of the RIVD are denoted by m_2, k_2, k_3 and c_2 respectively as shown in Fig. 6.14. The primary mass is subjected to external harmonic force excitation $F_1 \cos(\Omega_1 t)$. This external harmonic excitation may be developed due to the unwanted disturbances by the feedback gains or due to the tremors. It may be noted that in this case RIVD is used which is similar to the active feedback gain generated by using the PZT based actuator discussed in the previous Chapters. Hence, the analyses of nonlinear RIDTMD are carried out in this section. The governing equations of motion of the system are obtained using d'Alembert principle which is given below.

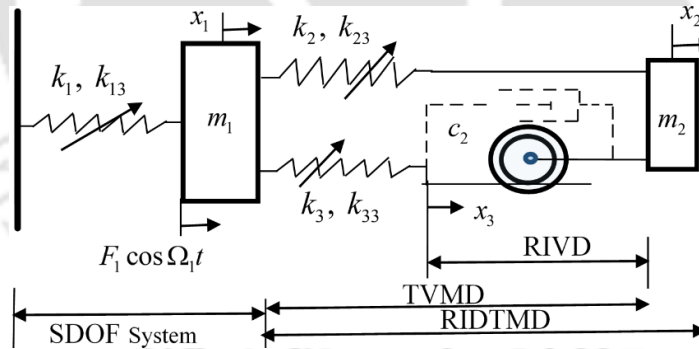


Fig. 6.14: SDOF nonlinear primary system incorporating a nonlinear RIDTMD.

$$m_1 \ddot{x}_1 + k_1 x_1 + k_{13} x_1^3 + k_2 (x_1 - x_2) + k_{23} (x_1 - x_2)^3 + k_3 (x_1 - x_3) + k_{33} (x_1 - x_3)^3 = F_1 \cos \Omega_1 t \quad (6.21)$$

$$m_2 \ddot{x}_2 + m_3 (\ddot{x}_2 - \ddot{x}_3) + c_2 (\dot{x}_2 - \dot{x}_3) + k_2 (x_2 - x_1) + k_{23} (x_2 - x_1)^3 = 0 \quad (6.22)$$

$$m_3 (\ddot{x}_3 - \ddot{x}_2) + c_2 (\dot{x}_3 - \dot{x}_2) + k_3 (x_3 - x_1) + k_{33} (x_3 - x_1)^3 = 0 \quad (6.23)$$

It may be noted that in the RIDTMD following the work of [340] the translational masses of the flywheel, rack, pinion and the immersed mass are assumed to be relatively small and can therefore be combined into m_1 . Assuming $\omega_1 = \sqrt{\frac{k_1}{m_1}}$ and $\tau = \omega_1 t$, Eqs. (6.21), (6.22) and (6.23) are non-dimensionalized as given below.

$$\ddot{x}_1 + x_1 + \alpha_{13}x_1^3 + \mu_1\omega_{21}^2(x_1 - x_2) + \alpha_{23}(x_1 - x_2)^3 + \mu_2\omega_{31}^2(x_1 - x_3) + \alpha_{33}(x_1 - x_3)^3 = F \cos \Omega \tau \quad (6.24)$$

$$\mu_1\ddot{x}_2 + \mu_2(\ddot{x}_2 - \ddot{x}_3) + 2\zeta_2\mu_1\omega_{21}(\dot{x}_2 - \dot{x}_3) + \mu_1\omega_{21}^2(x_2 - x_1) + \alpha_{23}(x_1 - x_2)^3 = 0 \quad (6.25)$$

$$\mu_2(\ddot{x}_3 - \ddot{x}_2) + 2\zeta_2\mu_1\omega_{21}(\dot{x}_3 - \dot{x}_2) + \mu_2\omega_{31}^2(x_3 - x_1) + \alpha_{33}(x_3 - x_1)^3 = 0 \quad (6.26)$$

where

$$\mu_1 = \frac{m_2}{m_1}, \mu_2 = \frac{m_3}{m_1}, \zeta_2 = \frac{c_2}{2m_2\omega_2}, \omega_1 = \sqrt{\frac{k_1}{m_1}}, \omega_2 = \sqrt{\frac{k_2}{m_2}}, \omega_3 = \sqrt{\frac{k_3}{m_3}}, \omega_{21} = \frac{\omega_2}{\omega_1}, \omega_{31} = \frac{\omega_3}{\omega_1},$$

$$\alpha_{13} = \frac{k_{13}x_0^2}{m_1\omega_1^2}, \alpha_{23} = \frac{k_{23}x_0^2}{m_1\omega_1^2}, \alpha_{33} = \frac{k_{33}x_0^2}{m_1\omega_1^2}, F = \frac{F_{11}}{m_1\omega_1^2x_0}, \Omega = \frac{\Omega_1}{\omega_1}, x_0 = \text{reference length}$$

Equations (6.24) to (6.26) are solved using HBM in the following section.

6.4.1 Solution by HBM

Similar to the previous section, the approximate solutions of Eq. (6.24) to (6.26) using HBM can be assumed as follows.

$$x_1(\tau) = A(\tau) \cos(\Omega \tau + \phi_1(\tau)) \quad (6.27)$$

$$x_1(\tau) - x_2(\tau) = B(\tau) \cos(\Omega \tau + \phi_2(\tau)) \quad (6.28)$$

$$x_1(\tau) - x_3(\tau) = C(\tau) \cos(\Omega \tau + \phi_3(\tau)) \quad (6.29)$$

where $A(\tau)$, $B(\tau)$, $C(\tau)$, $\phi_1(\tau)$, $\phi_2(\tau)$ and $\phi_3(\tau)$ are slowly-varying functions of time τ , such that one can neglect the following terms: $\ddot{A}$, $\ddot{B}$, $\ddot{C}$, $\ddot{\phi}_1$, $\ddot{\phi}_2$, $\ddot{\phi}_3$, $\dot{A}\dot{\phi}_1$, $\dot{B}\dot{\phi}_1$, $\dot{C}\dot{\phi}_1$, $\dot{\phi}_1^2$,

$\dot{\varphi}_2^2$ and $\dot{\varphi}_3^2$. Substituting Eqs. (6.27) to (6.29) into Eqs. (6.24) to (6.26) and equating the coefficient of $\sin\Omega\tau$ and $\cos\Omega\tau$ terms separately to zero, yields six sets of algebraic equations. The steady-state solutions are obtained from these equations by considering $\dot{A}$, $\dot{B}$, $\dot{C}$, $\dot{\varphi}_1$, $\dot{\varphi}_2$ and $\dot{\varphi}_3$ equal to zero, which are given below.

$$-A\Omega^2 \sin \varphi_1 + A \sin \varphi_1 + 0.75\alpha_{13}A^3 \sin \varphi_1 + \mu_1\omega_{21}^2 B \sin \varphi_2 + 0.75\alpha_{23}B^3 \sin \varphi_2 + \mu_2\omega_{31}^2 C \sin \varphi_3 + 0.75\alpha_{32}C^3 \sin \varphi_3 = 0 \quad (6.30)$$

$$A \cos \varphi_1 (-\Omega^2 + 1) + 0.75\alpha_{13}A^3 \cos \varphi_1 + \mu_1\omega_{21}^2 B \cos \varphi_2 + 0.75\alpha_{23}B^3 \cos \varphi_2 + \mu_2\omega_{31}^2 C \cos \varphi_3 + 0.75\alpha_{32}C^3 \cos \varphi_3 - F = 0 \quad (6.31)$$

$$\mu_1 (-B\Omega^2 \cos \varphi_2 - A\Omega^2 \cos \varphi_1) + \mu_2 (-B\Omega^2 \cos \varphi_2 + C\Omega^2 \cos \varphi_3) + 2\zeta_2\mu_1\omega_{21} (B\Omega \sin \varphi_2 - C\Omega \cos \varphi_3) + \mu_1\omega_{21}^2 B \cos \varphi_2 + 0.75\alpha_{23}B^3 \cos \varphi_2 = 0 \quad (6.32)$$

$$\mu_1 (-B\Omega^2 \sin \varphi_2 - A\Omega^2 \sin \varphi_1) + \mu_2 (-B\Omega^2 \sin \varphi_2 + C\Omega^2 \sin \varphi_3) + 2\zeta_2\mu_1\omega_{21} (-B\Omega \cos \varphi_2 + C\Omega \cos \varphi_3) + \mu_1\omega_{21}^2 B \sin \varphi_2 + 0.75\alpha_{23}B^3 \sin \varphi_2 = 0 \quad (6.33)$$

$$\mu_2 (-C\Omega^2 \cos \varphi_3 + B\Omega^2 \cos \varphi_2) + 2\zeta_2\mu_1\omega_{21} (-B\Omega \sin \varphi_2 + C\Omega \cos \varphi_3) + \mu_2\omega_{31}^2 C \cos \varphi_3 + 0.75\alpha_{32}C^3 \cos \varphi_3 = 0 \quad (6.34)$$

$$\mu_2 (-C\Omega^2 \sin \varphi_3 + B\Omega^2 \sin \varphi_2) + 2\zeta_2\mu_1\omega_{21} (B\Omega \cos \varphi_2 - C\Omega \cos \varphi_3) + \mu_2\omega_{31}^2 C \sin \varphi_3 + 0.75\alpha_{32}C^3 \sin \varphi_3 = 0 \quad (6.35)$$

The steady-state frequency response of the system is obtained by solving Eq. (6.30) to (6.35) using Newton's method. In the following section, the numerical results are discussed.

6.4.2 Results and discussions

The performance of the nonlinear rotational inertial double-tuned mass damper (RIDTMD) is studied by considering the following non-dimensional parameters for the system. The mass ratios μ_1 and μ_2 are taken as 0.05, 0.005, respectively. The damping ratio ζ_2 of the

absorber is considered 0.241 and the frequency ratios ω_{21} and ω_{31} are taken as 0.943 and 1.043, respectively. The external harmonic excitation (F) and the reference length x_0 are taken equal to 1. It may be noted that these system parameters are optimum, which are considered from the work of Garrido et al. [340]. In the next section, the time response of the system is studied with the variation in the cubic nonlinear stiffness (α_{13} , α_{23} and α_{33}).

6.4.2.1 Time responses of the system with RIDTMD

The time response of the primary mass and the rotational inertia mass are shown in Fig. 6.15 by solving the governing equation of motion (Eqs. (6.24) to (6.26)) by the fourth-order Runge Kutta method considering linear and nonlinear stiffness in the system at the resonating frequency ($\Omega = 1$). Fig. 6.15(a-d) shows a comparison of responses between linear and nonlinear spring stiffness (α_{13}) in the primary system. It can be observed from these figures that for $\alpha_{13} = 0.5$, there is a significant reduction (78%) in the amplitude of the primary system than for $\alpha_{13} = 0$ (linear). Figs. 6.15(a) and 6.15(c) show beating type responses in the primary and secondary systems for the linear case, while quasiperiodic type responses are seen in both linear and nonlinear cases. The phase portraits of the system (Fig. 6.15(b) and Fig. 6.15(d)) show stable responses for both the linear and nonlinear cases.

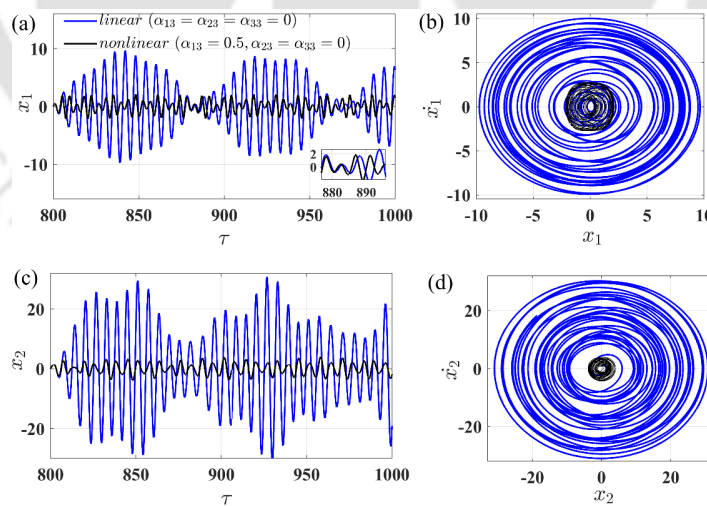


Fig. 6.15: Time response and phase portrait for $\alpha_{13} = 0$ (linear) $\alpha_{13} = 0.5$ (nonlinear) of the (a, b) primary mass (c, d) auxiliary mass of RIDTMD.

Fig. 6.16(a-d), shows the comparisons between the time responses and phase portraits of the primary and secondary system by considering only nonlinear spring stiffness present in the secondary system ($\alpha_3 = 0$, $\alpha_{23} = 0.5$) and considering nonlinear spring stiffness present both in the primary and secondary system ($\alpha_{13} = \alpha_{23} = 0.5$). The consideration of the nonlinear spring in both the primary and secondary system produces high responses than with the nonlinear spring present only in the secondary system. A 90% reduction in the

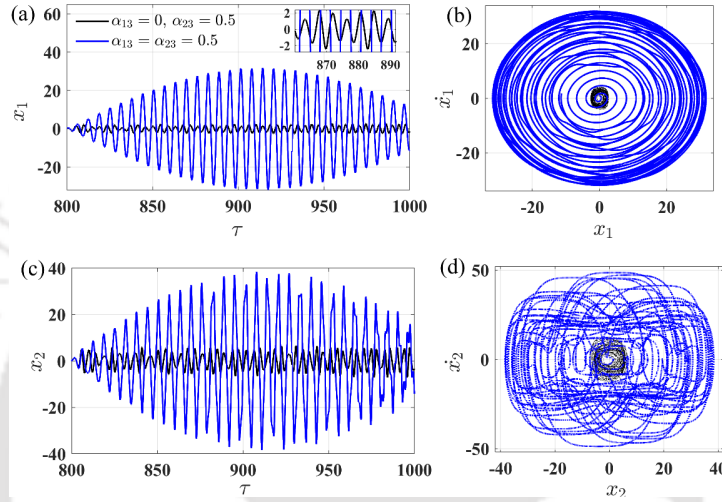


Fig. 6.16: Time response and phase portrait for the (a, b) primary mass (c, d) auxiliary mass of RIDTMD.

responses is observed when $\alpha_{13} = 0$, $\alpha_{23} = 0.5$ than $\alpha_{13} = 0.5$, $\alpha_{23} = 0.5$. However, comparing Fig. 6.15(a) with 6.16(a), the response amplitude of the primary system in the case of Fig. 6.15(a) is found to be slightly better (12.5%) than Fig. (6.21(a)). The corresponding phase portraits are shown in Figs. 6.16(b) and 6.16(d), respectively.

6.4.2.2 Frequency responses of the system with RIDTMD

In Fig. 6.17 the frequency responses of the primary system are shown by solving Eqs. (7.30) to (7.35) by Newton's method. In these figures, the frequency response of the primary system is studied considering various nonlinear spring stiffness in the system for $F = 0.1$, while all other system parameters are kept the same as mentioned before (section 6.4.2). Fig. 6.17(a) and Fig. 6.17(b) show the frequency responses of the primary system for the linear ($\alpha_{13} = 0$, $\alpha_{23} = \alpha_{33} = 0$) and nonlinear primary system ($\alpha_{13} = 0.5$, $\alpha_{13} = \alpha_{23} = 0$).

From Fig. 6.17(a), one can observe for the linear system three high resonating peaks in the response amplitude at $\Omega = 0.88, 1.04$ and 1.1 . The corresponding response amplitude considering only nonlinear stiffness in the primary system ($\alpha_{13} = 0.5, \alpha_{23} = \alpha_{33} = 0$) is shown in Fig. 6.17(b). From this figure, one observes a hardening effect in the frequency response but due to this effect, the response amplitude of the primary system at the peaks is significantly reduced at the above-mentioned frequencies. In Fig. 6.17 (c) the frequency response of the system is shown considering ($\alpha_{13} = 0.5, \alpha_{23} = 0.5$). From this figure more hardening effect in the frequency response of the primary system can be observed than shown in Fig. 6.17(a). These hardening effects in frequency response amplitude show multiple solutions of stable and unstable branches at these peaks. From Fig. 6.17(a-b) it can be observed that the consideration of only cubic nonlinear spring stiffness in the primary system is useful in suppressing vibration of the primary system at the three resonating peaks than the linear or nonlinear spring present both in the primary and secondary system. The frequency response amplitudes shown in Figs. 6.17(a) and 6.17(b) are also well matched with the time responses shown in Figs. 6.15 and 6.16, respectively. It may be noted that in

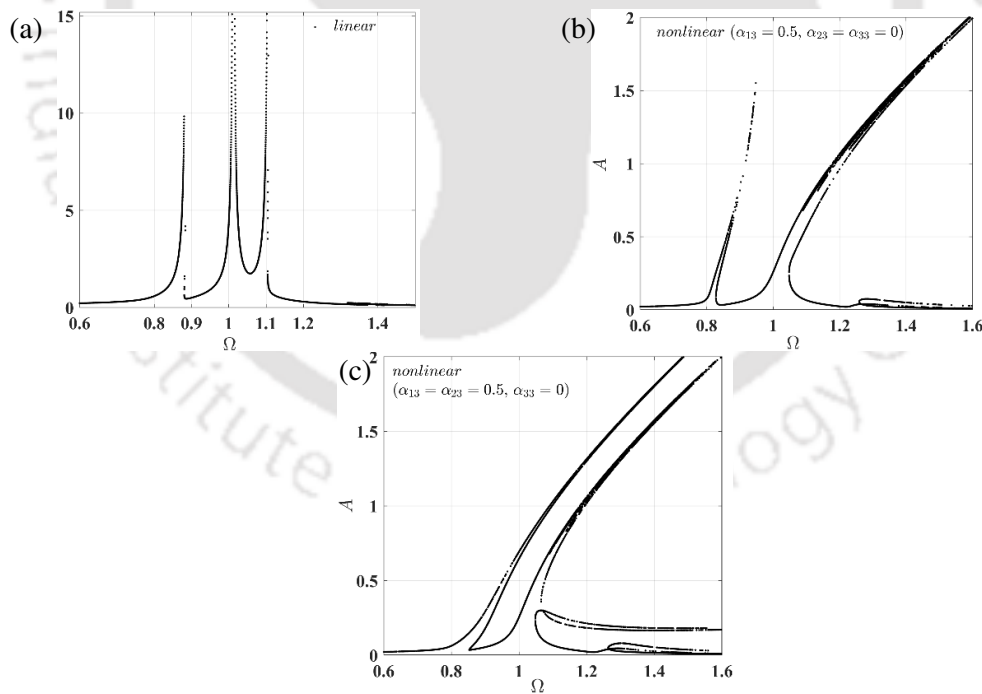


Fig. 6.17: Frequency response of the primary system for (a) linear system ($\alpha_{13} = 0, \alpha_{23} = \alpha_{33} = 0$) (b) nonlinear primary system ($\alpha_{13} = 0.5, \alpha_{23} = \alpha_{33} = 0$) (c) nonlinear system ($\alpha_{13} = 0.5, \alpha_{23} = 0.5, \alpha_{33} = 0$).

Figs. 6.15 and 6.16 the external amplitude of excitation (F) is considered 10 times less than that shown in Fig. 6.17. In Fig. 6.18(a) and Fig. 6.18(b) the frequency response amplitude of the primary system is compared with the variation of three nonlinear stiffness (α_{13} , α_{23} and α_{33}) present in the system by considering $F = 0.5$. Fig 6.18(a) shows higher α_{13} values produce more hardening effect in the frequency response than with other stiffness nonlinearities (α_{23} and α_{33}) present in the system. The higher α_{33} values shift the peak of the frequency response to the lower side of operating frequency as observed from Fig. 6.18(b) and with the negative nonlinear stiffness the frequency response tilt to the left side. In the next section, the conclusion of the various application of the nonlinear vibration absorber is discussed.

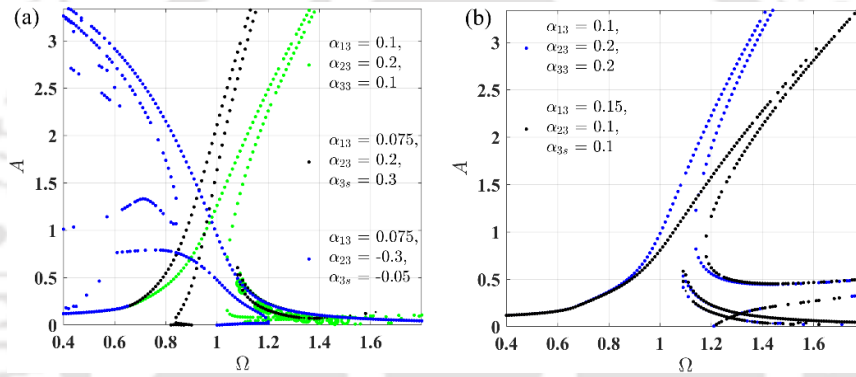


Fig. 6.18: Frequency response of the nonlinear system for (a) variation in α_{33} (b) variation in α_{13} , α_{23} and α_{33} .

6.5 Summary

In this chapter, three different primary systems with modifications in the absorber configurations have been studied. In the first problem, the primary system parameters are considered as the forearm of the human hand which is subjected to tremors due to Parkinson's disease. In the second problem, a beam system with various boundaries is considered. In order to suppress the vibration of these two primary systems, various parametric analyses were carried out considering the effects of nonlinear spring stiffness and active force applied by the vibration absorber. In the third problem, the active part of the vibration absorber is replaced with a two degree of freedom absorber, which is known as a rotational inertial double-tuned mass damper (RIDTMD) and its effects on the

frequency responses of the nonlinear primary system is studied with the variation in the nonlinear spring stiffness. In all these systems the governing nonlinear equations of motion are solved using HBM and the obtained results are also in good agreement with those by directly solving numerically the governing equations.

It is observed from the first system that with the various active feedback gain in the absorber the tremors in the forearm can be significantly reduced for a broader range of operating frequencies. The time responses, phase portraits, and the Poincare sections show the effectiveness of the active acceleration feedback gain by the active vibration absorber in suppressing the tremor when the forearm is assumed to undergo single and multi-excitations.

From the analyses of the second system where the vibration suppression of a general Euler Bernoulli beam system under a point load with fixed-fixed, simply supported, and cantilevered type boundary conditions are carried out, the following observations are noticed. The nonlinear stiffness in the absorber reduces the vibration of the beams system at the first three modal frequencies. In the active linear and nonlinear vibration absorber, by suitably taking the optimized displacement and velocity gains one may reduce the vibration of beams with fixed-fixed, simply supported and cantilevered beams for different modal frequencies.

In the last system, the efficacy of the nonlinear RIDTMD in suppressing the vibration of a single degree of freedom nonlinear spring-mass primary system is analysed. It is shown that with cubic nonlinear stiffness in the primary system the system amplitude reduces significantly at the resonating operating frequency due to the hardening effect of the frequency response.



CHAPTER 7

Conclusions

7.1 Introduction

In this section, the overall conclusions of the five problems considered in this work are given below. The present work focuses on vibration suppression of several linear and nonlinear primary systems under external harmonic, parametric, and base excitations by using a modified design in the absorber configuration. To investigate the vibration suppression of the primary system for a wider band of operating frequencies parametric studies have been carried out for a wide range of system parameters. The general conclusions for the five problems considered in this work are given below.

7.2 General conclusions

7.2.1 Linear PZT based active vibration absorber with acceleration feedback under external harmonic excitation.

- In the first problem, analyses are carried out considering linear active vibration absorber (AVA). Here, the primary system is modeled as a single degree of freedom (SDOF) linear spring-mass-damper system subjected to external harmonic forcing. The secondary system is modeled as a linear spring-mass damper system with PZT stack actuator in combination with a linear spring.
- The active force is produced by considering the acceleration feedback of the primary system. The governing equation of motion is solved using the Laplace transform method and Routh stability criteria are used to obtain the range of active force for which the system is stable.
- Fixed point theory (FPT) and Genetic algorithms (GA) are used to obtain the optimum system parameters viz., tuning ratio and damping ratio for the proposed AVA.

- The optimal active force is also evaluated at which the total vibration energy of the primary system is minimum with and without optimal tuning and damping ratio.
- The equal peaks in the frequency response of the system are observed with various active forces from the obtained optimal parameters of the AVA. The response amplitude of the primary system with the active force reduces close to the static deflection.
- The closed-form optimal equations obtained by the FPT for the optimal damping ratio and tuning ratio of the AVA can be utilized by the designer to easily suppress the vibration of the primary system effectively with less converging time than GA.

7.2.2 Dynamics of ANVA using Time Delay Acceleration Feedback

- In the second problem, a similar system is studied to the first problem but here cubic nonlinear springs are used in both the primary system and the absorber. Also here, the primary system is subjected to both external harmonic force and base excitations. The time delay is introduced in the acceleration feedback and the governing equations of motion are solved by the modified harmonic balancing method (HBM).
- Fundamentally based on nonlinearity the considered systems are divided into four cases viz., case-1: both primary and absorber are linear, case-2: primary system linear and absorber nonlinear, case-3: primary nonlinear and absorber linear, case-4: both primary and absorber nonlinear. Further considering the presence and absence of PZT stack actuator (active force), delay and base excitation twenty different cases have been investigated.
- The steady-state response of the system is studied through frequency responses, time responses, phase portraits, Poincare' sections, basin of attractions, force responses and different time delay effects in the feedback.
- From these analyses, the optimum range of nonlinear stiffness, active force by the acceleration feedback and time delay in the feedback are evaluated for both the passive vibration absorber (PVA) and ANVA at which the primary system response amplitude is minimum and stable.
- It is also shown that vibration suppression of both the primary system and the secondary system can be simultaneously achieved by the AVA.

- The Den Hartog's equal peak in the frequency response is also obtained for the first time by the use of AVA when the primary system is subjected to both harmonic and base excitations.

7.2.3 Traditional and Non-Traditional ANVA with Time Delay Combination Feedback

- In the third problem, the undamped primary system is considered and it is subjected to external harmonic force and parametric excitations. In the absorber, part modification is carried out by considering equivalent cubic nonlinear stiffness between the PZT stack actuator and the spring in series. Here time delayed displacement, velocity and acceleration feedback of the primary is considered.
- Two types of absorber models are considered viz., traditional vibration absorber (TVA) and non-traditional vibration absorber (NVA).
- In this case primary, principal parametric and simultaneous primary and principal parametric resonance conditions are studied.
- Here the method of multiple scales (MMS) is used which are in good agreement with HBM Laplace transform method and numerical method. From the reduced equations obtained by MMS, here for the first time Den Hartog's equal peaks in the frequency response of the system are shown for both linear and nonlinear systems.
- It is shown that the trivial state instability region is remarkably decreased with the time-delayed displacement and acceleration feedback by the TVA than the passive vibration absorber (PVA). But for the NVA the time delay does not decrease the instability region of the system as much as without delay.
- The trivial state region with TVA and NVA for the simultaneous force and parametric excitation shows an unstable region even at zero amplitude of parametric excitation with the PVA. But with displacement and acceleration feedback by the ANVA the critical value of parametric excitation is increased.
- For the linear and nonlinear systems, the safe operating frequency range is established for various system parameters and active forces.
- It is shown that with the proposed active gain complete vibration suppression of the linear and nonlinear primary system at the resonating peaks is achieved and also for

broadband of operating frequencies the vibration suppression of the primary system is effectively controlled.

7.2.4 Traditional and Non-Traditional ANVA with Time Delay Combination Feedback for Hard Excitations

- In the fourth problem, the SDOF primary system is subjected to hard harmonic, parametric and base excitations. The time delay in the damping in the TVA and NVA is taken into consideration and various combinations of time-delayed feedbacks are studied.
- The MMS is used to obtain the reduced equations and the system's trivial and non-trivial steady-state responses are analyzed for external and internal resonance conditions.
- The external resonance conditions considered in the present problems are simultaneous primary, principal parametric, sub and superharmonic resonance conditions along with 1:1, 1:2, 1:3 and 3:1 internal resonance conditions.
- The performance of the ANVA is studied by considering four different linear stiffness in the absorber with zero control gains. These four types of zero control gains by the ANVA mentioned in this work are *active₁*, *active₂*, *active₃* and *active₄* TVA, where the stiffness of the absorber can be actively changed.
- For simultaneous superharmonic, principal parametric, primary and 1:1 internal resonance conditions, it is shown that the trivial state instability region shows an unstable region even at zero external amplitude of parametric excitation for the PVA. But with actively changing the stiffness of the absorber the frequency ranges of the trivial state instability region increased. Further, it is shown that with velocity feedback and actively changing the stiffness of the absorber the instability region is remarkably reduced and the critical value of the amplitude of parametric excitation (below which the system is stable) improves from 0 N to 6.4 N.
- It is shown that with zero control force and *active₁* TVA, 100% vibration suppression for the nonlinear primary system is achieved near the slightly higher side of the superharmonic and primary resonance conditions.
- It is also shown that with the combination of velocity and acceleration feedback the vibration of the secondary system can also be reduced close to the zero response

amplitude but at that point, the response amplitude of the nonlinear primary system is comparatively higher (40%) than with velocity feedback.

- It is shown that one can almost completely absorb (98%) the response amplitude of both the nonlinear primary system as well as the secondary system by suitably using the velocity and acceleration feedback combination with the proposed ANVA.
- Considering time delay in the damper of the nonlinear secondary system it is shown that with the PVA the response amplitude of the nonlinear system is comparatively higher in comparison with the ANVA at the simultaneous resonating frequencies.
- The consideration of time delay in both the damper and in the feedback makes the system unstable in most of the operating frequencies. But considering small time delay either in the feedback or in the damping at a certain range of frequencies near the simultaneous resonance the response amplitudes of both the nonlinear primary system and the nonlinear secondary system are zero, i.e. 100% absorption.
- For the 3:1 internal resonance condition, it is observed that the response amplitude of the system becomes very large when the absorber modal frequency is close to the forcing frequency. But at other operating frequencies the response amplitude of the system is minimum.
- The control force by acceleration feedback with *active₃* TVA comparatively reduces the high response amplitude of the system and shows two peaks in the frequency response. But with the control force the unstable range of operating increases.
- For the 1:2 internal resonance condition with the active force, the response amplitude of the system is unstable but without active force, the response amplitude is close to zero. It may be noted that for 1:2 and 3:1 internal resonance conditions one can actively change the absorber stiffness to maintain the above internal resonance conditions.
- For simultaneous subharmonic, principal parametric, primary and 1:1 internal resonance conditions it is shown that with the control force by the combination feedback the trivial response is always stable than the PVA.
- For 1:3 internal resonance conditions, the time response of the system shows a better reduction in the response of the system at the higher side of the operating frequency for the simultaneous resonate conditions mentioned above.
- For the 1:2 internal resonance condition, it is shown that without control force and time delay the system response is unstable. However, with higher time delay both

in the damping and in the feedback and with the combination feedback force the response amplitude of the primary is almost zero at the steady-state.

7.2.5 Application of ANVA for vibration suppression

- In the fifth problem, the application of the ANVA is carried out considering three different systems.
- In the first system, the tremor suppression of the forearm by the ANVA is analyzed. The first system considered in this study is the same as the second problem (7.2.2) but here the primary system parameters are considered the same as a human forearm.
- It is shown that with the ANVA with acceleration feedback gain around 50% to 90% of vibration reduction in the response amplitude of the forearm is achieved than PVA.
- The time responses, phase portraits, and the Poincare sections show the effectiveness of active acceleration feedback gain by the absorber (bracelet) in suppressing the tremor, when the forearm is assumed to undergo single, two, and multi-base and harmonic excitations.
- In the second problem the general Euler-Bernoulli beam is considered as the primary system with fixed-fixed, simply supported and cantilevered type boundary conditions.
- The beam is subjected to a harmonically excited forcing at a point and its vibration suppression for the first three modal frequencies is analyzed with the ANVA.
- It is shown that with the cubic nonlinear spring in the absorber the 24% of vibration reduction in the beam is achieved without any control force than the linear PVA.
- Further, it is shown that with suitable active forces vibration reduction in the response amplitude of the beam is achieved for the first three modal frequencies.
- In the last or third system, a rotational inertial double-tuned mass damper (RIDTMD) is considered for vibration suppression of an SDOF nonlinear primary system.
- It is shown that the cubic nonlinear stiffness in the system reduces the vibration but shows multiple stable and unstable branches in the frequency response. The nonlinear system is not useful for broadband frequency of operation but at closer to

the resonating frequency of operation the system outperforms the linear RIDTMD. In the next section-specific conclusions related to the five problems considered in this work are presented.

7.3 Specific conclusions

7.3.1 Linear PZT based active vibration absorber with acceleration feedback under external harmonic excitation.

- The closed-form equation for the optimal tuning ratio (ω_{opt}) and damping ratio (ζ_{opt}) for the AVA with acceleration feedback obtained by the FPT technique is given as $\omega_{opt} = \sqrt{(1-F_a)/(1+\mu)}$ and $\zeta_{opt} = \sqrt{\frac{\zeta_2^2(\omega_a) + \zeta_2^2(\omega_b)}{2}}$. Here F_a and μ denote the active force and mass ratio between the absorber and the primary system. The term $\zeta_2^2(\omega_a)$ and $\zeta_2^2(\omega_b)$ are given in Chapter 2.
- From the Routh criterion, the range of active force at which the system is found to be stable is $-\mu < F_a < 1$.
- The optimization parameters obtained by FPT and GA give almost the same result in the frequency responses but GA takes more time to converge. So, one can use the closed-form equations shown above for the optimal parameters of AVA.
- 60% to 90% reduction in the maximum response amplitude of the primary system is achieved by varying the active feedback gain compared to the PVA.
- The optimal active force is found to be 0.19 with optimal parameters for the PVA mentioned in previous literature. The optimal active force obtained for the proposed AVA with acceleration feedback is 0.38.
- The step responses show the primary system response with active force settles down more quickly and the steady-state error is 94% less than that with PVA.

7.3.2 Dynamics of ANVA using Time Delay Acceleration Feedback

- For the PVA with cubic nonlinear spring stiffness equal to 2.2 % of its linear stiffness value, the first peak in the frequency response increases by 38% while the second peak reduces by 15% more than the linear PVA.
- Higher nonlinear stiffness in the PVA (22% of its linear stiffness) produces a very high response amplitude in the system for external harmonic forcing frequency (Ω) in the range of 0.9 to 0.972. Hence the cubic nonlinear stiffness value in the PVA must be below or equal to 2.2% of its linear stiffness to operate the system near or away from the resonating frequency ($\Omega = 1$). It may be noted that here Ω is a non-dimensional forcing frequency.
- The PVA with cubic nonlinear spring stiffness in the primary system is equal to 10% of linear stiffness, the system frequency response is similar to the linear PVA. But with higher nonlinear stiffness in the primary system (100% of its linear stiffness) shows jump up and jump down phenomena with saddle-node bifurcation points at $\Omega = 1.18$ and $\Omega = 1.8$. The jump up and jump down lengths are found to be 0.42 and 1.68 during sweeping down and sweeping up the forcing frequencies. The nonlinear stiffness in the primary system, however, reduces the response amplitude of the primary system to 37% near the first peak ($\Omega = 0.89$). In this case, the primary system is subjected to only harmonic force excitation.
- It is shown that with the active force 21% reduction in the peaks of the response amplitude of the primary system is achieved for lower nonlinear stiffness (10% of linear stiffness) present in the primary system when it is under both harmonic force and base excitation. Also, 72% reduction is achieved at resonating frequency. For higher nonlinear stiffness in the primary system (100% of linear stiffness), a 77% reduction in the second peak ($\Omega = 1.1$) is observed. Moreover, the highly nonlinear primary system behaves like that of a linear system with no hardening effect in the frequency response.
- In the case of the highly nonlinear absorber, by applying suitable active force the response amplitude of the primary system is shown to reduce significantly near the resonant frequency. After the resonant frequency, supercritical Hopf bifurcation is observed giving rise to periodic, quasiperiodic and chaotic responses. With the change

in the active force, one can control these responses to suppress the vibration of the system.

- With the increase in the active force by the ANVA the second peak in the frequency response shifts to the higher side of operating frequencies, so for a wider range of operating frequencies vibration suppression of the primary system is achieved. Also at the resonating frequency with the active force around 80%, vibration reduction is observed but shows both stable and unstable branches in the frequency response curve. Hence one has to choose a suitable control force to suppress the vibration of the primary system effectively.
- For the linear primary and secondary system around a 90% reduction in the response amplitude of the primary system is observed for forcing frequency (Ω) in the range of 0.85 to 1.1. It may be noted that here both harmonic and force excitations are acting on the primary system.
- It is shown that for the higher external amplitude of harmonic excitation on the nonlinear primary system the linear PVA shows unstable responses with jump up and jump down phenomena, Hopf bifurcation points in the force responses at $\Omega = 1.2$. But with the suitable active force by the ANVA the system response shows stable minimum response amplitude even for the very high amplitude of excitations. The effectiveness of the active force is observed also at $\Omega = 1$, reducing a minimum of 60% of response amplitude at different forcing amplitudes.
- The active force with a lower time delay (0.1) in the acceleration feedback performs marginally better (10% to 20% vibration reduction) than with zero-time delay for both linear and nonlinear primary systems.
- The time delay equal to 0.1 reduces 65% of response amplitude when the lower nonlinear stiffness (2.2%) is present only in the absorber. But for higher nonlinear stiffness in the absorber (22%), a 90% reduction in the response amplitude is achieved with a time delay of 1 in the active force.
- The higher time delay equal to 10, in the feedback makes the linear and nonlinear system unstable for a wide range of operating frequencies.
- It is also shown that with a comparative higher nonlinear stiffness in the spring attached to the PZT stack actuator vibration suppression of both the primary system and secondary system can be achieved. In this case, it is shown that 87% of vibration

suppression in the secondary system is achieved while for the primary system 31% reduction in the response is observed with the active force.

- The Den Hartog's equal peaks in the frequency response are the first time achieved for the AVA with acceleration feedback when the primary system is subjected to both harmonic and base excitations.
- In the proposed model, as both spring and actuator provide the controlling force, one can obtain a higher active force by the PZT stack actuator due to the blocking force produced by the spring which is in series connection with the PZT stack actuator. With the proposed ANVA, one can change the frequency of the absorber actively by applying the voltage, which can be used to tune the frequency of the absorber for a wider range of operating frequencies.
- From these analyses, it is shown that by suitably applying the controlling force by the spring and PZT stack actuator combination (30 - 90%) vibration suppression of maximum response amplitude of the primary system can be achieved for various nonlinear stiffness in the system and amplitude of external force and base excitations.

7.3.3 Traditional and Non-Traditional ANVA with Time Delay Combination Feedback

- For the nonlinear system, linear and nonlinear system Den Hartog's equal peaks in the frequency response are obtained by using MMS reduced equations which are not reported in earlier literature.
- The proposed active traditional vibration absorber (TVA) and non-traditional vibration absorber (NVA), reduce a significant reduction in the trivial state instability regions and around a 200% increase in the critical value of the amplitude of parametric excitations are found, which are not present in the earlier literature.
- The higher nonlinear stiffness in the main system (50% of linear stiffness) does not affect the stability of the system as much as the nonlinear stiffness in the absorber (10% of linear stiffness).
- For the linear system it is shown that with the various combination of active control forces by displacement, velocity and acceleration feedback complete vibration suppression (100% reduction) of the primary system can be achieved at the primary resonance condition.

- For the linear system, the vibration suppression for a wider band of operating frequencies is also achieved with a minimum reduction of 58% response amplitude at the peaks to a maximum of 100% for harmonic forcing frequency in the range of 0 to 1.6 rad/sec. It may be noted that in this problem primary resonance is at 1 rad/sec.
- The vibration suppression by the nontraditional vibration absorber (NVA) is comparatively better both in the case of with and without active force than the traditional vibration absorber (TVA). For NVA the complete vibration suppression is observed at a forcing frequency equal to 1.12 rad/sec. A 10% more reduction in the vibration of the primary system by NVA is achieved than TVA for forcing frequency in the range 1.12 to 1.6 rad/sec.
- A higher time delay (greater than $\pi/10$ sec) in the feedback makes the system unstable. However, lower time delay ($\pi/10$ sec) shows better vibration suppression near resonating frequency than without delay.
- For both primary and principal parametric resonance conditions complete (100% reduction) in the vibration of the primary system is achieved by the different combination feedback used in the TVA. Also for a wider band of operating frequency i.e 1 rad/sec to 1.6 rad/sec and 2 rad/sec to 3.2 rad/sec a minimum of 75% decrease in the response amplitude of the primary system is observed than the PVA. It may be noted that primary and principal parametric resonance condition occurs at 1 rad/sec and 2 rad/sec, respectively.
- For NVA complete vibration suppression is at 1.12 rad/sec for both primary and principal parametric resonance conditions and also for a broad range (1.2 rad/sec 1.6 rad/sec and 2.4 rad/sec to 3.2 rad/sec) a minimum of 90% reduction in vibration of the primary system is achieved than PVA.
- For the nonlinear system, 100% vibration reduction is achieved at 0.95 rad/sec and 1.9 rad/sec and also for a wide range (0.625 rad/sec to 1.12 rad/sec and 1.3 rad/sec to 2.24 rad/sec) a minimum of 75% reduction in the response amplitude of the primary system is observed than PVA.
- However, the active feedback gain induces a more unstable range of responses showing saddle-node bifurcations, Hopf bifurcation and chaotic responses at various operating frequencies for various nonlinear stiffness in comparison with the PVA. So, at these operating frequencies, one can use zero active force which makes

the system the same as PVA and thereby can suppress the vibration of the primary system for a wide range of operating frequencies.

- The force responses show with the active feedback gains by the ANVA the jump up and jump down phenomena is not observed in the primary system with the variation in the external harmonic and parametric excitations. Whereas with PVA two Hopf bifurcation points and two saddle node bifurcation points are observed. The saddle-node bifurcation points which lead to jump up and jump down phenomena are observed with the PVA are at 1 rad/sec and 2 rad/sec and forcing amplitude equals 0.82 N, 0.91 N, 1.06 and 2.29 N.
- This sudden decrease or increase in the response amplitude of the main system may fail the system, so the active TVA is more useful in these conditions. Similar observations are also noticed with the increase in the nonlinear stiffness coefficient of the system.
- The time delay in the range of $3\pi/4$ sec to $5\pi/4$ sec almost completely reduces the vibration of the nonlinear system with the combination feedback. Outside this region, unstable responses in the system are observed.
- The efficacy of the active feedback gain is observed in reducing the response amplitude of the nonlinear primary system to a minimum of 66% to a maximum of 97% at the resonating primary and principal parametric resonance conditions with the variation of the amplitude of force and parametric excitations from 0.2 N to 1.6 N for a mass ratio equal to 0.05.
- The efficacy of active force is also observed for the mass ratio of 0.025 (absorber mass is 40 times less than primary mass) with a reduction of 72% response amplitude for the primary and principal parametric resonance for the amplitude of harmonic and parametric excitations equals to 0.4 N.
- The efficacy of the active force is observed for a mass ratio equal to 0.01 also but in this case, one has to increase the active force to suppress the vibration.
- From this study, it is shown that one can completely suppress the vibration of the linear and nonlinear primary system under primary, principal parametric and simultaneous primary and principal parametric resonance conditions. The analyses show the effectiveness of the proposed ANVA especially for a mass ratio equal to 0.05 but its efficacy can be observed at a higher mass ratio and higher forcing amplitude with the increase in the active forces by the ANVA.

7.3.4 Traditional and Non-Traditional ANVA with Time Delay Combination Feedback for Hard Excitations

- For the system primary system under simultaneous superharmonic, principal parametric and primary resonance conditions with 1:1 internal resonance conditions, it is shown that the response amplitude of the primary system can be effectively reduced close to the 100% absorption at a certain range of frequencies at the above-mentioned resonance conditions.
- The 100% vibration suppression is achieved at various operating frequencies by many combinations of different feedback forces for various nonlinear stiffness in the system.
- The time delay both in damper and feedback makes the system more prone to unstable responses with the active forces. So, in this case, zero active force can be utilized and actively changing the frequency of the absorber the vibration suppression can be achieved.
- The non-traditional vibration absorber performs marginally better than the traditional vibration absorber considering both linear and nonlinear stiffness in the system.
- In the ANVA with 1:2 and 3:1 internal resonance conditions, a 100% reduction in the response amplitude of the primary system is achieved for external harmonic forcing frequency Ω_1 in the range of 0.18 rad/sec to 0.88 rad/sec without any feedback gain.
- The nontrivial state undergoes a transition from stable to unstable region by saddle-node bifurcation for both passive and active cases.
- For simultaneous subharmonic, principal parametric, primary and 1:1 internal resonance conditions the nontrivial state undergoes a transition from stable to unstable region by saddle-node bifurcation for both passive and active cases at operating frequency 1.36 rad/sec.
- The trivial solution with the PVA for the above resonance conditions shows unstable solutions for operating frequency in the range of 1.15 rad/sec to 1.65 rad/sec. But with the active force by the ANVA the trivial state always becomes stable.

- For 1:3 internal resonance conditions, the efficacy of the active controlling force is observed in the time responses when the operating frequencies are on the higher side (Ω_1 (rad/sec) = $3\omega_1 + 0.4$, Ω_2 (rad/sec) = $2\omega_1 + 0.4$, and Ω_3 (rad/sec) = $\omega_1 + 0.4$). It is shown that a 62% reduction in the response of the primary system is achieved with the active gain. It may be noted that here Ω_1 , Ω_2 and Ω_3 refers to the external operating frequencies for external hard harmonic excitation, parametric excitation and base excitation, respectively.
- For the 1:2 internal resonance condition, from the time responses plots, it can be observed that with active combination feedback along with time delay in the damper and feedback almost 100% vibration reduction is achieved. It may be noted that in the passive case the system time response at these internal and external resonance conditions is unstable.

7.3.5 Application of ANVA for vibration suppression

- It is shown that with the active acceleration feedback gain by the active linear vibration absorber (active bracelet) the first peak in the response amplitude of the primary system (forearm) is decreased by 52% and the second peak is decreased by 96% than with PVA (passive bracelet). Also, considering parameters of the human forearm the active bracelet with the F_{ac} gain reduces the response amplitude by 72.5% in the first peak (3.83 Hz) and 92.6% in the second peak (6.95 Hz) than the PVA.
- It is shown that the F_{ac} gain by the AVA (active bracelet) significantly suppresses the tremor suppression of the forearm both for the linear and nonlinear system for $\Omega > 3.5$ Hz than the PVA. For the lower frequency region, one can provide zero control gain and use the active bracelet passively to effectively suppress the tremors for a wide range of operating frequencies.
- The nonlinear stiffness in the system produces unstable responses, so one has to avoid these regions of operating frequencies to have less response amplitude of the forearm.
- The effectiveness of the acceleration feedback gain is also observed at the resonating frequencies in suppressing vibration than the PVA when the forearm is subjected to multi-base and harmonic excitations. active

- For the second system, one can observe that with cubic nonlinear stiffness in the absorber (nonlinear PVA) the response amplitude of the fixed-fixed beam at the first modal frequency decreases by 24% and for the nonlinear absorber 50% reduction in the response amplitude is observed.
- Further considering velocity feedback in the nonlinear AVA, 52% and 28% reduction in the response amplitude of the fixed-fixed beam and the absorber are observed in comparison with nonlinear PVA at the first modal frequency.
- For the third system, it is shown that with nonlinear stiffness in the primary system 78% to 90% reduction in the response amplitude of the primary system can be achieved than considering linear stiffness but for a narrow range of operating frequencies.
- The three high resonating peaks in the primary system are observed at 0.86 rad/sec, 1 rad/sec and 1.1 rad/sec, respectively with the linear PVA. The nonlinear stiffness in the primary system or the absorber reduces the response amplitude for a very narrow band of operating frequencies close to the resonating peaks. So one has to operate the system within the mentioned range of operating frequencies to suppress the vibration of the primacy system effectively.

This study will help the designer or the operator to identify the safe operating zone and the system responses for various system parameters such as various nonlinear stiffness values in the system, active forces through various combination feedbacks, actively changing the frequency of the absorber, time delays in the damping and in the feedback for both passive and active traditional and non-traditional vibration absorbers. From the above study, one can know the system responses considering various external and internal resonance conditions.

7.4 Scope of Future work

In the present study vibration, suppression by ANVA through various feedbacks and resonance conditions are studied. There are many scopes to extend the present work to enhance the vibration suppression of the various system. Some of them are given below.

- One can obtain the closed-form formulae for the optimal displacement, velocity and acceleration feedback control gains and time delay by an active nonlinear vibration

absorber to suppress the vibration of the primary system for many resonance conditions as considered in the present study.

- Similarly, one can obtain the formulae for the optimum nonlinear stiffness and feedback gain required both in the primary system and the secondary system by considering active nonlinear vibration absorbers (ANVA) for various feedback and time delays for a base excited primary system with external harmonic excitation.
- The optimal time delay for equal peaks in response amplitude may also be the focus of future research.
- The present work on ANVA attached to the beam can be extended considering damping in the beam and evaluating the optimal location and time-delayed feedback gains required by the ANVA to suppress the vibration of the beam for a wider range of operating frequencies considering various excitations and resonance conditions acting on the beam.
- The performance analyses between the nonlinear RIDTMD and ANVA with time-delay feedback can be extended to study the vibration suppression of the linear and nonlinear primary systems. The close form relation for the optimal nonlinear stiffness in the RIDTMD can be evaluated to achieve Den Hartog's equal peaks in the response.
- The present work of ANVA on suppression of forearm tremors can be extended to find the optimal time delayed active feedback gains required to suppress tremors. Experimental analyses can also be extended to study the effectiveness of the ANVA on patients.
- The application of ANVA is now widening in many fields such as in sports cars, wind turbines, biomedical devices, structural response under wind excitations, especially for flexible structures, etc. The present study can be extended into many other applications considering the actual system parameters.
- The vibration of the absorber can be utilized for energy harvesting purposes.
- The present work can be extended to an active nonlinear multi DOF vibration absorber with the spatial distribution and following optimality criterion.
- Experimental analyses for the active nonlinear vibration absorber for various resonance conditions studied in this work can also be implemented in future research work.

Appendix-B

Traditional and Non-Traditional ANVA with Time Delay Combination Feedback for Primary and Principal Parametric Resonance Conditions

$$\begin{aligned}
 2i\omega_1 D_1 A_1 &= iz_{11} A_1 \omega_1 - iz_{12} A_2 \omega_2 \exp(i(\sigma_1 - \sigma_2)T_1) - \\
 p_{11} \bar{k}_{13} &\left(p_{11}^3 N_1 + 3p_{11}^2 p_{12} N_2 + \right. \\
 &\left. 3p_{11} p_{12}^2 N_3 + p_{12}^3 N_4 \right) + (p_{21} - p_{11}) \bar{k}_{23} \left(N_{11} N_1 + N_{12} N_2 + \right. \\
 &\left. N_{13} N_3 + N_{14} N_4 \right) \\
 &+ (p_{21} - p_{11}) k_2 p_{11} A_{1d} Q_1 \exp(-i\omega_1 t_d) + (p_{21} - p_{11}) \bar{k}_{23} N \\
 &+ (p_{21} - p_{11}) k_2 p_{12} A_{2d} Q_2 \exp(-i\omega_2 t_d + i(\sigma_1 - \sigma_2)T_1) + 0.5 p_{11} F_{11} \exp(i\sigma_1 T_1) \\
 &+ 0.5 p_{11} F_{12} \left(p_{11} \bar{A}_1 \exp(i\sigma_3 T_1) + p_{12} \bar{A}_2 \exp(i(\sigma_3 - (\sigma_1 - \sigma_2)T_1)) \right)
 \end{aligned} \tag{B.1}$$

$$\begin{aligned}
 2i\omega_2 D_1 A_2 &= iz_{21} A_1 \omega_1 \exp(-i(\sigma_1 - \sigma_2)T_1) - iz_{12} A_2 \omega_2 - \\
 p_{12} \bar{k}_{13} &\left(p_{11}^3 N_{11} + 3p_{11}^2 p_{12} N_{12} \right. \\
 &\left. + 3p_{11} p_{12}^2 N_{13} + p_{12}^3 N_{14} \right) + (p_{22} - p_{12}) \bar{k}_{23} \left(N_{11} N_{11} + N_{12} N_{12} \right. \\
 &\left. + N_{13} N_{13} + N_{14} N_{14} \right) \\
 &+ (p_{22} - p_{12}) k_2 p_{11} A_{1d} Q_1 \exp(-i(\omega_1 t_d + (\sigma_1 - \sigma_2)T_1)) \\
 &+ (p_{22} - p_{12}) k_2 p_{12} A_{2d} Q_2 \exp(-i\omega_2 t_d) + (p_{22} - p_{12}) \bar{k}_{23} \tilde{N} + 0.5 p_{12} F_{11} \exp(i\sigma_2 T_1) \\
 &+ 0.5 p_{12} F_{12} \left(p_{12} \bar{A}_2 \exp(i\sigma_4 T_1) + p_{11} \bar{A}_1 \exp(i(\sigma_4 + (\sigma_1 - \sigma_2)T_1)) \right)
 \end{aligned} \tag{B.2}$$

Coefficients Q_1 , Q_2 , N_{11} to N_{24} , N and $\tilde{N}$ are given below.

$$\begin{aligned}
 Q_1 &= f_d - i\omega_1 f_v - \omega_1^2 f_a, \quad Q_2 = f_d - i\omega_2 f_v - \omega_2^2 f_a, \\
 N_{11} &= (p_{11} - p_{21})^3, \quad N_{12} = 3(p_{11}^2 + p_{21}^2)(p_{12} - p_{22}) + 6p_{11}(p_{21}p_{22} - p_{12}p_{21}), \\
 N_{13} &= 3(p_{12}^2 + p_{22}^2)(p_{11} - p_{21}) + 6p_{12}p_{22}(p_{21} - p_{11}), \quad N_{14} = (p_{12} - p_{22})^3, \\
 N_{21} &= 3p_{11}(p_{11}^2 + 2p_{11}p_{21} + p_{21}^2), \quad N_{22} = 6p_{11}(p_{12} - p_{22})(p_{11} - p_{21}), \\
 N_{23} &= 3p_{21}(p_{11}^2 - 2p_{11}p_{21} + 3p_{21}^2), \quad N_{24} = 6p_{21}^2(p_{22} - p_{12}) + 6p_{11}p_{21}(p_{12} + p_{22}), \\
 N_{25} &= 3p_{11}(p_{12}^2 - 2p_{12}p_{22} + p_{22}^2), \quad N_{26} = 3p_{21}(p_{12}^2 - 2p_{12}p_{22} + p_{22}^2),
 \end{aligned}$$

$$\begin{aligned}
N = & N_{21}Q_1A_1 \exp(i\omega_1 t_d) \left(A_1 \bar{A}_{1d} + 2\bar{A}_1 A_{1d} \exp(-2i\omega_1 t_d) \right) + \\
& N_{22}Q_1 \exp(-i\omega_1 t_d - i(\sigma_1 - \sigma_2)T_1) \left(\frac{A_1 \bar{A}_2 A_{1d} + \bar{A}_1 A_2 A_{1d} \exp(2i(\sigma_1 - \sigma_2)T_1)}{A_1 A_2 \bar{A}_{1d} \exp(2i\omega_1 t_d + 2i(\sigma_1 - \sigma_2)T_1)} + \right) + \\
& N_{23}Q_2 A_1 \exp(i\omega_2 t_d) \left(A_1 \bar{A}_{2d} + 2\bar{A}_1 A_{2d} \exp(-2i\omega_2 t_d + i(\sigma_1 - \sigma_2)T_1) \right) + \\
& N_{24}Q_2 \exp(-i\omega_2 t_d) \left(\frac{A_1 \bar{A}_2 A_{2d} + \bar{A}_1 A_2 A_{2d} \exp(2i(\sigma_1 - \sigma_2)T_1)}{A_1 A_2 \bar{A}_{2d} \exp(2i\omega_2 t_d)} + \right) + \\
& N_{25}Q_1 A_2 \exp(-i\omega_1 t_d) \left(\bar{A}_2 A_{1d} + A_2 A_{1d} \exp(2i\omega_1 t_d + 2i(\sigma_1 - \sigma_2)T_1) \right) + \\
& N_{26}Q_2 A_2 \exp(i\omega_2 t_d + i(\sigma_1 - \sigma_2)T_1) \left(A_2 \bar{A}_{2d} + \bar{A}_2 A_{2d} \exp(-2i\omega_2 t_d) \right),
\end{aligned}$$

$$\begin{aligned}
\tilde{N} = & N_{21}Q_1 A_1 \exp(i(\omega_1 t_d - (\sigma_1 - \sigma_2)T_1)) \left(A_1 \bar{A}_{1d} + 2\bar{A}_1 A_{1d} \exp(-2i\omega_1 t_d) \right) + \\
& N_{22}Q_1 \exp(-i\omega_1 t_d - 2i(\sigma_1 - \sigma_2)T_1) \left(\frac{A_1 \bar{A}_2 A_{1d} + \bar{A}_1 A_2 A_{1d} \exp(2i(\sigma_1 - \sigma_2)T_1)}{A_1 A_2 \bar{A}_{1d} \exp(2i\omega_1 T_d + 2i(\sigma_1 - \sigma_2)T_1)} + \right) + \\
& N_{23}Q_2 A_1 \exp(i(\omega_2 t_d - 2(\sigma_1 - \sigma_2)T_1)) \left(A_1 \bar{A}_{2d} + 2\bar{A}_1 A_{2d} \exp(-2i\omega_2 t_d) \right) + \\
& N_{24}Q_2 \exp(-i(\omega_2 t_d + (\sigma_1 - \sigma_2)T_1)) \left(\frac{A_1 \bar{A}_2 A_{2d} + \bar{A}_1 A_2 A_{2d} \exp(2i(\sigma_1 - \sigma_2)T_1)}{A_1 A_2 \bar{A}_{2d} \exp(2i\omega_2 t_d)} + \right) + \\
& N_{25}Q_1 A_2 \exp(-i\omega_1 t_d) \left(\bar{A}_2 A_{1d} + A_2 A_{1d} \exp(2i\omega_1 t_d + 2i(\sigma_1 - \sigma_2)T_1) \right) + \\
& N_{26}Q_2 A_2 \exp(i\omega_2 t_d + i(\sigma_1 - \sigma_2)T_1) \left(A_2 \bar{A}_{2d} + \bar{A}_2 A_{2d} \exp(-2i\omega_2 t_d) \right)
\end{aligned}$$

Terms Q_{11} , Q_{12} , $\tilde{Q}_{11}$, $\tilde{Q}_{12}$, q_{11} , q_{12} , $\tilde{q}_{11}$, $\tilde{q}_{12}$, G_1 to G_6 , $\tilde{G}_1$ to $\tilde{G}_6$ and h_{11} to h_{14} are given below.

$$\begin{aligned}
Q_{11} = & (f_d - f_a \omega_1^2) \sin(\omega_1 t_d + \beta_1 \tau_d) - f_v \omega_1 \cos(\omega_1 t_d + \beta_1 \tau_d), \\
Q_{12} = & (f_d - f_a \omega_1^2) \sin(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d) - f_v \omega_2 \cos(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d), \\
\tilde{Q}_{11} = & (f_d - f_a \omega_1^2) \cos(\omega_1 t_d + \beta_1 \tau_d) + f_v \omega_1 \sin(\omega_1 t_d + \beta_1 \tau_d), \\
\tilde{Q}_{12} = & (f_d - f_a \omega_1^2) \cos(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d) - f_v \omega_2 \sin(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d), \\
q_{11} = & (f_d - f_a \omega_1^2) \sin(\gamma_2 - \gamma_1 - \omega_1 t_d - \beta_1 \tau_d) + f_v \omega_2 \cos(\gamma_2 - \gamma_1 - \omega_1 T_d - \beta_1 \tau_d), \\
q_{12} = & -(f_d - f_a \omega_1^2) \sin(\omega_2 T_d + \beta_2 \tau_d) - f_v \omega_2 \cos(\omega_2 t_d + \beta_2 \tau_d), \\
\tilde{q}_{11} = & (f_d - f_a \omega_1^2) \cos(\gamma_2 - \gamma_1 - \omega_1 t_d - \beta_1 \tau_d) - f_v \omega_1 \cos(\gamma_2 - \gamma_1 - \omega_1 t_d - \beta_1 \tau_d), \\
\tilde{q}_{12} = & (f_d - f_a \omega_1^2) \cos(\omega_2 t_d + \beta_2 \tau_d) - f_v \omega_2 \cos(\omega_2 t_d + \beta_2 \tau_d),
\end{aligned}$$

$$\begin{aligned}
G_1 &= \sin(\gamma_1 - \gamma_2) (p_{11}^2 a_1^2 + p_{12}^2 a_2^2) + p_{11} p_{12} a_1 a_2 \sin(2(\gamma_1 - \gamma_2)), \\
G_2 &= (N_{12} a_1^2 + N_{14} a_2^2) \sin(\gamma_1 - \gamma_2) + N_{13} a_1 a_2 \sin(2(\gamma_1 - \gamma_2)), \\
G_3 &= N_{21} a_1^2 a_{1d} \left((f_d - f_a \omega_1^2) (\sin(\omega_1 t_d - \beta_1 \tau_d) - 2 \sin(\omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad \left(+ f_v \omega_1 (\cos(\omega_1 t_d - \beta_1 \tau_d) + 2 \cos(\omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad N_{22} a_1 a_{1d} a_2 \left((f_d - f_a \omega_1^2) \left(\sin \left(\frac{\gamma_2 - \gamma_1 -}{\omega_1 t_d - \beta_1 \tau_d} \right) + 2 \sin(\gamma_1 - \gamma_2 - \beta_1 \tau_d) \cos(\omega_1 t_d) \right) \right) + \\
&\quad \left(+ f_v \omega_1 (\cos(\gamma_2 - \gamma_1 - \omega_1 t_d - \beta_1 \tau_d) + 2 \cos(\gamma_1 - \gamma_2 - \beta_1 \tau_d) \cos(\omega_1 t_d)) \right) + \\
&\quad N_{23} a_1^2 a_{2d} \left((f_d - f_a \omega_2^2) (\sin(\gamma_2 - \gamma_1 + \omega_2 t_d - \beta_2 \tau_d) + 2 \sin(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d)) \right) + \\
&\quad \left(+ f_v \omega_2 (\cos(\gamma_2 - \gamma_1 + \omega_2 t_d - \beta_2 \tau_d) + 2 \cos(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d)) \right) + \\
&\quad N_{24} a_1 a_{2d} a_2 \left((f_d - f_a \omega_2^2) (\sin(2(\gamma_1 - \gamma_2) - \omega_2 t_d - \beta_2 \tau_d) - 2 \cos(\omega_2 t_d) \sin(\beta_2 \tau_d)) \right) + \\
&\quad \left(+ f_v \omega_2 (2 \cos(\omega_2 t_d) \cos(\beta_2 \tau_d) + \cos(2(\gamma_1 - \gamma_2) - \omega_2 t_d - \beta_2 \tau_d)) \right) + \\
&\quad N_{25} a_2^2 a_{1d} \left((f_d - f_a \omega_1^2) (\sin(2(\gamma_1 - \gamma_2) + \omega_1 t_d + \beta_1 \tau_d) - \sin(\omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad \left(- f_v \omega_1 (\cos(\omega_1 t_d + \beta_1 \tau_d) + \cos(2(\gamma_1 - \gamma_2) + \omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad 2 N_{26} a_2^2 a_{2d} \cos(\omega_2 t_d + \beta_2 \tau_d) ((f_d - f_a \omega_2^2) \sin(\gamma_1 - \gamma_2) - f_v \omega_2 \cos(\gamma_1 - \gamma_2)), \\
\tilde{G}_1 &= p_{11}^3 a_1^3 + (p_{11}^2 p_{12} a_1^2 a_2 + p_{12}^3 a_2^3) \cos(\gamma_1 - \gamma_2) + 2 p_{11} p_{12}^2 a_1 a_2^2 (1 + \cos(2(\gamma_1 - \gamma_2))), \\
\tilde{G}_2 &= 3 N_{11} a_1^3 + (N_{12} a_1^2 a_2 + N_{14} a_2^3) \cos(\gamma_1 - \gamma_2) + 2 N_{13} a_1 a_2^2 (1 + \cos(2(\gamma_1 - \gamma_2))), \\
\tilde{G}_3 &= N_{21} a_1^2 a_{1d} \left((f_d - f_a \omega_1^2) (\cos(\omega_1 t_d - \beta_1 \tau_d) - 2 \sin(\omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad \left(- f_v \omega_1 (\sin(\omega_1 t_d - \beta_1 \tau_d) + 2 \sin(\omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad N_{22} a_1 a_{1d} a_2 \left((f_d - f_a \omega_1^2) \left(\cos \left(\frac{\gamma_2 - \gamma_1 -}{\omega_1 t_d - \beta_1 \tau_d} \right) + 2 \cos(\gamma_1 - \gamma_2 - \beta_1 \tau_d) \cos(\omega_1 t_d) \right) \right) + \\
&\quad \left(- f_v \omega_1 (\sin(\gamma_2 - \gamma_1 - \omega_1 t_d - \beta_1 \tau_d) + 2 \sin(\gamma_1 - \gamma_2 - \beta_1 \tau_d) \cos(\omega_1 t_d)) \right) + \\
&\quad N_{23} a_1^2 a_{2d} \left((f_d - f_a \omega_2^2) \left(\cos \left(\frac{\gamma_2 - \gamma_1 +}{\omega_2 t_d - \beta_2 \tau_d} \right) + 2 \cos(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d) \right) \right) + \\
&\quad \left(- f_v \omega_2 (\sin(\gamma_2 - \gamma_1 + \omega_2 t_d - \beta_2 \tau_d) + 2 \sin(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d)) \right) + \\
&\quad N_{24} a_1 a_{2d} a_2 \left((f_d - f_a \omega_2^2) (2 \cos(\omega_2 t_d) \cos(\beta_2 \tau_d) + \cos(2(\gamma_1 - \gamma_2) - \omega_2 t_d - \beta_2 \tau_d)) \right) + \\
&\quad \left(- f_v \omega_2 (-2 \cos(\omega_2 t_d) \sin(\beta_2 \tau_d) + \sin(2(\gamma_1 - \gamma_2) - \omega_2 t_d - \beta_2 \tau_d)) \right) + \\
&\quad + N_{25} a_2^2 a_{1d} \left((f_d - f_a \omega_1^2) (\cos(-\omega_1 t_d - \beta_1 \tau_d) + \cos(2(\gamma_1 - \gamma_2) + \omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad \left(+ f_v \omega_1 (\sin(-\omega_1 t_d - \beta_1 \tau_d) + \sin(2(\gamma_1 - \gamma_2) + \omega_1 t_d + \beta_1 \tau_d)) \right) + \\
&\quad 2 N_{26} a_2^2 a_{2d} \cos(\omega_2 t_d + \beta_2 \tau_d) ((f_d - f_a \omega_2^2) \cos(\gamma_1 - \gamma_2) + f_v \omega_2 \sin(\gamma_1 - \gamma_2)),
\end{aligned}$$

$$\begin{aligned}
\tilde{G}_3 = & N_{21}a_1^2a_{1d} \left((f_d - f_a\omega_1^2)(\cos(\omega_1t_d - \beta_1\tau_d) - 2\sin(\omega_1t_d + \beta_1\tau_d)) \right) + \\
& N_{22}a_1a_{1d}a_2 \left((f_d - f_a\omega_1^2) \left(\cos\left(\frac{\gamma_2 - \gamma_1 - \omega_1t_d - \beta_1\tau_d}{\omega_1t_d - \beta_1\tau_d}\right) + 2\cos(\gamma_1 - \gamma_2 - \beta_1\tau_d)\cos(\omega_1t_d) \right) \right) + \\
& N_{23}a_1^2a_{2d} \left((f_d - f_a\omega_2^2) \left(\cos\left(\frac{\gamma_2 - \gamma_1 + \omega_2t_d - \beta_2\tau_d}{\omega_2t_d - \beta_2\tau_d}\right) + 2\cos(\gamma_1 - \gamma_2 - \omega_2t_d - \beta_2\tau_d) \right) \right) + \\
& N_{24}a_1a_{2d}a_2 \left((f_d - f_a\omega_2^2) (2\cos(\omega_2t_d)\cos(\beta_2\tau_d) + \cos(2(\gamma_1 - \gamma_2) - \omega_2t_d - \beta_2\tau_d)) \right) + \\
& N_{25}a_2^2a_{1d} \left((f_d - f_a\omega_1^2) (\cos(-\omega_1t_d - \beta_1\tau_d) + \cos(2(\gamma_1 - \gamma_2) + \omega_1t_d + \beta_1\tau_d)) \right) + \\
& 2N_{26}a_2^2a_{2d} \cos(\omega_2t_d + \beta_2\tau_d) ((f_d - f_a\omega_2^2)\cos(\gamma_1 - \gamma_2) + f_v\omega_2\sin(\gamma_1 - \gamma_2)), \\
G_4 = & (p_{11}^2a_1^2 + p_{12}^2a_2^2)\sin(\gamma_2 - \gamma_1) + p_{11}p_{12}a_1a_2\sin(2(\gamma_2 - \gamma_1)), \\
G_5 = & (3N_{11}a_1^2 + N_{13}a_2^2)\sin(\gamma_1 - \gamma_2) + N_{12}a_1a_2\sin(2(\gamma_2 - \gamma_1)), \\
G_6 = & N_{21}a_1^2a_{1d} \left((f_d - f_a\omega_1^2) (\sin(\gamma_2 - \gamma_1 + \omega_1t_d - \beta_1\tau_d) + 2\sin(\gamma_2 - \gamma_1 - \omega_1t_d + \beta_1\tau_d)) \right) + \\
& N_{22}a_1a_{1d}a_2 \left((f_d - f_a\omega_1^2) (\sin(2(\gamma_2 - \gamma_1) - \omega_1t_d - \beta_1\tau_d) - 2\cos(\omega_1t_d)\sin(\beta_1\tau_d)) \right) + \\
& N_{23}a_1^2a_{2d} \left((f_d - f_a\omega_2^2) (\sin(2(\gamma_2 - \gamma_1) + \omega_2t_d - \beta_2\tau_d) - 2\sin(\omega_2t_d - \beta_2\tau_d)) \right) + \\
& N_{24}a_1a_{2d}a_2 \left((f_d - f_a\omega_2^2) (2\sin(\gamma_2 - \gamma_1 - \beta_2\tau_d)\cos(\omega_2t_d) + \sin(\gamma_1 - \gamma_2 - \omega_2t_d - \beta_2\tau_d)) \right) + \\
& + 2N_{35}a_2^2a_{1d} ((f_d - f_a\omega_1^2)\sin(\gamma_1 - \gamma_2 + \omega_1t_d + \beta_1\tau_d) - f_v\omega_1\cos(\gamma_1 - \gamma_2 + \omega_1t_d + \beta_1\tau_d)) + \\
& + 2N_{36}a_2^2a_{2d} ((f_d - f_a\omega_1^2)\sin(\omega_2t_d + \beta_2\tau_d) - f_v\omega_2\cos(\omega_2t_d + \beta_2\tau_d)), \\
\tilde{G}_4 = & (p_{11}^3a_1^3 + p_{11}p_{12}^2a_1a_2^2)\cos(\gamma_2 - \gamma_1) + p_{11}^2p_{12}a_1^2a_2(\cos(2(\gamma_2 - \gamma_1)) + 2) + p_{12}^3a_2^3, \\
\tilde{G}_5 = & (3N_{11}a_1^3 + N_{13}a_1a_2^2)\cos(\gamma_2 - \gamma_1) + N_{12}a_1^2a_2(\cos(2(\gamma_2 - \gamma_1)) + 2) + 3N_{14}a_2^3,
\end{aligned}$$

$$\begin{aligned}
\tilde{G}_6 = & N_{21} a_1^2 a_{1d} \left((f_d - f_a \omega_1^2) (\cos(\gamma_2 - \gamma_1 + \omega_1 t_d - \beta_1 \tau_d) + 2 \cos(\gamma_2 - \gamma_1 - \omega_1 t_d + \beta_1 \tau_d)) \right. \\
& \left. - f_v \omega_1 (\sin(\gamma_2 - \gamma_1 + \omega_1 t_d - \beta_1 \tau_d) + 2 \sin(\gamma_2 - \gamma_1 - \omega_1 t_d + \beta_1 \tau_d)) \right) + \\
& N_{22} a_1 a_{1d} a_2 \left((f_d - f_a \omega_1^2) (\cos(2(\gamma_2 - \gamma_1) - \omega_1 t_d - \beta_1 \tau_d) + 2 \cos(\omega_1 t_d) \cos(\beta_1 \tau_d)) \right. \\
& \left. - f_v \omega_1 (\sin(2(\gamma_2 - \gamma_1) - \omega_1 t_d - \beta_1 \tau_d) + \sin(\omega_1 t_d) \cos(\beta_1 \tau_d)) \right) \\
& + N_{23} a_1^2 a_{2d} \left((f_d - f_a \omega_2^2) (\cos(2(\gamma_2 - \gamma_1) + \omega_2 t_d - \beta_2 \tau_d) - 2 \cos(\omega_2 t_d + \beta_2 \tau_d)) \right. \\
& \left. - f_v \omega_2 (\sin(2(\gamma_2 - \gamma_1) + \omega_2 t_d - \beta_2 \tau_d) - 2 \sin(\omega_2 t_d + \beta_2 \tau_d)) \right) + \\
& N_{24} a_1 a_{2d} a_2 \left((f_d - f_a \omega_2^2) (2 \cos(\gamma_2 - \gamma_1 - \beta_2 t_d) \cos(\omega_2 t_d) + \cos(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d)) \right. \\
& \left. - f_v \omega_2 (2 \sin(\gamma_2 - \gamma_1 - \beta_2 t_d) \cos(\omega_2 t_d) + \sin(\gamma_1 - \gamma_2 - \omega_2 t_d - \beta_2 \tau_d)) \right) \\
& + 2 N_{35} a_2^2 a_{1d} \left((f_d - f_a \omega_1^2) \cos(\gamma_1 - \gamma_2 + \omega_1 t_d + \beta_1 \tau_d) + f_v \omega_1 \sin(\gamma_1 - \gamma_2 + \omega_1 t_d + \beta_1 \tau_d) \right) + \\
& + 2 N_{36} a_2^2 a_{2d} \left((f_d - f_a \omega_1^2) \cos(\omega_2 t_d + \beta_2 \tau_d) + f_v \omega_2 \sin(\omega_2 t_d + \beta_2 \tau_d) \right), \\
h_{11} = & \sin(\omega_1 t_d + \beta_1 \tau_d) p_1 + \cos(\omega_1 t_d + \beta_1 \tau_d) q_1, \\
h_{12} = & \cos(\omega_1 t_d + \beta_1 \tau_d) p_1 - \sin(\omega_1 t_d + \beta_1 \tau_d) q_1, \\
h_{13} = & \cos(\omega_2 t_d + \beta_2 \tau_d) q_2 + \sin(\omega_2 t_d + \beta_2 \tau_d) p_2, \\
h_{14} = & \cos(\omega_2 t_d + \beta_2 \tau_d) p_2 - \sin(\omega_2 t_d + \beta_2 \tau_d) q_2,
\end{aligned}$$

The steady state response of the system is studied by considering $a_1 = a_{1d}$, $a_2 = a_{2d}$, $\omega_1 t_d = \beta_1 \tau_d$ and $\omega_2 t_d = \beta_2 \tau_d$.

The MMS second order reduced equations are given below.

$$\begin{aligned}
a_1' = & \varepsilon \left(-0.5 z_{11} a_1 - 0.5 z_{12} a_2 \frac{\omega_2}{\omega_1} \cos(\gamma_1 - \gamma_2) + 0.5 \frac{p_{11} F_{11}}{\omega_1} \sin \gamma_1 \right) + \\
\varepsilon^2 & \left[-\frac{z_{11}}{2 \omega_1} \left(-0.25 z_{11} a_1 - z_{12} \frac{a_2 \omega_2}{4 \omega_1} \cos(\gamma_1 - \gamma_2) + \frac{p_{11} F_{11}}{4 \omega_1} \sin \gamma_1 \right) - \right. \\
& \frac{z_{12} \omega_2}{2 \omega_1^2} \left(0.25 z_{22} a_2 \sin(\gamma_1 - \gamma_2) + \frac{p_{12} F_{11}}{4 \omega_2} \cos \gamma_1 \right) - \frac{p_{11} F_{11} \sigma_1}{4 \omega_1^2} \sin \gamma_1 + \\
& \frac{z_{11}}{\omega_1} \left(z_{12} a_2 \frac{\omega_2}{4 \omega_1} \sin(\gamma_1 - \gamma_2) + \frac{p_{11} F_{11}}{4 \omega_1} \cos \gamma_1 \right) + \\
& \left. \frac{z_{12}}{\omega_1} \left(0.25 z_{22} a_2 \sin(\gamma_1 - \gamma_2) + \frac{p_{12} F_{12}}{4 \omega_2} \cos \gamma_1 \right) \right] \quad (B.3)
\end{aligned}$$

$$\gamma_1' = \varepsilon \left(\sigma_1 + z_{12} \frac{a_2 \omega_2}{2a_1 \omega_1} \sin(\gamma_1 - \gamma_2) + \frac{p_{11} F_{11}}{2a_1 \omega_1} \cos \gamma_1 \right) - \varepsilon^2 \left[\begin{aligned} & \frac{z_{11}}{2\omega_1} \left(\frac{z_{12} a_2 \omega_2}{4a_1 \omega_1} \sin(\gamma_1 - \gamma_2) + \frac{p_{11} F_{11}}{4a_1 \omega_1} \cos \gamma_1 \right) + \frac{z_{12} \omega_2}{2\omega_1^2} \left(z_{21} \frac{a_1 \omega_1}{4\omega_2} + z_{22} \frac{a_2}{4} \cos(\gamma_1 - \gamma_2) \right) \\ & + \frac{z_{12} p_{12} F_{11}}{8\omega_1^2} \sin \gamma_1 - \frac{p_{11} F_{11} \sigma_1}{4a_1 \omega_1^2} \cos \gamma_1 \\ & - \frac{z_{11}}{a_1 \omega_1} \left(\frac{z_{12} a_2 \omega_2}{4\omega_1} \cos(\gamma_1 - \gamma_2) - \frac{p_{11} F_{11}}{4\omega_1} \sin \gamma_1 + 0.25 z_{11} a_1 \right) \\ & - \frac{z_{12}}{a_1 \omega_1} \left(z_{22} \frac{a_2 \omega_2}{4\omega_1} \cos(\gamma_1 - \gamma_2) - \frac{p_{12} F_{11}}{4\omega_2} \sin \gamma_1 + z_{21} \frac{a_1 \omega_1}{4\omega_2} \right) \end{aligned} \right] \quad (\text{B.4})$$

$$a_2' = \varepsilon \left(-0.5 z_{21} a_1 \frac{\omega_1}{\omega_2} \cos(\gamma_2 - \gamma_1) - 0.5 z_{22} a_2 + 0.5 \frac{p_{12} F_{11}}{\omega_2} \sin \gamma_2 \right) + \varepsilon^2 \left[\begin{aligned} & \frac{z_{21} \omega_1}{2\omega_2^2} \left(0.25 z_{11} a_1 \cos(\gamma_2 - \gamma_1) + 0.25 z_{12} a_2 \frac{\omega_2}{\omega_1} - \frac{p_{11} F_{11}}{4\omega_1} \sin \gamma_2 \right) \\ & - \frac{z_{22}}{2\omega_2} \left(\frac{z_{21} a_1 \omega_1}{4\omega_2} \sin(\gamma_2 - \gamma_1) + \frac{p_{12} F_{11}}{4\omega_2} \cos \gamma_2 \right) - \frac{p_{12} F_{11} \sigma_2}{4\omega_2^2} \sin \gamma_2 + \\ & \frac{z_{21}}{\omega_2} \left(\frac{z_{11} a_1}{4} \sin(\gamma_2 - \gamma_1) + \frac{p_{11} F_{11}}{4\omega_1} \cos \gamma_2 \right) + \frac{z_{22}}{\omega_2} \left(\frac{z_{21} a_1}{4\omega_1} \sin(\gamma_2 - \gamma_1) + \frac{p_{12} F_{11}}{4\omega_2} \cos \gamma_2 \right) \end{aligned} \right] \quad (\text{B.5})$$

$$\gamma_2' = \varepsilon \sigma_2 - \varepsilon \left(-0.5 z_{21} \frac{a_1 \omega_1}{a_2 \omega_2} \sin(\gamma_2 - \gamma_1) - 0.5 \frac{p_{12} F_{11}}{a_2 \omega_2} \cos \gamma_2 \right) + \varepsilon^2 \left[\begin{aligned} & - \frac{z_{21} \omega_1}{2\omega_2^2} \left(\frac{z_{11} a_1}{4a_2} \sin(\gamma_2 - \gamma_1) + \frac{p_{11} F_{11}}{4a_2 \omega_1} \cos \gamma_2 \right) - \frac{p_{12} F_{11}}{4a_2 \omega_2^2} \sigma_2 \cos \gamma_2 \\ & - \frac{z_{22}}{2\omega_2} \left(z_{21} \frac{a_1 \omega_1}{4a_2 \omega_2} \cos(\gamma_2 - \gamma_1) + 0.25 z_{22} - 0.25 \frac{p_{12} F_{11}}{4a_2 \omega_2} \sin \gamma_2 \right) \\ & + \frac{z_{21}}{a_2 \omega_2} \left(\frac{z_{11} a_1}{4} \cos(\gamma_2 - \gamma_1) + z_{12} \frac{a_2 \omega_2}{4\omega_1} - \frac{p_{11} F_{11}}{4\omega_1} \sin \gamma_2 \right) + \\ & \frac{z_{22}}{a_2 \omega_2} \left(\frac{z_{21} a_1}{4\omega_1} \cos(\gamma_2 - \gamma_1) + 0.25 z_{22} a_2 - \frac{p_{12} F_{11}}{4\omega_2} \sin \gamma_2 \right) \end{aligned} \right] \quad (\text{B.6})$$

where $\gamma_1 = \sigma_1 T_1 - \beta_1$, $\gamma_2 = \sigma_2 T_1 - \beta_2$.

Appendix-C

Traditional and Non-Traditional ANVA with Time Delay

Combination Feedback for Hard Excitations

The terms $D_1, D_2, D_3, D_4, D_5, D_6$ and D_7 are given as

$$\begin{aligned}
 D_1 &= \frac{2\Omega_1 \sin(\Omega_1(T_0 - \tau_1))}{\omega_1^2 - (\Omega_1(T_0 - \tau_1))^2}, \\
 D_2 &= -\frac{A_1^3 \cos(3\omega_1 T_0)}{4\omega_1^2} + \frac{2(3\Lambda_1^3 + 6A_1\bar{A}_1\Lambda_1)}{\omega_1^2 - \Omega_1^2} \cos(\Omega_1 T_0) + \frac{6\bar{A}_1\Lambda_1^2 \cos((2\Omega_1 - \omega_1)T_0)}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} \\
 &\quad + \frac{6A_1\Lambda_1^2 \cos((\omega_1 + 2\Omega_1)T_0)}{\omega_1^2 - (\omega_1 + 2\Omega_1)^2} + \frac{6A_1^2\Lambda_1 \cos((2\omega_1 + \Omega_1)T_0)}{\omega_1^2 - (2\omega_1 + \Omega_1)^2} + \frac{6A_1^2\Lambda_1 \cos((2\omega_1 - \Omega_1)T_0)}{\omega_1^2 - (2\omega_1 - \Omega_1)^2}, \\
 D_3 &= \frac{2A_1^2 A_2 \cos((2\omega_1 + \omega_2)T_0)}{\omega_1^2 - (2\omega_1 + \omega_2)^2} + \frac{2A_1^2 \Lambda_2 \cos((2\omega_1 + \Omega_1)T_0)}{\omega_1^2 - (2\omega_1 + \Omega_1)^2} + \frac{2A_1^2 \Lambda_2 \cos((2\omega_1 - \Omega_1)T_0)}{\omega_1^2 - (2\omega_1 - \Omega_1)^2} \\
 &\quad + \frac{2A_2\Lambda_1^2 \cos((2\Omega_1 + \omega_2)T_0)}{\omega_1^2 - (2\Omega_1 + \omega_2)^2} + \frac{2\bar{A}_2\Lambda_1^2 \cos((2\Omega_1 - \omega_2)T_0)}{\omega_1^2 - (2\Omega_1 - \omega_2)^2} + \frac{2\Lambda_1^2 A_2 \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2} + \\
 &\quad \frac{4A_1 A_2 \Lambda_1 \cos((\omega_1 + \omega_2 + \Omega_1)T_0)}{\omega_1^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{4A_1 \Lambda_1 \Lambda_2 \cos((\omega_1 + 2\Omega_1)T_0)}{\omega_1^2 - (\omega_1 + 2\Omega_1)^2} + \\
 &\quad \frac{4A_1 \Lambda_1 \bar{A}_2 \cos((\omega_1 - \omega_2 + \Omega_1)T_0)}{\omega_1^2 - (\omega_1 - \omega_2 + \Omega_1)^2} + \frac{4\bar{A}_1 A_2 \Lambda_1 \cos((\Omega_1 - \omega_1 + \omega_2)T_0)}{\omega_1^2 - (\Omega_1 - \omega_1 + \omega_2)^2} + \\
 &\quad 4 \left(\frac{\bar{A}_1 \Lambda_1 \Lambda_2 \cos((2\Omega_1 - \omega_1)T_0)}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} + \frac{\bar{A}_1 \bar{A}_2 \Lambda_1 \cos((\Omega_1 - \omega_1 - \omega_2)T_0)}{\omega_1^2 - (\Omega_1 - \omega_1 - \omega_2)^2} + \frac{(\bar{A}_1 A_2 \Lambda_1 + \Lambda_1^2 \Lambda_2) \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2} \right), \\
 D_4 &= \frac{2A_2^2 A_1 \cos((2\omega_2 + \omega_1)T_0)}{\omega_1^2 - (2\omega_2 + \omega_1)^2} + \frac{2A_2^2 \Lambda_1 \cos((2\omega_2 + \Omega_1)T_0)}{\omega_1^2 - (2\omega_2 + \Omega_1)^2} + \frac{2A_2^2 \Lambda_1 \cos((2\omega_2 - \Omega_1)T_0)}{\omega_1^2 - (2\omega_2 - \Omega_1)^2} +
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2A_1\Lambda_2^2 \cos((\omega_1 + 2\Omega_1)T_0)}{\omega_1^2 - (2\Omega_1 + \omega_2)^2} + \frac{2\bar{A}_1\Lambda_2^2 \cos((2\Omega_1 - \omega_1)T_0)}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} + \frac{2\Lambda_2^2\Lambda_1 \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2} \\
& + \frac{4A_2A_1\Lambda_2 \cos((\omega_1 + \omega_2 + \Omega_1)T_0)}{\omega_1^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{4A_2\Lambda_1\Lambda_2 \cos((\omega_2 + 2\Omega_1)T_0)}{\omega_1^2 - (\omega_2 + 2\Omega_1)^2} \\
& + 4\Lambda_2 \left(\frac{A_2\bar{A}_1 \cos((\omega_2 + \Omega_1 - \omega_1)T_0)}{\omega_1^2 - (\omega_2 + \Omega_1 - \omega_1)^2} + \frac{\bar{A}_2A_1 \cos((\Omega_1 - \omega_2 + \omega_1)T_0)}{\omega_1^2 - (\Omega_1 - \omega_2 + \omega_1)^2} + \frac{\bar{A}_2\Lambda_1 \cos((2\Omega_1 - \omega_2)T_0)}{\omega_1^2 - (2\Omega_1 - \omega_2)^2} \right) \\
& + \frac{4\bar{A}_1\bar{A}_2\Lambda_1 \cos((\Omega_1 - \omega_1 - \omega_2)T_0)}{\omega_1^2 - (\Omega_1 - \omega_1 - \omega_2)^2} + \frac{(4\bar{A}_1A_1\Lambda_2 + 4\Lambda_1^2\Lambda_2) \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2},
\end{aligned}$$

$$\begin{aligned}
D_5 = & \frac{2A_2^3 \cos(3\omega_2 T_0)}{\omega_1^2 - 9\omega_2^2} + \frac{2(3\Lambda_2^3 + 6A_2\bar{A}_2\Lambda_2)}{\omega_1^2 - \Omega_2^2} \cos(\Omega_2 T_0) + \frac{6\Lambda_2^2\bar{A}_2 \cos((2\Omega_1 - \omega_2)T_0)}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} \\
& + \frac{6A_2\Lambda_2^2 \cos((\omega_2 + 2\Omega_1)T_0)}{\omega_1^2 - (\omega_2 + 2\Omega_1)^2} + \frac{6A_2^2\Lambda_2 \cos((2\omega_2 + \Omega_1)T_0)}{\omega_1^2 - (2\omega_2 + \Omega_1)^2} + \frac{6A_2^2\Lambda_2 \cos((2\omega_2 - \Omega_1)T_0)}{\omega_1^2 - (2\omega_2 - \Omega_1)^2},
\end{aligned}$$

$$D_6 = \frac{2p_1 k_e}{\omega_1^2 - (\Omega_1(T_0 - \tau_2))^2} \left(\cos(\Omega_1(T_0 - \tau_2)) \left((p_{11}\Lambda_1 + p_{12}\Lambda_2)(f_d - f_a\Omega_1^2) \right) \right. \\
\left. - f_v\Omega_1(p_{11}\Lambda_1 + p_{12}\Lambda_2) \sin(\Omega_1(T_0 - \tau_2)) \right),$$

$$D_7 = \left(\frac{p_{11}A_1 \cos((\omega_1 + \Omega_2)T_0)}{\omega_1^2 - (\omega_1 + \Omega_2)^2} + \frac{p_{12}A_2 \cos((\omega_2 + \Omega_2)T_0)}{\omega_1^2 - (\omega_2 + \Omega_2)^2} \right. \\
\left. + \frac{(p_{11}\Lambda_1 + p_{12}\Lambda_2) \cos((\Omega_1 + \Omega_2)T_0)}{\omega_1^2 - (\Omega_1 + \Omega_2)^2} + \frac{(p_{11}\Lambda_1 + p_{12}\Lambda_2) \cos((- \Omega_1 + \Omega_2)T_0)}{\omega_1^2 - (- \Omega_1 + \Omega_2)^2} \right).$$

The terms $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$, $\tilde{D}_4$, $\tilde{D}_5$, $\tilde{D}_6$ and $\tilde{D}_7$ are given as

$$\tilde{D}_1 = \frac{2\Omega_1 \sin(\Omega_1(T_0 - \tau_2))}{\omega_2^2 - (\Omega_1(T_0 - \tau_2))^2},$$

$$\begin{aligned}
\tilde{D}_2 = & \frac{2A_1^3}{\omega_2^2 - 9\omega_1^2} \cos(3\omega_1 T_0) + \frac{2(3\Lambda_1^3 + 6A_1\bar{A}_1\Lambda_1)}{\omega_2^2 - \Omega_1^2} \cos(\Omega_1 T_0) + \frac{6\bar{A}_1\Lambda_1^2 \cos((2\Omega_1 - \omega_1)T_0)}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} \\
& + \frac{6A_1\Lambda_1^2 \cos((\omega_1 + 2\Omega_1)T_0)}{\omega_2^2 - (\omega_1 + 2\Omega_1)^2} + \frac{6A_1^2\Lambda_1 \cos((2\omega_1 + \Omega_1)T_0)}{\omega_2^2 - (2\omega_1 + \Omega_1)^2} + \frac{6A_1^2\Lambda_1 \cos((2\omega_1 - \Omega_1)T_0)}{\omega_2^2 - (2\omega_1 - \Omega_1)^2},
\end{aligned}$$

$$\begin{aligned}
\tilde{D}_3 = & \frac{2A_1^2 A_2 \cos((2\omega_1 + \omega_2)T_0)}{\omega_2^2 - (2\omega_1 + \omega_2)^2} + \frac{2A_1^2 \Lambda_2 \cos((2\omega_1 + \Omega_1)T_0)}{\omega_2^2 - (2\omega_1 + \Omega_1)^2} + \frac{2A_1^2 \Lambda_2 \cos((2\omega_1 - \Omega_1)T_0)}{\omega_2^2 - (2\omega_1 - \Omega_1)^2} \\
& + \frac{2A_2 \Lambda_1^2 \cos((2\Omega_1 + \omega_2)T_0)}{\omega_2^2 - (2\Omega_1 + \omega_2)^2} + \frac{2\bar{A}_2 \Lambda_1^2 \cos((2\Omega_1 - \omega_2)T_0)}{\omega_2^2 - (2\Omega_1 - \omega_2)^2} + \frac{2\Lambda_1^2 A_2 \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2} \\
& + \frac{4A_1 A_2 \Lambda_1 \cos((\omega_1 + \omega_2 + \Omega_1)T_0)}{\omega_2^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{4A_1 \Lambda_1 \Lambda_2 \cos((\omega_1 + 2\Omega_1)T_0)}{\omega_2^2 - (\omega_1 + 2\Omega_1)^2} \\
& + \frac{4A_1 \Lambda_1 \bar{A}_2 \cos((\omega_1 - \omega_2 + \Omega_1)T_0)}{\omega_2^2 - (\omega_1 - \omega_2 + \Omega_1)^2} + \frac{4\bar{A}_1 A_2 \Lambda_1 \cos((\Omega_1 - \omega_1 + \omega_2)T_0)}{\omega_2^2 - (\Omega_1 - \omega_1 + \omega_2)^2} \\
& + \frac{4\bar{A}_1 \Lambda_1 \Lambda_2 \cos((2\Omega_1 - \omega_1)T_0)}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} + \frac{4\bar{A}_1 \bar{A}_2 \Lambda_1 \cos((\Omega_1 - \omega_1 - \omega_2)T_0)}{\omega_2^2 - (\Omega_1 - \omega_1 - \omega_2)^2} \\
& + \frac{(4\bar{A}_1 A_2 \Lambda_1 + 4\Lambda_1^2 \Lambda_2) \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2},
\end{aligned}$$

$$\begin{aligned}
\tilde{D}_4 = & \frac{2A_2^2 A_1 \cos((2\omega_2 + \omega_1)T_0)}{\omega_2^2 - (2\omega_2 + \omega_1)^2} + \frac{2A_2^2 \Lambda_1 \cos((2\omega_2 + \Omega_1)T_0)}{\omega_2^2 - (2\omega_2 + \Omega_1)^2} + \frac{2A_2^2 \Lambda_1 \cos((2\omega_2 - \Omega_1)T_0)}{\omega_2^2 - (2\omega_2 - \Omega_1)^2} \\
& + \frac{2A_1 \Lambda_2^2 \cos((\omega_1 + 2\Omega_1)T_0)}{\omega_2^2 - (2\Omega_1 + \omega_2)^2} + \frac{2\bar{A}_1 \Lambda_2^2 \cos((2\Omega_1 - \omega_1)T_0)}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} + \frac{2\Lambda_2^2 \Lambda_1 \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2} \\
& + \frac{4A_2 A_1 \Lambda_2 \cos((\omega_1 + \omega_2 + \Omega_1)T_0)}{\omega_2^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{4A_2 \Lambda_1 \Lambda_2 \cos((\omega_2 + 2\Omega_1)T_0)}{\omega_2^2 - (\omega_2 + 2\Omega_1)^2} \\
& + \frac{4A_2 \bar{A}_1 \Lambda_2 \cos((\omega_2 + \Omega_1 - \omega_1)T_0)}{\omega_2^2 - (\omega_2 + \Omega_1 - \omega_1)^2} + \frac{4\bar{A}_2 A_1 \Lambda_2 \cos((\Omega_1 - \omega_2 + \omega_1)T_0)}{\omega_2^2 - (\Omega_1 - \omega_2 + \omega_1)^2} \\
& + \frac{4\bar{A}_2 \Lambda_2 \Lambda_1 \cos((2\Omega_1 - \omega_2)T_0)}{\omega_2^2 - (2\Omega_1 - \omega_2)^2} + \frac{4\bar{A}_1 \bar{A}_2 \Lambda_1 \cos((\Omega_1 - \omega_1 - \omega_2)T_0)}{\omega_2^2 - (\Omega_1 - \omega_1 - \omega_2)^2} \\
& + \frac{(4\bar{A}_1 A_1 \Lambda_2 + 4\Lambda_1^2 \Lambda_2) \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2},
\end{aligned}$$

$$\begin{aligned}
\tilde{D}_5 = & -\frac{A_2^3 \cos(3\omega_2 T_0)}{4\omega_2^2} + \frac{2(3\Lambda_2^3 + 6A_2 \bar{A}_2 \Lambda_2)}{\omega_2^2 - \Omega_2^2} \cos(\Omega_2 T_0) + \frac{6\Lambda_2^2 \bar{A}_2 \cos((2\Omega_1 - \omega_2)T_0)}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} \\
& + \frac{6A_2 \Lambda_2^2 \cos((\omega_2 + 2\Omega_1)T_0)}{\omega_2^2 - (\omega_2 + 2\Omega_1)^2} + \frac{6A_2^2 \Lambda_2 \cos((2\omega_2 + \Omega_1)T_0)}{\omega_2^2 - (2\omega_2 + \Omega_1)^2} + \frac{6A_2^2 \Lambda_2 \cos((2\omega_2 - \Omega_1)T_0)}{\omega_2^2 - (2\omega_2 - \Omega_1)^2},
\end{aligned}$$

$$\begin{aligned}\tilde{D}_7 = & \frac{p_{11}A_1 \cos((\omega_1 + \Omega_2)T_0)}{\omega_2^2 - (\omega_1 + \Omega_2)^2} + \frac{p_{12}A_2 \cos((\omega_2 + \Omega_2)T_0)}{\omega_2^2 - (\omega_2 + \Omega_2)^2} \\ & + \frac{(p_{11}\Lambda_1 + p_{12}\Lambda_2) \cos((\Omega_1 + \Omega_2)T_0)}{\omega_2^2 - (\Omega_1 + \Omega_2)^2} + \frac{(p_{11}\Lambda_1 + p_{12}\Lambda_2) \cos((- \Omega_1 + \Omega_2)T_0)}{\omega_2^2 - (-\Omega_1 + \Omega_2)^2}.\end{aligned}$$

The solution by MMS method are given as

$$\begin{aligned}X_1 = & X_{10} + \varepsilon X_{11} \\ = & a_1 \cos(3\Omega_1 T_0 - \gamma_1) + 2\Lambda_1 \cos(\Omega_1 T_0) + \varepsilon(c_{11}\Lambda_1 + c_{12}\Lambda_2)D_1 - \varepsilon k_{n1}D_{21} - 3\varepsilon k_{n2}D_{31} \\ & - 3\varepsilon k_{n3}D_{41} - \varepsilon k_{n4}D_{51} - \varepsilon D_6 + \varepsilon p_{11}F_{12}D_{71}\end{aligned}\quad (C.1)$$

$$\begin{aligned}X_2 = & X_{20} + \varepsilon X_{21} \\ = & a_2 \cos(3\Omega_1 T_0 - \gamma_2) + 2\Lambda_2 \cos(\Omega_1 T_0) + \varepsilon(c_{21}\Lambda_1 + c_{22}\Lambda_2)\tilde{D}_1 - \varepsilon k_{n1}\tilde{D}_{21} - 3\varepsilon k_{n2}\tilde{D}_{31} \\ & - 3\varepsilon k_{n3}\tilde{D}_{41} - \varepsilon k_{n4}\tilde{D}_{51} - \varepsilon \tilde{D}_6 + \varepsilon p_{11}F_{12}\tilde{D}_{71}\end{aligned}\quad (C.2)$$

where

$$\begin{aligned}D_{21} = & -\frac{a_1^3}{32\omega_1^2} \cos(9\Omega_1 T_0 - 3\gamma_1) + \frac{2(3\Lambda_1^3 + 1.5a_1^2\Lambda_1)}{\omega_1^2 - \Omega_1^2} \cos(\Omega_1 T_0) + \frac{3a_1\Lambda_1^2 \cos(\Omega_1 T_0 - \gamma_1)}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} \\ & + \frac{3a_1\Lambda_1^2 \cos(5\Omega_1 T_0 - \gamma_1)}{\omega_1^2 - (\omega_1 + 2\Omega_1)^2} + \frac{1.5a_1^2\Lambda_1 \cos((7\Omega_1 T_0 - 2\gamma_1))}{\omega_1^2 - (2\omega_1 + \Omega_1)^2} + \frac{1.5a_1^2\Lambda_1 \cos((5\Omega_1 T_0 - 2\gamma_1))}{\omega_1^2 - (2\omega_1 - \Omega_1)^2}, \\ D_{31} = & \frac{a_1^2 a_2 \cos((9\Omega_1 T_0 - \gamma_2))}{4(\omega_1^2 - (2\omega_1 + \omega_2)^2)} + \frac{a_1^2 \Lambda_2 \cos((7\Omega_1 T_0 - 2\gamma_1))}{2(\omega_1^2 - (2\omega_1 + \Omega_1)^2)} + \frac{a_1^2 \Lambda_2 \cos((5\Omega_1 T_0 - 2\gamma_1))}{2(\omega_1^2 - (2\omega_1 - \Omega_1)^2)} \\ & + \frac{a_2 \Lambda_1^2 \cos((5\Omega_1 T_0 - \gamma_2))}{\omega_1^2 - (2\Omega_1 + \omega_2)^2} + \frac{a_2 \Lambda_1^2 \cos((\Omega_1 - \gamma_2)T_0)}{\omega_1^2 - (2\Omega_1 - \omega_2)^2} + \frac{2\Lambda_1^2 \Lambda_2 \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2} \\ & + \frac{a_1 a_2 \Lambda_1 \cos((7\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_1^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{2a_1 \Lambda_1 \Lambda_2 \cos((5\Omega_1 T_0 - \gamma_1))}{\omega_1^2 - (\omega_1 + 2\Omega_1)^2} \\ & + \frac{a_1 a_2 \Lambda_1 \cos((\Omega_1 T_0 - \gamma_1 + \gamma_2))}{\omega_1^2 - (\omega_1 - \omega_2 + \Omega_1)^2} + \frac{a_1 a_2 \Lambda_1 \cos((\Omega_1 T_0 + \gamma_1 - \gamma_2))}{\omega_1^2 - (\Omega_1 - \omega_1 + \omega_2)^2} \\ & + \frac{2a_1 \Lambda_1 \Lambda_2 \cos((\Omega_1 T_0 - \gamma_1))}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} + \frac{a_1 a_2 \Lambda_1 \cos((5\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_1^2 - (\Omega_1 - \omega_1 - \omega_2)^2} + \frac{(a_1^2 \Lambda_1 + 4\Lambda_1^2 \Lambda_2) \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2},\end{aligned}$$

$$\begin{aligned}
D_{41} = & \frac{0.25a_2^2 a_1 \cos((9\Omega_1 T_0 - \gamma_1 - 2\gamma_2))}{\omega_1^2 - (2\omega_2 + \omega_1)^2} + \frac{0.5a_2^2 \Lambda_1 \cos((7\Omega_1 T_0 - 2\gamma_2))}{\omega_1^2 - (2\omega_2 + \Omega_1)^2} \\
& + \frac{0.5a_2^2 \Lambda_1 \cos((5\Omega_1 T_0 - 2\gamma_2))}{\omega_1^2 - (2\omega_2 - \Omega_1)^2} + \frac{a_1 \Lambda_2^2 \cos((5\Omega_1 T_0 - \gamma_1))}{\omega_1^2 - (2\Omega_1 + \omega_2)^2} + \frac{a_1 \Lambda_2^2 \cos((\Omega_1 T_0 - \gamma_1))}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} \\
& + \frac{2\Lambda_2^2 \Lambda_1 \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2} + \frac{a_2 a_1 \Lambda_2 \cos((7\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_1^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{2a_2 \Lambda_1 \Lambda_2 \cos((5\Omega_1 T_0 - \gamma_2))}{\omega_1^2 - (\omega_2 + 2\Omega_1)^2} \\
& + \frac{a_2 a_1 \Lambda_2 \cos((\Omega_1 T_0 + \gamma_1 - \gamma_2))}{\omega_1^2 - (\omega_2 + \Omega_1 - \omega_1)^2} + \frac{a_2 a_1 \Lambda_2 \cos((\Omega_1 T_0 - \gamma_1 + \gamma_2))}{\omega_1^2 - (\Omega_1 - \omega_2 + \omega_1)^2} + \frac{2a_2 \Lambda_2 \Lambda_1 \cos((\Omega_1 T_0 - \gamma_2))}{\omega_1^2 - (2\Omega_1 - \omega_2)^2} \\
& + \frac{a_1 a_2 \Lambda_1 \cos((5\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_1^2 - (\Omega_1 - \omega_1 - \omega_2)^2} + \frac{(a_1^2 \Lambda_2 + 4\Lambda_1^2 \Lambda_2) \cos(\Omega_1 T_0)}{\omega_1^2 - \Omega_1^2}, \\
D_{51} = & \frac{0.25a_2^3 \cos(9\Omega_1 T_0 - 3\gamma_2)}{\omega_1^2 - 9\omega_2^2} + \frac{2(3\Lambda_2^3 + 1.5a_2^2 \Lambda_2)}{\omega_1^2 - \Omega_2^2} \cos(\Omega_2 T_0) + \frac{3\Lambda_2^2 a_2 \cos((\Omega_1 T_0 - \gamma_2))}{\omega_1^2 - (2\Omega_1 - \omega_1)^2} \\
& + \frac{3a_2 \Lambda_2^2 \cos(5\Omega_1 T_0 - \gamma_2)}{\omega_1^2 - (\omega_2 + 2\Omega_1)^2} + \frac{1.5a_2^2 \Lambda_2 \cos(7\Omega_1 T_0 - 2\gamma_2)}{\omega_1^2 - (2\omega_2 + \Omega_1)^2} + \frac{1.5a_2^2 \Lambda_2 \cos(5\Omega_1 T_0 - 2\gamma_2)}{\omega_1^2 - (2\omega_2 - \Omega_1)^2}, \\
D_{71} = & \frac{p_{11} a_1 \cos(9\Omega_1 T_0 - \gamma_1)}{2(\omega_1^2 - (\omega_1 + \Omega_2)^2)} + \frac{p_{12} a_2 \cos(9\Omega_1 T_0 - \gamma_2)}{2(\omega_1^2 - (\omega_2 + \Omega_2)^2)} \\
& + \frac{(p_{11} \Lambda_1 + p_{12} \Lambda_2) \cos((\Omega_1 + \Omega_2) T_0)}{\omega_1^2 - (\Omega_1 + \Omega_2)^2} + \frac{(p_{11} \Lambda_1 + p_{12} \Lambda_2) \cos((-\Omega_1 + \Omega_2) T_0)}{\omega_1^2 - (-\Omega_1 + \Omega_2)^2}, \\
\tilde{D}_{21} = & \frac{a_1^3}{4(\omega_2^2 - 9\omega_1^2)} \cos(9\Omega_1 T_0 - 3\gamma_1) + \frac{2(3\Lambda_1^3 + 1.5a_1^2 \Lambda_1)}{\omega_2^2 - \Omega_1^2} \cos(\Omega_1 T_0) + \frac{3a_1 \Lambda_1^2 \cos(\Omega_1 T_0 - \gamma_1)}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} \\
& + \frac{3a_1 \Lambda_1^2 \cos(5\Omega_1 T_0 - \gamma_1)}{\omega_2^2 - (\omega_1 + 2\Omega_1)^2} + \frac{1.5a_1^2 \Lambda_1 \cos((7\Omega_1 T_0 - 2\gamma_1))}{\omega_2^2 - (2\omega_1 + \Omega_1)^2} + \frac{1.5a_1^2 \Lambda_1 \cos((5\Omega_1 T_0 - 2\gamma_1))}{\omega_2^2 - (2\omega_1 - \Omega_1)^2}, \\
\tilde{D}_{31} = & \frac{a_1^2 a_2 \cos((9\Omega_1 T_0 - \gamma_2))}{4(\omega_2^2 - (2\omega_1 + \omega_2)^2)} + \frac{a_1^2 \Lambda_2 \cos((7\Omega_1 T_0 - 2\gamma_1))}{2(\omega_2^2 - (2\omega_1 + \Omega_1)^2)} + \frac{a_1^2 \Lambda_2 \cos((5\Omega_1 T_0 - 2\gamma_1))}{2(\omega_2^2 - (2\omega_1 - \Omega_1)^2)} \\
& + \frac{a_2 \Lambda_1^2 \cos((5\Omega_1 T_0 - \gamma_2))}{\omega_2^2 - (2\Omega_1 + \omega_2)^2} + \frac{a_2 \Lambda_1^2 \cos((\Omega_1 - \gamma_2) T_0)}{\omega_2^2 - (2\Omega_1 - \omega_2)^2} + \frac{2\Lambda_1^2 \Lambda_2 \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2} \\
& + \frac{a_1 a_2 \Lambda_1 \cos((7\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_2^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{2a_1 \Lambda_1 \Lambda_2 \cos((5\Omega_1 T_0 - \gamma_1))}{\omega_2^2 - (\omega_1 + 2\Omega_1)^2} +
\end{aligned}$$

$$\frac{a_1 a_2 \Lambda_1 \cos((\Omega_1 T_0 - \gamma_1 + \gamma_2))}{\omega_2^2 - (\omega_1 - \omega_2 + \Omega_1)^2} + \frac{a_1 a_2 \Lambda_1 \cos((\Omega_1 T_0 + \gamma_1 - \gamma_2))}{\omega_2^2 - (\Omega_1 - \omega_1 + \omega_2)^2} \\ + \frac{2a_1 \Lambda_1 \Lambda_2 \cos((\Omega_1 T_0 - \gamma_1))}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} + \frac{a_1 a_2 \Lambda_1 \cos((5\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_2^2 - (\Omega_1 - \omega_1 - \omega_2)^2} + \frac{(a_1^2 \Lambda_1 + 4\Lambda_1^2 \Lambda_2) \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2},$$

$$\tilde{D}_{41} = \frac{a_2^2 a_1 \cos((9\Omega_1 T_0 - \gamma_1 - 2\gamma_2))}{4(\omega_2^2 - (2\omega_2 + \omega_1)^2)} + \frac{a_2^2 \Lambda_1 \cos((7\Omega_1 T_0 - 2\gamma_2))}{2(\omega_2^2 - (2\omega_2 + \Omega_1)^2)} + \frac{a_2^2 \Lambda_1 \cos((5\Omega_1 T_0 - 2\gamma_2))}{2(\omega_2^2 - (2\omega_2 - \Omega_1)^2)} \\ + \frac{a_1 \Lambda_2^2 \cos((5\Omega_1 T_0 - \gamma_1))}{\omega_2^2 - (2\Omega_1 + \omega_2)^2} + \frac{a_1 \Lambda_2^2 \cos((\Omega_1 T_0 - \gamma_1))}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} + \frac{2\Lambda_2^2 \Lambda_1 \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2} \\ + \frac{a_2 a_1 \Lambda_2 \cos((7\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_2^2 - (\omega_1 + \omega_2 + \Omega_1)^2} + \frac{2a_2 \Lambda_1 \Lambda_2 \cos((5\Omega_1 T_0 - \gamma_2))}{\omega_2^2 - (\omega_2 + 2\Omega_1)^2} \\ + \frac{a_2 a_1 \Lambda_2 \cos((\Omega_1 T_0 + \gamma_1 - \gamma_2))}{\omega_2^2 - (\omega_2 + \Omega_1 - \omega_1)^2} + \frac{a_2 a_1 \Lambda_2 \cos((\Omega_1 T_0 - \gamma_1 + \gamma_2))}{\omega_2^2 - (\Omega_1 - \omega_2 + \omega_1)^2} \\ + \frac{2a_2 \Lambda_2 \Lambda_1 \cos((\Omega_1 T_0 - \gamma_2))}{\omega_2^2 - (2\Omega_1 - \omega_2)^2} + \frac{a_1 a_2 \Lambda_1 \cos((5\Omega_1 T_0 - \gamma_1 - \gamma_2))}{\omega_2^2 - (\Omega_1 - \omega_1 - \omega_2)^2} + \frac{(a_1^2 \Lambda_2 + 4\Lambda_1^2 \Lambda_2) \cos(\Omega_1 T_0)}{\omega_2^2 - \Omega_1^2},$$

$$\tilde{D}_{51} = -\frac{a_2^3 \cos(3\omega_2 T_0)}{32\omega_2^2} + \frac{2(3\Lambda_2^3 + 1.5a_2^2 \Lambda_2)}{\omega_2^2 - \Omega_2^2} \cos(\Omega_2 T_0) + \frac{3\Lambda_2^2 a_2 \cos((\Omega_1 T_0 - \gamma_2))}{\omega_2^2 - (2\Omega_1 - \omega_1)^2} \\ + \frac{3a_2 \Lambda_2^2 \cos(5\Omega_1 T_0 - \gamma_2)}{\omega_2^2 - (\omega_2 + 2\Omega_1)^2} + \frac{1.5a_2^2 \Lambda_2 \cos(7\Omega_1 T_0 - 2\gamma_2)}{\omega_2^2 - (2\omega_2 + \Omega_1)^2} + \frac{1.5a_2^2 \Lambda_2 \cos(5\Omega_1 T_0 - 2\gamma_2)}{\omega_2^2 - (2\omega_2 - \Omega_1)^2},$$

$$\tilde{D}_{71} = \frac{p_{11} a_1 \cos(9\Omega_1 T_0 - \gamma_1)}{2(\omega_2^2 - (\omega_1 + \Omega_2)^2)} + \frac{p_{12} a_2 \cos(9\Omega_1 T_0 - \gamma_2)}{2(\omega_2^2 - (\omega_2 + \Omega_2)^2)} \\ + \frac{(p_{11} \Lambda_1 + p_{12} \Lambda_2) \cos((\Omega_1 + \Omega_2) T_0)}{\omega_2^2 - (\Omega_1 + \Omega_2)^2} + \frac{(p_{11} \Lambda_1 + p_{12} \Lambda_2) \cos((-\Omega_1 + \Omega_2) T_0)}{\omega_2^2 - (-\Omega_1 + \Omega_2)^2}$$

Appendix-D

Applications of ANVA for Vibration Suppression:

Forearm Tremor

$$F_{act} = \omega_r (x_1 - x_2) + F_{ac} \ddot{x}_1 + F_v \dot{x}_1 + F_d x_1, \quad \omega_r = k_r / M, \quad k_r = k_p^E k_3 / (k_p^E + k_3),$$

$$a_1 = -2\Omega \cos \theta_1 + 2\zeta_1 \omega_p \sin \theta_1 - 2F_{ac} \Omega \cos \theta_1 + F_v \sin \theta_1,$$

$$a_2 = 2\Omega A \sin \theta_1 + 2\zeta_1 \omega_p A \cos \theta_1 + 2F_{ac} A \Omega \sin \theta_1 + F_v A \cos \theta_1,$$

$$a_3 = -2\mu \zeta_2 \omega_a \sin \theta_2, \quad a_4 = -2\mu \zeta_2 \omega_a B \cos \theta_2,$$

$$a_5 = 2\Omega \sin \theta_1 + 2\zeta_1 \omega_p \cos \theta_1 + 2F_{ac} \Omega \sin \theta_1 + F_v \cos \theta_1,$$

$$a_6 = 2A \Omega \cos \theta_1 - 2\zeta_1 \omega_p A \sin \theta_1 + 2F_{ac} A \Omega \cos \theta_1 - F_v A \sin \theta_1,$$

$$a_7 = -2\mu \zeta_2 \omega_a \cos \theta_2, \quad a_8 = 2\mu \zeta_2 \omega_a B \sin \theta_2,$$

$$a_9 = -2\Omega \cos \theta_1 + (2F_{ac} \Omega \cos \theta_1 - F_v \sin \theta_1) / \mu,$$

$$a_{10} = 2\Omega A \sin \theta_1 - (2F_{ac} A \Omega \sin \theta_1 + F_v A \cos \theta_1) / \mu,$$

$$a_{11} = -2\Omega \cos \theta_2 + 2\zeta_2 \omega_a \sin \theta_2,$$

$$a_{12} = 2\Omega B \sin \theta_2 + 2\zeta_2 \omega_a B \cos \theta_2,$$

$$a_{13} = 2\Omega \sin \theta_1 - (2F_{ac} \Omega \sin \theta_1 + F_v \cos \theta_1) / \mu,$$

$$a_{14} = 2\Omega A \cos \theta_1 - (2F_{ac} A \Omega \cos \theta_1 - F_v A \sin \theta_1) / \mu,$$

$$a_{15} = 2\Omega \sin \theta_2 + 2\zeta_2 \omega_a B \cos \theta_2,$$

$$a_{16} = 2\Omega B \cos \theta_2 - 2\zeta_2 \omega_a B \sin \theta_2,$$

$$b_1 = (\omega_p^2 - \Omega^2) A \sin \theta_1 - 2\zeta_1 \omega_p A \Omega \cos \theta_1 - \mu \omega_a^2 B \sin \theta_2 + 2\mu \zeta_2 \omega_a B \Omega \cos \theta_2 + 2\zeta_1 \omega_p \Omega X_0$$

$$+ 0.75\alpha A^3 \sin \theta_1 + 0.75\alpha A X_0^2 \sin \theta_1 - 0.75\mu \beta B^3 \sin \theta_2 - \omega_r B \sin \theta_2$$

$$- F_{ac} A \Omega^2 \sin \theta_1 - F_v A \Omega \cos \theta_1 + F_d A \sin \theta_1,$$

$$b_2 = (\omega_p^2 - \Omega^2) A \cos \theta_1 + 2\zeta_1 \omega_p A \Omega \sin \theta_1 - \mu \omega_a^2 B \cos \theta_2 - 2\mu \zeta_2 \omega_a B \Omega \sin \theta_2 - \omega_p^2 X_0 - F$$

$$+ 0.75\alpha A^3 \cos \theta_1 - 0.75\alpha X_0^3 + 1.5\alpha X_0 A^2 + 0.75\alpha X_0 A^2 (\cos \theta_1 + \sin \theta_1) + 2.25\alpha A X_0^2 \cos \theta_1$$

$$- 0.75\mu \beta B^3 \cos \theta_2 - \omega_r B \cos \theta_2 - F_{ac} A \Omega^2 \cos \theta_1 + F_v A \Omega \sin \theta_1 - F_d A \cos \theta_1,$$

$$b_3 = -A \Omega^2 \sin \theta_1 - B \Omega^2 \sin \theta_2 - 2\zeta_2 \omega_a B \Omega \cos \theta_2 + \omega_a^2 B \sin \theta_2$$

$$+ 0.75\beta B^3 \sin \theta_2 + (\omega_r B \sin \theta_2 + F_{ac} A \Omega^2 \sin \theta_1 + F_v A \Omega \cos \theta_1 - F_d A \sin \theta_1) / \mu,$$

$$b_4 = -A \Omega^2 \cos \theta_1 - B \Omega^2 \cos \theta_2 + 2\zeta_2 \omega_a B \Omega \sin \theta_2 + \omega_a^2 B \cos \theta_2$$

$$+ 0.75\beta B^3 \cos \theta_2 + (\omega_r B \cos \theta_2 + F_{ac} A \Omega^2 \cos \theta_1 - F_v A \Omega \sin \theta_1 - F_d A \cos \theta_1) / \mu$$

Harmonically Excited Beam

$$\begin{aligned}
a_{11} &= 2\Omega \sin(\varphi_1), a_{12} = 2A_1\Omega \cos(\varphi_1), a_{13} = a_{14} = a_{15} = a_{16} = 0, a_{17} = -\phi_1(a)k_v \cos(\varphi_4), \\
a_{18} &= \phi_1(a)k_v B \sin(\varphi_4), a_{21} = -2\Omega \cos(\varphi_1), a_{22} = 2A_1\Omega \sin(\varphi_1), a_{23} = a_{24} = a_{25} = a_{26} = 0, \\
a_{27} &= -\phi_1(a)k_v \sin(\varphi_4), a_{28} = -\phi_1(a)k_v B \cos(\varphi_4), a_{31} = a_{32} = a_{35} = a_{36} = 0, a_{33} = 2\Omega \sin(\varphi_2), \\
a_{34} &= 2A_2\Omega \cos(\varphi_2), a_{37} = -\phi_2(a)k_v \cos(\varphi_4), a_{38} = \phi_2(a)k_v B \sin(\varphi_4), a_{41} = a_{42} = a_{45} = a_{46} = 0, \\
a_{43} &= -2\Omega \cos(\varphi_1), a_{44} = 2A_2\Omega \sin(\varphi_2), a_{47} = -\phi_2(a)k_v \sin(\varphi_4), a_{48} = -\phi_2(a)k_v B \cos(\varphi_4), \\
a_{51} &= a_{52} = a_{53} = a_{54} = 0, a_{55} = 2\Omega \sin(\varphi_3), a_{56} = 2A_3\Omega \cos(\varphi_3), a_{57} = -\phi_3(a)k_v \cos(\varphi_4), \\
a_{58} &= \phi_3(a)k_v B \sin(\varphi_4), a_{61} = a_{62} = a_{63} = a_{64} = 0, a_{65} = -2\Omega \cos(\varphi_3), a_{66} = 2A_3\Omega \sin(\varphi_3), \\
a_{67} &= -\phi_3(a)k_v \sin(\varphi_4), a_{68} = -\phi_3(a)k_v B \cos(\varphi_4), a_{71} = a_{72} = a_{73} = a_{74} = a_{75} = a_{76} = 0, \\
a_{77} &= 2\mu\Omega \sin(\varphi_4) + k_v \cos(\varphi_4), a_{78} = 2\mu\Omega B \cos(\varphi_4) - k_v B \sin(\varphi_4), a_{81} = a_{82} = a_{83} = 0, a_{84} = 0, \\
a_{85} &= 0, a_{86} = 0, a_{87} = -2\mu\Omega \cos(\varphi_4) + k_v \sin(\varphi_4), a_{88} = 2\mu\Omega B \sin(\varphi_4) + k_v B \cos(\varphi_4) \\
b_1 &= b_{l1} + b_{n1}, b_2 = b_{l2} + b_{n2}, b_3 = b_{l3} + b_{n3}, b_4 = b_{l4} + b_{n4}, \\
b_5 &= b_{l5} + b_{n5}, b_6 = b_{l6} + b_{n6}, b_7 = b_{l7} + b_{n7}, b_8 = b_{l8} + b_{n8}
\end{aligned}$$

The terms $b_{l1}, b_{l2}, b_{l3}, b_{l4}, b_{l5}, b_{l6}, b_{l7}, b_{l8}$ and $b_{n1}, b_{n2}, b_{n3}, b_{n4}, b_{n5}, b_{n6}, b_{n7}, b_{n8}$ are given below.

$$\begin{aligned}
b_{l1} &= -A_1\Omega^2 \cos(\varphi_1) + \omega_{n1}^2 A_1 \cos(\varphi_1) + \mu\omega_a^2 (\phi_1(a))^2 A_1 \cos(\varphi_1) + \mu\omega_a^2 \phi_1(a)\phi_2(a)A_2 \cos(\varphi_2) \\
&\quad + \mu\omega_a^2 \phi_1(a)\phi_3(a)A_3 \cos(\varphi_3) - \phi_1(a)(\mu\omega_a^2 - k_p)B \cos(\varphi_4) - \phi_1(a)k_v B\Omega \sin(\varphi_4) - \phi_1(b)F
\end{aligned}$$

$$b_{n1} = \alpha\mu\omega_1^2 \phi_1(a)b_{n11}$$

$$\begin{aligned}
b_{n11} &= 0.75(\phi_1(a))^3 A_1^3 \cos(\varphi_1) + 0.75(\phi_2(a))^3 A_2^3 \cos(\varphi_2) + 0.75(\phi_3(a))^3 A_3^3 \cos(\varphi_3) + \\
&\quad 0.75B^3 \cos(\varphi_4) + 3(\phi_1(a))^2 \phi_2(a)A_1^2 A_2 (0.5 \cos(\varphi_2) + 0.25 \cos(\varphi_2 - 2\varphi_1)) + \\
&\quad 3(\phi_1(a))^2 \phi_3(a)A_2^2 A_3 \left(\frac{0.5 \cos(\varphi_3) +}{0.25 \cos(\varphi_3 - 2\varphi_1)} \right) - 3(\phi_1(a))^2 A_1^2 B \left(\frac{0.5 \cos(\varphi_4) +}{0.25 \cos(\varphi_4 - 2\varphi_1)} \right) + \\
&\quad 3(\phi_2(a))^2 \phi_1(a)A_2^2 A_1 \left(\frac{0.5 \cos(\varphi_4) +}{0.25 \cos(\varphi_1 - 2\varphi_2)} \right) + 3(\phi_2(a))^2 \phi_3(a)A_2^2 A_3 \left(\frac{0.5 \cos(\varphi_3) +}{0.25 \cos(\varphi_3 - 2\varphi_2)} \right) - \\
&\quad 3(\phi_2(a))^2 A_2^2 B \left(\frac{0.5 \cos(\varphi_4) +}{0.25 \cos(\varphi_4 - 2\varphi_2)} \right) + 3(\phi_3(a))^2 \phi_1(a)A_3^2 A_1 \left(\frac{0.5 \cos(\varphi_1) +}{0.25 \cos(\varphi_1 - 2\varphi_3)} \right) +
\end{aligned}$$

$$\begin{aligned}
& 3(\phi_3(a))^2 \phi_2(a) A_3^2 A_2 \left(\frac{0.5 \cos(\varphi_2) +}{0.25 \cos(\varphi_4 - 2\varphi_3)} \right) - 3(\phi_3(a))^2 A_2^2 B \left(\frac{0.5 \cos(\varphi_4) +}{0.25 \cos(\varphi_4 - 2\varphi_3)} \right) + \\
& 1.5\phi_1(a)\phi_2(a)\phi_3(a) A_1 A_2 A_3 (\cos(-\varphi_2 - \varphi_3 + \varphi_1) - \cos(-\varphi_1 - \varphi_2 + \varphi_3) - \cos(-\varphi_1 - \varphi_3 + \varphi_2)) - \\
& 1.5\phi_1(a)\phi_3(a) A_1 A_3 B (\cos(-\varphi_3 - \varphi_4 + \varphi_1) - \cos(-\varphi_1 - \varphi_4 + \varphi_3) - \cos(-\varphi_1 - \varphi_3 + \varphi_4)) - \\
& 1.5\phi_2(a)\phi_3(a) A_2 A_3 B (\cos(-\varphi_3 - \varphi_4 + \varphi_2) - \cos(-\varphi_1 - \varphi_3 + \varphi_4) - \cos(-\varphi_2 - \varphi_4 + \varphi_3)) - \\
& 1.5\phi_1(a)\phi_2(a) A_1 A_2 B (\cos(-\varphi_2 - \varphi_4 + \varphi_1) - \cos(-\varphi_1 - \varphi_2 + \varphi_4) - \cos(-\varphi_1 - \varphi_4 + \varphi_2))
\end{aligned}$$

$$\begin{aligned}
b_{l_2} = & -A_1 \Omega^2 \sin(\varphi_1) + \omega_{n1}^2 A_1 \sin(\varphi_1) + \mu \omega_a^2 (\phi_1(a))^2 A_1 \sin(\varphi_1) + \mu \omega_a^2 \phi_1(a) \phi_2(a) A_2 \sin(\varphi_2) \\
& + \mu \omega_a^2 \phi_1(a) \phi_3(a) A_3 \sin(\varphi_3) - \phi_1(a) (\mu \omega_a^2 - k_p) B \sin(\varphi_4) + \phi_1(a) k_v B \Omega \cos(\varphi_4)
\end{aligned}$$

$$b_{n2} = \alpha \mu \omega_1^2 \phi_1(a) b_{n12}$$

$$\begin{aligned}
b_{n12} = & 0.75(\phi_1(a))^3 A_1^3 \sin(\varphi_1) + 0.75(\phi_2(a))^3 A_2^3 \sin(\varphi_2) + 0.75(\phi_3(a))^3 A_3^3 \sin(\varphi_3) + \\
& 0.75B^3 \sin(\varphi_4) + 3(\phi_1(a))^2 \phi_2(a) A_1^2 A_2 (0.5 \sin(\varphi_2) + 0.25 \sin(\varphi_2 - 2\varphi_1)) + \\
& 3(\phi_1(a))^2 \phi_3(a) A_2^2 A_3 \left(\frac{0.5 \sin(\varphi_3) +}{0.25 \sin(\varphi_3 - 2\varphi_1)} \right) - 3(\phi_1(a))^2 A_1^2 B \left(\frac{0.5 \sin(\varphi_4) +}{0.25 \sin(\varphi_4 - 2\varphi_1)} \right) + \\
& 3(\phi_2(a))^2 \phi_1(a) A_2^2 A_1 \left(\frac{0.5 \sin(\varphi_4) +}{0.25 \sin(\varphi_1 - 2\varphi_2)} \right) + \frac{3}{4} (\phi_2(a))^2 \phi_3(a) A_2^2 A_3 \left(\frac{2 \sin(\varphi_3) +}{\sin(\varphi_3 - 2\varphi_2)} \right) \\
& - 3(\phi_2(a))^2 A_2^2 B \left(\frac{0.5 \sin(\varphi_4) +}{0.25 \sin(\varphi_4 - 2\varphi_2)} \right) + 3(\phi_3(a))^2 \phi_1(a) A_3^2 A_1 \left(\frac{0.5 \sin(\varphi_1) +}{0.25 \sin(\varphi_1 - 2\varphi_3)} \right) \\
& + 3(\phi_3(a))^2 \phi_2(a) A_3^2 A_2 \left(\frac{0.5 \sin(\varphi_2) +}{0.25 \sin(\varphi_4 - 2\varphi_3)} \right) - 3(\phi_3(a))^2 A_2^2 B \left(\frac{0.5 \sin(\varphi_4) +}{0.25 \sin(\varphi_4 - 2\varphi_3)} \right) \\
& + 1.5\phi_1(a)\phi_2(a)\phi_3(a) A_1 A_2 A_3 \left(\frac{\sin(-\varphi_2 - \varphi_3 + \varphi_1) - \cos(-\varphi_1 - \varphi_2 + \varphi_3)}{-\sin(-\varphi_1 - \varphi_3 + \varphi_2)} \right) \\
& - 1.5\phi_1(a)\phi_3(a) A_1 A_3 B (\sin(-\varphi_3 - \varphi_4 + \varphi_1) - \cos(-\varphi_1 - \varphi_4 + \varphi_3) - \sin(-\varphi_1 - \varphi_3 + \varphi_4)) \\
& - 1.5\phi_2(a)\phi_3(a) A_2 A_3 B (\sin(-\varphi_3 - \varphi_4 + \varphi_2) - \cos(-\varphi_1 - \varphi_3 + \varphi_4) - \sin(-\varphi_2 - \varphi_4 + \varphi_3)) \\
& - 1.5\phi_1(a)\phi_2(a) A_1 A_2 B (\sin(-\varphi_2 - \varphi_4 + \varphi_1) - \sin(-\varphi_1 - \varphi_2 + \varphi_4) - \sin(-\varphi_1 - \varphi_4 + \varphi_2))
\end{aligned}$$

$$\begin{aligned}
b_{l_3} = & -A_2 \Omega^2 \cos(\varphi_2) + \omega_{n1}^2 A_2 \cos(\varphi_2) + \mu \omega_a^2 (\phi_2(a))^2 A_2 \cos(\varphi_2) + \mu \omega_a^2 \phi_1(a) \phi_2(a) A_1 \cos(\varphi_1) \\
& + \mu \omega_a^2 \phi_2(a) \phi_3(a) A_3 \cos(\varphi_3) - \phi_2(a) (\mu \omega_a^2 - k_p) B \cos(\varphi_4) - \phi_2(a) k_v B \Omega \sin(\varphi_4) - \phi_2(b) F
\end{aligned}$$

$$b_{n3} = \alpha \mu \omega_1^2 \phi_2(a) b_{n11}$$

$$b_{l4} = -A_2 \Omega^2 \sin(\varphi_2) + \omega_{n2}^2 A_2 \sin(\varphi_2) + \mu \omega_a^2 (\phi_2(a))^2 A_2 \sin(\varphi_2) + \mu \omega_a^2 \phi_1(a) \phi_2(a) A_1 \sin(\varphi_1) \\ + \mu \omega_a^2 \phi_2(a) \phi_3(a) A_3 \sin(\varphi_3) - \phi_2(a) (\mu \omega_a^2 - k_p) B \sin(\varphi_4) + \phi_2(a) k_v B \Omega \cos(\varphi_4)$$

$$b_{n4} = \alpha \mu \omega_1^2 \phi_2(a) b_{n12}$$

$$b_{l5} = -A_3 \Omega^2 \cos(\varphi_3) + \omega_{n3}^2 A_3 \cos(\varphi_3) + \mu \omega_a^2 (\phi_3(a))^2 A_3 \cos(\varphi_3) + \mu \omega_a^2 \phi_1(a) \phi_3(a) A_1 \cos(\varphi_1) \\ + \mu \omega_a^2 \phi_2(a) \phi_3(a) A_3 \cos(\varphi_3) - \phi_3(a) (\mu \omega_a^2 - k_p) B \cos(\varphi_4) - \phi_3(a) k_v B \Omega \sin(\varphi_4) - \phi_3(b) F$$

$$b_{n5} = \alpha \mu \omega_1^2 \phi_3(a) b_{n11}$$

$$b_{l6} = -A_3 \Omega^2 \sin(\varphi_3) + \omega_{n3}^2 A_3 \sin(\varphi_3) + \mu \omega_a^2 (\phi_3(a))^2 A_3 \sin(\varphi_3) + \mu \omega_a^2 \phi_1(a) \phi_3(a) A_1 \sin(\varphi_1) \\ + \mu \omega_a^2 \phi_2(a) \phi_3(a) A_3 \sin(\varphi_3) - \phi_3(a) (\mu \omega_a^2 - k_p) B \sin(\varphi_4) + \phi_3(a) k_v B \Omega \cos(\varphi_4)$$

$$b_{n6} = \alpha \mu \omega_1^2 \phi_3(a) b_{n12}$$

$$b_{l7} = -\mu \omega_a^2 \phi_1(a) A_1 \cos(\varphi_1) - \mu \omega_a^2 \phi_2(a) A_2 \cos(\varphi_2) - \mu \omega_a^2 \phi_3(a) A_3 \cos(\varphi_3) - \\ \mu \Omega^2 B \cos(\varphi_4) + k_v B \Omega \sin(\varphi_4) + (\mu \omega_a^2 - k_p) B \cos(\varphi_4)$$

$$b_{n7} = \alpha \mu \omega_1^2 b_{n11}$$

$$b_{l8} = -\mu \omega_a^2 \phi_1(a) A_1 \sin(\varphi_1) - \mu \omega_a^2 \phi_2(a) A_2 \sin(\varphi_2) - \mu \omega_a^2 \phi_3(a) A_3 \sin(\varphi_3) - \\ \mu \Omega^2 B \sin(\varphi_4) - k_v B \Omega \cos(\varphi_4) + (\mu \omega_a^2 - k_p) B \sin(\varphi_4)$$

$$b_{n8} = \alpha \mu \omega_1^2 b_{n12}$$



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