Regularity Results for the Schrödinger Equation on Rank-One Symmetric Spaces of Non-Compact Type

by

Manali Sajjan

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI-781039, INDIA
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Regularity Results for the Schrödinger Equation on Rank-One Symmetric Spaces of Non-Compact Type

A thesis submitted
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by

Manali Sajjan

(Roll Number: 206123013)

to the

DEPARTMENT OF MATHEMATICS
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Declaration

I hereby declare that the work contained in the thesis entitled **”‘Regularity Results for the Schrödinger Equation on Rank-One Symmetric Spaces of Non-Compact Type”** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under the guidance of **Dr. Pratyoosh Kumar**, Indian Institute of Technology Guwahati, for the award of **Doctor of Philosophy** and that this work has not been submitted elsewhere for a degree.

Manali Sajjan
Roll No: 206123013
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati-781039, India

Guwahati
January 2026

Certificate

It is certified that the work contained in the thesis titled **‘Regularity Results for the Schrödinger Equation on Rank-One Symmetric Spaces of Non-Compact Type”** by **Manali Sajjan (206123013)**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of **Doctor of Philosophy** has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

Guwahati
January 2026

Dr. Pratyoosh Kumar
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati-781039, India

Dedicated

To

My Family

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Manali Sajjan


भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
Indian Institute of Technology Guwahati

Abstract

In this thesis, we investigate the pointwise convergence of solutions to the Schrödinger equation on real rank one symmetric spaces of noncompact type. The study of pointwise convergence of solutions to the Schrödinger equation to their initial data is a classical problem in harmonic analysis and partial differential equations. Given the solution $u(x,t)$ of the Schrödinger equation, a fundamental question is; whether the solution $u(x,t)$ converges pointwise to the given initial data $f(x)$ as $t \to 0$. This problem originated from the question posed by Carleson, which asks how much regularity to be imposed on the initial data to ensure the pointwise convergence. On the Euclidean spaces the problems were studied extensively and sharp results were established under various regularity assumptions. However much less is known in non-Euclidean settings. In this context, Sjölin has studied the regularity of the fractional Schrödinger equation in $\mathbb{R}^n$.

In a part of this thesis, we focus on the fractional Schrödinger equation on rank-one symmetric spaces of non-compact type, with radial initial data. The standard method for proving pointwise convergence involves obtaining an estimate for the corresponding maximal operator. We also proved the boundedness of the Schrödinger maximal operator.

Finally, we study well-posedness and local regularity results for the Schrödinger equation with a non-zero potential V on these symmetric spaces.

Abbreviation and Notation

$\mathbb{N}$	The set of all natural numbers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, $n \geq 1$ and $\mathbb{R}$ the set of all real numbers
$\mathbb{C}^n$	$\{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}\}$, $n \geq 1$ and $\mathbb{C}$ the set of all complex numbers
$f \asymp g$	There exists positive constants C_1 and C_2 such that $C_1 f \leq g \leq C_2 f$
p'	The conjugate exponent of p, page 15
S^{n-1}	The unit sphere in $\mathbb{R}^n$, page 4
$\mathcal{X}_E$	The characteristic function of the set E, page 20
$C_c(\cdot)$	Set of compactly supported continuous function
$C_c^\infty(\cdot)$	Set of compactly supported infinitely differentiable functions
$L^p(X)$	$\{f : X \to \mathbb{C} \mid f$ is measurable and $\int\limits_X \lvert f\rvert^p dx < \infty\}$
$L^p_{loc}(X)$	$\{f : X \to \mathbb{C} \mid f$ is measurable and $\int\limits_B \lvert f\rvert^p dx < \infty\}$, for every ball B in the Banach space X
$\lVert T\rVert_{p\to p}$	The operator norm of T
Δ	Laplacian operator on $\mathbb{R}^n$ page 1
$H^s(\mathbb{R}^n)$	Sobolev space in $\mathbb{R}^n$ page 1
G	semisimple Lie group, page 5
K	maximal compact subgroup of G, page 5

$\mathbb{X}$	A rank one symmetric space of non-compact type, page 5
$\mathcal{L}$	Laplace-Beltrami operator on $\mathbb{X}$, page 15
$H^s(\mathbb{X})$	Sobolev space in $\mathbb{X}$, page 18
$H(x)$	page 13
$H^s_{loc}(X)$	$\{f : X \to \mathbb{C} \mid \lVert f\rVert_{H^s(B)} < \infty\}$ for every ball B in the Banach space X, page 7
ρ	half sum of positive roots, page 12
ϕ_λ	The elementary spherical function, page 14
$\widetilde{f}(\lambda, k)$	Helgason Fourier transform of a function defined on X, page 13
$\hat{f}(\lambda)$	The spherical transform of f defined on $\mathbb{X}$, page 14
$L^p_x(L^q_t[a,b])$	$\left\{ f(x,t) : \left\lVert \left(\int_a^b \lvert f(\cdot,t)\rvert^q dt \right)^{1/q} \right\rVert_{L^p(\mathbb{R}^n)} < \infty \right\}$, page 8
$L^p([-T,T] : L^q_x)$	$\left\{ f(x,t) : \left\lVert \left(\int_{\mathbb{R}^n} \lvert f(x,\cdot)\rvert^q dx \right)^{1/q} \right\rVert_{L^p(\mathbb{R}^n)} < \infty \right\}$, page 8
$L^2(H^s)$	page 27
$L^2([-T,T] : H^{s+\alpha}_{u;loc}(X))$	$\left\{ f(x,t) : \int_{-T}^{T} \lVert f(\cdot,t)\rVert^2_{H^{s+\alpha}_{u;loc}(X)} dt < \infty \right\}$, page 7
$C(I : X)$	space of all continuous function from the interval I to the Banach space X, pa
$L^\alpha_r H^\beta_\omega$	page 33
$H^\alpha_r H^\beta_\omega$	page 33
$L^1_{loc}(\mathbb{R}, H^{s,q}(\mathbb{R}^n))$	page 7

Contents

CHAPTER 1

Introduction

The study of pointwise convergence of solutions to the Schrödinger equation to their initial data is a classical problem in harmonic analysis and partial differential equations. In 1976, Carleson [6] made a remarkable contribution to the study of the regularity of solutions to Schrödinger equation on $\mathbb{R}$. He first raised a question, how much regularity to be imposed on the initial data f, such that $u(x,t) \to f(x)$ pointwise a.e as $t \to 0$, where $u(x,t)$ is the solution of the Schrödinger equation:

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(x,0) = f(x). \end{cases} \tag{1.0.1}$$

For $n = 1$, he proved that, if $s \geq 1/4$, and f has compact support then

$$u(x,t) \to f \quad \text{pointwise a.e as} \quad t \to 0, \quad \text{for} \quad f \in H^s(\mathbb{R}^n). \tag{1.0.2}$$

Here the Sobolev space $H^s(\mathbb{R}^n)$ is defined by:

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\widehat{f}(\xi)|^2 \, d\xi < \infty \right\}$$

The main idea in the proof was to obtain an a priori estimate for the corresponding maximal function, a standard approach for establishing a pointwise convergence result. There are various methods to obtain bounds for the maximal function; one such method, used by Carleson, is the $t \to t(x)$ method, known as the "Kolmogorov-Seliverstov-Plessner" method. In this method, we define the operator $S_t f$ by

$$u(x,t) = S_t f(x) = \int_{\mathbb{R}^n} e^{ix\xi} e^{it|\xi|^2} \widehat{f}(\xi)\, d\xi.$$

and the corresponding maximal function is

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|.$$

In the above-mentioned method, Carleson has proved the norm inequality

$$\|S^* f\|_{L^1[-1,1]} \leq C \|f\|_{H^{1/4}(\mathbb{R})}$$

by showing the integral of the linearized operator is controlled by the Sobolev norm of f. That is:

$$\left| \int_{-1}^{1} S_{t(x)} f(x)\, dx \right| \leq C \|f\|_{H^{1/4}(\mathbb{R})}. \tag{1.0.3}$$

The key step in evaluating this integral is obtaining an estimate for an oscillatory integral. The estimate is the following:

$$\left| \int_{\mathbb{R}} e^{i(a\xi + b\xi^2)} \frac{d\xi}{|\xi|^\alpha} \right| \leq c \left(|b|^{\alpha - \frac{1}{2}} |a|^{-\alpha} + |a|^{\alpha-1} \right).$$

Later in [12], Dahlberg and Kenig established that this result is sharp in the one dimensional case. In this paper, it is also mentioned that the results can be extended for $n > 1$. Thus, it was known that $s \geq n/4$ is a sufficient condition for all $n \geq 1$.

Although the inequality (1.0.3) ensures almost everywhere convergence, it does not describe the mapping property of the operator $S_t f$. Kenig and Ruiz [21] have shown

$$\int_I |S^* f|^2\, dx \leq C \|f\|^2_{H^{1/4}(\mathbb{R})}, \quad \text{for any finite interval } I \subset \mathbb{R}.$$

which is obviously a stronger estimate in the sense of L^p inclusions. Their approach is a bit different: Instead of treating the operator $S^* f$, for $f \in H^{1/4}(\mathbb{R})$ they have considered the operator $T^* f = \sup_{0<t<1} |T_t f(x)|$, where $T_t f(x) = \int_{\mathbb{R}} e^{ix\xi} e^{it|\xi|^2} \widehat{f}(\xi) \dfrac{d\xi}{|\xi|^{1/4}}$ and for $f \in L^2(\mathbb{R})$, they have shown

$$\int_I |T^* f|^2\, dx \leq C \|f\|^2_{L^2(\mathbb{R})}.$$

As in Carleson's paper, this method also relies on the "Kolmogorov-Seliverstov-Plessner" method. They considered the composition of the linearized operator $T_{t(x)}$ with its adjoint $T^*_{t(x)}$ (abuse of notation) and bound the L^2- norm of $T_{t(x)}T^*_{t(x)}$ by using some estimate for oscillatory integrals.

In 1982, Cowling [9] considered this problem in a much more general set-up and made a significant contribution to this problem. Instead of the Laplacian Δ, he took a self-adjoint operator T on $L^2(M)$ and showed that

$$\lim_{t\to 0} \psi(t) = \phi \ \text{ where } \ \psi(t) = \exp(itT)\phi$$

if $|T|^\alpha\phi$ is in $L^2(M)$, for some α in $(1/2, \infty)$ and for some measure space M.

In 1987-88, Vega [31], and Sjölin [27] independently proved by a new method that $s > 1/2$ is a sufficient condition for the pointwise convergence (1.0.2), for all n. So they have improved the results for the case $n \geq 3$, and better regularity has been obtained in view of the Sobolev embedding:

$$s_1 < s_2 \implies H^{s_2} \subset H^{s_1}.$$

Later, many authors like Bourgain, Moyua- Vargas- Vega, Tao-Vargas, and Lee [4, 14, 24, 22] have studied this problem in other higher dimensions. They have improved the regularity condition for the solution and obtained sharp results in many cases.

Bourgain showed that for $n \geq 2$, the pointwise convergence result holds for $s > 1/2 - 1/4n$, using a method known as multilinear methods, thereby improving the result of Sjölin and Vega. In particular, this yields the sufficient condition $s > 3/8$ in the case $n = 2$, which was also independently proved by Lee [22] using Tao-Wolff's bilinear restriction method, which was further improved by Du- Guth- Li [14] to $s > 1/3$.

In 2012, Bourgain [5] also proved that $s \geq 1/2 - 1/n$ is necessary for (1.0.2) to hold. Later, Luca-Rogers [23] constructed a sharper counterexample and improved the necessary condition to $s > \frac{1}{2} - \frac{1}{n+2}$ for $n \geq 2$. Further in [4], Bourgain revisited the result of Luca-Rogers and established a stronger version of their result within the same range.

We now discuss the more general case, the fractional Schrödinger equation. In 1987, Sjölin [27] and Vega [31] studied the pointwise convergence results for the solution of the fractional Schrödinger equation on $\mathbb{R}^n$.

Let $a > 1$. For $n \geq 3$, Sjölin proved that if $s > 1/2$,

$$u(x,t) \to f \ \text{ pointwise a.e as } \ t \to 0, \quad \text{for} \ \ f \in H^s(\mathbb{R}^n) \tag{1.0.4}$$

where $u(x,t)$ is the solution of the fractional Schrodinger equation:

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} = \Delta^{a/2} u(x,t), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(x,0) = f(x). & a > 1 \end{cases} \tag{1.0.5}$$

Also, it was shown in [27], that for the case $n = 2$ (1.0.4) holds if $s \geq 1/2$. For $n = 1$ the sufficient condition is $s \geq 1/4$.

To establish the pointwise convergence (1.0.4), Sjölin [27] also followed the well-established approach and obtained the following bound for the corresponding maximal function:

Theorem 1.0.1. *If $n \geq 3$ and $a > 1$, then*

$$\|S^* f\|_{L^2(B)} \leq C_B \, \|f\|_{H^s(\mathbb{R}^n)}, \ \ f \in C_c^\infty(\mathbb{R}^n) \tag{1.0.6}$$

holds for $s > 1/2$, and B is any ball of arbitrary radius in $\mathbb{R}^n$

Here $S^* f$ is the maximal operator corresponding to operator $S_t f$, defined as

$$S^* f = \sup_{0<t<1} |S_t f|,$$

and

$$S_t f(x) = u(x,t) = \int_{\mathbb{R}^n} e^{ix\xi} e^{it|\xi|^a} \widehat{f}(\xi) \, d\xi.$$

Clearly, the choice $a = 2$ recovers the usual Schrödinger equation.

Theorem 1.0.2. *If $n = 1, 2$ and $a > 1$, then*

$$\|S^* f\|_{L^2(B)} \leq C_B \, \|f\|_{H^s(\mathbb{R}^n)}, \ \ f \in C_c^\infty(\mathbb{R}^n)$$

holds for $s \geq n/4$, and B is any ball of arbitrary radius in $\mathbb{R}^n$

Moreover, for $n = 1$, Sjölin established the sharpness of the result in the above Theorem 1.0.2. He proved the following theorem :

Theorem 1.0.3. *Let $n = 1$, and $a > 1$, then (1.0.6) holds if and only if $s \geq 1/4$.*

Now, since the expression of the solution $S_t f(x)$ involves the Fourier transform of the initial data f, it's natural to ask whether the results regarding the pointwise convergence (1.0.4) can be improved for radial initial data.

In 1989, Prestini [25] first addressed this problem 1.0.5 with radial data f, and proved (1.0.4) holds iff $s \geq 1/4$ for all $n \geq 1$. Her idea of proof was in the spirit of Carleson's [6] method. Using the asymptotic expansion for the Bessel function and by linearization of $S_t f$, she proved the maximal estimate:

$$\|S^* f\|_{L^1(B)} \leq C\|f\|_{H^{1/4}}$$

Later, in [28] Sjölin has studied the local estimate as well as global estimate for the operator $S^* f$, hence derived a detailed mapping property for $S^* f$, with radial f. The local estimate is as follows:

$$\|S^* f\|_{L^q(B)} \leq C\|f\|_{H^s(\mathbb{R}^n)}, \tag{1.0.7}$$

$1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Assuming that f is radial he improved upon the results obtained in [27]. The theorems are as follows:

Theorem 1.0.4. *If $s < 1/4$, then (1.0.7) holds for no q.*

Theorem 1.0.5. *If $1/4 \leq s < n/2$, then (1.0.7) holds if and only if $q \leq \frac{2n}{n-2s}$*

In this thesis, we study these problems on real rank-one symmetric spaces of non-compact type. To formulate the problem, we need some notation. For details, we refer to chapter 2. Let G be a connected, non-compact, semisimple Lie group and K be a maximal compact subgroup of G. Then $\mathbb{X} = G/K$ is a real rank-one symmetric space of non-compact type with origin K. If we take G as $SO_e(n, 1)$ and $K = SO(n)$, where $SO_e(n, 1)$ is the connected Lie group leaving invariant the bilinear form on $\mathbb{R}^{n+1}$ and $K = SO(n)$ is the compact rotation subgroup of $SO_e(n, 1)$, then the homogeneous space G/K reduces to the n dimensional real hyperbolic space model $\mathbb{H}^n$.

In 2019, Wang and Zhang [32] addressed this problem in a manifold setting and proved the pointwise convergence result for the solution of the usual Schrödinger equation in $\mathbb{H}^n$. They have proved the following theorem:

Theorem 1.0.6. *If $f \in H^s(\mathbb{H}^n)$ and $s > 1/2$, then*

$$u(x,t) \to f \ \text{ pointwise a.e as } \ t \to 0,$$

where $u(x,t)$ is the solution of usual Schrödinger equation on $\mathbb{H}^n$.

Now, since $\mathbb{X}$ is the generalization of $\mathbb{H}^n$, it is natural to ask whether the same result holds for $\mathbb{X}$, and it is also relevant to investigate the sharpness of the result.

One of the goals of this thesis is to establish the pointwise convergence result for the fractional Schrödinger equation in the context of rank-one symmetric space $\mathbb{X}$. We also endeavour to obtain the analogue of Sjölin's result (Theorem 1.0.1) on $\mathbb{X}$ with radial initial data, which generalizes Theorem 1.0.6. Moreover, we will establish the sharpness of our result.

The last part of the thesis addresses the well-posedness of certain dispersive equations, the motivation comes from the local smoothing property of solutions to this equations. A time-reversible and conservative evolution equation preserves Sobolev regularity; that is,

$$\|u(\cdot,t)\|_{H^s} = \|u_0\|_{H^s}, \quad \text{for each } \ t > 0.$$

Hence, no global or pointwise-in-time smoothing is possible. But, locally in space, certain derivatives of u can be better behaved when averaged in time. This extra regularity is known as local smoothing, which often looks like:

$$\int_0^T \int_{|x| \leq R} |\nabla u(x,t)|^2 dx\, dt \leq C\|u_0\|^2_{L^2(\mathbb{R}^n)},$$

which shows a gain of one spatial derivative locally in space and after averaging in time.

In 1983, Kato [20] discovered a local smoothing property of the Korteweg-de Vries equation:

$$\begin{cases} du/dt + D^3u + uDu = 0, & t > 0 \\ u(x,0) = \phi(x), \end{cases} \tag{1.0.8}$$

where, $D = \frac{d}{dx}$.

He showed that the solution of the Cauchy problem (1.0.8) is locally one derivative smoother than the initial data by proving the following theorem:

Theorem 1.0.7. *Let* $s > 3/2$, $0 < T < \infty$, *if* $u \in C([0,T]\,; H^s)$ *is the solution of the equation* (1.0.8) *for* $\phi \in H^s$, *then*

$$u \in L^2\left([0,T]\,; H^{s+1}(-R,R)\right) \quad \textit{for any} \;\; R < \infty. \tag{1.0.9}$$

In other words, (1.0.9) can be interpreted as "u enjoys a local smoothing property." In fact, the result (1.0.7) has nothing to do with the sense of time. It is equally true for $-T \le t \le 0$.

Now, before discussing the smoothing properties of the solution, it is necessary to establish the well-posedness of the system.

In the same paper, Kato also established the well-posedness of the system (1.0.8). The theorem in this context is as follows:

Theorem 1.0.8. *Let* $s > 3/2$, *for each* $\phi \in H^s$ *there exists a unique* $t > 0$ *and a unique solution* u *to* (1.0.8), *such that*

$$u \in C([0,T] : H^s). \tag{1.0.10}$$

The map $\phi \to u$ *is continuous from* H^s *to the space* (1.0.10).

Later, Constantin and Saut [10] generalized this local smoothing effect for dispersive equations such as K-dV, Benjamin-Ono, intermediate long wave, various Boussinesq, Schrödinger equation, etc. They studied the systems of the form

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + P(D)u(x,t) = F(x,t), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\[2ex] u(x,0) = u_0(x). \end{cases} \tag{1.0.11}$$

where, $P(D)u$ is defined via the symbol $P(\xi)$ such that $P(\xi)$ behaves like $|\xi|^m$ as $|\xi| \to \infty, m > 1$ and $D = \frac{1}{i}\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$.

The typical result of local smoothing they obtained is: If u_0 belongs to $H^s(\mathbb{R}^n)$ then, for almost every $t \neq 0$, the solution $u(t,\cdot)$ belongs to $H^{s+d}(\mathbb{R}^n)$. That is, the solution gained some extra smoothness compared to the initial data.

For $s \ge -(m-1)/2$ and a certain value of α, they have proved the following theorem:

Theorem 1.0.9. *Let,* $u_0 \in H^s(\mathbb{R}^n), F \in L^1_{loc}(\mathbb{R}, H^{s,q}(\mathbb{R}^n)), 1 \le q \le 2$. *Then the solution* u *of the equation* (5.1.1) *belongs to* $L^2([-T,T] : H^{s+\alpha}_{u;loc}(\mathbb{R}^n))$ *for every* $T > 0$.

Moreover, for every $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ *which is supported in* $[-T,T]\times\mathbb{R}^n$*, one has the smoothing estimate:*

$$\left(\int_{\mathbb{R}^{n+1}} \chi^2(x,t)|(I-\Delta)^{(s+\alpha)/2}u(t,x)|^2 dx\,dt\right)^{1/2} \le C_\chi\left(\|u_0\|_{H^s(\mathbb{R}^n)} + \|F\|_{L^1(-T,T;H^{s,q}(\mathbb{R}^n))}\right) \tag{1.0.12}$$

As an application of this Theorem (1.0.9), they considered the Schrödinger equation with real time dependent potential $V(x,t)$:

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + \Delta u(x,t) + Vu(x,t) = 0, & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ u(x,0) = u_0(x). \end{cases} \tag{1.0.13}$$

For a certain value of real α,β they have the corollary:

Corollary 1.0.1. *Suppose* $V = V_1+V_2, V_1 \in L^p_{loc}(\mathbb{R}, L^\alpha(\mathbb{R}^n)), V_2 \in L^\infty_{loc}(\mathbb{R}, L^\beta(\mathbb{R}^n))$*. Then the solution* u *of* (1.0.13) *corresponding to* $u_0 \in L^2(\mathbb{R}^n)$ *satisfies*

$$u \in L^2([-T,T] : H^{1/2}_{loc}(\mathbb{R}^n)) \text{ for every } T > 0.$$

So, in particular, Corollary (1.0.1) gives a local smoothing effect for the Schrödinger equation with non-zero potential V.

Sjölin [27] and Vega [31] considered this equation (1.0.13) in the free case ($V = 0$). They established that the solutions of the equation (1.0.13) have locally 1/2 derivative more than the initial data almost every time.

In 1993, Kenig and Ruiz [26] addressed this problem (1.0.13) with nonzero potential V. Also they considered V as $V = V_1+V_2, V_1 \in L_x^{n/2}(L_t^\infty(\mathbb{R}))$, and $V_2 \in L^r([-T,T] : L_x^\infty), r > 1$. They have established the well-posedness and local smoothing for (1.0.13) by proving the following theorem:

Theorem 1.0.10. *Let* $\|V_1\|_{L_x^{n/2}(L_t^\infty(\mathbb{R}))}$ *be small enough. Then there exists a unique solution* $u(x,t)$ *such that*

$$\|u\|_{L_x^{2n/(n-2)}(L_t^2[-T,T])} + \sup_{|t|<T}\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le C\|u_0\|_{L^2(\mathbb{R}^n)} \tag{1.0.14}$$

Moreover,

$$\sup_{x_0,R}\frac{1}{R}\int_{B(x_0,R)}\int_{-T}^{T}|D_x^{1/2}u|^2 dt\,dx \le C(T)\|u_0\|^2_{L^2(\mathbb{R}^n)},$$

where $\widehat{(D_x^s f)}(\xi) = |\xi|^s\widehat{f}(\xi)$.

We aim to study this problem in the realm of the symmetric space $\mathbb{X}$. We obtain the estimate (1.0.14) in this setting and consequently establish the well-posedness of the equation (1.0.13) in $\mathbb{X}$.

1.1 Outline of the Thesis

In Chapter 2, we introduce all the notations and conventions that we will follow throughout the thesis. By appropriate references, we also quote all necessary results on the symmetric space, which will be used later. However, for the sake of simplicity, we will present the proof of a few of them, which we were unable to locate in the literature.

Chapter 3 mainly deals with a sufficient condition for the pointwise convergence of the solution to the fractional Schrödinger equation in $\mathbb{X}$. Here we extend Sjölin's result (1.0.4) by proving Theorem A. The proof of Theorem A is based on an appropriate maximal function estimate which we will prove in Theorem B. Moreover, Theorem B can be viewed as an analogue of Theorem 1.0.1 in $\mathbb{X}$.

In chapter 4, we improve the regularity result obtained in chapter 3. First, we prove an analogue of Sjölin's result (Theorem (1.0.2)), stated here as Theorem D. Using this result, we prove Theorem C, which improves the result of Theorem A obtained in the previous chapter.

Chapter 5 focuses on certain local regularity properties and well-posedness of some dispersive equations. To be more precise, we establish a well-posedness result by proving an estimate similar to (1.0.14), originally obtained by Ruiz and Vega. This result is proved for Schrödinger equation with nonzero potential V in $\mathbb{X}$.

In chapter 6, the last chapter of this thesis, we address some of the future problems related to the problem discussed in the thesis.

CHAPTER 2

Rank-One Symmetric Space of non-compact Type

2.1 Introduction

In this chapter, we collect the basic definitions and notation that will be used throughout this thesis. Most of the work in this thesis is carried out on Riemannian symmetric spaces of non-compact type with real rank one. The main aim of this chapter is to present details on elementary spherical functions and their estimates. We also recall some fundamental definitions, such as the Helgason–Fourier transform, the spherical transform, and the Harish-Chandra's c-function, along with their properties. Most of our notation is standard and can be found in [15, 17, 29, 18].

2.2 Basic Structure

In this section, we will discuss the basic facts of real rank-one symmetric space of non-compact type. Let G be a connected non-compact semisimple Lie group with finite center, and $\mathfrak{g}$ be the Lie algebra of G. Let B be the killing form on $\mathfrak{g}$, defined by

$$B(X,Y) = Tr(adX \circ adY), \quad X, Y \in \mathfrak{g}.$$

Let θ be a Cartan involution of $\mathfrak{g}$, then θ is an isomorphism of $\mathfrak{g}$ such that $\theta^2 = -I$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition, where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces of θ corresponding to the eigenvalue 1 and -1, that is

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}, \mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}.$$

Let $K = \exp\ \mathfrak{k}$ be a maximal compact subgroup of G and let $\mathbb{X} = G/K$ be the associated Riemannian symmetric space. If $o = eK$ denotes the identity coset, then for $g \in G$ the quantity $r(g)$ denotes the Riemannian distance of the element $g.o$ from the identity coset. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, $A = \exp\ \mathfrak{a}$ be the corresponding subgroup of G, and M the centralizer of A in K.

The rank of G is defined as the dimension of the real vector space $\mathfrak{a}$. From this point onward, we will assume that the group G has real rank one, which means that dim $\mathfrak{a} = 1$. Let $a^*_{\mathbb{R}}$ be the real dual of $\mathfrak{a}$ and for $\alpha \in a^*_{\mathbb{R}}$, we define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \ \text{ for all } H \in \mathfrak{a}\}.$$

We say that α is a root if $\dim_{\mathbb{R}} \mathfrak{g}_\alpha > 0$. Let $\mathfrak{a} = \text{Span}\{H_0\}$, $H_0 \in \mathfrak{a}$; α is called a positive root if $\alpha(H_0) > 0$. In the rank one case, it is well known that the set of roots is either of the form $\{-\alpha, \alpha\}$ or $\{-\alpha, -2\alpha, \alpha, 2\alpha\}$ [18] and the Lie algebra $\mathfrak{g}$ admits the decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

Let $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. The Iwasawa decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

We assume that $\alpha(H_0) = 1$ and identify $\mathfrak{a}$ with $\mathbb{R}$ as $t \mapsto tH_0$ via this identification. Then $\mathfrak{a}_+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0\}$ is identified with the set of positive real numbers. The complexification $\mathfrak{a}^*_{\mathbb{C}}$ of $\mathfrak{a}^*$ is defined as the vector space of complex valued real linear functionals on $\mathfrak{a}$. We identify $\mathfrak{a}^*$ (the real dual of $\mathfrak{a}$) and $\mathfrak{a}^*_{\mathbb{C}}$ (the complex dual of $\mathfrak{a}$) with $\mathbb{R}$ and $\mathbb{C}$ via the identification $t \mapsto t\alpha$ and $z \mapsto z\alpha$, $t \in \mathbb{R}$ and $z \in \mathbb{C}$ respectively.

Let $m_1 = \dim\ \mathfrak{g}_\alpha$, $m_2 = \dim\ \mathfrak{g}_{2\alpha}$ and $\rho = \frac{1}{2}(m_1 + 2m_2)\alpha$ be the half sum of positive roots. We will denote dim$\mathbb{X}$ by n, $n = m_1 + m_2 + 1$. By abuse of notation we will denote $\rho(H_0) = \frac{1}{2}(m_1 + 2m_2)$ by ρ.

Since $A = \exp\ \mathfrak{a}$, the space $A = \{a_s : a_s = \exp\ (sH_0), s \in \mathbb{R}\}$ is diffeomorphic to $\mathbb{R}$. If $N = \exp \mathfrak{n}$, we have the Iwasawa decomposition for the group G given by $G = KAN$. It is known that the Iwasawa decomposition is unique and hence any $g \in G$ can be uniquely written as

$$x = \mathbf{k}(g) \exp H(g) n(g), \tag{2.2.1}$$

where $H(g) \in \mathfrak{a}$, $\mathbf{k}(g) \in K$ and $n(g) \in N$. If $A^+ = \exp \mathfrak{a}_+$ and $\overline{A^+} = \exp \overline{\mathfrak{a}_+}$, then we also have the polar decomposition of G given by $G = K\overline{A^+}K$. The above decomposition is also unique and hence every element $g \in G$ can be uniquely written as

$$g = k_1 a_t k_2, \quad k_1, k_2 \in K, a_t = \exp tH_0 \in A, t \geq 0. \tag{2.2.2}$$

The functions defined on $\mathbb{X}$ can be viewed as right K-invariant functions on G. Furthermore, the radial functions on $\mathbb{X}$ are K-biinvariant functions on G. Let $G = K\overline{A^+}K$ be the polar decomposition G, where $\overline{A^+} = \{a_t \in A : t \geq 0\}$. In view of this decomposition, it is worth noting that any radial function on $\mathbb{X}$ depends only on a_t. The Haar measure associated with the two decompositions (2.2.2) and (2.2.1) are given by:

$$\int_G f(g)\, dg = C \int_K \int_0^\infty \int_K f(k_1 a_t k_2) D(t)\, dk\, dt\, dk, \tag{2.2.3}$$

$$\int_G f(g) dg = C_1 \int_K \int_{\mathbb{R}} \int_N f(k a_t n) e^{2\rho t}\, dk\, dt\, dn \tag{2.2.4}$$

where $D(t) = (\sinh t)^{m_1} (\sinh 2t)^{m_2}$. and dk is the Haar measure of K.

2.3 Helgason-Fourier transform and spherical Fourier Transform on $\mathbb{X}$

It can be seen easily from the Iwasawa decomposition that the function $g \mapsto e^{-(i\lambda+\rho)H(g^{-1}k)}$ is right invariant under the action of K and hence defines a function on the symmetric space $\mathbb{X}$. For $\lambda \in \mathbb{C}$, we define the following function on $\mathbb{X}$

$$e_\lambda(x) = e^{-(i\lambda+\rho)H(x^{-1}k)} = e^{-(i\lambda+\rho)H(g^{-1}k)}, \quad x = gK.$$

Now for $f \in C_c^\infty(\mathbb{X})$, the Helgason Fourier transform $\widetilde{f}$ of f is defined by

$$\widetilde{f}(\lambda, k) = \int_X f(x) e^{-(i\lambda+\rho)H(x^{-1}k)} dx, \quad \lambda \in \mathbb{C}.$$

We also have the following Fourier inversion on $C_c^\infty(\mathbb{X})$:

$$f(x) = C \int_{\mathfrak{a}^* \times K/M} \widetilde{f}(\lambda, k) e^{(i\lambda+\rho)H(x^{-1}k)} |c(\lambda)|^{-2} d\lambda \, dk_M. \tag{2.3.5}$$

The Plancherel formula is given by

$$\|f\|^2_{L^2(X)} = C \int_{\mathbb{R}} \int_K |\widetilde{f}(\lambda, k)|^2 |c(\lambda)|^{-2} dk d\lambda, \quad \text{for } f \in C_c^\infty(\mathbb{X}). \tag{2.3.6}$$

The $c(\lambda)$ in the above expressions is the famous Harish-Chandra's **c**-function.

2.3.1 Spherical Fourier Transform

For $\lambda \in \mathbb{C}$, the elementary spherical function ϕ_λ on $\mathbb{X}$ is defined as

$$\phi_\lambda(x) = \int_K e^{-(i\lambda+\rho)H(x^{-1}k)} dk, \tag{2.3.7}$$

where $H(g)$ is defined as in (2.2.1).

For $\lambda \in \mathbb{R}$, the spherical Fourier transform of a suitable radial function f on $\mathbb{X}$ is defined by

$$\widehat{f}(\lambda) = \int_{\mathbb{X}} f(x) \phi_\lambda(x) dx. \tag{2.3.8}$$

We now observed that for a suitable radial function f on $\mathbb{X}$ the Helgason Fourier transform and the spherical Fourier transform coincides. Suppose $f \in C_c(\mathbb{X})$ and radial then,

$$\begin{aligned}
\widetilde{f}(\lambda, k) &= \int_G f(g) e^{-(i\lambda+\rho)H(g^{-1}k)} dg \\
&= \int_G \int_K f(k_1 g) e^{-(i\lambda+\rho)H(g^{-1}k)} dk_1 \, dg \\
&= \int_K \int_G f(k_1 g) e^{-(i\lambda+\rho)H(g^{-1}k)} dg \, dk_1 \\
&= \int_G \left(\int_K f(g_1) e^{-(i\lambda+\rho)H(g_1^{-1}k_1 k)} dk \right) dg \\
&= \int_G f(g) \phi_\lambda(g) dg = \widehat{f}(\lambda).
\end{aligned}$$

In the above calculation, we put $k_1 g = g_1$ and used the fact that the groups G and K are unimodular.

The Plancherel and inversion formulas for the radial function f on $\mathbb{X}$ related to the spherical Fourier transform are given by:

$$f(x) = C \int_0^\infty \widehat{f}(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda, \quad \text{for } f \in C_c(\mathbb{X}) \quad \textbf{(Inversion formula)} \tag{2.3.9}$$

$$\|f\|^2_{L^2(\mathbb{X})} = C\int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda, \quad \text{for } f \in C_c(\mathbb{X}). \text{ (\textbf{Plancherel formula})} \tag{2.3.10}$$

For details about the spherical Fourier transform, Harish-Chandra's **c**−function, and related results, we refer the reader to (Chapter IV, [17]).

2.4 Elementary spherical function

Let $\mathcal{L}$ denote the Laplace-Beltrami operator on $\mathbb{X}$. We will now describe an important class of eigenfunctions of $\mathcal{L}$. For $\lambda \in \mathbb{C}$ we define the following function on $\mathbb{X}$

$$e_\lambda(x) = e^{-(i\lambda+\rho)H(x^{-1}k)} = e^{-(i\lambda+\rho)H(g^{-1}k)}, \quad x = gK.$$

These functions turn out to be the basic eigenfunctions of $\mathcal{L}$ in fact, they satisfy the differential equation

$$\mathcal{L}e_\lambda = -(\lambda^2 + \rho^2)e_\lambda.$$

(see [19] for details). This implies that the functions ϕ_λ also satisfy the equation

$$\mathcal{L}\phi_\lambda = -(\lambda^2 + \rho^2)\phi_\lambda. \tag{2.4.11}$$

Moreover, we have the characterization for spherical function on $\mathbb{X}$ due to Harish-Chandra [17, Chapter II, §4]:

Theorem 2.4.1. *As λ runs through $\mathfrak{a}^*_{\mathbb{C}}$, the functions*

$$\phi_\lambda(g) = \int_{\mathbb{K}} e^{-(i\lambda+\rho)(H(g^{-1}k))} dk, \quad g \in G$$

exhaust the class of spherical functions on G. Moreover, two such functions ϕ_λ and ϕ_η are identically equal iff $\lambda = s\eta$, for some s in the Weyl group. In particular, $\lambda = \pm\eta$ for a rank one symmetric space.

We now discuss some important properties of ϕ_λ (See (Chapter IV, [17]) and (Chapter 3, [15]) for more detail), which are valid for $\lambda \in \mathbb{C}$:

Theorem 2.4.2. i) *ϕ_λ is continuous and K-biinvariant, that is,*

$$\phi_\lambda(k_1 g k_2) = \phi_\lambda(g), \quad k_1, k_2 \in K, g \in G.$$

We also have $\phi_\lambda(e) = 1$.

ii) *For all* $g \in G$,

$$\phi_\lambda(g) = \phi_\lambda(g^{-1}) = \phi_{-\lambda}(g). \tag{2.4.12}$$

iii) *If* $Im\,\lambda < 0$ *then*

$$\lim_{t\to\infty} e^{(-i\lambda+\rho)(tH)}\phi_\lambda(a_t) = c(\lambda), \tag{2.4.13}$$

where $c(\lambda)$ *is Harish-Chandra's* **c***-function.*

iv) *The Harish-Chandra series for the elementary spherical function is given by:*

$$\phi_\lambda(\exp\ tH_0) = c(\lambda)e^{(i\lambda-\rho)t}\varphi_\lambda(t) + c(-\lambda)e^{(-i\lambda-\rho)t}\varphi_{-\lambda}(t), \tag{2.4.14}$$

where $\varphi_\lambda(t) = \sum_{n=0}^{\infty}\Gamma_n(\lambda)e^{-nt}$ *and* $\Gamma_n(\lambda)$ *has the following estimate:*

$$|\Gamma_n(\lambda)| \leq C\frac{n^v}{1+|\lambda|}, \qquad \text{[1, Lemma 2.1]} \tag{2.4.15}$$

2.4.1 Some estimates of ϕ_λ

We now describe some estimates of the elementary spherical function, which will be crucial for our proof. These estimates will be derived from the series expansion of ϕ_λ. Most of them are given in [18, 29]. We also need the following well-known estimate of the Plancherel density [1]:

$$|c(\lambda)|^{-2} \leq C|\lambda|^2(1+|\lambda|)^{(n-3)}, \ \ \forall\lambda\in\mathbb{R}. \tag{2.4.16}$$

Lemma 2.4.3. *Let* $\lambda \in \mathbb{R}$. *Then* $|\phi_\lambda(x)| \leq C$, *for all* $x \in \mathbb{X}$.

Proof. By [17, Theorem 8.1] ϕ_λ is bounded if and only if λ belongs to the strip $\{\lambda \in \mathbb{C} : |Im\lambda| \leq \rho\}$. Also, from the integral expression of ϕ_λ (2.2.1), it is clear that

$$|\phi_\lambda(x)| \leq \phi_0(x) \leq Ce^{-\rho t}(1+t) \leq C, \ \text{ for all } x \in \mathbb{X} \ \ [18]$$

□

Lemma 2.4.4. *Let* $\lambda \in \mathbb{R}$ *and* $t > 1$, *the elementary spherical function* ϕ_λ *has the following estimate :*

$$|\phi_\lambda(t)| \leq C|c(\lambda)| \leq C\frac{1}{|\lambda|(1+|\lambda|)^{\frac{m_1+m_2-2}{2}}}. \tag{2.4.17}$$

Proof. We use the following Harish-Chandra series of ϕ_λ given in [18]:

$$\phi_\lambda(t) = e^{-\rho t}(e^{i\lambda t}c(\lambda)a_2(\lambda,t) + e^{-i\lambda t}c(-\lambda)a_2(-\lambda,t)). \tag{2.4.18}$$

where, $a_2(\lambda,t) = \sum_{n=0}^{\infty}\Gamma_n(\lambda)e^{-nt}$ and $c(\lambda)$ is the Harish-Chandra's **c**-function. Γ_n satisfies the following recursion:

$$\Gamma_0(\lambda) = 1;$$

$$n(n+1-i\lambda)\Gamma_{n+1} = \sum_{j=0}^{n}\frac{m_1}{2}(\rho+2j-i\lambda)\Gamma_j + \sum_{\substack{j=n+1-2l\\ l>0,j\geq 0}} m_2(\rho+2j-i\lambda)\Gamma_j$$

The function a_2 satisfies the inequality $|a_2(\lambda,t)| \leq C$ for all $t \geq 1$ and $\lambda \in \mathbb{R}$ [18, Proposition A2]. Therefore from the above expression ϕ_λ, we have $|\phi_\lambda(t)| \leq C|c(\lambda)|$ for $t > 1$. Now the estimate (2.4.17) follows from (2.4.16). □

The proof of the following lemma mostly depends on information given in [29].

Lemma 2.4.5. *For $t \leq 1$ and $\lambda > 1$, we have:*

$$|\phi_\lambda(t)| \leq \frac{C}{(\lambda t)^{\frac{n-1}{2}}}. \tag{2.4.19}$$

Proof. It is follow from [29, Theorem 2.1], that there exist $R_0, R_1(>1)$ such that for any t with $0 \leq t \leq R_0$ and any $M \geq 0$,

$$\phi_\lambda(\exp tH_0) = c_0\left[\frac{t^{n-1}}{D(t)}\right]^{\frac{1}{2}}\sum_{0}^{\infty} t^{2m}a_m(t)\mathcal{J}_{\frac{n-2}{2}+m}(\lambda t), \tag{2.4.20}$$

where, $\mathcal{J}_\mu(z) = \frac{J_\mu(z)}{z^\mu}\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{\mu-1}$, and $J_\mu(z)$ is the standard Bessel function and $D(t) = (\sinh t)^{m_1}(\sinh 2t)^{m_2}$. The above expression of ϕ_λ can also be written as:

$$\phi_\lambda(\exp tH_0) = c_0\left[\frac{t^{n-1}}{D(t)}\right]^{\frac{1}{2}}\sum_{0}^{M} t^{2m}a_m(t)\mathcal{J}_{\frac{n-2}{2}+m}(\lambda t) + E_{M+1}(\lambda t), \tag{2.4.21}$$

where,

$$a_0(t) = 1, \text{and } \ |a_m(t)| \leq cR_1^{-m}.$$

$$|E_{M+1}(\lambda t)| \leq \begin{cases} c_M t^{2(M+1)}, & \text{if } \ |\lambda t| \leq 1, \\ c_M t^{2(M+1)}(\lambda t)^{-(\frac{n-1}{2}+M+1)}, & \text{if } \ |\lambda t| > 1. \end{cases} \tag{2.4.22}$$

First assume, $\lambda > 1$ and $t < \frac{1}{\lambda} < 1$. Using the fact $|\mathcal{J}_\mu(t)| \leq C$ for all μ and for all $t \leq 1$ and the bound of a_m, we obtain from (2.4.20):

$$|\phi_\lambda(t)| \leq \frac{C}{(D(t))^{\frac{1}{2}}} \frac{1}{\lambda^{\frac{n-1}{2}}} \sum_{m=0}^{\infty} t^{2m} \frac{1}{R_1} < \frac{C}{(\lambda t)^{\frac{n-1}{2}}}.$$

Next, consider the case $\lambda t \geq 1$. As $D(t) \asymp t^{n-1}$, for $0 < t \leq 1$; thus by using (2.4.21), we can write

$$|\phi_\lambda(t)| \leq \left(\frac{t^{n-1}}{D(t)}\right)^{\frac{1}{2}} \left[\mathcal{J}_{\frac{n-2}{2}}(\lambda t) + E_1(\lambda t)\right] \leq C \left[\mathcal{J}_{\frac{n-2}{2}}(\lambda t) + E_1(\lambda t)\right].$$

From a estimate of the Bessel function, we have $|\mathcal{J}_\mu(t)| \leq \dfrac{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2})2^{\mu-1}}{t^{\mu+\frac{1}{2}}}$, (for $\mu > -\frac{1}{2}$ and $t \geq 1$). Now by using the estimate of E_M given in (2.4.22), we get

$$|\phi_\lambda(t)| \leq \frac{C}{(\lambda t)^{\frac{n-1}{2}}}.$$

□

2.5 Sobolev space in $\mathbb{X}$

We conclude this chapter by recalling the definition of Sobolev spaces given in [1]. For $s \in \mathbb{R}$, we define $H^s(\mathbb{X})$ as the image of $L^2(\mathbb{X})$ under $(-\mathcal{L})^{-\frac{s}{2}}$. That is

$$\begin{aligned} H^s(\mathbb{X}) &= \left\{f = (-\mathcal{L})^{-\frac{s}{2}} g : g \in L^2(\mathbb{X})\right\} \\ &= \{f : (-\mathcal{L})^{\frac{s}{2}} f \in L^2(\mathbb{X})\} \end{aligned}$$

The norm on $H^s(\mathbb{X})$ is defined by

$$\|f\|_{H^s(\mathbb{X})} = \|(-\mathcal{L})^{\frac{s}{2}} f\|_{L^2(\mathbb{X})}$$

Now, by using the Plancherel formula for radial functions (2.3.10), we have

$$\|(-\mathcal{L})^{\frac{s}{2}} f\|^2_{L^2(\mathbb{X})} = \int_0^\infty \widehat{(-\mathcal{L})^{\frac{s}{2}} f}(\lambda) |c(\lambda)|^{-2} d\lambda.$$

Since $\widehat{(-\mathcal{L})^{\frac{s}{2}} f}(\lambda) = (\lambda^2 + \rho^2)^{s/2} \widehat{f}(\lambda)$,

$$\|f\|_{H^s(\mathbb{X})} = \left(\int_0^\infty (\lambda^2+\rho^2)^s |\widehat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda\right)^{1/2}.$$

CHAPTER 3

Regularity of Solutions to Schrödinger equation on rank-one symmetric space of non-compact type

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3.1 Introduction

Let $u(x,t)$ be the solution of the Schrödinger equation

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ \\ u(x,0) = f(x). \end{cases} \tag{3.1.1}$$

Taking the Fourier transform in the x-variable, the solution can be written as

$$u(x,t) = S_t f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{f}(\xi)\, d\xi,$$

where $\widehat{f}(\xi)$ is the Fourier transform of f, and S_t defines a one-parameter family of unitary operators on $L^2(\mathbb{R}^n)$ such that,

$$\widehat{S_t f}(\xi) = e^{it|\xi|^2} \hat{f}(\xi).$$

By Plancherel theorem $S_t f \to f$, in L^2-norm as $t \to 0$. A natural question, however, is under what additional regularity conditions on the initial data f, this convergence holds pointwise almost everywhere.

In a seminal paper [6], Carleson posed this question and for the case $n = 1$, proved that if $f \in H^s(\mathbb{R})$, $s \geq \frac{1}{4}$ and the support of f is compact, then $S_t f(x) \to f$ *a.e.* $x \in \mathbb{R}$, as t goes to 0. Moreover, Dahlberg and Kening [12] proved that the condition $s \geq \frac{1}{4}$ is sharp. This result has been further improved by many authors, including Sjölin, Bourgain, Moyua, Vargas, Vega, and Lee, in higher dimensions [4, 14, 22, 24], leading to a sharp result. Similar results were given in [32], where Xing Wang and Chunjie Zhang addressed the same problem in a manifold setting and obtained results analogous to those in Euclidean space.

In this chapter, we study the fractional Schrödinger equation in rank-one symmetric spaces of non-compact type. We will consider the case when the initial data is radial. Our result is an extension of Sjölin's result on $\mathbb{R}^n$ and further generalizes the work of Wang and Zhang on real hyperbolic spaces.

3.2 Statement of Main Results

In [27], Sjölin considered the fractional Schrödinger equation on $\mathbb{R}^n$:

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} = \Delta^{\frac{a}{2}} u(x,t), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, a > 1, \\ u(x,0) = f(x). \end{cases} \tag{3.2.2}$$

and proved the following result:

Theorem 3.2.1. *For $n > 2$, if $f \in H^s(\mathbb{R}^n)$ with $s > \frac{1}{2}$, and also support of f is compact, then*

$$S_t f(x) \to f \textit{ a.e. } x \in \mathbb{R}^n; \textit{ as } t \textit{ goes to } 0.$$

Here $S_t f(x)$ is the solution of (3.2.2) defined by

$$S_t f(x) = u(x,t) = \int_{\mathbb{R}^n} e^{ix\xi} e^{it|\xi|^a} \widehat{f}(\xi) d\xi,$$

which reduces to the usual Schrödinger equation discussed earlier for $a = 2$.

We recall the definition of maximal operator

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|. \tag{3.2.3}$$

In the same paper, Sjölin proved the following result:

Theorem 3.2.2. *If* $n \geq 3$ *and* $a > 1$, *then* $\|S^* f\|_{L^2(B)} \leq C_B \|f\|_{H^s(\mathbb{R}^n)}$ *holds for* $s > 1/2$.

However, in [32], Wang and Zhang addressed the same problem for real hyperbolic space, and proved an analogous maximal estimate like the above theorem for $a = 2$. In this chapter, we generalize both the results to the broader context of rank-one symmetric spaces of non-compact type, and for the full fractional range for all $a > 1$.

Now we formulate the problem on the rank-one symmetric spaces of noncompact type. Let $\mathbb{X}$ be a rank-one symmetric space and let $\mathcal{L}$ be the Laplace-Beltrami operator on $\mathbb{X}$. We consider the fractional Schrödinger equation on $\mathbb{X}$ is given by:

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} = (-\mathcal{L})^{\frac{a}{2}} u(x,t), & (t,x) \in \mathbb{R} \times \mathbb{X}, \ a > 1, \\ u(x,0) = f(x), & f \text{ radial.} \end{cases} \tag{3.2.4}$$

Now we state the main results of this chapter:

Theorem A. *Let* $\mathbb{X}$ *be a rank-one symmetric space of non-compact type. Suppose* $a > 1$ *and let* $f \in H^s(\mathbb{X})$ *be radial with compact support. Then the solution* $u(x,t)$ *of the fractional Schrödinger equation* (3.2.4) *satisfies*

$$\lim_{t \to 0} u(x,t) \to f(x), \quad a.e.\ x \in \mathbb{X},$$

provided $s > \frac{1}{2}$.

The standard method for proving pointwise convergence involves obtaining an estimate for the corresponding maximal operator. Here also a main ingredient of the proof of Theorem A is the following maximal estimate. The proof of Theorem A is mainly based on the following maximal estimate:

Theorem B. *Let* $a > 1$, *and let* B *be any ball of arbitrary radius in* $\mathbb{X}$. *Then for all radial functions* $f \in C_c^\infty(\mathbb{X})$ *we have,*

$$\|S^* f\|_{L^2(B)} \leq C_B \|f\|_{H^s(\mathbb{X})}, \quad \text{for} \quad s > 1/2, \tag{3.2.5}$$

where $S^* f$ *is defined as in* (3.2.3).

Our proof will use Sjölin's [27] ideas to establish some mixed norm estimates and then use interpolation and inclusion results of Sobolev spaces to derive the required estimate of the maximal operator.

3.3 Proof of Theorem A and Theorem B

Using the spherical transform in x-variable, the solution $u(x,t)$ of equation (3.2.4) is given by

$$u(x,t) = S_t f(x) = c \int_0^\infty \widehat{f}(\lambda) e^{it(\lambda^2+\rho^2)^{\frac{a}{2}}} \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda.$$

The corresponding maximal operator is given by

$$S^* f(x) = \sup_{0<t<1} |S_t f(x)|.$$

We first prove Theorem B, in particular the estimate (3.2.5). Let B be any ball in $\mathbb{X}$ and choose

- $\alpha_0 \in C_c^\infty(\mathbb{X})$, a real valued radial function such that $\alpha_0 \equiv 1$ on $B \subset \text{Supp}\ (\alpha_0)$, and
- $\psi_0 \in C_c^\infty(\mathbb{R})$, a real valued even function such that $\psi_0 \equiv 1$ in $[0,1]$.

Now set

$$Sf(x,t) = \alpha_0(x)\psi_0(t) S_t f(x) = c\ \alpha_0(x)\psi_0(t) \int_0^\infty \widehat{f}(\lambda) e^{it(|\lambda|^2+|\rho|^2)^{\frac{a}{2}}} \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda. \tag{3.3.6}$$

First we shall obtain the crucial norm estimate of Sf in the following Proposition:

Proposition 3.3.1. *Let Sf be defined as above. Then*

$$\|Sf\|_{L^2(\mathbb{X}\times\mathbb{R})} \leq C \|f\|_{\mathbb{H}^{-s}(\mathbb{X})}\ ,\ s = \frac{a-1}{2}. \tag{3.3.7}$$

Proof. We have,

$$\begin{aligned}
&\int_{\mathbb{X}}\int_{\mathbb{R}} |Sf(x,t)|^2 dxdt = \int_{\mathbb{X}}\int_{\mathbb{R}} Sf(x,t)\overline{Sf(x,t)} dxdt \\
&= c \int_{\mathbb{X}}\int_{\mathbb{R}} \alpha_0(x)^2 \psi_0(t)^2 \left(\int_0^\infty \widehat{f}(\lambda) e^{it(\lambda^2+\rho^2)^{\frac{a}{2}}} \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda \right) \\
&\qquad\qquad \left(\int_0^\infty \overline{\widehat{f}(\eta)} e^{-it(\eta^2+\rho^2)^{\frac{a}{2}}} \phi_\eta(x) |c(\eta)|^{-2} d\eta \right) dxdt \\
&= c \int_0^\infty \int_0^\infty \left(\int_{\mathbb{X}} \alpha_0(x)^2 \phi_\lambda(x) \phi_\eta(x) dx \right) \left(\int_{\mathbb{R}} \psi(t) e^{-it((\eta^2+\rho^2)^{\frac{a}{2}} - (\lambda^2+\rho^2)^{\frac{a}{2}})} dt \right) \\
&\qquad\qquad \widehat{f}(\lambda)\overline{\widehat{f}(\eta)} |c(\lambda)|^{-2} |c(\eta)|^{-2} d\lambda d\eta \\
&= c \int_0^\infty \int_0^\infty \left(\int_{\mathbb{X}} \alpha(x) \phi_\lambda(x) \phi_\eta(x) dx \right) \widehat{\psi} \left((\eta^2+\rho^2)^{\frac{a}{2}} - (\lambda^2+\rho^2)^{\frac{a}{2}}) \right)
\end{aligned}$$

$$\widehat{f}(\lambda)\overline{\widehat{f}(\eta)}|c(\lambda)|^{-2}|c(\eta)|^{-2}d\lambda d\eta, \tag{3.3.8}$$

In the above calculation, we use $\overline{\phi_\eta}(x) = \phi_{-\eta}(x) = \phi_\eta(x)$, which follows from the definition of ϕ_λ (2.2.1) and put $\alpha = {\alpha_0}^2$, $\psi = {\psi_0}^2$. For $\lambda, \eta \in (0, \infty)$, we define:

$$K(\lambda,\eta) = (\rho^2+|\lambda|)^s(\rho^2+|\eta|)^s\Big(\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\Big)\widehat{\psi}((\eta^2+\rho^2)^{\frac{a}{2}}-(\lambda^2+\rho^2)^{\frac{a}{2}}). \tag{3.3.9}$$

Given a suitable function h, we define an operator

$$Th(\lambda) = \int_{\mathbb{R}} K(\lambda,\eta)h(\eta)|c(\eta)|^{-2}d\eta.$$

Now we have,

$$\begin{aligned}\int_{\mathbb{X}}\int_{\mathbb{R}} |Sf(x,t)|^2dxdt &= \int_0^\infty\int_0^\infty K(\lambda,\eta)(|\rho|^2+|\lambda|)^{-s}(|\rho|^2+|\eta|)^{-s}\widehat{f}(\lambda)\overline{\widehat{f}(\eta)} \\ &\qquad |c(\lambda)|^{-2}|c(\eta)|^{-2}d\lambda d\eta \\ &= \int_0^\infty\Big(\int_0^\infty K(\lambda,\eta)\overline{h(\eta)}|c(\eta)|^{-2}d\eta\Big)h(\lambda)|c(\lambda)|^{-2}d\lambda \\ &= \int_0^\infty T(\overline{h(\lambda)})h(\lambda)|c(\lambda)|^{-2}d\lambda \end{aligned} \tag{3.3.10}$$

where, $h(\lambda) = \widehat{f}(\lambda)(|\rho|^2+|\lambda|)^{-s}$. If T is a bounded operator on $L^2(\mathbb{R}, |c(\lambda)|^{-2}d\lambda)$, then by Hölder's inequality we have,

$$\int_{\mathbb{X}}\int_{\mathbb{R}} |Sf(x,t)|^2dxdt \leq \|T\bar{h}\|_2\|h\|_2 \leq C\|h\|_2^2 \leq C\|f\|^2_{H^{-s}(\mathbb{X})}. \tag{3.3.11}$$

Now, to show T is bounded on $L^2(\mathbb{R}, |c(\lambda)|^{-2}d\lambda)$, it is sufficient to show that there exists a constant $C > 0$ such that

$$\int_0^\infty |K(\lambda,\eta)||c(\lambda)|^{-2}d\lambda \leq C \ \ ,\forall \eta \in (0,\infty), \tag{3.3.12}$$

and

$$\int_0^\infty |K(\lambda,\eta)||c(\eta)|^{-2}d\eta \leq C \ \ \forall \lambda \in (0,\infty). \tag{3.3.13}$$

Indeed by using the Cauchy-Schwarz inequality and Fubini's theorem we have,

$$\begin{aligned}
\int_{\mathbb{R}} |Th(\lambda)|^2 |c(\lambda)|^{-2} d\lambda &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} K(\lambda,\eta) h(\eta) |c(\eta)|^{-2} d\eta \right|^2 |c(\lambda)|^{-2} d\lambda \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K(\lambda,\eta)|^{1/2} |K(\lambda,\eta)|^{1/2} |h(\eta)| |c(\eta)|^{-2} d\eta \right)^2 |c(\lambda)|^{-2} d\lambda \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K(\lambda,\eta)| |c(\eta)|^{-2} d\eta \right) \left(\int_{\mathbb{R}} |K(\lambda,\eta)| |h(\eta)|^2 |c(\eta)|^{-2} d\eta \right) |c(\lambda)|^{-2} d\lambda \\
&\leq C \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K(\lambda,\eta)| |h(\eta)|^2 |c(\eta)|^{-2} d\eta \right) |c(\lambda)|^{-2} d\lambda \\
&\leq C \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K(\lambda,\eta)| |c(\lambda)|^{-2} d\lambda \right) |h(\eta)|^2 |c(\eta)|^{-2} d\eta \leq C \|h\|_2^2.
\end{aligned}$$

In the above calculation and throughout this chapter, C is a generic constant.

Now we will prove the boundedness stated in (3.3.12) and (3.3.13). Due to the symmetry in the expression for $K(\lambda,\eta)$, that is $K(\lambda,\eta) = K(\eta,\lambda)$, it is sufficient to prove (3.3.12), and (3.3.13) will follow. Performing a change of variables

$$u = (\lambda^2 + \rho^2)^{\frac{a}{2}}, \quad \text{and} \quad b = (\eta^2 + \rho^2)^{\frac{a}{2}},$$

using boundedness of ϕ_λ, the estimate (2.4.16) of $|c(\lambda)|$ and the fact that $\rho \geq 1/2$, we have

$$\begin{aligned}
&\int_0^\infty |K(\lambda,\eta)| |c(\lambda)|^{-2} d\lambda \\
&\leq C \int_{\rho^a}^\infty |\widehat{\psi}(u-b)| \left| \int_{\mathbb{X}} \alpha(x) \phi_\lambda(x) \phi_\eta(x) dx \right| \left(1 + \sqrt{b^{\frac{2}{a}} - \rho^2} \right)^s \\
&\qquad\qquad \left(1 + \sqrt{u^{\frac{2}{a}} - \rho^2} \right)^{s+m_1+m_2-1} u^{\frac{2}{a}-1} du. \qquad (3.3.14)
\end{aligned}$$

We now estimate $K(\lambda,\eta)$ into two cases:

Case I ($\eta \leq 2$) :

Given that $\eta \leq 2$, it follows that $b \leq (4+\rho^2)^{\frac{a}{2}}$. For $\beta = (4+\rho^2)^{a/2} + 1$ the integral in (3.3.14) can be expressed as

$$\begin{aligned}
\int_0^\infty |K(\lambda,\eta)| |c(\lambda)|^{-2} d\lambda &\leq C \int_{\rho^a}^{\beta+\rho^a} |\widehat{\psi}(u-b)| u^{\frac{s+m_1+m_2+1-a}{a}} du + C \int_{\beta+\rho^a}^\infty |\widehat{\psi}(u-b)| u^{\frac{s+m_1+m_2+1-a}{a}} du \\
&= I_1 + I_2.
\end{aligned}$$

In the above estimate we have used the fact, that the integral

$$\left|\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right|$$

is bounded as $\alpha \in C_c^\infty(\mathbb{X})$ and both ϕ_λ and ϕ_η are bounded functions. Moreover, as $\hat{\psi}$ is rapidly decreasing and bounded, we have

$$I_1 = \left|C\int_{\rho^a}^{\beta+\rho^a} |\widehat{\psi}(u-b)|u^{\frac{s+m_1+m_2+1-a}{a}}du\right| \leq C\left|\int_{\rho^a}^{\beta+\rho^a} u^{\frac{s+m_1+m_2+1-a}{a}}du\right|$$

which is bounded by a constant C, independent of λ and η.

For any large positive integer m, $\frac{u^m}{(u-b)^m}$ is uniformly bounded, we have

$$|\widehat{\psi}(u-b)| \leq C\frac{1}{(u-b)^m} \leq C\frac{1}{u^m} \quad \text{and} \quad I_2 \leq C\int_{\beta+\rho^a}^{\infty} \frac{u^\gamma}{u^m}du \leq C$$

where $\gamma = \frac{s+m_1+m_2+1-a}{a}$.

Case II ($\eta > 2$) :

Since $b = (\eta^2+\rho^2)^{\frac{a}{2}}$ and $\eta > 2$, we have $\left(1+\sqrt{b^{\frac{2}{a}} - \rho^2}\right)^s \leq Cb^{\frac{s}{a}}$, and $b > (2^2+\rho^2)^{\frac{a}{2}} > (\frac{17}{4})^{\frac{a}{2}}$, since $\rho \geq 1/2$. Now by using these estimates in (3.3.14) we get,

$$\begin{aligned}
&\int_0^\infty |K(\lambda,\eta)||c(\lambda)|^{-2}d\lambda \\
\leq &Cb^{\frac{s}{a}}\int_{\rho^a}^\infty \left|\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right| \left|\widehat{\psi}(u-b)\right| \left(1+\sqrt{u^{\frac{2}{a}} - \rho^2}\right)^{s+m_1+m_2-1} u^{\frac{2}{a}-1}du \\
= &Cb^{\frac{s}{a}}\int_{\rho^a}^{(3+\rho^2)^{\frac{a}{2}}} \left|\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right| \left|\widehat{\psi}(u-b)\right| \left(1+\sqrt{u^{\frac{2}{a}} - \rho^2}\right)^{s+m_1+m_2-1} u^{\frac{2}{a}-1}du \\
&+ Cb^{\frac{s}{a}}\int_{(3+\rho^2)^{\frac{a}{2}}}^{\frac{3b}{2}} \left|\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right| \left|\widehat{\psi}(u-b)\right| \left(1+\sqrt{u^{\frac{2}{a}} - \rho^2}\right)^{s+m_1+m_2-1} u^{\frac{2}{a}-1}du \\
&+ Cb^{\frac{s}{a}}\int_{\frac{3b}{2}}^{\infty} \left|\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right| \left|\widehat{\psi}(u-b)\right| \left(1+\sqrt{u^{\frac{2}{a}} - \rho^2}\right)^{s+m_1+m_2-1} u^{\frac{2}{a}-1}du \\
= &I_1 + I_2 + I_3.
\end{aligned}$$

Now we will bound the expressions I_1, I_2 and I_3. The boundedness of I_1 and I_3 is easy. Using again the decay property of $\widehat{\psi}$ and the boundedness of ϕ_λ, like in the previous case for large positive integer N, we have

$$I_1 \leq Cb^{\frac{s}{a}}\int_{\rho^a}^{(3+\rho^2)^{\frac{a}{2}}} \frac{du}{[b-(3+\rho^2)^{\frac{a}{2}}]^N} \leq C\frac{b^{\frac{s}{a}}}{b^N} \leq C, \tag{3.3.15}$$

and for $\theta = \frac{s+m_1+m_2+1}{a}$,

$$I_3 \leq C b^{\frac{s}{a}} \int_{\frac{3b}{2}}^{\infty} \frac{u^{\theta}}{|u-b|^{N+1}} du \leq C b^{\frac{s}{a}} \int_{\frac{3b}{2}}^{\infty} \frac{u^{\theta}}{u^{N}} du \leq C. \tag{3.3.16}$$

As $b > 1$, the constant C is independent of λ and η. The main and crucial step is to bound the expression I_2. To bound I_1 and I_3 we have used that the integral $\left(\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right)$ is bounded as $\alpha \in C_c^\infty(\mathbb{X})$. For I_2 we will use the bounds of $\phi_\lambda(t)$ in both variable t and λ. In this case we estimate the integral $\left(\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right)$ into two parts.

Since α, ϕ_λ and ϕ_η are radial functions, we have,

$$\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx = \int_0^\infty \alpha(s)\phi_\lambda(s)\phi_\eta(s)D(s)ds.$$

Let $d = m_1 + m_2 - 1$. By putting $\lambda = (u^{\frac{2}{a}} - \rho^2)^{\frac{1}{2}}$ and $\eta = (b^{\frac{2}{a}} - \rho^2)^{\frac{1}{2}}$ in (2.4.17) we obtain,

$$|\phi_\lambda(t)| \leq \frac{C}{u^{\frac{d+1}{2a}}} \; ; \; |\phi_\eta(t)| \leq \frac{C}{b^{\frac{d+1}{2a}}}. \tag{3.3.17}$$

Now by using estimates (3.3.17) and (2.4.19), we obtain,

$$\begin{aligned}
\left|\int_{\mathbb{X}} \alpha(x)\phi_\lambda(x)\phi_\eta(x)dx\right| &\leq \left|\int_{s>1} \alpha(s)\phi_\lambda(s)\phi_\eta(s)D(s)ds\right| + \left|\int_{s\leq 1} \alpha(s)\phi_\lambda(s)\phi_\eta(s)D(s)ds\right| \\
&\leq \frac{C}{u^{\frac{d+1}{2a}} b^{\frac{d+1}{2a}}} \left(\int_{s>1} |\alpha(s)||D(s)|ds + \int_{s\leq 1} |\alpha(s)| \frac{D(s)}{s^{n-1}} ds\right) \\
&\leq \frac{C}{u^{\frac{d+1}{2a}} b^{\frac{d+1}{2a}}}.
\end{aligned}$$

Finally, using the above bound, we get,

$$\begin{aligned}
I_2 &\leq b^{\frac{s}{a}} \int_{(3+\rho^2)^{\frac{a}{2}}}^{\frac{3b}{2}} |\widehat{\psi}(u-b)| \frac{u^{\frac{s}{a}+\frac{\alpha}{a}+\frac{2}{a}-1}}{u^{\frac{d+1}{2a}} b^{\frac{d+1}{2a}}} du, \quad \text{where } \alpha = m_1 + m_2 - 1. \\
&\leq C b^{\frac{s}{a}} \int_{(3+\rho^2)^{\frac{a}{2}}}^{\frac{3b}{2}} |\widehat{\psi}(u-b)| \frac{u^{\frac{s}{a}+\frac{1}{a}-1}}{u^{\frac{d+1}{2a}} b^{\frac{d+1}{2a}}} du \\
&\leq C b^{\frac{s}{a}} \int_{(3+\rho^2)^{\frac{a}{2}}}^{\frac{3b}{2}} |\widehat{\psi}(u-b)| b^{\frac{s}{a}+\frac{1}{a}-1} du \\
&\leq C b^{\frac{2s}{a}+\frac{1}{a}-1} \int_{(3+\rho^2)^{\frac{a}{2}}}^{\frac{3b}{2}} |\widehat{\psi}(u-b)| du \leq C.
\end{aligned}$$

This proves (3.3.12), and completes the proof of the Proposition.

$\square$

Given any suitable function g on $\mathbb{R}$, we define

$$\|g\|_{H^s} = \left(\int_{\mathbb{R}} |\widehat{g}(t)|^2 (1+t^2)^s dt \right)^{1/2}.$$

Now, we define

$$\|Sf\|_{L^2(H^s)} = \left(\int_{\mathbb{X}} \|Sf(x,\cdot)\|^2_{H^s} \, dx \right)^{1/2}. \tag{3.3.18}$$

We need the norm estimate given in the following lemma:

Lemma 3.3.2. *Let* $a > 1$ *and* $s = \frac{a-1}{2}$*. Then for* $f \in C_c^\infty(\mathbb{X})$*, we have*

$$\|Sf\|_{L^2(H^1)} \leq C \|f\|_{H^{-s+a}(\mathbb{X})}. \tag{3.3.19}$$

Proof. Differentiating $Sf(x,t)$ with respect to t-variable, we have

$$\begin{aligned} \frac{\partial}{\partial t} Sf(x,t) &= \alpha_0(x)\psi_0(t) \int_{\mathbb{X}} i(\lambda^2+\rho^2)^{\frac{a}{2}} e^{it(\lambda^2+\rho^2)^{\frac{a}{2}}} \widehat{f}(\lambda)\phi_\lambda(x)|c(\lambda)|^{-2} d\lambda \\ &\quad + \alpha_0(x)\psi_0{}'(t) \int_{\mathbb{X}} e^{it(\lambda^2+\rho^2)^{\frac{a}{2}}} \widehat{f}(\lambda)\phi_\lambda(x)|c(\lambda)|^{-2} d\lambda \\ &= S_1 f(x,t) + S_2 f(x,t). \end{aligned} \tag{3.3.20}$$

Now by equivalence of norms in Sobolev spaces [30, Lemma 3, p. 136], we have

$$\begin{aligned} \|Sf(x,\cdot)\|_{H^1(\mathbb{R})} &\leq C \left(\|Sf(x,\cdot)\|_{H^0(\mathbb{R})} + \|\frac{\partial}{\partial t} Sf(x,\cdot)\|_{H^0(\mathbb{R})} \right) \\ &\leq C \left(\|Sf(x,\cdot)\|_{H^0(\mathbb{R})} + \|S_1 f(x,\cdot)\|_{H^0(\mathbb{R})} + \|S_2 f(x,\cdot)\|_{H^0(\mathbb{R})} \right). \end{aligned}$$

Again, by triangle inequality we have,

$$\|Sf\|_{L^2(H^1)} \leq C \left(\|Sf\|_{L^2(H^0)} + \|S_1 f\|_{L^2(H^0)} + \|S_2 f\|_{L^2(H^0)} \right). \tag{3.3.21}$$

If we repeat the argument given in (3.3.8) and (3.3.10) in proof of the above Proposition, and substitute $h(\eta) = \widehat{f}(\eta)(\rho^2+\eta)^{-s}(\eta^2+\rho^2)^{\frac{a}{2}}$, then by (3.3.11) we have

$$\|S_1 f\|^2_{L^2(H^0)} \leq C\|h\|_2^2 \leq C \int_{\mathbb{R}} |\widehat{f}(\eta)|^2 (\eta^2+\rho^2)^{-s+a} |c(\eta)|^{-2} d\eta = C\|f\|^2_{H^{-s+a}(\mathbb{X})}. \tag{3.3.22}$$

Since $S_2 f$ is defined as Sf, by (3.3.7)

$$\|S_2 f\|_{L^2(H^0)} \le C\|f\|_{H^{-s}(\mathbb{X})}. \tag{3.3.23}$$

Since $H^{-s+a} \subset H^{-s}$, thus from (3.3.21), (3.3.22) and (3.3.23), we obtain (3.3.19). This completes the proof. $\square$

From the above Proposition and Lemma, we have

$$\|Sf\|_{L^2(H^0)} \le C\|f\|_{H^{-s}(\mathbb{X})}, \quad \text{and} \quad \|Sf\|_{L^2(H^1)} \le C\|f\|_{H^{-s+a}(\mathbb{X})}.$$

Interpolating between these two yields

$$\|Sf\|_{L^2(H^r)} \le C\|f\|_{H^{-s+ra}(\mathbb{X})}, \quad 0 \le r \le 1.$$

We refer to [3, Theorem 5.1.2 and page 153, (7)] for details about interpolation results.

By taking $r = \frac{1}{2} + \epsilon$, we have

$$\|Sf\|_{L^2\left(H^{\frac{1}{2}+\epsilon}\right)} \le C\|f\|_{H^{\frac{1}{2}+a\epsilon}(\mathbb{X})}, \quad 0 \le \epsilon \le \frac{1}{2}.$$

As, $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, for $s > \frac{1}{2}$, it follows that

$$\|Sf\|_{L^2(L^\infty)} \le C\|f\|_{H^s} \quad \text{for } s > \frac{1}{2}.$$

Now, $\|Sf\|_{L^2(L^\infty)} = \|\alpha_0(x)\psi_0(t)S_t f(x)\|_{L^2(L^\infty)} = \left(\int_{\mathbb{X}} \|\alpha_0(x)\psi_0(\cdot)S_{(\cdot)}f(x)\|^2_{L^\infty_t} dx\right)^{1/2}$ and

$$\int_B \|\alpha_0(x)\psi_0(\cdot)S_{(\cdot)}f(x)\|^2_{L^\infty_t} dx \le \int_{\mathbb{X}} \|\alpha_0(x)\psi_0(\cdot)S_{(\cdot)}f(x)\|^2_{L^\infty_t} dx.$$

Using the above inequality and the fact that $\alpha_0 \equiv 1$ on B and $\psi_0 \equiv 1$ we get,

$$\int_B \|S_{(\cdot)}f(x)\|^2_{L^\infty_t} dx \le \|Sf\|_{L^2(L^\infty)}, \quad \text{when } 0 \le t \le 1.$$

Since $\sup\limits_{0<t<1} |S_t f(x)| = \|S_{(\cdot)}f(x)\|_{L^\infty_t}$, finally we obtain

$$\left\|\sup_{0<t<1} |S_t f(x)|\right\|_{L^2(B)} \le C\|f\|_{H^s(\mathbb{X})} \quad \text{for } s > \frac{1}{2}.$$

This proves Theorem B.

Now we will prove Theorem A.

If $f \in C_c^\infty(\mathbb{X})$ and radial then it can be easily shown that $\lim\limits_{t\to 0} S_t f(x) = f(x)$. Now suppose $f \in H^s(\mathbb{X})$, radial and has compact support, then there exist a sequence $\{f_n\}$ in $C_c^\infty(\mathbb{X})$ of radial functions such that $f_n \to f$ in $H^s(\mathbb{X})$ and support of all f_n contained in the support of f. Now, for a particular subsequence $\{f_{n_k}\}$ of $\{f_n\}$,

$$S_t f(x) = S_t\left(\lim_{k\to\infty} f_{n_k}(x)\right) = \lim_{k\to\infty} S_t f_{n_k}(x) \leq \lim_{k\to\infty} |S^* f_{n_k}(x)|.$$

Taking supremum in t, we have

$$S^* f(x) \leq \lim_{k\to\infty} |S^* f_{n_k}(x)| = \lim_{k\to\infty} S^* f_{n_k}(x).$$

After squaring and integrating over B, we get

$$\int_B |S^* f(x)|^2 dx \leq \int_B \lim_{k\to\infty} (S^* f_{n_k}(x))^2$$

Applying Fatou's lemma and the estimate (3.2.5), we get

$$\int_B (S^* f(x))^2 dx \leq \lim_{n\to\infty} \int_B (S^* f_n(x))^2 dx \leq \lim_{n\to\infty} C\|f_n\|^2_{H^s} = C\|f\|^2_{H^s}, \tag{3.3.24}$$

for every ball B in $\mathbb{X}$. Now, if $f \in H^s(\mathbb{X})$ has compact support and $g \in C_c^\infty(\mathbb{X})$, then we have

$$\begin{aligned}\lim_{t\to 0} |S_t f(x) - f(x)| &= \lim_{t\to 0} |S_t(f-g)(x) - (S_t g(x) - g(x)) - (f(x) - g(x))| \\ &\leq S^*(f-g)(x) + |f(x) - g(x)|.\end{aligned}$$

Taking L^2-norm on both sides and using (3.3.24), one obtain

$$\int_B \left(\lim_{t\to 0} |S_t f(x) - f(x)|\right)^2 dx \leq C\|f-g\|^2_{H^s}.$$

By density, we can choose g, such that $\|f-g\|_{H^s}$ can be made arbitrarily small, consequently

$$\lim_{t\to 0} |S_t f(x) - f(x)| = 0.$$

This complete proof of the Theorem A.

CHAPTER 4

Sharp Regularity Results for the Schrödinger Equation on Rank-One Symmetric Spaces of Non-Compact Type

4.1 Introduction

In this chapter, we aim to improve the result obtained in the previous chapter and to establish the sharpness of the result. Our result is an extension of Sjölin's result on $\mathbb{R}^n$. In [27], Sjölin proved that for $n > 2$, if $f \in H^s(\mathbb{R}^n)$ with $s > \frac{1}{2}$ then $S_t f(x) \to f$ (as t goes to 0) a.e. $x \in \mathbb{R}^n$. For the one-dimensional case $n = 1$, he improved this result to the condition $s \geq \frac{1}{4}$. Moreover, the range of s obtained here is sharp, where $u(x,t) = S_t f(x)$ is the solution of the fractional Schrödinger equation (1.0.5).

Let $S^* f$ be defined as earlier in (3.2.3), in [27] Sjölin proved the following result:

Theorem 4.1.1. *If $n = 1$ and $a > 1$, then $\|S^* f\|_{L^2(B)} \leq C_B \|f\|_{H^s(\mathbb{R}^n)}$ holds iff $s \geq 1/4$; where B is a ball of any arbitrary radius in $\mathbb{R}$.*

Also, Sjölin addressed the same problem for radial initial data in $\mathbb{R}^n$ in [28]. He proved that, if $n \geq 2$ and f is radial, then $\|S^* f\|_{L^2(B)} \leq C_B \|f\|_{H^s(\mathbb{R}^n)}$ holds if $s \geq 1/4$.

In 2005, Cho, Lee and Shim considered this problem in a more generalized way [8]. They have established the following result:

Theorem 4.1.2. *For any* $\epsilon > 0$ *and* $b > 1$, *if* $f \in H_r^{1/4} H_\omega^{(n-1)/2-1/4+\epsilon}$, *then there exist a constant* C *such that,*

$$\|u^*\|_{L^2((1+|x|)^{-b}dx)} \leq C\|f\|_{H_r^{1/4} H_\omega^{(n-1)/2-1/4+\epsilon}} \tag{4.1.1}$$

where, $u(x,t)$ is the solution of the free Schrödinger-type equation given below:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = i\Omega(D)u(x,t), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, n \geq 2, \\ u(x,0) = f(x), & \end{cases} \tag{4.1.2}$$

and, $u^*(x) = \sup_{t>0} |u(x,t)|$ is the maximal function corresponding to $u(x,t)$.
In the above equation, $\Omega(D)$ is a generalized differential operator defined by a C^2-function Ω and $D = (-\Delta)^{1/2}$ and f has H^s regularity for $s > 0$ as well as some regularity in angular direction. For $\alpha, \beta \geq 0$ the initial data space $H_r^\alpha H_\omega^\beta$ has been defined by

$$H_r^\alpha H_\omega^\beta = \{f : \|f\|_{H_r^\alpha H_\omega^\beta} = \|(1-\Delta)^{\alpha/2} f\|_{L_r^2 H_\omega^\beta} < \infty\},$$

where $\|g\|^2_{L_r^2} = \int_0^\infty |g(r)|^2 r^{n-1} dr$, $\|g\|_{L_r^2 H_\omega^\beta} = \big\| \|(1-\Delta_\omega)^{\beta/2} g(r\omega)\|_{L_\omega^2} \big\|_{L_r^2}$.

In fact, it can be shown that if the initial data f is radial, then the norm estimate (4.1.1) reduces to

$$\|u^*\|_{L^2((1+|x|)^{-b}dx)} \leq C\|f\|_{H_r^{1/4}}. \tag{4.1.3}$$

In the previous chapter, we proved an analogous result of Sjölin (Theorem 3.2.2) in a rank-one symmetric space of non-compact type. There we showed that, if a radial function $f \in H^s(\mathbb{X})$, with $s > \frac{1}{2}$, then $S_t f(x) \to f$ *a.e.* $x \in \mathbb{X}$ (as t goes to 0). In this chapter, we improve the range of s up to $\frac{1}{4}$. We came to know that a similar result with respect to regularity in s was obtained independently by Ustav Dewan in [13]; he also established the sharpness of this result. However, our proof is quite different and shorter. We will follow the proof method used in the paper [8].

4.2 Formulation and Statement of Main Results

Let $\mathbb{X}$ be a rank-one symmetric space of non-compact type and let $\mathcal{L}$ be the Laplace-Beltrami operator in $\mathbb{X}$. The fractional Schrödinger equation in $\mathbb{X}$ is given by

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} = (-\mathcal{L})^{\frac{a}{2}}u(x,t), \qquad (t,x) \in \mathbb{R} \times \mathbb{X},\ a > 1, \\ \\ u(x,0) = f(x), \qquad f \text{ radial.} \end{cases} \tag{4.2.4}$$

We now state the main result of this paper:

Theorem C. *Let $\mathbb{X}$ be a rank-one symmetric space of non-compact type. Suppose $a > 1$ and let $f \in H^s(\mathbb{X})$ be radial with compact support. Then the solution $u(x,t)$ of the fractional Schrödinger equation (4.2.4) stasfies*

$$\lim_{t\to 0} u(x,t) \to f(x), \quad a.e.\ x \in \mathbb{X},$$

provided $s \geq \frac{1}{4}$.

As mentioned in Chapter 3, the standard method for proving pointwise convergence involves obtaining an estimate for the corresponding maximal operator. We will follow this approach as well. Our proof will align with the idea of the paper [8]; to establish some mixed norm estimates of the dual operator, derive the required estimate for the corresponding maximal operator. Proof of Theorem C is mainly based on the following maximal estimate:

Theorem D. *If $a > 1$, and B is ball of any arbitrary radius in $\mathbb{X}$, then*

$$\|S^*f\|_{L^2(B)} \leq C_B \|f\|_{H^s(\mathbb{X})}, \tag{4.2.5}$$

holds for $s \geq 1/4$, where S^*f is same as previously defined in (3.2.3) and $f \in C_c^\infty(\mathbb{X})$, radial.

4.3 Proof of Theorem C and Theorem D

In this section, we first prove the Theorem D and then Theorem C. Let B be a fixed radius ball in X and choose $\alpha_0 \in C_c^\infty(\mathbb{X})$, a radial, real cut-off function, and $\alpha_0 \equiv 1$ on the ball B contained in the support of α_0, . Now fix $s = \frac{1}{4}$ and set

$$Tf(x,t) = \alpha_0(x) \int_0^\infty \widehat{f}(\lambda) e^{it(\lambda^2+\rho^2)^{\frac{a}{2}}} \phi_\lambda(x) \frac{|c(\lambda)|^{-2}}{(\lambda^2+\rho^2)^{\frac{s}{2}}} d\lambda, \tag{4.3.6}$$

and the corresponding maximal operator is given by:

$$T^* f(x) = \sup_{t\in\mathbb{R}} |Tf(x,t)|. \tag{4.3.7}$$

To prove (4.2.5) it is enough to show that,

$$\|T^* f\|_{L^2(\mathbb{X})} \leq C \, \|f\|_{L^2(\mathbb{X})}. \tag{4.3.8}$$

In fact,

$$\begin{aligned} \|S^* f\|_{L^2(\mathbb{X})} &\leq \|T^* g\|_{L^2(\mathbb{X})}, \quad \text{where } \widehat{g}(\lambda) = \widehat{f}(\lambda)(\lambda^2+\rho^2)^{\frac{s}{2}} \\ &\leq \|g\|_{L^2(\mathbb{X})} \\ &= \left(\int_0^\infty |\widehat{f}(\lambda)|^2 (\lambda^2+\rho^2)^s \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda \right)^{1/2}, \quad \text{by Plancherel formula (2.3.6)} \\ &= \|f\|_{H^s(\mathbb{X})} \end{aligned}$$

Let $\Omega(\lambda)$ be a C^2-function satisfying the relation:

$$c_1|\lambda|^{a-k} \leq |\Omega^{(k)}(\lambda)| \leq c_2|\lambda|^{a-k}, \ \ (k=0,1,2) \ \text{ if } \lambda \geq N \tag{4.3.9}$$

for some $c_1, c_2, a > 0$ and for a large $N > 0$. Clearly $\Omega(\lambda) := (\lambda^2+\rho^2)^{\frac{a}{2}}$ satisfies the above condition.

For any operator $T : X \to Y$, the dual operator $T^d : Y^* \to X^*$ is defined by

$$< Tf, g > = < f, T^d g >$$

Now the dual operator T^d of T is given by:

$$T^d g(\lambda) = \frac{1}{(\lambda^2+\rho^2)^{\frac{s}{2}}} \int_{\mathbb{X}} \int_0^\infty e^{-it\Omega(\lambda)} \overline{\phi_\lambda(x)} \, \overline{\alpha_o(x)} g(x,t) dx dt$$

for $g \in C_c^\infty(\mathbb{X} \times (0,\infty))$ and radial. To prove (4.2.5), by duality it is enough to show that:

$$\|T^d g\|_{L^2} \leq C \|g\|_{L^2 L^1}, \ \text{ for } g \in C_c^\infty(\mathbb{X} \times (0,\infty)) \tag{4.3.10}$$

Using the boundedness of ϕ_λ (2.4.3) and Holder's inequality, we have

$$|T^d g(\lambda)| \leq \frac{1}{(\lambda^2+\rho^2)^{\frac{s}{2}}} \int_{\mathbb{X}} \|g(x,\cdot)\|_{L^1} |\overline{\alpha_0(x)}|\, dx \leq \frac{C}{(\lambda^2+\rho^2)^{\frac{s}{2}}} \|g\|_{L^2L^1}.$$

Now, squaring and integrating over $\{\lambda : 0<\lambda\leq N\}$, we get

$$\int_{\lambda\leq N} |T^d g(\lambda)|^2 |c(\lambda)|^{-2} d\lambda \leq C\left(\int_{\lambda\leq N} \frac{|c(\lambda)|^{-2}}{(\lambda^2+\rho^2)^s} d\lambda\right) \|g\|^2_{L^2L^1}$$

where, N is the integer chosen in (4.3.9). By using the estimate of $c(\lambda)$ given in (2.4.16),

$$\int_{\lambda\leq N} |T^d g(\lambda)|^2 |c(\lambda)|^{-2} d\lambda \leq C \|g\|^2_{L^2L^1}. \tag{4.3.11}$$

Now, we will consider the region $\{\lambda : \lambda > N\}$.

$$\begin{aligned}
&\int_{\lambda>N} |T^d g(\lambda)|^2 |c(\lambda)|^{-2} d\lambda \qquad (4.3.12)\\
&= \int_{\lambda>N} \frac{1}{(\lambda^2+\rho^2)^s} \left(\int_{\mathbb{X}}\int_0^\infty e^{-it\Omega(\lambda)}\, \overline{\phi_\lambda(x)}\, \overline{\alpha_0(x)}\, g(x,t)\, dx\, dt\right)\\
&\qquad \left(\int_{\mathbb{X}}\int_0^\infty e^{it'\Omega(\lambda)}\, \phi_\lambda(y)\, \alpha_0(y)\, \overline{g(y,t')}\, dy\, dt'\right) |c(\lambda)|^{-2}\, d\lambda\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \left(\int_{\lambda>N} \frac{1}{(\lambda^2+\rho^2)^s} e^{i\Omega(\lambda)(t'-t)} \overline{\phi_\lambda(r)}\phi_\lambda(r')|c(\lambda)|^{-2} d\lambda\right) \overline{\alpha_0(r)}\alpha_0(r') g(r,t)\overline{g(r',t')}\\
&\qquad D(r)D(r')\, dr\, dr'\, dt\, dt'.
\end{aligned}$$

Using the Harish-Chandra series for spherical functions ϕ_λ given in (2.4.14) we have

$$\begin{aligned}
\overline{\phi_\lambda(r)}\phi_\lambda(r') &= \overline{c(\lambda)}c(\lambda)e^{i\lambda(r'-r)}e^{-\rho(r'+r)}\overline{\varphi_\lambda(r)}\varphi_\lambda(r') + \overline{c(\lambda)}c(-\lambda)e^{-i\lambda(r'+r)}e^{-\rho(r'+r)}\overline{\varphi_\lambda(r)}\varphi_{-\lambda}(r') +\\
&\quad \overline{c(-\lambda)}c(\lambda)e^{i\lambda(r'+r)}e^{-\rho(r'+r)}\overline{\varphi_{-\lambda}(r)}\varphi_\lambda(r') + \overline{c(-\lambda)}c(-\lambda)e^{i\lambda(r-r')}e^{-\rho(r'+r)}\overline{\varphi_{-\lambda}(r)}\varphi_{-\lambda}(r')\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We will estimate the integral (4.3.12) for I_1 and I_2; the estimates for the other two integrals will follow similarly. Now, using the expression for $\varphi_\lambda(r)$ given in Harish- Chandra series expansion (2.4.14) of ϕ_λ we get,

$$\begin{aligned}
I_1 &= \overline{c(\lambda)}\, c(\lambda)\, e^{i\lambda(r'-r)}\, e^{-\rho(r'+r)} \Bigg[\Gamma_0(\lambda)\,\overline{\Gamma_0(\lambda)} + \overline{\Gamma_0(\lambda)}\sum_{n\geq 1}\Gamma_n(\lambda)\, e^{-2nr'}\\
&\qquad + \Gamma_0(\lambda)\sum_{n\geq 1}\overline{\Gamma_n(\lambda)}\, e^{-2nr} + \sum_{n\geq 1}\overline{\Gamma_n(\lambda)}\, e^{-2nr}\sum_{m\geq 1}\Gamma_m(\lambda)\, e^{-2mr'}\Bigg]\\
&= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}.
\end{aligned}$$

By putting the value of I_1 in (4.3.12) we obtain,

$$\begin{aligned}
&\int_{\lambda>N} |T^d g(\lambda)|^2 |c(\lambda)|^{-2} d\lambda \qquad (4.3.13)\\
&\leq \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left(\int_{\lambda>N} \frac{1}{(\lambda^2+\rho^2)^s} e^{i\Omega(\lambda)(t'-t)} \left(I_{1,1} + \sum_{i\geq 2} I_{1,i} \right) \; |c(\lambda)|^{-2} d\lambda \right)\\
&\qquad \overline{\alpha_0(r)} \alpha_0(r') g(r,t) \overline{g(r',t')} D(r) D(r')\, dr\, dr'\, dt\, dt'.
\end{aligned}$$

For $i = 1, 2, 3$ and 4, the expressions for the kernels K_i's are as follow:

$$K_i(r, r', t, t') = \int_{\lambda>N} \frac{1}{(\lambda^2+\rho^2)^s} e^{i\Omega(\lambda)(t'-t)} \; I_{1,i} \; |c(\lambda)|^{-2} d\lambda.$$

In particular,

$$K_1(r, r', t, t') = \int_{\lambda>N} \frac{1}{(\lambda^2+\rho^2)^s} e^{i\Omega(\lambda)(t'-t)} e^{i\lambda(r'-r)} d\lambda, \text{ since } \Gamma_0(\lambda) = 1 \text{ by Lemma 2.4.4.}$$

To get the estimate for K_1, we use the following lemma [8, Lemma 2.3] which gives a uniform bound for kernel K_1 in r, r', t, t'.

Lemma 4.3.1. *[8, Lemma 2.3] For any real number A, B there exists a constant C, independent of A and B such that*

$$\left| \int_{\lambda>N} e^{i(A\Omega(\lambda)+B\lambda)} \frac{d\lambda}{(\lambda^2+\rho^2)^s} \right| \leq C|B|^{-(1-2s)}.$$

where $\Omega(\lambda)$ is a function satisfying (4.3.9).

By the above lemma, we have the estimate:

$$|K_1(r, r', t, t')| \leq \frac{e^{-\rho(r'+r)}}{|r'-r|^{1-2s}},$$

and also by using the estimate for $\Gamma_n(\lambda)$ given in (2.4.15), we deduce the following bounds for the kernels $K_i's$:

For $i = 2$,

$$K_2(r, r', t, t') = \int_{\lambda>N} \frac{1}{(\lambda^2+\rho^2)^s} e^{i\Omega(\lambda)(t'-t)} \; I_{1,2} \; |c(\lambda)|^{-2} d\lambda$$

and

$$I_{1,2} = c(\lambda)\overline{c(\lambda)} e^{i\lambda(r'-r)} e^{-\rho(r+r')} \overline{\Gamma_0(\lambda)} \sum_{n=1}^{\infty} \Gamma_n(\lambda) e^{-2nr'}$$

$$\begin{aligned}
|K_2(r,r',t,t')| &\le e^{-\rho(r+r')}\sum_{n=1}^{\infty}\Gamma_n(\lambda)e^{-2nr'}\int_{\lambda>N}\frac{1}{(\lambda^2+\rho^2)^s}e^{i\Omega(\lambda)(t'-t)}e^{i\lambda(r'-r)}d\lambda\\
&\le e^{-\rho(r+r')}\sum_{n=1}^{\infty}n^v e^{-2nr'}\int_{\lambda>N}\frac{1}{(\lambda^2+\rho^2)^s}\frac{1}{(1+|\lambda|)}d\lambda\\
&\le e^{-\rho(r+r')}\int_{\lambda>N}\frac{d\lambda}{(\lambda^2+\rho^2)^s(1+|\lambda|)}\le Ce^{-\rho(r'+r)}.
\end{aligned}$$

By the similar computations as above, we can conclude

$$|K_i(r,r',t,t')|\le e^{-\rho(r+r')}\int_{\lambda>N}\frac{d\lambda}{(\lambda^2+\rho^2)^s(1+|\lambda|)}\le Ce^{-\rho(r'+r)};\quad \text{for}\ \ i=3,4.$$

Using these estimates in (4.3.13), we have

$$\begin{aligned}
&\int_{\lambda>N}|T^d g(\lambda)|^2|c(\lambda)|^{-2}d\lambda\\
&\le \int_0^\infty\int_0^\infty \frac{e^{-\rho(r'+r)}}{|r'-r|^{1-2s}}|\alpha_0(r)||\alpha_0(s)|\|g(r,\cdot)\|_{L^1(dt)}\|g(r',\cdot)\|_{L^1(dt')}D(r)D(r')\,dr\,dr'\\
&+\int_0^\infty\int_0^\infty e^{-\rho(r'+r)}|\alpha_0(r)||\alpha_0(r')|\|g(r,\cdot)\|_{L^1(dt)}\|g(r',\cdot)\|_{L^1(dt')}D(r)D(r')\,dr\,dr'\\
&\le \int_0^\infty I_{2s}\left(e^{-\rho r}|\alpha_0(r)|\|g(r,\cdot)\|_{L^1(dt)}D(r)\right)(r')\;e^{-\rho r'}|\alpha_0(r')|\|g(r',\cdot)\|_{L^1(dt)}D(r')\,dr'\\
&+\int_0^\infty e^{-\rho r}|\alpha_0(r)|\|g(r,\cdot)\|_{L^1(dt)}D(r)\,dr\int_0^\infty e^{-\rho r'}|\alpha_0(r')|\|g(r',\cdot)\|_{L^1(dt)}D(r')\,dr'\\
&=A+B,\quad \text{where } I_{2s} \text{ is the Riesz potential of order } 2s;\, I_{2s}f(x)=\int_{\mathbb{R}}\frac{f(y)}{|x-y|^{1-2s}}dy.
\end{aligned}$$

Now, using the boundedness of the Riesz potential and applying Hölder's inequality repeatedly, we obtain the following estimates for the integral expressions:

$$\begin{aligned}
A&\le\left\|I_{2s}\left(e^{-\rho r}|\alpha_0(r)|\|g(r,\cdot)\|_{L^1(dt)}D(r)\right)\right\|_{L^6}\left\|e^{-\rho r'}|\alpha_0(r')|\|g(r',\cdot)\|_{L^1(dt)}D(r')\right\|_{L^{\frac{6}{5}}}\\
&\le C\|e^{-\rho r}|\alpha_0(r)|\|g(r,\cdot)\|_{L^1(dt)}D(r)\|_{L^{3/2}}\|e^{-\rho r}|\alpha_0(r)|\|g(r,\cdot)\|_{L^1(dt)}D(r)\|_{L^{6/5}}\\
&=C\left(\int_0^\infty e^{-\rho r3/2}D(r)^{3/4}|\alpha_0(r)|^{3/2}\|g(r,\cdot)\|^{3/2}_{L^1(dt)}D(r)^{3/4}\,dr\right)^{2/3}\\
&\quad\left(\int_0^\infty e^{-\rho r6/5}D(r)^{6/10}|\alpha_0(r)|^{6/5}\|g(r,\cdot)\|^{6/5}_{L^1(dt)}D(r)^{6/10}\,dr\right)^{5/6}
\end{aligned}$$

$$\leq C \left(\int_0^\infty e^{-6\rho r} D(r)^3 |\alpha_0(r)|^6 \, dr\right)^{1/6} \left(\int_0^\infty \|g(r,\cdot)\|^2_{L^1(dt)} D(r)\, dr\right)^{1/2}$$

$$\left(\int_0^\infty e^{-3\rho r} D(r)^{3/2} |\alpha_0(r)|^3 \, dr\right)^{1/3} \left(\int_0^\infty \|g(r,\cdot)\|^2_{L^1(dt)} D(r)\, dr\right)^{1/2}$$

$$\leq C \left(\int_0^\infty \|g(r,\cdot)\|^2_{L^1(dt)} D(r)\, dr\right) \left(\int_0^\infty e^{-6\rho r} D(r)^3 |\alpha_0(r)|^6 \, dr\right)^{1/6}$$

$$\left(\int_0^\infty e^{-3\rho r} D(r)^{3/2} |\alpha_0(r)|^3 \, dr\right)^{1/3}$$

$$= C \, \|g\|^2_{L^2L^1}.$$

And

$$B \leq \left(\int_0^\infty \|g(r,\cdot)\|^2_{L^1(dt)} D(r)\, dr\right) \left(\int_0^\infty e^{-2\rho r} D(r) |\alpha_0(r)|^2 \, dr\right)^{1/2} \left(\int_0^\infty e^{-2\rho r'} D(r') |\alpha_0(r')|^2 \, dr'\right)^{1/3}$$

$$= C \, \|g\|^2_{L^2L^1}.$$

Finally, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left(\int_{\lambda > N} \frac{1}{(\lambda^2+\rho^2)^s} e^{i\Omega(\lambda)(t'-t)} I_1 c(\lambda)|^{-2} d\lambda\right) \overline{\alpha_0(r)} \alpha_0(r') \overline{g(r,t)} g(r',t')$$

$$D(r) D(r') dr \, dr' \, dt \, dt'$$

$$\leq C \, \|g\|^2_{L^2L^1}.$$

Again, we have the expression of I_2 as

$$I_2 = \overline{c(\lambda)}\, c(-\lambda)\, e^{-i\lambda(r'+r)}\, e^{-\rho(r'+r)} \Bigg[\Gamma_0(\lambda)\, \overline{\Gamma_0(-\lambda)} + \Gamma_0(\lambda) \sum_{n\geq 1} \overline{\Gamma_n(-\lambda)}\, e^{-2nr'}$$

$$+ \overline{\Gamma}_0(-\lambda) \sum_{n\geq 1} \overline{\Gamma_n(\lambda)} e^{-2nr} + \sum_{n\geq 1} \overline{\Gamma_n(\lambda)}\, e^{-2nr} \sum_{m\geq 1} \Gamma_m(-\lambda)\, e^{-2mr'}\Bigg]$$

$$= I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4}.$$

Since, $\dfrac{c(\lambda)}{c(-\lambda)}$ is a continuous function on $\mathbb{R}$ and $\left|\dfrac{c(\lambda)}{c(-\lambda)}\right| = 1$, by applying Lemma 4.3.1

and by the similar computations as above, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left(\int_{\lambda > N} \frac{1}{(\lambda^2+\rho^2)^s} e^{i\Omega(\lambda)(t'-t)} I_i \, |c(\lambda)|^{-2} d\lambda \right) \overline{\alpha_0(r)} \alpha_0(r') \overline{g(r,t)} g(r',t') \\ D(r) D(r') \; dr \; dr' \; dt \; dt'$$

$$\leq C \; \|g\|^2_{L^2 L^1}$$

Now, the estimates for I_3 and I_4 follow in a similar way and we obtain,

$$\int_{\lambda > N} |T^d g(\lambda)|^2 |c(\lambda)|^{-2} d\lambda \leq C \; \|g\|^2_{L^2 L^1}. \tag{4.3.14}$$

Combining (4.3.11) and (4.3.14) we conclude

$$\|T^d g\|_{L^2} \leq C \|g\|_{L^2 L^1}.$$

By using the duality argument, we get the estimate (4.3.8) and this proves Theorem D. The proof of Theorem C will follow exactly in the same way as Theorem A in the previous chapter.

CHAPTER 5

On Local Regularity and Well-posedness of Schrödinger Equations on Rank-One Symmetric Spaces of noncompact type

5.1 Introduction

Let $u(x,t)$ be the solution of the initial value problem:

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + P(D)u(x,t) = F(x,t), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(x,0) = u_0(x), \end{cases} \tag{5.1.1}$$

where $P(D)u$ is defined via the symbol $P(\xi)$ such that $P(\xi)$ behaves like $|\xi|^m$ as $|\xi| \to \infty$, $m > 1$ and $D = \frac{1}{i}\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$. In 1988, Constantin and Saut [10] studied a large number of dispersive equations of the form (5.1.1) and showed that solutions to this system enjoy a local smoothing property.

In particular, for $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ which is supported in $[-T,T] \times \mathbb{R}^n$, $u(x,t)$ satisfies the estimate:

$$\left(\int_{\mathbb{R}^{n+1}} \chi^2(x,t)|(I-\Delta)^{(s+\alpha)/2}u(x,t)|^2 dx\, dt\right)^{1/2} \leq C(T)\left(\|u_0\|_{H^s(\mathbb{R}^n)} + \|F\|_{L^1(-T,T;H^{s,q}(\mathbb{R}^n))}\right). \tag{5.1.2}$$

Thus, if u_0 belongs to $H^s(\mathbb{R}^n)$, then for almost every $t \neq 0$, the function $u(\cdot,t)$ belongs to $H^{s+\alpha}(\mathbb{R}^n)$; this kind of gain in smoothness is known as local smoothing. This property was discovered by T. Kato [20] for the Korteweg-de Vries equation: the solution of the initial value problem is locally one derivative smoother than the initial data. Independently, P. Sjölin [27] and Vega [31] studied the pointwise convergence of the solution of the equation (5.1.1) and proved that it has locally $1/2$ derivative more than the initial data for almost every time with $P(D) = -\Delta$ and $F = 0$. Later in [11], Constantin and Saut again studied this type of equation with non-zero potential V on $\mathbb{R}^n$ with certain properties and obtained local smoothing properties for the solution $u(x,t)$ of the equation

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + \Delta u(x,t) + Vu(x,t) = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ \\ u(x,0) = u_0(x). \end{cases} \tag{5.1.3}$$

The smoothing estimate is the following: For any $R, T > 0$,

$$\int_{-T}^{T} \int_{|x| \leq R} |(I - \Delta)^{1/4} u(x,t)|^2 dx\, dt \leq C(R,T) \|u_0\|^2_{L^2(\mathbb{R}^n)} \tag{5.1.4}$$

Further, in the same paper, they addressed the problem for a more generalized operator $P(x,D)$ and potential $V(x,D)$, instead of Δ and V respectively, and obtained the same kind of smoothing estimate (5.1.4).

5.2 Main Result

Now, before discussing the local smoothing property, it is necessary to establish the well-posedness of the corresponding system. In 1993, A. Ruiz and L. Vega [26] proved well-posedness and local smoothing estimate for the solution of Schrödinger equation with non zero potential $V \in L_x^{n/2} L_t^\infty + L_x^\infty L_t^r, r > 1$. One of the main results in [26] is the following:

Theorem 5.2.1. *Let V be a potential in $\mathbb{R}^n$ which can be written as $V(x,t) = V_1(x,t) + V_2(x,t)$ with $V_1 \in L_x^{n/2}(L_t^\infty(\mathbb{R})), V_2 \in L^r([-T,T] : L_x^\infty), r > 1$ and $\|V_1\|_{L_x^{n/2}(L_t^\infty(\mathbb{R}))}$ small enough. Then there exists a unique solution $u(x,t)$ such that*

$$\|u\|_{L_x^{2n/(n-2)}(L_t^2[-T,T])} + \sup_{|t|<T} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq C(T) \|u_0\|_{L^2(\mathbb{R}^n)}. \tag{5.2.5}$$

Moreover,

$$\sup_{x_0,R} \frac{1}{R} \int_{B(x_0,R)} \int_{-T}^{T} |D_x^{1/2} u|^2 dt\, dx \leq C(T) \|u_0\|^2_{L^2(\mathbb{R}^n)}, \tag{5.2.6}$$

where $\widehat{(D_x^s f)}(\xi) = |\xi|^s \widehat{f}(\xi)$.

In this chapter, our goal is to extend this result in symmetric space setting.

Let $\mathbb{X}$ be a symmetric space of noncompact type, with any arbitrary rank. Let $u(x,t)$ be the solution of the following Schrödinger equation with a nontrivial potential V in $\mathbb{X}$:

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) + V(x,t)u(x,t) = 0, & (t,x) \in \mathbb{R}\times\mathbb{X}, \\ u(x,0) = u_0(x). \end{cases} \tag{5.2.7}$$

We establish the well-posedness of the solution of equation (5.2.7).

Although there is no precise definition of well-posedness, the following definition is given in Kato's [20] paper: Consider an abstract Cauchy problem

$$du/dt = f(u), t > 0, \;\; u(0) = \phi. \tag{5.2.8}$$

Suppose there are two Banach spaces $Y \subset X$, where the injection $f : Y \to X$ is continuous. Suppose that for each $\phi \in Y$ there is a real number $T > 0$ and a unique function $u \in C([0,T];Y)$ satisfying (5.2.8) for $t \in (0,T]$. Suppose, moreover, that the map $\phi \to u$ is continuous from Y to $C([0,T];Y)$. Then we may say that the problem (5.2.8) is locally well posed in Y. If T can be taken arbitrarily large, the problem is globally well posed in Y. We will prove the following theorem:

Theorem E. *Let V be a potential in $\mathbb{X}$ and $V(x,t) = V_1(x,t) + V_2(x,t)$ with $V_1 \in L_x^{n/2}(L_t^\infty(\mathbb{R})), V_2 \in L^r([-T,T] : L_x^\infty), r > 1$ and $\|V_1\|_{L_x^{n/2}(L_t^\infty(\mathbb{R}))}$ is small enough. Then there exist a unique solution $u(x,t)$ such that*

$$\|u\|_{L_x^{2n/(n-2)}(L_t^2[-T,T])} + \sup_{|t|<T} \|u(\cdot,t)\|_{L^2(\mathbb{X})} \leq C(T)\|u_0\|_{L^2(\mathbb{X})}. \tag{5.2.9}$$

The proof is based on some mixed norm estimates of the solution of Schrödinger equations on $\mathbb{X}$.

5.3 Proof of Theorem E

To prove the Theorem E we need some estimates regarding the free Schrödinger propagator, which we will prove in the following Lemmas. These estimates will follow from the following Strichartz estimate obtained by Anker et al. in [2]:

Theorem 5.3.1. *(Strichartz inequality). Let* $\mathbb{X}$ *be a symmetric space of any arbitrary rank and* (p, q) *and* $(\tilde{p}, \tilde{q})$ *be two admissible pairs corresponding to the triangle*

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right] \times \left(0, \frac{1}{2}\right] : \frac{2}{p} + \frac{n}{q} \geq \frac{n}{2}\right\} \bigcup \left\{\left(0, \frac{1}{2}\right)\right\}$$

Then there exists a constant $C > 0$ *such that, for any bounded or unbounded* $I \subset \mathbb{R}$*, the solution to the inhomogeneous Schrödinger equation*

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = F(x,t), & (t,x) \in \mathbb{R} \times \mathbb{X}, \\ \\ u(x,0) = f(x) \end{cases} \tag{5.3.10}$$

satisfies

$$\|u\|_{L^p(I;L^q(\mathbb{X}))} \leq C\left(\|f\|_{L^2(\mathbb{X})} + \|F\|_{L^{\tilde{p}'}(I;L^{\tilde{q}'}(\mathbb{X}))}\right) \tag{5.3.11}$$

where, $\tilde{p}', \tilde{q}'$ *satisfies* $\frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1$ *and* $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1.$

Lemma 5.3.2. *Let* $u = e^{it\mathcal{L}}u_0$ *be the solution of the Schrödinger equation*

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = 0, & (t,x) \in \mathbb{R} \times \mathbb{X}, \\ \\ u(x,0) = u_0(x). \end{cases} \tag{5.3.12}$$

Then there exists a constant $C > 0$ *such that*

$$\|e^{it\mathcal{L}}u_0\|_{L^q_x L^2_t(\mathbb{R})} \leq C\|u_0\|_{L^2(\mathbb{X})}, \tag{5.3.13}$$

whenever $q \leq \frac{2n}{n-2}$.

Proof. The proof of this lemma is based on the Strichartz estimate proved in the above Theorem. Using that estimate (5.3.11) we have

$$\|u\|_{L^q_x L^p_t(\mathbb{R})} \leq c\|u_0\|_{L^2(\mathbb{X})}, \quad p \geq 2, q > 2 \; ; \; \frac{2}{p} + \frac{n}{q} \geq \frac{n}{2}$$

Choosing $p = 2$, we get $q \leq \frac{2n}{n-2}$. □

Lemma 5.3.3.

$$\left\|\int_0^T e^{it\mathcal{L}} f(\cdot,t)dt\right\|_{L^2(\mathbb{X})} \leq C\|f\|_{L^p_x L^2_t(\mathbb{R})} \tag{5.3.14}$$

with $p \geq \frac{2n}{n+2}$.

Proof. For $g \in L^2(\mathbb{X})$,

$$\begin{aligned}
&\left|\int_{\mathbb{X}}\int_0^T e^{it\mathcal{L}} f(x,t)dt\overline{g(x)}dx\right| \\
&= \left|\int_0^T \int_{\mathbb{X}} e^{it\mathcal{L}} f(x,t)\overline{g(x)}dxdt\right| \\
&\leq \int_0^T \int_{\mathbb{X}} \left|f(x,t)e^{-it\mathcal{L}}\overline{g(x)}\right| dxdt \\
&= \int_0^T \int_{\mathbb{X}} \left|f(x,t)\overline{e^{it\mathcal{L}} g(x)}\right| dxdt \\
&\leq \int_0^T \|f(\cdot,t)\|_{L^{q'}_x} \|e^{it\mathcal{L}} g(x)\|_{L^q_x} dt, \qquad \frac{1}{q}+\frac{1}{q'}=1; 1+\frac{n}{q} \geq \frac{n}{2} \\
&\leq \|f\|_{L^{q'}_x L^2_t(\mathbb{R})} \|e^{it\mathcal{L}} g(x)\|_{L^q_x L^2_t(\mathbb{R})} \\
&\leq C\|g\|_{L^2_x} \|f\|_{L^{q'}_x L^2_t(\mathbb{R})}, \quad \text{by Lemma 5.3.2.}
\end{aligned}$$

Taking supremum over $g \in L^2(\mathbb{X})$ such that $\|g\|_{L^2(\mathbb{X})} = 1$ and using duality we have the required estimate (5.3.14). ☐

Lemma 5.3.4. *Let $u(x,t)$ be a solution of*

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = f(x,t), & (t,x) \in \mathbb{R} \times \mathbb{X}, \\ u(x,0) = 0. \end{cases} \tag{5.3.15}$$

Then

$$\|u\|_{L^q_x L^2_t(\mathbb{R})} \leq C\,\|f\|_{L^{\tilde{q}'}_x L^2_t(\mathbb{R})} \tag{5.3.16}$$

with $q \leq \frac{2n}{n-2}$ and $\tilde{q}' \geq \frac{2n}{n+2}$.

Proof. From the Strichartz estimate (5.3.11) we have,

$$\|u\|_{L^q_x L^p_t(\mathbb{R})} \leq c\|f\|_{L^{\tilde{q}'}_x L^{\tilde{p}'}_t(\mathbb{R})}$$

where $p, \tilde{p}' \geq 2$; $q, \tilde{q}' > 2$ and $\frac{2}{p}+\frac{n}{q} \geq \frac{n}{2}$, $\frac{2}{\tilde{p}'}+\frac{n}{\tilde{q}'} \geq \frac{n}{2}$.

Now choosing $p = 2 = \tilde{p}'$, we get $q \leq \frac{2n}{n-2}$ and $\tilde{q}' \geq \frac{2n}{n+2}$, and proof of the lemma follows. ☐

Now we prove our main result, Theorem E.

By Duhamel's formula, the solution $u(x,t)$ of (5.2.7) can be written as (see [2]):

$$u(x,t) = e^{it\mathcal{L}}u_0 + i\int_0^t e^{i(t-s)\mathcal{L}}V(\cdot,s)u(\cdot,s)(x)ds \tag{5.3.17}$$

We define the space X_T, and the operator $\mathcal{T}_V$ on X_T by

$$X_T = \left\{ f \ : \ \|f\|_{X_T} = \max\left(\|f\|_{L^{p'}_x(L^2_t([-T,T]))}, \sup_{|t|<T} \|f(\cdot,t)\|_{L^2(\mathbb{X})} \right) < \infty, \quad \text{with } \frac{1}{2} - \frac{1}{p'} = \frac{1}{n} \right\}.$$

$$\mathcal{T}_V f(x,t) = i\int_0^t e^{i(t-s)\mathcal{L}}V(x,s)f(x,s)ds$$

So, we have

$$u = e^{it\mathcal{L}}u_0 + \mathcal{T}_V u \tag{5.3.18}$$

Now, suppose $u_0 \in L^2(\mathbb{X})$, then by Lemma 5.3.2

$$\|e^{it\mathcal{L}}u_0\|_{L^{p'}_x L^2_t[-T,T]} \leq C\|u_0\|_{L^2(\mathbb{X})}, \tag{5.3.19}$$

and by Plancherel theorem, we have

$$\|e^{it\mathcal{L}}u_0\|_{L^2(\mathbb{X})} = \|u_0\|_{L^2(\mathbb{X})}. \tag{5.3.20}$$

Combining the norm estimates (5.3.19) and (5.3.20), we have

$$e^{it\mathcal{L}}u_0 \in X_T.$$

In particular, from the definition of the space X_T, we can say

$$u_0 \in L^2(\mathbb{X}) \implies e^{it\mathcal{L}}u_0 \in X_T. \tag{5.3.21}$$

The equation (5.3.18) can be written in the form $(I - \mathcal{T}_V)u = e^{it\mathcal{L}}u_0$. So, if we can prove that the operator $(I - \mathcal{T}_V)$ is invertible, uniqueness of the solution of equation (5.2.7) will easily follow.

Now take $f \in X_T$. As $V = V_1 + V_2$ so we have,

$$\|\mathcal{T}_V f\|_{L^{p'}_x L^2_t[-T,T]} \leq \|\mathcal{T}_{V_1} f\|_{L^{p'}_x L^2_t([-T,T])} + \|\mathcal{T}_{V_2} f\|_{L^{p'}_x L^2_t[-T,T]} \tag{5.3.22}$$

Note that if $u_0 = 0$, then $u = \mathcal{T}_{V_1} u$ is the solution of

$$\begin{cases} i\dfrac{\partial u(x,t)}{\partial t} + \mathcal{L}u(x,t) = V_1(x,t)u(x,t), \qquad (t,x) \in \mathbb{R} \times \mathbb{X}, \\ \\ u(x,0) = 0. \end{cases} \tag{5.3.23}$$

It follows from the estimate proved in Lemma 5.3.4, that

$$\|\mathcal{T}_{V_1} u\|_{L^q_x L^2_t[-T,T]} \le C \|V_1 u\|_{L^{\tilde{q}'}_x L^2_t[-T,T]}$$

for $q \le \frac{2n}{n-2}$ and $\tilde{q}' \ge \frac{2n}{n+2}$. In particular,

$$\|\mathcal{T}_{V_1} f\|_{L^{p'}_x L^2_t[-T,T]} \le C \|V_1 f\|_{L^p_x L^2_t[-T,T]} \tag{5.3.24}$$

where $p' = \frac{2n}{n-2}, p = \frac{2n}{n+2}$.

Now, $\mathcal{T}_{V_2} f$ can be written as

$$\mathcal{T}_{V_2} f(x,t) = i \int_{-T}^{T} \chi_{[0,t]}(s) e^{i(t-s)\mathcal{L}} V_2(\cdot,s) f(\cdot,s) ds = \int_{-T}^{T} F(s,t)(x) ds$$

where $F(s,t) = \chi_{[0,t]}(s) e^{i(t-s)\mathcal{L}} V_2(\cdot,s) f(\cdot,s)$. Applying Minkowski's inequality twice, we get

$$\begin{aligned} \|\mathcal{T}_{V_2} f\|_{L^{p'}_x L^2_t[-T,T]} &\le \int_{-T}^{T} \|F(s,t)(x)\|_{L^{p'}_x L^2_t(\mathbb{R})} ds \\ &\le \int_{-T}^{T} \|e^{i(t-s)\mathcal{L}} V_2(\cdot,s) f(\cdot,s)\|_{L^{p'}_x L^2_t(\mathbb{R})} ds \\ &= C \int_{-T}^{T} \|V_2(\cdot,s) f(\cdot,s)\|_{L^2(\mathbb{X})} ds, \quad \text{by Lemma 5.3.2.} \end{aligned} \tag{5.3.25}$$

Now, applying Holder's inequality and using the fact that $V_1 \in L^{n/2}_x L^\infty_t(\mathbb{R})$, we obtain

$$\|V_1 f\|_{L^p_x L^2_t[-T,T]} \le \|V_1\|_{L^{n/2}_x L^\infty_t[-T,T]} \|f\|_{L^{p'}_x L^2_t[-T,T]} \tag{5.3.26}$$

Since $f \in X_T$, therefore $\sup\limits_{|t|<T} \|f(\cdot,t)\|_{L^2_x} < \infty$; that is $\|f(\cdot,s)\|_{L^2_x}$ is finite for each s. Using this fact, together with $V_2 \in L^r([-T,T] : L^\infty_x), r > 1$ and applying Holder's inequality again, we have

$$\int_{-T}^{T} \|V_2(\cdot,s) f(\cdot,s)\|_{L^2(\mathbb{X})} ds \le T^{1/r'} \|V_2\|_{L^r([-T,T]:L^\infty_x)} \sup_{|t|<T} \|f(\cdot,t)\|_{L^2_x} \tag{5.3.27}$$

Now substituting the estimates (5.3.26) and (5.3.27) into (5.3.24) and (5.3.25) respectively, and using (5.3.22), we finally obtain

$$\|\mathcal{T}_V f\|_{L^{p'}_x L^2_t[-T,T]} \leq C \left[\|V_1\|_{L^{n/2}_x L^\infty_t[-T,T]} + T^{1/r'} \|V_2\|_{L^r([-T,T]:L^\infty_x)} \right] \|f\|_{X_T}. \tag{5.3.28}$$

Now, in order to show $\mathcal{T}_V f \in X_T$, we have to compute the L^2-norm of $\mathcal{T}_V f$ in the x variable.

$$\|\mathcal{T}_V f(\cdot, t)\|_{L^2(\mathbb{X})} \leq \|\mathcal{T}_{V_1} f(\cdot, t)\|_{L^2(\mathbb{X})} + \|\mathcal{T}_{V_2} f(\cdot, t)\|_{L^2(\mathbb{X})} \tag{5.3.29}$$

By Lemma 5.3.3, we have

$$\|\mathcal{T}_{V_1} f(\cdot, t)\|_{L^2(\mathbb{X})} \leq C \|V_1 f\|_{L^p_x L^2_t(\mathbb{R})}, \quad \text{for} \quad p \geq \frac{2n}{n+2} \tag{5.3.30}$$

Using the definition of $\mathcal{T}_{V_2} f(x, t)$ and applying Minkowski inequality, we have

$$\|\mathcal{T}_{V_2} f(\cdot, t)\|_{L^2(\mathbb{X})} \leq \int_{-T}^{T} \|V_2(\cdot, s) f(\cdot, s)\|_{L^2(\mathbb{X})} ds \tag{5.3.31}$$

.

Again substituting (5.3.30) and (5.3.31) in (5.3.29), we finally obtain

$$\sup_{|t| \leq T} \|\mathcal{T}_V f(\cdot, t)\|_{L^2(\mathbb{X})} \leq C \|V_1 f\|_{L^p_x L^2_t(\mathbb{R})} + \int_{-T}^{T} \|V_2(\cdot, s) f(\cdot, s)\|_{L^2(\mathbb{X})} ds. \tag{5.3.32}$$

By using estimates (5.3.26) and (5.3.27) in the above inequality we get

$$\sup_{|t| \leq T} \|\mathcal{T}_V f(\cdot, t)\|_{L^2(\mathbb{X})} \leq C \left[\|V_1\|_{L^{n/2}_x L^\infty_t[-T,T]} + T^{1/r'} \|V_2\|_{L^r([-T,T]:L^\infty_x)} \right] \|f\|_{X_T}. \tag{5.3.33}$$

So, combining the above estimate and (5.3.32), we can say

$$\mathcal{T}_V f \in X_T, \text{ whenever } f \in X_T, \tag{5.3.34}$$

and the operator norm of $\mathcal{T}_V$ is dominated by

$$C \left[\|V_1\|_{L^{n/2}_x L^\infty_t[-T,T]} + T^{1/r'} \|V_2\|_{L^r([-T,T]:L^\infty_x)} \right].$$

Hence choosing $\|V_1\|_{L^{n/2}_x L^\infty_t[-T,T]}$ and T small enough, we conclude that the operator norm of $\mathcal{T}$ is less than 1. Hence $I - \mathcal{T}$ is invertible, which assures the uniqueness of the solution of the equation (5.2.7). Also, combining (5.3.34) and (5.3.21), we can say

$$u(x, t) \in X_T, \text{ whenever } u_0 \in L^2(\mathbb{X})$$

Now, we will prove the inequality (5.2.9). The estimates (5.3.19) and (5.3.20) together give,

$$\|e^{it\mathcal{L}}u_0\|_{L^{p'}_x\left(L^2_t[-T,T]\right)} + \|e^{it\mathcal{L}}u_0\|_{L^2(\mathbb{X})} \leq C\|u_0\|_{L^2(\mathbb{X})} \leq C\|u\|_{X_T},. \tag{5.3.35}$$

From (5.3.18), we have

$$\|u\|_{X_T} \leq \|(I-\mathcal{T}_V)^{-1}\|_{op}\|e^{it\mathcal{L}}u_0\|_{X_T} \leq C\|u_0\|_{L^2(\mathbb{X})}$$

(5.3.28) and (5.3.33) together imply

$$\|\mathcal{T}_V u\|_{L^{p'}_x\left(L^2_t[-T,T]\right)} + \sup_{|t|\leq T}\|\mathcal{T}_V u(\cdot,t)\|_{L^2(\mathbb{X})} \leq C\|u\|_{X_T} \leq C\|u_0\|_{L^2(\mathbb{X})} \tag{5.3.36}$$

Finally, combining (5.3.36) and (5.3.35), and using the expression of u given in (5.3.17), we obtain the required estimate (5.2.9).

From the estimate (5.2.9), we can say $u_0 \to u(x,t)$ is a continuous map from $L^2(\mathbb{X})$ to the space X_T. This completes the proof of Theorem E.

CHAPTER 6

Concluding Remarks

In this chapter, we would like to address some open problems that are closely related to the topics discussed throughout the thesis.

Problem 1: In chapter 3 and chapter 4, we have proved the pointwise convergence problem related to the fractional Schrödinger equation on $\mathbb{X}$ assuming the initial data to be radial. One of our future plans is to extend this result to the case with non-radial initial data and to find the admissible range of s in this case.

In that case, the solution of (3.2.4) will be

$$u(x,t) = S_t f(x) = \int_{\mathbb{R}} \int_K \widetilde{f}(\lambda, k) e^{it(\lambda^2+\rho^2)^{\frac{a}{2}}} e^{(i\lambda+\rho)H(x^{-1}k)} |c(\lambda)|^{-2} dk\, d\lambda, \quad \text{by } (2.3.5)$$

If we proceed as in the proof of Theorem B, we have

$$\begin{aligned}\int_X \int_{\mathbb{R}} |Sf(x,t)|^2\, dx\, dt = c \int_{\mathbb{R}} \int_{\mathbb{R}} \int_K \int_K & \Big(\int_X \alpha(x) e^{(i\lambda+\rho)H(x^{-1}k_1)} e^{(-i\eta-\rho)H(x^{-1}k_2)}\, dx \Big) \\ & \widehat{\psi}\left((\eta^2+\rho^2)^{\frac{a}{2}} - (\lambda^2+\rho^2)^{\frac{a}{2}} \right) \widetilde{f}(\lambda, k_1)\, \overline{\widetilde{f}(\eta, k_2)} \\ & |c(\lambda)|^{-2}\, |c(\eta)|^{-2}\, d\lambda\, d\eta\, dk_1\, dk_2.\end{aligned}$$

Thus here,

$$K(\lambda,\eta) = (\rho^2+|\lambda|)^s (\rho^2+|\eta|)^s \Big(\int_{\mathbb{X}} \alpha(x) e^{(i\lambda+\rho)H(x^{-1}k_1)} e^{(-i\eta-\rho)H(x^{-1}k_2)} dx \Big) \widehat{\psi}((\eta^2+\rho^2)^{\frac{a}{2}} - (\lambda^2+\rho^2)^{\frac{a}{2}})).$$

The main difference is that the terms ϕ_λ and ϕ_η appearing in the expression of $K(\lambda, \eta)$ in the proof of Theorem B is replaced by $e^{(i\lambda+\rho)H(x^{-1}k_1)}$ and $e^{(-i\eta-\rho)H(x^{-1}k_2)}$ respectively. Unlike the radial case, these terms do not provide any decay in terms of λ and η. Therefore it is not clear if $\int_{\mathbb{X}} \int_{\mathbb{R}} |Sf(x,t)|^2 dxdt$ can be convergent.

Problem 2: So far, we have considered the problems in the rank one symmetric space setting. It would be interesting to know to what extent these results remain valid for the higher rank case.

In the higher rank case, dim $\mathfrak{a}^*_{\mathbb{R}} > 1$, therefore the spectral parameter λ, η belongs to $\mathbb{R}^n$. It is therefore necessary to analyze how the results for spherical functions and Harish-Chandra's **c**-function extend to this case and to obtain an appropriate decay estimate for ϕ_λ and $c(\lambda)$ in terms of λ.

Problem 3: In Chapter 5, we have only established the well-posedness of (5.2.7).To fully resolve this problem, it remains to prove a local smoothing estimate of the type (5.2.6), due to Ruiz and Vega.

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भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
Indian Institute of Technology Guwahati

List of Articles from Thesis Work

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List of Communicated Articles

- Pratyoosh Kumar and Manali Sajjan, *Sharp Regularity Results for the Schrödinger Equation on Rank-One Symmetric Spaces of Non-Compact Type*