UNIQUENESS OF THE FOURIER TRANSFORM ON EUCLIDEAN SPACES AND CERTAIN LOCALLY COMPACT LIE GROUPS

by

DEB KUMAR GIRI

DEPARTMENT OF MATHEMATICS

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UNIQUENESS OF THE FOURIER TRANSFORM ON EUCLIDEAN SPACES AND CERTAIN LOCALLY COMPACT LIE GROUPS

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DECLARATION

I do hereby declare that this thesis entitled **"UNIQUENESS OF THE FOURIER TRANSFORM ON EUCLIDEAN SPACES AND CERTAIN LOCALLY COMPACT LIE GROUPS"** is a presentation of my original research work done under the supervision of **Dr. Rajesh K. Srivastava**, Assistant Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of doctor of philosophy. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

April, 2018

Deb Kumar Giri
Roll No. 136123005
Department of Mathematics
Indian Institute of Technology Guwahati

CERTIFICATE

This is certified that the work contained in the thesis entitled **"UNIQUENESS OF THE FOURIER TRANSFORM ON EUCLIDEAN SPACES AND CERTAIN LOCALLY COMPACT LIE GROUPS"**, by **Mr. Deb Kumar Giri (Roll No. 136123005)** has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

Dr. Rajesh Kumar Srivastava
Assistant Professor
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati, India.

April 2018

Abstract

Name of the Student	:	**Deb Kumar Giri**
Roll Number	:	**136123005**
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Let $X(\Gamma)$ be the space of all finite Borel measures μ in $\mathbb{R}^2$ which is supported on the curve Γ in $\mathbb{R}^2$ and absolutely continuous with respect to the arc length of Γ. For $\Lambda \subset \mathbb{R}^2$, the pair (Γ, Λ) is called a Heisenberg uniqueness pair for $X(\Gamma)$ if any $\mu \in X(\Gamma)$ satisfying $\hat{\mu}|_\Lambda = 0$, implies $\mu = 0$.

Heisenberg uniqueness pair has been first introduced by Hedenmalm and Montes-Rodríguez as a version of the uncertainty principle (see [18]). We would like to mention that Heisenberg uniqueness pair up to some extent is similar to an annihilating pair of Borel measurable sets of positive measure as described by Havin and Jöricke (see [17]). Let $S, \Sigma \subseteq \mathbb{R}$ be a pair of Borel measurable sets. Then (S, Σ) form a mutually annihilating pair if any $\varphi \in L^2(S)$ whose Fourier transform $\hat{\varphi}$ supported on Σ, implies $\varphi = 0$.

Heisenberg uniqueness pair has a close relation with the long-standing problem of determining the *exponential types* for a finite measure which is eventually about

exploring the density of the set $\{e^{i\lambda t} : \lambda \in \Lambda\}$ in $L^2(\mu)$. For more details, we refer to Poltoratski [35, 36]. In particular, the question of HUP can be thought as a dual problem of *gap problem* (see [35]). Let μ be a finite Borel measure which is supported on a closed set $\Gamma \subset \mathbb{R}$ and

$$G_\Gamma = \sup\left\{a > 0 : \exists\, \mu \neq 0,\ \text{supp}\ \mu \subset \Gamma \text{ satisfies } \hat{\mu}|_{[0,a]} = 0\right\}.$$

Let $\Lambda \subset \mathbb{R}$. Then (Γ, Λ) would be a HUP as long as the determining set Λ intersect $[0, G_\Gamma]$ at most on a set of measure zero.

Further, the concept of Heisenberg uniqueness pair has a sharp contrast to the determining sets for measures by Sitaram et al. [6, 38], due to the fact that the determining set Λ for the function $\hat{\mu}$ has also been considered a thin set. In particular, if Γ is compact, then $\hat{\mu}$ is a real analytic function having exponential growth and it can vanish on a very delicate set. Hence, in this case, finding the Heisenberg uniqueness pairs becomes little easier. However, this question becomes immensely difficult when the measure is supported on a non-compact curve.

Hedenmalm and Montes-Rodríguez have shown that the pair (hyperbola, some discrete set) is a Heisenberg uniqueness pair. As a dual problem, a weak* dense subspace of $L^\infty(\mathbb{R})$ has been constructed to solve the Klein-Gordon equation. Further, a complete characterization of the Heisenberg uniqueness pairs corresponding to any two parallel lines has been given by Hedenmalm and Montes-Rodríguez (see [18]).

Lev [27] and Sjölin [43] have independently shown that circle and certain system of lines are HUP corresponding to the unit circle S^1. Further, Vieli [53] has generalized HUP corresponding to circle in the higher dimension and shown that a sphere whose radius does not lie in the zero set of the Bessel functions $J_{(n+2k-2)/2}$; $k \in \mathbb{Z}_+$, the set of non-negative integers, is a HUP corresponding to the unit sphere S^{n-1}. In a recent article [46], it has been shown that a cone is a Heisenberg uniqueness pair corresponding to the sphere as long as the cone does not completely lay on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$. Thereafter, a sense of evidence emerged that the exceptional sets for the HUPs corresponding to the sphere are eventually contained in the zero sets of the spherical harmonics and the Bessel functions, though we yet to resolve it (see [46]).

Sjölin [44] has investigated some of the Heisenberg uniqueness pairs corresponding to the parabola. Subsequently, Babot [5] has given a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of three parallel lines. Thereafter, we have given some necessary and sufficient conditions for the Heisenberg uniqueness pairs corresponding to a system of four parallel lines. However, an exact analogue for the finitely many lines as compared to three parallel lines result is still open.

In a major development, Jaming and Kellay [20] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$. Further, Gröchenig and Jaming [15] have worked out some of the Heisenberg uniqueness pairs corresponding to the quadratic surface.

In the thesis, we explored the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle, cross, exponential curves, and surfaces. Then, we prove a characterization of the Heisenberg uniqueness pairs corresponding to four parallel lines. We observe that the size of the determining sets Λ for $X(\Gamma)$ depends on the number of lines and their irregular distribution that further relates to a phenomenon of interlacing of the zero sets of certain trigonometric polynomials.

In an interesting article, M. Benedicks [7] had extended the classical Paley-Wiener theorem to the class of integrable functions. That is, support of an integrable function

f and its Fourier transform $\hat{f}$ both cannot be of finite measure simultaneously.

Thereafter, a series of analogous results to the Benedicks theorem had been explored in various set ups, including the Heisenberg group and the Euclidean motion groups (see [31, 42]). In particular, Narayanan and Ratnakumar [31] had worked out an analogous result to the Benedicks theorem for the partially compactly supported functions on the Heisenberg group in terms of the finite rank of the Fourier transform of the function.

In a recent article, Vemuri [52] has relaxed the compact support of the function by finite Lebesgue measure. The latter result can be thought as nearly close to the original Benedicks theorem. However, it would be a good question to consider the case when the spectrum of the Fourier transform of an integrable function will be supported on a thin uncountable set.

In this thesis, our main focus is to explore analogue of Benedicks theorem on the Euclidean motion groups, Heisenberg motion group and step two nilpotent Lie groups. We prove that if the group Fourier transform of finitely supported certain integrable functions is of finite rank, then the function has to vanish identically.

The thesis is organized as follows:

In Chapter 1, we briefly introduce the problems considered in the thesis and the recent developments on the topics related to Heisenberg uniqueness pairs and Benedicks theorem.

In Chapter 2, we consider the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle, cross, exponential curves, and some surfaces.

In Chapter 3, we prove a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of four parallel lines and in this case, we observe the phenomenon of interlacing of three totally disconnected dispensable sets.

In Chapter 4, we prove that if the Fourier transform of a integrable function on the Euclidean motion group $M(n)$ is compactly supported, then the function has to vanish identically. Further, we explore the Heisenberg uniqueness pairs for the Fourier transform on $M(n)$ as well as on the product group $\mathbb{R}^n \times K$, where K is a compact group.

In Chapter 5, we prove that if the group Fourier transform of certain integrable functions on the Heisenberg motion group (or step two nilpotent Lie groups) is of finite rank, then the function is identically zero. These results can be thought as an analogue to the Benedicks theorem that dealt with the uniqueness of the Fourier transform of integrable functions on the Euclidean spaces. spaces.

Dedicated to

my

Parents and Sister

Acknowledgement

It is a pleasure for me to acknowledge my teachers, friends, and parents whose continuous support helped me to carry this thesis work.

Foremost, I would like to express my utmost sincere gratitude to my thesis advisor Dr. Rajesh K. Srivastava for his continuous support during my Ph.D. work, patience, motivation, enthusiasm. His guidance helped me in all the time of research and writing of this thesis.

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Last but not the least, I would like to thank my family members, especially my parents and my sister, Sujata for their love and enormous support throughout my life. My sincere thanks also go to my uncle Mr. Bharat Chandra Giri for supporting me spiritually throughout my career.

Finally, I would like to acknowledge the immeasurable learning experience the IIT Guwahati grants its students – an experience for which I will be forever grateful.

Deb Kumar Giri

Contents

List of Symbols

$\mathbb{C}^n = \{(z_1, \ldots, z_n) : z_i \in \mathbb{C}\},\ n \geq 1$ and $\mathbb{C}$ the field of complex numbers

$z.\bar{w} = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,\ z_i, w_i \in \mathbb{C},\ i = 1, \ldots, n$

$|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, where $z_i \in \mathbb{C}$

$\mathrm{Im} z$: the imaginary part of $z \in \mathbb{C}$

$\mathbb{B}^n$: the open unit ball in $\mathbb{R}^n$ $(n \geq 2)$

S^{d-1} : the unit sphere in $\mathbb{R}^d$ $(d \geq 2)$

$\mathbb{Z}_+$: set of all non-negative integers

$\mathbb{N}$: set of all positive integers

$\min(p, q)$: stands for the minimum of $p, q \in \mathbb{Z}_+$

$gl_n(\mathbb{R})$: space of all $n \times n$ matrices over $\mathbb{R}$

$C(X)$: space of continuous functions on topological space X

$C_c(X)$: space of compactly supported functions in $C(X)$

$C_c^\sharp(\mathbb{R}^n)$: space of radial functions in $C_c(X)$

$L^1_{\text{loc}}(X)$: space of locally integrable functions on measure space X

$L^p(X)$: space of functions f with $\int_X |f|^p < \infty,\ 1 \leq p < \infty$

Chapter 1

Introduction

Let $X(\Gamma)$ be the space of all finite Borel measures μ in $\mathbb{R}^2$ which is supported on the curve Γ and absolutely continuous with respect to the arc length of Γ. For $\Lambda \subset \mathbb{R}^2$, the pair (Γ, Λ) is called a Heisenberg uniqueness pair for $X(\Gamma)$ if any $\mu \in X(\Gamma)$ satisfying $\hat{\mu}|_\Lambda = 0$, implies $\mu = 0$.

The concept of the Heisenberg uniqueness pair has been first introduced in an influential article by Hedenmalm and Montes-Rodríguez (see [18]). We would like to mention that Heisenberg uniqueness pair up to a certain extent is similar to an annihilating pair of Borel measurable sets of positive measure as described by Havin and Jöricke [17]. Let $S, \Sigma \subseteq \mathbb{R}$ be a pair of Borel measurable sets. Then (S, Σ) form a mutually annihilating pair if any $\varphi \in L^2(S)$ whose Fourier transform $\hat{\varphi}$ supported on Σ, implies $\varphi = 0$. Further, the notion of Heisenberg uniqueness pair has a sharp contrast to the known results about determining sets for measures by Sitaram et al. [6, 38], due to the fact that the determining set Λ for the function $\hat{\mu}$ has also been considered a thin set.

Heisenberg uniqueness pair has a close relation with the long-standing problem of determining the *exponential types* for a finite measure which is eventually about exploring the density of the set $\{e^{i\lambda t} : \ \lambda \in \Lambda\}$ in $L^2(\mu)$. For more details, we refer to Poltoratski [35, 36]. In particular, the question of HUP can be thought as a dual problem of *gap problem* (see [35]). Let μ be a finite Borel measure which is supported on a closed set $\Gamma \subset \mathbb{R}$ and

$$G_\Gamma = \sup\left\{a > 0 : \ \exists\, \mu \neq 0, \ \text{supp}\ \mu \subset \Gamma \text{ satisfies } \hat{\mu}|_{[0,a]} = 0\right\}.$$

Let $\Lambda \subset \mathbb{R}$. Then (Γ, Λ) would be a HUP as long as the determining set Λ intersect $[0, G_\Gamma]$ at most on a set of measure zero.

In addition, the question of determining the Heisenberg uniqueness pair for a class of finite measures has also a significant similarity with the celebrated result due to M. Benedicks (see [7]). That is, support of a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\hat{f}$ cannot be of finite measure simultaneously. Later, various analogues of the Benedicks theorem have been investigated in different set ups, including the Heisenberg group and Euclidean motion groups (see [31, 32, 42]).

In particular, if Γ is compact, then $\hat{\mu}$ is a real analytic function having exponential growth and it can vanish on a very delicate set. Hence, in this case, finding the Heisenberg uniqueness pairs becomes little easier. However, this question becomes immensely difficult when the measure is supported on a non-compact curve. Eventually, the Heisenberg uniqueness pair is a natural invariant to the theme of the well studied the uncertainty principle for the Fourier transform.

In the article [18], Hedenmalm and Montes-Rodríguez have shown that the pair (hyperbola, some discrete set) is a Heisenberg uniqueness pair. As a dual problem, a weak* dense subspace of $L^\infty(\mathbb{R})$ has been constructed to solve the Klein-Gordon equation. Further, a complete characterization of the HUPs corresponding to any two parallel lines has been given by Hedenmalm and Montes-Rodríguez (see [18]).

Afterward, a considerable amount of work has been done pertaining to the Heisenberg uniqueness pair in the plane as well as in the higher dimensional Euclidean spaces.

Recently, Lev [27] and Sjölin [43] have independently shown that circle and certain system of lines are HUP corresponding to the unit circle S^1. Further, Vieli [53] has generalized HUP corresponding to circle in the higher dimension and shown that a sphere whose radius does not lie in the zero set of the Bessel functions $J_{(n+2k-2)/2};\ k \in \mathbb{Z}_+$, the set of non-negative integers, is a HUP corresponding to the unit sphere S^{n-1}. In a recent article [46], the author has shown that a cone is a Heisenberg uniqueness pair corresponding to the sphere as long as the cone does not completely lay on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$. Thereafter, a sense of evidence emerged that the exceptional sets for the HUPs corresponding to the sphere are eventually contained in the zero sets of the spherical harmonics and the Bessel functions, though we yet resolve it (see [46]).

Sjölin [44] has investigated some of the Heisenberg uniqueness pairs corresponding to the parabola. Subsequently, Babot [5] has given a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of three parallel lines. However, an exact analogue for the finitely many parallel lines is still open.

In a major development, Jaming and Kellay [20] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$, via dynamical system approach. Further, Gröchenig and Jaming [15] have worked out some of the Heisenberg uniqueness pairs corresponding to the quadratic surface.

Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measure μ in $\mathbb{R}^2$ which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of μ is defined by

$$\hat{\mu}(\xi,\eta)=\int_{\Gamma}e^{-i\pi(x\cdot\xi+y\cdot\eta)}d\mu(x,y).$$

In the above context, the function $\hat{\mu}$ becomes a uniformly continuous bounded function on $\mathbb{R}^2$. Thus, we can analyze the pointwise vanishing nature of the function $\hat{\mu}$.

Definition 1.0.1. *Let Λ be a set in $\mathbb{R}^2$. The pair (Γ,Λ) is called a Heisenberg uniqueness pair for $X(\Gamma)$ if any $\mu\in X(\Gamma)$ satisfying $\hat{\mu}|_{\Lambda}=0$, implies $\mu=0$.*

Since the Fourier transform is invariant under translation and rotation, one can easily deduce the following *invariance properties* about the Heisenberg uniqueness pair.

(i) Let $u_o, v_o\in\mathbb{R}^2$. Then the pair (Γ,Λ) is a HUP if and only if the pair $(\Gamma+u_o,\Lambda+v_o)$ is a HUP.

(ii) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be an invertible linear transform whose adjoint is denoted by T^*. Then (Γ,Λ) is a HUP if and only if $(T^{-1}\Gamma,T^*\Lambda)$ is a HUP.

Now, we state first known results on the Heisenberg uniqueness pair due to Hedenmalm and Montes-Rodríguez [18]. After that, we briefly indicate the progress on this recent problem.

Theorem 1.0.1. [18] *Let $\Gamma=L_1\cup L_2$, where $L_j;\ j=1,2$ are any two parallel straight lines and Λ a subset of $\mathbb{R}^2$ such that $\overline{\pi(\Lambda)}=\mathbb{R}$. Then (Γ,Λ) is a Heisenberg uniqueness pair if and only if the set*

$$\widetilde{\Lambda}=\pi_1^a(\Lambda)\cup\left[\pi_1^b(\Lambda)\smallsetminus\pi_1^c(\Lambda)\right] \tag{1.0.1}$$

is dense in $\mathbb{R}$.

Here we avoid mentioning the notations appeared in (1.0.1) as they are bit involved, however, we have written down the same notations as in the article [18]. Though, their main features can be perceived in Section 3.2.

Theorem 1.0.2. [18] *Let* Γ *be the hyperbola* $x_1x_2 = 1$ *and* $\Lambda_{\alpha,\beta}$ *a lattice-cross defined by*

$$\Lambda_{\alpha,\beta} = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α, β *are positive reals. Then* $(\Gamma, \Lambda_{\alpha,\beta})$ *is a Heisenberg uniqueness pair if and only if* $\alpha\beta \leq 1$.

For $\xi \in \Lambda$, define a function e_ξ on Γ by $e_\xi(x) = e^{i\pi x\cdot\xi}$. As a dual problem to Theorem 1.0.2, Hedenmalm and Montes-Rodríguez [18] have proved the following density result which in turn solve the one-dimensional Klein-Gordon equation.

Theorem 1.0.3. [18] *The pair* (Γ, Λ) *is a Heisenberg uniqueness pair if and only if the set* $\{e_\xi : \ \xi \in \Lambda\}$ *is a weak* dense subspace of* $L^\infty(\Gamma)$.

Remark 1.1. *In particular, for* Γ *to be an algebraic curve, the question of Heisenberg uniqueness pair can be understood through a partial differential equation (PDE). That is, if* Γ *is the zero set of a polynomial* P *on* $\mathbb{R}^2$*, then* $\hat{\mu}$ *satisfies the PDE*

$$P\left(\frac{\partial_1}{i\pi}, \frac{\partial_2}{i\pi}\right)\hat{\mu} = 0$$

with initial condition $\hat{\mu}|_\Lambda = 0$. *This formulation may help potentially in determining the geometrical structure of the set* $Z(\hat{\mu})$*, the zero set of the function* $\hat{\mu}$*. If we consider* Λ *to be contained in* $Z(\hat{\mu})$*, then* (Γ, Λ) *is not a HUP. Hence the question of the HUP arises when* Λ *has located away from* $Z(\hat{\mu})$.

In the case when μ is supported on a circle, the function $\hat{\mu}$ becomes real analytic and hence it could vanish at most on a very thin set. Thus, there are an enormous number of candidates for Λ such that (Γ, Λ) is a HUP. Some of the Heisenberg uniqueness pairs corresponding to circle has been independently investigated by Lev and Sjölin. Following are their main results. For more details, we refer to [27, 43].

Theorem 1.0.4. [27, 43] *Let $\Gamma = S^1$ be the unit circle.*

(i) Let Λ be a circle of radius r. Then (Γ, Λ) is a HUP if and only if $J_k(r) \neq 0$ for all $k \in \mathbb{Z}_+$.

(ii) Let Λ be a straight line. Then (Γ, Λ) is not a HUP.

(iii) Let $\Lambda = L_1 \cup L_2$, where $L_j;\ j = 1, 2$ are two straight lines. If L_1 and L_2 are parallel, then (Γ, Λ) is a HUP.

(iv) Let $L_j;\ j = 1, \ldots, N$ be the N different straight lines which intersect at one point and the angle between any of two lines out of these N lines is of the form $\pi\alpha$. Let $\Lambda_N = \bigcup_{j=1}^{N} L_j$. Then (Γ, Λ_N) is not a HUP if and only if α is rational.

In contrast to the case of finitely many straight lines, Sjölin [43] has shown that if $\Lambda = \bigcup_{k=1}^{\infty} L_k$, where $\{L_k\}$ is a sequence of straight lines which intersect at one point, then (S^1, Λ) is a HUP.

Remark 1.2. *Since we know that any homogeneous harmonic polynomial on $\mathbb{R}^2$ can be expressed as $Ar^j \sin(j\theta + \delta)$ for some $j \in \mathbb{N}$ and $\delta \in [0, 2\pi)$ (see [11]), up to some rotation and translation, we can think of $\Lambda_N = \bigcup_{k=1}^{N} L_k$, appeared in Theorem 1.0.4 (iv), as the zero set of some homogeneous harmonic polynomial. If (S^1, Λ) is a Heisenberg uniqueness pair, then the set Λ must be away from the zero set of any homogeneous harmonic polynomial. However, the converse is not true. Since (S^1, Λ) is not a HUP if Λ is a circle whose radius lie in the zero set of some Bessel function. Thus, it is an interesting question to examine the exceptional sets for the Heisenberg uniqueness pairs corresponding to circle.*

Subsequently, some of the Heisenberg uniqueness pairs corresponding to the parabola have been obtained by Sjölin [44]. Let $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}$.

Theorem 1.0.5. [44] *Let Γ denote the parabola $y = x^2$.*

(i) Let $\Lambda = L$ be a straight line. Then (Γ, Λ) is a HUP if and only if L is parallel to the X-axis.

(ii) Let $\Lambda = L_1 \cup L_2$, where $L_j;\ j = 1, 2$ are two different straight lines. Then (Γ, Λ) is a HUP.

(iii) Let $L_j;\ j = 1, 2$ be two different straight lines which are not parallel to the X-axis. Let $E_j \subset L_j$ and $|E_j| > 0;\ j = 1, 2$. If $\Lambda = E_1 \cup E_2$, then (Γ, Λ) is a HUP.

The question of Heisenberg uniqueness pair in the higher dimension has been first taken up by Vieli [53, 54].

Theorem 1.0.6. [53] *Let $\Gamma = S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and Λ a sphere of radius r. Then (Γ, Λ) is a HUP if and only if $J_{(n+2k-2)/2}(r) \neq 0$ for all $k \in \mathbb{Z}_+$.*

Theorem 1.0.7. [54] *Let Γ be the paraboloid $x_n = x_1^2 + x_2^2 + \cdots + x_{n-1}^2$ in $\mathbb{R}^n$ and Λ an affine hyperplane in $\mathbb{R}^n$ of dimension $n - 1$. Then (Γ, Λ) is a HUP if and only if Λ is parallel to the hyperplane $x_n = 0$.*

Let Γ denote a system of three parallel lines in the plane that can be expressed as $\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\}$, where $\alpha < \beta < \gamma$ and $(\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}$. By the invariance properties of HUP, one can assume that $\Gamma = \mathbb{R} \times \{0, 1, p\}$, for some $p \in \mathbb{N}$ with $p \geq 2$. The following characterization for the Heisenberg uniqueness pairs corresponding to the above mentioned three parallel lines has been given by Babot [5].

Theorem 1.0.8. [5] *Let $\Gamma = \mathbb{R} \times \{0, 1, p\}$, for some $p \in \mathbb{N}$ with $p \geq 2$ and $\Lambda \subset \mathbb{R}^2$ a closed set which is 2-periodic with respect to the second variable. Then (Γ, Λ) is a HUP if and only if the set*

$$\widetilde{\Lambda} = \Pi^3(\Lambda) \cup \left[\Pi^2(\Lambda) \setminus \Pi^{2^*}(\Lambda)\right] \cup \left[\Pi^1(\Lambda) \setminus \Pi^{1^*}(\Lambda)\right] \tag{1.0.2}$$

is dense in $\mathbb{R}$.

For the notations appeared in Equation (1.0.2), we would like to refer the article [5], as those notations are quite involved. However, the nature of their occurrence can be understood in the beginning of Section 3.2 when we formulate the four lines problem.

Further, Jaming and Kellay [20] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to certain algebraic curves.

Theorem 1.0.9. [20] *Let Γ be any of the following curves:*

(i) *the graph of $\psi(t) = |t|^\alpha,\ t \in \mathbb{R},\ \alpha > 0$;*

(ii) *a hyperbola;*

(iii) *a polygon;*

(iv) *an ellipse.*

Then there exists a set $E \subset (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ of positive Lebesgue measure such that for $(\theta_1, \theta_2) \in E$, the pair $(\Gamma, L_{\theta_1} \cup L_{\theta_2})$ is a HUP.

In the thesis, we explored the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle, cross, exponential curves, and surfaces. Then, we prove a characterization of the Heisenberg uniqueness pairs corresponding to four parallel lines.

We observe that the size of the determining sets Λ for $X(\Gamma)$ depends on the number of lines and their irregular distribution that further relates to a phenomenon of interlacing of the zero sets of certain trigonometric polynomials.

In an interesting article, M. Benedicks [7] had extended the classical Paley-Wiener theorem to the class of integrable functions. Let $\hat{f}$ denote the Fourier transform of $f \in L^1(R^n)$. Let $\mathcal{E} = \{x \in \mathbb{R}^n : f(x) \neq 0\}$ and $\mathcal{F} = \{\xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0\}$. If both the sets $\mathcal{E}$ and $\mathcal{F}$ have finite Lebesgue measure, then $f = 0$.

Thereafter, a series of analogous results to the Benedicks theorem and related problems had been explored in various set ups, including the Heisenberg group and the Euclidean motion groups (see [31–33, 37, 39, 40, 42]). In particular, Narayanan and Ratnakumar [31] had worked out an analogous result to the Benedicks theorem for the partial compactly supported functions on the Heisenberg group in terms of the finite rank of Fourier transform of the function. Further, in a recent article [52], Vemuri has relaxed the compact support condition on the function by the finite support. Since the group Fourier transform on the Heisenberg group is operator valued, the latter result seems close to the classical Benedicks theorem. However, it would be a good question to consider the case when the spectrum of the Fourier transform of an integrable function would be supported on a thin uncountable set.

In this thesis, our main focus is to explore analogue of Benedicks theorem on the Heisenberg motion group and step two nilpotent Lie groups. We prove that if the group Fourier transform of finitely supported certain integrable functions on the Euclidean motion groups (or Heisenberg motion group, step two nilpotent Lie groups) is of finite rank, then the function has to vanish identically.

The thesis is organized as follows:

In Chapter 2, we extensively consider the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle, cross, exponential curves, and certain exponential surfaces.

In Chapter 3, we prove a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of four parallel lines and in this case, we observe the phenomenon of interlacing of three totally disconnected dispensable sets. Let Γ denotes a system of four parallel straight lines in $\mathbb{R}^2$ which can be represented by $\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma, \delta\}$, where $\alpha < \beta < \gamma < \delta$ with $(\gamma - \alpha)/(\beta - \alpha) = 2$ and $(\delta - \alpha)/(\beta - \alpha) \in \mathbb{N}$. By the invariance properties of HUP, one can assume that $\Gamma = \mathbb{R} \times \{0, 1, 2, p\}$ for some $p \geq 3$.

Let $\Lambda \subset \mathbb{R}^2$ be a closed set which is 2-periodic with respect to the second variable. Suppose $\Pi(\Lambda)$ is dense in $\mathbb{R}$. If (Γ, Λ) is a Heisenberg uniqueness pair, then the set

$$\widetilde{\Pi}(\Lambda) = \Pi^4(\Lambda) \bigcup_{j=0}^{2} \left[\Pi^{(3-j)}(\Lambda) \smallsetminus \Pi^{(3-j)^*}(\Lambda)\right]$$

is dense in $\mathbb{R}$. Conversely, if the set

$$\widetilde{\Pi_p}(\Lambda) = \Pi^4(\Lambda) \bigcup_{j=2}^{3} \left[\Pi^j(\Lambda) \smallsetminus \Pi^p_{j^*}(\Lambda)\right] \cup \left[\Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)\right] \tag{1.0.3}$$

is dense in $\mathbb{R}$, then (Γ, Λ) is a Heisenberg uniqueness pair.

For the notations appeared in (1.0.3), we refer Chapter 3, Section 3.2.

In Chapter 4, we prove that if the Fourier transform of a integrable function on the Euclidean motion group $M(n)$ is compactly supported, then the function has to vanish identically. Further, we explore the Heisenberg uniqueness pairs for the Fourier transform on $M(n)$ as well as on the product group $G' = \mathbb{R}^n \times K$, where K is a compact group. We observed a one to one correspondence between the class of HUP's on $\mathbb{R}^n$ and the class of HUP's on G'.

In Chapter 5, we prove that if the group Fourier transform of certain integrable functions on the Heisenberg motion group (or step two nilpotent Lie groups) is of finite rank, then the function is identically zero. These results can be thought as an analogue to the Benedicks theorem that dealt with the uniqueness of the Fourier transform of integrable functions on the Euclidean spaces.

Chapter 2

Heisenberg uniqueness pairs for regular curves and surfaces

2.1 Introduction

In this chapter, we work out some of the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle, cross and certain exponential curves by using the basic tools of the Fourier analysis. Though, a complete characterization for the Heisenberg uniqueness pairs corresponding to either of the above algebraic curves is still unsolved.

2.1.1 HUPs corresponding to the spiral

First, we prove that the spiral is a Heisenberg uniqueness pair for the anti-spiral.

Theorem 2.1.1. *Suppose* $\Gamma = \{(e^{-t}\cos t, e^{-t}\sin t) : t \geq 0\}$ *is a spiral and let* $\Lambda = \{(e^{s}\cos s, e^{s}\sin s) : s \leq 0\}$. *Then* (Γ, Λ) *is a Heisenberg uniqueness pair.*

In order to prove Theorem 2.1.1, we need the following results from [6, 10].

Theorem 2.1.2. [10] *Let* h *be a bounded measurable function and* $g \in L^1(\mathbb{R}^n)$. *If* $h * g$ *vanishes identically, then* $\hat{h}$ *vanishes on the support of* $\hat{g}$.

Let $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq 0;\ j = 1, \ldots, n\}$. The following result had appeared in the article [6] by Bagchi and Sitaram, p. 421, as a part of the proof of Proposition 2.1.

Proposition 2.1.1. [6] *Let h be a non-zero bounded Borel measurable function which is supported on $\mathbb{R}_+^n$. Then supp $\hat{h} = \mathbb{R}^n$.*

Proof of Theorem 2.1.1. Since μ is absolutely continuous with respect to the arc length measure on Γ, by Radon-Nikodym theorem there exists $f \in L^1[0,\infty)$ such that $d\mu = \sqrt{2} f(t) e^{-t} dt$. Let $g(t) = \sqrt{2} f(t) e^{-t}$. Then by the finiteness of μ, it follows that $g \in L^1[0,\infty)$. By hypothesis, $\hat{\mu}|_\Lambda = 0$ implies

$$\hat{\mu}(\xi,\eta) = \int_0^\infty e^{-i\pi e^{-t}(\xi\cos t + \eta \sin t)} d\mu(t) = \int_0^\infty e^{-i\pi e^{(s-t)}\cos(t-s)} g(t) dt = 0 \qquad (2.1.1)$$

for all $(\xi,\eta) \in \Lambda$. Let $H(t) = e^{-i\pi e^t \cos t}\chi_{[0,\infty)}(t)$ and $G(t) = g(t)\chi_{(0,\infty)}(t)$. Then from (2.1.1), we get $\hat{\mu}(\xi,\eta) = (H * G)(s) = 0\ \forall\, s \in \mathbb{R}$. In view of Theorem 2.1.2, we infer that supp $\hat{H} \subset Z(\hat{G})$, where $Z(\hat{G})$ denotes the zero set of $\hat{G}$. As H is a non-zero bounded Borel measurable function supported in $[0,\infty)$, by Proposition 2.1.1 it follows that supp $\hat{H} = \mathbb{R}$ and hence $\hat{G} = 0$. Thus, $\mu = 0$. □

2.1.2 HUPs corresponding to the exponential curves

In this section, we work out some of the Heisenberg uniqueness pairs corresponding to certain exponential curves in the plane. Though, the result is true for a large class of exponential curves, for the sake of simplicity we prove only for a particular one.

Let μ be a finite Borel measure having support on $\Gamma = \{(t, e^{t^2}) : t \in \mathbb{R}\}$ which is absolutely continuous with respect to the arc length on Γ. Then there exists $f \in L^1(\mathbb{R})$

such that $d\mu = g(t)dt$, where $g(t) = f(t)\sqrt{1+4t^2e^{2t^2}}$. Hence by finiteness of μ, it follows that $g \in L^1(\mathbb{R})$ and

$$\hat{\mu}(x,y) = \int_{\mathbb{R}} e^{-i\pi\left(xt+ye^{t^2}\right)} g(t)dt. \tag{2.1.2}$$

Then $\hat{\mu}$ satisfies the PDE

$$\left(1+\mathcal{T}_x\right)\hat{\mu} = \alpha\partial_y\hat{\mu} \tag{2.1.3}$$

in the distributional sense, where $\mathcal{T}_x = \sum\limits_{n=0}^{\infty}\left(\sum\limits_{k=1}^{4}\dfrac{\alpha^{4n+k}}{(4n+k)!}\partial_x^{4n+k}\right)$ and $\alpha = \frac{i}{\pi}$.

Theorem 2.1.3. *Let $\Gamma = \{(t,\alpha(t)) : \ t\in\mathbb{R}\}$, where $\alpha : \mathbb{R}\to\mathbb{R}_+$ be defined by $\alpha(t) = e^{t^2}$.*

(i) Let Λ be a straight line. Then (Γ,Λ) is a Heisenberg uniqueness pair if and only if Λ is parallel to the x-axis.

(ii) Let $\Lambda = L_1\cup L_2$, where $L_j;\ j=1,2$ are any two straight lines parallel to the Y-axis. Then (Γ,Λ) is a HUP.

(iii) Let $L_j;\ j=1,2$ be two parallel lines which are not parallel to either of the axes. Then $(\Gamma, L_1\cup L_2)$ is a Heisenberg uniqueness pair.

In order to prove Theorem 2.1.3, we need the following two important results about the uniqueness of Fourier transform. First, we state a result which can be found in Havin and Jöricke [17], p. 36.

Lemma 2.1.1. [17] *If $\varphi \in L^1(\mathbb{R})$ is supported in $[0,\infty)$ and $\int\limits_{\mathbb{R}} \log|\hat{\varphi}|\frac{dx}{1+x^2} = -\infty$, then $\varphi = 0$.*

As a consequence of Lemma 2.1.1, we prove the following result.

Next, we state the following form of Radon-Nikodym derivative theorem (see [12], p.91).

Proposition 2.1.2. *Let ν be a σ-finite signed measure which is absolutely continuous with respect to a σ-finite measure μ on the measure space $(X, \mathcal{M})$. If $g \in L^1(\nu)$, then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and $\int g d\nu = \int g\frac{d\nu}{d\mu} d\mu$.*

As a consequence of Lemma 2.1.1 and Proposition 2.1.2, we prove the following result. Let $h_c : \mathbb{R} \to \mathbb{R}$ defined by $h_c(t) = \left(t + \frac{1}{2c}\right)^2 + e^{t^2} - t^2$, where c is a non-zero constant. Let $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}$.

Lemma 2.1.2. *Let $g \in L^1(\mathbb{R})$. Suppose $E \subset \mathbb{R}$ and $|E| > 0$. Then*

$$\int_{\mathbb{R}} e^{-i\pi c x h_c(t)} g(t) dt = 0 \tag{2.1.4}$$

for all $x \in E$ if and only if $\psi_c(u) = g \circ h_c^{-1}(u^2) \frac{2u}{h_c' \circ h_c^{-1}(u^2)}$ is an odd function.

Proof. By Proposition 2.1.2, we can write the left-hand side of (2.1.4) as

$$\begin{aligned} I &= \int_{\mathbb{R}} e^{-i\pi c x h_c(t)} g(t) dt \\ &= \int_{\mathbb{R}} e^{-i\pi c x u^2} g \circ h_c^{-1}(u^2) \frac{2u du}{h_c' \circ h_c^{-1}(u^2)} \\ &= \int_{\mathbb{R}} e^{-i\pi c x u^2} \psi_c(u) du. \end{aligned} \tag{2.1.5}$$

Hence by Proposition 2.1.2, it follows that $\psi_c \in L^1(\mathbb{R})$ and we get

$$\begin{aligned} I &= \int_0^\infty e^{-i\pi c x u^2} \psi_c(u) du + \int_0^\infty e^{-i\pi c x u^2} \psi_c(-u) du \\ &= \int_0^\infty e^{-i\pi c x u^2} F_c(u) du, \end{aligned}$$

where $F_c(u) = \psi_c(u) + \psi_c(-u)$ for all $u > 0$. Clearly $F_c \in L^1(0, \infty)$ and by the change of variables $u^2 = v$, we have

$$I = \int_0^\infty e^{-i\pi c x v} F_c(\sqrt{v}) \frac{dv}{2\sqrt{v}}. \tag{2.1.6}$$

Let $\varphi(v) = F_c(\sqrt{v})/2\sqrt{v}\ \chi_{(0,\infty)}(v)$. Then $\varphi \in L^1(\mathbb{R})$ and from (2.1.6) we obtain $I = \hat{\varphi}(cx) = 0$ for all $x \in E$. That is, $\hat{\varphi}$ vanishes on the set cE of positive measure. Thus, by Lemma 2.1.1 we conclude that $\varphi = 0$ a.e. Hence, it follows that $F_c = 0$ a.e. on $[0, \infty)$.

Conversely, if ψ_c is an odd function, then (2.1.4) trivially holds.

□

As a corollary to Lemma 2.1.2, we can derive the following result.

Corollary 2.1.1. *Let $g \in L^1(\mathbb{R})$ and $\alpha : \mathbb{R} \to \mathbb{R}_+$ be defined by $\alpha(t) = e^{t^2}$. Suppose $E \subset \mathbb{R}$ and $|E| > 0$. Then*

$$\int_{\mathbb{R}} e^{-i\pi x\alpha(t)} g(t)dt = 0 \tag{2.1.7}$$

for all $x \in E$ if and only if g is an odd function.

Proof. The left-hand side of Equation (2.1.7) can be expressed as

$$\begin{aligned} I &= \int_{-\infty}^{0} e^{-i\pi y\alpha(t)} g(t)dt + \int_{0}^{\infty} e^{-i\pi y\alpha(t)} g(t)dt \\ &= \int_{0}^{\infty} e^{-i\pi y\alpha(t)} (g(t) + g(-t))dt \\ &= \int_{0}^{\infty} e^{-i\pi y\alpha(t)} F(t)dt, \end{aligned}$$

where $F(t) = g(t) + g(-t)$ for all $t \geq 0$. Clearly $F \in L^1(0, \infty)$ and hence by the change of variables $u = \alpha(t)$, we have

$$I = \int_{1}^{\infty} e^{-i\pi xu} F(\sqrt{\log u}) \frac{du}{2u\sqrt{\log u}}. \tag{2.1.8}$$

Let $\varphi(u) = F(\sqrt{\log u})/2u\sqrt{\log u}\ \chi_{(1,\infty)}(u)$. Then $\varphi \in L^1(\mathbb{R})$ and from (2.1.8) we have $I = \hat{\varphi}(y) = 0$ for all $y \in E$. Since $\hat{\varphi}$ vanishes on a set E of positive Lebesgue measure, by Lemma 2.1.1 it follows that $\varphi = 0$. That is, $F = 0$ and hence g is an odd function.

Conversely, if g is an odd function, then (2.1.7) trivially holds. $\square$

Proposition 2.1.3. *Suppose $E \subset \mathbb{R}$ and $|E| > 0$. Assume $c, d \in \mathbb{R}$ with $c, d \neq 0$. Then (i) $\widehat{\mu}(x, cx) = 0$ for all $x \in E$ if and only if ψ_c is an odd function.*
(ii) $\widehat{\mu}(x, cx + d) = 0$ for all $x \in E$ if and only if $\phi_c(u) = \chi \circ h_c^{-1}(u^2)\dfrac{2u}{h_c' \circ h_c^{-1}(u^2)}$ is an odd function of u, where $\chi(t) = e^{-i\pi d e^{t^2}} g(t)$.

Proof. (i). From (2.1.2) we can express

$$\hat{\mu}(x, cx) = \int_{\mathbb{R}} e^{-i\pi x\left(t + ce^{t^2}\right)} g(t)dt = e^{i\pi x/4c} \int_{\mathbb{R}} e^{-i\pi cx h_c(t)} g(t)dt.$$

By Lemma 2.1.2, ψ_c is odd if and only if $\widehat{\mu}(x, cx) = 0$ for all $x \in E$.

(ii). By a simple computation, we get

$$\hat{\mu}(x, cx + d) = \int_{\mathbb{R}} e^{-i\pi x\left(t + ce^{t^2}\right)} \chi(t)dt = e^{i\pi x/4c} \int_{\mathbb{R}} e^{-i\pi cx h_c(t)} \chi(t)dt.$$

As similar to the above case, ϕ_c is odd if and only if $\widehat{\mu}(x, cx + d) = 0$ for all $x \in E$. $\square$

Proof of Theorem 2.1.3. (i). In view of the invariance property (i), we can assume that Λ is the x-axis. Recall that $\hat{\mu}$ satisfies

$$\hat{\mu}(x, y) = \int_{\mathbb{R}} e^{-i\pi\left(xt + ye^{t^2}\right)} g(t)dt.$$

Hence $\hat{\mu}|_\Lambda = 0$ implies that $\hat{g}(x) = 0$ for all $x \in \mathbb{R}$. Thus, we conclude that $\mu = 0$.

Conversely, suppose Λ is not parallel to the x-axis. If Λ is parallel to the y-axis, then by Corollary 2.1.1, it follows that (Γ, Λ) is not a HUP. Hence we can assume that Λ of the form $y = cx$, where $c \neq 0$. Choose a non-zero odd function $\varphi \in L^1(\mathbb{R})$ and let

$g(t) = \frac{\varphi(\sqrt{h_c(t)})h_c'(t)}{2\sqrt{h_c(t)}}$. Then by Proposition 2.1.3, it follows that (Γ, Λ) is not a Heisenberg uniqueness pair.

(ii). By invariance property (i), we can assume L_1 is the Y-axis and L_2 the line $x = x_o$, where $x_o \neq 0$. Since $\hat{\mu}$ vanishes on L_1, by Corollary 2.1.1 it follows that g is odd. Also, $\hat{\mu}$ vanishes on the line L_2 implies that

$$\int_{\mathbb{R}} e^{-i\pi(x_o t + ye^{t^2})} g(t)dt = 0$$

for all $y \in \mathbb{R}$. In view of Corollary 2.1.1, it follows that $e^{-i\pi x_o t}g(t)$ is an odd function. Hence $e^{-i\pi x_o t}g(t) = -e^{i\pi x_o t}g(-t)$. Since g is odd, it implies that $(e^{2i\pi x_o t} - 1)g(t) = 0$. As the identity $e^{2i\pi x_o t} = 1$ holds only for the countably many t, we conclude that $g = 0$. Thus, $\mu = 0$.

(iii). Let $L_1 = \{(x, cx) : x \in \mathbb{R}\}$ and $L_2 = \{(x, cx + d) : x \in \mathbb{R}\}$, where $c, d \neq 0$. Since $\hat{\mu}|_{L_j} = 0;\ j = 1, 2$, by Proposition 2.1.3, it follows that ψ_c and ϕ_c are odd functions. Let u_+ and u_- be the square roots of h_c. Since ϕ_c is an odd function, it implies that

$$\left[e^{i\pi d\left\{ e^{\left(h_c^{-1}(u_-^2)\right)^2} - e^{\left(h_c^{-1}(u_+^2)\right)^2} \right\}} - 1 \right] \psi_c(u) = 0.$$

That is, $\psi_c = 0$ a.e. and hence $g \circ h_c^{-1}(u^2) = 0$ almost all u. Thus, the pair $(\Gamma, L_1 \cup L_2)$ is a Heisenberg uniqueness pair. □

Remark 2.1. *Let $\alpha : \mathbb{R} \to \mathbb{R}_+$ be an even smooth function having finitely many local extrema and $\Gamma = \{(t, \alpha(t)) : t \in \mathbb{R}\}$. Then the conclusions of Theorem 2.1.3 would also hold.*

2.1.3 HUPs corresponding to the circle

In this section, we work out some of the Heisenberg uniqueness pairs corresponding to the circle. We show that (circle, spiral) is a HUP.

Let $\Gamma = S^1$ denote the unit circle in $\mathbb{R}^2$. If for $f \in L^1(\Gamma)$, we write $f(\theta)$ instead of $f(e^{i\theta})$, then f is a 2π periodic function and $f \in L^1[0, 2\pi)$. Let μ be a finite complex-valued Borel measure in $\mathbb{R}^2$ which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. Then there exists $f \in L^1(S^1)$ such that $d\mu = f(\theta)d\theta$. Now, we prove the following result.

Theorem 2.1.4. *Let $\Gamma = S^1$ and $\Lambda = \{(e^t \cos t, e^t \sin t) : t \leq 0\}$ be the spiral. Then (Γ, Λ) is a Heisenberg uniqueness pair.*

Proof. Since μ is supported on the unit circle Γ, we can write the Fourier transform of μ by

$$\hat{\mu}(x, y) = \int_{-\pi}^{\pi} e^{-i\pi(x\cos\theta + y\sin\theta)} f(\theta)d\theta.$$

Hence $\hat{\mu}$ can be extended holomorphically to $\mathbb{C}^2$. Thus, the function F defined by

$$F(z_1, z_2) = \int_{-\pi}^{\pi} e^{-i\pi(z_1\cos\theta + z_2\sin\theta)} f(\theta)d\theta,$$

is holomorphic on $\mathbb{C}^2$. In particular, $\hat{\mu} = F|_{\mathbb{R}^2}$ is a real analytic function. Since $\hat{\mu}$ vanishes on the spiral Λ, for any line L which passes through the origin, $\hat{\mu}|_{\Lambda\cap L} = 0$. As $(0, 0)$ is a limit point of the set $\Lambda \cap L$, it follows that $\hat{\mu}|_L = 0$. Since L is arbitrary, we infer that $\hat{\mu}(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$.

Let $S_r = \{(r\cos t, r\sin t) : 0 \leq t < 2\pi\}$, where $J_k(r) \neq 0$ for all $k \in \mathbb{Z}$. Then $\hat{\mu}(r\cos t, r\sin t) = 0$ implies $h * f(t) = 0$, where $h(t) = e^{-i\pi r\cos t}$. As we know that the Fourier coefficients of h satisfying $\hat{h}(k) = i^k(-1)^k J_k(r)$, it follows that $\hat{f}(k)J_k(r) = 0$

for all $k \in \mathbb{Z}$. Since $J_k(r) \neq 0$ for all $k \in \mathbb{Z}$, $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ and hence $f = 0$. $\square$

Remark 2.2. *A set which is determining set for any real analytic function is called NA - set. For instance, the spiral is an NA - set in the plane (see [34]). If μ is a finite Borel measure supported on a closed and bounded curve Γ, then $\hat{\mu}$ is real analytic. Thus, $(\Gamma, NA$ - set$)$ is a Heisenberg uniqueness pair. However, the converse is not true.*

Hence, in view of Remarks 1.2 and 2.2, we expect that the exceptional sets for the Heisenberg uniqueness pairs corresponding to the unit circle $\Gamma = S^1$ are eventually contained in the zero sets of all homogeneous harmonic polynomials union with the countably many circles whose radii are lying in the zero set of the certain class of Bessel functions. On the basis of these credible observations, we are trying to find out a complete characterization of the Heisenberg uniqueness pairs corresponding to circle which may be presented somewhere else.

2.1.4 HUPs corresponding to the hyperbola

In this section, we work out some of the Heisenberg uniqueness pairs corresponding to the hyperbola. Though in this case, Hedenmalm and Montes-Rodríguez [18] have found that some discrete set $\Lambda_{\alpha,\beta}$ is enough for $(\Gamma, \Lambda_{\alpha,\beta})$ to be a Heisenberg uniqueness pair. However, our approach is to consider those sets Λ which are essentially a union of continuous curves and located somewhere else than the set $\Lambda_{\alpha,\beta}$.

Theorem 2.1.5. *Let $\Gamma = \{(\cosh t, \sinh t) : t \geq 0\}$ be a branch of the hyperbola and $\Lambda = \{(\cosh s, -\sinh s) : s \in \mathbb{R}\}$. Then (Γ, Λ) is a HUP.*

Proof. Since μ is supported on Γ, there exists $f \in L^1[0, \infty)$ such that $d\mu = f(t)\sqrt{\cosh 2t}\, dt$.

If we write $g(t) = f(t)\sqrt{\cosh 2t}$, then $g \in L^1[0,\infty)$ and

$$\hat{\mu}(x,y) = \int_0^\infty e^{-i\pi(x\cosh t + y\sinh t)} g(t)dt.$$

By hypothesis, $\hat{\mu}|_\Lambda = 0$ implies

$$\hat{\mu}(x,y) = \int_0^\infty e^{-i\pi\cosh(t-s)} g(t)dt = 0 \tag{2.1.9}$$

for all $(x,y) \in \Lambda$. Let $H(t) = e^{-i\pi\cosh t}\chi_{[0,\infty)}(t)$ and $G(t) = g(t)\chi_{(0,\infty)}(t)$. Then from (2.1.9) we get $\hat{\mu}(x,y) = (H * G)(s) = 0$ for all $s \in \mathbb{R}$. In view of Theorem 2.1.2, it follows that supp $\hat{H} \subset Z(\hat{G})$. Hence by Proposition 2.1.1, supp $\hat{H} = \mathbb{R}$. Thus, we conclude that $G = 0$. □

Theorem 2.1.6. *Let $\Gamma = \{(\cosh t, \sinh t) : t \in \mathbb{R}\}$ and $\Lambda = L_1 \cup L_2$, where $L_j;\ j = 1,2$ are any two lines parallel to the X-axis. Then (Γ, Λ) is a HUP.*

We need the following result for proving Theorem 2.1.6.

Lemma 2.1.3. *Let $g \in L^1(\mathbb{R})$ and $E \subset \mathbb{R}$ such that $|E| > 0$. Then*

$$\int_{\mathbb{R}} e^{-i\pi x\cosh t} g(t)dt = 0 \tag{2.1.10}$$

for all $x \in E$ if and only if g is an odd function.

Proof. The left-hand side of (2.1.10) can be expressed as

$$\begin{aligned} I &= \int_{-\infty}^0 e^{-i\pi x\cosh t} g(t)dt + \int_0^\infty e^{-i\pi x\cosh t} g(t)dt \\ &= \int_0^\infty e^{-i\pi x\cosh t}(g(t) + g(-t))dt \\ &= \int_0^\infty e^{-i\pi x\cosh t} F(t)dt, \end{aligned}$$

where $F(t) = g(t)+g(-t)$ for all $t \geq 0$. Clearly $F \in L^1(0,\infty)$. By the change of variables $u = \cosh t$, we get

$$I = \int_1^\infty e^{-i\pi x u} F(\cosh^{-1} u) \frac{du}{\sqrt{u^2-1}}. \tag{2.1.11}$$

If we substitute $\varphi(u) = F(\cosh^{-1} u)/\sqrt{u^2-1}\,\chi_{(1,\infty)}$, then $\varphi \in L^1(\mathbb{R})$ and $I = \hat{\varphi}(x) = 0$ for all $x \in E$. Hence by Lemma 2.1.1, it follows that $\varphi = 0$. Thus, we infer that g is an odd function.

Conversely, suppose g is an odd function, then (2.1.10) trivially holds. □

Proof of Theorem 2.1.6. By invariance property (i), we can assume that L_1 is the X-axis and L_2 the line $y = y_o$, where $y_o \neq 0$. Since μ is supported on the hyperbola Γ, there exists $f \in L^1(\mathbb{R})$ such that $d\mu = f(t)\sqrt{\cosh 2t}\, dt$. Let $g(t) = f(t)\sqrt{\cosh 2t}$, then $g \in L^1(\mathbb{R})$. Hence in view of Lemma 2.1.3, $\hat{\mu}$ vanishes on L_1 implies that g is an odd function. Further, $\hat{\mu}|_{L_2} = 0$ implies that

$$\int_{\mathbb{R}} e^{-i\pi(x\cosh t + y_o \sinh t)} g(t)dt = 0$$

for all $x \in \mathbb{R}$. Then by Lemma 2.1.3 the function $e^{-i\pi y_o \sinh t} g(t)$ will be an odd function. Hence $e^{-i\pi y_o \sinh t} g(t) = -e^{i\pi y_o \sinh t} g(-t)$. As g is an odd function, it follows that $\left(e^{2i\pi y_o \sinh t} - 1\right) g(t) = 0$. Using the fact that $e^{2i\pi y_o \sinh t} = 1$ holds only for the countably many values of t, we conclude that $g = 0$. □

Theorem 2.1.7. *Let $\Gamma = \{(\cosh t, \sinh t) : \ t \in \mathbb{R}\}$ and $\Lambda = L_1 \cup L_2$, where $L_j;\ j = 1,2$ are any two straight lines which intersect at an angle $\alpha \in (0, \frac{\pi}{4})$. Then (Γ, Λ) is a HUP.*

Proof. Without loss of generality, we can assume that L_1 is the X-axis and $L_2 = \{(s\cosh t_o, -s\sinh t_o) : s \in \mathbb{R}\}$, where $\tan\alpha = -\tanh t_o$. Since μ is supported on the hyperbola Γ, as similar to Theorem 2.1.6 there exists $g \in L^1(\mathbb{R})$ such that $d\mu = g(t)dt$.

Suppose $\hat{\mu} = 0$ on Λ, then we have

$$\int_{\mathbb{R}} e^{-i\pi(x\cosh t + y\sinh t)} g(t)dt = 0$$

for all $(x, y) \in L_2$. This in turn implies

$$\int_{\mathbb{R}} e^{-i\pi s\cosh t} g(t + t_o)dt = 0$$

for all $s \in \mathbb{R}$. In view of Lemma 2.1.3, it follows that $g(t_o + \cdot)$ must be an odd function. Since $\hat{\mu}$ is also vanishing on the X-axis, g will be odd. Hence $g\,(2t_o \pm t) = g(t)$ for all $t \in \mathbb{R}$. That is, g is a periodic function contained in $L^1(\mathbb{R})$. Thus, we conclude that $g = 0$. $\square$

Remark 2.3. (**a**). *Let Γ be the hyperbola and Λ a straight line parallel to the X-axis. Then (Γ, Λ) is not a HUP. Consider $g(t) = \sqrt{\cosh 2t}\sin t\ \chi_{(-\pi,\pi)}(t)$ and $d\mu = g(t)dt$. Then $\hat{\mu}$ vanishes on Λ.*

(**b**). *We would like to mention that Theorem 2.1.7 is well contained in the case (ii) of Theorem 1.0.9 due to Jaming and Kellay* [20]. *However, our approach for a proof of Theorem 2.1.7 is quite different.*

2.2 HUPs corresponding to certain surfaces

The question of HUP in the higher dimension has been taken up by Vieli [53, 54] and worked out some of the HUP's corresponding to the sphere and the paraboloid.

The following is the main result of this section.

Theorem 2.2.1. *Let Γ be the surface $x_{n+1} = e^{x_1^2} + \cdots + e^{x_n^2}$ in $\mathbb{R}^{n+1}$ and Λ an affine* *hyperplane in $\mathbb{R}^{n+1}$ of dimension n. Then (Γ, Λ) is a Heisenberg uniqueness pair if and only if Λ is parallel to the hyperplane $x_{n+1} = 0$.*

For $u = (u_1, \ldots, u_n)$, denoting $\varphi(u) = e^{u_1^2} + \cdots + e^{u_n^2}$. Let μ be a finite Borel measure which is supported on $\Gamma = \{(u, \varphi(u)) : \; u \in \mathbb{R}^n\}$ and absolutely continuous with respect to the surface measure on Γ. Then by Radon-Nikodym theorem, there exists an integrable function f on $\mathbb{R}^n$ such that $d\mu = g(u)du$, where $g(u) = f(u)\sqrt{1 + |\text{grad } \varphi(u)|^2}$. Then by the finiteness of μ, it follows that $g \in L^1(\mathbb{R}^n)$. Denote $u' = (u_2, \ldots, u_n)$ and $x' = (x_2, \ldots, x_n)$. Then the Fourier transform of μ can be expressed as

$$\hat{\mu}(x) = \int_{\mathbb{R}^n} e^{-\pi i(x'.u' + x_{n+1}\varphi(u))} g(u) du \tag{2.2.1}$$

for $x \in \mathbb{R}^{n+1}$.

Proof of Theorem 2.2.1. Since Λ is an affine hyperplane in $\mathbb{R}^{n+1}$ of dimension n, by the invariance properties of HUP, we can assume that Λ is a linear subspace of $\mathbb{R}^{n+1}$ which can be considered as either $x_{n+1} = cx_1$, where $c \in \mathbb{R}$ or $x_1 = 0$.

If $\Lambda = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$, then by the hypothesis, $\hat{\mu}|_\Lambda = 0$ implies $\hat{g} = 0$ on $\mathbb{R}^n$. Thus, it follows that (Γ, Λ) is a HUP.

Conversely, suppose Λ is not parallel to the hyperplane $x_{n+1} = 0$. Consider a non-zero compactly supported odd function $\psi \in L^1(\mathbb{R})$ together with a non-zero compactly supported function $h \in L^1(\mathbb{R}^{n-1})$. Then we have the following two cases.

(i). If $\Lambda = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = 0\}$ and $g(u) = \psi(u_1)h(u')$, then for $x \in \Lambda$, we have

$$\begin{aligned}
\hat{\mu}(x) &= \int_{\mathbb{R}^n} e^{-\pi i(x'.u'+x_{n+1}\varphi(u))} g(u)du \\
&= \int_{\mathbb{R}^n} e^{-\pi i(x'.u'+x_{n+1}\varphi(u))} \psi(u_1)h(u')du \\
&= \int_{\mathbb{R}^{n-1}} e^{-\pi i\left\{x'.u'+x_{n+1}\left(\varphi(u)-e^{u_1^2}\right)\right\}} \left(\int_{\mathbb{R}} e^{-\pi i x_{n+1}e^{u_1^2}} \psi(u_1)du_1\right) h(u')du' \\
&= 0.
\end{aligned}$$

Thus, (Γ, Λ) is not a Heisenberg uniqueness pair.

(ii). Suppose $\Lambda = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = cx_1, c \neq 0\}$. Consider a function $\tau(t) = \frac{\psi\left(\sqrt{h_c(t)}\right)h_c'(t)}{2\sqrt{h_c(t)}}$ and write $g(u) = \tau(u_1)h(u')$. If we denote $x'' = (x_1, \ldots, x_n)$. Then for $x \in \Lambda$, we have

$$\begin{aligned}
\hat{\mu}(x) &= \int_{\mathbb{R}^n} e^{-\pi i(x''.u+cx_1\varphi(u))} g(u)du \\
&= \int_{\mathbb{R}^n} e^{-\pi i(x''.u+cx_1\varphi(u))} \tau(u_1)h(u')du \\
&= \int_{\mathbb{R}^{n-1}} e^{-\pi i\left\{x'.u'+cx_1\left(\varphi(u)-e^{u_1^2}\right)\right\}} \left(\int_{\mathbb{R}} e^{-\pi i\left(x_1u_1+cx_1e^{u_1^2}\right)} \tau(u_1)du_1\right) h(u')du'.
\end{aligned}$$

By Lemma 2.1.2, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\pi i x_1(u_1+ce^{u_1^2})} \tau(u_1)du_1 &= \int_{\mathbb{R}} e^{-\pi i c x_1 h_c(u_1)} \tau(u_1)du_1 \\
&= \int_{\mathbb{R}} e^{-\pi i c x_1 t^2} \psi(t)dt \\
&= 0.
\end{aligned}$$

Thus, we conclude that (Γ, Λ) is not a Heisenberg uniqueness pair. ☐

2.3 $\mathcal{C}$- Heisenberg uniqueness pair

Let $\Gamma = (\mathbb{R} \times F_n) \cup (F_n \times \mathbb{R})$, where $F_n = \{0, 1, \ldots, n\}$. Consider a subset $\mathcal{C} \subset X(\Gamma)$ such that for any $\mu \in \mathcal{C}$, there exist $f_j \in L^1(\mathbb{R});\ j \in F_n$ satisfying

$$d\mu(x, y) = \sum_{k=0}^{n} \left(f_k(x)dxd\delta_k(y) - f_k(y)d\delta_k(x)dy\right), \tag{2.3.1}$$

where δ_k denotes the point mass measure at k. Then the pair (Γ, Λ) is said to be a $\mathcal{C}$-HUP if any measure $\mu \in \mathcal{C}$ satisfies $\hat{\mu}|_\Lambda = 0$, implies μ is identically zero.

For $n = 0$, let $\alpha_j \in \mathbb{R};\ j = 1, 2$ be independent over $\mathbb{Q}$, the set of rational numbers. Consider the lines $L_{\alpha_j} = \{(\xi, \eta) \in \mathbb{R}^2 : \eta - \xi = \alpha_j\};\ j = 1, 2$.

Theorem 2.3.1. *Let $\Gamma = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ and $\Lambda = L_{\alpha_1} \cup L_{\alpha_2}$, where $\alpha_j;\ j = 1, 2$ are independent over $\mathbb{Q}$. Then (Γ, Λ) is a $\mathcal{C}$-HUP.*

Proof. For $\mu \in \mathcal{C}$ there exist $f \in L^1(\mathbb{R})$ such that

$$d\mu(x, y) = f(x)dxd\delta_o(y) - f(y)d\delta_o(x)dy.$$

By taking the Fourier transform of both the sides, we get

$$\begin{aligned}
\hat{\mu}(\xi, \eta) &= \int_\Gamma e^{-i\pi(x\xi + y\eta)} d\mu(x, y) \\
&= \int_{\mathbb{R}} e^{-i\pi x\xi} f(x)dx - \int_{\mathbb{R}} e^{-i\pi y\eta} f(y)dy \\
&= \hat{f}(\xi) - \hat{f}(\eta).
\end{aligned}$$

Now, $\hat{\mu}|_{L_{\alpha_1}} = 0$ implies that $\hat{f}(t) = \hat{f}(t + \alpha_1)$ for all $t \in \mathbb{R}$. Using the invariance property (ii), we can assume that $\alpha_j > 0;\ j = 1, 2$. Hence, $\hat{f}$ is a periodic function with some period $p_o > 0$. By the simple recursions, we can derive that $\hat{f}(t) = \hat{f}(t + m\alpha_1)$ for all

$m \in \mathbb{Z}$ and $t \in \mathbb{R}$. Similarly, by the condition $\hat{\mu}|_{L_{\alpha_2}} = 0$, it follows that $\hat{f}(t) = \hat{f}(t+n\alpha_2)$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Thus, we infer that

$$\hat{f}(t) = \hat{f}(t + m\alpha_1 + n\alpha_2)$$

whenever, $m, n \in \mathbb{Z}$ and $t \in \mathbb{R}$. By Krönecker approximation theorem (see [3]), we can always find $m_o, n_o \in \mathbb{Z}$ such that $|m_o\alpha_1 + n_o\alpha_2| < p_o$, which contradicts the fact that p_o is the period of f. This, in turn, implies that $\hat{f}$ must be a constant function. Thus, by Riemann-Lebesgue lemma $f = 0$. □

Remark 2.4. (i). *Let* $\Gamma_o = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ *and* $\Lambda = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \alpha\xi,\ |\alpha| < 1\}$. *Then by Riemann-Lebesgue lemma, it can be easily deduced that* (Γ_o, Λ) *is a* $\mathcal{C}$*-HUP. On the other hand, if* $\Lambda \subseteq \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \xi\}$, *then* (Γ_o, Λ) *is not a* $\mathcal{C}$*-HUP. Next, for* $\Lambda = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \xi^2\}$, *one can easily verify that* (Γ_o, Λ) *is a* $\mathcal{C}$*-HUP.*

(ii). *Consider* $\Gamma = (\mathbb{R} \times F_n) \cup (F_n \times \mathbb{R})$ *and* $\Lambda = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \xi\}$. *Then* (Γ, Λ) *is not a* $\mathcal{C}$*-HUP. For this, we can choose non-zero functions* $f_j \in L^1(\mathbb{R});\ j \in F_n$ *such that*

$$\hat{\mu}(\xi, \eta) = \sum_{k=0}^{n} \left(e^{-ik\pi\eta} \hat{f}_k(\xi) - e^{-ik\pi\xi} \hat{f}_k(\eta) \right) \tag{2.3.2}$$

and hence $\hat{\mu}(\xi, \eta) = 0$, *whenever* $\xi = \eta$. *For* $n = 1$, *if* Λ *is the* ξ*-axis, then* $(\hat{f}_0 + \hat{f}_1)(\xi) = \hat{f}_0(0) + e^{-i\pi\xi}\hat{f}_1(0)$ *for all* $\xi \in \mathbb{R}$. *By Riemann-Lebesgue lemma, it is enough to consider* $f_1 = -f_0$ *and* $\hat{f}_0(0) = 0$. *Thus,* (Γ, Λ) *is not a* $\mathcal{C}$*-HUP.*

(iii). In view of injectivity of the Fourier transform on $L^1(\mathbb{R})$, for a pair (Γ_o, Λ) to be a $\mathcal{C}$-HUP it is necessary that at least one of the orthogonal projections $\pi_j(\Lambda);\ j = 1, 2$ on the axes must be dense in $\mathbb{R}$. It would be an interesting question to get a sufficient condition for the $\mathcal{C}$-Heisenberg uniqueness pairs corresponding to Γ_o.

Chapter 3

Heisenberg uniqueness pairs for four parallel lines

3.1 Introduction

A characterization of the Heisenberg uniqueness pairs corresponding to any two parallel straight lines has been done by Hedenmalm and Montes-Rodríguez [18]. Further, Babot [5] has worked out an analogous result for a certain system of three parallel lines. In this chapter, we prove a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of four parallel lines. In the above case, we observe the phenomenon of interlacing of three totally disconnected sets.

3.2 Preliminaries and auxiliary results

Let Γ_o denote a system of four parallel lines that can be expressed as $\Gamma_o = \mathbb{R} \times \{\alpha, \beta, \gamma, \delta\}$, where $\alpha < \beta < \gamma < \delta$, $p = (\delta - \alpha)/(\beta - \alpha) \in \mathbb{N} \smallsetminus \{1, 2\}$ and $(\gamma - \alpha)/(\beta - \alpha) = 2$. If (Γ_o, Λ_o) is a HUP, then by using invariance property (i), (Γ_o, Λ_o) can be reduced to $(\Gamma_o - (0, \alpha), \Lambda_o)$. Since scaling can be thought as a diagonal matrix, by using invariance property (ii), $(\Gamma_o - (0, \alpha), \Lambda_o)$ can be reduced to $(T^{-1}(\Gamma_o - (0, \alpha)), T^{*}\Lambda_o)$, where $T = \text{diag}\{(\beta-\alpha),(\beta-\alpha)\}$. Let $\Lambda = T^{*}\Lambda_o$ and $\Gamma = T^{-1}(\Gamma_o - (0,\alpha))$. Then $\Gamma = \mathbb{R}\times\{0,1,2,p\}$, where $p\in\mathbb{N}$ with $p\geq 3$. Thus, (Γ_o,Λ_o) is a HUP if and only if (Γ,Λ) is a HUP.

Before we state our main result of this section, we need to set up some necessary notations and the subsequent auxiliary results.

Let μ be a finite Borel measure which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. Then there exist functions $f_k\in L^1(\mathbb{R});\ k = 0,1,2,3$ such that

$$d\mu = f_0(x)dxd\delta_0(y) + f_1(x)dxd\delta_1(y) + f_2(x)dxd\delta_2(y) + f_3(x)dxd\delta_p(y), \tag{3.2.1}$$

where δ_t denotes the point mass measure at t. By taking the Fourier transform of both sides of (3.2.1) we get

$$\hat{\mu}(\xi,\eta) = \hat{f}_0(\xi) + e^{\pi i\eta}\hat{f}_1(\xi) + e^{2\pi i\eta}\hat{f}_2(\xi) + e^{p\pi i\eta}\hat{f}_3(\xi). \tag{3.2.2}$$

Notice that for each fixed $(\xi,\eta)\in\Lambda$, the right-hand side of (3.2.2) is a trigonometric polynomial of degree p that could have preferably some missing terms. Therefore, it is an interesting question to find out the smallest set Λ that determines the above trigonometric polynomial. We observe that the size of Λ depends on the choice of a number of lines as well as irregular separation among themselves. That is, a larger number of lines or value of p would force smaller size of Λ. Eventually, the problem would become immensely difficult for a large value of p.

Observe that $\hat{\mu}$ is a 2-periodic function in the η-variable. Hence, for any set $\Lambda\subset\mathbb{R}^2$, it is enough to consider the set

$$\mathcal{L}(\Lambda) = \{(\xi, \eta) : (\xi, \eta + 2k) \in \Lambda, \text{ for some } k \in \mathbb{Z}\}$$

for the purpose of HUP. Also, it is easy to verify that (Γ, Λ) is a HUP if and only if $(\Gamma, \overline{\mathcal{L}(\Lambda)})$ is a HUP, where $\overline{\mathcal{L}(\Lambda)}$ denotes the closure of $\mathcal{L}(\Lambda)$ in $\mathbb{R}^2$. In view of the above facts, it is enough to work with the closed set $\Lambda \subset \mathbb{R}^2$ which is 2-periodic with respect to the second variable.

It is evident from the Riemann-Lebesgue lemma that the exponential functions, which appeared in (3.2.2), cannot be expressed as the Fourier transform of functions in $L^1(\mathbb{R})$. However, they can locally agree with the Fourier transform of functions in $L^1(\mathbb{R})$. Hence, in view of the condition $\hat{\mu}|_\Lambda = 0$, we can classify these related exponential functions.

Given a set $E \subset \mathbb{R}$ and a point $\xi \in E$, let I_ξ denote an interval containing ξ. We define three functions spaces in the following way.

(A). $L^{E,\xi}_{loc} = \{\psi : E \to \mathbb{C}$ such that there is an interval I_ξ and a function $\varphi \in L^1(\mathbb{R})$ which satisfies $\psi = \hat{\varphi}$ on $I_\xi \cap E\}$.

(B). $P^{1,2}[L^{E,\xi}_{loc}] = \{\psi : E \to \mathbb{C}$ such that there is an interval I_ξ and $\varphi_j \in L^1(\mathbb{R});\ j = 0, 1$ which satisfies $\psi^2 + \hat{\varphi}_1\psi + \hat{\varphi}_0 = 0$ on $I_\xi \cap E\}$.

Now, for $p \in \mathbb{N}$ with $p \geq 3$, we define the third functions space as follows.

(C). $P^{1,p}[L^{E,\xi}_{loc}] = \{\psi : E \to \mathbb{C}$ such that there is an interval I_ξ and functions $\varphi_j \in L^1(\mathbb{R});\ j = 0, 1, 2$ which satisfy $\psi^p + \hat{\varphi}_2\psi^2 + \hat{\varphi}_1\psi + \hat{\varphi}_0 = 0$ on $I_\xi \cap E\}$.

We will frequently use the following Wiener's lemma that plays a key role in the rest part of the arguments for proofs.

Lemma 3.2.1. [22] *Let* $\psi \in L^{E,\xi}_{loc}$ *and* $\psi(\xi) \neq 0$. *Then* $1/\psi \in L^{E,\xi}_{loc}$.

For more details, see [22], p.57.

In view of Lemma 3.2.1, we derive the following relation among the sets which are described by **(A)**, **(B)** and **(C)**. We would like to mention that the integral choice of p in Lemma 3.2.2 has been considered for a convenience.

Lemma 3.2.2. *For* $p \geq 3$, *the following inclusions hold.*

$$L^{E,\xi}_{loc} \subset P^{1,2}[L^{E,\xi}_{loc}] \subset P^{1,p}[L^{E,\xi}_{loc}]. \tag{3.2.3}$$

Proof. (**a**) If $\psi \in L^{E,\xi}_{loc}$, then by the Wiener's lemma $1/\psi \in L^{E,\xi}_{loc}$. By definition, there exist intervals I_1, I_2 containing ξ and functions $f, g \in L^1(\mathbb{R})$ such that $\psi = \hat{f}$ on $I_1 \cap E$ and $\frac{1}{\psi} = \hat{g}$ on $I_2 \cap E$. Hence we can extract an interval $I_3 \subset I_1 \cap I_2$ containing ξ such that $\psi^2 = \frac{\hat{f}}{\hat{g}}$ on $I_3 \cap E$. As $\hat{g}(\xi) \neq 0$, there exists an interval I_4 containing ξ and a function $h \in L^1(\mathbb{R})$ such that $\frac{1}{\hat{g}} = \hat{h}$ on $I_4 \cap E$. Further, we can extract an interval $I_5 \subset I_3 \cap I_4$ containing ξ such that

$$\psi^2 = \hat{f}\,\hat{h} = \widehat{f * h} = \hat{\varphi} \tag{3.2.4}$$

on $I_5 \cap E$, where $\varphi = f * h \in L^1(\mathbb{R})$. This implies $\psi^2 \in L^{E,\xi}_{loc}$. Hence by the induction argument, it can be shown that $\psi^p \in L^{E,\xi}_{loc}$, whenever $p \in \mathbb{N}$. Now, consider a function $f_0 \in L^1(\mathbb{R})$ such that $I_1 \subset \text{supp } \hat{f}_0$. Since $\psi = \hat{f}$ on $I_1 \cap E$, it follows that

$$\hat{f}_0 \psi = \hat{f}_0 \hat{f} = \widehat{f_0 * f} \tag{3.2.5}$$

on $I_1 \cap E$. Hence from (3.2.4) and (3.2.5) we conclude that

$$\psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0 \tag{3.2.6}$$

on $I_\xi \cap E$, where $I_\xi \subset I_1 \cap I_5$, $\varphi_0 = -(f_0 * f + \varphi)$ and $\varphi_1 = f_0$. Thus, $\psi \in P^{1,2}[L_{loc}^{E,\xi}]$. By applying induction, we can show that $\psi^p + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$, whenever $p \in \mathbb{N}$.

(**b**) If $\psi \in P^{1,2}[L_{loc}^{E,\xi}]$, then there exists an interval I_ξ containing ξ and functions $f, g \in L^1(\mathbb{R})$ such that

$$\psi^2 + \hat{f}\psi + \hat{g} = 0 \tag{3.2.7}$$

on $I_\xi \cap E$. Now, consider a function $f_0 \in L^1(\mathbb{R})$ such that $I_\xi \subset \text{supp } \hat{f}_0$. After multiplying (3.2.7) by ψ and $\hat{f}_0$ separately and adding the resultant equations, we can write

$$\psi^3 + \left(\hat{f}_0 + \hat{f}\right)\psi^2 + \left(\hat{f}_0 \hat{f} + \hat{g}\right)\psi + \hat{f}_0 \hat{g} = 0.$$

Hence for the appropriate choice of $\varphi_j;\ j = 0, 1, 2$, we have

$$\psi^3 + \hat{\varphi}_2 \psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0 \tag{3.2.8}$$

on $I_\xi \cap E$. Further by induction, it follows that $\psi^p + \hat{\varphi}_2 \psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$ on $I_\xi \cap E$, whenever $p \in \mathbb{N}$. Thus, $\psi \in P^{1,p}[L_{loc}^{E,\xi}]$. □

Let $\Pi(\Lambda)$ be the projection of Λ on $\mathbb{R} \times \{0\}$. For $\xi \in \Pi(\Lambda)$, we denote the corresponding image on the η - axis by

$$\Sigma_\xi = \{\eta \in [0, 2) : (\xi, \eta) \in \Lambda\}.$$

Now, we require analyzing the set $\Pi(\Lambda)$ to know its basic geometrical structure in accordance with the Heisenberg uniqueness pair. Since it is expected that the set Σ_ξ may consist one or more image points depending upon the order of its winding, the set $\Pi(\Lambda)$ can be decomposed into the following four disjoint sets. For the sake of convenience, we denote $F_o = \{0,1,2,3\}$.

$(\mathbf{P_1})$. $\Pi^1(\Lambda) = \{\xi \in \Pi(\Lambda) :$ there is a unique $\eta_0 \in \Sigma_\xi\}$.

$(\mathbf{P_2})$. $\Pi^2(\Lambda) = \{\xi \in \Pi(\Lambda) :$ there are only two distinct $\eta_j \in \Sigma_\xi;\ j = 0,1\}$.

In order to describe the rest of the two partitioning sets, we will use the notion of the symmetric polynomial. For each $k \in \mathbb{Z}_+$, the complete homogeneous symmetric polynomial H_k of degree k is the sum of all monomials of degree k. That is,

$$H_k\left(x_1, \ldots, x_n\right) = \sum_{l_1+\cdots+l_n=k;\ l_i \geq 0} x_1^{l_1} \ldots x_n^{l_n}.$$

For more details, we refer to [28].

Consider four distinct image points $\eta_j \in [0,2)$ and denote $a_j = e^{\pi i \eta_j};\ j \in F_0$. For $p \geq 3$, we define the remaining two sets as follows:

$(\mathbf{P_3})$. $\Pi^3(\Lambda) = \{\xi \in \Pi(\Lambda) :$ there are at least three distinct $\eta_j \in \Sigma_\xi$ for $j = 0,1,2$ and if there is another $\eta_3 \in \Sigma_\xi$, then $H_{p-2}(a_0,a_1,a_2) = H_{p-2}(a_0,a_1,a_3)\}$.

$(\mathbf{P_4})$. $\Pi^4(\Lambda) = \{\xi \in \Pi(\Lambda) :$ there are at least four distinct $\eta_j \in \Sigma_\xi;\ j \in F_o$ which satisfy $H_{p-2}(a_0,a_1,a_2) \neq H_{p-2}(a_0,a_1,a_3)\}$.

In this way, we get the desired decomposition as $\Pi(\Lambda) = \bigcup_{j=1}^{4} \Pi^j(\Lambda)$.

Now, for three distinct image points $\eta_j \in [0,2);\ j = 0,1,2$, denote $a = e^{\pi i \eta_0}$,

$b = e^{\pi i \eta_1}$ and $c = e^{\pi i \eta_2}$. Consider the system of equations $A^3_\xi X = B^p_\xi$, where

$$A^3_\xi = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}, \tag{3.2.9}$$

$X_\xi = (\tau_0, \tau_1, \tau_2)$ and $B^p_\xi = -(a^p, b^p, c^p)$. Since $\det A^3_\xi = (a-b)(b-c)(c-a) \neq 0$, $A^3_\xi X = B^p_\xi$ has a unique solution. A simple calculation gives

$$\begin{aligned} \tau_o &= -abcH_{p-3}(a,b,c), \\ \tau_1 &= H_{p-1}(a,b,c) - \left(a^{p-1} + b^{p-1} + c^{p-1}\right) + \sum_{\substack{l+m+n=p-1 \\ l,m,n\geq 1}} a^l b^m c^n, \\ \tau_2 &= -H_{p-2}(a,b,c). \end{aligned} \tag{3.2.10}$$

Since measure in the question is supported on a certain system of four parallel lines and the exponential functions which have appeared in (3.2.2) can locally agree with the Fourier transform of some functions in $L^1(\mathbb{R})$, the following sets sitting in $\Pi(\Lambda)$ seems to be dispensable in the process of getting the Heisenberg uniqueness pairs.

$(\mathbf{P_{1^*}})$. As each $\xi \in \Pi^1(\Lambda)$ has a unique image in Σ_ξ, we can define a function χ_0 on $\Pi^1(\Lambda)$ by $\chi_0(\xi) = e^{\pi i \eta_0}$, where $\eta_0 = \eta_0(\xi) \in \Sigma_\xi$. Now, the first dispensable set can be defined by

$$\Pi^{1^*}(\Lambda) = \left\{\xi \in \Pi^1(\Lambda) : \chi_0 \in P^{1,p}[L^{\Pi^1(\Lambda),\xi}_{loc}]\right\}.$$

Next, for $\xi \in \Pi^2(\Lambda)$, let $\chi_j(\xi) = e^{\pi i \eta_j}$, where $\eta_j = \eta_j(\xi) \in \Sigma_\xi;\ j = 0, 1.$

$(\mathbf{P'_{2^*}})$. Since each $\xi \in \Pi^2(\Lambda)$ has two distinct image points in Σ_ξ, we define two functions

δ_j on $\Pi^2(\Lambda); j=0,1$ such that $X_\xi = (\delta_0(\xi), \delta_1(\xi))$ is the solution of $A^2_\xi X_\xi = B^2_\xi$, where

$$A^2_\xi = \begin{pmatrix} 1 & \chi_0(\xi) \\ 1 & \chi_1(\xi) \end{pmatrix}$$

and $B^2_\xi = -\left(\chi_0(\xi)^2, \chi_1(\xi)^2\right)$. In this way, an auxiliary dispensable set can be defined by

$$\Pi^{2^*}(\Lambda) = \left\{\xi \in \Pi^2(\Lambda) : \delta_j \in L^{\Pi^2(\Lambda),\xi}_{loc};\ j=0,1\right\}.$$

($\mathbf{P''_{2^*}}$). Further, we define three functions ρ_j on $\Pi^2(\Lambda)$; $j=0,1,2$ such that $X_\xi = (\rho_0(\xi), \rho_1(\xi), \rho_2(\xi))$ becomes a solution of $A^p_\xi X_\xi = B^p_\xi$, where

$$A^p_\xi = \begin{pmatrix} 1 & \chi_0(\xi) & \chi_0(\xi)^2 \\ 1 & \chi_1(\xi) & \chi_1(\xi)^2 \end{pmatrix}$$

and $B^p_\xi = -\left(\chi_0(\xi)^p, \chi_1(\xi)^p\right)$. Hence the second dispensable set can be defined by

$$\Pi^p_{2^*}(\Lambda) = \left\{\xi \in \Pi^2(\Lambda) : \rho_j \in L^{\Pi^2(\Lambda),\xi}_{loc};\ j=0,1,2\right\}.$$

For $\xi \in \Pi^3(\Lambda)$, let $\chi_j(\xi) = e^{\pi i \eta_j}$, where $\eta_j = \eta_j(\xi) \in \Sigma_\xi;\ j=0,1,2$.

($\mathbf{P'_{3^*}}$). For each $\xi \in \Pi^3(\Lambda)$ has three distinct image points in Σ_ξ, we define three functions e_j on $\Pi^3(\Lambda)$; $j=0,1,2$ such that $X_\xi = (e_0(\xi), e_1(\xi), e_2(\xi))$ is the solution of $A^3_\xi X_\xi = B^3_\xi$, where A^3_ξ is the matrix given by Equation (3.2.9) and $B^3_\xi = -\left(\chi_0(\xi)^3, \chi_1(\xi)^3, \chi_2(\xi)^3\right)$. Hence another auxiliary dispensable set can be defined by

$$\Pi^{3^*}(\Lambda) = \left\{\xi \in \Pi^3(\Lambda) : e_j \in L^{\Pi^3(\Lambda),\xi}_{loc};\ j=0,1,2\right\}.$$

($\mathbf{P}''_{\mathbf{3}^*}$). Once again we define three functions τ_j on $\Pi^3(\Lambda)$; $j=0,1,2$ such that $X_\xi = (\tau_0(\xi),\tau_1(\xi),\tau_2(\xi))$ is the solution of $A^3_\xi X_\xi = B^p_\xi$, where $B^p_\xi = -\left(\chi_0(\xi)^p, \chi_1(\xi)^p, \chi_2(\xi)^p\right)$. Hence the third dispensable set can be defined by

$$\Pi^p_{3^*}(\Lambda) = \left\{\xi \in \Pi^3(\Lambda) : \tau_j \in L^{\Pi^3(\Lambda);\xi}_{loc};\ j=0,1,2\right\}.$$

Now, we prove the following two lemmas that speak about a sharp contrast in the pattern of dispensable sets as compared to dispensable sets which appeared in two lines and three lines results. That is, a larger value of p will increase the size of dispensable sets in case of four lines problem. Further, we observe that dispensable sets are eventually those sets contained in $\Pi(\Lambda)$ where we could not solve (3.2.2). For more details, we refer to [5, 18].

Lemma 3.2.3. *For $p \geq 3$, the following inclusion holds.*

$$\Pi^{2^*}(\Lambda) \subset \Pi^p_{2^*}(\Lambda).$$

Proof. If $\xi_o \in \Pi^{2^*}(\Lambda)$, then $\delta_j \in L^{\Pi^2(\Lambda),\xi_o}_{loc}$. Hence there exists an interval I_{ξ_o} containing ξ_o and $\varphi_j \in L^1(\mathbb{R})$ such that $\delta_j = \hat{\varphi}_j$; $j=0,1$ satisfy

$$\hat{\varphi}_0 + \hat{\varphi}_1\chi_j + {\chi_j}^2 = 0$$

on $I_{\xi_o} \cap \Pi^2(\Lambda)$, whenever $j=0,1$. Now, by the similar iteration as in the proof of Lemma 3.2.2(b), we infer that there exist a common set of $\psi_j \in L^1(\mathbb{R})$; $j=0,1,2$ such that

$$\hat{\psi}_0 + \hat{\psi}_1\chi_j + \hat{\psi}_2{\chi_j}^2 + {\chi_j}^p = 0$$

on $I_{\xi_o} \cap \Pi^2(\Lambda)$, whenever $j=0,1$. If we denote $\hat{\psi}_j = \rho_j$, then it is easy to see that $\xi_o \in \Pi^p_{2^*}(\Lambda)$. ∎

Lemma 3.2.4. *For $p \geq 3$, the following inclusion holds.*

$$\Pi^{3^*}(\Lambda) \subseteq \Pi^p_{3^*}(\Lambda).$$

Moreover, equality holds for $p = 3$.

Proof. If $\xi_o \in \Pi^{3^*}(\Lambda)$, then $e_j \in L^{\Pi^3(\Lambda),\xi_o}_{loc}$. Hence there exists an interval I_{ξ_o} containing ξ_o and $\varphi_j \in L^1(\mathbb{R})$ such that $e_j = \hat{\varphi}_j;\ j = 0, 1, 2$ satisfy

$$\hat{\varphi}_0 + \hat{\varphi}_1\chi_j + \hat{\varphi}_2{\chi_j}^2 + {\chi_j}^3 = 0$$

on $I_{\xi_o} \cap \Pi^3(\Lambda)$, whenever $j = 0, 1, 2$. By the similar iteration as in the proof of Lemma 3.2.2(b), it follows that there exist $\psi_j \in L^1(\mathbb{R});\ j = 0, 1, 2$ such that

$$\hat{\psi}_0 + \hat{\psi}_1\chi_j + \hat{\psi}_2{\chi_j}^2 + {\chi_j}^p = 0$$

on $I_{\xi_o} \cap \Pi^3(\Lambda)$, whenever $j = 0, 1, 2$. If we denote $\hat{\psi}_j = \tau_j$, then it is easy to verify that $\xi_o \in \Pi^p_{3^*}(\Lambda)$. □

On the basis of structural properties of the dispensable sets, we observe that these sets are essentially minimizing the size of projection $\Pi(\Lambda)$. Now, we can state our main result of this chapter about the Heisenberg uniqueness pairs corresponding to the above described system of four parallel straight lines.

3.3 The main result

Theorem 3.3.1. *Let $\Gamma = \mathbb{R} \times \{0, 1, 2, p\}$, where $p \in \mathbb{N}$ and $p \geq 3$. Let $\Lambda \subset \mathbb{R}^2$ be a closed set which is 2-periodic with respect to the second variable. Suppose $\Pi(\Lambda)$ is dense* *in* $\mathbb{R}$. *If* (Γ, Λ) *is a Heisenberg uniqueness pair, then the set*

$$\widetilde{\Pi}(\Lambda) = \Pi^4(\Lambda) \bigcup_{j=0}^{2} \left[\Pi^{(3-j)}(\Lambda) \smallsetminus \Pi^{(3-j)^*}(\Lambda)\right]$$

is dense in $\mathbb{R}$. *Conversely, if the set*

$$\widetilde{\Pi_p}(\Lambda) = \Pi^4(\Lambda) \cup \left[\Pi^3(\Lambda) \smallsetminus \Pi^p_{3^*}(\Lambda)\right] \cup \left[\Pi^2(\Lambda) \smallsetminus \Pi^p_{2^*}(\Lambda)\right] \cup \left[\Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)\right]$$

is dense in $\mathbb{R}$, *then* (Γ, Λ) *is a Heisenberg uniqueness pair.*

Remark 3.1. *In view of Lemma 3.2.3, we infer that* $\Pi^{2^*}(\Lambda)$ *is a proper subset of* $\Pi^p_{2^*}(\Lambda)$ *for any* $p \geq 3$. *However, for* $p = 3$, *Lemma 3.2.4 yields* $\Pi^{3^*}(\Lambda) = \Pi^p_{3^*}(\Lambda)$. *Hence for any* $p \geq 3$, *the set* $\widetilde{\Pi_p}(\Lambda)$ *is properly contained in* $\widetilde{\Pi}(\Lambda)$. *Thus, an analogous result for four lines problem as compared to three lines result is still unsolved.*

We need the following two lemmas which are required to prove the necessary part of Theorem 3.3.1. The main idea behind these lemmas is to pull down an interval from some of the partitioning sets of the projection $\Pi(\Lambda)$. The above argument helps to negate the assumption that $\widetilde{\Pi}(\Lambda)$ is not dense in $\mathbb{R}$.

Lemma 3.3.1. *Suppose* I *is an interval such that* $I \cap \Pi^{2^*}(\Lambda)$ *is dense in* I. *Then there exists an interval* $I' \subset I$ *such that* $I' \subset \bigcup_{j=2}^{4} \Pi^j(\Lambda)$.

Proof. If $\bar{\xi} \in I \cap \Pi^{2^*}(\Lambda)$, then $\delta_j \in L^{\Pi^2(\Lambda),\bar{\xi}}_{loc};\ j = 0, 1$. By hypothesis, $I \cap \Pi^{2^*}(\Lambda)$ is dense in I, therefore there exists an interval $I_{\bar{\xi}} \subset I$ containing $\bar{\xi}$ such that δ_j can be extended continuously on $I_{\bar{\xi}}$. In addition, δ_1 satisfies

$$|\delta_1(\bar{\xi})| = \left|e^{\pi i \bar{\eta}_0} + e^{\pi i \bar{\eta}_1}\right| < 2, \tag{3.3.1}$$

whenever $\bar{\xi} \in I \cap \Pi^{2^*}(\Lambda)$. Since δ_1 is continuous on $I_{\bar{\xi}}$, we can extract an interval $I' \subset I_{\bar{\xi}}$

containing $\bar{\xi}$ such that $|\delta_1(\xi)| < 2$ for all $\xi \in I'$.

Consequently, $I' \cap \Pi^{2^*}(\Lambda)$ is dense in I'. Now for $\xi \in I'$, there exists a sequence $\xi_n \in I' \cap \Pi^{2^*}(\Lambda)$ such that $\xi_n \to \xi$. Hence the corresponding image sequences $\eta_j^{(n)} \in \Sigma_{\xi_n} \subseteq [0,2)$ will have convergent subsequences, say $\eta_j^{(n_k)}$ which converge to η_j; $j = 0, 1$. Since the set Λ is closed, $(\xi, \eta_j) \in \Lambda$ for $j = 0, 1$. Now, we only need to show that $\eta_0 \neq \eta_1$. If possible, suppose $\eta_0 = \eta_1$, then by the continuity of δ_1 on I', it follows that $|\delta_1(\xi_n)| \to |\delta_1(\xi)|$. However,

$$|\delta_1(\xi_{n_k})| = \left| e^{\pi i \eta_0^{(n_k)}} + e^{\pi i \eta_1^{(n_k)}} \right| \to 2.$$

That is, $|\delta_1(\xi)| = 2$, which contradicts the fact that $|\delta_1(\xi)| < 2$ for all $\xi \in I'$. Thus, we infer that $I' \subset \bigcup_{j=2}^{4} \Pi^j(\Lambda)$. □

Lemma 3.3.2. *Let I be an interval such that $I \cap \Pi^{3^*}(\Lambda)$ is dense in I. Then there exists an interval $I' \subset I$ such that I' is contained in $\Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$.*

Proof. Let $\bar{\xi} \in I \cap \Pi^{3^*}(\Lambda)$, then $e_j \in L_{loc}^{\Pi^3(\Lambda),\bar{\xi}}$; $j = 0, 1, 2$. For $p = 3$, Equation (3.2.10) yields

$$(e_0, e_1, e_2) = (-abc, (ab + bc + ca), -(a + b + c)),$$

where $(a, b, c) = (\chi_0, \chi_1, \chi_2)$. Hence e_j; $j = 0, 1, 2$ are constant multiples of the elementary symmetric polynomials. Now, we define a function ρ on $\Pi^3(\Lambda)$ by

$$\rho = \left(a^3(b - c) + b^3(c - a) + c^3(a - b)\right)^2.$$

Since ρ is a symmetric polynomial in a, b, c, by the fundamental theorem of symmetric polynomials, ρ can be expressed as a polynomial in e_j; $j = 0, 1, 2$. Moreover, $\rho(\bar{\xi}) \neq 0$. Hence it follows that $\rho \in L_{loc}^{\Pi^3(\Lambda),\bar{\xi}}$. By hypothesis, $I \cap \Pi^{3^*}(\Lambda)$ is dense in I, there exists an interval $I_{\bar{\xi}} \subset I$ containing $\bar{\xi}$ such that ρ can be continuously extended on $I_{\bar{\xi}}$. Thus, by the continuity of ρ on $I_{\bar{\xi}}$, there exists an interval $J \subset I_{\bar{\xi}}$ containing $\bar{\xi}$ such that $\rho(\xi) \neq 0$ for all $\xi \in J$.

Consequently, $J \cap \Pi^{3^*}(\Lambda)$ is dense in J and hence for $\xi \in J$, there exists a sequence $\xi_n \in J \cap \Pi^{3^*}(\Lambda)$ such that $\xi_n \to \xi$. Thus, the corresponding image sequences $\eta_j^{(n)} \in \Sigma_{\xi_n} \subseteq [0,2)$ will have convergent subsequences, say $\eta_j^{(n_k)}$ which converge to $\eta_j;\ j = 0,1,2$. Since the set Λ is closed, $(\xi, \eta_j) \in \Lambda$ for $j = 0,1,2$.

Next, we claim that all of $\eta_j;\ j = 0,1,2$ are distinct. On the contrary, suppose all are equal or any two of them are equal. Then by the continuity of ρ on J, it follows that $\rho(\xi) = 0$, which contradicts the fact that $\rho(\xi) \neq 0$ for all $\xi \in J$. Hence we infer that $J \subset \bigcup_{j=3}^{4} \Pi^j(\Lambda)$. Further, using the facts that $e_j \in L_{loc}^{\Pi^3(\Lambda),\bar{\xi}}$ and $J \cap \Pi^{3^*}(\Lambda)$ is dense in J, e_j can be extended continuously on an interval $I' \subset J$ containing $\bar{\xi}$ such that $e_j(\xi) \neq 0$ for all $\xi \in I'$. That is, if $\xi \in I' \cap \Pi^3(\Lambda)$, then $e_j \in L_{loc}^{\Pi^3(\Lambda),\xi}$ and hence $\xi \in \Pi^{3^*}(\Lambda)$. Thus, we conclude that $I' \subset \Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$. □

3.4 Proof of Theorem 3.3.1

Proof of Theorem 3.3.1. We first prove the sufficient part of Theorem 3.3.1. Suppose the set $\widetilde{\Pi_p}(\Lambda)$ is dense in $\mathbb{R}$. Then we show that (Γ, Λ) is a Heisenberg uniqueness pair. For $\hat{\mu}|_\Lambda = 0$, we claim that $\hat{f}_k|_{\widetilde{\Pi_p}(\Lambda)} = 0$, whenever $k \in F_o$. Since $\hat{f}_k$ is a continuous function which vanishes on a dense set $\widetilde{\Pi_p}(\Lambda)$, it follows that $\hat{f}_k \equiv 0$ for all $k \in F_o$. Thus, $\mu = 0$.

As the projection $\Pi(\Lambda)$ is decomposed into the four pieces, the proof of the above assertion will be carried out in the following four cases.

$(\mathbf{S_1})$. If $\xi \in \Pi^4(\Lambda)$, then there exist at least four distinct $\eta_j \in \Sigma_\xi$ such that $\hat{\mu}(\xi, \eta_j) = 0$

for all $j \in F_o$. Hence $\hat{f}_k(\xi);\ k \in F_o$ satisfy a homogeneous system of four equations. As $\xi \in \Pi^4(\Lambda)$, by using the property that $H_{p-2}(a_0, a_1, a_2) \neq H_{p-2}(a_0, a_1, a_3)$, we infer that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$.

($\mathbf{S_2}$). If $\xi \in \Pi^3(\Lambda)$, then there exist at least three distinct $\eta_j \in \Sigma_\xi$ which satisfy $\hat{\mu}(\xi, \eta_j) = 0$; $j = 0, 1, 2$. If $\hat{f}_3(\xi) = 0$, then we get $\hat{f}_k(\xi) = 0$ for $k = 0, 1, 2$. On the other hand, if $\hat{f}_3(\xi) \neq 0$, then we can substitute

$$\hat{f}_j(\xi) = \tau_j(\xi)\hat{f}_3(\xi), \tag{3.4.1}$$

where τ_j are defined on $\Pi^3(\Lambda)$ for $j = 0, 1, 2$. Hence $X_\xi = (\tau_0(\xi), \tau_1(\xi), \tau_2(\xi))$ will satisfy the system of equations $A^3_\xi X_\xi = B^p_\xi$. By applying the Wiener lemma to Equations (3.4.1), we infer that $\tau_j \in L^{\Pi^3(\Lambda),\xi}_{loc};\ j = 0, 1, 2$. That is, $\xi \in \Pi^p_{3^*}(\Lambda)$. Thus for $\xi \in \Pi^3(\Lambda) \smallsetminus \Pi^p_{3^*}(\Lambda)$, we conclude that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$.

($\mathbf{S_3}$). If $\xi \in \Pi^2(\Lambda)$, then there exist two distinct $\eta_j \in \Sigma_\xi$ for which $\hat{\mu}(\xi, \eta_j) = 0$, whenever $j = 0, 1$. That is,

$$\hat{f}_0(\xi) + \chi_j(\xi)\hat{f}_1(\xi) + \chi^2_j(\xi)\hat{f}_2(\xi) + \chi^p_j(\xi)\hat{f}_3(\xi) = 0, \tag{3.4.2}$$

where $\chi_j(\xi) = e^{\pi i \eta_j};\ j = 0, 1$. If $\hat{f}_3(\xi) \neq 0$, then by applying the Wiener lemma to Equations (3.4.2), it follows that $\xi \in \Pi^p_{2^*}(\Lambda)$. That is, if $\xi \in \Pi^2(\Lambda) \smallsetminus \Pi^p_{2^*}(\Lambda)$, then $\hat{f}_3(\xi) = 0$.

Further, if $\hat{f}_3(\xi) = 0$ and $\hat{f}_2(\xi) \neq 0$, then an application of the Wiener lemma to Equations (3.4.2), it follows that $\xi \in \Pi^{2^*}(\Lambda)$. By Lemma 3.2.3, $\xi \in \Pi^p_{2^*}(\Lambda)$. Thus for $\xi \in \Pi^2(\Lambda) \smallsetminus \Pi^p_{2^*}(\Lambda)$, we infer that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$.

$(\mathbf{S_4})$. If $\xi \in \Pi^1(\Lambda)$, then there exists a unique $\eta_0 \in \Sigma_\xi$ for which $\hat{\mu}(\xi, \eta_0) = 0$. That is,

$$\hat{f}_0(\xi) + \chi_0(\xi)\hat{f}_1(\xi) + \chi_0^2(\xi)\hat{f}_2(\xi) + \chi_0^p(\xi)\hat{f}_3(\xi) = 0, \tag{3.4.3}$$

where $\chi_0(\xi) = e^{\pi i \eta_0}$. If $\hat{f}_3(\xi) \neq 0$, then by applying the Wiener lemma to Equation (3.4.3), it implies that $\chi_0 \in P^{1,p}[L_{loc}^{\Pi^1(\Lambda),\xi}]$. That is, $\xi \in \Pi^{1^*}(\Lambda)$. Thus for $\xi \in \Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)$, we have $\hat{f}_3(\xi) = 0$.

Further, if $\hat{f}_3(\xi) = 0$ and $\hat{f}_2(\xi) \neq 0$, then an application of the Wiener lemma to Equation (3.4.3), yields $\chi_0 \in P^{1,2}[L_{loc}^{\Pi^1(\Lambda),\xi}]$. By Lemma 3.2.2, it follows that $\xi \in \Pi^{1^*}(\Lambda)$. That is, if $\xi \in \Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)$, then $\hat{f}_k(\xi) = 0$ for $k = 2, 3$.

Finally, if $\hat{f}_k(\xi) = 0$ for $k = 2, 3$ and $\hat{f}_1(\xi) \neq 0$, then by applying the Wiener lemma to Equation (3.4.3), we infer that $\chi_0 \in L_{loc}^{\Pi^1(\Lambda),\xi}$. By Lemma 3.2.2, it follows that $\xi \in \Pi^{1^*}(\Lambda)$. Thus for $\xi \in \Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)$, we conclude that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$. ☐

Now, we prove the necessary part of Theorem 3.3.1. Suppose (Γ, Λ) is a Heisenberg uniqueness pair. Then we claim that the set $\widetilde{\Pi}(\Lambda)$ is dense in $\mathbb{R}$. We observe that this is possible if the dispensable sets $\Pi^{j^*}(\Lambda);\ j = 1, 2, 3$ interlace to each other, though these sets are disjoint among themselves.

If possible, suppose $\widetilde{\Pi}(\Lambda)$ is not dense in $\mathbb{R}$. Then there exists an open interval $I_o \subset \mathbb{R}$ such that $I_o \cap \widetilde{\Pi}(\Lambda)$ is empty. This, in turn, implies that

$$\Pi(\Lambda) \cap I_o = \left(\bigcup_{j=1}^{3} \Pi^{j^*}(\Lambda) \right) \cap I_o. \tag{3.4.4}$$

Thus from (3.4.4), it follows that I_o intersects only the dispensable sets. Now, the remaining part of the proof of Theorem 3.3.1 is a consequence of the following two lemmas which provide the interlacing property of the dispensable sets $\Pi^{j^*}(\Lambda);\ j =$

$1, 2, 3$.

Lemma 3.4.1. *There does not exist any interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J$ is contained in $\Pi^{j^*}(\Lambda)$; $j = 1, 2, 3$.*

Proof. On the contrary, suppose there exists an interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J \subset \Pi^{j^*}(\Lambda)$, for some $j \in \{1, 2, 3\}$. Since $\Pi^{j^*}(\Lambda)$; $j = 1, 2, 3$ are disjoint among themselves, there could be three possibilities.

(**a**). If $\xi \in \Pi(\Lambda) \cap J \subset \Pi^{1^*}(\Lambda)$, then $\chi_0 \in P^{1,p}[L_{loc}^{\Pi^1(\Lambda),\xi}]$. Hence there exists an interval $I_\xi \subset J$ containing ξ and $\varphi_k \in L^1(\mathbb{R})$; $k = 0, 1, 2$ such that

$$\chi_0^p + \hat{\varphi}_2 \chi_0^2 + \hat{\varphi}_1 \chi_0 + \hat{\varphi}_0 = 0$$

on $I_\xi \cap \Pi^1(\Lambda)$. Now, consider a function $f_3 \in L^1(\mathbb{R})$ such that $\hat{f}_3(\xi) \neq 0$ and supp $\hat{f}_3 \subset I_\xi$. Let $f_k = f_3 * \varphi_k$; $k = 0, 1, 2$. Then we can construct a Borel measure μ which is supported on Γ such that

$$\hat{\mu}(\overline{\xi}, \overline{\eta}) = \hat{f}_0(\overline{\xi}) + \chi_0(\overline{\xi}) \hat{f}_1(\overline{\xi}) + \chi_0^2(\overline{\xi}) \hat{f}_2(\overline{\xi}) + \chi_0^p(\overline{\xi}) \hat{f}_3(\overline{\xi}) = 0$$

for all $\overline{\xi} \in I_\xi \cap \Pi^{1^*}(\Lambda)$, where $\overline{\eta} \in \Sigma_{\overline{\xi}}$. Since (3.4.4) yields $I_\xi \cap \Pi(\Lambda) = I_\xi \cap \Pi^{1^*}(\Lambda)$, it implies that $\hat{\mu}|_\Lambda = 0$. However, μ is a non-zero measure which contradicts the fact that (Γ, Λ) is a HUP.

(**b**). If $\xi \in \Pi(\Lambda) \cap J \subset \Pi^{2^*}(\Lambda)$, then by Lemma 3.2.3, $\xi \in \Pi^p_{2^*}(\Lambda)$. Hence there exists an interval $I_\xi \subset J$ containing ξ and $\varphi_k \in L^1(\mathbb{R})$; $k = 0, 1, 2$ such that

$$\chi_j^p + \hat{\varphi}_2 \chi_j^2 + \hat{\varphi}_1 \chi_j + \hat{\varphi}_0 = 0$$

on $I_\xi \cap \Pi^2(\Lambda)$ for $j = 0, 1$. Let $f_3 \in L^1(\mathbb{R})$ be such that $\hat{f}_3(\xi) \neq 0$ and supp $\hat{f}_3 \subset I_\xi$.

Denote $f_k = f_3 * \varphi_k;\ k = 0,1,2$. Then we can construct a Borel measure μ that satisfies

$$\hat{\mu}(\bar{\xi}, \bar{\eta_j}) = \hat{f}_0(\bar{\xi}) + \chi_j(\bar{\xi})\hat{f}_1(\bar{\xi}) + \chi_j^2(\bar{\xi})\hat{f}_2(\bar{\xi}) + \chi_j^p(\bar{\xi})\hat{f}_3(\bar{\xi}) = 0$$

for all $\bar{\xi} \in I_\xi \cap \Pi^{2^*}(\Lambda)$ and $j = 0,1$. Since $I_\xi \cap \Pi(\Lambda) = I_\xi \cap \Pi^{2^*}(\Lambda)$, it follows that $\hat{\mu}|_\Lambda = 0$, though μ is a non-zero measure.

(**c**). If $\xi \in \Pi(\Lambda) \cap J \subset \Pi^{3^*}(\Lambda)$, then by Lemma 3.2.4, it follows that $\xi \in \Pi^p_{3^*}(\Lambda)$. As $\tau_k \in L^{\Pi^3(\Lambda),\xi}_{loc};\ k = 0,1,2$, there exists an interval $I_\xi \subset J$ containing ξ and $\varphi_k \in L^1(\mathbb{R})$ such that $\hat{\varphi}_k = \tau_k$ on $I_\xi \cap \Pi^3(\Lambda)$ for $k = 0,1,2$. Let $f_3 \in L^1(\mathbb{R})$ be such that $\hat{f}_3(\xi) \neq 0$ and supp $\hat{f}_3 \subset I_\xi$. Denote $f_k = f_3 * \varphi_k;\ k = 0,1,2$. Since $X_\xi = (\tau_0(\xi), \tau_1(\xi), \tau_2(\xi))$ satisfies $A^3_\xi X_\xi = B^p_\xi$, we have

$$\tau_0 + \chi_j\tau_1 + \chi_j^2\tau_2 + \chi_j^p = 0$$

on $I_\xi \cap \Pi^3(\Lambda)$ for $j = 0,1,2$. Hence we can construct a Borel measure μ such that

$$\hat{\mu}(\bar{\xi}, \bar{\eta_j}) = \hat{f}_0(\bar{\xi}) + \chi_j(\bar{\xi})\hat{f}_1(\bar{\xi}) + \chi_j^2(\bar{\xi})\hat{f}_2(\bar{\xi}) + \chi_j^p(\bar{\xi})\hat{f}_3(\bar{\xi}) = 0$$

for all $\bar{\xi} \in I_\xi \cap \Pi^{3^*}(\Lambda)$ and $j = 0,1,2$. As $I_\xi \cap \Pi(\Lambda) = I_\xi \cap \Pi^{3^*}(\Lambda)$, we infer that $\hat{\mu}|_\Lambda = 0$, even though μ is a non-zero measure. □

The next lemma is to deal with the situation that any interval $J \subset I_o$ can not contain only the points of any pair of dispensable sets.

Lemma 3.4.2. *There does not exist any interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J$ is contained in $\Pi^{j^*}(\Lambda) \cup \Pi^{k^*}(\Lambda)\ \forall\ j \neq k$ and $j,k \in \{1,2,3\}$.*

Proof. On the contrary, suppose there exists an interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J \subset \Pi^{j^*}(\Lambda) \cup \Pi^{k^*}(\Lambda)$ for some $j \neq k$ and $j,k \in \{1,2,3\}$. Then we have the following three cases:

(a). If $\Pi(\Lambda) \cap J \subset \Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)$, then Equation (3.4.4) yields

$$J \cap \Pi(\Lambda) = J \cap \left(\Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)\right). \tag{3.4.5}$$

We claim that $J \cap \Pi^{2^*}(\Lambda)$ is dense in J. If possible, suppose there exists an interval $I \subset J$ such that $\Pi^{2^*}(\Lambda) \cap I = \varnothing$. Then from (3.4.5), we get $I \cap \Pi(\Lambda) = I \cap \Pi^{1^*}(\Lambda) \subset \Pi^{1^*}(\Lambda)$ which contradicts Lemma 3.4.1. By Lemma 3.3.1, there exists an interval $I' \subset J$ such that $I' \subset \bigcup_{j=2}^{4} \Pi^j(\Lambda)$. This contradicts the assumption that I_o intersects only the dispensable sets.

(b). If $\Pi(\Lambda) \cap J \subset \Pi^{1^*}(\Lambda) \cup \Pi^{3^*}(\Lambda)$, then $J \cap \Pi(\Lambda) = J \cap \left(\Pi^{1^*}(\Lambda) \cup \Pi^{3^*}(\Lambda)\right)$. As similar to the case **(a)**, $J \cap \Pi^{3^*}(\Lambda)$ is also dense in J. Hence by Lemma 3.3.2, there exists an interval $I' \subset J$ such that I' is contained in $\Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$. Thus in view of Lemma 3.4.1, we have arrived at a contradiction to the assumption that I_o intersects only the dispensable sets.

(c). If $\Pi(\Lambda) \cap J \subset \Pi^{2^*}(\Lambda) \cup \Pi^{3^*}(\Lambda)$, then $J \cap \Pi(\Lambda) = J \cap (\Pi^{2^*}(\Lambda) \cup \Pi^{3^*}(\Lambda))$. Hence it follows that $J \cap \Pi^{3^*}(\Lambda)$ is dense in J. By using Lemma 3.3.2, there exists an interval $I' \subset J$ such that I' is contained in $\Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$, which contradict the assumption that I_o intersects only the dispensable sets. □

Finally, since $\Pi(\Lambda)$ is a dense subset of $\mathbb{R}$, in view of Lemmas 3.4.1 and 3.4.2, the only possibility that any interval $J \subset I_o$ would intersect all the dispensable sets $\Pi^{j^*}(\Lambda)$; $j = 1, 2, 3$. We claim that $\Pi^{2^*}(\Lambda) \cap I_o$ is dense in I_o. Otherwise, there exists an interval $I \subset I_o$ such that $\Pi^{2^*}(\Lambda) \cap I = \varnothing$. Then from (3.4.4), we get $I \cap \Pi(\Lambda) \subset \left(\Pi^{1^*}(\Lambda) \cup \Pi^{3^*}(\Lambda)\right)$ which contradicts Lemma 3.4.2. Hence by Lemma 3.3.1, there exists an interval $I' \subset I_o$ such that I' is contained in $\Pi^2(\Lambda) \cup \Pi^3(\Lambda) \cup \Pi^4(\Lambda)$ which contradicts the assumption that I_o intersects only the dispensable sets.

3.5 Remarks and open problems

(a). We observe a phenomenon of interlacing of three totally disconnected disjoint dispensable sets $\Pi^{(3-j)^*}(\Lambda): \ j = 0, 1, 2$ which are essentially derived from zero sets of four trigonometric polynomials.

(b). If the measure in question is supported on an arbitrary number of parallel lines, then the size of the dispensable sets would be larger. Indeed, the method used for the proof of Theorem 3.3.1 would be highly implicit for a large number of parallel lines. Since the dispensable sets are totally disconnected, it would be an interesting question to analyze Heisenberg uniqueness pairs corresponding to the finite number of parallel lines in terms of Hausdorff dimension of the dispensable sets.

(c). If we consider countably many parallel lines, then whether the projection $\Pi(\Lambda)$ would be still relevant after deleting the countably many dispensable sets, seems to be a reasonable question. We leave these questions open for the time being.

(d). For $p = 3$, in Lemma 3.3.2 we have used the fact that any symmetric polynomial in a, b, c can be expressed as a polynomial in $\tau_j; \ j = 0, 1, 2$. This enables us to define a function $\rho \in L_{loc}^{\Pi^3(\Lambda),\bar{\xi}}$, which is crucial in the proof of Lemma 3.3.2. However, for $p \geq 4$, the functions $\tau_j; \ j = 0, 1, 2$ appeared in (3.2.10) are away from the elementary symmetric polynomials. If we could identify the space of symmetric polynomials generated by $\tau_j; \ j = 0, 1, 2$, then we can think to modify the Lemma 3.3.2 in terms of $\Pi^p_{3^*}(\Lambda)$ that would help in minimizing the size of the set $\widetilde{\Pi}(\Lambda)$. Hence a characterization of Λ for four lines problem might be obtained that would be closed to three lines result. However, an exact analogue of three lines result for a large number of lines is still open.

Chapter 4

Uniqueness of the Fourier transform on the Euclidean motion group

4.1 Introduction

In this chapter, we work out an analogue of the Benedicks theorem to the Euclidean motion group $M(n)$. We prove that if the Fourier transform of certain integrable functions is of finite rank, then the function has to vanish identically. Further, we explore the possibility of the Heisenberg uniqueness pairs for the Fourier transform on $M(n)$ as well as on the product group $G' = \mathbb{R}^n \times K$. In the latter case, we observed a one to one correspondence between the class of HUP's on $\mathbb{R}^n$ and the class of HUP's on G'.

4.2 Notation and preliminaries

Euclidean motion group $G = M(n)$ is the group of isometries of $\mathbb{R}^n$ that leaves invariant the Laplacian. Since the action of the special orthogonal group $K = SO(n)$ defines a group of automorphisms on $\mathbb{R}^n$ via $y \mapsto ky + x$, where $x \in \mathbb{R}^n$ and $k \in K$, the group $M(n)$ can be identified as the semidirect product of $\mathbb{R}^n$ and K. Hence the group law on G can be expressed as

$$(x, s) \cdot (y, t) = (x + sy, st).$$

Since a right K-invariant function on G can be thought as a function on $\mathbb{R}^n$, we infer that the Haar measure on G can be written as $dg = dxdk$, where dx and dk are the normalized Haar measures on $\mathbb{R}^n$ and K respectively.

Let $\mathbb{R}_+ = (0, \infty)$ and $M = SO(n-1)$ be the subgroup of K that fixes the point $e_n = (0, 0, \ldots, 1)$. Let $\hat{M}$ be the unitary dual group of M. Given a unitary irreducible representation $\sigma \in \hat{M}$ realized on the Hilbert space $\mathcal{H}_\sigma$ of dimension d_σ, we consider the space $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ consisting of $d_\sigma \times d_\sigma$ complex matrices valued functions φ on K such that $\varphi(uk) = \sigma(u)\varphi(k)$, where $u \in M,\ k \in K$ and satisfying

$$\int_K \|\varphi(k)\|^2 dk = \int_K \operatorname{tr}(\varphi(k)^* \varphi(k)) dk.$$

It is easy to see that $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ is a Hilbert space under the inner product

$$\langle \varphi, \psi \rangle = \int_K \operatorname{tr}(\varphi(k)\psi(k)^*) dk.$$

For each $(a, \sigma) \in \mathbb{R}_+ \times \hat{M}$, defines a unitary representation $\pi_{a,\sigma}$ of G by

$$\pi_{a,\sigma}(g)(\varphi)(k) = e^{-ia\langle x, k \cdot e_n \rangle} \varphi(s^{-1}k), \tag{4.2.1}$$

where $\varphi \in L^2\left(K, \mathbb{C}^{d_\sigma \times d_\sigma}\right)$. Let $\varphi = (\varphi_1, \ldots, \varphi_{d_\sigma})$, where φ_j are the column vectors of φ. Then $\varphi_j(uk) = \sigma(u)\varphi_j(k)$. Now, consider the space

$$H\left(K, \mathbb{C}^{d_\sigma}\right) = \left\{ \varphi : K \to \mathbb{C}^{d_\sigma}, \int_K |\varphi(k)|^2 dk < \infty, \varphi(uk) = \sigma(u)\varphi(k), u \in M \right\}.$$

Then $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ is the direct sum of d_σ copies of the Hilbert space $H(K, \mathbb{C}^{d_\sigma})$ equipped with the inner product

$$\langle \varphi, \psi \rangle = d_\sigma \int_K (\varphi(k), \psi(k))\, dk.$$

Then it can be shown that an infinite dimensional unitary irreducible representation of G is the restriction of $\pi_{a,\sigma}$ to $H(K, \mathbb{C}^{d_\sigma})$. In other words, each of $(a, \sigma) \in \mathbb{R}_+ \times \hat{M}$ defines a principal series representation $\pi_{a,\sigma}$ of G via (4.2.1).

Besides the principal series representations, there are finite-dimensional unitary irreducible representations of G which can be parameterized by $\hat{K}$, however, these unitary representations do not take part in the Plancherel formula. For more details, we refer to Kumahara [25] and Sugiura [48].

Now, we define the group Fourier transform of a function $f \in L^1(G)$ by

$$\hat{f}(a, \sigma) = \int_G f(g)\pi_{a,\sigma}(g^{-1})dg$$

and

$$\hat{f}(\delta) = \int_G f(x, k)\delta(k^{-1})dxdk,$$

where $\delta \in \hat{K}$. Further, the operator $\hat{f}(a, \sigma)$ can be explicitly written as

$$\begin{aligned} (\hat{f}(a, \sigma)\varphi)(k) &= \int_{\mathbb{R}^n} \int_K f(x, s)e^{-i\langle x, ak\cdot e_n\rangle}\varphi(s^{-1}k)dxds, \\ &= \int_K \mathcal{F}_1 f(ak \cdot e_n, s)\varphi(s^{-1}k)ds, \end{aligned} \tag{4.2.2}$$

where $\mathcal{F}_1$ stands for the usual Fourier transform in the first variable and $\varphi \in H(K, \mathbb{C}^{d_\sigma})$. For more details, we refer to [12, 16, 26].

Now, if $f \in L^1 \cap L^2(G)$, then the operator $\hat{f}(a, \sigma)$ will be a Hilbert-Schmidt operator on $H(K, \mathbb{C}^{d_\sigma})$. Since the Plancherel measure μ_σ on $\hat{G}$ can be expressed as $d\mu_\sigma = c_n a^{n-1} da$, where c_n depends only on n, the corresponding Plancheral formula is given by

$$\int_0^\infty \left(\sum_{\sigma \in \hat{M}} d_\sigma \|\hat{f}(a,\sigma)\|_{HS}^2 \right) d\mu_\sigma(a) = \|f\|_2^2. \tag{4.2.3}$$

Further, in view of the Fourier inversion formula ([48], p. 175) for function in $L^2(M(2))$, an inversion formula for the function $f \in L^1 \cap L^2(M(n))$, can be expressed as

$$f(x,s) = c_n \sum_{\sigma \in \hat{M}} \int_0^\infty \operatorname{tr} \left(\pi_{a,\sigma}(x,s) \hat{f}(a,\sigma) \right) a^{n-1} da. \tag{4.2.4}$$

We would like to mention the following Wiener's theorem on motion group due to R. Gangolli [14]. For a function f on G, defining the two-sided translate by ${}^{g_1}f^{g_2}(g) = f(g_1 g g_2^{-1})$, where $g_j \in G; j = 1, 2$.

Theorem 4.2.1. [14] *Let $f \in L^1(G)$ and $S =$ span $\{{}^{g_1}f^{g_2} : g_j \in G; j = 1, 2\}$. Then the space S is dense in $L^1(G)$ if and only if $\hat{f}(a,\sigma) \neq 0$ and $\hat{f}(\delta) \neq 0$ for all $(a,\sigma) \in \mathbb{R}_+ \times \hat{M}$ and $\delta \in \hat{K}$.*

A close observation of Theorem 4.2.1 shows that if $\hat{f}(a,\sigma)$ is a finite rank operator, then $\overline{S}$ can be a proper subspace of $L^1(G)$. Hence, it opens a window to look at the determining properties of $\hat{f}$.

Next, we recall certain facts about the spherical harmonics. Let $\hat{K}_M$ denote the set of all equivalence classes of irreducible unitary representations of K which have a nonzero M-fixed vector. It is well known that each representation in $\hat{K}_M$ has, in fact, a unique nonzero M-fixed vector, up to a scalar multiple.

For a $\delta \in \hat{K}_M$, which is realized on V_δ, let $\{e_1, \ldots, e_{d_\delta}\}$ be an orthonormal basis of V_δ with e_1 as the M-fixed vector. Let $\varphi_{ij}^\delta(k) = \langle e_i, \delta(k) e_j \rangle$, $k \in K$. Then by the

Peter-Weyl theorem, it follows that $\{\sqrt{d_\delta}\varphi^{\delta}_{1j} : 1 \leq j \leq d_\delta, \delta \in \hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$.

We would further need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way.

For $l \in \mathbb{Z}_+$, denote the set of all non-negative integers, let P_l denote the space of all homogeneous polynomials P in n variables of degree l.

Let $H_l = \{P \in P_l : \Delta P = 0\}$, where Δ is the standard Laplacian on $\mathbb{R}^n$. The elements of H_l are called solid spherical harmonics of degree l. It is easy to see that the natural action of K leaves the space H_l invariant. In fact, the corresponding unitary representation π_l is in $\hat{K}_M$. Moreover, $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_l : l \in \mathbb{Z}_+\}$.

Define $Y_{lj}(\omega) = \sqrt{d_l}\varphi^{1j}_{\pi_l}(k)$, where $\omega = k \cdot e_n \in S^{n-1}$, $k \in K$ and d_l is the dimension of H_l. Then the set $\widetilde{H}_l = \{Y_{lj} : 1 \leq j \leq d_l \text{ and } l \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{n-1})$. Thus, a suitable function g on S^{n-1} can be expanded as

$$g(\omega) = \sum_{l=0}^{\infty}\sum_{j=1}^{d_l} a_{lj}Y_{lj}(\omega). \tag{4.2.5}$$

These spherical functions Y_{lj} are called the spherical harmonics on the unit sphere S^{n-1}. For more details, see [51], p. 11.

Next, we consider an orthogonality relation among the matrix coefficients of the irreducible unitary representations in $\hat{K}_M$.

Lemma 4.2.1. *For $\delta \in \widehat{K}$, denoting $\phi^{\delta}_{ij}(k) = \langle e_i, \delta(k)e_j\rangle$. Then for $\delta_1, \delta_2 \in \hat{K}$, there* *exists $\alpha \in \mathbb{Z}_+$ such that*

$$\int_M \phi_{ij}^{\delta_1}(km)\overline{\phi_{lu}^{\delta_2}(km)}dm = \sum_{v=0}^{\alpha} c_v Y_v(k \cdot e_n). \tag{4.2.6}$$

Proof. Since we know that the matrix coefficients of $\delta \in \hat{K}_M$ satisfy the functional relation

$$\phi_{ij}^{\delta}(km) = \sum_{p=1}^{d_\delta} \phi_{ip}^{\delta}(k)\phi_{pj}^{\delta}(m) \tag{4.2.7}$$

and M is a compact subgroup of K, it follows that each of $\delta \in \hat{K}$ will be the direct sum of finitely many irreducible unitary representations of M. Hence each of ϕ_{pj}^{δ} satisfies

$$\phi_{pj}^{\delta} = \sum_{q=1}^{d_{\delta,\beta}} \phi_{pj}^{\delta_q},$$

where $\delta_q \in \hat{M}$. By orthogonality of the coefficients $\phi_{pj}^{\delta_q}$'s and the fact that the left-hand side of (4.2.6) is M-invariant, we infer that it is a finite sum of the product of some spherical harmonics. Further, a homogeneous polynomial can be uniquely decomposed in terms of homogeneous harmonics polynomials, it follows that (4.2.6) holds. $\square$

For a fixed $\xi \in S^{n-1}$, we define a linear functional on H_l by $\xi \mapsto Y_l(\xi)$. Then there exists a unique spherical harmonic, say $Z_\xi^{(l)} \in H_l$ such that

$$Y_l(\xi) = \int_{S^{n-1}} Z_\xi^{(l)}(\eta)Y_l(\eta)d\sigma(\eta). \tag{4.2.8}$$

The spherical harmonic $Z_\xi^{(l)}$ is a K bi-invariant real-valued function which is constant on the geodesics those are orthogonal to the line joining the origin and ξ. The spherical harmonic $Z_\xi^{(l)}$ is called the zonal harmonic of the space $\widetilde{H}_l$ for the above and various other peculiar reasons. For more details, see [47].

Since the zonal harmonic $Z_\xi^{(l)}(\eta)$ is K bi-invariant, there exists a reasonable function F on $(-1,1)$ such that $Z_{\xi}^{(l)}(\eta) = F(\xi \cdot \eta)$. Hence, the extension of the formula (4.2.8) is inevitable. For $F \in L^1(-1,1)$, the Funk-Hecke identity is

$$\int_{S^{n-1}} F(\xi \cdot \eta) Y_l(\eta) d\sigma(\eta) = c_l Y_l(\xi), \tag{4.2.9}$$

where the constant c_l is given by

$$c_l = \alpha_l \int_{-1}^{1} F(t) G_l^{\frac{n-2}{2}}(t)(1-t^2)^{\frac{n-3}{2}}\, dt$$

and G_l^{β} stands for the Gegenbauer polynomial of order β and degree l.

Let f be a function in $L^1(S^{n-1})$. For each $l \in \mathbb{Z}_+$, we define the l^{th} spherical harmonic projection of the function f by

$$\Pi_l f(\xi) = \int_{S^{n-1}} Z_{\xi}^{(l)}(\eta) f(\eta) d\sigma(\eta). \tag{4.2.10}$$

Then $\Pi_l f$ is a spherical harmonic of degree k. Now, for $\delta > (n-2)/2$, if we denote $A_l^p = \binom{p-l+\delta}{\delta}\binom{p+\delta}{\delta}^{-1}$, then the spherical harmonic expansion $\sum\limits_{l=0}^{\infty} \Pi_l f$ is δ- Cesaro summable to f. In other words,

$$f = \lim_{p\to\infty} \sum_{l=0}^{p} A_l^p \, \Pi_l f, \tag{4.2.11}$$

where the limit on the right-hand side of (4.2.11) exists in $L^1\left(S^{n-1}\right)$. For more details, we refer to [22, 45]. We need the following lemma.

Lemma 4.2.2. *Let $f \in L^1\left(S^{n-1}\right)$ be such that $\int\limits_{S^{n-1}} e^{-ix\cdot\eta} f(\eta) d\sigma(\eta) = 0$. Then*

$$\lim_{p\to\infty} \sum_{l=0}^{p} i^l A_l^p \, \frac{J_{l+(n-2)/2}(r)}{r^{(n-2)/2}} \, \Pi_l f(\xi) = 0, \tag{4.2.12}$$

where $x = r\xi$, for some $r > 0$ and $\xi \in S^{n-1}$.

Proof. We have

$$\begin{aligned}
&\left|\sum_{l=0}^{p} A_l^p \int_{S^{n-1}} e^{-ix\cdot\eta}\Pi_l f(\eta)d\sigma(\eta)\right| \\
&= \left|\sum_{l=0}^{p} \int_{S^{n-1}} e^{-ix\cdot\eta}(A_l^p\Pi_l f(\eta) - f(\eta))\, d\sigma(\eta)\right| \\
&\leq \int_{S^{n-1}} \left|\sum_{l=0}^{p} A_l^p\Pi_l f(\eta) - f(\eta)\right| d\sigma(\eta)
\end{aligned}$$

By using the δ- Cesaro summability of f, it follows that

$$\lim_{p\to\infty}\sum_{l=0}^{p} A_l^p \int_{S^{n-1}} e^{-ix\cdot\eta}\Pi_l f(\eta)d\sigma(\eta) = 0. \tag{4.2.13}$$

Further, using the Funk-Hecke identity, it can be shown that

$$\int_{S^{n-1}} e^{-ix\cdot\eta}Y_j(\eta)d\sigma(\eta) = i^j\, \frac{J_{j+(n-2)/2}(r)}{r^{(n-2)/2}}Y_j(\xi), \tag{4.2.14}$$

whenever $Y_j \in \widetilde{H}_l$. For a proof of the identity (4.2.14), we refer [2], p. 464. This, in turn, implies that (4.2.12) holds. □

4.3 Results on the Euclidean motion group $M(n)$

In this section, we work out some of the uniqueness results for the Fourier transform on the Euclidean motion group $G = M(n)$ as an analogue to the Benedicks' theorem. We prove the group Fourier transform of a non-zero function in $L^1(G)$ cannot be compactly supported in $(0, \infty)$.

In order to prove this result, we need the following result from [6]. Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq 0;\ j = 1, \ldots, n\}$. The following result had appeared in the article [6] by Bagchi and Sitaram, p. 421.

Proposition 4.3.1. [6] *Let h be a non-zero function in $L^1(\mathbb{R}^n)$ which is supported on $\mathbb{R}^n_+$. Then supp $\hat{h} = \mathbb{R}^n$.*

Theorem 4.3.1. *Let $f \in L^1(G)$ be supported in $\mathbb{R}^n_+$ in the first variable. If $\hat{f}(.,\sigma)$ is compactly supported in $\mathbb{R}_+$, then $f = 0$.*

Proof. Suppose f is a radial function in the first variable. Then $\mathcal{F}_1 f(., s)$ will be radial and hence

$$(\hat{f}(a,\sigma)\varphi)(k) = \int_K \mathcal{F}_1 f(a,s)\varphi(s^{-1}k)ds, \tag{4.3.1}$$

where $\varphi \in H(K, \mathbb{C}^{d_\sigma})$. Since $\hat{f}(.,\sigma)$ is compactly supported in $\mathbb{R}_+$, it follows from (4.3.1) that $\mathcal{F}_1 f(., s)$ is compactly supported in $\mathbb{R}_+$, for almost all $s \in K$. This, in turn, contradicts Proposition 4.3.1. Thus, we conclude that $f = 0$.

Since $f \in L^1(G)$, in view of (4.2.11), we can write the spherical harmonic decomposition of f in the first variable $x = |x|t,\ t \in S^{n-1}$ as

$$f(x,s) = \lim_{p\to\infty} \sum_{l=0}^{p} A_l^p\, f_l(|x|, s)\Pi_l f(t,s), \tag{4.3.2}$$

where the series on the right-hand side is δ-Cesaro summable. Now, an application of the Hecke-Bochner identity to (4.3.1) yields

$$(\hat{f}(a,\sigma)\varphi)(k) = \int_K \lim_{p\to\infty} \sum_{l=0}^{p} A_l^p\, i^{-l} a^l \mathcal{F}_{n+2l} H_l(a,s)\Pi_l f(t,s)\varphi(s^{-1}k)ds,$$

where $\mathcal{F}_{n+2l}$ is the $(n+2l)$-dimensional Fourier transform of $H_l = \frac{f_l}{|x|^l}$. Since $\hat{f}$ is compactly supported, it follows that

$$\lim_{p\to\infty} \sum_{l=0}^{p} A_l^p\, i^{-l} a^l \mathcal{F}_{n+2l} H_l(a,s)\Pi_l f(t,s) = 0. \tag{4.3.3}$$

We know that the set $\{\Pi_l f(\cdot, s) : \; l \in \mathbb{Z}_+\}$ form an orthogonal set in $L^2(S^{n-1})$, from (4.3.3), it is easy to see that

$$\mathcal{F}_{2+2l} H_l(a, s) \, \|\Pi_l f(., s)\|_2^2 = 0.$$

If $\mathcal{F}_{2+2l} H_l(a, s) = 0$, then by the radial case, we infer that $H_l = 0$. Otherwise, we get $\|\Pi_l f(., s)\|_2 = 0$. Thus, it follows from (4.3.2) that $f = 0$. □

Further, we prove that a radial function on G can be determined by its Fourier transform at a single point.

Proposition 4.3.2. *Let $f \in L^1(G)$ be a radial function in the first variable such that sign$(J_{\frac{n-2}{2}} f) \geq 0$. If $\hat{f}(a_o, \sigma) = 0$ for some $a_o \in \mathbb{R}_+$ and a fixed $\sigma \in \hat{M}$, then $f = 0$.*

Proof. For $\varphi \in H(K, \mathbb{C}^{d_\sigma})$, we have

$$(\hat{f}(a_o, \sigma)\varphi)(k) = \int_K \mathcal{F}_1 f(a_o, s) \varphi(s^{-1}k) ds.$$

By the hypothesis, $\hat{f}(a_o, \sigma)\varphi = 0$, it follows that $\mathcal{F}_1 f(a_o, .) = 0$. Hence

$$\begin{aligned} \mathcal{F}_1 f(a_o, s) &= \int_0^\infty \int_{S^{n-1}} f(|t\omega|, s) e^{-ia_o p \cdot t\omega} d\omega t^{n-1} dt \\ &= \int_0^\infty J_{\frac{n-2}{2}}(a_o t) f(t, s) t^{n-1} dt = 0. \end{aligned}$$

Since sign$(J_{\frac{n-2}{2}} f) \geq 0$ and the Bessel function $J_{\frac{n-2}{2}}$ can vanish only at the countably many points, we conclude that $f = 0$. □

4.4 Some auxiliary results on compact group

In this section, we observe some of the properties of a Weyl type transform on the space $L^1(K)$ as analogous to the Weyl transform on the Heisenberg group, (see [50]). We use it to work out some uniqueness result for the Fourier transform on the motion group G.

Let K be a compact group. For a function $g \in L^1(K)$, we define an operator W on $H(K, \mathbb{C}^{d_\sigma})$ by

$$W(g) = \int_K g(t)\pi(t)dt,$$

where π is the left regular representation of K. Then $W(g)$ maps $H(K, \mathbb{C}^{d_\sigma})$ into $H(K, \mathbb{C}^{d_\sigma})$. Now, we derive the Plancheral formula and the Fourier inversion formula for the transform W.

Plancherel formula. For $g \in L^2(K)$ and $\varphi \in H(K, \mathbb{C}^{d_\sigma})$, we have

$$\begin{aligned} (W(g)\varphi)(k) &= \int_K g(t)(\pi(t)\varphi)(k)dt = \int_K g(t)\varphi(t^{-1}k)dt \\ &= \int_K g(s^{-1}k)\varphi(s)ds. \end{aligned}$$

Write $\mathcal{K}_g(s,k) = g(s^{-1}k)$. Then $W(g)$ is an integral operator with the kernel $\mathcal{K}_g \in L^2(K \times K)$. Hence $W(g)$ is a Hilbert-Schmidt operator that satisfying

$$\|(W(g)\|_{HS}^2 = \int_{K\times K} |\mathcal{K}_g(s,k)|^2 \, dsdk = \int_{K\times K} \left|g(s^{-1}k)\right|^2 dsdk = \|g\|_2^2.$$

In other words, W maps $L^1(K)$ onto S_2, the space of Hilbert-Schmidt operators on $H(K, \mathbb{C}^{d_\sigma})$.

Next, we prove the Fourier inversion formula for the transform W.

Lemma 4.4.1. *If $g \in C^2(K)$, then the transform W satisfies the inversion formula*

$g(t) = tr(\pi(t)^* W(g))$.

Proof. Given that $g \in C^2(K)$,

$$\begin{aligned}(\pi(t))^* W(g) &= \int_K g(s)(\pi(t))^* \pi(s) ds = \int_K g(s)\pi(t^{-1})\pi(s) ds \\ &= \int_K g^t(p)\pi(p) dp = W(g^t),\end{aligned}$$

where $g^t(p) = g(tp)$. That is, $W(g)$ is an integral operator with kernel $\mathcal{K}_g$. Since the kernel $\mathcal{K}_{g^t}$ satisfies $\mathcal{K}_{g^t}(s,k) = g^t(s^{-1}k)$, we obtain $\mathcal{K}_{g^t}(s,s) = g(t)$ and hence

$$\begin{aligned}\mathrm{tr}[\pi(t)^* W(g)] &= \mathrm{tr}(W(g^t)) = \int_K \mathcal{K}_{g^t}(s,s) ds \\ &= \int_K g(t) ds = g(t).\end{aligned}$$

□

Further, by using the Peter-Weyl theorem, we prove that if $g \in L^1(K)$, then the operator $W(g)$ has finite rank as long as g is a trigonometric polynomial. For $\delta \in \hat{K}$, a finite linear combination of matrix coefficients φ_{ij}^{δ}'s is known as a trigonometric polynomial.

Proposition 4.4.1. *Let $g \in L^1(K)$. Then the operator $W(g)$ is of finite rank if and only if g is a trigonometric polynomial on K.*

Proof. Consider the function $h = g * g^*$, where $g^*(t) = \overline{g(t^{-1})}$. Now, we show that

$W(h) = W(g)^*W(g)$. For this, we have

$$\begin{aligned} W(h) &= \int_K h(t)\pi(t)dt = \int_K (g * g^*)(t)\pi(t)dt \\ &= \int_K \int_K g(s)g^*(ts^{-1})\pi(t)dtds \\ &= \int_K g(s) \left(\int_K g^*(ts^{-1})\pi(t)dt \right) ds. \end{aligned}$$

By the change of variables $ts^{-1} = p$ in the inner integral, we get

$$\begin{aligned} W(h) &= \int_K g(s) \left(\int_K g^*(p)\pi(ps)dp \right) ds \\ &= W(g^*)W(g). \end{aligned}$$

Further, we require proving $W(g)^* = W(g^*)$. For $\varphi, \psi \in H(K, \mathbb{C}^{d_\sigma})$, consider

$$\langle W(g^*)\varphi, \psi \rangle = \int_K \overline{g(t^{-1})} \langle \pi(t)\varphi, \psi \rangle \, dt = \int_K \overline{g(s)} \left\langle \pi(s^{-1})\varphi, \psi \right\rangle ds.$$

Since π is the left regular representation of K, the operator $\pi(s)$ will be unitary. Hence

$$\langle W(g^*)\varphi, \psi \rangle = \int_K \langle \varphi, g(s)\pi(s)\psi \rangle \, ds = \langle \varphi, W(g)\psi \rangle = \langle W(g)^*\varphi, \psi \rangle \, .$$

This, in turn, implies that $W(h) = W(g)^*W(g)$ is a positive finite rank operator. Thus, by the spectral theorem, there exists an orthonormal set $\{\varphi_j \in H(K, \mathbb{C}^{d_\sigma}) : j = 1, \ldots, m\}$ and scalars $\lambda_j \geq 0$ such that

$$W(h)\varphi = \sum_{j=1}^{m} \lambda_j \langle \varphi, \varphi_j \rangle \varphi_j, \tag{4.4.1}$$

whenever $\varphi \in H(K, \mathbb{C}^{d_\sigma})$. Let $\varphi_j = (\varphi_{j,1}, \ldots, \varphi_{j,d_\sigma})$. Then by (4.4.1), it follows that

$h * \varphi_{j,q} = \lambda_j \varphi_{j,q}$. By taking Fourier coefficient of both the sides, we get

$$\widehat{\varphi^{\delta}}_{j,q}(\widehat{h^{\delta}} - \lambda_j I) = 0,$$

where $\delta \in \hat{K}$. Then $\widehat{h^{\delta}} = \lambda_j I$ for finitely many $\delta \in \hat{K}$, otherwise, by the Riemann-Lebesgue lemma $\lambda_j = 0$. Hence $\hat{\varphi}_{j,q}(\delta) \neq 0$ at most for finitely many $\delta \in \hat{K}$. Thus, by the Peter-Weyl theorem, we infer that $\varphi_{j,q}$ is a trigonometric polynomial. Since $h = g^* * g$, it follows that $\hat{h}(\delta) = |\hat{g}(\delta)|^2$. Thus, from (4.4.1), we conclude that g is a trigonometric polynomial.

Conversely, suppose g is a trigonometric polynomial, then without loss of generality, we can assume that $g = \varphi_{ij}^{\delta}$. Now, we can write

$$\varphi_{ij}^{\delta}(t^{-1}s) = \left\langle \delta(t^{-1}s)e_j, e_i \right\rangle = \left\langle \delta(s)e_j, \delta(t)e_i \right\rangle.$$

Since H_δ is π-invariant, it follows that

$$\varphi_{ij}^{\delta}(t^{-1}s) = \sum_{l=1}^{d_\delta} \varphi_{lj}^{\delta}(s)\overline{\varphi_{li}^{\delta}(t)}.$$

A straightforward calculation leads to $W(g)\varphi(s) = d_\delta \left\langle \varphi, \varphi_{ij}^{\delta} \right\rangle \varphi_{ij}^{\delta}$. Thus, we conclude that $W(g)$ is of finite rank. □

Remark 4.1. *In view of the Minkowski integral inequality, it can be easily seen that* $\|W(g)\| \leq \|g\|_1$. *Hence, the spectral radius of the operator* $W(g)$ *will satisfy the condition* $\lambda[\sigma(W(g))] \leq \|g\|_1$.

Next, by using Proposition 4.4.1, we prove that a radial function on the motion group G can be determined by its group Fourier transform at a single point. However, for a sake of simplicity, we prove the result for $G = M(2)$. For proving this result, we need the following lemma.

Lemma 4.4.2. *Let $f \in L^1(G)$ be such that $f(x,s) = f(|x|,s)$. If $\hat{f}(a_o\,,\sigma)$ is of finite rank for some $a_o \in \mathbb{R}_+$, then*

$$\int_0^\infty J(a_o t) F_t(s) t\, dt = \sum_{|n|\leq \alpha_o} \int_0^\infty J(a_o t) \hat{F}_t(n) Y_n(s) t\, dt,$$

where $F_t(s) = f(t,s)$.

Proof. We know that for $\varphi \in L^2(K, \mathbb{C}^{d_\sigma})$ we have

$$\begin{aligned}(\hat{f}(a_o\,,\sigma)\varphi)(k) &= \int_{\mathbb{R}^n}\int_K f(x,s) e^{-i\langle x, a_o k\rangle} \varphi(s^{-1}k) dx ds \\ &= \int_K \mathcal{F}_1 f(a_o k, s) \varphi(s^{-1}k) ds.\end{aligned}$$

Since f is radial in the first variable, then it follows that

$$(\hat{f}(a_o\,,\sigma)\varphi)(k) = (W\,(\mathcal{F}_1 f(a_o,.))\,\varphi)(k).$$

By the hypothesis, $\hat{f}(a_o\,,\sigma)$ is a finite rank operator, $W\,(\mathcal{F}_1 f(a_o,.))$ must be of finite rank. From Proposition 4.4.1, we conclude that $\mathcal{F}_1 f(a_o,.)$ is a trigonometric polynomial. That is,

$$\mathcal{F}_1 f(a_o, s) = \sum_{|m|\leq \alpha_o} \hat{G}_{a_o}(m) \chi_m(s), \tag{4.4.2}$$

where $G_{a_o}(s) = \mathcal{F}_1 f(a_o, s)$. On the other hand, we have

$$\begin{aligned}\mathcal{F}_1 f(a_o, s) &= \int_0^\infty \int_K f(|t\omega|, s) e^{-iap\cdot t\omega} d\omega\, tdt \\ &= \int_0^\infty J_0(a_o t) f(t,s) tdt. \qquad (4.4.3)\end{aligned}$$

Now, we have

$$\begin{aligned}
\hat{G}_a(m) &= \int_K \mathcal{F}_1 f(a_o,k)\chi_{-m}(k)dk \\
&= \int_0^\infty J_0(a_o t)\left(\int_K f(t,k)\chi_{-m}(k)dk\right)tdt \\
&= \int_0^\infty J_0(a_o t)\hat{F}_t(m)tdt,
\end{aligned}$$

where $F_t(k) = f(t,k)$. Hence from (4.4.2) we get

$$\mathcal{F}_1 f(a_o,s) = \sum_{|m|\le\alpha_o}\int_0^\infty J_0(a_o t)\hat{F}_t(m)\chi_m(s)tdt. \tag{4.4.4}$$

By comparing (4.4.3) with (4.4.4), we get the required identity. □

Remark 4.2. *Notice that by taking inverse Fourier transform in both the sides of (4.4.2), we can assume f is trigonometric polynomial as long as $\hat{f}(a_o,\sigma)$ is a finite rank operator for some $a_o\in\mathbb{R}_+$ and $\sigma\in\hat{M}$.*

Theorem 4.4.1. *Let $f\in L^1(G)$ be a radial function in the first variable which integrates zero in the second variable. If $\hat{f}(a_o,\sigma)$ is a finite rank operator for some $a_o\in\mathbb{R}_+$ and sign$(J_0 f)\ge 0$, then $f=0$.*

Proof. In view of Remark 4.2, from Lemma 4.4.2 we infer that

$$\int_0^\infty \hat{F}_t(o)tdt = \int_0^\infty f(t,s)tdt.$$

This, in tern, implies that

$$\int_0^\infty\left(\int_K f(t,k)dk\right)tdt = \int_0^\infty f(t,s)tdt.$$

Since f integrates zero on K and sign$(J_0 f)\ge 0$, we conclude that $f=0$. □

4.5 Some results on the Heisenberg uniqueness pairs

In this section, we explore the Heisenberg uniqueness pairs for the Fourier transform on the motion group G as well as on the product group $G' = \mathbb{R}^n \times K$, where K is a compact group. Further, we observed a one to one correspondence between the class of HUP's on $\mathbb{R}^n$ and the class of HUP's on G'.

Let Γ be a smooth surface (or a finite union of smooth surfaces) in $\mathbb{R}^n$ and $\Gamma' = \Gamma \times K$. Let $X(\Gamma')$ be the space of all finite complex-valued Borel measures μ in the motion group G which is supported on Γ' and absolutely continuous with respect to the surface measure on Γ'.

We define the Fourier transform of μ on G by

$$(\hat{\mu}(a,\sigma)\varphi)(k) = \int_\Gamma \int_K f(x,s) e^{-i\langle x, ak\cdot e_n\rangle} \varphi(s^{-1}k) d\mu(x) ds, \tag{4.5.1}$$

where $a \in \mathbb{R}^+$ and $\varphi \in H(K, \mathbb{C}^{d_\sigma})$.

Theorem 4.5.1. *Let $\Gamma' = S^{n-1} \times K$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $\mu \in X(\Gamma')$. If there exists a_o such that $J_{(n+2l-2)/2}(a_o) \neq 0$ for every $l \in \mathbb{Z}_+$ such that $\hat{\mu}(a_o, \sigma) = 0$ for all $\sigma \in \hat{M}$, then $\mu = 0$.*

Proof. Since μ is absolutely continuous with respect to the surface measure on Γ', by Radon-Nikodym theorem, there exists a function $f \in L^1(\Gamma')$ such that $d\mu = f ds dt$. By hypothesis, we have

$$(\hat{\mu}(a_o,\sigma)\varphi)(k) = \int_{S^{n-1}} \int_K f(t,s) e^{-i\langle t, a_o k\cdot e_n\rangle} \varphi(s^{-1}k) dt ds = 0,$$

whenever $\varphi \in C(K, \mathbb{C}^{d_\sigma})$. Now, by Fubini's theorem, we can write

$$\int_K \left(\int_{S^{n-1}} f(t,s) e^{-i\langle t, a_o k \cdot e_n \rangle} dt \right) \varphi(s^{-1}k) ds = \int_K \mathcal{F}_1 f(a_o k \cdot e_n, s) \varphi(s^{-1}k) ds = 0.$$

Hence $\mathcal{F}_1 f(a_o k \cdot e_n, s) = 0$ for almost all $s, k \in K$. Since $SO(n)$ can be identified with S^{n-1} via $k \to k \cdot e_n$, it follows that $\mathcal{F}_1 f(y, s) = 0$ for almost all $y \in S^{n-1}_{a_o}(o)$ and $s \in K$. Since we know from Theorem 1.0.6 that the pair $\left(S^{n-1}, S^{n-1}_{a_o}(o)\right)$ is a HUP as long as $J_{(n+2l-2)/2}(a_o) \neq 0$ for all $l \in \mathbb{Z}_+$, we conclude that $\mu = 0$. □

Remark 4.3. *Let (Γ, K) be a HUP in $\mathbb{R}^n$ and suppose $\mu \in X(\Gamma')$ is such that $\hat{\mu}(a_o) = 0$ for some $a_o \notin J^{-1}_{(n+2l-2)/2}(0)$ and $\forall\ l \in \mathbb{Z}_+$, then $\mu = 0$.*

The Haar measure on the product group G' is given by $dg = dxdk$, where dx is Lebesgue measure on $\mathbb{R}^n$ and dk is normalized Haar measure on K. Since the unitary dual of G' can be parameterized by $\hat{G'} = \mathbb{R}^n \times \hat{K}$, for each $(y, \gamma) \in \hat{G}$, the map $(x, k) \mapsto e^{-2\pi i x \cdot y} \gamma(k)$ is a unitary operator on the Hilbert space $\mathcal{H}_\gamma$.

Hence we can define the Fourier transform of the function $f \in Ł^1(G')$ by

$$\hat{f}(y, \gamma) = \int_{\mathbb{R}^n} \int_K f(x, k) e^{-2\pi i x \cdot y} \gamma(k^{-1}) dx dk. \tag{4.5.2}$$

Let $\Gamma' = \Gamma \times K$, where Γ is a smooth surface (or a finite union of smooth surfaces) in $\mathbb{R}^n$. Let $X(\Gamma')$ be the space of all finite complex-valued Borel measure μ in G' which is supported on Γ' and absolutely continuous with respect to the surface measure on Γ'. Then by the Radon-Nikodym theorem, there exists a function $f \in L^1(\Gamma')$ such that $d\mu = f d\nu dk$, where ν is the surface measure on Γ.

Now, the Fourier transform of the measure μ can be defined by

$$\begin{aligned}\hat{\mu}(y,\gamma) &= \int_\Gamma\int_K e^{-2\pi i x\cdot y}\gamma(k^{-1})d\mu(x,k)\\ &= \int_\Gamma\int_K f(x,k)e^{-2\pi i x\cdot y}\gamma(k^{-1})d\nu(x)dk. \qquad (4.5.3)\end{aligned}$$

Theorem 4.5.2. *The pair* (Γ,Λ) *is a Heisenberg uniqueness pairs in* $\mathbb{R}^n$ *if and only if* $(\Gamma',\Lambda\times\hat{K})$ *is a Heisenberg uniqueness pairs in* G'.

Proof. Suppose (Γ,Λ) is a Heisenberg uniqueness pair in $\mathbb{R}^n$ and $\mu\in X(\Gamma')$. Then by Fubini's theorem, the map $x\mapsto f(x,k)$ belongs to $L^1(\Gamma,d\nu)$ for almost all fixed $k\in K$. Hence for $(k,\sigma)\in K\times\hat{K}$, we can define the projection $f_{k,\sigma}$ of f by

$$f_{k,\sigma}(x)=\int_K f(x,kh^{-1})\chi_\sigma(h)dh, \qquad (4.5.4)$$

where $\chi_\sigma=\operatorname{tr}\sigma(\,\cdot\,)$, the character of the representation σ. Thus, the Euclidean Fourier transform of the projection $f_{k,\sigma}$ gives

$$\begin{aligned}\hat{f}_{k,\sigma}(y) &= \int_\Gamma\int_K f(x,kh^{-1})e^{-2\pi i x\cdot y}\chi_\sigma(h)dh\,d\nu(x)\\ &= \operatorname{tr}\int_\Gamma\int_K f(x,kh^{-1})\sigma(h)dh\,e^{-2\pi i x\cdot y}d\nu(x)\\ &= \operatorname{tr}\int_\Gamma\int_K f(x,h)\sigma(h^{-1})dh\,e^{-2\pi i x\cdot y}d\nu(x)\sigma(k)\\ &= \operatorname{tr}\left(\hat{\mu}(y,\sigma)\sigma(k)\right). \qquad (4.5.5)\end{aligned}$$

Suppose $\hat{\mu}|_{\Lambda\times\hat{K}}=0$. Since (Γ,Λ) is a Heisenberg uniqueness pair in $\mathbb{R}^n$, from

(4.5.5), it follows that $f_{k,\sigma} = 0$. Hence by the uniqueness of the Fourier series

$$f(x,k) = \sum_{\sigma \in \hat{K}} d_\sigma f_{k,\sigma}(x)$$

we conclude that $f = 0$.

Conversely, suppose $(\Gamma', \Lambda \times \hat{K})$ is a Heisenberg uniqueness pair in G'. Then for $\mu \in X(\Gamma)$, there exists a function $f \in L^1(\Gamma)$ such that $d\mu = f d\nu$. Suppose $\hat{\mu}|_\Lambda = 0$. Then

$$\int_\Gamma f(x) e^{-2\pi i x \cdot y} d\nu(x) = 0$$

for each $y \in \Lambda$. This, in turn, implies

$$\int_\Gamma \int_K f(x) e^{-2\pi i y \cdot x} \gamma(k^{-1}) dk \, d\nu(x) = 0. \tag{4.5.6}$$

Now, if we write $d\rho = f d\nu dk$, then $\rho \in X(\Gamma')$. Since $(\Gamma', \Lambda \times \hat{K})$ is a Heisenberg uniqueness pair, by (4.5.6), it follows that $\rho = 0$. Thus, using the fact that group compact group K is unimodular, we conclude that the measure $\mu = 0$. $\square$

Remarks and open problems: The main motive of this chapter is to consider the following problem.

Let $f \in L^1 \cap L^2(G)$ be such that $\hat{f}(a,\sigma)$ is a finite rank operator for each $a > 0$ and some $\sigma \in \hat{M}$. If $\hat{f}(\delta) \neq 0$ except for finitely many $\delta \in \hat{K}$, then $f = 0$.

We would like to mention the necessity of the non-vanishing conditions on the Fourier coefficients in the above problem. Since $M(2)$ is the semidirect product of $\mathbb{R}^2$ and $SO(2)$, each of $a \in \mathbb{R}_+$ defines a unitary irreducible representation π^a of $M(2)$ on

$L^2([0,2\pi])$. That is, for $(x,\theta) \in \mathbb{R}^2 \times [0,2\pi]$, the action of π^a is given by

$$(\pi^a(x,\theta)\varphi)(\omega) = e^{-i\langle x, ae^{i\omega}\rangle}\varphi(\omega-\theta),$$

where $\varphi \in L^2([0,2\pi])$. For $g \in L^1(\mathbb{R}^2)$, let $f(x,\theta) = g(x)e^{in\theta}$. Then

$$(\hat{f}(a)\varphi)(\omega) = \int_{\mathbb{R}^2}\int_0^{2\pi} f(x,\theta)e^{-i\langle x, ae^{i\omega}\rangle}\varphi(\omega-\theta)dxd\theta = \hat{g}(ae^{i\omega})\hat{\varphi}(n)e^{in\omega}.$$

Hence we infer that, if $\delta_n(f) = \int_{M(2)} f(x,\theta)e^{in\theta}dxd\theta \neq 0$ for finitely many $n \in \mathbb{Z}$, then $\hat{f}(a)$ is a rank one operator, however, f need not be the zero function.

Chapter 5

Uniqueness of the Fourier transform on certain Lie groups

5.1 Introduction

In an interesting article, M. Benedicks [7] had extended the classical Paley-Wiener theorem for compactly supported function to the class of integrable functions. In other words, support of an integrable function and its Fourier transform both cannot be of finite measure simultaneously. Thereafter, a series of analogous results to the Benedicks theorem has been explored in various contexts, including the Heisenberg group and the Euclidean motion groups (see [31–33, 37, 39, 42]). In article [31], an analogous result on the Heisenberg group has worked out for the partial compactly supported functions in terms of finite rank of Fourier transform of the function. Further, Vemuri [52] has relaxed the compact support condition on the functions by finite Lebesgue measure.

In this article, we explore analogous results to the Amrein-Berthier and Benedicks theorem on the Heisenberg motion group and step two nilpotent Lie groups. We prove that if the group Fourier transform of finitely supported certain integrable functions on the Heisenberg motion group (or step two nilpotent Lie groups) is of finite rank, then the function has to vanish identically. However, it would be a reasonable to consider the case when the spectrum of the Fourier transform of an integrable function will be supported on a thin uncountable set.

5.2 Preliminaries on the Heisenberg motion group

The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a step two nilpotent Lie group having center $\mathbb{R}$ that equipped with the group law

$$(z,t)\cdot(w,s) = \left(z+w, t+s+\frac{1}{2}\mathrm{Im}(z\cdot\bar{w})\right).$$

By Stone-von Neumann theorem, the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ can be parameterized by $\mathbb{R}^* = \mathbb{R}\smallsetminus\{0\}$. That is, each of $\lambda\in\mathbb{R}^*$ defines a Schrödinger representation π_λ of $\mathbb{H}^n$ by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}e^{i\lambda(x\cdot\xi+\frac{1}{2}x\cdot y)}\varphi(\xi+y),$$

where $z = x+iy$ and $\varphi\in L^2(\mathbb{R}^n)$. Let

$$T = \frac{\partial}{\partial t},\ X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t} \text{ and } Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}.$$

Then $\{T, X_j, Y_j : j = 1,\ldots,n\}$ forms a basis for the Lie algebra $\mathfrak{h}^n$ consists of all left-invariant vector fields on $\mathbb{H}^n$ and the representation π_λ induces a representation π_λ^* of

$\mathfrak{h}^n$ on the space of C^∞ vectors in $L^2(\mathbb{R}^n)$ via

$$\pi_\lambda^*(X)f = \frac{d}{dt}\Big|_{t=0} \pi_\lambda(\exp tX)f.$$

It is easy to see that $\pi_\lambda^*(X_j) = i\lambda x_j$ and $\pi_\lambda^*(Y_j) = \frac{\partial}{\partial x_j}$. Hence for the sub-Laplacian $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$, it follows that $\pi_\lambda^*(\mathcal{L}) = -\Delta_x + \lambda^2|x|^2 =: H_\lambda$, the scaled Hermite operator. Let $\phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}}\phi_\alpha(\sqrt{|\lambda|}x)$; $\alpha \in \mathbb{Z}_+^n$, where ϕ_α are the Hermite functions on $\mathbb{R}^n$. Then ϕ_α^λ's are the eigenfunctions of H_λ with eigenvalue $(2|\alpha| + n)|\lambda|$. Hence the entry functions $E_{\alpha\beta}^\lambda$'s of the representation π_λ are eigenfunctions of the sub-Laplacian $\mathcal{L}$ satisfying

$$\mathcal{L}E_{\alpha\beta}^\lambda = (2|\alpha| + n)|\lambda| E_{\alpha\beta}^\lambda,$$

where $E_{\alpha\beta}^\lambda(z,t) = \left\langle \pi_\lambda(z,t)\phi_\alpha^\lambda, \phi_\beta^\lambda \right\rangle$. Since $E_{\alpha\beta}^\lambda(z,t) = e^{i\lambda t}\left\langle \pi_\lambda(z)\phi_\alpha^\lambda, \phi_\beta^\lambda \right\rangle$, the eigenfunctions $E_{\alpha\beta}^\lambda$'s are not in $L^2(\mathbb{H}^n)$. However, for a fix t, they are in $L^2(\mathbb{C}^n)$. Now, define an operator L_λ by $\mathcal{L}\left(e^{i\lambda t}f(z)\right) = e^{i\lambda t}L_\lambda f(z)$. Then the special Hermite functions

$$\phi_{\alpha\beta}^\lambda(z) = (2\pi)^{-\frac{n}{2}} \left\langle \pi_\lambda(z)\phi_\alpha^\lambda, \phi_\beta^\lambda \right\rangle$$

are eigenfunctions of L_λ with eigenvalue $2|\alpha| + n$. We summarize by noting that the special Hermite functions $\phi_{\alpha\beta}^\lambda$'s forms an orthonormal basis for $L^2(\mathbb{C}^n)$ (see [51], Theorem 2.3.1).

Heisenberg motion group G is the group of isometries of $\mathbb{H}^n$ that leaves invariant the sub-Laplacian $\mathcal{L}$. Since the action of the unitary group $K = U(n)$ defines a group of automorphism on $\mathbb{H}^n$ via $k \cdot (z,t) = (kz, t)$, where $k \in K$, the group G can be expressed as the semidirect product of $\mathbb{H}^n$ and K. Hence the group law on G can be understood by

$$(k_1, z, t) \cdot (k_2, w, s) = \left(k_1 k_2, z + k_1 w, t + s - \frac{1}{2}\mathrm{Im}(k_1 w \cdot \bar{z})\right).$$

Since a right K-invariant function on G can be thought as a function on $\mathbb{H}^n$, we infer that the Haar measure on G can be written as $dg = dkdzdt$, where dk and $dzdt$ are the normalized Haar measure on K and $\mathbb{H}^n$ respectively.

For $k \in K$, define another set of representations of the Heisenberg group $\mathbb{H}^n$ by $\pi_{\lambda,k}(z,t) = \pi_\lambda(kz,t)$. Since $\pi_{\lambda,k}$ agrees with π_λ on the center of $\mathbb{H}^n$, it follows by the Stone-Von Neumann theorem for the Schrödinger representation that $\pi_{\lambda,k}$ is equivalent to π_λ. Hence there exists an intertwining operator $\mu_\lambda(k)$ satisfying

$$\pi_\lambda(kz,t) = \mu_\lambda(k)\pi_\lambda(z,t)\mu_\lambda(k)^*. \tag{5.2.1}$$

The operator-valued function μ_λ can be thought as a unitary representation of the group K on $L^2(\mathbb{R}^n)$ and it is known as metaplectic representation. Since for $\lambda \in \mathbb{R}^*$, the set $\{\phi_\alpha^\lambda : \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$, let $P_m = \{\phi_\alpha^\lambda : \ |\alpha| = m\}$. Then $\mu_\lambda|_{P_m}$ is an irreducible representation of K and the action of μ_λ on $L^2(\mathbb{R}^n)$ can be realized by

$$\mu_\lambda(k)\phi_\alpha^\lambda = \sum_{|\gamma|=|\alpha|} \eta_{\alpha\gamma}^\lambda(k)\phi_\gamma^\lambda. \tag{5.2.2}$$

For more details about the metaplectic representations and the spherical functions on $\mathbb{H}^n$, we refer the article by Benson et al. [8]. Let $(\sigma, \mathcal{H}_\sigma)$ be an irreducible unitary representation of K and $\mathcal{H}_\sigma = \mathrm{span}\{e_j^\sigma : 1 \le j \le d_\sigma\}$. For $k \in K$, the matrix coefficients of the representation $\sigma \in \hat{K}$, are define

$$\varphi_{ij}^\sigma(k) = \langle \sigma(k)e_j^\sigma, e_i^\sigma\rangle.$$

Define a bilinear form $\phi_\alpha^\lambda \otimes e_i^\sigma$ on $L^2(\mathbb{R}^n) \times \mathcal{H}_\sigma$ by $\phi_\alpha^\lambda \otimes e_i^\sigma = \phi_\alpha^\lambda\ e_i^\sigma$. Then the set

$\{\phi_\alpha^\lambda \otimes e_i^\sigma : 1 \leq i \leq d_\sigma, \alpha \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$. Denote $\mathcal{H}_\sigma^2 = L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$.

For $\lambda \neq 0$, we define a representation ρ_σ^λ of G on the space $\mathcal{H}_\sigma^2$ by

$$\rho_\sigma^\lambda(z,t,k) = \pi_\lambda(z,t)\mu_\lambda(k) \otimes \sigma(k).$$

In the article [41], it has been shown that ρ_σ^λ are the only irreducible unitary representations of G which appears in the Plancherel formula. Thus, in view of the above argument, we denote the partial dual of the group G by $G' \cong \mathbb{R}^* \times \hat{K}$.

Now, we define the Fourier transform of the function $f \in L^1(G)$ by

$$\hat{f}(\lambda,\sigma) = \int_K \int_{\mathbb{R}} \int_{\mathbb{C}^n} f(z,t,k)\rho_\sigma^\lambda(z,t,k)dzdtdk.$$

Let f^λ be the inverse Fourier transform of the function f in t variable. Then

$$f^\lambda(z,k) = \int_{\mathbb{R}} f(z,t,k)e^{i\lambda t}dt.$$

Thus,

$$\hat{f}(\lambda,\sigma) = \int_K \int_{\mathbb{C}^n} f^\lambda(z,k)\rho_\sigma^\lambda(z,k)dzdk,$$

where $\rho_\sigma^\lambda(z,k) = \rho_\sigma^\lambda(z,0,k)$. For $f \in L^1 \cap L^2(G)$, the following Plancherel formula derived in [41].

$$\int_K \int_{\mathbb{H}^n} f(z,t,k)dzdtdk = (2\pi)^{-n} \sum_{\sigma\in\hat{K}} \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|_{HS}^2 |\lambda|^2 d\lambda.$$

Further, the set $\{\phi_\alpha^\lambda \otimes e_i^\sigma : \alpha \in \mathbb{N}^n,\ 1 \leq i \leq d_\sigma\}$ forms an orthonormal basis for $\mathcal{H}_\sigma^2$, we can write

$$\hat{f}(\lambda,\sigma)(\phi_\gamma^\lambda \otimes e_i^\sigma) = \sum_{|\alpha|=|\gamma|} \int_K \eta_{\alpha\gamma}^\lambda(k) \int_{\mathbb{C}^n} f^\lambda(z,k) \left(\pi_\lambda(z)\phi_\alpha^\lambda \otimes \sigma(k)e_i^\sigma\right) dzdk.$$

5.3 Uniqueness results on the Heisenberg motion group

In this section, we work out some of the results pertaining to the uniqueness of the Fourier transform on the Heisenberg motion group $G = \mathbb{H}^n \ltimes K$. Those results can be thought as an analogue to the Benedicks theorem.

Weyl transform. For proving the main result of this section, we need to derive some of the properties of the Weyl type transform on $G^\times = \mathbb{C}^n \times K$. For more details on the Wely transform on the Heisenberg group, see [50].

For $(\lambda,\sigma) \in G'$, we define the Weyl transform W_σ^λ on $L^1(G^\times)$ by

$$W_\sigma^\lambda(F) = \int_K \int_{\mathbb{C}^n} F(z,k)\rho_\sigma^\lambda(z,k)dzdk.$$

Now, we define the λ-twisted convolutions of $F,H \in L^1 \cap L^2(G^\times)$ by

$$F \times_\lambda H(g) = \int_{G^\times} F\left(gg'^{-1}\right) H(g')e^{-\frac{i}{2}\lambda \mathrm{Im}(kw.\bar{z})}dg',$$

where $g=(z,k)$ and $g'=(w,s)$. For $\lambda = 1$, we simply call the λ-twisted convolutions as twisted convolutions and denote it by $F \times H$. We derive the following properties of the Weyl transform W_σ^λ.

Proposition 5.3.1. *If $F,H \in L^1 \cap L^2(G^\times)$, then* *(i)* $W_\sigma^\lambda(F^*) = W_\sigma^\lambda(F)^*$, *where* $F^*(z,k) = \overline{F\left((z,k)^{-1}\right)}$,

(ii) $W_\sigma^\lambda(F \times_\lambda H) = W_\sigma^\lambda(F) W_\sigma^\lambda(H)$.

Proof. By the scaling argument, it is enough to prove these results for the case $\lambda = 1$.

(i) If $\phi, \psi \in \mathcal{H}_\sigma^2$, then we have

$$\begin{aligned}
\langle W_\sigma(F^*)\phi, \psi\rangle &= \int_K \int_{\mathbb{C}^n} F^*(z,k) \langle \rho_\sigma(z,k)\phi, \psi\rangle \, dzdk \\
&= \int_K \int_{\mathbb{C}^n} \left\langle \phi, F\left((z,k)^{-1}\right) \rho_\sigma\left((z,k)^{-1}\right) \psi \right\rangle dzdk \\
&= \langle \phi, W_\sigma(F)\psi\rangle = \langle W_\sigma(F)^*\phi, \psi\rangle .
\end{aligned}$$

(ii) Let $dg = dzdk$, then

$$\begin{aligned}
\langle W_\sigma(F)W_\sigma(H)\phi, \psi\rangle &= \int_{G^\times} F(z,k) \langle \rho_\sigma(z,k)W_\sigma(H)\phi, \psi\rangle \, dg \\
&= \int_{G^\times} \int_{G^\times} F(g)H(g')e^{-\frac{i}{2}\mathrm{Im}(kw.\bar{z})} \langle \rho_\sigma(z+kw, ks)\phi, \psi\rangle \, dg'dg \\
&= \int_{G^\times} \int_{G^\times} F\left(gg'^{-1}\right) H(g')e^{-\frac{i}{2}\mathrm{Im}(kw.\bar{z})} \langle \rho_\sigma(g)\phi, \psi\rangle \, dg'dg \\
&= \int_{G^\times} (F \times H)(z,k) \langle \rho_\sigma(z,k)\phi, \psi\rangle \, dg \\
&= \langle W_\sigma(F \times H)\phi, \psi\rangle .
\end{aligned}$$

□

Next, we derive the Plancherel formula for the Weyl transform W_σ^λ on $L^2(G^\times)$ corresponding to $\lambda = 1$.

Proposition 5.3.2. *If* $F \in L^2(G^\times)$, *then the following holds.*

$$\sum_{\sigma \in \hat{K}} d_\sigma \, ||W_\sigma(F)||_{HS}^2 = (2\pi)^n \int_K \int_{\mathbb{C}^n} |F(z,k)|^2 dzdk.$$

Proof. Since $L^1 \cap L^2(G^\times)$ is dense in $L^2(G^\times)$, it is enough to prove the result for $L^1 \cap L^2(G^\times)$. For the sake of convenience, let $\phi^\sigma_{\alpha,i} = \phi^\lambda_\alpha \otimes e^\sigma_i$ and $\phi_{\alpha\beta} = (2\pi)^{\frac{n}{2}}\phi^\lambda_{\alpha\beta}$ when $\lambda = 1$. Then the set $\{\phi^\sigma_{\gamma,i} : \gamma \in \mathbb{N}^n, 1 \leq i \leq d_\sigma\}$ forms an orthonormal basis for $\mathcal{H}^2_\sigma$. By the Parseval identity, we have

$$\left\|W_\sigma(F)\phi^\sigma_{\gamma,i}\right\|^2_{\mathcal{H}^2_\sigma} = \sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^{d_\sigma}\left|\left\langle W_\sigma(F)\phi^\sigma_{\gamma,i}, \phi^\sigma_{\beta,j}\right\rangle\right|^2 =$$

$$(2\pi)^n \sum_{\beta\in\mathbb{N}^n}\sum_{j=1}^{d_\sigma}\left|\sum_{|\alpha|=|\gamma|}\int_K \eta_{\alpha\gamma}(k)\int_{\mathbb{C}^n} F(z,k)\phi_{\alpha\beta}(z)\varphi^\sigma_{ji}(k)dzdk\right|^2.$$

It is easy to see that the matrix coefficients $\eta_{\alpha\gamma}$ of the representation μ_λ satisfy the identity

$$\sum_{|\alpha|=m}\left|\sum_{|\gamma|=m} c_\alpha\eta_{\alpha\gamma}(k)\right|^2 = \sum_{|\alpha|=m}|c_\alpha\eta_{\alpha\gamma}(k)|^2, \tag{5.3.1}$$

where $k \in K$ and $c_\alpha \in \mathbb{C}$. Now, by Plancherel theorem for the compact group K and the identity (5.3.1), we infer that

$$\begin{aligned}\sum_{\sigma\in\hat{K}} d_\sigma \left\|W_\sigma(F)\right\|^2_{HS} &= (2\pi)^n \sum_{\beta,\gamma\in\mathbb{N}^n}\int_K\left|\sum_{|\alpha|=|\gamma|}\eta_{\alpha\gamma}(k)\int_{\mathbb{C}^n}F(z,k)\phi_{\alpha\beta}(z)dz\right|^2 dk\\ &= (2\pi)^n\sum_{\alpha,\beta\in\mathbb{N}^n}\int_K\left|\int_{\mathbb{C}^n}F(z,k)\phi_{\alpha\beta}(z)dz\right|^2dk\\ &= (2\pi)^n\int_K\int_{\mathbb{C}^n}|F(z,k)|^2dzdk.\end{aligned}$$

□

For $\sigma \in \hat{K}$, we defining a Fourier-Wigner type transform V^g_f of functions $f, g \in \mathcal{H}^2_\sigma$ on $G^\times$ by

$$V^g_f(z,k) = \langle\rho_\sigma(z,k)f, g\rangle.$$

Lemma 5.3.1. *For $f_l, g_l \in \mathcal{H}_\sigma^2$, $l = 1, 2$, the following identity holds.*

$$\int_K \int_{\mathbb{C}^n} V_{f_1}^{g_1}(z,k)\overline{V_{f_2}^{g_2}(z,k)}dzdk = (2\pi)^n \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}.$$

Proof. Since the set $\{\phi_\alpha \otimes e_i^\sigma : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma\}$ form an orthonormal basis for $\mathcal{H}_\sigma^2$ and $f_l, g_l \in \mathcal{H}_\sigma^2$, we can write

$$f_l = \sum_{\gamma\in\mathbb{N}^n} \sum_{1\leq i\leq d_\sigma} f_{\gamma,i}^l \phi_\gamma \otimes e_i^\sigma, \text{ and } g_l = \sum_{\beta\in\mathbb{N}^n} \sum_{1\leq j\leq d_\sigma} g_{\beta,j}^l \phi_\beta \otimes e_j^\sigma, \ l = 1, 2,$$

where $f_{\gamma,i}^l$ and $g_{\beta,j}^l$ are constants. Thus,

$$V_{f_l}^{g_l}(z,k) = (2\pi)^{\frac{n}{2}} \sum_{\alpha,\beta\in\mathbb{N}^n} \sum_{1\leq i,j\leq d_\sigma} \sum_{|\gamma|=|\alpha|} f_{\gamma,i}^l \overline{g_{\beta,j}^l} \eta_{\alpha\gamma}(k)\phi_{\alpha\beta}(z)\varphi_{ji}^\sigma(k),$$

By the orthogonality of the special Hermite functions $\phi_{\alpha\beta}$ together with the identity (5.3.1), it follows that

$$\int_{\mathbb{C}^n} V_{f_1}^{g_1}(z,k)\overline{V_{f_2}^{g_2}(z,k)}dz =$$

$$(2\pi)^n \sum_{\gamma,\beta\in\mathbb{N}^n} \left[\sum_{i,j=1}^{d_\sigma} \left(f_{\gamma,i}^1 \overline{g_{\beta,j}^1} \right) \phi_{ji}^\sigma(k) \sum_{i,j=1}^{d_\sigma} \left(\overline{f_{\gamma,i}^2} g_{\beta,j}^2 \right) \overline{\phi_{ji}^\sigma(k)} \right].$$

Finally, by integrating both the sides with respect to k, we get

$$\int_K \int_{\mathbb{C}^n} V_{f_1}^{g_1}(z,k)\overline{V_{f_2}^{g_2}(z,k)}dzdk =$$

$$(2\pi)^n \left(\sum_{\gamma\in\mathbb{N}^n} \sum_{1\leq i\leq d_\sigma} f_{\gamma,i}^1 \overline{f_{\gamma,i}^2} \right) \left(\sum_{\beta\in\mathbb{N}^n} \sum_{1\leq j\leq d_\sigma} g_{\beta,j}^2 \overline{g_{\beta,j}^1} \right) = (2\pi)^n \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}.$$

□

Notice that, if $f, g \in \mathcal{H}_\sigma^2$, then as particular case of Lemma 5.3.1, it follows that

$V_f^g \in L^2(G^\times)$. Let $V_\sigma = \overline{\text{span}}\left\{V_f^g : f, g \in \mathcal{H}_\sigma^2\right\}$. Since the set

$$B_\sigma = \left\{\psi_{\alpha,i}^\sigma = \phi_\alpha \otimes e_i^\sigma : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma\right\}$$

form an orthonormal basis for $\mathcal{H}_\sigma^2$, by Lemma 5.3.1, we infer that the set

$$V_{B_\sigma} = \left\{V_{\psi_{\alpha,i}^\sigma}^{\psi_{\beta,j}^\sigma} : \ \psi_{\alpha,i}^\sigma, \psi_{\beta,j}^\sigma \in B_\sigma\right\}$$

is an orthonormal basis for V_σ. Next, we recall the Peter-Weyl theorem which is crucial for the proof of Proposition 5.3.3. For more details, see [48].

Theorem 5.3.1. (Peter-Weyl). Let $\hat{K}$ be the unitary dual of the compact Lie group K. Then the set $\left\{\sqrt{d_\sigma}\phi_{ij}^\sigma : 1 \leq i, j \leq d_\sigma, \sigma \in \hat{K}\right\}$ is an orthonormal basis for the space $L^2(K)$.

Proposition 5.3.3. *The set* $\left\{V_{B_\sigma} : \sigma \in \hat{K}\right\}$ *is an orthonormal basis for* $L^2(G^\times)$.

Proof. By Theorem 5.3.1, it follows that $\left\{V_{B_\sigma} : \sigma \in \hat{K}\right\}$ is an orthonormal set. It only remains to prove the completeness. For this, suppose $F \in V_{B_\sigma}^\perp$, then

$$\begin{aligned} \left\langle W_\sigma(\overline{F})\psi_{\alpha,i}^\sigma, \psi_{\beta,j}^\sigma\right\rangle &= \int_K \int_{\mathbb{C}^n} \overline{F}(z,k) V_{\psi_{\alpha,i}^\sigma}^{\psi_{\beta,j}^\sigma}(z,k)\,dz dk \\ &= \left\langle F, V_{\psi_{\alpha,i}^\sigma}^{\psi_{\beta,j}^\sigma}\right\rangle = 0, \end{aligned}$$

whenever $\psi_{\alpha,i}^\sigma, \psi_{\beta,j}^\sigma \in B_\sigma$. Hence, it follows that $W_\sigma(\overline{F}) = 0$ for all $\sigma \in \hat{K}$. Thus, by Proposition 5.3.2, we conclude that $F = 0$. □

Moreover, by using the fact that V_{B_σ} is an orthonormal basis for V_σ, as a corollary to Proposition 5.3.3, we infer that $L^2(G^\times) = \bigoplus_{\sigma \in \hat{K}} V_\sigma$.

Now, we state our main result of this section. Let A and B be Lebesgue measurable subsets of $\mathbb{R}^n$ such that $0 < m(\mathrm{A})m(\mathrm{B}) < \infty$, where m denotes the Lebesgue measure.

Theorem 5.3.2. *Let $F \in L^1 \cap L^2(G)$ be supported on $(\Sigma \times \mathbb{R}) \times K$.*

(ii) If Σ has finite Lebesgue measure and $\hat{F}(\lambda, \sigma)$ is a rank one operator for all $(\lambda, \sigma) \in \mathbb{R}^* \times \hat{K}$, then $F = 0$.

(ii) If $\Sigma = A \times B$ and $\hat{F}(\lambda, \sigma)$ is a finite rank operator rank for all $(\lambda, \sigma) \in \mathbb{R}^* \times \hat{K}$, then $F = 0$.

Given $\phi, \psi \in L^2(\mathbb{R}^n)$, we define the Fourier-Wigner transform by

$$T(\phi, \psi)(z) = \langle \pi(z)\phi, \psi \rangle ,$$

where π is the Schrödinger representation corresponding to $\lambda = 1$. Next, we state the following result from [19, 21].

Theorem 5.3.3. *For $\phi, \psi \in L^2(\mathbb{R}^n)$, write $X = T(\phi, \psi)$. If $\{z \in \mathbb{C}^n : \ X(z) \neq 0\}$ has finite Lebesgue measure, then $X = 0$.*

In view of Theorem 5.3.3, we prove the following analogous result for the Fourier-Wigner transform. In fact, it says that the Fourier-Wigner transform of a pair of non-zero functions cannot be finitely supported.

Proposition 5.3.4. *For $f_j \in \mathcal{H}^2_\sigma; j = 1, 2$, denote $F = V^{f_2}_{f_1}$. If $\{z \in \mathbb{C}^n : F(z, k) \neq 0\}$ has finite Lebesgue measure for all $k \in K$, then $F = 0$.*

Proof. Since $f_j \in \mathcal{H}^2_\sigma$, we can express $f_j = \phi_j \otimes h_j$, where $\phi_j \in L^2(\mathbb{R}^n)$ and $h_j \in \mathcal{H}_\sigma$.

Then

$$\begin{aligned}
F(z,k) &= \langle \rho_\sigma(z,k) f_1, f_2\rangle \\
&= \langle \pi(z)\mu(k)\otimes\sigma(k)(\phi_1\otimes h_1), \phi_2\otimes h_2\rangle \\
&= \langle \pi(z)\mu(k)\phi_1, \phi_2\rangle \langle \sigma(k)h_1, h_2\rangle \\
&= \langle \pi(z)\psi_1, \phi_2\rangle \langle \sigma(k)h_1, h_2\rangle \\
&= X(z)\langle \sigma(k)h_1, h_2\rangle,
\end{aligned}$$

where $\psi_1 = \mu(k)\phi_1$ and $X = T(\psi_1, \phi_2)$. If $\langle \sigma(k)h_1, h_2\rangle = 0$ for some $k \in K$, then $F(.,k) = 0$. On the other hand, if $\langle \sigma(k)h_1, h_2\rangle \neq 0$, then X is a non-zero function that supported on a set of finite Lebesgue measure. Thus, in view of Theorem 5.3.3, we conclude that $F = 0$. □

Next, we prove that if for $F \in L^1 \cap L^2(G^\times)$, the operator $W_\sigma(F)$ is of finite rank for each $\sigma \in \hat{K}$, then $F = 0$. For proving this, we require the following crucial results.

For $k \in K$, define $b_j(k) = \langle \sigma(k)\psi_j, \psi_j\rangle$, where $\psi_j \in H_\sigma$. Then $b_j(e) = \|\psi_j\|^2$. Set $\|\psi_j\| = \alpha_j$.

Proposition 5.3.5. *For $\phi_j \in L^2(\mathbb{R}^n)$; $j \in \{1,\ldots,N\}$, define the function ψ on $\mathbb{C}^n$ by $\psi(z,k) = \sum_{j=1}^{N} b_j(k)\langle \pi(z)\mu(k)\phi_j, \phi_j\rangle$, where $k \in K$. If ψ is supported on a subset $\mathcal{E}\times\mathcal{F}$ of $\mathbb{C}^n$ such that $0 < m(\mathcal{E})m(\mathcal{F}) < \infty$, then $\psi \equiv 0$.*

Proof. Let e be the identity element of the group K. Then $\mu(e) = I$ is the identity operator on $L^2(\mathbb{R}^n)$. For $z = x + iy \in \mathbb{C}^n$, we write $\psi_y(x) = \psi(z,e)$. Since $\phi_j \in L^2(\mathbb{R}^n)$, there exists a set A of measure zero such that $|\phi_j|$ is finite on $\mathbb{R}^n \smallsetminus A$. Denote $K_y(\xi) = \sum_{j=1}^{N} \alpha_j^2 \phi_j(\xi+y)\overline{\phi_j(\xi)}$ for almost all $\xi \in \mathbb{R}^n$. Then by the hypothesis, ψ can be expressed as

$$\psi_y(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \frac{1}{2}x\cdot y)} K_y(\xi) d\xi. \tag{5.3.2}$$

Since ψ is supported on $\mathcal{E} \times \mathcal{F}$ of finite Lebesgue measure, it follows that $\psi_y = 0$ for all $y \in \mathbb{R}^n \smallsetminus \mathcal{F}$. Hence we infer that $K_y = 0$, whenever $y \in \mathbb{R}^n \smallsetminus \mathcal{F}$.

Define the function χ on $\mathbb{R}^n \smallsetminus A$ by $\chi = (\alpha_1\phi_1, \ldots, \alpha_N\phi_N)$. If $\chi = 0$ on $\mathbb{R}^n \smallsetminus A$, then result will follow. Suppose $\chi \neq 0$, then there exists $\xi_1 \in \mathbb{R}^n \smallsetminus A$ such that $\chi(\xi_1) \neq 0$.

Now, if it happens that $\chi = 0$ on $\mathbb{R}^n \smallsetminus (B(\xi_1) \cup A)$, where $B(\xi_1)$ is the set $\xi_1 + (\mathcal{F} \cup \{0\})$, then ϕ_j's are finitely supported.

Otherwise, we can choose $\xi_l \in \mathbb{R}^n \smallsetminus \bigcup_{i=1}^{l-1} B(\xi_i) \cup A$, where $B(\xi_i) = \xi_i + (\mathcal{F} \cup \{0\})$ such that $\chi(\xi_l) \neq 0$, whenever $l \leq N$. For $l \neq m$, we have $\xi_l - \xi_m \notin \mathcal{F}$. By the hypothesis, $K_{\xi_l - \xi_m}(\xi) = \sum_{j=1}^{N} \alpha_j^2 \phi_j(\xi + \xi_l - \xi_m)\overline{\phi_j(\xi)} = 0$, whenever $\xi \in \mathbb{R}^n \smallsetminus A$. Hence it follows that $\chi(\xi_l)$ and $\chi(\xi_m)$ are orthogonal. Thus, the set $S = \{\chi(\xi_1), \ldots, \chi(\xi_N)\}$ is an orthogonal set in $\mathbb{C}^N$.

Therefore, if $\xi \in \mathbb{R}^n \smallsetminus \bigcup_{l=1}^{N} (B(\xi_l) \cup A)$, then $\chi(\xi) \perp S$, and hence $\chi(\xi) = 0$. Thus, each of ϕ_j is supported on a set of finite Lebesgue measure.

Now, for $k \in K$, ψ can be expressed as

$$\psi_y(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \frac{1}{2}x\cdot y)} \left(\sum_{j=1}^{N} b_j(k)\chi_j(\xi + y)\overline{\phi_j(\xi)} \right) d\xi,$$

where $\chi_j = \mu(k)\phi_j \in L^2(\mathbb{R}^n)$. Let $H_y(\xi) = \sum_{j=1}^{N} b_j(k)\chi_j(\xi + y)\overline{\phi_j(\xi)}$. Then H_y is finitely supported for all $y \in \mathbb{R}^n$. By the Benedicks theorem, H_y and its Fourier transform both cannot be finitely supported simultaneously. Hence we conclude that $\psi_y \equiv 0$ for all $y \in \mathbb{R}^n$. □

Remark 5.1. *Instead of the rectangle $\mathcal{E} \times \mathcal{F}$ in $\mathbb{R}^{2n}$ if we consider a set E of finite Lebesgue measure in $\mathbb{R}^{2n}$, then the projection of E on $\mathbb{R}^n$ need not be a set of finite measure. Hence the above proof of Proposition 5.3.5 will not work.*

Let $\mathcal{E}$ and $\mathcal{F}$ be Lebesgue measurable subsets of $\mathbb{R}^n$ satisfying $0 < m(\mathcal{E})m(\mathcal{F}) < \infty$. Denote $\Sigma = \mathcal{E} \times \mathcal{F}$.

Theorem 5.3.4. *Let $F \in L^1 \cap L^2(G^\times)$ be supported on $\Sigma \times K$. If for each $\sigma \in \hat{K}$, the operator $W_\sigma(F)$ has finite rank, then $F = 0$.*

Proof. Let $\bar{\tau} = F^* \times F$, where $F^*(v) = \overline{F(v^{-1})}$. Then $W_\sigma(\bar{\tau}) = W_\sigma(F)^* W_\sigma(F)$ is a positive and finite rank operator on $\mathcal{H}^2_\sigma$. By the spectral theorem, it follows that

$$W_\sigma(\bar{\tau})f = \sum_{j=1}^{N} a_j \langle f, f_j \rangle f_j, \tag{5.3.3}$$

where $\{f_1, \ldots, f_N\}$ is an orthonormal basis for the range of $W_\sigma(\bar{\tau})$ which satisfies $W_\sigma(\bar{\tau})f_j = a_j f_j$ with $a_j \geq 0$. Now, for $f, g \in \mathcal{H}^2_\sigma$, we have

$$\begin{aligned} \langle W_\sigma(\bar{\tau})f, g \rangle &= \sum_{j=1}^{N} a_j \langle f, f_j \rangle \langle f_j, g \rangle \qquad (5.3.4) \\ &= (2\pi)^{-n} \sum_{j=1}^{N} a_j \int_K \int_{\mathbb{C}^n} V_f^g(z,k) \overline{V_{f_j}^{f_j}(z,k)} dz dk. \end{aligned}$$

Since $\tau \in L^2(G^\times)$, by Proposition 5.3.3, we can write $\tau = \bigoplus_{\sigma \in \hat{K}} \tau_\sigma$. In view of the above decomposition and by the definition of $W_\sigma(\bar{\tau})$, we can write

$$\begin{aligned} \langle W_\sigma(\bar{\tau})f, g \rangle &= \int_K \int_{\mathbb{C}^n} \bar{\tau}(z,k) \left\langle \rho^1_\sigma(z,k) f, g \right\rangle dz dk \qquad (5.3.5) \\ &= \int_K \int_{\mathbb{C}^n} \overline{\tau_\sigma}(z,k) V_f^g(z,k) dz dk. \end{aligned}$$

Hence, by comparing (5.3.4) with (5.3.5) in view of the orthogonality relation for the

Fourier-Wigner transform as in Lemma 5.3.1, it follows that

$$\tau_\sigma = \sum_{j=1}^{N} V_{h_j}^{h_j}, \tag{5.3.6}$$

where $h_j = (2\pi)^{-\frac{n}{2}} \sqrt{a_j} f_j \in \mathcal{H}_\sigma^2$. Now, let $h_j = \phi_j \otimes \psi_j$ for some $\phi_j \in L^2(\mathbb{R}^n)$ and $\psi_j \in \mathcal{H}_\sigma$. Then from (5.3.6) we have

$$\begin{aligned} \tau_\sigma(z,k) = \sum_{j=1}^{N} \langle \rho_\sigma(z,k)h_j, h_j \rangle &= \sum_{j=1}^{N} \langle \pi(z)\mu(k)\phi_j, \phi_j \rangle \langle \sigma(k)\psi_j, \psi_j \rangle \\ &= \sum_{j=1}^{N} b_j(k) \langle \pi(z)\chi_j, \phi_j \rangle , \end{aligned}$$

where $\chi_j = \mu(k)\phi_j \in L^2(\mathbb{R}^n)$ and $b_j(k) = \langle \sigma(k)\psi_j, \psi_j \rangle$. Since $\bar{\tau}$ is finitely supported in $\mathbb{C}^n$ variable, by Proposition 5.3.5, it follows that $\tau_\sigma = 0$, whenever $\sigma \in \hat{K}$. In view of Plancherel formula for the Weyl transform as mentioned in Proposition 5.3.2, we conclude that $F = 0$. □

Next, we prove Theorem 5.3.2 in the following two cases.

Proof of Theorem 5.3.2. (i). Since $F \in L^1 \cap L^2(G)$, we can write

$$\begin{aligned} \hat{F}(\lambda,\sigma) &= \int_K \int_{\mathbb{C}^n} \int_{\mathbb{R}} F(z,t,k)\rho_\sigma(z,t,k)dtdzdk \\ &= \int_K \int_{\mathbb{C}^n} F^\lambda(z,k)\rho_\sigma(z,k)dzdk \\ &= W_\sigma(F^\lambda). \end{aligned} \tag{5.3.7}$$

Suppose the operator $W_\sigma(F^\lambda)$ has rank one. Then it is enough to show that $F^\lambda = 0$. Consider the case when $\lambda = 1$. Since by hypothesis, $W_\sigma(F^1)$ has rank one, there exist $f_j \in \mathcal{H}_\sigma^2; j = 1, 2$ such that $W_\sigma(\bar{\tau})f = \langle f, f_1 \rangle f_2$ for all $f \in \mathcal{H}_\sigma^2$, where $\bar{\tau} = F^1$. Hence for $f, g \in \mathcal{H}^2_\sigma$, Lemma 5.3.1 yields

$$\begin{aligned} \langle W_\sigma(\bar\tau) f, g\rangle &= \langle f, f_1\rangle \langle f_2, g\rangle \\ &= (2\pi)^{-n} \int_K \int_{\mathbb{C}^n} V_f^g(z,k) \overline{V_{f_1}^{f_2}(z,k)} dz dk. \end{aligned} \tag{5.3.8}$$

Let $\tau = \bigoplus_{\sigma \in \hat{K}} \tau_\sigma$, where $\tau_\sigma \in V_{B_\sigma}$. Then by definition of $W_\sigma(\bar\tau)$, it follows that

$$\langle W_\sigma(\bar\tau) f, g\rangle = \int_K \int_{\mathbb{C}^n} \overline{\tau_\sigma}(z,k) V_f^g(z,k) dz dk. \tag{5.3.9}$$

Now, by comparing (5.3.8) with (5.3.9) in view of Proposition 5.3.3, we infer that $\tau_\sigma = (2\pi)^{-n} V_{f_1}^{f_2}$. Finally, by Proposition 5.3.4, it follows that $\tau_\sigma = 0$ for all $\sigma \in \hat{K}$. That is, $\tau = 0$ and hence we conclude that $F = 0$.

(ii). Suppose the operator $W_\sigma(F^\lambda)$ has finite rank. We prove the result for $\lambda = 1$ and the general case will be followed by the scaling argument. Since $\hat{F}(1,\sigma) = W_\sigma(F^1)$, by Theorem 5.3.4, it follows that $F^1 = 0$. Similarly, it can be shown that $F^\lambda = 0$ for all $\lambda \in \mathbb{R}^*$. Thus, we conclude that $F = 0$. $\square$

5.4 Preliminaries on step two nilpotent group

In this section, we prove an analogous result of the Benedick's theorem for the Euclidean Fourier transform on the step two nilpotent Lie groups. However, for the sake simplicity, we derive the result for the class of groups introduced by G. Métivier (see [29]). These groups are step two nilpotent Lie groups when quotiented with the hyperplane in the center becomes the Heisenberg group. The Heisenberg-type groups introduced by A. Kaplan (see [22]) are examples of Métivier group. However, there are Métivier groups which are distinct from the Heisenberg-type groups. For more details, see [30].

Let G be connected, simply connected Lie group with real step two nilpotent Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has the orthogonal decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp : \mathfrak{g} \to G$ is surjective. Thus, G can be parameterized by $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, endowed with the exponential coordinates.

Let $\{V_i : i = 1, \dots, m\}$ and $\{Z_j : j = 1, \dots, k\}$ be orthonormal bases of $\mathfrak{b}$ and $\mathfrak{z}$ respectively. Then for $V + Z \in \mathfrak{b} \oplus \mathfrak{z}$, we can identify $g \in G$ with the point $(V, Z) \in \mathbb{R}^m \times \mathbb{R}^k$ such that $g = \exp(V + Z)$. Since $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{z}$ and $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = \{0\}$, by the Baker-Campbell-Hausdorff formula, the group law on G can be expressed as

$$(V, Z)\,(V', Z') = \left(V + V', Z + Z' + \frac{1}{2}[V, V']\right).$$

Let dV and dZ be the Lebesgue measures on $\mathfrak{b}$ and $\mathfrak{z}$ respectively. Then the left-invariant Haar measure on G can be expressed as $dg = dV\,dZ$.

Now, for $\omega \in \mathfrak{z}^*$, consider the skew-symmetric bilinear form B_ω on $\mathfrak{b}$ by

$$B_\omega(X, Y) = \omega\left([X, Y]\right).$$

Let m_ω be the orthogonal complement of

$$r_\omega = \{X \in \mathfrak{b} : B_\omega(X, Y) = 0, \ \forall\, Y \in \mathfrak{b}\}$$

in $\mathfrak{b}$. Then B_ω is called a non-degenerate bilinear form when r_ω is trivial. If B_ω is non-degenerate for all $\omega \neq 0$, then G is called Métivier group.

Since m_ω is invariant under the skew-symmetric bilinear form B_ω, it follows that the dimension of m_ω is even. Let $\Lambda = \{\omega \in \mathfrak{z}^* : \dim m_\omega \text{ is maximum}\}$. Then Λ is a Zariski open subset of $\mathfrak{z}^*$ and for $\omega \in \Lambda$, there exists an orthonormal almost symplectic basis $\{X_i(\omega), Y_j(\omega) : i = 1, \ldots, n\}$ of $\mathfrak{b}$ and $d_i(\omega) > 0$ such that

$$\omega[X_i(\omega), Y_j(\omega)] = \begin{cases} \delta_{ij} d_i(\omega), & \text{when } X \neq Y; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\zeta_\omega = \text{span}\{X_i(\omega) : i = 1, \ldots, n\}$ and $\eta_\omega = \text{span}\{Y_j(\omega) : j = 1, \ldots, n\}$. Then we can write $\mathfrak{b} = \zeta_\omega \oplus \eta_\omega$ and each $(X, Y, Z) \in G$ can be represented by

$$(X, Y, Z) = \sum_{i=1}^{n} x_i(\omega) X_i(\omega) + \sum_{i=1}^{n} y_i(\omega) Y_i(\omega) + \sum_{i=1}^{k} t_i(\omega) Z_i(\omega).$$

Hence a typical element of G can be written as (x, y, t), where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^k$. For more details, we refer to [9, 24, 29].

Next, we briefly describe the irreducible representation of the Métivier group G which can be parameterized by Λ. That is, each $\omega \in \Lambda$ induces an irreducible unitary representation π_ω of G by

$$\left(\pi_\omega(x, y, t)\phi\right)(\xi) = e^{i \sum_{j=1}^{k} \omega_j t_j + i \sum_{j=1}^{n} d_j(\omega)\left(x_j \xi_j + \frac{1}{2} x_j y_j\right)} \phi(\xi + y),$$

whenever $\phi \in L^2(\eta_\omega)$. For the sake of simplicity, we write $v = (x, y)$. Then the group Fourier transform of $f \in L^1(G)$ can be defined by

$$\hat{f}(\omega) = \int_{\mathfrak{z}} \int_{\mathfrak{b}} f(v, t) \pi_\omega(v, t) \, dv dt,$$

where $\omega \in \Lambda$. Now, we define the Fourier inversion of f in the t variable by

$$f^\omega(v) = \int_{\mathfrak{z}} e^{i \sum_{j=1}^{k} \omega_j t_j} f(v, t) \, dt.$$

Then for the suitable functions f and g on $\mathfrak{b}$, we can define the ω-twisted convolution of f and g by

$$f *_\omega g(v) = \int_{\mathfrak{b}} f(v - v')g(v')e^{\frac{i}{2}\omega([v,v'])}dv'.$$

Here it is immediate that $(f * g)^\omega = f^\omega *_\omega g^\omega$. Let $p(\omega) = \Pi_{i=1}^n d_i(\omega)$ be the symmetric function of degree n corresponding to B_ω. For $f \in L^1 \cap L^2(G)$, the operator $\hat{f}(\omega)$ is a Hilbert-Schmidt operator that satisfies

$$p(\omega)\|\hat{f}(\omega)\|_{HS}^2 = (2\pi)^n \int_{\mathfrak{b}} |f^\omega(v)|^2 dv.$$

Denote $\pi_\omega(v) = \pi_\omega(v, o)$. Then the Fourier inversion f^ω can be determined by the formula

$$f^\omega(v) = (2\pi)^{-n} p(\omega) \operatorname{tr}(\pi_\omega(v)^* \hat{f}(\omega)).$$

5.5 Uniqueness results on step two nilpotent group

For $\omega \in \Lambda$ and $h \in L^1 \cap L^2(\mathfrak{b})$, the Weyl transform $W_\omega(h)$ is defined by

$$W_\omega(h) = \int_{\mathfrak{b}} h(v)\pi_\omega(v)dv. \tag{5.5.1}$$

The Weyl transform $W_\omega(h)$ is a Hilbert-Schmidt operator on $L^2(\eta_\omega)$ that satisfies the following Plancherel formula, (see [33]).

Theorem 5.5.1. *For $h \in L^2(\mathfrak{b})$, the following equality holds:*

$$p(\omega)\|W_\omega(h)\|_{HS}^2 = (2\pi)^n \int_{\mathfrak{b}} |h(v)|^2 dv.$$

Proposition 5.5.1. *For $h \in L^1 \cap L^2(\mathfrak{b})$, we have the identities:*

(i) $W_\omega(h^*) = W_\omega(h)^*$, *where* $h^*(v) = \overline{h(v^{-1})}$,

(ii) $W_\omega(h^* *_\omega h) = W_\omega(h)^* W_\omega(h)$.

Proof. (i) For $\phi, \psi \in L^2(\eta_\omega)$, we can write

$$\begin{aligned}\langle W_\omega(h^*)\phi, \psi\rangle &= \int_{\mathfrak{b}} h^*(v)\, \langle \pi_\omega(v)\phi, \psi\rangle\, dv \\ &= \int_{\mathfrak{b}} \left\langle \phi, h\left(v^{-1}\right) \pi_\omega\left(v^{-1}\right) g\right\rangle dv \\ &= \langle \phi, W_\omega(h)\psi\rangle = \langle W_\omega(h)^*\phi, \psi\rangle\,.\end{aligned}$$

(ii) Further, we have

$$\begin{aligned}\langle W_\omega(h)^* W_\omega(h)\phi, \psi\rangle &= \int_{\mathfrak{b}}\int_{\mathfrak{b}} h^*(v)h(v')\, \langle \pi_\omega(v)\pi_\omega(v')\phi, \psi\rangle\, dvdv' \\ &= \int_{\mathfrak{b}}\int_{\mathfrak{b}} h^*(v-v')h(v')e^{\frac{i}{2}\omega([v,v'])}\, \langle \pi_\omega(v)\phi, \psi\rangle\, dvdv' \\ &= \int_{\mathfrak{b}} (h^* *_\omega h)(v)\, \langle \pi_\omega(v)\phi, \psi\rangle\, dv \\ &= \langle W_\omega(h^* *_\omega h)\phi, \psi\rangle\,.\end{aligned}$$

□

Now, we state our main result of this section. Let A and B be Lebesgue measurable subsets of ζ_ω and η_ω respectively such that $0 < m(\mathrm{A})m(\mathrm{B}) < \infty$, where m denotes the Lebesgue measure.

Theorem 5.5.2. *Suppose $f \in L^1(G)$ is supported on the set $\Sigma \oplus \mathfrak{z}$, where Σ is a subset of $\mathfrak{b}$.*

(i) If Σ has finite Lebesgue measure and $\hat{f}(\omega)$ is a rank one operator for all $\omega \in \Lambda$, then $f = 0$.

(ii) If $\Sigma = A \times B$ and $\hat{f}(\omega)$ has finite rank for all $\omega \in \Lambda$, then $f = 0$.

In order to prove Theorem 5.5.2, we need the following crucial results. Let $\phi, \psi \in$

$L^2(\eta_\omega)$. Then the Fourier-Wigner transform of ϕ and ψ is a function on $\mathfrak{b}$ defined by

$$T(\phi,\psi)(v) = \langle \pi_\omega(v)\phi, \psi\rangle\,.$$

As a consequence of the Schur's orthogonality relation, these functions $T(\phi,\psi)$'s are orthogonal among themselves. For more details, we refer to Wolf [55].

Lemma 5.5.1. [55] *Let $\phi_j, \psi_j \in L^2(\eta_\omega);\ j = 1, 2$. Then*

$$\int_{\mathfrak{b}} T(\phi_1,\psi_1)(v)\overline{T(\phi_2,\psi_2)(v)}dv = c(\omega)\,\langle\phi_1,\phi_2\rangle\,\overline{\langle\psi_1,\psi_2\rangle},$$

where $c(\omega) = (2\pi)^n\ p(\omega)^{-1}$.

We observe that these functions $T(\phi,\psi)$'s generate an orthonormal basis for $L^2(\mathfrak{b})$. Let $\{\varphi_j : j \in \mathbb{N}\}$ be an orthonormal basis for $L^2(\eta_\omega)$.

Proposition 5.5.2. *The set $\{T(\varphi_i,\varphi_j) : i, j \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\mathfrak{b})$.*

Proof. In view of Lemma 5.5.1, it is clear that $\{T(\varphi_i,\varphi_j) : i, j \in \mathbb{N}\}$ is an orthonormal set. Now, it only remains to verify the completeness. For this, let $f \in L^2(\mathfrak{b})$ be such that $\langle f, T(\varphi_i,\varphi_j)\rangle = 0$, whenever $i, j \in \mathbb{N}$. Then

$$\begin{aligned}\left\langle W_\omega(\bar{f})\phi_i,\phi_j\right\rangle &= \int_{\mathfrak{b}} \bar{f}(v)\,\langle\pi_\omega(v)\phi_i,\phi_j\rangle\,dv \\ &= \langle f, T(\phi_i,\phi_j)\rangle = 0.\end{aligned} \tag{5.5.2}$$

Hence, we infer that $W_\omega(\bar{f}) = 0$. Thus, by the Plancherel Theorem 5.5.1, we conclude that $f = 0$. □

Proposition 5.5.3. *Let $F = T(\phi,\psi)$, where $\phi,\psi \in L^2(\eta_\omega)$. If the set $\{v \in \mathfrak{b} : F(v) \neq 0\}$ has a finite Lebesgue measure, then F has to vanish identically.*

Proof. We would like to mention that the proof of Proposition 5.5.3 is almost similar to Theorem 5.3.3 and hence we omit it here. □

Let $\mathcal{E}$ and $\mathcal{F}$ be Lebesgue measurable subsets of ζ_ω and η_ω respectively such that $0 < m(\mathcal{E})m(\mathcal{F}) < \infty$. Denote $\Sigma = \mathcal{E} \times \mathcal{F}$.

Lemma 5.5.2. *For $h_j \in L^2(\eta_\omega)$, write $K_y(\xi) = \sum_{j=1}^{N} h_j(\xi + y)\overline{h_j(\xi)}$, where $y \in \eta_\omega$. If $K_y(\xi) = 0$ for all $y \in \eta_\omega \setminus \mathcal{F}$ and for almost all $\xi \in \eta_\omega$, then each of h_j is finitely supported.*

Proof. Since $h_j \in L^2(\eta_\omega)$, there exists a set A of Lebesgue measure zero such that $|h_j|$ is finite on $\eta_\omega \setminus A$. Define a function χ on $\eta_\omega \setminus A$ by $\chi = (h_1, \ldots, h_N)$. If h_j is non-vanishing on $\eta_\omega \setminus A$ for some j, then we can choose $\xi_1 \in \eta_\omega \setminus A$ such that $\chi(\xi_1) \neq 0$. Let $B(\xi_1)$ be the set $\xi_1 + (\mathcal{F} \cup \{0\})$. If χ vanishes on $\eta_\omega \setminus B(\xi_1) \cup A$, then the result follows. Otherwise, by induction, we can choose $\xi_j \in \eta_\omega \setminus \bigcup_{i=1}^{j-1} (B(\xi_i) \cup A)$ such that $\chi(\xi_j) \neq 0$, whenever $j \leq N$, where $B(\xi_i) = \xi_i + (\mathcal{F} \cup \{0\})$. Thus by the hypothesis, the set $S = \{\chi(\xi_j) : j = 1, \ldots, N\}$ is an orthogonal set in $\mathbb{C}^N$. Now, if $\xi \in \eta_\omega \setminus \bigcup_{j=1}^{N} (B(\xi_j) \cup A)$, then $\chi(\xi) \in S^\perp$, and hence $\chi(\xi) = 0$.. □

Proposition 5.5.4. *Let $h \in L^1 \cap L^2(\mathfrak{b})$ be supported on Σ in $\mathfrak{b}$. If $W_\omega(h)$ is a finite rank operator, then $h = 0$.*

Proof. Let $\bar{\tau} = h^* *_\omega h$, where $h^*(v) = \overline{h(v^{-1})}$. Then $W_\omega(\bar{\tau}) = W_\omega(h)^* W_\omega(h)$ is a positive and finite rank operator on $L^2(\eta_\omega)$. By the spectral theorem, there exists an orthonormal set $\{\phi_j \in L^2(\eta_\omega) : j = 1, \ldots, N\}$ and scalars $a_j \geq 0$ such that

$$W_\omega(\bar{\tau})\phi = \sum_{j=1}^{N} a_j \langle \phi, \phi_j \rangle \phi_j,$$

whenever $\phi \in L^2(\eta_\omega)$. Now, for $\psi \in L^2(\eta_\omega)$, we have

$$\begin{aligned} \langle W_\omega(\bar{\tau})\phi, \psi\rangle &= \sum_{j=1}^{N} a_j \langle \phi, \phi_j\rangle \langle \phi_j, \psi\rangle \\ &= c(\omega)^{-1}\sum_{j=1}^{N} a_j \int_{\mathfrak{b}} T(\phi,\psi)(v)\overline{T(\phi_j,\phi_j)(v)}dv. \end{aligned} \tag{5.5.3}$$

Further, by definition of $W_\omega(\bar{\tau})$, we have

$$\langle W_\omega(\bar{\tau})\phi, \psi\rangle = \int_{\mathfrak{b}} \bar{\tau}(v)T(\phi,\psi)(v)dv. \tag{5.5.4}$$

Hence, by comparing (5.5.3) with (5.5.4) in view of Proposition 5.5.2, it follows that

$$\tau = \sum_{j=1}^{N} T(h_j, h_j), \tag{5.5.5}$$

where $h_j = c(\omega)^{-\frac{1}{2}}\sqrt{a_j}\ \phi_j \in L^2(\eta_\omega)$. Now, for $v = (x, y)$, write $\tau_y(x) = \tau(x, y)$. Then Equation (5.5.5) becomes

$$\tau_y(x) = \int_{\eta_\omega} e^{i\sum\limits_{j=1}^{n} d_j(\omega)(x_j\xi_j + \frac{1}{2}x_jy_j)} K_y(\xi)d\xi. \tag{5.5.6}$$

Since $\bar{\tau}$ is supported on $\mathcal{E}\times\mathcal{F}$, it follows that $K_y(\xi) = 0$ for almost every ξ and for all $y \in \eta_\omega \smallsetminus \mathcal{F}$. Then in view of Lemma 5.5.2, it follows that each of h_j is finitely supported and hence each of K_y is finitely supported. Since τ_y is is supported on $\mathcal{E}$, whenever $y \in \eta_\omega$, we infer that τ_y is zero for all $y \in \eta_\omega$. Now, by Plancherel Theorem 5.5.1, we conclude that $h = 0$. □

Proof of Theorem 5.5.2. (i). By a simple calculation, we get

$$\hat{f}(\omega) = \int_{\mathfrak{b}} f^\omega(v)\pi_\omega(v)dv = W_\omega(f^\omega).$$

Since f^ω is finitely supported and the operator $W_\omega(f^\omega)$ has finite rank, by Proposition 5.5.4, it follows that $f^\omega = 0$, whenever $\omega \in \Lambda$. Hence we infer $f = 0$.

(ii). It is enough to prove that if $W_\omega(f^\omega)$ has rank one, then $f^\omega = 0$. Let $W_\omega(f^\omega)$ be a rank one operator. Then there exist $\phi_j \in L^2(\eta_\omega)$; $j = 1, 2$ such that

$$W_\omega(\bar{\tau})\phi = \langle \phi, \phi_1 \rangle \, \phi_2$$

for all $\phi \in L^2(\eta_\omega)$, where $\bar{\tau} = f^\omega$. Thus, for $\psi \in L^2(\eta_\omega)$, it follows that

$$\begin{aligned} \langle W_\omega(\bar{\tau})\phi, \psi \rangle &= \langle \phi, \phi_1 \rangle \langle \phi_2, \psi \rangle \\ &= c(\omega)^{-1} \int_{\eta_\omega} \int_{\zeta_\omega} T(\phi, \psi)(v) \overline{T(\phi_1, \phi_2)(v)} dv. \end{aligned} \tag{5.5.7}$$

Further, by definition, we get

$$\langle W_\omega(\bar{\tau})\phi, \psi \rangle = \int_{\zeta_\omega} \int_{\eta_\omega} \bar{\tau}(v) T(\phi, \psi)(v) dv. \tag{5.5.8}$$

Hence by comparing (5.5.7) with (5.5.8) in view of Lemma 5.5.1, we infer that

$$\tau(v) = c(\omega)^{-1} T(\phi_1, \phi_2)(v).$$

Thus, from Proposition 5.5.3, it follows that $\tau \equiv 0$. $\square$

Remarks and open problems:

If the Fourier transform of a compactly supported function f on $\mathbb{H}^n \ltimes U(n)$ (or step two nilpotent Lie groups) lands into the space of compact operators, then f might be zero. However, it would be a good question to consider the case when the spectrum of the Fourier transform of a compactly supported function is supported on a thin uncountable set.

Bibliography

[1] W.O. Amrein and A.M. Berthier, *On support properties of L^p functions and their Fourier transforms*, J. Funct. Anal. 24 (1977), 258-267.

[2] G.E. Andrews, R. Askey and R. Roy, *Special Functions,* Cambridge University Press, Cambridge, 1999.

[3] T.M. Apostol, *Modular Functions and Dirichlet Series in Number Theory,* 2nd ed. New York: Springer-Verlag, 1997.

[4] F. Astengo, M. Cowling, B.D. Blasio, and M. Sundari, *Hardy's uncertainty principle on certain Lie groups,* J. Lond. Math. Soc. 62 (2000), pp. 461-472.

[5] D.B. Babot, *Heisenberg uniqueness pairs in the plane, Three parallel lines,* Proc. Amer. Math. Soc. 141 (2013), no. 11, 3899-3904.

[6] S.C. Bagchi and A. Sitaram, *Determining sets for measures on* $\mathbb{R}^n$, Illinois J. Math. 26 (1982), no. 3, 419-422.

[7] M. Benedicks, *On Fourier transforms of functions supported on sets of finite Lebesgue measure,* J. Math. Anal. Appl. 106 (1985), no.1, 180-183.

[8] C. Benson, J. Jenkins and G. Ratcliff, *Bounded K-spherical functions on Heisenberg groups,* J. Funct. Anal. 105 (1992), no. 2, 409-443.

[9] L.J. Corwin and F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications*. Cambridge Studies in Advanced Mathematics, 18. Cambridge University Press, Cambridge, 1990.

[10] W.F. Donoghue, Jr., *Distribution and Fourier transforms,* Academic Press, New York, 1969.

[11] L. Flatto, D.J. Newman and H. S. Shapiro, *The level curves of harmonic polynomials,* Trans. Amer. Math. Soc. 123 (1966), 425-436.

[12] G.B. Folland, *A course in abstract harmonic analysis,* Stud. Adv. Math. CRC Press, Boca Raton, Fla, 1995.

[13] G.B. Folland, *Real analysis. Modern techniques and their applications*, Second edition, John Wiley and Sons, Inc., New York, 1999.

[14] R. Gangolli, *On the symmetry of L1 algebras of locally compact motion groups, and theWiener Tauberian theorem,* J. Funct. Anal. 25 (1977), no. 3, 244-252.

[15] K. Gröchenig and P. Jaming, *The Cramér-Wold theorem on quadratic surfaces and Heisenberg uniqueness pairs,* arXiv:1608.06738.

[16] K.I. Gross and R.A. Kunze, *Fourier decompositions of certain representations in Symmetric Spaces,* Boothby WandWeiss G, Eds., Marcel-Dekker, New York, 1972, 119-139.

[17] V. Havin and B. Jöricke, *The Uncertainty Principle in Harmonic Analysis,* Springer-Verlag, Berlin, 1994.

[18] H. Hedenmalm and A. Montes-Rodríguez, *Heisenberg uniqueness pairs and the Klein-Gordon equation,* Ann. of Math. (2) 173 (2011), no. 3, 1507-1527.

[19] P. Jaming, *Principe d'incertitude qualitatif et reconstruction de phase pour la transformee de Wigner,* C. R. Acad. Sci. Paris Ser. I Math. 327 (1998), no. 3, 249-254.

[20] P. Jaming and K. Kellay, *A dynamical system approach to Heisenberg uniqueness pairs,* J. Anal. Math., Springer, 2015, http://link.springer.com/journal/11854

[21] A.J.E.M. Janssen, *Proof of a conjecture on the supports of Wigner distributions,* J. Fourier Anal. Appl. 4 (1998), no. 6, 723-726.

[22] J.P. Kahane, *Sries de Fourier absolument convergentes,* (French) Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 50 Springer-Verlag, Berlin-New York, 1970

[23] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms,* Trans. Amer. Math. Soc. 258 (1980), no. 1, 147-153.

[24] A. Kirillov, *Unitary representations of nilpotent Lie groups,* Russian Math. Sureys 17 (1962), 53-104.

[25] K. Kumahara and K. Okamoto, *An analogue of the Paley-Wiener theorem for the Euclidean motion group,* Osaka J. Math. 10 (1973), 77-91.

[26] K. Kumahara, *Fourier transforms on the motion groups,* J. Math. Soc. Japan 28 (1976), no. 1, 18-32.

[27] N. Lev, *Uniqueness theorem for Fourier transform. Bull. Sc. Math.,* 135 (2011), 134-140.

[28] I.G. Macdonald, *Symmetric Functions and Hall Polynomials,* Second ed., Oxford University Press, New York, 1995.

[29] G. Métivier, *Hypoellipticité analytique sur des groupes nilpotents de rang 2,* Duke Math. J. 47 (1980), no. 1, 195-221.

[30] D. Müller and A. Seeger, *Singular spherical maximal operators on a class of two step nilpotent Lie groups,* Israel J. Math. 141 (2004), 315-340.

[31] E.K. Narayanan and P.K. Ratnakumar, *Benedicks theorem for the Heisenberg group,* Proc. Amer. Math. Soc. 138 (2010), no. 6, 2135-2140.

[32] S. Parui and R.P. Sarkar, *Beurling's theorem and $L^p - L^q$ Morgan's theorem for step two nilpotent Lie groups*. Publ. Res. Inst. Math. Sci. 44 (2008), no. 4, 1027-1056.

[33] S. Parui and S. Thangavelu, *On theorems of Beurling and Hardy for certain step two nilpotent groups,* Integral Transforms Spec. Funct. 20 (2009), no. 1-2, 127-145.

[34] V. Pati and A. Sitaram, *Some questions on integral geometry on Riemannian manifolds,* Ergodic theory and harmonic analysis (Mumbai, 1999). Sankhya Ser. A 62 (2000), no. 3, 419-424.

[35] A. Poltoratski, *Spectral gaps for sets and measures,* Acta Math. 208 (2012), no. 1, 151-209.

[36] A. Poltoratski, *A problem on completeness of exponentials,* Ann. of Math. (2) 178 (2013), no. 3, 983-1016.

[37] J.F. Price and A. Sitaram, *Functions and their Fourier transforms with supports of finite measure for certain locally compact groups*, J. Funct. Anal. 79 (1988), no. 1, 166-182.

[38] R. Rawat and A. Sitaram, *Injectivity sets for spherical means on $\mathbb{R}^n$ and on symmetric spaces,* Jour. Four. Anal. Appl., 6 (2000), 343-348.

[39] S.K. Ray, *Uncertainty principles on two step nilpotent Lie groups.* Proc. Indian Acad. Sci. Math. Sci. 111 (2001), no. 3, 293-318.

[40] R.P. Sarkar and S. Thangavelu, *On theorems of Beurling and Hardy for the Euclidean motion group,* Tohoku Math. J. (2) 57 (2005), no. 3, 335-351.

[41] S. Sen, *Segal-Bargmann transform and Paley-Wiener theorems on Heisenberg motion groups*, Adv. Pure Appl. Math. 7 (2016), no. 1, 13-28.

[42] A. Sitaram, M. Sundari and S. Thangavelu, *Uncertainty principles on certain Lie groups,* Proc. Indian Acad. Sci. Math. Sci. 105 (1995), no. 2, 135-151.

[43] P. Sjölin, *Heisenberg uniqueness pairs and a theorem of Beurling and Malliavin,* Bull. Sc. Math., 135 (2011), 125-133.

[44] P. Sjölin, *Heisenberg uniqueness pairs for the parabola. Jour. Four. Anal. Appl.,* 19 (2013), 410-416.

[45] C.D. Sogge, *Oscillatory integrals and spherical harmonics,* Duke Math. J. 53 (1986), 43-65.

[46] R.K. Srivastava, *Non-harmonic cones are Heisenberg uniqueness pairs for the Fourier transform on* $\mathbb{R}^n$, J. Fourier Anal. Appl. 20th February (online), 2018.

[47] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces,* Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.

[48] M. Sugiura, *Unitary representations and harmonic analysis*, North-Holland Mathematical Library, 44. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1990.

[49] G. Szegö, *Orthogonal polynomials,* AMS, Colloq. Publ. vol. 23, New York, 1939.

[50] S. Thangavelu, *Harmonic analysis on the Heisenberg group. Progress in Mathematics,* 159. Birkhuser Boston, Inc., Boston, 1998.

[51] S. Thangavelu, *An introduction to the uncertainty principle*, Prog. Math. 217, Birkhauser, Boston, 2004.

[52] M.K. Vemuri, *Benedicks theorem for the Weyl transform*, J. Math. Anal. Appl. 452 (2017), no. 1, 209-217.

[53] F.J.G. Vieli, *A uniqueness result for the Fourier transform of measures on the sphere,* Bull. Aust. Math. Soc. 86 (2012), 78-82.

[54] F.J.G. Vieli, *A uniqueness result for the Fourier transform of measures on the paraboloid,* Mat. Vesnik, Vol. 67 (2015), No. 1, 52-55.

[55] J.A. Wolf, *Classical Analysis and Nilpotent Lie Groups*, Colloquium De Giorgi 2009, 1932, Colloquia, 3, Ed. Norm., Pisa, 2012.

Conclusion

In this thesis, we explored the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle, cross, exponential curves, and surfaces. Then, we prove a characterization of the Heisenberg uniqueness pairs corresponding to four parallel lines. We observe that the size of the determining sets Λ for $X(\Gamma)$ depends on the number of lines and their irregular distribution that further relates to a phenomenon of interlacing of the zero sets of certain trigonometric polynomials.

M. Benedicks [7] had extended the classical Paley-Wiener theorem (about uncertainty principle) to the class of integrable functions. That is, support of an integrable function f and its Fourier transform $\hat{f}$ both can not be of finite measure simultaneously.The Fourier transform on a non-commutative group becomes a linear operator of large rank in contrast to the Euclidean spaces $\mathbb{R}^n$ where the Fourier transform has rank one and hence it can be think of function on $\mathbb{R}^n$.

In this thesis, we prove that if the group Fourier transform of certain integrable functions on the Euclidean motion groups (or Heisenberg motion group/ step two nilpotent Lie groups) is of finite rank, then the function has to identically zero. These results can be thought as an analogue to the Benedicks theorem that dealt with the uniqueness of the Fourier transform of integrable functions on the Euclidean spaces.

List of communicated papers

1. D.K. Giri and R.K. Srivastava, Heisenberg uniqueness pairs for some algebraic curves in the plane, Adv. Math., 310 (2017), 993-1016.

2. A. Chattopadhyay, D.K. Giri and R.K. Srivastava, Uniqueness of the Fourier transform on certain Lie groups, (submitted).

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