Control Over Erasure Communication Networks

A

Thesis Submitted

in Partial Fulfilment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

By

Abhijit Mazumdar

Department of Electronics and Electrical Engineering
Indian Institute of Technology Guwahati
Guwahati - 781 039, INDIA.

June, 2021

Dedicated to

My Beloved Parents,

My Brother,

and

My Sister

Certificate

This is to certify that the thesis entitled "Control Over Erasure Communication Networks", submitted by Abhijit Mazumdar (146102019), a research scholar in the Department of Electronics and Electrical Engineering, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy, is a record of an original research work carried out by him under our supervision and guidance. The thesis has fulfilled all requirements as per the regulations of the institute and in our opinion has reached the standard needed for submission. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

Date:

Place: Guwahati

Dr. Srinivasan Krishnaswamy
Dept. of Electronics & Electrical Engg.
Indian Institute of Technology Guwahati
Guwahati - 781039, Assam, India.

Date:

Place: Guwahati

Prof. Somanath Majhi
Dept. of Electronics & Electrical Engg.
Indian Institute of Technology Guwahati
Guwahati - 781039, Assam, India.

Declaration

I declare that this written submission represents my ideas in my own words, and where others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Date:

Place: Guwahati.

Abhijit Mazumdar

Roll No: 146102019

Acknowledgements

First and foremost, I would like to express my deep gratitude to my thesis supervisors, Dr. Srinivasan Krishnaswamy and Professor Somanath Majhi, for their endless support and patience during this work. I am indebted to them for sharing their invaluable knowledge and experiences, making it possible to complete this thesis. Dr. Srinivasan Krishnaswamy has always motivated me to work on exciting and challenging problems, and also inspired me to work with mathematical rigor. Prof. Somanath Majhi Sir has, on the other hand, provided critical insights into my research. This work would not have been possible without their constant and untiring guidance.

I would like to sincerely thank my doctoral committee members Professor Chitralekha Mahanta, Dr. Indrani Kar, and Dr. Hanumant Singh Shekhawat, for evaluating the progress of my work. Their critique and suggestions have helped me immensely.

I wish to thank my dearest friend/brother Nayanjyoti Kakati for his unconditional support and consistent encouragement throughout this journey. His faith in me, even at times when I had doubts in myself, made me keep going. I am deeply grateful to him for motivating me during the rough period throughout this journey. I also extend my deep appreciation to Abhishek, Shashank, and Amit for their joyful company and making this journey a memorable one. A heartfelt thanks to Dibya Da for his unconditional help and for inspiring me to be a better person. I wish to thank Nayan Moni, Riju, Govind, Saswati, and Tamen for their joyful company. A special thanks to Pranjal Barman Da and Bhaskar Da for all the help and wonderful experiences.

I would like to convey my sincere gratitude to my colleagues from Control & Instrumentation Laboratory, Arghya Da, Trusna, Kamakshi, Sumi, Gautam, Mriganka, Raju, Sami, Manmohan, Mandar Da, Sushanta Da, and Suman Da for various discussions and joyful experience. I also wish to thank Uddipana Ba and Kasturi for all the help and useful discussions.

I am grateful to the technical and office staff members of the department, Mr. Mukut Baruah, Late Mr. Uday Shankar Uzir, Mr. Dasarath Das, Mr. Sundeep Borah, Syed Samimul Mazid, Mr. Sanjib Das, Mr. Sidananda Sonowal, Mr. Pranab Jyoti Goswami, Mrs. Chayanika Borah Majumdar, and Ms. Khurshida Yasmin for their help throughout my work.

Last but definitely not the least, I would like to express my heartfelt thanks to my parents for their endless sacrifices and relentless hard work. A sincere thanks to my brother Aniruddha, my sister Torali, my brother-in-law Jitu, my little nephew Sannidhya, my cousins Jagadish and Simanta for their constant and untiring support. Without their unmatched love and support, this journey would not have been possible.

Date: **Abhijit Mazumdar**

“Every great and deep difficulty bears in itself its own solution. It forces us to change our thinking in order to find it.”

Niels Bohr

Abstract

A control system architecture where its different subsystems, viz: the controller, actuators, and sensors, are connected through a communication network is called a networked control system (NCS). It provides many advantages, such as modular structure and flexibility in design. However, communication networks, which are generally lossy, bring some serious issues, e.g., time-delays, packet loss, and quantization into NCSs. These issues make the design of NCSs very complicated. In this thesis, we are concerned with investigating a few important control problems considering the packet loss issue. They are as follows.

To begin with, the H_∞ optimal control problem of a linear time-invariant (LTI) system operating over multiple communication channels is investigated. In order to account for the temporal correlation in packet loss, the Markovian packet loss model is considered. We formulate the problem in a dynamic game setting. The existence conditions for the finite horizon controller and the infinite horizon controller are derived in terms of control packet arrival probabilities and the disturbance attenuation level. The infinite horizon controller is designed by analyzing the convergence of the associated coupled algebraic Riccati equations (CAREs). The closed-loop system's stability is investigated for three cases: 1) without any external disturbance, 2) with finite energy disturbance, and 3) with the worst-case disturbance.

We then consider Markovian jump linear systems (MJLS). For such systems, the jump linear quadratic (JLQ) optimal controller over multiple lossy channels is designed. The existence of a positive definite stabilizing solution of the infinite horizon CAREs associated with the JLQ problem is established by using the weak observability notion.

Next, the results for the H_∞ optimal controller design problem over multiple lossy channels are extended to an MJLS. The existence conditions for the finite horizon controller and the in-

finite horizon controller are derived. Various results pertaining to the infinite horizon controller are established by using a relaxed notion of observability, namely weak observability. Furthermore, the condition for the closed-loop system's stability with the worst-case disturbance is derived.

Then, a smooth control-affine nonlinear system is considered. For such a system, a state-feedback H_∞ controller is designed. Subsequently, we extend these results to a smooth nonlinear system with the Markovian control packet loss. In our approach, we partition a subspace of the state-space into a number of polyhedral cells and approximate the nonlinear system by a piecewise affine (PWA) system in each of these cells. The main characteristic of the approach is that one can design the controller by solving a set of linear matrix inequalities (LMIs), which need to satisfy certain nonlinear constraints. Once the LMIs are solved, it is trivial to check whether the solutions satisfy the nonlinear constraints. As LMIs are easy to solve, our approach provides an efficient way of the controller design.

Finally, by using the PWA approximation approach, we investigate the problem of local feedback passivation of a smooth nonlinear system. First, the local feedback passivation of a classical nonlinear system is considered. After that, the problem of local feedback passivity with Markovian packet loss is studied. The controller is designed by solving a set of LMIs that are subject to certain nonlinear constraints.

সাৰাংশ

(Abstract in Assamese Language)

Networked control system (NCS) হ'ল এক control system architecture, য'ত ইয়াৰ বিভিন্ন subsystems, যেনে controller, actuators আৰু sensorsবোৰ এটা communication networkৰ জৰিয়তে সংযোগ কৰা থাকে। এই architectureটোৰ বহুতো সুবিধা আছে, যেনে: ভিন ভিন সৰু অংশৰে সহজে গঠন কৰিব পৰা গুণ (modular structure) আৰু ৰূপাঙ্কনৰ নমনীয়তা (flexibility in design)। যিহেতু communication networksবোৰ সাধাৰণতে lossy, NCSত কিছুমান গুৰুত্বৰ সমস্যা, যেনে: time-delay, packet loss আৰু quantizationএ দেখা দিয়ে। এই সমস্যাবোৰে NCSৰ ৰূপাঙ্কন প্ৰক্ৰিয়াটো বহুত জটিল কৰি তোলে। এই গৱেষণা গ্ৰন্থখনত, packet loss সমস্যাটোক বিবেচনা কৰি আমি কেইটামান গুৰুত্বপূৰ্ণ control problemsৰ তদন্ত কৰো। এই problems কেইটা তলত উল্লেখ কৰা ধৰণৰ।

প্ৰথমতে, এটা Linear time-invariant systemৰ বাবে H∞ optimal control over multiple communication channels problemটোৰ তদন্ত কৰা হৈছে। Packet lossৰ সময়গত পাৰস্পৰিক সম্পৰ্ক (temporal correlation)ক বিবেচনা কৰিবৰ বাবে, Markovian packet loss আৰ্হিটো বিবেচনা কৰা হৈছে। আমি problemটোক dynamic game setting ত উপস্থাপন কৰো। Finite horizon আৰু infinite horizon controllerৰ অস্তিত্বৰ চৰ্তবোৰ (existence conditions) control arrival probabilities আৰু disturbance attenuation levelৰ ৰূপত আহৰণ কৰা হয়। Infinite horizon controllerটো ৰূপাঙ্কন কৰিবৰ লৈ তাৰ লগত সংযুক্ত coupled algebraic Riccati equations (CAREs)ৰ সংমিলন (convergence)ৰ বিশ্লেষন কৰা হয়। Closed-loop systemৰ স্থিৰতা (stability)ৰ তদন্ত তিনিটা পৰিস্থিতিৰ বাবে কৰা হয় : ১) কোনো external disturbance নোহোৱাকৈ, ২) সীমিত শক্তিৰ disturbanceৰ সৈতে, আৰু ৩) worst-case disturbance ৰ সৈতে।

তাৰ পাছত আমি Markovian jump linear systemsৰ বিবেচনা কৰো। এনে systemsৰ বাবে, jump linear quadratic (JLQ) optimal controller over multiple lossy channelsৰ ৰূপাঙ্কন কৰা হয়। Weak observability ধাৰণা (notion)ৰ সহায়ত JLQ problemৰ লগত সংযুক্ত infinite horizon CAREsৰ positive definite stabilizing solutionৰ অস্তিত্ব প্ৰতিষ্ঠা কৰা হয়।

পৰৱৰ্তী পৰ্য্যায়ত, H∞ optimal controller design problem over multiple lossy channelsৰ সিদ্ধান্তবোৰ MJLSsলৈ প্ৰসাৰ কৰা হয়। Finite horizon controller আৰু infinite horizon controllerৰ অস্তিত্বৰ চৰ্তবোৰ (existence conditions) আহৰণ কৰা হয়। Infinite horizon controllerৰ সৈতে সম্পৰ্কিত বিভিন্ন সিদ্ধান্তবোৰ এক relaxed ধাৰণাৰে প্ৰতিষ্ঠা কৰা হয়। তদুপৰি, worst-case disturbanceৰ সৈতে closed-loop systemৰ স্থিৰতাৰ চৰ্ত আহৰণ কৰা হয়। ইয়াৰ পিছত এক smooth control-affine nonlinear systemৰ বিবেচনা কৰা হৈছে। এনেকুৱা systemৰ বাবে, এক state-feedback H∞ controllerৰ ৰূপাঙ্কন কৰা হৈছে।

পৰৱৰ্তীকালত, এই সিদ্ধান্তবোৰ আমি morkovian control packet lossৰ সৈতে এক smooth nonlinear systemলৈ প্ৰসাৰ কৰো। এই পদ্ধতিটোত আমি state-space ৰ ভিতৰৰ এক subspaceক কিছুমান বহুভুজ

আকাৰৰ (polyhedral) কক্ষ (cell)ত বিভাজন কৰো আৰু প্ৰতিটো কক্ষত nonlinear systemটোক এক piecewise affine system (PWA)ৰে আসন্ন (approximate)কৰা হয়। এই পদ্ধতিটোৰ প্ৰধান বৈশিষ্ট্য হ'ল যে কেইটামান linear matrix inequalities (LMIs) সমাধান কৰি controllerৰ ৰূপাঙ্কন কৰিব পাৰি, যদিহে কিছুমান অৰৈখিক চৰ্ত (nonlinear constraints)ৰ পালন হয়। যদি এই LMIs কেইটা সমাধান কৰা যায়, অৰৈখিক চৰ্তকেইটা পালন হৈছে নে নাই তাক অতি সহজে পৰীক্ষা কৰি চাব পাৰি। যিহেতু LMI সহজে সমাধান কৰিব পাৰি, আমাৰ পদ্ধতিটো ব্যৱহাৰ কৰি কাৰ্য্যক্ষমতাৰে controllerৰ ৰূপাঙ্কন কৰিব পাৰি।

অৱশেষত, PWA আসন্নতা ব্যৱহাৰ কৰি আমি smooth nonlinear systemsৰ local feedback passivation problemটোৰ তদন্ত কৰো। প্ৰথমতে classical nonlinear systemsৰ বাবে local feedback passivationৰ বিবেচনা কৰা হৈছে। তাৰ পিছত Morkovian packet lossৰ সৈতে local feedback passivation problemৰ অধ্যয়ন কৰা হয়। Controllerৰ ৰূপাঙ্কন কৰিবৰ বাবে কেইটামান LMI সমাধান কৰা হৈছে আৰু লগতে কিছুমান অৰৈখিক চৰ্ত (nonlinear constraints)ৰ পালন হৈছে নে নাই তাৰ পৰীক্ষা কৰা হৈছে।

Contents

List of Figures

List of Abbreviations

NCS	: Networked Control System
i.i.d.	: independent and identically distributed
CARE	: Coupled Algebraic Riccati Equation
PWA	: Piecewise Affine
PWL	: Piecewise Linear
MJLS	: Markovian Jump Linear System
JLQ	: Jump Linear Quadratic

List of Notations

$\mathbb{R}$	: The field of real numbers
$\mathbb{Z}^{+}$	: The set of nonnegative integers
$\mathbb{R}^{n}$	: The n-dimensional vector space over $\mathbb{R}$
$\mathbb{R}^{n\times m}$	: The space $n \times m$ dimensional matrices with their elements as real numbers
$\|\|x\|\|$	: The Euclidean norm of a vector x
$\|\|x\|\|_P^2$	: $x^T Px$ for a symmetric positive semidefinite matrix $P \geq 0$ and a vector x.
$\|\|W\|\|$	: The induced 2-norm of a matrix W.
$\mathbb{E}[.]$	: The expected value
$\mathbb{E}[.\|I]$	: The conditional expectation given I
$Pr(.)$	: The probability of the argument
$Pr(.\|I)$	: The conditional probability of the argument given I
$diag\{a_1, a_2, ..., a_m\}$	: Diagonal matrix with a_1, a_2,...,a_m as its diagonal elements
I_n	: The $n \times n$ identity matrix
0_n	: The $n \times n$ zero matrix
$0_{n\times m}$	: The $n \times m$ zero matrix
$\{Z_k\}$	: A sequence of matrices Z_k of appropriate dimensions with $k = 0, 1, ...$
$l_2\big([0, N], \mathbb{R}^n\big)$	: The space of square-summable sequences of functions that map the interval $[0, N]$ into $\mathbb{R}^n$
$B_\tau(x)$	: A ball of radius $\tau > 0$ around x.

List of Publications

Journal

- **A. Mazumdar**, S. Krishnaswamy and S. Majhi, "H_∞ Optimal Control of Jump Systems Over Multiple Lossy Communication Channels," *IMA Journal of Mathematical Control and Information, 2021, https://doi.org/10.1093/imamci/dnab013*.

- **A. Mazumdar**, S. Krishnaswamy, and S. Majhi, "H_∞ optimal control over multiple Gilbert-Elliott type communication channels," *IFAC Journal of Systems and Control, vol. 16, June, 2021. https://doi.org/10.1016/j.ifacsc.2020.100134*.

Conference

- **A. Mazumdar**, S. Krishnaswamy and S. Majhi, "Linear Quadratic Optimal Control of Jump System over Multiple Erasure Channels," in *Proceedings of the 23rd International Symposium on Mathematical Theory of Networks and Systems (MTNS2018), Hong Kong University of Science and Technology, Hong Kong (2018).*

- **A. Mazumdar**, S. Krishnaswamy and S. Majhi, "H_∞ - Optimal Control over erasure channel", *IFAC-PapersOnLine 50 (1) (2017) 349–354*.

1

Introduction

1.1 Networked control systems

As communication and networking technologies have become more sophisticated and cost-effective, it is simple and economical to connect many devices through a communication network. This has fueled the emergence of a new kind of control system architecture called the networked control system (NCS) architecture. In an NCS, different components of the system, viz. sensors, controllers, and actuators, which could potentially be spatially separated, are connected by a communication network, possibly a wireless one. This brings in modularity and flexibility in the design of NCSs while reducing the weight of the system [1]. Because of numerous advantages, NCSs have found applications in many areas such as supervisory control and data acquisition (SCADA) systems, automotive vehicles, formation control, smart home technology, etc. [2–4]

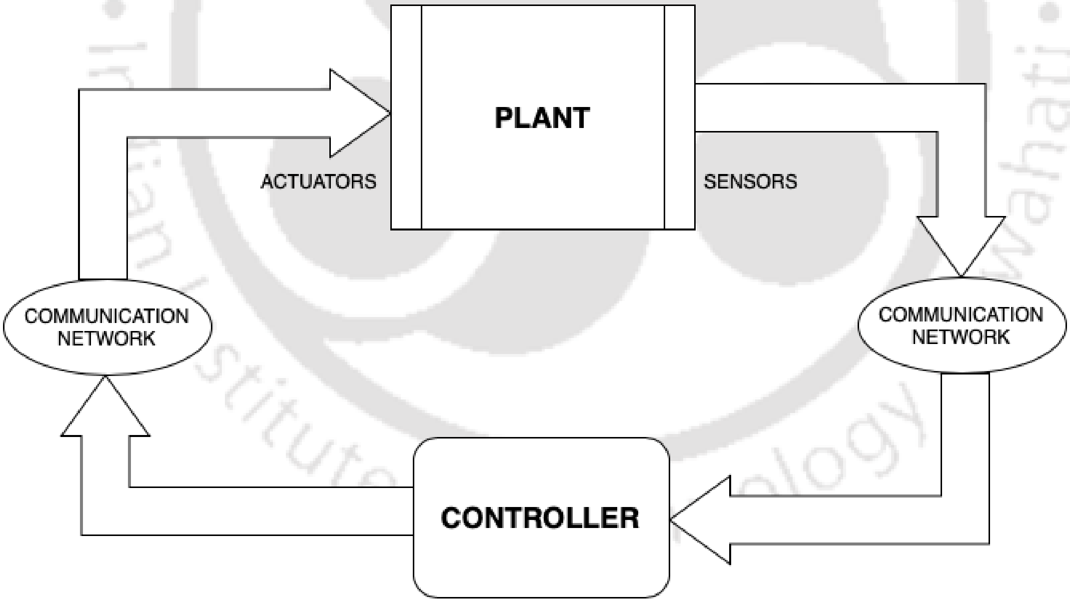

Figure 1.1: Schematic diagram of an NCS

Despite the various advantages that NCSs could provide, the insertion of a network between different sub-systems introduces many challenges, such as data packet loss, time-delay, and quantization. This thesis is primarily concerned with the effect of the data packet loss on an NCS.

1.2 Data packet losses in NCSs:

Packet loss occurs when a data packet fails to reach the destination node for various reasons such as transmission errors, transmission time-outs, buffer overflows due to congestion, etc. This degrades the performance of an NCS and can even destroy the system's stability. Thus, one has to explicitly consider the effect of the packet loss while analyzing the stability and performance of an NCS. Various models have been proposed in the literature to mathematically characterize the random data packet loss. These include the Bernoulli packet loss model [2] and the Markovian packet loss model [5].

- **Bernoulli packet loss model:** It is a simplistic packet loss model. In this model, the packet drop process is assumed to be independent and identically distributed (i.i.d.) Bernoulli process. Because of its mathematical tractability, the Bernoulli packet loss model has been extensively used in the literature [2].

- **Markovian packet loss model:** Although the Bernoulli packet loss model is widely used in the context of NCSs, it fails to capture the temporal correlation in packet loss that occurs in a realistic communication network. In practice, packet losses are usually temporally correlated. The probability of a packet drop at the current time instant is generally related to the packet drop delivery status at the previous time instant. This relation is effectively captured in the Markovian packet loss model [5].

1.3 Different Protocols used in NCSs:

In the context of NCSs, two different communication network protocols, namely the transmission control protocol (TCP) and the user datagram protocol (UDP), are usually being used. In the TCP-like protocol, a control packet's reception is acknowledged, while there is no such acknowledgment available in the UDP-like protocol [2].

This thesis is concerned with developing a few control strategies for linear systems, jump linear systems and nonlinear systems over the Gilbert-Elliott type communication network. For linear systems and jump linear systems, the H_∞ and jump linear quadratic (JLQ) optimal controllers are designed for the multiple lossy channels with the Markovian packet loss model. Finally, for smooth nonlinear systems operating over a Gilbert-Elliott type channel, controllers that ensure the desired $\mathcal{L}_2$ gain along with the local feedback passivity, respectively, are designed using a piecewise affine (PWA) approximation approach.

1.4 Literature Review

The design of the linear quadratic Gaussian (LQG) controller under a TCP-like protocol has been studied in [2, 6]. For the Bernoulli packet loss model, it has been shown that there exists a critical probability for packet arrival below which the closed-loop system cannot be stabilized. In [6], the design of the LQG controller with a UDP-like protocol has been considered. It has been demonstrated that for such a case, the separation principle does not hold. Extending this work, [5] studies the LQG control problem with the Markovian packet loss model. Explicit condition for the convergence of the infinite horizon cost function in terms of the control packet loss probability is also presented. The LQG cheap control over imperfect communication channels is investigated in [7–9]. Stabilization of an LTI system over multiple channels is considered in [10]. The LQG controller design problem for an LTI system over multiple lossy channels is discussed in [11], where the packet loss model is assumed to be Bernoulli. Estimation and control with a UDP-like protocol has been studied in [12–14]. It has been proved that the separation principle holds for the multi-channel case if the communication protocol is TCP-like.

Using the theory of Markovian jumped linear systems (MJLSs), the H_∞ controller design problem for an LTI system with random packet loss is considered in [15–18]. In [15–17], time-invariant controllers are designed for the Bernoulli packet loss model. On the other hand, [18] considers the Markovian packet loss model and designs both fixed-gain and variable-gain controllers for the single-channel case. The problem of designing a robust controller, considering an attack on the scheduling of packets, is considered in [19]. The Minimax (H_∞ optimal) control problem with the Bernoulli packet loss is investigated in [20–22]. Considering a TCP-like protocol, a state feedback minimax controller is designed in [20] with control packet erasures. Considering both control and sensor packet losses, [21] deals with the design of an output feedback minimax controller and a minimax estimator. For both TCP-like and UDP-like protocols, minimax controllers are designed in [22]. Generalizing this work, [23] designs minimax controller for an NCS with multiple lossy communication channels. The H_∞ control over multiple communications channels with networked induced delay is considered in [24]. Over unreliable acknowledgment, [25] deals with the design of an H_∞ controller with both the control packet loss and the sensor packet loss. Considering random packet losses as disturbance, an H_∞ controller is designed in [26]. It is also shown that the proposed controller outperforms the LQG controller. [27] addresses the resilient control problem against a denial-of-service (DoS) attack over a single channel. Robust H_∞ control of an LTI system over a lossy channel and with unreliable acknowledgment is addressed in [28].

The H_∞ control problem for nonlinear systems has been extensively studied in the recent past [29–32]. These works depend on the solubility of a few nonlinear inequalities in order to design the controller. In [33], a Taylor series based approach is used to design the state-feedback H_∞ controller for a smooth nonlinear system around a small neighbourhood of the origin. Control of smooth nonlinear systems using PWA approximations was introduced in [34]. Therein, the controllability and the stabilization of PWA systems and nonlinear systems are studied. The $\mathcal{L}_2$ gain of a PWA system is investigated in [35, 36]. The stability and the H_∞ controller design problem for uncertain piecewise linear systems are investigated in [37, 38]. In [39], the reliable H_∞ control problem for a piecewise linear (PWL) system with time-delays and actuator failure is considered. The robust H_∞ controller problem of an uncertain time-delay nonlinear system with missing measurements is studied in [40]. The nonlinear functions considered therein are sector-bound nonlinearities, and the packet loss is assumed to be described by the Bernoulli packet loss model.

The passivity of smooth discrete-time nonlinear systems has been investigated in great detail [41–45]. The passivity of PWA systems are studied in [35, 46–48]. For smooth nonlinear systems, [49] analyzes the passivity by linearizing around the origin. However, the results presented there are valid only inside a small neighborhood of the origin. The passivity of nonlinear systems by approximating the nonlinear dynamics through Taylor's theorem and a multivariate generalization of Bernstein polynomials is considered in [50]. However, the controller design for the feedback passivity has not been considered therein. The feedback passivity of a nonlinear system with packet losse has been studied in [51], although the packet loss model considered is not a realistic one.

1.5 Research motivations

Although there has been a surge in research on control over lossy communication channels or NCSs over the last few years, it is still in its infancy. There are still quite a few open and challenging problems that need proper attention. In view of the literature survey presented in the preceding section, some of these open problems are identified. Following are a few such issues that draw our attention.

Most of the works in the literature deal with LTI systems over a single communication channel with Bernoulli packet losses. There are certain issues with this kind of NCSs architecture. Firstly, the single-channel case may not always be sufficient. For example, for a team of mobile robots with a large number of agents, sending control commands to each of the agents through a single channel might lead to congestion in the channel. However, using multiple independent channels to send different control commands will reduce congestion, hence increases the probability of successful packet delivery [52]. Similarly, in a safety critical system where reliability is of great concern, such as the aircraft system, multiple actuators can be used to perform one specific task [53]. Sending control commands to the actuators through independent channels will increase the chances of receiving the required control command as compared to sending control commands to all the actuators through a single channel. The second issue stems from the consideration of the Bernoulli packet model. The Bernoulli packet loss model is a simple packet loss model and does not capture the temporal correlation in packet loss in realistic communication network.

The H_∞ optimal control problem for LTI systems operating over multiple lossy channels with Markovian packet losses has not been investigated as yet despite its practical importance. One probable reason for this could be the increase in complexity introduced by multiple channels with Markov packet losses. Although the H_∞ control problem with Markovian packet loss has been investigated recently for the single-channel case, results are only suboptimal due to the information structure considered therein. These results depend on the availability of the current packet loss status which is not accessible with a TCP-like protocol. Further, in the linear quadratic setting, results pertaining to the optimal controller design problem over multiple Gilbert-Elliott channels have not been derived yet.

It is well known that a linear system is a simplistic approximation model for various practical systems. Further, in many practical systems, the system dynamics changes randomly due to various factors such as component failure and environmental changes. Many such systems are best described by a jump linear system model or a more general nonlinear system model. However, there isn't much literature that consider jump linear systems and nonlinear systems operating over lossy communication networks. In view of the above observation, this thesis investigates a few control problem for linear systems, jump linear systems, and nonlinear systems over the Gilbert-Elliott type communication network.

1.6 Organization and contributions of the thesis

The organization and contributions of the thesis are as follows.

- **Chapter 2: H_∞ optimal control of linear time-invariant (LTI) systems over multiple lossy channels**

  In this chapter, the problem of H_∞ optimal controller design of an LTI system over multiple Gilbert-Elliott type communication channels is addressed. We have considered an appropriate information structure by introducing a TCP-like protocol. Formulating the problem in a dynamic game setting [54], conditions are derived for the existence of the finite horizon controller with random packet losses. The existence conditions for the infinite horizon controller are established by studying the convergence of the infinite horizon cost function. Important properties of the associated coupled algebraic Riccati equation (CAREs) such as monotonicity, positive definiteness are shown. Conditions for the convergence of the CAREs in terms of packet arrival probabilities are investigated. Stability of the closed-loop system in the face of random packet losses is established for three different scenarios: (a) with no disturbance, (b) with disturbance with finite energy, and (c) with the worst-case disturbance. A weaker notion of observability is used to prove stability of the closed-loop system with the worst-case disturbance. The condition for stability is found to be a relaxed one in comparison to the condition found in related literature [20–22].

- **Chapter 3: Jump linear quadratic optimal control of Markovian jump linear systems (MJLSs) over multiple lossy channels**

  The design of the jump linear quadratic (JLQ) optimal controller for a Markovian jump linear system (MJLS) operating over Gilbert-Elliott type channels is considered in chapter 3. The results generalize the existing works on the linear quadratic (LQ) optimal control over lossy channels. The motivation for considering the problem lies in the fact that the JLQ optimal controller is widely adopted for MJLSs. Thus, in the NCS setting, the JLQ optimal control problem over multiple channels with Markovian packet losses has practical significance. Both finite and infinite horizon controllers are designed using the dynamic programming approach. The Existence of the infinite horizon controller is investigated by analyzing the convergence of the infinite horizon cost function. Stability of the system with the infinite horizon optimal controller is shown. Using a weaker notion of observability, the positive definiteness of the fixed-point solution of the infinite horizon CAREs and stability of the closed-loop system are proved.

- **Chapter 4: H_∞ optimal control of Markovian jump linear systems (MJLSs) over multiple lossy channels**

  In chapter 4, results on the H_∞ optimal control of an LTI system with Markovian packet losses are extended to Markovian jump linear systems (MJLSs) operating over Gilbert-Elliott type communication channels. Unlike earlier works on the classical H_∞ optimal control of MJLSs such as [55], it is shown that a relatively weaker notion of observability is sufficient to establish various results for the infinite horizon part. Various properties of the associated coupled algebraic Riccati equations (CAREs) are established. Using the notion of weak observability, positive definiteness of the fixed-point solution of the infinite horizon CAREs is proved. Conditions for the convergence of the finite horizon CAREs, and hence the optimal cost function are presented. Further, conditions for the mean-square stability of the closed-loop system with the worst-case disturbance and without any disturbance are presented. The corresponding results for the classical H_∞ optimal control of an MJLS (i.e., with perfect communication between the controller and the actuators) become evident, and can be derived easily from the results of this chapter.

- **Chapter 5: H_∞ control of smooth nonlinear systems over lossy channel**

  Chapter 5 addresses the H_∞ controller design problem for smooth nonlinear systems over a Gilbert-Elliott type communication channel. A novel controller design approach is presented, which hinges on a piecewise affine (PWA) approximation [35] approach. The H_∞ control problem is addressed using the theory of dissipativity [32]. First, approximating the nonlinear system by a PWA system, conditions for an autonomous nonlinear system to have a finite $\mathcal{L}_2$ gain with respect to the disturbance input, while maintaining asymptotic stability, are derived. Then, the H_∞ state-feedback controller is designed for the nonlinear system such that, with the control law, the $\mathcal{L}_2$ gain of the closed-loop system is less than or equal to a prescribed disturbance attenuation level $\gamma > 0$ with asymptotic stability. Further, the H_∞ controller for a smooth discrete-time nonlinear system with Markovian packet losses is designed.

- **Chapter 6: Local feedback passivity of smooth nonlinear systems over lossy channel**
  In chapter 6, we investigate the problem of local feedback passivation of a smooth nonlinear system with random packet losses. A piecewise affine (PWA) approximation [35] method is employed for the analysis. To begin with, conditions for the local passivity of an autonomous nonlinear system are derived. Then, the local feedback passivation problem of a classical nonlinear system is investigated. Extending this work, the local feedback passivation of a nonlinear system over a Gilbert-Elliott type communication channel is addressed. Using the PWA approximation approach, sufficient conditions are derived for the local feedback passivity with Markovian packet loss.

- **Chapter 7: Conclusion and future works**
  Finally, in chapter 7, we summarize and conclude the thesis. Further, possible directions for future research works are also discussed.

2

H_∞ optimal control of linear time-invariant (LTI) systems over multiple lossy channels

2.1 Introduction

The theory of the H_∞ control is a crucial part of the robust control theory. While the literature of the classical H_∞ controller design is quite old and substantial, the H_∞ control over lossy channel counterpart is a recent and developing area. This chapter extends the work on the H_∞ controller design problem for classical LTI systems to LTI systems over a Gilbert-Elliott type communication network. Results that are presented in this chapter generalize the existing works on the H_∞ controller design over communication channels with Bernoulli packet losses [15, 16, 18, 20–23].

Specifically, the problem of the H_∞ optimal controller design over multiple channels with Markovian packet losses is dealt with. It is assumed that the communication protocol for each channel is TCP-like. The corresponding results for the single channel case with Markovian packet losses can easily be derived as a special case. It is assumed that the network between the controller and the actuators is lossy, while the network between the sensors and the controller is lossless. As given in [56], one can think of a practical system with this configuration.

The chapter is organized as follows. In Section 2.2, the problem is formulated as a dynamic game. Section 2.3 deals with the design of the finite and the infinite horizon controllers, and conditions for their existence. In Section 2.4, results are demonstrated using a numerical example. Finally, results are summarised in Section 2.5.

2.2 Problem Formulation

Consider the following discrete-time linear system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k^a + D_1 w_k \\ z_k &= Cx_k + Du_k^a, \end{aligned} \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k^a \in \mathbb{R}^m$ is the control input to the actuators, $w_k \in \mathbb{R}^s$ is the disturbance input having finite energy, i.e., $w_k \in l_2([0,\infty),\mathbb{R}^s)$, $z_k \in \mathbb{R}^p$ is the controlled output.

Since the goal of this chapter is to design a state-feedback controller, it is assumed that the state of the system x_k is directly accessible to the controller.

The architecture of the network between the controller and the actuators is such that the control unit exchanges information with each actuator through a different channel. In each of the channels, in the event of a packet loss, the zero-input strategy is used [57]. In the zero-input strategy, if a packet gets lost in a channel, the actuator connected to that particular channel becomes idle. If u_k is the controller output, then one can relate u_k^a and u_k as:

$$u_k^a = \xi_k u_k, \tag{2.2}$$

where $\xi_k = diag\{v_k^1, v_k^2, ..., v_k^m\}$, and v_k^i $(i \in \{1,2,..,m\})$ is a random variable which corresponds to the packet loss condition in the i^{th} channel. $v_k^i = 0$ and $v_k^i = 1$ imply a packet-loss and a successful packet reception in the i^{th} channel, respectively. In the Gilbert-Elliott channel model, packet losses are assumed to be governed by a two-state Markov chain. In this model, a successful packet arrival represents the good state and a packet loss represents the bad state of the Markov chain [5,58].

Remark 2.2.1. *In the system dynamics (2.1), the control input is u_k^a and is related to the actual controller output u_k by Equation (2.2) which takes into account the Gilbert-Elliott channel model. Thus Equation (2.2) introduces the Gilbert-Elliott channel model for packet loss in the system dynamics (2.1).*

Remark 2.2.2. *Note that although it is assumed that the controller sends control commands to each of the actuators through a distinct and independent channel, one can modify (2.2) such that the case where the controller sends control commands to a subset of the actuators through one* *channel can be taken into consideration. For example, if u_k^1 and u_k^2, two elements of the control vector $u_k = [u_k^1, u_k^2, ..., u_k^m]'$, are sent through one channel, then v_k^1 and v_k^2 would be represented by one random variable.*

The following notations will be used in the sequel:

(a) At a time instant $k(\geq 1)$, probabilities of successful packet reception, denoted by $\bar{v}^i$ and $\bar{\mu}^i$ are given by:

$$\bar{v}^i = Pr(v_k^i = 1 \big| v_{k-1}^i = 1) \text{ and } \bar{\mu}^i = Pr(v_k^i = 1 \big| v_{k-1}^i = 0), \ \forall i \in \{1, 2, ..., m\}.$$

(b) The actuators are indexed by the set $\mathscr{G} = \{1, 2, ..., m\}$. For every subset $\mathscr{I}$ of $\mathscr{G}$, we define the following matrix $\mathcal{N}(\mathscr{I})$.

$$\mathcal{N}(\mathscr{I}) = diag\{a_{jj}\}; \text{ where } a_{jj} = \begin{cases} 1, & \text{if } j \in \mathscr{I} \\ 0, & \text{if } j \notin \mathscr{I} \end{cases} \quad \text{for } j = 1, 2, ..., m.$$

At any instant k, the value of the random variable ξ_k is equal to one of the $\mathcal{N}(\mathscr{I})$s. This means that, at the instant k, the actuators that successfully receive the control command are those indexed by the elements of the set $\mathscr{I}$.

(c) For a TCP-like protocol, the information set $\mathcal{I}_k$ available to the controller at k^{th} time-index is defined as follows:

$$\mathcal{I}_k = \{x_0, ..., x_k, \xi_0, ..., \xi_{k-1}\}. \tag{2.3}$$

(d) $\mathcal{P}_k\big(\mathcal{N}(\mathscr{I})\big)$ denotes the probability, that the packet loss status in the network at the stage k is given by $\xi_k = \mathcal{N}(\mathscr{I})$, conditional to the information set $\mathcal{I}_k$, i.e.,

$$\mathcal{P}_k\big(\mathcal{N}(\mathscr{I})\big) = Pr\big(\xi_k = \mathcal{N}(\mathscr{I}) \big| \mathcal{I}_k\big)$$

At the stage $k = 0$, as the previous packet loss status does not exist, the probability

$\mathcal{P}_0\big(\mathcal{N}(\mathscr{I})\big)$ does not depend on the previous packet loss condition, and is given by:

$$\mathcal{P}_0\big(\mathcal{N}(\mathscr{I})\big) = \prod_{j\in\mathscr{I}} Pr\big(v_0^j = 1\big) \prod_{r\notin\mathscr{I}} Pr\big(v_0^r = 0\big);\ \ \forall \mathscr{I} \subseteq \mathscr{G}.$$

(e) The symbol $\mathbb{E}[.|\mathcal{I}_k]$ denotes the expected value of any function of packet loss given the information set $\mathcal{I}_k$. Suppose $X(.)$ is a map from $2^{\mathscr{G}}$ to a space which is closed under addition and scalar multiplication.

Then $\mathbb{E}[.|\mathcal{I}_k]$ is given by:

$$\mathbb{E}[X(\xi_k)\big|\mathcal{I}_k] = \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_k\big(\mathcal{N}(\mathscr{L})\big)X(\mathcal{N}(\mathscr{L})).$$

When the previous packet loss condition is given by $\xi_{k-1} = \mathcal{N}(\mathscr{I})$, the expected value of $X(.)$ is denoted by $\mathbb{E}^{\mathscr{I}}[X(\xi_k)]$ and takes the following form.

$$\mathbb{E}^{\mathscr{I}}[X(\xi_k)] = \mathbb{E}[X(\xi_k)\big|\xi_{k-1} = \mathcal{N}(\mathscr{I})]$$

Note 2.2.1. *At the stage $k = 0$, the previous packet loss condition is not known. In this case, the probability of a packet-loss and a successful packet arrival are given as follows [5]:*

$$Pr(v_0^i = 0) = \frac{1-\bar{\nu}^i}{1+\bar{\mu}^i-\bar{\nu}^i} \ \ \textit{and}\ \ Pr(v_0^i = 1) = \frac{\bar{\mu}^i}{1+\bar{\mu}^i-\bar{\nu}^i}.$$

The control policy $\zeta_{0:k}$ for a horizon k is a sequence $\zeta_{0:k} = \{\zeta_0, ..., \zeta_k\}$, where ζ_i maps the information set $\mathcal{I}_i$ to the control space $\mathcal{U}$, i.e., $u_i = \zeta_i(\mathcal{I}_i)$. Similarly, the disturbance policy $\eta_{0:k}$ for a horizon k is a sequence $\eta_{0:k} = \{\eta_0, ..., \eta_k\}$, where η_i maps the information set $\mathcal{I}_i$ to the disturbance space $\mathcal{W}$, i.e., $w_i = \eta_i(\mathcal{I}_i)$. $\zeta^*_{0:k}$ and $\eta^*_{0:k}$ denote the optimal control policy and the optimal disturbance policy, respectively.

The following notion of stability shall be followed in the chapter.

Definition 2.2.1. *System (2.1) is said to be mean-square stable if, with $u_k \equiv 0$ and $w_k \equiv 0$, we have $\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2 \Big| \mathcal{I}_0\Big] = 0$ for every x_0.* □

As in the standard H_∞ controller design problem, the main objective of this chapter is to design a state-feedback controller $u_k = F(x_k)$ with which the closed-loop system (2.1) attains the following two requirements:

R.1) The $\mathcal{L}_2$ gain from the disturbance input $w_k \in l_2([0, N], \mathbb{R}^s)$ to the controlled output z_k is less than or equal to some $\gamma > 0$, i.e., for zero initial condition ($x_0 = 0$),

$$\mathbb{E}\Big[\sum_{k=0}^{N} \|z_k\|^2 \Big| \mathcal{I}_0\Big] \leq \gamma^2 \sum_{k=0}^{N} \|w_k\|^2, \forall N \in \mathbb{Z}^+.$$

R.2) The closed-loop system is mean-square stable.

The theory of dynamic games has been widely used in solving the problem of H_∞ optimal controller design [22, 54, 59]. The results that are presented in this chapter shall also be derived using the theory dynamic games. To this end, consider a two player zero-sum game with the cost function as given below:

$$J_N(\zeta_{0:N-1}, \eta_{0:N-1}) = \mathbb{E}\Big[\|x_N\|_{W_N} + \sum_{k=0}^{N-1} \|z_k\|^2 - \gamma^2 \|w_k\|^2 \Big| \mathcal{I}_0\Big]. \tag{2.4}$$

It is assumed that C and D in (2.1) satisfy the following:

(a) $C^T D = 0$; (to ensure that no cross-product terms are present in the cost function),

(b) $R = D^T D > 0$; (to ensure the well-posedness of the optimal control problem).

Then, equation (2.4) transforms to the following:

$$J_N(\zeta_{0:N-1}, \eta_{0:N-1}) = \mathbb{E}\Big[\|x_N\|_{W_N} + \sum_{k=0}^{N-1} \|x_k\|_W^2 + \|u_k^a\|_R^2 - \gamma^2 \|w_k\|^2 \Big| \mathcal{I}_0\Big], \tag{2.5}$$

where $W_N \geq 0$, $W = C^T C$ and $R = D^T D$.

Note 2.2.2. *The assumptions made above are the same as those made in [54].*

For the game with cost function (2.5), the sequence of the control inputs $u_{0:k}$ is the minimizing player, and the sequence of the disturbance inputs $w_{0:k}$ is the maximizing player. The game admits a minimax solution or the value of the game exists if it has a Nash equilibrium or a saddle-point. The Nash equilibrium or the saddle-point policy $(\zeta^*_{0:N-1}, \eta^*_{0:N-1})$ satisfies the following inequality:

$$J_N(\zeta^*_{0:N-1}, \eta_{0:N-1}) \leq J_N(\zeta^*_{0:N-1}, \eta^*_{0:N-1}) \leq J_N(\zeta_{0:N-1}, \eta^*_{0:N-1}). \tag{2.6}$$

2.3 Main Results

In this section, we first solve the H_∞ optimal control problem for the finite horizon case and then extend the results to the infinite horizon case.

A. Finite horizon control:

Substituting (2.2) in (2.5), the cost function is written as follows:

$$J_N(\zeta_{0:N-1}, \eta_{0:N-1}) = \mathbb{E}\Big[x_N^T W_N x_N + \sum_{k=0}^{N-1} x_k^T W x_k + u_k^T \xi_k^T R \xi_k u_k - \gamma^2 w_k^T w_k \Big| \mathcal{I}_0\Big]. \tag{2.7}$$

The value function at the stage k, which is the optimal cost-to-go from the stage k to the stage N, is defined as:

$$V_{k,N}\big(x_k, \xi_{k-1}\big) = \min_{u_{k:N-1}} \max_{w_{k:N-1}} \mathbb{E}\left[x_N^T W_N x_N + \sum_{r=k}^{N-1} x_r^T W x_r + u_r^T \xi_r^T R \xi_r u_r - \gamma^2 w_r^T w_r \Big| \mathcal{I}_k\right]. \tag{2.8}$$

As a consequence of the principle of optimality, the value function at the stage k can be expressed by the following Isaacs Equation:

$$V_{k,N}\big(x_k, \xi_{k-1}\big) = \min_{u_k} \max_{w_k} \mathbb{E}\left[x_k^T W x_k + u_k^T \xi_k^T R \xi_k u_k - \gamma^2 w_k^T w_k + V_{k+1,N}(x_{k+1}, \xi_k)\Big| \mathcal{I}_k\right]. \tag{2.9}$$

The following lemma states conditions for the existence of the finite horizon saddle-point along with an expression for the finite horizon optimal control law.

Lemma 2.3.1. *For $k \in [0, N-1]$ and for all $\mathscr{I} \subseteq \mathscr{G}$, consider the following coupled algebraic Riccati equations (CAREs):*

$$\begin{aligned}\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big) &= W + \Big(\Gamma_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^T \mathbb{E}\Big[\xi_k R \xi_k \Big| \mathcal{I}_k\Big]\Gamma_{k,N}\big(\mathcal{N}(\mathscr{I})\big) - \gamma^2\Big(\Psi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^T \Psi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\\ &+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_k\big(\mathcal{N}(\mathscr{L})\big)\Big[\Big(A - B\mathcal{N}(\mathscr{L})\Gamma_{k,N}\big(\mathcal{N}(\mathscr{I})\big) + D_1\Psi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^T\\ &\times \Xi_{k+1,N}\big(\mathcal{N}(\mathscr{L})\big)\Big(A - B\mathcal{N}(\mathscr{L})\Gamma_{k,N}\big(\mathcal{N}(\mathscr{I})\big) + D_1\Psi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)\Big],\end{aligned} \tag{2.10}$$

where

$$\Gamma_{k,N}\big(\mathcal{N}(\mathscr{I})\big) = \Big(\Lambda_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^{-1}\Big(\mathbb{E}\Big[\xi_k B^T \Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big]\Big(A + D_1\Psi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)\Big), \tag{2.11a}$$

$$\begin{aligned}&\Psi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\\ &= \Big[I_s + \Big(\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^{-1} D_1^T \mathbb{E}\Big[\Xi_{k+1,N}(\xi_k) B \xi_k\Big|\mathcal{I}_k\Big]\Big(\Lambda_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^{-1}\mathbb{E}\Big[\xi_k B^T \Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big] D_1\Big]^{-1}\\ &\times \Big(\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^{-1}\Big[D_1^T \mathbb{E}\Big[\Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big] - D_1^T\mathbb{E}\Big[\Xi_{k+1,N}(\xi_k)B\xi_k\Big|\mathcal{I}_k\Big]\Big(\Lambda_{k,N}\big(\mathcal{N}(\mathscr{I})\big)\Big)^{-1}\\ &\times \mathbb{E}\Big[\xi_k B^T \Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big]\Big]A,\end{aligned} \tag{2.11b}$$

$$\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big) = \gamma^2 I_s - D_1^T\mathbb{E}\Big[\Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big]D_1, \tag{2.11c}$$

$$\Lambda_{k,N}\big(\mathcal{N}(\mathscr{I})\big) = \mathbb{E}\Big[\xi_k\Big(R + B^T\Xi_{k+1,N}(\xi_k)B\Big)\xi_k\Big|\mathcal{I}_k\Big], \tag{2.11d}$$

$$\textit{with}\ \ \Xi_{N,N}\big(\mathcal{N}(\mathscr{L})\big) = W_N, \ \ \forall \mathscr{L}\subseteq\mathscr{G}. \tag{2.11e}$$

Now, for the Isaacs equation (2.9), if $\xi_{k-1} = \mathcal{N}(\mathscr{I})$ $(k \geq 1)$*, the following claims are true:*

(a) At the stage $k \in [0, N-1]$*, a unique saddle-point exists if and only if:*

(i)

$$\Theta_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big) = \gamma^2 I_s - D_1^T \mathbb{E}\Big[\Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big]D_1 > 0. \tag{2.12}$$

(ii)

$$\begin{aligned}\Theta_{t,N}\Big(\mathcal{N}(\mathscr{L})\Big) = \gamma^2 I_s - D_1^T \mathbb{E}^{\mathscr{L}}\Big[\Xi_{t+1,N}(\xi_k)\Big]D_1 > 0;\\ \textit{for } \ k+1 \leq t \leq N-1, \ \textit{ and } \ \forall \mathscr{L} \subseteq \mathscr{G}.\end{aligned} \tag{2.13}$$

(b) If the saddle-point conditions (2.12) and (2.13) are satisfied, then the value function at the stage $k \in [0, N]$ *is given by:*

$$V_{k,N}\Big(x_k, \mathcal{N}(\mathscr{I})\Big) = x_k^T \Xi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_k. \tag{2.14}$$

(c) At $k = 0$*, the value function is expressed as:*

$$\begin{aligned}V_{0,N}(x_0) &= V_{0,N}\Big(x_0, \mathcal{N}(\mathscr{I})\Big), \ \ \forall \mathscr{I} \subseteq \mathscr{G}\\ &= x_0^T \Xi_{0,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_0,\end{aligned} \tag{2.15}$$

(d) $\Xi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big) \geq 0$ *for* $k \in [0, N]$ *and* $\forall \mathscr{I} \subseteq \mathscr{G}$.

(e) If the saddle-point conditions (2.12) and (2.13) are satisfied, then the finite horizon saddle-point at the stage $k \in [0, N-1]$ *stage is given by:*

$$u_k^* = \zeta_k^*(\mathcal{I}_k) = -\Gamma_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_k, \ \ w_k^* = \eta_k^*(\mathcal{I}_k) = \Psi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_k, \tag{2.16}$$

where $\Gamma_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ *and* $\Psi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ *for all* $\mathscr{I} \subseteq \mathscr{G}$ *are finite.*

Also, the value of the game with the cost function (2.7) and system dynamics (2.1)-(2.2) is given by:

$$J_N(\zeta^*_{0:N-1}, \eta^*_{0:N-1}) = x_0^T \Xi_{0,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_0. \tag{2.17}$$

Proof: We prove the lemma using induction. We start by proving that the lemma is true for the stage $N-1$ (we consider the base case as $k = N-1$, since $\Theta_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ is not defined for $k = N$).

At the stage $k = N$, observe that $V_{N,N}\Big(x_N, \xi_{N-1}\Big) = x_N^T W_N x_N$ for all ξ_{N-1}. So, with information set $\mathcal{I}_N$, if $\xi_{N-1} = \mathcal{N}(\mathscr{I})$, we can represent $V_{N,N}\Big(x_N, \xi_{N-1}\Big)$ as:

$$V_{N,N}\Big(x_{k+1}, \mathcal{N}(\mathscr{I})\Big) = x_N^T \Xi_{N,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_N, \tag{2.18}$$

where $\Xi_{N,N}\Big(\mathcal{N}(\mathscr{I})\Big) = W_N$ for all $\mathscr{I} \subseteq \mathscr{G}$.

Now, with information set $\mathcal{I}_{N-1}$, if $\xi_{N-2} = \mathcal{N}(\mathscr{I})$:

$$\begin{aligned}&\mathbb{E}\Big[V_{N,N}\Big(x_N, \xi_{N-1}\Big)\Big|\mathcal{I}_{N-1}\Big]\\ &= \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_{N-1}\Big(\mathcal{N}(\mathscr{L})\Big)\mathbb{E}\Big[\Big(Ax_{N-1} + B\mathcal{N}(\mathscr{L})u_{N-1} + D_1 w_{N-1}\Big)^T W_N\Big(Ax_{N-1} + B\mathcal{N}(\mathscr{L})u_{N-1} + D_1 w_{N-1}\Big)\Big].\end{aligned} \tag{2.19}$$

Consider the following functional:

$$\begin{aligned}H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1}) = \mathbb{E}\Big[x_{N-1}^T W x_{N-1} + u_{N-1}^T \xi_{N-1}^T R \xi_{N-1} u_{N-1} - \gamma^2 w_{N-1}^T w_{N-1}\\ + V_{N,N}\Big(x_N, \xi_{N-1}\Big)\Big|\mathcal{I}_{N-1}\Big].\end{aligned} \tag{2.20}$$

Note that $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is quadratic in u_{N-1}, w_{N-1}, and x_{N-1}. Hence, it admits a unique saddle-point if and only if all the following conditions are satisfied:

(i) $\frac{\partial^2 H_{N-1,N}(x_{N-1},u_{N-1},w_{N-1})}{\partial u_{N-1}^2} > 0$, i.e., $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is convex in u_{N-1}.

(ii) $\frac{\partial^2 H_{N-1,N}(x_{N-1},u_{N-1},w_{N-1})}{\partial w_{N-1}^2} < 0$, i.e., $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is concave in w_{N-1}.

(iii) There exist finite u_{N-1}^* and w_{N-1}^* such that

$$\begin{aligned} \frac{\partial H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})}{\partial u_{N-1}}\Big|_{(u_{N-1}^*, w_{N-1}^*)} &= 0 \\ \frac{\partial H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})}{\partial w_{N-1}}\Big|_{(u_{N-1}^*, w_{N-1}^*)} &= 0. \end{aligned} \tag{2.21}$$

Now, from (2.19) and (2.20):

$$\begin{aligned} \frac{\partial^2 H_{N-1,N}\big(x_{N-1}, u_{N-1}, w_{N-1}\big)}{\partial u_{N-1}^2} &= \mathbb{E}^{\mathscr{I}}\Big[\xi_{N-1}\Big(R + B^T \Xi_{N,N}(\xi_{N-1})B\Big)\xi_{N-1}\Big], \\ \frac{\partial^2 H_{N-1,N}\big(x_{N-1}, u_{N-1}, w_{N-1}\big)}{\partial w_{N-1}^2} &= -\gamma^2 I_s + D_1^T \mathbb{E}^{\mathscr{I}}\Big[\Xi_{N,N}(\xi_{N-1})\Big]D_1 = -\Theta_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big). \end{aligned}$$

Observe that $R > 0$ and $\Xi_{N,N}\Big(\mathcal{N}(\mathscr{I})\Big) = W_N \geq 0$, $\forall \mathscr{I} \subseteq \mathscr{G}$. Hence,

$$\mathbb{E}^{\mathscr{I}}\Big[\xi_{N-1}\Big(R + B^T \Xi_{N,N}(\xi_{N-1})B\Big)\xi_{N-1}\Big] > 0.$$

Therefore,

$$\frac{\partial^2 H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})}{\partial u_{N-1}^2} = \mathbb{E}^{\mathscr{I}}\Big[\xi_{N-1}\Big(R + B^T \Xi_{N,N}(\xi_{N-1})B\Big)\xi_{N-1}\Big] > 0.$$

Thus, $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is convex in u_{N-1}. Further, $\Theta_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big) > 0$ is equivalent to $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ being concave in w_{N-1}. Also, (2.21) is satisfied when,

$$\begin{aligned} u_{N-1}^* &= \zeta_{N-1}^*(\mathcal{I}_{N-1}) = -\Gamma_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_{N-1} \\ w_{N-1}^* &= \eta_{N-1}^*(\mathcal{I}_{N-1}) = \Psi_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_{N-1}, \end{aligned} \tag{2.22}$$

where $\Gamma_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ and $\Psi_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ are given by (2.11a) and (2.11b), respectively. The finiteness and positive semidefiniteness of W_N along with the positive definiteness of R ensure the invertibility of $\Lambda_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)$, $\forall \mathscr{I} \subseteq \mathscr{G}$. Further, the invertibility of $\Theta_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ and

$\Lambda_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big)$ ensure the finiteness of $\Psi_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big)$. Consequently, $\Gamma_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big)$ is finite. Substituting for u^*_{N-1} and w^*_{N-1} in (2.9) with $k = N-1$, and using (2.19), we get:

$$V_{N-1,N}\Big(x_{N-1}, \mathcal{N}(\mathscr{I})\Big) = x^T_{N-1}\Xi_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_{N-1},$$

where $\Xi_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big)$ is as given in equation (2.10).

Clearly, as $W_N \geq 0$, $W \geq 0$, and $R > 0$, $H_{N-1,N}(x_{N-1}, u^*_{N-1}, w_{N-1} = 0) \geq 0$. Since (u^*_{N-1}, w^*_{N-1}) constitutes a saddle-point,

$$H_{N-1,N}(x_{N-1}, u^*_{N-1}, w^*_{N-1}) \geq H_{N-1,N}(x_{N-1}, u^*_{N-1}, w_{N-1} = 0) \geq 0. \tag{2.23}$$

Hence,

$$\begin{aligned} &V_{N-1,N}\Big(x_{N-1}, \mathcal{N}(\mathscr{I})\Big) = H_{N-1,N}(x_{N-1}, u^*_{N-1}, w^*_{N-1}) \geq 0 \\ &\implies x^T_{N-1}\Xi_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_{N-1} \geq 0. \end{aligned} \tag{2.24}$$

Since (2.24) is true for all x_{N-1}, $\Xi_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big) \geq 0$. Thus the lemma is true for $k = N-1$.

Assume that the lemma is true for the stage $(p+1)$. This implies the following:

(i) If $\xi_p = \mathcal{N}(\mathscr{I})$, a unique saddle-point exists if and only if:

(i)

$$\Theta_{p+1,N}\Big(\mathcal{N}(\mathscr{I})\Big) = \gamma^2 I_s - D_1^T\mathbb{E}^{\mathscr{I}}\Big[\Xi_{p+2,N}\Big(\xi_{p+1}\Big)\Big]D_1 > 0.$$

(ii)

$$\Theta_{t,N}\Big(\mathcal{N}(\mathscr{L})\Big) = \gamma^2 I_s - D_1^T\mathbb{E}^{\mathscr{L}}\Big[\Xi_{t+1,N}\Big(\xi_t\Big)\Big]D_1 > 0;$$

$$\text{for } \; p+2 \leq t \leq N-1, \;\; and \;\; \forall \mathscr{L} \subseteq \mathscr{G}.$$

(ii) If the previous conditions are satisfied:

$$V_{t,N}\Big(x_t, \mathcal{N}(\mathscr{I})\Big) = x_t^T \Xi_{t,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_t, \text{ where, } \Xi_{t,N}\Big(\mathcal{N}(\mathscr{I})\Big) \geq 0; \\ \text{for } p+1 \leq t \leq N \text{ and } \forall \mathscr{I} \subseteq \mathscr{G}. \tag{2.25}$$

For $k = p$, the necessary part of statement (a) can be proved using the same line of argument as used in the proof of Theorem 3.2 in [54]. We now proceed to prove the sufficiency part.

Let $\Theta_{r,N}\Big(\mathcal{N}(\mathscr{J})\Big) > 0$ for $p+1 \leq r \leq N-1$, $\forall \mathscr{J} \subseteq \mathscr{G}$. With information set $\mathcal{I}_p$, if $\xi_{p-1} = \mathcal{N}(\mathscr{I})$,

$$\mathbb{E}\Big[V_{p+1,N}\Big(x_{p+1}, \xi_p\Big)\Big|\mathcal{I}_p\Big] \\ = \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_p\Big(\mathcal{N}(\mathscr{L})\Big)\mathbb{E}\Big[\Big(Ax_p + B\mathcal{N}(\mathscr{L})u_p + D_1 w_p\Big)^T \Xi_{p+1,N}\Big(\mathcal{N}(\mathscr{L})\Big)\Big(Ax_p + B\mathcal{N}(\mathscr{L})u_p + D_1 w_p\Big)\Big]. \tag{2.26}$$

Consider the following functional:

$$H_{p,N}(x_p, u_p, w_p) = \mathbb{E}\Big[x_p^T W x_p + u_p^T \xi_p^T R \xi_p u_p - \gamma^2 w_p^T w_p + V_{p+1,N}\Big(x_{p+1}, \xi_p\Big)\Big|\mathcal{I}_p\Big]. \tag{2.27}$$

By our hypothesis $\Xi_{p+1,N}\Big(\mathcal{N}(\mathscr{J})\Big) \geq 0$ for all $\mathscr{J} \subseteq \mathscr{G}$. Using the same line of argument as for the base case, one can show that $H_{p,N}(x_p, u_p, w_p)$ is convex in u_{N-1}, and it is concave in w_{N-1} if $\Theta_{p,N}\Big(\mathcal{N}(\mathscr{I})\Big) > 0$, i.e.,

$$H_{p,N}(x_p, u_p^*, w_p) \leq H_{p,N}(x_p, u_p^*, w_p^*) \leq H_{p,N}(x_p, u_p, w_p^*). \tag{2.28}$$

Hence, at the stage $k = p$, a unique saddle-point exists if and only the conditions given in (2.12) and (2.13) are satisfied. Further, the saddle-point at the stage $k = p$ is given by:

$$u_p^* = \zeta_p^*(\mathcal{I}_p) = -\Gamma_{p,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_p; \;\; w_p^* = \eta_p^*(\mathcal{I}_p) = \Psi_{p,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_p.$$

Therefore, substituting for u_p^* and w_p^* in Equations (2.9) (with $k = p$) and (2.26): $V_{p,N}\big(x_p, \mathcal{N}(\mathscr{I})\big) = x_p^T \Xi_{p,N}\big(\mathcal{N}(\mathscr{I})\big)x_p$, where $\Xi_{p,N}\big(\mathcal{N}(\mathscr{I})\big)$ is given in Equation (2.10). The proof for $\Xi_{p,N}\big(\mathcal{N}(\mathscr{I})\big) \geq 0$, $\forall \mathscr{I} \subseteq \mathscr{G}$, follows the same line of argument as for the base case.

At the stage $k = 0$, as there is no information available regarding the past packet loss condition, the value function will only be a function of x_0. Futher, $\mathcal{P}_0\big(\mathcal{N}(\mathscr{L})\big)$ will be the same for all $\mathscr{L} \subseteq \mathscr{G}$. Thus, $\Gamma_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$ will take same value for all $\mathscr{I} \subseteq \mathscr{G}$, so will $\Psi_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$, $\Theta_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$, $\Lambda_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$ and $\Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$. Hence, $V_{0,N}\big(x_0, \mathcal{N}(\mathscr{I})\big) = V_{0,N}\big(x_0\big) = x_0^T \Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big)x_0$.

Also, the value of the game with the cost function (2.7) is given as follows:

$$J_N(\zeta_{0:N-1}^*, \eta_{0:N-1}^*) = V_{0,N}\big(x_0\big) = x_0^T \Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big)x_0.$$

□

From *Lemma* 2.3.1, we infer that, for all $\mathscr{I} \subseteq \mathscr{G}$, $\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$, $\Gamma_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$, $\Psi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$, $\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$ and $\Lambda_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$ are unique.

Note 2.3.1. *From analysis presented in the above proof, we have that, at the stage $k = 0$, $\Xi_{0,N}(\mathcal{N}(\mathscr{I}))$ will same for all $\mathscr{I} \subseteq \mathscr{G}$.*

Remark 2.3.1. *If, for all packet loss conditions, a unique saddle-point exists at the stage $(k+1)$, then $\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big) > 0$ ensures existence of a finite solution of the CAREs (2.10) at the stage k. Using this argument recursively, it can be inferred that, for all $\mathscr{I} \subseteq \mathscr{G}$, if the conditions given by (2.12) and (2.13) are satisfied then the finite horizon CAREs (2.10) admit a unique finite solution.*

Remark 2.3.2. *If one considers v_k^is to be identical $\forall i \in \{1, 2, ..., m\}$ then the CAREs (2.10) becomes equivalent to the ones for an LTI over single Gilbert-Elliott type channel. Moreover, if it is assumed that $\bar{\nu}^i = \bar{\mu}^i$ for all $i \in \{1, 2, ..., m\}$ then the CAREs (2.10) become equivalent to* *the ones corresponding to the multiple channel with Bernoulli packet loss case, which are given in [22].*

Remark 2.3.3. *For the special case when ξ_k takes only two values 0_m or I_m, the results for the multi-channel case become equivalent to the results for the single-channel case.*

Remark 2.3.4. *It is easy to observe that the probability that the system becomes completely open-loop is lower with the multi-channel architecture than with the single-channel architecture. To see this, one can consider a control vector having two elements, i.e., $u_k = [u_k^1 \; u_k^2]'$. If both of them are sent by through different channels, each having a packet loss probability $\bar{p}'$, then the probability that both the control elements do not reach their respective actuators is $(\bar{p}')^2$. If a single channel was used, the probability would have been $\bar{p}'$.*

In the following lemma, we prove that if the saddle-point conditions are satisfied, the $\mathcal{L}_2$ gain from the disturbance w_k to the controlled output z_k is maintained.

Lemma 2.3.2. *Suppose, γ is chosen such that a unique saddle-point exists at the stage $k = 0$. Then, with the optimal control $u^*_{0:N-1}$, the $\mathcal{L}_2$ gain from the disturbance w_k to the controlled output z_k of the closed loop system is less than or equal to γ.*

Proof: As the game admits a saddle-point,

$$J_N(\zeta^*_{0:N-1}, \eta_{0:N-1}) \leq J_N(\zeta^*_{0:N-1}, \eta^*_{0:N-1}) \leq J_N(\zeta_{0:N-1}, \eta^*_{0:N-1}).$$

Or,

$$J_N(\zeta^*_{0:N-1}, \eta_{0:N-1}) \leq x_0^T \Xi_{0,N}\Big(\mathcal{N}(\mathscr{I})\Big)x_0. \tag{2.29}$$

Now, from (2.4), (2.29), and considering zero initial condition:

$$\mathbb{E}\Big[\|x_N\|^2_{W_N} + \sum_{k=0}^{N-1} \|z_k\|^2 - \gamma^2\|w_k\|^2 \Big| \mathcal{I}_0\Big] \leq 0$$
$$\implies \sum_{k=0}^{N-1} \mathbb{E}\Big[\|z_k\|^2 \Big| \mathcal{I}_0\Big] \leq \gamma^2 \sum_{k=0}^{N-1} \Big[\|w_k\|^2\Big].$$

Hence, the $\mathcal{L}_2$ gain from disturbance w_k to the controlled output z_k is less than or equal to γ. □

In the sequel it is assumed that $W_N = W$.

Lemma 2.3.3. *For $k \geq 1$, suppose a unique saddle-point exists. Then, for a fixed N, $\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big) \geq \Xi_{k+1,N}\big(\mathcal{N}(\mathscr{I})\big)$, for all $\mathscr{I} \subseteq \mathscr{G}$.*

Proof: This lemma is proved using induction.

From (2.19) and (2.20):

$$\begin{aligned}
&H_{N-1,N}(x, u^*_{N-1}, 0) \\
&= \min_{u_{N-1}}\Big[x^T W x + u^T_{N-1}\mathbb{E}^{\mathscr{I}}[\xi_{N-1} R \xi_{N-1}]u_{N-1} \\
&+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_{N-1}\big(\mathcal{N}(\mathscr{L})\big)\Big\{\big(Ax + B\mathcal{N}(\mathscr{L})u_{N-1}\big)^T \Xi_{N,N}\big(\mathcal{N}(\mathscr{L})\big)\big(Ax + B\mathcal{N}(\mathscr{L})u_{N-1}\big)\Big\}\Big] \qquad (2.30)\\
&\geq x^T W x \ \Big(As\ \ \mathbb{E}^{\mathscr{I}}[\xi_{N-1} R \xi_{N-1}] > 0 \ \text{ and } \ \Xi_{N,N}\big(\mathcal{N}(\mathscr{I})\big) = W_N \geq 0\Big) \\
&= V_{N,N}\big(x, \mathcal{N}(\mathscr{I})\big).
\end{aligned}$$

Further,

$$\begin{aligned}
V_{N-1,N}\big(x, \mathcal{N}(\mathscr{I})\big) &= x^T \Xi_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big)x \\
&= H_{N-1,N}(x, u^*_{N-1}, w^*_{N-1}) \\
&\geq H_{N-1,N}(x, u^*_{N-1}, 0) \\
&(Since\ w_{N-1}\ is\ the\ maximizing\ player) \\
&\geq V_{N,N}\big(x, \mathcal{N}(\mathscr{I})\big)\ \ (by\ (2.30)) \\
&= x^T \Xi_{N,N}\big(\mathcal{N}(\mathscr{I})\big)x.
\end{aligned} \tag{2.31}$$

Since (2.31) is true for all $x \neq 0$, $\Xi_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big) \geq \Xi_{N,N}\big(\mathcal{N}(\mathscr{I})\big)$; $\forall \mathscr{I} \subseteq \mathscr{G}$.

Assume that the lemma is true for the stage $k = p+1$. Therefore, $\Xi_{p+1,N}\big(\mathcal{N}(\mathscr{I})\big) \geq \Xi_{p+2,N}\big(\mathcal{N}(\mathscr{I})\big)$, $\forall \mathscr{I} \subseteq \mathscr{G}$.

From (2.9) with $k = p$, and (2.26):

$$\begin{aligned}
&V_{p,N}\big(x, \mathcal{N}(\mathscr{I})\big) \\
&= x^T \Xi_{p,N}\big(\mathcal{N}(\mathscr{I})\big)x \\
&= \min_{u_p} \max_{w_p}\Big[x^T W x + u_p^T \mathbb{E}^{\mathscr{I}}[\xi_p R \xi_p]u_p - \gamma^2 w_p^T w_p \\
&+ \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_p\big(\mathcal{N}(\mathscr{L})\big)\Big\{\big(Ax + B\mathcal{N}(\mathscr{L})u_p + D_1 w_p\big)^T \Xi_{p+1,N}\big(\mathcal{N}(\mathscr{L})\big)\big(Ax + B\mathcal{N}(\mathscr{L})u_p + D_1 w_p\big)\Big\}\Big].
\end{aligned} \tag{2.32}$$

TH-2488_146102019

Also,

$$\begin{aligned}
&V_{p+1,N}\Big(x, \mathcal{N}(\mathscr{I})\Big) \\
&= x^T \Xi_{p+1,N}\Big(\mathcal{N}(\mathscr{I})\Big)x \\
&= \min_{u_{p+1}} \max_{w_{p+1}}\Big[x^T W x + u_{p+1}^T \mathbb{E}^{\mathscr{I}}[\xi_{p+1} R \xi_{p+1}]u_{p+1} - \gamma^2 w_{p+1}^T w_{p+1} \\
&+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_{p+1}\Big(\mathcal{N}(\mathscr{L})\Big)\Big\{\Big(Ax + B\mathcal{N}(\mathscr{L})u_{p+1} + D_1 w_{p+1}\Big)^T \Xi_{p+2,N}\Big(\mathcal{N}(\mathscr{L})\Big) \\
&\times \Big(Ax + B\mathcal{N}(\mathscr{L})u_{p+1} + D_1 w_{p+1}\Big)\Big\}\Big].
\end{aligned} \tag{2.33}$$

For any two functions $f_1(u, w)$ and $f_2(u, w)$, we have:

$$f_1(u, w) \geq f_2(u, w) \implies \min_u \max_w f_1(u, w) \geq \min_u \max_w f_2(u, w). \tag{2.34}$$

Observe that, when $\xi_{p-1} = \xi_p = \mathcal{N}(\mathscr{I})$, we get that $\mathcal{P}_p(\mathcal{N}(\mathscr{L})) = \mathcal{P}_{p+1}(\mathcal{N}(\mathscr{L}))$, $\forall \mathscr{L} \subseteq \mathscr{G}$. Therefore, $\forall p \geq 1$, from Equations (2.32), (2.33), and (2.34):

$$V_{p,N}\Big(x, \mathcal{N}(\mathscr{I})\Big) \geq V_{p+1,N}\Big(x, \mathcal{N}(\mathscr{I})\Big).$$

Consequently, $\Xi_{p,N}\Big(\mathcal{N}(\mathscr{I})\Big) \geq \Xi_{p+1,N}\Big(\mathcal{N}(\mathscr{I})\Big)$, $\forall p \in [1, N]$, $\mathscr{I} \subseteq \mathscr{G}$. □

Lemma 2.3.4. *For $k \geq 1$ and $\forall \mathscr{I} \subseteq \mathscr{G}$, $\Xi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big) = \Xi_{k+1,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)$.*

Proof: This lemma is proved using induction.

Observe that: $V_{N,N}\Big(x, \mathcal{N}(\mathscr{I})\Big) = V_{N+1,N+1}\Big(x, \mathcal{N}(\mathscr{I})\Big) = x^T W x$. Therefore, $\Xi_{N,N}\Big(\mathcal{N}(\mathscr{I})\Big) = \Xi_{N+1,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)$.

Suppose, the lemma holds true for $k = p + 1$ stage. Hence, $\Xi_{p+1,N}\Big(\mathcal{N}(\mathscr{I})\Big) = \Xi_{p+2,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)$.

For the stage $k = p$,

$$\begin{aligned} & V_{p,N}\Big(x, \mathcal{N}(\mathscr{I})\Big) \\ & = x^T \Xi_{p,N}\Big(\mathcal{N}(\mathscr{I})\Big)x \\ & = \min_{u_p} \max_{w_p}\Big[x^T W x + u_p^T \mathbb{E}^{\mathscr{I}}[\xi_p R \xi_p]u_p - \gamma^2 w_p^T w_p \\ & + \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_p\Big(\mathcal{N}(\mathscr{L})\Big)\Big\{\Big(Ax + B\mathcal{N}(\mathscr{L})u_p + D_1 w_p\Big)^T \Xi_{p+1,N}\Big(\mathcal{N}(\mathscr{L})\Big)\Big(Ax + B\mathcal{N}(\mathscr{L})u_p + D_1 w_p\Big)\Big\}\Big], \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} & V_{p+1,N+1}\Big(x, \mathcal{N}(\mathscr{I})\Big) \\ & = x^T \Xi_{p+1,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)x \\ & = \min_{u_{p+1}} \max_{w_{p+1}}\Big[x^T W x + u_{p+1}^T \mathbb{E}^{\mathscr{I}}[\xi_{p+1} R \xi_{p+1}]u_{p+1} - \gamma^2 w_{p+1}^T w_{p+1} \\ & + \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_{p+1}\Big(\mathcal{N}(\mathscr{L})\Big)\Big\{\Big(Ax + B\mathcal{N}(\mathscr{L})u_{p+1} + D_1 w_{p+1}\Big)^T \Xi_{p+2,N+1}\Big(\mathcal{N}(\mathscr{L})\Big) \\ & \times \Big(Ax + B\mathcal{N}(\mathscr{L})u_{p+1} + D_1 w_{p+1}\Big)\Big\}\Big]. \end{aligned} \tag{2.36}$$

Therefore, if $\Xi_{p+1,N}\Big(\mathcal{N}(\mathscr{I})\Big) = \Xi_{p+2,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)$, for all $\mathscr{I} \subseteq \mathscr{G}$, then

$$V_{p,N}\Big(x, \mathcal{N}(\mathscr{I})\Big) = V_{p+1,N+1}\Big(x, \mathcal{N}(\mathscr{I})\Big), \quad \forall x; \quad (from\ (2.35)\ and\ (2.36))$$

Hence, $\Xi_{p,N}\Big(\mathcal{N}(\mathscr{I})\Big) = \Xi_{p+1,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)$. □

Remark 2.3.5. *For $k \geq 1$, both $x^T \Xi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)x$ and $x^T \Xi_{k+1,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)x$ define the optimal cost incurred when the system evolves from a state x for $(N - k)$ steps.* □

Note 2.3.2. *If a unique saddle-point exists at a stage $k \geq 1$, then given k, the sequence of solutions $\Big\{\Xi_{k,c}\Big(\mathcal{N}(\mathscr{I})\Big)\Big\}_{c=k+1}^{N} := \Big\{\Xi_{k,k+1}\Big(\mathcal{N}(\mathscr{I})\Big), \Xi_{k,k+2}\Big(\mathcal{N}(\mathscr{I})\Big), ..., \Xi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)\Big\}$ of CAREs (2.10) increases monotonically with N, for all $\mathscr{I} \subseteq \mathscr{G}$, i.e., $\Xi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big) \leq \Xi_{k,N+1}\Big(\mathcal{N}(\mathscr{I})\Big)$ (In light of Lemma 2.3.3 and Lemma 2.3.4)* □

Lemma 2.3.5. *If a unique saddle-point exists at the stage* $k = 0$*, the sequence* $\left\{\Xi_{0,c}\big(\mathcal{N}(\mathscr{I})\big)\right\}_{c=1}^{N}$ *is an increasing sequence in* N*, i.e.,* $\Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big) \leq \Xi_{0,N+1}\big(\mathcal{N}(\mathscr{I})\big)$.

Proof: From (2.9), we express the value function at the stage $k = 0$ as follows:

$$V_{0,N}(x_0) = \mathbb{E}\Big[x_0^T W x_0 + u_0^T \mathbb{E}[\xi_0 R \xi_0 \big| \mathcal{I}_0] u_0 - \gamma^2 w_0^T w_0 + V_{1,N}\big(x_1, \xi_0\big)\Big]. \tag{2.37}$$

Substituting $N = N + 1$ in the above Equation:

$$V_{0,N+1}(x_0) = \mathbb{E}\Big[x_0^T W x_0 + u_0^T \mathbb{E}[\xi_0 R \xi_0 \big| \mathcal{I}_0] u_0 - \gamma^2 w_0^T w_0 + V_{1,N+1}\big(x_1, \xi_0\big)\Big]. \tag{2.38}$$

From Note 2.3.2, $\Xi_{1,N}\big(\mathcal{N}(\mathscr{J})\big) \leq \Xi_{1,N+1}\big(\mathcal{N}(\mathscr{J})\big)$, $\forall\, \mathscr{J} \subseteq \mathscr{G}$. Hence,

$$\mathbb{E}\Big[V_{1,N}\big(x_1, \xi_0\big)\Big] \leq \mathbb{E}\Big[V_{1,N+1}\big(x_1, \xi_0\big)\Big].$$

Therefore, from (2.37) and (2.38):

$$\begin{aligned} &\mathbb{E}\Big[V_{0,N}(x_0)\Big] \leq \mathbb{E}\Big[V_{0,N+1}(x_0)\Big] \\ &\implies x_0^T \Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big) x_0 \leq x_0^T \Xi_{0,N+1}\big(\mathcal{N}(\mathscr{I})\big) x_0 \end{aligned} \tag{2.39}$$

As (2.39) is true for all x_0, $\Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big) \leq \Xi_{0,N+1}\big(\mathcal{N}(\mathscr{I})\big)$. □

B. Infinite horizon control:

In this section, we deal with the case when $N \to \infty$. The cost function for the infinite horizon case is defined as follows:

$$J_\infty(\zeta_{0:\infty}, \eta_{0:\infty}) = \mathbb{E}\Big[\sum_{k=0}^{\infty} x_k^T W x_k + u_k^T \xi_k^T R \xi_k u_k - \gamma^2 w_k^T w_k \Big| \mathcal{I}_0\Big]. \tag{2.40}$$

The following lemma states the condition for the convergence of the sequence $\left\{\Xi_{k,c}\big(\mathcal{N}(\mathscr{I})\big)\right\}_{c=k+1}^{N}$,

$\forall \mathscr{I} \subseteq \mathscr{G}$ as $N \to \infty$.

Lemma 2.3.6. *Suppose, the control packet arrival probabilities* $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$, *and* γ *are such that the following conditions are satisfied for all finite* $N \in \mathbb{Z}^+$, $k \in [0, N]$, $\mathscr{I} \subseteq \mathscr{G}$*:*

(a) A saddle-point exists, i.e., $\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big) > 0$.

(b) $\exists c < \infty$ *such that* $\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big) < cI_n$.

Then, there exist $\hat{\bar{\Xi}} < \infty$, *and* $\bar{\Xi}\big(\mathcal{N}(\mathscr{I})\big) < \infty$, $\forall \mathscr{I} \subseteq \mathscr{G}$, *such that* $\Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big) \to \hat{\bar{\Xi}}$, *and for* $k \in [1, \infty)]$, $\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big) \to \bar{\Xi}\big(\mathcal{N}(\mathscr{I})\big)$ *as* $N \to \infty$.

Proof: The positive definiteness of $\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$ ensures the existence and monotonicity of the sequence $\Big\{\Xi_{k,c}\big(\mathcal{N}(\mathscr{I})\big)\Big\}_{c=k+1}^{N}$. Since the parameters are such that, for all finite $N \in \mathbb{Z}^+$, $k \in [0, \infty)$, $\mathscr{I} \subseteq \mathscr{G}$, $\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big) < cI_n$ for some $c < \infty$, the sequence $\Big\{\Xi_{k,c}\big(\mathcal{N}(\mathscr{I})\big)\Big\}_{c=k+1}^{N}$ converges as $N \to \infty$ (in view of Theorem 3.14 in [60]). Similarly, monotonicity of the sequence $\Big\{\Xi_{0,c}\big(\mathcal{N}(\mathscr{I})\big)\Big\}_{c=1}^{N}$ (from Lemma 2.3.5) and condition (b) ensure that the sequence $\Big\{\Xi_{0,c}\big(\mathcal{N}(\mathscr{I})\big)\Big\}_{c=1}^{N}$ converges as $N \to \infty$. Therefore, there exist $\hat{\bar{\Xi}}$ and $\bar{\Xi}\big(\mathcal{N}(\mathscr{I})\big)$, such that $\Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big) \to \hat{\bar{\Xi}}$, and for $k \geq 1$, $\forall \mathscr{I} \subseteq \mathscr{G}$, $\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big) \to \bar{\Xi}\big(\mathcal{N}(\mathscr{I})\big)$ as $N \to \infty$.

□

Remark 2.3.6. *Note that the sequence* $\Big\{\Theta_{k,c}\big(\mathcal{N}(\mathscr{I})\big) : \Theta_{k,c}\big(\mathcal{N}(\mathscr{I})\big) > 0\Big\}_{c=k+1}^{N}$, *for each* $\mathscr{I} \subseteq \mathscr{G}$, *forms a decreasing sequence in N. If the conditions given in Lemma 2.3.6 are satisfied, then* $\Theta_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$ *converges to a positive definite value as* $N \to \infty$. *Thus, the conditions given in Lemma 2.3.6 ensure the existence of a unique saddle-point as* $N \to \infty$. *Further, if condition (a) in Lemma 2.3.6 does not get satisfied the value function* $V_{k,N}(.)$ *will not be well defined. In that scenario, Isaacs equation given by (2.9) will not be well defined, and thus the analysis using the value function will not be valid. Hence, the existence of a unique saddle-point is central to this work.* □

For $N \to \infty$ and $k \geq 1$, if the sequence $\Big\{\Xi_{k,c}\big(\mathcal{N}(\mathscr{I})\big)\Big\}_{c=k+1}^{N}$ converges, $\Xi_{k,N}\big(\mathcal{N}(\mathscr{I})\big)$ will no longer be a function of k. Hence, for $k \geq 1$, the CAREs (2.10) transform to the following

CAREs:

$$\begin{aligned}\bar{\Xi}\big(\mathcal{N}(\mathscr{I})\big) &= W + \big(\bar{\Gamma}\big(\mathcal{N}(\mathscr{I})\big)\big)^T \mathbb{E}^{\mathscr{I}}\big[\xi_k R \xi_k\big]\bar{\Gamma}\big(\mathcal{N}(\mathscr{I})\big) - \gamma^2\big(\bar{\Psi}\big(\mathcal{N}(\mathscr{I})\big)\big)^T\bar{\Psi}\big(\mathcal{N}(\mathscr{I})\big)\\ &+ \sum_{\mathscr{L}\subseteq\mathscr{G}} Pr\big(\xi_k = \mathcal{N}(\mathscr{L})\big|\xi_{k-1} = \mathcal{N}(\mathscr{I})\big)\Big[\big(A - B\mathcal{N}(\mathscr{L})\bar{\Gamma}\big(\mathcal{N}(\mathscr{I})\big) + D_1\bar{\Psi}\big(\mathcal{N}(\mathscr{I})\big)\big)^T \\ &\times \bar{\Xi}\big(\mathcal{N}(\mathscr{L})\big)\big(A - B\mathcal{N}(\mathscr{L})\bar{\Gamma}\big(\mathcal{N}(\mathscr{I})\big) + D_1\bar{\Psi}\big(\mathcal{N}(\mathscr{I})\big)\big)\Big],\end{aligned} \tag{2.41}$$

where $\bar{\Gamma}\big(\mathcal{N}(\mathscr{I})\big)$, $\bar{\Psi}\big(\mathcal{N}(\mathscr{I})\big)$, $\bar{\Theta}\big(\mathcal{N}(\mathscr{I})\big)$, and $\bar{\Lambda}\big(\mathcal{N}(\mathscr{I})\big)$ are given by:

$$\bar{\Gamma}\big(\mathcal{N}(\mathscr{I})\big) = \big(\bar{\Lambda}\big(\mathcal{N}(\mathscr{I})\big)\big)^{-1}\Big[\mathbb{E}^{\mathscr{I}}\big[\xi_k B^T\bar{\Xi}\big(\xi_k\big)\big]\big(A + D_1\bar{\Psi}\big(\mathcal{N}(\mathscr{I})\big)\big)\Big], \tag{2.42a}$$

$$\begin{aligned}&\bar{\Psi}\big(\mathcal{N}(\mathscr{I})\big)\\ &= \Big[I_s + \big(\bar{\Theta}\big(\mathcal{N}(\mathscr{I})\big)\big)^{-1}D_1^T\mathbb{E}^{\mathscr{I}}\big[\bar{\Xi}\big(\xi_k\big)B\xi_k\big]\big(\bar{\Lambda}\big(\mathcal{N}(\mathscr{I})\big)\big)^{-1}\mathbb{E}^{\mathscr{I}}\big[\xi_k B^T\bar{\Xi}\big(\xi_k\big)\big]D_1\Big]^{-1}\\ &\times\big(\bar{\Theta}\big(\mathcal{N}(\mathscr{I})\big)\big)^{-1}\Big[D_1^T\mathbb{E}^{\mathscr{I}}\big[\bar{\Xi}\big(\xi_k\big)\big] - D_1^T\mathbb{E}^{\mathscr{I}}\big[\bar{\Xi}(\xi_k)B\xi_k\big]\big(\bar{\Lambda}\big(\mathcal{N}(\mathscr{I})\big)\big)^{-1}\mathbb{E}^{\mathscr{I}}\big[\xi_k B^T\bar{\Xi}(\xi_k)\big]\Big]A,\end{aligned} \tag{2.42b}$$

$$\bar{\Theta}\big(\mathcal{N}(\mathscr{I})\big) = \gamma^2 I_s - D_1^T\mathbb{E}^{\mathscr{I}}\big[\bar{\Xi}(\xi_k)\big]D_1, \tag{2.42c}$$

$$\bar{\Lambda}\big(\mathcal{N}(\mathscr{I})\big) = \mathbb{E}^{\mathscr{I}}\big[\xi_k\big(R + B^T\bar{\Xi}(\xi_k)B\big)\xi_k\big], \tag{2.42d}$$

It is to be noted that as $\mathbb{E}^{\mathscr{I}}\big[\xi_k R\xi_k\big]$ and $P\big(\xi_k = \mathcal{N}(\mathscr{L})\big|\xi_{k-1} = \mathcal{N}(\mathscr{I})\big)$ are time-invariant, the CAREs (2.41) are also time-invariant.

Also, the infinite horizon Isaacs equation takes the following form:

$$V\big(x_k, \xi_{k-1}\big) = \min_{u_k}\max_{w_k}\mathbb{E}\left[x_k^T W x_k + u_k^T\xi_k^T R\xi_k u_k - \gamma^2 w_k^T w_k + V(x_{k+1}, \xi_k)|\mathcal{I}_k\right]. \tag{2.43}$$

By applying $N \to \infty$ in Lemma 2.3.1, we get the following result.

Lemma 2.3.7. *Suppose, the control packet arrival probabilities* $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$, *and* γ *are such that conditions (a) and (b) given in Lemma 2.3.6 are satisfied for all* $N \in \mathbb{Z}^+$, $k \in [0, \infty)$, *and* $\mathscr{I} \subseteq \mathscr{G}$. *If* $\xi_{k-1} = \mathcal{N}(\mathscr{I})$, *we get the following results.*

(a) The value function at any stage $k \in [1, \infty)$, is given by:

$$V\Big(x_k, \mathcal{N}(\mathscr{I})\Big) = x_k^T \bar{\Xi}\Big(\mathcal{N}(\mathscr{I})\Big) x_k. \tag{2.44}$$

And the infinite horizon saddle-point at the stage $k \in [1, \infty)$ is given by:

$$u_k^* = \zeta_k^*(\mathcal{I}_k) = -\bar{\Gamma}\Big(\mathcal{N}(\mathscr{I})\Big)x_k, \quad w_k^* = \eta_k^*(\mathcal{I}_k) = \bar{\Psi}\Big(\mathcal{N}(\mathscr{I})\Big)x_k, \tag{2.45}$$

(b) For $k = 0^{th}$ stage, the value function is expressed as follows:

$$V\Big(x_0\Big) = x_0^T \hat{\hat{\Xi}} x_0, \tag{2.46}$$

at $k = 0$, the infinite horizon saddle-point is given by:

$$\begin{aligned} u_0^* &= \zeta_0^*(\mathcal{I}_0) = -\hat{\hat{\Gamma}} x_0 \\ w_0^* &= \eta_0^*(\mathcal{I}_0) = \hat{\hat{\Psi}} x_0, \end{aligned} \tag{2.47}$$

wherein, $\hat{\hat{\Xi}}$, $\hat{\hat{\Gamma}}$, $\hat{\hat{\Psi}}$, $\hat{\hat{\Theta}}$ and $\hat{\hat{\Lambda}}$ are defined as follows:

$$\begin{aligned} \hat{\hat{\Xi}} &= W + \Big(\hat{\hat{\Gamma}}\Big)^T \mathbb{E}\Big[\xi_0 R \xi_0 \Big| \mathcal{I}_0\Big]\hat{\hat{\Gamma}} - \gamma^2(\hat{\hat{\Psi}})^T\hat{\hat{\Psi}} \\ &\quad + \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_0\Big(\mathcal{N}(\mathscr{L})\Big)\Big[\Big(A - B\mathcal{N}(\mathscr{L})\hat{\hat{\Gamma}} + D_1\hat{\hat{\Psi}}\Big)^T \bar{\Xi}\Big(\mathcal{N}(\mathscr{L})\Big)\Big(A - B\mathcal{N}(\mathscr{L})\hat{\hat{\Gamma}} + D_1\hat{\hat{\Psi}}\Big)\Big], \end{aligned} \tag{2.48}$$

$$\hat{\hat{\Gamma}} = \Big(\hat{\hat{\Lambda}}\Big)^{-1}\mathbb{E}\Big[\xi_0 B^T \bar{\Xi}(\xi_0)\Big|\mathcal{I}_0\Big]\Big(A + D_1\hat{\hat{\Psi}}\Big) \tag{2.49a}$$

$$\begin{aligned} \hat{\hat{\Psi}} &= \Bigg[I_s + \Big(\hat{\hat{\Theta}}\Big)^{-1} D_1^T\mathbb{E}[\bar{\Xi}(\xi_0)B\xi_0\Big|\mathcal{I}_0]\Big(\bar{\Lambda}\Big)^{-1}\mathbb{E}[\xi_0 B^T\bar{\Xi}(\xi_0)\Big|\mathcal{I}_0]D_1\Bigg]^{-1}\Big(\bar{\Theta}\Big)^{-1} \\ &\times \Bigg[D_1^T\mathbb{E}[\bar{\Xi}(\xi_0)\Big|\mathcal{I}_0] - D_1^T\mathbb{E}[\bar{\Xi}(\xi_0)B\xi_0\Big|\mathcal{I}_0]\Big(\hat{\hat{\Lambda}}\Big)^{-1}\mathbb{E}[\xi_0 B^T\bar{\Xi}(\xi_0)\Big|\mathcal{I}_0]\Bigg]A, \end{aligned} \tag{2.49b}$$

$$\hat{\hat{\Theta}} = \gamma^2 I_s - D_1^T\mathbb{E}\Big[\bar{\Xi}(\xi_0)\Big|\mathcal{I}_0\Big]D_1. \tag{2.49c}$$

$$\hat{\hat{\Lambda}} = \mathbb{E}\Big[\xi_0\Big(R + B^T\bar{\Xi}(\xi_0)B\Big)\xi_0\Big|\mathcal{I}_0\Big], \tag{2.49d}$$

(c) The infinite horizon optimal cost is given by:

$$J_\infty(\zeta^*_{0:\infty}, \eta^*_{0:\infty}) = x_0^T \hat{\bar{\Xi}} x_0,$$

where $\zeta^*_{0:\infty} = \{\zeta^*_0, \zeta^*_1, ...\}$ *and* $\eta^*_{0:\infty} = \{\eta^*_0, \eta^*_1, ...\}$. □

The following symbols shall be used in the sequel.

$$\begin{aligned}&\text{For } k \geq 1, \ \ \Gamma_k = \bar{\Gamma}\Big(\mathcal{N}(\mathscr{I})\Big) \ \text{ and } \ \Psi_k = \bar{\Psi}\Big(\mathcal{N}(\mathscr{I})\Big); \ \text{ if } \ \xi_{k-1} = \mathcal{N}(\mathscr{I}),\\ &\text{for } \ k = 0, \ \ \Gamma_0 = \hat{\bar{\Gamma}} \ \text{ and } \ \Psi_0 = \hat{\bar{\Psi}}.\end{aligned}$$

The following lemma establishes the positive definiteness of the fixed-point solution $\bar{\Xi}\Big(\mathcal{N}(\mathscr{I})\Big)$; $\forall \mathscr{I} \subseteq \mathscr{G}$ of CAREs (2.41).

Lemma 2.3.8. *For each* $\mathscr{I} \subseteq \mathscr{G}$, $\bar{\Xi}\Big(\mathcal{N}(\mathscr{I})\Big)$ *is positive definite if the following conditions are satisfied:*

(a) $(A, W^{1/2})$ *is observable.*

(b) $\{\bar{v}^1, \bar{v}^2, ..., \bar{v}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$, *and* γ *are such that conditions (a) and (b) given in Lemma 2.3.6 are satisfied for all* $N \in \mathbb{Z}^+$, $k \in [1, \infty)$, *and* $\mathscr{I} \subseteq \mathscr{G}$.

Proof: Statement (b) of Lemma 2.3.8 ensures that CAREs (2.41) have a finite solution, and a unique saddle-point of the game with the cost function (2.40) exists. For $k \geq 1$, if $\xi_{k-1} = \mathcal{N}(\mathscr{I})$, observe that,

$$\begin{aligned}&H_{k,N}(x_k, u^*_k, w^*_k)\\ &= V_{k,N}\Big(x_k, \mathcal{N}(\mathscr{I})\Big)\\ &= \min_{u_{k:N-1}} \max_{w_{k:N-1}} \mathbb{E}\left[x_N^T W x_N + \sum_{r=k}^{N-1} x_r^T W x_r + u_r^T \xi_r^T R \xi_r u_r - \gamma^2 w_r^T w_r \Big| \mathcal{I}_k\right].\end{aligned} \tag{2.50}$$

Taking limit as $N \to \infty$ in the above Equation:

$$\begin{aligned}\lim_{N\to\infty} H_{k,N}(x_k, u_k^*, w_k^*) &= \lim_{N\to\infty} \sum_{j=k}^{N} x_j^T W x_j + x_j^T \Gamma_j^T \mathbb{E}\Big[\xi_j R \xi_j \Big| \mathcal{I}_j\Big] \Gamma_j x_j - \gamma^2 x_j^T \Psi_j^T \Psi_j x_j \\ &\geq \lim_{N\to\infty} H_{k,N}(x_k, u_k^*, 0) \Big(As\ w_k^*\ is\ the\ maximizing\ player\Big),\end{aligned} \tag{2.51}$$

where

$$\begin{aligned}&\lim_{N\to\infty} H_{k,N}(x_k, u_k^*, 0) \\ &= \lim_{N\to\infty} \sum_{j=k}^{N} x_j^T W x_j + x_j^T \Gamma_j^T \mathbb{E}\Big[\xi_j R \xi_j \Big| \mathcal{I}_j\Big] \Gamma_j x_j,\end{aligned} \tag{2.52}$$

We have that $W > 0$ and $\mathbb{E}[\xi_j R \xi_j \big| \mathcal{I}_j] > 0$ for all j. Thus, if $\lim\limits_{N\to\infty} \Gamma_j x_j \neq 0$ for any j, then from (2.52):

$$\lim_{N\to\infty} H_{k,N}(x_k, u_k^*, 0) > x_k^T W x_k.$$

Now, consider the case when $\lim_{N\to\infty}\Gamma_j x_j = 0$ for all j. Then, from (2.52), one gets:

$$
\begin{aligned}
&\lim_{N\to\infty} H_{k,N}(x_k, u_k^*, 0)\\
&= \lim_{N\to\infty}\sum_{j=k}^{N} x_j^T W x_j + x_j^T\Gamma_j^T\mathbb{E}\Big[\xi_j R\xi_j\Big|\mathcal{I}_j\Big]\Gamma_j x_j\\
&= x_k^T W x_k + x_{k+1}^T W x_{k+1} + x_{k+1}^T A^T W A x_{k+1} + \ldots\\
&+ x_{k+1}^T (A^T)^{(n-1)} W A^{(n-1)} x_{k+1} + \lim_{N\to\infty}\sum_{r=k+n+1}^{N} x_r^T W x_r\\
&= x_k^T W x_k + x_{k+1}^T \begin{bmatrix} W^{1/2}\\ W^{1/2}A\\ W^{1/2}A^2\\ .\\ .\\ .\\ W^{1/2}A^{n-1}\end{bmatrix}^T \begin{bmatrix} W^{1/2}\\ W^{1/2}A\\ W^{1/2}A^2\\ .\\ .\\ .\\ W^{1/2}A^{n-1}\end{bmatrix} x_{k+1} + \lim_{N\to\infty}\sum_{r=k+n+1}^{N} x_r^T W x_r.
\end{aligned}
\tag{2.53}
$$

$(A, W^{1/2})$ being observable implies that the second term in (2.53) is strictly greater than 0 (> 0) for all $x_k \neq 0$. Further, $\mathbb{E}[\xi_j R\xi_j|\mathcal{I}_j] > 0$ (as $R > 0$) for all j, and $W \geq 0$. Therefore, $\forall x_k \neq 0$,

$$\lim_{N\to\infty} H_{k,N}(x_k, u_k^*, 0) > x_k^T W x_k.$$

Consequently,

$$
\begin{aligned}
V\Big(x_k, \mathcal{N}(\mathscr{I})\Big) &= x_k^T\Xi\Big(\mathcal{N}(\mathscr{I})\Big)x_k\\
&= \lim_{N\to\infty} H_{k,N}(x_k, u_k^*, w_k^*)\\
&\geq \lim_{N\to\infty} H_{k,N}(x_k, u_k^*, 0) \quad \Big(using\ (4.44)\Big)\\
&> x_k^T W x_k.
\end{aligned}
\tag{2.54}
$$

Since (2.54) is satisfied for all $x_k \neq 0$, $\Xi\Big(\mathcal{N}(\mathscr{I})\Big) > W \geq 0$; $\forall \mathscr{I} \subseteq \mathscr{G}$.

□

If the conditions (a) and (b) of Lemma 2.3.8 are satisfied for all $k \in [0, \infty)$, then using the same line of argument as in the proof of Lemma 2.3.8, one can prove that $\hat{\hat{\Xi}}$ is positive definite.

We shall now show the stability of the closed-loop system. Following analysis is required to show stability of the closed-loop system with the saddle-point policy. System (2.1)-(2.2) with the infinite horizon saddle-point transform to the following:

$$x_{k+1} = \mathscr{A}(\xi_k)x_k, \tag{2.55}$$

where

$$\mathscr{A}(\xi_k) = A - B\xi_k\Gamma_k + D_1\Psi_k.$$

Also, consider a dummy output (which will only be used to prove a part of final result) as given by:

$$y_k = \mathscr{C}(\xi_k)x_k, \tag{2.56}$$

where

$$\mathscr{C}(\xi_k) = \mathbb{E}\Big[\Big(W + \Gamma_k^T\xi_k R\xi_k\Gamma_k - \gamma^2\Psi_k^T\Psi_k\Big)^{1/2}\Big|\mathcal{I}_k\Big].$$

We shall also need the notion of weak observability for a Markovian jump linear system. The following definition is in line with the one given in [61].

Definition 2.3.1. *Consider the system (2.55)-(2.56). Take any initial Markov process state l_0, and any two initial system states x_0^1 and x_0^2. Suppose, for a known input u_k, $y_k(x_0 = x_0^1) = y_k(x_0 = x_0^2)$, $\forall k \leq S$ implies $Pr(x_0^1 = x_0^2) > 0$. The system is said to be weakly observable if* $\mathbb{E}\Big[S\Big] < \infty$.

□

As shown by the following lemma, an algebraic condition to check weak observability is also presented in [61].

Lemma 2.3.9. *System (2.55)-(2.56) is said to be weakly observable if and only if, with $S < \infty$, there exists a transition path $\left\{N(\mathscr{I}_0), N(\mathscr{I}_1), ..., N(\mathscr{I}_{S-1})\right\}$ where $\mathscr{I}_j \subseteq \mathscr{G}$, $\forall\ j \in \{1, 2, ..., S-1\}$, for which the jump observability matrix $O(\mathscr{I}_0, \mathscr{I}_1, ..., \mathscr{I}_{S-1})$ has*

$$rank\, O(\mathscr{I}_0, \mathscr{I}_1, ..., \mathscr{I}_{S-1}) = \ rank \begin{bmatrix} \mathscr{C}\big(\mathscr{N}(\mathscr{I}_0)\big) \\ \mathscr{C}\big(\mathscr{N}(\mathscr{I}_1)\big)\mathscr{A}\big(\mathscr{N}(\mathscr{I}_0)\big) \\ . \\ . \\ . \\ \mathscr{C}\big(\mathscr{N}(\mathscr{I}_{S-1})\big)\Pi_{r=0}^{S-2}\mathscr{A}\big(\mathscr{N}(\mathscr{I}_r)\big) \end{bmatrix} = n.$$

□

Note that weak observability of system (2.55)-(2.56) is a weaker notion than observability of individual $\big(\mathscr{A}(N(\mathscr{I})), \mathscr{C}(N(\mathscr{I}))\big)$ for each $\mathscr{I} \subseteq \mathscr{G}$. We shall show that to prove mean-square stability of the close-loop system with the optimal input and the optimal disturbance, weak observability is sufficient.

We now proceed to prove that the optimal control law u_k^* stabilizes the system.

Theorem 2.3.10. *If $\{\bar{v}^1, \bar{v}^2, ..., \bar{v}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$, and γ are such that the conditions (a) and (b) of Lemma 2.3.8 are satisfied for all finite $N \in \mathbb{Z}^+$, $k \in [0, \infty)$, and $\mathscr{I} \subseteq \mathscr{G}$, then the following claims are true.*

(a) With the optimal control law u_k^*, the $\mathcal{L}_2$ gain from the disturbance input w_k to the controlled output z_k of the closed loop system is less than or equal to γ.

(b) With the optimal control law $u_k^* = -\Gamma_k x_k$, expected value of the state response of the system (2.1) decays to zero asymptotically for arbitrary $w = (w_1, w_2, ...) \in l_2([0, \infty), \mathbb{R}^s)$, i.e.,

$\lim_{k\to\infty} \mathbb{E}[\|x_k\|^2 \big| \mathcal{I}_0] = 0.$

(c) With the optimal control law $u_k^* = -\Gamma_k x_k$, expected value of the state-response of the system (2.1) for a bounded disturbance w_k satisfies $\mathbb{E}[\|x_k\|^2 \big| \mathcal{I}_0] < \infty$ for all k.

(d) The system $x_{k+1} = \Big(A - B\xi_k\Gamma_k + D_1\Psi_k\Big)x_k$ is mean-square stable if the system (2.55)-(2.56) is weakly observable.

Proof: Part (a) directly follows from Lemma 2.3.2 by taking limit as $N \to \infty$. $\bar{\Theta}\Big(\mathcal{N}(\mathscr{I})\Big)$ being positive definite ensures the existence of the saddle-point defined by the optimal strategy $(\zeta^*_{0:\infty}, \eta^*_{0:\infty})$. So,

$$\begin{aligned}
&J_\infty(\zeta^*_{0:\infty}, \eta_{0:\infty}) \leq J_\infty(\zeta^*_{0:\infty}, \eta^*_{0:\infty}) \leq J_\infty(\zeta_{0:\infty}, \eta^*_{0:\infty})\\
&\implies J_\infty(\zeta^*_{0:\infty}, \eta_{0:\infty}) \leq J_\infty(\zeta^*_{0:\infty}, \eta^*_{0:\infty}) = V\Big(x_0\Big) < \infty.\\
&\implies \mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_W + \|\xi_k u_k^*\|^2_R - \gamma^2\|w_k\|^2 \Big| \mathcal{I}_0\Big] < \infty \;\Big(from\ (2.40)\Big)\\
&\implies \mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_W + \|\xi_k u_k^*\|^2_R \Big| \mathcal{I}_0\Big] < \infty \quad \Big(since\ w_k \in \mathcal{L}_2([0,\infty), \mathbb{R}^s)\Big)\\
&\implies \mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_W + \|x_k\|^2_{(\Gamma_k)^T\xi_k R\xi_k\Gamma_k} \Big| \mathcal{I}_0\Big] < \infty.
\end{aligned} \tag{2.57}$$

As $W \geq 0$, and $R > 0$, (2.57) implies that both the infinite series $\mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_W \Big| \mathcal{I}_0\Big]$ and $\mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_{(\Gamma_k)^T\xi_k R\xi_k\Gamma_k} \Big| \mathcal{I}_0\Big]$ converge. Also, as $R > 0$, in view of Theorem 3.23 in [60], convergence of the infinite series $\mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_{(\Gamma_k)^T\xi_k R\xi_k\Gamma_k} \Big| \mathcal{I}_0\Big]$ implies $\lim_{k\to\infty} \mathbb{E}\Big[\Gamma_k x_k \Big| \mathcal{I}_0\Big] = 0$.
Now, we claim that (2.57) implies $\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2 \Big| \mathcal{I}_0\Big] = 0$. This claim is proved using contradiction as follows.

Let us assume that $\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2 \Big| \mathcal{I}_0\Big] \neq 0$. Now, using the system dynamics given by (2.1) with the optimal control input $u_k^* = -\Gamma_k x_k$,

$$\begin{aligned}
&\lim_{k\to\infty} \mathbb{E}\Big[x_{k+1}^T W x_{k+1} + x_{k+1}^T (\Gamma_{k+1})^T \xi_{k+1} R \xi_{k+1} \Gamma_{k+1} x_{k+1} \Big| \mathcal{I}_0\Big] \\
&= \lim_{k\to\infty}\Big\{\mathbb{E}\Big[x_k^T A^T W\Big(A x_k - 2B\xi_k\Gamma_k x_k + 2D_1 w_k\Big) - 2x_k^T \Gamma_k^T \xi_k B^T W D_1 w_k \\
&+ x_k^T \Gamma_k^T \xi_k B^T W B \xi_k \Gamma_k x_k + w_k^T D_1^T W D_1 w_k + x_{k+1}^T \Gamma_{k+1}^T \xi_{k+1} R \xi_{k+1} \Gamma_{k+1} x_{k+1} \Big| \mathcal{I}_0\Big]\Big\},
\end{aligned} \tag{2.58}$$

Since $\lim_{k\to\infty} \mathbb{E}\Big[\Gamma_k x_k \Big| \mathcal{I}_0\Big] = 0$ and $w \in l_2([0,\infty), \mathbb{R}^s)$, $\lim_{k\to\infty} \mathbb{E}\Big[x_{k+1} \Big| \mathcal{I}_0\Big] = A\mathbb{E}\Big[x_k \Big| \mathcal{I}_0\Big]$. Therefore, from (2.58):

$$\begin{aligned}
&\lim_{k\to\infty} \mathbb{E}\Bigg[x_{k+1}^T W x_{k+1} + x_{k+1}^T (\Gamma_{k+1})^T \xi_{k+1} R \xi_{k+1} \Gamma_{k+1} x_{k+1} \Big| \mathcal{I}_0\Big] \\
&= \lim_{k\to\infty} \mathbb{E}\Big[x_k^T A^T W A x_k \Big| \mathcal{I}_0\Bigg].
\end{aligned} \tag{2.59}$$

Similarly,

$$\begin{aligned}
&\lim_{k\to\infty} \mathbb{E}\Big[x_{k+2}^T W x_{k+2} + x_{k+2}^T (\Gamma_{k+2})^T \xi_{k+2} R \xi_{k+2} \Gamma_{k+2} x_{k+2} \Big| \mathcal{I}_0\Big] \\
&= \lim_{k\to\infty} \mathbb{E}\Big[x_k^T (A^2)^T W A^2 x_k \Big| \mathcal{I}_0\Big],
\end{aligned} \tag{2.60}$$

and so on.

Using equations (2.59), (2.60),... , and the fact that $(A, W^{1/2})$ is observable:

$$\begin{aligned}
&\lim_{k\to\infty} \mathbb{E}\Bigg[[x_k^T W x_k + x_k^T (\Gamma_k)^T \xi_k R \xi_k \Gamma_k x_k] \\
&+ [x_{k+1}^T W x_{k+1} + x_{k+1}^T (\Gamma_{k+1})^T \xi_{k+1} R \xi_{k+1} \Gamma_{k+1} x_{k+1}] \\
&+ [x_{k+2}^T W x_{k+2} + x_{k+2}^T (\Gamma_{k+2})^T \xi_{k+2} R \xi_{k+2} \Gamma_{k+2} x_{k+2}] + ... \\
&+ [x_{k+n-1}^T W x_{k+n-1} + x_{k+n-1}^T (\Gamma_{k+n-1})^T \xi_{k+n-1} R \xi_{k+n-1} \Gamma_{k+n-1} x_{k+n-1})] \Big| \mathcal{I}_0\Bigg] \\
&= \lim_{k\to\infty} \mathbb{E}\Big[x_k^T W x_k + x_k^T A^T W A x_k + x_k^T (A^T)^2 W A^2 x_k + ... + x_k^T (A^T)^{n-1} W A^{n-1} x_k \Big| \mathcal{I}_0\Big] \\
&= \lim_{k\to\infty} \mathbb{E}\Big[x_k^T \mathcal{O} x_k \Big| \mathcal{I}_0\Big] > 0,
\end{aligned} \tag{2.61}$$

where,

$$\mathscr{O} = \begin{bmatrix} W^{1/2} \\ W^{1/2}A \\ W^{1/2}A^2 \\ . \\ . \\ . \\ W^{1/2}A^{n-1} \end{bmatrix}^T \begin{bmatrix} W^{1/2} \\ W^{1/2}A \\ W^{1/2}A^2 \\ . \\ . \\ . \\ W^{1/2}A^{n-1} \end{bmatrix}.$$

So, from (2.61), if $\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2 \Big| \mathcal{I}_0\Big] \neq 0$, we get that $\lim_{k\to\infty} \mathbb{E}\Big[x_k^T W x_k \Big| \mathcal{I}_0\Big] \neq 0$. Hence, as a consequence of Theorem 3.23 in [60], the infinite series $\sum_{k=0}^{\infty} \mathbb{E}\Big[\|x_k\|_W^2 \Big| \mathcal{I}_0\Big]$ does not converge. Therefore, with $R > 0$, if $\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2 \Big| \mathcal{I}_0\Big] \neq 0$:

$$\mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|_W^2 + \|x_k\|^2_{(\Gamma_k)^T \xi_k R \xi_k \Gamma_k} \Big| \mathcal{I}_0\Big] \to \infty.$$

Thus, we arrive at a contradiction. Therefore, $\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2 \Big| \mathcal{I}_0\Big] = 0$.

Proof for part (c): Consider system (2.1) with the optimal control law $u_k^* = -\Gamma_k x_k$ and with a bounded disturbance w_k as follows:

$$x_{k+1} = \Big(A - B\xi_k \Gamma_k\Big) x_k + D_1 w_k. \tag{2.62}$$

From the argument given for the proof of part (b), one gets that the following system is mean-square stable.

$$x_{k+1} = \Big(A - B\xi_k \Gamma_k\Big) x_k.$$

Thus, the state-responses of system (2.62) are bounded for all bounded disturbance w_k.

Proof for part (d): As $\bar{\Theta}\Big(\mathcal{N}(\mathscr{I})\Big)$ for all $\mathscr{I} \subseteq \mathscr{G}$:

$$
\begin{aligned}
& J_\infty(\zeta^*_{0:\infty}, \eta^*_{0:\infty}) < \infty \\
& \Longrightarrow \mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_W + \|x_k\|^2_{(\Gamma_k)^T \xi_k R \xi_k \Gamma_k} - \gamma^2 \|x_k\|^2_{(\Psi_k)^T \Psi_k} \Big| \mathcal{I}_0\Big] < \infty \\
& \Longrightarrow \mathbb{E}\Big[\sum_{k=0}^{\infty} x_k^T\Big(W + (\Gamma_k)^T \xi_k R \xi_k \Gamma_k - \gamma^2 (\Psi_k)^T \Psi_k\Big) x_k \Big| \mathcal{I}_0\Big] < \infty.
\end{aligned}
\tag{2.63}
$$

Thus, one gets:

$$
\begin{aligned}
& \lim_{k\to\infty} \mathbb{E}\Big[x_k^T\Big(W + (\Gamma_k)^T \xi_k R \xi_k \Gamma_k - \gamma^2 (\Psi_k)^T \Psi_k\Big) x_k \Big| \mathcal{I}_0\Big] = 0 \\
& \Longrightarrow \lim_{k\to\infty} \mathbb{E}\Big[x_k^T (\mathscr{C}^{1/2}(\xi_k))^T \mathscr{C}^{1/2}(\xi_k) x_k \Big| \mathcal{I}_0\Big] = 0 \\
& \text{similarly,} \quad \lim_{k\to\infty} \mathbb{E}\Big[x_{k+1}^T (\mathscr{C}^{1/2}(\xi_{k+1}))^T \mathscr{C}^{1/2}(\xi_{k+1}) x_{k+1} \Big| \mathcal{I}_0\Big] \\
& = \lim_{k\to\infty} \mathbb{E}\Big[x_k^T \mathscr{A}^T(\xi_k) (\mathscr{C}^{1/2}(\xi_{k+1}))^T \mathscr{C}^{1/2}(\xi_{k+1}) \mathscr{A}(\xi_k) x_k \Big| \mathcal{I}_0\Big] = 0 \\
& \vdots \\
& \text{Also,} \quad \lim_{k\to\infty} \mathbb{E}\Big[x_{k+\mathscr{T}}^T (\mathscr{C}^{1/2}(\xi_{k+\mathscr{T}}))^T \mathscr{C}^{1/2}(\xi_{k+\mathscr{T}}) x_{k+\mathscr{T}} \Big| \mathcal{I}_0\Big] \\
& = \lim_{k\to\infty} \mathbb{E}\Big[x_k^T \Big(\Pi_{l=k}^{k+\mathscr{T}-1} \mathscr{A}(\xi_l)\Big)^T (\mathscr{C}^{1/2}(\xi_{k+\mathscr{T}}))^T \mathscr{C}^{1/2}(\xi_{k+\mathscr{T}}) \Big(\Pi_{l=k}^{k+\mathscr{T}-1} \mathscr{A}(\xi_l)\Big) x_k\Big] = 0.
\end{aligned}
\tag{2.64}
$$

Now,

$$
\begin{aligned}
& \lim_{k\to\infty} \mathbb{E}\Big[x_k^T (\mathscr{C}^{1/2}(\xi_k))^T \mathscr{C}^{1/2}(\xi_k) x_k + x_{k+1}^T (\mathscr{C}^{1/2}(\xi_{k+1}))^T \mathscr{C}^{1/2}(\xi_{k+1}) x_{k+1} \\
& + \ldots + x_{k+\mathscr{T}}^T (\mathscr{C}^{1/2}(\xi_{k+\mathscr{T}}))^T \mathscr{C}^{1/2}(\xi_{k+\mathscr{T}}) x_{k+\mathscr{T}} \Big| \mathcal{I}_0\Big] = 0 \\
& \Longrightarrow \lim_{k\to\infty} \mathbb{E}\Bigg[x_k^T \begin{bmatrix} \mathscr{C}(\xi_k) \\ \mathscr{C}(\xi_{k+1})\mathscr{A}(\xi_k) \\ \cdot \\ \cdot \\ \cdot \\ \mathscr{C}(\xi_{k+\mathscr{T}})\Pi_{l=k}^{k+\mathscr{T}-1} \mathscr{A}(\xi_l) \end{bmatrix}^T \begin{bmatrix} \mathscr{C}(\xi_k) \\ \mathscr{C}(\xi_{k+1})\mathscr{A}(\xi_k) \\ \cdot \\ \cdot \\ \cdot \\ \mathscr{C}(\xi_{k+\mathscr{T}})\Pi_{l=k}^{k+\mathscr{T}-1} \mathscr{A}(\xi_l) \end{bmatrix} x_k \Bigg| \mathcal{I}_0\Bigg] = 0
\end{aligned}
\tag{2.65}
$$

Since system (2.55) is weakly observable and the Markov chain $\{\xi_k\}$ is irreducible, there always exist a k and a finite $\mathscr{T}$ such that the condition given in Lemma 2.3.9 always gets satisfied. Thus, from (2.65), we get that $\lim_{k\to\infty}\mathbb{E}\Big[\|x_k\|^2\Big|\mathcal{I}_0\Big] = 0$. □

Remark 2.3.7. *The special case where* $\bar{\nu}^1 = \bar{\nu}^2 = ... = \bar{\nu}^m = \bar{\nu}$, $\bar{\mu}^1 = \bar{\mu}^2 = ... = \bar{\mu}^m = \bar{\mu}$, *and* $\bar{\nu}^l = \bar{\mu}^l$ *for all* $l \in \{1, 2, ..., m\}$, *models an LTI system over a single channel with a Bernoulli packet loss model. In this case,* $\mathscr{A}(\xi_k)$ *and* $\mathscr{C}(\xi_k)$ *in Equations (2.55)-(2.56) are replaced by* $\mathscr{A}(\xi_k) = A - B\xi_k\Gamma_k + D\Psi_k$ *and* $\mathscr{C}(\xi_k) = (W + \bar{\nu}\Gamma_k^T R\Gamma_k - \gamma^2\Psi_k^T\Psi_k)^{1/2}$, $\forall k$. *The sufficient condition for stability in [22] is* $(\bar{\nu}\Gamma_k^T R\Gamma_k - \gamma^2\Psi_k^T\Psi_k) > 0$. *As* $W \geq 0$, *this implies that* $\mathscr{C}^T\mathscr{C} > 0$ *and therefore* $\mathscr{C}$ *has rank n. On the other hand, the weak observability condition considered in Claim (d) of Theorem 2.3.10 only requires* O *to have rank n. Since the rows of* $\mathscr{C}$ *are a subset of the rows of* O, *this will always be satisfied if the condition given in [22] is satisfied. On the other hand,* O *having rank n does not guarantee that* $\mathscr{C}$ *has rank n. Therefore, a system that satisfies the condition given in Claim (d) in Theorem 2.3.10 may not satisfy the condition given in [22]. Thus, for an LTI system over a single channel with a Bernoulli packet loss model, the stability condition given in Claim (d) is relaxed as compared to the one given in [22]. Further, unlike [22], as we have considered the realistic Gilbert-Elliott channel model, with m independent channels, the number of CAREs we get is* 2^m, *each CAREs having dimension n.* □

2.4 Numerical Example

Consider an LTI system with the following system parameters:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

with

$$W = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

To see the effect of packet arrival probabilities on the optimal cost, we have evaluated the optimal cost for different packet arrival probabilities. In order to find the fixed-point solution of the CAREs (2.41) for the infinite horizon case, we solve a sequence of finite horizon CAREs (2.10) using dynamic programming and generate a sequence of $\Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$s, for each $\mathscr{I} \subseteq \mathscr{G}$. If this sequence converges, then the solution of the infinite horizon CAREs (2.41) is taken as the limit point of the sequence, i.e., $\bar{\Xi}\big(\mathcal{N}(\mathscr{I})\big) = \lim_{N\to\infty} \Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$. Figure 2.1 shows the convergence of the optimal infinite horizon cost function for $\bar{\nu}^1 = 0.92$, $\bar{\nu}^2 = 0.89$, $\bar{\mu}^1 = 0.91$, $\bar{\mu}^2 = 0.9$ (orange graph), and $\bar{\nu}^1 = 0.88$, $\bar{\nu}^2 = 0.86$, $\bar{\mu}^1 = 0.89$, $\bar{\mu}^2 = 0.87$ (blue graph). It is observed that the optimal infinite horizon cost is more for the second case (where the control packet arrival probabilities are lower). Figure 2.2 shows that if control packet arrival probabilities are reduced to $\bar{\nu}^1 = 0.4$, $\bar{\nu}^2 = 0.38$, $\bar{\mu}^1 = 0.41$, $\bar{\mu}^2 = 0.3$, the optimal infinite horizon cost function does not converge. Figure 2.3 demonstrates the variation of the critical disturbance attenuation level γ_c with respect to $\bar{\nu}^1$ for ($\bar{\nu}^2 = 0.8$, $\bar{\mu}^1 = 0.83$, $\bar{\mu}^2 = 0.818$), ($\bar{\nu}^2 = 0.8$, $\bar{\mu}^1 = 0.83$, $\bar{\mu}^2 = 0.82$), ($\bar{\nu}^2 = 0.8$, $\bar{\mu}^1 = 0.83$, $\bar{\mu}^2 = 0.83$) and ($\bar{\nu}^2 = 0.8$, $\bar{\mu}^1 = 0.83$, $\bar{\mu}^2 = 0.84$). Likewise, Figure 2.4 depicts the variation of the critical disturbance attenuation level γ_c with respect to $\bar{\mu}^2$ for ($\bar{\nu}^1 = 0.86$,$\bar{\nu}^2 = 0.87$, $\bar{\mu}^1 = 0.88$), ($\bar{\nu}^1 = 0.86$,$\bar{\nu}^2 = 0.88$, $\bar{\mu}^1 = 0.88$), ($\bar{\nu}^1 = 0.86$,$\bar{\nu}^2 = 0.89$, $\bar{\mu}^1 = 0.88$) and ($\bar{\nu}^1 = 0.86$,$\bar{\nu}^2 = 0.9$, $\bar{\mu}^1 = 0.88$). The regions above the curves in Figure 2.3 and Figure 2.4 are the feasible regions for the existence of unique fixed-point solution of CAREs (2.41). One can see that for a given value of γ_c, if the value of $\bar{\nu}^1$ (and $\bar{\mu}^2$) drops below a critical value, the sequence of $\Xi_{0,N}\big(\mathcal{N}(\mathscr{I})\big)$s does not converge. Hence, the infinite horizon CAREs (2.41) do not admit a finite fixed-point solution. Expected value of the closed-loop state response with a disturbance $w_k = \sin(0.5\pi k)e^{-k/2}$ is shown in Figure 2.5. Further, in Figure 2.6, expected value of the state response of the closed-loop system with a persistent bounded disturbance $w_k = 0.025\sin(0.5\pi k)$ is displayed. In Figure 2.7, we have shown the state-response of the system without taking the expectation of the state x_k for a disturbance $w_k = 0.025\sin(0.5\pi k)$. To get the response, we have first simulated two Markov chains for two independent channels with 0, 1 as its two states according to the probabilities $\bar{\nu}^1 = 0.9$, $\bar{\nu}^2 = 0.91$, $\bar{\mu}^1 = 0.93$, $\bar{\mu}^2 = 0.92$. Suppose, at a time index k, a zero appears in the Markov chain which is used to model the packet loss in channel-1, it is assumed the control packet in channel-1 is lost, and first element of the control vector u_k is set to be zero for computation of x_{k+1}. Similarly, if 1 appears in the Markov chain at the time index k, the optimal control law is applied for the corresponding element of the control vector to compute x_{k+1}. Unlike Figure 2.6, which shows the expected value of the state response $\mathbb{E}[x_k | \mathcal{I}_0]$, Figure 2.7 demonstrates the instantaneous state response considering random packet loss. One can see the random fluctuation of the state response in Figure 2.7 due to random packet losses, which can not be seen in Figure 2.6. From Figure 2.8, we see that if the packet arrival probabilities are reduced to $\bar{\nu}^1 = 0.4$, $\bar{\nu}^2 = 0.38$, $\bar{\mu}^1 = 0.41$, $\bar{\mu}^2 = 0.3$, the optimal controller fails to stabilize the system. In Figure 2.9, we have demonstrated the optimal cost for different γ and the cost associated with LQ control. It can be seen that as the disturbance attenuation level γ increases, the infinite horizon optimal cost for H_∞ optimal control converges to the infinite horizon optimal cost for LQ control.

The following example demonstrates that, for an LTI system over a single channel network, the stability condition given in Theorem 2.3.10 is more relaxed than the one given in [22]. Consider an LTI system with the following system parameters.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$W = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \; R = 1.$$

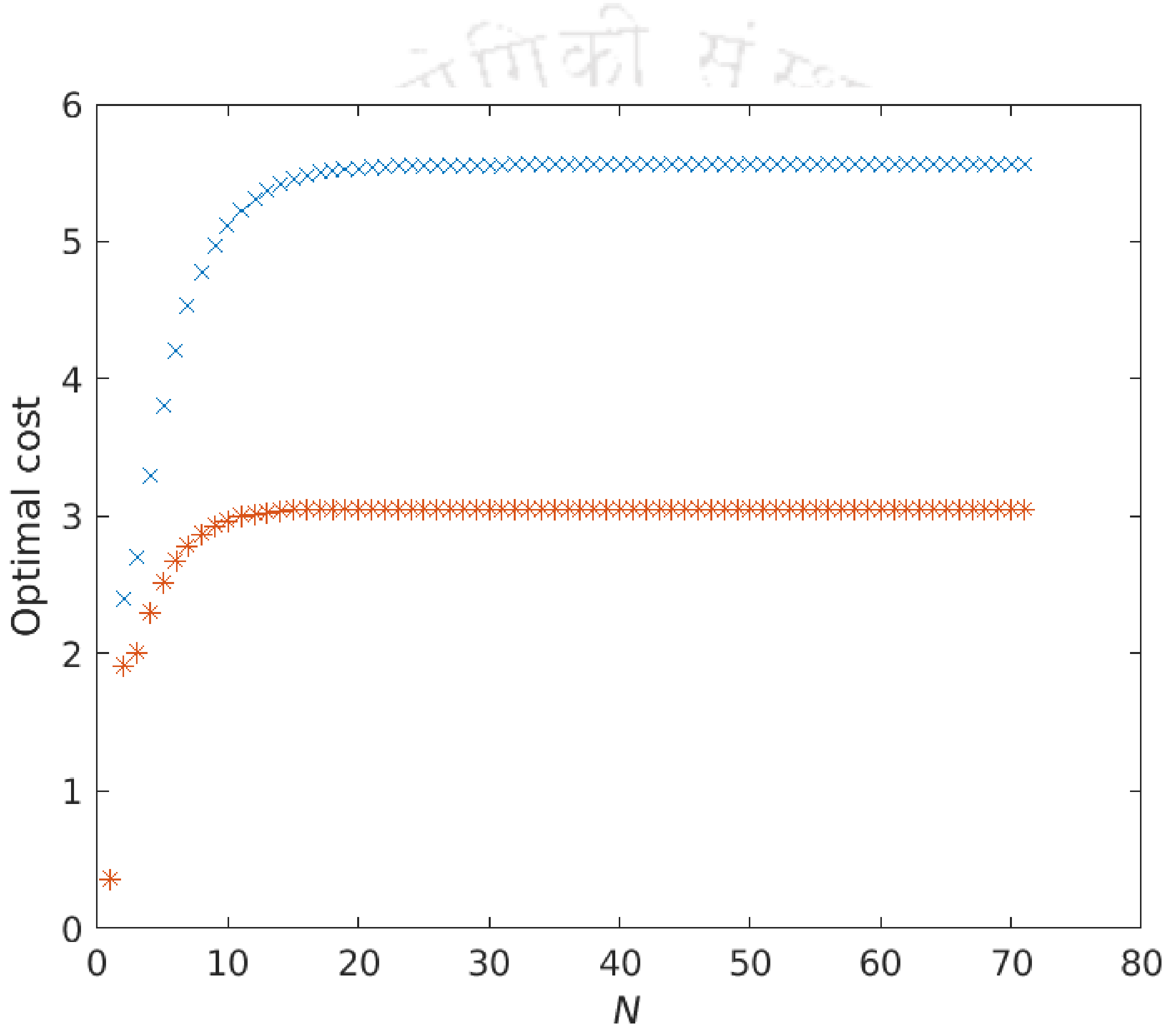

Figure 2.1: Behaviour of the optimal cost as N increases with $\bar{\nu}^1 = 0.92$, $\bar{\nu}^2 = 0.89$, $\bar{\mu}^1 = 0.91$, $\bar{\mu}^2 = 0.9$ (orange graph), and $\bar{\nu}^1 = 0.88$, $\bar{\nu}^2 = 0.86$, $\bar{\mu}^1 = 0.89$, $\bar{\mu}^2 = 0.87$ (blue graph).

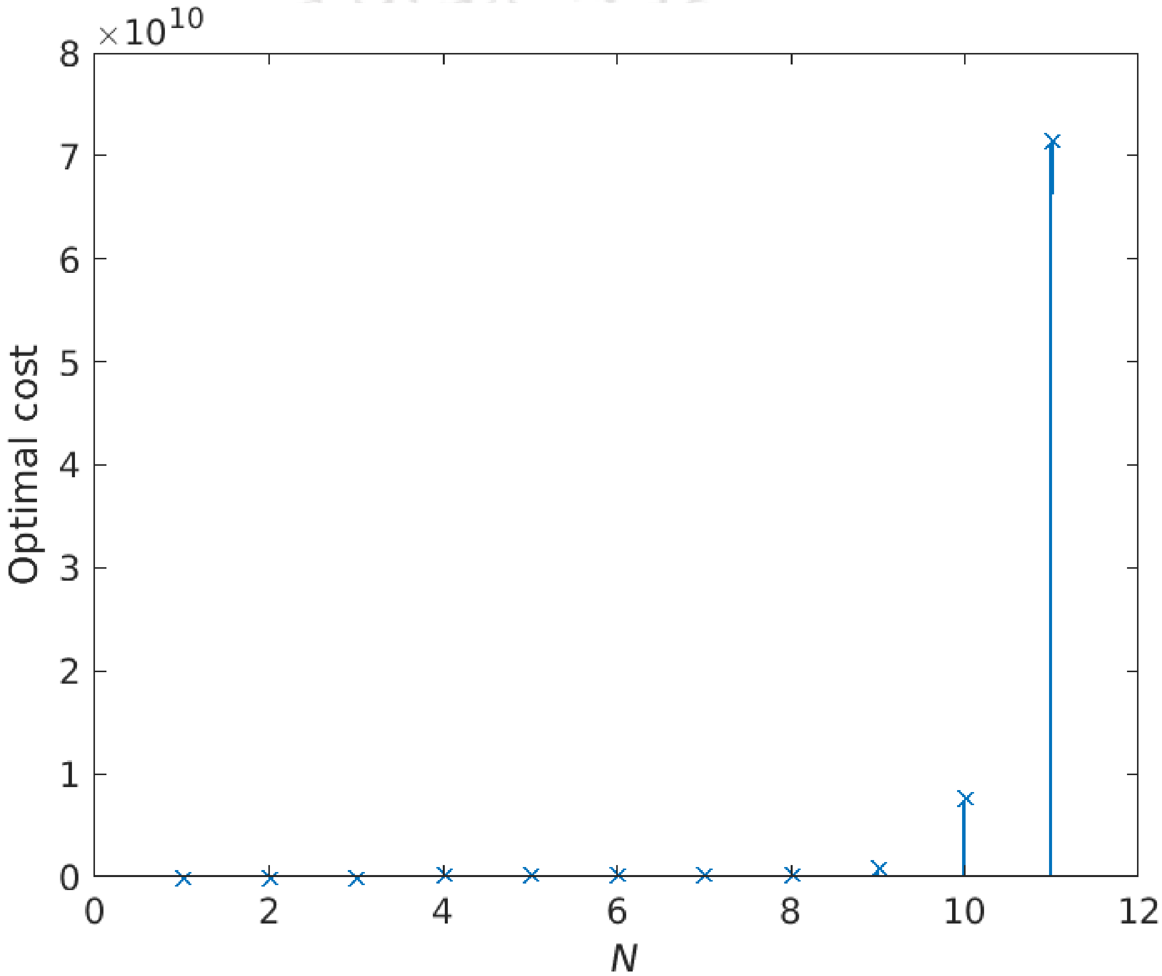

Figure 2.2: Behaviour of the optimal cost as N increases with $\bar{v}^1 = 0.4$, $\bar{v}^2 = 0.38$, $\bar{\mu}^1 = 0.41$, $\bar{\mu}^2 = 0.3$.

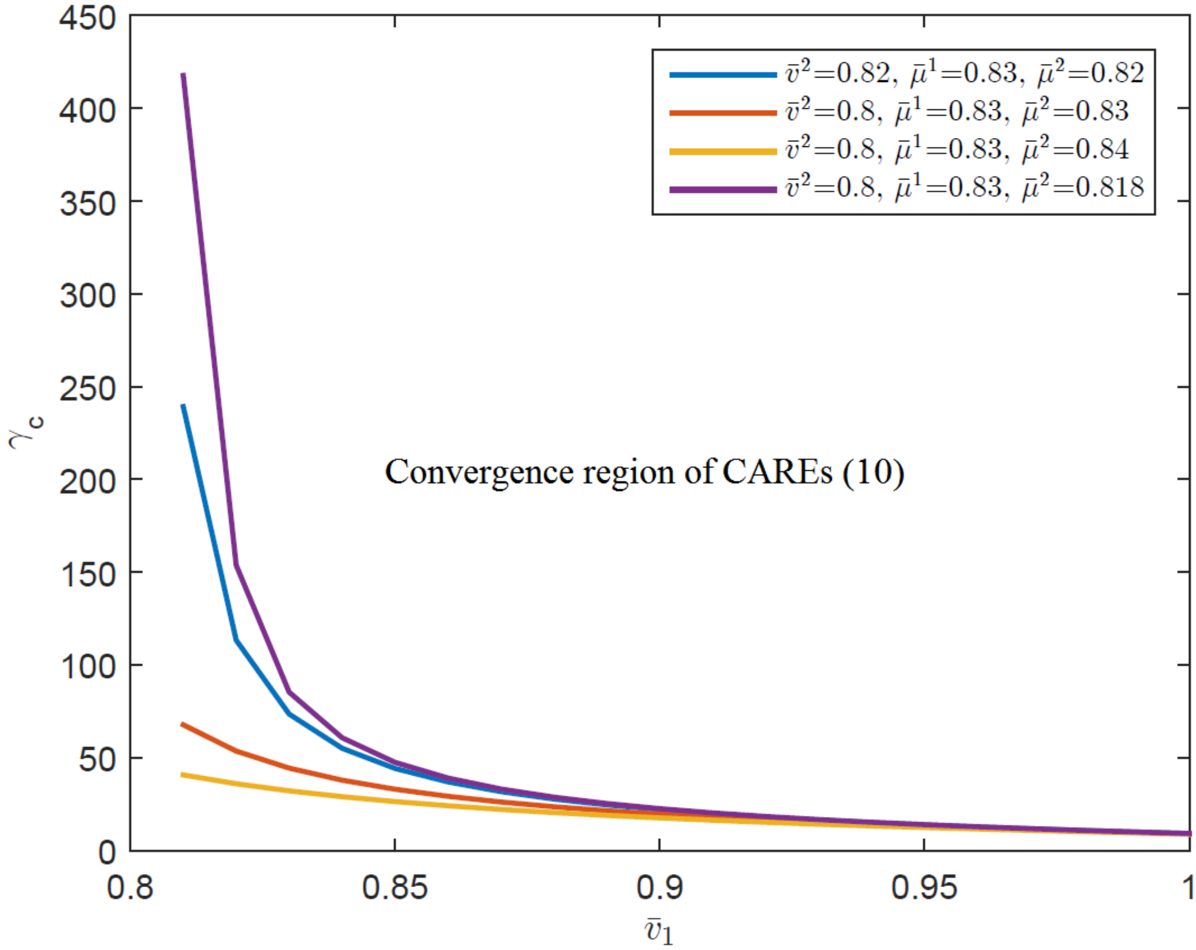

Figure 2.3: Behaviour of γ_c for different $\bar{v}^1$ with ($\bar{v}^2$ = 0.8, $\bar{\mu}^1$ = 0.83, $\bar{\mu}^2$ = 0.818), ($\bar{v}^2$ = 0.8, $\bar{\mu}^1$ = 0.83, $\bar{\mu}^2$ = 0.82), ($\bar{v}^2$ = 0.8, $\bar{\mu}^1$ = 0.83, $\bar{\mu}^2$ = 0.83), ($\bar{v}^2$ = 0.8, $\bar{\mu}^1$ = 0.83, $\bar{\mu}^2$ = 0.84).

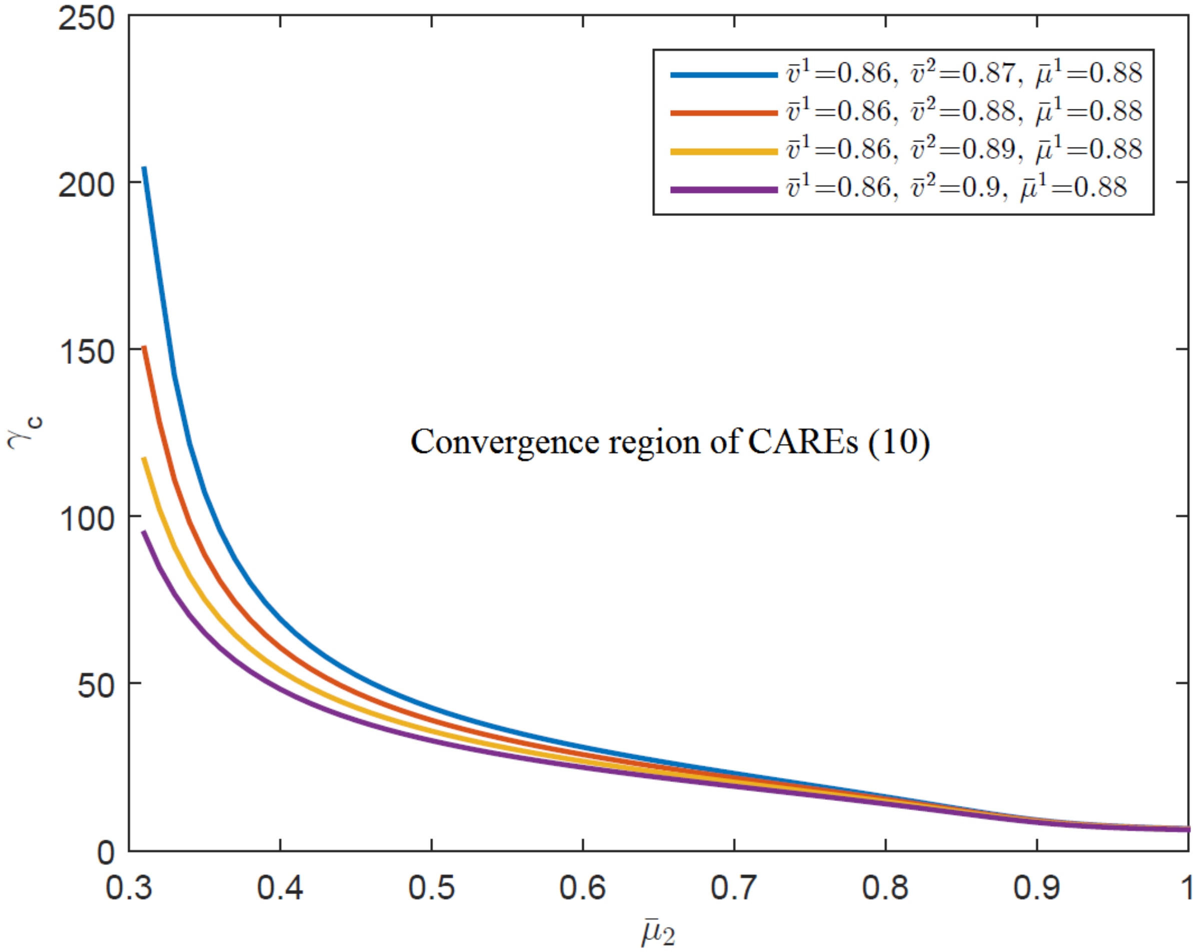

Figure 2.4: Behaviour of γ_c for different $\bar{\mu}^2$ with ($\bar{v}^1$ = 0.86,$\bar{v}^2$ = 0.87, $\bar{\mu}^1$ = 0.88), ($\bar{v}^1$ = 0.86,$\bar{v}^2$ = 0.88, $\bar{\mu}^1$ = 0.88), ($\bar{v}^1$ = 0.86,$\bar{v}^2$ = 0.89, $\bar{\mu}^1$ = 0.88), ($\bar{v}^1$ = 0.86,$\bar{v}^2$ = 0.9, $\bar{\mu}^1$ = 0.88).

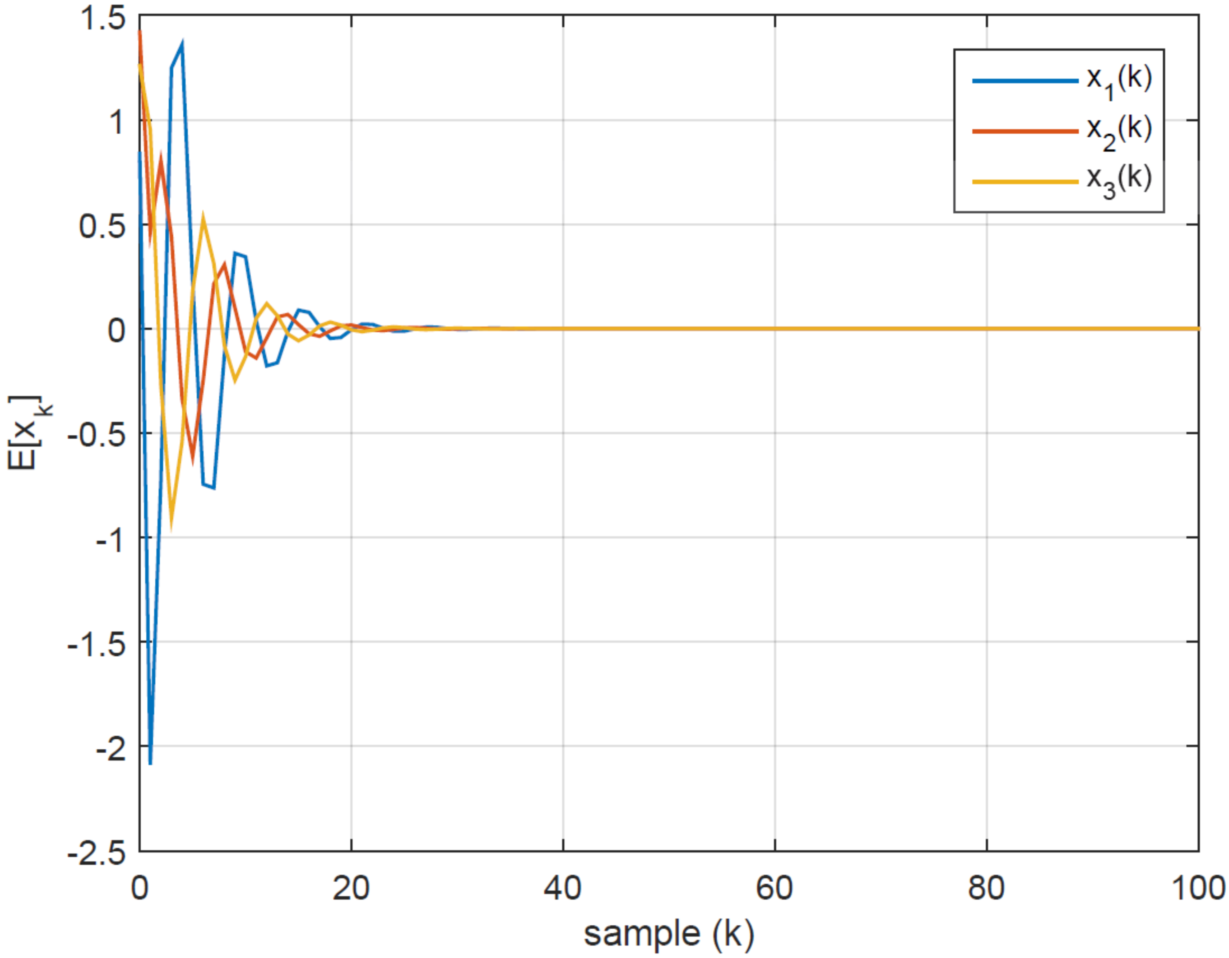

Figure 2.5: Expected value of the state response with disturbance $w_k = \sin(0.5\pi k)e^{-k/2}$ for $\bar{\nu}^1 = 0.9$, $\bar{\nu}^2 = 0.91$, $\bar{\mu}^1 = 0.93$, $\bar{\mu}^2 = 0.92$.

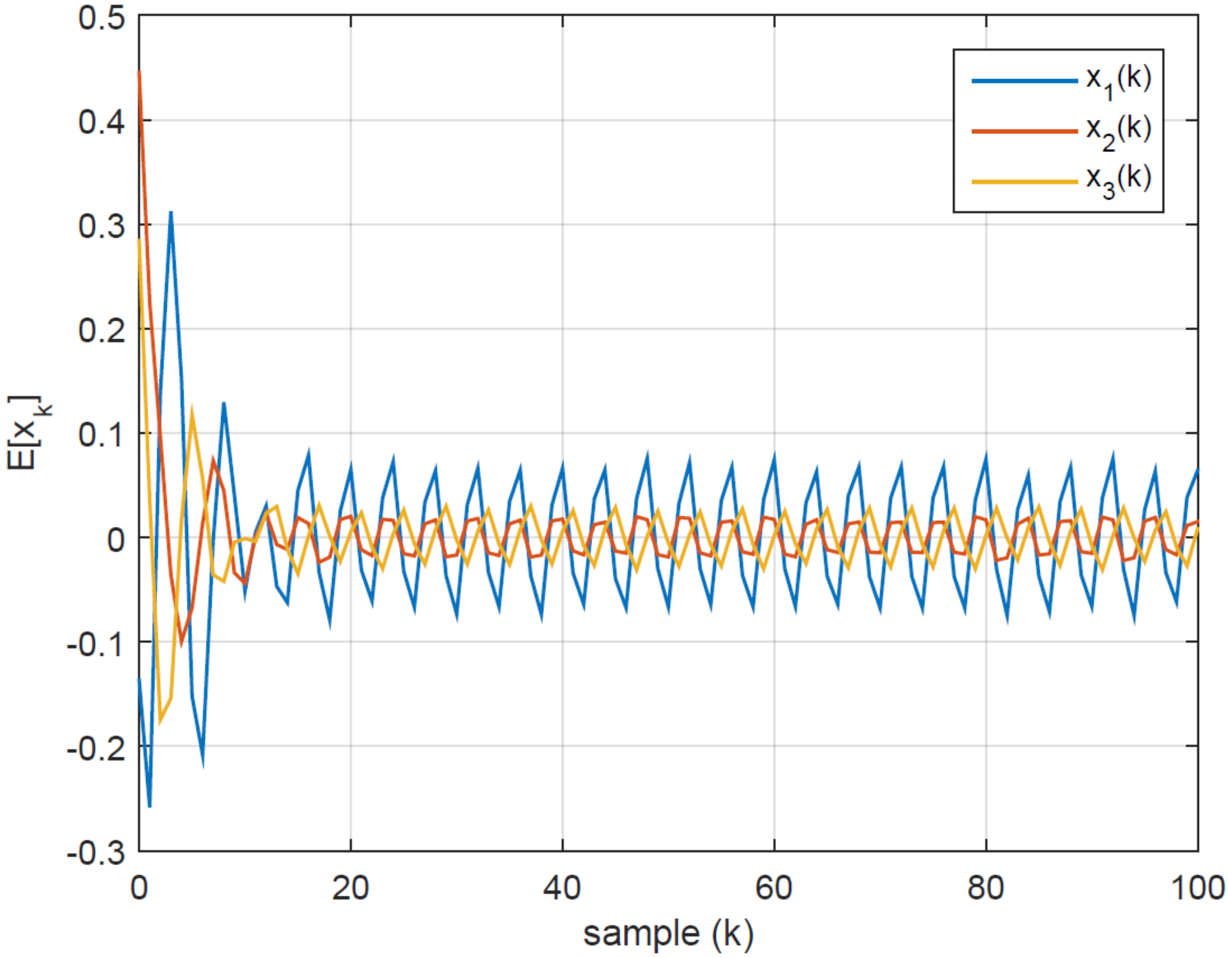

Figure 2.6: Expected value of the state response with persistent disturbance $w_k = 0.025\ sin(0.5\pi k)$ for $\bar{v}^1 = 0.9$, $\bar{v}^2 = 0.91$, $\bar{\mu}^1 = 0.93$, $\bar{\mu}^2 = 0.92$.

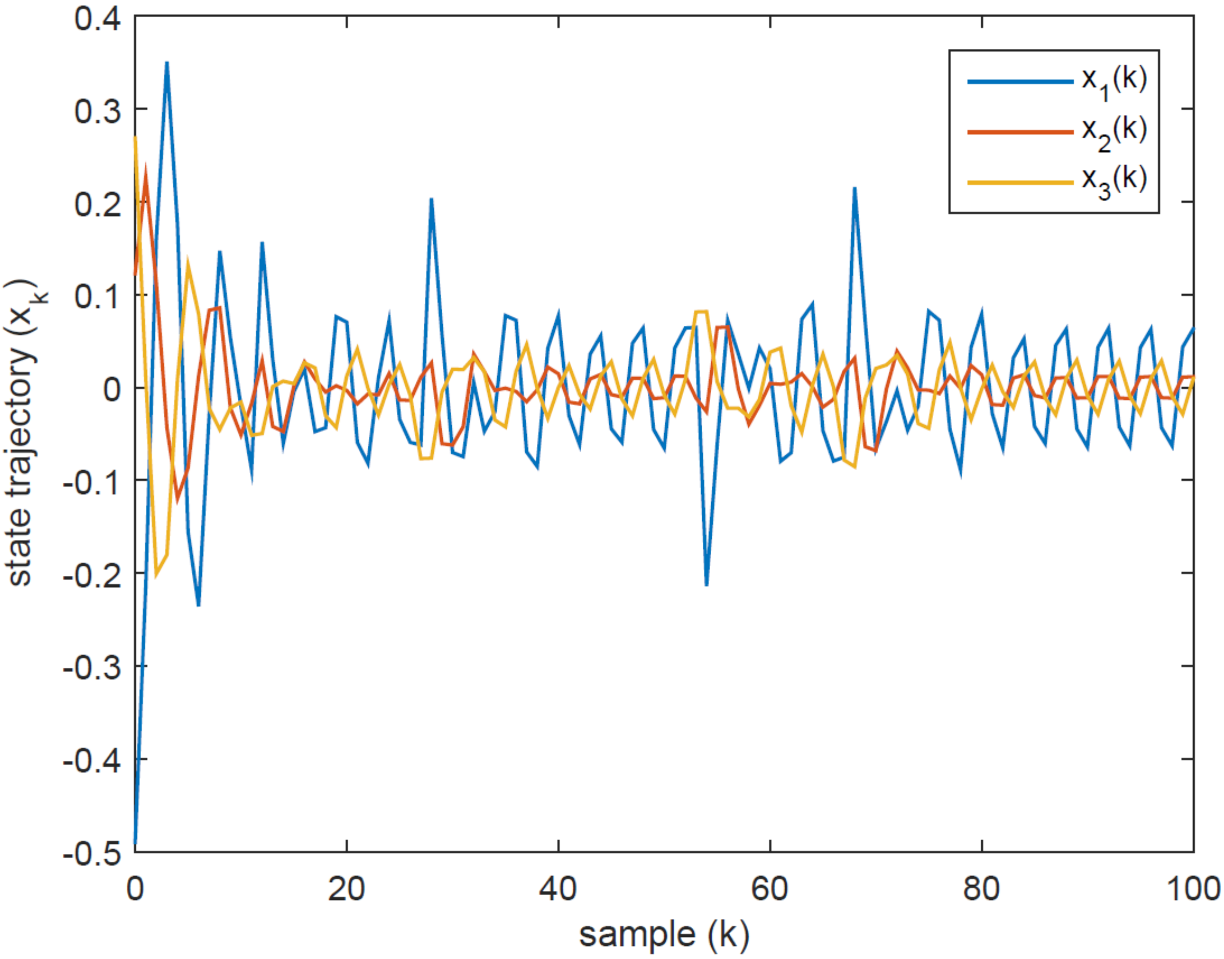

Figure 2.7: State response with random packet losses and persistent disturbance $w_k = 0.025\ sin(0.5\pi k)$ for $\bar{\nu}^1 = 0.9$, $\bar{\nu}^2 = 0.91$, $\bar{\mu}^1 = 0.93$, $\bar{\mu}^2 = 0.92$.

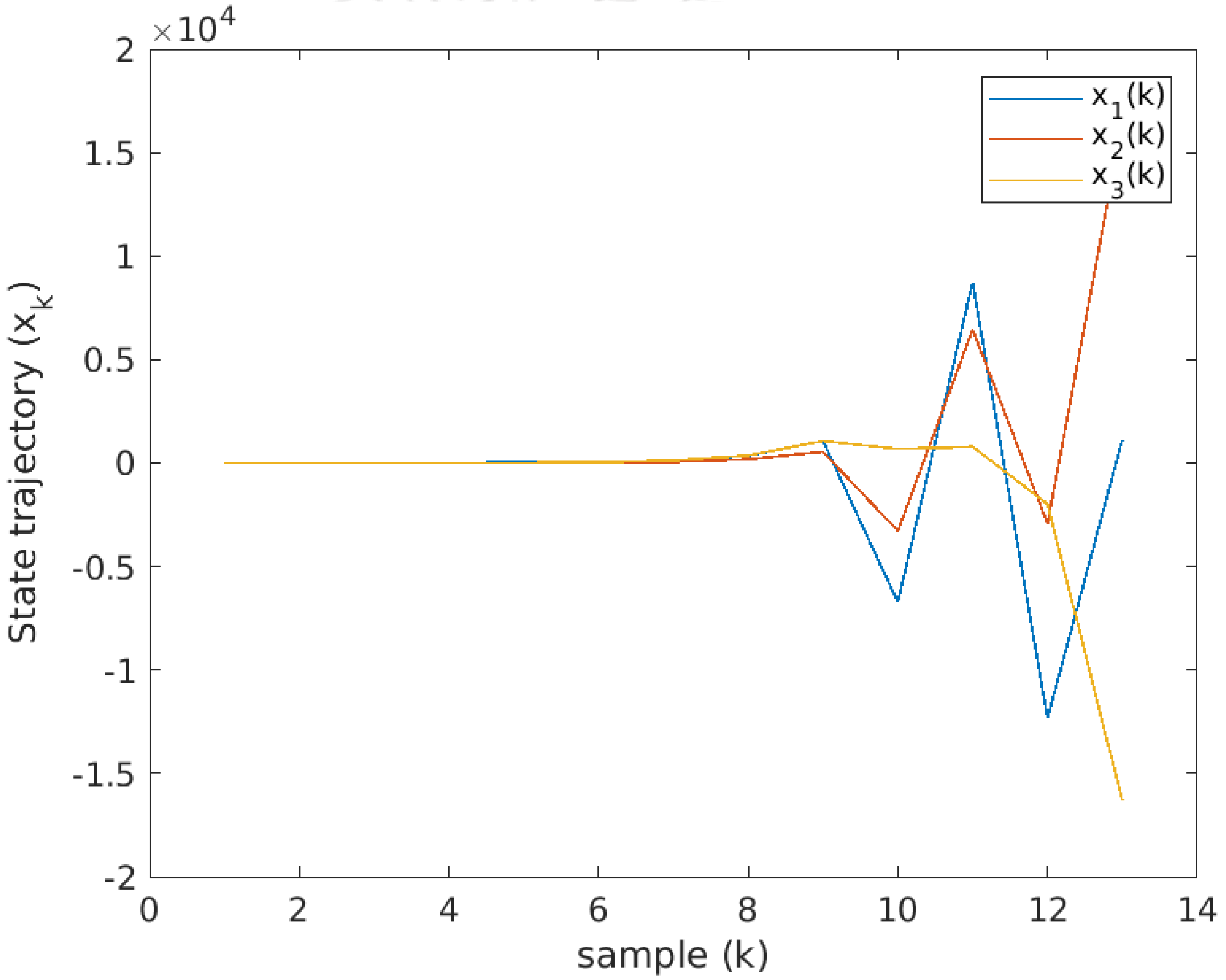

Figure 2.8: State response with random packet losses and persistent disturbance w_k = 0.025 $sin(0.5\pi k)$ for $\bar{v}^1 = 0.4$, $\bar{v}^2 = 0.38$, $\bar{\mu}^1 = 0.41$, $\bar{\mu}^2 = 0.3$.

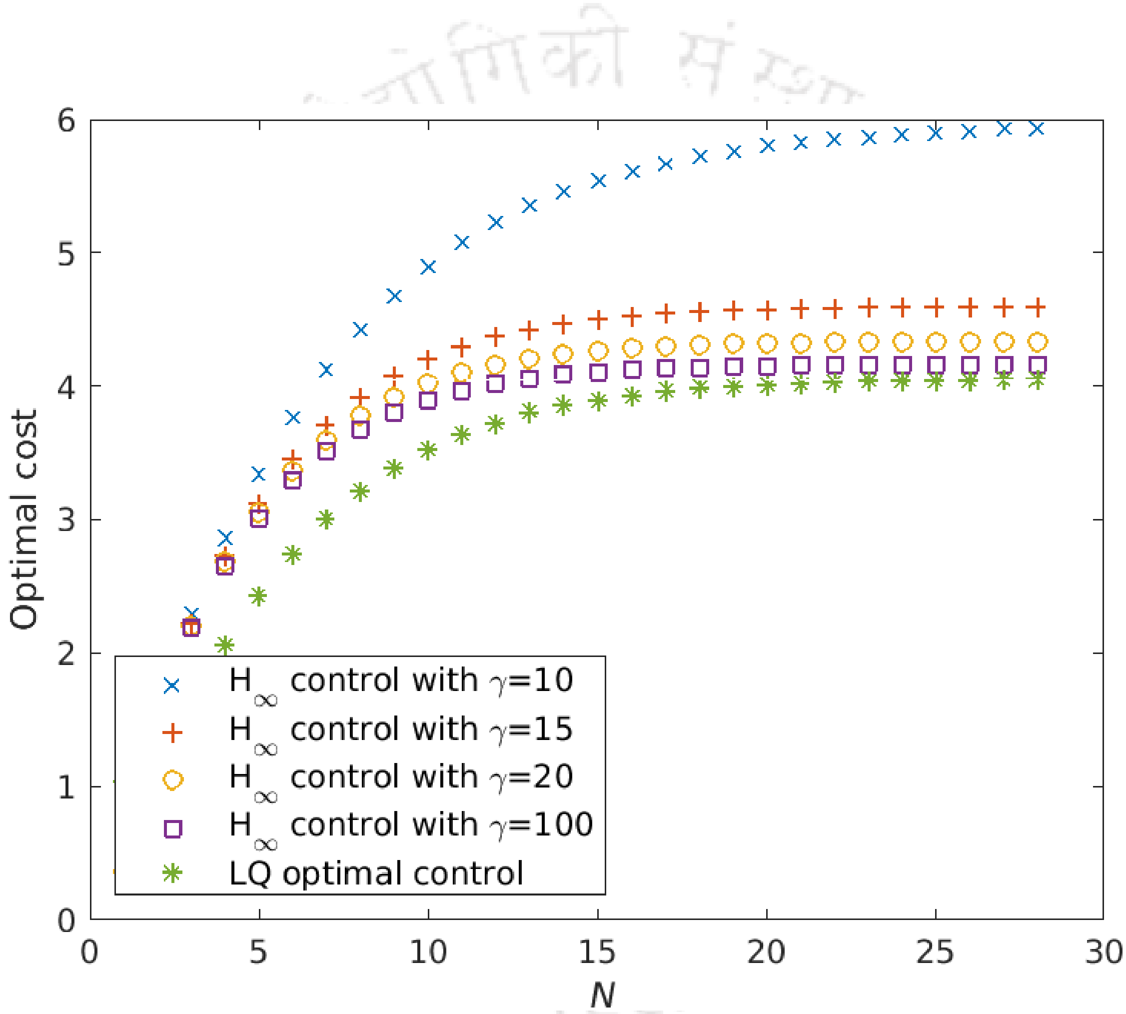

Figure 2.9: Comparison of optimal costs associated with H_∞ control and LQ control with $\bar{\nu}^1 = 0.88$, $\bar{\nu}^2 = 0.86$, $\bar{\mu}^1 = 0.89$, $\bar{\mu}^2 = 0.87$.

The arrival probability of the control signal is considered to be $\bar{\nu} = 0.95$. As the control input u_k and the disturbance input w_k are scalars, Γ_k and Ψ_k are row vectors. Further, $\mathbb{E}[\xi_k R \xi_k] = \bar{\nu} R$ where R is a scalar. Therefore,

$$\mathbb{E}\Big[\Gamma_k^T \xi_k R \xi_k \Gamma_k - \gamma^2 \Psi_k^T \Psi_k \Big| \mathcal{I}_k\Big] = \bar{\nu} R \Gamma_k^T \Gamma_k - \gamma^2 \Psi_k^T \Psi_k.$$

Since the matrices $\Gamma_k^T \Gamma_k$ and $\Psi_k^T \Psi_k$ have rank 1 (because Γ_k and Ψ_k are row matrices), the rank of the matrix $\bar{\nu} R \Gamma_k^T \Gamma_k - \gamma^2 \Psi_k^T \Psi_k$ can not be 3. Hence, the condition given in [22] is not satisfied.

Now, if we consider a transition path $\{\xi_k = 0, \xi_{k+1} = 1, \xi_{k+2} = 1\}$, we get the following,

$$\Gamma_k = \Gamma_{k+1} = \Gamma_{k+2} = \begin{bmatrix} 1.6177 & 1.6459 & 3.1014 \end{bmatrix}, \quad \Psi_k = \Psi_{k+1} = \Psi_{k+2} = \begin{bmatrix} 0.0826 & 0.0646 & 0.1498 \end{bmatrix},$$

$$\mathscr{A}(\xi_k) = \begin{bmatrix} 1.0826 & 2.0646 & 1.1498 \\ 0.0826 & 1.0646 & 1.1498 \\ 1.0826 & 0.0646 & 2.1498 \end{bmatrix}, \quad \mathscr{A}(\xi_{k+1}) = \begin{bmatrix} -0.5351 & 0.4186 & -1.9516 \\ -1.5351 & -0.5814 & -1.9516 \\ 1.0826 & 0.0646 & 2.1498 \end{bmatrix},$$

$$\mathscr{C}^T(\xi_k)\mathscr{C}(\xi_k) = \mathscr{C}^T(\xi_{k+1})\mathscr{C}(\xi_{k+1}) = \mathscr{C}^T(\xi_{k+2})\mathscr{C}(\xi_{k+2}) = \begin{bmatrix} 2.1677 & 2.4986 & 3.3749 \\ 2.4986 & 2.7676 & 3.9796 \\ 3.3749 & 3.9796 & 5.8001 \end{bmatrix}.$$

Thus,

$$\begin{aligned} & O^T O \\ &= \mathscr{C}^T(\xi_k)\mathscr{C}(\xi_k) + \mathscr{A}^T(\xi_k)\mathscr{C}^T(\xi_{k+1})\mathscr{C}(\xi_{k+1})\mathscr{A}(\xi_k) + \mathscr{A}^T(\xi_k)\mathscr{A}^T(\xi_{k+1})\mathscr{C}^T(\xi_{k+2})\mathscr{C}(\xi_{k+2})\mathscr{A}(\xi_{k+1})\mathscr{A}^T(\xi_k) \\ &= \begin{bmatrix} 4.8238 & 3.3274 & 6.7758 \\ 3.3274 & 3.2050 & 4.4559 \\ 6.7758 & 4.4559 & 12.1221 \end{bmatrix} \end{aligned} \tag{2.66}$$

As the matrix $O^T O$ in Equation (2.66) has rank 3, we get that the jump observability matrix $O^T O$ corresponding to the transition $\{\xi_k = 0, \xi_{k+1} = 1, \xi_{k+2} = 1\}$ has rank 3. Thus, the condition given in Claim (d) in Theorem 2.3.10 gets satisfied. Hence, the closed-loop system with the saddle-point policy is stable. Figure 2.10 shows the response of the closed loop system with the optimal control law and worst case disturbance.

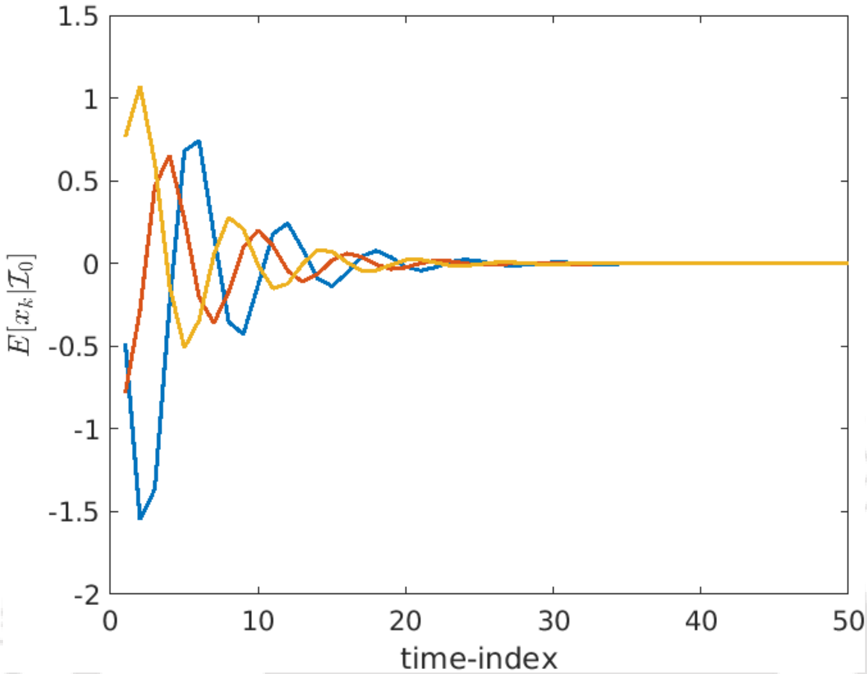

Figure 2.10: State response with the optimal control law and worst-case (optimal) disturbance for $\bar{v} = 0.95$.

Therefore, even though the above system does not satisfy the condition given in [22], it satisfies the condition given in Claim (d) of Theorem 2.3.10.

2.5 Summary

This chapter presents the results for the H_∞ optimal control problem over multiple erasure channels for both finite horizon and infinite horizon cases. Various properties of the associated CAREs are also investigated. Further, the stability of the closed-loop system with the optimal controller is established for three cases: a) with finite energy disturbance, b) without any disturbance, and c) with worst-case disturbance.

3

Jump linear quadratic optimal control of Markovian jump linear systems (MJLSs) over multiple lossy channels

3.1 Introduction

In this chapter, we investigate the optimal jump linear quadratic control problem of a Markovian jump linear system (MJLS) over multiple channels with a TCP-like protocol. Similar to the previous chapter, communication channels between the controller and the actuators are assumed to be lossy, while it is assumed that the channels between sensors and the controller are lossless. Further, the channels are modeled by the Gilbert-Elliott channel model.

We have designed both finite horizon and infinite horizon controllers with random packet losses. Various properties of the associated CAREs are established. Using the weak observability notion, the positive definiteness of the fixed-point solution of the infinite horizon CAREs is shown. Further, it is also demonstrated that the weak observability assumption is sufficient to show the stability of the closed-loop system in the mean-square sense with the optimal controller. All the corresponding results for the linear quadratic (LQ) optimal controller design problem for LTI systems over multiple channels with Markov packet loss are presented as a special case.

The results presented in this chapter generalizes the existing works on the LQ optimal control of linear systems over lossy channels such as [5] and [11]. To the best of our knowledge, jump linear quadratic optimal control of MJLSs over lossy channel has not been investigated. Further, the weak observability assumption which is considered to show the positive definiteness of the solution of the CAREs and stability of the system is less stringent than the one considered in the works on classical JLQ such as [61].

The chapter is organized as follows. In Section 3.2, the problem is formulated. Section 3.3 deals with design of finite horizon controller and infinite horizon controller. The convergence of the infinite horizon cost and stability of the closed-loop system have also been investigated. In section 3.4 with a numerical example we demonstrate our results. Finally, section 3.5 presents the summary of works presented in this chapter.

3.2 Problem Formulation

Let us consider the following discrete-time Markovian jumped linear system:

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k^a \tag{3.1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k^a \in \mathbb{R}^m$ is the control input to the actuators, $r_k \in \mathcal{D} \triangleq \{1, 2, ..., \mathcal{M}\}$ with $\mathcal{M} < \infty$ is an irreducible, aperiodic and time-homogeneous Markov chain. Transition probabilities of the Markov chain are expressed in the transition probability matrix $\Delta = [p_{ij}]$, where

$$p_{ij} = P(r_{k+1} = j|r_k = i); \forall i, j \in \mathcal{D}, k = 0, 1, 2, ...$$

Following terminology related to a Markov chain is followed throughout the work.

(i) A Markov chain is said to be *time-homogeneous* if $P(r_{k+1} = j\big|r_k = i) = P(r_{l+1} = j\big|r_l = i)$; $\forall k, l \in \mathbb{Z}^+$.

(ii) A state j is *accessible* from state i (written as $i \to j$) if there exists an integer $n(ij) \geq 0$ such that

$$P(r_{n(ij)} = j|r_0 = i) > 0.$$

(iii) A state i is said to *communicate* with state j if both $i \to j$ and $j \to i$.

(iv) A *communicating class* is a maximal set of states where every state communicates with each other.

(v) A communicating class is *closed* if there does not exist any transition with nonzero probability from the class to a state outside it.

(vi) A Markov chain is *irreducible* if its state space is a single communicating class.

(vii) A state i is said to be a *aperiodic* or has period 1 if $p_{ii} > 0$.

(viii) A Markov chain is *aperiodic* if all of its states are aperiodic.

(ix) Suppose $r_0 = i \in \mathcal{D}$. Consider a random variable $\mathcal{H}_i$ such that $\mathcal{H}_i = \inf\{n \geq 1 : r_n = i\}$. Now, the state i is called *recurrent* if

$$P\Big(\mathcal{H}_i < \infty \Big| r_0 = i\Big) = 1.$$

(x) A Markov chain is called *recurrent* if all of its states are recurrent.

Throughout the chapter, it is assumed that state of the system x_k and Markov chain state r_k are directly accessible to the controller.

Let $u_k \in \mathbb{R}^m$ be the controller output, which has been sent to the actuators through lossy channels. Following expression relates u_k and u_k^a :

$$u_k^a = \xi_k u_k, \quad \text{where } \xi_k \textit{ is as } \text{defined in Chapter 2.} \tag{3.2}$$

Note 3.2.1. *It should be clearly noted that* $\{r_k\}$ *and* $\{v_k^i\}$, $\forall i \in \{1, 2, ..., m\}$ *are independent Markov processes.*

For a TCP-like protocol, the information set $\mathcal{I}_k$ available to the controller at k^{th} time-index is expressed as:

$$\mathcal{I}_k = \{x_0, ..., x_k, r_0, ..., r_k, \xi_0, ...\xi_{k-1}\}.$$

The control policy $\zeta_{0:k} = \{\zeta_0, \zeta_1, ..., \zeta_k\}$ is defined as a sequence of maps ζ_0, ζ_1,...,ζ_k from the information set $\mathcal{I}_k$ to the control input set $\mathcal{U}$, i.e., $\zeta_k : \mathcal{I}_k \rightarrow \mathcal{U}$.

We now focus on finding control policy ζ, such that $u_k = \zeta_k(\mathcal{I}_k)$ minimizes the following cost function:

$$\begin{aligned} J_N(\zeta_{0:N-1}) &= \mathbb{E}\Big[\|x_N\|^2_{W_N} + \sum_{k=0}^{N-1} \|x_k\|^2_{W_k} + \|u^a_k\|^2_{R_k}\Big|\mathcal{I}_0\Big] \\ &= \mathbb{E}\Big[\|x_N\|^2_{W_N} + \sum_{k=0}^{N-1} \|x_k\|^2_{W_k} + \|\xi_k u_k\|^2_{R_k}\Big|\mathcal{I}_0\Big] \end{aligned} \tag{3.3}$$

where W_k and R_k are symmetric matrices such that $W_k \geq 0$ and $R_k > 0$ for all k.

3.3 Main Results

In this section, using dynamic programming, we shall find the optimal control law and the optimal value of the cost function given in (3.3).

A. Finite horizon control:

One can write the value function, i.e., the cost-to-go from k^{th} stage as follows:

$$V_{k,N}(x_k, r_k, \xi_{k-1}) = \min_{u_k} \mathbb{E}\Big[\|x_N\|^2_{W_N} + \sum_{j=k}^{N-1} \|x_j\|^2_{W_j} + \|\xi_j u_j\|^2_{R_j}\Big|\mathcal{I}_k\Big] \tag{3.4}$$

Using Bellman's principle of optimality, the value function can further be expressed as:

$$V_{k,N}(x_k, r_k, \xi_{k-1}) = \min_{u_k} \mathbb{E}\left[x_k^T W_k x_k + u_k^T \xi_k^T R_k \xi_k u_k + V_{k+1,N}(x_{k+1}, \xi_k, r_{k+1})\Big|\mathcal{I}_k\right] \tag{3.5}$$

Lemma 3.3.1. *For the cost function (3.3), subject to system dynamics (3.1), the following claims are true.*

(a) Suppose at the $(k-1)^{th}$ time index, packet loss status in the controller-to-actuator path is $\xi_{k-1} = \mathcal{N}(\mathscr{I})$, $\mathscr{I} \subseteq \mathscr{G}$, and $r_k = i \in \mathcal{D}$, then the value function at stage $k \in [0, N]$ can be expressed as follows:

$$V_{k,N}\Big(x_k, i, \mathcal{N}(\mathscr{I})\Big) = x_k^T \Xi_{k,N}\Big(i, \mathcal{N}(\mathscr{I})\Big)x_k, \tag{3.6}$$

where $\Xi_{k,N}\big(i, \mathcal{N}(\mathscr{I})\big)$, *for* $k \in [0, N-1]$, *is a symmetric matrix and is generated by the following coupled algebraic Riccati equations (CAREs):*

$$\begin{aligned}\Xi_{k,N}\big(i, \mathcal{N}(\mathscr{I})\big) &= W_k + A^T(i)\mathbb{E}\Big[\mathbf{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]A(i)\\ &- A^T(i)\Big(\mathbb{E}\Big[\xi_k B^T(i)\mathbf{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]\Big)^T\Big(\mathbb{E}\Big[\xi_k\Big(R_k + B^T(i)\mathbf{X}_{k+1,N}(i,\xi_k)B(i)\Big)\xi_k\Big|\mathcal{I}_k\Big]\Big)^{-1}\\ &\times \mathbb{E}\Big[\xi_k B^T(i)\mathbf{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]A(i),\end{aligned} \tag{3.7}$$

herein $\mathscr{I}$, $\mathscr{G}$, $\mathcal{N}(.)$, $\mathbb{E}(.|\mathcal{I}_k)$ *are as defined in chapter 2, and* $\mathbf{X}_{k+1,N}\big(i,\xi_k\big)$ *is defined as:*

$$\mathbf{X}_{k+1,N}(i,\xi_k) = \sum_{t=1}^{\mathcal{M}}\Big(p_{it}\Xi_{k+1,N}(t,\xi_k)\Big). \tag{3.8}$$

(b) The optimal control law, at a stage $k \in [0, N-1]$, *is given by*

$$u_k^* = -\Big(\mathbb{E}\Big[\xi_k\Big(R_k + B^T(i)\mathbf{X}_{k+1,N}(i,\xi_k)B(i)\Big)\xi_k\Big|\mathcal{I}_k\Big]\Big)^{-1}\mathbb{E}\Big[\xi_k B^T(i)\mathbf{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]A(i)x_k. \tag{3.9}$$

(c) The optimal cost for finite horizon is given by

$$J_N(\zeta^*_{0:N-1}) = x_0^T\Xi_{0,N}(r_0, \mathcal{N}(\mathscr{I}))x_0.$$

Proof: To prove the lemma, we use induction.

We assume the base case to be $k = N-1$. For the stage $k = N$, it is trivial to see that $V_{N,N}\big(x_N, i, \mathcal{N}(\mathscr{I})\big) = x_N^T\Xi_{N,N}(i, \mathcal{N}(\mathscr{I}))x_N$ where $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) = W_N$, $\forall i \in \mathcal{D}$ and $\forall \mathscr{I} \subseteq \mathscr{G}$.

With information set $\mathcal{I}_{N-1}$, if $r_{N-1} = i \in \mathcal{D}$ and $\xi_{N-2} = \mathcal{N}(\mathscr{I})$,

$$\begin{aligned}
&\mathbb{E}[V_{N,N}(x_N, \xi_{N-1}, r_N)\big|\mathcal{I}_{N-1}] \\
&= \sum_{\mathscr{L}\in\mathscr{G}} \mathcal{P}_{N-1}(\mathcal{N}(\mathscr{L}))\Big\{\Big(A(i)x_{N-1} + B(i)\mathcal{N}(\mathscr{L})u_{N-1}\Big)^T \sum_{l=1}^{\mathcal{M}} \Big(p_{il}\Xi_{N,N}(l, \mathcal{N}(\mathscr{L}))\Big) \\
&\times \Big(A(i)x_{N-1} + B(i)\mathcal{N}(\mathscr{L})u_{N-1}\Big)\Big\} \\
&= \sum_{\mathscr{L}\in\mathscr{G}} \mathcal{P}_{N-1}(\mathcal{N}(\mathscr{L}))\Big\{x_{N-1}^T A^T(i)\mathbf{X}_{N,N}(i, \mathcal{N}(\mathscr{L}))A(i)x_{N-1} \\
&+ u_{N-1}^T \mathcal{N}(\mathscr{L})B^T(i)\mathbf{X}_{N,N}(i, \mathcal{N}(\mathscr{L}))B(i)\mathcal{N}(\mathscr{L})u_{N-1} \\
&+ 2u_{N-1}^T \mathcal{N}(\mathscr{L})B^T(i)\mathbf{X}_{N,N}(i, \mathcal{N}(\mathscr{L}))A(i)x_{N-1}\Big\},
\end{aligned} \tag{3.10}$$

where, $\mathcal{P}_k(.)$ is as defined in Chapter 2.

Now, using (3.5) and (3.10), we can expressed $V_{N-1,N}\Big(x_{N-1}, i, \mathcal{N}(\mathscr{I})\Big)$ as follows:

$$\begin{aligned}
&V_{N-1,N}\Big(x_{N-1}, i, \mathcal{N}(\mathscr{I})\Big) \\
&= \min_{u_{N-1}}\Big\{ x_{N-1}^T W_{N-1}x_{N-1} + x_{N-1}^T A^T(i)\mathbb{E}[\mathbf{X}_{N,N}(i, \xi_{N-1})\big|\mathcal{I}_{N-1}]A(i)x_{N-1} \\
&+ u_{N-1}^T \mathbb{E}[\xi_{N-1}(R_{N-1} + B^T(i)\mathbf{X}_{N,N}(i, \xi_{N-1})B(i))\xi_{N-1}\big|\mathcal{I}_{N-1}]u_{N-1} \\
&+ 2u_{N-1}^T \mathbb{E}[\xi_{N-1}B^T(i)\mathbf{X}_{N,N}(i, \xi_{N-1})\big|\mathcal{I}_{N-1}]A(i)x_{N-1}\Big\}
\end{aligned} \tag{3.11}$$

From the above equation, the optimal control law can be derived as follows:

$$\begin{aligned}
u_{N-1}^* = &-\Big[\mathbb{E}[\xi_{N-1}(R_{N-1} + B^T(i)\mathbf{X}_{N,N}(i, \xi_{N-1})B(i))\xi_{N-1}\big|\mathcal{I}_{N-1}]\Big]^{-1} \\
&\times \mathbb{E}[\xi_{N-1}B^T(i)\mathbf{X}_{N,N}(i, \xi_{N-1})\big|\mathcal{I}_{N-1}]A(i)x_{N-1}
\end{aligned} \tag{3.12}$$

Substituting the optimal control law back in equation (3.11),

$$V_{N-1,N}\Big(x_{N-1}, i, \mathcal{N}(\mathscr{I})\Big) = x_{N-1}^T \Xi_{N-1,N}\Big(i, \mathcal{N}(\mathscr{I})\Big)x_{N-1},$$

where $\Xi_{N-1,N}(i, \mathcal{N}(\mathscr{I}))$ is given by CAREs (3.7).

Let us now assume that Claim (a) is true for the $(k+1)^{th}$ stage. So, with information set $\mathcal{I}_{k+1}$, if $r_{k+1} = i \in \mathcal{D}$ and $\xi_k = \mathcal{N}(\mathscr{I})$, one can represent $V_{k+1,N}(x_{k+1}, r_{k+1}, \xi_k)$ as:

$$V_{k+1,N}\Big(x_{k+1}, i, \mathcal{N}(\mathscr{I})\Big) = x_{k+1}^T \Xi_{k+1,N}(i, \mathcal{N}(\mathscr{I}))x_{k+1} \tag{3.13}$$

Now, with information set $\mathcal{I}_k$:

$$\begin{aligned}
&\mathbb{E}\Big[V_{k+1,N}(x_{k+1}, \xi_k, r_{k+1})\Big|\mathcal{I}_k\Big] \\
&= \sum_{\mathscr{L}\in\mathscr{G}} \mathcal{P}_k(\mathcal{N}(\mathscr{L}))\Big\{\Big(A(i)x_k + B(i)\mathcal{N}(\mathscr{L})u_k\Big)^T \sum_{l=1}^{\mathcal{M}} \Big(p_{il}\Xi_{k+1,N}(l, \mathcal{N}(\mathscr{L}))\Big)\Big(A(i)x_k + B(i)\mathcal{N}(\mathscr{L})u_k\Big)\Big\} \\
&= \sum_{\mathscr{L}\in\mathscr{G}} \mathcal{P}_k(\mathcal{N}(\mathscr{L}))\Big\{x_k^T A^T(i)\mathbf{X}_{k+1,N}(i, \mathcal{N}(\mathscr{L}))A(i)x_k + u_k^T \mathcal{N}(\mathscr{L})B^T(i)\mathbf{X}_{k+1,N}(i, \mathcal{N}(\mathscr{L}))B(i)\mathcal{N}(\mathscr{L})u_k \\
&+ 2u_k^T \mathcal{N}(\mathscr{L})B^T(i)\mathbf{X}_{k+1,N}(i, \mathcal{N}(\mathscr{L}))A(i)x_k\Big\}.
\end{aligned} \tag{3.14}$$

Combining (3.5) and (3.14):

$$\begin{aligned}
&V_{k,N}\Big(x_k, i, \mathcal{N}(\mathscr{I})\Big) \\
&= \min_{u_k}\Big\{ x_k^T W_k x_k + x_k^T A^T(i)\mathbb{E}[\mathbf{X}_{k+1,N}(i, \xi_k)\Big|\mathcal{I}_k]A(i)x_k + u_k^T \mathbb{E}[\xi_k(R_k + B^T(i)\mathbf{X}_{k+1,N}(i, \xi_k)B(i))\xi_k\Big|\mathcal{I}_k]u_k \\
&+ 2u_k^T \mathbb{E}[\xi_k B^T(i)\mathbf{X}_{k+1,N}(i, \xi_k)\Big|\mathcal{I}_k]A(i)x_k\Big\}
\end{aligned} \tag{3.15}$$

The optimal control law is given as:

$$u_k^* = -\Big[\mathbb{E}[\xi_k(R_k + B^T(i)\mathbf{X}_{k+1,N}(i, \xi_k)B(i))\xi_k\Big|\mathcal{I}_k]\Big]^{-1} \mathbb{E}[\xi_k B^T(i)\mathbf{X}_{k+1,N}(i, \xi_k)\Big|\mathcal{I}_k]A(i)x_k \tag{3.16}$$

Substituting the optimal control law back in equation (3.15),

$$V_{k,N}\Big(x_k, i, \mathcal{N}(\mathscr{I})\Big) = x_k^T \Xi_{k,N}\Big(i, \mathcal{N}(\mathscr{I})\Big) x_k.$$

The optimal cost is given by,

$$J_N(\zeta^*_{0:N-1}) = V_{0,N}\Big(x_0, r_0, \mathcal{N}(\mathscr{I})\Big) = x_0^T \Xi_{0,N}\Big(r_0, \mathcal{N}(\mathscr{I})\Big) x_0 \tag{3.17}$$

□

As one does not have knowledge of the packet loss status ξ_k at the stage $k = 0$, similar to Note 2.3.1, we get the following.

Note 3.3.1. *At the stage $k = 0$, for a $r_0 \in \mathcal{D}$, $\Xi_{0,N}\Big(r_0, \mathcal{N}(\mathscr{I})\Big)$s are identical for all $\mathscr{I} \subseteq \mathscr{G}$.*

□

Corollary 3.3.2. *The finite horizon linear quadratic optimal controller for a linear time-invariant (LTI) system can be derived from Lemma 3.3.1 by considering the special case where $\mathcal{D} = \{1\}$. The optimal controller can be expressed as follows:*

$$u_k^* = -\Big[\mathbb{E}[\xi_k\Big(R_k + B^T \Xi_{k+1,N}(\xi_k) B\Big)\xi_k \Big| \mathcal{I}_k]\Big]^{-1} \mathbb{E}\Big[\xi_k B^T \Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big] A x_k$$

where, $A(1) = A$, $B(1) = B$. For $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$, $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$ is generated by the following CAREs:

$$\begin{aligned}
&\Xi_{k,N}(\mathcal{N}(\mathscr{I})) \\
&= W_k + A^T \mathbb{E}\Big[\Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big]A - A^T\Big[\mathbb{E}\Big[\xi_k B^T \Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big]\Big]^T \Big[\mathbb{E}\Big[\xi_k\Big(R_k + B^T \Xi_{k+1,N}(\xi_k) B\Big)\xi_k\Big|\mathcal{I}_k\Big]\Big]^{-1} \\
&\times \mathbb{E}\Big[\xi_k B^T \Xi_{k+1,N}(\xi_k)\Big|\mathcal{I}_k\Big]A
\end{aligned} \tag{3.18}$$

Remark 3.3.1. *For the case where $\bar{\nu}^l = \bar{\mu}^l$, $\forall l \in \{1, 2, ..., m\}$, the packet loss model becomes equivalent to the Bernoulli packet loss model and hence, CAREs (3.18) coincide with the CAREs given in [11].*

In the sequel, we shall assume that $W_k = W$ and $R_k = R$ for all k.

The following lemma establishes the monotonicity of $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$; $\forall i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$, which will be used in the infinite horizon part.

Lemma 3.3.3. *For $k \geq 1$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$, $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{k+1,N}(i, \mathcal{N}(\mathscr{I}))$.*

Proof: The lemma is proved using the induction.

We have $\Xi_{N,N}(i, \mathcal{N}(\mathscr{I})) \geq 0 = \Xi_{N+1,N}(i, \mathcal{N}(\mathscr{I}))$. Let us now assume that $\Xi_{k+1,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{k+2,N}(i, \mathcal{N}(\mathscr{I}))$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Hence, from (3.8): $\mathbf{X}_{k+1,N}(i, \mathcal{N}(\mathscr{I})) \geq \mathbf{X}_{k+2,N}(i, \mathcal{N}(\mathscr{I}))$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$.

Therefore, using (3.5) and (3.14)

$$
\begin{aligned}
&V_{k,N}(x, i, \mathcal{N}(\mathscr{I})) \\
&= x^T \Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))x \\
&= \min_u \Big[x^T W x + u^T \mathbb{E}^{\mathscr{I}}\big[\xi_k R \xi_k\big] u + \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_k(\mathcal{N}(\mathscr{L}))\Big\{\big(A(i)x + B(i)\xi_k u\big)^T \mathbf{X}_{k+1,N}(i, \mathcal{N}(\mathscr{L})) \\
&\times \big(A(i)x + B(i)\xi_k u\big)\Big\}\Big] \\
&\geq \min_u \Big[x^T W x + u^T \mathbb{E}^{\mathscr{I}}\big[\xi_k R \xi_k\big] u + \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_{k+1}(\mathcal{N}(\mathscr{L}))\Big\{\big(A(i)x + B(i)\xi_k u\big)^T \mathbf{X}_{k+2,N}(i, \mathcal{N}(\mathscr{L})) \\
&\times \big(A(i)x + B(i)\xi_k u\big)\Big\}\Big] \quad \text{as } \mathcal{P}_k(\mathcal{N}(\mathscr{L})) = \mathcal{P}_{k+1}(\mathcal{N}(\mathscr{L})) \text{ if } \xi_{k-1} = \xi_k = \mathcal{N}(\mathscr{I}) \\
&= V_{k+1,N}(x, i, \mathcal{N}(\mathscr{I})) \\
&= x^T \Xi_{k+1,N}(i, \mathcal{N}(\mathscr{I}))x
\end{aligned}
\tag{3.19}
$$

As (3.19) is true for all $x \neq 0$, one can infer that $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{k+1,N}(i, \mathcal{N}(\mathscr{I}))$. □

Note 3.3.2. *As the Markov chain is time-homogeneous, and it is assumed that $W_k = W$, $R_k = R$, $\forall k$, it is easy to see that $V_{k,N}(x, i, \mathcal{N}(\mathscr{I})) = V_{k-r,N-r}(x, i, \mathcal{N}(\mathscr{I}))$ for all $r \leq k-1$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Hence, $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) = \Xi_{k-r,N-r}(i, \mathcal{N}(\mathscr{I}))$. Therefore, from Lemma 3.3.3, it can be concluded that $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{k,N-1}(i, \mathcal{N}(\mathscr{I}))$ for all $k \geq 1$.* □

Now, using the same line of argument as used in the proof of Lemma 2.3.5 in Chapter 2 and Note 3.3.2, we get the following lemma.

Lemma 3.3.4. $\Xi_{0,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{0,N-1}(i, \mathcal{N}(\mathscr{I}))$, *for all $\mathscr{I} \subseteq \mathscr{G}$ and $i \in \mathcal{D}$.* □

B. Infinite horizon control:

In this section, we obtain the infinite horizon optimal controller by considering $N \to \infty$. For doing this, we shall use the concept of stochastic stabilizability. We extend the definition of stochastic stabilizability for classical MJLSs given in [62] to MJLSs with packet losses as follows.

Definition 3.3.1. *System (3.1) is said to be stochastically stabilizable for the control arrival probabilities $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$ and $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ if there exists a gain $\mathcal{K}$ such that, with control input $u_k = -\mathcal{K}x_k$, the following inequality is satisfied:*

$$\sum_{k=0}^{\infty} \mathbb{E}\left[\|x_k\|^2 \middle| \mathcal{I}_0\right] < \infty$$

□

By the following lemma, the convergence of the cost function (3.3) as $N \to \infty$ will be established.

Lemma 3.3.5. *Suppose system (3.1) is stochastically stabilizable for the control arrival probabilities* $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$ *and* $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$*, then the infinite horizon problem has a well define solution.*

Proof: Since system (3.1) is stochastically stabilizable, there exists a control input $u_k = -\mathcal{K} x_k$ such that $\sum_{k=0}^{\infty} \mathbb{E}\Big[||x_k||^2 \Big| \mathcal{I}_0\Big] < \infty$. The infinite horizon cost incurred with this control input is given by:

$$\begin{aligned} J_\infty(\zeta_{0:\infty}) &= \mathbb{E}\Big[\sum_{k=0}^{\infty} ||x_k||_W^2 + ||\xi_k u_k||_R^2 \Big| \mathcal{I}_0\Big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{\infty} \Big[x_k^T \Big(W + \mathbb{E}[\xi_k \mathcal{K}^T R \mathcal{K} \xi_k | \mathcal{I}_k]\Big) x_k\Big] \Big| \mathcal{I}_0\Big] \\ &\leq \delta \mathbb{E}\Big[\sum_{k=0}^{\infty} ||x_k||^2 \Big| \mathcal{I}_0\Big] \\ &< \infty \end{aligned} \tag{3.20}$$

where, $\delta = \rho_{max}\Big\{W + \mathbb{E}^{\mathscr{I}}\Big[\xi_k \mathcal{K}^T R \mathcal{K} \xi_k\Big]\Big\}$.

Clearly, we have: $J_\infty(\zeta^*_{0:\infty}) \leq J_\infty(\zeta_{0:\infty}) < \infty$.

Now, from (3.17) and considering $N \to \infty$, we can write the infinite horizon cost as follows:

$$J_\infty(\zeta^*_{0:\infty}) = \lim_{N\to\infty} x_0^T \Xi_{0,N}(i, \mathcal{N}(\mathscr{L})) x_0 \tag{3.21}$$

As, $J_\infty(\zeta^*_{0:\infty}) < \infty$ for all x_0, we get that $\Xi_{0,N}(i, \mathcal{N}(\mathscr{I}))$, $\forall i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$, is bounded as $N \to \infty$. This, in turn, implies that $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$, for $k \geq 1$, is bounded as well. Also, from Note 3.3.2, $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$ is a monotonically increasing function in N. Therefore, there exists $\bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$ such that, for all $k \geq 1$, $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) \to \bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$ as $N \to \infty$, where $\bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$ is the unique fixed-point solution of the following CAREs.

$$\bar{\Xi}(i, \mathcal{N}(\mathscr{I})) = W + A^T(i)\mathbb{E}^{\mathscr{I}}\Big[\bar{\mathbf{X}}(i, \xi_k)\Big]A(i)$$
$$- A^T(i)\Big(\mathbb{E}^{\mathscr{I}}\Big[\xi_k\Big(B^T(i)\bar{\mathbf{X}}(i, \xi_k)\Big]\Big)^T\Big(\mathbb{E}^{\mathscr{I}}\Big[\xi_k\Big(R + B^T(i)\bar{\mathbf{X}}(i, \xi_k)B(i)\Big)\xi_k\Big]\Big)^{-1}\mathbb{E}^{\mathscr{I}}\Big[\xi_k B^T(i)\bar{\mathbf{X}}(i, \xi_k)\Big]A(i) \tag{3.22}$$

herein, $\bar{\mathbf{X}}(i, \mathcal{N}(\mathscr{I})) = \sum_{t=1}^{\mathcal{M}}\Big\{p_{it}\bar{\Xi}(t, \mathcal{N}(\mathscr{I}))\Big\}$.

Similarly, there exists $\hat{\bar{\Xi}}(i)$ such that, $\Xi_{0,N}(i, \mathcal{N}(\mathscr{I})) \rightarrow \hat{\bar{\Xi}}(i)$, as $N \rightarrow \infty$, where $\hat{\bar{\Xi}}(i)$ is the unique fixed-point solution of the following CAREs.

$$\hat{\bar{\Xi}}(i) = W + A^T(i)\mathbb{E}\Big[\bar{\mathbf{X}}(i, \xi_0)\Big|\mathcal{I}_0\Big]A(i)$$
$$- A^T(i)\Big(\mathbb{E}\Big[\xi_0 B^T(i)\bar{\mathbf{X}}(i, \xi_0)\Big|\mathcal{I}_0\Big]\Big)^T\Big(\mathbb{E}\Big[\xi_0\Big(R + B^T(i)\bar{\mathbf{X}}(i, \xi_0)B(i)\Big)\xi_0\Big|\mathcal{I}_0\Big]\Big)^{-1}\mathbb{E}\Big[\xi_0 B^T(i)\bar{\mathbf{X}}(i, \xi_0)\Big|\mathcal{I}_0\Big]A(i) \tag{3.23}$$

□

The following Lemma presents the infinite horizon version of Lemma 3.3.1

Lemma 3.3.6. *If the control arrival probabilities* $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$ *and* $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ *are such that system (3.1) is stochastically stabilizable, then:*

(a) For $\xi_{k-1} = \mathcal{N}(\mathscr{I})$ *and* $r_k \in \mathcal{D}$*, the infinite horizon value function, at a stage* $k \geq 1$*, is given by:*

$$V\Big(x_k, i, \mathcal{N}(\mathscr{I})\Big) = x_k^T\bar{\Xi}(i, \mathcal{N}(\mathscr{I}))x_k \tag{3.24}$$

(b) The infinite horizon optimal control law, for $k \geq 0$ *is given by*

$$\bar{u}_k^* = -\Big[\mathbb{E}\Big[\xi_k\Big(R + B^T(i)\bar{\mathbf{X}}(i, \xi_k)B(i)\Big)\xi_k\Big|\mathcal{I}_k\Big]\Big]^{-1}\mathbb{E}\Big[\xi_k B^T(i)\bar{\mathbf{X}}(i, \xi_k)\Big|\mathcal{I}_k\Big]A(i)x_k \tag{3.25}$$

(c) Optimal infinite horizon cost is given by:

$$J_\infty(\zeta^*_{0:\infty}) = V(x_0, r_0) = x_0^T \hat{\bar{\Xi}}(r_0) x_0 \tag{3.26}$$

Corollary 3.3.7. *If the packet loss status is given by $\xi_{k-1} = \mathcal{N}(\mathscr{I})$, $k \geq 1$, then, for a stage $k \geq 0$, the infinite horizon optimal controller for an LTI system is expressed as:*

$$\bar{u}^*_k = -\Big[\mathbb{E}\Big[\xi_k\Big(R + B^T\bar{\Xi}(\xi_k)B\Big)\xi_k\Big|\mathcal{I}_k\Big]\Big]^{-1}\mathbb{E}\Big[\xi_k B^T\bar{\Xi}(\xi_k)\Big|\mathcal{I}_k\Big]Ax_k \tag{3.27}$$

where $\bar{\Xi}(\xi_k)$, $\forall \mathscr{I} \subseteq \mathscr{G}$, is the fixed-point solution of the following CAREs:

$$\begin{aligned}&\bar{\Xi}(\mathcal{N}(\mathscr{I}))\\ &= W + A^T\mathbb{E}^{\mathscr{I}}\Big[\bar{\Xi}(\xi_k)\Big]A - A^T\Big[\mathbb{E}^{\mathscr{I}}\Big[\xi_k B^T\bar{\Xi}(\xi_k)\Big]\Big]^T\Big[\mathbb{E}^{\mathscr{I}}\Big[\xi_k\Big(R + B^T\bar{\Xi}(\xi_k)B\Big)\xi_k\Big]\Big]^{-1}\mathbb{E}^{\mathscr{I}}\Big[\xi_k B^T\bar{\Xi}(\xi_k)\Big]A\end{aligned} \tag{3.28}$$

We shall now state the definition of weak observability of system (3.1) similar to Definition 2.3.1, which is in line with one given in [61]. This notion shall be used in the subsequent results. We shall use the following dummy output in the sequel.

$$y_k = \mathscr{C}(r_k)x_k, \quad \text{where } \mathscr{C}(x_k) = W^{1/2}(r_k). \tag{3.29}$$

Definition 3.3.2. *[61] Consider system (3.1)-(3.29) without disturbance ($w_k \equiv 0$). Take any initial Markov process state r_0, and any two initial system states x_0^1 and x_0^2. Suppose, for a known input u_k, $y_k(x_0 = x_0^1) = y_k(x_0 = x_0^2)$, $\forall k \leq S$, implies that $Pr(x_0^1 = x_0^2) > 0$. The system is said to be weakly observable if $\mathbb{E}\Big[S\Big] < \infty$.* □

An algebraic condition to test weak observability as given by lemma below.

Lemma 3.3.8. *[61] System (3.1)-(3.29), without disturbance ($w_k \equiv 0$), is said to be weakly observable if and only if there exists a transition path $\{r_0, r_1, ..., r_{S-1}\}$ inside $\mathcal{D}$ with $S < \infty$, for* *which the jump observability matrix* $\mathscr{O}(r_0, r_1, ..., r_{S-1})$ *has*

$$rank\ \mathscr{O}(r_0, r_1, ..., r_{S-1}) = rank \begin{bmatrix} \mathscr{C}(r_0) \\ \mathscr{C}(r_1)A(r_0) \\ . \\ . \\ . \\ \mathscr{C}(r_{S-1})\Pi_{\tau=0}^{S-2}A(r_\tau) \end{bmatrix} = n.$$

□

Remark 3.3.2. *In [61], the condition for weak observability is presented for the case when the Markov chain has more than one closed communicating class. In our case, as the Markov chain* $\{r_k\}$ *is irreducible, we have only one closed communicating class* $\mathcal{D}$*. Hence, the system will be weakly observable if there exists a transition path* $\{r_0, r_1, ..., r_{S-1}\}$ *inside* $\mathcal{D}$ *with* $S < \infty$ *for which the condition given in Lemma 3.3.8 is satisfied.* □

Lemma 3.3.9. *Suppose* $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$ *and* $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ *are such that system (3.1) is stochastically stabilizable. Further, if system (3.1)-(3.29), with* $u_k \equiv 0$*, is weakly observable, then, for all* $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$, $\hat{\bar{\Xi}}(i) > 0$ *and* $\bar{\Xi}(i, \mathcal{N}(\mathscr{I})) > 0$*.*

Proof: As $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$ and $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ are such that system (3.1) is stochastically stabilizable, from Lemma 3.3.6, one gets that $V_{k,N}(x_k, i, \mathcal{N}(\mathscr{I}))$ converges to $V(x_k, i, \mathcal{N}(\mathscr{I}))$ as $N \to \infty$.

Then, taking limit as $N \to \infty$ in Equation (3.4),

$$V(x_k, r_k, \xi_{k-1}) = \lim_{N\to\infty} V_{k,N}(x_k, r_k, \xi_{k-1}) = \sum_{f=k}^{\infty} \mathbb{E}\Big[x_f^T W(r_f) x_f + x_f^T \Gamma_f^T \xi_f R(r_f) \xi_f \Gamma_f x_f \Big| \mathcal{I}_k\Big], \quad (3.30)$$

where, if $r_f = i$,

$$\Gamma_f = \Big[\mathbb{E}\Big[\xi_k\Big(R + B^T(i)\bar{\mathbf{X}}(i, \xi_k)B(i)\Big)\xi_k\Big|\mathcal{I}_k\Big]\Big]^{-1}\mathbb{E}\Big[\xi_k B^T(i)\bar{\mathbf{X}}(i, \xi_k)\Big|\mathcal{I}_k\Big]A(i). \quad (3.31)$$

Suppose $\Gamma_f x_f \neq 0$ for any $r_f \in \mathcal{D}$. Then as $\mathbb{E}[\xi_f R(i)\xi_f \big| \mathcal{I}_f] > 0$ for all f, and $W(r_f) \geq 0$ for all $r_f \in \mathcal{D}$, one gets:

$$V(x_k, i, \mathcal{N}(\mathscr{I})) = x_k^T \Xi_k x_k > 0,$$

where, for $k \geq 1$, $\Xi_k = \bar{\Xi}(i, \mathscr{I})$ if $\xi_{k-1} = \mathcal{N}(\mathscr{I})$, and $\Xi_0 = \hat{\bar{\Xi}}(i)$.

Now, consider the case when $\Gamma_f x_f = 0$ for all $r_f \in \mathcal{D}$. Then, for $w_k \equiv 0$, the state equation (3.1) for all k transforms into:

$$x_{k+1} = A(r_k)x_k. \tag{3.32}$$

Therefore, from (3.30) and (3.32):

$$\begin{aligned}
&V(x_k, i, \mathcal{N}(\mathscr{I}))\\
&= \sum_{f=k}^{\infty} \mathbb{E}\Big[x_f^T W(r_f)x_f \Big| \mathcal{I}_k\Big]\\
&= \mathbb{E}\Big[x_k^T W(i)x_k + x_{k+1}^T W(r_{k+1})x_{k+1} + \ldots + x_p^T W(r_p)x_p + x_p^T A^T(r_p)W(r_{p+1})A(r_p)x_p + \ldots\\
&+ x_p^T\Big(\Pi_{l=p}^{p+\mathscr{F}-1} A(r_l)\Big)^T W(r_{p+\mathscr{F}})\Big(\Pi_{l=p}^{p+\mathscr{F}-1} A(r_l)\Big)x_p + \sum_{t=p+\mathscr{F}+1}^{\infty} x_t^T W(r_t)x_t \Big| \mathcal{I}_k\Big]\\
&= \mathbb{E}\Big[x_k^T W(i)x_k + x_{k+1}^T W(r_{k+1})x_{k+1} + \ldots. + x_p^T \mathscr{Y}\Big(r_p, r_{p+1}, \ldots, r_{p+\mathscr{F}}\Big)x_p + \sum_{t=p+\mathscr{F}+1}^{\infty} [x_t^T W(t)x_t]\Big| \mathcal{I}_k\Big],
\end{aligned} \tag{3.33}$$

where

$$\mathscr{Y}\Big(r_p, r_{p+1}, \ldots, r_{p+\mathscr{F}}\Big) = \mathscr{O}^T\Big(r_p, r_{p+1}, \ldots, r_{p+\mathscr{F}}\Big)\mathscr{O}\Big(r_p, r_{p+1}, \ldots, r_{p+\mathscr{F}}\Big),$$

$\mathscr{O}\Big(r_p, r_{p+1}, \ldots, r_{p+\mathscr{F}}\Big)$ is as defined in Lemma 3.3.8.

Suppose, for a given Markov chain state r_{k-1}, $\mathcal{S}_{k:\mathscr{T}}^{r_k}$ is the set of all transition paths of length $(\mathscr{T} - k)$ that the Markov chain $\{r_k\}$ follows with nonzero probability. Let $\mathscr{J}(r_k, \ldots, r_{k+\mathscr{T}} \big| r_{k-1})$ be the probability that, from the stage k to $k + \mathscr{T}$, the Markov chain $\{r_k\}$ follows a transition path $\{r_k, r_{k+1}, ..., r_{k+\mathscr{T}}\}$ given r_{k-1}, and $\mathscr{J}(r_p, ..., r_{p+\mathscr{F}} \Big| \{r_k, ..., r_{p-1}\})$ be the probability that, from the stage p to $p + \mathscr{F}$, the Markov chain $\{r_k\}$ follows a transition path $\{r_p, r_{p+1}, ..., r_{p+\mathscr{F}}\}$ given that it has followed the transition path $\{r_k, r_{k+1}, ..., r_{p-1}\}$ from the stage k to $p-1$. Then,

$$\begin{aligned}
&\mathbb{E}\Big[x_p^T \mathscr{Y}\Big(r_p, r_{p+1}, .., r_{p+\mathscr{F}}\Big)x_p \Big| \mathcal{I}_k\Big] \\
&= \sum_{(r_{k+1},..,r_{p-1}) \in \mathcal{S}^{r_k}_{k+1:p-1}} \Big[\mathscr{J}(r_{k+1}, .., r_{p-1} \Big| r_k)\Big(\Pi_{l=k}^{p-1} A(r_l)x_k\Big)^T \sum_{(r_p,..,r_{p+\mathscr{F}-1}) \in \mathcal{S}^{r_{p-1}}_{p:p+\mathscr{F}-1}} [\mathscr{J}(r_p, .., r_{p+\mathscr{F}-1} \Big| \{r_{k+1}, .., r_{p-1}\}) \\
&\quad \times \mathscr{Y}\Big(r_p, .., r_{p+\mathscr{F}}\Big)]\Big(\Pi_{l=k}^{p-1} A(r_l)x_k\Big)\Big].
\end{aligned} \tag{3.34}$$

By our assumption, system (3.1)-(3.29), with $u_p \equiv 0$, $w_p \equiv 0$, is weakly observable. Thus, there exists a transition path of finite length $\{i_p, i_{p+1}, ..., i_{p+\mathscr{F}}\}$ such that the jump observability matrix with respect to that particular transition path has full column rank. Due to the irreducibility of the Markov chain $\{r_k\}$, one can choose a finite p such that the probability of such a transition path occurring is nonzero. Hence, the probablity that the matrix $\mathscr{Y}\Big(i_p, i_{p+1}, ..., i_{p+\mathscr{F}}\Big)$ has full rank is nonzero. Therefore, from (3.34):

$$\mathbb{E}\Big[x_p^T \mathscr{Y}\Big(r_p, r_{p+1}, ..., r_{p+\mathscr{F}}\Big)x_p \Big| \mathcal{I}_k\Big] > 0 \tag{3.35}$$

Thus, as $W(r_k) \geq 0$ for all $r_k \in \mathcal{D}$, from (3.33), (3.35), and using lemma 3.3.6:

$$V(x_k, i, \mathcal{N}(\mathscr{I})) = x_k^T \Xi_k x_k > 0 \tag{3.36}$$

Since (3.36) is true for all $x_k \neq 0$, $\Xi_k > 0$. Therefore, $\hat{\bar{\Xi}}(i) > 0$ and $\bar{\Xi}(i, \mathscr{I}) > 0$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. □

The following two results is used to prove the stability of the closed-loop system.

Lemma 3.3.10. *[63] An irreducible Markov chain with a finite number of states is always recurrent.* □

Lemma 3.3.11. *[64] For an irreducible, recurrent and aperiodic Markov chain, the limiting distribution for each state is nonzero, i.e.,*

$$\lim_{n\to\infty} Pr\Big(r_n = j\Big|r_0 = i\Big) > 0, \quad \forall i, j \in \mathcal{D}.$$

□

We use the following notion of stability in this chapter.

Definition 3.3.3. *System (3.1)-(3.2), with $u_k \equiv 0$, is said to be stable in the mean-square sense if*

$$\lim_{k\to\infty} \mathbb{E}\Big[||x_k||^2\Big|\mathcal{I}_0\Big] = 0, \ \ \textit{for all} \ \ x_0 \ \ \textit{and} \ \ r_0.$$

□

Theorem 3.3.12. *Suppose $\{\bar{v}^1, \bar{v}^2, ..., \bar{v}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ are such that system (3.1) is stochastically stabilizable. Further, if system (3.1)-(3.29), with $u_k \equiv 0$, is weakly observable, then, system (3.1)-(3.29) with the optimal control input (3.25) is stable in the mean-square sense.*

Proof: As $\{\bar{v}^1, \bar{v}^2, ..., \bar{v}^m\}$ and $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ are such that system (3.1)-(3.29) is stochastically stabilizable, the infinite horizon cost is well-defined. Thus,

$$\begin{aligned} & J_\infty(\zeta^*_{0:\infty}) < \infty \\ & \implies \mathbb{E}\Big[\sum_{k=0}^{\infty} ||x_k||^2_{W(r_k)} + ||u^*_k||^2_{\xi_k R(r_k)\xi_k}\Big|\mathcal{I}_0\Big] < \infty \\ & \implies \mathbb{E}\Big[\sum_{k=0}^{\infty} \Big(||x_k||^2_{W(r_k)} + ||x_k||^2_{\Gamma_k^T \xi_k R(r_k)\xi_k \Gamma_k}\Big|\mathcal{I}_0\Big] < \infty, \end{aligned} \tag{3.37}$$

where, Γ_k is as defined in (3.31).

Since $W(r_k) \geq 0$, $\mathbb{E}[\xi_k R(r_k)\xi_k\Big|\mathcal{I}_k] > 0$ (as $R(r_k) > 0$) for all k, $r_k \in \mathcal{D}$, (3.37) guaranties the convergence of the infinite series $\sum_{k=0}^{\infty} \mathbb{E}\Big[||x_k||^2_{W(r_k)} + ||x_k||^2_{\Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k}\Big|\mathcal{I}_0\Big]$. Now, this in turn implies that the infinite series $\sum_{k=0}^{\infty} \mathbb{E}\Big[||x_k||^2_{W(r_k)}\Big|\mathcal{I}_0\Big]$ and $\sum_{k=0}^{\infty} \mathbb{E}\Big[||x_k||^2_{\Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k}\Big|\mathcal{I}_0\Big]$ also converge. In view of Theorem 3.23 in [60], convergence of the infinite series $\sum_{k=0}^{\infty}\mathbb{E}\Big[\|x_k\|^2_{\Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k}\Big|\mathcal{I}_0\Big]$ implies that $\lim\limits_{k\to\infty}\mathbb{E}\Big[\|x_k\|^2_{\Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k}\Big|\mathcal{I}_0\Big] = 0$. Hence, as $\mathbb{E}\Big[\xi_k R(r_k)\xi_k\Big|\mathcal{I}_k\Big] > 0$ for all $r_k \in \mathcal{D}$, one gets that $\lim\limits_{k\to\infty}\Gamma_k x_k = 0$ with probability 1.

We now claim that if the infinite horizon cost is well-defined, then $\lim\limits_{k\to\infty}\mathbb{E}\Big[\|x_k\|^2\Big|\mathcal{I}_0\Big] = 0$. We shall use contradiction in order to prove the claim. Assume that $\lim\limits_{k\to\infty}\mathbb{E}\Big[\|x_k\|^2\Big|\mathcal{I}_0\Big] \neq 0$.

Using the system dynamics (3.1) with optimal control input $u_k^* = -\Gamma_k x_k$:

$$\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}\Big[x_{k+1}^T W(r_{k+1})x_{k+1}\Big|\mathcal{I}_0\Big]\\
&= \lim_{k\to\infty}\mathbb{E}\Big[x_k^T A^T(r_k)W(r_{k+1})\Big(A(r_k)x_k - 2B(r_k)\xi_k\Gamma_k x_k\Big) + x_k^T\Gamma_k^T\xi_k B^T(r_k)W(r_{k+1})B(r_k)\Gamma_k x_k\Big|\mathcal{I}_0\Big].
\end{aligned} \tag{3.38}$$

As $\lim\limits_{p\to\infty}\Gamma_k x_k = 0$ with probability 1, from (3.1), $\lim\limits_{k\to\infty}\Big[x_{k+1}\Big] = \lim\limits_{k\to\infty}\Big[A(r_k)x_k\Big]$. Thus, (3.38) implies:

$$\lim_{k\to\infty}\mathbb{E}\Big[x_{k+1}^T W(r_{k+1})x_{k+1}\Big|\mathcal{I}_0\Big] = \lim_{k\to\infty}\mathbb{E}\Big[x_k^T A^T(r_k)W(r_{k+1})A(r_k)x_k\Big|\mathcal{I}_0\Big]. \tag{3.39}$$

In similar fashion, we get:

$$\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}\Big[x_{k+2}^T W(r_{k+2})x_{k+2}\Big|\mathcal{I}_0\Big]\\
&= \lim_{k\to\infty}\mathbb{E}\Big[x_{k+1}^T A^T(r_{k+1})W(r_{k+2})A(r_{k+1})x_{k+1}\Big|\mathcal{I}_0\Big]\\
&= \lim_{k\to\infty}\mathbb{E}\Big[x_k^T A^T(r_k)A^T(r_{k+1})W(r_{k+2})A(r_{k+1})A(r_k)x_k\Big|\mathcal{I}_0\Big].
\end{aligned} \tag{3.40}$$

and so on.

From equations (3.39), (3.40),..., one gets for $\mathscr{F} \geq n$:

$$\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}\Big[x_k^T W(r_k)x_k + x_{k+1}^T W(r_{k+1})x_{k+1} + x_{k+2}^T W(r_{k+2})x_{k+2} + \ldots + x_{k+\mathscr{F}}^T W(r_{k+\mathscr{F}})x_{k+\mathscr{F}}\Big|\mathcal{I}_0\Big] \\
&= \lim_{k\to\infty}\mathbb{E}\Big[x_k^T W(r_k)x_k + x_k^T A^T(r_k)W(r_{k+1})A(r_k)x_k + x_k^T A^T(r_k)A^T(r_{k+1})W(r_{k+2})A(r_{k+1})A(r_k)x_k + \ldots \\
&+ x_k^T\Big(\Pi_{l=k}^{k+\mathscr{F}-1}A(r_l)\Big)^T W(r_{k+\mathscr{F}})\Pi_{l=k}^{k+\mathscr{F}-1}A(r_l)x_k\Big|\mathcal{I}_0\Big] \\
&= \lim_{k\to\infty}\mathbb{E}\Big[x_k^T\mathscr{Y}\Big(r_k, r_{k+1}, \ldots, r_{k+\mathscr{F}}\Big)x_k\Big|\mathcal{I}_0\Big].
\end{aligned} \tag{3.41}$$

Note that the Markov chain $\{r_k\}$ is irreducible and has a finite number of states. Thus, from Lemma 3.3.10, it is recurrent. Hence, in light of Lemma 3.3.11, it has a nonzero limiting distribution for all the states. Therefore, one can prove the following inequality by using the similar line of argument as used in the proof for Lemma 3.3.9.

$$\lim_{k\to\infty}\mathbb{E}\Big[x_k^T\mathscr{Y}\Big(r_k, r_{k+1}, \ldots, r_{k+\mathscr{F}}\Big)x_k\Big|\mathcal{I}_0\Big] > 0. \tag{3.42}$$

Then, from (3.41) and (3.42), one easily gets that, for certain r_k, $\lim_{k\to\infty}\mathbb{E}\Big[x_k^T W(r_k)x_k\Big|\mathcal{I}_0\Big] \neq 0$. Therefore, considering Theorem 3.23 in [60], we can infer that the infinite series $\sum_{k=0}^{\infty}\mathbb{E}\Big[x_k^T W(r_k)x_k\Big|\mathcal{I}_0\Big]$ does not converge. Further, $\mathbb{E}\Big[\xi_k R(i)\xi_k\Big|\mathcal{I}_k\Big] > 0$ for all $k \in [0, \infty)$, $i \in \mathcal{D}$. Thus,

$$\mathbb{E}\Big[\sum_{k=0}^{\infty}||x_k||^2_{W(r_k)} + ||x_k||^2_{\Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k}\Big|\mathcal{I}_0\Big] \to \infty.$$

Hence, we arrive at a contradiction. Therefore, we must have $\lim_{k\to\infty}\mathbb{E}\Big[||x_k||^2\Big|\mathcal{I}_0\Big] = 0$.

3.4 Numerical Example

Consider MJLS (3.1) with the following system parameters:

$$
\begin{aligned}
&r_k = \{1, 2\}\\
&A(1) = 2.1, \quad A(2) = 2.3,\\
&B(1) = B(2) = \begin{bmatrix} 1 & 2 \end{bmatrix},\\
&W = 1, \quad R = I_2.
\end{aligned}
$$

Switching probabilities for the Markov chain $\{r_k\}$ is considered as: $p_{11} = 0.4$, $p_{12} = 0.6$, $p_{22} = 0.75$ and $p_{21} = 0.35$. Figure 3.1 shows that with two sets of control packet arrival probabilities $\bar{\nu}^1 = 0.7; \bar{\nu}^2 = 0.73; \bar{\mu}^1 = 0.55; \bar{\mu}^2 = 0.65$ and $\bar{\nu}^1 = 0.72; \bar{\nu}^2 = 0.75; \bar{\mu}^1 = 0.57; \bar{\mu}^2 = 0.67$, the optimal cost converges as the horizon N increases. Thus, the infinite horizon optimal controller is well-defined for these two sets of control packet arrival probabilities. If the control packet arrival probabilities are reduced to $\bar{\nu}^1 = 0.52; \bar{\nu}^2 = 0.55; \bar{\mu}^1 = 0.47; \bar{\mu}^2 = 0.47$, then it can be observed that optimal cost does not converge as the N increases, which is shown in Figure 3.2. Therefore, an infinite horizon optimal controller can not be designed for these control packet arrival probabilities. Figure 3.3 demonstrates that the closed-loop system with the optimal controller is stable in the mean-square sense with control arrival probabilities $\bar{\nu}^1 = 0.9$, $\bar{\nu}^2 = 0.93$, $\bar{\mu}^1 = 0.85$, $\bar{\mu}^2 = 0.75$.

By the following example, we show that, unlike observability of each subsystem as given in [61], weak observability is sufficient for stability. Consider an MJLS with the following parameters.

$$
\mathcal{D} = \{1, 2\},
$$

$$
A(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; C(1) = \begin{bmatrix} 1 & 1 \end{bmatrix}.
$$

$$W(1) = C^T(1)C(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; R(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B(2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; C(2) = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

$$W(2) = C^T(2)C(2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; R(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

For this system $\left(A(1), C(1)\right)$ is not observable. Thus the results given in [61] can not infer stability for the system. However, as the system is weakly observable (since $\left(A(2), C(2)\right)$ is observable) our analysis can be used to infer stability of the system. Figure shows that with the optimal controller the closed-loop system is stable in the mean-square sense.

3.5 Summary

In this chapter, we have investigated the jump linear quadratic optimal control of an MJLS over multiple channels considering correlated packet losses. Finite horizon and infinite horizon controllers are designed considering a TCP-like case. The convergence of the infinite horizon cost function and the resulting existence of the infinite horizon controller are also investigated. It is observed that if the control packet arrival probabilities are greater than certain critical values, then the infinite horizon CAREs converge to the unique fixed-point solution. Moreover, as a special case, finite horizon and infinite horizon controllers for an LTI system are also derived.

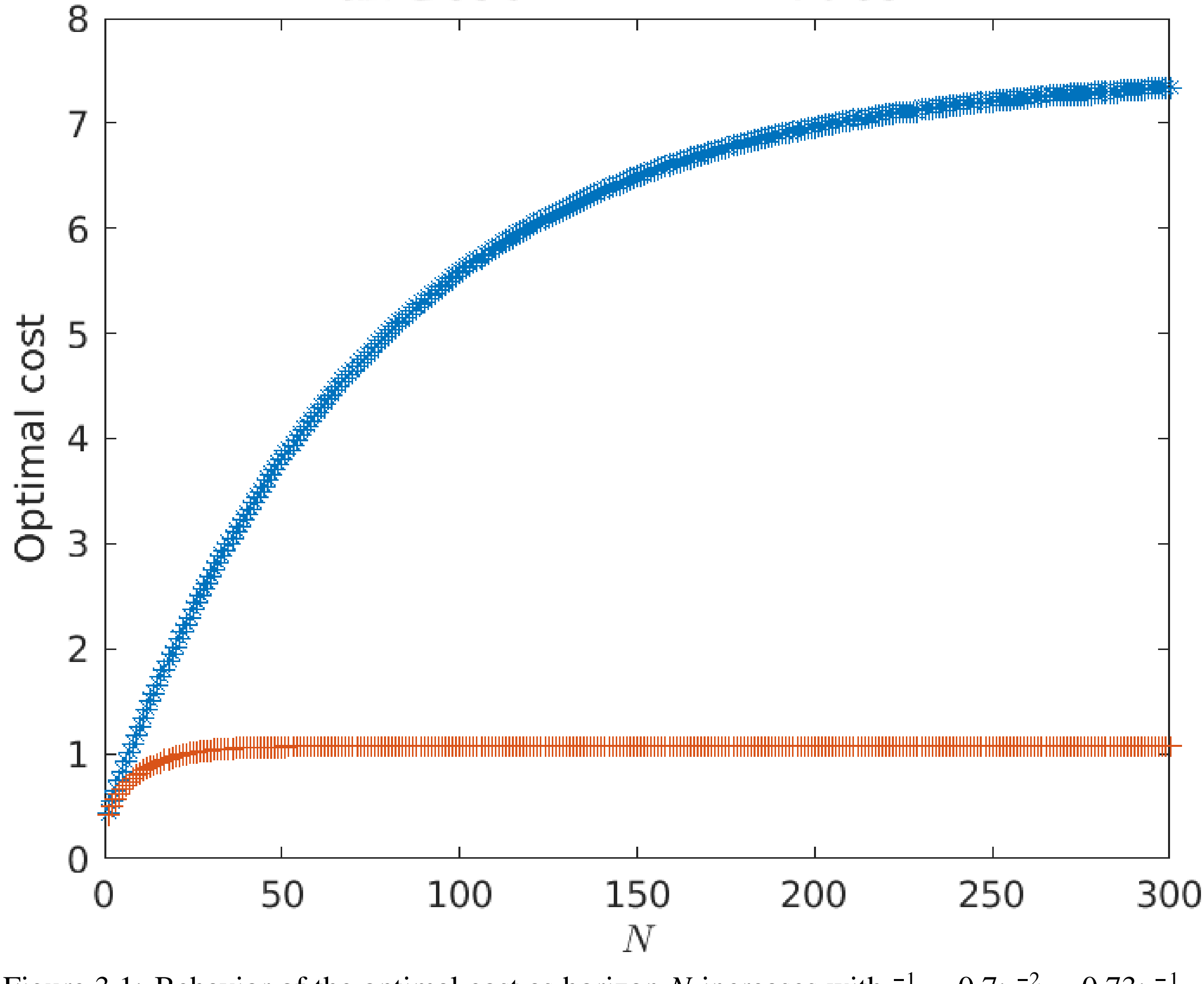

Figure 3.1: Behavior of the optimal cost as horizon N increases with $\bar{\nu}^1 = 0.7$; $\bar{\nu}^2 = 0.73$; $\bar{\mu}^1 = 0.55$; $\bar{\mu}^2 = 0.65$ (blue graph) and $\bar{\nu}^1 = 0.72$; $\bar{\nu}^2 = 0.75$; $\bar{\mu}^1 = 0.57$; $\bar{\mu}^2 = 0.67$ (orange graph).

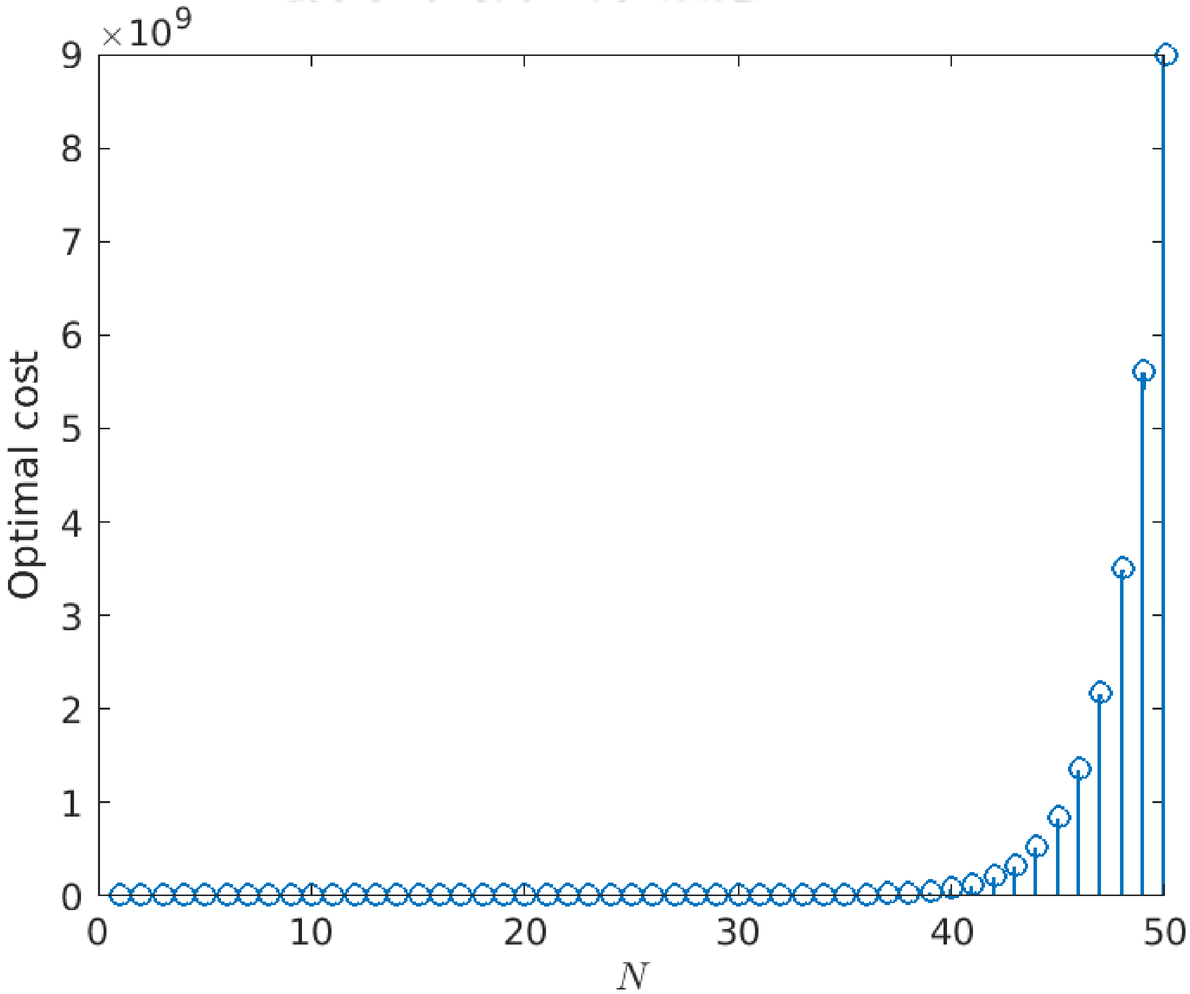

Figure 3.2: Behavior of the optimal cost as horizon N increases with $\bar{v}^1 = 0.52$; $\bar{v}^2 = 0.55$; $\bar{\mu}^1 = 0.47$; $\bar{\mu}^2 = 0.47$.

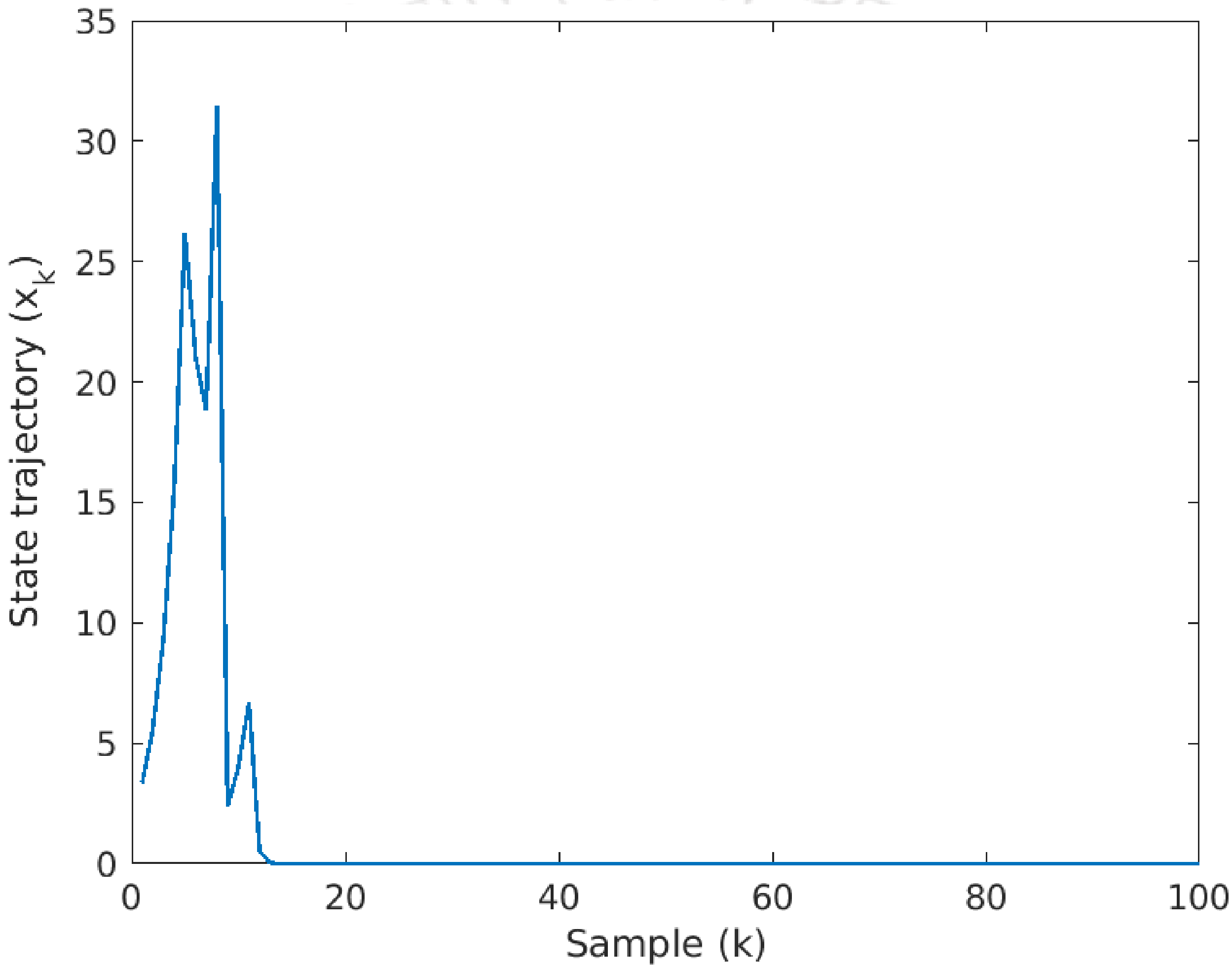

Figure 3.3: State response of the closed-loop system with $\bar{\nu}^1 = 0.9$; $\bar{\nu}^2 = 0.93$; $\bar{\mu}^1 = 0.85$; $\bar{\mu}^2 = 0.75$.

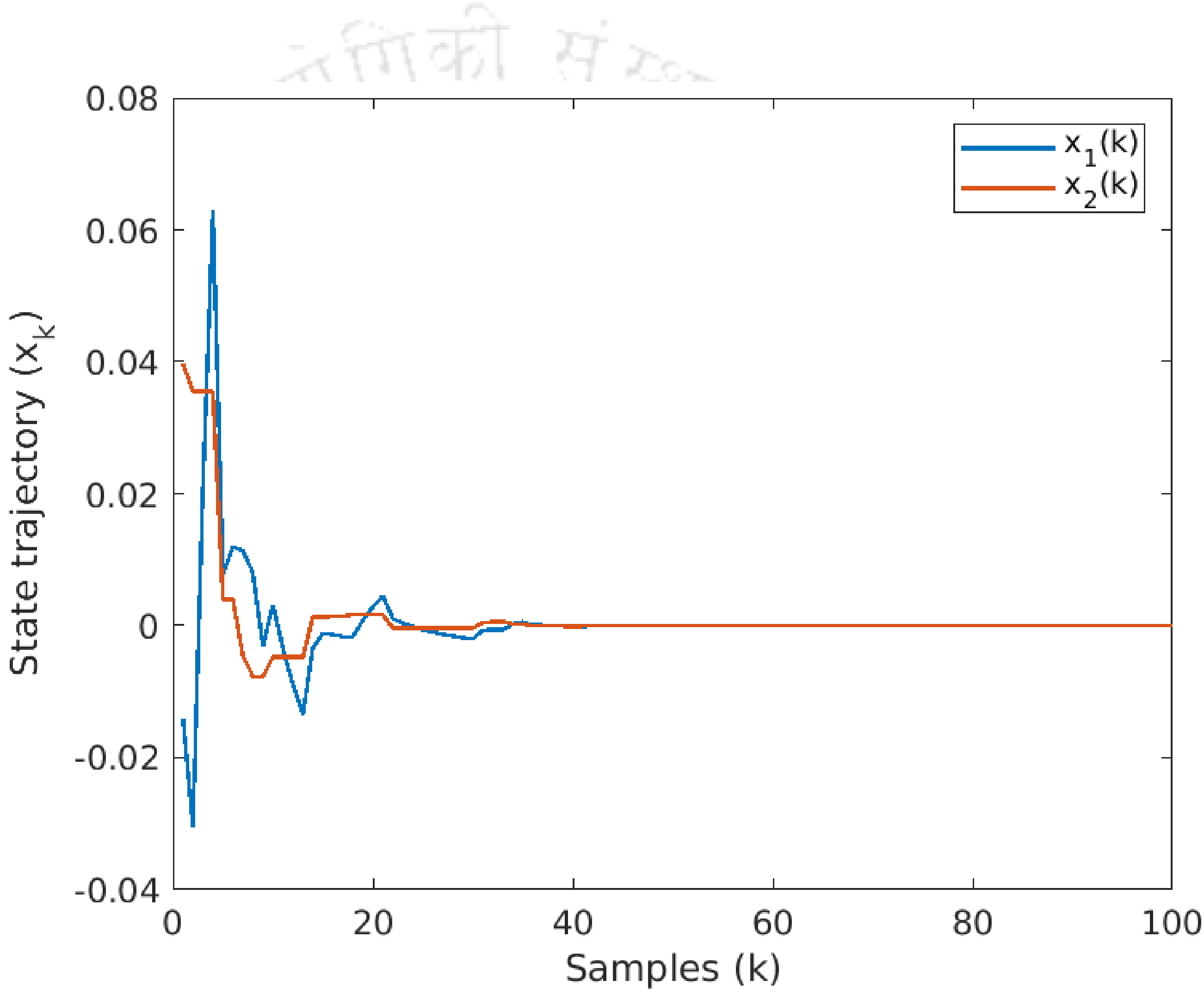

Figure 3.4: State response of the closed-loop system with $\bar{\nu}^1 = 0.94$; $\bar{\nu}^2 = 0.93$; $\bar{\mu}^1 = 0.95$; $\bar{\mu}^2 = 0.95$

4

H_∞ optimal control of Markovian jump linear systems (MJLSs) over multiple lossy channels

4.1 Introduction

In this chapter, we extend the results of Chapter 2 to an MJLS. Similar to Chapter 2, each channel is modeled as a Gilbert-Elliott type communication channel. Sensor-to-controller channels are assumed to be lossless.

To the best of our knowledge, H_∞ control of MJLSs over multiple lossy channels has not been investigated yet. Further, works on the classical H_∞ control of an MJLS such as [55] and [65] consider a more stringent observability notion to establish various results. However, in this chapter, we consider the less stringent weak observability notion which is sufficient to prove the positive definiteness of the solution of the associated infinite horizon coupled Riccati equations, and the stability of the closed-loop system.

The chapter is structured as follows. Section 4.2 describes the H_∞ control problem with packet losses. Section 4.3 contains the solution to the problem for both finite and infinite horizon cases. The convergence of the infinite horizon cost function and the stability of the closed-loop system are also investigated. Simulation results are presented in Section 4.4 followed by a summary in section 4.5.

4.2 Problem Formulation

Consider the following discrete-time MJLS:

$$\begin{aligned} x_{k+1} &= A(r_k)x_k + B(r_k)u_k^a + D_1(r_k)w_k \\ z_k &= C(r_k)x_k + D(r_k)u_k^a, \end{aligned} \tag{4.1}$$

where $x_k \in \mathbb{R}^n$ denotes the state vector, $u_k^a \in \mathbb{R}^m$ denotes the control input to the actuators, $w_k \in \mathbb{R}^s$ denotes the disturbance input, $z_k \in \mathbb{R}^p$ is the controlled output. $\{r_k\}$ is an irreducible, aperiodic and time-homogeneous Markov chain where $r_k \in \mathcal{D} \triangleq \{1, 2, ..., \mathcal{M}\}$ ($\mathcal{M} < \infty$). The transition probability matrix of the Markov chain is given by $\Delta = [p_{ij}]$, where

$$p_{ij} = Pr(r_{k+1} = j | r_k = i); \forall i, j \in \mathcal{D}, k = 0, 1, 2, ...$$

Throughout the chapter, it is assumed that state of the system x_k and Markov chain state r_k are directly accessible to the controller. We also assume that $A(r_k)$ is full rank for all $r_k \in \mathcal{D}$.

Suppose u_k is the controller output which is sent to the actuators through the lossy network. If u_k^a is the input to the actuators, then:

$$u_k^a = \xi_k u_k, \tag{4.2}$$

The control policy $\zeta_{0:k}$ and the disturbance policy $\eta_{0:k}$ for a horizon k are the sequences $\zeta_{0:k} = \{\zeta_0, ..., \zeta_k\}$ and $\eta_{0:k} = \{\eta_0, ..., \eta_k\}$, respectively. Here, ζ_i maps the information set $\mathcal{I}_i$ to the control input at the i^{th} time-index, i.e., $u_i = \zeta_i(\mathcal{I}_i)$. Similarly, η_i maps the information set $\mathcal{I}_i$ to the disturbance input at the i^{th} time-index, i.e., $w_i = \eta_i(\mathcal{I}_i)$. ζ_k^* and η_k^* are the optimal control and disturbance policy, respectively.

In this chapter, the following notion of stability shall be followed.

Definition 4.2.1. *System (4.1)-(4.2), with $u_k \equiv 0$, $w_k \equiv 0$, is said to be stable in the mean-square if*

$$\lim_{k\to\infty} \mathbb{E}\Big[||x_k||^2 \Big| \mathcal{I}_0\Big] = 0, \;\; for\ all \;\; x_0 \;\; and \;\; r_0.$$

□

Similar to Chapter 2, the objective of this chapter is to design state-feedback control policies for system (4.1), with network induced constraint (4.2), such that with a state-feedback control law, the closed-loop system attains the following requirements:

R.1) The $\mathcal{L}_2$ gain from the disturbance input w_k to the controlled output z_k must be less than or equal to some $\gamma > 0$, i.e., with zero initial condition $x_0 = 0$,

$$\sum_{k=0}^{N} \mathbb{E}\Big[\|z_k\|^2 \Big| \mathcal{I}_0\Big] \leq \gamma^2 \sum_{k=0}^{N} \|w_k\|^2, \forall N \in \mathbb{Z}^+.$$

R.2) The closed loop system is mean-square stable.

Subject to the constraints defined by the system dynamics (4.1), one can formulate a zero-sum game with the following cost function:

$$J_N(\zeta_{0:N-1}, \eta_{0:N-1}) = \mathbb{E}\Big[\|x_N\|_W^2 + \sum_{k=0}^{N-1} \|z_k\|^2 - \gamma^2 \|w_k\|^2 \Big| \mathcal{I}_0\Big]. \tag{4.3}$$

Using Equation (4.1) in (4.3), the cost function becomes:

$$J_N(\zeta_{0:N-1}, \eta_{0:N-1}) = \mathbb{E}\Big[\|x_N\|_W^2 + \sum_{k=0}^{N-1} \|x_k\|_{W(r_k)}^2 + \|u_k^a\|_{R(r_k)}^2 - \gamma^2 \|w_k\|^2 \Big| \mathcal{I}_0\Big], \tag{4.4}$$

where $W \geq 0$, $W(r_k) = C^T(r_k)C(r_k)$ and $R(r_k) = D^T(r_k)D(r_k)$. Further, $C(r_k)$ and $D(r_k)$ satisfy the following assumptions:

(a) $C^T(r_k)D(r_k) = 0, \forall r_k \in \mathcal{D}$; implying that there are no cross product terms in the cost function (4.4).

(b) $R(r_k) > 0, \forall r_k \in \mathcal{D}$; implies nonsingularity of the optimal control problem.

In the game with the cost function (4.4), the control input u_k acts as the minimizing player and the disturbance w_k acts as the maximizing player. As stated in Chapter 2, the game admits a solution if one can find a saddle-point policy $(\zeta^*_{0:N-1}, \eta^*_{0:N-1})$ which satisfies the inequality:

$$J_N(\zeta^*_{0:N-1}, \eta_{0:N-1}) \leq J_N(\zeta^*_{0:N-1}, \eta^*_{0:N-1}) \leq J_N(\zeta_{0:N-1}, \eta^*_{0:N-1}).$$

Remark 4.2.1. *If one considers the particular case when $\mathcal{D} = \{1\}$, the results presented in this chapter becomes equivalent to the results for an LTI system, which are presented chapter 2.* □

4.3 Main Results

In this section, we deal with the design of finite horizon and infinite horizon controllers.

A. Finite horizon control:

By substituting (4.2) in cost function (4.4), we get the following:

$$J_N(\zeta_{0:N-1}, \eta_{0:N-1}) = \mathbb{E}\Big[x_N^T W x_N + \sum_{k=0}^{N-1} x_k^T W(r_k) x_k + u_k^T \xi_k^T R(r_k)\xi_k u_k - \gamma^2 w_k^T w_k \Big| \mathcal{I}_0\Big]. \tag{4.5}$$

The optimal cost-to-go or value function at the stage k is given by:

$$V_{k,N}(x_k, r_k, \xi_{k-1}) = \min_{u_{k:N-1}} \max_{w_{k:N-1}} \mathbb{E}\left[x_N^T W x_N + \sum_{p=k}^{N-1} x_p^T W(r_p) x_p + u_p^T \xi_p^T R(r_p)\xi_p u_p - \gamma^2 w_p^T w_p \Big| \mathcal{I}_k\right] \tag{4.6}$$

Using principle of optimality, one can express (4.6) as:

$$V_{k,N}(x_k, r_k, \xi_{k-1}) = \min_{u_k} \max_{w_k} \mathbb{E}\left[x_k^T W(r_k) x_k + u_k^T \xi_k^T R(r_k)\xi_k u_k - \gamma^2 w_k^T w_k + V_{k+1,N}(x_{k+1}, r_{k+1}, \xi_k) | \mathcal{I}_k\right] \tag{4.7}$$

We now proceed to derive conditions under which the value of the game with cost function (4.7) has a well defined solution.

Lemma 4.3.1. *For $k \in [0, N-1]$, $r_k = i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$, consider the following coupled* *algebraic Riccati equations (CAREs):*

$$\begin{aligned}
&\Xi_{k,N}(i,\mathcal{N}(\mathscr{I}))\\
&= W(i) + \Gamma^T_{k,N}(i,\mathcal{N}(\mathscr{I}))\mathbb{E}\Big[\xi_k R(i)\xi_k\Big|\mathcal{I}_k\Big]\Gamma_{k,N}(i,\mathcal{N}(\mathscr{I})) - \gamma^2\Psi^T_{k,N}(i,\mathcal{N}(\mathscr{I}))\Psi_{k,N}(i,\mathcal{N}(\mathscr{I}))\\
&+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_k(\mathcal{N}(\mathscr{L}))\mathbb{E}\Big[\Big(A(i) - B(i)\mathcal{N}(\mathscr{L})\Gamma_{k,N}(i,\mathcal{N}(\mathscr{I})) + D_1(i)\Psi_{k,N}(i,\mathcal{N}(\mathscr{I}))\Big)^T\\
&\times \mathscr{X}_{k+1,N}(i,\mathcal{N}(\mathscr{L}))\Big(A(i) - B(i)\mathcal{N}(\mathscr{L})\Gamma_{k,N}(i,\mathcal{N}(\mathscr{I})) + D_1(i)\Psi_{k,N}(i,\mathcal{N}(\mathscr{I}))\Big)\Big],
\end{aligned} \tag{4.8}$$

where,

$$\Gamma_{k,N}(i,\mathcal{N}(\mathscr{I})) = \Big(\Lambda_{k,N}(i,\mathcal{N}(\mathscr{I}))\Big)^{-1}\mathbb{E}\Big[\xi_k B^T(i)\mathscr{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]\Big(A(i) + D_1(i)\Psi_{k,N}(i,\mathcal{N}(\mathscr{I}))\Big), \tag{4.9a}$$

$$\begin{aligned}
&\Psi_{k,N}(i,\mathcal{N}(\mathscr{I}))\\
&= \Big[I_s + (\Theta_{k,N}(i,\mathcal{N}(\mathscr{I})))^{-1}D_1^T(i)\mathbb{E}\Big[\mathscr{X}_{k+1,N}(i,\xi_k)B(i)\xi_k\Big|\mathcal{I}_k\Big](\Lambda_{k,N}(i,\mathcal{N}(\mathscr{I})))^{-1}\\
&\times\mathbb{E}\Big[\xi_k B^T(i)\mathscr{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]D_1(i)\Big]^{-1}\Theta^{-1}_{k,N}(i,\mathcal{N}(\mathscr{I}))\Big[D_1^T(i)\mathbb{E}\Big[\mathscr{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]\\
&- D_1^T(i)\mathbb{E}\Big[\mathscr{X}_{k+1,N}(i,\xi_k)B(i)\xi_k\Big|\mathcal{I}_k\Big]\Big(\Lambda_{k,N}(i,\mathcal{N}(\mathscr{I}))\Big)^{-1}\mathbb{E}\Big[\xi_k B^T(i)\mathscr{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]\Big]A(i),
\end{aligned} \tag{4.9b}$$

$$\Theta_{k,N}(i,\mathcal{N}(\mathscr{I})) = \gamma^2 I_s - D_1^T(i)\mathbb{E}\Big[\mathscr{X}_{k+1,N}(i,\xi_k)\Big|\mathcal{I}_k\Big]D_1(i), \tag{4.9c}$$

$$\Lambda_{k,N}(i,\mathcal{N}(\mathscr{I})) = \mathbb{E}\Big[\xi_k\Big(R(i) + B^T(i)\mathscr{X}_{k+1,N}(i,\xi_k)B(i)\Big)\xi_k\Big|\mathcal{I}_k\Big], \tag{4.9d}$$

$$\mathscr{X}_{k+1,N}(i,\xi_k) = \sum_{d=1}^{\mathcal{M}} p_{id}\Xi_{k+1,N}(d,\xi_k), \tag{4.9e}$$

$$\textit{with}\ \ \Xi_{N,N}(i,\mathcal{N}(\mathscr{I})) = W;\ \ \forall i\in\mathcal{D}, \mathscr{I}\subseteq\mathscr{G}. \tag{4.9f}$$

Now, suppose at the $(k-1)^{th}$ time index $(k \geq 1)$, the actuators which successfully receive the control signals are those indexed by the elements of $\mathscr{I}$, i.e., $\xi_{k-1} = \mathcal{N}(\mathscr{I})$ and $r_k = i$. Then, for the Isaacs equation (4.7) the following claims are true:

(a) The value function at the stage $k \in [0, N]$ is well-defined if and only if:

(i)

$$\Theta_{k,N}(i, \mathcal{N}(\mathscr{I})) > 0. \tag{4.10}$$

(ii)

$$\begin{gathered}\Theta_{l,N}(e, \mathcal{N}(\mathscr{L})) > 0;\\ k+1 \le l \le N+1, \forall e \in \mathcal{D}, \forall \mathscr{L} \subseteq \mathscr{G}.\end{gathered} \tag{4.11}$$

(b) *The value function is given by:*

$$V_{k,N}\Big(x_k, i, \mathcal{N}(\mathscr{I})\Big) = x_k^T \Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))x_k, \tag{4.12}$$

where $\Xi_{p,N}(i, \mathcal{N}(\mathscr{I}))$ *for all* $p \in [0, N]$ *is defined in (4.8).*

(c) $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) \ge 0$ *for all* $k \in [0, N]$, $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$.

(d) *If the saddle-point conditions (4.10) and (4.11) are satisfied, then the finite horizon saddle-point at the stage* $k \in [0, N-1]$ *stage is given by:*

$$u_k^* = \zeta_k^*(\mathcal{I}_k) = -\Gamma_{k,N}\Big(\mathcal{N}(i, \mathscr{I})\Big)x_k, \quad w_k^* = \eta_k^*(\mathcal{I}_k) = \Psi_{k,N}\Big(i, \mathcal{N}(\mathscr{I})\Big)x_k, \tag{4.13}$$

where $\Gamma_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ *and* $\Psi_{k,N}\Big(\mathcal{N}(\mathscr{I})\Big)$ *for all* $\mathscr{I} \subseteq \mathscr{G}$ *are finite.*

Proof: The lemma is proved using induction.

Similar to the proof of Lemma 2.3.1 in Chapter 2, we consider the base case as $k = N-1$, since $\Theta_{k,N}\Big(i, \mathcal{N}(\mathscr{I})\Big)$ is not defined for $k = N$.

At the stage $k = N$, observe that $V_{N,N}\Big(x_N, r_N, \xi_{N-1}\Big) = x_N^T W x_N$ for all r_N and ξ_{N-1}. So, with information set $\mathcal{I}_N$, if $r_N = i$ and $\xi_{N-1} = \mathcal{N}(\mathscr{I})$, we can represent $V_{N,N}\Big(x_N, r_N, \xi_{N-1}\Big)$ as:

$$V_{N,N}\Big(x_{k+1}, i, \mathcal{N}(\mathscr{I})\Big) = x_N^T \Xi_{N,N}\Big(i, \mathcal{N}(\mathscr{I})\Big)x_N, \tag{4.14}$$

where $\Xi_{N,N}\big(i, \mathcal{N}(\mathscr{I})\big) = W$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$.

Now, with information set $\mathcal{I}_{N-1}$, if $r_{N-1} = i$ and $\xi_{N-2} = \mathcal{N}(\mathscr{I})$:

$$\begin{aligned} &\mathbb{E}\Big[V_{N,N}\big(x_N, r_N, \xi_{N-1}\big)\Big|\mathcal{I}_{N-1}\Big] \\ &= \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_{N-1}\big(\mathcal{N}(\mathscr{L})\big)\mathbb{E}\Big[\big(A(i)x_{N-1} + B(i)\mathcal{N}(\mathscr{L})u_{N-1} + D_1(i)w_{N-1}\big)^T W \\ &\times \big(A(i)x_{N-1} + B(i)\mathcal{N}(\mathscr{L})u_{N-1} + D_1(i)w_{N-1}\big)\Big]. \end{aligned} \tag{4.15}$$

Consider the following functional:

$$\begin{aligned} &H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1}) \\ &= \mathbb{E}\Big[x_{N-1}^T W(r_{N-1})x_{N-1} + u_{N-1}^T\xi_{N-1}^T R(r_{N-1})\xi_{N-1}u_{N-1} - \gamma^2 w_{N-1}^T w_{N-1} + V_{N,N}\big(x_N, r_N, \xi_{N-1}\big)\Big|\mathcal{I}_{N-1}\Big]. \end{aligned} \tag{4.16}$$

Note that $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is quadratic in u_{N-1}, w_{N-1}, and x_{N-1}. Hence, it admits a unique saddle-point if and only if all the following conditions are satisfied:

(i) $\frac{\partial^2 H_{N-1,N}(x_{N-1},u_{N-1},w_{N-1})}{\partial u_{N-1}^2} > 0$, i.e., $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is convex in u_{N-1}.

(ii) $\frac{\partial^2 H_{N-1,N}(x_{N-1},u_{N-1},w_{N-1})}{\partial w_{N-1}^2} < 0$, i.e., $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is concave in w_{N-1}.

(iii) There exist finite u_{N-1}^* and w_{N-1}^* such that

$$\begin{aligned} \frac{\partial H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})}{\partial u_{N-1}}\Big|_{(u_{N-1}^*, w_{N-1}^*)} &= 0 \\ \frac{\partial H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})}{\partial w_{N-1}}\Big|_{(u_{N-1}^*, w_{N-1}^*)} &= 0. \end{aligned} \tag{4.17}$$

Now, using (4.15) in (4.16), and then differentiating it twice:

$$\begin{aligned} \frac{\partial^2 H_{N-1,N}\big(x_{N-1}, u_{N-1}, w_{N-1}\big)}{\partial u_{N-1}^2} &= \mathbb{E}^{\mathscr{I}}\Big[\xi_{N-1}\big(R(i) + B^T(i)\mathscr{X}_{N,N}(i, \xi_{N-1})B(i)\big)\xi_{N-1}\Big] \\ \frac{\partial^2 H_{N-1,N}\big(x_{N-1}, u_{N-1}, w_{N-1}\big)}{\partial w_{N-1}^2} &= -\gamma^2 I_s + D_1^T\mathbb{E}^{\mathscr{I}}\Big[\mathscr{X}_{N,N}(i, \xi_{N-1})\Big]D_1 = -\Theta_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big). \end{aligned}$$

Since $\Xi_{N,N}\big(i, \mathcal{N}(\mathscr{I})\big) = W \geq 0$, $\forall\ i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$, one gets that $\mathscr{X}_{N,N}(i, \mathcal{N}(\mathscr{I})) \geq 0$. Hence, as $R(i) > 0$, we have:

$$\mathbb{E}^{\mathscr{I}}\Big[\xi_{N-1}\Big(R(i) + B^T(i)\mathscr{X}_{N,N}(i, \xi_{N-1})B\Big)\xi_{N-1}\Big] > 0.$$

Therefore,

$$\frac{\partial^2 H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})}{\partial u_{N-1}^2} = \mathbb{E}^{\mathscr{I}}\Big[\xi_{N-1}\Big(R(i) + B^T(i)\mathscr{X}_{N,N}(i, \xi_{N-1})B\Big)\xi_{N-1}\Big] > 0.$$

Thus, $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ is convex in u_{N-1}. Further, $\Theta_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big) > 0$ is equivalent to $H_{N-1,N}(x_{N-1}, u_{N-1}, w_{N-1})$ being concave in w_{N-1}. Also, solving (4.17), we get

$$\begin{aligned} u_{N-1}^* &= \zeta_{N-1}^*(\mathcal{I}_{N-1}) = -\Gamma_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)x_{N-1} \\ w_{N-1}^* &= \eta_{N-1}^*(\mathcal{I}_{N-1}) = \Psi_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)x_{N-1}, \end{aligned} \tag{4.18}$$

where $\Gamma_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$ and $\Psi_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$ are given by (4.9a) and (4.9b), respectively. The finiteness and positive semidefiniteness of W along with the positive definiteness of $R(i)$ ensure the invertibility of $\Lambda_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$, $\forall\ i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Further, the invertibility of $\Theta_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$ and $\Lambda_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$ ensure the finiteness of $\Psi_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$. Consequently, $\Gamma_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$ is finite. Substituting for u_{N-1}^* and w_{N-1}^* in (4.7) with $k = N-1$, and using (4.15), we get:

$$V_{N-1,N}\big(x_{N-1}, i, \mathcal{N}(\mathscr{I})\big) = x_{N-1}^T \Xi_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)x_{N-1},$$

where $\Xi_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)$ is as given in equation (4.8).

Clearly, as $W \geq 0$, $W(i) \geq 0$, and $R(i) > 0$, $H_{N-1,N}(x_{N-1}, u_{N-1}^*, w_{N-1} = 0) \geq 0$. Since (u_{N-1}^*, w_{N-1}^*) constitutes a saddle-point,

$$H_{N-1,N}(x_{N-1}, u_{N-1}^*, w_{N-1}^*) \geq H_{N-1,N}(x_{N-1}, u_{N-1}^*, w_{N-1} = 0) \geq 0. \tag{4.19}$$

Hence,

$$\begin{aligned} &V_{N-1,N}\Big(x_{N-1}, i, \mathcal{N}(\mathscr{I})\Big) = H_{N-1,N}(x_{N-1}, u^*_{N-1}, w^*_{N-1}) \geq 0 \\ &\implies x^T_{N-1}\Xi_{N-1,N}\Big(i, \mathcal{N}(\mathscr{I})\Big)x_{N-1} \geq 0. \end{aligned} \tag{4.20}$$

Since (4.20) is true for all x_{N-1}, $\Xi_{N-1,N}\Big(\mathcal{N}(\mathscr{I})\Big) \geq 0$. Thus the lemma is true for $k = N-1$.

Let us now assume that the lemma is true for the stage $p+1$.

The necessary part of statement (a) can be proved in exactly the same way as Theorem 3.2 in [54]. We now prove the sufficiency part.

Assume that $\Theta_{l,N}(i, \mathcal{N}(\mathscr{I})) > 0$ for $p+1 \leq l \leq N$, $\forall i \in \mathcal{D}$, and $\forall \mathscr{I} \subseteq \mathscr{G}$. Then, with information set $\mathcal{I}_p$, if $\xi_{p-1} = \mathcal{N}(\mathscr{I})$:

$$\begin{aligned} &\mathbb{E}\Big[V_{p+1,N}(x_{p+1}, r_{p+1}, \xi_p)\Big|\mathcal{I}_p\Big] = \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_p(\mathcal{N}(\mathscr{L}))\Big[\Big(A(i)x_p + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\Big)^T \\ &\times \mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{L}))\Big(A(i)x_p + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\Big)\Big]. \end{aligned} \tag{4.21}$$

From (4.7) with $k = p$, and using (4.21):

$$\begin{aligned} V_{p,N}\Big(x_p, r_p, \xi_{p-1}\Big) &= \min_{u_p}\max_{w_p} \mathbb{E}\Big[x_p^T W(i)x_p + u_p^T\xi_p R(i)\xi_p u_p - \gamma^2 w_p^T w_p \\ &+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_p(\mathcal{N}(\mathscr{L}))\Big[\Big(A(i)x_p + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\Big)^T \mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{L})) \\ &\times \Big(A(i)x_p + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\Big)\Big]. \end{aligned} \tag{4.22}$$

Consider again the following functional:

$$H_{p,N}\Big(x_p, u_p, w_p\Big) = \mathbb{E}\Big[\|x_p\|^2_{W(i)} + \|u_p\|^2_{\xi_p R(i)\xi_p} - \gamma^2\|w_p\|^2 + V_{p+1,N}\Big(x_{p+1}, r_{p+1}, \xi_p\Big)\Big|\mathcal{I}_p\Big]. \tag{4.23}$$

Observe that $H_{p,N}(x_p, u_p, w_p)$ is quadratic in u_p, w_p and x_p for all $i \in \mathcal{D}$. Thus, a unique saddle-point exists if and only if $H_{p,N}\big(x_p, u_p, w_p, r_p = i, \xi_{p-1}\big)$ is convex in u_p and concave in w_p. Differentiating (4.23) and using (4.21), we get

$$\frac{\partial^2 H_{p,N}\big(x_p, u_p, w_p\big)}{\partial u_p^2} = \mathbb{E}\Big[\xi_p\Big(R(i) + B^T(i)\mathscr{X}_{p+1,N}(i, \xi_p)B(i)\Big)\xi_p\Big|\mathcal{I}_p\Big], \tag{4.24a}$$

$$\frac{\partial^2 H_{p,N}\big(x_p, u_p, w_p\big)}{\partial w_p^2} = D_1^T(i)\mathbb{E}\Big[\mathscr{X}_{p+1,N}(i, \xi_p)\Big|\mathcal{I}_p\Big]D_1(i) - \gamma^2 I_s = -\Theta_{p,N}(i, j). \tag{4.24b}$$

By our assumption $\Xi_{p+1,N}(i, \mathcal{N}(\mathscr{I})) \geq 0$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Thus, from (4.9e) and as $p_{il} \geq 0$ for all $i, l \in \mathcal{D}$, we get $\mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{I})) \geq 0$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Note that there exists a $\mathscr{L} \subseteq \mathscr{G}$ such that $\mathcal{N}(\mathscr{L}) = diag\{1, 1, ..., 1\}$. Hence, as $R(i) > 0$, one gets that $\mathcal{N}(\mathscr{L})\Big(R(i) + B^T(i)\mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{L}))B(i)\Big)\mathcal{N}(\mathscr{L}) > 0$. This, in turn, implies that $\mathbb{E}\Big[\xi_p\Big(R(i) + B^T(i)\mathscr{X}_{p+1,N}(i, \xi_k)B(i)\Big)\xi_p\Big|\mathcal{I}_p\Big] > 0$. So, $\frac{\partial^2 H_{p,N}\left(x_p, u_p, w_p\right)}{\partial u_p^2} > 0$ and hence, $H_{p,N}(x_p, u_p, w_p)$ is convex in u_p. Also, if $\Theta_{p,N}(i, j) > 0$, then $H_{p,N}(x_p, u_p, w_p)$ will be concave in w_p. Now, similar to Chapter 2, one can derive the saddle-point (u_p^*, w_p^*) as:

$$u_p^* = \zeta_p^*(\mathcal{I}_p) = -\Gamma_{p,N}(i, \mathcal{N}(\mathscr{I}))x_p \tag{4.25a}$$

$$w_p^* = \eta_p^*(\mathcal{I}_p) = \Psi_{p,N}(i, \mathcal{N}(\mathscr{I}))x_p, \tag{4.25b}$$

where, $\Gamma_{p,N}(i, \mathcal{N}(\mathscr{I}))$ and $\Psi_{p,N}(i, \mathcal{N}(\mathscr{I}))$ are as defined in (4.9a) and (4.9b), respectively. Substituting the saddle-point (u_p^*, w_p^*) from (4.25a) and (4.25b) in (4.22) with $k = p$, one gets:

$$V_{p,N}(x_p, i, \mathcal{N}(\mathscr{I})) = x_p^T \Xi_{p,N}(i, \mathcal{N}(\mathscr{I}))x_p,$$

where $\Xi_{p,N}(i, \mathcal{N}(\mathscr{I}))$ is got by solving (4.8).

Observe that $H_{p,N}(x_p, u_p^*, w_p = 0) \geq 0$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$ as $\Xi_{p+1,N}(i, \mathcal{N}(\mathscr{I})) \geq 0$.

Also, as w_p is the maximizing player:

$$H_{p,N}(x_p, u_p^*, w_p^*) \geq H_{p,N}(x_p, u_p^*, 0) \geq 0.$$

Therefore,

$$V_{p,N}(x_p, i, \mathcal{N}(\mathscr{I})) = x_p^T \Xi_{p,N}(i, \mathcal{N}(\mathscr{I}))x_p = H_{p,N}(x_p, u_p^*, w_p^*) \geq 0. \tag{4.26}$$

As (4.26) holds for all $x_p \neq 0$, $\Xi_{p,N}(i, \mathcal{N}(\mathscr{I})) \geq 0$. Hence, the lemma is true for the stage $k = p$.

At the stage $k = 0$, since the previous packet condition (ξ_{k-1}) is not available, $\Xi_{k,N}$ will not a function of ξ_{k-1}. Thus, one can observe that for a fixed r_0, $\Xi_{0,N}(r_0, \mathcal{N}(\mathscr{I})) = \Xi_{0,N}(r_0, \mathcal{N}(\mathscr{L}))$, for all $\mathscr{I}, \mathscr{L} \subseteq \mathscr{G}$. Hence, if a unique saddle-point exists at the stage $k = 0$, then, $V_{0,N}\big(x_0, r_0, \mathcal{N}(\mathscr{I})\big)$ $= V_{0,N}\big(x_0, r_0, \mathcal{N}(\mathscr{L})\big)$, for all $\mathscr{I}, \mathscr{L} \subseteq \mathscr{G}$. The value of the game with the cost function (4.5) is then given by:

$$J_N(\zeta_{0:N-1}^*, \eta_{0:N-1}^*) = V_{0,N}\big(x_0, r_0, \mathcal{N}(\mathscr{I})\big) = x_0^T \Xi_{0,N}(r_0, \mathcal{N}(\mathscr{I}))x_0, \tag{4.27}$$

□

As a direct consequence of Lemma 4.3.1, we get the following result.

Corollary 4.3.2. *At the stage $k \in [0, N-1]$, a unique saddle-point exists if and only if conditions (i) and (ii) given in (a) in Lemma 4.3.1 are satisfied. Also, the saddle-point at the stage $k \in [0, N-1]$ is as given by (4.25a) and (4.25b). Further, if conditions (i) and (ii) given in (a) in Lemma 4.3.1 are satisfied, $\Gamma_{k,N}(i, \mathcal{N}(\mathscr{I}))$ and $\Psi_{k,N}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$ are finite.*

□

Lemma 4.3.3. *Consider that a unique saddle-point exists at the stage $k = 0$. Then, with the optimal control sequence $u_{0:N-1}^*$, the $\mathcal{L}_2$ gain from disturbance w_k to controlled output z_k of the closed loop system is less than or equal to γ.*

Proof: If a unique saddle-point exists at the stage $k = 0$, then,

$$
\begin{aligned}
& J_N(\zeta^{*N-1}, \eta^{N-1}) \leq J_N(\zeta^{*N-1}, \eta^{*N-1}) \leq J_N(\zeta^{N-1}, \eta^{*N-1}) \\
& \implies J_N(\zeta^{*N-1}, \eta^{N-1}) \leq J_N(\zeta^{*N-1}, \eta^{*N-1}) = x_0^T \Xi_{0,N}(r_0, \mathcal{N}(\mathscr{I}))x_0 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big(\text{using}(4.27)\big) \\
& \implies J_N(\zeta^{*N-1}, \eta^{N-1}) \leq 0 \\
& \qquad\qquad\qquad\qquad\qquad \big(\text{considering zero initial condition}\big) \\
& \implies \mathbb{E}\Big[\|x_N\|_W^2 + \sum_{k=0}^{N-1} \|z_k\|^2 - \gamma^2\|w_k\|^2 \Big| \mathcal{I}_0\Big] \leq 0 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big(\text{using}(4.3)\big) \\
& \implies \sum_{k=0}^{N-1} \mathbb{E}\Big[\|z_k\|^2 \Big| \mathcal{I}_0\Big] \leq \gamma^2 \sum_{k=0}^{N-1} \|w_k\|^2.
\end{aligned}
$$

Therefore, the $\mathcal{L}_2$ gain from the disturbance input w_k to the controlled output z_k is less than or equal to γ. It completes the proof. □

In the subsequent results, we shall assume that $W(r_k) \geq W$ for all r_k.

Lemma 4.3.4. *If a unique saddle-point exists at the stage $k \in [1, N]$, then $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{k+1,N}(i, \mathcal{N}(\mathscr{I}))$ for all $k \geq 1$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$.*

Proof: This result is proved using induction.

From (4.15) and (4.16):

$$\begin{aligned}
&H_{N-1,N}(x, u^*_{N-1}, 0)\\
&= \min_{u_{N-1}}\Big[x^T W(i)x + u^T_{N-1}\mathbb{E}^{\mathscr{I}}[\xi_{N-1}R(i)\xi_{N-1}]u_{N-1}\\
&+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_{N-1}\big(\mathcal{N}(\mathscr{L})\big)\Big\{\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_{N-1}\big)^T \mathscr{X}_{N,N}\big(i, \mathcal{N}(\mathscr{L})\big)\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_{N-1}\big)\Big\}\Big]\\
&\geq x^T W(i)x \ \Big(As\ \mathbb{E}^{\mathscr{I}}[\xi_{N-1}R(i)\xi_{N-1}] > 0 \ \text{ and } \ \mathscr{X}_{N,N}\big(i, \mathcal{N}(\mathscr{I})\big) = W_N \geq 0\Big)\\
&= V_{N,N}\big(x, i, \mathcal{N}(\mathscr{I})\big).
\end{aligned} \tag{4.28}$$

Further,

$$\begin{aligned}
V_{N-1,N}\big(x, i, \mathcal{N}(\mathscr{I})\big) &= x^T\Xi_{N-1,N}\big(i, \mathcal{N}(\mathscr{I})\big)x\\
&= H_{N-1,N}(x, u^*_{N-1}, w^*_{N-1})\\
&\geq H_{N-1,N}(x, u^*_{N-1}, 0)\\
&\text{(Since } \mathrm{w_{N-1}} \text{ is the maximizing player)}\\
&\geq V_{N,N}\big(x, i, \mathcal{N}(\mathscr{I})\big) \ \ (by\ (4.28))\\
&= x^T\Xi_{N,N}\big(i, \mathcal{N}(\mathscr{I})\big)x.
\end{aligned} \tag{4.29}$$

Since (4.29) is true for all $x \neq 0$, $\Xi_{N-1,N}\big(\mathcal{N}(\mathscr{I})\big) \geq \Xi_{N,N}\big(\mathcal{N}(\mathscr{I})\big)$; $\forall \mathscr{I} \subseteq \mathscr{G}$.

For the stage $k = p$, from (4.22):

$$\begin{aligned}
&V_{p,N}\big(x, i, \mathcal{N}(\mathscr{I})\big)\\
&= x^T\Xi_{p,N}(i, \mathcal{N}(\mathscr{I}))x\\
&= \min_{u_p}\ \max_{w_p}\Big[x_p^T W(i)x + u_p^T\mathbb{E}\Big[\xi_p R(i)\xi_p\Big|\mathcal{I}_p\Big]u_p - \gamma^2 w_p^T w_p\\
&+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_p(\mathcal{N}(\mathscr{L}))\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\big)^T\\
&\times \mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{L}))\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\big)\Big].
\end{aligned} \tag{4.30}$$

Replacing p by $p+1$ in the above equation:

$$
\begin{aligned}
& V_{p+1,N}\Big(x, i, \mathcal{N}(\mathscr{I})\Big) \\
& = x^T \Xi_{p+1,N}(i, \mathcal{N}(\mathscr{I}))x \\
& = \min_{u_{p+1}} \max_{w_{p+1}} \Big[x^T W(i) x + u_{p+1}^T \mathbb{E}\Big[\xi_{p+1} R(i) \xi_{p+1} \Big| \mathcal{I}_{p+1}\Big] u_{p+1} - \gamma^2 w_{p+1}^T w_{p+1} \\
& + \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_{p+1}(\mathcal{N}(\mathscr{L})) \Big(A(i)x + B(i)\mathcal{N}(\mathscr{L}) u_{p+1} + D_1(i) w_{p+1}\Big)^T \\
& \times \mathscr{X}_{p+2,N}(i, \mathcal{N}(\mathscr{L})) \Big(A(i)x + B(i)\mathcal{N}(\mathscr{L}) u_{p+1} + D_1(i) w_{p+1}\Big)\Big].
\end{aligned} \tag{4.31}
$$

Suppose, the lemma is true for the stage $k = p+1$. Thus, $\Xi_{p+1,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{p+2,N}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Which, in turn, implies that $\mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{I})) \geq \mathscr{X}_{p+2,N}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Note that, $\mathcal{P}_p(\mathcal{N}(\mathscr{I})) = \mathcal{P}_{p+1}(\mathcal{N}(\mathscr{I}))$ for all $\mathscr{I} \subseteq \mathscr{G}$. Further, for two functions $g_1(u, w)$ and $g_2(u, w)$, if $g_1(u, w) \geq g_2(u, w)$ for all u, w, then one can show that

$$
\min_u \max_w \; g_1(u, w) \geq \min_u \max_w \; g_2(u, w).
$$

Therefore, from (4.30) and (4.31):

$$
\begin{aligned}
& V_{p,N}\Big(x, i, \mathcal{N}(\mathscr{I})\Big) \geq V_{p+1,N}\Big(x, i, \mathcal{N}(\mathscr{I})\Big) \\
& \implies x^T \Xi_{p,N}(i, \mathcal{N}(\mathscr{I}))x \geq x^T \Xi_{p+1,N}(i, \mathcal{N}(\mathscr{I}))x
\end{aligned} \tag{4.32}
$$

Since (4.32) holds true for all $x \neq 0$, $\Xi_{p,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{p+1,N}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$.

□

Lemma 4.3.5. *Suppose a unique saddle-point exists for the stage $k \geq 1$. Then, $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) = \Xi_{k+1,N+1}(i, \mathcal{N}(\mathscr{I}))$ for all $k \geq 1$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$.*

Proof: To prove the lemma, we again use induction.

Clearly:

$$
V_{N,N}\Big(x, i, \mathcal{N}(\mathscr{I})\Big) = V_{N+1,N+1}\Big(x, i, \mathcal{N}(\mathscr{I})\Big) = x^T W x.
$$

Assume that $\Xi_{p+1,N}(i, \mathcal{N}(\mathscr{I})) = \Xi_{p+2,N+1}(i, \mathcal{N}(\mathscr{I}))$, $\forall i \in \mathcal{D}$ and $\forall \mathscr{I} \subseteq \mathscr{G}$. So, $\mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{I})) = \mathscr{X}_{p+2,N+1}(i, \mathcal{N}(\mathscr{I}))$. Now,

$$\begin{aligned} &V_{p,N}\big(x, i, \mathcal{N}(\mathscr{I})\big) \\ &= x^T \Xi_{p,N}(i, \mathcal{N}(\mathscr{I}))x \\ &= \min_{u_p} \max_{w_p} \Big[x^T W(i)x + u_p^T \mathbb{E}\big[\xi_p R(i)\xi_p \big| \mathcal{I}_p\big]u_p - \gamma^2 w_p^T w_p \\ &+ \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_p(\mathcal{N}(\mathscr{L}))\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\big)^T \\ &\times \mathscr{X}_{p+1,N}(i, \mathcal{N}(\mathscr{L}))\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_p + D_1(i)w_p\big)\Big], \end{aligned} \tag{4.33}$$

and,

$$\begin{aligned} &V_{p+1,N+1}\big(x, i, \mathcal{N}(\mathscr{I})\big) \\ &= x^T \Xi_{p+1,N+1}(i, \mathcal{N}(\mathscr{I}))x. \\ &= \min_{u_{p+1}} \max_{w_{p+1}} \Big[x^T W(i)x + u_{p+1}^T \mathbb{E}\big[\xi_{p+1} R(i)\xi_{p+1} \big| \mathcal{I}_{p+1}\big]u_{p+1} - \gamma^2 w_{p+1}^T w_{p+1} \\ &+ \sum_{\mathscr{L} \subseteq \mathscr{G}} \mathcal{P}_{p+1}(\mathcal{N}(\mathscr{L}))\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_{p+1} + D_1(i)w_{p+1}\big)^T \\ &\times \mathscr{X}_{p+2,N+1}(i, \mathcal{N}(\mathscr{L}))\big(A(i)x + B(i)\mathcal{N}(\mathscr{L})u_{p+1} + D_1(i)w_{p+1}\big)\Big]. \end{aligned} \tag{4.34}$$

Note that $\mathcal{P}_p(\mathcal{N}(\mathscr{L})) = \mathcal{P}_{p+1}(\mathcal{N}(\mathscr{L}))$. Therefore, from (4.33) and (4.34), one gets that $\Xi_{p,N}(i, \mathcal{N}(\mathscr{I})) = \Xi_{p+1,N+1}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. □

Note 4.3.1. *From Lemma 4.3.4 and Lemma 4.3.5, we get that* $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) \geq \Xi_{k+1,N}(i, \mathcal{N}(\mathscr{I})) = \Xi_{k,N-1}(i, \mathcal{N}(\mathscr{I}))$ *or* $\Xi_{k,N-1}(i, \mathcal{N}(\mathscr{I})) \leq \Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$. *Thus, for all* $i \in \mathcal{D}$ *and* $\mathscr{I} \subseteq \mathscr{G}$, *the sequence* $\big\{\Xi_{k,k+1}(i, \mathcal{N}(\mathscr{I})), \Xi_{k,k+2}(i, \mathcal{N}(\mathscr{I})), ..., \Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))\big\}$, *denoted by* $\big\{\Xi_{k,c}(i, \mathcal{N}(\mathscr{I}))\big\}_{c=k+1}^{N}$, *monotonically increases with c.* □

Lemma 4.3.6. *Suppose a unique saddle-point exists at the stage* $k = 0$. *The sequence* $\big\{\Xi_{0,c}(i, \mathcal{N}(\mathscr{I}))\big\}_{c=1}^{N}$ *monotonically increases with c.*

Proof: Using (4.7), for a horizon $l \in [1, N]$, one can write the value function at the stage

$k = 0$ as:

$$V_{0,l}(x_0, i) = \mathbb{E}\Big[x_0^T W(i) x_0 + u_0^T \xi_0 R(i) \xi_0 u_0 - \gamma^2 w_0^T w_0 + V_{1,l}\Big(x_1, r_1, \xi_0\Big)\Big|\mathcal{I}_0\Big]. \tag{4.35}$$

Considering the horizon to be $l + 1$, we can write (4.35) as:

$$V_{0,l+1}(x_0, i) = \mathbb{E}\Big[x_0^T W(i) x_0 + u_0^T \xi_0 R(i) \xi_0 u_0 - \gamma^2 w_0^T w_0 + V_{1,l+1}\Big(x_1, r_1, \xi_0\Big)\Big|\mathcal{I}_0\Big]. \tag{4.36}$$

From Note 4.3.1,

$$\mathbb{E}\Big[V_{1,l}\Big(x_1, r_1, \xi_0\Big)\Big|\mathcal{I}_0\Big] \leq \mathbb{E}\Big[V_{1,l+1}\Big(x_1, r_1, \xi_0\Big)\Big|\mathcal{I}_0\Big].$$

Thus, from (4.35) and (4.36):

$$\begin{aligned} &V_{0,l}(x_0, i) \leq V_{0,l+1}(x_0, i) \\ &\implies x_0^T \Xi_{0,l}(i, \mathcal{N}(\mathscr{I})) x_0 \leq x_0^T \Xi_{0,l+1}(i, \mathcal{N}(\mathscr{I})) x_0. \end{aligned} \tag{4.37}$$

Since (4.37) is satisfied for all $x_0 \neq 0$, $\Xi_{0,l}(i, \mathcal{N}(\mathscr{I})) \leq \Xi_{0,l+1}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. Therefore, the sequence $\Big\{\Xi_{0,c}(i, \mathcal{N}(\mathscr{I}))\Big\}_{c=1}^{N}$ monotonically increases with c.

□

B. Infinite horizon control:

For the infinite horizon case, we consider the following cost function:

$$J_\infty(\zeta_{0:\infty}, \eta_{0:\infty}) = \mathbb{E}\Big[\sum_{k=0}^{\infty} x_k^T W(r_k) x_k + u_k^T \xi_k^T R(r_k) \xi_k u_k - \gamma^2 w_k^T w_k \Big|\mathcal{I}_0\Big]. \tag{4.38}$$

In the next lemma, we provide certain conditions for the convergence of the sequence $\Big\{\Xi_{k,c}(i, j)\Big\}_{c=k+1}^{N}$ as $N \to \infty$.

Lemma 4.3.7. *Suppose γ and $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ are such that for all finite $N \in \mathbb{Z}^+$,*

$k \in [0, N]$, $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$,

(i) An unique saddle-point exists, i.e., $\Theta_{k,N}(i, \mathcal{N}(\mathscr{I})) > 0$.

(ii) $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) < cI_n$, for some finite $c < \infty$.

Then, there exist $\hat{\bar{\Xi}}(i) < \infty$ and $\bar{\Xi}(i, \mathcal{N}(\mathscr{I})) < \infty$ such that $\lim_{N\to\infty} \Xi_{0,N}(i, \mathcal{N}(\mathscr{I})) = \hat{\bar{\Xi}}(i)$, and for $k \geq 1$, $\lim_{N\to\infty} \Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) = \bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$.

Proof: From Note 4.3.1 and Lemma 4.3.6, monotonicity of the sequence $\left\{\Xi_{k,c}(i, \mathcal{N}(\mathscr{I}))\right\}_{c=k+1}^{N}$ is guaranteed if condition (i) is satisfied. Further, suppose γ and $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ are such that for all finite $N \in \mathbb{Z}^+$, $k \in [0, N]$, $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$ we have $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) < cI_n$. Hence, in view of Theorem 3.14 in [60], $\left\{\Xi_{0,c}(i, \mathcal{N}(\mathscr{I}))\right\}_{c=1}^{N}$, and $\left\{\Xi_{k,c}(i, \mathcal{N}(\mathscr{I}))\right\}_{c=k+1}^{N}$ converge as $N \to \infty$. Therefore, there exist $\hat{\bar{\Xi}}(i) < \infty$ and $\bar{\Xi}(i, \mathcal{N}(\mathscr{I})) < \infty$ such that $\lim_{N\to\infty} \Xi_{0,N}(i, \mathcal{N}(\mathscr{I})) = \hat{\bar{\Xi}}(i)$, and for $k \in [1, N]$, $\lim_{N\to\infty} \Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) = \bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$ for all $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$. $\square$

Remark 4.3.1. *If matrices $D_1(i)$ in (4.1) are square and full rank, $\Theta_{k,N}(i, \mathcal{N}(\mathscr{I})) > 0$, for all $i \in \mathcal{D}$ ensures that $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I})) < \infty$ for all N. In such a scenario, condition (i) in Lemma (4.3.7) alone is sufficient for convergence of the sequence $\left\{\Xi_{k,c}(i, \mathcal{N}(\mathscr{I}))\right\}_{c=k+1}^{N}$ as $N \to \infty$.*

$\square$

If for all $k \in [1, N-1]$, $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$, the sequence $\left\{\Xi_{k,c}(i, \mathcal{N}(\mathscr{I}))\right\}_{c=k+1}^{N}$ converges as $N \to \infty$, $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$, $\Gamma_{k,N}(i, \mathcal{N}(\mathscr{I}))$, $\Theta_{k,N}(i, \mathcal{N}(\mathscr{I}))$, etc., will no longer be functions of k. The CAREs (4.8) then transform to the following:

$$\begin{aligned}
&\bar{\Xi}(i, \mathcal{N}(\mathscr{I})) = W(i) + \bar{\Gamma}^T(i, \mathcal{N}(\mathscr{I}))\mathbb{E}^{\mathscr{I}}\left[\xi_k R(i)\xi_k\right]\bar{\Gamma}(i, \mathcal{N}(\mathscr{I})) - \gamma^2\bar{\Psi}^T(i, \mathcal{N}(\mathscr{I}))\bar{\Psi}(i, \mathcal{N}(\mathscr{I}))\\
&+ \sum_{\mathscr{L}\subseteq\mathscr{G}} Pr\Big(\xi_k = \mathcal{N}(\mathscr{L})\Big|\xi_{k-1} = \mathcal{N}(\mathscr{I})\Big)\Big[\Big(A(i) - B(i)\mathcal{N}(\mathscr{L})\bar{\Gamma}(i, \mathcal{N}(\mathscr{I})) + D_1(i)\bar{\Psi}(i, \mathcal{N}(\mathscr{I}))\Big)^T\\
&\times \bar{\mathscr{X}}(i, \mathcal{N}(\mathscr{L}))\Big(A(i) - B(i)\mathcal{N}(\mathscr{L})\bar{\Gamma}(i, \mathcal{N}(\mathscr{I})) + D_1(i)\bar{\Psi}(i, \mathcal{N}(\mathscr{I}))\Big)\Big],
\end{aligned} \tag{4.39}$$

where $\bar{\Gamma}(i, \mathcal{N}(\mathscr{I}))$ and $\bar{\Psi}(i, \mathcal{N}(\mathscr{I}))$ are the infinite horizon counterparts of $\Gamma_{k,N}(i, \mathcal{N}(\mathscr{I}))$ and $\Psi_{k,N}(i, \mathcal{N}(\mathscr{I}))$, respectively. Replacing $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$ by $\bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$ in (4.9a) and (4.9b), $\bar{\Gamma}(i, \mathcal{N}(\mathscr{I}))$

and $\bar{\Psi}(i, \mathcal{N}(\mathscr{I}))$ can be derived easily.

Similarly, for the stage $k = 0$, infinite horizon CAREs takes the following form:

$$\begin{aligned}\hat{\bar{\Xi}}(i) &= W(i) + \hat{\bar{\Gamma}}^T(i)\mathbb{E}\Big[\xi_0 R(i)\xi_0\Big|\mathcal{I}_0\Big]\hat{\bar{\Gamma}}(i) - \gamma^2\hat{\bar{\Psi}}^T(i)\hat{\bar{\Psi}}(i)\\ &+ \sum_{\mathscr{L}\subseteq\mathscr{G}} \mathcal{P}_k(\mathcal{N}(\mathscr{L}))\Big[\Big(A(i) - B(i)\mathcal{N}(\mathscr{L})\hat{\bar{\Gamma}}(i) + D_1(i)\hat{\bar{\Psi}}(i)\Big)^T \bar{\mathscr{X}}(i, \mathcal{N}(\mathscr{L}))\\ &\times \Big(A(i) - B(i)\mathcal{N}(\mathscr{L})\hat{\bar{\Gamma}}(i) + D_1(i)\hat{\bar{\Psi}}(i)\Big)\Big],\end{aligned} \tag{4.40}$$

where $\hat{\bar{\Gamma}}(i)$ and $\hat{\bar{\Psi}}(i)$ can be got from (4.9a) and (4.9b) by replacing $\Xi_{k,N}(i, \mathcal{N}(\mathscr{I}))$ by $\bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$.

Further, the infinite horizon Isaacs Equation is as given below:

$$V(x_k, r_k, \xi_{k-1}) = \min_{u_k} \max_{w_k} \mathbb{E}\left[x_k^T W(r_k)x_k + u_k^T\xi_k^T R(r_k)\xi_k u_k - \gamma^2 w_k^T w_k + V(x_{k+1}, r_{k+1}, \xi_k)|\mathcal{I}_k\right]. \tag{4.41}$$

One gets the following result by applying limit $N \to \infty$ in Lemma 4.3.1.

Lemma 4.3.8. *Suppose* $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ *and* γ *are such that for all finite* $N \in \mathbb{Z}^+$, $k \in [0, N]$, $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$, *conditions (i) and (ii) given in the Lemma 4.3.7 are satisfied. Then,*

(a) The value function at the stage $k \in [1, \infty]$ *is expressed as:*

$$V\Big(x_k, i, N(j)\Big) = x_k^T\bar{\Xi}(i, \mathcal{N}(\mathscr{I}))x_k.$$

(b) The value function at the $k = 0$ *stage is expressed as:*

$$V\Big(x_0, r_0\Big) = x_0^T\hat{\bar{\Xi}}(r_0)x_0.$$

(c) The infinite horizon saddle-point is given by:

$$u_k^* = -\Gamma_k x_k; \quad w_k^* = \Psi_k x_k,$$

where

$$\begin{aligned}&For\ \ k \geq 1\ \ ,\ \ if\ \ r_k = i\ \ and\ \ \xi_{k-1} = \mathcal{N}(\mathscr{I}), then\ \ \Gamma_k = \bar{\Gamma}(i, \mathcal{N}(\mathscr{I})),\ \ \Psi_k = \bar{\Psi}(i, \mathcal{N}(\mathscr{I})).\\ &For\ \ k = 0,\ \ if\ \ r_0 = i, then\ \ \Gamma_0 = \hat{\bar{\Gamma}}(i),\ \ \Psi_0 = \hat{\bar{\Psi}}(i).\end{aligned} \tag{4.42}$$

(d) The infinite horizon value of the game with the cost function (4.38) is expressed as follows:

$$J_\infty(\zeta^*_{0:\infty}, \eta^*_{0:\infty}) = x_0^T \hat{\bar{\Xi}}(r_0) x_0. \qquad \square$$

Note 4.3.2. *In the sequel, we assume that the system (4.1)-(4.2) with $u_k \equiv 0$, $w_k \equiv 0$ is weakly observable. This condition is a relaxed one in comparison to the assumption taken in [55, 61], where it is assumed that $\big(C(i), A(i)\big)$ is observable for each $i \in \mathcal{D}$.*

The following lemma establishes the positive definiteness of the fixed-point solutions of CAREs (4.39) and (4.40).

Lemma 4.3.9. *Suppose, γ, $\{\bar{\nu}^1, \bar{\nu}^2, ..., \bar{\nu}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ are such that for all $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$, and finite $N \in \mathbb{Z}^+$, $k \in [0, N]$, the conditions (i) and (ii) of Lemma 4.3.7 are satisfied. Further, if system (4.1)-(4.2), with $u_k \equiv 0$, $w_k \equiv 0$, is weakly observable, then, for all $i \in \mathcal{D}$, $\mathscr{I} \subseteq \mathscr{G}$, $\hat{\bar{\Xi}}(i) > 0$ and $\bar{\Xi}(i, \mathcal{N}(\mathscr{I})) > 0$.*

Proof: Consider the following functional:

$$H_{k,N}\Big(x_k, u_k, w_k\Big) = \mathbb{E}\Big[\|x_k\|^2_{W(i)} + \|u_k\|^2_{\xi_k R(i) \xi_k} - \gamma^2 \|w_k\|^2 + V_{k+1,N}\Big(x_{k+1}, r_{k+1}, \xi_k\Big)\Big| \mathcal{I}_k\Big]. \tag{4.43}$$

As (u_k^*, w_k^*) constitutes a saddle-point,

$$H_{k,N}\Big(x_k, u_k^*, w_k^*\Big) \geq H_{k,N}\Big(x_k, u_k^*, 0\Big).$$

Note that as the the conditions (i) and (ii) of Lemma 4.3.7 are satisfied, the following analysis is well define as $N \to \infty$. Taking limit as $N \to \infty$:

$$\lim_{N\to\infty} H_{k,N}\big(x_k, u_k^*, w_k^*\big) \geq \lim_{N\to\infty} H_{k,N}\big(x_k, u_k^*, 0\big). \tag{4.44}$$

Observe that,

$$\begin{aligned}
&\lim_{N\to\infty} H_{k,N}\big(x_k, u_k^*, w_k^*\big)\\
&= V\big(x_k, i, \mathcal{N}(\mathscr{I})\big)\\
&= \sum_{f=k}^{\infty} \mathbb{E}\Big[x_f^T W(i) x_f + x_f^T \Gamma_f^T \xi_f R(i) \xi_f \Gamma_f x_f - \gamma^2 w_f^{*T} w_f^* \Big| \mathcal{I}_k\Big]. \ \big(\text{using (4.6)}\big)
\end{aligned} \tag{4.45}$$

It is assumed that Markov chain $\{r_k\}$ is irreducible. Thus, using the same line of argument as used in the proof of Lemma 3.3.9 presented in Chapter 3, for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$, one can show that

$$\begin{aligned}
&\lim_{N\to\infty} H_{k,N}\big(x_k, u_k^*, 0\big) > x_k^T W(i) x_k\\
&\implies \lim_{N\to\infty} H_{k,N}\big(x_k, u_k^*, w_k^*\big) > x_k^T W(i) x_k.
\end{aligned} \tag{4.46}$$

Hence, from (4.45) and (4.46):

$$\begin{aligned}
V\big(x_k, i, \mathcal{N}(\mathscr{I})\big) &= x_k^T \Xi_k x_k \quad \big(\text{using Lemma 4.3.8}\big)\\
&= \lim_{N\to\infty} H_{k,N}\big(x_k, u_k^*, w_k^*\big)\\
&\geq \lim_{N\to\infty} H_{k,N}\big(x_k, u_k^*, 0\big) \ \big(\text{as } w_k \text{ is the maximizing player}\big)\\
&> x_k^T W(i) x_k,
\end{aligned} \tag{4.47}$$

where, for $k \geq 1$, $\Xi_k = \bar{\Xi}(i, \mathcal{N}(\mathscr{I}))$ if $r_k = i$, $\xi_{k-1} = \mathcal{N}(\mathscr{I})$, and $\Xi_0 = \bar{\bar{\Xi}}(i)$ if $r_0 = i$. Since (4.47) is true for all $x_k \neq 0$, $\Xi_k > 0$. Therefore, $\bar{\bar{\Xi}}(i) > 0$ and $\bar{\Xi}(i, \mathcal{N}(\mathscr{I})) > 0$ for all $i \in \mathcal{D}$ and $\mathscr{I} \subseteq \mathscr{G}$. □

Following analysis is required to show stability of the closed-loop system with the saddle-point policy. State dynamics of the system (4.1)-(4.2) with the infinite horizon saddle-point transform to the following:

$$x_{k+1} = \mathscr{A}(q_k)x_k, \tag{4.48}$$

where

$$\mathscr{A}(q_k) = A(r_k) - B(r_k)\xi_k\Gamma_k + D_1(r_k)\Psi_k.$$

and $\{q_k\}$ is a Markov chain whose states are (r_k, ξ_k). Thus, the number of Markov states q_k is $\mathscr{M} \times 2^m$.

Also, consider a dummy output (which will only be used in the proof for the following Theorem) given by:

$$y_k = \mathscr{C}(q_k)x_k, \tag{4.49}$$

where

$$\mathscr{C}(q_k) = \mathbb{E}\Big[\Big(W(r_k) + \Gamma_k^T\xi_k R(i)\xi_k\Gamma_k - \gamma^2\Psi_k^T\Psi_k\Big)^{1/2}\Big|\mathcal{I}_k\Big].$$

In the following result, we shall show that the optimal controller stabilizes the closed-loop system while maintaining a prescribed $\mathcal{L}_2$ gain.

Theorem 4.3.10. *Suppose* $\{\bar{v}^1, \bar{v}^2, ..., \bar{v}^m\}$, $\{\bar{\mu}^1, \bar{\mu}^2, ..., \bar{\mu}^m\}$ *and* γ *are such that* $\forall i \in \mathcal{D}$ *and* $\forall \mathscr{I} \subseteq \mathscr{G}$, *and for all finite* $N \in \mathbb{Z}^+$, $k \in [0, N]$, *the conditions (i) and (ii) of Lemma 4.3.7 are satisfied. Then,*

(a) With the optimal control law $u^*_{0:\infty}$, *the* $\mathcal{L}_2$ *gain from the disturbance input* w_k *to the controlled output* z_k *of the closed loop system is less than or equal to* γ.

(b) The optimal control law $u_k^* = -\Gamma_k x_k$ *stabilizes system (4.1) in the mean-square sense with arbitrary disturbance* $w_k \in l_2([0,\infty), \mathbb{R}^s)$ *if system (4.1)-(4.2) with* $u_k \equiv 0$ *and* $w_k \equiv 0$ *is weakly observable.*

(c) The state-response of the closed-loop system (4.1) with the optimal controller is bounded in the mean-square sense (i.e. $\mathbb{E}\Big[\|x_k\|\Big|\mathcal{I}_0\Big] < \infty$, for all k) with bounded disturbance if the system (4.1)-(4.2), with $u_k \equiv 0$ and $w_k \equiv 0$, is weakly observable.

(d) The closed-loop system (4.1) with optimal controller and optimal disturbance is stable in the mean-square sense if the system (4.48)-(4.49) is weakly observable.

Proof: Proof for Claim (a) follows the same line of argument as the proof for Lemma 4.3.3.

Proof of Claim (b): Since the conditions (i) and (ii) of Lemma 4.3.7 are assumed to be satisfied for all finite $N \in \mathbb{Z}^+$, $k \in [0, N]$, there exist $\hat{\bar{\Xi}}(i) < \infty$ such that $\lim_{N\to\infty} \hat{\Xi}_{0,N}(i) = \hat{\bar{\Xi}}(i)$. Thus, we get:

$$
\begin{aligned}
&J_\infty(\zeta^*_{0:\infty}, \eta_{0:\infty}) \le J_\infty(\zeta^*_{0:\infty}, \eta^*_{0:\infty}) = V(x_0, i) = x_0^T\hat{\bar{\Xi}}(i)x_0 < \infty\\
&\implies \mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_{W(r_k)} + \|u_k^*\|^2_{\xi_k R(r_k)\xi_k} - \gamma^2\|w_k\|^2\Big|\mathcal{I}_0\Big] < \infty.
\end{aligned}
$$

As $w_k \in l_2([0,\infty), \mathbb{R}^s)$ or $\sum_{k=0}^\infty \|w_k\|^2 < \infty$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_{W(r_k)} + \|u_k^*\|^2_{\xi_k R(r_k)\xi_k}\Big|\mathcal{I}_0\Big] < \infty\\
&\implies \mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_{W(r_k)} + \|x_k\|^2_{\Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k}\Big|\mathcal{I}_0\Big] < \infty.
\end{aligned}
\tag{4.50}
$$

If system (4.1)-(4.2), with $u_k \equiv 0$ and $w_k \equiv 0$, is weakly observable, then using the same reasoning as presented in the proof of Theorem 3.3.12 in Chapter 3, it can be shown that (4.50) implies $\lim_{k\to\infty}\mathbb{E}\Big[\|x_k\|^2\Big|\mathcal{I}_0\Big] = 0$.

Proof of Claim (c): Consider system (4.1)-(4.2) with the optimal control law $u_k^* = -\tilde{\Gamma}_k x_k$ and with a bounded disturbance w_k as follows:

$$
x_{k+1} = \Big(A - B\xi_k\Gamma_k\Big)x_k + D_1 w_k. \tag{4.51}
$$

From the argument given for the proof of part (b), one gets that the following system is mean-square stable.

$$x_{k+1} = \Big(A - B\xi_k\Gamma_k\Big)x_k.$$

Thus, the state-responses of system (4.51) are bounded for bounded disturbance w_k.

Proof of Claim (d): As the conditions given in Lemma 4.3.7 are satisfied:

$$\begin{aligned} &J_\infty(\zeta^*_{0:\infty}, \eta^*_{0:\infty}) = x_0^T\hat{\hat{\Xi}}(i)x_0 < \infty \\ &\implies \mathbb{E}\Big[\sum_{k=0}^{\infty} \|x_k\|^2_{W(r_k)} + \|x_k\|^2_{\Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k} - \gamma^2\|x_k\|^2_{(\Psi_k)^T\Psi_k}\Big|\mathcal{I}_0\Big] < \infty \\ &\implies \mathbb{E}\Big[\sum_{k=0}^{\infty} x_k^T\Big(W(r_k) + \Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k - \gamma^2(\Psi_k)^T\Psi_k\Big)x_k\Big|\mathcal{I}_0\Big] < \infty. \end{aligned} \tag{4.52}$$

Convergence of the above infinite series imply that the limiting terms of the series will be zero. Thus:

$$\begin{aligned} &\lim_{k\to\infty}\mathbb{E}\Big[x_k^T\Big(W(r_k) + \Gamma_k^T\xi_k R(r_k)\xi_k\Gamma_k - \gamma^2(\Psi_k)^T\Psi_k\Big)x_k\Big|\mathcal{I}_0\Big] = 0, \\ &\implies \lim_{k\to\infty}\mathbb{E}\Big[x_k^T(\mathscr{C}^{1/2}(q_k))^T\mathscr{C}^{1/2}(q_k)x_k\Big|\mathcal{I}_0\Big] = 0, \end{aligned} \tag{4.53}$$

similarly,

$$\lim_{k\to\infty}\mathbb{E}\Big[x_{k+1}^T(\mathscr{C}^{1/2}(q_{k+1}))^T\mathscr{C}^{1/2}(q_{k+1})x_{k+1}\Big|\mathcal{I}_0\Big] = 0, \tag{4.54}$$

$$\vdots$$

Also,

$$\lim_{k\to\infty}\mathbb{E}\Big[x_{k+\mathscr{T}}^T(\mathscr{C}^{1/2}(q_{k+\mathscr{T}}))^T\mathscr{C}^{1/2}(q_{k+\mathscr{T}})x_{k+\mathscr{T}}\Big|\mathcal{I}_0\Big] = 0. \tag{4.55}$$

Now, similar to the proof of Theorem 2.3.10 in Chapter 2, we get the following:

$$\begin{aligned} &\lim_{k\to\infty}\mathbb{E}\Big[x_k^T(\mathscr{C}^{1/2}(q_k))^T\mathscr{C}^{1/2}(q_k)x_k + x_{k+1}^T(\mathscr{C}^{1/2}(q_{k+1}))^T\mathscr{C}^{1/2}(q_{k+1})x_{k+1} \\ &+ \ldots + x_{k+\mathscr{T}}^T(\mathscr{C}^{1/2}(q_{k+\mathscr{T}}))^T\mathscr{C}^{1/2}(q_{k+\mathscr{T}})x_{k+\mathscr{T}}\Big|\mathcal{I}_0\Big] = 0 \\ &\implies \lim_{k\to\infty}\mathbb{E}\Big[x_k^T\mathscr{Y}_1(q_k, \ldots, q_{k+\mathscr{T}})x_k\Big|\mathcal{I}_0\Big] = 0 \end{aligned} \tag{4.56}$$

where

$$\mathscr{Y}_1(q_k, ..., q_{k+\mathscr{T}}) = \mathscr{O}_1^T(q_k, ..., q_{k+\mathscr{T}})\mathscr{O}_1(q_k, ..., q_{k+\mathscr{T}})$$
$$= \begin{bmatrix} \mathscr{C}(q_k) \\ \mathscr{C}(q_{k+1})\mathscr{A}(q_k) \\ \cdot \\ \cdot \\ \cdot \\ \mathscr{C}(q_{k+\mathscr{T}})\Pi_{t=k}^{k+\mathscr{T}-1}\mathscr{A}(q_t) \end{bmatrix}^T \begin{bmatrix} \mathscr{C}(q_k) \\ \mathscr{C}(q_{k+1})\mathscr{A}(q_k) \\ \cdot \\ \cdot \\ \cdot \\ \mathscr{C}(q_{k+\mathscr{T}})\Pi_{t=k}^{k+\mathscr{T}-1}\mathscr{A}(q_t) \end{bmatrix}$$

Since the system (4.48)-(4.49) is weekly observable and the Markov chain $\{q_k\}$ is irreducible, there always exist a k and a finite $\mathscr{T}$ such that rank $\mathscr{O}_1(q_k, ..., q_{k+\mathscr{T}}) = n$. Thus, from (4.56), we get that $\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2\Big|\mathcal{I}_0\Big] = 0$. □

4.4 Numerical Example

Let us consider an MJLS with the following system parameters:

$$\mathcal{D} = \{1, 2\},$$

$$A(1) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \ B(1) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ D_1(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ D(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \ B(2) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ D_1(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C(2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; D(2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

State transition matrix for the Markov chain $\{r_k\}$ is given by:

$$\Delta = \begin{bmatrix} 0.45 & 0.55 \\ 0.4 & 0.6 \end{bmatrix}.$$

Also, consider the following parameters:

$$W(1) = W(2) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and

$$R(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; R(2) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

In order to demonstrate the influence of the control packet arrival probabilities on the infinite horizon optimal cost, we compute the optimal cost for different horizon. With $x_0 = [0.1 \; 0.2 \; 0.3]^T$ as the initial state vector, optimal cost is computed using (4.27). Figure 4.1 shows the behavior of the optimal cost $J_N(\zeta^*_{0:N-1}, \eta^*_{0:N-1})$ with the probabilities $\bar{\nu}^1 = 0.88$, $\bar{\nu}^2 = 0.86$, $\bar{\mu}^1 = 0.89$, $\bar{\mu}^2 = 0.87$ and $\bar{\nu}^1 = 0.82$, $\bar{\nu}^2 = 0.81$, $\bar{\mu}^1 = 0.83$, $\bar{\mu}^2 = 0.85$. One can observe that as packet arrival probabilities are reduced, the optimal cost $J_N(\zeta^*_{0:N-1}, \eta^*_{0:N-1})$ converges to a higher value. If the probabilities are reduced to $\bar{\nu}^1 = 0.72$, $\bar{\nu}^2 = 0.76$, $\bar{\mu}^1 = 0.77$, $\bar{\mu}^2 = 0.67$, it can be observed, as shown in Figure 4.2, that the optimal cost $J_N(\zeta^*_{0:N-1}, \eta^*_{0:N-1})$ does not converge. In Figure 4.3, the dependence of the critical value of the H_∞ disturbance attenuation level γ_c on $\bar{\nu}^1$ is demonstrated while keeping $\bar{\nu}^2 = 0.85$, $\bar{\mu}^1 = 0.82$ and $\bar{\mu}^2 = 0.8$. Similarly, Figure 4.4 shows the variation of γ_c with respect to $\bar{\mu}^1$ while keeping $\bar{\nu}^1 = 0.85$, $\bar{\nu}^2 = 0.83$, $\bar{\mu}^2 = 0.82$. The regions above the curve in Figure 4.3 and Figure 4.4 correspond to the feasible region for

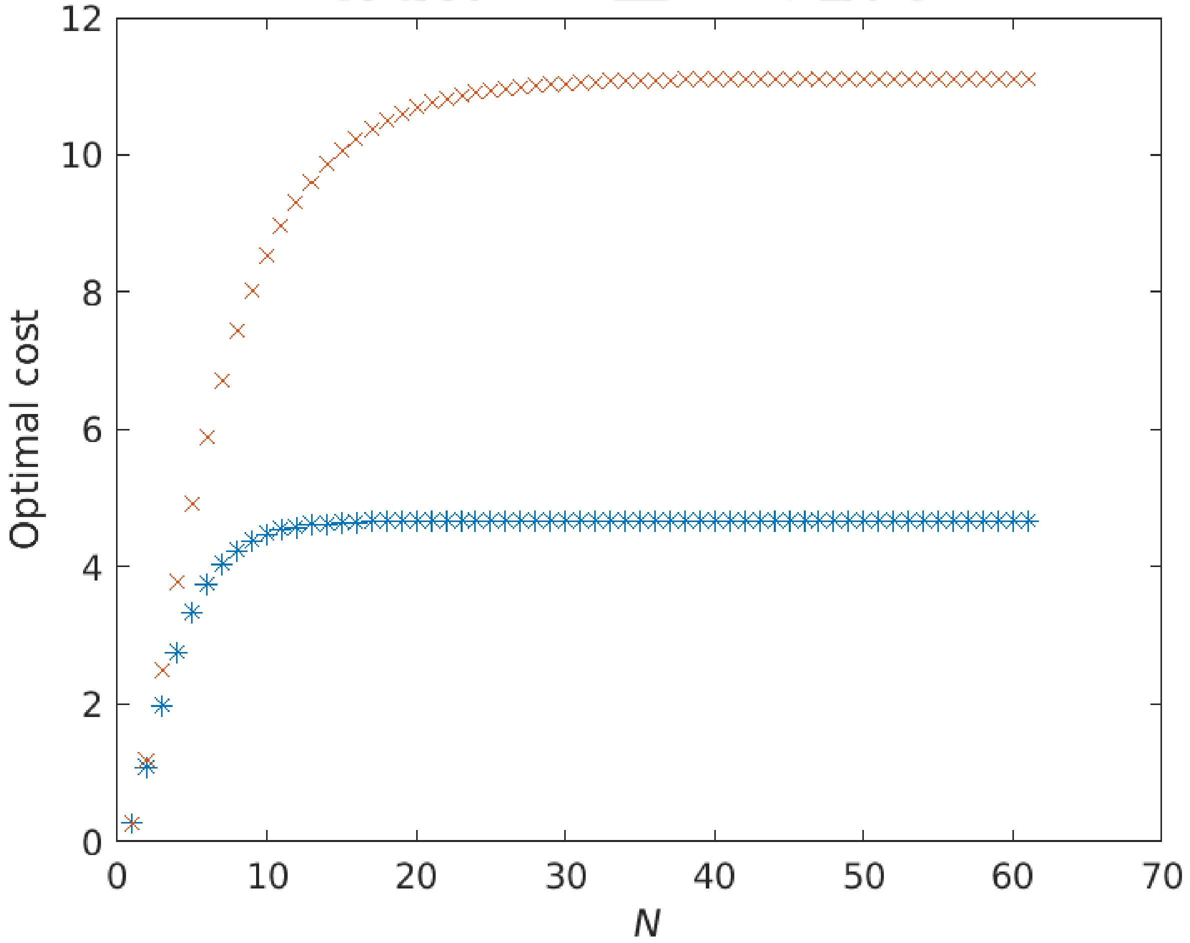

Figure 4.1: Optimal cost with $\bar{\nu}^1 = 0.88$, $\bar{\nu}^2 = 0.86$, $\bar{\mu}^1 = 0.89$, $\bar{\mu}^2 = 0.87$ (blue graph) and $\bar{\nu}^1 = 0.82$, $\bar{\nu}^2 = 0.81$, $\bar{\mu}^1 = 0.83$, $\bar{\mu}^2 = 0.85$ (orange graph).

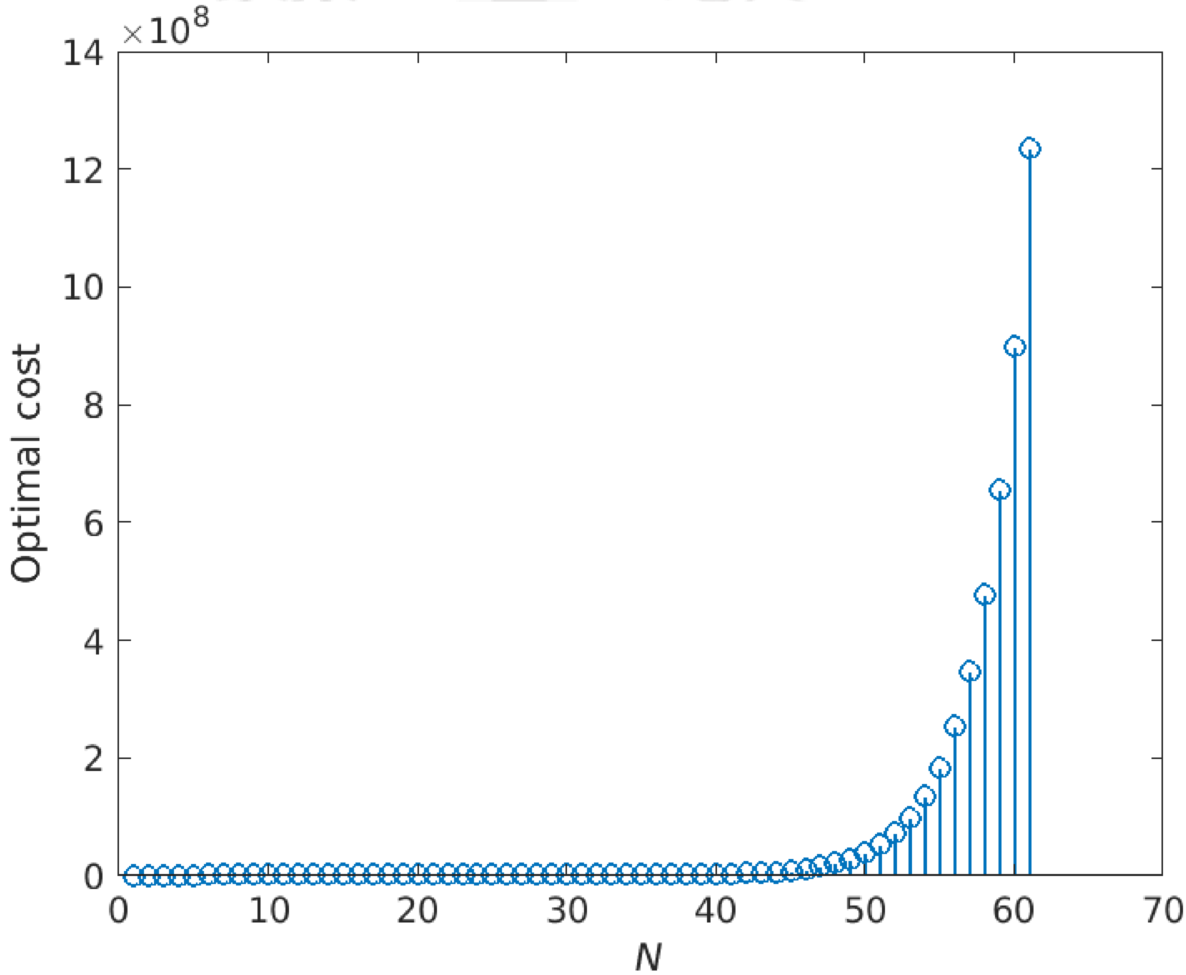

Figure 4.2: Optimal cost with $\bar{v}^1 = 0.72$, $\bar{v}^2 = 0.76$, $\bar{\mu}^1 = 0.77$, $\bar{\mu}^2 = 0.67$.

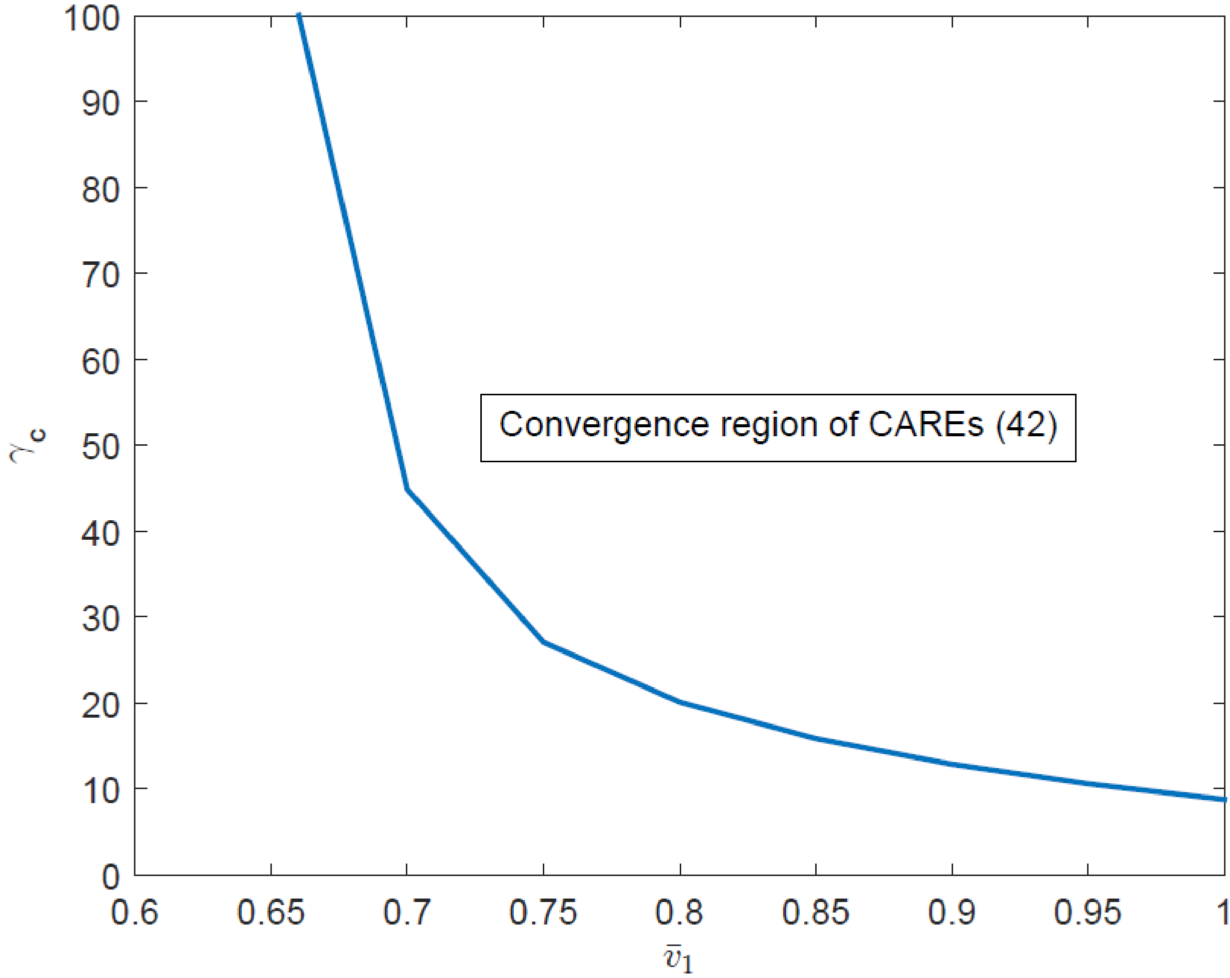

Figure 4.3: Variation of γ_c with different $\bar{v}^1$, and $\bar{v}^2 = 0.85$, $\bar{\mu}^1 = 0.82$, $\bar{\mu}^2 = 0.8$.

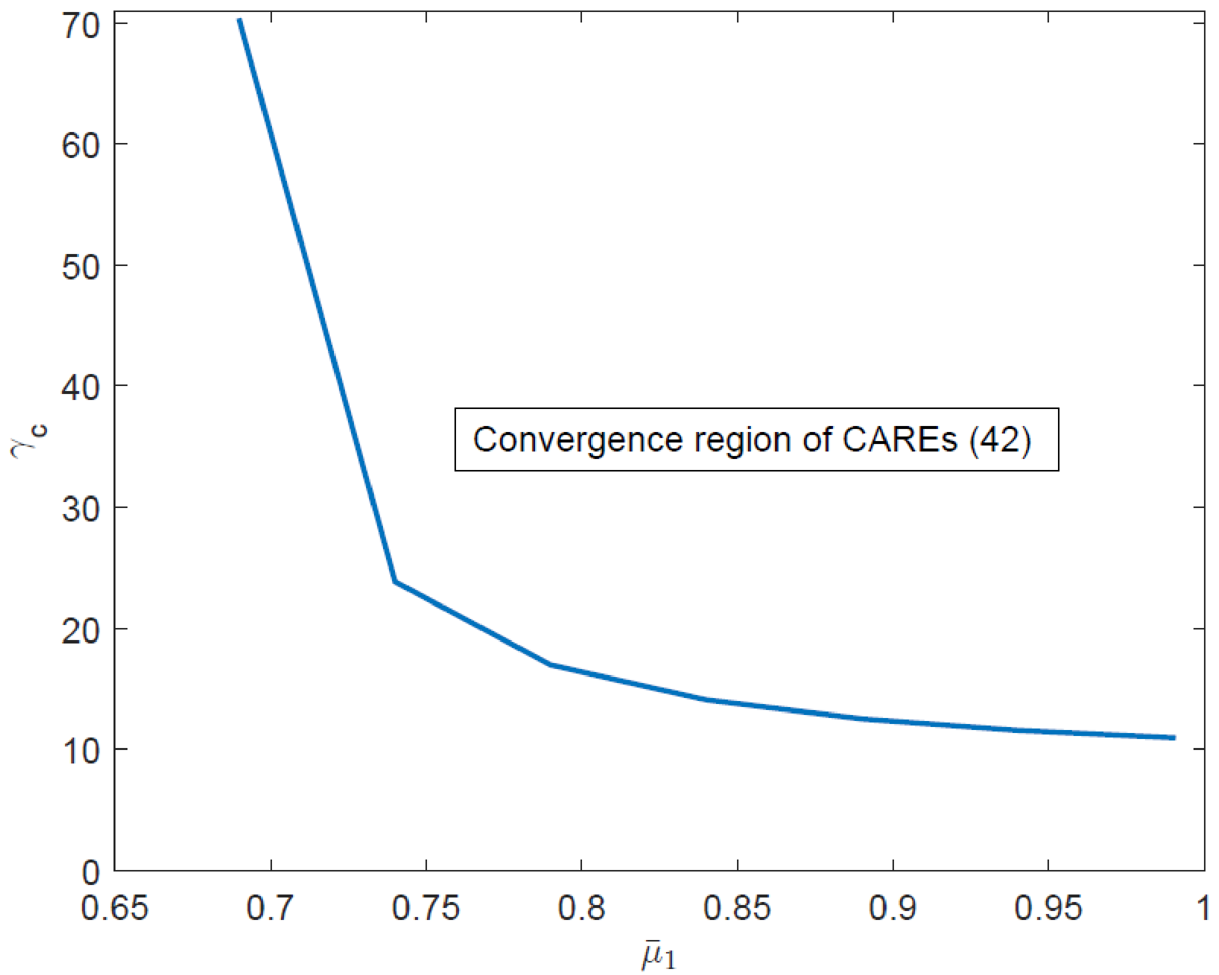

Figure 4.4: Variation of γ_c with different $\bar{\mu}^1$, and $\bar{v}^1 = 0.85$, $\bar{v}^2 = 0.83$, $\bar{\mu}^2 = 0.82$.

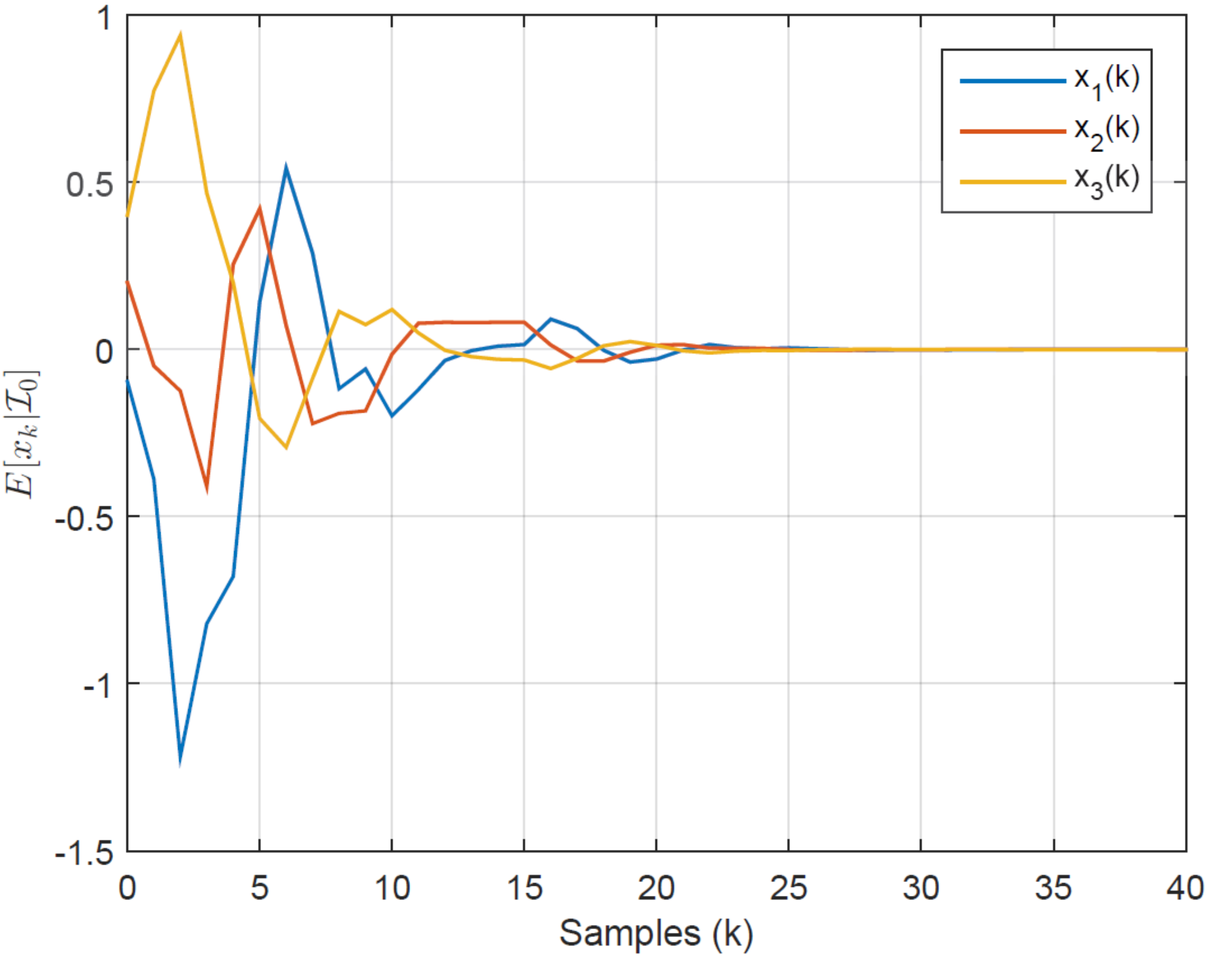

Figure 4.5: Expected value of the state response with the optimal control law for $\bar{\nu}^1 = 0.81$, $\bar{\nu}^2 = 0.8$, $\bar{\mu}^1 = 0.81$, $\bar{\mu}^2 = 0.79$.

convergence of the CAREs (4.8). Figure 4.3 and Figure 4.4 suggest that as the control packet arrival probabilities ($\bar{\nu}^1$ and $\bar{\mu}^1$, respectively) reduce, γ_c goes on increasing. Further, below a critical value of the control packet arrival probability, CAREs (4.39) do not admit any unique fixed-point solution. Expected value of the state responses with the optimal controller is shown in Figure 4.5 with disturbance input $w_k = sin(0.2\pi k)cos(0.2\pi k)e^{-k/2}$. With a persistent bounded disturbance $w_k = 0.02sin(0.2\pi k)cos(0.2\pi k)$ expected value of the state responses are shown in Figure 4.6. One can see that with the optimal control law the closed-loop responses are bounded for a persistent and bounded input. Figure 4.7 shows the state response of the closed-loop system without taking mathematical expectation with disturbance $w_k = 0.02sin(0.2\pi k)cos(0.2\pi k)$. To get the response shown in Figure 4.7, we have first simulated two Markov chains for two independent channels. The control arrival probabilities are $\bar{\nu}^1 = 0.81$, $\bar{\nu}^2 = 0.8$, $\bar{\mu}^1 = 0.81$, $\bar{\mu}^2 = 0.79$. Unlike in Figure 4.6, one can see random fluctuations in Figure 4.7 due to random packet losses.

Let us now consider an MJLS with system parameters (except for $A(2)$) same as those given at the starting of section 4.4. $A(2)$ is given as below:

$$A(2) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Now, one can easily verify that $(A(2), C(2))$ is not observable. Thus, from the analysis given in [55], stability can not be guaranteed. On the other hand, as the system is clearly weakly observable (since $(A(1), C(1))$ is observable), from Theorem 4.3.10 stability can be guaranteed. Figure 4.8 shows that the closed-loop system is stable.

4.5 Summary

In this chapter, we have designed the optimal H_∞ controller for a Markovian jumped linear system over multiple communication channels. Existence conditions for the finite-horizon controller are derived. It is observed that the convergence of the infinite-horizon cost function

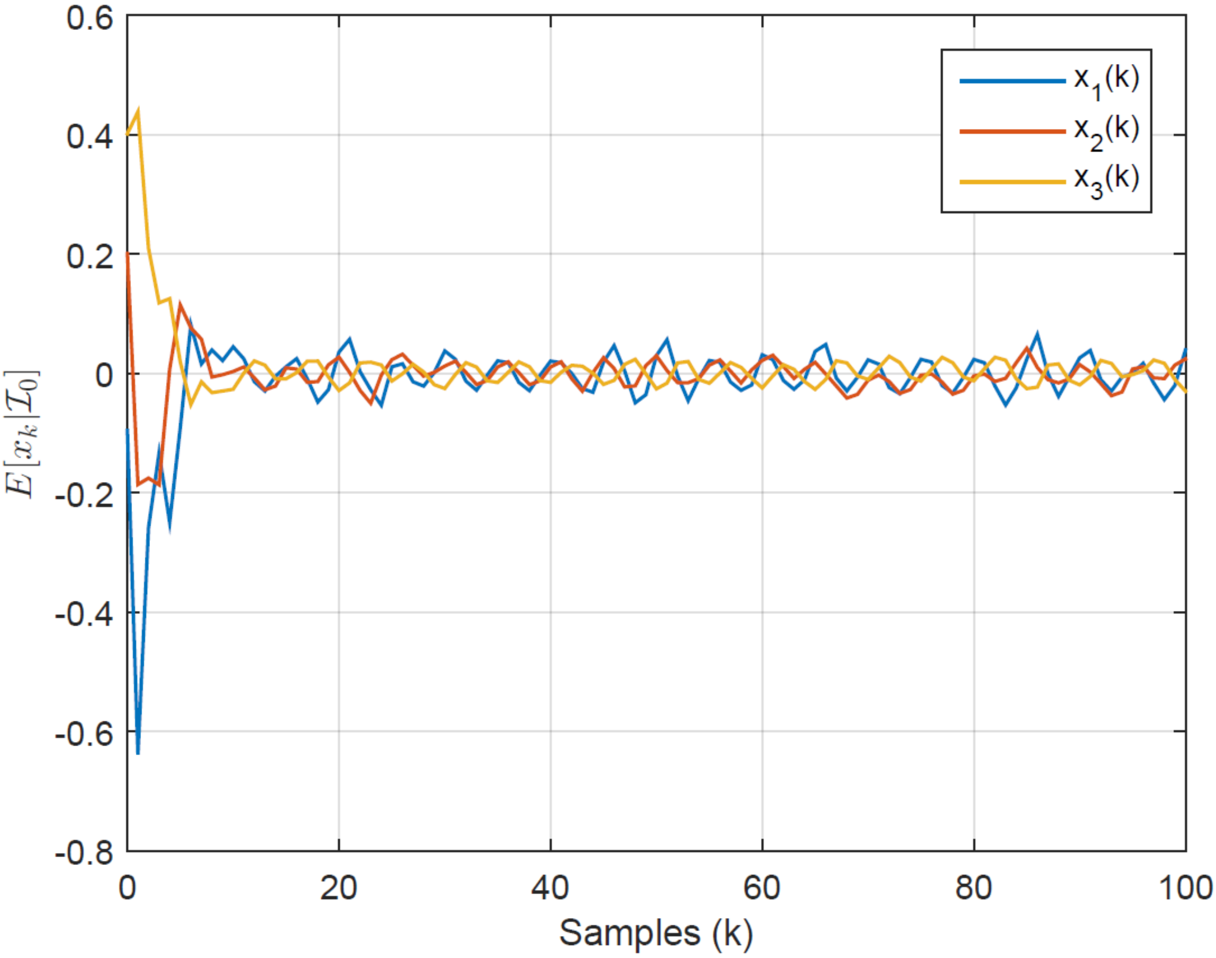

Figure 4.6: Expected value of the state response with the optimal control law and persistent disturbance for $\bar{v}^1 = 0.81$, $\bar{v}^2 = 0.8$, $\bar{\mu}^1 = 0.81$, $\bar{\mu}^2 = 0.79$.

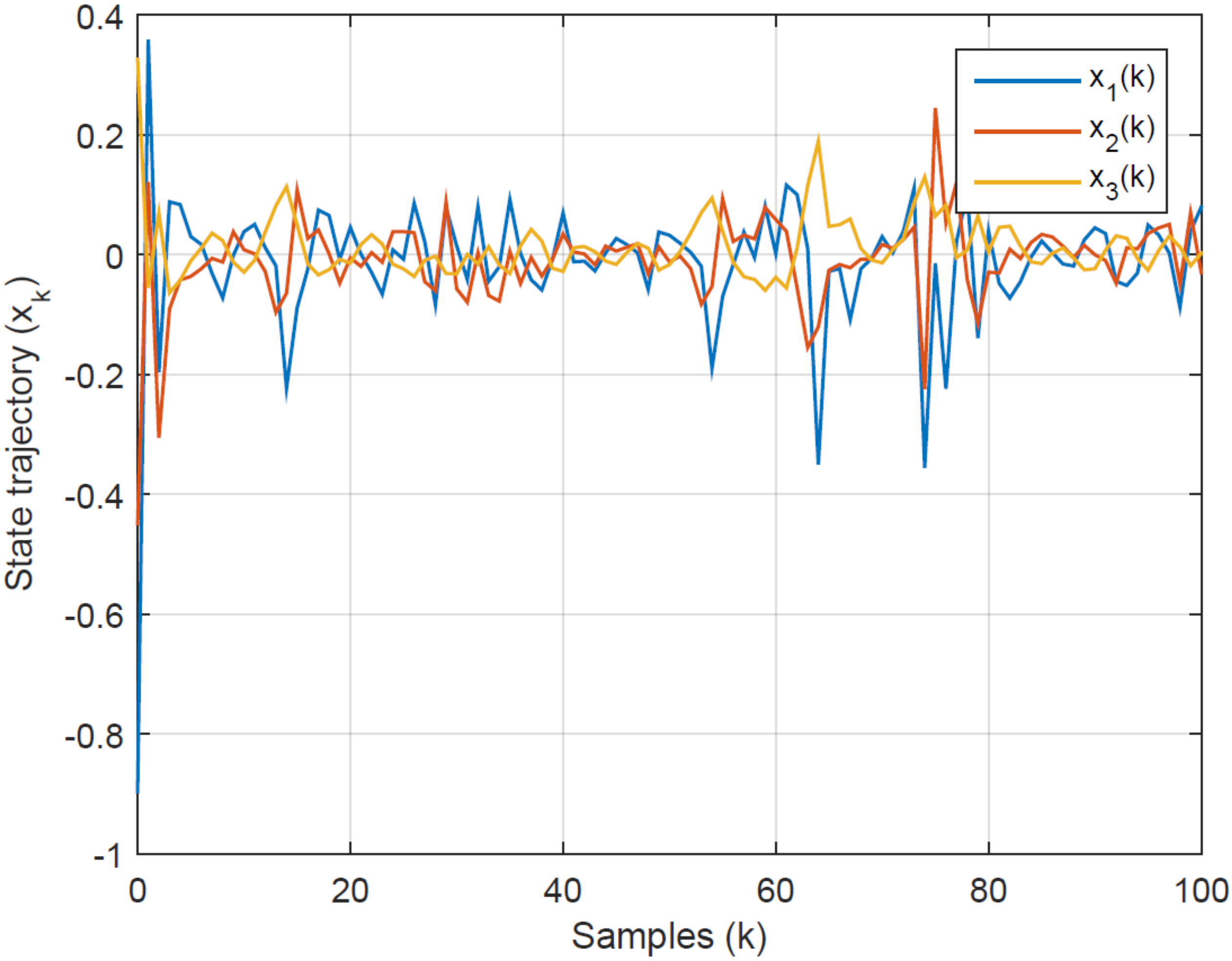

Figure 4.7: State response with the optimal control law and persistent disturbance for $\bar{v}^1 = 0.81$, $\bar{v}^2 = 0.8$, $\bar{\mu}^1 = 0.81$, $\bar{\mu}^2 = 0.79$.

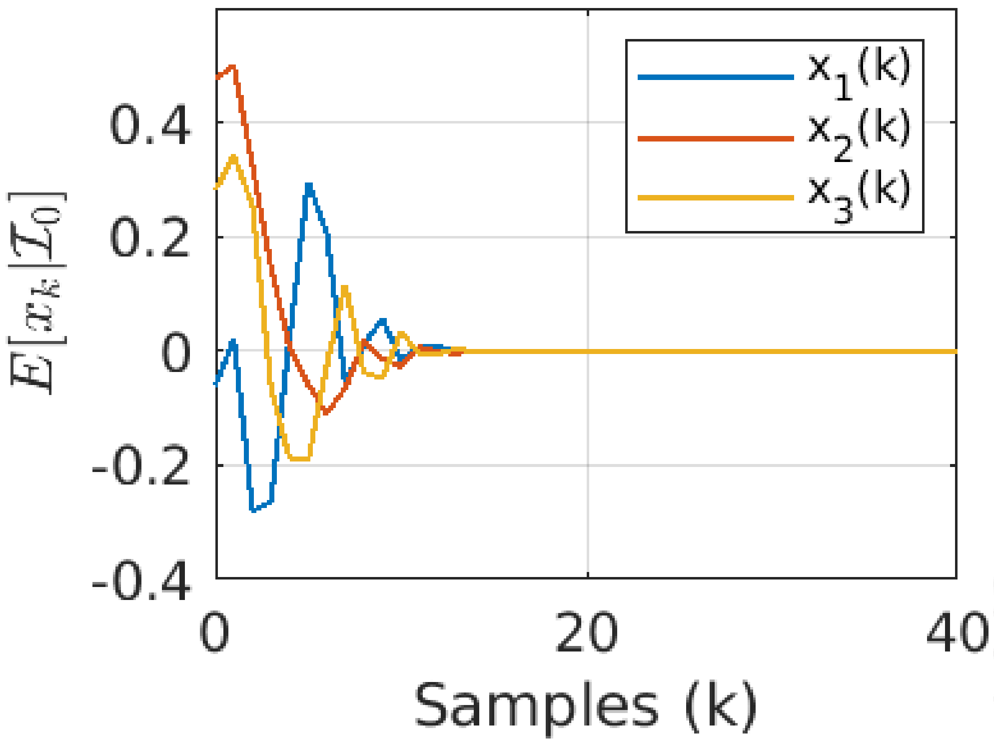

Figure 4.8: State response with the optimal control law and persistent disturbance for $\bar{\nu}^1 = 0.98$, $\bar{\nu}^2 = 0.96$, $\bar{\mu}^1 = 0.99$, $\bar{\mu}^2 = 0.97$.

depends on the control packet arrival probabilities. Stability of the closed-loop system with the optimal controller in the face of random packet loss has also been established.

5

H_∞ control of smooth nonlinear systems over lossy channel

5.1 Introduction

In this chapter, we extend the problem of H_∞ controller design over erasure channel to smooth nonlinear systems by considering a piecewise affine (PWA) approximation approach. The intuition for using such an approach stems from the fact that any smooth map can be locally approximated by an affine map with an arbitrary accuracy [35].

Firstly, by considering a PWA approximation approach, we derive conditions for a smooth autonomous nonlinear system to have an $\mathcal{L}_2$ gain less than or equal to a prescribed $\gamma > 0$ with internal stability. Then, a piecewise linear state-feedback H_∞ controller is designed for a smooth nonlinear system. Finally, we extend these results to a smooth nonlinear system with random control packet losses. Results pertaining to the H_∞ controller design problem of a PWA system with random packet losses can easily be derived as a special case of the final result of this chapter.

For the PWA approximation approach, the state-space is partitioned into a number of polyhedral cells. Each nonlinear function is then approximated by a distinct affine function in each of the cells. One of the main features of the PWA approach, presented in this thesis, is that it enables us to design the controllers by solving certain linear matrix inequalities (LMIs), which are subject to certain additional constraints. Although the results for the controller design can not be completely converted to LMIs, one can get the desired results by solving certain LMIs, and then checking whether the solutions satisfy a few nonlinear constraints. It will be shown that, once the LMIs are solved, it will be trivial to check whether the nonlinear constraints get satisfied or not. As LMIs are easy to solve, the approach thus provides an efficient and systematic way of controller design.

The chapter is structured as follows. Section 5.2 presents the results for the $\mathcal{L}_2$ gain analysis and the H_∞ controller design problem. Section 5.3, contains the results for the problem of H_∞ controller design with packet losses. In section 5.4, we demonstrate our results using a numerical example. Finally, section 5.5 presents the summary.

5.2 The $\mathcal{L}_2$ gain analysis and H_∞ control of smooth nonlinear systems:

In this section, we present results relating to the $\mathcal{L}_2$ gain analysis and the H_∞ control of a smooth nonlinear system.

Consider a smooth discrete-time nonlinear system

$$\begin{aligned} x_{k+1} &= f(x_k) + B_1 u_k + D_1 w_k \\ z_k &= h(x_k) + B_2 u_k + D_2 w_k, \end{aligned} \tag{5.1}$$

wherein $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input to the actuators, $w_k \in l_2([0,\infty), \mathbb{R}^s)$ is the disturbance input to be rejected, $z_k \in \mathbb{R}^p$ is the controlled output, $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^p$ are smooth maps, $B_1 \in \mathbb{R}^{n\times m}$, $D_1 \in \mathbb{R}^{n\times s}$, $B_2 \in \mathbb{R}^{p\times m}$, $D_2 \in \mathbb{R}^{p\times s}$ are constant matrices. The system dynamics given by Equation (5.1) describes many practical systems like missile dynamics [33], power networks [66], permanent magnet synchronous motors [67], and nonlinear RLC circuits [68].

The main goal of this chapter is as follows.

Objective: We design a state-feedback control law of the form $u_k = K(i)x_k$, for $x_k \in \mathscr{X}_i$, that attains the following requirements:

(G.1) The $\mathcal{L}_2$ gain from the disturbance w_k to the controlled output z_k is less than or equal to a certain prescribed $\gamma > 0$, i.e.,

$$\sum_{k=0}^{N} \|z_k\|^2 \le \gamma^2 \sum_{k=0}^{N} \|w_k\|^2, \ \forall N \in \mathbb{Z}^+$$

(G.2) The closed-loop system, with $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense. □

The following notion of stability is in this chapter.

Definition 5.2.1. *System (5.1), $u_k \equiv 0$ and $w_k \equiv 0$, is said to be locally asymptotically stable in the mean-square sense if, for all initial states within a ball around the origin, i.e., $x_0 \in B_\tau(0)$,*

$$\lim_{k\to\infty} \|x_k\|^2 = 0.$$

□

It is well known that any smooth map can be locally approximated by an affine map with arbitrary accuracy [35]. Thus, by considering an appropriate partition of the state space, it is possible to approximate the maps $f(.)$ and $h(.)$ by two affine maps, in each cell of the partition, respectively. In this work, we shall consider a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$ of a subspace $\mathcal{X}$ of the state space, where $\mathscr{X}_i$ denotes a cell and $\mathcal{N}$ is the index set of the cells. One can characterize each polyhedral cell as [35]:

$$E(i)x_k + e(i) \geq 0 \;\; x_k \in \mathscr{X}_i, \text{ or } \bar{E}(i)\bar{x}_k \geq 0, \tag{5.2}$$

where $\bar{E}(i) := [E(i) \;\; e(i)]$, $\bar{x}(k) := \begin{bmatrix} x(k) \\ 1 \end{bmatrix}$. If $\mathcal{N}_0$ denotes the cells that contains the origin, we have that $e(i) = 0$, for all $i \in \mathcal{N}_0$. Inequality (5.2) denotes that each element of the vector $\bar{E}(i)\bar{x}_k$ is nonnegative.

With a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$, one can find matrices $A(i)$, $C(i)$, and vectors $a(i)$, $c(i)$ for all $i \in \mathcal{N}$ such that $f(x_k)$ and $h(x_k)$ can be approximated by piecewise affine functions. Let, for $x_k \in \mathscr{X}_i$, $m(x_k, i)$ and $n(x_k, i)$ be the error terms in the approximations of $f(x_k)$ and $h(x_k)$, respectively, i.e.,

$$f(x_k) = A(i)x_k + a(i) + m(x_k, i)$$

$$h(x_k) = C(i)x_k + c(i) + n(x_k, i).$$

Note that, for a cell $i \in \mathcal{N}_0$, we have $a(i) = 0$ and $c(i) = 0$.

Let $\epsilon(i)$ and $\delta(i)$ are such that:

$$
\begin{aligned}
&\|f(x_k) - A(i)x_k - a(i)\| = \|m(x_k, i)\| \leq \epsilon(i)\|x_k\| \\
&\|h(x_k) - C(i)x_k - c(i)\| = \|n(x_k, i)\| \leq \delta(i)\|x_k\| \\
&\text{for } x_k \in \mathscr{X}_i \text{ with } \; a(i) = 0 \text{ and } c(i) = 0 \text{ if } i \in \mathcal{N}_0,
\end{aligned}
\tag{5.3}
$$

There are many ways to find the matrices $A(i)$ and $C(i)$, for each $i \in \mathcal{N}$. For example one can use Taylor series expansion.

With the help of Equation (5.3), the nonlinear system (5.1) can be expressed as follows:

$$
\begin{aligned}
x_{k+1} &= \bar{A}(i)\bar{x}_k + B_1 u_k + D_1 w_k + m(x_k, i) \\
z_k &= \bar{C}(i)\bar{x}_k + B_2 u_k + D_2 w_k + n(x_k, i),
\end{aligned}
\tag{5.4}
$$

where

$$
\bar{A}(i) = \begin{bmatrix} A(i) & a(i) \end{bmatrix}, \quad \bar{C}(i) := \begin{bmatrix} C(i) & c(i) \end{bmatrix}.
\tag{5.5}
$$

System (5.4) can further be expressed in terms of the variable $\bar{x}_k$ as:

$$
\begin{aligned}
\bar{x}_{k+1} &= \hat{A}(i)\bar{x}_k + \hat{B}_1 u_k + \hat{D}_1 w_k + \hat{m}(x_k, i) \\
z_k &= \bar{C}(i)\bar{x}_k + B_2 u_k + D_2 w_k + n(x_k, i),
\end{aligned}
\tag{5.6}
$$

where $\hat{A}(i)$, $\hat{B}_1$, $\hat{D}_1$, $\hat{m}(x_k, i)$ are given as:

$$\hat{A}(i) := \begin{bmatrix} A(i) & a(i) \\ 0_{1\times n} & 1 \end{bmatrix}, \; \hat{m}(x_k, i) := \begin{bmatrix} m(x_k, i) \\ 0 \end{bmatrix}, \; \hat{B}_1(i) := \begin{bmatrix} B_1(i) \\ 0_{1\times m} \end{bmatrix}, \; \hat{D}_1(i) := \begin{bmatrix} D_1(i) \\ 0_{1\times s} \end{bmatrix}. \tag{5.7}$$

The theory of dissipativity has played a central role in solving the H_∞ control problem [30, 32]. Our approach shall also be based on dissipativity. In the sequel, we will utilize the following local notion of dissipativity, which is motivated by the one defined in [50].

Definition 5.2.2. *Consider system (5.1) with $u_k \equiv 0$, $w_k \in \mathcal{W} \subseteq \mathbb{R}^s$ and $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, such that the state trajectories, with every disturbance input $w_k \in \mathcal{W}$, always remain in $\mathcal{X}$. Now, system (5.1), with $u_k \equiv 0$, is said to be locally dissipative with respect to a supply rate $s(z_k, w_k)$, if there exists a nonnegative function $V : \mathcal{X} \to \mathbb{R}^+$ with $V(0) = 0$, called the storage function, such that for all $w \in \mathcal{W}$, $x_k \in \mathcal{X}$, and for all $k \in \mathbb{Z}^+$,*

$$V(x_{k+1}) - V(x_k) \le s(z_k, w_k). \tag{5.8}$$

□

One can relate local dissipativity and the $\mathcal{L}_2$ gain analysis by the following lemma, which is in line with the one found in [32, 69].

Lemma 5.2.1. *Suppose that system (5.1), with $u_k \equiv 0$ and $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense. Now, if system (5.1), with $u_k \equiv 0$, is locally dissipative with respect to the supply rate $s(z_k, w_k) = \frac{1}{2}\left(\gamma^2\|w_k\|^2 - \|z_k\|^2\right)$, then it has the $\mathcal{L}_2$ gain less than or equal to γ.* □

As a PWA approximation of the nonlinear system is used in this work, we consider a piecewise quadratic storage function in order to study the dissipativity. The storage function is considered to be of the following form.

$$V(x_k) = \frac{1}{2}\bar{x}_k^T \bar{P}(i)\bar{x}_k, \text{ if } x_k \in \mathscr{X}_i, \tag{5.9}$$

where $\bar{P}(i)$ is such that $V(x_k) > 0$ for all $x_k \neq 0$, and $V(x_k) = 0$ if $x_k = 0$. Further, if $i \in \mathcal{N}_0$, then $\bar{P}(i)$ is of the form $\bar{P}(i) = diag\{P(i), 0\}$. For each $i \in \mathcal{N}_0$, $P(i)$ is chosen such that $x_k^T P(i) x_k > 0$ for all $x_k \neq 0$.

Note that for the storage function $V(x_k)$ to be positive, for all $x_k \neq 0$, it is not necessary for the $\bar{P}(i)$s to be positive definite (or positive semidefinite if $i \in \mathcal{N}_0$). Instead $\bar{x}_k^T \bar{P}(i)\bar{x}_k$ should be greater than zero only for those x_k that belong to the corresponding $\mathscr{X}_i$.

Before going on to present the results for the H_∞ controller design problem, we present the conditions for system (5.1), with $u_k \equiv 0$, to have the $\mathcal{L}_2$ gain less than or equal to a prescribed $\gamma > 0$ with internal stability as follows.

Theorem 5.2.2. *Consider system (5.1), with $u_k \equiv 0$, a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i \in \mathcal{N}}$ of a subspace $\mathcal{X}$ of the stae-space with (5.3), and the disturbance input $w_k \in \mathcal{W}$ for all k. Suppose, there exist symmetric matrices $W(i)$ and $R(i)$ with nonnegative entries, and symmetric matrices $\bar{P}(i)$ such that the following inequalities are satisfied for all $i, j \in \mathcal{N}$*

$$\begin{gathered} \bar{P}(i) - \bar{E}^T(i)R(i)\bar{E}(i) > 0 \\ \begin{bmatrix} \Lambda_{11}(i,j) & \Lambda_{12}(i,j) \\ \Lambda_{12}^T(i,j) & \Lambda_{22}(i,j) \end{bmatrix} \leq 0, \end{gathered} \tag{5.10}$$

where

$$\bar{P}(i) = [I_n \;\; 0_{n\times 1}]^T P(i)[I_n \;\; 0_{n\times 1}] \quad \text{if} \quad i \in \mathcal{N}_0$$

$$\Lambda_{11} = \hat{A}^T(i)\bar{P}(j)\hat{A}(i) - \bar{P}(i) + \bar{C}^T(i)\bar{C}(i) + \bar{E}^T(i)W(i)\bar{E}(i)$$

$$+ (\rho_1(i,j) + \rho_2(i,j) + \rho_3(i,j) + \rho_4(i) + \rho_5(i) + \rho_6(i))I_{n+1}$$

$$\Lambda_{12} = \hat{A}^T(i)\bar{P}(j)\hat{D}_1(i) + \bar{C}^T(i)D_2(i)$$

$$\Lambda_{22} = \hat{D}_1^T\bar{P}(j)\hat{D}_1 + D_2^T(i)D_2(i) - \gamma^2 I_s + (\rho_2(i,j) + \rho_5(i))I_s$$

$$\rho_1(i,j) = 2\epsilon(i)\|\bar{P}(j)\|\|\hat{A}(i)\|$$

$$\rho_2(i,j) = \epsilon(i)\|\bar{P}(j)\|\|\hat{D}_1\|$$

$$\rho_3(i,j) = \epsilon^2(i)\|\bar{P}(j)\|$$

$$\rho_4(i) = 2\delta(i)\|\bar{C}(i)\|$$

$$\rho_5(i) = \delta(i)\|D_2(i)\|$$

$$\rho_6(i) = \delta^2(i)$$

Then, system (5.1), with $u_k \equiv 0$ and $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense. Further, with $u \equiv 0$, it has an $\mathcal{L}_2$ gain less than or equal to γ.

Proof: With a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$, one can find matrices $E(i)$ and $\bar{E}(i)$, for $i \in \mathcal{N}$, such that $E(i)x_k \geq 0$ and $\bar{E}(i)\bar{x}_k \geq 0$. Now, assume that there exist symmetric matrices $W(i)$ and $R(i)$ with positive entries, and symmetric matrices $\bar{P}(i)$ such that the inequalities given in (5.10) are satisfied for all $i, j \in \mathcal{N}$.

In order to show dissipativity, we consider a storage function as given in (5.9). Then, using

(5.6) with $u_k \equiv 0$, one gets:

$$
\begin{aligned}
&V(x_{k+1}) - V(x_k) + \frac{1}{2}[\|z_k\|^2 - \gamma^2\|w_k\|^2] \\
&= \frac{1}{2}[\bar{x}_{k+1}^T\bar{P}(j)\bar{x}_{k+1} - \bar{x}_k^T\bar{P}(i)\bar{x}_k + z_k^T z_k - \gamma^2 w_k^T w_k] \\
&= \frac{1}{2}\Big[(\hat{A}(i)\bar{x}_k + \hat{m}(x_k,i) + \hat{D}_1 w_k)^T\bar{P}(j)\Big(\hat{A}(i)\bar{x}_k + \hat{m}(x_k,i) + \hat{D}_1 w_k\Big) - \bar{x}_k^T\bar{P}(i)\bar{x}_k \\
&+ (\bar{C}(i)\bar{x}_k + n(x_k,i) + D_2 w_k)^T(\bar{C}(i)\bar{x}_k + n(x_k,i) + D_2 w_k) - \gamma^2 w_k^T w_k\Big] \\
&= \frac{1}{2}\Big[(\hat{A}(i)\bar{x}_k + \hat{D}_1 w_k)^T\bar{P}(j)\Big(\hat{A}(i)\bar{x}_k + \hat{D}_1 w_k\Big) - \bar{x}_k^T\bar{P}(i)\bar{x}_k + (\bar{C}(i)\bar{x}_k + D_2 w_k)^T(\bar{C}(i)\bar{x}_k + D_2 w_k) - \gamma^2 w_k^T w_k \\
&+ 2\bar{x}_k^T\hat{A}^T(i)\bar{P}(j)\hat{m}(x_k,i) + 2w_k^T\hat{D}_1^T\bar{P}(j)\hat{m}(x_k,i) + \hat{m}^T(x_k,i)\bar{P}(j)\hat{m}(x_k,i) + 2\bar{x}_k^T\bar{C}^T(i)n(x_k,i) \\
&+ 2w_k^T D_2^T n(x_k,i) + n^T(x_k,i)n(x_k,i)\Big].
\end{aligned}
\tag{5.11}
$$

Due to the properties of 2-norm and induced 2-norm, the following holds true.

$$
\begin{aligned}
2\bar{x}_k^T\hat{A}(i)\bar{P}(j)\hat{m}(x_k,i) &\le \|2\bar{x}_k^T\hat{A}(i)\bar{P}(j)\hat{m}(x_k,i)\| \\
&\le 2\|\bar{P}(j)\|\,\|\hat{A}(i)\|\,\|\hat{m}(x_k,i)\|\,\|\bar{x}_k\| \\
&\le 2\|\bar{P}(j)\|\,\|\hat{A}(i)\|\,\epsilon(i)\,\|\bar{x}_k\|^2 \\
&\quad \text{(using (5.3) with the fact that } \|\hat{m}(x_k,i)\| = \|m(x_k,i)\| \text{ and } \|x_k\| < \|\bar{x}_k\|) \\
&= \rho_1(i,j)\|\bar{x}_k\|^2.
\end{aligned}
\tag{5.12}
$$

And,

$$2w_k^T\hat{D}_1^T\bar{P}(j)\hat{m}(x_k,i) \le 2\epsilon(i)\|\bar{P}(j)\|\,\|\hat{D}_1\|\,\|\bar{x}_k\|\,\|w_k\|$$

Note that $2\|\bar{x}_k\|\,\|w_k\| \leq (\|\bar{x}_k\|^2\| + \|w_k\|^2)$. Thus,

$$\begin{aligned} 2w_k^T\hat{D}_1^T\bar{P}(j)\hat{m}(x_k,i) &\leq 2\epsilon(i)\|\bar{P}(j)\|\,\|\hat{D}_1\|\|\bar{x}_k\|\,\|w_k\| \\ &\leq \epsilon(i)\,\|\bar{P}(j)\|\,\|\hat{D}_1\|\,(\|\bar{x}_k\|^2 + \|w_k\|^2) \\ &\leq \rho_2(i,j)\,(\|\bar{x}_k\|^2 + \|w_k\|^2) \end{aligned} \tag{5.13}$$

Also,

$$\begin{aligned} \hat{m}^T(x_k,i)\bar{P}(j)\hat{m}(x_k,i) &\leq \|\hat{m}^T(x_k,i)\bar{P}(j)\hat{m}(x_k,i)\| \\ &\leq \|\hat{m}(x_k,i)\|\,\|\bar{P}(j)\|\,\|\hat{m}(x_k,i)\| \\ &\leq \epsilon^2(i)\,\|\bar{P}(j)\|\,\|x_k\|^2 \\ &\leq \rho_3(i,j)\,\|\bar{x}_k\|^2. \end{aligned} \tag{5.14}$$

Similarly, one can show the following:

$$2\bar{x}_k^T\bar{C}^T(i)n(x_k,i) \leq \rho_4(i)\,\|\bar{x}_k\|^2 \tag{5.15}$$

$$2w_k^TD^T(i)n(x_k,i) \leq \rho_5(i)\,(\|\bar{x}_k\|^2 + \|w_k\|^2) \tag{5.16}$$

$$n^T(x_k,i)n(x_k,i) \leq \rho_6(i)\,\|\bar{x}_k\|^2 \tag{5.17}$$

Now, using (5.12), (5.13), (5.14), (5.15), (5.16), and (5.17) in (5.11):

$$\begin{aligned} &V(x_{k+1}) - V(x_k) + \frac{1}{2}[\|z_k\|^2 - \gamma^2\|w_k\|^2] \\ &\leq \frac{1}{2}\begin{bmatrix}\bar{x}_k \\ w_k\end{bmatrix}^T \begin{bmatrix}\Lambda_{11}(i,j) & \Lambda_{12}(i,j) \\ \Lambda_{12}^T(i,j) & \Lambda_{22}(i,j)\end{bmatrix}\begin{bmatrix}\bar{x}_k \\ w_k\end{bmatrix} \end{aligned} \tag{5.18}$$

Thus, if (5.10) gets satisfied, we get $V(x_{k+1}) - V(x_k) - \frac{1}{2}[\gamma^2\|w_k\|^2 - \|z_k\|^2] \leq 0$, i.e., $V(x_{k+1}) - V(x_k) \leq \frac{1}{2}[\gamma^2\|w_k\|^2 - \|z_k\|^2]$, for all $w_k \in \mathcal{W}$ and k. This implies that system (5.1), with $u_k \equiv 0$, is locally dissipative with respect to the supply rate $s(z_k, w_k) = \frac{1}{2}\left[\gamma^2\|w_k\|^2 - \|z_k\|^2\right]$.

We now proceed to show stability of system (5.1), with $u_k \equiv 0$, $w_k \equiv 0$. One can easily verify that $V(x_k) = \frac{1}{2}\bar{x}_k \bar{P}(i)\bar{x}_k$, for $x_k \in \mathscr{X}_i$, is well suited to be a Lyapunuov function. Now, using system dynamics (5.6), with $u_k \equiv 0$, $w_k \equiv 0$, one gets:

$$\begin{aligned} & V(x_{k+1}) - V(x_k) \\ & = \frac{1}{2}\Big[\Big(\hat{A}(i)\bar{x}_k + \hat{m}(x_k, i)\Big)^T \bar{P}(j)\Big(\hat{A}(i)\bar{x}_k + \hat{m}(x_k, i)\Big) - \bar{x}_k^T \bar{P}(i)\bar{x}_k\Big] \\ & = \frac{1}{2}\Big[\bar{x}_k^T \hat{A}^T(i)\bar{P}(j)\hat{A}(i)\bar{x}_k - \bar{x}_k^T \bar{P}(i)\bar{x}_k + 2\bar{x}_k^T \hat{A}^T(i)\bar{P}(j)\hat{m}(x_k, i) + \hat{m}^T(x_k, i)\bar{P}(j)\hat{m}(x_k, i)\Big] \end{aligned} \tag{5.19}$$

Then, (5.19) implies:

$$V(x_{k+1}) - V(x_k) \leq \frac{1}{2}\Big[\bar{x}_k^T\Big(\hat{A}^T(i)\bar{P}(j)\hat{A}(i) - \bar{P}(i) + \rho_1(i, j) + \rho_3(i, j)\Big)\bar{x}_k\Big]. \tag{5.20}$$

If the second inequality given in (5.10) holds true, one gets that $\Lambda_{11}(i, j) \leq 0$ for all $i, j \in \mathcal{N}$. Then,

$$\begin{aligned} & \hat{A}^T(i)\bar{P}(j)\hat{A}(i) - \bar{P}(i) + (\rho_1(i, j) + \rho_3(i, j))I_{n+1} \\ & \leq -(\rho_2(i, j) + \rho_4(i) + \rho_5(i) + \rho_6(i))I_{n+1} \end{aligned}$$

As $\rho_2(i, j)$, $\rho_4(i)$, $\rho_5(i)$ and $\rho_6(i)$ are positive scalars, (5.20) implies that $V(x_{k+1}) - V(x_k) < 0$ for all $x_k \in \mathcal{X}$. Thus, system (5.1), with $u_k \equiv 0$ and $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense. Further, Lemma (5.2.1) implies that the $\mathcal{L}_2$ gain from w_k to z_k is less than or equal to γ. □

Remark 5.2.1. *Note that, for a specific $i \in \mathcal{N}$, the conditions given by (5.10) need to be satisfied for all $j \in \mathcal{N}$. This is due to the fact that one may not have any knowledge about the external input w_k. Consequently, x_{k+1} is unknown, and hence the cell where x_{k+1} lies in is also unkown.*

□

In what follows next, we derive sufficient conditions which ensure that system (5.1), with a piecewise linear state-feedback control law, has the $\mathcal{L}_2$ gain less than or equal to γ while maintaining the closed-loop stability.

Now, with the control law of the type $u_k = K(i)x_k$, for $x_k \in \mathscr{X}_i$, using (5.4), one can express system (5.1) system as follows:

$$\begin{aligned} x_{k+1} &= \bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k, i) \\ z_k &= \bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k + n(x_k, i), \end{aligned} \tag{5.21}$$

where $\bar{\mathscr{A}}_K(i) = \begin{bmatrix} A(i) + B_1 W(i)U^{-1}(i) & a(i) \end{bmatrix}$ and $\bar{\mathscr{C}}_K(i) = \begin{bmatrix} C(i) + B_2 W(i)U^{-1}(i) & c(i) \end{bmatrix}$.

For the design of the H_∞ controller, we consider a storage function of the form $V(x_k) = \frac{1}{2}x_k^T P(i)x_k$, where, $P(i)$ is a positive definite matrix. Although, the storage given by (5.9) is less restrictive than this one, the controller design becomes very complicated with the storage function given by (5.9).

Theorem 5.2.3. *Consider system (5.1) with the disturbance input $w_k \in \mathcal{W}$, states $x_k \in \mathcal{X}$, for all k and a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$ equipped with (5.3). Suppose, there exist matrices $T(i) > 0$, $U(i)$, $W(i)$, $R(i) > 0$, and positive scalars γ, q, r, h, g such that for all $i, j \in \mathcal{N}$:*

$$\hat{T}(j) = \begin{bmatrix} T(j) & 0_{n\times 1} \\ 0_{1\times n} & h \end{bmatrix} > 0, \tag{5.22a}$$

$$\Omega(i,j) = \begin{bmatrix} \Omega_{11}(i) & 0_{n\times 1} & 0_{n\times s} & \Omega_{14}(i) & 0_{n\times 1} & \Omega_{16}(i) \\ 0_{1\times n} & \Omega_{22} & 0_{1\times s} & qa^T(i) & q & qc^T(i) \\ 0_{s\times n} & 0_{s\times 1} & \Omega_{33} & D_1^T & 0_{s\times 1} & D_2^T \\ \Omega_{14}^T(i) & qa(i) & D_1 & T(j) & 0_{n\times 1} & 0_{n\times p} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & h & 0_{1\times p} \\ \Omega_{16}^T & qc(i) & D_2 & 0_{p\times n} & 0_{p\times 1} & I_p \end{bmatrix} \geq 0, \tag{5.22b}$$

$$\Big[\rho_1(i,j) + \rho_2(i,j) + \rho_3(i,j) + \rho_4(i) + \rho_5(i) + \rho_6(i)\Big] I_{n+1} \leq \mathscr{L}(i), \tag{5.22c}$$

$$\Big[\rho_2(i,j) + \rho_5(i)\Big] \leq g, \tag{5.22d}$$

where

$$\Omega_{11}(i) = \Big[U(i) + U^T(i) - T(i)\Big] - R(i)$$

$$\Omega_{14}(i) = U^T(i)A^T(i) + W^T(i)B_1^T$$

$$\Omega_{16}(i) = U^T(i)C^T(i) + W^T(i)B_2^T$$

$$\Omega_{22} = 2q - (h + r)$$

$$\Omega_{33} = \gamma^2 I_s - gI_s$$

$$\rho_1(i,j) = 2\epsilon(i)\|\bar{\mathscr{A}}_K(i)\|\|T^{-1}(j)\|$$

$$\rho_2(i,j) = \epsilon(i)\|T^{-1}(j)\|\|D_1\|$$

$$\rho_3(i,j) = \epsilon^2(i)\|T^{-1}(j)\|$$

$$\rho_4(i) = 2\delta(i)\|\bar{\mathscr{C}}_K(i)\|$$

$$\rho_5(i) = \delta(i)\|D_2\|$$

$$\rho_6(i) = \delta^2(i)$$

$$\mathscr{L}(i) = \begin{bmatrix} U^{-T}(i)R(i)U^{-1}(i) & 0_{n\times1} \\ 0_{1\times n} & \frac{r}{q^2} \end{bmatrix}$$

$$\hat{\mathscr{A}}_K(i) := \begin{bmatrix} A(i) + B_1W(i)U^{-1}(i) & a(i) \\ 0_{1\times n} & 1 \end{bmatrix}$$

Then, system (5.1), with $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense with the control law $u_k = K(i)x_k = W(i)U^{-1}(i)x_k$. Further, with the control law $u_k = K(i)x_k = W(i)U^{-1}(i)x_k$, system (5.1) has the $\mathcal{L}_2$ gain, from w_k to z_k, that is less than or equal to a prescribed $\gamma > 0$.

Proof: Suppose, LMIs (5.22a) and (5.22b) are satisfied. Then, as all principal submatrices of a positive semidefinite symmetric matrix are also positive semidefinite, from (5.22b), one gets:

$$\begin{bmatrix} \Omega_{11}(i) & 0_{n\times1} \\ 0_{1\times n} & \Omega_{22} \end{bmatrix} \geq 0$$

$$\implies \hat{U}(i) + \hat{U}^T(i) \geq \hat{T}(i) + \begin{bmatrix} R(i) & 0_{n\times1} \\ 0_{1\times n} & r \end{bmatrix},$$

where, $\hat{U}(i) := diag\{U(i), q\}, \forall i \in \mathcal{N}$.

So, $\forall i \in \mathcal{N}$, one gets that $\hat{U}(i)$ is non-singular as $\hat{T}(i) > 0$. The following inequality can be proved easily.

$$\hat{U}^T(i)\hat{T}^{-1}(i)\hat{U}(i) \geq \left[\hat{U}(i) + \hat{U}^T(i) - \hat{T}(i)\right].$$

Thus,

$$\hat{U}^T(i)\hat{T}^{-1}(i)\hat{U}(i) - \begin{bmatrix} R(i) & 0_{n\times1} \\ 0_{1\times n} & r \end{bmatrix} \geq \left[\hat{U}(i) + \hat{U}^T(i) - \hat{T}(i)\right] - \begin{bmatrix} R(i) & 0_{n\times1} \\ 0_{1\times n} & r \end{bmatrix}.$$

which implies:

$$\begin{aligned} &U^T(i)T^{-1}(i)U(i) - R(i) \geq \left[U(i) + U^T(i) - T(i)\right] - R(i),\\ &\text{and } \frac{q^2}{h} - r \geq 2q - (h + r). \end{aligned} \tag{5.24}$$

Using (5.24) in (5.22b),

$$\Omega'(i, j) = \begin{bmatrix} \mathcal{O}(i) & 0_{n\times 1} & 0_{n\times s} & \Omega_{14}(i) & 0_{n\times 1} & \Omega_{16}(i) \\ 0_{1\times n} & \frac{q^2}{h} - r & 0_{1\times s} & qa^T(i) & q & qc^T(i) \\ 0_{s\times n} & 0_{s\times 1} & \Omega_{33} & D_1^T & 0_{s\times 1} & D_2^T \\ \Omega_{14}^T(i) & qa(i) & D_1 & T(j) & 0_{n\times 1} & 0_{n\times p} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & h & 0_{1\times p} \\ \Omega_{16}^T & qc(i) & D_2 & 0_{p\times n} & 0_{p\times 1} & I_p \end{bmatrix}$$

where $\mathcal{O}(i) = \left[U^T(i)T^{-1}(i)U(i)\right] - R(i)$.

Consider the matrix given by: $\mathscr{P}(i) := diag\{U^{-1}(i), \frac{1}{q}, I_s, I_n, 1, I_p\}$. Then,

$$\begin{aligned} &\mathscr{P}^T(i)\Omega'(i, j)\mathscr{P}(i) \geq 0 \\ \implies &\begin{bmatrix} \hat{T}^{-1}(i) - \mathscr{L}(i) & 0_{n+1\times s} & \hat{\mathscr{A}}_K^T(i) & \bar{\mathscr{C}}_K^T(i) \\ 0_{s\times n+1} & \Omega_{33} & \hat{D}_1^T & D_2^T \\ \hat{\mathscr{A}}_K(i) & \hat{D}_1 & \hat{T}(j) & 0_{n+1\times p} \\ \bar{\mathscr{C}}_K(i) & D_2 & 0_{p\times n+1} & I_p \end{bmatrix} \geq 0, \end{aligned} \tag{5.25}$$

Using Schur compliment in (5.25), we get the following:

$$\begin{bmatrix} \hat{\mathscr{A}}_K^T(i) & \bar{\mathscr{C}}_K^T(i) \\ \hat{D}_1^T & D_2^T \end{bmatrix} \begin{bmatrix} \hat{T}^{-1}(j) & 0_{n+1\times p} \\ 0_{p\times n+1} & I_p \end{bmatrix} \begin{bmatrix} \hat{\mathscr{A}}_K(i) & \hat{D}_1 \\ \bar{\mathscr{C}}_K(i) & D_2 \end{bmatrix} - \begin{bmatrix} \hat{T}^{-1}(i) - \mathscr{L}(i) & 0_{n+1\times s} \\ 0_{s\times n+1} & \Omega_{33} \end{bmatrix} \leq 0 \tag{5.26}$$

This implies that

$$\mathscr{S}(i,j) = \begin{bmatrix} \mathscr{S}_{11}(i,j) & \mathscr{S}_{12}(i,j) \\ \mathscr{S}_{12}^T(i,j) & \mathscr{S}_{22}(i,j) \end{bmatrix} \leq 0, \tag{5.27}$$

where

$$\mathscr{S}_{11}(i,j) = \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) + \bar{\mathscr{C}}_K^T(i)\bar{\mathscr{C}}_K(i) - \hat{T}^{-1}(i) + \mathscr{L}(i)$$

$$\mathscr{S}_{12}(i,j) = \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{D}_1 + \bar{\mathscr{C}}_K^T(i)D_2$$

$$\mathscr{S}_{22}(i,j) = \hat{D}_1^T\hat{T}^{-1}(j)\hat{D}_1 + D_2^TD_2 - \gamma^2 I_s + gI_s.$$

We now proceed to show dissipativity of the system (5.1) with respect to the supply rate $s(z_k, w_k) = \frac{1}{2}[\gamma^2\|w_k\|^2 - \|z_k\|^2]$. For that, consider the storage function: $V(x_k) = \frac{1}{2}x_k^T T^{-1}(i)x_k$, for $x_k \in \mathscr{X}_i$. Then, using (5.21), we get the following:

$$\begin{aligned}
&V(x_{k+1}) - V(x_k) + \frac{1}{2}[z_k^T z_k - \gamma^2 w_k^T w_k] \\
&= \frac{1}{2}\Big[\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big)^T T^{-1}(j)\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big) - x_k^T T^{-1}(i)x_k \\
&+ \Big(\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k + n(x_k,i)\Big)^T \Big(\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k + n(x_k,i)\Big) - \gamma^2 w_k^T w_k\Big] \\
&= \frac{1}{2}\Big[\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big)^T T^{-1}(j)\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big) - x_k^T T^{-1}(i)x_k + 2\bar{x}_k^T\bar{\mathscr{A}}_K^T(i)T^{-1}(j)m(x_k,i) \\
&+ 2w_k^T D_1^T T^{-1}(j)m(x_k,i) + m^T(x_k,i)m(x_k,i) + 2\bar{x}_k^T\bar{\mathscr{C}}_K^T(i)n(x_k,i) + 2w_k^T D_2^T n(x_k,i) + n^T(x_k,i)n(x_k,i)\Big].
\end{aligned} \tag{5.29}$$

It is easy to show that

$$\begin{aligned}
&\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big)^T T^{-1}(j)(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k) - x_k^T T^{-1}(i)x_k \\
&= \Big(\hat{\mathscr{A}}_K(i)\bar{x}_k + \hat{D}_1 w_k\Big)^T \hat{T}^{-1}(j)(\hat{\mathscr{A}}_K(i)\bar{x}_k + \hat{D}_1 w_k) - \bar{x}_k^T\hat{T}^{-1}(i)\bar{x}_k
\end{aligned} \tag{5.30}$$

Further, using the same reasoning as used in the proof of Theorem 5.2.2, one can show the following.

$$
\begin{aligned}
&2\bar{x}_k^T \bar{\mathscr{A}}_K T^{-1}(j) m(x_k, i) \leq \rho_1(i, j) \|\bar{x}_k\|^2 \\
&2w_k^T D_1^T T^{-1}(j) m(x_k, i) \leq \rho_2(i, j)(\|\bar{x}_k\|^2 + \|w_k\|^2) \\
&m^T(x_k, i) T^{-1}(j) m(x_k, i) \leq \rho_3(i, j) \|\bar{x}_k\|^2 \\
&2\bar{x}_k^T \bar{\mathscr{C}}_K^T(i) n(x_k, i) \leq \rho_4(i) \|\bar{x}_k\|^2 \\
&2w_k^T D_2^T n(x_k, i) \leq \rho_5(i)(\|\bar{x}_k\|^2 + \|w_k\|^2) \\
&n^T(x_k, i) n(x_k, i) \leq \rho_6(i) \|\bar{x}_k\|^2
\end{aligned}
\tag{5.31}
$$

Now, using (5.30), (5.31), (5.22c), (5.22d) in (5.29), and then utilizing (5.27) one gets that

$$
\begin{aligned}
V(x_{k+1}) - V(x_k) \leq{}& \frac{1}{2}\Big[\Big(\hat{\mathscr{A}}_K(i)\bar{x}_k + \hat{D}_1 w_k\Big)^T \hat{T}^{-1}(j)\Big(\hat{\mathscr{A}}_K(i)\bar{x}_k + \hat{D}_1 w_k\Big) - \bar{x}_k^T \hat{T}^{-1}(i)\bar{x}_k \\
&+ \Big(\rho_1(i, j) + \rho_2(i, j) + \rho_3(i, j) + \rho_4(i) + \rho_5(i) + \rho_6(i)\Big)\bar{x}_k^T \bar{x}_k + \Big(\rho_2(i, j) + \rho_5(i)\Big) w_k^T w_k\Big] \\
\leq{}& \frac{1}{2}\begin{bmatrix} \bar{x}_k \\ w_k \end{bmatrix}^T \begin{bmatrix} \mathscr{S}_{11}(i, j) & \mathscr{S}_{12}(i, j) \\ \mathscr{S}_{12}^T(i, j) & \mathscr{Y}_{22}(i, j) \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ w_k \end{bmatrix} \leq 0.
\end{aligned}
$$

Thus, system (5.1) with $u_k = K(i)x_k$ is locally dissipative with respect to the supply rate $s(z_k, w_k) = \frac{1}{2}[\gamma^2\|w_k\|^2 - \|z_k\|^2]$.

To study the asymptotic stability of closed-loop system (5.21), we consider a Lyapunov function $V(x_k) = \frac{1}{2}x_k^T T^{-1}(i) x_k$, for $x_k \in \mathscr{X}_i$, $i \in \mathcal{N}$. Then,

$$
\begin{aligned}
&V(x_{k+1}) - V(x_k) \\
&= \frac{1}{2}\Big[(\bar{\mathscr{A}}_K(i)\bar{x}_k + m(x_k, i))^T T^{-1}(j)(\bar{\mathscr{A}}_K(i)\bar{x}_k + m(x_k, i)) - x_k^T T^{-1}(i) x_k\Big] \\
&= \frac{1}{2}\Big[\bar{x}_k^T\Big(\bar{\mathscr{A}}_K^T(i) T^{-1}(j)\bar{\mathscr{A}}_K(i) - T^{-1}(i)\Big)\bar{x}_k + 2\bar{x}_k^T \bar{\mathscr{A}}_K^T(i) T^{-1}(j) m(x_k, i) + m^T(x_k, i) T^{-1}(j) m(x_k, i)\Big]
\end{aligned}
\tag{5.32}
$$

Then, using (5.30) with $w_k \equiv 0$, (5.31) and (5.22c),

$$
\begin{aligned}
&V(x_{k+1}) - V(x_k)\\
&= \frac{1}{2}\Big[\bar{x}_k^T\Big(\hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) - \hat{T}^{-1}(i)\Big)\bar{x}_k + 2\bar{x}_k^T\bar{\mathscr{A}}_K^T(i)T^{-1}(j)m(x_k,i) + m^T(x_k,i)m(x_k,i)\Big]\\
&\le \frac{1}{2}\bar{x}_k^T\Big(\hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) - \hat{T}^{-1}(i) + \rho_1(i,j) + \rho_3(i,j)\Big)\bar{x}_k\\
&\le \frac{1}{2}\bar{x}_k^T\Big(\hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) - \hat{T}^{-1}(i) + \mathscr{L}(i)\Big)\bar{x}_k - \frac{1}{2}\bar{x}_k^T\Big(\rho_2(i,j) + \rho_4(i) + \rho_5(i) + \rho_6(i)\Big)\bar{x}_k
\end{aligned}
$$

Inequality (5.27) implies that

$$
\begin{aligned}
&\mathscr{S}_{11}(i,j) \le 0\\
&\implies \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) - \hat{T}^{-1}(i) + \mathscr{L}(i) \le 0, \quad \forall i,j \in \mathcal{N}.
\end{aligned}
\tag{5.33}
$$

Since $\rho_2(i,j), \rho_4(i), \rho_5(i), \rho_6(i)$ are positive, as a consequence of (5.33), we get: $V(x_{k+1}) - V(x_k) < 0$ for all $x_k(\neq 0) \in \mathcal{X}_i$. Hence, system (5.1), with $u_k = K(i)x_k$ and $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense.

Therefore, system (5.1) has the $\mathcal{L}_2$ gain less than or equal to γ. □

Note 5.2.1. *To get the desired $K(i)$, $T(i)$ in a specific cell, we first solve (5.22b). Then, it is checked whether the conditions (5.22c) and (5.22d) get satisfied. If these two conditions are satisfied, then the cell is a valid one. If they are not satisfied, we opt for a finer cell with smaller $\epsilon(i)$ and $\delta(i)$ such that the conditions get satisfied.* □

5.3 H_∞ control of smooth nonlinear system with random packet losses:

In this section, we consider the case when the controller is spatially separated from the plant, and the controller sends control commands to the actuators through a Gilbert-Elliott type communication channel. Suppose u'_k is the controller output and is sent to the actuators through a lossy network. Then, under the zero-input scheme [57], one can relate u_k (defined in (5.1)) and u'_k as:

$$u_k = v_k u'_k, \tag{5.34}$$

where v_k is binary random variable, and can either be 0 or 1. It represents the packet loss condition in the channel. The arrival probability of the control packet, at a stage $k \geq 1$, is assumed to be $P(v_k = 1|v_{k-1} = 0) = \alpha$ and $P(v_k = 1|v_{k-1} = 1) = 1 - \beta$. For the stage $k = 0$, using Note 2.2.1 of Chapter 2, the arrival probability can be expressed as $P(v_0 = 1) = \alpha/(\alpha + \beta)$.

Consider again a piecewise linear state-feedback control law $u'_k = K(i)x_k$, for $x_k \in \mathscr{X}_i$. Then, with a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$ satisfying (5.3), nonlinear system (5.1) subject to the network induced constraint given by (5.34) can be written as follows:

$$\begin{aligned} x_{k+1} &= \Big(A(i) + v_k B_1 K(i)\Big)x_k + a(i) + D_1 w_k + m(i) \\ z_k &= \Big(C(i) + v_k B_2 K(i)\Big)x_k + c(i) + D_2 w_k + n(i). \end{aligned} \tag{5.35}$$

Objective: The main purpose of this section is to design an controller that makes the closed-loop system (5.35) locally asymptotically stable in the mean-square sense at the same time keeping the $\mathcal{L}_2$ gain from w_k to z_k less than or equal to a prescribed $\gamma > 0$. □

For system (5.35), the following notion of stability shall be followed.

Definition 5.3.1. *System (5.35), with $w_k \equiv 0$, is said to be locally asymptotically stable in the mean-square sense if, for all initial states within a ball around the origin, i.e., $x_0 \in B_\tau(0)$,*

$$\lim_{k\to\infty} \mathbb{E}\Big[\|x_k\|^2 \Big| \mathcal{I}_0\Big] = 0.$$

□

As all the states are accessible to the controller, one can define an information set, at a time instant k, as:

$$\mathcal{I}_k = \{x_0, x_1, ..., x_k, v_0, v_1, ..., v_{k-1}\}.$$

Note that closed-loop system (5.35) becomes stochastic in nature due to the random of packet losses (v_k). Thus, to analyze the $\mathcal{L}_2$ gain of the system (5.35), one needs a notion of local dissipativity in the stochastic sense. We now present one such notion which is motivated by the one given in [70] and [50].

Definition 5.3.2. *Consider system (5.35) with $w_k \in \mathcal{W} \subseteq \mathbb{R}^s$ and $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, such that the state trajectories, with every disturbance input $w_k \in \mathcal{W}$, always remain in $\mathcal{X}$. System (5.35), with a supply rate $s(z_k, w_k)$, is said to be locally dissipative in the stochastic sense if there exists a nonnegative function $V : \mathcal{X} \times \mathcal{N} \to \mathbb{R}^+$, with $V(0, .) = 0$, called the storage function, such that for all $w \in \mathcal{W}$, $x_k \in \mathcal{X}$, and for all $k \in \mathbb{Z}^+$:*

$$\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, e_k) \leq \mathbb{E}\Big[s(z_k, w_k)\Big|\mathcal{I}_k\Big],$$

where, $e_k \in \mathcal{N}$ denotes the cell in which x_k lies. □

Similar to Lemma 5.2.1, one can relate the local stochastic dissipativity and $\mathcal{L}_2$ gain analysis as follows. The following result is motivated from [71].

Lemma 5.3.1. *System (5.35) has the $\mathcal{L}_2$ gain less than or equal to a prescribed $\gamma > 0$ if system (5.35) is locally dissipative in the stochastic sense with respect to the supply rate $s(z_k, w_k) = \mathbb{E}\Big[\frac{1}{2}\Big(\gamma^2\|w_k\|^2 - \|z_k\|^2\Big)\Big|\mathcal{I}_k\Big]$, and the system, with $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense.* □

The following theorem presents results for the H_∞ controller design problem over a Gilbert-Elliott type communication channel.

Theorem 5.3.2. *Consider system (5.35) with the disturbance input $w_k \in \mathcal{W}$, states $x_k \in \mathcal{X}$, for all k and a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$ equipped with (5.3). Suppose, there exist matrices $T(i) > 0$, $U(i)$, $W(i)$, $R(i) > 0$ scalars $\gamma, q, r, h, g > 0$ such that for all $i, j \in \mathcal{N}$*

$$\hat{T}(j) = \begin{bmatrix} T(j) & 0_{n\times 1} \\ 0_{1\times n} & h \end{bmatrix} > 0, \tag{5.36a}$$

$$\Omega(i,j)=\begin{bmatrix}\Omega_{11}(i) & 0_{n\times 1} & 0_{n\times s} & \Omega_{14}(i) & 0_{n\times 1} & \Omega_{16}(i) & A^T(i) & 0_{n\times 1} & C^T(i)\\ 0_{1\times n} & \Omega_{22} & 0_{1\times s} & qa^T(i) & q & qc^T(i) & qa^T(i) & q & qc^T(i)\\ 0_{s\times n} & 0_{s\times 1} & \Omega_{33} & D_1^T & 0_{s\times 1} & D_2^T & D_1^T & 0_{s\times 1} & D_2^T\\ \Omega_{14}^T(i) & qa(i) & D_1 & \frac{1}{\bar{p}_k}T(j) & 0_{n\times 1} & 0_{n\times p} & 0_{n\times n} & 0_{n\times 1} & 0_{n\times p}\\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & \frac{h}{\bar{p}_k} & 0_{1\times p} & 0_{1\times n} & 0 & 0_{1\times p}\\ \Omega_{16}^T(i) & qc(i) & D_2 & 0_{p\times n} & 0_{p\times 1} & \frac{1}{\bar{p}_k}I_p & 0_{p\times n} & 0_{p\times 1} & 0_{p\times p}\\ A(i) & qa(i) & D_1 & 0_{n\times n} & 0_{n\times 1} & 0_{n\times p} & \frac{1}{1-\bar{p}_k}T(l) & 0_{n\times 1} & 0_{n\times p}\\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & 0 & 0_{1\times p} & 0_{1\times n} & \frac{h}{1-\bar{p}_k} & 0_{1\times p}\\ C(i) & qc(i) & D_2 & 0_{p\times n} & 0_{p\times 1} & 0_{p\times p} & 0_{p\times n} & 0_{p\times 1} & \frac{1}{1-\bar{p}_k}I_p\end{bmatrix}\geq 0 \tag{5.36b}$$

$$\Big[\rho_1(i,j)+\rho_2(i,j)+\rho_3(i,j)+\rho_4(i)+\rho_5(i)+\rho_6(i)+\rho_7(i,j)+\rho_8(i,j)+\rho_9(i,j)+\rho_{10}(i)\Big]I_{n+1}\leq \mathscr{L}(i), \tag{5.36c}$$

$$\Big[\rho_2(i,j)+\rho_5(i)+\rho_8(i,l)\Big]\leq g, \tag{5.36d}$$

where

$$\Omega_{11}(i)=\Big[U(i)+U^T(i)-T(i)\Big]-R(i)$$

$$\Omega_{14}(i)=U^T(i)A^T(i)+W^T(i)B_1^T$$

$$\Omega_{16}(i)=U^T(i)C^T(i)+W^T(i)B_2^T$$

$$\Omega_{22}=2q-(h+r)$$

$$\Omega_{33}=\gamma^2 I-gI$$

$$\Omega_{34}=D_1^T.$$

$$\rho_1(i,j)=2\bar{p}_k\epsilon(i)||\bar{\mathscr{A}}_K(i)||||T^{-1}(j)||$$

$$\rho_2(i,j)=\bar{p}_k\epsilon(i)||T^{-1}(j)||||D_1||$$

$$\rho_3(i, j) = \bar{p}_k \epsilon^2(i) \|T^{-1}(j)\|$$

$$\rho_4(i) = 2\bar{p}_k \delta(i) \|\bar{\mathscr{C}}_K(i)\|$$

$$\rho_5(i) = \delta(i) \|D_2\|$$

$$\rho_6(i) = \delta^2(i)$$

$$\rho_7(i, l) = 2(1 - \bar{p}_k)\epsilon(i) \|\bar{A}(i)\| \|T^{-1}(l)\|$$

$$\rho_8(i, l) = (1 - \bar{p}_k)\epsilon(i) \|T^{-1}(l)\| \|D_1\|$$

$$\rho_9(i, l) = (1 - \bar{p}_k)\epsilon^2(i) \|T^{-1}(l)\|$$

$$\rho_{10}(i) = 2(1 - \bar{p}_k)\delta(i) \|\bar{C}(i)\|$$

$$\text{for } k \geq 1, \quad \bar{p}_k = \begin{cases} \alpha & \text{if } v_{k-1} = 0 \\ 1 - \beta & \text{if } v_{k-1} = 1 \end{cases}$$

$$\text{and, } \bar{p}_0 = \frac{\alpha}{\alpha + \beta}.$$

Then, for $K(i) = W(i)U^{-1}(i)$, the closed-loop system (5.35), with $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense. Further, with $K(i) = W(i)U^{-1}(i)$, system (5.35) has the $\mathcal{L}_2$ gain less than or equal to a prescribed γ.

Proof: Assume that LMIs (5.36a) and (5.36b) are satisfied. Using the same line of argument as used in the proof for Theorem 5.2.3, one gets that $\hat{U}(i) := diag\{U(i), q\}$ is nonsingular. Further, it can also be shown that

$$\Omega'(i,j) = \begin{bmatrix} \mathscr{O}_1(i) & 0_{n\times 1} & 0_{n\times s} & \Omega_{14}(i) & 0_{n\times 1} & \Omega_{16}(i) & A^T(i) & 0_{n\times 1} & C^T(i) \\ 0_{1\times n} & \mathscr{O}_2(i) & 0_{1\times s} & qa^T(i) & q & qc^T(i) & qa^T(i) & q & qc^T(i) \\ 0_{s\times n} & 0_{s\times 1} & \Omega_{33} & D_1^T & 0_{s\times 1} & D_2^T & D_1^T & 0_{s\times 1} & D_2^T \\ \Omega_{14}^T(i) & qa(i) & D_1 & \frac{1}{\bar{p}_k}T(j) & 0_{n\times 1} & 0_{n\times p} & 0_{n\times n} & 0_{n\times 1} & 0_{n\times p} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & \frac{h}{\bar{p}_k} & 0_{1\times p} & 0_{1\times n} & 0 & 0_{1\times p} \\ \Omega_{16}^T(i) & qc(i) & D_2 & 0_{p\times n} & 0_{p\times 1} & \frac{1}{\bar{p}_k}I_p & 0_{p\times n} & 0_{p\times 1} & 0_{p\times p} \\ A(i) & qa(i) & D_1 & 0_{n\times n} & 0_{n\times 1} & 0_{n\times p} & \frac{1}{1-\bar{p}_k}T(l) & 0_{n\times 1} & 0_{n\times p} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & 0 & 0_{1\times p} & 0_{1\times n} & \frac{h}{1-\bar{p}_k} & 0_{1\times p} \\ C(i) & qc(i) & D_2 & 0_{p\times n} & 0_{p\times 1} & 0_{p\times p} & 0_{p\times n} & 0_{p\times 1} & \frac{1}{1-\bar{p}_k}I_p \end{bmatrix} \geq 0, \tag{5.38}$$

where $\mathscr{O}_1(i) = \left[U^T(i)T^{-1}(i)U(i)\right] - R(i)$, $\mathscr{O}_2 = \frac{q^2}{h} - r$.

Then, with $\mathscr{P}(i) = diag\left\{U^{-1}(i), \frac{1}{q}, I_s, I_n, 1, I_p, I_n, 1, I_p\right\}$, one gets

$$\mathscr{P}^T\Omega'(i,j)\mathscr{P} \quad \geq 0$$

$$\Longrightarrow \begin{bmatrix} \hat{T}^{-1}(i) - \mathscr{L}(i) & 0_{n+1\times s} & \hat{\mathscr{A}}_K^T(i) & \bar{\mathscr{C}}_K^T(i) & \hat{A}^T(i) & \bar{C}^T(i) \\ 0_{s\times n+1} & \Omega_{33} & \hat{D}_1^T & D_2^T & \hat{D}_1^T & D_2^T \\ \hat{\mathscr{A}}_K(i) & \hat{D}_1 & \frac{1}{\bar{p}_k}\hat{T}(j) & 0_{n+1\times p} & 0_{n+1\times n+1} & 0_{n+1\times p} \\ \bar{\mathscr{C}}_K(i) & D_2 & 0_{p\times n+1} & \frac{1}{\bar{p}}I_p & 0_{p\times n+1} & 0_{p\times p} \\ \hat{A}(i) & \hat{D}_1 & 0_{n+1\times n+1} & 0_{n+1\times p} & \frac{1}{1-\bar{p}_k}\hat{T}(l) & 0_{n+1\times p} \\ \bar{C}(i) & D_2 & 0_{n\times n+1} & 0_{n\times p} & 0_{p\times n+1} & \frac{1}{1-\bar{p}_k}I_p \end{bmatrix} \geq 0, \tag{5.39}$$

As shown in Theorem 5.2.3, one can show that

$$\begin{bmatrix} \mathscr{S}_{11}(i,j) & \mathscr{S}_{12}(i,j) \\ \mathscr{S}_{12}^T(i,j) & \mathscr{S}_{22}(i,j) \end{bmatrix} \leq 0, \tag{5.40}$$

where,

$$\mathscr{S}_{11}(i,j) = \bar{p}_k \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) + (1-\bar{p}_k)\hat{A}^T(i)\hat{T}^{-1}(l)\hat{A}(i) + \bar{p}_k\bar{\mathscr{C}}_K^T(i)\bar{\mathscr{C}}_K(i) + (1-\bar{p}_k)\bar{C}^T(i)\bar{C}(i) - \hat{T}^{-1}(i) + \mathscr{L}(i)$$

$$\mathscr{S}_{12}(i,j) = \bar{p}_k \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{D}_1 + (1-\bar{p}_k)\hat{A}^T(i)\hat{T}^{-1}(l)\hat{D}_1 + \bar{p}_k(\bar{\mathscr{C}}_K(i))^T D_2 + (1-\bar{p}_k)(\bar{C}(i))^T D_2$$

$$\mathscr{S}_{22}(i,j) = \bar{p}_k \hat{D}_1^T\hat{T}^{-1}(j)\hat{D}_1 + (1-\bar{p}_k)\hat{D}_1^T\hat{T}^{-1}(l)\hat{D}_1 + D_2^T D_2 - \Omega_{33}.$$

To show dissipativity of system (5.35), consider a piecewise quadratic storage function of the form $V(x_k,i) = \frac{1}{2}x_k^T T^{-1}(i)x_k$ if $x_k \in \mathscr{X}_i$. Assume that $x_{k+1} \in \mathscr{X}_j$ if $v_k = 1$, and $x_{k+1} \in \mathscr{X}_l$ if $v_k = 0$. Note that e_{k+1} can either be j or l depending on v_k. Thus, with a control law $u_k = K(i)x_k = W(i)U^{-1}(i)x_k$, for $x_k \in \mathscr{X}_i$, $i \in \mathcal{N}$:

$$\begin{aligned}
&\mathbb{E}\Big[V(x_{k+1},e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k,i) + \mathbb{E}\Big[\frac{1}{2}\Big(z_k^T z_k - \gamma^2 w_k^T w_k\Big)\Big|\mathcal{I}_k\Big] \\
&= \frac{1}{2}\Big\{\bar{p}_k\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big]^T T^{-1}(j)\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big] \\
&+ \Big(1-\bar{p}_k\Big)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big]^T T^{-1}(l)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big] - x_k^T T^{-1}(i)x_k \\
&+ \bar{p}_k[\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k + n(x_k,i)]^T[\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k + n(x_k,i)] \\
&+ (1-\bar{p}_k)[\bar{C}(i)\bar{x}_k + D_2 w_k + n(x_k,i)]^T[\bar{C}(i)\bar{x}_k + D_2 w_k + n(x_k,i)] - \gamma^2 w_k^T w_k\Big\} \\
&= \frac{1}{2}\Big\{\bar{p}_k\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big]^T T^{-1}(j)\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big] + \Big(1-\bar{p}_k\Big)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k\Big]^T T^{-1}(l)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k\Big] \\
&- x_k^T T^{-1}(i)x_k + \bar{p}_k[\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k]^T[\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k] + (1-\bar{p}_k)[\bar{C}(i)\bar{x}_k + D_2 w_k]^T[\bar{C}(i)\bar{x}_k + D_2 w_k] \\
&- \gamma^2 w_k^T w_k + \bar{p}_k\Big[2\bar{x}_k^T\bar{\mathscr{A}}_K^T(i)T^{-1}(j)m(x_k,i) + 2w_k^T D_1^T T^{-1}(j)m(x_k,i) + m^T(x_k,i)T^{-1}(j)m(x_k,i) \\
&+ 2\bar{x}_k\bar{\mathscr{C}}_K^T(i)n(x_k,i) + 2w_k^T D_2^T n(x_k,i) + n^T(x_k,i)n(x_k,i)\Big] + (1-\bar{p}_k)\Big[2\bar{x}_k^T\bar{A}^T(i)T^{-1}(l)m(x_k,i) \\
&+ 2w_k^T D_1^T T^{-1}(l)m(x_k,i) + m^T(x_k,i)T^{-1}(l)m(x_k,i) + 2\bar{x}_k\bar{C}^T(i)n(x_k,i) + 2w_k^T D_2^T n(x_k,i) + n^T(x_k,i)n(x_k,i)\Big]\Big\}
\end{aligned} \tag{5.41}$$

The following are true.

$$
\begin{aligned}
&2\bar{p}_k\bar{x}_k^T\bar{\mathscr{A}}_K(i)T^{-1}(j)m(x_k,i) \le \rho_1(i,j)\|\bar{x}_k\|^2\\
&2\bar{p}_k w_k^T D_1^T T^{-1}(j)m(x_k,i) \le \rho_2(i,j)(\|\bar{x}_k\|^2+\|w_k\|^2)\\
&\bar{p}_k m^T(x_k,i)T^{-1}(j)m(x_k,i) \le \rho_3(i,j)\|\bar{x}_k\|^2\\
&2\bar{p}_k\bar{x}_k^T\bar{\mathscr{C}}_K^T(i)n(x_k,i) \le \rho_4(i)\|\bar{x}_k\|^2\\
&2w_k^T D_2^T n(x_k,i) \le \rho_5(i)(\|\bar{x}_k\|^2+\|w_k\|^2)\\
&n^T(x_k,i)n(x_k,i) \le \rho_6(i)\|\bar{x}_k\|^2\\
&2(1-\bar{p}_k)\bar{x}_k^T\bar{A}(i)T^{-1}(l)m(x_k,i) \le \rho_7(i,j)\|\bar{x}_k\|^2\\
&2(1-\bar{p}_k)w_k^T D_1^T T^{-1}(l)m(x_k,i) \le \rho_8(i,j)(\|\bar{x}_k\|^2+\|w_k\|^2)\\
&(1-\bar{p}_k)m^T(x_k,i)T^{-1}(l)m(x_k,i) \le \rho_9(i,j)\|\bar{x}_k\|^2\\
&2(1-\bar{p}_k)\bar{x}_k^T\bar{C}^T(i)n(x_k,i) \le \rho_{10}(i)\|\bar{x}_k\|^2
\end{aligned}
\tag{5.42}
$$

Now, applying the same line of argument as presented in the proof of Theorem 5.2.3, from (5.41), (5.42), and (5.40), one can show that

$$\mathbb{E}\Big[V(x_{k+1},e_{k+1})\Big|\mathcal{I}_k\Big]-V(x_k,i)\le\mathbb{E}\Big[\frac{1}{2}\Big(\gamma^2w_k^Tw_k-z_k^Tz_k\Big)\Big|\mathcal{I}_k\Big]$$

To show the stability of the closed-loop system (5.35), with a control law $u_k = K(i)x_k =$

$W(i)U^{-1}(i)x_k$, for $x_k \in \mathscr{X}_i$, consider the Lyapunov function $V(x_k, i) = \frac{1}{2}x_k^T T^{-1}(i)x_k$.. Then,

$$
\begin{aligned}
&\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, i) \\
&= \frac{1}{2}\Big\{\bar{p}_k\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + m(x_k, i)\Big]^T T^{-1}(j)\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + m(x_k, i)\Big] \\
&+ \Big(1 - \bar{p}_k\Big)\Big[\bar{A}(i)\bar{x}_k + m(x_k, i)\Big]^T T^{-1}(l)\Big[\bar{A}(i)\bar{x}_k + m(x_k, i)\Big] - x_k^T T^{-1}(i)x_k\Big\} \\
&= \frac{1}{2}\Big\{\bar{p}_k\bar{x}_k^T\bar{\mathscr{A}}_K^T(i)T^{-1}(j)\bar{\mathscr{A}}_K(i)\bar{x}_k + \Big(1 - \bar{p}_k\Big)\bar{x}_k^T\bar{A}^T(i)T^{-1}(j)\bar{A}(i)\bar{x}_k - x_k^T T^{-1}(i)x_k \\
&+ \bar{p}_k\Big[2\bar{x}_k^T\bar{\mathscr{A}}_K^T(i)T^{-1}(j)m(x_k, i) + m^T(x_k, i)T^{-1}(j)m(x_k, i)\Big] + (1 - \bar{p}_k)\Big[2\bar{x}_k^T\bar{A}^T(i)T^{-1}(l)m(x_k, i) \\
&+ m^T(x_k, i)T^{-1}(l)m(x_k, i)\Big]\Big\}
\end{aligned}
$$

Using the same reasoning as used in the proof of Theorem 5.2.3, one can show that

$$
\begin{aligned}
&\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, i) \\
&\leq \frac{1}{2}\bar{x}_k^T\Big(\bar{p}_k\hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) + (1 - \bar{p}_k)\hat{A}^T(i)\hat{T}^{-1}(l)\hat{A}(i) - \hat{T}^{-1}(i) + \mathscr{L}(i)\Big)\bar{x}_k \\
&- \frac{1}{2}\bar{x}_k^T\Big(\rho_2(i, j) + \rho_4(i) + \rho_5(i) + \rho_6(i) + \rho_8(i, j) + \rho_{10}(i)\Big)\bar{x}_k.
\end{aligned}
$$

From (5.40), $\mathscr{S}_{11}(i, j) \leq 0$. Thus,

$$
\bar{p}_k\hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) + (1 - \bar{p}_k)\hat{A}^T(i)\hat{T}^{-1}(l)\hat{A}(i) - \hat{T}^{-1}(i) + \mathscr{L}(i) \leq 0.
$$

As $\rho_2(i, j), \rho_4(i), \rho_5(i), \rho_6(i), \rho_8(i, j)$ and $\rho_{10}(i)$ are positive, for all $x_k \neq 0$, we have

$$
\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, i) < 0
$$

Thus, for $K(i) = W(i)U^{-1}(i)$, the closed-loop system (5.35), with $w_k \equiv 0$, is locally asymptotically stable in the mean-square sense. Now with local stochastic dissipativity, we can infer that, with $K(i) = W(i)U^{-1}(i)$, system (5.35) has an $\mathcal{L}_2$ gain less than or equal to a prescribed γ.

□

Note 5.3.1. *If one puts $\epsilon(i) = 0$ and $\delta(i) = 0$ in the above theorem, for all $i \in \mathcal{N}$, then the result corresponds to the result for H_∞ controller design problem of a PWA system over a Gilbert-*

Elliott type channel.

5.4 Numerical Example

Consider the nonlinear system (5.1) with the following system parameters:

$$f(x_k) = \begin{bmatrix} 4sin(x_k^1) + x_k^2 \\ x_k^1 + x_k^3 \\ x_k^1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \ D_1 = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}, \ h(x_k) = x_k^1, \ B_2 = 0.1, \ D_2 = 2.$$

where $x_k = \begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix}$. External input is assumed to be $w_k = 0.02sin(0.2\pi k)exp(-k/25)$.

To demonstrate the results presented in Theorem 5.2.3, we consider the region $-0.3 \leq x_k^1 \leq 0.3$ (note that nonlinearity exists only in x_k^1). The region $-0.3 \leq x_k^1 \leq 0.3$ is partitioned into 20 cells, which are given by: $-0.3 \leq x_k^1 < -0.28$, $-0.28 \leq x_k^1 < -0.26$, $-0.26 \leq x_k^1 < -0.24$, $-0.24 \leq x_k^1 < -0.22$, $-0.22 \leq x_k^1 < -0.2$,$-0.2 \leq x_k^1 < -0.18$, $-0.18 \leq x_k^1 < -0.15$, $-0.15 \leq x_k^1 < -0.12$, $-0.12 \leq x_k^1 < -0.08$, $-0.08 \leq x_k^1 < 0$, $0 \leq x_k^1 < 0.08$, $0.08 \leq x_k^1 < 0.12$, $0.12 \leq x_k^1 < 0.15$, $0.15 \leq x_k^1 < 0.18$, $0.18 \leq x_k^1 < 0.2$, $0.2 \leq x_k^1 < 0.22$, $0.22 \leq x_k^1 < 0.24$, $0.24 \leq x_k^1 < 0.26$, $0.26 \leq x_k^1 < 0.28$, $0.28 \leq x_k^1 < 0.3$. Thus, we have to solve 20 LMIs, each with 8 unknown variables. $A(i)$ and $a(i)$ for the piecewise linear approximations are computed using Taylor series expansion. The error coefficient $\epsilon(i)$ is computed as the maximum value of $\frac{\|f(x_k)-A(i)x_k-a(i)\|}{\|\bar{x}_k\|}$, for $x_k \in \mathcal{X}_i$. As $h(x_k) = x_k^1$, one gets that $C(i)$ is equal to $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, for all $i \in \mathcal{N}$. We use the *lmisolver* function available in SCILAB to solve the LMIs in each of the cells. Solving (5.22b) we then calculate the controller gain $K(i)$ such that the closed system is locally asymptotically stable in the mean-square sense and has the $\mathcal{L}_2$ gain less than or equal to γ. The given cells are designed in such a way that the corresponding error coefficient $\epsilon(i)$, in each $i \in \mathcal{N}$, is almost the maximum value that satisfies the conditions given by (5.22c) and (5.22d). From Figure 5.1, one can see that difference in storage function at each stage is less than or equal to the supply rate $s(z_k, w_k) = \frac{1}{2}[\gamma^2\|w_k\|^2 - \|z_k\|^2]$. Stability of the closed-loop system is demonstrated in Figure 5.2.

To design the H_∞ controller with packet loss, we consider the region $-0.29 \leq x_k^1 \leq 0.29$. The partition is given by: $-0.29 \leq x_k^1 < -0.27$, $-0.27 \leq x_k^1 < -0.25$, $-0.25 \leq x_k^1 < -0.23$, $-0.23 \leq x_k^1 < -0.21$, $-0.21 \leq x_k^1 < -0.19$, $-0.19 \leq x_k^1 < -0.17$, $-0.17 \leq x_k^1 < -0.14$, $-0.14 \leq x_k^1 < -0.11$, $-0.11 \leq x_k^1 < -0.07$, $-0.07 \leq x_k^1 < 0$, $0 \leq x_k^1 < 0.07$, $0.07 \leq x_k^1 < 0.11$, $0.11 \leq x_k^1 < 0.14$, $0.14 \leq x_k^1 < 0.17$, $0.17 \leq x_k^1 < 0.19$, $0.19 \leq x_k^1 < 0.21$, $0.21 \leq x_k^1 < 0.23$, $0.23 \leq x_k^1 < 0.25$, $0.25 \leq x_k^1 < 0.27$, $0.27 \leq x_k^1 < 0.29$. Then, solving LMI (5.36b) in Theorem 5.3.2, we calculate the controller gain $K(i)$ in each cells with a control packet arrival probability $\alpha = 0.95$ and $1 - \beta = 0.96$. As the number of cells is 20, again we solve 20 LMIs, each having 8 unknown variables. Figure 5.3 demonstrates that the closed-loop system is locally dissipative with the quadratic storage function $V(x_k) = \frac{1}{2}x_k^T T^{-1}(i)x_k$. The decaying state response is shown in Figure 5.4.

It is to be noted that, to design the H_∞ controller with packet losses, we have to consider smaller cells as compared to the ones used to design the H_∞ controller without packet losses. This is due to the fact that condition (5.36c) contains the terms which come from the open-loop system dynamics.

5.5 Summary

In this chapter, we have used a piecewise affine approximation approach to study the $\mathcal{L}_2$ gain of a smooth nonlinear system. We have shown that with our approach it is possible to design a state-feedback controller by solving a set of LMIs that are subject to certain nonlinear constraints. Further, results for the H_∞ controller design problem for a PWA system can be easily derived from the results presented by the final theorem.

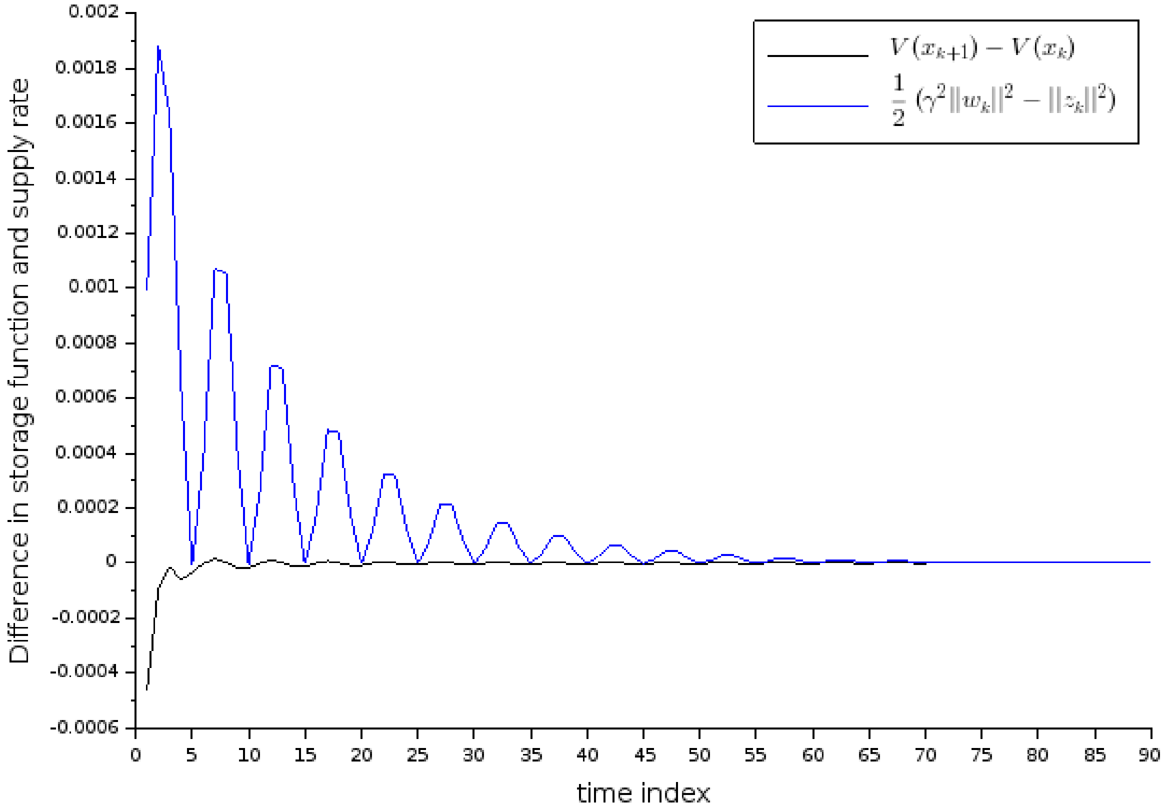

Figure 5.1: Difference in storage function and supply rate

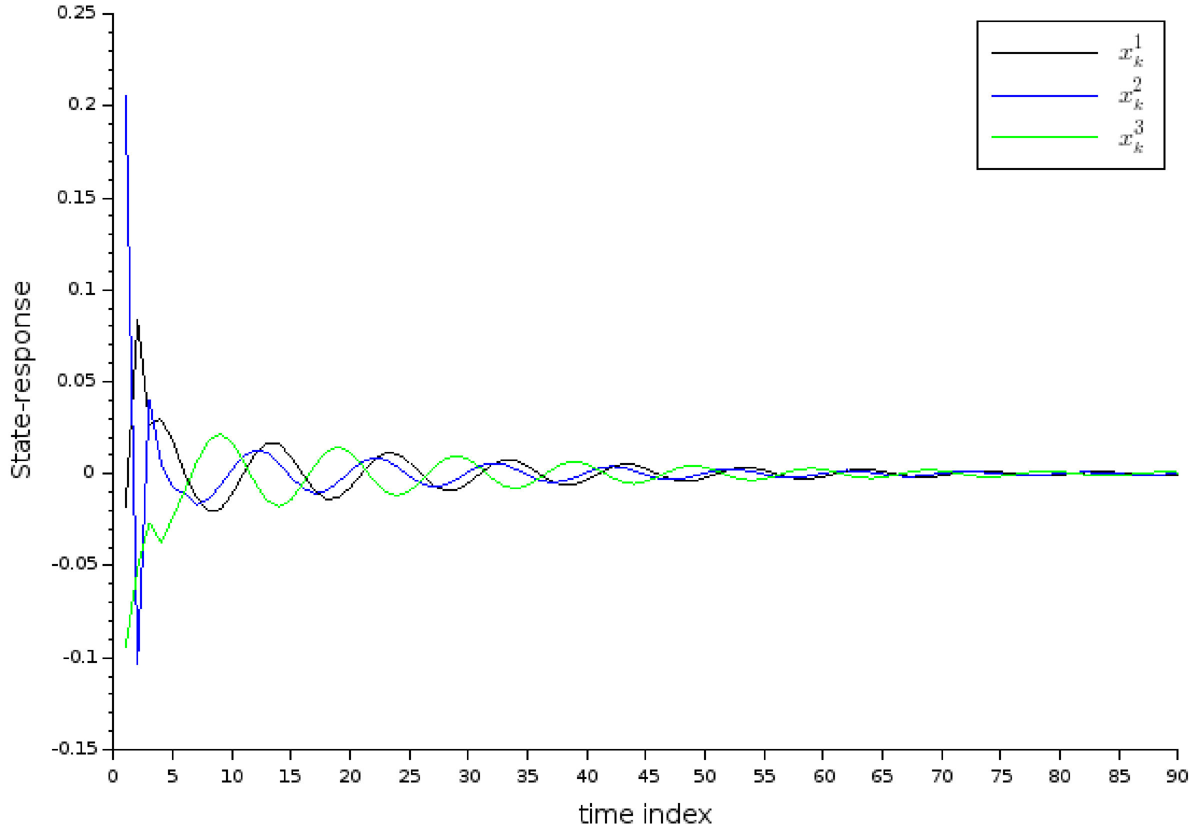

Figure 5.2: State response

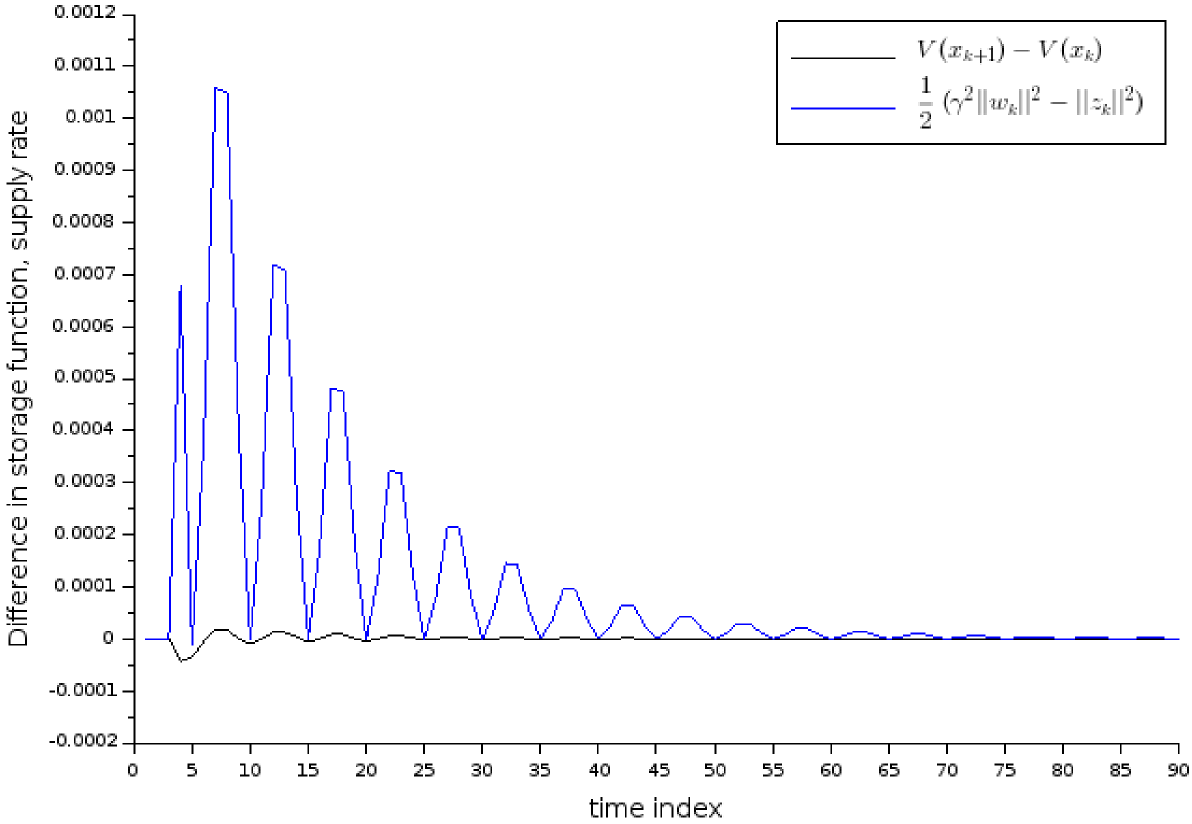

Figure 5.3: Difference in storage function and supply rate

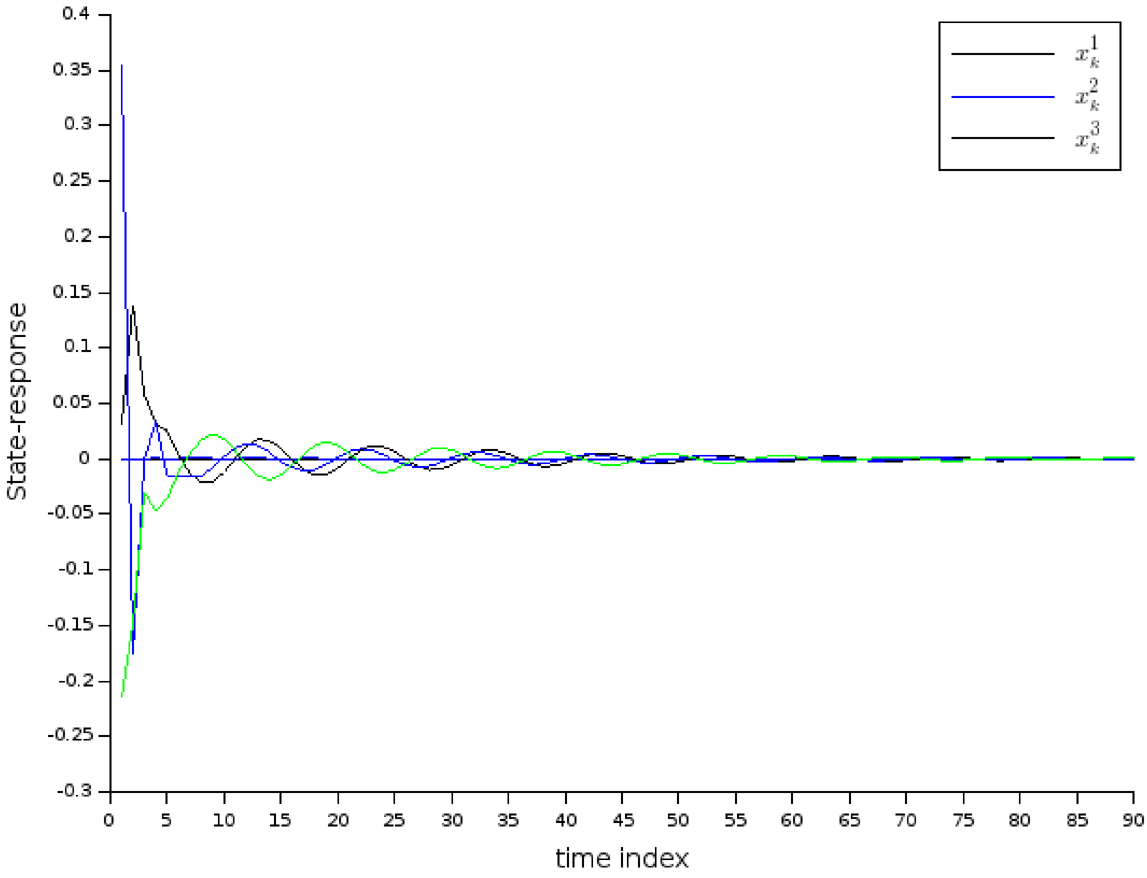

Figure 5.4: State response

6

Local feedback passivity of smooth nonlinear systems over lossy channel

6.1 Introduction

Passivity is an important tool to assess stability of nonlinear systems. In the recent past, passivity of smooth discrete-time nonlinear systems has been investigated in great detail [41–45].

In this chapter, we consider the local passivity problem of smooth nonlinear systems, which is a specific dissipativity problem. We address the problem using a PWA approximation. To start with, we derive conditions under which a smooth discrete-time nonlinear system becomes locally passive. Then, we derive conditions under which a piecewise linear state-feedback control law is sufficient to ensure local feedback passivity. Finally, we consider the problem of local feedback passivity of smooth nonlinear systems over a lossy communication network. Results for the feedback passivity of a PWA system with packet losses can easily be derived as a special case of this analysis.

The chapter is structured as follows. Section 6.2 presents the results pertaining to local passivity and local feedback passivity of smooth nonlinear systems. Section 6.3, contains the results for the local feedback passivity over erasure channel. In section 6.4, we demonstrate our results using a numerical example. Finally, section 6.5 summarizes the chapter.

6.2 Local passivity and local feedback passivity of smooth nonlinear systems

In this section, we investigate local passivity and local feedback passivation of a smooth nonlinear system.

Consider the smooth discrete-time nonlinear system:

$$\begin{aligned} x_{k+1} &= f(x_k) + B_1 u_k + D_1 w_k \\ z_k &= h(x_k) + B_2 u_k + D_2 w_k, \end{aligned} \tag{6.1}$$

wherein $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input to the actuators, $w_k \in \mathbb{R}^s$ is the external input (or disturbance), $z_k \in \mathbb{R}^s$ is the output, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h : \mathbb{R}^n \rightarrow \mathbb{R}^s$ are smooth maps, $B_1 \in \mathbb{R}^{n\times m}$, $D_1 \in \mathbb{R}^{n\times s}$, $B_2 \in \mathbb{R}^{s\times m}$, $D_2 \in \mathbb{R}^{s\times s}$ are constant matrices.

Similar to the previous chapter, we consider a PWA approximation approach by partitioning a subspace $\mathcal{X}$ of the state-space into a number of polyhedral cells. One can characterize each polyhedral cell as [35]:

$$E(i)x_k + e(i) \geq 0 \;\; x_k \in \mathscr{X}_i, \; or \; \bar{E}(i)\bar{x}_k \geq 0, \tag{6.2}$$

where $\bar{E}(i) := [E(i) \quad e(i)]$, $\bar{x}(k) := [x(k) \quad 1]^T$.

Now, with a polyhedral partition equipped with a PWA approximation as given by (5.3) in Chapter 5, the nonlinear system (6.1) can be expressed as follows:

$$\begin{aligned} x_{k+1} &= A(i)x_k + a(i) + B_1u_k + D_1w_k + m(x_k, i) \\ z_k &= C(i)x_k + c(i) + B_2u_k + D_2w_k + n(x_k, i), \quad \text{for } x_k \in \mathscr{X}_i. \end{aligned} \tag{6.3}$$

One can write (6.3) in terms of the variable $\bar{x}_k$ as:

$$\begin{aligned} \bar{x}_{k+1} &= \hat{A}(i)\bar{x}_k + \hat{B}_1u_k + \hat{D}_1w_k + \hat{m}(x_k, i) \\ z_k &= \bar{C}(i)\bar{x}_k + B_2u_k + D_2w_k + n(x_k, i), \end{aligned} \tag{6.4}$$

where $A(i)$, $a(i)$, B_1, D_1, $m(x_k, i)$, $C(i)$, $c(i)$, B_2, D_2, $n(x_k, i)$, $\hat{A}(i)$, $\hat{B}_1$, $\hat{D}_1$, $\hat{m}(x_k, i)$, $\bar{C}(x_k, i)$ are as defined in Chapter 5.

The following notion of passivity for nonlinear system (6.1), with $u_k \equiv 0$, shall be used in this section. This notion is inspired by the one given in [50].

Definition 6.2.1. *Consider system (6.1) with $u_k \equiv 0$, $w_k \in \mathcal{W} \subseteq \mathbb{R}^s$ and $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, such that the state trajectories, with every disturbance input $w_k \in \mathcal{W}$, always remain in $\mathcal{X}$. Now, system (6.1), with $u_k \equiv 0$, is said to be locally passive if there exists a nonnegative function $V : \mathcal{X} \to \mathbb{R}^+$ with $V(0) = 0$, called the storage function, such that for all $x_k \in \mathcal{X}$, $w_k \in \mathcal{W}$, and for all $k \in \mathbb{Z}^+$:*

$$V(x_{k+1}) - V(x_k) \leq z_k^T w_k. \tag{6.5}$$

Similar to Chapter 5, we again consider a piecewise quadratic storage function of the form

$$V(x_k) = \frac{1}{2}\bar{x}_k^T \bar{P}(i)\bar{x}_k \text{ for } x_k \in \mathscr{X}_i. \tag{6.6}$$

If $i \in \mathcal{N}_0$ then $\bar{P}(i)$ is given by $\bar{P}(i) = diag\{P(i), 0\}$. Matrices $P(i)$, $i \in \mathcal{N}_0$, and $\bar{P}(i)$, $i \in \mathcal{N} \setminus \mathcal{N}_0$ are such that $x_k^T P(i) x_k > 0$, when $x_k(\neq 0) \in \mathscr{X}_i$, $i \in \mathcal{N}_0$, and $\bar{x}_k^T \bar{P}(i)\bar{x}_k > 0$, when $x_k(\neq 0) \in \mathscr{X}_i$, $i \in \mathcal{N} \setminus \mathcal{N}_0$, respectively. It will ensure nonnegativeness of $V(x_k)$.

Following theorem presents conditions for local passivity of system (6.1) with $u_k \equiv 0$.

Theorem 6.2.1. *Consider system (6.1) with $u_k \equiv 0$, a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$ equipped with (5.3), and the disturbance input $w_k \in \mathcal{W}$ for all k. System (6.1), with $u_k \equiv 0$, is locally passive if there exist symmetric matrices with positive entries $W(i)$, $R(i)$, and symmetric matrices $\bar{P}(i)$ such that the following inequalities are satisfied for all $i, j \in \mathcal{N}$:*

$$\begin{aligned} &\bar{P}(i) - \bar{E}^T(i)R(i)\bar{E}(i) > 0, \\ &\begin{bmatrix} \Lambda_{11}(i,j) & \Lambda_{12}(i,j) \\ \Lambda_{12}^T(i,j) & \Lambda_{22}(i,j) \end{bmatrix} \leq 0, \end{aligned} \tag{6.7}$$

where

$$
\begin{aligned}
\Lambda_{11} &= \hat{A}^T(i)\bar{P}(j)\hat{A}(i) - \bar{P}(i) + \bar{E}^T(i)W(i)\bar{E}(i) \\
&\quad + \Big(\rho_1(i,j) + \rho_2(i,j) + \rho_3(i,j) + \rho_4(i)\Big)I_{n+1} \\
\Lambda_{12} &= \hat{A}^T(i)\bar{P}(j)\hat{D}_1(i) - \bar{C}^T(i) \\
\Lambda_{22} &= \Big(\hat{D}_1\Big)^T\bar{P}(j)\hat{D}_1 - \Big[D_2^T + D_2\Big] + \Big(\rho_2(i,j) + \rho_4(i)\Big)I_s \\
\rho_1(i,j) &= 2\epsilon(i)\|\bar{P}(j)\|\|\hat{A}(i)\|, \\
\rho_2(i,j) &= \epsilon(i)\|\bar{P}(j)\|\|\hat{D}_1\| \\
\rho_3(i,j) &= \epsilon^2(i)\|\bar{P}(j)\|, \\
\rho_4(i) &= \delta(i).
\end{aligned}
$$

Proof: With a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$, one can find matrices $E(i)$ and $\bar{E}(i)$, for $i \in \mathcal{N}$, such that $E(i)x_k \geq 0$ and $\bar{E}(i)\bar{x}_k \geq 0$. As a piecewise affine approximation is being employed, one can consider a piecewise quadratic storage function of the form $V(x_k) = \frac{1}{2}\bar{x}_k^T\bar{P}(i)\bar{x}_k$, for $x_k \in \mathscr{X}_i$, to study the local passivity of for the system (6.1).

The first inequality in (6.7) ensures that $V(x_k)$ is positive definite. Now, using (6.4), we get:

$$
\begin{aligned}
&V(x_{k+1}) - V(x_k) - z_k^T w_k \\
&= \frac{1}{2}\bar{x}_{k+1}^T\bar{P}(j)\bar{x}_{k+1} - \frac{1}{2}\bar{x}_k^T\bar{P}(i)\bar{x}_k - z_k^T w_k \\
&= \frac{1}{2}\Big[\Big(\hat{A}(i)\bar{x}_k + \hat{D}_1 w_k + \hat{m}(x_k,i)\Big)^T\bar{P}(j)\Big(\hat{A}(i)\bar{x}_k + \hat{D}_1 w_k + \hat{m}(x_k,i)\Big) - \bar{x}_k^T\bar{P}(i)\bar{x}_k \\
&- \Big(\bar{C}(i)\bar{x}_k + D_2 w_k + n(x_k,i)\Big)^T w_k - w_k^T\Big(\bar{C}(i)\bar{x}_k + D_2 w_k + n(x_k,i)\Big)\Big] \\
&= \frac{1}{2}\Big[\Big(\hat{A}(i)\bar{x}_k + \hat{D}_1 w_k\Big)^T\bar{P}(j)(\hat{A}(i)\bar{x}_k + \hat{D}_1 w_k) - \bar{x}_k^T\bar{P}(i)\bar{x}_k - \Big(\bar{C}(i)\bar{x}_k + D_2 w_k\Big)^T w_k \\
&- w_k^T\Big(\bar{C}(i)\bar{x}_k + D_2 w_k\Big) + 2\bar{x}_k^T\hat{A}^T(i)\bar{P}(j)\hat{m}(x_k,i) + 2w_k^T\hat{D}_1^T\bar{P}(j)\hat{m}(x_k,i) \\
&+ \hat{m}^T(i)\bar{P}(j)\hat{m}(x_k,i) - 2w_k^T n(x_k,i)\Big].
\end{aligned}
\tag{6.8}
$$

Using basic properties of 2-norm and induced 2-norm:

$$\begin{aligned}2\bar{x}_k\hat{A}^T(i)\bar{P}(j)\hat{m}(x_k,i) &\le \|2\bar{x}_k\hat{A}^T(i)\bar{P}(j)\hat{m}(x_k,i)\| \\ &\le 2\|\bar{P}(j)\|\|\hat{A}(i)\|\|\bar{x}_k\|\epsilon(i)\|x_k\| \\ &\le \rho_1(i,j)\|\bar{x}_k\|^2 \text{ as } \|x_k\|\le\|\bar{x}_k\|\end{aligned} \tag{6.9}$$

$$\begin{aligned}2w_k^T\hat{D}_1^T\bar{P}(j)\hat{m}(x_k,i) &\le \|2w_k^T\hat{D}_1^T\bar{P}(j)\hat{m}(i)\| \\ &\le 2\|\bar{P}(j)\|\|\hat{D}_1\|\|w_k\|\|\hat{m}(x_k,i)\| \\ &\le 2\epsilon(i)\|\bar{P}(j)\|\|\hat{D}_1\|\|\bar{x}_k\|\|w_k\|\end{aligned} \tag{6.10}$$

Note that $2\|\bar{x}_k\|\|w_k\| \le \left(\|\bar{x}_k\|^2 + \|w_k\|^2\right)$. Thus, (6.10) implies:

$$2w_k^T\hat{D}_1^T\bar{P}(j)\hat{m}(x_k,i) \le \rho_2(i,j)\left(\|\bar{x}_k\|^2 + \|w_k\|^2\right) \tag{6.11}$$

$$\text{Similarly, } \hat{m}^T(x_k,i)\bar{P}(j)\hat{m}(x_k,i) \le \rho_3\|\bar{x}_k\|^2 \tag{6.12}$$

$$-2w_k^Tn(x_k,i) \le \rho_4(i)\left(\|\bar{x}_k\|^2 + \|w_k\|^2\right) \tag{6.13}$$

From (6.8), (6.9), (6.11), (6.12), (6.13), and using the fact that $W(i)$ has only positive elements:

$$\begin{aligned}&V(x_{k+1}) - V(x_k) - z_k^Tw_k \\ &\le \frac{1}{2}\begin{bmatrix}\bar{x}_k \\ w_k\end{bmatrix}^T\begin{bmatrix}\Lambda_{11}(i,j) & \Lambda_{12}(i,j) \\ \Lambda_{12}^T(i,j) & \Lambda_{22}(i,j)\end{bmatrix}\begin{bmatrix}\bar{x}_k \\ w_k\end{bmatrix}\end{aligned}$$

If the second inequality in (6.7) is satisfied for $i, j \in \mathcal{N}$, then:

$$V(x_{k+1}) - V(x_k) \le z_k^Tw_k$$

Hence, the nonlinear system (6.1), with $u_k \equiv 0$, is locally passive. □

For a piecewise state feedback law $u_k = K(i)x_k$, where i denotes the index of the state space partition, let $\mathscr{A}_K(i)$, $\bar{\mathscr{A}}_K(i)$ and $\hat{\mathscr{A}}_K(i)$ denote the matrices $A(i) + B_1(i)K(i)$, $\begin{bmatrix}\mathscr{A}_K(i) & a(i)\end{bmatrix}$ and $\begin{bmatrix}\mathscr{A}_K(i) & a(i)\\ 0_{1\times n} & 1\end{bmatrix}$ respectively. Let $\mathscr{C}_K(i)$ and $\bar{\mathscr{C}}_K(i)$ denote the matrices $C(i) + B_2(i)K(i)$ and $\begin{bmatrix}\mathscr{C}_k(i) & c(i)\end{bmatrix}$, respectively.

In what follows next, we derive sufficient conditions under which a piecewise linear state-feedback controller is able to make the closed-loop system locally passive.

Theorem 6.2.2. *Consider nonlinear system (6.1) with the disturbance input $w_k \in \mathcal{W}$, states $x_k \in \mathcal{X}$, for all k and a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$ equipped with (5.3). Now, system (6.1), with $u_k = K(i)x_k = W(i)U^{-1}(i)x_k$, is locally passive if there exist matrices $T(i) > 0$, $U(i)$, $W(i)$, $R(i) > 0$, $G(i) > 0$ and scalars $q, r, h > 0$ such that for all $i, j \in \mathcal{N}$*

$$\hat{T}(j) = \begin{bmatrix} T(j) & 0_{n\times 1} \\ 0_{1\times n} & h \end{bmatrix} > 0, \tag{6.14a}$$

$$\begin{bmatrix} \Omega_{11}(i) & 0_{n\times 1} & \Omega_{13}(i) & \Omega_{14}(i) & 0_{n\times 1} \\ 0_{1\times n} & \Omega_{22} & qc^T(i) & qa^T(i) & q \\ \Omega_{13}^T(i) & qc(i) & \Omega_{33} & \Omega_{34} & 0_{s\times 1} \\ \Omega_{14}^T(i) & qa(i) & \Omega_{34}^T & T(j) & 0_{n\times 1} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & h \end{bmatrix} \geq 0, \tag{6.14b}$$

$$\left[\gamma_1(i,j) + \gamma_2(i,j) + \gamma_3(i,j) + \gamma_4(i)\right] I_{n+1} \leq \mathscr{L}(i) \tag{6.14c}$$

$$\left[\gamma_2(i,j) + \gamma_4(i)\right] I_s \leq G(i), \tag{6.14d}$$

where

$$\Omega_{11}(i) = \left[U(i) + U^T(i) - T(i)\right] - R(i)$$

$$\Omega_{13}(i) = U^T(i)C^T(i) + W^T(i)B_2^T$$

TH-2488_146102019

$$\Omega_{14}(i) = U^T(i)A^T(i) + W^T(i)B_1^T$$

$$\Omega_{22} = 2q - (h + r),\ \Omega_{33} = \left(D_2^T + D_2\right) - G(i),\ \ \Omega_{34} = D_1^T.$$

$$\gamma_1(i, j) = 2\epsilon(i)\|\bar{\mathscr{A}}_K(i)\|\|T^{-1}(j)\|$$

$$\gamma_2(i, j) = \epsilon(i)\|T^{-1}(j)\|\|D_1\|$$

$$\gamma_3(i, j) = \epsilon^2(i)\|T^{-1}(j)\|,\ \ \gamma_4(i) = \delta(i).$$

$$\mathscr{L}(i) = \begin{bmatrix} U^{-T}(i)R(i)U^{-1}(i) & 0_{n\times 1} \\ 0_{1\times n} & \frac{r}{q^2} \end{bmatrix}.$$

Proof: Suppose, LMIs (6.14a) and (6.14b) are satisfied. As all principal submatrices of a positive semidefinite symmetric matrix are also postive semidefinite, from (6.14b), one gets:

$$\begin{bmatrix} \Omega_{11}(i) & 0_{n\times 1} \\ 0_{1\times n} & \Omega_{22} \end{bmatrix} \geq 0$$

$$\implies \hat{U}(i) + \hat{U}^T(i) \geq \hat{T}(i) + \begin{bmatrix} R(i) & 0_{n\times 1} \\ 0_{1\times n} & r \end{bmatrix},$$

where, $\hat{U}(i) := diag\{U(i), q\}, \forall i \in \mathcal{N}$.

So, $\forall i \in \mathcal{N}$, one gets that $\hat{U}(i)$ is non-singular as $\hat{T}(i) > 0$. The following inequality can be proved easily.

$$\hat{U}^T(i)\hat{T}^{-1}(i)\hat{U}(i) \geq \left[\hat{U}(i) + \hat{U}^T(i) - \hat{T}(i)\right].$$

This implies:

$$U^T(i)T^{-1}(i)U(i) - R(i) \geq \left[U(i) + U^T(i) - T(i)\right] - R(i),$$

$$\text{and } \frac{q^2}{h} - r \geq 2q - (h + r).$$

Therefore,

$$\Omega'(i,j) = \begin{bmatrix} \mathscr{O}(i) & 0_{n\times1} & \Omega_{13}(i) & \Omega_{14}(i) & 0_{n\times1} \\ 0_{1\times n} & \frac{q^2}{h}-r & qc^T(i) & qa^T(i) & q \\ \Omega_{13}^T(i) & qc(i) & \Omega_{33} & \Omega_{34} & 0_{s\times1} \\ \Omega_{14}^T(i) & qa(i) & \Omega_{34}^T & T(j) & 0_{n\times1} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & h \end{bmatrix} \geq 0, \tag{6.16}$$

where $\mathscr{O}(i) = \left[U^T(i)T^{-1}(i)U(i)\right] - R(i)$.

Define $\mathscr{P}(i)$ as $\mathscr{P}(i) = diag\left\{U^{-1}(i), \frac{1}{q}, I_s, I_n, 1\right\}$. Then from (6.16):

$$\mathscr{P}^T(i)\Omega'(i,j)\mathscr{P}(i) \geq 0$$
$$\Longrightarrow \begin{bmatrix} \hat{T}^{-1}(i) - \mathscr{L}(i) & \bar{\mathscr{C}}_K^T(i) & \hat{\mathscr{A}}_K^T(i) \\ \bar{\mathscr{C}}_K(i) & \Omega_{33} & \hat{D}_1^T \\ \hat{\mathscr{A}}_K(i) & \hat{D}_1 & \hat{T}(j) \end{bmatrix} \geq 0, \tag{6.17}$$

As $\hat{T}(j) > 0$, from (6.17), we get the following inequality using Schur complement:

$$\mathscr{S}(i,j) = \begin{bmatrix} \mathscr{S}_{11}(i,j) & \mathscr{S}_{12}(i,j) \\ \mathscr{S}_{12}^T(i,j) & \mathscr{S}_{22}(i,j) \end{bmatrix} \leq 0, \tag{6.18}$$

where

$$\mathscr{S}_{11}(i,j) = \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) - \hat{T}^{-1}(i) + \mathscr{L}(i)$$

$$\mathscr{S}_{12}(i,j) = \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{D}_1 - \bar{\mathscr{C}}_K^T(i)$$

$$\mathscr{S}_{22}(i,j) = \hat{D}_1^T\hat{T}^{-1}(j)\hat{D}_1 - (D_2^T + D_2) + G(i).$$

Now, consider a piecewise quadratic storage function of the form $V(x_k) = \frac{1}{2}x_k^T T^{-1}(i)x_k$ for

$x_k \in \mathscr{X}_i$. Substituting $u_k = K(i)x_k$ for $x_k \in \mathscr{X}_i$, and using (6.3) we get:

$$\begin{aligned}
&V(x_{k+1}) - V(x_k) - z_k^T w_k \\
&= \frac{1}{2}x_{k+1}^T T^{-1}(j)x_{k+1} - \frac{1}{2}x_k^T T^{-1}(i)x_k - z_k^T w_k \\
&= \frac{1}{2}\Big[\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big)^T T^{-1}(j)\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k,i)\Big) - x_k^T T^{-1}(i)x_k \\
&- \Big(\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k + n(x_k,i)\Big)^T w_k - w_k^T\Big(\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k + n(x_k,i)\Big)\Big] \\
&= \frac{1}{2}\Big[\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big)^T T^{-1}(j)(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k) - x_k^T T^{-1}(i)x_k - \Big(\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k\Big)^T w_k \\
&- w_k^T\Big(\bar{\mathscr{C}}_K(i)\bar{x}_k + D_2 w_k\Big) + 2\bar{x}_k^T \bar{\mathscr{A}}_K^T(i)T^{-1}(j)m(x_k,i) + 2w_k^T D_1^T T^{-1}(j)m(x_k,i) \\
&+ m^T(x_k,i)T^{-1}(j)m(x_k,i) - 2w_k^T n(x_k,i)\Big].
\end{aligned} \tag{6.20}$$

Similar to (6.9), (6.11), (6.12), (6.13), the following inequalities can be derived.

$$2\bar{x}_k^T \bar{\mathscr{A}}_K^T(i)T^{-1}(j)m(x_k,i) \leq \gamma_1(i,j)\|\bar{x}_k\|^2 \tag{6.21}$$

$$2w_k^T D_1^T T^{-1}(j)m(x_k,i) \leq \gamma_2(i,j)\Big(\|\bar{x}_k\|^2 + \|w_k\|^2\Big) \tag{6.22}$$

$$m^T(x_k,i)T^{-1}(j)m(x_k,i) \leq \gamma_3(i,j)\|\bar{x}_k\|^2 \tag{6.23}$$

$$-2w_k^T n(x_k,i) \leq \gamma_4(i)\Big(\|\bar{x}_k\|^2 + \|w_k\|^2\Big) \tag{6.24}$$

Also, it is easy to show that:

$$\begin{aligned}
&\Big(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big)^T T^{-1}(j)(\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k) - x_k^T T^{-1}(i)x_k \\
&= \Big(\hat{\mathscr{A}}(i)\bar{x}_k + \hat{D}_1 w_k\Big)^T \hat{T}^{-1}(j)(\hat{\mathscr{A}}(i)\bar{x}_k + \hat{D}_1 w_k) - \bar{x}_k^T \hat{T}^{-1}(i)\bar{x}_k
\end{aligned} \tag{6.25}$$

Then, using (6.21), (6.22), (6.23), (6.24) and (6.25) in (6.20), one gets the following:

$$\begin{aligned}
&V(x_{k+1}) - V(x_k) - z_k^T w_k \\
&= \frac{1}{2}\begin{bmatrix}\bar{x}_k \\ w_k\end{bmatrix}^T \begin{bmatrix}\mathscr{R}_{11}(i,j) & \mathscr{R}_{12}(i,j) \\ \mathscr{R}_{12}^T(i,j) & \mathscr{R}_{22}(i,j)\end{bmatrix}\begin{bmatrix}\bar{x}_k \\ w_k\end{bmatrix},
\end{aligned} \tag{6.26}$$

where,

$$\mathscr{R}_{11}(i,j) = \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{\mathscr{A}}_K(i) - \hat{T}^{-1}(i) + \Big(\gamma_1(i,j) + \gamma_2(i,j) + \gamma_3(i,j) + \gamma_4(i)\Big)I_{n+1}$$

$$\mathscr{R}_{12}(i,j) = \hat{\mathscr{A}}_K^T(i)\hat{T}^{-1}(j)\hat{D}_1 - \bar{\mathscr{C}}_K^T(i)$$

$$\mathscr{R}_{22}(i,j) = \hat{D}_1^T\hat{T}^{-1}(j)\hat{D}_1 - (D_2^T + D_2) + (\gamma_2(i,j) + \gamma_4(i))I_s.$$

Now, using (6.14c), (6.14d) and (6.18) in (6.26), we get:

$$V(x_{k+1}) - V(x_k) - z_k^T w_k \leq \frac{1}{2}\begin{bmatrix} \bar{x}_k \\ w_k \end{bmatrix}^T \mathscr{S}(i,j)\begin{bmatrix} \bar{x}_k \\ w_k \end{bmatrix} \leq 0$$

Therefore, with $u_k = K(i)x_k$, the system (6.4) is locally passive. □

6.3 Local feedback passivity over erasure network

In this section, we deal with the problem of local feedback passivation over a Gilbert-Elliott type communication channel.

Similar to preceding chapters, in this chapter as well we consider a TCP-like protocol. Under a TCP-like protocol, with perfect state knowledge, one can define an information set given by: $\mathcal{I}_k = \{x_0, x_1, .., x_k, v_0, v_1, ..., v_{k-1}\}$.

Suppose u'_k is the controller output and is sent to the actuators through a lossy network. Then, under the zero-input scheme [57], similar to chapter 5, one can relate u_k (as defined in (6.1)) with u'_k by the expression $u_k = v_k u'_k$.

Consider a piecewise linear state-feedback control law $u'_k = K(i)x_k$. Then, with a polyhedral partition $\{\mathscr{X}_i\}_{i\in\mathcal{N}}$ satisfying (5.3) (given in Chapter 5), nonlinear system (6.1) over a Gilbert-

Elliott type channel takes the form:

$$\begin{aligned} x_{k+1} &= \Big(A(i) + v_k B_1 K(i)\Big)x_k + a(i) + D_1 w_k + m(x_k, i) \\ z_k &= \Big(C(i) + v_k B_2 K(i)\Big)x_k + c(i) + D_2 w_k + n(x_k, i), \text{ if } x_k \in \mathscr{X}_i. \end{aligned} \tag{6.27}$$

Observe that due to the randomness of packet losses (v_k), the closed-loop systems (6.27) becomes stochastic in nature. Thus, to analyze the local feedback passivity of the system (6.27), one needs a notion of local stochastic passivity. Local stochastic passivity, in the spirit of [70] and [50], is defined as follows:

Definition 6.3.1. *Consider system (6.27) with $w_k \in \mathcal{W} \subseteq \mathbb{R}^s$ and $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, such that the state trajectories, with every disturbance input $w_k \in \mathcal{W}$, always remain in $\mathcal{X}$. The system (6.27) is said to be locally passive in the stochastic sense if there exists a nonnegative function $V : \mathcal{X} \times \mathcal{N} \to \mathbb{R}^+$, with $V(0, .) = 0$, called the storage function, such that for all $x_k \in \mathcal{X}$, $w \in \mathcal{W}$ and for all $k \in \mathbb{Z}^+$:*

$$\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, e_k) \leq \mathbb{E}\Big[z_k^T w_k\Big|\mathcal{I}_k\Big],$$

where, $e_k \in \mathcal{N}$ denotes the cell in which x_k lies in. □

Following theorem presents results for feedback passivity with random packet losses.

Theorem 6.3.1. *Consider system (6.27) with the disturbance input $w_k \in \mathcal{W}$, states $x_k \in \mathcal{X}$, for all k, and a polyhedral partition $\left\{\mathscr{X}_i\right\}_{i\in\mathcal{N}}$ equipped with (5.3). Consider the nonlinear system (6.27) with given control packet arrival probabilities α and $1 - \beta$. With a control law $u'_k = K(i)x_k = W(i)U^{-1}(i)x_k$, the nonlinear system becomes locally stochastically passive if, for all $i, j \in \mathcal{N}$, there exist matrices $T(i)$, $U(i)$, $W(i)$, $R(i) > 0$, $G(i) > 0$, and positive scalars h, r, q such that:*

$$\hat{T}(j) = \begin{bmatrix} T(j) & 0_{n\times 1} \\ 0_{1\times n} & h \end{bmatrix} > 0, \tag{6.28a}$$

$$\begin{bmatrix} \Omega_{11}(i) & 0_{n\times 1} & \Omega_{13}(i) & \Omega_{14}(i) & 0_{n\times 1} & \Omega_{16}(i) & 0_{n\times 1} \\ 0_{1\times n} & \Omega_{22}(i) & qc^T(i) & qa^T(i) & q & qa^T(i) & q \\ \Omega_{13}^T(i) & qc(i) & \Omega_{33} & D_1^T & 0_{s\times 1} & D_1^T & 0_{s\times 1} \\ \Omega_{14}^T(i) & qa(i) & D_1 & \Omega_{44}(j) & 0_{n\times 1} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & \Omega_{55} & 0_{1\times n} & 0 \\ \Omega_{16}^T(i) & qa(i) & D_1 & 0_{n\times n} & 0_{n\times 1} & \Omega_{66}(l) & 0_{n\times 1} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & 0 & 0_{1\times n} & \Omega_{77} \end{bmatrix} \geq 0 \tag{6.28b}$$

$$\Big[\rho_1(i,j) + \rho_2(i,j) + \rho_3(i,j) + \rho_4(i,l) + \rho_5(i,l) + \rho_6(i,l) + \rho_7(i)\Big] I_{n+1} \leq \mathscr{L}(i), \tag{6.28c}$$

$$\Big[\rho_2(i,j) + \rho_5(i,l) + \rho_7(i)\Big] I_s \leq G(i), \tag{6.28d}$$

where

$$\Omega_{11}(i) = \Big[U(i) + U^T(i) - T(i)\Big] - R(i)$$

$$\Omega_{22} = 2q - (h + r)$$

$$\Omega_{13}(i) = U^T(i)C^T(i) + \bar{p}_k W^T(i) B_2^T$$

$$\Omega_{14}(i) = U^T(i)A^T(i) + W^T(i)B_1^T, \quad \Omega_{16}(i) = U^T(i)A^T(i)$$

$$\Omega_{33} = \left(D_2^T + D_2\right) - G(i), \Omega_{44}(j) = \frac{1}{\bar{p}_k} T(j), \quad \Omega_{55} = \frac{h}{\bar{p}_k}$$

$$\Omega_{66}(l) = \frac{1}{1-\bar{p}_k} T(l), \quad \Omega_{77} = \frac{h}{1-\bar{p}_k}$$

$\mathscr{L}(i)$ is as defined in Theorem 6.2.2,

$$\rho_1(i,j) = 2\bar{p}_k \epsilon(i) \|\bar{\mathscr{A}}_K(i)\| \|T^{-1}(j)\|$$

$$\rho_2(i,j) = \bar{p}_k \epsilon(i) \|T^{-1}(j)\| \|D_1\|$$

$$\rho_3(i,j) = \bar{p}_k \epsilon^2(i) \|T^{-1}(j)\|$$

$$\rho_4(i,l) = 2(1-\bar{p}_k)\epsilon(i) \|\bar{A}(i)\| \|T^{-1}(l)\|$$

$$\rho_5(i,l) = (1-\bar{p}_k)\epsilon(i)\|T^{-1}(l)\|\|D_1\|$$

$$\rho_6(i,j) = (1-\bar{p}_k)\epsilon^2(i)\|T^{-1}(l)\|$$

$$\rho_7(i) = \delta(i),$$

$$\text{for } k \geq 1,\ \ \bar{p}_k = \begin{cases} \alpha, & if\ \ v_{k-1} = 0 \\ 1-\beta, & if\ \ v_{k-1} = 1 \end{cases}$$

and

$$\bar{p}_0 = \frac{\alpha}{\alpha+\beta}$$

Proof: Assume that LMIs (6.28a) and (6.28b) are satisfied. Using the same line of argument as used in the proof for Theorem 6.2.2, one gets that $\hat{U}(i) := diag\{U(i), q\}$ is non singular. Consider the following:

$$\Omega'(i,j) =$$

$$\begin{bmatrix} \mathscr{O}_1(i) & 0_{n\times 1} & \Omega_{13}(i) & \Omega_{14}(i) & 0_{n\times 1} & \Omega_{16}(i) & 0_{n\times 1} \\ 0_{1\times n} & \mathscr{O}_2 & qc^T(i) & qa^T(i) & q & qa^T(i) & q \\ \Omega_{13}^T(i) & qc(i) & \Omega_{33} & D_1^T & 0_{s\times 1} & D_1^T & 0_{s\times 1} \\ \Omega_{14}^T(i) & qa(i) & D_1 & \Omega_{44}(j) & 0_{n\times 1} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & \Omega_{55} & 0_{1\times n} & 0 \\ \Omega_{16}^T(i) & qa(i) & D_1 & 0_{n\times n} & 0_{n\times 1} & \Omega_{66}(l) & 0_{n\times 1} \\ 0_{1\times n} & q & 0_{1\times s} & 0_{1\times n} & 0 & 0_{1\times n} & \Omega_{77} \end{bmatrix}$$

where

$$\mathscr{O}_1(i) = \left[U^T(i)T^{-1}(i)U(i)\right] - R(i), \quad \mathscr{O}_2 = \frac{q^2}{h} - r.$$

From (6.28b), using same reasoning as used in the proof for Theorem 2, it can be proved that $\Omega'(i,j) \geq 0$.

Then, with $\mathscr{P}(i) = diag\left\{U^{-1}(i), \frac{1}{q}, I_s, I_n, 1, I_n, 1\right\}$,

$$\begin{aligned} \mathscr{P}^T \Omega'(i,j) \mathscr{P} &\geq 0 \\ Or \begin{bmatrix} \hat{T}^{-1}(i) - \mathscr{L}(i) & (\bar{\mathscr{C}}'_K(i))^T & \hat{\mathscr{A}}^T_K(i) & \hat{A}^T(i) \\ \bar{\mathscr{C}}'_K(i) & \Omega_{33} & \hat{D}_1^T & \hat{D}_1^T \\ \hat{\mathscr{A}}_K(i) & \hat{D}_1 & \Lambda_1(j) & \Lambda_3 \\ \hat{A}(i) & \hat{D}_1 & \Lambda_3 & \Lambda_2(l) \end{bmatrix} &\geq 0, \end{aligned} \tag{6.30}$$

where,

$$\Lambda_1(j) = \bar{p}_k^{-1} \hat{T}(j), \;\; \Lambda_2(l) = \left(1 - \bar{p}_k\right)^{-1} \hat{T}(l), \;\; \Lambda_3 = 0_{n+1},$$

$$\mathscr{C}'_K(i) = C(i) + \bar{p}_k B_2(i) K(i), \;\; \bar{\mathscr{C}}'_K(i) = \begin{bmatrix} \mathscr{C}'_K(i) & c(i) \end{bmatrix}.$$

Using Schur complement, as $\hat{T}(j) > 0$ for all $j \in \mathcal{N}$, (6.30) implies:

$$\begin{aligned} &\begin{bmatrix} \hat{\mathscr{A}}^T_K(i) & \hat{A}^T(i) \\ \hat{D}_1^T(i) & \hat{D}_1^T(i) \end{bmatrix} \begin{bmatrix} \Lambda_1^{-1} & \Lambda_3 \\ \Lambda_3 & \Lambda_2^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathscr{A}}_K(i) & \hat{D}_1(i) \\ \hat{A}(i) & \hat{D}_1(i) \end{bmatrix} - \begin{bmatrix} \hat{T}^{-1}(i) - \mathscr{L}(i) & (\bar{\mathscr{C}}'_K(i))^T \\ \bar{\mathscr{C}}'_K(i) & \Omega_{33} \end{bmatrix} \leq 0 \\ &\implies \begin{bmatrix} \mathscr{S}_{11}(i,j) & \mathscr{S}_{12}(i,j) \\ \mathscr{S}_{12}^T(i,j) & \mathscr{S}_{22}(i,j) \end{bmatrix} \leq 0 \end{aligned}, \tag{6.31}$$

where,

$$\mathscr{S}_{11}(i,j) = \bar{p}_k \hat{\mathscr{A}}^T_K(i) \hat{T}^{-1}(j) \hat{\mathscr{A}}_K(i) + (1 - \bar{p}_k) \hat{A}^T(i) \hat{T}^{-1}(l) \hat{A}(i) - \hat{T}^{-1}(i) + \mathscr{L}(i)$$

$$\mathscr{S}_{12}(i,j) = \bar{p}_k \hat{\mathscr{A}}^T_K(i) \hat{T}^{-1}(j) \hat{D}_1 + (1 - \bar{p}_k) \hat{A}^T(i) \hat{T}^{-1}(l) \hat{D}_1 - (\bar{\mathscr{C}}'_K(i))^T$$

$$\mathscr{S}_{22}(i,j) = \bar{p}_k \hat{D}_1^T \hat{T}^{-1}(j) \hat{D}_1 + (1 - \bar{p}_k) \hat{D}_1^T \hat{T}^{-1}(l) \hat{D}_1 - \Omega_{33}.$$

Consider a piecewise quadratic storage function of the form $V(x_k, i) = \frac{1}{2} x_k^T T^{-1}(i) x_k$ if $x_k \in \mathscr{X}_i$.

Assume that $x_{k+1} \in \mathscr{X}_j$ if $v_k = 1$, and $x_{k+1} \in \mathscr{X}_l$ if $v_k = 0$. Note that e_{k+1} can either be j or l depending on v_k. Thus, with a control law $u_k = W(i)U^{-1}(i)x_k = K(i)x_k$ if $x_k \in \mathscr{X}_i$, $i \in \mathcal{N}$:

$$\begin{aligned}
&\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, i) - \mathbb{E}\Big[z_k^T w_k\Big|\mathcal{I}_k\Big] \\
&= \frac{1}{2}\Big\{\bar{p}_k\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k, i)\Big]^T T^{-1}(j)\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k + m(x_k, i)\Big] \\
&+ \Big(1 - \bar{p}_k\Big)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k + m(x_k, i)\Big]^T T^{-1}(l)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k + m(x_k, i)\Big] - x_k^T T^{-1}(i)x_k \\
&- [\bar{\mathscr{C}}'_K(i)\bar{x}_k + D_2 w_k + n(x_k, i)]^T w_k - w_k^T[\bar{\mathscr{C}}'_K(i)\bar{x}_k + D_2 w_k + n(x_k, i)]\Big\} \\
&= \frac{1}{2}\Big\{\bar{p}_k\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big]^T T^{-1}(j)\Big[\bar{\mathscr{A}}_K(i)\bar{x}_k + D_1 w_k\Big] \\
&+ \Big(1 - \bar{p}_k\Big)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k\Big]^T T^{-1}(l)\Big[\bar{A}(i)\bar{x}_k + D_1 w_k\Big] \\
&- x_k^T T^{-1}(i)x_k - [\bar{\mathscr{C}}'_K(i)\bar{x}_k + D_2 w_k]^T w_k - w_k^T[\bar{\mathscr{C}}'_K(i)\bar{x}_k + D_2 w_k] \\
&+ \bar{p}_k\Big[2\bar{x}_k^T \bar{\mathscr{A}}_K^T(i)T^{-1}(j)m(x_k, i) + 2w_k^T D_1^T T^{-1}(j)m(x_k, i) + m^T(x_k, i)T^{-1}(j)m(x_k, i)\Big] \\
&+ (1 - \bar{p}_k)\Big[2\bar{x}_k^T \bar{A}^T(l)T^{-1}(l)m(x_k, i) + 2w_k^T D_1^T T^{-1}(l)m(x_k, i) + m^T(x_k, i)T^{-1}(l)m(x_k, i)\Big] - 2w_k^T n(x_k, i)\Big\}
\end{aligned}$$

Using the same line of argument as used in the proof for Theorem 5.3.2 in Chapter 5, we get:

$$\begin{aligned}
&\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, i) - \mathbb{E}\Big[z_k^T w_k\Big|\mathcal{I}_k\Big] \\
&\leq \begin{bmatrix} \bar{x}_k^T & w_k^T \end{bmatrix} \begin{bmatrix} \mathscr{S}_{11}(i, j) & \mathscr{S}_{12}(i, j) \\ \mathscr{S}_{12}^T(i, j) & \mathscr{S}_{22}(i, j) \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ w_k \end{bmatrix}.
\end{aligned}$$

Now, from (6.31):

$$\mathbb{E}\Big[V(x_{k+1}, e_{k+1})\Big|\mathcal{I}_k\Big] - V(x_k, i) - \mathbb{E}\Big[z_k^T w_k\Big|\mathcal{I}_k\Big] \leq 0.$$

Hence, closed-loop system (6.27) is locally passive in the stochastic sense. □

Remark 6.3.1. *Although the feedback passivity of a nonlinear system with packet loss is considered in [51], results are derived by considering the frequency of the packet losses rather than considering an appropriate stochastic packet loss model. Our analysis is however based on a* *more realistic packet loss model. Further, as our approach contains LMIs for the controller design, it is more systematic than that of [51].*

Note 6.3.1. *It is straight forward to see that, if one puts $\epsilon(i) = 0$ and $\delta(i) = 0$ in the above theorem, for all $i \in \mathcal{N}$, then the result corresponds to the result for feedback passivity of an PWA system over a Gilbert-Elliott type channel.*

6.4 Numerical Example

Consider the nonlinear system (6.1) with the following system parameters:

$$f(x_k) = \begin{bmatrix} 4sin(x_k^1) + x_k^2 \\ x_k^1 + x_k^3 \\ x_k^1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}, \quad h(x_k) = x_k^1, \quad B_2 = 0.1, \quad D_2 = 2.$$

where $x_k = \begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix}$. External input is assumed to be $w_k = 0.02sin(0.2\pi k)exp(-k/25)$.

To demonstrate the results presented in Theorem 6.2.2, we calculate a piecewise linear state-feedback law that passifies the system in the region $-0.82 \leq x_k^1 \leq 0.82$ (note that nonlinearity exists only in x_k^1). The region $-0.82 \leq x_k^1 \leq 0.82$ is partitioned into 26 cells, which are given by: $-0.82 \leq x_k^1 < -0.78$, $-0.78 \leq x_k^1 < -0.74$, $-0.74 \leq x_k^1 < -0.7$, $-0.7 \leq x_k^1 < -0.65$, $-0.65 \leq x_k^1 < -0.6$, $-0.6 \leq x_k^1 < -0.55$, $-0.55 \leq x_k^1 < -0.5$, $-0.5 \leq x_k^1 < -0.45$, $-0.45 \leq x_k^1 < -0.4$, $-0.4 \leq x_k^1 < -0.34$, $-0.34 \leq x_k^1 < -0.28$, $-0.28 \leq x_k^1 < -0.13$, $-0.13 \leq x_k^1 < 0$, $0 \leq x_k^1 < 0.13$, $0.13 \leq x_k^1 < 0.28$, $0.28 \leq x_k^1 < 0.34$, $0.34 \leq x_k^1 < 0.4$, $0.4 \leq x_k^1 < 0.45$, $0.45 \leq x_k^1 < 0.5$, $0.5 \leq x_k^1 < 0.55$, $0.55 \leq x_k^1 < 0.6$, $0.6 \leq x_k^1 < 0.65$, $0.65 \leq x_k^1 < 0.7$, $0.7 \leq x_k^1 < 0.74$, $0.74 \leq x_k^1 < 0.78$, $0.78 \leq x_k^1 < 0.82$. Similar to chapter 5, $A(i)$ and $a(i)$ for the piecewise linear approximations are computed using Taylor series expansion. We again use the *lmisolver* function available in SCILAB to solve the LMIs in each of the cells.

We solve 26 LMIs each 8 variables. Solving (6.14b) we then calculate the controller gain $K(i)$ such that the closed system becomes locally feedback passive. From Figure 6.1, one can see that difference in storage function at each stage is less than or equal to the supply rate.

For the local feedback passivity with packet losses, we consider the region $-0.5 \leq x_k^1 \leq 0.5$. The region $-0.5 \leq x_k^1 \leq 0$ is partition into cells: $-0.5 \leq x_k^1 < -0.47$, $-0.47 \leq x_k^1 < -0.44$, $-0.44 \leq x_k^1 < -0.41$, $-0.41 \leq x_k^1 < -0.38$, $-0.38 \leq x_k^1 < -0.35$, $-0.35 \leq x_k^1 < -0.32$, $-0.32 \leq x_k^1 < -0.29$, $-0.29 \leq x_k^1 < -0.26$, $-0.26 \leq x_k^1 < -0.23$, $-0.23 \leq x_k^1 < -0.2$, $-0.2 \leq x_k^1 < -0.17$, $-0.17 \leq x_k^1 < -0.14$, $-0.14 \leq x_k^1 < -0.11$, $-0.11 \leq x_k^1 < -0.07$, $-0.07 \leq x_k^1 \leq 0$. The region $0 \leq x_k^1 \leq 0.5$ is partitioned in the similar fashion as the partition of the region $-0.5 \leq x_k^1 \leq 0$, i.e., $0 \leq x_k^1 < 0.07$, $0.07 \leq x_k^1 < 0.11$ and so on. Then, solving LMI (6.28b) having 8 variables in each of the 30 cells, we calculate the controller gain $K(i)$ in each cells with a control packet arrival probability $\alpha = 0.95$ and $1 - \beta = 0.96$. Figure 6.2 demonstrates that the closed-loop system is locally passive with the quadratic storage function $V(x_k) = \frac{1}{2}x_k^T T^{-1}(i)x_k$.

Note that, for the local feedback passivity with packet losses, we have to consider smaller cells as compared to feedback passivity. This is due to the fact that condition (6.28c) contains the term $\rho_4(i,l)$ which comes from the open loop system dynamics.

6.5 Summary

In this chapter, using a PWA approximation approach, we have first derived sufficient conditions for the local passivity of a smooth nonlinear system. Then, we have designed a piecewise linear state-feedback control law that makes the closed-loop system to be locally passive. Finally, for a system with random control packet losses, a piecewise linear state-feedback control law is derived which makes the closed-loop system locally passive. Moreover, the problem of controller design for feedback passivation of a PWA system over a lossy communication channel can easily be derived as a special case.

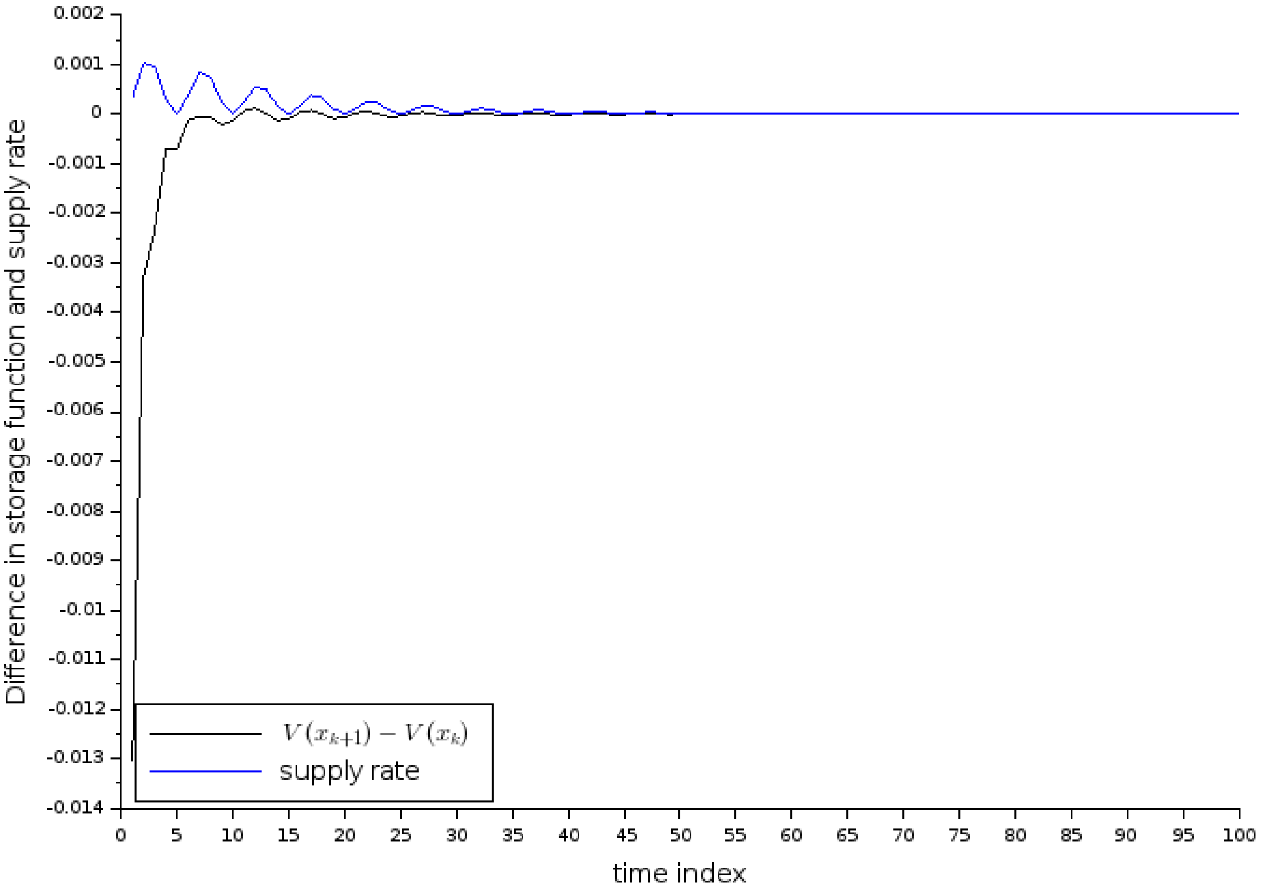

Figure 6.1: Difference in storage function and supply rate

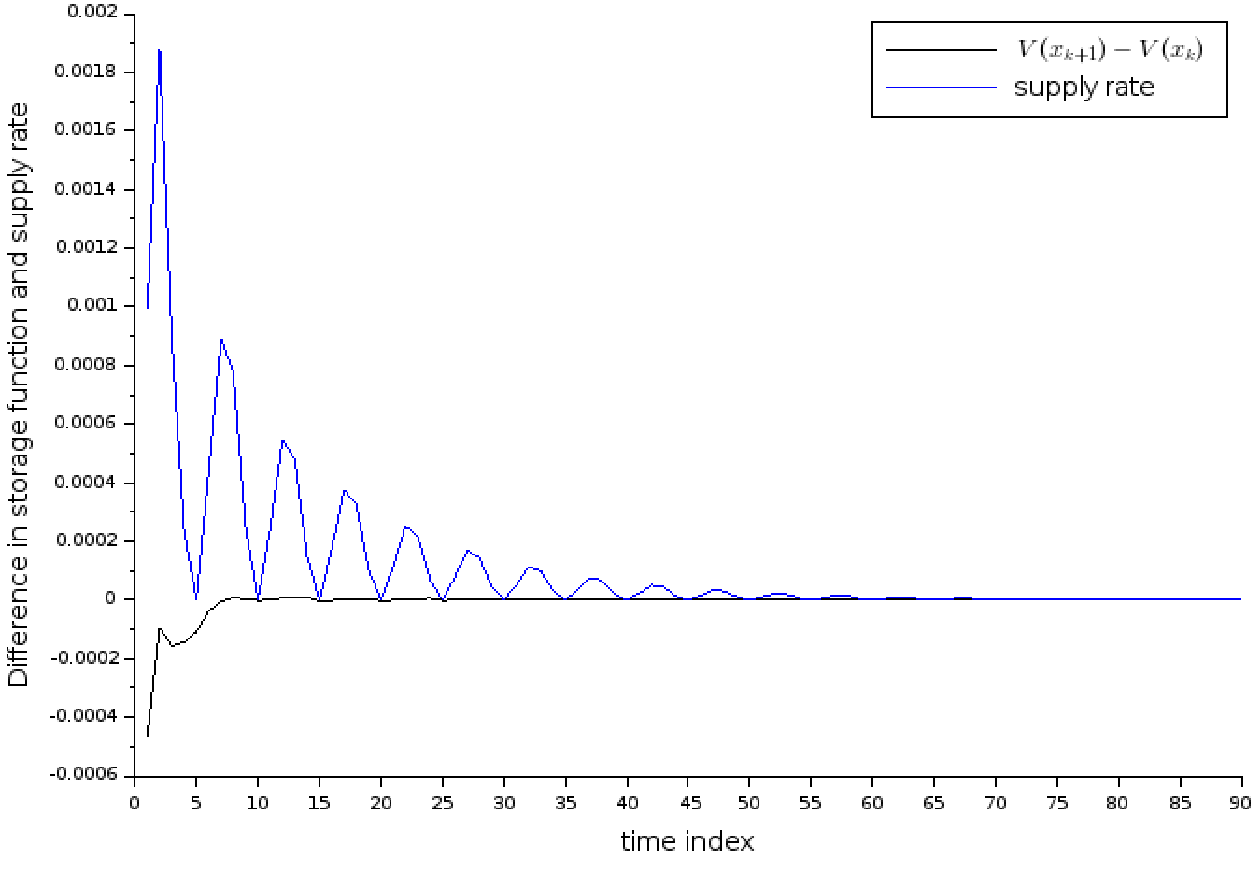

Figure 6.2: Difference in storage function and supply rate

7

Conclusions and Future Works

This chapter summarizes the thesis and provides possible directions for research works.

7.1 Conclusions

In this thesis, we have investigated a few control problems for systems operating over an erasure communication network. The thesis can be summarized as follows.

In Chapter 2, we have investigated the H_∞ optimal control problem for an LTI system operating over multiple Gilbert-Elliott type communication channels. The existence conditions for both finite horizon controller and infinite horizon controller are derived in terms of the disturbance attenuation level and the control packet arrival probabilities. The convergence of the infinite horizon cost function and the associated coupled algebraic Riccati equations are investigated. Further, stability of the closed-loop system with the optimal controller is established.

Chapter 3 considers the jump linear quadratic optimal control problem for Markovian jump linear systems over multiple Gilbert-Elliott type channels. The existence of the infinite horizon controller is established by studying the convergence of the infinite horizon cost function and the associated CAREs. Further, stability of the closed-loop system with the optimal controller is proved using a less stringent observability notion as compared to the ones used in related literature.

Chapter 4 extends the works presented in Chapter 2 to an MJLS.

Chapter 5 and Chapter 6 study the H_∞ controller design problem and the feedback passivation problem, respectively, for a smooth nonlinear system by considering a piecewise affine approximation. This approach reduces the respective controller design problems to solving a set of LMIs with nonlinear constraints. These results are then extended for a smooth nonlinear system with random Markovian packet losses.

7.2 Direction for Future Research Work

Following are a few directions to which the results of this thesis can be extended.

(i) Throughout this thesis, it is assumed that the acknowledgment of the control packet reception is sent through perfect communication channels, and thus, it is available all the time. In certain cases, however, the acknowledgment might have to be sent through lossy channels, which may cause random acknowledgment loss. All of the results presented in the thesis can be extended to the case where acknowledgment packets are also sent through Gilbert-Elliott type communication channels.

(ii) This thesis presents results pertaining to the state-feedback controller design problem. These results can further be extended to the output-feedback case, where the output may get lost randomly.

(iii) In Chapter 5 and Chapter 6, we have considered smooth nonlinear system dynamics where the coefficient matrices of the control input u_k and the external or disturbance input w_k, respectively, are constant matrices. These results can be generalized for nonlinear system dynamics where the coefficient matrices are also smooth functions of the states.

Bibliography

[1] J. Nilsson, "Real-time control systems with delays," Ph.D. dissertation, Lund Institute of Technology, Sweden, 1998.

[2] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, 2007.

[3] P. Millán, L. Orihuela, I. Jurado, and F. R. Rubio, "Formation control of autonomous underwater vehicles subject to communication delays," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 2, pp. 770–777, 2014.

[4] S. Oncu, N. Van de Wouw, W. M. H. Heemels, and H. Nijmeijer, "String stability of interconnected vehicles under communication constraints," in *Proceedings of the 51st IEEE Annual Conference on Decision and Control (CDC)*, 2012, pp. 2459–2464.

[5] Y. Mo, E. Garone, and B. Sinopoli, "LQG control with Markovian packet loss," in *Proceedings of the European Control Conference (ECC)*, 2013, pp. 2380–2385.

[6] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, and S. S. Sastry, "Optimal control with unreliable communication: the TCP case," in *Proceedings of the American Control Conference (ACC)*, Portland, OR, 2005, pp. 3354–3359.

[7] A. Chiuso, N. Laurenti, L. Schenato, and A. Zanella, "LQG cheap control over SNR-limited lossy channels with delay," in *Proceedings of the 52nd IEEE Annual Conference on Decision and Control (CDC)*, 2013, pp. 3988–3993.

[8] ——, “LQG cheap control subject to packet loss and SNR limitations,” in *Proceedings of the European Control Conference (ECC)*, 2013, pp. 2374–2379.

[9] ——, “LQG-like control of scalar systems over communication channels: The role of data losses, delays and SNR limitations,” *Automatica*, vol. 50, no. 12, pp. 3155–3163, 2014.

[10] D. Du, M. Fei, and T. Jia, “Modelling and stability analysis of MIMO networked control systems with multi-channel random packet losses,” *Transactions of the Institute of Measurement and Control*, vol. 35, no. 1, pp. 66–74, 2013.

[11] E. Garone, B. Sinopoli, A. Goldsmith, and A. Casavola, “LQG control for MIMO systems over multiple erasure channels with perfect acknowledgment,” *IEEE Transactions on Automatic Control*, vol. 57, no. 2, pp. 450–456, 2012.

[12] H. Lin, H. Su, Z. Shu, Z.-G. Wu, and Y. Xu, “Optimal estimation in UDP-like networked control systems with intermittent inputs: stability analysis and suboptimal filter design,” *IEEE Transactions on Automatic Control*, vol. 61, no. 7, pp. 1794–1809, 2015.

[13] H. Lin, H. Su, P. Shi, Z. Shu, R. Lu, and Z.-G. Wu, “Optimal estimation and control for lossy network: stability, convergence, and performance,” *IEEE Transactions on Automatic Control*, vol. 62, no. 9, pp. 4564–4579, 2017.

[14] H. Lin, J. Lam, M. Z. Chen, Z. Shu, and Z.-G. Wu, “Interacting multiple model estimator for networked control systems: stability, convergence, and performance,” *IEEE Transactions on Automatic Control*, vol. 64, no. 3, pp. 928–943, 2018.

[15] P. Seiler and R. Sengupta, “An H_∞ approach to networked control,” *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356–364, 2005.

[16] Z. Wang, F. Yang, D. W. Ho, and X. Liu, “Robust H_∞ control for networked systems with random packet losses,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 37, no. 4, pp. 916–924, 2007.

[17] J. C. Geromel, A. P. Gonçalves, and A. R. Fioravanti, "Dynamic output feedback control of discrete-time Markov jump linear systems through linear matrix inequalities," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 573–593, 2009.

[18] D. Wang, J. Wang, and W. Wang, "H_∞ controller design of networked control systems with Markov packet dropouts," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 43, no. 3, pp. 689–697, 2013.

[19] Y. Shoukry, J. Araujo, P. Tabuada, M. Srivastava, and K. H. Johansson, "Minimax control for cyber-physical systems under network packet scheduling attacks," in *Proceedings of the 2nd ACM International Conference on High Confidence Networked Systems*, 2013, pp. 93–100.

[20] J. Moon and T. Başar, "Control over TCP-like lossy networks: A dynamic game approach," in *Proceedings of the American Control Conference (ACC)*, Washington, DC, 2013, pp. 1578–1583.

[21] ——, "Control over lossy networks: A dynamic game approach," in *Proceedings of the American Control Conference (ACC)*, Portland, OR, 2014, pp. 5367–5372.

[22] ——, "Minimax control over unreliable communication channels," *Automatica*, vol. 59, pp. 182–193, 2015.

[23] ——, "Minimax control of MIMO systems over multiple TCP-like lossy networks," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 110–115, 2014.

[24] Z. Li, X. Yin, I. Kolmanovsky, J. Lu, D. Filev, and E. Atkins, "Robust H_∞ control for a class of networked uncertain systems with multiple channels subject to Markovian switching," in *Proceedings of the 54th IEEE Annual Conference on Decision and Control (CDC)*, 2015, pp. 6856–6861.

[25] J. Moon and T. Başar, "Robust control of LTI systems over unreliable communication channels with unreliable acknowledgments," in *Proceedings of the IEEE Region 10 Conference (TENCON)*. IEEE, 2016, pp. 3390–3393.

[26] T. Ogura, K. Kobayashi, H. Okada, and M. Katayama, "H-infinity control design considering packet loss as a disturbance for networked control systems," *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, vol. 100, no. 2, pp. 353–360, 2017.

[27] H. Yuan, Y. Xia, H. Yang, and Y. Yuan, "Resilient control for wireless networked control systems under DoS attack via a hierarchical game," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 15, pp. 4604–4623, 2018.

[28] J. Moon, "Robust control over unreliable communication channels with unreliable acknowledgements," *Journal of Institute of Control, Robotics and Systems*, vol. 25, no. 6, pp. 572–577, 2019.

[29] J. A. Ball and J. Helton, "H^∞ control for nonlinear plants: connections with differential games," in *Proceedings of the 28th IEEE Conference on Decision and Control(CDC)*. IEEE, 1989, pp. 956–962.

[30] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 9, pp. 1283–1293, 1992.

[31] J. A. Ball, J. W. Helton, and M. L. Walker, "H^∞ control for nonlinear systems with output feedback," *IEEE Transactions on Automatic Control*, vol. 38, no. 4, pp. 546–559, 1993.

[32] W. Lin and C. I. Byrnes, "H_∞-control of discrete-time nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 41, no. 4, pp. 494–510, 1996.

[33] J. Huang and C.-F. Lin, "Numerical approach to computing nonlinear H_∞ control laws," *Journal of Guidance, Control, and Dynamics*, vol. 18, no. 5, pp. 989–994, 1995.

[34] E. Sontag, “Nonlinear regulation: The piecewise linear approach,” *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 346–358, 1981.

[35] M. Johansson, “Piecewise linear control systems,” Ph.D. dissertation, Lund Institute of Technology, Sweden, 1999.

[36] A. Rantzer and M. Johansson, “Piecewise linear quadratic optimal control,” *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 629–637, 2000.

[37] G. Feng, “Controller design and analysis of uncertain piecewise-linear systems,” *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 2, pp. 224–232, 2002.

[38] H. Zhang, G. Feng, and C. Dang, “Stability analysis and H_∞ control for uncertain stochastic piecewise-linear systems,” *IET Control Theory & Applications*, vol. 3, no. 8, pp. 1059–1069, 2009.

[39] Z. Wang, G. Wei, and G. Feng, “Reliable H_∞ control for discrete-time piecewise linear systems with infinite distributed delays,” *Automatica*, vol. 45, no. 12, pp. 2991–2994, 2009.

[40] Z. Wang, D. W. Ho, Y. Liu, and X. Liu, “Robust H_∞ control for a class of nonlinear discrete time-delay stochastic systems with missing measurements,” *Automatica*, vol. 45, no. 3, pp. 684–691, 2009.

[41] C. I. Byrnes, A. Isidori, and J. C. Willems, “Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 36, no. 11, pp. 1228–1240, 1991.

[42] W. Lin and C. I. Byrnes, “Passivity and absolute stabilization of a class of discrete-time nonlinear systems,” *Automatica*, vol. 31, no. 2, pp. 263–267, 1995.

[43] A. Van Der Schaft, *$\mathcal{L}_2$-gain and passivity techniques in nonlinear control*. Springer, 2000, vol. 2.

[44] E. M. Navarro-López and E. Fossas-Colet, "Feedback passivity of nonlinear discrete-time systems with direct input–output link," *Automatica*, vol. 40, no. 8, pp. 1423–1428, 2004.

[45] E. M. Navarro-López, "Local feedback passivation of nonlinear discrete-time systems through the speed-gradient algorithm," *Automatica*, vol. 43, no. 7, pp. 1302–1306, 2007.

[46] L. Iannelli, F. Vasca, and K. Camlibel, "Complementarity and passivity for piecewise linear feedback systems," in *Proceedings of the 45th IEEE Conference on Decision and Control (CDC)*, 2006, pp. 4212–4217.

[47] W. J. Chang, C. H. Huang, and C. C. Ku, "Synthesis of discrete nonlinear passive systems via affine T-S fuzzy models with input energy constraints," in *Proceedings of the International Conference on Mechatronics and Automation*, 2007, pp. 1950–1955.

[48] A. Bemporad, G. Bianchini, and F. Brogi, "Passivity analysis and passification of discrete-time hybrid systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1004–1009, 2008.

[49] M. Xia, P. J. Antsaklis, V. Gupta, and M. J. McCourt, "Determining passivity using linearization for systems with feedthrough terms," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2536–2541, 2014.

[50] H. Zakeri and P. J. Antsaklis, "Passivity and passivity indices of nonlinear systems under operational limitations using approximations," *International Journal of Control*, pp. 1–11, 2019.

[51] Y. Wang, M. Xia, V. Gupta, and P. J. Antsaklis, "On feedback passivity of discrete-time nonlinear networked control systems with packet drops," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2434–2439, 2014.

[52] M. Sahebsara, T. Chen, and S. L. Shah, "Optimal H_∞ filtering in networked control systems with multiple packet dropouts," *Systems & Control Letters*, vol. 57, no. 9, pp. 696–702, 2008.

[53] A. Khelassi, D. Theilliol, and P. Weber, "Control design for over-actuated systems based on reliability indicators," in *Proceedings of the International Conference on Control (UKACC '10)*. IET, 2010, pp. 1–6.

[54] T. Başar and P. Bernhard, *H_∞ optimal control and related minimax design problems: a dynamic game approach*. Springer Science & Business Media, 2008.

[55] Z. Pan and T. Başar, "H_∞-control of Markovian jump systems and solutions to associated piecewise-deterministic differential games," in *New trends in dynamic games and applications*. Springer, 1995, pp. 61–94.

[56] C. Kawan and J.-C. Delvenne, "Network entropy and data rates required for networked control," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 57–66, 2016.

[57] L. Schenato, "To zero or to hold control inputs with lossy links?" *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1093–1099, 2009.

[58] G. Haßlinger and O. Hohlfeld, "The Gilbert-Elliott model for packet loss in real time services on the internet," in *Proceedings of the GI/ITG Conference on Measuring, Modelling and Evaluation Computer and Communication Systems*. VDE, 2008, pp. 1–15.

[59] M. Aliyu and E. Boukas, "H_∞ control for Markovian jump nonlinear systems," in *Proceedings of the 37th IEEE Conference on Decision and Control*, vol. 1. IEEE, 1998, pp. 766–771.

[60] W. Rudin, *Principles of mathematical analysis*. McGraw-hill New York, 1976, vol. 3, no. 4.2.

[61] Y. Ji and H. J. Chizeck, "Controllability, observability and discrete-time markovian jump linear quadratic control," *International Journal of Control*, vol. 48, no. 2, pp. 481–498, 1988.

[62] Y.-Y. Cao and J. Lam, "Stochastic stabilizability and H_∞ control for discrete-time jump linear systems with time delay," *Journal of the Franklin Institute*, vol. 336, no. 8, pp. 1263–1281, 1999.

[63] M. Iosifescu, *Finite Markov processes and their applications*. Courier Corporation, 2014.

[64] S. M. Ross, *Introduction to probability models*. Academic press, 2014.

[65] L. Li and V. A. Ugrinovskii, "On necessary and sufficient conditions for H_∞ output feedback control of Markov jump linear systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1287–1292, 2007.

[66] T. Stegink, C. De Persis, and A. van der Schaft, "A unifying energy-based approach to stability of power grids with market dynamics," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 2612–2622, 2016.

[67] R. Delpoux, L. Hetel, and A. Kruszewski, "Permanent magnet synchronous motor control via parameter dependent relay control," in *Proceedings of the American Control Conference (ACC)*, 2014, pp. 5230–5235.

[68] B. Jayawardhana, R. Ortega, E. Garcia-Canseco, and F. Castanos, "Passivity of nonlinear incremental systems: application to PI stabilization of nonlinear RLC circuits," *Systems & Control Letters*, vol. 56, no. 9-10, pp. 618–622, 2007.

[69] W. Lin and C. I. Byrnes, "Dissipativity, L_2 gain and H_∞ control for discrete-time nonlinear systems," in *Proceedings of the American Control Conference (ACC)*, vol. 2. IEEE, 1994, pp. 2257–2260.

[70] Y. Wang, V. Gupta, and P. J. Antsaklis, "Stochastic passivity of discrete-time Markovian jump nonlinear systems," in *Proceedings of the American Control Conference (ACC)*, 2013, pp. 4879–4884.

[71] M. Aliyu and E. Boukas, "H_∞ control for Markovian jump nonlinear systems," in *Proceedings of the 37th IEEE Conference on Decision and Control (CDC)*, vol. 1. IEEE, 1998, pp. 766–771.