Density Results in $C(S^n, S^n)$ for Lower Dimensions and Riemannian Morphisms

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Certificate

It is certified that the work contained in this thesis titled *"Density Results in $C(S^n, S^n)$ for Lower Dimensions and Riemannian Morphisms"* by Raj Bhawan Yadav, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

September 15, 2014

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Dedicated

to

My Parents

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Abstract

A standard paradigm in the expansion of knowledge is to describe nearby familiar objects of a new object. Density results in topology may be viewed as fitting into this paradigm. Another such paradigm is to give relations between new and relatively familiar objects. Morphisms between objects of a category may be viewed as falling under this paradigm. In this thesis, we have considered problems from these two paradigms. We have studied density results in the spaces of continuous functions $C(S^1, S^1)$ and $C(S^2, S^2)$. In the second half of the thesis we have defined and explored morphisms between Riemannian manifolds.

Contents

Chapter 1

Introduction

1.1 Background and Motivation

M. H. Stone gave the Stone-Weierstrass Theorem [Sto48a], which is a density result with respect to uniform topology. This result is a generalization of the Weierstrass Approximation Theorem by lightening the restrictions imposed on the domain over which the given functions are defined. By taking the co-domain of the given functions to be the complex plane in place of real line, he went further and gave a complex version of the latter result as Stone-Weierstrass Theorem [Sto48b]. Our aim is to seek density results for $C(M, N)$, the class of continuous functions from a topological manifold M to another topological manifold N. In particular, we have sought and reported on such results in $C(S^1, S^1)$ and $C(S^2, S^2)$. In our study of density results in $C(S^1, S^1)$, we found the need for a version of the Stone-Weierstrass Theorem with constraints. We call this result as 'Stone-Weierstrass with finitely many interpolatory constraints' (Theorem 2.2.1). Although developed independently by us, this result is available in the literature [HPR76, BCH01]. We add that our proof is different and more direct in comparison to that available in the literature. With the latter result (Theorem 2.2.1) as background, we give a generalization in Theorems 2.4.1 and 2.4.2.

The second half of the thesis is devoted to an entirely different question about morphisms in the category of Riemannian manifolds. In mathematics, we not only study objects with mathematical structures, but also consider functions which relate structures of the two objects. For example, in topology we consider continuous functions, in group theory we study homomorphisms, in linear algebra – linear transformations, in the theory of smooth manifolds – smooth functions and in category theory – functors are the 'right' kind of functions which are used for this purpose.

However such functions or more appropriately such morphisms are lacking in the study of Riemannian manifolds. Though we have isometries and conformal maps, the strong conditions they impose restrict wider applicability. When one attempts to define such morphisms between Riemannian manifolds, one realizes a need for morphisms between inner product spaces. Consequently, we devote the fourth chapter to developing a theory of morphisms between inner product spaces. Finally, in the fifth chapter, we define morphisms between Riemannian manifolds and pursue some of their properties.

1.2 Thesis Overview

There are five chapters in the thesis. Here we provide its chapter-wise content.

1.2.1 Chapter 1

The first chapter contains motivation behind this thesis and an overview of the same.

1.2.2 Chapter 2

In the second chapter we provide our proof of 'Stone-Weierstrass with finitely many interpolatory constraints' (Theorem 2.2.1). We give a generalization of this theorem by replacing the finite interpolating set by an arbitrary closed set. We also provide complex versions of these two results in this chapter.

1.2.3 Chapter 3

In this chapter we give some density results in $C(S^1, S^1)$ and $C(S^2, S^2)$. Also we pose some problems in the direction of generalizing the Stone-Weierstrass Theorem to $C(M, N)$, the class of continuous functions from a topological manifold M to another topological manifold N.

1.2.4 Chapter 4

In this chapter we define the concept of morphisms between inner product spaces and study some of their properties.

1.2.5 Chapter 5

In this chapter we introduce the notion of morphisms between Riemannian manifolds and study some of their properties.

Chapter 2

Generalized Stone-Weierstrass Theorem

2.1 Introduction

The classical Stone-Weierstrass Theorem [Sto48a] has been generalized in different directions. We recommend [Pin05] as a general survey on such density results. Theorem 1 of [HPR76] is one such generalization involving finitely many interpolatory constraints. The same result is also to be found in [BCH01]. An independent proof of this theorem is provided in Section 2.2. A natural generalization of this theorem would be to allow for an arbitrary closed $S \subset X$, as our interpolating set. To proceed, we examine the notion of a separating unital algebra in Section 2.3. A version of the Stone-Weierstrass Theorem with an arbitrary closed subset of the metric space as the interpolating set is stated and proved in Section 2.4 as Theorem 2.4.1.

2.1.1 Notations

Let X be a compact metric space. $\mathbb{F}$ denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. $C(X, \mathbb{F})$ denotes the collection of $\mathbb{F}$–valued continuous functions on X with the sup–norm. And, let $\mathcal{A}$ denote a subset of $C(X, \mathbb{F})$.

Let k be any natural number. Let $S = \{x_1, x_2, \cdots, x_k\} \subset X$ (with $x_i \neq x_j$ for distinct i and j) and let $V = \{v_1, v_2, \cdots, v_k\} \subset \mathbb{F}$. Define

$$C_S^V(X, \mathbb{F}) = \{f \in C(X, \mathbb{F}) \mid f(x_1) = v_1, f(x_2) = v_2, \cdots, f(x_k) = v_k\} \text{ and}$$
$$\mathcal{A}_S^V(X, \mathbb{F}) = \{f \in \mathcal{A} \mid f(x_1) = v_1, f(x_2) = v_2, \cdots, f(x_k) = v_k\}.$$

2.2 Finitely many Interpolatory Constraints

Theorem 2.2.1 (Stone-Weierstrass with finitely many interpolatory constraints). *For a natural number k, let $S = \{x_1, x_2, \cdots, x_k\} \subset X$ (with $x_i \neq x_j$ for distinct i and j) and let $V = \{v_1, v_2, \cdots, v_k\} \subset \mathbb{R}$. If $\mathcal{A}$ is a unital sub–algebra of $C(X, \mathbb{R})$ which separates points then $\mathcal{A}_S^V(X, \mathbb{R})$ is dense in $C_S^V(X, \mathbb{R})$.*

Proof. Case (a) Assume that the $\{v_i\}_1^k$ are all distinct, that is, $v_i \neq v_j$ for every $i \neq j$, where $i, j = 1, \cdots, k$.

Let $f \in C_S^V(X, \mathbb{R})$. By the Stone-Weierstrass theorem, there is a sequence $\{p_n\}$ in $\mathcal{A}$ such that $p_n \to f$ uniformly. Define a new sequence of functions $f_n : X \to \mathbb{R}$ via

$$f_n(x) = \sum_{1 \le i \le k} v_i \, \frac{\prod_{1 \le j \le k, j \neq i} \big(p_n(x) - p_n(x_j)\big)}{\prod_{1 \le j \le k, j \neq i} \big(p_n(x_i) - p_n(x_j)\big)}.$$

Dropping finitely many functions f_n, if necessary, ensures that the sequence is well-defined. Further,

$$f_n \to \sum_{1 \le i \le k} v_i \, \frac{\prod_{1 \le j \le k, j \neq i} \big(f(x) - f(x_j)\big)}{\prod_{1 \le j \le k, j \neq i} \big(f(x_i) - f(x_j)\big)} \text{ uniformly.} \tag{2.1}$$

However, we note that

$$f(x) - \sum_{1 \le i \le k} v_i \, \frac{\prod_{1 \le j \le k, j \neq i} \big(f(x) - f(x_j)\big)}{\prod_{1 \le j \le k, j \neq i} \big(f(x_i) - f(x_j)\big)}$$

is a polynomial in $f(x)$ of degree $k - 1$ which has at least k roots viz., $v_1, v_2, \cdots, v_k$. Hence for each $x \in X$,

$$f(x) = \sum_{1 \le i \le k} v_i \, \frac{\prod_{1 \le j \le k, j \neq i} \big(f(x) - f(x_j)\big)}{\prod_{1 \le j \le k, j \neq i} \big(f(x_i) - f(x_j)\big)}. \tag{2.2}$$

From 2.1 and 2.2 we conclude that f_n converges uniformly to the given f. Clearly, $f_n(x_i) = v_i$ for every $i = 1, 2, \cdots, k$ and $f_n \in \mathcal{A}_S^V(X, \mathbb{R})$.

Case (b) When two or more of v_i become equal, we assume without loss of generality that the prescribed values $\{v_i\}_1^k$ are ordered as $v_1 \leq v_2 \leq \cdots \leq v_k$. For some positive real number α, let $\{w_j\}_1^k$ be the strictly increasing sequence of real numbers given by $w_j = v_j + j\alpha$ for each j in $1, 2, \cdots, k$. By Urysohn's Lemma, there exists an $h \in C(X, \mathbb{R})$ such that $h(x_i) = w_i$ for every i in $1, 2, \cdots, k$. Now, define $f^{\pm} = \frac{1}{2}f \pm h$. Clearly, $f^{\pm}(x_i) \neq f^{\pm}(x_j)$ for every $i \neq j$ with $i, j = 1, 2, \cdots, k$.

Applying our conclusion in Case (a) to $f^{\pm}$, we get two sequences $f_n^{\pm} : X \to \mathbb{R}$ such that $f_n^{\pm} \to f^{\pm}$ uniformly on X satisfying $f_n^{\pm}(x_i) = f^{\pm}(x_i)$ for each $i = 1, 2, \cdots, k$. Let $f_n = f_n^+ + f_n^-$. Clearly $f_n \to f$ uniformly X and $f_n \in \mathcal{A}_S^V(X, \mathbb{R})$. □

Theorem 2.2.2 (Complex Stone-Weierstrass with finitely many interpolatory constraints)**.** *For a natural number k, let $S = \{x_1, x_2, \cdots, x_k\} \subset X$ (with $x_i \neq x_j$ for distinct i and j) and let $V = \{a_1 + ib_1, a_2 + ib_2, \cdots, a_k + ib_k\} \subset \mathbb{C}$. Let $\mathcal{A}$ be a unital sub–algebra of $C(X, \mathbb{C})$ such that if $f \in \mathcal{A}$, then $\bar{f} \in \mathcal{A}$. Further assume that $\mathcal{A}$ separates points of X. Then $\mathcal{A}_S^V(X, \mathbb{C})$ is dense in $C_S^V(X, \mathbb{C})$.*

Proof. Let $\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} | f(x) \in \mathbb{R} \text{ for all } x \in X\}$. Clearly $\mathcal{A}_{\mathbb{R}}$ is a unital sub–algebra of $\mathcal{A}$ over $\mathbb{R}$. Further, if $f \in \mathcal{A}$ separates points $x, y \in X$, we note that either $\Re(f)(x) \neq \Re(f)(y)$ or $\Im(f)(x) \neq \Im(f)(y)$. Since both $\Re(f), \Im(f) \in \mathcal{A}_{\mathbb{R}}$, we conclude that $\mathcal{A}_{\mathbb{R}}$ separates points.

Applying Theorem 2.2.1, we get two sequences $\{g_n\}_1^\infty, \{h_n\}_1^\infty$ in $\mathcal{A}_{\mathbb{R}}$ such that $g_n \to \Re(f)$ uniformly and $h_n \to \Im(f)$ uniformly with $g_n(x_j) = a_j$ and $h_n(x_j) = b_j$ for $j = 1, 2, \cdots, k$. Let $f_n = g_n + ih_n$ and we have $f_n \to f$ uniformly with $f_n \in \mathcal{A}_S^V(X, \mathbb{C})$. □

2.3 Generalizing the notion of a Separating Algebra

The following definitions are immediate when we attempt to generalize the notion of a separating subset of $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$.

Definition 2.3.1. *Let $k \geq 2$ be a fixed natural number. For $\mathcal{A} \subset C(X, \mathbb{F})$, $\mathcal{A}$ is k–separating, if, given any k distinct $x_1, x_2, \cdots, x_k \in X$, there exists an $f \in \mathcal{A}$ such that $f(x_i) \neq f(x_j)$ for every $i \neq j$ with i, j in $1, 2, \cdots, k$. Further, $\mathcal{A}$ is k–interpolating if given any k distinct $x_1, x_2, \cdots, x_k \in X$, and arbitrary $v_1, v_2, \cdots, v_k \in \mathbb{F}$, there exists an $f \in \mathcal{A}$ such that $f(x_i) = v_i$ for every i in $1, 2, \cdots, k$.*

Although a 2–separating subset $\mathcal{A}$ is equivalent to a separating $\mathcal{A}$, it is of interest to know whether requiring $\mathcal{A}$ to be k–separating or k–interpolating is more stringent than requiring $\mathcal{A}$ to be separating. The next proposition answers this question.

Proposition 2.3.1. *If $\mathcal{A}$ is a separating unital sub–algebra of $C(X, \mathbb{F})$ and $k \geq 2$ is any natural number, then $\mathcal{A}$ is k–separating and k–interpolating.*

Proof. We proceed by induction. For $k = 2$, let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Further, let $v_1, v_2 \in \mathbb{F}$ be given. Since $\mathcal{A}$ is separating, there exists $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$. Now,

$$f(x) = v_1 + (v_2 - v_1)\frac{g(x) - g(x_1)}{g(x_2) - g(x_1)}$$

is in $\mathcal{A}$ and takes values v_1 and v_2 at x_1 and x_2 respectively. Thus $\mathcal{A}$ is 2–interpolating.

Assume that $\mathcal{A}$ is k–interpolating for $k = m$, a natural number. And suppose we are given distinct $x_1, x_2, \cdots, x_{m+1} \in X$ and $v_1, v_2, \cdots, v_{m+1} \in \mathbb{F}$. Then, by induction hypothesis, there exists functions f_1, f_2 and $f_3 \in \mathcal{A}$ satisfying

$$\begin{array}{lllll} f_1(x_1) = 0, & f_1(x_2) = 0, & \cdots, & f_1(x_{m-1}) = 0, & f_1(x_{m+1}) = 1; \\ f_2(x_2) = 0, & f_2(x_3) = 0, & \cdots, & f_2(x_m) = 0, & f_2(x_{m+1}) = 1; \quad \text{and} \\ f_3(x_1) = v_1, & f_3(x_2) = v_2, & \cdots, & f_3(x_{m-1}) = v_{m-1}, & f_3(x_m) = v_m. \end{array}$$

Now, the function $f(x) = f_3(x) + f_1(x)f_2(x)\left(v_{m+1} - f_3(x_{m+1})\right) \in \mathcal{A}$ and has the desired interpolating properties. This completes the induction.

We have proved that $\mathcal{A}$ is k–interpolating for each natural number k and hence it is k–separating. □

The conclusion of the above Proposition 2.3.1 is implicit in the conclusion of Theorem 2.2.1.

2.4 Arbitrary Interpolatory Constraints

Definition 2.4.1. *Let S be a closed subset of X and $\mathcal{A} \subset C(X, \mathbb{F})$. We say that $\mathcal{A}$ is separating mod S, if, for any T such that $S \subset T \subset X$ where $T \setminus S$ is finite and any continuous $g : T \to \mathbb{F}$, there exists an $f \in \mathcal{A}$ such that $f_{|T} = g$.*

By Proposition 2.3.1, a unital sub–algebra $\mathcal{A} \subset C(X, \mathbb{F})$ is separating if and only if it is separating mod $\varnothing$, the empty set. Thus, the notion of a separating subset of $C(X, \mathbb{F})$ is subsumed by Definition 2.4.1, at least for unital sub–algebras.

Our main Theorem 2.4.1 necessitates the following definition.

Definition 2.4.2. *Let S be a closed subset of X and $\mathcal{A} \subset C(X, \mathbb{F})$. We say that $\mathcal{A}$ is interpolating mod S, if, for any $f \in \bar{\mathcal{A}}$ and for any T such that $S \subset T \subset X$ where $T \setminus S$ is finite, there is a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that $f_n \to f$ uniformly on X and $f_n(x) = f(x)$ for all $x \in T$.*

Theorem 2.4.1 (Stone-Weierstrass with arbitrary interpolatory constraints)**.** *Let X be a compact metric space and S be a closed subset of X with $\mathcal{A} \subset C(X, \mathbb{R})$. Further, assume that $\mathcal{A}$ is a unital sub–algebra of $C(X, \mathbb{R})$ which is separating mod S and interpolating mod S. Then for any $f \in C(X, \mathbb{R})$, there exists a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that $f_n \to f$ uniformly on X with $f_n(x) = f(x)$ for every $x \in S$.*

Proof. It suffices to produce, for each natural number n, an $f_n \in \mathcal{A}$ satisfying $f_n(x) = f(x)$ for every $x \in S$ and $||f_n - f|| < \frac{1}{n}$.

Pick any $u, v \in X \setminus S$. Since $\mathcal{A}$ is separating mod S, there exists $f_{S;u,v} \in \mathcal{A}$ such that $f_{S;u,v}(x) = f(x)$ for every $x \in S \cup \{u, v\}$.

Fix $u \in X \setminus S$. For a natural number n and $v \in X \setminus S$, set

$$U_v = \left\{ w \in X | f_{S;u,v}(w) < f(w) + \tfrac{1}{4n} \right\}.$$

U_v is an open set containing v. Hence, $\{U_v\}_{v \in X}$ is an open cover of X. By compactness of X, there exists finitely many $v_1, v_2, \cdots, v_k$ such that $X = \bigcup_{i=1}^{k} U_{v_i}$. Define $h_{S;u} = \min_{1 \le i \le k} f_{S;u,v_i}$. Then, $h_{S;u} \in \bar{\mathcal{A}}$, $h_{S;u}(x) = f(x)$ for every $x \in S \cup \{u\}$ and $h_{S;u}(x) < f(x) + \frac{1}{4n}$ for every $x \in X$. Since $\mathcal{A}$ is interpolating mod S, there exists $g_{S;u} \in \mathcal{A}$ such that $g_{S;u}(x) < h_{S;u}(x) + \frac{1}{4n} < f(x) + \frac{1}{2n}$ for every $x \in X$ and $g_{S;u}(x) = h_{S;u}(x) = f(x)$ for every $x \in S \cup \{u\}$.

Next, set

$$V_u = \left\{ w \in X | g_{S;u}(w) > f(w) - \tfrac{1}{2n} \right\}.$$

Then, $\{V_u\}_{u \in X}$ is an open cover of X which admits a finite subcover, say, $X = \cup_{i=1}^{l} V_{u_i}$. Define $q_n = \max_{1 \le i \le l} g_{S;u_i}$. Then, $q_n \in \bar{\mathcal{A}}$ with $q_n(x) = f(x)$ for every $x \in S$ and $||q_n - f|| < \frac{1}{2n}$. Again, since $\mathcal{A}$ is interpolating mod S, there exists an $\{f_n\}$ in $\mathcal{A}$ such that $||f_n - q_n|| < \frac{1}{2n}$ and $f_n(x) = q_n(x) = f(x)$ for every $x \in S$. By triangle inequality, $||f_n - f|| < \frac{1}{n}$. This completes the proof. □

Theorem 2.4.2 (Complex Stone-Weierstrass with arbitrary interpolatory constraints). *Let X be a compact metric space and S be a closed subset of X with $\mathcal{A}$ being a unital subalgebra of $C(X, \mathbb{C})$ such that if $f \in \mathcal{A}$ then $\bar{f} \in \mathcal{A}$. Further, assume that $\mathcal{A}$ is separating mod S and interpolating mod S. Then for any $f \in C(X, \mathbb{C})$, there exists a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that $f_n \to f$ uniformly on X with $f_n(x) = f(x)$ for every $x \in S$.*

Proof. Let $\mathcal{A}_\mathbb{R} = \{f \in \mathcal{A} | f(x) \in \mathbb{R} \text{ for all } x \in X\}$. Clearly $\mathcal{A}_\mathbb{R}$ is a unital sub–algebra of $\mathcal{A}$ over $\mathbb{R}$. To prove that $\mathcal{A}_\mathbb{R}$ is separating mod S, take a continuous $g : T \to \mathbb{R}$ for some T such that $S \subset T \subset X$ such that $T \setminus S$ is finite. Since $\mathcal{A}$ is separating mod S, we get an $f \in \mathcal{A}$ such that $f_{|T} = g$. Now, $\Re(f) \in \mathcal{A}_\mathbb{R}$ and $\Re(f)_{|T} = g$ which proves that $\mathcal{A}_\mathbb{R}$ is separating mod S.

Next, suppose $f \in \bar{\mathcal{A}}_\mathbb{R}$ and $S \subset T \subset X$ such that $T \setminus S$ is finite. Since $\bar{\mathcal{A}}_\mathbb{R} \subset \bar{\mathcal{A}}$ and $\mathcal{A}$ is interpolating mod S, we get a sequence $\{f_n\}_1^\infty$ in $\mathcal{A}$ such that $f_n \to f$ uniformly on X and $f_n(x) = f(x)$ for all $x \in T$. Consequently we get the sequence $\{\Re(f_n)\}_1^\infty$ in $\mathcal{A}_\mathbb{R}$ which converges to $\Re(f) = f$ and satisfies $\Re(f_n(x)) = f(x)$ for all $x \in T$. This proves that $\mathcal{A}_\mathbb{R}$ is interpolating mod S.

Applying Theorem 2.4.1, we get two sequences $\{g_n\}_1^\infty, \{h_n\}_1^\infty$ in $\mathcal{A}_\mathbb{R}$ such that $g_n \to \Re(f)$ uniformly and $h_n \to \Im(f)$ uniformly with $g_n(x) = \Re(f)$ and $h_n(x) = \Im(f)$ for all x in S. Let $f_n = g_n + ih_n$ and we have $f_n \to f$ uniformly with $f_n(x) = f(x)$ for all $x \in S$. □

The results developed in this chapter have been submitted for publication as [SYb].

Chapter 3

Some Density Results in $C(S^1, S^1)$ and $C(S^2, S^2)$

3.1 Introduction

When one starts looking for dense subsets of $C(S^1, S^1)$ and $C(S^2, S^2)$ first question that arises in one's mind is about the topology to be used. In some sense this topology should be natural. In this chapter whatever topologies we shall choose, either will be easily seen to be quite natural or we shall justify them to be so. Also the dense subsets should be either familiar classes of functions or there should a specified way to obtain them. In section 3.3 of this article, we give many dense subsets of $C(S^1, S^1)$ by using PL functions, polynomials and Stone-Weierstrass theorem. In section 3.4, by using uniformity, we give theorems 3.4.4, 3.4.5, 3.4.9, 3.4.8, 3.4.7 and 3.4.6 which are a sort of justification that compact open topology is one of the topologies with respect to which dense subsets should be searched. Also, in this section we give a dense subset of a subclass of $C(S^2, S^2)$ with respect to a topology which is slightly weaker than compact open topology. In section 3.5 we pose problems of finding dense subsets with respect to compact open topology and one other problem.

3.2 Preliminaries

3.2.1 Covering Spaces

Let E and X be topological spaces, and let $q : E \to X$ be a continuous map.

Definition 3.2.1. *An open set $U \subset X$ is said to be* ***evenly covered by q*** *if $q^{-1}(U)$ is a disjoint union of connected open subsets of E (called the* ***sheets of the covering over*** *U), each of which is mapped homeomorphically onto U by q.*

Definition 3.2.2. *A* ***covering map or projection map*** *is a continuous surjective map $q : E \to X$ such that E is connected and locally path-connected, and every point of X has an evenly covered neighborhood. If $q : E \to X$ is a covering map, we call E a* ***covering space of*** *X and X the* ***base of the covering****.*

Example 3.2.1.1. *The exponential quotient map $p : \mathbb{R} \to S^1$ given by $p(x) = \exp(2\pi\imath x)$ is a covering map.*

Example 3.2.1.2. *The n^{th} power map $p_n : S^1 \to S^1$ given by $p_n(z) = z^n$ is also a covering map.*

Example 3.2.1.3. *Let $\mathbb{T}^n = \underbrace{S^1 \times S^1 \times ... \times S^1}_{n\ times}$. Define $p^n : \mathbb{R}^n \to \mathbb{T}^n$ by*

$$p^n(x_1, ..., x_n) = (p(x_1), ..., p(x_n)),$$

where p is exponential map of Example 3.2.1.1. It can be verified that p^n is a covering map.

Definition 3.2.3. *If $q : E \to X$ is a covering map and $\phi : Y \to X$ is any continuous map, a* ***lift of*** *ϕ is a continuous map $\tilde{\phi} : Y \to E$ such that $q \circ \tilde{\phi} = \phi$.*

From [Lee00] we have following result called as **Path Lifting Property**:

Lemma 3.2.1. *Let $q : E \to X$ be a covering map. Suppose $f : [0, 1] \to X$ is any path, and $e \in E$ is any point in the fiber of q over $f(0)$. Then there exists a unique lift $\tilde{f} : I \to E$ of f such that $\tilde{f}(0) = e$.*

3.2.2 Uniform Spaces

We give following definitions and results from [Kel09].

Definition 3.2.4. *A uniformity for a set X is a nonempty family $\mathscr{U}$ of subsets of $X \times X$ such that*

(a) each member of $\mathscr{U}$ contains the diagonal $\triangle$;

(b) if $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$;

(c) if $U \in \mathscr{U}$, then $VV \subset U$ for some $V \in \mathscr{U}$;

(d) if U and V are members of $\mathscr{U}$, then $U \cap V \in \mathscr{U}$;

(e) if $U \in \mathscr{U}$ and $U \subset V \subset X \times X$, then $V \in \mathscr{U}$;

The pair $(X, \mathscr{U})$ is called a uniform space.

A subfamily $\mathscr{B}$ of a uniformity $\mathscr{U}$ is a **base** for $\mathscr{U}$ iff each member of $\mathscr{U}$ contains a member of $\mathscr{B}$. A subfamily $\mathscr{S}$ is a **subbase** for a uniformity $\mathscr{U}$ if the family of finite intersections of members of $\mathscr{S}$ forms a base for $\mathscr{U}$.

Theorem 3.2.1. *A nonempty family $\mathscr{B}$ of subsets of $X \times X$ is a base for some uniformity for X iff*

(a) each member of $\mathscr{B}$ contains the diagonal $\triangle$;

(b) if $U \in \mathscr{B}$ then U^{-1} contains a member of $\mathscr{B}$;

(c) if $U \in \mathscr{B}$ then $VV \subset U$ for some $V \in \mathscr{B}$; and

(d) intersection of two members of $\mathscr{B}$ contains a member.

Theorem 3.2.2. *A subfamily $\mathscr{S}$ of subsets of $X \times X$ is subbase for some uniformity for X if*

(a) each member of $\mathscr{S}$ contains the diagonal $\triangle$;

(b) if $U \in \mathscr{S}$ then U^{-1} contains a member of $\mathscr{S}$;

(c) for each $U \in \mathscr{S}$, $VV \subset U$ for some $V \in \mathscr{S}$.

Let $(X, \mathscr{U})$ be a uniform space. Given any $U \in \mathscr{U}$, define $U[x] = \{y|(x, y) \in U\}$. Collection of all $T \subset X$ such that for all $x \in T$, there exists $U \in \mathscr{U}$ with $U[x] \subset T$, is a topology on X. This topology is called *uniform* topology.

Remark 3.2.1. *(1) There can be more than one uniformities which has same uniform topology.*
(2) Different metrics may give rise to same uniformity.

Theorem 3.2.3. *A topological space is uniformizable iff it is completely regular.*

Theorem 3.2.4. *A uniformity is pseudometrizable iff it has a countable base.*

From above two theorems and two remarks we can conclude that, on the level of abstraction, uniform spaces lie in between metric spaces and topological spaces. Also we have the following theorem

Theorem 3.2.5. *If X is a compact regular topological space then there exists a unique uniformity whose uniform topology is the topology of the space.*

Definition 3.2.5. *Let X be any set and $\mathfrak{a}$ be family of subsets of X. Also let F be a family of functions from X to a uniform space $(Y, \mathscr{V})$. Then the family of all sets of the form $\{(f, g) : (f(x), g(x)) \in V, \forall x \in A\}$, for $V \in \mathscr{V}$ and $A \in \mathfrak{a}$ is a subbase for a uniformity. This uniformity is called uniformity of **uniform convergence on members** of $\mathfrak{a}$. If $\mathfrak{a}$ is singleton containing X then the resulting uniformity is called uniformity of **uniform convergence**. The uniform topology of this uniformity is called the topology of **uniform convergence** .*
*If X is a topological space and $\mathfrak{a}$ is collection of all compact subsets of X then the resulting uniformity is called uniformity of **uniform convergence on compacta**. And the uniform topology of this uniformity is called topology of **uniform convergence on compacta**.*

Let X and Y be topological spaces and F be a family of functions from X to Y. For each subset K of X and U of Y, define

$$W(K, U) = \{f : f \in F \text{ and } f(K) \subset U\}.$$

The topology obtained by taking the family of all sets of the form $W(K, U)$, for K a compact subset of X and U an open subset of Y, as a subbase, is called compact open topology.

Theorem 3.2.6. *Let F be a family of continuous functions from a topological space X to a uniform space (Y, v). Then the topology of uniform convergence on compacta is the compact open topology.*

3.3 Density in $C(S^1, S^1)$

Let $C(S^1, S^1)$ denote the collection of all continuous functions from S^1 to S^1. Further let $I = [0, 1]$ and $\widehat{C}(I, \mathbb{R}) = \{f | f : I \to \mathbb{R} \text{ is continuous with } f(0) \in [0, 1) \text{ and } f(1) - f(0) \in \mathbb{Z}\}$. Consider the function $\alpha : [0, 1] \to S^1$ given by $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ and the covering space $\mathbb{R}$ of S^1 with projection map p given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$.
For $f \in C(S^1, S^1)$, let $\tilde{f}_\alpha$ be the unique lift of $f \circ \alpha$ such that $\tilde{f}_\alpha(0) \in [0, 1)$. Clearly $\tilde{f}_\alpha \in \widehat{C}(I, \mathbb{R})$.
For $f \in C(S^1, S^1)$ and $g \in C(S^1, S^1)$, define

$$d_0(f, g) = 2\pi \sup_{\substack{x \in S^1 \\ x \neq (1,0)}} |\tilde{f}_\alpha(\alpha^{-1}(x)) - \tilde{g_\alpha}(\alpha^{-1}(x))|.$$

Remark 3.3.1. *If X is any compact space then the set $C(X, \mathbb{R})$ of all continuous real valued functions on X is a normed linear space with the* sup *norm given by $\|f\| = \sup_{x \in X} |f(x)|$, for all $f \in C(X, \mathbb{R})$. Let d_1 be the metric induced by this norm. With respect to this metric on $C(X, \mathbb{R})$, $\widehat{C}(I, \mathbb{R})$ is a metric space.*

Proposition 3.3.1. *d_0 is a metric on $C(S^1, S^1)$ and there exists a homeomorphism $\phi : C(S^1, S^1) \to \widehat{C}(I, \mathbb{R})$ such that $d_0(f, g) = 2\pi d_1(\phi(f), \phi(g))$, for all f and g in $C(S^1, S^1)$.*

Proof. Define $\phi : C(S^1, S^1) \to \bar{C}(I, \mathbb{R})$ by $\phi(f)(t) = \tilde{f}_\alpha(t), \forall t \in I$.

Now,

$$\begin{aligned}
&\phi(f) = \phi(g)\\
\Rightarrow\ &\phi(f)(t) = \phi(g)(t),\ \forall t \in I\\
\Rightarrow\ &\tilde{f}_\alpha(t) = \tilde{g}_\alpha(t),\ \ \forall t \in I\\
\Rightarrow\ &p \circ \tilde{f}_\alpha(t) = p \circ \tilde{g}_\alpha(t),\ \forall t \in I, \text{ where p is projection map}\\
\Rightarrow\ &f \circ \alpha(t) = g \circ \alpha(t),\ \forall t \in I\\
\Rightarrow\ &f(x) = g(x),\ x \in S^1\\
\Rightarrow\ &f = g.
\end{aligned}$$

So ϕ is injective.

Given any $g \in \widehat{C}(I, \mathbb{R})$, define f by

$$f(x) = \begin{cases} p \circ g \circ \alpha^{-1}(x) & \text{if } x \in S^1 - \{(1,0)\}\\ p \circ g(0) = p \circ g(1) & \text{otherwise.}\end{cases}$$

Clearly $f \in C(S^1, S^1)$ and $\phi(f) = g$. So ϕ is onto.

$$\begin{aligned}
d_0(f,g) &= 2\pi \sup_{\substack{x\in S^1\\ x\neq(1,0)}} |\tilde{f}_\alpha(\alpha^{-1}(x)) - \tilde{g}_\alpha(\alpha^{-1}(x))|\\
&= 2\pi \sup_{\substack{x\in S^1\\ x\neq(1,0)}} |\phi(f)(\alpha^{-1}(x)) - \phi(g)(\alpha^{-1}(x))|\\
&= 2\pi \sup_{t\in(0,1)} |\phi(f)(t) - \phi(g)(t)|\\
&= 2\pi \sup_{t\in[0,1]} |\phi(f)(t) - \phi(g)(t)|\\
&= 2\pi d_1(\phi(f), \phi(g)).
\end{aligned}$$

So d_0 is a metric space and ϕ is a homeomorphism. □

3.3.1 Density Results using PL-functions and Polynomials

Definition 3.3.1. *A function $f : [a,b] \to \mathbb{R}$ is said to be piecewise linear (PL) if there exist sets of points $\{p_i\}_{i=1}^k$ in $[a,b]$ and $\{q_i\}_{i=1}^k$ in $\mathbb{R}$ such that $p_1 = a$, $p_k = b$ and for all $t \in [p_i, p_{i+1}]$, $f(t) = q_i + (q_{i+1} - q_i)\frac{(t-p_i)}{p_{i+1}-p_i}$, for every $1 \leq i \leq k-1$.*

Define

$$\widehat{PL}(I,\mathbb{R}) = \{f|f : I \to \mathbb{R} \text{ is PL and } f(0) \in [0,1),\ f(1) - f(0) \in \mathbb{Z}\}.$$

Theorem 3.3.1. *The set $\widehat{PL}(I,\mathbb{R})$ is dense in $\widehat{C}(I,\mathbb{R})$.*

Proof. Let $f \in \widehat{C}(I,\mathbb{R})$. Since a continuous function on a compact set is uniformly continuous and [0,1] is compact, f is uniformly continuous. So, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{4}$, whenever $|x - y| < \delta$.
Let $\{p_i|i = 1, 2, ..., k\}$ be a partition of [0,1] such that its norm is less than δ. Let $f(p_i) = q_i$ for every $1 \le i \le k$. Define a PL function P by
$P(t) = q_i + (q_{i+1} - q_i)\frac{(t-p_i)}{p_{i+1}-p_i}$, $\forall t \in [p_i, p_{i+1}]$, for every $1 \le i \le k - 1$.
Now, for every $1 \le i \le k - 1$, we have

$$\begin{aligned}\sup_{t\in[p_i,p_{i+1}]} |f(t) - P(t)| &\le \sup_{t\in[p_i,p_{i+1}]} (|f(t) - q_i| + |(q_{i+1} - q_i)\frac{(t - p_i)}{p_{i+1} - p_i}|)\\ &\le \sup_{t\in[p_i,p_{i+1}]} (|f(t) - q_i| + |(q_{i+1} - q_i)|)\\ &\le \frac{\epsilon}{4} + \frac{\epsilon}{4}\\ &\le \frac{\epsilon}{2}.\end{aligned}$$

So $\|f - P\| < \epsilon$. This completes the proof. □

For $a, b \in \mathbb{R}$ define

$$C_{a,b}(I,\mathbb{R}) = \{f \in C(I,\mathbb{R})|f(0) = a,\ f(1) = b\} \quad \text{and}$$
$$P_{a,b}(I,\mathbb{R}) = \{p(x)|p(x) \text{ is a polynomial on } \mathbb{R} \text{ with } p(0) = a \text{ and } p(1) = b\}.$$

Lemma 3.3.1.1. *Given $a \in \mathbb{R}$, the set $P_{0,a}(I,\mathbb{R})$ is dense in $C_{0,a}(I,\mathbb{R})$ with* sup *metric.*

Proof. Let $f \in C_{0,a}(I,\mathbb{R})$. By Weierstrass approximation theorem a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly. Now define $\tilde{p}_n$ by

$$\tilde{p}_n(x) = p_n(x) - p_n(0) + x(f(1) - p_n(1) + p_n(0)).$$

Clearly $\tilde{p}_n \to f$ uniformly and $\tilde{p}_n(0) = 0$, $\tilde{p}_n(1) = a$. □

Theorem 3.3.2. *Given $a, b \in \mathbb{R}$, the set $P_{a,b}(I, \mathbb{R})$ is dense in $C_{a,b}(I, \mathbb{R})$ with* sup *metric.*

Proof. Define $\psi : C_{a,b}(I, \mathbb{R}) \to C_{0,b-a}(I, \mathbb{R})$ via $\psi(f)(x) = f(x) - a$
Clearly ψ is a homeomorphism and $\psi(P_{a,b}(I, \mathbb{R})) = P_{0,b-a}(I, \mathbb{R})$. So, since $P_{0,b-a}(I, \mathbb{R})$ is dense in $C_{0,b-a}(I, \mathbb{R})$, $P_{a,b}(I, \mathbb{R})$ is dense in $C_{a,b}(I, \mathbb{R})$. □

Definition 3.3.2. *Let $f : S^1 \to S^1$ be a continuous function. Then winding number $W(f)$ of f is the integer $\tilde{f}_\alpha(1) - \tilde{f}_\alpha(0)$.*

For $q \in S^1$ and $m \in \mathbb{Z}$ define

$$C^q_m(S^1, S^1) = \{f \in C(S^1, S^1) | f((1, 0)) = q \text{ and } W(f) = m\}$$
$$PL^q_m(S^1, S^1) = \{f \in C(S^1, S^1) | \tilde{f}_\alpha \in PL_{\tilde{q},\tilde{q}+m}(I, \mathbb{R}),\ p \circ \tilde{f}_\alpha(0) = q\} \quad \text{and}$$
$$P^q_m(S^1, S^1) = \{f \in C(S^1, S^1) | \tilde{f}_\alpha \in P_{\tilde{q},\tilde{q}+m}(I, \mathbb{R}),\ p \circ \tilde{f}_\alpha(0) = q\}.$$

Theorem 3.3.3. *The set $PL^q_m(S^1, S^1)$ is dense in $C^q_m(S^1, S^1)$.*

Proof. If $\tilde{f}_\alpha(0) = \tilde{q}$, then clearly $\phi^{-1}(C_{\tilde{q},\tilde{q}+m}(I, \mathbb{R})) = C^q_m(S^1, S^1)$ and $\phi^{-1}(PL_{\tilde{q},\tilde{q}+m}(I, \mathbb{R})) = PL^q_m(S^1, S^1)$. Since $PL_{\tilde{q},\tilde{q}+m}(I, \mathbb{R})$ is dense in $C_{\tilde{q},\tilde{q}+m}(I, \mathbb{R})$ and ϕ is a homeomorphism, $PL^q_m(S^1, S^1)$ is dense in $C^q_m(S^1, S^1)$. □

Theorem 3.3.4. *The set $P^q_m(S^1, S^1)$ is dense in $C^q_m(S^1, S^1)$.*

Proof. Proof is similar to the proof of Theorem3.3.3. □

3.3.2 Stone-Weierstrass type theorem for $C(S^1, S^1)$

Let X be a compact metric space. Fix $u, v \in X$ with $u \neq v$, $a, b \in \mathbb{R}$ and $\mathscr{A} \subset C(X, \mathbb{R})$. Define

$$C^{a,b}_{u,v}(X, \mathbb{R}) = \{f \in C(X, \mathbb{R}) | f(u) = a,\ f(v) = b\} \quad \text{and}$$
$$\mathscr{A}^{a,b}_{u,v}(X, \mathbb{R}) = \{f \in \mathscr{A} | f(u) = a,\ f(v) = b\}.$$

Lemma 3.3.2.1. *If $\mathscr{A}$ is a unital sub-algebra of $C(X, \mathbb{R})$ which separates points and a, b are any two distinct real numbers then $\mathscr{A}^{a,b}_{u,v}(X, \mathbb{R})$ is dense in $C^{a,b}_{u,v}(X, \mathbb{R})$.*

Proof. Since $\mathscr{A}$ is a unital sub-algebra of $C(X, \mathbb{R})$ which separates points, by Stone-Weierstrass theorem $\mathscr{A}$ is dense in $C(X, \mathbb{R})$. So, for each $f \in C(X, \mathbb{R})$ there exists a sequence f_n in $\mathscr{A}$ such that $f_n \to f$ uniformly. Since $a \neq b$ we can choose f_n in such a way that $f_n(v) - f_n(u) \neq 0$,for all n.
Now define

$$\tilde{f}_n = a + \frac{(b-a)(f_n(x) - f_n(u))}{f_n(v) - f_n(u)}.$$

Clearly $\tilde{f}_n(u) = a$ and $\tilde{f}_n(v) = b$, for all n and $\tilde{f}_n \to f$ uniformly. □

Theorem 3.3.5. *If $\mathscr{A}$ is a unital sub-algebra of $C(X, \mathbb{R})$ which separates points and a, b are any two real number then $\mathscr{A}^{a,b}_{u,v}(X, \mathbb{R})$ is dense in $C^{a,b}_{u,v}(X, \mathbb{R})$.*

Proof. When $a \neq b$ the result is true by above lemma. So let $f \in C^{a,a}_{u,v}(X, \mathbb{R})$. First let $a \neq 0$. By Urysohn Lemma there exists $g \in C(X, \mathbb{R})$ such that $g(u) = -\frac{a}{4}$ and $g(v) = \frac{a}{4}$. Now define f_1 and f_2 by $f_1 = \frac{1}{2}f + g$ and $f_2 = \frac{1}{2}f - g$. Clearly $f_1 \in C^{\frac{1}{4}a, \frac{3}{4}a}_{u,v}(X, \mathbb{R})$ and $f_2 \in C^{\frac{3}{4}a, \frac{1}{4}a}_{u,v}(X, \mathbb{R})$ and $f_1 + f_2 = f$. So by above lemma there exist sequences $f_{1,n} \in \mathscr{A}^{\frac{1}{4}a, \frac{3}{4}a}_{u,v}(X, \mathbb{R})$ and $f_{2,n} \in \mathscr{A}^{\frac{3}{4}a, \frac{1}{4}a}_{u,v}(X, \mathbb{R})$ such that $f_{1,n} \to f_1$ and $f_{2,n} \to f_2$ uniformly. Now define $f_n = f_{1,n} + f_{2,n}$. Clearly $f_n \in \mathscr{A}^{a,a}_{u,v}(X, \mathbb{R})$ and $f_n \to f$ uniformly. Again proof for the case when $a = 0$, can be given in the same way as was given in Theorem 3.3.2. □

For $q \in S^1$, $m \in \mathbb{Z}$ and a unital sub-algebra $\mathscr{A}$ of $C(I, \mathbb{R})$ which separates points, define

$$P^{\mathscr{A}}_{m,q}(S^1, S^1) = \{f \in C(S^1, S^1) | \tilde{f}_\alpha \in \mathscr{A}^{\tilde{q}, \tilde{q}+m}_{0,1}(I, \mathbb{R}),\ p \circ \tilde{f}_\alpha(0) = q\}.$$

Theorem 3.3.6. *For $m \in \mathbb{Z}$ and $q \in S^1$, $P^{\mathscr{A}}_{m,q}(S^1, S^1)$ is dense in $C^q_m(S^1, S^1)$.*

Proof. Proof is similar to the proof of Theorem 3.3.3. □

3.4 Density in $C(S^2, S^2)$

Let $S \subset \mathbb{C}$, $\mathscr{A} \subset C(\mathbb{C}, \mathbb{C})$. Define $\mathscr{A}(S) = \{f_{|S} | f \in \mathscr{A}\}$. Define

$$\mathscr{A}_1 = \{p(z, \bar{z}) \mid p(z, \bar{z}) \text{ is a polynomial with complex coefficients in } z, \bar{z}\}.$$

Let $K \subset \mathbb{C}$ be compact. Define

$$\mathscr{A}_1(K) = \{f | f : K \to \mathbb{C} \text{ is continuous with } f(z) = p(z, \bar{z}), \text{for some } p(z, \bar{z}) \in \mathscr{A}_1\}.$$

Lemma 3.4.1. *$\mathscr{A}_1(K)$ is a unital algebra which separates points and is closed under conjugation.*

Proof. Clearly $\mathscr{A}_1(K)$ is a unital algebra which is closed under conjugation. Since $\mathscr{A}_1(K)$ contains polynomials in z and the collection of polynomials in z separates points, $\mathscr{A}_1(K)$ separates points of K. □

Theorem 3.4.1. *$\mathscr{A}_1(K)$ is dense in $C(K, \mathbb{C})$.*

Proof. The result follows from the complex version of Stone-Weierstrass theorem and previous lemma. □

Let α be any positive integer. Define

$$\mathscr{A}_2 = \{p(z) \mid p(z) = \sum_{\substack{i=1 \\ d_{ij}, t_i \in \mathbb{N} \cup \{0\} \\ a_{ij}, b_{ij}, c_i \in \mathbb{C}}}^{k} c_i \prod_{j=1}^{t_i} (((a_{ij}z + b_{ij})(\overline{a_{ij}z + b_{ij}}))^\alpha + a_{ij}z + b_{ij} + \overline{a_{ij}z + b_{ij}})^{d_{ij}}, \text{ for some } k \in \mathbb{N}\}.$$

Lemma 3.4.0.2. *$\mathscr{A}_2(S)$ is a unital algebra which separates points of $\mathbb{C}$ and is closed under conjugation.*

Proof. Clearly $\mathscr{A}_2(S)$ is a unital algebra. Since for all $p(z) \in \mathscr{A}_2$, $\overline{p(z)} \in \mathscr{A}_2$, $\mathscr{A}_2(S)$ is closed under conjugation. Let $z_1, z_2 \in S$ with $z_1 \neq z_2$. Let $z_3, z_4 \in \mathbb{C}$ such that $\|z_3\| = \|z_4\|$ and $Re(z_3) \neq Re(z_4)$. Note that there exist $a, b \in \mathbb{C}$ such that $az_1 + b = z_3$, $az_2 + b = z_4$. So $((az_1 + b)(\overline{az_1 + b}))^\alpha + az_1 + b + \overline{az_1 + b} \neq ((az_2 + b)(\overline{az_2 + b}))^\alpha + az_2 + b + \overline{az_2 + b}$. So $\mathscr{A}_2(S)$ separates points of S.

□

Theorem 3.4.2. *If $K \subset \mathbb{C}$ is compact then $\mathscr{A}_2(K)$ is dense in $C(K, \mathbb{C})$.*

Proof. The result follows from the complex version of Stone-Weierstrass theorem and the previous lemma. □

Note that our goal is to give dense subsets of continuous functions between surfaces. In particular, we would like to study the case of continuous functions between spheres. And to extend the above results we need a complete investigation into the behaviour of elements of $\mathscr{A}_2$ at ∞.

Remark 3.4.1. *Given any $p(z) \in \mathscr{A}_2$, there exists $1 \leq l \leq k$ such that $2\alpha \sum_{j=1}^{t_i} d_{ij} = 2\alpha \sum_{j=1}^{t_r} d_{rj} = \deg(p(z))$, for all $l \leq i, r \leq k$. There are two possible cases. In the first case, we have*

$$\sum_{i=l}^{k} c_i \prod_{j=1}^{t_i} \|a_{ij}\|^{2\alpha d_{ij}} = 0$$

and in second case we have $\sum_{i=l}^{k} c_i \prod_{j=1}^{t_i} \|a_{ij}\|^{2\alpha d_{ij}} \neq 0$. Since $\lim_{z\to\infty} \frac{a}{z^r \bar{z}^s} = 0$, for all $a \in \mathbb{C}$, $r, s \in \mathbb{N}$ and $\lim_{z\to\infty} z\bar{z} = \infty$, it is clear that in latter case $\lim_{z\to\infty} p(z) = \infty$.

Define

$$C(\mathbb{C}, \mathbb{C})^{\diamondsuit} = \{f \in C(\mathbb{C}, \mathbb{C}) | \lim_{z\to\infty} f(z) = \infty\} \text{ and}$$
$$\mathscr{A}_2^{\diamondsuit} = \{f \in C(\mathbb{C}, \mathbb{C})^{\diamondsuit} | f \in \mathscr{A}_2\}.$$

Theorem 3.4.3. *If $K \subset \mathbb{C}$ is compact , then $\mathscr{A}_2^{\diamondsuit}(K)$ is dense in $C(\mathbb{C}, \mathbb{C})^{\diamondsuit}$.*

Proof. Let $m = \sup_{z\in K} \|z\|$. So, for any $a, b \in \mathbb{C}$ and $d \in \mathbb{N}$. Consider

$$\begin{aligned}\|(((az+b)(\overline{az+b}))^{\alpha} + az + b + \overline{az+b})^d\| &\leq (\|((az+b)^{\alpha}\|\|(\overline{az+b}))^{\alpha}\| \\ &\quad + \|az+b\| + \|\overline{az+b}\|)^d \\ &\leq (\|((az+b)^{2\alpha}\| + 2\|az+b\|)^d \\ &\leq ((\|a\|m + \|b\|)^{2\alpha} + 2(\|a\|m + \|b\|))^d.\end{aligned}$$

So, from above inequality we can conclude that, for any $\epsilon > 0$, there exists $a, b \in \mathbb{C}$ such that $\|(((az+b)(\overline{az+b}))^{\alpha} + az + b + \overline{az+b})^d\| < \epsilon$.

Now, let $f \in C(\mathbb{C}, \mathbb{C})^{\diamondsuit}$. By lemma 3.4.2 there exists $q(z) \in \mathscr{A}_2(K)$ such that $\|q(z) - f\| < \frac{\epsilon}{2}$. Now choose $a, b \in \mathbb{C}$ and $d \in \mathbb{N}$ with $2\alpha d > \deg(p(z))$ such that $\|(((az+b)(\overline{az+b}))^{\alpha} + az + b + \overline{az+b})^{d}\| < \frac{\epsilon}{2}$. So, from the remark 3.4.1, if we take $p(z) = (((az+b)(\overline{az+b}))^{\alpha} + az + b + \overline{az+b})^{d} + q(z)$, $p(z) \in \mathscr{A}_2^{\diamondsuit}(K)$. Also $\|p(z) - f\| < \epsilon$. □

Next, we invoke Theorems 3.4.4 – 3.4.6 as a sort of justification that compact open topology is the right topology with respect to which one should seek dense subsets in $C(M, N)$, where M and N are topological manifolds. We give a dense subset of a subclass of $C(S^2, S^2)$ with respect to a topology which is slightly weaker than compact open topology.

Remark 3.4.2. *(1) When X is compact the uniformity of uniform convergence coincides with the uniformity of uniform convergence on compacta. And hence, coincide their uniform topologies.*
(2) The Stone-Weierstrass theorem tells about the density of a subset, with some property, of the class of continuous functions in the space with the topology of uniform convergence.

From the Remark 3.4.2 and Theorem 3.2.6 we can formulate the two Stone-Weierstrass theorems (real and complex versions) in the following form:

Theorem 3.4.4. *Let X be a compact hausdorff space. If $\mathscr{A}$ is a unital sub-algebra of $C(X, \mathbb{R})$, which separates points then $\mathscr{A}$ is dense in $C(X, \mathbb{R})$ in compact open topology.*

Theorem 3.4.5. *Let X be a compact hausdorff space. If $\mathscr{A}$ is a unital sub-algebra of $C(X, \mathbb{C})$, which separates points and is closed under conjugation, then $\mathscr{A}$ is dense in $C(X, \mathbb{C})$ with compact open topology.*

We have the following four theorems:

Theorem 3.4.6. *Let $\mathscr{A}^{\clubsuit}$ be a unital subalgebra of $C(\mathbb{C}, \mathbb{C})$ which separates points and is closed under conjugation. Then, $\mathscr{A}^{\clubsuit}$ is dense in $C(\mathbb{C}, \mathbb{C})$ in compact open topology.*

Proof. To show that $\mathscr{A}^\clubsuit$ is dense it is enough to show that every member of a neighbourhood base at each point of $C(\mathbb{C}, \mathbb{C})$ intersects $\mathscr{A}^\clubsuit$. From Example 35.3 (a) of [Wil70], the set $\{D_\epsilon : \epsilon > 0 \text{ and } D_\epsilon = \{(x, y) \in \mathbb{C} \times \mathbb{C} : \|x - y\| < \epsilon\}\}$ is a base for the usual uniformity on $\mathbb{C}$. Since from Theorem 11 of Function Spaces in [Kel09] compact open topology on $C(\mathbb{C}, \mathbb{C})$ is same as the topology of uniform convergence on compacta, it is enough to work with topology of uniform convergence on compacta.

Let $\mathscr{V}$ be the usual uniformity on $\mathbb{C}$. By definition

$$\{\langle K, V\rangle : \langle K, V\rangle = \{(f, g) : (f(x), g(x)) \in V, \forall x \in K\}, K \subset \mathbb{C} \text{ is compact}, V \in \mathscr{V}\}$$

is a subbasis for the uniformity of uniform convergence on compacta on $C(\mathbb{C}, \mathbb{C})$. Now let $f \in C(\mathbb{C}, \mathbb{C})$, $K_1, \cdots, K_r$ be compact subsets of $\mathbb{C}$ and $V_1, \cdots, V_r$ be members of $\mathscr{V}$. Note that from Theorem 5 of Uniform Spaces in [Kel09] it is enough to show that $\cap_{i=1}^r \langle K_i, V_i\rangle (f)$ intersects $\mathscr{A}^\clubsuit$. Now since $\{D_\epsilon : \epsilon > 0 \text{ and } D_\epsilon = \{(x, y) \in \mathbb{C} \times \mathbb{C} : \|x - y\| < \epsilon\}\}$ is a basis for $\mathscr{V}$ we can choose $\epsilon > 0$ such that $D_\epsilon \subset V_i, \forall i = 1, 2, ..., r$. Let $K = \cup_{i=1}^r K_i$. Note that $\langle K, D_\epsilon\rangle \subset \cap_{i=1}^r \langle K_i, V_i\rangle$. Hence, $\langle K, D_\epsilon\rangle (f) \subset \cap_{i=1}^r \langle K_i, V_i\rangle (f)$. We have $\langle K, D_\epsilon\rangle (f) = \{g : g \in C(\mathbb{C}, \mathbb{C}), \|g(x) - f(x)\| < \epsilon, \forall x \in K\}$. By Stone-Weierstrass Theorem there exists a sequence f_n in $\mathscr{A}^\clubsuit$ such that $f_n \to f$ uniformly on K. So, there exists $g \in \mathscr{A}^\clubsuit$ such that $g \in \langle K, D_\epsilon\rangle (f) \subset \cap_{i=1}^r \langle K_i, V_i\rangle (f)$. Hence we are done. □

The next theorem is direct application of Theorem 3.4.6.

Theorem 3.4.7. *$\mathscr{A}_1$ is dense in $C(\mathbb{C}, \mathbb{C})$ in compact open topology.*

Proof of the next theorem is similar to the proof of Theorem 3.4.6.

Theorem 3.4.8. *Let $\mathscr{A}^*$ be a unital subalgebra of $C(\mathbb{R}, \mathbb{R})$ which separates points. Then $\mathscr{A}^*$ is dense in $C(\mathbb{R}, \mathbb{R})$ in compact open topology.*

The next theorem follows from Theorem 3.4.8.

Theorem 3.4.9. *The set of polynomials is dense in $C(\mathbb{R}, \mathbb{R})$ in compact open topology.*

Next, we give in Theorem 3.4.10 a dense subset of a subclass of $C(S^2, S^2)$ with respect to a topology which is slightly weaker than compact open topology.

Let $\pi : S^2 - \{(0, 0, 1)\} \to \mathbb{C}$ be the stereographic projection given by

$$\pi(x, y, z) = \frac{x}{1-z} + i\frac{y}{1-z}.$$

Since π is a homeomorphism, a topology can be given to the set $\mathbb{C} \cup \{\infty\}$ so that S^2 is homeomorphic to $\mathbb{C} \cup \{\infty\}$. Define

$$\mathscr{F}^\star_{\mathbb{C}} = \{f | f : \mathbb{C} \to \mathbb{C} \text{ is continuous and } \lim_{z\to\infty} f(z) = \infty\}$$

and

$$\mathscr{F}^\star_{S^2} = \{f | f : S^2 \to S^2 \text{ is continuous, } f(\infty) = \infty \textit{ and } f^{-1}(\infty) = \infty\}.$$

Now note that

$$\psi : \mathscr{F}^\star_{\mathbb{C}} \to \mathscr{F}^\star_{S^2}$$

given by

$$\psi(f)(x) = \begin{cases} \pi^{-1} \circ f \circ \pi(x) & \text{if } x \neq \infty \\ \infty & \text{if } x = \infty \end{cases}$$

is well defined and bijective.

For $p, q \in S^2$, define

$$\mathscr{F}^{p,q}_{S^2} = \{f | f : S^2 \to S^2 \text{ is continuous, } f(p) = q \text{ and } f^{-1}(q) = \{p\}\}.$$

For $a, b \in S^2$, let $R_{a\to b}$ be the plane rotation of S^2 in the plane containing the great circle through a, b and sending a to b.

Define $\eta : \mathscr{F}^{p,q}_{S^2} \to \mathscr{F}^\star_{S^2}$ by

$$\eta(f) = R_{q\to\infty} \circ f \circ R_{\infty\to p}.$$

Clearly, for all p,q $\in S^2$, η is a bijection.

For sets $K \subset S^2 - \{p\}$ and $U \subset S^2$, define

$$(K, U) = \{f | f \in \mathscr{F}^{p,q}_{S^2} \text{ and } f(K) \subset U\}.$$

Consider the collection

$$\{(K, U)| K \subset S^2 - \{p\} \text{ is compact and } U \subset S^2 \text{ is open}\}$$

as a subbase for a topology $\mathscr{T}_{S^2}^{p,q}$ on $\mathscr{F}_{S^2}^{p,q}$. Note that $\mathscr{F}_{S^2}^{p,q}$ with this topology is homeomorphic to $\mathscr{F}_{\mathbb{C}}^{\star}$ with compact open topology. Now, since $\mathscr{A}_2^{\diamondsuit}$ is dense in $\mathscr{F}_{\mathbb{C}}^{\star}$ with compact open topology, we have the following theorem:

Theorem 3.4.10. *$\eta^{-1} \circ \psi(\mathscr{A}_2^{\diamondsuit})$ is dense in $\mathscr{F}_{S^2}^{p,q}$ with the topology $\mathscr{T}_{S^2}^{p,q}$.*

3.5 Some Problems

Theorems 3.4.4, 3.4.5, 3.4.9, 3.4.8, 3.4.7 and 3.4.6 provide a way to pose problems in the direction of generalizing the Stone-Weierstrass Theorem for the class of continuous functions $C(M, N)$, where M and N are topological manifolds. Of course here dense subsets can't be expected to be an algebra.

Problem 3.5.1. *To find dense subsets of $C(M, N)$, where M and N are topological manifolds, with respect to compact open topology. To find a method to generate dense subsets of $C(M, N)$ with respect to compact open topology.*

Problem 3.5.2. *To find dense subsets of $C(S^2, S^2)$ with respect to compact open topology.*

Problem 3.5.3. *To know if, in Remark 3.4.1, limit exists in the first case also.*

Problem 3.5.4. *Is $\mathscr{A}_2 \subsetneq \mathscr{A}_1$?*

Some of the results developed in this chapter have been submitted for publication as [SYd].

Chapter 4

Inner-Product Morphism

4.1 Introduction

In Sec. 4.2 we give the definition of an *inner-product morphism* between two inner-product spaces. We provide examples of such morphisms and develop some of their properties. This will be foundational to the development of the concept of Riemannian morphisms in the next chapter. In Sec. 4.3 we give a geometric description for any invertible operator on a finite dimensional inner-product space. With the aid of such a description, we are able to decompose any given conformal transformation as a product of a scalar multiplication, two dimensional rotations and reflections. Also, we are able to conclude that an orthogonal transformation is a product of two dimensional rotations and reflections.

4.2 Definition and Examples

Definition 4.2.1. *Let V and W be inner-product spaces. Then a linear transformation $f : V \to W$ is called an **inner-product morphism**, if there exists two subspaces K and I of V such that $V = K \oplus I$, $K = Ker(f)$ and*

$$f_{|I} : I \to R(f)$$

is an isometry onto the image of f, $R(f)$ viz., $\langle u,v\rangle_V = \langle f(u), f(v)\rangle_W$ for all $u, v \in I$. I is called the **isometry subspace** *of V wrt f. Further, if $I = K^{\perp}$, we say that f is an* ***orthogonal inner-product morphism****.*

We will abbreviate 'inner-product morphism' by ipm. Also, we will denote the set of all inner-product morphisms from V to W by IPM(V, W).

Example 4.2.0.1. *Isometry and zero transformations are inner-product morphisms.*

Example 4.2.0.2. *Define $f : \mathbb{R}^3 \to \mathbb{R}^4$ by*

$$f(e_1) = \frac{e_1 + e_4}{\sqrt{2}},\ f(e_2) = e_3,\ f(e_3) = 0.$$

Its an easy task to verify that f is an inner-product morphism.

Proposition 4.2.1. *Let V, W be inner-product spaces of the same dimension and let*

$$Iso(V,W) = \{f | f : V \to W \text{ is linear, bijective, inner} - \text{product morphism}\}.$$

Then the set of possible isometries from V to W, $Iso(V,W)$ is non-empty. Further, if $f, g \in Iso(V,W)$ then there exist unique $\alpha \in Iso(V,V)$ and $\beta \in Iso(W,W)$ such that $f = g \circ \alpha$ and $\beta \circ f = g$.

Proof. Let $\{e_i\}_1^n$ and $\{p_i\}_1^n$ be orthonormal bases of V and W respectively. Any element of $v \in V$ can be written as $v = \sum_{i=1}^n \delta_i e_i$ for $\delta_i \in \mathbb{R}$. Then, we define $f : V \to W$ via $f(v) = \sum_{i=1}^n \delta_i p_i$. Clearly, $f \in Iso(V,W)$ and the latter is non-empty.
If we take $\alpha = g^{-1} \circ f$ and $\beta = g \circ f^{-1}$ then $\alpha \in Iso(V,V)$ and $\beta \in Iso(W,W)$ and $f = g \circ \alpha$ and $\beta \circ f = g$. Uniqueness follows similarly. □

Remark 4.2.1. 1. *Given two subspaces of equal dimension $X \subset V$ and $Y \subset W$ of inner-product spaces V, W, there exists an inner-product morphism $f : V \to W$, with X as the isometry subspace of f and Y as the range of f. Further, if $g : V \to W$ is another such ipm, there exist $\alpha \in Iso(X)$ and $\beta \in Iso(Y)$ such that $f_{|X} = g_{|X} \circ \alpha$ and $\beta \circ f_{|X} = g_{|X}$.*

2. *Given linearly independent sets of k vectors $\{c_i\}_1^k$ in V and $\{d_i\}_1^k$ in W such that $\langle c_i, c_j\rangle = \langle d_i, d_j\rangle$ for all $1 \leq i, j \leq k$, there exists a unique ipm $f : V \to W$ such that $f(c_i) = d_i$ for all $1 \leq i \leq k$ and $Ker(f) = (span\{c_i\}_1^k)^{\perp}$.*

3. *IPM(V, W), the set of all inner-product morphisms from V to W is not a vector space. For example, if $f \in \mathrm{IPM}(V, W)$ and $f \neq 0$, then for $\alpha \notin \{0, 1\}$, $\alpha f \notin \mathrm{IPM}(V, W)$.*

4. *Composition of ipms need not be an ipm, i.e., if $f : V \to W$ and $g : W \to U$ are ipms for inner-product spaces V, W and U, then $g \circ f$ need not be an ipm. For example take the ipms f and g from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by*

$$f((x, y)) = \frac{1}{\sqrt{2}}(x, x) \text{ and } g((x, y)) = (x, 0).$$

Then, since $g \circ f(e_1) = \frac{1}{\sqrt{2}}(1, 0)$, we know that $g \circ f$ is not a zero map. Also, since $g \circ f(e_2) = 0$, $\ker(f)$ has dimension one. So, if $g \circ f$ is an ipm then isometry subspace of V with respect to $g \circ f$ has to have dimension one. Hence for some θ we should have

$$\|g \circ f((cos\theta, sin\theta))\| = 1 \;\Rightarrow\; \|\frac{cos\theta}{\sqrt{2}}\| = 1,$$

which is not possible. So $g \circ f$ is not an ipm.

Proposition 4.2.2. *Let $f : V \to W$ and $g : W \to U$ be ipms such that range of f is a subspace of isometry subspace of W wrt g. Then if we define $h : V \to U$ as the composition $h = g \circ f$, then, h is an ipm.*

Proof. Suppose that B and C are the isometry subspaces of f and g respectively. We remark that $C \cap \ker(g) = \{0\}$ and by hypothesis, $\mathrm{range}(f) \subset C$. Consequently, $\mathrm{range}(f) \cap \ker(g) = \{0\}$ which implies that $\ker(g \circ f) = \ker(f)$.
Further, $h_{|B} = g_{|f(B)} \circ f_{|B}$ and since restriction and composition of isometries is an isometry, $h_{|B}$ is an isometry onto its range. Conclude that h is an ipm. □

Proposition 4.2.3. *If $f : V \to W$ is an ipm, then there exists an ipm $g : W \to V$ such that the composition $h = g \circ f$ is a projection on V.*

Proof. Since $f : V \to W$ is an ipm, there exists subspaces K and I of V such that $V = K \oplus I$, $K = \ker(f)$ and $f_* := f_{|I}$ is an isometry on I. Define $g : W \to V$ by

$$g(x) = \begin{cases} f_*^{-1}(x) & \text{if } x \in R(f) \\ 0 & \text{if } x \in R(f)^{\perp}. \end{cases}$$

Clearly $\ker(g) = R(f)^{\perp}$ and $g_{|R(f)} : R(f) \to I$ is an isometry. Thus, g is an ipm. Since $V = K \oplus I$, any $v \in V$ can be written as $v = k + i$ for $k \in K$ and $i \in I$. Thus, $h^2(v) = h[h(k + i)] = h[g(f(k)) + h(i)] = h[h(i)]$ as $k \in \ker(f)$. Further, $h[h(i)] = h[g(f(i))] = h[f_*^{-1}(f(i))] = h(i) = h(k + i) = h(v)$, as $h(k) = 0$. We conclude that h is a projection. □

The converse to the above proposition is not true in general. For example, consider the functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(e_1) = e_1$, $f(e_2) = 0$ and $g(e_1) = e_1$, $g(e_2) = \frac{1}{2}e_2$. Clearly f is an ipm but g is not an ipm, and $g \circ f$ is a projection.

In the definition of inner-product morphisms we have used isometry transformations (orthogonal transformations). So it is worth exploring orthogonal transformations. In fact geometric decomposition of orthogonal transformations as composition of rotations and reflections, is well known. We have gone further back and considered the question of geometrical decomposition of invertible and conformal transformations.

4.3 Geometric Decomposition of Invertible and Conformal Transformations

4.3.1 Preliminaries

For the remainder of this chapter, V denotes a finite dimensional inner-product space of dimension $n \in \mathbb{N}$ over the field of real numbers $\mathbb{R}$. The set of all invertible operators on V shall be denoted by $\mathrm{GL}(V)$.

Definition 4.3.1. *Let W be a two dimensional subspace of V. Suppose that $\{u, v\} \subset$*

V is an orthonormal basis of W. For any fixed real number θ, the assignments

$$u \mapsto (\cos\theta)\,u + (\sin\theta)\,v \text{ and } v \mapsto (-\sin\theta)\,u + (\cos\theta)\,v$$

determine a unique linear operator on W termed as a ***rotation*** *of W.*

Similarly the assignments

$$u \mapsto u \text{ and } v \mapsto -v$$

determine a unique linear operator on W termed as a ***reflection*** *of W.*

Definition 4.3.2. *Let T be a linear operator on V of the form $\rho \oplus id$ where ρ is an operator on a two dimensional subspace W of V and id is the identity operator on $W^{\perp}$. We say that T is a* ***planar rotation*** *or a* ***planar reflection*** *accordingly as ρ is a rotation or a reflection of W. A planar rotation and a planar reflection are also termed as* ***rotational*** *and* ***reflectional*** *operators respectively.*

Remark 4.3.1. *When V is uni-dimensional, we allow the identity operator on V to be termed a rotational operator.*

4.3.2 Axis of a Basis

Definition 4.3.3. *A basis $\{u_i\}_{i=1}^{n}$ of the inner-product space V is* ***equimodular*** *if there exists a real $\delta > 0$ such that $\|u_i\| = \delta$, for each i in $\{1, 2, \ldots, n\}$. When such a $\delta = 1$, we say that the basis is* ***unimodular****.*

Definition 4.3.4. *Let α be a non-zero vector in an inner-product space V and $\{u_i\}_{i=1}^{n}$ be a basis of V. If*

$$\langle u_i, \alpha\rangle \langle u_j, u_j\rangle^{\frac{1}{2}} = \langle u_j, \alpha\rangle \langle u_i, u_i\rangle^{\frac{1}{2}} \text{ for all } i, j \in \{1, 2, \ldots, n\},$$

we say that α is an ***axial-vector*** *of the given basis $\{u_i\}_{i=1}^{n}$.*

Remark 4.3.2. *Let α be an axial-vector of a basis $\{u_i\}_{i=1}^{n}$ of an inner-product space V.*

1. *Clearly, the ratio $\frac{\langle u_i, \alpha\rangle}{\|u_i\|\|\alpha\|}$ is independent of the choice of $i \in \{1, 2, \ldots, n\}$. Thus, the axial-vector α makes the same angle with each of the basis vectors u_i for $i \in \{1, 2, \ldots, n\}$.*

2. *When* $n = 1$, $\frac{\langle u_1, \alpha \rangle}{\|u_1\|\|\alpha\|}$ *is either* -1 *or* 1.

3. *However, when* $n \geq 2$, *if* $\frac{\langle u_i, \alpha \rangle}{\|u_i\|\|\alpha\|}$ *is either* -1, 0 *or* $+1$, *it forces* $\{u_i\}_{i=1}^n$ *to be linearly dependent. In other words, the common angle between an axial vector and the basis vectors can not be* 0, $\frac{\pi}{2}$ *or* π.

Lemma 4.3.1. *Every equimodular basis of a finite dimensional inner-product space has an axial-vector. Further, any two axial-vectors of a given equimodular basis are linearly dependent.*

Proof. Suppose $\{u_i\}_{i=1}^n$ is an equimodular basis of V. Given a non-zero real ω, we prove the existence of a non-zero $\alpha \in V$ such that $\langle u_i, \alpha \rangle = \omega$ for all $i \in \{1, 2, \ldots, n\}$. Such an α would be an axial-vector of the given equimodular basis.

Let A be the $n \times n$ matrix with $A_{ij} = \langle u_i, u_j \rangle$ for $i, j \in \{1, 2, \ldots, n\}$. Let $\alpha = \sum_{i=1}^n x_i u_i$ for undetermined real numbers $\{x_i\}_{i=1}^n$. Further, let X denote the column vector $(x_1, x_2, ..., x_n)^{\mathrm{T}}$. Then for $i \in \{1, 2, \ldots, n\}$, the collection of n equations $\langle u_i, \alpha \rangle = \langle u_i, \sum_{j=1}^n x_j u_j \rangle = \omega$ is the system $AX = \Omega$, where Ω is the column vector $(\omega, \omega, \ldots, \omega)^T$. Existence of a solution X to the latter system suffices to prove the existence of α. In fact, we show that for each Ω there is a unique solution X by proving that A is invertible.

Suppose to the contrary that A is not invertible. Then, there exists a non-zero column vector $Y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ such that $AY = 0$. Set $s = \sum_{i=1}^n y_i u_i$. Then, $s \neq 0$ and consequently $\|s\| \neq 0$. However, $\|s\|^2 = \langle \sum_{i=1}^n y_i u_i, \sum_{j=1}^n y_j u_j \rangle = Y^T A Y = 0$, a contradiction.

If ω is non-zero, it is evident from $X = A^{-1}\Omega$ that $X \neq 0$ and hence α is non-zero. This proves the existence of an axial vector for the given basis.

Suppose that α and $\tilde{\alpha}$ are two axial-vectors of a given equimodular basis $\{u_i\}_{i=1}^n$ of V. Let $\alpha = \sum_{i=1}^n x_i u_i$ and $\tilde{\alpha} = \sum_{i=1}^n \tilde{x}_i u_i$ for some real numbers $\{x_i\}_{i=1}^n$ and $\{\tilde{x}_i\}_{i=1}^n$. Set $X = (x_1, x_2, \ldots, x_n)^T$ and $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$. Corresponding to the axial vectors α and $\tilde{\alpha}$, there are two non-zero real numbers ω and $\tilde{\omega}$ such that $\langle u_i, \alpha \rangle = \omega$ and $\langle u_i, \tilde{\alpha} \rangle = \tilde{\omega}$ for all $i \in \{1, 2, \ldots, n\}$. These two sets of n equations are the two systems $AX = \Omega$ and $A\tilde{X} = \tilde{\Omega}$ where $\Omega = (\omega, \omega, \ldots, \omega)^T$ and $\tilde{\Omega} = (\tilde{\omega}, \tilde{\omega}, \ldots, \tilde{\omega})^T$. Clearly the column vectors Ω and $\tilde{\Omega}$ are linearly dependent. Hence the corresponding solutions $X = A^{-1}\Omega$ and $\tilde{X} = A^{-1}\tilde{\Omega}$ are linearly dependent. We can now conclude that the axial-vectors α and $\tilde{\alpha}$ are linearly dependent. □

Theorem 4.3.1. *Every basis of a finite dimensional inner-product space has an axial-vector. Any two axial-vectors of a given basis are linearly dependent.*

Proof. If $\{v_i\}_{i=1}^n$ is a basis of V, then the collection $u_i := \frac{v_i}{\|v_i\|}$ is an equimodular basis. The latter has an axial-vector α by Lemma 4.3.2. Consequently $\langle u_i, \alpha\rangle = \langle u_j, \alpha\rangle$ for all $i, j \in \{1, 2, \ldots, n\}$. Substituting for u_i, we get that $\langle v_i, \alpha\rangle\langle v_j, v_j\rangle = \langle v_j, \alpha\rangle\langle v_i, v_i\rangle$ for all $i, j \in \{1, 2, \ldots, n\}$. Thus, α is an axial-vector of the given basis $\{v_i\}_{i=1}^n$.

Further, if α and $\tilde{\alpha}$ are two axial-vectors of the given basis $\{v_i\}_{i=1}^n$, then α and $\tilde{\alpha}$ are also axial-vectors of the equimodular $\{u_i\}_{i=1}^n$ and again by Lemma 4.3.2 are linearly dependent. □

Theorem 4.3.1 ensures the existence and uniqueness of the **axis** of a basis defined below.

Definition 4.3.5. *Let α be an axial-vector of a basis $\mathcal{B}$ of a finite dimensional inner-product space. The **axis** of $\mathcal{B}$ is defined to be the span of $\{\alpha\}$.*

Definition 4.3.6. *Given a non-zero $\alpha \in V$ and a real number $\theta \in [0, \pi]$, we define the **cone** around α of **vertex angle** θ by*

$$\Lambda_\alpha^\theta = \{x \in V | \langle x, \alpha\rangle = \|x\|\|\alpha\| \cos\theta\}.$$

*The span of $\{\alpha\}$ shall be termed as the **axis** of the cone Λ_α^θ.*

Remark 4.3.3.

1. *When V is one dimensional, the cone around any non-zero vector is empty if the angle $\theta \neq 0, \pi$.*
2. *When the dimension of V is at least two, every such cone Λ_α^θ is non-empty.*
3. *From Theorem 4.3.1, if $\mathcal{B}$ is any basis of V, there exists a cone in V whose vertex is the zero vector, whose axis is the axis of the basis and whose vertex*

angle is the common angle between each of the basis elements and an axial-vector of $\mathcal{B}$. Such a cone is termed as the ***associated cone*** *of basis $\mathcal{B}$.*

4.3.3 Geometric Description of Invertible Operators

Definition 4.3.7. *An invertible linear operator $T : V \to V$ is called* ***axonal*** *if it maps an equimodular basis $\mathcal{B}$ to an equimodular basis $\mathcal{B}'$ such that $\mathcal{B}$ and $\mathcal{B}'$ share a common axis.*

We denote the set of all axonal operators on V by $\mathrm{AX}(V)$. The latter is a subset of $\mathrm{GL}(V)$.

Remark 4.3.4.

1. *It would be of interest to know examples of axonal operators on V. Indeed, given any two equimodular bases sharing a common axis, we have an example of an axonal operator which would map one of these bases to the other.*

2. *Axonal linear operators which map an equimodular basis $\mathcal{B}$ to $\mathcal{B}'$ may be classified into two kinds: those for which the associated cones of $\mathcal{B}$ and $\mathcal{B}'$ are same and those for which the associated cones are distinct. Proposition 4.3.1 below provides examples of axonal operators of the latter kind.*

Proposition 4.3.1. *Suppose that V is an inner-product space of dimension at least two. Let $\{u_i\}_{i=1}^{n}$ be a basis of V with α as its axial-vector. Assume that for each $i \in \{1, 2, \ldots, n\}$, u_i is rotated in the space spanned by u_i and α to get v_i such that each v_i makes the same angle ϕ with α. If $\phi \notin \{0, \frac{1}{2}\pi, \pi\}$, then $\{v_i\}_{i=1}^{n}$ is a basis of V.*

Proof. Without any loss of generality assume that $\{u_i\}_{i=1}^{n}$ is an equimodular basis. Let S be the span of $\{\alpha\}$. Since each of the $u_i's$ have the same norm and make the same angle with α, they have the same component, say s, in S. For each $i \in \{1, 2, \ldots, n\}$, orthogonally decompose the two collections of vectors to get

$$u_i = s + w_i, \text{where } s \in S \text{ and } w_i \in S^{\perp} \tag{4.1}$$

$$v_i = ks + hw_i, \text{ for some real numbers } h, k \neq 0. \tag{4.2}$$

We note that if either h or k equals 0, angle $\phi \in \{0, \frac{1}{2}\pi, \pi\}$, contradicting hypothesis.

Next, assume a linear relation of the form $\sum_{i=1}^{n} \lambda_i v_i = 0$ for some real numbers $\{\lambda_i\}_{i=1}^{n}$. Using Eq. 4.2 from above, we get

$$k\left(\sum_{i=1}^{n} \lambda_i\right) s + h\left(\sum_{i=1}^{n} \lambda_i w_i\right) = 0.$$

The summands are in orthogonal complements S and $S^{\perp}$ while $h, k \neq 0$. Hence we conclude

$$\left(\sum_{i=1}^{n} \lambda_i\right) s = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i w_i = 0.$$

Adding the two equalities, we get $\sum_{i=1}^{n} \lambda_i (s + w_i) = 0$ and hence $\sum_{i=1}^{n} \lambda_i u_i = 0$. From the linear independence of $\{u_i\}_{i=1}^{n}$, we conclude $\lambda_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. This proves that $\{v_i\}_{i=1}^{n}$ are linearly independent and hence form a basis of V. □

Definition 4.3.8. *Let k be a positive integer with $k \leq n$. A linear transformation $S : V \to V$ is called a $k-$**shear** if there exists a $k-$dimensional subspace W of V such that the restriction $S_{|W^{\perp}}$ is the identity while $S_{|W}$ is an axonal transformation which maps an equimodular basis $\{u_i\}_{i=1}^{k}$ to an equimodular basis $\{v_i\}_{i=1}^{k}$ of W with the same axis so that each v_i is obtained by rotating u_i by the same angle θ in the two dimensional subspace containing u_i and the axis of $\{u_i\}_{i=1}^{k}$.*

Remark 4.3.5. *Any axonal transformation $A : V \to V$ can be written as $A = A' \circ S$, where S is a n-shear and A' is an axonal transformation which maps an equimodular basis to another equimodular basis such that the two bases have the same associated cones.*

Theorem 4.3.2. *Let $T : V \to V$ be any invertible linear operator on V. Then, there exist a diagonal operator D, an axonal operator A and a rotational operator R on V such that $T = D \circ A \circ R$.*

Proof. Suppose $\{u_i\}_{i=1}^n$ is any equimodular basis of V. Let $v_i = T(u_i)$ and $w_i = \frac{v_i}{\|v_i\|}$ for all $i \in \{1, 2, \ldots, n\}$. Since T is invertible, $\{w_i\}_{i=1}^n$ is a unimodular basis of V. Let L_1 be the axis of the basis $\{u_i\}_{i=1}^n$ and L_2 be the common axis of the bases $\{v_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$. There exists a rotational operator R of V which maps L_1 to L_2. Clearly, every rotational operator is invertible and norm-preserving and hence $\{R(u_i)\}_{i=1}^n$ is a unimodular basis whose axis is L_2. Further the linear operator A which maps $R(u_i)$ to w_i for all $i \in \{1, 2, \ldots, n\}$ is axonal. Let D be the diagonal operator on V which maps w_i to v_i. Since both T and $D \circ A \circ R$ map the basis $\{u_i\}_{i=1}^n$ to the basis $\{v_i\}_{i=1}^n$ we can conclude that $T = D \circ A \circ R$. □

Remark 4.3.6. *From Remark 4.3.5, a further characterization of axonal transformations which maps an equimodular basis to another equimodular basis sharing the same associated cone would provide a finer description of invertible transformations on the lines of Theorem 4.3.2.*

4.3.4 Decomposition of Conformal and Orthogonal Operators

Remark 4.3.7. *Recall that a linear $f : V \to V$ is said to be conformal if there exists a real $\lambda > 0$ such that $\langle f(u), f(v)\rangle = \lambda\langle u, v\rangle$ for all $u, v \in V$.*

Remark 4.3.8. *A linear function is conformal if and only if it is angle preserving.*

Theorem 4.3.3. *Given any conformal $T \in GL(V)$, we can write*

$$T = D \circ \mathcal{R} \circ R_{n-2} \circ R_{n-3} \circ \cdots \circ R_2 \circ R_1,$$

where R_k is a rotational operator on V for each $k \in \{1, 2, \ldots, n-2\}$, $\mathcal{R}$ is either a rotational or a reflectional operator on V and D is a scalar operator on V.

Proof. We use induction on n – the dimension of V. For unidimensional V, every T is a scalar operator. When dimension of V is two, it is easily verified that every conformal T is of the form $D \circ \mathcal{R}$ where D is a scalar operator on V and $\mathcal{R}$ is either a rotation or a reflection of V.

Assume next that V has dimension at least three. By Theorem 4.3.2, we have $T = D \circ A_1 \circ R_1$ for linear operators D, A_1 and R_1 on V such that D is diagonal, A_1 is axonal and R_1 is rotational. Suppose the axonal A_1 maps an equimodular basis $\{u_i\}_{i=1}^n$ to an equimodular basis $\{v_i\}_{i=1}^n$ which share a common axis. By rescaling $\{u_i\}_{i=1}^n$, we can assume that the basis $\{u_i\}_{i=1}^n$ is unimodular. By rescaling D, if necessary, we can assume that $\{v_i\}_{i=1}^n$ is unimodular. Assume that for some real numbers λ_i the diagonal operator D maps v_i to $\lambda_i v_i$ for each $i \in \{1, 2, \ldots, n\}$. The scalars $\{\lambda_i\}_{i=1}^n$ are non-zero as T is invertible.

Write $A_1 = D^{-1} \circ T \circ R_1^{-1}$ and in the latter composition R_1^{-1} and T are angle preserving. Now $T \circ R_1^{-1}$ maps u_i to $\lambda_i v_i$ and hence the angle between u_i and u_j equals the angle between $\lambda_i v_i$ and $\lambda_j v_j$ for $i, j \in \{1, 2, \ldots, n\}$. Note that D^{-1} being diagonal along $\{v_i\}_{i=1}^n$ keeps the angle between $\lambda_i v_i$ and $\lambda_j v_j$ equal to the angle between v_i and v_j for $i, j \in \{1, 2, \ldots, n\}$. Consequently, if we take $\{u_i\}_{i=1}^n$ to be orthonormal, then $\{v_i\}_{i=1}^n$ is an orthonormal basis. We conclude that A_1 is orthogonal.

Now that A_1 and R_1 are conformal and T is given to be conformal, we conclude that $D = T \circ R_1^{-1} \circ A_1^{-1}$ is conformal. Hence, D is a scalar operator.

Suppose α is an axial-vector of the basis $\{u_i\}_{i=1}^n$. If the common angle between α and each of these basis vectors is θ, then $A_1(\alpha)$ makes the same angle θ with each of the vectors from the basis $\{v_i\}_{i=1}^n$ as A_1 is orthogonal. Hence $A_1(\alpha)$ is an axial-vector for the basis $\{v_i\}_{i=1}^n$. However, these two bases share a common axis, say, W. Since $W = \text{Span}\{\alpha\} = \text{Span}\{A_1(\alpha)\}$ and A_1 is orthogonal, $A_1(\alpha) = \pm\alpha$. By replacing D by $-D$ if necessary, we assume $A_1(\alpha) = \alpha$ and hence A_1 is the identity on W. Since A_1 is orthogonal, $W^\perp$ is an invariant subspace of A_1. Thus we may write $A_1 = id \oplus T_2$ where id is the identity operator on W and T_2 is an orthogonal operator on $W^\perp$ of dimension $n-1$. We apply induction hypothesis to the orthogonal T_2 to realize this as a composition

$$T_2 = \tilde{D}_2 \circ \tilde{\mathcal{R}} \circ \tilde{R}_{n-2} \circ \tilde{R}_{n-3} \circ \cdots \circ \tilde{R}_2,$$

where $\tilde{D}_2$ is scalar while $\tilde{\mathcal{R}}$ is either reflectional or rotational and $\tilde{R}_2, \tilde{R}_3, \ldots, \tilde{R}_{n-2}$ are rotational operators on $W^\perp$. Since T_2 is orthogonal, $\tilde{D}_2$ is the identity on $W^\perp$. We extend $\tilde{\mathcal{R}}, \tilde{R}_2, \tilde{R}_3, \ldots \tilde{R}_{n-2}$ respectively to $\mathcal{R}, R_2, R_3, \ldots R_{n-2}$ by declaring the latter operators to be identity on W. We now have

$$T = D \circ \mathcal{R} \circ R_{n-2} \circ R_{n-3} \circ \cdots \circ R_2 \circ R_1.$$ □

Theorem 4.3.4. *Given any orthogonal $T \in GL(V)$, we can write*

$$T = \mathcal{R} \circ R_{n-2} \circ R_{n-3} \circ \cdots \circ R_2 \circ R_1,$$

where R_k is a rotational operator on V for each $k \in \{1, 2, \ldots, n-2\}$ and $\mathcal{R}$ is either a rotational or a reflectional operator on V.

Proof. T being orthogonal is conformal. From Theorem 4.3.3, we may write

$$T = D \circ \mathcal{R} \circ R_{n-2} \circ R_{n-3} \circ \cdots \circ R_2 \circ R_1,$$

where R_k is a rotational operator on V for each $k \in \{1, 2, \ldots, n-2\}$, $\mathcal{R}$ is either a rotational or a reflectional operator on V and D is a scalar operator on V. Since T is orthogonal, the scalar operator D has to be the identity and hence

$$T = \mathcal{R} \circ R_{n-2} \circ R_{n-3} \circ \cdots \circ R_2 \circ R_1.$$ □

The material developed in this chapter is available at `http://arxiv.org/abs/1309.5805`.

Chapter 5

Riemannian Morphisms

5.1 Introduction

In Sec. 5.2 of this chapter we discuss Riemannian pre-morphisms and their properties. This definition of pre-morphisms requires a result from linear algebra which is quoted and proved in this section. We take a detour and include a few interesting applications to manifolds. In Sec. 5.3 we define Riemannian morphisms and discuss some of their properties.

5.2 Riemannian Pre-morphism and its properties

Definition 5.2.1. *Let M and N be Riemannian manifolds. A smooth function $f : M \to N$ is called a* ***Riemannian pre-morphism*** *or simply a* ***pre-morphism*** *if $df_p : T_pM \to T_{f(p)}N$ is an inner-product morphism for all $p \in M$.*

5.2.1 Some Linear Algebra and its Applications

Lemma 5.2.1. *Let m and n be two natural numbers. Further, let $M(m \times n, \mathbb{R})$ denote the inner-product space of all $m \times n$ matrices with real entries. Let r be any* *integer. Define*

$$O_r = \{A | A \in M(m \times n, \mathbb{R}),\ \operatorname{rank}(A) \geqslant r\}\ \&$$
$$C_r = \{A | A \in M(m \times n, \mathbb{R}),\ \operatorname{rank}(A) \leqslant r\}.$$

Then, for each r, O_r and C_r are respectively open and closed in $M(m \times n, \mathbb{R})$.

Proof. Since $C_r = O^c_{r+1}$, it suffices to show that O_r is open in $M(m \times n, \mathbb{R})$ for each integer r. We dispose off the special cases when $r \leqslant 0$ or $r > \min\{m, n\}$. If $r \leqslant 0$, then $O_r = M(m \times n, \mathbb{R})$ which is open in $M(m \times n, \mathbb{R})$. On the other hand if $r > \min\{m, n\}$, then O_r is the empty set which too is open in $M(m \times n, \mathbb{R})$.

Next, suppose $0 < r \leqslant \min\{m, n\}$. In any $m \times n$ matrix the number of minors of size r is given by $\binom{m}{r}\binom{n}{r}$. Denote this number by k. Let $p_i : M(m \times n, \mathbb{R}) \to M(r \times r, \mathbb{R})$ for $i \in \{1, 2, \ldots, k\}$ be the projection maps corresponding to these k minors. Let $\delta : M(r \times r, \mathbb{R}) \to \mathbb{R}$ be the determinant function. Since p_i and δ are continuous functions, the composition $\delta \circ p_i$ is a continuous function from $M(m \times n, \mathbb{R})$ to $\mathbb{R}$ for each $i \in \{1, 2, \ldots, k\}$. Consequently, for each $i \in \{1, 2, \ldots, k\}$, the set $U_i = (\delta \circ p_i)^{-1}(\mathbb{R} \backslash \{0\})$ is open. Now, every matrix in $M(m \times n, \mathbb{R})$ of rank at least r belongs to at least one of U_i for $i \in \{1, 2, \ldots, k\}$. Thus, $O_r = \bigcup_{i=1}^{k} U_i$ is open in $M(m \times n, \mathbb{R})$. □

Let V and W be inner-product spaces of dimension n and m respectively. Then V and W can be identified by $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. And, so $\operatorname{Hom}(V, W)$ can be identified by $M(m \times n, \mathbb{R})$ and be given the usual topology of $M(m \times n, \mathbb{R})$. Lemma 5.2.1 leads to the following theorem.

Theorem 5.2.1. *Let V and W be finite-dimensional real inner-product spaces. Let r be any integer. Define*

$$O_r = \{f \,|\, f : V \to W,\ f \text{ is linear},\ \operatorname{rank}(f) \geqslant r\}\ \&$$
$$C_r = \{f \,|\, f : V \to W,\ f \text{ is linear},\ \operatorname{rank}(f) \leqslant r\}.$$

Then, for each r, O_r and C_r are respectively open and closed in Hom(V, W).

As another application of Theorem 5.2.1 we have the following lemma.

Lemma 5.2.2. *Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^m$ be a smooth function, that is its partial derivatives of all orders exist and are continuous on U. Then for any non-negative integer r the set $\{x \in U | \text{rank}(df_p) \geqslant r\}$ is an open subset of $\mathbb{R}^n$.*

Proof. Since f is smooth the function $df : U \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ given by $df(x) = df_x$ is a smooth function. For any non-negative integer r, by Thm. 5.2.1 the set $A = \{T \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) | \text{rank}(T) \geqslant r\}$ is an open set. Consequently, $\{x \in U | \text{rank}(df_p) \geqslant r\} = df^{-1}(A)$ is an open subset U. Since U is open in $\mathbb{R}^n$, we conclude that $\{x \in U | \text{rank}(df_p) \geqslant r\}$ is an open set in $\mathbb{R}^n$. □

Theorem 5.2.2. *Let M and N be two smooth manifolds. Let $f : M \to N$ be a smooth function. Then for any non-negative integer r, the set $\{x \in M | rank(df_x) \geqslant r\}$ is an open subset of M.*

Proof. For $x \in M$ choose co-ordinate charts (U, ϕ) and (V, ψ) at x and $f(x)$ respectively such that $f(U) \subseteq V$. Now the function $\psi \circ f \circ \phi^{-1} = g : \phi(U) \to \psi(V)$ is a smooth function. So by Lemma 5.2.2 the set $A = \{p \in \phi(U) | \text{rank}(dg_p) \geqslant r\}$ is an open set in $\phi(U)$. So $\phi^{-1}(A)$ is an open set in U and hence is an open set in M. Note that since ϕ and ψ are diffeomorphisms for any $p \in \phi(U)$, dg_p has rank r iff $df_{\phi^{-1}(p)}$ has rank r. So $\phi^{-1}(A) = \{x \in U | \text{rank}(df_x) \geqslant r\}$. Now we can cover M by a collection of co-ordinate charts (U_i, ϕ_i) such that there is a collection of co-ordinate charts (V_i, ψ_i) in N satisfying $f(U_i) \subseteq V_i$. So if $\psi_i \circ f \circ {\phi_i}^{-1} = g_i$ and $A_i = \{p \in \phi(U_i) | \text{rank}(d(g_i)_p) \geqslant r\}$ then $\bigcup_i \phi_i^{-1}(A_i)$ is open and $\bigcup_i {\phi_i}^{-1}(A_i) = \{x \in M | \text{rank}(df_x) \geqslant r\}$. This completes the proof. □

Lemma 5.2.3. *Let V and W be finite-dimensional real inner-product spaces and let $T : V \to W$ be a linear transformation. Suppose $\{T_k\}$ is a sequence in IPM (V, W) which converges to T. Then, terms of the sequence $\{T_k\}$ eventually have the same rank as T.*

Proof. Let the dimension of V be n and rank(T) be r. For each natural number k, let X_k and Y_k denote, respectively, the kernel and isometry subspaces of T_k.

Since $T_k \to T$, there exists a natural number K_1 such that for any natural number $k > K_1$, we have $\|T_k - T\| < \frac{1}{2}$. We claim that for each natural number $k > K_1$, the intersection $\ker(T) \cap Y_k = \{0\}$. To prove this, it suffices to show that if $\alpha \in \ker(T)$ with $\|\alpha\| = 1$, then $\alpha \notin Y_k$. We have $\|T_k(\alpha)\| = \|T_k(\alpha) - T(\alpha)\| \leqslant \|T_k - T\|\|\alpha\| < \frac{1}{2}$. Thus, $\alpha \notin Y_k$, the isometry subspace of T_k.

Fix a natural number $k > K_1$ and suppose that $\{u_i\}_{i=1}^{n-r}$ is a basis of $\ker(T)$. Since each T_k is an inner-product morphism, we have $V = X_k \oplus Y_k$. Thus, we can find $v_i \in X_k$ and $w_i \in Y_k$ such that $u_i = v_i + w_i$ for all $i \in \{1, 2, \ldots, n-r\}$. We claim that $\{v_i\}_{i=1}^{n-r}$ is linearly independent. Suppose that for some scalars β_i, we have $\sum_{i=1}^{n-r} \beta_i v_i = 0$. Since each $v_i = u_i - w_i$, we get $\sum_{i=1}^{n-r} \beta_i u_i = \sum_{i=1}^{n-r} \beta_i w_i$ which implies that $\sum_{i=1}^{n-r} \beta_i u_i \in Y_k$. From our previous observation that $\ker(T) \cap Y_k = \{0\}$, we have $\sum_{i=1}^{n-r} \beta_i u_i = 0$ which by linear independence implies that $\beta_i = 0$ for all $i \in \{1, 2, \ldots, n-r\}$. Conclude that $\{v_i\}_{i=1}^{n-r}$ is linearly independent and hence $\operatorname{rank}(T_k) \leqslant \operatorname{rank}(T)$. Finally, from Theorem 5.2.1, there exists a natural number K such that for all $k > K$, we have $\operatorname{rank}(T_k) = \operatorname{rank}(T)$. □

Proposition 5.2.1. *Let $f : M \to N$ be a Riemannian pre-morphism between Riemannian manifolds M and N. Define $\rho : M \to \mathbb{Z}$ by $x \mapsto \operatorname{rank}(df_x)$. Then, the function ρ is constant on each connected component of M.*

Proof. Let the dimensions of M and N be m and n respectively. We shall first show that ρ is a locally constant function on M.

On the contrary suppose that ρ is not locally constant at a $p \in M$. Choose co-ordinate charts (U, ϕ) and (V, ψ) around p and $f(p)$ respectively. Let g denote the function $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$. Since f is a Riemannian pre-morphism, for each $y \in \phi(U)$ we may view $dg_y : \mathbb{R}^m \to \mathbb{R}^n$ as an inner-product morphism between real inner-product spaces. And since ϕ and ψ are diffeomorphisms, the function $y \mapsto \operatorname{rank}(dg_y)$ is not locally constant at $q = \phi(p)$. Hence, there exists a sequence $\{q_k\}$ in $\phi(U)$ such that $q_k \to q$ but has infinitely many terms q_l with $\operatorname{rank}(dg_{q_l}) \neq \operatorname{rank}(dg_q)$, i.e., the sequence $\operatorname{rank}(dg_{q_k})$ is not eventually constant. This contradicts Lemma 5.2.3 and makes untenable our assumption that ρ is not locally constant at p.

Next, every connected component of M is path-connected. Hence any two points in a component of M can be joined by a path which is compact and connected. On each such path ρ is locally constant and hence constant on the entire path. Consequently, ρ is constant on every connected component of M. □

Corollary 5.2.1. *Let $f : M \to N$ be a Riemannian pre-morphism between Riemannian manifolds M and N. Then the dimensions of the kernel and isometry subspaces of df_x for $x \in M$ are constants on each connected component of M.*

Corollary 5.2.2. *Let M and N be Riemannian manifolds such that M is connected. Further, let $f : M \to N$ be a Riemannian pre-morphism. Then, for each $a \in f(M)$, we have that $f^{-1}(a)$ is a submanifold of M.*

Definition 5.2.2. *Let $f : M \to N$ be a pre-morphism from a connected Riemannian manifold M to a Riemannian manifold N. Let k denote the constant rank (df_x) for $x \in M$. We say that f is a **pre-morphism of rank** k.*

5.3 Riemannian Morphism and its Properties

Definition 5.2.1 of Riemannian pre–morphism incorporates the idea of an inner–product morphism without integrating the kernel and isometry subspaces of the differentials with the smooth structure on the manifold. Our next definition rectifies this defect.

Definition 5.3.1. *Let M and N be Riemannian manifolds and $f : M \to N$ a Riemannian pre-morphism. For each connected component C of M, let κ_C and ι_C denote the dimensions of the kernel and isometry subspaces respectively of the differentials df_x for $x \in C$. We say that f is a **Riemannian morphism** or simply a **morphism** if for every $p \in C$, there exist an open $U \subset C$ with $p \in U$ and smooth sections*

$$v_k : U \to TM \text{ and } w_i : U \to TM, \quad \text{for } k \in \{1, 2, \ldots, \kappa_C\} \text{ and } i \in \{1, 2, \ldots, \iota_C\}$$

such that for all $x \in U$, $\{v_k(x)\}_1^{\kappa_C}$ and $\{w_i(x)\}_1^{\iota_C}$ are respectively a basis for the kernel and isometry subspace of df_x.

*If we can take $U = C$ above, we say that f is a **global Riemannian morphism** or simply a **global morphism**.*

From [Lee09] we have following lemma:

Lemma 5.3.1. *Let $E \to M$ be a rank n vector bundle. Suppose that a subspace $E_p^{'}$ of E_p is given for each $p \in M$ and consider $E^{'} = \cup_{p \in M} E_p^{'}$. Then $E^{'}$ is the total space of a rank l vector subbundle if and only if for each $p \in M$, there is an open neighbourhood U of p on which smooth sections $\sigma_1, \sigma_2, ..., \sigma_l$ are defined such that for each $q \in U$ the set $\{\sigma_1(q), \sigma_2(q), ..., \sigma_l(q)\}$ is a basis of the subspace $E_q^{'}$.*

Proof. Suppose that for each $p \in M$, there is an open neighborhood U of p on which smooth sections $\sigma_1, \sigma_2, ..., \sigma_l$ are defined such that for each $q \in U$ the set $\{\sigma_1(q), \sigma_2(q), ..., \sigma_l(q)\}$ is a basis of the subspace.

First we will show that there is a manifold structure on $E^{'}$. Choose a bundle chart (U, ϕ) of E such that (U, ψ) is a chart of M and there exist smooth sections $\sigma_1, \sigma_2, ..., \sigma_l$ defined on U such that for each $q \in U$ the set $\{\sigma_1(q), \sigma_2(q), ..., \sigma_l(q)\}$ is a basis of the subspace $E_q^{'}$. Let $\{U_i, \phi_i\}_{i \in I}$ be a collection of such bundle charts such that $\{U_i\}_{i \in I}$ is an open covering of M. Define

$\gamma_i : \psi_i(U_i) \times \mathbb{R}^l \to \cup_{q \in U_i} E_q^{'}$ by $\gamma_i(u, a) = \sum_{j=1}^{l} a_j \sigma_j^i(\psi_i^{-1}(u))$, *where* $a = \sum_{j=1}^{l} a_j e_j$ and $\sigma_1^i, ..., \sigma_l^i$ are the smooth sections defined on U_i. Now topologize $E^{'}$ by considering the collection $\{\gamma_i(V) : i \in I \text{ and } V \text{ is open in } \psi_i(U_i) \times \mathbb{R}^l\}$ as subbase. It can be shown that this makes $E^{'}$ into a topological manifold. Suppose that we have chosen $\{U_i, \phi_i\}_{i \in I}$ in such a way that$(\psi_i, \mathscr{I}) \circ \phi_i \circ \gamma_i(u, a) = (u, \overline{a})$, where $\overline{a} = (a_1, a_2, ..., a_l, 0..., 0)$ and $(\psi_i, \mathscr{I}) : U_i \times \mathbb{R}^n \to \psi(U_i) \times \mathbb{R}^n$ is given by $(\psi_i, \mathscr{I})(p, a) = (\psi_i(p), a)$. Now suppose that U_r and U_s be two members in $\{U_i\}_{i \in I}$ such that $U_r \cap U_s$ is nonempty. Since the map $\gamma_s^{-1} \circ \gamma_r : \psi_r(U_r) \times \mathbb{R}^l \to \psi_r(U_s) \times \mathbb{R}^l$ is given by $\gamma_s^{-1} \circ \gamma_r(u, a) = (\psi_s, \mathscr{I}) \circ \phi_s \circ \gamma_r(u, a)$ and each of $(\psi_s, \mathscr{I})$, ϕ_s and γ_r are smooth, $\gamma_s^{-1} \circ \gamma_r$ is also smooth. So $E^{'}$ is a smooth manifold. Note that with this structure $E^{'}$ can be imbedded in E. Also since $\phi_i^{-1}(U_i) \cap E^{'} = \gamma_i(\psi_i(U_i) \times \mathbb{R}^l)$ and $\phi_i(\gamma_i(\psi_i(U_i) \times \mathbb{R}^l)) = U_i \times \tilde{\mathbb{R}}^l$, where $\tilde{\mathbb{R}}^l = \{(x, 0, ..., 0) \in \mathbb{R}^n | x \in \mathbb{R}^l\}$, $E^{'}$ is a subbundle of E.

Conversely suppose that $E^{'}$ is a subbundle of E. Since $E^{'}$ is a subbundle for all $p \in M$, there exists a vector bundle chart (U, ϕ) such that $\phi(\phi^{-1}(U) \cap E^{'}) = U \times \mathbb{R}^l$. Define $\theta = \phi_{|\phi^{-1}(U) \cap E'}$. Define $\sigma_i : U \to E$ by $\sigma_i(q) = \theta^{-1}(q, e_i)$, for each i. Since

θ^{-1} is smooth, σ_i is smooth. Clearly σ_i's are smooth sections satisfying the requisite properties. □

Remark 5.3.1. *Let M and N be two smooth manifolds of dimensions m and n respectively. If $f : M \to N$ is a Riemannian morphism of rank k then from Lemma 5.3.1 there exist two distributions on M of ranks k and m-k.*

Remark 5.3.2. *We will call the distributions given by above lemma and corresponding to isometry subspace as* ***Isometric Distribution*** *and the one corresponding to kernel subspaces as* ***Null Distribution****.*

Lemma 5.3.2. *A smooth function f on a connected manifold M such that rank of df_p is zero for all $p \in M$, is constant.*

Proof. Let $f : M \to N$ be a smooth function such that rank of df_p is zero for all $p \in M$. Choose co-ordinate charts (ϕ, U) and (ψ, V) at p and $f(p)$ respectively such that $f(U) \subseteq V$. For all $x \in \phi(U)$ $d(\psi \circ f \circ \phi^{-1})_x$ has rank zero and hence $\psi \circ f \circ \phi^{-1}$ is constant on $\phi(U)$. Since ϕ and ψ are bijections, f is a constant map on U. We have proved that f is a locally constant function on a connected manifold M and consequently $f$is constant on M. □

Corollary 5.3.1. *If f is a non-constant Riemannian pre-morphism on a connected smooth manifold M then rank(df_p) is nonzero for all $p \in M$.*

Proof. By previous lemma rank(df_p) cannot be zero everywhere. But by Prop. 5.2.1 rank(df_p) is constant on M, so it is nonzero everywhere on M. □

Remark 5.3.3. *If M is any smooth manifold and $f : M \to N$ is a global Riemannian morphism with dimension of its isometry subspaces nowhere zero then there exists a non-vanishing smooth vector field on M.*

Theorem 5.3.1. *There is no non-constant global Riemannian morphism from a compact oriented manifold with non-zero Euler characteristic to any manifold.*

Proof. Let M be a compact orientable manifold with non-zero Euler characteristic and suppose if possible, there is a non-constant global Riemannian morphism from M to any other Riemannian manifold. So by the 5.3.3, there exists a non-vanishing locally smooth vector field on M. But from Poincare-Hopf Index theorem (see [GP10]) we know that there cannot be a non-vanishing smooth vector field on a compact oriented manifold with non-zero Euler characteristic. Hence our assumption is wrong. □

Corollary 5.3.2. *There is no non-constant global Riemannian morphism from sphere or any tori of genus more than one to any manifold.*

Definition 5.3.2. *Let $E \to M$ be a distribution on an n-manifold M and let X be a vector field defined on an open set $U \subset M$. We say that X lies in the distribution if $X(p) \in E_p$ for each p in the domain of X; that is, if X takes values in E.*

Definition 5.3.3. *If for every pair of locally defined vector fields X and Y with common domain that lie in a distribution $E \to M$, the bracket $[X, Y]$ also lies in the distribution, we say that the distribution is involutive.*

From [Lee09] we have Theorem 5.3.2

Theorem 5.3.2. *A distribution is involutive if and only if it is completely integrable and if and only if it is integrable.*

From Theorem 5.3.2 we can conclude following theorem:

Theorem 5.3.3. *Let $f : M \to N$ be a surjective Riemannian morphism whose Isometric distribution is involutive. Then f maps a submanifold of M isometrically onto N.*

The material developed in this chapter has been submitted for publication as article [SYc].

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[SYa] K. V. Srikanth and Raj Bhawan Yadav. Decomposition of invertible and conformal transformations. `http://arxiv.org/abs/1309.5805`.

[SYb] K. V. Srikanth and Raj Bhawan Yadav. On a generalization of the stone-weierstrass theorem.

[SYc] K. V. Srikanth and Raj Bhawan Yadav. On morphisms between riemannian manifolds.

[SYd] K. V. Srikanth and Raj Bhawan Yadav. Some density results in $c(s^1, s^1)$ and $c(s^2, s^2)$.

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List of published and communicated papers

Based on the work in this thesis, the following research articles are published/have been communicated.

- K. V. Srikanth, Raj Bhawan Yadav., Some Density Results in $C(S^1, S^1)$ and $C(S^2, S^2)$, submitted.

- K. V. Srikanth, Raj Bhawan Yadav, On an extension of the Stone-Weierstrass Theorem, accepted.

- K. V. Srikanth, Raj Bhawan Yadav, On Morphisms between Riemannian manifolds, submitted.

- K.V. Srikanth, Raj Bhawan Yadav, Decomposition Of Invertible And Conformal Transformations, http://arxiv.org/abs/1309.5805.

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