STATE TRANSFER ON NEPS AND CAYLEY GRAPHS

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November, 2016

STATE TRANSFER ON NEPS AND CAYLEY GRAPHS

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of

Doctor of Philosophy

by

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to the

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November, 2016

DECLARATION

I do hereby declare that this thesis entitled **“State Transfer on NEPS and Cayley Graphs”** is a presentation of my original research work done under the supervision of **Dr. Bikash Bhattacharjya**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is to certify that the work contained in this thesis entitled **"State Transfer on NEPS and Cayley Graphs"** has been carried out by **Hiranmoy Pal,** a student at the Department of Mathematics, Indian Institute of Technology Guwahati, under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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To My Parents

Ramkrishna Pal

&

Rekha Pal

Acknowledgement

I would like to acknowledge all those people who helped me to carry out the research work and shared their precious experience.

I am very grateful to my thesis supervisor **Dr. Bikash Bhattacharjya** for his generous guidance throughout the years. Also I would like to express my deepest gratitude for his patience, understanding and support. He helped me to overcome many obstacles that emerged while working on several problems. In a good way, he always used to push for better results. He was deeply involved and helped me in every possible ways. His continuous encouragement and advice made it possible for me to work on this thesis.

I would like to express my gratitude to the doctoral committee members, **Prof. B. K. Sarma**, **Prof. S. Pati** and **Dr. A. K. Chakrabarty** for their encouragement, precious comments and discussions on several occasions.

I would also like to convey my deep gratitude to **Prof. Chris Godsil** for his insightful comments and advice. I would take this opportunity to thank him for his lectures given at **RMMC Summer School 2013** held at University of Wyoming, USA. The interesting topics discussed in those lectures left a lasting impression on me and so I started my research work on those topics.

I am also indebted to my friends, colleagues and the faculties of Department of Mathematics, IIT Guwahati, who have motivated and helped me in many situations directly or indirectly.

I sincerely acknowledge **Indian Institute of Technology Guwahati** for providing a very nice educational environment. I am highly grateful to the **Ministry of Human Resource Development**, Government of India for the necessary financial supports. I would also like to appreciate the travel grant provided by **National Board for Higher Mathematics** to attend **RMMC Summer School 2013** held at University of Wyoming, USA.

I would like to mention the love and enormous support provided by my family members, especially my parents and my brothers, Chiranjit Pal and Tanmoy Pal. I am also very grateful to my uncle Prof. Surath Ch. Pal for the encouragement and motivation he gave throughout my career.

I am also very thankful to all those peoples who shared some of their noble views and countless ideas about mathematics and helped me to better understand the subject.

November, 2016

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Abstract

The study of state transfer in quantum communication networks received a lot of attention over the past few decades. A quantum network can be modelled by a graph with the adjacency matrix of the graph as the Hamiltonian of such system. In that case, we identify a quantum system by its underlying graph. If there is no external dynamic control over the system then some physical properties of the quantum system depend only on the underlying graph. In this thesis, we are concerned with two such properties: perfect state transfer and pretty good state transfer. We find some major classes of graphs exhibiting either of those properties.

Let G be a graph with adjacency matrix A. The transition matrix of G relative to A is defined by $H(t) := \exp(-itA)$, $t \in \mathbb{R}$. The graph G is said to have perfect state transfer from a vertex u to another vertex v if there exists $\tau\,(\neq 0) \in \mathbb{R}$ such that the uv-th entry of $H(\tau)$ has unit modulus. In case $u = v$, we say that G is periodic at the vertex u at time τ. The graph G is said to be periodic if it is periodic at all vertices at the same time. Perfect state transfer is a rare phenomena so we also consider an approximation called pretty good state transfer. The graph G is said to admit pretty good state transfer between a pair of vertices u and v if there exists a sequence of real numbers $\{t_k\}$ and a complex number γ of unit modulus such that $\lim_{k\to\infty} H(t_k)\mathbf{e}_u = \gamma\mathbf{e}_v$. In Chapter 1, we introduce these topics in detail. Along with that we discuss some relevant definitions and basic results.

We mainly consider two classes of graphs. One is NEPS (non-complete extended P-sum) of the path on three vertices and the other one is Cayley graph. In both classes, we investigate graphs for perfect state transfer and pretty good state transfer.

It is well known that a path on three vertices exhibits perfect state transfer and so we investigate some NEPS of the path on three vertices in Chapter 2. A sufficient condition is found for a NEPS of path on three vertices to have perfect state transfer. Using these NEPS, some other graphs are also constructed that admit perfect state transfer. The results of this chapter are published in [40].

In Chapter 3, we also find that NEPS of the path on three vertices whose basis contains tuples with hamming weights of both parities do not exhibit perfect state transfer. But these NEPS admit pretty good state transfer with an additional condition. Further, we investigate pretty good state transfer on Cartesian product of graphs and we find that a graph can have PGST from a vertex u to two different vertices v and w. The results of this chapter are published in [42].

A gcd-graph is a Cayley graph over a finite abelian group defined by greatest common divisors. In Chapter 4, we establish a sufficient condition for a gcd-graph to have periodicity and PST. Using this we deduce that there exists gcd-graph having PST over any abelian group of order divisible by 4. Also, we find a necessary and sufficient condition for a class of gcd-graphs to be periodic at π. Using this, we characterize a class of gcd-graphs not exhibiting PST at $\frac{\pi}{2^k}$ for any positive integer k. The results of this chapter appear in [39].

In Chapter 5, we find that pretty good state transfer occurs in a cycle on n vertices if and only if n is a power of two and it occurs between every pair of antipodal vertices. In addition, we look for pretty good state transfer in more general circulant graphs. We prove that union (edge disjoint) of an integral circulant graph with a cycle, each on 2^k $(k \geq 3)$ vertices, admits pretty good state transfer. The complement of such union also admits pretty good state transfer. This enables us to find some non-circulant graphs admitting pretty good state transfer. Among the complement of cycles we also find a class of graphs not exhibiting pretty good state transfer. The results of this chapter appear in [41].

In Chapter 6, we describe a few directions for future research based on the work of this thesis.

Publications

Based on the work in this thesis, the following research articles are published.

1. H. Pal and B. Bhattacharjya. *A class of gcd-graphs having Perfect State Transfer.* Electronic Notes in Discrete Mathematics, **53**:319-329, 2016. [Proceedings of International Conference on Graph Theory and its Applications 2015, Amrita School of Engineering, Coimbatore, India].

2. H. Pal and B. Bhattacharjya. *Perfect State Transfer on gcd-graphs.* Linear and Multilinear Algebra, doi:10.1080/03081087.2016.1267105.

3. H. Pal and B. Bhattacharjya. *Perfect state transfer on NEPS of the path on three vertices.* Discrete Mathematics, **339**(2):831-838, 2016.

4. H. Pal and B. Bhattacharjya. *Pretty Good State Transfer on Circulant Graphs.* The Electronic Journal of Combinatorics, **24**(2):# P2.23, 2017.

5. H. Pal and B. Bhattacharjya. *Pretty Good State Transfer on Some NEPS.* Discrete Mathematics, **340**(4):746-752, 2017.

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Chapter 1

Introduction

Graph theory has applications in many areas of natural sciences. In this thesis, we study a topic of algebraic graph theory that has application in quantum physics (see [8, 12, 21]). The graphs we consider are mostly NEPS of the path on three vertices and Cayley graphs over abelian group. In the following sections, we discuss some definitions and basic results of graph theory that will be required later in the thesis. For any other graph theoretic, group theoretic and linear algebra terms and notations, that have been used in the thesis but not defined, we refer the reader to [10, 20, 29, 30] and [50]. All unreferenced results appearing in this thesis are the authors own contribution. Due references are given for all results that have been taken from other sources.

1.1 Definitions and Basic Results

A graph G is an ordered pair $(V(G), E(G))$ consisting of a non-empty vertex set $V(G)$ and an edge set $E(G)$ disjoint from $V(G)$. Each edge is associated with two vertices, called endpoints. Two vertices (not necessarily distinct) are called adjacent if there is an edge with those two endpoints. An edge with same endpoints is called a loop. Edges having same pair of endpoints are called multiple (parallel) edges. A simple graph is a graph having no loops or multiple edges. A graph is called finite if both its vertex set and edge set are finite. An undirected graph is a graph where its edges have no direction. Throughout the thesis all graphs are assumed to be simple, undirected and finite, unless otherwise stated. Let the vertex set $V(G)$ of G be $\{v_1, \ldots, v_n\}$. The adjacency matrix A associated to G is the $n \times n$ matrix in which the ij-th entry is the number of edges with endpoints v_i and v_j.

Certain types of graphs appear frequently in the theory of graphs. A complete graph is a simple graph in which every pair of vertices are adjacent. A complete graph with n number of vertices is denoted by K_n. A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if and only if they are consecutive in the sequence. A path with n number of vertices is denoted by P_n.

Graph isomorphism plays an important role in the theory of graphs. An isomorphism from a graph G to another graph H is a bijection f that maps $V(G)$ onto $V(H)$ such that two vertices u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. An automorphism of G is an isomorphism from G to itself. A graph G is called vertex-transitive if for every pair of vertices u and v in G there exists an automorphism that maps u to v. A very well known class of vertex-transitive graphs is the class of Cayley graphs. If every vertex of a graph have same degree then it is called a regular graph. It is also well established that all vertex-transitive graphs are regular graphs.

1.1.1 Cayley Graph

Now we introduce Cayley graphs over abelian group and discuss some properties of their adjacency matrices.

Let $(\Gamma, +)$ be a finite abelian group and consider $S \subseteq \Gamma$ with the property that $\{-s : s \in S\} = S$. Such a set S is called a symmetric subset of Γ. A Cayley graph over Γ with symmetric set S has the vertex set Γ, where two vertices $a, b \in \Gamma$ are adjacent if and only if $a - b \in S$. The graph is denoted by $Cay(\Gamma, S)$ and the set S is called the connection set of $Cay(\Gamma, S)$. Notice that if the additive identity $0 \in S$ then $Cay(\Gamma, S)$ has loops at each of its vertices. In Chapter 4, a part of the discussion involve looped Cayley graphs.

Let $\mathbb{Z}_n$ be the cyclic group of order n. A circulant graph is a Cayley graph over $\mathbb{Z}_n$. A cycle C_n, in particular, is a circulant graph over $\mathbb{Z}_n$ with the connection set $\{1, n-1\}$.

The eigenvalues and eigenvectors of a Cayley graph over an abelian group are known in terms of characters of the abelian group. A character [47] of an abelian group Γ is a group homomorphism from Γ to the multiplicative group of non-zero complex numbers. A character is called irreducible if it corresponds to an irreducible representation of Γ. The eigenvalues and eigenvectors corresponding to the adjacency matrix of a Cayley graph over an abelian group Γ can be determined in terms of irreducible characters of Γ. The following comes as a consequence to the results proved in [36]. Also see [35] for some more useful informations in this regard.

Theorem 1.1.1. *[36] Let Γ be a finite abelian group and also let S be a symmetric subset of Γ. Suppose A is the adjacency matrix of $Cay(\Gamma, S)$. If χ is an irreducible character of Γ then the column vector $(\chi(v))_{v\in\Gamma}$ is an eigenvector of A with eigenvalue $\sum\limits_{s\in S}\chi(s)$.*

Proof. Consider that $A = (a_{u,v})$, where $u, v \in \Gamma$. For $u \in \Gamma$, we have

$$\begin{aligned}\sum_{v\in\Gamma} a_{u,v}\chi(v) = \sum_{v\in\Gamma, u-v\in S}\chi(v) &= \sum_{s\in S}\chi(s+u)\\ &= \sum_{s\in S}\chi(s)\chi(u)\\ &= \left(\sum_{s\in S}\chi(s)\right)\chi(u).\end{aligned}$$

Hence we have the desired result. □

Consider a finite abelian group Γ with the cyclic group decomposition $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. We denote the primitive n-th root $\exp\left(\frac{2\pi i}{n}\right)$ of unity by ω_n. For each $(i_1, \ldots, i_k) \in \Gamma$, the map $\chi_{i_1,\ldots,i_k} : \Gamma \longrightarrow \mathbb{C}\setminus\{0\}$ defined by

$$\chi_{i_1,\ldots,i_k}(s_1, \ldots, s_k) = \omega_{n_1}^{i_1 s_1} \ldots \omega_{n_k}^{i_k s_k},\ (s_1, \ldots, s_k) \in \Gamma$$

is an irreducible character of Γ and all the irreducible characters of Γ are of this form. See [47] for details. It is well known that irreducible characters of a finite group form an orthonormal set. Hence the eigenvectors (in Theorem 1.1.1) corresponding to irreducible characters of Γ are linearly independent. Notice that each element of S can be written uniquely as $(s_1, s_2, \ldots, s_k)$ where $s_i \in \mathbb{Z}_{n_i}$ for $i = 1, 2, \ldots, k$. Therefore all the eigenvalues of $Cay(\Gamma, S)$ can be evaluated explicitly as

$$\lambda_{i_1,\ldots,i_k} = \sum_{(s_1,\ldots,s_k)\in S} \omega_{n_1}^{i_1 s_1} \ldots \omega_{n_k}^{i_k s_k},\ (i_1, \ldots, i_k) \in \Gamma. \tag{1.1}$$

Consider two symmetric subsets S_1 and S_2 in Γ. Observe that the eigenvector in Theorem 1.1.1 is independent of the connection set. Therefore the set of eigenvectors of both graphs $Cay(\Gamma, S_1)$ and $Cay(\Gamma, S_2)$, obtained by using Theorem 1.1.1, are equal. Thus we have the following result.

Proposition 1.1.2. *If S_1 and S_2 are symmetric subsets of an abelian group Γ then adjacency matrices of $Cay(\Gamma, S_1)$ and $Cay(\Gamma, S_2)$ commute.*

Proof. Let $|\Gamma| = n$ and suppose that the adjacency matrices of $Cay(\Gamma, S_1)$ and $Cay(\Gamma, S_2)$ are A and B, respectively. Consider $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to be the set of all the orthogonal eigenvectors, as obtained using Theorem 1.1.1, of both graphs $Cay(\Gamma, S_1)$ and $Cay(\Gamma, S_2)$. It is easy to see that $(AB)\,\mathbf{v}_i = (BA)\,\mathbf{v}_i$ for $i = 1, \ldots, n$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ forms a basis of $\mathbb{R}^n$, we have $(AB)\,\mathbf{v} = (BA)\,\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. Hence A and B commute. ☐

For more details on the eigenvalues and eigenvectors of a Cayley graph we refer the reader to [35, 36].

1.1.2 NEPS of Graphs

Now we introduce NEPS (non-complete extended P-sum) of graphs and mention some of the relevant results regarding its adjacency matrix, eigenvalues and eigenvectors. A NEPS [20] is a graph product defined as follows.

Let $G_1, \ldots, G_n$ be n graphs and suppose that $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$. The NEPS of $G_1, \ldots, G_n$ associated to Ω is denoted by $NEPS\,(G_1, \ldots, G_n; \Omega)$, which has the vertex set $V\,(G_1) \times \cdots \times V\,(G_n)$. Two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent in $NEPS\,(G_1, \ldots, G_n; \Omega)$ if and only if there is an n-tuple $(\beta_1, \ldots, \beta_n)$ in Ω such that for $j = 1, \ldots, n$

- $x_j = y_j$ exactly when $\beta_j = 0$; and
- x_j is adjacent to y_j in G_j exactly when $\beta_j = 1$.

The graphs $G_1, \ldots, G_n$ are called factor graphs and Ω is called basis of the NEPS. If all the factor graphs of a NEPS with basis Ω are identical to a graph G then, for the sake of simplicity, we denote the NEPS by $NEPS_n\,(G, \Omega)$ where n is the number of factor graphs.

If the basis Ω is $\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$ then the corresponding NEPS is called the Cartesian product of the associated graphs. If the basis Ω is $\{(1, 1, \ldots, 1)\}$ then the corresponding NEPS is called the Kronecker product of the associated graphs.

The adjacency matrix of a NEPS can be obtained in terms of the adjacency matrices of its factors. The tensor product of two matrices $A = \left(a_{i,j}\right)$ and B is denote by $A \otimes B$, which is defined as the block matrix $A \otimes B := \left(a_{i,j}B\right)$. The following result gives the adjacency matrix of a NEPS of some graphs.

Theorem 1.1.3. *[20] Let $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ and suppose the graphs $G_1, \ldots, G_n$ have the adjacency matrices $A_1, \ldots, A_n$, respectively. Then the adjacency matrix of $NEPS\,(G_1, \ldots, G_n; \Omega)$ is*

$$A_\Omega = \sum_{(\beta_1, \ldots, \beta_n) \in \Omega} A_1^{\beta_1} \otimes \cdots \otimes A_n^{\beta_n}. \tag{1.2}$$

In Theorem 1.1.3, it is considered that for a matrix A, $A^0 = I$ where the identity matrix I has same order as that of A. In this setting, it can be observed that the sum in Equation 1.2 is indeed a $0/1$-matrix. The eigenvalues and eigenvectors of a NEPS are also well known in terms of those of the factor graphs. The following result gives the eigenvalues and eigenvectors of a NEPS of some graphs.

Theorem 1.1.4. *[20] For $i = 1, 2, \ldots, n$, let G_i be a graph with m_i vertices. Also suppose that G_i has the eigenvalues $\lambda_{i1}, \ldots, \lambda_{im_i}$, not necessarily distinct, and let the corresponding eigenvectors be $\mathbf{x}_{i1}, \ldots, \mathbf{x}_{im_i}$. Then the NEPS with basis Ω has thc cigcnvalucs*

$$\Lambda_{j_1 \cdots j_n} = \sum_{(\beta_1,\ldots,\beta_n)\in\Omega} \lambda_{1j_1}^{\beta_1} \cdots \lambda_{nj_n}^{\beta_n},\ j_k = 1, \ldots, m_k,\ k = 1, \ldots, n. \tag{1.3}$$

Also the eigenvector corresponding to $\Lambda_{j_1 \cdots j_n}$ is $\mathbf{x}_{1j_1} \otimes \cdots \otimes \mathbf{x}_{nj_n}$.

More information on the adjacency matrix, eigenvalues and eigenvectors of NEPS can be found in [20].

1.2 Perfect State Transfer (PST)

Perfect state transfer (PST) has great significance due to its applications in quantum information processing and cryptography (see [8, 12, 21]). The phenomenon of PST in quantum communication network was originally introduced by Bose in [11]. Throughout the thesis, we consider PST with respect to adjacency matrix of a graph. In that setting, PST is defined as follows. The transition matrix[1] of a graph G with adjacency matrix A is defined by

$$H(t) := \exp(-itA) = \sum_{k\geq 0} \frac{(-i)^k}{k!} t^k A^k,\ t \in \mathbb{R}.$$

We denote the characteristic vector corresponding to a vertex u of G by $\mathbf{e}_u$. The graph G is said to exhibit PST from a vertex u to another vertex v if there exists a real number $\tau (\neq 0)$ and a complex number γ with $|\gamma| = 1$ such

[1] Later we have used some other convenient notations for the transition matrix according to our requirement.

that $H(\tau)\mathbf{e}_u = \gamma\mathbf{e}_v$, *i.e*, the vu-th entry of $H(\tau)$ has unit modulus. In case $H(\tau)\mathbf{e}_u = \gamma\mathbf{e}_u$, we say that G is periodic at the vertex u at time τ. Moreover, the graph G is said to be periodic if it is periodic at all vertices at the same time. We illustrate this by the following example.

Figure 1.1: The path P_2 with vertices u and v.

Example 1.2.1. Consider the path P_2 on two vertices u and v, as shown in Figure 1.1. The adjacency matrix A of P_2 is given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we calculate the transition matrix $H(t)$ of P_2 using the series expansions of $\cos t$ and $\sin t$. It is clear that $A^2 = I$ and therefore

$$\begin{aligned} \exp(-itA) &= I - itA - \frac{1}{2!}t^2 I + \frac{1}{3!}it^3 A + \frac{1}{4!}t^4 I + \cdots \\ &= \cos(t)I - i\sin(t)A. \end{aligned}$$

Hence the transition matrix of P_2 is given by

$$H(t) = \begin{pmatrix} \cos(t) & -i\sin(t) \\ -i\sin(t) & \cos(t) \end{pmatrix}.$$

Note that if $t = \frac{\pi}{2}$ then uv-th entry of $H(\frac{\pi}{2})$ is $-i$ which has unit modulus. This implies that P_2 has PST at $\frac{\pi}{2}$ from u to v. Moreover, we have $H(\pi) = -I$ and hence P_2 is periodic at π.

We will see later that the path P_3 on three vertices also exhibits PST. However, in [14], Christandl *et al.* proved that PST does not occur between the end vertices of a path on n vertices whenever $n \geq 4$.

In most of the published papers, we see that PST has been considered with respect to the adjacency matrix. But PST with respect to the Laplacian matrix can be considered as well (see [1, 3, 19]). For an r-regular graph G with adjacency matrix A, the Laplacian matrix is given by $L := rI - A$, where I is the identity matrix. Now observe that for $t \in \mathbb{R}$,

$$\begin{aligned} \exp(-itL) = \exp[-it(rI - A)] &= \exp(-irtI)\exp(itA) \\ &= [\exp(-irt)I]\exp[-i(-t)A] \\ &= \exp(-irt)\exp[-i(-t)A]. \end{aligned} \tag{1.4}$$

Since $\exp(-irt)$ is of unit modulus for every real number t, it follows that a regular graph G has PST with respect to adjacency matrix if and only if PST occurs in G with respect to Laplacian matrix.

1.2.1 Properties of Transition Matrix

Now we discuss some useful properties of the transition matrix of a graph G. Before that we briefly introduce the spectral decomposition of a real symmetric matrix. Let A be an $n \times n$ real symmetric matrix. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of A and also let the projections (idempotents) onto the corresponding eigenspaces be $E_1, \ldots, E_m$. Then we necessarily have $E_r^2 = E_r$ for $1 \leq r \leq m$. Since the eigenvectors corresponding to distinct eigenvalues are orthogonal, it follows that $E_r E_s = 0$ for $r \neq s$, $1 \leq r, s \leq m$. It is also well known that $\mathbb{R}^n$ has a basis containing only the eigenvectors of A and hence we have $E_1 + \cdots + E_m = I$. This further implies that

$$A = \sum_{r=1}^{m} \lambda_r E_r,$$

which is known as the spectral decomposition of A. For more details on spectral decomposition, we refer to [29].

Notice that the adjacency matrix A of a graph G is a real symmetric matrix.

Therefore, using the spectral decomposition of A, we compute the transition matrix of G by

$$H(t) = \exp(-itA) = \sum_{r=1}^{m} \exp(-it\lambda_r)E_r.$$

We now establish that the transition matrix is in fact a polynomial in A. By Lagrange interpolation, there is a polynomial $p(x)$ of degree at most $m-1$ such that $p(\lambda_r) = \exp(-it\lambda_r)$, where $1 \leq r \leq m$. This implies that

$$p(A) = \sum_{r=1}^{m} p(\lambda_r) E_r = \sum_{r=1}^{m} \exp(-it\lambda_r)E_r = H(t).$$

So $H(t)$ is a polynomial in A and hence $H(t)$ is symmetric. Another implication to this fact is that all matrices that commute with A must commute with the transition matrix $H(t)$. Again notice that

$$\begin{aligned} H(t)\,(H(t))^* &= H(t)\overline{H(t)}, \text{ since } H(t) \text{ is symmetric} \\ &= \exp(-itA)\exp(itA) \\ &= I, \text{ using properties of exponential function.} \end{aligned}$$

Hence $H(t)$ is also a unitary matrix.

It is well known that periodicity is necessary for a graph to exhibit PST, which follows from the next lemma.

Lemma 1.2.2. *[25] If a graph G admits perfect state transfer from a vertex u to another vertex v at time τ then G is periodic at u and v with period 2τ.*

Proof. Let $H(t)$ be the transition matrix of G. Since G exhibits PST from u to v, we have $H(\tau)\mathbf{e}_u = \gamma\mathbf{e}_v$ for some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$. As $H(\tau)$ is symmetric, we also have $H(\tau)\mathbf{e}_v = \gamma\mathbf{e}_u$. Further, using the properties of exponential function, we obtain

$$H(2\tau)\mathbf{e}_u = H(\tau)\,[H(\tau)\mathbf{e}_u] = H(\tau)\,[\gamma\mathbf{e}_v] = \gamma^2\mathbf{e}_u.$$

Therefore G is periodic at u. Similarly, G is also periodic at v. □

Finding whether a given graph has PST is quite difficult, especially when the graph is large. Remarkably, in [17], Coutinho *et al.* showed that one can decide whether a graph admits PST in polynomial time with respect to the size of the graph on a classical computer.

In initial papers (see [14, 15]), Christandl *et al.* proved that PST occurs on Cartesian powers of the path P_2 on two vertices and Cartesian powers of the path P_3 on three vertices. Further, the results have been generalized for NEPS of P_2 in [9, 13]. PST on several other graph operations have been considered in [16, 24, 26]. Moreover, some results regarding PST on corona product of two graphs appear in [2]. Several results on the existence of PST in circulant graphs can be found in [7, 43, 45]. Also, in [18], the authors investigated PST on distance-regular graphs and association schemes. In [27], Godsil provided some characterizing results on the existence of PST in a graph. A deep analysis on the sensitivity of PST appears in [33].

In Chapter 2 of this thesis, we consider the problem of finding PST on NEPS of P_3. This problem was raised by Stevanović in [48]. We solved the problem to some extent in Chapter 2. We establish that there are NEPS of P_3 having PST at infinitesimal time. The results established in this regard generalize the results of Christandl *et al.*[14, 15] for the path P_3.

In Chapter 4, we consider the problem of classifying PST in a subclass of Cayley graphs known as gcd-graphs. In [26], it is observed that a regular graph is periodic if and only if its eigenvalues are integers. As a consequence, if PST occurs on a Cayley graph then it must be integral. In [35], a characterization of integral Cayley graph over finite abelian groups is given. A well known class of integral Cayley graphs are gcd-graphs. It can be noted from [46] that all integral circulant graphs are gcd-graphs. A classification of PST in integral circulant graphs appears in [7]. A cubelike graph is a Cayley graph over $\mathbb{Z}_2^n$. Bernasconi *et al.* [9] showed that PST occurs on certain cubelike graphs, which are actually gcd-graphs over a direct product of finitely many copies of the group $\mathbb{Z}_2$. In Chapter 4, we classify some of the gcd-graphs admitting or not admitting PST. This generalizes some of the results included in [7, 9].

1.3 Pretty Good State Transfer (PGST)

In [27], Godsil demonstrated the fact that there are only finitely many connected graphs with maximum valency at most k where PST occurs. Since there are less number of graphs having PST, we consider a relaxation to PST called pretty good state transfer.

The notion of pretty good state transfer (PGST) was initially introduced by Godsil in [26]. A graph G with transition matrix $H(t)$ is said to have PGST between a pair of vertices u and v if there is a sequence $\{t_k\}$ of real numbers and a complex number γ of unit modulus such that

$$\lim_{k\to\infty} H(t_k)\mathbf{e}_u = \gamma\mathbf{e}_v. \tag{1.5}$$

In such a case, we also say that G exhibits PGST from u to v with respect to the sequence $\{t_k\}$. It is clear from Equation (1.5) that for each $\epsilon > 0$, there exists $t \in \mathbb{R}$ such that

$$\left|\mathbf{e}_v^T H(t)\mathbf{e}_u - \gamma\right| = \left|\mathbf{e}_v^T \left[H(t)\mathbf{e}_u - \gamma\mathbf{e}_v\right]\right| \leq ||H(t)\mathbf{e}_u - \gamma\mathbf{e}_v|| < \epsilon,$$

where the norm is the usual l_2 norm. Conversely, let us suppose, for given any $\epsilon > 0$ there exist $t \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that $|\mathbf{e}_v^T H(t)\mathbf{e}_u - \gamma| < \epsilon$ for fixed u, v. Then we can find a sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} \mathbf{e}_v^T H(t_k)\mathbf{e}_u = \gamma. \tag{1.6}$$

Moreover

$$\lim_{k\to\infty} \left|\mathbf{e}_v^T H(t_k)\mathbf{e}_u\right|^2 = 1.$$

Since the transition matrix $H(t)$ is a unitary matrix for every real values of t, every column of $H(t)$ has unit norm and hence

$$\sum_{w\in V(G)} \left|\mathbf{e}_w^T H(t_k)\mathbf{e}_u\right|^2 = 1, \text{ for all } k \in \mathbb{N}.$$

This further implies that for all $w \in V(G)$ with $w \neq v$,

$$\lim_{k\to\infty} \left| \mathbf{e}_w^T H(t_k)\mathbf{e}_u \right|^2 = 0, \ i.e, \ \lim_{k\to\infty} \mathbf{e}_w^T H(t_k)\mathbf{e}_u = 0. \tag{1.7}$$

Finally (1.6) and (1.7) together imply that

$$\lim_{k\to\infty} H(t_k)\mathbf{e}_u = \gamma \mathbf{e}_v.$$

Thus we have the following equivalent definition of PGST. There exists PGST between the vertices u and v of G if and only if, for $\epsilon > 0$, there exist $t \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that

$$\left| \mathbf{e}_v^T H(t)\mathbf{e}_u - \gamma \right| < \epsilon. \tag{1.8}$$

Next we make another statement which can be considered as an alternative definition of PGST. A graph G has PGST between u and v if and only if, for each $\epsilon > 0$, there exists $t \in \mathbb{R}$ such that

$$\left| |\mathbf{e}_v^T H(t)\mathbf{e}_u| - 1 \right| < \epsilon. \tag{1.9}$$

The equivalence of (1.8) and (1.9) can be shown in the following way. Notice that (1.9) follows directly from (1.8). On the other hand, if (1.9) holds, then there exists a sequence $\{t_m\}$ of real numbers such that $\{|\mathbf{e}_v^T H(t_m)\mathbf{e}_u|\}$ converges to 1. Now by Bolzano-Weierstrass theorem, there exists a subsequence $\{t_{m_k}\}$ of $\{t_m\}$ such that $\{\mathbf{e}_v^T H(t_{m_k})\mathbf{e}_u\}$ is convergent. Further, it follows from (1.9) that the sequence $\{\mathbf{e}_v^T H(t_{m_k})\mathbf{e}_u\}$ must converge to a complex number γ of unit modulus. Hence the inequality (1.8) is also satisfied.

As an example, we see in [26] that the path on four vertices exhibits PGST between the end vertices.

As in case of PST, for an r-regular graph, PGST occurs with respect to adjacency matrix A if and only if PGST occurs with respect to the Laplacian matrix L. Recall that in (1.4) we obtained $\exp[-itL] = \exp(-irt)\exp[-i(-t)A]$.

Since $\exp(-irt)$ has unit modulus, for $u, v \in V(G)$ we have

$$\left|\mathbf{e}_v^T \exp[-itL]\mathbf{e}_u\right| = \left|\mathbf{e}_v^T \exp[-i(-t)A]\mathbf{e}_u\right|, \ t \in \mathbb{R}.$$

Therefore for $\epsilon > 0$, we observe that

$$\left||\mathbf{e}_v^T \exp[-itL]\mathbf{e}_u| - 1\right| < \epsilon \text{ if and only if } \left||\mathbf{e}_v^T \exp[-i(-t)A]\mathbf{e}_u| - 1\right| < \epsilon.$$

Hence, for an r-regular graph, finding PGST with respect to adjacency matrix is equivalent to finding PGST with respect to Laplacian matrix.

Now we include the proof the following proposition which was mentioned in [23] without a proof. The result gives a relation between PST and PGST in periodic graphs.

Proposition 1.3.1. *If a graph is periodic, then it admits perfect state transfer if and only if it admits pretty good state transfer.*

Proof. It is easy to observe that if a graph G has PST then it must also have PGST. On the other hand, suppose that a graph is periodic at τ and therefore for a vertex u of G there exists $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ so that $H(\tau)\mathbf{e}_u = \gamma\mathbf{e}_u$. For $u, v \in V(G)$, the uv-th entry of $H(t)$ is a continuous function of real numbers. Also observe that, for $t \in \mathbb{R}$,

$$\begin{aligned} \left|\mathbf{e}_v^T H(t+\tau)\mathbf{e}_u\right| = \left|\mathbf{e}_v^T H(t)H(\tau)\mathbf{e}_u\right| &= \left|\mathbf{e}_v^T H(t)\left(\gamma\mathbf{e}_u\right)\right|, \text{ as } H(\tau)\mathbf{e}_u = \gamma\mathbf{e}_u \\ &= \left|\mathbf{e}_v^T H(t)\mathbf{e}_u\right|. \end{aligned}$$

Therefore, the modulus of the uv-th entry of $H(t)$ is a periodic continuous function and so its image is a compact set. Since $H(t)$ is unitary, the modulus of each entry of $H(t)$ is bounded by 1 for all $t \in \mathbb{R}$. Finally, if (1.9) holds then by extreme value theorem we can find $\tau_0 \in \mathbb{R}$ so that $|\mathbf{e}_v^T H(\tau_0)\mathbf{e}_u| = 1$, *i.e,* the graph G has PST between u and v. Hence it follows that if a graph is periodic then the graph has PGST if and only if it has PST. ☐

There are a few published papers which discuss PGST. Godsil *et al.* [28]

showed that the path P_n exhibits PGST if and only if $n+1$ equals to either 2^m or p or $2p$, where p is an odd prime. We also see in [23] that a double star $S_{k,k}$ admits PGST if and only if $4k+1$ is not a perfect square. The double star can be realized as a corona product of the complete graph K_2 and an empty graph. PGST in more general corona products has been studied in [1, 2].

In Chapter 3, we describe some NEPS admitting PGST. We prove that if the basis of a NEPS of P_3 contains tuples with hamming weights of both parities then those graphs do not have PST. However, we find that some of those graphs possesses PGST. Apart from NEPS of P_3, we find some more NEPS exhibiting PGST.

Circulant graph appears frequently in communication networks. We study PGST in circulant graphs in Chapter 5. In that chapter, we provide a complete characterization of PGST in cycles. Further, we find PGST in some other circulant graphs where there is no PST.

1.4 Graph Product and Transition Matrix

Now we introduce two widely known graph products and then we discuss the transition matrices associated to those products of graph.

1.4.1 Transition Matrix of Cartesian Product

Recall that Cartesian product of two graphs is a NEPS with basis $\{(1,0),(0,1)\}$. Thus the Cartesian product of two graphs G_1 and G_2 with vertex sets V_1 and V_2 is the graph $G_1 \square G_2$, with vertex set $V_1 \times V_2$. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \square G_2$ if and only if either u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$, or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2. If G_1 and G_2 have adjacency matrices A and B, respectively, then the Cartesian product $G_1 \square G_2$ has the adjacency matrix $A \otimes I + I \otimes B$.

The transition matrix of a Cartesian product of two graphs is given by the following lemma. Using this result we can construct many graphs admitting PST and PGST, which we will see in successive chapters.

Lemma 1.4.1. *[14] Let G_1 and G_2 be two graphs having transition matrices $H_{G_1}(t)$ and $H_{G_2}(t)$, respectively. Then the transition matrix of $G_1 \square G_2$ is $H_{G_1 \square G_2}(t) = H_{G_1}(t) \otimes H_{G_2}(t)$.*

Consequently, if a graph have PST then all its Cartesian powers also have PST. We demonstrate this by the following example which appears in [14].

Example 1.4.2. We know that a hypercube can be realised as a Cartesian power of P_2. Since P_2 admits PST at $\frac{\pi}{2}$, all hypercubes also admit PST at $\frac{\pi}{2}$.

Observe that the time for admitting PST in a hypercube does not depend on its size. An explanation, from a physical point of view, of the fact that the transfer time remains constant even though the size of the hypercube is increased, would have been interesting. However, we fail to find such an explanation.

1.4.2 Transition Matrix of Kronecker Product

Recall that Kronecker product of two graphs is a NEPS with basis $\{(1, 1)\}$. Thus the Kronecker product of two graphs G_1 and G_2 with vertex sets V_1 and V_2 is the graph $G_1 \times G_2$, whose vertex set is $V_1 \times V_2$. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ whenever u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2. If G_1 and G_2 have adjacency matrices A and B, respectively, then $G_1 \times G_2$ has the adjacency matrix $A \otimes B$.

The next result enables us to find the transition matrix of Kronecker product of graphs when transition matrix of one of the graphs is known. The following result appears in [24, 26]. We include a simple proof of this result.

Proposition 1.4.3. *[24, 26] Let G_1 and G_2 be two graphs having adjacency matrices A and B, respectively. Also let $H_A(t)$ be the transition matrix of G_1. If spectral decomposition of B is $\sum\limits_{s=1}^{q} \mu_s F_s$ then $G_1 \times G_2$ has the transition matrix $\sum\limits_{s=1}^{q} H_A(\mu_s t) \otimes F_s$.*

Proof. Suppose spectral decomposition of A is $\sum_{r=1}^{p} \lambda_r E_r$. Then the spectral decomposition of $A \otimes B$ is given by $\sum_{r=1}^{p}\sum_{s=1}^{q} \lambda_r \mu_s (E_r \otimes F_s)$. Therefore transition matrix of $G_1 \times G_2$ is

$$\begin{aligned}\sum_{r=1}^{p}\sum_{s=1}^{q} \exp(-it\lambda_r\mu_s)(E_r \otimes F_s) &= \sum_{s=1}^{q}\left(\sum_{r=1}^{p} \exp(-it\lambda_r\mu_s)E_r\right) \otimes F_s \\ &= \sum_{s=1}^{q} H_A(\mu_s t) \otimes F_s.\end{aligned}$$

Hence the result follows. □

Using Proposition 1.4.3, we can find many graphs admitting PST. We include an example.

Example 1.4.4. Suppose the d-th Cartesian powers of P_2 and P_3 are Q_d and R_d, respectively. In [24, 26], we find that if X is a graph with odd integral eigenvalues then $X \times Q_d$ has PST when d is even and PST occurs in $X \times R_d$ for each d.

1.5 Kronecker Approximation Theorem

Kronecker approximation theorem has several applications in mathematics. We will see in Chapter 3 and Chapter 5 that this theorem plays a crucial role in finding PGST. The next result is one dimensional version of Kronecker approximation theorem.

Theorem 1.5.1. *[6] Let θ be an irrational number and suppose α is a real number. For every $\delta > 0$ there are integers p and q such that $|p\theta - q - \alpha| < \delta$.*

Now we introduce Kronecker approximation theorem on simultaneous approximation of numbers.

Theorem 1.5.2. *[6] If $\alpha_1, \ldots, \alpha_l$ are arbitrary real numbers and if $1, \theta_1, \ldots, \theta_l$ are real, algebraic numbers linearly independent over $\mathbb{Q}$ then for $\epsilon > 0$ there exist $q, p_1, \ldots, p_l \in \mathbb{Z}$ such that*

$$\left|q\theta_j - p_j - \alpha_j\right| < \epsilon.$$

A bound on q is given in [37] relative to the precision ϵ and some other given constraints.

Chapter 2

PST on NEPS

We mentioned in Chapter 1 that PST occurs in Cartesian powers of P_2 and P_3 (see [14, 15]). It is worth mentioning that a cubelike graph can be realised as a NEPS of P_2. We see several characterizations of PST on cubelike graphs in [9, 13], which generalize the results in [14, 15] for P_2. So we have a natural question whether there is any NEPS of P_3 admitting PST. This problem was asked by Stevanović in [48]. In this chapter, we find that some restrictions on the basis Ω yields PST on $NEPS_n(P_3, \Omega)$. This generalizes the results in [14, 15] for P_3. Moreover, we find some other graphs admitting PST. The results of this chapter are published in [40].

2.1 Preliminaries

We start with an example that a path on three vertices exhibits PST. Consider the graph P_3, a path of length two with three vertices 1, 2 and 3, where both vertices 1 and 3 are adjacent to the vertex 2. With this settings, the adjacency matrix of P_3 becomes

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

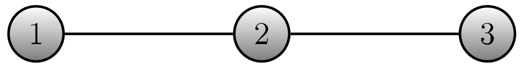

Figure 2.1: The path P_3 with vertices 1, 2 and 3.

The eigenvalues of A are $-\sqrt{2},\ 0,\ \sqrt{2}$ with the corresponding normalised eigenvectors

$$\mathbf{x}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix},\quad \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\quad \mathbf{x}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

The spectral decomposition of A is therefore $A = -\sqrt{2}\cdot E_1 + 0\cdot E_2 + \sqrt{2}\cdot E_3$, where the idempotents are

$$E_1 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},\quad E_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},\quad E_3 = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}.$$

Now the transition matrix of P_3 can be calculated as

$$H(t) = \exp\left(-i(-\sqrt{2})t\right)E_1 + E_2 + \exp\left(-i\sqrt{2}t\right)E_3.$$

At $t = \frac{\pi}{\sqrt{2}}$ we find that

$$H\left(\frac{\pi}{\sqrt{2}}\right) = -E_1 + E_2 - E_3 \ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

This shows that P_3 exhibits PST between the vertices 1 and 3 at time $t = \frac{\pi}{\sqrt{2}}$ and it is periodic at the vertex 2 at the same time.

The set of all automorphisms of a graph G is denoted by $Aut\,(G)$. If G admits PST between the vertices u and v at time τ then $H(\tau)\mathbf{e}_u = \gamma\mathbf{e}_v$, where γ is a complex number of unit modulus. Let $f \in Aut\,(G)$ and Q be the permutation matrix corresponding to f. Then Q commutes with the adjacency matrix A of G and hence Q commutes with $H(\tau)$, as $H(\tau)$ is a polynomial in A. Therefore we have $H(\tau)Q\mathbf{e}_u = QH(\tau)\mathbf{e}_u = \gamma Q\mathbf{e}_v$. Note that $Q\mathbf{e}_u = \mathbf{e}_{f(u)}$ and $Q\mathbf{e}_v = \mathbf{e}_{f(v)}$. Thus we have the following result.

Lemma 2.1.1. *Let f be an automorphism of a graph G and assume that u, v are two vertices of G. If perfect state transfer occurs between u and v then perfect state transfer occurs between $f(u)$ and $f(v)$.*

Let G and H be two graphs and suppose $f : G \to H$ is an isomorphism. Using Lemma 2.1.1, we find that, if there is PST between two vertices u and v in G then there is PST between the vertices $f(u)$ and $f(v)$ in H as well.

For information, the principal idea of Lemma 2.1.1 also appears in [27].

2.2 Definitions and Basic Results

In this section, we provide some definitions and results that will be used to find PST on NEPS of P_3.

Definition 2.2.1. The *center* of a square matrix A of odd order n, where $A = \left(a_{i,j}\right)$ with $1 \leq i, j \leq n$, is defined by $\mathscr{C}(A) := a_{\frac{n+1}{2},\frac{n+1}{2}}$.

Definition 2.2.2. Let $A = \left(a_{i,j}\right)$ be a square matrix of odd order $n \geq 3$. We define $\mathscr{M}_3(A)$ to be the 3×3 principal sub-matrix of A that lies in the rows indexed by $\left\{\frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}\right\}$.

It is easy to see that both $\mathscr{C}$ and $\mathscr{M}_3$ are linear functions on the set of all matrices of odd order n where $n \geq 3$, *i.e.*, if A and B are two matrices of odd order n ($n \geq 3$) and α is a scalar then we have

1. $\mathscr{C}(\alpha A + B) = \alpha\mathscr{C}(A) + \mathscr{C}(B)$; and

2. $\mathscr{M}_3(\alpha A + B) = \alpha\mathscr{M}_3(A) + \mathscr{M}_3(B)$.

In the following result we find the center $\mathscr{C}$ and the principal sub-matrix $\mathscr{M}_3$ of tensor product of some matrices with appropriate orders.

Proposition 2.2.3. *Let $B_1, \ldots, B_n$ be square matrices such that order of each matrix is an odd number greater than or equal to 3. If $A = B_1 \otimes \cdots \otimes B_n$ then*

1. $\mathscr{C}(A) = \prod_{i=1}^{n} \mathscr{C}(B_i)$ *and*

2. $\mathscr{M}_3(A) = \left(\prod_{i=1}^{n-1} \mathscr{C}(B_i)\right) \mathscr{M}_3(B_n)$.

Proof. For $n = 2$, the result follows directly from the definition of tensor product. Suppose that $A' = B_1 \otimes \cdots \otimes B_k$ and consider $C = B_1 \otimes \cdots \otimes B_{k-1}$ so that $A' = C \otimes B_k$. Assume that the result holds for tensor product of $k-1$ matrices. Then we have

$$\mathscr{C}(A') = \mathscr{C}(C)\mathscr{C}(B_k) = \left(\prod_{i=1}^{k-1} \mathscr{C}(B_i)\right) \mathscr{C}(B_k) = \prod_{i=1}^{k} \mathscr{C}(B_i).$$

Also we have

$$\mathscr{M}_3(A') = \mathscr{C}(C)\mathscr{M}_3(B_k) = \left(\prod_{i=1}^{k-1} \mathscr{C}(B_i)\right) \mathscr{M}_3(B_k).$$

Hence the result follows by induction. □

Let W_1, W_2 be two unitary matrices of odd order n with $n \geq 3$. If both $\mathscr{M}_3(W_1)$ and $\mathscr{M}_3(W_2)$ are also unitary then we see that

$$\begin{aligned} W_1W_2 &= \begin{pmatrix} * & \mathbf{0} & * \\ \mathbf{0} & \mathscr{M}_3(W_1) & \mathbf{0} \\ * & \mathbf{0} & * \end{pmatrix} \begin{pmatrix} * & \mathbf{0} & * \\ \mathbf{0} & \mathscr{M}_3(W_2) & \mathbf{0} \\ * & \mathbf{0} & * \end{pmatrix} \\ &= \begin{pmatrix} * & \mathbf{0} & * \\ \mathbf{0} & \mathscr{M}_3(W_1)\mathscr{M}_3(W_2) & \mathbf{0} \\ * & \mathbf{0} & * \end{pmatrix}. \end{aligned}$$

Notice that $\mathscr{M}_3(W_1)\mathscr{M}_3(W_2)$ is also a unitary matrix. Therefore we have the following result.

Proposition 2.2.4. *Let $W_1, \ldots, W_k$ be unitary matrices of odd order n with $n \geq 3$. If, for each $1 \leq j \leq k$, the principal sub-matrix $\mathscr{M}_3(W_j)$ is also unitary then*

$$\mathscr{M}_3\left(\prod_{j=1}^{k} W_j\right) = \prod_{j=1}^{k} \mathscr{M}_3(W_j).$$

Proof. We prove the result by induction on the number of matrices in the product. It is clear that the result holds for $k = 2$. Assume that the result holds for $k = l - 1$ matrices. Now, consider l unitary matrices $W_1, \ldots, W_l$ of odd order n with $n \geq 3$. Also assume that, for each $1 \leq j \leq k$, the principal sub-matrix $\mathscr{M}_3(W_j)$ is unitary. Now we have

$$\begin{aligned}
\mathscr{M}_3\left(\prod_{j=1}^{l} W_j\right) &= \mathscr{M}_3\left(\prod_{j=1}^{l-1} W_j\right) \mathscr{M}_3(W_l) \\
&= \left(\prod_{j=1}^{l-1} \mathscr{M}_3(W_j)\right) \mathscr{M}_3(W_l) \\
&= \prod_{j=1}^{l} \mathscr{M}_3(W_j).
\end{aligned}$$

This proves the result. □

2.3 PST on NEPS of Path on Three Vertices

We begin with the discussion that the transition matrix of a NEPS can be written as a product of transition matrices of some of its spanning subgraphs. Consider the following result.

Proposition 2.3.1. *Let $G_1, \ldots, G_n$ be n graphs and consider $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$. For $\beta \in \Omega$, let $H_\beta(t)$ be the transition matrix of NEPS $(G_1, \ldots, G_n; \{\beta\})$.*

Then $NEPS\,(G_1,\ldots,G_n;\Omega)$ *has the transition matrix*

$$H_\Omega(t)=\prod_{\beta\in\Omega}H_\beta(t).$$

Proof. Suppose $G_1,\ldots,G_n$ have adjacency matrices $A_1,\ldots,A_n$, respectively. For $\beta=(\beta_1,\ldots,\beta_n)\in\Omega$, let us consider

$$A_\beta=A_1^{\beta_1}\otimes\cdots\otimes A_n^{\beta_n}.$$

Here A_β can be considered as adjacency matrix of $NEPS\,(G_1,\ldots,G_n;\{\beta\})$ [see Equation (1.2)]. The adjacency matrix of $NEPS\,(G_1,\ldots,G_n;\Omega)$ is

$$A_\Omega=\sum_{\beta\in\Omega}A_\beta.$$

Note that if $\beta,\delta\in\Omega$, then $A_j^{\beta_j}$ and $A_j^{\delta_j}$ are either A_j or I and so $A_j^{\beta_j}$, $A_j^{\delta_j}$ commute. Hence, due to properties of tensor product of matrices, we find that A_β commutes with A_δ. Thus the transition matrix of $NEPS\,(G_1,\ldots,G_n;\Omega)$ can be obtained as

$$\begin{aligned}H_\Omega(t)=\exp\left(-it\sum_{\beta\in\Omega}A_\beta\right)&=\prod_{\beta\in\Omega}\exp\left(-itA_\beta\right),\text{ as }A_\beta A_\delta=A_\delta A_\beta\\&=\prod_{\beta\in\Omega}H_\beta(t).\end{aligned}$$

Hence the result follows. □

Thus it is evident that if we can find the transition matrices $H_\beta(t)$ of each of the spanning subgraphs $NEPS\,(G_1,\ldots,G_n;\{\beta\})$, then we can find the transition matrix of $NEPS\,(G_1,\ldots,G_n;\Omega)$ quite easily.

From now onwards, we consider NEPS having factor graph P_3 only. The hamming weight $wt(\beta)$ of an n-tuple β in $\mathbb{Z}_2^n$ is the number of non-zero entries of β. Let us denote $\tau_n:=\frac{\pi}{(\sqrt{2})^n}$, $n\in\mathbb{N}$ so that $\sqrt{2}\tau_{n+1}=\tau_n$ for every $n\in\mathbb{N}$.

Consider the matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Recall that the transition matrix of P_3 at $\tau_1 = \frac{\pi}{\sqrt{2}}$ is $-P$. For $\beta \in \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$, let $H_\beta(t)$ be the transition matrix of $NEPS_n(P_3, \{\beta\})$. At a time t depending on β, we now evaluate the principal submatrix $\mathscr{M}_3\left(H_\beta(t)\right)$.

Lemma 2.3.2. *Let $\beta = (\beta_1, \ldots \beta_n) \in \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ and also let the transition matrix of $NEPS_n(P_3, \{\beta\})$ be $H_\beta(t)$. If $wt(\beta) = k$ then $\mathscr{M}_3\left(H_\beta(\tau_k)\right)$ is $-I$ or $-P$ according as β_n is 0 or 1 with $H_\beta(-\tau_k) = H_\beta(\tau_k)$.*

Proof. We prove this by induction on n, the length of β. For $n = 1$ we have $wt(\beta) = 1$ as $\beta \neq \mathbf{0}$. In this case the NEPS is the graph P_3 itself. Therefore $\mathscr{M}_3\left(H_\beta(\tau_1)\right) = \mathscr{M}_3(-P) = -P$ and also $H_\beta(-\tau_1) = H_\beta(\tau_1)$ as $P^{-1} = P$. So the result holds for $n = 1$.

Assume that the result is true for any β of length $n = l$. Consider $\beta = (\beta_1, \ldots, \beta_l, \beta_{l+1})$ and let $\beta^* = (\beta_1, \ldots \beta_l)$. If $wt(\beta^*) = k^*$ then by our assumption $\mathscr{M}_3\left(H_{\beta^*}(\tau_{k^*})\right)$ is $-I$ or $-P$ according as β_l is equal to 0 or 1 with $H_{\beta^*}(-\tau_{k^*}) = H_{\beta^*}(\tau_{k^*})$. Now we consider two cases according as β_{l+1} is 1 or 0.
Case I: Let us assume $\beta_{l+1} = 1$ and therefore $wt(\beta) = k^* + 1$. In that case $NEPS_{l+1}(P_3, \{\beta\})$ is actually the Kronecker product $NEPS_l(P_3, \{\beta^*\}) \times P_3$. Recall that spectral decomposition of the adjacency matrix of P_3 is given by $A = -\sqrt{2} \cdot E_1 + 0 \cdot E_2 + \sqrt{2} \cdot E_3$, where E_1, E_2 and E_3 are the idempotents. Using Proposition 1.4.3, we have

$$\begin{aligned} H_\beta(\tau_{k^*+1}) &= H_{\beta^*}(-\sqrt{2}\tau_{k^*+1}) \otimes E_1 + H_{\beta^*}(0) \otimes E_2 + H_{\beta^*}(\sqrt{2}\tau_{k^*+1}) \otimes E_3 \\ &= H_{\beta^*}(-\tau_{k^*}) \otimes E_1 + I \otimes E_2 + H_{\beta^*}(\tau_{k^*}) \otimes E_3, \text{ as } \sqrt{2}\tau_{k^*+1} = \tau_{k^*} \\ &= H_{\beta^*}(\tau_{k^*}) \otimes (E_1 + E_3) + I \otimes E_2 \\ &= H_{\beta^*}(\tau_{k^*}) \otimes (E_2 + P) + I \otimes E_2, \text{ as } E_1 - E_2 + E_3 = P \\ &= \left(H_{\beta^*}(\tau_{k^*}) + I\right) \otimes E_2 + H_{\beta^*}(\tau_{k^*}) \otimes P. \end{aligned} \tag{2.1}$$

Since $\mathscr{C}\left(H_{\beta^*}(\tau_{k^*})\right) = -1$, we have $\mathscr{M}_3\left(H_\beta(\tau_{k^*+1})\right) = -P$. Also it is evident that $H_\beta(-\tau_{k^*+1}) = H_\beta(\tau_{k^*+1})$ as the negative sign can be absorbed in (2.1).

Case II: Let $\beta_{l+1} = 0$ so that $wt(\beta) = k^*$. Suppose A_{β^*} is the adjacency matrix of $NEPS_l(P_3, \{\beta^*\})$. Adjacency matrix of $NEPS_{l+1}(P_3, \{\beta\})$ is therefore $A_\beta = A_{\beta^*} \otimes I$. This implies that

$$H_\beta(\tau_{k^*}) = \exp\left(-i\tau_{k^*}\left(A_{\beta^*} \otimes I\right)\right) = H_{\beta^*}(\tau_{k^*}) \otimes I.$$

Therefore $\mathscr{M}_3\left(H_\beta(\tau_{k^*})\right) = \mathscr{C}\left(H_{\beta^*}(\tau_{k^*})\right) I = -I$ with $H_\beta(-\tau_{k^*}) = H_\beta(\tau_{k^*})$. Hence we have the desired result. □

In the following theorem we provide a sufficient condition for a NEPS of P_3 to exhibit PST. Later on, we will generalize this theorem and at the end of this section we will provide a sufficient condition for a NEPS to admit PST. For $1 \le j \le n$, let U_j and V_j be the vertices of $NEPS_n(P_3, \Omega)$, where the j-th entry of U_j and V_j are 1 and 3, respectively, and the remaining entries are 2.

Theorem 2.3.3. *Let k be an integer. Assume $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ such that for each $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \Omega$, the hamming weight $wt(\beta) = k$. Then the following holds for $NEPS_n(P_3, \Omega)$ at time τ_k.*

1. *If $\sum\limits_{\beta\in\Omega} \beta_j \neq 0 \pmod 2$ for some j then the graph exhibits perfect state transfer between the pair of vertices U_j and V_j.*
2. *If $\sum\limits_{\beta\in\Omega} \beta_j = 0 \pmod 2$ for some j then the graph is periodic at U_j and V_j.*
3. *The graph is periodic at $(2, \ldots, 2)$.*

Proof. Consider $\Omega_j = \left\{\delta(\beta) = \left(\beta_1, \ldots, \beta_n, \ldots, \beta_j\right) : \beta \in \Omega\right\}$, *i.e*, each $\delta(\beta)$ in Ω_j is defined by interchanging the j-th entry and n-th entry of $\beta \in \Omega$. Let us denote $G := NEPS_n(P_3, \Omega)$ and $G_j := NEPS_n\left(P_3, \Omega_j\right)$. Note that both graphs G and G_j have the common vertex set $V(P_3) \times \ldots \times V(P_3)$. It is easy to check that the map $f : V(G) \to V\left(G_j\right)$ defined by

$$f(v_1, \ldots, v_j, \ldots, v_n) = (v_1, \ldots, v_n, \ldots, v_j)$$

is an isomorphism. Thus PST occurs between U_j and V_j in G if and only if PST occurs between U_n and V_n in G_j. Similarly, the graph G is periodic at U_j and V_j if and only if G_j is periodic at U_n and V_n. Note that if $\sum\limits_{\beta\in\Omega}\beta_j \neq 0 \pmod 2$ in G then $\sum\limits_{\delta(\beta)\in\Omega_j}\delta(\beta)_n \neq 0 \pmod 2$ in G_j. Similarly, if $\sum\limits_{\beta\in\Omega}\beta_j = 0 \pmod 2$ in G then $\sum\limits_{\delta(\beta)\in\Omega_j}\delta(\beta)_n = 0 \pmod 2$ in G_j as well. Therefore it is enough to prove the result for $j = n$.

Let r be the number of β in Ω for which $\beta_n = 1$. By Lemma 2.3.2, $\mathscr{M}_3\left(H_\beta(\tau_k)\right)$ is equal to $-I$ or $-P$ according as β_n is 0 or 1. So $\mathscr{M}_3\left(H_\beta(\tau_k)\right)$ is unitary for all $\beta \in \Omega$. Let $H_\Omega(t)$ be transition matrix of $NEPS_n\left(P_3, \Omega\right)$. By Proposition 2.2.4 and Proposition 2.3.1, we have

$$\begin{aligned}\mathscr{M}_3\left(H_\Omega(\tau_k)\right) = \mathscr{M}_3\left(\prod_{\beta\in\Omega} H_\beta(\tau_k)\right) &= \prod_{\beta\in\Omega}\mathscr{M}_3\left(H_\beta(\tau_k)\right)\\ &= (-1)^m P^r, \text{ where } m = |\Omega|. \quad (2.2)\end{aligned}$$

Note that the first row of $\mathscr{M}_3\left(H_\Omega(\tau)\right)$, $\tau \in \mathbb{R}$, corresponds to the $\frac{3^n-1}{2}$-th row of $H_\Omega(\tau)$. Also there are 3^{n-1} vertices preceding to the vertex $(2, 1, \ldots, 1)$ in dictionary ordering. Thus position of the row in $H_\Omega(\tau)$ corresponding to the vertex $(2, \ldots, 2, 1)$ is $3^{n-1}+3^{n-2}+\ldots+3^1+1 = \frac{3^n-1}{2}$. Similarly, position of the column in $H_\Omega(\tau)$ corresponding to the vertex $(2, \ldots, 2, 3)$ is $\frac{3^n-1}{2}+2 = \frac{3^n+3}{2}$, as $(2, \ldots, 2, 1)$, $(2, \ldots, 2, 2)$ and $(2, \ldots, 2, 3)$ are consecutive vertices in dictionary ordering. So $(1, 3)$-th entry of $\mathscr{M}_3\left(H_\Omega(\tau)\right)$ is actually U_nV_n-th entry of $H_\Omega(\tau)$.

(**1**) If $\sum\limits_{\beta\in\Omega}\beta_n \neq 0 \pmod 2$ then r is odd. As $P^2 = I$, using (2.2), we see that $(1, 3)$-th entry of $\mathscr{M}_3\left(H_\Omega(\tau_k)\right)$ is $(-1)^m$. Hence PST occurs between the vertices U_n and V_n at time τ_k.

(**2**) If $\sum\limits_{\beta\in\Omega}\beta_n = 0 \pmod 2$ then r is even. In that case, again using (2.2), we have $\mathscr{M}_3\left(H_\Omega(\tau_k)\right) = (-1)^m I$. Now $(1, 1)$ and $(3, 3)$-th entries of $\mathscr{M}_3\left(H_\Omega(\tau_k)\right)$ correspond to U_nU_n and V_nV_n-th entries of $H_\Omega(\tau_k)$, respectively. Therefore the graph is periodic at U_n and V_n at time τ_k.

(**3**) In both cases (**1**) and (**2**), the $(2, 2)$-th entry of $\mathscr{M}_3\left(H_\Omega(\tau_k)\right)$ is $(-1)^m$.

Hence the graph is periodic at $(2, \ldots, 2)$ at time τ_k. $\square$

Suppose J is the matrix in which all entries are equal to 1. The following example shows that the NEPS of n copies of P_3 with basis Ω containing all the rows of $J - I$ exhibits PST when n is even. We will find later that this graph is indeed connected.

Example 2.3.4. Let us consider the NEPS of n copies of P_3 with basis Ω containing all the rows of $J - I$. In this case $wt(\beta) = n - 1$ for all $\beta \in \Omega$. Considering the sum over $\mathbb{Z}_2^n$, we have $\sum\limits_{\beta \in \Omega} \beta = (1, \ldots, 1)$ or $(0, \ldots, 0)$ according as n is even or odd. By Theorem 2.3.3, we observe that when n is even, the graph admits PST at $\tau_{n-1} = \frac{\pi}{(\sqrt{2})^{n-1}}$ between U_j and V_j for each $1 \leq j \leq n$. Again if n is odd then the graph is periodic at $\tau_{n-1} = \frac{\pi}{(\sqrt{2})^{n-1}}$ at the vertices U_j, V_j for each $1 \leq j \leq n$. In either case the graph is periodic at $(2, \ldots, 2)$.

The following result which is indeed proved in [14, 15], can also be obtained as a corollary to Theorem 2.3.3.

Corollary 2.3.5. *Cartesian power of n copies of P_3 exhibits perfect state transfer at time $\frac{\pi}{\sqrt{2}}$ between the pair vertices U_j and V_j for $1 \leq j \leq n$.*

Proof. Cartesian power of n copies of P_3 is actually the NEPS with basis Ω containing all the rows of the identity matrix I. See [20] for details. It is easy to verify that for $\beta \in \Omega$, the number $wt(\beta) = 1$ and $\sum\limits_{\beta \in \Omega} \beta = (1, \ldots, 1)$ in $\mathbb{Z}_2^n$. By Theorem 2.3.3, the graph admits PST at time $\tau_1 = \frac{\pi}{\sqrt{2}}$ between the vertices U_j and V_j for each $1 \leq j \leq n$. $\square$

We now extend Theorem 2.3.3 to construct more general NEPS of P_3 exhibiting PST. In the next result we show that, for a certain type of NEPS of P_3, the transition matrix for a spanning subgraph is same as that of the whole graph at a fixed time τ (say). Now if the spanning subgraph admits PST at time τ then so does the whole graph between the same pair of vertices. That is, in some sense, adding extra edges (following certain rules) does not disturb the property of PST in some NEPS of P_3.

Theorem 2.3.6. *Let $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ be such that for all β in Ω, the hamming weight $wt(\beta)$ is of the same parity. Also suppose that $k = \min_{\beta\in\Omega} wt(\beta)$ and let $\Omega^* = \{\beta \in \Omega : wt(\beta) = k\}$. If the transition matrices of $NEPS_n(P_3, \Omega)$ and $NEPS_n(P_3, \Omega^*)$ are $H_\Omega(t)$ and $H_{\Omega^*}(t)$, respectively, then $H_\Omega(\tau_k) = H_{\Omega^*}(\tau_k)$.*

Proof. Assume that $\beta \in \Omega\backslash\Omega^*$ and $wt(\beta) = k'$. By our assumption $k' - k (\neq 0)$ is even and this implies that

$$\tau_k = \frac{\pi}{\left(\sqrt{2}\right)^k} = \left(\sqrt{2}\right)^{k'-k} \cdot \frac{\pi}{\left(\sqrt{2}\right)^{k'}} = 2^{\frac{k'-k}{2}} \tau_{k'} = 2m_{k'}\tau_{k'},$$

where $m_{k'}$ is a positive integer. By Lemma 2.3.2 we have $H_\beta(-\tau_{k'}) = H_\beta(\tau_{k'})$, *i.e.*, $H_\beta(2\tau_{k'}) = I$. Thus for each $\beta \in \Omega \setminus \Omega^*$, we find

$$H_\beta(\tau_k) = H_\beta(2m_{k'}\tau_{k'}) = \left[H_\beta(2\tau_{k'})\right]^{m_{k'}} = I.$$

Therefore, by Proposition 2.3.1, we obtain

$$\begin{aligned} H_\Omega(\tau_k) = \prod_{\beta\in\Omega} H_\beta(\tau_k) &= \prod_{\beta\in\Omega^*} H_\beta(\tau_k) \prod_{\beta\in\Omega\backslash\Omega^*} H_\beta(\tau_k) \\ &= \prod_{\beta\in\Omega^*} H_\beta(\tau_k) \\ &= H_{\Omega^*}(\tau_k). \end{aligned}$$

Hence the theorem follows. □

Note that if $\sum_{\beta\in\Omega^*} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$ then by Theorem 2.3.3, PST occurs in $NEPS_n(P_3, \Omega^*)$. Further, Theorem 2.3.6 implies that $NEPS_n(P_3, \Omega)$ also exhibits PST between the same pair of vertices as in $NEPS_n(P_3, \Omega^*)$.

Corollary 2.3.7. *Let Ω satisfy all the conditions of Theorem 2.3.6, and also let $\sum_{\beta\in\Omega^*} \beta \neq \mathbf{0}$. Then both $NEPS_n(P_3, \Omega^*)$ and $NEPS_n(P_3, \Omega)$ admits perfect state transfer. Moreover, PST occurs in $NEPS_n(P_3, \Omega)$ between the same pair of vertices as in $NEPS_n(P_3, \Omega^*)$ at time τ_k.*

We illustrate Theorem 2.3.6 by the following example.

Example 2.3.8. Consider the graph $NEPS_3(P_3, \Omega)$, where Ω is given by

$$\Omega = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}.$$

Note that number of non-zero entries in each tuple contained in Ω is odd and minimum of those numbers is 1. For $\Omega^* = \{(1,0,0), (0,1,0), (0,0,1)\}$, the graph $NEPS_3(P_3, \Omega^*)$ is actually a Cartesian power of P_3. By Corollary 2.3.5, the graph $NEPS_3(P_3, \Omega^*)$ exhibits PST at $\frac{\pi}{\sqrt{2}}$. Finally, Theorem 2.3.6 implies that $NEPS_3(P_3, \Omega)$ admits PST at $\frac{\pi}{\sqrt{2}}$ between the same pair of vertices as in $NEPS_3(P_3, \Omega^*)$.

We generalize the observation given in Example 2.3.8 as a corollary which is a direct consequence of Theorem 2.3.6.

Corollary 2.3.9. *Let Ω be the set containing all the rows of the identity matrix of order n. Suppose $\Omega' = \{\beta \in \mathbb{Z}_2^n \setminus \{\mathbf{0}\} : wt(\beta) \text{ is odd and } wt(\beta) \neq 1\}$. Then for any $S \subseteq \Omega'$, the graph $NEPS_n(P_3, S \cup \Omega)$ exhibits perfect state transfer between the same pair of vertices as in $NEPS_n(P_3, \Omega)$.*

Note that if a graph is disconnected then its vertices can be arranged in such a way that the adjacency matrix becomes a block diagonal matrix. Consequently, all powers of the adjacency matrix are also block diagonal. Hence the transition matrix is also block diagonal with each block having the same size as that of the adjacency matrix. We thus conclude that PST cannot occur between vertices belonging to distinct connected components. Therefore we focus in finding PST on connected graphs. The NEPS in Theorem 2.3.3 and Theorem 2.3.6 may not always be connected. The following result determines when a NEPS of P_3 is connected.

Consider a NEPS with basis Ω. Let $M(\Omega)$ be the matrix formed by writing the elements in Ω as its rows. We denote the rank of $M(\Omega)$ over $\mathbb{Z}_2$ by $r(\Omega)$.

Theorem 2.3.10. *[49] Let $B_1, \ldots, B_n$ be connected bipartite graphs. Then $NEPS(B_1, \ldots, B_n; \Omega)$ is connected if and only if the rank $r(\Omega) = n$.*

The graph P_3 is connected and also it is a bipartite graph. So if we impose the extra condition $r(\Omega) = n$ in Theorem 2.3.3 and Theorem 2.3.6 then $NEPS_n\left(P_3, \Omega\right)$ is connected and exhibits PST. In the following theorem, we use induction on the number of factor graphs to establish the existence of a basis $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ with $r(\Omega) = n$ that satisfies the conditions of Theorem 2.3.3 and Theorem 2.3.6. This, in turn, gives a recursive construction of a basis Ω such that $NEPS_n\left(P_3, \Omega\right)$ exhibits PST.

Theorem 2.3.11. *For every $n \in \mathbb{N} \setminus \{1\}$ and an odd positive integer $k < n$, there is a basis $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ such that $NEPS_n\left(P_3, \Omega\right)$ is connected and exhibits perfect state transfer at time $\tau_k = \frac{\pi}{(\sqrt{2})^k}$.*

Proof. Let $n \in \mathbb{N} \setminus \{1\}$ and $k < n$ be an odd positive integer. It is enough to show that there is a matrix $M(\Omega)$ of size n over $\mathbb{Z}_2$ such that each of its rows have exactly k non-zero entries with $r(\Omega) = n$. Note that $r(\Omega) = n$ will imply $\sum\limits_{\beta \in \Omega} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$. Then Theorem 2.3.3 will imply that $NEPS_n\left(P_3, \Omega\right)$ has PST at time $\tau_k = \frac{\pi}{(\sqrt{2})^k}$.

We prove this by induction on n. For the initial case $n = 2$, the only possibility for k is 1. In this case

$$M(\Omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

serves our purpose. Assume that for $n = l$ and any odd positive integer $k < l$, there exists such a matrix $M(\Omega)$. Now suppose that $n = l + 1$. We consider the following two cases.

Case I.(**l is even**)**:** Let k be an odd positive integer and $k < l + 1$ so that $k < l$ as well. So, by induction hypothesis, there is a matrix $M(\Omega)$ of size l over $\mathbb{Z}_2$ such that each of its rows have exactly k non-zero entries with $r(\Omega) = l$. Suppose that δ is an l-tuple with $k - 1$ non-zero entries. Now consider the block matrix

$$M(\Omega') = \begin{pmatrix} M(\Omega) & \mathbf{0} \\ \delta & 1 \end{pmatrix}.$$

Clearly each row of $M(\Omega')$ contains exactly k nonzero entries with $r(\Omega') = l+1$.
Case II.(l is odd): Let k be an odd positive integer and $k < l+1$ so that either $k < l$ or $k = l$. If $k < l$ we use the method described in the previous case to find such an $M(\Omega')$. For $k = l$, consider the matrix $M(\Omega') = J - I$ of order $l+1$. As $l+1$ is even, we have

$$M(\Omega')^T M(\Omega') = (J-I)^T(J-I) = (J-I)^2 = (l+1)J - 2J + I = I, \text{ over } \mathbb{Z}_2.$$

This further implies that $r(\Omega') = l+1$ over $\mathbb{Z}_2$. This completes the proof of the theorem. □

In this section we have developed a sufficient condition for a NEPS of P_3 to exhibit PST. We now provide that condition as a theorem.

Theorem 2.3.12 (Sufficient Condition). *Let $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ be such that $wt(\beta)$ is of the same parity, for each $\beta \in \Omega$. Suppose that $k = \min_{\beta\in\Omega} wt(\beta)$ and $\Omega^* = \{\beta \in \Omega : wt(\beta) = k\}$. If $\sum_{\beta\in\Omega^*} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$ then $NEPS_n(P_3, \Omega)$ admits perfect state transfer at time $\frac{\pi}{(\sqrt{2})^k}$.*

Applying Theorem 2.3.12 we can construct several, in fact infinitely many, NEPS of P_3 admitting PST. Notice that, for $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$, if each row of $M(\Omega)$ contains even number of 1's, then sum of the columns of $M(\Omega)$ is $\mathbf{0}$ over $\mathbb{Z}_2$. Accordingly, columns of $M(\Omega)$ are linearly dependent, and so $r(\Omega) < n$. Therefore, even if $NEPS_n(P_3, \Omega)$ satisfies all the conditions of Theorem 2.3.12, where hamming weight of each tuple in Ω is even, then $r(\Omega) \neq n$. Hence, even though PST occurs in $NEPS_n(P_3, \Omega)$ in this case, the graph becomes disconnected. Despite this fact, we will see in Chapter 3 that these graphs are useful to obtain connected NEPS of P_3 exhibiting PGST.

In [9, 13], we find some cubelike graphs $X(C)$ having PST at $\frac{\pi}{2}$ or $\frac{\pi}{4}$. Using those graphs we can produce some more graphs, apart from NEPS of P_3, admitting PST. We use some techniques given in [14] and present a corollary of Theorem 2.3.12.

Corollary 2.3.13. *If $NEPS_n(P_3, \Omega)$ satisfies all the conditions of Theorem 2.3.12 with k even and the cubelike graph $X(C)$ admits PST at $\frac{\pi}{2}$ or $\frac{\pi}{4}$ then the Cartesian product $NEPS_n(P_3, \Omega) \square X(C)$ exhibits PST.*

Proof. Note that if $k = 2$ then by Theorem 2.3.12, the graph $NEPS_n(P_3, \Omega)$ exhibits PST at $\frac{\pi}{2}$. If $X(C)$ admits PST at $\frac{\pi}{2}$ then by Lemma 1.4.1, we find that $NEPS_n(P_3, \Omega) \square X(C)$ exhibits PST at $\frac{\pi}{2}$. On the other hand, if $X(C)$ admits PST at $\frac{\pi}{4}$ then using Lemma 1.2.2, we find that $X(C)$ has a vertex at which it is periodic at $\frac{\pi}{2}$. In that case, again applying Lemma 1.4.1, we observe that $NEPS_n(P_3, \Omega) \square X(C)$ exhibits PST at $\frac{\pi}{2}$.
Similarly, we prove the result for $k \geq 4$. Consider the following two cases.
Case I: Let $X(C)$ admits PST at $\frac{\pi}{2}$. By Lemma 1.2.2 and Theorem 2.3.12, we find that $NEPS_n(P_3, \Omega)$ has a vertex at which it is periodic at $\frac{\pi}{2}$. Then using Lemma 1.4.1, we find that $NEPS_n(P_3, \Omega) \square X(C)$ exhibits PST at $\frac{\pi}{2}$.
Case II: Let $X(C)$ admits PST at $\frac{\pi}{4}$. By Theorem 2.3.12, we find that $NEPS_n(P_3, \Omega)$ has PST at $\frac{\pi}{4}$, when $k = 4$. Otherwise, again using Lemma 1.2.2, there is a vertex at which $NEPS_n(P_3, \Omega)$ is periodic at $\frac{\pi}{4}$, when $k > 4$. In either case, using Lemma 1.4.1, we find that $NEPS_n(P_3, \Omega) \square X(C)$ exhibits PST at $\frac{\pi}{4}$. □

The next result gives some more graphs admitting PST.

Corollary 2.3.14. *Let G be a graph and suppose that there is an $r \in \mathbb{R}$ such that for every eigenvalue λ of G, the number $\frac{\lambda}{r}$ is an odd integer. If $NEPS_n(P_3, \Omega)$ satisfies the conditions of Theorem 2.3.12 then perfect state transfer occurs in $NEPS_n(P_3, \Omega) \times G$ at time $\frac{\tau_k}{r}$.*

Proof. Let $NEPS_n(P_3, \Omega)$ satisfy all the conditions of Theorem 2.3.12. By Theorem 2.3.6 we have $H_\Omega(\tau_k) = H_{\Omega^*}(\tau_k)$, where $H_\Omega(t)$ and $H_{\Omega^*}(t)$ are transition matrices of $NEPS_n(P_3, \Omega)$ and $NEPS_n(P_3, \Omega^*)$, respectively. By Lemma 2.3.2, we have $H_\beta(-\tau_k) = H_\beta(\tau_k)$ for each $\beta \in \Omega^*$. Therefore Proposition 2.3.1 implies that $H_{\Omega^*}(-\tau_k) = H_{\Omega^*}(\tau_k)$, *i.e.*, $(H_{\Omega^*}(\tau_k))^2 = I$. For every integer m, we have

$$H_\Omega((2m+1)\tau_k) = (H_\Omega(\tau_k))^{2m+1} = (H_{\Omega^*}(\tau_k))^{2m+1} = H_{\Omega^*}(\tau_k) = H_\Omega(\tau_k).$$

Let $\sum_{s=1}^{l} \lambda_s F_s$ be the spectral decomposition of the adjacency matrix of G. By Proposition 1.4.3, the transition matrix of $NEPS_n(P_3, \Omega) \times G$ at time $\frac{\tau_k}{r}$ can be obtained as

$$\begin{aligned} H\left(\frac{\tau_k}{r}\right) = \sum_{s=1}^{l} H_\Omega\left(\frac{\lambda_s}{r}\tau_k\right) \otimes F_s &= \sum_{s=1}^{l} H_\Omega(\tau_k) \otimes F_s \\ &= H_\Omega(\tau_k) \otimes \sum_{s=1}^{l} F_s \\ &= H_\Omega(\tau_k) \otimes I. \end{aligned}$$

By Theorem 2.3.12, the graph $NEPS_n(P_3, \Omega)$ admits PST at τ_k. Hence PST occurs on $NEPS_n(P_3, \Omega) \times G$ at $\frac{\tau_k}{r}$. □

Corollary 2.3.14 is in fact an extension of the following proposition in [24]. Let $Spec(G)$ denote the set of eigenvalues of G.

Proposition 2.3.15. *[24] Let G be a graph that admits perfect state transfer at time τ. If $\tau Spec(G) \subseteq \pi\mathbb{Z}$ and H is a circulant graph with odd eigenvalues then the graph $G \times H$ has perfect state transfer.*

We illustrate Corollary 2.3.14 by the following example.

Example 2.3.16. The complete graph K_m has eigenvalues $-1, \ldots, -1, m-1$. If m is even then all eigenvalues of K_m are odd. Let Ω be the set containing all the rows of $J - I$, where both I and J has order n. If n is even then $NEPS_n(P_3, \Omega)$ is connected and admits PST at $\frac{\pi}{(\sqrt{2})^{n-1}}$ (by Theorem 2.3.12). Consequently, by Corollary 2.3.14, the graph $NEPS_n(P_3, J-I) \times K_m$ admits PST at $\frac{\pi}{(\sqrt{2})^{n-1}}$.

2.4 Conclusion

In this chapter we have developed a method to find some NEPS of P_3 admitting PST. We have found that NEPS of P_3 with basis Ω exhibits PST whenever the following holds:

- For each tuple in Ω, the hamming weight is even (or odd);
- $\sum_{\beta \in \Omega^*} \beta \neq \mathbf{0}$ over $\mathbb{Z}_2^n$, where $\Omega^* = \{\beta \in \Omega : wt(\beta) = k\}$ and $k = \min_{\beta \in \Omega} wt(\beta)$.

In particular, we have seen that the Cartesian power of P_3 exhibits PST, which was also shown by Christandl et al. in [14, 15]. But not only the Cartesian powers, there are several other, in fact infinitely many, NEPS of P_3 admitting PST. It is always preferable to find graphs where PST occurs at arbitrarily less time. Given any time τ, we have established that we can actually find infinite number of graphs admitting PST at time less than τ. Also we have found that for every $n \in \mathbb{N} \setminus \{1\}$ and an odd positive integer $k < n$, there is a basis Ω such that $NEPS_n(P_3, \Omega)$ is connected and exhibits PST at time $\tau_k = \frac{\pi}{(\sqrt{2})^k}$. Besides NEPS of P_3, using Cartesian product and Kronecker product of graphs, we have constructed some other graphs exhibiting PST.

Chapter 3

PGST on NEPS

In the previous chapter, we considered NEPS of P_3 with basis Ω, where the hamming weight of each element in Ω have the same parity. We found that an additional condition on the basis Ω guarantees PST in those NEPS. Now we show that if we allow the basis set to contain elements with hamming weights of both parities then there is no PST in those NEPS. However, we find that PGST occurs on a subclass of those NEPS. Further, we study Cartesian product of graphs and find some more graphs admitting PGST. As a special case we obtain a class of NEPS with factor graphs P_2 and P_3 admitting PGST. The results of this chapter are published in [42].

3.1 Preliminaries

Let A be the adjacency matrix of a graph G. Recall that if $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of A and if the projections to the respective eigenspaces are $E_1, \ldots, E_m$ then the transition matrix of G is given by

$$H(t) = \sum_{r=1}^{m} \exp(-it\lambda_r)E_r.$$

Suppose the graph G has n vertices. The eigenvalue support of a vector $\mathbf{x} \in \mathbb{R}^n$ is a set containing those eigenvalues λ_r of A for which $E_r\mathbf{x} \neq \mathbf{0}$. Here we include a simple observation. Suppose $E_r = F_1 + \cdots + F_k$, where $F_iF_j = \delta_{ij}F_i$ and $F_i^T = F_i$ for $1 \leq i, j \leq k$. Here δ_{ij} is the usual Kronecker delta function. Now for any vector $\mathbf{x}$, we obtain

$$(F_i\mathbf{x})^T E_r\mathbf{x} = (F_i\mathbf{x})^T (F_1\mathbf{x} + \cdots + F_k\mathbf{x}) = ||F_i\mathbf{x}||^2, \text{ for } 1 \leq i \leq k.$$

Therefore if $F_i\mathbf{x} \neq \mathbf{0}$ for some i then the eigenvalue λ_r belongs to the eigenvalue support of the vector $\mathbf{x}$. For instance, if $\{\mathbf{w_1}, \ldots, \mathbf{w_k}\}$ is an orthogonal basis of the eigenspace corresponding to λ_r then

$$E_r = \frac{\mathbf{w_1}\mathbf{w_1^T}}{\mathbf{w_1^T}\mathbf{w_1}} + \cdots + \frac{\mathbf{w_k}\mathbf{w_k^T}}{\mathbf{w_k^T}\mathbf{w_k}}.$$

Therefore, if $\left(\mathbf{w_i}\mathbf{w_i^T}\right)\mathbf{x} \neq \mathbf{0}$ for some i $(1 \leq i \leq k)$ then the eigenvalue λ_r belongs to the eigenvalue support of the vector $\mathbf{x}$. The following theorem concerning the eigenvalues of a periodic graph will be used in the next section.

Theorem 3.1.1. *[25] Let a graph G be periodic at a vertex u. If $\lambda_k, \lambda_l, \lambda_r, \lambda_s$ are eigenvalues in the eigenvalue support of $\mathbf{e}_u$ and $\lambda_r \neq \lambda_s$ then*

$$\frac{\lambda_k - \lambda_l}{\lambda_r - \lambda_s} \in \mathbb{Q}. \tag{3.1}$$

3.2 NEPS of Path on Three Vertices having no PST

Assume that $\Omega \subseteq \mathbb{Z}_2^n \backslash \{\mathbf{0}\}$ and Ω contains tuples with hamming weights of both parities. We denote Ω_e and Ω_o to be the subsets of Ω such that the hamming weight of each tuple in Ω_e and Ω_o are even and odd, respectively. Consequently, we have $\Omega = \Omega_e \cup \Omega_o$. Now we find that the graph $NEPS_n(P_3, \Omega)$ does not exhibit perfect state transfer. Recall that $wt(\beta)$ denotes the hamming weight of a tuple β.

Theorem 3.2.1. *Let Ω be a non-empty subset of $\mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ such that both Ω_e and Ω_o are also non-empty. Then the graph $NEPS_n(P_3, \Omega)$ does not exhibit perfect state transfer.*

Proof. Recall that the eigenvalues of the path P_3 are $-\sqrt{2},\ 0,\ \sqrt{2}$ with corresponding normalised eigenvectors

$$\mathbf{x}_1 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{2} & 1 \end{bmatrix}^T,\ \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T,\ \mathbf{x}_3 = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \end{bmatrix}^T.$$

By Theorem 1.1.4, the eigenvectors of $NEPS_n(P_3, \Omega)$ are $\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}$, where $j_1, \ldots, j_n \in \{1,\ 2,\ 3\}$. Since all entries of $\mathbf{x_1}$ and $\mathbf{x_3}$ are non-zero, all entries of $\left(\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}\right)$ are also non-zero for $j_1, \ldots, j_n \in \{1,\ 3\}$. This further implies that all columns of $\left(\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}\right)\left(\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}\right)^T$ are non-zero whenever, $j_1, \ldots, j_n \in \{1,\ 3\}$. Therefore, for any standard unit vector $\mathbf{e}_u$ in $\mathbb{R}^{3^n}$, we have

$$\left[\left(\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}\right)\left(\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}\right)^T\right]\mathbf{e}_u \neq \mathbf{0}.$$

Now, using the observation preceding Theorem 3.1.1, we find that the eigenvalue support of $\mathbf{e}_u$ contains the eigenvalue corresponding to $\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}$. Hence, the eigenvalue support of all standard unit vectors in $\mathbb{R}^{3^n}$ contains all the eigenvalues corresponding to $\mathbf{x}_{j_1} \otimes \cdots \otimes \mathbf{x}_{j_n}$ for $j_1, \ldots, j_n \in \{1,\ 3\}$.

Observe that the hamming weight $wt(\beta)$ is either even or odd according as $\beta \in \Omega_e$ or $\beta \in \Omega_o$. Using Equation (1.3), the eigenvalue corresponding to $\mathbf{x}_1 \otimes \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_1$ can be calculated as

$$\begin{aligned}
\sum_{\beta\in\Omega}\left(-\sqrt{2}\right)^{wt(\beta)} &= \sum_{\beta\in\Omega_e}\left(-\sqrt{2}\right)^{wt(\beta)} + \sum_{\beta\in\Omega_o}\left(-\sqrt{2}\right)^{wt(\beta)} \\
&= \sum_{\beta\in\Omega_e}\left(\sqrt{2}\right)^{wt(\beta)} - \sum_{\beta\in\Omega_o}\left(\sqrt{2}\right)^{wt(\beta)} \\
&= a - b\sqrt{2}, \quad \text{where } a \text{ and } b \text{ are positive integers.}
\end{aligned}$$

Similarly the eigenvalue corresponding to the vector $\mathbf{x}_3 \otimes \mathbf{x}_3 \otimes \cdots \otimes \mathbf{x}_3$ can be obtained in terms of a and b as

$$\begin{aligned}\sum_{\beta\in\Omega}\left(\sqrt{2}\right)^{wt(\beta)} &= \sum_{\beta\in\Omega_e}\left(\sqrt{2}\right)^{wt(\beta)} + \sum_{\beta\in\Omega_o}\left(\sqrt{2}\right)^{wt(\beta)} \\ &= a + b\sqrt{2}.\end{aligned}$$

For $1 \leq j \leq n$, consider $\Omega_e^j = \left\{(\beta_1, \ldots, \beta_n) \in \Omega_e : \beta_j = 1\right\}$. Since Ω_e is a non-empty subset of $\mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ there is at least one j for which Ω_e^j is also non-empty. Without loss of generality, let $j = 1$ and hence the integral part of the eigenvalue corresponding to $\mathbf{x}_1 \otimes \mathbf{x}_3 \otimes \cdots \otimes \mathbf{x}_3$ is

$$\sum_{\beta\in\Omega_e^1}\left(-\sqrt{2}\right)\left(\sqrt{2}\right)^{wt(\beta)-1} + \sum_{\beta\in\Omega_e\setminus\Omega_e^1}\left(\sqrt{2}\right)^{wt(\beta)},$$

which is clearly not equal to the integer a. Let the eigenvalue corresponding to $\mathbf{x}_1 \otimes \mathbf{x}_3 \otimes \cdots \otimes \mathbf{x}_3$ be $c + d\sqrt{2}$, for some integers $c(\neq a)$ and d.

Now if $NEPS_n(P_3, \Omega)$ exhibits PST then by Lemma 1.2.2 the graph is periodic at some vertex, say u. Hence by Theorem 3.1.1, the eigenvalues in the eigenvalue support of $\mathbf{e}_u$ must satisfy the ratio condition (3.1). The eigenvalues $a + b\sqrt{2}$, $a - b\sqrt{2}$ and $c + d\sqrt{2}$ lie in the eigenvalue support of $\mathbf{e}_u$. Note that

$$\begin{aligned}\frac{\left(a+b\sqrt{2}\right)-\left(a-b\sqrt{2}\right)}{\left(a+b\sqrt{2}\right)-\left(c+d\sqrt{2}\right)} &= \frac{2b\sqrt{2}}{(a-c)+(b-d)\sqrt{2}} \\ &= \frac{2b\sqrt{2}\left[(a-c)-(b-d)\sqrt{2}\right]}{(a-c)^2-2(b-d)^2} \notin \mathbb{Q}.\end{aligned}$$

Therefore $NEPS_n(P_3, \Omega)$ is not periodic at any vertex and hence the graph does not exhibit PST. □

Theorem 3.2.1 gives a partial characterization of PST in the class of all NEPS of P_3. In the next section, we investigate PGST on NEPS of P_3 with basis Ω containing tuples with hamming weights of both parities.

3.3 PGST on NEPS of Path on Three Vertices

Recall that $\tau_k = \frac{\pi}{(\sqrt{2})^k}$ for $k \in \mathbb{N}$. In the following result, we find that a NEPS with basis Ω containing tuples of hamming weights of same parity is essentially periodic. Suppose $k = \min_{\beta\in\Omega} wt(\beta)$ and $\Omega^* = \{\beta \in \Omega : wt(\beta) = k\}$. Let the transition matrices of $NEPS_n(P_3, \Omega)$ and $NEPS_n(P_3, \Omega^*)$ be $H_\Omega(t)$ and $H_{\Omega^*}(t)$, respectively.

Theorem 3.3.1. *Let $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ and $k = \min_{\beta\in\Omega} wt(\beta)$. If the hamming weight of each tuple in Ω have the same parity then $NEPS_n(P_3, \Omega)$ is periodic at $2\tau_k$.*

Proof. Using Lemma 2.3.2, we find that $H_\beta(-\tau_k) = H_\beta(\tau_k)$, for $\beta \in \Omega^*$. Further, Proposition 2.3.1 implies that

$$H_{\Omega^*}(-\tau_k) = \prod_{\beta\in\Omega^*} H_\beta(-\tau_k) = \prod_{\beta\in\Omega^*} H_\beta(\tau_k) = H_{\Omega^*}(\tau_k).$$

Therefore we obtain $H_{\Omega^*}(2\tau_k) = I$. Now Theorem 2.3.6 gives

$$H_\Omega(2\tau_k) = (H_\Omega(\tau_k))^2 = (H_{\Omega^*}(\tau_k))^2 = H_{\Omega^*}(2\tau_k) = I.$$

Hence $NEPS_n(P_3, \Omega)$ is periodic at $2\tau_k$. □

Note that if k is even then $\tau_k = \frac{\pi}{(\sqrt{2})^k} = \frac{\pi}{2^{k/2}}$. Also observe that if a graph is periodic at τ then it is periodic at $2^r\tau$ for all non-negative integer r. As an implication, we have the following obvious corollary which will be used to find PGST in some NEPS of P_3.

Corollary 3.3.2. *Let $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ be such that the hamming weight of β for each $\beta \in \Omega$ is even. Then $NEPS_n(P_3, \Omega)$ is periodic at π.*

Suppose $\Omega \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$. Recall that $r(\Omega)$ is the binary rank of the matrix whose rows are the elements of Ω. Also, we have discussed earlier that if $r(\Omega) = n$ then $NEPS_n(P_3, \Omega)$ is connected. In the following theorem, we find some NEPS of P_3 which are connected and exhibit PGST.

Theorem 3.3.3. *Let $\Omega = \Omega_e \cup \Omega_o \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ be such that both Ω_e, Ω_o are non-empty and $r(\Omega) = n$. Assume that $k = \min\limits_{\beta\in\Omega_e} wt(\beta)$ and $l = \min\limits_{\beta\in\Omega_o} wt(\beta)$ with $\Omega_e^* = \{\beta \in \Omega_e : wt(\beta) = k\}$ and $\Omega_o^* = \{\beta \in \Omega_o : wt(\beta) = l\}$. Then pretty good state transfer occurs in the graph $NEPS_n(P_3, \Omega)$ if one of the following conditions holds:*

1. $\sum\limits_{\beta\in\Omega_o^*} \beta \neq \mathbf{0}$ *in* $\mathbb{Z}_2^n$*; or*
2. $\sum\limits_{\beta\in\Omega_e^*} \beta \neq \mathbf{0}$ *in* $\mathbb{Z}_2^n$*.*

Proof. Suppose the transition matrices of $NEPS_n(P_3, \Omega), NEPS_n(P_3, \Omega_e)$ and $NEPS_n(P_3, \Omega_o)$ are $H_\Omega(t)$, $H_{\Omega_e}(t)$ and $H_{\Omega_o}(t)$, respectively. Proposition 2.3.1 implies that $H_\Omega(t) = H_{\Omega_e}(t)H_{\Omega_o}(t)$. As k is even, by Corollary 3.3.2, we find that $NEPS_n(P_3, \Omega_e)$ is periodic at π and hence for all integer q we have $H_{\Omega_e}(q\pi) = I$. Let us denote $\tau = \frac{\pi}{(\sqrt{2})^l}$ and $\eta = \frac{\pi}{(\sqrt{2})^k}$.
Case I: Assume that $\sum\limits_{\beta\in\Omega_o^*} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$. Notice that $NEPS_n(P_3, \Omega_o)$ satisfies all the conditions of Theorem 2.3.12. Hence $NEPS_n(P_3, \Omega_o)$ admits PST at τ, say, between the pair of vertices u and v, *i.e*, we have $|\mathbf{e}_u^T H_{\Omega_o}(\tau)\mathbf{e}_v| = 1$. Let us consider the function

$$f(t) = \left|\mathbf{e}_u^T H_{\Omega_o}(t)\mathbf{e}_v\right|,$$

which is necessarily a continuous function. By Theorem 3.3.1, the function $f(t)$ is also periodic with period 2τ and therefore $f(t)$ is uniformly continuous on $\mathbb{R}$. So for $\epsilon > 0$, there exists $\delta > 0$ such that $|t - t'| < \delta$ implies $|f(t) - f(t')| < \epsilon$. Consider $\alpha = \frac{1}{(\sqrt{2})^l}$ and $\theta = 2\alpha$. Since l is an odd number the chosen number θ is indeed an irrational number. By Theorem 1.5.1, for $\delta > 0$ there exist integers p and q such that $|p\theta - q - \alpha| < \frac{\delta}{\pi}$, *i.e*, $|(2p-1)\alpha - q| < \frac{\delta}{\pi}$. Now $|(2p-1)\tau - q\pi| < \delta$ implies that

$$|f((2p-1)\tau) - f(q\pi)| < \epsilon.$$

Notice that $f((2p-1)\tau) = f(\tau) = 1$ and hence $|f(q\pi) - 1| < \epsilon$. Therefore we obtain $H_\Omega(q\pi) = H_{\Omega_e}(q\pi)H_{\Omega_o}(q\pi) = H_{\Omega_o}(q\pi)$, which implies that

$f(q\pi) = |\mathbf{e}_u^T H_\Omega(q\pi)\mathbf{e}_v|$. Hence for every $\epsilon > 0$ there exists $q\pi \in \mathbb{R}$ such that $\Big||\mathbf{e}_u^T H_\Omega(q\pi)\mathbf{e}_v| - 1\Big| < \epsilon$. So $NEPS_n\,(P_3, \Omega)$ admits PGST between u and v.
Case II: Suppose that $\sum\limits_{\beta\in\Omega_e^*} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$. Notice that $NEPS_n\,(P_3, \Omega_e)$ satisfies all the conditions of Theorem 2.3.12. Therefore $NEPS_n\,(P_3, \Omega_e)$ exhibits PST at time η, say, between the pair of vertices u and v. Now consider

$$g(t) = \left|\mathbf{e}_u^T H_{\Omega_e}(t)\mathbf{e}_v\right|.$$

By the same argument as in Case I, the function $g(t)$ is uniformly continuous. Hence for $\epsilon > 0$, there exists $\delta > 0$ so that $|t - t'| < \delta$ implies $|g(t) - g(t')| < \epsilon$. Notice that for every integer q, we have $g(q\pi + \eta) = g(\eta) = 1$. Consider $\theta = \frac{2}{(\sqrt{2})^l}$, which is an irrational number as l is an odd natural number. Also let $\alpha = \frac{1}{(\sqrt{2})^k}$. Then by Theorem 1.5.1, for $\delta > 0$ there exist integers p and q such that $|p\theta - q - \alpha| < \frac{\delta}{\pi}$. Now $|2p\tau - (q\pi + \eta)| < \delta$ implies that

$$|g\,(2p\tau) - g(q\pi + \eta)| < \epsilon,\ i.e,\ |g(2p\tau) - 1| < \epsilon.$$

From the proof of Theorem 3.3.1, we get $H_{\Omega_o}(2\tau) = I$ and thus $H_{\Omega_o}(2p\tau) = I$. Finally we have $H_\Omega(2p\tau) = H_{\Omega_e}(2p\tau)H_{\Omega_o}(2p\tau) = H_{\Omega_e}(2p\tau)$. So, for each $\epsilon > 0$ there exists $2p\tau \in \mathbb{R}$ such that $\Big||\mathbf{e}_u^T H_\Omega(2p\tau)\mathbf{e}_v| - 1\Big| < \epsilon$. Hence we conclude that PGST occurs in $NEPS_n\,(P_3, \Omega)$ between u and v. □

From Theorem 3.3.3, we find an infinite class of graphs admitting PGST. In the following section we find some other graphs exhibiting PGST.

3.4 PGST on Cartesian Product

Now we investigate PGST on Cartesian product of two graphs. The results appearing in this section help us to find some other graphs admitting PGST. For instance, we find some NEPS with factor graphs P_2 and P_3 having PGST.

Theorem 3.4.1. *Let G_1 and G_2 be two graphs. Suppose G_1 is periodic at a vertex u at time τ and G_2 exhibits perfect state transfer at time η between* *two vertices v and w. If τ and η are linearly independent over the rational numbers then the graph $G_1\square G_2$ admits pretty good state transfer between the pair of vertices (u,v) and (u,w).*

Proof. Since τ and η are independent over $\mathbb{Q}$, the number $\frac{\tau}{2\eta}$ is irrational. By Theorem 1.5.1 (Kronecker approximation theorem), for $\delta>0$, there exist $m,n\in\mathbb{Z}$ so that

$$\left|m\frac{\tau}{2\eta}-n-\frac{1}{2}\right|<\frac{\delta}{2|\eta|}.$$

This in turn implies that $|m\tau-(2n+1)\eta|<\delta$. Let $H_{G_1}(t)$ and $H_{G_2}(t)$ be the transition matrices of G_1 and G_2, respectively. Since G_1 is periodic at the vertex u at time τ, we have $|\mathbf{e}_u^T H_{G_1}(m\tau)\mathbf{e}_u|=1$ for all integer m. Now consider $f(t)=\left|\mathbf{e}_v^T H_{G_2}(t)\mathbf{e}_w\right|$. Since G_2 exhibits PST at time η between v and w, we have $H_{G_2}(\eta)\,\mathbf{e}_w=\gamma\mathbf{e}_v$ for some $\gamma\in\mathbb{C}$ with $|\gamma|=1$. Therefore, for each integer n, we get

$$\begin{aligned} f((2n+1)\eta)=\left|\mathbf{e}_v^T H_{G_2}((2n+1)\eta)\,\mathbf{e}_w\right| &= \left|\mathbf{e}_v^T H_{G_2}(2n\eta)\,H_{G_2}(\eta)\,\mathbf{e}_w\right| \\ &= \left|\mathbf{e}_v^T H_{G_2}(2n\eta)\,\gamma\mathbf{e}_v\right| \\ &= \left|\mathbf{e}_v^T H_{G_2}(2n\eta)\,\mathbf{e}_v\right|. \end{aligned}$$

Using Lemma 1.2.2, we find that G_2 is periodic at the vertex v at time 2η and therefore we obtain $1=\left|\mathbf{e}_v^T H_{G_2}(2n\eta)\,\mathbf{e}_v\right|=f((2n+1)\eta)$. Since $f(t)$ is uniformly continuous, for given any $\epsilon>0$ there exist $m,n\in\mathbb{Z}$ such that $|f(m\tau)-f((2n+1)\eta)|<\epsilon$, *i.e.* $|f(m\tau)-1|<\epsilon$. Using Lemma 1.4.1 and a property of tensor product, we obtain

$$\begin{aligned} \left|(\mathbf{e}_u\otimes\mathbf{e}_v)^T H_{G_1\square G_2}(m\tau)\,(\mathbf{e}_u\otimes\mathbf{e}_w)\right| &= \left|(\mathbf{e}_u\otimes\mathbf{e}_v)^T\left(H_{G_1}(m\tau)\otimes H_{G_2}(m\tau)\right)(\mathbf{e}_u\otimes\mathbf{e}_w)\right| \\ &= \left|\left(\mathbf{e}_u^T H_{G_1}(m\tau)\mathbf{e}_u\right)\otimes\left(\mathbf{e}_v^T H_{G_2}(m\tau)\mathbf{e}_w\right)\right| \\ &= f(m\tau). \end{aligned}$$

Hence $G_1\square G_2$ admits PGST between the pair of vertices (u,v) and (u,w). □

As a corollary, using some of the NEPS of P_3 satisfying the conditions of

Theorem 2.3.12, we find some other graphs exhibiting PGST.

Corollary 3.4.2. *Let G be an integral graph. If $NEPS_n(P_3, \Omega)$ satisfies all the conditions of Theorem 2.3.12 with k odd then the Cartesian product $G \square NEPS_n(P_3, \Omega)$ admits pretty good state transfer.*

Proof. Since G is integral, we deduce from the spectral decomposition of the transition matrix that G is periodic at 2π. Also Theorem 2.3.12 infers that $NEPS_n(P_3, \Omega)$ admits PST at $\frac{\pi}{(\sqrt{2})^k}$. Since k is odd and $\sqrt{2}$ is irrational, the numbers 2π and $\frac{\pi}{(\sqrt{2})^k}$ are independent over $\mathbb{Q}$. By applying Theorem 3.4.1, we obtain the desired result. □

We provide an example supporting Corollary 3.4.2.

Example 3.4.3. Since all complete graphs are integral, Cartesian product of a complete graph and a NEPS of P_3, as in Corollary 3.4.2, exhibits PGST.

Suppose u, v and w are three vertices of a graph G. It is well known that if PST occurs in G from u to v as well as from u to w then we must have $v = w$ (see [26, 32]). However, it is interesting to see that a graph can have PGST from a vertex u to two different vertices v and w. To support this fact, consider the following example.

Example 3.4.4. The path P_2 with vertices, say u, v, exhibits PST between u and v at $\frac{\pi}{2}$. It is also well known that P_2 is periodic at π. On the other hand the path P_3 with vertices, say 1, 2 and 3, admits PST between the pair of vertices 1 and 3 at $\frac{\pi}{\sqrt{2}}$. The path P_3 is also periodic at $\frac{2\pi}{\sqrt{2}}$. Since π and $\frac{\pi}{\sqrt{2}}$ are independent over $\mathbb{Q}$, by Theorem 3.4.1, the Cartesian product $P_2 \square P_3$ admits PGST between $(u, 1)$ and $(u, 3)$. Using similar argument, we find that $P_3 \square P_2$ admits PGST between $(1, u)$ and $(1, v)$. Since $P_3 \square P_2 \cong P_2 \square P_3$, we see that $P_2 \square P_3$ also exhibits PGST between $(u, 1)$ and $(v, 1)$.

Now using Theorem 3.4.1, we find that some of the NEPS with factor graphs P_2 and P_3, which can be realized as a Cartesian product of a NEPS of P_2 and a NEPS of P_3, exhibit PGST. We will use the fact from [26] that for

$\Omega' \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$, the graph $NEPS_n\left(P_2, \Omega'\right)$ is periodic at π and if $\sum\limits_{\beta\in\Omega'} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$ then $NEPS_n\left(P_2, \Omega'\right)$ admits PST at $\frac{\pi}{2}$.

Corollary 3.4.5. *Let $\Omega \subseteq \mathbb{Z}_2^m \setminus \{\mathbf{0}\}$ and $\Omega' \subseteq \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$. Also, let the hamming weight of each tuple in Ω be odd, $k = \min\limits_{\beta\in\Omega} wt(\beta)$ and $\Omega^* = \{\beta \in \Omega : wt(\beta) = k\}$. Then the Cartesian product $NEPS_n\left(P_3, \Omega\right) \square NEPS_n\left(P_2, \Omega'\right)$ exhibits PGST if either $\sum\limits_{\beta\in\Omega^*} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^m$ or $\sum\limits_{\beta\in\Omega'} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$.*

Proof. Assume that $\sum\limits_{\beta\in\Omega^*} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^m$. Then by Theorem 2.3.12, we find that $NEPS_n\left(P_3, \Omega\right)$ admits PST at $\frac{\pi}{\left(\sqrt{2}\right)^k}$. Also $NEPS_n\left(P_2, \Omega'\right)$ is periodic at π. Since k is odd, the numbers π and $\frac{\pi}{\left(\sqrt{2}\right)^k}$ are independent over $\mathbb{Q}$. Therefore, applying Theorem 3.4.1, we see that $NEPS_n\left(P_2, \Omega'\right) \square NEPS_n\left(P_3, \Omega\right)$ exhibits PGST. Hence we have PGST in $NEPS_n\left(P_3, \Omega\right) \square NEPS_n\left(P_2, \Omega'\right)$.

On the other hand, suppose $\sum\limits_{\beta\in\Omega'} \beta \neq \mathbf{0}$ in $\mathbb{Z}_2^n$. Then $NEPS_n\left(P_2, \Omega'\right)$ admits PST at $\frac{\pi}{2}$. Also, by Theorem 3.3.1, the graph $NEPS_n\left(P_3, \Omega\right)$ is periodic at $\frac{2\pi}{\left(\sqrt{2}\right)^k}$. Finally, Theorem 3.4.1 implies that $NEPS_n\left(P_3, \Omega\right) \square NEPS_n\left(P_2, \Omega'\right)$ exhibits PGST. □

3.5 Conclusion

In Chapter 2, we have seen that there are many NEPS of P_3 admitting PST. In this chapter, we have considered the case when the basis of a NEPS of P_3 contains tuples with hamming weights of both parities. We have found that in such a case there is no PST. Despite the fact, we have found that many of them admit PGST. Finally, in Theorem 3.4.1, we have found a general result regarding PGST on Cartesian product of graphs. Using that we have seen that there are NEPS with factor graphs P_2 and P_3 exhibiting PGST. Following this, we can construct many graphs admitting PGST.

Chapter 4

PST on gcd-Graph

The gcd-graphs are well known class of integral Cayley graphs, which are defined over finite abelian groups. In this chapter, we study PST on gcd-graphs. Some authors have already done some work in this direction. A characterization of PST in integral circulant graphs appears in [7, 43]. Also Bernasconi *et al.* [9] showed that PST occurs on certain cubelike graphs. Both integral circulant graphs and cubelike graphs can be realised as gcd-graphs. We however consider more general gcd-graphs. We obtain some characterizations of periodicity and PST in gcd-graphs. These results generalize some of the results appearing in [7, 9]. The results of this chapter appear in [39].

4.1 Preliminaries

We are going to introduce gcd-graphs. Before that, for the sake of convenience, let us recall the definition of a Cayley graph over a finite abelian group $(\Gamma, +)$. The Cayley graph over Γ with a symmetric connection set S, denoted by $Cay(\Gamma, S)$, has the vertex set Γ, where two vertices $a, b \in \Gamma$ are adjacent if and only if $a - b \in S$. If the additive identity $0 \in S$ then $Cay(\Gamma, S)$ has loops at each of its vertices. In that case, we use the convention that **each loop contributes one** to the corresponding diagonal entry of the adjacency matrix. However while discussing PST on gcd-graphs, we consider only simple graphs, *i.e*, we make sure that $0 \notin S$.

The greatest common divisor of two non-negative integers m, n is denoted by $gcd(m, n)$. For every non-negative integer n we use the convention that $gcd(0, n) = gcd(n, 0) = n$. Let us consider two r-tuples of non-negative integers $\mathbf{m} = (m_1, \ldots, m_r)$ and $\mathbf{n} = (n_1, \ldots, n_r)$. For $i = 1, \ldots, r$, suppose $gcd(m_i, n_i) = d_i$ and we write $\mathbf{d} = (d_1, \ldots, d_r)$. We define gcd of $\mathbf{m}, \mathbf{n}$ to be $\mathbf{d}$ and write $gcd(\mathbf{m}, \mathbf{n}) = \mathbf{d}$.

Let $\mathbb{Z}_n$ be the cyclic group of order n. We know that every finite abelian group $(\Gamma, +)$ has a cyclic group decomposition

$$\Gamma = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r}, \text{ where } r \geq 1 \text{ and } m_i \geq 1 \text{ for } i = 1, \ldots, r.$$

For each $i = 1, \ldots, r$, assume that d_i is a divisor of m_i with $1 \leq d_i \leq m_i$. For the divisor tuple $\mathbf{d} = (d_1, \ldots, d_r)$ of $\mathbf{m} = (m_1, \ldots, m_r)$, define

$$S_\Gamma(\mathbf{d}) = \{\mathbf{x} \in \Gamma : gcd(\mathbf{x}, \mathbf{m}) = \mathbf{d}\}.$$

Let $\mathbf{D}$ be a set of divisor tuples of $\mathbf{m}$ and define

$$S_\Gamma(\mathbf{D}) = \bigcup_{\mathbf{d} \in \mathbf{D}} S_\Gamma(\mathbf{d}).$$

Note that the union is actually a disjoint union. The sets $S_\Gamma(\mathbf{D})$ are called gcd-sets of Γ. A Cayley graph over a finite abelian group whose connection set is a gcd-set is called a gcd-graph. For example, the cycle on four vertices is a gcd-graph over $\mathbb{Z}_4$ with $\mathbf{D} = \{1\}$. More information regarding gcd-graphs can be found in [34, 35].

4.2 PST on Cubelike Graph

A cubelike graph $X(\mathbf{C})$ is a Cayley graph over $\mathbb{Z}_2^n$ with a connection set $\mathbf{C}$. PST on cubelike graphs (simple) has already been discussed in [9, 13]. Here we discuss some relevant results from [13], which remain valid for looped cubelike graphs. Recall that, according to our convention, each loop contributes one to the adjacency matrix. Finally, we deduce a simple result that will be used later to characterize PST in gcd-graphs over general abelian groups.

For each $\mathbf{x} \in \mathbb{Z}_2^n$, the map $P_{\mathbf{x}} : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2^n$ defined by $P_{\mathbf{x}}(\mathbf{y}) = \mathbf{x} + \mathbf{y}$ is a permutation of the elements of $\mathbb{Z}_2^n$ and hence it can be realized as a permutation matrix of appropriate order. It is easy to see that $P_{\mathbf{0}} = I$ and $P_{\mathbf{x}}P_{\mathbf{y}} = P_{\mathbf{x}+\mathbf{y}}$, which imply $P_{\mathbf{x}}^2 = I$. The following result gives the adjacency matrix of a cubelike graph.

Lemma 4.2.1. *[13] If $\mathbf{C} \subseteq \mathbb{Z}_2^n$ and $X(\mathbf{C})$ is the cubelike graph with connection set $\mathbf{C}$ then $X(\mathbf{C})$ has the adjacency matrix $\sum\limits_{\mathbf{x}\in\mathbf{C}} P_{\mathbf{x}}$.*

Note that $P_{\mathbf{x}}P_{\mathbf{y}} = P_{\mathbf{x}+\mathbf{y}} = P_{\mathbf{y}}P_{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_n$. Therefore, using a property of matrix exponential function, we find that

$$\exp\big(-it(P_{\mathbf{x}} + P_{\mathbf{y}})\big) = \exp(-itP_{\mathbf{x}})\exp(-itP_{\mathbf{y}}).$$

Using this the transition matrix of a cubelike graph can be obtained as follows.

Lemma 4.2.2. *[13] Let $X(\mathbf{C})$ be a cubelike graph. If $H(t)$ is the transition matrix of $X(\mathbf{C})$ then*

$$H(t) = \prod_{\mathbf{x}\in\mathbf{C}} \exp(-itP_{\mathbf{x}}).$$

We already have $P_{\mathbf{x}}^2 = I$ and this implies that

$$\begin{aligned} \exp(-itP_{\mathbf{x}}) &= I - itP_{\mathbf{x}} - \frac{t^2}{2!}I + i\frac{t^3}{3!}P_{\mathbf{x}} + \frac{t^4}{4!}I + \dots \\ &= \cos(t)I - i\sin(t)P_{\mathbf{x}}. \end{aligned}$$

Observe that, if σ is the sum of the elements in $\mathbf{C} \subseteq \mathbb{Z}_2^n$ then $\prod\limits_{\mathbf{x}\in\mathbf{C}} P_{\mathbf{x}} = P_{\sigma}$. Therefore using Lemma 4.2.2, we deduce that

$$H\left(\frac{\pi}{2}\right) = \prod_{\mathbf{x}\in\mathbf{C}} (-i)P_{\mathbf{x}} = (-i)^{|\mathbf{C}|}P_{\sigma}.$$

The next result determines periodicity and PST in cubelike graphs at $t = \frac{\pi}{2}$.

Theorem 4.2.3. *[13] Let $\mathbf{C} \subseteq \mathbb{Z}_2^n$ and let σ be the sum of the elements of $\mathbf{C}$. If $\sigma \neq \mathbf{0}$ then PST occurs in $X(\mathbf{C})$ from $\mathbf{x}$ to $\mathbf{x}+\sigma$ at $\frac{\pi}{2}$. If $\sigma = \mathbf{0}$ then $X(\mathbf{C})$ is periodic with period $\frac{\pi}{2}$.*

Now we find a sufficient condition for periodicity in a class of graphs constructed from cubelike graphs.

Proposition 4.2.4. *Let $\mathbf{C} \subseteq \mathbb{Z}_2^n$ and let $X(\mathbf{C})$ be a cubelike graph with the connection set $\mathbf{C}$. Also assume that the sum of the elements of $\mathbf{C}$ is $\mathbf{0}$ and $|\mathbf{C}| \equiv 0 \pmod 4$. Then for every integral graph G, the transition matrix of $X(\mathbf{C}) \times G$ at $\frac{\pi}{2}$ is the identity matrix.*

Proof. If $H(t)$ is the transition matrix of $X(\mathbf{C})$ and if σ is the sum of the elements of $\mathbf{C}$, then recall that $H\left(\frac{\pi}{2}\right) = (-i)^{|\mathbf{C}|} P_\sigma$. As $\sigma = \mathbf{0}$, we have $P_\sigma = I$ and hence $|\mathbf{C}| \equiv 0 \pmod 4$ implies that $H\left(\frac{\pi}{2}\right) = I$. Therefore, for every integer μ, we obtain

$$H\left(\frac{\pi\mu}{2}\right) = \left(H\left(\frac{\pi}{2}\right)\right)^{\mu} = I.$$

Now consider $\sum\limits_{s=1}^{q} \mu_s F_s$ to be the spectral decomposition of adjacency matrix of G. As the graph G is assumed to be integral, the eigenvalues μ_s of G are integers. By Proposition 1.4.3, the transition matrix of the graph $X(\mathbf{C}) \times G$ at $\frac{\pi}{2}$ is obtained as

$$\sum_{s=1}^{q} H\left(\frac{\pi\mu_s}{2}\right) \otimes F_s = I \otimes \sum_{s=1}^{q} F_s = I \otimes I = I,$$

where the identity matrices have appropriate orders. □

We now provide an example supporting the result.

Example 4.2.5. Let $\mathbf{C} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\} \subseteq \mathbb{Z}_2^3$ and consider the cubelike graph $X(\mathbf{C})$ over $\mathbb{Z}_2^3$. Notice here that sum of the elements in $\mathbf{C}$ is $\mathbf{0}$. Also it is clear that $|\mathbf{C}| \equiv 0 \pmod 4$. Since a complete graphs K_n

on n vertices is integral, by Proposition 4.2.4, we conclude that the transition matrix of $X(\mathbf{C}) \times K_n$ at $\frac{\pi}{2}$ is the identity matrix. This further implies that the graph is periodic at $\frac{\pi}{2}$.

4.3 Periodicity and PST on gcd-Graphs

Now we investigate gcd-graphs for periodicity and PST. First we construct some periodic gcd-graphs which are not necessarily connected. Also we find some gcd-graphs exhibiting PST and then we club them to obtain connected gcd-graph having PST. Consider the following characterization of PST in vertex-transitive graphs as given in [26].

Theorem 4.3.1. *[26] Suppose G is a connected vertex-transitive graph having perfect state transfer at τ between two vertices u and v. Then the transition matrix of G is a scalar multiple of a permutation matrix of order two, having no fixed points, and it lies in the center of the automorphism group of G.*

Consequently, if a vertex-transitive graph admits PST then it must have an even number of vertices. We therefore have the following obvious characterization for PST on Cayley graphs and, in particular, on gcd-graphs.

Corollary 4.3.2. *A Cayley graph over a group of odd order does not exhibit perfect state transfer.*

Now we discuss periodicity of a vertex transitive graph G. Let $H(t)$ be the transition matrix of G which is periodic at a vertex u. Then there exist $\tau \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ so that $H(\tau)\mathbf{e}_u = \gamma\mathbf{e}_u$. Since G is vertex transitive, for a vertex v of G, there exists an automorphism f so that $v = f(u)$. If Q is the permutation matrix corresponding to f then $\mathbf{e}_v = Q\mathbf{e}_u$. Recall that $H(t)$ is a polynomial of the adjacency matrix and hence Q commutes with $H(t)$. Now observe that

$$H(\tau)\mathbf{e}_v = H(\tau)\left(Q\mathbf{e}_u\right) = Q\left(H(\tau)\mathbf{e}_u\right) = Q\left(\gamma\mathbf{e}_u\right) = \gamma\mathbf{e}_v.$$

Hence G is periodic at all vertices and also notice that $H(\tau) = \gamma I$, where I is the identity matrix of appropriate order.

The following result allows us to find the transition matrix of a union of two edge disjoint Cayley graphs.

Proposition 4.3.3. *Let Γ be a finite abelian group and consider two disjoint and symmetric subsets $S, T \subset \Gamma$. Suppose the transition matrices of $Cay(\Gamma, S)$ and $Cay(\Gamma, T)$ are $H_S(t)$ and $H_T(t)$, respectively. Then $Cay(\Gamma, S \cup T)$ has the transition matrix $H_S(t)H_T(t)$.*

Proof. The group Γ is a finite abelian group. By Proposition 1.1.2, the adjacency matrices of the graphs $Cay(\Gamma, S)$ and $Cay(\Gamma, T)$ commute. For any two commuting square matrices A and B, we have $\exp(A + B) = \exp(A)\exp(B)$. Using this we get the desired result. □

In [45], the authors showed that a circulant graph, which is actually a Cayley graph over $\mathbb{Z}_n$, is periodic if and only if the graph is integral. Further the result has been generalised. In [26], we find that a regular graph is periodic if and only if its eigenvalues are integers. As a consequence, if PST occurs on a Cayley graph then it must be integral. However, we provide an alternative proof of this result for Cayley graph over a finite abelian group. We use some of the techniques from [45] to establish the result.

Theorem 4.3.4. *If a Cayley graph (connected) over a finite abelian group is periodic then the graph has integral spectrum.*

Proof. We have already concluded in Corollary 4.3.2 that a Cayley graph over a group of odd order does not exhibit PST. Therefore, consider a finite abelian group $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ of even order and let S be a symmetric subset of Γ. Suppose the spectral decomposition of adjacency matrix of $G = Cay(\Gamma, S)$ is $\sum_s \mu_s E_s$. So the transition matrix of G is given by

$$H(t) = \sum_s \exp(-it\mu_s)E_s.$$

Suppose G is periodic at τ, and therefore $H(\tau) = \gamma I$ for some $\gamma \in \mathbb{C}$. Using the property $\sum\limits_{s} E_s = I$ of idempotents, we find that

$$\gamma \sum_{s} E_s = H(\tau) = \sum_{s} \exp(-i\tau\mu_s) E_s.$$

Multiplying both sides of the above equation by E_s we get $\exp(-i\tau\mu_s) = \gamma$, for all s. If μ_r and μ_s are two eigenvalues of G then note that $\tau(\mu_r - \mu_s) = 2l\pi$ for some integer l. Therefore, for eigenvalues μ_i, μ_j, μ_r and μ_s of G with $\mu_r \neq \mu_s$, we have

$$\frac{\mu_i - \mu_j}{\mu_r - \mu_s} \in \mathbb{Q}. \tag{4.1}$$

The eigenvalues of G are given by (1.1). Note that $\lambda_{0,0,\ldots,0} = |S| = k$ (say) and all the other eigenvalues are different from k.

Suppose that G has exactly two distinct eigenvalues. Since the adjacency matrix is diagonalizable, its minimal polynomial has only two linear factors. Since the minimal polynomial has integer coefficients and k is an eigenvalue of G, the other eigenvalue of G must also be an integer.

Therefore, assume that G has at least three distinct eigenvalues, say, μ_i, μ_r and k. Since the eigenvalues of G satisfy (4.1), we have

$$\frac{\mu_i - k}{\mu_r - k} \in \mathbb{Q}.$$

This implies that $\mu_i = a\mu_r + b$ for some $a, b \in \mathbb{Q}$. Hence, if one of the eigenvalues of G, other than k, is rational then all other eigenvalues will also be rational. We know that eigenvalues of a graph are algebraic integers and the rational numbers which are algebraic integers are integers. Hence, if the eigenvalues of G are in $\mathbb{Q}$, then they are integers. Now we prove that a periodic Cayley graph over an abelian group has a rational eigenvalue other than k. Consider the following two possible cases.

Case I: Suppose $|\Gamma| = n_1 \ldots n_k$ has two distinct prime factors p and q. Without loss of generality, let p and q divide n_1 and n_2, respectively. Following

(1.1), let us denote $\mu_p = \lambda_{\frac{n_1}{p},0,0,\ldots,0}$ and $\mu_q = \lambda_{0,\frac{n_2}{q},0,\ldots,0}$. Suppose $\mathbb{Q}\left(\mu_p\right)$ is the smallest field containing $\mathbb{Q}$ and μ_p. Similarly, we assume that $\mathbb{Q}\left(\mu_q\right)$ is the smallest field containing $\mathbb{Q}$ and μ_q. Since $\mu_p = a\mu_q + b$ for some $a, b \in \mathbb{Q}$, we conclude that $\mathbb{Q}\left(\mu_p\right) = \mathbb{Q}\left(\mu_q\right)$. Observe that

$$\mu_p = \lambda_{\frac{n_1}{p},0,0,\ldots,0} = \sum_{(s_1,\ldots,s_k)\in S} \omega_{n_1}^{s_1\left(\frac{n_1}{p}\right)} = \sum_{(s_1,\ldots,s_k)\in S} \omega_p^{s_1} \in \mathbb{Q}\left(\omega_p\right).$$

Similarly, we find that $\mu_q \in \mathbb{Q}\left(\omega_q\right)$. Therefore $\mu_p \in \mathbb{Q}\left(\omega_p\right) \cap \mathbb{Q}\left(\omega_q\right) = \mathbb{Q}$. Thus μ_p is a rational eigenvalue of G other than k.

Case II: Suppose $|\Gamma| = 2^r$, $r \geq 1$.

It is clear that $\lambda_{\frac{n_1}{2},0,\ldots,0} \in \mathbb{Q}(\omega_2)$. Since $\omega_2 = -1$, we have $\lambda_{\frac{n_1}{2},0,\ldots,0} \in \mathbb{Q}$. We thus find a rational eigenvalue of G other than k.

This completes the proof. $\square$

A characterization of integral Cayley graph over finite abelian group is given in [35]. Among these integral graphs, we study gcd-graphs for PST.

Suppose the prime factorization of an integer $n(\geq 2)$ is $n = p_1 \ldots p_k$, where the primes are not necessarily distinct. In [34], the authors showed that every gcd-graph with n vertices is isomorphic to a gcd-graph over $\Gamma = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}$. If n is a power of 2 then the gcd-graph is actually a cubelike graph. The next theorem is thus a special case to that result. Still we include the result with proof, whenever n is a power of 2, so as to learn a definite structure of the associated connection set of the cubelike graph, which will be used to characterize PST on gcd-graphs. We make use of techniques from [34] and consider looped graphs.

Assume that $X(\mathbf{C}_1)$ and $X(\mathbf{C}_2)$ are two cubelike graphs. It is easy to see that there is a natural isomorphism between $X(\mathbf{C}_1) \times X(\mathbf{C}_2)$ and $X(\mathbf{C}_1 \times \mathbf{C}_2)$.

Theorem 4.3.5. *[34] A gcd-graph over an abelian group of order 2^n is isomorphic to a cubelike graph.*

Proof. Consider an abelian group $\Gamma = \mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_r}}$. For each set of divisor tuples $\mathbf{D}$, we show that $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is isomorphic to a cubelike graph. Let $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbf{D}$. For $i = 1, \ldots, r$ we have $d_i = 2^{k_i}$ for some $k_i \leq n_i$. If $\mathbf{x} = (x_1, \ldots, x_r), \mathbf{y} = (y_1, \ldots, y_r) \in \Gamma$ and $\mathbf{n} = (2^{n_1}, \ldots, 2^{n_r})$ then $gcd(\mathbf{x} - \mathbf{y}, \mathbf{n}) = \mathbf{d}$ if and only if $gcd(x_i - y_i, 2^{n_i}) = d_i$ for each i. That is $\mathbf{x} \sim \mathbf{y}$ in $Cay(\Gamma, S_\Gamma(\mathbf{d}))$ if and only if $x_i \sim y_i$ in $Cay(\mathbb{Z}_{2^{n_i}}, S_{\mathbb{Z}_{2^{n_i}}}(d_i))$ for each i. Note that, if $d_i = 2^{n_i}$ then $Cay(\mathbb{Z}_{2^{n_i}}, S_{\mathbb{Z}_{2^{n_i}}}(d_i))$ is the graph with loops at each of its vertices and no other edges. We therefore have the following:

$$Cay(\Gamma, S_\Gamma(\mathbf{d})) \cong Cay(\mathbb{Z}_{2^{n_1}}, S_{\mathbb{Z}_{2^{n_1}}}(d_1)) \times \cdots \times Cay(\mathbb{Z}_{2^{n_r}}, S_{\mathbb{Z}_{2^{n_r}}}(d_r)).$$

Now for a fixed i, if $z \in \mathbb{Z}_{2^{n_i}}$ then there exists a unique 2-adic representation

$$z = \sum_{j=0}^{n_i - 1} z_j 2^j, \text{ where } z_j \in \{0, 1\} \text{ for } j = 0, 1, \ldots, n_i - 1. \tag{4.2}$$

We write $\tilde{\mathbf{z}} = \left(z_0, \ldots, z_{n_i - 1}\right) \in \mathbb{Z}_2^{n_i}$. We show that the map $z \mapsto \tilde{\mathbf{z}}$ gives an isomorphism from $Cay(\mathbb{Z}_{2^{n_i}}, S_{\mathbb{Z}_{2^{n_i}}}(d_i))$ to the cubelike graph $X(\mathbf{C}_{d_i})$ over $\mathbb{Z}_2^{n_i}$ where the connection set $\mathbf{C}_{d_i}$ with $d_i = 2^{k_i}$ is given by

$$\mathbf{C}_{d_i} = \left\{ \left(c_0, c_1, \ldots, c_{n_i - 1}\right) \in \mathbb{Z}_2^{n_i} : c_j = 0 \text{ for every } j < k_i \text{ and } c_{k_i} = 1 \right\}.$$

Note that for $j > k_i$, the value of c_j can be either 0 or 1. It is enough to show that $u \sim v$ in $Cay(\mathbb{Z}_{2^{n_i}}, S_{\mathbb{Z}_{2^{n_i}}}(d_i))$ if and only if $\tilde{\mathbf{u}} - \tilde{\mathbf{v}} \in \mathbf{C}_{d_i}$. Now $u \sim v$ in $Cay(\mathbb{Z}_{2^{n_i}}, S_{\mathbb{Z}_{2^{n_i}}}(d_i))$ if and only if $gcd(u - v, 2^{n_i}) = 2^{k_i}$. Observe from (4.2) that $gcd(u - v, 2^{n_i}) = 2^{k_i}$ if and only if $u_j - v_j = 0$ for every $j < k_i$ and $u_{k_i} - v_{k_i} = 1$. Thus we have

$$Cay(\mathbb{Z}_{2^{n_i}}, S_{\mathbb{Z}_{2^{n_i}}}(d_i)) \cong X(\mathbf{C}_{d_i}), \text{ for each } i = 1, \ldots, r.$$

Therefore for $\mathbf{d} \in \mathbf{D}$ we find that

$$Cay(\Gamma, S_\Gamma(\mathbf{d})) \cong X(\mathbf{C}_{d_1}) \times \cdots \times X(\mathbf{C}_{d_r}) \cong X(\mathbf{C}_{d_1} \times \cdots \times \mathbf{C}_{d_r}).$$

The isomorphism that is exhibited between the vertices of $Cay(\Gamma, S_\Gamma(\mathbf{d}))$ and $X(\mathbf{C}_{d_1} \times \cdots \times \mathbf{C}_{d_r})$ works for all divisors $\mathbf{d} \in \mathbf{D}$. Hence we have

$$Cay(\Gamma, S_\Gamma(\mathbf{D})) \cong X(\mathbf{C}), \text{ where } \mathbf{C} = \bigcup_{\mathbf{d}\in\mathbf{D}} \left(\mathbf{C}_{d_1} \times \cdots \times \mathbf{C}_{d_r}\right).$$

This completes the proof. □

Assume that Γ is a finite abelian group. We write $\Gamma = \Gamma_1 \oplus \Gamma_2$, where Γ_1 is an abelian group of order 2^n and Γ_2 is an abelian group of odd order. Also, consider the cyclic group decomposition of Γ_1 and Γ_2 as follows:

$$\Gamma_1 = \mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_r}} \text{ and } \Gamma_2 = \mathbb{Z}_{p_1^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{m_s}},$$

where $p_1, \ldots, p_s$ are odd prime numbers, not necessarily distinct. We write $\mathbf{m}_\Gamma := (2^{n_1}, \ldots, 2^{n_r}, p_1^{m_1}, \ldots, p_s^{m_s})$ associated to the group Γ. In what follows, we will always consider a finite abelian group Γ in the form $\Gamma_1 \oplus \Gamma_2$, as described above. The next lemma is on the structure of certain gcd-graphs.

Lemma 4.3.6. *Let $\Gamma_1 = \mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_r}}$ and $\Gamma_2 = \mathbb{Z}_{p_1^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{m_s}}$ be such that $\Gamma = \Gamma_1 \oplus \Gamma_2$. Assume that $d_{r+1}, \ldots, d_{r+s}$ are fixed divisors of $p_1^{m_1}, \ldots, p_s^{m_s}$, respectively. Consider a set of divisor tuples $\mathbf{D}$ such that $\mathbf{m}_\Gamma \notin \mathbf{D}$. If the last s components of each $\mathbf{d} \in \mathbf{D}$ are $d_{r+1}, \ldots, d_{r+s}$, respectively, then there exists a cubelike graph $X(\mathbf{C})$ such that*

$$Cay(\Gamma, S_\Gamma(\mathbf{D})) \cong X(\mathbf{C}) \times Cay(\Gamma_2, S_{\Gamma_2}(\{(d_{r+1}, \ldots, d_{r+s})\})).$$

Proof. Let $\mathbf{D}^* = \{(d_1, \ldots, d_r) : \mathbf{d} = (d_1, \ldots, d_r, \ldots, d_{r+s}) \in \mathbf{D}\}$. Then we find

$$Cay(\Gamma, S_\Gamma(\mathbf{D})) \cong Cay(\Gamma_1, S_{\Gamma_1}(\mathbf{D}^*)) \times Cay(\Gamma_2, S_{\Gamma_2}(\{(d_{r+1}, \ldots, d_{r+s})\})).$$

By Theorem 4.3.5, we have $Cay(\Gamma_1, S_{\Gamma_1}(\mathbf{D}^*)) \cong X(\mathbf{C})$ for some cubelike graph $X(\mathbf{C})$. Hence we have the desired result. □

Now we find a sufficient condition for a gcd-graph to be periodic at $\frac{\pi}{2}$. Let $\mathscr{D}'$ be the collection of all divisors sets $\mathbf{D}$ satisfying all the conditions of Lemma 4.3.6 as well as the following two conditions:

1. the sum of the elements in $\mathbf{C}$ is $\mathbf{0}$ in $\mathbb{Z}_2^{n_1+\cdots+n_r}$; and
2. $|\mathbf{C}| \equiv 0 \pmod 4$ whenever $d_{r+i} < p_i^{m_i}$ for some $i = 1, \ldots, s$.

Now suppose $\mathscr{D}$ is the collection of all disjoint union of members of $\mathscr{D}'$. The following result determines a class of periodic gcd-graphs.

Theorem 4.3.7. *If* Γ *is a finite abelian group and* $\mathbf{D} \in \mathscr{D}$ *then* $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ *is periodic at* $\frac{\pi}{2}$.

Proof. Let $\mathbf{D} = \mathbf{D}_1 \cup \cdots \cup \mathbf{D}_k \in \mathscr{D}$, where $\mathbf{D}_l \in \mathscr{D}'$ for all $l = 1, \ldots, k$. Suppose d_{r+j} is a fixed divisor of $p_j^{m_j}$ for $j = 1, \ldots, s$. For a fixed l, assume that the last s components of each $\mathbf{d} \in \mathbf{D}_l$ are $d_{r+1}, \ldots, d_{r+s}$, respectively. Using Lemma 4.3.6, we find that the graph $Cay(\Gamma, S_\Gamma(\mathbf{D}_l))$ is isomorphic to $X(\mathbf{C}) \times Cay(\Gamma_2, S_{\Gamma_2}(\{(d_{r+1}, \ldots, d_{r+s})\}))$ for some cubelike graph $X(\mathbf{C})$. Note that $Cay(\Gamma_2, S_{\Gamma_2}(\{(d_{r+1}, \ldots, d_{r+s})\}))$ is a gcd-graph and therefore it is an integral graph. Now consider the following two cases.

Case I: ($d_{r+j} < p_j^{m_j}$ for some j) By our assumption on $\mathscr{D}$, the connection set $\mathbf{C}$ is such that $|\mathbf{C}| \equiv 0 \pmod 4$ and the sum of the elements in $\mathbf{C}$ is zero. Therefore by Proposition 4.2.4, the transition matrix of $Cay(\Gamma, S_\Gamma(\mathbf{D}_l))$ is the identity matrix at $\frac{\pi}{2}$.

Case II: ($d_{r+j} = p_j^{m_j}$ for all j) In this case the sum of the elements of $\mathbf{C}$ is zero. Note that $Cay(\Gamma_2, S_{\Gamma_2}(\{(d_{r+1}, \ldots, d_{r+s})\}))$ has loops at each of its vertices and no more edges. Recall that, according to our convention, each loop contributes 1 to the adjacency matrix. If A is the adjacency matrix of $X(\mathbf{C})$ then by Lemma 4.3.6, we find that the adjacency matrix of $Cay(\Gamma, S_\Gamma(\mathbf{D}_l))$ is $A \otimes I$. Therefore the transition matrix of $Cay(\Gamma, S_\Gamma(\mathbf{D}_l))$ can be calculated as

$$\exp\left(-it(A \otimes I)\right) = \exp\left(-itA\right) \otimes I.$$

Note that transition matrix of $X(\mathbf{C})$ is $\exp(-itA)$. By Theorem 4.2.3, the graph $X(\mathbf{C})$ is periodic at $\frac{\pi}{2}$ and hence $Cay(\Gamma, S_\Gamma(\mathbf{D}_l))$ is also periodic at $\frac{\pi}{2}$.

In both cases the graph $Cay(\Gamma, S_\Gamma(\mathbf{D}_l))$ is periodic at $\frac{\pi}{2}$. Finally, for $\mathbf{D} \in \mathscr{D}$, we apply Proposition 4.3.3 to have the desired result. □

We illustrate Theorem 4.3.7 by the following example. Here we find a periodic graph over $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

Example 4.3.8. Consider $\Gamma = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $\mathbf{D} = \{(1,1,1),(1,2,1)\}$, a set of divisor tuples of $(4,2,3)$. We show that the graph $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is periodic at $\frac{\pi}{2}$. Suppose $\Gamma' = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and consider $\mathbf{D}^* = \{(1,1),(1,2)\}$. Now we follow the proof of Theorem 4.3.5 to find the cubelike graph isomorphic to $Cay(\Gamma', S_{\Gamma'}(\mathbf{D}^*))$. Notice that $\mathbf{d} = (d_1, d_2) = (1,1)$ is a divisor of $\left(2^2, 2\right)$, where $d_1 = d_2 = 2^0$. So we have

$$\mathbf{C}_{d_1} = \left\{(c_0, c_1) \in \mathbb{Z}_2^2 : c_j = 0 \text{ for every } j < 0 \text{ and } c_0 = 1\right\}$$

and thus $\mathbf{C}_{d_1} = \{(1,0),(1,1)\}$. Similarly, we find that $\mathbf{C}_{d_2} = \{(1)\}$. Therefore $\mathbf{C}_{d_1} \times \mathbf{C}_{d_2} = \{(1,0,1),(1,1,1)\}$. Again, for $\mathbf{d}' = (d_1', d_2') = (1,2)$, we find that $\mathbf{C}_{d_1'} \times \mathbf{C}_{d_2'} = \{(1,0,0),(1,1,0)\}$. Hence $Cay(\Gamma', S_{\Gamma'}(\mathbf{D}^*))$ is isomorphic to the cubelike graph $X(\mathbf{C})$, where

$$\mathbf{C} = \left(\mathbf{C}_{d_1} \times \mathbf{C}_{d_2}\right) \cup \left(\mathbf{C}_{d_1'} \times \mathbf{C}_{d_2'}\right) = \{(1,0,1),(1,1,1),(1,0,0),(1,1,0)\}.$$

Clearly $|\mathbf{C}| \equiv 0 \pmod 4$ and the sum of the elements in $\mathbf{C}$ is $\mathbf{0}$ in $\mathbb{Z}_2^3$. This implies that $\mathbf{D} \in \mathscr{D}$ and hence Theorem 4.3.7 further implies that the graph $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is periodic at $\frac{\pi}{2}$.

In this way, we can construct many more gcd-graphs which are periodic. Our next motive is to add extra edges to these graphs so that the resulting graphs exhibit PST. Generally periodicity and PST are considered for connected graphs. Observe that whenever $\mathbf{D}$ generates the whole group Γ then the associated gcd-graph is connected. We use this idea in the following result to find a sufficient condition for a gcd-graph to exhibit PST.

Consider a set of divisor tuples $\mathbf{D}$ of $\mathbf{m}_\Gamma$ with $\mathbf{m}_\Gamma \notin \mathbf{D}$. Assume for each tuple $d = (d_1, \ldots, d_r, \ldots, d_{r+s})$ in $\mathbf{D}$ that $d_{r+j} = p_j^{m_j}$ for $j = 1, \ldots, s$. Let

$X(\mathbf{C})$ be the cubelike graph, as in Lemma 4.3.6, associated to $Cay(\Gamma, S_\Gamma(\mathbf{D}))$. We denote $\tilde{\mathscr{D}}$ to be the set of all such $\mathbf{D}$ so that sum of the elements in the associated set $\mathbf{C}$ is non-zero.

Theorem 4.3.9. *Let Γ be a finite abelian group. Suppose $\mathbf{D}_1 \in \mathscr{D}$ and $\mathbf{D}_2 \in \tilde{\mathscr{D}}$ and $\mathbf{D}_1 \cap \mathbf{D}_2 = \emptyset$. If $\mathbf{D} := \mathbf{D}_1 \cup \mathbf{D}_2$ generates Γ then $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is connected and admits perfect state transfer at $\frac{\pi}{2}$.*

Proof. If $\mathbf{D}$ generates the group Γ then the graph $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is clearly connected. By Theorem 4.3.7, the graph $Cay(\Gamma, S_\Gamma(\mathbf{D}_1))$ is periodic at $\frac{\pi}{2}$. Thus due to Proposition 4.3.3, it is enough to show that $Cay(\Gamma, S_\Gamma(\mathbf{D}_2))$ exhibits PST at $\frac{\pi}{2}$. Using Lemma 4.3.6, we find that

$$Cay(\Gamma, S_\Gamma(\mathbf{D}_2)) \cong X(\mathbf{C}) \times Cay(\Gamma_2, S_{\Gamma_2}(\{(p_1^{m_1}, \ldots, p_s^{m_s})\})).$$

Since $G = Cay(\Gamma_2, S_{\Gamma_2}(\{(p_1^{m_1}, \ldots, p_s^{m_s})\}))$ has loops at each of its vertices and no more edges, the adjacency matrix of G is I. Suppose $X(\mathbf{C})$ has the adjacency matrix A. The adjacency matrix of $Cay(\Gamma, S_\Gamma(\mathbf{D}_2))$ is therefore $A \otimes I$. Now the transition matrix of $Cay(\Gamma, S_\Gamma(\mathbf{D}_2))$ can be calculated as $\exp(-it(A \otimes I)) = \exp(-itA) \otimes I$. Observe that $\exp(-itA)$ is the transition matrix of $X(\mathbf{C})$. Since $\mathbf{D}_2 \in \tilde{\mathscr{D}}$, the sum of the elements in $\mathbf{C}$ is non-zero. Therefore, by Theorem 4.2.3, the cubelike graph $X(\mathbf{C})$ exhibits PST at $\frac{\pi}{2}$. Hence $Cay(\Gamma, S_\Gamma(\mathbf{D}_2))$ also exhibits PST at $\frac{\pi}{2}$. □

We illustrate Theorem 4.3.9 by the following example.

Example 4.3.10. Suppose $\Gamma = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and consider

$$\mathbf{D} = \{(1,1,1), (1,2,1), (2,2,3), (4,1,3)\}.$$

We show that $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ admits PST at $\frac{\pi}{2}$. Let $\mathbf{D}_1 = \{(1,1,1), (1,2,1)\}$. We have shown in Example 4.3.8 that $\mathbf{D}_1 \in \mathscr{D}$. Let $\mathbf{D}_2 = \{(2,2,3), (4,1,3)\}$. Here the connection set $\mathbf{C}$ in $X(\mathbf{C})$ associated to $Cay(\Gamma, S_\Gamma(\mathbf{D}_2))$ can be evaluated as $\mathbf{C} = \{(0,1,0), (0,0,1)\}$. Thus we have $\mathbf{D}_2 \in \tilde{\mathscr{D}}$. Note here that $\mathbf{D}$

generates Γ. Hence, by Theorem 4.3.9, we find that $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is connected and exhibits PST at $\frac{\pi}{2}$.

Now we find a characterization of gcd-graphs having PST. We show that if $|\Gamma| \equiv 0 \pmod 4$ then there is a gcd-graph over Γ having perfect state transfer. Recall that $\mathbf{m}_\Gamma$ is the tuple $(2^{n_1}, \ldots, 2^{n_r}, p_1^{m_1}, \ldots, p_s^{m_s})$ so that $(\mathbf{m}_\Gamma)_i = 2^{n_i}$ or $p_{i-r}^{m_{i-r}}$ according as $1 \leq i \leq r$ or $r+1 \leq i \leq r+s$.

Lemma 4.3.11. *If $\mathbb{Z}_8$ is a subgroup of Γ then there exists a connected gcd-graph over Γ exhibiting perfect state transfer at $\frac{\pi}{2}$.*

Proof. Suppose $\Gamma_1 = \mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_r}}$, $n_1 > 2$ and $\Gamma_2 = \mathbb{Z}_{p_1^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{m_s}}$, $p_i > 2$ for $1 \leq i \leq s$ are such that $\Gamma = \Gamma_1 \oplus \Gamma_2$. Consider the set $\mathbf{D}$ of divisors $(d_1, \ldots, d_{r+s})$ such that $d_1 = 1$ and for $i \geq 2$, let $d_i = 1$ for at most one i, otherwise $d_i = (\mathbf{m}_\Gamma)_i$. Here the set $\mathbf{D}$ generates Γ and so $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is connected. Now for $\mathbf{d} = (d_1, \ldots, d_{r+s}) \in \mathbf{D}$, we show that the cubelike graph $X(\mathbf{C})$ associated to $Cay(\Gamma, S_\Gamma(\{\mathbf{d}\}))$ (as in Lemma 4.3.6) has the connection set $\mathbf{C}$ which has the property that $|\mathbf{C}| \equiv 0 \pmod 4$ and the sum of the elements in $\mathbf{C}$ is $\mathbf{0}$. Notice that $d_1 = 1 = 2^0$ and $n_1 > 2$. Therefore

$$\mathbf{C}_{d_1} = \left\{ \left(c_0, \ldots, c_{n_1-1}\right) \in \mathbb{Z}_2^{n_1} : c_0 = 1 \text{ and for } i \geq 1,\ c_i = 0 \text{ or } 1 \right\}.$$

Observe that $\mathbf{C}_{d_1}$ has 2^{n_1-1} elements and hence $|\mathbf{C}_{d_1}| \equiv 0 \pmod 4$. Also it is clear that the sum of the elements in $\mathbf{C}_{d_1}$ is $\mathbf{0}$ in $\mathbb{Z}_2^{n_1}$. Notice that for any subset S of $\mathbb{Z}_2^k$, we have $|\mathbf{C}_{d_1} \times S| \equiv 0 \pmod 4$ and sum of the elements in $\mathbf{C}_{d_1} \times S$ is $\mathbf{0}$ in $\mathbb{Z}_2^{n_1+k}$. Here the connection set $\mathbf{C}$ in $X(\mathbf{C})$ is given by $\mathbf{C} = \mathbf{C}_{d_1} \times \mathbf{C}_{d_2} \times \cdots \times \mathbf{C}_{d_r}$. So $|\mathbf{C}| \equiv 0 \pmod 4$ and the sum of the elements in $\mathbf{C}$ is $\mathbf{0}$ in $\mathbb{Z}_2^{n_1+\cdots+n_r}$. Therefore $\{\mathbf{d}\} \in \mathscr{D}$ for each $\mathbf{d} \in \mathbf{D}$ and hence $\mathbf{D} \in \mathscr{D}$.

Now consider $\mathbf{D}' = \left\{ \left(2^{n_1-1}, 2^{n_2}, \ldots, 2^{n_r}, p_1^{m_1}, \ldots, p_s^{m_s}\right) \right\}$. Here we show that $\mathbf{D}' \in \tilde{\mathscr{D}}$. Let $X(\mathbf{C}')$ be the cubelike graph associated to $Cay(\Gamma, S_\Gamma(\mathbf{D}'))$ (as in Lemma 4.3.6). Note that $\mathbf{C}'_{d_1} = \{(0, \ldots, 0, 1)\}$ and for $2 \leq i \leq r$ we have $\mathbf{C}'_{d_i} = \{(0, \ldots, 0)\}$. So the set $\mathbf{C}'$, which is $\mathbf{C}'_{d_1} \times \cdots \times \mathbf{C}'_{d_r}$, contains exactly one element (non zero) and hence $\mathbf{D}' \in \tilde{\mathscr{D}}$. Finally, by Theorem 4.3.9, we find that $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D}'))$ is connected and exhibits PST at $\frac{\pi}{2}$. ◻

Following the idea behind the proof of Lemma 4.3.11, we can construct many more gcd-graphs admitting PST at $\frac{\pi}{2}$.

Lemma 4.3.12. *Let $\mathbb{Z}_4$ be a subgroup of Γ and also suppose that $\mathbb{Z}_8$ is not a subgroup of Γ. Then there exists a connected gcd-graph over Γ exhibiting perfect state transfer at $\frac{\pi}{2}$.*

Proof. Suppose $\Gamma_1 = \mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_r}}$ and $\Gamma_2 = \mathbb{Z}_{p_1^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{m_s}}$, $p_i > 2$ for $1 \leq i \leq s$ are such that $\Gamma = \Gamma_1 \oplus \Gamma_2$. Since $\mathbb{Z}_4$ is a subgroup of Γ and $\mathbb{Z}_8$ is not a subgroup of Γ, without loss of generality we assume that $n_1 = 2$.

Consider the set $\mathbf{D}$ of divisors $(d_1, \ldots, d_{r+s})$ such that $d_1 \in \{1, 2, 4\}$ and for $i \geq 2$, let $d_i = 1$ for exactly one i, otherwise $d_i = (\mathbf{m}_\Gamma)_i$. Here the set $\mathbf{D}$ generates the group Γ and hence $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is connected. Notice that

$$\mathbf{C}_{d_1} = \begin{cases} \{(1,0), (1,1)\}, & \text{if } d_1 = 1; \\ \{(0,1)\}, & \text{if } d_1 = 2; \\ \{(0,0)\}, & \text{if } d_1 = 4. \end{cases}$$

Now for a fixed divisor tuple $(d_2, \ldots, d_{r+s})$, consider the set of divisor tuples $\mathbf{D}_1 = \{1, 2, 4\} \times \{(d_2, \ldots, d_{r+s})\}$. Let $X(\mathbf{C})$ be the cubelike graph associated to $Cay(\Gamma, S_\Gamma(\mathbf{D}_1))$ (as in Lemma 4.3.6). The connection set $\mathbf{C}$ is therefore

$$\mathbf{C} = \left(\bigcup_{d_1 \in \{1,2,4\}} \mathbf{C}_{d_1} \right) \times \mathbf{C}_{d_2} \times \cdots \times \mathbf{C}_{d_r}.$$

Observe that $|\mathbf{C}| \equiv 0 \pmod 4$ and the sum of the elements in $\mathbf{C}$ is $\mathbf{0}$ in $\mathbb{Z}_2^{n_1+\cdots+n_r}$. Hence we find that $\mathbf{D}_1 \in \mathscr{D}$ and accordingly $\mathbf{D} \in \mathscr{D}$.

Now consider $\mathbf{D}' = (2, 2^{n_2}, \ldots, 2^{n_r}, p_1^{m_1}, \ldots, p_s^{m_s})$. Here we show that $\mathbf{D}'$ is in $\tilde{\mathscr{D}}$. Let $X(\mathbf{C}')$ be the cubelike graph associated to $Cay(\Gamma, S_\Gamma(\mathbf{D}'))$ (as in Lemma 4.3.6). Note that $\mathbf{C}'_{d_1} = \{(0,1)\}$ and for $2 \leq i \leq r$ we have $\mathbf{C}'_{d_i} = \{(0, \ldots, 0)\}$. So the set $\mathbf{C}'$, which is $\mathbf{C}'_{d_1} \times \cdots \times \mathbf{C}'_{d_r}$, contains exactly one element which is non-zero and hence $\mathbf{D}' \in \tilde{\mathscr{D}}$. Finally, by Theorem 4.3.9, we find that $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D'}))$ is connected and exhibits PST at $\frac{\pi}{2}$. $\square$

Now we consider the remaining possible case so that $|\Gamma| \equiv 0 \pmod 4$.

Lemma 4.3.13. *Let Γ be a group such that $|\Gamma| \equiv 0$ $(mod\ 4)$. If $\mathbb{Z}_2$ is a subgroup of Γ and $\mathbb{Z}_4$ is not a subgroup of Γ then there exists a connected gcd-graph over Γ exhibiting perfect state transfer at $\frac{\pi}{2}$.*

Proof. Suppose $\Gamma_1 = \mathbb{Z}_{2^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{n_r}}$ and $\Gamma_2 = \mathbb{Z}_{p_1^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{m_s}}$, $p_i > 2$ for $1 \leq i \leq s$ are such that $\Gamma = \Gamma_1 \oplus \Gamma_2$. Since $|\Gamma| \equiv 0 \pmod 4$, $\mathbb{Z}_2$ is a subgroup of Γ and $\mathbb{Z}_4$ is not a subgroup of Γ, we have $n_i = 1$ for $1 \leq i \leq r$ and $r \geq 2$.

Consider the set $\mathbf{D}$ of divisors $(d_1, \ldots, d_{r+s})$ such that $d_1, d_2 \in \{1, 2\}$ and for $i \geq 3$, let $d_i = 1$ for exactly one i, otherwise $d_i = (\mathbf{m}_\Gamma)_i$. For $1 \leq i \leq r$, if $d_i = 2$ then $\mathbf{C}_{d_i} = \{(0)\}$ and when $d_i = 1$ we have $\mathbf{C}_{d_i} = \{(1)\}$. Now for a fixed divisor tuple $(d_3, \ldots, d_{r+s})$ consider $\mathbf{D}_1 = \{1, 2\} \times \{1, 2\} \times \{(d_3, \ldots, d_{r+s})\}$. Suppose $X(\mathbf{C})$ is the cubelike graph associated to $Cay(\Gamma, S_\Gamma(\mathbf{D}_1))$ (as in Lemma 4.3.6). The connection set $\mathbf{C}$ is therefore

$$\mathbf{C} = \left(\bigcup_{d_1 \in \{1,2\}} C_{d_1} \right) \times \left(\bigcup_{d_2 \in \{1,2\}} C_{d_2} \right) \times C_{d_3} \times \cdots \times C_{d_r}.$$

Observe that $|\mathbf{C}| \equiv 0 \pmod 4$ and the sum of the elements in $\mathbf{C}$ is $\mathbf{0}$ in $\mathbb{Z}_2^{n_1+\cdots+n_r}$. Hence we find that $\mathbf{D}_1 \in \mathscr{D}$ and accordingly $\mathbf{D} \in \mathscr{D}$.

Consider $\mathbf{D'} = \{(1, 2, 2, \ldots, 2, p_1^{m_1}, \ldots, p_s^{m_s}), (2, 1, 2, \ldots, 2, p_1^{m_1}, \ldots, p_s^{m_s})\}$. Here it is easy to observe that $\mathbf{D'} \in \tilde{\mathscr{D}}$. Also note that $\mathbf{D} \cup \mathbf{D'}$ generates the group Γ and hence $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ is connected. Finally, by Theorem 4.3.9, we find that $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D'}))$ is connected and exhibits PST at $\frac{\pi}{2}$. $\square$

We now combine Lemma 4.3.11, Lemma 4.3.12 and Lemma 4.3.13 to state the following theorem.

Theorem 4.3.14. *Let an abelian group Γ be such that $|\Gamma| \equiv 0$ $(mod\ 4)$. Then there exists a connected gcd-graph over Γ exhibiting perfect state transfer at $\frac{\pi}{2}$.*

We have already seen that gcd-graphs over abelian groups of odd order do not exhibit PST. In the following theorem we find a necessary and sufficient condition for a class of gcd-graphs to be periodic at π. Here we add some more edges to a graph $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ admitting perfect PST at $\frac{\pi}{2^k}, k \in \mathbb{N}$ and observe the behavior of the transition matrix.

Theorem 4.3.15. *Let Γ and $\mathbf{D}$ be as defined in Lemma 4.3.6 and assume that the graph $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ admits perfect state transfer at $\frac{\pi}{2^k}$, for some $k \in \mathbb{N}$. Suppose $d_{r+1}, \ldots, d_{r+s}$ are fixed divisors of $p_1^{m_1}, \ldots, p_s^{m_s}$, respectively. Consider $\mathbf{D}' = \{\mathbf{d} = (d_1, \ldots, d_{r+1}, \ldots, d_{r+s}) \in \Gamma \, : \, \mathbf{d}$ divides $\mathbf{m}_\Gamma\}$, so that $\mathbf{D}$ and $\mathbf{D}'$ are disjoint. Also, for an integral graph G, let $Cay(\Gamma, S_\Gamma(\mathbf{D}')) \cong X(\mathbf{C}') \times G$ (as in Lemma 4.3.6) with $|\mathbf{C}'| \not\equiv 0 \pmod 2$. The graph $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D}'))$ is periodic at π if and only if the eigenvalues of G have same parity.*

Proof. Using Theorem 4.3.1, we find that if a vertex transitive graph exhibits PST at time τ then for all $k \in \mathbb{N}$, the transition matrix $H\left(2^k\tau\right)$ is a scalar multiple of identity. Since $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ admits PST at $\frac{\pi}{2^k}$, the associated transition matrix at π is a scalar multiple of the identity matrix. By Proposition 4.3.3, the transition matrix of $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D}'))$ is the product of transition matrices of $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ and $Cay(\Gamma, S_\Gamma(\mathbf{D}'))$. So $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D}'))$ is periodic at π if and only if $Cay(\Gamma, S_\Gamma(\mathbf{D}'))$ is periodic at π.

Now we have $Cay(\Gamma, S_\Gamma(\mathbf{D}')) \cong X(\mathbf{C}') \times G$. If the sum of the elements in $\mathbf{C}'$ is σ then by Lemma 4.2.2, the transition matrix of $X(\mathbf{C}')$ at $\frac{\pi}{2}$ can be calculated as $(-i)^{|\mathbf{C}'|} P_\sigma$. At $\tau = \pi$, the transition matrix becomes $[(-i)^{|\mathbf{C}'|} P_\sigma]^2 = i^{2|\mathbf{C}'|} I$ as $P_\sigma^2 = I$. If $\sum\limits_s \mu_s F_s$ is the spectral decomposition of adjacency matrix of G then, by Proposition 1.4.3, the transition matrix of $X(\mathbf{C}') \times G$ becomes

$$\sum_s \left(i^{2|\mathbf{C}'|}\right)^{\mu_s} I \otimes F_s = I \otimes \sum_s i^{2|\mathbf{C}'|\mu_s} F_s.$$

If $Cay(\Gamma, S_\Gamma(\mathbf{D}'))$ is periodic at π then $\exists \gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that

$$I \otimes \sum_s i^{2|\mathbf{C}'|\mu_s} F_s = \gamma I, \tag{4.3}$$

where the identity matrices have appropriate orders. Note that $F_s^2 = F_s$ and

$F_rF_s = 0$ for $r \neq s$. Multiplying both sides of Equation (4.3) by $I \otimes F_s$, we find that $i^{2|\mathbf{C}'|\mu_s} = \gamma$ for all s. If μ_s and $\mu_{s'}$ are two distinct eigenvalues of G then we have $e^{i|\mathbf{C}'|(\mu_s - \mu_{s'})\pi} = 1$. By our assumption $|\mathbf{C}'| \not\equiv 0 \pmod 2$ and therefore the eigenvalues μ_s and $\mu_{s'}$ must have same parity.

Conversely, suppose the eigenvalues of G have same parity and assume $i^{2|\mathbf{C}'|\mu_s} = \gamma$. Then for any other eigenvalue $\mu_{s'}$, we see that

$$i^{2|\mathbf{C}'|\mu_{s'}} = i^{2|\mathbf{C}'|\mu_s} \cdot i^{2|\mathbf{C}'|(\mu_{s'} - \mu_s)} = \gamma.$$

Thus $i^{2|\mathbf{C}'|\mu_s} = \gamma$ for all μ_s and this implies that $I \otimes \sum_s i^{2|\mathbf{C}'|\mu_s} F_s = \gamma I$. Therefore $Cay(\Gamma, S_\Gamma(\mathbf{D}'))$ is periodic at π. Hence $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D}'))$ is periodic at π. □

We thus have a necessary condition for PST at $\frac{\pi}{2^k}$, $k \in \mathbb{N}$ for the class of gcd-graphs given in Theorem 4.3.15. We include this as a corollary.

Corollary 4.3.16. *Suppose the conditions of Theorem 4.3.15 hold. If perfect state transfer occurs in $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D}'))$ at $\frac{\pi}{2^k}$, $k \in \mathbb{N}$ then the eigenvalues of G have same parity.*

In the following example, we use Corollary 4.3.16 to find a gcd-graph not having PST at $\frac{\pi}{2^k}$ for any $k \in \mathbb{N}$.

Example 4.3.17. Suppose $\Gamma = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and consider the set of divisors $\mathbf{D} = \{(1, 1, 1), (1, 2, 1), (2, 2, 3), (4, 1, 3)\}$ and $\mathbf{D}' = \{(2, 2, 1)\}$. We have already witnessed in Example 4.3.10 that $Cay(\Gamma, S_\Gamma(\mathbf{D}))$ admits PST at $\frac{\pi}{2}$. Applying Lemma 4.3.6, we find that $Cay(\Gamma, S_\Gamma(\mathbf{D}')) \cong X(\mathbf{C}) \times G$, where $G = Cay(\mathbb{Z}_3, S_{\mathbb{Z}_3}(1))$. Note that $Cay(\mathbb{Z}_3, S_{\mathbb{Z}_3}(1))$ is actually the complete graph K_3 whose eigenvalues are $-1, -1$ and 2. Clearly the eigenvalues of G do not have same parity. Hence, by Corollary 4.3.16, we conclude that $Cay(\Gamma, S_\Gamma(\mathbf{D} \cup \mathbf{D}'))$ does not exhibit PST at $\frac{\pi}{2^k}$ for any $k \in \mathbb{N}$.

It turns out that we can strike out many gcd-graphs that do not have PST at $\frac{\pi}{2^k}$ for any $k \in \mathbb{N}$.

4.4 Conclusion

In this chapter, we have studied gcd-graphs for PST. Here we find a method to construct gcd-graphs having PST. We have shown that if an abelian group has an order divisible by four then there exists a connected gcd-graph over that group exhibiting PST. In fact there are many such graphs having PST. In Lemma 4.3.11, Lemma 4.3.12 and Lemma 4.3.13, we observe that there are many other choices for the set $\mathbf{D} \in \mathscr{D}$. In particular, in Lemma 4.3.11, if we consider a set of divisor tuples $\mathbf{D}$ where each tuple in $\mathbf{D}$ has first component 1 and the remaining components are any divisor, then also $\mathbf{D} \in \mathscr{D}$. Finally, in Theorem 4.3.15, a necessary and sufficient condition is given for a certain class of gcd-graphs to exhibit periodicity at time π. From this we find a necessary condition to have PST in a class of gcd-graphs at some specific times. This gives a partial characterization of gcd-graphs admitting PST.

Chapter 5

PGST on Circulant Graphs

Circulant graphs arise frequently in communication networks. Among the circulant graphs, only integral circulant graphs are periodic (see [26]). In Chapter 1, we have seen that if a graph is periodic then the graph has PGST if and only if it has PST. Since a complete characterization of PST in integral circulant graphs is known (see [7]), we consider PGST in circulant graphs that are not integral. In this chapter, we completely classify the cycles exhibiting PGST. Beside cycles, we also find two classes of non-integral circulant graphs one of which exhibits PGST and the other does not exhibit PGST. Apart from circulant graphs, we also find some non-circulant graphs admitting PGST. The results of this chapter appear in [41].

5.1 Preliminaries

Recall that $\mathbb{Z}_n$ is the cyclic group of order n and that a circulant graph is a Cayley graph over $\mathbb{Z}_n$. A cycle C_n, in particular, is a circulant graph over $\mathbb{Z}_n$ with the connection set $\{1, n-1\}$. Eigenvalues and eigenvectors of a cycle are very well known. Recall that $\omega_n = \exp\left(\frac{2\pi i}{n}\right)$ is the primitive n-th root of unity. Then the eigenvalues of C_n are

$$\lambda_l = 2\cos\left(\frac{2l\pi}{n}\right),\ 0 \leq l \leq n-1, \tag{5.1}$$

and the corresponding eigenvectors are $\mathbf{v}_l = \left[1, \omega_n^l, \ldots, \omega_n^{l(n-1)}\right]^T$.

Now we introduce some notations useful for our discussion. If d is a proper divisor of n then we define

$$S_n(d) = \{x \in \mathbb{Z}_n : gcd(x, n) = d\}.$$

For any set D containing proper divisors of n, we define

$$S_n(D) = \bigcup_{d \in D} S_n(d).$$

The set $S_n(D)$ is called a gcd-set of $\mathbb{Z}_n$. A gcd-graph over $\mathbb{Z}_n$ is a circulant graph whose connection set is a gcd-set. We denote a gcd-graph with the connection set $S_n(D)$ by $G(n, D)$.

A graph is called integral if all its eigenvalues are integers. The following theorem determines circulant graphs which are integral.

Theorem 5.1.1. *[46] A circulant graph is integral if and only if the connection set is a gcd-set.*

5.2 PGST on Circulant Graphs

We begin with the discussion that the odd cycles never exhibit PGST. Moreover, if an even cycle admits PGST then it must occur only between the antipodal vertices. Suppose G is a graph with adjacency matrix A. If P is the matrix of an automorphism of G then P must commute with A, and consequently P commutes with the transition matrix $H(t)$. Suppose G admits PGST between two vertices u and v. Then there exists a sequence of real numbers $\{t_k\}$ and a complex number γ of unit modulus such that

$$\lim_{k \to \infty} H(t_k)\mathbf{e}_u = \gamma \mathbf{e}_v.$$

This implies that

$$\lim_{k\to\infty} H(t_k)\left(P\mathbf{e}_u\right) = \gamma\left(P\mathbf{e}_v\right). \tag{5.2}$$

Since the sequence $\{H(t_k)\mathbf{e}_u\}$ cannot have two different limits, we conclude that if P fixes $\mathbf{e}_u$ then P must fix $\mathbf{e}_v$ as well. Note that an odd cycle has automorphisms fixing a single vertex and an even cycle has automorphisms fixing only a pair of antipodal vertices. Hence we have the following result.

Lemma 5.2.1. *If pretty good state transfer occurs in a cycle C_n or in the complement of C_n then n is even and it occurs only between the pair of vertices u and $u+\frac{n}{2}$, where $u, u+\frac{n}{2} \in \mathbb{Z}_n$.*

From now onwards, we only consider even cycles. As C_n is vertex-transitive and (5.2) holds, it is enough to find PGST in C_n between the pair of vertices 0 and $\frac{n}{2}$. We thus calculate the $(0, \frac{n}{2})$-th entry of the transition matrix of C_n. Recall that if $\sum\limits_{l=0}^{n-1} \lambda_l E_l$ is the spectral decomposition of the adjacency matrix of C_n then the corresponding transition matrix is given by

$$H(t) = \sum_{l=0}^{n-1} \exp\left(-i\lambda_l t\right) E_l.$$

Notice that $E_l = \frac{\mathbf{v}_l\mathbf{v}_l^*}{||\mathbf{v}_l||^2}$, and therefore $(0, \frac{n}{2})$-th entry of E_l is obtained as

$$\left(E_l\right)_{0,\frac{n}{2}} = \frac{1}{n}\left(\bar{\omega}_n\right)^{\frac{nl}{2}} = \frac{1}{n}\left(\exp\left(-\frac{2\pi i}{n}\right)\right)^{\frac{nl}{2}} = \frac{1}{n}\exp\left(-il\pi\right).$$

Hence we evaluate the $(0, \frac{n}{2})$-th entry of $H(t)$ as

$$H(t)_{0,\frac{n}{2}} = \sum_{l=0}^{n-1} \exp\left(-i\lambda_l t\right)\cdot\left(E_l\right)_{0,\frac{n}{2}} = \frac{1}{n}\sum_{l=0}^{n-1} \exp\left[-i\left(\lambda_l t + l\pi\right)\right]. \tag{5.3}$$

It is well known that a cycle on four vertices has PST between the antipodal vertices. Thus the cycle C_4 admits PGST. In the following result, we distinguish a class of cycles admitting PGST. Later, we will show that these are the only cycles having PGST.

Lemma 5.2.2. *If $n = 2^k$ $(k \geq 3)$, then the cycle C_n exhibits pretty good state transfer with respect to a sequence in $2\pi\mathbb{Z}$.*

Proof. First we show that the distinct positive eigenvalues of C_n are linearly independent over $\mathbb{Q}$. The eigenvalues of C_n can be realized as

$$\lambda_l = 2\cos\left(\frac{2l\pi}{n}\right) = \omega_n^l + \omega_n^{-l}.$$

It is well known that the minimal polynomial of ω_n over $\mathbb{Q}$ has degree $\phi(n)$, where ϕ is the Euler's phi-function (see [22]). It is evident that λ_l is positive only when $-\frac{\pi}{2} < \frac{2l\pi}{n} < \frac{\pi}{2}$. Consequently, the distinct positive eigenvalues are λ_l where $0 \leq l < \frac{n}{4}$. If the distinct positive eigenvalues are linearly dependent over $\mathbb{Q}$ then ω_n will be a root of a polynomial of degree at most $m = 2^{k-1} - 1$, as $\omega_n^{-l} = -\omega_n^{m-l+1}$ for $0 \leq l < \frac{n}{4}$. But since $2^{k-1} - 1 < 2^{k-1} = \phi(2^k)$, we conclude that the distinct positive eigenvalues are linearly independent over $\mathbb{Q}$. Thus the distinct positive eigenvalues are

$$\lambda_0, \lambda_1, \ldots, \lambda_{2^{k-2}-1}.$$

For $1 \leq l \leq 2^{k-2} - 1$, consider the following real numbers

$$\alpha_l = \begin{cases} 0, & \text{if } l \text{ is even;} \\ \frac{1}{2}, & \text{if } l \text{ is odd.} \end{cases}$$

By Theorem 1.5.2 (Kronecker approximation theorem), we find that for $\delta > 0$ there exist $q, m_1, \ldots, m_{2^{k-2}-1} \in \mathbb{Z}$ such that for $l = 1, \ldots, 2^{k-2} - 1$

$$|q\lambda_l - m_l - \alpha_l| < \frac{\delta}{2\pi}, \ \ i.e, \ |2q\pi\lambda_l - 2m_l\pi - 2\alpha_l\pi| < \delta. \tag{5.4}$$

Each of the eigenvalues of C_{2^k} is repeated twice except the eigenvalues 2 and -2. For $1 \leq l \leq 2^{k-2} - 1$, we see that

$$\lambda_l = 2\cos\left(\frac{2l\pi}{n}\right) = 2\cos\left(2\pi - \frac{2l\pi}{n}\right) = \lambda_{n-l}.$$

Similarly, for $1 \leq l \leq 2^{k-2} - 1$, we have $\lambda_l = -\lambda_{\frac{n}{2}-l} = -\lambda_{\frac{n}{2}+l}$. Since $\lambda_0, \lambda_{2^{k-2}}, \lambda_{2^{k-1}}$ and $\lambda_{3.2^{k-2}}$ are integers, considering $t = 2q\pi$, we observe that for each $l = 0,\ 2^{k-2},\ 2^{k-1},\ 3\cdot 2^{k-2}$ there exists an integer l' such that

$$\left|(\lambda_l t + l\pi) - 2l'\pi\right| < \delta.$$

Also for each $l = 1, \ldots, 2^{k-2} - 1$, considering $t = 2q\pi$ and using (5.4), we find

$$l' = \begin{cases} \frac{2m_l+l+1}{2}, & \text{if } l \text{ is odd} \\ \frac{2m_l+l}{2}, & \text{if } l \text{ is even} \end{cases}$$

such that $\left|(\lambda_l t + l\pi) - 2l'\pi\right| < \delta$. Since the eigenvalues of C_{2^k} satisfy $\lambda_l = -\lambda_{\frac{n}{2}-l} = -\lambda_{\frac{n}{2}+l} = \lambda_{n-l}$ where $1 \leq l \leq 2^{k-2} - 1$, considering $t = 2q\pi$, we conclude that for each $l = 0, \ldots, n-1$, there is an integer l' such that

$$\left|(\lambda_l t + l\pi) - 2l'\pi\right| < \delta.$$

Thus by uniform continuity of the exponential function $\exp(-ix)$, for $\epsilon > 0$ there exists $q \in \mathbb{Z}$ so that $t = 2q\pi$ and

$$\left|\exp\left[-i\left(\lambda_l t + l\pi\right)\right] - 1\right| < \epsilon.$$

Finally, from Equation (5.3), we observe that

$$\left|H(t)_{0,\frac{n}{2}} - 1\right| = \frac{1}{n}\left|\sum_{l=0}^{n-1}\left[\exp\left[-i\left(\lambda_l t + l\pi\right)\right] - 1\right]\right| < \epsilon.$$

This leads to the conclusion that C_n, for $n > 4$, admits PGST whenever n is a power of two, with respect to a sequence in $2\pi\mathbb{Z}$. □

Now using Lemma 5.2.2, we find more general circulant graphs admitting PGST. We present this as a theorem.

Theorem 5.2.3. *Let $n = 2^k$ with $k \geq 3$. If D is a set of proper divisors of n not containing 1 then $C_n \cup G(n, D)$ and its complement admit pretty good state transfer with respect to the same sequence in $2\pi\mathbb{Z}$.*

Proof. Notice that both graphs C_n and $G(n, D)$ have the same vertex set $\mathbb{Z}_n$ but the edge sets are disjoint as $1 \notin D$. Suppose that A and B are the adjacency matrices of C_n and $G(n, D)$, respectively. Notice that the connection sets of C_n and $G(n, D)$ are $\{1, n-1\}$ and $S_n(D)$, respectively. As $1 \notin D$, we have $\{1, n-1\} \cap S_n(D) = \emptyset$. Therefore, by Proposition 4.3.3, the transition matrix of $C_n \cup G(n, D)$ is $H(t) = H_A(t)H_B(t)$, where $H_A(t)$ and $H_B(t)$ are transition matrices of C_n and $G(n, D)$, respectively. Suppose B has the spectral decomposition $\sum\limits_{j=1}^{r} \theta_j E_j$. Since $G(n, D)$ is integral, the eigenvalue θ_j is an integer for all j. This implies that if $t \in 2\pi\mathbb{Z}$ then

$$H_B(t) = \exp\left(-itB\right) = \sum_{j=1}^{r} \exp\left(-it\theta_j\right) E_j = I.$$

Hence $H(t) = H_A(t)$ whenever $t \in 2\pi\mathbb{Z}$. Finally, by Lemma 5.2.2, we observe that $C_n \cup G(n, D)$ exhibits PGST.

It remains to show that complement of $C_n \cup G(n, D)$ also admits PGST. The adjacency matrix of the complement is $J - I - (A + B)$. Since all circulant graphs are regular, the matrix $A + B$ commutes with $J - I$. If $\widetilde{H}(t)$ is the transition matrix of the complement then

$$\widetilde{H}(t) = \exp\left(-it\left(J - I\right)\right) H(-t).$$

Note that the eigenvalues of $J - I$ are integers. Hence, following the same argument as given in the previous part, we have the desired result. □

In Theorem 5.2.3, the graph $C_n \cup G(n,D)$ can never be a gcd graph and so it cannot be integral. The reason is that if $C_n \cup G(n,D) = G(n,D')$ for some divisor set D' then $1 \in D'$. Therefore $\{1,n-1\} \cup S_n(D)$ contains all odd numbers in $\mathbb{Z}_n$ as $\{1,n-1\} \cup S_n(D) = S_n(D')$. As $n = 2^k$ $(k \geq 3)$, we see, in particular, that $\{1,n-1\} \cup S_n(D)$ can never contain 3.

Notice that taking $D = \emptyset$ in Theorem 5.2.3, we find that complement of a cycle also admits PGST. In the proof of Theorem 5.2.3, we also observe that for a fixed n, there exists a fixed sequence with respect to which all graphs of the form $C_n \cup G(n,D)$ as well as their complements exhibit PGST.

It turns out that there are some more graphs admitting PGST apart from the circulant graphs we already mentioned. Before finding those graphs we introduce the following notations. For $k \geq 3$, we denote

$$\mathcal{G}_k = \left\{ C_{2^k} \cup G\left(2^k, D\right) \;:\; D \text{ is a set of proper divisors of } 2^k \text{ and } 1 \notin D \right\}.$$

Also the set of the complements of graphs in $\mathcal{G}_k$ is denoted by $\bar{\mathcal{G}}_k$. Further we define the set $\mathcal{G}$ by

$$\mathcal{G} = \bigcup_{k \geq 3} \left(\mathcal{G}_k \cup \bar{\mathcal{G}}_k\right).$$

Now we find the following corollaries regarding PGST in Cartesian products.

Corollary 5.2.4. *Let $G_1, G_2 \in \mathcal{G}_k \cup \bar{\mathcal{G}}_k$. Then the Cartesian product $G_1 \square G_2$ as well as its complement admit pretty good state transfer.*

Proof. In the proof of Theorem 5.2.3, we see that if $G_1, G_2 \in \mathcal{G}_k \cup \bar{\mathcal{G}}_k$ then both G_1 and G_2 have PGST with respect to the same sequence $\{t_m\}$ in $2\pi\mathbb{Z}$. Suppose G_1 admits PGST between the vertices u_1 and v_1 and G_2 admits PGST between the vertices u_2 and v_2. Also assume that $H_{G_1}(t)$ and $H_{G_2}(t)$ are the transition matrices of G_1 and G_2, respectively. Therefore there exist $\gamma_1, \gamma_2 \in \mathbb{C}$ with $|\gamma_1| = |\gamma_2| = 1$ such that

$$\lim_{m\to\infty} \mathbf{e}_{u_1}^T H_{G_1}(t_m) \mathbf{e}_{v_1} = \gamma_1 \text{ and } \lim_{m\to\infty} \mathbf{e}_{u_2}^T H_{G_2}(t_m) \mathbf{e}_{v_2} = \gamma_2.$$

Using the property of transition matrix of a Cartesian product, we have

$$\begin{aligned}\left(\mathbf{e}_{u_1}\otimes\mathbf{e}_{u_2}\right)^T\left(H_{G_1\Box G_2}(t_m)\right)\left(\mathbf{e}_{v_1}\otimes\mathbf{e}_{v_2}\right) &= \left(\mathbf{e}_{u_1}\otimes\mathbf{e}_{u_2}\right)^T\left(H_{G_1}(t_m)\otimes H_{G_2}(t_m)\right)\left(\mathbf{e}_{v_1}\otimes\mathbf{e}_{v_2}\right)\\ &= \left(\mathbf{e}_{u_1}^T H_{G_1}(t_m)\,\mathbf{e}_{v_1}\right)\cdot\left(\mathbf{e}_{u_2}^T H_{G_2}(t_m)\,\mathbf{e}_{v_2}\right).\end{aligned}$$

Now taking limits on both sides we find that $G_1\Box G_2$ admits PGST.

It remains to show that the complement of $G_1\Box G_2$ exhibits PGST. Notice that both G_1 and G_2 are regular graphs and therefore $G_1\Box G_2$ is also a regular graph. Now proceeding as in the proof of second part of Theorem 5.2.3, we arrive at the desired conclusion. □

Remark 5.2.5. *More generally, following the proof of Corollary 5.2.4, we can deduce that if two graphs have PGST with respect to the same sequence then their Cartesian product also admits PGST with respect to that sequence.*

We already witnessed that there are a handful of graphs having periodicity at some of its vertices. Using those graphs we construct many more graphs exhibiting PGST. Consider the following result.

Corollary 5.2.6. *Let a graph G_1 be periodic at a vertex at time 2π. If $G_2\in\mathcal{G}$ then the Cartesian product $G_1\Box G_2$ admits pretty good state transfer. If G_1 is regular then the complement of $G_1\Box G_2$ also exhibits pretty good state transfer.*

Proof. Suppose G_1 is periodic at a vertex u at time 2π. If $H_{G_1}(t)$ is the transition matrix of G_1 then there exists $\gamma_1\in\mathbb{C}$ with $|\gamma|=1$ such that $\mathbf{e}_u^T H_{G_1}(2\pi)\mathbf{e}_u=\gamma_1$. Hence for $q\in\mathbb{Z}$, we have $\mathbf{e}_u^T H_{G_1}(2q\pi)\mathbf{e}_u=\gamma_1^q$. Since $G_2\in\mathcal{G}$ there is a sequence $\{t_m\}$ in $2\pi\mathbb{Z}$ with respect to which G_2 exhibits PGST between two vertices v and w, say. Since the unit circle is compact there is a subsequence $\{t'_m\}$ of $\{t_m\}$ such that $\left\{\mathbf{e}_u^T H_{G_1}(t'_m)\mathbf{e}_u\right\}$ is convergent. If $H_{G_2}(t)$ is the transition matrix of G_2 then

$$(\mathbf{e}_u\otimes\mathbf{e}_v)^T H_{G_1\Box G_2}(t'_m)(\mathbf{e}_u\otimes\mathbf{e}_w)=\left(\mathbf{e}_u^T H_{G_1}(t'_m)\,\mathbf{e}_u\right)\cdot\left(\mathbf{e}_v^T H_{G_2}(t'_m)\,\mathbf{e}_w\right).$$

Now taking limits on both sides, we find that $G_1\Box G_2$ admits PGST.

In case G_1 is regular then $G_1 \square G_2$ is also regular. Therefore the complement of $G_1 \square G_2$ also exhibits PGST. □

Remark 5.2.7. *If a graph is integral then it is periodic at 2π. So Cartesian product of an integral graph and a graph in $\mathcal{G}$ admits PGST. This gives a large number of graphs having PGST.*

5.3 Circulant Graphs having no PGST

In Lemma 5.2.2, we found that if n is a power of two then the cycle C_n exhibits PGST. Now we investigate PGST in the remaining class of cycles. The only possibility we need to consider is the case when n has an odd prime factor, in which case we show that there is no PGST. We use some of the techniques from [28] to prove the following result.

Lemma 5.3.1. *Let $m \in \mathbb{N}$ and p be an odd prime such that $n = mp$. Then the cycle C_n does not exhibit pretty good state transfer.*

Proof. Notice that if m is an odd number then, by Lemma 5.2.1, we have the desired result. Hereafter we assume that m is even. We have the following identity involving the primitive p-th root ω_p of unity:

$$1 + \omega_p + \omega_p^2 + \ldots + \omega_p^{p-1} = 0.$$

This further yields

$$1 + 2\sum_{r=1}^{\frac{p-1}{2}} \cos\left(\frac{2r\pi}{p}\right) = 0. \tag{5.5}$$

Multiplying both sides of (5.5) by $2\cos\left(\frac{2\pi}{n}\right)$, we obtain the following relation of eigenvalues (as given in (5.1)) of C_n.

$$\lambda_1 + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr+1} + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr-1} = 0. \tag{5.6}$$

Similarly, multiplying (5.5) by $2\cos\left(\frac{4\pi}{n}\right)$ gives

$$\lambda_2 + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr+2} + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr-2} = 0. \tag{5.7}$$

Now from Equation (5.6) and (5.7), we get

$$(\lambda_2 - \lambda_1) + \sum_{r=1}^{\frac{p-1}{2}} (\lambda_{mr+2} - \lambda_{mr+1}) + \sum_{r=1}^{\frac{p-1}{2}} (\lambda_{mr-2} - \lambda_{mr-1}) = 0. \tag{5.8}$$

If C_n admits PGST then, by Equation (5.3), we have a sequence of real numbers $\{t_k\}$ and a complex number γ with $|\gamma| = 1$ such that

$$\lim_{k\to\infty} \sum_{l=0}^{n-1} \exp\left[-i\left(\lambda_l t_k + l\pi\right)\right] = n\gamma. \tag{5.9}$$

Since the unit circle is compact, there is a subsequence $\left\{t_k^{(0)}\right\}$ of $\{t_k\}$ such that $\left\{\exp\left[-i\left(\lambda_0 t_k^{(0)} + 0\cdot\pi\right)\right\}\right.$ is convergent. Similarly, there is a subsequence $\left\{t_k^{(1)}\right\}$ of $\left\{t_k^{(0)}\right\}$ such that $\left\{\exp\left[-i\left(\lambda_1 t_k^{(1)} + 1\cdot\pi\right)\right\}\right.$ is also convergent. Continuing in this process, we find a sequence $\{t_k'\}$ such that for $0 \le l \le n-1$, the sequence $\left\{\exp\left[-i\left(\lambda_l t_k' + l\pi\right)\right\}\right.$ is convergent to a limit γ_l, say, where $|\gamma_l| = 1$. As a consequence, from (5.9) we obtain $\gamma_0 + \cdots + \gamma_{n-1} = n\gamma$, which further implies that $\gamma_l = \gamma$ for all l. Thus we have

$$\lim_{k\to\infty} \exp\left[-i\left(\lambda_l t_k' + l\pi\right)\right. = \gamma, \text{ for } 0 \le l \le n-1. \tag{5.10}$$

This further implies that

$$\lim_{k\to\infty} \exp\left[-i\left(\lambda_{l+1} - \lambda_l\right) t_k'\right] = -1.$$

Denoting the term in left side of Equation (5.8) as L, it is evident that

$$\lim_{k\to\infty} \exp\left(-iLt_k'\right) = -1.$$

But this is not possible as $L = 0$. Hence there is no pretty good state transfer in C_n, whenever n has an odd prime factor. □

So far we have developed a complete characterization for PGST on cycles. We state the result as a theorem.

Theorem 5.3.2. *A cycle C_n admits pretty good state transfer if and only if $n = 2^k$ for some $k \geq 2$.*

In the next result we see that complement of some cycles do not admit PGST. This gives another class of circulant graphs not admitting PGST.

Theorem 5.3.3. *Let $m \in \mathbb{N}$ with $m \neq 2$ such that $n = mp$ for some odd prime p. Then the complement of the cycle C_n does not exhibit pretty good state transfer.*

Proof. For $m = 1$, by Lemma 5.2.1, we conclude that complement of C_n does not admit PGST. Now consider the case $m \geq 3$.

The cycles are regular graphs. Therefore the eigenvalues of the complement of C_n are $\lambda_0' = n - \lambda_0 - 1$ and $\lambda_l' = -\lambda_l - 1$, whenever $1 \leq l \leq n-1$, corresponding to the same set of eigenvectors as that of C_n. This means that $(0, \frac{n}{2})$-th entry of the transition matrix of the complement graph can be obtained from Equation (5.3) by replacing the eigenvalues λ_l by λ_l'. Now along the line of proof of Lemma 5.3.1, we find that

$$\lambda_1' + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr+1}' + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr-1}' = -p.$$

Since $m \geq 3$, we also have

$$\lambda_2' + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr+2}' + \sum_{r=1}^{\frac{p-1}{2}} \lambda_{mr-2}' = -p.$$

The above two identities give

$$\left(\lambda_2' - \lambda_1'\right) + \sum_{r=1}^{\frac{p-1}{2}} \left(\lambda_{mr+2}' - \lambda_{mr+1}'\right) + \sum_{r=1}^{\frac{p-1}{2}} \left(\lambda_{mr-2}' - \lambda_{mr-1}'\right) = 0,$$

which is similar to (5.8). Hence, following the same argument as in Lemma 5.3.1, we conclude that the complement of C_n does not exhibit PGST. □

We have some more observations which we include as a remark.

Remark 5.3.4. *Suppose that p is a prime number and consider the complement of the cycle C_{2p}. In this case, proceeding as in the proof of Theorem 5.3.3, we find the identities*

$$\lambda_1' + \sum_{r=1}^{\frac{p-1}{2}} \lambda'_{mr+1} + \sum_{r=1}^{\frac{p-1}{2}} \lambda'_{mr-1} = -p \text{ and } \lambda_2' + \sum_{r=1}^{\frac{p-1}{2}} \lambda'_{mr+2} + \sum_{r=1}^{\frac{p-1}{2}} \lambda'_{mr-2} = n - p.$$

Therefore we evaluate

$$(\lambda_2' - \lambda_1') + \sum_{r=1}^{\frac{p-1}{2}} (\lambda'_{mr+2} - \lambda'_{mr+1}) + \sum_{r=1}^{\frac{p-1}{2}} (\lambda'_{mr-2} - \lambda'_{mr-1}) = n.$$

Further, following as in the proof of Lemma 5.3.1, we conclude that C_{2p} does not admit PGST with respect to any sequence in $\pi\mathbb{Z}$.

5.4 Conclusion

It is always preferable to find graphs having PST between vertices at a long distance. So far, the best known graphs in this regard are the hypercubes. In a hypercube with n vertices, PST occurs between vertices at a distance $\log_2(n)$. It is thus desirable to have graphs admitting PST between vertices at a distance of $O(n)$. Some lucrative classes of graphs in this regard are the paths P_n and the cycles C_n, as both of them have large diameter. But it is well known that P_n does not exhibit PST whenever $n \geq 4$ and C_n admits PST only when $n = 4$.

Meanwhile the study of PGST got some interest. In [28], the authors presented a remarkable result which classifies the paths P_n admitting PGST between the end vertices. This serves as an example where PGST takes place between vertices at a distance n. In this chapter, we have found that C_n exhibits PGST if and only if n is a power of two and that PGST occurs between any pair of antipodal vertices. This gives another class of graphs having PGST between vertices at a distance of $O(n)$. We also have found a good number of circulant graphs admitting or not admitting PGST. Apart from the circulant graphs, we have found some other graphs admitting PGST.

Chapter 6

Future Work

We list some of the problems arising in the previous chapters that we can address in future works.

PST on NEPS of P_3

In [48], Stevanović asked a problem, which NEPS of P_3 exhibits PST. In Chapter 2, we found that a NEPS of P_3 with basis Ω exhibits PST whenever the following holds:

- for each tuple in Ω, the hamming weight is even (or odd);
- $\sum_{\beta \in \Omega^*} \beta \neq \mathbf{0}$ over $\mathbb{Z}_2^n$, where $\Omega^* = \{\beta \in \Omega : wt(\beta) = k\}$ and $k = \min_{\beta \in \Omega} wt(\beta)$.

In Chapter 3, we established the fact that a NEPS of P_3 with basis Ω which contains tuples with hamming weights of both parities, does not admit PST. So the only case that remains to check for PST in NEPS of P_3 is those NEPS with basis Ω for which $\sum_{\beta \in \Omega^*} \beta = \mathbf{0}$ over $\mathbb{Z}_2^n$, where $\Omega^* = \{\beta \in \Omega : wt(\beta) = k\}$ and $k = \min_{\beta \in \Omega} wt(\beta)$. We also found in Chapter 3 that these graphs are periodic (see Theorem 3.3.1). It will thus be interesting to find whether there are any graphs exhibiting PST in that subclass.

PGST on NEPS of P_3

In Chapter 3, we considered those NEPS of P_3 with basis Ω for which Ω contains tuples with hamming weights of both parities. In that scenario we found that the associated NEPS does not exhibit PST and therefore we have initiated our investigation for PGST in that class of NEPS. In Theorem 3.3.3, we found that PGST occurs in NEPS of P_3 if

$$\text{either} \sum_{\beta \in \Omega_o^*} \beta \neq \mathbf{0} \text{ or } \sum_{\beta \in \Omega_e^*} \beta \neq \mathbf{0} \text{ in } \mathbb{Z}_2^n.$$

So there is a natural question to ask whether PGST occurs in NEPS of P_3 when both $\sum_{\beta \in \Omega_o^*} \beta = \mathbf{0}$ and $\sum_{\beta \in \Omega_e^*} \beta = \mathbf{0}$ in $\mathbb{Z}_2^n$.

PST on Cayley Graphs

We know that among the Cayley graphs only integral graphs are qualified to have PST. Since gcd-graphs are well known class of integral Cayley graphs, we studied gcd-graphs for PST in Chapter 4. There are few scopes for further research in this direction.

- In Chapter 4, we proved results which find PST in gcd-graphs at time $\frac{\pi}{2}$. Also we find some gcd-graphs not having PST at $\frac{\pi}{2^k}$ for any $k \in \mathbb{N}$. Thus we can try to find PST in these gcd-graphs at other possible times.

- We can also attempt to find PST on other possible gcd-graphs that were not covered in Chapter 4. A complete classification of all gcd-graphs admitting PST is most desirable.

- More generally, we can make efforts to find PST in those integral Cayley graphs which are not gcd-graphs and, if possible, we can characterize all such graphs having PST.

PGST on Cayley Graphs

In Chapter 5, we have considered PGST on circulant graphs. We found that the circulant graph $C_{2^k} \cup G\left(2^k, D\right)$ for $k \geq 3, 1 \notin D$, admits PGST. We list a few possible problems that can be explored in this direction.

- We can try to find whether there are any other circulant graphs, apart from those we already discussed, admitting PGST.
- In Remark 5.3.4, we see that the complement of C_{2p}, where p is a prime, does not have PGST with respect to any sequence in $\pi\mathbb{Z}$. In that case, it would be interesting to find if the complement of those cycles have PGST at all. Moreover, it is most desirable to classify which circulant graphs exhibit PGST.
- More generally, it is preferable to have a characterization of PGST in Cayley graphs.

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