Strichartz Estimates Associated with Certain Self-Adjoint Operators

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Declaration

I do hereby declare that this thesis entitled **"Strichartz Estimates Associated with Certain Self-Adjoint Operators"** is a presentation of my original research work done under the supervision of **Dr. Jitendriya Swain**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of doctor of philosophy. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

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Certificate

This is certified that the work contained in the thesis entitled **"Strichartz Estimates Associated with Certain Self-Adjoint Operators"** by **Mr. Sunit Ghosh** (Roll No. 206123108) has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other university or institute for the award of degree or diploma.

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To my Parents

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Sunit Ghosh

Abstract

The main focus of this thesis is the study of Strichartz estimates and their extensions to systems of orthonormal functions associated with certain self-adjoint operators. In addition, we investigate a sharp Heisenberg-Pauli-Weyl uncertainty principle for the fractional Dunkl transform. We begin with the Fourier analysis on the Euclidean space, discuss some well known results, basic definitions, and and recent developments that motivate the problems discussed in the thesis.

We establish anisotropic Strichartz estimates associated with the Grushin operator $G = -\Delta - |x|^2\partial_t^2$ on $\mathbb{R}^{n+1}$. It is well known that the Grushin-Schrödinger equation is totally non-dispersive and hence the classical approach to obtain Strichartz estimates fails. Instead, we employ restriction estimates associated with the scaled Hermite–Fourier transform on $\mathbb{R}^{n+2}$ for certain surfaces in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$.

Let $\mathcal{L}$ be the special Hermite operator on $\mathbb{C}^n$. We establish Strichartz estimates for systems of orthonormal functions associated with general flows of the form $e^{-it\phi(\mathcal{L})}$, where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function. Our approach relies on restriction estimates for the Fourier-special Hermite transform on the class of surfaces $\{(\lambda, \mu, \nu) \in \mathbb{R} \times \mathbb{N}_0^n \times \mathbb{N}_0^n : \lambda = \phi(2|\nu|+n)\}$. We then address the optimality of the Schatten exponent and endpoint case of the orthonormal Strichartz estimate for the Schrödinger propagator $e^{-it\mathcal{L}}$. Furthermore, we investigate restriction estimates for the special Hermite spectral projections in the context of Schatten spaces.

Next, we derive a necessary condition on the Schatten exponent for the orthonormal Strichartz estimates for the Schrödinger equation associated to the Dunkl Laplacian and the Dunkl-Hermite operator, which turns out to be optimal for the Schrödinger equations associated with Laplacian and Hermite operator as a particular case. The proof uses coherent states in the Dunkl setting and semiclassical analysis.

Finally, we establish an L^p-type Heisenberg-Pauli-Weyl uncertainty principle for the fractional Dunkl transform, with $1 \leq p \leq 2$. For the case $p = 2$, we further derive a sharper uncertainty principle for the fractional Dunkl transform. Furthermore, we derive conditions leading to equality in both the uncertainty principles obtained.

Abbreviation and Notation

$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}$	The set of all integers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{R}^*$	$\mathbb{R}\backslash\{0\}$
$\mathbb{N}_0^n$	$\{(k_1, k_2, \ldots, k_n) \mid k_i \in \mathbb{Z}, i = 1, 2, \cdots, n\}$, $n \geq 1$
$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$, $n \geq 1$
$\mathbb{C}^n$	$\{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}, i = 1, 2, \cdots, n\}$, $n \geq 1$
$\operatorname{Re} z$	The real part of $z \in \mathbb{C}$
$\operatorname{Im} z$	The imaginary part of $z \in \mathbb{C}$
$\mathbb{S}^{n-1}$	The unit sphere in $\mathbb{R}^n$
χ_E	The characteristic function of the set E
$L^p(S)$	$\{f : S \to \mathbb{C} \mid f$ is measurable and $\int\limits_S \lvert f\rvert^p ds < \infty\}$
Δ	Laplacian on $\mathbb{R}^n$
G	Grushin operator on $\mathbb{R}^{n+1}$
$\mathcal{L}$	Special-Hermite operator on $\mathbb{C}^n$
$f * g$	Convolution of f and g

$f \times g$	Twisted convolution of f and g
A^*	The adjoint of the operator A
$\mathrm{Tr}(A)$	Trace of an (trace class) operator A defined on some Hilbert space

Contents

CHAPTER 1

Introduction

The main focus of this thesis is the study of Strichartz estimates and their extensions to systems of orthonormal functions associated with certain self-adjoint operators. In addition, we investigate a sharp Heisenberg-Pauli-Weyl uncertainty principle for the fractional Dunkl transform.

In this chapter, we introduce the basic definitions, notation, and preliminary concepts that will be used throughout the thesis. To motivate the work presented here, we also briefly outline the historical background and some key developments related to the topics under consideration, without aiming at a comprehensive survey.

1.1 Some definitions and basic results

In this section, we review some definitions and basic results, most of which are assumed to be well known to the reader.

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the Euclidean norm of x is denoted by $|x| = \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}}$. If $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ the open ball centered at x with radius r. The Lebesgue measure on $\mathbb{R}^n$ is denoted by dx. If E is a subset of $\mathbb{R}^n$, then $|E|$ denotes its Lebesgue measure and χ_E its characteristic function: $\chi_E(x) = 1$

if $x \in E$ and 0 otherwise.

Let X and Y be two measurable spaces with measures μ and ν, respectively. For $1 \leq p < \infty$, the space $L^p(X)$ denotes the Banach space of complex-valued measurable functions on X whose $p-$th powers are integrable; the norm of $f \in L^p(X)$ is

$$\|f\|_{L^p(X)} = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

The space $L^\infty(X)$ denotes the Banach space of essentially bounded complex-valued on X; more precisely, functions f such that for some $C > 0$, $\mu(\{x \in X : |f(x)| > C\}) = 0$. The norm $\|f\|_{L^\infty(X)}$ is defined as the infimum of all such constants C.

For $1 \leq p, q \leq \infty$, the mixed Lebesgue space $L^{p,q}(X \times Y) = L^p(X, L^q(Y))$ denotes the Banach space of complex-valued measurable functions on $X \times Y$ with the norm

$$\|f\|_{L^p(X,L^q(Y))} = \left(\int_X \left(\int_Y |f(x,y)|^q \, d\nu \right)^{p/q} d\mu \right)^{1/p},$$

with the usual modifications (as above) when $p = \infty$ or $q = \infty$. In the case $X = \mathbb{R}^n$ and $d\mu = dx$, we often write the mixed norm simply as $L^p_x L^q_y$. The conjugate exponent of p is always denoted by p' and satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

We also have Minkowski's integral inequality: if the measures μ and ν are σ-finite and $1 \leq q \leq p \leq \infty$, then for any measurable function $f : X \times Y \to \mathbb{C}$,

$$\|f\|_{L^p(X,L^q(Y))} \leq \|f\|_{L^q(Y,L^p(X))}.$$

The convolution of functions f and g on $\mathbb{R}^n$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \, dy.$$

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space, consisting of all functions $f \in C^\infty(\mathbb{R}^n)$ that decay rapidly, together with all derivatives, i.e.

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta f(x) \right| < \infty, \quad \forall \, \alpha, \beta \in \mathbb{N}_0^n, \tag{1.1.1}$$

where $x = (x_1, ..., x_n)$; $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$; $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; $\partial^\beta = \partial_1^\beta \cdots \partial_n^\beta$, with $\partial_j = \frac{\partial}{\partial x_j}$. The quantities appearing on the left-hand side of (1.1.1) define a countable collection of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ turning it into a Fréchet space.

Let $C_0(\mathbb{R}^n)$ denote the space of continuous functions that vanish at infinity. Then $\mathcal{S}(\mathbb{R}^n)$ is dense in both $C_0(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.

We denote by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions, defined as the space of continuous linear functionals on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

1.2 The Fourier transform

The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ (or $L^1(\mathbb{R}^n)$) is defined as

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^n. \tag{1.2.1}$$

It is well known that the operator $\mathcal{F}$ defines a linear topological isomorphism on $\mathcal{S}(\mathbb{R}^n)$ and the inverse is given by the inversion formula

$$(\mathcal{F}^{-1}\hat{f})(x) = f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi. \tag{1.2.2}$$

Moreover, it is clear that $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, it follows that $\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$, since $C_0(\mathbb{R}^n)$ is the L^∞-closure of $\mathcal{S}(\mathbb{R}^n)$. This inclusion is known as the Riemann–Lebesgue lemma.

The Fourier transform extends to a unitary operator on $L^2(\mathbb{R}^n)$, that is, $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$. This result is known as Plancherel's theorem and it follows from the following Parseval's formula

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)\overline{g(x)}dx\, d\xi = \langle \hat{f}, \hat{g}\rangle_{L^2},$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$. By the Riesz–Thorin interpolation theorem, one then obtains the Hausdorff–Young inequality which states that for $1 \le p \le 2$ we have

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{1.2.3}$$

where p' is the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

The inequality (1.2.3) allows us to extend the definition of the Fourier transform of functions $f \in L^p(\mathbb{R}^n)$ for $1 < p \le 2$ as an $L^{p'}(\mathbb{R}^n)$-limit. More precisely, if (f_j) is a sequence of Schwartz functions converging to f in $L^p(\mathbb{R}^n)$, then the Fourier transform of f can be viewed as $\widehat{f} := \lim_{n\to\infty} \widehat{f_j}$, in the norm of $L^{p'}(\mathbb{R}^n)$. Since the convergence is in $L^{p'}(\mathbb{R}^n)$, the Fourier transform is not defined on sets of Lebesgue measure zero.

1.3 The restriction problem

Let S be a hypersurface endowed with a smooth measure $d\sigma$. Then S is a subset of Lebesgue measure zero. The restriction problem raises the following question: for which exponents p, q with $1 \leq p \leq 2$, $1 \leq q \leq \infty$, is it true that $f \in L^p(\mathbb{R}^n)$ implies $\widehat{f}$ has a well-defined restriction to S in $L^q(S, d\sigma)$ with

$$\left(\int_S |\widehat{f}(\xi)|^q \, d\sigma(\xi)\right)^{\frac{1}{q}} \leq C_{p,q}\|f\|_{L^p(\mathbb{R}^n)} \ ? \tag{1.3.1}$$

The problem was originally considered by Stein in the late 1960s for the unit sphere $\mathbb{S}^{n-1}$.

When $p = 1$ and S is compact, the Riemann–Lebesgue lemma implies that $\widehat{f}$ is continuous and hence can always be restricted to S as a bounded function with (1.3.1) any $1 \leq q \leq \infty$. When $p > 1$, the question of restriction onto subsets of Lebesgue measure zero does not, a priori, make sense, since the Fourier transform of an L^p-function is defined as the $L^{p'}$-limit of Schwartz functions. In particular, if $f \in L^2(\mathbb{R}^n)$, then by the Plancherel theorem, $\widehat{f}$ may be any function in $L^2(\mathbb{R}^n)$, and hence there is no well-defined way to restrict $\widehat{f}$ to S.

However, for $1 < p < 2$, the difficulty that functions in $L^p(\mathbb{R}^n)$ are not defined pointwise on sets of measure zero can be avoided by working with Schwartz functions. If the restriction estimate (1.3.1) holds for all $f \in \mathcal{S}(\mathbb{R}^n)$, then by density and completeness arguments it extends to all functions in $L^p(\mathbb{R}^n)$.

Let $\mathcal{R}_S$ denote the restriction operator associated with $(S, d\sigma)$, defined by $\mathcal{R}_S = \widehat{f}|_S$, $f \in \mathcal{S}(\mathbb{R}^n)$. The operator dual to $\mathcal{R}_S$ is called the extension operator, denoted by $\mathcal{E}_S$, and satisfies the identity

$$\mathcal{E}_S f(x) = (2\pi)^{-\frac{n}{2}} \int_S f(\xi) e^{i\xi\cdot x} d\sigma(\xi), \quad x \in \mathbb{R}^n.$$

for all $f \in L^1(S, d\sigma)$. By duality, the restriction problem can be reformulated in terms of an extension problem: for which exponents p', q' with $2 \leq p' \leq \infty$, $1 \leq q' \leq \infty$, is it true that $f \in L^{q'}(S, \sigma)$ implies $\mathcal{E}_S f \in L^{p'}(\mathbb{R}^n)$ with

$$\|\mathcal{E}_S f\|_{L^{p'}(\mathbb{R}^n)} \leq C_{p',q'}\|f\|_{L^{q'}(S,d\sigma)} \ ? \tag{1.3.2}$$

For a general survey on these questions we refer to the book of Stein [100] and the text of Tao [103]. A model case of the restriction problem which is often considered in the literature is the case $q = 2$. Since $L^2(S, d\sigma)$ is a Hilbert space, we may compose the restriction and extension operators. This yields a third formulation of the restriction problem: for which exponents $1 \leq p \leq 2$, the operator $\mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, that is,

$$\|\mathcal{E}_S(\mathcal{E}_S)^* f\|_{L^{p'}(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \ ? \tag{1.3.3}$$

For smooth compact hypersurfaces with non-zero Gauss curvature, the celebrated Stein– Tomas theorem asserts that the following restriction estimate holds.

Theorem 1.3.1. *(Stein-Tomas) [100, 108] Let S be a smooth compact hypersurface in $\mathbb{R}^n$ with non vanishing Gaussian curvature at every point, and let $d\sigma$ be a smooth measure on S. Then for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq p \leq \frac{2(n+1)}{(n+3)}$, we have*

$$\|\mathcal{R}_S f\|_{L^2(S,d\sigma)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.3.4}$$

The sharpness of this result was proved with a counterexample due to A. Knapp [103]. For quadratic surfaces, a complete characterization was obtained by Strichartz [102], with admissible range of exponents depending on the type of surface, such as the paraboloid, cone, or sphere. The restriction problem has attracted a considerable attention over the years and is closely connected to several challenging problems in harmonic analysis and Partial Differential Equations (PDEs), such as the Bochner-Riesz means, Strichartz estimates, and the Kakeya conjecture, and the local smoothing conjecture [35, 104, 112].

Restriction problems are also closely connected to several phenomena in spectral theory. Let $S = \mathbb{S}^{n-1}$ be the unit sphere and let $\sigma_{\mathbb{S}^{n-1}}$ be the associated surface measure. For $f \in \mathcal{S}(\mathbb{R}^n)$, the Fourier inversion formula, written in polar coordinates, takes the form

$$f(x) = \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\lambda x\cdot\omega} \widehat{f}(\lambda\omega)\, d\sigma_{\mathbb{S}^{n-1}}(\omega)\, \lambda^{n-1}\, d\lambda.$$

The inner integral can be interpreted as a convolution with a Bessel kernel. More precisely, define

$$Q_\lambda f(x) = \int_{\mathbb{S}^{n-1}} \hat{f}(\lambda\omega)\, d\sigma_{\mathbb{S}^{n-1}}(\omega) = f * \varphi_\lambda(x)$$

where $\varphi_\lambda(x) = (2\pi)^{-\frac{n}{2}} |\lambda x|^{-\frac{n}{2}+1} J_{\frac{n}{2}-1}(\lambda|x|)$, and $J_{\frac{n}{2}-1}$ denotes the Bessel function of order $\frac{n}{2} - 1$. Then $Q_\lambda f$ is an eigenfunction of the Laplacian Δ with eigenvalue $-\lambda^2$. In order to prove the restriction estimate (1.3.3) for $\mathbb{S}^{n-1}$, it is equivalent to show that

$$\|Q_\lambda f\|_{L^{p'}(\mathbb{R}^n)} \leq C_\lambda \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p \leq \frac{2(n+1)}{n+3}. \tag{1.3.5}$$

It is therefore natural to study analogues of $Q_\lambda f$ for a more general positive differential operator L in place of $-\Delta$, which allows the Fourier restriction problem to be formulated as a spectral problem in broader settings. On compact Riemannian manifolds, spectral restriction problems (or cluster estimates) for Laplace–Beltrami operators were established by Sogge in [96, 97]. In the non-compact setting, restriction problems for Hermite and special Hermite projection operators have been investigated in several works [73, 89, 101]. For the sublaplacian on the Heisenberg group we refer to [85].

Restriction problems analogous to quadratic surfaces (such as paraboloid and cone) associated to general positive differential operators L in connection with PDEs have been extensively studied in the literature. Such results have been established in a variety of settings, including compact manifolds [87, 110], the Hermite operator [83], the Heisenberg group [10], and H-type groups [12].

In this thesis, we study several restriction problems associated to the Grushin operator on $\mathbb{R}^{n+1}$ and the special Hermite operator on $\mathbb{C}^n$.

1.4 Strichartz estimates

Strichartz estimates date back to the 1970s, originating from the seminal work of Strichartz [102]. Over the subsequent decades, Strichartz estimates for linear dispersive evolution equations, such as the Schrödinger and wave equations, have become a central tool in the analysis of semilinear and quasilinear partial differential equations, arising in many physical applications.

In this subsection, we briefly describe the different strategies used to establish Strichartz estimates, which may be viewed as another manifestation of the Stein-Tomas restriction problem. Consider the free Schrödinger equation on $\mathbb{R}^n$,

$$\begin{aligned} i\partial_s u(t,x) - \Delta u(t,x) &= 0, \quad x \in \mathbb{R}^n,\ s \in \mathbb{R}\setminus\{0\}, \\ u(0,x) &= f(x). \end{aligned} \tag{1.4.1}$$

This equation was introduced by Schrödinger in 1925 in the context of quantum mechanics. It describes the time evolution of a quantum system: given the state of the system at an initial time, the Schrödinger equation determines its state at all subsequent times.

If $f \in L^2(\mathbb{R}^n)$, the solution can be written explicitly as

$$u(t,x) = e^{it\Delta} f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)}\, \widehat{f}(\xi)\, d\xi. \tag{1.4.2}$$

Formula (1.4.2) can be interpreted as the restriction of the Fourier transform on the paraboloid in $\mathbb{R}^{n+1}$, defined by $S := \{(w,\xi) \in \mathbb{R}\times\mathbb{R}^n : w = |\xi|^2\}$ with the measure $d\sigma = d\xi$ induced by the projection $\pi : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ onto the second factor.

Given $f : \mathbb{R}^n \to \mathbb{C}$, define $F : S \to \mathbb{C}$ by $F = \widehat{f}\circ \pi\big|_S$, that is, $F(|\xi|^2,\xi) = \widehat{f}(\xi)$. Clearly, one has $\|f\|_{L^2(\mathbb{R}^n)} = \|F\|_{L^2(S,d\mu)}$. Then

$$e^{it\Delta} f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{f}(\xi)\, d\xi = \int_S e^{i(t,x)\cdot(w,\xi)} F(w,\xi)\, d\sigma(w,\xi).$$

Assume now that $\widehat{f}$ is supported in a unit ball. This assumption translates into a compact support condition on the surface measure $d\sigma$. Then, by the Stein-Tomas theorem, we get

$$\|e^{it\Delta} f\|_{L^{p'}(\mathbb{R}^{n+1})} \le C\|f\|_{L^2(\mathbb{R}^n)}, \tag{1.4.3}$$

for all $p' \ge \frac{2(n+2)}{n}$. A scaling argument, together with the density of spectrally localized functions in $L^2(\mathbb{R}^n)$, yields that for $p' = \frac{2n+4}{n}$, and for all $f \in L^2(\mathbb{R}^n)$, one has

$$\|e^{it\Delta} f\|_{L^{\frac{2n+4}{n}}\left(\mathbb{R}, L^{\frac{2n+4}{n}}(\mathbb{R}^n)\right)} \le C\|f\|_{L^2(\mathbb{R}^n)}. \tag{1.4.4}$$

This estimate was first obtained by Strichartz [102] as a restriction theorem for paraboloid. The solution (1.4.2) can also be written as

$$e^{it\Delta} f(x) = \frac{e^{i\frac{|\cdot|^2}{4s}}}{(4\pi i s)^{\frac{n}{2}}} * f(x). \tag{1.4.5}$$

Applying Young's convolution inequality to (1.4.5) implies that this solution satisfies the so-called dispersive estimate

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{(4\pi|t|)^{\frac{n}{2}}}\|f\|_{L^1(\mathbb{R}^n)}, \quad \forall t \neq 0. \tag{1.4.6}$$

In the late 1990s, Ginibre and Velo [59] and later Keel and Tao [70] for the endpoint case, used dispersive estimates (1.4.6) together with the TT^* method to extend Strichartz estimate (1.4.4) to a wider range of exponents. Using this approach, the Strichartz estimate in the mixed-norm setting can be stated as follows:

Theorem 1.4.1 (Strichartz estimates)**.** *Let* $n \geq 1$*. If* $p, q \geq 1$ *satisfy* $(p, q, n) \neq (1, \infty, 2)$ *and* $\frac{2}{p} + \frac{n}{q} = n$*, then* $e^{it\Delta} f \in L_t^{2p} L_x^{2q} (\mathbb{R} \times \mathbb{R}^n)$ *and satisfies the estimate*

$$\|e^{it\Delta} f\|_{L^{2p}(\mathbb{R}, L^{2q}(\mathbb{R}^n))} \leq C\|f\|_{L^2(\mathbb{R}^n)}. \tag{1.4.7}$$

In the case of the wave equation on $\mathbb{R}^n$

$$\begin{aligned} &i\partial_{tt} u(t,x) - \Delta u(t,x) = 0, \quad x \in \mathbb{R}^n, \, t \in \mathbb{R} \setminus \{0\}, \\ &u(0,x) = \phi(x), \quad \partial_t u(0,x) = \psi(x). \end{aligned} \tag{1.4.8}$$

the solution can be expressed as $u = u_+ + u_-$, where $u_\pm = e^{\pm it\sqrt{\Delta}} f_\pm$ and $f_\pm$ satisfy $(f_+ + f_-, i\sqrt{\Delta}(f_+ - f_-)) = (\phi, \psi)$. Consequently, the Strichartz estimates for the wave equation are usually given by those for the one-sided propagator.

Theorem 1.4.2. *Let* $n \geq 2$*. If* $p, q \geq 1$ *satisfying* $(p, q, n) \neq (1, \infty, 2)$ *and* $\frac{2}{p} + \frac{n-1}{q} = n-1$*, then* $e^{it\sqrt{\Delta}} f \in L_t^{2p} L_x^{2q} (\mathbb{R} \times \mathbb{R}^n)$ *and satisfies the inequality*

$$\|e^{it\sqrt{\Delta}} f\|_{L^{2p}(\mathbb{R}, L^{2q}(\mathbb{R}^n))} \leq C\|f\|_{\dot{H}^s}, \quad s = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{2q}\right). \tag{1.4.9}$$

Here $\dot{H}^s$ denotes the homogeneous Sobolev space. The estimate (1.4.9) is basically corresponds to the Stein–Tomas adjoint restriction estimate for the cone. Strichartz estimates have also been studied in many other settings. For results concerning more general dispersive semigroups, we refer to Keel and Tao [70].

Dispersion may fail or become significantly weaker in certain settings. For instance, for the wave equation on the Heisenberg group, where dispersive estimates hold with the optimal decay rate $|t|^{-1/2}$ independent of the dimension (see [9]). Similar phenomena occur on compact Riemannian manifolds and on some bounded domains. An even more striking situation was pointed out by Bahouri, Gérard and Xu [9] for the Schrödinger operator on the Heisenberg group, where it is shown that dispersion fails completely. In such cases the Euclidean approach described above is no longer effective, and then establishing Strichartz estimates becomes substantially more delicate.

Despite these difficulties, Strichartz estimates on compact Riemannian manifolds and bounded domains, often involving a loss of derivatives, have been obtained in several works; see, for instance, Bourgain [23], Burq, Gérard, and Tzvetkov [25], Ivanovici, Lebeau, and Planchon [66], and the references therein. For results on the Heisenberg group, see [8]. The case of hyperbolic space, which is noncompact and has negative curvature, has also been investigated; see [3].

It is well known that the Grushin-Schrödinger equation is an example of totally non-dispersive equation [54]. In this thesis, we establish anisotropic Strichartz estimates associated with the Grushin-Schrödinger and Grushin wave equation. This is achieved using a Stein–Tomas type restriction approach adapted to the Grushin operator setting.

1.5 Strichartz estimates for orthonormal system of functions

Strichartz estimate (1.4.7) has been substantially generalized for a system of orthonormal functions in the works of Frank-Lewin-Lieb-Seiringer [48] and Frank-Sabin [49]. We briefly recall the formulation of Strichartz estimates for orthonormal system of functions and summarize their results. Let $p, q \geq 1$ and $\beta \geq 1$, consider the estimate

$$\left\| \sum_j n_j \left| e^{it\Delta} f_j \right|^2 \right\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C_{n,q} \left(\sum_j |n_j|^\beta \right)^{\frac{1}{\beta}}, \tag{1.5.1}$$

for any orthonormal system $(f_j)_j$ in $L^2(\mathbb{R}^n)$ and all sequence $(n_j)_j \subset \mathbb{C}$. Clearly, the case $\beta = 1$ follows directly from the triangle inequality combined with the Strichartz estimate (1.4.7), without using the orthonormal hypothesis. Hence, in view of the inclusion relation of ℓ^β space, the problem is to determine the largest possible exponent β for which the estimate (1.5.1) holds for a given pair p, q. We call this exponent β the Schatten exponent of (1.5.1), as it is connected via duality to Schatten norm estimates.

Theorem 1.5.1. [48, 49] *Let $n \geq 1$. Assume that $p, q \geq 1$ satisfy $\frac{2}{p} + \frac{n}{q} = n$.*

(i) If $1 \leq q < \frac{n+1}{n-1}$, then the estimate (1.5.1) holds for any orthonormal system $(f_j)_j$ in $L^2(\mathbb{R}^n)$ and all sequence $(n_j)_j \subset \mathbb{C}$ whenever $\beta \leq \frac{2q}{q+1}$. This is sharp in the sense that, the estimate fails for all $\beta > \frac{2q}{q+1}$. Further, if $q = \frac{n+1}{n-1}$ the estimate (1.5.1) holds for all $\beta < \frac{2q}{q+1}$ and fails at $\beta = \frac{2q}{q+1}$.

(ii) If $d = 2$ and $6 \leq q < \infty$, or if $d \geq 3$ and $\frac{2(d+1)}{d-1} \leq q \leq \frac{2d}{d-2}$, then the estimate (1.5.1) holds for all $\beta < p$. This estimate is sharp in the sense that it fails for $\beta > p$.

The motivation for studying estimates of the form (1.5.1) arises in the context of many-body quantum mechanics. A system of N independent fermions is described by a collection of N orthonormal functions $f_1, \dots, f_N$ in L^2. Consequently, functional inequalities that incorporate a significant number of orthonormal functions are highly valuable for the mathematical analysis of large-scale quantum systems. The inequalities have applications to the Hartree equation modeling infinitely many fermions in a quantum system, see Lewin-Sabin [74, 75], Frank-Sabin [49]. The first fundamental work of such generalization goes back to the famous work due to Lieb-Thirring [77, 78], where the Gagliardo-Nirenberg-Sobolev inequality was generalized to the orthonormal inequality, the so-called Lieb-Thirring's inequality.

It is also important to study the nature of the Schatten exponent in orthonormal estimates. In particular, the orthonormal estimates with optimal Schatten exponent played a crucial role to prove the stability of matter [77, 78, 93]. Also the sharp orthonormal Strichartz estimate as in Theorem 1.5.1 was employed crucially to establish well-posedness and the scattering theory for the Hartree equations [74, 75].

For the wave equation, Frank–Sabin [49] obtained a substantial generalization of (1.4.9) to families of orthonormal functions $(f_j)_j$ in the homogeneous Sobolev space $\dot{H}^s$.

Theorem 1.5.2. *[49] Let $n \geq 2$. Assume that $p, q \geq 1$ satisfies $1 \leq q \leq \frac{n+1}{n-1}$ and $\frac{2}{p} + \frac{n-1}{q} = n - 1$. Then the following estimate*

$$\left\| \sum_j n_j \left| e^{it\sqrt{\Delta}} f_j \right|^2 \right\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^n))} \leq C_{n,q} \left(\sum_j |n_j|^\beta \right)^{\frac{1}{\beta}}, \tag{1.5.2}$$

holds for all orthonormal system $(f_j)_j$ in $\dot{H}^s$ with $s = \frac{n+1}{2}(\frac{1}{2} - \frac{1}{2q})$ and all sequence $(n_j)_j \subset \mathbb{C}$, whenever $1 \leq \beta \leq \frac{2q}{q+1}$.

More recent developments, including orthonormal Strichartz estimates associated with the Klein–Gordon equation and the fractional Schrödinger equation, can be found in [20, 49].

Another interesting development is the extension of dual Stein-Tomas inequality (1.3.4) to system of orthonormal system by Frank-Sabin [49].

Theorem 1.5.3. *[49, 51] Let $n \geq 2$ and $S \subset \mathbb{R}^n$ a compact hypersurface with non-vanishing Gauss curvature. Then, for any $\frac{n+1}{n-1} \leq p \leq \infty$ and $1 \leq \beta \leq \frac{p(n-1)}{2n}$, the* *following estimate*

$$\left\| \sum_j n_j \left|\mathcal{E}_S f_j\right|^2 \right\|_{L^p(\mathbb{R}^n)} \leq C_{n,q} \left(\sum_j |n_j|^\beta \right)^{\frac{1}{\beta}}, \tag{1.5.3}$$

holds for all orthonormal system $(f_j)_j$ *in* $L^2(S, d\sigma)$ *and all sequence* $(n_j)_j \subset \mathbb{C}$*. Moreover, this is sharp in the sense that, the estimate fails for all* $\beta > \frac{p(n-1)}{2n}$*.*

Estimates of this type for systems of orthonormal functions have subsequently been studied in various other settings. Notably, Frank and Sabin extended Sogge's L^p spectral cluster estimates for the Laplace–Beltrami operator on compact Riemannian manifolds to the orthonormal setting [50]; see also the related work of Nguyen [88].

Strichartz estimates for orthonormal systems associated with the Schrödinger propagator associated with several self-adjoint operators have been obtained in several works. For instance, we refer to Mondal–Swain [83] for Hermite operator, Ghosh-Mondal-Swain [57] for special Hermite operator, Feng–Song [44] for the Laguerre operator, Mondal–Song [82] for (k, a)-generalized Laguerre operators, and Nakamura [87], Wang-Zhang-Zhang [110] for Laplace Beltrami operator on compact manifolds. For more general results covering a wide class of dispersive equations see Hoshiya [65], Feng-Mondal-Song-Wu [43].

In this thesis, we establish Strichartz estimates for systems of orthonormal functions associated with general flows of the form $e^{-it\phi(\mathcal{L})}$, where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function. We then address the optimality of the Schatten exponent and endpoint case of the orthonormal Strichartz estimate for the Schrödinger propagator $e^{-it\mathcal{L}}$. Furthermore, we generalize restriction estimates for the special Hermite spectral projections in the context of Schatten spaces.

Finally, we derive a necessary condition on the Schatten exponent for the the orthonormal Strichartz estimates for the Schrödinger equation associated to the Dunkl Laplacian and the Dunkl-Hermite operator, which turns out to be optimal for the Schrödinger equations associated with Laplacian and Hermite operator as a particular case.

1.6 Heisenberg–Pauli–Weyl uncertainty principle

The phrase "uncertainty principle" refers to a wide class of theorems, all of which express the idea that a nonzero function and its Fourier transform cannot both be sharply localized. This phenomenon is among the most famous and exciting research areas at the interface of mathematics and physics, and has attracted continuous attention for nearly a century. We refer to Folland and Sitaram [46] for a detailed survey.

The study of uncertainty principles originated with Heisenberg's seminal work in 1927 [63], and the mathematical formulation was later developed independently by Kennard [71] and Weyl [111]. In its classical form, the principle states that for $f \in L^2(\mathbb{R})$ with $\|f\|_{L^2(\mathbb{R})} = 1$,

$$\Delta^2_{f,2}\Delta^2_{\hat{f},2} \geq \frac{1}{4}, \tag{1.6.1}$$

where $\Delta^2_{f,2} = \int_{-\infty}^{+\infty} |(x-(x)_f)f(x)|^2 dx, (x)_f = \int_{-\infty}^{+\infty} x|f(x)|^2 dx$. This formulation is popularly known as the Heisenberg–Pauli–Weyl uncertainty principle (HPWUP).

In 1946, Gabor [52] introduced the HPWUP in the context of signal analysis. Since then, numerous different forms of the HPWUP have emerged through various mathematical formulations [41, 62]. It has significant applications in the field of signal processing, optics and so on (see [36, 113, 114, 116, 117]). Although inequality (1.6.1) is the simplest form, it is not the most refined. Subsequently, Cohen [31, 32] established a lower bound sharper than (1.6.1):

$$\Delta^2_{f,2}\Delta^2_{\hat{f},2} \geq \frac{1}{4} + Cov^2(f), \tag{1.6.2}$$

where $Cov(f) = \int_{-\infty}^{+\infty}(x-(x)_f)(\varphi'(x)-(x)_{\hat{f}})|f(x)|^2 dx$ is the covariance of f, defined in [32], where, as usual, $f(x) = \rho(x)e^{i\varphi(x)}$, $\rho(x)$ is a real-valued function and $\varphi'(x)$ is the classical derivative of $\varphi(x)$. Later, Dang-Deng-Qian [36] provided a stronger result, that is

$$\Delta^2_{f,2}\Delta^2_{\hat{f},2} \geq \frac{1}{4} + COV^2(f), \tag{1.6.3}$$

where $COV(f) = \int_{-\infty}^{+\infty} |(x-(x)_f)(\varphi'(x)-(x)_{\hat{f}})||f(x)|^2 dx$ is the absolute covariance of f. Note that the lower bound in (1.6.3) is bigger than that in (1.6.2). An interesting extension was provided by Cowling-Price [33], who refined the classical HPWUP for L^p-type functions with $1 \leq p \leq 2$, yielding the following inequality

$$\Delta^2_{f,p}\Delta^2_{\hat{f},p} \geq \frac{1}{4}, \tag{1.6.4}$$

where $\Delta^p_{f,p} = \int_{-\infty}^{+\infty} |(x-(x)_f)f(x)|^p dx$, is the $p-$th covariance. Later, Zhang [115] made further improvements to estimate (1.6.4):

$$\Delta^2_{f,p}\Delta^2_{\hat{f},p} \geq \frac{1}{4} + COV^2(f). \tag{1.6.5}$$

The fractional Fourier transform is an extension of the classical Fourier Transform. We refer to [72, 86, 114, 116] for its properties and applications. The fractional Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}^\alpha f(\omega) = \begin{cases} A_\alpha \int\limits_{-\infty}^{+\infty} e^{\frac{i}{2}(x^2+\omega^2)\cot\alpha} e^{-i\omega x \csc\alpha} f(x)dx, & (2n-1)\pi < x < (2n+1)\pi, \\ f(x), & x = 2n\pi, \\ f(-x), & x = (2n+1)\pi, \end{cases}$$

where $A_\alpha = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$. For $\alpha \notin \pi\mathbb{Z}$, we have the following relation

$$\mathcal{F}^\alpha f(\omega) = \sqrt{1-i\cot\alpha}\, e^{\frac{i}{2}u^2\cot\alpha}\, \mathcal{F}\Big(e^{\frac{i}{2}x^2\cot\alpha} f(x)\Big)(u\csc\alpha)\,.$$

In the literature, many extensions of the HPWUP have been established for the fractional Fourier transform. Unlike the classical Fourier transform, the HPWUP for the fractional Fourier transform is formulated between two fractional Fourier domains of different orders (see [116]). In particular, one has

$$\Delta^2_{\mathcal{F}^\alpha f,\,2}\,\Delta^2_{\mathcal{F}^\beta f,\,2} \geq \frac{\sin^2(\alpha-\beta)}{4}, \tag{1.6.6}$$

where $\Delta^2_{\mathcal{F}^\alpha f,2} = \int_{-\infty}^{+\infty} \left|\big(x-(x)_{\mathcal{F}^\alpha f}\big)\mathcal{F}^\alpha f(x)\right|^2 dx$, $(x)_{\mathcal{F}^\alpha f} = \int_{-\infty}^{+\infty} x\,|\mathcal{F}^\alpha f(x)|^2\,dx$. Furthermore, L^p-type versions of HPWUP for the fractional Fourier transform, with $1 \leq p \leq 2$, were established in [6]. These results were later refined in [36] and [37], where sharper lower bounds were obtained.

The Dunkl transform and the Fractional Dunkl Transform generalize the Fourier transform and the Fractional Fourier Transform, respectively. The HPWUP for the Dunkl transform and the Fractional Dunkl Transform are introduced by Rösler-Voit [92] and Ghazouani-Bouzeffour [56], respectively and later improved in [42]. In this thesis, we establish an L^p-type Heisenberg-Pauli-Weyl uncertainty principle for the fractional Dunkl transform, with $1 \leq p \leq 2$. For the case $p = 2$, we further derive a sharper uncertainty principle for the fractional Dunkl transform. Moreover, we derive equality conditions for both the uncertainty principles obtained, thereby strengthening previously known results.

1.7 Outline of the thesis

This thesis consists of five chapters. The present chapter deals with the basic definitions, review of recent developments, and our motivation to consider the problems discussed in the thesis.

In Chapter 2, we consider the Schrödinger and wave equation associated with the Grushin operator $G = -\Delta - |x|^2\partial_t^2$ on $\mathbb{R}^{n+1}$. Since the associated Grushin–Schrödinger equation is totally non-dispersive, the classical dispersive approach to Strichartz estimates does not apply. We prove a restriction theorem with respect to the scaled Hermite–Fourier transform on $\mathbb{R}^{n+2}$ for suitable surfaces in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$, and as an application derive anisotropic Strichartz estimates for the Grushin–Schrödinger equation and the Grushin wave equation.

In Chapter 3, we consider the special Hermite operator $\mathcal{L}$ on $\mathbb{C}^n$. We establish new orthonormal Strichartz estimates for systems of orthonormal functions associated with general flows of the form $e^{-it\phi(\mathcal{L})}$, where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function. Our approach relies on restriction estimates for the Fourier–special Hermite transform on suitable spectral surfaces, and we also discuss the endpoint case for the Schrödinger propagator $e^{-it\mathcal{L}}$. We also investigate restriction estimates for special Hermite spectral projections in the framework of trace ideals (Schatten classes).

In Chapter 4, we study orthonormal Strichartz estimates for Schrödinger equations associated with the Dunkl Laplacian and the Dunkl–Hermite operator. By constructing coherent states in the Dunkl setting and applying semi-classical analysis, we derive a necessary condition on the Schatten exponent, which is shown to be optimal and recovers the classical Laplacian and Hermite cases as special instances.

In Chapter 5, we investigate uncertainty principles for the fractional Dunkl transform. We first obtain an explicit characterization of the functions for which equality holds in known uncertainty inequalities, and then establish an L^p-type Heisenberg–Pauli–Weyl uncertainty principle for $1 \leq p \leq 2$. In the case $p = 2$, we derive a sharper uncertainty principle and characterize the corresponding equality cases.

CHAPTER 2

Strichartz estimates associated with the Grushin operator

2.1 Introduction

The Grushin operator G on $\mathbb{R}^{n+1}$ is defined by

$$G = -\Delta - |x|^2 \partial_t^2, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. The study of the Grushin operator dates back to the work of Baouendi and Grushin [11, 60, 61]. Since then, several authors have studied this operator extensively in various contexts, including the classification of solutions to elliptic equations, free boundary problems in partial differential equations, and well-posedness issues in Sobolev spaces; see, for example, [2, 34, 53, 69].

In this chapter, we establish anisotropic Strichartz estimates associated with the Grushin-Schrödinger and Grushin wave equations.

2.2 Preliminaries

In this section, we discuss the spectral theory for the Grushin operator and the Fourier analysis tools associated with it.

2.2.1 The Grushin operator and its spectral theory:

Let H_k denote the Hermite polynomial on $\mathbb{R}$, defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}, \quad k = 0, 1, 2, \cdots ,$$

and h_k denote the normalized Hermite functions on $\mathbb{R}$ defined by

$$h_k(x) = (2^k\sqrt{\pi}k!)^{-\frac{1}{2}} H_k(x) e^{-\frac{1}{2}x^2}, \quad k = 0, 1, 2, \cdots .$$

The higher dimensional Hermite functions denoted by Φ_α are then obtained by taking tensor product of one dimensional Hermite functions. Thus for any multi-index $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we define $\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$. For $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the scaled Hermite functions are defined by $\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}} \; \Phi_\alpha(\sqrt{|\lambda|}x)$, they are the eigenfunctions of the (scaled) Hermite operator $H(\lambda) = -\Delta + \lambda^2|x|^2$ with eigenvalues $(2|\alpha| + n)|\lambda|$, where $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha \in \mathbb{N}_0^n$. For each $\lambda \in \mathbb{R}^*$, the family $\{\Phi_\alpha^\lambda : \alpha \in \mathbb{N}_0^n\}$ is then an orthonormal basis for $L^2(\mathbb{R}^n)$. For each $k \in \mathbb{N}_0$, let $P_k(\lambda)$ stand for the orthogonal projection of $L^2\left(\mathbb{R}^n\right)$ onto the eigenspace of $H(\lambda)$ spanned by $\left\{\Phi_\alpha^\lambda : |\alpha| = k\right\}$. More precisely, for $f \in L^2(\mathbb{R}^n)$

$$P_k(\lambda)f = \sum_{|\alpha|=k} \langle f, \Phi_\alpha^\lambda\rangle \, \Phi_\alpha^\lambda \, , \tag{2.2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $L^2(\mathbb{R}^n)$. Then the spectral decomposition of $H(\lambda)$ is explicitly given as

$$H(\lambda)f = \sum_{k=0}^{\infty}(2k+n)|\lambda|P_k(\lambda)f. \tag{2.2.2}$$

Note that

$$P_k(\lambda)f(x) = P_k(1)(f \circ d_{|\lambda|^{-\frac{1}{2}}}) \circ d_{|\lambda|^{\frac{1}{2}}}(x), \tag{2.2.3}$$

where the dilations d_r on $\mathbb{R}^n$ are defined by $d_r(x) = rx$ for $r > 0$.

For a Schwartz function f on $\mathbb{R}^{n+1}$, let $f^\lambda(x) = \int_{\mathbb{R}} f(x,t)e^{i\lambda t}dt$ denote the inverse Fourier transform of $f(x,t)$ in the t variable. Applying the operator G to the Fourier expansion $f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t} f^\lambda(x)d\lambda$, we see that

$$Gf(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^*} e^{-i\lambda t} H(\lambda) f^\lambda(x) d\lambda. \tag{2.2.4}$$

The Grushin operator belongs to the wide class of subelliptic operators studied by Franchi et al. in [47]. Moreover, it is positive, self-adjoint, and hypoelliptic. The operator G possesses a natural family of anisotropic dilations, namely

$$\delta_r(x,t) = (rx, r^2 t) \quad \text{for } r > 0. \tag{2.2.5}$$

and this anisotropic dilation structure introduces homogeneous norm on $\mathbb{R}^{n+1}$ $\rho := \rho(x,t) = (\sum_{i=1}^{n} |x_i|^4 + t^2)^{\frac{1}{4}}$. With the norm ρ, we define the ball centered at $w_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and of radius $R \geq 0$ by $B(w_0, R) = \{(x,t) \in \mathbb{R}^{n+1} : \rho(x - x_0, t - t_0) < R\}$.

Using (2.2.2), the spectral decomposition of the Grushin operator is given by

$$Gf(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda t} \left(\sum_{k=0}^{\infty} (2k+n)|\lambda| P_k(\lambda) f^{\lambda}(x) \right) d\lambda. \tag{2.2.6}$$

We refer to [84] and the references therein for detailed information about the Grushin operator.

2.2.2 The scaled Hermite-Fourier transform on $\mathbb{R}^{n+1}$:

For a reasonable function f, the scaled Fourier-Hermite transform is defined by

$$\hat{f}(\alpha, \lambda) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{i\lambda t} f(x,t) \Phi_{\alpha}^{\lambda}(x) dt dx = \langle f^{\lambda}, \Phi_{\alpha}^{\lambda} \rangle, \quad (\alpha, \lambda) \in \mathbb{N}_0^n \times \mathbb{R}^*. \tag{2.2.7}$$

If $f \in L^2(\mathbb{R}^{n+1})$ then $\hat{f} \in L^2(\mathbb{N}_0^n \times \mathbb{R}^*)$ and satisfies the Plancherel formula

$$\|f\|_{L^2(\mathbb{R}^{n+1})} = \frac{1}{2\pi} \|\hat{f}\|_{L^2(\mathbb{N}_0^n \times \mathbb{R}^*)}. \tag{2.2.8}$$

The inversion formula is given by

$$f(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda t} \sum_{\alpha \in \mathbb{N}_0^n} \hat{f}(\alpha, \lambda)\, \Phi_{\alpha}^{\lambda}(x)\, d\lambda. \tag{2.2.9}$$

If $f \in L^1(\mathbb{R}^{n+1})$, it can be seen that for $r > 0$,

$$\widehat{(f \circ \delta_r)}(\alpha, \lambda) = r^{-(\frac{n}{2}+1)} \hat{f}(\alpha, r^{-2}\lambda), \tag{2.2.10}$$

where δ_r is defined in (2.2.5).

Replacing f by Gf in (2.2.9) and comparing (2.2.7) with (2.2.6), we get

$$\widehat{(Gf)}(\alpha, \lambda) = (2|\alpha| + n)|\lambda|\, \hat{f}(\alpha, \lambda), \quad (\alpha, \lambda) \in \mathbb{N}_0^n \times \mathbb{R}^*. \tag{2.2.11}$$

2.2.3 A frequency decomposition adapted to the Grushin operator

Definition 2.2.1. *A function* $f \in \mathcal{S}(\mathbb{R}^{n+1})$ *is said to be frequency localized in a ball* $\mathcal{B}_R$*, centered at* 0 *of radius* R *if there exists a smooth, even function* ψ *supported in* $\{\tau \in \mathbb{R} : |\tau| \leq 2\}$ *and equal to* 1 *near* 0 *such that*

$$f = \psi(\, R^{-2}\, G\,)\, f, \tag{2.2.12}$$

which is equivalent to saying that for all $(\alpha, \lambda) \in \mathbb{N}_0^n \times \mathbb{R}^*$,

$$\hat{f}(\alpha, \lambda) = \psi(R^{-2}(2|\alpha| + n)|\lambda|)\hat{f}(\alpha, \lambda). \tag{2.2.13}$$

By construction it is clear that collection of frequency localized functions of all radii is dense in $L^2(\mathbb{R}^{n+1})$.

Definition 2.2.2. *A function* $f \in \mathcal{S}(\mathbb{R}^{n+1})$ *is said to be frequency localized in a ring* $\mathcal{C}_R$*, centered at* 0 *of inner radius* $R/2$ *and outer radius* R *if there exists a smooth, even function* ψ *supported in* $\{\tau \in \mathbb{R} : 1/2 \leq |\tau| \leq 2\}$ *and equal to* 1 *in a ring* $\mathcal{C}' \subset \{\tau \in \mathbb{R} : 1/2 \leq |\tau| \leq 2\}$ *such that*

$$f = \psi(\, R^{-2}\, G\,)\, f, \tag{2.2.14}$$

which is equivalent to saying that for all $(\alpha, \lambda) \in \mathbb{N}_0^n \times \mathbb{R}^*$,

$$\hat{f}(\alpha, \lambda) = \psi(R^{-2}(2|\alpha| + n)|\lambda|)\hat{f}(\alpha, \lambda). \tag{2.2.15}$$

Suppose f is frequency localized in a ring $\mathcal{C}_R$. Then, for any $s \in \mathbb{R}$, it follows from (2.2.12) and (2.2.11) that

$$C_1 R^s \|f\|_{L^2(\mathbb{R}^{n+1})} \leq \|G^{s/2} f\|_{L^2(\mathbb{R}^{n+1})} \leq C_2 R^s \|f\|_{L^2(\mathbb{R}^{n+1})}. \tag{2.2.16}$$

Let $\tilde{\psi} \in C^\infty(\mathbb{R})$ be an even function such that $0 \leq \tilde{\psi} \leq 1$, $\tilde{\psi} = 1$ in $[0, 1]$ and 0 in $[2, \infty)$. Let $\psi(\tau) = \tilde{\psi}(\tau) - \tilde{\psi}(2\tau)$ such that *supp* $\psi \subset (1/2, 2)$ and generate a Littlewood-Paley decomposition

$$\tilde{\psi}(\tau) + \sum_{j \geq 0} \psi(2^{-j}\tau) = 1, \;\; \forall \tau \in \mathbb{R}.$$

Now define

$$\Delta_{-1} f = \tilde{\psi}(G) f \quad \text{and} \quad \Delta_j f = \psi(2^{-j} G) f, \; j \geq 0. \tag{2.2.17}$$

Then $\Delta_{-1}f$ is frequency localized in ball $\mathcal{B}_1$ and $\Delta_j f$ are frequency localized in ring $\mathcal{C}_{2^{j/2}}$, $j \geq 0$. Moreover, any $f \in \mathcal{S}(\mathbb{R}^{n+1})$ admits the frequency decomposition

$$f = \sum_{j \geq -1} \Delta_j f \tag{2.2.18}$$

and

$$\|f\|^2_{L^2(\mathbb{R}^{n+1})} \sim \sum_{j \geq -1} \|\Delta_j f\|^2_{L^2(\mathbb{R}^{n+1})}. \tag{2.2.19}$$

Here, $A \sim B$ means that there exists a constants $C_1, C_2 > 0$ such that $A \leq C_1 B$ and $B \leq C_2 A$.

Next, we consider the Grushin Sobolev space $H^s_G = \{f \in L^2(\mathbb{R}^{n+1}) : \int_{\mathbb{R}^{n+1}} |(I + G)^{s/2} f(x)|^2 \, dx < \infty\}$, equipped with the norm

$$\|f\|_{H^s_G} = \|(I+G)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^{n+1})}.$$

On the support of $\psi(2^{-j}\cdot)$, we have $1 + (2|\alpha| + n)|\lambda| \sim 2^j$ and consequently,

$$\|f\|^2_{H^s_G} \sim \sum_{j \geq 0} \|(I+G)^{\frac{s}{2}} \Delta_j f\|^2_{L^2(\mathbb{R}^{n+1})} \sim \sum_{j \geq 0} 2^{sj} \|\Delta_j f\|^2_{L^2(\mathbb{R}^{n+1})}. \tag{2.2.20}$$

For a similar frequency decomposition for the Grushin operator, we refer to [53].

2.3 The Grushin-Schrödinger equation

Consider the following free Grushin-Schrödinger equation:

$$\begin{aligned} i\partial_s u(x,t,s) - Gu(x,t,s) &= 0, \quad s \in \mathbb{R}, \; (x,t) \in \mathbb{R}^{n+1}, \\ u(x,t,0) &= f(x,t). \end{aligned} \tag{2.3.1}$$

For our convenience, we use a different notation and denote the time variable by s. As in the Euclidean case, (2.2.11) allows us to solve (2.3.1) explicitly. For $f \in L^2(\mathbb{R}^{n+1})$, taking the scaled Hermite-Fourier transform with respect to (x,t) variable in (2.3.1), we get

$$\begin{aligned} i\frac{d}{ds}\hat{u}(\alpha,\lambda,s) - (2|\alpha|+n)\,|\lambda|\,\hat{u}(\alpha,\lambda,s) &= 0, \\ \hat{u}(\alpha,\lambda,0) &= \hat{f}(\alpha,\lambda). \end{aligned} \tag{2.3.2}$$

Solving the ordinary differential equation (2.3.2), we get $\hat{u}(\alpha,\lambda,s)=e^{-is(2|\alpha|+n)|\lambda|}\hat{f}(\alpha,\lambda)$. Now applying the inversion formula (2.2.9) the solution of the IVP (2.3.1) can be written as

$$u(x,t,s)=e^{-isG}f(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}^*}e^{-i\lambda t}\sum_{\alpha\in\mathbb{N}^n}e^{-is(2|\alpha|+n)|\lambda|}\hat{f}(\alpha,\lambda)\Phi_\alpha^\lambda(x)d\lambda. \tag{2.3.3}$$

Unlike the Euclidean case, the IVP (2.3.1) is totally non-dispersive (see [54]) for $n=1$. A similar phenomenon is observed for $n\geq 1$ in the following proposition.

Proposition 2.3.1. *There exists a function $f\in\mathcal{S}(\mathbb{R}^{n+1})$, the space of all Schwartz class functions on $\mathbb{R}^{n+1}$, such that the solution to the IVP (2.3.1) with initial data f satisfies*

$$u(x,t,s)=f(x,t+sn),\quad \forall s\in\mathbb{R},\quad \forall(x,t)\in\mathbb{R}^{n+1}. \tag{2.3.4}$$

Proof. Fix a function $Q\in C_c^\infty((1,\infty))$ and consider

$$f(x,t)=\frac{1}{2\pi}\int_1^\infty e^{-i\lambda t}\Phi_0^\lambda(x)Q(\lambda)d\lambda. \tag{2.3.5}$$

Thus $f\in\mathcal{S}(\mathbb{R}^{n+1})$ and comparing (2.3.5) with the inversion formula (2.2.9) we have

$$\hat{f}(\alpha,\lambda)=\begin{cases}0, & \text{if } \alpha\neq 0,\lambda\in\mathbb{R}^*\\ Q(\lambda), & \text{if } \alpha=0,\lambda\in\mathbb{R}^*.\end{cases}$$

By (2.2.9), the solution of the IVP (2.3.1) can be written as

$$u(x,t,s)=e^{-isG}f(x,t)=\frac{1}{2\pi}\int_1^\infty e^{-i\lambda(t+ns)}\Phi_0^\lambda(x)Q(\lambda)d\lambda=f(x,t+ns).$$

□

Notice that $\|u(\cdot,s)\|_p=\|f\|_p$ for all $1\leq p\leq\infty$, hence no global dispersive estimate of the type (1.4.8) can hold. Due to this loss of dispersion, the standard Euclidean approach to deriving Strichartz estimates fails. A similar phenomenon occurs for the Schrödinger operator on the Heisenberg group: Bahouri, Gérard, and Xu [9] emphasized that, in this setting, the Schrödinger flow exhibits no dispersion at all. Further, Bahouri-Barilari-Gallagher [10] derived anisotropic Strichartz estimates for the Schrödinger and the wave equations on the Heisenberg group involving the sublaplacian, only for the radial initial data, by adapting the restriction approach.

The Grushin operator is closely linked with the sublaplacian on the Heisenberg group, following the strategy introduced in [10], we obtain Strichartz estimates for the Grushin-Schrödinger equation (2.3.1) and the Grushin wave equation (2.6.1) for initial data that belongs to a more general class of functions.

2.3.1 A restriction theorem for the scaled Hermite-Fourier transform on $\mathbb{R}^{n+1}$

For $\mu > 0$, consider the surface

$$\mathbb{S}^n(\mu) = \{(\alpha, \lambda) \in \mathbb{N}_0^n \times \mathbb{R}^* : (2|\alpha| + n)|\lambda| = \mu\},$$

with the measure $d\sigma_\mu$ on $\mathbb{S}^n(\mu)$ defined by

$$\int_{\mathbb{S}^n(\mu)} \Theta(\alpha, \lambda)\, d\sigma_\mu = \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{2|\alpha|+n} \left(\Theta(\alpha, \frac{\mu}{2|\alpha|+n}) + \Theta(\alpha, \frac{-\mu}{2|\alpha|+n}) \right),$$

for suitable functions Θ on $\mathbb{S}^n(\mu)$. The surface $\mathbb{S}^n(\mu)$ can be viewed as an analogue of the sphere of radius μ in $\mathbb{N}_0^n \times \mathbb{R}^*$ with surface measure $d\sigma_\mu$, in the sense that for any $F \in L^1(\mathbb{N}_0^n \times \mathbb{R}^*)$, we have

$$\sum_{\alpha\in\mathbb{N}_0^n} \int_{\mathbb{R}^*} F(\alpha, \lambda) d\lambda = \int_0^\infty \left(\int_{\mathbb{S}^n(\mu)} F(\alpha, \lambda)\, d\sigma_\mu \right) d\mu.$$

In [80], Liu-Song derived a restriction theorem associated to Grushin operator on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, analogous to the seminal work of Müller [85]. Specifically, by setting $d_1 = n, d_2 = 1, q = p$ and $r = p'$, with $\frac{1}{p} + \frac{1}{p'} = 1$, Theorem 2 in [80] can be reframed as follows:

Theorem 2.3.2. *[80] If $1 \le p < 2$, then*

$$\|\hat{f}|_{\mathbb{S}^n(\mu)}\|_{L^2(\mathbb{S}^n(\mu), d\sigma_\mu)} \le C\mu^{n(\frac{1}{p}-\frac{1}{2})} \|f\|_{L^1_t L^p_x},$$

for all functions $f \in \mathcal{S}(\mathbb{R}^{n+1})$ and $\mu > 0$.

In order to obtain Strichartz estimates via the Fourier restriction method for evolution PDEs, one applies the result to specific surfaces in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, such as the paraboloid for the Schrödinger equation and the cone for the wave equation (see [102]).

When dealing with evolution equations associated to the Grushin operator G on $\mathbb{R}^{n+1}$, one is naturally led to consider surfaces in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$. Consequently, restriction theorems in $\mathbb{N}_0^n \times \mathbb{R}^*$ alone are not sufficient. Thus we adapt the scaled Hermite-Fourier transform on $\mathbb{R}^{n+2}$ and establish a restriction theorem (Theorem 2.4.1) for surfaces in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$.

2.4 Restriction theorem for the scaled Hermite-Fourier transform on $\mathbb{R}^{n+2}$

For $f \in \mathcal{S}(\mathbb{R}^{n+2})$, the space of all Schwartz class functions on $\mathbb{R}^{n+2}$, let

$$f^{\lambda,\nu}(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,t,s)e^{i\lambda t}e^{i\nu s}dtds \tag{2.4.1}$$

stands for the inverse Fourier transform of $f(x,t,s)$ in the (t,s) variable. We define the scaled Hermite-Fourier transform of f on $\mathbb{R}^{n+2}$ as

$$\hat{f}(\alpha,\lambda,\nu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda t}e^{i\nu s}f(x,t,s)\Phi_\alpha^\lambda(x)dsdtdx = \langle f^{\lambda,\nu}, \Phi_\alpha^\lambda\rangle, \tag{2.4.2}$$

for any $(\alpha,\lambda,\nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$. If $f \in L^2(\mathbb{R}^{n+2})$ then $\hat{f} \in L^2(\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R})$ and satisfies the Plancherel formula

$$\|f\|_{L^2(\mathbb{R}^{n+2})} = \frac{1}{(2\pi)^2}\|\hat{f}\|_{L^2(\mathbb{N}_0^n\times\mathbb{R}^*\times\mathbb{R})}. \tag{2.4.3}$$

The inversion formula is given by

$$f(x,t,s) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\nu s}e^{-i\lambda t}\sum_{\alpha\in\mathbb{N}_0^n}\hat{f}(\alpha,\lambda,\nu)\,\Phi_\alpha^\lambda(x)\;d\lambda d\nu. \tag{2.4.4}$$

Given a surface S in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$ endowed with an induced measure $d\sigma$, we define the restriction operator $\mathcal{R}_S : L^2(\mathbb{R}^{n+2}) \to L^2(S,d\sigma)$ as

$$\mathcal{R}_S f = \hat{f}|_S, \tag{2.4.5}$$

on the surface S and the operator dual to $\mathcal{R}_S$ (called the extension operator) as

$$\mathcal{E}_S(\Theta)(x,t,s) = \frac{1}{(2\pi)^2}\int_S e^{-i\nu s}e^{-i\lambda t}\Theta(\alpha,\lambda,\nu)\,\Phi_\alpha^\lambda(x)\;d\sigma, \tag{2.4.6}$$

$\Theta \in L^2(S,d\sigma)$.

2.4.1 A surface measure

Let us consider the surface

$$S = \{(\alpha,\lambda,\nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} : \nu = (2|\alpha|+n)|\lambda|\}. \tag{2.4.7}$$

We endow S with the measure $d\sigma$ induced by the projection $\pi : \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} \to \mathbb{N}_0^n \times \mathbb{R}^*$ onto the first two factors, where $\mathbb{N}_0^n \times \mathbb{R}^*$ endowed with the measure $d\mu \otimes d\lambda$, $d\mu$ and $d\lambda$ denote the counting measure on $\mathbb{N}_0^n$ and Lebesgue measure on $\mathbb{R}^*$ respectively. More explicitly, for any integrable function Θ on S, we have

$$\int_S \Theta \, d\sigma = \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} \Theta(\alpha, \lambda, (2|\alpha| + n)|\lambda|) \, d\lambda.$$

By construction it is clear that if $\Theta = \hat{f} \circ \pi|_S$, where $\hat{f}$ is a function on $\mathbb{N}_0^n \times \mathbb{R}^*$, then for all $1 \leq p \leq \infty$

$$\|\Theta\|_{L^p(S,d\sigma)} = \|\hat{f}\|_{L^p(\mathbb{N}_0^n \times \mathbb{R}^*)}. \tag{2.4.8}$$

2.4.2 Restriction Theorem

Our purpose here is to show that every (appropriate) function f (on $\mathbb{R}^{n+2}$) has a scaled Hermite-Fourier transform $\hat{f}$ that can be restricted to the surface S. In view of Fourier restriction theorem due to Thomas [108], such restriction property is best dealt with compact subsets in the Euclidean space. Therefore, we consider the surface S endowed with the surface measure $d\sigma_{loc} = \psi(\nu) d\sigma$ defined by

$$\int_S \Theta \, d\sigma_{loc} = \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} \Theta(\alpha, \lambda, (2|\alpha| + n)|\lambda|) \, \psi((2|\alpha| + n)|\lambda|) \, d\lambda. \tag{2.4.9}$$

with ψ any smooth, even, compactly supported function in $\mathbb{R}$ with an L^∞ norm at most 1. Let $S_{\sigma_{loc}}$ be the support of σ_{loc} in S, i.e., $S_{\sigma_{loc}} = \{(\alpha, \lambda, \nu) \in S : \psi(\nu) \neq 0\}$. The restriction operator, $\mathcal{R}_{S_{\sigma_{loc}}}$ and the extension operator, $\mathcal{E}_{S_{\sigma_{loc}}}$ with respect to the surface $(S, d\sigma_{\sigma_{loc}})$ can be computed as $\mathcal{R}_{S_{\sigma_{loc}}} f = \hat{f}|_{\sigma_{S_{loc}}}$ and

$$\mathcal{E}_{S_{\sigma_{loc}}}(\Theta)(x, t, s) = \frac{1}{(2\pi)^2} \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} e^{-i(2|\alpha|+n)|\lambda| s} e^{-i\lambda t} \Theta(\alpha, \lambda, (2|\alpha| + n)|\lambda|) \Phi_\alpha^\lambda(x) \psi((2|\alpha| + n)|\lambda|) d\lambda. \tag{2.4.10}$$

Let $S_{\sigma_{loc}}$ be the support of $d\sigma_{loc}$ in S. We obtain the following restriction theorem for Scaled Hermite-Fourier transform for $S_{\sigma_{loc}}$.

Theorem 2.4.1 (Scaled Hermite-Fourier restriction theorem). *Let* $n \geq 1$.

1. *If* $1 \leq q \leq p < 2$, *then*

$$\|\mathcal{R}_{S_{\sigma_{loc}}} f\|_{L^2(S, d\sigma_{loc})} \leq C(p,q)\|f\|_{L^1_t L^q_s L^p_x}, \tag{2.4.11}$$

   for all functions $f \in \mathcal{S}(\mathbb{R}^{n+2})$.

2. *For* $n = 1$, *the inequality (2.4.11) holds for all* $f \in \mathcal{S}(\mathbb{R}^3)$, *when* $p = 2$ *and* $1 \leq q \leq 2$.

3. *For* $n \geq 2$, *the inequality (2.4.11) holds for all* $f \in \mathcal{S}_{rad}(\mathbb{R}^{n+2})$, *the space of all radial*[2] *Schwartz class functions on* $\mathbb{R}^{n+2}$, *when* $p = 2$ *and* $1 \leq q \leq 2$.

Proof. We prove each case in Theorem 2.4.1 separately. First we prove the case $1 \leq q \leq p < 2$. Before proceeding to the proof, we need to observe the following:

Lemma 2.4.2. *Let* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\lambda \in \mathbb{R}^*$, *then for all* $1 \leq p \leq 2$,

$$\|P_k(\lambda)\phi\|_{L^{p'}(\mathbb{R}^n)} \leq C|\lambda|^{\frac{n}{2}(1-\frac{2}{p'})}(2k+n)^{\frac{n-1}{2}(1-\frac{2}{p'})}\|\phi\|_{L^p(\mathbb{R}^n)}, \tag{2.4.12}$$

where p' *is the conjugate exponent of* p, *i.e.,* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Since, $\{P_k(\lambda)\}_{k\geq 0}$ are orthogonal projections on $L^2(\mathbb{R}^n)$, so we have

$$\|P_k(\lambda)\phi\|_{L^2(\mathbb{R}^n)} \leq \|\phi\|_{L^2(\mathbb{R}^n)}. \tag{2.4.13}$$

Using the relation (2.2.3) and the $L^1 - L^\infty$ estimate in the proof of Proposition 4.4.2 in [106], we have

$$\|P_k(\lambda)\phi\|_{L^\infty(\mathbb{R}^n)} \leq |\lambda|^{\frac{n}{2}}(2k+n)^{\frac{n-1}{2}}\|\phi\|_{L^1(\mathbb{R}^n)}. \tag{2.4.14}$$

This estimate can also be found in the proof of Proposition 1 in [80]. Thus, the Lemma 2.4.2 follows by interpolating (2.4.13) and (2.4.14). □

Proof of the case $1 \leq q \leq p < 2$**:** By duality argument, it is enough to show that the boundedness of the operator $\mathcal{E}_{S_{\sigma_{loc}}}$ from $L^2(S, d\sigma_{loc})$ to $L^\infty_t(\mathbb{R}; L^{q'}_s(\mathbb{R}; L^{p'}_x(\mathbb{R}^n)))$. Equivalently, we show that the operator $\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^*$ is bounded from $L^1_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ to $L^\infty_t(\mathbb{R}; L^{q'}_s(\mathbb{R}; L^{p'}_x(\mathbb{R}^n)))$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

---

[2] A function f on $\mathbb{R}^{n+2}$ (resp. $\mathbb{R}^{n+1}$) is said to be radial if $f(x,t,s) = f(|x|,t,s)$ (resp. $f(x,t) = f(|x|,t)$) for all $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$.

Let $f \in \mathcal{S}(\mathbb{R}^{n+2})$. From (2.4.10) and (2.4.9), we have

$$
\begin{aligned}
&\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f(x,t,s) \\
&\quad = \frac{1}{(2\pi)^2} \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} e^{-i(2|\alpha|+n)|\lambda| s} e^{-i\lambda t} \hat{f}(\alpha, \lambda, (2|\alpha|+n)|\lambda|)\, \Phi_\alpha^\lambda(x) \psi((2|\alpha|+n)|\lambda|)\, d\lambda \\
&\quad = \frac{1}{(2\pi)^2} \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{2|\alpha|+n} \int_{\mathbb{R}^*} e^{-i|\lambda| s} e^{-\frac{i\lambda t}{2|\alpha|+n}} \hat{f}(\alpha, \frac{\lambda}{2|\alpha|+n}, |\lambda|)\, \Phi_\alpha^{\frac{\lambda}{2|\alpha|+n}}(x) \psi(|\lambda|)\, d\lambda,
\end{aligned}
$$

where the last term obtained by performing the change of variables $(2|\alpha|+n)\lambda \mapsto \lambda$ in each integral. Using (2.4.2), (2.2.1) and writing $a_k = \frac{1}{2k+n}$, we obtain

$$
\begin{aligned}
\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f(x,t,s) &= \frac{1}{(2\pi)^2} \sum_{k=0}^{\infty} \frac{1}{2k+n} \sum_{\pm} \int_0^\infty e^{-i\lambda s} e^{\mp i a_k \lambda t} P_k(a_k\lambda) f^{\pm a_k \lambda, \lambda}(x) \psi(\lambda)\, d\lambda \\
&= C \sum_{k=0}^{\infty} \sum_{\pm} \frac{1}{2k+n} \mathcal{F}_{\lambda \to s} \left(e^{\mp i a_k \lambda t} P_k(a_k\lambda) f^{\pm a_k \lambda, \lambda}(x) \psi_+(\lambda) \right),
\end{aligned}
\tag{2.4.15}
$$

where $\psi_+(\lambda) = \psi(\lambda)\mathbf{1}_{\lambda > \mathbf{0}}$. For fixed $t \in \mathbb{R}$, Hausdorff-Young inequality on the right-hand side of (2.4.15) with respect to $s-$variable gives

$$
\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f\|_{L_s^{q'}} \leq C \sum_{k=0}^{\infty} \sum_{\pm} \frac{1}{2k+n} \|\psi_+(\lambda) e^{\mp i a_k \lambda t} P_k(a_k\lambda) f^{\pm a_k \lambda, \lambda}(x)\|_{L_\lambda^q}. \tag{2.4.16}
$$

Now for any function g defined on $\mathbb{R}^{n+1}$ and for $q' \geq p' > 2$, applying Minkowski's inequality followed by Hausdorff-Young inequality and again applying Minkowski's inequality, we get

$$
\|\mathcal{F}_{\lambda \to s} g\|_{L_s^{q'} L_x^{p'}} \leq \|\mathcal{F}_{\lambda \to s} g\|_{L_x^{p'} L_s^{q'}} \leq C \|g\|_{L_x^{p'} L_\lambda^q} \leq C \|g\|_{L_\lambda^q L_x^{p'}}. \tag{2.4.17}
$$

In view of (2.4.17) and (2.4.16), we deduce that

$$
\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f\|_{L_t^\infty L_s^{q'} L_x^{p'}} \leq C \sum_{k=0}^{\infty} \sum_{\pm} \frac{1}{2k+n} \|\psi(\lambda) P_k(a_k\lambda) f^{\pm a_k \lambda, \lambda}(x)\|_{L_\lambda^q L_x^{p'}}.
$$

But, by Lemma 2.4.2, we have

$$
\|P_k(a_k\lambda) f^{\pm a_k \lambda, \lambda}\|_{L_x^{p'}} \leq C |a_k \lambda|^{\frac{n}{2}(1-\frac{2}{p'})} (2k+n)^{\frac{n-1}{2}(1-\frac{2}{p'})} \|\mathcal{F}_{s \to -\lambda} f(\cdot,\cdot,s)\|_{L_x^p L_t^1},
$$

which implies that

$$
\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f\|_{L_t^\infty L_s^{q'} L_x^{p'}} \leq C \sum_{k=0}^{\infty} \frac{1}{(2k+n)^{1+\frac{1}{2}(1-\frac{2}{p'})}} \left\| \|\mathcal{F}_{s \to -\lambda} f(\cdot,\cdot,s)\|_{L_x^p L_t^1} \psi(\lambda) \lambda^{\frac{n}{2}(1-\frac{2}{p'})} \right\|_{L_\lambda^q}
$$

$$\begin{aligned}
&\leq C \left\| \|\mathcal{F}_{s\to-\lambda} f(\cdot,\cdot,s)\|_{L^p_x L^1_t} \psi(\lambda) \lambda^{\frac{n}{2}(1-\frac{2}{p'})} \right\|_{L^q_\lambda} \\
&\leq C \|\mathcal{F}_{s\to-\lambda} f(\cdot,\cdot,s)\|_{L^a_\lambda L^p_x L^1_t} \|\psi(\lambda)\lambda^{\frac{n}{2}(1-\frac{2}{p'})}\|_{L^b_\lambda(\mathbb{R})}, \qquad (2.4.18)
\end{aligned}$$

where the last step is justified by an application of Hölder's inequality in (2.4.18) with $a \geq 2$, $\frac{1}{a}+\frac{1}{a'}=1$ and $\frac{1}{a}+\frac{1}{b}=\frac{1}{q}$. Then, taking $a=q'$ and applying Minkowski's inequality followed by Hausdorff-Young inequality in $\lambda-$ variable, we get

$$\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f\|_{L^\infty_t L^{q'}_s L^{p'}_x} \leq C\|\psi(\lambda)\lambda^{\frac{n}{2}(1-\frac{2}{p'})}\|_{L^b_\lambda(\mathbb{R})} \|f\|_{L^p_x L^1_t L^q_s}. \qquad (2.4.19)$$

Thus, (2.4.11) follows from (2.4.19) by Minkowski's integral inequality for all $1 \leq q \leq p < 2$. ☐

Proof of the case $n=1, p=2, 1\leq q\leq 2$**:** Note that for $n=1$,

$$\|\mathcal{R}_{S_{\sigma_{loc}}} f\|^2_{L^2(S,d\sigma_{loc})} = \frac{1}{(2\pi)^2}\sum_{\pm}\sum_{k=0}^{\infty}\int_0^\infty \frac{1}{2k+1}\left\|P_k(\pm a_k\lambda) f^{\pm a_k\lambda,\lambda}\right\|^2_{L^2(\mathbb{R})}\psi(\lambda)d\lambda. \quad (2.4.20)$$

Consider the Hilbert space $L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$, with respect to the inner product $\langle\tilde{\alpha},\tilde{\beta}\rangle' = \sum_{k=0}^{\infty}\int_{\mathbb{R}^+}\langle\tilde{\alpha}(k,\lambda),\tilde{\beta}(k,\lambda)\rangle\psi(\lambda)d\lambda$, for all $\tilde{\alpha},\ \tilde{\beta}\in L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$, where $\mathbb{R}_+$ denote the set of all positive reals. In view of (2.4.20) it is enough to prove that the operator T defined on $\mathcal{S}(\mathbb{R}^3)$ by

$$Tf = \frac{1}{(2k+1)^{\frac{1}{2}}} P_k(a_k\lambda) f^{a_k\lambda,\lambda},$$

is bounded from $L^1_t(\mathbb{R};L^q_s(\mathbb{R};L^2_x(\mathbb{R}^n)))$ into $L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$ or equivalently that its adjoint T^* is bounded from $L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$ into $L^\infty_t(\mathbb{R};L^{q'}_s(\mathbb{R};L^2_x(\mathbb{R}^n)))$ to obtain (2.4.11).

For $\tilde{\alpha}\in L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$, the operator T^* can be computed to be

$$T^*(\tilde{\alpha})(x,t,s) = \sum_{k=0}^{\infty}\int_{\mathbb{R}^+}\frac{1}{(2k+1)^{\frac{1}{2}}} e^{-ia_k\lambda t} e^{-i|\lambda|s} P_k(a_k\lambda)(\tilde{\alpha}(k,\lambda))(x)\psi(\lambda)d\lambda.$$

Using Minkowski's inequality together with the Hausdorff-Young inequality (see (2.4.17)), for any fixed $t\in\mathbb{R}$, we have

$$\|T^*(\tilde{\alpha})(\cdot,t,\cdot)\|_{L^{q'}_s L^2_x} \leq C\|g\|_{L^q_\lambda L^2_x},$$

where $g(x,\lambda)=\psi(\lambda)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\frac{1}{2}}}P_k(a_k\lambda)(\tilde{\alpha}(k,\lambda))(x)$. Now

$$\|g(\cdot,\lambda)\|^2_{L^2(\mathbb{R})} = \psi(\lambda)^2\sum_{k,l\geq 0}\frac{1}{(2k+1)^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}}\langle P_k(a_k\lambda)\tilde{\alpha}(k,\lambda), P_l(a_k\lambda)\tilde{\alpha}(l,\lambda)\rangle$$

$$\le C\psi(\lambda)^2 \sum_{k\le l} \frac{\|\tilde{\alpha}(k,\lambda)\|_{L^2(\mathbb{R})}\|\tilde{\alpha}(l,\lambda)\|_{L^2(\mathbb{R})}}{(2k+1)^{\frac{3}{4}}(2l+1)^{\frac{3}{4}}} \int_{\mathbb{R}} \left|h_k\left(\frac{x}{\sqrt{2k+1}}\right)\right|\left|h_l\left(\frac{x}{\sqrt{2l+1}}\right)\right| dx, \tag{2.4.21}$$

where the last line obtained by Cauchy-Schwarz inequality and a change of variable $x \mapsto \lambda x$. Using Proposition 2.7.2 (see appendix), (2.4.21) turns out to be

$$\|g(\cdot,\lambda)\|^2_{L^2(\mathbb{R})} \le C\psi(\lambda)^2 \sum_{l} \|\tilde{\alpha}(l,\lambda)\|_{L^2(\mathbb{R})} \left(\frac{1}{l}\sum_{k=0}^{l} \|\tilde{\alpha}(k,\lambda)\|_{L^2(\mathbb{R})}\right).$$

By Hardy's inequality (see [5,85]), we get

$$\|g(\cdot,\lambda)\|_{L^2(\mathbb{R})} \le C\psi(\lambda) \left(\sum_{k=0}^{\infty} \|\tilde{\alpha}(k,\lambda)\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}.$$

Further, applying Hölder's inequality, we have

$$\|g\|_{L^q_\lambda L^2_x} \le C\|\psi(\lambda)\|_{L_\lambda^{\frac{2q}{2-q}}(\mathbb{R}^+)} \|\tilde{\alpha}\|_{L^2(\mathbb{N}_0\times\mathbb{R}^+)}.$$

Proposition 2.7.2 plays a decisive role in the proof presented above. However, we could not find such estimate for the higher dimensional Hermite functions ($n \ge 2$). Nonetheless, we prove the restriction inequality (2.4.11) for $n \ge 2$ and $p = 2$ for the radial functions. Recall that a function f on $\mathbb{R}^{n+2}$ is said to be radial if $f(x,t,s) = f(|x|,t,s)$ for all $x \in \mathbb{R}^n$ and $t,s \in \mathbb{R}$. If f is radial on $\mathbb{R}^{n+2}$ then $f^{\lambda,\nu}$ is radial on $\mathbb{R}^n$ for any $\lambda \in \mathbb{R}^*$ and $\nu \in \mathbb{R}$. Thus by Corollary 3.4.1 in [106] and the relation (2.2.3), for all $k \in \mathbb{N}_0$, we get

$$P_{2k+1}(\lambda)(f^{\lambda,\nu}) = 0 \quad \text{and} \quad P_{2k}(\lambda)(f^{\lambda,\nu})(x) = R_{2k}(f^{\lambda,\nu})L_k^{\frac{n}{2}-1}(|\lambda||x|^2)e^{-\frac{|\lambda|}{2}|x|^2},$$

where

$$R_{2k}(f^{\lambda,\nu}) = \frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})}|\lambda|^{\frac{n}{2}} \int_{\mathbb{R}^n} f^{\lambda,\nu}(x)L_k^{\frac{n}{2}-1}(|\lambda||x|^2)e^{-\frac{|\lambda|}{2}|x|^2}dx$$

and L_k^δ denote the Laguerre polynomials, defined by $L_k^\delta(r) = \frac{1}{k!}e^r r^{-\delta}\frac{d^k}{dx^k}(e^{-r}r^{k+\delta})$, $r > 0$, of type $\delta\,(> -1)$.

Proof of the case $n \ge 2, p = 2, 1 \le q \le 2$**:** Let $f \in \mathcal{S}_{rad}(R^{n+2})$. To prove (2.4.11) for $n \ge 2$ and $p = 2$ (proceeding as in (2.4.20) for $n = 1$ case), it suffices to show

$$\sum_{k=0}^{\infty}\int_0^{\infty} \left(\left|R(k, \frac{\lambda}{4k+n}, \lambda)\right|^2 + \left|R(k, \frac{-\lambda}{4k+n}, \lambda)\right|^2\right) \lambda^{\frac{n}{2}}\phi(\lambda)d\lambda \le C\|f\|^2_{L^1_t L^q_s L^p_x}, \tag{2.4.22}$$

where

$$R(k,\lambda,\nu)=\left(\frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})(4k+n)^{\frac{n}{2}+1}}\right)^{\frac{1}{2}}\int_{\mathbb{R}^n} f^{\lambda,\nu}(x)L_k^{\frac{n}{2}-1}(|\lambda||x|^2)e^{-\frac{|\lambda|}{2}|x|^2}dx. \tag{2.4.23}$$

Consider the operator $T:\mathcal{S}_{rad}(R^{n+2})\to L^2(\mathbb{N}_0\times\mathbb{R}^+)$ defined by

$$(Tf)(k,\lambda)=R(k,a_{2k}\lambda,\lambda),$$

where f is related to R through (2.4.23) and the space $L^2(\mathbb{N}_0\times\mathbb{R}^+)$ endowed with the measure $\ell^2(\mathbb{N}_0)\otimes L^2(\mathbb{R}^+,\lambda^{\frac{n}{2}}\phi(\lambda)d\lambda)$. To prove (2.4.22), it is enough to show the adjoint T^* is bounded from $L^2(\mathbb{N}_0\times\mathbb{R}^+)$ into $L^\infty_t(\mathbb{R};L^{q'}_s(\mathbb{R};L^2_x(\mathbb{R}^n)))$. For $\alpha\in L^2(\mathbb{N}_0\times\mathbb{R}^+)$, the operator T^* is given by

$$T^*(\alpha)(x,t,s)=\sum_{k=0}^{\infty}\int_{\mathbb{R}^+}\alpha(k,\lambda)e^{-ia_k\lambda t}e^{-i|\lambda|s}\mathcal{L}_k\left(a_{2k}\lambda\right)(x)\lambda^{\frac{n}{2}}\psi(\lambda)d\lambda,$$

with

$$\mathcal{L}_k(\lambda)(x)=\left(\frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})(4k+n)^{\frac{n}{2}+1}}\right)^{\frac{1}{2}}L_k^{\frac{n}{2}-1}(|\lambda||x|^2)e^{-\frac{|\lambda|}{2}|x|^2}.$$

Again using Minkowski's inequality together with the Hausdorff-Young inequality (see (2.4.17)), for any fixed $t\in\mathbb{R}$, we have

$$\|T^*(\alpha)(\cdot,t,\cdot)\|_{L^{q'}_sL^2_x}\le C\|g\|_{L^q_\lambda L^2_x},$$

where $g(x,\lambda)=\lambda^{\frac{n}{2}}\psi(\lambda)\sum_{k=0}^{\infty}\alpha(k,\lambda)\,\mathcal{L}_k\left(a_{2k}\lambda\right)(x)$. By an obvious change of variable, we get

$$\|g(\cdot,\lambda)\|^2_{L^2(\mathbb{R})}\le\lambda^n\psi(\lambda)^2\sum_{k,l\ge0}|\alpha(k,\lambda)||\alpha(l,\lambda)|\int_{\mathbb{R}^n}|\mathcal{L}_k(a_k)(x)|\,|\mathcal{L}_l(a_l)(x)|\,dx. \tag{2.4.24}$$

Now, by Lemma 4.2 in [85], there exists $C>0$ such that for all $k,l\in\mathbb{N}_0$,

$$\int_{\mathbb{R}^n}\left|\mathcal{L}_k\left(\frac{1}{4k+n}\right)(x)\right|\left|\mathcal{L}_l\left(\frac{1}{4l+n}\right)(x)\right|dx\le\frac{C}{\max(k,l)}. \tag{2.4.25}$$

Note that the above result is stated in Lemma 4.2 of [85] for even n, but a same idea works for odd n as well. Once we have (2.4.25), applying Hardy's inequality (see [5, 85]) in (2.4.21) and after using Hölder's inequality (arguing as in the proof of $n=1$ case), we obtain (2.4.22). ☐

By a duality argument, Theorem 2.4.1 can be reframed as follows: For any $2 \leq p' \leq q' \leq \infty$, the following estimate

$$\|\mathcal{E}_{S_{\sigma_{loc}}}(\Theta)\|_{L_t^\infty L_s^{q'} L_x^{p'}} \leq C(p,q)\|\Theta\|_{L^2(S,d\sigma_{loc})} \tag{2.4.26}$$

holds for all $\Theta \in L^2(S, d\sigma_{loc})$. The end point case $p' = 2$ holds with an appropriate modification.

Remark 2.4.3. *We consider the surfaces*

$$S_\pm = \{(\alpha, \lambda, \nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} : \nu^2 = (2|\alpha| + n)|\lambda|, \pm\nu > 0\}, \tag{2.4.27}$$

to obtain Strichartz estimate for the wave equation (2.6.1). The induced measure $d\sigma_\pm$ by the projection $\pi : \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} \to \mathbb{N}_0^n \times \mathbb{R}^*$ onto the first two factors, for the surfaces $S_\pm$ are given by

$$\int_{S_\pm} \Theta \, d\sigma_\pm = \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} \Theta(\alpha, \lambda, \pm\sqrt{(2|\alpha| + n)|\lambda|}\ ) \, d\lambda,$$

for any integrable function Θ on $S_\pm$.

Arguing as in the proof of Theorem (2.4.1), the restriction inequality (2.4.11) can be achieved for the surface $S_w = S_+ \cup S_-$ endowed with the corresponding localized measure.

2.5 Anisotropic Strichartz estimates for the Grushin-Schrödinger equation

Consider the following free Grushin-Schrödinger equation:

$$\begin{aligned} i\partial_s u(x,t,s) - Gu(x,t,s) &= 0, \quad s \in \mathbb{R},\ (x,t) \in \mathbb{R}^{n+1}, \\ u(x,t,0) &= f(x,t). \end{aligned} \tag{2.5.1}$$

Now, realizing the solution of (2.5.1) as the extension operator $\mathcal{E}_{S_{\sigma_{loc}}}$ acting on a suitable function on S and using (2.4.26), we prove the following anisotropic Strichartz estimate for the solution of the free Grushin-Schrödinger equation. More generally, we obtain the following result.

Theorem 2.5.1. *Let $n, p, q \geq 1$. If (p, q) lies in the admissible set*

$$A = \left\{ (p,q) : 2 < p \leq q \leq \infty \quad \text{and} \quad \frac{2}{q} + \frac{n}{p} \leq \frac{n+2}{2} \right\},$$

then the solution $u(x,t,s)$ of the IVP (2.5.1) is in $L_t^\infty(\mathbb{R}; L_s^q(\mathbb{R}; L_x^p(\mathbb{R}^n)))$ and satisfies the estimate:

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \leq C\|f\|_{H_G^\sigma}, \tag{2.5.2}$$

where $\sigma > \frac{n+2}{2} - \frac{2}{q} - \frac{n}{p}$. Moreover, at the end point $(p,q) = (2,2)$, the estimate (2.5.2) is valid with $\sigma = 0$ for all functions f and h when $n = 1$, and for radial functions f and h when $n \geq 2$.

Proof. First, suppose $f \in \mathcal{S}(\mathbb{R}^{n+1})$ is frequency localized in the unit ball $\mathcal{B}_1$ (or ring $\mathcal{C}_1$), i.e., there exists a smooth, even function ψ supported in $(-2,2)$ such that $\hat{f}(\alpha,\lambda) = \psi((|\alpha|+n)|\lambda|)\hat{f}(\alpha,\lambda)$. Let $\Theta = \hat{f} \circ \pi|_S$ and the localized measure on S be $d\sigma_{loc} = \psi d\sigma$ defined in (2.4.9). In view of (2.3.3) and (2.4.10) we can write

$$e^{-isG}f(x,t) = \mathcal{E}_{S_{\sigma_{loc}}}(\Theta)(x,t,s).$$

By the restriction inequality (2.4.26), we have for $2 < p \leq q \leq \infty$

$$\|e^{-isG}f\|_{L_t^\infty L_s^q L_x^p} \leq C\|\Theta\|_{L^2(S,d\sigma_{loc})} = C\|\hat{f}\circ\pi|_S\|_{L^2(S,d\sigma)} = C\|f\|_{L^2(\mathbb{R}^{n+1})}, \tag{2.5.3}$$

where the last equality is obtained by (2.4.8) and the Plancherel formula (2.4.3).

Next, assume that f is frequency localized in the ball $\mathcal{B}_R$ (or ring $\mathcal{C}_R$). By (2.2.10) one can check that the function $f_R := f \circ \delta_{R^{-1}}$ is frequency localized in $\mathcal{B}_1$ (or ring $\mathcal{C}_1$) and hence applying (2.5.3) we get

$$\|e^{-isG}f_R(x,t)\|_{L_t^\infty L_s^q L_x^p} \leq C\|f_R\|_{L^2(\mathbb{R}^{n+1})} = CR^{\frac{n}{2}+1}\|f\|_{L^2(\mathbb{R}^{n+1})}. \tag{2.5.4}$$

Again using (2.5.3), we have $e^{-isG}f_R(x,t) = e^{-iR^{-2}sG}f(R^{-1}x, R^{-2}t)$, thus from (2.5.4) we obtain

$$\|e^{-isG}f\|_{L_t^\infty L_s^q L_x^p} = R^{-\frac{2}{q}-\frac{n}{p}}\|e^{-iR^{-2}sG}f(R^{-1}x,R^{-2}t)\|_{L_t^\infty L_s^q L_x^p} \leq CR^{\frac{n+2}{2}-\frac{2}{q}-\frac{n}{p}}\|f\|_{L^2(\mathbb{R}^{n+1})}.$$

Consequently, for any $f \in \mathcal{S}(\mathbb{R}^{n+1})$, we have

$$\|e^{-isG}\Delta_j f\|_{L_t^\infty L_s^q L_x^p} \leq 2^{\frac{\sigma j}{2}}\|\Delta_j f\|_{L^2(\mathbb{R}^{n+1})}, \quad j \geq -1,$$

where $\sigma = \frac{n+2}{2} - \frac{2}{q} - \frac{n}{p}$. Let $0 < \varepsilon \ll 1$. Applying the triangle inequality to the frequency decomposition (2.2.18), followed by the Cauchy–Schwarz inequality, we obtain

$$\|e^{-isG}f\|_{L_t^\infty L_s^q L_x^p} \leq \sum_{j\geq -1} \|e^{-isG}\Delta_j f\|_{L_t^\infty L_s^q L_x^p} \leq C_\varepsilon \left(\sum_{j\geq -1} 2^{(\sigma+\varepsilon)j} \|\Delta_j f\|^2_{L^2(\mathbb{R}^{n+1})} \right)^{\frac{1}{2}}.$$

Therefore, estimate (2.5.2) follows from (2.2.20).

At the end point $(p,q) = (2,2)$, we observe that $\frac{n+2}{2} - \frac{2}{q} - \frac{n}{p} = 0$. Using Theorem 2.4.1 for $(p,q) = (2,2)$ and following the preceding argument, we obtain that if f is frequency localized in the ball $\mathcal{B}_R$, then

$$\|e^{-isG}f\|_{L_t^\infty L_s^2 L_x^2} \leq C\|f\|_{L^2(\mathbb{R}^{n+1})}.$$

Consequently, estimate (2.5.2) at the end point follows by density of frequency localized functions in $L^2(\mathbb{R}^{n+1})$. □

Remark 2.5.2. *The Strichartz estimate (2.5.2) is not the usual ones in terms of order of Lebesgue norms. Note that the usual Strichartz estimate, i.e., the semigroup* e^{-isG} *is bounded from* $L^2(\mathbb{R}^{n+1})$ *to* $L_s^q(\mathbb{R}; L_t^r(\mathbb{R}; L_x^p(\mathbb{R}^n)))$ *only when* $(q,r,p) = (\infty, 2, 2)$*, by Proposition 2.3.1.*

We now consider the following inhomogeneous Grushin–Schrödinger equation:

$$\begin{aligned} i\partial_s u(x,t,s) - Gu(x,t,s) &= h(x,t,s), \quad s \in \mathbb{R},\ (x,t) \in \mathbb{R}^{n+1}, \\ u(x,t,0) &= f(x,t). \end{aligned} \tag{2.5.5}$$

Using Duhamel's principle together with standard arguments, we derive the following anisotropic Strichartz estimate for the solution of the IVP (2.5.5).

Theorem 2.5.3. *Let* $n \geq 1$*. If* (p,q) *lies in the admissible set*

$$A = \left\{ (p,q) : 2 < p \leq q \leq \infty \quad \text{and} \quad \frac{2}{q} + \frac{n}{p} \leq \frac{n+2}{2} \right\},$$

then the solution $u(x,t,s)$ *of the IVP (2.5.5) is in* $L_t^\infty(\mathbb{R}; L_s^q(\mathbb{R}; L_x^p(\mathbb{R}^n)))$ *and satisfies the estimate:*

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \leq C\left(\|f\|_{H_G^\sigma} + \|h\|_{L_s^1(\mathbb{R}; H_G^\sigma)}\right). \tag{2.5.6}$$

where $\sigma > \frac{n+2}{2} - \frac{2}{q} - \frac{n}{p}$*. Moreover, at the end point* $(p,q) = (2,2)$*, the estimate (2.5.6) is valid for all functions* f *and* h *when* $n = 1$*, and for radial functions* f *and* h *when* $n \geq 2$*.*

Proof. The solution of inhomogeneous Grushin-Schrödinger equation (2.5.5) is given by the Duhamel's formula:

$$u(x,t,s) = e^{-isG}f(x,t) - i\int_0^s e^{-i(s-s')G}h(x,t,s')ds'. \tag{2.5.7}$$

Let $v(x,t,s) = i\int_0^s e^{-i(s-s')G}h(x,t,s')ds'$. Clearly we have

$$\|v(\cdot,\cdot,\cdot)\|_{L^\infty_t L^q_s L^p_x} \leq \int_{\mathbb{R}} \|e^{-i(\cdot)G}e^{is'G}h(\cdot,\cdot,s')\|_{L^\infty_t L^q_s L^p_x}ds'. \tag{2.5.8}$$

First assume that, for all s', $h(\cdot,\cdot,s')$ is frequency localized in unit ball $\mathcal{B}_1$ (or ring $\mathcal{C}_1$) in $\mathbb{R}^{n+1}$. For each s', using (2.5.3) and the unitarity of $e^{is'G}$, (2.5.8) yields

$$\|v\|_{L^\infty_t L^q_s L^p_x} \leq C\int_{\mathbb{R}} \|e^{is'G}h(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds' = C\int_{\mathbb{R}}\|h(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds'. \tag{2.5.9}$$

Now assume, for all s, $h(\cdot,\cdot,s)$ is frequency localized in $\mathcal{B}_R$ (or in $\mathcal{C}_R$). Letting

$$h_R(x,t,s) = R^{-2}h(\cdot,\cdot,R^{-2}s)\circ\delta_{R^{-1}} \quad \text{and} \quad v_R(x,t,s) = i\int_0^s e^{-i(s-s')G}h_R(x,t,s')ds',$$

we find that $h_R(\cdot,\cdot,s)$ is frequency localized in ball $\mathcal{B}_1$ (or ring $\mathcal{C}_1$) for all s and $v_R(x,t,s) = v(R^{-1}x, R^{-2}t, R^{-2}s)$. Applying (2.5.9) to h_R and using $\|v_R\|_{L^\infty_t L^q_s L^p_x} = R^{\frac{2}{q}+\frac{n}{p}}\|v\|_{L^\infty_t L^q_s L^p_x}$ with $\int_{\mathbb{R}}\|h_R(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds' = R^{\frac{n}{2}+1}\int_{\mathbb{R}}\|h(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds'$, we obtain

$$\|v\|_{L^\infty_t L^q_s L^p_x} \leq CR^{\frac{n+2}{2}-\frac{2}{q}-\frac{n}{p}}\int_{\mathbb{R}}\|h(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds'. \tag{2.5.10}$$

For $j \geq -1$, let $h_j(\cdot,\cdot,s) = \Delta_j h(\cdot,\cdot,s)$ for all s, and let

$$v_j(x,t,s) = i\int_0^s e^{-i(s-s')G}h_j(x,t,s')ds'.$$

Clearly, $h(x,t,s) = \sum_{j\geq -1} h_j(x,t,s)$ and $v(x,t,s) = \sum_{j\geq -1} v_j(x,t,s)$, for all (x,t,s). Now, for $j \geq -1$, from (2.5.10) we have

$$\|v_j\|_{L^\infty_t L^q_s L^p_x} \leq C\int_{\mathbb{R}} 2^{\frac{\sigma j}{2}}\|\Delta_j h(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds',$$

where $\sigma = \frac{n+2}{2} - \frac{2}{q} - \frac{n}{p}$. Applying the triangle inequality, followed by the Cauchy–Schwarz inequality, we obtain

$$\|v\|_{L^\infty_t L^q_s L^p_x} \leq C\int_{\mathbb{R}}\sum_{j\geq -1} 2^{\frac{\sigma j}{2}}\|\Delta_j h(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds'$$

$$\leq C_\varepsilon \int_{\mathbb{R}} \left(\sum_{j\geq -1} 2^{(\sigma+\varepsilon)j} \|\Delta_j h(\cdot,\cdot,s')\|^2_{L^2(\mathbb{R}^{n+1})} \right)^{\frac{1}{2}} ds'.$$

Therefore, from (2.2.20), we obtain

$$\|v\|_{L^\infty_t L^q_s L^p_x} \leq C\|h\|_{L^1_s(\mathbb{R};H^\sigma_G)}, \tag{2.5.11}$$

where $\sigma > \frac{n+2}{2} - \frac{2}{q} - \frac{n}{p}$. Combining the estimate for the first term in (2.5.7) from Theorem 2.5.1 together with (2.5.11), we get (2.5.6).

At the end point $(p,q) = (2,2)$, we observe that $\frac{n+2}{2} - \frac{2}{q} - \frac{n}{p} = 0$. Using density of frequency localized functions on balls, (2.5.10) turns out to be

$$\|v\|_{L^\infty_t L^2_s L^2_x} \leq C\|h\|_{L^1_s(\mathbb{R};L^2_{x,t}(\mathbb{R}^{n+1}))}, \tag{2.5.12}$$

and holds for all $h \in L^1(\mathbb{R}; L^2(\mathbb{R}^{n+1}))$. Combining the estimate for the first term in (2.5.7) from Theorem 2.5.1 at the end point, we obtain (2.5.6) at the end point. ☐

2.6 Anisotropic Strichartz estimate for the Grushin wave equation

Consider the following free Grushin wave equation:

$$\begin{aligned} \partial_s^2 u(x,t,s) + Gu(x,t,s) &= 0 \quad s \in \mathbb{R},\ (x,t) \in \mathbb{R}^{n+1}, \\ u(x,t,0) = f(x,t), &\quad \partial_s u(x,t,0) = g(x,t). \end{aligned} \tag{2.6.1}$$

Solution to the above IVP (2.6.1) can be realized as the extension operator $\mathcal{E}_{S_w}$ acting on a suitable function on the surface S_w (defined in Remark 2.4.3). Using of scaled Hermite-Fourier restriction theorem for the surface S_w, we prove the following an anisotropic Strichartz estimate for the solution of the free Grushin wave equation.

Theorem 2.6.1. *Let $n, p, q \geq 1$. If (p,q) lies in the admissible set*

$$A_w = \left\{ (p,q) : 2 < p \leq q \leq \infty \quad \textit{and} \quad \frac{1}{q} + \frac{n}{p} \leq \frac{n+2}{2} - 1 \right\},$$

then the solution $u(x,t,s)$ of the IVP (2.6.1) is in $L^\infty_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ and satisfies the estimate:

$$\|u(x,t,s)\|_{L^\infty_t L^q_s L^p_x} \leq C\left(\|G^{1/2} f\|_{H^\sigma_G} + \|g\|_{H^\sigma_G} \right), \tag{2.6.2}$$

where $\sigma > \frac{n}{2} - \frac{1}{q} - \frac{n}{p}$.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$ with $G^{-1/2}g \in L^2(\mathbb{R}^{n+1})$. Using (2.2.11) and the inversion formula (2.2.9), the solution of (2.6.1) (with $h = 0$) is given by

$$u(x,t,s) = \sum_{\pm} \frac{1}{2\pi} \int_{\mathbb{R}^*} e^{-i\lambda t} \sum_{\alpha \in \mathbb{N}^n} e^{\mp is\sqrt{(2|\alpha|+n)|\lambda|}} \widehat{\varphi_{\pm}}(\alpha, \lambda) \Phi_\alpha^\lambda(x) d\lambda, \tag{2.6.3}$$

where $\widehat{\varphi_{\pm}} = \frac{1}{2}\left(\hat{f} \mp i\widehat{G^{-1/2}g}\right)$.

Let the surface $S_w = S_+ \cup S_-$ endowed with the measure $d\sigma_{\pm}$, where $S_{\pm}$, $d\sigma_{\pm}$ are defined in Remark 2.4.3 and $\mathit{\Theta} = \widehat{\varphi_{\pm}} \circ \pi|_{S_{\pm}}$ on each sheet. With this (2.6.3) can be written as $u(x,t,s) = \mathcal{E}_{S_w}(\mathit{\Theta})(x,t,s)$. Assume that $\varphi\pm$ are frequency localized in $\mathcal{C}_1$. Proceeding as in proof of Theorem 2.5.1 for the surface $(S_w, d\sigma_{\pm})$ and using (2.4.8), we obtain

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \leq C\|\mathit{\Theta}\|_{L^2(S,d\sigma_{\pm})} = \|\widehat{\varphi_{\pm}}\|_{L^2(\mathbb{N}_0^n \times \mathbb{R}^*)} = \|\varphi_{\pm}\|_{L^2(\mathbb{R}^{n+1})}, \tag{2.6.4}$$

for $2 < p \leq q \leq \infty$.

If $\varphi_{\pm}$ are frequency localized in $\mathcal{C}_R$, then the functions $\varphi_{\pm,R} = \varphi_{\pm} \circ \delta_{R^{-1}}$ are frequency localized in $\mathcal{C}_1$ and give rise to the solution $u_R(x,t,s) = u(R^{-1}x, R^{-2}t, R^{-1}s)$. Thus, using (2.6.4) we obtain

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \leq CR^{\frac{n+2}{2} - \frac{1}{q} - \frac{n}{p}} \|\varphi_{\pm}\|_{L^2(\mathbb{R}^{n+1})} \leq CR^{\frac{n+2}{2} - 1 - \frac{1}{q} - \frac{n}{p}} \|G^{1/2}\varphi_{\pm}\|_{L^2(\mathbb{R}^{n+1})},$$

where the last inequality follows from (2.2.16). By the Plancherel formula, we have

$$\|G^{1/2}\varphi_{\pm}\|^2_{L^2(\mathbb{R}^{n+1})} = \|G^{1/2}\varphi_+\|^2_{L^2(\mathbb{R}^{n+1})} + \|G^{1/2}\varphi_-\|^2_{L^2(\mathbb{R}^{n+1})} = \|G^{1/2}f\|^2_{L^2(\mathbb{R}^{n+1})} + \|g\|^2_{L^2(\mathbb{R}^{n+1})}.$$

Consequently, if f, g are frequency localized in $\mathcal{C}_R$, then

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \leq C\left(R^\sigma \|G^{1/2}f\|_{L^2(\mathbb{R}^{n+1})} + R^\sigma \|g\|_{L^2(\mathbb{R}^{n+1})}\right),$$

where $\sigma = \frac{n+2}{2} - 1 - \frac{1}{q} - \frac{n}{p}$. Now using frequency decomposition of f and g (as in the proof of Theorem 2.5.1), we obtain (2.6.2). □

Remark 2.6.2. *In [9], Bahouri-Gérard-Xu derived a (usual) Strichartz estimate for the wave equation associated with the sublaplacian on the Heisenberg group, we can expect a analogue result in case of the Grushin operator. However, the above theorem may be viewed as an extension of Theorem 1.1 in [9] in the context of Grushin operator.*

Now consider the following free Grushin wave equation:

$$\begin{aligned} \partial_s^2 u(x,t,s) + Gu(x,t,s) &= h(x,t,s) \quad s \in \mathbb{R},\ (x,t) \in \mathbb{R}^{n+1}, \\ u(x,t,0) = f(x,t), \quad & \partial_s u(x,t,0) = g(x,t). \end{aligned} \tag{2.6.5}$$

We obtain the following anisotropic Strichartz estimate for the solution of the IVP (2.6.5).

Theorem 2.6.3. *Let $n,p,q \geq 1$. If (p,q) lies in the admissible set*

$$A_w = \left\{ (p,q) : 2 < p \leq q \leq \infty \quad \textit{and} \quad \frac{1}{q} + \frac{n}{p} \leq \frac{n+2}{2} - 1 \right\},$$

then the solution $u(x,t,s)$ of the IVP (2.6.5) is in $L_t^\infty(\mathbb{R}; L_s^q(\mathbb{R}; L_x^p(\mathbb{R}^n)))$ and satisfies the estimate:

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \leq C \left(\|G^{1/2} f\|_{H_G^\sigma} + \|g\|_{H_G^\sigma} + \|h\|_{L_s^1(\mathbb{R}; H_G^\sigma)} \right), \tag{2.6.6}$$

where $\sigma > \frac{n}{2} - \frac{1}{q} - \frac{n}{p}$.

Proof. One can apply Duhamel's principle to the IVP (2.6.5) and follow arguments similar to those used for the inhomogeneous Grushin–Schrödinger equation (2.5.5) discussed above to establish Theorem 2.5.3. This yields the bound (2.6.6), which establishes Theorem 2.6.3. The details of the proof are left to the reader. ☐

2.7 Appendix

Let us recall a simplified pointwise estimates for the Hermite functions $\{h_k\}_{k \in \mathbb{N}_0}$ (see [53], Corollary 2.8). For $k \in \mathbb{N}_0$, we denote $\lambda_k = \sqrt{2k+1}$.

Lemma 2.7.1 (Rough pointwise estimates for Hermite functions)**.** *There exits $C > 0$ such that for any $k \in \mathbb{N}_0$ and $x \in \mathbb{R}$,*

$$|h_k(x)| \leq C \begin{cases} \lambda_k^{-\frac{1}{2}} & \textit{if} \quad |x| \leq \frac{\lambda_k}{2} \\ \left(\lambda_k^{\frac{2}{3}} + |x^2 - \lambda_k^2| \right)^{-\frac{1}{4}} & \textit{if} \quad \frac{\lambda_k}{2} \leq |x| \leq 2\lambda_k \\ e^{-\frac{x^2}{8}} & \textit{if} \quad |x| \geq 2\lambda_k. \end{cases}$$

Using the previous lemma, we derive the following proposition, which plays a crucial role in proving the end point case for $n=1$ in Theorem 2.4.1.

Proposition 2.7.2. *There exists $C>0$ such that for any $k,l\in\mathbb{N}$,*

$$\frac{1}{(2k+1)^{\frac{3}{4}}(2l+1)^{\frac{3}{4}}}\int_{\mathbb{R}}\left|h_k\left(\frac{x}{\sqrt{2k+1}}\right)\right|\left|h_l\left(\frac{x}{\sqrt{2l+1}}\right)\right|dx\le\frac{C}{\max\{k,l\}}. \tag{2.7.1}$$

Proof. Let $k\le l$. We split the region of the integration in (2.7.1) into three parts and estimate each part separately.

(1) In the region $\{x\in\mathbb{R}:|x|\le 2\lambda_k^2\}$, applying Hölder inequality and using the estimate $\left\|h_k\left(\lambda_k^{-1}\cdot\right)h_l\left(\lambda_l^{-1}\cdot\right)\right\|_{L^2(\mathbb{R})}\le\frac{\lambda_k^{\frac{1}{2}}}{(2l+1)^{\frac{1}{4}}}$, in Corollary 5.2 of [53], we obtain

$$\int_{|x|\le 2\lambda_k^2}\left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right|dx\le 2\lambda_k\left\|h_k\left(\lambda_k^{-1}\cdot\right)h_l\left(\lambda_l^{-1}\cdot\right)\right\|_{L^2(\mathbb{R})}\le\frac{(2k+1)^{\frac{3}{4}}}{(2l+1)^{\frac{1}{4}}}.$$

(2) In the region $\{x\in\mathbb{R}:2\lambda_k^2\le|x|\le 2\lambda_l^2\}$, we use the pointwise estimates in Lemma 2.7.1.

Case I: Assume $\frac{1}{2}\lambda_l^2\le 2\lambda_k^2$. Then

$$\begin{aligned}\int_{2\lambda_k^2\le|x|\le 2\lambda_l^2}\left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right|dx&\le C\int_{2\lambda_k^2}^{2\lambda_l^2}e^{-\frac{x^2}{8\lambda_k^2}}\frac{1}{\left(\lambda_l^{\frac{2}{3}}+|\lambda_l^{-2}x^2-\lambda_l^2|\right)^{\frac{1}{4}}}dx\\&=C\,\lambda_l^{\frac{3}{2}}\int_{2\frac{\lambda_k^2}{\lambda_l^2}}^{2}e^{-\frac{\lambda_l^4}{8\lambda_k^2}x^2}\frac{1}{\left(\lambda_l^{-\frac{4}{3}}+|x^2-1|\right)^{\frac{1}{4}}}dx\\&\le C\,\lambda_l^{\frac{3}{2}}\int_{\frac{1}{2}}^{2}e^{-\frac{\lambda_l^4}{8\lambda_k^2}x^2}\frac{1}{\left(\lambda_l^{-\frac{4}{3}}+|x^2-1|\right)^{\frac{1}{4}}}dx\\&\le C\,\lambda_l^{\frac{3}{2}}\,e^{-\frac{\lambda_l^4}{32\lambda_k^2}}\int_{\frac{1}{2}}^{2}\frac{1}{|x^2-1|^{\frac{1}{4}}}dx\\&\le\frac{C}{(2l+1)^{\frac{1}{4}}},\end{aligned}$$

where the second equality is obtained by changing the variable $x\mapsto\lambda_l^2x$ and the last inequality follows from the fact that $e^{-\frac{\lambda_l^4}{32\lambda_k^2}}\le\frac{32\lambda_k^2}{\lambda_l^4}$.

Case II: Assume $2\lambda_k^2\le\frac{1}{2}\lambda_l^2$. Then

$$\begin{aligned}\int_{2\lambda_k^2\le|x|\le\frac{1}{2}\lambda_l^2}\left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right|dx&\le C\,\lambda_l^{-\frac{1}{2}}\int_{2\lambda_k^2}^{\frac{1}{2}\lambda_l^2}e^{-\frac{x^2}{8\lambda_k^2}}dx\\&\le C\,\lambda_l^{-\frac{1}{2}}\int_{2\lambda_k^2}^{\frac{1}{2}\lambda_l^2}\frac{8\lambda_k^2}{x^2}dx\end{aligned}$$

$$\le \frac{C}{(2l+1)^{\frac{1}{4}}},$$

and arguing as in the Case I, we obtain $\displaystyle\int_{\frac{1}{2}\lambda_l^2\le|x|\le 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right| \left|h_l\left(\lambda_l^{-1}x\right)\right| dx \le \frac{C}{(2l+1)^{\frac{1}{4}}}$.

Thus

$$\int_{2\lambda_k^2\le|x|\le 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right| \left|h_l\left(\lambda_l^{-1}x\right)\right| dx \le \frac{C}{(2l+1)^{\frac{1}{4}}}.$$

(3) In the region $\{x\in\mathbb{R} : |x|\ge 2\lambda_l^2\}$, again we use the Lemma 2.7.1. We obtain

$$\int_{|x|\ge 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right| \left|h_l\left(\lambda_l^{-1}x\right)\right| dx \le C\int_{2\lambda_l^2}^{\infty} e^{-\left(\frac{\lambda_k^2+\lambda_l^2}{8\lambda_k^2\lambda_l^2}\right)x^2} dx.$$

Then writing $A=\frac{\lambda_k^2+\lambda_l^2}{8\lambda_k^2\lambda_l^2}$ and $X=2\lambda_l^2$, we have

$$\int_X^{\infty} e^{-Ax^2}dx \le \frac{1}{2AX}\int_X^{\infty} 2Axe^{-Ax^2}dx = \frac{1}{2AX}e^{-AX^2} \le \frac{1}{2AX}\left(\frac{1}{\sqrt{AX}}\right).$$

Thus

$$\int_{|x|\ge 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right| \left|h_l\left(\lambda_l^{-1}x\right)\right| dx \le C\left(\frac{\lambda_k^2}{\lambda_k^2+\lambda_l^2}\right)^{\frac{3}{2}}\frac{1}{\lambda_l} \le \frac{C}{(2l+1)^{\frac{1}{2}}}.$$

After combining the estimates obtained in each case we get (2.7.1). □

CHAPTER 3

Strichartz estimate for orthonormal functions associated with the special Hermite operator

3.1 Introduction

Let $\mathcal{L}$ be the special Hermite operator on $\mathbb{C}^n$, and let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function. The function $e^{-it\phi(\mathcal{L})}f$ denotes the solution to the initial value problem

$$\begin{aligned} i\partial_t u(t,z) - \phi(\mathcal{L})u(t,z) &= 0, \quad z \in \mathbb{C}^n,\, t \in \mathbb{R}\setminus\{0\}, \qquad (3.1.1)\\ u(0,z) &= f(z). \end{aligned}$$

This chapter is concerned with extended versions of the Strichartz estimates of the form

$$\left\| \sum_{j\in J} n_j \left| e^{-it\phi(\mathcal{L})} f_j \right|^2 \right\|_{L^p(I,L^q(\mathbb{C}^n))} \leqslant CN^\sigma \left(\sum_{j\in J} |n_j|^\beta \right)^{1/\beta}, \qquad (3.1.2)$$

for all orthonormal system $(f_j)_j$ in $L^2(\mathbb{C}^n)$ with $supp \hat{f}_j \subset \{(\mu,\nu) : \sqrt{2|\nu|+n} \le N\}, N \geq 2n$ and all sequence $(n_j)_j \subset \mathbb{C}$, for a bounded interval $I \subset \mathbb{R}$.

More precisely, we prove (3.1.2) under the assumption that ϕ is of power type near ∞, see (3.3.4). Moreover, the estimate (3.1.2) can be upgraded to a global-type estimate (see Theorem 3.4.3), provided that the orthonormal system $(f_j)_j$ lies in the Sobolev space

$H^s(\mathcal{L})$ with $s > \frac{\sigma}{2}$. We then address the optimality of the Schatten exponent and the end point case of the orthonormal Strichartz estimate for the Schrödinger propagator $e^{-it\mathcal{L}}$. Furthermore, we generalize restriction estimates for the special Hermite spectral projections in the context of Schatten spaces.

3.2 Preliminaries

In this section, we provide some basic definitions and discuss certain dispersive semigroups associated with the special Hermite operator.

3.2.1 The special Hermite operator and the special Hermite functions

The special Hermite functions are defined as the Fourier-Wigner transform of the Hermite functions Φ_μ and Φ_ν on $\mathbb{R}^n$ (defined in Section 2.2), namely,

$$\Phi_{\mu,\nu}(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Phi_\mu\left(\xi + \frac{y}{2}\right) \Phi_\nu\left(\xi - \frac{y}{2}\right) d\xi, \quad z = x + iy \in \mathbb{C}^n.$$

The family of functions $\{\Phi_{\mu,\nu}\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$, leading to the special Hermite expansion

$$f(z) = \sum_{\mu,\nu\in\mathbb{N}_0^n} \langle f, \Phi_{\mu,\nu}\rangle \Phi_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \left(\sum_{|\nu|=k} \sum_{\mu\in\mathbb{N}_0^n} \langle f, \Phi_{\mu,\nu}\rangle \Phi_{\mu,\nu}(z) \right). \tag{3.2.1}$$

The spectral projection operator $\mathcal{Q}_k$ onto the eigenspace of $\mathcal{L}$ associated to the eigenvalue $2k + n$ is given by

$$\mathcal{Q}_k f = \sum_{|\nu|=k} \sum_{\mu\in\mathbb{N}_0^n} \langle f, \Phi_{\mu,\nu}\rangle \Phi_{\mu,\nu}(z), \quad f \in \mathcal{S}(\mathbb{C}^n). \tag{3.2.2}$$

Consider the special Hermite operator (also called twisted Laplacian) $\mathcal{L}$ on $\mathbb{C}^n, n \geq 1$, defined by

$$\mathcal{L} = -\Delta_z + \frac{1}{4}|z|^2 - i \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right),$$

with $z = x + iy \in \mathbb{C}^n$. The spectrum of this operator is discrete and consists of points $2k + n, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the eigenspaces associated to each of these eigenvalues are infinite-dimensional. These eigenspaces are spanned by $\Phi_{\mu,\nu}, \mu, \nu \in \mathbb{N}_0^n$, $|\nu| = \nu_1 + \cdots + \nu_n = k$, where $\Phi_{\mu,\nu}$ are the special Hermite functions. For $\mu, \nu \in \mathbb{N}_0^n$, let $\hat{f}(\mu, \nu) = \langle f, \Phi_{\mu,\nu}\rangle$ denote the special Hermite coefficients.

The spectral decomposition of $\mathcal{L}$ can be written as

$$\mathcal{L}f = \sum_{k=0}^{\infty}(2k+n)\mathcal{Q}_k f.$$

The twisted convolution of two functions f and g on $\mathbb{C}^n$ is defined by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\operatorname{Im}(z\cdot\bar{w})}dw = \int_{\mathbb{C}^n} f(w)g(z-w)e^{-\frac{i}{2}\operatorname{Im}(z\cdot\bar{w})}dw, \quad z \in \mathbb{C}^n. \tag{3.2.3}$$

The family $\{\Phi_{\mu,\nu}\}$ satisfies the following orthogonality properties

$$\Phi_{\mu,\nu} \times \Phi_{\alpha,\beta} = \begin{cases} (2\pi)^{n/2}\Phi_{\mu,\beta}, & \text{if } \nu = \alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.4}$$

The special Hermite functions $\Phi_{\nu,\nu}$ are related to the Laguerre functions $\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)e^{-\frac{1}{4}|z|^2}$, where L_k^{n-1} is the Laguerre polynomial of type $(n-1)$, by the following relation

$$(2\pi)^{n/2}\sum_{|\nu|=k}\Phi_{\nu,\nu} = \varphi_k. \tag{3.2.5}$$

Now taking twisted convolution on both sides of (3.2.1) with $\Phi_{\alpha,\alpha}$ and using the orthogonality property (3.2.4), we have

$$f \times \Phi_{\alpha,\alpha} = (2\pi)^{n/2}\sum_{\mu\in\mathbb{N}_0^n}\langle f, \Phi_{\mu,\alpha}\rangle\,\Phi_{\mu,\alpha}, \quad \alpha \in \mathbb{N}_0^n. \tag{3.2.6}$$

Summing both sides of (3.2.6) with respect to all α such that $|\alpha| = k$ and using (3.2.5), the spectral projection $\mathcal{Q}_k$ has the simpler representation

$$\mathcal{Q}_k f(z) = (2\pi)^{-\frac{n}{2}}\sum_{|\alpha|=k} f \times \Phi_{\alpha,\alpha}(z) = (2\pi)^{-n} f \times \varphi_k(z), \quad z \in \mathbb{C}^n.$$

For a detailed study on the special Hermite operator and its associated functions, we refer the reader to the monograph by Thangavelu [106].

3.2.2 Dispersive semigroups for the Special Hermite Operator

Consider the Schrödinger equation on $\mathbb{C}^n$:

$$i\partial_t u(t,z) - \mathcal{L}u(t,z) = 0, \quad z \in \mathbb{C}^n,\, t \in \mathbb{R} \setminus \{0\}, \tag{3.2.7}$$
$$u(0,z) = f(z).$$

If $f \in L^2(\mathbb{C}^n)$, then the solution to the IVP (3.2.7) is $e^{-it\mathcal{L}}f$. The Schrödinger propagator can be expressed by using the spectral decomposition of $\mathcal{L}$, that is

$$e^{-it\mathcal{L}}f = \sum_{k=0}^{\infty} e^{-(2k+n)it}\mathcal{Q}_k f. \tag{3.2.8}$$

So, we clearly have

$$\|e^{-it\mathcal{L}}f\|_{L^2(\mathbb{C}^n)} = \|f\|_{L^2(\mathbb{C}^n)}, \quad t \in \mathbb{R}. \tag{3.2.9}$$

The Schrödinger propagator $e^{-it\mathcal{L}}$ also has the following kernel representation:

$$e^{-it\mathcal{L}}f(z) = \int_{\mathbb{C}^n} f(z-w)K_{it}(w)e^{\frac{i}{2}\operatorname{Im}(z\cdot\bar{w})}dw, \tag{3.2.10}$$

where the kernel is given by

$$K_{it}(w) = (2\pi i \sin t)^{-n} e^{i\cot t \frac{|w|^2}{4}}.$$

This can be easily deduced from the corresponding kernel formula for the heat operator $e^{-t\mathcal{L}}$ by replacing t with it (see [106], page 29). Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function satisfying (3.3.4) with $0 < m \leq 1$. Consider the following Schrödinger equation on $\mathbb{C}^n$:

$$i\partial_t u(t,z) - \phi(\mathcal{L})u(t,z) = 0, \quad z \in \mathbb{C}^n,\, t \in \mathbb{R} \setminus \{0\}, \tag{3.2.11}$$
$$u(0,z) = f(z).$$

If $f \in L^2(\mathbb{C}^n)$, then the solution to the IVP (3.2.11), is given by $u(t,z) = e^{-it\phi(\mathcal{L})}f(z)$, where

$$e^{-it\phi(\mathcal{L})}f = \sum_{k=0}^{\infty} e^{-it\phi(2k+n)}\mathcal{Q}_k f. \tag{3.2.12}$$

From Theorem 3.3 of [26] (see also [45]), it follows that the following local dispersive estimate holds:

$$\left\| \psi(h^{-1}\sqrt{\mathcal{L}})e^{-it\phi(\mathcal{L})}f \right\|_{L^\infty(\mathbb{C}^n)} \leq |t|^{-\frac{2n-1}{2}} h^{(1-m)2n+m} \|f\|_1, \tag{3.2.13}$$

where $\psi \in C^\infty(\mathbb{R})$ supported in $[1/2, 2]$, $h \geq 1$ and $|t| \leq 2T_0 < \pi$. As a consequence of the dispersive estimate (3.2.13) and the Keel–Tao result [70], the following Strichartz estimate holds for a single function.

Theorem 3.2.1. *Let $f \in L^2(\mathbb{C}^n)$. Suppose $p \geq 1$ and $1 \leq q < \infty$ satisfy $\frac{2}{p}+\frac{2n-1}{q} = 2n-1$. Then, for $\psi \in C^\infty(\mathbb{R})$ supported in $[1/2, 2]$ and $h \geq 1$, we have*

$$\left\|\psi\left(h^{-1}\sqrt{\mathcal{L}}\right) e^{-it\phi(\mathcal{L})} f\right\|_{L^{2p}((-T_0,T_0);\, L^{2q}(\mathbb{C}^n))} \leq h^{\frac{2n(1-m)+m}{2}\left(1-\frac{1}{q}\right)} \|f\|_{L^2(\mathbb{C}^n)}. \tag{3.2.14}$$

3.2.3 Analytic families of operators

We briefly recall the notion of analytic families of operators in the sense of Stein, which plays a central role in our approach.

A family of operators (G_α) on $\mathbb{C}^n$, defined on a strip $a \leq \operatorname{Re}\alpha \leq b$ in the complex plane (with $a < b$) is said to be analytic in the sense of Stein if, for all simple functions f, g on $\mathbb{C}^n$ (that is, functions that take a finite number of non-zero values on sets of finite measure in $\mathbb{C}^n$), the map

$$\alpha \mapsto \langle g, G_\alpha f\rangle$$

is analytic in the strip $a < \operatorname{Re}\alpha < b$, continuous on the closure $a \leq \operatorname{Re}\alpha \leq b$, and if

$$\sup_{a\leq\lambda\leq b} \left|\langle g, G_{\lambda+i\tau} f\rangle\right| \leq C(\tau),$$

for some function $C(\tau)$ with at most (double) exponential growth in τ.

3.2.4 Schatten class

Let X be a measure space. Let $A : L^2(X) \to L^2(X)$ be a compact operator and let A^* denotes the adjoint of A. For $1 \leq r < \infty$, $\mathcal{G}^r(L^2(X))$ denotes the Schatten space based on $L^2(X)$ that is the space of all compact operators A on $L^2(X)$ such that $Tr|A|^r < \infty$, where $|A| = \sqrt{A^*A}$, and its norm is defined by $\|A\|_{\mathcal{G}^r(L^2(X))} = (Tr|A|^r)^{\frac{1}{r}}$. If $r = \infty$, we define

$$\|A\|_{\mathcal{G}^\infty(L^2(X))} = \|A\|_{L^2(X)\to L^2(X)}.$$

Also, the case $r = 2$ is special in the sense that $\mathcal{G}^2(L^2(X))$ is the Hilbert-Schmidt class, equipped with the norm

$$\|A\|_{\mathcal{G}^2(L^2(X))} = \|K\|_{L^2(X\times X)},$$

if K is the integral kernel of A. For more details on Schatten classes, we refer the reader to Simon [95].

3.3 Restriction theorem for the Fourier-special Hermite transform

For $F \in L^1(\mathbb{R} \times \mathbb{C}^n)$ the Fourier-special Hermite transform of F is given by

$$\hat{F}(\lambda, \mu, \nu) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{C}^n} F(t, w)\Phi_{\mu,\nu}(w)e^{i\lambda t}\, dwdt, \quad \forall \mu, \nu \in \mathbb{N}_0^n, \lambda \in \mathbb{R}. \tag{3.3.1}$$

If $F \in L^2(\mathbb{R}\times\mathbb{C}^n)$, then $\hat{F} \in L^2(\mathbb{R}\times\mathbb{N}_0^{2n}, d\lambda\times d\sigma)$, where $d\lambda$ and $d\sigma$ denotes the Lebesgue measure on $\mathbb{R}$ and the counting measure on $\mathbb{N}_0^{2n}$, respectively. The Plancherel formula is of the form

$$\|F\|_{L^2(\mathbb{R}\times\mathbb{C}^n)} = \|\hat{F}(\lambda, \mu, \nu)\|_{L^2(\mathbb{R}\times\mathbb{N}_0^{2n})}$$

and the inverse Fourier–special Hermite transform is given by

$$F(t, z) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \sum_{(\mu,\nu)\in\mathbb{N}_0^{2n}} \hat{F}(\lambda, \mu, \nu)\Phi_{\mu,\nu}(z)e^{-it\lambda}d\lambda.$$

Given a surface S in $\mathbb{R} \times \mathbb{N}_0^{2n}$ with a positive measure $d\Sigma$, we define the restriction operator $(\mathcal{R}_S F) := \{\hat{F}(\lambda, \mu, \nu)\}_{(\lambda,\mu,\nu)\in S}$ and the operator dual to $\mathcal{R}_S$ (called the extension operator) as

$$\mathcal{E}_S(\{\hat{F}(\lambda, \mu, \nu)\})(t, z) := (2\pi)^{-\frac{1}{2}} \int_S e^{-it\lambda}\hat{F}(\lambda, \mu, \nu)\Phi_{\mu,\nu}(z)d\Sigma.$$

We consider the following special Hermite restriction problem: Let $I \subset \mathbb{R}$ be an interval.
Problem 1: For which exponents $1 \leq p, q \leq 2$ is the Fourier-Hermite restriction operator $\mathcal{R}_S$ bounded from $L^p(I, L^q(\mathbb{C}^n))$ to $L^2(S, d\Sigma)$, i. e.,

$$\|\mathcal{R}_S f\|_{L^2(S,d\Sigma)} \leq \|f\|_{L^p(I,L^q(\mathbb{C}^n))}?$$

By duality, it is not difficult to prove that the boundedness of $\mathcal{R}_S$ from $L^p(I, L^q(\mathbb{C}^n))$ to $L^2(S, d\sigma)$ is equivalent to the boundedness of the operator $\mathcal{E}_S(\mathcal{E}_S)^*$ from $L^p(I, L^q(\mathbb{C}^n))$ to $L^{p'}(I, L^{q'}(\mathbb{C}^n))$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, Problem 1 can be re-written as follows:

Problem 2: For which exponents $1 \leq p, q \leq 2$, the operator $\mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p(I, L^q(\mathbb{C}^n))$ to $L^{p'}(I, L^{q'}(\mathbb{C}^n))$

$$\|\mathcal{E}_S(\mathcal{E}_S)^* f\|_{L^{p'}(I,L^{q'}(\mathbb{C}^n))} \leq \|f\|_{L^p(I,L^q(\mathbb{C}^n))}?$$

Again, by Hölder's inequality, the boundedness of $\mathcal{E}_S(\mathcal{E}_S)^*$ from $L^p(I, L^q(\mathbb{C}^n))$ to $L^{p'}(I, L^{q'}(\mathbb{C}^n))$ is equivalent to the following: for any $W_1, W_2 \in L^{2p/(2-p)}(I, L^{2q/(2-q)}(\mathbb{C}^n))$, the operator $W_1\mathcal{E}_S(\mathcal{E}_S)^*W_2$ is bounded from $L^2(I \times \mathbb{C}^n)$ to $L^2(I \times \mathbb{C}^n)$ with the estimate

$$\|W_1\mathcal{E}_S(\mathcal{E}_S)^*W_2\|_{L^2(I\times\mathbb{C}^n)\to L^2(I\times\mathbb{C}^n)} \leq C\|W_1\|_{L^{2p/(2-p)}(I,L^{2q/(2-q)}(\mathbb{C}^n))}\|W_2\|_{L^{2p/(2-p)}(I,L^{2q/(2-q)}(\mathbb{C}^n))} \tag{3.3.2}$$

with $C > 0$ independent of W_1, W_2.

For certain surface S, a stronger result than (3.3.2) holds: the operator $W_1\mathcal{E}_S(\mathcal{E}_S)^*W_2$ belongs to Schatten class $\mathcal{G}^\alpha(L^2((-\pi, \pi) \times \mathbb{C}^n))$, for some $\alpha > 0$ (see Subsection 3.2.4 for definition of Schatten class). More precisely, the following theorem is proved in [57].

Theorem 3.3.1. *[57] Let $n, p, q \geq 1$ and let the surface $S = \{(\lambda, \mu, \nu) \in \mathbb{R} \times \mathbb{N}_0^n \times \mathbb{N}_0^n : \lambda = 2|\nu| + n\}$ with respect to the counting measure. Suppose $q > 2n + 1$ and $\frac{2}{p} + \frac{2n}{q} = 1$, then the estimate*

$$\|W_1\mathcal{E}_S(\mathcal{E}_S)^*W_2\|_{\mathcal{G}^q(L^2((-\pi,\pi)\times\mathbb{C}^n))} \leq C\|W_1\|_{L^p((-\pi,\pi),L^q(\mathbb{C}^n))}\|W_2\|_{L^p((-\pi,\pi),L^q(\mathbb{C}^n))} \tag{3.3.3}$$

holds for all W_1, W_2 with a constant $C > 0$ independent of W_1, W_2.

We now introduce a more general discrete surface in order to derive a localized restriction estimate. Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function such that there exists $0 < m \leq 1$ for which

$$\phi'(r) \sim r^{m-1} \quad \text{and} \quad |\phi''(r)| \geq r^{m-2}, \ r \geq 1. \tag{3.3.4}$$

Consider the discrete surface $S_\phi = \{(\lambda, \mu, \nu) \in \mathbb{R} \times \mathbb{N}_0^n \times \mathbb{N}_0^n : \lambda = \phi(2|\nu| + n)\}$ with respect to the localized measure $d\Sigma_N$, $N \geq 2n$ defined by

$$\int_{S_\phi} F(\lambda, \mu, \nu)\, d\Sigma_N = \sum_{\mu,\nu\in\mathbb{N}_0^n} F(\phi(2|\nu| + n), \mu, \nu)\psi\left(\frac{\sqrt{2|\nu| + n}}{N}\right), \tag{3.3.5}$$

where $\psi \in C_0^\infty(\mathbb{R})$ such that $\chi_{[-1,1]} \leq \psi \leq \chi_{[-2,2]}$, where χ_A denotes the characteristic function of the set $A \subset \mathbb{R}$.

Let $\mathcal{E}_{S_{\phi,N}}$ be the corresponding extension operator. We obtain the following Schatten estimate.

Theorem 3.3.2. *Let $n, p, q \geq 1$ and $N \geq 2n$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function satisfying (3.3.4) with $0 < m \leq 1$. Let $S_\phi = \{(\lambda, \mu, \nu) \in \mathbb{R} \times \mathbb{N}_0^n \times \mathbb{N}_0^n : \lambda = \phi(2|\nu| + n)\}$ with respect to the measure $d\Sigma_N$. Suppose that the pair (p, q) satisfies $\frac{2}{p} + \frac{2n-1}{q} = 1$ and let $\sigma = \frac{2(2n(1-m)+m)}{q}$. Then*

$$\|W_1 \mathcal{E}_{S_{\phi,N}} \mathcal{E}^*_{S_{\phi,N}} W_2\|_{\mathcal{G}^\alpha(L^2((-T_0,T_0)\times\mathbb{C}^n))} \leq C N^\sigma \|W_1\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \|W_2\|_{L^p((T_0,T_0),L^q(\mathbb{C}^n))} \tag{3.3.6}$$

holds for all W_1, W_2 with a constant $C > 0$ independent of W_1, W_2, provided that α satisfies the following conditions corresponding to the pair (p, q):

(i) If $n \geq 1$, $2n < q \leq \infty$, then $\alpha \geq q$.

(ii) If $n \geq 2$, $2n - 1 < q \leq 2n$, then $\alpha > \dfrac{q}{q - 2n + 1}$; and if $q = 2n - 1$, one has $\alpha = \infty$.

Proof. For all f such that $\hat{f} \in \ell^1(S_\phi)$ and for all $(t, z) \in \mathbb{R} \times \mathbb{C}^n$, the extension operator can be written as

$$\mathcal{E}_{S_{\phi,N}} f(t, z) = \sum_{(\lambda,\mu,\nu)\in S} \psi(\sqrt{\lambda}/N) e^{-it\lambda} \hat{f}(\lambda, \mu, \nu) \Phi_{\mu,\nu}(z), \tag{3.3.7}$$

where $\hat{f}(\lambda, \mu, \nu)$ is defined in (3.3.1). Now,

$$\begin{aligned}
\mathcal{E}_{S_{\phi,N}} \mathcal{E}^*_{S_{\phi,N}} f(t, z) &= \sum_{(\lambda,\mu,\nu)\in S_\phi} \psi^2 \left(\sqrt{\lambda}/N\right) e^{-it\lambda} \hat{f}(\lambda, \mu, \nu) \Phi_{\mu,\nu}(z) \\
&= (2\pi)^{-\frac{1}{2}} \sum_{(\mu,\nu,\lambda)\in S_\phi} \int_{\mathbb{R}} \psi^2 \left(\sqrt{\lambda}/N\right) e^{-it\lambda} \langle f(t', \cdot), \Phi_{\mu,\nu}\rangle \Phi_{\mu,\nu}(z)\, dwdt' \\
&= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \sum_{\mu,\nu} \psi^2 \left(\frac{\sqrt{2|\nu| + n}}{N}\right) e^{-i(t-t')\phi(2|\nu|+n)} \langle f(t', \cdot), \Phi_{\mu,\nu}\rangle \Phi_{\mu,\nu}(z)\, dwdt'.
\end{aligned}$$

Now from (3.2.6) and (3.2.5), we write

$$\begin{aligned}
\mathcal{E}_{S_{\phi,N}} \mathcal{E}^*_{S_{\phi,N}} f(t, z) &= (2\pi)^{-(n+\frac{1}{2})} \int_{\mathbb{R}} f(t', \cdot) \times \sum_{k=0}^{\infty} \psi^2 \left(\frac{\sqrt{2k + n}}{N}\right) e^{-i(t-t')\phi(2k+n)} \varphi_k(z) dt' \\
&= (2\pi)^{-(n+\frac{1}{2})} \int_{\mathbb{R}} \int_{\mathbb{C}^n} H_{\phi,N}(t - t', z - w) f(t', w) e^{-\frac{i}{2} \operatorname{Im}(z\cdot\bar{w})} dt' dw,
\end{aligned}$$

where

$$H_{\phi,N}(t, z) = \sum_{k=0}^{\infty} \psi^2 \left(\frac{\sqrt{2k + n}}{N}\right) e^{-it\phi(2k+n)} \varphi_k(z). \tag{3.3.8}$$

Let $\Psi \in C^\infty(\mathbb{R})$ be an even function such that $0 \leq \Psi \leq 1$, $\Psi = 1$ in $[0, 1]$ and $\Psi = 0$ in $[2, \infty)$. Let $\mathit{\Psi}(s) = \Psi(s) - \Psi(2s)$ such that *supp* $\mathit{\Psi} \subset (1/2, 2)$ and generate a Littlewood-Paley decomposition $\sum_{j\in\mathbb{Z}} \mathit{\Psi}_j(s) = 1$, for all $s > 0$, where $\mathit{\Psi}_j(s) = \mathit{\Psi}(2^{-j}s)$. It follows that

$$\sum_{j\geq 0} \mathit{\Psi}_j(s) = 1, \quad \forall s \geq 1.$$

For any small $\varepsilon > 0$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \in [-r/2, 0]$ and $j \geq 0$, we consider the following analytic family of operators

$$G_{\alpha,j,\varepsilon} f(t, z) = \int_{\mathbb{C}^n} \int_I H_{\phi,N,j,\varepsilon}(t - t', z - w) e^{-\frac{i}{2}\operatorname{Im}(z\cdot\bar{w})} f(t', w)\, dt' dw,$$

where the kernel

$$H_{\phi,N,j,\varepsilon}(t, z) = 1_{\varepsilon<|T_0|}\, t^{-1-\alpha} \sum_{k=0}^{\infty} \psi^2 \left(\frac{\sqrt{2k + n}}{N} \right) \mathit{\Psi}_j \left(\sqrt{2k + n} \right) e^{-it\phi(2k+n)} \varphi_k(z),$$

and $I = [-T_0, T_0]$. But from (3.2.13), for every $t \in [-2T_0, 2T_0]$ and $z \in \mathbb{C}^n$, we have

$$|H_{\phi,N,j,\varepsilon}(t, z)| \leq C|t|^{-\frac{2\operatorname{Re}(\alpha)+2n+1}{2}} 2^{((1-m)2n+m)j}. \tag{3.3.9}$$

Let $z_1 = -\frac{r}{2} + i\tau, \tau \in \mathbb{R}$, using the Hardy–Littlewood–Sobolev inequality, (see page 39 in [13]) along with (3.3.9) yields

$$\begin{aligned}
\|W_1 G_{z_1,j,\varepsilon} W_2\|^2_{\mathcal{G}^2(L^2(I\times\mathbb{C}^n))} &= \int_{I\times I} \int_{\mathbb{C}^{2n}} |W_1(t, z)|^2 |H_{\phi,N,j,\varepsilon}(t-t', z-w)|^2 |W_2\,(t', w)\,|^2 dz dw dt dt' \\
&\leq C 2^{((1-m)2n+m)2j} \int_I \int_I \frac{\|W_1(t)\|^2_{L^2(\mathbb{C}^n)}\, \|W_2\,(t')\|^2_{L^2(\mathbb{C}^n)}}{|t - t'|^{2n+1-r}} dt dt' \\
&\leq C 2^{((1-m)2n+m)2j} \left\| \|W_1\|^2_{L^2_w(\mathbb{C}^n)} \right\|_{L^u_t(I)} \left\| \|W_2\|^2_{L^2_w(\mathbb{C}^n)} \right\|_{L^u_t(I)},
\end{aligned}$$

provided we have $0 \leq 2n + 1 - r < 1$ and $\frac{1}{u} + \frac{2n+1-r}{2} = 1$. Thus, for $2n < r \leq 2n + 1$, we obtain

$$\|W_1 G_{z_1,j,\varepsilon} W_2\|_{\mathcal{G}^2(L^2(I\times\mathbb{C}^n))} \leq C 2^{((1-m)2n+m)j} \|W_1\|_{L^{\frac{4}{r-2n+1}}(I, L^2(\mathbb{C}^n))} \|W_2\|_{L^{\frac{4}{r-2n+1}}(I, L^2(\mathbb{C}^n))}$$

with the constant C independent of ε and τ.

Next, we consider the case $z_2 = i\tau, \tau \in \mathbb{R}$. We show that $G_{z_2,j,\varepsilon} : L^2\,(I \times \mathbb{C}^n) \to L^2\,(I \times \mathbb{C}^n)$ is bounded with some constant that only depends on the dimension n and τ exponentially. After a simple calculation, we write

$$G_{z_2,j,\varepsilon} f(t, z) = (2\pi)^n \sum_{\mu,\nu} \Phi_{\mu,\nu}(z) \psi \left(\frac{\sqrt{2|\nu| + n}}{N} \right) \mathit{\Psi}_j \left(\sqrt{2|\nu| + n} \right)$$

$$\cdot \int_{\varepsilon<|t'|<T_0} e^{-it'(2|\nu|+n)^{\varrho}} t'^{-1-\alpha} \hat{f}_2(t-t',\cdot)(\mu,\nu)dt',$$

where $\hat{f}_2$ denotes the special Hermite transform of f with respect to the second variable. Then using Plancherel's theorem, for each $t \in I$, we have

$$\|G_{z_2,j,\varepsilon}f(t,\cdot)\|^2_{L^2(\mathbb{C}^n)} \leq C \sum_{\mu,\nu} \left| \int_{\varepsilon<|t'|<T_0} t'^{-1+i\tau} \Theta_{\mu,\nu}(t-t') \; dt' \right|^2, \tag{3.3.10}$$

where $\Theta_{\mu,\nu}(t) = e^{-it\phi(2|\nu|+n)} \hat{f}_2(t,\cdot)(\mu,\nu)$. If we define

$$\Gamma_{z_2,\varepsilon} : \Theta(t) \mapsto \int_{\varepsilon<|t'|<T_0} t'^{-1+i\tau} \Theta(t-t') \; dt',$$

then (3.3.10) becomes

$$\|G_{z_2,j,\varepsilon}f\|_{L^2(I\times\mathbb{C}^n)} \leq (2\pi)^{2n} \sum_{\mu,\nu} \|\Gamma_{z_2,\varepsilon}\Theta_{\mu,\nu}\|^2_{L^2(I)}. \tag{3.3.11}$$

Since, the operator $\Gamma_{z_2,\varepsilon}$ is just a Hilbert transform up to $i\tau$, from [109], the operator $\Gamma_{z_2,\varepsilon} : L^2 \to L^2$ is bounded with constant depends only on τ exponentially. Thus, using the boundedness of

$$G_{z_2,j,\varepsilon} : L^2(I\times\mathbb{C}^n) \to L^2(I\times\mathbb{C}^n)$$

and the fact that S_∞-norm is the operator norm, we have

$$\|W_1 G_{z_2,j,\varepsilon} W_2\|_{\mathcal{G}^\infty(L^2(I\times\mathbb{C}^n))} \leq C(\tau) \|W_1\|_{L^\infty(I,L^\infty(\mathbb{C}^n))} \|W_2\|_{L^\infty(I,L^\infty(\mathbb{C}^n))}.$$

Now, applying Stein's analytic interpolation result [14, 19], we get

$$\|W_1 G_{-1,j,\varepsilon} W_2\|_{\mathcal{G}^r(L^2(I\times\mathbb{C}^n))} \leq C 2^{\frac{((1-m)2n+m)2j}{r}} \|W_1\|_{L^{\frac{2r}{r-2n+1}}(I,L^r(\mathbb{C}^n))} \|W_2\|_{L^{\frac{2r}{r-2n+1}}(I,L^r(\mathbb{C}^n))},$$

with the constant independent of ε and j. Summing over $2^j \leq N$ and letting $\varepsilon \to 0$, we obtain (3.3.6) for $2n < q \leq 2n+1$, since $\mathcal{E}_{S_{\phi,N}}\mathcal{E}^*_{S_{\phi,N}} = \sum_{\{j:\,2^j\leq N\}} G_{-1,j,\varepsilon}$ in the limit as $\varepsilon \to 0$.

Again, it is easy to check that

$$\left\| \sum_{j\in J} n_j \left|\mathcal{E}_{S_{\phi,N}} f_j\right|^2 \right\|_{L^\infty(I,L^1(\mathbb{C}^n))} \leq \sum_{j\in J} |n_j| \; \sup_{t\in I} \|\mathcal{E}_{S_{\phi,N}} f_j\|^2_{L^2(\mathbb{C}^n))} \leq \sum_{j\in J} |n_j|.$$

Hence by duality Lemma 3.4.1, we have

$$\|W_1 \mathcal{E}_{S_{\phi,N}} \mathcal{E}^*_{S_{\phi,N}} W_2\|_{\mathcal{G}^\infty(L^2(I\times\mathbb{C}^n))} \leq C \|W_1\|_{L^2(I,L^\infty(\mathbb{C}^n))} \|W_2\|_{L^2(I,L^\infty(\mathbb{C}^n))}. \tag{3.3.12}$$

Interpolating (3.3.6) with $q = 2n+1$ and (3.3.12), we get (3.3.6) for $2n+1 \le q \le \infty$.

It remains to establish (3.3.6) for $2n-1 \le q \le 2n$. Assume $n \ge 2$. Using the Littlewood–Paley decomposition as above together with Theorem 3.2.1 at the end point $(p,q) = (1, \frac{2n-1}{2n-3})$, we have

$$\Big\|\sum_{j\in J} n_j |\mathcal{E}_{S_{\phi,N}} f_j|^2\Big\|_{L^1(I,L^{\frac{2n-1}{2n-3}}(\mathbb{C}^n))} \le \sum_{j\in J} |n_j| \|\mathcal{E}_{S_{\phi,N}} f_j\|^2_{L^2(I,L^{\frac{2(2n-1)}{2n-3}}(\mathbb{C}^n))} \le N^{\frac{2((1-m)2n+m)}{2n-1}} \sum_{j\in J} |n_j| .$$

Applying again the duality Lemma 3.4.1, we deduce (3.3.6) for $q = 2n-1$, that is,

$$\|W_1 \mathcal{E}_{S_{\phi,N}} \mathcal{E}^*_{S_{\phi,N}} W_2\|_{\mathcal{G}^\infty(L^2(I\times\mathbb{C}^n))} \le C N^{\frac{2((1-m)2n+m)}{2n-1}} \|W_1\|_{L^\infty(I,L^{2n-1}(\mathbb{C}^n))} \|W_2\|_{L^\infty(I,L^{2n-1}(\mathbb{C}^n))} . \tag{3.3.13}$$

Finally, interpolating between (3.3.6) for $q > 2n$ (with $\alpha = q$) and (3.3.13) yields (3.3.6) for $2n-1 < q \le 2n$. ☐

3.4 Strichartz estimate for system of orthonormal functions associated with $e^{-it\phi(\mathcal{L})}$

In order to obtain the Strichartz estimate for a system of orthonormal functions we need a duality principle in our setting. Following the arguments in the proof of Lemma 3 in [49], we obtain the following result.

Lemma 3.4.1. *(Duality principle) Let $\mathcal{H}$ be a separable Hilbert space. Suppose A is a bounded linear operator from $\mathcal{H}$ to $L^{p'}(I, L^{q'}(\mathbb{C}^n))$, where $1 \le p, q \le 2$, $I \subset \mathbb{R}$, and let $\alpha \ge 1$. Then the following statements are equivalent:*

1. *There is a constant $C > 0$ such that*

$$\|WAA^*\overline{W}\|_{\mathcal{G}^\alpha(L^2(I\times\mathbb{C}^n))} \le C\|W\|^2_{L^{\frac{2p}{2-p}}(I,L^{\frac{2q}{2-q}}(\mathbb{C}^n))} \tag{3.4.1}$$

   for all $W \in L^{\frac{2p}{2-p}}(I, L^{\frac{2q}{2-q}}(\mathbb{C}^n))$, where the function W is interpreted as an operator which acts by multiplication.

2. *For any orthonormal system $(f_j)_{j\in J}$ in $L^2(\mathbb{C}^n)$ and any sequence $(n_j)_{j\in J} \subset \mathbb{C}$, there is a constant $C' > 0$ such that*

$$\Big\|\sum_{j\in J} n_j |Af_j|^2\Big\|_{L^{\frac{p'}{2}}(I,L^{\frac{q'}{2}}(\mathbb{C}^n))} \le C' \Big(\sum_{j\in J} |n_j|^{\alpha'}\Big)^{1/\alpha'} . \tag{3.4.2}$$

Proof. We note that (3.4.2) is equivalent to itself but with the additional condition on the scalars are that $n_j \geq 0, j \in J$, i.e.,

$$\left\| \sum_{j\in J} n_j \left|Af_j\right|^2 \right\|_{L^{\frac{p'}{2}}(I,L^{\frac{q'}{2}}(\mathbb{C}^n))} \leq C' \left(\sum_{j\in J} |n_j|^{\alpha'} \right)^{1/\alpha'}, \quad \forall n_j \geq 0, j \in J. \tag{3.4.3}$$

Consequently, to establish the equivalence of (3.4.1) and (3.4.2), it suffices to show that (3.4.1) is equivalent to (3.4.3).

We first prove that (3.4.1) implies (3.4.3). Let $(f_j)_{j\in J}$ be an orthonormal system in $\mathcal{H}$ and $n_j \geq 0, j \in J$.

We define an operator γ on $\mathcal{H}$ as

$$\gamma := \sum_{j} n_j \left|f_j\right\rangle \left\langle f_j\right|, \tag{3.4.4}$$

where the Dirac's notation $|u\rangle\langle v|$ stands for the rank-one operator $f \mapsto \langle v, f\rangle u$. Then (f_j) are the eigenfunctions of γ corresponding to the eigenvalues (n_j).

Moreover, the estimate (3.4.1) is equivalent to

$$\left\|A^*|W|^2A\right\|_{\mathcal{G}^{\alpha}(\mathcal{H})} \leq C\|W\|^2_{L^{\frac{2p}{2-p}}(I,L^{\frac{2q}{2-q}}(\mathbb{C}^n))}, \tag{3.4.5}$$

for all $W \in L^{\frac{2p}{2-p}}(I, L^{\frac{2q}{2-q}}(\mathbb{C}^n))$. Using (3.4.5) and Hölder's inequality for Schatten spaces, we get

$$\begin{aligned}
\operatorname{Tr}_{L^2(I\times\mathbb{C}^n)}\left(WA\gamma(WA)^*\right) &= \operatorname{Tr}_{\mathcal{H}}\left(\gamma A^*|W|^2A\right) \\
&\leq \|\gamma\|_{\mathcal{G}^{\alpha'}(\mathcal{H})} \left\|A^*|W|^2A\right\|_{\mathcal{G}^{\alpha}(L^2(\mathcal{H}))} \\
&= C\left(\sum_{j\in J}|n_j|^{\alpha'}\right)^{1/\alpha'} \|W\|^2_{L^{\frac{2p}{2-p}}(I,L^{\frac{2q}{2-q}}(\mathbb{C}^n))}.
\end{aligned}$$

On the other hand, using the notation in (3.4.4), we have the following identity

$$\operatorname{Tr}_{L^2(I\times\mathbb{C}^n)}\left(WA\gamma(WA)^*\right) = \int_I \int_{\mathbb{C}^n} \left(\sum_{j\in J} n_j \left|\left(Af_j\right)(t,z)\right|^2 \right) |W(t,z)|^2 \, dz \, dt.$$

So we can infer that, for all $V \in L^{\frac{p}{2-p}}(I, L^{\frac{q}{2-q}}(\mathbb{C}^n))$ with $V \geq 0$,

$$\begin{aligned}
&\int_{(-\pi,\pi)} \int_{\mathbb{R}^n} \left(\sum_{j\in J} n_j \left|\left(Af_j\right)(t,z)\right|^2 \right) |V(t,z)| dx \, dt \\
&\leq C\left(\sum_{j\in J}|n_j|^{\alpha'}\right)^{1/\alpha'} \|V\|_{L^{\frac{p}{2-p}}(I,L^{\frac{q}{2-q}}(\mathbb{C}^n))}.
\end{aligned}$$

Since $\left(\frac{p}{2-p}\right)' = \frac{p'}{2}$, the duality principle for L^p-spaces leads to (3.4.3). The proof of other part is similar. $\square$

As a consequence of Lemma 3.4.1, we derive the orthonormal Strichartz estimate for a more general class of semigroups associated with the special Hermite operator in the following theorem.

Theorem 3.4.2. *Let* $n, p, q \geq 1$ *and* $N \geq 2n$. *Let* $\phi : \mathbb{R}^+ \to \mathbb{R}$ *be a smooth function satisfying (3.3.4) with* $0 < m \leq 1$. *Suppose that the pair* (p, q) *satisfies* $\frac{2}{p} + \frac{2n-1}{q} = 2n - 1$, *and let* $\sigma = (2n(1-m) + m)(1 - \frac{1}{q})$. *Then*

$$\left\| \sum_{j \in J} n_j \left| e^{-it\phi(\mathcal{L})} f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leqslant CN^{\sigma} \left(\sum_{j \in J} |n_j|^{\beta} \right)^{1/\beta}, \tag{3.4.6}$$

holds for all orthonormal system $(f_j)_j$ *in* $L^2(\mathbb{C}^n)$ *with supp* $\hat{f}_j \subset \{(\mu, \nu) : \sqrt{2|\nu| + n} \leq N\}$, *and all sequence* $(n_j)_j \subset \mathbb{C}$, *provided that* β *satisfies the following conditions corresponding to the pair* (p, q):

(i) If $n \geq 1$, $1 \leq q < \frac{n}{n-1}$, *then* $\beta \leq \frac{2q}{q+1}$.

(ii) If $n \geq 2$, $\frac{n}{n-1} \leq q < \frac{2n-1}{2n-3}$, *then* $\beta < p$; *and if* $q = \frac{2n-1}{2n-3}$, *one has* $\beta = 1$.

Proof. Consider the discrete surface $S_\phi = \{(\lambda, \mu, \nu) \in \mathbb{R} \times \mathbb{N}_0^n \times \mathbb{N}_0^n : \lambda = \phi(2|\nu| + n)\}$ with respect to the measure σ_N defined in (3.3.5). The extension operator $\mathcal{E}_{S_{\phi,N}}$ is given by (3.3.7). Using the fact that

$$f \times \Phi_{\mu\mu} = (2\pi)^{\frac{n}{2}} \sum_{\nu} \langle f, \Phi_{\mu,\nu} \rangle \Phi_{\mu,\nu},$$

and choosing

$$\hat{f}(\lambda, \mu, \nu) = \begin{cases} (2\pi)^n \langle u, \Phi_{\mu,\nu} \rangle, & \text{if } \lambda = \phi(2|\nu| + n) \text{ and } \sqrt{2|\nu| + n} \leq N \\ 0, & \text{otherwise,} \end{cases}$$

for some $u : \mathbb{C}^n \to \mathbb{C}$ in (3.3.7), we get

$$\begin{aligned} \mathcal{E}_{S_{\phi,N}} f(t, z) &= (2\pi)^n \sum_{\nu} \left(\sum_{\mu} \langle u, \Phi_{\mu,\nu} \rangle \Phi_{\mu,\nu}(z) \right) e^{-it\phi(2|\nu|+n)} \\ &= (2\pi)^{\frac{n}{2}} \sum_{\nu} e^{-it\phi(2|\nu|+n)} u \times \Phi_{\nu,\nu}(z) \end{aligned}$$

$$= \sum_{k=0}^{\infty} e^{-it\phi(2k+n)} u \times \varphi_k(z) = e^{-it\phi(\mathcal{L})} u(z). \tag{3.4.7}$$

Thus, Theorem 3.4.2 follows from Theorem 3.3.2 together with the duality principle (Lemma 3.4.1). □

Using the vector-valued version of the Littlewood–Paley inequality (see [93], Lemma 1), the frequency-localized estimate (3.4.6) can be extended to a frequency-global estimate:

Theorem 3.4.3. *Let $n, p, q \geq 1$ and $N \geq 2n$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function satisfying (3.3.4) with $0 < m \leq 1$. Suppose that the pair (p, q) satisfies $\frac{2}{p} + \frac{2n-1}{q} = 2n - 1$, and let $s > \frac{2n(1-m)+m}{2}(1 - \frac{1}{q})$. Then*

$$\left\| \sum_{j \in J} n_j \left| e^{-it\phi(\mathcal{L})} f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leqslant C \left(\sum_{j \in J} |n_j|^\beta \right)^{1/\beta}, \tag{3.4.8}$$

holds for all orthonormal system $(f_j)_j$ in the Sobolev space $H^s(\mathcal{L}) := \left\{ f \in L^2(\mathbb{C}^n) : \mathcal{L}^{\frac{s}{2}} f \in L^2(\mathbb{C}^n) \right\}$ and all sequence $(n_j)_j \subset \mathbb{C}$, provided that β satisfies the following conditions corresponding to the pair (p, q):

(i) If $n \geq 1$, $1 \leq q < \frac{n}{n-1}$, then $\beta \leq \frac{2q}{q+1}$.

(ii) If $n \geq 2$, $\frac{n}{n-1} \leq q < \frac{2n-1}{2n-3}$, then $\beta < p$; and if $q = \frac{2n-1}{2n-3}$, one has $\beta = 1$.

Proof. We adopt an approach based on [87]. Let $\varPhi \in C_c^\infty$ such that *supp* $\varPhi \subset [1/2, 2]$ and $h \geq 1$. Then Theorem 3.4.2 can be rephrased as

$$\left\| \sum_{j \in J} n_j \left| e^{-it\phi(\mathcal{L})} \varPhi(h^{-1}\sqrt{\mathcal{L}}) f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leqslant C h^\sigma \left(\sum_{j \in J} |n_j|^\beta \right)^{1/\beta}. \tag{3.4.9}$$

Let $\{\varPsi_\ell\}_{\ell \geq 0}$ be defined as in the proof of Theorem 3.3.2. Using the vector-valued version of the Littlewood–Paley inequality (see [93], Lemma 1), triangle inequality, we obtain

$$\left\| \sum_{j \in J} n_j \left| e^{-it\phi(\mathcal{L})} \mathcal{L}^{-\frac{\sigma}{4}-\varepsilon} f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))}$$
$$\leqslant \sum_{\ell=0}^{\infty} \left\| \sum_{j \in J} n_j \left| \mathcal{L}^{-\frac{\sigma}{4}-\varepsilon} e^{-it\phi(\mathcal{L})} \varPsi_\ell(\sqrt{\mathcal{L}}) f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))}$$

$$= \sum_{\ell=0}^{\infty} 2^{-\ell(\sigma+4\varepsilon)} \left\| \sum_{j\in J} n_j \left| e^{-it\phi(\mathcal{L})} \tilde{\Psi}_\ell(\sqrt{\mathcal{L}}) f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))},$$

where $\tilde{\Psi}_\ell(s) = \tilde{\Psi}(2^{-\ell}s)$ with $\tilde{\Psi}(s) = s^{-\frac{\sigma}{2}-2\varepsilon}\Psi(s)$. Clearly, $\tilde{\Psi} \in C_c^\infty(\mathbb{R})$ and $supp\,\tilde{\Psi} \subset [1/2, 2]$, thus from (3.4.9) we obtain

$$\left\| \sum_{j\in J} n_j \left| e^{-it\phi(\mathcal{L})} \mathcal{L}^{-\frac{\sigma}{4}-\varepsilon} f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leq C_\varepsilon \left(\sum_{j\in J} |n_j|^\beta \right)^{1/\beta}.$$

This completes the proof of the theorem. ☐

The wave equation: It is well–known that the solution of the special Hermite wave equation

$$i\partial_{tt}u(t,z) - \mathcal{L}u(t,z) = 0, \quad z \in \mathbb{C}^n,\ t \in \mathbb{R}\setminus\{0\}, \tag{3.4.10}$$
$$u(0,z) = f(z)$$
$$\partial_t u(0,z) = g(z).$$

can be expressed as a superposition of waves generated by the propagators $e^{\pm it\sqrt{\mathcal{L}}}$. This corresponds to the case $\phi(r) = \sqrt{r}$ which satisfies condition (3.3.4) with $m = \frac{1}{2}$. We obtain the following Strichartz estimate by Theorem 3.4.2.

Theorem 3.4.4. *Let $n, p, q \geq 1$ and $N \geq 2n$. Suppose that (p,q) satisfying $\frac{2}{p} + \frac{2n-1}{q} = 2n-1$, and let $\sigma = \frac{2n+1}{2}(1-\frac{1}{q})$. Then,*

$$\left\| \sum_{j\in J} n_j \left| e^{-it\sqrt{\mathcal{L}}} f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leqslant CN^\sigma \left(\sum_{j\in J} |n_j|^\beta \right)^{1/\beta}$$

holds for all orthonormal system $(f_j)_j$ in $L^2(\mathbb{C}^n)$ with $supp\hat{f_j} \subset \{(\mu,\nu) : \sqrt{2|\nu|+n} \leq N\}$ and all sequence $(n_j)_j \subset \mathbb{C}$,provided that β satisfies the following conditions:

(i) If $n \geq 1$, $1 \leq q < \frac{n}{n-1}$, then $\beta \leq \frac{2q}{q+1}$.

(ii) If $n \geq 2$, $\frac{n}{n-1} \leq q < \frac{2n-1}{2n-3}$, then $\beta < p$; and if $q = \frac{2n-1}{2n-3}$, one has $\beta = 1$.

The Klein-Gordon equation: The solution of the special Hermite Klein-Gordon equation

$$i\partial_{tt}u(t,z) - \mathcal{L}u(t,z) - u(t,z) = 0, \quad z \in \mathbb{C}^n,\ t \in \mathbb{R}\setminus\{0\}, \tag{3.4.11}$$

$$u(0,z) = f(z)$$
$$\partial_t u(0,z) = g(z).$$

can be expressed as a superposition of waves generated by the propagators $e^{\pm it\sqrt{1+\mathcal{L}}}$, which corresponds to the case $\phi(r) = \sqrt{r}$ satisfying (3.3.4) with $m = \frac{1}{2}$. We obtain the following Strichartz estimate by Theorem 3.4.2.

Theorem 3.4.5. *Let $n, p, q \geq 1$ and $N \geq 2n$. Suppose that (p,q) satisfying $\frac{2}{p} + \frac{2n-1}{q} = 2n-1$, and let $\sigma = \frac{2n+1}{2}(1-\frac{1}{q})$. Then,*

$$\left\| \sum_{j\in J} n_j \left| e^{-it\sqrt{1+\mathcal{L}}} f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leqslant CN^{\sigma} \left(\sum_{j\in J} |n_j|^{\beta} \right)^{1/\beta}$$

holds for all orthonormal system $(f_j)_j$ in $L^2(\mathbb{C}^n)$ with supp$\hat{f}_j \subset \{(\mu,\nu) : \sqrt{2|\nu|+n} \leq N\}$, and all sequence $(n_j)_j \subset \mathbb{C}$, provided that β satisfies the following conditions:

(i) If $n \geq 1$, $1 \leq q < \frac{n}{n-1}$, then $\beta \leq \frac{2q}{q+1}$.

(ii) If $n \geq 2$, $\frac{n}{n-1} \leq q < \frac{2n-1}{2n-3}$, then $\beta < p$; and if $q = \frac{2n-1}{2n-3}$, one has $\beta = 1$.

The fractional Schrödinger equation: For $0 \leq \varrho < 1$, the solution to the special Hermite fractional Schrödinger equation

$$i\partial_t u(t,z) - \mathcal{L}^{\varrho} u(t,z) = 0, \quad z \in \mathbb{C}^n, t \in \mathbb{R}\setminus\{0\}, \tag{3.4.12}$$
$$u(0,z) = f(z)$$

is described by the unitary flow $e^{-it\mathcal{L}^{\varrho}}$. This corresponds to the case $\phi(r) = r^{\varrho}$ satisfying (3.3.4) with $m = \varrho$. We obtain the following Strichartz estimate by Theorem 3.4.2.

Theorem 3.4.6. *Let $n, p, q \geq 1$ and $N \geq 2n$. Suppose that (p,q) satisfying $\frac{2}{p} + \frac{2n-1}{q} = 2n-1$, and let $\sigma = (2n(1-\varrho)+\varrho)(1-\frac{1}{q})$. Then*

$$\left\| \sum_{j\in J} n_j \left| e^{-it\mathcal{L}^{\varrho}} f_j \right|^2 \right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leqslant CN^{\sigma} \left(\sum_{j\in J} |n_j|^{\beta} \right)^{1/\beta} \tag{3.4.13}$$

holds for all orthonormal system $(f_j)_j$ in $L^2(\mathbb{C}^n)$ with supp$\hat{f}_j \subset \{(\mu,\nu) : \sqrt{2|\nu|+n} \leq N\}$, and all sequence $(n_j)_j \subset \mathbb{C}$, provided that β satisfies the following conditions:

(i) If $n \geq 1$, $1 \leq q < \frac{n}{n-1}$, then $\beta \leq \frac{2q}{q+1}$.

(ii) If $n \geq 2$, $\frac{n}{n-1} \leq q < \frac{2n-1}{2n-3}$, then $\beta < p$; and if $q = \frac{2n-1}{2n-3}$, one has $\beta = 1$.

Remark 3.4.7. *Theorems 3.5.1, 3.4.2 and 3.4.3 can be extended for a wider range of* $p, q \geq 1$ *such that* $\frac{2}{p}+\frac{2n-1}{q} \geq 2n-1$, *using the inclusion properties of the* $L^p(-T_0,T_0)$*–spaces (see [58]). When* $1<q<\frac{n}{n-1}$, *then there exists a* $0<\beta\leq 1$ *such that* $\frac{2\beta}{p}+\frac{2n-1}{q} = 2n-1$. *Since* $\frac{p}{\beta} \geq p \geq 1$ *the desired estimate follows from the fact that*

$$\left\|\sum_j n_j \left|e^{-it\phi(\mathcal{L})} f_j\right|^2\right\|_{L^p((-T_0,T_0),L^q(\mathbb{C}^n))} \leq \left\|\sum_j n_j \left|e^{-it\phi(\mathcal{L})} f_j\right|^2\right\|_{L^{\frac{p}{\beta}}((-T_0,T_0),L^q(\mathbb{C}^n))}.$$

The case $p=1$ *follows directly from the triangle inequality.*

3.5 Strichartz estimate for system of orthonormal functions associated with $e^{-it\mathcal{L}}$

The following orthonormal Strichartz estimate for the Schrödinger semigroup associated to the special Hermite operator is obtained in [57].

Theorem 3.5.1. *[57] Let* $n \geq 1$. *Let* $p, q \geq 1$ *satisfies* $1 \leq q < \frac{2n+1}{2n-1}$ *and* $\frac{1}{p}+\frac{n}{q}=n$ *and* $1 \leq \beta \leq \frac{2q}{q+1}$. *Then the following estimate*

$$\left\|\sum_{j\in J} n_j \left|e^{-it\mathcal{L}} f_j\right|^2\right\|_{L^p((-\pi,\pi),L^q(\mathbb{C}^n))} \leqslant C\left(\sum_{j\in J} |n_j|^\beta\right)^{1/\beta}, \tag{3.5.1}$$

for all orthonormal system $(f_j)_j$ *in* $L^2(\mathbb{C}^n)$ *and all sequence* $(n_j)_j \subset \mathbb{C}$.

3.5.1 Optimality of the Schatten exponent

Using a semiclassical analysis on coherent states, in this section, we prove that the exponent $\beta = \frac{2q}{q+1}$ in (3.5.1) is optimal, which cannot be further reduced.

Theorem 3.5.2. *The estimate (3.5.1) fails for all* $\beta > \frac{2q}{q+1}$.

Proof. The inequality (3.5.1) can also be written in terms of the operator

$$\gamma_0 := \sum_j n_j \left|u_j\right\rangle \left\langle u_j\right| \tag{3.5.2}$$

on $L^2(\mathbb{C}^n)$, where the Dirac's notation $|u\rangle\langle v|$ stands for the rank-one operator $f \mapsto \langle v, f\rangle u$. For such γ_0, let

$$\gamma(t) := e^{-it\mathcal{L}}\gamma_0 e^{it\mathcal{L}} = \sum_j n_j \left|e^{-it\mathcal{L}}u_j\right\rangle \left\langle e^{-it\mathcal{L}}u_j\right|.$$

Then the density of the operator $\gamma(t)$ is given by

$$\rho_{\gamma(t)} := \sum_j n_j \left|e^{-it\mathcal{L}}u_j\right|^2. \tag{3.5.3}$$

With these notations, (3.5.1) can be rewritten as

$$\left\|\rho_{\gamma(t)}\right\|_{L^p_t L^q_z((-\pi,\pi)\times\mathbb{C}^n)} \leq C_{n,q}\|\gamma_0\|_{\mathcal{G}^{\frac{2q}{q+1}}}, \tag{3.5.4}$$

where $\|\gamma_0\|_{\mathcal{G}^{\frac{2q}{q+1}}} = \left(\sum_j |n_j|^{\frac{2q}{q+1}}\right)^{\frac{q+1}{2q}}$ and $\mathcal{G}^r$ is the Schatten r class defined in Section 3.2.

In view of (3.5.4), the optimality of the power $\frac{2q}{q+1}$ on the right hand side in (3.5.1) follows from the following theorem. ☐

Theorem 3.5.3. [Optimality of the Schatten exponent] *Assume that $n, p, q \geq 1$ satisfy $\frac{1}{p} + \frac{n}{q} = n$. Then*

$$\sup_{\gamma_0\in\mathcal{G}^r} \frac{\left\|\rho_{e^{-it\mathcal{L}}\gamma_0 e^{it\mathcal{L}}}\right\|_{L^p_t L^q_z((-\pi,\pi)\times\mathbb{C}^n)}}{\|\gamma_0\|_{\mathcal{G}^r}} = +\infty,$$

for all $r > \frac{2q}{q+1}$.

Proof. Depending on the positive parameters β, τ and μ, we construct the family of operators

$$\gamma_0 = \frac{1}{(2\pi)^{2n}} \iint_{\mathbb{C}^n\times\mathbb{C}^n} e^{-\frac{x^2}{L^2}-\frac{\xi^2}{\mu}} \left|F_{x,\xi}\right\rangle \left\langle F_{x,\xi}\right| dx d\xi,$$

where $F_{x,\xi}(z) = (2\pi\beta)^{-\frac{n}{2}} e^{-\frac{|z-x|^2}{4\beta}} e^{\frac{i}{2}Im(z\cdot\bar{\xi})}$. The functions $F_{x,\xi}$ are normalized and satisfy

$$\iint_{\mathbb{C}^n\times\mathbb{C}^n} \frac{dx d\xi}{(2\pi)^{2n}} \left|F_{x,\xi}\right\rangle \left\langle F_{x,\xi}\right| = 1.$$

By Mehler's formula, we get

$$e^{-it\mathcal{L}}F_{x,\xi}(z) = (2\pi i\sin t)^{-n}(2\pi\beta)^{-\frac{n}{2}} \int_{\mathbb{C}^n} e^{-\frac{|z-w-x|^2}{4\beta}} e^{\frac{i}{2}Im\left((z-w)\cdot\bar{\xi}\right)} e^{\frac{i}{4}\cot t\,|w|^2} e^{\frac{i}{2}Im(z\cdot\bar{w})} dw.$$

On simplifying and taking modulus, we have

$$
\begin{aligned}
\left|e^{-it\mathcal{L}}F_{x,\xi}(z)\right| &= \left(\frac{2\beta\pi}{(\beta^2\cos^2 t+\sin^2 t)}\right)^{\frac{n}{2}} e^{-\frac{|z-x|^2}{4\beta}} e^{-\frac{\left(\beta^2|z+\xi|^2-|z-x|^2\right)\sin^2 t}{(\beta^2\cos^2 t+\sin^2 t)}} e^{\frac{\beta Im\left((z-x)\cdot\overline{(z+\xi)}\right)\cos t\sin t}{2(\beta^2\cos^2 t+\sin^2 t)}}\\
&= \left(\frac{2\beta\pi}{(\beta^2\cos^2 t+\sin^2 t)}\right)^{\frac{n}{2}} e^{-\frac{\beta|(z-x)\cos t-i(z+\xi)\sin t|^2}{4(\beta^2\cos^2 t+\sin^2 t)}},
\end{aligned}
$$

where the exponent term is obtained by suitable adjustments in the previous step.

Now

$$
\rho_{\gamma(t)}(z) := \rho_{e^{it\mathcal{L}}\gamma_0 e^{-it\mathcal{L}}}(z) = \iint_{\mathbb{C}^n\times\mathbb{C}^n} \frac{dxd\xi}{(2\pi)^{2n}} e^{-\frac{x^2}{L^2}-\frac{\xi^2}{\mu}} \left|e^{it\mathcal{L}}F_{x,\xi}(z)\right|^2.
$$

Writing $x = x_1 + ix_2, \xi = \xi_1 + i\xi_2, z = z_1 + iz_2$, and integrating with respect to ξ_1, ξ_2 respectively, we get

$$
\begin{aligned}
&\rho_{\gamma(t)}(z)\\
=&\frac{(\mu\beta)^n}{\left(2\beta^2\cos^2 t+(2+\mu\beta)\sin^2 t\right)^n} \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\frac{2\beta((z_1-x_1)\cos t+z_2\sin t)^2}{(2\beta^2\cos^2 t+(2+\mu\beta)\sin^t)}} e^{-\frac{2\beta((z_2-x_2)\cos t-z_1\sin t)^2}{(2\beta^2\cos^2 t+(2+\mu\beta)\sin^t)}} dx_1dx_2\\
=&\frac{(\pi\mu L^2\beta)^n}{\left((2\beta^2+2\beta L^2)\cos^2 t+(2+\mu\beta)\sin^2 t\right)^n} e^{-\frac{2\beta(z_1\cos t+z_2\sin t)^2}{(2\beta^2+2\beta L^2)\cos^2 t+(2+\mu\beta)\sin^2 t}} e^{-\frac{2\beta(z_2\cos t-z_1\sin t)^2}{(2\beta^2+2\beta L^2)\cos^2 t+(2+\mu\beta)\sin^2 t}}\\
=&\frac{(\pi\mu L^2\beta)^n}{\left((2\beta^2+2\beta L^2)\cos^2 t+(2+\mu\beta)\sin^2 t\right)^n} e^{-\frac{2\beta|z|^2}{(2\beta^2+2\beta L^2)\cos^2 t+(2+\mu\beta)\sin^2 t}}.
\end{aligned}
$$

Therefore,

$$
\|\rho_{\gamma(t)}\|^q_{L^q_z(\mathbb{C}^n)} = \left(\frac{\pi}{q}\right)^n (\mu L^2)^{nq} \left(\frac{\beta}{(4\beta^2+2\beta L^2)\cos^2 2t+(1+2\mu\beta)\sin^2 2t}\right)^{n(q-1)}.
$$

Using the fact that $n(q-1)p = q$, we have

$$
\begin{aligned}
&\|\rho_{\gamma(t)}\|^p_{L^p_tL^q_z((-\pi,\pi)\times\mathbb{C}^n)}\\
&= \left(\frac{\pi}{q}\right)^{\frac{np}{q}} (\mu L^2)^{np} \int_{(-\pi,\pi)} \frac{\beta}{(2\beta^2+2\beta L^2)\cos^2 2t+(2+\mu\beta)\sin^2 2t}dt\\
&= \sqrt{2}\pi \left(\frac{\pi}{q}\right)^{\frac{np}{q}} (\mu L^2)^{np} \frac{\beta}{\sqrt{\beta^2+\beta L^2}\sqrt{2+\mu\beta}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|\rho_{\gamma(t)}\|_{L^p_tL^q_z((-\pi,\pi)\times\mathbb{C}^n)}\\
&= A_{n,p}\left(\mu L^2\right)^n (L^2)^{-\frac{1}{2p}}\mu^{-\frac{1}{2p}} \frac{1}{\left(\frac{\beta}{L^2}+1\right)^{\frac{1}{2p}}\left(\frac{2}{\mu\beta}+1\right)^{\frac{1}{2p}}}
\end{aligned}
$$

$$= A_{n,p}\left(\mu L^2\right)^{n-\frac{1}{2p}} \frac{1}{\left(\frac{\beta}{L^2}+1\right)^{\frac{1}{2p}}\left(\frac{2}{\mu\beta}+1\right)^{\frac{1}{2p}}}.$$

Using the fact that $\frac{n}{2}\left(1+\frac{1}{q}\right) = n-\frac{1}{2p}$ and choosing $1/\mu < \beta < L^2$, we obtain

$$\|\rho_{\gamma(t)}\|_{L^p_t L^q_z((-\pi,\pi)\times\mathbb{C}^n)} \geq A_{n,p}\left(\mu L^2\right)^{\frac{n}{2}\left(1+\frac{1}{q}\right)} = A_{n,p}\left(\mu L^2\right)^{n\left(\frac{1+q}{2q}\right)} = A_{n,p}N^{\frac{1+q}{2q}},$$

where

$$\begin{aligned} N &= \int_{\mathbb{C}^n} \gamma_0(z,z)dz \\ &= \iiint_{\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^n} \frac{dxd\xi}{(2\pi)^{2n}} e^{-\frac{|x|^2}{2}-\frac{|\xi|^2}{\mu}} \left|F_{x,\xi}(z)\right|^2 dz \\ &= \iint_{\mathbb{C}^n\times\mathbb{C}^n} \frac{dxd\xi}{(2\pi)^{2n}} e^{-\frac{|x|^2}{2}-\frac{|\xi|^2}{\mu}} \\ &= A_n L^{2n}\mu^n. \end{aligned}$$

An application of Berezin-Lieb inequality [18, 76] gives that

$$\operatorname{Tr}\gamma_0^r \leq \iint_{\mathbb{C}^n\times\mathbb{C}^n} \frac{dxd\xi}{(2\pi)^{2n}} e^{-\frac{r|x|^2}{2}-\frac{r|\xi|^2}{\mu}} = r^{-2n}N,$$

where $r \geq 1$ and $N = \frac{(\mu L^2)^n}{2^{2n}}$. Therefore

$$\frac{\left\|\rho_{e^{-it\mathcal{L}}\gamma_0 e^{it\mathcal{L}}}\right\|_{L^p_t L^q_z((-\pi,\pi)\times\mathbb{C}^n)}}{\|\gamma_0\|_{\mathcal{G}^r}} \geq \frac{A_{n,p}}{r^{-\frac{2n}{r}}} N^{\left(\frac{1+q}{2q}-\frac{1}{r}\right)}.$$

□

3.5.2 The end point case

In this section, we show that the estimate (3.5.1) fails when $q = \frac{2n+1}{2n-1}$ and $\beta = \frac{2q}{q+1}$. Consequently, this implies that the range of q cannot be improved for the choice $\beta = \frac{2q}{q+1}$.

Theorem 3.5.4. *The estimate (3.5.1) fails at the end point* $(q,\beta) = (\frac{2n+1}{2n-1}, \frac{2q}{q+1})$.

For a trace-class operator γ and bounded function V of compact support

$$\operatorname{Tr}(V(z)\gamma) = \int_{\mathbb{C}^n} V(z)\rho_\gamma(z)dz, \tag{3.5.5}$$

where $V(z)$ is identified with the corresponding multiplication operator on $L^2(\mathbb{C}^n)$ and for a time-dependent potential $V(t,z) \in L^\infty_c([-\pi,\pi]\times\mathbb{C}^n)$, we have

$$\left|\operatorname{Tr}\left(\int_{[-\pi,\pi]} e^{it\mathcal{L}}V(t,z)e^{-it\mathcal{L}}dt\right)\gamma\right| = \left|\int_{[-\pi,\pi]} \operatorname{Tr}\left(Ve^{-it\mathcal{L}}\gamma e^{it\mathcal{L}}\right)dt\right|$$

$$
\begin{aligned}
&= \left| \int_{[-\pi,\pi]} \int_{\mathbb{R}^n} V(t,z)\rho_{\gamma(t)}(z)dzdt \right| \\
&\leq \int_{[-\pi,\pi]} \left(\int_{\mathbb{R}^n} |V(t,z)|^{q'} dz \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^n} \left|\rho_{\gamma(t)}(z)\right|^{q} dz \right)^{\frac{1}{q}} dt \\
&\leq \|V\|_{L_t^{p'}L_x^{q'}((-\pi,\pi)\times\mathbb{R}^n)} \left\|\rho_{\gamma(t)}\right\|_{L_t^{p}L_x^{q}((-\pi,\pi)\times\mathbb{C}^n)},
\end{aligned}
$$

where p' and q' are the dual to p and q respectively. By duality in Schatten spaces (see [48] for details), Theorem 3.5.1 can be equivalently restated in the following dual version.

Theorem 3.5.5. *Assume that* $p', q', d \geq 1$ *satisfy*

$$\frac{2n+1}{2} < p' \leq \infty \quad \textit{and} \quad \frac{1}{q'} + \frac{n}{p'} = 1.$$

Then we have

$$\left\| \int_{-\pi}^{\pi} e^{-it\mathcal{L}} V(t,z) e^{it\mathcal{L}}\, dt \right\|_{\mathcal{G}^{2q'}} \leq C\|V\|_{L^{p'}((-\pi,\pi),L^{q'}(\mathbb{C}^n))}. \tag{3.5.6}$$

The estimate (3.3.6) at the end point $(q,\beta) = (\frac{2n+1}{2n-1}, \frac{2q}{q+1})$ corresponds to the estimate (3.5.6) at $2q' = 2n+1$. Hence, Theorem 3.5.4 follows from the following result.

Theorem 3.5.6. *There exists* $0 \neq V \in L^{2n+1}((-\pi,\pi), L^{\frac{2n+1}{2}}(\mathbb{C}^n))$ *such that*

$$Tr \left(\int_{-\pi}^{\pi} e^{it\mathcal{L}} V(t,z) e^{-it\mathcal{L}}\, dt \right)^{2n+1} = \infty. \tag{3.5.7}$$

Proof. Define the operator

$$B_V := \int_{-\pi}^{\pi} e^{it\mathcal{L}} V(t,z) e^{-it\mathcal{L}}\, dt, \tag{3.5.8}$$

whose kernel can be calculated as

$$B_V(z,w) = \int_{-\pi}^{\pi} (2\pi i \sin t)^{-2n} \int_{\mathbb{C}^n} e^{-\frac{i}{2}\Im(z\cdot\bar{\zeta})} e^{-i\cot t\frac{|z-\zeta|^2}{4}} V(t,\zeta) e^{\frac{i}{2}\Im(w\cdot\bar{\zeta})} e^{i\cot t\frac{|w-\zeta|^2}{4}}\, d\zeta\, dt,$$

for a $V(t,z)$ such that the integral makes sense, we choose $V(t,z)$ precisely later. For $f \in \mathcal{S}(\mathbb{C}^n)$ (the Schwartz space on $\mathbb{C}^n$), let $\mathcal{F}_s$ be the symplectic Fourier transform on $\mathbb{C}^n$ given by

$$\mathcal{F}_s f(z) = \int_{\mathbb{C}^n} f(w)\, e^{-\frac{i}{2}\Im(z\cdot\bar{w})}\, dw.$$

Now the kernel of B_V in the symplectic Fourier space is given by

$$\widehat{B_V}(p,q) = C \int_{-\pi}^{\pi} (\cos t)^{-2n} \int_{\mathbb{C}^n} V(t,\zeta) e^{i\tan t\frac{|p+\zeta|^2}{4}} e^{-i\tan t\frac{|q+\zeta|^2}{4}} e^{-\frac{i}{2}\Im((q-p)\cdot\bar{\zeta})}\, d\zeta\, dt$$

$$=C\int_{-\pi}^{\pi}(\cos t)^{-2n}e^{i\tan t(|p|^2-|q|^2)}\int_{\mathbb{C}^n}V(t,\zeta)e^{-\frac{i}{2}\Im((q-p)\cdot\bar{\eta})}\,d\zeta\,dt \tag{3.5.9}$$

where $\eta=\eta_1+i\eta_2$ is given by

$$\eta_1=\zeta_1+\tan t\,\zeta_2,\quad \eta_2=-\tan t\,\zeta_1+\zeta_2. \tag{3.5.10}$$

Now, let $0\neq V_1(t,z)\in L^\infty_c(\mathbb{R}\times\mathbb{C}^n)$ be a non-negative function such that $\widehat{V_1}=\mathcal{F}_t\mathcal{F}_{s,z}V_1$ is non-negative. Here $\mathcal{F}_{s,z}$ is the symplectic Fourier transform with respect to z-variable, and $\mathcal{F}_t$ denotes the Fourier transform with respect to t-variable. Now consider

$$V(t,z)=\chi_{(-\frac{\pi}{2},\frac{\pi}{2})}V_1\left(\frac{\tan t}{4},\zeta_1+\tan t\,\zeta_2+i(-\tan t\,\zeta_1+\zeta_2)\right)\sec^2 t,$$

one can check that $V\in L^{2n+1}((-\pi,\pi),L^{\frac{2n+1}{2}}(\mathbb{C}^n))$. Performing the change of variable (3.5.10) in (3.5.9), we obtain

$$\begin{aligned}\widehat{B_V}(p,q)=&C\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{i\frac{\tan t}{4}(|p|^2-|q|^2)}\sec^2 t\int_{\mathbb{C}^n}V_1\left(\frac{\tan t}{4},\eta\right)e^{-\frac{i}{2}\Im((q-p)\cdot\bar{\eta})}\,d\zeta\,dt\\ =&\widehat{V_1}(|q|^2-|p|^2,q-p),\end{aligned}$$

where the last equality is obtained by an obvious change of variable. Hence, we deduce that

$$Tr(B_V^{2n+1})=\int_{\mathbb{C}^n}dp\int_{\mathbb{C}^n}dp_1\cdots\int_{\mathbb{C}^n}dp_{2n}\;\widehat{B_V}(p,p_1)\widehat{B_V}(p_1,p_2)\cdots\widehat{B_V}(p_{2n},p).$$

Now, proceeding as in the proof of Proposition 2 in [48], we obtain $Tr(B_V^{2n+1})=+\infty$. ☐

3.6 Restriction estimates for special Hermite spectral projections

The following restriction estimate for the special Hermite spectral projections is well-known in the literature.

Theorem 3.6.1. *Let $n\geq 1$ and $k\geq 1$. Then*

$$\|\mathcal{Q}_k f\|_2\leq C_p k^{\varrho(p)}\|f\|_p \tag{3.6.1}$$

holds with the exponent $\varrho(p)$ *is given by*

$$\varrho(p) = \begin{cases} n(1/p - 1/2) - 1/2, & \text{if } 1 \leq p \leq \frac{2(2n+1)}{2n+3}, \\ -\frac{1}{2}(1/p - 1/2), & \text{if } \frac{2(2n+1)}{2n+3} \leq p \leq 2, \end{cases} \tag{3.6.2}$$

and the estimate (3.6.1) *is optimal in the sense that exponent* $\varrho(p)$ *cannot be improved.*

The estimate (3.6.1) was first established by Thangavelu [107] for the range $1 \leq p \leq \frac{2n}{n+1}$. Subsequently, Ratnakumar, Rawat, and Thangavelu [89] extended this range to $1 \leq p < \frac{2(3n+1)}{3n+4}$. Later, Stempak and Zienkiewicz [101] proved (3.6.1) for all $1 \leq p \leq 2$ except for $p = \frac{2(2n+1)}{2n+3}$. Finally, Koch and Ricci [73] settled the endpoint case $p = \frac{2(2n+1)}{2n+3}$, and showed that the estimate (3.6.1) is optimal. A local version of this endpoint estimate was obtained earlier by Thangavelu [105].

Using a duality argument, one can show that (3.6.1) is equivalent to

$$\|\mathcal{Q}_k f\|_{p'} \leq C k^{2\varrho(p)} \|f\|_p, \tag{3.6.3}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. By Hölder's inequality, (3.6.3) holds if and only if for any $W_1, W_2 \in L^{2p/(2-p)}(\mathbb{C}^n)$, the operator $W_1 \mathcal{Q}_k W_2$ is bounded on $L^2(\mathbb{C}^n)$ with the estimate

$$\|W_1 \mathcal{Q}_k W_2\|_{L^2(\mathbb{C}^n) \to L^2(\mathbb{C}^n)} \leq C k^{2\varrho(p)} \|W_1\|_{L^{2p/(2-p)}(\mathbb{C}^n)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{C}^n)}, \tag{3.6.4}$$

with $C > 0$ independent of W_1 and W_2.

In this section, we upgrade the restriction estimate (3.6.4) in the context of Schatten spaces. More precisely, we prove that

$$\|W_1 \mathcal{Q}_k W_2\|_{\mathcal{G}^\alpha(L^2(\mathbb{C}^n))} \leq k^{2\varrho(p)} \|W_1\|_{L^{2p/(2-p)}(\mathbb{C}^n)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{C}^n)}, \tag{3.6.5}$$

for some $\alpha \geq 1$.

3.6.1 Representation formula for the special Hermite spectral projections

Using the fact that $\displaystyle\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{it(2k-2\ell)}\, dt = \delta(k-\ell)$, $k, \ell \in \mathbb{N}_0$, we obtain

$$\mathcal{Q}_k f = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sum_{\ell \in \mathbb{N}} e^{it(2k+n)} e^{-it(2\ell+n)}\, \mathcal{Q}_\ell f\, dt, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

since the series converges uniformly. By (3.2.8), it follows that

$$\mathcal{Q}_k f(z) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{it(2k+n)} e^{-it\mathcal{L}} f(z)\,dt, \qquad f \in \mathcal{S}(\mathbb{C}^n). \tag{3.6.6}$$

Using the kernel representation (3.2.10) of the semigroup $e^{-it\mathcal{L}}$, we may write

$$\mathcal{Q}_k f(z) = C_n \int_{\mathbb{C}^n}\int_{-\pi/2}^{\pi/2} (\sin t)^{-n} e^{i\left(t(2k+n)+\frac{|z-w|^2}{4}\cot t - \frac{1}{2}\operatorname{Im}(z\cdot\overline{w})\right)} f(w)\,dt\,dw. \tag{3.6.7}$$

The kernel of $e^{-it\mathcal{L}}$ has singularity at $t = 0$. We decompose the operator away from the singularity using a partition of unity introduced in Jeong–Lee–Ryu [67].

For any function $\eta \in C^\infty(\mathbb{R})$, define the operator $\mathcal{Q}_k[\eta]$ by

$$\mathcal{Q}_k[\eta] f = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \eta(t)\, e^{it(2k+n)} e^{-it\mathcal{L}} f\,dt. \tag{3.6.8}$$

Let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth function supported in $\left(-\frac{\pi}{16}, \frac{\pi}{16}\right)$ and equals 1 on $\left(-\frac{\pi}{32}, \frac{\pi}{32}\right)$. For $j \geq 1$, define

$$\psi_j(t) = \psi(2^{j-1}t) - \psi(2^j t).$$

Then $\sum_{j=1}^\infty \psi_j(t) = \psi(t)$ for all $t \neq 0$. Next, we define periodic functions φ_0 and φ_j of period π by setting

$$\varphi_j(t) = \psi_j(t - \tfrac{\pi}{2}), \qquad \varphi_0(t) = 1 - \psi(t) - \psi\big(2^2(t - \tfrac{\pi}{2})\big), \tag{3.6.9}$$

for $t \in [-\pi/4, 3\pi/4]$. It is clear that φ_0 and φ_k are smooth functions. Moreover, they form a partition of unity

$$\sum_{j=1}^\infty \psi_j(t) \;+\; \sum_{j=3}^\infty \varphi_j(t) \;+\; \varphi_0(t) = 1, \qquad t \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \setminus \{0\}. \tag{3.6.10}$$

Using this partition of unity, we decompose the spectral projection operator as

$$\mathcal{Q}_k = \sum_{j\geq 1} \mathcal{Q}_k[\psi_j] + \sum_{j\geq 3} \mathcal{Q}_k[\varphi_j] + \mathcal{Q}_k[\varphi_0]. \tag{3.6.11}$$

3.6.2 Estimates for the oscillatory integral

Let the phase function ϕ be defined by

$$\phi(t) := \phi(z,w,t) := t + \frac{|z-w|^2}{4}\cot t - \frac{1}{2}\Im(z\cdot\overline{w}), \qquad z, w \in \mathbb{C}^n. \tag{3.6.12}$$

For $j \in \mathbb{Z}$, $\mu \in \mathbb{R}$, $z, w \in \mathbb{C}^n$ and η is a function supported in $[-\pi/8, -\pi/32] \cup [\pi/32, \pi/8]$, we consider the oscillatory integrals I_j, J_j, and J_0 defined by

$$\begin{aligned} I_j(z,w,\mu) &:= \int \eta(2^j t)\, e^{i\mu\phi(z,w,t)}\, dt, \\ J_j(z,w,\mu) &:= \int \eta\big(2^j(t-\tfrac{\pi}{2})\big)\, e^{i\mu\phi(z,w,t)}\, dt, \\ J_0(z,w,\mu) &:= \int_0^{\pi} \varphi_0(t)\, e^{i\mu\phi(z,w,t)}\, dt. \end{aligned}$$

Lemma 3.6.2. *[67] Let $n \geq 1$, $\mu \geq 1$ and $j \geq 1$. Let η be a C^1-function supported in $[-\pi/8, -\pi/32] \cup [\pi/32, \pi/8]$. Then we have*

$$|I_j(z,w,\mu)| \leq C\, \mu^{-1/2}\, 2^{-j/2}\, \|\eta\|_{C^1}, \tag{3.6.13}$$

$$|J_j(z,w,\mu)| \leq C\, \mu^{-1/2}\, 2^{j/2}\, \|\eta\|_{C^1}, \tag{3.6.14}$$

$$|J_0(z,w,\mu)| \leq C\, \mu^{-1/2}, \tag{3.6.15}$$

with C independent of $z, w \in \mathbb{C}^n$, j, μ.

3.6.3 Restriction estimates for special Hermite spectral projections in Schatten spaces

To obtain the estimate (3.6.5), we use a complex interpolation estimate in Schatten spaces. The following Schatten space estimate obtained in [49].

Proposition 3.6.3. *[49] Let (G_α) be an analytic family of operators on $\mathbb{C}^n$ in the sense of Stein, defined on the strip $-\lambda_0 \leq \operatorname{Re}\alpha \leq 0$ for some $\lambda_0 > 1$. Assume that we have the bounds*

$$\|G_{i\tau}\|_{L^2\to L^2} \leq M_0 e^{a|\tau|}, \qquad \|G_{-\lambda_0+i\tau}\|_{L^1\to L^\infty} \leq M_1 e^{b|\tau|}, \qquad \text{for all } \tau \in \mathbb{R},$$

for some $a, b \geq 0$ and some $M_0, M_1 \geq 0$. Then, for all $W_1, W_2 \in L^{2\lambda_0}(\mathbb{C}^n)$, the operator $W_1 G_{-1} W_2$ belongs to the Schatten class $\mathcal{G}^{2\lambda_0}(L^2(\mathbb{C}^n))$ and we have the estimate

$$\|W_1 G_{-1} W_2\|_{\mathcal{G}^{2\lambda_0}(L^2(\mathbb{C}^n))} \leq M_0^{1-\frac{1}{\lambda_0}} M_1^{\frac{1}{\lambda_0}} \|W_1\|_{L^{2\lambda_0}(\mathbb{C}^n)} \|W_2\|_{L^{2\lambda_0}(\mathbb{C}^n)}.$$

We are now in a position to state and prove our main result of this section.

Theorem 3.6.4. *Let $n \geq 1$, $k \geq 1$. and $\frac{2(2n+1)}{2n+3} < p \leq 2$. Then there exists $C > 0$ such that for all $W_1, W_2 \in L^{2p/(2-p)}(\mathbb{C}^n)$, we have the estimate*

$$\|W_1 \mathcal{Q}_k W_2\|_{\mathcal{G}^{2p/(2-p)}(L^2(\mathbb{C}^n))} \leq k^{2\varrho(p)} \|W_1\|_{L^{2p/(2-p)}(\mathbb{C}^n)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{C}^n)}, \tag{3.6.16}$$

where $\varrho(p)$ is defined in (3.6.2). Moreover, the exponent $\varrho(p)$ in (3.6.16) is optimal.

Proof. For any function $\eta \in C^\infty(\mathbb{R})$, we set

$$G_k^\alpha[\eta]f(z) = C \int_{\mathbb{C}^n} \int_{-\pi/2}^{\pi/2} (\sin t)^{-\alpha-n-1} \eta(t) e^{i\left(t(2k+n) + \frac{|z-w|^2}{4}\cot t - \frac{1}{2}\operatorname{Im}(z\cdot\overline{w})\right)} f(w)\, dt\, dw. \tag{3.6.17}$$

The family $(G_k^\alpha[\eta])_\alpha$ forms an analytic family of operators in the strip $-n - \frac{1}{2} - \varepsilon \leq \operatorname{Re}\alpha \leq 0$, with $0 < \varepsilon \ll \frac{1}{2}$.

Now take $\alpha = -n - \frac{1}{2} - \varepsilon + i\tau$, $\tau \in \mathbb{R}$. Let $\mu = 2k + n$. By scaling, we see that

$$G_k^{-n-1/2-\epsilon+i\tau}[\eta] f(\sqrt{\mu}\, z) = \frac{C_n}{\pi} \int_{\mathbb{C}^n} \left(\int_{-\pi/2}^{\pi/2} (\sin t)^{-\frac{1}{2}+\epsilon-i\tau}\, \eta(t)\, e^{i\mu\phi(z,w,t)}\, dt \right) f(\sqrt{\mu}\, w)\, \mu^n\, dw, \tag{3.6.18}$$

where ϕ is defined in (3.6.12). Take $\tau = 0$. Since $\|f(\sqrt{\mu}\,\cdot)\mu^n\|_1 = \|f\|_1$, to obtain the $L^1 - L^\infty$ bound for $G_k^{-n-1/2+\epsilon}[\eta]$ we need only consider the kernel

$$\int_{-\pi/2}^{\pi/2} (\sin t)^{-\frac{1}{2}+\epsilon} \eta(t) e^{i(2k+n)\phi(z,w,t)}\, dt.$$

On the support of ψ_j, $|\sin t| \gtrsim 2^{-j}$, thus by (3.6.13),

$$\|G_k^{-n-1/2-\epsilon+i\tau}[\psi_j]\|_{L^1 \to L^\infty} \leq C e^{a|\tau|} k^{-1/2} 2^{-j\epsilon}\ \|\psi\|_{C^1}.$$

For $\alpha = i\tau, \tau \in \mathbb{R}$, we can write

$$G_k^{i\tau}[\psi_j] f(z) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \psi_j(t)\, e^{it(2k+n)} (\sin t)^{-i\tau-1} e^{-itL} f(z)\, dt.$$

By (3.2.9) and Minkowski inequality, we obtain

$$\|G_k^{i\tau}[\psi_j]\|_{L^2 \to L^2} \leq e^{b|\tau|}\ \|\psi_j(t)(\sin t)^{-1}\|_{L^1} \leq e^{a|\tau|}\ \|\psi\|_{L^1}.$$

Now, by Proposition 3.6.3 and using the fact that $\mathcal{Q}_k[\psi_j] = G_k^{-1}[\psi_j]$, we obtain

$$\|W_1 \mathcal{Q}_k[\psi_j] W_2\|_{\mathcal{G}^{2n+1+2\varepsilon}(L^2(\mathbb{C}^n))} \leq C k^{-\frac{1}{2n+1+2\varepsilon}} 2^{-\frac{2\varepsilon}{2n+1+2\varepsilon} j} \|W_1\|_{L^{2n+1+2\varepsilon}(\mathbb{C}^n)} \|W_2\|_{L^{2n+1+2\varepsilon}(\mathbb{C}^n)}. \tag{3.6.19}$$

Next, on the support of φ_j, $|\sin t| \gtrsim 1$, thus by (3.6.14) we have

$$\|G_k^{-n-1/2-\varepsilon+i\tau}[\varphi_j]\|_{L^1\to L^\infty} \leq Ce^{a|\tau|}k^{-1/2}2^{j/2}\ \|\psi\|_{C^1}.$$

As before using (3.2.9) and Minkowski inequality, we get

$$\|G_k^{i\tau}[\varphi_j]\|_{L^2\to L^2} \leq e^{a|\tau|}\ \|\varphi_j(t)(\sin t)^{-1}\|_{L^1} \leq 2^{-j}e^{b|\tau|}\ \|\psi\|_{L^1}.$$

Now, by Proposition 3.6.3, and using the fact that $\mathcal{Q}_k[\varphi_j] = G_k^{-1}[\varphi_j]$, we obtain

$$\|W_1\mathcal{Q}_k[\varphi_j]W_2\|_{\mathcal{G}^{2n+1+2\varepsilon}(L^2(\mathbb{C}^n))} \leq Ck^{-\frac{1}{2n+1+2\varepsilon}}2^{-j(1-\frac{3}{2n+1+2\varepsilon})}\|W_1\|_{L^{2n+1+2\varepsilon}(\mathbb{C}^n)}\|W_2\|_{L^{2n+1+2\varepsilon}(\mathbb{C}^n)}. \tag{3.6.20}$$

Again, as before, $|\sin t| \gtrsim 1$ on the support of φ_j, thus by (3.6.15) we have

$$\|G_k^{-n-1/2-\varepsilon+i\tau}[\varphi_j]\|_{L^1\to L^\infty} \leq Ce^{a|\tau|}k^{-1/2}\ \|\psi\|_{C^1}.$$

Again,

$$\|G_k^{i\tau}[\varphi_0]\|_{L^2\to L^2} \leq\ e^{b|\tau|}\|\psi\|_{L^1}.$$

Now, by Proposition 3.6.3 and using the fact that $\mathcal{Q}_k[\psi_0] = G_k^{-1}[\psi_0]$, we obtain

$$\|W_1\mathcal{Q}_k[\varphi_0]W_2\|_{\mathcal{G}^{2n+1+2\epsilon}(L^2(\mathbb{C}^n))} \leq Ck^{-\frac{1}{2n+1+2\epsilon}}\|W_1\|_{L^{2n+1+2\epsilon}(\mathbb{C}^n)}\|W_2\|_{L^{2n+1+2\epsilon}(\mathbb{C}^n)}. \tag{3.6.21}$$

Hence, using the decomposition (3.6.11) together with the estimates (3.6.19), (3.6.20), and (3.6.21), we obtain

$$\|W_1\mathcal{Q}_kW_2\|_{\mathcal{G}^{2n+1+2\varepsilon}(L^2(\mathbb{C}^n))} \leq Ck^{-\frac{1}{2n+1+2\epsilon}}\|W_1\|_{L^{2n+1+2\varepsilon}(\mathbb{C}^n)}\|W_2\|_{L^{2n+1+2\varepsilon}(\mathbb{C}^n)}. \tag{3.6.22}$$

Furthermore, we have $\|\mathcal{Q}_kf\|_{L^2(\mathbb{C}^n)} \leq \|f\|_{L^2(\mathbb{C}^n)}$. If $W_1, W_2 \in L^\infty(\mathbb{C}^n)$, then applying Hölder's inequality, we obtain

$$\|W_1\mathcal{Q}_kW_2\|_{\mathcal{G}^\infty(L^2(\mathbb{C}^n))} \leq \|W_1\|_{L^\infty(\mathbb{C}^n)}\|W_2\|_{L^\infty(\mathbb{C}^n)}. \tag{3.6.23}$$

By complex interpolation between (3.6.23) and (3.6.22) with sufficiently small ε, we obtain (3.6.16) for $\frac{2(2n+1)}{2n+3} < p \leq 2$. ☐

By applying the duality principle (see Lemma 3 in [49]) to (3.6.16), we immediately deduce the following result for orthonormal systems.

Theorem 3.6.5. *Let $n \geq 1$, $k \geq 1$. Assume $1 \leq p < \frac{2n+1}{2n-1}$ and $1 \leq \beta \leq \frac{2p}{p+1}$. Then*

$$\left\| \sum_{j \in J} n_j \left| \mathcal{Q}_k f_j \right|^2 \right\|_{L^p(\mathbb{C}^n)} \leq C_B k^{-\frac{1}{2}(1-1/p)} \left(\sum_{j \in J} |n_j|^\beta \right)^{1/\beta}, \tag{3.6.24}$$

holds for any orthonormal system $(f_j)_{j \in J}$ in $L^2(\mathbb{C}^n)$ and any sequence $(n_j)_{j \in J} \subset \mathbb{C}$. Moreover, the exponent $-\frac{1}{2}(1-1/p)$ in (3.6.24) is optimal.

The estimate (3.6.24) reduces to the dual of (3.6.1) when the orthonormal system is reduced to one function (and the corresponding coefficient $n = 1$).

For $1 \leq p < \frac{2(3n+1)}{3n+4}$, by using the analytic family of operators introduced in [89], we obtain the following result.

Theorem 3.6.6. *Let $n \geq 1$ and $k \geq 1$. Then for any $1 \leq p < \frac{2(3n+1)}{3n+4}$, there exists $C > 0$ such that for all $W_1, W_2 \in L^{2p/(2-p)}(\mathbb{C}^n)$, we have*

$$\|W_1 \mathcal{Q}_k W_2\|_{\mathcal{G}^{\frac{(3n-2)p}{6n-1-(3n+1)p}}(L^2(\mathbb{C}^n))} \leq C k^{2\varrho(p)} \|W_1\|_{L^{2p/(2-p)}(\mathbb{C}^n)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{C}^n)}, \tag{3.6.25}$$

where $\varrho(p)$ is defined in (3.6.2). Moreover, the exponent $\varrho(p)$ in (3.6.25) is optimal.

Proof. For $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > -1$, consider

$$\psi_k^\alpha(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha \left(\frac{1}{2}|z|^2 \right) e^{-\frac{1}{4}|z|^2}. \tag{3.6.26}$$

We then set

$$G_k^\alpha f(z) = f \times \psi_k^{\alpha+n}(z), \quad f \in \mathcal{S}(\mathbb{C}^n).$$

The family $(G_k^\alpha)_\alpha$ forms an analytic family of operators in the strip $-n - \frac{1}{2} \leq \operatorname{Re} \alpha \leq 0$ (see [89]). From [89] and [107], we have the following estimates

$$\|G_k^{-\lambda_0 + i\tau} f\|_{L^\infty(\mathbb{C}^n)} \leq C(1+|\tau|)^{2/3} \|f\|_{L^1(\mathbb{C}^n)}, \quad 0 \leq \lambda_0 < n + 1/3, \tag{3.6.27}$$

and

$$\|G_k^{i\tau} f\|_{L^2(\mathbb{C}^n)} \leq C(1+|\tau|)^n k^{-n} \|f\|_{L^2(\mathbb{C}^n)}. \tag{3.6.28}$$

Also we have

$$\mathcal{Q}_k f(z) = \frac{\Gamma(k+n)\Gamma(n)}{\Gamma(k+1)} G_k^{-1} f(z)$$

and $\frac{\Gamma(k+n)\Gamma(n)}{\Gamma(k+1)} \leq Ck^{n-1}$. Thus by Proposition 3.6.3, we obtain

$$\|W_1 \mathcal{Q}_k W_2\|_{\mathcal{G}^{2\lambda_0}(L^2(\mathbb{C}^n))} \leq Ck^{\frac{n}{\lambda_0}-1} \|W_1\|_{L^{2\lambda_0}(\mathbb{C}^n)} \|W_2\|_{L^{2\lambda_0}(\mathbb{C}^n)}, \tag{3.6.29}$$

for $1 \leq \lambda_0 < n + \frac{1}{3}$.

To complete the proof of the theorem, we observe that $W_1 \mathcal{Q}_k W_2 = (W_1 \mathcal{Q}_k)(\overline{W_2}\mathbb{Q}_k)^*$. The operator $W\mathcal{Q}_k$ acts from $L^2(\mathbb{C}^n)$ to $L^2(\mathbb{C}^n)$ as an integral operator with integral kernel $K(w,z) = W(z)\varphi_k(z-w)e^{iIm(z\bar{w})}$, where $z, w \in \mathbb{C}^n$. If $W \in L^2(\mathbb{C}^n)$, then $W\mathcal{Q}_k$ is Hilbert-Schmidt, in fact,

$$\|W\mathcal{Q}_k\|^2_{\mathcal{G}^2(L^2(\mathbb{C}^n))} = \int_{\mathbb{C}^n} |W(z)|^2 \left(\int_{\mathbb{C}^n} |\varphi_k(z-w)|^2 \, dw \right) dz \leq Ck^{n-1} \|W\|^2_{L^2(\mathbb{C}^n)},$$

since $\|\varphi_k\|_{L^2(\mathbb{C}^n)} \leq Ck^{\frac{n-1}{2}}$. Now, by Hölder's inequality for trace ideals, we get

$$\|W_1 \mathcal{Q}_k W_2\|_{\mathcal{G}^1(L^2(\mathbb{C}^n))} \leq Ck^{n-1} \|W_1\|_{L^2(\mathbb{C}^n)} \|W_2\|_{L^2(\mathbb{C}^n)}. \tag{3.6.30}$$

By complex interpolation between (3.6.30) and (3.6.29) with λ_0 close to $n + \frac{1}{3}$, we get

$$\|W_1 \mathcal{Q}_k W_2\|_{\mathcal{G}^{\frac{(6n-4)q}{6n-1-3q}}(L^2(\mathbb{C}^n))} \leq Ck^{\frac{n}{q}-1} \|W_1\|_{L^{2q}(\mathbb{C}^n)} \|W_2\|_{L^{2q}(\mathbb{C}^n)}.$$

for $1 \leq q < \frac{3n+1}{3}$. By setting $2q = \frac{2p}{2-p}$, we obtain the estimate (3.6.25). $\square$

Applying the duality principle (see Lemma 3 in [49]) to (3.6.25), we deduce the following result for orthonormal systems.

Theorem 3.6.7. *Let $n \geq 1$, $k \geq 1$. Suppose $\frac{3n+1}{3n-2} < p \leq \infty$ and $1 \leq \beta \leq \frac{2p(3n-2)}{6n-1}$. Then*

$$\left\| \sum_{j \in J} n_j \left| \mathcal{Q}_k f_j \right|^2 \right\|_{L^p(\mathbb{C}^n)} \leq Ck^{n(1-1/p)-1} \left(\sum_{j \in J} |n_j|^\beta \right)^{1/\beta}, \tag{3.6.31}$$

for any orthonormal system $(f_j)_{j \in J}$ in $L^2(\mathbb{C}^n)$ and any sequence $(n_j)_{j \in J} \subset \mathbb{C}$. Moreover, the exponent $n(1-1/p)-1$ in (3.6.31) is optimal.

We expect that the Schatten exponent in (3.6.25) can be improved to $\frac{(2n-1)p}{4n-(2n+1)p}$. However, we obtain such an estimate only locally, in the following theorem.

Theorem 3.6.8. *Let B be a fixed compact subset of $\mathbb{C}^n$. Let $n \geq 1$, $k \geq 1$ and $1 \leq p \leq \frac{2(2n+1)}{2n+3}$. Then there exists $C_B > 0$ depending only on B, such that for all $W_1, W_2 \in L^{2p/(2-p)}(\mathbb{C}^n)$, we have the estimate*

$$\|W_1 \chi_B \mathcal{Q}_k \chi_B W_2\|_{\mathcal{G}^{\frac{(2n-1)p}{4n-(2n+1)p}}(L^2(\mathbb{C}^n))} \leq C_B k^{2\varrho(p)} \|W_1\|_{L^{2p/(2-p)}(\mathbb{C}^n)} \|W_2\|_{L^{2p/(2-p)}(\mathbb{C}^n)}, \tag{3.6.32}$$

Moreover, the exponent $\varrho(p)$ in (3.6.32) is optimal.

Proof. Let B be a fixed compact set of $\mathbb{C}^n$. We consider the following analytic family of operators, for $\alpha \in \mathbb{C}$, let

$$G_k^{\alpha} f(z) = (\chi_B f) \times \psi_k^{\alpha+n}(z), \quad f \in \mathcal{S}(\mathbb{C}^n).$$

We have the following estimates

$$\|G_k^{-(n+\frac{1}{2})+i\tau} f\|_{L^\infty(\mathbb{C}^n)} \leq C_B (1+|\tau|)^{1/2} \|f\|_{L^1(\mathbb{C}^n)}, \tag{3.6.33}$$

$$\|G_k^{i\tau} f\|_{L^2(\mathbb{C}^n)} \leq C_B (1+|\tau|)^n k^{-n} \|f\|_{L^2(\mathbb{C}^n)}. \tag{3.6.34}$$

Estimate (3.6.33) can be derived from estimate (3.7) in [105], while estimate (3.6.34) follows from the better estimate (3.6.28). For $\alpha = -1$, we have

$$\mathcal{Q}_k \chi_B f(z) = \frac{\Gamma(k+n)\Gamma(n)}{\Gamma(k+1)} G_k^{-1} f(z).$$

Now, by Proposition 3.6.3, we obtain

$$\|W_1 \chi_B \mathcal{Q}_k \chi_B W_2\|_{\mathcal{G}^{2n+1}(L^2(\mathbb{C}^n))} \leq C k^{-\frac{1}{2n+1}} \|W_1\|_{L^{2n+1}(\mathbb{C}^n)} \|W_2\|_{L^{2n+1}(\mathbb{C}^n)}. \tag{3.6.35}$$

We observe that $W_1 \chi_B \mathcal{Q}_k \chi_B W_2 = (W_1 \chi_B \mathcal{Q}_k)(\overline{W_2} \chi_B \mathbb{Q}_k)^*$. The operator $W \chi_B \mathcal{Q}_k$ acts on $L^2(\mathbb{C}^n)$ as an integral operator with kernel $W(z)\chi_B(z)\varphi_k(z-w)e^{iIm(z\bar{w})}$, where $z, w \in \mathbb{C}^n$. If $W \in L^2(\mathbb{C}^n)$, then $W\chi_B \mathcal{Q}_k$ is Hilbert-Schmidt, with

$$\|W \chi_B \mathcal{Q}_k\|^2_{\mathcal{G}^2(L^2(\mathbb{C}^n))} \leq C_B k^{n-1} \|W\|^2_{L^2(\mathbb{C}^n)}.$$

Now, by Hölder's inequality for trace ideals, it follows that

$$\|W_1 \chi_B \mathcal{Q}_k \chi_B W_2\|_{\mathcal{G}^1(L^2(\mathbb{C}^n))} \leq C_B k^{n-1} \|W_1\|_{L^2(\mathbb{C}^n)} \|W_2\|_{L^2(\mathbb{C}^n)}. \tag{3.6.36}$$

By complex interpolation between (3.6.35) and (3.6.36) we get the estimate (3.6.32) for $1 \leq p \leq \frac{2(2n+1)}{2n+3}$, which completes the proof of the theorem. □

Again, by the duality principle, (3.6.32) is equivalent to the following local version of the restriction estimate for orthonormal systems.

Theorem 3.6.9. *Let B be a fixed compact subset of $\mathbb{C}^n$. Let $n \geq 1$, $k \geq 1$. Suppose $\frac{2n+1}{2n-1} \leq p \leq \infty$ and $1 \leq \beta \leq \frac{p(2n-1)}{2n}$. Then*

$$\left\| \sum_{j \in J} n_j \left| \mathcal{Q}_k \chi_B f_j \right|^2 \right\|_{L^p(\mathbb{C}^n)} \leq C_B k^{n(1-1/p)-1} \left(\sum_{j \in J} |n_j|^\beta \right)^{1/\beta}, \tag{3.6.37}$$

holds for any orthonormal system $(f_j)_{j \in J}$ in $L^2(\mathbb{C}^n)$ and any sequence $(n_j)_{j \in J} \subset \mathbb{C}$. Moreover, the exponent $n(1-1/p)-1$ in (3.6.37) is optimal.

Remark 3.6.10. *We believe that Theorem 3.6.9 remains valid in the global setting. This is suggested by interpolating the estimate (3.6.16) at the end point* $p = \frac{2(2n+1)}{2n+3}$ *with the estimate (3.6.30). However, establishing the estimate (3.6.16) at this end point* $p = \frac{2(2n+1)}{2n+3}$ *appears to be challenging.*

A natural direction for further research is to establish the endpoint estimate (3.6.16) at $p = \frac{2(2n+1)}{2n+3}$ *and and to investigate the optimality of the exponent* β *appearing in the estimates (3.6.24) and (3.6.37) are optimal. For similar estimates in the orthonormal setting, we refer to the work of Frank and Sabin [50], which extends Sogge's* L^p *spectral cluster estimates for the Laplace–Beltrami operator on compact Riemannian manifolds to systems of orthonormal functions; see also the related work of Nguyen [88].*

CHAPTER 4

On the Schatten exponent in orthonormal Strichartz estimates for the Dunkl operators

4.1 Introduction

The main objective of this chapter is to analyze the Schatten exponent appearing in the orthonormal Strichartz estimates associated with the Dunkl Laplacian Δ_k and the Dunkl-Hermite operator $\mathcal{H}_k$ studied by Senapati-Pradeep-Mondal-Mejjaoli [94] and Mondal-Song [82] respectively.

Theorem 4.1.1. *[94]. (Orthonormal Strichartz estimate for the Dunkl Laplacian) Suppose $k \geq 0$ and $p, q, n \geqslant 1$ such that $1 \leqslant p < \frac{2\gamma+n+1}{2\gamma+n-1}$, $\frac{2}{q} + \frac{2\gamma+n}{p} = 2\gamma+n$ and $1 \leq \beta \leq \frac{2p}{p+1}$. Then*

$$\left\| \sum_j \lambda_j \left| e^{-it\Delta_k} f_j \right|^2 \right\|_{L^q(\mathbb{R}, L^p_k(\mathbb{R}^n))} \leqslant C \left(\sum_{j \in J} |n_j|^\beta \right)^{1/\beta}, \tag{4.1.1}$$

holds for any orthonormal system $(f_j)_j$ in $L^2_k(\mathbb{R}^n)$ and all sequence $(n_j)_j \subset \mathbb{C}$.

Theorem 4.1.2. *[82].(Orthonormal Strichartz estimate for the Dunkl-Hermite operator) Suppose $k \geq 0$ and $p, q, n \geqslant 1$ such that $1 \leqslant p < \frac{2\gamma+n+1}{2\gamma+n-1}$, $\frac{2}{q} + \frac{2\gamma+n}{p} = 2\gamma + n$ and*

$1 \leq \beta \leq \frac{2p}{p+1}$. Then

$$\left\| \sum_j \lambda_j \left| e^{-it\mathcal{H}_k} f_j \right|^2 \right\|_{L^q((-\frac{\pi}{2},\frac{\pi}{2}),L^p_k(\mathbb{R}^n))} \leqslant C \left(\sum_{j\in J} |n_j|^\beta \right)^{1/\beta}, \tag{4.1.2}$$

holds for any orthonormal system $(f_j)_j$ in $L^2_k(\mathbb{R}^n)$ and all sequence $(n_j)_j \subset \mathbb{C}$.

We provide a necessary condition on β for the validity of (4.1.1) and (4.1.2), which turns out to be optimal for the Schrödinger equations associated with Laplacian and Hermite operator as a particular case. The proof is based on the construction of suitable coherent states in the Dunkl setting and a semiclassical argument.

4.2 Preliminaries

In this section we provide some essential background information from Dunkl theory.

4.2.1 The Dunkl operators

Let $\mathcal{R}$ be a (reduced) root system on $\mathbb{R}^n$, normalized so that $\langle \alpha, \alpha \rangle = \|\alpha\|^2 = 2$ for all $\alpha \in \mathcal{R}$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product and $\|\cdot\|$ its induced norm. Let $G \subset O(n, \mathbb{R})$ be the associated reflection group and $k : \mathcal{R} \to [0, \infty)$ a G-invariant nonnegative multiplicity function on $\mathcal{R}$. The Dunkl operators $T^k_j, j = 1, ..., n$, associated with $\mathcal{R}$ and k, which were introduced in [38], are defined by

$$T^k_j f(x) := \partial_j f(x) + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \alpha_j \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^n),$$

here ∂_j denotes the jth partial derivative, $\mathcal{R}^+$ is a fixed positive subsystem of $\mathcal{R}$, and r_α denotes the reflection in the hyperplane orthogonal to α. The Dunkl operators $T^k_j, j = 1, ..., n$ form a commutative algebra of differential-difference operators, and reduce to $\partial_j, j = 1, ..., n$, when $k \equiv 0$. An analog of the Laplace operator, the Dunkl Laplacian is defined by

$$\Delta_k = \sum_{j=1}^n (T^k_j)^2.$$

Associated to this operator the Dunkl-Hermite operator is defined by

$$\mathcal{H}_k = -\frac{1}{2}(\Delta_k - \|x\|^2).$$

Note that the operators Δ_k and $\mathcal{H}_k$ turn out to be the usual Laplacian Δ and the classical Hermite operator $\mathcal{H} = -\frac{1}{2}(\Delta - \|x\|^2)$ on $\mathbb{R}^n$ respectively, when $k \equiv 0$.

We introduce the weighted measure $dw_k(x) = \prod\limits_{\alpha\in\mathcal{R}^+} |\langle\alpha, x\rangle|^{2k(\alpha)}dx$ on $\mathbb{R}^n$, which is G-invariant and homogeneous of degree 2γ with $\gamma = \sum\limits_{\alpha\in\mathcal{R}^+} k(\alpha)$. For $1 \leq p < \infty$, the space $L^p_k(\mathbb{R}^n)$ consists of all functions f on $\mathbb{R}^n$ such that $\|f\|_{L^p_k} = \left(c_k \int_{\mathbb{R}^n} |f|^p dw_k\right)^{1/p} < \infty$, where $c_k^{-1} = \int_{\mathbb{R}^n} e^{-\|x\|^2/2}dw_k(x)$; and $L^\infty_k(\mathbb{R}^n)$ is defined in the usual way.

4.2.2 The Dunkl kernel

For $y \in \mathbb{R}^n$, the initial value problem

$$\begin{cases} T^k_j u(x, y) = y_j u(x, y), \quad j = 1, \cdots, n, \\ u(0, y) = 1; \end{cases} \tag{4.2.1}$$

admits a unique analytic solution on $\mathbb{R}^n$, denoted by $E_k(x, y)$ and called the Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^n \times \mathbb{C}^n$. The Dunkl kernel possesses the following properties: for $z, w \in \mathbb{C}^n$, $\beta \in \mathbb{C}$,

$$E_k(z, w) = E_k(w, z), \ E_k(\beta z, w) = E_k(z, \beta w), \ \overline{E_k(z, w)} = E_k(\bar{z}, \bar{w}). \tag{4.2.2}$$

For more details, we refer to [38, 39]. Moreover, the Dunkl kernel $E_k(x, y)$ satisfies the following upper and lower bound estimates (see [4]): for every $\epsilon > 0$, there exists $C_1, C_2 > 0$ such that

$$\frac{C_1\, t^\gamma}{\prod\limits_{\alpha\in\mathcal{R}^+} (|\langle\alpha, x\rangle| + \sqrt{t})^{2k(\alpha)}} \leq e^{-\frac{\|x\|^2+\|y\|^2}{4t}} E_k\left(\frac{x}{2t}, y\right) \leq \frac{C_2\, t^\gamma}{\prod\limits_{\alpha\in\mathcal{R}^+} (|\langle\alpha, x\rangle| + \sqrt{t})^{2k(\alpha)}}, \tag{4.2.3}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$ satisfying $\|x - y\| < \epsilon\sqrt{t}$.

The following proposition is crucial in Dunkl's theory and its applications.

Proposition 4.2.1. *[39] For $k \geq 0$, $z, w \in \mathbb{C}^n$*

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2} E_k(x, z)E_k(x, w)w_k(x)dx = c_k e^{(\ell(z)+\ell(w))/2} E_k(z, w). \tag{4.2.4}$$

4.2.3 Generalized Hermite polynomials

Let $\mathcal{P} = \mathbb{C}[\mathbb{R}^n]$ be the algebra of polynomial functions on $\mathbb{R}^n$ and $\mathcal{P}_l$, $l \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, the subspace of homogeneous polynomials of degree l. In [38], Dunkl introduced the bilinear form on $\mathcal{P}$ by $[p, q]_k := (p(T)q)(0)$, $p, q \in \mathcal{P}$, here $p(T)$ is the operator derived from $p(x)$ by replacing x_j by T_j. Then for a given orthonormal basis $\{\varphi_\nu : \nu \in \mathbb{N}_0^n\}$ of $\mathcal{P}$ with respect to the scalar product $[\cdot,\cdot]_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$ with real coefficients, the generalized Hermite polynomials $\{H_\nu : \nu \in \mathbb{N}_0^n\}$ and Hermite functions $\{h_\nu : \nu \in \mathbb{N}_0^n\}$ are defined by

$$H_\nu(x) := 2^{|\nu|} e^{-\Delta_k/4}\varphi_\nu(x), \quad \text{and} \quad h_\nu(x) := 2^{-|\nu|/2} e^{-\|x\|^2/2} H_\nu(x), \quad x \in \mathbb{R}^n, \tag{4.2.5}$$

where $|\nu| = \sum_{j=1}^n \nu_j$. We refer to [91] for the above results. We recall some properties of H_ν and h_ν in the following proposition, which can be found in [91], pp. 525–531.

Proposition 4.2.2. *1. The generalized Hermite functions $\{h_\nu : \nu \in \mathbb{N}_0^n\}$ form a complete set of eigenfunctions for the Dunkl-Hermite operator $\mathcal{H}_k$ with $\mathcal{H}_k h_\nu = (2|\nu| + 2\gamma + n)h_\nu$.*

2. The set $\{h_\nu : \nu \in \mathbb{N}_0^n\}$ is an orthonormal basis of $L_k^2(\mathbb{R}^n)$.

3. For all $z, w \in \mathbb{C}^n$, there is a generating function for the generalized Hermite polynomials,

$$e^{-\ell(w)} E_k(2z, w) = \sum_{\nu \in \mathbb{N}_0^n} H_\nu(z)\varphi_\nu(w). \tag{4.2.6}$$

4. For $r \in \mathbb{C}$ with $|r| < 1$ and for all $x, y \in \mathbb{R}^n$, the Mehler formula for the generalized Hermite polynomials,

$$\sum_{\nu \in \mathbb{N}_0^n} \frac{H_\nu(x)H_\nu(y)}{2^{|\nu|}} r^{|\nu|} = \frac{1}{(1-r^2)^{\gamma+n/2}} e^{-\frac{r^2(x^2+y^2)}{1-r^2}} E_k\left(\frac{2rx}{1-r^2}, y\right). \tag{4.2.7}$$

4.2.4 Generalized Fock space

In [98], Soltani introduced (independently by Ben Saïd–Ørsted around the same time [17]) the generalized Fock space $\mathcal{A}_k$ associated with the Dunkl operators. This is a Hilbert space of holomorphic functions on $\mathbb{C}^n$ with the reproducing kernel $E_k(z, \bar{w})$, for $z, w \in \mathbb{C}^n$. More precisely, $\mathcal{A}_k = \{f(z) = \sum_{\nu \in \mathbb{N}_0^n} a_\nu \varphi_\nu(z) : \|f\|_k^2 := \sum_{\nu \in \mathbb{N}_0^n} |a_\nu|^2 < \infty\}$. For $f(z) = \sum_{\nu \in \mathbb{N}_0^n} a_\nu \varphi_\nu(z)$, $g(z) = \sum_{\nu \in \mathbb{N}_0^n} b_\nu \varphi_\nu(z) \in \mathcal{A}_k$, the inner product in $\mathcal{A}_k$ is given by $(f, g)_k = \sum_{\nu \in \mathbb{N}_0^2} a_\nu \bar{b_\nu}$. The chaotic transform (also called the generalized Segal-Bargmann transform) is the transformation defined on $L_k^2(\mathbb{R}^n)$, by

$$\mathcal{C}_k(f)(z) := \int_{\mathbb{R}^n} e^{-(\ell(z)+\ell(x))/2} E_k(\sqrt{2}z, x) f(x) w_k(x) dx, \tag{4.2.8}$$

where $\ell(z)=\sum_{j=1}^{n} z_j^2$, $z=(z_1,z_2\cdots,z_n)\in\mathbb{C}^n$. The chaotic transform $\mathcal{C}_k$ is a unitary mapping of $L_k^2(\mathbb{R}^n)$ onto $\mathcal{A}_k$. Moreover, the basis elements are related by $\mathcal{C}_k(h_\nu)=\varphi_\nu$. We refer to [99] and the references therein for additional properties and applications of the chaotic transform.

4.2.5 Schrödinger semigroup associated to Dunkl-Hermite and Dunkl Laplacian

In [1], Amri–Hammi have studied the Dunkl-Hermite Schrödinger semigroup $e^{-it\mathcal{H}_k}$ $(t\in\mathbb{R})$ on $L_k^2(\mathbb{R}^n)$ (see also [15]). For $t\in\mathbb{R}\setminus\pi\mathbb{Z}$, the operator $e^{-it\mathcal{H}_k}$ can be expressed as

$$e^{-it\mathcal{H}_k}f(x)=c_k\int_{\mathbb{R}^n}\Lambda_k(x,y;t)f(y)w_k(y)dy, \tag{4.2.9}$$

with the kernel

$$\Lambda_k(x,y;t)=\frac{1}{(i\sin t)^{\gamma+\frac{n}{2}}}e^{-\frac{i}{2}\cot t\left(\|x\|^2+\|y\|^2\right)}E_k\left(\frac{ix}{\sin t},y\right). \tag{4.2.10}$$

The Dunkl-Schrödinger semigroup $e^{-i\frac{t}{2}\Delta_k}$ $(t\in\mathbb{R})$ is given on $L_k^2(\mathbb{R}^n)$ by the following integral representation

$$u(t,x)=e^{-i\frac{t}{2}\Delta_k}f(x)=c_k\int_{\mathbb{R}^n}\Gamma_k(x,y;t)f(y)w_k(y)dy, \tag{4.2.11}$$

where

$$\Gamma_k(x,y;t)=\frac{1}{(it)^{\gamma+\frac{n}{2}}}e^{-\frac{it}{2}\left(\|x\|^2+\|y\|^2\right)}E_k\left(\frac{ix}{t},y\right). \tag{4.2.12}$$

Applying the change of variable $s=\tan(t)$ with $t\in(-\pi/2,\pi/2)$, we get

$$\Lambda_k(x,y;\tan^{-1}s)=c_k^{-1}\left(1+s^2\right)^{\frac{2\gamma+n}{4}}\exp\left(-is\frac{\|x\|^2}{2}\right)\Gamma_k\left(\left(1+s^2\right)^{\frac{1}{2}}x,y;s\right). \tag{4.2.13}$$

For a detailed study, we refer the reader to [17]. Using kernel relation (4.2.13) the following lemma is obtained in the proof of Theorem 6.1 in [82].

Lemma 4.2.3. *Suppose* $n\geq 1$. *If* $p,q\geq 1$ *satisfy* $\frac{1}{q}+\frac{2\gamma+n}{2p}=\frac{2\gamma+n}{2}$, *then*

$$\left\|\sum_j n_j\left|e^{it\Delta_k}f_j\right|^2\right\|_{L^q(\mathbb{R},L_k^p(\mathbb{R}^n))}=\left\|\sum_j n_j\left|e^{-it\mathcal{H}_k}f_j\right|^2\right\|_{L^q((-\frac{\pi}{2},\frac{\pi}{2}),L_k^p(\mathbb{R}^n))}, \tag{4.2.14}$$

for any system $(f_j)_j$ *of orthonormal functions in* $L_k^2(\mathbb{R}^n)$ *and any coefficients* $(n_j)_j\subset\mathbb{C}$.

4.3 Coherent states and the Dunkl-Hermite Schrödinger semigroup

In this section we construct a parameterized family of functions on $L^2_k(\mathbb{R}^n)$ that plays the role of coherent states and study their properties.

Definition 4.3.1. *A set of coherent states in a Hilbert space* H *is a subset* $\{\Phi_x\}_{x\in X}$ *of* H *such that*

1. X *is a locally compact topological space and the mapping* $x \mapsto \Phi_x : X \to H$ *is continuous.*
2. *there is a positive Borel measure* $d\mu$ *on* X *such that, for* $f \in H$*, we have*

$$\int_X |\langle \Phi_x, f\rangle_H|^2 d\mu(x) = \|f\|_H^2. \tag{4.3.1}$$

For $z \in \mathbb{C}^n$, let

$$F_z(x) = e^{-(\ell(z)+\ell(x))/2} E_k(\sqrt{2}z, x), \quad x \in \mathbb{R}^n. \tag{4.3.2}$$

It is easy to check that $F_z \in L^2_k(\mathbb{R}^n)$ with $\|F_z\|_{L^2_k} = \sqrt{E_k(z,\bar{z})}$ (see Lemma 3 in [98]). Since the chaotic transform $\mathcal{C}_k : L^2_k(\mathbb{R}^n) \to \mathcal{A}_k$ is unitary, i. e., $\|\mathcal{C}_k(f)\|_k^2 = \|f\|_{L^2_k}^2$, the bilinear extension can be obtained by polarization in the following:

$$(\mathcal{C}_k(f), \mathcal{C}_k(g))_k = \int_{\mathbb{R}^n} f(x)\overline{g(x)} dw_k(x), \tag{4.3.3}$$

for all $f, g \in L^2_k(\mathbb{R}^n)$.

Consider the linear operator γ_1 on $L^2_k(\mathbb{R}^n)$ defined by

$$\gamma_1(f)(x) := (\mathcal{C}_k(f)(z), F_z(x))_k = \left(\int_{\mathbb{R}^n} F_z(y) f(y) dw_k(y), F_z(x)\right)_k. \tag{4.3.4}$$

Now (4.3.3) shows that $\gamma_1 = \mathbf{1}$ weakly.

For $k \equiv 0$ and $n \geq 1$, the inner product on $\mathcal{A}_0$ is given by $(\phi, \psi)_0 = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \phi(z)\overline{\psi(z)} e^{-|z|^2} dz$, for $\phi, \psi \in \mathcal{A}_0$ and hence (4.3.1) follows from (4.3.4). Thus the family $\{F_z\}_{z\in\mathbb{C}^n}$ defines a set of coherent states in $L^2(\mathbb{R}^n)$ (see [95]). In particular, when $G = \mathbb{Z}_2^n$, the inner product on $\mathcal{A}_k$ is given by $(\phi, \psi)_k = \int_{\mathbb{C}^n} \phi(z)\overline{\psi(z)} d\mu_k(z)$, for $\phi, \psi \in \mathcal{A}_k$, where the measure $d\mu_k$ can be found explicitly in [17, 99], and in this case the family $\{F_z\}_{z\in\mathbb{C}^n}$ gives rise to a set of coherent states in $L_k^2(\mathbb{R}^n)$. We also refer to Ghazouani [55] for the study of one dimensional coherent states in Dunkl setting. However, the existence of the positive Borel measure $d\mu_k$ such that $(\phi,\psi)_k = \int_{\mathbb{C}^n} \phi(z)\overline{\psi(z)}d\mu_k(z)$ is not known, in general, for $n \geq 2$. (See the discussion below Lemma 3.9 in [17]). In view of (4.3.3) and the fact that $\gamma_1 = \mathbf{1}$ weakly, the family $\{F_z\}_{z\in\mathbb{C}^n}$ plays the role of coherent states in $L_k^2(\mathbb{R}^n)$.

Now we calculate the image of F_z under the Dunkl-Hermite Schrödinger semigroup. To do this, we first generalize (4.2.4) involving the function $e^{-\delta\|x\|^2}$ for $\delta \in \mathbb{C}$ and obtain the following lemma.

Lemma 4.3.2. *Let $k \geq 0$ and $\delta, z, w \in \mathbb{C}^n$ such that $Re(\delta) > 0$. Then*

$$\int_{\mathbb{R}^n} e^{-\delta\|x\|^2} E_k(x,z)E_k(x,w)w_k(x)dx = \frac{c_k}{(2\delta)^{\gamma+\frac{n}{2}}} e^{\frac{\ell(z)+\ell(w)}{4\delta}} E_k\left(\frac{z}{2\delta}, w\right). \tag{4.3.5}$$

Proof. If $\delta \in \mathbb{R}$ and $\delta > 0$, using the change of variable $x \mapsto \frac{1}{\sqrt{2\delta}}x$ in (4.2.4), we obtain (4.3.2).

For $\delta \in \mathbb{C}$, we consider

$$F(\delta) = \int_{\mathbb{R}^n} e^{-\delta\|x\|^2} E_k(x,z)E_k(x,w)w_k(x)dx \tag{4.3.6}$$

and

$$G(\delta) = \frac{c_k}{(2\delta)^{\gamma+\frac{n}{2}}} e^{\frac{\ell(z)+\ell(w)}{4\delta}} E_k\left(\frac{z}{2\delta}, w\right). \tag{4.3.7}$$

For $Re(\delta) > 0$, the integral in (4.3.6) converges and can be differentiated with respect to δ, thus F is analytic in $Re(\delta) > 0$. On the other hand G is also analytic in $Re(\delta) > 0$ as $e^{\ell(z)}, E_k(z,w)$ are analytic in $z-$variable (see (4.2.6)). Since F and G coincide on the positive real line, identity theorem gives $F(\delta) = G(\delta)$ for all δ with Re $(\delta) > 0$. ☐

We show that the Dunkl-Hermite Schrödinger semigroup maps the image of F_z to another coherent state through a time-dependent label change, as stated in the following proposition.

Proposition 4.3.3. *Let $F_z(x) = e^{-(\ell(z)+\ell(x))/2}E_k(\sqrt{2}z, x), x \in \mathbb{R}^n$. Then*

$$e^{-it\mathcal{H}_k}F_z(x) = \frac{c_k^2}{(ie^{it})^{\gamma+\frac{n}{2}}} F_{e^{it}z}(x). \tag{4.3.8}$$

Proof. By (4.2.11) and using Lemma 4.3.2, the image of F_z under the Dunkl-Hermite Schrödinger semigroup can be computed as

$$\begin{aligned} e^{-it\mathcal{H}_k}F_z(x) =& \frac{c_k e^{-\frac{i}{2}\cot t\|x\|^2}e^{-\ell(z)/2}}{(i\sin t)^{\gamma+\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-(\frac{1}{2}+\frac{i}{2}\cot t)\|y\|^2}E_k\left(\frac{ix}{\sin t}, y\right)E_k(\sqrt{2}z, y)w_k(y)dy \\ =& \frac{c_k e^{-\frac{i}{2}\cot t\|x\|^2}e^{-\ell(z)/2}}{(i\sin t)^{\gamma+\frac{n}{2}}}\frac{c_k}{(1+i\cot t)^{\gamma+\frac{n}{2}}}e^{\frac{\ell(z)}{1+i\cot t}}e^{-\frac{\ell(x)}{2\sin^2 t(1+i\cot t)}}E_k(\sqrt{2}e^{it}z, x). \end{aligned}$$

After a simple calculation, we find that

$$e^{-\frac{\ell(x)}{2\sin^2 t(1+i\cot t)}} = e^{-\frac{\|x\|^2}{2}}e^{\frac{i}{2}\cot t\|x\|^2} \quad \text{and} \quad e^{\frac{\ell(z)}{1+i\cot t}} = e^{\frac{\ell(z)}{2}}e^{-\frac{\ell(e^{it}z)}{2}}.$$

Thus

$$e^{-it\mathcal{H}_k}F_z(x) = \frac{c_k^2}{(ie^{it})^{\gamma+\frac{n}{2}}}e^{-\frac{\ell(e^{it}z)}{2}}e^{-\frac{\ell(x)}{2}}E_k(\sqrt{2}e^{it}z, x).$$

□

In particular, when $G = \mathbb{Z}_2^n$, the Proposition 4.3.3 can also be derived from Theorem 5.3 in [99].

The coherent state F_z can be expressed in terms of the orthonormal basis of $\mathcal{A}_k$ as shown below.

Lemma 4.3.4. *For $w, z \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$, we have*

$$F_{wz}(x) = \sum_{\nu\in\mathbb{N}_0^n} h_\nu(x)\varphi_\nu(z)w^{|\nu|}. \tag{4.3.9}$$

Proof. Using (4.2.2) and (4.2.6) we can write

$$F_{wz}(x) = e^{-x^2/2}e^{-\ell(wz)/2}E_k(2x, \frac{w}{\sqrt{2}}z) = e^{-\ell(x)/2}\sum_{\nu\in\mathbb{N}_0^n} H_\nu(x)\varphi_\nu(\frac{w}{\sqrt{2}}z).$$

Since φ_ν is homogeneous of degree $|\nu|$, i.e., $\varphi_\nu(\frac{w}{\sqrt{2}}z) = \frac{w^{|\nu|}}{2^{\frac{|\nu|}{2}}}\varphi_\nu(z)$, we have

$$F_{wz}(x) = \sum_{\nu\in\mathbb{N}_0^n} 2^{-|\nu|/2}e^{-\|x\|^2/2}H_\nu(x)\varphi_\nu(z)w^{|\nu|} = \sum_{\nu\in\mathbb{N}_0^n} h_\nu(x)\varphi_\nu(z)w^{|\nu|}.$$

□

Using the coherent states F_z, $z \in \mathbb{C}^n$, we define a parameterized family of self-adjoint operators on $L_k^2(\mathbb{R}^n)$ belonging to $\mathcal{G}^r$, for all $1 \le r \le \infty$ as follows: For $0 < \epsilon < 1$, consider

$$\gamma_\epsilon(f)(x) := (\mathcal{C}_k(f)(\epsilon z), F_{\epsilon z}(x))_k = \left(\int_{\mathbb{R}^n} F_{\epsilon z}(y)f(y)dw_k(y), F_{\epsilon z}(x)\right)_k. \tag{4.3.10}$$

Let $f \in L^2_k(\mathbb{R}^n)$. A simple calculation shows that

$$\|\gamma_\epsilon(f)\|^2_{L^2_k} = \sum_{\nu\in\mathbb{N}_0^n} \left| \int_{\mathbb{R}^n} f(x) h_\nu(x) dw_k(x) \right|^2 \epsilon^{2|\nu|} \leq \|f\|^2_{L^2_k},$$

thus

$$\|\gamma_\epsilon\|_{\mathcal{G}^\infty} = \|\gamma_\epsilon\|_{L^2_k \to L^2_k} \leq 1.$$

By Lemma 4.3.4, the density of the operator γ_ϵ can be written as

$$\rho_{\gamma_\epsilon}(x) = (F_{\epsilon z}(x), F_{\epsilon z}(x))_k = e^{-\|x\|^2} \sum_{\nu\in\mathbb{N}_0^n} \frac{H_\nu(x) H_\nu(x)}{2^{|\nu|}} \epsilon^{2|\nu|}. \tag{4.3.11}$$

Here, the density function $\rho_\gamma : \mathbb{R}^n \to \mathbb{R}$ of an operator γ is formally defined by $\rho_\gamma(x) = \gamma(x,x)$, where $\gamma(x,y)$ denotes the integral kernel of γ. Applying the Mehler-formula (4.2.7) in (4.3.11), we get

$$\rho_{\gamma_\epsilon}(x) = \frac{1}{(1-\epsilon^4)^{\gamma+n/2}} e^{-\frac{1+\epsilon^4}{1-\epsilon^4}\|x\|^2} E_k\left(\frac{2\epsilon^2 x}{1-\epsilon^4}, x \right).$$

The trace norm of γ_ϵ can be computed as

$$\|\gamma_\epsilon\|_{\mathcal{G}^1} = Tr(\gamma_\epsilon) = \frac{1}{(1-\epsilon^4)^{\gamma+n/2}} \int_{\mathbb{R}^n} e^{-\frac{1+\epsilon^4}{1-\epsilon^4}\|x\|^2} E_k\left(\frac{2\epsilon^2 x}{1-\epsilon^4}, x \right) dw_k(x). \tag{4.3.12}$$

Applying the change of variable $x \mapsto \sqrt{\frac{1+\epsilon^2}{1-\epsilon^2}}x$ and using (4.2.3), we get

$$\begin{aligned}
\|\gamma_\epsilon\|_{\mathcal{G}^1} =& \frac{1}{(1-\epsilon^2)^{2\gamma+n}} \int_{\mathbb{R}^n} e^{-\left(1+\frac{2\epsilon^2}{(1-\epsilon^2)^2}\right)\|x\|^2} E_k\left(\frac{2\epsilon^2 x}{(1-\epsilon^2)^2}, x \right) dw_k(x) \\
\leq& \frac{C_2}{(1-\epsilon^2)^{2\gamma+n}} \left(\frac{1-\epsilon^2}{2\epsilon} \right)^{2\gamma} \int_{\mathbb{R}^n} \frac{Ce^{-\|x\|^2}}{\prod_{\alpha\in\mathcal{R}^+} (|\langle\alpha, x\rangle| + \frac{1-\epsilon^2}{2\epsilon})^{2k(\alpha)}} dw_k(x) \\
\leq& \frac{C_2}{(2\epsilon)^{2\gamma}(1-\epsilon^2)^n} \int_{\mathbb{R}^n} e^{-\|x\|^2} dx.
\end{aligned}$$

By Hölder's inequality in Schatten spaces, we deduce that $\gamma_\epsilon \in \mathcal{G}^r(L^2_k(\mathbb{R}^n))$ for all $1 \leq r \leq \infty$ and

$$\|\gamma_\epsilon\|_{\mathcal{G}^r} \leq \frac{C}{\epsilon^{\frac{2\gamma}{r}}(1-\epsilon^2)^{\frac{n}{r}}}. \tag{4.3.13}$$

Remark 4.3.5. *For $k \equiv 0$, the operator γ_ϵ coincides with the operator defined in the proof of Proposition 1 in [48] with $L^2 = \mu = \frac{\epsilon^2}{1-\epsilon^2}, \beta = \frac{1}{2}$ and $z = \frac{x+i\xi}{\sqrt{2}}$.*

4.4 Necessary condition on the Schatten exponent

Now we are in a position to obtain the necessary condition on the Schatten exponent for the estimates (4.1.1) and (4.1.2).

Theorem 4.4.1. *(Necessary condition on Schatten exponent) Suppose $k \geq 0$ and $p, q, n \geqslant 1$ satisfies $2\gamma < \frac{n(p+1)}{p-1}$. Then the estimate (4.1.2) fails for all $\beta > \frac{2pn}{(p+1)n-(p-1)2\gamma}$.*

Proof. The estimate (4.1.2) can also be written in terms of the operator

$$\gamma_0 := \sum_j \lambda_j \left|u_j\right\rangle \left\langle u_j\right| \tag{4.4.1}$$

on $L_k^2(\mathbb{R}^n)$, where the Dirac's notation $|u\rangle\langle v|$ stands for the rank-one operator $f \mapsto \langle v, f\rangle u$. For such γ_0, let

$$\gamma(t) := e^{-it\mathcal{H}_k}\gamma_0 e^{it\mathcal{H}_k} = \sum_j \lambda_j \left|e^{-it\mathcal{H}_k}u_j\right\rangle \left\langle e^{-it\mathcal{H}_k}u_j\right|.$$

Then the density of the operator $\gamma(t)$ is given by

$$\rho_{\gamma(t)} := \sum_j \lambda_j \left|e^{-it\mathcal{H}_k}u_j\right|^2. \tag{4.4.2}$$

With these notations (4.1.2) can be rewritten as

$$\left\|\rho_{\gamma(t)}\right\|_{L^q((-\frac{\pi}{2},\frac{\pi}{2}),L_k^p(\mathbb{R}^n))} \leq C_{n,q}\|\gamma_0\|_{\mathcal{G}^{\frac{2p}{p+1}}}. \tag{4.4.3}$$

In view of (4.4.3), the proof of the Theorem 4.4.1 follows from the following proposition.

□

Proposition 4.4.2. *Suppose $k \geq 0$ and $p, q, n \geqslant 1$ satisfies $2\gamma < \frac{n(p+1)}{p-1}$. Then we have*

$$\sup_{\gamma_0 \in \mathcal{G}^r} \frac{\left\|\rho_{e^{-it\mathcal{H}_k}\gamma_0 e^{it\mathcal{H}_k}}\right\|_{L^q((-\frac{\pi}{2},\frac{\pi}{2}),L_k^p(\mathbb{R}^n))}}{\|\gamma_0\|_{\mathcal{G}^r}} = +\infty, \tag{4.4.4}$$

for all $r > \frac{2pn}{(1+p)n-(p-1)2\gamma}$.

Proof. For $0 < \epsilon < 1$, consider the operators γ_ϵ defined in (4.3.10). After a simple computation and applying Proposition (4.3.3), we obtain

$$\rho_{\gamma_\epsilon(t)}(x) := \rho_{e^{-it\mathcal{H}_k}\gamma_\epsilon e^{it\mathcal{H}_k}}(x) = \left(e^{it\mathcal{H}_k}F_{\epsilon z}(x), e^{it\mathcal{H}_k}F_{\epsilon z}(x)\right)_k$$

$$= \left(F_{\epsilon e^{-it}z}(x), F_{\epsilon e^{-it}z}(x)\right)_k .$$

Applying Lemma 4.3.4 and (4.2.7) we get $F_{\epsilon e^{-it}z}(x) = \sum_{\nu \in \mathbb{N}_0^n} h_\nu(x)\varphi_\nu(z)e^{-i|\nu|t}\epsilon^{|\nu|}$, and

$$\rho_{\gamma_\epsilon(t)}(x) = e^{-\|x\|^2} \sum_{\nu \in \mathbb{N}_0^n} \frac{H_\nu(x)H_\nu(x)}{2^{|\nu|}}\epsilon^{2|\nu|} = \frac{1}{(1-\epsilon^4)^{\gamma+n/2}} e^{-\frac{1+\epsilon^4}{1-\epsilon^4}\|x\|^2} E_k\left(\frac{2\epsilon^2 x}{1-\epsilon^4}, x\right).$$

Therefore,

$$\begin{aligned}\|\rho_{\gamma_\epsilon(t)}\|^p_{L^p_k(\mathbb{R}^n)} &= \frac{c_k^{4p}}{(1-\epsilon^4)^{(\gamma+n/2)p}} \int_{\mathbb{R}^n} e^{-p\left(\frac{1+\epsilon^4}{1-\epsilon^4}\right)\|x\|^2} E_k\left(\frac{2\epsilon^2 x}{1-\epsilon^4}, x\right)^p dw_k(x) \\ &= \frac{c_k^{4p}(1+\epsilon^2)^{(\gamma+\frac{n}{2})(1-p)}}{(1-\epsilon^2)^{(\gamma+\frac{n}{2})(p+1)}} \int_{\mathbb{R}^n} e^{-p\left(1+\frac{2\epsilon^2}{(1-\epsilon^2)^2}\right)\|x\|^2} E_k\left(\frac{2\epsilon^2 x}{(1-\epsilon^2)^2}, x\right)^p dw_k(x),\end{aligned}$$

where the last equality is obtained by changing the variable $x \mapsto \sqrt{\frac{1+\epsilon^2}{1-\epsilon^2}}x$. Using (4.2.3), we get

$$\begin{aligned}\|\rho_{\gamma_\epsilon(t)}\|_{L^q((-\frac{\pi}{2},\frac{\pi}{2}),L^p_k(\mathbb{R}^n))} \geq & \frac{c_k^4(1+\epsilon^2)^{\frac{(2\gamma+n)(1-p)}{2p}}}{(2\epsilon)^{2\gamma}(1-\epsilon^2)^{\frac{(1+p)n+(1-p)2\gamma}{2p}}} \left(\int_{\mathbb{R}^n} \frac{C_1 e^{-p\|x\|^2}}{\prod\limits_{\alpha\in\mathcal{R}^+}(|\langle\alpha,x\rangle| + \frac{1-\epsilon^2}{2\epsilon})^{2pk(\alpha)}} dw_k(x)\right)^{\frac{1}{p}} \\ \geq & \frac{A_{n,p}}{(1-\epsilon^2)^{\frac{(1+p)n+(1-p)2\gamma}{2p}}} \left(\int_{\mathbb{R}^n} \frac{e^{-p\|x\|^2}}{\prod\limits_{\alpha\in\mathcal{R}^+}(|\langle\alpha,x\rangle| + \frac{1-\epsilon^2}{2\epsilon} + 1)^{2pk(\alpha)}} dw_k(x)\right)^{\frac{1}{p}}.\end{aligned}$$

From (4.3.13) we write

$$\begin{aligned}&\frac{\left\|\rho_{e^{-it\mathcal{H}_k}\gamma_\epsilon e^{it\mathcal{H}_k}}\right\|_{L^q((-\frac{\pi}{2},\frac{\pi}{2}),L^p_k(\mathbb{R}^n))}}{\|\gamma_\epsilon\|_{\mathcal{G}^r}} \\ &\geq A_{n,p}\,\epsilon^{\frac{2\gamma}{r}} \left(\frac{1}{1-\epsilon^2}\right)^{n\left(\frac{(1+p)n+(1-p)2\gamma}{2pn} - \frac{1}{r}\right)} \left(\int_{\mathbb{R}^n} \frac{e^{-p\|x\|^2}}{\prod\limits_{\alpha\in\mathcal{R}^+}(|\langle\alpha,x\rangle| + \frac{1-\epsilon^2}{2\epsilon} + 1)^{2pk(\alpha)}} dw_k(x)\right)^{\frac{1}{p}}.\end{aligned}$$

Since

$$0 < \int_{\mathbb{R}^n} \frac{e^{-p\|x\|^2}}{\prod\limits_{\alpha\in\mathcal{R}^+}(|\langle\alpha,x\rangle| + 1)^{2pk(\alpha)}} dw_k(x) \leq \int_{\mathbb{R}^n} C^{-1}e^{-p\|x\|^2} dw_k(x),$$

letting $\epsilon \to 1^-$ we get (4.4.4) for $r > \frac{2pn}{(1+p)n+(1-p)2\gamma}$. □

Remark 4.4.3. *Note that if* $p, q, n \geq 1$ *satisfy* $\frac{2}{q} + \frac{2\gamma+n}{p} = 2\gamma + n$, *then* $\frac{(p+1)n-(p-1)2\gamma}{2pn} = 1 - \frac{1}{nq} > 0$, *thus for such pair* p, q *in Theorem 4.1.2 the estimate (4.1.2) fails for all* $r > \frac{2pn}{(p+1)n-(p-1)2\gamma}$.

Using the kernel relation between the semigroups $e^{-it\mathcal{H}_k}$ and $e^{it\Delta_k}$, we obtain:

Theorem 4.4.4. *(Necessary condition on Schatten exponent) Suppose* $k \geq 0$ *and* $p, q, n \geqslant 1$ *satisfies* $\frac{2}{q} + \frac{2\gamma+n}{p} = 2\gamma + n$. *Then the estimate (4.1.1) fails for all* $\beta > \frac{2pn}{(p+1)n-(p-1)2\gamma}$.

Proof. In view of Theorem 4.4.1 the proof follows directly from Lemma 4.2.3 and the elementary fact that the orthonormality of $(f_j)_j$ is preserved under complex conjugation. ☐

Further, using the inclusion relation of $L^q(-\frac{\pi}{2}, \frac{\pi}{2})$- spaces, Theorem 4.1.2 can be extended to a wider range of p, q (also generalizes Theorem A of Ben Saïd-Nandakumaran-Ratnakumar [16] to orthonormal systems) in the following.

Theorem 4.4.5. *Suppose* $k \geq 0$ *and* $p, q, n \geqslant 1$ *such that*

$$1 \leq p < \frac{2\gamma+n+1}{2\gamma+n-1} \quad \text{and} \quad \frac{2}{q} + \frac{2\gamma+n}{p} \geq 2\gamma + n.$$

Then the estimate (4.1.2) holds for any orthonormal system $(f_j)_j$ *in* $L^2_k(\mathbb{R}^n)$ *and all sequence* $(n_j)_j \in \ell^r(\mathbb{C})$, *if* $1 \leq \beta \leq \frac{2p}{p+1}$; *and fails for all* $\beta > \frac{2pn}{(p+1)n-(p-1)2\gamma}$.

Proof. Let $p, q \geq 1$ such that $\frac{2}{q} + \frac{2\gamma+n}{p} \geq 2\gamma + n$. If $1 < p < \frac{2\gamma+n+1}{2\gamma+n-1}$, then there exists a $0 < \alpha \leq 1$ such that $\frac{2\alpha}{q} + \frac{2\gamma+n}{p} = 2\gamma + n$. Since $\frac{q}{\alpha} \geq q \geq 1$, by the inclusion relation of $L^q(-\frac{\pi}{2}, \frac{\pi}{2})$-spaces, we have

$$\left\| \sum_j n_j \left| e^{-it\mathcal{H}_k} f_j \right|^2 \right\|_{L^q((-\frac{\pi}{2},\frac{\pi}{2}),L^p(\mathbb{R}^n))} \leq \left\| \sum_j n_j \left| e^{-it\mathcal{H}_k} f_j \right|^2 \right\|_{L^{\frac{q}{\alpha}}((-\frac{\pi}{2},\frac{\pi}{2}),L^p(\mathbb{R}^n))}.$$

Clearly the pair $p, \frac{q}{\alpha}$ satisfies the conditions of Theorem 4.1.2, the estimate (4.1.2) follows from Theorem 4.1.2. Since $\frac{(1+p)n+(1-p)2\gamma}{2pn} = 1 - \frac{\alpha}{nq} > 0$, the necessary part follows from Theorem 4.4.1.

For $p = 1$, the estimate (4.1.2) follows from triangle inequality and Theorem A of [16], and the necessary part follows from Theorem 4.4.1. ☐

Conclusion

We mainly obtain Theorems 4.4.1 and 4.4.4, by extending the semiclassical argument based on coherent states of Frank-Lewin-Lieb-Seiringer [48] (see also [83]) in the Dunkl setting. As discussed in Section 4.3, the existence of the measure $d\mu_k$ on $\mathbb{C}^n$ for general $k > 0$ is not known. Even if it does exist in some special cases, defining operators using functional calculus, as shown in the proof of Proposition 1 in [48], is difficult. To overcome this difficulty we define the class of operators γ_ϵ, $0 < \epsilon < 1$, which plays the same role in proving Proposition 4.4.2. However, this method does not answer the validity of Theorems 4.1.1 and 4.1.2 when the Schatten exponent $r \in \left(\frac{2p}{p+1}, \frac{2pn}{(p+1)n-(p-1)2\gamma}\right]$.

As an immediate consequence of Theorems 4.4.1 and 4.4.4, we cannot expect the global well-posedness for the Hartree equation, as presented in [49], in the context of the Dunkl Laplacian and the Dunkl-Hermite operator, if the initial data $\gamma_0 \in \mathcal{G}^r$, for $r > \frac{2pn}{(p+1)n-(p-1)2\gamma}$.

CHAPTER 5

Heisenberg-Pauli-Weyl uncertainty principles for the fractional Dunkl transform on the real line

5.1 Introduction

One-dimensional Dunkl operators have received considerable attention, and several recent works have developed harmonic analysis associated with the one-dimensional Dunkl operator.

In this chapter, we improve the Heisenberg-Pauli-Weyl uncertainty principle (HPWUP) for the Dunkl transform [42, 92] and the fractional Dunkl transform [56] in one dimension, obtaining sharper bounds than those previously known.

Throughout this chapter, we adopt the following notations. Let $\mu \geq -1/2$. For $1 \leq p < \infty$, the space $L^p_\mu(\mathbb{R})$ consists of all complex functions on $\mathbb{R}$ such that

$$\|f\|_{\mu,p} = \left(\int_{-\infty}^{+\infty} |f(x)|^p \, |x|^{2\mu+1} \, dx \right)^{\frac{1}{p}} < \infty$$

and $L^\infty_\mu(\mathbb{R})$ is defined in the usual way. For $f \in L^2_\mu(\mathbb{R}) \cap L^p_\mu(\mathbb{R})$, $1 \leq p \leq 2$, we denote

$$\langle x\rangle_f = \int_{-\infty}^{+\infty} x|f(x)|^2\,|x|^{2\mu+1}\,dx, \quad \text{and}$$

$$\Delta_{\mu,p}(f) = \left(\int_{-\infty}^{+\infty} |(x-\langle x\rangle_f)f(x)|^p\,|x|^{2\mu+1}\,dx\right)^{\frac{1}{p}}. \tag{5.1.1}$$

For a complex function f defined on $\mathbb{R}$, we write its even and odd parts by $f_e(x) = \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$ respectively.

5.2 Preliminaries

In this section, we introduce the Dunkl transform and the fractional Dunkl transform on the real line and outline some important results related to these transforms.

5.2.1 Dunkl transform and the Dunkl operator on the real line

The Dunkl transform on the real line is a generalization of the classical Fourier transform. It was introduced by C. F. Dunkl [39], where many of its fundamental properties were first established. For $f \in L^1_\mu(\mathbb{R})$, the Dunkl transform is given by

$$D_\mu f(w) = \frac{1}{2^{\mu+1}\Gamma(\mu+1)} \int_{-\infty}^{+\infty} f(x)E_\mu(-iwx)\,|x|^{2\mu+1}\,dx, \tag{5.2.1}$$

where E_μ denotes the one dimensional Dunkl kernel, given by

$$E_\mu(z) = j_\mu(iz) + \frac{z}{2(\mu+1)}j_{\mu+1}(iz), \tag{5.2.2}$$

with the normalized spherical Bessel function $j_\mu(z) = \Gamma(\mu+1)\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{z}{2})^{2n}}{n!\Gamma(n+\mu+1)}$, $z \in \mathbb{C}$. Notice that when $\mu = -1/2$, we just have $E_{-1/2}(-iwx) = e^{-iwx}$, hence $D_{-1/2}$ coincides with the Fourier transform on $\mathbb{R}$. It is well known that the functions for $\mu \geq -1/2$, $E_\mu(w.)$ is the unique solution of the initial value problem

$$T_\mu f = wf, \;\; f(0) = 1, \tag{5.2.3}$$

where

$$T_\mu f(x) = f'(x) + \left(\mu + \frac{1}{2}\right)\frac{f(x)-f(-x)}{x}, \;\; f \in \mathcal{C}^1(\mathbb{R}), \;\; \mu \geq -1/2 \tag{5.2.4}$$

is the Dunkl operator with parameter μ associated with the reflection group $\mathbb{Z}_2$.

For any function $f \in L^1_\mu(\mathbb{R})$, $D_\mu(f)$ belongs to $C_0(\mathbb{R})$ and satisfies

$$\|D_\mu(f)\|_{\mu,\infty} \leq \frac{1}{2^{\mu+1}\Gamma(\mu+1)}\|f\|_{\mu,1}. \tag{5.2.5}$$

If $f \in L^2_\mu(\mathbb{R})$, then $D_\mu(f)$ belongs to $L^2_\mu(\mathbb{R})$ and satisfies the following Plancherel formula

$$\|D_\mu(f)\|_{\mu,2} = \|f\|_{\mu,2}. \tag{5.2.6}$$

Interpolating these two inequalities gives the Hausdorff-Young inequality: if $f \in L^p_\mu(\mathbb{R})$, where $1 \leq p \leq 2$ and $\frac{1}{p}+\frac{1}{q}=1$, then $D_\mu(f) \in L^q_\mu(\mathbb{R})$, and

$$\|D_\mu(f)\|_{\mu,q} \leq \frac{1}{(2^{\mu+1}\Gamma(\mu+1))^{\frac{2}{p}-1}}\|f\|_{\mu,p}, \tag{5.2.7}$$

with equality if and only if $p=q=2$.

Suppose $f, g \in L^2_\mu(\mathbb{R})$ such that $T_\mu f, T_\mu g \in L^2_\mu(\mathbb{R})$, then we have

$$\int_{-\infty}^{+\infty} T_\mu f(x)g(x)|x|^{2\mu+1}dx = -\int_{-\infty}^{+\infty} f(x)T_\mu g(x)|x|^{2\mu+1}dx \tag{5.2.8}$$

and

$$D_\mu(T_\mu f)(w) = iwD_\mu(f)(w). \tag{5.2.9}$$

For $f, g \in C^1(\mathbb{R}\setminus\{0\})$ with g being even, the following product formula holds

$$T_\mu(fg) = (T_\mu f)\cdot g + f\cdot(T_\mu g). \tag{5.2.10}$$

We refer to [39, 68, 90, 92] for a detailed study on Dunkl transform and its properties.

5.2.2 Fractional Dunkl transform

The fractional Dunkl transform is a natural generalization of the Dunkl transform introduced by Ghazouani-Bouzeffour [56].

Let $\alpha \in \mathbb{R}$. For $f \in L^1_\mu(\mathbb{R})$, we consider the fractional Dunkl transform $D^\alpha_\mu f$ defined as

$$D^\alpha_\mu f(\omega) = \begin{cases} N_{\mu,n}\int\limits_{-\infty}^{+\infty} e^{\frac{i}{2}(x^2+\omega^2)\cot\alpha}E_\mu\left(\frac{-i\omega x}{\sin\alpha}\right)f(x)|x|^{2\mu+1}dx, & (2n-1)\pi < x < (2n+1)\pi, \\ f(x), & x = 2n\pi, \\ f(-x), & x = (2n+1)\pi, \end{cases} \tag{5.2.11}$$

where

$$N_{\mu,n} = \frac{e^{i(\mu+1)(\frac{\hat{\alpha}\pi}{2}-(\alpha-2n\pi))}}{\Gamma(\mu+1)(2|\sin\alpha|)^{\mu+1}}, \quad \text{with} \quad \hat{\alpha} = sgn(\sin(\alpha)).$$

The operator $D_\mu^\alpha f$ initially defined on $L^1_\mu(\mathbb{R})$, admits a unique extension to operator on $L^2_\mu(\mathbb{R})$. If we denote this extension by D_μ^α, the family $\{D_\mu^\alpha\}_{\alpha\in\mathbb{R}}$ forms a group structure. Specifically, for any $f \in L^2_\mu(\mathbb{R})$, we have the following property:

$$D_\mu^\alpha \circ D_\mu^\beta(f) = D_\mu^{\alpha+\beta}(f) \tag{5.2.12}$$

with D_μ^0 is the identity, and the inverse is given by $(D_\mu^\alpha)^{-1} = D_\mu^{-\alpha}$. The fractional Dunkl transform D_μ^α also has the following exponential form

$$D_\mu^\alpha = e^{-i\alpha(\mu+1)}e^{-i\frac{\alpha}{2}(T_\mu^2-x^2)}.$$

Let $\alpha \notin \pi\mathbb{Z}$. The following relation holds between the fractional Dunkl transform and the Dunkl transform

$$D_\mu^\alpha(f)(w) = \frac{e^{i(\mu+1)(\frac{\hat{\alpha}\pi}{2}-(\alpha-2n\pi))}}{|\sin\alpha|^{\mu+1}} e^{\frac{iw^2\cot\alpha}{2}} D_\mu(g)\left(\frac{w}{\sin\alpha}\right), \tag{5.2.13}$$

where $g(x) = e^{\frac{ix^2\cot\alpha}{2}} f(x)$ and $\hat{\alpha} = sgn(\sin(\alpha))$.

5.2.3 Covariance and absolute covariance

Suppose that the complex function f is of the form $f(x) = \rho(x)e^{i\varphi(x)}$, where the classical derivatives $\rho'(x)$ and $\varphi'(x)$ exist for all $x \in \mathbb{R}$. The covariance and the absolute covariance of f are defined in [42] as follows

$$Cov_\mu(f) = \int_{-\infty}^{+\infty} x\phi'(x)|x|^{2\mu+1}dx - \langle x\rangle_f \langle x\rangle_{D_\mu(f)}, \tag{5.2.14}$$

and

$$\begin{aligned} COV_\mu(f) = \int_{-\infty}^{+\infty} \Bigg| (x-\langle x\rangle_f)\Bigg\{T_\mu(\varphi)(x) + \frac{\mu+\frac{1}{2}}{x}\Bigg(\frac{\sin[\varphi(x)-\varphi(-x)]\rho(-x)}{\rho(x)} - (\varphi(x) \\ - \varphi(-x))\Bigg) - \langle x\rangle_{D_\mu(f)}\Bigg\}\Bigg|\rho^2(x)|x|^{2\mu+1}dx. \end{aligned} \tag{5.2.15}$$

The covariance of f can also be expressed as in the following way

$$Cov_\mu(f) = \int_{-\infty}^{+\infty} (x-\langle x\rangle_f)\Bigg\{T_\mu(\varphi)(x) + \frac{\mu+\frac{1}{2}}{x}\Bigg(\frac{\sin[\varphi(x)-\varphi(-x)]\rho(-x)}{\rho(x)} - (\varphi(x)$$

$$-\varphi(-x))\Big)-\langle x\rangle_{D_\mu(f)}\Big\}\rho^2(x)|x|^{2\mu+1}dx. \tag{5.2.16}$$

From the fact $\int_{-\infty}^{+\infty}|f(x)||x|^{2\mu+1}dx \geq \int_{-\infty}^{+\infty} f(x)|x|^{2\mu+1}dx$, it follows that $COV_\mu(f) \geq Cov_\mu(f)$. Assuming that $\varphi'(x)$ is continuous, it is clear from equations (5.2.16) and (5.2.15) that $COV_\mu(f) = Cov_\mu(f)$ if and only if $(x-\langle x\rangle_f)$ and the expression

$$T_\mu(\varphi)(x)+\frac{\mu+\frac{1}{2}}{x}\left(\frac{\sin[\varphi(x)-\varphi(-x)]\rho(-x)}{\rho(x)}-(\varphi(x)-\varphi(-x))\right)-\langle x\rangle_{D_\mu(f)}$$

have the same or opposite signs.

Suppose $xf(x)\in L^2_\mu(\mathbb{R})$ and $xD_\mu(f)(x)\in L^2_\mu(\mathbb{R})$ with $\|f\|_{\mu,2}=1$. Then we have the following relation (see [92]):

$$Re\int_{-\infty}^{+\infty} xf(x)\overline{T_\mu f(x)}|x|^{2\mu+1}dx = -\left(\|f_e\|^2_{\mu,2}-\|f_0\|^2_{\mu,2}\right)-\frac{1}{2}, \tag{5.2.17}$$

where left hand side denotes the real part of the integral. Also we can write

$$\begin{aligned}\int_{-\infty}^{+\infty} xf(x)\overline{T_\mu f(x)}|x|^{2\mu+1}dx &= \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\overline{(xT_\mu-T_\mu x)f(x)}|x|^{2\mu+1}dx\\ &\quad+\frac{1}{2}\int_{-\infty}^{+\infty} f(x)\overline{(xT_\mu+T_\mu x)f(x)}|x|^{2\mu+1}dx.\end{aligned} \tag{5.2.18}$$

A straightforward calculation (see the proof of Theorem 3.2 in [42]) yields,

$$(xT_\mu-T_\mu x)f(x) = -f(x)-(2\mu+1)f(-x),$$

and

$$(xT_\mu+T_\mu x)f(x) = 2xf'(x)+2(\mu+1)f(x).$$

Now, assume that $f(x)=\rho(x)e^{i\varphi(x)}$, where the derivatives $\rho'(x)$ and $\varphi'(x)$ exists for all $x\in\mathbb{R}$, then, from (5.2.18), we obtain

$$\int_{-\infty}^{+\infty} xf(x)\overline{T_\mu f(x)}|x|^{2\mu+1}dx = -\mathcal{A}-i\left(Cov_\mu(f)+\langle x\rangle_f\langle x\rangle_{D_\mu(f)}\right), \tag{5.2.19}$$

where

$$\mathcal{A}=\left(\mu+\frac{1}{2}\right)\int_{-\infty}^{+\infty}\rho(x)\rho(-x)\cos[\varphi(x)-\varphi(-x)]|x|^{2\mu+1}dx+\frac{1}{2}. \tag{5.2.20}$$

By comparing (5.2.17) and (5.2.19), we conclude that

$$\mathcal{A}=\left(\|f_e\|^2_{\mu,2}-\|f_0\|^2_{\mu,2}\right)+\frac{1}{2}. \tag{5.2.21}$$

5.3 HPWUP for the Dunkl transform

The HPWUP for the Dunkl transform, established by Rösler-Voit [92], has attracted significant attention in the literature and is stated as follows:

Theorem 5.3.1. *[92]. Let $xf(x) \in L^2_\mu(\mathbb{R})$ and $xD_\mu(f)(x) \in L^2_\mu(\mathbb{R})$ with $\|f\|_{\mu,2} = 1$. Then*

$$\Delta^2_{\mu,2}(f)\Delta^2_{\mu,2}(D_\mu(f)) \geq \left\{ \left(\mu + \frac{1}{2}\right) \left(\|f_e\|^2_{\mu,2} - \|f_0\|^2_{\mu,2}\right) + \frac{1}{2} \right\}^2, \tag{5.3.1}$$

equality holds in (5.3.1) if and only if f has the form $f(x) = de^{-\frac{1}{2\zeta}x^2}E_\mu(bx)$, with $b \in \mathbb{C}$ and $\zeta > 0$.

In [42], Fei-Wang-Yang improved the inequality (5.3.1) by incorporating covariance and absolute covariance, leading to a strengthened formulation. The best lower bound they obtained is given in the following theorem.

Theorem 5.3.2. *[42]. Assume that $f(x) = \rho(x)e^{i\varphi(x)}$ with $\|f\|_{\mu,2} = 1$ and the classical derivatives $\rho'(x)$ and $\varphi'(x)$ exist for all $x \in \mathbb{R}$, with $xf(x) \in L^2_\mu(\mathbb{R})$ and $xD_\mu(f)(x) \in L^2_\mu(\mathbb{R})$. Then*

$$\Delta^2_\mu(f)\Delta^2_\mu(D_\mu(f)) \geq \mathcal{A}^2 + COV^2_\mu(f), \tag{5.3.2}$$

where $\mathcal{A}$ is defined in (5.2.20). Moreover, if $\varphi'(x)$ is continuous and ρ is nonzero almost everywhere, then the equality in (5.3.2) holds if and only if there exist $\zeta \in \mathbb{R} \setminus \{0\}$ and $\xi > 0$ such that

$$(x - \langle x \rangle_f)\rho(x) = -\zeta \left(T_\mu \rho(x) + \frac{\mu + \frac{1}{2}}{x} \rho(-x) \left(1 - \cos[\varphi(x) - \varphi(-x)]\right) \right) \tag{5.3.3}$$

and

$$|x - \langle x \rangle_f| = \xi \left| T_\mu \varphi(x) + \frac{\mu + \frac{1}{2}}{x} \left(\frac{\sin[\varphi(x) - \varphi(-x)]}{\rho(x)} \rho(-x) - (\varphi(x) - \varphi(-x)) \right) - \langle x \rangle_{D_\mu(f)} \right|. \tag{5.3.4}$$

Remark 5.3.3. *(1) If f is of the form $f(x) = \rho(x)e^{i\varphi(x)}$, then $\mathcal{A}^2$ coincides with the right-hand side of (5.3.1). (See Section 5.2).*

(2) For the equality in (5.3.2), the authors overlooked the fact that $(x - \langle x \rangle_f)\rho(x)$ and

$T_\mu\rho(x) + \frac{\mu+\frac{1}{2}}{x}\rho(-x)\left(1-\cos[\varphi(x)-\varphi(-x)]\right)$ *must have the same or opposite sign. This follows from the second-to-last line on page 891 of [42]. Combining this with (3.18) of [42] yields (5.3.3).*

We solve the differential equations (5.3.3) and (5.3.4) and, as a result, provide an explicit form of the functions that attain equality in (5.3.2), thereby completing the results in Theorem 3.3 of [42].

Theorem 5.3.4. *Assume that* $f(x) = \rho(x)e^{i\varphi(x)}$ *with* $\|f\|_{\mu,2} = 1$ *and the classical derivatives* $\rho'(x)$ *and* $\varphi'(x)$ *exist for all* $x \in \mathbb{R}$, *with* $\varphi'(x)$ *continuous and* ρ *nonzero almost everywhere. Then (5.3.3) and (5.3.4) holds if and only if* f *has one of the following forms*

$$f(x) = d_1 e^{-\frac{1}{2\zeta}x^2} e^{\frac{i}{2\xi}x^2} E_\mu(bx), \tag{5.3.5}$$

$$f(x) = d_2 e^{-\frac{1}{2\zeta}x^2} e^{-\frac{i}{2\xi}x^2} E_\mu(b'x), \tag{5.3.6}$$

$$f(x) = \begin{cases} d_3 e^{-\frac{1}{2\zeta}x^2} e^{\frac{i}{2\xi}x^2} E_\mu(bx), & x \geq \langle x\rangle_f, \\ d_4 e^{-\frac{1}{2\zeta}x^2} e^{-\frac{i}{2\xi}x^2} E_\mu(b'x), & x < \langle x\rangle_f, \end{cases} \tag{5.3.7}$$

or

$$f(x) = \begin{cases} d_5 e^{-\frac{1}{2\zeta}x^2} e^{-\frac{i}{2\xi}x^2} E_\mu(b'x), & x \geq \langle x\rangle_f, \\ d_6 e^{-\frac{1}{2\zeta}x^2} e^{\frac{i}{2\xi}x^2} E_\mu(bx), & x < \langle x\rangle_f, \end{cases} \tag{5.3.8}$$

where $b = \left(\frac{1}{\zeta} - \frac{i}{\xi}\right)\langle x\rangle_f + i\langle x\rangle_{D_\mu(f)}$ *and* $b' = \left(\frac{1}{\zeta} + \frac{i}{\xi}\right)\langle x\rangle_f + i\langle x\rangle_{D_\mu(f)}$ *for some* $\zeta, \xi > 0$ *and* $d_1, d_2, d_3, d_4, d_5, d_6 \in \mathbb{C}$ *chosen such that* $\|f\|_{\mu,2} = 1$.

Proof. Let $A(x) = T_\mu\varphi(x) + \frac{\mu+\frac{1}{2}}{x}\left(\frac{\sin[\varphi(x)-\varphi(-x)]}{\rho(x)}\rho(-x) - (\varphi(x)-\varphi(-x))\right)\ \forall x \in \mathbb{R}$. As the left-hand side of equation (5.3.4) corresponds to the absolute value of a linear function, there are four possible scenarios:

$$A(x) = \frac{1}{\xi}(x - \langle x\rangle_f) + \langle x\rangle_{D_\mu(f)}, \tag{5.3.9}$$

$$A(x) = -\frac{1}{\xi}(x - \langle x\rangle_f) + \langle x\rangle_{D_\mu(f)}, \tag{5.3.10}$$

$$A(x)=\begin{cases}\frac{1}{\xi}(x-\langle x\rangle_f)+\langle x\rangle_{D_\mu(f)}, & x\geq\langle x\rangle_f,\\ -\frac{1}{\xi}(x-\langle x\rangle_f)+\langle x\rangle_{D_\mu(f)}, & x<\langle x\rangle_f,\end{cases}\tag{5.3.11}$$

or

$$A(x)=\begin{cases}-\frac{1}{\xi}(x-\langle x\rangle_\mu)+\langle x\rangle_{D_\mu(f)}, & x\geq\langle x\rangle_f,\\ \frac{1}{\xi}(x-\langle x\rangle_\mu)+\langle x\rangle_{D_\mu(f)}, & x<\langle x\rangle_f.\end{cases}\tag{5.3.12}$$

First we assume that

$$A(x)=\frac{1}{\xi}(x-\langle x\rangle_\mu)+\langle x\rangle_{D_\mu(f)}.\tag{5.3.13}$$

It follows that $f\in\mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$ solves (5.3.3) and (5.3.13), if $r(x)e^{i\phi(x)}=F(x)=e^{\frac{1}{2\zeta}x^2}e^{\frac{i}{2\xi}x^2}f(x)$ satisfies

$$T_\mu r(x)+\frac{\mu+\frac{1}{2}}{x}r(-x)\left(1-\cos[\phi(x)-\phi(-x)]\right)=\frac{1}{\zeta}\langle x\rangle_f\, r(x)\tag{5.3.14}$$

and

$$T_\mu\phi(x)+\frac{\mu+\frac{1}{2}}{x}\left(\frac{r(-x)\sin[\phi(x)-\phi(-x)]}{r(x)}-(\phi(x)-\phi(-x))\right)=-\frac{\langle x\rangle_f}{\xi}+\langle x\rangle_{D_\mu(f)},\tag{5.3.15}$$

which is of the form

$$T_\mu F=bF,\quad\text{where}\quad b=\left(\frac{1}{\zeta}-\frac{i}{\xi}\right)\langle x\rangle_f+i\langle x\rangle_{D_\mu(f)}.\tag{5.3.16}$$

In [92], the authors derived the explicit solution to the equation (5.3.16) for $F\in\mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$ which is given by

$$F(x)=d_1E_\mu(bx),\ \text{for some}\, d_1\in\mathbb{C}.$$

Therefore $f(x)=d_1e^{-\frac{1}{2\zeta}x^2}e^{\frac{i}{2\xi}x^2}E_\mu(bx)$. Now, if $\zeta<0$, then $f(x)=d_1e^{-\frac{1}{2\zeta}x^2}e^{\frac{i}{2\xi}x^2}E_\mu(bx)$ fails to be in $L^2_\mu(\mathbb{R})$. Hence we must have $\zeta>0$ and choose $d_1\in\mathbb{C}$ such that $\|f\|_{\mu,2}=1$.

When $A(x)=-\frac{1}{\xi}(x-\langle x\rangle_\mu)+\langle x\rangle_{D_\mu(f)}$, then following a similar reasoning as before, we can conclude that $f(x)=d_2e^{-\frac{1}{2\zeta}x^2}e^{\frac{i}{2\xi}x^2}E_\mu(b'x)$, where $b'=\left(\frac{1}{\zeta}+\frac{i}{\xi}\right)\langle x\rangle_f+i\langle x\rangle_{D_\mu(f)}$, $\zeta>0$ and $d_2\in\mathbb{C}$.

When

$$A(x)=\begin{cases}\frac{1}{\xi}(x-\langle x\rangle_f)+\langle x\rangle_{D_\mu(f)}, & x\geq\langle x\rangle_f,\\ -\frac{1}{\xi}(x-\langle x\rangle_\mu)+\langle x\rangle_{D_\mu(f)}, & x<\langle x\rangle_f,\end{cases}$$

then, we must have

$$f(x) = \begin{cases} d_3 e^{-\frac{1}{2\zeta}x^2} e^{\frac{i}{2\xi}x^2} E_\mu(bx), & x \geq \langle x \rangle_f, \\ d_4 e^{-\frac{1}{2\zeta}x^2} e^{-\frac{i}{2\xi}x^2} E_\mu(b'x), & x < \langle x \rangle_f. \end{cases}$$

Finally, when

$$A(x) = \begin{cases} -\frac{1}{\xi}(x - \langle x \rangle_\mu) + \langle x \rangle_{D_\mu(f)}, & x \geq \langle x \rangle_f, \\ \frac{1}{\xi}(x - \langle x \rangle_\mu) + \langle x \rangle_{D_\mu(f)}, & x < \langle x \rangle_f, \end{cases}$$

then

$$f(x) = \begin{cases} d_5 e^{-\frac{1}{2\zeta}x^2} e^{-\frac{i}{2\xi}x^2} E_\mu(b'x), & x \geq \langle x \rangle_f, \\ d_6 e^{-\frac{1}{2\zeta}x^2} e^{\frac{i}{2\xi}x^2} E_\mu(bx), & x < \langle x \rangle_f. \end{cases}$$

Conversely, if f has one of the forms (5.3.5), (5.3.6), (5.3.7) or (5.3.8), with $\|f\|_{\mu,2} = 1$, then it is clear that (5.3.3) and (5.3.4) hold. □

5.4 HPWUP for the fractional Dunkl transform

In [56], Ghazouani and Bouzeffour established the following uncertainty principle for the fractional Dunkl transform.

Theorem 5.4.1. *[56]. Suppose $\alpha, \beta \in \mathbb{R}$ such that $\beta - \alpha \notin \pi\mathbb{Z}$, and let $xD_\mu^\alpha f(x) \in L_\mu^2(\mathbb{R})$ and $xD_\mu^\beta f(x) \in L_\mu^2(\mathbb{R})$ with $\|f\|_{\mu,2} = 1$. Then*

$$\Delta_{\mu,2}^2(D_\mu^\alpha f)\Delta_{\mu,2}^2(D_\mu^\beta(f)) \geq \sin^2(\alpha - \beta)\left\{\left(\mu + \frac{1}{2}\right)\left(\|f_e\|_{\mu,2}^2 - \|f_0\|_{\mu,2}^2\right) + \frac{1}{2}\right\}^2. \qquad (5.4.1)$$

Moreover, equality holds if and only if $f(x) = de^{-\frac{1}{2\zeta}x^2}E_\mu(bx)$ with $b, d \in \mathbb{C}$ and $\zeta > 0$.

We improve the uncertainty inequality in (5.5.1) by deriving a truly sharp lower bound. We start with the following lemmas.

Lemma 5.4.2. *[42] Let $f(x) = \rho(x)e^{i\varphi(x)} \in L_\mu^2(\mathbb{R})$ with $\|f\|_{\mu,2} = 1$. Suppose that the classical derivatives $\rho'(x)$, $\varphi'(x)$ exists for all $x \in \mathbb{R}$ and $f'(x) \in L_\mu^2(\mathbb{R})$, then*

$$\Delta_{\mu,2}^2(D_\mu(g)) = \int_{-\infty}^{+\infty}\left\{T_\mu\rho(x) + \frac{\mu + \frac{1}{2}}{x}\rho(-x)\big(1 - \cos[\varphi(x) - \varphi(-x)]\big)\right\}^2 |x|^{2\mu+1}dx +$$

$$\int_{-\infty}^{+\infty}\left\{T_\mu\varphi(x) + \frac{\mu + \frac{1}{2}}{x}\left(\frac{\sin[\varphi(x) - \varphi(-x)]}{\rho(x)}\rho(-x) - (\varphi(x) - \varphi(-x))\right) - \langle x \rangle_{D_\mu(f)}\right\}^2 \rho^2(x)|x|^{2\mu+1}dx. \qquad (5.4.2)$$

Lemma 5.4.3. *Let* $f(x)=\rho(x)e^{i\varphi(x)}$ *with* $\|f\|_{\mu,2}=1$ *and* $g(x)=e^{i\frac{x^2\cot\alpha}{2}}f(x)$; $\alpha\in\mathbb{R}\backslash\pi\mathbb{Z}$. *Suppose that the classical derivatives* $\rho'(x)$, $\varphi'(x)$ *exist for all* $x\in\mathbb{R}$ *and* $f'(x)\in L^2_\mu(\mathbb{R})$, *then*

$$\langle x\rangle_{D_\mu(g)}=\cot\alpha\,\langle x\rangle_f+\langle x\rangle_{D_\mu(f)}, \tag{5.4.3}$$

and

$$\Delta^2_{\mu,2}(D_\mu(g))=\Delta^2_{\mu,2}(D_\mu(f))+2\cot\alpha\;Cov_\mu(f)+\cot\alpha\,\Delta^2_{\mu,2}(f). \tag{5.4.4}$$

Proof. We have

$$\langle x\rangle_{D_\mu(g)}=\int\limits_{-\infty}^{+\infty}xD_\mu(g)(x)\overline{D_\mu(g)(x)}|x|^{2\mu+1}dx=\int\limits_{-\infty}^{+\infty}-iT_\mu(g)(x)\overline{g(x)}|x|^{2\mu+1}dx.$$

On substituting $T_\mu(g)(x)=e^{i\frac{x^2\cot\alpha}{2}}\left(ix\cot\alpha\,f(x)+T_\mu f(x)\right)$, we get

$$\begin{aligned}\langle x\rangle_{D_\mu(g)}&=\cot\alpha\int\limits_{-\infty}^{+\infty}x\,|f(x)|^2\,|x|^{2\mu+1}\,dx+\int\limits_{-\infty}^{+\infty}-iT_\mu f(x)\overline{f(x)}\,|x|^{2\mu+1}\,dx\\&=\cot\alpha\,\langle x\rangle_f+\langle x\rangle_{D_\mu(f)}.\end{aligned}$$

Applying Lemma 5.4.2, we get

$$\begin{aligned}\Delta^2_{\mu,2}(D_\mu(g))=&\int\limits_{-\infty}^{+\infty}\left\{T_\mu\rho(x)+\frac{\mu+\frac{1}{2}}{x}\rho(-x)\left(1-\cos[\varphi(x)-\varphi(-x)]\right)\right\}^2\rho^2(x)\,|x|^{2\mu+1}\,dx\\&+\int\limits_{-\infty}^{+\infty}\left\{\varphi'(x)+x\cot\alpha+\frac{\mu+\frac{1}{2}}{x}\left(\frac{\sin[\varphi(x)-\varphi(-x)]}{\rho(x)}\rho(-x)\right)-\cot\alpha\,\langle x\rangle_f\right.\\&\left.-\langle x\rangle_{D_\mu(f)}\right\}^2\rho^2(x)\,|x|^{2\mu+1}\,dx\\=&\Delta^2_{\mu,2}(D_\mu(f))+\cot^2\alpha\,\Delta^2_{\mu,2}(f)+2\cot\alpha\int\limits_{-\infty}^{+\infty}(x-\langle x\rangle_f)\\&\times\left\{\varphi'(x)+\frac{\mu+\frac{1}{2}}{x}\left(\frac{\sin[\varphi(x)-\varphi(-x)]}{\rho(x)}\rho(-x)\right)-\langle x\rangle_{D_\mu(f)}\right\}\rho^2(x)\,|x|^{2\mu+1}\,dx\\=&\Delta^2_{\mu,2}(D_\mu(f))+\cot\alpha\Delta^2_{\mu,2}(f)+2\cot\alpha\left(\int\limits_{-\infty}^{+\infty}x\varphi'(x)\rho^2(x)|x|^{2\mu+1}dx-\langle x\rangle_f\langle x\rangle_{D_\mu(f)}\right)\\=&\Delta^2_{\mu,2}(D_\mu(f))+2\cot\alpha\;Cov_\mu(f)+\cot\alpha\,\Delta^2_{\mu,2}(f).\end{aligned}$$

□

Lemma 5.4.4. *Let* $\alpha \in \mathbb{R}$ *and let* $f(x) = \rho(x)e^{i\varphi(x)}$ *with* $\|f\|_{\mu,2} = 1$. *Suppose that the classical derivatives* $\rho'(x)$, $\varphi'(x)$ *exist for all* $x \in \mathbb{R}$. *If* $xf(x) \in L^2_\mu(\mathbb{R})$ *and* $xD^\alpha_\mu f(x) \in L^2_\mu(\mathbb{R})$, *then*

$$\langle x \rangle_{D^\alpha_\mu(f)} = \cos\alpha \, \langle x \rangle_f + \sin\alpha \, \langle x \rangle_{D_\mu(f)} \tag{5.4.5}$$

and

$$\Delta^2_{\mu,2}(D^\alpha_\mu(f)) = \cos^2\alpha \, \Delta^2_{\mu,2}(f) + 2\cos\alpha \, \sin\alpha \, Cov_\mu(f) + \sin^2\alpha \, \Delta^2_{\mu,2}(D_\mu(f)). \tag{5.4.6}$$

Proof. Using (5.2.13), we can write

$$\langle x \rangle_{D^\alpha_\mu(f)} = \frac{1}{|\sin\alpha|^{(\mu+1)2}} \int\limits_{-\infty}^{+\infty} x \left| D_\mu(g)\left(\frac{x}{\sin\alpha}\right)\right| |x|^{2\mu+1} \, dx = \sin\alpha \, \langle x \rangle_{D_\mu(g)}, \tag{5.4.7}$$

where $g(x) = e^{\frac{ix^2 \cot\alpha}{2}} f(x)$. Using Lemma 5.4.3, we get

$$\langle x \rangle_{D^\alpha_\mu(f)} = \cos\alpha \, \langle x \rangle_f + \sin\alpha \, \langle x \rangle_{D_\mu(f)}.$$

By using (5.2.13) and (5.4.7), we have

$$\begin{aligned}
\Delta^2_{\mu,2}(D^\alpha_\mu(f)) &= \int\limits_{-\infty}^{+\infty} \left(x - \langle x \rangle_{D^\alpha_\mu(f)}\right)^2 \left|D^\alpha_\mu(f)(x)\right|^2 |x|^{2\mu+1} \, dx \\
&= \int\limits_{-\infty}^{+\infty} \left(x \sin\alpha - \langle x \rangle_{D^\alpha_\mu(f)}\right)^2 |D_\mu(g)(x)|^2 |x|^{2\mu+1} \, dx \\
&= \sin^2\alpha \; \Delta^2_{\mu,2}(D_\mu(g)).
\end{aligned}$$

Therefore, Lemma 5.4.4 follows from Lemma 5.4.3. □

We are now in a position to state and prove the improved Heisenberg–Pauli–Weyl uncertainty principle for the fractional Dunkl transform.

Theorem 5.4.5. *Assume that* $f(x) = \rho(x)e^{i\varphi(x)}$ *with* $\|f\|_{\mu,2} = 1$ *and the derivatives* $\rho'(x)$, $\varphi'(x)$ *exist for all* $x \in \mathbb{R}$. *Let* $\alpha, \beta \in \mathbb{R}$ *be such that* $\beta - \alpha \notin \pi\mathbb{Z}$, *and let* $xD^\alpha_\mu f(x) \in L^2_\mu(\mathbb{R})$ *and* $xD^\beta_\mu f(x) \in L^2_\mu(\mathbb{R})$, *then*

$$\Delta^2_{\mu,2}(D^\alpha_\mu f)\Delta^2_{\mu,2}(D^\beta_\mu(f)) \geq \sin^2(\alpha - \beta)\Bigg(\bigg\{\left(\mu + \frac{1}{2}\right)\left(\|f_e\|^2_{\mu,2} - \|f_0\|^2_{\mu,2}\right) + \frac{1}{2}\bigg\}^2 + COV^2_\mu(f)$$

$$-Cov_{\mu}^2(f)\Big)+\Big(\cos\alpha\,\cos\beta\Delta_{\mu,2}^2(f)+\sin(\alpha+\beta)Cov_{\mu}(f)+\sin\alpha\,\sin\beta\Delta_{\mu,2}^2(D_{\mu}(f))\Big)^2. \tag{5.4.8}$$

Moreover, if $\varphi'(x)$ is continuous and ρ is nonzero almost everywhere, then the equality in (5.4.8) holds if and only if f is of the form (5.3.5), (5.3.6), (5.3.7) or (5.3.8).

Proof. By using Lemma 5.4.3 and Theorem 5.3.2, we have

$$\begin{aligned}\Delta_{\mu,2}^2(D_{\mu}^{\alpha}f)\Delta_{\mu,2}^2(D_{\mu}^{\beta}f) &= \sin^2(\alpha-\beta)\,\big(\Delta_{\mu,2}^2(f)\Delta_{\mu,2}^2(D_{\mu}(f))-Cov_{\mu}^2(f)\big)\\ &\quad+\Big(\cos\alpha\,\cos\beta\,\Delta_{\mu,2}^2(f)+\sin\alpha\,\sin\beta\,\Delta_{\mu,2}^2(D_{\mu}(f))+\sin(\alpha+\beta)\,Cov_{\mu}(f)\Big)^2\\ &\geq\sin^2(\alpha-\beta)\left(\left\{\left(\mu+\frac{1}{2}\right)\left(\|f_e\|_{\mu,2}^2-\|f_0\|_{\mu,2}^2\right)+\frac{1}{2}\right\}^2+COV_{\mu}^2(f)-Cov_{\mu}^2(f)\right)\\ &\quad+\Big(\cos\alpha\,\cos\beta\,\Delta_{\mu,2}^2(f)+\sin\alpha\,\sin\beta\,\Delta_{\mu,2}^2(D_{\mu}(f))+\sin(\alpha+\beta)\,Cov_{\mu}(f)\Big)^2.\end{aligned} \tag{5.4.9}$$

Notice that equality holds in (5.4.9) if and only if the equality holds in (5.3.2). ☐

5.5 L^p-type HPWUP for the fractional Dunkl transform

For the fractional Dunkl transform, we obtain the following L^p-type HPWUP, $1\leq p\leq 2$, in a more general situation.

Theorem 5.5.1. *Assume that $f(x)=\rho(x)e^{i\varphi(x)}$ with $\|f\|_{\mu,2}=1$ and the derivatives $\rho'(x)$, $\varphi'(x)$ exist for all $x\in\mathbb{R}$. Let $\alpha,\beta\in\mathbb{R}$ be such that $\beta-\alpha\notin\pi\mathbb{Z}$ and let $xD_{\mu}^{\alpha}f(x)\in L_{\mu}^2(\mathbb{R})\cap L_{\mu}^p(\mathbb{R})$, $xD_{\mu}^{\beta}f(x)\in L_{\mu}^p(\mathbb{R}), 1\leq p\leq 2$, then*

$$\begin{aligned}\Delta_{\mu,p}^2(D_{\mu}^{\alpha}f)\Delta_{\mu,p}^2(D_{\mu}^{\beta}(f))\geq\frac{|\sin(\beta-\alpha)|^{2(\mu+1)(\frac{2}{p}-1)}}{(2^{\mu+1}\Gamma(\mu+1))^{2(\frac{2}{p}-1)}}&\left\{\sin^2(\beta-\alpha)\left((\mu+\frac{1}{2})(\|f_e\|_{\mu,2}^2-\|f_o\|_{\mu,2}^2)+\frac{1}{2}\right)^2\right.\\ &\left.+\left(\cos\alpha\,\cos\beta\,\Delta_{\mu,2}^2(f)+\sin(\alpha+\beta)\,Cov_{\mu}(f)+\sin\alpha\sin\beta\,\Delta_{\mu,2}^2(D_{\mu}(f))\right)^2\right\}.\end{aligned} \tag{5.5.1}$$

Moreover, if $\varphi'(x)$ is continuous and ρ is nonzero almost everywhere, then the equality in (5.5.1) holds if and only if $p=2$ and f is of the form (5.3.5) or (5.3.6).

Remark 5.5.2. *(1) For $p = 2$, Theorem 5.5.1 provides a stronger lower bound than the one presented in equation (5.4.1) (see Theorem 6.1 of [56]).*
(2) When $\alpha = 0, \beta = \pi/2$ and $p = 2$, Theorem 5.5.1 coincides with Theorem 3.2 of [42] and provides a necessary and sufficient condition for the uncertainty inequality to hold with equality.

Proof. Assume that $\beta \notin \pi\mathbb{Z}$ and let $a = \langle x \rangle_{D_\mu^\alpha(f)}$ and $c = \langle x \rangle_{D_\mu^\beta(f)}$. Now using Hausdorff-Young inequality (5.2.7), we obtain

$$\Delta_{\mu,p}(D_\mu^\beta(f)) = \left(\int_{-\infty}^{+\infty} \left|(x-c) D_\mu^\beta(f)(x)\right|^p |x|^{2\mu+1}\, dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{-\infty}^{+\infty} \left|(x-c) D_\mu^{\beta-\alpha}(D_\mu^\alpha f(u))(x)\right|^p |x|^{2\mu+1}\, dx \right)^{\frac{1}{p}}$$

$$= \frac{1}{|\sin(\beta-\alpha)|^{\mu+1}} \left(\int_{-\infty}^{+\infty} \left|(x-c) D_\mu \left(e^{\frac{iu^2 \cot(\beta-\alpha)}{2}} D_\mu^\alpha f(u) \right) \left(\frac{x}{\sin(\beta-\alpha)} \right) \right|^p |x|^{2\mu+1} dx \right)^{\frac{1}{p}}$$

$$= |\sin(\beta-\alpha)|^{-\mu+\frac{2(\mu+1)}{p}} \left(\int_{-\infty}^{+\infty} |(x - c\csc(\beta-\alpha)) D_\mu(g)(x)|^p |x|^{2\mu+1} dx \right)^{\frac{1}{p}}$$

$$= |\sin(\beta-\alpha)|^{-\mu+\frac{2(\mu+1)}{p}} \left(\int_{-\infty}^{+\infty} |D_\mu((-iT_\mu - c')g)(x)|^p |x|^{2\mu+1} dx \right)^{\frac{1}{p}}$$

$$\geq \frac{|\sin(\beta-\alpha)|^{-\mu+\frac{2(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{\frac{2}{p}-1}} \left(\int_{-\infty}^{+\infty} |(-iT_\mu - c')g(x)|^q |x|^{2\mu+1} dx \right)^{\frac{1}{q}},$$

where $c' = b\csc(\beta-\alpha)$, $\quad g(x) = e^{\frac{ix^2\cot(\beta-\alpha)}{2}} D_\mu^\alpha f(x), \quad x \in \mathbb{R}$. Thus by Hölder's inequality,

$$\Delta_{\mu,p}(D_\mu^\alpha(f))\Delta_{\mu,p}(D_\mu^\beta(f))$$
$$\geq \frac{|\sin(\beta-\alpha)|^{-\mu+\frac{2(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{\frac{2}{p}-1}} \left(\int_{-\infty}^{+\infty} |(x-a) D_\mu^\alpha f(x)|^p |x|^{2\mu+1} dx \right)^{\frac{1}{p}} \left(\int_{-\infty}^{+\infty} |(-iT_\mu - c')g(x)|^q |x|^{2\mu+1} dx \right)^{\frac{1}{q}}$$
$$\geq \frac{|\sin(\beta-\alpha)|^{-\mu+\frac{2(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{\frac{2}{p}-1}} \left| \int_{-\infty}^{+\infty} e^{\frac{ix^2\cot(\beta-\alpha)}{2}} (x-a) D_\mu^\alpha f(x) \overline{(-iT_\mu - c')g(x)} |x|^{2\mu+1} dx \right|.$$

Now using

$$T_\mu g(x) = ix\cot(\beta-\alpha) e^{\frac{ix^2\cot(\beta-\alpha)}{2}} D_\mu^\alpha f(x) + e^{\frac{ix^2\cot(\beta-\alpha)}{2}} T_\mu D_\mu^\alpha f(x),$$

we have

$$\Delta_{\mu,p}(D^\alpha_\mu(f))\Delta_{\mu,p}(D^\beta_\mu(f)) \geq \frac{|\sin(\beta-\alpha)|^{-\mu+\frac{2(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{\frac{2}{p}-1}}\left|\mathbf{I}+\mathbf{II}\right|,$$

where

$$\mathbf{I} = \int\limits_{-\infty}^{+\infty} x(x-a)\;\cot(\beta-\alpha)\,|D^\alpha_\mu f(x)|^2|x|^{2\mu+1}dx = \cot(\beta-\alpha)\,\Delta^2_{\mu,2}(D^\alpha_\mu f) \tag{5.5.2}$$

and

$$\mathbf{II} = \int\limits_{-\infty}^{+\infty} (x-a)D^\alpha_\mu f(x)\overline{(-iT_\mu - c')D^\alpha_\mu f(x)}|x|^{2\mu+1}dx. \tag{5.5.3}$$

Case 1: Assume that $\alpha \neq n\pi$, for some $n \in \mathbb{Z}$, then

$$D^\alpha_\mu f(x) = \frac{1}{|\sin\alpha|^{\mu+1}}e^{\frac{ix^2\cot\alpha}{2}}D_\mu(h)\left(\frac{x}{\sin\alpha}\right),\ \text{where}\,h(x) = e^{\frac{ix^2\cot\alpha}{2}}f(x). \tag{5.5.4}$$

Now,

$$T_\mu(D_\mu f(x)) = \frac{1}{|\sin\alpha|^{\mu+1}}e^{i\frac{x^2\cot\alpha}{2}}\left[ix\cot\alpha\,D_\mu(h)\left(\frac{x}{\sin\alpha}\right) + T_\mu(D_\mu h)\left(\frac{x}{\sin\alpha}\right)\right].$$

Therefore,

$$\begin{aligned}\mathbf{II} &= \frac{\cot\alpha}{|\sin\alpha|^{(\mu+1)2}}\int\limits_{-\infty}^{+\infty} x(x-a)\left|D_\mu(h)\left(\frac{x}{\sin\alpha}\right)\right|^2|x|^{2\mu+1}dx \\ &+ \frac{1}{|\sin\alpha|^{2(\mu+1)}}\int\limits_{-\infty}^{+\infty}(x-a)D_\mu(h)\left(\frac{x}{\sin\alpha}\right)(-iT_\mu - c')\left[D_\mu(h)\left(\frac{x}{\sin\alpha}\right)\right](x)|x|^{2\mu+1}dx \\ &= \mathbf{II}_1 + \mathbf{II}_2.\end{aligned}$$

Now,

$$\begin{aligned}\mathbf{II}_1 &= \sin\alpha\,\cos\alpha\int\limits_{-\infty}^{+\infty} x\left(x-\frac{a}{\sin\alpha}\right)|D_\mu(h)(x)|^2\,|x|^{2\mu+1}dx \\ &= \sin\alpha\,\cos\alpha\int\limits_{-\infty}^{+\infty} x^2\,|D_\mu(h)(x)|^2\,|x|^{2\mu+1}\,dx - a\cos\alpha\int\limits_{-\infty}^{+\infty} x\,|D_\mu(h)(x)|^2\,|x|^{2\mu+1}\,dx \\ &= \sin\alpha\,\cos\alpha\int\limits_{-\infty}^{+\infty}(x-\langle x\rangle_{D_\mu(h)})^2\,|D_\mu(h)(x)|^2|x|^{2\mu+1}\,dx + \langle x\rangle_{D_\mu(h)}\cos\alpha\left(\sin\alpha\langle x\rangle_{D_\mu(h)} - a\right).\end{aligned}$$

By Lemma (5.4.3) and Lemma (5.4.4), we have $\sin\alpha\,\langle x\rangle_{D_\mu(h)} = a$. So

$$\mathbf{II}_1 = \sin\alpha\ \cos\alpha\Delta^2_{\mu,2}(D_\mu(h)).$$

Also,

$$\mathbf{II}_2 = \frac{1}{\sin\alpha\,|\sin\alpha|^{(\mu+1)2}}\int_{-\infty}^{+\infty}(x-a)D_\mu(h)\left(\frac{x}{\sin\alpha}\right)(-iT_\mu-\sin\alpha\,c')\,D_\mu(h)\left(\frac{x}{\sin\alpha}\right)|x|^{2\mu+1}dx$$

$$= \int_{-\infty}^{+\infty}\left(x-\frac{a}{\sin\alpha}\right)D_\mu(h)(x)\overline{(-iT_\mu-\sin\alpha\,c')D_\mu(h)(x)}|x|^{2\mu+1}dx$$

$$= \int_{-\infty}^{+\infty}xD_\mu(h)(x)\overline{(-iT_\mu)D_\mu(h)(x)}|x|^{2\mu+1}dx + ac'\int_{-\infty}^{+\infty}|D_\mu(h)(x)|^2\,|x|^{2\mu+1}dx$$

$$-\sin\alpha\,c'\int_{-\infty}^{+\infty}x\,|D_\mu(h)(x)|^2\,|x|^{2\mu+1}dx-\frac{a}{\sin\alpha}\int_{-\infty}^{+\infty}D_\mu(h)(x)\overline{(-iT_\mu(D_\mu(h)(x)))}\,|x|^{2\mu+1}\,dx.$$

Clearly, $\int_{-\infty}^{+\infty}|D_\mu(h)(x)|^2\,|x|^{2\mu+1}dx = 1$, and from Lemma 5.4.3 and Lemma 5.4.4, we have $a = \sin\alpha\langle x\rangle_{D_\mu(h)}$ and

$$\int_{-\infty}^{+\infty}D_\mu(h)(x)\overline{(-iT_\mu(D_\mu(h)(x)))}\,|x|^{2\mu+1}\,dx = \int_{-\infty}^{+\infty}h(-x)\overline{xh(-x)}\,|x|^{2\mu+1}\,dx$$

$$= -\int_{-\infty}^{+\infty}x\,|f(x)|^2\,|x|^{2\mu+1}dx = -\langle x\rangle_f.$$

Substituting this back into $\mathbf{II}_2$ and using (5.2.19) we have

$$\mathbf{II}_2 = \int_{-\infty}^{+\infty}iT_\mu h(x)\overline{xh(x)}\,|x|^{2\mu+1}\,dx+\frac{a}{\sin\alpha}\langle x\rangle_f$$

$$= -\int_{-\infty}^{+\infty}\cot\alpha\,x^2\,|f(x)|^2\,|x|^{2\mu+1}\,dx + i\int_{-\infty}^{+\infty}T_\mu f(x)\overline{xf(x)}\,|x|^{2\mu+1}\,dx+\frac{a}{\sin\alpha}\langle x\rangle_f$$

$$= -\cot\alpha\int_{-\infty}^{+\infty}(x-\langle x\rangle_f)^2\,|f(x)|^2\,|x|^{2\mu+1}\,dx-\cot\alpha\langle x\rangle_f^2$$

$$+\,i\left[-\mathcal{A}+i\left(Cov_\mu(f)+\langle x\rangle_f\langle x\rangle_{D_\mu(f)}\right)\right]+\frac{a}{\sin\alpha}\langle x\rangle_f$$

$$= -\cot\alpha\Delta^2_{\mu,2}(f)-Cov_\mu(f)-i\mathcal{A}+\frac{\langle x\rangle_f}{\sin\alpha}\left[-\cos\alpha\langle x\rangle_f+a\right]-\langle x\rangle_f\langle x\rangle_{D_\mu(f)}$$

$$= -\cot\alpha\,\Delta^2_{\mu,2}(f) - Cov_\mu(f) - i\mathcal{A},$$

where $\mathcal{A}$ is given in (5.2.21). Therefore,

$$\begin{aligned}\mathbf{II} &= \mathbf{II}_1 + \mathbf{II}_2\\ &= \left(\frac{\cos^3\alpha - \cos\alpha}{\sin\alpha}\right)\Delta^2_{\mu,2}(f) + (2\cos^2\alpha - 1)Cov_\mu(f) + \sin\alpha\,\cos\alpha\Delta^2_{\mu,2}(D_\mu(f)) - i\mathcal{A}.\end{aligned}$$

So, we have

$$\begin{aligned}&\mathbf{I}+\mathbf{II}\\ &= \left(\cot(\beta-\alpha)\cos^2\alpha + \frac{\cos^3\alpha - \cos\alpha}{\sin\alpha}\right)\Delta^2_{\mu,2}(f) + \Big(2\sin\alpha\,\cos\alpha\cot(\beta-\alpha) + 2\cos^2\alpha - 1\Big)\\ &\quad\times Cov_\mu(f) + \Big(\sin^2\alpha\cot(\beta-\alpha) + \sin\alpha\,\cos\alpha\Big)\Delta^2_{\mu,2}(D_\mu(f)) - i\mathcal{A}\\ &= \left(\frac{\cos\alpha\,\cos\beta}{\sin(\beta-\alpha)}\right)\Delta^2_{\mu,2}(f) + \left(\frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}\right)Cov_\mu(f) + \left(\frac{\sin\alpha\,\sin\beta}{\sin(\beta-\alpha)}\right)\Delta^2_{\mu,2}(D_\mu(f)) - i\mathcal{A}.\end{aligned}$$

Thus,

$$\begin{aligned}&|\mathbf{I}+\mathbf{II}|\\ &= \left\{\left[\left(\frac{\cos\alpha\cos\beta}{\sin(\beta-\alpha)}\right)\Delta^2_{\mu,2}(f) + \left(\frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}\right)Cov_\mu(f) + \left(\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\right)\Delta^2_{\mu,2}(D_\mu(f))\right]^2 + \mathcal{A}^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Now,

$$\begin{aligned}\Delta^2_{\mu,p}(D^\alpha_\mu f)\Delta^2_{\mu,p}(D^\beta_\mu f) &\geq \frac{|\sin(\beta-\alpha)|^{-2\mu+\frac{4(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{2(\frac{2}{p}-1)}}\,|\mathrm{I}+\mathrm{II}|^2\\ &= \frac{|\sin(\beta-\alpha)|^{-2\mu+\frac{4(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{2(\frac{2}{p}-1)}}\left\{\left[\left(\frac{\cos\alpha\cos\beta}{\sin(\beta-\alpha)}\right)\Delta^2_{\mu,2}(f) + \left(\frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}\right)Cov_\mu(f)\right.\right.\\ &\qquad \left.\left. + \left(\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}\right)\Delta^2_{\mu,2}(D_\mu(f))\right]^2 + \mathcal{A}^2\right\}.\end{aligned}$$

Consequently, we obtain (5.5.1).

Case 2: Assume that $\alpha = 2n\pi$, for some $n \in \mathbb{Z}$. Then from (5.5.2), we have $\mathbf{I} = \cot(\beta)\,\Delta^2_{\mu,2}(f)$ and

$$\mathbf{II} = \int_{-\infty}^{+\infty}(x-a)f(x)\overline{(-iT_\mu - b')\,f(x)}|x|^{2\mu+1}dx$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} x f(x)\overline{(-iT_\mu)\, f(x)}|x|^{2\mu+1}dx + ab' \int_{-\infty}^{+\infty} |f(x)|^2|x|^{2\mu+1}dx \\
&\quad - a \int_{-\infty}^{+\infty} f(x)\overline{(-iT_\mu)\, f(x)}|x|^{2\mu+1}dx - b' \int_{-\infty}^{+\infty} x|f(x)|^2|x|^{2\mu+1}dx \\
&= \int_{-\infty}^{+\infty} x f(x)\overline{(-iT_\mu)\, f(x)}|x|^{2\mu+1}dx - \langle x\rangle_f \langle x\rangle_{D_\mu(f)} \\
&= Cov_\mu(f) + i\mathcal{A}.
\end{aligned}
$$

Therefore,

$$
\Delta^2_{\mu,p}(D^\alpha_\mu f)\Delta^2_{\mu,p}(D^\beta_\mu f) \geq \frac{|\sin(\beta-\alpha)|^{-2\mu+\frac{4(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{2(\frac{2}{p}-1)}}\left(\mathcal{A}^2 + \left(\cot\beta\, \Delta^2_{\mu,2} + Cov_\mu(f)\right)^2\right),
$$

thus (5.5.1) follows.

Case 3: Assume that $\alpha = (2n+1)\pi$, for some $n \in \mathbb{Z}$. Arguing as in the previous case, we obtain

$$
\Delta^2_{\mu,p}(D^\alpha_\mu f)\Delta^2_{\mu,p}(D^\beta_\mu f) \geq \frac{|\sin(\beta-\alpha)|^{-2\mu+\frac{4(\mu+1)}{p}}}{(2^{\mu+1}\Gamma(\mu+1))^{2(\frac{2}{p}-1)}}\left(\mathcal{A}^2 + \left(-\cot\beta\, \Delta^2_{\mu,2} + Cov_\mu(f)\right)^2\right).
$$

As a result, we get (5.5.1).

Now, it remains to prove the conditions under which equality holds in Theorem 5.5.1. Since we are applying the Hausdorff-Young inequality to prove equation (5.5.1), it follows that the condition $p = 2$ is necessary for equality to hold in (5.5.1). If $p = 2$, then equality holds in (5.5.1) if and only if equality holds in (5.4.8) with $COV_\mu(f) = Cov_\mu(f)$. Moreover, $COV_\mu(f) = Cov_\mu(f)$ if and only if $(x - \langle x\rangle_f)$ and $T_\mu(\varphi)(x) + \frac{\mu+\frac{1}{2}}{x}\left(\frac{\sin[\varphi(x)-\varphi(-x)]\rho(-x)}{\rho(x)} - (\varphi(x)-\varphi(-x))\right) - \langle x\rangle_{D_\mu(f)}$ have the same or opposite signs (see Subsection 5.2.3).

Thus, equality holds in (5.5.1) if and only if $f(x) = \rho(x)e^{i\varphi(x)}$ satisfies either equations (5.3.3) and (5.3.9), or equations (5.3.3) and (5.3.10). Therefore, f must be of the form (5.3.5) or (5.3.6). This completes the proof of Theorem 5.5.1. □

Remark 5.5.3. *(1) The constants d_1, d_2 in Theorem 5.3.4 can be chosen such that (see Lemma 4.3 in [92])* $d_1 = e^{i\theta}\left(\zeta^{\mu+1}\Gamma(\mu+1)e^{-Re\left(\frac{b^2\zeta}{2}\right)}E_\mu\left(\frac{|b|^2\zeta}{2}\right)\right)^{-\frac{1}{2}}$ *and*

$$d_2 = e^{i\theta}\left(\zeta^{\mu+1}\Gamma(\mu+1)e^{-Re\left(\frac{b'^2\zeta}{2}\right)}E_\mu\left(\frac{|b'|^2\zeta}{2}\right)\right)^{-\frac{1}{2}}.$$

(2) For $\mu = -\frac{1}{2}$*, the results presented in Theorems 5.5.1 and 5.4.5 are obtained by Chen-Dang-Mai [28] and Dang-Deng-Qian [37], respectively.*

(3) For $\mu > -\frac{1}{2}$*, the weighted measure* $|x|^{2\mu+1}dx$ *is not translation-invariant. As a result, Theorem 5.5.1 cannot be recovered by first proving it for the cases where* $\langle x\rangle_f = 0, \langle x\rangle_{D_\mu(f)} = 0$ *as in Chapter 6 of [32] or [28].*

(4) If the function f *is of the form given in (5.3.7) or (5.3.8), it is clear from the proof of Theorem 5.3.4 that* $(x - \langle x\rangle_f)$ *and* $T_\mu(\varphi)(x) + \frac{\mu+\frac{1}{2}}{x}\left(\frac{\sin[\varphi(x)-\varphi(-x)]\rho(-x)}{\rho(x)} - (\varphi(x) - \varphi(-x))\right) - \langle x\rangle_{D_\mu(f)}$ *do not have the same or opposite signs. Hence, it follows that* $COV_\mu(f) > Cov_\mu(f)$ *(see Subsection 5.2.3).*

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Publications

Based on the work presented in this thesis, the following research articles are published or communicated.

1. S. Ghosh, S. S. Mondal and J. Swain, Strichartz estimates associated with the Grushin operator, *New York J. Math.* 30, 1621–1642 (2024).
   http://nyjm.albany.edu/j/2024/30-69.html

2. S. Ghosh and J. Swain, Restriction theorem and Strichartz estimate for orthonormal functions associated with the Special Hermite operator, (2025).
   arXiv preprint arXiv:2306.10298

3. S. Ghosh, S. S. Mondal and J. Swain, Strichartz inequality for orthonormal functions associated with special Hermite operator, *Forum Math.* 36(3), 655–669 (2024).
   https://doi.org/10.1515/forum-2023-0115

4. S. Ghosh and J. Swain, On the Schatten exponent in orthonormal Strichartz estimate for the Dunkl operators, *Anal. Math. Phys.* 14, 111 (2024).
   https://doi.org/10.1007/s13324-024-00970-7

5. S. Ghosh, Y. A. Bhat and J. Swain, Heisenberg-Pauli-Weyl uncertainty principles for the fractional Dunkl transform on the real line, *J. Math. Anal. Appl.* 553(1), Paper No. 129890, 16pp. (2026).
   https://doi.org/10.1016/j.jmaa.2025.129890