Spectra of Weighted Directed Graphs

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

by

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to the

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March, 2012

Declaration

I do hereby declare that the work contained in this thesis entitled **"Spectra of Weighted Directed Graphs'** has done by me, under the supervision of **Dr. Sukanta Pati**, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in this thesis entitled **"Spectra of Weighted Directed Graphs"** by **Debajit Kalita**, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

March, 2012

Dr. Sukanta Pati
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Dedicated

to

My Parents

Acknowledgements

In the first place I would like to express my gratitude to Dr. Sukanta Pati, for his supervision, advice, and guidance from the very first day of my research work. His Socratic questioning, constructive criticism and incredible experience have helped me to enrich my growth as a researcher in Mathematics. I will be always indebted to him for his unflinching encouragement and support in various ways.

It is a great pleasure to express my sincere thanks to Prof. R. B. Bapat for his invaluable suggestions on my research papers and his great influence to my work.

I am deeply indebted to Prof. Meenaxi Bhattacharjee for her encouragements during the most critical period of my academic career, which exceptionally inspired me to enrich my basic mathematics skills. Dr. S. N. Bora, Prof. B. K. Sarma, Prof. K. C. Chowdhury, and Prof. H. K. Saikia are some of the personalities I would like to acknowledge for giving me the motivation, inspiration and information required to enter into the research world of mathematics.

The interesting feedback, useful suggestions, support and friendship from Dr. Anjan Kr. Chakrabarty, Dr. Bikash Bhattacharjya and Biswajit da throughout this journey, has been invaluable to me on both academic and personal level, for which I am extremely grateful to them.

I sincerely acknowledge Indian Institute of Technology Guwahati for providing me with the various facilities necessary to carry out my research. I am most grateful to CSIR, India for providing me with the financial assistance during the Ph.D process.

I express my special and heartiest thanks to my best friends ever Manoj, Ujwal and Kuki, who always stand beside me with their support and helping hands in my difficult times. Besides, I thank all my research scholar friends of the Department of Mathematics, IIT Guwahati for their love and company during my stay in the IIT campus.

Finally, I thank my parents, sisters Pranita and Binita, brother Nabajit for their unequivocal support, unwavering love, quiet patience, and most importantly allowing me to be as ambitious as I wanted.

March, 2012

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Abstract

The study of mixed graph and its Laplacian matrix have gained quite a bit of interest among the researchers. Mixed graphs are very important for the study of graph theory as they provide a setup where one can have directed and undirected edges in the graph. In this thesis we present a more general structure than that of mixed graphs, namely the weighted directed graphs. We supply appropriate generalizations of several existing results in the literature for mixed graphs. We also prove many new combinatorial results relating the Laplacian (resp. adjacency) matrix and the graph structure. The notion of 3-colored digraphs is introduced here. This notion naturally generalizes the notion of mixed graphs but is much restricted in comparison to the weighted directed graph. Our main objective is to study the spectral properties of the adjacency and the Laplacian matrix of these graphs.

We establish that the Laplacian matrix of weighted directed graphs are not always singular. A weighted directed graph is said to be singular (resp. non-singular) if its Laplacian matrix is singular (resp. non-singular). We give several characterizations of singularity of the weighted directed graphs. Apart from these, we provide some additional characterization of singularity of the connected 3-colored digraphs. A combinatorial description of the determinant of the Laplacian matrix of weighted directed graphs is supplied here.

We prove that the adjacency (resp. Laplacian) spectrum of a 3-colored digraph can be realized as a subset of the adjacency (resp. Laplacian) spectrum of a suitable undirected graph. In order to achieve this some graph operations similar to that in [17] are introduced. Using these graph operations we show that for a connected 3-colored digraph on n vertices, there exists a mixed graph on $2n$ vertices whose adjacency and Laplacian eigenvalues are precisely those of the 3-colored digraph with multiplicities doubled. We also show that for a connected mixed graph G on n vertices, there is an unweighted undirected graph H on $2n$ vertices whose adjacency (resp. Laplacian) spectrum contains the adjacency (resp. Laplacian) spectrum of the

mixed graph. Moreover, a description of the remaining adjacency (resp. Laplacian) eigenvalues of H is supplied. We observe that the graph H may be viewed as the result of a special case of a new graph operation on unweighted undirected graph introduced here. We show that the adjacency (resp. Laplacian) spectrum of the graph resulting from such an operation is completely determined by the adjacency (resp. Laplacian) spectra of some closely related weighted directed graphs.

The Laplacian spectrum of the class of connected 3-colored digraphs containing exactly one non-singular cycle is studied here. Mainly, we study the smallest Laplacian eigenvalue and the corresponding eigenvectors of such graphs. We show that the smallest Laplacian eigenvalue of such a graph can be realized as the algebraic connectivity (second smallest Laplacian eigenvalue) of a suitable undirected graph. We determine the non-singular unicyclic 3-colored digraph on n vertices, which minimize the smallest Laplacian eigenvalue over all such graphs. A class of non-singular unicyclic 3-colored digraphs maximizing the smallest Laplacian eigenvalue over all such graphs is also supplied. We give a complete characterization of non-singular unicyclic 3-colored digraphs that have 1 as the second smallest Laplacian eigenvalue.

A combinatorial description of the coefficients of characteristic polynomial of the adjacency matrix of 3-colored digraphs is supplied here. We obtain a relationship between these coefficients and the structural properties of the graph, generalizing Sachs theorem. A graph G is said to have SR-property if $A(G)$ is non-singular and λ is an eigenvalue of $A(G)$ of multiplicity k if and only if $\frac{1}{\lambda}$ is an eigenvalue of $A(G)$ with the same multiplicity. Finally, we supply the structure of unicyclic 3-colored digraphs satisfying SR-property.

Contents

Chapter 1

Introduction

1.1 Preamble

The study of graph spectra is an important part of graph theory. It has found its applications in several subjects like Biology, Geography, Economics, Social Sciences, computer science, information and communication technologies, see for example [15] and references there in. Several researchers have studied various spectral properties of the adjacency and Laplacian matrices of graphs. We refer the reader to the classical book by Cvetković, Doob and Sachs [13] and the survey articles by Merris [35] and Mohar [38], for more background on these matrices.

All our graphs are simple. All our directed graphs have simple **underlying** undirected graphs (except in Remark 3.3.2 and Definition 4.1.8). At times we use $V(G)$ (resp. $E(G)$) to denote the set of vertices (resp. edges) of a graph G (directed or undirected). In the absence of any specification $V(G)$ is assumed to be $\{1, 2, \ldots, n\}$. We write $(i, j) \in E(G)$ to mean the existence of the directed edge from the vertex i to the vertex j. Throughout the thesis $\mathrm{i} = \sqrt{-1}$.

Definition 1.1.1. Let G be a directed graph. With each edge (i, j) in $E(G)$ we associate a complex number w_{ij} of unit absolute value and non-negative imaginary part. We call it the weight of that edge. We call the directed graph G with such a weight function w a *weighted directed graph.*

Definition 1.1.2. Let G be a weighted directed graph. We define the *adjacency matrix* $A(G)$ of G as the matrix with ij-th entry

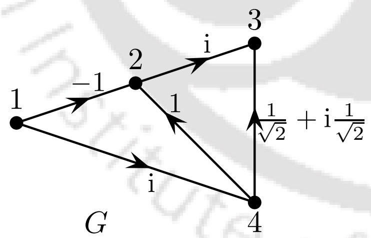
$$a_{ij} = \begin{cases} \mathsf{w}_{ij} & \text{if } (i,j) \in E(G), \\ \overline{\mathsf{w}}_{ji} & \text{if } (j,i) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.1.3. Note that choosing the weights only from the 'upper half part of the unit circle' in Definition 1.1.1 is not really a restriction for the study of adjacency matrices. For example, if G has an edge (i,j) of weight $x+y\mathrm{i}$, then we may replace that edge by an edge (j,i) of weight $x-y\mathrm{i}$ while the adjacency matrix remains unchanged.

Let G be a weighted directed graph. In defining subgraph, walk, path, component, connectedness, matching and degree of a vertex in G we focus only on the underlying unweighted undirected graph of G. The *degree* d_i of a vertex i in a weighted directed graph G may be viewed as the sum of absolute values of the weights of the edges incident with the vertex i.

Definition 1.1.4. Let G be a weighted directed graph. We define the *Laplacian matrix* $L(G)$ of G as the matrix $D(G)-A(G)$, where $D(G)$ is the diagonal matrix with d_i as the i-th diagonal entry.

Example 1.1.5. Consider the weighted directed graph G as shown below. Weights of the edges are written beside them. The adjacency and the Laplacian matrix of G are also supplied.



$$A(G)=\begin{bmatrix} 0 & -1 & 0 & \mathrm{i} \\ -1 & 0 & \mathrm{i} & 1 \\ 0 & -\mathrm{i} & 0 & \frac{1}{\sqrt{2}}-\mathrm{i}\frac{1}{\sqrt{2}} \\ -\mathrm{i} & 1 & \frac{1}{\sqrt{2}}+\mathrm{i}\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$L(G)=\begin{bmatrix} 2 & 1 & 0 & -\mathrm{i} \\ 1 & 3 & -\mathrm{i} & -1 \\ 0 & \mathrm{i} & 2 & -\frac{1}{\sqrt{2}}+\mathrm{i}\frac{1}{\sqrt{2}} \\ \mathrm{i} & -1 & -\frac{1}{\sqrt{2}}-\mathrm{i}\frac{1}{\sqrt{2}} & 3 \end{bmatrix}$$

Figure 1.1: $G, A(G)$ and $L(G)$

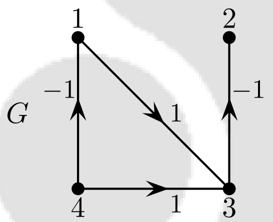
Remark 1.1.6. Let G be a weighted directed graph.

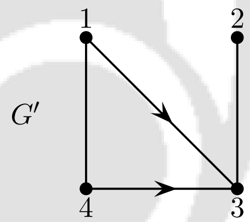
(a) Notice that if weight of each edge in G is 1, then our definition of $L(G)$ coincides with the usual Laplacian matrix of an unweighted undirected graph. This has motivated us to use $D(G)-A(G)$ rather than $D(G)+A(G)$ for the Laplacian matrix.

(b) If weights of the edges in G are ± 1, then (viewing the edges with weight 1 as directed and the edges with weight -1 as undirected) our definition of $L(G)$ coincides with the Laplacian matrix of a mixed graph as defined in [4].

(c) If weight of each edge in G is -1, then our definition of $L(G)$ coincides with the well studied signless Laplacian matrix (see for example, Cvetkovic, Rowlinson and Simic [14]) of undirected graph G.

Example 1.1.7. Consider the weighted directed graphs G shown below. Weights of the edges are written beside them. Their adjacency and Laplacian matrices are also supplied. Observe that in the graph G if we view the edges having weight 1 as directed and the edges having weight -1 as undirected, then $A(G)$ (resp. $L(G)$) is the same as the adjacency (resp. Laplacian) matrix of the mixed graph G'.





$$A(G)=\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}; \qquad L(G)=\begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 3 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

Note that with this set-up the Laplacian matrix of a weighted directed graph is positive semi-definite. The justification is as follows.

Definition 1.1.8. We define the *vertex edge incidence* matrix $M=M(G)=[m_{i,e}]$ of a weighted directed graph G as the matrix with rows labelled by the vertices and columns labelled by the edges in G satisfying

$$m_{i,e}=\begin{cases} 1 & \text{if } e=(i,j) \text{ for some vertex } j, \\ -\overline{\mathsf{w}}_{ij} & \text{if } e=(j,i) \text{ for some vertex } j, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $(MM^*)_{ii} = d_i$, and for $i \neq j$, $(MM^*)_{ij} = -\mathsf{w}_{ij}$ if $(i,j) \in E(G)$; $(MM^*)_{ij} = -\overline{\mathsf{w}}_{ji}$ if $(j,i) \in E(G)$; $(MM^*)_{ij} = 0$ otherwise. Thus we see that $L(G) = MM^*$, which implies the Laplacian matrix of a weighted directed graph is positive semi-definite.

Observe that $(M^*x)_e = (x(i) - \mathsf{w}_{ij}x(j))$, for any $x \in \mathbb{C}^n$ and edge $e = (i,j)$. It follows that

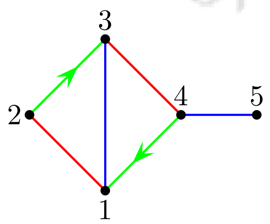
$$x^*L(G)x = (M^*x)^*(M^*x) = \sum_{(i,j)\in E(G)} |x(i) - \mathsf{w}_{ij}x(j)|^2. \tag{1.1.1}$$

Definition 1.1.9. Let G be a directed graph with edges having colors red, blue, or green. We assign each red edge the weight 1, each blue edge the weight -1 and each green edge the weight i. We call this graph a 3-*colored digraph*.

Note that the class of 3-colored digraphs is a very small subclass of the weighted directed graphs and is still a larger class than the mixed graphs.

Remark 1.1.10. Let G be a 3-colored digraph and $(i,j) \in E(G)$ have a color red or blue. Then $a_{ij} = a_{ji} = \pm 1$. Thus $A(G)$ is indifferent about the orientations of the red and blue edges. In view of this we keep the red or blue edges in the figures unoriented. We write $ij \in E(G)$ to mean the existence of the red or the blue edge between the vertices i and j in G. We write $(i,j) \in E(G)$ to mean the existence of the green edge directed from the vertex i to the vertex j in G.

Example 1.1.11. The graph as shown in the following picture is a 3-colored digraph G, whose adjacency matrix $A(G)$ is also supplied.



$$A(G) = \begin{bmatrix} 0 & 1 & -1 & -\mathrm{i} & 0 \\ 1 & 0 & \mathrm{i} & 0 & 0 \\ -1 & -\mathrm{i} & 0 & 1 & 0 \\ \mathrm{i} & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Figure 1.2: G, $A(G)$.

Note that the usual Laplacian matrix of an unweighted undirected graph G, that is, the Laplacian matrix of a weighted directed graph G with all edges having a weight 1 is always singular. Fiedler [23] proved that 0 is a simple eigenvalue of $L(G)$ if and only if G is connected. Thus the second smallest eigenvalue of $L(G)$ is positive if and only if G is connected. Fiedler [23] termed the second smallest eigenvalue of $L(G)$ as the *algebraic connectivity* of G, henceforth we denote it by $a(G)$. Here we see a relationship between the spectral and structural properties of a graph. As the term algebraic connectivity suggests, $a(G)$ provides an algebraic measure of how connected the graph G is. There is a wealth of results to support that statement, beginning with the pioneering work of Fielder on the subject. An eigenvector of $L(G)$ corresponding to the algebraic connectivity is popularly known as a *Fiedler vector* of G.

Definition 1.1.12. Let G be a weighted directed graph. The *weight* of a i_1-i_k-walk $W = [i_1, \ldots, i_k]$ in G, denoted by w_W is $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k}$, where a_{ij} are the entries of $A(G)$. For $1 \leq p \leq k-1$, if $e = (i_p, i_{p+1}) \in E(G)$, then we say e is directed **along** the walk, otherwise we say e is directed **opposite** to the walk.

Let G be a weighted directed graph and $D = \text{diag}(d_{11}, \ldots, d_{nn})$ with $|d_{ii}| = 1$, for each i. Then $D^* A(G) D$ (resp. $D^* L(G) D$) is the adjacency (resp. Laplacian) matrix of another weighted directed graph which we denote by ${}^D G$. Observe that if $(i, j) \in E(G)$ has a weight w_{ij}, then it has the weight $\overline{d}_{ii} \mathsf{w}_{ij} d_{jj}$ in ${}^D G$.

Definition 1.1.13. Let G and H be weighted directed graphs. We say H is D-*similar* to G if there exists a diagonal matrix D (with $|d_{ii}| = 1$, for each i) such that $H = {}^D G$. Thus, both of them have the same undirected unweighted underlying graph.

1.2 Organization of the Thesis

The thesis is organized as follows. There are five chapters in the thesis. Chapter 1 contains a brief introduction of the thesis and a few lines for motivation.

Chapter 2 is devoted mainly to the study of singularity of the Laplacian matrix of weighted directed graphs. We show that singularity of the Laplacian matrix of weighted directed graphs have close connection with the graph structure. We provide a combinatorial description of the determinant of the Laplacian matrix of weighted directed graphs relating the graph structure.

Chapter 3 deals with the adjacency and the Laplacian spectra of 3-colored digraphs. We show the realizability of the adjacency (resp. Laplacian) spectrum of a 3-colored digraph as a subset of the adjacency (resp. Laplacian) spectrum of a suitable undirected graph constructed by some graph operations on the 3-colored digraph.

In Chapter 4 we study the smallest Laplacian eigenvalue and the corresponding eigenvectors of 3-colored digraphs containing exactly one non-singular cycle. We discuss the non-singular unicyclic 3-colored digraphs, which minimize (resp. maximize) the smallest Laplacian eigenvalue over all such graphs. Further, we characterize the non-singular unicyclic 3-colored digraphs which have 1 as the second smallest Laplacian eigenvalue.

An unweighted undirected graph G is bipartite if and only if $-\lambda$ is an eigenvalue of $A(G)$ whenever λ is an eigenvalue of $A(G)$, (see [13]). In contrast to this property of bipartite graphs, Barik, Pati and Sarma [9] introduced the notion of graphs with *property* (R), that is, the graphs satisfying the property that $\frac{1}{\lambda}$ is an eigenvalue of $A(G)$ whenever λ is an eigenvalue of $A(G)$. Further, when λ and $\frac{1}{\lambda}$ are eigenvalues of $A(G)$ with the same multiplicity, then G is said to have SR-property. In [9], the authors characterized all trees with SR-property and proved that a tree has SR-property if and only if it is a simple corona tree. Barik et al. [7] studied the structure of a unicyclic unweighted undirected graph with SR-property.

In Chapter 5 we determine the coefficients of the characteristic polynomial of the adjacency matrix of 3-colored digraphs in terms of the graph structure. We supply the structure of unicyclic 3-colored digraphs satisfying SR-property.

Chapter 2

Laplacian singularity of weighted directed graphs

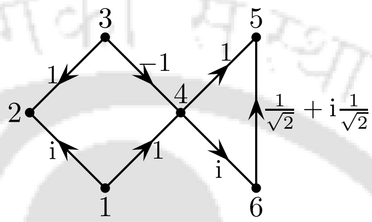
In this chapter our focus is on the Laplacian matrix of weighted directed graphs and its singularity. In Section 2.1 we supply several characterizations of singularity of the Laplacian matrix of weighted directed graphs. This provides a better combinatorial insight. Many results in this section generalize the known results related to Laplacian singularity of the mixed graphs in the literature. We provide a characterization of the connected weighted directed graphs which are D-similar to mixed graphs, which is new of its kind. Tan and Fan [40] have introduced and studied the parameter edge singularity of a mixed graph. In Section 2.2 we continue to study the edge singularity for weighted directed graphs. The problem of characterizing mixed graphs with a fixed edge singularity has never been addressed. We provide a combinatorial characterization of connected weighted directed graphs having a fixed edge singularity. In Section 2.3 we consider the class of 3-colored digraphs and supply some additional informations on the structure of singular connected 3-colored digraphs, apart from that in section 2.1. In Section 2.4, we establish a relationship between the determinant of the Laplacian matrix of weighted directed graphs and the graph structure.

2.1 D-similarity and Laplacian singularity

It was first observed in [4], that unlike the usual Laplacian matrix of an undirected graph, the Laplacian matrix of a mixed graph is sometimes non-singular. Several characterizations of singularity for mixed graphs were provided in [4]. It is natural to ask for similar characterization of singularity for the weighted directed graphs.

Definition 2.1.1. We call a weighted directed graph **singular** (resp. non-singular) if its Laplacian matrix is singular (resp. non-singular).

Example 2.1.2. Consider the weighted directed graph G shown below. Observe that $W_1 : 1, 4, 5, 6$ and $W_2 : 1, 4, 6$ are two different 1-6-walks in G with the weights $\mathsf{w}_{W_1} = \frac{1}{\sqrt{2}} - \mathrm{i}\frac{1}{\sqrt{2}}$ and $\mathsf{w}_{W_2} = \mathrm{i}$, respectively. Clearly $\mathsf{w}_{W_1} \neq \mathsf{w}_{W_2}$.



In view of Example 2.1.2 above, a natural question is the following: Does there exist a weighted directed graph G such that each u-v-walk in G has the same weight, for each fixed $u, v \in V(G)$?

The answer to this question is in the affirmative, for example, we consider a weighted directed graph G with all the edges having weight 1. Note that such a weighted directed graph is always singular. So it is natural to ask the following question : Does there exist a non-singular weighted directed graph H such that each u-v-walk in H has the same weight, for each fixed $u, v \in V(H)$?

Let G be a connected weighted directed graph. Assume that weight of any 1-i-walk is the same. By $\mathbf{n}$ we denote the vector of size n defined by $\mathbf{n}(1) = 1$ and $\mathbf{n}(i) =$ conjugate of the weight of a 1-i-walk which is the same as the weight of a i-1-walk. The following result answers the previous question in the negative.

Lemma 2.1.3. *Let G be a connected weighted directed graph. Then $L(G)$ is singular if and only if the weight of any 1-i-walk is the same. Furthermore, when $L(G)$ is singular, 0 is a simple eigenvalue with an eigenvector* $\mathbf{n}$.

Proof. Suppose that $L(G)$ is singular. Let $x \neq 0$ be a null vector of $L(G)$. Then using equation 1.1.1, we have $x(u) = \mathsf{w}_{uv} x(v)$ whenever (u, v) is an edge. Note that if $x(u) = 0$, then for each neighbor w of u we have $x(w) = 0$. As G is connected, this implies that $x = 0$. Hence the eigenvalue 0 has multiplicity one. Let W be any 1-i-walk. Using equation 1.1.1, we have $x(1) = \mathsf{w}_W x(i)$. Hence each 1-i-walk has the same weight and $x = x(1)\mathbf{n}$.

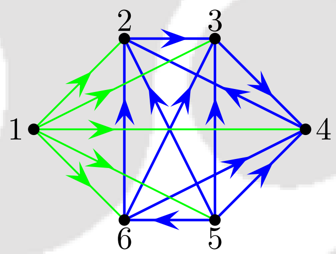
Conversely, suppose that the weight of any 1-i-walk is the same. Note that if $(i,j) \in E(G)$, then $\mathbf{n}(j) = \overline{\mathsf{w}}_{ij}\mathbf{n}(i)$. Using equation (1.1.1), we have

$$\mathbf{n}^* L(G)\mathbf{n} = \sum_{(i,j)\in E(G)} |\mathbf{n}(i) - \mathsf{w}_{ij}\mathbf{n}(j)|^2 = 0.$$

Therefore $\|M^*\mathbf{n}\|^2 = 0$ and $L(G)\mathbf{n} = MM^*\mathbf{n} = 0$. So $L(G)$ is singular. ∎

It follows that the class of singular connected weighted directed graphs is same as the class of connected weighted directed graphs G satisfying the property that each u-v-walk in G has the same weight, for each fixed $u, v \in V(G)$.

Example 2.1.4. The graph in the following picture is a weighted directed graph. Here the blue edges have a weight -1 and the green edges have a weight i. Note that the graph is non-singular and the smallest Laplacian eigenvalue has multiplicity 5.



The following result tells that a singular connected weighted directed graph is nothing but an unweighted undirected graph up to D-similarity.

Lemma 2.1.5. *Let G be a connected weighted directed graph. Then $L(G)$ is singular if and only if G is D-similar to the underlying unweighted undirected graph of G.*

Proof. Suppose that $L(G)$ is singular. By Lemma 2.1.3, the vector $\mathbf{n}$ is well defined. Take D to be the diagonal matrix with $d_{ii} = \mathbf{n}(i)$, for each i. We have $(D^*L(G)D)_{ij} = \overline{\mathbf{n}(i)}\, l_{ij}\, \mathbf{n}(j)$. If $(i,j) \in E(G)$, then $l_{ij} = -\mathsf{w}_{ij} = -\mathbf{n}(i)/\mathbf{n}(j)$ and so $\overline{\mathbf{n}(i)}\, l_{ij}\, \mathbf{n}(j) = -1$. If $(j,i) \in E(G)$, then $l_{ij} = -\overline{\mathsf{w}}_{ji} = -\overline{\mathbf{n}(j)}/\overline{\mathbf{n}(i)}$ and so

$\overline{\mathbf{n}(i)}\, l_{ij}\, \mathbf{n}(j) = -1$. Furthermore, $l_{ii} = d_i$ implies $\overline{\mathbf{n}(i)}\, l_{ii}\, \mathbf{n}(i) = d_i$. The converse is trivial. ■

Remark 2.1.6. Notice that when G is a singular mixed graph, $\mathbf{n}$ is the vector with entries 1 or -1. Hence in this case the diagonal matrix D in Lemma 2.1.5 is nothing but a signature matrix.

Next result characterizes the singular cycles in a weighted directed graph. It will be used to give another characterization of a non-singular weighted directed graph.

Lemma 2.1.7. *Let C be a weighted directed graph whose underlying undirected graph is a cycle. Then C is singular if and only if $\mathsf{w}_C = 1$.*

Proof. If C is singular then by Lemma 2.1.3, we have $1 = \mathsf{w}_C$. Conversely let $\mathsf{w}_C = 1$ and W_1 be a 1-i-path, $i \neq 1$. Let W_2 be the other 1-i-path. Denote by W_3 the i-1 path obtained by tracing back W_2. Then $1 = \mathsf{w}_C = \mathsf{w}_{W_1}\mathsf{w}_{W_3}$, which implies that $\mathsf{w}_{W_1} = 1/\mathsf{w}_{W_3} = \mathsf{w}_{W_2}$. Hence by Lemma 2.1.3, C is singular. ■

► In view of Lemma 2.1.7, we call a cycle C in a weighted directed graph **singular** if its weight $\mathsf{w}_C = 1$. Otherwise we call it a **non-singular** cycle.

Remark 2.1.8. Notice that if we consider mixed graphs, then a cycle C is singular if and only if $\mathsf{w}_C = 1$, that is there are an even number of undirected edges (viewing the edges of weight -1 as undirected) on the cycle. That is the cycle is non-singular if and only if it has an odd number undirected edges. So the previous lemma generalizes Lemma 1 of [4].

The following result gives another characterization of singularity of a connected weighted directed graph.

Lemma 2.1.9. *Let G be a connected weighted directed graph. Then $L(G)$ is singular if and only if there exist a partition $V(G) = V_1 \cup V_2 \cdots \cup V_k$ such that the following conditions are satisfied.*

(i) There are distinct complex numbers w_i of unit modulus associated with each V_i, for $i = 1, \ldots, k$,

(ii) Any edge between V_i and V_j, $i < j$ is either directed from V_i to V_j with a weight $\mathsf{w}_i\overline{\mathsf{w}}_j \neq 1$ or is directed from V_j to V_i with a weight $\overline{\mathsf{w}}_i\mathsf{w}_j \neq 1$,

(iii) Each edge within V_i has a weight 1, for $i = 1\ldots, k$.

Proof. Suppose that $L(G)$ is singular. By Lemma 2.1.3, 0 is a simple eigenvalue and $\mathbf{n}$ is a null vector of $L(G)$. Let $V_i = \{j \in V(G) : \mathbf{n}(j) = \mathbf{n}(i)\}$. Let $u \in V_i$, $v \in V_j$ and $i < j$ such that (u, v) is an edge. If $\mathsf{w}_{uv} = 1$, then $\mathbf{n}(u) = \mathbf{n}(v)$, which is not possible. Since $\mathbf{n}(u) = \mathbf{n}(i)$ and $\mathbf{n}(v) = \mathbf{n}(j)$, we must have $\mathsf{w}_{uv} = \mathbf{n}(i)\overline{\mathbf{n}(j)} \neq 1$, by Lemma 2.1.3 and the definition of $\mathbf{n}$. Similarly, if (v, u) is an edge, then we must have $\mathsf{w}_{vu} = \mathbf{n}(j)\overline{\mathbf{n}(i)} \neq 1$. So with each V_i we associate the complex number $\mathsf{w}_i = \mathbf{n}(i)$. By definition of $\mathbf{n}$, it is easy to see that edges within V_i have weights 1.

Conversely, suppose that $V(G) = V_1 \cup V_2 \cdots \cup V_k$, and (i), (ii), (iii) are satisfied. Put $D = \operatorname{diag}(d_{11}, \ldots, d_{nn})$, where $d_{uu} = \mathsf{w}_i$ if $u \in V_i$ for some i. Note that $(D^*L(G)D)_{uv} = \overline{d}_{uu}l_{uv}d_{vv}$. If $(u, v) \in E(G)$ has a weight 1, then (as the edges of weight 1 appear only inside a V_i) both $u, v \in V_i$, for some i, where $1 \leq i \leq k$. In that case $d_{uu} = d_{vv}$ and $l_{uv} = -1$ which implies $\overline{d}_{uu}l_{uv}d_{vv} = -1$. If $(u, v) \in E(G)$ has a weight other than 1, then $u \in V_i$, $v \in V_j$, for some i, j, $i \neq j$. In that case $\mathsf{w}_{uv} = \mathsf{w}_i\overline{\mathsf{w}}_j$, by (ii). Thus $\overline{d}_{uu}l_{uv}d_{vv} = \overline{\mathsf{w}}_i(-\mathsf{w}_i\overline{\mathsf{w}}_j)\mathsf{w}_j = -1$. Furthermore, $\overline{d}_{uu}l_{uu}d_{uu} = l_{uu}$. Since $D^*L(G)D$ is Hermitian, we see that $D^*L(G)D$ is the Laplacian matrix of the underlying unweighted undirected graph of G. Hence $L(G)$ is singular, by Lemma 2.1.5. ∎

Remark 2.1.10. Notice that if we have mixed graph in Lemma 2.1.9, then we have only two types of weights. Hence a connected mixed graph is singular if and only if there exist a partition $V(G) = V_1 \cup V_2$ such that edges inside V_i have weights 1 and edges between V_1 and V_2 have weights -1.

The following theorem which is a summary of the previous discussions and is a generalization of [4, Theorem 4].

Theorem 2.1.11. *Let G be a connected weighted directed graph. Then the following are equivalent.*

(a) $L(G)$ is singular.

(b) G is D-similar to the underlying unweighted undirected graph of G.

(c) Each cycle C in G has weight $\mathsf{w}_C = 1$.

(d) There exist a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that the following conditions are satisfied.

(i) There are distinct complex numbers w_i of unit modulus associated with each V_i, for $i = 1, \ldots, k$,

(ii) Any edge between V_i and V_j, $i < j$ is either directed from V_i to V_j with a weight $\mathsf{w}_i\overline{\mathsf{w}}_j \neq 1$ or is directed from V_j to V_i with a weight $\overline{\mathsf{w}}_i\mathsf{w}_j \neq 1$,

(iii) Each edge within V_i has a weight 1, for $i = 1\ldots, k$.

Proof. (a) $\Leftrightarrow$ (b). Follows from Lemma 2.1.5.

(b)$\Leftrightarrow$(c). Suppose that G is D-similar to the underlying unweighted undirected graph of G. Consider ${}^DG$ for this D. Note that if $(i,j) \in E(G)$ has a weight w_{ij}, then it has the weight $\overline{d}_{ii}\mathsf{w}_{ij}d_{jj}$ in ${}^DG$. So the weight of a cycle C in G remains the same in ${}^DG$. Note that each cycle in ${}^DG$ has weight 1. Hence the result holds.

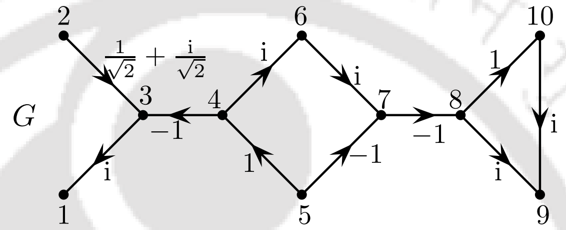
Conversely suppose that each cycle in G has weight equal to 1. Let T be a weighted directed spanning tree of G. Put $d_{11} = 1$ and for $i > 1$, $d_{ii} = \mathsf{w}_{P_i}$, where P_i is the unique i-1-path in T. Let $D = \text{diag}(d_{11}, \ldots, d_{nn})$. Consider the graph ${}^DG$ whose Laplacian matrix is $D^*L(G)D$. Take an edge $(i,j) \in E(G)$. If $(i,j) \in E(T)$, then $d_{jj} = \overline{\mathsf{w}}_{ij}d_{ii}$. In that case $(D^*L(G)D)_{ij} = \overline{d}_{ii}l_{ij}d_{jj} = -1$. Thus weight of (i,j) is 1 in ${}^DG$. If $(i,j) \in E(G) - E(T)$, then consider the cycle $C = P + (i,j)$ in G, where P is the unique i-j-path in T. Thus $\mathsf{w}_C = \overline{\mathsf{w}}_{ij}\mathsf{w}_P$. Observe that, weight of a cycle in G remains the same in ${}^DG$. Thus weight of (i,j) must be 1 in ${}^DG$, as w_P is equal to 1 in ${}^DG$ and $\mathsf{w}_C = 1$. Hence ${}^DG$ is the underlying unweighted undirected graph of G.

(b)$\Leftrightarrow$(d). Follows from Lemma 2.1.9. ∎

The following result is an immediate consequence.

Corollary 2.1.12. *Let G be a connected weighted directed graph. Then G is non-singular if and only if it contains a non-singular cycle. In particular, a weighted directed tree is always singular.*

Example 2.1.13. Consider G as in the following picture. Note that there are two cycles in G and both of them have weight 1. Hence the graph is singular. Indeed one can check that $\mathbf{n} = \begin{bmatrix}1 & -\frac{1}{\sqrt{2}}+\frac{\mathrm{i}}{\sqrt{2}} & \mathrm{i} & -\mathrm{i} & -\mathrm{i} & -1 & \mathrm{i} & -\mathrm{i} & -1 & -\mathrm{i}\end{bmatrix}^t$ is a null vector of $L(G)$.



Observe that in the above picture, if we take the directed edge $(9, 8)$ instead of $(8, 9)$, then the weight of the cycle $[8, 10, 9, 8]$ becomes -1. Hence by Corollary 2.1.12, the graph is non-singular.

Note that by Lemma 2.1.5, a connected weighted directed graph is singular if and only if it is D-similar to an unweighted undirected graph. The following is a natural question: which connected weighted directed graphs are D-similar to mixed graphs? Next result characterizes those graphs.

Theorem 2.1.14. *Let G be a connected weighted directed graph. Then G is D-similar to a mixed graph if and only if G does not contain a cycle of non-real weight.*

Proof. Suppose that G does not contain a cycle of non-real weight. Then each of the cycle contained in G has a weight ± 1, as the weights of the edges have absolute value 1. Let T be a weighted directed spanning tree of G. By Corollary 2.1.12, T is singular. By Lemma 2.1.5, there is a diagonal matrix D, such that ${}^{D}T$ is an unweighted undirected tree. Consider the graph ${}^{D}G$ for this D. Take an edge $(i, j) \in E(G)$. If $(i, j) \in E(T)$, then weight of (i, j) is equal to 1 in ${}^{D}G$. If $(i, j) \in E(G) - E(T)$, then consider the cycle $C = P + (i, j)$, where P is the unique

i-j-path in T. Since the edges in ${}^DG$ corresponding to the edges in P have weight 1, we see that w_P is equal to 1 in ${}^DG$. Observe that the weight of a cycle in G remains the same in ${}^DG$. Thus the weight of (i, j) is either 1 or -1 in ${}^DG$, as $\mathsf{w}_C = \pm 1$. Hence the ${}^DG$ is a mixed graph.

Conversely, suppose that G is D-similar to a mixed graph H. So $L(H) = D^*L(G)D$ and $H = {}^DG$. As the weight of a cycle is the same in both G and ${}^DG$, we see that the weights of the cycles are real. ■

2.2 Edge singularity of weighted directed graph

The edge singularity of mixed graphs was studied in [40]. We continue the same study in the context of weighted directed graphs.

Definition 2.2.1. The *edge singularity* $\varepsilon s(G)$ of a weighted directed graph is the minimum number of edges whose removal results a weighted directed graph containing no non-singular cycles or cycles of weight different from 1 (by Lemma 2.1.7). That is, all components of the resulting graph are singular.

The following result is very fundamental in nature and it relates the edge singularity with connectivity.

Lemma 2.2.2. *Let G be a connected weighted directed graph. Let F be a set of $\varepsilon s(G)$ edges in G such that $G - F$ does not contain a cycle of weight different from 1. Then $G - F$ is connected.*

Proof. If G is singular, then the result holds obviously. Suppose that G is non-singular and $G - F$ is disconnected. Let $G_1, G_2, \ldots G_r, (r \geq 2)$ be the components of $G - F$. As the graph G is connected, we can choose $r - 1$ edges $e_1, e_2, \ldots e_{r-1}$ from F such that the graph

$$H := G_1 \cup G_2 \cup \ldots G_r + \{e_1, e_2, \ldots e_{r-1}\}$$

is connected. So each edge $e_1, \ldots, e_{r-1}$ must be a bridge in H. By Corollary 2.1.12, as G_i's do not contain non-singular cycles, we see that H does not contain a non-singular cycle. Thus H is singular, by Corollary 2.1.12. Hence $\varepsilon s(G) \le |F|-(r-1) < |F|$, a contradiction. ■

The following result generalizes [40, Theorem 2.1] obtained by Tan and Fan for mixed graphs.

Lemma 2.2.3. *Let G be a connected weighted directed graph on n vertices and m edges. Then $0 \le \varepsilon s(G) \le m-n+1$. In particular, $\varepsilon s(G) = m-n+1$ if and only if all the cycles contained in G are non-singular.*

Proof. Clearly, $\varepsilon s(G) \ge 0$. Let T be a spanning tree of G. By Corollary 2.1.12, T is singular. Thus removal of the $m-n+1$ edges which are not in T from the graph G makes the resulting graph singular. Hence $\varepsilon s(G) \le m-n+1$.

Suppose that $\varepsilon s(G) = m-n+1$ and G contains a singular cycle C. Let H be the unicyclic spanning subgraph of G containing the cycle C. By Corollary 2.1.12, H is singular. Thus by deleting the $m-n$ edges from G we obtain a singular weighted directed graph. Hence $\varepsilon s(G) \le m-n < m-n+1$, a contradiction.

Conversely, suppose that each of the cycles contained in G are non-singular and $\varepsilon s(G) < m-n+1$. Let F be a set of $\varepsilon s(G)$ edges in G such that the graph $G-F$ has each component singular. By Lemma 2.2.2, $G-F$ is a connected graph and $|E(G-F)| = m-\varepsilon s(G) > n-1$. Thus $G-F$ contains a cycle, and by the assumption this cycle is non-singular, a contradiction. Hence the result holds. ■

We have two natural questions.

a) Given a non-negative integer k, is it possible to find a graph G with $\varepsilon s(G) = k$?

b) Given n, m and an integer $0 \le k \le m-n+1$, does there exist a graph G with n vertices and m edges for which $\varepsilon s(G) = k$?

The following example answers the first question in the affirmative.

Example 2.2.4. Let k be a given non-negative integer. Consider the weighted directed star H on $2k+1$ vertices with all the edges having a weight 1. Let $1, \ldots, 2k$ be the pendent vertices and v be the vertex of degree $2k$ on H. We construct the weighted directed graph G from H by inserting the new directed edges $(j, j+1)$ with an weight i, for $j = 1, 3, \ldots, 2k-1$. Notice that G contains k cycles of length 3 formed by the vertices v, j and $j+1$, for each $j = 1, 3, \ldots, 2k-1$. Let F be any set of edges in G with $|F| < k$. Then $G - F$ contains at least one cycle of the form $[v, j_0, j_0+1, v]$, for some $j_0 \in \{1, 3, \ldots, 2k-1\}$. Hence $\varepsilon s(G) = k$. For $k = 6$, our graph G is shown in figure 2.1.

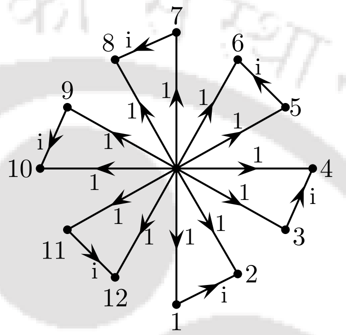
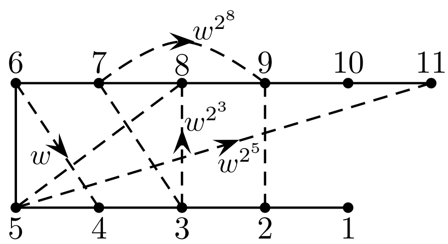


Figure 2.1: G with $\varepsilon s(G) = 6$

Let m, n be any given positive integers with $m \leq \binom{n}{2}$. Let $0 \leq k \leq m - n + 1$. Consider the weighted directed path P_n on n vertices with each edge having a weight 1. Let $w = e^{\mathrm{i}\frac{2\pi}{2^p}}$, where $p \geq k$. Construct a weighted directed graph obtained from P_n by inserting $m - n + 1$ new directed edges e_i. To k of these edges assign weights $w^{2^{r_i}}$, where $0 \leq r_i \leq p - 1$ are distinct, $i = 1, \ldots, k$. Assign a weight 1 to the remaining edges. Denote the class of all such graphs by $P(n; m; k)$.

Example 2.2.5. Here we give an example of a graph in $P(11; 17; 4)$. It is obtained from the path P_{11} by adding the dotted edges. We choose $p = 9$ and $w = e^{\mathrm{i}\frac{2\pi}{2^9}}$. The undirected edges have weights 1.



In the next result we prove that edge singularity of any graph in $P(n;m;k)$ is k, which answers the second question raised in this section in the affirmative.

Lemma 2.2.6. *Let $G \in P(n;m;k)$. Then $\varepsilon s(G) = k$.*

Proof. Let C be a cycle in G which contains l edges of weight different from 1. Then the weight of C is $\mathsf{w}_C = w^K$, where $K = \sum_{i=1}^{l} \pm 2^{r_i}$ and $0 \le r_i \le p-1$ are distinct for $i = 1, \ldots, l$. Since $0 < |K| < 2^p$ and $w = e^{\mathrm{i}\frac{2\pi}{2^p}}$, we see that $\mathsf{w}_C = w^K \neq 1$. Thus any cycle in G which contains an edge of weight different from 1 is non-singular by Lemma 2.1.7. We shall use induction to show that $\varepsilon s(G) = k$. For $k = 1$, G contains exactly one edge say, e having a weight $w^{2^{r_1}}$ and e must be contained in a non-singular cycle of G. Hence $\varepsilon s(G) = 1$. Assume that any graph $H \in P(n;m;k_0), k_0 < k$ has $\varepsilon s(H) = k_0$. Let $G \in P(n;m;k)$, $k > 1$. Let $F = \{e_1, \ldots, e_k\}$ be the set of edges in G such that e_i has a weight $w^{2^{r_i}}$, for $i = 1, \ldots, k$. Since each of the remaining $m-k$ edges in G has an weight 1, $G-F$ does not contain a non-singular cycle, by Theorem 2.1.11. Notice further that, $G-F$ is connected. Thus $\varepsilon s(G) \le k$. If possible, suppose that $\varepsilon s(G) < k$. Let F' be a set of edges in G such that $|F'| = \varepsilon s(G)$ and $G - F'$ does not contain a non-singular cycle.

Claim. $F' \cap F = \emptyset$. Suppose that our claim is not true. Put $r = |F \cap F'|$. Consider $G - (F \cap F')$. Observe that $\varepsilon s(G - (F \cap F')) \le \varepsilon s(G) - r < k - r$. But, as the graph $G - (F \cap F') \in P(n; m-r; k-r)$, by induction hypothesis, $\varepsilon s(G - (F \cap F')) = k - r$. This is a contradiction. Hence our claim is valid.

Recall that $G - F'$ does not contain a non-singular cycle. By the observation given in the beginning of the proof we see that each edge $e_i \in F$ must be a bridge in $G - F'$. As $|F| = k$, we see that $G - F' - F$ has at least $k+1$ components. On the other hand, as the graph $G - F$ is connected and as $|F'| < k$, the graph $G - F - F'$ can have at most k components. This is a contradiction. Hence $\varepsilon s(G) = k$. Our proof is complete. ∎

Remark 2.2.7. In Lemma 2.2.6, we only used the fact that the graphs in $P(n;m;k)$ are created from a connected graph. So the statement of the lemma will remain true for graphs in $T(n; m; k)$ which are created from a tree T in a similar way.

The graphs in $P(n; m; k)$ may be viewed as some graphs obtained from a connected undirected graph by adding k edges of weight different from 1. So a natural question is the following: is it true that each connected weighted directed graph G with $\varepsilon s(G) = k$ can be created from a connected unweighted undirected graph by adding k directed edges of weight different from 1?

The answer is in the affirmative as shown below.

Theorem 2.2.8. *Let G be a connected weighted directed graph with $\varepsilon s(G) = k$. Then G is D-similar to a graph H, obtained from the underlying unweighted graph of G by assigning weights different from 1 to some k edges.*

Proof. Let F be a set of edges in G such that $|F| = \varepsilon s(G)$ and $G - F$ has each component singular. By Lemma 2.2.2, the graph $G - F$ is connected. Let D be the diagonal matrix with i-th diagonal entry $d_{ii} = \mathbf{n}(i)$, where $\mathbf{n}$ is the null vector of $G - F$. By Lemma 2.1.5, $G - F$ is D-similar to the unweighted undirected graph $H_0 := {}^DG - {}^DF$, where ${}^DF$ is the set of edges in ${}^DG$ corresponding to F. Note that H_0 is connected as $G - F$ is connected. As $\varepsilon s(G) = \varepsilon s({}^DG)$, we see that edges in ${}^DF$ must have weights other than 1. Put $H={}^DG$. Then the graph G is D-similar to H which can be obtained from the connected unweighted graph H_0 by adding the k directed edges contained in ${}^DF$. ■

2.3 3-colored digraphs and their singularity

Recall that the class of 3-colored digraphs contains the mixed graphs but is a small subclass of the class of weighted directed graphs. In Section 2.1, we have given some characterizations of a singular connected weighted directed graphs. In this section we supply some additional characterizations of singularity of connected 3-colored digraphs. Further information on the structure of a singular connected 3-colored digraph is obtained.

Remark 2.3.1. In particular, if a 3-colored digraph G does not contain a green edge, then G is nothing but a mixed graph. In that case an edge with color red corresponds to a directed edge and an edge with color blue corresponds to an undirected edge.

The following theorem provides some additional information on the the structure of singular connected 3-colored digraphs in comparison to Lemma 2.1.9. It generalizes the result about the structure of a singular mixed graph obtained in [4].

Theorem 2.3.2. *Let G be a connected 3-colored digraph. Then $L(G)$ is singular if and only if there exist a partition $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ such that the following conditions are satisfied.*

(i) Edges between V_1 and V_3 are blue. Edges between V_2 and V_4 are blue,

(ii) Edges between V_i and V_{i+1} are green and are directed from V_i to V_{i+1}, for each $i \in \mathbb{Z}_4 = \{1, \ldots, 4\}$(with addition modulo 4),

(iii) Edges within V_i are red, $i \in \mathbb{Z}_4$. (See figure 2.2).

Proof. Suppose that $L(G)$ is singular. By Lemma 2.1.3, 0 is a simple eigenvalue and $\mathbf{n}$ is a null vector of $L(G)$. Note that entries of $\mathbf{n}$ are from $\{\pm 1, \pm \mathrm{i}\}$. Let V_1, V_2, V_3, V_4 be the set of those vertices of G which corresponds to the entries 1, $-\mathrm{i}$, -1 and i, respectively in $\mathbf{n}$. Let $u \in V_1$, $v \in V_3$ such that e is an edge in G with u and v as the end vertices. Since $\mathbf{n}(u) = 1$ and $\mathbf{n}(v) = -1$, we see that $\mathsf{w}_{uv} = -1$, by Lemma 2.1.3 and the definition of $\mathbf{n}$. Hence any edge connecting V_1 and V_3 must be blue. Similarly any edge connecting V_2 and V_4 must be blue. Similarly edges connecting V_i and V_{i+1} must be green, directed from V_i to V_{i+1}, for each $i \in \mathbb{Z}_4$. It is easy to see that edges within V_i must be red.

Conversely, suppose that $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, and (i), (ii), (iii) are satisfied. We associate the complex numbers $1, -\mathrm{i}, -1$ and i with V_1, V_2, V_3 and V_4, respectively. Then by Theorem 2.1.11, G is singular. ∎

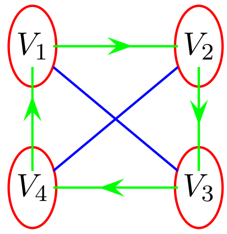
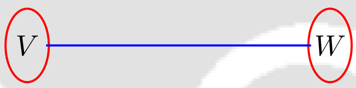


Figure 2.2: The structure of singular 3-colored digraph.

Remark 2.3.3. (a) Notice that in Theorem 2.3.2, some of the V_i's could be empty. For example, taking G an unweighted undirected graph, we have $V(G) = V_1$. Hence the structure of a connected singular 3-colored digraph naturally extends that of the unweighted undirected graph.

(b) Notice further that, as a mixed graph does not have green edges, the components V_2 and V_4 in Theorem 2.3.2 are empty. Hence the structure of a connected singular mixed graph is as shown in the following picture.



(c) Observe that if we consider all edges blue (edge weights -1), then $L(G)$ is the signless Laplacian. As we do not have red edges and green edges, we see the following well known result: *the signless Laplacian of a connected undirected graph is singular if and only if the graph is bipartite.*

Next result says that a connected singular 3-colored digraph is nothing but a 3-colored digraph with all edges red, up to D-similarity.

Theorem 2.3.4. *Let G be a connected 3-colored digraph. Then G is singular if and only if it is D-similar to the underlying unweighted undirected uncolored graph of G.*

Proof. Using Lemma 2.1.5 and the information about the entries of D, the proof easily follows. ■

Remark 2.3.5. Notice that in the case of mixed graphs we do not have green edges. Hence a singular mixed graph has a null vector $\mathbf{n}$ with entries ± 1. In that case the diagonal matrix D in Theorem 2.3.4 is nothing but a signature matrix. Thus Theorem 2.3.4 is a generalization of Theorem 4 (iii) of [4].

Let $C = [i_1, \ldots, i_k, i_1]$ be a cycle contained in a 3-colored digraph G. Let $n_b(C)$ denote the number of blue edges in C. Let $n_g^+(C)$ and $n_g^-(C)$ denote the number of green edges in C which are directed along the cycle and the number of green edges in C which are directed opposite to the cycle, respectively. The following result is crucial for another characterization of singularity for 3-colored digraphs which is done next.

Lemma 2.3.6. *Let G be a 3-colored digraph whose underlying undirected graph is a cycle C. Then G is singular if and only if*

(a) either $n_b(C)$ is even and $n_g^+(C) - n_g^-(C) \equiv 0 \pmod 4$, or

(b) $n_b(C)$ is odd and $n_g^+(C) - n_g^-(C) \equiv 2 \pmod 4$.

Proof. Using Lemma 2.1.7, $L(G)$ is singular if and only if

$$1 = \mathsf{w}_C = (-1)^{n_b(C)} \mathrm{i}^{n_g^+(C)} (-\mathrm{i})^{n_g^-(C)} = (-1)^{n_b(C)} \mathrm{i}^{n_g^+(C) - n_g^-(C)},$$

which implies the result. ∎

Remark 2.3.7. Note that Theorem 2.3.2, Theorem 2.3.4 and Lemma 2.3.6 together naturally generalizes [4, Theorem 4].

Next theorem gives a characterization of connected non-singular 3-colored digraphs.

Theorem 2.3.8. *Let G be a connected 3-colored digraph. Then G is non-singular if and only if G contains a cycle C satisfying one of the following conditions:*

(a) $n_g^+(C) - n_g^-(C) \equiv 1 \pmod 2$,

(b) $n_b(C)$ is even and $n_g^+(C) - n_g^-(C) \equiv 2 \pmod 4$, or

(c) $n_b(C)$ is odd and $n_g^+(C) - n_g^-(C) \equiv 0 \pmod 4$.

Proof. Suppose that G is non-singular. By Corollary 2.1.12, G contains a non-singular cycle, say C. Hence by Lemma 2.3.6, it follows that the cycle C satisfies one of the conditions (a), (b) or (c).

Conversely, suppose that G contains a cycle C satisfying one of the conditions (a), (b) or (c). Then by Lemma 2.3.6, the cycle C is non-singular. Hence G is non-singular, by Corollary 2.1.12. ∎

Remark 2.3.9. Notice that in the case of mixed graphs we do not have green edges. Hence a mixed graph whose underlying undirected graph is a cycle is non-singular if and only if $n_b(C)$ is odd. Thus in view of remark 2.3.1, Theorem 2.3.8 is a generalization of Lemma 1 of [4].

2.4 Determinant of the Laplacian matrix of weighted directed graph

In this section we describe the determinant of the Laplacian matrix of a weighted directed graph. The following lemma gives the determinant of the Laplacian matrix of a cycle in a weighted directed graph.

Lemma 2.4.1. *Let C be a weighted directed graph whose underlying unweighted undirected graph is a cycle. Then* $\det(L(C)) = 2(1 - \mathsf{Re}\,\mathsf{w}_C)$.

Proof. Consider the vertex edge incident matrix $M(C)$ corresponding to C. We may assume $C = [1, 2, \ldots, n, 1]$ such that the edges e_i in C has end vertices i and $i + 1$, for $i \in \mathbb{Z}_n$ and $m_{1,e_1} = 1$, after a relabelling of the vertices if necessary. Note that the nonzero entries of $M(C)$ occurs precisely at the positions m_{i,e_i} and m_{i+1,e_i} for each i. In that case expanding along the first row of $M(C)$, we see that

$$\det(M(C)) = \prod_{\substack{(i+1,i)\in E(C)\\ i\in\mathbb{Z}_n}} (-\overline{\mathsf{w}}_{i+1,i}) - (-1)^n \prod_{\substack{(i,i+1)\in E(C)\\ i\in\mathbb{Z}_k}} (-\overline{\mathsf{w}}_{i,i+1})$$

Since $L(C) = M(C)M(C)^*$, we see that $\det(L(C)) = 2(1 - \mathsf{Re}\,\mathsf{w}_C)$. ∎

Next lemma gives the determinant of the Laplacian matrix of a unicyclic weighted directed graph.

Lemma 2.4.2. *Let G be a connected unicyclic weighted directed graph with the cycle C. Then* $\det(L(G)) = 2(1 - \mathsf{Re}\,\mathsf{w}_{\mathsf{C}})$.

Proof. If G is the cycle C itself then the result follows immediately from Lemma 2.4.1. Otherwise, G has a pendent vertex say i. Let j be the vertex adjacent to i in G with an edge e of weight w. We may assume, after a permutation similarity that the first row and the first column of $M(G)$ correspond to the vertex i and the edge e, respectively. Then expanding along the first row, we see that if $e = (i, j)$ then $\det(M(G)) = \det(M(G'))$, otherwise $\det(M(G)) = (-\overline{\mathsf{w}})\det(M(G'))$, where G' is the weighted directed graph obtained from G by deleting the vertex i. Hence in any case $\det(L(G)) = \det(L(G'))$. Continuing similarly, after finitely many steps we see that $\det(L(G)) = \det(L(C))$. Hence the result holds, by Lemma 2.4.1. ∎

Definition 2.4.3. Let G be a connected non-singular weighted directed graph. We call a subgraph H an *essential spanning subgraph* of G if $V(G) = V(H)$ and every component of H is a non-singular unicyclic weighted directed graph. By $\mathcal{E}(G)$ we denote the class of all essential spanning subgraphs of G.

Next result describes the determinant of the Laplacian matrix of a weighted directed graph in terms of the determinants of its essential spanning subgraphs.

Lemma 2.4.4. *Let G be a connected weighted directed graph. Then* $\det(L(G)) = \sum_{H\in\mathcal{E}(G)} \det L(H)$.

Proof. Since $L(G) = MM^*$, by Cauchy-Binet Theorem we know that

$$\det L(G) = \sum_{\substack{E'\subseteq E\\ |E'|=n}} \det M[V, E'] \det M[V, E']^*,$$

where $M[V, E']$ is a square submatrix of M. Note that $M[V, E']$ is the vertex edge incident matrix of the spanning subgraph say $H_{E'}$ with the edge set $E' \subseteq E$. Thus $L(H_{E'}) = M[V, E']M[V, E']^*$. Note that $\det L(H_{E'}) \neq 0$ if and only if each component of $H_{E'}$ is non-singular. Thus $\det L(H'_E) \neq 0$ if and only if $H_{E'} \in \mathcal{E}(G)$, as $|V| = |E'|$. Hence the result holds. ∎

Remark 2.4.5. $L(G)$ is non-singular if and only if G contains a non-singular cycle.

▶ Let G be a weighted directed graph and let H be an essential spanning subgraph of G. We denote the number of components of H by $\omega(H)$ and a cycle contained in H by $C_i(H)$, for $1 \leq i \leq \omega(H)$.

Next, we give our main result of this section, which generalizes [4, Corollary 2].

Theorem 2.4.6. *Let G be a connected non-singular weighted directed graph. Then* $\det(L(G)) = \sum_{H \in \mathcal{E}(G)} 2^{\omega(H)} \prod_{i=1}^{\omega(H)} (1 - \mathsf{Re}\,\mathsf{w}_{C_i(H)})$.

Proof. Proof follows from Lemma 2.4.4 and Lemma 2.4.2. ∎

The following result is an immediate consequence.

Corollary 2.4.7. *Let G be a connected 3-colored digraph. Then*

$$\det L(G) = \sum_{H \in \mathcal{E}(G)} 2^{2\omega_1(H)+\omega_2(H)},$$

where $\omega_1(H)$ and $\omega_2(H)$ denotes the number of cycles of weight -1 and $\pm \mathrm{i}$ in H, respectively.

Chapter 3

Spectra of 3-colored Digraphs

In this chapter we discuss the adjacency and Laplacian spectra of 3-colored digraphs. In Section 3.1, we study the realizability of the adjacency (resp. Laplacian) spectrum of a 3-colored digraph as a subset of the adjacency (resp. Laplacian) spectrum of a mixed graph. Using a graph operation (on 3-colored digraphs) similar to that in [17] we show that given a connected 3-colored digraph G on n vertices, there is a mixed graph $\mathsf{G}[\mathsf{g}]$ on $2n$ vertices, which satisfy both these requirements simultaneously.

In Section 3.2 we study the realizability of the adjacency (resp. Laplacian) spectrum of a mixed graph as a subset of the adjacency (resp. Laplacian) spectrum of an unweighted undirected graph. Note that some study on the Laplacian spectrum has been done by Zhang and Luo[42] and Fan [17, 18, 19, 20]. Using the graph operation (on mixed graphs) given in [17] we show that given a connected mixed graph G on n vertices, there is an unweighted undirected graph $\mathsf{G}[\mathsf{b}]$ on $2n$ vertices, which satisfy both these requirements. We establish a relationship between the singularity of $L(G)$ and the connectedness of the graph $\mathsf{G}[\mathsf{b}]$. We give complete characterization of the adjacency and Laplacian spectrum of $\mathsf{G}[\mathsf{b}]$. Denote by $\lambda_i(B)$ the ith smallest eigenvalue of a Hermitian matrix B. A family of mixed graphs G for which $L(G)$ is non-singular and $\lambda_1(L(G)) = \lambda_2(L(\mathsf{G}[\mathsf{b}]))$ was provided by Fan [17]. A larger family was supplied by Tan and Fan[40]. We provide a more general class of such mixed graphs.

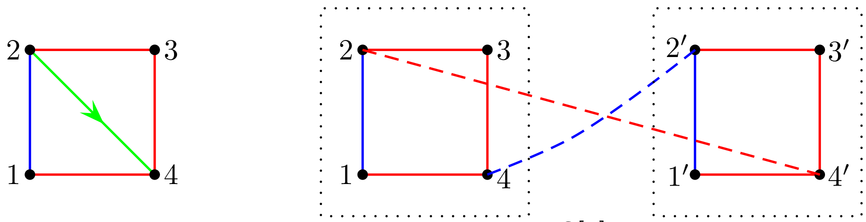
Combining the results of sections 3.1 and 3.2, we see that given a 3-colored digraph G, there is an unweighted undirected graph H such that the adjacency (resp. Laplacian) spectrum of H contains the adjacency (resp. Laplacian) spectrum of G. We observe that the graph H may also be viewed as the result of a special case of a graph operation on unweighted undirected graphs which we introduce in Section 3.3. This graph operation is more general than those discussed in sections 3.1 and 3.2 and is a new of its kind. We give a complete characterization of the adjacency (resp. Laplacian) spectrum of the graph resulting from such operation.

3.1 Mixed Graphs whose spectrum contains the spectrum of a given 3-colored digraph

In this section we ask the following question: Let G be a 3-colored digraph. Is it possible to find a mixed graph H_1 whose adjacency spectrum is the same as that of G, not considering multiplicities? Is it possible to find a mixed graph H_2 whose Laplacian spectrum is the same as that of G, not considering multiplicities? In case both the answers are 'yes', is it possible to have $H_1 = H_2$? In this section we show that the answers to all these questions are in the affirmative.

Definition 3.1.1. Let G be a 3-colored digraph on vertices $1, \ldots, n$ and let F_g be the set of green edges in G. Note that $G - F_g$ is a mixed graph. Let G' be a copy of G, in which we replace the label of the vertex i by i', for each $i = 1, \ldots, n$. Let F_g' denote the green edges in G' corresponding to F_g. Construct the mixed graph on $2n$ vertices obtained from $(G - F_g) \cup (G' - F_g')$ by inserting a red edge uv' and a blue edge $u'v$, for each green edge $(u, v) \in F_g$. We denote this graph by $\mathsf{G[g]}$. We call an edge e of $\mathsf{G[g]}$ a *pivotal* edge if one end vertex of e is from $\{1, \ldots, n\}$ and the other is from $\{1', \ldots, n'\}$.

Example 3.1.2. Let G be a 3-colored digraph with the green edge $(2, 4)$. The mixed graph $\mathsf{G[g]}$ is shown in the following picture.



G $\mathsf{G[g]}$

For two disjoint subsets of vertices U and V of a graph G, let $E(U,V)$ denote the set of edges in G with one end in U and the other end in V.

Lemma 3.1.3. *Let G be a connected 3-colored digraph such that $F_g \neq \emptyset$ and that G does not contain a cycle of weight $\pm\mathrm{i}$. Then $G - F_g$ is disconnected. Furthermore, there is a partition $V(G) = V_1 \cup V_2$ such that $E(V_1, V_2) = F_g$.*

Proof. If possible, suppose that $G - F_g$ is connected. Let $e \in F_g$. Then the graph $G - F_g + e$ has a cycle, say C which contains the green edge e. Thus we get a cycle in G of weight $\pm\mathrm{i}$, which is a contradiction.

Let $G_1, \ldots, G_k$ be the connected components of $G - F_g$. Note that there cannot be any green edge between two vertices of G_1, otherwise we get a cycle of weight $\pm\mathrm{i}$ in G.

Consider the graph H obtained from $G - F_g$ by compressing G_i to a vertex v_i, adding an edge between v_i and v_j if there is a green edge between a vertex of G_i and a vertex of G_j in G. The graph H cannot have an odd cycle, otherwise we would get a cycle in G of weight $\pm\mathrm{i}$. Hence H is bipartite, say $V(H) = W_1 \cup W_2$.

Put $V_1 = \bigcup\limits_{v_i \in W_1} V(G_i)$ and $V_2 = \bigcup\limits_{v_i \in W_2} V(G_i)$. Then $E(V_1, V_2) = F_g$. ■

Remark 3.1.4. The converse of the first part of Lemma 3.1.3 is not true in view of a cycle $C = [1, 2, 3, 1]$ with all edges green.

The following are some crucial observations.

Lemma 3.1.5. *Let G be a connected 3-colored digraph and P be a 1-u-path in G. Then the following statements hold.*

(a) If P has an odd number of green edges, then there is a 1-u'-path $P_{1u'}$ in $\mathsf{G}[\mathsf{g}]$ with $\mathsf{w}_{P_{1u'}} = -\mathrm{i}\mathsf{w}_P$. Also there is a $1'$-u-path $P_{1'u}$ in $\mathsf{G}[\mathsf{g}]$ with $\mathsf{w}_{P_{1'u}} = \mathrm{i}\mathsf{w}_P$.

(b) If P has an even number of green edges, then we have a 1-u-path P_{1u} and a $1'$-u'-path $P_{1'u'}$ in $\mathsf{G}[\mathsf{g}]$ with $\mathsf{w}_{P_{1u}} = \mathsf{w}_{P_{1'u'}} = \mathsf{w}_P$.

Proof. (a) Let e_i be the green edges in P with end vertices u_i and v_i, for $i = 1, \ldots, 2l+1$. For vertices x, y on P, let $P(x,y)$ denote the subpath from x to y. Let

G' denote the graph as defined in Definition 3.1.1. By $P'(x,y)$ we mean the path in G' corresponding to $P(x,y)$. Then the path P may be viewed as

$$P = P(1,u_1) + e_1 + P(v_1,u_2) + e_2 + \cdots + e_{2l+1} + P(v_{2l+1},u).$$

Then

$$P_{1u'} = P(1,u_1) + u_1v_1' + P'(v_1,u_2) + u_2'v_2 + \cdots + u_{2l+1}v_{2l+1}' + P'(v_{2l+1},u)$$

is a 1-u'-path in $\mathsf{G}[\mathsf{g}]$ and

$$P_{1'u} = P'(1,u_1) + u_1'v_1 + P(v_1,u_2) + u_2v_2' + \cdots + u_{2l+1}'v_{2l+1} + P(v_{2l+1},u)$$

is a $1'$-u-path in $\mathsf{G}[\mathsf{g}]$.

To prove the second part of (a), consider the finite sequence $e_1, e_2, \ldots, e_{2l+1}$ of the green edges in P. Notice that the pivotal edge in $P_{1u'}$ corresponding to e_j has color red if and only if either j is odd and e_j is directed along P or j is even and e_j is directed opposite to P. Hence

$$\mathsf{w}_{P_{1u'}} = \mathsf{w}_P \frac{1}{\mathrm{i}^{l+1}} \frac{1}{(-\mathrm{i})^l},$$

where the middle and last terms in the RHS account for all changes required due to e_j's with j odd and even, respectively. Hence $\mathsf{w}_{P_{1,u'}} = -\mathrm{i}\mathsf{w}_P$. Similarly, we can show that $\mathsf{w}_{P_{1',u}} = \mathrm{i}\mathsf{w}_P$.

(b) Proof is similar to that of part (a). ∎

Theorem 3.1.6. *Let G be a connected 3-colored digraph. Then G contains a cycle of weight $\pm\mathrm{i}$ if and only if the mixed graph $\mathsf{G}[\mathsf{g}]$ is connected.*

Proof. Suppose that G contains a cycle $C = [1,2,\ldots,k,1]$ of weight $\mathsf{w}_C = \pm\mathrm{i}$. Clearly C contains an odd number of green edges. Hence by Lemma 3.1.5(a), there is a 1-$1'$-path in $\mathsf{G}[\mathsf{g}]$. Let u be any vertex in G. Since G is connected, there is a 1-u-path in G. Hence by Lemma 3.1.5, there is a path from u either to 1 or to $1'$. Moreover, there is also a path from u' either to 1 or to $1'$. Hence $\mathsf{G}[\mathsf{g}]$ is connected.

Conversely, suppose that G does not contain a cycle of weight $\pm\mathrm{i}$. By Lemma 3.1.3, there is a partition $V(G) = V_1 \cup V_2$ such that $E(V_1, V_2) = F_g$. Then in $\mathsf{G}[\mathsf{g}]$ there cannot be a path from a vertex of V_1 to any vertex of V_1'.

An alternate way to see the converse is the following. Suppose that $\mathsf{G}[\mathsf{g}]$ is connected. Take a 1-1$'$-path, say P in $\mathsf{G}[\mathsf{g}]$. Note that there are an odd number of pivotal edges on P. The existence of a pivotal edge ij' or $i'j$ in $\mathsf{G}[\mathsf{g}]$ implies that either the edge $(i,j) \in F_g$ or $(j,i) \in F_g$. Hence P yields a 1-1-walk in G containing an odd number of green edges. So there must be a cycle in G containing an odd number of green edges, implying that G has a cycle of weight $\pm\mathrm{i}$. ∎

Corollary 3.1.7. *Let G be a connected 3-colored digraph. Then G is D-similar to a mixed graph if and only if $\mathsf{G}[\mathsf{g}]$ is disconnected.*

Proof. Follows from Theorem 2.1.14 and Theorem 3.1.6. ∎

Definition 3.1.8. Let G be a 3-colored digraph. Recall that F_g denotes the set of green edges in G. We define $B_g = [b_{ij}]$ to be the $n \times n$ matrix with

$$b_{ij} = \begin{cases} 1 & \text{if } (i,j) \in F_g, \\ -1 & \text{if } (j,i) \in F_g, \\ 0 & \text{otherwise.} \end{cases}$$

By D_g we denote the diagonal matrix with i-th diagonal entry equal to the number of green edges incident with the vertex i in G.

The proof of the next result follows from the construction of $\mathsf{G}[\mathsf{g}]$.

Proposition 3.1.9. *Let G be a 3-colored digraph. Then*

$$\begin{aligned} A(\mathsf{G}[\mathsf{g}]) &= I_2 \otimes A(G - F_g) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes B_g, \\ A(G) &= A(G - F_g) + \mathrm{i}B_g, \\ L(\mathsf{G}[\mathsf{g}]) &= I_2 \otimes L(G - F_g) + I_2 \otimes D_g - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes B_g, \\ L(G) &= L(G - F_g) + D_g - \mathrm{i}B_g. \end{aligned}$$

Next, we relate the Laplacian (resp. adjacency) spectrum of G with that of $\mathsf{G}[\mathsf{g}]$. By $\mathsf{Re}\,x, \mathsf{Im}\,x$ we mean the real part and the imaginary part of a vector x, respectively.

Lemma 3.1.10. *Let G be a 3-colored digraph and λ be a Laplacian (resp. adjacency) eigenvalue of G with an eigenvector x. Then λ is a Laplacian (resp. adjacency) eigenvalue of* $\mathsf{G}[\mathsf{g}]$ *with linearly independent eigenvectors* $\begin{bmatrix} \mathsf{Re}\, x \\ -\mathsf{Im}\, x \end{bmatrix}, \begin{bmatrix} \mathsf{Im}\, x \\ \mathsf{Re}\, x \end{bmatrix}$.

Proof. We prove the statement only for $L(\mathsf{G}[\mathsf{g}])$. The proof of the statement for $A(\mathsf{G}[\mathsf{g}])$ is similar. Put $y = \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}$. Observe that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y = \mathrm{i}y$. Hence

$$\begin{aligned} L(\mathsf{G}[\mathsf{g}])(y \otimes x) &= y \otimes L(G - F_g)x + y \otimes D_g x - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y \otimes B_g x \\ &= y \otimes [L(G - F_g) + D_g - \mathrm{i}B_g]x \\ &= y \otimes L(G)x \qquad \text{using Proposition (3.1.9)} \\ &= y \otimes \lambda x \\ &= \lambda(y \otimes x). \end{aligned}$$

So $L(\mathsf{G}[\mathsf{g}]) \begin{bmatrix} x \\ \mathrm{i}x \end{bmatrix} = \lambda \begin{bmatrix} x \\ \mathrm{i}x \end{bmatrix}$. Hence $\begin{bmatrix} \mathsf{Re}\, x \\ -\mathsf{Im}\, x \end{bmatrix}$ and $\begin{bmatrix} \mathsf{Im}\, x \\ \mathsf{Re}\, x \end{bmatrix}$ are linearly independent eigenvectors of $L(\mathsf{G}[\mathsf{g}])$ corresponding to λ, as $L(\mathsf{G}[\mathsf{g}])$ is real. ∎

The following result whose proof follows immediately from Lemma 3.1.10 is the main result of this section. This answers the questions raised in the beginning of this section.

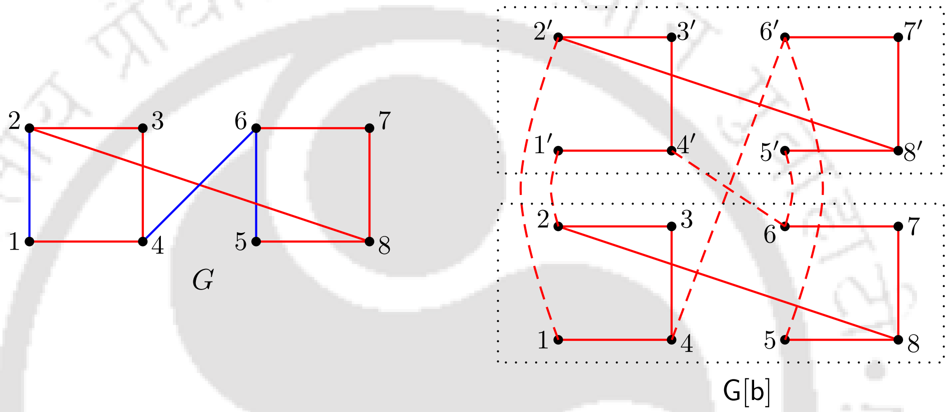
Theorem 3.1.11. *Let G be a 3-colored digraph. Then the adjacency (resp. Laplacian) spectrum of G, without multiplicities, are the same as that of* $\mathsf{G}[\mathsf{g}]$.

3.2 Realizability of the spectrum of a mixed graph by an unweighted undirected graph.

Let us ask the following question: Let G be a mixed graph. Is it possible to find an unweighted undirected graph H_1 whose adjacency spectrum contains the the adjacency spectrum of G. Is it possible to find an unweighted undirected graph H_2 whose Laplacian spectrum contains the Laplacian spectrum of G? In case both the answers are 'yes', is it possible to have $H_1 = H_2$?

Definition 3.2.1. [17] Let G be a mixed graph on vertices $1, \ldots, n$ and F_b be the set of blue edges in G. Let G' be a copy of G, in which we replace the label of the vertex i by i', for each $i = 1, \ldots, n$. Let F'_b be the set of blue edges in G' corresponding to F_b. Construct the unweighted undirected graph on $2n$ vertices obtained from $(G - F_b) \cup (G' - F_b')$ by inserting the red edges uv' and $u'v$, for each blue edge $e = uv \in F_b$. We denote this graph by $\mathsf{G[b]}$. We call an edge e of $\mathsf{G[b]}$ a *pivotal* edge if one end vertex of e is from $\{1, \ldots, n\}$ and the other is from $\{1', \ldots, n'\}$.

Example 3.2.2. Let G be a mixed graph with $F_b = \{12, 46, 56\}$. The graph $\mathsf{G[b]}$ is as shown in the following figure.



Fan [17] has proved that for a connected mixed graph G with exactly one non-singular cycle, $\lambda_1(L(G)) = \lambda_2(L(\mathsf{G[b]}))$. We note that the statement holds even if connectedness is not assumed. In this section we first show that much more is true. Indeed, for a mixed graph G, the adjacency (resp. Laplacian) spectrum of G is contained in that of $\mathsf{G[b]}$. We supply a combinatorial argument to show that for a connected mixed graph G, the non-singularity of $L(G)$ is equivalent to the connectedness of $\mathsf{G[b]}$. Later on we characterize the remaining adjacency (resp. Laplacian) eigenvalues of $\mathsf{G[b]}$. Tan and Fan [40] showed that a mixed graph G with $\varepsilon s(G) = 1$ satisfies $\lambda_1(L(G)) = \lambda_2(L(\mathsf{G[b]}))$. Observe that if G is a mixed graph with exactly one non-singular cycle, then G is D-similar to a mixed graph H with $\varepsilon s(H) = 1$. Hence the class of mixed graphs with $\varepsilon s(G) = 1$ is a larger class than the mixed graphs with exactly one non-singular cycle. Here we supply a further larger class of mixed graphs G satisfying $\lambda_1(L(G)) = \lambda_2(L(\mathsf{G[b]}))$.

The following result is essentially contained in [4, Theorem 4].

Lemma 3.2.3. *Let G be a connected mixed graph with $F_b \neq \emptyset$. If $L(G)$ is singular, then $G - F_b$ is disconnected. Furthermore, there is partition $V(G) = V_1 \cup V_2$ such that $E(V_1, V_2) = F_b$.*

The following are some crucial observations.

Lemma 3.2.4. *Let G be a connected mixed graph and P be a 1-u-path in G. Then the following statements hold.*

(a) If P contains an odd number of blue edges, then we have a 1-u'-path $P_{1u'}$ and a $1'$-u-path $P_{1'u}$ in $\mathsf{G}[\mathsf{b}]$.

(b) If P contains an even number of blue edges, then we have a 1-u-path P_{1u} and a $1'$-u'-path $P_{1'u'}$ in $\mathsf{G}[\mathsf{b}]$.

Proof. The proof is similar to the proof of Lemma 3.1.5 ∎

Next, we give a characterization of the non-singular mixed graphs.

Theorem 3.2.5. *Let G be a connected mixed graph. Then $L(G)$ is non-singular if and only if $\mathsf{G}[\mathsf{b}]$ is connected.*

Proof. Assume that $L(G)$ is non-singular. Then G contains a non-singular cycle, say $C = [1, \ldots, k, 1]$. As C contains an odd number of blues edges, by Lemma 3.2.4(b), there is 1-$1'$-path in $\mathsf{G}[\mathsf{b}]$. Let u be any vertex in G. Since G is connected, there is a 1-u-path from 1 to u. Again by Lemma 3.2.4, there is a path from u either to 1 or to $1'$. Moreover, there is also a path from u' either to 1 or to $1'$. Hence $\mathsf{G}[\mathsf{b}]$ is connected.

Conversely, suppose that G does not contain a cycle of weight -1. By Lemma 3.2.3, there is a partition $V(G) = V_1 \cup V_2$ such that $E(V_1, V_2) = F_b$. Then in $\mathsf{G}[\mathsf{g}]$ there cannot be a path from a vertex of V_1 to any vertex of V_1'.

An alternate way to see the converse is the following. Suppose that $\mathsf{G}[\mathsf{b}]$ is connected. Take a 1-$1'$-path P in $\mathsf{G}[\mathsf{b}]$. The path P can only contain an odd number of pivotal edges due to the structure of $\mathsf{G}[\mathsf{b}]$. The existence of a pivotal edge ij' or $i'j$ in $\mathsf{G}[\mathsf{b}]$ implies that the edge ij in G has color blue. Hence P describes a 1-1-walk in G containing an odd number of blue edges. Thus there must be a cycle in G containing an odd number of blue edges, implying that G has a cycle of weight -1. So $L(G)$ is non-singular. ∎

Definition 3.2.6. Let G be a mixed graph on vertices $1, \ldots, n$. Recall that F_b denotes the set of blue edges in G. We define $B_b = [c_{ij}]$ to be the $n \times n$ matrix with $c_{ij} = c_{ji} = 1$ if $ij \in F_b$, and 0 otherwise. By D_b we denote the diagonal matrix with i-th diagonal entry equal to the number of blue edges incident with the vertex i in G.

The proof of the next result follows from the construction of $\mathsf{G}[\mathsf{b}]$.

Proposition 3.2.7. *Let G be a mixed graph. Then*

$$\begin{aligned} A(\mathsf{G}[\mathsf{b}]) &= I_2 \otimes A(G - F_b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes B_b, \\ A(G) &= A(G - F_b) - B_b, \\ L(\mathsf{G}[\mathsf{b}]) &= I_2 \otimes L(G - F_b) + I_2 \otimes D_b - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes B_b, \\ L(G) &= L(G - F_b) + D_b + B_b. \end{aligned}$$

Next, we relate the Laplacian (resp. adjacency) spectrum of the mixed graph G with that of $\mathsf{G}[\mathsf{b}]$.

Lemma 3.2.8. *Let G be a mixed graph and λ be a Laplacian (resp. adjacency) eigenvalue of G with an eigenvector x. Then λ is a Laplacian (resp. adjacency) eigenvalue of $\mathsf{G}[\mathsf{b}]$ with an eigenvector $\begin{bmatrix} x \\ -x \end{bmatrix}$.*

Proof. We prove the statement only for $L(\mathsf{G}[\mathsf{b}])$. The proof of the statement for $A(\mathsf{G}[\mathsf{b}])$ is similar. Put $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Observe that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y = -y$. Hence

$$\begin{aligned} L(\mathsf{G}[\mathsf{b}])(y \otimes x) = &\; y \otimes L(G - F_b)x + y \otimes D_b x - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y \otimes B_b x \\ = &\; y \otimes [L(G - F_b) + D_b + B_b]x \\ = &\; y \otimes L(G)x, \qquad \text{using Proposition 3.2.7} \\ = &\; y \otimes \lambda x \\ = &\; \lambda(y \otimes x). \end{aligned}$$

So $L(\mathsf{G[b]})\begin{bmatrix} x \\ -x \end{bmatrix} = \lambda \begin{bmatrix} x \\ -x \end{bmatrix}$. ∎

The following result whose proof follows immediately from Lemma 3.2.8 is the main result of this section. This answers the questions raised in the beginning of this section.

Theorem 3.2.9. *Let G be a mixed graph. Then the adjacency (resp. Laplacian) spectrum of the unweighted undirected graph* $\mathsf{G[b]}$ *contains the adjacency (resp. Laplacian) spectrum of G.*

In view of Theorem 3.2.9, we have the following natural question: Does there exist an unweighted undirected graph whose adjacency (resp. Laplacian) eigenvalues are the remaining n adjacency (resp. Laplacian) eigenvalues of $\mathsf{G[b]}$?

Definition 3.2.10. Let G be a connected mixed graph. By $\mathsf{G_b}^{\Rsh\mathsf{r}}$ we mean the unweighted undirected graph obtained from G, changing the color of each blue edge in G to red.

Using the same notations as in Definition 3.2.6 we observe that

$$\begin{aligned} A(\mathsf{G_b}^{\Rsh\mathsf{r}}) &= A(G - F_b) + B_b \\ D(\mathsf{G_b}^{\Rsh\mathsf{r}}) &= D(G) \\ L(\mathsf{G_b}^{\Rsh\mathsf{r}}) &= L(G - F_b) + D_b - B_b. \end{aligned} \tag{3.2.1}$$

Next, we show that the adjacency (resp. Laplacian) eigenvalues of $\mathsf{G_b}^{\Rsh\mathsf{r}}$ are indeed the adjacency (resp. Laplacian) eigenvalues of $\mathsf{G[b]}$.

Lemma 3.2.11. *Let G be a mixed graph and λ be a Laplacian (resp. adjacency) eigenvalue of* $\mathsf{G_b}^{\Rsh\mathsf{r}}$ *with an eigenvector x. Then λ is a Laplacian (resp. adjacency) eigenvalue of* $\mathsf{G[b]}$ *with an eigenvector* $\begin{bmatrix} x \\ x \end{bmatrix}$.

Proof. We prove the statement only for $L(\mathsf{G_b}^{\Rsh\mathsf{r}})$. The proof of the statement for $A(\mathsf{G_b}^{\Rsh\mathsf{r}})$ is similar. Put $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Observe that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y = y$. Hence

$$\begin{aligned} L(\mathsf{G[b]})(y \otimes x) &= y \otimes [L(G - F_b) + D_b - B_b]x \\ &= y \otimes L(\mathsf{G_b}^{\Rsh\mathsf{r}})x, \qquad \text{using Proposition 3.2.7 and equation (3.2.1)} \\ &= y \otimes \lambda x \\ &= \lambda(y \otimes x). \end{aligned}$$

So $L(\mathsf{G}[\mathsf{b}])\begin{bmatrix} x \\ x \end{bmatrix} = \lambda \begin{bmatrix} x \\ x \end{bmatrix}$. ∎

Using Theorem 3.2.9 and Lemma 3.2.11 we obtain the following result which is one of the main results of this section. This also answers our earlier question.

Theorem 3.2.12. *Let G be a connected mixed graph. Then the Laplacian (resp. adjacency) spectrum of $\mathsf{G}[\mathsf{b}]$ is the union of the Laplacian (resp. adjacency) spectra of G and $\mathsf{G_b}^{\Rsh\mathsf{r}}$.*

The second smallest Laplacian eigenvalue $\lambda_2(L(G))$ of an undirected graph G is popularly known as the algebraic connectivity of G, denoted by $a(G)$. The following is an immediate corollary.

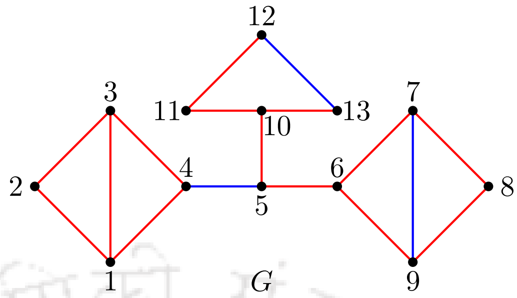
Corollary 3.2.13. *Let G be a non-singular connected mixed graph. Then $a(\mathsf{G}[\mathsf{b}]) = \min\{\lambda_1(L(G)), a(\mathsf{G_b}^{\Rsh\mathsf{r}})\}$.*

It is natural to ask for a characterization of non-singular mixed graphs G such that $a(\mathsf{G}[\mathsf{b}]) = \lambda_1(L(G))$. Fan[17], Tan and Fan[40] have provided some class of such non-singular mixed graphs. In the next theorem, we provide a more general class (than that of Fan, Tan and Fan) of non-singular mixed graphs G with $a(\mathsf{G}[\mathsf{b}]) = \lambda_1(L(G))$.

Theorem 3.2.14. *Let G be a non-singular connected mixed graph such that $a(\mathsf{G_b}^{\Rsh\mathsf{r}})$ has multiplicity k. Let $W \subset V(G)$ be such that $0 < |W| \leq k$ and that $G - W$ has all components singular. Then $a(\mathsf{G}[\mathsf{b}]) = \lambda_1(L(G))$.*

Proof. Let L_W and L'_W be the principal submatrices of $L(G)$ and $L(\mathsf{G_b}^{\Rsh\mathsf{r}})$ corresponding to the graphs $G - W$ and $\mathsf{G_b}^{\Rsh\mathsf{r}} - W$, respectively. By Theorem 2.3.4, L_W is D-similar to L'_W. By interlacing theorem, $\lambda_1(L(G)) \leq \lambda_1(L_W) = \lambda_1(L'_W) \leq a(\mathsf{G_b}^{\Rsh\mathsf{r}})$. Hence $a(\mathsf{G}[\mathsf{b}]) = \lambda_1(L(G))$, by Corollary 3.2.13. ∎

Example 3.2.15. Consider G as shown below. Note that the $a(\mathsf{G_b}^{\Rsh\mathsf{r}})$ has multiplicity 2 and $G - W$ is singular, where $W = \{9, 13\}$. Thus $a(\mathsf{G}[\mathsf{b}]) = \lambda_1(L(G))$.



Corollary 3.2.16. *Let G be a connected non-singular mixed graph such that all the blue edges of G have a common end vertex i. Then $a(\mathsf{G}[\mathsf{b}]) = \lambda_1(L(G))$.*

In view of Theorem 3.2.14 and Theorem 2.2.8, we have the following corollary, which is a generalization of Lemma 2.5 [40].

Corollary 3.2.17. *Let G be a non-singular connected mixed graph such that the multiplicity of $a(\mathsf{G}_{\mathsf{b}}{}^{\Rsh\mathsf{r}})$ is k. If $\varepsilon s(G) \le k$, then $\lambda_1(L(G)) = a(\mathsf{G}[\mathsf{b}])$.*

3.3 Spectrum of an unweighted undirected graph resulting from a more general graph operation

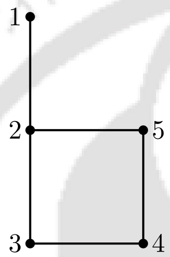
Throughout this section a graph means an unweighted undirected graph, unless otherwise specified.

Definition 3.3.1. Let G be a graph on vertices $1, \ldots, n$ and $F' \subseteq E(G^c)$. Give some orientations to the edges in F' and call it F. Let Γ denote the cycle $[1, \ldots, p, 1]$ where $(i, i+1) \in E(\Gamma)$ for each $i \in \mathbb{Z}_p = \{1, \ldots, p\}$ (with addition modulo p). Consider the disjoint union of p copies $G^{(1)}, \ldots, G^{(p)}$ of G. The label of each vertex u in $G^{(i)}$ is replaced by $u^{(i)}$. Add edges $u^{(i)}v^{(j)}$ whenever $(i, j) \in E(\Gamma)$ and $(u, v) \in F$. Call these edges *pivotal* and denote the resulting graph by $G_p[F]$.

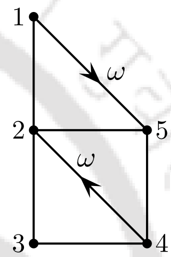
Remark 3.3.2. Let us reconsider the Example 3.2.2. Then $\mathsf{G}[\mathsf{b}]$ is nothing but $K_2[F]$ where $K = G - F_b$, $F = \{(1,2), (4,6), (5,6)\}$. This motivates us to study $G_2[F]$ in general. We note that only in this discussion we have a directed graph (namely Γ) whose underlying undirected graph is not simple.

Definition 3.3.3. Let G and F be as defined in Definition 3.3.1. We may view G as a weighted directed graph where each edge has a weight 1. The orientations of these edges are immaterial. Assign a weight w to each edge in F. By G_w we denote the weighted directed graph $G + F$.

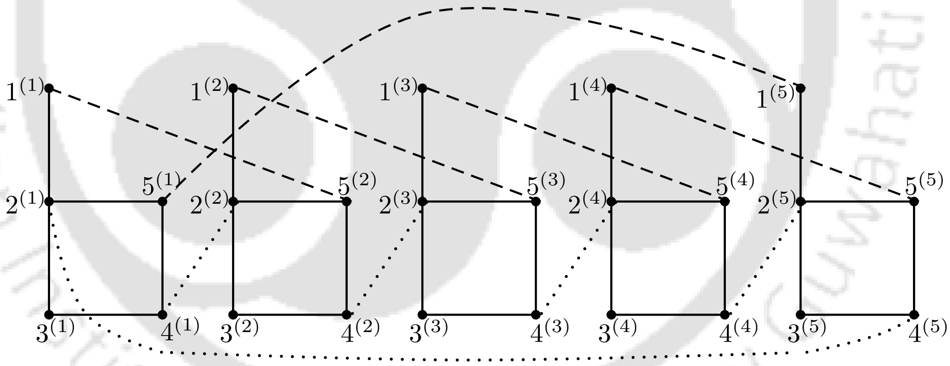
Example 3.3.4. Let G be the graph as shown in the following picture (a). Let $F = \{(4, 2), (1, 5)\}$. The graph $G_5[F]$ is supplied in picture (c). A weighted directed graph G_ω is supplied in picture (b).



(a) G



(b) G_ω



(c) The graph $G_5[F]$

Definition 3.3.5. Put $\mathsf{B} = [b_{ij}]_{n\times n}$ where $b_{ij} = 1$ if $(i, j) \in F$ and $b_{ij} = 0$ otherwise. Let D_{B} be the diagonal matrix with i-th diagonal entry equal to the number of edges in F that are incident with the vertex i. Let $\mathsf{P} = [p_{ij}]$ denote the $p \times p$ basic circulant permutation matrix, that is $p_{ii+1} = 1$ for $i \in \mathbb{Z}_p$ and $p_{ij} = 0$ otherwise.

Henceforth we assume that $p \geq 3$ while considering $G_p[F]$.

Remark 3.3.6. Note that ω is an eigenvalue of P with an eigenvector $z_\omega = \begin{bmatrix} 1 \\ \omega \\ \vdots \\ \omega^{p-1} \end{bmatrix}$ if and only if ω is a p-th root of unity. Moreover $\overline{\omega}$ is an eigenvalue of P^T with the same eigenvector z_ω.

Next we give the structure the adjacency and Laplacian matrix of $G_p[F]$. The proof is routine.

Proposition 3.3.7. *Let $G_p[F]$ be as in Definition 3.3.1. Then*

$$\begin{aligned}
A(G_p[F]) &= I_p \otimes A(G) + \mathsf{P} \otimes \mathsf{B} + \mathsf{P}^T \otimes \mathsf{B}^T, \\
L(G_p[F]) &= I_p \otimes L(G) + I_p \otimes D_{\mathsf{B}} - \mathsf{P} \otimes \mathsf{B} - \mathsf{P}^T \otimes \mathsf{B}^T, \\
A(G_w) &= A(G) + w\mathsf{B} + \overline{w}\mathsf{B}^T, \\
L(G_w) &= L(G) + D_{\mathsf{B}} - w\mathsf{B} - \overline{w}\mathsf{B}^T.
\end{aligned}$$

Lemma 3.3.8. *Consider $G_p[F]$ as in Definition 3.3.1. Let ω be a p-th root of unity and λ be a Laplacian (resp. adjacency) eigenvalue of G_ω with an eigenvector x. Then λ is a Laplacian (resp. adjacency) eigenvalue of $G_p[F]$ with an eigenvector $\mathbf{x} = z_\omega \otimes x$. Furthermore, if $\omega \neq \pm 1$, then λ is a Laplacian (resp. adjacency) eigenvalue of $G_p[F]$ with linearly independent eigenvectors $\mathsf{Re}\,\mathbf{x}$ and $\mathsf{Im}\,\mathbf{x}$.*

Proof. We prove the statement only for $L(G_p[F])$. The proof of the statement for $A(G_p[F])$ is similar. Now

$$\begin{aligned}
L(G_p[F])(z_\omega \otimes x) &= z_\omega \otimes L(G)x + z_\omega \otimes D_{\mathsf{B}}x - \mathsf{P}z_\omega \otimes \mathsf{B}x - \mathsf{P}^T z_\omega \otimes \mathsf{B}^T x \\
&= z_\omega \otimes \left[L(G) + D_{\mathsf{B}} - \omega\mathsf{B} - \overline{\omega}\mathsf{B}^T\right] x \\
&= z_\omega \otimes L(G_\omega)x \quad \text{using Proposition (3.3.7)} \\
&= z_\omega \otimes \lambda x = \lambda(z_\omega \otimes x).
\end{aligned}$$

Observe that $L(G_{\overline{\omega}}) = \overline{L(G_\omega)}$. So λ is a Laplacian eigenvalue of $G_{\overline{\omega}}$ with an eigenvector $\overline{x}$. It follows that λ is a Laplacian eigenvalue of $G_p[F]$ with eigenvectors $\mathbf{x}$ and $\overline{\mathbf{x}}$. Since $\omega \notin \mathbb{R}$, the vectors $\mathbf{x}$ and $\overline{\mathbf{x}}$ are linearly independent. Hence $\mathsf{Re}\,\mathbf{x}$ and $\mathsf{Im}\,\mathbf{x}$ are linearly independent eigenvectors of $L(G_p[F])$ corresponding to λ, as $L(G_p[F])$ is real. ■

The following result whose proof follows immediately from Lemma 3.3.8 is the main result of this section.

Theorem 3.3.9. *Consider $G_p[F]$ as in Definition 3.3.1. Let ω be a p-th root of unity such that $\omega^k \neq 1$, for $0 < k < p$. Then the Laplacian (resp. adjacency) spectrum of $G_p[F]$ is the union of the Laplacian (resp. adjacency) spectrum of the weighted directed graphs $G_{\omega^1}, \ldots, G_{\omega^p}$.*

Remark 3.3.10. Note that the Laplacian (resp. adjacency) spectrum of G_ω and $G_{\overline{\omega}}$ are the same. So for each p-th root of unity $\omega(\neq \pm 1)$, each Laplacian (resp. adjacency) eigenvalue of G_ω appears at least twice in the Laplacian (resp. adjacency) spectrum of $G_p[F]$.

Chapter 4

3-colored digraphs with exactly one non-singular cycle

In Section 4.1 we describe the structure of 3-colored digraphs containing exactly one non-singular cycle. We show that given a connected 3-colored digraph G containing exactly one non-singular cycle, there is a graph H with $a(H) = \lambda_1(L(G))$. In Section 4.2 we establish a monotonicity property on the real and imaginary parts of the eigenvectors corresponding to $\lambda_1(L(G))$, which is analogous to Fiedler's monotonicity Theorem. We describe the sign structure of the real and imaginary parts of the eigenvectors corresponding to $\lambda_1(L(G))$. In Section 4.3 we prove that among all non-singular unicyclic 3-colored digraphs on n vertices, the smallest Laplacian eigenvalue is minimized by the cycle of weight $\pm \mathrm{i}$ and length n. The non-singular unicyclic 3-colored digraphs which maximizes the smallest Laplacian eigenvalue over all such graphs is discussed in Section 4.4. In Section 4.5 we characterize the non-singular unicyclic 3-colored digraphs G having $\lambda_2(L(G)) = 1$.

4.1 Smallest Laplacian eigenvalue

The following lemmas are crucial in describing the structure of a 3-colored digraph containing exactly one non-singular cycle.

Lemma 4.1.1. *Let G be a connected 3-colored digraph containing exactly one non-singular cycle C. Then the following holds:*

(i) If $\mathsf{w}_C = -1$, then G is D-similar to ${}^D G$ with all edges red except one edge on C which is blue.

(ii) If $\mathsf{w}_C = \pm\mathrm{i}$*, then* G *is* D*-similar to* ${}^DG$ *with all edges red except one edge on* C *which is green.*

Proof. Let e be an edge on the cycle C in G. Take $G' = G - e$. By Lemma 2.1.11, G' is singular and by Lemma 2.3.4, each edge in ${}^DG'$ has color red. Consider ${}^DG$ for this D. Note that all the edges in ${}^DG$ except the edge corresponding to e have color red. Since weight of a cycle does not change in D-similar copies, we see that weight of the edge corresponding to e in ${}^DG$ is w_C or $\overline{\mathsf{w}}_C$. Hence the result holds. ∎

Lemma 4.1.2. *Let* G *be a* 3*-colored digraph containing exactly one non-singular cycle* C*. Then the subgraph induced by* C *is* C *itself.*

Proof. Consider $C = [1, \ldots, m, 1]$. If possible, suppose that C has a chord joining the vertices i and j with $1 \le i < j \le m$. Take the cycles $C_1 = [1, i, j \ldots, m, 1]$ and $C_2 = [i, i+1, \ldots, j, i]$. Note that $\mathsf{w}_{C_1}\mathsf{w}_{C_2} = \mathsf{w}_C \neq 1$, which implies one of C_1 and C_2 has weight other than 1. Hence G contains at least two non-singular cycles, which is a contradiction. ∎

Lemma 4.1.3. *Let* G *be a connected* 3*-colored digraph containing exactly one non-singular cycle* C*. Let* u *be a vertex of* G *not on* C*. Then there is a vertex* v *on the cycle* C *such that* $G - v$ *is disconnected with at least two components, one containing* u *and another containing the remaining vertices of* C*.*

Proof. In view of Lemma 4.1.1, we assume that all the edges of G have color red except an edge e on the cycle C. Since G is connected, let v be a vertex in the cycle for which the distance $d(v, u)$ is minimum. Let P_{uv} be a shortest u-v-path in G. Then the vertex v is on every u-w-path, for each vertex w in C. If not, suppose G contains a u-w-path, say P_{uw} which does not contain v, for some vertex w in C. Let $P_{vw}(e)$ be the v-w-path on the cycle C containing the edge e. Take the cycle $C' = P_{uv} + P_{vw}(e) + P_{uw}$. Note that $\mathsf{w}_{C'} \neq 1$. So the cycle C' is non-singular, by Theorem 2.1.11, which is a contradiction. Hence $G - v$ is disconnected with at least two components, one containing u and another containing the remaining vertices of C. ∎

Next corollary follows immediately, which generalizes [17, Lemma 2.1].

Corollary 4.1.4. *Let G be a connected 3-colored digraph with exactly one non-singular cycle. If G has no cut vertex, then G is exactly the non-singular cycle.*

Definition 4.1.5. Let G be a connected 3-colored digraph containing exactly one non-singular cycle C and let u be a vertex on C. In view of Lemma 4.1.3, let H be the component of $G-u$, which contains the remaining vertices of C. Notice that if $G-u$ is connected, then $H = G-u$. We define G_u to be the graph $G-V(H)$.

Next lemma characterizes the structure of connected 3-colored digraphs containing exactly one non-singular cycle.

Lemma 4.1.6. *Let G be a 3-colored connected digraph on vertices $1, \cdots, n$. Then the following two statements are equivalent.*

(a) G has exactly one non-singular cycle $C = [1, \ldots, m, 1]$.

(b) G is obtained from a non-singular cycle $C = [1, \ldots, m, 1]$ by appending a singular connected graph G_i to the vertex i of C while identifying a vertex of G_i with i, for each $i = 1, \ldots, m$.

Proof. $(a) \Rightarrow (b)$. Assume that G has exactly one non-singular cycle $C = [1, \ldots, m, 1]$. By Lemma 4.1.2, the subgraph induced by C is C itself. In view of Lemma 4.1.3, for each $i = 1, \ldots, m$, let $G_i = G - V(H_i)$, where H_i is the component of the graph containing the remaining vertices of C. As G_i does not contain a non-singular cycle, G_i is singular, for each $i = 1, \ldots, m$. Hence the result holds. (b)$\Rightarrow$(a) is trivial. ∎

Remark 4.1.7. Let G be a connected 3-colored digraph with exactly one non-singular cycle $C = [1, \ldots, m, 1]$. In view of Lemma 4.1.1 and Lemma 4.1.6, ${}^{D}G$ is as in figure 4.1 or figure 4.2, if $\mathsf{w}_C = \pm \mathrm{i}$ or $\mathsf{w}_C = -1$, respectively. We denote the graph in Figure 4.1, by G_g and the graph in Figure 4.2, by G_b.

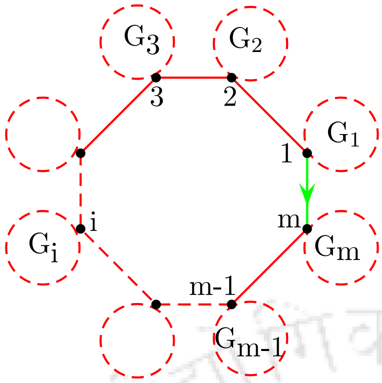


Figure 4.1: G_g has exactly one green edge, each G_i are connected.

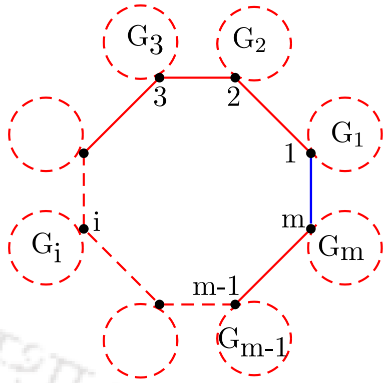


Figure 4.2: G_b has exactly one blue edge, each G_i are connected.

Let G be a connected 3-colored digraph with exactly one non-singular cycle. In view of Remark 4.1.7, in order to study the spectrum of G it is enough to study the spectrum of G_b or G_g. Note that G_b is nothing but a non-singular mixed graph with exactly one non-singular cycle. Fan has studied $\lambda_1(L(G_b))$ and its corresponding eigenvectors in [17]. He has proved that for the mixed graph G_b, there is an unweighted undirected graph H such that $a(H) = \lambda_1(L(G_b))$.

We now ask the following question : Does there exist an unweighted undirected graph H whose algebraic connectivity $a(H) = \lambda_1(L(G_g))$? In this section we answer this question in the affirmative.

Definition 4.1.8. Consider the 3-colored digraph G_g as shown in Figure 4.1. Take the graphs $K_2[F]$ and $K_4[F]$, where $K = G_g - F$, $F = \{(1, m)\}$ (as in Definition 3.3.1). We denote the graph $K_2[F]$ by $\widetilde{G}$ and $K_4[F]$ by $\widehat{G}$. By G_r we denote the graph obtained from G_b changing the color of each blue edge in G_b to red. Note that K_ω is nothing but G_g, G_b, or G_r if $\omega = \mathrm{i}$, $\omega = -1$, or $\omega = 1$, respectively.

Next Theorem follows immediately from Lemma 3.3.8.

Theorem 4.1.9. *Consider $\widehat{G}$ as in Definition 4.1.8. Then the following holds:*

(a) If λ is an eigenvalue of $L(G_g)$ with an eigenvector x, then λ is an eigenvalue

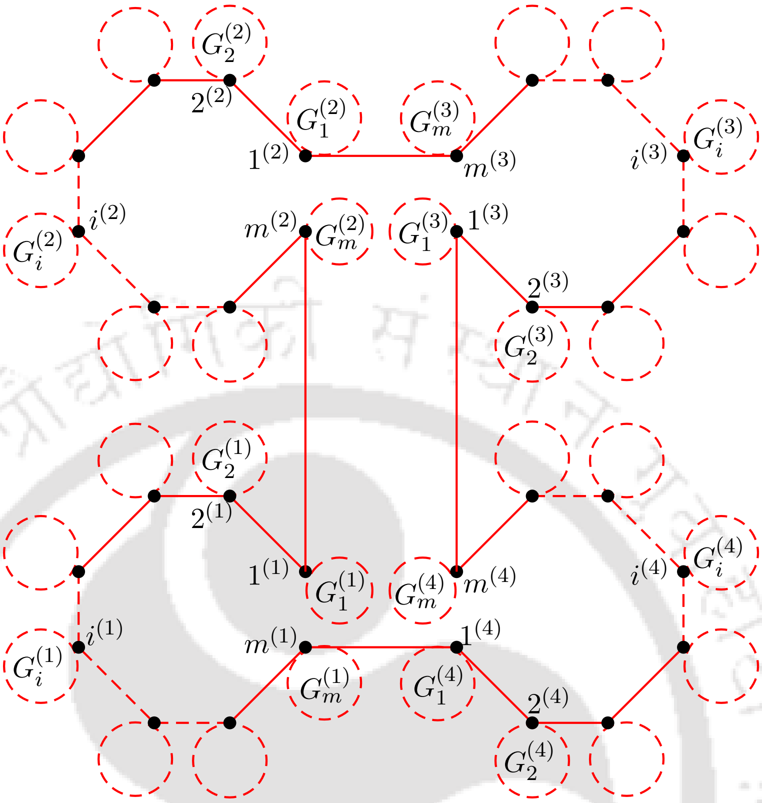


Figure 4.3: The graph $\widehat{G}$.

of $L(\widehat{G})$ with linearly independent eigenvectors $\begin{bmatrix} \operatorname{Re} x \\ -\operatorname{Im} x \\ -\operatorname{Re} x \\ \operatorname{Im} x \end{bmatrix}$ *and* $\begin{bmatrix} \operatorname{Im} x \\ \operatorname{Re} x \\ -\operatorname{Im} x \\ -\operatorname{Re} x \end{bmatrix}$.

(b) If μ is an eigenvalue of $L(G_b)$ with an eigenvector y, then μ is an eigenvalue of $L(\widehat{G})$ with an eigenvector $z_\omega \otimes y$, where $\omega = -1$.

(c) If ν is an eigenvalue of $L(G_r)$ with an eigenvector z, then ν is an eigenvalue of $L(\widehat{G})$ with an eigenvector $z_\omega \otimes z$, where $\omega = 1$.

Lemma 4.1.10. *Consider G_b as in figure 4.2. Then $\lambda_1(L(G_b)) \leq a(G_r)$.*

Proof. Let $e = 1m$ be the blue edge in G_b. Note that the principal submatrices of $L(G)$ and $L(G_r)$ obtained by deleting the 1-th row and column are same. Let L_1 be the principal submatrix of $L(G)$ and $L(G_r)$ formed by deleting its 1-st row and column. Then by interlacing theorem we have $\lambda_1(L(G_b)) \leq \lambda_1(L_1) \leq a(G_r)$. Hence $\lambda_1(L(G_b)) \leq a(G_r)$. ∎

Definition 4.1.11. Let G be a 3-colored digraph. We call an eigenvector corresponding to $\lambda_1(L(G))$ a *first eigenvector* of G.

Next Lemma follows immediately using Lemma 3.3.8 and Lemma 4.1.10.

Lemma 4.1.12. *[17, Lemma 2.4] Consider G_b as in figure 4.2 and let x be a first eigenvector of G_b. Then $a(\widetilde{G}) = \lambda_1(L(G_b))$ and $\begin{bmatrix} x \\ -x \end{bmatrix}$ is a Fiedler vector of $\widetilde{G}$.*

Remark 4.1.13. Let G be a connected 3-colored digraph with exactly one non-singular cycle C. Assume that C contains an even number of green edges. If x is a first eigenvector of G, then $\begin{bmatrix} \operatorname{Re} D^{-1}y \\ -\operatorname{Re} D^{-1}y \end{bmatrix}$, $\begin{bmatrix} \operatorname{Im} D^{-1}y \\ -\operatorname{Im} D^{-1}y \end{bmatrix}$ are Fiedler vectors of $\widetilde{G}$, for some diagonal matrix D with diagonal entries $\pm 1, \pm \mathrm{i}$.

Lemma 4.1.14. *[6, Theorem 12] Let G be a connected graph and Y be a Fiedler vector. Let $S = \mathcal{C}(G, Y)$. Suppose S lies in the block B. Then $1 \leq |S| \leq \mathcal{N}_{\mathcal{B}} + 1$.*

Next Lemma says non-negativity of the first eigenvector of G_b on the cycle.

Lemma 4.1.15. *Consider the mixed graph G_b as in figure 4.2. Then G_b is D-similar to a mixed graph H such that a first eigenvector of H is non-negative on the vertices of the non-singular cycle in H.*

Proof. Proof is similar to that of [22, Theorem 2.9]. ∎

Lemma 4.1.16. *Consider G_g and G_b as in figure 4.1 and figure 4.2, respectively. Then $\lambda_1(L(G_g)) \leq \lambda_1(L(G_b))$.*

Proof. In view of Lemma 4.1.15, let x be a first eigenvector of G_b such that x is non-negative on the vertices of the cycle containing the blue edge $e = ij$. Then

$$\begin{aligned} \lambda_1(L(G_b)) = x^* L(G_b) x &= \sum_{(u,v)\in E(G_b - e)} |x_u - x_v|^2 + (x_i + x_j)^2 \\ &\geq \sum_{(u,v)\in E(G_b - e)} |x_u - x_v|^2 + |x_i + \mathrm{i} x_j|^2 \\ &\geq \min x^* L(G_g) x \geq \lambda_1(L(G_g)). \end{aligned}$$

∎

Theorem 4.1.17. *Consider G_g as in figure 4.1. Let x be a first eigenvector of G_g. Then $a(\widehat{G}) = \lambda_1(L(G_g))$ and* $\hat{\mathbf{x}} = \begin{bmatrix} \mathsf{Re}\, x \\ -\mathsf{Im}\, x \\ -\mathsf{Re}\, x \\ \mathsf{Im}\, x \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} \mathsf{Im}\, x \\ \mathsf{Re}\, x \\ -\mathsf{Im}\, x \\ -\mathsf{Re}\, x \end{bmatrix}$ *are linearly independent Fiedler vectors.*

Proof. Note that $\lambda_1(L(G_g)) \le \lambda_1(L(G_b)) \le a(G_r)$, by Lemma 4.1.10 and Lemma 4.1.16. Hence $a(\widehat{G}) = \lambda_1(L(G_g))$, and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ are linearly independent Fiedler vectors of $\widehat{G}$, by Theorem 4.1.9(a). ■

Next corollary follows immediately from Lemma 4.1.1 and Theorem 4.1.17.

Corollary 4.1.18. *Let G be a 3-colored digraph containing exactly one non-singular cycle C. Assume that C contains an odd number of green edges. If x is a first eigenvector of G, then* $\begin{bmatrix} \mathsf{Re}\, D^{-1}x \\ -\mathsf{Im}\, D^{-1}x \\ -\mathsf{Re}\, D^{-1}x \\ \mathsf{Im}\, D^{-1}x \end{bmatrix}$ *and* $\begin{bmatrix} \mathsf{Im}\, D^{-1}x \\ \mathsf{Re}\, D^{-1}x \\ -\mathsf{Im}\, D^{-1}x \\ -\mathsf{Re}\, D^{-1}x \end{bmatrix}$ *are Fiedler vectors of $\widehat{G}$, for some diagonal matrix D with the diagonal entries $\pm 1, \pm \mathrm{i}$.*

4.2 First eigenvectors of G_g

Fan [17] has studied the structure of the first eigenvectors of G_b. In this section, we describe the structure of the real and imaginary parts of first eigenvectors of G_g. Finally, we prove that multiplicity of $\lambda_1(L(G_g))$ is one.

Proposition 4.2.1. *[25, Fiedler's monotonicity theorem] Let G be a connected graph and y be a Fiedler vector of G. Then exactly one of the following cases occurs:*

<u>Case A:</u> There is exactly one block B in G which contains both positively and negatively valuated vertices (with respect to y). Every other block has either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every pure path P which starts at a vertex i of B and contains no other vertex of B has the property that the valuations at points of articulation contained in P form either an increasing, or decreasing, or a zero sequence along this path according to whether $y(i) > 0$, $y(i) < 0$ or $y(i) = 0$; in the last case all vertices in P have valuation zero.

<u>*Case B:*</u> *No block of G contains both positively and negatively valuated vertices (with respect to y). In this case, there exists a unique vertex k which has valuation zero and is adjacent to a vertex with non-zero valuation. This vertex k is a point of articulation. Each block with respect to y contains (with exception of k) either vertices with positive valuation only, vertices with negative valuation only, or vertices with zero valuation only. Every pure path P which starts at k has the property that the valuations at its points of articulation either increase, and then all valuations of vertices on P are (with exception of k) positive, or decrease, and then all valuations of vertices on P are (with exception of k) negative, or all valuations of vertices on P are zero. Every path containing both positively and negatively valued vertices passes through k.*

Theorem 4.2.2. *Consider G_g as in figure 4.1. Let $C = [1, \ldots, m, 1]$ be the non-singular cycle with the green edge $e = (1, m)$. Let x be a first eigenvector of G_g. Then the following holds:*

(a) The cycle C contains a nonzero valuated vertex (with respect to x) and $\widehat{G}$ satisfies Case A of Proposition 4.2.1.

(b) Each block B of G_g other than C has either vertices of valuations with positive real part (resp. imaginary part) only, or vertices with negative real part (resp. imaginary part) only, or vertices with zero real part (resp. imaginary part) only.

(c) Every path P which starts at a vertex v of C and contains no other vertex of C has the property that the real part (resp. imaginary part) of the valuations at the points of articulation contained in P form either an increasing, decreasing, or zero sequence along this path according to whether $\mathsf{Re}\, x(v) > 0$ (resp. $\mathsf{Im}\, x(v) > 0$), $\mathsf{Re}\, x(v) < 0$ (resp. $\mathsf{Im}\, x(v) < 0$), or $\mathsf{Re}\, x(v) = 0$ (resp. $\mathsf{Im}\, x(v) = 0$), respectively. In the last case all vertices in P have real part (resp. imaginary part) zero.

Proof. (a) If possible, suppose that the valuations of x at the vertices of C are all zero. Let $P^{(i)}$ denote the $1^{(i)}$-$m^{(i)}$-path in $\widehat{G}$ corresponding to the 1-m-path in

$G_g - e$, for $i = 1, 2, 3, 4$. Take the cycle

$$\widehat{C} = P^{(1)} + 1^{(1)}m^{(2)} + P^{(2)} + 1^{(2)}m^{(3)} + P^{(3)} + 1^{(3)}m^{(4)} + P^{(4)} + 1^{(4)}m^{(1)}.$$

in $\widehat{G}$. Note that $\hat{\mathbf{x}}$ (as in Theorem 4.1.17) is a Fiedler vector of $\widehat{G}$. Thus valuations of $\hat{\mathbf{x}}$ at the vertices of $\widehat{C}$ are all zero.

Case 1: If $\widehat{G}$ satisfies Case A of Proposition 4.2.1 with respect to $\hat{\mathbf{x}}$, then there exist a single block B_0 in $\widehat{G}$ containing both positively and negatively valuated vertices and each other block has either vertices with positive valuation, or negative valuation, or vertices zero valuations only. Clearly $\widehat{C}$ can not be the block B_0. So B_0 must be be contained in some $G_k^{(i)}$ (where G_k is as in Definition 4.1.5), for some k with $1 \leq k \leq m$. Thus $\widehat{G}$ has at least four blocks containing both positively and negatively valuated vertices (with respect to $\hat{\mathbf{x}}$), which is a contradiction.

Case 2: If $\widehat{G}$ satisfies Case B of Proposition 4.2.1 with respect to $\hat{\mathbf{x}}$, then there exist a unique vertex u^i in $\widehat{G}$ such that $\hat{\mathbf{x}}(u) = 0$ and $u^{(i)}$ is adjacent to a vertex $w^{(j)}$ with $\hat{\mathbf{x}}(w^{(j)}) \neq 0$, for some i, j with $1 \leq i, j \leq 4$. So $w^{(j)}$ can not be a vertex in $\widehat{C}$. Thus $w^{(j)}$ must be contained in $G_k^{(j)}$, for some k with $1 \leq k \leq m$. From the structure of $\widehat{G}$ and G_g it follows that $i = j$, that is, $u^{(i)}$ is a vertex in $G^{(j)}$, where $G^{(j)}$ is as in Definition 4.1.8. Thus $\hat{\mathbf{x}}(u^{(i)}) = 0$, $\hat{\mathbf{x}}(w^{(i)}) \neq 0$, and $u^{(i)}$ is adjacent to $w^{(i)}$ in $\widehat{G}$, for $i = 1, 2, 3, 4$, which is contradiction. Hence the cycle C contains at least one nonzero valuated vertex (with respect to $\hat{\mathbf{x}}$).

To prove the second part of (a), let $x(j) \neq 0$, for some vertex j in C. Thus from the structure of $\widehat{G}$ and $\hat{\mathbf{x}}$, it follows that $\widehat{C}$ contains both positively and negatively valuated vertices (with respect to $\hat{\mathbf{x}}$). Hence the graph $\widehat{G}$ satisfies Case A of Proposition 4.2.1, with respect to $\hat{\mathbf{x}}$.

(b) Let $B \neq C$ be a block of G_g. Then B is contained in some G_i, for $i = 1, \ldots, m$. Note that the blocks $B^{(i)}$ in $\widehat{G}$ corresponding to B, for $i = 1, 2, 3, 4$ contains vertices either with positive valuation only, vertices with negative valuation only, or vertices with zero valuation only (with respect to $\hat{\mathbf{x}}$), by part (a). Hence with respect to $\hat{\mathbf{x}}$, the block B contains vertices with positive real part (resp. imaginary part) only or vertices with negative real part (resp. imaginary part) only or vertices with zero real part (resp. imaginary part) only.

(c) Follows immediately by using part(a), Case A of Proposition 4.2.1 to $\widehat{G}$ and the structure of $\hat{\mathbf{x}}$. ■

The following example illustrates the Theorem 4.2.2.

Example 4.2.3. Consider G as shown below. Real and imaginary parts of a first eigenvector x of G is supplied here.

$$\begin{aligned}\mathsf{Re}\,x &= [1.160 \quad 1.106 \quad 1 \quad 0.906 \quad 0.986 \quad 0.940 \quad 0.770 \quad .964 \quad 1.030 \quad 1.047 \quad 1.099]^t \\ \mathsf{Im}\,x &= [0 \quad 0 \quad 0 \quad -0.404 \quad 0.424 \quad 0.404 \quad -0.790 \quad -.989 \quad -1.057 \quad -1.074 \quad -1.127]^t\end{aligned}$$

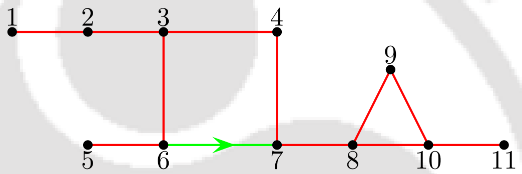


Figure 4.4:

Remark 4.2.4. *By Theorem 4.2.2, the characteristic set $\mathcal{C}(\widehat{G}, \hat{\mathbf{x}})$ lies on the cycle $\widehat{C}$ and by Lemma 4.1.14, $|\mathcal{C}(\widehat{G}, \hat{\mathbf{x}})| \leq 2$.*

Next, we describe the sign structure of the real and imaginary parts of first first eigenvectors of G_g.

Theorem 4.2.5. *Consider G_g as in figure 4.1. Let $C = [1, \ldots, m, 1]$ be the cycle with the green edge $e = (1, m)$. Let x be a first eigenvector of G_g. Then x satisfies the following properties:*

(a) On the vertices of the cycle C, $\mathsf{Re}\,x \neq 0$ and $\mathsf{Im}\,x \neq 0$,

(b) No entry of x at the vertices of the cycle C is zero,

(c) $\mathsf{Re}\,x$, (resp. $\mathsf{Im}\,x$) can be zero on at most one vertex of C,

(d) If there exist a vertex v on C such that $\mathsf{Re}\,x(v) = 0$ (resp. $\mathsf{Im}\,x(v) = 0$), then $\mathsf{Im}\,x$ (resp. $\mathsf{Re}\,x$) is positive or negative on the vertices of C.

(e) *If there exist vertices u and v on C such that* $\mathrm{Re}\,x(u)\,\mathrm{Re}\,x(v) < 0$ *(resp.* $\mathrm{Im}\,x(u)\,\mathrm{Im}\,x(v) < 0$*), then* $\mathrm{Im}\,x$ *(resp.* $\mathrm{Re}\,x$*) is positive or negative on the vertices of* C*,*

(f) *Let* $\mathrm{Re}\,x(1)\,\mathrm{Re}\,x(m) < 0$ *(resp.* $\mathrm{Im}\,x(1)\,\mathrm{Im}\,x(m) < 0$*). Then* $\mathrm{Im}\,x$ *is either positive or negative, according as* $\mathrm{Re}\,x(1)$ *is negative or positive.*

(g) *If* $\mathrm{Re}\,x(1)\,\mathrm{Re}\,x(m) > 0$ *(resp.* $\mathrm{Im}\,x(1)\,\mathrm{Im}\,x(m) > 0$*), then* $\mathrm{Re}\,x$ *(resp.* $\mathrm{Im}\,x$*) is either positive or negative, according as* $\mathrm{Re}\,x(1)$ *(resp.* $\mathrm{Im}\,x(1)$*) is positive or negative, respectively.*

Proof. Consider the graph $\widehat{G}$ corresponding to G_g as in Definition 4.1.8. Note that $\hat{\mathbf{x}}$ (as in Theorem 4.1.17) is a Fiedler vector of $\widehat{G}$. For vertices y, z on C, let $P_{yz}^{(i)}$ denotes the $y^{(i)}$-$z^{(i)}$-path in $\widehat{G}$ corresponding to the y-z-path in $G_g - e$, for $i = 1, \ldots, 4$.

(a) If possible, suppose that $\mathrm{Re}\,x = 0$ on the vertices of C. By Theorem 4.2.2, x is non-zero on C. Let $\mathrm{Im}\,x(u) \neq 0$, for some vertex u on C such that length of the m-u-path in $G_g - e$ is shortest. Observe that $\hat{\mathbf{x}}(u^{(i)}) \neq 0$, for $i = 2, 4$ and $\hat{\mathbf{x}}(j^{(i)}) = 0$, on every vertex j of C, for $i = 1, 3$. Thus the paths $P_{mu}^{(2)} + 1^{(1)}m^{(2)}$, $P_{u1}^{(2)} + 1^{(2)}m^{(3)}$ and $P_{mu}^{(4)} + 1^{(3)}m^{(4)}$ in $\widehat{G}$ contains one characteristic vertex of $\widehat{G}$ each, which is a contradiction to $|\mathcal{C}(\widehat{G}, \hat{\mathbf{x}})| \leq 2$. Similarly we can prove that $\mathrm{Im}\,x \neq 0$ on C.

(b) If possible, let u be a vertex on the cycle C such that $x(u) = 0$. Then $\mathrm{Re}\,x(u) = 0, \mathrm{Im}\,x(u) = 0$. In view of part(a), without loss of generality, let w be a vertex on C such that $\mathrm{Re}\,x(w) \neq 0$. Thus $\hat{\mathbf{x}}(u^{(i)}) = 0$, $\hat{\mathbf{x}}(w^{(i)}) \neq 0$, for $i = 1, 3$ and $\hat{\mathbf{x}}(u^{(2)}) = 0 = \hat{\mathbf{x}}(u^{(4)})$. Hence if w lies on the 1-u-path in $G_g - e$, then the paths $P_{u1}^{(1)} + 1^{(1)}m^{(2)} + P_{mu}^{(2)}$ and $P_{uw}^{(3)}$ contains at least three characteristic vertices of $\widehat{G}$, which is a contradiction. Similarly if w lies on the m-u-path in $G_g - e$, we get a contradiction.

(c) If possible, let u and v be two vertices on C such that $\mathrm{Re}\,x(u) = 0 = \mathrm{Re}\,x(v)$. By part(b), $\mathrm{Im}\,x(u) \neq 0$ and $\mathrm{Im}\,x(v) \neq 0$, with v is on the u-1-path in $G_g - e$. Then the paths

$$P_{v1}^{(1)} + 1^{(1)}m^{(2)} + P_{m1}^{(2)} + 1^{(2)}m^{(3)} + P_{mu}^{(3)} \text{ and } P_{v1}^{(3)} + 1^{(3)}m^{(4)} + P_{mu}^{(4)}$$

contains at least three characteristic vertices of $\widehat{G}$, which is a contradiction.

Similarly we can prove for $\operatorname{Im} x$.

(d) Let v be a vertex on C such that $\operatorname{Re} x(v) = 0$. If possible, let $\operatorname{Im} x(w) = 0$, for some vertex w on C. By part(b), $\operatorname{Im} x(v) \neq 0$, $\operatorname{Re} x(w) \neq 0$. So $\hat{\mathbf{x}}(v^{(i)}) \neq 0$ & $\hat{\mathbf{x}}(w^{(i)}) = 0$ for $i = 2, 4$ and $\hat{\mathbf{x}}(v^{(i)}) = 0$ & $\hat{\mathbf{x}}(w^{(i)}) \neq 0$, for $i = 1, 3$. Thus $|\mathcal{C}(\widehat{G}, \hat{\mathbf{x}})| > 2$, which is impossible. Similarity, if $\operatorname{Im} x(i) \operatorname{Im} x(j) < 0$ for some vertex i, j on C, we see that $|\mathcal{C}(\widehat{G}, \hat{\mathbf{x}})| > 2$, which is contradiction.

(e) Suppose that $\operatorname{Re} x(u) \operatorname{Re} x(v) < 0$, for some vertices u, v on the cycle C. Then $\hat{\mathbf{x}}(u^{(i)})\hat{\mathbf{x}}(v^{(i)}) < 0$, for $i = 1, 3$. Thus the paths $P_{uv}^{(i)}$, for $i = 1, 3$ contains at least two characteristic elements of $\widehat{G}$. Hence the result follows from the fact that $|\mathcal{C}(\widehat{G}, \hat{\mathbf{x}})| \leq 2$.

(f) Proof follows with similar argument.

(g) Suppose that $\operatorname{Re} x(1) \operatorname{Re} x(m) > 0$ and $\operatorname{Re} x(1) > 0$. Then the paths $1^{(1)}m^{(2)} + P_{1m}^{(2)} + 1^{(2)}m^{(3)}$ and $1^{(3)}m^{(4)} + P_{1m}^{(4)} + 1^{(4)}m^{(1)}$ contains at least two characteristic elements of $\widehat{G}$. Hence the result follows from the fact that $|\mathcal{C}(\widehat{G}, \hat{\mathbf{x}})| \leq 2$. ∎

Next, we prove the non-negativity of the real and imaginary parts on the non-singular cycle of the first eigenvectors of G_g.

Theorem 4.2.6. *Consider G_g as in figure 4.1. Then G_g is D-similar to a 3-colored digraph H, such that a first eigenvector of H has non-negative real and imaginary parts on the vertices of the non-singular cycle in H.*

Proof. Let x be a first eigenvector of G_g.

If $\operatorname{Re} x \geq 0$, $\operatorname{Im} x \leq 0$ on the vertices of C, then we take $D = -\mathrm{i}I_n$. If $\operatorname{Re} x \leq 0$, $\operatorname{Im} x \geq 0$ on the vertices of C, then we take $D = \mathrm{i}I_n$. In these cases D^*x is non-negative on the non-singular cycle of ${}^{D}G_g$.

First we assume that $\operatorname{Im} x$ is not non-negative on C. Let u, v be two vertices on C, such that $\operatorname{Im} x(u) > 0$ and $\operatorname{Im} x(v) < 0$ with $1 \leq u < v \leq m$. By Lemma 4.2.5(e), $\operatorname{Re} x$ is either positive or negative on C. Without loss of generality suppose that $\operatorname{Re} x > 0$ on C.

case 1: If $\operatorname{Im} x(i) \neq 0$ for $i = 1, \ldots, m$, then the u-v-path in $G_g - e$ has an edge f joining vertices i_0, i_0+1 with $\mathsf{Im}\, x(i_0) > 0$ and $\mathsf{Im}\, x(i_0+1) < 0$. Since $|\mathcal{C}(\widehat{G}, x)| \leq 2$, we see that $\mathsf{Im}\, x(j) > 0$ for each $j = 1, \ldots, i_0$ and $\mathsf{Im}\, x(j) < 0$ for each $j = i_0+1, \ldots, m$. Let G_1 and G_2 be the two components of $G_b - f - e$, such that G_1 contains the vertex 1 and G_2 contains the vertex m. Take $D = -\mathrm{i} I_{|V(G_2)|} \oplus I_{|V(G_1)|}$. Then $\mathsf{Re}\, D^*x$ and $\mathsf{Im}\, D^*x$ are nonnegative on the vertices of C, and D^*x is a first eigenvector of ${}^{D}G_g$ with the green edge f.

case 2: If $\mathsf{Im}\, x(i) = 0$, for some vertex i on C. Let f be the edge on C, joining the vertices i_0, i_0+1. Since $|\mathcal{C}(\widehat{G}, x)| \leq 2$, we see that $\mathsf{Im}\, x(j) \neq 0$, for each vertex $j \neq i_0$ on C. Moreover, $\mathsf{Im}\, x(j) > 0$ for each $j = 1, \ldots, i_0 - 1$ and $\mathsf{Im}\, x(j) < 0$ for each $j = i_0+1, \ldots, m$. Let G_1 and G_2 be the two components of $G_b - f - e$, such that G_1 contains the vertex 1 and G_2 contains the vertex m. Take $D = -\mathrm{i} I_{|V(G_2)|} \oplus I_{|V(G_1)|}$. Then $\mathsf{Re}\, D^*x$ and $\mathsf{Im}\, D^*x$ are non-negative on the vertices of C, and D^*x is a first eigenvector of ${}^{D}G_g$ with the green edge f. Rest of the proof follows with similar argument. ■

Next, we prove that $\lambda_1(L(G_g))$ has multiplicity one.

Theorem 4.2.7. *Consider G_g as in figure 4.1. Let $C = [1, \ldots, m, 1]$ be the cycle in G_g with the green edge $e = (1, m)$. Then multiplicity of $\lambda_1(L(G_g))$ is one.*

Proof. Proof follows immediately from Theorem 4.2.5(a). ■

Next, we discuss the multiplicity of $\lambda_1(L(G_b))$.

Remark 4.2.8. Consider G_b as in figure 4.2 and take $\widetilde{G}$. Let x be a first eigenvector of G_b. In view of Lemma 4.1.12 and the structure of $\widetilde{G}$, we see that there exist a vertex u on the non-singular cycle in G_b such that $x(u) \neq 0$.

The following result is a generalization of [22, Corollary 2.8].

Lemma 4.2.9. *Consider G_b as in figure 4.2. Let x be a first eigenvector of G_b. Then there exist at most one vertex i on the non-singular cycle C such that $x(i) = 0$.*

Proof. Let $C = [1, \ldots, m, 1]$ and $e = 1m$ be the blue edge. If possible, suppose that there exist two vertices u and v with $u < v$ on C such that $x(u) = x(v) = 0$. By

Remark 4.2.8, $x(w) \neq 0$, for some vertex w on C. If w is on the u-v-path in $G_b - e$ path, then by Lemma 4.1.12, $\widetilde{G}$ has at least four characteristic vertices, which is not possible. For vertices y, z on C, let $P_{yz}^{(i)}$ denotes the $y^{(i)}$-$z^{(i)}$-path in $\widetilde{G}$ corresponding to the y-z-path in $G_b - e$, for $i = 1, 2$. If w is not on u-v-path in $G_b - e$, then the paths $P_{u1}^{(1)} + 1^{(1)}m^{(2)} + P_{mv}^{(2)}$ and $P_{vm}^{(1)} + m^{(1)}1^{(2)} + P_{1u}^{(2)}$ in $\widetilde{G}$ contains at least two characteristic vertex each, which is a contradiction. Hence the result holds. ∎

In [22], the authors proved that multiplicity of $\lambda_1(L(G_b)$ is at most two for G_b is unicyclic. Here G_b is not necessarily unicyclic. Hence the following result generalizes [22, Theorem 2.14].

Lemma 4.2.10. *Consider G_b as in figure 4.2. Then $\lambda_1(L(G_b))$ has multiplicity at most two.*

Proof. Proof is similar to that of [22, Theorem 2.14] using Lemma 4.2.9. ∎

4.3 Minimizing the smallest Laplacian eigenvalue over unicyclic 3-colored digraphs

In this section we consider the non-singular unicyclic 3-colored digraphs. In view of Lemma 4.1.1, throughout this section all our non-singular unicyclic 3-colored digraphs have all edges red except one blue or green edge on the cycle. Fan [18] has obtained the non-singular unicyclic mixed graphs on n vertices with a fixed girth, which minimizes the smallest Laplacian eigenvalue over all such graphs. The unique graph which minimizes the smallest Laplacian eigenvalue over all non-singular unicyclic mixed graphs on n vertices was determined in [22].

We ask the following question: Does there exist a unique non-singular unicyclic 3-colored digraph on n vertices, which minimizes the smallest Laplacian eigenvalue over all such graphs? In this section we answer this question in the affirmative.

Next lemma is an immediate application of the interlacing Theorem [32].

Lemma 4.3.1. *Let G be a 3-colored digraph and let G' be the graph obtained from G by adding a pendent vertex. Then $\lambda_1(L(G')) \leq \lambda_1(L(G))$.*

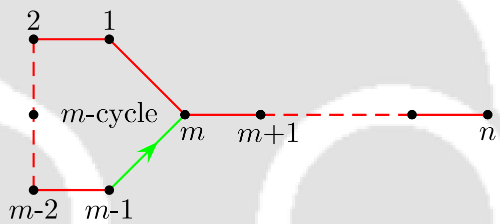
Lemma 4.3.2. *[20] Consider the cycle C on n vertices with $\mathsf{w}_C = -1$. Then the Laplacian spectrum of C is* $\{2[1 - \cos\frac{(2i-1)\pi}{n}] : i = 1, \ldots, n\}$

Lemma 4.3.3. *Consider the cycle C on n vertices with $\mathsf{w}_C = \pm\mathrm{i}$. Then the Laplacian spectrum of C is* $\{2[1 - \cos\frac{(2i-1)\pi}{2n}] : i = 1, \ldots, n\}$

Proof. Proof follows from Lemma 3.1.10 and Lemma 4.3.2.

Definition 4.3.4. Consider the 3-colored digraph obtained by appending a cycle $C = [1, \ldots, m, 1]$ to a pendent vertex of the undirected path on $n - m$ vertices. Call this 3-colored digraph a lollipop graph, b-lollipop graph or g-lollipop graph if $\mathsf{w}_C = 1$, $\mathsf{w}_C = -1$ or $\mathsf{w}_C = \pm\mathrm{i}$, respectively. By $\mathsf{C_{n,m}}$, ${}^{\mathsf{b}}\mathsf{C_{n,m}}$ and ${}^{\mathsf{g}}\mathsf{C_{n,m}}$ we mean a lollipop graph, b-lollipop graph and g-lollipop graph, respectively.

Example 4.3.5. The following picture is an example of a g-lollipop graph ${}^{\mathsf{g}}\mathsf{C_{n,m}}$



Note that $\mathsf{C_{n,m}}$ is singular and henceforth, we assume that all the edges in $\mathsf{C_{n,m}}$ are red, in veiw of Theorem 2.3.4. Fallat, Kirkland and Pati [16] have proved that for $n \geq \frac{3m-1}{2}$ and $m \geq 4$, $a(\mathsf{C_{n,m}}) > a(\mathsf{C_{n,m-1}})$. Recently Guo, Shiu and Li [30] have proved that for $m \geq 4$, $a(\mathsf{C_{n,m}}) > a(\mathsf{C_{n,m-1}})$ is true for all n. Here we give a proof of the result '$a(\mathsf{C_{n,m}}) > a(\mathsf{C_{n,m-1}})$ for all n', which is completely different from that in [30]. The proof is based on the following proposition.

Proposition 4.3.6. *Let G be an unweighted undirected graph and y be a Fiedler vector of G. Let i and j be two non adjacent vertices of G such that $y(i) = y(j)$ and $y(i) \neq y(k)$, where k is vertex adjacent to i. Then $a(G - \{i,k\} + \{i,j\}) < a(G)$.*

Proof. Assume that $y^t y = 1$. Since $[y(i) - y(k)]^2 > 0 = [y(i) - y(j)]^2$, we see that $a(G) = y^t L(G) y > y^t L(G - \{i,k\} + \{i,j\}) y \geq a(G - \{i,k\} + \{i,j\})$. ∎

Definition 4.3.7. Let G be a weighted directed graph and let x be an eigenvector of $L(G)$ corresponding to an eigenvalue λ. By *eigen-condition* at a vertex i, we mean the following equation.

$$(d_i - \lambda)x(i) = \sum_{(i,v)\in E(G)} \mathsf{w}_{iv}x(v) + \sum_{(w,i)\in E(G)} \overline{\mathsf{w}}_{wi}x(w)$$

Lemma 4.3.8. *[30, Theorem 2.11] Let $\mathsf{C}_{\mathsf{n,m}}$ be the lollipop graph on n vertices with a girth m. Then $a(\mathsf{C}_{\mathsf{n,m-1}}) < a(\mathsf{C}_{\mathsf{n,m}})$, for $m \geq 4$.*

Proof. First we prove the result for $m = n$. Note that $\mathsf{C}_{\mathsf{n,n}}$ is just the cycle C_{n} of length n. It is well known that $a(\mathsf{C}_{\mathsf{n}}) = 2(1 - \cos\frac{2\pi}{n})$ and a vector $y \in \mathbb{R}^n$ satisfying $y(i) = \cos\frac{(i-1)2\pi}{n}$, for $i = 1\ldots, n$ is a Fiedler vector of $\mathsf{C}-\mathsf{n}$. Note that $[y(1) - y(n)]^2 > 0 = [y(2) - y(n)]^2$. It follows that for $n \geq 4$,

$$a(\mathsf{C}_{\mathsf{n}}) = y^T L(\mathsf{C}_{\mathsf{n,n}})y > y^T L(\mathsf{C}_{\mathsf{n,n-1}})y \geq a(\mathsf{C}_{\mathsf{n,n-1}}). \tag{4.3.1}$$

Let $4 \leq m \leq n-1$ and y be a fiedler vector of $\mathsf{C}_{\mathsf{n,m}}$. If possible, suppose that $y(n) = 0$. Then by the eigen-condition at the vertices $n, n-1, \ldots, m+1$, we see that $y(n) = y(n-1) = \cdots = y(m) = 0$. In that case, $a(\mathsf{C}_{\mathsf{n,m}})$ is an eigenvalue of the cycle C_{m} of length m with an eigenvector $[y(1), \cdots, y(m)]^t$. Thus $a(\mathsf{C}_{\mathsf{n,m}}) \geq a(\mathsf{C}_{\mathsf{m}}) = 2(1 - \cos\frac{2\pi}{m})$. Note that by Lemma 4.3.1, $a(\mathsf{C}_{\mathsf{n,m}}) \leq a(\mathsf{C}_{\mathsf{m+1,m}})$. Hence

$$a(\mathsf{C}_{\mathsf{n,m}}) \leq a(\mathsf{C}_{\mathsf{m+1,m}}) < a(\mathsf{C}_{\mathsf{m+1}}) = 2(1 - \cos\frac{2\pi}{m+1}), \text{ using equation 4.3.1,}$$

which implies $a(\mathsf{C}_{\mathsf{n,m}}) < 2(1 - \cos\frac{2\pi}{m})$, a contradiction. So $y(n) \neq 0$ and hence $a(\mathsf{C}_{\mathsf{n,m}})$ has multiplicity one.

Let $z \in \mathbb{R}^n$ be a vector such that $z(i) = y(m-i)$, for $i = 1, \ldots, m-1$ and $z(i) = y(i)$, for $i = m, m+1, \ldots, n$. Observe that z is also an eigenvector of $\mathsf{C}_{\mathsf{n,m}}$ corresponding to the eigenvalue $a(\mathsf{C}_{\mathsf{n,m}})$. Thus $y = z$ and hence $y(1) = y(m-1)$. Since $y(n) \neq 0$, we assume that $y(n) > 0$.

If $y(m) > 0$, then the characteristic set of $\mathsf{C}_{\mathsf{n,m}}$ lies on the cycle. In that case $y(1) \neq y(m)$, otherwise using eigen-condition at the vertex m of $\mathsf{C}_{\mathsf{n,m}}$ we have $y(m) - y(m+1) = a(\mathsf{C}_{\mathsf{n,m}})y(m) > 0$, which is a contradiction to Proposition 4.2.1.

If $y(m) < 0$, then the characteristic set of $\mathsf{C_{n,m}}$ lies on the path joining the vertices m and n. In that case $y(1) \neq y(m)$, otherwise using eigencondition at the vertices of the cycle we see that $y(1) < y(2) < \cdots < y(m)$, a contradiction.

If $y(m) = 0$, then the vertex m has to be a characteristic vertex of $\mathsf{C_{n,m}}$, otherwise $y(1) = y(m-1) = y(m+1) = 0$ implies $y = 0$. So $y(1) = y(m-1) \neq 0 = y(m)$.

Thus we have $y(1) = y(m-1)$ and $y(1) \neq y(m)$. Hence by Proposition 4.3.6, $a(\mathsf{C_{n,m-1}}) < a(\mathsf{C_{n,m}})$. ∎

Lemma 4.3.9. *Let G be a unicyclic 3-colored digraph of girth m with the cycle C of weight $\pm \mathrm{i}$. Then $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{2m}) < \frac{1}{2}$. Equality holds if and only if G is a cycle of weight $\pm\mathrm{i}$ on m vertices.*

Proof. Note that G can be obtained from C by adding pendent vertices repeatedly. By Lemma 4.3.1 and Lemma 4.3.3, $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{2m}) < \frac{1}{2}$.

If possible, suppose that G is not a cycle. By Theorem 4.1.17, $\lambda_1(L(G)) = a(\widehat{G})$, where $\widehat{G}$ is as in Definition 4.1.8, constructed from G_g (as in figure 4.1) corresponding to G. Since G has at least one pendent vertex, we see that $\widehat{G}$ can be obtained from the lollipop graph $C_{4m+1,4m}$ by adding pendents sequentially. Thus by Lemma 4.3.1 and 4.3.8, $a(\widehat{G}) \leq a(C_{4m+1,4m}) < 2(1 - \cos\frac{2\pi}{4m+1}) < 2(1 - \cos\frac{\pi}{2m})$. Hence $\lambda_1(L(G)) < 2(1 - \cos\frac{\pi}{2m})$. So the equality holds if and only if G has no pendent vertices. ∎

Lemma 4.3.10. *[18, Theorem 2.6] Let G be a non-singular unicyclic mixed graph with a fixed girth m. Then $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{m})$, with equality holds if and only if G is a non-singular cycle on m vertices.*

Remark 4.3.11. *Let G be a unicyclic 3-colored digraph with fixed girth m and the cycle contains an even number of green edges. Then G is D-similar to a non-singular unicyclic mixed graph. Hence by Lemma 4.3.10, $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{3})$.*

Lemma 4.3.12. *Let G be a unicyclic 3-colored digraph on n vertices with a fixed girth m. Let T be the tree on k vertices attached to a vertex j on the cycle in G. If G' is the unicyclic 3-colored digraph obtained from G by replacing T with the path P_k, then $\lambda_1(L(G)) \geq \lambda_1(L(G'))$.*

Proof. Let $i_1, \ldots, i_k$ be the vertices T with $j = i_1$. Let x be a normalized eigenvector of $L(G)$ corresponding to $\lambda_1(L(G))$ with $\mathsf{Im}\, x(j) = 0$. By Theorem 4.2.2, $\mathsf{Im}\, x(i_r) = 0$, for $1 \le r \le k$. By Theorem 4.2.5(d), we have $\mathsf{Re}\, x > 0$. In view of Theorem 4.2.2, we can arrange the vertices of T with $0 < \mathsf{Re}\, x(i_1) \le \ldots \le \mathsf{Re}\, x(i_k)$. Now

$$\begin{aligned}
\lambda_1(L(G)) &= x^* L(G - E(T))x + \sum_{ij \in E(T)} |x(i) - x(j)|^2 \\
&\ge x^* L(G - E(T))x + \sum_{j=1}^{k-1} |x(i_j) - x(i_{j+1})|^2 \\
&= x^* L(G - E(T) + P_k)x = x^* L(G')x \ge \lambda_1(L(G')).
\end{aligned}$$

∎

Next Lemma generalizes [18, Theorem 3.2], which follows from Lemma 4.3.12.

Lemma 4.3.13. *Among all non-singular unicyclic 3-colored digraphs on n vertices with a fixed girth m, the smallest Laplacian eigenvalue is minimized by a non-singular unicyclic 3-colored digraph with girth m having the following property: there are at most two connected components at every vertex on the cycle, and the components not including the vertices on the cycle (if exists) is a path.*

Next Lemma generalizes [22, Lemma 3.2].

Lemma 4.3.14. *Let G be a non-singular unicyclic 3-colored digraph on n vertices with a fixed girth m obtained from a cycle C by attaching at most one path to each vertex i of C. Let $P_i = i, i_1, i_2 \ldots, i_r$ $(r \ge 1)$ and $P_j = j, j_1, \ldots, j_s$, $(s \ge 1)$ be the paths attached to the vertex i and j on C, respectively. Let x be a first eigenvector of G with $|x(j)| \ge |x(i)|$. If $G_1 = G - ii_1 + i_s i_1$, then $\lambda_1(L(G)) \ge \lambda_1(L(G_1))$.*

Proof. Without loss of generality we assume that $\mathsf{Im}\, x(i) = 0$. Then by Theorem 4.2.5(d), $\mathsf{Re}\, x > 0$ and by Theorem 4.2.2, $x(i_k)$ are real, for $k = 1, \ldots, r$ and

$$x(i) \le x(i_1) \le \ldots \le x(i_r)$$

Let $y \in \mathbb{C}^n$ be defined on the vertices of G_1 such that

$$\begin{cases} y(i_k) = x(i_k) + x(j_s) - x(i), \text{ for } k = 1, \dots r; \\ y(v) = x(v), \text{ for } v \neq i_k, k = 1, \dots r. \end{cases}$$

Then we have

$$x^* L(G) x = y^* L(G_1) y.$$

Now

$$\begin{aligned} y^* y &= \sum_{v \in V(G) \setminus \{i_1, i_2, \dots i_r\}} |x(v)|^2 + \sum_{k=1}^{r} |x(i_k) + x(j_s) - x(i)|^2 \\ &= x^* x + r\,|x(j_s) - x(i)|^2 + 2\,\mathsf{Re}\left(\Big(x(j_s) - x(i)\Big) \sum_{k=1}^{r} \overline{x(i_k)}\right) \\ &\geq x^* x + r\Big(|x(j_s)|^2 + |x(i)|^2 - 2x(i)\,\mathsf{Re}\,x(j_s)\Big) + 2rx(i)\Big(\,\mathsf{Re}\,x(j_s) - x(i)\Big) \\ &\geq x^* x + r\Big(|x(j_s)|^2 - |x(i)|^2\Big). \end{aligned}$$

By Theorem 4.2.2, $|x(i)| \leq |x(j)| \leq |x(j_1)| \leq \dots \leq |x(j_s)|$. So we have $y^* y \geq x^* x$. Hence

$$\lambda_1(L(G)) = \frac{x^* L(G) x}{x^* x} \geq \frac{y^* L(G_1) y}{y^* y} \geq \lambda_1(L(G_1)).$$

∎

► By ${}^{\mathsf{g}}\mathsf{C}_\mathsf{n}$ and ${}^{\mathsf{b}}\mathsf{C}_\mathsf{n}$ we mean a cycle of weight $\pm i$ and a cycle of weight -1, on n vertices, respectively.

Next, we prove that among all non-singular unicyclic 3-colored digraphs on n vertices with a fixed girth m, ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}$ minimizes the smallest eigenvalue.

Theorem 4.3.15. *Let G be a non-singular unicyclic 3-colored digraph on n vertices with a fixed girth m. Then $\lambda_1(L(G) \geq \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}))$.*

Proof. Let $C = [1, \dots, m, 1]$ be the cycle in G and let $H := C + \sum_{i=1}^{k} w_i v_i + P_{n_i}$, where P_{n_i} is a path on n_i vertices with pendent vertices u_i and v_i, w_i is a vertex on C, for $i = 1 \dots, k$ and $n = m + \sum_{i=1}^{k} n_i$. Let x be a first eigenvector of H such that $\max\limits_{1 \leq i \leq k} |x(w_i)| = |x(w_0)|$. Thus by Theorem 4.3.14, $\lambda_1(L(H)) \geq \lambda_1(L(H_1))$, where $H_1 = (H - w_0 v_0) + w_1 v_0 + P_{n_0}$. Note that the graph H_1 has exactly $k - 1$

paths attached to $k-1$ vertices of the cycle. By Lemma 4.1.16, $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}})) \geq \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}))$. Thus with similar argument, after a finite number of steps we have $\lambda_1(L(H)) \geq \lambda_1(L(H_1)) \geq \cdots \geq \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}))$. Note that $\lambda_1(L(G)) \geq \lambda_1(L(H))$, by Lemma 4.3.14. Hence the result holds. ∎

Definition 4.3.16. Consider the path $P_n, n \geq 3$ on vertices $1, \ldots, n$. Add the blue colored edge joining the vertices k and $n-k+1$ of P_n, for $1 \leq k < \frac{n}{2}$ to obtain a mixed graph. We denote this graph by $G(n,k)$.

Example 4.3.17. Mixed graphs $G(7,3)$ and $G(8,4)$ are supplied in the following picture.

$G(7,3)$ $G(8,2)$

The following Lemma is essentially contained in [22, Theorem 3.4].

Lemma 4.3.18. *Consider the b-lollipop graph* ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}$. *Then the following holds:*

(i) Multiplicity of the smallest Laplacian eigenvalue of ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}$ *is one, for* $m < n$.

(ii) If x *is a first eigenvector of* ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}$, *then* $x(i) = -x(m-i)$, *for* $i = 1, \ldots, m-1$.

(iii) $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m-1}})) < \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}))$, *for* $m > 3$.

In the next lemma, we prove that $\lambda_1(L(G(n,k))) = a(P_n)$, for each k.

Lemma 4.3.19. *Consider the mixed graph* $G(n,k)$ *as in Definition 4.3.16*

(i) $\lambda_1(L(G(n,k)))$ *has multiplicity one, for each* $k \neq 1$.

(ii) $\lambda_1(L(G(n,k)) = a(P_n)$,

Proof. (i) Let Y be a first eigenvector of $G(n,k)$. If possible, suppose that $Y(n) = 0$. Then by the eigen-conditions at the vertices $n, n-1, \ldots, n-k+2$, it follows that $Y(n-k+1) = Y(n-k+2) = \cdots = Y(n) = 0$. So $\lambda_1(L(G(n,k))$ is an eigenvalue of ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}}$ with $Y' = [Y(1), Y(2), \cdots, Y(n-k+1)]^t$ as an eigenvector. Thus $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}})) \leq \lambda_1(L(G(n,k)))$. By Lemma 4.3.1,

$$\lambda_1(L(G(n,k))) \leq \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}})).$$

Hence $\lambda_1(L(G(n,k))) = \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}}))$ and Y' is an eigenvector corresponding to $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}}))$. Observe that $Y'(k+1) = -Y'(n-k+1)$, by Lemma 4.3.18(ii). Thus $Y(k+1) = -Y(n-k+1) = 0$, a contradiction to Lemma 4.2.9. So $Y(n) \neq 0$ and hence $\lambda_1(L(G(n,k)))$ has multiplicity one, for $k \neq 1, n$.

(ii) Let y be a Fiedler vector of P_n. Then $y(k) = -y(n-k+1)$. Hence y is an eigenvector of $L(G(n,k))$ corresponding to $a(P_n)$. Thus $\lambda_1(L(G(n,k))) \leq a(P_n)$. If possible, suppose that $\lambda_1(L(G(n,k))) \neq a(P_n)$. Let z be an eigenvector of $L(G(n,k))$ with $z^tz = 1$ corresponding to $\lambda_1(L(G(n,k)))$. Then $a(P_n)y^tz = y^tL(G(n,k))z = \lambda_1(L(G(n,k)))y^tz$ and hence $y^tz = 0$. Let $\frac{1}{\sqrt{n}}\mathbb{1}, y_2, \ldots, y_n$ be the normalized eigenvectors of $L(P_n)$ corresponding to eigenvalues $0, \lambda_2 = a(P_n), \ldots, \lambda_n$. As $y^tz = 0$ and $z \neq \alpha\mathbb{1}$, for $\alpha \in \mathbb{R}$, we see that $z = \alpha_1\frac{\mathbb{1}}{\sqrt{n}} + \alpha_3 y_3 + \ldots + \alpha_n y_n$, where at least one $\alpha_i \neq 0$, for $2 < i \leq n$. Observe that

$$\begin{aligned}\lambda_1(L(G(n,k))) &= z^tL(P_n)z + [z(k) + z(n-k+1)]^2\\ &= [\alpha_3^2\lambda_3 + \ldots \alpha_n^2\lambda_n] + [z(k) + z(n-k+1)]^2\\ &> a(P_n)[1-\alpha_1^2] + \alpha_1^2\frac{4}{n} = a(P_n) + \alpha_1^2\left[\frac{4}{n} - a(P_n)\right]\\ &> a(P_n), \text{ as it can be seen that } \frac{4}{n} > 2(1-\cos\frac{\pi}{n}) = a(P_n),\end{aligned}$$

a contradiction. Hence the result holds. ■

Lemma 4.3.20. $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) \geq \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}))$.

Proof. Let $x \in \mathbb{C}^n$ be a first eigenvector of ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}$ such that $\mathsf{Im}\, x(m) = 0$. Thus by Theorem 4.2.5, $\mathsf{Re}\, x(m) \neq 0$ and $\mathsf{Im}\, x(m) = \mathsf{Im}\, x(m+1) = \cdots = \mathsf{Im}\, x(n) = 0$. By Theorem 3.1.10, $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) = \lambda_1(L(\mathsf{G[g]}))$, and $Y = \begin{bmatrix}\mathsf{Re}\, x\\ -\mathsf{Im}\, x\end{bmatrix}$ is a first eigenvector of $\mathsf{G[g]}$, where G=${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}$. Note that $Y(m') = Y(m'+1) = \cdots = Y(n') = 0$. Thus $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) = \lambda_1(L(\mathsf{G[g]})) = \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}}))$. Using Lemma 4.3.1 and

Lemma 4.3.19(ii) we have, $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}})) \geq \lambda_1(L(G(2n, n-m+1))) = a(P_{2n})$. Hence $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) = \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}})) \geq a(P_{2n}) = 2(1 - \cos\frac{\pi}{2n})$. By Lemma 4.3.3, $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}})) = 2(1 - \cos\frac{\pi}{2n})$. Hence $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) \geq \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}))$. ∎

The next lemma follows from Theorem 4.3.15 and Lemma 4.3.20.

Lemma 4.3.21. *Let G be a unicyclic 3-colored digraph on n vertices. Then*

$$\lambda_1(L(G)) \geq 2(1 - \cos\frac{\pi}{2n}).$$

Equality holds if and only if G is a cycle ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}$.

Next Theorem is our main result of this section which provides the unique graph minimzing the smallest Laplacian eigenvalue over all non-singular unicyclic 3-colored digraphs on n vertices.

Theorem 4.3.22. *Among all non-singular unicyclic 3-colored digraphs on n vertices, the smallest Laplacian eigenvalue is minimized by the cycle ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}$.*

Proof. Proof follows from Lemma 4.3.21. ∎

4.4 Maximizing the smallest Laplacian eigenvalue over unicyclic 3-colored digraphs

In this section we obtain the non-singular unicyclic 3-colored digraph maximizing the smallest Laplacian eigenvalue over all non-singular unicyclic 3-colored digraph on n vertices with a fixed girth.

Lemma 4.4.1. *Let G be a non-singular unicyclic 3-colored digraph on $n > 4$ vertices. Let T be a tree on k vertices which is not a star attached to a vertex v on the cycle. Replace T with a star $K_{1,k}$ on k vertices in G in such a way that v is the center of $K_{1,k}$ and call the resulting graph G'. Then $\lambda_1(L(G')) > \lambda_1(L(G))$.*

Proof. Let $v = v_1, v_2, \ldots, v_k$ be the vertices of T such that $vv_2 \in E(T)$. Let x be a first eigenvector of G'. By Lemmas 4.3.9 and 4.3.10, $\lambda_1(L(G')) < 1$. Hence by eigen-condition at the vertices v_i, we have $x(v_2) = \cdots = x(v_k)$. Let d_v and d'_v be the degrees of the vertex v in G and G', respectively. Note that $d_v < d'_v$. Thus

$$\begin{aligned}
\lambda_1(L(G')) &= x^*L(G')x \\
&= x^*L(G' - E(K_{1,k}))x + \sum_{r=2}^{k} |x(v) - x(v_2)|^2 \\
&= x^*L(G' - E(K_{1,k}))x + (d'_v - 2)|x(v) - x(v_2)|^2 \\
&> x^*L(G' - E(K_{1,k}))x + (d_v - 2)|x(v) - x(v_2)|^2 \\
&= x^*L(G' - E(K_{1,k}))x + \sum_{v_iv_j \in E(T)} |x_(v_i) - x(v_j)|^2 = x^*L(G)x \\
&\geq \lambda_1(L(G))
\end{aligned}$$

■

Applying Lemma 4.3.14 repeatedly and then using Lemma 4.1.16, we obtain the following result which is a generalization of [18, Theorem 3.4].

Theorem 4.4.2. *Among all non-singular unicyclic 3-colored digraphs with a fixed girth m, the smallest Laplacian eigenvalue is maximized by a non-singular unicyclic 3-colored digraph of girth m with the following property: weight of the cycle is -1 and each vertex on the cycle is adjacent only to a non-negative number of pendent vertices out side the cycle.*

4.5 Unicyclic 3-colored digraph with second smallest Laplacian eigenvalue equal to 1

In this section we obtain the non-singular unicyclic 3-colored digraphs G that have second smallest Laplacian eigenvalue greater than 1, less than 1 and equal to 1.

Lemma 4.5.1. *Let G be a non-singular unicyclic 3-colored digraph with a fixed girth m. Let C be the non-singular cycle in G. Then the following holds.*

(i) If $m \geq 4$ and $\mathsf{w}_C = -1$, then $\lambda_2(L(G)) < 1$.

(ii) If $m \geq 5$ and $\mathsf{w}_C = \pm\mathrm{i}$, then $\lambda_2(L(G)) < 1$.

Proof. (i) By Lemma 4.3.2, $\lambda_1(L(C)) = \lambda_2(L(C)) = 2(1 - \cos\frac{\pi}{m})$. It follows from Lemma 4.3.1, $\lambda_2(L(G)) \leq \lambda_2(L(C)) = 2(1 - \cos\frac{\pi}{m})$. As $m \geq 4$, we see that $2(1 - \cos\frac{\pi}{m}) < 1$ and hence $\lambda_2(L(G)) < 1$.

(ii) By Lemmas 4.3.3 and 4.3.1, $\lambda_2(L(G)) \leq \lambda_2(L(C)) = 2(1 - \cos\frac{3\pi}{2m}) < 1$. ∎

Next lemma says that a non-singular unicyclic 3-colored digraph G of girth $m \geq 4$ can not have $\lambda_2(L(G) = 1$.

Lemma 4.5.2. *Let G be a non-singular unicyclic 3-colored digraph with a fixed girth $m \geq 4$. Then $\lambda_2(L(G)) \neq 1$.*

Proof. Observe that $\lambda_2(L(^{\mathsf{g}}\mathsf{C}_4)) = 2\left(1 - \cos(\frac{3\pi}{8})\right) > 1$ and $\lambda_2(L(^{\mathsf{g}}\mathsf{C}_{4,1})) \approx .7586$. Using Lemma 4.3.1 and Lemma 4.5.1, we see that $\lambda_2(L(G)) \neq 1$. ∎

Remark 4.5.3. Note that $\lambda_2(L(^{\mathsf{g}}\mathsf{C}_3)) = 2\left(1 - \cos(\frac{3\pi}{6})\right) = 2$, using Lemma 4.3.3.

Next we prove that the 3-colored digraphs as shown in figure 4.5, figure 4.6, figure 4.7 and figure 4.8 has second smallest eigenvalue equal to 1.

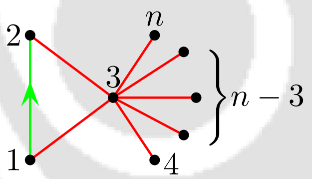


Figure 4.5: $^{\mathsf{g}}G_{\mathsf{n,n-3}}, n \geq 4$.

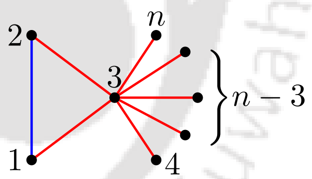


Figure 4.6: $^{\mathsf{b}}G_{\mathsf{n,n-3}}, n \geq 3$.

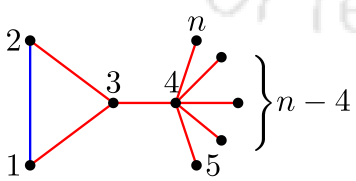


Figure 4.7: $G_{\mathsf{n,1,n-4}}, n \geq 5$.

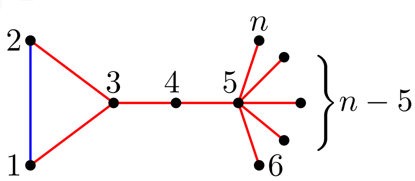


Figure 4.8: $G_{\mathsf{n,2,n-5}}, n \geq 6$.

Lemma 4.5.4. *Consider* ${}^{\mathsf{g}}G_{\mathsf{n,n-3}}$ *on vertices* $1,\ldots,n$ $(n \geq 4)$ *as shown in figure 4.5. Then* $\lambda_2(L({}^{\mathsf{g}}G_{\mathsf{n,n-3}})) = 1$.

Proof. Observe that 1 is an eigenvalue of $L({}^{\mathsf{g}}G_{\mathsf{n,n-3}})$ corresponding to an eigenvector $[1 \quad -\mathrm{i} \quad 0 \quad \mathrm{i}-1 \quad 0 \quad \cdots \quad 0]$. By Lemma 4.3.9, $\lambda_1(L({}^{\mathsf{g}}G_{\mathsf{n,n-3}})) < 1$. Note that $\lambda_1(\hat{L}) = 1$, where $\hat{L}$ is the principal submatrix of $L({}^{\mathsf{g}}G_{\mathsf{n,n-3}})$ corresponding to ${}^{\mathsf{g}}G_{\mathsf{n,n-3}} - 3$. Hence by interlacing Theorem, $1 \leq \lambda_2(L({}^{\mathsf{g}}G_{\mathsf{n,n-3}}))$ which implies $\lambda_2(L({}^{\mathsf{g}}G_{\mathsf{n,n-3}})) = 1$. ∎

Lemma 4.5.5. *[18] Let G be a non-singular unicyclic mixed graph on n vertices with a fixed girth m. Then* $\lambda_1(L(G)) \leq \frac{2m}{n(m-1)} \leq \frac{3}{n}$.

Lemma 4.5.6. *Consider* ${}^{\mathsf{b}}G_{\mathsf{n,n-3}}$, $G_{\mathsf{n},1,\mathsf{n}-4}$ *and* $G_{\mathsf{n},2,\mathsf{n}-5}$ *on vertices* $1,\ldots,n$ *as shown in figure 4.6, figure 4.7 and figure 4.8, respectively. Then* $\lambda_2(L({}^{\mathsf{b}}G_{\mathsf{n,n-3}})) = 1$, $\lambda_2(L(G_{\mathsf{n},1,\mathsf{n}-4})) = 1$, $\lambda_2(L(G_{\mathsf{n},2,\mathsf{n}-5})) = 1$.

Proof. For $n = 3$ the result holds. Let $n \geq 4$ and $\hat{L}$ be the principal submatrix of $L(G_{\mathsf{n,n-3}})$ corresponding to ${}^{\mathsf{b}}G_{\mathsf{n,n-3}} - 3$. Note that $\lambda_1(\hat{L}) = 1$. Hence by interlacing Theorem, $1 \leq \lambda_2(L({}^{\mathsf{b}}G_{\mathsf{n,n-3}}))$. Observe that 1 is an eigenvalue of $L({}^{\mathsf{b}}G_{\mathsf{n,n-3}})$ with an eigenvector $[1 \quad -1 \quad 0 \quad \cdots \quad 0]$. By Lemma 4.5.5, $\lambda_1(L({}^{\mathsf{b}}G_{\mathsf{n,n-3}})) < 1$, which implies $\lambda_2(L({}^{\mathsf{b}}G_{\mathsf{n,n-3}})) = 1$. With similar argument we can show $\lambda_2(L(G_{\mathsf{n},1,\mathsf{n}-4})) = 1$ and $\lambda_2(L(G_{\mathsf{n},2,\mathsf{n}-5})) = 1$. ∎

Using eigen-condition we have the following Proposition.

Proposition 4.5.7. *Let G be a 3 colored digraph and u be a vertex adjacent to a pendent vertex v in G. Let y be an eigenvector of $L(G)$ corresponding to an eigenvalue 1. Then $y(u) = 0$.*

Next, we show that the non-singular unicyclic 3-colored digraphs in figure 4.9(a), 4.9(b), 4.9(c), 4.9(d), 4.9(e) and 4.9(f) have second smallest Laplacian eigenvalue less than 1.

Lemma 4.5.8. *Let G be a non-singular unicyclic 3-colored digraph as in figure 4.9(a), 4.9(b), or 4.9(c). Then* $\lambda_2(L(G)) < 1$.

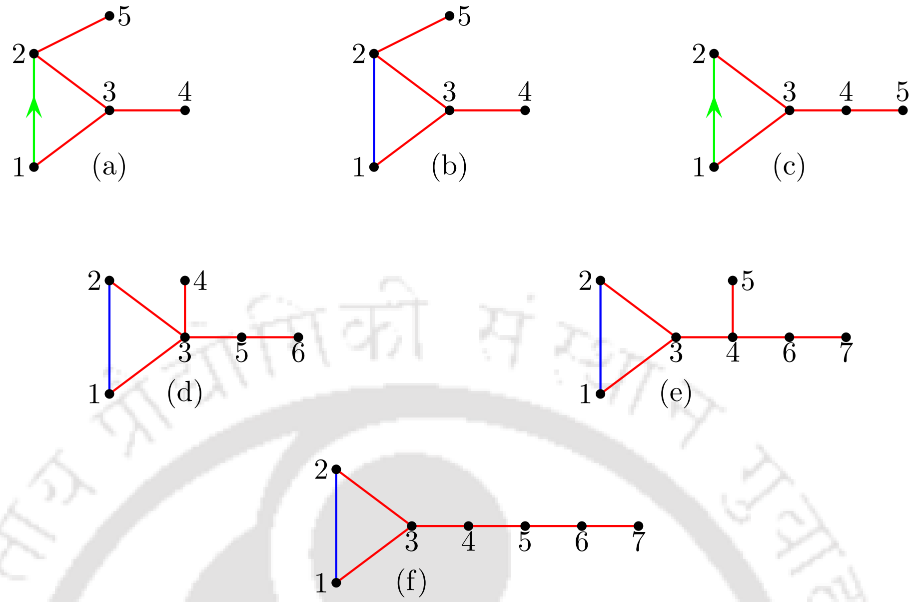


Figure 4.9: Non-singular unicyclic 3-colored digraphs with $\lambda_2(L(G)) < 1$.

Proof. Consider G as in figure 4.9(c). Note that $\lambda_2(L(G)) \leq \lambda_2(L({}^{\mathfrak{g}}G_{4,1})) = 1$, by Lemma 4.3.1 and Lemma 4.5.4. If possible, let 1 be an eigenvalue of $L(G)$ with an eigenvector y. Then $y(4) = 0$ by Proposition 4.5.7 and hence eigen-conditions at the vertices we see that $y(1) = y(2) = y(3) = -y(5) = 0$, which is impossible. Thus $\lambda_2(L(G)) < 1$. Proofs for G as in figure 4.9(a) and 4.9(b) follows with similar arguments. ∎

Lemma 4.5.9. *Let* $A = \begin{bmatrix} B & C \\ C^t & d \end{bmatrix}$ *be a symmetric matrix, where* C *is a vector and* d *is real. Let there exist a vector* U *such that* $BU = 0$ *and* $C^tU \neq 0$. *Then* $In(A) = In(B) + (1, 1, -1)$, *where* $In(A), In(B)$ *denotes inertia of* A, B, *respectively.*

Lemma 4.5.10. *Let* G *be a non-singular unicyclic* 3*-colored digraph as shown in figure 4.9(d) or figure 4.9(e). Then* $\lambda_2(L(G)) < 1$.

Proof. Consider G as in figure 4.9(d). Note that $\lambda_1(\hat{L}) < 1$, where $\hat{L}$ is the principal submatrix of $L(G)$ corresponding to $G - 3$. After a permutation similarity operation we have,

$$L(G) = \begin{bmatrix} \hat{L} & C \\ C^t & 4 \end{bmatrix} \text{ and } \hat{L} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } C^t = [-1, -1, -1, 0, -1].$$

Observe that $\hat{L}y = y$ and $C^t y \neq 0$, where $y = [0 \quad 0 \quad 0 \quad 0 \quad 1]$. Thus by Lemma 4.5.9, $L(G) - I$ has at least 2 negative eigenvalues, as $\hat{L} - I$ has at least one negative eigenvalue. Hence $\lambda_2(L(G)) < 1$. Similarly, if G is as in figure 4.9(e), we have $\lambda_2(L(G)) < 1$. ■

Lemma 4.5.11. *Let G be a non-singular unicyclic 3-colored digraph as shown in figure 4.9(f). Then $\lambda_2(L(G)) < 1$.*

Proof. Note that $\lambda_1(\hat{L}) < 1$, where $\hat{L}$ is the principal submatrix of $L(G)$ corresponding to $G - 5$. After a permutation similarity operation we have,

$$L(G) = \begin{bmatrix} \hat{L} & C \\ C^t & 2 \end{bmatrix} \text{ and } \hat{L} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}, \text{ where } C^t = [0 \quad 0 \quad 0 \quad -1 \quad -1 \quad 0].$$

Observe that $\hat{L}y = y$ and $C^t y \neq 0$, where $y = [1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0]$. Thus by Lemma 4.5.9, $L(G) - I$ has at least 2 negative eigenvalues, as $\hat{L} - I$ has at least one negative eigenvalue. Hence $\lambda_2(L(G)) < 1$. ■

We summarise our discussions to state the main result of this section.

Theorem 4.5.12. *Let G be a non-singular unicyclic 3-colored digraph on n vertices. Then $\lambda_2(L(G)) = 1$ if and only if G is one of ${}^{\mathsf{g}}G_{\mathsf{n,n-3}}$, ${}^{\mathsf{b}}G_{\mathsf{n,n-3}}$, $G_{\mathsf{n,1,n-4}}$ and $G_{\mathsf{n,2,n-5}}$.*

Proof. Assume that $\lambda_2(L(G)) = 1$ and G is not any one of ${}^{\mathsf{g}}G_{\mathsf{n,n-3}}$, ${}^{\mathsf{b}}G_{\mathsf{n,n-3}}$, $G_{\mathsf{n,1,n-4}}$, $G_{\mathsf{n,2,n-5}}$. By Lemma 4.5.2, girth of G is 3. Also, $G \neq {}^{\mathsf{g}}\mathsf{C}_3$, by Remark 4.5.3. Thus G can be obtained from one of the 3-colored digraphs as shown in figure 4.9(a), 4.9(b), 4.9(c), 4.9(d), 4.9(e) and 4.9(f) by adding pendents repeatedly. Thus by Lemma 4.3.1 and Lemma 4.5.8, Lemma 4.5.10 or Lemma 4.5.11, $\lambda_2(G) < 1$, a contradiction. Converse follows from Lemma 4.5.4 and Lemma 4.5.6. ■

Chapter 5

Unicyclic 3-colored digraphs with SR-property

In Chapter 2 we have observed that the notion of 3-colored digraph is a generalization of the notion of mixed graph. The adjacency matrix of a 3-colored digraph distinguishes the orientations of the green edges and hence spectral properties of $A(G)$ may supply some additional structural information about the graph. The characteristic polynomial $P(G; x)$ of an undirected graph has been studied by many researchers (see for example [13]). The coefficients in $P(G; x)$ have a combinatorial interpretation in terms of the graph structure. In Section 5.1 we supply a combinatorial interpretation of the coefficients of $P(G; x)$ in terms of the graph structure, for a 3-colored digraph G.

Definition 5.0.13. Let G be a 3-colored digraph and $\sigma(G)$ be the multiset of eigenvalues of $A(G)$. We say G has *SR-property* if $\lambda \in \sigma(G)$ if and only if $\frac{1}{\lambda} \in \sigma(G)$ and both have the same multiplicity. It is understood that $A(G)$ is non-singular.

In Section 5.2 we discuss the 3-colored digraphs satisfying SR-property. In Section 5.3 we supply the structure of the unicyclic 3-colored digraphs containing the cycle of weight -1 satisfying SR-property. In Section 5.4 we supply the structure of the unicyclic 3-colored digraphs containing the cycle of weight $\pm$i satisfying SR-property.

5.1 Characteristic polynomial of $A(G)$

In this section we study the coefficients of the characteristic polynomial of a 3-colored digraph. Recall that in defining subgraph, walk, path, matching, component, connectedness, and degree of a vertex in G we focus only on the underlying unweighted undirected graph of G.

Definition 5.1.1. We call a subgraph H *linear* if each component of H is either an edge or a cycle. We call a component of a linear subgraph *non-singular*, if it is a cycle of weight other than 1. All other components are called singular.[1]

▶ We use C_H to mean the number of cycles in a linear subgraph H. By S_H and N_H we mean the number of singular components and the number of non-singular components in H, respectively.

▶ We use $\mathcal{C}_k$ to denote the set $\{H \mid H$ is a linear subgraph of order k of G in which no cycle has weight $\pm\mathrm{i}\}$.

Lemma 5.1.2. *Let G be a 3-colored digraph. Then* $\det A(G) = \sum\limits_{H \in \mathcal{C}_n} (-1)^{n-S_H} 2^{C_H}$.

Proof. Note that $\det A(G) = \sum_\pi \operatorname{sgn}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}$, where the summation is taken over all permutations of $\{1,\ldots,n\}$. Consider a permutation π giving a nonzero contribution to $\det A(G)$. Note that each term in the cycle decomposition of π, corresponds either to an edge or to a cycle of length at least 3 in G. As $\pi(i) \neq i$, for any i, we see that π corresponds to a spanning linear subgraph of G.

Let H' be a linear subgraph of G containing a cycle $C = [i_1 \cdots i_k, i_1]$ of weight $\pm\mathrm{i}$. Let π be a permutation corresponding to H'. Take the permutation τ, satisfying $\tau(i_2) = i_1, \tau(i_3) = i_2, \ldots, \tau(i_k) = i_{k-1}, \tau(i_1) = i_k$, and $\tau(i) = \pi(i)$ for $i \notin \{i_1, i_2 \ldots i_k\}$. Then $a_{1,\pi(1)} \cdots a_{n,\pi(n)} = -\left(a_{1,\tau(1)} \ldots a_{n,\tau(n)}\right)$. Thus the contribution from linear subgraphs of G which contains at least one cycle of weight $\pm\mathrm{i}$, to $\det A(G)$ is 0. Observe that $a_{1,\pi(1)} \ldots a_{n,\pi(n)} = (-1)^{N_H}$ and $\operatorname{sgn}(\pi) = (-1)^{n-S_H-N_H}$, for each permutation π corresponding to a linear subgraph H of G without a cycle of weight $\pm\mathrm{i}$. Since each linear subgraph H of G corresponds to exactly 2^{C_H} permutations of $\{1\ldots,n\}$, we see that $\det A(G) = \sum_{H\in\mathcal{C}_n}(-1)^{n-S_H}2^{C_H}$. ■

▶ Henceforth, we understand that $P(G;x) = \sum_{i=0}^{n} a_i x^{n-i}$, $a_0 = 1$.

Next, we describe the coefficients a_i of $P(G;x)$ generalizing Sachs theorem [13].

Theorem 5.1.3. *Let G be a 3-colored digraph. Then* $a_i = \sum\limits_{H\in\mathcal{C}_i} (-1)^{S_H} 2^{C_H}$.

[1]They are actually Laplacian singular, see Theorem 2.1.11.

Proof. Note that a_i is $(-1)^i E_i(A)$, where $E_i(A)$ is the sum of the principal minors of $A = A(G)$ of order i. Hence by Lemma 5.1.2,

$$a_i = (-1)^i \sum_{H \in \mathcal{C}_i} (-1)^{i-S_H} 2^{C_H} = \sum_{H \in \mathcal{C}_i} (-1)^{S_H} 2^{C_H}.$$

∎

The following lemma is crucial for further developments which follows from Theorem 2.1.11 and Lemma 4.1.1.

Lemma 5.1.4. *Let G be a connected unicyclic 3-colored digraph with the cycle C. Then the following holds:*

(i) If $\mathsf{w}_C = 1$, then G is D-similar to a unicyclic graph with all edges red.

(ii) If $\mathsf{w}_C = -1$, then G is D-similar to a unicyclic graph with all edges red except one edge on C which is blue.

(iii) If $\mathsf{w}_C = \pm\mathrm{i}$, then G is D-similar to a unicyclic graph with all edges red except one edge on C which is green.

5.2 3-colored digraphs with SR-property

In view of Lemma 5.1.4, unicyclic 3-colored digraphs with all edges red satisfying SR-property was characterized in [7]. Let us denote the cycle in a unicyclic 3-colored digraph G by C. Henceforth, all our unicyclic 3-colored digraphs have all edges red except possibly **one edge** on C which is either blue or green.

Recall that an unweighted undirected graph G is bipartite if and only if G satisfies the negation property:

(N) $\qquad -\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$.

Definition 5.2.1. We say a 3-colored digraph G has the *strong negative property*

(SN) $\qquad$ if $-\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$ and both have equal multiplicities.

Definition 5.2.2. We call a 3-colored digraph ±1-*bipartite* if it does not contain odd cycles of weight ±1.

Lemma 5.2.3. *Let G be a 3-colored digraph which is ± 1-bipartite. Then G has SN-property.*

Proof. Consider $P(G;x)$. By Theorem 5.1.3, $a_{2i+1} = 0$. Thus $P(G;x) = x^k q(x^2)$, for some integer k and polynomial $q(x)$. Hence G has SN-property. ∎

▶ Unlike the undirected case, converse of Lemma 5.2.3 is false. For example consider G as shown in Figure 5.1. It has an odd cycle of weight ± 1 and still has SN-property.

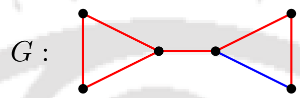


Figure 5.1: G has an odd cycle of weight ± 1 and it has SN-property

However the situation is better for unicyclic graphs.

Lemma 5.2.4. *Let G be a unicyclic 3-colored digraph. Then G has SN-property if and only if G is ± 1-bipartite.*

Proof. Assume that G has SN-property. Hence $a_{2i}a_{2i+1} = 0$, for each i. If possible, suppose that $\mathsf{w}_C = \pm 1$ and girth $g = 2k+1$. Then by Theorem 5.1.3, $a_{2k+1} = \pm 2$ and $a_{2k} = (-1)^k |\mathcal{C}_{2k}|$. Clearly $\mathcal{C}_{2k} \neq \emptyset$. Thus $a_{2k}a_{2k+1} \neq 0$, a contradiction. Hence g is even. The converse follows from Lemma 5.2.3. ∎

Definition 5.2.5. [26] Let H be a 3-colored digraph. For each $j \in V(H)$ add a new vertex j' and add the edge (j, j') of weight $w_j = \pm 1, \mathrm{i}$. We call the resulting 3-colored digraph a *simple corona* and denote it by $H \circ K_1$. A *non-corona* graph means a graph that is not a simple corona.

Remark 5.2.6. Let H be a 3-colored and $G = H \circ K_1$. Put $D = \mathrm{diag}(w_1, \ldots, w_n)$ and $\hat{D} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & D^* \end{bmatrix}$. Then $A(G) = \begin{bmatrix} A(H) & D \\ D^* & 0 \end{bmatrix}$ and $B = \hat{D}^* A(G) \hat{D} = \begin{bmatrix} A(H) & I \\ I & \mathbf{0} \end{bmatrix}$. Thus G is D-similar to the 3-colored digraph obtained from H by adding a pendent vertex to each vertex i of H with an edge of color red. Henceforth, the newly added edges in a simple corona are taken to be red. By [8, Lemma 2.1] $-\frac{1}{\lambda} \in \sigma(B)$ whenever $\lambda \in \sigma(B)$.

Lemma 5.2.7. *Let H be a 3-colored, ± 1-bipartite. Then $H \circ K_1$ has SR-property.*

Proof. Follows using Lemma 5.2.3 and Remark 5.2.6. ∎

It is natural to wonder whether a 3-colored digraph with SR-property is necessarily a simple corona.

Lemma 5.2.8. *Let G be a 3-colored digraph with SR-property. Then $|a_i| = |a_{n-i}|$, for each i.*

Proof. Similar to [7, Lemma 2.1]. ∎

▶ We denote the number of perfect matchings of a graph H by $m_0(H)$. For a graph with a unique perfect matching we **denote** the perfect matching by M.

Definition 5.2.9. Let G be a 3-colored digraph with a unique perfect matching. An *alternating path* P in G is a path of odd length such that alternate edges of P (including the terminating edges) are in M. By $\mathcal{P}_G^k$ we denote the set of all alternating paths of length at least k in G. By $m_1(G)$ we denote the number of $(\frac{n}{2} - 1)$-matchings of G.

Definition 5.2.10. Let G be a connected unicyclic 3-colored digraph. For $v \in V(G)$, a component T of $G - v$ which does not contain any other vertex of C is called a *tree branch* at v. We say a tree branch is *odd* or *even*, if the order of the tree branch is odd or even, respectively.

Lemma 5.2.11. *Let G be a unicyclic 3-colored digraph satisfying SR-property. Then $m_0(G) = 1$. Furthermore, there is an odd tree branch of G at a vertex of the cycle C.*

Proof. If $\mathsf{w}_C = 1$, then in view of Lemma 5.1.4, this result is precisely Lemma 2.2 of [7]. If $\mathsf{w}_C = -1$, then the proof is similar to that of Lemma 2.2 of [7].

If $\mathsf{w}_C = \pm\mathrm{i}$, then by Theorem 5.1.2, $\pm 1 = \det A(G) = \pm m_0(G)$. Thus $m_0(G) = 1$ and so $n = 2m$, for some integer m. Note that, if $m_0(G - C) > 0$, then C must be an even cycle, and hence $m_0(G) \geq 2$, not possible. Thus $m_0(G - C) = 0$, which implies the existence of a vertex u on C at which there is an odd tree branch. ∎

Lemma 5.2.12. *Let G be a connected 3-colored digraph on $n = 2m$ vertices with $m_0(G) = 1$. Then $m_1(G) = m + |\mathcal{P}_G^3| = |E(G)| + |\mathcal{P}_G^5|$.*

Proof. Put $M = \{e_1, e_2, \ldots, e_m\}$ and let $f_1, f_2, \ldots, f_s$ be the remaining edges of G. Note that any $(m-1)$-subset of M contributes 1 to $m_1(G)$. Take an alternating path in $\mathcal{P}_G^3$, say, $e_1 f_1 e_2 f_2 \ldots f_{2k} e_{2k+1}$. Then $\{f_{j_1}, \ldots, f_{j_{r-1}}\} \cup M - \{e_{i_1}, \ldots, e_{i_r}\}$ is an $(m-1)$-matching of G. Thus $m_1 \geq n + |\mathcal{P}_G^3|$.

Conversely, take an $(m-1)$-matching, say,

$$(*) \qquad e_1, \ldots, e_p, f_1, \ldots, f_q.$$

Assume that e_is have end vertices u_i, v_i. Put $V = \{u_{p+1}, \ldots, u_m, v_{p+1}, \ldots, v_m\}$ and $E = \{e_{p+1}, \ldots, e_m, f_1, \ldots, f_q\}$. Consider $H = (V, E)$. Note that $|E| = |V| - 1$.

Observation. No two e_is in E have a common end vertex. Similarly, no two f_js in E have a common end vertex.

Assume that H contains a cycle Γ. Then, in view of the previous observation, Γ must be an even cycle containing e_is and f_js alternately. This implies that G has more than one perfect matchings. Not possible.

It follows that H is a tree. Then, in view of the previous observation, the maximum degree of a vertex in H is 2. Hence H is path. Obviously, it is an alternating path.

Observe that H is nothing but the subgraph of G induced by the edges (and their end vertices) in E. If there is another alternating path say H' which results in the same $(m-1)$-matching as in $(*)$, then $E(H')$ must be $\{e_{p+1}, \ldots, e_m, f_1, \ldots, f_q\}$. Hence H' is nothing but the subgraph induced by E. So $H = H'$.

Hence $m_1(G) = m + |\mathcal{P}_G^3|$. The other equality follows as the number of alternating paths of length 3 equals the number of non-matching edges. ∎

► Let G be a unicyclic 3-colored digraph. By $\mathcal{L}_G$ we mean the collection of all linear subgraphs of G on $n-2$ vertices containing C as one of its components.

Lemma 5.2.13. *Let G be a 3-colored digraph with $m_0(G) = 1$. The following statements hold.*

(a) If G is a simple corona, then $\mathcal{P}_G^5 = \emptyset$.

(b) If G is a unicyclic simple corona, then $\mathcal{L}_G = \emptyset$.

(c) If $\mathcal{P}_G^5 = \emptyset$ and girth $g > 3$, then G is a simple corona.

Proof. (a)-(b) Trivial as the pendant edges form the unique perfect matching.

(c) Suppose that G is not a simple corona. Then there is an edge $\{u, v\} \in M$ such that $d_u, d_v \geq 2$. So we have a path $p : u_1, u, v, v_1$ in G such that $\{u_1, u\}, \{v, v_1\} \notin M$. Note that $u_1 \neq v_1$, as $g > 3$. Further, if $\{u_1, v_1\} \in M$, then G has more than one perfect matchings, which is not possible. Thus there exists $u_2, v_2 \in V(G)$ such that $\{u_2, u_1\}, \{v_1, v_2\} \in M$. So $q : u_2, u_1, u, v, v_1, v_2 \in \mathcal{P}_G^5$, a contradiction. ∎

Remark 5.2.14. Let G be a connected non-corona unicyclic 3-colored digraph satisfying SR-property not containing a green edge. Assume that $\mathcal{P}_G^5 = \emptyset$ and girth $g = 3$. Consider proof (c) of Lemma 5.2.13. Noting that $2 = |a_3| = |a_{n-3}|$, we see that G is as shown in Figure 5.2 with possibly one edge on the cycle being blue. We can verify that G does not satisfy SR-property.

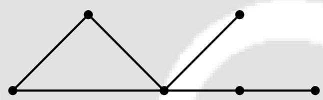


Figure 5.2: Only possible structure of a non-corona G with $\mathcal{P}_G^5 = \emptyset$ and girth $g = 3$. Here at most one edge on the cycle can be blue.

Theorem 5.2.15. *Let G be a connected unicyclic 3-colored digraph satisfying SR-property. Assume that either $\mathsf{w}_C = \pm \mathrm{i}$ or $\mathcal{L}_G = \emptyset$. Then $\mathcal{P}_G^5 = \emptyset$.*

Furthermore, if girth $g > 3$, then G is a simple corona.

Proof. By Theorem 5.1.3, $a_{n-2}(G) = (-1)^{m-1} m_1(G)$ and $|a_2| = |E(G)| = n$. By Lemma 5.2.8, $|a_2| = |a_{n-2}|$. Hence $m_1(G) = n$. By Lemma 5.2.12, $\mathcal{P}_G^5 = \emptyset$. The next conclusion follows from Lemma 5.2.13. ∎

5.3 Unicyclic 3-colored digraphs with a blue edge on C

Throughout this section G_b denotes a connected unicyclic 3-colored digraph with a cycle C on which there is one blue edge and all other edges are red.

Lemma 5.3.1. *Let $G = G_b$ satisfy SR-property. The following statements hold.*
(a) Girth g is even. Furthermore, if $g \equiv 0 \pmod 4$, then G is a simple corona.
(b) $|\mathcal{P}_G^5| = 2|\mathcal{L}_G|$.
(c) If G is not a simple corona, then G has exactly two odd tree branches at two different vertices of the cycle.

Proof. (a) Similar to that of [7, Theorem 2.7].
(b) Similar to that of [7, Lemma 2.9].
(c) Similar to that of [7, Lemma 2.8]. ■

Lemma 5.3.2. *[7, Lemma 2.11] Let T be a tree such that $T - v$ has a perfect matching M_v and u be another vertex in T. Suppose that $[v = v_1, \ldots, v_r = u]$ is the unique path from v to u in T. Then $T - u$ has a perfect matching M_u if and only if $r = 2k + 1$, for some k and the edges $\{v_{2i}, v_{2i+1}\} \in M_v$.*

Lemma 5.3.3. *Let T be a tree with $|T| \geq 3$ odd, such that $T - v$ has a perfect matching M. Then there exist a pendent vertex u such that on the v-u-path $[v = v_0, v_1, \ldots, v_{2k} = u]$ in T, the edges $\{v_{2i+1}, v_{2i+2}\} \in M$, for $i = 0, \ldots, k - 1$.*

Proof. Take a vertex v_1 adjacent to v in T. Then there exist a vertex v_2 in $T - v$ such that $\{v_1, v_2\} \in M$, as $T - v$ has perfect matching. If v_2 is pendent then we take $u = v_2$. If not, take a vertex v_3 adjacent to v_2. Continuing similarly, after finitely many steps we obtain u satisfying the statement. ■

Lemma 5.3.4. *Let $G = G_b$ satisfy SR-property. Assume that G is not a simple corona. In view of Lemma 5.3.1, let T_u and T_v be the odd tree branches at vertices u and v on C. Then $|T_u| = |T_v| = 1$ and the girth $g = 6$.*

Proof. Let M be the unique perfect matching of G and let $\{u, u_0\}, \{v, v_0\} \in M$ where $u_0 \in T_u, v_0 \in T_v$. Note that each vertex $w \neq u, v$ on C is matched to another vertex of C and hence both the u_0-v_0-paths in G are alternating. Let p_1, p_2 be these two paths. By Lemma 5.3.1, we see that $g \equiv 2 \pmod 4$. Thus $g \geq 6$ and hence either <u>case 1:</u> one of p_1, p_2 has length at least 7 (say p_1) or <u>case 2:</u> both have length at least 5.

Claim $|\mathcal{L}_G| = 1$. If possible, suppose that $|\mathcal{L}_G| > 1$. Let $D \in \mathcal{L}_G$ such that D misses a vertex $w \in T_u, w \neq u_0$. As $T_u - u_0$ has a perfect matching (it is part of M), by Lemma 5.3.2, the w-u_0-path is alternating. If case 1 (resp. case 2) holds then we can extend this alternating path along p_1 (resp. along p_1 and p_2) to obtain two alternating paths of lengths at least 5 starting from w and ending at some vertex of the cycle. Furthermore, we have an alternating w-v_0-path which has length at least 5. Thus $|\mathcal{P}^5_G| \geq 3[|\mathcal{L}_G| - 1] + 2 > 2|\mathcal{L}_G|$ which is not possible by Lemma 5.3.1.

Suppose that $|T_u| > 1$. By Lemma 5.3.3, there are two vertices $u_0, x \in T_u$ such that $T_u - u_0$ and $T_u - x$ have perfect matchings. Therefore $|\mathcal{L}_G| > 1$, not possible.

If $g \geq 10$, then $|\mathcal{P}^5_G| > 2$. This is not possible. ∎

Corollary 5.3.5. *Let $G = G_b$ have girth $g \neq 6$. Then G has SR-property if and only if G is a simple corona with girth even.*

Proof. Follows from Lemmas 5.3.1, 5.3.4, and 5.2.7. ∎

We shall now investigate the non-corona unicyclic graphs G_b with girth $g = 6$ which satisfy SR property. The graph in Figure 5.3 is such an example. In that figure F_u, F_v are forests consisting of corona trees and T^*_u, T^*_v are trees induced by $V(F_u) \cup \{u, u_0\}$ and $V(F_v) \cup \{v, v_0\}$ respectively.

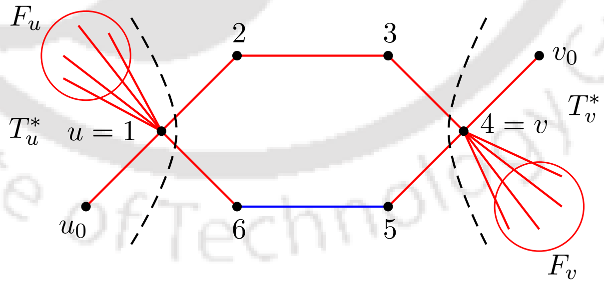


Figure 5.3: A non-corona unicyclic G_b of girth 6 which satisfy SR-property

Lemma 5.3.6. *Let $G, u, v, T_u, T_v, u_0, v_0$ be as in Lemma 5.3.4. Then G has the structure as in figure 5.3, where T^*_u, T^*_v are corona trees. Any edge on the cycle may be blue.*

Proof. The proof here may be seen as a continuation of proof of Lemma 5.2.4. Note that $|\mathcal{P}_G^5| = 2$, by Lemma 5.3.1. Hence 'case 1' does not arise.

Since $[u_0, u, 2, 3]$ is an alternating path of length 3, we cannot have any tree branch at 3 otherwise $|\mathcal{P}_G^5|$ will be more than 2. Similarly we conclude that there are no tree branches at the vertices $2, 5, 6$.

The tree T_u^* cannot have an alternating path of length more than 3. Hence it must be a corona tree. The same is true for T_v^*. ∎

► Let $\alpha, \beta \subset \{1, \ldots, n\}$. By $B(\alpha|\beta)$ we denote the submatrix of $B_{n\times n}$ obtained by deleting the rows and columns corresponding to α and β, respectively.

Lemma 5.3.7. *Let $G = G_b$ be the 3-colored digraph in Figure 5.3. Then*

$$P(G;x) = xP(G-2;x) - P(G-2-3;x) - P(G-1-2;x) + 2P(G-C;x).$$

Proof. Put $B = xI - A(G)$ and put $E_{\alpha|\beta} = \det B(\alpha|\beta)$, for $|\alpha| = |\beta|$. Using Laplace expansion along the second row of B, we see that

$$\begin{aligned}
P(G;x) &= E_{2|1} + xE_{2|2} + E_{2|3} \\
&= -E_{1,2|1,2} + E_{2,3|1,2} + xP(G-2;x) - E_{2,3|2,3} + E_{2,3|3,4} \\
&= -P(G-1-2;x) + E_{2,3,4|1,2,3} + xP(G-2;x) - P(G-2-3;x) \\
&\quad + E_{1,2,3|2,3,4} \\
&= xP(G-2;x) - P(G-1-2;x) - P(G-2-3;x) + 2E_{1,2,3|2,3,4}.
\end{aligned}$$

Note that

$$\begin{aligned}
E_{1,2,3|2,3,4} &= xE_{1,2,3,5|2,3,4,5} - E_{1,2,3,5|2,3,4,6} \\
&= (x^2-1)E_{1,2,3,5,6|2,3,4,5,6} + E_{1,2,3,4,5,6|1,2,3,4,5,6} \\
&= P(G-C;x),
\end{aligned}$$

as after a permutation similarity operation we have

$$B(1,2,3,5,6|2,3,4,5,6) = \left[\begin{array}{cc|c} 0 & C_1^t & 0 \\ \mathbf{0} & B_1 & \mathbf{0} \\ \hline C_2 & \mathbf{0} & B_2 \end{array}\right],$$

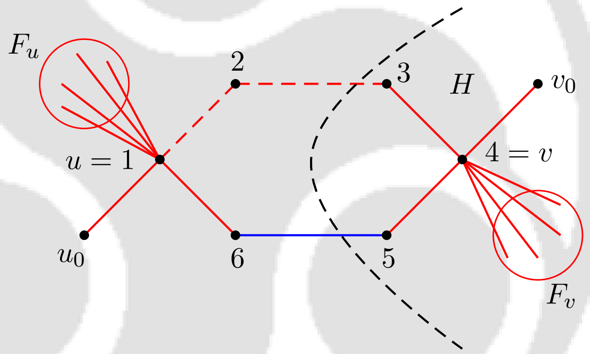
which implies $E_{1,2,3,5,6|2,3,4,5,6} = 0$. Hence the result holds. ■

Next two lemmas are crucial for us which is contained in [13].

Lemma 5.3.8. *[13] Let u be a vertex of degree 1 in a graph G and v be the vertex adjacent to u. Then $P(G;x) = xP(G-u;x) - P(G-u-v;x)$.*

Lemma 5.3.9. *[13] Let G be the graph obtained by joining the vertex u_1 of the graph G_1 to the vertex u_2 of the graph G_2 by an edge. Let G_i' be the induced subgraph of G_i obtained by deleting the vertex u_i from G_i, for $i = 1, 2$. Then*

$$P(G;x) = P(G_1;x)P(G_2;x) - P(G_1';x)P(G_2';x).$$



Next, we show that the 3-colored digraphs as in figure 5.3 has the SR-property.

Lemma 5.3.10. *The 3-colored digraph $G = G_b$ be in Figure 5.3 has SR-property.*

Proof. By Lemma 5.3.7,

$$P(G;x) = xP(G-2;x) - P(G-2-3;x) - P(G-1-2;x) + 2P(G-C;x) \quad \boxed{5.3.1}$$

Let G_u, G_v be the subgraphs of G induced by $V(T_u^*) \cup \{6\}$, and $V(T_v^*) \cup \{5\}$, respectively. Note that G_u, G_v are same as the graphs induced by $V(T_u^*) \cup \{2\}$, and $V(T_v^*) \cup \{3\}$, respectively. Let H be the subgraph of G induced by $V(T_v^*) \cup \{3, 5\}$.

Using Lemma 5.3.8 and Lemma 5.3.9, we get

$$\begin{aligned}
P(G_u;x) &= xP(T_u^*;x) - xP(F_u;x),\\
P(G_v;x) &= xP(T_v^*;x) - xP(F_v;x),\\
P(H;x) &= xP(G_v;x) - x^2P(F_v;x) = x^2P(T_v^*;x) - 2x^2P(F_v;x),\\
P(G-2;x) &= P(G_u;x)P(H;x) - P(T_u^*;x)P(G_v;x),\\
P(G-2-3;x) &= P(G_u;x)P(G_v;x) - P(T_u^*;x)P(T_v^*;x),
\end{aligned}$$

$$\begin{aligned}
P(G-1-2;x) &= xP(G-1-2-6;x) - P(G-1-2-6-5;x)\\
&= x^2P(F_u;x)P(H;x) - xP(F_u;x)P(G_v;x),\\
\text{and } P(G-C;x) &= x^2P(F_u;x)P(F_v;x)
\end{aligned}$$

Thus from equation 5.3.1, we see that

$$\begin{aligned}
P(G;x) = \quad & (x^2-1)^2P(T_u^*;x)P(T_v^*;x) + 4x^4P(F_u;x)P(F_v;x)\\
& -2x^2(x^2-1)\left[P(T_u^*;x)P(F_v;x) + P(T_v^*;x)P(F_u;x)\right].
\end{aligned}$$

By Lemma 5.2.7, T_u^*, T_v^* and the trees in F_u, F_v have SR-property, as they are simple corona trees. Let n=2m. Thus

$$\begin{aligned}
P(T_u^*;x)P(T_v^*;x) &= (-1)^{m-2}x^{n-4}P\left(T_u^*;\frac{1}{x}\right)P\left(T_v^*;\frac{1}{x}\right),\\
P(T_u^*;x)P(F_v;x) &= (-1)^{m-3}x^{n-6}P\left(T_u^*;\frac{1}{x}\right)P\left(F_v;\frac{1}{x}\right),\\
P(T_v^*;x)P(F_u;x) &= (-1)^{m-3}x^{n-6}P\left(T_v^*;\frac{1}{x}\right)P\left(F_u;\frac{1}{x}\right),\\
\text{and } P(F_u;x)P(F_v;x) &= (-1)^{m-4}x^{n-8}P\left(F_u;\frac{1}{x}\right)P\left(F_v;\frac{1}{x}\right).
\end{aligned}$$

This gives

$$(-1)^m x^n P\left(G;\frac{1}{x}\right) = P(G;x).$$

Hence G has SR-property. ■

Theorem 5.3.11. *Let G be a unicyclic 3-colored digraph with $\mathsf{w}_C = -1$. Then G has SR-property if and only if either G is a simple corona of girth $g = 2k$, $k \geq 3$ or G has the structure as shown in figure 5.3, up to D-similarity.*

Proof. Proof follows from Lemmas 5.3.1, 5.3.4, 5.2.7, 5.3.6 and 5.3.10. ■

5.4 Unicyclic 3-colored digraph with a green edge on C

Throughout this section G_g denotes a connected unicyclic 3-colored digraph with a cycle C on which there is one green edge and all other edges are red. By Theorem 5.2.15, if G_g has girth $g > 3$ and SR-property, then it is a simple corona. By Lemma 5.2.7, if G_g is a simple corona, then it has SR-property. In this section we characterize those non-corona G_g of girth 3 which satisfy SR-property.

Remark 5.4.1. Consider a non-corona G_g of girth 3 which satisfy SR-property. Note that $|a_3| = |a_{n-3}| = 0$ here. So $\mathcal{P}_G^5 = \emptyset$. By an argument similar to that of Remark 5.2.14, we see that G_g must have the structure as shown in Figure 5.4. Here any edge on the cycle could be green. Furthermore, as $\mathcal{P}_G^5 = \emptyset$, the tree T^* induced by $V(F) \cup \{v_0, 3\}$ must be a corona tree, where F is a forest consisting of corona trees.

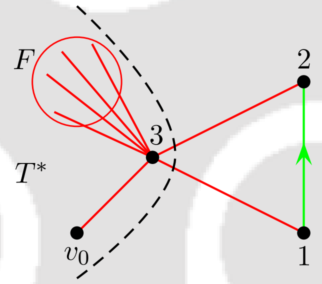


Figure 5.4: The structure of a non-corona G_g of girth 3 which satisfy SR-property

Lemma 5.4.2. *Consider G as shown in figure 5.4. Then G has SR-property.*

Proof. Put $B = xI - A(G)$ and put $E_{\alpha|\beta} = \det B(\alpha|\beta)$, for $|\alpha| = |\beta|$. Using Laplace expansion along the row corresponding the vertex 1 of G, we see that

$$\begin{aligned} P(G;x) &= xE_{1|1} + \mathrm{i}E_{1|2} - E_{1|3} \\ &= xP(G-1;x) + \mathrm{i}(\mathrm{i}E_{1,2|1,2} + E_{1,2|2,3}) - (\mathrm{i}E_{1,2|1,3} + E_{1,3|1,3}) \\ &= xP(G-1;x) - E_{1,2|1,2} + \mathrm{i}(E_{1,2|2,3} - E_{1,2|1,3}) - xE_{1,2,3|1,2,3} \\ &= xP(G-1;x) - P(G-1-2) - xP(G-1-2-3) \\ &= xP(G-1;x) - P(T^*, x) - x^2 P(F, x). \end{aligned}$$

Note that $P(G-1;x) = xP(T^*;x) - xP(F;x)$, using Lemma 5.3.8. Hence

$$P(G;x) = (x^2-1)P(T^*;x) - 2x^2P(F;x).$$

By Lemma 5.2.7, F and T^* have SR-property. Let $n = 2m$. Then

$$\begin{aligned} P(T^*;x) &= (-1)^{m-1}x^{n-2}P\left(T^*;\frac{1}{x}\right) \\ \text{and } P(F;x) &= (-1)^{m-2}x^{n-4}P\left(F;\frac{1}{x}\right) \end{aligned}$$

Observe that $(-1)^m x^n P\left(G;\frac{1}{x}\right) = P(G;x)$. Hence G has SR-property. ∎

Theorem 5.4.3. *Let G be a unicyclic 3-colored digraph with $\mathsf{w}_C = \pm\mathrm{i}$. Then G has SR-property if and only if either G is a simple corona or G has the structure as in figure 5.4 up to D-similarity.*

Proof. Proof follows from Theorem 5.2.15, Remark 5.4.1, Lemma 5.2.7 and Lemma 5.4.2 ∎

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