FIEDLER LINEARIZATIONS FOR LTI STATE-SPACE SYSTEMS AND FOR RATIONAL EIGENVALUE PROBLEMS

by

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To

My Family

Certificate

It is certified that the work contained in this thesis entitled **"Fiedler linearizations for LTI state-space systems and for rational eigenvalue problems"** by **Namita Behera,** a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

May 2014

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Abstract

The primary aim of this thesis is to develop a framework for direct methods for solutions of rational eigenvalue problems. To achieve this goal, we propose to reformulate the problem of solving a rational eigenvalue problem $G(\lambda)u = 0$ for $\lambda \in \mathbb{C}$ and nonzero vector $u \in \mathbb{C}^n$ to that of computation of transmission zeros and zero directions of a linear time invariant (LTI) system Σ given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + P(\tfrac{d}{dt})u(t) \end{aligned}$$

for which $G(\lambda)$ is the transfer function. Then the eigenvalues of $G(\lambda)$ form a subset of the transmission zeros of the LTI system Σ. On the other hand, transmission zeros of Σ are subset of the invariant zeros of the system Σ and the invariant zeros are eigenvalues of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ given by

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right].$$

The transmission zeros of Σ coincide with the invariant zeros when the LTI system Σ is controllable as well as observable. We therefore develop a framework for computing eigenvalues and eigenvectors of the the Rosenbrock system matrix $\mathcal{S}(\lambda)$. For this purpose, we introduce three families of linearizations - which we refer to as Fiedler pencils, Generalized Fiedler (GF) pencils and Generalized Fiedler pencils with repetition (GFPR) - of the Rosenbrock system matrix $\mathcal{S}(\lambda)$. We solve the eigenvalue problem for these Fiedler pencils to obtain invariant zeros of the LTI system Σ. Thus, schematically, our strategy for numerical solution of rational eigenvalue problem is as follows:

Rational matrix function $\longrightarrow$ State-space realization $\longrightarrow$ Linearization $\longrightarrow$ Solution.

There are efficient methods for computing a (minimal) state-space realization of a rational matrix function. We therefore focus on the construction of Fiedler pencils of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ and show that these Fiedler pencils are *linearizations* for $\mathcal{S}(\lambda)$. Further, we show that the Fiedler pencils of $\mathcal{S}(\lambda)$ are *linearizations* of $G(\lambda)$ when the LTI system Σ is both controllable and observable, that is, when Σ is a minimal state-space realization of the transfer function $G(\lambda)$. We also describe eigenvector recovery of $G(\lambda)$ from that of Fiedler pencils of

the system matrix $\mathcal{S}(\lambda)$. In fact, we show that the eigenvectors of $G(\lambda)$ could be recovered from those of the Fiedler pencils of $\mathcal{S}(\lambda)$ without incurring any additional computational cost. Further, we show that the state-space framework so developed could be gainfully used to *linearize* a higher order LTI state-space system so as to analyze and solve the higher order LTI system. Furthermore, we show that the linearized systems are *strict system equivalent* to the higher order systems and hence preserve system characteristics of the original systems.

We also develop a framework for sensitivity analysis of eigenvalues of the system matrix $\mathcal{S}(\lambda)$ and $G(\lambda)$. We define condition numbers for simple eigenvalues of $\mathcal{S}(\lambda)$ and $G(\lambda)$, and obtain explicit computable expressions for the condition numbers. We also analyze the effect of linearization on the conditioning of the eigenvalues of the system matrix $\mathcal{S}(\lambda)$.

Contents

List of Symbols

$\mathbb{N}$	the set of the natural numbers
$\mathbb{Z}$	the set of the integers
$\mathbb{R}$	the field of the real numbers
$\mathbb{C}$	the field of the complex numbers
$\mathbb{C}^{m\times n}$	the space of complex matrices of size $m \times n$
$\mathbb{C}[\lambda]^{m\times n}$	the vector space of $m \times n$ matrix polynomials
$\mathbb{C}(\lambda)$	the field of rational functions
$\mathbb{C}(\lambda)^{n\times n}$	the vector space of $n \times n$ rational matrix functions
e_k	the vector with the kth component equal to 1 and all the other entries are zero.
A^T	the transpose of real A
A^{-1}	the inverse of A
A^*	complex conjugate transpose of A
$\text{rank}(A)$	rank of matrix A
$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$	the identity matrix of order n
$A \otimes B$	Kronecker product of matrix A and B
$A^{\mathcal{B}}$	block transpose of matrix A

$\mathrm{adj}(A)$	adjugate of matrix $A \in \mathbb{C}^{n\times n}$
$\det(A)$	the determinant of $A \in \mathbb{C}^{n\times n}$
$\mathrm{Sp}(A)$	the spectrum of $A \in \mathbb{C}^{n\times n}$
$\sigma_{\max}$	the largest nonzero singular value of $A \in \mathbb{C}^{m\times n}$
$\sigma_{\min}$	the smallest nonzero singular value of $A \in \mathbb{C}^{m\times n}$
$\\|x\\|_p := (\sum\limits_{i=1}^{n} \|x_i\|^p)^{1/p}$	the Hölder p-norm on $\mathbb{C}^n$
$\\|A\\|_2 := \max_{\\|x\\|_2=1} \\|Ax\\|_2$	the spectral norm $\\|.\\|_2$ on $\mathbb{C}^{n\times n}$
$\\|A\\|_F := (\mathrm{Trace} A^* A)^{1/2}$	the Frobenius norm $\\|.\\|_F$ on $\mathbb{C}^{n\times n}$
$\langle A, B\rangle$	inner product of matrix A and B
$\\|A\\|_*$	dual norm of matrix norm $\\|A\\|$
$Df(a)$	the derivative of f at a
$\nabla f(a)$	gradient of f at a
$P(\lambda) = \sum\limits_{j=0}^{m} A_j \lambda^j$	matrix polynomial of degree m
$\mathbf{SF}(P(\lambda))$	Smith form of $P(\lambda)$
$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$	transfer function with $P(\lambda)$ of degree m
$\mathbf{SM}(G(\lambda))$	Smith-McMillan form of $G(\lambda)$
$\mathcal{S}(\lambda) = \left[\begin{array}{c\|c} P(\lambda) & C \\ \hline B & (A-\lambda E) \end{array}\right]$	Rosenbrock system matrix associated with system
M_i	Fiedler matrices for matrix polynomial
$\mathbb{M}_i$	Fiedler matrices for system matrix
$C_1(\lambda)$	first companion matrix of $P(\lambda)$
$C_2(\lambda)$	second companion matrix of $P(\lambda)$
$\mathcal{C}_1(\lambda)$	first companion matrix of $\mathcal{S}(\lambda)$
$\mathcal{C}_2(\lambda)$	second companion matrix of $\mathcal{S}(\lambda)$
$\mathcal{N}_r(P)$	right nullspace of singular matrix polynomial $P(\lambda)$
$\mathcal{N}_l(P)$	left nullspace of singular matrix polynomial $P(\lambda)$
$\mathcal{N}_r(P(\lambda))$	right nullspace of regular matrix polynomial $P(\lambda)$
$\mathcal{N}_l(P(\lambda))$	left nullspace of regular matrix polynomial $P(\lambda)$
GF	generalized Fiedler pencil

PGF	proper generalized Fiedler pencil
GFPR	generalized Fiedler pencil with repetition
cond(λ, A)	condition number of simple eigenvalue of matrix A

1 Introduction

Linear and nonlinear eigenvalue problems arise frequently in various problems in science and engineering. For example, the polynomial eigenvalue problem $P(\lambda)x = 0$, where $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ and A_j's are n-by-n matrices, arises in the study of acoustics and linear stability of flows in fluid mechanics, the vibration analysis of buildings, machines, and vehicles [11, 39]. Nonlinear eigenvalue problems also arise from certain differential equations such delay differential equations. Consider the delay differential equations ($DDEs$):

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i), \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, $A_i \in \mathbb{R}^{n\times n}$, and $0 < \tau_1 < \tau_2 \cdots, \tau_m$ represent the time-delays [28]. The substitution of a sample solution of the form $e^{\lambda t}v$, with $v \neq 0$, leads to the nonlinear eigenvalue problem

$$H(\lambda)v := (\lambda I - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda\tau_i})v = 0,$$

where $H(\lambda)$ is holomorphic in λ.

Rational eigenvalue problems, that is, eigenvalue problems for rational matrix functions, also arise in wide range of applications such as in acoustic emissions of high speed trains, calculations of quantum dots, free vibration of plates with elastically attached masses, vibrations of fluid-solid structures and in control theory, see [23, 34, 17, 46, 38, 29, 7, 44] and the references therein. For example, the rational eigenvalue problem

$$G(\lambda)x := -Ax + \lambda Bx + \lambda^2 \sum_{j=1}^{k} \frac{1}{\omega_j - \lambda} C_j x = 0,$$

arises when a generalized linear eigenproblem is condensed exactly [27]. The rational eigenproblem

$$G(\lambda)x := (K - \lambda M + \sum_{j=1}^{k} \frac{\lambda}{\lambda - \sigma_j} C_j)x = 0, \tag{1.2}$$

where $K = K^T$ and $M = M^T$ are positive definite and $C_j = C_j^T$ are matrices of small ranks, arises in the study of the vibrations of fluid solid structures [29, 7, 44, 26] and in the study of the free vibration of plates with elastically attached masses [25, 33]. Also the analysis of free vibrations of a structure leads to the eigenproblem

$$G(\lambda)x := (\lambda^2 M + K - \sum_{j=1}^{k} \frac{1}{1 + b_j\lambda} \Delta T_j)x = 0, \tag{1.3}$$

where K and M are symmetric positive definite, k denotes the number of regions with different relaxation parameters b_j and ΔT_j is an assemblage of element stiffness matrices over the region with the distinct relaxation constants [27]. Further, the rational matrix eigenvalue problem

$$G(\lambda)x := -Ax + \lambda Bx + \sum_{j=1}^{k} \frac{\rho\lambda}{k_j - \lambda m_j} C_j x = 0, \tag{1.4}$$

where the matrices A, B and C_j are symmetric and positive (semi-) definite, arises when an elliptic eigenvalue problem with non-local boundary conditions is discretized by finite elements [7, 42]. Computation of eigenvalues and eigenvectors of a rational matrix function is an emerging area of research. We therefore consider the following problem.

PROBLEM-I. Let $G(\lambda)$ be an n-by-n regular rational matrix function. Compute $\lambda \in \mathbb{C}$ and nonzero vectors x and y in $\mathbb{C}^n$ such that $G(\lambda)x = 0$ and $y^T G(\lambda) = 0$.

The rational eigenvalue problem (REP) in PROBLEM-I could be solved by employing general nonlinear eigensolvers such as Newton method [31, 38], inverse iteration [31, 38], and nonlinear Rayleigh-Ritz methods (e.g., nonlinear Arnoldi, rational Krylov, Jacobi-Davidson), see [43, 20, 45] and the references therein. We mention, however, that the convergence analysis of a general nonlinear eigensolver is a challenging task. Alternatively, we could convert the REP into a polynomial eigenvalue problem by clearing out the denominators in $G(\lambda)$ and then "linearizing" the resulting polynomial eigenvalue problem to obtain a generalized eigenvalue problem. A downside of this brute-force approach is that the resulting generalized eigenvalue problem may be of very large dimension especially when $G(\lambda)$ has a large number of poles [37].

On the other hand, considering a minimal realization of $G(\lambda)$ of the form

$$G(\lambda) = \sum_{j=0}^{m} \lambda^j A_j + C(\lambda E - A)^{-1}B =: P(\lambda) + C(\lambda E - A)^{-1}B,$$

where $C \in \mathbb{C}^{n\times r}, A, E \in \mathbb{C}^{r\times r}, B \in \mathbb{C}^{r\times n}$, it is shown in [37] that the eigenvalues and eigenvectors of $G(\lambda)$ could be computed by solving the generalized eigenvalue problem for the pencil

$$\mathcal{C}(\lambda) := \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] + \left[\begin{array}{cccc|c} A_{m-1} & A_{m-2} & \cdots & A_0 & C \\ -I_n & 0 & \cdots & 0 & \\ & \ddots & & \vdots & \\ & & -I_n & 0 & \\ \hline & & & B & A \end{array}\right]. \tag{1.5}$$

The pencil $\mathcal{C}(\lambda)$ is referred to as a *companion form* of $G(\lambda)$ [37] and could be thought of as an extension of the Frobenius companion form $C(\lambda)$ of the matrix polynomial $P(\lambda) := \sum_{j=0}^{m} A_j \lambda^j$ given by

$$C(\lambda) := \lambda \begin{bmatrix} A_m & 0 & \cdots & 0 \\ 0 & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_n \end{bmatrix} + \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_n & 0 \end{bmatrix} \tag{1.6}$$

to the case of $G(\lambda)$. The pencil $C(\lambda)$ is also known as the first companion form of $P(\lambda)$ [24, 11, 14]. An advantage of realization based approach to solution of the REP is that often the size of the campanion form $\mathcal{C}(\lambda)$ of $G(\lambda)$ could be much smaller compared with the brute-force approach especially when the coefficient matrices of $G(\lambda)$ have low ranks. It is well known that the companion form $C(\lambda)$ of the matrix polynomial $P(\lambda)$ is a *linearization* of $P(\lambda)$ [24, 11, 14]. This raises a natural question.

Problem-II. Can the companion form $\mathcal{C}(\lambda)$ be considered as a *linearization* of the rational matrix function $G(\lambda)$? And what does it mean to say that $\mathcal{C}(\lambda)$ is a linearization of $G(\lambda)$?

Note that we could view a realization of $G(\lambda)$ as a transfer function of a linear time invariant (LTI) state-space system Σ given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + P(\tfrac{d}{dt})u(t) \end{aligned} \tag{1.7}$$

and compute "transmission zeros" of Σ, where $P(\lambda)$ is an n-by-n matrix polynomial and A, E, C, B are constant matrices of appropriate dimensions with E being nonsingular. In such case, an eigenvalue of $G(\lambda)$ is a transmission zero of the LTI system Σ. Consequently, the eigenvalues and eigenvectors of $G(\lambda)$ could be obtained from transmission zeros and zero directions of the LTI system Σ. We therefore consider the following problem.

PROBLEM-III. Let Σ be an LTI system in state-space form as in (1.7). Compute transmission zeros, invariant zeros, and decoupling zeros of the system Σ.

Observe that the rational eigenvalue problem in PROBLEM-I could be reformulated as a special case of PROBLEM-III. Thus, we develop a general framework for solving PROBLEM-III. The Rosenbrock system matrix of the LTI system Σ is given by

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right].$$

The eigenvalues of $\mathcal{S}(\lambda)$ are invariant zeros of the LTI system Σ. The transmission zeros of the LTI sytem Σ form a subset of the invariant zeros of the LTI system and the transmission zeros are the same as the invariant zeros when the LTI system Σ is controllable as well as observable. We therefore develop a framework for computing eigenvalues of the the Rosenbrock system matrix $\mathcal{S}(\lambda)$. For this purpose, we introduce three classes of linearizations, which we refer to as Fiedler pencils, Generalized Fiedler (GF) pencils and Generalized Fiedler pencils with repetition (GFPR) of the Rosenbrock system matrix $\mathcal{S}(\lambda)$. We solve the eigenvalue problem for these Fiedler pencils to obtain invariant zeros of the LTI system Σ. Thus, schematically, our strategy for numerical solution of PROBLEM-I is as follows:

$$\text{Rational matrix function} \longrightarrow \text{State-space realization} \longrightarrow \text{Linearization} \longrightarrow \text{Solution.}$$

There are efficient methods for computing a (minimal) state-space realization of a rational matrix function. We, therefore, focus on solving PROBLEM-III and address the issues in PROBLEM-II. More precisely, we describe construction of families of Fiedler pencils of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ and show that these Fiedler pencils are *linearizations* for $\mathcal{S}(\lambda)$. Further, we show that the Fiedler pencils of $\mathcal{S}(\lambda)$ are *linearizations* of $G(\lambda)$ when the LTI system Σ is both controllable and observable, that is, when Σ is a minimal state-space realization of the transfer function $G(\lambda)$. We also describe eigenvector recovery of $G(\lambda)$ from that of Fiedler pencils of the system matrix $\mathcal{S}(\lambda)$. In fact, we show that the eigenvectors of $G(\lambda)$ could be recovered from those of the Fiedler pencils of $\mathcal{S}(\lambda)$ without incurring any additional computational cost. Further, we show that the state-space framework so developed could be gainfully used to *linearize* higher order LTI state-space systems so as to analyze and solve the higher order LTI systems. Furthermore, we show that the linearized systems are *strict system equivalent* to the higher order systems and hence preserve various system characteristics of the original systems.

Errors are unavoidable in numerical computations. Sensitivity and perturbation analysis of eigenvalue problems play an important role in the accuracy assessment of computed eigenvalues and eigenvectors. Thus, for accuracy assessment of computed eigenvalues, we develop a general framework for sensitivity analysis of eigenvalues of the system matrix $\mathcal{S}(\lambda)$ and $G(\lambda)$. We define condition numbers for simple eigenvalues of $\mathcal{S}(\lambda)$ and $G(\lambda)$ and obtain explicit computable expressions for the condition numbers. We also analyze the effect of linearization on the conditioning of the eigenvalues of the system matrix $\mathcal{S}(\lambda)$ and the transfer function $G(\lambda)$.

The thesis is organized as follows. The rest of this chapter is devoted to basic results on matrices, matrix polynomials, rational matrix functions, and LTI state-space systems. Chapter 2 develops a general framework for construction of Fiedler pencils of the Rosenbrock system matrix associated with an LTI system in state-space form and shows that Fiedler pencils are linearizations of the system matrix. Chapter 3 analyzes recovery property of eigenvectors of the system matrix and transfer function from those of Fiedler pencils of the system matrix. Chapter 4 describes construction of generalized Fiedler (GF) pencils of the Rosenbrock system and the eigenvector recovery property of the system matrix and the transfer function from those of the GF pencils. Chapter 5 is devoted to construction of generalized Fiedler pencils with repetition (GFPR) of the Rosenbrock system matrix and their eigenvector recovery property. Chapter 6 develops a general framework for sensitivity analysis of eigenvalues of the Rosenbrock system matrix and the transfer function. Finally, Chapter 7 presents conclusions.

1.1 Preliminaries

Throughout this thesis, we use $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$ to denote the vector space of n-tuples $[x_1, \cdots, x_n]^T, x_i \in \mathbb{C}$, and the vector space of $m \times n$ matrices with entries from $\mathbb{C}$.

Let $A \in \mathbb{C}^{n\times m}$. Then the *kernel* or the *null space* of A is defined as

$$N(A) = \{x \in \mathbb{C}^m : Ax = 0\}.$$

We denote the dimension of $N(A)$ by $\dim N(A)$. The *range* of A is given by

$$R(A) = \{Ax : x \in \mathbb{C}^m\}$$

and dimension of $R(A)$ is called the rank of A. We denote the rank of A by $\text{rank}(A)$.

Now we define the Kronecker product of matrices which will be used in the subsequent development.

Definition 1.1.1. *([35]) Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}$. Then the Kronecker product (tensor product) of A and B is defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}$$

Some properties of the kronecker product are as follows:

- Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{r\times s}, C \in \mathbb{C}^{n\times p}$ and $D \in \mathbb{C}^{s\times t}$. Then

$$(A \otimes B)(C \otimes D) = AC \otimes BD \in \mathbb{C}^{mr\times pt}.$$

- For all A and B, $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)^* = A^* \otimes B^*$.
- If A and B are nonsingular then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

1.1.1 Polynomial Eigenvalue Problem(PEP)

We denote by $\mathbb{C}[\lambda]^{m\times n}$ the vector space of $m \times n$ matrix polynomials. Let

$$P(z) := \sum_{i=0}^{m} z^i A_i, \text{ where } A_i \in \mathbb{C}^{n\times n} \text{ with } A_m \neq 0, \tag{1.8}$$

be a matrix polynomial of degree m. Denote $\mathbb{L}_m(\mathbb{C}^{n\times n})$ is the set of $n \times n$ matrix polynomial of degree less than equal to m of the form (1.8). Then $\mathbb{L}_m(\mathbb{C}^{n\times n})$ is a vector space. Then the polynomial eigenvalue problem is to find $\lambda \in \mathbb{C}$ and non zero vector $x, y \in \mathbb{C}^n$ such that $P(\lambda)x = 0$ and $y^T P(\lambda) = 0$. In such a case, λ is called an eigenvalue of $P(\lambda)$, and x, y are called right and left eigenvector respectively, of P corresponding to the eigenvalue λ. If $m = 1$, then $L(z) := A_0 + zA_1$ is referred to as $n \times n$ *matrix pencil* and the eigenvalue problem $L(\lambda)x = 0$, is called a generalized eigenvalue problem.

A matrix polynomial $P(\lambda)$ is said to be regular if $\det(P(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular matrix polynomial P, denoted by $\mathrm{Sp}(P)$, is given by

$$\mathrm{Sp}(P) := \{\lambda \in \mathbb{C} : \mathrm{rank}(P(\lambda)) < n\}.$$

i.e., for regular matrix polynomials the finite eigenvalues are precisely the roots of the scalar polynomial $\det P(\lambda)$. It is possible for P to have an infinite eigenvalue when the leading coefficient of the matrix polynomial P is singular. An easy way of studying an ∞ eigenvalue of matrix polynomial $P(\lambda) \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ can also be resolved by defining the reversal of matrix polynomial $P(\lambda)$, which is given by [24]

$$rev P(\lambda) := \lambda^m P(1/\lambda) = \sum_{i=0}^{m} \lambda^i A_{m-i}. \tag{1.9}$$

for $\lambda \in \mathbb{C}$. Note that λ_0 is an eigenvalue of $P(\lambda)$ iff $1/\lambda_0$ is an eigenvalue of rev $P(\lambda)$ with 0 and ∞ considered as reciprocals, see [24]. Here we are not dealing with the infinite eigenvalue.

Now we state the Smith form of a matrix polynomial $P(\lambda)$ which will be used in the subsequent development.

Theorem 1.1.1 (Smith form, [11])**.** *Every $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$ admits the representation*

$$P(\lambda) = U(\lambda)\,\mathbf{SF}(P(\lambda))\,V(\lambda), \tag{1.10}$$

where

$$\mathbf{SF}(P(\lambda)) = \mathrm{diag}\,(\varepsilon_1(\lambda), \varepsilon_2(\lambda), \ldots, \varepsilon_r(\lambda), 0, \ldots, 0) \in \mathbb{C}[\lambda]^{m\times n}$$

with monic scalar polynomials $\varepsilon_i(\lambda)$ such that $\varepsilon_i(\lambda)$ is divisible by $\varepsilon_{i-1}(\lambda)$; $U(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ and $V(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ with constant nonzero determinants.

The representation (1.10) given in Theorem 1.1.1 with $\mathbf{SF}(P(\lambda))$ is called the **Smith form** of the matrix polynomial $P(\lambda)$. Let $D_i(\lambda)$ be the monic greatest common divisor of all nonzero minors of $P(\lambda)$ of order i, if not all of them are zeros, and let $D_i(\lambda) \equiv 0$ if all the minors of order i of $P(\lambda)$ are zeros. Let $D_0(\lambda) = 1$ and $\mathbf{SF}(P(\lambda))$ be the Smith form of $P(\lambda)$ given in Theorem 1.1.1. Then r is the maximal integer such that $D_r(\lambda) \neq 0$, and

$$\varepsilon_i(\lambda) = \frac{D_i(\lambda)}{D_{i-1}(\lambda)}, \qquad i \in \{1, 2, \ldots, r\},$$

are called the **invariant factors** of $P(\lambda)$, see [11].

Example 1.1. Consider the matrix polynomial

$$P(\lambda) := \begin{bmatrix} \lambda(\lambda+2) & 0 \\ 0 & (\lambda+1)^2 \\ (\lambda+1)(\lambda+2) & \lambda+1 \\ 0 & \lambda(\lambda+1) \end{bmatrix}.$$

Then the Smith form of $P(\lambda)$ is given by

$$\mathbf{SF}(P(\lambda)) = \begin{bmatrix} 1 & 0 \\ 0 & (\lambda+1)(\lambda+2) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The right and left null spaces of $P(\lambda)$ are defined as follows.

Definition 1.1.2. *Let $P(\lambda) \in \mathbb{C}[\lambda]^{n\times n}$ be regular. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(\lambda)$. Then the* right *and the* left nullspaces *of $P(\lambda)$ are defined by*

$$\mathcal{N}_r(P(\lambda)) := \{x \in \mathbb{C}^n : P(\lambda)x = 0\}$$

and

$$\mathcal{N}_l(P(\lambda)) := \{y \in \mathbb{C}^n : y^T P(\lambda) = 0\}$$

respectively.

Definition 1.1.3. *The right and the left nullspaces of a singular matrix polynomial $P(\lambda)$, denoted by $\mathcal{N}_r(P)$ and $\mathcal{N}_l(P)$, respectively are given by*

$$\mathcal{N}_r(P) := \{x(\lambda) \in \mathbb{C}[\lambda]^n : P(\lambda)x(\lambda) = 0\}$$

and

$$\mathcal{N}_l(P) := \{y(\lambda) \in \mathbb{C}[\lambda]^n : y(\lambda)^T P(\lambda) = 0\}.$$

The standard method for solving polynomial eigenvalue problem $P(\lambda)x = 0$ is to convert the matrix polynomial P to a linear polynomial (pencil) $L(\lambda) = \lambda X + Y$ with the same eigenvalue, and solve the problem for the pencil L, see [11, 24].

Definition 1.1.4 ([24], Unimodular matrix polynomials)**.** *Matrix polynomial $E(\lambda)$ is said to be unimodular if* $\det E(\lambda)$ *is a nonzero constant, independent of λ. Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are said to be* equivalent *if there exists unimodular matrices $U(\lambda)$ and $V(\lambda)$, such that $Q(\lambda) = U(\lambda)P(\lambda)V(\lambda)$. If $U(\lambda), V(\lambda)$ are constant matrices, then $P(\lambda)$ and $Q(\lambda)$ are said to be strictly equivalent.*

Definition 1.1.5 (Linearization, [24])**.** *Let $P(\lambda)$ be an $n\times n$ matrix polynomial of degree m with $m \geq 1$ given in (1.8). A pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{C}^{mn\times mn}$ is called a linearization of $P(\lambda)$ if there exist unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$ such that*

$$E(\lambda)L(\lambda)F(\lambda) = \left[\begin{array}{c|c} P(\lambda) & 0 \\ \hline 0 & I_{(m-1)n} \end{array}\right]$$

for all $\lambda \in \mathbb{C}$. If $L(\lambda)$ is a linearization for $P(\lambda)$ and rev $L(\lambda)$ is a linearization for rev $P(\lambda)$, then $L(\lambda)$ is said to be a strong linearization *for $P(\lambda)$.*

From the above definition it is clear that $\alpha\det(L(\lambda)) = \det(P(\lambda))$ for some nonzero constant α. Hence L and P have the same spectrum.

For a given matrix polynomial, there are infinitely many linearizations exists [24] and the most common examples in practice are the companion forms also called the Frobenius forms. The first and second companion forms of $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ are given by [11, 24]

$$C_1(\lambda) = \lambda \begin{bmatrix} A_m & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_n & 0 \end{bmatrix} \qquad (1.11)$$

and

$$C_2(\lambda) = \lambda \begin{bmatrix} A_m & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{m-1} & -I_n & & 0 \\ A_{m-2} & 0 & \ddots & \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \cdots & 0 \end{bmatrix}. \qquad (1.12)$$

Generalizing the first and second companion forms of P, Mackey et al. [24] developed two classes of vector space linearizations.

$$\mathbb{L}_1(P) = \{L(\lambda) : \; L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda), \; v \in \mathbb{C}^m\}, \qquad (1.13)$$

$$\mathbb{L}_2(P) = \left\{L(\lambda) : \; (\Lambda^T \otimes I_n)L(\lambda) = w^T \otimes P(\lambda), \; w \in \mathbb{C}^m\right\}, \qquad (1.14)$$

where $L(\lambda)$ is an $mn \times mn$ matrix pencil and $\Lambda = [\lambda^{m-1}, \lambda^{m-2}, \ldots, \lambda, 1]^T \in \mathbb{C}^m$, $\otimes$ is the Kronecker product, v is called the right ansatz vector for $L(\lambda) \in \mathbb{L}_1(P)$, and w is the left ansatz vector for $L(\lambda) \in \mathbb{L}_2(P)$. The spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ are generalizations of first and second companion form of matrix polynomial.

Recently, a new classes of linearizations for matrix polynomial - referred to as Fiedler linearizations, Generalized Fiedler (GF) linearization and Generalized Fiedler pencil with repetition (GFPR) have been introduced and analyzed in [3], [9], [6], [41], [18].

1.2 Fiedler linearizations for matrix polynomial

Let $P(\lambda)$ be the matrix polynomial in (1.8). Define $nm \times nm$ Fiedler matrices associated with $P(\lambda)$ by

$$M_m = \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, M_0 := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \tag{1.15}$$

and

$$M_i := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \; i = 1, \ldots, m-1. \tag{1.16}$$

The Fiedler matrices $M_i, i = 0 : m$ have the following properties, see, [9].

- For $i = 1, \ldots, m-1$, each M_i is invertible and

$$M_i^{-1} = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}. \tag{1.17}$$

- $M_iM_j = M_jM_i$ for $|i-j| > 1$, $M_i^{-1}M_j = M_jM_i^{-1}$ and $M_i^{-1}M_j^{-1} = M_j^{-1}M_i^{-1}$ for $|i-j| > 1$ [6].

If $\sigma : \{0, 1, \ldots, m-1\} \rightarrow \{1, 2, \ldots, m\}$ is a bijection, then we define $M_\sigma := M_{\sigma^{-1}(1)}M_{\sigma^{-1}(2)} \cdots M_{\sigma^{-1}(m)}$ and $M_\emptyset := I_{nm}$. Note that $\sigma(i)$ describes the position of the factor M_i in the product M_σ; i.e., $\sigma(i) = j$ means that M_i is the jth factor in the product.

Definition 1.2.1 ([9], Fiedler pencil)**.** *Let $M_0, \ldots, M_m$ be Fiedler matrices associated with $P(\lambda)$. Given any bijection $\sigma : \{0, 1, \ldots, m-1\} \rightarrow \{1, 2, \ldots, m\}$, the $mn \times mn$ matrix pencil*

$$L_\sigma(\lambda) := \lambda M_m - M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(m)} =: \lambda M_m - M_\sigma. \tag{1.18}$$

is called the Fiedler pencil *of $P(\lambda)$ associated with σ.*

The first and second companion forms given in (1.11) and (1.12) are Fiedler pencils of $P(\lambda)$ [3]. Indeed, we have

$$C_1(\lambda) = \lambda M_m - M_{m-1} \cdots M_1 M_0 \tag{1.19}$$

$$C_2(\lambda) = \lambda M_m - M_0 \cdots M_{m-2} M_{m-1}. \tag{1.20}$$

Now consider the Fiedler pencil $L_\sigma(\lambda)$ of $P(\lambda)$ associated with a bijection σ. Then for showing $L_\sigma(\lambda)$ is a linearizations of $P(\lambda)$, we need the following definitions.

Definition 1.2.2. *[9] Let $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ be a bijection.*

(1) For $d = 0, \ldots, m-2$, we say that σ has a consecution at d if $\sigma(d) < \sigma(d+1)$ and σ has an inversion at d if $\sigma(d) > \sigma(d+1)$.

(2) We denote the total number of consecutions in σ by $c(\sigma)$ and the total number of inversions in σ by $i(\sigma)$.

(3) The consecution-inversion structure sequence of σ is the tuple $(c_1, i_1, c_2, i_2, \ldots, c_l, i_l)$, where σ has c_1 consecutive consecutions at $0, 1, \ldots, c_1 - 1$; i_1 consecutive inversions at $c_1, c_1 + 1, \ldots, c_1 + i_1 - 1$ and so on, up to i_l inversions at $m - 1 - i_l, \ldots, m-2$, and is denoted by CISS(σ).

Definition 1.2.3. *[9] The reversal revσ of a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ is a bijection from $\{0, 1, \ldots, m-1\}$ into $\{1, 2, \ldots, m\}$, defined by rev$\sigma(i) = m + 1 - \sigma(i)$ for $0 \leq i \leq m-1$. Equivalently, if $\sigma = (\sigma(0), \ldots, \sigma(m-1))$ then rev $\sigma = (\sigma(m-1), \ldots, \sigma(0))$.*

Observe that if $M_\sigma = M_{\sigma^{-1}(1)} M_{\sigma^{-1}(2)} \cdots M_{\sigma^{-1}(m)}$ then

$$M_{\text{rev}\sigma} = M_{\sigma^{-1}(m)} M_{\sigma^{-1}(m-1)} \cdots M_{\sigma^{-1}(1)}. \tag{1.21}$$

Definition 1.2.4. *[9] Let $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^m A_m$ be a matrix polynomial of degree m. For $k = 0, \ldots m$, the degree k Horner shift of $P(\lambda)$ is the matrix polynomial $P_k(\lambda) := A_{m-k} + \lambda A_{m-k+1} + \cdots + \lambda^k A_m$. These Horner shifts satisfy the following:*

$$\begin{aligned} &P_0(\lambda) = A_m, \\ &P_{k+1}(\lambda) = \lambda P_k(\lambda) + A_{m-k-1}, \ \textit{for } 0 \leq k \leq m-1, \\ &P_m(\lambda) = P(\lambda). \end{aligned} \tag{1.22}$$

Definition 1.2.5. *Let $\mathcal{H} = (\mathcal{H}_{ij})$ be a block $m \times n$ matrix with $p \times q$ blocks $\mathcal{H}_{ij}$. The* block transpose *of $\mathcal{H}$ is the block $n \times m$ matrix with $p \times q$ blocks, and defined by $\mathcal{H}^{\mathcal{B}} = (\mathcal{H}_{ji})_{n \times m}$.*

Definition 1.2.6. *[9] Let $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ be an $n \times n$ matrix polynomial, and let $P_i(\lambda)$ be the degree i Horner shift of $P(\lambda)$. For $1 \leq i \leq m-1$, define the following $nm \times nm$ matrix polynomials:*

$$Q_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & \lambda I_n & \\ & 0_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix},$$

$$R_i(\lambda) := \begin{bmatrix} I_{(i-1)n} & & & \\ & 0_n & I_n & \\ & I_n & P_i(\lambda) & \\ & & & I_{(m-i-1)n} \end{bmatrix} = R_i(\lambda)^{\mathcal{B}},$$

$$T_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0_n & \lambda P_{i-1}(\lambda) & \\ & \lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(m-i-1)n} \end{bmatrix},$$

$$D_i(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & & \\ & P_{i-1}(\lambda) & 0_n & \\ & 0_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix},$$

and $D_m(\lambda) := \operatorname{diag}\left[0_{(m-1)n}, P_{m-1}(\lambda)\right]$. Sometimes we may write Q_i, R_i, T_i, D_i in place of $Q_i(\lambda), R_i(\lambda), T_i(\lambda), D_i(\lambda)$. Note that $D_1(\lambda) = M_m$, and $Q_i(\lambda)$, $R_i(\lambda)$ are unimodular for all $i = 1, \ldots, m-1$.

The following result shows that every Fiedler pencil is a linearization.

Theorem 1.2.1. *[9] Let $P(\lambda)$ be a matrix polynomial (singular or regular) of degree $m \geq 1$ and $L_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ. Let Q_i, R_i for $i = 1, \ldots, m-1$, be the matrices introduced in Definition 1.2.6. Then*

$$U(\lambda) L_\sigma(\lambda) V(\lambda) = \begin{bmatrix} -I_{(m-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

where $U(\lambda)$ and $V(\lambda)$ are the following $nm \times nm$ unimodular matrix polynomials:

$$U(\lambda) = U_0 U_1 \dots U_{m-3} U_{m-2} \textit{ with } U_i = \begin{cases} Q^{\mathcal{B}}_{m-(i+1)} & \textit{if } \sigma \textit{ has a consecution at } i \\ R^{\mathcal{B}}_{m-(i+1)} & \textit{if } \sigma \textit{ has an inversion at } i, \end{cases}$$

$$V(\lambda) = V_{m-2} V_{m-3} \dots V_1 V_0 \textit{ with } V_i = \begin{cases} R_{m-(i+1)} & \textit{if } \sigma \textit{ has a consecution at } i \\ Q_{m-(i+1)} & \textit{if } \sigma \textit{ has an inversion at } i. \end{cases}$$

Thus $L_\sigma(\lambda)$ is a linearization of $P(\lambda)$.

Let σ be a permutation. Suppose that $\text{CISS}(\sigma) = (c_1, i_1, \dots, c_l, i_l)$. Define

$$s_0 := 0, \qquad s_j := \sum_{p=1}^{j} (c_p + i_p), \text{ for } j = 1, \dots, l.$$

Note that $s_l = m - 1$. Also define

$$m_0 := 0, \qquad m_j := i_1 + i_2 + \cdots + i_j \text{ for } j = 1, \dots, l.$$

Note that m_l is the total number of inversions in σ. Also define $\Lambda_{\sigma,j}(P)$ for $j = 1, \dots, l$ and $\widehat{\Lambda}_{\sigma,j}(P)$ for $j = 1, \dots, l-1$ as follows (see [9]):

$$\Lambda_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ P_{m-s_j-1-c_j} \\ \vdots \\ P_{m-s_j-1-2} \\ P_{m-s_j-1-1} \end{bmatrix} \text{ and } \widehat{\Lambda}_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j-1} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ P_{m-s_j-1-c_j} \\ \vdots \\ P_{m-s_j-1-2} \\ P_{m-s_j-1-1} \end{bmatrix} \text{ if } c_1 \geq 1. \qquad (1.23)$$

If $c_1 = 0$, then

$$\Lambda_{\sigma,1}(P) := [\lambda^{i_j} I_n, \dots, \lambda I_n, I_n]^{\mathcal{B}}, \qquad \widehat{\Lambda}_{\sigma,1}(P) := [\lambda^{i_1-1} I_n, \dots, \lambda I_n, I_n]^{\mathcal{B}}$$

and $\Lambda_{\sigma,j}(P), \widehat{\Lambda}_{\sigma,j}(P)$ as in (1.23) for $j > 1$, where $P_d(\lambda)$ is the degree d Horner shift of matrix polynomial $P(\lambda)$. For the simplicity, we omit the λ from the Horner shifts $P_d(\lambda)$.

Theorem 1.2.2. *[9] Let $L(\lambda)$ be a linearization of an $n \times n$ matrix polynomial $P(\lambda)$, and let $U(\lambda)$ and $V(\lambda)$ be unimodular matrices such that*

$$U(\lambda)L(\lambda)V(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right].$$

Viewing $U(\lambda)$ and $V(\lambda)$ as block $m \times m$ matrices with $n \times n$ blocks, let $U^L = U^L(\lambda)$ be the last block row of $U(\lambda)$ and $V^R = V^R(\lambda)$ the last block column of $V(\lambda)$. Then

(a) the linear map

$$\mathcal{L} : \mathcal{N}_l(P) \to \mathcal{N}_l(L)$$
$$w^T(\lambda) \mapsto w^T(\lambda).U^L$$

is an isomorphism.

(b) the linear map

$$\mathcal{R} : \mathcal{N}_r(P) \to \mathcal{N}_r(L)$$
$$v(\lambda) \mapsto V^R.v(\lambda)$$

is an isomorphism.

Let $L_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ, and let $V(\lambda)$ be as in Theorem 1.2.1. If CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, then the last block column $V^R(\lambda)$ of $V(\lambda)$ is given by

$$V_\sigma^R(\lambda) := \begin{bmatrix} \lambda^{m_{l-1}}\Lambda_{\sigma,l}(P) \\ \lambda^{m_{l-2}}\widehat{\Lambda}_{\sigma,l-1}(P) \\ \vdots \\ \lambda^{m_1}\widehat{\Lambda}_{\sigma,2}(P) \\ \widehat{\Lambda}_{\sigma,1}(P) \end{bmatrix} \quad \text{if } l > 1, \tag{1.24}$$

and $V^R(\lambda) = \Lambda_{\sigma,1}(P)$ if $l = 1$. Further, $\mathcal{R}_\sigma(P) : \mathcal{N}_r(P) \to \mathcal{N}_r(L_\sigma)\, v \mapsto V_\sigma^R(\lambda)v$ is an isomorphism. Furthermore, let $V^R_{rev\,\sigma}(\lambda)$ be, for the reversal bijection rev σ, the $nm \times n$ matrix defined in (3.6). Then the linear map $\mathcal{L}_\sigma : \mathcal{N}_l(P) \to \mathcal{N}_l(L_\sigma)\, u^T \mapsto u^T U_\sigma^L(\lambda)$, where $U_\sigma^L(\lambda) := [V^R_{rev\,\sigma}(\lambda)]^{\mathcal{B}}$, is an isomorphism.

Definition 1.2.7. *[18] An ordered tuple of indices consisting of consecutive integers is called a* string *and denoted by $(t : p)$ for the string of integers from t to p, i.e.,*

$$(t : p) := \begin{cases} (t, t+1, \ldots, p), & \text{if } t \le p \\ \phi, & \text{if } t > p. \end{cases}$$

Remark 1.2.1. *In the above definition, if $t_1 > p$ and $t_2 > p$, then both index tuples $(t_1 : p)$ and $(t_2 : p)$ correspond to the empty index tuple. To avoid this notation, we will adapt the notation $(\infty : p)$ for any tuple of the form $(t : p)$ having $t > p$ where applicable.*

Definition 1.2.8. *[41] Let h be a non-negative integer and $\boldsymbol{q}$ be an index tuple containing indices from $\{0, 1, \ldots, h\}$. Then $\boldsymbol{q}$ is said to be in* column standard form *if*

$$\boldsymbol{q} = (a_p : b_p, a_{p-1} : b_{p-1}, \ldots, a_2 : b_2, a_1 : b_1),$$

with $0 \leq b_1 < b_2 < \ldots < b_{p-1} < b_p \leq h$ and $0 \leq a_j \leq b_j$, for all $j = 1, \ldots, p$. We denote this tuple by $csf(\boldsymbol{q})$.

Lemma 1.2.1. *[18] Let σ be a permutation of $\{h_0, h_0 + 1, \ldots, h\}$, with $0 \leq h_0 \leq h$. Then σ is in column standard form if and only if*

$$\sigma = (t_{\alpha-1} + 1 : h, t_{\alpha-2} + 1 : t_{\alpha-1}, \ldots, t_2 + 1 : t_3, t_1 + 1 : t_2, h_0 : t_1),$$

for some positive integers $h_0 \leq t_1 < t_2 < \cdots < t_{\alpha-1} < h$.

Denote $t_0 = h_0 - 1$ and $t_\alpha = h$. We call each sequence of consecutive integers $(t_{i-1} + 1 : t_i)$, for $i = 1, \ldots, \alpha$, a string in σ.

The following results gives the eigenvector formula for Fiedler pencil of $P(\lambda)$.

Theorem 1.2.3. *[18] Let $P(\lambda)$ be an matrix polynomial of degree m and $P_0, \ldots, P_m$ be the Horner shifts of $P(\lambda)$. Let σ be a permutation of $\{0, 1, \ldots, m-1\}$ with $csf(\sigma) = (\boldsymbol{b}_\beta, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_k = (t_{k-1} + 1 : t_k)$, for $k = 1, \ldots, \beta$. Let $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ. Then*

$$E_\sigma(P) := \begin{bmatrix} B_0 \, B_1 \, \ldots \, B_{m-1} \end{bmatrix}^{\mathcal{B}} \tag{1.25}$$

where, if $\sigma(i) \in \boldsymbol{b}_k$, for some $k = 1, \ldots, \beta$, then

$$B_i = \begin{cases} \lambda^{k-1} I_n, & \text{if } i = m - t_k - 1 \\ \lambda^{k-1} P_i, & \text{otherwise.} \end{cases} \tag{1.26}$$

Let $H_\sigma(P) := E_{rev\,\sigma}(P^T)$. Then

(a) $v \in \mathcal{N}_r(L_\sigma(\lambda))$ associated with the eigenvalue λ iff $v = E_\sigma(P)x$, where $x \in \mathcal{N}_r(P(\lambda))$ associated with the eigenvalue λ.

(b) $w \in \mathcal{N}_l(L_\sigma(\lambda))$ associated with the eigenvalue λ_0 iff $w = H_\sigma(P)y$, where $y \in \mathcal{N}_l(P(\lambda))$ associated with the eigenvalue λ.

Moreover, if σ has c_0 consecutions at 0, then the $(m-c_0)$th block of $E_\sigma(P)$ is equal to I_n, and if σ has i_0 inversions at 0, then the $(m-i_0)$th block of $H_\sigma(P)$ is equal to I_n.

Next, we consider another class of pencils for matrix polynomial referred to as generalized Fiedler (GF) pencils.

Definition 1.2.9. *(GF pencil) [6] Let $\{C_0, C_1\}$ be a partition of $\{0, 1, \ldots, m\}$ with $k_i = \#(C_i)$ for $i = 0, 1$. Given any pair of bijections $\mu_i : C_i \to \{0, 1, \ldots, k_i\}$, $i = 0, 1$, we denote $\mu = (\mu_0, \mu_1)$. Then the GF pencil of $P(\lambda)$ associated with μ is the $nm \times nm$ pencil $T_\mu(\lambda) := \lambda T_{\mu_1} - T_{\mu_0}$ with*

$$T_{\mu_i} := \widetilde{M}_{\mu_i^{-1}(1)} \widetilde{M}_{\mu_i^{-1}(2)} \cdots \widetilde{M}_{\mu_i^{-1}(k_i)}, i = 0, 1,$$

where the factors $\widetilde{M}_j$ are defined, in a different way for $i = 0$, than for $i = 1$ as follows:
(a) if $i = 0$ and $j \in C_0$, then

$$\begin{cases} \widetilde{M}_j = M_m^{-1} & \text{for } j = m \\ \widetilde{M}_j = M_j & \text{for } j \neq m \end{cases}$$

(b) if $i = 1$ and $j \in C_1$, then

$$\begin{cases} \widetilde{M}_j = M_m & \text{for } j = m \\ \widetilde{M}_j = M_j^{-1} & \text{for } j \neq m. \end{cases}$$

Note that $\mu_i(j)$ describes the position of $\widetilde{M}_j$ in the product $\widetilde{M}_{\mu_i^{-1}(1)} \widetilde{M}_{\mu_i^{-1}(2)} \cdots \widetilde{M}_{\mu_i^{-1}(k_i)}$. If $0 \in C_0$ and $m \in C_1$, then the pencil $T_\mu(\lambda)$ is said to be a PGF pencil of $P(\lambda)$.

Note that GF pencils that are not proper are defined only if A_m and/or A_0 are non-singular. Also note that by using the commutativity relation it is easy to see that every GF pencil of matrix polynomial is strictly equivalent to a Fiedler pencil of matrix polynomial.

Theorem 1.2.4. *[6] Let $P(\lambda)$ be an $n \times n$ matrix polynomial. Then any GF pencil of $P(\lambda)$ is a strong linearization for $P(\lambda)$.*

Recall that $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$. Following [18], we consider the Fiedler matrices

$$M_0 := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \quad M_{-m} := \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, \tag{1.27}$$

and

$$M_{-i} := M_i^{-1} \text{ for } i = 0, 1, \ldots, m-1 \text{ and } M_m := M_{-m}^{-1}, \tag{1.28}$$

where

$$M_i := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \quad \text{and } M_i^{-1} = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}$$

for $i = 1 : m-1$. Note that M_{-m} and M_0 are invertible if and only if A_m and A_0 are invertible. Then as before $M_i M_j = M_j M_i$ if $||i| - |j|| > 1$.

With this new notation, the GF pencil can be defined as follows.

Definition 1.2.10. *Let $P(\lambda)$ be the matrix polynomial. Let $\{C_0, C_1\}$ be a partition of $\{0, 1, \ldots, m\}$ (C_0 or C_1 may be empty set). Let σ and τ be permutations of C_0 and $-C_1$, respectively. Then the pencil*

$$T_\omega(\lambda) := \lambda M_\tau - M_\sigma$$

is said to be a generalized Fiedler (GF) pencil of $P(\lambda)$ associated with $\omega = (\tau, \sigma)$. If $0 \in C_0$ and $m \in C_1$, then the pencil $T_\omega(\lambda)$ is said to be a Proper Generalized Fiedler (PGF) pencil of $P(\lambda)$.

We use the boldface small letters, such as $\mathbf{t}, \mathbf{q}, \mathbf{s}, \ldots$ for index tuples (or ordered tuples of indices).

Definition 1.2.11. *[18] Let $\mathbf{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple containing indices from $\{0, 1, \ldots, m, -0, -1, \ldots, -m\}$. Then i_j is said to be a simple index of $\mathbf{q}$ if $i_j \neq i_k$ for $k = 1 : s$ and $k \neq j$. We say that $\mathbf{q}$ is* simple index tuple *if each index $i_j, j = 1 : s$, is a simple index of $\mathbf{q}$.*

Definition 1.2.12. *[18] Let $d \geq 1$ be an integer and $\mathbf{q}$ be a simple index tuple containing all indices from $\{0, 1, \ldots, d\}$ or all from $\{-d, -d+1, \ldots, -1\}$.*

(a) Then $\mathbf{q}$ has a consecution at k if both $k, k+1 \in \mathbf{q}$ and k is to the left of $k+1$ in $\mathbf{q}$. We say that $\mathbf{q}$ has an inversion at k if both $k, k+1 \in \mathbf{q}$ and k is to the right of $k+1$ in $\mathbf{q}$.

(b) We say that $\mathbf{q}$ has c_k (resp. i_k) consecutions (res. inversions) at k if $\mathbf{q}$ has consecutions (resp. inversions) at $k, k+1, \ldots, k+c_k-1$ (res. at $k, k+1, \ldots, k+i_k-1$) and $\mathbf{q}$ has no consecution (res. inversion) at $k+c_k$ (res. $k+i_k$).

Example 1.2. Let $\mathbf{q} = (7 : 10, 6, 5, 3 : 4, 0 : 2)$. Then this tuple has consecutions at $0, 1, 3, 7, 8, 9$ and inversions at $5, 6, 4, 2$. Moreover, $\mathbf{q}$ has two consecutions at 0, one consecutions at 3 and, three consecutions at 7. It has one inversion at 2 and three inversions at 4. ■

Definition 1.2.13. *[41, 18] Let $\boldsymbol{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple (or an ordered tuple of indices). Then $\boldsymbol{q}$ is said to satisfy the* Successor Infix Property (SIP) *if for every pair of indices $i_a, i_b \in \boldsymbol{q}$ with $1 \leq a < b \leq s$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ such that $a < c < b$.*

It is clear that if $\mathbf{q}$ satisfies SIP, then for any partition $(\mathbf{q}_l, \mathbf{q}_m, \mathbf{q}_r)$ of $\mathbf{q}$, the tuples $\mathbf{q}_l, \mathbf{q}_m$, and $\mathbf{q}_r$ also satisfy SIP.

Definition 1.2.14. *[41, 18] Let d be a non-negative integer and $\boldsymbol{q}$ be an index tuple containing indices from $\{0, 1, \ldots, d\}$. Then $\boldsymbol{q}$ is said to be in column standard form if*

$$\boldsymbol{q} = (a_p : b_p, a_{p-1} : b_{p-1}, \ldots, a_2 : b_2, a_1 : b_1),$$

with $0 \leq b_1 < b_2 < \ldots < b_{p-1} < b_p \leq d$ and $0 \leq a_j \leq b_j$, for all $j = 1, \ldots, p$. Let $\boldsymbol{t}$ be an index tuple containing indices from $\{-d, -d+1, \ldots, -1\}$. Then $\boldsymbol{t}$ is said to be in column standard form if $d + \boldsymbol{t}$ is in column standard form.

Lemma 1.2.2. *[18] Let $\boldsymbol{q} = (i_1, \ldots, i_s)$ be an index tuple containing indices from $\{0, 1, \ldots, d\}$ or from $\{-d, -d+1, \ldots, -1\}$, for some $d \geq 1$, then $\boldsymbol{q}$ satisfies the SIP if and only if $\boldsymbol{q}$ is equivalent to a (unique) tuple in column standard form.*

Definition 1.2.15. *[18] Let $\boldsymbol{q} = (i_1, \ldots, i_s)$ be an index tuple containing indices from $\{0, 1, \ldots, d\}$ or from $\{-d, -d+1, \ldots, -1\}$, for some $d \geq 1$. The unique index tuple in column standard form equivalent to an index tuple $\boldsymbol{q}$ satisfying the SIP is called the column standard form of $\boldsymbol{q}$. We denote this tuple by $csf(\boldsymbol{q})$.*

If $\mathbf{q}$ is simple then $\mathbf{q}$ satisfies the SIP. So $\mathbf{q}$ is equivalent to a tuple in column standard form.

Theorem 1.2.5. *[18] Let $P(\lambda)$ be an matrix polynomial of degree m. Let P_i for $i = 0, 1, \ldots, m$, be the ith Horner shift of matrix polynomial P. Let $\omega := (\omega_0, \omega_1)$ and $T_\omega(\lambda) = \lambda M_{\omega_1} - M_{\omega_0}$ be a PGF pencil of $P(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(\lambda)$. Assume that ω_1 has c_{-m} consecutions at $-m$, and csf$(\omega_1) = (\tau, -m : -m + c_{-m})$. Set $\xi := csf(-rev\tau, \omega_0) = (\boldsymbol{b}_\alpha, \boldsymbol{b}_{\alpha-1}, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_k = (t_{k-1} + 1 : t_k)$, for $k = 1, \ldots, \alpha$. Define $E_{\omega_0,\omega_1}(P) := E_\xi(P)$, if $c_{-m} = 0$ and*

$$E_{\omega_0,\omega_1}(P) := \begin{bmatrix} \lambda^\alpha [P_0 \quad P_1 \quad \ldots \quad P_{c_{-m}-1}] & B_{c_{-m}} & B_{c_{-m}+1} & \ldots & B_{m-1} \end{bmatrix}^{\mathcal{B}}, \qquad (1.29)$$

if $c_{-m} > 0$, where $E_\xi(P)$ is as in Theorem 1.2.3 and if $\xi(i) \in \boldsymbol{b}_j$, for some $j = 1, 2, \ldots, \alpha$, then the block $B_{i+c_{-m}}$ is as in (1.26). Then $E_{\omega_0,\omega_1}(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(T_\omega(\lambda))$ is an isomorphism.

Further, set $rev\,\omega := (rev\,\omega_0, rev\,\omega_1)$ and define $H_{\omega_0,\omega_1}(P) := E_{rev\,\omega_0, rev\,\omega_1}(P^T)$. Then $H_{\omega_0,\omega_1}(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(T_\omega(\lambda))$ is an isomorphism. Furthermore, if ω_0 has c_0 consecutions at 0, then the $(m - c_0)$th block of $E_{\omega_0,\omega_1}(P)$ is equal to I_n, and if ω_0 has i_0 inversions at 0, then the $(m - i_0)$th block of $H_{\omega_0,\omega_1}(P)$ is equal to I_n.

Definition 1.2.16 ([41], GFPR)**.** *Let $P(\lambda)$ be the matrix polynomial, where A_0 and A_m are non-singular matrices. Let $0 \leq h \leq m-1$, and let $\boldsymbol{q}$ and $\boldsymbol{s}$ be permutations of $\{0, 1, \ldots, h\}$ and $\{-m, -m+1, \ldots, -h-1\}$, respectively. Assume that $\boldsymbol{l}_q$ and $\boldsymbol{r}_q$ are index tuples with elements from $\{0, 1, \ldots, h-1\}$ such that $(\boldsymbol{l}_q, \boldsymbol{q}, \boldsymbol{r}_q)$ satisfies the SIP. Similarly, let $\boldsymbol{l}_s$ and $\boldsymbol{r}_s$ be index tuples with elements from $\{-m, -m+1, \ldots, -h-2\}$ such that $(\boldsymbol{l}_s, \boldsymbol{s}, \boldsymbol{r}_s)$ satisfies the SIP. Then, the pencil*

$$L(\lambda) = \lambda M_{\boldsymbol{l}_s} M_{\boldsymbol{l}_q} M_{\boldsymbol{s}} M_{\boldsymbol{r}_q} M_{\boldsymbol{r}_s} - M_{\boldsymbol{l}_s} M_{\boldsymbol{l}_q} M_{\boldsymbol{q}} M_{\boldsymbol{r}_q} M_{\boldsymbol{r}_s}$$

is called a generalized Fiedler pencil with repetition (GFPR) of $P(\lambda)$.

Note that if $\mathbf{l}_q, \mathbf{r}_q, \mathbf{l}_s, \mathbf{r}_s$ are all empty index tuple then $L(\lambda)$ is a PGF pencil.

Theorem 1.2.6. *[41] Every GFPR of $P(\lambda)$ is a linearization of $P(\lambda)$.*

1.3 Rational matrix function

Let $\mathbb{C}(\lambda)$ denote the field of rational functions of the form $p(\lambda)/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are scalar polynomials in $\mathbb{C}[\lambda]$. Let $\mathbb{C}(\lambda)^n$ denotes the vector space of column n-tuples over the field $\mathbb{C}(\lambda)$, and $\mathbb{C}(\lambda)^{m\times n}$ denote the vector space of m-by-n matrices over the field $\mathbb{C}(\lambda)$, that is, elements of $\mathbb{C}(\lambda)^{m\times n}$ are m-by-n rational matrix functions.

Definition 1.3.1. *[5] Let $G(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$. Then the* normal rank *of G, denoted by* nrank(G), *is defined by* $\operatorname{nrank}(G) := \max_{\lambda} \operatorname{rank}(G(\lambda))$, *where the maximum is taken over all λ which are not poles of $G(\lambda)$.*

Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ be a rational matrix function. Then $G(\lambda)$ is said to be regular if $\operatorname{nrank}(G) = n$. Equivalently, $G(\lambda)$ is regular if the rank of $G(\lambda)$ over the field $\mathbb{C}(\lambda)$ is n.

Any rational matrix function $G(\lambda) \in \mathbb{C}(\lambda)^{n\times m}$ can be written as

$$G(\lambda) = \frac{Q(\lambda)}{d(\lambda)},$$

where $Q(\lambda) \in \mathbb{C}[\lambda]^{n\times m}$ is a matrix polynomial and $d(\lambda)$ is the monic least common denominator of all entries of $G(\lambda)$.

Theorem 1.3.1 (Smith-McMillan form, [40, 21])**.** *Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times m}$ with normal rank r. Then there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ of sizes $n \times n$ and $m \times m$, respectively, such that*

$$G(\lambda) = U(\lambda)\,\mathbf{SM}(G(\lambda))\,V(\lambda),$$

where

$$\mathbf{SM}(G(\lambda)) = \operatorname{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \frac{\phi_2(\lambda)}{\psi_2(\lambda)}, \dots, \frac{\phi_r(\lambda)}{\psi_r(\lambda)}, 0, \dots, 0\right)$$

is called the Smith-McMillan form of $G(\lambda)$. The scalar polynomials $\phi_i(\lambda), \psi_i(\lambda)$ are monic (have the highest degree coefficient equal to 1), are pairwise coprime (do not have common divisors) for $i = 1, 2, \dots, r-1$ and satisfy the divisibility properties: $\phi_i(\lambda)/\phi_{i+1}(\lambda)$ and $\psi_{i+1}(\lambda)/\psi_i(\lambda), i = 1, 2, \dots, r-1$. Moreover, the polynomials $\phi_i(\lambda)$ and $\psi_i(\lambda)$ are uniquely defined by $G(\lambda)$ and are called elementary divisors of G.

Remark 1.3.1. *[13] Note that $\frac{\phi_i(\lambda)}{\psi_i(\lambda)}$ in the diagonal of the Smith-McMillan form are not necessarily proper.*

Example 1.3. [13] Consider the rational matrix function

$$G(\lambda) = \begin{bmatrix} \frac{\lambda+2}{\lambda+1} & 0 \\ 0 & \frac{\lambda+1}{\lambda} \\ \frac{\lambda+2}{\lambda} & \frac{1}{\lambda} \\ 0 & 1 \end{bmatrix} = \frac{Q(\lambda)}{\lambda(\lambda+1)}, \text{ where } Q(\lambda) = \begin{bmatrix} \lambda(\lambda+2) & 0 \\ 0 & (\lambda+1)^2 \\ (\lambda+1)(\lambda+2) & \lambda+1 \\ 0 & \lambda(\lambda+1) \end{bmatrix}.$$

The Smith form of $Q(\lambda)$ is given in Example 1.1. Hence the Smith-McMillan form of $G(\lambda)$ is given by

$$\mathbf{SM}(G(\lambda)) = \frac{\mathbf{SF}(Q(\lambda))}{\lambda(\lambda+1)} = \begin{bmatrix} \frac{1}{\lambda(\lambda+1)} & 0 \\ 0 & \frac{\lambda+2}{\lambda} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition 1.3.2 (Poles and zeros)**.** *For a rational matrix function $G(\lambda) \in \mathbb{C}(\lambda)^{n\times m}$ with Smith-McMilla form given in Theorem 1.3.1, the polynomials*

$$\psi_G(\lambda) := \psi_1(\lambda)\psi_2(\lambda)\dots\psi_r(\lambda) \tag{1.30}$$

and

$$\phi_G(\lambda) := \phi_1(\lambda)\phi_2(\lambda)\dots\phi_r(\lambda) \tag{1.31}$$

are called the pole (characteristic) polynomial *and* zero polynomial *of* $G(\lambda)$, *respectively. The zeros (i.e., roots) of* $\psi_G(\lambda)$ *and* $\phi_G(\lambda)$, *are called finite* poles *and* zeros *of the rational matrix* $G(\lambda)$ *respectively.*

Example 1.4. Consider the rational matrix function given in Example 1.3 and its Smith-McMillan form. Then the pole and zero polynomials are

$$\psi_G(\lambda) = \lambda^2(\lambda+1), \qquad \phi_G(\lambda) = \lambda + 2.$$

Hence it has three poles $0, 0, -1$ and one transmission zero -2. ■

Remark 1.3.2. *[13] For regular rational matrix function* $G(\lambda)$ *one can study the pole and zero polynomials without computing the Smith-Mcmillan form since by Theorem 1.3.1, we have*

$$\det G(\lambda) = \det U(\lambda) \det \mathbf{SM}(G(\lambda)) \det V(\lambda) = c \det \mathbf{SM}(G(\lambda)) = c\frac{\phi(\lambda)}{\psi(\lambda)}, \tag{1.32}$$

where $c = \det U(\lambda) \det V(\lambda)$. *Unfortunately, when the pole and the zero polynomials have common roots, then the corresponding zeros and poles do not appear in* $\det G(\lambda)$. *For example, the rational matrix function*

$$G(\lambda) = \begin{bmatrix} \frac{1}{\lambda(\lambda-2)} & 0 \\ 0 & \frac{\lambda-2}{\lambda} \end{bmatrix}$$

has a zero at 2 *and poles at* $0, 0, 2$, *but* $\det G(\lambda) = \frac{1}{\lambda^2}$, *which gives double pole at* 0, *but hides the pole and zero at* 2.

We denote the set of poles of $G(\lambda)$ by $\text{Poles}(G)$ and the set of zeros of $G(\lambda)$ by $\text{Sp}(G)$ and are given by

$$\text{Poles}(G) := \{\lambda \in \mathbb{C} : \psi(\lambda) = 0\} \tag{1.33}$$

$$\text{Sp}(G) := \{\lambda \in \mathbb{C} : \phi(\lambda) = 0\}. \tag{1.34}$$

1.4 State space theory

Consider a linear, time invariant (LTI), multivariable system Σ given by

$$\begin{aligned} A(\tfrac{d}{dt})x(t) &= B(\tfrac{d}{dt})u(t) \\ y(t) &= C(\tfrac{d}{dt})x(t) + D(\tfrac{d}{dt})u(t), \ t \geq 0 \end{aligned}$$

where $\frac{d}{dt}$ is the differential operator, $A(\lambda) \in \mathbb{C}[\lambda]^{r\times r}$ is regular, $B(\lambda) \in \mathbb{C}[\lambda]^{r\times m}$, $C(\lambda) \in \mathbb{C}[\lambda]^{p\times r}$, $D(\lambda) \in \mathbb{C}[\lambda]^{p\times m}$, $u(t) : \mathbb{R}^+ \to \mathbb{R}^m$ is the input vector, $x(t) : \mathbb{R}^+ \to \mathbb{R}^r$ is the state vector and $y(t) : \mathbb{R}^+ \to \mathbb{R}^p$ is the output vector [40].

Definition 1.4.1. *[30] The matrix polynomial $\mathcal{S}(\lambda)$ given by*

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} -A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right] \in \mathbb{C}[\lambda]^{(r+p)\times(r+m)}$$

is called the Rosenbrock system matrix or the Rosenbrock system polynomial or simply called system matrix of the system Σ. The rational matrix $G(\lambda)$ given by

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$$

is called the transfer function of the system Σ.

The system Σ is said to be in *state-space form* if it is given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + P(\lambda)u(t), \end{aligned} \tag{1.35}$$

where $P(\lambda) \in \mathbb{C}[\lambda]^{p\times m}$ is a matrix polynomial and $A, E \in \mathbb{C}^{n\times n}$ with E being nonsingular, $B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{p\times n}$ are constant matrices, see [40]. We denote the system defined in (1.35) denoted by $(E, A, B, C, P(\lambda))$.

A complex number λ is said to be an **eigenvalue** of the system matrix $\mathcal{S}(\lambda)$ if $\text{rank}(\mathcal{S}(\lambda)) < \text{nrank}(\mathcal{S})$. Equivalently, $\lambda \in \mathbb{C}$ an eigenvalue of $\mathcal{S}(\lambda) \Leftrightarrow \lambda$ is a root of the zero polynomial $\phi_{\mathcal{S}}(\lambda)$ of $\mathcal{S}(\lambda)$ given by $\phi_{\mathcal{S}}(\lambda) = \phi_1(\lambda)\cdots\phi_k(\lambda)$, where $\phi_1(\lambda), \ldots, \phi_k(\lambda)$ are invariant polynomials of $\mathcal{S}(\lambda)$. An eigenvalue λ of $\mathcal{S}(\lambda)$ is called an **invariant zero** of the system Σ. We denote the set of eigenvalues of $\mathcal{S}(\lambda)$ by $\text{Sp}(\mathcal{S})$. Recall the zero polynomial $\phi_G(\lambda)$ and pole polynomial $\psi_G(\lambda)$ given in (1.31) and (4.6). Then the zeros (i.e., roots) of $\phi_G(\lambda)$ are called **transmission zeros** of the system Σ. The zeros of $\psi_G(\lambda)$ are called **poles** of the system Σ.

Let $P(\lambda), Q(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$. Then $Z(\lambda) \in \mathbb{C}^{m\times m}$ is said to be a **common left divisor** of $P(\lambda)$ and $Q(\lambda)$ if $P(\lambda) = Z(\lambda)R(\lambda)$ and $Q(\lambda) = Z(\lambda)T(\lambda)$, for some $R(\lambda), T(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$. Equivalently, $\begin{bmatrix} P(\lambda) & Q(\lambda) \end{bmatrix} = Z(\lambda)\begin{bmatrix} R(\lambda) & T(\lambda) \end{bmatrix}$. Further, $Z(\lambda)$ is said to be the **greatest common left divisor** of $P(\lambda)$ and $Q(\lambda)$ if $U(\lambda)$ is a common left divisor of $P(\lambda)$ and $Q(\lambda)$ then $Z(\lambda) = U(\lambda)V(\lambda)$ for some $V(\lambda) \in \mathbb{C}^{m\times m}$, see [40].

The matrix polynomial $P(\lambda)$ and $Q(\lambda)$ are said to be **left coprime** if the greatest common left divisors of $P(\lambda)$ and $Q(\lambda)$ is unimodular. The greatest common right divisor of $P(\lambda)$ and $Q(\lambda)$ is defined similarly. Then $P(\lambda)$ and $Q(\lambda)$ are said to be **right coprime** if the greatest common right divisor of $P(\lambda)$ and $Q(\lambda)$ is unimodular, see [40].

Definition 1.4.2. *[40] The LTI system Σ is said to be of least order (or minimal) if $A(\lambda)$ and $B(\lambda)$ are left coprime, and $A(\lambda)$ and $C(\lambda)$ are right coprime.*

If $A(\lambda) = \lambda I - A, B(\lambda) = B$ and $C(\lambda) = C$ then:

(i) $A(\lambda)$ and $B(\lambda)$ are left coprime $\Leftrightarrow \operatorname{rank}[\lambda I - A, B] = n$ for $\lambda \in \mathbb{C}$,

(ii) $A(\lambda)$ and $C(\lambda)$ are right coprime $\Leftrightarrow \operatorname{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for $\lambda \in \mathbb{C}$.

Definition 1.4.3. *[40] The degree of a matrix polynomial $P(\lambda) \in \mathbb{C}[\lambda]^{p\times m}$, denoted by* $\deg P(\lambda)$*, is defined as the maximum degree among the degrees of all its maximum order (non-zero) minors.*

Note that if $P(\lambda)$ is square and regular, then $\deg P(\lambda) = \det(P(\lambda))$.

Definition 1.4.4. *[40] Let $\mathcal{S}_1(\lambda)$ and $\mathcal{S}_2(\lambda)$ be system matrices with extended forms*

$$\mathcal{S}_i^e(\lambda) = \left[\begin{array}{cc|c} I_{q-r_i} & 0 & 0 \\ 0 & A_i(\lambda) & B_i(\lambda) \\ \hline 0 & C_i(\lambda) & D_i(\lambda) \end{array}\right] \qquad i = 1, 2, \tag{1.36}$$

where $A_i(\lambda) \in \mathbb{C}[\lambda]^{r_i\times r_i}$, $B_i(\lambda) \in \mathbb{C}[\lambda]^{r_i\times m}, C(\lambda) \in \mathbb{C}[\lambda]^{p\times r_i}, D(\lambda) \in \mathbb{C}[\lambda]^{p\times m}$, $n_i =$ $\deg |(A_i(\lambda))|$ *and $q \geq \max\{n_1, n_2\}$. Then $\mathcal{S}_1(\lambda)$ and $\mathcal{S}_2(\lambda)$ are said to be* strict system equivalent *(SSE) if there exists unimodular $U(\lambda), V(\lambda) \in \mathbb{C}[\lambda]^{q\times q}$, and $X(\lambda) \in \mathbb{C}[\lambda]^{p\times q}$, $Y(\lambda) \in \mathbb{C}[\lambda]^{q\times m}$ such that*

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & I_p \end{bmatrix} \begin{bmatrix} I_{q-r_1} & 0 & 0 \\ 0 & A_1(\lambda) & B_1(\lambda) \\ 0 & C_1(\lambda) & D_1(\lambda) \end{bmatrix} = \begin{bmatrix} I_{q-r_2} & 0 & 0 \\ 0 & A_2(\lambda) & B_2(\lambda) \\ 0 & C_2(\lambda) & D_2(\lambda) \end{bmatrix} \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & I_m \end{bmatrix}.$$

Theorem 1.4.1. *[40] Every extended system matrix $\mathcal{S}_i^e(\lambda) \in \mathbb{C}[\lambda]^{(n+p)\times(n+m)}$ is a strict system equivalent to a system matrix in state space form.*

Theorem 1.4.2. *[40] Suppose that the LTI system Σ is of minimal order. Consider the system matrix*

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} -A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right]$$

and the transfer function

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) \in \mathbb{C}(\lambda)^{p\times m}.$$

Suppose that the Smith-McMillan form of $G(\lambda)$ is given by

$$\mathbf{SM}(G(\lambda)) = \mathrm{diag}[\phi_1(\lambda)/\psi_1(\lambda), \ldots, \phi_q(\lambda)/\psi_q(\lambda), 0_{p-q,m-q}].$$

Then the Smith forms of $A(\lambda)$ and $\mathcal{S}(\lambda)$ are given by

$$\mathbf{SF}(A(\lambda)) = \mathrm{diag}[I_{r-q}, \psi_q(\lambda), \psi_{q-1}(\lambda) \ldots, \psi_1(\lambda)], \tag{1.37}$$

$$\mathbf{SF}(\mathcal{S}(\lambda)) = \mathrm{diag}[I_r, \phi_1(\lambda), \phi_2(\lambda) \ldots, \phi_q(\lambda), 0_{p-q,m-q}]. \tag{1.38}$$

Thus we have Poles$(G) =$ Sp(A) and Sp$(\mathcal{S}) =$ Sp(G).

Definition 1.4.5. *([8]) The system given in (1.35) is said to be* controllable *if starting from any initial state $x(0)$, the system can be driven to any final state $x_1 = x(t_1)$ in some finite time t_1, choosing the input variable $u(t), 0 \leq t \leq t_1$ appropriately.*

Theorem 1.4.3. *[8] The following statements are equivalent:*

(a) *The system (1.35) is controllable.*

(c) $\mathrm{rank}[\lambda E - A, B] = n$ *for every* $\lambda \in \mathbb{C}$.

Definition 1.4.6. *([8]) The system given in (1.35) is said to be* observable *if there exists $t_1 > 0$ such that the initial condition $x(0)$ can be uniquely determined by the input $u(t)$ and the output $y(t)$, for all $t, 0 \leq t < t_1$*

Theorem 1.4.4. *[8] The following statements are equivalent:*

(a) *The system (1.35) is observable.*

(c) $\mathrm{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n$ *for all* $\lambda \in \mathbb{C}$.

1.4.1 Minimal realizations

Let $G(\lambda) \in \mathbb{C}(\lambda)^{p\times m}$ be a rational matrix. Then $G(\lambda)$ is said to be **proper** if $G(\lambda) \to D \in \mathbb{C}^{p\times m}$ as $|\lambda| \to \infty$. If $D = 0$ then $G(\lambda)$ is called **strictly proper** [8].

Definition 1.4.7 (Realization, [8])**.** *Let $G \in \mathbb{C}^{p\times m}(\lambda)$ be a proper rational matrix function. If there exists matrices $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{p\times m}$ such that*

$$G(\lambda) = C(\lambda I - A)^{-1}B + D \tag{1.39}$$

then (A, B, C, D) is called a realization of G.

Theorem 1.4.5 ([12])**.** *Every $p \times m$ proper rational matrix function has a realization.*

A realization of a rational matrix is not unique, that is, the realization of a rational function can be represented in different forms. For example, if $G(\lambda) = C(\lambda I - A)^{-1}B + D$ is a realization of G then direct computation shows that

$$\left(\begin{bmatrix} A & 0 \\ 0 & \widehat{A} \end{bmatrix}, \begin{bmatrix} B \\ \widehat{B} \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

where $\widehat{A}$ and $\widehat{B}$ are arbitrary matrices of appropriate sizes, is also a realization of $G(\lambda)$. A realization of $\frac{1}{(\sigma-\lambda)^2}$ is given by [37]

$$\frac{1}{(\sigma - \lambda)^2} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} \sigma & -1 \\ 0 & \sigma \end{bmatrix} - \lambda I \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On the other hand, a symmetric realization is given by

$$\frac{1}{(\sigma - \lambda)^2} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & \sigma \\ \sigma & -1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Definition 1.4.8 (Minimal Realization, [4])**.** *Let $G(\lambda)$ be a proper rational matrix function. A realization (A, B, C, D) of $G(\lambda)$ is said to be minimal if the order of A is the smallest, that is, if (A', B', C', D') is any other realization of $G(\lambda)$, then the order of A' is greater than or equal to the order of A.*

Consider a realization $G(\lambda) = C(\lambda I - A)^{-1}B + D$. It is easy to see that every pole of $G(\lambda)$ is an eigenvalue of $\lambda I - A$. The converse does not hold in general but if the realization is minimal then the converse is true. The state space associated with $G(\lambda)$ is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1.40}$$

Then the pair (A, B) is said to be **controllable** if the system (1.40) is controllable. The pair (A, C) is said to be **observable** if the system (1.40) is observable.

Theorem 1.4.6. *([22]) Let (A, B, C, D) be a realization of $G(\lambda)$.*

(a) The (A, B, C, D) is minimal if and only if the pair (A, B) is controllable and the pair (A, C) is observable.

(b) If (A, C, B, D) is a minimal realization of $G(\lambda)$ then the eigenvalues of A coincide with the poles of G.

Next, one may ask whether it is possible to construct directly a minimal realization. There are several algorithms available for construction of minimal realization such as *Gilbert's realization*, see [22] and "Silverman Realization Algorithm" ([4], Theorem. 4.41). An outline of Gilbert algorithm is as follows.

Let $G(\lambda)$ be a $n \times m$ transfer matrix. We write it in the form

$$G(\lambda) = \frac{Q(\lambda)}{d(\lambda)},$$

where $d(\lambda)$ is a scalar polynomial and $Q(\lambda)$ is a matrix polynomial. We denote the zeros of $d(\lambda)$ by $\lambda_1, \ldots, \lambda_m$ and assume for simplicity that they are pairwise distinct and real. Then

$$d(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_m)$$

and $G(\lambda)$ has the following partial fractional expansion (with $m \times m$-matrices W_k):

$$G(\lambda) = D + \sum_{k=1}^{m} \frac{1}{\lambda - \lambda_k} W_k.$$

Suppose $\text{rank}(W_k) = \rho_k$. Let $B_k \in \mathbb{R}^{\rho_k \times r}$ and $C_k \in \mathbb{R}^{n \times \rho_k}$ be such that $W_k = C_k B_k$. Then a realization of $G(\lambda)$ is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{ccc|c} \lambda_1 I_{\rho_1} & \cdots & 0 & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_r I_{\rho_r} & B_r \\ \hline C_1 & \cdots & C_r & D \end{array}\right].$$

Using Theorem 1.4.3 and Theorem 1.4.4 it is easy to see that this realization is controllable and observable and hence it is minimal. Note that each pole λ_k is of multiplicity 1 by assumption, whereas in the state space realization the eigenvalue λ_k is of multiplicity ρ_k, which may be larger than 1. The method can be extended to cases with complex or multiple roots of $d(\lambda)$ [22].

Theorem 1.4.7. *[40] Let $\Sigma_i, i = 1, 2$ be LTI systems of minimal order with system matrix*

$$\mathcal{S}_i(\lambda) = \left[\begin{array}{c|c} -A_i(\lambda) & B_i(\lambda) \\ \hline C_i(\lambda) & D_i(\lambda) \end{array}\right],$$

and transfer functions $G_i(\lambda), i = 1, 2$. Then Σ_1 is strict system equivalent to $\Sigma \Leftrightarrow G_1(\lambda) = G_2(\lambda)$.

1.5 Sensitivity analysis of eigenvalues

Norms of vectors, matrices and matrix polynomials will play an important role in the sensitivity analysis of eigenvalue problems.

Vector and matrix norm. A function $\|.\| : \mathbb{C}^n \to \mathbb{R}$ is said to be norm on $\mathbb{C}^n$ (or a vector norm) if $\|.\|$ satisfies the following properties:

(i) $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in \mathbb{C}$ and $x \in \mathbb{C}^n$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in \mathbb{C}^n$

For $x \in \mathbb{C}^n$ and $1 \leq p < \infty$, the Hölder p-norm on $\mathbb{C}^n$ is given by $\|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}$, and $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$. Let $\|.\|$ be a norm on $\mathbb{C}^n$. Define $\|.\|_* : \mathbb{C}^n \to \mathbb{R}$ by

$$\|y\|_* := \sup\{|y^* x| : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

Then $\|.\|_*$ is a norm and is called the dual norm of the norm $\|.\|$. It is easy to see that for $x, y \in \mathbb{C}^n$, we have $|y^* x| \leq \|x\|\|y\|_*$.

Definition 1.5.1. *([36]) Let $x = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$. We define $|x| := [|x_1|, \ldots, |x_n|]^T$. We say that $|x| \leq |y|$ if $|x_i| \leq |y_i|$ for all $i = 1 : n$. A vector norm on $\mathbb{C}^n$ is said to be monotone if $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ for all $x, y \in \mathbb{C}^n$.*

Next we consider a matrix norm. Let $\|.\|$ be a norm on $\mathbb{C}^n$. Define $\|.\| : \mathbb{C}^{n\times n} \to \mathbb{R}$ by

$$\|A\| := \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

Then $\|.\|$ is a norm on $\mathbb{C}^{n\times n}$ and is referred to as subordinate norm or induced operator norm. The operator norm induced by the 2-norm $\|.\|_2$ on $\mathbb{C}^n$ is referred to as the 2-norm or the spectral norm on $\mathbb{C}^{n\times n}$. We denote the spectral norm on $\mathbb{C}^{n\times n}$ by $\|.\|_2$. Thus

$$\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|_2.$$

Define $\langle X, Y\rangle := \text{Trace}(Y^*X)$ for $X, Y \in \mathbb{C}^{n\times n}$. Then $\langle .,.\rangle$ defines an inner product on $\mathbb{C}^{n\times n}$ and $\|X\|_F := \sqrt{\langle X, X\rangle} = (\text{Trace}X^*X)^{1/2}$ is the Frobenius norm on $\mathbb{C}^{n\times n}$, where $\text{Trace}(A)$ is the trace of A. Now let $\|.\|$ be a norm on $\mathbb{C}^{n\times n}$. Then $\|.\|_* : \mathbb{C}^{n\times n} \to \mathbb{R}$ given by

$$\|Y\|_* = \sup\{|\langle X, Y\rangle| : X \in \mathbb{C}^{n\times n}, \|X\| = 1\}$$

defines a norm and is referred to as the dual norm of $\|.\|$.

Definition 1.5.2. *A norm $\|.\|$ on $\mathbb{C}^{n\times n}$ is said to be a max-norm if* $\|\text{diag}(A, B)\| = \max(\|A\|, \|B\|)$.

A matrix $U \in \mathbb{C}^{n\times n}$ is said to be unitary if $U^*U = UU^* = I$, where U^* is the conjugate transpose of U. The spectral and Frobenius norms satisfy the following properties:

- $\|Ux\|_2 = \|x\|_2$ for any unitary matrix U.
- $\|UAV^*\|_F = \|A\|_F$ for any unitary matrix $Uand\ V$.
- $\|A \otimes B\|_2 = \|A\|_2\|B\|_2$, for $A \in \mathbb{R}^{n\times n}$ and , $B \in \mathbb{R}^{m\times m}$.

Let $A \in \mathbb{C}^{n\times m}$ be a nonzero matrix with rank r. Then the singular value decomposition (SVD) of A is defined as

$$A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, with $\sigma_1 \geq \sigma_1 \geq \cdots \geq \sigma_r > 0$. Here $\sigma_1, \ldots, \sigma_r$ are called singular values of A. The columns of U and V are left and right singular vectors of A. We denote by $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$, respectively, the largest and the smallest nonzero singular values of A [47]. Note that

$$\|A\|_2 = \sigma_{\max}(A) = \sigma_1 \qquad \text{and} \qquad \|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2.$$

Let H be a finite dimensional innerproduct space with an innerproduct $\langle .,.\rangle$. Let $\|.\|$ be a norm on H. Then

$$\|x\|_* := \sup_{\|z\|=1} \{|\langle z, x\rangle| : z \in H\}$$

defines a norm on H. The norm $\|.\|_*$ on H is called the dual norm of $\|.\|$ relative to the innerproduct $\langle .,.\rangle$. Note that $|\langle x, y\rangle| \leq \|x\|\|y\|_*$ and that $\|x\| = \sup_{\|y\|_*=1} \{|\langle x, y\rangle| : y \in H\}$. Also note that there exists x_{opt} and y_{opt} in H such that $\|x_{opt}\| = 1 = \|y_{opt}\|_*$ and that

$$\langle x, y_{opt}\rangle = \|x\| \text{ and } \langle x_{opt}, y\rangle = \|y\|_*.$$

Let H_j be a finite dimensional innerproduct space with an innerproduct $\langle , \rangle_{H_j}$, $j = 1, \ldots, m$. Consider the cartesian product $H = H_1 \times H_2 \times \cdots \times H_m$ and the innerproduct $\langle ., . \rangle_H : H \times H \to \mathbb{C}$ given by

$$\langle x, y \rangle_H = \langle x_1, y_1 \rangle_{H_1} + \cdots + \langle x_m, y_m \rangle_{H_m},$$

where $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m) \in H$. Let $\|.\|_{H_j}$ be a norm on H_j and $\|.\|_{H_j^*}$ be the dual norm of $\|.\|_{H_j}$ for $j = 1, \ldots, m$. Then $|||.||| : H \to \mathbb{R}$ defined by

$$|||x||| := \|(\|x_1\|_{H_1}, \cdots, \|x_m\|_{H_m})\|_p, \text{ where } 1 \le p \le \infty,$$

is a norm on H. Further,

$$|||x|||_* := \|(\|x_1\|_{H_1^*}, \cdots, \|x_m\|_{H_m^*})\|_q, \text{ where } 1/p + 1/q = 1, \tag{1.41}$$

is the dual norm of $|||.|||$ relative to the innerproduct $\langle ., . \rangle_H$. Furthermore, we have $|\langle x, y \rangle_H| \le |||x||| |||y|||_*$. Also, for each x and y in H there exists x_{opt} and y_{opt} in H such that $|||x_{opt}||| = 1 = |||y_{opt}|||_*$ and that

$$\langle x, y_{opt} \rangle = |||x||| \text{ and } \langle x_{opt}, y \rangle = |||y|||_*.$$

Indeed, choose $x_{j,opt} \in H_j$ such that $\|x_{j,opt}\| = 1$ and $\langle x_{j,opt}, y_j \rangle = \|y\|_{H_{j^*}}$ for $j = 1, \ldots, m$. Set $x_{opt} := (x_{1,opt}, \ldots, x_{m,opt})$ when $p = \infty$ and $x_{opt} := (\widehat{x_1}, \ldots, \widehat{x_m})$ when $1 < p < \infty$, where

$$\widehat{x_j} := \frac{(\|y_j\|_{H_{j^*}})^{q-1} x_{j,opt}}{(|||y|||_*)^{q/p}}, \qquad j = 1, \ldots, m. \tag{1.42}$$

Then $|||x_{opt}||| = 1$ and $\langle x_{opt}, y \rangle = |||y|||_*$ for $1 < p \le \infty$. When $p = 1$, define $x_{opt} := (\widehat{x_1}, \ldots, \widehat{x_m})$, where

$$\widehat{x_j} = \begin{cases} x_{j,opt} & if \ \|y_j\|_{H_{j^*}} = |||y|||_* \\ 0, & else. \end{cases} \tag{1.43}$$

Then $|||x_{opt}||| = 1$ and $\langle x_{opt}, y \rangle = |||y|||_*$.

Norms of matrix polynomials. We briefly review norms of matrix polynomial, see [2, 1] for details. Let $\|.\|$ be a norm on $\mathbb{C}^{n \times n}$. For $1 \le p \le \infty$, define $|||.||| : \mathbb{L}_m(\mathbb{C}^{n \times n}) \to \mathbb{R}$ by

$$|||P||| := \|(\|A_0\|, \cdots, \|A_m\|)\|_p, \qquad 1 \le p \le \infty,$$

where $P(z) = \sum_{i=0}^{m} z^i A_i$ and $\|.\|_p$ is the Hölder's p-norm. Then $|||.|||$ is a norm on $\mathbb{L}_m(\mathbb{C}^{n \times n})$ and $\langle ., . \rangle_m : \mathbb{L}_m(\mathbb{C}^{n \times n}) \times \mathbb{L}_m(\mathbb{C}^{n \times n}) \to \mathbb{C}$ given by

$$\langle P_1, P_2 \rangle_m := \sum_{i=0}^{m} \langle A_i, B_i \rangle$$

is an innerproduct on $\mathbb{L}_m(\mathbb{C}^{n\times n})$, where $P_1(\lambda) = \sum_{j=0}^{m} \lambda^i A_i$ and $P_2(\lambda) = \sum_{j=0}^{m} \lambda^i B_i$. Then the dual norm $|||.|||_*$ of $|||.|||$ is given by

$$|||Y|||_* = \max\{|\langle X, Y\rangle_m| : |||X||| = 1\}.$$

Let $P(z) = \sum\limits_{i=0}^{m} z^i A_i \in \mathbb{L}_m(\mathbb{C}^{n\times n})$. Then there exists $Q \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ such that $|||Q||| = 1$ and $|||P|||_* = \langle Q, P\rangle_m = \|(\|A_0\|_*, \cdots, \|A_m\|_*)\|_q$, where $1/p + 1/q = 1$. In particular, for Hölder p-norm on $\mathbb{C}^n$, we have

$$|||P|||_p := \|(\|A_0\|_p, \cdots, \|A_m\|_p)\|_p.$$

Then

$$|||P|||_q = \|(\|A_0\|_q, \cdots, \|A_m\|_q)\|_q$$

is the dual norm of $|||P|||_p$, where $1/p + 1/q = 1$.

The condition number of a simple eigenvalue of a matrix polynomial is defined as follows:

Definition 1.5.3. *[1] Let λ denote a simple, finite eigenvalue of $P(z) = \sum\limits_{i=0}^{m} A_i z^i$. Then the condition number* $\mathrm{cond}(\lambda, P)$ *of λ is given by*

$$\mathrm{cond}(\lambda, P) := \limsup_{|||\Delta P|||\to 0} \left\{ \frac{\mathrm{dist}(\lambda, Sp(P + \Delta P))}{|||\Delta P|||} : \Delta P \in \mathbb{L}_m(\mathbb{C}^{n\times n}) \right\}.$$

Definition 1.5.4. *[16] The adjugate of A is the transpose of the matrix of cofactors of A and is given by* $\mathrm{adj}(A) = C^T$.

The explicit formula for condition number of simple eigenvalue of matrix polynomial $P(\lambda)$ is as follows.

Theorem 1.5.1. *[1] Let $P \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ be regular and (λ, x, y) be a simple eigentriple. Set $\Lambda_m := (1, \lambda, \ldots, \lambda^m)$ and $p(z) := \det(P(z))$. Then λ as a function of P is differentiable at P and the derivative $D\lambda(P)$ is given by*

$$D\lambda(P)X = \frac{-\mathit{Trace}\,(\mathrm{adj}P(\lambda)X(\lambda))}{p'(\lambda)} = \langle P, \nabla\lambda(P)\rangle,$$

where

$$\nabla\lambda(P) = \frac{-\bar{\Lambda}_m \otimes \mathrm{adj}(P(\lambda))^*}{\overline{p'(\lambda)}} = \frac{-\bar{\Lambda}_m \otimes yx^*}{\overline{y^*\partial_\lambda P(\lambda)x}}.$$

Hence

$$\mathrm{cond}(\lambda, P) = \frac{\|\Lambda_m\|_q \|\mathrm{adj}(P(\lambda))^*\|_*}{|p'(\lambda)|} = \frac{\|\Lambda_m\|_q \|yx^*\|_*}{|y^*\partial_\lambda P(\lambda)x|}.$$

For the Spectral or the Frobenius norm on $\mathbb{C}^{n\times n}$, we have

$$\mathrm{cond}(\lambda, P) = \frac{\|\Lambda_m\|_q \prod_{j=1}^{n-1} \sigma_j(P(\lambda))}{|p'(\lambda)|} = \frac{\|\Lambda_m\|_q \|x\|_2 \|y\|_2}{|y^*\partial_\lambda P(\lambda)x|}.$$

2 Fiedler Linearizations for system matrix of LTI system

In this chapter, we introduce a family of linearizations, which we refer to as Fiedler linearizations, of the Rosenbrock system matrix of an LTI system in state space form for computation of transmission and invariant zeros of the system. We define linearizations for the transfer function of the LTI system and show that under appropriate assumptions a Fiedler linearization of the system matrix is also a linearization of the transfer function. Thus, given a rational eigenvalue problem, we reformulate the problem of computing eigenvalues of a rational matrix function to that of computation of transmission zeros of an LTI state space system. Hence we compute the eigenvalues by solving a generalized eigenvalue problem for Fiedler pencil of the system matrix.

2.1 Introduction

Rational eigenvalue problems (REP) arise in many applications such as in acoustic emissions of high speed trains, calculations of quantum dots, free vibration of plates with elastically attached masses, vibrations of fluid-solid structures and in control theory, see [23, 34, 17, 46, 38, 29, 7, 44] and the references therein. Computation of eigenvalues and eigenvectors of a rational matrix function is an emerging area of research. We therefore consider the following problem.

Problem-I. Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ be a regular rational matrix function. Compute $\lambda \in \mathbb{C}$ and nonzero vectors x and y in $\mathbb{C}^n$ such that $G(\lambda)x = 0$ and $y^T G(\lambda) = 0$.

We could solve the REP in Problem-I by converting it into a polynomial eigenvalue problem by clearing out the denominators in $G(\lambda)$ and then "linearizing" the resulting polynomial eigenvalue problem to obtain a generalized eigenvalue problem. A downside of this brute-force approach is that the resulting generalized eigenvalue problem may be of very large dimension especially when $G(\lambda)$ has a large number of poles. On the other hand, we could consider a state-space realization of $G(\lambda)$, that is, view $G(\lambda)$ as a transfer function of a linear time invariant (LTI) state-space system Σ given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + P(\tfrac{d}{dt})u(t) \end{aligned} \tag{2.1}$$

and compute "transmission zeros" of Σ, where $P(\lambda)$ is an n-by-n matrix polynomial and A, E, C, B are constant matrices of appropriate dimensions with E being nonsingular. Indeed, $G(\lambda)$ can be written uniquely as $G(\lambda) = P(\lambda) + Q(\lambda)$, where $P(\lambda)$ is a matrix polynomial and $Q(\lambda)$ is a strictly proper rational matrix function, that is, $Q(\lambda) \to 0$ as $|\lambda| \to \infty$. Since $Q(\lambda)$ is strictly proper there exist matrices $A \in \mathbb{C}^{r\times r}, B \in \mathbb{C}^{r\times n}, C \in \mathbb{C}^{n\times r}$ such that $Q(\lambda) = C(\lambda I - A)^{-1}B$. Alternatively, $Q(\lambda)$ could also be realized as

$$Q(\lambda) = C(\lambda E - A)^{-1}B,$$

where $E \in \mathbb{C}^{r\times r}$ is nonsingular. Then $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ gives the transfer function of an LTI system Σ of the form (2.1). In such case, an eigenvalue of $G(\lambda)$ is a transmission zero of the LTI system Σ. Consequently, the eigenvalues and eigenvectors of $G(\lambda)$ could be obtained from transmission zeros and zero directions of the LTI system Σ. We therefore consider the following problem.

Problem-II. Let Σ be an LTI system in state-space form as in (2.1). Compute transmission zeros, invariant zeros, and decoupling zeros of the system Σ.

Observe that Problem-I could be reformulated as a special case of Problem-II. We, therefore, develop a framework for solving Problem-II. For this purpose, we introduce a family of linearizations, which we refer to as Fiedler pencils, of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ associated with the LTI system Σ. We solve the eigenvalue problem for the Fiedler pencils and obtain invariant zeros of the LTI system Σ. We show that the Fiedler pencils of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ are in a sense "linearizations" of $G(\lambda)$ when the LTI system Σ is both controllable and observable, that is, when Σ is a minimal state-space realization of the transfer function $G(\lambda)$. Schematically, our strategy for numerical solution of Problem-I is as follows:

Rational matrix function $\longrightarrow$ State-space realization $\longrightarrow$ Linearization $\longrightarrow$ Solution.

There are efficient methods for computing a (minimal) state-space realization of a rational matrix function. We, therefore, focus on solving Problem-II. More precisely, in this chapter, we describe construction of Fiedler pencils of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ and show that Fiedler pencils are linearizations for $\mathcal{S}(\lambda)$.

2.2 Zeros and poles

Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ be a rational matrix function. Suppose that the Smith-McMillan form $\mathbf{SM}(G(\lambda))$ of $G(\lambda)$ is given by

$$\mathbf{SM}(G(\lambda)) = \operatorname{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \cdots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0_{n-k,n-k}\right).$$

where the scalar polynomials $\phi_i(\lambda)$ and $\psi_i(\lambda)$ are monic, are pairwise coprime and, $\phi_i(\lambda)$ divides $\phi_{i+1}(\lambda)$ and $\psi_{i+1}(\lambda)$ divides $\psi_i(\lambda)$, for $i = 1, 2, \ldots, k-1$. Recall that the zero polynomial $\phi_G(\lambda)$ and the pole polynomial $\psi_G(\lambda)$ of $G(\lambda)$ are given by

$$\phi_G(\lambda) := \prod_{j=1}^{k} \phi_j(\lambda) \tag{2.2}$$

and

$$\psi_G(\lambda) := \prod_{j=1}^{k} \psi_j(\lambda). \tag{2.3}$$

Also recall that the set of *zeros* of $G(\lambda)$ or the *spectrum* of $G(\lambda)$ is given by

$$\mathrm{Sp}(G) = \{\lambda \in \mathbb{C} : \phi_G(\lambda) = 0\} \tag{2.4}$$

and the poles of $G(\lambda)$ are given by

$$\mathrm{Poles}(G) = \{\lambda \in \mathbb{C} : \psi_G(\lambda) = 0\}. \tag{2.5}$$

Definition 2.2.1. *Let $\lambda_0 \in Sp(G)$. Then $\phi_G(\lambda_0) = 0$ and $\phi_i(\lambda) = (\lambda-\lambda_0)^{\gamma_i} d_i(\lambda)$ with $d_i(\lambda_0) \neq 0$ and $\gamma_i \geq 0$ for $i = 1, 2, \ldots, k$. The index tuple $Ind_s(\lambda_0, G) := (\gamma_1, \ldots, \gamma_k)$ is called the* multiplicity index *of G at λ_0 and satisfies the condition $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k$. The nonzero components in $Ind_s(\lambda_0, G)$ are called the* partial multiplicities *of λ_0 as a zero of $G(\lambda)$. The factors $(\lambda-\lambda_0)^{\gamma_i}$ with $\gamma_i \neq 0$ are called* elementary divisors *of $G(\lambda)$ at λ_0. The* algebraic multiplicity *of λ_0 is defined by*

$$m_s(\lambda_0) := \gamma_1 + \gamma_2 + \cdots + \gamma_k = \textit{multiplicity of } \lambda_0 \textit{ as a root of } \phi_G(\lambda).$$

If $m_s(\lambda_0) = 1$ then λ_0 is called a simple eigenvalue *of $G(\lambda)$.*

Definition 2.2.2. *Let $\lambda_0 \in Poles(G)$. Then $\psi_G(\lambda_0) = 0 \Rightarrow \psi_i(\lambda) = (\lambda-\lambda_0)^{\alpha_i} q_i(\lambda)$ with $q_i(\lambda_0) \neq 0$ and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, k$. The index tuple $Ind_p(\lambda_0, G) := (\alpha_k, \alpha_{k-1} \ldots, \alpha_1)$ is called the* multiplicity index *of G at λ_0 and satisfies the condition $\alpha_k < \alpha_{k-1} < \cdots < \alpha_1$. The nonzero components in $Ind_p(\lambda_0, G)$ are called the* partial multiplicities *of λ_0 as* *a pole of* $G(\lambda)$*. The factors* $(\lambda-\lambda_0)^{\alpha_i}$ *with* $\alpha_i \neq 0$ *are called* elementary divisors *of* $G(\lambda)$ *at the pole* λ_0*. The* algebraic multiplicity *of* λ_0 *is defined by*

$$m_p(\lambda_0) := \alpha_1 + \alpha_2 + \cdots + \alpha_k = \textit{multiplicity of } \lambda_0 \textit{ as a root of } \psi_G(\lambda).$$

If $m_p(\lambda_0) = 1$ *then* λ_0 *is called a* simple pole *of* $G(\lambda)$*.*

Remark 2.2.1. *It is possible that* $Sp(G) \cap poles(G) \neq \emptyset$*. If* $\lambda_0 \in Sp(G) \cap poles(G)$ *then* $Ind_s(\lambda_0, G) \neq 0$ *and* $Ind_p(\lambda_0, G) \neq 0$*. In such a case,* λ_0 *has two multiplicity indices, namely, multiplicity index as a zero and multiplicity index as a pole. In general* $Ind_s(\lambda_0, G) \neq Ind_p(\lambda_0, G)$*.*

Example 2.1. Consider $G(\lambda) = \begin{bmatrix} \frac{1}{\lambda(\lambda-2)^2} & \\ & \frac{\lambda-2}{\lambda} \end{bmatrix}$. Then $\mathrm{Sp}(G) = \{2\}$ and $\mathrm{poles}(G) = \{0, 0, 2, 2\}$. So $2 \in \mathrm{Sp}(G) \cap \mathrm{poles}(G), \mathrm{Ind}_s(2, G) = (0, 1)$ and $\mathrm{Ind}_p(2, G) = (0, 2)$. Hence $\mathrm{Ind}_s(2, G) \neq \mathrm{Ind}_p(2, G)$. ■

Example 2.2. Consider $G(\lambda) = \begin{bmatrix} \frac{1}{(\lambda-1)^2(\lambda+1)^2} & 0 \\ 0 & \frac{(\lambda-1)(\lambda+2)}{1} \end{bmatrix}$. Then $\mathrm{Sp}(G) = \{1, -2\}$ and $\mathrm{poles}(G) = \{1, 1, -1, -1\}$. So $1 \in \mathrm{Sp}(G) \cap \mathrm{poles}(G), \mathrm{Ind}_s(1, G) = (0, 1)$ and $\mathrm{Ind}_p(1, G) = (0, 2)$. Hence $\mathrm{Ind}_s(1, G) \neq \mathrm{Ind}_p(1, G)$. ■

Definition 2.2.3. *A complex number* $\lambda_0 \in \mathbb{C}$ *is said to be an* eigenvalue *of* $G(\lambda)$ *if* $\mathrm{rank}(G(\lambda_0)) < \mathrm{nrank}(G)$*. We denote the* eigenspectrum *of* $G(\lambda)$ *by* $Sp_e(G)$ *given by*

$$Sp_e(G) := \{\lambda \in \mathbb{C} : \mathrm{rank}(G(\lambda)) < \mathrm{nrank}(G)\}.$$

Note that $\mathrm{Sp}_e(G) \cap \mathrm{poles}(G) = \emptyset$ and $\mathrm{Sp}_e(G) \subset \mathrm{Sp}(G)$ where the inclusion may be strict.

Definition 2.2.4. *A complex number* λ_0 *is said to be an* eigenpole *of* $G(\lambda)$ *if* λ_0 *is a pole of* $G(\lambda)$ *and there exists* $v(\lambda) \in \mathbb{C}^n[\lambda]$ *with* $v(\lambda_0) \neq 0$ *such that* $\lim_{\lambda \to \lambda_0} G(\lambda)v(\lambda) = 0$*. We denote the set of eigenpoles of* $G(\lambda)$ *by* $Sp_{ep}(G)$ *given by*

$$Sp_{ep}(G) := \{\lambda \in \mathbb{C} : \lambda \textit{ is an eigenpole of } G(\lambda).\}$$

Observe that that $\mathrm{Sp}_{ep}(G) \subset \mathrm{Poles}(G)$. More generally, we have the following result.

Theorem 2.2.1. *Let* $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$*. Then* $Sp(G) = Sp_e(G) \cup Sp_{ep}(G)$*. In particular,* $Sp_{ep}(G) = Sp(G) \cap Poles(G)$*.*

Proof. Suppose that $\mathbf{SM}(G(\lambda)) = \mathrm{diag}\left(\phi_1(\lambda)/\psi_1(\lambda), \ldots, \phi_k(\lambda)/\psi_k(\lambda), 0, \ldots, 0\right)$. Let $\phi_G(\lambda)$ and $\psi_G(\lambda)$ be zero and pole polynomials. Let $\lambda_0 \in \mathrm{Sp}_e(G) \Rightarrow \mathrm{rank}(G(\lambda_0)) < k = \mathrm{nrank}(G) \Rightarrow \phi_i(\lambda_0) = 0$ for some $i \Rightarrow \phi_G(\lambda_0) = 0 \Rightarrow \lambda_0 \in \mathrm{Sp}(G)$. Thus $\mathrm{Sp}_e(G) \subset \mathrm{Sp}(G)$.

Let $\lambda_0 \in \mathrm{Sp}_{ep}(G)$. Then there exists $v(\lambda) \in \mathbb{C}^n[\lambda]$ such that $v(\lambda_0) \neq 0$ and $\lim_{\lambda\to\lambda_0} G(\lambda)v(\lambda) = 0$. Now by Smith McMillan form we have $G(\lambda) = U(\lambda)\,\mathbf{SM}(G(\lambda))\,V(\lambda)$ with $U(\lambda)$ and $V(\lambda)$ being unimodular. So

$$U^{-1}(\lambda)G(\lambda)v(\lambda) = \mathbf{SM}(G(\lambda))V(\lambda)v(\lambda). \tag{2.6}$$

Set $x(\lambda) = V(\lambda)v(\lambda)$. Then $x(\lambda) \to x(\lambda_0) = V(\lambda_0)v(\lambda_0) \neq 0$. So from (2.6) we have $\frac{\phi_i(\lambda)}{\psi_i(\lambda)}x_i(\lambda) \to 0$ for $i = 1 : k$. Since $x(\lambda_0) \neq 0$, there is some i such that $x_i(\lambda_0) \neq 0$. Since $\frac{1}{x_i(\lambda)} \to \frac{1}{x_i(\lambda_0)}$, we have

$$\frac{1}{x_i(\lambda)}\frac{\phi_i(\lambda)}{\psi_i(\lambda)}x_i(\lambda) = \frac{\phi_i(\lambda)}{\psi_i(\lambda)},$$

and

$$\lim_{\lambda\to\lambda_0}\frac{1}{x_i(\lambda)}\frac{\phi_i(\lambda)}{\psi_i(\lambda)}x_i(\lambda) = 0.$$

So $\frac{\phi_i(\lambda)}{\psi_i(\lambda)} \to 0$ as $\lambda \to \lambda_0$. Since ϕ_i and ψ_i are coprime, $\psi_i(\lambda_0) \neq 0$ and $\phi_i(\lambda) \to \phi_i(\lambda_0) = 0$ for some i. Thus $\phi_G(\lambda_0) = 0$. So $\lambda_0 \in \mathrm{Sp}(G)$. Thus $\mathrm{Sp}_e(G) \cup \mathrm{Sp}_{ep}(G) \subset \mathrm{Sp}(G)$.

Conversely, let $\lambda_0 \in \mathrm{Sp}(G) \Rightarrow \phi_G(\lambda_0) = 0 \Rightarrow \phi_i(\lambda_0) = 0$ for some i. So $\psi_i(\lambda_0) \neq 0$, since $\gcd(\phi_i, \psi_i) = 1$. If $\psi_G(\lambda_0) \neq 0$, then $\mathrm{rank}(\mathbf{SM}(G(\lambda_0))) < k \Rightarrow \mathrm{rank}G(\lambda_0) < \mathrm{nrank}(G) \Rightarrow \lambda_0 \in \mathrm{Sp}_e(G)$. If $\psi_G(\lambda_0) = 0$ then define $u(\lambda) = V^{-1}(\lambda)e_i$, where $G(\lambda) = U(\lambda)\,\mathbf{SM}(G(\lambda))\,V(\lambda)$ with $U(\lambda)$ and $V(\lambda)$ being unimodular. Then

$$G(\lambda)u(\lambda) = U(\lambda)\mathbf{SM}(G(\lambda))e_i = U(\lambda)\frac{\phi_i(\lambda)}{\psi_i(\lambda)}e_i \to 0 \text{ as } \lambda \to \lambda_0$$

$\Rightarrow \lambda_0 \in \mathrm{Sp}_{ep}(G)$. This shows that $\mathrm{Sp}(G) \subset \mathrm{Sp}_e(G) \cup \mathrm{Sp}_{ep}(G)$. Hence the proof. □

Remark 2.2.2. *When $G(\lambda)$ is a transfer function of an LTI state-space system, the zeros of G are called* transmission *zeros of the state-space system.*

Suppose that $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ is a realization of $G(\lambda)$. Then recall that the realization is said to be minimal if the size of A is the smallest. A realization $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ is minimal if and only if the associated LTI system is controllable as well as observable [19].

Now suppose that $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ is the transfer function of the LTI system Σ given in (2.1). Then recall that the matrix polynomial

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right] \in \mathbb{C}^{(n+r)\times(n+r)}$$

is the Rosenbrock system matrix of the LTI system Σ.

Definition 2.2.5. *A complex number λ is said to be an invariant zero of the LTI system Σ associated with $\mathcal{S}(\lambda)$ if $\operatorname{rank}(\mathcal{S}(\lambda)) < \operatorname{nrank}(\mathcal{S})$. The spectrum of $\mathcal{S}$ given by*

$$Sp(\mathcal{S}) := \{\lambda \in \mathbb{C} : \operatorname{rank}(\mathcal{S}(\lambda)) < \operatorname{nrank}(\mathcal{S})\}$$

is the set of invariant zeros of the LTI system Σ. The vector $\boldsymbol{v} = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^{n+r}$ with $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^r$ such that $\mathcal{S}(\lambda)v = 0$ is called an eigenvector or an invariant direction *of system Σ. The vector u is called an input direction and v is called a state direction of the LTI system Σ.*

The following result provides structural information about $G(\lambda)$ and a state-space realization of $G(\lambda)$. The proof is immediate and follows from Theorem 1.4.2.

Theorem 2.2.2. *Suppose that $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ is minimal and that*

$$\mathbf{SM}(G(\lambda)) = \operatorname{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \frac{\phi_2(\lambda)}{\psi_2(\lambda)}, \ldots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0, \ldots, 0\right).$$

Set $L(\lambda) = \lambda E - A$. Then the following results hold

(a) The Smith form $\mathbf{SF}(L(\lambda))$ of $L(\lambda)$ is given by

$$\mathbf{SF}(L(\lambda)) = \operatorname{diag}\left(I_{r-k}, \psi_k, \psi_{k-1}, \ldots, \psi_1\right).$$

The Smith form $\mathbf{SF}(\mathcal{S}(\lambda))$ of $\mathcal{S}(\lambda)$ is given by

$$\mathbf{SF}(\mathcal{S}(\lambda)) = \operatorname{diag}\left(I_r, \phi_1, \phi_2, \ldots, \phi_k, 0, \ldots, 0\right).$$

(b) $Poles(G) \subset Sp(L)$, and the equality holds if the realization of $G(\lambda)$ is minimal. In such a case, $Ind_p(\lambda_0, G) = Ind(\lambda_0, L)$, for $\lambda_0 \in Poles(G)$.

(c) $Sp(G) \subset Sp(\mathcal{S})$ and the equality holds if the realization is minimal. In such a case $Ind_s(\lambda, G) = Ind_s(\lambda, \mathcal{S})$ for $\lambda \in Sp(G)$.

(d) $Sp(\mathcal{S}) \cap Sp(L) \supset Sp_{ep}(G)$, and the equality holds if G is minimal.

(e) $Sp(\mathcal{S}) \cap \rho(L) = Sp_e(G)$, where $\rho(L)$ is the resolvent of L.

2.3 Fiedler pencils for Rosenbrock system matrix

Consider a linear time invariant (LTI) system Σ given by

$$E\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + P\left(\frac{d}{dt}\right)u(t),$$

where $E \in \mathbb{C}^{r\times r}$ is nonsingular and $P(\lambda) \in \mathbb{C}^{n\times n}[\lambda]$. For the rest of the chapter, we assume that

$$P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j \qquad \text{and } A_m \neq 0.$$

The Rosenbrock system matrix associated with Σ is given by

$$\mathcal{S}(\lambda) := \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A-\lambda E) \end{array}\right]. \tag{2.7}$$

The transfer function of the LTI system Σ is given by

$$G(\lambda) := P(\lambda) + C(\lambda E - A)^{-1}B. \tag{2.8}$$

Define $(nm+r)\times(nm+r)$ matrices

$$\mathbb{M}_0 := \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right], \qquad \mathbb{M}_m := \left[\begin{array}{c|c} M_m & 0 \\ \hline 0 & -E \end{array}\right] \tag{2.9}$$

and

$$\mathbb{M}_i := \left[\begin{array}{c|c} M_i & 0 \\ \hline 0 & I_r \end{array}\right], \qquad i = 1, \ldots, m-1, \tag{2.10}$$

where $M_i, i = 0, 1, \ldots, m$ are Fiedler matrices associated with $P(\lambda)$. We refer to the matrices $\mathbb{M}_i, i = 0, 1, \ldots, m$, as the *Fiedler matrices* associated with the system Σ or simply *Fiedler matrices* associated with $\mathcal{S}(\lambda)$. The Fiedler matrices $\mathbb{M}_i, i = 0, \ldots, m$ have the following properties.

(a) We have

$$\mathbb{M}_i\mathbb{M}_j = \mathbb{M}_j\mathbb{M}_i \text{ for } |i-j| > 1, \text{ except for } \mathbb{M}_m \text{ and } \mathbb{M}_0. \tag{2.11}$$

(b) The matrices $\mathbb{M}_i$ for $i = 1, \ldots, m-1$ are always invertible and

$$\mathbb{M}_i^{-1} = \left[\begin{array}{c|c} M_i^{-1} & 0 \\ \hline 0 & I_r \end{array}\right], \text{ where } M_i^{-1} \text{ is given in (1.17).} \tag{2.12}$$

Further,

$$\mathbb{M}_i^{-1}\mathbb{M}_j^{-1} = \mathbb{M}_j^{-1}\mathbb{M}_i^{-1} \text{ for } |i-j|>1, \text{ except for } \mathbb{M}_0^{-1} \text{ and } \mathbb{M}_m^{-1}. \tag{2.13}$$

If $\sigma : \{0,1,\ldots,m-1\} \to \{1,2,\ldots,m\}$ is a bijection, then we define $\mathbb{M}_\sigma := \mathbb{M}_{\sigma^{-1}(1)}\mathbb{M}_{\sigma^{-1}(2)}\cdots\mathbb{M}_{\sigma^{-1}(m)}$. Note that $\sigma(i)$ denotes the position of the factor $\mathbb{M}_i$ in the product $\mathbb{M}_\sigma$; i.e., $\sigma(i)=j$ means that $\mathbb{M}_i$ is the jth factor in the product $\mathbb{M}_\sigma$.

Definition 2.3.1 (Fiedler Pencil)**.** *Let $\mathcal{S}(\lambda)$ be the system matrix associated with LTI system Σ, and let $\mathbb{M}_0,\ldots,\mathbb{M}_m$ be the Fiedler matrices given in (2.9) and (2.10). Given any bijection $\sigma : \{0,1,\ldots,m-1\} \to \{1,2,\ldots,m\}$, we define the pencil $\mathbb{L}_\sigma(\lambda)$ by*

$$\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_m - \mathbb{M}_{\sigma^{-1}(1)}\mathbb{M}_{\sigma^{-1}(2)}\cdots\mathbb{M}_{\sigma^{-1}(m)} = \lambda\mathbb{M}_m - \mathbb{M}_\sigma. \tag{2.14}$$

We refer to $\mathbb{L}_\sigma(\lambda)$ as Fiedler pencil of $\mathcal{S}(\lambda)$ (or $G(\lambda)$) associated with σ.

Definition 2.3.2. *Let $\sigma, \tau : \{0,1,\ldots,m-1\} \to \{1,2,\ldots,m\}$ be bijections. Then σ is said to be equivalent to τ ($\sigma \sim \tau$) if $\mathbb{M}_\sigma = \mathbb{M}_\tau$.*

Example 2.3. Consider $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \lambda^2 A_2 + \lambda A_1 + A_0 & C \\ \hline B & 0-\lambda I_r \end{array}\right]$ and $G(\lambda) = \lambda^2 A_3 + \lambda A_2 + A_1 + C(\lambda I_r - 0)^{-1}B$, where $C \in \mathbb{C}^{n\times r}$ and $B \in \mathbb{C}^{r\times n}$. Then the Fiedler matrices associated with $\mathcal{S}(\lambda)$ are

$$\mathbb{M}_0 = \left[\begin{array}{cc|c} I_n & 0 & 0 \\ 0 & -A_0 & -C \\ \hline 0 & -B & 0 \end{array}\right], \quad \mathbb{M}_1 = \left[\begin{array}{cc|c} -A_1 & I_n & 0 \\ I_n & 0 & 0 \\ \hline 0 & 0 & I_r \end{array}\right] \text{ and } \mathbb{M}_2 = \left[\begin{array}{cc|c} A_2 & & \\ & I_n & \\ \hline & & -I_r \end{array}\right].$$

Then the pencil

$$\begin{aligned} \mathbb{L}_\sigma(\lambda) &= \lambda\mathbb{M}_2 - \mathbb{M}_1\mathbb{M}_0 \\ &= \lambda \left[\begin{array}{cc|c} A_3 & & \\ & I_n & \\ \hline & & -I_r \end{array}\right] - \left[\begin{array}{cc|c} -A_2 & -A_1 & -C \\ I_n & 0 & 0 \\ \hline 0 & -B & 0 \end{array}\right] \end{aligned}$$

is a Fiedler pencil of $\mathcal{S}(\lambda)$. ■

Example 2.4. Consider the transfer function $G(\lambda) = \lambda^4 A_4 + \cdots + A_0 + C(\lambda E - A)^{-1}B$ and the Fiedler pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_1\mathbb{M}_0\mathbb{M}_2\mathbb{M}_3$. Then

$$\mathbb{L}_\sigma(\lambda) = \lambda \left[\begin{array}{cc|c} A_4 & & \\ & I_{3n} & \\ \hline & & -E \end{array}\right] - \left[\begin{array}{cccc|c} -A_3 & I_n & 0 & 0 & 0 \\ -A_2 & 0 & I_n & 0 & 0 \\ -A_1 & 0 & 0 & -A_0 & -C \\ I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -B & -A \end{array}\right].$$

Next, consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_3 = \lambda\mathbb{M}_4 - \mathbb{M}_\sigma$, where $\sigma^{-1} = (2, 0, 1, 3)$ and $\sigma = (2, 3, 1, 4)$. Then

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{cccc|c} A_4 & & & & \\ & I_n & & & \\ & & I_n & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] - \left[\begin{array}{cccc|c} -A_3 & I_n & 0 & 0 & 0 \\ -A_2 & 0 & -A_1 & I_n & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_0 & 0 & -C \\ \hline 0 & 0 & -B & 0 & -A \end{array}\right].$$

Note that if we take another bijection $\tau^{-1} = (0, 2, 3, 1)$, then $\mathbb{L}_\tau(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_3\mathbb{M}_1$

$$= \left[\begin{array}{cccc|c} A_4 & & & & \\ & I_n & & & \\ & & I_n & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] - \left[\begin{array}{cccc|c} -A_3 & I_n & 0 & 0 & 0 \\ -A_2 & 0 & -A_1 & I_n & 0 \\ I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_0 & 0 & -C \\ \hline 0 & 0 & -B & 0 & -A \end{array}\right] = \mathbb{L}_\sigma(\lambda).$$

This shows that $\mathbb{L}_\sigma(\lambda) = \mathbb{L}_\tau(\lambda)$ i.e., $\sigma \sim \tau$. ■

Let $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ be a bijection. Then we have the following.

(1) σ has a consecution at d if and only if $\mathbb{M}_d$ is to the left of $\mathbb{M}_{d+1}$ in $\mathbb{M}_\sigma$, while σ has an inversion at d if and only if $\mathbb{M}_d$ is to the right of $\mathbb{M}_{d+1}$ in $\mathbb{M}_\sigma$.

(2) For $i, j \in \sigma$, we have $M_iM_j = M_jM_i \Leftrightarrow \mathbb{M}_i\mathbb{M}_j = \mathbb{M}_j\mathbb{M}_i$.

Example 2.5. Consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_6 - \mathbb{M}_\sigma$ where $\mathbb{M}_\sigma = \mathbb{M}_1\mathbb{M}_3\mathbb{M}_5\mathbb{M}_0\mathbb{M}_2\mathbb{M}_4$. Then we have $\sigma^{-1} = (1, 3, 5, 0, 2, 4), \sigma = (4, 1, 5, 2, 6, 3)$, and $\mathrm{CISS}(\sigma) = (0, 1, 1, 1, 1, 1)$.

Similarly if we consider $\mathbb{M}_\tau = \mathbb{M}_0\mathbb{M}_1\mathbb{M}_3\mathbb{M}_5\mathbb{M}_2\mathbb{M}_4$, then $\tau^{-1} = (0,1,3,5,2,4), \tau = (1,2,5,3,6,4)$, and $\text{CISS}(\tau) = (2,1,1,1)$.

Next, consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_3$. Then we have $\sigma^{-1} = (2,0,1,3), \sigma = (2,3,1,4)$, and $\text{CISS}(\sigma) = (1,1,1,0)$. ■

Now, we show that there is a bijection from set of Fiedler pencils of $P(\lambda)$ to set of Fiedler pencils of $\mathcal{S}(\lambda)$ which can be used to construct a Fiedler pencil of $\mathcal{S}(\lambda)$ from that of $P(\lambda)$ and vice-versa.

Theorem 2.3.1. *Let $\sigma : \{0,1,\ldots,m-1\} \to \{1,2,\ldots,m\}$ be a bijection. Let $L_\sigma(\lambda)$ and $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencils of $P(\lambda)$ and $\mathcal{S}(\lambda)$, respectively, given by $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ and $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$. If $\sigma^{-1} = (\sigma_1^{-1}, 0, \sigma_2^{-1})$ for some bijections σ_1 and σ_2 then*

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & M_{\sigma_1}(e_m \otimes C) \\ \hline (e_m^T \otimes B)M_{\sigma_2} & (A - \lambda E) \end{array}\right].$$

Further, if CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$ then

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_m \otimes C \\ \hline e^T_{(m-c_1)} \otimes B & A - \lambda E \end{array}\right], \qquad \text{if } c_1 > 0$$

and

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & e^T_{m-i_1} \otimes C \\ \hline e^T_m \otimes B & A - \lambda E \end{array}\right], \qquad \text{if } c_1 = 0.$$

Thus the map $Fiedler(P) \to Fiedler(\mathcal{S})$, $\lambda M_m - M_\sigma \mapsto \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ is a bijection, where $Fiedler(P)$ and $Fiedler(\mathcal{S})$, respectively, denote the set of Fiedler pencils of $P(\lambda)$ and $\mathcal{S}(\lambda)$.

Proof. Consider $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma = \lambda\mathbb{M}_m - \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}$

$$= \lambda \left[\begin{array}{c|c} M_m & 0 \\ \hline 0 & -E \end{array}\right] - \left[\begin{array}{c|c} M_{\sigma_1} & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right] \left[\begin{array}{c|c} M_{\sigma_2} & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \lambda \left[\begin{array}{c|c} M_m & 0 \\ \hline 0 & -E \end{array}\right] - \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & -M_{\sigma_1}(e_m \otimes C) \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right]$$

$$= \left[\begin{array}{c|c} L_\sigma(\lambda) & M_{\sigma_1}(e_m \otimes C) \\ \hline (e_m^T \otimes B)M_{\sigma_2} & (A - \lambda E) \end{array}\right],$$

which is Fiedler pencil of $\mathcal{S}(\lambda)$. Next, suppose that $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$.

Case I : Suppose that $c_1 > 0$. Then by commutativity relation we have $\mathbb{M}_\sigma = \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_1 \cdots \mathbb{M}_{c_1}$ with $c_1 + 1 \in \sigma_1$. Thus $\mathbb{M}_\sigma = \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}$, where $\mathbb{M}_{\sigma_2} = \mathbb{M}_1 \cdots \mathbb{M}_{c_1}$. Hence

$$\mathbb{M}_\sigma = \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right] \left[\begin{array}{c|c} M_{\sigma_2} & \\ \hline & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & M_{\sigma_1}(-e_m \otimes C) \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right].$$

Since $j \in \sigma_1$ implies that $j \geq c_1 + 1$, we have $\mathbb{M}_{\sigma_1} = \left[\begin{array}{c|c} * & \\ \hline & I_{c_1 n} \end{array}\right]$. This shows that $\mathbb{M}_{\sigma_1}(e_m \otimes I_n) = e_m \otimes I_n$. Hence $M_{\sigma_1}(-e_m \otimes C) = -e_m \otimes C$. Next, we have

$$(e_m^T \otimes I_n)\mathbb{M}_1 = (e_m^T \otimes I_n) \begin{bmatrix} I_{(m-2)n} & & \\ & -A_1 & I_n \\ & I_n & 0 \end{bmatrix} = (e_{m-1}^T \otimes I_n).$$

$$(e_m^T \otimes I_n)\mathbb{M}_1\mathbb{M}_2 = (e_{m-1}^T \otimes I_n) \begin{bmatrix} I_{(m-3)n} & & & \\ & -A_2 & I_n & \\ & I_n & 0 & \\ & & & I_n \end{bmatrix} = (e_{m-2}^T \otimes I_n).$$

Thus $(e_m^T \otimes I_n)\mathbb{M}_1\mathbb{M}_2 \cdots \mathbb{M}_{c_1} = (e_{m-c_1}^T \otimes I_n)$. Hence $(e_m^T \otimes I_n)\mathbb{M}_{\sigma_2} = (e_{m-c_1}^T \otimes I_n)$ and $(-e_m^T \otimes B)M_{\sigma_2} = -(e_{m-c_1}^T \otimes B)$. Consequently, we have

$$\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma = \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_m \otimes C \\ \hline e_{(m-c_1)}^T \otimes B & A - \lambda E \end{array}\right].$$

Case II : Suppose that $c_1 = 0$. Then σ has i_1 inversions at 0. Hence by commutativity relations we have $\mathbb{M}_\sigma = \mathbb{M}_{i_1} \cdots \mathbb{M}_1\mathbb{M}_0\mathbb{M}_{\sigma_2}$ with $i_1 + 1 \in \sigma_2$. Thus $\mathbb{M}_\sigma = \mathbb{M}_{\sigma_1}\mathbb{M}_0 M_{\sigma_2}$, where $M_{\sigma_1} = \mathbb{M}_{i_1} \cdots \mathbb{M}_1$. Hence

$$\mathbb{M}_\sigma = \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right] \left[\begin{array}{c|c} M_{\sigma_2} & \\ \hline & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & M_{\sigma_1}(-e_m \otimes C) \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right].$$

Since $j \in \sigma_2$ implies that $j \geq i_1 + 1$, we have $\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c} * & \\ \hline & I_{i_1 n} \end{array}\right]$. This shows that $(e_m^T \otimes I_n)\mathbb{M}_{\sigma_2} = e_m^T \otimes I_n$. Hence $M_{\sigma_2}(-e_m^T \otimes B) = -e_m^T \otimes B$. Next, we have

$$\mathbb{M}_1(e_m \otimes I_n) = \begin{bmatrix} I_{(m-2)n} & & \\ & -A_1 & I_n \\ & I_n & 0 \end{bmatrix} (e_m \otimes I_n) = (e_{m-1} \otimes I_n).$$

$$\mathbb{M}_2\mathbb{M}_1(e_m \otimes I_n) = \begin{bmatrix} I_{(m-3)n} & & & \\ & -A_2 & I_n & \\ & I_n & 0 & \\ & & & I_n \end{bmatrix} (e_{m-1} \otimes I_n) = (e_{m-2} \otimes I_n).$$

Thus $\mathbb{M}_{i_1} \cdots \mathbb{M}_2\mathbb{M}_1(e_m \otimes I_n) = (e_{m-i_1} \otimes I_n)$. Hence $\mathbb{M}_{\sigma_1}(e_m \otimes I_n) = (e_{(m-i_1)} \otimes I_n)$ and $M_{\sigma_1}(-e_m \otimes B) = -(e_{(m-i_1)} \otimes B)$. Consequently, we have

$$\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma = \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_{m-i_1}^T \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right].$$

For each $i, j \in \sigma$, $M_iM_j = M_jM_i \Leftrightarrow \mathbb{M}_i\mathbb{M}_j = \mathbb{M}_j\mathbb{M}_i$. Hence $\#(Fiedler(P)) = \#(Fiedler(\mathcal{S}))$. This completes the proof. $\square$

Consider the Fiedler pencil of $\mathcal{S}(\lambda)$ given by

$$\mathcal{C}_1(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{m-1}\mathbb{M}_{m-2} \cdots \mathbb{M}_1\mathbb{M}_0. \tag{2.15}$$

Observe that $\mathcal{C}_1(\lambda) = \mathbb{L}_{\sigma_1}(\lambda)$, where $\sigma_1 = (m-1, \ldots, 2, 1, 0)$ and $\mathrm{CISS}(\sigma_1) = (0, m-1)$. Since $L_{\sigma_1}(\lambda) = \lambda M_m - M_{m-1} \cdots M_1M_0 = C_1(\lambda)$ given in (1.19) is the first companion form of $P(\lambda)$, we conclude that

$$\mathcal{C}_1(\lambda) = \left[\begin{array}{c|c} C_1(\lambda) & e_1 \otimes C \\ \hline e_m^T \otimes B & (A - \lambda E) \end{array}\right]$$

$$= \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] - \left[\begin{array}{cccc|c} -A_{m-1} & -A_{m-2} & \cdots & -A_0 & -C \\ I_n & 0 & \cdots & 0 & \\ & \ddots & & \vdots & \\ & & I_n & 0 & \\ \hline & & & -B & -A \end{array}\right] \tag{2.16}$$

is companion form of $\mathcal{S}(\lambda)$. The pencil $\mathcal{C}_1(\lambda)$ is derived in [37] and is referred to as companion form of $G(\lambda)$. We refer to $\mathcal{C}_1(\lambda)$ as the *first companion form* of $\mathcal{S}(\lambda)$ or $G(\lambda)$. Next, consider the pencil

$$\mathcal{C}_2(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_0\mathbb{M}_1 \cdots \mathbb{M}_{m-2}\mathbb{M}_{m-1}.$$

Observe that $\mathcal{C}_2(\lambda) = \mathbb{L}_{\sigma_2}(\lambda)$, where $\sigma_2 = (0, 1, \ldots, m-1)$ and $\text{CISS}(\sigma_2) = (m-1, 0)$. Since $L_{\sigma_2}(\lambda) = \lambda M_m - M_0 \cdots M_{m-2}M_{m-1} = C_2(\lambda)$ given in (1.20) is the second companion form of $P(\lambda)$, we have

$$\mathcal{C}_2(\lambda) = \lambda \left[\begin{array}{c|c} C_2(\lambda) & e_m \otimes C \\ \hline e_1^T \otimes B & (A - \lambda E) \end{array}\right]$$
$$= \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] - \left[\begin{array}{ccccc|c} -A_{m-1} & I_n & & & & \\ -A_{m-2} & 0 & \ddots & & & \\ \vdots & \ddots & & & I_n & \\ -A_0 & \cdots & 0 & & 0 & -C \\ \hline -B & & & & & -A \end{array}\right]. \tag{2.17}$$

We refer to $\mathcal{C}_2(\lambda)$ as the *second companion form* of $\mathcal{S}(\lambda)$ or $G(\lambda)$.

There are Fiedler pencils of $\mathcal{S}(\lambda)$ such as block pentadiagonal pencils which have lower bandwidth than the first companion form $\mathcal{C}_1(\lambda)$ and second companion form $\mathcal{C}_2(\lambda)$.

Theorem 2.3.2. *Let $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ be a block pentadiagonal Fiedler pencil of $P(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ having CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$. Then $\mathbb{L}_\sigma(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_\sigma$ is a block pentadiagonal Fiedler pencils of $\mathcal{S}(\lambda)$ provided that $c_1 \leq 1$ and $i_1 \leq 1$. In such a case, we have*

$$\mathbb{L}_\sigma(\lambda) := \left[\begin{array}{c|c} L_\sigma(\lambda) & e_m \otimes C \\ \hline e_{m-1}^T \otimes B & (A - \lambda E) \end{array}\right] \quad \text{if } c_1 > 0$$

$$\text{and} \quad \mathbb{L}_\sigma(\lambda) := \left[\begin{array}{c|c} L_\sigma(\lambda) & e_{m-1} \otimes C \\ \hline e_m^T \otimes B & (A - \lambda E) \end{array}\right] \quad \text{if } c_1 = 0.$$

Proof. If $c_1 > 0$ then $c_1 = 1$ and $e_{m-c_1}^T \otimes B = e_{m-1}^T \otimes B$. So by the Theorem 2.3.1 we have

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_m \otimes C \\ \hline e_{m-c_1}^T \otimes B & (A - \lambda E) \end{array}\right] = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_m \otimes C \\ \hline e_{m-1}^T \otimes B & (A - \lambda E) \end{array}\right]$$

is block pentadiagonal, since $L_\sigma(\lambda)$ is block pentadiagonal. Similar argument holds when $c_1 = 0$ and $i_1 = 1$. □

Let $\mathcal{O} = M_1 M_3 \cdots$ be the product of odd M_i factors and $\mathcal{E} = M_2 M_4 \cdots$ be the product of the even M_i factors, excluding M_0 and M_m. Then it is shown in [9] that the product of $\mathcal{O}, \mathcal{E}$ and M_0 in any order gives a pentadiagonal Fiedler pencil of $P(\lambda)$.

Corollary 2.3.3. *Let $\mathbb{M}_{\sigma_1}$ be the product of odd $\mathbb{M}_i$ factors and $\mathbb{M}_{\sigma_2}$ be the product of even $\mathbb{M}_i$ factors, excluding $\mathbb{M}_0$ and $\mathbb{M}_m$. Then $\mathbb{L}_\sigma(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}$ and $\mathbb{L}_{\widetilde{\sigma}}(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_0\mathbb{M}_{\sigma_2}\mathbb{M}_{\sigma_1}$ are block pentadiagonal pencils of $\mathcal{S}(\lambda)$. However, the pencils $\mathbb{L}(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{\sigma_2}\mathbb{M}_{\sigma_1}\mathbb{M}_0$ and $\mathbb{L}(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma_2}$ are not block pentadiagonal.*

Proof. Consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}$. Then $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots) = (0, 1, \ldots)$. Hence by Theorem 2.3.1, we have

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_{m-i_1} \otimes C \\ \hline e_m^T \otimes B & (A - \lambda E) \end{array}\right] = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_{m-1} \otimes C \\ \hline e_m^T \otimes B & (A - \lambda E) \end{array}\right],$$

where $L_\sigma(\lambda) = \lambda M_m - M_{\sigma_1} M_0 M_{\sigma_2}$. It is well known that the product of $M_{\sigma_1}, M_{\sigma_2}, M_0$ in any order gives a block pentadiagonal matrix ([9], Example 3.2). Hence $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ is a block pentadiagonal and consequently $\mathbb{L}_\sigma(\lambda)$ is block pentadiagonal. For the pencil $\mathbb{L}_{\widetilde{\sigma}}(\lambda)$ we have $\mathrm{CISS}(\widetilde{\sigma}) = (c_1, i_1, \ldots) = (1, 1, \ldots)$. Hence by Theorem 2.3.1, $\mathbb{L}_{\widetilde{\sigma}(\lambda)}$ is block pentadiagonal. For $\mathbb{L}(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{\sigma_2}\mathbb{M}_{\sigma_1}\mathbb{M}_0$, we have $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots) = (2, \ldots)$. By Theorem 2.3.1, $\mathbb{L}(\lambda)$ has more than two subdiagonal blocks and hence is not block pentadiagonal. Similar argument holds for the pencil $\mathbb{L}(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma_2}$. ☐

Recall that $\mathbb{M}_i = \left[\begin{array}{c|c} M_i & 0 \\ \hline 0 & I_r \end{array}\right], i = 1 : m-1$ are the Fiedler matrices associated with $\mathcal{S}(\lambda)$, where $M_i = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}$. Since $\mathbb{M}_i$ are invertible for $i = 1 : m-1$ then $\mathbb{M}_{\sigma_1} = \mathbb{M}_1\mathbb{M}_3\cdots$ and $\mathbb{M}_{\sigma_2} = \mathbb{M}_2\mathbb{M}_4\cdots$ excluding $\mathbb{M}_0$ and $\mathbb{M}_m$ are always invertible. Note that $\mathbb{M}_{\sigma_1}$ and $\mathbb{M}_{\sigma_2}$ are block tridiagonal matrices, since the differences between each pair of indices in σ_1 and σ_2 is greater than equal to two.

We illustrate the above corollary by the following example.

Example 2.6. Let $G(\lambda) = A_6\lambda^6 + \cdots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B$. Let $\sigma_1 = (1, 3, 5)$ and $\sigma_2 = (2, 4)$.

Case I: Consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_6 - \mathbb{M}_1\mathbb{M}_3\mathbb{M}_5\mathbb{M}_0\mathbb{M}_2\mathbb{M}_4 = \lambda\mathbb{M}_6 - \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2} = \lambda\mathbb{M}_6 - \mathbb{M}_\sigma$, where $\mathbb{M}_{\sigma_1} := \mathbb{M}_1\mathbb{M}_3\mathbb{M}_5$ and $\mathbb{M}_{\sigma_2} := \mathbb{M}_2\mathbb{M}_4$. Now

$$\mathbb{M}_{\sigma_1} = \left[\begin{array}{cccccc|c} -A_5 & I_n & & & & & \\ I_n & 0 & & & & & \\ & & -A_3 & I_n & & & \\ & & I_n & 0 & & & \\ & & & & -A_1 & I_n & \\ & & & & I_n & 0 & \\ \hline & & & & & & I \end{array}\right], \quad \mathbb{M}_0\mathbb{M}_{\sigma_2} = \left[\begin{array}{cccccc|c} I_n & & & & & & \\ & -A_4 & I_n & & & & \\ & I_n & 0 & & & & \\ & & & -A_2 & I_n & & \\ & & & I_n & 0 & & \\ & & & & & -A_0 & -C \\ \hline & & & & & -B & -A \end{array}\right].$$

So

$$\mathbb{M}_\sigma = \left[\begin{array}{cccccc|c} -A_5 & -A_4 & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 & -C \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -B & -A \end{array}\right].$$

This shows that $\mathbb{L}_\sigma(\lambda)$ is block pentadiagonal. Note that the number of consecutions at 0 in $\mathbb{M}_\sigma$ is zero and the number of inversion at 0 in $\mathbb{M}_\sigma$ is 1.

Case II: Consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_6 - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_4\mathbb{M}_1\mathbb{M}_3\mathbb{M}_5 = \lambda\mathbb{M}_6 - \mathbb{M}_0\mathbb{M}_{\sigma_2}\mathbb{M}_{\sigma_1} = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$. Then

$$\mathbb{M}_0\mathbb{M}_{\sigma_2}\mathbb{M}_{\sigma_1} = \left[\begin{array}{cccccc|c} -A_5 & I_n & 0 & 0 & 0 & 0 & 0 \\ -A_4 & 0 & -A_3 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_2 & 0 & -A_1 & I_n & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_0 & 0 & -C \\ \hline 0 & 0 & 0 & 0 & -B & 0 & -A \end{array}\right]$$

is block pentadiagonal. Thus $\mathbb{L}_\sigma(\lambda)$ is block pentadiagonal and low bandwidth pencil. Note that the number of consecutions at 0 in $\mathbb{M}_\sigma$ is 1, i.e., $c_1 = 1$.

Case III: Consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_6 - \mathbb{M}_0\mathbb{M}_1\mathbb{M}_3\mathbb{M}_5\mathbb{M}_2\mathbb{M}_4 = \lambda\mathbb{M}_6 - \mathbb{M}_0\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma_2}$, where $\sigma_1^{-1} = (1,3,5)$ and $\sigma_2^{-1} = (2,4)$. Then this pencil is not pentadiagonal, since the number of consecutions at 0 in $\mathbb{M}_\sigma$ is 2. Now $\mathbb{M}_0\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma_2}$ is equal to

$$\left[\begin{array}{cc|c} I_{5n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right] \left[\begin{array}{cccccc|c} -A_5 & I_n & & & & & \\ I_n & 0 & & & & & \\ & & -A_3 & I_n & & & \\ & & I_n & 0 & & & \\ & & & & -A_1 & I_n & \\ & & & & I_n & 0 & \\ \hline & & & & & & I \end{array}\right] \left[\begin{array}{cccccc|c} I_n & & & & & & \\ & -A_4 & I_n & & & & \\ & I_n & 0 & & & & \\ & & & -A_2 & I_n & & \\ & & & I_n & 0 & & \\ & & & & & I_n & \\ \hline & & & & & & I \end{array}\right]$$

$$= \left[\begin{array}{cc|c} I_{5n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right] \left[\begin{array}{cccccc|c} -A_5 & -A_4 & I_n & & & & \\ I_n & 0 & 0 & & & & \\ & -A_3 & 0 & -A_2 & I_n & 0 & \\ & I_n & 0 & 0 & 0 & 0 & \\ & & & -A_1 & 0 & I_n & \\ & & & I_n & 0 & 0 & \\ \hline & & & & & & I \end{array}\right]$$

$$= \left[\begin{array}{cccccc|c} -A_5 & -A_4 & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & I_n & 0 \\ 0 & 0 & 0 & -A_0 & 0 & 0 & -C \\ \hline 0 & 0 & 0 & -B & 0 & 0 & -A \end{array}\right],$$

which is not block pentadiagonal but the first block which is M_σ is pentadiagonal. Similarly, $\mathbb{M}_\sigma = \mathbb{M}_{\sigma_2}\mathbb{M}_{\sigma_1}\mathbb{M}_0$ is also not block pentadiagonal, since the number of inversions at 0 in $\mathbb{M}_\sigma$ is 2. ■

Given a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$, the following algorithm constructs the product $\mathbb{M}_\sigma$ of Fiedler matrices.

Theorem 2.3.4. *Let $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ be a bijection. Let $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ be the Fiedler pencil of system matrix $\mathcal{S}(\lambda)$ associated with σ. Consider the matrices $W_0, W_1, \ldots, W_{m-2}$ constructed by Algorithm 1 below. Then $\mathbb{M}_\sigma = W_{m-2}$.*

Algorithm 1 Construction of $\mathbb{M}_\sigma$ for $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$.

Input: $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \sum_{i=0}^{m} \lambda^i A_i & C \\ \hline B & A - \lambda E \end{array}\right]$ and a bijection σ.

Output: $\mathbb{M}_\sigma$

if σ has a consecution at 0 **then**

$$W_0 = \left[\begin{array}{cc|c} -A_1 & I_n & 0 \\ -A_0 & 0 & -C \\ \hline -B & 0 & -A \end{array}\right]$$

else

$$W_0 = \left[\begin{array}{cc|c} -A_1 & -A_0 & -C \\ I_n & 0 & 0 \\ \hline 0 & -B & -A \end{array}\right]$$

end if

for $i = 1 : m-2$

if σ has a consecution at i **then**

$$W_i = \begin{bmatrix} -A_{i+1} & I_n & 0 & 0 \\ W_{i-1}(:,1) & 0 & W_{i-1}(:,2:i+1) & W_{i-1}(:,i+2) \end{bmatrix}$$

else

$$W_i = \begin{bmatrix} -A_{i+1} & W_{i-1}(1,:) \\ I_n & 0 \\ 0 & W_{i-1}(2:i+1,:) \\ 0 & W_{i-1}(i+2,:) \end{bmatrix}$$

end if

endfor

$\mathbb{M}_\sigma = W_{m-2}$

Proof. We prove the result by induction on the degree m. For $m = 2$ the proof is obvious, since for $m = 2$ there are only three Fiedler factors and two possibility for $\mathbb{M}_\sigma$, either $\mathbb{M}_\sigma = \mathbb{M}_0\mathbb{M}_1$ if σ has a consecution at 0 or $\mathbb{M}_\sigma = \mathbb{M}_1\mathbb{M}_0$ if σ has an inversion at

0. Now

$$\mathbb{M}_0\mathbb{M}_1 = \left[\begin{array}{cc|c} -A_1 & I_n & 0 \\ -A_0 & 0 & -C \\ \hline -B & 0 & -A \end{array}\right] \text{ and } \mathbb{M}_1\mathbb{M}_0 = \left[\begin{array}{cc|c} -A_1 & -A_0 & -C \\ I_n & 0 & 0 \\ \hline 0 & -B & -A \end{array}\right].$$

This proves the case for $m = 2$. Suppose that the result is true for $m - 1 \geq 2$. We show that it is true for m and the bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$. Note that the Fiedler matrices $\mathbb{M}_i$ associated with $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \sum_{j=0}^{m} \lambda^j A_j & C \\ \hline B & A - \lambda E \end{array}\right]$ satisfies

$$\mathbb{M}_i = \text{diag}(I_n, \widetilde{M_i}), \text{ for } i = 0 : m-2, \tag{2.18}$$

where $\widetilde{M_i}$ are the $(n(m-1)+r) \times (n(m-1)+r)$ Fiedler matrices associated with $\widetilde{S}(\lambda) = \left[\begin{array}{c|c} \sum_{j=0}^{m-1} \lambda^j A_j & C \\ \hline B & A - \lambda E \end{array}\right]$.

Case I. If σ has a consecution at $m-2$, then using the commutativity relations of the matrices $\mathbb{M}_i$ we can write

$$\mathbb{M}_\sigma = \mathbb{M}_{i_0} \cdots \mathbb{M}_{i_{m-2}} \mathbb{M}_{m-1},$$

where $(i_0, i_1, \ldots, i_{m-2})$ is a permutation of $(0, 1, \ldots, m-2)$. By (2.18) we can write

$$\mathbb{M}_\sigma = \text{diag}(I_n, \widetilde{M_\tau})\mathbb{M}_{m-1}, \tag{2.19}$$

where $\tau : \{0, 1, \ldots, m-2\} \to \{1, \ldots, m-1\}$ is a bijection such that for $i = 0 : m-3$, τ has a consecution (resp., inversion) at i if and only if σ has a consecution (resp., inversion) at i. So by the induction hypothesis, $\widetilde{M_\tau} = W_{m-3}$. Finally the product given in (2.19) is

$$\mathbb{M}_\sigma = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & W_{m-3}(:,1) & W_{m-3}(:,2:m-1) & W_{m-3}(:,m) \end{bmatrix} \left[\begin{array}{ccc|c} -A_{m-1} & I_n & & \\ I_n & 0 & & \\ & & I_{(m-2)n} & \\ \hline & & & I_r \end{array}\right]$$

$$= \begin{bmatrix} -A_{m-1} & I_n & 0 & 0 \\ W_{m-3}(:,1) & 0 & W_{m-3}(:,2:m-1) & W_{m-3}(:,m) \end{bmatrix},$$

which is equal to W_{m-2} constructed when σ has a consecution at $m-2$.

Case II. If σ has an inversion at $m-2$ the proof is similar, but

$$\mathbb{M}_\sigma = \mathbb{M}_{m-1}\mathbb{M}_{i_0}\cdots\mathbb{M}_{i_{m-2}} = \mathbb{M}_{m-1}\mathrm{diag}(I_n, \widetilde{M}_\tau),$$

since $\mathbb{M}_{m-1}$ lies to the left of $\mathbb{M}_{m-2}$. □

2.4 Fielder pencils are linearizations

Let $\mathbb{L}_\sigma(\lambda)$ be a Fiedler pencil of the system matrix $\mathcal{S}(\lambda)$. We show that $\mathbb{L}_\sigma(\lambda)$ is a linearization for $\mathcal{S}(\lambda)$. Note that a unimodular equivalence transformation of the system matrix $\mathcal{S}(\lambda)$ need not be a system matrix of an LTI system in state space form. We therefore introduce a notion of linearization of the system matrix by imposing additional conditions on the choice of unimodular equivalence transformations.

Definition 2.4.1 (Linearization for system matrix)**.** *An $(nm+r)\times(nm+r)$ matrix pencil $\mathbb{L}(\lambda)$ of the form $\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & X \\ \hline Y & A-\lambda E\end{array}\right]$, where $L(\lambda)$ is an $nm\times nm$ matrix pencil, is said to be a linearization of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ if there exist $nm\times nm$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that*

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r\end{array}\right]\mathbb{L}(\lambda)\left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r\end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda)\end{array}\right].$$

If, in addition, $U(\lambda)$ and $V(\lambda)$ are constant matrices then $\mathbb{L}(\lambda)$ is said to be a strict linearization *of $\mathcal{S}(\lambda)$.*

Theorem 2.4.1. *Let $L(\lambda)$ be a linearization of $P(\lambda)$ (regular or singular), that is,*

$$U(\lambda)L(\lambda)V(\lambda) = \mathrm{diag}[I_{(m-1)n}, P(\lambda)],$$

for some unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$. If $Q := U^{-1}(\lambda)(e_m\otimes C)$ and $Z := (e_m^T\otimes B)V^{-1}(\lambda)$ are independent of λ, then

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & Q \\ \hline Z & (A-\lambda E)\end{array}\right]$$

is a linearization of $\mathcal{S}(\lambda)$. In particular, if $L(\lambda)$ is a strict linearization of $P(\lambda)$ then $\mathbb{L}(\lambda)$ is a strict linearization of $\mathcal{S}(\lambda)$.

Proof. We have

$$\left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] = \left[\begin{array}{cc|c} I_{(m-1)n} & & 0_{n\times r} \\ & P(\lambda) & C \\ \hline 0_{r\times n} & B & (A-\lambda E) \end{array}\right].$$

Since

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(m-1)n} & \\ & P(\lambda) \end{bmatrix},$$

we have

$$\left[\begin{array}{c|c} U(\lambda)L(\lambda)V(\lambda) & e_m \otimes C \\ \hline e_m^T \otimes B & (A-\lambda E) \end{array}\right] =$$

$$\left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} L(\lambda) & U(\lambda)^{-1}(e_m \otimes C) \\ \hline (e_m^T \otimes B)V(\lambda)^{-1} & (A-\lambda E) \end{array}\right] \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right].$$

So if $(e_m^T \otimes B)V^{-1}(\lambda) = Z$ and $U^{-1}(\lambda)(e_m \otimes C) = Q$ are independent of λ then $\left[\begin{array}{c|c} L(\lambda) & Q \\ \hline Z & (A-\lambda E) \end{array}\right]$ is a linearization of $\mathcal{S}(\lambda)$. □

Next, we recall $Q_i(\lambda), R_i(\lambda), T_i(\lambda), D_i(\lambda)$ for $1 \leq i \leq m-1$, and $D_m(\lambda)$ given in Definition 1.2.6. We use these matrices frequently in this section to calculate the last block column of $U^{-1}(\lambda)$ and last block row of $V^{-1}(\lambda)$.

Note that

$$Q_i^{\mathcal{B}}(\lambda) = \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & 0_n & \\ & \lambda I_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix}, \tag{2.20}$$

$$Q_i^{-1}(\lambda) = \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & -\lambda I_n & \\ & 0_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix}, \tag{2.21}$$

$$(Q_i^{\mathcal{B}}(\lambda))^{-1} = \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & 0_n & \\ & -\lambda I_n & I_n & \\ & & & I_{(m-i-1)n} \end{bmatrix} = (Q_i(\lambda)^{-1})^{\mathcal{B}}, \tag{2.22}$$

$$R_i^{-1}(\lambda) = \begin{bmatrix} I_{(i-1)n} & & & \\ & -P_i(\lambda) & I_n & \\ & I_n & 0_n & \\ & & & I_{(m-i-1)n} \end{bmatrix} = (R_i(\lambda)^{\mathcal{B}})^{-1}, \tag{2.23}$$

Lemma 2.4.1. *Let $P_d(\lambda)$ for $d = 0, \ldots, m$ be the Horner shifts of the matrix polynomial $P(\lambda)$ and $(Q_i^{\mathcal{B}})^{-1}$ and R_i^{-1} for $i = 1, \ldots, m-1$ be as in (2.22) and (2.23). Then for each $i = 1, \ldots, m-1$ and $j = 1, \ldots, m-i$, we have*

$$Q(i,j) := (Q_i^{\mathcal{B}})^{-1}(Q_{i+1}^{\mathcal{B}})^{-1} \cdots (Q_{i+j-1}^{\mathcal{B}})^{-1} = \left[\begin{array}{c|cccccc|c} I_{(i-1)n} & & & & & & & \\ \hline & I_n & 0 & 0 & \ldots & 0 & & \\ & -\lambda I_n & I_n & 0 & & 0 & & \\ & & -\lambda I_n & I_n & \ddots & \vdots & & \\ & & & \ddots & \ddots & 0 & & \\ & & & & -\lambda I_n & I_n & & \\ \hline & & & & & & & I_{(m-(i+j))n} \end{array}\right]$$

and

$$R(i,j) := R_i^{-1} R_{i+1}^{-1} \cdots R_{i+j-1}^{-1} = \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & -P_i(\lambda) & -P_{i+1}(\lambda) & \ldots & -P_{i+j-1}(\lambda) & I & \\ & I_n & 0 & \ldots & 0 & 0 & \\ & & \ddots & \ddots & \vdots & \vdots & \\ & & & \ddots & 0 & 0 & \\ & & & & I_n & 0 & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right].$$

Proof. We have

$$R_i^{-1}.R_{i+1}^{-1} = \left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & -P_i & I_n & & \\ & I_n & 0 & & \\ & & & I_n & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right] \cdot \left[\begin{array}{c|ccc|c} I_{(i-1)n} & & & & \\ \hline & I_n & & & \\ & & -P_{i+1} & I_n & \\ & & I_n & 0 & \\ \hline & & & & I_{(m-i-2)n} \end{array}\right]$$

$$= \left[\begin{array}{c|ccc|c}
I_{(i-1)n} & & & & \\ \hline
 & -P_i & -P_{i+1} & I_n & \\
 & I_n & 0 & 0 & \\
 & 0 & I_n & 0 & \\ \hline
 & & & & I_{(m-i-2)n}
\end{array}\right]$$

and $R_i^{-1}.R_{i+1}^{-1}.R_{i+2}^{-1} =$

$$\left[\begin{array}{c|cccc|c}
I_{(i-1)n} & & & & & \\ \hline
 & -P_i & -P_{i+1} & I_n & & \\
 & I_n & 0 & 0 & & \\
 & 0 & I_n & 0 & & \\
 & & & & I_n & \\ \hline
 & & & & & I_{(m-i-3)n}
\end{array}\right] . \left[\begin{array}{c|cccc|c}
I_{(i-1)n} & & & & & \\ \hline
 & I_n & & & & \\
 & & I_n & & & \\
 & & & -P_{i+2} & I_n & \\
 & & & I_n & 0 & \\ \hline
 & & & & & I_{(m-i-3)n}
\end{array}\right]$$

$$= \left[\begin{array}{c|cccc|c}
I_{(i-1)n} & & & & & \\ \hline
 & -P_i & -P_{i+1} & -P_{i+2} & I_n & \\
 & I_n & 0 & 0 & 0 & \\
 & 0 & I_n & 0 & 0 & \\
 & 0 & 0 & I_n & 0 & \\ \hline
 & & & & & I_{(m-i-3)n}
\end{array}\right].$$

Similarly, we have

$$(Q_i^{\mathcal{B}})^{-1}.(Q_{i+1}^{\mathcal{B}})^{-1} = \left[\begin{array}{c|ccc|c}
I_{(i-1)n} & & & & \\ \hline
 & I_n & 0 & & \\
 & -\lambda I_n & I_n & & \\
 & & & I_n & \\ \hline
 & & & & I_{(m-i-2)n}
\end{array}\right] . \left[\begin{array}{c|ccc|c}
I_{(i-1)n} & & & & \\ \hline
 & I_n & & & \\
 & & I_n & 0 & \\
 & & -\lambda I_n & I_n & \\ \hline
 & & & & I_{(m-i-2)n}
\end{array}\right]$$

$$= \left[\begin{array}{c|ccc|c}
I_{(i-1)n} & & & & \\ \hline
 & I_n & 0 & & \\
 & -\lambda I_n & I_n & & \\
 & & -\lambda I_n & I_n & \\ \hline
 & & & & I_{(m-i-2)n}
\end{array}\right].$$

Thus the proof is a straight forward induction on the number of factors. $\square$

The following result shows that the last block column of $U(\lambda)^{-1}$ is independent of λ.

Lemma 2.4.2. *Let $L_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ, and let $U(\lambda)$ be as in Theorem 1.2.1. If CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, then*

$$U(\lambda)^{-1}(e_m \otimes I_n) = \begin{cases} e_m \otimes I_n, & \textit{if } c_1 > 0, \\ e_{m-i_1} \otimes I_n, & \textit{if } c_1 = 0. \end{cases}$$

Proof. Note that

$$U^{-1}(\lambda) = U_{m-2}^{-1}U_{m-3}^{-1}\ldots U_1^{-1}U_0^{-1} \text{ with } U_i^{-1} = \begin{cases} (Q^{\mathcal{B}}_{m-(i+1)})^{-1} & \text{if } \sigma \text{ has a consecution at } i \\ (R^{\mathcal{B}}_{m-(i+1)})^{-1} & \text{if } \sigma \text{ has an inversion at } i, \end{cases} \tag{2.24}$$

where $(Q_i^{\mathcal{B}})^{-1}$ and $(R_i^{\mathcal{B}})^{-1}$ are given in (2.22) and (2.23). As CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, the factors defining $U^{-1}(\lambda)$ in (2.24) can be sorted together into the form $U^{-1}(\lambda) = \widetilde{U}_l\widetilde{U}_{l-1}\ldots\widetilde{U}_2\widetilde{U}_1$, where

$$\widetilde{U}_j = R(m - s_j, i_j).Q(m - s_j + i_j, c_j)$$

is associated with the pair (c_j, i_j) from CISS(σ) and consists of i_j consecutive R^{-1}-factors and c_j consecutive $(Q^{\mathcal{B}})^{-1}$-factors are defined in Lemma 2.4.1. Recall that $s_0 = 0, s_j = \sum_{k=1}^{j}(c_k + i_k)$, for $j = 1, \ldots, l$, and $s_l = m - 1$. A direct multiplication gives

$$\widetilde{U}_j = \left[\begin{array}{c|cccc|c|ccccc|c}
I_{(m-s_j-1)n} & & & & & & & & & & & \\ \hline
 & -P_{m-s_j} & \cdots & \cdots & -P_{m-s_j+i_j-1} & I_n & & & & & & \\
 & I_n & 0 & \cdots & 0 & 0 & & & & & & \\
 & & \ddots & \ddots & \vdots & \vdots & & & & & & \\
 & & & I_n & 0 & 0 & & & & & & \\ \hline
 & & & & I_n & 0 & & & & & & \\ \hline
 & & & & & -\lambda I_n & I_n & & & & & \\
 & & & & & & -\lambda I_n & I_n & & & & \\
 & & & & & & & -\lambda I_n & I_n & & & \\
 & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & -\lambda I_n & I_n & \\ \hline
 & & & & & & & & & & & I_{(s_j-1)n}
\end{array}\right]$$

The central part of $\widetilde{U}_j$ is a $(i_j + 1 + c_j) \times (i_j + 1 + c_j)$ block matrix with $n \times n$ blocks. i.e, the order of the second diagonal block matrix in $\widetilde{U}_j$ is $(i_j \times i_j)n$ and the order of the fourth diagonal block matrix is $(c_j \times c_j)n$. To find out the last block column of $U^{-1}(\lambda)$, we calculate $\widetilde{U}_l\widetilde{U}_{l-1}\ldots\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n)$. If $c_1 > 0$ then for $j = 1$ we have $\widetilde{U}_1(e_m \otimes I_n) = e_m \otimes I_n$, since for $j = 1$ we have $s_0 = 0$. Now for $j = 2$, the order of last diagonal block is $s_1 n$. So $\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_m \otimes I_n$. Applying induction on $j = 1, \ldots, l$ we have

$$U^{-1}(e_m \otimes I_n) = \widetilde{U}_j\widetilde{U}_{j-1}\ldots\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_m \otimes I_n.$$

If $c_1 = 0$ then in $\widetilde{U}_j$ the fourth diagonal block matrix is not there. So for $j = 1$ we have $s_0 = 0$, and hence $\widetilde{U}_1(e_m \otimes I_n) = e_{m-i_1} \otimes I_n$. Similarly, $\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_{m-i_1} \otimes I_n$. Applying induction on $j = 1, \ldots, l$, we have

$$U^{-1}(e_m \otimes I_n) = \widetilde{U}_j\widetilde{U}_{j-1}\ldots\widetilde{U}_2\widetilde{U}_1(e_m \otimes I_n) = e_{m-i_1} \otimes I_n.$$

Hence the result follows. $\square$

We have seen that the last block column of $U^{-1}(\lambda)$ is independent of λ. Now we show that the last block row of $V^{-1}(\lambda)$ is independent of λ.

Lemma 2.4.3. *Let $P_k(\lambda)$ for $k = 0, \ldots, m$ be the Horner shifts of the matrix polynomial $P(\lambda)$, and Q_i^{-1} and R_i^{-1} for $i = 1, \ldots, m-1$ be as in (2.23). Then for each $i = 1, \ldots, m-1$ and $j = 1, \ldots, m-i$, we have*

$$\mathbf{Q}^{-1}(i,j) := Q_{i+j-1}^{-1}\cdots Q_{i+1}^{-1}Q_i^{-1} = \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & I_n & -\lambda I_n & 0 & \ldots & 0 & \\ & & I_n & -\lambda I_n & & 0 & \\ & & & I_n & \ddots & \vdots & \\ & & & & \ddots & -\lambda I_n & \\ & & & & & I_n & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right]$$

and

$$\mathbf{R}^{-1}(i,j) := R_{i+j-1}^{-1}\cdots R_{i+1}^{-1}R_i^{-1} = \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & -P_i(\lambda) & I_n & & & & \\ & -P_{i+1}(\lambda) & 0 & I_n & & & \\ & \vdots & \ddots & \ddots & \ddots & & \\ & -P_{i+j-1}(\lambda) & & \ddots & 0 & I_n & \\ & I & \ldots & \ldots & 0 & 0 & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right]$$

Proof. Proof is similar to that of Lemma 2.4.1 and follows by induction on the number of factors. $\square$

Lemma 2.4.4. *Let $L_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with a bijection σ, and let $V(\lambda)$ be as in Theorem 1.2.1. If CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, then $(e_m^T \otimes I_n)V(\lambda)^{-1} = e_{m-c_1}^T \otimes I_n$.*

Proof. We have

$$V^{-1}(\lambda) = V_0^{-1}V_1^{-1}\ldots V_{m-3}^{-1}V_{m-2}^{-1} \text{ with } V_i^{-1} = \begin{cases} R_{m-(i+1)}^{-1} & \text{if } \sigma \text{ has a consecution at } i \\ Q_{m-(i+1)}^{-1} & \text{if } \sigma \text{ has an inversion at } i, \end{cases} \tag{2.25}$$

where Q_i^{-1} and R_i^{-1} are given in (2.22) and (2.23). Since CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, the factors of $V^{-1}(\lambda)$ given in (2.25) can be grouped together so that $V^{-1}(\lambda) = \widetilde{V}_1\widetilde{V}_2\ldots\widetilde{V}_l$, where

$$\widetilde{V}_j = \mathbf{R}^{-1}(m - s_j + i_j, c_j).\mathbf{Q}^{-1}(m - s_j, i_j)$$

is associated with the pair (c_j, i_j) from CISS(σ), and consists of c_j consecutive R^{-1}-factors and i_j consecutive Q^{-1}-factors defined in Lemma 2.4.3. A direct multiplication gives

$$\widetilde{V}_j = \left[\begin{array}{c|cccc|cccccc|c}
I_{(m-s_j-1)n} & & & & & & & & & & & \\ \hline
 & I_n & -\lambda I_n & \cdots & 0 & 0 & & & & & & \\
 & & I_n & \ddots & \vdots & \vdots & & & & & & \\
 & & & \ddots & -\lambda I_n & 0 & & & & & & \\
 & & & & I_n & -\lambda I_n & & & & & & \\ \hline
 & & & & & -P_{m-s_j+i_j} & I_n & & & & & \\
 & & & & & -P_{m-s_j+i_j+1} & 0 & I_n & & & & \\
 & & & & & \vdots & & \ddots & \ddots & & & \\
 & & & & & \vdots & \ddots & & \ddots & \ddots & & \\
 & & & & & -P_{m-s_j+i_j+c_j-1} & & \ddots & & \ddots & I_n & \\
 & & & & & I_n & \cdots & & \cdots & 0 & 0 & \\ \hline
 & & & & & & & & & & & I_{(s_j-1)n}
\end{array}\right]$$

The central four blocks of $\widetilde{V}_j$ is a $(i_j+c_j+1)\times(i_j+c_j+1)$ block matrix with $n\times n$ blocks. To find out the last block row of $V^{-1}(\lambda)$, we calculate $(e_m^T \otimes I_n)\widetilde{V}_1\widetilde{V}_2\ldots\widetilde{V}_l$. Applying the induction on $j = 1, \ldots, l$, we have

$$(e_m^T \otimes I_n).\widetilde{V}_1\widetilde{V}_2\ldots\widetilde{V}_j = e_{m-c_1}^T \otimes I_n.$$

Hence the follows. $\square$

Now we show that Fiedler pencils of $\mathcal{S}(\lambda)$ are linearizations for $\mathcal{S}(\lambda)$.

Theorem 2.4.2. *Let $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ and $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ and $P(\lambda)$, respectively, associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$. Then $\mathbb{L}_\sigma(\lambda)$ is a linearization of $\mathcal{S}(\lambda) \Leftrightarrow L_\sigma(\lambda)$ is a linearization of $P(\lambda)$. Further, $\mathbb{L}_\sigma(\lambda)$ is a strict linearization of $\mathcal{S}(\lambda) \Leftrightarrow L_\sigma(\lambda)$ is a strict linearization of $P(\lambda)$.*

Proof. Suppose that $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$. Then by Theorem 2.3.1, we have

$$\mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_m \otimes C \\ \hline e_{m-c_1}^T \otimes B & (A - \lambda E) \end{array}\right] \quad \text{if } c_1 > 0$$

$$\text{and} \quad \mathbb{L}_\sigma(\lambda) = \left[\begin{array}{c|c} L_\sigma(\lambda) & e_{m-i_1} \otimes C \\ \hline e_m^T \otimes B & (A - \lambda E) \end{array}\right] \quad \text{if } c_1 = 0.$$

Suppose that $L_\sigma(\lambda)$ is a linearization of $P(\lambda)$. By Theorem 1.2.1, we have

$$\begin{bmatrix} -I_{(m-1)n} & \\ & P(\lambda) \end{bmatrix} = U(\lambda) L_\sigma(\lambda) V(\lambda).$$

Now we have

$$\left[\begin{array}{c|c} -I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right] = \left[\begin{array}{cc|c} -I_{(m-1)n} & & 0 \\ & P(\lambda) & C \\ \hline 0 & B & A - \lambda E \end{array}\right] = \left[\begin{array}{c|c} U(\lambda) L_\sigma(\lambda) V(\lambda) & e_m \otimes C \\ \hline e_m^T \otimes B & (A - \lambda E) \end{array}\right]$$

$$= \left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} L_\sigma(\lambda) & U(\lambda)^{-1}(e_m \otimes C) \\ \hline (e_m^T \otimes B) V(\lambda)^{-1} & (A - \lambda E) \end{array}\right] \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right].$$

By Lemma 2.4.2 and Lemma 2.4.4, we have

$$U(\lambda)^{-1}(e_m \otimes I_n) = \begin{cases} e_m \otimes I_n, & \text{if } c_1 > 0, \\ e_{m-i_1} \otimes I_n, & \text{if } c_1 = 0. \end{cases}$$

and $(e_m^T \otimes I_n) V(\lambda)^{-1} = e_{m-c_1}^T \otimes I_n$. Consequently, we have

$$\left[\begin{array}{c|c} L_\sigma(\lambda) & U(\lambda)^{-1}(e_m \otimes C) \\ \hline (e_m^T \otimes B) V(\lambda)^{-1} & (A - \lambda E) \end{array}\right] = \mathbb{L}_\sigma(\lambda).$$

Hence $\mathbb{L}_\sigma(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$.

Conversely, let $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ be a linearization of $\mathcal{S}(\lambda)$. Then

$$\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right] = \left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right]$$

for some unimodular $U(\lambda)$ and $V(\lambda)$ in $\mathbb{C}^{nm\times nm}[\lambda]$. Thus, we have

$$\left[\begin{array}{cc|c} I_{(m-1)n} & & 0 \\ & P(\lambda) & C \\ \hline 0 & B & A-\lambda E \end{array}\right] = \left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} L_\sigma(\lambda) & * \\ \hline * & * \end{array}\right] \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right].$$

This shows that

$$\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right] = U(\lambda)L_\sigma(\lambda)V(\lambda).$$

Hence the proof. □

The Fiedler pencils of the system matrix under appropriate assumptions can be thought of as *linearizations* of the transfer function $G(\lambda)$. For this purpose, we now introduce a notion of linearization for rational matrix functions. The degree of a matrix polynomial $U(\lambda)$ is the highest power of λ in $U(\lambda)$ and is denoted by $\deg U(\lambda)$.

Definition 2.4.2 (Linearization for rational matrix function)**.** *Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ be rational matrix function such that $G(\lambda) = P(\lambda) + Q(\lambda)$, where $P(\lambda)$ is a matrix polynomial and $Q(\lambda)$ is a strictly proper rational matrix function. Suppose that*

$$\mathbf{SM}(G(\lambda)) = \mathrm{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \frac{\phi_2(\lambda)}{\psi_2(\lambda)}, \ldots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0, \ldots, 0\right),$$

where $k = \mathrm{nrank}(G)$. Set $m := \max(\deg P(\lambda), 1)$ and $r := \deg \psi_G(\lambda)$, where $\psi_G(\lambda)$ is the pole polynomial. Then a pencil $\mathbb{L}(\lambda) = \lambda\mathcal{X} + \mathcal{Y} \in \mathbb{C}^{(nm+r)\times(nm+r)}[\lambda]$ is said to be a linearization of $G(\lambda)$ if the Smith form of $\mathbb{L}(\lambda)$ is given by

$$\mathbf{SF}(\mathbb{L}(\lambda)) = \mathrm{diag}\left(I_{(m-1)n+r}, \phi_1(\lambda), \cdots, \phi_k(\lambda), 0_{n-k,n-k}\right).$$

Theorem 2.4.3. *Let $\mathbb{L}_\sigma(\lambda)$ be a Fiedler pencil of the Rosenbrock system matrix $\mathcal{S}(\lambda)$. If the LTI system Σ is controllable and observable (realization of $G(\lambda)$ is minimal) then $\mathbb{L}_\sigma(\lambda)$ is a linearization of the transfer function $G(\lambda)$.*

Proof. Suppose that the Smith form of $G(\lambda)$ is given by

$$\mathbf{SM}(G(\lambda)) = \mathrm{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \frac{\phi_2(\lambda)}{\psi_2(\lambda)}, \ldots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0, \ldots, 0\right).$$

Since $G(\lambda)$ is minimal, by Theorem 2.2.2, the Smith form $\mathbf{SF}(\mathcal{S}(\lambda))$ of $\mathcal{S}(\lambda)$ is given by

$$\mathbf{SF}(\mathcal{S}(\lambda)) = \operatorname{diag}(I_r, \phi_1, \phi_2, \ldots, \phi_k, 0, \ldots, 0).$$

Since $\mathbb{L}_\sigma(\lambda)$ is a Fiedler linearization of $\mathcal{S}(\lambda)$, we have

$$\mathbf{SF}(\mathbb{L}_\sigma(\lambda)) = \operatorname{diag}\left(I_{(m-1)n+r}, \phi_1, \phi_2, \ldots, \phi_k, 0, \ldots, 0\right).$$

Hence $\mathbb{L}_\sigma(\lambda)$ is a linearization of $G(\lambda)$. $\square$

We have already seen that Fiedler pencils of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ are linearizations for $\mathcal{S}(\lambda)$. We now give a direct proof of this fact without any recourse to Fiedler pencils of matrix polynomials. We adapt the method of proof given in [9] for matrix polynomials to the case of Rosenbrock system matrix.

Lemma 2.4.5. *[9] Let Q_i, R_i, T_i, D_i be as in Definition 1.2.6. Then the following relations hold for $i = 1, \ldots, m-1$.*

(a) $Q_i^{\mathcal{B}}(\lambda D_i)R_i = \lambda D_{i+1} + T_i$, and $Q_i^{\mathcal{B}}(M_{m-(i+1)}M_{m-i})R_i = M_{m-(i+1)} + T_i$.

(b) $R_i^{\mathcal{B}}(\lambda D_i)Q_i = \lambda D_{i+1} + T_i^{\mathcal{B}}$, and $R_i^{\mathcal{B}}(M_{m-i}M_{m-(i+1)})Q_i = M_{m-(i+1)} + T_i^{\mathcal{B}}$.

(c) $T_iM_j = M_jT_i = T_i$ and $T_i^{\mathcal{B}}M_j = M_jT_i^{\mathcal{B}} = T_i^{\mathcal{B}}$ for all $j \leq m-i-2$.

It is also shown in [[9], Lemma 5.2] that for $i = 1, \ldots, m-1$ and $j = 1, \ldots, m-i$,

$$\mathbf{Q}(i,j) := Q_iQ_{i+1}\cdots Q_{i+j-1} = \left[\begin{array}{c|cccc|c} I_{(i-1)n} & & & & & \\ \hline & I_n & \lambda I_n & \ldots & \lambda^j I_n & \\ & & \ddots & \ddots & \vdots & \\ & & & \ddots & \lambda I_n & \\ & & & & I_n & \\ \hline & & & & & I_{(m-(i+j))n} \end{array}\right] \tag{2.26}$$

and

$$\mathbf{R}(i,j) := R_iR_{i+1}\cdots R_{i+j-1} = \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & 0 & 0 & \ldots & 0 & I_n & \\ & I_n & 0 & \ldots & 0 & P_i(\lambda) & \\ & & \ddots & \ddots & \vdots & \vdots & \\ & & & \ddots & 0 & P_{i+j-2(\lambda)} & \\ & & & & I_n & P_{i+j-1(\lambda)} & \\ \hline & & & & & & I_{(m-(i+j))n} \end{array}\right], \tag{2.27}$$

where 0 is the zero block matrix of size $n \times n$, and $P_i(\lambda)$ is the degree i Horner shift of $P(\lambda)$.

Definition 2.4.3. *Let* $\mathcal{A} = \left[\begin{array}{c|c} A & e_i \otimes X \\ \hline e_j^T \otimes Y & Z \end{array}\right] \in \mathbb{C}^{(nm+r)\times(nm+r)}$*, where* $A = [A_{ij}]$ *is an* $m \times m$ *block matrix with* $A_{ij} \in \mathbb{C}^{n\times n}$*,* $e_j \in \mathbb{C}^m$*,* $X \in \mathbb{C}^{n\times r}, Y \in \mathbb{C}^{r\times n}$ *and* $Z \in \mathbb{C}^{r\times r}$*. We define the* block transpose *of* $\mathcal{A}$ *by* $\mathcal{A}^{\mathcal{B}} := \left[\begin{array}{c|c} A^{\mathcal{B}} & e_j \otimes X \\ \hline e_i^T \otimes Y & Z \end{array}\right]$.

For $i = 1, \ldots, m-1$, we define

$$\mathcal{Q}_i(\lambda) := \left[\begin{array}{c|c} Q_i(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right], \quad \mathcal{R}_i(\lambda) := \left[\begin{array}{c|c} R_i(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right], \tag{2.28}$$

$$\mathcal{T}_i(\lambda) := \left[\begin{array}{c|c} T_i(\lambda) & 0 \\ \hline 0 & 0_r \end{array}\right], \quad \mathcal{D}_i(\lambda) := \left[\begin{array}{c|c} D_i(\lambda) & 0 \\ \hline 0_{r\times nm} & -E \end{array}\right], \tag{2.29}$$

$$\mathcal{D}_m(\lambda) := \left[\begin{array}{c|c} D_m & 0 \\ \hline 0 & -E \end{array}\right], \quad \mathcal{Q}_i^{\mathcal{B}}(\lambda) := \left[\begin{array}{c|c} Q_i^{\mathcal{B}}(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right], \tag{2.30}$$

$$\mathcal{R}_i^{\mathcal{B}}(\lambda) := \left[\begin{array}{c|c} R_i^{\mathcal{B}}(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] = \mathcal{R}_i(\lambda), \quad \mathcal{T}_i^{\mathcal{B}}(\lambda) := \left[\begin{array}{c|c} T_i^{\mathcal{B}}(\lambda) & 0 \\ \hline 0 & 0_r \end{array}\right], \tag{2.31}$$

where $Q_i(\lambda), R_i(\lambda), T_i(\lambda)$, and $D_i(\lambda)$ are as in Definition 1.2.6.

Observe that $\mathcal{D}_1(\lambda) = \left[\begin{array}{c|c} D_1(\lambda) & 0 \\ \hline 0 & -E \end{array}\right] = \left[\begin{array}{c|c} M_m & 0 \\ \hline 0 & -E \end{array}\right] = \mathbb{M}_m$, and for all $i = 1, \ldots, m-1$, $\mathcal{Q}_i(\lambda)$ and $\mathcal{R}_i(\lambda)$ are unimodular matrices.

Lemma 2.4.6. *Let* $\mathcal{Q}_i, \mathcal{R}_i, \mathcal{T}_i, \mathcal{D}_i$ *be the matrices defined in (2.28) and (2.29). Then the following unimodular equivalence relations hold for* $i = 1, \ldots, m-1$.

(a) $\mathcal{Q}_i^{\mathcal{B}}(\lambda\mathcal{D}_i)\mathcal{R}_i = \lambda\mathcal{D}_{i+1} + \mathcal{T}_i$*, and* $\mathcal{Q}_i^{\mathcal{B}}(\mathbb{M}_{m-(i+1)}\mathbb{M}_{m-i})\mathcal{R}_i = \mathbb{M}_{m-(i+1)} + \mathcal{T}_i$.

(b) $\mathcal{R}_i^{\mathcal{B}}(\lambda\mathcal{D}_i)\mathcal{Q}_i = \lambda\mathcal{D}_{i+1} + \mathcal{T}_i^{\mathcal{B}}$*, and* $\mathcal{R}_i^{\mathcal{B}}(\mathbb{M}_{m-i}\mathbb{M}_{m-(i+1)})\mathcal{Q}_i = \mathbb{M}_{m-(i+1)} + \mathcal{T}_i^{\mathcal{B}}$.

(c) $\mathcal{T}_i\mathbb{M}_j = \mathbb{M}_j\mathcal{T}_i = \mathcal{T}_i$ *and* $\mathcal{T}_i^{\mathcal{B}}\mathbb{M}_j = \mathbb{M}_j\mathcal{T}_i^{\mathcal{B}} = \mathcal{T}_i^{\mathcal{B}}$ *for all* $j \leq m-i-2$.

Proof. (a) We have

$$\mathcal{Q}_i^{\mathcal{B}}(\lambda\mathcal{D}_i)\mathcal{R}_i = \left[\begin{array}{c|c} Q_i^{\mathcal{B}} & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} \lambda D_i & 0 \\ \hline 0 & -\lambda E \end{array}\right]\left[\begin{array}{c|c} R_i & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} Q_i^{\mathcal{B}}\lambda D_i R_i & 0 \\ \hline 0 & -\lambda E \end{array}\right] = \left[\begin{array}{c|c} \lambda D_{i+1} + T_i & 0 \\ \hline 0 & -\lambda E \end{array}\right] = \lambda\mathcal{D}_{i+1} + \mathcal{T}_i$$

and

$$\mathcal{Q}_i^{\mathcal{B}}(\mathbb{M}_{m-(i+1)}\mathbb{M}_{m-i})\mathcal{R}_i = \left[\begin{array}{c|c} Q_i^{\mathcal{B}} & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} M_{m-(i+1)} & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} M_{m-i} & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} R_i & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} Q_i^{\mathcal{B}} M_{m-(i+1)} M_{m-i} R_i & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{m-(i+1)} + T_i & 0 \\ \hline 0 & I_r \end{array}\right] = \mathbb{M}_{m-(i+1)} + \mathcal{T}_i.$$

For $i = m-1$, we have $\mathcal{Q}_{m-1}^{\mathcal{B}}(\mathbb{M}_0\mathbb{M}_1)\mathcal{R}_{m-1}$

$$= \left[\begin{array}{c|c} Q_{m-1}^{\mathcal{B}} & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right]\left[\begin{array}{c|c} M_1 & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} R_{m-1} & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} Q_{m-1}^{\mathcal{B}} & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} M_0 M_1 R_{m-1} & -e_m \otimes C \\ \hline -e_m^T \otimes BM_1R_{m-1} & -A \end{array}\right]$$

$$= \left[\begin{array}{c|c} Q_{m-1}^{\mathcal{B}} M_0 M_1 R_{m-1} & Q_{m-1}^{\mathcal{B}}(-e_m \otimes C) \\ \hline -e_m^T \otimes BM_1R_{m-1} & -A \end{array}\right] = \left[\begin{array}{c|c} M_0 + T_{m-1} & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right]$$

$= \mathbb{M}_0 + \mathcal{T}_{m-1}$, since $\begin{bmatrix} I_{(m-2)n} & & \\ & I_n & \lambda I_n \\ & 0 & I_n \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ -C \end{bmatrix} = -e_m \otimes C$ and

$$-e_m^T \otimes BM_1R_{m-1} = -e_m^T \otimes B \begin{bmatrix} I_{(m-2)n} & & \\ & -A_1 & I_n \\ & I_n & 0 \end{bmatrix}\begin{bmatrix} I_{(m-2)n} & & \\ & 0 & I_n \\ & I_n & P_{m-1}(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & -B \end{bmatrix}\begin{bmatrix} I_{(m-2)n} & & \\ & I_n & -A_1 + P_{m-1}(\lambda) \\ & 0 & I_n \end{bmatrix} = -e_m^T \otimes B.$$

(b) Now

$$\begin{aligned}
\mathcal{R}_i^{\mathcal{B}}(\lambda\mathcal{D}_i)\mathcal{Q}_i &= \left[\begin{array}{c|c} R_i & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} \lambda D_i & 0 \\ \hline 0 & -\lambda E \end{array}\right] \left[\begin{array}{c|c} Q_i & 0 \\ \hline 0 & I_r \end{array}\right] \\
&= \left[\begin{array}{c|c} R_i\lambda D_i Q_i & 0 \\ \hline 0 & -\lambda E \end{array}\right] = \left[\begin{array}{c|c} \lambda D_{i+1} + T_i^{\mathcal{B}} & 0 \\ \hline 0 & -\lambda E \end{array}\right] \\
&= \lambda\mathcal{D}_{i+1} + \mathcal{T}_i^{\mathcal{B}}.
\end{aligned}$$

Similarly $\mathcal{R}_i^{\mathcal{B}}(\mathbb{M}_{m-i}\mathbb{M}_{m-(i+1)})\mathcal{Q}_i = \mathbb{M}_{m-(i+1)} + \mathcal{T}_i^{\mathcal{B}}$.

(c) We have

$$\begin{aligned}
\mathcal{T}_i\mathbb{M}_j &= \left[\begin{array}{c|c} T_i & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} M_j & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} T_i M_j & 0 \\ \hline 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} M_j T_i & 0 \\ \hline 0 & 0 \end{array}\right] = \mathbb{M}_j\mathcal{T}_i = \left[\begin{array}{c|c} T_i & 0 \\ \hline 0 & 0 \end{array}\right] = \mathcal{T}_i.
\end{aligned}$$

Similarly $\mathcal{T}_i^{\mathcal{B}}\mathbb{M}_j = \mathbb{M}_j\mathcal{T}_i^{\mathcal{B}} = \mathcal{T}_i^{\mathcal{B}}$. For $i = m-2$ we have

$$\mathcal{T}_{m-2}^{\mathcal{B}}\mathbb{M}_0 = \left[\begin{array}{c|c} T_{m-2}^{\mathcal{B}} & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right]$$

$$= \left[\begin{array}{c|c} T_{m-2}^{\mathcal{B}}M_0 & T_{m-2}^{\mathcal{B}}(-e_m \otimes C) \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} T_{m-2}^{\mathcal{B}}M_0 & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} T_{m-2}^{\mathcal{B}} & 0 \\ \hline 0 & 0 \end{array}\right] = \mathcal{T}_{m-2}^{\mathcal{B}},$$

since $T_{m-2}^{\mathcal{B}}(-e_m \otimes C) = \begin{bmatrix} 0_{(m-3)n} & & & \\ & 0 & \lambda I & \\ & \lambda P_{m-3} & \lambda^2 P_{m-3} & \\ & & & 0_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -C \end{bmatrix} = 0$

This completes the proof. $\square$

The next definition gives the intermediate steps in the unimodular transformation of a Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ into $\left[\begin{array}{cc|c} I_{(m-1)} & & \\ & P(\lambda) & C \\ \hline & B & (A-\lambda E) \end{array}\right]$.

Definition 2.4.4. *Let $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection σ. For $j = 1, 2, \ldots, m$, define*

$$\mathbb{M}_\sigma^{(j)} := \prod_{\sigma^{-1}(i) \leq m-j} \mathbb{M}_{\sigma^{-1}(i)},$$

where the factors $\mathbb{M}_{\sigma^{-1}(i)}$ are in the same relative order as they are in $\mathbb{M}_\sigma$. Observe that $\mathbb{M}_\sigma^{(1)} = \prod_{\sigma^{-1}(i)\leq m-1} \mathbb{M}_{\sigma^{-1}(i)} = \mathbb{M}_\sigma$, and $\mathbb{M}_\sigma^{(m)} = \mathbb{M}_0$. Also for $j = 1, 2, \ldots, m$ define $(nm + r) \times (nm + r)$ pencils

$$\mathbb{L}_\sigma^{(j)}(\lambda) := \lambda\mathcal{D}_j(\lambda) - \mathbb{M}_\sigma^{(j)}.$$

Notice that

$$\mathbb{L}_\sigma^{(1)}(\lambda) = \lambda\mathcal{D}_1 - \mathbb{M}_\sigma^{(1)} = \lambda\mathbb{M}_m - \mathbb{M}_\sigma = \mathbb{L}_\sigma$$

and

$$\mathbb{L}_\sigma^{(m)}(\lambda) = \lambda\mathcal{D}_m - \mathbb{M}_\sigma^{(m)} = \lambda \left[\begin{array}{c|c} D_m & 0 \\ \hline 0 & -E \end{array}\right] - \mathbb{M}_0$$

$$= \lambda \left[\begin{array}{c|c} D_m & 0 \\ \hline 0 & -E \end{array}\right] - \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right] = \left[\begin{array}{c|c} \begin{matrix} -I_{(m-1)n} & \\ & P(\lambda) \end{matrix} & C \\ \hline B & A - \lambda E \end{array}\right].$$

The following lemma demonstrates, how to transform $\mathbb{L}_\sigma^{(i)}(\lambda)$ into $\mathbb{L}_\sigma^{(i+1)}(\lambda)$ by unimodular transformations and adapt the proof given in [9].

Lemma 2.4.7. $\mathbb{L}_\sigma^{(i)}(\lambda)$ *is unimodularly equivalent to* $\mathbb{L}_\sigma^{(i+1)}(\lambda)$ *for* $i = 1, 2, \ldots, m-1$. *Particularly, if $\mathcal{Q}_i$ and $\mathcal{R}_i$ are the unimodular matrices defined in (2.28), then*

$$\mathbb{L}_\sigma^{(i+1)}(\lambda) = \begin{cases} \mathcal{Q}_i^{\mathcal{B}}\mathbb{L}_\sigma^{(i)}(\lambda)\mathcal{R}_i, & \text{if } \sigma \text{ has a consecution at } m-i-1 \\ \mathcal{R}_i^{\mathcal{B}}\mathbb{L}_\sigma^{(i)}\mathcal{Q}_i, & \text{if } \sigma \text{ has an inversion at } m-i-1. \end{cases}$$

Proof. Suppose that σ has a consecution at $m - i - 1$, then $\mathbb{M}_{m-i-1}$ is to the left of $\mathbb{M}_{m-i}$ in the product $\mathbb{M}_\sigma^{(i)}$. Since $\mathbb{M}_{m-i}$ is the highest index among all factors in $\mathbb{M}_\sigma^{(i)}$, so by using the commutativity relations given in (2.11) we can shift $\mathbb{M}_{m-i}$ up to adjacent to $\mathbb{M}_{m-i-1}$, that is $\mathbb{M}_\sigma^{(i)} = \cdots\mathbb{M}_{m-i-1}\mathbb{M}_{m-i}\cdots$. Now $\mathcal{Q}_i^{\mathcal{B}}$ and $\mathcal{R}_i$ commute with $\mathbb{M}_j$ for all $j \leq m - i - 2$, so we have

$$\begin{aligned} \mathcal{Q}_i^{\mathcal{B}}\mathbb{M}_\sigma^{(i)}\mathcal{R}_i &= \cdots(\mathcal{Q}_i^{\mathcal{B}}\mathbb{M}_{m-i-1}\mathbb{M}_{m-i}\mathcal{R}_i)\cdots \\ &= \cdots(\mathbb{M}_{m-i-1} + \mathcal{T}_i)\cdots, \qquad \text{by Lemma 2.4.6(a)} \\ &= \mathbb{M}_\sigma^{(i+1)} + \mathcal{T}_i, \qquad \text{by Lemma 2.4.6(c).} \end{aligned}$$

So,

$$\begin{aligned}\mathcal{Q}_i^{\mathcal{B}}\mathbb{L}_\sigma^{(i)}\mathcal{R}_i &= \mathcal{Q}_i^{\mathcal{B}}(\lambda\mathcal{D}_i - \mathbb{M}_\sigma^{(i)})\mathcal{R}_i = \mathcal{Q}_i^{\mathcal{B}}(\lambda\mathcal{D}_i)\mathcal{R}_i - \mathcal{Q}_i^{\mathcal{B}}(\mathbb{M}_\sigma^{(i)})\mathcal{R}_i\\ &= (\lambda\mathcal{D}_{i+1} + \mathcal{T}_i) - (\mathbb{M}_\sigma^{(i+1)} + \mathcal{T}_i) \text{ by Lemma 2.4.6(a)}\\ &= \lambda\mathcal{D}_{i+1} - \mathbb{M}_\sigma^{(i+1)} = \mathbb{L}_\sigma^{(i+1)}.\end{aligned}$$

The proof of (b) is similar. □

The next result shows that a Fiedler pencil is a linearization of $\mathcal{S}(\lambda)$.

Theorem 2.4.4. *Let $\mathbb{L}_\sigma(\lambda)$ be a Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$. Then $\mathbb{L}_\sigma(\lambda)$ is a linearization for $\mathcal{S}(\lambda)$. Further, if $G(\lambda)$ is minimal then $\mathbb{L}_\sigma(\lambda)$ is a linearization for $G(\lambda)$.*

Proof. To show that $\mathbb{L}_\sigma(\lambda)$ is a linearization for $\mathcal{S}(\lambda)$, we have to find out unimodular matrices $\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right]$ and $\left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right]$ such that

$$\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right]\mathbb{L}_\sigma(\lambda)\left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{cc|c} -I_{(m-1)n} & & \\ & P(\lambda) & C \\ \hline & B & (A-\lambda E) \end{array}\right]. \tag{2.32}$$

Using Lemma 2.4.7 we can construct an equivalence relation as the composition of a sequence of $m-1$ unimodular equivalences as follows:

$$\mathbb{L}_\sigma(\lambda) = \mathbb{L}_\sigma^{(1)}(\lambda) \to \mathbb{L}_\sigma^{(2)}(\lambda) \to \cdots \to \mathbb{L}_\sigma^{(m)}(\lambda) = \left[\begin{array}{cc|c} -I_{(m-1)n} & & \\ & P(\lambda) & C \\ \hline & B & (A-\lambda E) \end{array}\right], \tag{2.33}$$

where

$$\mathbb{L}_\sigma^{(i+1)}(\lambda) = \begin{cases} \mathcal{Q}_i^{\mathcal{B}}\mathbb{L}_\sigma^{(i)}(\lambda)\mathcal{R}_i, & \text{if } \sigma \text{ has a consecution at } m-i-1,\\ \mathcal{R}_i^{\mathcal{B}}\mathbb{L}_\sigma^{(i)}(\lambda)\mathcal{Q}_i, & \text{if } \sigma \text{ has an inversion at } m-i-1.\end{cases}$$

As the product of unimodular matrices is again unimodular and $\mathcal{R}_i$ and $\mathcal{Q}_i$ have the desired block diagonal form, taking the final product together with premultiplication by $\mathrm{diag}[-I_{(m-1)n}, I_n, I_r]$, we have (2.32). This completes the proof that $\mathbb{L}_\sigma(\lambda)$ is a linearization for $\mathcal{S}(\lambda)$. □

The following corollary gives precisely the $m-1$ unimodular equivalences given in (2.33).

Corollary 2.4.5. *Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection σ, and $\mathcal{Q}_i, \mathcal{R}_i, \mathcal{Q}_i^{\mathcal{B}}$ for $i = 1, 2, \ldots m-1$, be as in (2.28) and (2.31). Then*

$$\mathcal{U}(\lambda)\mathbb{L}_\sigma(\lambda)\mathcal{V}(\lambda) = \left[\begin{array}{cc|c} -I_{(m-1)n} & & \\ & P(\lambda) & C \\ \hline & B & (A-\lambda E) \end{array}\right], \tag{2.34}$$

where $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$ are $(nm+r)\times(nm+r)$ unimodular matrix polynomials given by

$$\mathcal{U}(\lambda) := \mathcal{U}_0\mathcal{U}_1\cdots\mathcal{U}_{m-3}\mathcal{U}_{m-2}, \text{ with } \mathcal{U}_i = \begin{cases} \mathcal{Q}^{\mathcal{B}}_{m-(i+1)}, & \text{if } \sigma \text{ has a consecution at } i, \\ \mathcal{R}^{\mathcal{B}}_{m-(i+1)}, & \text{if } \sigma \text{ has an inversion at } i, \end{cases}$$

$$\mathcal{V}(\lambda) := \mathcal{V}_{m-2}\mathcal{V}_{m-3}\cdots\mathcal{V}_1\mathcal{V}_0, \text{ with } \mathcal{V}_i = \begin{cases} \mathcal{R}_{m-(i+1)}, & \text{if } \sigma \text{ has a consecution at } i, \\ \mathcal{Q}_{m-(i+1)}, & \text{if } \sigma \text{ has an inversion at } i. \end{cases}$$

Note that the indexing of $\mathcal{U}_i$ and $\mathcal{V}_i$ factors in $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$ in the above corollary has been chosen here for simplification of notation.

Example 2.7. Consider the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_3$ of a rational matrix function $G(\lambda) = A_4\lambda^4 + \cdots + A_1\lambda + A_0 + C(\lambda E - A)^{-1}B$. By Corollary 2.4.5, $\mathcal{Q}_3^{\mathcal{B}}(\mathcal{R}_2\mathcal{Q}_1^{\mathcal{B}})\mathbb{L}_\sigma(\lambda)(\mathcal{R}_1\mathcal{Q}_2\mathcal{R}_3)$ transforms the pencil to the form (2.34). ■

3 Eigenvector Recovery From Fiedler Pencils

The main objective of this chapter is to analyze recovery of eigenvector from Fiedler pencils of system matrix. We also describe recovery of eigenvector of transfer function from that of Fiedler pencil of system matrix. In fact, we establish isomorphisms between null spaces of transfer functions, system matrix, and Fiedler pencils.

3.1 Introduction

Consider the LTI system Σ :

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + P\left(\frac{d}{dt}\right)u(t), \end{aligned} \tag{3.1}$$

the associated system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right] \tag{3.2}$$

and the transfer function

$$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B = \sum_{j=0}^{m} \lambda^j A_j + C(\lambda E - A)^{-1}B, \tag{3.3}$$

where $A_j \in \mathbb{C}^{n\times n}$, $A, E \in \mathbb{C}^{r\times r}$, $C \in \mathbb{C}^{n\times r}$ and $B \in \mathbb{C}^{r\times n}$.

Recall that $\mathrm{Sp}(\mathcal{S})$ is the set of invariant zeros of Σ and $\mathrm{Sp}(G)$ is the set of transmission zeros of Σ. When Σ is both controllable and observable we have $\mathrm{Sp}(\mathcal{S}) = \mathrm{Sp}(G)$. Also recall that if $\mathbb{L}_\sigma(\lambda)$ is a Fiedler linearization of $\mathcal{S}(\lambda)$ then $\mathrm{Sp}(\mathcal{S}) = \mathrm{Sp}(\mathbb{L}_\sigma)$, and $\mathrm{Ind}(\lambda, \mathbb{L}_\sigma) = \mathrm{Ind}(\lambda, \mathcal{S})$ for $\lambda \in \mathrm{Sp}(\mathcal{S})$. We now describe recovery of invariant directions of Σ, that is, eigenvectors of $\mathcal{S}(\lambda)$ from the eigenvectors of $\mathbb{L}_\sigma(\lambda)$. We also derive eigenvectors of $G(\lambda)$ from that of $\mathbb{L}_\sigma(\lambda)$.

For any bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$, consider the Fiedler pencils $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$. We wish to recover eigenvectors of $\mathcal{S}(\lambda)$ and $G(\lambda)$ from the eigenvectors of $\mathbb{L}_\sigma(\lambda)$. For the special case of companion pencil $\mathcal{C}_1(\lambda)$ given by

$$\mathcal{C}_1(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{m-1}\mathbb{M}_{m-2}\cdots\mathbb{M}_1\mathbb{M}_0 = \lambda\mathcal{X} - \mathcal{Y}, \tag{3.4}$$

where

$$\mathcal{Y} = \left[\begin{array}{cccc|c} -A_{m-1} & -A_{m-2} & \cdots & -A_0 & -C \\ I_n & 0 & \cdots & 0 & \\ & \ddots & & \vdots & \\ & & I_n & 0 & \\ \hline & & & -B & -A \end{array}\right] \quad \text{and } \mathcal{X} = \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right],$$

it is shown in [37] that eigenvalues and eigenvectors of $G(\lambda)$ can be computed from those of $\mathcal{C}_1(\lambda)$. Indeed, if λ is an eigenvalue of $G(\lambda)$ then $G(\lambda)x = 0$ if and only if $\mathcal{C}_1(\lambda)v = 0$, where

$$v = \left[\begin{array}{c} \lambda^{m-1}x \\ \lambda^{m-2}x \\ \vdots \\ x \\ \hline (\lambda E - A)^{-1}Bx \end{array}\right]. \tag{3.5}$$

Equivalently, let $\mathcal{N}(G(\lambda))$ and $\mathcal{N}(\mathcal{C}_1(\lambda))$ be nullspaces of $G(\lambda)$ and $\mathcal{C}_1(\lambda)$ respectively, then the map

$$\mathcal{N}(G(\lambda)) \to \mathcal{N}(\mathcal{C}_1(\lambda)), \qquad x \mapsto \left[\begin{array}{c} \lambda^{m-1}x \\ \lambda^{m-2}x \\ \vdots \\ x \\ \hline (\lambda E - A)^{-1}Bx \end{array}\right]$$

is an isomorphism.

It is also easy to recover left eigenvector of $G(\lambda)$ from that of $\mathcal{C}_1(\lambda)$. Indeed, we have the following result.

Lemma 3.1.1. *The vector $y \in \mathbb{C}^n$ is a left eigenvector of $G(\lambda)$ corresponding to an eigenvalue λ if and only if*

$$u = \begin{bmatrix} I_n \\ (\lambda A_m + A_{m-1})^T \\ \vdots \\ (\lambda^{m-1}A_m + \lambda^{m-2}A_{m-1} + \cdots + A_1)^T \\ (C(\lambda E - A)^{-1})^T \end{bmatrix} y,$$

is a left eigenvector of $\mathcal{C}_1(\lambda)$ corresponding to λ.

Proof. We have

$$u^T\mathcal{C}_1(\lambda) = y^T \begin{bmatrix} I_n & (\lambda A_m + A_{m-1}) & \cdots & (\lambda^{m-1}A_m + \lambda^{m-2}A_{m-1} + \cdots + A_1) & C(\lambda E - A)^{-1} \end{bmatrix}$$
$$\left[\begin{array}{cccc|c} \lambda A_m + A_{m-1} & A_{m-2} & \cdots & A_0 & C \\ -I_n & \lambda I_n & \cdots & 0 & \\ \vdots & \ddots & \ddots & \vdots & \\ & & -I_n & \lambda I_n & 0 \\ \hline 0 & & & B & A - \lambda E \end{array}\right]$$
$$= y^T \begin{bmatrix} \lambda A_m + A_{m-1} - (\lambda A_m + A_{m-1}) & \cdots & (\lambda^m A_m + \lambda^{m-1}A_{m-1} + \lambda^{m-2}A_{m-2} + \cdots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B) & C - C \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \cdots & y^T G(\lambda) & 0 \end{bmatrix}.$$

This shows that $u^T\mathcal{C}_1(\lambda) = 0$ if and only if $y^T G(\lambda) = 0$. Hence proved. ☐

This shows that eigenvector of $G(\lambda)$ can be recovered from that of $\mathcal{C}_1(\lambda)$. We now show how to recover right and left eigenvectors of $G(\lambda)$ and $\mathcal{S}(\lambda)$ from those of Fiedler pencils $\mathbb{L}_\sigma(\lambda)$.

3.2 Eigenvector recovery for system matrix

First, we define left and right null spaces of a rational matrix function.

Definition 3.2.1. *Let $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$ be regular. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Then the* right *and the* left nullspaces *of $G(\lambda)$ are defined by*

$$\mathcal{N}_r(G(\lambda)) := \{x \in \mathbb{C}^n : G(\lambda)x = 0\}$$

and

$$\mathcal{N}_l(G(\lambda)) := \{y \in \mathbb{C}^n : y^T G(\lambda) = 0\}$$

respectively.

Definition 3.2.2. *The right and the left nullspaces of a singular $G(\lambda) \in \mathbb{C}(\lambda)^{n\times n}$, denoted by $\mathcal{N}_r(G)$ and $\mathcal{N}_l(G)$, respectively, are given by*

$$\mathcal{N}_r(G) := \{x(\lambda) \in \mathbb{C}^n[\lambda] : G(\lambda)x(\lambda) = 0\}$$

and

$$\mathcal{N}_l(G) := \{y(\lambda) \in \mathbb{C}^n[\lambda] : y(\lambda)^T G(\lambda) = 0\}.$$

Now, we define left inverse of a matrix which will play an important role in recovering eigenvector of $\mathcal{S}(\lambda)$ from its linearizations.

Definition 3.2.3. *Let $E : \mathbb{C}^n \to \mathbb{C}^m$, $m > n$. Then E is said to be* left invertible *if there exists $E_\ell : \mathbb{C}^m \to \mathbb{C}^n$ such that $E_\ell E = I_n$. In such a case, E_ℓ is called a left inverse of E.*

Lemma 3.2.1. *Let $E : \mathbb{C}^n \to \mathbb{C}^m$ be left invertible with left inverse E_ℓ. Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$. If $E : \mathcal{N}(A) \to \mathcal{N}(B)$ is an isomorphism then $E_\ell : \mathcal{N}(B) \to \mathcal{N}(A)$ is an isomorphism.*

Proof. Let $y \in \mathcal{N}(B) \Rightarrow y = Ex$ for some $x \in \mathcal{N}(A) \Rightarrow E_\ell y = E_\ell Ex = x \in \mathcal{N}(A) \Rightarrow E_\ell y \in \mathcal{N}(A)$. This shows that $E_\ell : \mathcal{N}(B) \to \mathcal{N}(A)$. Let $y \in \mathcal{N}(B)$ and suppose that $E_\ell y = 0$. Then $E_\ell y = E_\ell Ex = 0$ for some $x \in \mathcal{N}(A) \Rightarrow x = 0 \Rightarrow y = Ex = 0 \Rightarrow E_\ell \restriction_{\mathcal{N}(B)}$ is one to one. Next, let $x \in \mathcal{N}(A)$. Let $y = Ex$. Then $E_\ell y = E_\ell Ex = x \Rightarrow E_\ell \restriction_{\mathcal{N}(B)}$ is onto. Hence E_ℓ is an isomorphism. □

Recall that $P(\lambda)$ is a matrix polynomial of degree m. Let $L(\lambda)$ be a linearization of $P(\lambda)$. We show that there is an isomorphism between nullspaces of $P(\lambda)$ and $L(\lambda)$.

Theorem 3.2.1. *Suppose that $P(\lambda)$ is regular. Let $L(\lambda)$ be a linearization of $P(\lambda)$, and let $U(\lambda)$ and $V(\lambda)$ be unimodular matrices such that*

$$U(\lambda)L(\lambda)V(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right].$$

(a) Define $E(P) : \mathbb{C}^n \to \mathbb{C}^{nm}$ by $E(P) := V(\lambda)(e_m \otimes I_n)$. Then $E(P)$ is left invertible and $F(P) : \mathbb{C}^{nm} \to \mathbb{C}^n$ defined by $F(P) := (e_m^T \otimes I_n)V(\lambda)^{-1}$ is a left inverse of $E(P)$.

(b) Further the linear map $E(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L(\lambda))$ is an isomorphism and $F(P) : \mathcal{N}_r(L(\lambda)) \to \mathcal{N}_r(P(\lambda))$ is an isomorphism.

Proof. (a) We have $F(P)E(P) = (e_m^T \otimes I_n)V(\lambda)^{-1}V(\lambda)(e_m \otimes I_n) = (e_m^T \otimes I_n)I_{mn}(e_m \otimes I_n) = (e_m^T \otimes I_n)(e_m \otimes I_n) = I_n$. Hence $F(P)$ is a left inverse of $E(P)$.

(b) Let $x \in \mathcal{N}_r(P(\lambda)) \Rightarrow P(\lambda)x = 0$. Now

$$L(\lambda)E(P)x = L(\lambda)V(\lambda)(e_m \otimes I_n)x = U(\lambda)^{-1}U(\lambda)L(\lambda)V(\lambda)(e_m \otimes I_n)x$$
$$= U(\lambda)^{-1}\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)\end{array}\right]\begin{bmatrix} 0 \\ x\end{bmatrix} = U(\lambda)^{-1}\begin{bmatrix} 0 \\ P(\lambda)x\end{bmatrix} = 0.$$

Thus $L(\lambda)E(P)x = 0$ implies that $E(P)x \in \mathcal{N}_r(L(\lambda))$. This shows that $E(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L(\lambda))$. Let $x \in \mathcal{N}_r(P(\lambda))$ and suppose that $E(P)x = 0$. Since $F(P)E(P) = I_n$ we have $F(P)E(P)x = 0 \Rightarrow I_n x = 0 \Rightarrow x = 0$. Hence $E(P)$ is one-one.

Let $y \in \mathcal{N}_r(L(\lambda)) \Rightarrow L(\lambda)y = 0$. Define $u = F(P)y$. We have

$$L(\lambda)y = 0 \Rightarrow L(\lambda)V(\lambda)V(\lambda)^{-1}y = 0 \Rightarrow U(\lambda)^{-1}\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)\end{array}\right]V(\lambda)^{-1}y = 0$$
$$\Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)\end{array}\right]V(\lambda)^{-1}y = 0 \Rightarrow P(\lambda)(e_m^T \otimes I_n)V^{-1}(\lambda)y = 0$$
$$\Rightarrow P(\lambda)F(P)y = 0 \Rightarrow P(\lambda)u = 0.$$

So $u = F(P)y \in \mathcal{N}_r(P(\lambda))$. This shows that $F(P) : \mathcal{N}_r(L(\lambda)) \to \mathcal{N}_r(P(\lambda))$. Let $y \in \mathcal{N}_r(L(\lambda))$ and suppose that $F(P)y = 0$. Then $V(\lambda)^{-1}y = \left[\begin{array}{c} z \\ \hline F(P)y\end{array}\right] = \left[\begin{array}{c} z \\ \hline 0\end{array}\right]$, for some $z \in \mathbb{C}^{(m-1)n}$. Now $L(\lambda)y = 0 \Rightarrow L(\lambda)V(\lambda)V(\lambda)^{-1}y = 0 \Rightarrow U(\lambda)^{-1}\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)\end{array}\right]\begin{bmatrix} z \\ 0\end{bmatrix} = 0 \Rightarrow z = 0 \Rightarrow V(\lambda)^{-1}y = 0 \Rightarrow y = 0$. Hence $F(P)$ is one-one. Since $E(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L(\lambda))$ is one-one and $F(P) : \mathcal{N}_r(L(\lambda)) \to \mathcal{N}_r(P(\lambda))$ is one-one, we have $\dim \mathcal{N}_r(P(\lambda)) \leq \dim \mathcal{N}_r(L(\lambda))$ and $\dim \mathcal{N}_r(L(\lambda)) \leq \dim \mathcal{N}_r(P(\lambda))$. Therefore $\dim \mathcal{N}_r(P(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$. Hence $E(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L(\lambda))$ is an isomorphism and $F(P) : \mathcal{N}_r(L(\lambda)) \to \mathcal{N}_r(P(\lambda))$ is an isomorphism. Hence proved.

□

We mention that $E(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L(\lambda))$ is an isomorphism has also been shown in [9]. Since $F(P)$ is the left inverse of $E(P)$, by Lemma 3.2.1, $F(P) : \mathcal{N}_r(L(\lambda)) \to \mathcal{N}_r(P(\lambda))$ is also an isomorphism.

Theorem 3.2.2. *Suppose that $P(\lambda)$ is regular. Let $L(\lambda)$ be a linearization of $P(\lambda)$, and let $U(\lambda)$ and $V(\lambda)$ be unimodular matrices such that*

$$U(\lambda)L(\lambda)V(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right].$$

(a) Define $H(P) : \mathbb{C}^n \to \mathbb{C}^{nm}$ by $H(P) := (U(\lambda))^T(e_m \otimes I_n)$. Then $H(P)$ is left invertible and $K(P) : \mathbb{C}^{nm} \to \mathbb{C}^n$ defined by $K(P) := (e_m^T \otimes I_n)(U(\lambda)^{-1})^T$ is a left inverse of $H(P)$.

(b) Further the linear map $H(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(L(\lambda))$ is an isomorphism and $K(P) : \mathcal{N}_l(L(\lambda)) \to \mathcal{N}_l(P(\lambda))$ is an isomorphism.

Proof. (a) We have $K(P)H(P) = (e_m^T \otimes I_n)(U(\lambda)^{-1})^T(U(\lambda))^T(e_m \otimes I_n) = (e_m^T \otimes I_n)I_{mn}(e_m \otimes I_n) = (e_m^T \otimes I_n)(e_m \otimes I_n) = I_n$. Hence $K(P)$ is a left inverse of $H(P)$.

(b) Let $y \in \mathcal{N}_l(P(\lambda)) \Rightarrow y^T P(\lambda) = 0$. Now $(H(P)y)^T L(\lambda) = y^T (H(P))^T L(\lambda)$

$$= y^T(e_m \otimes I_n)^T U(\lambda)L(\lambda) = y^T(e_m^T \otimes I_n)U(\lambda)L(\lambda)$$

$$= y^T(e_m^T \otimes I_n)U(\lambda)L(\lambda)V(\lambda)V(\lambda)^{-1} = y^T(e_m^T \otimes I_n)\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right]V(\lambda)^{-1}$$

$$= \begin{bmatrix} 0 & y^T \end{bmatrix}\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right]V(\lambda)^{-1} = \begin{bmatrix} 0 & y^T P(\lambda) \end{bmatrix}V(\lambda)^{-1}$$

$$= \begin{bmatrix} 0 & 0 \end{bmatrix}V(\lambda)^{-1} = 0.$$

Thus $(H(P)y)^T L(\lambda) = 0$ imply that $H(P)y \in \mathcal{N}_l(L(\lambda))$. This shows that $H(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(L(\lambda))$. Let $y \in \mathcal{N}_l(P(\lambda))$ and suppose that $H(P)y = 0$. Since $K(P)H(P) = I_n$ we have $K(P)H(P)y = 0 \Rightarrow I_n y = 0 \Rightarrow y = 0$. Hence $H(P)$ is one-one.

Let $w \in \mathcal{N}_l(L(\lambda)) \Rightarrow w^T L(\lambda) = 0$. Define $u = K(P)w$. We have

$$w^T L(\lambda) = 0 \Rightarrow w^T U(\lambda)^{-1}U(\lambda)L(\lambda)V(\lambda)V(\lambda)^{-1} = 0$$

$$\Rightarrow w^T U(\lambda)^{-1}\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right]V(\lambda)^{-1} = 0$$

$$\Rightarrow w^T U(\lambda)^{-1}\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right] = 0 \Rightarrow w^T U(\lambda)^{-1}(e_m \otimes I_n)P(\lambda) = 0$$

$$\Rightarrow w^T (K(P))^T P(\lambda) = 0 \Rightarrow (K(P)w)^T P(\lambda) = 0 \Rightarrow u^T P(\lambda) = 0.$$

So $u = K(P)w \in \mathcal{N}_l(P(\lambda))$. This shows that $K(P) : \mathcal{N}_l(L(\lambda)) \to \mathcal{N}_l(P(\lambda))$. Let $w \in \mathcal{N}_l(L(\lambda))$ and suppose that $K(P)w = 0$. Then $(U(\lambda)^T)^{-1}w = \left[\begin{array}{c} z \\ \hline K(P)w \end{array}\right] = \left[\begin{array}{c} z \\ \hline 0 \end{array}\right]$, for some $z \in \mathbb{C}^{(m-1)n}$. Now

$$\begin{aligned} w^T L(\lambda) = 0 \Rightarrow (L(\lambda))^T w = 0 \Rightarrow L(\lambda)^T U(\lambda)^T (U(\lambda)^T)^{-1} w = 0 \\ \Rightarrow (V(\lambda)^T)^{-1} \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)^T \end{array}\right] (U(\lambda)^T)^{-1} w = 0 \\ \Rightarrow (V(\lambda)^T)^{-1} \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)^T \end{array}\right] \left[\begin{array}{c} z \\ 0 \end{array}\right] = 0 \Rightarrow z = 0 \end{aligned}$$

So $(U(\lambda)^T)^{-1}w = 0 \Rightarrow w = 0$. Hence $K(P)$ is one-one. Since $H(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(L(\lambda))$ is one-one and $K(P) : \mathcal{N}_l(L(\lambda)) \to \mathcal{N}_l(P(\lambda))$ is one-one, we have $\dim \mathcal{N}_l(P(\lambda)) \leq \dim \mathcal{N}_l(L(\lambda))$ and $\dim \mathcal{N}_l(L(\lambda)) \leq \dim \mathcal{N}_l(P(\lambda))$. Therefore $\dim \mathcal{N}_l(P(\lambda)) = \dim \mathcal{N}_l(L(\lambda))$. Hence $H(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(L(\lambda))$ is an isomorphism and $K(P) : \mathcal{N}_l(L(\lambda)) \to \mathcal{N}_l(P(\lambda))$ is an isomorphism. Hence proved.

$\square$

Similar results hold for singular polynomial. Indeed, we have the following.

Corollary 3.2.3. *Suppose that $P(\lambda)$ is singular. Let $L(\lambda)$ be a linearization of $P(\lambda)$. Let $E(P), F(P)$ and $H(P), K(P)$ be as in Theorem 3.2.1 and Theorem 3.2.2 respectively. Then we have the following:*

(a) The linear map

$$\begin{aligned} E(P) : \mathcal{N}_r(P) \to \mathcal{N}_r(L) \\ x(\lambda) \mapsto E(P)x(\lambda) \end{aligned}$$

is an isomorphism. Further, the linear map

$$\begin{aligned} F(P) : \mathcal{N}_r(L) \to \mathcal{N}_r(P) \\ v(\lambda) \mapsto F(P)v(\lambda) \end{aligned}$$

is an isomorphism.

(b) The linear map

$$H(P) : \mathcal{N}_l(P) \to \mathcal{N}_l(L)$$
$$y(\lambda) \mapsto H(P)y(\lambda)$$

is an isomorphism. Further, the linear map

$$K(P) : \mathcal{N}_l(L) \to \mathcal{N}_l(P)$$
$$w(\lambda) \mapsto K(P)w(\lambda)$$

is an isomorphism.

Remark 3.2.1. *Note that $E(P), F(P), H(P)$ and $K(P)$ depend on λ, i.e., $E(P) := E(P(\lambda))$ and so on. We refer to $F(P)$ as the right eigenvector recovery map and $K(P)$ as the left eigenvector recovery map.*

To further specify the eigenvector recovery maps, we recall the following.

Let σ be a permutation. Suppose that $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$. Recall that

$$s_0 := 0, \qquad s_j := \sum_{p=1}^{j}(c_p + i_p), \text{ for } j = 1, \ldots, l.$$

Note that $s_l = m - 1$. Define

$$m_0 := 0, \qquad m_j := i_1 + i_2 + \cdots + i_j \text{ for } j = 1, \ldots, l.$$

Note that m_l is the total number of inversions in σ. Also recall from (1.23) that $\Lambda_{\sigma,j}(P)$ for $j = 1, \ldots, l$ and $\widehat{\Lambda}_{\sigma,j}(P)$ for $j = 1, \ldots, l-1$ are given by

$$\Lambda_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ P_{m-s_j-1-c_j} \\ \vdots \\ P_{m-s_j-1-2} \\ P_{m-s_j-1-1} \end{bmatrix} \text{ and } \widehat{\Lambda}_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j-1} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ P_{m-s_j-1-c_j} \\ \vdots \\ P_{m-s_j-1-2} \\ P_{m-s_j-1-1} \end{bmatrix} \text{ if } c_1 \geq 1.$$

If $c_1 = 0$, then

$$\Lambda_{\sigma,1}(P) := [\lambda^{i_j} I_n, \ldots, \lambda I_n, I_n]^{\mathcal{B}}, \qquad \widehat{\Lambda}_{\sigma,1}(P) := [\lambda^{i_1-1} I_n, \ldots, \lambda I_n, I_n]^{\mathcal{B}}$$

and $\Lambda_{\sigma,j}(P), \widehat{\Lambda}_{\sigma,j}(P)$ as in (1.23) for $j > 1$, where $P_d(\lambda)$ is the degree d Horner shift of matrix polynomial $P(\lambda)$. For the simplicity, we omit the λ from the Horner shifts $P_d(\lambda)$.

Theorem 3.2.4. *Suppose that $P(\lambda)$ is regular. Let $L_\sigma(\lambda)$ be the linearization of $P(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$. Let $U(\lambda)$ and $V(\lambda)$ be unimodular matrices such that*

$$U(\lambda)L_\sigma(\lambda)V(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right].$$

(a) Define $E_\sigma(P) : \mathbb{C}^n \to \mathbb{C}^{mn}$ by $E_\sigma(P) = V(\lambda)(e_m \otimes I_n)$ and $F_\sigma(P) : \mathbb{C}^{mn} \to \mathbb{C}^n$ by $F_\sigma(P) = (e_m^T \otimes I_n)V(\lambda)^{-1}$. Then $F_\sigma(P)E_\sigma(P) = I_n$.

(b) Further, if CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$. Then

$$E_\sigma(P) = \begin{bmatrix} \lambda^{m_{l-1}}\Lambda_{\sigma,l}(P) \\ \lambda^{m_{l-2}}\widehat{\Lambda}_{\sigma,l-1}(P) \\ \vdots \\ \lambda^{m_1}\widehat{\Lambda}_{\sigma,2}(P) \\ \widehat{\Lambda}_{\sigma,1}(P) \end{bmatrix} \tag{3.6}$$

and

$$F_\sigma(P) = e^T_{(m-c_1)} \otimes I_n. \tag{3.7}$$

(c) Furthermore, let $P_0, \ldots, P_m$ be the Horner shifts of $P(\lambda)$ and $csf(\sigma) = (\boldsymbol{b}_\beta, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_k = (t_{k-1}+1 : t_k)$, for $k = 1, \ldots, \beta$. Then

$$E_\sigma(P) = \begin{bmatrix} B_0\, B_1\, \ldots\, B_{m-1} \end{bmatrix}^{\mathcal{B}} \tag{3.8}$$

where, if $\sigma(i) \in \boldsymbol{b}_k$, for some $k = 1, \ldots, \beta$, then

$$B_i = \begin{cases} \lambda^{k-1} I_n, & \text{if } i = m - t_k - 1 \\ \lambda^{k-1} P_i, & \text{otherwise.} \end{cases} \tag{3.9}$$

Thus $E_\sigma(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L_\sigma(\lambda))$ and $F_\sigma(P) : \mathcal{N}_r(L_\sigma(\lambda)) \to \mathcal{N}_r(P(\lambda))$ are isomorphisms.

Proof. Part (a) follows from Theorem 3.2.1. The equality in (3.6) given in Theorem 1.2.2. Since CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, by Lemma 2.4.4 we have $F_\sigma(P) = e^T_{(m-c_1)} \otimes I_n$. This proves part (b). The equality in (3.8) given in Theorem 1.2.3. $E_\sigma(P) : \mathcal{N}_r(P(\lambda)) \to \mathcal{N}_r(L_\sigma(\lambda))$ and $F_\sigma(P) : \mathcal{N}_r(L_\sigma(\lambda)) \to \mathcal{N}_r(P(\lambda))$ are isomorphisms follow from Theorem 3.2.1. □

The reversal of a bijection plays an important role in showing that there is an isomorphism between the left nullspace of $\mathcal{S}(\lambda)$ and the left nullspace of pencil $\mathbb{L}_\sigma(\lambda)$.

Definition 3.2.4. *[9] The reversal* $rev\,\sigma$ *of a bijection* $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ *is a bijection from* $\{0, 1, \ldots, m-1\}$ *into* $\{1, 2, \ldots, m\}$ *defined by* $rev\,\sigma(i) = m+1-\sigma(i)$ *for* $0 \leq i \leq m-1$*. Equivalently, if* $\sigma = (\sigma(0), \ldots, \sigma(m-1))$ *then* $rev\ \sigma = (\sigma(m-1), \ldots, \sigma(0))$.

Observe that if $\mathbb{M}_\sigma = \mathbb{M}_{\sigma^{-1}(1)}\mathbb{M}_{\sigma^{-1}(2)} \cdots \mathbb{M}_{\sigma^{-1}(m)}$ then

$$\mathbb{M}_{\text{rev}\,\sigma} = \mathbb{M}_{\sigma^{-1}(m)}\mathbb{M}_{\sigma^{-1}(m-1)} \cdots \mathbb{M}_{\sigma^{-1}(1)}. \tag{3.10}$$

For the left nullspace of $P(\lambda)$ and $L_\sigma(\lambda)$, we have the following.

Theorem 3.2.5. *Let* $L_\sigma(P) := \lambda M_m - M_\sigma$ *be the Fiedler linearization of* $P(\lambda)$ *associated with a bijection* σ*. Let* $E_\sigma(P)$ *and* $F_\sigma(P)$ *be as in Theorem 3.2.4. Define* $H_\sigma(P) : \mathbb{C}^n \to \mathbb{C}^{mn}$ *and* $K_\sigma(P) : \mathbb{C}^{mn} \to \mathbb{C}^n$ *by* $H_\sigma(P) = U(\lambda)^T(e_m \otimes I_n)$ *and* $K_\sigma(P) = (e_m^T \otimes I_n)(U(\lambda)^{-1})^T$*. Then* $K_\sigma(P)H_\sigma(P) = I_n$*. Further,* $H_\sigma(P) = E_{rev\,\sigma}(P^T)$ *and* $K_\sigma(P) = F_{rev\,\sigma}(P^T)$*. Furthermore, if* $CISS(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$*, then*

$$K_\sigma(P) = \begin{cases} e_m^T \otimes I_n, & \text{if } c_1 > 0 \\ e_{(m-i_1)}^T \otimes I_n, & \text{if } c_1 = 0. \end{cases}$$

Thus $H_\sigma(P) : \mathcal{N}_l(P(\lambda)) \to \mathcal{N}_l(L_\sigma(\lambda))$ *and* $K_\sigma(P) : \mathcal{N}_l(L_\sigma(\lambda)) \to \mathcal{N}_l(P(\lambda))$ *are isomorphisms.*

Proof. By Theorem 3.2.2, we have $K_\sigma(P)H_\sigma(P) = I_n$. It is easy to see that $L_\sigma(P)^T = \lambda M_m^T - M_\sigma^T = L_{rev\,\sigma}(P^T)$. Since $U(\lambda)L_\sigma(P)V(\lambda) = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda) \end{array}\right]$, we have

$$V(\lambda)^T L_\sigma(P)^T U(\lambda)^T = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & P(\lambda)^T \end{array}\right] = V(\lambda)^T L_{rev\,\sigma}(P^T)U(\lambda)^T.$$

Hence $H_\sigma(P) = U(\lambda)^T(e_m \otimes I_n) = E_{rev\,\sigma}(P^T)$ and $K_\sigma(P) = (e_m^T \otimes I_n)(U(\lambda)^{-1})^T = F_{rev\,\sigma}(P^T)$. Given that $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$. If $c_1 = 0$ then $\text{CISS}(rev\,\sigma) = (i_1, \ldots)$, and if $c_1 > 0$ then $\text{CISS}(rev\,\sigma) = (0, i_1, \ldots)$. Therefore by Theorem 3.2.4, we have

$$K_\sigma(P) = F_{rev\,\sigma}(P) = \begin{cases} e_m^T \otimes I_n, & \text{if } c_1 > 0 \\ e_{(m-i_1)}^T \otimes I_n, & \text{if } c_1 = 0. \end{cases}$$

Note that $\mathcal{N}_l(P(\lambda)) = \mathcal{N}_r(P(\lambda)^T)$ and $\mathcal{N}_l(L_\sigma(P)) = \mathcal{N}_r(L_\sigma(P)^T)$. Since $L_\sigma(P)^T = L_{rev\,\sigma}(P^T)$, we have

$$\mathcal{N}_r(L_\sigma(P)^T) = \mathcal{N}_r(L_{rev\,\sigma}(P^T)).$$

Since $H_\sigma(P)u = E_{rev\,\sigma}(P^T)u$, by Theorem 3.2.4, we have $H_\sigma(P) : \mathcal{N}_r(P(\lambda)^T) \rightarrow \mathcal{N}_r(L_{rev\,\sigma}(P^T))$ is an isomorphism. Hence $H_\sigma(P) : \mathcal{N}_l(P(\lambda)) \rightarrow \mathcal{N}_l(L_\sigma)$ is an isomorphism. Since $K_\sigma(P)$ is a left inverse of $H_\sigma(P)$, so by Lemma 3.2.1, $K_\sigma(P) : \mathcal{N}_l(L_\sigma(P)) \rightarrow \mathcal{N}_l(P(\lambda))$ is an isomorphism. $\square$

Next, we consider the system matrix $\mathcal{S}(\lambda)$ and the associated transfer function $G(\lambda)$. Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection σ. We show that there is an isomorphism between nullspaces of $G(\lambda)$, $\mathcal{S}(\lambda)$, and $\mathbb{L}_\sigma(\lambda)$.

Theorem 3.2.6. *Let $\mathcal{S}(\lambda)$ and $G(\lambda)$ be as in (3.2) and (3.3). Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Define $f : \mathbb{C}^n \rightarrow \mathbb{C}^{n+r}$ by $f(x) = \begin{bmatrix} x \\ (\lambda E - A)^{-1}Bx \end{bmatrix}$ and $g : \mathbb{C}^n \rightarrow \mathbb{C}^{n+r}$ by $g(x) = \begin{bmatrix} x \\ (C(\lambda E - A)^{-1})^T x \end{bmatrix}$. Then the maps $f : \mathcal{N}_r(G(\lambda)) \rightarrow \mathcal{N}_r(\mathcal{S}(\lambda))$ and $g : \mathcal{N}_l(G(\lambda)) \rightarrow \mathcal{N}_l(\mathcal{S}(\lambda))$ are isomorphisms.*

Proof. Let $x \in \mathcal{N}_r(G(\lambda)) \Rightarrow G(\lambda)x = 0$. Now

$$\begin{aligned}\mathcal{S}(\lambda)\begin{bmatrix} x \\ (\lambda E - A)^{-1}Bx \end{bmatrix} &= \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & A - \lambda E \end{array}\right]\begin{bmatrix} x \\ (\lambda E - A)^{-1}Bx \end{bmatrix} \\ &= \begin{bmatrix} G(\lambda)x \\ Bx - Bx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\end{aligned}$$

So $\begin{bmatrix} x \\ (\lambda E - A)^{-1}Bx \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$. This shows that $f : \mathcal{N}_r(G(\lambda)) \rightarrow \mathcal{N}_r(\mathcal{S}(\lambda))$. It is easy to check $f : \mathcal{N}_r(G(\lambda)) \rightarrow \mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism. The proof is similar for g. $\square$

Next, we consider Fiedler linearization of $\mathcal{S}(\lambda)$ and determine isomorphism between $\mathcal{N}_r(\mathcal{S}(\lambda))$ and $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$.

Theorem 3.2.7. *Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler linearization of $\mathcal{S}(\lambda)$ associated with a bijection σ. Let $E_\sigma(P)$ and $F_\sigma(P)$ be as in Theorem 3.2.4. Then $\mathbb{E}_\sigma(\mathcal{S}) : \mathbb{C}^{(n+r)} \rightarrow \mathbb{C}^{nm+r}$ defined by*

$$\mathbb{E}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} E_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right]$$

is left invertible and $\mathbb{F}_\sigma(\mathcal{S}) : \mathbb{C}^{nm+r} \to \mathbb{C}^{n+r}$ *given by*

$$\mathbb{F}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} F_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right]$$

is a left inverse of $\mathbb{E}_\sigma(\mathcal{S})$. *Further,* $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ *is an isomorphism and* $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ *is an isomorphism.*

Proof. We have

$$\mathbb{F}_\sigma(\mathcal{S})\mathbb{E}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} F_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} E_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} F_\sigma(P)E_\sigma(P) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} I_n & \\ \hline & I_r \end{array}\right],$$

since $F_\sigma(P)$ is the left inverse of $E_\sigma(P)$. Hence $\mathbb{F}_\sigma$ is a left inverse of $\mathbb{E}_\sigma$. Let $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$ with $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^r$. Then $\mathcal{S}(\lambda)\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right] \begin{bmatrix} 0 \\ u \\ v \end{bmatrix} = 0$. Since $\mathbb{L}_\sigma(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$, we have

$$\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right].$$

Consequently, we have

$$\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] \begin{bmatrix} 0 \\ u \\ v \end{bmatrix} = 0.$$

Since $\left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right]$ is unimodular, we have

$$\mathbb{L}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] \begin{bmatrix} 0 \\ u \\ v \end{bmatrix} = 0 \Rightarrow \mathbb{L}_\sigma(\lambda) \begin{bmatrix} V(\lambda)\begin{bmatrix} 0 \\ u \end{bmatrix} \\ v \end{bmatrix} = 0$$

$$\Rightarrow \mathbb{L}_\sigma(\lambda) \begin{bmatrix} E_\sigma(P)u \\ v \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} E_\sigma(P)u \\ v \end{bmatrix} \in \mathcal{N}(\mathbb{L}_\sigma(\lambda)).$$

This shows that $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$. Let $w = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$ with $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^r$. Suppose that $\mathbb{E}_\sigma(\mathcal{S})w = 0$. Since $\mathbb{F}_\sigma(\mathcal{S})\mathbb{E}_\sigma(\mathcal{S}) = I_{n+r}$ we have $\mathbb{F}_\sigma(\mathcal{S})\mathbb{E}_\sigma(\mathcal{S})w = 0 \Rightarrow I_{n+r}w = 0$ imply that $w = 0$. Hence $\mathbb{E}_\sigma(\mathcal{S})$ is one-one.

Let $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$. Then $\mathbb{L}_\sigma(\lambda)\begin{bmatrix} u \\ v \end{bmatrix} = 0$. Define $\begin{bmatrix} z \\ w \end{bmatrix} = \mathbb{F}_\sigma(\mathcal{S})\begin{bmatrix} u \\ v \end{bmatrix}$. Then

$$\begin{aligned}
&\mathbb{L}_\sigma(\lambda)\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow \mathbb{L}_\sigma(\lambda)\left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = 0 \\
&\Rightarrow \left[\begin{array}{c|c} U(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = 0 \\
&\Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = 0 \\
&\Rightarrow \mathcal{S}(\lambda)\left[\begin{array}{c|c} (e_m^T \otimes I_n)V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow \mathcal{S}(\lambda)\left[\begin{array}{c|c} F_\sigma(P) & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = 0 \\
&\Rightarrow \mathcal{S}(\lambda)\mathbb{F}_\sigma(\mathcal{S})\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow \mathcal{S}(\lambda)\begin{bmatrix} z \\ w \end{bmatrix} = 0.
\end{aligned}$$

So $\begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$. This shows that $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$. Let $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and suppose that $\mathbb{F}_\sigma(\mathcal{S})\begin{bmatrix} u \\ v \end{bmatrix} = 0$. Then $F_\sigma(P)u = 0$ and $v = 0$. Consequently,

$$\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} V(\lambda)^{-1}u \\ v \end{bmatrix} = \begin{bmatrix} z \\ F_\sigma(P)u \\ v \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}.$$

Now

$$\mathbb{L}_\sigma(\lambda)\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow \left[\begin{array}{c|c} U(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$\Rightarrow \left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right]\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow z = 0.$$

So $\left[\begin{array}{c|c} V(\lambda)^{-1} & \\ \hline & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = 0$. Hence $\mathbb{F}_\sigma(\mathcal{S})$ is one-one. Since $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ is one-one and $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is one-one, we have $\dim \mathcal{N}_r(\mathcal{S}(\lambda)) \leq \dim \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\dim \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \leq \mathcal{N}_r(\mathcal{S}(\lambda))$. Therefore $\dim \mathcal{N}_r(\mathcal{S}(\lambda)) = \dim \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$. Hence $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ are isomorphisms.

□

We refer to $\mathbb{F}_\sigma(\mathcal{S})$ as the eigenvector recovery map for the system matrix $\mathcal{S}(\lambda)$.

The next theorem shows an isomorphism between $\mathcal{N}_l(\mathcal{S}(\lambda))$ and $\mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$.

Theorem 3.2.8. *Let $\mathbb{L}_\sigma(\mathcal{S}) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ be the Fiedler linearization of $\mathcal{S}(\lambda)$. Let $H_\sigma(P)$ and $K_\sigma(P)$ be as in Theorem 3.2.5. Then $\mathbb{H}_\sigma(\mathcal{S}) : \mathbb{C}^{(n+r)} \to \mathbb{C}^{nm+r}$ defined by*

$$\mathbb{H}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} H_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right]$$

is left invertible and $\mathbb{K}_\sigma(\mathcal{S}) : \mathbb{C}^{nm+r} \to \mathbb{C}^{n+r}$ defined by

$$\mathbb{K}_\sigma(\mathcal{S}) := \left[\begin{array}{c|c} K_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right]$$

is a left inverse of $\mathbb{H}_\sigma(\mathcal{S})$. Further, we have $\mathbb{H}_\sigma(\mathcal{S}) = \mathbb{E}_{rev\,\sigma}(\mathcal{S}^T)$ and $\mathbb{K}_\sigma(\mathcal{S}) = \mathbb{F}_{rev\,\sigma}(\mathcal{S}^T)$. Furthermore, $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S}))$ is an isomorphism and $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S})) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ is an isomorphism.

Proof. We have $\mathbb{K}_\sigma(\mathcal{S})\mathbb{H}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} K_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} H_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} K_\sigma(P)H_\sigma(P) & \\ \hline & I_r \end{array}\right]$.
Since $K_\sigma(P)$ is the left inverse of $H_\sigma(P)$, we have

$$\mathbb{K}_\sigma(\mathcal{S})\mathbb{H}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} I_n & \\ \hline & I_r \end{array}\right].$$

Hence $\mathbb{K}_\sigma(\mathcal{S})$ is a left inverse of $\mathbb{H}_\sigma(\mathcal{S})$. Note that $\mathcal{N}_l(\mathcal{S}(\lambda)) = \mathcal{N}_r(\mathcal{S}(\lambda)^T)$ and $\mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S})) = \mathcal{N}_r(\mathbb{L}_\sigma(\mathcal{S})^T)$. It is easy to see that $\mathbb{L}_\sigma(\mathcal{S})^T = \lambda\mathbb{M}_m^T - \mathbb{M}_\sigma^T = \mathbb{L}_{rev\,\sigma}(\mathcal{S}^T)$. Therefore

$$\mathcal{N}_r(\mathbb{L}_\sigma(\mathcal{S})^T) = \mathcal{N}_r(\mathbb{L}_{rev\,\sigma}(\mathcal{S}^T)).$$

Since $\mathbb{H}_\sigma(\mathcal{S})u = \mathbb{E}_{rev\,\sigma}(\mathcal{S}^T)u$, by Theorem 3.2.7, we have $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)^T) \to \mathcal{N}_r(\mathbb{L}_{rev\,\sigma}(\mathcal{S}^T))$ is an isomorphism. Hence we have $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S}))$ is an isomorphism. Since $\mathbb{K}_\sigma(\mathcal{S})$ is a left inverse of $\mathbb{H}_\sigma(\mathcal{S})$, by Lemma 3.2.1, $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S})) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ is also an isomorphism. □

We refer to $\mathbb{K}_\sigma(\mathcal{S})$ as the left eigenvector recovery map for system matrix.

Example 3.1. Let $G(\lambda) = \lambda^6 A_6 + \ldots + A_0 + C(\lambda E - A)^{-1}B$ and $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_6 - \mathbb{M}_0\mathbb{M}_1\mathbb{M}_3\mathbb{M}_5\mathbb{M}_2\mathbb{M}_4 = \lambda\mathbb{M}_6 - \mathbb{M}_\sigma$ be the Fiedler pencil of $G(\lambda)$. Here CISS $(\sigma) = (c_1, i_1, c_2, i_2) = (2,1,1,1)$. So by the Theorem 3.2.4, we have $l = 2$, $s_{l-1} = s_1 = 3$, $m_{l-1} = m_1 = i_1 = 1$, and $m_l = 2$. Thus we have

$$\widehat{\Lambda}_{\sigma,1}(P) = \begin{bmatrix} I_n \\ P_{m-2} \\ P_{m-1} \end{bmatrix}, \qquad \Lambda_{\sigma,2}(P) = \begin{bmatrix} \lambda I_n \\ I_n \\ P_{m-4} \end{bmatrix}$$

and

$$\mathbb{E}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} E_\sigma(P) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} \lambda^{m_1}\Lambda_{\sigma,2}(P) \\ \widehat{\Lambda}_{\sigma,1}(P) \end{bmatrix} & \\ \hline & I_r \end{array}\right]$$
$$= \left[\begin{array}{c|c} \begin{bmatrix} \lambda^2 I_n & \lambda I_n & \lambda P_2(\lambda) & I_n & P_4(\lambda) & P_5(\lambda) \end{bmatrix}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right].$$

Theorem 3.2.9 (Eigenvector recovery from Fiedler pencils)**.** *Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection σ having CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$.*

(a) If $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ *with* $u \in \mathbb{C}^{nm}$ *and* $v \in \mathbb{C}^r$ *then* $x := \begin{bmatrix} (e^T_{(m-c_1)} \otimes I_n)u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$.

(b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *is a basis of* $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$, *and* $\mathbf{v}_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$ *with* $u_j \in \mathbb{C}^{nm}$ *and* $v_j \in \mathbb{C}^r$ *then* $\{x_1, \ldots, x_p\}$ *is a basis of* $\mathcal{N}_r(\mathcal{S}(\lambda))$, *where* $x_j := \begin{bmatrix} (e^T_{(m-c_1)} \otimes I_n)u_j \\ v_j \end{bmatrix}$ *for* $j = 1, \ldots, p$.

(c) Let $\begin{bmatrix} u \\ w \end{bmatrix} \in \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ *with* $u \in \mathbb{C}^{nm}$ *and* $w \in \mathbb{C}^r$. *Define* $y := \begin{bmatrix} (e^T_m \otimes I_n)u \\ w \end{bmatrix}$ *if* $c_1 > 0$, *and* $y := \begin{bmatrix} (e^T_{(m-i_1)} \otimes I_n)u \\ w \end{bmatrix}$ *if* $c_1 = 0$. *Then* $y \in \mathcal{N}_l(\mathcal{S}(\lambda))$.

(d) If $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ *is a basis of* $\mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ *and* $\mathbf{w}_j = \begin{bmatrix} u_j \\ w_j \end{bmatrix}$ *with* $u_j \in \mathbb{C}^{nm}$ *and* $w_j \in \mathbb{C}^r$ *then* $\{y_1, \ldots, y_p\}$ *is a basis of* $\mathcal{N}_l(\mathcal{S}(\lambda))$, *where* $y_j := \begin{bmatrix} (e^T_{(m-i_1)} \otimes I_n)u_j \\ w_j \end{bmatrix}$ *if* $c_1 = 0$, *and* $y_j := \begin{bmatrix} (e^T_m \otimes I_n)u_j \\ w_j \end{bmatrix}$ *if* $c_1 > 0$, *for* $j = 1, \ldots, p$.

Proof. (a) Since $x = \mathbb{F}_\sigma(\mathcal{S}) \begin{bmatrix} u \\ v \end{bmatrix}$ and $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism, it follows that $x \in \mathcal{N}_r(\mathcal{S}(\lambda))$.

(b) Define $\mathcal{B}_r := \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, and $\mathcal{E}_r := \{x_1, \ldots, x_p\}$. Since $\mathbb{F}_\sigma(\mathcal{S})$ is an isomorphism, so $\mathcal{E}_r = \mathbb{F}_\sigma(\mathcal{S})(\mathcal{B}_r)$ is a basis for $\mathcal{N}_r(\mathcal{S}(\lambda))$.

(c) Since $y = \mathbb{K}_\sigma(\mathcal{S}) \begin{bmatrix} u \\ w \end{bmatrix}$ and $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S})) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ is an isomorphism, it follows that $y \in \mathcal{N}_l(\mathcal{S}(\lambda))$.

(d) Define $C_l := \{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$, and $D_l := \{y_1, \ldots, y_p\}$. Since $\mathbb{K}_\sigma(\mathcal{S})$ is an isomorphism, so $D_l = \mathbb{K}_\sigma(\mathcal{S})(C_l)$ is a basis for $\mathcal{N}_l(\mathcal{S}(\lambda))$.

□

Consider the first companion linearization of $\mathcal{S}(\lambda)$ given by

$$\mathcal{C}_1(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{m-1}\mathbb{M}_{m-2}\cdots\mathbb{M}_1\mathbb{M}_0 = \lambda\mathbb{M}_m - \mathbb{M}_\sigma.$$

Then we have CISS$(\sigma) = (c_1, i_1) = (0, m-1)$ for $\mathcal{C}_1(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$. Then $\mathcal{C}_1(\lambda)\begin{bmatrix} u \\ v \end{bmatrix} = 0$ for some nonzero $\begin{bmatrix} u \\ v \end{bmatrix}, u \in \mathbb{C}^{mn}, v \in \mathbb{C}^r$. By Theorem 3.2.9, we have $x = \mathbb{F}_\sigma(\mathcal{S})\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (e_{m-c_1}^T \otimes I_n)u \\ v \end{bmatrix} = \begin{bmatrix} (e_m^T \otimes I_n)u \\ v \end{bmatrix}$ is a right eigenvector of $\mathcal{S}(\lambda)$.

The next result gives an isomorphism between nullspaces of $G(\lambda)$ and a Fiedler linearization of $\mathcal{S}(\lambda)$.

Theorem 3.2.10. *Let $\mathbb{L}_\sigma(\lambda)$ be the linearization of $\mathcal{S}(\lambda)$ associated with a bijection σ. Let $E_\sigma(P)$ be as in Theorem 3.2.4 and $H_\sigma(P)$ be as in Theorem 3.2.5. Let $\mathbb{F}_\sigma(\mathcal{S})$ be as in Theorem 3.2.7 and $\mathbb{K}_\sigma(\mathcal{S})$ be as in Theorem 3.2.8. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $G(\lambda)$.*

(a) Then $\mathbb{E}_\sigma(G) : \mathbb{C}^n \to \mathbb{C}^{nm+r}$ defined by

$$\mathbb{E}_\sigma(G)x = \begin{bmatrix} E_\sigma(P)x \\ (\lambda E - A)^{-1}Bx \end{bmatrix}$$

is left-invertible and $\mathbb{F}_\sigma(G) : \mathbb{C}^{mn+r} \to \mathbb{C}^n$ defined by $\mathbb{F}_\sigma(G) := \Pi_1 \circ \mathbb{F}_\sigma(\mathcal{S})$ is a left inverse of $\mathbb{E}_\sigma(G)$, where $\Pi_1 : \mathbb{C}^{n+r} \to \mathbb{C}^n$ is given by $\Pi_1\left(\begin{bmatrix} x \\ z \end{bmatrix}\right) = x$. Further, $\mathbb{E}_\sigma(G) : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{F}_\sigma(G) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(G(\lambda))$ are isomorphisms.

(b) Define $\mathbb{H}_\sigma(G) : \mathbb{C}^n \to \mathbb{C}^{nm+r}$ by

$$\mathbb{H}_\sigma(G)y = \begin{bmatrix} H_\sigma(P)y \\ (C(\lambda E - A)^{-1})^T y \end{bmatrix}.$$

Then $\mathbb{H}_\sigma(G)$ is left-invertible and $\mathbb{K}_\sigma(G) : \mathbb{C}^{mn+r} \to \mathbb{C}^n$ defined by $\mathbb{K}_\sigma(G) := \Pi_1 \circ \mathbb{K}_\sigma(\mathcal{S})$ is a left inverse of $\mathbb{H}_\sigma(G)$. Further, $\mathbb{H}_\sigma(G) : \mathcal{N}_l(G(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{K}_\sigma(G) : \mathcal{N}_l(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_l(G(\lambda))$ are isomorphisms.

Proof. By Theorem 3.2.6, we have $f : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ and $g : \mathcal{N}_l(G(\lambda)) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ are isomorphisms and by Theorem 3.2.7, we have $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ are isomorphisms. It is easy to see that $\mathbb{E}_\sigma(G) = \mathbb{E}_\sigma(\mathcal{S}) \circ f$ and $\mathbb{H}_\sigma(G) = \mathbb{H}_\sigma(\mathcal{S}) \circ g$. Hence $\mathbb{E}_\sigma(\mathcal{S}) \circ f$ and $\mathbb{H}_\sigma(\mathcal{S}) \circ g$ are isomorphisms. Thus $\mathbb{E}_\sigma(G) : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(G(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$

are isomorphisms. Since $\mathbb{F}_\sigma(\mathcal{S})\mathbb{E}_\sigma(\mathcal{S}) = I_{n+r}$, we have $\mathbb{F}_\sigma(G)\mathbb{E}_\sigma(G) = \Pi_1\mathbb{F}_\sigma(\mathcal{S})\mathbb{E}_\sigma(\mathcal{S})f = \Pi_1 f$. By Theorem 3.2.6, we have $\Pi_1 f = I_n$. Therefore $\mathbb{F}_\sigma(G)\mathbb{E}_\sigma(G) = I_n$. Hence by Lemma 3.2.1, $\mathbb{F}_\sigma(G) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(G(\lambda))$ is an isomorphism. Similarly, $\mathbb{K}_\sigma(G)\mathbb{H}_\sigma(G) = I_n$. Hence by Lemma 3.2.1, $\mathbb{K}_\sigma(G) : \mathcal{N}_l(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_l(G(\lambda))$ is an isomorphism. $\square$

The following examples illustrates the eigenvectors of Fiedler pencils of $G(\lambda)$.

Example 3.2. Consider the transfer function $G(\lambda) = A_4\lambda^4 + \cdots + A_0 + C(\lambda E - A)^{-1}B$, and $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_3 = \lambda\mathbb{M}_4 - \mathbb{M}_\sigma$. The right and left eigenvectors of $\mathbb{L}_\sigma(\lambda)$ are

$$\mathbb{E}_\sigma(G)x = \left[\begin{array}{c} \lambda x \\ \lambda P_1 x \\ x \\ P_3 x \\ \hline (\lambda E - A)^{-1}Bx \end{array}\right], \quad \mathbb{H}_\sigma(G)y = \left[\begin{array}{c} \lambda^2 y \\ \lambda y \\ \lambda P_2^T y \\ y \\ \hline (C(\lambda E - A)^{-1})^T y \end{array}\right],$$

where x and y are right and left eigenvector of $G(\lambda)$.

Example 3.3. Let $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ with $\deg(P(\lambda)) = 13$. Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencil associated with $\sigma = (10 : 12, 9, 8, 6 : 7, 5, 2 : 4, 0 : 1)$. So $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_{13} - \mathbb{M}_\sigma$ and

$$\mathbb{E}_\sigma(G) = \begin{bmatrix} \begin{bmatrix} \lambda^6 I_n & \lambda^6 P_1 & \lambda^6 P_2 & \lambda^5 I_n & \lambda^4 I_n & \lambda^3 I_n & \lambda^3 P_6 & \lambda^2 I_n & \lambda I_n & \lambda P_9 & \lambda P_{10} & I_n & P_{12} \end{bmatrix}^{\mathcal{B}} \\ (\lambda E - A)^{-1}B \end{bmatrix}.$$

Now $csf(\operatorname{rev}\sigma) = (12, 11, 7 : 10, 4 : 6, 3, 1 : 2, 0)$ and

$$\mathbb{H}_\sigma(G) = \begin{bmatrix} \begin{bmatrix} \lambda^6 I_n & \lambda^5 I_n & \lambda^4 I_n & \lambda^4 P_3^T & \lambda^4 P_4^T & \lambda^4 P_5^T & \lambda^3 I_n & \lambda^3 P_7^T & \lambda^3 P_8^T & \lambda^2 I_n & \lambda I_n & \lambda P_{11}^T & I_n \end{bmatrix}^{\mathcal{B}} \\ (C(\lambda E - A)^{-1})^T \end{bmatrix}.$$

Corollary 3.2.11. *Let $\mathbb{L}_\sigma(\lambda)$ be a Fiedler pencil of $G(\lambda)$ associated with a bijection σ having CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of $G(\lambda)$.*

(a) If $\boldsymbol{v} = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$ then $x := (e^T_{(m-c_1)} \otimes I_n)u \in \mathcal{N}_r(G(\lambda))$.

(b) If $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is a basis of $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\boldsymbol{v}_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$ with $u_j \in \mathbb{C}^{nm}$ and $v_j \in \mathbb{C}^r$ then $\{x_1, \ldots, x_p\}$ is a basis of $\mathcal{N}_r(G(\lambda))$, where $x_j := (e^T_{(m-c_1)} \otimes I_n)u_j$ for $j = 1, \ldots, p$.

(c) Let $\boldsymbol{w} = \begin{bmatrix} u \\ w \end{bmatrix} \in \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ *with* $u \in \mathbb{C}^{nm}$ *and* $w \in \mathbb{C}^r$*. Define* $y := (e_m^T \otimes I_n)u$ *if* $c_1 > 0$*, and* $y := (e_{(m-i_1)}^T \otimes I_n)u$ *if* $c_1 = 0$*. Then* $y \in \mathcal{N}_l(G(\lambda))$*.*

(d) If $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_p\}$ *is a basis of* $\mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ *and* $\boldsymbol{w}_j = \begin{bmatrix} u_j \\ w_j \end{bmatrix}$ *with* $u_j \in \mathbb{C}^{nm}$ *and* $w_j \in \mathbb{C}^r$ *then* $\{y_1, \ldots, y_p\}$ *is a basis of* $\mathcal{N}_l(G(\lambda))$*, where* $y_j := (e_{(m-i_1)}^T \otimes I_n)u_j$ *if* $c_1 = 0$*, and* $y_j := (e_m^T \otimes I_n)u_j$ *if* $c_1 > 0$*, for* $j = 1, \ldots, p$*.*

Consider the first and second companion linearization of $G(\lambda)$ given by

$$\mathcal{C}_1(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{m-1}\mathbb{M}_{m-2}\cdots\mathbb{M}_1\mathbb{M}_0 = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$$

and

$$\mathcal{C}_2(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_1\cdots\mathbb{M}_{m-2}\mathbb{M}_{m-1} = \lambda\mathbb{M}_m - \mathbb{M}_{rev\,\sigma}.$$

Then we have $\mathrm{CISS}(\sigma) = (c_1, i_1) = (0, m-1)$ for $\mathcal{C}_1(\lambda)$ and $\mathrm{CISS}(rev\,\sigma) = (c_1, i_1) = (m-1, 0)$ for $\mathcal{C}_2(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Then $\mathcal{C}_1(\lambda)\begin{bmatrix} u \\ v \end{bmatrix} = 0$ for some nonzero $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^{nm+r}$, $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$. By Theorem 3.2.11, we have $x = (e_{m-c_1}^T \otimes I_n)u = (e_m^T \otimes I_n)u$ is a right eigenvector of $G(\lambda)$ which is consistent with (3.5). Let $\begin{bmatrix} u \\ v \end{bmatrix}^T \mathcal{C}_1(\lambda) = 0$ for some nonzero $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^{nm+r}$. Then by Theorem 3.2.11, $y = (e_{m-c_1}^T \otimes I_n)u = (e_{m-m+1}^T \otimes I_n)u = (e_1^T \otimes I_n)u$ is a left eigenvector of $G(\lambda)$ which is consistent with Lemma 3.1.1. For $\mathcal{C}_2(\lambda)$, we have

$$\mathcal{C}_2(\lambda) = \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] - \left[\begin{array}{cccc|c} -A_{m-1} & I_n & & & \\ -A_{m-2} & 0 & \ddots & & \\ \vdots & \ddots & & I_n & \\ -A_0 & \cdots & 0 & 0 & -C \\ \hline -B & & & & -A \end{array}\right].$$

It is easy to see that

$$\left[\begin{array}{c} v \\ \hline w \end{array}\right] = \left[\begin{array}{c} x \\ (\lambda A_m + A_{m-1})x \\ \vdots \\ (\lambda^{m-1}A_m + \lambda^{m-2}A_{m-1} + \cdots + A_1)x \\ \hline (\lambda E - A)^{-1}Bx \end{array}\right] \text{ and } \left[\begin{array}{c} u \\ \hline z \end{array}\right] = \left[\begin{array}{c} \lambda^{m-1}y \\ \lambda^{m-2}y \\ \vdots \\ y \\ \hline (C(\lambda E - A)^{-1})^T y \end{array}\right]$$

are such that $\mathcal{C}_2(\lambda)\left[\begin{array}{c} v \\ \hline w \end{array}\right] = 0$ and $\left[\begin{array}{c} u \\ \hline z \end{array}\right]^T \mathcal{C}_2(\lambda) = 0$. Hence $x = (e_{m-c_1}^T \otimes I_n)v = (e_{m-m+1}^T \otimes I_n)v = (e_1^T \otimes I_n)v$ is a right eigenvector of $G(\lambda)$ and $y = (e_{m-c_1}^T \otimes I_n)u = (e_m^T \otimes I_n)u$ is a left eigenvector of $G(\lambda)$.

Next, we consider singular system matrix and singular transfer function, i.e., $\mathcal{S}(\lambda)$ and $G(\lambda)$ are singular and then analyze isomorphism between $\mathcal{N}(\mathcal{S}(\lambda))$ and $\mathcal{N}(\mathbb{L}_\sigma(\lambda))$, where $\mathbb{L}_\sigma(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$.

Recall that the *right* and *left nullspaces* of the singular system matrix $\mathcal{S}(\lambda)$ are given by

$$\mathcal{N}_r(\mathcal{S}) := \left\{v(\lambda) \in \mathbb{C}^{(n+r)}[\lambda] : \mathcal{S}(\lambda)v(\lambda) = 0\right\},$$
$$\mathcal{N}_l(\mathcal{S}) := \left\{u(\lambda) \in \mathbb{C}^{(n+r)}[\lambda] : u^T(\lambda)\mathcal{S}(\lambda) = 0\right\}.$$

Theorem 3.2.12. *Let $\mathcal{S}(\lambda)$ be singular. Let $\mathbb{L}_\sigma(\lambda)$ be the Fiedler linearization of $\mathcal{S}(\lambda)$ associated with a bijection σ. Let $\mathbb{E}_\sigma(\mathcal{S})$ and $\mathbb{F}_\sigma(\mathcal{S})$ be as in Theorem 3.2.7, and let $\mathbb{H}_\sigma(\mathcal{S})$ and $\mathbb{K}_\sigma(\mathcal{S})$ be as in Theorem 3.2.8.*

(a) The map $\mathbb{E}_\sigma(\mathcal{S}) : \mathbb{C}^{(n+r)}[\lambda] \to \mathbb{C}^{nm+r}[\lambda]$ defined by $v(\lambda) \mapsto \mathbb{E}_\sigma(\mathcal{S})v(\lambda)$ is left invertible and $\mathbb{F}_\sigma(\mathcal{S}) : \mathbb{C}^{nm+r}[\lambda] \to \mathbb{C}^{n+r}[\lambda]$ given by $u(\lambda) \mapsto \mathbb{F}_\sigma(\mathcal{S})u(\lambda)$ is a left inverse of $\mathbb{E}_\sigma(\mathcal{S})$. Further, $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma) \to \mathcal{N}_r(\mathcal{S})$ is an isomorphism and $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma) \to \mathcal{N}_r(\mathcal{S})$ is an isomorphism.

(b) The map $\mathbb{H}_\sigma(\mathcal{S}) : \mathbb{C}^{(n+r)}[\lambda] \to \mathbb{C}^{nm+r}[\lambda]$ defined by $v(\lambda) \mapsto \mathbb{H}_\sigma(\mathcal{S})v(\lambda)$ is left invertible and $\mathbb{K}_\sigma(\mathcal{S}) : \mathbb{C}^{nm+r}[\lambda] \to \mathbb{C}^{n+r}[\lambda]$ given by $u(\lambda) \mapsto \mathbb{K}_\sigma(\mathcal{S})u(\lambda)$ is a left inverse of $\mathbb{H}_\sigma(\mathcal{S})$. Further, $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma) \to \mathcal{N}_l(\mathcal{S})$ is an isomorphism and $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma) \to \mathcal{N}_l(\mathcal{S})$ is an isomorphism.

Similar results hold for singular transfer function $G(\lambda)$.

Next, we consider an LTI state space system and associated system matrix. We wish to derive a linearized state space system of the given LTI system. Also, we consider a higher order state space system and derive a linearized state space system.

3.3 Linearized state-space system

Consider the time invariant state space system Σ given in (3.1). Recall that the Rosenbrock system matrix associated with Σ is given by

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & -(\lambda E - A) \end{array}\right] \in \mathbb{C}^{(n+r)\times(n+r)},$$

and the transfer function is given by $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$. The system Σ can be written as

$$\mathcal{S}(\lambda)\begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}.$$

The nonzero vector u for which $G(\lambda)u = 0$ is called a *zero direction* of Σ. The nonzero vector $\begin{bmatrix} u \\ x \end{bmatrix}$ such that $\mathcal{S}(\lambda)\begin{bmatrix} u \\ x \end{bmatrix} = 0$ is called an *invariant direction* of Σ, where u is referred as *input zero direction* and x is referred to as *state zero direction* associated with λ [40]. We have discussed recovery of zero directions of Σ from eigenvector recovery of $\mathbb{L}_\sigma(\lambda)$. Now, we derive the linearized state-space system.

Theorem 3.3.1. *Let Σ be the LTI state-space system given in (3.1) and $\mathcal{S}(\lambda)$ be the associated system matrix polynomial. Let $E_\sigma(P)$ and $K_\sigma(P)$ be as in Theorem 3.2.4 and Theorem 3.2.5 respectively. Set $\mathcal{U} = E_\sigma(P)u$. If $\mathbb{L}_\sigma(\lambda)$ is the Fiedler linearization of $\mathcal{S}(\lambda)$ associated with the bijection σ, then the state-space system $\widehat{\Sigma}$ associated with $\mathbb{L}_\sigma(\lambda)$ is given by*

$$\begin{aligned} \lambda E x &= Ax + BF_\sigma(P)\mathcal{U} \\ y &= Cx + K_\sigma(P)L_\sigma(\lambda)\mathcal{U}. \end{aligned} \tag{3.11}$$

Further, if CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, then the state space system (3.11) is given by

$$\left.\begin{aligned} \lambda E x &= Ax + (e_{m-c_1}^T \otimes B)\mathcal{U} \\ y &= (e_m \otimes C)x + L_\sigma(\lambda)\mathcal{U}, \end{aligned}\right. \quad \text{if } c_1 > 0$$

and

$$\left.\begin{aligned} \lambda E x &= Ax + (e_m^T \otimes B)\mathcal{U} \\ y &= (e_{m-i_1} \otimes C)x + L_\sigma(\lambda)\mathcal{U} \end{aligned}\right. \quad \text{if } c_1 = 0.$$

Proof. We have

$$\left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & -(\lambda E - A) \end{array}\right]\begin{bmatrix} u \\ x \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}. \tag{3.12}$$

Considering the extended system matrix, we have

$$\left[\begin{array}{c|cc} I_{(m-1)n} & & \\ \hline & P(\lambda) & C \\ & B & A-\lambda E \end{array}\right]\begin{bmatrix} 0 \\ u \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right]\mathbb{L}_\sigma(\lambda)\left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right]\begin{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} \\ x \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbb{L}_\sigma(\lambda)\begin{bmatrix} V(\lambda)(e_m \otimes I_n)u \\ x \end{bmatrix} = \begin{bmatrix} U(\lambda)^{-1} & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} (e_m \otimes I_n)y \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbb{L}_\sigma(\lambda)\begin{bmatrix} E_\sigma(P)u \\ x \end{bmatrix} = \begin{bmatrix} K(P)^T y \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{c|c} L_\sigma(\lambda) & K_\sigma(P)^T C \\ \hline BF_\sigma(P) & A-\lambda E \end{array}\right]\begin{bmatrix} E_\sigma(P)u \\ x \end{bmatrix} = \begin{bmatrix} K_\sigma(P)^T y \\ 0 \end{bmatrix}.$$

This gives

$$\begin{aligned} (A-\lambda E)x + BF_\sigma(P)(E_\sigma(P)u) &= 0 \\ K_\sigma(P)^T Cx + L_\sigma(\lambda)(E_\sigma(P)u) &= K(P)^T y. \end{aligned} \tag{3.13}$$

Since $K_\sigma(P)K_\sigma(P)^T = I_n$, the state-space system associated with $\mathbb{L}_\sigma(\lambda)$ is given by

$$\begin{aligned} (A-\lambda E)x + F_\sigma(P)B\mathcal{U} &= 0 \\ Cx + K_\sigma(P)L_\sigma(\lambda)\mathcal{U} &= y, \end{aligned} \tag{3.14}$$

where $\mathcal{U} = E_\sigma(P)u$. Hence

$$\begin{aligned} \lambda Ex &= Ax + BF_\sigma(P)\mathcal{U} \\ y &= Cx + K_\sigma(P)L_\sigma(\lambda)\mathcal{U} \end{aligned}$$

is the required state-space system associated with $\mathbb{L}_\sigma(\lambda)$. Since $\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$, by Theorem 3.2.4 and Theorem 3.2.5, we have $F_\sigma(P) = e^T_{(m-c_1)} \otimes I_n$ and

$$K_\sigma(P) = \begin{cases} e^T_m \otimes I_n, & \text{if } c_1 > 0 \\ e^T_{(m-i_1)} \otimes I_n, & \text{if } c_1 = 0. \end{cases}$$

Hence the state space system associated with (3.11) is given by

$$\begin{aligned} \lambda E x &= Ax + (e^T_{m-c_1} \otimes B)\mathcal{U} \\ y &= (e_m \otimes C)x + L_\sigma(\lambda)\mathcal{U}, \end{aligned} \quad \text{if } c_1 > 0$$

and

$$\begin{aligned} \lambda E x &= Ax + (e^T_m \otimes B)\mathcal{U} \\ y &= (e_{m-i_1} \otimes C)x + L_\sigma(\lambda)\mathcal{U} \end{aligned} \quad \text{if } c_1 = 0,$$

where $\mathcal{U} = E_\sigma(P)u$. □

Consider the system $\widehat{\Sigma}$ given in Theorem 3.3.1. Then the associated transfer function $\mathbb{G}(\lambda)$ is given by

$$\mathbb{G}(\lambda) = L_\sigma(\lambda) + (e_m \otimes C)(\lambda E - A)^{-1}(e^T_{m-c_1} \otimes B) \;\; \text{if } c_1 > 0$$

and

$$\mathbb{G}(\lambda) = L_\sigma(\lambda) + (e_{m-i_1} \otimes C)(\lambda E - A)^{-1}(e^T_m \otimes B) \;\; \text{if } c_1 = 0.$$

Theorem 3.3.2. *Let Σ and $\widehat{\Sigma}$ be as in Theorem 3.3.1. Let $G(\lambda)$ and $\mathbb{G}(\lambda)$ be associated transfer functions of Σ and $\widehat{\Sigma}$ respectively. Then we have the following.*

(a) The system Σ is observable if and only if the system $\widehat{\Sigma}$ is observable.

(b) The system Σ is controllable if and only if the system $\widehat{\Sigma}$ is controllable.

(c) The transfer function $\mathbb{G}(\lambda)$ is unimodularly equivalent to $\begin{bmatrix} I_{(m-1)n} & \\ & G(\lambda) \end{bmatrix}$.

(d) Let $\lambda \in \mathbb{C}$. Then λ is an input (output, input-output) decoupling zero of Σ if and only if λ is an input (output, input-output) decoupling zero of $\widehat{\Sigma}$.

Proof. Since $\text{rank}[\lambda E - A,\ B] = \text{rank}[\lambda E - A,\ BF_\sigma(P)]$ and

$$\text{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = \text{rank}\begin{bmatrix} \lambda E - A \\ K_\sigma(P)^T C \end{bmatrix}$$

for all $\lambda \in \mathbb{C}$, the desired results in (a), (b) and (d) follow. The result in (c) is immediate. □

Next, we consider a higher order state space system and derive linearized state space systems which are strict system equivalent to the higher order system.

3.3.1 Higher order state space system

Now, we consider a higher order LTI system. We wish to compute zeros of higher order systems. Let $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j \in \mathbb{C}^{n\times n}[\lambda]$ be regular. Consider the *higher order LTI state space system* Σ_1 given by

$$\begin{aligned} P\left(\frac{d}{dt}\right) x(t) &= Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3.15}$$

where $D \in \mathbb{C}^{r\times r}, C \in \mathbb{C}^{r\times n}, B \in \mathbb{C}^{n\times r}$. The associated system matrix is given by

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} -P(\lambda) & B \\ \hline C & D \end{array}\right] \in \mathbb{C}^{(n+r)\times(n+r)} \tag{3.16}$$

and the transfer function of Σ_1 is given by

$$G(\lambda) = CP(\lambda)^{-1}B + D \in \mathbb{C}^{r\times r}. \tag{3.17}$$

Now, we define system linearization of $S(\lambda)$.

Definition 3.3.1 (System linearization)**.** *A pencil of the form* $\mathbb{L}(\lambda) = \left[\begin{array}{c|c} L(\lambda) & X \\ \hline Y & D \end{array}\right] \in \mathbb{C}^{(nm+r)\times(nm+r)}$*, where* $L(\lambda)$ *is an* $nm\times nm$ *matrix pencil, is called a system linearization of* $\mathcal{S}(\lambda)$ *given in (3.16) if* $\mathcal{S}(\lambda)$ *is strict system equivalent to* $\mathbb{L}(\lambda)$*.*

Now, we define the Fiedler pencil of $\mathcal{S}(\lambda)$ given in (3.16). Define $(nm+r)\times(nm+r)$ matrices

$$\mathbb{M}_0 := \left[\begin{array}{c|c} M_0 & -e_m \otimes B \\ \hline -e_m^T \otimes C & -D \end{array}\right], \quad \mathbb{M}_m := \left[\begin{array}{c|c} M_m & 0 \\ \hline 0 & 0_r \end{array}\right] \tag{3.18}$$

and

$$\mathbb{M}_i := \left[\begin{array}{c|c} M_i & 0 \\ \hline 0 & I_r \end{array}\right] \qquad i = 1, \ldots, m-1, \tag{3.19}$$

where

$$M_m := \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \tag{3.20}$$

and

$$M_i := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, i = 1, \ldots, m-1. \tag{3.21}$$

are the Fiedler matrices associated with $P(\lambda)$. The Fiedler matrices $\mathbb{M}_i$ associated with $\mathcal{S}(\lambda)$ have the following properties.

- The matrix $\mathbb{M}_m$ is always singular.
- Since $M_iM_j = M_jM_i$ for $|i-j| > 1$ so it follows that

$$\mathbb{M}_i\mathbb{M}_j = \mathbb{M}_j\mathbb{M}_i \text{ for } |i-j| > 1, \text{ except for } \mathbb{M}_m \text{ and } \mathbb{M}_0. \tag{3.22}$$

For any bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$, we define the pencil

$$\mathbb{S}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma.$$

We refer to $\mathbb{S}_\sigma(\lambda)$ as a Fiedler pencil of $\mathcal{S}(\lambda)$.

Theorem 3.3.3. *Let $\mathcal{S}(\lambda)$ be the system matrix given in (3.16). Let $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ be the Fiedler linearization of $P(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ having CISS(σ) $= (c_1, i_1, \ldots, c_l, i_l)$. Then the Fiedler pencil $\mathbb{S}_\sigma(\lambda)$ associated with σ defined by*

$$\mathbb{S}_\sigma(\lambda) := \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_m \otimes B \\ \hline e^T_{m-c_1} \otimes C & D \end{array}\right] \qquad \text{if } c_1 > 0$$

and

$$\mathbb{S}_\sigma(\lambda) := \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_{m-i_1} \otimes B \\ \hline e^T_{m} \otimes C & D \end{array}\right] \qquad \text{if } c_1 = 0$$

is a system linearization of $\mathcal{S}(\lambda)$.

Proof. By Theorem 2.3.1, $\mathbb{S}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$ is the Fiedler pencil of $\mathcal{S}(\lambda)$ associated with σ. Now, we have

$$\left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] = \left[\begin{array}{cc|c} I_{(m-1)n} & & 0_{n\times r} \\ & -P(\lambda) & B \\ \hline 0_{r\times n} & C & D \end{array}\right]. \tag{3.23}$$

Since

$$U(\lambda)(-L_\sigma(\lambda))V(\lambda) = \begin{bmatrix} I_{(m-1)n} & \\ & -P(\lambda) \end{bmatrix}, \tag{3.24}$$

we have

$$\left[\begin{array}{c|c} U(\lambda)(-L_\sigma(\lambda))V(\lambda) & e_m \otimes B \\ \hline e_m^T \otimes C & D \end{array}\right]$$
$$= \left[\begin{array}{c|c} U(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} -L_\sigma(\lambda) & U(\lambda)^{-1}(e_m \otimes B) \\ \hline (e_m^T \otimes C)V(\lambda)^{-1} & D \end{array}\right] \left[\begin{array}{c|c} V(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right]. \tag{3.25}$$

By Lemma 2.4.2 and Lemma 2.4.4 we have

$$U(\lambda)^{-1}(e_m \otimes I_n) = \begin{cases} e_m \otimes I_n, & \text{if } c_1 > 0, \\ e_{m-i_1} \otimes I_n, & \text{if } c_1 = 0, \end{cases}$$

and $(e_m^T \otimes I_n)V(\lambda)^{-1} = e_{m-c_1}^T \otimes I_n$. Hence by (3.25) we have

$$\left[\begin{array}{c|c} -L_\sigma(\lambda) & U(\lambda)^{-1}(e_m \otimes B) \\ \hline (e_m^T \otimes C)V(\lambda)^{-1} & D \end{array}\right] = \mathbb{S}_\sigma(\lambda)$$

and

$$\left[\begin{array}{c|c} I_{(m-1)n} & 0 \\ \hline 0 & \mathcal{S}(\lambda) \end{array}\right] = \left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{S}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right].$$

This shows that $\mathbb{S}_\sigma(\lambda)$ is a system linearization of $S(\lambda)$. $\square$

Example 3.4. Consider the pencil $\mathbb{S}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{m-1}\ldots\mathbb{M}_1\mathbb{M}_0$ of $\mathcal{S}(\lambda)$ given in (3.16). Then CISS$(\sigma) = (0, m-1)$. Since $L_\sigma(\lambda) = \lambda M_m - M_{m-1}\cdots M_1 M_0$ is the first companion form of $P(\lambda)$, by the Theorem 3.3.3 we have

$$\mathbb{S}_\sigma(\lambda) = \left[\begin{array}{c|c} C_1(\lambda) & e_1 \otimes B \\ \hline e_m^T \otimes C & D \end{array}\right]$$

$$= \lambda \left[\begin{array}{cccc|c} A_m & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ \hline & & & & 0_r \end{array}\right] - \left[\begin{array}{cccc|c} -A_{m-1} & -A_{m-2} & \cdots & -A_0 & -B \\ I_n & 0 & \cdots & 0 & \\ & \ddots & & \vdots & \\ & & I_n & 0 & \\ \hline & & & -C & -D \end{array}\right]$$

is a system linearization of $\mathcal{S}(\lambda)$. Therefore we refer to $\mathcal{C}_1(\lambda)$ as *first companion form* of $\mathcal{S}(\lambda)$. ■

As before, we construct low band-width linearization of $\mathcal{S}(\lambda)$.

Theorem 3.3.4. *Let $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ be a pentadiagonal Fiedler pencil of $P(\lambda)$ associated with a bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ having CISS$(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$. Then $\mathbb{S}_\sigma(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_\sigma$ is pentadiagonal Fiedler pencil of $\mathcal{S}(\lambda)$ provided that $c_1 \leq 1$ and $i_1 \leq 1$. In such a case, we have*

$$\mathbb{S}_\sigma(\lambda) := \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_m \otimes B \\ \hline e_{m-1}^T \otimes C & D \end{array}\right] \quad \text{if } c_1 > 0$$

$$\text{and} \quad \mathbb{S}_\sigma(\lambda) := \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_{m-1} \otimes B \\ \hline e_m^T \otimes C & D \end{array}\right] \quad \text{if } c_1 = 0.$$

Proof. If $c_1 > 0$ then $c_1 = 1$ and $e_{m-c_1}^T \otimes C = e_{m-1}^T \otimes C$. So by the Theorem 3.3.3 we have

$$\mathbb{S}_\sigma(\lambda) = \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_m \otimes B \\ \hline e_{m-c_1}^T \otimes C & D \end{array}\right] = \left[\begin{array}{c|c} -L_\sigma(\lambda) & e_m \otimes B \\ \hline e_{m-1}^T \otimes C & D \end{array}\right]$$

is a pentadiagonal, since $L_\sigma(\lambda)$ is pentadiagonal. Similar argument holds for $c_1 = 0$ and $i_1 = 1$. □

Example 3.5. Let $P(\lambda) = \lambda^6 A_6 + \cdots + \lambda A_1 + A_0$ be regular and $G(\lambda) = CP(\lambda)^{-1}B + D$. Consider the pencil $\mathbb{S}_\sigma(\lambda) = \lambda \mathbb{M}_6 - \mathbb{M}_1\mathbb{M}_3\mathbb{M}_5\mathbb{M}_0\mathbb{M}_2\mathbb{M}_4 = \lambda \mathbb{M}_6 - \mathbb{M}_\sigma$. Now

$$\mathbb{M}_\sigma = \left[\begin{array}{cccccc|c} -A_5 & -A_4 & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 & -C \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -B & -D \end{array}\right]$$

shows that $\mathbb{S}_\sigma(\lambda)$ is pentadiagonal. ■

Theorem 3.3.5. *Let Σ_1 be the LTI state-space system given in (3.15) and $\mathcal{S}(\lambda)$ be the associated system matrix in (3.16). Let $E_\sigma(P)$ and $K_\sigma(P)$ be as in Theorem 3.2.4* *and Theorem 3.2.5 respectively. If* $\mathbb{S}_\sigma(\lambda)$ *is the Fiedler pencil of* $\mathcal{S}(\lambda)$ *associated with a bijection* σ*, then the state-space system* Σ_2 *associated with* $\mathbb{S}_\sigma(\lambda)$ *is given by*

$$\begin{aligned} \lambda M_m X &= M_\sigma X + K_\sigma(P)^T B u \\ y &= C F_\sigma(P) X + D u \end{aligned}$$

where $L_\sigma(\lambda) = \lambda M_m - M_\sigma$ *is the Fiedler pencil of* $P(\lambda)$ *associated with the bijection* σ*. The state vector* x *of* Σ_1 *is given by* $x = F_\sigma(P)X$*. Further, if CISS* $(\sigma) = (c_1, i_1, \ldots, c_l, i_l)$*, then* Σ_2 *is given by*

$$\left.\begin{aligned} \lambda M_m X = M_\sigma X + (e_m \otimes B)u, \\ y = (e_{m-c_1}^T \otimes C)X + Du \end{aligned}\right. \quad \text{if } c_1 > 0$$

and

$$\left.\begin{aligned} \lambda M_m X = M_\sigma X + (e_{m-i_1} \otimes B)u, \\ y = (e_m^T \otimes C)X + Du \end{aligned}\right. \quad \text{if } c_1 = 0.$$

Further, the state vector x *of* Σ_1 *is given by* $x = (e_{(m-c_1)}^T \otimes I_n)X$*. Furthermore, the system* Σ_1 *is strict system equivalent to the system* Σ_2*.*

Proof. We have

$$\left[\begin{array}{c|c} -P(\lambda) & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

Considering the extended system, we have

$$\left[\begin{array}{c|cc} I_{(m-1)n} & & \\ \hline & -P(\lambda) & B \\ & C & D \end{array}\right] \begin{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} \\ u \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ y \end{bmatrix}$$

$$= \left[\begin{array}{c|c} U(\lambda) & \\ \hline & I_r \end{array}\right] \mathbb{S}_\sigma(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & I_r \end{array}\right] \begin{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} \\ u \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ y \end{bmatrix}$$

$$\Rightarrow \mathbb{S}_\sigma(\lambda) \begin{bmatrix} V(\lambda)(e_m \otimes I_n)x \\ u \end{bmatrix} = \left[\begin{array}{c|c} U(\lambda)^{-1} & 0 \\ \hline 0 & I_r \end{array}\right] \begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$\Rightarrow \mathbb{S}_\sigma(\lambda) \begin{bmatrix} E_\sigma(P)x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{c|c} -L_\sigma(\lambda) & K_\sigma(P)^T B \\ \hline CF_\sigma(P) & D \end{array}\right] \begin{bmatrix} E_\sigma(P)x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

This gives

$$\begin{aligned} -L_\sigma(\lambda)(E_\sigma(P)x) + K_\sigma(P)^T Bu &= 0 \\ CF_\sigma(P)(E_\sigma(P)x) + Du &= y. \end{aligned}$$

Thus the state-space system associated with $\mathbb{S}_\sigma(\lambda)$ is given by

$$\begin{aligned} -L_\sigma(\lambda)X + K_\sigma(P)^T Bu &= 0 \\ CF_\sigma(P)X + Du &= y. \end{aligned}$$

where $X := E_\sigma(P)x$. Hence

$$\lambda M_m X = M_\sigma X + K_\sigma(P)^T Bu$$

$$y = CF_\sigma(P)X + Du$$

is the required state-space system associated with $\mathbb{S}_\sigma(\lambda)$. Since $F_\sigma(P)E_\sigma(P) = I_n$, we have $F_\sigma(P)X = F_\sigma(P)E_\sigma(P)x = x$. Second part of the proof follows from Theorem 3.3.3. $\square$

Consider the system Σ_2 given in Theorem 3.3.5. Then the associated transfer function $\mathbb{G}(\lambda)$ is given by

$$\mathbb{G}(\lambda) = (e_{m-c_1}^T \otimes C)L_\sigma(\lambda)^{-1}(e_m \otimes B) + D, \qquad \text{if } c_1 > 0$$

and

$$\mathbb{G}(\lambda) = (e_m^T \otimes C)L_\sigma(\lambda)^{-1}(e_{m-i_1} \otimes B) + D \qquad \text{if } c_1 = 0.$$

It is easy to see that $\mathbb{G}(\lambda) = G(\lambda)$, which is consistent with the fact that $\mathcal{S}(\lambda)$ is SSE to $\mathbb{S}_\sigma(\lambda)$.

Next, we consider Fiedler linearization $\mathbb{S}_\sigma(\lambda)$ of the system matrix $\mathcal{S}(\lambda)$ in (3.16) and determine isomorphism between nullspaces of $\mathcal{S}(\lambda)$ and $\mathbb{S}_\sigma(\lambda)$. Recall $\mathbb{E}_\sigma(\mathcal{S}), \mathbb{F}_\sigma(\mathcal{S})$ and $\mathbb{H}_\sigma(\mathcal{S}), \mathbb{K}_\sigma(\mathcal{S})$ defined in Theorem 3.2.7 and Theorem 3.2.8 respectively. Then it is also easy to see that $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{S}_\sigma(\lambda))$ and $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{S}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ are isomorphisms. Furthermore, $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{S}_\sigma(\lambda))$ and $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{S}_\sigma(\lambda)) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ are isomorphisms. The next result gives an isomorphism between nullspaces of $G(\lambda)$ and $\mathcal{S}(\lambda)$.

Theorem 3.3.6. *Let $\mathcal{S}(\lambda)$ and $G(\lambda)$ be as in (3.16) and (3.17). Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Define $f : \mathbb{C}^r \to \mathbb{C}^{n+r}$ by $f(x) = \begin{bmatrix} P(\lambda)^{-1}Bx \\ x \end{bmatrix}$ and $g : \mathbb{C}^r \to \mathbb{C}^{n+r}$ by $g(x) = \begin{bmatrix} CP(\lambda)^{-1}x \\ x \end{bmatrix}$. Then the maps $f : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ and $g : \mathcal{N}_l(G(\lambda)) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ are isomorphisms.*

Proof. Let $x \in \mathcal{N}_r(G(\lambda)) \Rightarrow G(\lambda)x = 0$. Now

$$\mathcal{S}(\lambda)\begin{bmatrix} P(\lambda)^{-1}Bx \\ x \end{bmatrix} = \left[\begin{array}{c|c} -P(\lambda) & B \\ \hline C & D \end{array}\right]\begin{bmatrix} P(\lambda)^{-1}Bx \\ x \end{bmatrix}$$
$$= \begin{bmatrix} -Bx + Bx \\ G(\lambda)x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $\begin{bmatrix} P(\lambda)^{-1}Bx \\ x \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$. This shows that $f : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$. It is easy to check $f : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism. The proof is similar for g. $\square$

Consider $G(\lambda)$ and $S(\lambda)$ given in (3.17) and (3.16) respectively. Then we show that there is an isomorphism between nullspaces of $G(\lambda)$ and a Fiedler linearization of $\mathcal{S}(\lambda)$.

Theorem 3.3.7. *Let $\mathbb{L}_\sigma(\lambda)$ be the linearization of $\mathcal{S}(\lambda)$ associated with a bijection σ. Let $E_\sigma(P)$ be as in Theorem 3.2.4 and $H_\sigma(P)$ be as in Theorem 3.2.5. Suppose that $G(\lambda)$ is given in (3.17) and $\lambda \in \mathbb{C}$ is an eigenvalue of $G(\lambda)$.*

(a) Then $\mathbb{E}_\sigma(G) : \mathbb{C}^r \to \mathbb{C}^{nm+r}$ defined by

$$\mathbb{E}_\sigma(G)x = \begin{bmatrix} E_\sigma(P)P(\lambda)^{-1}Bx \\ x \end{bmatrix}$$

is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$

(b) Define $\mathbb{H}_\sigma(G) : \mathbb{C}^r \to \mathbb{C}^{nm+r}$ by

$$\mathbb{H}_\sigma(G)y = \begin{bmatrix} H_\sigma(P)CP(\lambda)^{-1}y \\ y \end{bmatrix}.$$

Then $\mathbb{H}_\sigma(G)$ is an isomorphism from $\mathcal{N}_l(G(\lambda))$ to $\mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$.

Proof. By Theorem 3.3.6, we have $f : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ and $g : \mathcal{N}_l(G(\lambda)) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ are isomorphisms. Since $\mathbb{E}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ are isomorphisms, then it is easy to see that $\mathbb{E}_\sigma(G) = \mathbb{E}_\sigma(\mathcal{S}) \circ f$ and $\mathbb{H}_\sigma(G) = \mathbb{H}_\sigma(\mathcal{S}) \circ g$. Hence $\mathbb{E}_\sigma(\mathcal{S}) \circ f$ and $\mathbb{H}_\sigma(\mathcal{S}) \circ g$ are isomorphisms. Thus $\mathbb{E}_\sigma(G) : \mathcal{N}_r(G(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$ and $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(G(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ are isomorphisms. $\square$

4 Generalized Fiedler Pencils For System Matrix

We introduce generalized Fiedler (GF) pencil of system matrix and show that GF pencils are also linearizations for system matrix. It is well known that Fiedler pencils of matrix polynomials do not preserve structures such as symmetric or Hermitian structure that may be present in the matrix polynomials. With a view to preserving structures, generalized Fiedler pencils have been proposed in the literature for matrix polynomials. We generalize these results to the case of system matrix. In particular, we construct self-adjoint linearizations of self-adjoint system matrix. We analyze eigenvector recovery of system matrix from GF pencils.

4.1 Generalized Fiedler pencil

Consider the system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right] \tag{4.1}$$

and the transfer function

$$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1} B, \tag{4.2}$$

where $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$. Recall that for any bijection $\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \cdots, m\}$, the Fiedler pencil $\mathbb{L}_\sigma(\lambda) = \lambda \mathbb{M}_m - \mathbb{M}_\sigma$ is a linearization of $\mathcal{S}(\lambda)$. Note that for any subsets σ_1 and σ_2 of $\{1, 2, \cdots, m-1\}$, the pencil $\mathbb{T}(\lambda) = \mathbb{M}_{\sigma_1}^{-1} \mathbb{L}_\sigma(\lambda) \mathbb{M}_{\sigma_2}^{-1}$ is strictly equivalent to $\mathbb{L}_\sigma(\lambda)$ and hence a linearization for $\mathcal{S}(\lambda)$. Note however that $\mathbb{T}(\lambda)$ may not possess a distinctive feature of $\mathbb{L}_\sigma(\lambda)$, namely, that the block entries of $\mathbb{T}(\lambda)$ may not consist of $0, A_i, A, B, C$ and E as is the case for $\mathbb{L}_\sigma(\lambda)$. We now analyze pencils which are strictly equivalent to Fiedler pencils and the block entries of the pencils consist of $0, A_i, A, B, C, E$. We refer to these pencils as Generalized Fiedler (GF) pencil. We also discuss eigenvector recovery from GF pencils.

Recall that the $(nm+r)\times(nm+r)$ Fiedler matrices associated with $\mathcal{S}(\lambda)$ are given by

$$\mathbb{M}_0 := \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right], \qquad \mathbb{M}_m := \left[\begin{array}{c|c} M_m & 0 \\ \hline 0 & -E \end{array}\right] \tag{4.3}$$

and

$$\mathbb{M}_i := \left[\begin{array}{c|c} M_i & 0 \\ \hline 0 & I_r \end{array}\right] \qquad i = 1, \ldots, m-1, \tag{4.4}$$

where $M_i, i = 0 : m$ are Fiedler matrices of $P(\lambda)$. The $\mathbb{M}_i$ matrices given in (4.4) are always invertible for $i = 1, \ldots, m-1$.

Consider the pencil

$$\mathbb{L}_\sigma(\lambda) = (\lambda\mathbb{M}_3 - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_1)\mathbb{M}_1^{-1} := \left[\begin{array}{ccc|c} A_3 & & & \\ & 0 & I_n & \\ & I_n & A_1 & \\ \hline & & & -E \end{array}\right] - \left[\begin{array}{ccc|c} -A_2 & I_n & & \\ I_n & 0 & & \\ & & -A_0 & -C \\ \hline & & -B & -A \end{array}\right].$$

Note that $\mathbb{L}_\sigma(\lambda) = (\lambda\mathbb{M}_3 - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_1)\mathbb{M}_1^{-1}$ is not a Fiedler pencil.

Generalized Fiedler pencils are further classified into two groups, namely, proper generalized Fiedler pencils (PGF pencils) and non-proper generalized Fiedler pencils (NPGF pencils). These are defined as follows. If $\tau : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}$ is an injective then we define

$$\mathbb{M}_\tau := \mathbb{M}_{\tau^{-1}(1)}\mathbb{M}_{\tau^{-1}(2)}\ldots\mathbb{M}_{\tau^{-1}(m)} \qquad \text{and } \mathbb{M}_\emptyset := I_{nm+r}. \tag{4.5}$$

Definition 4.1.1. *Let $\{\tau_0, \tau_1\}$ be a partition of $\{0, 1, \ldots, m\}$ with $k_i = \#(\tau_i)$ for $i = 0, 1$. For any pair of bijections $\omega_i : \tau_i \to \{1, \ldots, k_i\}$, $i = 0, 1$, we set $\omega := (\omega_0, \omega_1)$. Then the pencil $\mathbb{T}_\omega(\lambda) := \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is called a GF pencil of $\mathcal{S}(\lambda)$ associated with ω. Here*

$$\mathbb{M}_{\omega_i} := \widetilde{\mathbb{M}}_{\omega_i^{-1}(1)}\widetilde{\mathbb{M}}_{\omega_i^{-1}(2)}\ldots\widetilde{\mathbb{M}}_{\omega_i^{-1}(k_i)}, \; i = 0, 1,$$

and the factors $\widetilde{\mathbb{M}}_j$ are defined as follows:

(a) For $\mathbb{M}_{\omega_0}$, we have $\widetilde{\mathbb{M}}_m = \mathbb{M}_m^{-1}$ and $\widetilde{\mathbb{M}}_j = \mathbb{M}_j$ for $j \neq m$.

(b) For $\mathbb{M}_{\omega_1}$, we have $\widetilde{\mathbb{M}}_m = \mathbb{M}_m$ and $\widetilde{\mathbb{M}}_j = \mathbb{M}_j^{-1}$ for $j \neq m$.

If $0 \in \tau_0$ and $m \in \tau_1$, then the pencil $\mathbb{T}_\omega(\lambda)$ is said to be a proper generalized Fiedler (PGF) pencil of $\mathcal{S}(\lambda)$ associated with ω.

It is clear that any Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ of $\mathcal{S}(\lambda)$ is a particular case of GF pencil with $\tau_0 = \{0, 1, \ldots, m-1\}, \tau_1 = \{m\}, \omega_0 = \sigma$ and $\omega_1(m) = 1$. Note that non-proper GF (NPGF) pencils are defined only when $\mathbb{M}_m$ and/or $\mathbb{M}_0$ are non singular. Also note that a non PGF pencil $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is not operation-free, that is, formation of $\mathbb{T}_\omega(\lambda)$ requires operations on the matrices A_i, A, B, C, and E unless $E = I_r$. For example, the NPGF pencil $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_3^{-1}\mathbb{M}_1^{-1}\mathbb{M}_2^{-1} - \mathbb{M}_4^{-1}\mathbb{M}_0 = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ of $\mathbb{S}(\lambda)$ is not operation-free. Indeed, we have

$$\mathbb{M}_4^{-1}\mathbb{M}_0 = \left[\begin{array}{cc|c} A_4^{-1} & & \\ & I_{3n} & \\ \hline & & -E^{-1} \end{array}\right] \left[\begin{array}{cccc|c} I_n & & & & \\ & I_n & & & \\ & & I_n & & \\ & & & -A_0 & -C \\ \hline & & & -B & -A \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} A_4^{-1} & & & & \\ & I_n & & & \\ & & I_n & & \\ & & & -A_0 & -C \\ \hline & & & E^{-1}B & E^{-1}A \end{array}\right]$$

which shows that $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_3^{-1}\mathbb{M}_1^{-1}\mathbb{M}_2^{-1} - \mathbb{M}_4^{-1}\mathbb{M}_0$ is not operation-free. Note however that $\mathbb{M}_{\omega_1}$ is operation-free, and $\mathbb{M}_{\omega_0}$ is operation-free when $E = I_r$. Thus a non PGF pencil is operation-free when $E = I_r$.

Remark 4.1.1. *Unless stated otherwise, we consider only GF pencils of the form* $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ *such that* $0 \in \omega_0$.

There are GF pencils of $\mathcal{S}(\lambda)$ which have lower bandwidth than pentadiagonal Fiedler pencil such as block tridiagonal pencil which may be useful for computations.

Example 4.1. Consider $G(\lambda) = A_6\lambda^6 + \cdots + A_0 + C(\lambda E - A)^{-1}B$ and the pencil $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_6 - \mathbb{M}_1\mathbb{M}_3\mathbb{M}_5\mathbb{M}_0\mathbb{M}_2\mathbb{M}_4 = \lambda\mathbb{M}_6 - \mathbb{M}_\sigma$. Define $\mathcal{O} := \mathbb{M}_1\mathbb{M}_3\mathbb{M}_5$ and $\mathcal{E} := \mathbb{M}_2\mathbb{M}_4$. Note that the matrices $\mathcal{O}$ and $\mathcal{E}$ are always nonsingular with block-tridiagonal inverses since $\mathbb{M}_i$ are invertible for $i = 1 : m-1$. Now, $\mathbb{L}_\sigma(\lambda) := \lambda\mathbb{M}_6 - \mathcal{O}\mathbb{M}_0\mathcal{E}$ is strictly equivalent to the pencils $\lambda\mathcal{O}^{-1}\mathbb{M}_6 - \mathbb{M}_0\mathcal{E}$ and $\lambda\mathbb{M}_6\mathcal{E}^{-1} - \mathcal{O}\mathbb{M}_0$. Although $\lambda\mathcal{O}^{-1}\mathbb{M}_6 - \mathbb{M}_0\mathcal{E} =$

$$\lambda\left[\begin{array}{cccccc|c} 0 & I_n & & & & & \\ A_6 & A_5 & & & & & \\ & & 0 & I_n & & & \\ & & I_n & A_3 & & & \\ & & & & 0 & I_n & \\ & & & & I_n & A_1 & \\ \hline & & & & & & -E \end{array}\right] - \left[\begin{array}{cccccc|c} I_n & & & & & & \\ & -A_4 & I_n & & & & \\ & I_n & 0 & & & & \\ & & & -A_2 & I_n & & \\ & & & I_n & 0 & & \\ & & & & & -A_0 & -C \\ \hline & & & & & -B & -A \end{array}\right]$$

is a block tridiagonal pencil, the pencil $\lambda\mathbb{M}_6\mathcal{E}^{-1} - \mathcal{O}\mathbb{M}_0$

$$= \lambda\left[\begin{array}{c|c} M_6 & 0 \\ \hline 0 & -E \end{array}\right]\left[\begin{array}{c|c} (M_2M_4)^{-1} & 0 \\ \hline 0 & I \end{array}\right] - \left[\begin{array}{c|c} M_1M_3M_5 & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} M_0 & -e_6 \otimes C \\ \hline -e_6^T \otimes B & -A \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda M_6 - (M_2M_4)^{-1} & M_1M_3M_5(e_6 \otimes C) \\ \hline e_6^T \otimes B & A - \lambda E \end{array}\right]$$

is not block tridiagonal. Indeed, we have $M_1M_3M_5(e_6 \otimes C) = \begin{bmatrix} e_5 \otimes C \\ 0 \end{bmatrix}$. Consequently, the pencil $\lambda\mathbb{M}_6\mathcal{E}^{-1} - \mathcal{O}\mathbb{M}_0 =$

$$\lambda\left[\begin{array}{cccccc|c} A_6 & & & & & & \\ & 0 & I_n & & & & \\ & I_n & A_4 & & & & \\ & & & 0 & I_n & & \\ & & & I_n & A_2 & & \\ & & & & & I_n & \\ \hline & & & & & & -E \end{array}\right] - \left[\begin{array}{cccccc|c} -A_5 & I_n & & & & & \\ I_n & 0 & & & & & \\ & & -A_3 & I_n & & & \\ & & I_n & 0 & & & \\ & & & & -A_1 & -A_0 & -C \\ & & & & I_n & 0 & 0 \\ \hline & & & & 0 & -B & -A \end{array}\right]$$

is not block tridiagonal unlike in the case of matrix polynomial, see[9]. ■

Theorem 4.1.1. *Let $\mathcal{S}(\lambda)$ be the system matrix in (4.1). Let $\mathbb{M}_{\sigma_1}$ be the product of odd $\mathbb{M}_i$ factors and $\mathbb{M}_{\sigma_2}$ be the product of even $\mathbb{M}_i$ factors for $i \neq 0, m$. Then the PGF pencil $\mathbb{T}(\lambda) = \lambda\mathbb{M}_{\sigma_1}^{-1}\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_{\sigma_2}$ is a block tridiagonal pencil.*

Proof. Let $\sigma_1 = (1,3,5,\ldots)$ and $\sigma_2 = (2,4,6,\ldots)$. Now

$$\lambda \mathbb{M}_{\sigma_1}^{-1}\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c} \lambda M_{\sigma_1}^{-1}M_m & \\ \hline & -\lambda E \end{array}\right] - \left[\begin{array}{c|c} M_0 M_{\sigma_2} & -e_m \otimes C \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right].$$

Since $M_{\sigma_2} = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & I_n \end{array}\right]$, we have $(-e_m^T \otimes B)M_{\sigma_2} = -e_m^T \otimes B$. Thus

$$\lambda \mathbb{M}_{\sigma_1}^{-1}\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c} \lambda M_{\sigma_1}^{-1}M_m - M_0 M_{\sigma_2} & e_m \otimes C \\ \hline e_m^T \otimes B & A - \lambda E \end{array}\right].$$

It is shown in [9] that $\lambda M_{\sigma_1}^{-1}M_m - M_0 M_{\sigma_2}$ is a block tridiagonal pencil. Hence proved. □

Definition 4.1.2. *[6] Let $\{\tau_0, \tau_1\}$ be a partition of $\{0,1,\ldots,m\}$, with $k_i = \#(\tau_i)$ for $i = 0,1$, and let $\omega_i : \tau_i \to \{1,2\ldots,k_i\}, i = 0,1$ be a pair of bijections.*

(a) We say that ω_i has a consecution at q if $\{q,q+1\} \subset \tau_i$ and $\omega_i(q) < \omega_i(q+1)$. We say that ω_i has an inversion at q if $\{q,q+1\} \subset \tau_i$ and $\omega_i(q) > \omega_i(q+1)$.

(b) We say that ω_i has c_q (resp., i_q) consecutions (resp., inversions) at q if ω_i has consecutions (resp., inversions) at $q, q+1, \ldots, q+c_q-1$ (resp., at $q, q+1, \ldots, q+i_q-1$) and it does not have a consecution (resp., inversion) at $q+c_q$ (resp., $q+i_q$).

(c) We say that ω_i has c_j (resp., i_j) final consecutions (resp., inversions) if ω_i has consecutions (resp., inversions) at $m-c_j, m-c_j+1, \ldots, m-2, m-1$ (resp., at $m-i_j, m-i_j+1, \ldots, m-2, m-1$) and it does not have a consecution (resp., inversion) at $m-c_j-1$ (resp., $m-i_j-1$).

Note that the bijection $\omega_i, i = 0,1$ in Definition 4.1.2 has a consecution (resp., an inversion) at q if and only if $\widetilde{\mathbb{M}}_q$ and $\widetilde{\mathbb{M}}_{q+1}$ are both factors of the product defining $\mathbb{M}_{\omega_i}$ and $\widetilde{\mathbb{M}}_q$ is to the left (resp., right) of $\widetilde{\mathbb{M}}_{q+1}$ in $\mathbb{M}_{\omega_i}$. Also note that if $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is a GF pencil, then the commutativity relations given in (2.11) may allow us to change the position of the factors $\widetilde{\mathbb{M}}_q$ in $\mathbb{M}_{\omega_1}$ and $\mathbb{M}_{\omega_0}$. The new position will be related to a pair of bijections $\tau = (\tau_0, \tau_1)$ such that $\tau \neq \omega$ and $\mathbb{T}_\tau(\lambda) = \mathbb{T}_\omega(\lambda)$. However, the use of the commutativity relations can not change the relative positions of $\widetilde{\mathbb{M}}_q$ and $\widetilde{\mathbb{M}}_{q+1}$ and so, for $i = 0,1, \tau_i$ has a consecution (res., inversion) at q if and only if ω_i has a consecution (res., inversion) at q, see [6].

It is well known [6] that for a matrix polynomial $P(\lambda)$ there are more than one Fiedler pencils which are strictly equivalent to a given GF pencil. Like matrix polynomial, there are more than one Fiedler pencil of $\mathcal{S}(\lambda)$ which are strictly equivalent to a given GF pencil. However, all these Fiedler pencils do not preserve consecutions at 0.

Example 4.2. Consider $G(\lambda) = \lambda^5 A_i + \cdots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B$ and the PGF pencil of $\mathcal{S}(\lambda)$ given by

$$\begin{aligned}\mathbb{T}_\omega(\lambda) &= \lambda\mathbb{M}_5\mathbb{M}_3^{-1}\mathbb{M}_1^{-1} - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_4 = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}\\ &= \lambda\mathbb{M}_1^{-1}\mathbb{M}_5\mathbb{M}_3^{-1} - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_4 \text{ by commutativity relation.}\end{aligned}$$

So $\mathbb{T}_\omega(\lambda)$ is strictly equivalent to the Fiedler pencil

$$\mathbb{L}_\sigma(\lambda) = \mathbb{T}_\omega(\lambda)\mathbb{M}_1\mathbb{M}_3 = \lambda\mathbb{M}_5 - \mathbb{M}_0\mathbb{M}_2\mathbb{M}_4\mathbb{M}_1\mathbb{M}_3$$

and is also strictly equivalent to the Fiedler pencil

$$\mathbb{L}_{\sigma'}(\lambda) = \mathbb{M}_1\mathbb{T}_\omega(\lambda)\mathbb{M}_3 = \lambda\mathbb{M}_5 - \mathbb{M}_1\mathbb{M}_0\mathbb{M}_2\mathbb{M}_4\mathbb{M}_3.$$

Now

$$\mathbb{M}_1\mathbb{M}_3 = \left[\begin{array}{ccccc|c} I_n & & & & & \\ & -A_3 & I_n & & & \\ & I_n & 0 & & & \\ & & & -A_1 & I_n & \\ & & & I_n & 0 & \\ \hline & & & & & I_r \end{array}\right] \text{ and } \mathbb{M}_0\mathbb{M}_2\mathbb{M}_4 = \left[\begin{array}{ccccc|c} -A_4 & I_n & & & & \\ I_n & 0 & & & & \\ & & -A_2 & I_n & & \\ & & I_n & 0 & & \\ & & & & -A_0 & -C \\ \hline & & & & -B & -A \end{array}\right].$$

So

$$\mathbb{M}_0\mathbb{M}_2\mathbb{M}_4\mathbb{M}_1\mathbb{M}_3 = \left[\begin{array}{ccccc|c} -A_4 & -A_3 & I_n & & & \\ I_n & 0 & 0 & & & \\ & -A_2 & 0 & -A_1 & I_n & \\ & I_n & 0 & 0 & 0 & \\ & & & -A_0 & 0 & -C \\ \hline & & & -B & 0 & -A \end{array}\right] \text{ and}$$

$$\mathbb{M}_1\mathbb{M}_0\mathbb{M}_2\mathbb{M}_4\mathbb{M}_3 = \left[\begin{array}{ccccc|c} -A_4 & -A_3 & I_n & & & \\ I_n & 0 & 0 & & & \\ & -A_2 & 0 & I_n & & \\ & -A_1 & 0 & 0 & -A_0 & -C \\ & I_n & 0 & 0 & 0 & 0 \\ \hline & & & & -B & -A \end{array}\right].$$

Thus $\mathbb{L}_\sigma(\lambda)$ and $\mathbb{L}_{\sigma'}(\lambda)$ are not same. Note that ω_0 has 0 consecutions at 0. But σ has 1 consecution at 0 and σ' has 0 consecution at 0. Thus $\mathbb{L}_{\sigma'}(\lambda)$ preserves the consecutions at 0 of $\mathbb{M}_{\omega_0}$, but $\mathbb{L}_\sigma(\lambda)$ does not. ■

Now our aim is to prove that any PGF pencil $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ of $\mathcal{S}(\lambda)$ is strictly equivalent to a Fiedler pencil that preserves the consecutions at 0 of ω_0, except in the following particular case:

$$\mathbb{T}_\delta(\lambda) = \lambda\mathbb{M}_m\mathbb{M}_{m-1}^{-1}\ldots\mathbb{M}_{c_0+1}^{-1} - \mathbb{M}_0\mathbb{M}_1\ldots\mathbb{M}_{c_0}, \tag{4.6}$$

where $c_0 \in \{0, 1, \ldots, m-2\}$. In (4.6) the bijections $\delta = (\delta_0, \delta_1)$ is defined by

$$\begin{aligned}(\delta_0^{-1}(1), \delta_0^{-1}(2), \ldots, \delta_0^{-1}(c_0+1)) &= (0, 1, \ldots, c_0)\\ (\delta_1^{-1}(1), \delta_1^{-1}(2), \ldots, \delta_1^{-1}(m-c_0)) &= (m, m-1, \ldots, c_0+1).\end{aligned} \tag{4.7}$$

Note that $\mathbb{M}_0\mathbb{M}_m \neq \mathbb{M}_m\mathbb{M}_0$. Therefore the GF pencils of $\mathcal{S}(\lambda)$ although similar to those of $P(\lambda)$ require a special treatment.

The following result describes PGF pencils which are equivalent to Fiedler pencils that preserve consecutions at 0. We adapt the proof from [6].

Lemma 4.1.1. *Let $\mathcal{S}(\lambda)$ be the system matrix given in (4.1), and $\mathbb{T}_\omega(\lambda)$ be the PGF pencil of $\mathcal{S}(\lambda)$ associated with the bijection $\omega = (\omega_0, \omega_1)$. If ω_0 has c_0 consecutions at 0 and $(\omega_0, \omega_1) \neq (\delta_0, \delta_1)$, where δ_0 and δ_1 are defined in (4.7), then there exist two ordered subsets ξ_1 and ξ_2 of $\{1, 2, \ldots, m-1\}$ such that*

(a) $c_0 \notin \xi_2$ and $c_0 + 1 \notin \xi_2$; and

(b) $\mathbb{L}_\sigma(\lambda) = \mathbb{M}_{\xi_1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\xi_2}$ is a Fiedler pencil of $\mathcal{S}(\lambda)$ associated with a bijection σ that has c_0 consecutions at 0.

Proof. If $\mathbb{T}_\omega(\lambda)$ is a Fiedler pencil of $\mathcal{S}(\lambda)$, then by taking $\xi_1 = \emptyset = \xi_2$, we are done since $\mathbb{M}_{\xi_1} = \mathbb{M}_{\xi_2} = I_{(nm+r)}$. Therefore we assume that $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is not a Fiedler pencil. Since it is given that $\omega_0 : \tau_0 \to \{1, \ldots, k_0\}$ has c_0 consecutions at 0 then by the Definition 4.1.2, we have $\{0, 1, \ldots, c_0\} \subseteq \tau_0$, and $\mathbb{M}_0, \mathbb{M}_1, \ldots, \mathbb{M}_{c_0}$ are the factors of $\mathbb{M}_{\omega_0}$ and appear in $\mathbb{M}_{\omega_0}$ in this order. Now $c_0 \leq m-2$, (otherwise if $c_0 > m-2$, then the pencil would be a Fiedler pencil, since in a Fiedler pencil, $\mathbb{M}_{\omega_0}$ contains $m-1$ number of $\mathbb{M}_i$ factors). Therefore we divide the proof into three cases:

Case I. If $c_0+1 \in \tau_0$ then $\mathbb{M}_{c_0+1}$ must be to the left of $\mathbb{M}_{c_0}$ in $\mathbb{M}_{\omega_0}$, because otherwise ω_0 should have more than c_0 consecutions at 0. So

$$\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\xi_1}^{-1}\mathbb{M}_m\mathbb{M}_{\xi_2}^{-1} - \mathbb{M}_{\omega_0},$$

where $c_0 \notin \xi_2$, $c_0 + 1 \notin \xi_2$, and $\{\xi_1, \xi_2, \tau_0\}$ is a partition of $\{0, 1, \ldots, m-1\}$. Then

$$\mathbb{L}_\sigma(\lambda) = \mathbb{M}_{\xi_1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\xi_2} = \lambda\mathbb{M}_m - \mathbb{M}_{\xi_1}\mathbb{M}_{\omega_0}\mathbb{M}_{\xi_2}$$

is a Fiedler pencil of $\mathcal{S}(\lambda)$ and preserves the consecution at 0 i.e., σ has also c_0 consecutions at 0, since the consecutions are determined only by the factors in $\mathbb{M}_{\omega_0}$.

Case II. If $c_0+1 \in \tau_1$ and $\omega_1(c_0+1) < \omega_1(m)$ then $\mathbb{M}_{c_0+1}^{-1}$ must be to the left of $\mathbb{M}_m$ in $\mathbb{M}_{\omega_1}$ (by the definition of PGF pencil, and $c_0+1 \neq m$, so $\mathbb{M}_{c_0+1}^{-1}$ appears in $\mathbb{M}_{\omega_1}$). Therefore

$$\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_{\xi_1}^{-1}\mathbb{M}_m\mathbb{M}_{\xi_2}^{-1} - \mathbb{M}_{\omega_0}, \tag{4.8}$$

where $c_0+1 \in \xi_1, c_0 \notin \xi_2$, and $c_0+1 \notin \xi_2$ and $\{\xi_1, \xi_2, \tau_0\}$ is a partition of $\{0, 1, \ldots, m-1\}$. Then

$$\mathbb{L}_\sigma(\lambda) = \mathbb{M}_{\xi_1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\xi_2} = \lambda\mathbb{M}_m - \mathbb{M}_{\xi_1}\mathbb{M}_{\omega_0}\mathbb{M}_{\xi_2}$$

is a Fiedler pencil of $\mathcal{S}(\lambda)$. Since consecutions are determined only by the factors in $\mathbb{M}_{\omega_0}$ and the fact that $\mathbb{M}_{c_0+1}$ is a factor of $\mathbb{M}_{\xi_1}$, so σ has an inversion at c_0. Therefore σ has c_0 consecutions at 0.

Case III. If $c_0+1 \in \tau_1$ and $\omega_1(m) < \omega_1(c_0+1)$ then $\mathbb{M}_{c_0+1}^{-1}$ must be to the right of $\mathbb{M}_m$ in $\mathbb{M}_{\omega_1}$ (by the definition of PGF pencil, and $c_0+1 \neq m$, $\mathbb{M}_{c_0+1}^{-1}$ appears in $\mathbb{M}_{\omega_1}$). Therefore

$$\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_{\xi_1'}^{-1}\mathbb{M}_m\mathbb{M}_{\xi_2'}^{-1} - \mathbb{M}_{\omega_0},$$

where $c_0+1 \in \xi_2', c_0 \notin \xi_2'$, and $\{\xi_1', \xi_2', \tau_0\}$ is a partition of $\{0, 1, \ldots, m-1\}$.

Our aim is to rearrange the order of the factors defining $\mathbb{M}_{\omega_1}$ and shift $\mathbb{M}_{c_0+1}^{-1}$ to the left of $\mathbb{M}_m$ using the commutativity relation. For this purpose let $d \geq 0$ be the integer such that ω_1 has inversion at $c_0+1, c_0+2, \ldots, c_0+d$. Note that $c_0+1+d < m$, (otherwise if $c_0+1+d \geq m$ then the pencil $\mathbb{T}_\omega(\lambda)$ must be of the form

$$\lambda\mathbb{M}_m\mathbb{M}_{m-1}^{-1}\ldots\mathbb{M}_{c_0+1}^{-1} - \mathbb{M}_0\mathbb{M}_1\ldots\mathbb{M}_{c_0}, \tag{4.9}$$

which is not possible, as $(\omega_0, \omega_1) \neq (\delta_0, \delta_1)$). So by commutativity relations we can shift the factors $\mathbb{M}_{c_0+1}^{-1}, \mathbb{M}_{c_0+2}^{-1}, \ldots, \mathbb{M}_{c_0+d}^{-1}$ in $\mathbb{M}_{\omega_1}$ to the left and set

$$\mathbb{M}_d^{-1} := \mathbb{M}_{c_0+1+d}^{-1}\mathbb{M}_{c_0+d}^{-1}\ldots\mathbb{M}_{c_0+2}^{-1}\mathbb{M}_{c_0+1}^{-1}.$$

In this shifting process it may happen that $\mathbb{M}_{c_0+1}^{-1}$ is to the left of $\mathbb{M}_m$ and then

$$\mathbb{M}_{\omega_1} = \ldots\mathbb{M}_d^{-1}\ldots\mathbb{M}_m\ldots, \tag{4.10}$$

or that $\mathbb{M}_{c_0+1}^{-1}$ goes to the right of $\mathbb{M}_m$ and then

$$\mathbb{M}_{\omega_1} = \ldots\mathbb{M}_m\ldots\mathbb{M}_d^{-1}\ldots. \tag{4.11}$$

Again from (4.11) we have $c_0+1+d < m-1$ (otherwise the pencil must be of the form (4.9), which is not possible) and $\mathbb{M}_{c_0+2+d}^{-1}$ is not between $\mathbb{M}_m$ and $\mathbb{M}_d^{-1}$ (since ω_1 has no inversion at c_0+1+d). So by commutativity relation, (4.11) reduces to (4.10). By (4.10), we have $\omega_1(c_0+1) < \omega_1(m)$. Hence the result follows from case II. $\square$

Theorem 4.1.2. *Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be a GF pencil of $\mathcal{S}(\lambda)$. If $\mathbb{T}_\omega(\lambda)$ is a PGF pencil then it is a linearization of $\mathcal{S}(\lambda)$. If $\mathbb{T}_\omega(\lambda)$ is a non PGF pencil then*

$$\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right] = \left[\begin{array}{c|c} U(\lambda) & \\ \hline & X \end{array}\right] \mathbb{T}_\omega(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & Y \end{array}\right],$$

where $X = -E$ and $Y = I_r$ or $X = I_r$ and $Y = -E$. Further, if $E = I_r$, then the operation-free NPGF pencil $\mathbb{T}_\omega(\lambda)$ is also a linearization of $\mathcal{S}(\lambda)$. Furthermore, If $G(\lambda)$ is minimal, then $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is also a linearization of $G(\lambda)$.

Proof. Case I : Suppose that $\mathbb{T}_\omega(\lambda)$ is a PGF pencil of $\mathcal{S}(\lambda)$. Then $0 \in \omega_0$ and $m \in \omega_1$. By definition of PGF pencil, we have $\mathbb{M}_{\omega_1} = \mathbb{M}_{\tau_1}^{-1}\mathbb{M}_m\mathbb{M}_{\tau_2}^{-1}$. Hence $\mathbb{M}_{\tau_1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\tau_2} = \lambda\mathbb{M}_m - \mathbb{M}_{\tau_1}\mathbb{M}_{\omega_0}\mathbb{M}_{\tau_2} =: \mathbb{L}_\sigma(\lambda)$ is a Fiedler pencil, where $\mathbb{M}_{\tau_i} = \left[\begin{array}{c|c} M_{\tau_i} & \\ \hline & I_r \end{array}\right], i = 1, 2.$

Since $\mathbb{M}_{\tau_i} = \left[\begin{array}{c|c} M_{\tau_i} & \\ \hline & I_r \end{array}\right]$ and $\mathbb{L}_\sigma(\lambda)$ is a Fiedler pencil of $\mathcal{S}(\lambda)$, it follows that $\mathbb{T}_\omega(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$. Therefore every PGF of $\mathcal{S}(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$.

Case II : Suppose that $\mathbb{T}_\omega(\lambda)$ is a non PGF. Note that in this case $\mathbb{M}_m$ is nonsingular. Since we are considering NPGF pencils where $0, m \in \omega_0$, we have $\mathbb{M}_{\omega_0} = \mathbb{M}_{\sigma_1}\mathbb{M}_m^{-1}\mathbb{M}_{\sigma_2}$ with $0 \in \sigma_1$ or $0 \in \sigma_2$. Now if $0 \in \sigma_1$, then $\mathbb{T}_\omega(\lambda)\mathbb{M}_{\sigma_2}^{-1}\mathbb{M}_m = \lambda\mathbb{M}_{\omega_1}\mathbb{M}_{\sigma_2}^{-1}\mathbb{M}_m - \mathbb{M}_{\sigma_1}$ is a PGF pencil. Hence $\mathbb{T}_\omega(\lambda)\left[\begin{array}{c|c} M_{\sigma_2}^{-1}M_m & \\ \hline & -E \end{array}\right]$ is a PGF pencil. If $0 \in \sigma_2$ then $\mathbb{M}_m\mathbb{M}_{\sigma_1}^{-1}\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_m\mathbb{M}_{\sigma_1}^{-1}\mathbb{M}_{\omega_1} - \mathbb{M}_{\sigma_2}$ is a PGF pencil. Hence $\left[\begin{array}{c|c} M_mM_{\sigma_1}^{-1} & \\ \hline & -E \end{array}\right]\mathbb{T}_\omega(\lambda)$ is a PGF, where E is nonsingular. Consequently, we have

$$\left[\begin{array}{c|c} I_{(m-1)n} & \\ \hline & \mathcal{S}(\lambda) \end{array}\right] = \left[\begin{array}{c|c} U(\lambda) & \\ \hline & X \end{array}\right] \mathbb{T}_\omega(\lambda) \left[\begin{array}{c|c} V(\lambda) & \\ \hline & Y \end{array}\right],$$

where $X = -E$ and $Y = I_r$ or $X = I_r$ and $Y = -E$. So GF pencils are also linearizations of $\mathcal{S}(\lambda)$ when $E = I_r$. □

4.2 Self-adjoint PGF pencil

We define adjoint of $\mathcal{S}(\lambda)$ by

$$\mathcal{S}^*(\lambda) = \left[\begin{array}{c|c} P^*(\lambda) & B^* \\ \hline C^* & (A^* - \lambda E^*) \end{array}\right],$$

where $P^*(\lambda) = \sum_{j=1}^{m} \lambda^j A_j^*$. The adjoint of the associated transfer function $G(\lambda)$ is given by $G^*(\lambda) = P^*(\lambda) + B^*(\lambda E^* - A^*)^{-1}C^*$. A transfer function $G(\lambda)$ is said to be self-adjoint if $G^*(\lambda) = G(\lambda)$. Note that $\mathcal{S}^*(\lambda) = \mathcal{S}(\lambda) \Leftrightarrow G^*(\lambda) = G(\lambda)$.

The next result analyzes the existance of self-adjoint linearizations of self-adjoint system matrix.

Theorem 4.2.1. *Let $\mathcal{S}(\lambda)$ be a self-adjoint/ symmetric system matrix. If m is odd, then*

$$\mathbb{L}(\lambda) = \lambda \mathbb{M}_m \mathbb{M}_{m-2}^{-1} \cdots \mathbb{M}_3^{-1}\mathbb{M}_1^{-1} - \mathbb{M}_0\mathbb{M}_2 \cdots \mathbb{M}_{m-3}\mathbb{M}_{m-1}$$

is a self-adjoint/ symmetric PGF pencil of $\mathcal{S}(\lambda)$. Hence $\mathbb{L}(\lambda)$ is a self-adjoint/ symmetric linearization of $\mathcal{S}(\lambda)$. In particular, if $G(\lambda)$ is minimal then $\mathbb{L}(\lambda)$ is a self-adjoint/ symmetric linearization of $G(\lambda)$.

Proof. Recall that the matrices $\mathbb{M}_i$ and $\mathbb{M}_j$ commute for $|i-j| > 1$ except for $\mathbb{M}_0$ and $\mathbb{M}_m$. Suppose that m is odd. Define $\mathcal{O} := \mathbb{M}_1\mathbb{M}_3 \ldots \mathbb{M}_{m-2}$ and $\mathcal{E} := \mathbb{M}_2\mathbb{M}_4 \cdots \mathbb{M}_{m-1}$. Consider the pencil

$$\mathbb{L}(\lambda) = \lambda \mathbb{M}_m \mathcal{O}^{-1} - \mathbb{M}_0 \mathcal{E}.$$

Suppose that $\mathcal{S}(\lambda)$ is self-adjoint. Then we have $A_i^* = A_i, C = B^*, A = A^*,$ and $E = E^*$. So $\mathbb{M}_i^* = \mathbb{M}_i$ for $i = 0, 1, \ldots, m$. Now

$$\begin{aligned}(\mathbb{M}_m \mathcal{O}^{-1})^* &= (\mathbb{M}_m \ldots \mathbb{M}_5^{-1}\mathbb{M}_3^{-1}\mathbb{M}_1^{-1})^* = (\mathbb{M}_1^{-1})^*(\mathbb{M}_3^{-1})^*(\mathbb{M}_5^{-1})^* \ldots (\mathbb{M}_m)^* \\ &= \mathbb{M}_1^{-1}\mathbb{M}_3^{-1}\mathbb{M}_5^{-1} \ldots \mathbb{M}_m \\ &= \mathbb{M}_m \ldots \mathbb{M}_5^{-1}\mathbb{M}_3^{-1}\mathbb{M}_1^{-1} \text{ by commutativity property given in (2.12)} \\ &= \mathbb{M}_m \mathcal{O}^{-1}.\end{aligned}$$

Similarly, again by commutativity relation we have

$$(\mathbb{M}_0\mathcal{E})^* = \ldots \mathbb{M}_4^*\mathbb{M}_2^*\mathbb{M}_0^* = \ldots \mathbb{M}_4\mathbb{M}_2\mathbb{M}_0 = \mathbb{M}_0\mathbb{M}_2\mathbb{M}_4 \ldots = \mathbb{M}_0\mathcal{E}.$$

Hence $\mathbb{L}(\lambda)^* = \mathbb{L}(\lambda)$. The proof is similar when $\mathcal{S}(\lambda)$ is symmetric. □

Remark 4.2.1. *We mention that $\mathcal{S}(\lambda)$ does not admit a self adjoint GF pencil when m is even. We also mention that for the matrix polynomial $P(\lambda)$, the GF pencil $T_\omega(\lambda) = \lambda\mathcal{O}^{-1} - M_0\mathcal{E}M_m^{-1}$ is self-adjoint/ symmetric when $P(\lambda)$ is self-adjoint/ symmetric, where $\mathcal{O} = M_1M_3 \ldots M_{m-1}$ and $\mathcal{E} = M_2M_4 \cdots M_{m-2}$, see [3, 6]. In contrast, $\mathbb{L}(\lambda) = \lambda\mathcal{O}^{-1} - \mathbb{M}_0\mathcal{E}\mathbb{M}_m^{-1}$ is not self-adjoint/ symmetric, where $\mathcal{E} = \mathbb{M}_2\mathbb{M}_4 \cdots \mathbb{M}_{m-2}$ and $\mathcal{O} = \mathbb{M}_1\mathbb{M}_3 \ldots \mathbb{M}_{m-1}$.*

Example 4.3. Let $G(\lambda) = \lambda^5 A_5 + \cdots + A_0 + C(\lambda E - A)^{-1}B$ be self-adjoint. Then the PGF pencil $\mathbb{L}(\lambda) = \lambda \mathbb{M}_5 \mathbb{M}_3^{-1} \mathbb{M}_1^{-1} - \mathbb{M}_0 \mathbb{M}_2 \mathbb{M}_4$ is equal to

$$\mathbb{L}(\lambda) = \lambda \left[\begin{array}{ccccc|c} A_5 & & & & & \\ & 0 & I_n & & & \\ & I_n & A_3 & & & \\ & & & 0 & I_n & \\ & & & I_n & A_1 & \\ \hline & & & & & -E \end{array}\right] - \left[\begin{array}{ccccc|c} -A_4 & I_n & & & & \\ I_n & 0 & & & & \\ & & -A_2 & I_n & & \\ & & I_n & 0 & & \\ & & & & -A_0 & -C \\ \hline & & & & -C^* & -A \end{array}\right].$$

So $\mathbb{L}(\lambda)$ is self-adjoint and has block-tridiagonal structure. ■

Example 4.4. Let $G(\lambda) = \lambda^6 A_6 + \cdots + A_0 + C(\lambda E - A)^{-1}B$ be self-adjoint. Then the GF pencil $\mathbb{L}(\lambda) = \lambda \mathbb{M}_5 \mathbb{M}_3^{-1} \mathbb{M}_1^{-1} - \mathbb{M}_0 \mathbb{M}_2 \mathbb{M}_4 \mathbb{M}_6^{-1}$ is not self-adjoint, since $\mathbb{M}_6^{-1}$ and $\mathbb{M}_0$ do not commute, we have $(\mathbb{M}_0 \mathbb{M}_2 \mathbb{M}_4 \mathbb{M}_6^{-1})^* \neq \mathbb{M}_0 \mathbb{M}_2 \mathbb{M}_4 \mathbb{M}_6^{-1}$. Hence $\mathbb{L}(\lambda)$ is not self-adjoint. ■

Next, we analyze eigenvector recovery from generalized Fiedler pencils of system matrix.

4.3 Eigenvector recovery from GF pencils

Consider the system matrix $\mathcal{S}(\lambda)$. Recall that $\mathcal{N}_r(\mathcal{S})$ and $\mathcal{N}_l(\mathcal{S})$, are the right and left nullspaces of singular $\mathcal{S}(\lambda)$, and $\mathcal{N}_r(\mathcal{S}(\lambda))$ and $\mathcal{N}_l(\mathcal{S}(\lambda))$ are the right and left nullspaces of regular $\mathcal{S}(\lambda)$.

First, consider two linearizations $\mathbb{L}(\lambda)$ and $\mathbb{K}(\lambda)$ of $\mathcal{S}(\lambda)$ that are strictly equivalent; i.e., $\mathbb{L}(\lambda) = \mathcal{E}_1 \mathbb{K}(\lambda) \mathcal{E}_2$ for some nonsingular constant matrices $\mathcal{E}_1, \mathcal{E}_1 \in \mathbb{C}^{(nm+r)\times(nm+r)}$. Assume that $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $\mathcal{S}(\lambda)$. Then the map

$$\mathcal{N}_r(\mathbb{L}(\lambda_0)) \to \mathcal{N}_r(\mathbb{K}(\lambda_0)), \qquad v \mapsto \mathcal{E}_2 v, \tag{4.12}$$

is an isomorphism.

Second, recall that any PGF pencil $\mathbb{T}_\omega(\lambda)$ of $\mathcal{S}(\lambda)$ is strictly equivalent to an appropriate Fiedler pencil $\mathbb{L}_\sigma(\lambda)$ of $\mathcal{S}(\lambda)$ (see Lemma 4.1.1). The main idea for recovery of eigenvector from GF pencils is as follows: Suppose that $\mathbb{T}_\omega(\lambda) = \mathbb{M}_{\sigma_1}^{-1} \mathbb{L}_\sigma(\lambda) \mathbb{M}_{\sigma_2}^{-1}$, then $\mathcal{N}_r(\mathbb{T}_\omega(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$, $v \mapsto \mathbb{M}_{\sigma_2}^{-1} v$ is an isomorphism. We have seen in chapter 3 that $\mathbb{F}_\sigma(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism. Hence the map $\mathcal{N}_r(\mathbb{T}_\omega(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$, $v \mapsto \mathbb{F}_\sigma(\mathcal{S}) \mathbb{M}_{\sigma_2}^{-1} v$ is an isomorphism. The next result gives eigenvector recovery from PGF pencils of $\mathcal{S}(\lambda)$.

Theorem 4.3.1 (Eigenvector recovery from PGF pencils)**.** *Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be the PGF pencil of $\mathcal{S}(\lambda)$ associated with a bijection $\omega = (\omega_0, \omega_1)$, where $\omega_i : \tau_i \to \{1, \ldots, k_i\}$, $i = 0, 1$. Suppose that $0 \in \tau_0$ and δ is given in (4.7). Let λ be a finite eigenvalue of $\mathcal{S}(\lambda)$.*

Right eigenvectors. *Let $\mathbf{z} \in \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$ and $\mathbf{z} = \begin{bmatrix} u \\ v \end{bmatrix}$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$. Assume that ω_0 has c_0 consecutions at 0. Then $x \in \mathcal{N}_r(\mathcal{S}(\lambda))$, where $x := \begin{bmatrix} (e^T_{(m-c_0)} \otimes I_n)u \\ v \end{bmatrix}$ when $\omega \neq \delta$ and $x := \begin{bmatrix} (e^T_1 \otimes I_n)u \\ v \end{bmatrix}$ when $\omega = \delta$.*

Left eigenvectors. *Let $\mathbf{w} \in \mathcal{N}_l(\mathbb{T}_\omega(\lambda))$, and $\mathbf{w} = \begin{bmatrix} u \\ w \end{bmatrix}$ with $u \in \mathbb{C}^{nm}$ and $w \in \mathbb{C}^r$ and assume that ω_0 has d_0 inversions at 0. Then $y \in \mathcal{N}_l(\mathcal{S}(\lambda))$, where $y := \begin{bmatrix} (e^T_{(m-d_0)} \otimes I_n)u \\ w \end{bmatrix}$ when $\omega \neq \delta$ and $y := \begin{bmatrix} (e^T_1 \otimes I_n)u \\ w \end{bmatrix}$ when $\omega = \delta$.*

Proof. Suppose that $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is a PGF pencils having c_0 consecutions at 0. Note that PGF pencil has always less than $m+1$ factors $\mathbb{M}_j$ in $\mathbb{M}_{\omega_0}$, since $\mathbb{M}_m$ is a factor of $\mathbb{M}_{\omega_1}$, i.e., $c_0 < m$. Assume that $\omega \neq \delta$, where δ is given in (4.7). By Lemma 4.1.1, we have $\mathbb{L}_\sigma(\lambda) = \mathbb{M}_{\sigma_1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\sigma_2}$, for some σ_1 and σ_2, is a Fiedler pencil associated with a bijection $\sigma = (\sigma_1, \omega_0, \sigma_2)$ with $m, c_0, c_0 + 1 \notin \sigma_2$ and has c_0 consecutions at 0. Hence $\mathrm{CISS}(\sigma) = (c_0, \ldots)$. Now $\mathbb{T}_\omega(\lambda) = \mathbb{M}^{-1}_{\sigma_1}\mathbb{L}_\sigma(\lambda)\mathbb{M}^{-1}_{\sigma_2}$. Since $\mathbf{z} = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, then $\mathbb{M}^{-1}_{\sigma_2}\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$. Let $\mathbb{F}_\sigma(\mathcal{S})$ be as in Theorem 3.2.7. We have to show that $\mathbb{F}_\sigma(\mathcal{S})\mathbb{M}^{-1}_{\sigma_2}\begin{bmatrix} u \\ v \end{bmatrix} = \mathbb{F}_\sigma(\mathcal{S})\begin{bmatrix} u \\ v \end{bmatrix}$. Since σ_2 does not contain $m, c_0, c_0 + 1$, then by the structure of the Fiedler matrices we have

$$\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & I_{(c_0+1)n} & 0 \\ \hline 0 & 0 & I_r \end{array}\right] \Rightarrow \mathbb{M}^{-1}_{\sigma_2} = \left[\begin{array}{c|c|c} \widehat{*} & 0 & 0 \\ \hline 0 & I_{(c_0+1)n} & 0 \\ \hline 0 & 0 & I_r \end{array}\right]. \tag{4.13}$$

Now

$$\mathbb{F}_\sigma(\mathcal{S})\mathbb{M}^{-1}_{\sigma_2} = \left[\begin{array}{c|c} (e^T_{(m-c_0)} \otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} M^{-1}_{\sigma_2} & \\ \hline & I_r \end{array}\right] = \mathbb{F}_\sigma(\mathcal{S}).$$

Hence

$$\mathbb{F}_{\sigma}(\mathcal{S})\mathbb{M}_{\sigma_2}^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \left[\begin{array}{c|c} (e_{(m-c_0)}^T \otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (e_{(m-c_0)}^T \otimes I_n)u \\ v \end{bmatrix}.$$

Thus $\begin{bmatrix} (e_{(m-c_0)}^T \otimes I_n)u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$.

Next assume that $\omega = \delta$ and define $\mathbb{M}_{\xi_2} := \mathbb{M}_{c_0+1}\mathbb{M}_{c_0+2}\dots\mathbb{M}_{m-1}$. Then by (4.6), we have

$$\mathbb{T}_{\omega}(\lambda)\mathbb{M}_{\xi_2} = \lambda\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_1\dots\mathbb{M}_{m-1} := \mathbb{L}_{\sigma'}(\lambda), \tag{4.14}$$

is a Fiedler pencil associated with a bijection $\sigma' = (\omega_0, \xi_2)$ with $c_0 + 1 \in \xi_2$ and has $m-1$ consecutions at 0. i.e., $CISS(\sigma') = (c_0, i_0, \dots, c_l, i_l) = (m-1, 0)$. Note that σ' does not preserve the consecution of ω_0 at 0. Since $\mathbf{z} = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_{\omega}(\lambda))$, then $\mathbb{M}_{\xi_2}^{-1}\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_{\sigma'}(\lambda))$. Note that ξ_2 has $m - c_0 - 2$ consecutions at $c_0 + 1$, hence by the structure of the Fiedler matrices we have

$$\mathbb{M}_{\xi_2} = \left[\begin{array}{ccccc|c|c} -A_{m-1} & I_n & \dots & 0 & 0 & & \\ -A_{m-2} & 0 & \ddots & 0 & 0 & & \\ \vdots & \vdots & & \ddots & 0 & & \\ -A_{c_0+1} & 0 & & & I_n & & \\ I_n & 0 & \dots & \dots & 0 & & \\ \hline & & & & & I_{c_0 n} & \\ \hline & & & & & & I_r \end{array}\right] \Rightarrow \mathbb{M}_{\xi_2}^{-1} = \left[\begin{array}{c|c|c} * & & \\ \hline & I_{c_0 n} & \\ \hline & & I_r \end{array}\right].$$

Now

$$\mathbb{F}_{\sigma'}(\mathcal{S})\mathbb{M}_{\xi_2}^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \left[\begin{array}{c|c} (e_{(m-c_0)}^T \otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right]\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (e_1^T \otimes I_n)u \\ v \end{bmatrix}.$$

Thus $\begin{bmatrix} (e_1^T \otimes I_n)u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$.

From proof of the Theorem 3.2.8 recall that $\mathcal{N}_l(\mathcal{S}(\lambda)) = \mathcal{N}_r(\mathcal{S}(\lambda)^T)$ and $\mathcal{N}_l(\mathbb{L}_{\sigma}(\mathcal{S})) = \mathcal{N}_r(\mathbb{L}_{\sigma}(\mathcal{S})^T)$. Also recall that $\mathbb{L}_{\sigma}(\mathcal{S})^T = \lambda\mathbb{M}_m^T - \mathbb{M}_{\sigma}^T = \mathbb{L}_{rev\,\sigma}(\mathcal{S}^T)$. Therefore

$$\mathcal{N}_r(\mathbb{L}_{\sigma}(\mathcal{S})^T) = \mathcal{N}_r(\mathbb{L}_{rev\,\sigma}(\mathcal{S}^T)).$$

By Theorem 3.2.8 $\mathbb{H}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{L}_\sigma(\lambda))$ is an isomorphism and $\mathbb{K}_\sigma(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}_\sigma(\mathcal{S})) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ is an isomorphism. Now $\mathbb{T}_\omega(\lambda)^T = (\mathbb{M}_{\sigma_2}^{-1})^T \mathbb{L}_\sigma(\lambda)^T (\mathbb{M}_{\sigma_1}^{-1})^T$. Note that since $\mathbb{T}_\omega(\lambda)$ is a PGF, and ω_0 has c_0 consecutions at 0, then $m \notin \sigma_1$ and contains the indices which are greater than equal to $c_0 + 1$. Hence

$$\mathbb{M}_{\sigma_1} = \left[\begin{array}{c|c|c} * & & \\ \hline & I_{c_0 n} & \\ \hline & & I_r \end{array}\right] \Rightarrow (\mathbb{M}_{\sigma_1}^{-1})^T = \left[\begin{array}{c|c|c} \widehat{*} & & \\ \hline & I_{c_0 n} & \\ \hline & & I_r \end{array}\right].$$

Hence the result follows. □

Remark 4.3.1. *Note that in Theorem 4.3.1, we are not considering the case $0 \in \tau_1$.*

Theorem 4.3.2 (Eigenvector recovery from GF pencils). *Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be the GF pencil of $\mathcal{S}(\lambda)$ associated with a bijection $\omega = (\omega_0, \omega_1)$, where $\omega_i : \tau_i \to \{1, \ldots, k_i\}$, $i = 0, 1$. Suppose that $0 \in \tau_0$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$.*

Right eigenvectors. *Let $\mathbf{z} \in \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$ and $\mathbf{z} = \begin{bmatrix} u \\ v \end{bmatrix}$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$. Assume that ω_0 has c_0 consecutions at 0.*

(i) If $c_0 < m$ and $\mathbb{M}_m$ is to the left of $\mathbb{M}_0$, then $\begin{bmatrix} (e^T_{(m-\widehat{c_0})} \otimes I_n)u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$.

If $\mathbb{M}_m$ is to the right of $\mathbb{M}_0$, then $\begin{bmatrix} (e^T_{(m-\widehat{c_0})} \otimes I_n)u \\ -E^{-1}v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$, where $\widehat{c_0} = \begin{cases} c_0, & \text{if } \omega \neq \delta \\ m-1, & \text{if } \omega = \delta \end{cases}$.

(ii) If $c_0 = m$, then $\begin{bmatrix} A_m^{-1}(e_1^T \otimes I_n)u \\ -E^{-1}v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$.

Left eigenvectors. *Let $\mathbf{w} \in \mathcal{N}_l(\mathbb{T}_\omega(\lambda))$, and $\mathbf{w} = \begin{bmatrix} u \\ w \end{bmatrix}$ with $u \in \mathbb{C}^{nm}$ and $w \in \mathbb{C}^r$. Assume that ω_0 has d_0 inversions at 0.*

(i) If $d_0 < m$ and $\mathbb{M}_m$ is to the left of $\mathbb{M}_0$, then $\begin{bmatrix} (e^T_{(m-\widehat{d_0})} \otimes I_n)u \\ w \end{bmatrix} \in \mathcal{N}_l(\mathcal{S}(\lambda))$.

If $\mathbb{M}_m$ *is to the right of* $\mathbb{M}_0$*, then* $\begin{bmatrix} (e^T_{(m-\widehat{d_0})} \otimes I_n)u \\ -E^{-1}v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$*, where* $\widehat{d_0} = \begin{cases} d_0, & \textit{if } \omega \neq \delta \\ m-1, & \textit{if } \omega = \delta \end{cases}$.

(ii) If $d_0 = m$*, then* $\begin{bmatrix} A_m^{-1}(e_1^T \otimes I_n)u \\ -E^{-1}w \end{bmatrix} \in \mathcal{N}_l(\mathcal{S}(\lambda))$.

Proof. Suppose that $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is a non proper GF (NPGF) pencils, i.e., $m \in \tau_0$. Since ω_0 has c_0 consecutions at 0, so $\mathbb{M}_{c_0+1}$ appears to the left of $\mathbb{M}_{c_0}$, otherwise ω_0 has more than c_0 consecutions at 0. By commutativity relation given in (2.11) and the fact that ω_0 has c_0 consecutions at 0 allow us to write $\mathbb{M}_{\omega_0}$ as

$$\mathbb{M}_{\omega_0} = \mathbb{M}_{\mathbf{s}}(\mathbb{M}_0\mathbb{M}_1, \ldots, \mathbb{M}_{c_0})\mathbb{M}_{\mathbf{t}}, \tag{4.15}$$

Note that $0 : c_0, c_0+1 \notin \mathbf{t}$. That is $\mathbb{M}_{\mathbf{t}}$ contains Fiedler matrices with indices greater than $c_0 + 1$ and also $0 \notin \mathbf{s}$. Also note that there two cases:

Case I : $\mathbb{M}_{\mathbf{s}} = \mathbb{M}_{i_1} \ldots \mathbb{M}_{i_l}\mathbb{M}_m^{-1}$ or $\mathbb{M}_{\mathbf{s}} = \mathbb{M}_m^{-1}\mathbb{M}_{i_1} \ldots \mathbb{M}_{i_l}$ and $\mathbb{M}_{\mathbf{t}} = \mathbb{M}_{j_1} \ldots \mathbb{M}_{j_k}$.

Case II : $\mathbb{M}_{\mathbf{s}} = \mathbb{M}_{i_1} \ldots \mathbb{M}_{i_l}$ and $\mathbb{M}_{\mathbf{t}} = \mathbb{M}_m^{-1}\mathbb{M}_{j_1} \ldots \mathbb{M}_{j_l}$ or $\mathbb{M}_{\mathbf{t}} = \mathbb{M}_{j_1} \ldots \mathbb{M}_{j_l}\mathbb{M}_m^{-1}$.

Assume that Case I is true, i.e., $\mathbb{M}_m$ is to the left of $\mathbb{M}_0$, then $\mathbb{M}_{\mathbf{s}}$ and $\mathbb{M}_{\mathbf{t}}$ are always nonsingular, and

$$\mathbb{T}_{\widetilde{\omega}}(\lambda) := \mathbb{M}_{\mathbf{s}}^{-1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\mathbf{t}}^{-1} = \lambda\mathbb{M}_{\mathbf{s}}^{-1}\mathbb{M}_{\omega_1}\mathbb{M}_{\mathbf{t}}^{-1} - \mathbb{M}_0 \ldots \mathbb{M}_{c_0}$$

has c_0 consecutions at 0. Since either $\mathbb{M}_{\mathbf{s}}^{-1}$ or $\mathbb{M}_{\mathbf{t}}^{-1}$ contains $\mathbb{M}_m$, hence $\mathbb{T}_{\widetilde{\omega}}(\lambda)$ is a PGF pencil having c_0 consecutions at 0. Note that $m, c_0, c_0+1 \notin \mathbf{t}$. Hence $\mathbb{M}_{\mathbf{t}}$ is of the form $\mathbb{M}_{\sigma_2}$ given in (4.13). Hence the result follows from the proof of PGF pencil in Theorem 4.3.1.

Assume that Case II is true, then $\mathbb{M}_m \in \mathbb{M}_{\mathbf{t}}^{-1}$. Since $c_0, c_0+1 \notin \mathbf{t}$, we have

$$\mathbb{M}_{\mathbf{t}} = \left[\begin{array}{c|c|c} * & & \\ \hline & I_{(c_0+1)n} & \\ \hline & & -E^{-1} \end{array}\right].$$

Let $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$ then $\mathbb{M}_{\mathbf{t}} \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_{\widetilde{\omega}}(\lambda))$. Hence by Theorem 4.3.1, we have

$$\left[\begin{array}{c|c} (e^T_{(m-c_0)} \otimes I_n) & 0 \\ \hline 0 & I_r \end{array}\right] \mathbb{M}_{\mathbf{t}} \begin{bmatrix} u \\ v \end{bmatrix} = \left[\begin{array}{c|c} (e^T_{(m-c_0)} \otimes I_n) & 0 \\ \hline 0 & -E^{-1} \end{array}\right] \begin{bmatrix} u \\ v \end{bmatrix}$$
$$= \begin{bmatrix} (e^T_{(m-c_0)} \otimes I_n)u \\ -E^{-1}v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda)).$$

(ii) Suppose that $c_0 = m$, that is the number of consecutions at 0 is m, then $\mathbb{T}_\omega(\lambda) = \lambda I_{nm+r} - \mathbb{M}_0\mathbb{M}_1 \dots \mathbb{M}_{m-1}\mathbb{M}_m^{-1}$ and $\mathbb{L}_\sigma(\lambda) = \mathbb{T}_\omega(\lambda)\mathbb{M}_m = \lambda\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_1 \dots \mathbb{M}_{m-1}$ is a Fiedler pencil associated with a bijection σ. Note that $\mathbb{L}_\sigma(\lambda)$ has $(m-1)$ consecutions at 0, i.e., $\mathrm{CISS}(\sigma) = (c_0, i_0, \dots, c_l, i_l) = (m-1, \dots)$. Since $\mathbb{T}_\omega(\lambda) = \mathbb{L}_\sigma(\lambda)\mathbb{M}_m$, if $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, then $\mathbb{M}_m^{-1} \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{L}_\sigma(\lambda))$. Let $\mathbb{F}_\sigma(\mathcal{S})$ be as in Theorem 3.2.7. Now

$$\mathbb{F}_\sigma(\mathcal{S}) = \left[\begin{array}{c|c} e^T_{(m-c_0)} \otimes I_n & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} (e^T_{(m-m+1)} \otimes I_n) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} e^T_1 \otimes I_n & \\ \hline & I_r \end{array}\right].$$

Hence

$$\mathbb{F}_\sigma(\mathcal{S})\mathbb{M}_m^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \left[\begin{array}{c|c} (e^T_1 \otimes I_n) & \\ \hline & I_r \end{array}\right] \left[\begin{array}{cc|c} A_m^{-1} & & \\ & I_{(m-1)n} & \\ \hline & & -E^{-1} \end{array}\right] \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} A_m^{-1} & \cdots & 0 \end{bmatrix} & 0 \\ \hline & -E^{-1} \end{array}\right] \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A_m^{-1}(e^T_1 \otimes I_n)u \\ -E^{-1}v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda)).$$

□

Example 4.5. Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0} = \lambda\mathbb{M}_6^{-1}\mathbb{M}_5^{-1} - \mathbb{M}_4\mathbb{M}_0\mathbb{M}_7^{-1}\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3$ be a GF pencil of $\mathcal{S}(\lambda)$, where $\mathbb{M}_{\omega_0}$ has 3 consecutions at 0, i.e, $c_0 = 3 < 7$. Note that $\mathbb{M}_7$ is to the right of $\mathbb{M}_0$. By commutativity relation given in (2.11) we have $\mathbb{M}_{\omega_0} = \mathbb{M}_4\mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3\mathbb{M}_7^{-1}$. Define $\mathbb{M}_{\mathbf{s}} = \mathbb{M}_4$, $\mathbb{M}_{\mathbf{t}} = \mathbb{M}_7^{-1}$, and

$$\mathbb{T}_{\widetilde{\omega}}(\lambda) = \mathbb{M}_{\mathbf{s}}^{-1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\mathbf{t}}^{-1} = \mathbb{M}_4^{-1}\mathbb{T}_\omega(\lambda)\mathbb{M}_7 = \lambda\mathbb{M}_4^{-1}\mathbb{M}_6^{-1}\mathbb{M}_5^{-1}\mathbb{M}_7 - \mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3.$$

Then $\mathbb{T}_{\widetilde{\omega}}(\lambda)$ is a PGF and preserves consecution at 0. If $z = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_{\omega}(\lambda_0))$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$, then by Theorem 4.3.2 we have $\begin{bmatrix} (e_4^T \otimes I_n)u \\ -E^{-1}v \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda_0))$. ■

Corollary 4.3.3. *Suppose that the transfer function $G(\lambda)$ is regular. Let $\mathbb{T}_{\omega}(\lambda) = \lambda \mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be the GF pencil of $G(\lambda)$ associated with the bijection $\omega = (\omega_0, \omega_1)$, where $\omega_i : \tau_i \to \{1, \ldots, k_i\}$, $i = 0, 1$. Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$.*

Right eigenvectors. *Let $z \in \mathcal{N}_r(\mathbb{T}_{\omega}(\lambda_0))$ and $z = \begin{bmatrix} u \\ v \end{bmatrix}$ with $u \in \mathbb{C}^{mn}$ and $v \in \mathbb{C}^r$. Suppose $0 \in \tau_0$ and assume that ω_0 has c_0 consecutions at 0.*

(i) If $c_0 < m$, then $(e^T_{(m-\widehat{c_0})} \otimes I_n)u \in \mathcal{N}_r(G(\lambda_0))$, where $\widehat{c_0} = \begin{cases} c_0, & \text{if } \omega \neq \delta \\ m-1, & \text{if } \omega = \delta \end{cases}$.

(ii) If $c_0 = m$, then $A_m^{-1}(e_1^T \otimes I_n)u \in \mathcal{N}_r(G(\lambda_0))$.

Left eigenvectors. *Let $w \in \mathcal{N}_l(\mathbb{T}_{\omega}(\lambda_0))$ and $w = \begin{bmatrix} u \\ v \end{bmatrix}$ with $u \in \mathbb{C}^{mn}$ and $v \in \mathbb{C}^r$. Suppose $0 \in \tau_0$ and assume that ω_0 has d_0 inversions at 0.*

(i) If $d_0 < m$, then $(e^T_{(m-\widehat{d_0})} \otimes I_n)u \in \mathcal{N}_l(G(\lambda_0))$, where $\widehat{d_0} = \begin{cases} d_0, & \text{if } \omega \neq \delta \\ m-1, & \text{if } \omega = \delta \end{cases}$.

(ii) If $d_0 = m$, then $A_m^{-1}(e_1^T \otimes I_n)u \in \mathcal{N}_l(G(\lambda_0))$.

We illustrate Corollary 4.3.3 by the following example.

Example 4.6. Let $G(\lambda) = \lambda^9 A_9 + \cdots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B$. Consider a GF pencil $\mathbb{T}_{\omega}(\lambda) = \lambda \mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0} = \lambda \mathbb{M}_6^{-1}\mathbb{M}_7^{-1}\mathbb{M}_4^{-1} - \mathbb{M}_0\mathbb{M}_9^{-1}\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3\mathbb{M}_5\mathbb{M}_8$, where ω_0 has 3 consecutions at 0. Now $c_0 = 3 < 9$. By commutativity relation given in (2.11) we have $\mathbb{M}_{\omega_0} = \mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3\mathbb{M}_9^{-1}\mathbb{M}_5\mathbb{M}_8$. Define $\mathbb{M}_{\mathbf{t}} := \mathbb{M}_9^{-1}\mathbb{M}_5\mathbb{M}_8$ and

$$\mathbb{T}_{\widetilde{\omega}}(\lambda) = \mathbb{T}_{\omega}(\lambda)\mathbb{M}_{\mathbf{t}}^{-1} = \lambda \mathbb{M}_6^{-1}\mathbb{M}_7^{-1}\mathbb{M}_4^{-1}\mathbb{M}_8^{-1}\mathbb{M}_5^{-1}\mathbb{M}_9 - \mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3.$$

Then $\mathbb{T}_{\widetilde{\omega}}(\lambda)$ is a PGF and preserves consecution at 0. If $z = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_{\omega}(\lambda_0))$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$, then by Corollary 4.3.3 we have $(e_6^T \otimes I_n)u \in \mathcal{N}_r(G(\lambda_0))$.

Consider $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0} = \lambda\mathbb{M}_6^{-1}\mathbb{M}_7^{-1}\mathbb{M}_4^{-1} - \mathbb{M}_9^{-1}\mathbb{M}_0\mathbb{M}_5\mathbb{M}_1\mathbb{M}_2\mathbb{M}_8\mathbb{M}_3$, where $\mathbb{M}_{\omega_0}$ has 3 consecutions at 0 Now $c_0 = 3 < 9$. By commutativity relation given in (2.11) we have $\mathbb{M}_{\omega_0} = \mathbb{M}_9^{-1}\mathbb{M}_5\mathbb{M}_8\mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3$. Define $\mathbb{M}_{\mathbf{s}} := \mathbb{M}_9^{-1}\mathbb{M}_5\mathbb{M}_8$ and

$$\mathbb{T}_{\widetilde{\omega}}(\lambda) = \mathbb{M}_{\mathbf{s}}^{-1}\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_8^{-1}\mathbb{M}_5^{-1}\mathbb{M}_9\mathbb{M}_6^{-1}\mathbb{M}_7^{-1}\mathbb{M}_4^{-1} - \mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3$$

Then $\mathbb{T}_{\widetilde{\omega}}(\lambda)$ is a PGF and preserves consecution at 0. Hence $\mathcal{N}_r(\mathbb{T}_{\widetilde{\omega}}(\lambda)) = \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.

Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0} = \lambda\mathbb{M}_6^{-1}\mathbb{M}_5^{-1} - \mathbb{M}_4\mathbb{M}_0\mathbb{M}_7^{-1}\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3$ be a GF pencil of $G(\lambda) = \lambda^7 A_7 + \cdots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B$. Now $\mathbb{M}_{\omega_0}$ has 3 consecutions at 0, has an inversion at $c_0 = 3$. By commutativity relation given in (2.11) we have $\mathbb{M}_{\omega_0} = \mathbb{M}_4\mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3\mathbb{M}_7^{-1}$. Define $\mathbb{M}_{\mathbf{s}} = \mathbb{M}_4$, $\mathbb{M}_{\mathbf{t}} = \mathbb{M}_7^{-1}$, and

$$\mathbb{T}_{\widetilde{\omega}}(\lambda) = \mathbb{M}_{\mathbf{s}}^{-1}\mathbb{T}_\omega(\lambda)\mathbb{M}_{\mathbf{t}}^{-1} = \lambda\mathbb{M}_4^{-1}\mathbb{M}_6^{-1}\mathbb{M}_5^{-1}\mathbb{M}_7 - \mathbb{M}_0\mathbb{M}_1\mathbb{M}_2\mathbb{M}_3.$$

Then $\mathbb{T}_{\widetilde{\omega}}(\lambda)$ is a PGF and preserves consecution at 0. If $z = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{N}_r(\mathbb{T}_\omega(\lambda_0))$ with $u \in \mathbb{C}^{nm}$ and $v \in \mathbb{C}^r$, then by Corollary 4.3.3 we have $(e_4^T \otimes I_n)u \in \mathcal{N}_r(G(\lambda_0))$. ■

Corollary 4.3.4 (Eigenvector recovery from symmetric pencils)**.** *Let $\mathbb{L}(\lambda)$ be the GF linearization of $\mathcal{S}(\lambda)$ defined in Theorem 4.2.1.*

Case I : Let $\mathcal{S}(\lambda)$ be regular and $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$.

(a) Let $\{z_1, \ldots, z_p\} \subset \mathbb{C}^{(nm+r)}$ be a basis of $\mathcal{N}_r(\mathbb{L}(\lambda_0))$ and $z_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$ with $u_j \in \mathbb{C}^{nm}$ and $v_j \in \mathbb{C}^r$. Then $\{x_1, \ldots, x_p\}$ is a basis for $\mathcal{N}_r(\mathcal{S}(\lambda_0))$, where $x_j := \begin{bmatrix} (e_m^T \otimes I_n)u_j \\ v_j \end{bmatrix}$ for $j = 1, \ldots, p$.

(b) Let $\{w_1, \ldots, w_p\} \subset \mathbb{C}^{(nm+r)}$ be a basis of $\mathcal{N}_l(\mathbb{L}(\lambda_0))$ and $w_j = \begin{bmatrix} u_j \\ s_j \end{bmatrix}$ with $u_j \in \mathbb{C}^{nm}$ and $s_j \in \mathbb{C}^r$. Then $\{y_1, \ldots, y_p\}$ is a basis for $\mathcal{N}_l(\mathcal{S}(\lambda_0))$, where $y_j := \begin{bmatrix} (e_m^T \otimes I_n)u_j \\ s_j \end{bmatrix}$ for $j = 1, \ldots, p$.

Case II : Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$.

(a) Let $\{z_1, \ldots, z_p\} \subset \mathbb{C}^{(nm+r)}$ be a basis of $\mathcal{N}_r(\mathbb{L}(\lambda_0))$ and $z_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$ with $u_j \in \mathbb{C}^{nm}$ and $v_j \in \mathbb{C}^r$. Then $\{x_1, \ldots, x_p\}$ is a basis for $\mathcal{N}_r(G(\lambda_0))$, where $x_j := (e_m^T \otimes I_n)u_j$ for $j = 1, \ldots, p$.

(b) Let $\{w_1, \ldots, w_p\} \subset \mathbb{C}^{(nm+r)}$ *be a basis of* $\mathcal{N}_l(\mathbb{L}(\lambda_0))$ *and* $z_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$ *with* $u_j \in \mathbb{C}^{nm}$ *and* $v_j \in \mathbb{C}^r$*. Then* $\{y_1, \ldots, y_p\}$ *is a basis for* $\mathcal{N}_l(G(\lambda_0))$*, where* $y_j := (e_m^T \otimes I_n)u_j$ *for* $j = 1, \ldots, p$*.*

Proof. Since $\mathbb{L}(\lambda)$ has 0 consecutions at 0, the desired results follow from Theorem 4.3.1 and Corollary 4.3.3.

$\square$

5

Generalized Fiedler Pencils With Repetition

This chapter is devoted to a new family of linearizations, referred to as generalized Fiedler pencil with repetition (GFPR), for system matrix. We define operation-free product of Fiedler matrices to study GFPR. We construct self-adjoint linearizations of self-adjoint system matrix. We also discuss eigenvector recovery of system matrix from GFPR.

5.1 Introduction

Given a system matrix of an LTI system, the main purpose of this chapter is to investigate a new family of linearizations referred to as generalized Fiedler pencil with repetition (GFPR) for system matrix and for the associated transfer function. These linearizations allow repetition of Fiedler matrices associated with system matrix. One important property of the companion form is that there is simple relationship between its eigenvectors and those of the system matrix $\mathcal{S}(\lambda)$ that it linearizes. We have already seen recovery of eigenvectors of system matrix from its Fiedler pencils and from GF pencils. In this chapter we also discuss eigenvector recovery of system matrix and of transfer function from GFPR.

We use the boldface small letters, such as $\mathbf{t}, \mathbf{q}, \mathbf{s}, \ldots$ for index tuples (or ordered tuples of indices). Recall that, if $\mathbf{q} = (i_1, i_2, \ldots, i_s)$ is an index tuple, then the *reverse* of $\mathbf{q}$, denoted by rev $\mathbf{q}$, is defined by *rev* $\mathbf{q} = (i_s, i_{s-1}, \ldots, i_1)$. Let $\mathbf{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple and d be an integer. Then we write

$$-\mathbf{q} = (-i_1, -i_2, \ldots, -i_s) \text{ and } d + \mathbf{q} = (d + i_1, d + i_2, \ldots, d + i_s).$$

Recall that $P(\lambda) = \sum\limits_{j=0}^{m} \lambda^j A_j$ and the Fiedler matrices

$$M_0 := \begin{bmatrix} I_{(m-1)n} & \\ & -A_0 \end{bmatrix}, \quad M_{-m} := \begin{bmatrix} A_m & \\ & I_{(m-1)n} \end{bmatrix}, \tag{5.1}$$

$$M_{-i} := M_i^{-1} \text{ for } i = 0, 1, \ldots, m-1 \text{ and } M_m := M_{-m}^{-1}, \tag{5.2}$$

where

$$M_i := \begin{bmatrix} I_{(m-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix} \text{ and } M_i^{-1} = \begin{bmatrix} I_{(m-i-1)n} & & & \\ & 0 & I_n & \\ & I_n & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}$$

for $i = 1 : m-1$. Note that M_{-m} and M_0 are invertible if and only if A_m and A_0 are invertible. Then as before $M_i M_j = M_j M_i$ if $||i| - |j|| > 1$.

Now, consider the system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right] \tag{5.3}$$

and the associated transfer function

$$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1} B. \tag{5.4}$$

We define the Fiedler matrices of $\mathcal{S}(\lambda)$ by

$$\mathbb{M}_0 := \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right], \quad \mathbb{M}_{-m} := \left[\begin{array}{c|c} M_{-m} & 0 \\ \hline 0 & -E \end{array}\right], \tag{5.5}$$

$$\mathbb{M}_{-i} := \mathbb{M}_i^{-1}, \text{ for } i = 0, 1, \ldots, m-1 \text{ and } \mathbb{M}_m := \mathbb{M}_{-m}^{-1},$$

where $\mathbb{M}_i^{-1} = \left[\begin{array}{c|c} M_i^{-1} & 0 \\ \hline 0 & I_r \end{array}\right], i = 1 : m-1$. It is easy to see that the commutativity relation satisfy

$$\mathbb{M}_i \mathbb{M}_j = \mathbb{M}_j \mathbb{M}_i \qquad \text{for } ||i| - |j|| \neq 1, \tag{5.6}$$

except for $\mathbb{M}_{-0}$ and $\mathbb{M}_{-m}$.

Let $\mathbf{q} = (i_1, i_2, \ldots, i_s)$ be an ordered tuple of indices from $\{-0, -1, \ldots, -m\}$. Then $\mathbb{M}_{\mathbf{q}}$ is defined by

$$\mathbb{M}_{\mathbf{q}} := \mathbb{M}_{i_1}\mathbb{M}_{i_2}\ldots\mathbb{M}_{i_s}, \qquad \mathbb{M}_{\emptyset} := I_{nm+r}.$$

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be two ordered tuple of indices from $\{0, 1\ldots, m, -0, -1\ldots, -m\}$. Recall that $\mathbf{q}_1$ is *equivalent* to $\mathbf{q}_2$ ($\mathbf{q}_1 \sim \mathbf{q}_2$) if $M_{\mathbf{q}_1} = M_{\mathbf{q}_2}$. Observe that this relation is an equivalence relation and if $M_{\mathbf{q}_2}$ can be obtained from $M_{\mathbf{q}_1}$ by applying the commutativity relations given in (5.6), then $\mathbf{q}_1$ is equivalent to $\mathbf{q}_2$.

With this new notation, the GF pencil can be defined as follows.

Definition 5.1.1. *Let $\mathcal{S}(\lambda)$ be the system matrix given in (5.3). Let $\{C_0, C_1\}$ be a partition of $\{0, 1, \ldots, m\}$ (C_0 or C_1 may be empty set). Let σ and τ be permutations of C_0 and $-C_1$, respectively. Then the pencil*

$$\mathbb{T}_{\omega}(\lambda) := \lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma}$$

is said to be a generalized Fiedler (GF) pencil of $\mathcal{S}(\lambda)$ associated with $\omega = (\tau, \sigma)$. If $0 \in C_0$ and $m \in C_1$, then the pencil $\mathbb{T}_{\omega}(\lambda)$ is said to be a Proper Generalized Fiedler (PGF) pencil of $\mathcal{S}(\lambda)$.

Note that if $C_0 = \emptyset$, then $\mathbb{M}_{\sigma} = I_{(nm+r)}$ and if $C_1 = \emptyset$, then $\mathbb{M}_{\tau} = I_{(nm+r)}$. It is clear that any Fiedler pencil $\mathbb{L}_{\sigma}(\lambda)$ of $\mathcal{S}(\lambda)$ is a special case of a GF pencil with $C_0 = \{0, 1, \ldots, m-1\}$ and $C_1 = \{m\}$.

We have seen that a self-adjoint system matrix may not have a self-adjoint GF pencil. Recall that any PGF pencil $\mathbb{T}_{\omega}(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is a linearization of $\mathcal{S}(\lambda)$. Note that for appropriate subsets σ_1 and σ_2 of $\{1, 2, \cdots, m-1\}$ and $\{-1, -2, \cdots, -(m-1)\}$ respectively, the pencil $\mathbb{L}(\lambda) = \mathbb{M}_{\sigma_1}\mathbb{T}_{\omega}(\lambda)\mathbb{M}_{\sigma_2}$ is strictly equivalent to $\mathbb{T}_{\omega}(\lambda)$ and hence a linearization for $\mathcal{S}(\lambda)$. Note however that $\mathbb{L}(\lambda)$ may not possess a distinctive feature of $\mathbb{T}_{\omega}(\lambda)$, namely, that the block entries of $\mathbb{L}(\lambda)$ may not consist of $0, I_n, A_i, i = 0 : m, A, B, C$ and E as is the case for $\mathbb{T}_{\omega}(\lambda)$. We now analyze the product of Fiedler matrices where the block entries of the product consist of $0, I_n, A_i, A, B, C, E$. We refer to these products as operation-free product.

5.2 Operation-free products of Fiedler matrices.

Definition 5.2.1. *A product $\mathbb{M}_{\boldsymbol{q}}$ corresponding to the index tuple $\boldsymbol{q} = (i_1, i_2, \ldots, i_m)$ is said to be* operation-free *if the block entries of $\mathbb{M}_{\boldsymbol{q}}$ are either $0, I_n, I_r, C, B, A, E,$ and $-A_k, k = 0, \ldots, m$.*

For example, when $\sigma = (0, 1, \ldots, m-1)$, the product $\mathbb{M}_\sigma = \mathbb{M}_{\sigma(0)} \cdots \mathbb{M}_{\sigma(m-1)}$ is operation-free. Thus a Fiedler pencil associated with a bijection σ is also operation-free. So the question is: if we allow to repeat the Fiedler matrices, will that product still be an operation-free product? Our aim is to analyze products of Fiedler matrices which are operation-free.

Lemma 5.2.1. *The product* $\mathbb{M}_i\mathbb{M}_i$, *is not operation-free for* $i = 0, \pm 1, \ldots, \pm(m-1), -m$.

Proof. For $i = 0$,

$$\mathbb{M}_0\mathbb{M}_0 = \left[\begin{array}{cc|c} I_{(m-1)n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right]\left[\begin{array}{cc|c} I_{(m-1)n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right]$$

$$= \left[\begin{array}{cc|c} I_{(m-1)n} & & \\ & A_0^2 + CB & A_0C + CA \\ \hline & BA_0 + B^2 & BC + A^2 \end{array}\right]$$

is not operation-free. For $1 \leq i \leq m-1$

$$\mathbb{M}_i\mathbb{M}_i = \left[\begin{array}{cccc|c} I_{(m-i-1)n} & & & & \\ & -A_i & I_n & & \\ & I_n & 0 & & \\ & & & I_{(i-1)n} & \\ \hline & & & & I_r \end{array}\right]\left[\begin{array}{cccc|c} I_{(m-i-1)n} & & & & \\ & -A_i & I_n & & \\ & I_n & 0 & & \\ & & & I_{(i-1)n} & \\ \hline & & & & I_r \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} I_{(m-i-1)n} & & & & \\ & A_i^2 + I_n & -A_i & & \\ & -A_i & I_n & & \\ & & & I_{(i-1)n} & \\ \hline & & & & I_r \end{array}\right]$$

is not operation-free. Similarly, for $-(m-1) \leq i \leq -1$, $\mathbb{M}_i\mathbb{M}_i$ is not operation-free, since

$$\mathbb{M}_{-i}\mathbb{M}_{-i} = (\mathbb{M}_i\mathbb{M}_i)^{-1} = \left[\begin{array}{cccc|c} I_{(m-i-1)n} & & & & \\ & I_n & A_i & & \\ & A_i & A_i^2 + I_n & & \\ & & & I_{(i-1)n} & \\ \hline & & & & I_r \end{array}\right].$$

Again for $i = -m$

$$\mathbb{M}_{-m}\mathbb{M}_{-m} = \left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right]\left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right] = \left[\begin{array}{cc|c} A_m^2 & & \\ & I_{(m-1)n} & \\ \hline & & E^2 \end{array}\right]$$

is not operation-free.

$\square$

Lemma 5.2.2. *The product* $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i$ *is operation-free for* $i = 1, 2, \ldots, m-2$ *but* $\mathbb{M}_0\mathbb{M}_1\mathbb{M}_0$ *is not operation-free. Further* $\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1}$ *is not operation-free for* $i = 0, 1, \ldots, m-2$.

Proof. For $i = 0$

$$\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i = \mathbb{M}_0\mathbb{M}_1\mathbb{M}_0 = \left[\begin{array}{ccc|c} I_{(m-2)n} & & & \\ & -A_1 & -A_0 & -C \\ & -A_0 & 0 & CA \\ \hline & -B & BA & A^2 \end{array}\right]$$

is not operation-free. For $1 \le i \le m-2$,

$$\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i = \left[\begin{array}{ccccc|c} I_{(m-i-2)n} & & & & & \\ & -A_{i+1} & -A_i & I_n & & \\ & -A_i & I_n & 0 & & \\ & I_n & 0 & 0 & & \\ & & & & I_{(i-1)n} & \\ \hline & & & & & I_r \end{array}\right].$$

So $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i$ is operation-free. Note that $\mathbb{M}_1\mathbb{M}_0\mathbb{M}_1 =$

$$\left[\begin{array}{ccc|c} I_{(m-2)n} & & & \\ & -A_1 & I_n & \\ & I_n & 0 & \\ \hline & & & I_r \end{array}\right] \left[\begin{array}{cc|c} I_{(m-1)n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right] \left[\begin{array}{ccc|c} I_{(m-2)n} & & & \\ & -A_1 & I_n & \\ & I_n & 0 & \\ \hline & & & I_r \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} I_{(m-2)n} & & & \\ & A_1^2 - A_0 & -A_1 & -C \\ & -A_1 & I_n & 0 \\ \hline & -B & 0 & -A \end{array}\right]$$

is not operation-free. For $1 \leq i \leq m-2$,

$$\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1} = \left[\begin{array}{ccccc|c} I_{(m-i-2)n} & & & & & \\ & -A_{i+1}^2 - A_i & -A_{i+1} & I_n & & \\ & -A_{i+1} & I_n & 0 & & \\ & I_n & 0 & 0 & & \\ & & & & I_{(i-1)n} & \\ \hline & & & & & I_r \end{array}\right]$$

is not operation-free. $\square$

Corollary 5.2.1. *The product $\mathbb{M}_i\mathbb{M}_{i+1}\mathbb{M}_i$ is operation-free for $i = -2, -3, \ldots, -(m-1)$ but $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$ is not operation-free. If $E = I_r$, then $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$ is operation-free. Further, $\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1}$ is not operation-free for $i = -2, -3, \ldots, -(m-1)$.*

Proof. We have $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$

$$= \left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right] \left[\begin{array}{ccc|c} 0 & I_n & & \\ I_n & A_{m-1} & & \\ & & I_{(m-2)n} & \\ \hline & & & I_r \end{array}\right] \left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0 & A_m & & \\ A_m & A_{m-1} & & \\ & & I_{(m-2)n} & \\ \hline & & & E^2 \end{array}\right]$$

is not operation-free. If $E = I_r$, then $\mathbb{M}_{-m}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}$ is operation-free. The proof for $\mathbb{M}_{i+1}\mathbb{M}_i\mathbb{M}_{i+1}$ is similar. $\square$

Lemma 5.2.3. *[41] Let $\mathcal{I}$ be an index tuple of indices from $\{0, 1, \ldots, m-1\}$ such that $M_{\mathcal{I}}$ is not operation-free. Then for any two other index tuples $\mathcal{I}_1$ and $\mathcal{I}_2$ from $\{0, 1, \ldots, m-1\}$, $M_{\mathcal{I}_1}M_{\mathcal{I}}M_{\mathcal{I}_2}$ is not operation-free, where $M_{\mathcal{I}}$ is the product of Fiedler matrices associated with $P(\lambda)$.*

As a consequence, we have the following.

Lemma 5.2.4. *Let $\boldsymbol{q} = (i_1, i_2, \ldots, i_s)$ be an index tuple of indices from $\{0, 1, \ldots, m-1\}$ such that $\mathbb{M}_{\boldsymbol{q}}$ is not operation-free. Then for any two other index tuples τ and σ from $\{0, 1, \ldots, m-1\}$ the product $\mathbb{M}_\tau\mathbb{M}_{\boldsymbol{q}}\mathbb{M}_\sigma$ is not operation-free.*

Lemma 5.2.5. *The product $\mathbb{M}_{(i:j)}$ is operation-free and is given by*

$$\mathbb{M}_{(i:j)} = \left[\begin{array}{c|c|c|c|c} I_{(m-j-1)n} & & & & \\ \hline & \begin{matrix} -A_j \\ -A_{j-1} \\ \vdots \\ -A_i \end{matrix} & I_{(j-i+1)n} & & \\ \hline & I_n & 0_{n\times(j-i+1)n} & & \\ \hline & & & I_{(i-1)n} & \\ \hline & & & & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{(i:j)} & \\ \hline & I_r \end{array}\right] \quad (5.7)$$

for $i > 0$, $i \leq j \leq m-1$ and

$$\mathbb{M}_{(0:j)} = \left[\begin{array}{c|c|c|c} I_{(m-j-1)n} & & & \\ \hline & \begin{matrix} -A_j \\ \vdots \\ -A_1 \end{matrix} & I_{jn} & \\ \hline & -A_0 & 0_{n\times jn} & -C \\ \hline & -B & & -A \end{array}\right] = \left[\begin{array}{c|c} M_{(0:j)} & -e_m \otimes C \\ \hline -e_{m-j}^T \otimes B & -A \end{array}\right]. \quad (5.8)$$

Proof. We prove the result by induction on j. Suppose that $i > 0$. Then for $j = i$, we have $\mathbb{M}_{(i:i)} = \mathbb{M}_i$ which is operation free. So assume that (5.7) is true for $j = s$. We have to show that (5.7) is true for $j = s + 1$. Now $\mathbb{M}_{(i:s+1)} = \mathbb{M}_{(i:s)}\mathbb{M}_{s+1} =$

$$\left[\begin{array}{c|c|cccc|c|c}
I_{(m-s-1)n} & & & & & & & \\ \hline
 & -A_s & I_n & & & & & \\
 & -A_{s-1} & 0 & I_n & & & & \\
 & \vdots & \vdots & & \ddots & & & \\
 & -A_i & 0 & 0 & \cdots & I_n & & \\ \hline
 & I_n & 0 & 0 & \cdots & 0 & 0 & \\ \hline
 & & & & & & & I_r
\end{array}\right]
\left[\begin{array}{cccc|c}
I_{(m-s-2)n} & & & & \\
 & -A_{(s+1)} & I_n & & \\
 & I_n & 0 & & \\
 & & & I_{sn} & \\ \hline
 & & & & I_r
\end{array}\right]$$

$$= \left[\begin{array}{c|c|cccc|c|c}
I_{(m-s-2)n} & & & & & & & \\ \hline
 & -A_{s+1} & I_n & & & & & \\
 & -A_s & 0 & I_n & & & & \\
 & \vdots & \vdots & & \ddots & & & \\
 & -A_i & 0 & 0 & \cdots & I_n & & \\ \hline
 & I_n & 0 & 0 & \cdots & 0 & 0 & \\ \hline
 & & & & & & & I_r
\end{array}\right]$$

$$= \left[\begin{array}{c|c|cccc|c|c}
I_{(m-s-2)n} & & & & & & & \\ \hline
 & -A_{s+1} & & & & & & \\
 & -A_s & & & & & & \\
 & \vdots & & I_{(s-i+2)n} & & & & \\
 & -A_i & & & & & & \\ \hline
 & I_n & 0 & 0 & \cdots & 0 & & \\ \hline
 & & & & & & I_{(i-1)n} & \\ \hline
 & & & & & & & I_r
\end{array}\right],$$

which satisfies (5.7). Clearly this is operation-free. Thus proves the case for $i > 0$.

For the case $i = 0$ multiply $\mathbb{M}_0$ with the case $i > 0$. So we have $\mathbb{M}_{(0:j)} = \mathbb{M}_0\mathbb{M}_{(1:j)}$, which is equal to

$$
\left[\begin{array}{ccc|c}
I_{(m-j-1)n} & & & \\
& I_{jn} & & \\
& & -A_0 & -C \\ \hline
& & -B & -A
\end{array}\right]
\left[\begin{array}{c|c|ccccc|c}
I_{(m-j-1)n} & & & & & & & \\ \hline
& -A_j & I_n & & & & & \\
& -A_{j-1} & 0 & I_n & & & & \\
& \vdots & \vdots & & \ddots & & & \\
& -A_1 & 0 & 0 & \cdots & I_n & & \\ \hline
& I_n & 0 & 0 & \cdots & 0 & 0 & \\ \hline
& & & & & & & I_r
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c|ccc|c}
I_{(m-j-1)n} & & & & & \\ \hline
& -A_j & & & & \\
& \vdots & & I_{jn} & & \\
& -A_1 & & & & \\ \hline
& -A_0 & 0 & \ldots & 0 & -C \\ \hline
& -B & & & & -A
\end{array}\right]
$$

is operation-free. $\square$

Remark 5.2.1. *Similarly the product* $\mathbb{M}_{(-i:-j)}$, *where* $1 \leq j \leq i \leq m$ *is operation-free and is given by*

$$
\mathbb{M}_{(-i:-j)} = \left[\begin{array}{c|c|c|c|c}
I_{(m-i-1)n} & & & & \\ \hline
& 0_{n\times(i-j+1)n} & I_n & & \\ \hline
& & A_i & & \\
& I_{(i-j+1)n} & A_{i-1} & & \\
& & \vdots & & \\
& & A_j & & \\ \hline
& & & I_{(j-1)n} & \\ \hline
& & & & I_r
\end{array}\right], \qquad j \leq i < m,
$$

$$\mathbb{M}_{(-m:-j)} = \left[\begin{array}{c|c|c|c} 0_{n\times(m-j)n} & A_m & & \\ \hline & A_{m-1} & & \\ I_{(m-j)n} & \vdots & & \\ & A_j & & \\ \hline & & I_{(j-1)n} & \\ \hline & & & -E \end{array}\right].$$

Theorem 5.2.2. *[41] Each product of the form*

$$\prod_{i=m-1}^{0} M_{(c_i:i)}, \text{ for } c_i \in (0:i) \cup \{\infty\}, \tag{5.9}$$

is operation-free. The product in (5.9) is in column standard form.

Theorem 5.2.3. *[41] Let $M_i, i = 0 : m-1$ be the Fiedler matrices associated with the matrix polynomial $P(\lambda)$. Let $\boldsymbol{q}$ be an index tuple of indices from $\{0, 1, \ldots, m-1\}$. Then the following statements are equivalent.*

(a) $M_{\boldsymbol{q}}$ *is operation-free.*

(b) $\boldsymbol{q}$ *satisfies SIP.*

(c) $M_{\boldsymbol{q}}$ *can be written in the column standard form as* $\prod_{i=m-1}^{0} M_{(c_i:i)}$, *for* $c_i \in (0 : i) \cup \{\infty\}$.

Further, let $M_{-i}, i = -1 : -m$ be as in (5.1) and (5.2). Let $\boldsymbol{q}$ be an index tuple of indices from $\{-m, , \ldots, -1\}$. Then the following statements are equivalent.

(a) $M_{\boldsymbol{q}}$ *is operation-free.*

(b) $\boldsymbol{q}$ *satisfies SIP.*

(c) $M_{\boldsymbol{q}}$ *can be written in the column standard form as* $\prod_{i=-1}^{-m} M_{(c_i:i)}$, *for* $c_i \in (-m : i) \cup \{\infty\}$.

The next theorem provides a canonical form of operation-free product, which helps us to construct structure preserving linearizations of system matrix.

Theorem 5.2.4. *Each product of the form*

$$\prod_{i=m-1}^{1} \mathbb{M}_{(c_i:i)}, \text{ for } c_i \in (1:i) \cup \{\infty\}, \tag{5.10}$$

is operation-free. The product in (5.10) is in column standard form.

Proof. We have

$$\prod_{i=m-1}^{1} \mathbb{M}_{(c_i:i)} = \prod_{i=m-1}^{1} \left[\begin{array}{c|c} M_{(c_i:i)} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} \prod_{i=m-1}^{1} M_{(c_i:i)} & \\ \hline & I_r \end{array}\right].$$

Since by Theorem 5.2.2, $\prod_{i=m-1}^{1} M_{(c_i:i)}$ is operation-free for $c_i \in (1:i) \cup \{\infty\}$, hence the result follows.

□

Theorem 5.2.5. *Let* $\boldsymbol{q}$ *be an index tuple of indices from* $\{0, 1, \ldots, m-1\}$.
Case I : Suppose that $0 \notin \boldsymbol{q}$. *Then the following are equivalent.*

(a) $\mathbb{M}_{\boldsymbol{q}}$ *is operation-free.*

(b) $\boldsymbol{q}$ *satisfies SIP.*

(c) $\mathbb{M}_{\boldsymbol{q}}$ *can be written in the column standard form given in (5.10).*

Case II : Suppose that $0 \in \boldsymbol{q}$ *is a simple index of* $\boldsymbol{q}$. *Then the following are equivalent.*

(a) $\mathbb{M}_{\boldsymbol{q}}$ *is operation-free.*

(b) $\boldsymbol{q}$ *satisfies SIP.*

(c) $\mathbb{M}_{\boldsymbol{q}}$ *can be written in the column standard form* $\prod_{i=m-1}^{1} \mathbb{M}_{(c_i:i)}$, *for* $c_i \in (0:i) \cup \{\infty\}$.

Proof. Case I : If $0 \notin \mathbf{q}$ then $\mathbb{M}_{\mathbf{q}} = \left[\begin{array}{c|c} M_{\mathbf{q}} & \\ \hline & I_r \end{array}\right]$. Hence $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 5.2.3.

Case II : Suppose that $0 \in \mathbf{q}$. Since 0 is a simple index of $\mathbf{q}$, we have $\mathbb{M}_{\mathbf{q}} = \mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2}$ for some index tuples σ_1 and σ_2 from $\{1, \ldots, m-1\}$. Now

$$\mathbb{M}_{\sigma_1}\mathbb{M}_0\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} M_0 & -e_m \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right] \left[\begin{array}{c|c} M_{\sigma_2} & \\ \hline & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} M_{\sigma_1}M_0M_{\sigma_2} & -M_{\sigma_1}(e_m \otimes C) \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right] \left[\begin{array}{c|c} M_{\mathbf{q}} & -M_{\sigma_1}(e_m \otimes C) \\ \hline (-e_m^T \otimes B)M_{\sigma_2} & -A \end{array}\right].$$

This shows that $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow M_{\mathbf{q}}, M_{\sigma_1}(e_m \otimes C)$, and $(e_m^T \otimes B)M_{\sigma_2}$ are operation-free. Now $M_{\mathbf{q}} = M_{\sigma_1}M_0M_{\sigma_2}$ is operation free implies that M_{σ_1} and M_{σ_2}

are operation-free. Hence σ_1 and σ_2 satisfies SIP. Since σ_1 and σ_2 are tuples from $\{1, 2, \ldots, m-1\}$, by Theorem 5.2.3, M_{σ_1} and M_{σ_2} can be written in the form $\prod_{i=m-1}^{1} M_{(c_i:i)}$, for $c_i \in (1 : i) \cup \{\infty\}$. Hence by Lemma 5.2.5, it follows that the block entries of $M_{\sigma_1}(e_m \otimes I_n)$ and $(e_m^T \otimes I_n)M_{\sigma_2}$ are either 0 or I_n. In fact, it is easy to see that $M_{\sigma_1}(e_m \otimes I_n) = e_k \otimes I_n$ and $(e_m^T \otimes I_n)M_{\sigma_2} = e_j^T \otimes I_n$ for some k and j. Hence $M_{\sigma_1}(e_m \otimes C)$ and $(e_m^T \otimes B)M_{\sigma_2}$ are operation-free. This shows that $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow$ $M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 5.2.3.

□

Remark 5.2.2. *The assumption that* 0 *is a simple index can not be relaxed in Theorem 5.2.5. For example,* $\prod_{i=3}^{0} \mathbb{M}_{(c_i:i)}$, *for* $c_i \in (0 : i) \cup \{\infty\}$ *may not be operation-free. Indeed, by Lemma 5.2.4, the product* $\mathbb{M}_{0:3}\mathbb{M}_{0:2}\mathbb{M}_{0:1}\mathbb{M}_{0:0}$. *is not operation-free since* $\mathbb{M}_{0:1}\mathbb{M}_{0:0}$ *is not operation-free.*

Considering index tuple from the set $\{-m, -(m-1), \ldots, -1\}$ and using similar arguments as above, we have the following result.

Theorem 5.2.6. *Let* $\boldsymbol{q} = (i_1, i_2, \ldots, i_p)$ *be an index tuple from the set* $\{-m, -(m-1), \ldots, -1\}$.
Case I : *If* $-m \notin \boldsymbol{q}$ *then the following are are equivalent.*

(a) $\mathbb{M}_{\boldsymbol{q}}$ *is operation-free.*

(b) $\boldsymbol{q}$ *satisfies the SIP.*

(c) $\mathbb{M}_{\boldsymbol{q}}$ *can be written in the column standard form* $\prod_{i=-1}^{-(m-1)} \mathbb{M}_{(c_i:i)}$, *for* $c_i \in (-(m-1) : i) \cup \{\infty\}$.

Case II : *If* $-m \in \boldsymbol{q}$ *is a simple index of* $\boldsymbol{q}$ *then the following are are equivalent.*

(a) $\mathbb{M}_{\boldsymbol{q}}$ *is operation-free.*

(b) $\boldsymbol{q}$ *satisfies the SIP.*

(c) $\mathbb{M}_{\boldsymbol{q}}$ *can be written in the column standard form* $\prod_{i=-1}^{-(m-1)} \mathbb{M}_{(c_i:i)}$, *for* $c_i \in (-m : i) \cup \{\infty\}$.

Proof. Case I : If $-m \notin \mathbf{q}$ then $\mathbb{M}_{\mathbf{q}} = \left[\begin{array}{c|c} M_{\mathbf{q}} & \\ \hline & I_r \end{array}\right]$. Hence $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow$ $M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 5.2.3.

Case II : Suppose that $-m \in \mathbf{q}$. Since $-m \in \mathbf{q}$ is a simple index of $\mathbf{q}$, then we have $\mathbb{M}_{\mathbf{q}} = \mathbb{M}_{\sigma_1}\mathbb{M}_{-m}\mathbb{M}_{\sigma_2}$ for some index tuples σ_1 and σ_2 from $\{-(m-1), \ldots, -1\}$. Now

$$\mathbb{M}_{\sigma_1}\mathbb{M}_m\mathbb{M}_{\sigma_2} = \left[\begin{array}{c|c} M_{\sigma_1} & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} M_{-m} & \\ \hline & -E \end{array}\right] \left[\begin{array}{c|c} M_{\sigma_2} & \\ \hline & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} M_{\sigma_1}M_{-m}M_{\sigma_2} & \\ \hline & -E \end{array}\right].$$

This shows that $\mathbb{M}_{\mathbf{q}}$ is operation-free $\Leftrightarrow$ $M_{\mathbf{q}}$ is operation-free. Hence the result follows from Theorem 5.2.3. $\square$

Remark 5.2.3. *If $E = I_r$, then the product $\prod_{i=-1}^{-m} \mathbb{M}_{(c_i:i)}$, for $c_i \in (-m : i) \cup \{\infty\}$ is operation-free, because $E = I_r$ implies that $\mathbb{M}_{-m}\mathbb{M}_{(-m-1)}\mathbb{M}_{-m}$ is operation-free.*

Remark 5.2.4. *The assumption that $-m$ is a simple index can not be relaxed in Theorem 5.2.6. For example, $\prod_{i=-1}^{-4} \mathbb{M}_{(c_i:i)}$, for $c_i \in (-4 : i) \cup \{\infty\}$ may not be operation-free. Indeed, by Lemma 5.2.4, the product $\mathbb{M}_{-4:-1}\mathbb{M}_{-4:-2}\mathbb{M}_{-4:-3}\mathbb{M}_{-4:-4}$ is not operation-free as $\mathbb{M}_{-4:-3}\mathbb{M}_{-4}$ is not operation-free.*

5.3 Generalized Fiedler pencil with repetition

We have seen that a self-adjoint $\mathcal{S}(\lambda)$ does not admit a self-adjoint GF pencil when m is even. It turns out that allowing Fiedler matrices to repeat in the products $\mathbb{M}_\sigma$ and $\mathbb{M}_\tau$ enables a pencil $\mathbb{L}(\lambda) = \lambda\mathbb{M}_\tau - \mathbb{M}_\sigma$ to have certain desired properties.

Example 5.1. Consider $\mathcal{S}(\lambda)$ with $m = 3$. Let $\mathbb{L}(\lambda) := \lambda\mathbb{M}_{-3}\mathbb{M}_1 - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0\mathbb{M}_1$. Then

$$\mathbb{L}(\lambda) = \lambda \left[\begin{array}{ccc|c} A_3 & & & \\ & -A_1 & I_n & \\ & I_n & 0 & \\ \hline & & & -E \end{array}\right] - \left[\begin{array}{ccc|c} -A_2 & -A_1 & I_n & 0 \\ -A_1 & -A_0 & 0 & -C \\ I_n & 0 & 0 & 0 \\ \hline 0 & -B & 0 & -A \end{array}\right].$$

Note that $\mathbb{L}(\lambda) = (\lambda\mathbb{M}_{-3} - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0)\mathbb{M}_1$ is not a GF pencil. Also note that in this pencil the Fiedler matrix $\mathbb{M}_1$ is repeated and at the same time the pencil is operation-free. Further, $\mathbb{L}(\lambda)$ is symmetric when $\mathcal{S}(\lambda)$ is symmetric. ■

So we are interested in finding the operation-free pencils which allow repetition of Fiedler matrices. We define generalized Fiedler pencils with repetition (GFPR), which are operation-free, as follows.

Definition 5.3.1 (GFPR). *Let $\mathcal{S}(\lambda)$ be the system matrix. Let $0 \leq h \leq m-1$, and let σ and τ be permutations of $\{0, 1, \ldots, h\}$ and $\{-m, -m+1, \ldots, -h-1\}$, respectively. Let σ_1 and σ_2 be index tuples with elements from $\{1, 2, \ldots, h-1\}$, such that $(\sigma_1, \sigma, \sigma_2)$ satisfies the SIP. Similarly, let τ_1 and τ_2 be index tuples with elements from $\{-m+1, \ldots, -h-2\}$ such that (τ_1, τ, τ_2) satisfies the SIP. Then the pencil*

$$\mathbb{L}(\lambda) := \lambda \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$$

is called a generalized Fiedler pencil with repetition (GFPR) of $\mathcal{S}(\lambda)$.

Remark 5.3.1. *Note that in the Definition 5.3.1, 0 is simple index and so $\mathbb{L}(\lambda)$ can always be expressed as $\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}(\lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma})\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$, where $\lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma}$ is a PGF pencil. Thus a GFPR is strictly equivalent to a PGF pencil.*

(1) If $\mathbb{M}_{\tau_1}, \mathbb{M}_{\tau_2}, \mathbb{M}_{\sigma_2}, \mathbb{M}_{\sigma_1}$ are all the empty index tuples, then $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma}$ is a PGF pencil and hence a GF pencil. Note, however, that not all GF pencils are GFPR. For example, $\mathbb{L}(\lambda) = \lambda\mathbb{M}_2^{-1}\mathbb{M}_1^{-1} - \mathbb{M}_3\mathbb{M}_6^{-1}\mathbb{M}_0\mathbb{M}_5$ is a GF pencil of $\mathcal{S}(\lambda)$ with $m = 6$, but not a GFPR.

(2) The constraint A_m being non-singular can be relaxed. We need A_m to be non-singular only if the index $-m$ is in τ_1 and τ_2.

Theorem 5.3.1. *Let $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ be a GFPR of $\mathcal{S}(\lambda)$. Then $\mathbb{L}(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$. If $G(\lambda)$ is minimal, then $\mathbb{L}(\lambda)$ is also a linearization of $G(\lambda)$.*

Proof. We have $\mathbb{L}(\lambda) = \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}(\lambda\mathbb{M}_{\tau} - \mathbb{M}_{\sigma})\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$. Since $\mathbb{M}_{\tau_i} = \left[\begin{array}{c|c} M_{\tau_i} & \\ \hline & I_r \end{array}\right]$, $\mathbb{M}_{\sigma_i} = \left[\begin{array}{c|c} M_{\sigma_i} & \\ \hline & I_r \end{array}\right]$, $i = 1, 2$, the pencil $\mathbb{L}(\lambda)$ is strictly equivalent to a PGF pencil. Therefore every GFPR $\mathbb{L}(\lambda)$ of $\mathcal{S}(\lambda)$ is a linearization of $\mathcal{S}(\lambda)$. $\square$

Example 5.2. Consider the system matrix $\mathcal{S}(\lambda)$ with $m = 12$. Consider $\sigma = (6, 1:5, 0)$, $\sigma_2 = (1:4), \tau = (-7, -8, -11:-9, -12), \tau_2 = (-11:-10), \sigma_1 = \{\emptyset\}, \tau_1 = (-8, -9)$. Then $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ is a GFPR of $\mathcal{S}(\lambda)$. ■

Next, recall the block transpose given in Definition 2.4.3.

Definition 5.3.2 (Block-symmetry). *A block $p \times p$ matrix A with $m \times n$ blocks is said to be* block-symmetric *if $A^{\mathcal{B}} = A$.*

Since $M_i^{\mathcal{B}} = M_i$ for $i = 0 : m$, by Definition 2.4.3, we have $(\mathbb{M}_i)^{\mathcal{B}} = \mathbb{M}_i$ for $i = 0 : m$. Thus the Fiedler matrices are block symmetric.

Lemma 5.3.1. *[41] Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two index tuples from the set $\{0, 1, \ldots, m-1\}$. Let $M_{\boldsymbol{q}_1}$ and $M_{\boldsymbol{q}_2}$ be two products of Fiedler matrices associated with the matrix polynomial $P(\lambda)$ such that $M_{\boldsymbol{q}_1} M_{\boldsymbol{q}_2}$ is operation-free. Then $(M_{\boldsymbol{q}_1} M_{\boldsymbol{q}_2})^{\mathcal{B}} = (M_{\boldsymbol{q}_2})^{\mathcal{B}} (M_{\boldsymbol{q}_1})^{\mathcal{B}}$. Further, the result also holds for indices from $\{-1, -2, \ldots - m\}$.*

We have the following result for Fiedler matrices associated with $\mathcal{S}(\lambda)$.

Lemma 5.3.2. *Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two index tuples from the set $\{0, 1, 2, \ldots, m-1\}$. Let $\mathbb{M}_{\boldsymbol{q}_1}$ and $\mathbb{M}_{\boldsymbol{q}_2}$ be two products of Fiedler matrices associated with the system matrix $\mathcal{S}(\lambda)$ such that $\mathbb{M}_{\boldsymbol{q}_1} \mathbb{M}_{\boldsymbol{q}_2}$ is operation-free. Then $(\mathbb{M}_{\boldsymbol{q}_1} \mathbb{M}_{\boldsymbol{q}_2})^{\mathcal{B}} = (\mathbb{M}_{\boldsymbol{q}_2})^{\mathcal{B}} (\mathbb{M}_{\boldsymbol{q}_1})^{\mathcal{B}}$. Further, the result also holds for indices from $\{-1, -2, \ldots - m\}$.*

Proof. Case I : Suppose that $0 \notin \mathbf{q}_1 \cup \mathbf{q}_2$. Then we have

$$\mathbb{M}_{\mathbf{q}_1} \mathbb{M}_{\mathbf{q}_2} = \left[\begin{array}{c|c} M_{\mathbf{q}_1} & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} M_{\mathbf{q}_2} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{\mathbf{q}_1} M_{\mathbf{q}_2} & \\ \hline & I_r \end{array}\right].$$

Thus

$$\left[\begin{array}{c|c} M_{\mathbf{q}_1} M_{\mathbf{q}_2} & \\ \hline & I_r \end{array}\right]^{\mathcal{B}} = \left[\begin{array}{c|c} (M_{\mathbf{q}_1} M_{\mathbf{q}_2})^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} M_{\mathbf{q}_1}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] = (\mathbb{M}_{\mathbf{q}_2})^{\mathcal{B}} (\mathbb{M}_{\mathbf{q}_1})^{\mathcal{B}}.$$

Case II : Suppose that $0 \in \mathbf{q}_1 \cup \mathbf{q}_2$. If $0 \in \mathbf{q}_1 \cap \mathbf{q}_2$ then $\mathbb{M}_{\mathbf{q}_1} \mathbb{M}_{\mathbf{q}_2}$ is not operation-free. Thus either $0 \in \mathbf{q}_1$ or $0 \in \mathbf{q}_2$. Without loss of generality assume that $0 \in \mathbf{q}_1$. Then $\mathbb{M}_{\mathbf{q}_1} = \mathbb{M}_p \mathbb{M}_0 \mathbb{M}_{p_1}$ for some indices p and p_1. Now $\mathbb{M}_{\mathbf{q}_1} \mathbb{M}_{\mathbf{q}_2} = \mathbb{M}_p \mathbb{M}_0 \mathbb{M}_{p_1} \mathbb{M}_{\mathbf{q}_2} = \mathbb{M}_p \mathbb{M}_0 \mathbb{M}_{\mathbf{s}} = \left[\begin{array}{c|c} M_p M_0 M_{\mathbf{s}} & -M_p(e_m \otimes C) \\ \hline -(e_m^T \otimes B) M_{\mathbf{s}} & -A \end{array}\right]$, where $M_{\mathbf{s}} = \mathbb{M}_{p_1} \mathbb{M}_{\mathbf{q}_2}$. Since $\mathbb{M}_{\mathbf{q}_1} \mathbb{M}_{\mathbf{q}_2}$ is operation-free, we have $M_{\mathbf{q}_1} M_{\mathbf{q}_2}$ is operation-free. Hence $M_{\mathbf{q}_1}$ and $M_{\mathbf{q}_2}$ are operation-free. Thus M_p is operation-free. Consequently, from the proof of Lemma 5.2.5, we have $M_p(e_m \otimes I_n) = e_k \otimes I_n$, $(e_m^T \otimes I_n) M_{p_1} = e_j^T \otimes I_n$ and $(e_m^T \otimes I_n) M_{\mathbf{s}} = e_h^T \otimes I_n$ for some k, j and h. Hence we have $\mathbb{M}_{\mathbf{q}_1} \mathbb{M}_{\mathbf{q}_2} = \left[\begin{array}{c|c} M_p M_0 M_{\mathbf{s}} & -e_k \otimes C \\ \hline -e_h^T \otimes B & -A \end{array}\right]$. This shows that

$$(\mathbb{M}_{\mathbf{q}_1} \mathbb{M}_{\mathbf{q}_2})^{\mathcal{B}} = \left[\begin{array}{c|c} (M_p M_0 M_{\mathbf{s}})^{\mathcal{B}} & -e_h \otimes C \\ \hline -e_k^T \otimes B & -A \end{array}\right].$$

Now

$$(\mathbb{M}_{\mathbf{q}_2})^{\mathcal{B}}(\mathbb{M}_{\mathbf{q}_1})^{\mathcal{B}} = \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] (\mathbb{M}_p\mathbb{M}_0\mathbb{M}_{p_1})^{\mathcal{B}}$$

$$= \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] \left(\left[\begin{array}{c|c} M_pM_0M_{p_1} & -M_p(e_m \otimes C) \\ \hline -(e_m^T \otimes B)M_{p_1} & -A \end{array}\right]\right)^{\mathcal{B}}$$

$$= \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] \left(\left[\begin{array}{c|c} M_pM_0M_{p_1} & -e_k \otimes C \\ \hline -(e_j^T \otimes B) & -A \end{array}\right]\right)^{\mathcal{B}} =$$

$$\left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}} & \\ \hline & I_r \end{array}\right] \left[\begin{array}{c|c} (M_pM_0M_{p_1})^{\mathcal{B}} & -(e_j \otimes C) \\ \hline -e_k^T \otimes B & -A \end{array}\right] = \left[\begin{array}{c|c} M_{\mathbf{q}_2}^{\mathcal{B}}(M_pM_0M_{p_1})^{\mathcal{B}} & -M_{\mathbf{q}_2}^{\mathcal{B}}(e_j \otimes C) \\ \hline -e_k^T \otimes B & -A \end{array}\right].$$

Since $(e_m^T \otimes I_n)M_{\mathbf{s}} = e_h^T \otimes I_n$ and $(e_m^T \otimes I_n)M_{p_1} = e_j^T \otimes I_n$, we have $(e_j^T \otimes I_n)M_{\mathbf{q}_2} = e_h^T \otimes I_n$. This shows that $M_{\mathbf{q}_2}^{\mathcal{B}}(e_j \otimes I_n) = e_h \otimes I_n$. Hence by Lemma 5.3.1, we have $(\mathbb{M}_{\mathbf{q}_2})^{\mathcal{B}}(\mathbb{M}_{\mathbf{q}_1})^{\mathcal{B}} =$

$$\left[\begin{array}{c|c} (M_pM_0M_{p_1}M_{\mathbf{q}_2})^{\mathcal{B}} & -M_{\mathbf{q}_2}^{\mathcal{B}}(e_j \otimes C) \\ \hline -e_k^T \otimes B & -A \end{array}\right] = \left[\begin{array}{c|c} (M_pM_0M_{\mathbf{s}})^{\mathcal{B}} & -e_h \otimes C \\ \hline -e_k^T \otimes B & -A \end{array}\right] = (\mathbb{M}_{\mathbf{q}_1}\mathbb{M}_{\mathbf{q}_2})^{\mathcal{B}}.$$

□

Consider the first companion form $\mathcal{C}_1(\lambda)$ of $\mathcal{S}(\lambda)$ given by

$$\mathcal{C}_1(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_{m-1}\mathbb{M}_{m-2}\cdots\mathbb{M}_1\mathbb{M}_0 = \left[\begin{array}{c|c} C_1(\lambda) & -e_1 \otimes C \\ \hline -e_m^T \otimes B & -A \end{array}\right],$$

where $C_1(\lambda)$ is the first companion form of $P(\lambda)$ given in (1.11). Note that $\mathcal{C}_1(\lambda)$ is operation-free. Hence by Lemma 5.3.2, we have

$$\mathcal{C}_1(\lambda)^{\mathcal{B}} = \lambda\mathbb{M}_m - \mathbb{M}_0\mathbb{M}_1\cdots\mathbb{M}_{m-2}\mathbb{M}_{m-1} = \left[\begin{array}{c|c} C_2(\lambda) & -e_m \otimes C \\ \hline -e_1^T \otimes B & -A \end{array}\right] = \mathcal{C}_2(\lambda)$$

is the second companion form of $\mathcal{S}(\lambda)$ given in (2.17), where $C_2(\lambda)$ is the second companion form of $P(\lambda)$ given in (1.12).

Let $\mathbf{q}_1$ be the index tuple containing indices from $\{0, 1, \ldots, m-1\}$. Let $\mathbb{M}_{\mathbf{q}_1}$ be the product of Fiedler matrices such that $\mathbb{M}_{\mathbf{q}_1}$ is operation-free. Since $\mathbb{M}_i^{\mathcal{B}} = \mathbb{M}_i$ then by Lemma 5.3.2, we have

$$\mathbb{M}_{rev\,\mathbf{q}_1} = \mathbb{M}_{\mathbf{q}_1}^{\mathcal{B}}$$

is also operation-free. Further, $\mathbb{M}_{\mathbf{q}_1}$ is operation-free and block symmetric if and only if

$$\mathbb{M}_{\mathbf{q}_1} = \mathbb{M}_{\mathbf{q}_1}^{\mathcal{B}} = \mathbb{M}_{rev\,\mathbf{q}_1} \Leftrightarrow \mathbf{q}_1 \sim rev\,\mathbf{q}_1.$$

For example, consider $\mathbf{q}_1 = (0, 2, 3, 2)$. Then

$$\mathbb{M}_{\mathbf{q}_1} = \mathbb{M}_0\mathbb{M}_2\mathbb{M}_3\mathbb{M}_2 = \left[\begin{array}{cc|c} I_{4n} & & \\ & -A_0 & -C \\ \hline & -B & -A \end{array}\right] \left[\begin{array}{ccccc|c} I_n & & & & & \\ & -A_3 & -A_2 & I_n & & \\ & A_2 & I_n & 0 & & \\ & I_n & 0 & 0 & & \\ & & & & I_n & \\ \hline & & & & & I_r \end{array}\right]$$

$$= \left[\begin{array}{ccccc|c} I_n & & & & & \\ & -A_3 & -A_2 & I_n & & \\ & -A_2 & I_n & 0 & & \\ & I_n & 0 & 0 & & \\ & & & & -A_0 & -C \\ \hline & & & & -B & -A \end{array}\right] = \mathbb{M}_2\mathbb{M}_3\mathbb{M}_2\mathbb{M}_0.$$

Since $\mathbb{M}_{\mathbf{q}_1}$ is operation-free and $(\mathbb{M}_i)^{\mathcal{B}} = \mathbb{M}_i$, we have $(\mathbb{M}_{\mathbf{q}_1})^{\mathcal{B}} = \mathbb{M}_2^{\mathcal{B}}\mathbb{M}_3^{\mathcal{B}}\mathbb{M}_2^{\mathcal{B}}\mathbb{M}_0^{\mathcal{B}} = \mathbb{M}_2\mathbb{M}_3\mathbb{M}_2\mathbb{M}_0 = \mathbb{M}_{rev\,\mathbf{q}_1} = \mathbb{M}_{\mathbf{q}_1}$, that is, $rev\,\mathbf{q}_1 \sim \mathbf{q}_1$. Thus $\mathbb{M}_{\mathbf{q}_1}$ is block symmetric.

The following examples illustrate symmetric GFPR of $\mathcal{S}(\lambda)$ when $\mathcal{S}(\lambda)$ is symmetric.

Example 5.3. Suppose that $\mathcal{S}(\lambda)$ is symmetric with $m = 3$. Consider $\sigma = (1 : 2), \tau = \emptyset, \sigma_2 = 1, \tau_2 = \emptyset, \sigma_1 = \emptyset, \tau_1 = \emptyset$. Then $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{-3}\mathbb{M}_1 - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0\mathbb{M}_1$ in Example 5.1, is symmetric, since $\mathcal{S}$ is symmetric. Note that $\mathbb{L}(\lambda)$ is block symmetric, since $rev\,(1, 2, 0, 1) \sim (1, 2, 0, 1)$ and $rev\,(-3, 1) \sim (-3, 1)$. ■

Example 5.4. Suppose that $\mathcal{S}(\lambda)$ is symmetric with $m = 5$. Consider $\sigma = (1 : 2), \tau = (-4 : -3), \sigma_1 = \emptyset, \tau_1 = \emptyset, \sigma_2 = 1, \tau_2 = -4$. Then $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{-4}\mathbb{M}_{-3}\mathbb{M}_{-5}\mathbb{M}_1\mathbb{M}_{-4} - \mathbb{M}_1\mathbb{M}_2\mathbb{M}_0\mathbb{M}_1\mathbb{M}_{-4}$ is operation-free symmetric pencil, since $\mathcal{S}$ is symmetric. Note that $\mathbb{L}(\lambda)$ is block symmetric, since

$$rev\,(-4, -3, -5, 1, -4) \sim (-4, -3, -5, 1, -4) \text{ and } rev\,(1, 2, 0, 1, -4) \sim (1, 2, 0, 1, -4).$$

■

The next example illustrates a symmetric linearization of a symmetric $\mathcal{S}(\lambda)$, which is not a GFPR.

Example 5.5. Suppose that $\mathcal{S}(\lambda)$ is symmetric with $m = 4$. Then

$$\mathbb{L}(\lambda) = \lambda \mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_{4}\mathbb{M}_{-2}\mathbb{M}_{-3} - \mathbb{M}_{1}\mathbb{M}_{0}\mathbb{M}_{1}$$

is symmetric, since $(\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_{4}\mathbb{M}_{-2}\mathbb{M}_{-3})^T = (\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_{4}\mathbb{M}_{-2}\mathbb{M}_{-3})$, $(\mathbb{M}_{1}\mathbb{M}_{0}\mathbb{M}_{1})^T = \mathbb{M}_{1}\mathbb{M}_{0}\mathbb{M}_{1}$, and $\mathcal{S}$ is symmetric. Note that $\mathbb{L}(\lambda)$ does not satisfy SIP. Hence $\mathbb{L}(\lambda)$ is not operation-free, but symmetric. Therefore, we have more pencils with repetition of Fiedler matrices which is not operation-free but have symmetric structure. ■

We have already established the eigenvector formula for Fiedler pencil of $\mathcal{S}(\lambda)$ and its associated transfer function $G(\lambda)$. Now, we derive the eigenvector formula for PGF and NPGF pencil of system matrix.

Theorem 5.3.2 (Eigenvector formula for PGF pencil of system matrix)**.** *Let P_i for $i = 0, 1, \ldots, m$, be the ith Horner shift of matrix polynomial $P(\lambda)$. Let $\omega := (\omega_0, \omega_1)$ and $\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be a PGF pencil of $\mathcal{S}(\lambda)$. Let $\lambda \in Sp(\mathcal{S})$.*

(a) Define

$$\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S}) := \left[\begin{array}{c|c} E_{\omega_0,\omega_1}(P) & 0 \\ \hline 0 & I_r \end{array}\right],$$

where $E_{\omega_0,\omega_1}(P)$ is as in Theorem 1.2.5. Then $\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \longrightarrow \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$ is an isomorphism.

(b) Set $rev\,\omega := (rev\,\omega_0, rev\,\omega_1)$ and define

$$\mathbb{H}_{\omega_0,\omega_1}(\mathcal{S}) := \mathbb{E}_{(rev\,\omega_0, rev\,\omega_1)}(\mathcal{S}^T) = \left[\begin{array}{c|c} E_{(rev\,\omega_0, rev\,\omega_1)}(P^T) & \\ \hline & I_r \end{array}\right].$$

Then $\mathbb{H}_{\omega_0,\omega_1}(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \longrightarrow \mathcal{N}_l(\mathbb{T}_\omega(\lambda))$ is an isomorphism.

Proof. Suppose that ω_1 and ω_0 are in column standard form. Assume that ω_1 has c_{-m} consecutions at $-m$. Then, there exists an index tuple τ such that

$$\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_\tau \mathbb{M}_{(-m:-m+c_{-m})} - \mathbb{M}_{\omega_0}. \tag{5.11}$$

Case I : If $c_{-m} = 0$, then $\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_\tau \mathbb{M}_{-m} - \mathbb{M}_{\omega_0}$ and $\mathbb{L}_\sigma(\lambda) = \mathbb{M}_{-rev\,\tau}\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_{-m} - \mathbb{M}_{(-rev\,\tau, \omega_0)}$ is a Fiedler pencil associated with a bijection $\sigma = (-rev\,\tau, \omega_0)$. Hence $\mathcal{N}_r(\mathbb{L}_\sigma(\lambda)) = \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Observe that the index tuple $(-rev\tau, \omega_0)$ is a permutation of $\{0, 1, \ldots, m-1\}$. Set $\xi = csf(\sigma) = csf(-rev\,\tau, \omega_0) = (\mathbf{b}_\alpha, \mathbf{b}_{\alpha-1}, \ldots, \mathbf{b}_1)$.

Then by Theorem 3.2.7, $\mathbb{E}_{\omega_0,\omega_1}(\mathcal{S}) := \mathbb{E}_{\xi}(\mathcal{S}) = \left[\begin{array}{c|c} E_{\xi}(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_{\omega}(\lambda))$.

Case II : Suppose that $c_{-m} \neq 0$. Since $c_{-m} \neq 0$, $\mathbb{L}_{\widehat{\sigma}}(\lambda) = \mathbb{M}_{-rev\,\tau}\mathbb{T}_{\omega}(\lambda)\mathbb{M}_{(m-c_{-m}:m-1)}$

$$= \lambda\mathbb{M}_{-rev\,\tau}\mathbb{M}_{\tau}\mathbb{M}_{(-m:-m+c_{-m})}\mathbb{M}_{(m-c_{-m}:m-1)} - \mathbb{M}_{-rev\,\tau}\mathbb{M}_{\omega_0}\mathbb{M}_{(m-c_{-m}:m-1)}$$

$$= \lambda\mathbb{M}_{(-m:-m+c_{-m})}\mathbb{M}_{(m-c_{-m}:m-1)} - \mathbb{M}_{-rev\,\tau}\mathbb{M}_{\omega_0}\mathbb{M}_{(m-c_{-m}:m-1)}$$

$$= \lambda\mathbb{M}_{-m} - \mathbb{M}_{(-rev\,\tau,\omega_0,m-c_{-m}:m-1)}$$

is a Fiedler pencil associated with a bijection $\widehat{\sigma} = (-rev\,\tau, \omega_0, m - c_{-m} : m-1)$. So the map $\mathcal{N}_r(\mathbb{L}_{\widehat{\sigma}}(\lambda)) \to \mathcal{N}_r(\mathbb{T}_{\omega}(\lambda))$, $\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \mathbb{M}_{(m-c_{-m}:m-1)}\begin{bmatrix} u \\ v \end{bmatrix}$ is an isomorphism. By Theorem 3.2.7, $\mathbb{E}_{\widehat{\sigma}}(\mathcal{S}) = \left[\begin{array}{c|c} E_{\widehat{\sigma}}(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{L}_{\widehat{\sigma}}(\lambda))$, where $E_{\widehat{\sigma}}(P)$ is as in Theorem 3.2.4 (c). Consequently, the map $\mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{T}_{\omega}(\lambda))$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_{\widehat{\sigma}}(\mathcal{S})\begin{bmatrix} x \\ y \end{bmatrix}$ is an isomorphism. Now our aim is to calculate $\mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_{\widehat{\sigma}}(\mathcal{S})$. Let $\sigma = csf(\widehat{\sigma}) = csf(-rev\,\tau, \omega_0, m - c_{-m} : m-1)$. Then $\mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_{\sigma}(\mathcal{S})$

$$= \left[\begin{array}{c|c} M_{(m-c_{-m}:m-1)} & \\ \hline & I_r \end{array}\right]\left[\begin{array}{c|c} E_{\sigma}(P) & \\ \hline & I_r \end{array}\right] = \left[\begin{array}{c|c} M_{(m-c_{-m}:m-1)}E_{\sigma}(P) & \\ \hline & I_r \end{array}\right].$$

By Theorem 1.2.5 [18], we have $M_{(m-c_{-m}:m-1)}E_{\sigma}(P) = E_{\omega_0,\omega_1}(P)$. Hence $\mathbb{M}_{(m-c_{-m}:m-1)}\mathbb{E}_{\sigma}(\mathcal{S}) = \mathbb{E}_{\omega_0,\omega_1}(\mathcal{S})$. This completes the proof.

Next, note that $\mathcal{N}_l(\mathcal{S}(\lambda)) = \mathcal{N}_r(\mathcal{S}(\lambda)^T)$ and $\mathcal{N}_l(\mathbb{T}_{\omega}(\mathcal{S})) = \mathcal{N}_r(\mathbb{T}_{\omega}(\mathcal{S})^T) = \mathcal{N}_r(\mathbb{T}_{rev\,\omega}(\mathcal{S}^T))$. This shows that $\mathbb{H}_{\omega_0,\omega_1}(\mathcal{S}) := \mathbb{E}_{rev\,\omega_0,rev\,\omega_1}(\mathcal{S}^T)$ is an isomorphism from $\mathcal{N}_l(\mathcal{S}(\lambda))$ to $\mathcal{N}_l(\mathbb{T}_{\omega}(\lambda))$. □

Corollary 5.3.3 (Eigenvector formula for PGF pencil of transfer function). *Let P_i for $i = 0, 1, \ldots, m$, be the ith Horner shift of matrix polynomial P. Let $\omega = (\omega_0, \omega_1)$ and $\mathbb{T}_{\omega}(\lambda) = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ be a PGF pencil of $G(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$.*

(a) Define

$$\mathbb{E}_{\omega_0,\omega_1}(G) := \begin{bmatrix} E_{\omega_0,\omega_1}(P) \\ (\lambda E - A)^{-1}B \end{bmatrix},$$

where $E_\omega(P)$ is as in Theorem 1.2.5. Then $\mathbb{E}_{\omega_0,\omega_1}(G) : \mathcal{N}_r(G(\lambda)) \longrightarrow \mathcal{N}_r(\mathbb{T}(\lambda))$ is an isomorphism.

(b) Set rev $\omega := (rev\,\omega_0, rev\,\omega_1)$ and define

$$\mathbb{H}_{\omega_0,\omega_1}(G) := \mathbb{E}_{(rev\,\omega_0, rev\,\omega_1)}(G^T) := \begin{bmatrix} E_{(rev\,\omega_0, rev\,\omega_1)}(P^T) \\ (C(\lambda E - A)^{-1})^T \end{bmatrix}.$$

Then $\mathbb{H}_{\omega_0,\omega_1}(G) : \mathcal{N}_l(G(\lambda)) \longrightarrow \mathcal{N}_l(\mathbb{T}_\omega(\lambda))$ is an isomorphism.

Example 5.6. Consider the system matrix $\mathcal{S}(\lambda)$ with $m = 3$ and the associated transfer function $G(\lambda)$. Let $\mathbb{K}_\omega(\lambda) = \lambda\mathbb{M}_{\omega_0} - \mathbb{M}_{\omega_0} = \lambda\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_{-1} - \mathbb{M}_0$ be the PGF pencil of $G(\lambda)$. Here ω_1 has 2 consecutions at -3, i.e., $c_{-3} = 2$. Now

$$\mathbb{M}_{-3}\mathbb{M}_{-2}\mathbb{M}_{-1} = \left[\begin{array}{ccc|c} 0 & 0 & A_3 & \\ I_n & 0 & A_2 & \\ 0 & I_n & A_1 & \\ \hline & & & -E \end{array}\right], \text{ so } \mathbb{K}_\omega(\lambda) = \left[\begin{array}{ccc|c} -I_n & 0 & \lambda A_3 & \\ \lambda I_n & -I_n & \lambda A_2 & \\ 0 & \lambda I_n & \lambda A_1 + A_0 & C \\ \hline & & B & A - \lambda E \end{array}\right].$$

Thus by the Corollary 5.3.3, $\begin{bmatrix} \lambda A_3 \\ \lambda^2 A_3 + \lambda A_2 \\ I_n \\ (\lambda E - A)^{-1} B \end{bmatrix} x \in \mathcal{N}_r(\mathbb{K}(\lambda))$, where $x \in \mathcal{N}_r(G(\lambda))$. ■

The next result describes eigenvector formula for GF pencils of $\mathcal{S}(\lambda)$ for the case when $\mathbb{M}_0^{-1}$ does not appear in the GF pencils.

Theorem 5.3.4 (Eigenvector formula for GF pencil of system matrix)**.** *Let $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\boldsymbol{t}} - \mathbb{M}_\sigma$ be a GF pencil of $\mathcal{S}(\lambda)$. Assume that $0, m \in \sigma$. Let $\sigma' = \sigma \setminus \{m\}$, $\xi := csf(-rev\boldsymbol{t}, \sigma') = (\boldsymbol{b}_\alpha, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_\alpha = (t_{k-1} + 1 : t_k)$ for $k = 1, 2, \ldots, \alpha$.*
Case I : Suppose that $m - 1$ is to the left of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) If m is to the right of 0 in σ then

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(\mathcal{S}) := \left[\begin{array}{c|c} \begin{bmatrix} A_m B_0 & B_1 & \cdots & B_{m-1} \end{bmatrix}^{\mathcal{B}} & \\ \hline & -E \end{array}\right] \tag{5.12}$$

is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\xi(i) \in \boldsymbol{b}_j$, for some $j = 1, 2, \ldots, \alpha$, and the block B_i's are as in Theorem 3.2.4.

(b) *If m is to the left of 0 in σ and s is the largest index such that $(m-s:m) \in \sigma$ then set $\tau_0 = \sigma \setminus \{m-s:m\}$. Then $\mathbb{E}_{\sigma,\boldsymbol{t}}(\mathcal{S}) = \left[\begin{array}{c|c} E_{\tau_0,\tau_1}(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\tau_1 = (-rev(m-s:m), \boldsymbol{t})$ and $E_{\tau_0,\tau_1}(P)$ is as in Theorem 1.2.5.*

Case II : Suppose that $m-1$ is to the right of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) *If m is to the left of 0 in σ, then*

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(\mathcal{S}) := \mathbb{E}_{\xi}(\mathcal{S}), \tag{5.13}$$

is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\mathbb{E}_\xi(\mathcal{S})$ is given in Theorem 3.2.7.

(b) *If m is to the right of 0 in σ and h is the largest index such that $(m, m-1, \ldots, m-h)$ in σ then set $\sigma'' = \sigma \setminus \{m, m-1, \ldots, m-h\}$, $\sigma_4 = (-rev\boldsymbol{t}, \sigma'')$, $\sigma_3 = (-(m-h), -(m-1), -m) =: (\boldsymbol{t}_1, -m)$ and $\xi' = csf(-rev\boldsymbol{t}_1, \sigma_4)$. Then*

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(\mathcal{S}) := \mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)} \mathbb{M}_{-m} \mathbb{E}_{\xi'}(\mathcal{S})$$

$$= \left[\begin{array}{c|c} B_1 & 0 \\ B_2 & \\ \vdots & \\ B_h & \\ A_m B_0 + A_{m-1} B_1 + \cdots + A_{m-h} B_h & \\ B_{h+2} & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & -E \end{array}\right].$$

is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\mathbb{E}_{\xi'}(\mathcal{S})$ is given in Theorem 3.2.7 and $B_i, i = 1 : m-1$, are given in Theorem 3.2.4.

Proof. Case I: If $m-1$ is to the left of m in $(-rev\mathbf{t}, \sigma)$, then $(-rev\mathbf{t}, \sigma)$ is equivalent to either $(-rev\mathbf{t}, \sigma', m)$ or $(-rev\mathbf{t}, m, \sigma')$, since $\mathbb{M}_0$ and $\mathbb{M}_m$ do not commute, so there is a possibility for m to be either to the left or to the right of 0.

(a) If m is right of 0 then $(-rev\,\mathbf{t},\sigma) \sim (-rev\mathbf{t},\sigma',m)$, so $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\mathbf{t}} - \mathbb{M}_\sigma = \lambda\mathbb{M}_{\mathbf{t}} - \mathbb{M}_{\sigma'}\mathbb{M}_m$ and hence

$$\mathbb{L}_\tau(\lambda) := \mathbb{M}_{-rev\mathbf{t}}\mathbb{T}_\omega(\lambda)\mathbb{M}_{-m} = \mathbb{M}_{-rev\mathbf{t}}(\lambda\mathbb{M}_{\mathbf{t}} - \mathbb{M}_{\sigma'}\mathbb{M}_m)\mathbb{M}_{-m} = \lambda\mathbb{M}_{-m} - \mathbb{M}_{-rev\mathbf{t}}\mathbb{M}_{\sigma'}$$

is a Fiedler pencil associated with a bijection $\tau = (-rev\mathbf{t},\sigma')$. So $v \mapsto \mathbb{M}_{-m}v$ is an isomorphism from $\mathcal{N}_r(\mathbb{L}_\tau(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. By Theorem 3.2.7, we have $\mathbb{E}_\tau(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}_\tau(\lambda))$ is an isomorphism. Thus for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}_r(\mathcal{S}(\lambda))$ we have $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \mathbb{M}_{-m}\mathbb{E}_\tau(\mathcal{S})\begin{bmatrix} x \\ y \end{bmatrix}$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Since $\xi = csf(\tau) = csf(-rev\,\mathbf{t},\sigma')$, we have $\mathbb{M}_{-m}\mathbb{E}_\xi(\mathcal{S})$

$$= \mathbb{M}_{-m}\left[\begin{array}{c|c} E_\xi(P) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{cc|c} A_m & & \\ & I_{(m-1)n} & \\ \hline & & -E \end{array}\right]\left[\begin{array}{c|c} B_0 & 0 \\ \vdots & 0 \\ B_{m-1} & 0 \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} A_mB_0 & B_1 & \cdots & B_{m-1} \end{bmatrix}^{\mathcal{B}} & \\ \hline & -E \end{array}\right].$$

(b) Since m is left of 0 in σ, we have $(-rev\mathbf{t},\sigma) \sim (-rev\mathbf{t}, m-s:m,\tau_0)$. Consequently, we have $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{\mathbf{t}} - \mathbb{M}_\sigma = \lambda\mathbb{M}_{\mathbf{t}} - \mathbb{M}_{m-s:m-2}\mathbb{M}_{m-1}\mathbb{M}_m\mathbb{M}_{\tau_0}$. This shows that

$$\begin{aligned} \mathbb{L}_\tau(\lambda) &:= \mathbb{M}_{-rev(m-s:m)}\mathbb{T}_\omega(\lambda) \\ &= \mathbb{M}_{-rev(m-s:m)}\left(\lambda\mathbb{M}_{\mathbf{t}} - \mathbb{M}_{m-s:m-2}\mathbb{M}_{m-1}\mathbb{M}_m\mathbb{M}_{\tau_0}\right) \\ &= \lambda\mathbb{M}_{-rev(m-s:m)}\mathbb{M}_{\mathbf{t}} - \mathbb{M}_{\tau_0} = \lambda\mathbb{M}_{\tau_1} - \mathbb{M}_{\tau_0} \end{aligned}$$

is a PGF pencil. Hence $\mathcal{N}_r(\mathbb{T}_\omega(\lambda)) = \mathcal{N}_r(\mathbb{L}_\tau(\lambda))$. By Theorem 5.3.2, we have $\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) = \left[\begin{array}{c|c} E_{\tau_0,\tau_1}(P) & \\ \hline & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.

CaseII: (a) If $m-1$ is to the right of m and m is to the left of 0 in $(-rev\mathbf{t},\sigma)$, then $(-rev\mathbf{t},\sigma)$ is equivalent to $(m,-rev\mathbf{t},\sigma')$. So $\mathbb{L}_\tau(\lambda) = \mathbb{M}_{-m}\mathbb{M}_{-rev\mathbf{t}}\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{-m} - \mathbb{M}_{-rev\mathbf{t}}\mathbb{M}_{\sigma'}$ is a Fiedler pencil associated with a bijection $\tau = (-rev\mathbf{t},\sigma')$. Hence $\mathcal{N}_r(\mathbb{L}_\tau(\lambda)) = \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Since $\xi = csf(\tau)$, by Theorem 3.2.7, $\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) := \mathbb{E}_\xi(\mathcal{S})$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$.

(b) If $m-1$ is to the right of m in $(-rev\mathbf{t},\sigma)$ and m is to the right of 0 in σ, then $(-rev\mathbf{t},\sigma) \sim (-rev\mathbf{t},\sigma'',m,m-1,\ldots,m-h)$. So

$$\begin{aligned}\mathbb{L}_\tau(\lambda) &= \mathbb{M}_{-rev\mathbf{t}}\mathbb{T}_\omega(\lambda)\mathbb{M}_{-(m-h)}\cdots\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}\\ &= \lambda\mathbb{M}_{-(m-h)}\cdots\mathbb{M}_{-(m-1)}\mathbb{M}_{-m} - \mathbb{M}_{-rev\mathbf{t}}\mathbb{M}_{\sigma''} = \lambda\mathbb{M}_{\sigma_3} - \mathbb{M}_{\sigma_4}\end{aligned}$$

is a PGF pencil. Hence $\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \mathbb{M}_{-(m-h)}\cdots\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}v$ is an isomorphism from $\mathcal{N}_r(\mathbb{L}_\tau)$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Since σ_3 has always 0 consecutions at $-m$, i.e., $c_{-m}=0$, by Theorem 5.3.2, $\mathbb{E}_{\xi'}(\mathcal{S}) = \left[\begin{array}{c|c} B_0 & \\ B_1 & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & I_r \end{array}\right]$ is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{L}_\tau(\lambda))$, where $B_i, i = 0:m-1$ are as in Theorem 3.2.4. This shows that

$$\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) = \mathbb{M}_{-(m-h)}\cdots\mathbb{M}_{-(m-1)}\mathbb{M}_{-m}\mathbb{E}_{\xi'}(\mathcal{S})$$

is an isomorphism from $\mathcal{N}_r(\mathcal{S}(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$. Now

$$\begin{aligned}\mathbb{M}_{-(m-1)}\mathbb{M}_{-m} &= \left[\begin{array}{ccc|c} 0 & I_n & & \\ I_n & A_{m-1} & & \\ & & I_{(m-2)n} & \\ \hline & & & I_r \end{array}\right]\left[\begin{array}{ccc|c} A_m & & & \\ & I_n & & \\ & & I_{(m-2)n} & \\ \hline & & & -E \end{array}\right]\\ &= \left[\begin{array}{ccc|c} 0 & I_n & & \\ A_m & A_{m-1} & & \\ & & I_{(m-2)n} & \\ \hline & & & -E \end{array}\right]\end{aligned}$$

and

$$\mathbb{M}_{-(m-h)} \cdots \mathbb{M}_{-(m-1)} \mathbb{M}_{-m} = \left[\begin{array}{cccc|c|c} 0 & I_n & & & & \\ 0 & 0 & I_n & & & \\ 0 & \vdots & 0 & I_n & & \\ A_m & A_{m-1} & \cdots & A_{m-h} & & \\ \hline & & & & I_{(m-h-1)n} & \\ \hline & & & & & -E \end{array}\right].$$

Hence

$$\mathbb{E}_{\sigma,\mathbf{t}}(\mathcal{S}) = \left[\begin{array}{cccc|c|c} 0 & I_n & & & & \\ 0 & 0 & I_n & & & \\ 0 & \vdots & 0 & I_n & & \\ A_m & A_{m-1} & \cdots & A_{m-h} & & \\ \hline & & & & I_{(m-h-1)n} & \\ \hline & & & & & -E \end{array}\right] \left[\begin{array}{c|c} B_0 & \\ B_1 & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & I_r \end{array}\right]$$

$$= \left[\begin{array}{c|c} B_1 & 0 \\ B_2 & \\ \vdots & \\ B_h & \\ A_m B_0 + A_{m-1} B_1 + \cdots + A_{m-h} B_h & \\ B_{h+2} & \\ \vdots & \\ B_{m-1} & \\ \hline 0 & -E \end{array}\right].$$

□

Remark 5.3.2. *Recovery of eigenvectors of $\mathcal{S}(\lambda)$ from GF pencils follows from Theorem 4.3.2.*

Corollary 5.3.5 (Eigenvector formula for GF pencil of transfer function)**.** *Let $\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_t - \mathbb{M}_\sigma$ ba GF pencil of a regular $G(\lambda)$ given in (5.4). Let $\lambda \in \mathbb{C}$ be an eigenvalue of $G(\lambda)$. Assume that $0, m \in \sigma$. Let $\sigma' = \sigma \setminus \{m\}$, $\xi = csf(-rev\boldsymbol{t}, \sigma')$.*
Case I : Suppose that $m-1$ is to the left of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) *If m is to the right of 0 then*

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(G) := \begin{bmatrix} A_m B_0 \\ B_1 \\ \vdots \\ B_{m-1} \\ -E(\lambda E - A)^{-1}B \end{bmatrix} \tag{5.14}$$

is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\xi(i) \in \boldsymbol{b}_j$, for some $j = 1, 2, \ldots, \alpha$, and the block B_i is as in Theorem 3.2.4.

(b) *If m is to the left of 0 in σ and s is the largest index such that $(m-s:m) \in \sigma$ then set $\tau_0 = \sigma \setminus \{m-s:m\}$. Then $\mathbb{E}_{\sigma,\boldsymbol{t}}(G) = \begin{bmatrix} E_{\tau_0,\tau_1}(P) \\ (\lambda E - A)^{-1}B \end{bmatrix}$ is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\tau_1 = (-rev(m-s:m), \boldsymbol{t})$ and $E_{\tau_0,\tau_1}(P)$ is as in Theorem 1.2.5.*

Case II : Suppose that $m-1$ is to the right of m in $(-rev\boldsymbol{t}, \sigma)$.

(a) *If m is to the left of 0, then*

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(G) := \mathbb{E}_\xi(G) \tag{5.15}$$

is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\mathbb{E}_\xi(G)$ is given in Theorem 3.2.10.

(b) *If m is to the right of 0 in σ and h is the largest index such that $(m, m-1, \ldots, m-h)$ in σ then set $\sigma'' = \sigma \setminus \{m, m-1, \ldots, m-h\}$, $\sigma_4 = (-rev\boldsymbol{t}, \sigma'')$, $\sigma_3 = (-(m-h), -(m-1), -m) =: (\boldsymbol{t}_1, -m)$ and $\xi' = csf(-rev\boldsymbol{t}_1, \sigma_4)$. Then*

$$\mathbb{E}_{\sigma,\boldsymbol{t}}(G) := \mathbb{M}_{-(m-1)}\mathbb{M}_{-m}\mathbb{E}_{\xi'}(\mathcal{S}) = \left[\begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_h \\ A_m B_0 + A_{m-1}B_1 + \cdots + A_{m-h}B_h \\ B_{h+1} \\ \vdots \\ B_{m-1} \\ \hline -E(\lambda E - A)^{-1}B \end{array}\right]$$

is an isomorphism from $\mathcal{N}_r(G(\lambda))$ to $\mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $\mathbb{E}_{\xi'}(G)$ is given in Theorem 3.2.10 and $B_i, i = 1 : m-1$, are given in Theorem 3.2.4.

The following examples illustrate the result in Corollary 5.3.5.

Example 5.7. Consider a rational matrix function $G(\lambda) = \lambda^6 A_6 + \ldots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B$ and the GF pencil $\mathbb{T}(\lambda) = \lambda \mathbb{M}_{\mathbf{t}} - \mathbb{M}_\sigma = \lambda \mathbb{M}_{-1}\mathbb{M}_{-5}\mathbb{M}_{-3} - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_6\mathbb{M}_4$. Now $\sigma = (2,0,6,4), \mathbf{t} = (-1,-5,-3), -rev\mathbf{t} = (3,5,1)$, and $(-rev\mathbf{t}, \sigma) = (3,5,1,2,0,6,4)$. Note that 5 is to the left of 6 in $(-rev\mathbf{t}, \sigma)$ and 6 is to the right of 0 in σ. This is the case $I(a)$ of Corollary 5.3.5. Hence by Corollary 5.3.5, we have $\sigma' = (2,0,4)$, $\xi = csf(-rev\,\mathbf{t}, \sigma') = (5, 3:4, 1:2, 0)$, and

$$\mathbb{E}_{\sigma,\mathbf{t}}(G) = \begin{bmatrix} \begin{bmatrix} A_6\lambda^3 I_n & \lambda^2 I_n & \lambda^2 P_2 & \lambda I_n & \lambda P_4 & I_n \end{bmatrix}^{\mathcal{B}} \\ -E(\lambda E - A)^{-1}B \end{bmatrix}.$$

Thus $\mathbb{E}_{\sigma,\mathbf{t}}(G)x$ is a right eigenvector of $\mathbb{T}(\lambda)$.

Now, consider the GF pencil $\mathbb{T}_\omega(\lambda) = \lambda \mathbb{M}_{-1}\mathbb{M}_{-5}\mathbb{M}_{-3} - \mathbb{M}_6\mathbb{M}_2\mathbb{M}_0\mathbb{M}_4 = \lambda \mathbb{M}_{\mathbf{t}} - \mathbb{M}_\sigma$. Then $\sigma = (6,2,0,4), \mathbf{t} = (-1,-5,-3)$, and $-rev\mathbf{t} = (3,5,1)$. By Corollary 5.3.5, we have $\sigma' = (2,0,4)$ and $(-rev\mathbf{t}, \sigma) = (3,5,1,6,2,0,4) \sim (3,5,1,6,2,0,4)$. Note that 5 is to the left of 6 in $(-rev\mathbf{t}, \sigma)$ and 6 is to the left of 0 in σ. This is the Case $I(b)$ of Corollary 5.3.5. Thus $\tau_1 = (-6,-1,-5,-3)$ and $\tau_0 = (2,0,4)$ Hence by corollary 5.3.3 we have $c_{-m} = 1$, $\xi = (3:4, 1:2, 0)$ and

$$\mathbb{E}_{\sigma,\mathbf{t}}(G) = \begin{bmatrix} \begin{bmatrix} \lambda^3 P_0 & \lambda^2 I_n & \lambda^2 P_2 & \lambda I_n & \lambda P_4 & I_n \end{bmatrix}^{\mathcal{B}} \\ (\lambda E - A)^{-1}B \end{bmatrix}.$$

Thus $\mathbb{E}_{\sigma,\mathbf{t}}(G)x$ is a right eigenvector of $\mathbb{T}_\omega(\lambda)$.

Next, consider the GF pencil $\mathbb{T}(\lambda) = \lambda \mathbb{M}_{-4}\mathbb{M}_{-1}\mathbb{M}_{-3} - \mathbb{M}_5\mathbb{M}_2\mathbb{M}_6\mathbb{M}_0 = \lambda \mathbb{M}_{\mathbf{t}} - \mathbb{M}_\sigma$. Now $\sigma = (5,2,6,0), \mathbf{t} = (-4,-1,-3)$, and $-rev\mathbf{t} = (3,1,4)$. Thus $\sigma' = (5,2,0)$ and $(-rev\mathbf{t}, \sigma) = (3,1,4,5,2,6,0) \sim (3,1,4,5,6,2,0)$. Note that 5 is to the left of 6 in $(-rev\mathbf{t}, \sigma)$ and 6 is to the left of 0 in σ. This is the case $I(b)$ of Corollary 5.3.5. Hence we have $\tau_1 = (-6,-5,-4,-1,-3)$ and $\tau_0 = (2,0)$. Consequently, by corollary 5.3.3 we have $c_{-m} = 2$, $\xi = (3, 1:2, 0)$ and

$$\mathbb{E}_{\sigma,\mathbf{t}}(G) = \begin{bmatrix} \begin{bmatrix} \lambda^3 P_0 & \lambda^3 P_1 & \lambda^2 I_n & \lambda P_3 & \lambda P_4 & I_n \end{bmatrix}^{\mathcal{B}} \\ (\lambda E - A)^{-1}B \end{bmatrix}.$$

Thus $\mathbb{E}_{\sigma,\mathbf{t}}(G)x$ is a right eigenvector of $\mathbb{T}(\lambda)$.

Finally, consider the GF pencil $\mathbb{T}(\lambda) = \lambda\mathbb{M}_{-1}\mathbb{M}_{-4}\mathbb{M}_{-3} - \mathbb{M}_2\mathbb{M}_6\mathbb{M}_0\mathbb{M}_5$. Now $\sigma = (2,6,0,5), \mathbf{t} = (-1,-4,-3)$, and $-rev\,\mathbf{t} = (3,4,1)$. Thus $\sigma' = (2,0,5)$ and $(-rev\,\mathbf{t},\sigma) = (3,4,1,2,6,0,5) \sim (6,3,4,1,2,0,5)$. Note that 5 is to the right of 6 in $(-rev\,\mathbf{t},\sigma)$, and 6 is to the left of 0 in σ. This is the case $II(a)$ of Corollary 5.3.5. Hence $\xi = csf(-rev\,\mathbf{t},\sigma') = (3:5,1:2,0)$ and by Corollary 5.3.5, we have

$$\mathbb{E}_{\sigma,\mathbf{t}}(G) = \begin{bmatrix} \begin{bmatrix} \lambda^2 I_n & \lambda^2 P_1 & \lambda^2 P_2 & \lambda I_n & \lambda P_4 & I_n \end{bmatrix}^{\mathcal{B}} \\ (\lambda E - A)^{-1}B \end{bmatrix}.$$

Thus $\mathbb{E}_{\sigma,\mathbf{t}}(G)x$ is a right eigenvector of $\mathbb{T}(\lambda)$.

■

Example 5.8. Consider $G(\lambda) = \lambda^4 A_4 + \ldots + \lambda A_1 + A_0 + C(\lambda E - A)^{-1}B$ and the GF pencil $\mathbb{T}_\omega(\lambda) = \lambda\mathbb{M}_{-1} - \mathbb{M}_2\mathbb{M}_0\mathbb{M}_4\mathbb{M}_3 = \lambda\mathbb{M}_{\mathbf{t}} - \mathbb{M}_\sigma$. Now $\sigma = (2,0,4,3), \mathbf{t} = (-1)$. Note that 3 is to the right of 4 in $(-rev\,\mathbf{t},\sigma) = (1,2,0,4,3)$ and 4 is to the right of 0 in σ. This is the Case $II(b)$ of Corollary 5.3.5. So $\sigma'' = \sigma \setminus \{4,3\} = (2,0)$, $-rev\mathbf{t} = (1)$ and $\sigma_4 = (-rev\,\mathbf{t},\sigma'') = (1)$. Again by Corollary 5.3.5, we have $\sigma_3 = (-3,-4) := (\mathbf{t}_1,-4)$, So $\xi' = csf(-rev\mathbf{t}_1,\sigma_4) = csf(3,1,2,0) = (3,1:2,0) = (\mathbf{b}_3,\mathbf{b}_2,\mathbf{b}_1)$, and $\alpha = 3$. Thus by Corollary 5.3.3, we have $\mathbb{E}_{\xi'}(\mathcal{S})x = \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ \lambda P_2 \\ I_n \\ (\lambda E - A)^{-1}B \end{bmatrix} x \in \mathcal{N}_r(\mathcal{K}_\tau(\lambda))$ and by Corollary 5.3.5, $\mathbb{E}_{\sigma,\mathbf{t}}(G)x = \mathbb{M}_{-3}\mathbb{M}_{-4}\mathbb{E}_{\xi'}(\mathcal{S})x$

$$= \left[\begin{array}{cccc|c} 0 & I_n & & & \\ A_4 & A_3 & & & \\ & & I_n & & \\ & & & I_n & \\ \hline & & & & -E \end{array}\right] \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ \lambda^3 A_4 + \lambda^2 A_3 + \lambda A_2 \\ I_n \\ (\lambda E - A)^{-1}B \end{bmatrix} x = \begin{bmatrix} \lambda I_n \\ \lambda^2 A_4 + \lambda A_3 \\ \lambda^3 A_4 + \lambda^2 A_3 + \lambda A_2 \\ I_n \\ -E(\lambda E - A)^{-1}B \end{bmatrix} x$$

is a right eigenvector of $\mathbb{T}_\omega(\lambda)$, i.e., $\mathbb{E}_{\sigma,\mathbf{t}}(G)x \in \mathcal{N}_r(\mathbb{T}_\omega(\lambda))$, where $x \in \mathcal{N}_r(G(\lambda))$. ■

Next, we recall some definitions given in [18] for recovery of eigenvectors from GFPR and the eigenvector formula of GFPR.

Definition 5.3.3 (Type 1 indices relative to a simple index tuple [18])**.** *Let h be a non-negative integer and σ be a permutation of $\{0, 1, \ldots, h\}$. Let s be an index in $\{0, 1, \ldots, h-1\}$. Then s is said to be a* right index of type 1 relative to σ *if there is a string $(t_{d-1}+1 : t_d)$ in $csf(\sigma)$ such that $s = t_{d-1}+1 < t_d$.*

Definition 5.3.4 (Associated simple tuple [18])**.** *Let h be a non-negative integer and σ be a permutation of $\{0, 1, \ldots, h\}$. Let $csf(\sigma) = (\boldsymbol{b}_{\alpha+1}, \boldsymbol{b}_{\alpha}, \ldots, \boldsymbol{b}_1)$, where $\boldsymbol{b}_i = (t_{i-1}+1 : t_i), i = 1, \ldots, \alpha+1$, are the strings of $csf(\sigma)$. Set $\mathfrak{s}(\sigma) := csf(\sigma)$ Then we say that $\mathfrak{s}(\sigma)$ is the* simple tuple associated with σ*. If s is an index of type 1 with respect to σ, say $s = t_{d-1}+1 < t_d$, then the* simple tuple associated with (σ, s) *is the simple tuple:*

$$\mathfrak{s}(\sigma, s) := (\boldsymbol{b}_{\alpha+1}, \boldsymbol{b}_{\alpha}, \ldots, \boldsymbol{b}_{d+1}, \tilde{\boldsymbol{b}}_d, \tilde{\boldsymbol{b}}_{d-1}, \boldsymbol{b}_{d-2}, \ldots, \boldsymbol{b}_1), \textit{where}$$

$$\tilde{\boldsymbol{b}}_d = (t_{d-1}+2 : t_d) \qquad \tilde{\boldsymbol{b}}_{d-1} = (t_{d-2}+1 : t_{d-1}+1) \textit{ if } s \neq 0$$

and

$$\mathfrak{s}(\sigma, 0) := (\boldsymbol{b}_{\alpha+1}, \boldsymbol{b}_{\alpha}, \ldots, \tilde{\boldsymbol{b}}_1, \tilde{\boldsymbol{b}}_0) \textit{where}$$

$$\tilde{\boldsymbol{b}}_1 = (1 : t_1) \qquad \tilde{\boldsymbol{b}}_0 = (0).$$

Definition 5.3.5 (Index tuple of type 1 [18])**.** *Let h be a non-negative integer and σ be a permutation of $\{0, 1, \ldots, h\}$. Let σ_2 and σ_1 be tuples with indices from $\{0, 1, \ldots, h-1\}$, possibly with repetitions. We say that $\sigma_2 = (s_1, \ldots, s_r)$, where s_i is the ith index of σ_2, is an index tuple of type 1 relative to σ if, for $i = 1, \ldots, r$, s_i is a right index of type 1 with respect to $\mathfrak{s}(\sigma, (s_1, \ldots, s_{i-1}))$, where $\mathfrak{s}(\sigma, (s_1, \ldots, s_{i-1})) := \mathfrak{s}(\mathfrak{s}(\sigma, (s_1, \ldots, s_{i-2})), s_{i-1})$ for $i > 2$.*

5.3.1 Eigenvector formula for GFPR

We derive eigenvector formula for GFPR, when the tuples are of type 1.

Lemma 5.3.3. *Let $0 \leq h \leq m-3$, and let σ and τ be permutations of $\{0, 1, \ldots, h\}$ and $\{-m, -m+1, \ldots, -h-1\}$ respectively. Assume that σ_1, σ_2 are index tuples with elements from $\{1, 2, \ldots, h-1\}$, and τ_1, τ_2 are index tuples with elements from $\{-m+1, -m+2 \ldots, -h-2\}$. Then $\mathbb{M}_{\tau_2}$ commutes with $\mathbb{M}_{\sigma}$ and $\mathbb{M}_{\sigma_2}$, and $\mathbb{M}_{\tau}$ commutes with $\mathbb{M}_{\sigma_2}$.*

Proof. Since the distance between each pair of indices in τ_2 and σ is greater than 1, by commutativity relation $\mathbb{M}_{\tau_2}$ commute with $\mathbb{M}_{\sigma}$. Similarly, the other cases follow. $\square$

Lemma 5.3.4. *Let $\sigma, \tau, \sigma_1, \sigma_2, \tau_1, \tau_2$ be as in Lemma 5.3.3 such that $(\sigma_1, \sigma, \sigma_2)$ and (τ_1, τ, τ_2) satisfy the SIP. Suppose that τ_2 and σ_2 are type 1 tuples relative to τ and σ, respectively. Let $\mathfrak{s}(\sigma, \sigma_2)$ and $\mathfrak{s}(\tau, \tau_2)$ be the simple tuple associated with (σ, σ_2) and (τ, τ_2). Then $(\sigma, \sigma_2) \sim (\sigma_2, \mathfrak{s}(\sigma, \sigma_2))$ and $(\tau, \tau_2) \sim (\tau_2, \mathfrak{s}(\tau, \tau_2))$.*

Proof. First we prove $(\sigma, \sigma_2) \sim (\sigma_2, \mathfrak{s}(\sigma, \sigma_2))$ by induction on the number of indices of σ_2, (other one follow similarly applying induction on τ_2). Assume that $\sigma_2 = (r_1, r_2, \ldots, r_p)$, where r_i denotes the ith index in σ_2, and $\sigma = (\mathbf{b}_\alpha, \mathbf{b}_{\alpha-1}, \ldots, \mathbf{b}_1)$, where $\mathbf{b}_i = (t_{i-1}+1, t_i)$, for $i = 1, 2, \ldots, \alpha$. Since σ_2 is of type 1 relative to σ, we have $r_1 = t_{d-1}+1 < t_d$ for some $1 \leq d \leq \alpha$. So $(\sigma, r_1) \sim (t_{d-1}+1, \mathbf{b}_\alpha, \ldots, \mathbf{b}_{d+1}, t_{d-1}+2 : t_d, \mathbf{b}_{d-1}, t_{d-1}+1, \ldots, \mathbf{b}_1)) = (r_1, \mathfrak{s}(\sigma, r_1))$, if $d > 1$. We mention that σ_2 does not contain 0. Hence $d \neq 1$. Applying induction on indices of σ_2 we have $(\sigma, \sigma_2) \sim (\sigma_2, \mathfrak{s}(\sigma, \sigma_2))$. Similarly applying induction τ_2 we get $(\tau, \tau_2) \sim (\tau_2, \mathfrak{s}(\tau, \tau_2))$. $\square$

Example 5.9. Consider $m = 10$. Let $\sigma = (7, 5 : 6, 2 : 4, 0 : 1) = (\mathbf{b}_4, \mathbf{b}_3, \mathbf{b}_2, \mathbf{b}_1), \sigma_2 = (5)$ such that σ_2 is of type 1 relative to σ. By Lemma 5.3.4, here $\alpha = 4$, and for $\sigma_2 = t_{d-1}+1 = 5$, $d = 3 > 1$. So $(\sigma, \sigma_2) = (7, 5 : 6, 2 : 4, 0 : 1, 5) \sim (5, 7, 6 : 6, 2 : 4, 5, 0 : 1) = (5, \mathbf{b}_4, 6 : 6, \mathbf{b}_2, 5, \mathbf{b}_1))$, and $\mathfrak{s}(\sigma, \sigma_2) = (7, 5 : 6, 2 : 4, 0 : 1, 5) = (5, 7, 6 : 6, 2 : 4, 0 : 1, 5) = (7, 6 : 6, 2 : 4, 0 : 1, 5)$, and $(\sigma_2, \mathfrak{s}(\sigma, \sigma_2)) = (5, 7, 6 : 6, 2 : 4, 0 : 1, 5) = (5, 7, 6 : 6, 2 : 4, 5, 0 : 1)$. Therefore $(\sigma, \sigma_2) \sim (\sigma_2, \mathfrak{s}(\sigma, \sigma_2))$. ■

Example 5.10. Consider $m = 12$. Let $\sigma = (6, 1 : 5, 0) = (\mathbf{b}_3, \mathbf{b}_2, \mathbf{b}_1), \sigma_2 = (1 : 4) = (r_1, r_2, r_3, r_4)$ such that σ_2 is of type 1 relative to σ. By Lemma 5.3.4, here $\alpha = 3$, $d = 2$, and $r_1 = t_{d-1}+1 = 1$. So $(\sigma, r_1) = (6, 1 : 5, 0, 1) \sim (1, 6, 2 : 5, 0, 1) = (1, \mathbf{b}_3, 2 : 5, \mathbf{b}_1, 1)$, and $\mathfrak{s}(\sigma, r_1) = (6, 1 : 5, 0, 1) = (1, 6, 2 : 5, 0, 1) = (6, 2 : 5, 0, 1)$, and $(r_1, \mathfrak{s}(\sigma, r_1)) = (1, 6, 2 : 5, 0, 1)$. Therefore $(\sigma, r_1) = (r_1, \mathfrak{s}(\sigma, r_1))$. Again $(\sigma, r_1, r_2) = (6, 1 : 5, 0, 1, 2) \sim (1, 2, 6, 3 : 5, 0, 1, 2) = (1, 2, \mathbf{b}_3, 3 : 5, \mathbf{b}_1, 1, 2)$, and $\mathfrak{s}(\sigma, (r_1, r_2)) = (6, 1 : 5, 0)(1, 2) = (1, 6, 2 : 5, 0, 1)(2) = (6, 2 : 5, 0, 1)(2) = (6, 3 : 5, 0, 1, 2)$, and $((r_1, r_2), \mathfrak{s}(\sigma, (r_1, r_2))) = (1, 2, 6, 3 : 5, 0, 1, 2)$. Therefore $(\sigma, r_1, r_2) = ((r_1, r_2)\mathfrak{s}(\sigma, (r_1, r_2)))$. Similarly we get $(\sigma, \sigma_2) \sim (\sigma_2, \mathfrak{s}(\sigma, \sigma_2))$. ■

The following result gives eigenvector formula for GFPR and eigenvector recovery from GFPR.

Theorem 5.3.6 (Eigenvector formula of GFPR)**.** *Let $\sigma, \tau, \sigma_1, \sigma_2, \tau_1, \tau_2$ satisfy the conditions in Lemma 5.3.3 and Lemma 5.3.4. Let $\mathbb{L}(\lambda) = \lambda \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ be a GFPR of $\mathcal{S}(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$ and δ be as in (4.7).*

(a) Set $\omega_0 = \mathfrak{s}(\sigma, \sigma_2)$, $\omega_1 = \mathfrak{s}(\tau, \tau_2)$ and $\omega = (\omega_0, \omega_1)$. Define $\mathbb{E}_{\mathbb{L}}(\mathcal{S}) := \mathbb{E}_{\omega_0, \omega_1}(\mathcal{S})$, where $\mathbb{E}_{\omega_0, \omega_1}(\mathcal{S})$ is as in Theorem 5.3.2. Then $\mathbb{E}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_r(\mathcal{S}(\lambda)) \to \mathcal{N}_r(\mathbb{L}(\lambda))$

is an isomorphism. Further, assume that ω_0 *has* c_0 *consecutions at* 0*. Define* $\mathbb{F}_{\mathbb{L}}(\mathcal{S}) := \mathbb{F}_{\omega_0,\omega_1}(\mathcal{S})$*, where* $\mathbb{F}_{\omega_0,\omega_1}(\mathcal{S}) := \left[\begin{array}{c|c} (e^T_{(m-c_0)} \otimes I_n) & \\ \hline & I_r \end{array}\right]$ *if* $\omega \neq \delta$ *and* $\mathbb{F}_{\omega_0,\omega_1}(\mathcal{S}) := \left[\begin{array}{c|c} (e^T_1 \otimes I_n) & \\ \hline & I_r \end{array}\right]$ *if* $\omega = \delta$*. Then* $\mathbb{F}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ *is an isomorphism.*

(b) *Assume that* $rev\,\tau_1$ *and* $rev\,\sigma_1$ *are type* 1 *tuples relative to* $rev\,\tau$ *and* $rev\,\sigma$*, respectively. Also suppose that* $\mathfrak{s}(rev\sigma, rev\sigma_1)$*,* $\mathfrak{s}(rev\tau, rev\tau_1)$ *are the simple tuples associated with* $(rev\sigma, rev\sigma_1)$ *and* $(rev\tau, rev\tau_1)$ *respectively. Set* $\xi_1 = \mathfrak{s}(rev\tau, rev\tau_1)$*,* $\xi_0 = \mathfrak{s}(rev\sigma, rev\sigma_1)$ *and* $\xi = (\xi_0, \xi_1)$*. Define* $\mathbb{H}_{\mathbb{L}}(\mathcal{S}) := \mathbb{E}_{\xi_0,\xi_1}(\mathcal{S}^T)$*, where* $\mathbb{E}_{\xi_0,\xi_1}(\mathcal{S}^T)$ *is as in Theorem 5.3.2. Then* $\mathbb{H}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_l(\mathcal{S}(\lambda)) \to \mathcal{N}_l(\mathbb{L}(\lambda))$ *is an isomorphism. Further, assume that* ξ_0 *has* c_0 *consecutions at* 0*. Define* $\mathbb{K}_{\mathbb{L}}(\mathcal{S}) := \mathbb{K}_{\xi_0,\xi_1}(\mathcal{S}^T)$*, where* $\mathbb{K}_{\xi_0,\xi_1}(\mathcal{S}^T) := \left[\begin{array}{c|c} (e^T_{(m-c_0)} \otimes I_n) & \\ \hline & I_r \end{array}\right]$ *when* $\xi \neq \delta$ *and* $\mathbb{K}_{\xi_0,\xi_1}(\mathcal{S}^T) := \left[\begin{array}{c|c} (e^T_1 \otimes I_n) & \\ \hline & I_r \end{array}\right]$ *when* $\xi = \delta$*. Then* $\mathbb{K}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}(\lambda)) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ *is an isomorphism.*

Proof. Given that $\mathbb{L}(\lambda) = \lambda \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$ is a GFPR of $\mathcal{S}(\lambda)$. By Lemma 5.3.3, $\mathbb{M}_{\tau_2}$ commutes with $\mathbb{M}_{\sigma}$ and $\mathbb{M}_{\sigma_2}$, and $\mathbb{M}_{\tau}$ commutes with $\mathbb{M}_{\sigma_2}$. Hence by Lemma 5.3.4, we have

$$\begin{aligned}\mathbb{L}(\lambda) &= \lambda \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}\mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)} \\ &= \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}(\lambda\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)} - \mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)}) = \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau_2}\mathbb{M}_{\sigma_2}\mathcal{K}_{\omega}(\lambda),\end{aligned}$$

where $\mathcal{K}_{\omega}(\lambda) = \lambda\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)} - \mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)} = \lambda\mathbb{M}_{\omega_1} - \mathbb{M}_{\omega_0}$ is a PGF pencil. This shows that $\mathcal{N}_r(\mathbb{L}(\lambda)) = \mathcal{N}_r(\mathcal{K}_{\omega}(\lambda))$. Hence by Theorem 5.3.2, $\mathbb{E}_{\mathbb{L}}(\mathcal{S}) := \mathbb{E}_{\omega_0,\omega_1}(\mathcal{S})$ is an isomorphism from $\mathcal{N}_l(\mathcal{S}(\lambda))$ to $\mathcal{N}_l(\mathbb{L}(\lambda))$. Since $\mathcal{K}_{\omega}(\lambda)$ is a PGF pencil and ω_0 has c_0 consecutions at 0, then by Theorem 4.3.1, $\mathbb{F}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_r(\mathbb{L}(\lambda)) \to \mathcal{N}_r(\mathcal{S}(\lambda))$ is an isomorphism.

Next, note that $\mathcal{N}_l(\mathcal{S}(\lambda)) = \mathcal{N}_r(\mathcal{S}(\lambda)^T)$ and $\mathcal{N}_l(\mathbb{L}(\lambda)) = \mathcal{N}_r(\mathbb{L}(\lambda)^T)$. Now

$$\begin{aligned}\mathbb{L}(\lambda)^T &= (\lambda \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2})^T - (\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2})^T \\ &= \lambda \mathbb{M}^T_{\tau_2}\mathbb{M}^T_{\sigma_2}\mathbb{M}^T_{\tau}\mathbb{M}^T_{\sigma_1}\mathbb{M}^T_{\tau_1} - \mathbb{M}^T_{\tau_2}\mathbb{M}^T_{\sigma_2}\mathbb{M}^T_{\sigma}\mathbb{M}^T_{\sigma_1}\mathbb{M}^T_{\tau_1} \\ &= \lambda \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{rev\,\tau_1} - \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\sigma}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{rev\,\tau_1},\end{aligned}$$

where $\widehat{\mathbb{M}}_j = \mathbb{M}_j(\mathcal{S}^T)$ is the Fiedler matrices associated with $\mathcal{S}^T(\lambda)$. Note that by Lemma 5.3.3, we have $\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma} = \mathbb{M}_{\sigma}\mathbb{M}_{\tau_1}$. Taking transpose we have $\widehat{\mathbb{M}}_{rev\,\sigma}\widehat{\mathbb{M}}_{rev\,\tau_1} =$

$\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma}$. Similarly, by Lemma 5.3.3, $\widehat{\mathbb{M}}_{rev\,\tau_1}$ commutes $\widehat{\mathbb{M}}_{rev\,\sigma_1}$, and $\widehat{\mathbb{M}}_{rev\,\tau}$ commutes with $\widehat{\mathbb{M}}_{rev\,\sigma_1}$. Since rev τ_1 and rev σ_1 are type 1 tuples relative to rev τ and rev σ, respectively, and $\mathfrak{s}(rev\sigma, rev\sigma_1)$ and $\mathfrak{s}(rev\tau, rev\tau_1)$ are the simple tuples associated with $(rev\sigma, rev\sigma_1)$ and $(rev\tau, rev\tau_1)$, respectively, by Lemma 5.3.4, we have $(rev\,\sigma, rev\,\sigma_1) \sim (rev\,\sigma_1, \xi_0)$ and $(rev\,\tau, rev\,\tau_1) \sim (rev\,\tau_1, \xi_1)$. Therefore

$$\begin{aligned}
\mathbb{L}(\lambda)^T &= \lambda\widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{\xi_1} - \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\widehat{\mathbb{M}}_{\xi_0} \\
&= \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}(\lambda\widehat{\mathbb{M}}_{\xi_1} - \widehat{\mathbb{M}}_{\xi_0}) \\
&= \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\left(\lambda\mathbb{M}_{\xi_1}(\mathcal{S}^T) - \mathbb{M}_{\xi_0}(\mathcal{S}^T)\right) \\
&= \widehat{\mathbb{M}}_{rev\,\tau_2}\widehat{\mathbb{M}}_{rev\,\sigma_2}\widehat{\mathbb{M}}_{rev\,\tau_1}\widehat{\mathbb{M}}_{rev\,\sigma_1}\mathcal{H}_\xi(\mathcal{S}^T(\lambda)),
\end{aligned}$$

where $\mathcal{H}_\xi(\mathcal{S}^T(\lambda)) = \lambda\mathbb{M}_{\xi_1}(\mathcal{S}^T) - \mathbb{M}_{\xi_0}(\mathcal{S}^T)$ is a PGF pencil. This shows that $\mathcal{N}_l(\mathbb{L}(\lambda)) = \mathcal{N}_r(\mathbb{L}(\lambda)^T) = \mathcal{N}_r(\mathcal{H}_\xi(\mathcal{S}^T(\lambda)))$. By part (a) it follows that $\mathbb{H}_{\mathbb{L}}(\mathcal{S}) := \mathbb{E}_{\xi_0,\xi_1}(\mathcal{S}^T)$ is an isomorphism from $\mathcal{N}_l(\mathcal{S}(\lambda))$ to $\mathcal{N}_l(\mathbb{L}(\lambda))$. Since $\mathcal{H}_\xi(\mathcal{S}^T(\lambda))$ is a PGF pencil and ξ_0 has c_0 consecutions at 0, then by Theorem 4.3.1, we have $\mathbb{K}_{\mathbb{L}}(\mathcal{S}) : \mathcal{N}_l(\mathbb{L}(\lambda)) \to \mathcal{N}_l(\mathcal{S}(\lambda))$ is an isomorphism. $\square$

Example 5.11. Consider the system matrix $\mathcal{S}(\lambda)$ with $m = 12$ and the associated transfer function $G(\lambda)$. Consider the GFPR $\mathbb{L}(\lambda) = \lambda\mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\tau}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2} - \mathbb{M}_{\tau_1}\mathbb{M}_{\sigma_1}\mathbb{M}_{\sigma}\mathbb{M}_{\sigma_2}\mathbb{M}_{\tau_2}$. Choose σ and σ_2 as in Example 5.10 and consider $\tau = (-7,-8,-11:-9,-12), \tau_2 = (-11:-10), \sigma_1 = \{\emptyset\}, \tau_1 = (-8,-9)$. Now $(\sigma,\sigma_2) = (6, 1:5, 0:4)$ and $\omega_0 = \mathfrak{s}(\sigma,\sigma_2) = (6,5,0:4)$, and $(\tau,\tau_2) = (-7,-8,-11:-9,-12:-10)$, $\omega_1 = \mathfrak{s}(\tau,\tau_2) = (-7,-8,-9,-12:-10)$. So $\mathcal{K}_\omega(\lambda) = \lambda\mathbb{M}_{\mathfrak{s}(\tau,\tau_2)} - \mathbb{M}_{\mathfrak{s}(\sigma,\sigma_2)}$ is a PGF pencil. By Theorem 5.3.2, we have $c_{-m} = 2$ and $csf(\mathfrak{s}(\tau,\tau_2)) = (\mathbf{s}_1, -m : -m + c_{-m}) = (-7,-8,-9,-12 : -10)$, where $\mathbf{s}_1 = (-7,-8,-9)$. So $\xi = (9,8,7,6,5,0:4)$. Hence

$$\begin{bmatrix} \begin{bmatrix} \lambda^6 P_0 & \lambda^6 P_1 & \lambda^5 I_n & \lambda^4 I_n & \lambda^3 I_n & \lambda^2 I_n & \lambda I_n & I_n & P_8 & P_9 & P_{10} & P_{11} \end{bmatrix}^{\mathcal{B}} \\ (\lambda E - A)^{-1}B \end{bmatrix} x \in \mathcal{N}_r(\mathbb{L}(\lambda)),$$

where $x \in \mathcal{N}_r(G(\lambda))$.

Now $(rev\sigma, rev\sigma_1) = (0,5,4,3,2,1,6) \sim (5:6,4,3,2,0:1)$ and $\xi_0 = \mathfrak{s}(rev\sigma, rev\sigma_1) = (5:6,4,3,2,0:1)$, $(rev\tau, rev\tau_1) = (-12,-9,-10,-11,-8,-7,-9,-8) \sim (-9:-7,-10,-12:-11,-9:-8)$, and $\xi_1 = \mathfrak{s}(rev\tau, rev\tau_1) = (-7,-10,-12:-11,-9,-8) \sim (-7,-10:-8,-12:-11)$. So $\mathcal{H}_\xi(\mathcal{S}^T(\lambda)) = \lambda\mathbb{M}_{\mathfrak{s}(rev\tau,rev\tau_1)}(\mathcal{S}^T) - \mathbb{M}_{\mathfrak{s}(rev\sigma,rev\sigma_1)}(\mathcal{S}^T)$ is a PGF pencil, and by Theorem 5.3.2, we have $c_{-m} = 1$, and $csf(\mathfrak{s}(rev\tau, rev\tau_1)) = (\tau_1, -m : -m + c_{-m})$, where $\tau_1 = (-7,-10:-8)$. So $\xi = csf(-rev\,\tau_1, \mathfrak{s}(rev\,\sigma, rev\sigma_1)) =$

$csf(8, 9, 10, 7, 5 : 6, 4, 3, 2, 0 : 1) = (8 : 10, 7, 5 : 6, 4, 3, 2, 0 : 1)$. Hence

$$\begin{bmatrix} \begin{bmatrix} \lambda^7 P_0^T & \lambda^6 I_n & \lambda^6 P_2^T & \lambda^6 P_3^T & \lambda^5 I_n & \lambda^4 I_n & \lambda^4 P_6^T & \lambda^3 I_n & \lambda^2 I_n & \lambda I_n & I_n & P_{11}^T \end{bmatrix}^{\mathcal{B}} \\ (C(\lambda E - A)^{-1})^T \end{bmatrix} y,$$

is a left eigenvector of $\mathbb{L}(\lambda)$, where $y \in \mathcal{N}_l(G(\lambda))$. ■

6

Sensitivity Analysis of Eigenvalues

We develop a general framework for sensitivity analysis of invariant zeros of LTI system and eigenvalues of rational matrix functions. We derive explicit computable expressions for condition numbers of invariant zeros of LTI system and hence of eigenvalues of rational matrix functions. We analyze the effect of linearization on the conditioning of invariant zeros of an LTI system and the eigenvalues of a rational matrix function.

6.1 Sensitivity of invariant zeros of LTI system

We develop a general framework for sensitivity analysis of invariant zeros of system matrix. When we restrict our zero $\lambda \in \mathrm{Sp}_e(G)$ then we get the condition number of simple eigenvalues of transfer function $G(\lambda)$.

Recall the LTI system Σ given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + P(\lambda)u(t), \end{aligned}$$

the Rosenbrock system matrix

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right]$$

and the associated matrix transfer function

$$G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B = P(\lambda) + CH(\lambda)^{-1}B,$$

where $H(\lambda) = \lambda E - A$. Consider the eigenvalue problem $G(\lambda)x = 0$. Also recall that $\mathrm{Sp}(\mathcal{S})$ is the set of invariant zeros of Σ and $\mathrm{Sp}(G)$ is the set of transmission zeros of Σ.

When Σ is both controllable and observable we have $\mathrm{Sp}(\mathcal{S}) = \mathrm{Sp}(G)$. Also recall that if $\mathbb{L}_\sigma(\lambda)$ is a Fiedler pencil of $\mathcal{S}(\lambda)$ then $\mathrm{Sp}(\mathcal{S}) = \mathrm{Sp}(\mathbb{L}_\sigma)$ and $\mathrm{Ind}(\lambda, \mathbb{L}_\sigma) = \mathrm{Ind}(\lambda, \mathcal{S})$ for $\lambda \in \mathrm{Sp}(\mathcal{S})$.

We analyze sensitivity of invariant zeros of Σ with respect to small perturbations in the coefficient matrices in the LTI system Σ. In particular, we analyze sensitivity of the eigenvalues of $G(\lambda)$ when the coefficient matrices $\{A_0, \ldots, A_m, A, B, C, E\}$ are subjected small perturbations. Since invariant zeros of Σ are the eigenvalues of the system matrix $\mathcal{S}(\lambda)$, for sensitivity analysis, we have to choose perturbations $\Delta\mathcal{S}$ such that $\mathcal{S}(\lambda) + \Delta\mathcal{S}(\lambda)$ is again a system matrix. For this purpose, we consider the space of Rosenbrock polynomials $\mathbb{S}$ given by

$$\mathbb{S} := \left\{ \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & L(\lambda) \end{array}\right] : P \in \mathbb{L}_m(\mathbb{C}^{n\times n}), L(\lambda) \in \mathbb{L}_1(\mathbb{C}^{r\times r}), C \in \mathbb{C}^{n\times r}, B \in \mathbb{C}^{r\times n} \right\}.$$

Note that $\mathbb{S}$ is a subspace of $\mathbb{L}_m\left(\mathbb{C}^{(n+r)\times(n+r)}\right)$ and hence inherits the norm and inner-product on $\mathbb{L}_m\left(\mathbb{C}^{(n+r)\times(n+r)}\right)$.

Subspace norm: Let $|||.|||$ be a norm on $\mathbb{L}_m\left(\mathbb{C}^{(n+r)\times(n+r)}\right)$ and $\mathcal{S}(\lambda) \in \mathbb{S}$ be given by $\mathcal{S}(\lambda) = \sum_{i=0}^{m} S_i\lambda^i$. Then

$$|||\mathcal{S}||| := \|(\|S_0\|, \|S_1\|, \cdots, \|S_m\|)\|_p, \quad \text{where } 1 \le p \le \infty, \tag{6.1}$$

is a norm on $\mathbb{S}$ and $\langle ., . \rangle_{\mathbb{S}} : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$ given by

$$\langle X, Y \rangle_{\mathbb{S}} := \langle X, Y \rangle_m \tag{6.2}$$

is an inner product on $\mathbb{S}$. We denote the dual norm of $|||.|||$ with respect to the inner-product $\langle ., . \rangle_{\mathbb{S}}$ by $|||.|||_*$, i.e., for $Y \in \mathbb{S}$

$$|||Y|||_* = \sup_{|||X|||=1} \{\langle X, Y \rangle_{\mathbb{S}} : X \in \mathbb{S}\}.$$

Then we have

$$|\langle X, Y \rangle_{\mathbb{S}}| \le |||X||| \, |||Y|||_*$$

and there exists $X_{opt} \in \mathbb{S}$ such that $|||X_{opt}||| = 1$ and $\langle X_{opt}, Y \rangle_{\mathbb{S}} = |||Y|||_*$.

Lemma 6.1.1. *Let $\mathbb{V} = \{\mathrm{diag}(A, B) : A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}\}$ be the space of block diagonal matrices. Let $\|.\|$ be a max-norm on $\mathbb{C}^{(n+m)\times(n+m)}$ and $\mathcal{A} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \in \mathbb{V}$.*

Then there exists $\mathcal{X} = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right] \in \mathbb{V}$ *such that* $\|\mathcal{X}\| = 1$ *and*

$$\|\mathcal{A}\|_* = \langle \mathcal{X}, \mathcal{A} \rangle = \|A\|_* + \|B\|_*.$$

Proof. Proof is easy. □

Lemma 6.1.2. *Let* $1 \leq p \leq \infty$ *and* $1/p + 1/q = 1$. *Let* $\|.\|$ *be a max-norm on* $\mathbb{C}^{(n+r)\times(n+r)}$. *Then for*

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \sum_{i=0}^{m} A_i\lambda^i & C \\ \hline B & A - \lambda E \end{array}\right] \in \mathbb{S},$$

we have

$$|||\mathcal{S}||| = \left\| \left(\left\| \left[\begin{array}{c|c} A_0 & C \\ \hline B & A \end{array}\right] \right\|, \left\| \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & -E \end{array}\right] \right\|, \|A_2\|, \ldots, \|A_m\| \right) \right\|_p \tag{6.3}$$

and there exists $\mathcal{X} \in \mathbb{S}$ *such that* $|||\mathcal{X}||| = 1$ *and*

$$|||\mathcal{S}|||_* = \langle \mathcal{X}, \mathcal{S} \rangle_{\mathbb{S}} = \left\| \left(\left\| \left[\begin{array}{c|c} A_0 & C \\ \hline B & A \end{array}\right] \right\|_*, (\|A_1\|_* + \|E\|_*), \|A_2\|_*, \ldots, \|A_m\|_* \| \right) \right\|_q.$$

Proof. The proof follows from Lemma 6.1.1 and (1.42), (1.43). □

In particular, for Hölder p-norm on $\mathbb{C}^n$, we have

$$|||\mathcal{S}|||_p = \left\| \left(\left\| \left[\begin{array}{c|c} A_0 & C \\ \hline B & A \end{array}\right] \right\|_p, \left\| \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & -E \end{array}\right] \right\|_p, \|A_2\|_p, \ldots, \|A_m\|_p \right) \right\|_p.$$

Then

$$|||\mathcal{S}|||_q = \left\| \left(\left\| \left[\begin{array}{c|c} A_0 & C \\ \hline B & A \end{array}\right] \right\|_q, (\|A_1\|_q + \|E\|_q), \|A_2\|_q, \ldots, \|A_m\|_q \| \right) \right\|_q.$$

is the dual norm of $|||\mathcal{S}|||_p$, where $1/p + 1/q = 1$.

Product norm on $\mathbb{S}$: Note that $\mathbb{S}$ can be identified with the product space $\mathbb{X} = \mathbb{L}_m(\mathbb{C}^{n\times n}) \times \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times n} \times \mathbb{L}_1(\mathbb{C}^{n\times n})$. Hence for $\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & -H(\lambda) \end{array}\right] \in \mathbb{S}$ we have the product norm

$$|||\mathcal{S}||| := \|(|||P|||, \|C\|, \|B\|, |||H|||)\|_p, \text{ where } 1 \leq p \leq \infty. \tag{6.4}$$

The innerproduct $\langle ., . \rangle_{\mathbb{S}}$ on $\mathbb{S}$ coincides with the innerproduct of the product space $\mathbb{X}$. Indeed, for $X, Y \in \mathbb{S}$ with $X = \left[\begin{array}{c|c} X_{11}(\lambda) & X_{12} \\ \hline X_{21} & X_{22}(\lambda) \end{array}\right]$ and $Y = \left[\begin{array}{c|c} Y_{11}(\lambda) & Y_{12} \\ \hline Y_{21} & Y_{22}(\lambda) \end{array}\right]$, we have

$$\langle X, Y \rangle_{\mathbb{X}} = \langle X_{11}, Y_{11} \rangle + \langle X_{12}, Y_{12} \rangle + \langle X_{21}, Y_{21} \rangle + \langle X_{22}, Y_{22} \rangle = \langle X, Y \rangle_{\mathbb{S}}.$$

The dual norm $|||.|||_*$ of $|||.|||$ with respect to the innerproduct $\langle ., . \rangle_{\mathbb{S}}$ is given by

$$|||\mathcal{S}|||_* := \|(|||P|||_*, \|C\|_*, \|B\|_*, |||H|||_*)\|_q, \text{ where } 1/p + 1/q = 1. \tag{6.5}$$

Thus

$$|\langle X, Y \rangle| \leq |||X||| |||Y|||_*$$

and there exists $X_{opt} \in \mathbb{S}$ such that $|||X_{opt}||| = 1$ and $\langle X_{opt}, Y \rangle_{\mathbb{S}} = |||Y|||_*$.

We denote the unstructured condition number of an eigenvalue λ of $\mathcal{S}(\lambda)$ by $\mathrm{cond}(\lambda, \mathcal{S})$. Recall that

$$\mathrm{cond}(\lambda, \mathcal{S}) := \limsup_{|||X|||\to 0} \left\{ \frac{\mathrm{dist}(\lambda, \mathrm{Sp}(\mathcal{S} + X))}{|||X|||} : X \in \mathbb{L}_m(\mathbb{C}^{(n+r)\times(n+r)}) \right\}.$$

We now restrict the perturbation X to $\mathbb{S}$ so that $\mathcal{S} + X \in \mathbb{S}$.

Definition 6.1.1. *Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$. Define*

$$\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S}) := \limsup_{|||\Delta\mathcal{S}|||\to 0} \left\{ \frac{\mathrm{dist}(\lambda, Sp(\mathcal{S} + \Delta\mathcal{S}))}{|||\Delta\mathcal{S}|||} : \Delta\mathcal{S} \in \mathbb{S} \right\}.$$

We refer to $\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S})$ as the structured condition number of the eigenvalue λ.

Observe that if λ is not a pole of the transfer function $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ then $\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S})$ measures the sensitivity of λ with respect to small perturbation in the coefficient matrices. Thus we have the following.

Definition 6.1.2. *For $\lambda \in Sp_e(G)$, we define $\mathrm{cond}(\lambda, G) := \mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S})$ and refer to $\mathrm{cond}(\lambda, G)$ as the condition of λ as an eigenvalue of $G(\lambda)$.*

Lemma 6.1.3. *Let* $\boldsymbol{x} = \begin{bmatrix} x \\ v \end{bmatrix}$ *and* $\boldsymbol{y} = \begin{bmatrix} y \\ u \end{bmatrix}$ *be right and left eigenvector of* $\mathcal{S}(\lambda)$ *with* $x, y \in \mathbb{C}^{nm}$ *and* $u, v \in \mathbb{C}^n$. *If* $\lambda \in Sp_e(G)$ *then* x *and* y *are right and left eigenvector of* $G(\lambda)$ *and* $\boldsymbol{y}^* \partial_\lambda \mathcal{S}(\lambda)\boldsymbol{x} = y^* \partial_\lambda G(\lambda)x$, *where* $\partial_\lambda G(\lambda)$ *is the derivative of the matrix function* $G(z)$ *evaluated at* λ.

Proof. We have

$$\mathbf{y}^* \partial_\lambda \mathcal{S}(\lambda)\mathbf{x} = \begin{bmatrix} y^* & u^* \end{bmatrix} \left[\begin{array}{c|c} \partial_\lambda P(\lambda) & 0 \\ \hline 0 & -E \end{array}\right] \begin{bmatrix} x \\ v \end{bmatrix} = y^* \partial_\lambda P(\lambda)x - u^* E v.$$

Since $\lambda \in \mathrm{Sp}_e(G)$ by Theorem 3.2.6, we have $\mathbf{x} = \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x \\ (\lambda E - A)^{-1} B x \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y \\ (C(\lambda E - A)^{-1})^* y \end{bmatrix}$. Now $y^* \partial_\lambda P(\lambda)x - u^* E v = y^* \partial_\lambda P(\lambda)x - y^* C(\lambda E - A)^{-1} E(\lambda E - A)^{-1} B x = y^* \partial_\lambda G(\lambda)x$. □

Lemma 6.1.4. *[32] For matrices* C, Q, R, *we have*

$$C(\mathrm{adj}C) = \mathrm{adj}(C)C = I \det C, \tag{6.6}$$

$$R[\mathrm{adj}(QCR)]Q = (\mathrm{adj}C)\det(RQ)$$

$$\mathrm{adj}(CQ) = \mathrm{adj}(Q)\mathrm{adj}(C)$$

Lemma 6.1.5. *([1]) Let* (λ, x, y) *be a simple eigentriple of* $G(\lambda)$. *Then*

$$\mathrm{adj}(G(\lambda)) = xy^*$$

for some right and left eigenvectors x *and* y *of* $G(\lambda)$ *corresponding to the eigenvalue* λ.

Now, we consider the system matrix $\mathcal{S}(\lambda)$ and derive an explicit computable expression for condition number of simple eigenvalue of $\mathcal{S}(\lambda)$.

Theorem 6.1.1. *Let* λ *be a simple eigenvalue of* $\mathcal{S}(z)$. *Let* $\boldsymbol{x} = \begin{bmatrix} x \\ v \end{bmatrix}$ *and* $\boldsymbol{y} = \begin{bmatrix} y \\ u \end{bmatrix}$, *respectively, be right and left eigenvector of* $\mathcal{S}(\lambda)$ *corresponding to* λ. *Set* $s(z) = \det(\mathcal{S}(z))$ *and* $\Lambda_m = (1, \lambda, \cdots, \lambda^m)$. *Then for* $X \in \mathbb{L}_m(\mathbb{C}^{(n+r)\times(n+r)})$

$$D\lambda(\mathcal{S})X = \frac{-\mathit{Trace}\,(\mathrm{adj}\mathcal{S}(\lambda)X(\lambda))}{s'(\lambda)} = \langle \mathcal{S}, \nabla\lambda(\mathcal{S})\rangle,$$

where

$$\nabla\lambda(\mathcal{S}) = \frac{-\bar{\Lambda}_m \otimes \operatorname{adj}(\mathcal{S}(\lambda))^*}{\overline{s'(\lambda)}} = \frac{-\bar{\Lambda}_m \otimes \boldsymbol{y}\boldsymbol{x}^*}{\overline{\boldsymbol{y}^*\partial_\lambda \mathcal{S}(\lambda)\boldsymbol{x}}}.$$

Hence $\operatorname{cond}(\lambda, \mathcal{S}) = \|\bar{\Lambda}_m \otimes \boldsymbol{y}\boldsymbol{x}^*\|_* \Big/ |\boldsymbol{y}^*\partial_\lambda \mathcal{S}(\lambda)\boldsymbol{x}|$. *Further, if* λ *is not a pole of* G *then*

$$\nabla\lambda(\mathcal{S}) = \frac{-\bar{\Lambda}_m \otimes \operatorname{adj}(\mathcal{S}(\lambda))^*}{\overline{s'(\lambda)}} = \left[\begin{array}{c|c} \operatorname{adj}G(\lambda) & \operatorname{adj}G(\lambda)CH(\lambda)^{-1} \\ \hline H(\lambda)^{-1}B\operatorname{adj}G(\lambda) & H(\lambda)^{-1}B\operatorname{adj}G(\lambda)CH(\lambda)^{-1} \end{array}\right] \Big/ \overline{r'(\lambda)},$$

where $H(\lambda) = \lambda E - A$, *and* $r(z) = \det(G(z))$.

Proof. First part follows from Theorem 1.5.1. Now

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & -H(\lambda) \end{array}\right] = \left[\begin{array}{c|c} I_n & -CH(\lambda)^{-1} \\ \hline 0 & I_r \end{array}\right] \left[\begin{array}{c|c} G(\lambda) & 0 \\ \hline 0 & -H(\lambda) \end{array}\right] \left[\begin{array}{c|c} I_n & 0 \\ \hline -H(\lambda)^{-1}B & I_r \end{array}\right].$$

Then by Lemma 6.1.4,

$$T := \operatorname{adj}\left[\begin{array}{c|c} G(\lambda) & 0 \\ \hline 0 & -H(\lambda) \end{array}\right] = \operatorname{adj}\left(\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & -H(\lambda) \end{array}\right]\left[\begin{array}{c|c} G(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right]\right) =$$

$$\operatorname{adj}\left[\begin{array}{c|c} G(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] \operatorname{adj}\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & -H(\lambda) \end{array}\right] = \operatorname{adj}\left[\begin{array}{c|c} G(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right]\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & -H(\lambda)^{-1} \end{array}\right] \det(-H(\lambda)).$$

By the definition of adjugate we have

$$\operatorname{adj}\left[\begin{array}{c|c} G(\lambda) & 0 \\ \hline 0 & I_r \end{array}\right] = \left[\begin{array}{c|c} \operatorname{adj}G(\lambda) & 0 \\ \hline 0 & 0 \end{array}\right].$$

Consequently, we have

$$T = \det(-H(\lambda))\left[\begin{array}{c|c} \operatorname{adj}G(\lambda) & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & -H(\lambda)^{-1} \end{array}\right] = \det(-H(\lambda))\left[\begin{array}{c|c} \operatorname{adj}G(\lambda) & 0 \\ \hline 0 & 0 \end{array}\right].$$

Hence by Lemma 6.1.4, we have

$$\operatorname{adj}(\mathcal{S}(\lambda)) = \left[\begin{array}{c|c} I_n & 0 \\ \hline H(\lambda)^{-1}B & I_r \end{array}\right] \det(-H(\lambda)) \left[\begin{array}{c|c} \operatorname{adj}G(\lambda) & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_n & CH(\lambda)^{-1} \\ \hline 0 & I_r \end{array}\right]$$

$$= \det(-H(\lambda))\left[\begin{array}{c|c} I_n & 0 \\ \hline H(\lambda)^{-1}B & I_r \end{array}\right]\left[\begin{array}{c|c} \operatorname{adj}G(\lambda) & \operatorname{adj}G(\lambda)CH(\lambda)^{-1} \\ \hline 0 & 0 \end{array}\right]$$

$$= \det(-H(\lambda))\left[\begin{array}{c|c} \mathrm{adj}G(\lambda) & \mathrm{adj}G(\lambda)CH(\lambda)^{-1} \\ \hline H(\lambda)^{-1}B\mathrm{adj}G(\lambda) & H(\lambda)^{-1}B\mathrm{adj}G(\lambda)CH(\lambda)^{-1} \end{array}\right].$$

Since $s(z) = \det \mathcal{S}(z) = \det(-H(z)) \det G(z)$, we have $s'(\lambda) = \det(-H(\lambda))r'(\lambda)$. Hence the proof. ☐

Theorem 6.1.2. *(a) Let $(\lambda, \boldsymbol{x}, \boldsymbol{y})$ be as in Theorem 6.1.1. Define*

$$\mathcal{N}_{\mathcal{S}(\lambda)}(z) = \left[\begin{array}{c|c} \sum_{i=0}^{m} z^i(\bar{\lambda}^i yx^*) & yv^* \\ \hline ux^* & (1 - z\bar{\lambda})(uv^*) \end{array}\right] \bigg/ \boldsymbol{y}^*\partial_\lambda \mathcal{S}(\lambda)\boldsymbol{x}.$$

Then $\mathcal{N}_{\mathcal{S}(\lambda)}(z) \in \mathbb{S}$ and $D\lambda(\mathcal{S})X = \langle X, \mathcal{N}_{\mathcal{S}(\lambda)}\rangle_{\mathbb{S}}$ for $X \in \mathbb{S}$.

(b) Set $r(z) := \det(G(z))$. If λ is not a pole of G then

$$\mathcal{N}_{\mathcal{S}(\lambda)}(z) = \left[\begin{array}{c|c} \sum_{i=0}^{m} z^i\bar{\lambda}^i \mathrm{adj}(G(\lambda))^* & (H(\lambda)^{-1}B\mathrm{adj}G(\lambda))^* \\ \hline (\mathrm{adj}G(\lambda)CH(\lambda)^{-1})^* & (1 + z\bar{\lambda})(H(\lambda)^{-1}B\mathrm{adj}G(\lambda)CH(\lambda)^{-1})^* \end{array}\right] \bigg/ \overline{r'(\lambda)}.$$

Proof. Let

$$X(\lambda) = \left[\begin{array}{c|c} X_{11}(\lambda) & X_{12} \\ \hline X_{21} & X_{22}(\lambda) \end{array}\right] \in \mathbb{S}.$$

Then by Theorem 6.1.1 and Lemma 6.1.5, we have

$$D\lambda(\mathcal{S})X = \frac{-\mathrm{Trace}\,(\mathrm{adj}\mathcal{S}(\lambda)X(\lambda))}{s'(\lambda)} = \frac{-\mathrm{Trace}\,((\mathbf{yx}^*)^*X(\lambda))}{s'(\lambda)} = \frac{-\mathbf{y}^*X(\lambda)\mathbf{x}}{s'(\lambda)}$$

$$= \left[\begin{array}{cc} y^* & u^* \end{array}\right]\left[\begin{array}{c|c} X_{11}(\lambda) & X_{12} \\ \hline X_{21} & X_{22}(\lambda) \end{array}\right]\left[\begin{array}{c} x \\ v \end{array}\right] = \langle X, \mathcal{N}_{\mathcal{S}(\lambda)}\rangle.$$

Again, by Theorem 6.1.1, we have

$$\nabla\lambda(\mathcal{S}) = \mathrm{trace}\left[\begin{array}{c|c} \mathrm{adj}G(\lambda) & \mathrm{adj}G(\lambda)CH(\lambda)^{-1} \\ \hline H(\lambda)^{-1}B\mathrm{adj}G(\lambda) & H(\lambda)^{-1}B\mathrm{adj}G(\lambda)CH(\lambda)^{-1} \end{array}\right] \bigg/ \overline{r'(\lambda)}.$$

It is easy to see that $\langle X, \nabla\lambda(\mathcal{S})\rangle = \langle X, \mathcal{N}_{\mathcal{S}(\lambda)}\rangle$ for $X \in \mathbb{S}$. Hence the proof. ☐

Theorem 6.1.3. *The condition number $\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S})$ with respect to the norm in (6.4) is given by*

$$\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S}) = \left\| \left(\|\bar{\Lambda}_m \otimes yx^*\|_*, \|yv^*\|_*, \|ux^*\|_*, \|(1, -\bar{\lambda}) \otimes uv^*\|_* \right) \right\|_q \Big/ |\boldsymbol{y}^*\partial_\lambda \mathcal{S}(\lambda)\boldsymbol{x}|.$$

Proof. Let $\mathcal{N}_{\mathcal{S}(\lambda)}$ be as in Theorem 6.1.2. Then we have

$$\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S}) = |||\mathcal{N}_{\mathcal{S}(\lambda)}|||_*.$$

Hence by (6.5) we have

$$|||\mathcal{N}_{\mathcal{S}(\lambda)}|||_* = \left\| \left(\|\bar{\Lambda}_m \otimes yx^*\|_*, \|yv^*\|_*, \|ux^*\|_*, \|(1, -\bar{\lambda}) \otimes uv^*\|_* \right) \right\|_q \Big/ |\mathbf{y}^* \partial_\lambda \mathcal{S}(\lambda) \mathbf{x}|.$$

□

Theorem 6.1.4. *The condition number* $\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S})$ *with respect to the norm in (6.3) is given by*

$$\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S}) = \|(\|\boldsymbol{yx}^*\|_*, |\lambda|(\|yx^*\|_* + \|uv^*\|_*), |\lambda|^2 \|yx^*\|_*, \ldots, |\lambda|^m \|yx^*\|_*)\|_q \Big/ |\boldsymbol{y}^* \partial_\lambda \mathcal{S}(\lambda) \boldsymbol{x}|.$$

Proof. Let $\mathcal{N}_{\mathcal{S}(\lambda)}(z)$ be as in Theorem 6.1.2. Then

$$\mathcal{N}_{\mathcal{S}(\lambda)}(z) = \left[\begin{array}{c|c} yx^* & yv^* \\ \hline ux^* & uv^* \end{array} \right] + z \left[\begin{array}{c|c} \bar{\lambda} yx^* & 0 \\ \hline 0 & \bar{\lambda} uv^* \end{array} \right] + \sum_{i=2}^{m} z^i \left[\begin{array}{c|c} \bar{\lambda}^i yx^* & 0 \\ \hline 0 & 0 \end{array} \right] \Big/ \overline{\mathbf{y}^* \partial_\lambda \mathcal{S}(\lambda) \mathbf{x}}$$

$$= \mathbf{yx}^* + z \left[\begin{array}{c|c} \bar{\lambda} yx^* & 0 \\ \hline 0 & \bar{\lambda} uv^* \end{array} \right] + \sum_{i=2}^{m} z^i \left[\begin{array}{c|c} \bar{\lambda}^i yx^* & 0 \\ \hline 0 & 0 \end{array} \right] \Big/ \overline{\mathbf{y}^* \partial_\lambda \mathcal{S}(\lambda) \mathbf{x}}$$

Hence by Lemma 6.1.2, we have $\mathrm{cond}^{\mathbb{S}}(\lambda, \mathcal{S}) = |||\mathcal{N}_{\mathcal{S}(\lambda)}|||_*$

$$= \|(\|\mathbf{yx}^*\|_*, |\lambda|(\|yx^*\|_* + \|uv^*\|_*), |\lambda|^2 \|yx^*\|_*, \ldots, |\lambda|^m \|yx^*\|_*)\|_q \Big/ |\mathbf{y}^* \partial_\lambda \mathcal{S}(\lambda) \mathbf{x}|.$$

□

Corollary 6.1.5. *Let* $\lambda \in Sp_e(G)$ *be simple.*

(a) *Then the condition number* $\mathrm{cond}(\lambda, G)$ *with respect to the norm in (6.4) is given by*

$$\mathrm{cond}(\lambda, G) = \Big\| \Big(\|\bar{\Lambda}_m \otimes \mathrm{adj}(G(\lambda))^*\|_*, \|(H(\lambda)^{-1} B \mathrm{adj} G(\lambda))^*\|_*, \|(\mathrm{adj} G(\lambda) C H(\lambda)^{-1})^*\|_*,$$
$$\|(-1, \bar{\lambda}) \otimes (H(\lambda)^{-1} B \mathrm{adj} G(\lambda) C H(\lambda)^{-1})^*\|_* \Big) \Big\|_q \Big/ |r'(\lambda)|.$$
$$= \left\| \left(\|\bar{\Lambda}_m \otimes yx^*\|_*, \|yv^*\|_*, \|ux^*\|_*, \|(-1, \bar{\lambda}) \otimes uv^*\|_* \right) \right\|_q \Big/ |y^* \partial_\lambda G(\lambda) x|,$$

where $v = (\lambda E - A)^{-1} B x$ *and* $u = (C(\lambda E - A)^{-1})^* y$.

(b) *The condition number* $\operatorname{cond}(\lambda, G)$ *with respect to the subspace norm in (6.3) is given by*

$$\operatorname{cond}(\lambda, G) = \|(\|\boldsymbol{yx}^*\|_*, |\lambda|(\|yx^*\|_*+\|uv^*\|_*), |\lambda|^2\|yx^*\|_*, \ldots, |\lambda|^m\|yx^*\|_*)\|_q \Big/ |y^*\partial_\lambda G(\lambda)x|,$$

where $v = (\lambda E - A)^{-1}Bx$ *and* $u = (C(\lambda E - A)^{-1})^* y$.

Proof. Since λ is not a pole, by Theorem 6.1.2, we have

$$\|\mathcal{N}_{\mathcal{S}(\lambda)}\|_* = \Bigg\|\Bigg(\|\bar{\Lambda}_m \otimes \operatorname{adj}(G(\lambda))^*\|_*, \|(H(\lambda)^{-1}B\operatorname{adj}G(\lambda))^*\|_*, \|(\operatorname{adj}G(\lambda)CH(\lambda)^{-1})^*\|_*, \\ \|(-1, \bar{\lambda}) \otimes (H(\lambda)^{-1}B\operatorname{adj}G(\lambda)CH(\lambda)^{-1})^*\|_*\Bigg)\Bigg\|_q \Big/ |r'(\lambda)|.$$

Also, since $\lambda \in \operatorname{Sp}_e(G)$, we have $\mathbf{x} = \begin{bmatrix} x \\ (\lambda E - A)^{-1}Bx \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y \\ (C(\lambda E - A)^{-1})^* y \end{bmatrix}$. Hence the result follows. $\square$

Next, we consider the system matrix and the Fiedler linearizations of system matrix. Then we compare the unstructure condition number of simple eigenvalue of system matrix with the condition number of Fiedler linearizations.

6.2 Condition number of Fiedler pencils

Let $\mathbb{L}_\sigma(\lambda)$ be a Fiedler pencil of $\mathcal{S}(\lambda)$. We now determine condition numbers of simple eigenvalues of $\mathbb{L}_\sigma(\lambda)$.

Theorem 6.2.1. *Let* $\mathbb{L}_\sigma(\lambda)$ *be a Fiedler pencil of* $\mathcal{S}(\lambda)$ *and let* $(\lambda, \boldsymbol{x}, \boldsymbol{y})$ *be a simple eigentriple of* $\mathcal{S}(\lambda)$ *be as in Theorem 6.1.1. Then*

$$\operatorname{cond}_p(\lambda, \mathbb{L}_\sigma) = \frac{\|(1, \lambda)\|_q \|\mathbb{E}_\sigma(\mathcal{S})\boldsymbol{x}\|_p \|\mathbb{H}_\sigma(\mathcal{S})\boldsymbol{y}\|_q}{|\boldsymbol{y}^*\partial_\lambda \mathcal{S}(\lambda)\boldsymbol{x}|},$$

where $\mathbb{E}_\sigma(\mathcal{S})$ *and* $\mathbb{H}_\sigma(\mathcal{S})$ *are as in Theorem 3.2.7 and Theorem 3.2.8, respectively, and* $1/p + 1/q = 1$.

Proof. By Theorem 6.1.1, we have

$$\operatorname{cond}_p(\lambda, \mathbb{L}_\sigma) = \frac{\|(1, \lambda)\|_q \|\mathbb{E}_\sigma(\mathcal{S})\mathbf{x}\|_p \|\mathbb{H}_\sigma(\mathcal{S})\mathbf{y}\|_q}{|(\mathbb{H}_\sigma(\mathcal{S})\mathbf{y})^*\partial_\lambda \mathbb{L}_\sigma(\lambda)(\mathbb{E}_\sigma(\mathcal{S})\mathbf{x})|}.$$

Let $\mathbb{L}_\sigma(\lambda) = \lambda\mathbb{M}_m - \mathbb{M}_\sigma$. Now

$$(\mathbb{H}_\sigma(\mathcal{S})\mathbf{y})^*\partial_\lambda \mathbb{L}_\sigma(\lambda)(\mathbb{E}_\sigma(\mathcal{S})\mathbf{x}) = \begin{bmatrix} H_\sigma(P)y \\ u \end{bmatrix}^* \mathbb{M}_m \begin{bmatrix} E_\sigma(P)x \\ u \end{bmatrix}$$

$$= \begin{bmatrix} (H_\sigma(P)y)^* & u^* \end{bmatrix} \begin{bmatrix} M_m & \\ & -E \end{bmatrix} \begin{bmatrix} E_\sigma(P)x \\ u \end{bmatrix} = (H_\sigma(P)y)^* M_m E_\sigma(P)x - u^*Ev$$
$$= y^*\partial_\lambda P(\lambda)x - u^*Ev = \mathbf{y}^*\partial_\lambda \mathcal{S}(\lambda)\mathbf{x},$$

where $E_\sigma(P)$ and $H_\sigma(P)$ are as in Theorem 3.2.4 and Theorem 3.2.5, respectively. $\square$

Theorem 6.2.2. *Let $1 \leq q \leq \infty$ and $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{S}(\lambda)$. Then for the Spectral norm on $\mathbb{C}^{n\times n}$, we have*

$$\frac{\|(1,\lambda)\|_q}{\|\Lambda_m\|_q}\sigma_{\min}(\mathbb{E}_\sigma(\mathcal{S}))\sigma_{\min}(\mathbb{H}_\sigma(\mathcal{S})) \leq \frac{\mathrm{cond}(\lambda,\mathbb{L}_\sigma)}{\mathrm{cond}(\lambda,\mathcal{S})} \leq \frac{\|(1,\lambda)\|_q}{\|\Lambda_m\|_q}\sigma_{\max}(\mathbb{E}_\sigma(\mathcal{S}))\sigma_{\max}(\mathbb{H}_\sigma(\mathcal{S}))$$

Proof. We have
$$\frac{\mathrm{cond}(\lambda,\mathbb{L}_\sigma)}{\mathrm{cond}(\lambda,\mathcal{S})} = \frac{\|(1,\lambda)\|_q\|\mathbb{E}_\sigma(\mathcal{S})\mathbf{x}\|_2\|\mathbb{H}_\sigma(\mathcal{S})\mathbf{y}\|_2}{\|\Lambda_m\|_q\|\mathbf{x}\|_2\|\mathbf{y}\|_2}$$
$$\leq \frac{\|(1,\lambda)\|_q}{\|\Lambda_m\|_q}\sigma_{\max}(\mathbb{E}_\sigma(\mathcal{S}))\sigma_{\max}(\mathbb{H}_\sigma(\mathcal{S})).$$

Note that
$$\frac{\mathrm{cond}(\lambda,\mathbb{L}_\sigma)}{\mathrm{cond}(\lambda,\mathcal{S})} \geq \frac{\|(1,\lambda)\|_q}{\|\Lambda_m\|_q}\sigma_{\min}(\mathbb{E}_\sigma(\mathcal{S}))\sigma_{\min}(\mathbb{H}_\sigma(\mathcal{S})).$$

Hence the proof. $\square$

Next, we compare the condition number of simple eigenvalue of $\mathcal{S}(\lambda)$ with condition number of first companion form. Suppose that $\mathcal{C}_1(\lambda)$ is the first companion form of $\mathcal{S}(\lambda)$ and $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{S}(\lambda)$. Then for matrix polynomial norm $p = 2$ and matrix 2-norm, we have

$$\frac{\mathrm{cond}(\lambda,\mathcal{C}_1)}{\mathrm{cond}(\lambda,\mathcal{S})} = \frac{\|(1,\lambda)\|_2 \left\| \begin{bmatrix} \Lambda\otimes x \\ v \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} Hy \\ u \end{bmatrix} \right\|_2}{\|\Lambda_m\|_2 \left\| \begin{bmatrix} x \\ v \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_2},$$

where $\Lambda = [\lambda^{m-1}, \dots, \lambda, 1]^T$ and $H = \begin{bmatrix} I_n \\ (\lambda A_m + A_{m-1})^T \\ \vdots \\ (\lambda^{m-1} A_m + \lambda^{m-2} A_{m-1} + \cdots + A_1)^T \end{bmatrix}$. Now

$$\frac{\mathrm{cond}(\lambda, \mathcal{C}_1)}{\mathrm{cond}(\lambda, \mathcal{S})} \leq \frac{\|(1,\lambda)\|_2 \|\Lambda\|_2}{\|\Lambda_m\|_2} \frac{\left\| \begin{bmatrix} x \\ v \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} Hy \\ u \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} x \\ v \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_2}.$$

Since $1 \leq \frac{\|(1,\lambda)\|_2 \|\Lambda\|_2}{\|\Lambda_m\|_2} \leq (\frac{2m}{m+1})^{1/2}$ [10], we have

$$\frac{\mathrm{cond}(\lambda, \mathcal{C}_1)}{\mathrm{cond}(\lambda, \mathcal{S})} \leq \left(\frac{2m}{m+1} \right)^{1/2} \frac{\left\| \begin{bmatrix} Hy \\ u \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_2} \leq \left(\frac{2m}{m+1} \right)^{1/2} \sigma_{\max}(H).$$

Again, note that

$$\frac{\mathrm{cond}(\lambda, \mathcal{C}_1)}{\mathrm{cond}(\lambda, \mathcal{S})} \geq \frac{\|(1,\lambda)\|_2}{\|\Lambda_m\|_2} \sigma_{\min} \left(\begin{bmatrix} \Lambda \otimes I_n \\ I_r \end{bmatrix} \right) \sigma_{\min} \left(\begin{bmatrix} H \\ I_r \end{bmatrix} \right).$$

Theorem 6.2.3. *Let $(\lambda, \boldsymbol{x}, \boldsymbol{y})$ be simple eigentriple of $\mathcal{S}(\lambda)$ as in Theorem 6.1.1 and $\mathbb{L}_\sigma(\lambda)$ be the Fiedler pencils associated with a bijection σ. Let $\mathbb{E}_\sigma(\mathcal{S})$ and $\mathbb{H}_\sigma(\mathcal{S})$ be as in Theorem 3.2.7 and Theorem 3.2.8, respectively. Set $H_\sigma(P)y = z$ and $E_\sigma(P)x = w$, where $E_\sigma(P)$ and $H_\sigma(P)$ are as in Theorem 3.2.4 and Theorem 3.2.5, respectively.*

(a) The condition number $\mathrm{cond}^{\mathbb{S}}(\lambda, \mathbb{L}_\sigma)$ with respect to the norm in (6.4) is given by

$$\mathrm{cond}^{\mathbb{S}}(\lambda, \mathbb{L}_\sigma) = \|(\|(1,\bar{\lambda}) \otimes z w^*\|_*, \|z v^*\|_*, \|u w^*\|_*, \|(1, -\bar{\lambda}) \otimes u v^*\|_*)\|_q \Big/ |\boldsymbol{y}^* \partial_\lambda \mathcal{S}(\lambda) \boldsymbol{x}|.$$

(b) The condition number $\mathrm{cond}^{\mathbb{S}}(\lambda, \mathbb{L}_\sigma)$ with respect to the norm in (6.3) is given by

$$\mathrm{cond}^{\mathbb{S}}(\lambda, \mathbb{L}_\sigma) = \|(\|\mathbb{H}_\sigma(\mathcal{S}) \boldsymbol{y} (\mathbb{E}_\sigma(\mathcal{S}) \boldsymbol{x})^*\|_*, |\lambda| \left(z w^*\|_* + \|u v^*\|_* \right))\|_q \Big/ |\boldsymbol{y}^* \partial_\lambda \mathcal{S}(\lambda) \boldsymbol{x}|.$$

Lemma 6.2.1. *If $B = \left[\begin{array}{c|c} C & I_m \\ \hline I_n & 0 \end{array} \right]$ or $B = \left[\begin{array}{c|c} 0 & I_m \\ \hline I_n & C \end{array} \right]$ then $\|B\|_2 \leq (1 + \|C\|_2^2)^{1/2}$.*

Proof. Define $A := \left[\begin{array}{c|c} C & I_m \\ \hline I_m & 0 \end{array}\right] = \left[\begin{array}{cc} C & I_m \\ I_n & 0 \\ \hline I_{(m-n)} & 0 \end{array}\right] = \begin{bmatrix} B \\ D \end{bmatrix}$, where $D = \begin{bmatrix} I_{(m-n)} & 0 \end{bmatrix}$.

Then $A^*A = \begin{bmatrix} B^* & D^* \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = B^*B + D^*D$, and $D^*D = \begin{bmatrix} I_{(m-n)} & 0 \\ 0 & 0 \end{bmatrix}$. Since A^*A is positive semidefinite matrix, so we have $\lambda_{\max}(A^*A) \geq \lambda_{\max}(B^*B) \Rightarrow (\lambda_{\max}(A^*A))^{1/2} \geq (\lambda_{\max}(B^*B))^{1/2}$. Consequently, we have

$$\|A\|_2 \geq \|B\|_2 \Rightarrow \|B\|_2 \leq (1 + \|C\|_2^2)^{1/2}.$$

□

Theorem 6.2.4. *Let $L(\lambda)$ and $T(\lambda)$ be $n \times n$ matrix pencils such that $L(\lambda) = NT(\lambda)F$, where N and F are non-singular matrices in $\mathbb{C}^{n\times n}$. Let $\lambda \in \mathbb{C}$ be a simple eigenvalue of $L(\lambda)$. Then*

$$\frac{1}{\|N\|_2\|F\|_2} \leq \frac{\mathrm{cond}(\lambda, L)}{\mathrm{cond}(\lambda, T)} \leq \|N^{-1}\|_2\|F^{-1}\|_2.$$

Proof. Let u and v are left and right eigenvector of $L(\lambda)$ associated with λ. So

$$\mathrm{cond}(\lambda, L) = \frac{\|(1,\lambda)\|_2\|u\|_2\|v\|_2}{|u^*\partial_\lambda L(\lambda)v|}.$$

Since $NT(\lambda)F = L(\lambda)$ and N is nonsingular then for $v \in \mathcal{N}_r(L)$, we have $L(\lambda)v = 0 \Leftrightarrow (NT(\lambda)F)v = 0 \Leftrightarrow T(\lambda)Fv = 0 \Leftrightarrow Fv \in \mathcal{N}_r(T)$. Similarly, since F is nonsingular we have, $u \in \mathcal{N}_l(L) \Leftrightarrow N^T u \in \mathcal{N}_l(T)$. Thus

$$\begin{aligned}\mathrm{cond}(\lambda, L) = \frac{\|(1,\lambda)\|_2\|u\|_2\|v\|_2}{|u^*\partial_\lambda L(\lambda)v|} &= \frac{\|(1,\lambda)\|_2\|(N^T)^{-1}N^Tu\|_2\|F^{-1}Fv\|_2}{|u^T\partial_\lambda(NT(\lambda)F)v|} \\ &\leq \frac{\|(1,\lambda)\|_2\|(N^T)^{-1}\|_2\|N^Tu\|_2\|F^{-1}\|_2\|Fv\|_2}{|u^T\partial_\lambda(NT(\lambda)F)v|} \\ &= \frac{\|(1,\lambda)\|_2\|N^{-1}\|_2\|F^{-1}\|_2\|N^Tu\|_2\|Fv\|_2}{(N^Tu)^T\partial_\lambda T(\lambda)(Fv)} = \|N^{-1}\|_2\|F^{-1}\|_2\mathrm{cond}(\lambda, T).\end{aligned}$$

So

$$\frac{\mathrm{cond}(\lambda, L)}{\mathrm{cond}(\lambda, T)} \leq \|N^{-1}\|_2\|F^{-1}\|_2.$$

Again

$$
\begin{aligned}
\operatorname{cond}(\lambda, L) &\geq \frac{\|(1,\lambda)\|_2 \sigma_{\min}(N^{-1})\|N^T u\|_2 \sigma_{\min}(F^{-1})\|Fv\|_2}{|(N^T u)^T \partial_\lambda T(\lambda)(Fv)|} \\
&= \sigma_{\min}(N^{-1})\sigma_{\min}(F^{-1})\operatorname{cond}(\lambda, T) \\
&= \frac{1}{\|N\|_2}\frac{1}{\|F\|_2}\operatorname{cond}(\lambda, T), \qquad \text{since } \sigma_{\min}(N^{-1}) = \frac{1}{\sigma_{\max}(N)} \\
&\Rightarrow \frac{\operatorname{cond}(\lambda, L)}{\operatorname{cond}(\lambda, T)} \geq \frac{1}{\|N\|_2\|F\|_2}.
\end{aligned}
$$

Thus

$$
\frac{1}{\|N\|_2\|F\|_2} \leq \frac{\operatorname{cond}(\lambda, L)}{\operatorname{cond}(\lambda, T)} \leq \|N^{-1}\|_2\|F^{-1}\|_2. \tag{6.7}
$$

$\square$

Lemma 6.2.2. *Let $\mathbb{M}_i, i = 1 : m-1$ be the Fiedler matrices associated with $\mathcal{S}(\lambda)$. Then for $j < i$, we have*

$$
\|\mathbb{M}_{(-i:-j)}\|_2 \leq \left(1 + \|Y_{(j:i)}\|_2^2\right)^{1/2}, \ \textit{where } Y_{(j:i)} = \begin{bmatrix} A_i \\ \vdots \\ A_j \end{bmatrix}.
$$

Proof. By Remark 5.2.1, we have

$$
\mathbb{M}_{(-i:-j)} = \left[\begin{array}{c|c|c|c|c}
I_{(m-i-1)n} & & & & \\ \hline
 & 0_{n\times(i-j+1)n} & I_n & & \\ \hline
 & & A_i & & \\
 & I_{(i-j+1)n} & A_{i-1} & & \\
 & & \vdots & & \\
 & & A_j & & \\ \hline
 & & & I_{(j-1)n} & \\ \hline
 & & & & I_r
\end{array}\right], 1 \leq j \leq i < m. \tag{6.8}
$$

Consider the middle block matrix of (6.8) and set $X = \left[\begin{array}{c|c} 0_{n\times n(i-j+1)} & I_n \\ \hline I_{i-j+1} & Y_{(j:i)} \end{array}\right]$, where

$Y_{(j:i)} = \begin{bmatrix} A_i \\ A_{i-1} \\ \vdots \\ A_j \end{bmatrix}$. Define $A = \left[\begin{array}{c|c} 0_{(i-j+1)n} & I_{(i-j+1)n} \\ \hline I_{(i-j+1)n} & Y_{(j:i)} \end{array}\right] = \begin{bmatrix} Z \\ X \end{bmatrix}$, where $Z = \begin{bmatrix} 0 & I_{(i-j+1)n} \end{bmatrix}$.

So by Lemma 6.2.1, we have

$$\|X\|_2 \leq (1 + \|Y_{(j:i)}\|_2^2)^{1/2}.$$

Therefore

$$\|\mathbb{M}_{(-i:-j)}\|_2 \leq \max\left(1, (1 + \|Y_{(i:j)}\|_2^2)^{1/2}\right) = \left(1 + \|Y_{(i:j)}\|_2^2\right)^{1/2}.$$

□

Theorem 6.2.5. *Let $\mathbb{T}_\mu(\lambda) = \lambda \mathbb{M}_{\mu_1} - \mathbb{M}_{\mu_0}$ be a PGF pencil of $\mathcal{S}(\lambda)$, where μ_1 and μ_0 are index tuples in column standard form. Suppose that μ_1 has c_{-m} consecutions at $-m$, and $csf(\mu_1) = (-a_s : -b_s, -a_{s-1} : -b_{s-1}, \ldots, -a_1 : -b_1, -m : -m + c_{-m}) := (\tau_1, \tau_2)$ where $\tau_1 = (-a_s : -b_s, -a_{s-1} : -b_{s-1}, \ldots, -a_1 : -b_1)$ and $\tau_2 = -m : -m + c_{-m}$.*

(a) Then $\mathbb{M}_{-rev\tau_1} \mathbb{T}_\mu(\lambda) \mathbb{M}_{\tilde{\tau}_2} = \mathbb{L}_\sigma(\lambda)$ is a Fiedler pencil of $\mathcal{S}(\lambda)$, where $\tilde{\tau}_2 = (m - c_{-m} : m-1)$.

(b) If $csf(-rev\tau_1) := (c_s : d_s, c_{s-1} : d_{s-1}, \ldots, c_1 : d_1) =: \tilde{\tau}_1$ and $|c_i - d_{i-1}| > 1$ for all $i \in \{2, \ldots, s\}$ then

$$\frac{1}{\alpha.\beta} \leq \frac{\mathrm{cond}(\lambda, \mathbb{L}_\sigma)}{\mathrm{cond}(\lambda, \mathbb{T}_\mu)} \leq \alpha.\beta,$$

where

$$\alpha = \max\left(\left(1 + \|Y_{(c_s:d_s)}\|_2^2\right)^{1/2} \cdots \left(1 + \|Y_{(c_1:d_1)}\|_2^2\right)^{1/2}\right), \beta = \left(1 + \|Y_{(m-c_{-m}:m-1)}\|_2^2\right)^{1/2},$$

and $$Y_{(c_s:d_s)} = \begin{bmatrix} A_{d_s} \\ A_{d_s-1} \\ \vdots \\ A_{c_s} \end{bmatrix}.$$

Proof. (a) Since μ_1 has c_{-m} consecutions at $-m$, part (a) follows from proof of Theorem 5.3.2.

(b) Since $csf(-rev\tau_1) := (c_s : d_s, c_{s-1} : d_{s-1}, \ldots, c_1 : d_1) =: \tilde{\tau}_1$, we have $c_s > d_{s-1} > \ldots > c_2 > d_1 > c_1$. Since $|c_i - d_{i-1}| > 1$ for all $i \in \{2, \ldots, s\}$, we have $\mathbb{M}_{\tilde{\tau}_1}^{-1}$ is a block diagonal matrix. Hence the result follows. □

Remark 6.2.1. *If $csf(-rev\tau_1) := (c_3 : d_3, c_2 : d_2, c_1 : d_1) := \tilde{\tau}_1$ with $|c_3 - d_2| > 1$ and $|c_2 - d_1| = 1$. Then*

$$\frac{1}{\ell.\beta} \leq \frac{\mathrm{cond}(\lambda, \mathbb{L}_\sigma)}{\mathrm{cond}(\lambda, \mathbb{T}_\mu)} \leq \ell.\beta,$$

where

$$\ell = \max\left[\left(1+\|Y_{(c_3:d_3)}\|_2^2\right)^{1/2}, \left(1+\|Y_{(c_2:d_2)}\|_2^2\right)^{1/2}\right].\left(1+\|Y_{(c_1:d_1)}\|_2^2\right)^{1/2}$$

One can generalize this to $csf(-rev\tau_1) := (c_s : d_s, c_{s-1} : d_{s-1}, \ldots, c_1 : d_1) = \tilde{\tau}_1$, *where some* $|c_{i+1} - d_i| > 1$ *and* $|c_i - d_{i-1}| = 1$.

Example 6.1. Consider the system matrix $\mathcal{S}(\lambda)$ with $m = 10$. Consider the PGF pencil

$$\mathbb{L}_\mu(\lambda) = \lambda\mathbb{M}_{-6}\mathbb{M}_{-4}\mathbb{M}_{-2}\mathbb{M}_{-1}\mathbb{M}_{-10} - \mathbb{M}_0\mathbb{M}_3\mathbb{M}_5\mathbb{M}_7\mathbb{M}_8\mathbb{M}_9 = \lambda\mathbb{M}_{\mu_1} - \mathbb{M}_{\mu_0}$$

of $\mathcal{S}(\lambda)$. Then $csf(\mu_1) = (-2 : -1, -4 : -4, -6 : -6, -10 : -10)$. Thus by the Theorem 6.2.5, we have $\mathbb{M}_{\tilde{\tau}_1} = \mathbb{M}_1\mathbb{M}_2\mathbb{M}_4\mathbb{M}_6$, where $\tilde{\tau}_1 = (6, 4, 1, 2) = (6, 4, 1 : 2)$ and $\mathbb{M}_{\tilde{\tau}_2} = I$. Now

$$\mathbb{M}_{\tilde{\tau}_1} = \begin{bmatrix} I_{3n} & & & & & & & & \\ & -A_6 & I_n & & & & & & \\ & I_n & 0 & & & & & & \\ & & & -A_4 & I_n & & & & \\ & & & I_n & 0 & & & & \\ & & & & & -A_2 & I_n & 0 \\ & & & & & -A_1 & 0 & I_n \\ & & & & & I_n & 0 & 0 \end{bmatrix}$$

and

$$\alpha = \max\left[(1+\|Y_{(1:2)}\|_2^2)^{1/2}, (1+\|A_6\|_2^2)^{1/2}, (1+\|A_4\|_2^2)^{1/2}\right].$$

Therefore

$$\frac{1}{\alpha} \le \frac{\operatorname{cond}(\lambda, \mathbb{L}_\sigma)}{\operatorname{cond}(\lambda, \mathbb{T}_\mu)} \le \alpha.$$

■

Example 6.2. Consider the system matrix $\mathcal{S}(\lambda)$ with $m = 13$ and the PGF pencil

$$\mathbb{L}_\mu(\lambda) = \lambda\mathbb{M}_{-2}\mathbb{M}_{-3}\mathbb{M}_{-6}\mathbb{M}_{-5}\mathbb{M}_{-13}\mathbb{M}_{-12}\mathbb{M}_{-11} - \mathbb{M}_0\mathbb{M}_4\mathbb{M}_1\mathbb{M}_7\mathbb{M}_8\mathbb{M}_9\mathbb{M}_{10} = \lambda\mathbb{M}_{\mu_1} - \mathbb{M}_{\mu_0}$$

of $\mathcal{S}(\lambda)$. Then $csf(\mu_1) = (-2 : -2, -3 : -3, -6 : -5, -13 : -11)$. Thus by the Theorem 6.2.5, we have $\mathbb{M}_{\tilde{\tau}_1} = \mathbb{M}_5\mathbb{M}_6\mathbb{M}_3\mathbb{M}_2$, where $\tilde{\tau}_1 = (c_3 : d_3, c_2 : d_2, c_1 : d_1) = (5 : 6, 3 : 3, 2 : 2)$ and $\mathbb{M}_{\tilde{\tau}_2} = \mathbb{M}_{11}\mathbb{M}_{12}$. Then $\mathbb{M}_{\tilde{\tau}_1}\mathbb{T}_\mu(\lambda)\mathbb{M}_{\tilde{\tau}_2} = \mathbb{L}_\sigma(\lambda)$ is a Fiedler pencil of $\mathcal{S}(\lambda)$. Since $|c_3 - d_2| > 1$ and $|c_2 - d_1| = 1$ we have

$$\ell = \max\left[\left(1+\|Y_{(5:6)}\|_2^2\right)^{1/2}, \left(1+\|A_3\|_2^2\right)^{1/2}\right].\left(1+\|A_2\|_2^2\right)^{1/2} \text{ and } \beta = \left(1+\|Y_{(11:12)}\|_2^2\right)^{1/2}.$$

Therefore

$$\frac{1}{\ell\beta} \leq \frac{\operatorname{cond}(\lambda, \mathbb{L}_\sigma)}{\operatorname{cond}(\lambda, \mathbb{T}_\mu)} \leq \ell\beta.$$

■

Example 6.3. Consider the system matrix $\mathcal{S}(\lambda)$ with $m = 13$ and the PGF pencil

$$\mathbb{L}_\mu(\lambda) = \lambda\mathbb{M}_{-1}\mathbb{M}_{-3}\mathbb{M}_{-5}\mathbb{M}_{-4}\mathbb{M}_{-8}\mathbb{M}_{-7}\mathbb{M}_{-13}\mathbb{M}_{-12} - \mathbb{M}_2\mathbb{M}_9\mathbb{M}_0\mathbb{M}_{10}\mathbb{M}_{11}\mathbb{M}_6 = \lambda\mathbb{M}_{\mu_1} - \mathbb{M}_{\mu_0}$$

of $\mathcal{S}(\lambda)$. Then $csf(\mu_1) = (-1 : -1, -3 : -3, -5 : -4, -8 : -7, -13 : -12)$. Thus by the Theorem 6.2.5, we have $\mathbb{M}_{\tilde{\tau}_1} = \mathbb{M}_7\mathbb{M}_8\mathbb{M}_4\mathbb{M}_5\mathbb{M}_3\mathbb{M}_1$, where $\tilde{\tau}_1 = (c_4 : d_4, c_3 : d_3, c_2 : d_2, c_1 : d_1) = (7 : 8, 4 : 5, 3 : 3, 1 : 1)$ and $\mathbb{M}_{\tilde{\tau}_2} = \mathbb{M}_{12}$. Then $\mathbb{M}_{\tilde{\tau}_1}\mathbb{T}_\mu(\lambda)\mathbb{M}_{\tilde{\tau}_2} = \mathbb{L}_\sigma(\lambda)$ is a Fiedler pencil of $\mathcal{S}(\lambda)$ and since $|c_3 - d_2| = 1$ we have

$$\ell = \max\left[\left(1 + \|Y_{(7:8)}\|_2^2\right)^{1/2}, \left(1 + \|Y_{(4:5)}\|_2^2\right)^{1/2}\right] . \max\left[\left(1 + \|A_3\|_2^2\right)^{1/2}, \left(1 + \|A_1\|_2^2\right)^{1/2}\right],$$

and $\beta = (1 + \|A_{12}\|_2^2)^{1/2}$. Therefore

$$\frac{1}{\ell\beta} \leq \frac{\operatorname{cond}(\lambda, \mathbb{L}_\sigma)}{\operatorname{cond}(\lambda, \mathbb{T}_\mu)} \leq \ell\beta.$$

■

Corollary 6.2.6. *Let* $\mathbb{S}(\lambda)$ *be the PGF pencil be as in Theorem 4.2.1. Then* $\mathbb{M}_{-rev\tau_1}\mathbb{S}(\lambda)\mathbb{M}_{\tilde{\tau}_2} = \mathbb{L}_\sigma$ *is the Fiedler pencil, where* $-rev\tau_1 = (1, 3, \ldots, m-2)$, $\tilde{\tau}_2 = \phi$ *and*

$$\frac{1}{\alpha} \leq \frac{\operatorname{cond}(\lambda, \mathbb{L}_\sigma)}{\operatorname{cond}(\lambda, \mathbb{S})} \leq \alpha,$$

where $\alpha = \max\left((1 + \|A_1\|_2^2)^{1/2}(1 + \|A_3\|_2^2)^{1/2} \cdots (1 + \|A_{m-2}\|_2^2)^{1/2}\right)$.

Proof. Note that $csf(-rev\tau_1) := (c_s : d_s, c_{s-1} : d_{s-1}, \ldots, c_1 : d_1) = (m-2, \ldots, 3, 1)$ and $|c_i - d_{i-1}| > 1$ for all $i \in \{2, \ldots, s\}$. Hence by Theorem 6.2.5(b), the result follows. □

Example 6.4. Consider $\mathbb{T}_\mu(\lambda) = \lambda\mathbb{M}_3^{-1}\mathbb{M}_1^{-1}\mathbb{M}_4 - \mathbb{M}_0\mathbb{M}_2 = \lambda\mathbb{M}_{-3}\mathbb{M}_{-1}\mathbb{M}_{-4} - \mathbb{M}_0\mathbb{M}_2$ is a block tridiagonal PGF pencil of $\mathcal{S}(\lambda)$. Then $\mathbb{M}_1\mathbb{M}_3\mathbb{T}_\mu(\lambda) = \lambda\mathbb{M}_4 - \mathbb{M}_1\mathbb{M}_3\mathbb{M}_0\mathbb{M}_2 = \mathbb{L}_\sigma(\lambda)$ is a Fiedler pencil of $G(\lambda)$. By Corollary 6.2.6, we have $\mathbb{M}_{\tilde{\tau}_1} = \mathbb{M}_1\mathbb{M}_3$ and $\mathbb{M}_{\tilde{\tau}_2} = I$. Since $\tilde{\tau}_1$ has no consecutions, $\alpha = \max\left((1 + \|A_1\|_2^2), (1 + \|A_3\|_2^2)\right)$. Hence

$$\frac{1}{\alpha} \leq \frac{\operatorname{cond}(\lambda, \mathbb{L}_\sigma)}{\operatorname{cond}(\lambda, \mathbb{T}_\mu)} \leq \alpha.$$

■

Now, we can compare the condition number of the first companion form with that of $G(\lambda)$. Define

$$J = \left[\|\Lambda_m\|_q^q \|y\|_2^q \|x\|_2^q + \|yw^*\|_2^q + \|sx^*\|_2^q + (1+|\lambda|^q)\|sw^*\|_2^q\right]^{1/q},$$

where $w = (\lambda E - A)^{-1}Bx$ and $s = (C(\lambda E - A)^{-1})^* y$. We have

$$\frac{\operatorname{cond}(\lambda, \mathcal{C}_1)}{\operatorname{cond}(\lambda, G)} = \frac{\|(1,\lambda)\|_q \|v\|_2 \|u\|_2}{J} \tag{6.9}$$

Theorem 6.2.7. *Let $1 \le q \le \infty$. Let (λ, x, y) be simple eigentriple of $G(\lambda)$ and (λ, u, v) be simple eigentriple of first companion form $\mathcal{C}_1(\lambda)$ of $G(\lambda)$. Then for $\|(C(\lambda E - A)^{-1})^* y\|_2 \neq 0$ and $\|(\lambda E - A)^{-1}Bx\|_2 \neq 0$ we have*

$$\hat{s}.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2} \le \frac{\operatorname{cond}(\lambda, \mathcal{C}_1)}{\operatorname{cond}(\lambda, G)} \le s_1.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2} \le$$
$$\hat{s}.\operatorname{cond}((\lambda E - A)^{-1}B)\operatorname{cond}(C(\lambda E - A)^{-1})\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2},$$

where

$$s_1 = \frac{\|(1,\lambda)\|_q}{[\|\Lambda_m\|_q^q + \sigma_{\min}^q((\lambda E - A)^{-1}B) + \sigma_{\min}^q(C(\lambda E - A)^{-1}) + (1+|\lambda|^q)(\sigma_{\min}^q(C(\lambda E - A)^{-1})\sigma_{\min}^q((\lambda E - A)^{-1}B))]^{1/q}},$$

$$\hat{s} = \frac{\|(1,\lambda)\|_q}{[\|\Lambda_m\|_q^q + \sigma_{\max}^q((\lambda E - A)^{-1}B) + \sigma_{\max}^q(C(\lambda E - A)^{-1}) + (1+|\lambda|^q)(\sigma_{\max}^p(C(\lambda E - A)^{-1})\sigma_{\max}^q((\lambda E - A)^{-1}B))]^{1/q}}$$

and

$$1 \le \frac{\|v\|_2}{\|x\|_2}\frac{\|u\|_2}{\|y\|_2} \le \sigma_{\max}\left(\begin{bmatrix} I \\ (\lambda A_m + A_{m-1})^* \\ \vdots \\ (\lambda^{m-1}A_m + \lambda^{m-2}A_{m-1} + \cdots + A_1)^* \\ (C(\lambda E - A)^{-1})^* \end{bmatrix}\right)\sigma_{\max}\left(\begin{bmatrix} \lambda^{m-1}I \\ \lambda^{m-2}I \\ \vdots \\ I \\ (\lambda E - A)^{-1}B \end{bmatrix}\right)$$

Proof. Since

$$\frac{1}{\|(\lambda E - A)^{-1}Bx\|_2^q} \le \frac{1}{\sigma_{\min}^q((\lambda E - A)^{-1}B)\|x\|_2^q},$$
$$\frac{1}{\|(C(\lambda E - A)^{-1})^* y\|_2^q} \le \frac{1}{\sigma_{\min}^q(C(\lambda E - A)^{-1})\|y\|_2^q},$$

by (6.9) we have

$$\frac{\operatorname{cond}(\lambda, \mathcal{C}_1)}{\operatorname{cond}(\lambda, G)} \le s_1.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2},$$

where $s_1 =$

$$\frac{\|(1,\lambda)\|_q}{[\|\Lambda_m\|_q^q + \sigma_{\min}^q((\lambda E - A)^{-1}B) + \sigma_{\min}^q(C(\lambda E - A)^{-1}) + (1+|\lambda|^q)(\sigma_{\min}^q(C(\lambda E - A)^{-1})\sigma_{\min}^q((\lambda E - A)^{-1}B))]^{1/q}}.$$

Again, since $\|(\lambda E-A)^{-1}Bx\|_2^q \leq \|(\lambda E-A)^{-1}B\|_2^q\|x\|_2^q = \sigma_{\max}^q((\lambda E-A)^{-1}B)\|x\|_2^q$,

$$\|(C(\lambda E-A)^{-1})^* y\|_2^q\| \leq \|C(\lambda E-A)^{-1}\|_2^q\|y\|_2^q \leq \sigma_{\max}^q(C(\lambda E-A)^{-1})\|y\|_2^q$$

then $\frac{1}{\|(\lambda E-A)^{-1}Bx\|_2^q} \geq \frac{1}{\sigma_{\max}^q((\lambda E-A)^{-1}B)\|x\|_2^q}$ and $\frac{1}{\|(C(\lambda E-A)^{-1})^* y\|_2^q} \geq \frac{1}{\sigma_{\max}^q(C(\lambda E-A)^{-1})\|y\|_2^q}$. So

$$\frac{\mathrm{cond}(\lambda, \mathcal{C}_1)}{\mathrm{cond}(\lambda, G)} \geq s.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2}$$

where

$$\hat{s} = \frac{\|(1,\lambda)\|_q}{[\|\Lambda_m\|_q^q + \sigma_{\max}^q((\lambda E-A)^{-1}B) + \sigma_{\max}^q(C(\lambda E-A)^{-1}) + (1+|\lambda|^q)(\sigma_{\max}^q(C(\lambda E-A)^{-1})\sigma_{\max}^q((\lambda E-A)^{-1}B))]^{1/q}}.$$

Hence

$$\hat{s}.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2} \leq \frac{\mathrm{cond}(\lambda, \mathcal{C}_1)}{\mathrm{cond}(\lambda, G)} \leq s_1.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2}, \tag{6.10}$$

where $\frac{s_1}{\hat{s}} \geq 1$. Let $\alpha = \sigma_{\max}((\lambda E-A)^{-1}B)$, $\beta = \sigma_{\max}(C(\lambda E-A)^{-1})$, $\alpha' = \sigma_{\min}((\lambda E-A)^{-1}B)$, $\beta' = \sigma_{\min}(C(\lambda E-A)^{-1})$. Then

$$\hat{s} = \frac{\|(1,\lambda)\|_q}{[\|\Lambda_m\|_q^q + \alpha^q + \beta^q + (1+|\lambda|^q)\alpha^q\beta^q]^{1/q}} \quad \text{and}$$

$$s_1 = \frac{\|(1,\lambda)\|_q}{[\|\Lambda_m\|_q^q + \alpha'^q + \beta'^q + (1+|\lambda|^q)\alpha'^q\beta'^q]^{1/q}}.$$

Now

$$\begin{aligned}\frac{\|(1,\lambda)\|_q}{\hat{s}} &= [\|\Lambda_m\|_q^q + \frac{\alpha^q}{\alpha'^q}\alpha'^q + \frac{\beta^q}{\beta'^q}\beta'^q + \frac{\alpha^q}{\alpha'^q}\frac{\beta^q}{\beta'^q}\alpha'^q\beta'^q(1+|\lambda|^q)]^{1/q} \\ &\leq \Big[[\mathrm{cond}((\lambda E-A)^{-1}B)]^q[\mathrm{cond}(C(\lambda E-A)^{-1})]^q \\ &\qquad \big(\|\Lambda_m\|_q^q + \alpha'^q + \beta'^q + (1+|\lambda|^q)\alpha'^q\beta'^q\big)\Big]^{1/q} \\ &= \mathrm{cond}((\lambda E-A)^{-1}B)\mathrm{cond}(C(\lambda E-A)^{-1})\frac{\|(1,\lambda)\|_q}{s_1} \\ \Rightarrow \frac{s_1}{\hat{s}} &\leq \mathrm{cond}((\lambda E-A)^{-1}B)\mathrm{cond}(C(\lambda E-A)^{-1}).\end{aligned}$$

Thus (6.10) becomes

$$\hat{s}.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2} \leq \frac{\mathrm{cond}(\lambda, \mathcal{C}_1)}{\mathrm{cond}(\lambda, G)} \leq s_1.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2} \leq \hat{s}.\mathrm{cond}((\lambda E-A)^{-1}B).\mathrm{cond}(C(\lambda E-A)^{-1})\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2}.$$

The last part follows from the Lemma 3.1.1 and (3.5).

□

Next, we compare the condition number of quadratic eigenvalue problem (QEP) with that of the rational eigenvalue problem obtained from the QEP.

6.2.1 Comparison of Condition Numbers for QEP

Consider the quadratic eigenvalue problem (QEP)

$$Q(\lambda)x = \left(\lambda^2 A_2 + \lambda A_1 + A_0\right) x = 0, \tag{6.11}$$

where $A_2, A_1,$ and $A_0 \in \mathbb{C}^{n\times n}$. The companion linearizations of $Q(\lambda)$ is given by

$$L(\lambda)v = \left(\lambda \begin{bmatrix} A_2 & 0 \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ -I_n & 0 \end{bmatrix}\right) \begin{bmatrix} \lambda x \\ x \end{bmatrix} = 0 \tag{6.12}$$

and the left eigenvector of $L(\lambda)$ is defined by $u = \begin{bmatrix} I_n \\ (\lambda A_2 + A_1)^* \end{bmatrix} y$, where y is the left eigenvector of $P(\lambda)$. Let

$$A_0 = CB$$

be the full-rank decomposition, where $C \in \mathbb{R}^{n\times r}, B \in \mathbb{R}^{r\times n}$ and r is the rank of A_0. Then the QEP in (6.11) can be written as [37]

$$G(\lambda)x = \left[\lambda A_2 + A_1 + C(\lambda I_n - 0)^{-1}B\right] x = 0. \tag{6.13}$$

Then we have

$$\mathcal{C}(\lambda)\tilde{v} = \left(\lambda \begin{bmatrix} A_2 & \\ & -I_n \end{bmatrix} + \begin{bmatrix} A_1 & C \\ B & 0 \end{bmatrix}\right) \begin{bmatrix} x \\ s \end{bmatrix} = 0, \tag{6.14}$$

where $\widetilde{v} = \begin{bmatrix} x \\ \lambda^{-1}Bx \end{bmatrix}$ is a right eigenvector of $\mathcal{C}(\lambda)$ and $\widetilde{u} = \begin{bmatrix} y \\ (\lambda^{-1}C)^* y \end{bmatrix}$ is a left eigenvector of $\mathcal{C}(\lambda)$.

The following result compares condition number of the quadratic eigenvalue problem with that of the companion form given in (6.14).

Theorem 6.2.8. *Let $Q(\lambda)$ be as in (6.11) and $\mathcal{C}(\lambda)$ be as in (6.14). Then*

$$\frac{|\lambda|}{\sqrt{1+|\lambda|^2}} \|\tilde{u}\|_2 \|\tilde{v}\|_2 \leq \frac{\mathrm{cond}(\lambda, \mathcal{C})}{\mathrm{cond}(\lambda, Q)} \leq \|\tilde{u}\|_2 \|\tilde{v}\|_2$$

and

$$\frac{|\lambda|}{\sqrt{1+|\lambda|^2}} \left(\left(1 + \frac{1}{|\lambda|^2}\sigma^2_{\min}(B)\right)\left(1 + \frac{1}{|\lambda|^2}\sigma^2_{\min}(C)\right)\right)^{1/2} \leq \frac{\mathrm{cond}(\lambda, \mathcal{C})}{\mathrm{cond}(\lambda, Q)} \leq \left(\left(1 + \frac{1}{|\lambda|^2}\sigma^2_{\max}(B)\right)\left(1 + \frac{1}{|\lambda|^2}\sigma^2_{\max}(C)\right)\right)^{1/2}.$$

Proof. Let x and y be the left and right eigenvector of $Q(\lambda)$ corresponding to the eigenvalue λ with $\|x\| = 1$ and $\|y\| = 1$. Then

$$\begin{aligned}\frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,Q)} &= \frac{\|(1,\lambda)\|_2\|\tilde{u}\|_2\|\tilde{v}\|_2}{|y^*\partial_\lambda G(\lambda)x|}\frac{|y^*\partial_\lambda Q(\lambda)x|}{\|(1,\lambda,\lambda^2)\|_2\|x\|_2\|y\|_2}\\ &= \frac{\|(1,\lambda)\|_2\|\tilde{u}\|_2\|\tilde{v}\|_2}{\|(1,\lambda,\lambda^2)\|_2}\frac{|y^*\partial_\lambda Q(\lambda)x|}{|y^*\partial_\lambda G(\lambda)x|}\\ &= \frac{\|(1,\lambda)\|_2\|\tilde{u}\|_2\|\tilde{v}\|_2}{\|(1,\lambda,\lambda^2)\|_2}|\lambda|\\ &= \frac{\|(\lambda,\lambda^2)\|_2\|\tilde{u}\|_2\|\tilde{v}\|_2}{\|(1,\lambda,\lambda^2)\|_2}\end{aligned}$$

Now

$$\frac{\|(\lambda,\lambda^2)\|_2^2}{\|(1,\lambda,\lambda^2)\|_2^2} = \frac{|\lambda|^2+|\lambda|^4}{1+|\lambda|^2+|\lambda|^4} \leq 1$$

and

$$\frac{\|(\lambda,\lambda^2)\|_2^2}{\|(1,\lambda,\lambda^2)\|_2^2} = \frac{|\lambda|^2+|\lambda|^4}{1+|\lambda|^2+|\lambda|^4} = 1 - \frac{1}{1+|\lambda|^2+|\lambda|^4} \geq 1 - \frac{1}{1+|\lambda|^2} = \frac{|\lambda|^2}{1+|\lambda|^2}.$$

So

$$\frac{|\lambda|}{\sqrt{1+|\lambda|^2}} \leq \frac{\|(\lambda,\lambda^2)\|_2}{\|(1,\lambda,\lambda^2)\|_2} \leq 1.$$

Hence

$$\frac{|\lambda|}{\sqrt{1+|\lambda|^2}}\|\tilde{u}\|_2\|\tilde{v}\|_2 \leq \frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,Q)} \leq \|\tilde{u}\|_2\|\tilde{v}\|_2.$$

Since $\tilde{u}$ and $\tilde{v}$ are left and right eigenvector of $\mathcal{C}(\lambda)$ given in (6.14), we have

$$\sqrt{1+\frac{1}{|\lambda|^2}\sigma_{\min}^2(C)} \leq \|\tilde{u}\|_2 \leq \sqrt{1+\frac{1}{|\lambda|^2}\sigma_{\max}^2(C)},$$

$$\sqrt{1+\frac{1}{|\lambda|^2}\sigma_{\min}^2(B)} \leq \|\tilde{v}\|_2 \leq \sqrt{1+\frac{1}{|\lambda|^2}\sigma_{\max}^2(B)}.$$

Consequently,

$$\frac{|\lambda|}{\sqrt{1+|\lambda|^2}}\left(\left(1+\frac{1}{|\lambda|^2}\sigma_{\min}^2(B)\right)\left(1+\frac{1}{|\lambda|^2}\sigma_{\min}^2(C)\right)\right)^{1/2} \leq \frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,Q)} \leq \left(\left(1+\frac{1}{|\lambda|^2}\sigma_{\max}^2(B)\right)\left(1+\frac{1}{|\lambda|^2}\sigma_{\max}^2(C)\right)\right)^{1/2}.$$

□

Theorem 6.2.9. *[15] Let λ be a simple, finite, and nonzero eigenvalue of the quadratic matrix polynomial $Q(\lambda)$ given in (6.11)and L be the corresponding companion linearization given in (6.12). Then*

$$\frac{\|u\|_2}{\|y\|_2} \leq \frac{\mathrm{cond}(\lambda,L)}{\mathrm{cond}(\lambda,Q)} \leq \frac{2}{\sqrt{3}}\frac{\|u\|_2}{\|y\|_2}.$$

Theorem 6.2.10. *Let $G(\lambda)$ be given in (6.13) and $\mathcal{C}(\lambda)$ be the corresponding companion linearization given in (6.14). Let λ be a simple, finite, and nonzero eigenvalue. Then*

$$\frac{\left[(|\lambda|^2+\sigma_{\min}^2(B))(|\lambda|^2+\sigma_{\min}^2(C))\right]^{1/2}}{\left[(|\lambda|^2+\sigma_{\max}^2(B))(|\lambda|^2+\sigma_{\max}^2(C))\right]^{1/2}} \leq \frac{\operatorname{cond}(\lambda,\mathcal{C})}{\operatorname{cond}(\lambda,G)} \leq \frac{\left[(|\lambda|^2+\sigma_{\max}^2(B))(|\lambda|^2+\sigma_{\max}^2(C))\right]^{1/2}}{\left[|\lambda|^4+\sigma_{\min}^2(B)\sigma_{\min}^2(C)\right]^{1/2}}.$$

Proof. By Theorem 6.2.7, we have

$$\hat{s}.\|\tilde{u}\|_2\|\tilde{v}\|_2 \leq \frac{\operatorname{cond}(\lambda,\mathcal{C})}{\operatorname{cond}(\lambda,G)} \leq s_1.\|\tilde{u}\|_2\|\tilde{v}\|_2,$$

where

$$\hat{s} = \frac{\|(1,\lambda)\|_2}{\left[\|(1,\lambda)\|_2^2+\sigma_{\max}^2(\lambda^{-1}B)+\sigma_{\max}^2(\lambda^{-1}C)+\|(1,\lambda)\|_2^2\left(\sigma_{\max}^2(\lambda^{-1}B)\sigma_{\max}^2(\lambda^{-1}C)\right)\right]^{1/2}},$$

$$s_1 = \frac{\|(1,\lambda)\|_2}{\left[\|(1,\lambda)\|_2^2+\sigma_{\min}^2(\lambda^{-1}B)+\sigma_{\min}^2(\lambda^{-1}C)+\|(1,\lambda)\|_2^2\left(\sigma_{\min}^2(\lambda^{-1}B)\sigma_{\min}^2(\lambda^{-1}C)\right)\right]^{1/2}}. \tag{6.15}$$

Now

$$\begin{aligned}\hat{s} &= \frac{1}{\left[1+\frac{\sigma_{\max}^2(\lambda^{-1}B)+\sigma_{\max}^2(\lambda^{-1}C)}{\|(1,\lambda)\|_2^2}+\sigma_{\max}^2(\lambda^{-1}B)\sigma_{\max}^2(\lambda^{-1}C)\right]^{1/2}}\\ &\geq \frac{1}{\left[1+\sigma_{\max}^2(\lambda^{-1}B)+\sigma_{\max}^2(\lambda^{-1}C)+\sigma_{\max}^2(\lambda^{-1}B)\sigma_{\max}^2(\lambda^{-1}C)\right]^{1/2}}\\ &= \frac{1}{\left[(1+\sigma_{\max}^2(\lambda^{-1}B))(1+\sigma_{\max}^2(\lambda^{-1}C))\right]^{1/2}}\end{aligned}$$

and

$$\|\tilde{u}\|_2\|\tilde{v}\|_2 \geq \left[\left(1+\sigma_{\min}^2(\lambda^{-1}B)\right)\left(1+\sigma_{\min}^2(\lambda^{-1}C)\right)\right]^{1/2}.$$

Hence

$$\begin{aligned}&\frac{\operatorname{cond}(\lambda,\mathcal{C})}{\operatorname{cond}(\lambda,G)} \geq \frac{\left[(1+\sigma_{\min}^2(\lambda^{-1}B))(1+\sigma_{\min}^2(\lambda^{-1}C))\right]^{1/2}}{\left[(1+\sigma_{\max}^2(\lambda^{-1}B))(1+\sigma_{\max}^2(\lambda^{-1}C))\right]^{1/2}}\\ \Rightarrow &\frac{\operatorname{cond}(\lambda,\mathcal{C})}{\operatorname{cond}(\lambda,G)} \geq \frac{\left[(|\lambda|^2+\sigma_{\min}^2(B))(|\lambda|^2+\sigma_{\min}^2(C))\right]^{1/2}}{\left[(|\lambda|^2+\sigma_{\max}^2(B))(|\lambda|^2+\sigma_{\max}^2(C))\right]^{1/2}}.\end{aligned}$$

Now, from equation (6.15) we have

$$\frac{1}{\left[1+\frac{\sigma_{\min}^2(\lambda^{-1}B)+\sigma_{\min}^2(\lambda^{-1}C)}{\|(1,\lambda)\|_2^2}+\sigma_{\min}^2(\lambda^{-1}B).\sigma_{\min}^2(\lambda^{-1}C)\right]^{1/2}}.$$

Note that $f(t) := \frac{t}{[(1+\alpha)t^2+\beta]^{1/2}}$ is strictly increasing function on $[0,\infty)$ and $f(t) \to \frac{1}{\sqrt{1+\alpha}}$ as $t \to \infty$. Hence

$$\frac{1}{(1+\alpha+\beta)^{1/2}} \leq f(t) \leq \frac{1}{(1+\alpha)^{1/2}}.$$

This shows that

$$\frac{1}{[1+\sigma^2_{\min}(\lambda^{-1}B)\sigma^2_{\min}(\lambda^{-1}C)+\sigma^2_{\min}(\lambda^{-1}B)+\sigma^2_{\min}(\lambda^{-1}C)]^{1/2}} \leq s_1 \leq \frac{1}{\sqrt{1+\sigma^2_{\min}(\lambda^{-1}B)\sigma^2_{\min}(\lambda^{-1}C)}}$$

Now

$$\|\tilde{u}\|_2\|\tilde{v}\|_2 \leq \sqrt{(1+\sigma^2_{\max}(\lambda^{-1}B))\,(1+\sigma^2_{\max}(\lambda^{-1}C))}.$$

Consequently,

$$\begin{aligned}\frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,G)} &\leq \frac{[(1+\sigma^2_{\max}(\lambda^{-1}B))\,(1+\sigma^2_{\max}(\lambda^{-1}C))]^{1/2}}{[1+\sigma^2_{\min}(\lambda^{-1}B)\sigma^2_{\min}(\lambda^{-1}C)]^{1/2}}\\ &= \frac{[(|\lambda|^2+\sigma^2_{\max}(B))\,(|\lambda|^2+\sigma^2_{\max}(C))]^{1/2}}{[|\lambda|^4+\sigma^2_{\min}(B)\sigma^2_{\min}(C)]^{1/2}}.\end{aligned}$$

This completes the proof. □

Theorem 6.2.11. *Let $G(\lambda)$ be given in (6.13) and $\mathcal{C}(\lambda)$ be the corresponding companion linearization given in (6.14). Let λ be a simple, finite, and nonzero eigenvalue. Then*

$$1 \leq \frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,G)} \leq \left[1+\frac{\frac{1}{|\lambda|^2}\|Bx\|_2^2+\frac{1}{|\lambda|^2}\|C^*y\|_2^2}{1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\frac{1}{|\lambda|^2}\|C^*y\|_2^2}\right]^{1/2}.$$

Proof. Let x and y be the left and right eigenvector of $G(\lambda)$ corresponding to the eigenvalue λ with $\|x\|=1$ and $\|y\|=1$. Then

$$\frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,G)} = \frac{\|(1,\lambda)\|_2\|\tilde{u}\|_2\|\tilde{v}\|_2}{[\|(1,\lambda)\|_2^2+\|(\lambda^{-1}Bx)^*\|_2^2+\|(\lambda^{-1}C)^*y\|_2^2+\|(1,\lambda)\|_2^2(\|(\lambda^{-1}Bx)^*\|_2^2\|(\lambda^{-1}C)^*y\|_2^2)]^{1/2}}.$$

Now $\|\widetilde{u}\|_2 = [\|x\|_2^2+\|\lambda^{-1}Bx\|_2^2]^{1/2} = \left[1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\right]^{1/2}$ and $\|\widetilde{v}\|_2 = \left[1+\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right]^{1/2}$.

$$\begin{aligned}\frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,G)} &= \frac{\|(1,\lambda)\|_2\left(1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\right)^{1/2}\left(1+\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)^{1/2}}{\left[\|(1,\lambda)\|_2^2+\frac{1}{|\lambda|^2}\|Bx\|_2^2+\frac{1}{|\lambda|^2}\|C^*y\|_2^2+(1+|\lambda|^2)\left(\frac{1}{|\lambda|^2}\|Bx\|_2^2.\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)\right]^{1/2}}\\ &= \frac{\left(1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\right)^{1/2}\left(1+\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)^{1/2}}{\left[1+\frac{\frac{1}{|\lambda|^2}\|Bx\|_2^2+\frac{1}{|\lambda|^2}\|C^*y\|_2^2}{\|(1,\lambda)\|_2^2}+\left(\frac{1}{|\lambda|^2}\|Bx\|_2^2.\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)\right]^{1/2}} \qquad (6.16)\\ &\geq \frac{\left(1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\right)^{1/2}\left(1+\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)^{1/2}}{\left[1+\frac{1}{|\lambda|^2}\|Bx\|_2^2+\frac{1}{|\lambda|^2}\|C^*y\|_2^2+\left(\frac{1}{|\lambda|^2}\|Bx\|_2^2.\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)\right]^{1/2}}\end{aligned}$$

$$= \frac{\left(1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\right)^{1/2}\left(1+\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)^{1/2}}{\left(1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\right)^{1/2}\left(1+\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)^{1/2}} = 1.$$

So

$$\frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,G)} \geq 1.$$

Again from equation (6.16) we have

$$\frac{\mathrm{cond}(\lambda,\mathcal{C})}{\mathrm{cond}(\lambda,G)} \leq \frac{\left(1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\right)^{1/2}\left(1+\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right)^{1/2}}{\left[1+\frac{1}{|\lambda|^2}\|Bx\|_2^2.\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right]^{1/2}}$$

$$= \frac{\left[1+\frac{1}{|\lambda|^2}\|Bx\|_2^2+\frac{1}{|\lambda|^2}\|C^*y\|_2^2+\frac{1}{|\lambda|^2}\|Bx\|_2^2\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right]^{1/2}}{\left[1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\frac{1}{|\lambda|^2}\|C^*y\|_2^2\right]^{1/2}}$$

$$= \left[1+\frac{\frac{1}{|\lambda|^2}\|Bx\|_2^2+\frac{1}{|\lambda|^2}\|C^*y\|_2^2}{1+\frac{1}{|\lambda|^2}\|Bx\|_2^2\frac{1}{|\lambda|^2}\|C^*y\|_2^2}\right]^{1/2}.$$

□

The following result compares the condition number of $Q(\lambda)$ given in (6.11) with that of $G(\lambda)$ given in (6.13).

Theorem 6.2.12. *Let $Q(\lambda)$ be given in (6.11) and $G(\lambda)$ given in (6.13). Let λ be a simple, finite, and nonzero eigenvalue. Then*

$$\frac{\sqrt{\sigma_{\min}^2(C)+\sigma_{\min}^2(B)}}{\|(1,\lambda,\lambda^2)\|_2} \leq \frac{\mathrm{cond}(\lambda,G)}{\mathrm{cond}(\lambda,Q)} \leq \left[\left(1+\sigma_{\max}^2(\lambda^{-1}B)\right)\left(1+\sigma_{\max}^2(\lambda^{-1}C)\right)\right]^{1/2}.$$

Proof. We have $\frac{\mathrm{cond}(\lambda,G)}{\mathrm{cond}(\lambda,Q)}$

$$= \frac{[\|(1,\lambda)\|_2^2+\|(\lambda^{-1}Bx)^*\|^2+\|(\lambda^{-1}C)^*y\|^2+(1+|\lambda|^2)(\|(\lambda^{-1}Bx)^*\|^2\|(\lambda^{-1}C)^*y\|^2)]^{1/2}}{\|(1,\lambda,\lambda^2)\|_2\|x\|_2\|y\|_2} \cdot \frac{|y^*\partial_\lambda Q(\lambda)x|}{|y^*\partial_\lambda G(\lambda)x|} \tag{6.17}$$

$$= \frac{[\|(1,\lambda)\|_2^2+\|\lambda^{-1}Bx\|_2^2+\|(\lambda^{-1}C)^*y\|_2^2+(1+|\lambda|^2)(\|\lambda^{-1}Bx\|_2^2\|(\lambda^{-1}C)^*y\|_2^2)]^{1/2}}{\|(1,\lambda,\lambda^2)\|_2}|\lambda|.$$

Hence

$$s \leq \frac{\mathrm{cond}(\lambda,G)}{\mathrm{cond}(\lambda,Q)} \leq s_1,$$

where

$$s = \frac{|\lambda| \left[\|(1,\lambda)\|_2^2 + \sigma_{\min}^2(\lambda^{-1}B) + \sigma_{\min}^2(\lambda^{-1}C) + (1+|\lambda|^2)(\sigma_{\min}^2(\lambda^{-1}B)\sigma_{\min}^2(\lambda^{-1}C))\right]^{1/2}}{\|(1,\lambda,\lambda^2)\|_2},$$

$$\begin{aligned}
s_1 &= \frac{|\lambda| \left[\|(1,\lambda)\|_2^2 + \sigma_{\max}^2(\lambda^{-1}B) + \sigma_{\max}^2(\lambda^{-1}C) + (1+|\lambda|^2)(\sigma_{\max}^2(\lambda^{-1}B)\sigma_{\max}^2(\lambda^{-1}C))\right]^{1/2}}{\|(1,\lambda,\lambda^2)\|_2} \\
&= \frac{|\lambda|\|(1,\lambda)\|_2 \left[1 + \frac{\sigma_{\max}^2(\lambda^{-1}B)+\sigma_{\max}^2(\lambda^{-1}C)}{\|(1,\lambda)\|_2^2} + (\sigma_{\max}^2(\lambda^{-1}B)\sigma_{\max}^2(\lambda^{-1}C))\right]^{1/2}}{\|(1,\lambda,\lambda^2)\|_2} \\
&\le \frac{|\lambda|\|(1,\lambda)\|_2}{(1,\lambda,\lambda^2)\|_2} \left[1 + \sigma_{\max}^2(\lambda^{-1}B) + \sigma_{\max}^2(\lambda^{-1}C) + (\sigma_{\max}^2(\lambda^{-1}B)\sigma_{\max}^2(\lambda^{-1}C))\right]^{1/2} \\
&= \frac{|\lambda|\|(1,\lambda)\|_2}{(1,\lambda,\lambda^2)\|_2} \left[(1 + \sigma_{\max}^2(\lambda^{-1}B))(1 + \sigma_{\max}^2(\lambda^{-1}C))\right]^{1/2}.
\end{aligned}$$

Now

$$\frac{|\lambda|\|(1,\lambda)\|_2}{(1,\lambda,\lambda^2)\|_2} = \frac{|\lambda|\sqrt{1+|\lambda|^2}}{\sqrt{1+|\lambda|^2+|\lambda|^4}} = \left(\frac{|\lambda|^2+|\lambda|^4}{1+|\lambda|^2+|\lambda|^4}\right)^{1/2} \le 1$$

So

$$\frac{\mathrm{cond}(\lambda, G)}{\mathrm{cond}(\lambda, Q)} \le \left[(1 + \sigma_{\max}^2(\lambda^{-1}B))(1 + \sigma_{\max}^2(\lambda^{-1}C))\right]^{1/2}$$

Now

$$s = \frac{\|(\lambda,\lambda^2)\|_2}{\|(1,\lambda,\lambda^2)\|_2} \left[1 + \frac{\sigma_{\min}^2(\lambda^{-1}B) + \sigma_{\min}^2(\lambda^{-1}C)}{\|(1,\lambda)\|_2^2} + \sigma_{\min}^2(\lambda^{-1}B).\sigma_{\min}^2(\lambda^{-1}C)\right]^{1/2}.$$

Thus

$$\frac{\sqrt{\sigma_{\min}^2(C) + \sigma_{\min}^2(B)}}{\|(1,\lambda,\lambda^2)\|_2} \le \frac{\mathrm{cond}(\lambda, G)}{\mathrm{cond}(\lambda, Q)} \le \left[\left(1 + \sigma_{\max}^2(\lambda^{-1}B)\right)\left(1 + \sigma_{\max}^2(\lambda^{-1}C)\right)\right]^{1/2}.$$

$\square$

Example 6.5 (Loaded elastic string, [37])**.** Consider the rational eigenvalue problem

$$\left(A_1 - \lambda A_2 + \frac{\lambda}{\lambda - \sigma} E\right) x = 0, \tag{6.18}$$

where A_1 and A_2 are tridiagonal and symmetric positive definite, and $E = e_n e_n^T$ is the last column of the identity matrix and $\sigma > 0$ is a parameter. Then we have

$$G(\lambda)x = \left[A_1 + e_n e_n^T - \lambda A_2 + e_n \left(\frac{\lambda}{\sigma} - 1\right)^{-1} e_n^T\right] x = 0, \tag{6.19}$$

and $C_1(\lambda)v = (\lambda\mathcal{X} - \mathcal{Y})v =$

$$\left(\lambda \begin{bmatrix} A_2 & \\ & -1/\sigma \end{bmatrix} - \begin{bmatrix} -(A_1 + e_n e_n^T) & -e_n \\ -e_n^T & -1 \end{bmatrix}\right) \begin{bmatrix} x \\ \left(\frac{\lambda}{\sigma} - 1\right)^{-1} e_n^T x \end{bmatrix} = 0. \tag{6.20}$$

The quadratic eigenvalue problem of (6.18) is given by

$$Q(\lambda)x = [\lambda^2 A_2 - \lambda(A_1 + \sigma A_2 + e_n e_n^T) + \sigma A_1]x = 0, \qquad \sigma > 0, \tag{6.21}$$

Here

$$\sigma_{\min}\left(\left(\frac{\lambda}{\delta}-1\right)^{-1} e_n^T\right) = \sigma_{\max}\left(\left(\frac{\lambda}{\delta}-1\right)^{-1} e_n^T\right) = \frac{1}{\left|\frac{\lambda}{\delta}-1\right|} = \sigma_{\min}\left(e_n\left(\frac{\lambda}{\delta}-1\right)^{-1}\right)$$
$$= \sigma_{\max}\left(e_n\left(\frac{\lambda}{\delta}-1\right)^{-1}\right)$$

By Theorem 6.2.7, we have $s = s_1$. So

$$\frac{\mathrm{cond}(\lambda, C_1)}{\mathrm{cond}(\lambda, G)} = s.\frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2},$$

where

$$\begin{aligned} s &= \frac{\|(1,\lambda)\|_q}{[\|(1,\lambda)\|_q^q + |\frac{\lambda}{\delta}-1|^{-q} + |\frac{\lambda}{\delta}-1|^{-q} + (1+|\lambda|^q)\left(|\frac{\lambda}{\delta}-1|^{-2q}\right)]^{1/q}} \\ &= \frac{(1+|\lambda|^q)^{1/q}}{[1+|\lambda|^q + \frac{\delta^q}{|\lambda-\delta|^q} + \frac{\delta^q}{|\lambda-\delta|^q} + (1+|\lambda|^q)\left(\frac{\delta^{2q}}{|\lambda-\delta|^{2q}}\right)]^{1/q}} \\ &= \frac{(1+|\lambda|^q)^{1/q}}{[(1+|\lambda|^q) + 2|\frac{\delta}{\lambda-\delta}|^q + (1+|\lambda|^q)\left(|\frac{\delta}{\lambda\delta}|^{2q}|\right)]^{1/q}} \leq 1 \end{aligned}$$

and $1 \leq \frac{\|u\|_2}{\|y\|_2}\frac{\|v\|_2}{\|x\|_2} \leq \sigma_{\max}\left(\begin{bmatrix} I_n \\ (e_n(\frac{\lambda}{\delta}-1)^{-1})^* \end{bmatrix}\right)\sigma_{\max}\left(\begin{bmatrix} I_n \\ (\frac{\lambda}{\delta}-1)^{-1}e_n^T \end{bmatrix} v\right) \leq 1 + |\lambda/\delta - 1|^{-2}$.
This shows that

$$\frac{\mathrm{cond}(\lambda, C_1)}{\mathrm{cond}(\lambda, G)} \leq 1 + |\lambda/\delta - 1|^{-2}.$$

Next, considering $G(\lambda)$ and $Q(\lambda)$ given in (6.19) and (6.21), respectively, by Theorem 6.2.12, we have

$$\frac{\mathrm{cond}(\lambda, G)}{\mathrm{cond}(\lambda, Q)} \leq \left(1 + \frac{\sigma^2}{|\lambda-\sigma|^2}\right).$$

7 Conclusions

In this thesis, we have developed a framework for solutions of rational eigenvalue problems. We have achieved this goal by reformulating a rational eigenvalue problem $G(\lambda)u = 0$ to that of computing transmission zeros and zero directions of a linear time invariant (LTI) system Σ given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + P(\tfrac{d}{dt})u(t) \end{aligned}$$

for which $G(\lambda)$ is the transfer function. Since eigenvalues of $G(\lambda)$ form a subset of the transmission zeros of the LTI system Σ and the transmission zeros form a subset of the invariant zeros of the system Σ, we have focused on computing invariant zeros and zero directions of the LTI system which are in fact eigenvalues and eigenvectors of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ given by

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} P(\lambda) & C \\ \hline B & (A - \lambda E) \end{array}\right].$$

Hence we have developed a framework for computing eigenvalues and eigenvectors of the Rosenbrock system matrix $\mathcal{S}(\lambda)$. For this purpose, we have introduced three families of linearizations - which we referred to as Fiedler pencils, Generalized Fiedler (GF) pencils and Generalized Fiedler pencils with repetition (GFPR) - of the Rosenbrock system matrix $\mathcal{S}(\lambda)$. Thus invariant zeros of the LTI system Σ could be computed by solving generalized eigenvalue problems for the Fiedler pencils.

We have described construction of Fiedler pencils of the Rosenbrock system matrix $\mathcal{S}(\lambda)$ and have shown that these Fiedler pencils are *linearizations* for $\mathcal{S}(\lambda)$. Further, we have shown that the Fiedler pencils of $\mathcal{S}(\lambda)$ are *linearizations* of $G(\lambda)$ when the LTI system Σ is both controllable and observable, that is, when Σ is a minimal state-space realization of the transfer function $G(\lambda)$. Furthermore, we have shown that GF pencils and GFPR could be utilized to obtain structure (such as symmetric and Hermitian) preserving linearizations of the Rosenbrock system matrix.

We have also described eigenvector recovery of $G(\lambda)$ from that of Fiedler pencils of the system matrix $\mathcal{S}(\lambda)$. In fact, we have shown that the eigenvectors of $G(\lambda)$ could be recovered from those of the Fiedler pencils of $\mathcal{S}(\lambda)$ without incurring any additional computational cost. Further, we have shown that the state-space framework so developed could be utilized to *linearize* a higher order LTI state-space system so as to analyze and solve the higher order LTI system. Furthermore, we have shown that the linearized systems so obtained are *strict system equivalent* to the higher order systems and hence preserve system characteristics of the original systems.

We have also developed a framework for sensitivity analysis of eigenvalues of the system matrix $\mathcal{S}(\lambda)$ and the transfer function $G(\lambda)$. We have defined condition numbers for simple eigenvalues of $\mathcal{S}(\lambda)$ and $G(\lambda)$, and have obtained explicit computable expressions for the condition numbers. We have also analyzed the effect of linearization on the conditioning of the eigenvalues of the system matrix $\mathcal{S}(\lambda)$.

Bibliography

[1] Sk. Safique Ahmad and Rafikul Alam. On the conditioning of polynomial eigenproblems. *Preprint*.

[2] Sk. Safique Ahmad and Rafikul Alam. Pseudospectra, critical points and multiple eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 430(4):1171–1195, 2009.

[3] E. N. Antoniou and S. Vologiannidis. A new family of companion forms of polynomial matrices. *Electron. J. Linear Algebra*, 11:78–87, 2004.

[4] Athanasios C. Antoulas. *Approximation of large-scale dynamical systems*. SIAM, Philadelphia, PA, 2005.

[5] Panos J. Antsaklis and Anthony N. Michel. *Linear systems*. Birkhäuser Boston Inc., Boston, MA, 2006.

[6] María I. Bueno, Fernando de Terán, and Froilán M. Dopico. Recovery of eigenvectors and minimal bases of matrix polynomials from generalized Fiedler linearizations. *SIAM J. Matrix Anal. Appl.*, 32(2):463–483, 2011.

[7] C. Conca, J. Planchard, and M. Vanninathan. Existence and location of eigenvalues for fluid-solid structures. *Comput. Methods Appl. Mech. Engrg.*, 77(3):253–291, 1989.

[8] Biswa Nath Datta. *Numerical methods for linear control systems*. Elsevier Academic Press, San Diego, CA, 2004.

[9] Fernando De Terán, Froilán M. Dopico, and D. Steven Mackey. Fiedler companion linearizations and the recovery of minimal indices. *SIAM J. Matrix Anal. Appl.*, 31(4):2181–2204, 2009/10.

[10] D.S.Mackey. Structured linearizations for matrix polynomials. *PhD Thesis, The University of Manchester, U.K, 2006.*

[11] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix polynomials*. Academic Press, Inc., New York, 1982.

[12] Israel Gohberg, Peter Lancaster, and Leiba Rodman. *Invariant subspaces of matrices with applications*. SIAM, Philadelphia, PA, 2006.

[13] João P. Hespanha. *Linear systems theory*. Princeton University Press, Princeton, NJ, 2009.

[14] Nicholas J. Higham, D. Steven Mackey, Niloufer Mackey, and Fran**C**coise Tisseur. Symmetric linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 29(1):143–159 (electronic), 2006/07.

[15] Nicholas J. Higham, D. Steven Mackey, and Fran**C**coise Tisseur. The conditioning of linearizations of matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):1005–1028, 2006.

[16] Kenneth Hoffman and Ray Kunze. *Linear algebra*. Second edition. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971.

[17] Tsung-Min Hwang, Wen-Wei Lin, Wei-Cheng Wang, and Weichung Wang. Numerical simulation of three dimensional pyramid quantum dot. *J. Comput. Phys.*, 196(1):208–232, 2004.

[18] Maria I.Bueno and Fernando De Teran. Eigenvectors and minimal bases for some families of Fiedler-like linearizations. *Linear and Multilinear Algebra*, pages 1–24, 2013.

[19] M.Sc. Viatcheslav I.Sokolov. Contributions to the minimal realization problem for descriptor systems. *PhD thesis, TU Chemnitz, 2006.*

[20] Elias Jarlebring and Heinrich Voss. Rational Krylov for nonlinear eigenproblems, an iterative projection method. *Appl. Math.*, 50(6):543–554, 2005.

[21] Thomas Kailath. *Linear systems*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1980.

[22] Uwe Mackenroth. *Robust control systems*. Springer-Verlag, Berlin, 2004.

[23] D. Steven Mackey, Niloufer Mackey, Christian Mehl, and Volker Mehrmann. Structured polynomial eigenvalue problems: good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[24] D. Steven Mackey, Niloufer Mackey, Christian Mehl, and Volker Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):971–1004, 2006.

[25] Lada Mazurenko and Heinrich Voss. On the number of eigenvalues of a rational eigenproblem. SIAM J. Numer. Anal, 2003.

[26] Lada Mazurenko and Heinrich Voss. Low rank rational perturbations of linear symmetric eigenproblems. *ZAMM Z. Angew. Math. Mech.*, 86(8):606–616, 2006.

[27] Volker Mehrmann and Heinrich Voss. Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods. *GAMM Mitt. Ges. Angew. Math. Mech.*, 27(2):121–152 (2005), 2004.

[28] Wim Michiels and Silviu-Iulian Niculescu. *Stability and stabilization of time-delay systems*. SIAM, Philadelphia, PA, 2007.

[29] J. Planchard. Eigenfrequencies of a tube bundle placed in a confined fluid. *Comput. Methods Appl. Mech. Engrg.*, 30(1):75–93, 1982.

[30] H. H. Rosenbrock. *State-space and multivariable theory*. John Wiley & Sons, Inc., New York, 1970.

[31] Axel Ruhe. Algorithms for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 10:674–689, 1973.

[32] R. A. Smith. The condition numbers of the matrix eigenvalue problem. *Numer. Math.*, 10:232–240, 1967.

[33] S. I. Solov'ev and S. I. Solov v. Eigenvibrations of a plate with elastically attached load, 2003.

[34] Sergey I. Solov′ëv. Preconditioned iterative methods for a class of nonlinear eigenvalue problems. *Linear Algebra Appl.*, 415(1):210–229, 2006.

[35] Willi-Hans Steeb. *Matrix calculus and Kronecker product with applications and C++ programs*. World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[36] G. W. Stewart and Ji Guang Sun. *Matrix perturbation theory*. Academic Press Inc., Boston, MA, 1990.

[37] Yangfeng Su and Zhaojun Bai. Solving rational eigenvalue problems via linearization. *SIAM J. Matrix Anal. Appl.*, 32(1):201–216, 2011.

[38] Volker Mehrmann Christian Schr Ader and Fran Axoise Tisseur Timo Betcke, Nicholas J. Higham. Nlevp: A collection of nonlinear eigenvalue problems, 2013.

[39] FranCcoise Tisseur and Karl Meerbergen. The quadratic eigenvalue problem. *SIAM Rev.*, 43(2):235–286, 2001.

[40] A. I. G. Vardulakis. *Linear multivariable control*. John Wiley & Sons Ltd., Chichester, 1991.

[41] S. Vologiannidis and E. N. Antoniou. A permuted factors approach for the linearization of polynomial matrices. *Math. Control Signals Systems*, 22(4):317–342, 2011.

[42] H. Voss. Solving a rational eigenvalue problem in fluid-structure interaction. In *Proceedings of the Seventh International Conference on Applications of High-Performance Computers in Engineering*, 2002.

[43] H. Voss. An Arnoldi method for nonlinear eigenvalue problems. *BIT*, 44(2):387–401, 2004.

[44] Heinrich Voss. A rational spectral problem in fluid-solid vibration. *Electron. Trans. Numer. Anal.*, 16:93–105, 2003.

[45] Heinrich Voss. A Jacobi-Davidson method for nonlinear eigenproblems. In *Computational science—ICCS 2004. Part II*, volume 3037 of *Lecture Notes in Comput. Sci.*, pages 34–41. Springer, Berlin, 2004.

[46] Heinrich Voss. Iterative projection methods for computing relevant energy states of a quantum dot. *J. Comput. Phys.*, 217(2):824–833, 2006.

[47] David S. Watkins. *Fundamentals of matrix computations*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2010.