A Study of Frames and Their Generalizations

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September, 2017

A Study of Frames and Their Generalizations

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of

Doctor of Philosophy

by

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to the

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Guwahati, India- 781039

September, 2017

DECLARATION

I do hereby declare that this thesis entitled **A Study of Frames and Their Generalizations** is a presentation of my original research work done under the supervision of **Dr. Jitendriya Swain**, Assistant Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is to certify that the work contained in this thesis entitled **A Study of Frames and Their Generalizations** has been carried out by **Anirudha Poria,** a student in the Department of Mathematics, Indian Institute of Technology Guwahati, under my supervision for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

September, 2017

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To My Parents

Acknowledgement

First and foremost, I am very grateful to my thesis supervisor Dr. Jitendriya Swain for his generous guidance throughout the years. Also I would like to express my deep appreciation for his patience, understanding and support. He helped me to overcome many obstacles that emerged while working on several problems. In a good way, he always used to push for better results. He was deeply involved and helped me in every possible way. His continuous encouragement and advice made it possible for me to work on this thesis.

I wish to express my warm and sincere thanks to Prof. Bhaba Kumar Sarma, former Head of the Department and Prof. Swaroop Nandan Bora, Head of the Department for providing excellent research facilities during my Ph.D programme. I would like to express my gratitude to the doctoral committee members, Prof. M. Guru Prem Prasad, Dr. Anjan K. Chakrabarty and Dr. Rajesh Srivastava for their encouragement, precious comments and discussions on several occasions. Also, I would like to thank Dr. Pratyoosh Kumar for his help and support during the tenure of the research.

I would also like to convey my deepest appreciation to Prof. Radu Balan, Prof. Hans G. Feichtinger, Prof. Kasso A. Okoudjou and Dr. Azita Mayeli for their several valuable suggestions, insightful comments and continuous encouragement during my research tenure.

I am deeply grateful to the United States-India Educational Foundation for providing the Fulbright-Nehru Doctoral Research Fellowship, and Department of Mathematics, University of Maryland, College Park, USA for the support provided during the period of my stay at Maryland. I would like to express my gratitude to the Norbert Wiener Center for Harmonic Analysis and Applications at the University of Maryland, College Park, USA and the Department of Mathematics and Computer Science, the Graduate Center, City University of New York, USA for the kind hospitality. I would also like to appreciate the travel grant and hospitality provided by National Science Foundation to attend the CBMS Conference and the 16th New Mexico Analysis Seminar held at New Mexico State University, New Mexico, USA.

I am also indebted to my friends, colleagues and the faculties of Department of Mathematics, IIT Guwahati for motivating, helping and providing me a stimulating and fun-filled environment. I would like to thank all the staff members of the Department of Mathematics and administration for providing me necessary help whenever required. I would take this opportunity to thank Swarup, Hiranmoy, Koyel, Abhishek, Anirban, Sougata, Balasubramani, Ankur and Swapnendu for their continuous encouragement and support.

I am highly grateful to the Ministry of Human Resource Development, Government of India for the necessary financial supports. I sincerely acknowledge Indian Institute of Technology Guwahati for providing a very nice educational environment and all kinds of support.

I would like to express my deepest gratitude to my parents for their unconditional trust, encouragement and patience. I would also like to mention the love and enormous support provided by my family members, especially my sisters.

I am also very thankful to all those peoples who shared some of their noble views, precious experiences and countless ideas about Mathematics and helped me to understand the subject better.

Finally, I would like to acknowledge everybody who is important to the successful completion of the thesis as well as express my apology that I could not mention each of them individually.

September, 2017

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Abstract

This thesis addresses some problems on frames and their generalizations viz Hilbert space valued Gabor frames and Hilbert-Schmidt frames. The main objectives of the thesis are to analyze Gabor frames on amalgam spaces, find solution of a Feichtinger problem and prove Balian-Low type theorems on $L^2(\mathbb{C})$. This thesis consists of six chapters.

In Chapter 1, we give a brief introduction of frame theory, discuss some well-known results, basic definitions and provide a literature survey.

In Chapter 2, we prove the convergence of Gabor expansions to identity operator in the operator norm as well as in weak* sense on $W(L^p, L^q)$ as the sampling density tends to infinity. Using it we show the validity of the Janssen's representation and the Wexler-Raz biorthogonality condition for Gabor frame operator on $W(L^p, L^q)$.

Let $\mathbb{H}$ be a separable Hilbert space. In Chapter 3, we generalize and extend the Walnut's representation and Janssen's representation of the $\mathbb{H}-$valued Gabor frame operator on $\mathbb{H}-$valued weighted amalgam spaces $W_{\mathbb{H}}(L^p, L^q_v)$, $1 \leq p, q \leq \infty$. Also we show that the frame operator is invertible on $W_{\mathbb{H}}(L^p, L^q_v)$, $1 \leq p, q \leq \infty$, if the window function is in the Wiener amalgam space $W_{\mathbb{H}}(L^\infty, L^1_w)$. Further, we obtain the Walnut's representation and invertibility of the frame operator corresponding to Gabor superframes and multi-window Gabor frames on $W_{\mathbb{H}}(L^p, L^q_v)$, $1 \leq p, q \leq \infty$, as a special case by choosing the appropriate Hilbert space $\mathbb{H}$.

In Chapter 4, we study the Hilbert$-$Schmidt frame (HS-frame) theory for separable Hilbert spaces. We first present some characterizations of HS-frames and prove that HS-frames share many important properties with frames. Then we show that the inverse of the HS-frame operator can be approximated using finite-dimensional methods. Also we present a classical perturbation result and prove that HS-frames are stable under small perturbations. Further, as an application we establish Parseval type identities and inequalities for HS-frames.

In Chapter 5, we answer to the spectral problem about positive semi-definite trace-class pseudodifferential operators on modulation spaces posed by Hans Feichtinger. Also, we discuss the solutions of several reformulated problems inspired by the original Feichtinger's question in Hilbert space operator theory that was posed by Heil and Larson. These results provide some connections between operator theory and the theory of modulation spaces.

In Chapter 6, we prove the Balian-Low type theorem (BLT) on $L^2(\mathbb{C})$ using the operators Z and $\bar{Z}$ i.e., we prove that $\|Zg\|_2$ and $\|\bar{Z}g\|_2$ cannot both be simultaneously finite if the twisted Gabor frame generated by $g \in L^2(\mathbb{C})$ forms an orthonormal

basis or an exact frame for $L^2(\mathbb{C})$. The operators

$$Z = \frac{d}{dz} + \frac{1}{2}\bar{z} \quad \text{and} \quad \bar{Z} = \frac{d}{d\bar{z}} - \frac{1}{2}z$$

are associated with the special Hermite operator

$$L = -\Delta_z + \frac{1}{4}|z|^2 - i\left(x\frac{d}{dy} - y\frac{d}{dx}\right)$$

on $\mathbb{C}$, where Δ_z is the standard Laplacian on $\mathbb{C}$ and $z = x + iy$. Also the amalgam version of BLT is proved using Weyl transform and the distinction between BLT and amalgam BLT is illustrated by examples. The twisted Zak transform is introduced and using it several versions of the Balian-Low type theorems on $L^2(\mathbb{C})$ are established.

In Chapter 7, we describe a few directions for future research based on the work of this thesis.

Contents

List of symbols

$\mathbb{R}$:	Set of all real numbers.
$\mathbb{N}$:	Set of all natural numbers.
$\mathbb{Z}$:	Set of all integers.
$\mathbb{Q}$:	Set of all rational numbers.
$\mathbb{C}$:	Set of all complex numbers.
$\mathbb{R}^d$:	The d-dimensional Euclidean space.
$Im(z)$:	The imaginary part of $z \in \mathbb{C}$.
$\mathbb{H}, \mathbb{K}$:	Hilbert spaces.
$L^p(\mathbb{R})$:	The space of all complex valued measurable functions on $\mathbb{R}$ whose p-th power is integrable.
$C(\mathbb{R})$:	The space of all complex valued continuous functions on $\mathbb{R}$.
$C^k(\mathbb{R})$:	The space of all complex valued k times differentiable functions whose k-th derivative is continuous on $\mathbb{R}$.
$\hat{f}$:	The Fourier transform of f.
ℓ^p :	The space of all complex sequences whose p-th power is summable.
$\|I\|$:	The Lebesgue measure of the Borel set I.
χ_E :	The indicator function for a set E, $\chi_E(x) = 1$ if $x \in E$, otherwise 0.
$\delta_{k,j}$:	The Kronecker delta: $\delta_{k,j} = 1$ if $k = j$, $\delta_{k,j} = 0$ if $k \neq j$.

Chapter 1

Introduction

Bases play a crucial role in the analysis of vector spaces as every vector in the considered space can be expressed in a unique way as a linear combination of the basis elements. But the conditions for a family of elements in the vector space to form a basis are very restrictive since the elements of a basis need to be linearly independent and some times we even want the elements to be orthogonal with respect to an inner product. This makes it hard or even impossible to find bases satisfying extra conditions. Due to this reason one might look for a more flexible tool. Frames are such tools where every element in the vector space equipped with an inner product can be written as linear combination of frame elements i.e. if we consider the sequence of elements $\{f_k\}$ in a Hilbert space $\mathbb{H}$, then every $f \in \mathbb{H}$ can be written as $f = \sum_{k=1}^{\infty} c_k(f) f_k$. However the coefficient $c_k(f)$ is not necessarily unique. Thus a frame might not be a basis but a generalization of a basis.

The concept of a frame in Hilbert spaces has been introduced in 1952 by Duffin and Schaeffer [48], in the context of nonharmonic Fourier series (see [143]). After the work of Daubechies et al. [41] frame theory got considerable attention outside signal processing and began to be more broadly studied (see [25, 35, 77]). A frame for a Hilbert space is a redundant set of vectors in Hilbert space which provides non-unique representations of vectors in terms of frame elements. The redundancy and flexibility offered by frames has spurred their application in several areas of mathematics, physics, and engineering such as sigma-delta quantization [20], neural networks [23], image processing [24], system modelling [47], quantum measurements

[53], sampling theory [59], wireless communications [126] and many other well known fields.

1.1 Frames in Hilbert Spaces

Let us denote $\mathbb{H}$ and $\mathbb{K}$ as Hilbert spaces, $\mathcal{L}(\mathbb{H})$ the algebra of all bounded linear operators on $\mathbb{H}$, I the identity operator on $\mathbb{H}$, and J as a countable index set. We start with the definition and some basic properties of frames in Hilbert spaces.

Definition 1.1.1. *A family $\{f_j : j \in J\}$ in $\mathbb{H}$ is called a frame for $\mathbb{H}$, if there exist constants $0 < A \leq B < \infty$ such that for all $f \in \mathbb{H}$*

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

The constants A and B are called *frame bounds.* If $A = B$, then this frame is called an *A-tight frame*, and if $A = B = 1$, then it is called a *Parseval frame.* If we only have the right hand inequality in (1.1), we call $\{f_j : j \in J\}$ a *Bessel sequence with Bessel bound B.* A frame $\{f_j : j \in J\}$ is *exact* if it ceases to be a frame when any single element f_n is deleted, that is, $\{f_j\}_{j\neq n}$ is not a frame for any n.

Example 1.1.2. Every orthonormal basis $\{e_j\}$ for $\mathbb{H}$ is a tight frame for $\mathbb{H}$ with $A = B = 1$. Moreover, $\{e_j\}$ is an exact frame since if we delete any element e_m, then $\{e_j\}_{j\neq m}$ cannot be a frame.

Example 1.1.3. Let $\{e_j\}$ be an orthonormal basis for a separable Hilbert space $\mathbb{H}$.

(i) The family $\{e_1, e_1, e_2, e_2, e_3, e_3, \cdots\}$ is a tight inexact frame for $\mathbb{H}$ with bounds $A = B = 2$, but it is not orthogonal and it is not a basis, although it does contain an orthonormal basis. Similarly, if $\{f_j\}$ is another orthonormal basis for $\mathbb{H}$ then $\{e_j\} \cup \{f_j\}$ is a tight inexact frame for $\mathbb{H}$.

(ii) The family $\{e_1, \frac{e_2}{2}, \frac{e_3}{3}, \cdots\}$ is a complete orthogonal sequence and it is a basis for $\mathbb{H}$, but it does not possess a lower frame bound and hence is not a frame for $\mathbb{H}$.

(*iii*) The family $\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \cdots\}$ is a tight inexact frame for $\mathbb{H}$ with bounds $A = B = 1$, and no nonredundant subsequence is a frame.

(*iv*) The family $\{2e_1, e_2, e_3, \cdots\}$ is a non-tight exact frame for $\mathbb{H}$ with bounds $A = 1$, $B = 2$.

(*v*) Let $J \subset \mathbb{N}$ be a proper subset. Then $\{e_j : j \in J\}$ cannot be a frame for $\mathbb{H}$. However, $\{e_j : j \in J\}$ is a frame for $\overline{\text{span}}\{e_j : j \in J\}$.

Since the family $\{f_j : j \in J\}$ is a frame for $\mathbb{H}$ with bounds A, B, it is also a Bessel sequence with Bessel bound B, the operator $T : \ell^2(J) \to \mathbb{H}$ is defined by $T(\{c_j\}_{j\in J}) = \sum_{j\in J} c_j f_j$ is well-defined, bounded and is called the *synthesis operator*. The adjoint operator $T^* : \mathbb{H} \to \ell^2(J)$ is given by $T^* f = \{\langle f, f_j\rangle\}_{j\in J}$ is called the *analysis operator*. The *frame operator* $S : \mathbb{H} \to \mathbb{H}$ is defined by

$$Sf = TT^* f = \sum_{j\in J} \langle f, f_j\rangle f_j.$$

Since $\{f_j : j \in J\}$ is a Bessel sequence for $\mathbb{H}$, the series defining S converges unconditionally for every $f \in \mathbb{H}$. We state some well known properties of S in the following:

Lemma 1.1.4. *([35], Lemma 5.1.5) Let $\{f_j : j \in J\}$ be a frame for $\mathbb{H}$ with frame operator S and frame bounds A, B. Then the following holds.*

(*i*) *S is bounded, invertible, positive, and self-adjoint.*

(*ii*) *The family $\{S^{-1} f_j : j \in J\}$ is a frame for $\mathbb{H}$ with bounds B^{-1}, A^{-1}. The corresponding frame operator for $\{S^{-1} f_j : j \in J\}$ is S^{-1}.*

The frame $\{\tilde{f}_j = S^{-1} f_j : j \in J\}$ is called the *canonical dual* of $\{f_j : j \in J\}$. If $\{f_j : j \in J\}$ is a frame with frame operator S, then the following reconstruction formula holds:

$$f = \sum_{j\in J} \langle f, f_j\rangle S^{-1} f_j = \sum_{j\in J} \langle f, S^{-1} f_j\rangle f_j, \quad \forall f \in \mathbb{H}. \tag{1.2}$$

The series converges unconditionally for every $f \in \mathbb{H}$. A frame $\{g_j : j \in J\}$ for $\mathbb{H}$ is called an *alternate dual* of $\{f_j : j \in J\}$ if for all $f \in \mathbb{H}$ the following equality holds:

$$f = \sum_{j \in J} \langle f, g_j \rangle f_j. \tag{1.3}$$

The two reconstruction formulas in (1.2) provide a non-orthogonal expansion of f with respect to the frame vectors f_j and the dual frame vectors $S^{-1}f_j$ with coefficients $\langle f, S^{-1}f_j \rangle$ and $\langle f, f_j \rangle$ respectively. But in contrast to orthonormal bases the coefficients in the frame expansions (1.2) are in general not unique. Whether the coefficients in the frame expansion (1.2) are uniquely determined under some additional assumption on the frame? This question is settled by the following proposition.

Proposition 1.1.5. *Suppose that $\{f_j : j \in J\}$ is a frame for $\mathbb{H}$. Then the following conditions are equivalent.*

(i) *The coefficients $c \in \ell^2(J)$ in the series expansion (1.2) are unique.*

(ii) *The analysis operator T^* maps onto $\ell^2(J)$.*

(iii) *There exist constants $A', B' > 0$ such that the inequalities*

$$A'\|c\|_2 \leq \Big\| \sum_{j \in J} c_j f_j \Big\| \leq B'\|c\|_2$$

hold for all finite sequences $c = (c_j)_{j \in J}$.

(iv) *The family $\{f_j : j \in J\}$ is the image of an orthonormal basis $\{g_j : j \in J\}$ under an invertible operator $T \in B(\mathbb{H})$.*

(v) *The Gram matrix G, given by $G_{jm} = \langle f_m, f_j \rangle$, $m, j \in J$, defines a positive invertible operator on $\ell^2(J)$.*

A frame that satisfies any one of the conditions of Proposition 1.1.5 is called a Riesz basis of $\mathbb{H}$. We formally define the Riesz basis in the following.

Definition 1.1.6. *A Riesz basis for a Hilbert space $\mathbb{H}$ is a family of the form $\{Tf_j : j \in J\}$, where $\{f_j : j \in J\}$ is an orthonormal basis for $\mathbb{H}$ and $T : \mathbb{H} \to \mathbb{H}$ is a bounded invertible operator.*

Since the omission of one element results in an incomplete set (can be observed from Proposition 1.1.5(iv)), Riesz bases are sometimes referred to as exact frames. Notice that every orthonormal basis is automatically a Riesz basis for the Hilbert space $\mathbb{H}$. But a Riesz basis need not be an orthonormal basis. If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{H}$, then $f_n = (1+n^2)^{-1/2}(ne_1 + e_n)$ is a basis of normalized vectors which is not a Riesz basis for $\mathbb{H}$.

We refer to [25, 35, 40, 77, 88, 91, 143] for basic results on frames. A frame can be approached in various ways. One way is to consider frame theory as a branch of functional analysis and investigate general frames in general Hilbert spaces. Also, one can consider a particular class of frames and explore it. Most articles concentrate on one of these two aspects. We are interested in Gabor frames or Weyl-Heisenberg frames generated by time-frequency shifts of a single window function in $L^2(\mathbb{R}^d)$. In the next section, we discuss Gabor frames in details.

1.2 Gabor Frames

The theory for Gabor analysis in $L^2(\mathbb{R}^d)$ is based on two classes of operators on $L^2(\mathbb{R}^d)$, namely translation and modulation operators. Fix $s, t \in \mathbb{R}^d$ and $g \in L^2(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$, the translation operator $T_s : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by $T_s g(x) = g(x-s)$ and the modulation operator $M_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by $M_t g(x) = e^{2\pi i \langle t,x \rangle} g(x)$. Gabor analysis aims at representing functions $f \in L^2(\mathbb{R}^d)$ as superpositions of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R}^d)$. This can be thought in two ways. First is to ask for integral representations involving all possible translations and modulations i.e. to find $c_f(a,b)$ such that

$$f(x) = \iint_{\mathbb{R}^{2d}} c_f(a,b) e^{2\pi i b \cdot x} g(x-a) \, dadb \tag{1.4}$$

is valid. Note that we have to specify in which sense (1.4) is valid. Second approach is to restrict the translation and modulation parameters to a discrete subset $\Lambda \subset \mathbb{R}^{2d}$

and investigate for series representations of f in terms of the functions

$$\{M_t T_s g\}_{(t,s)\in\Lambda}. \tag{1.5}$$

Here we are interested in the second approach. To discuss some of the fundamental properties of Gabor frames, we first define a Gabor frame in the following:

Definition 1.2.1. *Given a non-zero window function $g \in L^2(\mathbb{R}^d)$ and lattice parameters $\alpha, \beta > 0$, the collection of functions $\mathcal{G}(g,\alpha,\beta) = \{M_{\beta n}T_{\alpha k}g : k, n \in \mathbb{Z}^d\}$ in $L^2(\mathbb{R}^d)$, is called a Gabor frame or a Weyl-Heisenberg frame if there exist constants $A, B > 0$ such that*

$$A\|f\|_2^2 \leq \sum_{k,n\in\mathbb{Z}^d} |\langle f, M_{\beta n}T_{\alpha k}g\rangle|^2 \leq B\|f\|_2^2, \ \ \forall f \in L^2(\mathbb{R}^d). \tag{1.6}$$

An example of a Gabor frame is $\{M_n T_k \chi_{[0,1]^d}\}_{n,k\in\mathbb{Z}^d}$, which is also an orthonormal basis for $L^2(\mathbb{R}^d)$. For a Gabor frame $\{M_{\beta n}T_{\alpha k}g : k, n \in \mathbb{Z}^d\}$, the points $\{(\alpha k, \beta n)\}_{k,n\in\mathbb{Z}^d}$ form a lattice in $\mathbb{R}^{2d}$ and we call $\{M_{\beta n}T_{\alpha k}g : k, n \in \mathbb{Z}^d\}$ a *regular Gabor frame*. For more general subsets $\{(\mu_n, \lambda_n) : n \in J \subseteq \mathbb{Z}^d\}$ of $\mathbb{R}^{2d}$, if the collection $\{M_{\lambda_n}T_{\mu_n}g : n \in J\}$ forms a Gabor frame for $L^2(\mathbb{R}^d)$, then $\{M_{\lambda_n}T_{\mu_n}g : n \in J\}$ is called an *irregular Gabor frame*.

1.2.1 Necessary and Sufficient Conditions

In this section, we provide a necessary and sufficient condition for a Gabor system $\mathcal{G}(g,\alpha,\beta)$ to be a Gabor frame. The natural question is how to choose $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$ such that the functions in (1.5) constitute a frame for $L^2(\mathbb{R}^d)$. The answer to this question is very difficult and we will mainly discuss the case where Λ is a lattice in $\mathbb{R}^{2d}$. In 1946, D. Gabor [72] considered the system $\{M_{\beta n}T_{\alpha k}g\}_{n,k\in\mathbb{Z}^d}$, where $\alpha\beta = 1$ and g is the Gaussian $g(x) = e^{-x^2/2}$. It was observed much later by Janssen [95, 96] that this particular Gabor system leads to unstable expansions. Gabor analysis took a new direction with the article [41] by Daubechies et al. from 1986. There one may find the initial idea of combining Gabor analysis with the frame theory. The following result obtained by Ron and Shen [122], provides a necessary and sufficient condition for $\mathcal{G}(g,\alpha,\beta)$ to be a frame for $L^2(\mathbb{R}^d)$.

Theorem 1.2.2. *Let $A, B > 0$ and the Gabor system $\mathcal{G}(g,\alpha,\beta)$ be given. Then $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A, B if and only if*

$$\beta AI \leq M(x)M(x)^* \leq \beta BI \quad \text{a.e. } x, \tag{1.7}$$

where

$$M(x) := (g(x - n\alpha - m/\beta))_{m,n\in\mathbb{Z}^d}, \quad x \in \mathbb{R}^d, \tag{1.8}$$

and I is the identity operator on $\ell^2(\mathbb{Z}^d)$.

One of the important and interesting concept in Gabor frame theory is to obtain the necessary condition on the lattice parameters α, β such that the Gabor system $\mathcal{G}(g,\alpha,\beta)$ constitute a frame. The algebraic structure of the lattice $\Lambda = \{(\alpha k, \beta n) : k, n \in \mathbb{Z}^d\}$ has been exploited to derive the necessary condition for a Gabor system $\mathcal{G}(g,\alpha,\beta)$ to be complete, a frame or an exact frame in terms of the product $\alpha\beta$. In this direction the following results are known for Gabor frames in one dimension case ($d = 1$) with a rectangular lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$. In [120], Rieffel proved that the Gabor system $\mathcal{G}(g,\alpha,\beta)$ is incomplete for any g if $\alpha\beta > 1$. Daubechies [39] proved Rieffel's result for the case when $\alpha\beta$ is rational and exceeds one. Assuming further decay on g and $\hat{g}$ Landau [105] proved that $\mathcal{G}(g,\alpha,\beta)$ cannot be a frame for $L^2(\mathbb{R})$ if $\alpha\beta > 1$.

For the Gaussian $g(x) = e^{-x^2}$, it is well known that the Gabor system $\mathcal{G}(g,\alpha,\beta)$, is a frame if and only if $\alpha\beta < 1$. This was proved in 1991 by Lyubarski and independently by Seip and Wallsten. The following results says that the product $\alpha\beta$ decides whether it is possible for $\mathcal{G}(g,\alpha,\beta)$ to be a frame for $L^2(\mathbb{R}^d)$.

Theorem 1.2.3. *([35], Theorem 8.3.1) Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$ be given. Then the following holds:*

(*i*) *If $\alpha\beta > 1$, then $\mathcal{G}(g,\alpha,\beta)$ is not a frame for $L^2(\mathbb{R}^d)$.*

(*ii*) *If $\mathcal{G}(g,\alpha,\beta)$ is a frame, then $\alpha\beta = 1 \Leftrightarrow \mathcal{G}(g,\alpha,\beta)$ is a Riesz basis.*

Therefore, only possibility for $\mathcal{G}(g,\alpha,\beta)$ to be a frame if $\alpha\beta \leq 1$, and the frame is *overcomplete* if $\alpha\beta < 1$. But the assumption $\alpha\beta \leq 1$ is not enough for $\mathcal{G}(g,\alpha,\beta)$

to be a frame, even if $g \neq 0$. For example, if $\alpha \in (\frac{1}{2}, 1)$ the functions $\mathcal{G}(\chi_{[0,\frac{1}{2}]^d}, \alpha, \beta)$ cannot form a frame for $L^2(\mathbb{R}^d)$. A necessary condition for $\mathcal{G}(g, \alpha, \beta)$ to be a frame for $L^2(\mathbb{R}^d)$ is given in the following proposition.

Proposition 1.2.4. *([35], Proposition 8.3.2) Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$ be given, and assume that $\mathcal{G}(g, \alpha, \beta)$ is a frame with bounds A, B. Then*

$$\beta A \leq \sum_{k \in \mathbb{Z}^d} |g(x - \alpha k)|^2 \leq \beta B, \ a.e. \tag{1.9}$$

Sufficient conditions for $\mathcal{G}(g, \alpha, \beta)$ to be a frame for $L^2(\mathbb{R}^d)$ are given by Daubechies [39], and Heil and Walnut [94]. We state that results in the following.

Theorem 1.2.5. *Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$ be given. Suppose that there exist $A, B > 0$ such that*

$$A \leq \sum_{k \in \mathbb{Z}^d} |g(x - \alpha k)|^2 \leq B, \ for\ a.e.\ x \in \mathbb{R}^d \tag{1.10}$$

and

$$\sum_{n \neq 0} \left\| \sum_{k \in \mathbb{Z}^d} T_{\alpha k} g T_{\alpha k + \frac{n}{\beta}} \overline{g} \right\|_{\infty} < A. \tag{1.11}$$

Then $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$.

In [39], Daubechies conjectured that if $f \in L^1(\mathbb{R})$ with positive Fourier transform $\hat{f} \in L^1(\mathbb{R})$ then $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 1$. This conjecture was disproved by Janssen in [99]. In [85], Gröchenig and Stöckler proved that the result holds if $g \in L^2(\mathbb{R})$ is a totally positive function of finite type at least 2. This means that $\hat{g}(\xi) = \prod_{\nu=1}^{M} (1 + 2\pi i \delta_\nu \xi)^{-1}$ for $\delta_\nu \in \mathbb{R}$, $\delta_\nu \neq 0$, and $M \geq 2$.

Irregular Gabor systems arise naturally from perturbations of lattice Gabor systems, e.g., due to jitter or noise. For the irregular setting new tools are needed, and these were first supplied by Ramanathan and Steger [119], who provided the extensions of the density theorem to irregular Gabor frames. For the lattice $\Lambda \subset \mathbb{R}^{2d}$, they proved the incompleteness of Gabor systems that are uniformly discrete (i.e. there is a minimum distance δ between elements of Λ) in terms of the Beurling

density defined as follows:
Let $\Lambda \subset \mathbb{R}^d$ be uniformly discrete. Let B be the ball of volume one in $\mathbb{R}^d$ centered at origin. For each $r > 0$, $\nu^+(r)$ and $\nu^-(r)$ denote the maximum and minimum number of points of Λ that lie in any translate of rB, i.e.

$$\nu^+(r) = \max_{x \in \mathbb{R}^d} \#\{\lambda \in \Lambda : \lambda \in \Lambda \cap (x + rB)\}$$

and

$$\nu^-(r) = \min_{x \in \mathbb{R}^d} \#\{\lambda \in \Lambda : \lambda \in \Lambda \cap (x + rB)\}.$$

Since Λ is uniformly discrete, both $\nu^+(r)$ and $\nu^-(r)$ are finite for every $r > 0$. The upper and lower densities are defined by

$$D^+(\Lambda) = \limsup_{r \to \infty} \frac{\nu^+(r)}{r^d} \quad \text{and} \quad D^-(\Lambda) = \liminf_{r \to \infty} \frac{\nu^-(r)}{r^d}.$$

In [104], Landau proved that these quantities are independent of the particular choice of the set B with measure 1. If $D^+(\Lambda) = D^-(\Lambda)$, then the set Λ is said to have uniform Beurling density $D(\Lambda) = D^+(\Lambda) = D^-(\Lambda)$. Ramanathan and Steger [119] proved the following result for irregular Gabor systems that are uniformly discrete.

Theorem 1.2.6. *(Density theorem) Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$ be a uniformly discrete set.*

(i) If $D^+(\Lambda) < 1$, then $\{\rho(p,q)g : (p,q) \in \Lambda\})$ is not a frame for $L^2(\mathbb{R}^d)$ where $\rho(p,q)g(x) = e^{2\pi i q \cdot x} g(x - p)$.

(ii) If $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ is a rectangular lattice with uniform Beurling density $D(\Lambda) < 1$ then $\{\rho(p,q)g : (p,q) \in \Lambda\})$ is incomplete in $L^2(\mathbb{R}^d)$.

(iii) If Λ has uniform Beurling density $D(\Lambda)$ such that $\{\rho(p,q)g : (p,q) \in \Lambda\}$ is a Riesz basis then $D(\Lambda) = 1$.

By the density theorem, there is a clear separation between "overcomplete" frames and "undercomplete" Riesz sequences with Riesz bases corresponding to the critical density lattices that satisfy $D(\Lambda) = 1$. A detailed study of the density theorem for Gabor frames can be found in [90].

1.2.2 Representations of Gabor Frame Operators

The structure of a Gabor frame turns out to have important implications for the Gabor frame operator, which can be written in several ways. The Gabor frame operator is defined by

$$S_g f := \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}g, \quad f \in L^2(\mathbb{R}^d). \tag{1.12}$$

If $g \in L^2(\mathbb{R}^d)$ generates a Gabor frame $\mathcal{G}(g,\alpha,\beta)$ then there exists a dual window (canonical dual window) $\gamma = S_g^{-1}(g) \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma,\alpha,\beta) = \{M_{\beta n}T_{\alpha k}\gamma : k,n \in \mathbb{Z}^d\}$ is also a frame for $L^2(\mathbb{R}^d)$ called the dual Gabor frame. We also denote the canonical dual window as $\tilde{g}$. Consequently every $f \in L^2(\mathbb{R}^d)$ possess the expansion

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}\gamma\rangle M_{\beta n}T_{\alpha k}g \tag{1.13}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. The inverse frame operator is given by

$$S_g^{-1} f = S_\gamma f = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}\gamma\rangle M_{\beta n}T_{\alpha k}\gamma. \tag{1.14}$$

The aim of this section is to investigate Gabor frames in detail and state the fundamental theorems on Gabor frames. In order to do so, we need to define a particular function space which can be used as a convenient class of window functions for time-frequency analysis on $L^2(\mathbb{R}^d)$.

Definition 1.2.7. *A function $g \in L^\infty(\mathbb{R}^d)$ belongs to the Wiener space $W = W(\mathbb{R}^d)$ if*

$$\|g\|_W = \sum_{k\in\mathbb{Z}^d} \|g \cdot T_k\chi_{[0,1)^d}\|_\infty < \infty.$$

For $1 \le p < \infty$, we have $W(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$. Further, $W(\mathbb{R}^d)$ is a Banach algebra under pointwise multiplication with respect to the norm $\int_{\mathbb{R}^d} \|f \cdot T_x \chi_{[0,1)^d}\|_\infty dx$.

There are several different approaches to understand S_g owing to Daubechies [39], Walnut [135], Ron and Shen [122], and Feichtinger and Zimmermann [68]. Before the detailed investigation of the Gabor frame operator, we define synthesis and analysis operators. For Gabor systems the synthesis operator

$$R_g : \ell^2(\mathbb{Z}^{2d}) \to L^2(\mathbb{R}^d), \text{ defined by } R_g((c_{kn})_{k,n\in\mathbb{Z}^d}) = \sum_{k,n\in\mathbb{Z}^d} c_{kn} M_{\beta n} T_{\alpha k} g \quad (1.15)$$

and the analysis operator

$$C_g : L^2(\mathbb{R}^d) \to \ell^2(\mathbb{Z}^{2d}), \text{ defined by } C_g f(k,n) = \langle f, M_{\beta n} T_{\alpha k} g\rangle, \quad k,n \in \mathbb{Z}^d. \quad (1.16)$$

If $g \in W(\mathbb{R}^d)$, then the operators R_g and C_g are well-defined and bounded. We can also define the Gabor frame operator as

$$S_{g,\gamma} f = R_\gamma C_g f = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k} \gamma.$$

The Gabor frame operator $S_{g,\gamma}$ can be written using the correlation functions G_n of the pair (g, γ), where

$$G_n(x) = \sum_{k\in\mathbb{Z}^d} \overline{g}(x - \frac{n}{\beta} - \alpha k)\gamma(x - \alpha k), \ n \in \mathbb{Z}^d.$$

Which is well known as *Walnut's representation* [135] of the Gabor frame operator.

Theorem 1.2.8. *([77], Theorem 6.3.2) Let $g, \gamma \in W(\mathbb{R}^d)$ and $\alpha, \beta > 0$. Then the frame operator $S_{g,\gamma}$ associated to $\mathcal{G}(g, \alpha, \beta)$ has the representation*

$$S_{g,\gamma} f = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f. \quad (1.17)$$

Moreover, $S_{g,\gamma}$ is bounded on all L^p-spaces, $1 \le p \le \infty$ with operator norm

$$\|S_{g,\gamma}\|_{B(L^p)} \le 2^d \left(\frac{1}{\alpha}+1\right)^d \left(\frac{1}{\beta}+1\right)^d \|g\|_W \|\gamma\|_W.$$

Further, expanding the Fourier series of α-periodic correlation functions gives

$$G_n(x) = \alpha^{-d} \sum_{l\in\mathbb{Z}^d} \left\langle \gamma, M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} g \right\rangle e^{2\pi i l\cdot x/\alpha}, \tag{1.18}$$

By substituting this expression into Walnut's representation (1.17) we obtain the informal expansion

$$\begin{aligned} S_{g,\gamma} f &= \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f \\ &= (\alpha\beta)^{-d} \sum\sum_{l,n\in\mathbb{Z}^d} \left\langle \gamma, M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} g \right\rangle M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} f, \end{aligned} \tag{1.19}$$

with the convergence of the series in the operator norm provided

$$\sum_{l,n\in\mathbb{Z}^d} \left| \left\langle \gamma, M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} g \right\rangle \right| < \infty.$$

The expression in (1.19) is called the *Janssen's representation* [98]. It represents the frame operator as a superposition of time-frequency shifts along the adjoint lattice $\frac{1}{\beta}\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$. The weak form of Janssen's representation was introduced by Tolimieri and Orr [134], and the strong forms were independently proved in [44] and [98].

For any Gabor frame $\mathcal{G}(g, \alpha, \beta)$, one may ask whether there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that $S_{g,\gamma} = S_{\gamma,g} = I$. A particular dual window which satisfies this identity is the canonical dual window. But in general the canonical dual is not the only dual window, there may be other dual windows also satisfying $S_{g,\gamma} = I$. In an attempt to find an alternative way to compute the dual window of a Gabor frame, Z. Wexler and S. Raz [138] discovered the biorthogonality relations. The following conditions, the Wexler-Raz biorthogonality relations characterize all dual windows.

Theorem 1.2.9. *Assume that R_g and R_γ are bounded on $\ell^2(\mathbb{Z}^{2d})$. Then the fol-*

lowing conditions are equivalent:

(*i*) $S_{g,\gamma} = S_{\gamma,g} = I$ *on* $L^2(\mathbb{R}^d)$.

(*ii*) $(\alpha\beta)^{-d}\left\langle \gamma, M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} g \right\rangle = \delta_{l0}\delta_{n0}$ *for* $l, n \in \mathbb{Z}^d$.

The relevance of these identities for the study of Gabor frame have pointed out in [60, 65, 68, 90]. A special case of the Wexler-Raz biorthogonality relations says that the two sets $\mathcal{G}(g, \frac{1}{\beta}, \frac{1}{\alpha})$ and $\mathcal{G}(\gamma, \frac{1}{\beta}, \frac{1}{\alpha})$ are biorthogonal to each other on $L^2(\mathbb{R}^d)$. A. Ron and Z. Shen were the first to obtain the relationship between the Gabor frame $\mathcal{G}(g, \alpha, \beta)$ and the dual Gabor frame $\mathcal{G}(g, \frac{1}{\beta}, \frac{1}{\alpha})$ in [122] which is well known as *Ron-Shen duality principle.* We state the Ron-Shen duality principle in the following theorem and refer to [77, 98] for more information on duality principle.

Theorem 1.2.10. *([77], Theorem 7.4.3) Let* $g \in L^2(\mathbb{R}^d)$ *and* $\alpha, \beta > 0$. *Then the Gabor system* $\mathcal{G}(g, \alpha, \beta)$ *is a frame for* $L^2(\mathbb{R}^d)$ *if and only if* $\mathcal{G}(g, \frac{1}{\beta}, \frac{1}{\alpha})$ *is a Riesz basis for its closed linear span.*

Next we discuss two important function spaces namely Wiener amalgam spaces and modulation spaces, which play a significant role in time-frequency analysis.

1.3 Wiener Amalgam Spaces

Let Q denote the unit cube $[0, 1)^d$ and $Q_a = [0, a)^d$. Let $1 \le p, q \le \infty$. A measurable function f belongs to the Wiener amalgam space $W(L^p, L^q)$ if

$$\|f\|_{W(L^p,L^q)} := \left(\sum_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_Q\|_p^q \right)^{\frac{1}{q}} < \infty,$$

with the obvious modification for $q = \infty$. Notice that if $p = \infty$ and $q = 1$, the amalgam space $W(L^\infty, L^1)$ is precisely the Wiener space $W(\mathbb{R}^d)$ as defined in Definition 1.2.7.

The closed subspace of $W(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, L^1)$ and is called the Wiener algebra. It is easy to observe that $W(L^p, L^p) =$

$L^p(\mathbb{R}^d)$. For $1 \le p_1, p_2, q_1, q_2 \le \infty$, we have

$$W(L^{p_1}, L^q) \hookleftarrow W(L^{p_2}, L^q), \quad p_1 \le p_2$$

and

$$W(L^p, L^{q_1}) \hookrightarrow W(L^p, L^{q_2}), \quad q_1 \le q_2.$$

Thus

$$W(L^\infty, L^1) \subset L^p(\mathbb{R}^d) \subset W(L^1, L^\infty), \quad 1 \le p \le \infty.$$

The Köthe dual of $W(L^p, L^q)$ is the space of all measurable functions g on $\mathbb{R}^d$ such that $g \cdot W(L^p, L^q) \subseteq L^1(\mathbb{R}^d)$. It is equal to $W(L^{p'}, L^{q'})$, where $1/p + 1/p' = 1/q + 1/q' = 1$ for all $1 \le p, q \le \infty$. The pairing

$$\langle \cdot, \cdot \rangle : W(L^p, L^q) \times W(L^{p'}, L^{q'}) \to \mathbb{C}, \qquad \langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx,$$

is bounded. The collection of all bounded linear operators from $W(L^p, L^q)$ to $W(L^p, L^q)$ is denoted by $B(W(L^p, L^q))$.

The dual and Köthe dual of the amalgam spaces are given in the following lemma.

Lemma 1.3.1. *Let* $1/p + 1/p' = 1/q + 1/q' = 1$. *Then*

(i) *For* $1 \le p, q < \infty$, *the dual space of* $W(L^p, L^q)$ *is* $W(L^{p'}, L^{q'})$.

(ii) *For* $1 \le p, q \le \infty$ *the Köthe dual of* $W(L^p, L^q)$ *is* $W(L^{p'}, L^{q'})$.

We refer to the paper of Fournier and Stewart [70] for detailed study on classical amalgam spaces.

1.4 The Schwartz Space and the Fourier Transform

Definition 1.4.1. *Let* $\mathcal{S}(\mathbb{R}^d)$ *denote the class of all infinitely differentiable functions on* $\mathbb{R}^d$ *such that*

$$\sup_{x \in \mathbb{R}^d} \left| x^\alpha D^\beta \varphi(x) \right| < \infty, \quad \forall\ \alpha, \beta \in \mathbb{N}^d,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_d)$, $\beta = (\beta_1, \beta_2 \cdots \beta_d)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ *and* $D^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}$ *for* $x = (x_1, x_2, \cdots x_d)$. *The space* $\mathcal{S}(\mathbb{R}^d)$ *is called Schwartz class of rapidly decreasing functions.*

Let $f \in L^1(\mathbb{R}^d)$. Then the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \xi . t} dt, \quad \xi \in \mathbb{R}^d.$$

For $f \in L^1 \cap L^2(\mathbb{R}^d)$, one has the Plancherel formula $\|f\|_2 = \|\hat{f}\|_2$. As $L^1 \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, the definition of Fourier transform is extended to functions in $L^2(\mathbb{R}^d)$. The Fourier transform $\mathcal{F}$ maps $L^2(\mathbb{R}^d)$ unitarily onto $L^2(\mathbb{R}^d)$. The inversion formula is given by

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \xi . t} d\xi, \quad a.e. \text{ for } t \in \mathbb{R}^d.$$

The Fourier transform $f \mapsto \hat{f}$ is a homeomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto itself. $\mathcal{S}(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$, the class of continuous functions vanishing at infinity, and in $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. The collection $\mathcal{S}'(\mathbb{R}^d)$ of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$ is called the space of tempered distributions.

1.5 Modulation Spaces

The modulation spaces were invented and extensively investigated by Feichtinger in [58, 60]. For a detailed development of the theory of modulation spaces we refer to the above mentioned references and to Gröchenig's text [77].

Given a non-zero function $g \in L^2(\mathbb{R})$, we recall the short-time Fourier transform (STFT) or windowed Fourier transform of a function $f \in L^2(\mathbb{R})$ with respect to g as

$$F_g f(x, w) = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i w t} dt.$$

It is well known that if $\|g\|_2 = 1$, then F_g is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$.

Thus we have the norm equality

$$\|f\|_2 = \|F_g f\|_2 = \left(\iint_{\mathbb{R}^2} |F_g f(x,w)|^2 dx\, dw \right)^{1/2}. \tag{1.20}$$

For more details of STFT and the inversion formula see [77].

Definition 1.5.1. *Fix a non-zero window $g \in \mathcal{S}(\mathbb{R})$, and $1 \leq p, q \leq \infty$. Then the modulation space $M^{p,q}(\mathbb{R})$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $F_g f \in L^{p,q}(\mathbb{R}^2)$. The norm on $M^{p,q}$ is*

$$\|f\|_{M^{p,q}} = \|F_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F_g f(x,w)|^p dx \right)^{q/p} dw \right)^{1/q} < \infty,$$

with the usual adjustments if p or q is infinite. If $p = q$, then we write M^p instead of $M^{p,p}$.

The definition of $M^{p,q}$ is independent of the choice of g in the sense that each different choice of g defines an equivalent norm for the same set $M^{p,q}$. Each modulation space is a Banach space. For $p = q = 2$, by equation (1.20), we have that $M^2 = L^2$. For other $p = q$, the space M^p is not L^p. In fact for $p = q > 2$, the space M^p is a superset of L^2. We have the following inclusion

$$\mathcal{S}(\mathbb{R}) \subset M^1(\mathbb{R}) \subset M^2(\mathbb{R}) = L^2(\mathbb{R}) \subset M^\infty(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

Among the modulation spaces, the space M^1 plays an important role. For example, see [80] for important applications of M^1 space in the theory of pseudodifferential operators. The space M^1 is often called the *Feichtinger algebra*. The Feichtinger algebra has many interesting properties: it forms a Banach algebra under pointwise products and convolution, and is invariant under the Fourier transform. Also it is the minimal non-trivial Banach space contained in L^1 which is isometrically invariant under both translations and modulations.

Let $g \in \mathcal{S}(\mathbb{R})$, or $g \in M^1(\mathbb{R})$ and $\mathcal{G}(g, \alpha, \beta)$ be a frame for $L^2(\mathbb{R})$. Then the Gabor frame expansions given in (1.13) converge not only in L^2 but in all the modulation spaces, as follows (see [77] for proofs).

Theorem 1.5.2. *Let $g \in M^1(\mathbb{R})$ and $\mathcal{G}(g, \alpha, \beta)$ be a frame for $L^2(\mathbb{R})$. Then the following statements hold.*

(i) The dual window $\tilde{g}$ belongs to $M^1(\mathbb{R})$.

(ii) For every $1 \le p \le \infty$ we have that

$$f = \sum_{k,n\in\mathbb{Z}} \langle f, M_{\beta n}T_{\alpha k}\tilde{g}\rangle M_{\beta n}T_{\alpha k}g = \sum_{k,n\in\mathbb{Z}} \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\tilde{g}, \ \ \forall f \in M^p(\mathbb{R}),$$

where these series converge unconditionally in the norm of M^p (weak* convergence if $p = \infty$).

(iii) For every $1 \le p \le \infty$ the Gabor frame coefficients provide an equivalent norm for the modulation space M^p, i.e.,

$$\|f\|_{M^p} = \left(\sum_{k,n\in\mathbb{Z}} |\langle f, M_{\beta n}T_{\alpha k}g\rangle|^p\right)^{1/p} \tag{1.21}$$

is an equivalent norm for M^p.

The theorem says that any Gabor frame for $L^2(\mathbb{R})$ whose window lies in $M^1(\mathbb{R})$ is a $Banach\ frame$ for every modulation space.

1.6 Wilson Bases

The equivalent norm for M^p given in (1.21) suggests that we should have $M^p \cong \ell^p$. There exists a remarkable construction of an orthonormal basis for $L^2(\mathbb{R})$, called a *Wilson basis*, which is simultaneously an unconditional basis for every modulation spaces (see [61]). Wilson bases were first suggested by Wilson in [140]. They provide orthonormal bases for $L^2(\mathbb{R})$, which was rigorously proved in [42]. We refer to [77] for more details on Wilson bases.

The construction of a Wilson basis starts with a twice redundant Parseval Gabor frame $\mathcal{G}(g, \frac{1}{2}\mathbb{Z}\times\mathbb{Z})$ whose generator satisfies a symmetry condition, then forms linear combinations of elements, namely $M_nT_{\frac{k}{2}}g \pm M_{-n}T_{\frac{k}{2}}g$ and finally extracts from the set of these linear combinations a subset which forms an orthonormal basis for

$L^2(\mathbb{R})$. Moreover, a Wilson basis will be an unconditional basis not only for $L^2(\mathbb{R})$, but also for all the modulation spaces, if the original window g has sufficient joint concentration in the time-frequency plane. This is summarized in the following result (see Theorem 8.5.1 in [77] for proof).

Theorem 1.6.1. *Let $\mathcal{G}(g, \frac{1}{2}\mathbb{Z}\times\mathbb{Z})$ be a Parseval Gabor frame for $L^2(\mathbb{R})$ and $g(x) = \overline{g(-x)}$. Define $\psi_{k,0} = T_k g,\ k \in \mathbb{Z}$ and*

$$\psi_{k,n}(x) = \begin{cases} \sqrt{2}\ \cos(2\pi nx)\ g(x - \frac{k}{2}), & \text{if } k+n \text{ is even,} \\ \sqrt{2}\ \sin(2\pi nx)\ g(x - \frac{k}{2}), & \text{if } k+n \text{ is odd,} \end{cases}$$

and set $\mathcal{W}(g) = \{\psi_{k,n}\}_{k\in\mathbb{Z},\ n\geq 0}$. Then $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$. In addition if we have $g \in M^1(\mathbb{R})$, then the following further statements hold.

(i) For every $1 \leq p \leq \infty$ we have $f = \sum\limits_{k\in\mathbb{Z}}\sum\limits_{n\geq 0}\langle f, \psi_{k,n}\rangle\psi_{k,n}$, for all $f \in M^p(\mathbb{R})$, where the series converges unconditionally in the norm of M^p (weak* convergence if $p = \infty$).

(ii) For every $1 \leq p \leq \infty$, the Wilson basis coefficients provide an equivalent norm for the modulation space M^p i.e., $\|f\|_{M^p} = \left(\sum\limits_{k\in\mathbb{Z}}\sum\limits_{n\geq 0}|\langle f, \psi_{k,n}\rangle|^p\right)^{1/p}$ is an equivalent norm for M^p.

Consequently, $f \mapsto \{\langle f, \psi_{k,n}\rangle\}_{k\in\mathbb{Z},n\geq 0}$ defines an isomorphism of M^p onto ℓ^p. When we consider Wilson bases in this thesis, we assume that they are constructed from M^1 windows. Also, tensor product of Wilson bases are unconditional bases for the higher dimensional modulation spaces.

1.7 Literature Review and Overview of Main Results

Since the frame theory has been studied by various authors under various settings, it is extremely difficult to provide the complete literature survey. However, we are giving a few important results which are necessary for this thesis. The literature survey has been presented in the order of the chapters of this thesis.

1.7.1 Gabor Frame Operators

The convergence property of Gabor expansions is a fundamental problem in time-frequency analysis. It has received much attention and therefore studied widely by many authors. It is well known that the Gabor expansion converges unconditionally in $L^2(\mathbb{R}^d)$. So it is natural to investigate the convergence of Gabor expansions in $L^p(\mathbb{R}^d)$, $p \geq 1$. In this direction Grafakos and Lennard [75] in 2001, proved that the Gabor expansions of L^p functions converge conditionally in $L^p(\mathbb{R}^d)$, $1 < p < \infty$. However, for $p = 1$ they have shown that the Gabor expansions converge to the functions almost everywhere in the sense of Cesaro. Simultaneously, Gröchenig and Heil in [81], established that Gabor expansions converge conditionally in $L^p(\mathbb{R}^d)$, $1 < p < \infty$. This result has been further extended to a much larger class of spaces than the L^p spaces, namely the weighted amalgam spaces $W(L^p, L^q_v)$, $1 < p < \infty$, $1 \leq q < \infty$, in [82] by Gröchenig et al. Further, Weisz [136] considered the Riemannian sums of the inverse short-time Fourier transform to reconstruct the original function. He considered the operator

$$S_{\alpha,\beta;g,\gamma}f = \frac{(\alpha\beta)^d}{\langle \gamma, g\rangle} \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma$$

and proved that it converges to f in $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$ whenever $g, \gamma \in S_0(\mathbb{R}^d) := \{g : F_g g \in L^1(\mathbb{R}^{2d})\}$ and in $W(L^p, L^q)(\mathbb{R}^d)$, $1 < p < \infty$, $1 \leq q < \infty$ whenever $g, \gamma \in W(\mathbb{R}^d)$. In [67], Feichtinger and Weisz used θ-summability method to prove the Gabor expansion of f converges to f in $W(L^p, L^q)(\mathbb{R}^d)$ norm, $1 < p < \infty$, $1 \leq q < \infty$ whenever $g, \gamma \in W(\mathbb{R}^d)$. Further, Sun [129] proved that if $g, \gamma \in W(\mathbb{R}^d)$, then the Gabor frame operator $S_{\alpha,\beta;g,\gamma}$ converges to the identity operator in L^p, $1 \leq p \leq \infty$ operator norm with some additional local integrability conditions on g, γ.

In Chapter 2, we extend Sun's [129] results and show if both g and γ are in the Wiener space then the Gabor frame operator $S_{\alpha,\beta;g,\gamma}$ converges to the identity operator in $W(L^p, L^q)$, $1 \leq p, q \leq \infty$ operator norm, assuming similar local integrability conditions on g, γ.

Another interesting property of Gabor frames is to know whether the generator

g and the canonical dual γ have similar properties for a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ from applications (see [66]) point of view. In this direction the following results are known: If g has compact support then in general γ is no longer compactly supported but has exponential decay [22] and in [116] Del Prete proved that if g has exponential decay, then γ also has exponential decay. If g can be estimated by $C(1+|t|)^{-s}$, then the same holds for γ (see [125]). It natural to ask if g is in a given function space whether its canonical dual is in the same space? The first result in this direction is due to Janssen [98] which ensures that if g is in the Schwartz space on $\mathbb{R}^d$ then its canonical dual is in the same space. Gröchenig and Leinert [83] proved that if g is an element in the Feichtinger's algebra and $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, then the canonical dual γ is in the same space. In [102], the authors proved that if $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame generated by $g \in W(L^\infty, L^1_v)$ then the Gabor frame operator S_g is invertible on the Wiener amalgam space $W(L^\infty, L^1_v)$, where v is an admissible weight function and hence $\gamma = S_g^{-1} g \in W(L^\infty, L^1_v)$. In [137], Weisz extended the analogous result on amalgam spaces $W(L^\infty, L^q_v)$ for $1 \leq q \leq 2$. Last two results were based on the reformulation of a non-commutative Wiener's lemma proved by Baskakov[14, 15]. In [102], Krishtal and Okoudjou has used a generalization of $1/f$ lemma to prove that for a Gabor frame with generator in the Wiener space $W(L^\infty, L^1)$ the frame operator is invertible and the canonical dual is also in $W(L^\infty, L^1)$. In [11], Balan et al. obtained a similar result for multi-window Gabor frames on Wiener amalgam spaces using a recent Wiener type result on non-commutative almost periodic Fourier series [12]. We refer to [6] and [88] for detailed study on multi-window Gabor frames.

The vector-valued Gabor frames or superframes were introduced by Balan [7] in the context of "multiplexing" and several well-known results for Gabor frames are extended to superframes in [6, 88]. The super wavelet and Gabor frames in $L^2(\mathbb{R}^d, \mathbb{C}^n)$ has various applications in mathematics and engineering (see [7, 6, 51, 52, 84, 86, 88, 106, 147, 148]). Using the growth estimates for the Weierstrass σ-function and a new type of interpolation problem for entire functions on Bargmann-Fock space, Gröchenig and Lyubarskii [84] obtained a complete characterization of all lattices $\Lambda \subset \mathbb{R}^2$ such that the Gabor system on first $n + 1$ Hermite functions is a frame on $L^2(\mathbb{R}, \mathbb{C}^n)$.

It is natural to investigate the following statements for superframes:
(i) Walnut's representation of superframe operator on vector valued amalgam spaces.
(ii) Convergence of Gabor expansions for the superframe on vector valued amalgam spaces.
(iii) Invertibility of the superframe operator on vector valued amalgam spaces.

In Chapter 3, we investigate the above statements when the window function is in the vector valued Wiener space. Therefor we consider Hilbert space valued Gabor frames on $W_{\mathbb{H}}(L^p, L^q_v), 1 \leq p, q \leq \infty$, in a general set up. As a result, we could address the above statements for Gabor superframes and multi window Gabor frames as a special case by choosing the appropriate separable Hilbert space $\mathbb{H}$ (see Remark 3.2.13 and Remark 3.4.3). Further we show that if the window function $\mathbf{g} \in W_{\mathbb{H}}(C_0, L^1_w)$, the subspace formed by the functions of $\mathbb{H}-$valued Wiener space that are continuous, then the canonical dual is in the same space. To obtain this result we show that the frame operator $S_{\mathbf{g}}(= S_{\mathbf{g},\mathbf{g}})$ is invertible on $W_{\mathbb{H}}(L^p, L^q_v), 1 \leq p, q \leq \infty$ and $S_{\mathbf{g}}^{-1} : W(L^p, L^q_v) \to W(L^p, L^q_v)$ is continuous both in $\sigma(W(L^p, L^q_v), W(L^{p'}, L^{q'}_{1/v}))$ and the norm topologies. Such type of invertibility results have been addressed for modulation spaces (see [78, 83]) and for L^p spaces (see [102]) by using Wiener's $1/f$ lemma (see [12, 14, 15]). We construct a Banach algebra of operators admitting an expansion like (3.14) and use a version of Wiener's $1/f$ lemma (see [12]) to prove that the algebra is spectral within the class of bounded linear operators on $L^p(\mathbb{R}^d, \mathbb{H}),\ 1 \leq p \leq \infty$.

1.7.2 Generalization of Frames

A study on generalization of frames is also an important and interesting branch in frame theory. Generalized frames or simply g-frames were introduced by Sun in [127], which include ordinary frame as well as many generalizations of frames, e.g., bounded quasi-projectors [69], frames of subspaces [30], pseudo-frames [107], oblique frames [36], outer frames [2], and time-frequency localization operators [46]. G-frames provide more choices on analyzing functions from frame expansion coefficients. Hilbert$-$Schmidt frames or simply HS-frames were introduced in [123] as a class of von Neumann$-$Schatten p-frames, which includes g-frames [127] and hence all the above classes of frames are a class of HS-frames. We refer to

[4, 29, 30, 38, 71, 100, 115, 123, 127] for results on generalizations of frames. It is well known that g-frames and g-Riesz bases in Hilbert spaces have some properties similar to those of frames and Riesz bases, but not all the properties are similar, e.g., exact g-frames are not equivalent to g-Riesz bases (see [127, 128]). The natural question to ask is: which properties of the frame, or the g-frame may be extended to the HS-frame for a Hilbert space? In Chapter 4, we establish some necessary and sufficient conditions for a HS-Bessel sequence, a HS-frame and a HS-Riesz basis in a Hilbert space. We also characterize HS-frames from the point of view of operator theory and discuss the relation between a HS-frame and a HS-Riesz basis.

The reconstruction formula for a frame allows to write every element in the Hilbert space as a linear combination of the frame elements with frame coefficients. Calculations of those coefficients require knowledge of the inverse frame operator. But in practice it is very difficult to invert the frame operator if the Hilbert space is infinite dimensional. Calculations of the inverse frame operator for HS-frames in infinite dimensional Hilbert space is also very difficult. Christensen introduced the projection method in [31] and the strong projection method in [33] to approximate the frame coefficients. In [26, 28, 34], Christensen and Casazza proved that the inverse frame operator can be approximated arbitrarily closely using finite-dimensional linear algebra. Using similar methods the authors of [1] proved approximation results for inverse g-frame operators. In Chapter 4, we generalize and extend this result to approximate the inverse HS-frame operator in the strong operator topology using finite subsets of the HS-frame.

Given a family $\{g_j : j \in J\} \subseteq \mathbb{H}$ which is close to the frame or Riesz basis $\{f_j : j \in J\} \subseteq \mathbb{H}$, finding conditions to ensure that $\{g_j : j \in J\}$ is also a frame or Riesz basis is called the stability problem. This problem is important in practice and therefore studied by many authors [32, 37, 54, 112, 128]. Since frames can be characterized in terms of operators, many results on perturbations of frames can also be characterized from the operator point of view (see [27, 87]). In Chapter 4, we study the stability of HS-frames. We first present a classical perturbation result of HS-frames. Then we give other perturbation results for HS-frames.

1.7.3 Identities and Inequalities for Frames

Let $\{f_j : j \in J\}$ be a frame for $\mathbb{H}$. For every $K \subset J$, we define the operator S_K by

$$S_K f = \sum_{j \in K} \langle f, f_j \rangle f_j, \tag{1.22}$$

and also we denote K^c as $J \setminus K$.

In [8], the authors proved a longstanding conjecture of the signal processing community: a signal can be reconstructed without information about the phase. While working on efficient algorithms for signal reconstruction, Balan et al. [10] discovered a remarkable new identity for Parseval frames, given in the following form. We refer to [9] for a discussion of the origins of this fundamental identity.

Theorem 1.7.1. *Let $\{f_j : j \in J\}$ be a Parseval frame for $\mathbb{H}$, then for every $K \subset J$ and every $f \in \mathbb{H}$, we have*

$$\sum_{j \in K} |\langle f, f_j \rangle|^2 - \left\| \sum_{j \in K} \langle f, f_j \rangle f_j \right\|^2 = \sum_{j \in K^c} |\langle f, f_j \rangle|^2 - \left\| \sum_{j \in K^c} \langle f, f_j \rangle f_j \right\|^2.$$

Theorem 1.7.2. *If $\{f_j : j \in J\}$ be a Parseval frame for $\mathbb{H}$, then for every $K \subset J$ and every $f \in \mathbb{H}$, we have*

$$\sum_{j \in K} |\langle f, f_j \rangle|^2 + \left\| \sum_{j \in K^c} \langle f, f_j \rangle f_j \right\|^2 \geq \frac{3}{4} \|f\|^2.$$

The authors in [10] mainly focused on Parseval frames and proved several interesting variants of Theorem 1.7.1. The identity that appears in Theorem 1.7.1 was obtained in [10] as a particular case of the following result for general frames.

Theorem 1.7.3. *Let $\{f_j : j \in J\}$ be a frame for $\mathbb{H}$ with canonical dual frame $\{\tilde{f}_j : j \in J\}$. Then for every $K \subset J$ and every $f \in \mathbb{H}$, we have*

$$\sum_{j \in K} |\langle f, f_j \rangle|^2 - \sum_{j \in J} |\langle S_K f, \tilde{f}_j \rangle|^2 = \sum_{j \in K^c} |\langle f, f_j \rangle|^2 - \sum_{j \in J} |\langle S_{K^c} f, \tilde{f}_j \rangle|^2.$$

The following results were obtained in [74], which generalized Theorems 1.7.1 and 1.7.2 to canonical and alternate dual frames:

Theorem 1.7.4. *Let $\{f_j : j \in J\}$ be a frame for $\mathbb{H}$ with canonical dual frame $\{\tilde{f}_j : j \in J\}$. Then for every $K \subset J$ and every $f \in \mathbb{H}$, we have*

$$\begin{aligned}\sum_{j\in K} |\langle f, f_j\rangle|^2 + \sum_{j\in J} |\langle S_{K^c} f, \tilde{f}_j\rangle|^2 &= \sum_{j\in K^c} |\langle f, f_j\rangle|^2 + \sum_{j\in J} |\langle S_K f, \tilde{f}_j\rangle|^2 \\ &\geq \frac{3}{4}\sum_{j\in J} |\langle f, f_j\rangle|^2.\end{aligned}$$

Theorem 1.7.5. *Let $\{f_j : j \in J\}$ be a frame for $\mathbb{H}$ and $\{g_j : j \in J\}$ be an alternate dual frame of $\{f_j : j \in J\}$, then for every $K \subset J$ and every $f \in \mathbb{H}$, we have*

$$\begin{aligned}& Re\Big(\sum_{j\in K} \langle f, g_j\rangle\overline{\langle f, f_j\rangle}\Big) + \Big\|\sum_{j\in K^c} \langle f, g_j\rangle f_j\Big\|^2 \\ =\ & Re\Big(\sum_{j\in K^c} \langle f, g_j\rangle\overline{\langle f, f_j\rangle}\Big) + \Big\|\sum_{j\in K} \langle f, g_j\rangle f_j\Big\|^2 \geq \frac{3}{4}\|f\|^2.\end{aligned}$$

Motivated by these interesting results the authors in [145] generalized Theorem 1.7.5 to a form that does not involve the real parts of the complex numbers is given below.

Theorem 1.7.6. *Let $\{f_j : j \in J\}$ be a frame for $\mathbb{H}$ and $\{g_j : j \in J\}$ be an alternate dual frame of $\{f_j : j \in J\}$. Then for every $K \subset J$ and every $f \in \mathbb{H}$, we have*

$$\Big(\sum_{j\in K} \langle f, g_j\rangle\overline{\langle f, f_j\rangle}\Big) - \Big\|\sum_{j\in K} \langle f, g_j\rangle f_j\Big\|^2 = \overline{\Big(\sum_{j\in K^c} \langle f, g_j\rangle\overline{\langle f, f_j\rangle}\Big)} - \Big\|\sum_{j\in K^c} \langle f, g_j\rangle f_j\Big\|^2.$$

Moreover, the authors in [108, 141] have extended Theorem 1.7.3 for g-frames and canonical dual g-frames in Hilbert spaces. Also, the authors in [142] have established an equality and an inequality for the alternate dual g-frame. Further, in [109], the authors generalized the equality and inequality for g-frame to a g-Bessel sequence in Hilbert spaces.

In Chapter 4, we generalize the above mentioned results for Hilbert-Schmidt frames. Also, we generalize the above inequalities to a more general form which involve a scalar $\lambda \in [0,1]$. As a particular case (for $\lambda = 1/2$) the above inequalities can be obtained. Since g-frames can be considered as a class of Hilbert-Schmidt frames, the previous equality and inequalities on g-frames can be obtained as a special case of the results we establish on Hilbert-Schmidt frames.

1.7.4 Feichtinger's Problem

In 2004, Feichtinger posed a spectral problem about positive semi-definite trace-class pseudodifferential operators on modulation spaces at an Oberwolfach mini-workshop on *Wavelets, Frames and Operator Theory* (see [64]). Later, Heil and Larson in [92], rephrased the problem in operator-theoretic terms to promote some connections between the operator theory and the theory of modulation spaces. In Chapter 5, we solve the Feichtinger's problem along with the solution of the reformulated problem. This solution consists of constructing a counterexample that solves Feichtinger's problem by first solving the reformulated problem.

1.7.5 The Balian-Low Theorem

The Balian-Low theorem (BLT) is one of the fundamental and interesting result in time-frequency analysis. It says that a function $g \in L^2(\mathbb{R})$ generating Gabor Riesz basis cannot be localized in both time and frequency domains. Precisely if $g \in L^2(\mathbb{R})$ and if a Gabor system $\mathcal{G}(g,\alpha,\beta) := \{e^{2\pi i m\beta t} g(t-n\alpha)\}_{m,n\in\mathbb{Z}}$ with $\alpha\beta = 1$ forms an orthonormal basis for $L^2(\mathbb{R})$, then

$$\left(\int_{-\infty}^{\infty} |tg(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} |\gamma\hat{g}(\gamma)|^2 d\gamma\right) = +\infty,$$

where $\hat{g}$ is the Fourier transform of g. This result was originally stated by Balian [13] and independently by Low in [110]. The proofs given by Balian and Low each contained a technical gap, which was filled by Coifman et al. [39] and extended the BLT to the case of Riesz bases. Battle [16] provided an elegant and entirely new proof based on the operator theory associated with the classical uncertainty principle. For general Balian-Low type results, historical comments and variations of BLT we refer to [19, 43].

The Balian-Low type results were proved for multi-window Gabor systems by Zibulski and Zeevi [147] and for superframes by Balan [5]. The BLT and its variations for symplectic lattices in higher dimensions (see [60, 79]), for the symplectic form on $\mathbb{R}^{2d}$ (see [17]) and on locally compact abelian groups (see [76]) are obtained in the literature. For further results on BLT we refer to [3, 18, 73, 93, 113, 114, 133].

In Chapter 6, we establish the BLT and some of its variations on $L^2(\mathbb{C})$ using the operators

$$Z = \frac{d}{dz} + \frac{1}{2}\bar{z} \quad \text{and} \quad \bar{Z} = \frac{d}{d\bar{z}} - \frac{1}{2}z.$$

These operators are associated with the special Hermite operator

$$L = -\Delta_z + \frac{1}{4}|z|^2 - i\left(x\frac{d}{dy} - y\frac{d}{dx}\right) = -\frac{1}{2}(Z\bar{Z} + \bar{Z}Z) \tag{1.23}$$

on $\mathbb{C}$, where Δ_z is the standard Laplacian on $\mathbb{C}$ and $z = x + iy$. We define twisted Gabor frames, twisted Zak transform and deduce some of its properties. Also we prove the amalgam BLT and provide examples illustrating the distinction between the BLT and the amalgam BLT. Then using the operators $Z, \bar{Z}$ and the continuity of twisted Zak transform we obtain a version of amalgam BLT. After that we prove the BLT for exact frames on $L^2(\mathbb{C})$ using the operators $Z, \bar{Z}$ and the calculations are non-distributional. Also we obtain a variation of Heisenberg uncertainty principle (Theorem 6.4.1), weaker version of BLT (Theorem 6.5.2) and show the equivalence of weak BLT and BLT (Theorem 6.5.5). Finally, we discuss several consequences of BLT and weak BLT in terms of the operators L, Z and $\bar{Z}$ in Remark 6.5.6.

Chapter 2

Gabor Frame Operators on Wiener Amalgam Spaces

In this chapter, we show that the Gabor expansions converge to identity operator in the operator norm as well as in weak* sense on amalgam spaces as the sampling density tends to infinity. Also we obtain the analogue of the Janssen's representation and the Wexler-Raz biorthogonality condition for Gabor frame operators on amalgam spaces.

2.1 Preliminaries

The time-frequency shift $\tau(t,\omega)$ for the function g on $\mathbb{R}^d$ is defined by

$$(\tau(t,\omega)g)(x) = g(x-t)e^{2\pi i\langle x,\omega\rangle}, \quad t,\omega \in \mathbb{R}^d,$$

and the windowed Fourier transform of $f \in L^2(\mathbb{R}^d)$ with respect to $g \in L^2(\mathbb{R}^d)$ is given by

$$(F_g f)(t,\omega) = \langle f, \tau(t,\omega)g\rangle. \tag{2.1}$$

The inversion formula for windowed Fourier transform is given by

$$f = \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^{2d}} (F_g f)(t,\omega)\tau(t,\omega)\gamma\, dt\, d\omega, \tag{2.2}$$

where $\gamma \in L^2(\mathbb{R}^d)$ satisfies $\langle \gamma, g\rangle \neq 0$ and the integral is convergent in $L^2(\mathbb{R}^d)$ norm (see [77], p. 48).

Definition 2.1.1. *Given a window $g \in L^2(\mathbb{R}^d)$ and $a, b > 0$, the collection of functions $\{\tau(na, mb)g : m, n \in \mathbb{Z}^d\}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that*

$$A\|f\|_2^2 \leq \sum_{n,m\in\mathbb{Z}^d} |\langle f, \tau(na, mb)g\rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d).$$

If $\{\tau(na, mb)g : m, n \in \mathbb{Z}^d\}$ is a Gabor frame then there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that $\{\tau(na, mb)\gamma : m, n \in \mathbb{Z}^d\}$ is also a Gabor frame for $L^2(\mathbb{R}^d)$ and

$$\begin{aligned} f &= \sum_{n,m\in\mathbb{Z}^d} \langle f, \tau(na, mb)g\rangle \tau(na, mb)\gamma \\ &= \sum_{n,m\in\mathbb{Z}^d} \langle f, \tau(na, mb)\gamma\rangle \tau(na, mb)g, \quad \forall f \in L^2(\mathbb{R}^d). \end{aligned} \tag{2.3}$$

The series in (2.3) converges unconditionally in L^2. We refer to [77] for a detailed study on Gabor expansions on $L^2(\mathbb{R}^d)$.

Consider the Gabor frame operator on $L^2(\mathbb{R}^d)$ of the form

$$S_{a,b;g,\gamma}f = \frac{(ab)^d}{\langle \gamma, g\rangle} \sum_{n,m\in\mathbb{Z}^d} \langle f, \tau(na, mb)g\rangle \tau(na, mb)\gamma. \tag{2.4}$$

Then $S_{a,b;g,\gamma}f$ can be regarded as a Riemannian sum of the integral in (2.2). Weisz [136] proved that if both g and γ are in Feichtinger's algebra $S_0(\mathbb{R}^d) := \{g : F_g g \in L^1(\mathbb{R}^{2d})\}$, then the series in (2.4) converges to f in various norms. In [129], Sun generalized Weisz's results to some extent: If g, γ are in the Wiener spaces $W(\mathbb{R}^d)$, then $S_{a,b;g,\gamma}f$ converges to f on $L^p(\mathbb{R}^d)$ as (a, b) tends to $(0, 0)$. Notice that $S_0(\mathbb{R}^d)$ is a proper subspace of $W(\mathbb{R}^d)$. Moreover, by putting additional condition on g and γ, $S_{a,b;g,\gamma}$ converges to identity operator in $B(L^p(\mathbb{R}^d))$, the space of all bounded linear operators on $L^p(\mathbb{R}^d)$, as (a, b) tends to $(0, 0)$.

In this chapter, we extend the above results for the amalgam spaces, and obtain the weak* convergence and the operator norm convergence of the Gabor frame operator on $W(L^p, L^q),\ 1 \leq p, q < \infty$.

2.2 Gabor Frame Operators on Amalgam Spaces

In this section, we consider the convergence of the Gabor frame operator $S_{a,b;g,\gamma}$ on $W(L^p, L^q)$, $1 \leq p, q < \infty$, whenever g, γ are in $W(\mathbb{R}^d)$. It was shown in [82] that the Gabor expansions converge in weighted amalgam spaces when both g, γ are in weighted Wiener spaces. In this context, the following well known results are used in the proof of our main result.

Proposition 2.2.1. *([77], Lemma 6.1.2) If $g \in W(\mathbb{R}^d)$ and $a > 0$, then*

$$\sum_{n\in\mathbb{Z}^d} |g(x - an)| \leq \Big(1 + \frac{1}{a}\Big)^d \parallel g \parallel_{W(\mathbb{R}^d)}, \quad a.e.$$

For $g, \gamma \in W(\mathbb{R}^d)$ and $a, b > 0$, define

$$G_{a,b;n}(x) = \sum_{k\in\mathbb{Z}^d} \overline{g}\Big(x - \frac{n}{b} - ak\Big)\gamma(x - ak), \quad n \in \mathbb{Z}^d.$$

Lemma 2.2.2. *([129], Lemma 3.3) For any $g, \gamma \in W(\mathbb{R}^d)$, we have*

$$\sum_{n\in\mathbb{Z}^d} \|G_{a,b;n}\|_\infty \leq \Big(1 + \frac{1}{a}\Big)^d (2 + 2b)^d \|g\|_{W(\mathbb{R}^d)} \|\gamma\|_{W(\mathbb{R}^d)}, \quad \forall a, b > 0, \tag{2.5}$$

and

$$\lim_{(a,b)\to(0,0)} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a^d \|G_{a,b;n}\|_\infty = 0. \tag{2.6}$$

The Walnut representation of the Gabor frame operator on $W(L^p, L^q)$ (see [82]) is of the form

$$(S_{a,b;g,\gamma} f)(x) = \frac{1}{\langle \gamma, g\rangle} \sum_{n\in\mathbb{Z}^d} a^d G_{a,b;n}(x) f\left(x - \frac{n}{b}\right),$$

where g, $\gamma \in W(\mathbb{R}^d)$. Using it in the following proposition, we show that the frame operator is bounded on $W(L^p, L^q)$.

Proposition 2.2.3. *Let g, $\gamma \in W(\mathbb{R}^d)$ and $a, b > 0$. Then the operator*

$$(S_{a,b;g,\gamma}f)(x) = \frac{1}{\langle \gamma, g\rangle} \sum_{n\in\mathbb{Z}^d} a^d G_{a,b;n}(x) f\Big(x - \frac{n}{b}\Big) \tag{2.7}$$

is bounded from $W(L^p, L^q)$ to $W(L^p, L^q)$, $1 \le p, q \le \infty$ with operator norm

$$\|S_{a,b;g,\gamma}\|_{B(W(L^p,L^q))} \le \frac{a^d}{|\langle \gamma, g\rangle|}\Big(1 + \frac{1}{a}\Big)^d (2+2b)^d \|g\|_{W(\mathbb{R}^d)} \|\gamma\|_{W(\mathbb{R}^d)}.$$

Proof. If $g, \gamma \in W(\mathbb{R}^d)$, then by Lemma 2.2.2, $\sum_{n\in\mathbb{Z}^d} \|G_{a,b;n}\|_\infty < \infty$. Let $1 \le p, q \le \infty$ and $f \in W(L^p, L^q)$. Then

$$\begin{aligned}
\|S_{a,b;g,\gamma}f\|_{W(L^p,L^q)} &= \Big\| \frac{1}{\langle \gamma, g\rangle} \sum_{n\in\mathbb{Z}^d} a^d G_{a,b;n} T_{\frac{n}{b}} f \Big\|_{W(L^p,L^q)} \\
&\le \frac{a^d}{|\langle \gamma, g\rangle|} \sum_{n\in\mathbb{Z}^d} \|G_{a,b;n} T_{\frac{n}{b}} f\|_{W(L^p,L^q)} \\
&= \frac{a^d}{|\langle \gamma, g\rangle|} \sum_{n\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} \|G_{a,b;n} T_{\frac{n}{b}} f \cdot T_k \chi_Q\|_p^q \Big)^{\frac{1}{q}} \\
&\le \frac{a^d}{|\langle \gamma, g\rangle|} \sum_{n\in\mathbb{Z}^d} \|G_{a,b;n}\|_\infty \Big(\sum_{k\in\mathbb{Z}^d} \|f \cdot T_{k-\frac{n}{b}} \chi_Q\|_p^q \Big)^{\frac{1}{q}} \\
&= \frac{a^d}{|\langle \gamma, g\rangle|} \|f\|_{W(L^p,L^q)} \sum_{n\in\mathbb{Z}^d} \|G_{a,b;n}\|_\infty \\
&\le C\, \|g\|_{W(\mathbb{R}^d)} \|\gamma\|_{W(\mathbb{R}^d)} \|f\|_{W(L^p,L^q)},
\end{aligned}$$

where $C = \frac{a^d}{|\langle \gamma, g\rangle|}\left(1 + \frac{1}{a}\right)^d (2+2b)^d$. □

Since $G_{a,b;0}$ is independent of b, we define

$$G_a(x) := \frac{a^d}{\langle \gamma, g\rangle} G_{a,b;0}(x) = \frac{a^d}{\langle \gamma, g\rangle} \sum_{k\in\mathbb{Z}^d} \overline{g}(x - ak)\gamma(x - ak), \;\; x \in \mathbb{R}^d. \tag{2.8}$$

By Proposition 2.2.1, we have

$$M_0 := \sup_{0<a\leq 1} \|G_a - 1\|_\infty \leq \sup_{0<a\leq 1} \frac{1}{|\langle \gamma, g\rangle|}(1+a)^d \|\overline{g}.\gamma\|_{W(\mathbb{R}^d)} < \infty. \tag{2.9}$$

Lemma 2.2.4. *Let $g, \gamma \in W(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$. Then*

(i) For any $f \in W(L^p, L^q)$,

$$\lim_{(a,b)\to(0,0)} (\|S_{a,b;g,\gamma} f - f\|_{W(L^p,L^q)} - \|(G_a - 1)f\|_{W(L^p,L^q)}) = 0.$$

(ii) $\lim_{(a,b)\to(0,0)} (\|S_{a,b;g,\gamma} - I\|_{B(W(L^p,L^q))} - \|G_a - 1\|_\infty) = 0.$

Proof. Let $1 \leq p, q \leq \infty$. Define operators $T_{a;g,\gamma}$ and $R_{a,b;g,\gamma}$ on $W(L^p, L^q)$ by

$$T_{a;g,\gamma} f = (G_a - 1)f \quad \text{and}$$

$$R_{a,b;g,\gamma} f = \frac{1}{\langle \gamma, g\rangle} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a^d G_{a,b;n} \cdot f\left(\cdot - \frac{n}{b}\right), \; f \in W(L^p, L^q).$$

Now we can rewrite the Walnut's representation as

$$S_{a,b;g,\gamma} f - f = T_{a;g,\gamma} f + R_{a,b;g,\gamma} f, \; \forall f \in W(L^p, L^q).$$

Then, we have

$$\|S_{a,b;g,\gamma} f - f\|_{W(L^p,L^q)} \leq \|T_{a;g,\gamma} f\|_{W(L^p,L^q)} + \|R_{a,b;g,\gamma} f\|_{W(L^p,L^q)}.$$

Since

$$\lim_{(a,b)\to(0,0)} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a^d \|G_{a,b;n}\|_\infty = 0,$$

by Lemma 2.2.2, we have

$$\lim_{(a,b)\to(0,0)} \|R_{a,b;g,\gamma}\|_{B(W(L^p,L^q))} \leq \lim_{(a,b)\to(0,0)} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} a^d \|G_{a,b;n}\|_\infty = 0.$$

Further, it is easy to observe that

$$\|T_{a;g,\gamma}\|_{B(W(L^p,L^q))} = \|G_a - 1\|_\infty.$$

□

To prove the strong and weak* convergence, we make use of the following two lemmas.

Lemma 2.2.5. *([129], Lemma 3.5) Suppose that $f \in W(\mathbb{R}^d)$ is locally Riemann integrable. Then we have*

$$\lim_{a\to 0}\sup_{y\in\mathbb{R}^d}\left|\sum_{n\in\mathbb{Z}^d} a^d f(y+na) - \int_{\mathbb{R}^d} f(x)dx\right| = 0.$$

Lemma 2.2.6. *([129]) If $f \in L^p(\mathbb{R}^d)$ and G_a is defined as in (2.8), then*

$$\lim_{a\to 0}\|(G_a - 1)f\|_p = 0.$$

Next we state and prove the convergence of the Gabor frame operator on $W(L^p, L^q)$.

Theorem 2.2.7. *Let $g, \gamma \in W(\mathbb{R}^d)$. Then we have:*

(i) *For any $f \in W(L^p, L^q)$, $1 \le p, q < \infty$,*

$$\lim_{(a,b)\to(0,0)} \|S_{a,b;g,\gamma}f - f\|_{W(L^p,L^q)} = 0, \tag{2.10}$$

and conclusion holds for $q = \infty$.

(ii) *Moreover, if $\overline{g}\cdot\gamma$ is locally Riemann integrable, then for any $1 \le p, q \le \infty$,*

$$\lim_{(a,b)\to(0,0)} \|S_{a,b;g,\gamma} - I\|_{B(W(L^p,L^q))} = 0. \tag{2.11}$$

Proof. (i) Let $f \in W(L^p, L^q)$, $1 \le p, q < \infty$. Then for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\sum_{\|k\|_\infty > N} \|f \cdot T_k\chi_Q\|_p^q < \epsilon^q.$$

We have

$$\begin{aligned} & \|(G_a-1)f\|^q_{W(L^p,L^q)} \\ = & \sum_{k\in\mathbb{Z}^d} \|(G_a-1)f\cdot T_k\chi_Q\|^q_p \\ = & \sum_{\|k\|_\infty>N} \|(G_a-1)f\cdot T_k\chi_Q\|^q_p + \sum_{\|k\|_\infty\leq N} \|(G_a-1)f\cdot T_k\chi_Q\|^q_p. \end{aligned} \tag{2.12}$$

$$\sum_{\|k\|_\infty>N} \|(G_a-1)f\cdot T_k\chi_Q\|^q_p \leq \|(G_a-1)\|^q_\infty \sum_{\|k\|_\infty>N} \|f\cdot T_k\chi_Q\|^q_p \leq {M_0}^q\varepsilon^q, \tag{2.13}$$

where M_0 is defined in (2.9). Since

$$\|f\cdot T_k\chi_Q\|_p \leq \|f\|_{W(L^p,L^q)} < \infty, \quad \forall f\in W(L^p,L^q), \quad \forall k\in\mathbb{Z}^d,$$

we have $f\cdot T_k\chi_Q \in L^p(\mathbb{R}^d)$ for every $k\in\mathbb{Z}^d$. Applying Lemma 2.2.6, on $f\cdot T_k\chi_Q$ we get

$$\lim_{a\to 0}\|(G_a-1)f\cdot T_k\chi_Q\|_p = 0,$$

i.e., for every $\varepsilon>0$ and sufficiently small $a>0$,

$$\|(G_a-1)f\cdot T_k\chi_Q\|_p < \frac{\varepsilon}{(2N+1)^{\frac{d}{q}}}.$$

Taking q-th power and summation on both sides of the inequality we get

$$\sum_{\|k\|_\infty\leq N} \|(G_a-1)f\cdot T_k\chi_Q\|^q_p \leq \sum_{\|k\|_\infty\leq N} \frac{\varepsilon^q}{(2N+1)^d} \;=\; \varepsilon^q. \tag{2.14}$$

Now putting (2.13), (2.14) in (2.12) we get

$$\|(G_a-1)f\|^q_{W(L^p,L^q)} \;\leq\; (M_0^q+1)\varepsilon^q.$$

Hence

$$\lim_{a\to 0}\|(G_a-1)f\|_{W(L^p,L^q)}=0.$$

Thus for any $f\in W(L^p,L^q),\ 1\le p,q<\infty$, we obtain

$$\lim_{(a,b)\to(0,0)}\|S_{a,b;g,\gamma}f-f\|_{W(L^p,L^q)}=0,$$

using Lemma 2.2.4. This proves (2.10) for $1\le p,q<\infty$. For $q=\infty$, with the obvious modification similar result can be obtained.

(*ii*) Further, if $\overline{g}\cdot\gamma$ is locally Riemann integrable, we have

$$\lim_{(a,b)\to(0,0)}\|G_a-1\|_\infty=0,$$

by Lemma 2.2.5. Again using Lemma 2.2.4, we get

$$\lim_{(a,b)\to(0,0)}\|S_{a,b;g,\gamma}-I\|_{B(W(L^p,L^q))}=0.$$

□

Now we show that (2.10) is not true for $p=\infty$ by producing a counterexample.

Example 2.2.8. For simplicity, we consider only the case of $d=1$. Let us take some $E\subset[0,1]$ such that E is nowhere dense and is of positive measure. Let $g=\gamma=\chi_E$. For any $a>0$, we have

$$\{x\in[0,1]:G_a(x)>0\}=\bigcup_{n\in\mathbb{Z}}(na+E)\cap[0,1]=\bigcup_{\|n\|_\infty\le\frac{1}{a}}(na+E)\cap[0,1].$$

Since each of $na+E$ is nowhere dense, so is $\bigcup_{\|n\|_\infty\le\frac{1}{a}}(na+E)$. Therefore $\{x\in[0,1]:G_a(x)>0\}$ is nowhere dense. Hence

$$|\{x\in[0,1]:|G_a(x)-1|=1\}|\ge|\{x\in[0,1]:G_a(x)=0\}|>0.$$

Let $f_0=\chi_{[0,1]}$. Then we have $\|(G_a-1)f_0\|_\infty\ge 1,\ \ \forall a>0$.

Now

$$\|(G_a - 1)f_0\|^q_{W(L^\infty,L^q)} = \sum_{k\in\mathbb{Z}} \|(G_a - 1)\chi_{[0,1]} \cdot \chi_{[k,k+1]}\|^q_\infty \geq \|(G_a - 1)\chi_{[0,1]}\|^q_\infty.$$

That is

$$\|(G_a - 1)f_0\|_{W(L^\infty,L^q)} \geq 1, \ \ \forall a > 0.$$

By Lemma 2.2.4, $\lim\limits_{(a,b)\to(0,0)} \|S_{a,b;g,\gamma}f_0 - f_0\|_{W(L^\infty,L^q)} \geq 1$, i.e. (2.10) fails for $p = \infty$.

Finally we illustrate by an example that the locally Riemann integrability condition in Theorem 2.2.7(ii) is not redundant. The following example can be found in [63], which also works in this case.

Example 2.2.9. Let us fix some $0 < \varepsilon < 1$. Let $g = \gamma = \chi_E$, where $E = (0,1) \setminus A$ and

$$A = \bigcup_{k\geq 1} \bigcup_{l\in\mathbb{Z}} \left(\frac{l}{3^k} - \frac{\varepsilon}{3^{2k}}, \frac{l}{3^k} + \frac{\varepsilon}{3^{2k}}\right).$$

Note that E is nowhere dense and of positive measure exceeds $1-\varepsilon$. For any $a, b > 0$, $\{\tau(na, mb)g : n, m \in \mathbb{Z}\}$ has no positive lower frame bound (see [63]). Therefore, $\|S_{a,b;g,g} - I\|_{B(L^2(\mathbb{R}^d))}$ cannot converge to 0. For $2 \leq p \leq \infty$ and $1 \leq q \leq 2$, we have $W(L^p, L^q) \subset W(L^2, L^2) = L^2(\mathbb{R}^d)$. Hence, $S_{a,b;g,g}$ cannot converge to the identity operator strongly on $W(L^p, L^q)$, $2 \leq p \leq \infty$ and $1 \leq q \leq 2$. But $S_{a,b;g,g}$ converges to the identity operator on $W(L^p, L^q)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, in weak* sense since $g \in W(\mathbb{R}^d)$. Since $E = \{x : g \text{ is not continuous at } x\}$ is of positive measure, in this example the window function $g = \bar{g} \cdot \gamma$ is not locally Riemann integrable.

2.3 The Structure of Gabor Systems

In this section, we extend the Janssen's representation of Gabor frame operators $S_{a,b;g,\gamma}$ on $W(L^p, L^q)$, $1 \leq p, q \leq \infty$, and the biorthogonality condition of Wexler-Raz on $W(L^p, L^q)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$. First, we expand $G_{a,b;n}$ into its Fourier series. The l-th Fourier coefficient of $G_{a,b;n}$ is

$$\hat{G}_{a,b;n}(l) \ = \ a^{-d} \int_{Q_a} G_{a,b;n}(x) e^{-2\pi i\langle l, x/a\rangle} dx$$

$$= a^{-d}\int_{Q_a}\sum_{k\in\mathbb{Z}^d}(T_{\frac{n}{b}}\overline{g}\cdot\gamma)(x-ak)e^{-2\pi i\langle l,x/a\rangle}dx$$
$$= a^{-d}\int_{\mathbb{R}^d}(T_{\frac{n}{b}}\overline{g}\cdot\gamma)(x)e^{-2\pi i\langle l,x/a\rangle}dx = a^{-d}\langle\gamma, M_{\frac{l}{a}}T_{\frac{n}{b}}g\rangle.$$

Since $G_{a,b;n}\in L^\infty(Q_a)\subseteq L^2(Q_a)$ by Lemma 2.2.2, $G_{a,b;n}$ has the Fourier series

$$G_{a,b;n}(x) = a^{-d}\sum_{l\in\mathbb{Z}^d}\langle\gamma, M_{\frac{l}{a}}T_{\frac{n}{b}}g\rangle e^{2\pi i\langle l,x/a\rangle}, \tag{2.15}$$

with convergence in $L^2(Q_a)$. Now substituting this into Walnut's representation (2.7), we obtain

$$S_{a,b;g,\gamma}f = \frac{a^d}{\langle\gamma,g\rangle}\sum_{n\in\mathbb{Z}^d}G_{a,b;n}\cdot T_{\frac{n}{b}}f = \frac{1}{\langle\gamma,g\rangle}\sum\sum_{l,n\in\mathbb{Z}^d}\langle\gamma, M_{\frac{l}{a}}T_{\frac{n}{b}}g\rangle M_{\frac{l}{a}}T_{\frac{n}{b}}f,$$

in operator notation,

$$S_{a,b;g,\gamma} = \frac{1}{\langle\gamma,g\rangle}\sum\sum_{l,n\in\mathbb{Z}^d}\langle\gamma, M_{\frac{l}{a}}T_{\frac{n}{b}}g\rangle M_{\frac{l}{a}}T_{\frac{n}{b}}. \tag{2.16}$$

The above expression is valid provided the series on the right-hand side converges. If $g,\gamma\in L^2(\mathbb{R}^d)$, then it is unclear how the Fourier series in (2.15) represents $G_{a,b;n}$. The convergence of the series (2.15) and (2.16) can be justified under some additional condition on the window functions g and γ. In the investigations of Gabor families by Tolimieri-Orr [134], and Janssen [98], the following technical condition on windows g,γ were used. Here that condition is provided as definition.

Definition 2.3.1. *A pair of window functions (g,γ) in $L^2(\mathbb{R}^d)$ satisfies condition (A') for the parameters $a,b>0$ if*

$$\sum_{l,n\in\mathbb{Z}^d}|\langle\gamma, M_{\frac{l}{a}}T_{\frac{n}{b}}g\rangle| < \infty. \tag{2.17}$$

If $g = \gamma$, then g is said to satisfy condition (A) for the parameters $a, b > 0$ if

$$\sum_{l,n\in\mathbb{Z}^d} |\langle g, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle| < \infty. \tag{2.18}$$

Now the condition (A') guarantees the absolute convergence of the series expansions (2.15) and (2.16). However condition (A') is not always satisfied even for $g, \gamma \in W(\mathbb{R}^d)$ (for example see p. 132 of [77]). So we have to put additional condition on the window function. If we consider g, γ as in Feichtinger's algebra $S_0(\mathbb{R}^d)$, then the condition (A') is satisfied together for all $a, b > 0$ (see [77], Corollary 12.1.12, p. 255). Now with this hypothesis we derive representation of the Gabor frame operator $S_{a,b;g,\gamma}$ on $W(L^p, L^q)$, $1 \le p, q \le \infty$. A version of Janssen's representation can be found in [62]. However, in the following we establish the Janssen's representation for frame operator $S_{a,b;g,\gamma}$ on $W(L^p, L^q)$, $1 \le p, q \le \infty$ by considering the window functions $g, \gamma \in S_0(\mathbb{R}^d)$.

Theorem 2.3.2. (Janssen's representation). *Suppose that $g, \gamma \in S_0(\mathbb{R}^d)$. Then for all $a, b > 0$ frame operator $S_{a,b;g,\gamma}$ on $W(L^p, L^q)$, $1 \le p, q \le \infty$, can be expressed as follows:*

$$S_{a,b;g,\gamma} = \frac{1}{\langle \gamma, g\rangle} \sum\sum_{l,n\in\mathbb{Z}^d} \langle \gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle M_{\frac{l}{a}} T_{\frac{n}{b}} = \frac{1}{\langle \gamma, g\rangle} \sum\sum_{k,n\in\mathbb{Z}^d} \langle \gamma, T_{\frac{k}{b}} M_{\frac{n}{a}} g\rangle T_{\frac{k}{b}} M_{\frac{n}{a}}$$

and converges absolutely in the operator norm.

Proof. Let

$$\tilde{S}_{a,b;g,\gamma} := \frac{1}{\langle \gamma, g\rangle} \sum\sum_{l,n\in\mathbb{Z}^d} \langle \gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle M_{\frac{l}{a}} T_{\frac{n}{b}}$$

and we want to show $S_{a,b;g,\gamma} = \tilde{S}_{a,b;g,\gamma}$. As $g, \gamma \in S_0(\mathbb{R}^d)$, so by condition (A') the series for $\tilde{S}_{a,b;g,\gamma}$ converges absolutely in operator norm and hence its expression is independent of the order of summations. Therefore

$$\tilde{S}_{a,b;g,\gamma} = \frac{a^d}{\langle \gamma, g\rangle} \sum_{n\in\mathbb{Z}^d} \Big(a^{-d} \sum_{l\in\mathbb{Z}^d} \langle \gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle e^{2\pi i \langle l, x/a\rangle} \Big) T_{\frac{n}{b}}$$

$$= \frac{a^d}{\langle \gamma, g\rangle} \sum_{n\in\mathbb{Z}^d} G_{a,b;n} \cdot T_{\frac{n}{b}} = S_{a,b;g,\gamma},$$

by using (2.15) and Walnut's representation. □

Next we state and prove the Wexler-Raz biorthogonality condition for frame operator $S_{a,b;g,\gamma}$ on $W(L^p, L^q)$. In [138], the authors found a exceptional relation between window g and dual window γ. Their conditions characterized all dual windows. Here we make use of Theorem 2.2.7 and present a version of that important result.

Theorem 2.3.3. (Wexler-Raz biorthogonality). *Assume $g, \gamma \in S_0(\mathbb{R}^d)$. Then for any $1 \leq p < \infty$ and $1 \leq q \leq \infty$, the following conditions are equivalent:*

(i) $\lim_{(a,b)\to(0,0)} S_{a,b;g,\gamma} f = \lim_{(a,b)\to(0,0)} S_{a,b;\gamma,g} f = f$ *on* $W(L^p, L^q)$.

(ii) $\lim_{(a,b)\to(0,0)} \frac{1}{\langle\gamma,g\rangle}\langle \gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle = \delta_{l0}\delta_{n0}$ *for* $l, n \in \mathbb{Z}^d$.

Proof. $(i) \Rightarrow (ii)$ Let $f \in W(L^p, L^q)$ and $h \in W(L^{p'}, L^{q'})$ (Köthe dual of $W(L^p, L^q)$) and assume that $\lim_{(a,b)\to(0,0)} S_{a,b;g,\gamma} f = f$. Let $l, m \in \mathbb{Z}^d$ be arbitrary. Then

$$\begin{aligned}
\delta_{lm}\langle f, h\rangle &= \left\langle \lim_{(a,b)\to(0,0)} S_{a,b;g,\gamma} T_{\frac{l}{b}} f, T_{\frac{m}{b}} h \right\rangle \\
&= \lim_{(a,b)\to(0,0)} \frac{a^d}{\langle \gamma, g\rangle} \left\langle \sum_{n\in\mathbb{Z}^d} G_{a,b;n} \cdot T_{\frac{n+l}{b}} f, T_{\frac{m}{b}} h \right\rangle \\
&= \lim_{(a,b)\to(0,0)} \frac{a^d}{\langle \gamma, g\rangle} \langle G_{a,b;m-l} \cdot T_{\frac{m}{b}} f, T_{\frac{m}{b}} h\rangle \\
&= \lim_{(a,b)\to(0,0)} \frac{a^d}{\langle \gamma, g\rangle} \langle (T_{-\frac{m}{b}} G_{a,b;m-l}) f, h\rangle.
\end{aligned}$$

So we conclude that $\lim_{(a,b)\to(0,0)} \frac{a^d}{\langle\gamma,g\rangle} G_{a,b;m-l}(\cdot + \frac{m}{b}) f = \delta_{lm} f$. Now varying $l, m \in \mathbb{Z}^d$, we get $\lim_{(a,b)\to(0,0)} \frac{a^d}{\langle\gamma,g\rangle} G_{a,b;0}(\cdot) f = f$ and $\lim_{(a,b)\to(0,0)} \frac{a^d}{\langle\gamma,g\rangle} G_{a,b;n}(\cdot) f = 0$, when $n \neq 0$.

Therefore

$$\lim_{(a,b)\to(0,0)} \frac{a^d}{\langle\gamma, g\rangle} G_{a,b;n}(x) f(x) = \lim_{(a,b)\to(0,0)} \frac{1}{\langle\gamma, g\rangle} \sum_{l\in\mathbb{Z}^d} \langle\gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle e^{2\pi i\langle l, x/a\rangle} f(x),$$

by (2.15). We conclude by using uniqueness of Fourier coefficients that

$$\lim_{(a,b)\to(0,0)} \frac{1}{\langle\gamma, g\rangle} \langle\gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle = \delta_{l0}\delta_{n0}.$$

The implication $(ii) \Rightarrow (i)$ follows from Janssen's representation: If the biorthogonality condition (ii) is satisfied then

$$\lim_{(a,b)\to(0,0)} \sum_{l,n\in\mathbb{Z}^d} |\langle\gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle| = \sum_{l,n\in\mathbb{Z}^d} |\langle\gamma, g\rangle \delta_{l0}\delta_{n0}| = |\langle\gamma, g\rangle| < \infty,$$

i.e., for arbitrary small $a, b > 0$, $\sum_{l,n\in\mathbb{Z}^d} |\langle\gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle| < \infty$, i.e., the pair (g, γ) satisfies condition (A') for arbitrary small $a, b > 0$ and hence the representation (2.16) converges in the operator norm. Therefore by Theorem 2.2.7, for any $1 \leq p < \infty$ and $1 \leq q \leq \infty$, $\lim_{(a,b)\to(0,0)} S_{a,b;g,\gamma} f = f$ on $W(L^p, L^q)$. □

The next corollary follows immediately from Theorem 2.3.3.

Corollary 2.3.4. *In the assumption of Theorem 2.3.3, if* $\overline{g} \cdot \gamma$ *is locally Riemann integrable then for any* $1 \leq p, q \leq \infty$*, the following conditions are equivalent:*

(*i*) $\lim_{(a,b)\to(0,0)} S_{a,b;g,\gamma} = \lim_{(a,b)\to(0,0)} S_{a,b;\gamma,g} = I$ *on* $B(W(L^p, L^q))$.

(*ii*) $\lim_{(a,b)\to(0,0)} \frac{1}{\langle\gamma, g\rangle} \langle\gamma, M_{\frac{l}{a}} T_{\frac{n}{b}} g\rangle = \delta_{l0}\delta_{n0}$ *for* $l, n \in \mathbb{Z}^d$.

Proof. $(i) \Rightarrow (ii)$ follows from the fact that $\lim_{(a,b)\to(0,0)} S_{a,b;g,\gamma} = I$ on $B(W(L^p, L^q))$ $\Rightarrow \lim_{(a,b)\to(0,0)} S_{a,b;g,\gamma} f = f$ on $W(L^p, L^q)$.

The implication $(ii) \Rightarrow (i)$ follows from Janssen's representation and Theorem 2.2.7: If the biorthogonality condition (ii) is satisfied then the pair (g, γ) satisfies condition (A') for arbitrary small $a, b > 0$ and the frame operator $S_{a,b;g,\gamma}$ converges in the operator norm. Since $\overline{g} \cdot \gamma$ is locally Riemann integrable then by using Theorem 2.2.7

we conclude that for any $1 \leq p, q \leq \infty$, $\lim\limits_{(a,b)\to(0,0)} S_{a,b;g,\gamma} = \lim\limits_{(a,b)\to(0,0)} S_{a,b;\gamma,g} = I$ on $B(W(L^p, L^q))$. □

Chapter 3

Hilbert Space Valued Gabor Frames in Weighted Amalgam Spaces

Let $\mathbb{H}$ be a separable Hilbert space. In this chapter, we establish a generalization of Walnut's representation and Janssen's representation of the $\mathbb{H}$-valued Gabor frame operator on $\mathbb{H}$-valued weighted amalgam spaces $W_{\mathbb{H}}(L^p, L^q_v)$, $1 \leq p, q \leq \infty$. Also we show that the frame operator is invertible on $W_{\mathbb{H}}(L^p, L^q_v)$, $1 \leq p, q \leq \infty$, if the window function is in the Wiener amalgam space $W_{\mathbb{H}}(L^\infty, L^1_w)$. Further, we obtain the Walnut's representation and invertibility of the frame operator corresponding to Gabor superframes and multi-window Gabor frames on $W_{\mathbb{H}}(L^p, L^q_v)$, $1 \leq p, q \leq \infty$, as a special case by choosing an appropriate Hilbert space $\mathbb{H}$.

3.1 Preliminaries

Let us begin with the definition of $\mathbb{H}-$valued Gabor frames on $L^2(\mathbb{R}^d, \mathbb{H})$. Throughout this chapter, $\mathbb{H}$ is denoted as separable complex Hilbert space. Let $\alpha, \beta > 0$ and $\mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{H})$. For $n, k \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$, we define

$$M_{\beta n}\mathbf{g}(x) = e^{2\pi i\langle \beta n, x\rangle}\mathbf{g}(x) \quad \text{and} \quad T_{\alpha k}\mathbf{g}(x) = \mathbf{g}(x - \alpha k).$$

Definition 3.1.1. *Given a non-zero window function* $\mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{H})$ *and lattice parameters* $\alpha, \beta > 0$*, the* $\mathbb{H}$*-valued Gabor system* $\mathcal{G}(\mathbf{g}, \alpha, \beta) = \{M_{\beta n} T_{\alpha k} \mathbf{g} : k, n \in \mathbb{Z}^d\}$ *is a frame for* $L^2(\mathbb{R}^d, \mathbb{H})$ *if there exist constants* $A, B > 0$ *such that for all* $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{H})$*,*

$$A\|\mathbf{f}\|^2_{L^2(\mathbb{R}^d,\mathbb{H})} \leq \sum_{k,n\in\mathbb{Z}^d} |\langle \mathbf{f}, M_{\beta n} T_{\alpha k} \mathbf{g}\rangle_{L^2(\mathbb{R}^d,\mathbb{H})}|^2 \leq B\|\mathbf{f}\|^2_{L^2(\mathbb{R}^d,\mathbb{H})}. \tag{3.1}$$

For $\mathbf{g}, \boldsymbol{\gamma} \in L^2(\mathbb{R}^d, \mathbb{H})$, the associated frame operator is given by

$$S_{\mathbf{g},\boldsymbol{\gamma}} \mathbf{f} = \sum_{k,n\in\mathbb{Z}^d} \langle \mathbf{f}, M_{\beta n} T_{\alpha k} \mathbf{g}\rangle M_{\beta n} T_{\alpha k} \boldsymbol{\gamma}, \quad \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{H}).$$

Definition 3.1.2. *For* $1 \leq p \leq \infty$ *and a strictly positive function* w *on* $\mathbb{R}^d$*,* $L^p_w(\mathbb{R}^d, \mathbb{H})$ *denotes the space of all equivalence classes of* $\mathbb{H}$*-valued Bochner integrable functions* $\mathbf{f}$ *defined on* $\mathbb{R}^d$ *with* $\int_{\mathbb{R}^d} \|\mathbf{f}(x)\|^p_{\mathbb{H}} w(x)^p dx < \infty$*, with the usual adjustment if* $p = \infty$*.*

Definition 3.1.3. *The Fourier transform of* $\mathbf{f} \in L^1(\mathbb{R}^d, \mathbb{H})$ *is defined as*

$$\hat{\mathbf{f}}(w) = \mathcal{F}\mathbf{f}(w) = \int_{\mathbb{R}^d} \mathbf{f}(t) e^{-2\pi i \langle w, t\rangle} dt, \quad w \in \mathbb{R}^d.$$

Let $\mathbf{f} \in L^{p'}(\mathbb{R}^d, \mathbb{H})$. Define $\Lambda_{\mathbf{f}} : L^p(\mathbb{R}^d, \mathbb{H}) \to \mathbb{C}$ by $\Lambda_{\mathbf{f}}(\mathbf{g}) = \int_{\mathbb{R}^d} \langle \mathbf{f}(x), \mathbf{g}(x)\rangle_{\mathbb{H}} dx$. Then the map $\mathbf{f} \mapsto \Lambda_{\mathbf{f}}$ defines an isometric isomorphism of $L^{p'}(\mathbb{R}^d, \mathbb{H})$ onto $[L^p(\mathbb{R}^d, \mathbb{H})]^*$. For a more detailed study of vector valued functions, we refer to Diestel and Uhl [45].

Definition 3.1.4. *For* $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{H})$ *the inner product on* $L^2(\mathbb{R}^d, \mathbb{H})$ *is defined by*

$$\langle \mathbf{f}, \mathbf{g}\rangle_{L^2(\mathbb{R}^d,\mathbb{H})} = \int_{\mathbb{R}^d} \langle \mathbf{f}(x), \mathbf{g}(x)\rangle_{\mathbb{H}} dx.$$

Definition 3.1.5. *If* $x, y \in \mathbb{H}$*, the operator* $x \odot y : \mathbb{H} \to \mathbb{H}$ *defined by*

$$(x \odot y)(z) = \langle z, y\rangle x, \quad z \in \mathbb{H}.$$

Clearly for non zero x and y, $x \odot y$ is a rank one operator with $\|x \odot y\| = \|x\|\|y\|$. If $x_1, x_2, y_1, y_2 \in \mathbb{H}$ then the following equalities hold:

$$(x_1 \odot x_2)(y_1 \odot y_2) = \langle y_1, x_2 \rangle (x_1 \odot y_2) \quad \text{and} \quad (x_1 \odot y_1)^* = (y_1 \odot x_1).$$

3.1.1 Gabor Frames in $L^2(\mathbb{R}^d, \mathbb{H})$

As in the case of Gabor frames for $L^2(\mathbb{R}^d)$, with the assumption that $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{H})$ we list out the basic properties of $\mathbb{H}$-valued Gabor frames for $L^2(\mathbb{R}^d, \mathbb{H})$ in the following theorem.

Theorem 3.1.6. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a Gabor frame for $L^2(\mathbb{R}^d, \mathbb{H})$ with frame bounds A, B and let $\mathbf{h} \in \mathbb{H}$ with unit norm. Then the following statements hold.*

(i) The analysis operator $C_{\mathbf{g},\mathbf{h}}\mathbf{f} = (\langle \mathbf{f}, M_{\beta n} T_{\alpha k} \mathbf{g} \rangle \mathbf{h})_{k,n \in \mathbb{Z}^d}$ is a bounded mapping $C_{\mathbf{g},\mathbf{h}} : L^2(\mathbb{R}^d, \mathbb{H}) \to \ell^2(\mathbb{Z}^{2d}, \mathbb{H})$, and we have the norm equivalence $\|\mathbf{f}\|_{L^2(\mathbb{R}^d,\mathbb{H})} \asymp \|C_{\mathbf{g},\mathbf{h}}\mathbf{f}\|_{\ell^2(\mathbb{Z}^{2d},\mathbb{H})}$.

(ii) The synthesis operator $R_{\mathbf{g},\mathbf{h}} d = \sum_{k,n \in \mathbb{Z}^d} \langle d_{kn}, \mathbf{h} \rangle_{\mathbb{H}} M_{\beta n} T_{\alpha k} \mathbf{g}$ is a bounded mapping $R_{\mathbf{g},\mathbf{h}} : \ell^2(\mathbb{Z}^{2d}, \mathbb{H}) \to L^2(\mathbb{R}^d, \mathbb{H})$. The series defining $R_{\mathbf{g},\mathbf{h}} d$ converges unconditionally in L^2 for every $d \in \ell^2$.

(iii) $R_{\mathbf{g},\mathbf{h}} = C^*_{\mathbf{g},\mathbf{h}}$, and the frame operator $S_{\mathbf{g}} = R_{\mathbf{g},\mathbf{h}} C_{\mathbf{g},\mathbf{h}} : L^2(\mathbb{R}^d, \mathbb{H}) \to L^2(\mathbb{R}^d, \mathbb{H})$ is strictly positive, invertible operator satisfies

$$A I_{L^2(\mathbb{R}^d,\mathbb{H})} \leq S_{\mathbf{g}} \leq B I_{L^2(\mathbb{R}^d,\mathbb{H})} \quad \text{and} \quad B^{-1} I_{L^2(\mathbb{R}^d,\mathbb{H})} \leq S_{\mathbf{g}}^{-1} \leq A^{-1} I_{L^2(\mathbb{R}^d,\mathbb{H})}.$$

(iv) The dual window $\boldsymbol{\gamma} = S_{\mathbf{g}}^{-1} \mathbf{g}$ generates a Gabor frame $\mathcal{G}(\boldsymbol{\gamma}, \alpha, \beta)$ for $L^2(\mathbb{R}^d, \mathbb{H})$ with frame bounds $1/B, 1/A$.

(v) $R_{\boldsymbol{\gamma},\mathbf{h}} C_{\mathbf{g},\mathbf{h}} = I$ on $L^2(\mathbb{R}^d, \mathbb{H})$ i.e., we have the Gabor expansions

$$\mathbf{f} = R_{\boldsymbol{\gamma},\mathbf{h}} C_{\mathbf{g},\mathbf{h}} \mathbf{f} = \sum_{k,n \in \mathbb{Z}^d} \langle \mathbf{f}, M_{\beta n} T_{\alpha k} \boldsymbol{\gamma} \rangle M_{\beta n} T_{\alpha k} \mathbf{g} \tag{3.2}$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{H})$, with unconditional convergence of the series.

3.1.2 Weight Functions

Weight functions play an important role in time-frequency analysis and occur in many problems and contexts. For our study we need the following types of weights.

Definition 3.1.7. *A function $w : \mathbb{R}^d \to (0, +\infty)$ is called a weight if it is continuous and symmetric (i.e. $w(x) = w(-x)$). A weight w is submultiplicative if*

$$w(x + y) \leq w(x)w(y), \quad x, y \in \mathbb{R}^d.$$

Definition 3.1.8. *Given a submultiplicative weight w, a second weight $v : \mathbb{R}^d \to (0, +\infty)$ is called w-moderate if there exists a constant $C_v > 0$ such that,*

$$v(x + y) \leq C_v w(x)v(y), \quad x, y \in \mathbb{R}^d. \tag{3.3}$$

Definition 3.1.9. *A weight w is called admissible if it is submultiplicative, $w(0) = 1$ and satisfies the Gelfand-Raikov-Shilov condition,*

$$\lim_{k\to\infty} w(kx)^{1/k} = 1,\ x \in \mathbb{R}^d.$$

By (3.3) and using symmetry of w one can conclude that the class of w-moderate weights is closed under reciprocals, and consequently the class of spaces L^p_v using w-moderate weights is closed under duality (with the usual exception for $p = \infty$). We refer to [89] for a detailed study on moderate weights.

Example 3.1.10. (i) For $s > 0$, $w(x) = (1 + |x|)^s$ is a submultiplicative weight which is a polynomially-growing function.

(ii) If we consider polynomial weights $v(x) = (1 + |x|)^t$, $w(x) = (1 + |x|)^s$, then v is w-moderate if $|t| \leq s$.

The translation-invariance property for moderate weights holds good for $L^p_v(\mathbb{R}^d, \mathbb{H})$ as in $L^p_v(\mathbb{R}^d)$ (see [82, 89]). Since the proof follows exactly as for $L^p_v(\mathbb{R}^d)$ we only state the translation-invariant property of $L^p_v(\mathbb{R}^d, \mathbb{H})$ with respect to moderate weights in the following lemma without proof.

Lemma 3.1.11. *(See [89]) Let w be a submultiplicative weight on $\mathbb{R}^d$, and fix $1 \leq p \leq \infty$. Then the following statements are equivalent.*

(i) v is w-moderate.

(ii) $L^p_v(\mathbb{R}^d, \mathbb{H})$ is translation-invariant (i.e., for each $x \in \mathbb{R}^d$, T_x is a continuous mapping of $L^p_v(\mathbb{R}^d, \mathbb{H})$ onto itself).

(iii) For each compact set $K \subset \mathbb{R}^d$, there exists a constant $C > 0$ such that

$$\sup_{t \in y+K} v(t) \leq C \inf_{t \in y+K} v(t), \quad \forall y \in \mathbb{R}^d.$$

Throughout this chapter, a submultiplicative weight function is denoted by w and a w-moderate function is denoted by v. Given a w-moderate weight v on $\mathbb{R}^d$, we will often use the notation $\tilde{v}$ to denote the weight on $\mathbb{Z}^d$ defined by $\tilde{v}(k) = v(\alpha k)$, and for a weight v on $\mathbb{R}^{2d}$ we define $\tilde{v}(k, n) = v(\alpha k, \beta n)$.

3.1.3 Weighted Amalgam Spaces

Let Q_α denote the cube $Q_\alpha = [0, \alpha)^d$.

Definition 3.1.12. *Given an w-moderate weight v on $\mathbb{R}^d$ and given $1 \leq p, q \leq \infty$, the weighted amalgam space $W(L^p(\mathbb{R}^d, \mathbb{H}), L^q_v)$ is the Banach space of all $\mathbb{H}$-valued measurable functions $\mathbf{f} : \mathbb{R}^d \to \mathbb{H}$ for which the norm*

$$\|\mathbf{f}\|_{W(L^p(\mathbb{R}^d,\mathbb{H}),L^q_v)} := \Big(\sum_{k \in \mathbb{Z}^d} \|\mathbf{f} \cdot T_{\alpha k} \chi_{Q_\alpha}\|^q_{L^p(\mathbb{R}^d,\mathbb{H})} v(\alpha k)^q \Big)^{1/q} < \infty, \tag{3.4}$$

with the obvious modification for $q = \infty$.

Throughout this chapter, the weighted amalgam space is denoted by $W_{\mathbb{H}}(L^p, L^q_v)$ instead of $W(L^p(\mathbb{R}^d, \mathbb{H}), L^q_v)$. We refer to [55, 56, 57, 70, 89] for discussions of weighted amalgam spaces and their applications. The space $W_{\mathbb{H}}(L^p, L^q_v)$ is independent of the value of α used in (3.4) in the sense that each different choice of α yields an equivalent norm for $W_{\mathbb{H}}(L^p, L^q_v)$ (as in the scalar valued case).

Weighted amalgam spaces are *solid*. This means that if $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$ and $\mathbf{m} \in L^{\infty}(\mathbb{R}^d, B(\mathbb{H}))$, then $\mathbf{m}(\mathbf{f}) \in W_{\mathbb{H}}(L^p, L^q_v)$ and

$$\|\mathbf{m}(\mathbf{f})\|_{W_{\mathbb{H}}(L^p,L^q_v)} \leq \|\mathbf{m}\|_{L^{\infty}(\mathbb{R}^d,B(\mathbb{H}))}\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)}. \tag{3.5}$$

In addition, $W_{\mathbb{H}}(L^p, L^q_v)$ is closed under translations and

$$\|T_x\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)} \leq C_v w(x)\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)}, \tag{3.6}$$

where C_v is the constant in (3.3). For each w-moderate weight v, and $p_1 \geq p_2$, $q_1 \leq q_2$ we have the following relations:

$$W_{\mathbb{H}}(L^{p_1}, L^{q_1}_w) \subset W_{\mathbb{H}}(L^{p_1}, L^{q_1}_v) \subset W_{\mathbb{H}}(L^{p_2}, L^{q_2}_v) \subset W_{\mathbb{H}}(L^{p_2}, L^{q_2}_{1/w}).$$

In particular, the inclusions $W_{\mathbb{H}}(L^{\infty}, L^1_w) \subset W_{\mathbb{H}}(L^p, L^q_v) \subset W_{\mathbb{H}}(L^1, L^{\infty}_{1/w})$ hold for all $1 \leq p, q \leq \infty$ and all w-moderate weights v.

The Köthe dual of $W_{\mathbb{H}}(L^p, L^q_v)$ is the space of all measurable functions $\mathbf{g} : \mathbb{R}^d \to \mathbb{H}$ such that $\mathbf{g} \cdot W_{\mathbb{H}}(L^p, L^q_v) \subseteq L^1(\mathbb{R}^d)$. It is equal to $W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v})$, where $1/p + 1/p' = 1/q + 1/q' = 1$ for all $1 \leq p, q \leq \infty$. The pairing

$$\langle \cdot, \cdot \rangle : W_{\mathbb{H}}(L^p, L^q_v) \times W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v}) \to \mathbb{C}, \quad \langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}^d} \langle \mathbf{f}(x), \mathbf{g}(x) \rangle_{\mathbb{H}} dx,$$

is bounded. As in the case of scalar valued amalgam spaces the dual and Köthe dual of the $\mathbb{H}$-valued amalgam spaces are obtained with necessary modifications in the following lemma:

Lemma 3.1.13. *Let v be an w-moderate weight and $1/p + 1/p' = 1/q + 1/q' = 1$. Then*

(i) For $1 \leq p, q < \infty$, the dual space of $W_{\mathbb{H}}(L^p, L^q_v)$ is $W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v})$.

(ii) For $1 \leq p, q \leq \infty$ the Köthe dual of $W_{\mathbb{H}}(L^p, L^q_v)$ is $W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v})$.

For detailed study on Köthe dual for scalar valued amalgam spaces we refer to the text of Bennett and Sharpley [21].

3.2 ℍ-Valued Gabor Expansions in Weighted Amalgam Spaces

The theory of Gabor expansions on Wiener amalgam spaces has been discussed in [67, 75, 81, 82] for a separable lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d,\ \alpha, \beta > 0$. Here, the Walnut's representation of the ℍ-valued frame operator is generalized and extended and the convergence of ℍ-valued Gabor expansions on $W_{\mathbb{H}}(L^p, L^q_v)$ is shown. Let us begin with some definitions.

Definition 3.2.1. *The Fourier transform of* $\mathbf{f} \in L^1(Q_{1/\beta}, \mathbb{H})$ *is the sequence* $\hat{\mathbf{f}}$ *defined by*

$$\hat{\mathbf{f}}(n) = \mathcal{F}\mathbf{f}(n) = \beta^d \int_{Q_{1/\beta}} e^{-2\pi i\beta\langle n,t\rangle}\mathbf{f}(t)dt, \quad n \in \mathbb{Z}^d.$$

For $1 \leq p, q \leq \infty$, let $\mathcal{F}L^p(Q_{1/\beta}, \mathbb{H})$ denote the image of $L^p(Q_{1/\beta}, \mathbb{H})$ under the Fourier transform. Since Fourier coefficients are unique in $L^p(Q_{1/\beta}, \mathbb{H})$, if $d = (d_n)_{n\in\mathbb{Z}^d} \in \mathcal{F}L^p(Q_{1/\beta}, \mathbb{H})$ then there exists a unique function $\mathbf{m} \in L^p(Q_{1/\beta}, \mathbb{H})$ such that $\hat{\mathbf{m}}(n) = d_n$ for every n, and the norm on $\mathcal{F}L^p(Q_{1/\beta}, \mathbb{H})$ is defined by

$$\|d\|_{\mathcal{F}L^p(Q_{1/\beta},\mathbb{H})} = \|\mathbf{m}\|_{p,Q_{1/\beta}}. \tag{3.7}$$

Definition 3.2.2. *Let* $\alpha, \beta > 0$ *be given. Then* $S^{p,q}_{\tilde{v}}(\mathbb{H}) = \ell^q_{\tilde{v}}(\mathcal{F}L^p(Q_{1/\beta}, \mathbb{H}))$ *denotes the space of all* $\mathcal{F}L^p(Q_{1/\beta}, \mathbb{H})$*-valued sequences which are* $\ell^q_{\tilde{v}}$*-summable. That is, a doubly-indexed sequence* $d = (d_{kn})_{k,n\in\mathbb{Z}^d}$ *lies in* $S^{p,q}_{\tilde{v}}(\mathbb{H})$ *if for each* $k \in \mathbb{Z}^d$ *there exists* $\mathbf{m}_k \in L^p(Q_{1/\beta}, \mathbb{H})$ *such that* $\hat{\mathbf{m}}_k(n) = d_{kn},\ k, n \in \mathbb{Z}^d$ *and*

$$\|d\|_{S^{p,q}_{\tilde{v}}(\mathbb{H})} = \Big(\sum_{k\in\mathbb{Z}^d} \|\mathbf{m}_k\|^q_{p,Q_{1/\beta}}\tilde{v}(k)^q\Big)^{1/q} < \infty,$$

with the usual change if $q = \infty$.

When $1 < p < \infty$, we can write $\mathbf{m}_k$ as a Fourier series

$$\mathbf{m}_k(x) = \sum_{n\in\mathbb{Z}^d} d_{kn}e^{2\pi i\beta\langle n,x\rangle}, \tag{3.8}$$

in the sense that the square partial sums of (3.8) converge to $\mathbf{m}_k$ in the norm of $L^p(Q_{1/\beta},\mathbb{H})$, cf. [101, 146]. Hence, for $1<p<\infty$ and $1\le q<\infty$ we can write the norm on $S^{p,q}_{\tilde{v}}(\mathbb{H})$ as

$$\|d\|_{S^{p,q}_{\tilde{v}}(\mathbb{H})}=\Big(\sum_{k\in\mathbb{Z}^d}\Big(\int_{Q_{1/\beta}}\|\sum_{n\in\mathbb{Z}^d}d_{kn}e^{2\pi i\beta\langle n,x\rangle}\|^p_{\mathbb{H}}dx\Big)^{q/p}\tilde{v}(k)^q\Big)^{1/q}.$$

The analysis and synthesis operators associated with the $\mathbb{H}$-valued Gabor frame are defined as follows: Take $\mathbf{h}\in\mathbb{H}$ such that $\|\mathbf{h}\|_{\mathbb{H}}=1$. Let $\alpha,\beta>0$, $1\le p,q\le\infty$ and fix $\mathbf{g},\boldsymbol{\gamma}\in W_{\mathbb{H}}(L^\infty,L^1_w)$. For $\mathbf{f}\in W_{\mathbb{H}}(L^p,L^q_v)$, define the analysis operator by

$$\begin{aligned}C_{\mathbf{g},\mathbf{h}}f(k,n) &= \langle\mathbf{f},M_{\beta n}T_{\alpha k}\mathbf{g}\rangle\mathbf{h}\\ &= \int_{\mathbb{R}^d}\langle\mathbf{f}(x),M_{\beta n}T_{\alpha k}\mathbf{g}(x)\rangle_{\mathbb{H}}\mathbf{h}dx\\ &= \int_{\mathbb{R}^d}\langle\mathbf{f}(x),T_{\alpha k}\mathbf{g}(x)\rangle_{\mathbb{H}}\mathbf{h}e^{-2\pi i\beta\langle n,x\rangle}dx\\ &= \mathcal{F}(\langle\mathbf{f},T_{\alpha k}\mathbf{g}\rangle\mathbf{h})(\beta n)=\mathcal{F}((\mathbf{h}\odot T_{\alpha k}\mathbf{g})\mathbf{f})(\beta n).\end{aligned}$$

So

$$\sum_{n\in\mathbb{Z}^d}\langle\mathbf{f},M_{\beta n}T_{\alpha k}\mathbf{g}\rangle\mathbf{h}e^{2\pi i\beta\langle n,x\rangle} = \sum_{n\in\mathbb{Z}^d}\mathcal{F}((\mathbf{h}\odot T_{\alpha k}\mathbf{g})\mathbf{f})(\beta n)e^{2\pi i\beta\langle n,x\rangle}.$$

By using Poisson summation formula we get,

$$\sum_{n\in\mathbb{Z}^d}\langle\mathbf{f},M_{\beta n}T_{\alpha k}\mathbf{g}\rangle\mathbf{h}e^{2\pi i\beta\langle n,x\rangle} = \beta^{-d}\sum_{n\in\mathbb{Z}^d}((\mathbf{h}\odot T_{\alpha k}\mathbf{g}(x))\mathbf{f})(x-\frac{n}{\beta})=\mathbf{m}_k(x).$$

Let $d=(d_{kn})\in S^{p,q}_{\tilde{v}}(\mathbb{H})$. The synthesis operator is defined by

$$\begin{aligned}R_{\mathbf{g},\mathbf{h}}d(x) &= \sum_{k,n\in\mathbb{Z}^d}\langle d_{kn},\mathbf{h}\rangle_{\mathbb{H}}M_{\beta n}T_{\alpha k}\mathbf{g}(x)\\ &= \sum_{k\in\mathbb{Z}^d}\Big\langle\sum_{n\in\mathbb{Z}^d}d_{kn}e^{2\pi i\beta\langle n,x\rangle},\mathbf{h}\Big\rangle_{\mathbb{H}}T_{\alpha k}\mathbf{g}(x) \qquad (3.9)\\ &= \sum_{k\in\mathbb{Z}^d}\langle\mathbf{m}_k(x),\mathbf{h}\rangle_{\mathbb{H}}T_{\alpha k}\mathbf{g}(x),\end{aligned}$$

where $\mathbf{m}_k(x) = \sum_{n\in\mathbb{Z}^d} d_{kn} e^{2\pi i\beta\langle n,x\rangle}$ is $1/\beta$-periodic.

From the above observations we obtain the analogue of the Walnut's representation for the $\mathbb{H}$-valued Gabor frames in the following theorem.

Theorem 3.2.3. *Let v be an w-moderate weight on $\mathbb{R}^d$. Let $\alpha, \beta > 0$ and $1 \leq p, q \leq \infty$ be given. Fix $\mathbf{g}, \boldsymbol{\gamma} \in W_{\mathbb{H}}(L^\infty, L^1_w)$ and $\mathbf{h} \in \mathbb{H}$ with $\|\mathbf{h}\|_{\mathbb{H}} = 1$. Then the following statements hold.*

(i) The analysis operator $C_{\mathbf{g},\mathbf{h}}\mathbf{f} = (\langle \mathbf{f}, M_{\beta n}T_{\alpha k}\mathbf{g}\rangle \mathbf{h})_{k,n\in\mathbb{Z}^d}$ is a bounded mapping $C_{\mathbf{g},\mathbf{h}} : W_{\mathbb{H}}(L^p, L^q_v) \to S^{p,q}_{\tilde{v}}(\mathbb{H})$, Moreover, there exist unique functions $\mathbf{m}_k \in L^p(Q_{1/\beta}, \mathbb{H})$ which satisfy $\hat{\mathbf{m}}_k(n) = C_{\mathbf{g},\mathbf{h}}\mathbf{f}(k,n)$ for all $k, n \in \mathbb{Z}^d$, and these are given explicitly by

$$\begin{aligned} \mathbf{m}_k(x) &= \beta^{-d} \sum_{n\in\mathbb{Z}^d} (\mathbf{h} \odot T_{\alpha k}\mathbf{g}(x))\mathbf{f}\Big(x - \frac{n}{\beta}\Big) \\ &= \beta^{-d} \sum_{n\in\mathbb{Z}^d} (\mathbf{h} \odot T_{\alpha k + \frac{n}{\beta}}\mathbf{g}(x)) T_{\frac{n}{\beta}}\mathbf{f}(x). \end{aligned} \tag{3.10}$$

The series on the right side of (3.10) converges unconditionally in $L^p(Q_{1/\beta}, \mathbb{H})$ (unconditionally in the $\sigma(L^\infty(Q_{1/\beta}, \mathbb{H}), L^1(Q_{1/\beta}, \mathbb{H}))$ topology if $p = \infty$).

(ii) Given $d \in S^{p,q}_{\tilde{v}}(\mathbb{H})$, let $\mathbf{m}_k \in L^p(Q_\alpha, \mathbb{H})$ be the unique functions satisfying $\hat{\mathbf{m}}_k(n) = d_{kn}$ for all $k, n \in \mathbb{Z}^d$. Then the series

$$R_{\mathbf{g},\mathbf{h}}d = \sum_{k\in\mathbb{Z}^d} \langle \mathbf{m}_k(\cdot), \mathbf{h}\rangle_{\mathbb{H}} T_{ak}\mathbf{g}, \tag{3.11}$$

converges unconditionally in $W_{\mathbb{H}}(L^p, L^q_v)$ (unconditionally in the $\sigma(W_{\mathbb{H}}(L^p, L^q_v), W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v}))$ topology if $p = \infty$ or $q = \infty$), and $R_{\mathbf{g},\mathbf{h}}$ is a bounded mapping $R_{\mathbf{g},\mathbf{h}} : S^{p,q}_{\tilde{v}}(\mathbb{H}) \to W_{\mathbb{H}}(L^p, L^q_v)$.

(iii) The Walnut's representation

$$R_{\boldsymbol{\gamma},\mathbf{h}}C_{\mathbf{g},\mathbf{h}}\mathbf{f} = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n\Big(T_{\frac{n}{\beta}}\mathbf{f}\Big), \tag{3.12}$$

holds for $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$, with the series on the right of (3.12) converging *absolutely in* $W_{\mathbb{H}}(L^p, L^q_v)$*, where*

$$G_n(x) = \sum_{k\in\mathbb{Z}^d} T_{\alpha k}\boldsymbol{\gamma}(x) \odot T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x) \in B(\mathbb{H}). \tag{3.13}$$

Remark 3.2.4. If $\mathbf{g}, \boldsymbol{\gamma} \in W_{\mathbb{H}}(L^\infty, L^1_w)$, then from the above theorem the Walnut's representation of the frame operator on $W_{\mathbb{H}}(L^p, L^q_v)$ is

$$S_{\mathbf{g},\boldsymbol{\gamma}}\mathbf{f}(x) = \beta^{-d}\sum_{n\in\mathbb{Z}^d} G_n(x)\Big(T_{\frac{n}{\beta}}\mathbf{f}(x)\Big). \tag{3.14}$$

Since we deal with vector valued functions, obtaining the above expression is bit technical. Notice that in the Walnut's representation for the scalar valued case (see [82]), the summation is taken over point-wise product of $G_n(x)$ with $T_{\frac{n}{\beta}}f(x)$, whereas in our case $G_n(x)$ is operating on the Hilbert space element $T_{\frac{n}{\beta}}\mathbf{f}(x)$.

To prove Theorem 3.2.3 we need the following results. Since the proof of these results follows in a similar way as in scalar valued case we only state them without proof.

Lemma 3.2.5. *([82], Lemma 5.1) Let* $\alpha, \beta > 0$ *be given. Let* $K_{\alpha\beta}$ *be the maximum number of* $\frac{1}{\beta}\mathbb{Z}^d$*-translates of* $Q_{1/\beta}$ *required to cover any* $\alpha\mathbb{Z}^d$*-translate of* Q_α*, i.e.,* $K_{\alpha\beta} = \max_{k\in\mathbb{Z}^d} \#\{\ell \in \mathbb{Z}^d : |(\frac{\ell}{\beta} + Q_{1/\beta}) \cap (\alpha k + Q_\alpha)| > 0\}$*. Then given* $1 \le p \le \infty$*, we have for any* $1/\beta$*-periodic function* $\mathbf{m} \in L^p(Q_{1/\beta}, \mathbb{H})$ *and any* $k \in \mathbb{Z}^d$ *that* $\|\mathbf{m}\|_{p,\alpha k+Q_\alpha} \le K_{\alpha\beta}^{1/p}\|\mathbf{m}\|_{p,Q_{1/\beta}}$*, where* $K_{\alpha\beta}^{1/\infty} = 1$*.*

Lemma 3.2.6. *If* $\mathbf{g} \in W(L^\infty(\mathbb{R}^d, B(\mathbb{H})), L^1)$ *and* $\alpha > 0$ *then*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} \|\mathbf{g}(x-\alpha n)\|_{B(\mathbb{H})} \le \Big(\frac{1}{\alpha}+1\Big)^d \|\mathbf{g}\|_{W(L^\infty(\mathbb{R}^d,B(\mathbb{H})),L^1)}. \tag{3.15}$$

Lemma 3.2.7. *If* $\mathbf{g}, \boldsymbol{\gamma} \in W(L^\infty, L^1)$*, then* G_n *are defined by (3.13) is in* $L^\infty(\mathbb{R}^d, B(\mathbb{H}))$*, and*

$$\sum_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))} \le \Big(\frac{1}{\alpha}+1\Big)^d (2\beta+2)^d\ \|\mathbf{g}\|_{W(L^\infty,L^1)}\|\boldsymbol{\gamma}\|_{W(L^\infty,L^1)}.$$

The next lemma is a weighted version of previous lemma that is useful in the Walnut's representation of the frame operator (see [135], Lemma 2.2).

Lemma 3.2.8. *Let w be a submultiplicative weight, and let $\alpha, \beta > 0$ be given. Then there exists a constant $C = C(\alpha, \beta, w) > 0$ such that if $\mathbf{g}, \boldsymbol{\gamma} \in W_{\mathbb{H}}(L^\infty, L^1_w)$ and the functions G_n are defined by (3.13), then*

$$\sum_{n \in \mathbb{Z}^d} \|G_n\|_{L^\infty(\mathbb{R}^d, B(\mathbb{H}))} w\Big(\frac{n}{\beta}\Big) \leq C \, \|\mathbf{g}\|_{W_{\mathbb{H}}(L^\infty, L^1_w)} \|\boldsymbol{\gamma}\|_{W_{\mathbb{H}}(L^\infty, L^1_w)}.$$

Following lemma is an estimate on the effect of translations on the amalgam space norm.

Lemma 3.2.9. *Let v be an w-moderate weight. Then for $1 \leq p, q \leq \infty$, we have for each $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$ and $n \in \mathbb{Z}^d$ that*

$$\|T_{\alpha n}\mathbf{f}\|_{W_{\mathbb{H}}(L^p, L^q_v)} \leq C_v w(\alpha n) \|\mathbf{f}\|_{W_{\mathbb{H}}(L^p, L^q_v)}.$$

The structural results about ℍ-valued Gabor frames can be derived from the corresponding well-known results for scalar valued Gabor frames. Now we will present some important results on ℍ-valued Gabor frames in the following remark.

Remark 3.2.10. The l-th Fourier coefficient of G_n is

$$\begin{aligned} \hat{G}_n(l) &= \alpha^{-d} \int_{Q_\alpha} G_n(x) e^{-2\pi i \langle l, x/\alpha \rangle} dx \\ &= \alpha^{-d} \int_{Q_\alpha} \Big(\sum_{k \in \mathbb{Z}^d} T_{ak}\boldsymbol{\gamma}(x) \odot T_{\alpha k + \frac{n}{\beta}} \mathbf{g}(x)\Big) e^{-2\pi i \langle l, x/\alpha \rangle} dx \\ &= \alpha^{-d} \int_{\mathbb{R}^d} (\boldsymbol{\gamma}(x) \odot T_{\frac{n}{\beta}} \mathbf{g}(x)) e^{-2\pi i \langle l, x/\alpha \rangle} dx \\ &= \alpha^{-d} \int_{\mathbb{R}^d} \boldsymbol{\gamma}(x) \odot M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} \mathbf{g}(x) dx := \alpha^{-d} [\boldsymbol{\gamma}, M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} \mathbf{g}]. \end{aligned}$$

Then Fourier series

$$G_n(x) = \alpha^{-d} \sum_{l \in \mathbb{Z}^d} [\boldsymbol{\gamma}, M_{\frac{l}{\alpha}} T_{\frac{n}{\beta}} \mathbf{g}] e^{2\pi i \langle l, x/\alpha \rangle}, \tag{3.16}$$

is convergent in $L^2(Q_\alpha, B(\mathbb{H}))$. By substituting this into Walnut's representation, we obtain the expression

$$S_{\mathbf{g},\boldsymbol{\gamma}}\mathbf{f} = \beta^{-d}\sum_{n\in\mathbb{Z}^d} G_n\Big(T_{\frac{n}{\beta}}\mathbf{f}\Big) = (\alpha\beta)^{-d}\sum_{n\in\mathbb{Z}^d}\Big(\sum_{l\in\mathbb{Z}^d}[\boldsymbol{\gamma}, M_{\frac{l}{\alpha}}T_{\frac{n}{\beta}}\mathbf{g}]\Big)\Big(M_{\frac{l}{\alpha}}T_{\frac{n}{\beta}}\mathbf{f}\Big),$$

or in operator notation,

$$S_{\mathbf{g},\boldsymbol{\gamma}} = (\alpha\beta)^{-d}\sum_{n\in\mathbb{Z}^d}\Big(\sum_{l\in\mathbb{Z}^d}[\boldsymbol{\gamma}, M_{\frac{l}{\alpha}}T_{\frac{n}{\beta}}\mathbf{g}]\Big)\Big(M_{\frac{l}{\alpha}}T_{\frac{n}{\beta}}\Big). \tag{3.17}$$

This is $\mathbb{H}$-valued analogue of *Janssen's representation* for the $\mathbb{H}$-valued frame operator $S_{\mathbf{g},\boldsymbol{\gamma}}$. Using Janssen's representation we obtain the $\mathbb{H}$-valued analogue of Wexler-Raz biorthogonality relation in the following theorem.

Theorem 3.2.11. (Wexler-Raz biorthogonality relation). *Assume that* $\mathcal{G}(\mathbf{g},\alpha,\beta)$, $\mathcal{G}(\boldsymbol{\gamma},\alpha,\beta)$ *are Bessel sequence in* $L^2(\mathbb{R}^d,\mathbb{H})$*. Then the following conditions are equivalent:*

(i) $S_{\mathbf{g},\boldsymbol{\gamma}} = S_{\boldsymbol{\gamma},\mathbf{g}} = I$ *on* $L^2(\mathbb{R}^d,\mathbb{H})$.

(ii) $(\alpha\beta)^{-d}[\boldsymbol{\gamma}, M_{\frac{l}{\alpha}}T_{\frac{n}{\beta}}\mathbf{g}] = \delta_{l0}\delta_{n0}I_{B(\mathbb{H})}$ *for* $l,n\in\mathbb{Z}^d$.

Proof. The implication $(ii)\Rightarrow(i)$ is trivial consequence of Janssen's representation. For the converse $(i)\Rightarrow(ii)$, assume that $S_{\mathbf{g},\boldsymbol{\gamma}} = I$. Let $\mathbf{f},\mathbf{h}\in L^\infty(Q_{1/\beta},\mathbb{H})$ and let $l,m\in\mathbb{Z}^d$ be arbitrary. Then

$$\begin{aligned}
\delta_{lm}[\mathbf{f},\mathbf{h}] &= \delta_{lm}\int_{\mathbb{R}^d}\mathbf{f}(x)\odot\mathbf{h}(x)dx = \delta_{lm}\int_{\mathbb{R}^d}S_{\mathbf{g},\boldsymbol{\gamma}}\mathbf{f}(x)\odot\mathbf{h}(x)dx\\
&= \int_{\mathbb{R}^d}S_{\mathbf{g},\boldsymbol{\gamma}}T_{\frac{l}{\beta}}\mathbf{f}(x)\odot T_{\frac{m}{\beta}}\mathbf{h}(x)dx\\
&= \beta^{-d}\int_{\mathbb{R}^d}\sum_{n\in\mathbb{Z}^d}G_n(x)\Big(T_{\frac{n+l}{\beta}}\mathbf{f}(x)\Big)\odot T_{\frac{m}{\beta}}\mathbf{h}(x)dx\\
&= \beta^{-d}\int_{\mathbb{R}^d}G_{m-l}(x)\Big(T_{\frac{m}{\beta}}\mathbf{f}(x)\Big)\odot T_{\frac{m}{\beta}}\mathbf{h}(x)dx\\
&= \beta^{-d}\int_{\mathbb{R}^d}(T_{-\frac{m}{\beta}}G_{m-l}(x))\mathbf{f}(x)\odot\mathbf{h}(x)dx\\
&= \beta^{-d}[(T_{-\frac{m}{\beta}}G_{m-l})(\mathbf{f}),\mathbf{h}].
\end{aligned}$$

By density this identity extends to $\mathbf{f},\mathbf{h}\in L^2(Q_{1/\beta},\mathbb{H})$, so we conclude that

$$\beta^{-d}G_{m-l}\Big(x+\frac{m}{\beta}\Big)=\delta_{lm}I_{B(\mathbb{H})},$$

for almost all $x\in Q_{1/\beta}$. Varying $l,m\in\mathbb{Z}^d$ it follows that $G_n(x)=\beta^d\delta_{n0}I_{B(\mathbb{H})}$, for almost all $x\in\mathbb{R}^d$. Therefore by (3.16) and uniqueness of Fourier coefficients we conclude that $(\alpha\beta)^{-d}[\boldsymbol{\gamma},M_{\frac{l}{\alpha}}T_{\frac{n}{\beta}}\mathbf{g}]=\delta_{l0}\delta_{n0}I_{B(\mathbb{H})}$. ☐

Now we will prove boundedness of the analysis and synthesis operator, and Walnut's representation in the following proof.

Proof of Theorem 3.2.3. (*i*) Given that $\mathbf{g}\in W_{\mathbb{H}}(L^\infty,L^1_w)$ and $1\le p,q\le\infty$. Let $\mathbf{f}\in W_{\mathbb{H}}(L^p,L^q_v)$, which is a subspace of $W(L^1,L^\infty_{1/w})$. First we show that the operators $\mathbf{m}_k$ given by (3.10) are well-defined. Observe that $\mathbf{m}_k$ is the $1/\beta$-periodization of the integrable $\mathbb{H}$-valued function $(\mathbf{h}\odot T_{\alpha k}\mathbf{g})\mathbf{f}$ and hence the series defining $\mathbf{m}_k$ converges in $L^1(Q_\alpha,\mathbb{H})$. To show that the periodization converges unconditionally in $L^p(Q_{1/\beta},\mathbb{H})$ (weakly if $p=\infty$) and to derive a useful estimate, fix any $1/\beta$-periodic function $\mathbf{h}'\in L^{p'}(Q_{1/\beta},\mathbb{H})$ and for each fixed k, we consider

$$\begin{aligned}
&\int_{Q_{1/\beta}}|\langle\sum_{n\in\mathbb{Z}^d}(\mathbf{h}\odot T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x))T_{\frac{n}{\beta}}\mathbf{f}(x),\mathbf{h}'(x)\rangle_{\mathbb{H}}|dx\\
&\le\int_{Q_{1/\beta}}\sum_{n\in\mathbb{Z}^d}|\langle\langle T_{\frac{n}{\beta}}\mathbf{f}(x),T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x)\rangle_{\mathbb{H}}\mathbf{h},\mathbf{h}'(x)\rangle_{\mathbb{H}}|dx\\
&=\int_{\mathbb{R}^d}|\langle\mathbf{f}(x),T_{\alpha k}\mathbf{g}(x)\rangle_{\mathbb{H}}\langle\mathbf{h},\mathbf{h}'(x)\rangle_{\mathbb{H}}|dx\\
&=\sum_{n\in\mathbb{Z}^d}\int_{Q_\alpha}|\langle\mathbf{f}(x),T_{\alpha k}\mathbf{g}(x)\rangle_{\mathbb{H}}\langle\mathbf{h},\mathbf{h}'(x)\rangle_{\mathbb{H}}|T_{\alpha k+\alpha n}\chi_{Q_\alpha}(x)dx\\
&\le\sum_{n\in\mathbb{Z}^d}\|T_{\alpha k}\mathbf{g}\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})}\|\mathbf{f}\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p\times\\
&\qquad\qquad\|\mathbf{h}'\|_{p',\alpha k+\alpha n+Q_\alpha}\frac{v(\alpha k+\alpha n-\alpha n)}{v(\alpha k)}\\
&\le\sum_{n\in\mathbb{Z}^d}\|\mathbf{g}\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})}\|\mathbf{f}\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p\times\\
&\qquad\qquad K_{\alpha\beta}^{1/p'}\|\mathbf{h}'\|_{p',Q_{1/\beta}}\frac{C_v v(\alpha k+\alpha n)w(\alpha n)}{v(\alpha k)}
\end{aligned}$$

$$= C_v K_{\alpha\beta}^{1/p'} \|\mathbf{h}'\|_{p',Q_{1/\beta}} \frac{1}{v(\alpha k)} \sum_{n\in\mathbb{Z}^d} \|\mathbf{g}\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})} w(\alpha n)\times$$
$$\|\mathbf{f}\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p v(\alpha k+\alpha n). \tag{3.18}$$

Taking the supremum in (3.18) over $\mathbf{h}'$ with unit norm we get,

$$\|\mathbf{m}_k\|_{p,Q_{1/\beta}} \leq \beta^{-d} C_v K_{\alpha\beta}^{1/p'} \frac{1}{v(\alpha k)} \sum_{n\in\mathbb{Z}^d} \|\mathbf{g}\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})} w(\alpha n)\times$$
$$\|\mathbf{f}\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p v(\alpha k+\alpha n), \tag{3.19}$$

where $K_{\alpha\beta} = \max_{k\in\mathbb{Z}^d} \#\{\ell \in \mathbb{Z}^d : |\left(\frac{\ell}{\beta} + Q_{1/\beta}\right) \cap (\alpha k + Q_\alpha)| > 0\}$. This shows the convergence of the series defining $\mathbf{m}_k$ in $L^p(Q_{1/\beta}, \mathbb{H})$. Note that,

$$\begin{aligned}
\hat{\mathbf{m}}_k(n) &= \beta^d \int_{Q_{1/\beta}} \mathbf{m}_k(x) e^{-2\pi i\beta\langle n,x\rangle} dx \\
&= \int_{Q_{1/\beta}} \sum_{m\in\mathbb{Z}^d} (\mathbf{h}\odot T_{\alpha k+\frac{m}{\beta}}\mathbf{g}(x)) T_{\frac{m}{\beta}}\mathbf{f}(x) e^{-2\pi i\beta\langle n,x\rangle} dx \\
&= \int_{Q_{1/\beta}} \sum_{m\in\mathbb{Z}^d} \langle T_{\frac{m}{\beta}}\mathbf{f}(x), T_{\alpha k+\frac{m}{\beta}}\mathbf{g}(x)\rangle_{\mathbb{H}} \mathbf{h} e^{-2\pi i\beta\langle n,x-m/\beta\rangle} dx \\
&= \int_{\mathbb{R}^d} \langle \mathbf{f}(x), T_{\alpha k}\mathbf{g}(x)\rangle_{\mathbb{H}} \mathbf{h} e^{-2\pi i\beta\langle n,x\rangle} dx \\
&= \langle \mathbf{f}, M_{\beta n}T_{\alpha k}\mathbf{g}\rangle \mathbf{h} = C_{\mathbf{g},\mathbf{h}}\mathbf{f}(k,n).
\end{aligned}$$

Finally, we show that $C_{\mathbf{g},\mathbf{h}}$ is a bounded mapping of $W_{\mathbb{H}}(L^p, L^q_v)$ into $S^{p,q}_{\tilde{v}}(\mathbb{H})$. Given $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$, to show that $C_{\mathbf{g},\mathbf{h}}\mathbf{f} \in S^{p,q}_{\tilde{v}}(\mathbb{H})$ it is enough to show that the sequence r given by $r(k) = \|\mathbf{m}_k\|_{p,Q_{1/\beta}}$, $k \in \mathbb{Z}^d$ lies in $\ell^q_{\tilde{v}}$. To do this, fix any sequence $a \in \ell^{q'}_{1/\tilde{v}}$. Then by using (3.19), we have

$$\begin{aligned}
|\langle r, a\rangle| &\leq \sum_{k\in\mathbb{Z}^d} \|\mathbf{m}_k\|_{p,Q_{1/\beta}} |a(k)| \\
&\leq \beta^{-d} C_v K_{\alpha\beta}^{1/p'} \sum_{n\in\mathbb{Z}^d} \|\mathbf{g}\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})} w(\alpha n)\times
\end{aligned}$$

$$\sum_{k\in\mathbb{Z}^d}\|\mathbf{f}\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p v(\alpha k+\alpha n)|a(k)|\frac{1}{v(\alpha k)}$$
$$\leq \beta^{-d}C_vK_{\alpha\beta}^{1/p'}\sum_{n\in\mathbb{Z}^d}\|\mathbf{g}\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})}w(\alpha n)\times$$
$$\Big(\sum_{k\in\mathbb{Z}^d}\|\mathbf{f}\cdot T_{\alpha k+\alpha n}\chi_{Q_\alpha}\|_p^q v(\alpha k+\alpha n)^q\Big)^{1/q}\Big(\sum_{k\in\mathbb{Z}^d}|a(k)|^{q'}\frac{1}{v(\alpha k)^{q'}}\Big)^{1/q'}$$
$$\leq \beta^{-d}C_vK_{\alpha\beta}^{1/p'}\|\mathbf{g}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)}\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)}\|a\|_{\ell^{q'}_{1/\tilde{v}}}. \tag{3.20}$$

Since $\ell^{q'}_{1/\tilde{v}}=(\ell^q_{\tilde{v}})^*$ when $q<\infty$ and is a norm-fundamental subspace when $q=\infty$, taking the supremum over sequences a with unit norm in (3.20) we get

$$\|C_{\mathbf{g},\mathbf{h}}\mathbf{f}\|_{S^{p,q}_{\tilde{v}}(\mathbb{H})}=\|r\|_{\ell^q_{\tilde{v}}}\leq \beta^{-d}C_vK_{\alpha\beta}^{1/p'}\|\mathbf{g}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)}\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)}.$$

Hence $C_{\mathbf{g},\mathbf{h}}$ is a bounded mapping of $W_{\mathbb{H}}(L^p,L^q_v)$ into $S^{p,q}_{\tilde{v}}(\mathbb{H})$. This proves (i).

(ii) Let $1\leq p,q<\infty$. Given $d\in S^{p,q}_{\tilde{v}}(\mathbb{H})$, we have $\sum_{k\in\mathbb{Z}^d}\|\mathbf{m}_k\|^q_{p,Q_{1/\beta}}\tilde{v}(k)^q<\infty$. That means for every $\varepsilon>0$, there exists a finite set F_0 such that

$$\sum_{k\notin F}\|\mathbf{m}_k\|^q_{p,Q_{1/\beta}}\tilde{v}(k)^q<\varepsilon^q,\quad \forall \text{ finite } F\supset F_0. \tag{3.21}$$

Since $1/v$ is also an w-moderate weight, for any fix $\mathbf{h}'\in W_{\mathbb{H}}(L^{p'},L^{q'}_{1/v})$, we have

$$\sum_{k\notin F}|\langle\langle\mathbf{m}_k(\cdot),\mathbf{h}\rangle_{\mathbb{H}}T_{\alpha k}\mathbf{g},\mathbf{h}'\rangle|$$
$$\leq\sum_{k\notin F}\int_{\mathbb{R}^d}|\langle\langle\mathbf{m}_k(x),\mathbf{h}\rangle_{\mathbb{H}}T_{\alpha k}\mathbf{g}(x),\mathbf{h}'(x)\rangle_{\mathbb{H}}|dx$$
$$=\sum_{k\notin F}\sum_{n\in\mathbb{Z}^d}\int_{Q_\alpha}|\langle\mathbf{m}_k(x),\mathbf{h}\rangle_{\mathbb{H}}\langle T_{\alpha k}\mathbf{g}(x),\mathbf{h}'(x)\rangle_{\mathbb{H}}|T_{\alpha n+\alpha k}\chi_{Q_\alpha}(x)dx$$
$$\leq\sum_{k\notin F}\sum_{n\in\mathbb{Z}^d}\|T_{\alpha k}\mathbf{g}\cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})}\|\mathbf{m}_k\|_{p,\alpha n+\alpha k+Q_\alpha}\times$$
$$\|\mathbf{h}'\cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|_{p'}\ \frac{v(\alpha k)}{v(\alpha n+\alpha k-\alpha n)}$$

$$\leq \sum_{n\in\mathbb{Z}^d} \|\mathbf{g}\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})}\times$$
$$\sum_{k\notin F} K_{\alpha\beta}^{1/p}\|\mathbf{m}_k\|_{p,Q_{1/\beta}}\|\mathbf{h}'\cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|_{p'}\ \frac{C_v v(\alpha k)w(\alpha n)}{v(\alpha n+\alpha k)}$$
$$\leq C_v K_{\alpha\beta}^{1/p}\sum_{n\in\mathbb{Z}^d}\|\mathbf{g}\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^\infty(\mathbb{R}^d,\mathbb{H})}w(\alpha n)\Big(\sum_{k\notin F}\|\mathbf{m}_k\|^q_{p,Q_{1/\beta}}v(\alpha k)^q\Big)^{1/q}\times$$
$$\Big(\sum_{k\in\mathbb{Z}^d}\|\mathbf{h}'\cdot T_{\alpha n+\alpha k}\chi_{Q_\alpha}\|^{q'}_{p'}\frac{1}{v(\alpha n+\alpha k)^{q'}}\Big)^{1/q'}. \tag{3.22}$$

Combining (3.21) and (3.22), we get

$$\sum_{k\notin F}|\langle\langle\mathbf{m}_k(\cdot),\mathbf{h}\rangle_{\mathbb{H}}T_{\alpha k}\mathbf{g},\mathbf{h}'\rangle| \leq \epsilon C_v K_{\alpha\beta}^{1/p}\|\mathbf{g}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)}\|\mathbf{h}'\|_{W_{\mathbb{H}}(L^{p'},L^{q'}_{1/v})}.$$

Taking the supremum over all $\mathbf{h}'$ of unit norm, we see that

$$R_{\mathbf{g},\mathbf{h}}d=\sum_{k\in\mathbb{Z}^d}\langle\mathbf{m}_k(\cdot),\mathbf{h}\rangle_{\mathbb{H}}T_{ak}\mathbf{g}$$

converges unconditionally. Now replacing F by $\mathbb{Z}^d$ in (3.22), we get

$$\begin{aligned}|\langle R_{\mathbf{g},\mathbf{h}}d,\mathbf{h}'\rangle| &\leq \sum_{k\in\mathbb{Z}^d}|\langle\langle\mathbf{m}_k(\cdot),\mathbf{h}\rangle_{\mathbb{H}}T_{\alpha k}\mathbf{g},\mathbf{h}'\rangle|\\ &\leq C_v K_{\alpha\beta}^{1/p}\|\mathbf{g}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)}\|d\|_{S^{p,q}_{\tilde{v}}(\mathbb{H})}\|\mathbf{h}'\|_{W_{\mathbb{H}}(L^{p'},L^{q'}_{1/v})}.\end{aligned} \tag{3.23}$$

Since $W_{\mathbb{H}}(L^{p'},L^{q'}_{1/v})$ is the dual space of $W_{\mathbb{H}}(L^p,L^q_v)$, taking the supremum over all $\mathbf{h}'$ of unit norm in (3.23) shows that

$$\begin{aligned}\|R_{\mathbf{g},\mathbf{h}}d\|_{W_{\mathbb{H}}(L^p,L^q_v)} &= \sup\{|\langle R_{\mathbf{g},\mathbf{h}}d,\mathbf{h}'\rangle| : \|\mathbf{h}'\|_{W_{\mathbb{H}}(L^{p'},L^{q'}_{1/v})}=1\}\\ &\leq C_v K_{\alpha\beta}^{1/p}\|\mathbf{g}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)}\|d\|_{S^{p,q}_{\tilde{v}}(\mathbb{H})},\end{aligned} \tag{3.24}$$

This completes the proof for the case $1\leq p,q<\infty$. Since $W_{\mathbb{H}}(L^{p'},L^{q'}_{1/v})$ is the Köthe dual of $W_{\mathbb{H}}(L^p,L^q_v)$ a similar argument as in (3.22), (3.23) will imply the convergence of $R_{\mathbf{g},\mathbf{h}}$ in the weak topology when $p = \infty$ or $q = \infty$ and the estimate $\|R_{\mathbf{g},\mathbf{h}}d\|_{W_{\mathbb{H}}(L^p,L^q_v)}$ as in (3.24). This proves part (*ii*).

(*iii*) Now we show the frame operator $R_{\boldsymbol{\gamma},\mathbf{h}}C_{\mathbf{g},\mathbf{h}}$ admits Walnut's representation on $W_{\mathbb{H}}(L^p, L^q_v)$. Given $\mathbf{g}, \boldsymbol{\gamma} \in W_{\mathbb{H}}(L^\infty, L^1_w)$ and $1 \le p, q \le \infty$. Notice that for a w-moderate weight v and $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$, $1 \le p, q \le \infty$ and for each $n \in \mathbb{Z}^d$, $\alpha > 0$ we have

$$\|T_{\alpha n}\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)} \le C_v w(\alpha n)\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)}. \tag{3.25}$$

Replacing α by $1/\beta$ in (3.25) we get,

$$\|T_{\frac{n}{\beta}}\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)} \le C_v w\Big(\frac{n}{\beta}\Big)\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)}.$$

Therefore, for $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$ consider

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d} \|G_n\left(T_{\frac{n}{\beta}}\mathbf{f}\right)\|_{W_{\mathbb{H}}(L^p,L^q_v)} \\
\le\ & \sum_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}\|T_{\frac{n}{\beta}}\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)} \\
\le\ & C_v\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)} \sum_{n\in\mathbb{Z}^d} \|G_n\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))} w\Big(\frac{n}{\beta}\Big) \\
\le\ & CC_v\|\mathbf{f}\|_{W_{\mathbb{H}}(L^p,L^q_v)}\|\mathbf{g}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)}\|\boldsymbol{\gamma}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)},
\end{aligned}$$

by Lemma 3.2.8. Therefore the series $\sum_{n\in\mathbb{Z}^d} G_n\left(T_{\frac{n}{\beta}}\mathbf{f}\right)$ converges absolutely in $W_{\mathbb{H}}(L^p, L^q_v)$. Now for fixed $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$, define $\mathbf{m}_k$ such that $c_{\mathbf{g},\mathbf{h}}\mathbf{f}(k, n) = \hat{\mathbf{m}}_k(n)$. For $\mathbf{h}' \in W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v})$ we have

$$\begin{aligned}
&\langle R_{\boldsymbol{\gamma},\mathbf{h}}C_{\mathbf{g},\mathbf{h}}\mathbf{f}, \mathbf{h}'\rangle \\
=\ & \sum_{k\in\mathbb{Z}^d} \langle\langle \mathbf{m}_k(\cdot), \mathbf{h}\rangle_{\mathbb{H}} T_{ak}\boldsymbol{\gamma}, \mathbf{h}'\rangle \\
=\ & \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \langle\langle \mathbf{m}_k(x), \mathbf{h}\rangle_{\mathbb{H}} T_{ak}\boldsymbol{\gamma}(x), \mathbf{h}'(x)\rangle_{\mathbb{H}} dx \\
=\ & \beta^{-d} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \sum_{n\in\mathbb{Z}^d} \langle\langle (\mathbf{h} \odot T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x))T_{\frac{n}{\beta}}\mathbf{f}(x), \mathbf{h}\rangle_{\mathbb{H}} T_{ak}\boldsymbol{\gamma}(x), \mathbf{h}'(x)\rangle_{\mathbb{H}} dx
\end{aligned}$$

$$= \beta^{-d}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\sum_{n\in\mathbb{Z}^d}\langle\langle\langle T_{\frac{n}{\beta}}\mathbf{f}(x), T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x)\rangle_{\mathbb{H}}\mathbf{h},\mathbf{h}\rangle_{\mathbb{H}}T_{ak}\boldsymbol{\gamma}(x),\mathbf{h}'(x)\rangle_{\mathbb{H}}dx$$
$$= \beta^{-d}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\sum_{n\in\mathbb{Z}^d}\langle\langle T_{\frac{n}{\beta}}\mathbf{f}(x), T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x)\rangle_{\mathbb{H}}T_{ak}\boldsymbol{\gamma}(x),\mathbf{h}'(x)\rangle_{\mathbb{H}}dx$$
$$= \beta^{-d}\sum_{n\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\langle(T_{ak}\boldsymbol{\gamma}(x)\odot T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x))T_{\frac{n}{\beta}}\mathbf{f}(x),\mathbf{h}'(x)\rangle_{\mathbb{H}}dx$$
$$= \beta^{-d}\sum_{n\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\langle G_n(x)\left(T_{\frac{n}{\beta}}\mathbf{f}(x)\right),\mathbf{h}'(x)\rangle_{\mathbb{H}}dx$$
$$= \beta^{-d}\sum_{n\in\mathbb{Z}^d}\langle G_n\left(T_{\frac{n}{\beta}}\mathbf{f}\right),\mathbf{h}'\rangle,$$

from which (3.12) follows. The interchanges of integration and summation can be justified by Lemma 3.2.8 and Fubini's theorem. □

Using the expression for $\mathbf{m}_k$ in (3.8), the synthesis operator $R_{\mathbf{g},\mathbf{h}}$ can be expressed as the iterated sum as

$$\sum_{k\in\mathbb{Z}^d}\langle\sum_{n\in\mathbb{Z}^d}d_{kn}e^{2\pi i\beta\langle n,x\rangle},\mathbf{h}\rangle_{\mathbb{H}}T_{\alpha k}\mathbf{g} = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\langle d_{kn},\mathbf{h}\rangle_{\mathbb{H}}M_{\beta n}T_{\alpha k}\mathbf{g}.$$

We show that this series of partial sums converge to $R_{\mathbf{g},\mathbf{h}}d$ in the $W_{\mathbb{H}}(L^p, L^q_v)$ norm in the following proposition.

Proposition 3.2.12. *Let v be an w-moderate weight on $\mathbb{R}^d$. Let $\alpha,\beta > 0$ and $1 < p < \infty$, $1 \le q < \infty$ be given. Assume that $\mathbf{g},\boldsymbol{\gamma} \in W_{\mathbb{H}}(L^\infty, L^1_w)$ are such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d,\mathbb{H})$ with dual window $\boldsymbol{\gamma}$ and take $\mathbf{h}\in\mathbb{H}$ with unit norm. Then the following statements hold.*

(i) If $d \in S^{p,q}_{\tilde{v}}(\mathbb{H})$, then the partial sums

$$S_{K,N}d = \sum_{\|k\|_\infty\le K}\sum_{\|n\|_\infty\le N}\langle d_{kn},\mathbf{h}\rangle_{\mathbb{H}}M_{\beta n}T_{\alpha k}\mathbf{g},\quad K,N>0,$$

converge to $R_{\mathbf{g},\mathbf{h}}d$ in the norm of $W_{\mathbb{H}}(L^p, L^q_v)$, i.e., for each $\varepsilon > 0$ there exist

$K_0, N_0 > 0$ such that

$$\forall K \geq K_0,\ \ \forall N \geq N_0,\ \ \|R_{\mathbf{g},\mathbf{h}}d - S_{K,N}d\|_{W_{\mathbb{H}}(L^p,L^q_v)} < \varepsilon.$$

(*ii*) *If* $\mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$, *then the partial sums*

$$S_{K,N}(R_{\boldsymbol{\gamma},\mathbf{h}}C_{\mathbf{g},\mathbf{h}}\mathbf{f}) = \sum_{\|k\|_\infty \leq K} \sum_{\|n\|_\infty \leq N} \langle \mathbf{f}, M_{\beta n}T_{\alpha k}\mathbf{g}\rangle M_{\beta n}T_{\alpha k}\boldsymbol{\gamma}$$

of the Gabor expansion of $\mathbf{f}$ *converge to* $\mathbf{f}$ *in the norm of* $W_{\mathbb{H}}(L^p, L^q_v)$ *and for* $1 \leq p, q \leq \infty$, *in the* $\sigma(W_{\mathbb{H}}(L^p, L^q_v), W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v}))$*-topology.*

Proof. The idea of the proof is similar to Proposition 4.6 of [82] with appropriate modifications. □

The Walnut's representation for superframe operator and the multi-window Gabor frame operator can be obtained by choosing an appropriate Hilbert space in Theorem 3.2.3. However, we list out some consequences of Theorem 3.2.3 and Proposition 3.2.12 in the following remark:

Remark 3.2.13. (*i*) If $\mathbb{H} = \mathbb{C}$ then the rank one operator $x \odot y$ turns out to the point-wise product xy and all the above results for the ℍ-valued Gabor frame viz. Walnut's representation of ℍ-valued Gabor frame operator, convergence of Gabor expansions, etc., coincides with the results for the scalar valued Gabor frames (see [67, 82, 135, 137]).

(*ii*) If $\mathbb{H} = \mathbb{C}^n$ then the ℍ-valued Gabor frame is the super Gabor frame (see [84]). The Gabor expansions for the Gabor super-frames on vector valued amalgam spaces also converge by Proposition 3.2.12.

(*iii*) Let $(\mathbb{H}_1, \langle\cdot,\cdot\rangle_1), (\mathbb{H}_2, \langle\cdot,\cdot\rangle_2), \cdots, (\mathbb{H}_r, \langle\cdot,\cdot\rangle_r)$ be r Hilbert spaces. If $\mathbb{H} = \bigoplus_{i=1}^r \mathbb{H}_i$ (i.e ℍ is the direct sum of r Hilbert spaces) then ℍ is also a Hilbert space with respect to the inner product $\langle x, y\rangle = \sum_{i=1}^r \langle x_i, y_i\rangle$ where $x = \oplus_{i=1}^r x_i,\ y = \oplus_{i=1}^r y_i,\ x, y \in \mathbb{H}, x_i, y_i \in \mathbb{H}_i, i = 1, 2\cdots, r$. If $f : \mathbb{R}^d \to \mathbb{H}$ then

$$f(x) = f_1(x) \bigoplus f_2(x) \bigoplus \cdots \bigoplus f_r(x) \quad \text{with} \quad f_i(x) \in \mathbb{H}_i,\ i = 1, 2\cdots, r.$$

Note that $f \in W_{\mathbb{H}}(L^p, L^q_w) \Leftrightarrow f_i \in W_{\mathbb{H}_i}(L^p, L^q_w)$ for all $i = 1, 2, \cdots, r$. The frame operator of the Gabor system on $W_{\mathbb{H}}(L^p, L^q_w)$ with respect to a single lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ is given by

$$\begin{aligned} S_{\mathbf{g},\boldsymbol{\gamma}}\mathbf{f} &= \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n\left(T_{\frac{n}{\beta}}\mathbf{f}\right) = \beta^{-d} \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} T_{\alpha k}\boldsymbol{\gamma}(x) \odot T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x)\left(T_{\frac{n}{\beta}}\mathbf{f}\right) \\ &= \beta^{-d} \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} T_{\alpha k}\Big(\bigoplus_{i=1}^r \boldsymbol{\gamma}_i(x)\Big) \odot T_{\alpha k+\frac{n}{\beta}}\Big(\bigoplus_{i=1}^r \mathbf{g}_i(x)\Big)\Big(T_{\frac{n}{\beta}}\Big(\bigoplus_{i=1}^r \mathbf{f}_i\Big)\Big) \\ &= \beta^{-d} \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{i=1}^r \Big(T_{\alpha k}\boldsymbol{\gamma}_i(x) \odot T_{\alpha k+\frac{n}{\beta}}\mathbf{g}_i(x)\Big)\left(T_{\frac{n}{\beta}}\mathbf{f}_i\right) = \sum_{i=1}^r S_{\mathbf{g}_i,\boldsymbol{\gamma}_i}\mathbf{f}_i, \end{aligned}$$

where $\mathbf{g}(x) = \bigoplus_{i=1}^r \mathbf{g}_i(x)$, $\boldsymbol{\gamma}(x) = \bigoplus_{i=1}^r \boldsymbol{\gamma}_i(x)$ and $S_{\mathbf{g}_i,\boldsymbol{\gamma}_i}$ is the frame operator of the Gabor system on $W_{\mathbb{H}_i}(L^p, L^q_w)$. If $\mathbb{H}_1 = \mathbb{H}_2 \cdots = \mathbb{H}_r = \mathbb{C}$ then the $\mathbb{H}-$valued Gabor frame turns out to be a Gabor superframe.

(iv) Let $\Lambda = \Lambda^1 \times ... \times \Lambda^r$ be the Cartesian product of separable lattices $\Lambda^i = \alpha_i\mathbb{Z}^d \times \beta_i\mathbb{Z}^d$ and let $\mathbf{g}_1, ..., \mathbf{g}_r, \boldsymbol{\gamma}_1, ..., \boldsymbol{\gamma}_r \in W_{\mathbb{H}_i}(L^\infty, L^1_w)$. Suppose the collection $\mathcal{G}_i(\mathbf{g}_i, \alpha_i, \beta_i)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{H}_i)$ with the corresponding frame operator $S^{\Lambda^i}_{\mathbf{g}_i,\boldsymbol{\gamma}_i}$. As in the previous set up (as in (iii)) we show that frame operator associated with the Gabor system on $W_{\mathbb{H}}(L^p, L^q_w)$ is the sum of frame operator associated with the Gabor systems on $W_{\mathbb{H}_i}(L^p, L^q_w)$. In this case we consider cartesian product of separable lattices $\Lambda^i = \alpha_i\mathbb{Z}^d \times \beta_i\mathbb{Z}^d$, $i = 1, 2, \cdots, r$ instead of a single lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$. For $1 \le p, q \le \infty$ the space $S^{p,q}_{\tilde{v}}(\mathbb{H})$ turns out to be $S^{p,q}_{\tilde{v}}(\mathbb{H}_1) \times S^{p,q}_{\tilde{v}}(\mathbb{H}_2) \times \cdots \times S^{p,q}_{\tilde{v}}(\mathbb{H}_r)$ with the norm $\|d\|_{S^{p,q}_{\tilde{v}}(\mathbb{H})} = \|(d^1, d^2 \cdots, d^r)\|_{S^{p,q}_{\tilde{v}}(\mathbb{H})} = \sum_{i=1}^r \|d^i\|_{S^{p,q}_{\tilde{v}}(\mathbb{H}_r)}$, where ${S_{\tilde{v}}}^{p,q}(\mathbb{H}_i)$is defined as in Definition 3.2.2 with respect to the lattice Λ^i and the Hilbert space H_i. For $x \in \mathbb{R}^d, k, n \in \mathbb{Z}^d$, $\alpha = (\alpha_1, \alpha_2 \cdots, \alpha_r)$ and $\beta = (\beta_1, \beta_2 \cdots, \beta_r)$ with $\alpha_i > 0, \beta_i > 0$ the translation operator $T_{\alpha k}$ and the modulation operator $M_{\beta n}$ of $\mathbf{f}(x) = \bigoplus_{i=1}^r \mathbf{f}_i(x)$ are defined as $T_{\alpha k}\mathbf{f}(x) = \bigoplus_{i=1}^r T_{\alpha_i k}\mathbf{f}_i(x)$ and $M_{\beta n}\mathbf{f}(x) = \bigoplus_{i=1}^r M_{\beta_i n}\mathbf{f}_i(x)$. The frame operator of the Gabor system on $W_{\mathbb{H}}(L^p, L^q_w)$ is given by

$$S^{\Lambda}_{\mathbf{g},\boldsymbol{\gamma}}\mathbf{f} = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n\left(T_{\frac{n}{\beta}}\mathbf{f}\right) = \beta^{-d} \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} T_{\alpha k}\boldsymbol{\gamma}(x) \odot T_{\alpha k+\frac{n}{\beta}}\mathbf{g}(x)\left(T_{\frac{n}{\beta}}\mathbf{f}\right)$$

$$= \beta^{-d} \sum_{n\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \Big(\bigoplus_{i=1}^{r} T_{\alpha_i k} \boldsymbol{\gamma}_i(x) \Big) \odot \Big(\bigoplus_{i=1}^{r} T_{\alpha_i k+\frac{n}{\beta_i}} \mathbf{g}_i(x) \Big) \Big(\Big(\bigoplus_{i=1}^{r} T_{\frac{n}{\beta_i}} \mathbf{f}_i \Big) \Big)$$

$$= \beta^{-d} \sum_{n\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \sum_{i=1}^{r} \Big(T_{\alpha_i k} \boldsymbol{\gamma}_i(x) \odot T_{\alpha_i k+\frac{n}{\beta_i}} \mathbf{g}_i(x) \Big) \Big(T_{\frac{n}{\beta_i}} \mathbf{f}_i \Big) = \sum_{i=1}^{r} S^{\Lambda^i}_{\mathbf{g}_i,\boldsymbol{\gamma}_i} \mathbf{f}_i,$$

where $\beta^{-d} = \prod_{i=1}^{r} \beta_i^{-d}$, $\mathbf{g}(x) = \bigoplus_{i=1}^{r} \mathbf{g}_i(x)$, $\boldsymbol{\gamma}(x) = \bigoplus_{i=1}^{r} \boldsymbol{\gamma}_i(x)$ and $S^{\Lambda^i}_{\mathbf{g}_i,\boldsymbol{\gamma}_i}$ is the frame operator of the Gabor system on $W_{\mathbb{H}_i}(L^p, L^q_w)$ with respect to the lattice Λ^i.

If $\mathbb{H}_1 = \mathbb{H}_2 \cdots = \mathbb{H}_r = \mathbb{C}$ and $\mathbf{f}(x) = f(x) \bigoplus f(x) \bigoplus \cdots \bigoplus f(x)$ (r-times tensor product of $f(x)$ with itself) where $f \in W(L^p, L^q_w)$ then the $\mathbb{H}-$valued Gabor frame turns out to be a "multi-window Gabor frame".

3.3 The Algebra of $\mathbb{H}$-Valued L^∞-Weighted Shifts

3.3.1 $\mathbb{H}$-Valued L^∞-Weighted Shifts

In this section, we mainly aim to prove the spectral invariance theorem for a sub-algebra weighted shift operators in $B(L^p(\mathbb{R}^d, \mathbb{H}))$, the algebra of bounded linear operators on $L^p(\mathbb{R}^d, \mathbb{H})$. For an admissible weight function w we construct a Banach $*-$algebra $\mathcal{A}_w$ of weighted shift operators in $B(L^p(\mathbb{R}^d, \mathbb{H}))$ and identify with $AP^p_w(\rho)$, the class of ρ-almost periodic elements, having w-summable Fourier coefficients. Finally we prove the spectral invariance theorem on $AP^p_w(\rho)$ which assures spectral invariance property on $\mathcal{A}_w$. Let us start with a definition of multiplication operator on $B(L^p(\mathbb{R}^d, \mathbb{H}))$.

Definition 3.3.1. *Let $\phi \in L^\infty(\mathbb{R}^d, B(\mathbb{H}))$, then we define the multiplication operator $T_\phi : L^p(\mathbb{R}^d, \mathbb{H}) \to L^p(\mathbb{R}^d, \mathbb{H})$ defined by $(T_\phi f)(x) = \phi(x)(f(x)),\ x \in \mathbb{R}^d$.*

Note that T_ϕ is linear, bounded and $\|T_\phi\| = \|\phi\|_{L^\infty(\mathbb{R}^d, B(\mathbb{H}))}$. For an admissible weight w define

$$\mathcal{A}_w := \{\mathcal{M} = (\mathbf{m}_x)_{x\in\mathbb{R}^d} \in L^\infty(\mathbb{R}^d, B(\mathbb{H})) : \sum_{x\in\mathbb{R}^d} \|\mathbf{m}_x\|_{L^\infty(\mathbb{R}^d, B(\mathbb{H}))} w(x) < +\infty\},$$

with norm

$$\|\mathcal{M}\|_{\mathcal{A}_w} = \sum_{x\in\mathbb{R}^d} \|\mathbf{m}_x\|_{L^\infty(\mathbb{R}^d, B(\mathbb{H}))} w(x) < +\infty.$$

If $\mathcal{M} = (\mathbf{m}_x)_{x\in\mathbb{R}^d} \in \mathcal{A}_w$, then $\mathcal{M} = (\mathbf{m}_x)_{x\in\mathbb{R}^d} \in \ell^1_w(\mathbb{R}^d, L^\infty(\mathbb{R}^d, B(\mathbb{H})))$ of the family $\mathcal{M} = (\mathbf{m}_x)_{x\in\mathbb{R}^d}$ has countable support. The identification of $\mathcal{A}_w$ with the subclass of bounded operators on $L^2(\mathbb{R}^d, \mathbb{H})$ is as follows: Given $(\mathbf{m}_x)_{x\in\mathbb{R}^d} \in \mathcal{A}_w$ define the operator

$$T : L^p(\mathbb{R}^d, \mathbb{H}) \to L^p(\mathbb{R}^d, \mathbb{H}) \quad \text{by} \quad T(\mathbf{f}) = \sum_{x\in\mathbb{R}^d} \mathbf{m}_x(T_x\mathbf{f}).$$

Clearly, T is well defined, linear and bounded on all $L^p(\mathbb{R}^d, \mathbb{H})$, $1 \le p \le \infty$ (by using admissibility of w). The identification $\mathbf{f} \mapsto \sum_{x\in\mathbb{R}^d} \mathbf{m}_x(T_x\mathbf{f})$ maps $\mathcal{A}_w$ into a closed subspace of $B(L^p(\mathbb{R}^d, \mathbb{H}))$. It is then convenient to write $\mathcal{M} \in \mathcal{A}_w$ as

$$\mathcal{M} = \sum_{x\in\mathbb{R}^d} \mathbf{m}_x(T_x),\ (\mathbf{m}_x)_{x\in\mathbb{R}^d} \in \ell^1_w(\mathbb{R}^d, L^\infty(\mathbb{R}^d, B(\mathbb{H}))).$$

If we endow $\mathcal{A}_w$ with the product and involution inherited from $B(L^2(\mathbb{R}^d, \mathbb{H}))$ then $\mathcal{A}_w$ is a Banach *-algebra which embeds continuously into $B(L^2(\mathbb{R}^d, \mathbb{H}))$: i.e., for $(\mathbf{m}_x)_{x\in\mathbb{R}^d}$, $(\mathbf{n}_x)_{x\in\mathbb{R}^d} \in \mathcal{A}_w$, define

$$\Big(\sum_{x\in\mathbb{R}^d} \mathbf{m}_x(T_x)\Big) \cdot \Big(\sum_{x\in\mathbb{R}^d} \mathbf{n}_x(T_x)\Big) = \sum_{x\in\mathbb{R}^d} \Big(\sum_{y\in\mathbb{R}^d} \mathbf{m}_y \mathbf{n}_{x-y}(\cdot - y)\Big)(T_x),$$

and the involution by

$$\Big(\sum_{x\in\mathbb{R}^d} \mathbf{m}_x(T_x)\Big)^* = \sum_{x\in\mathbb{R}^d} \overline{\mathbf{m}_x(\cdot + x)}(T_{-x}) = \sum_{x\in\mathbb{R}^d} \overline{\mathbf{m}_{-x}(\cdot - x)}(T_x).$$

Notice that the identification of families in $\mathcal{A}_w$ and operators on $B(L^p(\mathbb{R}^d, \mathbb{H}))$ is one-to-one. We simply write $\mathcal{A}_w \subset B(L^p(\mathbb{R}^d, \mathbb{H}))$ and we treat members of $\mathcal{A}_w$ as operators on $L^p(\mathbb{R}^d, \mathbb{H})$. The following result for $\mathbf{m} \in L^\infty(\mathbb{R}^d, B(\mathbb{H}))$ plays a crucial role in proving the spectral invariance theorem:

Lemma 3.3.2. *For $\mathbf{m} \in L^\infty(\mathbb{R}^d, B(\mathbb{H}))$ and $x, w \in \mathbb{R}^d$. The following relation hold.*

$$M_w \mathbf{m}(T_x M_{-w}) = e^{2\pi i \langle w,x\rangle} \mathbf{m}(T_x). \tag{3.26}$$

Proof. The proof of this result is trivial if $\mathbb{H} = \mathbb{C}$. Otherwise, for each $y \in \mathbb{R}^d$, $m(y)$ is a linear bounded operator on $\mathbb{H}$, which we view as an infinite matrix with scalar entries and prove the lemma. Since $\mathbb{H}$ is separable Hilbert space and for $y \in \mathbb{R}^d$, $\mathbf{m}(y) \in B(\mathbb{H})$ can be written as

$$\mathbf{m}(y)u = \mathbf{m}(y)\Big(\sum_n \langle u, e_n\rangle_{\mathbb{H}} e_n\Big) = \sum_n \langle u, e_n\rangle_{\mathbb{H}} \mathbf{m}(y)e_n = \sum_{n,j} \langle u, e_n\rangle_{\mathbb{H}} a_{nj}(y)e_j,$$

where $u \in \mathbb{H}$ and $(e_n)_n$ is orthonormal basis for $\mathbb{H}$. Now for $\mathbf{f} \in L^p(\mathbb{R}^d, \mathbb{H})$

$$\begin{aligned} M_w \mathbf{m}(y)(T_x M_{-w}\mathbf{f}(y)) &= e^{2\pi i\langle w,y\rangle} \sum_{n,j} \langle T_x M_{-w}\mathbf{f}(y), e_n\rangle_{\mathbb{H}} a_{nj}(y)e_j \\ &= e^{2\pi i\langle w,x\rangle} \sum_{n,j} \langle T_x \mathbf{f}(y), e_n\rangle_{\mathbb{H}} a_{nj}(y)e_j \\ &= e^{2\pi i\langle w,x\rangle} \mathbf{m}(y)(T_x\mathbf{f}(y)). \end{aligned}$$

This completes the proof. □

The continuity properties of operators defined by the family in $\mathcal{A}_w$ is established by the following proposition.

Proposition 3.3.3. *Let $1 \le p, q \le \infty$ and let v be a w-moderate weight. Then the following statements hold.*

(i) $\mathcal{A}_w \hookrightarrow B(W_{\mathbb{H}}(L^p, L^q_v))$. *More precisely, every $\mathcal{M} = \sum_{x\in\mathbb{R}^d} \mathbf{m}_x(T_x) \in \mathcal{A}_w$ defines a bounded operator on $W_{\mathbb{H}}(L^p, L^q_v)$ given by the formula*

$$\mathcal{M}(\mathbf{f}) := \sum_{x\in\mathbb{R}^d} \mathbf{m}_x(T_x\mathbf{f}).$$

The series defining $\mathcal{M} : W_{\mathbb{H}}(L^p, L^q_v) \to W_{\mathbb{H}}(L^p, L^q_v)$ converges absolutely in the norm of $W_{\mathbb{H}}(L^p, L^q_v)$ and $\|\mathcal{M}\|_{B(W_{\mathbb{H}}(L^p,L^q_v))} \le C_v \|\mathcal{M}\|_{\mathcal{A}_w}$, where C_v is a *constant in (3.3).*

(ii) *For every* $\mathcal{M} \in \mathcal{A}_w, \mathbf{f} \in W_{\mathbb{H}}(L^p, L^q_v)$ *and* $\mathbf{g} \in W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v})$, *Then* $\langle \mathcal{M}(\mathbf{f}), \mathbf{g}\rangle = \langle \mathbf{f}, \mathcal{M}^*(\mathbf{g})\rangle$.

(iii) *For every* $\mathcal{M} \in \mathcal{A}_w$, *the operator* $\mathcal{M} : W_{\mathbb{H}}(L^p, L^q_v) \to W_{\mathbb{H}}(L^p, L^q_v)$ *is continuous in the* $\sigma(W_{\mathbb{H}}(L^p, L^q_v), W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v}))$*-topology.*

Proof. Using the fact that v is w-moderate along with an appropriate change of variable, proves part (i). Part (ii) follows from the fact the involution in $\mathcal{A}_w$ coincides with taking adjoint. The interchange of summation and integration is justified by the absolute convergence in part (i). Part (iii) is follows immediately from (ii). □

3.3.2 Spectral invariance

In order to identify the class $\mathcal{A}_w \subset B(L^p(\mathbb{R}^d, \mathbb{H}))$ with $AP^p_w(\rho)$ which is the class of ρ-almost periodic elements having w-summable Fourier series, we require the following definitions and necessary theories.

Let $y \in \mathbb{R}^d$ and $\mathcal{M} \in B(L^p(\mathbb{R}^d, \mathbb{H}))$, $1 \leq p \leq \infty$. Define $\rho(y)\mathcal{M} := M_y \mathcal{M} M_{-y}$. Then,

$$\rho(y)\mathcal{M}\mathbf{f}(x) = e^{2\pi i\langle y,x\rangle}\mathcal{M}(\mathbf{g}(x)), \text{ where } \mathbf{g}(x) = e^{-2\pi i\langle y,x\rangle}\mathbf{f}(x).$$

Clearly ρ is a representation of $\mathbb{R}^d$ on the Banach space $B(L^p(\mathbb{R}^d, \mathbb{H}))$. For each $y \in \mathbb{R}^d$, $\rho(y)$ is an algebra automorphism and an isometry.

Definition 3.3.4. *A continuous map* $Y : \mathbb{R}^d \to B(L^p(\mathbb{R}^d, \mathbb{H}))$ *is almost-periodic in the sense of Bohr if for every* $\varepsilon > 0$ *there is a compact set* $K = K_\varepsilon \subset \mathbb{R}^d$ *such that for all* $x \in \mathbb{R}^d$

$$(x + K) \cap \{y \in \mathbb{R}^d : \|Y(g + y) - Y(g)\| < \varepsilon, \ \forall g \in \mathbb{R}^d\} \neq \emptyset$$

Then Y extends uniquely to a continuous map of the Bohr compactification $\hat{R}^d_c$ of $\mathbb{R}^d$, denoted by Y. Thus $Y : \hat{R}^d_c \to B(L^p(\mathbb{R}^d, \mathbb{H}))$, where $\hat{R}^d_c$ represents the topological dual group (i.e. the group of characters) of $\mathbb{R}^d$ when $\mathbb{R}^d$ is endowed with the discrete topology. The normalized Haar measure on $\hat{R}^d_c$ is denoted by $\overline{\mu}(dy)$.

For each $\mathcal{M} \in B(L^p(\mathbb{R}^d, \mathbb{H}))$, we consider the map $\widehat{\mathcal{M}} : \mathbb{R}^d \to B(L^p(\mathbb{R}^d, \mathbb{H}))$ defined by

$$\widehat{\mathcal{M}}(y) := \rho(y)\mathcal{M} = M_y \mathcal{M} M_{-y}. \tag{3.27}$$

If the map $\widehat{\mathcal{M}}$ is continuous and almost-periodic in the sense of Bohr then the operator $\mathcal{M} \in B(L^p(\mathbb{R}^d, \mathbb{H}))$ is called ρ-*almost periodic*. For every ρ-almost periodic operator $\mathcal{M}$, the function $\widehat{\mathcal{M}}$ admits a $B(L^p(\mathbb{R}^d, \mathbb{H}))$-valued Fourier series,

$$\widehat{\mathcal{M}}(y) \sim \sum_{x \in \mathbb{R}^d} e^{2\pi i \langle y,x \rangle} C_x(\mathcal{M}), \quad (y \in \mathbb{R}^d). \tag{3.28}$$

The coefficients $C_x(\mathcal{M}) \in B(L^p(\mathbb{R}^d, \mathbb{H}))$ in (3.28) are uniquely determined by $\mathcal{M}$ via

$$C_x(\mathcal{M}) = \int_{\hat{R}_c^d} \widehat{\mathcal{M}}(y) e^{-2\pi i \langle y,x \rangle} \overline{\mu}(dy) = \lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} \widehat{\mathcal{M}}(y) e^{-2\pi i \langle y,x \rangle} dy \tag{3.29}$$

and satisfy

$$\rho(y) C_x(\mathcal{M}) = e^{2\pi i \langle y,x \rangle} C_x(\mathcal{M}). \tag{3.30}$$

Therefore, they are eigenvectors of ρ (see [12] for details).

Within the class of ρ-almost periodic operators consider $AP_w^p(\rho)$, the subclass of those operators for which the Fourier series in (3.28) is w-summable, where w is an admissible weight. More precisely, a ρ-almost periodic operator $\mathcal{M}$ belongs to $AP_w^p(\rho)$ if its Fourier coefficients with respect to ρ satisfy

$$\|\mathcal{M}\|_{AP_w^p(\rho)} := \sum_{x \in \mathbb{R}^d} \|C_x(\mathcal{M})\|_{B(L^p(\mathbb{R}^d, \mathbb{H}))} w(x) < \infty. \tag{3.31}$$

Since w is submultiplicative, for $\mathcal{M} \in AP_w^p(\rho)$ the series

$$\widehat{\mathcal{M}}(y) = \sum_{x \in \mathbb{R}^d} e^{2\pi i \langle y,x \rangle} C_x(\mathcal{M}), \quad y \in \mathbb{R}^d, \tag{3.32}$$

converges absolutely to $\widehat{\mathcal{M}}(y)$ on $B(L^p(\mathbb{R}^d, \mathbb{H}))$, where each $C_x \in B(L^p(\mathbb{R}^d, \mathbb{H}))$ satisfies (3.29) and hence (3.30). In particular, for $y = 0$, each $\mathcal{M} \in AP_w^p(\rho)$ can be written as

$$\mathcal{M} = \sum_{x\in\mathbb{R}^d} C_x(\mathcal{M}). \tag{3.33}$$

Conversely, if $\mathcal{M}$ is given by (3.33) with the coefficients C_x satisfying (3.31) and (3.30), it follows from the theory of almost-periodic series that $\mathcal{M} \in AP^p_w(\rho)$ and C_x satisfy (3.29). Now we are in a position to establish connection between $\mathcal{A}_w$ and $AP^p_w(\rho)$ and prove spectral invariance result for $\mathcal{A}_w$. For that we first characterize the eigenvectors C_x of the representation ρ.

Lemma 3.3.5. *For any $1 \le p \le \infty$ and any $\mathbf{m} \in L^\infty(\mathbb{R}^d, B(\mathbb{H}))$ and $x \in \mathbb{R}^d$, $C_x = \mathbf{m}(T_x)$ is an eigenvector of $\rho : \mathbb{R}^d \to B(B(L^p(\mathbb{R}^d, \mathbb{H})))$. For $1 \le p < \infty$ these are the only eigenvectors.*

Proof. If $C_x = \mathbf{m}(T_x)$, then by (3.26), it satisfies (3.30).

The converse holds only for $1 \le p < \infty$. Suppose that $C_x \in B(L^p(\mathbb{R}^d, \mathbb{H}))$ satisfies (3.30). Using (3.26) once again we have,

$$\rho(y)C_x(T_{-x}) = M_y C_x M_{-y}(T_{-x}) = C_x(T_{-x}).$$

It follows that $C_x(T_{-x}M_y) = M_y C_x(T_{-x})$. Hence, $C_x(T_{-x})$ must be a multiplication operator $\mathbf{m}$, so $C_x = m(T_x)$. □

For $p = \infty$, there are eigenvectors of ρ which may not of the form $\mathbf{m}(T_x)$. An example of such an eigenvector is given in ([103], Section 5.1.11) for the case $\mathbb{H} = \mathbb{C}$. Hence $AP^p_w(\rho)$ consists of all the operators $\mathcal{M} = \sum_{x\in\mathbb{R}^d} C_x$, with C_x satisfying (3.31) and (3.30). The previous lemma says that for $1 \le p < \infty$ an operator C_x satisfies (3.30) if and only if it is of the form $C_x = \mathbf{m}(T_x)$, for some function $\mathbf{m} \in L^\infty(\mathbb{R}^d, B(\mathbb{H}))$. Note that $\|C_x\|_{B(L^p(\mathbb{R}^d,\mathbb{H}))} = \|\mathbf{m}\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}$ and thus

$$\|\mathcal{M}\|_{\mathcal{A}_w} = \sum_{x\in\mathbb{R}^d} \|\mathbf{m}_x\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))} w(x) = \sum_{x\in\mathbb{R}^d} \|C_x(\mathcal{M})\|_{B(L^p(\mathbb{R}^d,\mathbb{H}))} w(x) = \|\mathcal{M}\|_{AP^p_w(\rho)}.$$

This gives the identification of $\mathcal{A}_w$ with $AP^p_w(\rho)$.

Proposition 3.3.6. *For $p \in [1,\infty)$ the class $\mathcal{A}_w \subset B(L^p(\mathbb{R}^d,\mathbb{H}))$ coincides with $AP^p_w(\rho)$, the class of ρ-almost periodic elements, having w-summable Fourier coefficients.*

For $p=\infty$, the two classes are different. Now we are in a position to prove that the algebra $\mathcal{A}_w$ is spectral within the class of bounded operators on $L^p(\mathbb{R}^d,\mathbb{H})$. This means if an operator from $\mathcal{A}_w$ is invertible on $L^p(\mathbb{R}^d,\mathbb{H})$ then the inverse operator necessarily belongs to $\mathcal{A}_w$. In other words, invertibility in the bigger algebra implies invertibility in the smaller algebra. To prove these kinds of results one may use Wiener's $1/f$ lemma or its several versions. We resort to recent Wiener type result on non-commutative, almost periodic Fourier series ([12], Theorem 3.2) to obtain the following theorem.

Theorem 3.3.7. *Let w be an admissible weight. Then, the embedding $\mathcal{A}_w \hookrightarrow B(L^p(\mathbb{R}^d,\mathbb{H}))$, $p\in[1,\infty]$ is spectral. In other words, if $\mathcal{M}\in\mathcal{A}_w$ defines an invertible operator $\sum_{x\in\mathbb{R}^d}\mathbf{m}_x(T_x)\in B(L^p(\mathbb{R}^d,\mathbb{H}))$ for some $p\in[1,\infty]$, then $\mathcal{M}^{-1}\in\mathcal{A}_w$.*

Proof. We will prove $AP^p_w(\rho)$ is spectral. For $1\le p<\infty$ the result follows from Proposition 3.3.6 and Theorem 3.2 in [12].

For $p=\infty$, take

$$\mathcal{M}=\sum_{x\in\mathbb{R}^d}\mathbf{m}_x(T_x)\in\mathcal{A}_w\subset B(L^\infty(\mathbb{R}^d,\mathbb{H}))$$

with $\sum\limits_{x\in\mathbb{R}^d}\|\mathbf{m}_x\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}w(x)<\infty$. Define

$$\mathcal{N}=\sum_{x\in\mathbb{R}^d}(T_x(\mathbf{m}_{-x}))(T_x)=\sum_{x\in\mathbb{R}^d}\mathbf{m}_{-x}(\cdot-x)(T_x)\in\mathcal{A}_w\subset B(L^1(\mathbb{R}^d,\mathbb{H})).$$

Since $\|T_x(\mathbf{m}_{-x})\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}=\|\mathbf{m}_{-x}\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}$, $\mathcal{N}$ is well defined and a straight forward calculation shows that $\mathcal{M}$ is the transpose (Banach adjoint) of $\mathcal{N}$. Therefore $\mathcal{N}$ is invertible when $\mathcal{M}$ is invertible. Since $\mathcal{A}_w$ is spectral in $B(L^1(\mathbb{R}^d,\mathbb{H}))$ we get $\mathcal{M}^{-1}=(\mathcal{N}^{-1})'\in\mathcal{A}_w$. That means $\mathcal{M}^{-1}=\sum_{x\in\mathbb{R}^d}\mathbf{n}_x(T_x)$ for some $\mathbf{n}_x\in L^\infty(\mathbb{R}^d,B(\mathbb{H}))$ such that $\sum_{x\in\mathbb{R}^d}\|\mathbf{n}_x\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}w(x)<\infty$. □

3.4 Invertibility of the $\mathbb{H}$-Valued Gabor Frame Operator

Theorem 3.4.1. *Let w be an admissible weight, v be w-moderate weight and $\mathbf{g} \in W_{\mathbb{H}}(L^\infty, L^1_w)$. Suppose that the Gabor system $\mathcal{G}(\mathbf{g}, \alpha, \beta) = \{M_{\beta n} T_{\alpha k}\mathbf{g} : \ k, n \in \mathbb{Z}^d\}$ is a frame for $L^2(\mathbb{R}^d, \mathbb{H})$ with frame operator $S_{\mathbf{g}}$. Then the inverse operator*

$$S_{\mathbf{g}}^{-1} : W_{\mathbb{H}}(L^p, L^q_v) \to W_{\mathbb{H}}(L^p, L^q_v), \quad 1 \leq p, q \leq \infty$$

is continuous both in $\sigma(W_{\mathbb{H}}(L^p, L^q_v), W_{\mathbb{H}}(L^{p'}, L^{q'}_{1/v}))$ and the norm topologies.

Proof. As a consequence of the Walnut's representation in Theorem 3.2.3 the frame operator $S_{\mathbf{g}}$ belongs to the algebra $\mathcal{A}_w$. As $S_{\mathbf{g}}$ is invertible in $L^2(\mathbb{R}^d, \mathbb{H})$, Theorem 3.3.7 implies that $S_{\mathbf{g}}^{-1} \in \mathcal{A}_w$. Since $\mathcal{A}_w$ is continuously embedded in $B(W_{\mathbb{H}}(L^p, L^q_v))$, by Proposition 3.3.3 $S_{\mathbf{g}}^{-1} \in B(W_{\mathbb{H}}(L^p, L^q_v))$. Hence the theorem follows. □

Let $C_0(\mathbb{R}^d, B(\mathbb{H}))$ be the subspace formed by the functions of $L^\infty(\mathbb{R}^d, B(\mathbb{H}))$ that are continuous. The next corollary shows the continuity of the dual generator, provided the window function is continuous.

Corollary 3.4.2. *In the conditions of Theorem 3.4.1, if window $\mathbf{g}$ is continuous then the dual window $\tilde{\mathbf{g}} = S_{\mathbf{g}}^{-1}(\mathbf{g})$ is also continuous.*

Proof. Let

$$\tilde{\mathcal{A}_w} = \Big\{\mathcal{M} = (\mathbf{m}_x)_{x\in\mathbb{R}^d} \in \ell^1_w(\mathbb{R}^d, C_0(\mathbb{R}^d, B(\mathbb{H}))) \mid \sum_{x\in\mathbb{R}^d} \|\mathbf{m}_x\|_{C_0(\mathbb{R}^d, B(\mathbb{H}))} w(x) < \infty\Big\}.$$

Then $\tilde{\mathcal{A}_w} \subset \mathcal{A}_w \subset B(L^p(\mathbb{R}^d, \mathbb{H}))$. If $\mathbf{g} \in W_{\mathbb{H}}(C_0, L^1_w)$ then $S_{\mathbf{g}} = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n(T_{\frac{n}{\beta}}) \in \tilde{\mathcal{A}_w}$. Since $S_{\mathbf{g}}$ is invertible in $B(L^2(\mathbb{R}^d, \mathbb{H}))$, applying Theorem 3.3.7 on $\tilde{\mathcal{A}_w}$ we get $S_{\mathbf{g}}^{-1} \in \tilde{\mathcal{A}_w}$.

Let $\mathbf{g}$ be continuous. To show $S_{\mathbf{g}}^{-1}(\mathbf{g})$ is continuous it is enough to show $S_{\mathbf{g}}$ maps $W_{\mathbb{H}}(C_0, L^1_w)$ to $W_{\mathbb{H}}(C_0, L^1_w)$. Let $\mathbf{f} \in W_{\mathbb{H}}(C_0, L^1_w)$. Since $G_n(x)\left(T_{\frac{n}{\beta}}\mathbf{f}(x)\right)$ is continuous for each n, $\sum_{\text{finite}} G_n(x)\left(T_{\frac{n}{\beta}}\mathbf{f}(x)\right)$ is continuous. Again,

$$\begin{aligned}\left\|\sum_{n\in\mathbb{Z}^d} G_n(x)\left(T_{\frac{n}{\beta}}\mathbf{f}(x)\right)\right\|_{\mathbb{H}} &\leq \sum_{n\in\mathbb{Z}^d}\|G_n\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}\|T_{\frac{n}{\beta}}\mathbf{f}(x)\|_{\mathbb{H}}\\ &\leq \sum_{n\in\mathbb{Z}^d}\|G_n\|_{L^\infty(\mathbb{R}^d,B(\mathbb{H}))}\|\mathbf{f}\|_{W_{\mathbb{H}}(L^\infty,L^1_w)}<\infty.\end{aligned}$$

So by Weierstrass M-test $\sum_{n\in\mathbb{Z}^d} G_n(x)\left(T_{\frac{n}{\beta}}\mathbf{f}(x)\right)$ converges uniformly and hence $S_{\mathbf{g}}\mathbf{f}(x)=\beta^{-d}\sum_{n\in\mathbb{Z}^d} G_n(x)\left(T_{\frac{n}{\beta}}\mathbf{f}(x)\right)$ is continuous. □

Remark 3.4.3. (i) By Proposition 3.2.12 for all $\mathbf{f}\in W_{\mathbb{H}}(L^p,L^q_v)$

$$S_{\mathbf{g}}(\mathbf{f})=\lim_{K,N\to\infty}\sum_{\|k\|_\infty\leq K}\sum_{\|n\|_\infty\leq N}\langle\mathbf{f},M_{\beta n}T_{\alpha k}\mathbf{g}\rangle M_{\beta n}T_{\alpha k}\mathbf{g}, \tag{3.34}$$

with convergence in the $\sigma(W_{\mathbb{H}}(L^p,L^q_v),W_{\mathbb{H}}(L^{p'},L^{q'}_{1/v}))$-topology and for $p,q<\infty$ in the norm of $W_{\mathbb{H}}(L^p,L^q_v)$. Since $S_{\mathbf{g}}^{-1}\in\mathcal{A}_w$, using Proposition 3.3.3(c) and applying $S_{\mathbf{g}}^{-1}$ to both sides of (3.34) and we obtain

$$\begin{aligned}\mathbf{f} &= \lim_{K,N\to\infty}\sum_{\|k\|_\infty\leq K}\sum_{\|n\|_\infty\leq N}\langle\mathbf{f},M_{\beta n}T_{\alpha k}\mathbf{g}\rangle M_{\beta n}T_{\alpha k}\tilde{\mathbf{g}}\\ &= \lim_{K,N\to\infty}\sum_{\|k\|_\infty\leq K}\sum_{\|n\|_\infty\leq N}\langle\mathbf{f},M_{\beta n}T_{\alpha k}\tilde{\mathbf{g}}\rangle M_{\beta n}T_{\alpha k}\mathbf{g}.\end{aligned}$$

Similarly using Proposition 3.2.12 we get the convergence in the norm of $W_{\mathbb{H}}(L^p,L^q_v)$.

(ii) If $\mathcal{G}(\mathbf{g},\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{H})$ with dual window $\tilde{\mathbf{g}}=S_{\mathbf{g}}^{-1}(\mathbf{g})\in L^2(\mathbb{R}^d,\mathbb{H})$, then the inverse frame operator is given by

$$S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{f}=S_{\tilde{\mathbf{g}},\tilde{\mathbf{g}}}\mathbf{f}=\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\langle\mathbf{f},T_{\alpha k}M_{\beta n}\tilde{\mathbf{g}}\rangle T_{\alpha k}M_{\beta n}\tilde{\mathbf{g}}.$$

(iii) If $\mathbb{H}=\mathbb{C}$ then Theorem 3.4.1 coincides with Theorem 3.2 of [102] and Theorem 2 of [137]. Again if $\mathbb{H}=\mathbb{C}^n$ the invertibility of Gabor super frames on vector valued amalgam spaces is obtained. If we take $\mathbb{H}_1=\mathbb{H}_2\cdots=\mathbb{H}_r=\mathbb{C}$ and

$\mathbf{f}(x) = f(x) \bigoplus f(x) \bigoplus \cdots \bigoplus f(x)$ (r-times tensor product of $f(x)$ with itself), where $f \in W(L^p, L^q_w)$ as in Remark 3.2.13 (iv) we obtain the invertibility of multi-window Gabor frames on amalgam spaces (see Theorem 6 of [11]).

(iv) Since the frame operator $S_{\mathbf{g}} \in \mathcal{A}_w$ and $S_{\mathbf{g}}$ is invertible, the last line of the proof of Theorem 3.3.7, $S_{\mathbf{g}}^{-1}$ (as an operator on $L^2(\mathbb{R}^d, \mathbb{H})$) can be expressed as $S_{\mathbf{g}}^{-1}\mathbf{f}(x) = \sum\limits_{k \in \mathbb{Z}^d} G_n(x)(f(x - x_k))$, where the family of points $\{x_k\}$ may not lie in the lattice $\Lambda = \prod_{i=1}^{r} \alpha_i \mathbb{Z}^d \times \beta_i \mathbb{Z}^d$.

Chapter 4

Hilbert-Schmidt Frames

In this chapter, we study the HS-frame for separable Hilbert spaces. We present some characterizations of HS-frames and show that HS-frames share many important properties with frames. Then we show that the inverse of the HS-frame operator can be approximated using finite-dimensional methods. Also we present a classical perturbation result and prove that HS-frames are stable under small perturbations. Finally as an application we establish Parseval type identities and inequalities for HS-frames.

4.1 Preliminaries

We denote $\{\mathbb{K}_j : j \in J\} \subset \mathbb{K}$ as a sequence of Hilbert spaces and $\mathcal{L}(\mathbb{H}, \mathbb{K}_j)$ as the collection of all bounded linear operators from $\mathbb{H}$ to $\mathbb{K}_j$. Note that for any sequence $\{\mathbb{K}_j : j \in J\}$, we can always find a larger space $\mathbb{K}$ containing all the Hilbert space $\mathbb{K}_j$ by setting $\mathbb{K} = \bigoplus_{j\in J} \mathbb{K}_j$. The notion of a frame was extended to a g-frame by Sun [127]. We start with the definition of a g-frame.

Definition 4.1.1. *[127] A family $\{\Lambda_j \in \mathcal{L}(\mathbb{H}, \mathbb{K}_j) : j \in J\}$ is called a generalized frame, or simply a g-frame, for $\mathbb{H}$ with respect to $\{\mathbb{K}_j : j \in J\}$ if there are two constants $A, B > 0$ such that for all $f \in \mathbb{H}$*

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j(f)\|^2 \leq B\|f\|^2. \tag{4.1}$$

Suppose $\{\mathcal{X}_j : j \in J\}$ is a collection of normed spaces. Then $\prod\{\mathcal{X}_j : j \in J\}$ is a vector space if the linear operations are defined coordinatewise. We define

$$\bigoplus \mathcal{X}_j \equiv \Big\{x \in \prod_{j\in J} \mathcal{X}_j : \|x\| = \Big(\sum_{j\in J} \|x_j\|^2\Big)^{1/2} < \infty\Big\}.$$

with the inner product given by $\langle x, y\rangle = \sum_{j\in J} \langle x_j, y_j\rangle$. It is known that $\bigoplus \mathcal{X}_j$ is a Hilbert space if and only if it is so for each $\mathcal{X}_j$.

Let $\mathcal{L}(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on a complex separable Hilbert space $\mathbb{H}$. For a compact operator $T \in \mathcal{L}(\mathbb{H})$, the eigenvalues of the positive operator $|T| = (T^*T)^{1/2}$ are called the singular values of T and are denoted by $s_j(T)$. We arrange the singular values $s_j(T)$ in a decreasing order and these are repeated according to multiplicity, that is, $s_1(T) \geq s_2(T) \geq ... \geq 0$. For $1 \leq p < \infty$, the von Neumann−Schatten p-class C_p is defined to be the set of all compact operators T for which

$$\|T\|_p = (\tau|T|^p)^{\frac{1}{p}} = \left(\sum_{j=1}^{\infty} s_j^p(T)\right)^{\frac{1}{p}} < \infty, \tag{4.2}$$

where τ is the usual trace functional defined as $\tau(T) = \sum_{e\in E} \langle T(e), e\rangle$, and E is any orthonormal basis of $\mathbb{H}$. For $p = \infty$, let C_∞ denote the class of all compact operators with $\|T\|_\infty = s_1(T) < \infty$. For more information about a von Neumann−Schatten p-class we refer to [121]. We recall that C_2 is a Banach space with respect to $\|.\|_2$, and also it is a Hilbert space with the inner product defined by $\big[T, S\big]_\tau = \tau(S^*T)$. Moreover, C_2 is called the class of Hilbert−Schmidt operators on $\mathbb{H}$. An operator $T \in \mathcal{L}(\mathbb{H})$ belongs to the Hilbert−Schmidt class if and only if $\|T\|_{HS}^2 := \sum_{j\in J} \|Te_j\|^2 < \infty$, where $\{e_j\}_{j\in J}$ is any orthonormal basis for $\mathbb{H}$. Notice that $\|T\|_{HS} = \|T\|_2$.

Definition 4.1.2. *[123] A family $\{\mathcal{G}_j : j \in J\}$ of bounded linear operators from $\mathbb{H}$ to $C_2 \subseteq \mathcal{L}(\mathbb{K})$ is said to be a Hilbert−Schmidt frame, or simply a HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$, if there exist constants $A, B > 0$ such that for all $f \in \mathbb{H}$*

$$A\|f\|^2 \leq \sum_{j\in J} \|\mathcal{G}_j(f)\|_{HS}^2 \leq B\|f\|^2. \tag{4.3}$$

If $A = B = 1$, then $\{\mathcal{G}_j : j \in J\}$ is called the Parseval HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$. If the right-hand side of (4.3) holds, it is said to be a *HS-Bessel sequence* with bound B. If $\{f \in \mathbb{H} : \mathcal{G}_j(f) = 0, \forall j \in J\} = \{0\}$, then $\{\mathcal{G}_j : j \in J\}$ is called *HS-complete*. If $\{\mathcal{G}_j : j \in J\}$ is HS-complete and there are positive constants A and B such that for any finite subset $J_1 \subset J$ and $\mathcal{A}_j \in C_2, j \in J_1$,

$$A \sum_{j \in J_1} \|\mathcal{A}_j\|_{HS}^2 \leq \left\| \sum_{j \in J_1} \mathcal{G}_j{}^*(\mathcal{A}_j) \right\|^2 \leq B \sum_{j \in J_1} \|\mathcal{A}_j\|_{HS}^2, \tag{4.4}$$

where $\mathcal{G}_j^*$ is the adjoint operator of $\mathcal{G}_j$. Then $\{\mathcal{G}_j : j \in J\}$ is called a *HS-Riesz basis* for $\mathbb{H}$ with respect to $\mathbb{K}$.

Let $y_0 \in \mathbb{K}$ be an unit vector, the operator $\mathcal{U} : \mathbb{K} \to C_2 \subseteq \mathcal{L}(\mathbb{K})$, defined by $\mathcal{U}x = x \odot y_0$ is a linear isometry since $\|\mathcal{U}x\|_{HS} = \|x \odot y_0\|_{HS} = \|x\|$, where $\odot$ is defined in Definition 3.1.5. So we can consider $\mathbb{K}$ as subspace of C_2, and hence it is a subspace of $\mathcal{L}(\mathbb{K})$.

Lemma 4.1.3. *[123] Let $\{\Lambda_j : j \in J\}$ be a g-frame for $\mathbb{H}$ with respect to $\{\mathbb{K}_j : j \in J\}$. Then $\{\Lambda_j : j \in J\}$ is a HS-frame for $\mathbb{H}$ with respect to $\mathbb{K} = \bigoplus_{j \in J} \mathbb{K}_j$.*

In [127], Sun has shown that bounded quasi-projectors [69], frames of subspaces [30], pseudo-frames [107], oblique frames [36], outer frames [2], and time-frequency localization operators [46] are special classes of g-frames. Hence, Lemma 4.1.3 implies that each of these classes is also a class of HS-frames.

Remark 4.1.4. Each $\mathcal{G}_j \in \mathcal{L}(\mathbb{H}, C_2)$ is an operator-valued function. So HS-frames $\{\mathcal{G}_j : j \in J\}$, are operator-valued frames. In particular, if we consider $\mathbb{K}_j \subseteq \mathbb{K} \subseteq C_2 \subseteq \mathcal{L}(\mathbb{K})$, then g-frames for $\mathbb{H}$ with respect to $\{\mathbb{K}_j : j \in J\}$ can be considered as HS-frames for $\mathbb{H}$ with respect to $\mathbb{K}$. Thus HS-frames share many useful properties with g-frames.

4.2 Characterization of HS-Frames

In order to define the synthesis operator for a HS-frame, we need to show that the series appearing in the definition of a synthesis operator converges unconditionally. We start with the following lemma.

Lemma 4.2.1. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-Bessel sequence for $\mathbb{H}$ with bound B. Then for each sequence $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$, the series $\sum_{j\in J} {\mathcal{G}_j}^*(\mathcal{A}_j)$ converges unconditionally.*

Proof. Let $J_1 \subseteq J$ with $|J_1| < \infty$, then

$$\begin{aligned}
\Big\| \sum_{j\in J_1} {\mathcal{G}_j}^*(\mathcal{A}_j) \Big\| &= \sup_{h\in\mathbb{H},\, \|h\|=1} \Big| \Big\langle \sum_{j\in J_1} {\mathcal{G}_j}^*(\mathcal{A}_j), h \Big\rangle \Big| \\
&\leq \Big(\sum_{j\in J_1} \|\mathcal{A}_j\|_{HS}^2 \Big)^{1/2} \sup_{h\in\mathbb{H},\, \|h\|=1} \Big(\sum_{j\in J_1} \|\mathcal{G}_j(h)\|_{HS}^2 \Big)^{1/2} \\
&\leq \sqrt{B} \Big(\sum_{j\in J_1} \|\mathcal{A}_j\|_{HS}^2 \Big)^{1/2}.
\end{aligned}$$

It follows that $\sum_{j\in J} {\mathcal{G}_j}^*(\mathcal{A}_j)$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathbb{H}$. □

Definition 4.2.2. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$. Then the synthesis operator for $\{\mathcal{G}_j : j \in J\}$ is the operator $T : \bigoplus C_2 \to \mathbb{H}$ defined by $T(\{\mathcal{A}_j\}_{j\in J}) = \sum_{j\in J} {\mathcal{G}_j}^*(\mathcal{A}_j)$.*

The adjoint T^* of the synthesis operator is called the *analysis operator*. The following lemma provides a formula for the analysis operator.

Lemma 4.2.3. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$. Then the analysis operator $T^* : \mathbb{H} \to \bigoplus C_2$, given by $T^*(f) = \{\mathcal{G}_j(f)\}_{j\in J}$ is well defined.*

Proof. Let $f \in \mathbb{H}$ and $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$. Then

$$\begin{aligned}
\langle T^*(f), \{\mathcal{A}_j\}_{j\in J} \rangle &= \langle f, T\{\mathcal{A}_j\}_{j\in J} \rangle = \Big\langle f, \sum_{j\in J} {\mathcal{G}_j}^*(\mathcal{A}_j) \Big\rangle \\
&= \sum_{j\in J} [\mathcal{G}_j(f), \mathcal{A}_j]_\tau = \langle \{\mathcal{G}_j(f)\}_{j\in J}, \{\mathcal{A}_j\}_{j\in J} \rangle.
\end{aligned}$$

Hence $T^*(f) = \{\mathcal{G}_j(f)\}_{j\in J}$ is well defined. □

In the following proposition, we characterize the HS-Bessel sequence in terms of the synthesis operator.

Proposition 4.2.4. *A sequence $\{\mathcal{G}_j : j \in J\} \subseteq \mathcal{L}(\mathbb{H}, C_2)$ is a HS-Bessel sequence for $\mathbb{H}$ with bound B if and only if the synthesis operator T is a well defined bounded operator with $\|T\| \leq \sqrt{B}$.*

Proof. Let $\{\mathcal{G}_j : j \in J\}$ be a HS-Bessel sequence for $\mathbb{H}$ with bound B. Then by Lemma 4.2.1, T is a well defined bounded operator with $\|T\| \leq \sqrt{B}$.

Conversely, let T be a well defined and $\|T\| \leq \sqrt{B}$. Let $J_1 \subseteq J$ with $|J_1| < \infty$, then

$$\begin{aligned}\sum_{j\in J_1} \|\mathcal{G}_j(f)\|_{HS}^2 = \sum_{j\in J_1} \langle \mathcal{G}_j^* \mathcal{G}_j(f), f\rangle &= \left\langle T(\{\mathcal{G}_j(f)\}_{j\in J_1}), f\right\rangle \\ &\leq \|T\| \|\{\mathcal{G}_j(f)\}_{j\in J_1}\| \|f\|, \ \forall f \in \mathbb{H}.\end{aligned}$$

Therefore

$$\sum_{j\in J_1} \|\mathcal{G}_j(f)\|_{HS}^2 \leq \|T\| \left(\sum_{j\in J_1} \|\mathcal{G}_j(f)\|_{HS}^2 \right)^{1/2} \|f\| \leq \|T\|^2 \|f\|^2 \leq B\|f\|^2.$$

It follows that $\{\mathcal{G}_j : j \in J\}$ is a HS-Bessel sequence for $\mathbb{H}$ with bound B. □

Definition 4.2.5. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$. Then the HS-frame operator for $\{\mathcal{G}_j : j \in J\}$ is the operator $S : \mathbb{H} \to \mathbb{H}$ defined by*

$$Sf = TT^* f = \sum_{j\in J} \mathcal{G}_j^* \mathcal{G}_j(f), \quad f \in \mathbb{H}.$$

If $\{\mathcal{G}_j : j \in J\}$ is a HS-frame with bounds A and B, then we have

$$\langle Sf, f\rangle = \left\langle \sum_{j\in J} \mathcal{G}_j^* \mathcal{G}_j(f), f \right\rangle = \sum_{j\in J} \left[\mathcal{G}_j(f), \mathcal{G}_j(f)\right]_\tau = \sum_{j\in J} \|\mathcal{G}_j(f)\|_{HS}^2, \ \forall f \in \mathbb{H}.$$

Hence

$$A\langle f, f\rangle \leq \langle Sf, f\rangle \leq B\langle f, f\rangle, \text{ i.e., } AI \leq S \leq BI.$$

Therefore S is a bounded, invertible and positive self-adjoint operator. Also, the following reconstruction formula holds for all $f \in \mathbb{H}$

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j\in J} \mathcal{G}_j^*\mathcal{G}_j S^{-1}f = \sum_{j\in J} S^{-1}\mathcal{G}_j^*\mathcal{G}_j f. \tag{4.5}$$

Moreover, $\{\mathcal{G}_j S^{-1} : j \in J\}$ is a HS-frame with bounds B^{-1} and A^{-1}. We call $\{\tilde{\mathcal{G}}_j = \mathcal{G}_j S^{-1} : j \in J\}$ the *canonical dual HS-frame* of $\{\mathcal{G}_j : j \in J\}$. A HS-frame $\{\Gamma_j : j \in J\}$ is called an *alternate dual HS-frame* of $\{\mathcal{G}_j : j \in J\}$ if for all $f \in \mathbb{H}$ the following identity holds:

$$f = \sum_{j\in J} \mathcal{G}_j^*\Gamma_j f = \sum_{j\in J} {\Gamma_j}^*\mathcal{G}_j f. \tag{4.6}$$

Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame. For every $K \subset J$, define the bounded linear operators $S_K, S_{K^c} : \mathbb{H} \to \mathbb{H}$ by

$$S_K f = \sum_{j\in K} \mathcal{G}_j^*\mathcal{G}_j(f), \quad S_{K^c} f = \sum_{j\in K^c} \mathcal{G}_j^*\mathcal{G}_j(f).$$

Clearly, S_K and S_{K^c} are self-adjoint.

The following result provides a connection between a HS-frame and a HS operator.

Proposition 4.2.6. *Let $S \in \mathcal{L}(\mathbb{H})$ be a HS-frame operator. Then, S is a Hilbert-Schmidt operator if and only if $\mathbb{H}$ is finite-dimensional.*

Proof. Let $\{e_n\}_{n\in J}$ be an orthonormal basis for $\mathbb{H}$. By Lemma 4.2.1, we get

$$\begin{aligned} \|S\|_{HS}^2 = \sum_{n\in J} \|Se_n\|^2 = \sum_{n\in J} \left\| \sum_{j\in J} {\mathcal{G}_j}^*\mathcal{G}_j(e_n) \right\|^2 &\leq B \sum_{n\in J}\sum_{j\in J} \|\mathcal{G}_j(e_n)\|_{HS}^2 \\ &\leq B \sum_{n\in J} B\|e_n\|^2. \end{aligned}$$

If $\dim \mathbb{H} = \text{card } J < \infty$, we have $\|S\|_{HS}^2 \leq B^2 \text{ card } J < \infty$.

Conversely, let S be a Hilbert–Schmidt operator. Since Hilbert−Schmidt operators are compact, S is compact. Also, S is invertible on $\mathbb{H}$. Thus $SS^{-1} = I$ implies that the identity I must be a compact operator. Hence $\dim \mathbb{H} < \infty$. ☐

Lemma 4.2.7. *[35] Suppose that $U : \mathbb{K} \to \mathbb{H}$ is a bounded surjective operator. Then there exists a bounded operator (called the pseudo-inverse of U) $U^{\dagger} : \mathbb{H} \to \mathbb{K}$ for which*

$$UU^{\dagger} f = f, \ \ \forall f \in \mathbb{H}.$$

If U is a bounded invertible operator, then $U^{\dagger} = U^{-1}$.

In the following proposition we establish a relationship between a HS-frame and the associated synthesis operator.

Proposition 4.2.8. *A sequence $\{\mathcal{G}_j : j \in J\} \subseteq \mathcal{L}(\mathbb{H}, C_2)$ is a HS-frame for $\mathbb{H}$ if and only if the synthesis operator T is a well defined, bounded and surjective operator.*

Proof. If $\{\mathcal{G}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$, then $S = TT^*$ is invertible. So T is surjective. Conversely, let T be well defined, bounded and surjective operator. Then by Proposition 4.2.4, the sequence $\{\mathcal{G}_j : j \in J\}$ is a HS-Bessel sequence for $\mathbb{H}$. Since T is surjective, by Lemma 4.2.7, there exists an operator $T^{\dagger} : \mathbb{H} \to \bigoplus C_2$ such that $TT^{\dagger} = I$. Hence $(T^{\dagger})^* T^* = I$. Then for all $f \in \mathbb{H}$,

$$\|f\|^2 \leq \|(T^{\dagger})^*\|^2 \|T^* f\|^2 = \|T^{\dagger}\|^2 \|T^* f\|^2 = \|T^{\dagger}\|^2 \sum_{j \in J} \|\mathcal{G}_j(f)\|_{HS}^2.$$

It follows that $\{\mathcal{G}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$ with lower HS-frame bound $\|T^{\dagger}\|^{-2}$ and upper HS-frame bound $\|T\|^2$. ☐

Now we establish the relation between a HS-frame and a HS-Riesz basis. We first establish the following lemma.

Lemma 4.2.9. *A sequence $\{\mathcal{G}_j : j \in J\} \subseteq \mathcal{L}(\mathbb{H}, C_2)$ is a HS-Riesz basis for $\mathbb{H}$ with bounds A and B if and only if the synthesis operator T is a linear homeomorphism such that*

$$A \sum_{j \in J} \|\mathcal{A}_j\|_{HS}^2 \leq \|T(\{\mathcal{A}_j\}_{j \in J})\|^2 \leq B \sum_{j \in J} \|\mathcal{A}_j\|_{HS}^2, \ \ \forall \{\mathcal{A}_j\}_{j \in J} \in \bigoplus C_2. \tag{4.7}$$

Proof. If $\{\mathcal{G}_j : j \in J\}$ is a HS-Riesz basis for $\mathbb{H}$ with bounds A and B, then from the definition of HS-Riesz bases, the synthesis operator T is a bounded, injective operator with the closed range $T(\bigoplus C_2)$ and $\|T\| \le \sqrt{B}$. So, from Proposition 4.2.4, the sequence $\{\mathcal{G}_j : j \in J\}$ is a HS-Bessel sequence for $\mathbb{H}$. Let $f \in [T(\bigoplus C_2)]^{\perp}$, then $\{\mathcal{G}_j(f)\}_{j\in J} \in \bigoplus C_2$. Hence we get

$$0 = \langle T(\{\mathcal{G}_j(f)\}_{j\in J}), f\rangle = \left\langle \sum_{j\in J} \mathcal{G}_j^*\mathcal{G}_j(f), f \right\rangle = \sum_{j\in J}[\mathcal{G}_j(f), \mathcal{G}_j(f)]_\tau = \sum_{j\in J} \|\mathcal{G}_j(f)\|_{HS}^2.$$

It implies that $\mathcal{G}_j(f) = 0$, for all $j \in J$. Since $\{\mathcal{G}_j : j \in J\}$ is HS-complete, we obtain $f = 0$, which proves $T(\bigoplus C_2) = \mathbb{H}$. Hence T is a linear homeomorphism. Also, from Equation (4.4), for every $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$ we obtain

$$A\sum_{j\in J} \|\mathcal{A}_j\|_{HS}^2 \le \|T(\{\mathcal{A}_j\}_{j\in J})\|^2 \le B\sum_{j\in J} \|\mathcal{A}_j\|_{HS}^2.$$

Conversely, if T is a linear homeomorphism satisfying (4.7), then by Proposition 4.2.8, we find that $\{\mathcal{G}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$ with bounds $\|T^\dagger\|^{-2}$ and $\|T\|^2$. If $\mathcal{G}_j(f) = 0$ for $f \in \mathbb{H}$ and all $j \in J$, then $\|f\|^2 \le \|T^\dagger\|^2 \sum_{j\in J} \|\mathcal{G}_j(f)\|_{HS}^2 = 0$ implies $f = 0$. Thus $\{\mathcal{G}_j : j \in J\}$ is a HS-complete. Now by the definition of HS-Riesz bases and the inequalities (4.7), we conclude that $\{\mathcal{G}_j : j \in J\}$ is a HS-Riesz basis for $\mathbb{H}$ with bounds A and B. This completes the proof. □

Theorem 4.2.10. *Let $\{\mathcal{G}_j : j \in J\} \subseteq \mathcal{L}(\mathbb{H}, C_2)$. Then the following are equivalent:*

(*i*) *The sequence $\{\mathcal{G}_j : j \in J\}$ is a HS-Riesz basis for $\mathbb{H}$ with bounds A and B.*

(*ii*) *The sequence $\{\mathcal{G}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$ with bounds A and B, and $\{\mathcal{G}_j : j \in J\}$ is an $\bigoplus C_2$-linearly independent family, i.e., if $\sum_{j\in J} \mathcal{G}_j^*(\mathcal{A}_j) = 0$ for $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$, then $\mathcal{A}_j = 0$ for all $j \in J$.*

Proof. $(i) \Rightarrow (ii)$ From Lemma 4.2.9, the operator T is a linear homeomorphism with $\|T^\dagger\|^2 = \|T^{-1}\|^2 \le \frac{1}{A}$ and $\|T\|^2 \le B$. Thus the operator T is surjective with $\|T^\dagger\|^{-2} \ge A$ and

$$\ker T = \left\{\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2 : T(\{\mathcal{A}_j\}_{j\in J}) = \sum_{j\in J} \mathcal{G}_j^*(\mathcal{A}_j) = 0\right\} = \{0\}. \tag{4.8}$$

It follows that $\{\mathcal{G}_j : j \in J\}$ is an $\bigoplus C_2$-linearly independent family. Hence by Proposition 4.2.8, the statement (i) implies (ii).

$(ii) \Rightarrow (i)$ From Proposition 4.2.8 and (4.8), the operator T is a linear homeomorphism with $\|T\|^2 \leq B$, so is the adjoint T^*. Since $\{\mathcal{G}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$ with bounds A and B, $\|T^*(f)\|^2 = \sum_{j\in J} \|\mathcal{G}_j(f)\|_{HS}^2 \geq A\|f\|^2$. So, $\|T^{-1}\|^2 = \|(T^*)^{-1}\|^2 \leq A^{-1}$. Hence for all $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$, we have

$$\begin{aligned}
\|T(\{\mathcal{A}_j\}_{j\in J})\|^2 \leq \|T\|^2 \|\{\mathcal{A}_j\}_{j\in J}\|^2_{\bigoplus C_2} &\leq B\sum_{j\in J} \|\mathcal{A}_j\|^2_{HS},\\
\|\{\mathcal{A}_j\}_{j\in J}\|^2_{\bigoplus C_2} = \|T^{-1}T(\{\mathcal{A}_j\}_{j\in J})\|^2_{\bigoplus C_2} &\leq \|T^{-1}\|^2 \|T(\{\mathcal{A}_j\}_{j\in J})\|^2\\
&\leq \frac{1}{A}\|T(\{\mathcal{A}_j\}_{j\in J})\|^2.
\end{aligned}$$

From Lemma 4.2.9, the statement (ii) implies (i). This completes the proof. □

4.3 Approximation of the Inverse HS-Frame Operator

In this section, $\mathbb{H}$ denotes a finite dimensional Hilbert space and let $\{J_n\}_{n=1}^{\infty}$ be a family of finite subsets of J such that $J_1 \subseteq J_2 \subseteq ... \subseteq J_n \nearrow J$. Given a family $\{\mathcal{G}_j : j \in J\} \subseteq \mathcal{L}(\mathbb{H}, C_2)$, we define the space $\mathbb{H}_n = \text{span}\{\mathcal{G}_j{}^*(C_2) : j \in J_n\}$. Then it is easy to see that $\{\mathcal{G}_j : j \in J_n\}$ is a HS-frame for $\mathbb{H}_n$. The HS-frame operator for $\{\mathcal{G}_j : j \in J_n\}$ is

$$S_n : \mathbb{H}_n \to \mathbb{H}_n, \quad S_n f = \sum_{j\in J_n} \mathcal{G}_j{}^*\mathcal{G}_j f.$$

We show that the inverse HS-frame operator S^{-1} can be approximated by operators S_n^{-1} using finite dimensional methods. Here S_n is an operator on the finite dimensional space $\mathbb{H}_n$. In the following theorem, we generalize Theorem 3.1 in [31] from the setting of Hilbert space frames to HS-frames.

Theorem 4.3.1. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with bounds A and B. Then for every $f, g \in \mathbb{H}$*

$$\langle g, S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\rangle \to \langle g, S^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\rangle \quad \text{as} \quad n \to \infty, \tag{4.9}$$

if and only if for every $j \in J$ and every $f \in \mathbb{H}$ there exists a constant c_j such that

$$\|S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\| \le c_j, \quad \forall n \text{ such that } j \in J_n. \tag{4.10}$$

Proof. Assume that (4.9) is satisfied. Fix $f \in \mathbb{H}$ and $j \in J$. For every n with $j \in J_n$, define

$$F_n : \mathbb{H} \to \mathbb{C}, \quad F_n(g) = \langle g, S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\rangle.$$

Then each F_n is continuous, and by (4.9) the family $\{F_n\}$ converges pointwise. By Banach Steinhaus theorem there is a constant c_j such that $\|F_n\| = \|S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\| \le c_j$ for all n.

Conversely, suppose (4.10) is satisfied. Let $f \in \mathbb{H}$. Fix a $j \in J$, and take an N such that $j \in J_n$ for all $n \ge N$. Define $\Phi_n = S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f - S^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f$, $n \ge N$. Then

$$\begin{aligned} S\Phi_n &= SS_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f - \mathcal{G}_j{}^*\mathcal{G}_j f \\ &= S_nS_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f + \sum_{j\in J\setminus J_n} \mathcal{G}_j{}^*\mathcal{G}_j S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f - \mathcal{G}_j{}^*\mathcal{G}_j f \\ &= \sum_{j\in J\setminus J_n} \mathcal{G}_j{}^*\mathcal{G}_j S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f, \end{aligned}$$

thus $\Phi_n = \sum_{j\in J\setminus J_n} S^{-1}\mathcal{G}_j{}^*\mathcal{G}_j S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f$. Therefore, for $g \in \mathbb{H}$, we obtain

$$\begin{aligned} |\langle g, \Phi_n\rangle|^2 &= \left|\left\langle g, \sum_{j\in J\setminus J_n} S^{-1}\mathcal{G}_j{}^*\mathcal{G}_j S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\right\rangle\right|^2 \\ &= \left|\sum_{j\in J\setminus J_n} \langle \mathcal{G}_j S^{-1} g, \mathcal{G}_j S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\rangle\right|^2 \\ &\le \sum_{j\in J\setminus J_n} \|\mathcal{G}_j S^{-1} g\|_{HS}^2 \sum_{j\in J\setminus J_n} \|\mathcal{G}_j S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\|_{HS}^2 \\ &\le B\|S_n^{-1}\mathcal{G}_j{}^*\mathcal{G}_j f\|^2 \sum_{j\in J\setminus J_n} \|\mathcal{G}_j S^{-1} g\|_{HS}^2 \\ &\le Bc_j^2 \sum_{j\in J\setminus J_n} \|\mathcal{G}_j S^{-1} g\|_{HS}^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus $|\langle g, \Phi_n \rangle| \to 0$ as $n \to \infty$, i.e., $\langle g, S_n^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f\rangle \to \langle g, S^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f\rangle$ as $n \to \infty$. Hence we have the desired result. □

The orthogonal projection $P_n : \mathbb{H} \to \mathbb{H}_n$ is given by $P_n f = \sum_{j\in J_n} S_n^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f$ for all $f \in \mathbb{H}$. Since $\{P_n\}_{n=1}^{\infty}$ is increasing and $\overline{(\cup_{n=1}^{\infty}\mathbb{H}_n)} = \mathbb{H}$, we have $P_n f \to f = \sum_{j\in J} S^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f$, as $n \to \infty$. Following Christensen [33], we say that the *projection method* works if (4.9) is satisfied for every $f, g \in \mathbb{H}$ and the *strong projection method* works if

$$\sum_{j\in J_n} |\langle f, S_n^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f - S^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f\rangle|^2 \to 0 \quad \text{as} \quad n \to \infty,$$

is satisfied for every $f \in \mathbb{H}$. Note that the projection method works if the strong projection method works. Since for any $f \in \mathbb{H}$, we have

$$\begin{aligned}
& \sum_{j\in J_n} |\langle f, S_n^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f - S^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f\rangle|^2 \\
= & \sum_{j\in J_n} |\langle P_n f, S_n^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f\rangle - \langle f, S^{-1}{\mathcal{G}_j}^*\mathcal{G}_j f\rangle|^2 \\
= & \sum_{j\in J_n} |\langle \mathcal{G}_j(S_n^{-1}P_n f - S^{-1}f), \mathcal{G}_j f\rangle|^2 \\
\leq & \sum_{j\in J_n} \|\mathcal{G}_j(S_n^{-1}P_n f - S^{-1}f)\|_{HS}^2 \cdot \sum_{j\in J_n} \|\mathcal{G}_j f\|_{HS}^2 \\
\leq & B^2\|S_n^{-1}P_n f - S^{-1}f\|^2 \cdot \|f\|^2,
\end{aligned}$$

it follows that the strong projection method works if any one of the conditions appearing in Theorem 4.3.2 is satisfied. The result stated in the following can be found in ([33], Theorem 4.5) for Hilbert space frames. We generalize that result to HS-frames as follows.

Theorem 4.3.2. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with the upper bound B. Then the following are equivalent:*

(i) $\|S_n^{-1}P_n f - S^{-1}f\| \to 0$ *as* $n \to \infty$, $\forall f \in \mathbb{H}$.

(ii) $\|(S - S_n)S_n^{-1}P_n f\| \to 0$ *as* $n \to \infty$, $\forall f \in \mathbb{H}$.

(iii) $\sum_{j\in J\setminus J_n} \|\mathcal{G}_j S_n^{-1}P_n f\|_{HS}^2 \to 0$ *as* $n \to \infty$, $\forall f \in \mathbb{H}$.

Proof. $(i) \Leftrightarrow (ii)$ Let $f \in \mathbb{H}$. Then we have

$$\begin{aligned} S_n^{-1}P_nf - S^{-1}f &= S^{-1}(P_nf - f) + S^{-1}(S - S_n)S_n^{-1}P_nf \\ (S - S_n)S_n^{-1}P_nf &= S(S_n^{-1}P_nf - S^{-1}f) - (P_nf - f) \\ \Rightarrow \qquad \|S_n^{-1}P_nf - S^{-1}f\| &\leq \|S^{-1}\| \cdot \|P_nf - f\| + \|S^{-1}\| \cdot \|(S - S_n)S_n^{-1}P_nf\| \\ \|(S - S_n)S_n^{-1}P_nf\| &\leq \|S\| \cdot \|S_n^{-1}P_nf - S^{-1}f\| + \|P_nf - f\|. \end{aligned}$$

Since $\|P_nf - f\| \to 0$ as $n \to \infty$, we obtain that (i) and (ii) are equivalent.
$(i) \Rightarrow (iii)$ For every $f \in \mathbb{H}$, we have

$$\begin{aligned} &\Big(\sum_{j\in J\setminus J_n} \|\mathcal{G}_jS_n^{-1}P_nf\|_{HS}^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{j\in J\setminus J_n} \|\mathcal{G}_j(S_n^{-1}P_nf - S^{-1}f)\|_{HS}^2\Big)^{\frac{1}{2}} + \Big(\sum_{j\in J\setminus J_n} \|\mathcal{G}_jS^{-1}f\|_{HS}^2\Big)^{\frac{1}{2}} \\ &\leq \sqrt{B}\|S_n^{-1}P_nf - S^{-1}f\| + \Big(\sum_{j\in J\setminus J_n} \|\mathcal{G}_jS^{-1}f\|_{HS}^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Since $\sum_{j\in J\setminus J_n} \|\mathcal{G}_jS^{-1}f\|_{HS}^2 \to 0$ as $n \to \infty$, the result follows.
$(iii) \Rightarrow (ii)$ For every $f \in \mathbb{H}$, we obtain

$$\begin{aligned} \|(S - S_n)S_n^{-1}P_nf\|^2 &= \sup_{\|g\|=1} |\langle (S - S_n)S_n^{-1}P_nf, g\rangle|^2 \\ &= \sup_{\|g\|=1} \Big|\Big\langle \sum_{j\in J\setminus J_n} \mathcal{G}_j{}^*\mathcal{G}_jS_n^{-1}P_nf, g\Big\rangle\Big|^2 \\ &\leq \sup_{\|g\|=1} \sum_{j\in J\setminus J_n} \|\mathcal{G}_jS_n^{-1}P_nf\|_{HS}^2 \cdot \sum_{j\in J\setminus J_n} \|\mathcal{G}_jg\|_{HS}^2 \\ &\leq B \sum_{j\in J\setminus J_n} \|\mathcal{G}_jS_n^{-1}P_nf\|_{HS}^2. \end{aligned}$$

Since $\sum\limits_{j\in J\setminus J_n} \|\mathcal{G}_jS_n^{-1}P_nf\|_{HS}^2 \to 0$ as $n \to \infty$, we have the desired result. □

Now we derive a general method to approximate the inverse HS-frame operator. We first establish the following result, which generalizes Lemmas 3.1 and 3.2 in [28] to HS-frames in a more general form.

Proposition 4.3.3. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with bounds A and B. Let $\lambda > 1$ be a scalar. Then for any $n \in \mathbb{N}$ there exists a number $m(n)$ such that the following holds:*

(i) $\frac{A}{\lambda}\|f\|^2 \leq \sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j(f)\|^2_{HS}$ *for all* $f \in \mathbb{H}_n$.

(ii) $\{\mathcal{G}_j P_n\}_{j\in J_{n+m(n)}}$ *is a HS-frame for* $\mathbb{H}_n$ *with bounds* $\frac{A}{\lambda}$ *and* B. *Moreover, the HS-frame operator for* $\{\mathcal{G}_j P_n\}_{j\in J_{n+m(n)}}$ *is* $P_n S_{n+m(n)} : \mathbb{H}_n \to \mathbb{H}_n$, *with*

$$\|P_n S_{n+m(n)}\| \leq B, \quad \text{and} \quad \|(P_n S_{n+m(n)})^{-1}\| \leq \frac{\lambda}{A}.$$

Proof. (i) Let $n \in \mathbb{N}$ and $\lambda > \mu > 1$. Choose $\varepsilon > 0$ such that $\sqrt{A/\mu} - \sqrt{B}\varepsilon \geq \sqrt{A/\lambda}$. Since $\{f \in \mathbb{H}_n : \|f\| = 1\}$ is compact, there exist a finite set of elements $g_k \in \mathbb{H}_n$ with $\|g_k\| = 1$, for all k such that the balls $B(g_k, \varepsilon) = \{f \in \mathbb{H}_n : \|f - g_k\| \leq \varepsilon\}$ cover the set $\{f \in \mathbb{H}_n : \|f\| = 1\}$. Since $\{\mathcal{G}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$, we have $A \leq \sum_{j\in J} \|\mathcal{G}_j(g_k)\|^2_{HS}$ for all k. Hence we can choose $m(n)$ such that

$$\frac{A}{\mu} \leq \sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j(g_k)\|^2_{HS}, \quad \forall k.$$

Now let $f \in \mathbb{H}_n$ with $\|f\| = 1$. Choose k such that $f \in B(g_k, \varepsilon)$. Therefore

$$\begin{aligned}\Big(\sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j(f)\|^2_{HS}\Big)^{\frac{1}{2}} &\geq \Big(\sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j(g_k)\|^2_{HS}\Big)^{\frac{1}{2}} - \Big(\sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j(f-g_k)\|^2_{HS}\Big)^{\frac{1}{2}} \\ &\geq \sqrt{A/\mu} - \sqrt{B}\|f - g_k\| \geq \sqrt{A/\mu} - \sqrt{B}\varepsilon \geq \sqrt{A/\lambda}.\end{aligned}$$

(ii) Since $P_n f = f$ for all $f \in \mathbb{H}_n$, from (i) we get

$$\begin{aligned}\frac{A}{\lambda}\|f\|^2 \leq \sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j(f)\|^2_{HS} &= \sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j P_n(f)\|^2_{HS} \\ &\leq \sum_{j\in J} \|\mathcal{G}_j P_n(f)\|^2_{HS} \leq B\|f\|^2, \ \forall f \in \mathbb{H}_n.\end{aligned}$$

Hence $\{\mathcal{G}_j P_n\}_{j\in J_{n+m(n)}}$ is a HS-frame for $\mathbb{H}_n$ with bounds A/λ and B. Moreover,

$$P_n S_{n+m(n)} f = \sum_{j\in J_{n+m(n)}} P_n \mathcal{G}_j^* \mathcal{G}_j P_n f = \sum_{j\in J_{n+m(n)}} (\mathcal{G}_j P_n)^* (\mathcal{G}_j P_n) f, \ \ \forall f \in \mathbb{H}_n.$$

Therefore $P_n S_{n+m(n)}$ is the HS-frame operator for $\{\mathcal{G}_j P_n\}_{j\in J_{n+m(n)}}$. Now the norm estimates follow from the fact that

$$B = \sup_{\|f\|=1} \sum_{j\in J} \|\mathcal{G}_j P_n(f)\|_{HS}^2 = \sup_{\|f\|=1} \langle P_n S_{n+m(n)} f, f\rangle = \|P_n S_{n+m(n)}\|,$$

and $\|(P_n S_{n+m(n)})^{-1}\| = \frac{\lambda}{A}$, which again follows from the properties of dual HS-frames. □

Remark 4.3.4. If we consider $\lambda = 2$ in Proposition 4.3.3, then we obtain similar inequalities as in Lemmas 3.1 and 3.2 in [28].

Now we are ready to prove that S^{-1} can be approximated arbitrarily closely in the strong operator topology using the operators $(P_n S_{n+m(n)})^{-1} P_n$. We use Proposition 4.3.3 to prove the following theorem.

Theorem 4.3.5. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with bounds A and B. For a fixed $\lambda > 1$, and for any $n \in \mathbb{N}$, choose $m(n)$ such that for all $f \in \mathbb{H}_n$*

$$\frac{A}{\lambda}\|f\|^2 \leq \sum_{j\in J_{n+m(n)}} \|\mathcal{G}_j(f)\|_{HS}^2.$$

Then $(P_n S_{n+m(n)})^{-1} P_n f \to S^{-1} f$ as $n \to \infty$, for all $f \in \mathbb{H}$.

Proof. Let $f \in \mathbb{H}$. Since $(P_n - I)S^{-1} f \to 0$ as $n \to \infty$ and

$$(P_n S_{n+m(n)})^{-1} P_n f - S^{-1} f = (P_n S_{n+m(n)})^{-1} P_n f - P_n S^{-1} f + (P_n - I) S^{-1} f,$$

it is enough to show that $(P_n S_{n+m(n)})^{-1} P_n f - P_n S^{-1} f \to 0$ as $n \to \infty$. Using Proposition 4.3.3, we obtain

$$\|(P_n S_{n+m(n)})^{-1} P_n f - P_n S^{-1} f\|$$

$$\begin{aligned}
&\leq \|(P_nS_{n+m(n)})^{-1}\| \cdot \|P_nf - P_nS_{n+m(n)}P_nS^{-1}f\| \\
&\leq \frac{\lambda}{A}\|S_{n+m(n)}P_nS^{-1}f - f\| \\
&\leq \frac{\lambda}{A}\Big(\|S_{n+m(n)}(P_n - I)S^{-1}f\| + \|S_{n+m(n)}S^{-1}f - f\|\Big) \\
&\leq \frac{\lambda}{A}\Big(B\|(P_n - I)S^{-1}f\| + \Big\|\sum_{j\in J\setminus J_{n+m(n)}} {\mathcal{G}_j}^*\mathcal{G}_jS^{-1}f\Big\|\Big) \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Hence, we have the desired result. □

Now we are in a position to prove the generalization of Theorem 4 in [26] from the setting of Hilbert space frames to HS-frames.

Theorem 4.3.6. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$. Then the following are equivalent:*

(i) $\sum_{j\in J_n} S_n^{-1}{\mathcal{G}_j}^*(\mathcal{A}_j) \to \sum_{j\in J} S^{-1}{\mathcal{G}_j}^*(\mathcal{A}_j)$ *as* $n \to \infty$*, for all* $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$*.*

(ii) $S_n^{-1}\sum_{j\in J_n} {\mathcal{G}_j}^*(\mathcal{A}_j) \to 0$ *as* $n\to\infty$*, for all* $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$ *with*

$$\sum_{j\in J} {\mathcal{G}_j}^*(\mathcal{A}_j) = 0.$$

Proof. Let T be the synthesis operator for $\{\mathcal{G}_j : j \in J\}$. Since $\bigoplus C_2$ is the orthogonal sum of the range of T^* and the kernel of T, we can write any $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$ as $\{\mathcal{A}_j\}_{j\in J} = \{\mathcal{G}_j(g)\}_{j\in J} + \{\mathcal{F}_j\}_{j\in J}$ for some $g \in \mathbb{H}$ and $\{\mathcal{F}_j\}_{j\in J} \in \operatorname{Ker} T$. Then

$$\sum_{j\in J_n} S_n^{-1}{\mathcal{G}_j}^*(\mathcal{A}_j) = \sum_{j\in J_n} S_n^{-1}{\mathcal{G}_j}^*\mathcal{G}_j(g) + \sum_{j\in J_n} S_n^{-1}{\mathcal{G}_j}^*(\mathcal{F}_j) = P_ng + S_n^{-1}\sum_{j\in J_n} {\mathcal{G}_j}^*(\mathcal{F}_j).$$

Also, we have

$$\sum_{j\in J} S^{-1}{\mathcal{G}_j}^*(\mathcal{A}_j) = \sum_{j\in J} S^{-1}{\mathcal{G}_j}^*\mathcal{G}_j(g) + \sum_{j\in J} S^{-1}{\mathcal{G}_j}^*(\mathcal{F}_j) = g + S^{-1}\sum_{j\in J} {\mathcal{G}_j}^*(\mathcal{F}_j) = g,$$

from which (i) and (ii) are equivalent. □

4.4 Stability of HS-Frames

In this section, we study the stability results for HS-frames. Before we prove the main results of this section, we first need the following lemma.

Lemma 4.4.1. *[27] Let $\mathcal{X}$ be a Banach space, $U : \mathcal{X} \to \mathcal{X}$ is a linear operator. If there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Ux - x\| \leq \lambda_1 \|x\| + \lambda_2 \|Ux\|, \ \forall x \in \mathcal{X}.$$

Then U is a bounded invertible operator on $\mathcal{X}$, and

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \leq \|Ux\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|x\|, \ \frac{1-\lambda_2}{1+\lambda_1}\|x\| \leq \|U^{-1}x\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|x\|, \forall x \in \mathcal{X}.$$

The following is a fundamental result in the study of the stability of frames.

Proposition 4.4.2. (*[27],* Theorem 2) *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for some Hilbert space $\mathbb{H}$ with bounds A, B. Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathbb{H}$ and assume that there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that $\max(\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2) < 1$ and*

$$\left\| \sum_{i=1}^{n} c_i(f_i - g_i) \right\| \leq \lambda_1 \left\| \sum_{i=1}^{n} c_i f_i \right\| + \lambda_2 \left\| \sum_{i=1}^{n} c_i g_i \right\| + \mu \left[\sum_{i=1}^{n} |c_i|^2 \right]^{1/2} \tag{4.11}$$

for all $c_1, ..., c_n (n \in \mathbb{N})$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame for $\mathbb{H}$ with bounds

$$A \left(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1 + \lambda_2} \right)^2, \ B \left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1 - \lambda_2} \right)^2.$$

Similar to ordinary frames, HS-frames are stable under small perturbations. The stability of HS-frames is discussed in the following theorem.

Theorem 4.4.3. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$. Let A, B be the frame bounds. Suppose that $\Gamma_j \in \mathcal{L}(\mathbb{H}, C_2)$ and there exist constants $\lambda_1, \lambda_2, \mu \geq 0$*

such that $\max(\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2) < 1$ *and one of the following two conditions is satisfied:*

$$\begin{aligned} &\Big(\sum_{j\in J}\|(\mathcal{G}_j - \Gamma_j)f\|_{HS}^2\Big)^{1/2} \\ \leq\ & \lambda_1\Big(\sum_{j\in J}\|\mathcal{G}_j(f)\|_{HS}^2\Big)^{1/2} + \lambda_2\Big(\sum_{j\in J}\|\Gamma_j(f)\|_{HS}^2\Big)^{1/2} + \mu\|f\|, \end{aligned} \tag{4.12}$$

for every $f \in \mathbb{H}$, *or*

$$\begin{aligned} &\Big\|\sum_{j\in J_1}({\mathcal{G}_j}^* - {\Gamma_j}^*)\mathcal{A}_j\Big\| \\ \leq\ & \lambda_1\Big\|\sum_{j\in J_1}{\mathcal{G}_j}^*(\mathcal{A}_j)\Big\| + \lambda_2\Big\|\sum_{j\in J_1}{\Gamma_j}^*(\mathcal{A}_j)\Big\| + \mu\Big(\sum_{j\in J_1}\|\mathcal{A}_j\|_{HS}^2\Big)^{1/2}, \end{aligned} \tag{4.13}$$

for any finite subset $J_1 \subset J$ *and* $\mathcal{A}_j \in C_2$. *Then* $\{\Gamma_j : j \in J\}$ *is a HS-frame for* $\mathbb{H}$ *with bounds*

$$A\left(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1+\lambda_2}\right)^2, \ \ B\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)^2. \tag{4.14}$$

Proof. First, we assume that (4.12) is satisfied. Notice that

$$\sum_{j\in J}\|\mathcal{G}_j(f)\|_{HS}^2 \leq B\|f\|^2.$$

From (4.12) we see that

$$\Big(\sum_{j\in J}\|(\mathcal{G}_j - \Gamma_j)f\|_{HS}^2\Big)^{1/2} \leq \Big(\lambda_1\sqrt{B} + \mu\Big)\|f\| + \lambda_2\Big(\sum_{j\in J}\|\Gamma_j(f)\|_{HS}^2\Big)^{1/2}.$$

Using the triangle inequality, we get

$$\Big(\sum_{j\in J}\|(\mathcal{G}_j - \Gamma_j)f\|_{HS}^2\Big)^{1/2} \geq \Big(\sum_{j\in J}\|\Gamma_j(f)\|_{HS}^2\Big)^{1/2} - \Big(\sum_{j\in J}\|\mathcal{G}_j(f)\|_{HS}^2\Big)^{1/2}.$$

Hence

$$\begin{aligned}(1-\lambda_2)\Big(\sum_{j\in J}\|\mathit{\Gamma}_j(f)\|_{HS}^2\Big)^{1/2} &\leq \Big(\lambda_1\sqrt{B}+\mu\Big)\|f\|+\Big(\sum_{j\in J}\|\mathcal{G}_j(f)\|_{HS}^2\Big)^{1/2}\\ &\leq \sqrt{B}\Big(1+\lambda_1+\frac{\mu}{\sqrt{B}}\Big)\|f\|.\end{aligned}$$

Therefore,

$$\sum_{j\in J}\|\mathit{\Gamma}_j(f)\|_{HS}^2 \leq B\left(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)^2\|f\|^2.$$

Similarly we can prove that

$$\sum_{j\in J}\|\mathit{\Gamma}_j(f)\|_{HS}^2 \geq A\left(1-\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}}}{1+\lambda_2}\right)^2\|f\|^2.$$

Next, we assume that (4.13) is satisfied. Let T and S denote the synthesis operator and frame operator associated with $\{\mathcal{G}_j : j \in J\}$. Also, let V denote the synthesis operator associated with $\{\mathit{\Gamma}_j : j \in J\}$. Since $\{\mathcal{G}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$ with bounds A and B, by Proposition 4.2.4, T is a bounded operator with $\|T\| \leq \sqrt{B}$. From the inequality (4.13), using the triangle inequality, we get

$$\Big\|\sum_{j\in J_1}\mathit{\Gamma}_j^{\,*}(\mathcal{A}_j)\Big\| \leq \frac{1+\lambda_1}{1-\lambda_2}\Big\|\sum_{j\in J_1}\mathcal{G}_j^{\,*}(\mathcal{A}_j)\Big\|+\frac{\mu}{1-\lambda_2}\Big(\sum_{j\in J_1}\|\mathcal{A}_j\|_{HS}^2\Big)^{1/2}.$$

So for any $\{\mathcal{A}_j : j \in J\} \in \bigoplus C_2$, the series $\sum_{j\in J}\mathit{\Gamma}_j^{\,*}(\mathcal{A}_j)$ is convergent. Hence $\{\mathit{\Gamma}_j : j \in J\}$ is a HS-Bessel sequence for $\mathbb{H}$. Using the definition of a synthesis operator, we get

$$\begin{aligned}\|V(\{\mathcal{A}_j\}_{j\in J})\| &\leq \frac{1+\lambda_1}{1-\lambda_2}\|T(\{\mathcal{A}_j\}_{j\in J})\|+\frac{\mu}{1-\lambda_2}\|\{\mathcal{A}_j\}_{j\in J}\|\\ &\leq \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\|\{\mathcal{A}_j\}_{j\in J}\|\\ &= \sqrt{B}\left(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)\|\{\mathcal{A}_j\}_{j\in J}\|,\ \forall\{\mathcal{A}_j\}_{j\in J}\in\bigoplus C_2.\end{aligned}$$

It implies that $\{\Gamma_j : j \in J\}$ is a HS-Bessel sequence with bound $B\Big(1+\frac{\lambda_1+\lambda_2+\mu/\sqrt{B}}{1-\lambda_2}\Big)^2$. For any $f \in \mathbb{H}$, let $\{\mathcal{A}_j\}_{j\in J} = \{\mathcal{G}_j S^{-1}f\}_{j\in J} \in \bigoplus C_2$. Using the inequality (4.13) on the sequence $\{\mathcal{G}_j S^{-1}f\}_{j\in J}$ we obtain that

$$\begin{aligned} &\Big\|\sum_{j\in J}({\mathcal{G}_j}^* - {\Gamma_j}^*)\mathcal{G}_j S^{-1}f\Big\| \\ \leq\ & \lambda_1\Big\|\sum_{j\in J}{\mathcal{G}_j}^*\mathcal{G}_j S^{-1}f\Big\| + \lambda_2\Big\|\sum_{j\in J}{\Gamma_j}^*\mathcal{G}_j S^{-1}f\Big\| + \mu\Big(\sum_{j\in J}\|\mathcal{G}_j S^{-1}f\|_{HS}^2\Big)^{1/2}. \end{aligned}$$

Since for any $f \in \mathbb{H}$, we have

$$\sum_{j\in J}{\mathcal{G}_j}^*\mathcal{G}_j S^{-1}f = f, \sum_{j\in J}{\Gamma_j}^*\mathcal{G}_j S^{-1}f = VT^*S^{-1}f$$

and

$$\Big(\sum_{j\in J}\|\mathcal{G}_j S^{-1}f\|_{HS}^2\Big)^{1/2} \leq \frac{1}{\sqrt{A}}\|f\|,$$

from the above inequality we obtain

$$\|f - VT^*S^{-1}f\| \leq \Big(\lambda_1 + \frac{\mu}{\sqrt{A}}\Big)\|f\| + \lambda_2\|VT^*S^{-1}f\|, \forall f \in \mathbb{H}.$$

So, by Lemma 4.4.1, the operator VT^*S^{-1} is invertible, and

$$\|VT^*S^{-1}\| \leq \frac{1+\lambda_1+\frac{\mu}{\sqrt{A}}}{1-\lambda_2},\ \|(VT^*S^{-1})^{-1}\| \leq \frac{1+\lambda_2}{1-(\lambda_1+\frac{\mu}{\sqrt{A}})}.$$

Every $f \in \mathbb{H}$ can be written as

$$f = VT^*S^{-1}(VT^*S^{-1})^{-1}f = \sum_{j\in J}{\Gamma_j}^*\mathcal{G}_j S^{-1}(VT^*S^{-1})^{-1}f.$$

It implies that

$$\langle f, f\rangle \ = \ \Big\langle\sum_{j\in J}{\Gamma_j}^*\mathcal{G}_j S^{-1}(VT^*S^{-1})^{-1}f, f\Big\rangle$$

$$\begin{aligned}
&= \sum_{j\in J} [\mathcal{G}_j S^{-1}(VT^*S^{-1})^{-1}f, \Gamma_j f]_\tau \\
&\leq \sum_{j\in J} \|\mathcal{G}_j S^{-1}(VT^*S^{-1})^{-1}f\|_{HS} \cdot \|\Gamma_j f\|_{HS} \\
&\leq \Big(\sum_{j\in J} \|\mathcal{G}_j S^{-1}(VT^*S^{-1})^{-1}f\|_{HS}^2\Big)^{1/2} \cdot \Big(\sum_{j\in J} \|\Gamma_j f\|_{HS}^2\Big)^{1/2} \\
&\leq \frac{1}{\sqrt{A}} \|(VT^*S^{-1})^{-1}f\| \cdot \Big(\sum_{j\in J} \|\Gamma_j f\|_{HS}^2\Big)^{1/2} \\
&\leq \frac{1}{\sqrt{A}} \Big(\frac{1+\lambda_2}{1-(\lambda_1+\frac{\mu}{\sqrt{A}})}\Big) \|f\| \cdot \Big(\sum_{j\in J} \|\Gamma_j f\|_{HS}^2\Big)^{1/2}, \forall f \in \mathbb{H}.
\end{aligned}$$

Therefore, for every $f \in \mathbb{H}$

$$\sum_{j\in J} \|\Gamma_j f\|_{HS}^2 \geq A\Big(\frac{1-(\lambda_1+\frac{\mu}{\sqrt{A}})}{1+\lambda_2}\Big)^2 \|f\|^2 = A\Big(1-\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{A}}}{1+\lambda_2}\Big)^2 \|f\|^2.$$

This completes the proof. □

Remark 4.4.4. In general, the inequality (4.12) does not imply that $\{\Gamma_j : j \in J\}$ is a HS-frame regardless how small the parameters $\lambda_1, \lambda_2, \mu$ are. A counterexample for g-frames can be found in [128], and an example can be constructed similarly for HS-frames.

Corollary 4.4.5. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-Riesz basis for $\mathbb{H}$ with bounds A and B. Assume that the condition (4.13) in Theorem 4.4.3 is satisfied, then $\{\Gamma_j : j \in J\}$ is also a HS-Riesz basis for $\mathbb{H}$ with bounds given by (4.14).*

Proof. From Theorem 4.2.10 and Theorem 4.4.3, we obtain that $\{\Gamma_j : j \in J\}$ is a HS-frame for $\mathbb{H}$ with bounds given by (4.14). Let $\sum_{j\in J} {\Gamma_j}^*(\mathcal{A}_j) = 0$ for $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$. Since $\{\mathcal{G}_j : j \in J\}$ is a HS-Riesz basis for $\mathbb{H}$ with bounds A and B, from (4.13), we get

$$\sqrt{A}\Big(\sum_{j\in J} \|\mathcal{A}_j\|_{HS}^2\Big)^{1/2} \leq \Big\|\sum_{j\in J} {\mathcal{G}_j}^*(\mathcal{A}_j)\Big\| \leq \frac{\mu}{1-\lambda_1}\Big(\sum_{j\in J} \|\mathcal{A}_j\|_{HS}^2\Big)^{1/2}.$$

It implies that

$$\Big(1-\lambda_1-\frac{\mu}{\sqrt{A}}\Big)\Big(\sum_{j\in J}\|\mathcal{A}_j\|_{HS}^2\Big)^{1/2}\leq 0.$$

Since $1-\lambda_1-\frac{\mu}{\sqrt{A}}>0$, $\sum_{j\in J}\|\mathcal{A}_j\|_{HS}^2=0$. Hence $\mathcal{A}_j=0$ for all $j\in J$. It follows that $\{\varGamma_j : j\in J\}$ is an $\bigoplus C_2$-linearly independent family. From Theorem 4.2.10, we find that $\{\varGamma_j : j\in J\}$ is a HS-Riesz basis for $\mathbb{H}$ with bounds given by (4.14), which completes the proof. □

Corollary 4.4.6. *Let $\{\mathcal{G}_j : j\in J\}$ be a HS-frame for $\mathbb{H}$ with bounds A, B, and let $\{\varGamma_j : j\in J\}$ be a sequence in $\mathcal{L}(\mathbb{H}, C_2)$. Assume that there exists a constant $0<M<A$ such that $\sum\limits_{j\in J}\|(\mathcal{G}_j-\varGamma_j)f\|_{HS}^2\leq M\|f\|^2$, for all $f\in\mathbb{H}$, then $\{\varGamma_j : j\in J\}$ is a HS-frame for $\mathbb{H}$ with bounds $A[1-(M/A)^{1/2}]^2$ and $B[1+(M/B)^{1/2}]^2$.*

Proof. Let $\lambda_1=\lambda_2=0$ and $\mu=\sqrt{M}$. Since $M<A$, $\mu/\sqrt{A}=\sqrt{M/A}<1$. So, by Theorem 4.4.3, $\{\varGamma_j : j\in J\}$ is a HS-frame for $\mathbb{H}$ with bounds $A[1-(M/A)^{1/2}]^2$ and $B[1+(M/B)^{1/2}]^2$. □

In [87], the author established the various perturbation results on g-frames in Hilbert spaces. Motivated by these results, in the following, we discuss some interesting perturbation results for HS-frames.

Theorem 4.4.7. *Let $\{\mathcal{G}_j : j\in J\}$ be a HS-frame for $\mathbb{H}$ with bounds A, B and $\{\varGamma_j : j\in J\}\subseteq\mathcal{L}(\mathbb{H}, C_2)$ be a HS-Bessel sequence with bound D. Assume that there exist constants $\lambda_1,\lambda_2,\mu,\nu\geq 0$ such that $\max\{\lambda_1+\frac{\mu}{\sqrt{A}}+\frac{\nu}{A}\cdot\sqrt{D},\lambda_2\}<1$ and the following condition is satisfied,*

$$\Big\|\sum_{j\in J}({\mathcal{G}_j}^*\mathcal{G}_jf-{\varGamma_j}^*\varGamma_jf)\Big\| \leq \lambda_1\Big\|\sum_{j\in J}{\mathcal{G}_j}^*\mathcal{G}_jf\Big\|+\lambda_2\Big\|\sum_{j\in J}{\varGamma_j}^*\varGamma_jf\Big\| +\mu\Big(\sum_{j\in J}\|\mathcal{G}_jf\|_{HS}^2\Big)^{1/2}+\nu\Big(\sum_{j\in J}\|\varGamma_jf\|_{HS}^2\Big)^{1/2} \quad (4{,}15)$$

for every $f\in\mathbb{H}$. Then $\{\varGamma_j : j\in J\}$ is a HS-frame for $\mathbb{H}$.

Proof. Let $Sf=\sum_{j\in J}{\mathcal{G}_j}^*\mathcal{G}_jf$ and $Gf=\sum_{j\in J}{\varGamma_j}^*\varGamma_jf$. Since $\{\mathcal{G}_j : j\in J\}$ is a HS-frame and $\{\varGamma_j : j\in J\}$ is a HS-Bessel sequence, S is invertible and G is a bounded operator on $\mathbb{H}$. From the inequality (4.15), for each $f \in \mathbb{H}$ we have

$$\begin{aligned}\|Sf - Gf\| &\leq \lambda_1\|Sf\| + \lambda_2\|Gf\| + \mu\Big(\sum_{j\in J}\|\mathcal{G}_j f\|_{HS}^2\Big)^{1/2} + \nu\Big(\sum_{j\in J}\|\Gamma_j f\|_{HS}^2\Big)^{1/2} \\ &\leq \lambda_1\|Sf\| + \lambda_2\|Gf\| + \mu\Big(\sum_{j\in J}\|\mathcal{G}_j f\|_{HS}^2\Big)^{1/2} + \nu\cdot\sqrt{D}\|f\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\|f - GS^{-1}f\| \\ \leq\ & \lambda_1\|f\| + \lambda_2\|GS^{-1}f\| + \mu\Big(\sum_{j\in J}\|\mathcal{G}_j S^{-1}f\|_{HS}^2\Big)^{1/2} + \nu\sqrt{D}\|S^{-1}f\| \\ \leq\ & \lambda_1\|f\| + \left(\frac{\mu}{\sqrt{A}} + \frac{\nu}{A}\cdot\sqrt{D}\right)\|f\| + \lambda_2\|GS^{-1}f\| \\ =\ & \left(\lambda_1 + \frac{\mu}{\sqrt{A}} + \frac{\nu}{A}\cdot\sqrt{D}\right)\|f\| + \lambda_2\|GS^{-1}f\|.\end{aligned}$$

Since $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}} + \frac{\nu}{A}\cdot\sqrt{D}, \lambda_2\} < 1$, by Lemma 4.4.1, GS^{-1} is invertible and consequently G is invertible. It follows that $\{\Gamma_j : j \in J\}$ is a HS-frame for $\mathbb{H}$. □

Corollary 4.4.8. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with bounds A, B and $\{\Gamma_j : j \in J\} \subseteq \mathcal{L}(\mathbb{H}, C_2)$ be a family of operators. Assume that there exist a constant $0 < M < A$ such that*

$$\sum_{j\in J}\|\mathcal{G}_j{}^*\mathcal{G}_j f - \Gamma_j{}^*\Gamma_j f\| \leq M\|f\|,\ \forall f \in \mathbb{H},$$

then $\{\Gamma_j : j \in J\}$ is a HS-frame for $\mathbb{H}$.

Proof. For each $f \in \mathbb{H}$, we have

$$\Big\|\sum_{j\in J}\Gamma_j{}^*\Gamma_j f\Big\| \leq \Big\|\sum_{j\in J}(\mathcal{G}_j{}^*\mathcal{G}_j f - \Gamma_j{}^*\Gamma_j f)\Big\| + \Big\|\sum_{j\in J}\mathcal{G}_j{}^*\mathcal{G}_j f\Big\| \leq (M+B)\|f\|.$$

Thus $\sum_{j\in J} {\mathit{\Gamma}_j}^* \mathit{\Gamma}_j f$ is convergent for each $f \in \mathbb{H}$. Therefore for all $f \in \mathbb{H}$

$$\sum_{j\in J} \|\mathit{\Gamma}_j f\|_{HS}^2 = \sum_{j\in J} \langle {\mathit{\Gamma}_j}^* \mathit{\Gamma}_j f, f\rangle = \left\langle \sum_{j\in J} {\mathit{\Gamma}_j}^* \mathit{\Gamma}_j f, f \right\rangle \leq \left\| \sum_{j\in J} {\mathit{\Gamma}_j}^* \mathit{\Gamma}_j f \right\| \cdot \|f\| \leq (M+B)\|f\|^2.$$

It follows that $\{\mathit{\Gamma}_j : j \in J\}$ is a HS-Bessel sequence for $\mathbb{H}$. Also we have

$$\left\| \sum_{j\in J} ({\mathcal{G}_j}^* \mathcal{G}_j f - {\mathit{\Gamma}_j}^* \mathit{\Gamma}_j f) \right\| \leq M\|f\| \leq \frac{M}{\sqrt{A}} \left(\sum_{j\in J} \|\mathcal{G}_j(f)\|_{HS}^2 \right)^{1/2}, \ f \in \mathbb{H}.$$

Let $\lambda_1 = \lambda_2 = \nu = 0$ and $\mu = M/\sqrt{A}$. Since $M < A$, $\mu/\sqrt{A} = M/A < 1$. So, by Theorem 4.4.7, $\{\mathit{\Gamma}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$. □

Theorem 4.4.9. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with bounds A, B and $\{\mathit{\Gamma}_j : j \in J\} \subseteq \mathcal{L}(\mathbb{H}, C_2)$ be a HS-Bessel sequence for $\mathbb{H}$. Assume that there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ with $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$ such that*

$$\left\| \sum_{j\in J} ({\mathcal{G}_j}^* \mathcal{A}_j - {\mathit{\Gamma}_j}^* \mathcal{A}_j) \right\| \leq \lambda_1 \left\| \sum_{j\in J} {\mathcal{G}_j}^* (\mathcal{A}_j) \right\| + \lambda_2 \left\| \sum_{j\in J} {\mathit{\Gamma}_j}^* (\mathcal{A}_j) \right\| + \mu \left(\sum_{j\in J} \|\mathcal{A}_j\|_{HS}^2 \right)^{1/2}, \tag{4.16}$$

where $\{\mathcal{A}_j\}_{j\in J} \in \bigoplus C_2$, then $\{\mathit{\Gamma}_j : j \in J\}$ is a HS-frame for $\mathbb{H}$.

Proof. Let T and S denote the synthesis operator and frame operator respectively associated with $\{\mathcal{G}_j : j \in J\}$. Also, let V denote the synthesis operator associated with $\{\mathit{\Gamma}_j : j \in J\}$. From the inequality (4.16), we obtain

$$\|T(\{\mathcal{A}_j\}_{j\in J}) - V(\{\mathcal{A}_j\}_{j\in J})\| \leq \lambda_1 \|T(\{\mathcal{A}_j\}_{j\in J})\| + \lambda_2 \|V(\{\mathcal{A}_j\}_{j\in J})\| + \mu \left(\sum_{j\in J} \|\mathcal{A}_j\|_{HS}^2 \right)^{1/2}.$$

For any $f \in \mathbb{H}$, let $\{\mathcal{A}_j\}_{j\in J} = \{\mathcal{G}_j S^{-1} f\}_{j\in J} \in \bigoplus C_2$, then

$$\begin{aligned}
& \|T(\{\mathcal{G}_j S^{-1} f\}_{j\in J}) - V(\{\mathcal{G}_j S^{-1} f\}_{j\in J})\| \\
= \ & \|TT^*(S^{-1}f) - VT^*(S^{-1}f)\| \\
= \ & \|SS^{-1}f - VT^*S^{-1}f\| = \|f - VT^*S^{-1}f\| \\
\leq \ & \lambda_1\|f\| + \lambda_2\|VT^*S^{-1}f\| + \frac{\mu}{\sqrt{A}}\|f\| \\
= \ & \left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right)\|f\| + \lambda_2\|VT^*S^{-1}f\|.
\end{aligned}$$

Since $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$, by Lemma 4.4.1, the operator VT^*S^{-1} is invertible and hence V is surjective. Then by Proposition 4.2.8, the sequence $\{\varGamma_j : j \in J\}$ is a HS-frame for $\mathbb{H}$. $\square$

Remark 4.4.10. Since g-frames can be considered as a class of HS-frames, the previous results on g-frames can be obtained as a special case of the results we established for HS-frames.

4.5 Identities and Inequalities for HS-Frames

In this section, we establish Parseval type identities and new inequalities for HS-frames. These results generalize and improve the remarkable results which have been obtained by Balan et al. [9, 10] and Găvruţa [74]. We generalize the previous inequalities to a more general form which involve a parameter $\lambda \in [0, 1]$. As a particular case, for $\lambda = 1/2$, the previous inequalities can be obtained. We first state a simple result on operators, which can be found in [145].

Lemma 4.5.1. *If $P, Q \in \mathcal{L}(\mathbb{H})$ satisfying $P + Q = I$, then $P - P^*P = Q^* - Q^*Q$.*

Proof. We compute $P - P^*P = (I - P^*)P = Q^*(I - Q) = Q^* - Q^*Q$. $\square$

Now we state and prove a Parseval HS-frame identity.

Theorem 4.5.2. *Let $\{\mathcal{G}_j : j \in J\}$ be a Parseval HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$.*

Then for all $K \subset J$ and all $f \in \mathbb{H}$, we have

$$\sum_{j\in K} \|\mathcal{G}_j(f)\|_{HS}^2 - \Big\|\sum_{j\in K} \mathcal{G}_j^*\mathcal{G}_j f\Big\|^2 = \sum_{j\in K^c} \|\mathcal{G}_j(f)\|_{HS}^2 - \Big\|\sum_{j\in K^c} \mathcal{G}_j^*\mathcal{G}_j f\Big\|^2.$$

Proof. Since $\{\mathcal{G}_j : j \in J\}$ is a Parseval HS-frame, the corresponding frame operator $S = I$, and hence $S_K + S_{K^c} = I$. Note that S_{K^c} is a self-adjoint operator, and therefore $S_{K^c}^* = S_{K^c}$. Applying Lemma 4.5.1 to the operators S_K and S_{K^c}, we obtain that for all $f \in \mathbb{H}$

$$\begin{aligned}
&\langle S_K f, f\rangle - \langle S_K^* S_K f, f\rangle = \langle S_{K^c}^* f, f\rangle - \langle S_{K^c}^* S_{K^c} f, f\rangle \\
\text{i.e.}\quad &\langle S_K f, f\rangle - \|S_K f\|^2 = \langle S_{K^c} f, f\rangle - \|S_{K^c} f\|^2 \\
\text{i.e.}\quad &\sum_{j\in K} \|\mathcal{G}_j(f)\|_{HS}^2 - \Big\|\sum_{j\in K} \mathcal{G}_j^*\mathcal{G}_j f\Big\|^2 = \sum_{j\in K^c} \|\mathcal{G}_j(f)\|_{HS}^2 - \Big\|\sum_{j\in K^c} \mathcal{G}_j^*\mathcal{G}_j f\Big\|^2.
\end{aligned}$$

Hence we have the desired result. □

The next inequality for a Parseval HS-frame, appearing in Corollary 4.5.3, is a simple consequence of Theorem 1.7.2.

Corollary 4.5.3. *Let $\{\mathcal{G}_j : j \in J\}$ be a Parseval HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$. Then for all $K \subset J$ and all $f \in \mathbb{H}$, we have*

$$\sum_{j\in K} \|\mathcal{G}_j(f)\|_{HS}^2 + \Big\|\sum_{j\in K^c} \mathcal{G}_j^*\mathcal{G}_j f\Big\|^2 \geq \frac{3}{4}\|f\|^2.$$

Proof. Since $\{\mathcal{G}_j : j \in J\}$ is a Parseval HS-frame, $S_K + S_{K^c} = I$. A simple computation shows that

$$S_K^2 + S_{K^c}^2 = S_K^2 + (I - S_K)^2 = 2S_K^2 - 2S_K + I = 2\Big(S_K - \frac{1}{2}I\Big)^2 + \frac{1}{2}I,$$

and so $S_K^2 + S_{K^c}^2 \geq \frac{1}{2}I$. Since $S_K + S_{K^c} = I$, it follows that $S_K + S_{K^c}^2 + S_{K^c} + S_K^2 \geq \frac{3}{2}I$. Notice that operators S_K and S_{K^c} are self-adjoint and therefore $S_K^* = S_K$, $S_{K^c}^* = S_{K^c}$. Applying Lemma 4.5.1 to the operators $P = S_K$ and $Q = S_{K^c}$, we obtain $S_K - S_K^2 = S_{K^c} - S_{K^c}^2 \Rightarrow S_K + S_{K^c}^2 = S_{K^c} + S_K^2$. Thus

$$2(S_K + S_{K^c}^2) = S_K + S_{K^c}^2 + S_{K^c} + S_K^2 \geq \frac{3}{2}I.$$

Therefore for all $f \in \mathbb{H}$ we have

$$\begin{aligned} & \sum_{j\in K} \|\mathcal{G}_j(f)\|_{HS}^2 + \left\| \sum_{j\in K^c} \mathcal{G}_j^*\mathcal{G}_j f \right\|^2 \\ = \ & \langle S_K f, f\rangle + \langle S_{K^c} f, S_{K^c} f\rangle = \langle (S_K + S_{K^c}^2) f, f\rangle \geq \frac{3}{4}\|f\|^2. \end{aligned}$$

This completes the proof. □

Now we generalize Theorems 1.7.1 and 1.7.5 to dual HS-frames. We first establish the following result.

Proposition 4.5.4. *Let $P, Q \in \mathcal{L}(\mathbb{H})$ be two self-adjoint operators such that $P+Q = I$. Then for any $\lambda \in [0,1]$ and all $f \in \mathbb{H}$ we have*

$$\|Pf\|^2 + 2\lambda\langle Qf, f\rangle = \|Qf\|^2 + 2(1-\lambda)\langle Pf, f\rangle + (2\lambda - 1)\|f\|^2 \geq (1-(\lambda-1)^2)\|f\|^2.$$

Proof. We have

$$\|Pf\|^2 + 2\lambda\langle Qf, f\rangle = \langle P^2 f, f\rangle + 2\lambda\langle (I-P)f, f\rangle = \langle (P^2 - 2\lambda P + 2\lambda I)f, f\rangle,$$

and

$$\begin{aligned} & \|Qf\|^2 + 2(1-\lambda)\langle Pf, f\rangle + (2\lambda-1)\|f\|^2 \\ = \ & \langle (I-P)^2 f, f\rangle + 2(1-\lambda)\langle Pf, f\rangle + (2\lambda-1)\langle f, f\rangle \\ = \ & \langle (P^2 - 2\lambda P + 2\lambda I)f, f\rangle \\ = \ & \langle ((P-\lambda I)^2 - \lambda^2 I + 2\lambda I)f, f\rangle \\ = \ & \langle ((P-\lambda I)^2 + (1-(\lambda-1)^2)I)f, f\rangle \geq (1-(\lambda-1)^2)\|f\|^2. \end{aligned}$$

This proves the desired result. □

Theorem 4.5.5. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$ and $\{\tilde{\mathcal{G}}_j : j \in J\}$ be the canonical dual HS-frame of $\{\mathcal{G}_j : j \in J\}$. Then for any $\lambda \in [0,1]$, for all $K \subset J$ and all $f \in \mathbb{H}$, we have*

$$\begin{aligned}\sum_{j\in J}\|\tilde{\mathcal{G}}_j S_K f\|_{HS}^2 + \sum_{j\in K^c}\|\mathcal{G}_j(f)\|_{HS}^2 &= \sum_{j\in J}\|\tilde{\mathcal{G}}_j S_{K^c} f\|_{HS}^2 + \sum_{j\in K}\|\mathcal{G}_j(f)\|_{HS}^2 \\ &\geq (2\lambda-\lambda^2)\sum_{j\in K}\|\mathcal{G}_j(f)\|_{HS}^2 + (1-\lambda^2)\sum_{j\in K^c}\|\mathcal{G}_j(f)\|_{HS}^2.\end{aligned}$$

Proof. Let S be the frame operator for $\{\mathcal{G}_j : j \in J\}$. Since $S_k + S_{K^c} = S$, it follows that

$$S^{-1/2}S_K S^{-1/2} + S^{-1/2}S_{K^c}S^{-1/2} = I.$$

Considering

$$P = S^{-1/2}S_K S^{-1/2}, \quad Q = S^{-1/2}S_{K^c}S^{-1/2},$$

and $S^{1/2}f$ instead of f in Proposition 4.5.4, we obtain

$$\begin{aligned}
& \|S^{-1/2}S_K f\|^2 + 2\lambda\langle S^{-1/2}S_{K^c}f, S^{1/2}f\rangle \\
= \;& \|S^{-1/2}S_{K^c}f\|^2 + 2(1-\lambda)\langle S^{-1/2}S_K f, S^{1/2}f\rangle + (2\lambda-1)\|S^{1/2}f\|^2 \\
\geq \;& (1-(\lambda-1)^2)\|S^{1/2}f\|^2 \\
\text{i.e.} \quad & \langle S^{-1}S_K f, S_K f\rangle + 2\lambda\langle S_{K^c}f, f\rangle \\
= \;& \langle S^{-1}S_{K^c}f, S_{K^c}f\rangle + 2(1-\lambda)\langle S_K f, f\rangle + (2\lambda-1)\langle Sf, f\rangle \\
\geq \;& (2\lambda-\lambda^2)\langle Sf, f\rangle \\
\text{i.e.} \quad & \langle S^{-1}S_K f, S_K f\rangle \\
= \;& \langle S^{-1}S_{K^c}f, S_{K^c}f\rangle + 2\langle S_K f, f\rangle - 2\lambda\langle (S_K + S_{K^c})f, f\rangle + (2\lambda-1)\langle Sf, f\rangle \\
\geq \;& (2\lambda-\lambda^2)\langle Sf, f\rangle - 2\lambda\langle S_{K^c}f, f\rangle \\
\text{i.e.} \quad & \langle S^{-1}S_K f, S_K f\rangle = \langle S^{-1}S_{K^c}f, S_{K^c}f\rangle + 2\langle S_K f, f\rangle - \langle Sf, f\rangle \\
\geq \;& 2\lambda\langle S_K f, f\rangle - \lambda^2\langle Sf, f\rangle \\
\text{i.e.} \quad & \langle S^{-1}S_K f, S_K f\rangle + \langle S_{K^c}f, f\rangle = \langle S^{-1}S_{K^c}f, S_{K^c}f\rangle + \langle S_K f, f\rangle \\
\geq \;& (2\lambda-\lambda^2)\langle S_K f, f\rangle + (1-\lambda^2)\langle S_{K^c}f, f\rangle. \qquad (4.17)
\end{aligned}$$

We have

$$\begin{aligned}\langle S^{-1}S_Kf, S_Kf\rangle &= \langle SS^{-1}S_Kf, S^{-1}S_Kf\rangle = \Big\langle \sum_{j\in J}\mathcal{G}_j^*\mathcal{G}_jS^{-1}S_Kf, S^{-1}S_Kf\Big\rangle \\ &= \sum_{j\in J}\Big[\mathcal{G}_jS^{-1}S_Kf, \mathcal{G}_jS^{-1}S_Kf\Big]_\tau \\ &= \sum_{j\in J}\Big[\tilde{\mathcal{G}}_jS_Kf, \tilde{\mathcal{G}}_jS_Kf\Big]_\tau = \sum_{j\in J}\|\tilde{\mathcal{G}}_jS_Kf\|_{HS}^2. \end{aligned} \tag{4.18}$$

Similarly

$$\langle S^{-1}S_{K^c}f, S_{K^c}f\rangle = \sum_{j\in J}\|\tilde{\mathcal{G}}_jS_{K^c}f\|_{HS}^2, \tag{4.19}$$

$$\langle S_{K^c}f, f\rangle = \sum_{j\in K^c}\|\mathcal{G}_j(f)\|_{HS}^2 \quad \text{and} \quad \langle S_Kf, f\rangle = \sum_{j\in K}\|\mathcal{G}_j(f)\|_{HS}^2. \tag{4.20}$$

Using equations (4.18)-(4.20) in the inequality (4.17), we obtain

$$\begin{aligned}\sum_{j\in J}\|\tilde{\mathcal{G}}_jS_Kf\|_{HS}^2 + \sum_{j\in K^c}\|\mathcal{G}_j(f)\|_{HS}^2 &= \sum_{j\in J}\|\tilde{\mathcal{G}}_jS_{K^c}f\|_{HS}^2 + \sum_{j\in K}\|\mathcal{G}_j(f)\|_{HS}^2 \\ &\geq (2\lambda-\lambda^2)\sum_{j\in K}\|\mathcal{G}_j(f)\|_{HS}^2 + (1-\lambda^2)\sum_{j\in K^c}\|\mathcal{G}_j(f)\|_{HS}^2.\end{aligned}$$

This completes the proof. □

Proposition 4.5.6. *If $P, Q \in \mathcal{L}(\mathbb{H})$ satisfy $P + Q = I$, then for any $\lambda \in [0,1]$ we have*

$$P^*P + \lambda(Q^* + Q) = Q^*Q + (1-\lambda)(P^* + P) + (2\lambda - 1)I \geq (1 - (\lambda-1)^2)I.$$

Proof. We have

$$P^*P + \lambda(Q^* + Q) = P^*P + \lambda(I - P^* + I - P) = P^*P - \lambda(P^* + P) + 2\lambda I,$$

and

$$\begin{aligned}
& Q^*Q + (1-\lambda)(P^* + P) + (2\lambda - 1)I \\
= \; & (I - P^*)(I - P) + (1-\lambda)(P^* + P) + (2\lambda - 1)I \\
= \; & P^*P - \lambda(P^* + P) + 2\lambda I \\
= \; & (P - \lambda I)^*(P - \lambda I) + (1 - (\lambda - 1)^2)I \geq (1 - (\lambda - 1)^2)I.
\end{aligned}$$

Hence the result follows. □

Theorem 4.5.7. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$ and $\{\Gamma_j : j \in J\}$ be an alternate dual HS-frame of $\{\mathcal{G}_j : j \in J\}$. Then for any $\lambda \in [0,1]$, for all $K \subset J$ and all $f \in \mathbb{H}$, we have*

$$\begin{aligned}
& Re\Big\{ \sum_{j \in K^c} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau \Big\} + \Big\| \sum_{j \in K} \mathcal{G}_j^* \Gamma_j(f) \Big\|^2 \\
= \; & Re\Big\{ \sum_{j \in K} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau \Big\} + \Big\| \sum_{j \in K^c} \mathcal{G}_j^* \Gamma_j(f) \Big\|^2 \\
\geq \; & (2\lambda - \lambda^2) Re\Big\{ \sum_{j \in K} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau \Big\} + (1 - \lambda^2) Re\Big\{ \sum_{j \in K^c} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau \Big\}.
\end{aligned}$$

Proof. For $K \subset J$ and $f \in \mathbb{H}$, define the operator F_k by $F_k f = \sum_{j \in K} \mathcal{G}_j^* \Gamma_j f$. Then the series converges unconditionally and $F_K \in \mathcal{L}(\mathbb{H})$. By (4.6), we have $F_K + F_{K^c} = I$. By Proposition 4.5.6, we get

$$\begin{aligned}
& (1 - (\lambda - 1)^2)\|f\|^2 \leq \langle F_K^* F_K f, f \rangle + \lambda \langle (F_{K^c}^* + F_{K^c}) f, f \rangle \\
= \; & \langle F_{K^c}^* F_{K^c} f, f \rangle + (1 - \lambda)\langle (F_K^* + F_K) f, f \rangle + (2\lambda - 1)\|f\|^2 \\
\text{i.e.} \quad & (2\lambda - \lambda^2) Re(\langle If, f \rangle) \leq \|F_K f\|^2 + \lambda(\overline{\langle F_{K^c} f, f \rangle} + \langle F_{K^c} f, f \rangle) \\
= \; & \|F_{K^c} f\|^2 + (1 - \lambda)(\overline{\langle F_K f, f \rangle} + \langle F_K f, f \rangle) + (2\lambda - 1)\|f\|^2 \\
\text{i.e.} \quad & (2\lambda - \lambda^2) Re(\langle (F_K + F_{K^c}) f, f \rangle) \leq \|F_K f\|^2 + 2\lambda Re(\langle F_{K^c} f, f \rangle) \\
= \; & \|F_{K^c} f\|^2 + 2(1 - \lambda) Re(\langle F_K f, f \rangle) + (2\lambda - 1)\|f\|^2 \\
\text{i.e.} \quad & (2\lambda - \lambda^2) Re(\langle F_K f, f \rangle) - \lambda^2 Re(\langle F_{K^c} f, f \rangle) \leq \|F_K f\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|F_{K^c}f\|^2 + 2Re(\langle F_K f, f\rangle) - Re(\langle If, f\rangle)\\
\text{i.e.}\quad & (2\lambda-\lambda^2)Re(\langle F_K f, f\rangle) - \lambda^2 Re(\langle F_{K^c} f, f\rangle) \le \|F_K f\|^2\\
&= \|F_{K^c}f\|^2 + 2Re(\langle F_K f, f\rangle) - Re(\langle (F_K + F_{K^c})f, f\rangle)\\
\text{i.e.}\quad & (2\lambda-\lambda^2)Re(\langle F_K f, f\rangle) + (1-\lambda^2) Re(\langle F_{K^c} f, f\rangle)\\
&\le \|F_K f\|^2 + Re(\langle F_{K^c} f, f\rangle) = \|F_{K^c} f\|^2 + Re(\langle F_K f, f\rangle).
\end{aligned}
$$

We have

$$
\langle F_K f, f\rangle = \langle \sum_{j\in K} \mathcal{G}_j^* \Gamma_j f, f\rangle = \sum_{j\in K} [\Gamma_j f, \mathcal{G}_j f]_\tau.
$$
$$
\langle F_{K^c} f, f\rangle = \sum_{j\in K^c} [\Gamma_j f, \mathcal{G}_j f]_\tau.
$$

So finally

$$
\begin{aligned}
& Re\Big\{\sum_{j\in K^c} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau\Big\} + \Big\|\sum_{j\in K} \mathcal{G}_j^*\Gamma_j(f)\Big\|^2\\
= \;& Re\Big\{\sum_{j\in K} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau\Big\} + \Big\|\sum_{j\in K^c} \mathcal{G}_j^*\Gamma_j(f)\Big\|^2\\
\ge \;& (2\lambda-\lambda^2) Re\Big\{\sum_{j\in K} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau\Big\} + (1-\lambda^2) Re\Big\{\sum_{j\in K^c} [\Gamma_j(f), \mathcal{G}_j(f)]_\tau\Big\}.
\end{aligned}
$$

This completes the proof. □

Remark 4.5.8. If we consider $\lambda = 1/2$ in Theorem 4.5.5 and Theorem 4.5.7, then we obtain the similar inequalities as in Theorem 1.7.4 and Theorem 1.7.5 respectively, with scalar $3/4$.

Next we give a simplified presentation of Theorem 1.7.6 for HS-frames, which generalizes Theorem 4.5.7 to a more general form that does not involve the real parts of the complex numbers. We first establish the following result.

Lemma 4.5.9. *If $P, Q \in \mathcal{L}(\mathbb{H})$ such that $P + Q = I$, then $P + Q^*Q = Q^* + P^*P$.*

Proof. By simple computation, we obtain

$$P + Q^*Q = P + (I - P^*)(I - P) = (I - P^*) + P^*P = Q^* + P^*P,$$

which is as required. □

Theorem 4.5.10. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$ and $\{\Gamma_j : j \in J\}$ be an alternate dual HS-frame of $\{\mathcal{G}_j : j \in J\}$, then for every $K \subset J$ and every $f \in \mathbb{H}$, we have*

$$\begin{aligned} &\left(\sum_{j\in K^c} \left[\Gamma_j(f), \mathcal{G}_j(f)\right]_\tau\right) + \left\|\sum_{j\in K} \mathcal{G}_j^*\Gamma_j(f)\right\|^2 \\ =\ & \overline{\left(\sum_{j\in K} \left[\Gamma_j(f), \mathcal{G}_j(f)\right]_\tau\right)} + \left\|\sum_{j\in K^c} \mathcal{G}_j^*\Gamma_j(f)\right\|^2. \end{aligned}$$

Proof. For $K \subset J$ and $f \in \mathbb{H}$, we define the operator F_k as in Theorem 4.5.7. Therefore, we have $F_K + F_{K^c} = I$. By Lemma 4.5.9, we have

$$\begin{aligned} &\left(\sum_{j\in K^c} \left[\Gamma_j(f), \mathcal{G}_j(f)\right]_\tau\right) + \left\|\sum_{j\in K} \mathcal{G}_j^*\Gamma_j(f)\right\|^2 \\ =\ & \langle F_{K^c} f, f\rangle + \langle F_K^* F_K f, f\rangle \\ =\ & \langle F_K^* f, f\rangle + \langle F_{K^c}^* F_{K^c} f, f\rangle \\ =\ & \overline{\langle F_K f, f\rangle} + \|F_{K^c} f\|^2 \\ =\ & \overline{\left(\sum_{j\in K} \left[\Gamma_j(f), \mathcal{G}_j(f)\right]_\tau\right)} + \left\|\sum_{j\in K^c} \mathcal{G}_j^*\Gamma_j(f)\right\|^2. \end{aligned}$$

Hence the relation stated in the theorem holds. □

Theorem 4.5.11. *Let $\{\mathcal{G}_j : j \in J\}$ be a HS-frame for $\mathbb{H}$ with respect to $\mathbb{K}$ and $\{\Gamma_j : j \in J\}$ be an alternate dual HS-frame of $\{\mathcal{G}_j : j \in J\}$. Then for every bounded sequence $\{w_j : j \in J\}$ and every $f \in \mathbb{H}$, we have*

$$\left(\sum_{j\in J} w_j \left[\Gamma_j(f), \mathcal{G}_j(f)\right]_\tau\right) + \left\|\sum_{j\in J} (1 - w_j)\mathcal{G}_j^*\Gamma_j(f)\right\|^2$$

$$= \overline{\left(\sum_{j\in J}(1-w_j)\big[\mathit{\Gamma}_j(f),\mathcal{G}_j(f)\big]_\tau\right)} + \left\|\sum_{j\in J} w_j\mathcal{G}_j^*\mathit{\Gamma}_j(f)\right\|^2.$$

Proof. We define the operators $Ff = \sum_{j\in J} w_j\mathcal{G}_j^*\mathit{\Gamma}_j f$ and $Gf = \sum_{j\in J}(1-w_j)\mathcal{G}_j^*\mathit{\Gamma}_j f$. Note that both series converge unconditionally. Also we have $F, G \in \mathcal{L}(\mathbb{H})$ and $F+G=I$. By Lemma 4.5.9, we have

$$\begin{aligned} &\left(\sum_{j\in J} w_j\big[\mathit{\Gamma}_j(f),\mathcal{G}_j(f)\big]_\tau\right) + \left\|\sum_{j\in J}(1-w_j)\mathcal{G}_j^*\mathit{\Gamma}_j(f)\right\|^2 \\ = &\ \langle Ff, f\rangle + \langle G^*Gf, f\rangle \\ = &\ \langle G^*f, f\rangle + \langle F^*Ff, f\rangle \\ = &\ \overline{\langle Gf, f\rangle} + \|Ff\|^2 \\ = &\ \overline{\left(\sum_{j\in J}(1-w_j)\big[\mathit{\Gamma}_j(f),\mathcal{G}_j(f)\big]_\tau\right)} + \left\|\sum_{j\in J} w_j\mathcal{G}_j^*\mathit{\Gamma}_j(f)\right\|^2. \end{aligned}$$

Hence the relation holds. □

Observe that if we consider $K \subset J$ and

$$w_j = \begin{cases} 0 & \text{if } j \in K \\ 1 & \text{if } j \in K^c, \end{cases}$$

then Theorem 4.5.10 follows from Theorem 4.5.11.

Chapter 5

On a Feichtinger Problem

In this chapter, we solve a spectral problem about positive semi-definite trace-class pseudodifferential operators on modulation spaces which was posed by Hans Feichtinger. Later, C. Heil and D. Larson rephrased the problem in the broader setting of positive semi-definite trace-class operator on a separable Hilbert space. This solution consists of constructing a counterexample that solves Feichtinger's problem by first solving this second problem.

5.1 Preliminaries

Let $\mathbb{H}$ be an infinite-dimensional separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $T \in \mathcal{B}(\mathbb{H})$ be a compact operator and $s_n(T) = \lambda_n(T^*T)^{1/2}$ be the singular values of T. If T is self-adjoint then $s_n(T) = |\lambda_n(T)|$ and if T is positive then $s_n(T) = \lambda_n(T)$. The operator T is *trace-class* if $\|T\|_{\mathcal{I}_1} = \sum\limits_{n=1}^{\infty} s_n(T) < \infty$. The space $\mathcal{I}_1$ of trace-class operators is a Banach space under the norm $\|\cdot\|_{\mathcal{I}_1}$. For detailed study on trace-class operators see [49, 124].

Choose an orthonormal basis $\{w_n\}_{n\geq 1}$ for $\mathbb{H}$. We define a subspace $\mathbb{H}^1$ of $\mathbb{H}$ by

$$\mathbb{H}^1 = \Big\{ f \in \mathbb{H} : |||f||| := \sum_{n=1}^{\infty} |\langle f, w_n\rangle| < \infty \Big\}. \tag{5.1}$$

It follows that $|||w_n||| = \|w_n\| = 1$ for every n and that if $f \in \mathbb{H}^1$ then $f =$

$\sum_{n=1}^{\infty} \langle f, w_n \rangle w_n$ with convergence of this series in *both* norms $\| \cdot \|$ and $|||\cdot|||$.

We define an operator $T : \mathbb{H} \to \mathbb{H}$ by

$$T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} (w_m \otimes w_n), \tag{5.2}$$

where the scalars c_{mn} are such that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| < \infty$. It is easy to see that $T \in \mathcal{I}_1$ with

$$\|T\|_{\mathcal{I}_1} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \|c_{mn}(w_m \otimes w_n)\|_{\mathcal{I}_1} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |c_{mn}| < \infty.$$

In addition, note that the series defining T converges not only in the strong operator topology but also absolutely in trace-class norm.

Further, if $\mathbb{H} = L^2(\mathbb{R})$ and $\mathcal{W}(g)$ is a Wilson basis for $L^2(\mathbb{R})$, then the space $\mathbb{H}^1$ defined by equation (5.1) is precisely the modulation space $M^1(\mathbb{R})$.

5.1.1 Integral operators

Given a kernel function $k \in L^2(\mathbb{R}^2)$, the corresponding *integral operator* is

$$Tf(x) = \int_{\mathbb{R}} k(x, y)\, f(y)\, dy. \tag{5.3}$$

In terms of the kernel, T is self-adjoint if $k(x, y) = \overline{k(y, x)}$. Since $k \in L^2(\mathbb{R}^2)$, we know that T is a compact mapping of $L^2(\mathbb{R})$ onto itself. In fact, we have the equivalence that $k \in L^2(\mathbb{R}^2)$ if and only if T is Hilbert-Schmidt operator. Also if k lies in the two-dimensional version of the Feichtinger algebra, i.e., $k \in M^1(\mathbb{R}^2)$, then T is a trace-class operator. This result was proved in [92] and it is a special case of more general theorems proved in [80]. We discuss this result below, as given in [92]:

Let $k \in M^1(\mathbb{R}^2)$ and $\mathcal{W}(g)$ be a Wilson orthonormal basis for $L^2(\mathbb{R})$ such that $g \in M^1(\mathbb{R})$. For simplicity of notation, we index this basis as $\mathcal{W}(g) = \{w_n\}_{n \in \mathbb{N}}$. Construct an orthonormal basis for $L^2(\mathbb{R}^2)$ by forming tensor products, i.e., set

$$W_{mn}(x, y) := w_m \otimes w_n(x, y) = w_m(x)\, \overline{w_n(y)}.$$

Then $\mathcal{W} = \{W_{mn}\}_{m,n\in\mathbb{N}}$ is both an orthonormal basis for $L^2(\mathbb{R}^2)$ and an unconditional basis for $M^1(\mathbb{R}^2)$. Since $k \in M^1(\mathbb{R}^2)$, we have

$$k = \sum_{m,n\in\mathbb{Z}} \langle k, W_{mn}\rangle W_{mn}, \tag{5.4}$$

with convergence of the series in M^1-norm, and furthermore

$$\|k\|_{M^1} = \sum_{m,n\in\mathbb{Z}} |\langle k, W_{mn}\rangle| < \infty. \tag{5.5}$$

Substituting the expansion (5.4) into the definition of the integral operator in (5.3) yields

$$\begin{aligned} Tf(x) &= \int_{\mathbb{R}} \sum_{m,n\in\mathbb{Z}} \langle k, W_{mn}\rangle W_{mn}(x,y)\, f(y)\, dy \qquad (5.6)\\ &= \sum_{m,n\in\mathbb{Z}} \langle k, W_{mn}\rangle \int_{\mathbb{R}} w_m(x)\, \overline{w_n(y)}\, f(y)\, dy \\ &= \sum_{m,n\in\mathbb{Z}} \langle k, W_{mn}\rangle \langle f, w_n\rangle\, w_m(x) \\ &= \sum_{m,n\in\mathbb{Z}} \langle k, W_{mn}\rangle (w_m \otimes w_n)(f)(x) \\ &= \sum_{m,n\in\mathbb{Z}} \langle k, W_{mn}\rangle (W_{mn})(f)(x). \qquad (5.7) \end{aligned}$$

The discrete version of T is given by the matrix $K = (\langle k, W_{mn}\rangle)_{m,n\in\mathbb{Z}}$ or equivalently

$$T = \sum_{m,n\in\mathbb{Z}} \langle k, W_{mn}\rangle W_{mn}. \tag{5.8}$$

Since each operator $w_m \otimes w_n$ belongs to $\mathcal{I}_1$ and the scalars $\langle k, W_{mn}\rangle$ are summable, the series (5.8) converges absolutely in $\mathcal{I}_1$, and therefore $T \in \mathcal{I}_1$. Suppose in addition that T is a positive semi-definite. Then, by the spectral theorem,

$$T = \sum_{k=1}^{\infty} \lambda_k t_k \otimes t_k = \sum_{k=1}^{\infty} h_k \otimes h_k$$

where $\{\lambda_k\}_{k=1}^{\infty} \subset (0,\infty)$ is the set of eigenvalues of T and $\{t_k\}_{k=1}^{\infty}$ is the set of corresponding orthonormal basis of eigenfunctions, and $h_k = \sqrt{\lambda_k} t_k$ for each $k \geq 1$. It was proved in [92] that $h_k \in M^1(\mathbb{R})$.

5.1.2 Type A and Type B Operators

We fix now an orthonormal basis $\{w_n\}_{n\geq 1}$ for $\mathbb{H}$, once and for all. This basis induces the norm $|||\cdot|||$ on the dense subset $\mathbb{H}^1$ introduced in (5.1), and repeated here for convenience of the reader:

$$|||f||| = \sum_{n=1}^{\infty} |\langle f, w_n\rangle|, \quad \mathbb{H}^1 = \Big\{ f \in \mathbb{H} : \sum_{n=1}^{\infty} |\langle f, w_n\rangle| < \infty \Big\}.$$

Definition 5.1.1. *An operator T given by* (5.2) *is of Type A with respect to the orthonormal basis $\{w_n\}_{n\geq 1}$ if, for an orthogonal set of eigenvectors $\{g_n\}_{n\geq 1}$ of T such that $T = \sum_{n=1}^{\infty} g_n \otimes g_n$ with convergence in the strong operator topology, we have that $\sum_{n=1}^{\infty} |||g_n|||^2 < \infty$.*

The operator T is of Type B with respect to the orthonormal basis $\{w_n\}_{n\geq 1}$ if there is some sequence of vectors $\{v_n\}_{n\geq 1}$ in $\mathbb{H}$ such that $T = \sum_{n=1}^{\infty} v_n \otimes v_n$ with convergence in the strong operator topology, we have that $\sum_{n=1}^{\infty} |||v_n|||^2 < \infty$.

It is clear that if T is of Type A then it is of Type B. However, it was shown in [92, Example 2.2] that not every positive trace-class operator is of Type A or Type B, even when the operator is of finite-rank. We can characterize the finite-rank operators that are of Type A or Type B, as follows (see [92] for proof).

Theorem 5.1.2. *Let T be a positive finite-rank operator. Then the following statements are equivalent.*

(i) T is of Type A.

(ii) T is of Type B.

(iii) Each eigenvector of T corresponding to a nonzero eigenvalue belongs to $\mathbb{H}^1$.

(iv) $\mathrm{range}(T) \subseteq \mathbb{H}^1$.

5.2 The Feichtinger Problem

The problem posed by Hans Feichtinger at an Oberwolfach mini-workshop on *Wavelets, Frames, and Operator Theory*, in February 2004 [64] is the following.

Problem 5.2.1. *Let T be a positive semi-definite trace class operator on $L^2(\mathbb{R})$ given by $Tf(x) = \int_{\mathbb{R}} k(x,y)f(y)dy$ where $f \in L^2(\mathbb{R})$ and $k \in M^1(\mathbb{R}^2)$. Suppose that $T = \sum_{k=1}^{\infty} h_k \otimes h_k$, where $\{h_k\}_{k=1}^{\infty} \subset L^2(\mathbb{R})$ is a set of orthogonal eigenfunctions of T corresponding to the eigenvalues $\{\|h_k\|^2\}_{k=1}^{\infty}$, such that $\|h_k\|_{M^1(\mathbb{R})} < \infty$. In particular, trace$(T) = \sum_{k=1}^{\infty} \|h_k\|_2^2 < \infty$. Must we have: $\sum_{k=1}^{\infty} \|h_k\|_{M^1(\mathbb{R})}^2 < \infty$?*

Heil and Larson later put the problem in the broader setting of positive semi-definite trace-class operators on a separable Hilbert space $\mathbb{H}$ [92].

Suppose that the operator T given by (5.2) is positive semi-definite. Let $\{h_n\}_{n\geq 1}$ be an orthonormal basis of eigenvectors of T and $\{\lambda_n\}_{n\geq 1} \subset [0,\infty)$ be the corresponding eigenvalues. It follows that $T = \sum_{n=1}^{\infty} \lambda_n (h_n \otimes h_n) = \sum_{n=1}^{\infty} g_n \otimes g_n$, where $g_n = \lambda_n^{1/2} h_n$. In addition, $\|T\|_{\mathcal{I}_1} = \sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n \|h_n\|^2 < \infty$. Heil and Larson's generalization of Problem 5.2.1 is the following question [92].

Problem 5.2.2. *Must we have* $\sum_{n=1}^{\infty} \lambda_n |\!|\!| h_n |\!|\!|^2 < \infty$?

By Theorem 5.1.2, Type A and Type B are equivalent for positive finite-rank operators. But is it true in general? The reformulation of the previous problem is the following.

Problem 5.2.3. *[92] If T is of Type B with respect to an orthonormal basis $\{w_n\}_{n\in\mathbb{N}}$, must it be of Type A with respect to $\{w_n\}_{n\in\mathbb{N}}$?*

Next problem is an equivalent reformulation of original Feichtinger's question Problem 5.2.1, which was shown in [92].

Problem 5.2.4. *Let $\mathbb{H} = L^2(\mathbb{R})$ and $\{w_n\}_{n\in\mathbb{N}}$ be a Wilson orthonormal basis for $L^2(\mathbb{R})$. If T is one of the operators defined in (5.2) with respect to a Wilson basis, must T be of Type A?*

Now we show that the solutions to Problems 5.2.1, 5.2.2 and 5.2.3 are negative by providing counterexamples for each of them. We answer negatively Problems 5.2.2 and 5.2.3 by constructing a counterexample for the complex Hilbert space $\mathbb{H}$, in Proposition 5.2.5. This example is then modified to generate an example when the Hilbert space $\mathbb{H}$ is over the real field, in Proposition 5.2.7. From there, we answer the Feichtinger original problem in Theorem 5.2.8.

Proposition 5.2.5. *Let $\mathbb{H} = \ell^2(\{1, 2, ...\})$, and choose $p > 1$. Let $\{w_\ell\}_{\ell=1}^\infty$ denote the standard orthonormal basis of $\mathbb{H}$, i.e., $w_\ell = \delta_\ell$. Then $\mathbb{H}^1 = \ell^1(\{1, 2, ...\})$. For each $n \geq 1$, let $\{e_{n,k}\}_{k=0}^{n-1}$ be the Fourier ONB of $\mathbb{C}^n$ defined by*

$$e_{n,k} = \frac{1}{\sqrt{n}}\Big(e^{-\frac{2\pi ikl}{n}}\Big)_{l=0}^{n-1} = \frac{1}{\sqrt{n}}\Big(1, e^{-\frac{2\pi ik}{n}}, e^{-\frac{4\pi ik}{n}}, ..., e^{-\frac{2\pi ik(n-1)}{n}}\Big)^T,$$

and consider the $n \times n$ matrix T_n given by

$$T_n = \sum_{k=0}^{n-1} \lambda_{n,k}(e_{n,k} \otimes e_{n,k}) = \frac{1}{n^3}\sum_{k=0}^{n-1}\Big(1 + \frac{k}{n^p}\Big)^2 (e_{n,k} \otimes e_{n,k}) \in \mathbb{C}^{n\times n}$$

where $\lambda_{n,k} = \frac{1}{n^3}\Big(1 + \frac{k}{n^p}\Big)^2$. We define an infinite block-diagonal matrix T by $T = T_1 \oplus T_2 \oplus ... \oplus T_n \oplus ...$ Then, T is a positive semi-definite trace-class operator of Type B but not of Type A with respect to the orthonormal basis $\{w_\ell\}$.

Proof. By construction, the blocks T_n that make up T are pairwise orthogonal. Furthermore, for each $n \geq 1$, the spectrum of T_n consists of simple eigenvectors $e_{n,k}$ with corresponding eigenvalues $\lambda_{n,k}$ for $k = 0, \ldots, n-1$. Consequently, for each $n \geq 1$, and each $k \in \{0, \ldots, n-1\}$, $e_{n,k}$ generates a one-dimensional eigenspace of T corresponding to the eigenvalue $\lambda_{n,k}$. It is clear that T is positive semi-definite. Since, $\|e_{n,k}\|_2 = 1$ and $T = \oplus_{n=1}^\infty \sum_{k=0}^{n-1} \lambda_{n,k}(e_{n,k} \otimes e_{n,k})$, we see that

$$\begin{aligned}\|T\|_{\text{op}} &\leq \sum_{n=1}^\infty \sum_{k=0}^{n-1} \frac{1}{n^3}\Big(1 + \frac{k}{n^p}\Big)^2 \|e_{n,k} \otimes e_{n,k}\|_{\text{op}} \\ &= \sum_{n=1}^\infty \sum_{k=0}^{n-1} \frac{1}{n^3}\Big(1 + \frac{k}{n^p}\Big)^2 \|e_{n,k}\|^2 = \sum_{n=1}^\infty \sum_{k=0}^{n-1} \frac{1}{n^3}\Big(1 + \frac{k}{n^p}\Big)^2 < \infty.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|T\|_{\mathcal{I}_1} = \text{trace}(T) &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{1}{n^3}\Big(1+\frac{k}{n^p}\Big)^2 \\ &= \sum_{n=1}^{\infty}\frac{1}{n^3}\Big(n+\frac{n(n-1)}{n^p}+\frac{n(n-1)(2n-1)}{6n^{2p}}\Big) \\ &< \infty, \text{ since } p>1.\end{aligned}$$

Hence T is a well-defined trace-class operator on $\mathbb{H}$. We now show that T is of Type B. To this end we observe that for each $n \geq 1$, $\sum_{k=0}^{n-1} e_{n,k}\otimes e_{n,k} = I_n$, where I_n denotes the identity of order n. Then

$$\begin{aligned}T_n &= \frac{1}{n^3}\sum_{k=0}^{n-1}\Big(1+\frac{k}{n^p}\Big)^2(e_{n,k}\otimes e_{n,k}) \\ &= \frac{1}{n^3}\sum_{k=0}^{n-1}(e_{n,k}\otimes e_{n,k}) + \frac{1}{n^{3+p}}\sum_{k=0}^{n-1}\Big(2k+\frac{k^2}{n^p}\Big)(e_{n,k}\otimes e_{n,k}) \\ &= \frac{1}{n^3}I_n + \frac{1}{n^{3+p}}\sum_{k=0}^{n-1}\Big(2k+\frac{k^2}{n^p}\Big)(e_{n,k}\otimes e_{n,k}).\end{aligned}$$

Thus T can be written as

$$\begin{aligned}T &= \oplus_{n\geq 1}T_n = \oplus_{n\geq 1}\Big(\frac{1}{n^3}I_n + \frac{1}{n^{3+p}}\sum_{k=0}^{n-1}\Big(2k+\frac{k^2}{n^p}\Big)(e_{n,k}\otimes e_{n,k})\Big) \\ &= \oplus_{n\geq 1}\Big(\frac{1}{n^3}I_n\Big) + \oplus_{n\geq 1}\frac{1}{n^{3+p}}\sum_{k=0}^{n-1}\Big(2k+\frac{k^2}{n^p}\Big)(e_{n,k}\otimes e_{n,k}) \\ &= \oplus_{n\geq 1}\frac{1}{n^3}\sum_{k=1}^{n}(w_{\frac{n(n-1)}{2}+k}\otimes w_{\frac{n(n-1)}{2}+k}) \\ &\quad + \oplus_{n\geq 1}\,\frac{1}{n^{3+p}}\sum_{k=0}^{n-1}\Big(2k+\frac{k^2}{n^p}\Big)(e_{n,k}\otimes e_{n,k}).\end{aligned}$$

Then we have $\Big|\Big|\Big|w_{\frac{n(n-1)}{2}+k}\Big|\Big|\Big| = 1, \quad |||e_{n,k}||| = \sqrt{n}$, and

$$\begin{aligned} & \sum_{n\geq 1}\frac{1}{n^3}\cdot\sum_{k=1}^{n}1+\sum_{n\geq 1}\frac{1}{n^{3+p}}\sum_{k=0}^{n-1}\Big(2k+\frac{k^2}{n^p}\Big)\cdot(\sqrt{n})^2 \\ = & \sum_{n\geq 1}\frac{1}{n^2}+\sum_{n\geq 1}\frac{1}{n^{2+p}}\Big(2\cdot\frac{n(n-1)}{2}+\frac{1}{n^p}\cdot\frac{n(n-1)(2n-1)}{6}\Big) \\ = & \sum_{n\geq 1}\Big(\frac{1}{n^2}+\frac{n-1}{n^{1+p}}+\frac{(n-1)(2n-1)}{6n^{2+2p}}\Big)<\infty, \quad \text{for any } p>1. \end{aligned}$$

Hence, T is of Type B with respect to $\{w_\ell\}_{\ell\geq 1}$. We now show that T is not of Type A with respect to $\{w_\ell\}_\ell$. The key point is that T has only one-dimensional eigenspaces, so

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k}(e_{n,k}\otimes e_{n,k})=\frac{1}{n^3}\sum_{k=0}^{n-1}\Big(1+\frac{k}{n^p}\Big)^2(e_{n,k}\otimes e_{n,k})$$

is the unique decomposition of T using sum of rank one projections generated by orthogonal eigenfunctions of T. Note that $|||e_{n,k}|||=\frac{1}{\sqrt{n}}\cdot n=\sqrt{n}$, and

$$\lambda_{n,k}|||e_{n,k}|||=\frac{1}{n^3}\Big(1+\frac{k}{n^p}\Big)^2\cdot\sqrt{n}<\infty.$$

However,

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k}|||e_{n,k}|||^2 &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}\Big(1+\frac{k}{n^p}\Big)^2 \\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\Big(n+\frac{n(n-1)}{n^p}+\frac{n(n-1)(2n-1)}{6n^{2p}}\Big) \\ &\geq \sum_{n=1}^{\infty}\frac{1}{n}=\infty. \end{aligned}$$

□

We can modify the counter-example in Proposition 5.2.5 to deal with the case of a real Hilbert space $\mathbb{H}$. This amount to use a real-valued ONB for $\mathbb{R}^n$ instead of the ONB $\{e_{n,k}\}_{k=0}^{n-1}$.

For a fix $n\geq 1$, let $S_n:=\Big\{\frac{2\pi k}{n}:0\leq k\leq n-1\Big\}$. It is easy to see that for any

$0 \le l \le n-1$, we have

$$S_n = \left\{ \frac{2\pi kl}{n} \pmod{2\pi} : 0 \le k \le n-1 \right\} = \left\{ -\frac{2\pi k}{n} \pmod{2\pi} : 0 \le k \le n-1 \right\}.$$

Now for each $n \ge 1$, let $\{h_{n,k}\}_{k=0}^{n-1}$ denote the Hartley ONB basis for $\mathbb{R}^n$ (see [139]), where

$$h_{n,k} = \frac{1}{\sqrt{n}} \left(\cos\left(\frac{2\pi kl}{n}\right) + \sin\left(\frac{2\pi kl}{n}\right) \right)_{l=0}^{n-1} = \sqrt{\frac{2}{n}} \left(\cos\left(\frac{2\pi kl}{n} - \frac{\pi}{4}\right) \right)_{l=0}^{n-1}.$$

Lemma 5.2.6. *For a fixed $n \ge 1$ and any $0 \le k \le n-1$ we have*

$$|||h_{n,k}||| = \sum_{l=0}^{n-1} \left| \cos\left(\frac{2\pi kl}{n}\right) + \sin\left(\frac{2\pi kl}{n}\right) \right| \ge \frac{n}{\sqrt{2}}.$$

Proof. Let $E = \sum_{x \in S_n} |\cos x + \sin x|$. Then

$$\begin{aligned} 2E &= \sum_{x \in S_n} |\cos x + \sin x| + \sum_{-x \in S_n} |\cos x + \sin x| \\ &= \sqrt{2} \sum_{k=0}^{n-1} \left| \cos\left(\frac{2\pi k}{n} - \frac{\pi}{4}\right) \right| + \sqrt{2} \sum_{k=0}^{n-1} \left| \cos\left(\frac{2\pi k}{n} + \frac{\pi}{4}\right) \right| \\ &= \sqrt{2} \sum_{k=0}^{n-1} \left[\left| \cos\left(\frac{2\pi k}{n} - \frac{\pi}{4}\right) \right| + \left| \sin\left(\frac{2\pi k}{n} - \frac{\pi}{4}\right) \right| \right]. \qquad (5.9) \end{aligned}$$

Now for any $x \in \mathbb{R}$,

$$(|\sin x| + |\cos x|)^2 = |\sin x|^2 + |\cos x|^2 + 2|\sin x \cos x| = 1 + |\sin 2x| \ge 1,$$
$$\Rightarrow |\sin x| + |\cos x| \ge 1.$$

Therefore by (5.9) we get $2E \ge \sqrt{2} \sum_{k=0}^{n-1} 1 = \sqrt{2} n \Rightarrow E \ge \frac{n}{\sqrt{2}}$. ☐

Proposition 5.2.7. *Let $\mathbb{H} = \ell^2(\{1, 2, ...\})$, and choose $p > 1$. Let $\{w_\ell\}_{\ell=1}^{\infty}$ denote the standard orthonormal basis of $\mathbb{H}$, i.e., $w_\ell = \delta_\ell$. For each $n \ge 1$ let T_n denote* *the $n \times n$ matrix given by*

$$T_n = A_n \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right)^2 (h_{n,k} \otimes h_{n,k}) \in \mathbb{R}^{n \times n},$$

where $A_n = n^{-3}$. We define an infinite block-diagonal matrix T by $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots$ Then, T is a positive semi-definite trace-class operator of Type B but not of Type A with respect to the orthonormal basis $\{w_\ell\}_{\ell \geq 1}$.

Proof. The proof is identical to that of Proposition 5.2.5. But for the sake of completeness we provide the details.

By construction, the blocks T_n that make up T are pairwise orthogonal. Furthermore, for each $n \geq 1$, the spectrum of T_n consists of simple eigenvectors $h_{n,k}$ with corresponding eigenvalues $\lambda_{n,k} = \frac{1}{n^3}(1 + \frac{k}{n^p})^2$ for $k = 0, \ldots, n-1$. Thus T is positive semi-definite. Consequently, for each $n \geq 1$, and each $k \in \{0, \ldots, n-1\}$, $h_{n,k}$ generates a one-dimensional eigenspace of T corresponding to the eigenvalue $\lambda_{n,k}$. Since, $\|h_{n,k}\|_2 = 1$ and $T = \oplus_{n=1}^{\infty} \sum_{k=0}^{n-1} \lambda_{n,k}(e_{n,k} \otimes e_{n,k})$, we see that

$$\begin{aligned} \|T\|_{\mathcal{I}_1} = \operatorname{trace}(T) &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} A_n \left(1 + \frac{k}{n^p}\right)^2 \\ &= \sum_{n=1}^{\infty} A_n \left(n + \frac{n(n-1)}{n^p} + \frac{n(n-1)(2n-1)}{6n^{2p}}\right). \end{aligned}$$

The choice $A_n = 1/n^3$ guarantees that the series is convergent in trace-class norm i.e., $\|T\|_{\mathcal{I}_1} < \infty$. Thus T is a positive semi-definite trace-class operator on $\mathbb{H}$.

We now show that T is not of Type A. The key point is that T has only one-dimensional eigenspaces, so $\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} g_{n,k} \otimes g_{n,k}$, where $g_{n,k} = \sqrt{A_n} \left(1 + \frac{k}{n^p}\right) h_{n,k}$ is the unique decomposition of T using sum of rank one projections generated by orthogonal eigenfunctions of T. Then

$$\begin{aligned} |||g_{n,k}||| &= \sqrt{A_n} \left(1 + \frac{k}{n^p}\right) |||h_{n,k}||| \\ &= \sqrt{A_n} \left(1 + \frac{k}{n^p}\right) \cdot \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left| \cos\left(\frac{2\pi kl}{n}\right) + \sin\left(\frac{2\pi kl}{n}\right) \right| \end{aligned}$$

$$\geq \sqrt{A_n}\left(1+\frac{k}{n^p}\right)\cdot\frac{1}{\sqrt{n}}\frac{n}{\sqrt{2}} = \sqrt{\frac{n}{2}\cdot A_n\left(1+\frac{k}{n^p}\right)^2}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}|||g_{n,k}|||^2 &\geq \sum_{n=1}^{\infty}\frac{n}{2}\cdot A_n\sum_{k=0}^{n-1}\left(1+\frac{k}{n^p}\right)^2\\
&= \sum_{n=1}^{\infty}\frac{n}{2}\cdot A_n\left(n+\frac{n(n-1)}{n^p}+\frac{n(n-1)(2n-1)}{6n^{2p}}\right)\\
&\geq \frac{1}{2}\sum_{n=1}^{\infty}n^2A_n = \sum_{n=1}^{\infty}\frac{1}{2n} = \infty.
\end{aligned}$$

So T is not of Type A.

We now show that T is of Type B. Let $\lambda_k = \sqrt{A_n}\left(1+\frac{k}{n^p}\right)$ and let U be the unitary operator (matrix) given by $U = [h_{n,0}\ h_{n,1}\ ...\ h_{n,n-1}]$. It follows that

$$\begin{aligned}
T_n &= [\lambda_0 h_{n,0}\ \ \lambda_1 h_{n,1}\ ...\ \lambda_{n-1}h_{n,n-1}]\cdot(U^*U)\cdot[\lambda_0 h^*_{n,0}\ \ \lambda_1 h^*_{n,1}\ \ ...\ \ \lambda_{n-1}h^*_{n,n-1}]\\
&= [\lambda_0 h_{n,0}\ \ \lambda_1 h_{n,1}\ \ ...\ \ \lambda_{n-1}h_{n,n-1}]\,U^*\cdot\left([\lambda_0 h_{n,0}\ \ \lambda_1 h_{n,1}\ \ ...\ \ \lambda_{n-1}h_{n,n-1}]\,U^*\right)^*\\
&= G_n\cdot G_n^*,
\end{aligned}$$

where $G_n = [\lambda_0 h_{n,0}\ \ \lambda_1 h_{n,1}\ \ ...\ \ \lambda_{n-1}h_{n,n-1}]\,U^*$. Now we compute G_n

$$\begin{aligned}
G_n &= [\lambda_0 h_{n,0}\ \ \lambda_1 h_{n,1}\ \ ...\ \ \lambda_{n-1}h_{n,n-1}]\cdot[h^*_{n,0}\ h^*_{n,1}\ ...\ h^*_{n,n-1}]\\
&= \lambda_0 h_{n,0}h^*_{n,0}+\lambda_1 h_{n,1}h^*_{n,1}+...+\lambda_{n-1}h_{n,n-1}h^*_{n,n-1}\\
&= \left(\sqrt{A_n}+\lambda_0-\sqrt{A_n}\right)h_{n,0}h^*_{n,0}+\left(\sqrt{A_n}+\lambda_1-\sqrt{A_n}\right)h_{n,1}h^*_{n,1}+...\\
&\quad+\left(\sqrt{A_n}+\lambda_{n-1}-\sqrt{A_n}\right)h_{n,n-1}h^*_{n,n-1}\\
&= \sqrt{A_n}\left(h_{n,0}h^*_{n,0}+h_{n,1}h^*_{n,1}+...+h_{n,n-1}h^*_{n,n-1}\right)+\left(\lambda_0-\sqrt{A_n}\right)h_{n,0}h^*_{n,0}\\
&\quad+\left(\lambda_1-\sqrt{A_n}\right)h_{n,1}h^*_{n,1}+...+\left(\lambda_{n-1}-\sqrt{A_n}\right)h_{n,n-1}h^*_{n,n-1}\\
&= \sqrt{A_n}I_n+(\lambda_1-\lambda_0)\,h_{n,1}h^*_{n,1}+...+(\lambda_{n-1}-\lambda_0)\,h_{n,n-1}h^*_{n,n-1},
\end{aligned}$$

where I_n is the identity matrix of order n and $\sqrt{A_n} = \lambda_0$. Let $\Lambda_n = \lambda_n - \lambda_0$ then

$$\begin{aligned}
T_n &= G_n \cdot G_n^* \\
&= \left(\sqrt{A_n} I_n + \Lambda_1 h_{n,1} h_{n,1}^* + \ldots + \Lambda_{n-1} h_{n,n-1} h_{n,n-1}^*\right) \cdot \\
&\quad \left(\sqrt{A_n} I_n + \Lambda_1 h_{n,1} h_{n,1}^* + \ldots + \Lambda_{n-1} h_{n,n-1} h_{n,n-1}^*\right) \\
&= A_n I_n + 2\Lambda_1 \sqrt{A_n} h_{n,1} h_{n,1}^* + 2\Lambda_2 \sqrt{A_n} h_{n,2} h_{n,2}^* + \ldots + 2\Lambda_{n-1} \sqrt{A_n} h_{n,n-1} h_{n,n-1}^* \\
&\quad + \Lambda_1^2 h_{n,1} h_{n,1}^* + \Lambda_2^2 h_{n,2} h_{n,2}^* + \ldots + \Lambda_{n-1}^2 h_{n,n-1} h_{n,n-1}^* \\
&= A_n I_n + \left(2\Lambda_1 \sqrt{A_n} + \Lambda_1^2\right) h_{n,1} h_{n,1}^* + \ldots + \left(2\Lambda_{n-1} \sqrt{A_n} + \Lambda_{n-1}^2\right) h_{n,n-1} h_{n,n-1}^* \\
&= A_n \left(\delta_0 \delta_0^* + \delta_1 \delta_1^* + .. + \delta_{n-1} \delta_{n-1}^*\right) + \left(2\Lambda_1 \sqrt{A_n} + \Lambda_1^2\right) h_{n,1} h_{n,1}^* + \ldots \\
&\quad + \left(2\Lambda_{n-1} \sqrt{A_n} + \Lambda_{n-1}^2\right) h_{n,n-1} h_{n,n-1}^* \\
&= f_{n,0} f_{n,0}^* + \ldots + f_{n,n-1} f_{n,n-1}^* + f_{n,n} f_{n,n}^* + \ldots + f_{n,2n-2} f_{n,2n-2}^*,
\end{aligned}$$

where $f_{n,k} = \sqrt{A_n}\delta_k$, for $0 \le k \le n-1$ and $f_{n,k} = \sqrt{2\Lambda_{k-n+1}\sqrt{A_n} + \Lambda_{k-n+1}^2}\, h_{n,k-n+1}$, for $n \le k \le 2n-2$. Now we compute $|||\cdot|||$-norm of $f_{n,k}$

$$\begin{aligned}
|||f_{n,k}||| &= \begin{cases} \sqrt{A_n}|||\delta_k|||, & 0 \le k \le n-1 \\ \sqrt{2\Lambda_{k-n+1}\sqrt{A_n} + \Lambda_{k-n+1}^2}\,|||h_{n,k-n+1}|||, & n \le k \le 2n-2 \end{cases} \\
&= \begin{cases} \sqrt{A_n}, & 0 \le k \le n-1 \\ \sqrt{2\Lambda_{k-n+1}\sqrt{A_n} + \Lambda_{k-n+1}^2}\,|||h_{n,k-n+1}|||, & n \le k \le 2n-2. \end{cases}
\end{aligned}$$

Since $\Lambda_k = \lambda_k - \lambda_0 = \sqrt{A_n} \cdot \left(1 + \frac{k}{n^p}\right) - \sqrt{A_n} = \sqrt{A_n} \cdot \frac{k}{n^p}$, then

$$\begin{aligned}
\sqrt{2\Lambda_{k-n+1}\sqrt{A_n} + \Lambda_{k-n+1}^2} &= \sqrt{\sqrt{A_n} \cdot \frac{k-n+1}{n^p}\left(2\sqrt{A_n} + \sqrt{A_n} \cdot \frac{k-n+1}{n^p}\right)} \\
&= \sqrt{A_n} \cdot \sqrt{\frac{k-n+1}{n^p}\left(2 + \frac{k-n+1}{n^p}\right)} \\
&\le \sqrt{A_n} \cdot \sqrt{\frac{3n}{n^p}}.
\end{aligned}$$

Also $|||h_{n,k-n+1}||| \le \sqrt{2n}$. Therefore,

$$\sqrt{2\Lambda_{k-n+1}\sqrt{A_n}+\Lambda_{k-n+1}^2}|||h_{n,k-n+1}|||$$
$$\leq \quad \sqrt{A_n}\cdot\sqrt{\frac{3n}{n^p}}\cdot\sqrt{2n}=\sqrt{6A_n}\cdot\sqrt{\frac{n^2}{n^p}}=\sqrt{\frac{6}{n^{1+p}}}, \quad \text{since } A_n=\frac{1}{n^3}.$$

Now

$$\sum_{n=1}^{\infty}\sum_{k=0}^{2n-2}|||f_{n,k}|||^2 \leq \sum_{n=1}^{\infty}(2n-1)\cdot\frac{6}{n^{1+p}} \leq 12\sum_{n=1}^{\infty}\frac{1}{n^p}<\infty, \quad \text{since } p>1,$$

and $T=\oplus_{n=1}^{\infty}\Big(\sum_{k=0}^{2n-2} f_{n,k}\otimes f_{n,k}\Big)$. So T is of Type B. □

We can now give an answer to Feichtinger's question, i.e., Problem 5.2.2.

Theorem 5.2.8. *Suppose that $\{w_n\}_{n\geq 1}$ is a Wilson basis for $L^2(\mathbb{R})$ with $g\in M^1(\mathbb{R})$. Let $p\geq 2$, and for each $n\geq 1$ set $\lambda_{n,k}=\frac{1}{n^3}(1+\frac{k}{n^p})^2$. For fix $n\geq 1$ and any $0\leq k\leq n-1$, let $h_{n,k}\in L^2(\mathbb{R})$ where*

$$h_{n,k}=\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\left(\cos\left(\frac{2\pi kl}{n}\right)+\sin\left(\frac{2\pi kl}{n}\right)\right)w_{\frac{n(n-1)}{2}+l+1}.$$

Let T be an operator define by $T=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k}h_{n,k}\otimes h_{n,k}$. The following holds:

(i) $\{h_{n,k}:0\leq k\leq n-1, n\geq 1\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

(ii) T is a positive semi-definite trace-class operator on $L^2(\mathbb{R})$ that provides a counter-example to Problem 5.2.2.

Proof. (*i*) It is easy to see that for each $n\geq 1$, $\{h_{n,k}\}_{k=0}^{n-1}$ is an orthogonal set in $L^2(\mathbb{R})$. Also, $\langle h_{n,k},h_{n',k'}\rangle=0$, for $n\neq n'$. They are normalized in $L^2(\mathbb{R})$, since $\langle w_n,w_m\rangle=\delta_{n,m}$.

(*ii*) It is also easy to see that T is a well defined operator on $L^2(\mathbb{R})$. In fact, the series defining T converges in the operator norm. Furthermore, since $\|h_{n,k}\otimes$

$h_{n,k}\|_{\mathcal{I}_1} = 1$, it follows that

$$\begin{aligned}\|T\|_{\mathcal{I}_1} &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k} = \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=0}^{n-1}\left(1+\frac{k}{n^p}\right)^2 \\ &= \sum_{n=1}^{\infty}\frac{1}{n^3}\left(n+\frac{n(n-1)}{n^p}+\frac{n(n-1)(2n-1)}{6n^{2p}}\right) < \infty.\end{aligned}$$

Consequently, T is a trace-class operator. By Lemma 5.2.6,

$$\begin{aligned}\|h_{n,k}\|_{M^1} &= \sum_{m=1}^{\infty}|\langle h_{n,k}, w_m\rangle| \\ &= \frac{1}{\sqrt{n}}\sum_{m=1}^{\infty}\left|\left\langle \sum_{l=0}^{n-1}\left(\cos\left(\frac{2\pi kl}{n}\right)+\sin\left(\frac{2\pi kl}{n}\right)\right) w_{\frac{n(n-1)}{2}+l}, w_m\right\rangle\right| \\ &= \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\left|\cos\left(\frac{2\pi kl}{n}\right)+\sin\left(\frac{2\pi kl}{n}\right)\right| \geq \frac{1}{\sqrt{n}}\cdot\frac{n}{\sqrt{2}} = \sqrt{\frac{n}{2}}.\end{aligned}$$

Also each term

$$\begin{aligned}\lambda_{n,k}\|h_{n,k}\|_{M^1} &= \frac{1}{n^3}(1+\frac{k}{n^p})^2\cdot\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\left|\cos\left(\frac{2\pi kl}{n}\right)+\sin\left(\frac{2\pi kl}{n}\right)\right| \\ &\leq \frac{1}{n^3}(1+\frac{k}{n^p})^2\cdot 2\sqrt{n} < \infty,\end{aligned}$$

but

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\lambda_{n,k}\|h_{n,k}\|_{M^1}^2 &\geq \sum_{n=1}^{\infty}\frac{1}{2n^2}\sum_{k=0}^{n-1}(1+\frac{k}{n^p})^2 \\ &= \sum_{n=1}^{\infty}\frac{1}{2n^2}\left(n+\frac{n(n-1)}{n^p}+\frac{n(n-1)(2n-1)}{6n^{2p}}\right) \\ &\geq \sum_{n=1}^{\infty}\frac{1}{2n} = \infty.\end{aligned}$$

□

Chapter 6

Balian-Low Type Theorems on $L^2(\mathbb{C})$

In this chapter, we prove the Balian-Low type theorem (BLT) on $L^2(\mathbb{C})$ using the operators Z and $\bar{Z}$ i.e., we prove that $\|Zg\|_2$ and $\|\bar{Z}g\|_2$ cannot both be simultaneously finite if the twisted Gabor frame generated by $g \in L^2(\mathbb{C})$ forms an orthonormal basis or an exact frame for $L^2(\mathbb{C})$. The operators

$$Z = \frac{d}{dz} + \frac{1}{2}\bar{z} \quad \text{and} \quad \bar{Z} = \frac{d}{d\bar{z}} - \frac{1}{2}z$$

are associated with the special Hermite operator

$$L = -\Delta_z + \frac{1}{4}|z|^2 - i\left(x\frac{d}{dy} - y\frac{d}{dx}\right)$$

on $\mathbb{C}$, where Δ_z is the standard Laplacian on $\mathbb{C}$ and $z = x + iy$. Also we present the amalgam version of BLT using Weyl transform and illustrate the distinction between BLT and amalgam BLT by examples. We introduce the twisted Zak transform and using it we establish several versions of the Balian-Low type theorems on $L^2(\mathbb{C})$.

6.1 Preliminaries

6.1.1 Heisenberg Group and the Weyl Transform

One of the simple and natural example of non-abelian, non-compact groups is the famous Heisenberg group $\mathbf{H}$, which plays an important role in several branches of mathematics. The Heisenberg group $\mathbf{H}$ is a unimodular nilpotent Lie group whose underlying manifold is $\mathbb{C} \times \mathbb{R}$ and the group operation is defined by

$$(z,t)\cdot(w,s) = (z+w, t+s+\frac{1}{2}Im(z\bar{w})).$$

The Haar measure on $\mathbf{H}$ is given by $dzdt$. By Stone-von Neumann theorem, the only infinite dimensional unitary irreducible representations (up to unitary equivalence) are given by $\pi_\lambda,\ \lambda \in \mathbb{R}\setminus\{0\}$, where π_λ is defined by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{4\pi i\lambda t}e^{4\pi i\lambda(x\xi+\frac{1}{2}xy)}\varphi(\xi+y),$$

where $z = x+iy$ and $\varphi \in L^2(\mathbb{R})$.

The group Fourier transform of $f \in L^1(\mathbf{H})$ is defined as

$$\hat{f}(\lambda) = \int_{\mathbf{H}} f(z,t)\pi_\lambda(z,t)dzdt, \qquad \lambda \in \mathbb{R}\setminus\{0\}.$$

Note that for each $\lambda \in \mathbb{R}\setminus\{0\}$, $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R})$. Under the operation "group convolution" $L^1(\mathbf{H})$ turns out to be a non-commutative Banach algebra.

Let

$$f^\lambda(z) = \int_{\mathbb{R}} e^{4\pi i\lambda t} f(z,t)dt$$

denote the inverse Fourier transform of f in the $t-$variable. Therefore $\hat{f}(\lambda) = \int_{\mathbb{C}} f^\lambda(z)\pi_\lambda(z,0)dz$. Thus we are led to consider operators of the form

$$W_\lambda(g) = \int_{\mathbb{C}} g(z)\pi_\lambda(z)dz, \tag{6.1}$$

where $\pi_\lambda(z,0) = \pi_\lambda(z)$. For $\lambda = 1$ we call (6.1) as the Weyl transform of g. Thus for $g \in L^1(\mathbb{C})$ and writing $\pi(z)$ in place of $\pi_1(z)$ we have

$$W(g)\varphi(\xi) = \int_{\mathbb{C}} g(z)\pi(z)\varphi(\xi)dz, \qquad \varphi \in L^2(\mathbb{C}). \tag{6.2}$$

For $f, g \in L^1(\mathbb{C})$, the twisted convolution is defined by

$$f \times g(z) = \int_{\mathbb{C}} f(z-w)g(w)e^{-2\pi i Im(z\cdot\bar{w})}dw.$$

Under twisted convolution $L^1(\mathbb{C})$ is a non-commutative Banach algebra. For $f \in L^1 \cap L^2(\mathbb{C})$ the Weyl transform of f can be explicitly written as

$$W(f)\varphi(\xi) = \int_{\mathbb{C}} f(z)e^{4\pi i(x.\xi+\frac{1}{2}x.y)}\varphi(\xi+y)dz, \ \ \varphi \in L^2(\mathbb{R}), \ \ z = x+iy,$$

which maps $L^1(\mathbb{C})$ into the space of bounded operators on $L^2(\mathbb{R})$, denoted by $\mathcal{B}$. The Weyl transform $W(f)$ is an integral operator with kernel

$$K_f(\xi,\eta) = \int_{\mathbb{R}} f(x,\eta-\xi)e^{2\pi i x(\xi+\eta)}\,dx.$$

If $f \in L^2(\mathbb{C})$, then $W(f) \in C_2$, the space of all Hilbert-Schmidt operators on $L^2(\mathbb{R})$ and satisfies the Plancherel formula $\|W(f)\|_{C_2} = \frac{1}{2}\|f\|_{L^2(\mathbb{C})}$. In general, for $f, g \in L^2(\mathbb{C})$, we have $\langle W(f), W(g)\rangle_{C_2} = \frac{1}{2}\langle f, g\rangle_{L^2(\mathbb{C})}$. The inversion formula for Weyl transform is $f(z) = \tau(\pi(z)^* W(f))$, where $\pi(z)^*$ is the adjoint of $\pi(z)$ and τ is the usual trace on $\mathcal{B}$. For detailed study on Weyl transform we refer to the text of Thangavelu [131, 132].

6.1.2 Hermite Operators

Let H_k denote the Hermite polynomial on $\mathbb{R}$, defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}, \ k = 0, 1, 2, \cdots,$$

and h_k denote the normalized Hermite functions on $\mathbb{R}$ defined by

$$h_k(x) = (2^k\sqrt{\pi}k!)^{-\frac{1}{2}} H_k(x)e^{-\frac{1}{2}x^2}, \quad k = 0, 1, 2, \cdots .$$

Let $A = -\frac{d}{dx} + x$ and $A^* = \frac{d}{dx} + x$ denote the creation and annihilation operators in quantum mechanics respectively. The Hermite operator H is defined as

$$H = \frac{1}{2}(AA^* + A^*A) = -\frac{d^2}{dx^2} + x^2.$$

The Hermite functions $\{h_k\}$ are the eigenfunctions of the operator H with eigenvalues $2k+1, k = 0, 1, 2 \cdots$. Using the Hermite functions, the special Hermite functions on $\mathbb{C}$ are defined as follows:

$$\phi_{m,n}(z) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{i\xi x} h_m\left(\xi + \frac{1}{2}y\right) h_n\left(\xi - \frac{1}{2}y\right) d\xi,$$

where $z = x + iy \in \mathbb{C}$ and $m, n = 0, 1, 2 \cdots$. The functions $\{\phi_{m,n} : m, n = 0, 1, 2 \cdots\}$ form an orthonormal basis for $L^2(\mathbb{C})$. The special Hermite functions are the eigenfunctions of a second order elliptic operator L on $\mathbb{C}$. To define this operator L we need to define the operators Z and $\bar{Z}$ as follows: $Z = \frac{d}{dz} + \frac{1}{2}\bar{z}$ and $\bar{Z} = \frac{d}{d\bar{z}} - \frac{1}{2}z$. The functions $\phi_{m,n}$ are eigenfunctions of the special Hermite operator

$$L = -\Delta_z + \frac{1}{4}|z|^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -\frac{1}{2}(Z\bar{Z} + \bar{Z}Z), \tag{6.3}$$

with eigenvalues $(2n+1)$, where Δ_z denotes the Laplacian on $\mathbb{C}$. We list out some of the properties (see [131, 132]) of the operators Z and $\bar{Z}$ in the following proposition, which will be useful at several places.

Proposition 6.1.1. *(i)* $Z(\phi_{m,n}) = i\sqrt{2n}\,\phi_{m,n-1}$ *and* $\bar{Z}(\phi_{m,n}) = i\sqrt{2n+2}\,\phi_{m,n+1}$.

(ii) $W(Zf) = iW(f)A$ *and* $W(\bar{Z}f) = iW(f)A^*$ *for every Schwartz class function* f*. (This expression is similar to the relation* $(\frac{d}{dx}f)\hat{}\,(\gamma) = 2\pi i\gamma\hat{f}(\gamma)$*).*

(iii) $[Z, \bar{Z}] = -2I$, *where* $[Z, \bar{Z}] = Z\bar{Z} - \bar{Z}Z$ *is the commutator of* Z *and* $\bar{Z}$.

(iv) The adjoint Z^* *of* Z *is* $-\bar{Z}$.

The motivation to prove the BLT on $L^2(\mathbb{C})$ arises from the classical Heisenberg's uncertainty principle on $L^2(\mathbb{R})$:

Let P and M be the position and the momentum operators defined by $Pf(t) = tf(t)$ and $Mf(t) = \frac{d}{dt}f(t)$, on a suitable domain.

Theorem 6.1.2. (Classical Heisenberg's uncertainty principle on $L^2(\mathbb{R})$). *Let $f \in L^2(\mathbb{R})$. Then*

$$\|Pf\|_2 \, \|Mf\|_2 \geq \frac{1}{4\pi}\|f\|_2^2, \tag{6.4}$$

and equality holds in (6.4) if and only if $f(t) = Ce^{-st^2}$, $s > 0$ and $C \in \mathbb{C}$.

Observe that the Laplacian L_0 on $\mathbb{R}$ can be written as

$$L_0 = \frac{d^2}{dx^2} = \frac{1}{4}(A^* - A)(A^* - A) = \frac{1}{4}(A^*B + AB^*) \tag{6.5}$$

and satisfies $[A, B] = [A, A^*] = -2I$, where $B = A^* - A$. The expression for special Hermite operator L is similar to the Laplacian L_0 on $\mathbb{R}$ (see (1.23) and (6.5)) with $[Z, \bar{Z}] = -2I$. The classical uncertainty principle requires the operators P and M to be self-adjoint and uses the fact that $[P, M] = -I$, whereas the operators Z and $\bar{Z}$ are not self-adjoint. However, we obtain a variation of Heisenberg's uncertainty principle (see Theorem 6.4.1) involving the operators Z and $\bar{Z}$. Motivated by this result we prove the BLT on $L^2(\mathbb{C})$ for twisted Gabor frames using the operators Z and $\bar{Z}$ (see Theorem 6.3.4).

6.1.3 Twisted Gabor Frames

Definition 6.1.3. *Let $f \in L^2(\mathbb{C})$ and $a, b > 0$. For $(m, n) \in \mathbb{Z}^2$ we define twisted translation of f, denoted by $T^t_{(am,bn)}f$, as*

$$T^t_{(am,bn)}f(z) = e^{2\pi i(bnx - amy)}f(x - am, y - bn), \quad z = x + iy \in \mathbb{C}. \tag{6.6}$$

For $a = b = 1$ the properties of twisted translation are listed below (see [117]).

(*i*) The adjoint $(T^t_{(m,n)})^*$ of $T^t_{(m,n)}$ is $T^t_{(-m,-n)}$.

(*ii*) $T^t_{(m_1,n_1)}T^t_{(m_2,n_2)} = T^t_{(m_1+m_2,n_1+n_2)}$.

(*iii*) $T^t_{(m,n)}$ is a unitary operator on $L^2(\mathbb{C})$ for all $(m,n)\in\mathbb{Z}^2$.

(*iv*) The Weyl transform of $T^t_{(m,n)}f$ is given by $W(T^t_{(m,n)}f)=\pi(m,n)W(f)$.

Definition 6.1.4. *For $a,b>0$, $g\in L^2(\mathbb{C})$ the collection of functions $\mathcal{G}^t(g,a,b)=\{T^t_{(am,bn)}g:\ m,n\in\mathbb{Z}\}$ in $L^2(\mathbb{C})$, is called a twisted Gabor frame or a twisted Weyl-Heisenberg frame if there exist constants $A,B>0$ such that*

$$A\|f\|_2^2\le\sum_{k,n\in\mathbb{Z}}|\langle f,T^t_{(am,bn)}g\rangle|^2\le B\|f\|_2^2,\ \ \forall f\in L^2(\mathbb{C}). \tag{6.7}$$

Then one can define the twisted Gabor tight frames, Riesz basis and the frame operator analogously. It is natural to ask about the density result as in Theorem 1.2.6 for twisted Gabor frames. For $a,b>0$ and $g\in L^2(\mathbb{C})$, the sequence $\{T^t_{(am,bn)}g:\ m,n\in\mathbb{Z}\}$ is complete in $L^2(\mathbb{C})$ if and only if the system $\{\rho(p,q)g:(p,q)\in\Lambda\subset\mathbb{R}^4\}$ is complete in $L^2(\mathbb{R}^2)$, where $p=(am,bn),q=(bn,-am)$. In this case uniform Beurling density is $D(\Lambda)=1/(ab)^2$. So by Theorem 1.2.6, if $ab>1$ then the twisted Gabor system $\mathcal{G}^t(g,a,b)=\{T^t_{(am,bn)}g:\ m,n\in\mathbb{Z}\}$ is incomplete in $L^2(\mathbb{C})$. Therefore without loss of generality we consider the case when $a=b=1$ throughout the chapter.

6.1.4 Twisted Zak Transform

The Zak transform $\mathcal{Z}f$ on $L^2(\mathbb{R})$ is formally defined by

$$\mathcal{Z}f(x,t)=\sum_{k\in\mathbb{Z}}T_kf(x)\cdot M_k\mathbf{1}(t),\quad x,t\in\mathbb{R},$$

where $\mathbf{1}$ is the constant function 1. Since we are interested to obtain the BLT for twisted Gabor frames we define the twisted Zak transform with a slight modification in the following way.

Definition 6.1.5. *Let $f\in L^2(\mathbb{C})$. The twisted Zak transform Z^tf of f is the function on $\mathbb{C}^2$ defined by*

$$(Z^tf)(z,w)=\sum_{k\in\mathbb{Z}^2}T_kf(z)\cdot T^t_k\mathbf{1}(-w)=\sum_{k\in\mathbb{Z}^2}f(z-k)e^{2\pi iIm(w\bar{k})},\quad z,w\in\mathbb{C},$$

where $\bar{k}$ is the complex conjugate of k and $Im(w\bar{k})$ is the imaginary part of $w\bar{k}$.

Clearly $Z^t f$ is well-defined for continuous functions with compact support and converges in L^2-sense for $f \in L^2(\mathbb{C})$. In fact Z^t is a unitary map of $L^2(\mathbb{C})$ onto $L^2(Q \times Q)$, where $Q := [0,1) \times [0,1)$. The idea of the proof is similar to the Zak transform on $L^2(\mathbb{R})$ as in [35]. The unitary nature of twisted Zak transform allows to transfer certain conditions on frames for $L^2(\mathbb{C})$ into conditions on $L^2(Q \times Q)$. More precisely, $\{f_k\}$ is complete or a frame or an exact frame or an orthonormal basis for $L^2(\mathbb{C})$ if and only if the same is true for $\{Z^t f_k\}$ in $L^2(Q \times Q)$.

As in case of Zak transform on $L^2(\mathbb{R})$ we obtain the similar properties of twisted Zak transform on $L^2(\mathbb{C})$ in the following lemma. However, our main results are still valid if Zak transform on $L^2(\mathbb{R}^2)$ is applied in place of twisted Zak transform on $L^2(\mathbb{C})$.

Lemma 6.1.6. *Let $f \in L^2(\mathbb{C})$. Let $z = x + iy$, $w = r + is$ and $Q := [0,1) \times [0,1)$. Then the following holds:*

(i) $Z^t f(z+1, w) = e^{2\pi i s} Z^t f(z, w)$, $Z^t f(z+i, w) = e^{-2\pi i r} Z^t f(z, w)$ *and* $Z^t f(z, w+1) = Z^t f(z, w+i) = Z^t f(z, w)$.

(ii) $Z^t(T^t_{(m,n)} f)(z, w) = e^{2\pi i (x.n - y.m)} e^{2\pi i (r.n - s.m)} Z^t f(z, w)$.

(iii) $\{T^t_{(m,n)} f\}$ *is complete in* $L^2(\mathbb{C})$ *if and only if* $Z^t f \neq 0$ *a.e.*

(iv) $\{T^t_{(m,n)} f\}$ *is minimal and complete in* $L^2(\mathbb{C})$ *if and only if* $1/(Z^t f) \in L^2(Q \times Q)$.

(v) $\{T^t_{(m,n)} f\}$ *is a frame for* $L^2(\mathbb{C})$ *with frame bounds* A, B *if and only if* $0 < A^{1/2} \leq |Z^t f| \leq B^{1/2} < \infty$ a.e.. *In this case,* $\{T^t_{(m,n)} f\}$ *is an exact frame for* $L^2(\mathbb{C})$.

(vi) $\{T^t_{(m,n)} f\}$ *is an orthonormal basis for* $L^2(\mathbb{C})$ *if and only if* $|Z^t f|^2 = 1$, *a.e.*

(vii) $\{T^t_{(m,n)} f\}$ *is a Riesz basis for* $L^2(\mathbb{C})$ *with bounds* A, B *if and only if* $0 < A^{1/2} \leq |Z^t f| \leq B^{1/2} < \infty$ a.e.

(viii) *If* $Z^t f$ *is continuous on* $\mathbb{C}^2$ *then* $Z^t f$ *has a zero in* $Q \times Q$.

Proof. The proof of the lemma follows similarly as in the Zak transform for $L^2(\mathbb{R})$ (see [19, 35, 77] or [97]). We only prove part $(viii)$. Assume that $Z^t f(z,w) \neq 0$ for all $(z,w) \in \mathbb{C}^2$. Since $Z^t f$ is continuous on a simply connected domain $\mathbb{C}^2$, there is a continuous function $\varphi(z,w)$ such that

$$Z^t f(z,w) = |Z^t f(z,w)| e^{2\pi i \varphi(z,w)} \quad \text{for} \quad (z,w) \in [0,1]^2 \times [0,1]^2.$$

By part (i), we have

$$Z^t f(z+i,w) = e^{-2\pi i r} Z^t f(z,w) \quad \text{and} \quad Z^t f(z,w+1) = Z^t f(z,w+i).$$

Therefore for each z and w there are integers l_z and k_w such that

$$\varphi(z,1) = \varphi(z,i) + 2\pi l_z \quad \text{and} \quad \varphi(i,w) = \varphi(0,w) + 2\pi k_w - 2\pi r.$$

Since $\varphi(z,1) - \varphi(z,i)$ and $\varphi(i,w) - \varphi(0,w) + 2\pi r$ are continuous functions of z and w respectively, so $l_z = l$ (say) and $k_w = k$ (say), for all $z, w \in \mathbb{C}$. Therefore,

$$\begin{aligned} 0 &= \varphi(0,1) - \varphi(0,i) + \varphi(0,i) - \varphi(i,i) + \varphi(i,i) - \varphi(i,1) + \varphi(i,1) - \varphi(0,1) \\ &= 2\pi l - 2\pi k - 2\pi l + 2\pi k - 2\pi \\ &= -2\pi, \end{aligned}$$

contradicting our assumption. □

6.2 The Amalgam BLT

In this section, we prove a variation of the BLT called the *amalgam BLT* in terms of Wiener amalgam space using certain properties of twisted Zak transform.

For $p \geq 1$, consider the amalgam space defined by $W(C_0, L^p) = \{f \in W(L^\infty, L^p) : f \text{ is continuous}\}$. Clearly $W(C_0, L^1) \subseteq L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. The amalgam BLT in terms of $W(C_0, L^1)$ and a subspace of C_2 is obtained in the following.

Theorem 6.2.1. *(Amalgam BLT) Let $g \in L^2(\mathbb{C})$. If the twisted Gabor system*

$\mathcal{G}^t(g,1,1)$ is an exact frame for $L^2(\mathbb{C})$ then $g \notin W(C_0, L^1)$ and $W(g) \notin \mathcal{W}$, where $\mathcal{W} = \{T \in C_2 : h(z) = \tau(\pi(z)^*T)$ and $h \in W(C_0, L^1)\}$.

Proof. Suppose that $g \in W(C_0, L^1)$. Then by the definition of twisted Zak transform, $Z^t g$ is continuous. By Lemma 6.1.6 (viii), $Z^t g$ must have a zero. Therefore $|Z^t g|^{-1}$ is unbounded and by Lemma 6.1.6 (v), $\mathcal{G}^t(g,1,1)$ cannot be a frame. Again assume that $\mathcal{G}^t(g,1,1)$ is an exact frame and $W(g) \in \mathcal{W}$. So by the inversion formula for Weyl transform $g(z) = \tau(\pi(z)^* W(g))$ and $g \in W(C_0, L^1)$, leads to a contradiction. □

The BLT and amalgam BLT are two distinct results. There exists a function $g \in L^2(\mathbb{C})$ satisfying BLT but not amalgam BLT and vice-versa. The following examples illustrate the difference between the BLT and amalgam BLT.

Example 6.2.2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} e^{-\left[\frac{1}{x(1-x)} + \frac{1}{y(1-y)}\right]}, & (x,y) \in (0,1) \times (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $z = x + iy$, and define $g : \mathbb{C} \to \mathbb{R}$ by

$$g(z) = \sum_{(k_1,k_2) \in \mathbb{N}^2} \frac{1}{k_1^{\frac{3}{2}} k_2^{\frac{3}{2}}} f(x - k_1, y - k_2).$$

Then clearly $g \in W(C_0, L^1)$. Further,

$$W(g) = \sum_{(k_1,k_2) \in \mathbb{N}^2} \frac{1}{k_1^{\frac{3}{2}} k_2^{\frac{3}{2}}} Wf(x - k_1, y - k_2).$$

Clearly $W(g) \in C_2$. From the inversion formula for Weyl transform it follows that $W(g) \in \mathcal{W}$. Next we show that $\|\bar{Z}g\|_2 = \infty$. Consider

$$\begin{aligned} \|\bar{Z}g\|_2^2 &= \int_{\mathbb{C}} |\bar{Z}g(z)|^2 dz \\ \geq \sum_{m,n \in \mathbb{N}} &\int_{[m,m+1]\times[n,n+1]} \bar{Z}g(z) \cdot \overline{\bar{Z}g(z)}\, dz \end{aligned}$$

$$= \sum_{m,n\in\mathbb{N}} \int_{[m,m+1]\times[n,n+1]} \left[\left| \frac{dg(z)}{d\bar{z}} \right|^2 + \frac{1}{4}|zg(z)|^2 - \mathrm{Re}\left(\bar{z}g(z)\frac{dg(z)}{d\bar{z}} \right) \right] dz.$$

Note that for each $m, n \in \mathbb{N}$ and $(x, y) \in (m, m+1) \times (n, n+1)$, the integrand

$$\begin{aligned} &\left| \frac{dg(z)}{d\bar{z}} \right|^2 - \mathrm{Re}\left(\bar{z}g(z)\frac{dg(z)}{d\bar{z}} \right) \\ &= \frac{1}{4m^3n^3} e^{-\left[\frac{2}{x(1-x)} + \frac{2}{y(1-y)}\right]} \left[\frac{(2x-1)^2}{x^4(1-x)^4} + \frac{(2y-1)^2}{y^4(1-y)^4} + \frac{2(2x-1)}{x(1-x)^2} + \frac{2(2y-1)}{y(1-y)^2} \right] \\ &\geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} \|\bar{Z}g\|_2^2 &\geq \frac{1}{4} \sum_{m,n\in\mathbb{N}} \int_{[m,m+1]\times[n,n+1]} |zg(z)|^2 \, dz \\ &\geq \frac{1}{4} \sum_{m,n\in\mathbb{N}} \int_{[m,m+1]\times[n,n+1]} \frac{m^2+n^2}{m^3n^3} |f(x-m, y-n)|^2 \, dxdy \\ &\geq \frac{1}{4}\|f\|_2 \sum_{m\in\mathbb{N}} \frac{1}{m} = \infty. \end{aligned}$$

Example 6.2.3. We shall construct a function f such that Zf and $\bar{Z}f \in L^2(\mathbb{C})$ but $f \notin W(C_0, L^1)$ and $W(f) \notin \mathcal{W}$. For sufficiently large k (say $k > N$) choose $a_k \neq b_k$ such that $[a_k - \frac{1}{k}, b_k + \frac{1}{k}] \subset [k, k+1]$ and $b_k^3 - a_k^3 < k$. Define the continuous function g_k by

$$g_k(x) = \begin{cases} \dfrac{1}{\log k}(x - a_k + \frac{1}{k}), & x \in [a_k - \frac{1}{k}, a_k], \\ \dfrac{1}{k \log k}, & x \in [a_k, b_k], \\ \dfrac{1}{\log k}(b_k + \frac{1}{k} - x), & x \in [b_k, b_k + \frac{1}{k}], \\ 0, & x \notin [a_k - \frac{1}{k}, b_k + \frac{1}{k}]. \end{cases}$$

Clearly the function $g = \sum_{k=N}^{\infty} g_k$ is continuous on $\mathbb{R}$. Also

$$\|g\|_2 \le 2\sum_{k=N}^{\infty} \frac{1}{(k\log k)^2} < \infty, \quad \|xg\|_2 \le 3\sum_{k=N}^{\infty} \frac{1}{k(\log k)^2} < \infty,$$

and

$$\|g'\|_2 \le 2\sum_{k=N}^{\infty} \frac{1}{k(\log k)^2} < \infty,$$

where g' is the classical derivative of g, defined except at countably many points.

Define $f(z) = f(x,y) = g(x)g(y)$. Since $Zf = \frac{1}{2}(f_x - if_y + xf - iyf)$ we have

$$\begin{aligned}\|Zf\|_2 &\le \frac{1}{2}(\|f_x\|_2 + \|f_y\|_2 + \|xf\|_2 + \|yf\|_2)\\ &= \frac{1}{2}(\|g'\|_2\|g\|_2 + \|g'\|_2\|g\|_2 + \|xg\|_2\|g\|_2 + \|yg\|_2\|g\|_2) < \infty.\end{aligned}$$

Similarly $\|\bar{Z}f\|_2 < \infty$. Further,

$$\|f\|_{W(L^\infty,L^1)} = \sum_{k\in\mathbb{Z}^2} \|f\cdot T_k\chi_{[0,1)^2}\|_\infty = \sum_{k_1,k_2=N}^{\infty} \frac{1}{k_1\log k_1}\frac{1}{k_2\log k_2} = \infty.$$

Again, if $W(f) \in \mathcal{W}$ then the inversion formula for Weyl transform gives $f \in W(C_0, L^1)$, which is a contradiction.

Now we investigate the relationships between the operators $Z, \bar{Z}$ and the continuity of twisted Zak transform. A version of BLT assuming the Wiener amalgam condition is obtained in the following theorem:

Theorem 6.2.4. *If $g \in L^2(\mathbb{C})$ and*

$$Zg,\ \bar{Z}g \in W(C_0, L^2), \tag{6.8}$$

then $\{T^t_{(m,n)}g\}$ cannot be a twisted Gabor frame for $L^2(\mathbb{C})$.

Proof. Given that g is continuous and hence Fundamental theorem of calculus for complex variables and ML-inequality can be applied. Now we claim that $g \in W(C_0, L^2)$. To prove the claim it is sufficient to show

$$\sum_k |g(z_k + k)|^2 < \infty \tag{6.9}$$

for every sequence $\{z_k\} \in [0,1] \times [0,1]$. Since $g \in L^2(\mathbb{C})$ we have

$$\sum_k |g(z+k)|^2 < \infty, \; a.e. \text{ on } [0,1] \times [0,1]. \tag{6.10}$$

For fixed $z_0 \in [0,1] \times [0,1]$ and any sequence $\{z_k\} \in [0,1] \times [0,1]$ together with (6.10) gives

$$\begin{aligned}
&\left(\sum_k |g(z_k+k)|^2\right)^{\frac{1}{2}} \\
\leq \; & \left(\sum_k |g(z_k+k) - g(z_0+k)|^2\right)^{\frac{1}{2}} + \left(\sum_k |g(z_0+k)|^2\right)^{\frac{1}{2}} \\
= \; & \left(\sum_k \left|\int_{z_0}^{z_k} \partial g(z+k)\, dz\right|^2\right)^{\frac{1}{2}} + \left(\sum_k |g(z_0+k)|^2\right)^{\frac{1}{2}} \\
= \; & \left(\sum_k \left|\int_{\gamma_k} \left(Z - \frac{\bar{z}}{2}\right) g(z+k)\, dz\right|^2\right)^{\frac{1}{2}} + \left(\sum_k |g(z_0+k)|^2\right)^{\frac{1}{2}} \\
\leq \; & \sqrt{2}\left(\sum_k (M_k + m_k)^2\right)^{\frac{1}{2}} + \left(\sum_k |g(z_0+k)|^2\right)^{\frac{1}{2}} \\
\leq \; & \sqrt{2}\left[\left(\sum_k M_k^2\right)^{\frac{1}{2}} + \left(\sum_k m_k^2\right)^{\frac{1}{2}}\right] + \left(\sum_k |g(z_0+k)|^2\right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

where γ_k is the straight line joining the points z_0 and z_k, with

$$M_k = \operatorname*{ess\,sup}_{z \in \gamma_k} |Zg(z)|, \text{ and } 2m_k = \operatorname*{ess\,sup}_{z \in \gamma_k} |zg(z)|.$$

Observe that $\sum_k M_k^2$ and $\sum_k m_k^2$ are finite, since g satisfies (6.8). Without loss of generality we choose the curve γ_k, because Fundamental theorem of calculus assures that the complex line integral is independent of path. Therefore $g \in W(C_0, L^2)$. Using this fact and the definition of twisted Zak transform yields $Z^t g$ is continuous on $\mathbb{C}$. Thus $\{T^t_{(m,n)} g\}$ cannot be a twisted Gabor frame for $L^2(\mathbb{C})$ (see Lemma 6.1.6 (v) and $(viii)$). □

6.3 Non-Distributional Calculations and the BLT

In this section, we prove BLT using non-distributional calculations. Unlike the Fourier transform of functions in $L^1(\mathbb{R})$, the Weyl transform of functions in $L^1(\mathbb{C})$ are bounded operators on $L^2(\mathbb{R})$. Therefore it is difficult to estimate the suitable upper bound for the oscillation of the twisted Zak transform of $f \in L^2(\mathbb{C})$ in terms of $\|Zf\|_2$ and $\|\bar{Z}f\|_2$. We make use of Weyl transform to estimate the variation of twisted Zak transform of $f \in L^2(\mathbb{C})$ over small cubes of length $r < 1$. We start with the following lemma.

Lemma 6.3.1. *Let f, Zf and $\bar{Z}f \in L^2(\mathbb{C})$. Fix $\epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{C}$. If $\tilde{f}(z) = f(z)e^{2\pi i(y\epsilon_1 - x\epsilon_2)}, \tau_\epsilon f(z) = f(z-\epsilon)$ and $f_\epsilon(z) = f(z-\epsilon)e^{2\pi i(x\epsilon_2 - y\epsilon_1)}$, then there exists an $N_\epsilon \in \mathbb{N}$ such that*

(i) $\|\tilde{f} - f\|_2 \leq 2\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\|_2.$

(ii) $\|\tau_\epsilon f - f\|_2 \leq \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\|_2\right).$

(iii) $\|f_\epsilon - f\|_2 \leq \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\|_2\right).$

Proof. (i) Choose the smallest positive integer N_ϵ such that $\frac{1}{(1+|\epsilon|)^{N_\epsilon}} < |\epsilon|$. Therefore

$$\|\tilde{f} - f\|_2 \leq 2\|f\|_2 \leq 2\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\|_2.$$

(ii) Notice that $Zf \in L^2(\mathbb{C}) \Leftrightarrow Z\tilde{f} \in L^2(\mathbb{C})$ and $\bar{Z}f \in L^2(\mathbb{C}) \Leftrightarrow \bar{Z}\tilde{f} \in L^2(\mathbb{C})$. Since $f \in L^2(\mathbb{C})$ we have $\|\tau_\epsilon f - f\|_2 = \|W(\tau_\epsilon f) - W(f)\|_{C_2}$. Then

$$\begin{aligned} W(\tau_\epsilon f)\phi(\xi) &= \int_{\mathbb{C}} f(z-\epsilon)e^{4\pi i(x\xi + \frac{1}{2}xy)}\phi(\xi+y)\,dxdy \\ &= \int_{\mathbb{C}} f(z)e^{4\pi i[(x+\epsilon_1)\xi + \frac{1}{2}(x+\epsilon_1)(y+\epsilon_2)]}\phi(\xi+y+\epsilon_2)\,dxdy \\ &= e^{4\pi i(\epsilon_1\xi + \frac{1}{2}\epsilon_1\epsilon_2)}W(\tilde{f})\phi(\xi+\epsilon_2),\ \forall \phi \in L^2(\mathbb{R}). \end{aligned}$$

Applying mean value theorem on the Schwartz class function ϕ on $\mathbb{R}$ we have

$$|[W(\tau_\epsilon f) - W(\tilde{f})]\phi(\xi)|$$

$$
\begin{aligned}
&= |e^{4\pi i(\epsilon_1\xi+\frac{1}{2}\epsilon_1\epsilon_2)}W(\tilde{f})\phi(\xi+\epsilon_2) - W(\tilde{f})\phi(\xi)| \\
&\leq |[e^{4\pi i(\epsilon_1\xi+\frac{1}{2}\epsilon_1\epsilon_2)} - 1]W(\tilde{f})\phi(\xi+\epsilon_2)| + |W(\tilde{f})(\phi(\xi+\epsilon_2) - \phi(\xi))| \\
&\leq 4\pi|\epsilon||W(\tilde{f})(\xi+\epsilon_2)\phi(\xi+\epsilon_2)| + \pi|\epsilon||W(\tilde{f})\phi(\xi+\epsilon_2)| + |\epsilon||W(\tilde{f})\phi'(\xi+\theta\epsilon_2)|,
\end{aligned}
$$

for some $\theta \in (0,1)$. Writing $2\xi\phi(\xi) = (A + A^*)\phi(\xi)$ and $2\phi'(\xi) = (A^* - A)\phi(\xi)$ we get

$$
\begin{aligned}
\|[W(\tau_\epsilon f) - W(\tilde{f})]\phi\|_2 &\leq 4\pi|\epsilon|\|W(\tilde{f})\xi\phi\|_2 + \pi|\epsilon\|W(\tilde{f})\phi\|_2 + |\epsilon|\|W(\tilde{f})\phi'\|_2 \\
&\leq \frac{5}{2}\pi|\epsilon|(\|W(\tilde{f})A\phi\|_2 + \|W(\tilde{f})A^*\phi\|_2 + \|W(\tilde{f})\phi\|_2).
\end{aligned}
$$

But $\|W(\tau_\epsilon f) - W(f)\|_{C_2} \leq \|W(\tau_\epsilon f) - W(\tilde{f})\|_{C_2} + \|W(\tilde{f}) - W(f)\|_{C_2}$. Thus for any othonormal basis $\{\phi_j\}_{j\in\mathbb{N}}$ for $L^2(\mathbb{R})$ we have

$$
\begin{aligned}
&\|W(\tau_\epsilon f) - W(\tilde{f})\|_{C_2}^2 \\
&= \sum_{j=1}^{\infty} \|[W(\tau_\epsilon f) - W(\tilde{f})]\phi_j\|_2^2 \\
&\leq \frac{25}{4}\pi^2|\epsilon|^2 \sum_{j=1}^{\infty}(\|W(\tilde{f})A\phi_j\|_2 + \|W(\tilde{f})A^*\phi_j\|_2 + \|W(\tilde{f})\phi_j\|_2)^2 \\
&\leq \frac{75}{4}\pi^2|\epsilon|^2 \sum_{j=1}^{\infty}(\|W(\tilde{f})A\phi_j\|_2^2 + \|W(\tilde{f})A^*\phi_j\|_2^2 + \|W(\tilde{f})\phi_j\|_2^2) \\
&= \frac{75}{4}\pi^2|\epsilon|^2(\|W(\tilde{f})A\|_{C_2}^2 + \|W(\tilde{f})A^*\|_{C_2}^2 + \|W(\tilde{f})\|_{C_2}^2) \\
&= \frac{75}{4}\pi^2|\epsilon|^2\left(\|Z\tilde{f}\|_2^2 + \|\bar{Z}\tilde{f}\|_2^2 + \|\tilde{f}\|_2^2\right).
\end{aligned}
$$

Therefore by (i) we get

$$
\begin{aligned}
\|W(\tau_\epsilon f) - W(f)\|_{C_2} &\leq \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2\right) + 2\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\|_2 \\
&\leq \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\|_2\right).
\end{aligned}
$$

(iii) From (i) and (ii) we get

$$
\|f_\epsilon - f\|_2 \leq \|f_\epsilon - \tau_\epsilon f\|_2 + \|\tau_\epsilon f - f\|_2
$$

$$\leq \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2\right) + 4\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\|_2$$
$$\leq \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\|_2\right).$$

This completes the proof. □

We use the following notation to estimate the upper bound for the oscillation of the twisted Zak transform over small cubes. Let $x = (t, w) \in \mathbb{R}^2$ and $r > 0$. Then $Q(x; r)$ is the square centered at x with radius r, i.e.

$$\begin{aligned} Q(x;r) &= \left[t - \frac{r}{2}, t + \frac{r}{2}\right] \times \left[w - \frac{r}{2}, w + \frac{r}{2}\right] \\ &= \left\{(u,v) : u \in \left[t - \frac{r}{2}, t + \frac{r}{2}\right], v \in \left[w - \frac{r}{2}, w + \frac{r}{2}\right]\right\}. \end{aligned}$$

Thus the square $Q = [0,1) \times [0,1)$ can be represented as $Q(\frac{1}{2}, \frac{1}{2}; 1)$.

Theorem 6.3.2. *Let $f, Zf, \bar{Z}f \in L^2(\mathbb{C})$, $G = Z^t f$,*

$$\alpha_0 = (z_0, w_0) \in Q(z_0, 1) \times Q(w_0, 1) := Q[\alpha_0, 1],\ z_0 \in \left[-\frac{3}{2}, \frac{3}{2}\right] \times \left[-\frac{3}{2}, \frac{3}{2}\right]$$

and $w_0, \epsilon \in \mathbb{C}$ be given. Let $f_\epsilon, \tilde{f}$ be as in Lemma 6.3.1. Then there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned} &\|T^t_{\epsilon,1}G - G\|_{L^2(Q[\alpha_0,1])} \\ \leq\ & 8\pi|\epsilon|\|f\|_2 + \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\|_2\right), \end{aligned}$$

and

$$\|T^t_{\epsilon,2}G - G\|_{L^2(Q[\alpha_0,1])} \leq 2\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\|_2 + 8\pi|\epsilon|\|f\|_2,$$

where $T^t_{\epsilon,j}G(z,w)$ is the twisted translation of G in the jth variable for $j = 1, 2$.

Proof. Notice that $T^t_{\epsilon,1}G(z,w) = e^{2\pi i Im(\bar{z}\epsilon)} Z^t(\tau_\epsilon f)(z,w)$. Then by using the fact that the twisted Zak transform Z^t is an unitary operator of $L^2(\mathbb{C})$ onto $L^2(Q[\alpha_0, 1])$ we get,

$$\|T^t_{\epsilon,1}G - G\|_{L^2(Q[\alpha_0,1])}$$

$$\begin{aligned}
&\leq \|T^t_{\epsilon,1}G - Z^t(\tau_\epsilon f)\|_{L^2(Q[\alpha_0,1])} + \|Z^t(\tau_\epsilon f) - G\|_{L^2(Q[\alpha_0,1])} \\
&= \|\left(e^{2\pi i Im(\bar{z}\epsilon)} - 1\right) Z^t(\tau_\epsilon f)\|_{L^2(Q[\alpha_0,1])} + \|Z^t(\tau_\epsilon f) - Z^t f\|_{L^2(Q[\alpha_0,1])} \\
&\leq 8\pi|\epsilon|\|f\|_2 + \|\tau_\epsilon f - f\|_2 \\
&\leq 8\pi|\epsilon|\|f\|_2 + \frac{15}{2}\pi|\epsilon|\left(\|Z\tilde{f}\|_2 + \|\bar{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\|_2\right),
\end{aligned}$$

by Lemma 6.3.1 (ii). Observe that

$$\begin{aligned}
T^t_{\epsilon,2}G(z,w) &= G(z, w-\epsilon)e^{2\pi i Im(\bar{w}\epsilon)} = \sum_{k\in\mathbb{Z}^2} f(z-k)e^{2\pi i Im((w-\epsilon)\bar{k})}e^{2\pi i Im(\bar{w}\epsilon)} \\
&= e^{-2\pi i Im(\bar{z}\epsilon)} \sum_{k\in\mathbb{Z}^2} f(z-k)e^{2\pi i Im((\overline{z-k})\epsilon)}e^{2\pi i Im(w\bar{k})}e^{2\pi i Im(\bar{w}\epsilon)} \\
&= e^{2\pi i Im(\overline{w-z})\epsilon} \sum_{k\in\mathbb{Z}^2} h(z-k)e^{2\pi i Im(w\bar{k})} \\
&= e^{2\pi i Im(\overline{w-z})\epsilon} Z^t h(z,w),
\end{aligned}$$

where $h(z) = f(z)e^{2\pi i Im(\bar{z}\epsilon)}$. Therefore

$$\|T^t_{\epsilon,2}G - G\|_{L^2(Q[\alpha_0,1])} \leq \|(e^{2\pi i Im(\overline{w-z})\epsilon} - 1)Z^t h\|_{L^2(Q[\alpha_0,1])} + \|Z^t h - Z^t f\|_{L^2(Q[\alpha_0,1])}.$$

Choose same N_ϵ as in Lemma 6.3.1 (i) to get

$$|(e^{2\pi i Im(\overline{w-z})\epsilon} - 1)| \leq 2 \leq 2\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}.$$

Then

$$\|(e^{2\pi i Im(\overline{w-z})\epsilon} - 1)Z^t h\|_{L^2(Q[\alpha_0,1])} \leq 2\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\|_2.$$

Thus $\|T^t_{\epsilon,2}G - G\|_{L^2(Q[\alpha_0,1])} \leq 2\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\|_2 + 8\pi|\epsilon|\|f\|_2.$ □

Corollary 6.3.3. *Let $f, Zf, \bar{Z}f \in L^2(\mathbb{C})$, $G = Z^t f$. For $0 < r < 1$, fix $\alpha_0 = (z_0, w_0) \in Q(z_0, r) \times Q(w_0, r) := Q[\alpha_0, r]$, $z_0 \in [-\frac{3}{2}, \frac{3}{2}] \times [-\frac{3}{2}, \frac{3}{2}]$ and $w_0, \epsilon \in \mathbb{C}$. Then*

$$\int_{Q[\alpha_0,r]} |T^t_{\epsilon,1}G(z,w) - G(z,w)|dzdw \leq r^2|\epsilon|C^\epsilon_{1,f}(r) \tag{6.11}$$

and

$$\int_{Q[\alpha_0,r]} |T^t_{\epsilon,2}G(z,w) - G(z,w)|dzdw \leq r^2|\epsilon|C^{\epsilon}_{2,f}(r), \tag{6.12}$$

where $T^t_{\epsilon,j}G(z,w)$ *is the twisted translation of* G *in the* j*th variable with* $\lim_{r\to 0} C^{\epsilon}_{j,f}(r) = 0$ *for* $j = 1, 2$.

Proof. As in the proof of Theorem 6.3.2 we have

$$\begin{aligned}
& \|T^t_{\epsilon,1}G - G\|_{L^2(Q[\alpha_0,r])} \\
\leq\ & \| \left(e^{2\pi i Im(\bar{z}\epsilon)} - 1\right) Z^t(\tau_\epsilon f)\|_{L^2(Q[\alpha_0,r])} + \|Z^t(\tau_\epsilon f) - Z^t f\|_{L^2(Q[\alpha_0,r])} \\
=\ & \| \left(e^{2\pi i Im(\bar{z}\epsilon)} - 1\right) Z^t(\tau_\epsilon f)\cdot \chi_{Q(z_0,r)}\|_{L^2(Q[\alpha_0,1])} \\
& + \|(Z^t(\tau_\epsilon f) - Z^t f)\cdot \chi_{Q(z_0,r)}\|_{L^2(Q[\alpha_0,1])} \\
\leq\ & 8\pi|\epsilon|\|f\cdot\chi_{Q(z_0,r)}\|_2 + \frac{15}{2}\pi|\epsilon| \left(\|Z\tilde{f}\cdot\chi_{Q(z_0,r)}\|_2 + \|\bar{Z}\tilde{f}\cdot\chi_{Q(z_0,r)}\|_2 \right. \\
& \left. + \|\tilde{f}\cdot\chi_{Q(z_0,r)}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\cdot\chi_{Q(z_0,r)}\|_2\right).
\end{aligned}$$

Again

$$\begin{aligned}
& \|T^t_{\epsilon,2}G - G\|_{L^2(Q[\alpha_0,r])} \\
\leq\ & \|(e^{2\pi i Im(\overline{w-z})\epsilon} - 1)Z^t h\|_{L^2(Q[\alpha_0,r])} + \|Z^t h - Z^t f\|_{L^2(Q[\alpha_0,r])} \\
=\ & \|(e^{2\pi i Im(\overline{w-z})\epsilon} - 1)Z^t h\cdot\chi_{Q(z_0,r)}\|_{L^2(Q[\alpha_0,1])} \\
& + \|(Z^t h - Z^t f)\cdot\chi_{Q(z_0,r)}\|_{L^2(Q[\alpha_0,1])} \\
\leq\ & 8\pi|\epsilon|(1+|\epsilon|)^{N_\epsilon}\|f\cdot\chi_{Q(z_0,r)}\|_2 + 8\pi|\epsilon|\|f\cdot\chi_{Q(z_0,r)}\|_2.
\end{aligned}$$

Applying Cauchy–Schwartz inequality in the left hand side of (6.11) and (6.12) the proof follows immediately, where

$$\begin{aligned}
C^{\epsilon}_{1,f}(r) \ =\ & 8\pi\|f\cdot\chi_{Q(z_0,r)}\|_2 + \frac{15}{2}\pi \left(\|Z\tilde{f}\cdot\chi_{Q(z_0,r)}\|_2 + \|\bar{Z}\tilde{f}\cdot\chi_{Q(z_0,r)}\|_2 \right. \\
& \left. + \|\tilde{f}\cdot\chi_{Q(z_0,r)}\|_2 + (1+|\epsilon|)^{N_\epsilon}\|f\cdot\chi_{Q(z_0,r)}\|_2\right)
\end{aligned}$$

and

$$C^{\epsilon}_{2,f}(r) = 8\pi(1+|\epsilon|)^{N_\epsilon}\|f\cdot\chi_{Q(z_0,r)}\|_2 + 8\pi\|f\cdot\chi_{Q(z_0,r)}\|_2.$$

Further, using the fact that $\|f.\chi_{Q(z_0,r)}\|_2 \to 0$ as $r \to 0$, we have $\lim_{r\to 0} C^{\epsilon}_{j,f}(r) = 0$ for $j = 1, 2$. □

Now we are in a position to state and prove the BLT. Motivated by the proof of BLT on $L^2(\mathbb{R})$ (Coifman and Semmes [39], Benedetto et al. [19] etc.), we obtain the following proof of BLT on $L^2(\mathbb{C})$.

Theorem 6.3.4. *Let $g \in L^2(\mathbb{C})$. If the twisted Gabor system $\mathcal{G}^t(g,1,1) = \{T^t_{(m,n)}g : m,n \in \mathbb{Z}\}$ forms an exact frame for $L^2(\mathbb{C})$, then $\|Zg\|_2\,\|\bar{Z}g\|_2 = +\infty$.*

Proof. Assume that $\{T^t_{(m,n)}g\}$ is an exact frame for $L^2(\mathbb{C})$. Then by Lemma 6.1.6 (v) we have

$$0 < A^{1/2} \le |Z^t g| \le B^{1/2} < \infty \quad \text{a.e.} \tag{6.13}$$

Assume that both Zg and $\bar{Z}g \in L^2(\mathbb{C})$. We will show that our assumption together with (6.13) leads to a contradiction in the following three steps.

Step 1: (Construction of an continuous averaged function $G_r(z,w)$ that approximating $G(z,w) = Z^t g(z,w)$.)

Let $\rho(z,w) = \chi_{[0,1)^4}(z,w)$ and for $r > 0$, let $\rho_r(z,w) = \frac{1}{r^4}\rho\left(\frac{z}{r},\frac{w}{r}\right)$. Define

$$G_r(z,w) = G \times \rho_r(z,w) = \int_{[0,1)^4} G(z-z', w-w')\rho_r(z',w')e^{-2\pi i Im(z\bar{z'}+w\bar{w'})}\,dz'dw'.$$

Then G_r satisfies the following properties:

$$(a) \qquad |G_r(z_1,w_1) - G_r(z_2,w_2)| \le 2\left(\pi(r+\max\{|z_1|,|w_1|\}) + \frac{1}{r}\right)B^{\frac{1}{2}}\left(|z_1-z_2| + |w_1-w_2|\right).$$

Using (6.13) we have

$$|G_r(z_1,w_1) - G_r(z_2,w_2)|$$

$$
\begin{aligned}
&= \frac{1}{r^4}\left|\int_{Q[z_1^*,w_1^*;r]} G(u,v)e^{2\pi i Im(z_1\bar{u}+w_1\bar{v})}\,dudv \right.\\
&\qquad \left. -\int_{Q[z_2^*,w_2^*;r]} G(u,v)e^{2\pi i Im(z_2\bar{u}+w_2\bar{v})}\,dudv\right|\\
&\leq \frac{1}{r^4}\left|\int_{Q[z_1^*,w_1^*;r]} G(u,v)\left[e^{2\pi i Im(z_1\bar{u}+w_1\bar{v})}-e^{2\pi i Im(z_2\bar{u}+w_2\bar{v})}\right]dudv\right|\\
&\qquad +\frac{1}{r^4}\left|\int_{Q[z_1^*,w_1^*;r]} G(u,v)e^{2\pi i Im(z_2\bar{u}+w_2\bar{v})}\,dudv\right.\\
&\qquad\qquad \left. -\int_{Q[z_2^*,w_2^*;r]} G(u,v)e^{2\pi i Im(z_2\bar{u}+w_2\bar{v})}\,dudv\right|\\
&\leq \frac{2\pi}{r^4}B^{\frac{1}{2}}\left(|z_1-z_2|(r+|z_1|)+|w_1-w_2|(r+|w_1|)\right)|Q[z_1^*,w_1^*;r]|\\
&\qquad +\frac{1}{r^4}B^{\frac{1}{2}}\int_{Q[z_1^*,w_1^*;r]\Delta Q[z_2^*,w_2^*;r]} dudv\\
&\leq 2\pi B^{\frac{1}{2}}(r+\max\{|z_1|,|w_1|\})\left(|z_1-z_2|+|w_1-w_2|\right)\\
&\qquad +\frac{1}{r^4}B^{\frac{1}{2}}|Q[z_1^*,w_1^*;r]\Delta Q[z_2^*,w_2^*;r]|\\
&\leq 2\left(\pi(r+\max\{|z_1|,|w_1|\})+\frac{1}{r}\right)B^{\frac{1}{2}}\left(|z_1-z_2|+|w_1-w_2|\right),
\end{aligned}
$$

where Δ is the symmetric difference operator and

$$Q[z_j^*,w_j^*;r]=Q[z_j-\frac{r}{2}(1+i),w_j-\frac{r}{2}(1+i);r],\quad j=1,2.$$

(b) (i) $G_r(z,w+1)=G_r(z,w)+\psi_{1,r}(z,w)$ and $G_r(z,w+i)=G_r(z,w)+\psi_{2,r}(z,w)$,

(ii)

$$G_r(z+1,w)=e^{2\pi i Im(w)}G_r(z,w)+\psi_{3,r}(z,w)$$

and

$$G_r(z+i,w)=e^{-2\pi i Im(iw)}G_r(z,w)+\psi_{4,r}(z,w),$$

where $|\psi_{j,r}(z,w)|\leq 2\pi B^{1/2}r,\ j=1,2,3,4.$

$G_r(z,w+1)$

$$
\begin{aligned}
&= \int_{[0,1)^4} G(z-z',w+1-w')\rho_r(z',w')e^{-2\pi i Im(z\bar{z'}+(w+1)\bar{w'})}\,dz'dw' \\
&= \int_{[0,1)^4} G(z-z',w-w')\rho_r(z',w')e^{-2\pi i Im(z\bar{z'}+w\bar{w'})}\,dz'dw' + \psi_{1,r}(z,w) \\
&= G_r(z,w)+\psi_{1,r}(z,w),
\end{aligned}
$$

where

$$
\psi_{1,r}(z,w)=\int_{[0,1)^4}(e^{-2\pi i Im(\bar{w'})}-1)G(z-z',w-w')\rho_r(z',w')e^{-2\pi i Im(z\bar{z'}+w\bar{w'})}\,dz'dw'.
$$

Further

$$
\begin{aligned}
& |\psi_{1,r}(z,w)| \\
= & \left|\int_{[0,1)^4}(e^{-2\pi i Im(\bar{w'})}-1)G(z-z',w-w')\rho_r(z',w')e^{-2\pi i Im(z\bar{z'}+w\bar{w'})}\,dz'dw'\right| \\
\leq & B^{1/2}\int_{[0,1)^4}|2\pi Im(\bar{w'})|\rho_r(z',w')\,dz'dw' \leq 2\pi B^{1/2}r.
\end{aligned}
$$

Similarly we can obtain the other identities with $|\psi_{j,r}(z,w)|\leq 2\pi B^{1/2}r,\ j=2,3,4.$

(c) Fix $(z,w),(z',w')\in\mathbb{C}^2$ and using (a) one has

$$
\begin{aligned}
& |G(z,w)-G_r(z,w)| \\
\geq & |G(z,w)|-|G_r(z,w)-G_r(z',w')|-|G_r(z',w')| \\
\geq & A^{\frac{1}{2}}-2B^{\frac{1}{2}}\left(\pi(r+\max\{|z|,|w|\})+\frac{1}{r}\right)(|z-z'|+|w-w'|)-|G_r(z',w')|.
\end{aligned}
$$

In particular for fixed $(z,w)\in[0,1)^4, c<1$ and $(z,w)\in Q[z',w';cr]$ we have

$$
\begin{aligned}
& |G(z,w)-G_r(z,w)| \\
\geq & A^{\frac{1}{2}}-2cr\left(\pi(cr+\max\{|z|,|w|\})+\frac{1}{cr}\right)B^{\frac{1}{2}}-|G_r(z',w')|. \qquad (6.14)
\end{aligned}
$$

Step 2: For any $(z_0,w_0)\in[0,1)^4, c<1$ and $r<1$ we have

$$
c^4r^4(A^{\frac{1}{2}}-2crc^r_{z,w}B^{\frac{1}{2}}-|G_r(z',w')|)
$$

$$\begin{aligned} &\leq \int_{Q[z,w;cr]} |G(z,w) - G_r(z,w)| dzdw \\ &\leq c^2 r^4 C(r), \end{aligned} \tag{6.15}$$

where $c_{z,w}^r = \left(\pi(cr + \max\{|z|, |w|\}) + \frac{1}{cr}\right)$ and $C(r)$ is independent on the point (z,w) and

$$\lim_{r \to 0} C(r) = 0. \tag{6.16}$$

$$\begin{aligned} &\int_{Q[z_0,w_0;cr]} |G(z,w) - G_r(z,w)|\, dzdw \\ \leq &\int_{[0,1)^4} |\rho_r(z',w')| \int_{Q[z_0,w_0;cr]} |G(z,w) - G(z,w-w')e^{-2\pi i Im(w\bar{w}')}| dz'dw'dzdw \\ &+ \int_{[0,1)^4} |\rho_r(z',w')| \int_{Q[z_0,w_0;cr]} \Big| G(z,w-w') \\ &\qquad -G(z-z',w-w')e^{-2\pi i Im(z\bar{z}')}\Big| dz'dw'dzdw \\ \leq &\int_{[0,1)^4} |\rho_r(z',w')| \int_{Q[z_0,w_0;cr]} \left| G(z,w) - G(z,w-w')e^{-2\pi i Im(w\bar{w}')} \right| dzdwdz'dw' \\ &+ \int_{[0,1)^4} |\rho_r(z',w')| \int_{Q[z_0,w_0-w';cr]} \left| G(z,s) - G(z-z',s)e^{-2\pi i Im(z\bar{z}')} \right| dzdsdz'dw'. \end{aligned}$$

Using Corollary 6.3.3 and the fact that $|z'| < 1,\ |w'| < 1$, we have $|C_{1,g}^{z'}(r)| < C_{1,g}(r)$ and $|C_{2,g}^{w'}(r)| < C_{2,g}(r)$, where

$$\begin{aligned} C_{1,g}(r) \;&=\; 8\pi\|g \cdot \chi_{Q(z_0,r)}\|_2 + \frac{15}{2}\pi \left(\|Z\tilde{g} \cdot \chi_{Q(z_0,r)}\|_2 + \|\bar{Z}\tilde{g} \cdot \chi_{Q(z_0,r)}\|_2 \right. \\ &\left. + \|\tilde{g} \cdot \chi_{Q(z_0,r)}\|_2 + 2\|g \cdot \chi_{Q(z_0,r)}\|_2\right) \end{aligned}$$

and $C_{2,g}(r) = 16\pi\|g \cdot \chi_{Q(z_0,r)}\|_2 + 8\pi\|g \cdot \chi_{Q(z_0,r)}\|_2$. Putting $C(r) = C_{1,g}(r) + C_{2,g}(r)$ we get (6.16). Then the inequality (6.15) can be obtained by (6.14) and applying Cauchy-Schwartz inequality in the last term of the above calculation.

Step 3: Claim: $\inf\limits_{(z,w)\in[0,1)^4} |G(z,w)| = 0.$

From (6.15) we get $|G_r(z',w')| \geq A^{\frac{1}{2}} - 2crc_{z,w}^r B^{\frac{1}{2}} - \frac{C(r)}{c^2}$. Choose $c < 1$ such that

$A^{\frac{1}{2}} - 2crc_{z,w}^{r} B^{\frac{1}{2}} > \frac{A^{\frac{1}{2}}}{2}$ and letting $r \to 0$ we get $|G_r(z,w)| \geq \frac{A^{\frac{1}{2}}}{2}$. Since $G_r(z,w)$ is continuous real valued function on $[0,1)^4$ (see [118], pp. 377-385), there exists a continuous real valued function θ_r such that $G_r(z,w) = |G_r(z,w)|e^{i\theta_r(z,w)}$. Define

$$\begin{aligned}
\delta_{1,r}(z,w) &= 1 + \frac{\psi_{1,r}(z,w)}{G_r(z,w)},\\
\delta_{2,r}(z,w) &= 1 + \frac{\psi_{2,r}(z,w)}{G_r(z,w)},\\
\delta_{3,r}(z,w) &= 1 + \frac{\psi_{3,r}(z,w)}{e^{2\pi i Im(w)}G_r(z,w)},\\
\delta_{4,r}(z,w) &= 1 + \frac{\psi_{4,r}(z,w)}{e^{-2\pi i Im(iw)}G_r(z,w)}.
\end{aligned}$$

Clearly $\delta_{j,r}$ is continuous and non vanishing on $[0,1]^4$ for each $r > 0$ and every $j = 1,2,3,4$. Therefore there exists a continuous real valued function $\theta_{j,r}$ such that $\delta_{j,r}(z,w) = |\delta_{j,r}(z,w)|e^{i\theta_{j,r}(z,w)}$ for $j = 1,2,3,4$. Since

$$\begin{aligned}
G_r(z,w+1) &= G_r(z,w)\delta_{1,r}(z,w),\\
G_r(z,w+i) &= G_r(z,w)\delta_{2,r}(z,w),\\
G_r(z+1,w) &= e^{2\pi i Im(w)}G_r(z,w)\delta_{3,r}(z,w),\\
G_r(z+i,w) &= e^{-2\pi i Im(iw)}G_r(z,w)\delta_{4,r}(z,w),
\end{aligned}$$

for each $r > 0$ and for all $z, w \in [0,1] \times [0,1]$, there are integers I_r, J_r, K_r and L_r such that

$$\begin{aligned}
\theta_r(z,1) &= \theta_r(z,0) + \theta_{1,r}(z,0) + 2\pi I_r,\\
\theta_r(z,i) &= \theta_r(z,0) + \theta_{2,r}(z,0) + 2\pi J_r,\\
\theta_r(1,w) &= 2\pi Im(w) + \theta_r(0,w) + \theta_{3,r}(0,w) + 2\pi K_r,\\
\theta_r(i,w) &= -2\pi Im(iw) + \theta_r(0,w) + \theta_{4,r}(0,w) + 2\pi L_r.
\end{aligned}$$

Now

$$\begin{aligned}
0 &= [\theta_r(0,1) - \theta_r(0,i)] + [\theta_r(0,i) - \theta_r(i,i)] + [\theta_r(i,i) - \theta_r(i,1)]\\
&\quad +[\theta_r(i,1) - \theta_r(0,1)]\\
&= [\theta_{1,r}(0,0) - \theta_{2,r}(0,0) + 2\pi(I_r - J_r)] + [-\theta_{4,r}(0,i) - 2\pi L_r]\\
&\quad +[\theta_{2,r}(i,0) - \theta_{1,r}(i,0) + 2\pi(J_r - I_r)] + [-2\pi + \theta_{4,r}(0,1) + 2\pi L_r]\\
&= \theta_{1,r}(0,0) - \theta_{2,r}(0,0) - \theta_{4,r}(0,i) + \theta_{2,r}(i,0) - \theta_{1,r}(i,0) - 2\pi + \theta_{4,r}(0,1).
\end{aligned}$$

Letting $r \to 0$ we get $0 = -2\pi$, a contradiction. □

Remark 6.3.5. Using Plancherel formula for Weyl transform we have

$$\|Zg\|_2 \|W(g)A^*\|_{C_2} = \frac{1}{2}\|Zg\|_2 \, \|\bar{Z}g\|_2 = +\infty.$$

This expression is analogous to the conclusion of the classical BLT.

6.4 Uncertainty Principle Approach to BLT

Motivated by the proofs of BLT for orthonormal basis and Riesz basis (see [16, 43]), we prove several versions of BLT on $L^2(\mathbb{C})$ analogously. We start with a variation of Heigenberg's uncertainty inequality for $L^2(\mathbb{C})$.

Theorem 6.4.1. *Let $f \in L^2(\mathbb{C})$. Then*

$$\int_{\mathbb{C}} |Zf(z)|^2 \, dz + \int_{\mathbb{C}} |\bar{Z}f(z)|^2 \, dz \geq 2\|f\|_2^2.$$

Proof. Let $\mathcal{S}(\mathbb{C})$ be the collection of all Schwartz class functions on $\mathbb{C}$. Let $f \in \mathcal{S}(\mathbb{C}) \subset L^2(\mathbb{C})$. Therefore

$$\begin{aligned} \|Zf\|_2^2 &= \sum_{m,n=0}^{\infty} |\langle Zf, \phi_{m,n}\rangle|^2 = \sum_{m,n=0}^{\infty} |\langle f, \bar{Z}\phi_{m,n}\rangle|^2 \\ &= \sum_{m,n=0}^{\infty} (2n+2)|\langle f, \phi_{m,n+1}\rangle|^2 = \sum_{m,n=0}^{\infty} 2n|\langle f, \phi_{m,n}\rangle|^2 \end{aligned} \tag{6.17}$$

and

$$\begin{aligned} \|\bar{Z}f\|_2^2 &= \sum_{m,n=0}^{\infty} |\langle \bar{Z}f, \phi_{m,n}\rangle|^2 = \sum_{m,n=0}^{\infty} |\langle f, Z\phi_{m,n}\rangle|^2 \\ &= \sum_{m,n=0}^{\infty} 2n|\langle f, \phi_{m,n-1}\rangle|^2 = \sum_{m,n=0}^{\infty} (2n+2)|\langle f, \phi_{m,n}\rangle|^2. \end{aligned} \tag{6.18}$$

Using equations (6.17) and (6.18) we get

$$\|Zf\|_2^2 + \|\bar{Z}f\|_2^2 = \sum_{m,n=0}^{\infty} (4n+2)|\langle f, \phi_{m,n}\rangle|^2 \geq 2 \sum_{m,n=0}^{\infty} |\langle f, \phi_{m,n}\rangle|^2 = 2\|f\|_2^2.$$

Using the fact that $Z\phi_{m,0} = 0$ for $m = 0, 1, 2 \cdots$ we conclude that equality holds in the above inequality if and only if $n = 0$ i.e. $f = \sum_{m=0}^{\infty} c_m \phi_{m,0}$. □

6.5 The Weak BLT

Let us start with the following lemma.

Lemma 6.5.1. *The operator $T^t_{(m,n)}$ commutes with Z and $\bar{Z}$ i.e. $T^t_{(m,n)} Z = Z T^t_{(m,n)}$ and $T^t_{(m,n)} \bar{Z} = \bar{Z} T^t_{(m,n)}$.*

Proof. Enough to show that the commutators $[Z, T^t_{(m,n)}] = [\bar{Z}, T^t_{(m,n)}] = 0$. For a Schwartz class function f on $\mathbb{C}$ we have

$$\begin{aligned} \langle [\bar{Z}, T^t_{(m,n)}] f, f\rangle &= \langle \bar{Z} T^t_{(m,n)} f - T^t_{(m,n)} \bar{Z} f, f\rangle \\ &= \langle \bar{Z} T^t_{(m,n)} f, f\rangle - \langle T^t_{(m,n)} \bar{Z} f, f\rangle \\ &= -\langle f, T^t_{(-m,-n)} Z f\rangle + \langle f, Z T^t_{(-m,-n)} f\rangle \\ &= -\langle W(f), i\pi(-m,-n) W(f) A\rangle + \langle W(f), i\pi(-m,-n) W(f) A\rangle \\ &= 0. \end{aligned}$$

Similarly we can show that $[Z, T^t_{(m,n)}] = 0$. □

Hereafter let us denote $T^t_{(m,n)} g$ as $g_{m,n}$ for simplicity.

Theorem 6.5.2. *Assume $g \in L^2(\mathbb{C})$ is such that $\{g_{m,n}\}$ is an exact twisted Gabor frame for $L^2(\mathbb{C})$ and $\tilde{g}$ be the dual function. Then we cannot have all of $Zg, Z\tilde{g}, \bar{Z}g, \bar{Z}\tilde{g} \in L^2(\mathbb{C})$, i.e., we must have $\|Zg\|_2 \, \|Z\tilde{g}\|_2 \, \|\bar{Z}g\|_2 \, \|\bar{Z}\tilde{g}\|_2 = +\infty$.*

Proof. Assume that $Zg, Z\tilde{g}, \bar{Z}g, \bar{Z}\tilde{g} \in L^2(\mathbb{C})$. Since $\{g_{m,n}\}$ is a twisted Gabor frame for $L^2(\mathbb{C})$, any $f \in L^2(\mathbb{C})$ can be expressed as

$$f = \sum_{m,n} \langle f, g_{m,n} \rangle \tilde{g}_{m,n} = \sum_{m,n} \langle f, \tilde{g}_{m,n} \rangle g_{m,n}.$$

Using Lemma 6.5.1 we get

$$\begin{aligned} \langle Zg, Z\tilde{g} \rangle &= \sum_{m,n} \langle Zg, \tilde{g}_{m,n} \rangle \langle g_{m,n}, Z\tilde{g} \rangle = \sum_{m,n} \langle g_{-m,-n}, \bar{Z}\tilde{g} \rangle \langle \bar{Z}g, \tilde{g}_{-m,-n} \rangle \\ &= \sum_{m,n} \langle \bar{Z}g, \tilde{g}_{m,n} \rangle \langle g_{m,n}, \bar{Z}\tilde{g} \rangle = \langle \bar{Z}g, \bar{Z}\tilde{g} \rangle. \end{aligned} \tag{6.19}$$

Therefore, bi-orthogonality relation (Proposition 5.4.8 of [35], pp. 101) and the above equality gives

$$1 = \langle g, \tilde{g} \rangle = -\frac{1}{2} \left(\langle g, [Z, \bar{Z}]\tilde{g} \rangle \right) = -\frac{1}{2} (\langle Zg, Z\tilde{g} \rangle - \langle \bar{Z}g, \bar{Z}\tilde{g} \rangle) = 0,$$

a contradiction. □

Remark 6.5.3. If the twisted Gabor frame $\{g_{m,n}\}$ forms an orthonormal basis then $g = \tilde{g}$ and the above theorem is precisely analogue of Battle's proof of BLT in [16]. The BLT will follow from the weak BLT if $\bar{Z}g \in L^2(\mathbb{C}) \Leftrightarrow Z\tilde{g} \in L^2(\mathbb{C})$ and $Zg \in L^2(\mathbb{C}) \Leftrightarrow \bar{Z}\tilde{g} \in L^2(\mathbb{C})$. However we show that BLT and weak BLT are actually equivalent.

Proposition 6.5.4. *If $g \in L^2(\mathbb{C})$ and $\{g_{m,n}\}$ is an exact twisted Gabor frame for $L^2(\mathbb{C})$, then there is a unique $\tilde{g} \in L^2(\mathbb{C})$ such that $Z^t\tilde{g} = 1/\overline{Z^t g}$.*

Proof. Let $h = {Z^t}^{-1}\left(\frac{1}{\overline{Z^t g}}\right)$. By Lemma 6.1.6 (v), h is well defined and $h \in L^2(\mathbb{C})$. Let $z = x + iy$ and $w = r + is \in \mathbb{C}$. Using Lemma 6.1.6 (ii) and bi-orthogonality relation (Proposition 5.4.8 of [35], pp. 101) we have

$$\begin{aligned} & \langle h, g_{m,n} \rangle = \langle Z^t h, Z^t g_{m,n} \rangle \\ = & \int_{Q \times Q} \frac{1}{\overline{Z^t g}(z, w)} e^{-2\pi i (x.n - y.m)} e^{-2\pi i (s.n - t.m)} \overline{Z^t g}(z, w)\, dz dw \\ = & \delta_{m,0}\delta_{n,0} = \langle \tilde{g}, g_{m,n} \rangle, \quad \forall\ m, n \in \mathbb{Z}. \end{aligned}$$

Since $\{g_{m,n}\}$ is complete in $L^2(\mathbb{C})$ and $h, \tilde{g} \in L^2(\mathbb{C})$, it follows that $h = \tilde{g}$. $\square$

Theorem 6.5.5. *If $g \in L^2(\mathbb{C})$ and $\{g_{m,n}\}$ is an exact twisted Gabor frame for $L^2(\mathbb{C})$, then*

$$\bar{Z}g \in L^2(\mathbb{C}) \Leftrightarrow Z\tilde{g} \in L^2(\mathbb{C}) \text{ and } Zg \in L^2(\mathbb{C}) \Leftrightarrow \bar{Z}\tilde{g} \in L^2(\mathbb{C}).$$

Proof. Assume that $Zg \in L^2(\mathbb{C})$. Then

$$\begin{aligned} Z^t(Zg)(z,w) &= \sum_k Zg(z-k)e^{2\pi i Im(w.\bar{k})} = \sum_k \Big(\frac{d}{dz} + \frac{1}{2}\bar{z}\Big) g(z-k) e^{2\pi i Im(w.\bar{k})} \\ &= \partial_z (Z^t g)(z,w) + \frac{1}{2}\bar{z}(Z^t g)(z,w) - \frac{1}{2\pi}\partial_w (Z^t g)(z,w). \end{aligned} \quad (6.20)$$

Similarly,

$$Z^t(\bar{Z}g)(z,w) = \partial_{\bar{z}}(Z^t g)(z,w) - \frac{1}{2}z(Z^t g)(z,w) - \frac{1}{2\pi}\partial_{\bar{w}}(Z^t g)(z,w). \quad (6.21)$$

Now using Proposition 6.5.4, we compute

$$\begin{aligned} \overline{Z^t(Z\tilde{g})(z,w)} &= \overline{\partial_z(Z^t\tilde{g})(z,w)} + \frac{1}{2}z\overline{(Z^t\tilde{g})(z,w)} - \frac{1}{2\pi}\overline{\partial_w(Z^t\tilde{g})(z,w)} \\ &= \partial_{\bar{z}}\Big(1/Z^t g\Big)(z,w) + \frac{z/2}{(Z^t g)(z,w)} - \frac{1}{2\pi}\partial_{\bar{w}}(1/Z^t g)(z,w) \\ &= -\frac{\partial_{\bar{z}}(Z^t g)(z,w)}{(Z^t g)^2(z,w)} + \frac{z}{2}\frac{(Z^t g)(z,w)}{(Z^t g)^2(z,w)} + \frac{1}{2\pi}\frac{\partial_{\bar{w}}(Z^t g)(z,w)}{(Z^t g)^2(z,w)} \\ &= -\frac{\partial_{\bar{z}}(Z^t g)(z,w) - \frac{z}{2}(Z^t g)(z,w) - \frac{1}{2\pi}\partial_{\bar{w}}(Z^t g)(z,w)}{(Z^t g)^2(z,w)} \\ &= -\frac{Z^t(\bar{Z}g)(z,w)}{(Z^t g)^2(z,w)}. \end{aligned} \quad (6.22)$$

Thus it follows that $\bar{Z}g \in L^2(\mathbb{C}) \Leftrightarrow Z\tilde{g} \in L^2(\mathbb{C})$ provided all the calculations are justified in distribution point of view. Similarly the other equivalent relation can be obtained. $\square$

Remark 6.5.6. Let $g \in L^2(\mathbb{C})$ and $\{g_{m,n}\}$ be an exact twisted Gabor frame for

$L^2(\mathbb{C})$. Then the following statements hold.

(i) The functions $L^{\frac{1}{2}}g$ and $L^{\frac{1}{2}}\tilde{g}$ cannot both be in $L^2(\mathbb{C})$: If $L^{\frac{1}{2}}g \in L^2(\mathbb{C})$, then

$$\|L^{\frac{1}{2}}g\|_2^2 = \langle L^{\frac{1}{2}}g, L^{\frac{1}{2}}g\rangle = \langle g, Lg\rangle = \frac{1}{2}(\|Zg\|_2^2 + \|\bar{Z}g\|_2^2).$$

Therefore
$$L^{\frac{1}{2}}g \in L^2(\mathbb{C}) \Leftrightarrow Zg, \bar{Z}g \in L^2(\mathbb{C}).$$

Now if $L^{\frac{1}{2}}g$ and $L^{\frac{1}{2}}\tilde{g} \in L^2(\mathbb{C})$ then $Zg, \bar{Z}g, Z\tilde{g}, \bar{Z}\tilde{g} \in L^2(\mathbb{C})$, contradicting the Theorem 6.5.2.

(ii) The functions $Z\bar{Z}g$ and $\bar{Z}Zg$ cannot both be in $L^2(\mathbb{C})$: If $Z\bar{Z}g, \bar{Z}Zg \in L^2(\mathbb{C})$, then

$$Lg \in L^2(\mathbb{C}) \quad \text{and} \quad L^{\frac{1}{2}}g = L^{-\frac{1}{2}}(Lg) \in L^2(\mathbb{C}).$$

This would imply $Zg, \bar{Z}g \in L^2(\mathbb{C})$, a contradiction to Theorem 6.3.4.

(iii) The functions Lg and g cannot both be in $L^2(\mathbb{C})$.

(iv) Consider the operators R and $\bar{R}$ (Riesz transforms) defined by

$$Rg = ZL^{-\frac{1}{2}}g \quad \text{and} \quad \bar{R}g = \bar{Z}L^{-\frac{1}{2}}g.$$

Then the functions $\bar{Z}Rg$ and $Z\bar{R}g$ cannot both be in $L^2(\mathbb{C})$: If $\bar{Z}Rg, Z\bar{R}g \in L^2(\mathbb{C})$, then

$$L^{\frac{1}{2}}g = -\frac{1}{2}(\bar{Z}Z + Z\bar{Z})L^{-\frac{1}{2}}g = -\frac{1}{2}(\bar{Z}R + Z\bar{R})g \in L^2(\mathbb{C}),$$

leading to a contradiction.

The classical uncertainty inequality on $L^2(\mathbb{R})$ can be written as

$$\||x|f\|_2 \, \|(-\Delta)^{\frac{1}{2}}f\|_2 \geq \frac{1}{2}\|f\|_2,$$

where $(-\Delta)^{\frac{1}{2}}$ is defined by $((-\Delta)^{\frac{1}{2}}f)\hat{}\,(\xi) = |\xi|\hat{f}(\xi)$. If we consider the special Hermite operator L on $\mathbb{C}$, then the inequality on $L^2(\mathbb{C})$ has the form $\||z|f\|_2 \, \|L^{\frac{1}{2}}f\|_2 \geq$

$\frac{1}{2}\|f\|_2$ (see [130]). Therefore we consider the following:

Remark 6.5.7. Let $g \in L^2(\mathbb{C})$ and $\{g_{m,n}\}$ be an exact twisted Gabor frame for $L^2(\mathbb{C})$. Then the functions $L^{\frac{1}{2}}g$ and $|z|g$ cannot both be in $L^2(\mathbb{C})$: We have

$$L^{\frac{1}{2}}g \in L^2(\mathbb{C}) \Leftrightarrow Zg, \bar{Z}g \in L^2(\mathbb{C}).$$

If $|z|g \in L^2(\mathbb{C})$, then by Lemma 6.3.1(i), $\|\tilde{g} - g\|_2 \leq 2\pi|\epsilon|\||z|g\|_2$ and the upper bounds in Lemma 6.3.1(ii) and (iii) can be obtained accordingly. Then proceeding exactly as in Theorem 6.3.4, we get $\|L^{\frac{1}{2}}g\|_2 \; \||z|g\|_2 = \infty$.

Chapter 7

Future Work

In this chapter, we describe some problems related to the results discussed in this thesis. In [111], the generalized Fourier transform on $L^2(\mathbb{R})$ is defined in connection with the differential-difference operator. One can investigate Balian-Low type theorems (as discussed in Chapter 6) with respect to the generalized Fourier transform. We first introduce the differential-difference operator in the following section.

7.1 The Differential-Difference Operator

Let us consider the differential-difference operator Λ on $\mathbb{R}$, given by

$$\Lambda f = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right), \tag{7.1}$$

where

$$A(x) = |x|^{2\alpha+1}B(x), \quad \alpha > -\frac{1}{2}.$$

B being a positive C^∞ even function on $\mathbb{R}$.

For $\lambda \in \mathbb{C}$, the differential-difference equation:

$$\begin{cases} \Lambda u = i\lambda u, & \lambda \in \mathbb{C}, \\ u(0) = 1, \end{cases} \tag{7.2}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted by Φ_λ and given by

$$\Phi_\lambda(x) = \begin{cases} \phi_\lambda(x) + \frac{1}{i\lambda}\frac{d}{dx}\phi_\lambda(x) & \text{if } \lambda \neq 0, \\ 1 & \text{if } \lambda = 0, \end{cases} \tag{7.3}$$

where ϕ_λ denotes the solution of the differential equation

$$\Delta u = -\lambda^2 u, \quad u(0) = 1, \quad u'(0) = 1$$

and Δ being the second-order singular differential operator defined by

$$\Delta = \frac{1}{A(x)}\frac{d}{dx}\Big(A(x)\frac{d}{dx}\Big).$$

Moreover, Φ_λ is entire in λ. For a detailed study of the differential-difference operator, we refer to [50].

7.2 The Generalized Fourier Transform

Let us denote $\mathcal{D}(\mathbb{R})$ as the space of compactly supported C^∞ functions on $\mathbb{R}$.

Definition 7.2.1. *The generalized Fourier transform $\mathcal{F}_G$ of function $f \in \mathcal{D}(\mathbb{R})$ is defined by*

$$(\mathcal{F}_G f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-i\lambda}(x)A(x)\,dx, \quad \lambda \in \mathbb{C}. \tag{7.4}$$

The important properties of this generalized Fourier transform is given by the following theorem ([111], Theorem 2.2):

Theorem 7.2.2. (*Plancherel*) (*i*) *There is an even positive tempered measure σ (and only one) on $\mathbb{R}$ such that for all $f \in L^1 \cap L^2(\mathbb{R}, A(x)\,dx)$,*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)\,dx = \int_{\mathbb{R}} |(\mathcal{F}_G f)(\lambda)|^2 d\sigma(\lambda).$$

(*ii*) *The generalized Fourier transform $\mathcal{F}_G$ extends uniquely to a unitary isomorphism from $L^2(\mathbb{R}, A(x)\,dx)$ onto $L^2(\mathbb{R}, \sigma)$.*

Now we list few problems that can be explored using the generalized Fourier transform.

- **(BLT).** Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ satisfy $\alpha\beta = 1$. If the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is an exact frame for $L^2(\mathbb{R})$, then can we prove that

$$\|xg(x)\|_2 \|\lambda(\mathcal{F}_G g)(\lambda)\|_2 = +\infty?$$

- **(Amalgam BLT).** Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$ satisfy $\alpha\beta = 1$. If the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is an exact frame for $L^2(\mathbb{R})$, then can we prove that

$$g \notin W(C_0, L^1) \quad \text{and} \quad \mathcal{F}_G g \notin W(C_0, L^1)?$$

It is well known that if $\{e^{im\beta x} g(x - n\alpha) : m, n \in \mathbb{Z}\}$ is a Gabor frame for $L^2(\mathbb{R})$ with bounds A, B, then the following inequalities hold:

$$A \leq \frac{2\pi}{\beta} \sum_{n \in \mathbb{Z}} |g(x - n\alpha)|^2 \leq B, \quad \text{a.e.}$$

and

$$A \leq \frac{1}{\alpha} \sum_{m \in \mathbb{Z}} |\hat{g}(w - m\beta)|^2 \leq B, \quad \text{a.e.}$$

In [144], the authors proved the similar inequalities for multi-generated irregular Gabor frames. One can investigate the similar problem using the generalized Fourier transform.

Problem 5.2.3 in chapter 5, leads to the following problem, posed by Heil and Larson [92] and answer of this problem is still unknown.

Problem 7.2.3. *Let $\{w_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $\mathbb{H}$. Find a characterization of all positive semi-definite trace-class operators T that are of Type B with respect to $\{w_n\}_{n \in \mathbb{N}}$.*

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Publications:

Based on the work in this thesis, the following research articles are published or communicated for publication.

List of Papers Published

1. A. Poria, *Behavior of Gabor frame operators on Wiener amalgam spaces*, Int. J. Wavelets Multiresolut. Inf. Process., **14**(4):1650028, 15 pp. (2016).

2. A. Poria, *Some identities and inequalities for Hilbert-Schmidt frames*, Mediterr. J. Math., **14**(2):Art. 59, 14 pp. (2017).

3. A. Poria, *Approximation of the inverse frame operator and stability of Hilbert-Schmidt frames*, Mediterr. J. Math., **14**(4):Art. 153, 22 pp. (2017).

List of Papers Communicated

1. A. Poria and J. Swain, *Hilbert space valued Gabor frames in weighted amalgam spaces*, preprint arXiv:1508.01646 (2015).

2. A. Poria and J. Swain, *Balian-Low type theorems on* $L^2(\mathbb{C})$, preprint arXiv: 1706.00294 (2017).

3. R. Balan, K. A. Okoudjou and A. Poria, *On a Feichtinger problem*, preprint arXiv:1705.06392 (2017).