Indian Institute of Technology Guwahati

Doctoral Thesis

On Units in Group Algebras

Author:
Dishari Chaudhuri

Supervisor:
Dr. Anupam Saikia

Roll No. 10612306

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy

in the

Department of Mathematics

September 2016

Declaration of Authorship

I, Dishari Chaudhuri, declare that this thesis titled, 'On Units in Group Algebras' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this Institute.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this Institute or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ________________________________

Date: ________________________________

CERTIFICATE

I declare that I am satisfied with the thesis entitled On Units in Group Algebras presented by Ms. Dishari Chaudhuri (Roll No. 10612306) and consider it to be worthy for the award of the degree of Doctor of Philosophy. It is a record of the original bonafide research work carried out by her under my supervision and the results contained in it have not been submitted in part or full to any other university or institute for award of any degree/diploma.

Professor Anupam Saikia
(Thesis Supervisor)

September 2016
Department of Mathematics
Indian Institute of Technology Guwahati

Abstract

Department of Mathematics

IIT GUWAHATI

Doctor of Philosophy

On Units in Group Algebras

by Dishari Chaudhuri

The aim of this thesis is to study the relation between the derived length of the group of units in a group algebra of a finite group over a field of finite characteristic and the commutativity of the group. We have mostly studied group algebras with their unit groups having derived length at most four and proved their commutativity when the characteristic of the field $p \geq 17$. We also prove commutativity of G when the derived length of U is smaller than $\lceil \log_2(2p) \rceil$ under certain additional hypothesis. As the derived length of the group of units is related to the strong Lie derived length of the group algebra, we have studied group algebras with strong Lie derived length at most four as well.

Acknowledgements

I would like to express my heartfelt gratitude towards my supervisor Professor Anupam Saikia for his guidance and support throughout my time as a student at IIT Guwahati. I would also like to thank all the teachers in the Department of Mathematics, IIT Guwahati who extended a helping hand to me whenever required. I have been fortunate to find an incredible set of friends who have made my stay very enjoyable here. Finally, I would like to take this opportunity to thank both my family and extended family for their patience and continuous encouragement in every endeavor of my life.

Contents

Symbols

$\in$	belongs to
$\notin$	does not belong to
$\neq$	not equal to
$\subseteq$	subset or equal to
$\subsetneq$	proper subset
$\cup, \cap$	union, intersection
$\emptyset$	empty set
$\cong$	is isomorphic to
$\equiv$	is equivalent to
$\not\equiv$	not equivalent to
$\lvert X \rvert$	cardinality of the set X
$H \leq G$	H is a subgroup of the group G
$H \trianglelefteq G$	H is a normal subgroup of the group G
G/H	quotient of G by H

$G \times H$	direct product of G and H
$a\|b$	integer a divides integer b
$a \nmid b$	integer a does not divide integer b
$\lceil r \rceil$	the smallest integer greater than the real number r

Dedicated to my Family

Chapter 1

Introduction

1.1 An Overview

Group algebras are very interesting algebraic structures. At the beginning of the 20th century an algebraic structure consisting of a group and a field appeared in the works of G. Frobenius which was used for the observation of the representations of finite groups. Their prominence grew when T. Molien, I. Schur and H. Maschke also worked considerably on these structures during that time. In the period of 1927-1929, E. Noether and R. Brauer established the importance of these algebraic constructions by pointing out the central role they play in group representation theory. E. Noether named this algebraic construction as group algebra in 1930's. Group rings in general are involved in the studies of the theory of fields, linear algebra and algebraic number theory. They are also related to algebraic topology, homological algebra and algebraic K-Theory. In the last few decades they have also found applications in algebraic coding theory. Hence, the theory of group rings provides a subject where many branches of algebra come to a rich interplay. We denote by KG the group algebra of a group G over a field K.

Group algebras, after 1930's, were not so much observed for themselves but for the possibilities of their applications in representation theory and algebraic topology, etc. Group algebras played a vital role in the works of W. Magnus regarding lower central series of free groups. Again I. Kaplansky's works on ring theory brought group algebras over infinite groups into focus in 1960's. Newer directions in research were evolved due to several characterisation theorems proved during this period. Several survey articles were published which helped in the formation of this new branch of mathematics. The study of the ring theoretical properties of group algebras is now at the most advanced level. Significant results on their unit groups are known as well. Many researchers took interest in the units of group algebras because of their vivid topological applications and then again, after the description of simple groups in terms of finite p-groups. The study of modular group algebras , that is, group algebras over fields of finite characteristic p with the underlying group having at least one element of order p, was started by S.A. Jennings in 1940's. But as the solution of every single problem required the elaboration of a new method, the results came very slowly. In some interesting cases the unit group has such a high order that even the present day computers are not capable enough to deal with them.

The structure of a group algebra can be associated to a Lie algebra after introducing the usual bracket operation. In general it is very difficult to determine a unit in a group algebra. So finding out its inverse as well as computation of commutators of the unit group is not an easy task. But investigation of the Lie properties of a group algebra led to the discovery of certain relations between the underlying unit group and the associated Lie algebra in 1980's. The so called Lie commutator of an element can be determined without the knowledge of the inverse of that element. Considering the results connected to the series which are constructed with the help of Lie commutators we can have conclusions about the corresponding series of the group of units, for example, derived series, upper and lower central series, etc. This method was first applied by A.A. Bovdi and I.I. Khripta ([1]) in 1977 who obtained that the unit group of group algebras is solvable if and only if the group algebra is Lie solvable under the assumptions that the characteristic p of the field is greater than three and the basic group is nonabelian and if it is a non-torsion group, then its p-Sylow subgroup is infinite. Lie methods were used by A.A. Bovdi and J. Kurdics for the investigation of derived length, the nilpotency class and the Engel length of the group of units ([3],[22],[24]).

The investigation of necessary and sufficient conditions for the solvability of the group of units independently, that is, without results regarding the corresponding Lie series dates back to the 1970s with the works of J.M.Bateman and D.S. Passman ([25], [12]). However, Lie properties of group algebras were extensively used for obtaining these results. A lot of work has been done on this context with a complete solution of the problem being given by A. Bovdi ([4]) in 2005. As a consequence we know that for group algebras of finite groups over fields of characteristic $p > 3$, the unit group of the group algebra is solvable if and only if either the underlying group G is abelian or $G/O_p(G)$ is abelian, where $O_p(G)$ is the maximal normal p-subgroup of G.

We denote the multiplicative group of units of KG by $U(KG) = U$. Another interesting problem is to relate structural properties of G with those of U. The natural question is to ask about the derived length of U once it is assumed to be solvable. It seems quite difficult to give a general formula for the derived length of U. Only a few results have been proved. A. Shalev ([6]), J. Kurdics ([23]), M. Sahai and H. Chandra ([30],[31],[19] and [20]) have investigated group algebras with units having derived length at most two and three respectively over fields of finite characteristic. C. Bagiński [7] showed that if G is a finite nonabelian p-group such that G' is cyclic, then the derived length of U is $\lceil log_2(|G'|+1)\rceil$, where $\lceil r\rceil$ denotes the minimal integer not smaller than r for a real number r. This result was extended by Z.Balogh and Y.Li for an arbitrary group G with cyclic derived subgroup of p-power order $p > 2$ in [36]. From these results it easily followed that if G is a torsion nilpotent nonabelian group, then the derived length of U is at least $\lceil log_2(p+1)\rceil$. Finally F. Catino and E. Spinelli characterized group algebras over any torsion nilpotent group for which this lower bound is attained in [16]. The same for infinite groups was given recently by G.T. Lee, S.K. Sehgal and E. Spinelli in [18].

The analogous problem of finding the structure of G for a fixed Lie derived length of KG still remains open. F.Levin and G.Rosenberger in ([17]) have completely characterised the necessary and sufficient conditions for a group algebra over fields of finite characteristic to have Lie derived length at most two. M.Sahai in ([29])has given the full description of the strongly Lie solvable group algebra KG with strong Lie derived length at most three and shown that strong Lie derived length of the group algebra is at most three if and only if its Lie derived length is at most three when the characteristic of the field is greater than or equal to seven. Notable works on strong Lie derived length were done by T. Juhász ([32]), F. Catino and E. Spinelli ([15]). Eventually the group algebras whose strong Lie derived length is exactly $\lceil log_2(p+1)\rceil$ have been characterised by Z. Balogh and T. Juhász ([34],[35]).

We have dealt with the problems of characterising group algebras whose unit groups are solvable having derived length at most four as well as group algebras with strong Lie derived length at most four. Let $G', G^{(2)}$ and $G^{(3)}$ denote the first, second and third terms in the derived series of a group G.

Our **first** problem in Chapter 2 assumes that the unit group U of the group algebra satisfies the relation $(U^{(3)}, U') = 1$, that is, U belongs to the class of groups having derived length four. It is well known ([16]) that if G is a torsion nilpotent abelian group and K is a field of positive characteristic p, then the derived length of the unit group of the corresponding group algebra is at least $\lceil \log_2(p+1) \rceil$. Our first result extends this statement for groups of odd order for the case when characteristic of the field is greater than or equal to seventeen. The proof of this result is combinatorial in nature and contains ideas that might be generalised. The result has been published in [9].

The **second** problem in Chapter 3 deals with the case of units having derived length four with the most natural conditions, that is, $U^{(4)} = (U^{(3)}, U^{(3)}) = 1$. The elements involved are quite large and difficult to compute with. But we have simplified them as much as possible. In this chapter we resolve it completely for any G with U of derived length four without any condition on the order of G. Also the method of the proof followed is much simpler as we avoid combinatorial argument altogether and this is a much stronger version of our first result in Chapter 2. We also prove commutativity of G when the derived length of U is smaller than $\lceil \log_2(2p) \rceil$ under certain additional hypothesis.

Our **third** problem in Chapter 4 characterises group algebras with strong Lie derived length at most four and contains some generalised results too. Though already generalised version of this exists in works like ([35],[34]), we have given an independent proof of the strong Lie derived length four case.

Finally Chapter 5 contains problems which can be explored further in the future based on our work.

Before we proceed with the detailed description of the main results, we give a short survey of the basic concepts and notations in the next section.

1.2 Preliminaries

Let R be a ring. An element u in R is called a unit if there exists an element $v \in R$ such that $uv = vu = 1$. The set of all units in R form a group under multiplication and is called the unit group of R. It is denoted by $U(R)$.

1.2.1 Some Well-known Group Theoretic results

All groups considered are finite. The following definitions can be found in [28].

Definition 1.1. Given two elements x, y in a group G, the **commutator** of x and y is the element $(x, y) = x^{-1}y^{-1}xy \in G$.

More generally, a **commutator of weight** $n \geq 2$ is defined inductively by the rule:

$$(x_1, x_2, ..., x_n) = \big((x_1, x_2, ..., x_{n-1}), x_n\big).$$

Given two subsets H and K of a group G, we shall denote by (H, K) the subgroup of G generated by the set:

$$\big\{(h, k) : h \in H, k \in K\big\}.$$

In particular, the group $G' = (G, G)$ is called the **commutator subgroup** or the **derived subgroup** of G.

Lemma 1.2. *Let H be a normal subgroup of a group G. Then, the factor group G/H is abelian if and only if $G' \subset H$.*

Definition 1.3. A group G is called **solvable** if it contains a chain of subgroups:

$$\{1\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$$

such that each subgroup G_{i-1} is normal in G_i and the factor groups G_i/G_{i-1}, $1 \leq i \leq n$, are abelian.

A chain of subgroups of G with this property is called an **abelian subnormal series** of G.

We define inductively:

$$G^{(0)} = G, \qquad \text{and} \qquad G^{(n)} = \Big(G^{(n-1)}, G^{(n-1)}\Big).$$

Definition 1.4. The decreasing series of subgroups

$$G^{(0)} \supset G^{(1)} \supset \ldots \supset G^{(n)} \supset \ldots$$

is called the **derived series** of the group G. If the series terminates, i.e., if $G^{(n)} = \{1\}$ for some positive integer n then the smallest such integer is called the **derived length** of G.

Theorem 1.5. *A group G is solvable if and only if its derived series terminates.*

Definition 1.6. The **lower central series** of a group G is the chain of subgroups of G defined by:

$$\begin{aligned} \gamma_1(G) &= G \\ \gamma_{i+1}(G) &= (\gamma_i(G), G) \qquad for\ i \geq 1. \end{aligned}$$

Definition 1.7. A group G is said to be **nilpotent** if $\gamma_{c+1}(G) = 1$ for some c. The least such c is the **nilpotency class** of G.

Theorem 1.8. *Finite p-groups are nilpotent.*

Definition 1.9. Let G be a group with subgroups H and K. We say that G is a **(internal) semidirect product** of H by K and write $G = H \rtimes K$, if we have:

(i) $G = HK = KH$,

(ii) $H \cap K = \{1\}$,

(iii) $H \lhd G$.

We can also construct an *external* semidirect product of two groups. Let H, K be groups and assume that there exists a homomorphism $f : K \to Aut(H)$, where $Aut(H)$ denotes the group of automorhisms of H. Given elements $h \in H$ and $k \in K$ we have that $f(k)$ is an automorphism of H, so we can compute its value on the element $h \in H$. We shall denote the image of h under $f(k)$ by $h^{f(k)}$.

In the set of ordered pairs $G = \{(k,h) : h \in H, k \in K\}$ we define a product by:

$$(k_1, h_1)(k_2, h_2) = (k_1k_2, {h_1}^{f(k_2)}h_2)$$

It is easy to see that G, with the operations above, is a group. This is an important method of constructing new groups from given ones.

Definition 1.10. Let H, K be groups and assume that there exists a homomorphism $f : K \to Aut(H)$. Then, the group structure defined above on the set $G = \{(k, h) : h \in H, k \in K\}$ is called the **(external) semidirect product** of H by K.

It is to be noted that $\overline{H} = \{(1, h) : h \in H\}$ is a normal subgroup of G, $\overline{K} = \{(k, 1) : k \in K\}$ is a subgroup of G and $G = \overline{H} \rtimes \overline{K}$. We shall not distinguish between internal and external semidirect products.

Example 1.1. ***Dihedral group of order*** $2m$.
Let $H = \langle a \rangle$ *be a cyclic group of order* m *and let* $K = \langle b \rangle = \{1, b\}$ *be a cyclic group of order 2. To define* $f : K \to Aut(H)$ *we only need to give the value of* $f(b)$, *so we define it to be the automorphism of* H *given by* $a^{f(b)} = a^{-1}$. *Then, we define the* ***dihedral group of order*** $2m$ *to be the semidirect product* $D_m = \langle a \rangle \rtimes \langle b \rangle$.

Notice that D_m *consists of all elements of the form* $\{a^i b^j : 0 \le i \le m-1, j = 0, 1\}$ *and the following relations are satisfied* $a^m = 1, b^2 = 1, ba = a^{-1}b$. *Clearly* $|D_m| = 2m$.

Geometrically, D_m *can be defined as the group of isometries of the* m*-sided regular polygon. In this case,* a *represents the rotation of angle* $2\pi/m$ *(i.e., the one rotating each vertex one step to the next one) and* b *represents a reflection along the line determined by the centre of the polygon and one of its vertices.*

Definition 1.11. Let G be a finite group of order $|G| = p^n m$ where $p \nmid m$. A subgroup of G of order p^n is called a **Sylow** p**-subgroup** of G.

Theorem 1.12 (Sylow). *Let* G *be a finite group of order* $|G| = p^n m$, *where* p *is a prime integer which does not divide the positive integer* m. *Then:*

(i) G *contains Sylow* p*-subgroups and, moreover, every* p*-subgroup of* G *is contained in a Sylow* p*-subgroup of* G.

(ii) All the Sylow p*-subgroups of* G *are conjugate in* G.

(iii) If n_p *denotes the number of Sylow* p*-subgroups of* G, *then*

$$n_p \equiv 1 \pmod{p}.$$

Let $O_p(G)$ denote the maximal normal p-subgroup of G. We have the following result.

Lemma 1.13. *If $G/O_p(G)$ is abelian, then G has a normal Sylow p-subgroup.*

Proof. Let P be a Sylow p-subgroup of G such that $O_p(G) \leq P$. Let $x \in P$ and $g \in G$. Since, $G/O_p(G)$ is abelian, we have $G' \leq O_p(G)$. Therefore, $x^{-1}g^{-1}xg \in O_p(G)$, which implies that $g^{-1}xg \in P$ for every $x \in P$ and $g \in G$. Hence, $P \trianglelefteq G$. □

Theorem 1.14 (Schur-Zassenhaus). *If G is a finite group, and N is a normal subgroup of G whose order is coprime to the order of the quotient group G/N, then G is a semidirect product of N and G/N.*

The following can be found in ([11]). By a p'-element or a p'-automorphism of a group G we mean an element or an automorphism of G whose order is not divisible by p.

Theorem 1.15. *If A is a p'-group of automorphisms of the p-group Q, then $(Q, A, A) = (Q, A)$. In particular, if $(Q, A, A) = \{1\}$, then $A = \{1\}$.*

Note that $(Q, A) = \{q^{-1}\sigma_a(q) \mid q \in Q \text{ and } a \in A\}$, where σ_a denotes the automorphism of Q corresponding to $a \in A$.

Definition 1.16. Frattini subgroup of a group G is the intersection of all the maximal subgroups of G. We denote it by $\Phi(G)$. It has the following properties:

(i) $\Phi(G)$ is the set of all non-generators of G.

(ii) $\Phi(G)$ is a characteristic subgroup of G, that is, it remains invariant under all automorphisms of G. In particular, it is always a normal subgroup of G.

(iii) If G is finite, then $\Phi(G)$ is nilpotent.

(iv) The **Frattini factor group** $P/\Phi(P)$ of a p-group P is elementary abelian. Furthermore, $\Phi(P) = 1$ if and only if P is elementary abelian.

(v) If H and K are finite, then $\Phi(H \times K) = \Phi(H) \times \Phi(K)$.

Example 1.2. *An example of a group with nontrivial Frattini subgroup is the cyclic group G of order p^2, where p is a prime, generated by a, say. Here, $\Phi(G) = \langle a^p \rangle$.*

The following is a well known result by Burnside.

Theorem 1.17. *Let ψ be a p'-automorphism of a p-group P, which induces the identity on $P/\Phi(P)$. Then ψ is the identity automorphism on P.*

1.2.2 Group Algebras

Next we come to the definition of group algebras which can be found in [32] and [13].

Definition 1.18. Let G be a group and K a field. Denote by KG all the formal sums $\sum_{g\in G}\alpha_g g$, where only finitely many coefficients $\alpha_g \in K$ are nonzero. Clearly two formal sums are equal if and only if all corresponding coefficients of group elements are equal. Let us define the sum of $x = \sum_{g\in G}\alpha_g g \in KG$ and $y = \sum_{g\in G}\beta_g g \in KG$ as

$$x + y = \sum_{g\in G}(\alpha_g + \beta_g)g$$

and the product of $\beta \in K$ and x as

$$\beta \cdot x = x \cdot \beta = \sum_{g\in G}(\beta\alpha_g)g.$$

Then KG can be considered as a vector space over K and the elements of G form a K-basis for KG. The multiplication of formal sums are defined as follows:

$$xy = \sum_{g\in G}\Big(\sum_{h\in G}\alpha_h\beta_{h^{-1}g}\Big)g.$$

With these operations KG is an algebra over the field K which is called **group algebra** (of the group G over the field K).

In the special case when K is a field of characteristic $char(F) = p$ and G contains an element of order p, KG is called **modular group algebra**.

Definition 1.19. Let $x = \sum_{g\in G}\alpha_g g$ be a nonzero element of the group algebra KG. The subset $\{g \in G|\alpha_g \neq 0\}$ of the group G is said to be the **support** of x.

Definition 1.20. The homomorphism $\varepsilon : KG \to K$ given by

$$\varepsilon\Big(\sum_{g\in G}\alpha_g g\Big) = \sum_{g\in G}\alpha_g$$

is called the **augmentation mapping** of KG and its kernel

$$\Delta(G) = \{x \in KG \mid \varepsilon(x) = 0\}$$

is called the **augmentation ideal** of the group algebra KG.

Proposition 1.21. *The set* $\{g-1 : g \in G, g \neq 1\}$ *is a basis of* $\Delta(G)$ *over* K. *Thus, we can write*

$$\Delta(G) = \left\{ \sum_{g\in G} \alpha_g(g-1) : g \in G, g \neq 1, \alpha_g \in K \right\}$$

where as usual we assume that only finitely many coefficients α_g *are nonzero.*

Recall that an ideal I in a ring is said to be nilpotent if there exists a natural number k such that $I^k = \{\sum x_1 \dots x_k \ : x_i \in I\} = 0$ and its nilpotency index is the least positive integer n such that $I^n = 0$. The following is an important result by D.B.Coleman.

Theorem 1.22. *Let* K *be a field of characteristic* $p \geq 0$ *and* G *is an arbitrary group. Then the augmentation ideal* $\Delta(G)$ *of* KG *is nilpotent if and only if* $p > 0$ *and* G *is a finite* p*-group.*

As a corollary of the above theorem we have the following result.

Corollary 1.23. *If* G *has a finite normal* p*-subgroup* P*, where* p *is the characteristic of* K*, then* $\Delta(P)KG$ *is nilpotent.*

The next result is a very useful one.

Result 1.24. Let G be a finite p-group of order p^k. Then the nilpotency index of $\Delta(G)$ over a field of characteristic p is p^k if and only if G is cyclic. Further, if $G = P_1 \times P_2 \times \dots \times P_k$, where each P_i is a cyclic subgroup of order p^{t_i}, $t_i \in \mathbb{Z}$, for $i = 1, 2, \dots, k$, then the nilpotency index of $\Delta(G)$ is $(p^{t_1} + p^{t_2} + \dots + p^{t_k} - k + 1)$.

Result 1.25. The following are important results concerning ideals in group algebras.

(i) For any normal subgroup H of G the set

$$\mathfrak{J}(H) = \{(h-1)x | h \in H, x \in KG\}$$

is a two-sided ideal of KG.

(ii) Clearly, $\mathfrak{J}(G)$ coincides with $\Delta(G)$ and $\mathfrak{J}(H) = \Delta(H)KG$.

(iii) Let $T(G/H)$ be a transversal of the normal subgroup H in G. Then all the elements of the form $(h-1)u$, where $1 \neq h \in H$ and $u \in T(G/H)$ form a basis of the vector space $\mathfrak{J}(H)$.

(iv) The isomorphism $KG/\mathfrak{J}(H) \cong K(G/H)$ is valid, which is called the **isomorphism theorem** of group algebras.

1.2.3 Associated Lie Algebra of Group Algebras

Definition 1.26. Let $(L, +)$ be a vector space over the field K and assume that a second binary operation $[a, b]$ is defined in L such that the following identities hold in L for all $a, b, c \in L$ and $\alpha \in K$.

(i) $\alpha[a, b] = [\alpha a, b] = [a, \alpha b]$;

(ii) $[a + b, c] = [a, c] + [b, c]$ and $[a, b + c] = [a, b] + [a, c]$;

(iii) $[a, a] = 0$;

(iv) $\big[[a, b], c\big] + \big[[b, c], a\big] + \big[[c, a], b\big] = 0$

We say that L is a **Lie algebra** over the field K.

Let A be an associative algebra over the field K and $x, y \in A$. The element $[x, y] = xy - yx$ will be called the **Lie commutator** of x and y. Let us introduce in A the new operation $[x, y] = xy - yx$. Then A is a Lie algebra with respect to the operations $+$ and $[,]$, which is said to be the **associated Lie algebra** of A.

For the sequence (x_i) of elements of A we define the **left n-normed Lie commutator** by induction as

$$[x_1, x_2, \ldots, x_n] = \big[[x_1, x_2, \ldots, x_{n-1}], x_n\big].$$

The following are useful identities:

$$[x, yz] = [x, y]z + y[x, z]; \qquad [xy, z] = x[y, z] + [x, z]y \tag{1.1}$$

For units a, b,

$$[a, b] = ba\Big((a, b) - 1\Big) = \Big((a^{-1}, b^{-1}) - 1\Big)ba.$$

For the subsets $X, Y \subseteq A$ we denote by $[X, Y]$ the additive subgroup generated by all Lie commutators $[x, y]$ with $x \in X$ and $y \in Y$. The following properties hold:

- $[X, Y] = [Y, X]$
- $[X, Y, Z] \subseteq [X, Y]Z + Y[X, Z]$
- $[XY, Z] \subseteq X[Y, Z] + [X, Z]Y$

for any $X, Y, Z \subseteq A$.

1.2.4 Series in the Associated Lie Algebra

As before let A be an associative algebra over the field K.

Definition 1.27. The **Lie central series of** A is defined as follows: let $\gamma^{[0]}(A) = A$ and for $n \in \mathbb{N}$, $n \geq 0$, let $\gamma^{[n+1]}(A)$ be the *additive subgroup of* A generated by all Lie commutators x, y with $x \in \gamma^{[n]}(A), y \in A$, that is,

$$\gamma^{[n+1]}(A) = [\gamma^{[n]}(A), A]. \tag{1.2}$$

Definition 1.28. The **Lie derived series of** A is defined as follows: let $\delta^{[0]}(A) = A$ and for $n \in \mathbb{N}$, $n \geq 0$, let $\delta^{[n+1]}(A)$ be the *additive subgroup of* A generated by all Lie commutators x, y with $x, y \in \delta^{[n]}(A)$, that is,

$$\delta^{[n+1]}(A) = [\delta^{[n]}(A), \delta^{[n]}(A)]. \tag{1.3}$$

Clearly,

$$A = \delta^{[0]}(A) \supseteq \delta^{[1]}(A) \supseteq \ldots \supseteq \delta^{[m]}(A) \supseteq \ldots \tag{1.4}$$

Definition 1.29. The **strong Lie derived series of** A is introduced: let $\delta^{(0)}(A) = A$ and for $n \in \mathbb{N}$, $n \geq 0$, let $\delta^{(n+1)}(A)$ be the *ideal of* A generated by $\delta^{[n+1]}(A)$, that is, the ideal of A generated by all Lie commutators $[x, y]$ with $x, y \in \delta^{(n)}(A)$. So,

$$\delta^{(n+1)}(A) = [\delta^{(n)}(A), \delta^{(n)}(A)]A. \tag{1.5}$$

Evidently,

$$A = \delta^{(0)}(A) \supseteq \delta^{(1)}(A) \supseteq \ldots \supseteq \delta^{(m)}(A) \supseteq \ldots \tag{1.6}$$

Note 1.30. It is easy to see that $\delta^{[n]}(A) \subseteq \delta^{(n)}(A)$ for any n. But the equality does not always hold.

Definition 1.31. We say that A is **Lie solvable of derived length** n , if $\delta^{[n]}(A) = 0$ but $\delta^{[n-1]}(A) \neq 0$. The Lie derived length is denoted by $dl_L(KG)$.

A is said to be **Lie nilpotent of class** c, if $\gamma^{[c+1]}(A) = 0$, but $\gamma^{[c]}(A) \neq 0$.

Similarly, A is said to be **strongly Lie solvable of derived length** n, if $\delta^{(n)}(A) = 0$ but $\delta^{(n-1)}(A) \neq 0$. The strong Lie derived length is denoted by $dl^L(KG)$.

Note 1.32. Let a, b be elements of the unit group U of A. Then by the equality $(a, b) = 1 + a^{-1}b^{-1}[a, b]$, we can see that the m-th term of the derived series of U, that is, $U^{(m)}$, is contained in $1 + \delta^{(m)}(A)$. Thus for the investigation of the derived length of U, the strong Lie derived series provide an important tool.

Chapter 2

Commutativity of Group Algebras with Units Satisfying $(U^{(3)}, U') = 1$

2.1 Introduction

An interesting problem is to relate structural properties of G with those of $U(KG)$. The conditions under which U is solvable was started independently in 1970s by Motose and Tominaga ([26]) and Bateman ([25]) for finite groups. Further results were given by Motose and Ninomiya ([27]), Bovdi and Khripta ([2], [1]) and Taylor ([10]). Finally, Passman in [12] gave necessary and sufficient conditions to have U solvable when G is finite. The classification was completed for finite groups by Bovdi [4].The complete set of necessary and sufficient conditions for U to be solvable given by Bovdi are:

Theorem 2.1. *Let K be a field of finite characteristic p, and $O_p(G)$ a maximal normal p-subgroup of the finite group G. Then the group $U(KG)$ is solvable if and only if one of the following statements holds:*

(i) G is abelian.

(ii) $G/O_p(G)$ is abelian and K is a field of characteristic p.

(iii) $|K| = 2$ and $G/O_p(G)$ is an extension of an elementary abelian 3-group A by a group $\langle b \rangle$ of order 2, and $bab = a^{-1}$ for all $a \in A$.

(iv) $|K| = 3$ *and* $G/O_p(G)$ *is an extension of an elementary abelian* 2*-group* A *by a group* $\langle b \rangle$ *of order* 2*.*

(v) $|K| = 3$ *and* G *is an extension of an abelian group* A *of exponent* 4 *by a group* $\langle b \rangle$ *of order* 2 *and* $bab^{-1} = a^{-1}$ *for all* $a \in A$*.*

(vi) $|K| = 3$ *and* $G/O_p(G)$ *is an extension of an abelian group* A *of exponent* 8 *by a group* $\langle b \rangle$ *of order* 2 *and* $bab = a^3$ *for all* $a \in A$*.*

(vii) $|K| = 3$ *and* $G/O_p(G)$ *is a direct product of the group*

$$\langle a, b | a^4 = b^4 = 1, (ab)^2 = 1, (a, ba) = (a, bb) = 1 \rangle$$

of order 32 *and an elementary abelian* 2*-group.*

Necessary and sufficient conditions for $U(KG)$ to be solvable have been given by Bovdi ([4]) when the group G, not necessary finite, contains at least one element of order p, where p is the characteristic of the field K. The natural question is to ask about the derived length of U once it is assumed to be solvable. It seems quite difficult to give a general formula for the derived length of U. Only a few results have been proved. Shalev ([6]), Kurdics ([23]), Sahai and Chandra ([30],[31],[19] and [20]) have investigated group algebras with units having derived length at most two and three respectively over fields of finite characteristic.

Shalev gave the following result in [6].

Theorem 2.2. *Let* G *be a finite group and let* K *be a field of characteristic* p*.*

(i) *If* $p > 3$*, then* U *is meta-abelian, that is,* $U^{(2)} = 1$*, if and only if* G *is abelian.*

(ii) *If* $p = 3$*, then* U *is meta-abelian if and only if* G *is either abelian or nilpotent with* $|G'| = 3$*.*

The description of finite groups for which U is meta-abelian was completed for $p = 2$ by Coleman and Sandling in [8] and independently by Kurdics in [23]. Sahai under similar assumptions as the result of Shalev, obtained necessary and sufficient conditions for U to be centrally metabelian, that is, $(U^{(2)}, U) = 1$ and U' to be nilpotent of class at most two, that is, $(U^{(2)}, U') = 1$ were given in [30] and [31] respectively for odd characteristic. Finally Sahai and Chandra in [19] gave the characterization of group algebras with units having derived length three. Their result was as follows:

Theorem 2.3. *Let K be a field of characteristic $p \neq 2, 3$ and let G be a finite non abelian group. Then the following are equivalent.*

(i) $U^{(3)} = 1$;

(ii) *p and G satisfy the following conditions:*

(a) $p = 7$, $G' = C_7$ *and* $\gamma_3(G) = 1$;

(b) $p = 5$, $G' = C_5$ *and either* $\gamma_3(G) = 1$ *or* $\gamma_n(G) = G'$ *for all* $n \geq 3$ *with* $x^g = x^{-1}$ *for all* $x \in G'$ *and for all* $g \notin C_G(G')$.

The authors also characterized the same for characteristic three in [20].

Baginski in [7] gave the following result:

Theorem 2.4. *Let K be a field of characteristic p, $p > 2$, and let G be a finite p-group. If the derived subgroup G' of G is cyclic, then the derived length of U is equal to* $\lceil log_2(|G'| + 1)\rceil$.

This result was extended by Balogh and Li ([36]) for an arbitrary group G with cyclic derived subgroup G' of p-power order (and $p > 2$). An easy consequence of the theorem is the following result.

Corollary 2.5. *Let K be a field of characteristic p, $p > 2$, and let G be a finite nonabelian p-group. Then the derived length of U is not smaller than* $\lceil log_2(p+1)\rceil$.

Finally Catino and Spinelli characterized group algebras over any torsion nilpotent group in [16]. Their result was:

Theorem 2.6. *Let KG be a group algebra of a torsion nilpotent group G over a field K of positive characteristic p. Then derived length of U is* $\lceil \log_2(p + 1)\rceil$ *if and only if one of the following holds:*

(i) *G' has order p;*

(ii) $p = 2$, *and G' is central of order 4 and exponent 2.*

In this chapter we consider group algebras with unit group U which satisfies $(U^{(3)}, U') = 1$. Such U is obviously of derived length at most four, that is,

$U^{(4)} = 1$. Lie algebraic properties of KG play an important role in our investigation. As mentioned in Chapter 1, for $X, Y \subseteq KG$, we denote by $[X, Y]$ the additive subgroup generated by all Lie commutators $[x, y] = xy - yx$, where $x \in X$ and $y \in Y$. Also, $O_p(G)$ stands for the maximal normal p-subgroup of G, and $\Delta(G)$ denotes the augmentation ideal of the group algebra KG. For any two elements $x, h \in G$, x^h denotes the conjugation of x by h, that is, $h^{-1}xh$. We denote the Frattini subgroup of a group G by $\Phi(G)$, which is the intersection of all maximal subgroups of G. It is well-known that $\Phi(G)$ is a characteristic subgroup of G. By a p'-element or a p'-automorphism of a group G we mean an element or an automorphism of G whose order is not divisible by p. All groups considered are finite. Our main result for this chapter is as follows and can be found in [9]:

Theorem 2.7. *Let K be a field of characteristic $p \geq 17$ and let G be a group of odd order. Then G is abelian if and only if U satisfies $(U^{(3)}, U') = 1$.*

2.2 Useful Known Results

In this section we discuss a few important known results which provide useful tools for the proof of our theorem.

This result can be found in ([33], Theorem 3.5)

Theorem 2.8. *Let G be a group of order $p^a b$ and $(p, b) = 1$ and let K be a field of characteristic p. Assume that G has a normal Sylow p-subgroup P. Then the Jacobson radical $J = J(KG)$ of KG is $J = \Delta(P)KG$.*

Hence by the corollary of the theorem by Coleman (1.23) we know that J in the above case will be nilpotent.

For any two elements x, y in KG, it is easy to observe that

$$xy - 1 \;=\; (x-1)(y-1) + (x-1) + (y-1). \tag{2.1}$$

Let J be any ideal of KG and let $x, y \in KG$ be such that $x - 1 \in J^i$ and $y - 1 \in J^j$ for some $i, j > 0$. Then it can be easily established using (2.1) that

$$(x, y) \equiv\; 1 + [x, y] \quad (mod\ J^{i+j+1}). \tag{2.2}$$

2.3 Main Result

Proof of Theorem 2.7

- **Necessary conditions:**

  Let G be a group of odd order. When G is abelian, the result follows trivially.

- **Sufficient conditions:**

  Let KG be the group algebra of a finite group G over a field K of characteristic $p \geq 5$, such that, the unit group $U = U(KG)$ satisfies the condition $(U^{(3)}, U') = 1$. Then U is solvable and according to Theorem 2.1, $G/O_p(G)$ is abelian.

  If $G/O_p(G)$ is abelian then $O_p(G)$ is a Sylow p-subgroup of G by 1.13. Let $P = O_p(G)$. Now, $|P|$ and $[G : P]$ are relatively prime, hence by Schur-Zassenhaus Theorem (1.14), we have $G = P \rtimes H$, where H is a p'-prime subgroup of G. Also, by the above conditions, H is abelian.

Lemma 2.9. *Let Char* $K = p \geq 11$. *Let* G *be a group of odd order. Suppose that* U *satisfies* $(U^{(3)}, U') = 1$. *Then* $G = P \times H$, *where* P *is a* p*-group and* H *is an abelian* p'*-group, where* p' *is odd.*

Proof. We know from above that $G = P \rtimes H$, where P is a p-group and H is an abelian p'-group. Also $P \trianglelefteq G$. Since G is of odd order, we have p' is odd. We need to show that $(P, H) = 1$. We will show that if $(P, H) \neq 1$, then we can construct nontrivial element in $(U^{(3)}, U')$.

We first assume that P is elementary abelian. Suppose, $(P, h) \neq 1$ for some $h \in H$. Then, $h^2 \neq 1$, and $(P, h) \leq P$ (as $P \trianglelefteq G$) and hence (P, h) is a p-group. Since h induces a p'-automorphism on P, by Theorem 1.15, $(P, h, h) = (P, h)$. Let $L = \langle (P, h), h \rangle$. Then $L' = (P, h, h) = (P, h)$ and $(P, h) \trianglelefteq L$. So on replacing G with L if necessary, we may assume that $P = G' = (P, h)$. By Theorem 2.8, the Jacobson radical $J = J(KG) = \Delta(P)KG$. Now, since $(P, h) \neq 1$, we can find $x \in P$ such that $(x, h) \neq 1$. Put $\alpha = x - 1$. Then $u = 1 + h\alpha$ is a unit in KG. Also, $(x, h), (x, h)^h, (x, h^2) \in P$. By forming commutators of suitable elements in U, we obtain elements in

$U', U^{(2)}$ and then in $U^{(3)}$. Now consider $u_1 = (u, h) \in U'$ and $v_1 = (u, x) \in U'$. Keeping in mind that $x \equiv 1 \pmod{J}$, we have

$$\begin{aligned} u_1 = (u,h) &= 1 + u^{-1}h^{-1}[u,h] \\ &\equiv 1 + (1 - h\alpha)(\alpha h - h\alpha) \quad \big(mod\ J^2\big) \\ &\equiv 1 + \alpha h - h\alpha \quad \big(mod\ J^2\big) \\ &= 1 + hx\big((x,h) - 1\big) \quad \big(mod\ J^2\big) \\ &\equiv 1 + h\big((x,h) - 1\big) \quad \big(mod\ J^2\big) \end{aligned} \tag{2.3}$$

As $G' = P$, we use identity 2.2 to obtain the following:

$$\begin{aligned} v_1 = (u,x) &\equiv 1 + [u,x] \quad \big(mod\ J^3\big) \\ &= 1 + (x + h\alpha x - x - xh\alpha) \quad \big(mod\ J^3\big) \\ &= 1 + \{h(x-1)x - xh(x-1)\} \quad \big(mod\ J^3\big) \\ &= 1 + (hx - xh)(x-1) \quad \big(mod\ J^3\big) \\ &= 1 + hx\big(1 - (x,h)\big)(x-1) \quad \big(mod\ J^3\big) \\ &\equiv 1 - h\big((x,h) - 1\big)(x-1) \quad \big(mod\ J^3\big) \end{aligned} \tag{2.4}$$

Next we consider $u_2 = (u_1, x)$ and $v_2 = (v_1, x)$. As $x \in P = G' \subset U'$, u_2 and v_2 are in $U^{(2)}$.

$$\begin{aligned} u_2 = (u_1,x) &\equiv 1 + [u_1,x] \quad \big(mod\ J^3\big) \\ &= 1 + \big\{x + h\big((x,h) - 1\big)x - x - xh\big((x,h) - 1\big)\big\} \quad \big(mod\ J^3\big) \\ &= 1 + (hx - xh)\big((x,h) - 1\big) \quad \big(mod\ J^3\big) \\ &= 1 + hx\big(1 - (x,h)\big)\big((x,h) - 1\big) \quad \big(mod\ J^3\big) \\ &\equiv 1 - h\big((x,h) - 1\big)^2 \quad \big(mod\ J^3\big) \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
v_2 = (v_1, x) &\equiv 1 + [v_1, x] \quad \big(mod\ J^4\big)\\
&= 1 + \big\{x - h\big((x,h) - 1\big)(x-1)x\\
&\qquad -x + xh\big((x,h)-1\big)(x-1)\big\} \quad \big(mod\ J^4\big)\\
&= 1 + (xh - hx)\big((x,h)-1\big)(x-1) \quad \big(mod\ J^4\big)\\
&= 1 + hx\big((x,h)-1\big)^2(x-1) \quad \big(mod\ J^4\big)\\
&\equiv 1 + h\big((x,h)-1\big)^2(x-1) \quad \big(mod\ J^4\big) \qquad (2.6)
\end{aligned}
$$

Finally we obtain an element $w = (u_2, v_2)$ in $U^{(3)}$. We have

$$
\begin{aligned}
w &= (u_2, v_2)\\
&\equiv 1 + [u_2, v_2] \quad \big(mod\ J^6\big)\\
&= 1 + [u_2 - 1, v_2 - 1] \quad \big(mod\ J^6\big)\\
&= 1 + \big\{ - h\big((x,h)-1\big)^2 h\big((x,h)-1\big)^2(x-1)\\
&\qquad + h\big((x,h)-1\big)^2(x-1)h\big((x,h)-1\big)^2\big\} \quad \big(mod\ J^6\big)\\
&= 1 + h\big((x,h)-1\big)^2\{(x-1)h - h(x-1)\}\big((x,h)-1\big)^2 \quad \big(mod\ J^6\big)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + h\big((x,h)-1\big)^2 hx\big((x,h)-1\big)^3 \quad \big(mod\ J^6\big)\\
&\equiv 1 + h\big((x,h)-1\big)^2 h\big((x,h)-1\big)^3 \quad \big(mod\ J^6\big)\\
&= 1 + h^2\big((x,h)^h - 1\big)^2\big((x,h)-1\big)^3 \quad \big(mod\ J^6\big) \qquad (2.7)
\end{aligned}
$$

We now show that the element (w, x) in $(U^{(3)}, U')$ is nontrivial. It suffices to show that $V_1 = (w, x)$ is nontrivial modulo J^7. Now,

$$
\begin{aligned}
V_1 &= (w, x)\\
&\equiv 1 + [w, x] \quad \big(mod\ J^7\big)\\
&= 1 + [w - 1, x] \quad \big(mod\ J^7\big)\\
&= 1 + \Big\{h^2\big((x,h)^h - 1\big)^2\big((x,h)-1\big)^3 x\\
&\qquad - xh^2\big((x,h)^h - 1\big)^2\big((x,h)-1\big)^3\Big\} \quad \big(mod\ J^7\big)\\
&= 1 + (h^2x - xh^2)\big((x,h)^h - 1\big)^2\big((x,h)-1\big)^3 \quad \big(mod\ J^7\big)\\
&= 1 + h^2x\big(1 - (x,h^2)\big)\big((x,h)^h - 1\big)^2\big((x,h)-1\big)^3 \quad \big(mod\ J^7\big)\\
&\equiv 1 - h^2\big((x,h^2)-1\big)\big((x,h)^h - 1\big)^2\big((x,h)-1\big)^3 \qquad (2.8)
\end{aligned}
$$

Since P is elementary abelian, let $P = P_1 \times P_2 \times \ldots \times P_k$, where $k \geq 1$, each $P_i \cong C_p$, $i = 1, 2, \ldots k$ and C_p is a cyclic group of order p. Let x_i be the generator of P_i, for $i = 1, 2, \ldots, k$. Let

$$\begin{aligned}(x, h^2) &= {x_1}^{a_1}{x_2}^{a_2}\ldots{x_k}^{a_k}\\(x, h)^h &= {x_1}^{b_1}{x_2}^{b_2}\ldots{x_k}^{b_k}\\(x, h) &= {x_1}^{c_1}{x_2}^{c_2}\ldots{x_k}^{c_k}\end{aligned}$$

where a_i's, b_i's and c_i's are integers such that $1 \leq a_i, b_i, c_i \leq p$ for every $i = 1, 2, \ldots, k$, with at least one element in every set of a_i's, b_i's and c_i's being greater than or equal to one but strictly less than p. Then Eq. 2.8 can be written as:

$$\begin{aligned}V_1 = (w, x) \equiv\ & 1 - h^2({x_1}^{a_1}{x_2}^{a_2}\ldots{x_k}^{a_k} - 1)({x_1}^{b_1}{x_2}^{b_2}\ldots{x_k}^{b_k} - 1)^2\\&({x_1}^{c_1}{x_2}^{c_2}\ldots{x_k}^{c_k} - 1)^3 \quad (mod\ J^7)\end{aligned} \tag{2.9}$$

Now by repeated use of identity 2.1 and working modulo J^7, Eq.2.9 becomes

$$\begin{aligned}V_1 \equiv\ & 1 - h^2\{({x_1}^{a_1} - 1) + ({x_2}^{a_2} - 1) + \ldots + ({x_k}^{a_k} - 1)\}\\&\times\{({x_1}^{b_1} - 1) + ({x_2}^{b_2} - 1) + \ldots + ({x_k}^{b_k} - 1)\}^2\\&\times\{({x_1}^{c_1} - 1) + ({x_2}^{c_2} - 1) + \ldots + ({x_k}^{c_k} - 1)\}^3\end{aligned} \tag{2.10}$$

Now, whenever we have $d \in \mathbb{N}$, we can write:

$$\begin{aligned}\therefore\ x^d - 1 &= (x - 1)(1 + x + \ldots + x^{d-1})\\&= (x - 1)\{1 + ((x - 1) + 1) + \ldots + ((x^{d-1} - 1) + 1)\}\\&= (x - 1)(d + D_1)\ \text{ where } D_1 \in \Delta(P) \subseteq J\\&= (x - 1)D\end{aligned} \tag{2.11}$$

If $p \nmid d$, then $D = (d + D_1)$ is a unit in KG. With the help of this technique, the second term in RHS of Eq.(3.8) can be written as:

$$\begin{aligned}
M &= \{(x_1-1)A_1 + (x_2-1)A_2 + \ldots + (x_k-1)A_k\} \\
&\quad \times \{(x_1-1)B_1 + (x_2-1)B_2 + \ldots + (x_k-1)B_k\}^2 \\
&\quad \times \{(x_1-1)C_1 + (x_2-1)C_2 + \ldots + (x_k-1)C_k\}^3 \\
&= \Big\{(x_1-1)A_1 + (x_2-1)A_2 + \ldots + (x_k-1)A_k\Big\} \\
&\quad \times \Big\{\sum_{i=1}^{k}(x_i-1)^2 {B_i}^2 \;+\; 2\sum_{\substack{i,j=1\\ i\neq j}}^{k}(x_i-1)(x_j-1)B_iB_j\Big\} \\
&\quad \times \Big\{\sum_{i=1}^{k}(x_i-1)^3 {C_i}^3 \;+\; 3\sum_{\substack{i,j=1\\ i\neq j}}^{k}(x_i-1)^2(x_j-1){C_i}^2C_j \;+ \\
&\qquad 6\sum_{\substack{i,j,l=1\\ i\neq j\neq l}}^{k}(x_i-1)(x_j-1)(x_l-1)C_iC_jC_l\Big\} \qquad (2.12)
\end{aligned}$$

where all the A_i's, B_i's, C_i's, for $i = 1, 2, \ldots, k$, belong to KG with at least one element in every set of A_i's, B_i's, C_i's, for $i = 1, 2, \ldots, k$, is a unit in KG. Now, by Theorem 1.24, nilpotency index of $\Delta(P)$ as well as J in this case is $(kp - k + 1)$. Let, if possible, $V_1 \equiv 1 \pmod{J^7}$, that is , let $M \in J^7$. Let $I = \{1, 2, \ldots, k\}$. Let $I_A = \{t \in I |\ A_t \text{ is a unit}\}$, $I_B = \{t \in I |\ B_t \text{ is a unit}\}$ and $I_C = \{t \in I |\ C_t \text{ is a unit}\}$. Whenever there is an element i in the intersection of any of these sets, we apply a trick by adjusting the powers of the element $(x_i - 1)$ to get an element contradicting the nilpotency index of $\Delta(P)$. Now the following mutually exclusive cases may arise:

CASE (I) $I_C \cap I_B \neq \emptyset$. We consider the following mutually exclusive subcases:

(a) $I_C \cap I_B \cap I_A \neq \emptyset$. Let $r \in I_C \cap I_B \cap I_A$. We examine the term

$$\Big\{(x_r-1)^{p-7}\prod_{\substack{i=1\\ i\neq r}}^{k}(x_i-1)^{p-1}\Big\}M,$$

and find that $M \in J^7$ would imply

$$A_r B_r^2 C_r^3 \; (x_1-1)^{p-1}(x_2-1)^{p-1}\dots(x_k-1)^{p-1} \in J^{kp-k-6} \; J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_r B_r^2 C_r^3$ is a unit in KG.

(b) $I_C \cap I_B \cap I_A = \emptyset$. Pick $m \in I_C \cap I_B$ and $m_A \in I_A$, so $m \notin I_A$. We examine the term

$$\left\{ (x_m-1)^{p-6}(x_{m_A}-1)^{p-2} \prod_{\substack{i=1 \\ i\neq m,m_A}}^{k} (x_i-1)^{p-1} \right\} M$$

and find that $M \in J^7$ would imply

$$A_{m_A} B_m^2 C_m^3 \; (x_1-1)^{p-1}(x_2-1)^{p-1}\dots(x_k-1)^{p-1} \in J^{kp-k-6} \; J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_{m_A} B_m^2 C_m^3$ is a unit in KG.

CASE (II) $I_C \cap I_B = \emptyset$. Again, this has the following subcases:

(a) $I_C \cap I_A \neq \emptyset$. Pick $l \in I_C \cap I_A$ and $l_B \in I_B$, so $l \notin I_B$ and $l_B \notin I_C$. We examine the term

$$\left\{ (x_l-1)^{p-5}(x_{l_B}-1)^{p-3} \prod_{\substack{i=1 \\ i\neq l,l_B}}^{k} (x_i-1)^{p-1} \right\} M$$

and find that $M \in J^7$ would imply

$$A_l B_{l_B}^2 C_l^3 \; (x_1-1)^{p-1}(x_2-1)^{p-1}\dots(x_k-1)^{p-1} \in J^{kp-k-6} \; J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_l B_{l_B}^2 C_l^3$ is a unit in KG.

(b) $I_C \cap I_A = \emptyset$. This has the following subcases.

(i) $I_B \cap I_A \neq \emptyset$. Let $n \in I_B \cap I_A$, and $n_C \in I_C$, so $n \notin I_C$ and $n_C \notin I_A \cup I_B$. We examine the term

$$\left\{(x_n - 1)^{p-4}(x_{n_C} - 1)^{p-4} \prod_{\substack{i=1 \\ i \neq n, n_C}}^{k} (x_i - 1)^{p-1}\right\} M$$

and find that $M \in J^7$ would imply

$$A_n B_n^2 C_{n_C}^3 \, (x_1 - 1)^{p-1}(x_2 - 1)^{p-1} \ldots (x_k - 1)^{p-1} \in J^{kp-k-6} \, J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_n B_n^2 C_{n_C}^3$ is a unit in KG.

(ii) $I_B \cap I_A = \emptyset$, so that I_A, I_B and I_C are pairwise disjoint. Let $d \in I_A$, $e \in I_B$ and $f \in I_C$. Examining the term

$$\left\{(x_d - 1)^{p-2}(x_e - 1)^{p-3}(x_f - 1)^{p-4} \prod_{\substack{i=1 \\ i \neq d, e, f}}^{k} (x_i - 1)^{p-1}\right\} M$$

and find that $M \in J^7$ would imply

$$A_d B_e^2 C_f^3 \, (x_1 - 1)^{p-1}(x_2 - 1)^{p-1} \ldots (x_k - 1)^{p-1} \in J^{kp-k-6} \, J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_d B_e^2 C_f^3$ is a unit in KG.

Therefore, $V_1 \not\equiv 1 \pmod{J^7}$, which implies that V_1 is a nontrivial element in $(U^{(3)}, U') = 1$, a contradiction to our given condition. So, when P is elementary abelian, we get that $G = P \times H$.

Now, let P be any p-group. Assume $(P, h) \neq 1$ for some $h \in H$. As the Frattini subgroup $\Phi(P)$ is a characteristic subgroup of P, we have $h\Phi(P) = \Phi(P)$ and hence h induces an automorphism on $P/\Phi(P)$. Now $P/\Phi(P)$ is elementary abelian (by 1.16). We have already proved that h induces the identity automorphism on the elementary abelian group $P/\Phi(P)$. Hence by

Theorem 1.17, h induces the identity automorphism on P as well. Hence we get, $G = P \times H$.

□

Proposition 2.10. *Let Char* $K = p \geq 17$ *and let* G *be a finite* p*-group such that* U *satisfies* $(U^{(3)}, U') = 1$*, then* G *is abelian.*

Proof. A finite p-group is nilpotent and torsion. If G is non-abelian then by Result 2.6, the derived length of U for $p \geq 17$ is $\lceil \log_2(p+1) \rceil \geq \lceil \log_2(17+1) \rceil \approx \lceil 4.16 \rceil = 5$. Thus U can only satisfy $(U^{(3)}, U') = 1$, if G is abelian.

□

- **Conclusion:**

Combining lemma 2.9 and Proposition 2.10, we find that when $Char K \geq 17$ and G is a group of odd order such that U satisfies $(U^{(3)}, U') = 1$, then G is abelian. Hence, Theorem 2.7 is proved.

Chapter 3

Derived Length of Units and Commutativity of a Group Algebra

3.1 Introduction

In this chapter, we further extend our result obtained in the previous chapter. Our result extends the result of Catino and Spinelli (Theorem 2.6) from torsion nilpotent groups to groups for the case when derived length of $U = U(KG)$ is four. In chapter 2, we discussed a special case when $(U^{(3)}, U') = 1$ and G is of odd order using a combinatorial argument after considering several cases and subcases. In the present chapter we resolve it completely for any G with U of derived length four without any condition on the order of G. The present argument is far simpler and in particular, it avoids the combinatorial argument altogether. Our main result can be stated as follows:

Theorem 3.1. *Let K be a field of characteristic $p \geq 17$ and let G be finite group. Then G is abelian if and only if U satisfies $U^{(4)} = 1$.*

We also prove commutativity of the group G of odd order pm $[(p, m) = 1]$ when the derived length of the unit group U is small compared to the characteristic p of the field K.

Theorem 3.2. *Let K be a field of characteristic p and G be any group of odd order pm where m is co-prime to p. If the derived length of U is strictly less than $\lceil \log_2(2p) \rceil$, then G must be abelian.*

We denote the Frattini subgroup of a group G by $\Phi(G)$, which is the intersection of all maximal subgroups of G. It is well-known that $\Phi(G)$ is a characteristic subgroup of G. We can easily extend theorem 3.2 to any group G of odd order $p^n m$ with $(p, m) = 1$ provided the quotient of its p-Sylow subgroup by the Frattini subgroup is cyclic.

Theorem 3.3. *Let K be a field of characteristic p and G be any group of odd order $p^n m$ where m is co-prime to p. Let P be a p-Sylow subgroup and $\Phi(P)$ be the Frattini subgroup of P. If the quotient $P/\Phi(P)$ is cyclic and the derived length of U is strictly less than $\lceil \log_2(2p) \rceil$, then G must be abelian.*

3.2 Key Steps in the Proof of Theorem 3.1

We first consider the simpler case when G is a finite p-group. Then we briefly outline the key steps that we are going to adopt when G is not a finite p-group.

G is a finite p-group

From the result of Catino and Spinelli we know that, if KG is a non-commutative group algebra of a torsion nilpotent group G over a field K of positive characteristic p such that U is solvable, then the derived length of U is at least $\lceil \log_2(p+1) \rceil$. In our case KG is a group algebra such that $Char\ K = p \geq 17$ and G is a finite p-group such that U satisfies $U^{(4)} = 1$. A finite p-group is nilpotent and torsion. If G is non-abelian then by Catino and Spinelli's result, the derived length of U for $p \geq 17$ is $\lceil \log_2(p+1) \rceil \geq \lceil \log_2(17+1) \rceil \approx \lceil 4.16 \rceil = 5$. Thus U can only satisfy $U^{(4)} = 1$, if G is abelian.

G is not a finite p-group

Consider a group G which is not a p-group such that the group U of units in KG satisfies $U^{(4)} = 1$ and $char(K) = p \geq 17$. We will proceed as follows.

(i) First we will show that G can be written as a semidirect product of P and H where P is a p-group and H is an abelian p'-group.

(ii) Our aim is to then express G as a direct product of P and H. If not, then there will exist an element h in H that will induce a non-identity p'-automorphism on P.

(iii) When P is elementary abelian, we will show that the non-identity p'- automorphism on P induced by h can be used to construct a non trivial element in $U^{(4)}$.

(iv) We will then reduce the argument for general P to the case when P is elementary abelian by exploiting the Frattini subgroup as follows.

When P is any p-group, let us assume $(P, h) \neq 1$ for some $h \in H$. As the Frattini subgroup $\Phi(P)$ is a characteristic subgroup of P, we have $h\Phi(P) = \Phi(P)$ and hence h induces an automorphism on $P/\Phi(P)$. Now $P/\Phi(P)$ is elementary abelian (by definition 1.16). Now, we have already proved that h induces the identity automorphism on the elementary abelian group $P/\Phi(P)$. There is a well known result by Burnside (Theorem 1.17) which states that if ψ is a p'-automorphism of a p-group P, which induces the identity on $P/\Phi(P)$, then ψ is the identity automorphism on P.

Hence in our case, using the above result we conclude that h induces the identity automorphism on P as well. Thus we get, $G = P \times H$.

(v) The fact that P will also be abelian now easily follows from the above subcase regarding p-groups. Thus the sufficient part of the main result is proved.

(vi) The necessary part, that is, when G is abelian is a trivial case.

3.3 Proof of Theorem 3.1

In this section, we will provide proofs for the steps outlined in §3.2. In the first subsection, we will show that G is a semi-direct product of a p-group P and an abelian p'-group H. If that semi-direct product is not a direct product, then we will construct a non-trivial element in $U^{(4)} = (U^{(3)}, U^{(3)})$ in the second subsection. We may assume P to be elementary abelian as we indicated in §3.2.

Proof of the fact that G is a semidirect product of a p-group and an abelian p'-group

Let KG be the group algebra of a finite group G over a field K of characteristic $p \geq 5$, such that, the unit group $U = U(KG)$ satisfies the condition $U^{(4)} = 1$. Then U is solvable and according to Theorem 2.1, $G/O_p(G)$ is abelian.

So $O_p(G) = P$, a Sylow p-subgroup. Now, $|P|$ and $[G : P]$ are relatively prime, hence by Schur-Zassenhaus Theorem (1.14), we have $G = P \rtimes H$, where H is a p'-prime subgroup of G. Also, by the above conditions, H is abelian.

Proof of the fact that G will be abelian

In this subsection we want to prove that G will be abelian, that is, G will be a direct product the above p-group and the abelian p'-group. We first prove the following lemma, which is the most crucial ingredient in establishing theorem 3.1.

Lemma 3.4. *Let Char $K = p \geq 17$. Let G be a finite group. Suppose that $U = U(KG)$ satisfies $U^{(4)} = 1$. Then $G = P \times H$, where P is a p-group and H is an abelian p'-group, where p' is odd.*

We now outline the strategy for the proof of the above lemma. First observe that we may assume P to be elementary abelian by 3.2 (iv). We know from the last paragraph of §3.2 that $G = P \rtimes H$, where P is a p-group and H is an abelian p'-group. Also $P \trianglelefteq G$. We need to show that $(P, H) = 1$. We will show that if $(P, H) \neq 1$, then we can construct nontrivial element in $U^{(4)}$. Clearly it will be enough to show non-triviality modulo a suitable power of the Jacobson radical J.

Construction of a nontrivial element in $U^{(4)}$ when $(P, H) \neq 1$

Note that by theorem 1.15, $(P, H) = (P, H, H)$ and $(P, H) \leq P$ as P is normal in G. If $(P, H) = (P, H, H) \neq 1$, then there exists $x \in (P, H) \subset G'$ and $h \in H$ such that $(x, h) \neq 1$. In the rest of this section, we will show that $(x, h) \neq 1$ results in a nontrivial element u_4 in $U^{(4)}$. Let x_i denote $\big(x, \underbrace{h, h, \ldots, h}_{i \text{ times}}\big)$, for $i = 1, 2, \ldots$.

As $x-1 \in \Delta(P)$ is contained in Jacobson radical J, $u = 1 + h(x-1)$ is a unit in KG. We now proceed to form commutators of suitable elements in U and obtain elements in $U', U^{(2)}, U^{(3)}$ and finally in $U^{(4)}$. We first consider $u_1 = (u,h) \in U'$ and $v_1 = (u,x) \in U'$ and construct elements in $U^{(2)}$ using u_1, v_1, x and h. We also keep track of their behavior modulo a suitable power of the Jacobson radical J. We begin by observing u_1 and v_1 modulo J^2.

$$\begin{aligned} u_1 &= (u,h) = 1 + u^{-1}h^{-1}[u,h] \\ &\equiv 1 + (1 - h(x-1))((x-1)h - h(x-1)) \quad (mod\ J^2) \\ &\equiv 1 + (x-1)h - h(x-1) \quad (mod\ J^2) \\ &= 1 + hx((x,h)-1) \quad (mod\ J^2) \\ &\equiv 1 + h(x_1 - 1) \quad (mod\ J^2) \end{aligned} \tag{3.1}$$

As $x-1 \in J$, we use identity 2.2 to obtain the following:

$$\begin{aligned} v_1 &= (u,x) \equiv 1 + [u,x] \quad (mod\ J^3) \\ &= 1 + (x + h(x-1)x - x - xh(x-1)) \quad (mod\ J^3) \\ &= 1 + (hx - xh)(x-1) \quad (mod\ J^3) \\ &= 1 + hx(1 - (x,h))(x-1) \quad (mod\ J^3) \\ &\equiv 1 - h(x-1)(x_1 - 1) \quad (mod\ J^3) \end{aligned} \tag{3.2}$$

Next we consider $u_2 = (u_1, x)$ and $v_2 = (v_1, x)$. As $x \in (P,H) \subset G' \subset U'$, u_2 and v_2 are in $U^{(2)}$. Their residual properties modulo J can be observed as follows.

$$\begin{aligned} u_2 &= (u_1,x) \equiv 1 + [u_1,x] \quad (mod\ J^3) \\ &= 1 + \{x + h(x_1-1)x - x - xh(x_1-1) \\ &= 1 + (hx - xh)(x_1 - 1) \quad (mod\ J^3) \\ &= 1 + hx(1 - (x,h))(x_1 - 1) \quad (mod\ J^3) \\ &\equiv 1 - h(x_1-1)^2 \qquad (mod\ J^3) \end{aligned} \tag{3.3}$$

$$\begin{aligned} v_2 &= (v_1,x) \equiv 1 + [v_1,x] \quad (mod\ J^4) \\ &= 1 - \big\{h(x-1)(x_1-1)x - xh(x-1)(x_1-1) \\ &= 1 + (xh - hx)(x-1)(x_1-1) \quad (mod\ J^4) \\ &\equiv 1 + h(x_1-1)^2(x-1) \quad (mod\ J^4) \end{aligned} \tag{3.4}$$

Finally we obtain an element $u_3 = (u_2, v_2)$ in $U^{(3)}$. We have

$$\begin{aligned}
u_3 &= (u_2, v_2) \equiv 1 + [u_2, v_2] \quad (mod\ J^6) \\
&= 1 + \{ -h(x_1-1)^2 h(x_1-1)^2(x-1) \\
&\qquad + h(x_1-1)^2(x-1)h(x_1-1)^2\} \ (mod\ J^6) \\
&= 1 + h(x_1-1)^2\{(x-1)h \\
&\qquad - h(x-1)\}(x_1-1)^2 \qquad (mod\ J^6) \\
&= 1 + h(x_1-1)^2 hx(x_1-1)^3 \quad (mod\ J^6) \\
&\equiv 1 + h^2\big(x_1^h - 1\big)^2\big(x_1-1\big)^3 \quad (mod\ J^6) \qquad (3.5)
\end{aligned}$$

With $u_2 \in U^{(2)} \subseteq U'$ and $x \in U'$, we obtain $w_2 = (u_2, x) \in (U', U') = U^{(2)}$ and $v_3 = (u_2, w_2) \in (U^{(2)}, U^{(2)}) = U^{(3)}$. Noting that $x_1 \equiv 1 \ (mod\ J)$, we examine w_2 and v_3 modulo powers of J.

$$\begin{aligned}
w_2 &= (u_2, x) \\
&\equiv 1 + [1 - h(x_1-1)^2, x] \quad (mod\ J^4) \\
&= 1 - hx(x_1-1)^2 + xh(x_1-1)^2 \ (mod\ J^4) \\
&\equiv 1 + h(x_1-1)^3 \quad (mod\ J^4) \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
v_3 &= (u_2, w_2) \\
&\equiv 1 + [1 - h(x_1-1)^2, 1 + h(x_1-1)^3] \quad (mod\ J^6) \\
&= 1 + h(x_1-1)^3 h(x_1-1)^2 - h(x_1-1)^2 h(x_1-1)^3 \ (mod\ J^6) \\
&= 1 + h(x_1-1)^2\{(x_1-1)h - h(x_1-1)\}(x_1-1)^2 \ (mod\ J^6) \\
&\equiv 1 + h(x_1-1)^2 h\big((x_1,h)-1\big)(x_1-1)^2 \ (mod\ J^6) \\
&= 1 + h^2(x_1^h-1)^2(x_2-1)(x_1-1)^2 \ (mod\ J^6) \qquad (3.7)
\end{aligned}$$

From $u_3 \in U^{(3)} \subseteq U^{(2)}$ and $u_2 \in U^{(2)}$, we obtain $w_3 = (u_3, u_2) \in (U^{(2)}, U^{(2)}) = U^{(3)}$ and view it modulo J^8.

$$\begin{aligned}
w_3 &= (u_3, u_2) \\
&\equiv 1 + [1 + h^2\big(x_1^h-1\big)^2\big(x_1-1\big)^3, 1 - h(x_1-1)^2] \\
&= 1 + h(x_1-1)^2 h^2(x_1^h-1)^2(x_1-1)^3 - h^2(x_1^h-1)^2(x_1-1)^3 h(x_1-1)^2 \\
&= 1 + \big\{h^2(x_1^h-1)^2 h(x_1^h-1)^2(x_1-1) - h^2(x_1^h-1)^2(x_1-1)^3 h\big\}(x_1-1)^2 \\
&= 1 + \big\{h^3(x_1^{h^2}-1)^2(x_1^h-1)^2(x_1-1) - h^3(x_1^{h^2}-1)^2(x_1^h-1)^3\big\}(x_1-1)^2
\end{aligned}$$

$$\begin{aligned}
&= 1 + h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2\{x_1 - x_1^h\}(x_1 - 1)^2 \\
&= 1 + h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2 x_1\big(1 - (x_1, h)\big)(x_1 - 1)^2 \\
&\equiv 1 - h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_2 - 1)(x_1 - 1)^2 \qquad (3.8)
\end{aligned}$$

Finally we construct $u_4 = (v_3, w_3)$ in $U^{(4)}$ and examine it modulo J^{13}. Note that $x_2 \equiv 1 \ (mod\ J)$.

$$\begin{aligned}
u_4 &= (v_3, w_3) \equiv 1 + [v_3 - 1, w_3 - 1] \\
&\equiv 1 + [h^2\big(x_1^h - 1\big)^2\big(x_1 - 1\big)^2(x_2 - 1), -h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)] \\
&= 1 - h^2\big(x_1^h - 1\big)^2\big(x_1 - 1\big)^2(x_2 - 1)h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&\qquad + h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)h^2\big(x_1^h - 1\big)^2\big(x_1 - 1\big)^2(x_2 - 1) \\
&= 1 - \big\{h^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)h^3\big(x_1^{h^2} - 1\big)^2 \\
&\qquad - h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)h^2\big\}(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&= 1 - \big\{h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_2^{h^3} - 1)\big(x_1^{h^2} - 1\big)^2 \\
&\qquad - h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_1^{h^2} - 1)^2(x_2^{h^2} - 1)\big\}(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&= 1 - \big\{h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2\big(x_1^{h^2} - 1\big)^2\big\}(x_2^{h^3} - x_2^{h^2})(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&= 1 - \big\{h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2\big(x_1^{h^2} - 1\big)^2\big\}h^{-2}x_2((x_2, h) - 1)h^2 \\
&\qquad (x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&\equiv 1 - h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_1^{h^2} - 1)^2\big(x_3^{h^2} - 1\big)(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&\qquad\qquad (3.9)
\end{aligned}$$

As $x_1^{h^j}$ is a conjugate of x_1 and $(x_1 - 1) \in J - J^2$, it follows that $(x_1^{h^j} - 1) \in J - J^2$. Further, x_2 is an element of order p in P as $x_2 = (x_1, h) = (x, h, h) = 1$ would imply $x_1 = (x, h) = 1$ by an obvious application of theorem 1.15 with $Q = \langle x \rangle$ and $A = \langle h \rangle$. We claim $x_2 - 1 \notin J^2$. Suppose $x_2 = y_1^{e_1} y_2^{e_2} \ldots y_k^{e_k}$, where $y_1, y_2, \ldots, y_k$ are independent generators of P_i for the elementary abelian group $P = P_1 \times P_2 \times \cdots \times P_k$ and $e_1, e_2, \ldots, e_k$ are integers such that $1 \leq e_i \leq p$ for every $i = 1, 2, \ldots, k$, with at least one e_i being greater than or equal to one but strictly less than p. Now, For any natural number d, we have

$$\begin{aligned}
x^d - 1 &= (x - 1)\{1 + ((x - 1) + 1) + \ldots + ((x^{d-1} - 1) + 1)\} \\
&= (x - 1)(d + D_1) \qquad \text{(where } D_1 \in \Delta(P) \subseteq J) \\
&\equiv d(x - 1) \ \ (mod\ J^2) \qquad (3.10)
\end{aligned}$$

With the help of 3.10, we can write $x_2 - 1 \equiv e_1(y_1 - 1) + e_2(y_2 - 1) + \cdots + e_k(y_k - 1) \pmod{J^2}$. Without loss of generality, we can take y_1 to be the generator such that $y_1^{e_1} \neq 1$, that is, $1 \leq e_1 < p$. Now if we multiply the element $V = (y_1 - 1)^{p-2}(y_2 - 1)^{p-1} \ldots (y_k - 1)^{p-1}$ with $x_2 - 1$, then we get $V \times (x_2 - 1) \in J^{kp-k-1} \times J^2 = J^{kp-k+1} = \{0\}$ as the nilpotency index of J is $kp-k+1$ by theorem 1.24 and corollary 1.23. But $V \times (x_2 - 1) \equiv e_1(y_1 - 1)^{p-1}(y_2 - 1)^{p-1} \ldots (y_k - 1)^{p-1} = \{0\}$ which contradicts the nilpotency index of $\Delta(P)$. Therefore, it follows that $(x_2 - 1) \in J - J^2$. By the same argument, $(x_3 - 1) \in J - J^2$. Therefore the product on the right hand side in (3.9) belongs to $J^{12} - J^{13}$. Consequently by (3.9), u_4 is nontrivial modulo J^{13} which contradicts our assumption that $U^{(4)} = 1$. Therefore, we can not have $(x, h) \neq 1$ which we have assumed for the construction of u_4. It follows that H must act trivially on P. Therefore, G must be a direct product of P and H.

3.4 When the Derived Length of U is Smaller than $\lceil \log_2(2p) \rceil$

In this section we will mention how we can prove theorems 3.2 and 3.3. Let p be the characteristic of the field K and d be the derived length of the units $U(KG)$. As discussed in the beginning of §3.3, by theorem 2.1, we have $G/O_p(G)$ is abelian. Again if $G/O_p(G)$ is abelian then $O_p(G)$ is a Sylow p-subgroup of G by 1.13. Let $P = O_p(G)$. Now, $|P|$ and $[G : P]$ are relatively prime, hence by Schur-Zassenhaus Theorem (1.14) we can conclude that G has a normal p-Sylow subgroup P and $G = P \rtimes H$. Also we have H an abelian p'-subgroup of G. We are going to use the following result by Balogh and Li (Lemma 2.3, [36]).

Result 3.5. Let G be a group with derived subgroup $G' = \langle u | u^{p^n} = 1 \rangle$, where p is an odd prime, and let $char(K) = p$. Assume that the order of G/C, where $C = C_G(G')$=centralizer of G' in G, is divisible by an odd prime $q \neq p$. Then the derived length of U is greater than or equal to $\lceil log_2(2p^n) \rceil$.

Note that $C_G(G')$ is a normal subgroup of G since it contains G'.

3.4.1 Proof of Theorem 3.2

Suppose G is of odd order pm where $(m, p) = 1$. Then the p-Sylow subgroup P is cyclic of order p. We assume that the derived length d of $U(KG)$ satisfies $d < \lceil \log_2(2p) \rceil$. We want to show that $G = P \times H$, i.e., $(P, H) = \{1\}$. If possible, let $(P, H) \neq 1$. As P is normal in G, (P, H) is a subgroup of P and hence $(P, H) = P$. Therefore, we have $G' = (P, H) = P$.

We now apply the result 3.5 to G and $G' = P$. Let C be the centralizer of G', i.e., $C = \{x \in G \mid xy = yx \;\; \forall \;\; y \in G'\}$. Then $h \notin C$ and hC is a non-trivial element of G/C. Let l be the order of h, so l is coprime to p. Then hC is a nontrivial element of order dividing l in G/C and a suitable power of hC gives an element of prime order q in G/C. As $(l, p) = 1$ and the order of G is odd, q must be an odd prime other than p. By result 3.5, we can conclude that the derived length of $U(KG)$ must be at least $\lceil \log_2(2p) \rceil$. But it contradicts our assumption that the derived length of $U(KG)$ is smaller than $\lceil \log_2(2p) \rceil$. Hence, we can not have $(P, H) \neq 1$ and the proof of the theorem is complete.

3.4.2 Proof of theorem 3.3

Assume that G is of odd order $p^n m$, and the derived length d of $U(KG)$ satisfies $d < \lceil \log_2(2p) \rceil$. As before, the p-Sylow subgroup P of G is normal and $G = P \rtimes H$ where H an abelian p'-subgroup of G. By part (iv) of §2.2, it is enough to show that the induced conjugacy action of H on the group $P/\Phi(P)$ is trivial. Now $P/\Phi(P)$ is an elementary abelian group which is assumed to be cyclic in addition. Therefore $P/\Phi(P)$ is cyclic of order p and by theorem 3.2, we know that the conjugacy action of H on $P/\Phi(P)$ has to be trivial. Hence the conjugacy action of H on P itself is trivial and G must be abelian. $\square$

We conclude by observing that a group G of odd order with a cyclic p-Sylow subgroup must be abelian if the derived length of $U(KG)$ is smaller than $\lceil \log_2(2p) \rceil$ where p is the characteristic of the field K.

Chapter 4

Strongly Lie Solvable Group Algebras

Recall the discussions in section 1.2.3. A classical result of Passi, Passman and Sehgal ([21]) says that when $p > 2$ and G is a non-abelian group, the group algebra KG is Lie solvable if and only if KG is strongly Lie solvable. Unfortunately no general formula to compute the Lie derived length $dl_L(KG)$ is known, whereas it is possible to give a more accurate estimate of $dl^L(KG)$. Levin and Rosenberger in ([17]) have completely characterised KG with $dl_L(KG) = 2$. Sahai in ([29])has characterised KG with $dl_L(KG) = 3$. In fact as a consequence of results of Sahai ([29]) one has that

$$\lceil log_2(t(G') + 1)\rceil \leq dl^L(KG) \leq \lceil log_2(2t(G'))\rceil, \tag{4.1}$$

where $t(G')$ is the nilpotency index of the augmentation ideal $\Delta(G')$. Sahai's characterization of KG with $dl^L(KG) = 3$ was as follows:

Theorem 4.1. *Let K be a field of characteristic $p \neq 2$ and let G be a group. Then $\delta^{(3)}(KG) = 0$ if and only if one of the following holds:*

(i) G is abelian.

(ii) $p = 7$, $G' = C_7$ and $\gamma_3(G) = 1$.

(iii) $p = 5$, $G' = C_5$ and either $\gamma_3(G) = 1$ or $\gamma_n(G) = G'$ for all $n \geq 3$ with $x^g = x^{-1}$ for all $x \in G'$ and for all $g \notin C_G(G')$.

(iv) $p = 3$, G' *is a group of one of the following types:*

(a) $G' = C_3$.

(b) $G' = C_3 \times C_3$ *and either* $\gamma_3(G) = 1$ *or* $\gamma_3(G) = C_3$, $\gamma_4(G) = 1$ *or* $\gamma_n(G) = G'$ *for all* $n \geq 3$ *with* $x^g = x^{-1}$ *for all* $x \in G'$ *and for all* $g \notin C_G(G')$.

(c) $G' = C_3 \times C_3 \times C_3$, $\gamma_3(G) = 1$.

where the centralizer of a subset S of a group G is denoted by $C_G(S)$, that is, $C_G(S) = \{g \in G | sg = gs \ \forall \ s \in S\}$.

According to Theorem A of Shalev's paper([5]), for a field K of characteristic $p > 0$ and a non-abelian group G, $\lceil log_2(p+1) \rceil \leq dl_L(KG)$. Very simple calculations allow to conclude that this lower bound holds true for the strong Lie derived length of a group algebra as well ([32],[15]). Along this way, the group algebras whose strong Lie derived length is exactly $\lceil log_2(p+1) \rceil$ have been characterised by Balogh and Juhász ([34],[35]). This was also proved independently by Spinelli ([14], [**?**]). The conditions for group algebras to have minimal strong Lie derived length given by Balogh and Juhász in [35] was as follows:

Theorem 4.2. *Let* KG *be a strongly Lie solvable group algebra of positive characteristic* p*. Then* $dl^L(KG) = \lceil log_2(p+1) \rceil$ *if and only if one of the following conditions holds:*

(i) $p = 2$ *and* G' *is central elementary abelian subgroup of order* 4*;*

(ii) G' *has order* p, $G/C_G(G')$ *has order* $2^m p^r$*, and the minimal integer* d *such that* $s(d, m) \geq p$ *satisfies the inequality* $2^d - 1 < p$.

where for $m \geq 0$, $s(l, m)$ *is defined as:*

$$s(l,m) = \begin{cases} 1, & \textit{if} \quad l = 0; \\ 2s(l-1,m)+1, & \textit{if} \quad 2^m | s(l-1,m); \\ 2s(l-1,m), & \textit{otherwise}. \end{cases}$$

A simple consequence of all these results is:

- For $p > 13$, there are no strongly Lie solvable group algebras of strong Lie derived length 4. (by Shalev's bounds or simple observation in [15]).
- For $p \in \{11, 13\}$ strongly Lie solvable group algebras of strong Lie derived length 4 coincide with those of minimal strong Lie derived length (by [34], [35]).

So the problem is reduced to characteristics $p = 3, 5, 7$.

Now, in this chapter, we have given an independent proof of the characterization of KG when $dl^L(KG) \leq 4$ which does not involve Lie derived series of KG for the cases $p \geq 11$. For the cases $p = 3, 5, 7, 11$, we have given sufficient conditions for KG to be strongly Lie solvable with strong Lie derived length 4. For $p = 7$, we have given the necessary conditions also. Also we have given some generalized results.

As mentioned in Chapter 1, for subsets X, Y of a group G, we denote by (X, Y) the subgroup of G generated by all commutators $(x, y) = x^{-1}y^{-1}xy$ with $x \in X$ and $y \in Y$. The derived subgroups of G are defined as $G^{(0)} = G$, $G^{(1)} = G' = (G, G)$, and $G^{(i)} = (G^{(i-1)}, G^{(i-1)})$ for all $i > 0$. The lower central chain of G is defined by $\gamma_1(G) = G$, $\gamma_{n+1}(G) = (\gamma_n(G), G)$ for all $n \geq 1$. For any two elements $x, h \in G$, x^h denotes the conjugation of x by h, that is, $h^{-1}xh$. Also, $\Delta(G)$ denotes the augmentation ideal of the group algebra KG. All groups considered are finite. Our main results in this chapter are as follows, where theorems 4.3 and 4.4 $(I), (II)$ are already existing results as mentioned above but proved independently. Theorems 4.4 (III) and 4.5 are new results.

Theorem 4.3. *Let K be a field of characteristic $p \geq 13$ and let G be a group. Then $\delta^{(4)}(KG) = (0)$ if and only if one of the following holds:*

(i) G is abelian.

(ii) $p = 13$, $G' = C_{13}$ and $\gamma_3(G) = 1$.

Theorem 4.4. *Let K be a field of characteristic $7 \leq p \leq 11$ and let G be a group. If $\delta^{(4)}(KG) = (0)$, then one of the following holds:*

(I) G is abelian.

(II) $p = 11$, $G' = C_{11}$ *and either* $\gamma_3(G) = 1$ *or* $\gamma_n(G) = G'$ *for all* $n \geq 3$ *with* $x^g = x^i$, $i = 1, 3, 4, 5, 9$, $\forall g \in G$.

(III) $p = 7$, *then* G' *is any one of the following:*

(a) $G' = C_7$.

(b) $G' = C_7 \times C_7 = \langle x \rangle \times \langle y \rangle$ *and one of the following holds:*

(i) $\gamma_3(G) = 1$.

(ii) $\gamma_3(G) = C_7$ *with* $\gamma_4(G) = 1$.

(iii) $\gamma_n(G) = G'$, *for all* $n \geq 3$ *such that for all* $g \in G \backslash C_G(G')$, $x^g \in \langle x \rangle$ *and* $y^g \in \langle y \rangle$ *and also* $C_G(x) = C_G(y)$.

Theorem 4.5. *Let* K *be a field of characteristic* $3 \leq p \leq 11$ *and let* G *be a group and any one of the following holds.*

(i) G *is abelian.*

(ii) G' *is cyclic with* $|G'| \leq 11$ *and* $\gamma_3(G) = 1$.

(iii) $\gamma_n(G) = G'$ *for all* $n \geq 3$ *such that*

(a) G' *is cyclic with* $G' = \langle x \rangle$, $x \neq 1$,

(b) $|G'| \leq 11$, *and*

(c) $C_G(x)$ *has index* 2 *in* G, *i.e., for all* $g \notin C_G(G')$, $x^g = x^i$, *for some fixed* i, *where* $1 < i < \circ(x)$,

then $\delta^{(4)}(KG) = (0)$.

4.1 Background

In this section we discuss a few important known results which provide useful tools for the proof of our theorem. We first state the complete set of necessary and sufficient conditions for KG to be Lie solvable, given by Passi, Passman and Sehgal ([21]). We say a group G is p-abelian, if G' is a finite p-group.

Theorem 4.6. *Let* KG *be the group algebra of* G *over the field* K *with characteristic* $p \geq 0$. *Then*

(i) *KG is Lie nilpotent if and only if G is p-abelian and nilpotent;*

(ii) *for $p \neq 2$, KG is Lie solvable if and only if G is p-abelian;*

(iii) *for $p = 2$, KG is Lie solvable if and only if G has a 2-abelian subgroup of index at most 2.*

The following two results were proved by Sahai ([29], Remark 2.1 and Lemma 2.2 and the paragraph before these).

Result 4.7. When Char $K \geq 3$, then

(i) $\delta^{(1)}(KG) = [KG, KG]KG = \Delta(G')KG$.

(ii) $\delta^{(2)}(KG) = [\Delta(G')KG, \Delta(G')KG]KG$
$= \Delta(G'')KG + \Delta(G')^3KG + \Delta(\gamma_3(G))\Delta(G')KG + \Delta(G')\Delta(\gamma_3(G))KG.$

(iii) if G' is central, then $\delta^{(2)}(KG) = \Delta(G')^3KG$.

(iv) if $\gamma_3(G) = G'$, then $\delta^{(2)}(KG) = \Delta(G')^2KG$.

Result 4.8. For all $s \geq 1$, $s \in \mathbb{N}$, $\Delta(G')^{2^s-1}KG \subseteq \delta^{(s)}(KG) \subseteq \Delta(G')^{2^{s-1}}KG$.

The following are very useful identities.

For $\delta_1, \delta_2, g_1, g_2 \in KG$,

$$\delta_1\delta_2[g_1, g_2] = [\delta_1 g_1, \delta_2 g_2] - [\delta_1, \delta_2 g_2]g_1 - [\delta_1 g_1, \delta_2]g_2 + [\delta_1, \delta_2]g_1 g_2. \tag{4.2}$$

For $g, h \in G$, we can write

$$[g^{-1}, h] = h\{(h, g) - 1\}g^{-1}; \qquad [g, h^{-1}] = g\{1 - (g, h)\}h^{-1}. \tag{4.3}$$

As for a prime p, $(-1)^i\, {}^{p-i}C_i \equiv 1 \pmod p$, another useful observation is that

$$(x - 1)^{p-1} = 1 + x + \ldots + x^{p-1} \qquad \text{in } \mathbb{F}_p[x]. \tag{4.4}$$

4.2 Main Results

Some Useful Results

We first prove the following propositions:

Proposition 4.9. *If* $(G', G) = \gamma_3(G) = G'$, *then for* $Char K \neq 2$, $\Delta(G')^5 KG \subseteq \delta^{(3)}(KG)$.

Proof. By Result 4.7, part (iii), when $(G', G) = G'$, then $\delta^{(3)}(KG) = [\Delta(G')^2 KG, \Delta(G')^2 KG]KG$. By Result 4.7, $\Delta(G')KG = [KG, KG]KG$. Using Eqn. 4.2, for all $\delta_1, \delta_2 \in \Delta(G')^2$ and $g_1, g_2 \in G$, we have:

$$\begin{aligned} \Delta(G')^5 KG &= \Delta(G')^2 \Delta(G')^2 \Delta(G')KG \\ &= \Delta(G')^2 \Delta(G')^2 [KG, KG]KG \\ &\subseteq \left[\Delta(G')^2 KG, \Delta(G')^2 KG\right] KG = \delta^{(3)}(KG) \end{aligned}$$

Hence, we have, $\Delta(G')^5 KG \subseteq \delta^{(3)}(KG)$.

□

Proposition 4.10. *For* $Char K \neq 2$, $\Delta(\gamma_3(G))\Delta(G')^5 KG \subseteq \delta^{(3)}(KG)$.

Proof. By Result 4.7, we know that $\Delta(\gamma_3(G))\Delta(G')KG$, $\Delta(G')^3 KG \subseteq \delta^{(2)}(KG)$. It is known that $\Delta(G')KG = [KG, KG]KG$. Using Eqn. 4.2, for all $\delta_1 \in \Delta(\gamma_3(G))\Delta(G')$, $\delta_2 \in \Delta(G')^3$ and $g_1, g_2 \in G$, we have:

$$\begin{aligned} \Delta(\gamma_3(G))\Delta(G')^5 KG &= \Delta(\gamma_3(G))\Delta(G')\Delta(G')^3 \Delta(G')KG \\ &= \Delta(\gamma_3(G))\Delta(G')\Delta(G')^3 [KG, KG]KG \\ &\subseteq \left[\Delta(\gamma_3(G))\Delta(G')KG, \Delta(G')^3 KG\right] KG \\ &\subseteq [\delta^{(2)}(KG), \delta^{(2)}(KG)]KG = \delta^{(3)}(KG) \end{aligned}$$

Hence, we have, $\Delta(\gamma_3(G))\Delta(G')^5 KG \subseteq \delta^{(3)}(KG)$.

□

Proposition 4.11. *When G' is central, that is, when $(G', G) = 1$, for $Char K \neq 2$, and the nilpotency index $t(G')$ of $\Delta(G')$ greater than n, we have:*

$$\big[\Delta(G')^n KG, \Delta(G')^n KG\big] KG \subseteq \Delta(G')^{2n+1} KG \tag{4.5}$$

Proof. Let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in G'$. Then for $g, h \in G$, $(x_1 - 1)(x_2 - 1) \dots (x_n - 1)g, (y_1 - 1)(y_2 - 1) \dots (y_n - 1)h \in \Delta(G')^n KG$. Now,

$$\begin{aligned}
&\big[(x_1 - 1)(x_2 - 1) \dots (x_n - 1)g, (y_1 - 1)(y_2 - 1) \dots (y_n - 1)h\big] \\
&= (x_1 - 1)(x_2 - 1) \dots (x_n - 1)g(y_1 - 1)(y_2 - 1) \dots (y_n - 1)h \\
&\quad - (y_1 - 1)(y_2 - 1) \dots (y_n - 1)h(x_1 - 1)(x_2 - 1) \dots (x_n - 1)g \\
&= (x_1 - 1) \dots (x_n - 1)(y_1 - 1) \dots (y_n - 1)gh \\
&\quad - (x_1 - 1) \dots (x_n - 1)(y_1 - 1) \dots (y_n - 1)hg \\
&= (x_1 - 1) \dots (x_n - 1)(y_1 - 1) \dots (y_n - 1)(ghg^{-1}h^{-1} - 1)hg \\
&\in \Delta(G')^{2n+1} KG.
\end{aligned}$$

Thus, this result can be extended over the whole group algebra KG and our desired result is proved. □

The above proposition can be extended to get the following result.

Lemma 4.12. *For $Char K \neq 2$, and the nilpotency index $t(G')$ of $\Delta(G')$ greater than n, we have*

$$\begin{aligned}
\big[\Delta(G')^n KG, \Delta(G')^n KG\big] KG \subseteq\ & \Delta(G', G)\Delta(G')^{2n-1} KG + \\
& \Delta(G')\Delta(G', G)\Delta(G')^{2n-2} KG + \cdots \\
& \cdots + \Delta(G')^{2n-1}\Delta(G', G) KG \\
& + \Delta(G')^{2n+1} KG + \Delta(G'')\Delta(G')^{2n-2} KG \\
& + \Delta(G')\Delta(G'')\Delta(G')^{2n-3} KG \\
& + \cdots\cdots + \Delta(G')^{2n-2}\Delta(G'') KG.
\end{aligned}$$

Proof. Let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in G'$. Then for $g, h \in G$, $(x_1 - 1)(x_2 - 1) \dots (x_n - 1)g, (y_1 - 1)(y_2 - 1) \dots (y_n - 1)h \in \Delta(G')^n KG$. Now,

$$
\begin{aligned}
&\big[(x_1-1)(x_2-1)\dots(x_n-1)g,(y_1-1)(y_2-1)\dots(y_n-1)h\big]\\
&=(x_1-1)(x_2-1)\dots(x_n-1)g(y_1-1)(y_2-1)\dots(y_n-1)h\\
&\qquad-(y_1-1)(y_2-1)\dots(y_n-1)h(x_1-1)(x_2-1)\dots(x_n-1)g\\
&=\Big\{(x_1-1)\dots(x_n-1)g(y_1-1)(y_2-1)\dots(y_n-1)h\\
&\qquad-(x_1-1)\dots(x_n-1)(y_1-1)g(y_2-1)\dots(y_n-1)h\\
&+(x_1-1)\dots(x_n-1)(y_1-1)g(y_2-1)\dots(y_n-1)h\\
&\qquad-(x_1-1)\dots(x_n-1)(y_1-1)(y_2-1)g\dots(y_n-1)h\\
&+\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad+\\
&(x_1-1)\dots(x_n-1)(y_1-1)\dots(y_{n-1}-1)g(y_n-1)h\\
&\qquad-(x_1-1)\dots(x_n-1)(y_1-1)\dots(y_n-1)gh\Big\}\\
&+\Big\{(x_1-1)\dots(x_n-1)(y_1-1)\dots(y_n-1)gh\\
&\qquad-(x_1-1)\dots(x_n-1)(y_1-1)\dots(y_n-1)hg\Big\}\\
&+\Big\{(x_1-1)\dots(x_n-1)h(y_1^h-1)\dots(y_n^h-1)g\\
&\qquad-(x_1-1)\dots(x_{n-1}-1)h(x_n-1)(y_1^h-1)\dots(y_n^h-1)g\\
&+(x_1-1)\dots(x_{n-1}-1)h(x_n-1)(y_1^h-1)\dots(y_n^h-1)g\\
&\qquad-(x_1-1)\dots(x_{n-2}-1)h(x_{n-1}-1)(x_n-1)(y_1^h-1)\dots(y_n^h-1)g\\
&+\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad+\\
&(x_1-1)h(x_2-1)\dots(x_n-1)(y_1^h-1)\dots(y_n^h-1)g\\
&\qquad-h(x_1-1)\dots(x_n-1)(y_1^h-1)\dots(y_n^h-1)g\Big\}\\
&+\Big\{(x_1^{h^{-1}}-1)\dots(x_n^{h^{-1}}-1)(y_1-1)\dots(y_n-1)hg\\
&\qquad-(x_1^{h^{-1}}-1)\dots(x_{n-1}^{h^{-1}}-1)(y_1-1)(x_n^{h^{-1}}-1)(y_2-1)\dots(y_n-1)hg\\
&+(x_1^{h^{-1}}-1)\dots(x_{n-1}^{h^{-1}}-1)(y_1-1)(x_n^{h^{-1}}-1)(y_2-1)\dots(y_n-1)hg\\
&\qquad-(x_1^{h^{-1}}-1)...(x_{n-2}^{h^{-1}}-1)(y_1-1)(x_{n-1}^{h^{-1}}-1)(x_n^{h^{-1}}-1)(y_2-1)...(y_n-1)hg\\
&+\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad+\\
&(x_1^{h^{-1}}-1)(y_1-1)(x_2^{h^{-1}}-1)\dots(x_n^{h^{-1}}-1)(y_2-1)\dots(y_n-1)hg\\
&\qquad-(y_1-1)(x_1^{h^{-1}}-1)\dots(x_n^{h^{-1}}-1)(y_2-1)\dots(y_n-1)hg\\
&+\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad+\\
&(y_1-1)\dots(y_{n-1}-1)(x_1^{h^{-1}}-1)\dots(x_n^{h^{-1}}-1)(y_n-1)hg\\
&\qquad-(y_1-1)\dots(y_{n-1}-1)(x_1^{h^{-1}}-1)\dots(x_{n-1}^{h^{-1}}-1)(y_n-1)(x_n^{h^{-1}}-1)hg\\
&+\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad+
\end{aligned}
$$

$$+ (y_1 - 1)\dots(y_{n-1} - 1)(x_1^{h^{-1}} - 1)(y_n - 1)(x_2^{h^{-1}} - 1)\dots(x_n^{h^{-1}} - 1)hg$$
$$- (y_1 - 1)(y_2 - 1)\dots(y_n - 1)h(x_1 - 1)(x_2 - 1)\dots(x_n - 1)g\Big\} \quad (4.6)$$

Now for $i = 1, \dots, n$, the first n-pairs of terms of Eqn. 4.6 are of the form:

$$(x_1 - 1)\dots(x_n - 1)(y_1 - 1)\dots(y_{i-1} - 1)g(y_i - 1)\dots(y_n - 1)h$$
$$- (x_1 - 1)\dots(x_n - 1)(y_1 - 1)\dots(y_{i-1} - 1)(y_i - 1)g\dots(y_n - 1)h$$
$$= (x_1 - 1)\dots(x_n - 1)(y_1 - 1)\dots(y_{i-1} - 1)[g, y_i](y_{i+1} - 1)\dots(y_n - 1)h$$
$$= -(x_1 - 1)...(x_n - 1)(y_1 - 1)...(y_{i-1} - 1)gy_i\Big((y_i, g) - 1\Big)(y_{i+1} - 1)...(y_n - 1)h$$
$$\in \Delta(G')^n\Delta(G')^{i-1}\Delta(G', G)\Delta(G')^{n-i}KG \qquad (\because G' \trianglelefteq G, (G', G) \trianglelefteq G).$$

The next one, i.e., the $(n+1)^{th}$ term: $(x_1 - 1)\dots(x_n - 1)(y_1 - 1)\dots(y_n - 1)(gh - hg) \in \Delta(G')^{2n}[KG, KG]KG \subseteq \Delta(G')^{2n+1}KG$. Again in the same way it can be shown that the next n-pairs of terms for $i = 1, \dots, n$, which are of the form:

$$(x_1 - 1)\dots(x_{i-1} - 1)h(x_i - 1)\dots(x_n - 1)(y_1^h - 1)\dots(y_n^h - 1)g$$
$$- (x_1 - 1)\dots(x_{i-1} - 1)(x_i - 1)h(x_{i+1} - 1)\dots(x_n - 1)(y_1^h - 1)\dots(y_n^h - 1)g$$
$$\in \Delta(G')^{i-1}\Delta(G', G)\Delta(G')^{n-i}\Delta(G')^nKG.$$

Finally, the last n^2-pairs can similarly be shown to be contained in $\Delta(G'')\Delta(G')^{2n-2}KG + \Delta(G')\Delta(G'')\Delta(G')^{2n-3}KG + \dots + \Delta(G')^{2n-2}\Delta(G'')KG$ using the fact that $G'' \trianglelefteq G' \trianglelefteq G$. Thus, this element can be extended over the whole group algebra KG and our desired result is proved.

□

Proof of Theorem 4.3

§ **Necessary conditions:**

We have $\delta^{(4)}(KG) = (0)$. As strong Lie solvability implies Lie solvability, therefore, by Theorem 4.6, we have G' is a finite p-group as char $K \neq 2$. Let $|G'| = p^l$. Putting $n = 4$ in Result 4.8, we get $\Delta(G')^{15}KG \subseteq \delta^{(4)}(KG) = (0) \subseteq \Delta(G')^8KG$. So we get. $\Delta(G')^{15}KG = (0)$. Hence, the nilpotency index $t(G')$ of $\Delta(G')$ is less than or equal to 15. We consider the following cases:

Case (i) *When Char* $K = p \geq 17$*:*
In Theorem 1.24, putting $p \geq 17$, we get, $16l + 1 \leq t(G') \leq 17^l$. But $t(G') \leq 15$. Therefore, l must be 0. So, for $p \geq 17$, $|G'| = 1$. Hence, G is abelian.

Case (ii) *When Char* $K = p = 13$*:*
Again, in Theorem 1.24, putting $p = 13$, we get, $12l + 1 \leq t(G') \leq 13^l$. As $t(G') \leq 15$, we must have $l = 0$ or $l = 1$. Now, $l = 0$ again implies that G is abelian. When $l = 1$, for $p = 13$, we have $|G'| = 13$. Hence, $G' = C_{13}$.

Now, when G is abelian, we are through. So we discuss the non-abelian case.

<u>*Claim:*</u> G' is central, i.e., $\gamma_3(G) = (G', G) = 1$.

Let if possible, $(G', G) \neq 1$. Since, $G' = C_{13}$ and $G' \subseteq (G', G)$, we have $(G', G) = G'$. Therefore, by Proposition 4.9, we have $\Delta(G')^5 KG \subseteq \delta^{(3)}(KG)$. Let $G' = \langle x \rangle$, where $\circ(x)$, order of x, is 13 and $(x-1)^{12} = 1 + x + x^2 + \cdots + x^{12} = \hat{x}$ in KG by (4.4). Hence, for any $g \in G$, we have, $(x-1)^5$, $(x-1)^5 g^{-1} \in \Delta(G')^5 KG \subseteq \delta^{(3)}(KG)$. Hence, $\big[(x-1)^5, (x-1)^5 g^{-1}\big] \in \big[\delta^{(3)}(KG), \delta^{(3)}(KG)\big]KG = \delta^{(4)}(KG) = (0)$. So, making use of the fact that for $x, y \in G$, $[x-1, g^{-1}] = [x, g^{-1}]$ and (4.3), we have:

$$
\begin{aligned}
0 &= \big[(x-1)^5, (x-1)^5 g^{-1}\big] \\
&= (x-1)^{10} g^{-1} - (x-1)^5 g^{-1} (x-1)^5 \\
&= (x-1)^9 \big\{(x-1)g^{-1} - g^{-1}(x-1)\big\} \\
&\qquad\qquad + (x-1)^5 \big\{(x-1)^4 g^{-1} - g^{-1}(x-1)^4\big\}(x-1) \\
&= (x-1)^9 \big[x-1, g^{-1}\big] + (x-1)^5 \Big\{(x-1)^3 \big[x-1, g^{-1}\big] \\
&\qquad + \big\{(x-1)^3 g^{-1} - g^{-1}(x-1)^3\big\}(x-1)\Big\}(x-1) \\
&= (x-1)^9 \big[x, g^{-1}\big] + (x-1)^8 \big[x, g^{-1}\big](x-1) \\
&\quad + (x-1)^5 \Big\{(x-1)^2 \big[x-1, g^{-1}\big] + \big\{(x-1)^2 g^{-1} \\
&\qquad\qquad - g^{-1}(x-1)^2\big\}(x-1)\Big\}(x-1)^2 \\
&= (x-1)^9 \big[x, g^{-1}\big] + (x-1)^8 \big[x, g^{-1}\big](x-1) + (x-1)^7 \big[x, g^{-1}\big](x-1)^2 \\
&\quad + (x-1)^5 \times \Big\{(x-1)\big[x-1, g^{-1}\big] \\
&\qquad\qquad + \big\{(x-1)g^{-1} - g^{-1}(x-1)\big\}(x-1)\Big\} \times (x-1)^3
\end{aligned}
$$

$$\begin{aligned}
&= (x-1)^9\big[x,g^{-1}\big] + (x-1)^8\big[x,g^{-1}\big](x-1) + (x-1)^7\big[x,g^{-1}\big](x-1)^2 \\
&\quad +(x-1)^6\big[x,g^{-1}\big](x-1)^3 + (x-1)^5\big[x,g^{-1}\big](x-1)^4 \\
&= -x\Big\{(x-1)^9\{(x,g)-1\}g^{-1} + (x-1)^8\{(x,g)-1\}g^{-1}(x-1) \\
&\qquad +(x-1)^7\{(x,g)-1\}g^{-1}(x-1)^2 + (x-1)^6\{(x,g)-1\}g^{-1}(x-1)^3 \\
&\qquad\quad +(x-1)^5\{(x,g)-1\}g^{-1}(x-1)^4\Big\} \\
&= -x\Big\{(x-1)^9\{(x,g)-1\} + (x-1)^8\{(x,g)-1\}(x^g-1) \\
&\quad +(x-1)^7\{(x,g)-1\}(x^g-1)^2 + (x-1)^6\{(x,g)-1\}(x^g-1)^3 \\
&\quad +(x-1)^5\{(x,g)-1\}(x^g-1)^4\Big\}g^{-1} \qquad (4.7)
\end{aligned}$$

That is, finally

$$(x-1)^9\{(x,g)-1\} + (x-1)^8\{(x,g)-1\}(x^g-1) + (x-1)^7\{(x,g)-1\}(x^g-1)^2 \\ + (x-1)^6\{(x,g)-1\}(x^g-1)^3 + (x-1)^5\{(x,g)-1\}(x^g-1)^4 = 0 \qquad (4.8)$$

Now, $(x,g) \in G' = C_{13}$. Let $(x,g) = x^k$, where $k \in \mathbb{N};\ 1 \le k \le 11$, as $k = 12$ would imply $x^{-1}g^{-1}xg = x^{12}$, i.e., $g^{-1}xg = x^{13} = 1$, which means $x = 1$, a contradiction to our assumption. Also, $x^g = g^{-1}xg = x^{k+1}$. Substituting $(x,g) = x^k$ and $x^g = x^{k+1}$ in Eq. 4.8, we get:

$$\begin{aligned}
0 &= (x-1)^9(x^k-1) + (x-1)^8(x^k-1)(x^{k+1}-1) \\
&\quad +(x-1)^7(x^k-1)(x^{k+1}-1)^2 + (x-1)^6(x^{k+1}-1)(x^{k+1}-1)^3 \\
&\quad +(x-1)^5(x^k-1)(x^{k+1}-1)^4 \\
&= (x-1)^{10}(1+x+\ldots+x^{k-1}) \\
&\quad +(x-1)^{10}(1+x+\ldots+x^{k-1})(1+x+\ldots+x^k) \\
&\quad +(x-1)^{10}(1+x+\ldots+x^{k-1})(1+x+\ldots+x^k)^2 \\
&\quad +(x-1)^{10}(1+x+\ldots+x^{k-1})(1+x+\ldots+x^k)^3 \\
&\quad +(x-1)^{10}(1+x+\ldots+x^{k-1})(1+x+\ldots+x^k)^4 \\
&= (x-1)^{10}(1+x+\ldots+x^{k-1}) \\
&\quad \times\Big\{1+(1+x+\ldots+x^k)(2+x+\ldots+x^k) \\
&\qquad +(1+x+\ldots+x^k)^3(2+x+\ldots+x^k)\Big\} \\
&= (x-1)^{10}(1+x+\ldots+x^{k-1}) \\
&\quad \times\Big\{1+(1+x+\ldots+x^k)\Big(1+(1+x+\ldots+x^k)^2\Big)(2+x+\ldots+x^k)\Big\}
\end{aligned} \qquad (4.9)$$

Now, multiplying Eq. 4.9 by $(x-1)^2$, we get:

$$\begin{aligned}
&\hat{x}(1+x+\ldots+x^{k-1})\\
&\times\Big\{1+(1+x+\ldots+x^k)\Big(1+(1+x+\ldots+x^k)^2\Big)(2+x+\ldots+x^k)\Big\}=0\\
&\text{or, } k\{1+(k+1)\big(1+(k+1)^2\big)(2+k)\}\hat{x}=0\\
&\text{i.e., } k\big(k^4+5k^3+10k^2+10k+5\big)=0 \qquad \text{in } K. \qquad (4.10)
\end{aligned}$$

Now, there is no $k\in\mathbb{N}$ such that $k^4+5k^3+10k^2+10k+5$ is divisible by 13 . So $k=0$. Hence, $(x,g)=x^0=1$. But g was an arbitrary element of G. Hence $(x,g)=1$ for every $g\in G$. Thus, $(G',G)=1$, i.e., G' is central. We have thus proved that when $\delta^{(4)}(KG)=(0)$, then for any field K with characteristic 13, G' is central.

§ **Sufficient conditions:**

Case (i) When G is abelian, then clearly $\delta^{(4)}(KG)=(0)$.

Case (ii) When Char $K=13$ and $G'=C_{13}$ with $\gamma_3(G)=(G',G)=1$ and $|G'|=13$, then by Result 4.7(iii), we have $\delta^{(2)}(KG)=\Delta(G')^3KG$. Thus,

$\delta^{(3)}(KG)=[\delta^{(2)}(KG),\delta^{(2)}(KG)]KG=[\Delta(G')^3KG,\Delta(G')^3KG]KG\subseteq\Delta(G')^6KG$. Also, as $G'=C_{13}$, we get $t(G')=13$, and so, $\Delta(G')^{13}KG=(0)$. Hence, using Proposition 4.11, we have

$$\begin{aligned}
\delta^{(4)}(KG) &= [\delta^{(3)}(KG),\delta^{(3)}(KG)]KG\subseteq[\Delta(G')^6KG,\Delta(G')^6KG]KG\\
&\subseteq \Delta(G')^{13}KG = (0)
\end{aligned}$$

Hence, for this case too, $\delta^{(4)}(KG)=(0)$.

Hence, Theorem 4.3 is proved.

Proof of Theorem 4.4

(*I*) *When* $\delta^{(4)}(KG) = (0)$ *and char* $K = p = 11$:

Proceeding in the same way as in the proof of the necessary part of Theorem 4.3, using Result 4.8 we find that $\delta^{(4)}(KG) = (0)$ will imply that $\Delta(G')^{15}KG = (0)$ and so we get $t(G') \leq 15$. Suppose $|G'| = p^l$. So putting, $p = 11$, in Theorem 1.24, we have $10l + 1 \leq t(G') \leq 11^l$. So, we must have $l = 0$ or $l = 1$. Now, $l = 0$ again implies that G is abelian. When $l = 1$, for $p = 11$, we have $|G'| = 11$. Hence, $G' = C_{11}$.

Again, when G is abelian, we are through. So we discuss the non-abelian case.

<u>*Claim:*</u> Either G' is central i.e., $\gamma_3(G) = 1$ or $\gamma_n(G) = G'$ for all $n \geq 3$ with $x^g = x^i$, $i = 1, 3, 4, 5, 9$, $\forall g \in G$.

Let $G' = \langle x \rangle$, where $\circ(x)$ is 11. If G' is not central, then $\gamma_3(G) = G'$ and hence by Proposition 4.9, $\Delta(G')^5KG \subseteq \delta^{(3)}(KG)$. Let $(x, g) = x^k$, where $k \in \mathbb{N}$; $0 \leq k \leq 9$, as $k = 10$ would imply $x^{-1}g^{-1}xg = x^{10}$, i.e., $g^{-1}xg = x^{11} = 1$, which means $x = 1$, a contradiction to our assumption. If we start with $\left[(x-1)^5, (x-1)^5g^{-1}\right] = 0$ and proceeding exactly as in the $p = 13$ case, we get:

$$\begin{aligned} & k\{1 + (k+1)\big(1 + (k+1)^2\big)(2+k)\} = 0 \\ \text{or,} \quad & k\big(k^4 + 5k^3 + 10k^2 + 10k + 5\big) = 0 \qquad \text{in } K. \\ \text{or,} \quad & k(k-2)(k-3)(k-4)(k-8) = 0 \qquad \text{in } K. \end{aligned} \tag{4.11}$$

This gives $k = 0$ or 2 or 3 or 4 or 8. Thus $(x, g) = 1$ or x^2 or x^3 or x^4 or x^8. That is, $x^g = x^i$, $i = 1, 3, 4, 5, 9$, $\forall g \in G$, as desired.

(*II*) *When* $\delta^{(4)}(KG) = (0)$ *and char* $K = p = 7$:

Again, we proceed in the same way as the necessary part of Theorem 4.3 and obtain $t(G') \leq 15$. Suppose $|G'| = p^l$. Putting $p = 7$ in Theorem 1.24, we have $6l + 1 \leq t(G') \leq 7^l$. So, we must have $l = 0$ or $l = 1$ or $l = 2$. Now, $l = 0$ again implies that G is abelian. When $l = 1$, for $p = 7$, we have $|G'| = 7$. Hence, $G' = C_7$. When $l = 2$, $t(G')$ has to be 13 for characteristic 7, as a group of order p^2 is always abelian and the other choice for a group of order 7^2 would be C_{49} which will give $t(G') > 15$. Hence, $G' = C_7 \times C_7$.

Claim: $G' = C_7$ or $G' = C_7 \times C_7 = \langle x\rangle \times \langle y\rangle$ and either $\gamma_3(G) = 1$ or $\gamma_3(G) = C_7$ with $\gamma_4(G) = 1$ or $\gamma_n(G) = G'$, for all $n \geq 3$ such that for all $g \in G \setminus C_G(G')$, $x^g \in \langle x\rangle$ and $y^g \in \langle y\rangle$ and also $C_G(x) = C_G(y)$.

(a) We have already seen that G' can be C_7.

(b) So let us discuss the case when $G' = C_7 \times C_7$. So $\gamma_3(G) = (G', G)$ can be 1 or C_7 or $G' = C_7 \times C_7$. Now let us discuss these three cases separately.

(i) When $\gamma_3(G) = 1$, we are through.

(ii) When $\gamma_3(G) = C_7$, let $\gamma_3(G) = C_7 = \langle z\rangle$. Let $y \in G' \setminus \gamma_3(G)$. By, Proposition 4.10, $\Delta(\gamma_3(G))\Delta(G')^5KG \subseteq \delta^{(3)}(KG)$. Therefore, for all $g \in G$, $(z-1)(y-1)^5g^{-1}$ and also $(z-1)(z-1)^4(y-1) = (z-1)^5(y-1) \in \delta^{(3)}(KG)$. So,

$$\begin{aligned}
0 &= \left[(z-1)(y-1)^5g^{-1}, (z-1)^5(y-1)\right] \\
&= (z-1)(y-1)^5g^{-1}(z-1)^5(y-1) - (z-1)^6(y-1)^6g^{-1} \\
&= (z-1)(y-1)^5\{g^{-1}(z-1)^5(y-1) - (z-1)^5(y-1)g^{-1}\} \\
&= (z-1)(y-1)^5 \\
&\quad \times\Big\{g^{-1}(y-1)(z-1)^5 - (y-1)g^{-1}(z-1)^5 \\
&\quad +(y-1)g^{-1}(z-1)^5 - (y-1)(z-1)g^{-1}(z-1)^4 \\
&\quad +(y-1)(z-1)g^{-1}(z-1)^4 - (y-1)(z-1)^2g^{-1}(z-1)^3 \\
&\quad +(y-1)(z-1)^2g^{-1}(z-1)^3 - (y-1)(z-1)^3g^{-1}(z-1)^2 \\
&\quad +(y-1)(z-1)^3g^{-1}(z-1)^2 - (y-1)(z-1)^4g^{-1}(z-1) \\
&\quad +(y-1)(z-1)^4g^{-1}(z-1) - (y-1)(z-1)^5g^{-1}\Big\} \\
&= (z-1)(y-1)^5 \\
&\quad \times\Big\{y\big((y,g)-1\big)\big(z^g-1\big)^5g^{-1} \\
&\quad +(y-1)z\big((z,g)-1\big)\big(z^g-1\big)^4g^{-1} \\
&\quad +(y-1)(z-1)z\big((z,g)-1\big)\big(z^g-1\big)^3g^{-1} \\
&\quad +(y-1)(z-1)^2z\big((z,g)-1\big)\big(z^g-1\big)^2g^{-1}
\end{aligned}$$

$$+(y-1)(z-1)^3 z\big((z,g)-1\big)\big(z^g-1\big)g^{-1}$$
$$+(y-1)(z-1)^4 z\big((z,g)-1\big)g^{-1}\Big\} \qquad \text{(Using 4.3.)}$$

Now, $(y,g) \in (G',G) = \gamma_3(G)$. As, $\Delta(\gamma_3(G))^7 = (0)$, so $(z-1)\big((y,g)-1\big)\big(z^g-1\big)^5 = 0$. So the first term of the above equation becomes zero. As $(z,g) = z^{-1}z^g \in \gamma_3(G) = C_7$, let $(z,g) = z^k$, where $k \in \mathbb{N}$; $1 \leq k \leq 5$, as $k = 6$ would imply $z^{-1}g^{-1}zg = z^6$, i.e., $g^{-1}zg = z^7 = 1$, which means $z = 1$, a contradiction. Also, $z^g = g^{-1}zg = z^{k+1}$. Substituting $(z,g) = z^k$ and $z^g = z^{k+1}$ and $(z-1)^6 = \hat{z} = 1 + z + \ldots + z^6$, in the remaining terms of the above equation, we are left with:

$$\begin{aligned}
0 &= (y-1)^6(z-1)^6 z(1+z+\ldots+z^{k-1}) \\
&\quad \times\Big\{(1+z+\ldots+z^k)^4 + (1+z+\ldots+z^k)^3 \\
&\qquad + (1+z+\ldots+z^k)^2 + (1+z+\ldots+z^k) + 1\Big\}g^{-1} \\
&= (y-1)^6 \hat{z}(1+z+\ldots+z^{k-1}) \\
&\quad \times\Big\{(1+z+\ldots+z^k)^4 + (1+z+\ldots+z^k)^3 \\
&\qquad + (1+z+\ldots+z^k)^2 + (1+z+\ldots+z^k) + 1\Big\} \\
&= (y-1)^6 \hat{z} k\big\{(1+k)^4 + (1+k)^3 + (1+k)^2 + (1+k) + 1\big\} \\
&= k\big(k^4 + 5k^3 + 10k^2 + 10k + 5\big)(y-1)^6(z-1)^6
\end{aligned} \tag{4.12}$$

Now, $k^4 + 5k^3 + 10k^2 + 10k + 5 \neq 0$ in K for any $k \in \mathbb{N}$, as $7 \nmid k^4 + 5k^3 + 10k^2 + 10k + 5$. So $k = 0$. Because if $k \neq 0$, then $(y-1)^6(z-1)^6 = 0$ which is impossible as nilpotency index of $\Delta(G')$ in this case is 13, and this is a contradiction to the fact that $y \notin \gamma_3(G)$. This shows that $\gamma_3(G)$ is central, i.e., $\gamma_4(G) = 1$.

(iii) When $\gamma_n(G) = G'$ for all $n \geq 3$. Let $\gamma_3(G) = G' = C_7 \times C_7 = \langle x \rangle \times \langle y \rangle$. Let $g \in G$ such that $g \notin C_G(G')$. By Proposition 4.9, $\Delta(G')^5 KG \subseteq \delta^{(3)}(KG)$. So, $(x-1)^5$, $(x-1)^5 g^{-1} \in \delta^{(3)}(KG)$. Therefore,

$$\begin{aligned} 0 &= \big[(x-1)^5, (x-1)^5 g^{-1}\big] \\ &= (x-1)^{10} g^{-1} - (x-1)^5 g^{-1} (x-1)^5 \\ &= -(x-1)^5 (x^g - 1)^5 g^{-1} \qquad [\because\ (x-1)^{10} = 0] \end{aligned}$$

We get, $(x-1)^5(x^g-1)^5 = 0$. As $t(G') = 15$, we must have $x^g \in \langle x \rangle$ for all $g \notin C_G(G')$. Similarly, it can be proved that $y^g \in \langle y \rangle$ for all $g \notin C_G(G')$ with the help of the elements $(y-1)^5$ and $(y-1)^5 g^{-1}$. Now, let $g \in C_G(x)$, but $g \notin C_G(y)$, i.e., $x^g = x$ but $y^g \neq y$. Now, $(x-1)(y-1)^4 g^{-1}$, $(x-1)^4(y-1) \in \delta^{(3)}(KG)$. So,

$$\begin{aligned} 0 &= \big[(x-1)(y-1)^4, (x-1)^4(y-1)\big] \\ &= (x-1)(y-1)^4 g^{-1}(x-1)^4(y-1) - (x-1)^5(y-1)^5 g^{-1} \\ &= (x-1)(y-1)^4(x^g-1)^4 g^{-1}(y-1) - (x-1)^5(y-1)^5 g^{-1} \\ &= (x-1)^5(y-1)^4\big[g^{-1}, y\big] \qquad [\because\ x^g = x] \\ &= (x-1)^5(y-1)^4 y\big((y,g)-1\big)g^{-1} \\ &= (x-1)^5(y-1)^5 \qquad [\because\ (y,g) = y^{-1}y^g \in \langle y \rangle]. \end{aligned}$$

This contradicts the fact that $y \notin \langle x \rangle$. So we must have $y^g = y$. Similarly it can be shown that whenever $g \in C_G(y)$, then $g \in C_G(y)$. This shows that $C_G(x) = C_G(y)$.

Hence Theorem 4.4 is proved.

Proof of Theorem 4.5

Case (i) When G is abelian, then clearly $\delta^{(4)}(KG) = (0)$.

Case (ii) When Char $K \neq 2$ and G' is cyclic with $\gamma_3(G) = 1$ and $|G'| \leq 11$, then by Result 4.7 *(iii)*, we have $\delta^{(2)}(KG) = \Delta(G')^3 KG$. Thus, $\delta^{(3)}(KG) = [\delta^{(2)}(KG), \delta^{(2)}(KG)]KG = [\Delta(G')^3 KG, \Delta(G')^3 KG]KG$ $\subseteq \Delta(G')^6 KG$. Also, as $|G'| \leq 11$, we get $t(G') \leq 11$, and so, $\Delta(G')^{13} KG$ $= (0)$. Hence,

$$\begin{aligned}\delta^{(4)}(KG) &= [\delta^{(3)}(KG), \delta^{(3)}(KG)]KG \\ &\subseteq [\Delta(G')^6 KG, \Delta(G')^6 KG]KG \\ &\subseteq \Delta(G')^{13}KG \ = (0)\end{aligned}$$

(Using Proposition 4.11.)

Hence, for this case too, $\delta^{(4)}(KG) = (0)$.

Case (iii) We first prove the following proposition.

Proposition 4.13. *If $(G', G) = \gamma_3(G) = G'$, and G' is cyclic with $G' = \langle x \rangle$, $x \neq 1$ and $|G'| > 5$ such that $C_G(x)$ has index 2 in G, i.e., for all $g \notin C_G(G')$, $x^g = x^i$, where i is fixed and $1 < i < \circ(x)$, then for $Char K \neq 2$, $\delta^{(3)}(KG) = \Delta(G')^5 KG = (x-1)^5 KG$.*

Proof. By Result 4.7 (iv), we know that when $(G', G) = G'$, then $\delta^{(2)}(KG) = \Delta(G')^2 KG$. Here $G' = \langle x \rangle$ where $x \neq 1$ and $t(G') > 5$. So, $\Delta(G')^n KG = (x-1)^n KG$ for positive integer $n \leq 5$. So, $\delta^{(3)}(KG) = \big[(x-1)^2 KG, (x-1)^2 KG\big]$. Now, for $g, h \in G \setminus C_G(G')$, and using the fact that $[a, b-1] = [a, b]$ for $a, b \in G$

$$\begin{aligned}
&\big[(x-1)^2 g, (x-1)^2 h\big] \\
&= (x-1)^2 g(x-1)^2 h - (x-1)^2 h(x-1)^2 g \\
&= (x-1)^2 g(x-1)^2 h - (x-1)^3 g(x-1)h \\
&+ (x-1)^3 g(x-1)h - (x-1)^4 gh \\
&+ (x-1)^4 gh - (x-1)^4 hg + (x-1)^4 hg - (x-1)^3 h(x-1)g \\
&+ (x-1)^3 h(x-1)g - (x-1)^2 h(x-1)^2 g \\
&= (x-1)^2[g, x](x-1)h + (x-1)^3[g, x]h + (x-1)^4[g, h] \\
&\quad + (x-1)^3[x, h]g + (x-1)^2[x, h](x-1)g \\
&= (x-1)^4\Big(\big(g^{-1}, h^{-1}\big) - 1\Big)hg + (x-1)^3\Big(\big(g^{-1}, x^{-1}\big) - 1\Big)xgh \\
&\quad + (x-1)^2\Big(\big(g^{-1}, x^{-1}\big) - 1\Big)x\big(x^{g^{-1}} - 1\big)gh \\
&\quad + (x-1)^3\Big(\big(x^{-1}, h^{-1}\big) - 1\Big)x^{h^{-1}}hg \\
&\quad + (x-1)^2\Big(\big(x^{-1}, h^{-1}\big) - 1\Big)x^{h^{-1}}\big(x^{h^{-1}} - 1\big)hg
\end{aligned} \tag{4.13}$$

Now, let $\big(g^{-1}, h^{-1}\big) = x^j$, where j is a positive integer, $1 \leq j \leq \circ(x)$, as it belongs to $G' = \langle x \rangle$. So, $\big(g^{-1}, h^{-1}\big) - 1 = x^j - 1 = (x-1)X_1$, say, where $X_1 = 1 + x + \dots x^{j-1}$. According to the given conditions, $gxg^{-1} = hxh^{-1} = x^i$. So, $\left(g^{-1}, x^{-1}\right) - 1 = gxg^{-1}x^{-1} - 1 = x^i x^{-1} - 1 = x^{i-1} - 1 = (x-1)X_2$ and $\left(x^{-1}, h^{-1}\right) - 1 = xhx^{-1}h^{-1} - 1 = xx^{-i} - 1 = x^{1-i} - 1 = -x^{1-i}(x-1)X_2$, where i is fixed with $2 \leq i < \circ(x)$, and $X_2 = 1 + x + \dots x^{i-2}$. Putting all these in Eqn. 4.13, we get

$$\begin{aligned}
&\left[(x-1)^2 g, (x-1)^2 h\right] \\
&= (x-1)^5 X_1 hg + (x-1)^4 X_2 xgh + (x-1)^4 X_2 x(X_2 + x^{i-1})gh \\
&\quad - (x-1)^4 xX_2 hg - (x-1)^4 xX_2(X_2 + x^{i-1})hg \\
&= (x-1)^5 X_1 hg + (x-1)^4 xX_2(gh - hg) \\
&\qquad\qquad + (x-1)^4 xX_2(X_2 + x^{i-1})(gh - hg) \\
&= (x-1)^5 X_1 hg + (x-1)^5 X_1 xX_2 hg + (x-1)^5 X_1 xX_2(X_2 + x^{i-1})hg \\
&\in (x-1)^5 KG
\end{aligned}$$

So, we get $\delta^{(3)}(KG) \subseteq \Delta(G')^5 KG = (x-1)^5 KG$. The reverse containment has been proved in Proposition 4.9. Hence, for the above case, $\delta^{(3)}(KG) = (x-1)^5 KG$. □

Now, let us return to proving part (*iii*) of Theorem 4.5. Let $\gamma_n(G) = G' = \langle x \rangle \neq \{1\}$ for all $n \geq 3$ with $|G'| \leq 11$ and with the condition if $g \notin C_G(G')$, then $x^g = x^i$, where i is fixed and $1 < i < \circ(x)$. So, $\Delta(G')^m KG = (x-1)^m KG$, for some positive integer m. First, let us assume $|G'| \leq 8$. Then, by Result 4.8, $\delta^{(4)}(KG) \subseteq (x-1)^8 KG = (0)$. So, $\delta^{(4)}(KG) = (0)$.

Now, let $8 < |G'| \leq 11$. Hence for this case, by Proposition 4.13, $\delta^{(3)}(KG) = \Delta(G')^5 KG$. So, $\delta^{(4)}(KG) = \left[\Delta(G')^5 KG, \Delta(G')^5 KG\right]$KG. Let $g, h \in G \setminus C_G(G')$, then

$$\begin{aligned}
&\left[(x-1)^5 g, (x-1)^5 h\right] = (x-1)^5 g(x-1)^5 h - (x-1)^5 h(x-1)^5 g \\
&= (x-1)^5 g(x-1)^5 h - (x-1)^6 g(x-1)^4 h \\
&+ (x-1)^6 g(x-1)^4 h - (x-1)^7 g(x-1)^3 h \\
&+ (x-1)^7 g(x-1)^3 h - (x-1)^8 g(x-1)^2 h \\
&+ (x-1)^8 g(x-1)^2 h - (x-1)^9 g(x-1)h \\
&+ (x-1)^9 g(x-1)h - (x-1)^{10} gh + (x-1)^{10} gh - (x-1)^{10} hg
\end{aligned}$$

$$
\begin{aligned}
& +(x-1)^{10}hg-(x-1)^9h(x-1)g \\
& +(x-1)^9h(x-1)g-(x-1)^8h(x-1)^2g \\
& +(x-1)^8h(x-1)^2g-(x-1)^7h(x-1)^3g \\
& +(x-1)^7h(x-1)^3g-(x-1)^6h(x-1)^4g \\
& +(x-1)^6h(x-1)^4g-(x-1)^5h(x-1)^5g \\
= \; & (x-1)^5[g,x](x-1)^4h+(x-1)^6[g,x](x-1)^3h \\
& +(x-1)^7[g,x](x-1)^2h+(x-1)^8[g,x](x-1)h \\
& +(x-1)^9[g,x]h+(x-1)^{10}[g,h] \\
& +(x-1)^9[x,h]g+(x-1)^8[x,h](x-1)g \\
& +(x-1)^7[x,h](x-1)^2g+(x-1)^6[x,h](x-1)^3g \\
& +(x-1)^5[x,h](x-1)^4 \\
= \; & (x-1)^5\Big((g^{-1},x^{-1})-1\Big)x\big(x^{g^{-1}}-1\big)^4gh \\
& +(x-1)^6\Big((g^{-1},x^{-1})-1\Big)x\big(x^{g^{-1}}-1\big)^3gh \\
& +(x-1)^7\Big((g^{-1},x^{-1})-1\Big)x\big(x^{g^{-1}}-1\big)^2gh \\
& +(x-1)^8\Big((g^{-1},x^{-1})-1\Big)x\big(x^{g^{-1}}-1\big)gh \\
& +(x-1)^9\Big((g^{-1},x^{-1})-1\Big)xgh \\
& +(x-1)^{10}\Big((g^{-1},h^{-1})-1\Big)hg \\
& +(x-1)^9\Big((x^{-1},h^{-1})-1\Big)x^{h^{-1}}hg \\
& +(x-1)^8\Big((x^{-1},h^{-1})-1\Big)x^{h^{-1}}\big(x^{h^{-1}}-1\big)hg \\
& +(x-1)^7\Big((x^{-1},h^{-1})-1\Big)x^{h^{-1}}\big(x^{h^{-1}}-1\big)^2hg \\
& +(x-1)^6\Big((x^{-1},h^{-1})-1\Big)x^{h^{-1}}\big(x^{h^{-1}}-1\big)^3hg \\
& +(x-1)^5\Big((x^{-1},h^{-1})-1\Big)x^{h^{-1}}\big(x^{h^{-1}}-1\big)^4hg \\
= \; & (x-1)^5\Big((g^{-1},x^{-1})-1\Big)x \\
& \times\Big\{\big(x^{g^{-1}}-1\big)^4+(x-1)\big(x^{g^{-1}}-1\big)^3+(x-1)^2\big(x^{g^{-1}}-1\big)^2 \\
& \qquad +(x-1)^3\big(x^{g^{-1}}-1\big)+(x-1)^4\Big\}gh \\
& +(x-1)^5\Big((x^{-1},h^{-1})-1\Big)x^{h^{-1}} \\
& \times\Big\{\big(x^{h^{-1}}-1\big)^4+(x-1)\big(x^{h^{-1}}-1\big)^3+(x-1)^2\big(x^{h^{-1}}-1\big)^2 \\
& \qquad +(x-1)^3\big(x^{h^{-1}}-1\big)+(x-1)^4\Big\}hg \\
& +(x-1)^{10}\Big((g^{-1},h^{-1})-1\Big)hg
\end{aligned} \tag{4.14}
$$

Now, let $\left(g^{-1},h^{-1}\right)=x^{j}$, where j is a positive integer, $1\leq j\leq 11$, as it belongs to $G'=\langle x\rangle=C_m$, with $8<m\leq 11$. So, $\left(g^{-1},h^{-1}\right)-1=x^{j}-1=(x-1)X_1$, say, where $X_1=1+x+\ldots+x^{j-1}$. Hence the term $(x-1)^{10}\Big(\left(g^{-1},h^{-1}\right)-1\Big)hg$ in Eqn. 4.14 becomes $(x-1)^{11}X_1hg=0$ as $8<\circ(x)\leq 11$. According to the given conditions, $\left(g^{-1},x^{-1}\right)-1=gxg^{-1}x^{-1}-1=x^{i}x^{-1}-1=x^{i-1}-1=(x-1)A$ and $\left(x^{-1},h^{-1}\right)-1=xhx^{-1}h^{-1}-1=xx^{-i}-1=x^{1-i}-1=-x^{1-i}(x-1)A$, where i is fixed with $1<i<\circ(x)$, and $A=1+x+\ldots x^{i-2}$. Also, $\left(x^{h^{-1}}-1\right)^{k}=\left(x^{g^{-1}}-1\right)^{k}=(x^{i}-1)^{k}=(x-1)^{k}A_k$, where $A_k=(1+x+\ldots x^{i-1})^{k}$, where $k\in\mathbb{N}$. Putting all these in Eqn. 4.14, we get

$$
\begin{aligned}
&\left[(x-1)^5g,(x-1)^5h\right]\\
&=(x-1)^6Ax(x-1)^4\left\{A_4+A_3+A_2+A_1+1\right\}gh\\
&\quad-(x-1)^6Ax(x-1)^4\left\{A_4+A_3+A_2+A_1+1\right\}hg\\
&=(x-1)^{10}X(gh-hg)\qquad \text{where } X=Ax\left\{A_4+A_3+A_2+A_1+1\right\}\\
&=(x-1)^{11}XX_1hg\qquad (\because\ 8<t(G')\leq 11)\\
&=0
\end{aligned}
$$

Thus, we get for the above case, $\delta^{(4)}(KG)=(0)$. Hence, Theorem 4.5 is proved.

Thus the question remaining for this problem is what the necessary conditions will be for a strongly Lie solvable group algebra to have strong Lie derived length 4 over fields of characteristic 3 and 5.

Chapter 5

Future Work

Based on our work in the thesis, a few problems can be looked into in the future. The following question is the remaining part of our work in Chapter 3.

▶ What will be the structure of KG over fields of characteristic $p < 17$ with units having derived length four?

▶ Whether all our results can be extended to any **torsion** group?

As our work contains ideas for generalizing the problem of characterizing group algebras having units of derived length n, our ideas can be utilized to answer the following very important questions.

▶ What will be the minimum derived length of U for group algebras of any given **finite** non-abelian group ?

▶ Finally complete the above problem by investigating whether the result can be extended to any **infinite** group also?

Appendix A

Publication and Manuscripts

(i) D.Chaudhuri and A.Saikia. On group algebras with unit groups of derived length at most four. Publ. Math. Debrecen, Vol 86, 39-48, 2015.

(ii) D.Chaudhuri and A.Saikia. On the derived length of units in group algebra. Czechoslovak Mathematical Journal (To Appear).

(iii) D.Chaudhuri and A.Saikia. On group algebras with strong Lie derived length at most four (Communicated).

Bibliography

[1] A.A.Bovdi and I.I.Khripta. Group algebras of periodic groups with solvable unit groups. *Math. Zametki.*, 22:725–731, 1977.

[2] A.A.Bovdi and I.Khripta. Finite dimensional group algebras having solvable unit groups. *Trans. Science Conf. Uzhgorod State University*, 15:227–233, 1974.

[3] A.A.Bovdi and J.Kurdics. Lie properties of the group algebra and the nilpotency class of the group of units. *J. Algebra*, 212:28–64, 1999.

[4] A.Bovdi. Group Algebras with a solvable group of units. *Comm. Algebra*, 33:3725–3738, 2005.

[5] A.Shalev. The derived length of Lie soluble group rings I. *J. Pure Appl. Algebra*, 78:291–300.

[6] A.Shalev. Meta-abelian unit groups of group algebras are usually abelian. *J. Pure Appl. Algebra*, 72:295–302, 1991.

[7] C.Bagiński. A note on the derived length of the unit group of a modular group algebra. *Comm. Algebra*, 30:4905–4913, 2002.

[8] D.B.Coleman and R.Sandling. Mod 2 group algebras with metabelian unit groups. *J. Pure Appl. Algebra*, 131:25–36, 1998.

[9] D.Chaudhuri and A.Saikia. On group algebras with unit groups of derived length at most four. *Publ. Math. Debrecen*, 86:39–48, 2015.

[10] D.E.Taylor. Groups whose modular group rings have soluble unit groups. *Lectures Notes in Mathematics*, 573:112–117, 1977.

[11] D.Gorenstein. *Finite Groups*. Harper and Row, 1989.

[12] D.S.Passman. Observations on group rings. *Comm. Algebra*, 5:1119–1162, 1977.

[13] D.S.Passman. *The Algebraic Structure of Group Rings.* Wiley-Interscience, 1977.

[14] E.Spinelli. Group Algebras with minimal strong Lie derived length. *Canad. Math. Bull.*, 51:291–297, 2008.

[15] F.Catino and E.Spinelli. A note on strong Lie derived length of group algebras. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.*, 10:83–86, 2007.

[16] F.Catino and E.Spinelli. On the derived length of the unit group of a group algebra. *J. Group Theory*, 13:577–588, 2010.

[17] F.Levin and G.Rosenberger. Lie metabelian group rings. *North-Holland Math. Stud.*, 126:153–161, 1986.

[18] G.T.Lee, S.K.Sehgal, and E.Spinelli. Group Rings with Solvable Unit Groups of Minimal Derived Length. *Algbr. Represent. Theory*, 17:1597–1601, 2014.

[19] H.Chandra and M.Sahai. Group algebras with unit groups of derived length three. *J. Algebra Appl.*, 9:305–314, 2010.

[20] H.Chandra and M.Sahai. On Group algebras with unit groups of derived length three in characteristic three. *Publ. Math. Debrecen*, 82:697–708, 2013.

[21] I.B.S.Passi, D.S.Passman, and S.K. Sehgal. Lie solvable group rings. *Can. J. Math.*, 25:748–757, 1973.

[22] J.Kurdics. Engel properties of group algebras. I. *Publ. Math. Debrecen*, 49:183–192, 1996.

[23] J.Kurdics. On group algebras with metabelian unit groups. *Periodica Mathematica Hungarica*, 32:57–64, 1996.

[24] J.Kurdics. Engel properties of group algebras. II. *Ring Theory (Miskolc, 1996).,J. Pure Appl. Algebra*, 133:179–196, 1998.

[25] J.M.Bateman. On the solvability of unit groups of group algebra. *Trans. Amer. Math. Soc.*, 157:73–86, 1971.

[26] K.Motose and H.Tominaga. Group rings with solvable unit groups. *J. Okayama Univ*, 15:37–40, 1971.

[27] K.Motose and Y.Ninomiya. On the solvability of the unit groups of group rings. *J. Okayama Univ*, 15:209–214, 1972.

[28] C.P. Milies and S.K. Sehgal. *An Introduction to Group Rings (Algebras and Applications)*. Kluwer Academic Publishers, 2002.

[29] M.Sahai. Lie solvable group algebras of derived length three. *Publ. Math.*, 39:233–240, 1995.

[30] M.Sahai. Group algebras with centrally metabelian unit groups. *Publ. Math.*, 40:443–456, 1996.

[31] M.Sahai. On Group algebras KG with $U(KG)'$ Nilpotent of Class at most 2. *Contem. Math.*, 456:165–173, 2008.

[32] T.Juhász. On the derived length of Lie solvable group algebras. *Publ. Math. Debrecen*, 68:243–256, 2006.

[33] W.S. Yoo. The structure of the radical of the non semisimple group rings. *Korean J. Math.*, 18:97–103, 2010.

[34] Z.Balogh and T.Juhász. Lie derived lengths of group algebras of groups with cyclic derived subgroup. *Comm. Algebra*, 36:315–324, 2008.

[35] Z.Balogh and T.Juhász. Conditions for groups whose group algebras have minimal derived length. *Annales Mathematicae et Informaticae*, 37:13–20, 2010.

[36] Z.Balogh and Y.Li. On the derived length of the group of units of a group algebra. *J. Algebra Appl.*, 6:991–999, 2007.