Classifications of some Algebraically Positive, Diagonalizable and Stable Matrices with their Sign Patterns

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Declaration

I hereby declare that the work contained in the thesis entitled "**Classifications of some algebraically positive, diagonalizable and stable matrices with their sign patterns**" has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under the guidance of **Dr. Sriparna Bandopadhyay**, Indian Institute of Technology Guwahati, for the award of **Doctor of Philosophy** and that this work has not been submitted elsewhere for a degree.

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Certificate

It is certified that the work contained in the thesis titled "**Classifications of some algebraically positive, diagonalizable and stable matrices with their sign patterns**" by **Sunil Das**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of **Doctor of Philosophy** has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

Guwahati
March, 2021

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Sunil Das
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Dedicated

To

My Family

Abstract

A sign pattern matrix is a matrix with entries from $\{+, -, 0\}$. The qualitative class of a matrix A is denoted by $Q(A)$ and is defined by the set of all real matrices with the same sign pattern as A. A sign pattern matrix A allows a property P if at least one matrix in $Q(A)$ has the property P, and requires a property P if all matrices in $Q(A)$ have the property P. In this thesis, we find some sign patterns of algebraically positive, diagonalizable and stable matrices. We mainly focus on sign pattern matrices associated with trees.

The concept of an algebraically positive matrix was introduced by Kirkland, Qiao, and Zhan in 2016. A real square matrix A is said to be algebraically positive if there exists a real polynomial f such that $f(A)$ is a positive matrix. We prove that a real square matrix is algebraically positive if and only if it commutes with a unique (upto scalar multiplication) rank one positive matrix. We also show that for a real square matrix A, if $\text{adj}(A)$ is algebraically positive, then A is also algebraically positive. We characterize all tree sign pattern matrices that allow algebraic positivity, and all star and path sign pattern matrices that require algebraic positivity. We also identify all tree sign pattern matrices of order less than 6 requiring algebraic positivity.

We introduce the concept of an essential index for a tree sign pattern matrix. We observe that a tree sign pattern matrix requires singularity if and only if it has an essential index.

Further, we give a result regarding column spaces of matrices in the qualitative class of a tree sign pattern matrix. We use this result to obtain a sufficient condition for sign pattern matrices whose graphs are trees to allow diagonalizability. We also characterize sign pattern matrices allowing diagonalizability, whose graphs are either star or path. Moreover, we give a necessary condition for a sign pattern matrix requiring diagonalizability, and describe all star sign pattern matrices requiring diagonalizability.

A square matrix M is said to be stable if all eigenvalues of M have negative real part, and a sign pattern matrix A is said to be potentially stable if there exists a stable matrix in $Q(A)$. A sign pattern matrix A allows a properly signed nest if there exist $B \in Q(A)$ and a permutation matrix P such that the sign of the k-th leading principal minor of PBP^T is $(-)^k$ for all $k \in \{1, 2, \ldots, n\}$. We give some sufficient conditions for tree sign pattern matrices with all edges negative to allow a properly signed nest. In 1997, Johnson, Maybee, Olesky and van den Driessche proved that if a sign pattern matrix allows a properly signed nest, then it is potentially stable. However, the converse is not true, even for tree sign pattern matrices. We believe that the converse is true for tree sign pattern matrices with negative edges, which we propose as a conjecture. We prove that this conjecture is true for tree sign pattern matrices with negative edges of order at most 6. Further, we identify all potentially stable star and path sign pattern matrices with negative edges, and prove that the conjecture is valid for these classes. A sign pattern matrix A of order n is a spectrally arbitrary pattern if, for any given real monic polynomial $r(x)$ of degree n, there is a matrix in $Q(A)$ with characteristic polynomial $r(x)$. As a consequence of the results on potentially stable sign pattern matrices with negative edges, we describe all 5-by-5 spectrally arbitrary tree sign pattern matrices with negative edges.

List of Symbols

$\lfloor x \rfloor$	:	Largest integer less than or equal to x
$\|S\|$	:	Cardinality of a finite set S
$\emptyset$	:	Empty set
$\langle n \rangle$	:	$\{1, 2, \ldots, n\}$
$\mathbb{N}$	:	The set of all natural numbers
$\mathbb{R}$	:	The set of all real numbers
$\mathbb{R}^n$	:	The set of all real column vectors with n entries
$M_{m\times n}(\mathbb{R})$	:	The set of all $m \times n$ real matrices
I	:	Identity matrix
$\mathbf{e}_i$	:	i-th column of an identity matrix
O	:	A matrix with all entries zero
$\mathbf{0}$	:	A column vector with all entries zero
x_i	:	i-th entry of a vector $\mathbf{x}$
a_{ij}	:	(i, j)-th entry of a matrix A

$\mathcal{N}(A)$	:	Null space of A
$\mathcal{R}(A)$	:	Row space of A
$\mathcal{C}(A)$	:	Column space of A
$z(A)$	:	The algebraic multiplicity of the eigenvalue 0 of A
$\sigma(A)$	:	The set of all eigenvalues of A
$\sigma^*(A)$	:	The set of all nonzero eigenvalues of A
$i_+(A)$	:	Number of eigenvalues of A with positive real part
$i_-(A)$	:	Number of eigenvalues of A with negative real part
$i_0(A)$	:	Number of eigenvalues of A with zero real part
$i(A)$	:	$(i_+(A), i_-(A), i_0(A))$
$P_A(x)$	:	The characteristic polynomial of A in x
$A[\alpha, \beta]$	:	The submatrix of A with rows, columns corresponding to the indices in α, β respectively
$A[\alpha]$	:	The principal submatrix of A corresponding to the indices in α
$A(\alpha, \beta)$	:	The submatrix obtained from A by deleting rows, columns corresponding to the indices in α, β respectively
$A(\alpha)$	:	The principal submatrix obtained from A by deleting rows and columns corresponding to the indices in α
A^T	:	Transpose of a matrix A
$\mathrm{adj}(A)$	:	Adjugate of a square matrix A
$\det(A)$	:	Determinant of a square matrix A
$\mathrm{diag}(d_1, \ldots, d_n)$	:	Diagonal matrix of order n whose i-th diagonal entry is d_i for all $i \in \{1, 2, \ldots, n\}$
$A \circ B$	:	Hadamard product of two matrices A, B of the same order, page 89

$V(G)$	:	Vertex set of a graph G
$E(G)$	:	Edge set of a graph G
$[u,v]$	:	The edge joining two distinct vertices u,v of an undirected graph
(u,v)	:	The arc from vertex u to vertex v of a directed graph
$\deg(u)$	:	Degree of a vertex u in an undirected graph, page 4
$d(u,v)$	:	Distance between two vertices u,v in an undirected graph, page 4
$G(S)$	:	Subgraph of an undirected graph G induced by $S \subseteq V(G)$, page 108
$G(A)$	:	Graph of a square matrix A, page 4
$D(A)$	:	Digraph of a square matrix A, page 3
$c(A)$	:	Maximum cycle length in a square matrix A, page 3
$Q(A)$	:	Qualitative class of a sign pattern matrix A, page 1
$\mathrm{MR}(A)$	:	Maximum rank of a sign pattern matrix A, page 2
$\mathrm{mr}(A)$	:	Minimum rank of a sign pattern matrix A, page 2

Contents

CHAPTER 1

Introduction

1.1 Preliminaries

The origin of sign pattern matrices can be traced back to the book "Foundations of Economic Analysis" [47] by Nobel prize winner economist P. A. Samuelson. In this dissertation, we investigate some problems related to sign pattern matrices associated with trees. Let us recall some definitions from [3, 8, 26, 30, 48, 49] to be used throughout this dissertation.

- A sign pattern matrix is a matrix with entries from the set $\{+, -, 0\}$.
- Let $M_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ real matrices. If A is an $m \times n$ sign pattern matrix, then the qualitative class of A is denoted by $Q(A)$ and is defined by

$$Q(A) = \{B \in M_{m\times n}(\mathbb{R}) : \text{sign}(b_{ij}) = a_{ij} \text{ for all } i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}\}.$$

- A sign pattern matrix A is said to allow a property P if at least one matrix in its qualitative class has the property P, and A is said to require a property P if all matrices in its qualitative class have the property P.

- A subpattern $\tilde{A}$ of a sign pattern matrix A is a sign pattern matrix obtained from A by replacing some (possibly none) of the nonzero entries of A with 0. In this case, A is a super-pattern of $\tilde{A}$.

- For products of entries of sign pattern matrices, the rules are that multiplication is associative, commutative and

$$(+)(+) = +,\ (+)(-) = -,\ (-)(-) = +,\ (0)(+) = 0,\ (0)(-) = 0,\ (0)(0) = 0.$$

- A diagonal sign pattern matrix is a square sign pattern matrix with all off-diagonal entries 0. A signature sign pattern matrix is a diagonal sign pattern matrix with each diagonal entry $+$ or $-$. A signature similarity of a square sign pattern matrix A is a product SAS, where S is a signature sign pattern matrix.

- A permutation sign pattern matrix is a square sign pattern matrix with entries from $\{0, +\}$, where the entry $+$ occurs precisely once in each row and each column. A permutation similarity of a square sign pattern matrix A is a product P^TAP, where P is a permutation sign pattern matrix.

- A sign pattern matrix B is equivalent to another sign pattern matrix A if B is obtained from A by negation and/or permutation similarity and/or signature similarity.

- A square sign pattern matrix $A = [a_{ij}]$ is said to be combinatorially symmetric if for each i, j, either both a_{ij}, a_{ji} are zero or both a_{ij}, a_{ji} are nonzero.

- The minimum rank and maximum rank of a sign pattern matrix A are denoted by $\text{mr}(A)$ and $\text{MR}(A)$, respectively, and are defined by $\text{mr}(A) = \min\{\text{rank}(B) : B \in Q(A)\}$ and $\text{MR}(A) = \max\{\text{rank}(B) : B \in Q(A)\}$.

Let A be a square matrix of order n. If α, β are subsets of $\{1, 2, \ldots, n\}$, then $A[\alpha, \beta]$ denotes the submatrix of A with rows, columns corresponding to the indices in α, β respectively, and $A(\alpha, \beta)$ is the submatrix obtained from A by deleting rows, columns corresponding to the indices in α, β respectively. Further, $A[\alpha]$ denotes $A[\alpha, \alpha]$, and $A(\alpha)$ denotes $A(\alpha, \alpha)$.

A nonzero product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ in which the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices is called a simple cycle of length k (or simple k-cycle) in A. A composite k-cycle is a product of simple cycles whose total length is k, and whose index sets are mutually disjoint. The maximum cycle length in A is denoted by $c(A)$. A nonzero entry a_{ij} is a chord of a cycle γ (simple or composite), if i and j are two vertices of γ and a_{ij} is not in γ. A cycle γ is chordless if γ has no chords.

A directed graph D is a triple consisting of a vertex set $V(D)$, an arc set $E(D)$, and a function assigning each arc to an ordered pair of vertices. An ordered pair (i, j) is an arc from vertex i to vertex j. If $A = [a_{ij}]$ is a matrix of order n, then the digraph of A, denoted by $D(A)$, is defined as the directed graph with vertices $1, 2, \ldots, n$ such that $D(A)$ has the arc (i, j) if and only if $a_{ij} \neq 0$. A directed walk from i_0 to i_k is a sequence $(i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k)$ of arcs in $D(A)$. We also denote this walk by $i_0 \to i_1 \to i_2 \to \cdots \to i_{k-1} \to i_k$. If the indices $i_0, i_1, \ldots, i_k$ are distinct, then the walk is called a directed path. The digraph $D(A)$ is said to be strongly connected, if for each $i, j \in \{1, 2, \ldots, n\}$, there is a directed path from i to j.

A matrix A of order n is said to be reducible if it is permutationally similar to a matrix of the form

$$\begin{bmatrix} A_1 & O \\ A_3 & A_2 \end{bmatrix},$$

where A_1, A_2 are square matrices of order at least 1. If A is not reducible, then it is called irreducible. From [8, Theorem 3.2.1], we have the following result.

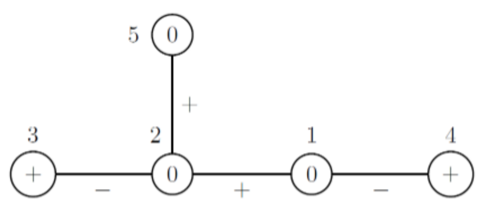
Theorem 1.1 ([8]). *Let A be a square matrix of order n. Then A is irreducible if and only if its digraph $D(A)$ is strongly connected.*

An undirected graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates each edge with two vertices (not necessarily distinct). An edge between two vertices $u, v \in V(G)$ is denoted by $[u, v]$, where u, v are called its endpoints. A loop is an edge whose endpoints are equal, and multiple edges are edges with the same pair of end vertices. A simple undirected graph is an undirected graph without loops and multiple edges. If $e = [u, v] \in E(G)$, then u, v are adjacent, and e is incident to both u, v. The degree of a vertex u is denoted by $\deg(u)$ and is defined by the number of distinct edges incident to u. If $\deg(u) = 1$ for a vertex u, then u is called pendant. A path between two vertices u, v is a sequence $(u = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v)$ of distinct vertices such that $[v_{i_{k-1}}, v_{i_k}] \in E(G)$ for all $i \in \{1, 2, \ldots, k\}$. The paths $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k)$ and $(v_k, v_{k-1}, \ldots, v_1, v_0)$ are considered to be identical. Distance between two vertices u, v is denoted by $d(u, v)$ and is defined by the minimum of lengths of all paths between u, v. A tree is a simple undirected graph T such that for any two vertices $u, v \in V(T)$, T has exactly one path between u, v. A tree T is a path if it has exactly two pendant vertices, and T is a star if T has at most one vertex of degree ≥ 2.

If $A = [a_{ij}]$ is a matrix of order n, then the graph of A, denoted by $G(A)$, is defined to be a simple undirected graph with vertices $1, 2, \ldots, n$, and for each $i \neq j$ it has the edge $[i, j]$ if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$. If this graph is a tree, then the matrix A is irreducible if and only if $a_{ij} \neq 0$, whenever $a_{ji} \neq 0$. An irreducible sign pattern matrix whose graph is a tree is called a tree sign pattern matrix. Therefore a sign pattern matrix A is a tree sign pattern matrix if A is combinatorially symmetric and $G(A)$ is a tree. In particular, if this graph is a star or a path, then the sign pattern matrix is called a star sign pattern matrix or a path sign pattern matrix, respectively.

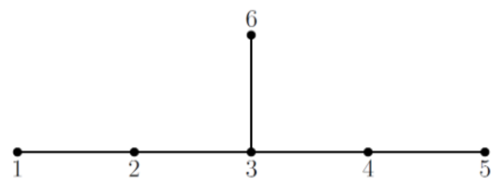
Eigenvalues of matrices in the qualitative class of a tree sign pattern matrix A depend on the signs a_{ii} and $a_{ij}a_{ji}$. Therefore each tree sign pattern matrix can be identified by a signed tree whose vertices may be signed $+, -, 0$ and whose edges may be signed $+, -$. Vertex i has the sign a_{ii}, and for $i \neq j$, if $a_{ij}a_{ji} \neq 0$, then the edge between vertices i and j has the sign $a_{ij}a_{ji}$. For example, a sign pattern matrix with its signed tree is as follows.

$$\begin{bmatrix} 0 & - & 0 & + & 0 \\ - & 0 & + & 0 & - \\ 0 & - & + & 0 & 0 \\ - & 0 & 0 & + & 0 \\ 0 & - & 0 & 0 & 0 \end{bmatrix}$$



A tree sign pattern matrix is symmetric if each edge of its signed tree is $+$, and skew-symmetric if each edge is $-$.

A set of edges in an undirected graph G is called a matching if no two edges in that set have a common end vertex. A perfect matching of a graph G is a matching such that each vertex of G is an end vertex of an edge in that matching. It is a well known fact that 'a tree has a perfect matching if and only if it has a unique perfect matching'. For example, if G is an undirected graph as follows, then $\{[1,2],[3,6],[4,5]\}$ is the perfect matching of G.



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1.2 Origin and Progression of Sign Pattern Matrices

Samuelson [47] posed the problem of determining the conditions under which the sign pattern of the solution vector of a system of linear equations can be determined only from the signs of the system matrix and the problem of determining the stability of a matrix from its sign pattern. The linear system in the first problem is called a sign-solvable linear system, formally defined as follows.

Let $A, \mathbf{x}, \mathbf{b}$ be sign pattern matrices of order $m \times n, n \times 1, m \times 1$ respectively. The linear system $A\mathbf{x} = \mathbf{b}$ is said to be sign-solvable if for each $\tilde{A} \in Q(A)$ and $\tilde{\mathbf{b}} \in Q(\mathbf{b})$, $\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$ is solvable and $\{\tilde{\mathbf{x}} :$ there exists $\tilde{A} \in Q(A)$ and $\tilde{\mathbf{b}} \in Q(\mathbf{b})$ with $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}\}$ is entirely contained in one qualitative class.

Lancaster [37] claimed that for a system $C\mathbf{x} = \mathbf{0}$ with n homogeneous equations and $(n+1)$ variables, the necessary and sufficient condition for all the ratios $\frac{x_i}{x_j}$ to be signed is that the system is, or can be brought into $\overline{C}\overline{\mathbf{x}} = 0$ by permutation and negation of rows and columns, where $\overline{\mathbf{x}}$ is related to $\mathbf{x}$ by sign changes only and $\overline{C}$ has the sign pattern

$$\begin{bmatrix} - & + & + & \cdots & + \\ 0 & - & + & \cdots & + \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & - & + \end{bmatrix}$$

or a degenerate version of it. Gorman [24] pointed out that the above condition is not necessary, and Lancaster [38] mentioned that $\overline{C}$ can be partitioned into

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \\ A_1 & O \\ O & A_2 \end{bmatrix},$$

where $\mathbf{a}_1$ is a $1 \times k$ matrix, $\mathbf{a}_2$ is a $1 \times (n-k+1)$ matrix, A_1 is a $(k-1) \times k$ matrix, A_2 is a $(n-k) \times (n-k+1)$ matrix, and either A_1, A_2 are of the above form or they can be partitioned again.

Let L_n be the vector space generated by n mutually orthogonal oriented axes intersecting at the origin. Lancaster [39] gave a necessary and sufficient condition for a system $A\mathbf{x} = \mathbf{0}$ to have a unique solution in the positive orthant of L_n, where A is an $n \times (n+1)$ sign pattern matrix.

Bassett, Maybee, and Quirk [2] described sign-solvable linear systems $A\mathbf{x} = \mathbf{b}$, where A is a square sign pattern matrix of order n. $A\mathbf{x} = \mathbf{b}$ is sign solvable if and only if after transformation into the system $\hat{A}\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{x}} \leq \mathbf{0}, \hat{\mathbf{b}} \geq \mathbf{0}$ and all diagonal elements of $\hat{A}$ are negative, the following conditions are satisfied:

1. every cycle of $\hat{A}$ is nonpositive, and
2. $\hat{b}_{i_r} > 0$ implies every chain $a(j \to i_r)$[1] in $\hat{A}$ is nonnegative.

A complete characterization of the sign-solvable linear systems is given in [9].

Theorem 1.2 ([9]). *Let A be an $m \times n$ matrix and $\mathbf{b}$ be an $m \times 1$ vector. Suppose $z = (z_1, \ldots, z_n)^T$ be a solution of $A\mathbf{x} = \mathbf{b}$. Let*

$$\beta = \{j : z_j \neq 0\} \text{ and } \alpha = \{i : a_{ij} \neq 0 \text{ for some } j \in \beta\}.$$

Then $A\mathbf{x} = \mathbf{b}$ is sign solvable if and only if the matrix

$$\begin{bmatrix} A[\alpha, \beta] & -\mathbf{b}[\alpha] \end{bmatrix}$$

is an S^*-matrix[2] *and the matrix $A(\alpha, \beta)^T$ is an L-matrix*[3].

[1] $a(i \to j)$ is the product $a_{ij_1}a_{j_1j_2} \cdots a_{j_{r-1}j_r}a_{j_rj}$, where the indices $i, j_1, \ldots, j_r, j$ are distinct.

[2] An $n \times (n+1)$ matrix is an S^*-matrix if each of the $(n+1)$ submatrices of order n requires nonsingularity.

[3] A matrix is an L-matrix provided every matrix in its qualitative class has linearly independent rows.

A square matrix A is said to be semi-stable if all eigenvalues of A have nonpositive real part, and stable if all eigenvalues of A have negative real part. A sign pattern matrix A is sign semi-stable (sign stable) if all matrices in $Q(A)$ are semi-stable (stable).

Quirk and Ruppert [46] described all sign semi-stable patterns. A sign pattern matrix A is sign semi-stable if and only if it satisfies the following conditions:

1. $a_{ii} \leq 0$ for all i;
2. $a_{ij}a_{ji} \leq 0$ for all $i \neq j$;
3. $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1} = 0$ for each sequence of $k(\geq 3)$ indices $i_1, i_2, \ldots, i_k$.

Jeffries, Klee, and van den Driessche [31] characterized all sign stable patterns, which was presented later in matrix form by Brualdi and Shader [9]. A sign pattern matrix A is sign stable if and only if it satisfies the following conditions:

1. $a_{ii} \leq 0$ for all i;
2. $a_{ij}a_{ji} \leq 0$ for all $i \neq j$;
3. The digraph $D(A)$ of A is a doubly directed tree;
4. A does not have an identically zero determinant;
5. There does not exist a non-empty subset β of $\{1, 2, \ldots, n\}$ such that diagonal element of $A[\beta]$ is zero, each row of $A[\beta]$ contains at least one nonzero entry, and no row of $A[\overline{\beta}, \beta]$ contains exactly one nonzero entry.

A sign pattern matrix A is said to be potentially stable if there exists a stable matrix in $Q(A)$. Potentially stable sign pattern matrices are studied in [5, 21, 25, 30, 32, 45].

Sign pattern matrices that require repeated or distinct eigenvalues have been studied in [16, 17, 34, 40]. Sign pattern matrices allowing diagonalizability have been studied in [19, 20, 48]. Sign patterns of matrices with some properties related to the leading principal minors have been studied in [33, 44]. Sign patterns of matrices diagonalizing an irreducible nonnegative matrix, or whose certain powers are nonnegative matrices have been studied in [10, 15, 35, 50]. Spectrally arbitrary patterns are studied in [1, 4, 13, 14, 22, 23, 43]. Sign pattern matrices have applications in different disciplines such as Chemistry, Biology, etc., which can be found in [6, 12, 28, 41].

Brualdi and Shader [9] summarized some of the above results in this topic in their book "Matrices of Sign-solvable Linear Systems". Further, Hall and Li [26] surveyed some of these works and applications in different disciplines in the book "Handbook of Linear Algebra".

1.3 A Brief Description of the Progress on the Topics of Research

In this dissertation, we focus on tree sign pattern matrices. We try to find the combinatorial structures of sign pattern matrices which allow or require algebraic positivity or diagonalizability. We also try to identify the associated graphs of certain sign pattern matrices which are potentially stable.

A matrix with each entry positive is said to be a positive matrix. Kirkland, Qiao, and Zhan [36] introduced the concept of algebraically positive matrices in 2016. A real square matrix is algebraically positive if there exists a real polynomial f such that $f(A)$ is a positive matrix. Kirkland, Qiao, and Zhan proved that a real square matrix is algebraically positive if and only if it has a simple real eigenvalue and corresponding left and right positive eigenvectors. They defined the index of algebraic positivity of A to be the least degree of a real polynomial f such that $f(A)$ is a positive matrix.

Let A be a real matrix of order n. If $h(x)$ is the characteristic polynomial of A in x, then we know by Cayley-Hamilton theorem that $h(A)$ is the zero matrix. Therefore any polynomial in A can be expressed as a polynomial in A of degree at most $n-1$. So it is enough to consider the real polynomials in A of degree at most $n-1$ to check whether A is algebraically positive. Therefore the index of algebraic positivity of a real square matrix A of order n is at most $n-1$. Kirkland, Qiao, and Zhan proved that for each n, k with $1 \leq k \leq n-1$, there exists an algebraically positive matrix of order n whose index of algebraic positivity is k. They identified all symmetric tridiagonal sign pattern matrices that allow algebraic positivity, and all 3×3 symmetric tridiagonal sign pattern matrices that require algebraic positivity.

The problem of characterizing sign pattern matrices allowing diagonalizability emerged from the study of sign patterns requiring repeated eigenvalues by Eschenbach and Johnson in [16]. For a sign pattern matrix A of order n, Eschenbach and Johnson [16] proved that if $c(A) \geq n-1$, then A allows diagonalizability. They proved that for a sign pattern matrix A of order n, $z(A) \leq n - \mathrm{mr}(A)$ is necessary for A to allow diagonalizability, where $z(A)$ denotes the minimum algebraic multiplicity of the eigenvalue 0 occurring among matrices in $Q(A)$. They also conjectured that this condition is also sufficient. However, Shao and Gao [48] showed that the conjecture is not true. They proved that if $c(A) = \mathrm{MR}(A)$ or A is combinatorially symmetric or A has some chordless k-cycle with $\mathrm{mr}(A) \leq k \leq \mathrm{MR}(A)$, then A allows diagonalizability.

It was conjectured by Feng et al. [18] and proved by Choi et al. [11] that if A is an invertible matrix, then there exists an invertible diagonal matrix D such that AD has distinct eigenvalues. Feng et al. [19] observed that D can be chosen so that all diagonal entries of D are positive. Therefore if a sign pattern matrix allows nonsingularity, then it allows diagonalizability. They also give two characterizations of sign pattern matrices allowing diagonalizability in terms of other allow problems.

Theorem 1.3 ([19]). *An $n \times n$ sign pattern matrix A allows diagonalizability if and only if there exists a real matrix $B \in Q(A)$ with $\mathrm{rank}(B) = k$, such that B has a nonsingular $k \times k$ principal submatrix.*

Theorem 1.4 ([19]). *An $n \times n$ sign pattern matrix A allows diagonalizability if and only if there exists a real matrix $B \in Q(A)$ such that $\mathrm{rank}(B) = \mathrm{rank}(B^2)$.*

Recently, Feng et al. [20] discussed the problem of sign patterns allowing diagonalizability with conditions on minimum rank.

Quirk [45] initiated the investigation of potentially stable patterns. He described all 3×3 potentially stable sign pattern matrices with nonpositive diagonal entries.

Let A be a tree sign pattern matrix of order n. The skew-symmetric factorization of A is $A = S_1A_1$, where S_1 is a signature sign pattern matrix with $(1,1)$ entry $+$ and A_1 is a skew- symmetric tree sign pattern matrix. The symmetric factorization of A is $A = S_2A_2$, where S_2 is a signature sign pattern matrix with $(1,1)$ entry $+$ and A_2 is a symmetric tree sign pattern matrix. Let $i_+(S)$ denotes the number of $+$, and $i_-(S)$ denotes the number of $-$ of a signature sign pattern matrix S. Jeffries and Johnson [30] gave the following results to identify some tree sign pattern matrices that are not potentially stable.

Theorem 1.5 ([30]). *Let $A = S_1A_1$ be the skew-symmetric factorization of an $n \times n$ tree sign pattern matrix A. If no diagonal entries of A_1 are $-$, then*

$$i_+(A) \leq i_+(S_1) \text{ and } i_-(A) \leq i_-(S_1)$$

for all matrices $B \in Q(A)$. If no diagonal entries of A_1 are $+$, then

$$i_+(A) \leq i_-(S_1) \text{ and } i_-(A) \leq i_+(S_1)$$

for all matrices $B \in Q(A)$.

Theorem 1.6 ([30]). *Let $A = S_2A_2$ be the symmetric factorization of an $n \times n$ tree sign pattern matrix A. If A is potentially stable, then there is a symmetric matrix $B_2 \in Q(A_2)$ such that*

$$i_+(B_2) = n - i_+(S_2).$$

Using these two results, Johnson and Summers [32] listed all potentially stable tree sign patterns of dimensions less than five. Gao and Li [21] produced a complete description of all potentially stable star sign pattern matrices. Few methods to construct higher order potentially stable sign pattern matrices from lower order potentially stable sign pattern matrices are given in [5, 25].

1.4 Thesis Overview

We describe the contents of the remaining part of the thesis briefly as follows.

Chapter 2. In this chapter, we give an alternate characterization of algebraically positive matrices, and give a sufficient condition for a matrix to be algebraically positive.

Chapter 3. In this chapter, we focus on the sign patterns of algebraically positive matrices. We identify all tree sign pattern matrices that allow algebraic positivity, and all star and path sign pattern matrices that require algebraic positivity. Further, we characterize all sign pattern matrices of order less than 6 requiring algebraic positivity.

Chapter 4. In this chapter, we define 'essential index' for a tree sign pattern matrix and characterize all tree sign pattern matrices requiring singularity in terms of the existence of essential indices. Further, we give a result regarding column spaces of matrices in the qualitative class of a tree sign pattern matrix. Moreover, we give a sufficient condition for a sign pattern matrix whose graph is a tree to allow diagonalizability in terms of connectivity of its essential indices. Also, we characterize sign pattern matrices whose associated graphs are either star or path to allow diagonalizability. Finally, we give a necessary condition for sign pattern matrices that require diagonalizability, and characterize star sign pattern matrices requiring diagonalizability.

Chapter 5. In this chapter, we give several sufficient conditions for a tree sign pattern matrix with negative edges to allow a properly signed nest. Johnson, Maybee, Olesky, and van den Driessche [33] proved that if a sign pattern matrix allows a properly signed nest, then it is potentially stable. However, the converse is not true, even for tree sign pattern matrices. For example,

$$\begin{bmatrix} -5 & 1 & 0 & 0 \\ 25 & 0 & 1 & 0 \\ 0 & -700 & 0 & 1 \\ 0 & 0 & 150 & 1 \end{bmatrix}$$

has eigenvalues $-3.8699, -0.1244$, and $-0.0029 \pm 22.7925i$, and thus is stable, but has no properly signed nest.

We believe that the converse is true for tree sign pattern matrices with negative edges, which we propose as a conjecture. We prove that this conjecture is true for tree sign pattern matrices with negative edges of order at most 6. Further, we identify all potentially stable star and path sign pattern matrices with negative edges, and prove that the conjecture is valid for these classes. Drew et al. [14] introduced the concept of spectrally arbitrary patterns. A sign pattern matrix A of order n is a spectrally arbitrary pattern if, for any given real monic polynomial $r(x)$ of degree n, there is a matrix in $Q(A)$ with characteristic polynomial $r(x)$. As a consequence of the results on potentially stable sign pattern matrices with negative edges, we describe all 5-by-5 spectrally arbitrary tree sign pattern matrices with negative edges.

Chapter 6. We conclude with a brief summary of the previous chapters. We also discuss possible future directions of our research.

CHAPTER 2

Algebraically Positive Matrices

2.1 Backgrounds

A positive matrix is a matrix all of whose entries are positive real numbers. If A is a positive matrix, then we write $A > 0$. Kirkland, Qiao, and Zhan [36] introduced the concept of algebraic positivity in 2016.

Definition 2.1 ([36]). *A square real matrix A is said to be algebraically positive if there exists a real polynomial f such that $f(A)$ is a positive matrix.*

For example,

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

is algebraically positive, since

$$3I + A - A^2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 4 & -2 & -1 \\ -1 & 2 & 0 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} > 0.$$

The authors of [36] give the following characterization of algebraic positive matrices.

Theorem 2.2 ([36], Theorem 1). *A real matrix A is algebraically positive if and only if it has a simple real eigenvalue and corresponding left and right positive eigenvectors.*

The following results are some necessary/sufficient conditions for an algebraically positive matrix.

Theorem 2.3 ([36]). *If A is a real square matrix, then we have the following.*

1. *If A is algebraically positive, then A is irreducible.*
2. *If there is a positive integer k such that A^k is algebraically positive, then A is algebraically positive.*
3. *If A is irreducible and all of its off-diagonal entries are nonnegative (or nonpositive), then A is algebraically positive.*

In the following section, we give an alternative characterization for algebraically positive matrices, and a sufficient condition for a matrix to be algebraically positive.

2.2 An Alternative Characterization of Algebraically Positive Matrices

Lemma 2.4. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. A real matrix A commutes with $\mathbf{x}\,\mathbf{y}^T$ if and only if there exists a $\lambda \in \mathbb{R}$ such that $A\,\mathbf{x} = \lambda\,\mathbf{x}$ and $\mathbf{y}^T A = \lambda\,\mathbf{y}^T$.*

Proof. Suppose $A\,\mathbf{x}\,\mathbf{y}^T = \mathbf{x}\,\mathbf{y}^T A$. Then $(\mathbf{x}^T A\,\mathbf{x})(\mathbf{y}^T\mathbf{y}) = (\mathbf{x}^T\mathbf{x})(\mathbf{y}^T A\,\mathbf{y})$. Then

$$\frac{\mathbf{x}^T A\,\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{\mathbf{y}^T A\,\mathbf{y}}{\mathbf{y}^T\mathbf{y}} = \lambda, \text{ say.}$$

Further, $A\,\mathbf{x}(\mathbf{y}^T\mathbf{y}) = \mathbf{x}(\mathbf{y}^T A\,\mathbf{y})$ that is, $A\,\mathbf{x} = \lambda\,\mathbf{x}$, and $(\mathbf{x}^T A\,\mathbf{x})\,\mathbf{y}^T = (\mathbf{x}^T\mathbf{x})\,\mathbf{y}^T A$ that is, $\mathbf{y}^T A = \lambda\,\mathbf{y}^T$.

Conversely, assume there exists a $\lambda \in \mathbb{R}$ such that $A\,\mathbf{x} = \lambda\,\mathbf{x}$ and $\mathbf{y}^T A = \lambda\,\mathbf{y}^T$. Therefore, $A\,\mathbf{x}\,\mathbf{y}^T = \lambda\,\mathbf{x}\,\mathbf{y}^T = \mathbf{x}(\lambda\,\mathbf{y}^T) = \mathbf{x}\,\mathbf{y}^T A$. □

Let for a square matrix A, the null space, row space and column space be denoted by $\mathcal{N}(A), \mathcal{R}(A)$ and $\mathcal{C}(A)$ respectively. For a set $S \subseteq \mathbb{R}^n$, we consider

$$S^{\perp} = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y}^T\mathbf{x} = 0 \text{ for all } \mathbf{x} \in S\}.$$

We have the following lemma from [29, pp. 78-80]. However, we give an alternate proof of this lemma.

Lemma 2.5 ([29]). *Let A be a real square matrix and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{y}^T A = \mu\mathbf{y}^T$ for some $\lambda, \mu \in \mathbb{R}$.*

1. *If $\lambda \neq \mu$, then $\mathbf{y}^T\mathbf{x} = 0$.*
2. *If $\lambda = \mu$ and λ has geometric multiplicity 1, then its algebraic multiplicity is 1 if and only if $\mathbf{y}^T\mathbf{x} \neq 0$.*

Proof.

1. Since $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{y}^T A = \mu\mathbf{y}^T$, $\lambda\mathbf{y}^T\mathbf{x} = \mathbf{y}^T A\mathbf{x} = \mu\mathbf{y}^T\mathbf{x}$. Thus $\lambda \neq \mu$ implies $\mathbf{y}^T\mathbf{x} = 0$.

2. Suppose $\lambda = \mu$ and λ has geometric multiplicity 1. Since $\mathbf{y}^T A = \lambda\mathbf{y}^T$,
$$\mathbf{y} \in \mathcal{N}(A^T - \lambda I) = [\mathcal{R}(A^T - \lambda I)]^\perp = [\mathcal{C}(A - \lambda I)]^\perp.$$
Let the algebraic multiplicity of λ be 1. Then $\mathcal{N}((A-\lambda I)^2) = \mathcal{N}(A-\lambda I)$. If $\mathbf{y}^T\mathbf{x} = 0$, then $\mathbf{x} \in \mathcal{C}(A-\lambda I)$ and thus there exists $\mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{x} = (A-\lambda I)\mathbf{z}$. Now $A\mathbf{x} = \lambda\mathbf{x}$ implies $\mathbf{x} \in \mathcal{N}(A - \lambda I)$. So $(A - \lambda I)^2\mathbf{z} = 0$ and thus $\mathbf{z} \in \mathcal{N}(A - \lambda I)^2 = \mathcal{N}(A - \lambda I)$. Therefore $\mathbf{x} = 0$, a contradiction. Thus algebraic multiplicity of λ is 1 implies $\mathbf{y}^T\mathbf{x} \neq 0$.

   If the algebraic multiplicity of λ is greater than 1, then since the geometric multiplicity of λ is equal to 1,
$$\mathcal{N}(A - \lambda I) \subsetneqq \mathcal{N}((A - \lambda I)^2).$$
So there exists $\mathbf{z} \in \mathbb{R}^n$ such that $(A - \lambda I)\mathbf{z} \neq 0$ but $(A - \lambda I)^2\mathbf{z} = 0$, which implies $(A - \lambda I)\mathbf{z} \in \mathcal{N}(A - \lambda I)$. Since $A\mathbf{x} = \lambda\mathbf{x}$ and the geometric multiplicity of λ is 1, there exists $\alpha \neq 0$ such that $(A - \lambda I)\mathbf{z} = \alpha\mathbf{x}$. Further, $(A - \lambda I)\mathbf{z} \in \mathcal{C}(A - \lambda I)$ and $\mathbf{y} \in [\mathcal{C}(A - \lambda I)]^\perp$ implies $\alpha\mathbf{y}^T\mathbf{x} = 0$ that is, $\mathbf{y}^T\mathbf{x} = 0$. Therefore $\mathbf{y}^T\mathbf{x} \neq 0$ implies algebraic multiplicity of λ is 1. □

The following result characterizes all algebraically positive matrices.

Theorem 2.6. *A real square matrix A is algebraically positive if and only if it commutes with a unique (up to scalar multiplication) rank one positive matrix.*

Proof. Let A be algebraically positive. Then by Theorem 2.2, there exists $\lambda \in \mathbb{R}$ and positive vectors $\mathbf{x}, \mathbf{y}$ such that $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{y}^T A = \lambda\mathbf{y}^T$, and λ is a simple eigenvalue of A. Then by Lemma 2.4, A commutes with $\mathbf{x}\mathbf{y}^T$, a rank one positive matrix. If there exist positive vectors $\mathbf{u}, \mathbf{v}$ such that A commutes with $\mathbf{u}\mathbf{v}^T$, then by Lemma 2.4, $A\mathbf{u} = \mu\mathbf{u}$ and $\mathbf{v}^T A = \mu\mathbf{v}^T$. Since $\mathbf{v}^T\mathbf{x} \neq 0$, by Lemma 2.5.1, $\lambda = \mu$. Since λ is simple, $\mathbf{u}, \mathbf{v}$ are scalar multiples of $\mathbf{x}, \mathbf{y}$ respectively. Hence the uniqueness follows.

Suppose A commutes with a unique (up to scalar multiplication) rank one positive matrix $\mathbf{x}\mathbf{y}^T$. Then by Lemma 2.4, A has an eigenvalue λ such that $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{y}^T A = \lambda\mathbf{y}^T$, and both $\mathbf{x}, \mathbf{y}$ are positive vectors. If the geometric multiplicity of λ is at least 2, then there exists a nonzero vector $\mathbf{z}$ such that $\{\mathbf{z}, \mathbf{x}\}$ is linearly independent and $A\mathbf{z} = \lambda\mathbf{z}$. Then there exists an $\alpha \neq 0$ such that $\mathbf{x} + \alpha\mathbf{z}$ is a positive right eigenvector of A and $\{\mathbf{x} + \alpha\mathbf{z}, \mathbf{x}\}$ is linearly independent. Again by Lemma 2.4, A commutes with $\mathbf{z}\mathbf{y}^T$. Therefore A commutes with $(\mathbf{x} + \alpha\mathbf{z})\mathbf{y}^T$, a rank one positive matrix which is not a scalar multiple of $\mathbf{x}\mathbf{y}^T$. This contradicts the uniqueness property. So the geometric multiplicity of λ is 1. Therefore from Lemma 2.5.2 we can conclude that λ is simple. Hence the result follows from Theorem 2.2. □

For any square matrix, the following result is quite well known. Nevertheless, for the sake of completeness, we give a proof of it.

Lemma 2.7. *For a square matrix A, $\operatorname{adj}(A) = (-1)^{n+1}(A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I)$, where $c_k = (-1)^k$ (sum of all $k \times k$ principal minors of A) for $k = 1, 2, \ldots, n-1$.*

Proof. Let A be a square matrix of order n. By Cayley-Hamilton theorem we have

$$A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I = O,$$

where $c_k = (-1)^k$ (sum of all $k \times k$ principal minors of A) for $k = 1, 2, \ldots, n$.

If A is invertible, then

$$\operatorname{adj}(A) = \det(A) \cdot A^{-1} = (-1)^n c_n A^{-1} = (-1)^{n+1}(A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I).$$

For every matrix A of order n, there exists a sequence (A_t) of invertible matrices of order n such that (A_t) converges to A. Again, $\operatorname{adj}(A), c_1, c_2, \ldots, c_{n-1}$ are continuous functions of A. Therefore for every square matrix A of order n,

$$\operatorname{adj}(A) = (-1)^{n+1}(A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I),$$

where $c_k = (-1)^k$ (sum of all $k \times k$ principal minors of A) for $k = 1, 2, \ldots, n-1$. □

Hence if A is a real square matrix, then $\operatorname{adj}(A)$ is a real polynomial in A.

Theorem 2.8. *If* $\operatorname{adj}(A)$ *is algebraically positive for some real square matrix A, then A is algebraically positive.*

Proof. From Lemma 2.7, $\operatorname{adj}(A) = p(A)$ for some real polynomial p. Since $\operatorname{adj}(A)$ is algebraically positive, $f(\operatorname{adj}(A))$ is a positive matrix for some real polynomial f. Further, $f(\operatorname{adj}(A))$ is a real polynomial in A. Hence A is algebraically positive. □

However, the following example shows that the converse of the above result is not valid.

Example 2.9. *Let us consider the real matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

If we consider the polynomial $f(x) = x$, then $f(A) = A$ is a positive matrix. Therefore A is algebraically positive. But we see that $\operatorname{adj}(A)$ *is the zero matrix. So any polynomial in* $\operatorname{adj}(A)$ *is a scalar multiple of the identity matrix of order* 3*. Thus* $\operatorname{adj}(A)$ *is not algebraically positive.*

CHAPTER 3

Sign Patterns that Allow or Require Algebraic Positivity

In this chapter, we classify certain sign pattern matrices associated with some graphs that allow or require algebraically positive matrices. In Section 3.1, we define $\mathbb{T}$-pattern matrices, and in Section 3.2, we characterize all tree sign pattern matrices allowing algebraic positivity. In Section 3.3, we characterize all star and path sign pattern matrices requiring algebraic positivity. We also characterize all 5-by-5 tree sign pattern matrices requiring algebraic positivity.

3.1 $\mathbb{T}$-pattern Matrices

Let $\mathbb{T} = \{+, -, 0, +_0, -_0, \#\}$. Any matrix with entries from $\mathbb{T}$ is said to be a $\mathbb{T}$-pattern matrix. The qualitative class of a $\mathbb{T}$-pattern matrix A is denoted by $Q(A)$ and is defined by the set of all real matrices obtained from A replacing $+, -, 0, +_0, -_0, \#$ by a positive, negative, zero, nonnegative, nonpositive, and an arbitrary real number, respectively. A $\mathbb{T}$-pattern matrix A is said to be symmetric if both a_{ij} and a_{ji} have the same sign for all i, j. A $\mathbb{T}$-pattern matrix A allows a property P if at least one matrix in $Q(A)$ satisfies the property P, and A requires a property P if all matrices in $Q(A)$ satisfy the property P.

Let $\mathcal{F}(A)$ denotes the set of all sign pattern matrices obtained from a $\mathbb{T}$-pattern matrix A by fixing a possible sign in each entry. It is clear that if a $\mathbb{T}$-pattern matrix A requires a property P, then all sign pattern matrices in $\mathcal{F}(A)$ require the property P.

3.2 Tree Sign Pattern Matrices that Allow Algebraic Positivity

Kirkland, Qiao, and Zhan [36] give the following necessary condition for a sign pattern matrix to allow algebraic positivity.

Theorem 3.1 ([36]). *If a sign pattern matrix allows algebraic positivity, then every row and column contains a* $+$*, or every row and column contains a* $-$*.*

Let A be a real square matrix. Let us recall from [27] that a walk W in $G(A)$ is a sequence $(u_0, u_1, u_2, \cdots, u_{t-1}, u_t)$ of vertices (need not be distinct) such that u_{p-1}, u_p are adjacent in $G(A)$ for $p = 1, 2, \ldots, t$. If initial and terminal vertices in a walk are same, then the walk is called a circuit. If $W = (u_0, u_1, u_2, \ldots, u_{t-1}, u_t)$ is a walk in $G(A)$, then we call the value of the walk W as the product $a_{u_0u_1}a_{u_1u_2}\cdots a_{u_{t-1}u_t}$. The following result gives a necessary condition for a tree sign pattern matrix to allow algebraic positivity.

Theorem 3.2. *Let $\mathcal{T}$ be a tree sign pattern matrix of order n. If $\mathcal{T}$ allows algebraic positivity, then $\mathcal{T}$ is a symmetric sign pattern matrix.*

Proof. Suppose $\mathcal{T}$ is not a symmetric sign pattern matrix and $A \in Q(\mathcal{T})$. Since $\mathcal{T}$ is a tree sign pattern matrix, without loss of generality, we may assume that both a_{12}, a_{21} are nonzero and of different signs.

Since $G(A)$ is a tree, for every $m \geq 2$,

$$(A^m)_{12} = a_{12}\sum_{k=0}^{m-1} S_{1,k}S_{2,m-k-1} \text{ and } (A^m)_{21} = a_{21}\sum_{k=0}^{m-1} S_{1,k}S_{2,m-k-1},$$

where $S_{1,0} = S_{2,0} = 1$, and for each $j \in \mathbb{N}$, $S_{1,j}$ is the sum of values of the circuits of length j starting and ending at vertex 1, and $S_{2,j}$ is the sum of values of the circuits of length j

starting and ending at vertex 2. So for any $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$,

$$\left(\sum_{i=0}^{n-1} \alpha_i A^i\right)_{12} = \sum_{i=1}^{n-1} \alpha_i \left(A^i\right)_{12} = a_{12} \sum_{i=1}^{n-1} \alpha_i \sum_{k=0}^{i-1} S_{1,k} S_{2,i-k-1};$$
$$\left(\sum_{i=0}^{n-1} \alpha_i A^i\right)_{21} = \sum_{i=1}^{n-1} \alpha_i \left(A^i\right)_{21} = a_{21} \sum_{i=1}^{n-1} \alpha_i \sum_{k=0}^{i-1} S_{1,k} S_{2,i-k-1}.$$

We know that any real polynomial in A can be written as a real polynomial in A of degree at most $n-1$. Since a_{12} and a_{21} have different signs, no real polynomial in A is a positive matrix. Thus A is not algebraically positive. So $\mathcal{T}$ does not allow algebraic positivity. Hence the result follows. □

Kirkland, Qiao, and Zhan [36] characterized all irreducible symmetric tridiagonal sign pattern matrices allowing algebraic positivity. We note that an irreducible tridiagonal sign pattern matrix is a path sign pattern matrix.

Theorem 3.3 ([36]). *Suppose that A is an $n \times n$ irreducible symmetric tridiagonal sign pattern matrix. Then A allows algebraic positivity if and only if every row and column of A contains a $+$, or every row and column of A contains a $-$.*

In the following lemma, we show that if an irreducible matrix A has all row sums equal to 0 such that $G(A)$ is a tree, then 0 is an eigenvalue of A with geometric multiplicity 1.

Lemma 3.4. *Let A be an irreducible matrix of order n such that each row sum of A is 0. If $G(A)$ is a tree and $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$, then $x_1 = x_2 = \cdots = x_n$.*

Proof. We will prove the result by induction on n. For $n = 1$, the result is true. Suppose the result is true for any matrix of order $n-1$, and let A be of order n.

Let u, v be two vertices in $G(A)$ such that u is pendant and adjacent to v. Now a_{uu}, a_{uv} are the only nonzero entries in the u-th row of A. So $a_{uu} + a_{uv} = 0$ and $a_{uu}x_u + a_{uv}x_v = 0$. Therefore $x_u = x_v$. Let $\tilde{A}$ be the matrix obtained from A replacing a_v by $a_u + a_v$, where a_u, a_v are respectively the u-th column and the v-th column of A; and $B = \tilde{A}(\{u\})$ be the principal submatrix of $\tilde{A}$ obtained from $\tilde{A}$ by deleting the u-th row and the u-th column. Then B is irreducible, each row sum of B is 0, $G(B)$ is a tree, and the order of B is $n-1$.

Let $\tilde{\mathbf{x}}$ be the vector obtained from $\mathbf{x}$ by deleting the entry x_u. Since $x_u = x_v$ and $B\tilde{\mathbf{x}} = \mathbf{0}$, by induction hypothesis $x_j = x_v$ for all $j \in \{1, 2, \ldots, n\} \setminus \{u\}$. Thus $x_1 = x_2 = \cdots = x_n$. Hence the result follows. □

The following result describes all tree sign pattern matrices that allow algebraic positivity.

Theorem 3.5. *Let $\mathcal{T}$ be a tree sign pattern matrix. Then $\mathcal{T}$ allows algebraic positivity if and only if $\mathcal{T}$ is a symmetric sign pattern matrix and every row of $\mathcal{T}$ contains a $+$, or every row of $\mathcal{T}$ contains a $-$.*

Proof. Suppose $\mathcal{T}$ is a symmetric tree sign pattern matrix and every row of $\mathcal{T}$ contains a $+$, otherwise every row of $-\mathcal{T}$ contains a $+$. Suppose $n_+(i), n_-(i)$ are respectively the number of $+$ and the number of $-$ in the i-th row of $\mathcal{T}$. In the i-th row, replace $-$ by -1 and $+$ by $\frac{1+n_-(i)}{n_+(i)}$, and let B be the new matrix. Then $B \in Q(\mathcal{T})$ and each row sum of B is 1. So B has an eigenvalue 1 with corresponding right eigenvector $(1, \ldots, 1)^T$. Consider the matrix $B - I$ and denote it by A. Suppose $A\mathbf{x} = 0$, where $\mathbf{x} = (x_1, \ldots, x_n)^T$. Now A is irreducible, each row sum of A is 0, and $G(A)$ is a tree. So by Lemma 3.4, $x_1 = x_2 = \cdots = x_n$, and thus the geometric multiplicity of 1 as an eigenvalue of B is 1.

Since $B \in Q(\mathcal{T})$ and $\mathcal{T}$ is a symmetric tree sign pattern matrix, there exists a diagonal matrix D with positive diagonal entries such that DBD^{-1} is symmetric. Let $M = DBD^{-1}$. Then $M \in Q(\mathcal{T})$ and the geometric multiplicity of 1 as an eigenvalue of M is 1. Further, M is diagonalizable. Therefore 1 is a simple eigenvalue of M. Let $D = \text{diag}(d_1, \ldots, d_n)$ and assume $\mathbf{u} = (d_1, \ldots, d_n)^T$. Then $M^T\mathbf{u} = M\mathbf{u} = \mathbf{u}$, and $\mathbf{u}$ is a positive vector. Therefore by Theorem 2.2, M is algebraically positive, and hence $\mathcal{T}$ allows algebraic positivity.

Proof of the converse part follows from Theorem 3.1 and Theorem 3.2. □

3.3 Tree Sign Pattern Matrices that Require Algebraic Positivity

A characterization of all irreducible 3×3 symmetric tridiagonal sign pattern matrices that require algebraic positivity was given by Kirkland, Qiao, and Zhan [36].

Theorem 3.6 ([36]). *Suppose that A is an irreducible 3×3 symmetric tridiagonal sign pattern matrix. Then A requires algebraic positivity if and only if one of the following holds.*

1. *All nonzero off-diagonal entries of A are $+$.*
2. *All nonzero off-diagonal entries of A are $-$.*
3. *A or $-A$ is permutationally similar to some sign pattern matrix from*

$$\left\{ \begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & + & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ + & + & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & - & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 \\ + & - & - \\ 0 & - & + \end{bmatrix} \right\}.$$

Let A be a real square matrix of order n such that $A\mathbf{u} = \lambda\mathbf{u}$, where $\mathbf{u} = (u_1, \ldots, u_n)^T$ is a positive vector and $\lambda \in \mathbb{R}$. Then $D^{-1}AD\mathbf{1} = \lambda\mathbf{1}$, where $D = \text{diag}(u_1, \ldots, u_n)$. So using the proof of Theorem 3.5, we have the following result.

Theorem 3.7. *If $\mathcal{T}$ is a tree sign pattern matrix, then $\mathcal{T}$ requires algebraic positivity if and only if $\mathcal{T}$ is a symmetric sign pattern matrix and every matrix in $Q(\mathcal{T})$ has a positive right eigenvector.*

Now we will find some combinatorial structure for some classes of tree sign pattern matrices requiring algebraic positivity. From Theorem 2.3.3, we have the following lemma.

Lemma 3.8. *If all the nonzero off-diagonal entries of an irreducible sign pattern matrix are of the same sign ($+$ or $-$), then that sign pattern matrix requires algebraic positivity.*

If a tree sign pattern matrix is symmetric in sign, then each matrix in its qualitative class is similar to a symmetric matrix through a diagonal matrix with each diagonal entry positive. So we have the following observation.

Remark 3.9. *A symmetric tree sign pattern matrix requires algebraic positivity if and only if every symmetric matrix in its qualitative class is algebraically positive.*

3.3.1 Star Sign Pattern Matrices Requiring Algebraic Positivity

For each $p, q, r \in \mathbb{N}$, let us consider a $\mathbb{T}$-pattern matrix

$$\mathcal{X}_{p,q,r} = \begin{bmatrix} \# & \overbrace{- \ \cdots \ -}^{p \text{ times}} & \overbrace{+ \ \cdots \ +}^{q \text{ times}} & \overbrace{+ \ \cdots \ +}^{r \text{ times}} \\ - & + & & \\ \vdots & \ddots & & \\ - & \quad + & & \\ + & & - & \\ \vdots & & \ddots & \\ + & & \quad - & \\ + & & & 0 \\ \vdots & & & \ddots \\ + & & & \quad 0 \end{bmatrix},$$

where all unspecified entries are zero.

Lemma 3.10. *For each $p, q, r \in \mathbb{N}$, $\mathcal{X}_{p,q,r}$ requires algebraic positivity.*

Proof. By Remark 3.9, it is enough to show that every symmetric matrix in $Q(\mathcal{X}_{p,q,r})$ is algebraically positive. Let

$$A = \begin{bmatrix} a & -b_1 & \cdots & -b_p & d_1 & \cdots & d_q & e_1 & \cdots & e_r \\ -b_1 & a_1 & & & & & & & & \\ \vdots & & \ddots & & & & & & & \\ -b_p & & & a_p & & & & & & \\ d_1 & & & & -c_1 & & & & & \\ \vdots & & & & & \ddots & & & & \\ d_q & & & & & & -c_q & & & \\ e_1 & & & & & & & 0 & & \\ \vdots & & & & & & & & \ddots & \\ e_r & & & & & & & & & 0 \end{bmatrix}$$

be a symmetric matrix in $Q(\mathcal{X}_{p,q,r})$, where a_i, b_i, c_i, d_i, e_is are positive and a is of arbitrary sign. The characteristic polynomial of A is given by

$$f(x) = x^r(x-a)\prod_{i=1}^{p}(x-a_i)\prod_{i=1}^{q}(x+c_i) - x^r\sum_{i=1}^{p} b_i^2 \prod_{\substack{j=1\\ j\neq i}}^{p}(x-a_j)\prod_{j=1}^{q}(x+c_j)$$
$$- x^r\sum_{i=1}^{q} d_i^2 \prod_{j=1}^{p}(x-a_j)\prod_{\substack{j=1\\ j\neq i}}^{q}(x+c_j) - x^{r-1}\sum_{i=1}^{r} e_i^2 \prod_{j=1}^{p}(x-a_j)\prod_{j=1}^{q}(x+c_j).$$

Now $\frac{1}{x^{r-1}}f(x)\big|_{x=0} = (-1)^{p+1}\sum_{i=1}^{r} e_i^2 \prod_{j=1}^{p} a_j \prod_{j=1}^{q} c_j \neq 0$ and x^{r-1} is a factor of $f(x)$. So the algebraic multiplicity of zero as an eigenvalue of A is $r-1$. Without loss of generality we may assume that $a_1 \leq a_2 \leq \cdots \leq a_m < a_{m+1} \leq \cdots \leq a_p$. Suppose $a_1 = \cdots = a_m$.

Now $\frac{1}{(x-a_1)^{m-1}}f(x)\Big|_{x=a_1} = -a_1^r \sum_{i=1}^{m} b_i^2 \prod_{j=m+1}^{p}(a_1 - a_j)\prod_{j=1}^{q}(a_1 + c_j) \neq 0$ and $(x-a_1)^{m-1}$ is a factor of $f(x)$. So the algebraic multiplicity of a_1 as an eigenvalue of A is $m-1$. Now eigenvalues of $A(\{1\})$ are $a_1, \ldots, a_p$, $-c_1, \ldots, -c_q$, 0 (r times) where a_1 occurs m times. So by Cauchy's interlacing theorem ([29, Theorem 4.3.17]), there exists a simple eigenvalue λ of A such that $0 < \lambda < a_1$. Now we will show that there is a positive right eigenvector of A corresponding to this eigenvalue λ.

We can rewrite A as

$$A = \begin{bmatrix} a & -\mathbf{x}^T & \mathbf{y}^T & \mathbf{z}^T \\ -\mathbf{x} & D_1 & O & O \\ \mathbf{y} & O & -D_2 & O \\ \mathbf{z} & O & O & O \end{bmatrix},$$

where $\mathbf{x} = (b_1, \ldots, b_p)^T, \mathbf{y} = (d_1, \ldots, d_q)^T, \mathbf{z} = (e_1, \ldots, e_r)^T, D_1 = \text{diag}(a_1, \ldots, a_p)$ and $D_2 = \text{diag}(c_1, \ldots, c_q)$.

Let $\begin{bmatrix} \alpha & \mathbf{u}^T & \mathbf{v}^T & \mathbf{w}^T \end{bmatrix}^T$ be a right eigenvector of A corresponding to the eigenvalue λ, where $\alpha \in \mathbb{R}, \mathbf{u} \in \mathbb{R}^p, \mathbf{v} \in \mathbb{R}^q, \mathbf{w} \in \mathbb{R}^r$. Then we have

$$-\alpha\,\mathbf{x} + D_1\mathbf{u} = \lambda\mathbf{u} \quad \text{or} \quad \mathbf{u} = \alpha(D_1 - \lambda I)^{-1}\,\mathbf{x},$$

$$\alpha\,\mathbf{y} - D_2\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad \mathbf{v} = \alpha(D_2 + \lambda I)^{-1}\,\mathbf{y},$$

$$\alpha\mathbf{z} = \lambda\mathbf{w} \quad \text{or} \quad \mathbf{w} = \frac{\alpha}{\lambda}\mathbf{z}.$$

If $\alpha = 0$, then $\mathbf{u} = \mathbf{0}, \mathbf{v} = \mathbf{0}, \mathbf{w} = \mathbf{0}$. So we can assume $\alpha = 1$. Then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are positive vectors, as $\lambda > 0$, and $D_1 - \lambda I, D_2 + \lambda I$ are diagonal matrices with positive diagonal entries, and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are positive vectors. Since for a symmetric matrix left and right eigenvectors corresponding to an eigenvalue are same, A has positive left and right eigenvectors corresponding to the simple eigenvalue λ. So A is algebraically positive. □

For each $p, q \in \mathbb{N}$, let us consider the $\mathbb{T}$-pattern matrices $\mathcal{Y}_{p,q}$ and $\mathcal{Z}_{p,q}$ as

$$\mathcal{Y}_{p,q} = \begin{bmatrix} \# & \overbrace{- \ \cdots \ -}^{p \text{ times}} & \overbrace{+ \ \cdots \ +}^{q \text{ times}} \\ - & + & \\ \vdots & \ddots & \\ - & \quad + & \\ + & & - \\ \vdots & & \ddots \\ + & & \quad - \end{bmatrix}, \quad \mathcal{Z}_{p,q} = \begin{bmatrix} \# & \overbrace{- \ \cdots \ -}^{p \text{ times}} & \overbrace{+ \ \cdots \ +}^{q \text{ times}} \\ - & + & \\ \vdots & \ddots & \\ - & \quad + & \\ + & & 0 \\ \vdots & & \ddots \\ + & & \quad 0 \end{bmatrix}.$$

Using similar arguments as in the proof of Lemma 3.10, we can show that the $\mathbb{T}$-pattern matrices $\mathcal{Y}_{p,q}, \mathcal{Z}_{p,q}$ $(p, q \geq 1)$ require algebraic positivity.

Let $\mathcal{X}^{+}_{p,q,r}$, $\mathcal{X}^{-}_{p,q,r}$, $\mathcal{X}^{0}_{p,q,r}$ be the sign pattern matrices obtained from $\mathcal{X}_{p,q,r}$ by replacing $(1, 1)$ entry with $+, -, 0$ respectively. Let $\mathcal{Y}^{+}_{p,q}$, $\mathcal{Y}^{-}_{p,q}$, $\mathcal{Y}^{0}_{p,q}$, $\mathcal{Z}^{+}_{p,q}$, $\mathcal{Z}^{-}_{p,q}$, $\mathcal{Z}^{0}_{p,q}$ be the same type of sign patterns obtained from $\mathcal{Y}_{p,q}, \mathcal{Z}_{p,q}$. Then we have the following result.

Lemma 3.11. *For each $p, q, r \in \mathbb{N}$, the sign pattern matrices $\mathcal{X}^{+}_{p,q,r}$, $\mathcal{X}^{-}_{p,q,r}$, $\mathcal{X}^{0}_{p,q,r}$, $\mathcal{Y}^{+}_{p,q}$, $\mathcal{Y}^{-}_{p,q}$, $\mathcal{Y}^{0}_{p,q}$, $\mathcal{Z}^{+}_{p,q}$, $\mathcal{Z}^{-}_{p,q}$, $\mathcal{Z}^{0}_{p,q}$ require algebraic positivity.*

The following theorem is a characterization of all star sign pattern matrices that require algebraic positivity.

Theorem 3.12. *Let A be a star sign pattern matrix. Then A requires algebraic positivity if and only if one of the following conditions hold.*

1. *All nonzero off-diagonal entries of A are $+$.*

2. *All nonzero off-diagonal entries of A are $-$.*

3. *A or $-A$ is permutationally similar to some sign pattern matrix from*

$$\{\mathcal{X}^{+}_{p,q,r},\ \mathcal{X}^{-}_{p,q,r},\ \mathcal{X}^{0}_{p,q,r},\ \mathcal{Y}^{+}_{p,q},\ \mathcal{Y}^{-}_{p,q},\ \mathcal{Y}^{0}_{p,q},\ \mathcal{Z}^{+}_{p,q},\ \mathcal{Z}^{-}_{p,q},\ \mathcal{Z}^{0}_{p,q}\}$$

for some suitable $p, q, r \in \mathbb{N}$.

Proof. Suppose A requires algebraic positivity. Then A allows algebraic positivity. So by Theorem 3.5, A is a symmetric sign pattern matrix, and each row of A contains a $+$ or each row of A contains a $-$. Let $B \in Q(A)$, and

$$B = \begin{bmatrix} a_1 & c_2 & c_3 & \dots & c_n \\ b_2 & a_2 & 0 & \cdots & 0 \\ b_3 & 0 & a_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_n & 0 & \cdots & 0 & a_n \end{bmatrix},$$

where at least one $b_i > 0$ and at least one $b_i < 0$.

Suppose each row of A contains a $+$. Then $b_i < 0$ implies $a_i > 0$ for $i = 2, 3, \dots, n$. Since B is algebraically positive, let $(x_1, \dots, x_n)^T$ be a positive right eigenvector of B corresponding to a real eigenvalue λ. Then

$$b_i x_1 + a_i x_i = \lambda x_i \quad \text{for } i = 2, \dots, n$$
$$\text{or, } b_i x_1 = (\lambda - a_i) x_i \quad \text{for } i = 2, \dots, n.$$

Then $b_i > 0$ implies $\lambda > a_i$, and $b_i < 0$ implies $\lambda < a_i$ for $i = 2, \dots, n$. Thus we have

$$\max\{a_i : b_i > 0,\ 2 \le i \le n\} < \lambda < \min\{a_i : b_i < 0,\ 2 \le i \le n\}.$$

Since A requires algebraic positivity, $b_i > 0$ implies $a_i \le 0$.

Further, each row of A contains a $-$ implies each row of $-A$ contains a $+$, and $B \in Q(A)$ is algebraically positive if and only if $-B \in Q(-A)$ is algebraically positive. Therefore the necessary condition is obtained.

Proof of the converse part follows from Lemma 3.8 and Lemma 3.11. □

3.3.2 Path Sign Pattern Matrices Requiring Algebraic Positivity

In this section, we characterize all path sign pattern matrices requiring algebraic positivity. Since any path sign pattern matrix of order at most 3 is also a star sign pattern matrix, we characterize all path sign pattern matrices of order at least 4 requiring algebraic positivity. If all nonzero off-diagonal entries of such a sign pattern matrix have the same sign, then by Lemma 3.8, that sign pattern matrix requires algebraic positivity. So we consider the path sign pattern matrices that have both positive and negative off-diagonal entries. Again by Theorem 3.5, such sign pattern matrices must be sign-symmetric to require algebraic positivity.

Lemma 3.13 ([36]). *Let A be a sign pattern matrix of order n and P be a permutation pattern of order n. Then the following conditions hold.*

1. *A requires algebraic positivity if and only if $-A$ requires algebraic positivity.*
2. *A requires algebraic positivity if and only if P^TAP requires algebraic positivity.*

Lemma 3.14. *Let A be an irreducible, real square matrix of order $n \geq 2$, and λ be an eigenvalue of A corresponding to a positive right eigenvector. Then for each $k \in \{1, 2, \dots, n\}$, the following conditions hold.*

1. *If $a_{ki} \leq 0$ for all $i \neq k$, then $\lambda < a_{kk}$.*
2. *If $a_{ki} \geq 0$ for all $i \neq k$, then $\lambda > a_{kk}$.*

Proof. Suppose A has a positive right eigenvector $\mathbf{x} = (x_1, \dots, x_n)^T$ corresponding to the eigenvalue λ. Then $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$, which implies

$$(a_{kk} - \lambda)x_k + \sum_{i \neq k} a_{ki}x_i = 0.$$

Since A is irreducible, $a_{ki} \neq 0$ for some $i \neq k$. Hence the result follows. □

Lemma 3.15. *Suppose A is an irreducible sign pattern matrix of order n, and let s, t be two indices such that all nonzero off-diagonal entries in s-th row are $+$ and all nonzero off-diagonal entries in t-th row are $-$. If A requires algebraic positivity, then the only possibilities of the ordered pair (a_{ss}, a_{tt}) are $(-, 0), (-, +), (0, +)$.*

Proof. Suppose $a_{ss} = +$. Then we can choose $B \in Q(A)$ such that $b_{ss} > b_{tt}$. Since B is algebraically positive, let λ be an eigenvalue of B corresponding to a positive right eigenvector. Since A is irreducible, by Lemma 3.14, $b_{ss} < \lambda < b_{tt}$, which is a contradiction. So $a_{ss} \in \{-, 0\}$. Since A requires algebraic positivity, by Lemma 3.14, the only possibilities of the ordered pair (a_{ss}, a_{tt}) are $(-, 0), (-, +), (0, +)$. □

Let $R_i(A)$ denotes the i-th row vector of the matrix A for $i = 1, 2, \dots, n$. Let us recall from Chapter 2 that $\mathcal{N}(A)$ denotes the null space of a square matrix A. We will use the following notations throughout this chapter.

$v(+) = 1, v(-) = -1, v(0) = 0$, and if a is a real number, then

$$\operatorname{sgn}(a) = \begin{cases} 1, & \text{if } a > 0; \\ -1, & \text{if } a < 0; \\ 0, & \text{if } a = 0. \end{cases}$$

Consider a sign pattern matrix of order $n \geq 4$ given by

$$\begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & & \vdots \\ 0 & b_2 & a_3 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & \cdots & \cdots & 0 & b_{n-1} & a_n \end{bmatrix}, \tag{3.16}$$

where $a_i \in \{+, -, 0\}$ and $b_j \in \{+, -\}$ for all $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, n-1\}$.

Lemma 3.17. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +$ and $b_k = -$ for some k. If A requires algebraic positivity, then a_1 cannot be $+$. Further, if $b_{n-1} = -$, then the only possibilities of the ordered pair (a_1, a_n) are $(-, 0), (-, +), (0, +)$.*

Proof. Suppose $a_1 = +$. Take $B \in Q(A)$ replacing

- a_1 by 10, a_j by $v(a_j)$ for all $j \in \{2, \ldots, n\}$ and
- b_j by $v(b_j)$ for all $j \in \{1, 2, \ldots, n-1\}$.

If λ is the eigenvalue of B corresponding to a positive right eigenvector, then $\lambda > 10$, by Lemma 3.14. Then $R_{k+1}(B - \lambda I) + \cdots + R_n(B - \lambda I)$ is a nonzero, nonpositive vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction. So a_1 cannot be $+$.

If $b_{n-1} = -$, then by Lemma 3.15, we can conclude that the only possibilities of the ordered pair (a_1, a_n) are $(-, 0), (-, +), (0, +)$. □

Lemma 3.18. *If A is a sign pattern matrix of order n as in (3.16) with $b_1 = +, b_{n-1} = -$, and A requires algebraic positivity, then n is odd and $b_k b_{k+1} = -$ for all k.*

Proof. **Case I:** Suppose $b_{l-1}, b_l = +$ and $b_i b_{i+1} = -$ for $i = 1, \ldots, l-2$. Then l is even. Take $B \in Q(A)$ replacing

- a_j by $v(a_j)$ for all $j \in \{1, 2, \ldots, n\}$ and
- $b_1, b_3, \cdots, b_{l-1}$ by 10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then by Lemma 3.14, $-1 < \lambda < 1$. So $R_1(B - \lambda I) + \cdots + R_l(B - \lambda I)$ is a nonzero, nonnegative vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction. So there exists no l such that $b_{l-1}, b_l = +$ and $b_i b_{i+1} = -$ for $i = 1, \ldots, l-2$.

Case II: Similar argument shows that there exist no l such that $b_{l-1}, b_l = -$ and $b_i b_{i+1} = -$ for $i = l, \ldots, n-2$

Case III: So assume $b_{l-1}, b_l = -$, $b_i b_{i+1} = -$ for $i = 1, \ldots, l-2$, and $b_m, b_{m+1} = +$ for some $m > l$, and $b_i b_{i+1} = -$ for $i = m+1, \ldots, n-2$. Then $l, n-m$ are odd, and by Lemma 3.15,

$$(a_1, a_l), (a_1, a_n), (a_{m+1}, a_l), (a_{m+1}, a_n) \in \{(-, 0), (-, +), (0, +)\}.$$

If at least one of a_1, a_{m+1} is 0, then $a_n = +, a_l = +$. Take $B \in Q(A)$ replacing

- a_n by 10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_{m+1}, b_{m+3}, \cdots, b_{n-2}$ by 10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then by Lemma 3.14, $0 < \lambda < 1$. So $R_{m+1}(B - \lambda I) + \cdots + R_n(B - \lambda I)$ is a nonzero, nonnegative vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction.

Suppose both a_1, a_{m+1} are $-$. Take $B \in Q(A)$ replacing

- a_1 by -10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_2, b_4, \cdots, b_{l-1}$ by -10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then by Lemma 3.14, $-1 < \lambda < 1$. Then $R_1(B - \lambda I) + \cdots + R_l(B - \lambda I)$ is a nonzero, nonpositive vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction.

So $b_k b_{k+1} = -$ for all $k \in \{1, 2, \ldots, n-2\}$, and hence n is odd. □

Lemma 3.19. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +$ and $b_{n-1} = -$. If A requires algebraic positivity, then $a_k = 0$ for all odd k except possibly for $k = 1, n$.*

Proof. By Lemma 3.18, n is odd and $b_k b_{k+1} = -$ for all k. Since $b_1 = +$, $b_k = +$ for all odd k and $b_k = -$ for all even k.

Suppose $a_k = +$ for some odd k other than $k = 1, n$. Take $B \in Q(A)$ replacing

- a_k by 10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_1, b_3, b_5, \cdots, b_{n-2}$ by 10, $b_2, b_4, b_6, \cdots, b_{n-1}$ by -1.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then by Lemma 3.14, $-1 < \lambda < 1$. So $R_1(B-\lambda I)+\cdots+R_k(B-\lambda I)$ is a nonzero nonnegative vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction. So a_k cannot be $+$ for any odd k except possibly for $k = 1, n$.

Suppose $a_k = -$ for some odd k other than $k = 1, n$. Then $n - k + 1$ is odd and the $(n - k + 1)$-th diagonal entry of $P^T(-A)P$ is $+$, where P is the anti diagonal permutation sign pattern matrix of order n. Since $b_k = +$ for all odd k and $b_k = -$ for all even k, off-diagonal entries of both $P^T(-A)P$ and A have the same sign pattern. So by Lemma 3.13 and the argument in the previous paragraph, a_k cannot be $-$ for any odd k except possibly for $k = 1, n$.

Hence $a_k = 0$ for all odd k except possibly for $k = 1, n$. □

Lemma 3.20. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +$ and $b_{n-1} = -$. If $a_n = +$ and A requires algebraic positivity, then a_k cannot be $+$ for any even k except possibly for $k = 2$.*

Proof. By Lemma 3.18, n is odd, $b_k = +$ for all odd k and $b_k = -$ for all even k.

Suppose $a_k = +$ for some even k other than $k = 2$. Take $B \in Q(A)$ replacing

- a_k, a_n by 10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_{k+1}, b_{k+3}, \cdots, b_{n-2}$ by 10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then by Lemma 3.14, $-1 < \lambda < 10$. If $2 \leq \lambda < 10$, then $R_1(B - \lambda I) + R_2(B - \lambda I)$ is a nonzero, nonpositive vector. If $-1 < \lambda < 2$, then $R_k(B - \lambda I) + \cdots + R_n(B - \lambda I)$ is a nonzero, nonnegative vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction. Thus a_k cannot be $+$ for any even k except possibly for $k = 2$. □

From Lemma 3.18, Lemma 3.19 and Lemma 3.20, we have the following corollary.

Corollary 3.21. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +$, and $b_r = -, b_{r+1} = -$ for some $r \in \{2, 3, \ldots, n-3\}$. If A requires algebraic positivity, then*

1. *$b_k b_{k+1} = -$ for all $k \in \{1, 2, \ldots, r-1\}$ and thus r is even, and*
2. *$a_k = 0$ for all $k \in \{3, 5, \ldots, r-1\}$.*

Moreover, if $a_{r+1} = +$, then a_k cannot be $+$ for any $k \in \{4, 6, \ldots, r\}$.

Lemma 3.22. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +$ and $b_{n-1} = -$. If A requires algebraic positivity, then the following conditions hold.*

1. *n is odd, $b_k = +$ for all odd k and $b_k = -$ for all even k.*
2. *If $a_1 = -, a_n = +$, then $a_k = 0$ for all k except possibly $k = 1, 2, n-1, n$.*
3. *If $a_1 = 0$, then $a_n = +$, $a_k = 0$ for all odd k except $k = n$, and a_k cannot be $+$ for any even k except possibly $k = 2$.*

Proof.

1. It follows from Lemma 3.18.
2. By Lemma 3.19, $a_k = 0$ for all odd k except possibly for $k = 1, n$. Since $a_n = +$, by Lemma 3.20, a_k cannot be $+$ for any even k except possibly for $k = 2$. Suppose $a_k = -$ for some even k other than $k = n - 1$. Since $a_1 = -$, the $(n - k + 1)$-th and n-th diagonal entries of $P^T(-A)P$ are $+$, where P is the anti-diagonal permutation pattern of order n. Further, n is odd implies $n - k + 1$ is even, and $b_k = +$ for all odd k and

$b_k = -$ for all even k imply that off-diagonal entries of both $P^T(-A)P$ and A have the same sign pattern. So by Lemma 3.13 and Lemma 3.20, a_k cannot be $-$ for any even k except possibly for $k = n-1$. Therefore $a_k = 0$ for all k except possibly for $k = 1, 2, n-1, n$.

3. Let $a_1 = 0$. Then by Lemma 3.17, $a_n = +$. Hence the other two conclusions follows from Lemma 3.19 and Lemma 3.20, respectively. □

Lemma 3.23. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +, b_2 = -, a_1 = -$. If A requires algebraic positivity, then a_2 cannot be $-$.*

Proof. Case I: Let $b_{n-1} = +$.

Suppose $a_2 = -$. Take $B \in Q(A)$ replacing

- a_1, a_2 by -10, a_j by $v(a_j)$ for all the remaining a_js and
- b_j by $v(b_j)$ for all $j \in \{1, 2, \dots, n-1\}$.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $\lambda > -1$, by Lemma 3.14. So $R_1(B-\lambda I) + R_2(B-\lambda I)$ is a nonzero, nonpositive vector. So $\mathcal{N}(B-\lambda I)$ does not contain any positive vector, which is a contradiction.

Case II: Let $b_{n-1} = -$. Then $b_{n-2} = +$, by Lemma 3.18.

Suppose $a_2 = -$. Take $B \in Q(A)$ replacing

- a_1, a_2 by -3, a_j by $v(a_j)$ for all the remaining a_js and
- b_j by $v(b_j)$ for all $j \in \{1, 2, \dots, n-1\}$.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $-3 < \lambda < 1$, by Lemma 3.14. If $-2 \le \lambda < 1$, then $R_1(B-\lambda I) + R_2(B-\lambda I)$ is a nonzero, nonpositive vector. If $-3 < \lambda < -2$, then $R_{n-1}(B-\lambda I) + R_n(B-\lambda I)$ is a nonzero, nonnegative vector. So $\mathcal{N}(B-\lambda I)$ does not contain any positive vector, which is a contradiction. □

Let $\mathcal{M}_1$ be the symmetric $\mathbb{T}$-pattern matrix of odd order n defined by

$$(\mathcal{M}_1)_{22} = \#,\ (\mathcal{M}_1)_{nn} = +;$$
$$(\mathcal{M}_1)_{ij} = 0, \qquad \text{if } |i-j| > 1;$$
$$(\mathcal{M}_1)_{ii} = \begin{cases} 0, & \text{if } i \neq n \text{ and } i \text{ is odd;} \\ -_0, & \text{if } i \neq 2 \text{ and } i \text{ is even;} \end{cases}$$
$$(\mathcal{M}_1)_{i,i+1} = \begin{cases} +, & \text{if } i \text{ is odd;} \\ -, & \text{if } i \text{ is even.} \end{cases}$$

Therefore,

$$\mathcal{M}_1 = \begin{bmatrix} 0 & + & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ + & \# & - & \ddots & & & & \vdots \\ 0 & - & 0 & + & \ddots & & & \vdots \\ \vdots & \ddots & + & -_0 & - & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & - & 0 & + & 0 \\ \vdots & & & & \ddots & + & -_0 & - \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & - & + \end{bmatrix}.$$

Lemma 3.24. *$\mathcal{M}_1$ requires algebraic positivity.*

Proof. Let $A \in Q(\mathcal{M}_1)$ and $j < k$. Then

$$\det(A(\{j\},\{k\})) = \det(A[\alpha]) \cdot \det(A[\beta]) \cdot \prod_{m=j+1}^{k} a_{m,m-1},$$

where $\alpha = \{1,\ldots,j-1\}$ and $\beta = \{k+1,\ldots,n\}$.

Observe the following details. Here n is odd.

$$\operatorname{sgn}\det(A[\alpha]) = \begin{cases} (-1)^{\frac{j-1}{2}}, & \text{if } j \text{ is odd;} \\ 0, & \text{if } j \text{ is even.} \end{cases}$$

$$\operatorname{sgn}\det(A[\beta]) = \begin{cases} (-1)^{\frac{n-k}{2}}, & \text{if } k \text{ is odd;} \\ (-1)^{\frac{n-k-1}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

$$\operatorname{sgn} \prod_{m=j+1}^{k} a_{m,m-1} = \begin{cases} (-1)^{\frac{k-j}{2}}, & \text{if both of } j,k \text{ are odd or even;} \\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{k-j+1}{2}}, & \text{if } j \text{ is even and } k \text{ is odd.} \end{cases}$$

Therefore

$$\operatorname{sgn}\det(A(\{j\},\{k\})) = \begin{cases} 0, & \text{if } j \text{ is even;} \\ (-1)^{\frac{n-3}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{n-1}{2}}, & \text{if } j \text{ is odd and } k \text{ is odd.} \end{cases}$$

Since A is sign-symmetric, $\operatorname{adj}(A)$ is also sign-symmetric. Thus for each $j \in \{1,3,5,\ldots,n\}$ and $k \in \{1,2,3,\ldots,n\}$ with $j < k$, the digraph of $\operatorname{adj}(A)$ have both arcs $(j,k),(k,j)$. So for any two distinct $j,k \in \{1,2,\ldots,n\}$ with $j < k$, there is a directed path

1. $j \to k$, if j is odd;
2. $j \to 1 \to k$, if j is even

from j to k in the digraph of $\operatorname{adj}(A)$. Since $\operatorname{adj}(A)$ is sign-symmetric, there is also a directed path from k to j in the digraph of $\operatorname{adj}(A)$. Therefore the digraph of $\operatorname{adj}(A)$ is strongly connected, and hence by Theorem 1.1, $\operatorname{adj}(A)$ is irreducible. Again all nonzero off-diagonal entries of $\operatorname{adj}(A)$ are of the same sign. So by Theorem 2.3.3, $\operatorname{adj}(A)$ is algebraically positive. Therefore by Theorem 2.8, A is algebraically positive. Hence the result follows. □

Let $\mathcal{M}_2$ be the symmetric $\mathbb{T}$-pattern matrix of odd order n defined by

$$(\mathcal{M}_2)_{11} = -,\ (\mathcal{M}_2)_{22} = +_0,\ (\mathcal{M}_2)_{n-1,n-1} = -_0,\ (\mathcal{M}_2)_{nn} = +;$$

$$(\mathcal{M}_2)_{ij} = 0, \quad \text{if } |i-j| > 1;$$

$$(\mathcal{M}_2)_{ii} = 0, \quad \text{if } i \neq 1,2,n-1,n;$$

$$(\mathcal{M}_2)_{i,i+1} = \begin{cases} +, & \text{if } i \text{ is odd;} \\ -, & \text{if } i \text{ is even.} \end{cases}$$

Therefore,

$$\mathcal{M}_2 = \begin{bmatrix} - & + & 0 & \cdots & \cdots & \cdots & 0 \\ + & +_0 & - & \ddots & & & \vdots \\ 0 & - & 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & + & 0 \\ \vdots & & & \ddots & + & -_0 & - \\ 0 & \cdots & \cdots & \cdots & 0 & - & + \end{bmatrix}.$$

Lemma 3.25. *$\mathcal{M}_2$ requires algebraic positivity.*

Proof. Let $A \in Q(\mathcal{M}_2)$ and $j < k$. Then

$$\det(A(\{j\},\{k\})) = \det(A[\alpha]) \cdot \det(A[\beta]) \cdot \prod_{m=j+1}^{k} a_{m,m-1},$$

where $\alpha = \{1, \ldots, j-1\}$ and $\beta = \{k+1, \ldots, n\}$.

Observe the following details. Here n is odd.

$$\operatorname{sgn} \det(A[\alpha]) = \begin{cases} (-1)^{\frac{j-1}{2}}, & \text{if } j \text{ is odd;} \\ (-1)^{\frac{j}{2}}, & \text{if } j \text{ is even.} \end{cases}$$

$$\operatorname{sgn} \det(A[\beta]) = \begin{cases} (-1)^{\frac{n-k}{2}}, & \text{if } k \text{ is odd;} \\ (-1)^{\frac{n-k-1}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

$$\operatorname{sgn} \prod_{m=j+1}^{k} a_{m,m-1} = \begin{cases} (-1)^{\frac{k-j}{2}}, & \text{if both of } j,k \text{ are odd or even;} \\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{k-j+1}{2}}, & \text{if } j \text{ is even and } k \text{ is odd.} \end{cases}$$

Therefore

$$\operatorname{sgn} \det(A(\{j\},\{k\})) = \begin{cases} (-1)^{\frac{n+1}{2}}, & \text{if } j \text{ is even and } k \text{ is odd;} \\ (-1)^{\frac{n-3}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{n-1}{2}}, & \text{if both } j,k \text{ are odd or even.} \end{cases}$$

Since A is sign-symmetric, $\text{adj}(A)$ is also sign-symmetric. Therefore for each $j, k \in \{1, 2, \ldots, n\}$ with $j < k$, the digraph of $\text{adj}(A)$ have both arcs $(j, k), (k, j)$. Therefore the digraph of $\text{adj}(A)$ is strongly connected, and hence by Theorem 1.1, $\text{adj}(A)$ is irreducible. Again all nonzero off-diagonal entries of $\text{adj}(A)$ are of the same sign. So by Theorem 2.3.3, $\text{adj}(A)$ is algebraically positive. Therefore by Theorem 2.8, A is algebraically positive. Hence the result follows. □

Lemma 3.26. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +$ and $b_{n-1} = -$. Then A requires algebraic positivity if and only if A or $-A$ is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where $\mathcal{S} \in \{\mathcal{M}_1, \mathcal{M}_2\}$.*

Proof. Suppose A requires algebraic positivity. Since $b_1 = +, b_{n-1} = -$, by Lemma 3.18, n is odd, $b_k = +$ for all odd k and $b_k = -$ for all even k; and by Lemma 3.17, the possibilities of the ordered pair (a_1, a_n) are $(0, +), (-, +), (-, 0)$. Now $P^T(-A)P$ has same sign pattern on the off-diagonal as A, where P is the anti-diagonal permutation sign pattern matrix of order n. So the possibility $(0, +)$ is equivalent to the possibility $(-, 0)$, by Lemma 3.13. Hence by Lemma 3.13, Lemma 3.22 and Lemma 3.23, A or $-A$ is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where $\mathcal{S} \in \{\mathcal{M}_1, \mathcal{M}_2\}$.

The converse part follows from Lemma 3.13, Lemma 3.24 and Lemma 3.25. □

Next we characterize the path sign pattern matrices requiring algebraic positivity, with $b_1, b_{n-1} = +$.

Lemma 3.27. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +, b_{n-1} = +, b_k = -$ for some k. If A requires algebraic positivity, then there exists no k such that $b_k, b_{k+1} = +$ and there exists at most one k such that $b_k, b_{k+1} = -$.*

Proof. `Case I`: Let $b_r, b_{r+1} = -$ for some $r \in \{2, 3, \ldots, n-2\}$.

Using Lemma 3.13 and Corollary 3.21, it can be shown that there is no k such that $b_k b_{k+1} = +$ except $k = r$. So there is no k such that $b_k, b_{k+1} = +$ and there is exactly one k such that $b_k, b_{k+1} = -$.

`Case II`: Assume there is no r such that $b_r, b_{r+1} = -$, and $b_{n-2} = -$.

Let $b_k, b_{k+1} = +$ for some $k < n-3$ and $\Lambda = \{j \in \{1, 2, \dots, k\} : b_j = +\}$. Take $B \in Q(A)$ replacing

- a_j by $v(a_j)$ for all $j \in \{1, 2, \dots, n\}$ and
- b_j by 10 for all $j \in \Lambda$, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $\lambda > -1$, by Lemma 3.14. If $\lambda \geq 3$, then $R_{n-1}(B - \lambda I) + R_n(B - \lambda I)$ is a nonzero, nonpositive vector. If $-1 < \lambda < 3$, then $R_1(B - \lambda I) + \cdots + R_{k+1}(B - \lambda I)$ is a nonzero, nonnegative vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction.

`Case III`: Assume there is no r such that $b_r, b_{r+1} = -$, and $b_{n-2} = +$.

Then $b_k = -$ for some $k \in \{2, 3, \dots, n-3\}$. Take $B \in Q(A)$ replacing

- a_j by $v(a_j)$ for all $j \in \{1, 2, \dots, n\}$ and
- b_{n-1} by 10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $\lambda > -1$, by Lemma 3.14. If $\lambda \geq 3$, then $R_1(B - \lambda I) + \cdots + R_k(B - \lambda I)$ is a nonzero, nonpositive vector. If $-1 < \lambda < 3$, then $R_{n-1}(B - \lambda I) + R_n(B - \lambda I)$ is a nonzero, nonnegative vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction. Hence the result follows. □

Lemma 3.28. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1, b_{n-1} = +$, $b_k = -$ for some k and A requires algebraic positivity. Then $b_k, b_{k+1} = -$ for exactly one $k \in \{2, 3, \dots, n-3\}$ if and only if n is odd and $a_1, a_n = 0$.*

Proof. Suppose $b_k, b_{k+1} = -$ for exactly one $k \in \{2, 3, \dots, n-3\}$. Using Lemma 3.13 and Corollary 3.21, it can be shown that $b_i b_{i-1} = -$ for all $i \in \{2, 3, \dots, n-1\} \setminus \{k+1\}$. So n is odd and k is even. By Lemma 3.15, a_1, a_n cannot be $+$.

Suppose $a_1 = -$. Take $B \in Q(A)$ replacing

- a_1 by -10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_2, b_4, \cdots, b_k$ by -10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $-1 < \lambda < 1$, by Lemma 3.14. Then $R_1(B-\lambda I)+\cdots+R_{k+1}(B-\lambda I)$ is a nonzero, nonpositive vector. So $\mathcal{N}(B-\lambda I)$ does not contain any positive vector, which is a contradiction. So $a_1 = 0$. Using similar arguments we can show that $a_n = 0$.

Conversely if $b_k, b_{k+1} = -$ does not hold for exactly one $k \in \{2, 3, \ldots, n-3\}$, then by Lemma 3.27, $b_k = +$ for all odd k and $b_k = -$ for all even k. So n is even, a contradiction. □

From Lemma 3.27 and Lemma 3.28, we have the following corollary.

Corollary 3.29. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +$, $b_{n-1} = +$ and $b_k = -$ for some k. If at least one of a_1, a_n is nonzero and A requires algebraic positivity, then $b_k = +$ for all odd k and $b_k = -$ for all even k, and thus n is even.*

Lemma 3.30. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1, b_{n-1} = +$ and $b_k = -$ for some k. If at least one of a_1, a_n is nonzero and A requires algebraic positivity, then a_k cannot be $+$ for any k except possibly $k = 1, 2, n-1, n$.*

Proof. By Corollary 3.29, $b_k = +$ for all odd k, $b_k = -$ for all even k and n is even.

Suppose $a_k = +$ for some odd k other than $k = 1, n-1$. Take $B \in Q(A)$ replacing

- a_k by 10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_1, b_3, \cdots, b_{k-2}$ by 10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $\lambda > -1$, by Lemma 3.14. If $\lambda \geq 2$, then $R_{n-1}(B-\lambda I) + R_n(B-\lambda I)$ is a nonzero, nonpositive vector. If $-1 < \lambda < 2$, then $R_1(B-\lambda I) + \cdots + R_k(B-\lambda I)$ is a nonzero, nonnegative vector. So $\mathcal{N}(B-\lambda I)$ does not contain any positive vector, which is a contradiction. So a_k cannot be $+$ for any odd k except possibly $k = 1, n-1$.

Suppose $a_k = +$ for some even k other than $k = 2, n$. Now n, k are even implies $(n-k+1)$ is odd and $(n-k+1)$-th diagonal entry of P^TAP is $+$, where P is the anti-diagonal permutation sign pattern matrix of order n. Since $b_k = +$ for all odd k, $b_k = -$ for all even k, off-diagonal entries of both P^TAP and A have the same sign pattern. So by Lemma 3.13 and the arguments in the previous paragraph, we can conclude that a_k cannot be $+$ for any even k except possibly $k = 2, n$. □

If however both a_1, a_n are zero, then by repeating the proof of the above lemma we get the following result.

Lemma 3.31. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_k = +$ for all odd k and $b_k = -$ for all even k. If n is even, $a_1, a_n = 0$ and A requires algebraic positivity, then a_k cannot be $+$ for any k except possibly for $k = 2, n-1$.*

Lemma 3.32. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1, b_{n-1} = +$ and $b_k = -$ for some k. If $a_1 = -$ and A requires algebraic positivity, then $a_k = 0$ for all even k except possibly for $k = 2, n$.*

Proof. By Corollary 3.29, $b_k = +$ for all odd k, $b_k = -$ for all even k and n is even. Again by Lemma 3.30, a_k cannot be $+$ for any even k except possibly $k = 2, n$.

Suppose $a_k = -$ for some even k other than $k = 2, n$. Take $B \in Q(A)$ replacing

- a_1, a_k by -10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_2, b_4, \cdots, b_{k-2}$ by -10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $\lambda > -1$, by Lemma 3.14. So $R_1(B - \lambda I) + \cdots + R_k(B - \lambda I)$ is a nonzero and nonpositive vector. So $\mathcal{N}(B - \lambda I)$ does not contain any positive vector, which is a contradiction. So a_k cannot be $-$ for any even k except possibly $k = 2, n$. Thus $a_k = 0$ for all even k except possibly $k = 2, n$. □

Lemma 3.33. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1 = +, b_{n-1} = +$ and $b_k = -$ for some k. If at least one of a_1, a_n is nonzero and A requires algebraic positivity, then the following conditions hold.*

1. *If $a_1 = -, a_n = -$, then $a_k = 0$ for all k except possibly for $k = 1, 2, n-1, n$.*
2. *If $a_1 = -, a_n = 0$, then*
   - *(a) a_k cannot be $+$ for any odd k except possibly for $k = n-1$, and*
   - *(b) $a_k = 0$ for all even k except possibly for $k = 2$.*

Proof.

1. By Corollary 3.29, n is even and by Lemma 3.30, a_k cannot be $+$ for any k except possibly $k = 1, 2, n-1, n$. Since $a_1 = -$, by Lemma 3.32, a_k cannot be $-$ for any even k except possibly for $k = 2, n$.

   Suppose $a_k = -$ for some odd k other than $k = 1, n-1$. Now n is even, k is odd implies the $(n-k+1)$ is even and $(n-k+1)$-th diagonal entry of $P^T AP$ is $-$, where P is the anti-diagonal permutation sign pattern matrix of order n. Since by Corollary 3.29, $b_k = +$ for all odd k and $b_k = -$ for all even k, off-diagonal entries of both $P^T AP$ and A have the same sign pattern. So by Lemma 3.13 and Lemma 3.32, we can conclude that a_k cannot be $-$ for any odd k except possibly $k = 1, n-1$.

   So $a_k = 0$ for all k except possibly $k = 1, 2, n-1, n$.

2. It follows from Lemma 3.30 and Lemma 3.32. □

Lemma 3.34. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1, b_{n-1} = +$ and $b_k = -$ for some k. Further assume n is even and $a_1 = 0, a_n = 0$. If A requires algebraic positivity and $a_k = -$ for some odd k other than $k = 1, n-1$, then $a_r = 0$ for all even r greater than k.*

Proof. Since n is even, by Lemma 3.27 and Lemma 3.28, $b_k = +$ for all odd k and $b_k = -$ for all even k. So by Lemma 3.31, a_k cannot be $+$ for any even k except possibly $k = 2, n-1$.

Assume $a_k = -$ for some odd k other than $k = 1, n-1$. Suppose $a_r = -$ for some even $r > k$. Take $B \in Q(A)$ replacing

- a_k, a_r by -10, a_j by $v(a_j)$ for all the remaining a_js and
- $b_{k+1}, b_{k+3}, \cdots, b_{n-2}$ by -10, b_j by $v(b_j)$ for all the remaining b_js.

Suppose λ be the eigenvalue of B corresponding to a positive right eigenvector. Then $\lambda > 0$, by Lemma 3.14. If $\lambda \geq 2$, then $R_1(B-\lambda I) + R_2(B-\lambda I)$ is a nonzero, nonpositive vector. If $0 < \lambda < 2$, then $R_k(B-\lambda I) + \cdots + R_r(B-\lambda I)$ is a nonzero, nonpositive vector. So $\mathcal{N}(B-\lambda I)$ does not contain any positive vector, which is a contradiction. Thus if $a_k = -$ for some odd k, then $a_r = 0$ for all even $r > k$. □

Lemma 3.35. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1, b_{n-1} = +$ and $b_k = -$ for some k. Further assume n is odd and $a_1 = 0, a_n = 0$. If A requires algebraic positivity, then there is exactly one m such that $b_m, b_{m+1} = -$. Moreover m is even, $a_{m+1} = +$, $a_k = 0$ for all odd k except $k = m+1$ and a_k cannot be $+$ for any even k except possibly for $k = 2, n-1$.*

Proof. Since n is odd and $a_1 = 0, a_n = 0$, from Lemma 3.28, it follows that there is exactly one m such that $b_m, b_{m+1} = -$. So by Corollary 3.21, m is even, and by Lemma 3.15, $a_{m+1} = +$. The other two conclusions follows from Lemma 3.13 and Corollary 3.21. □

Let $\mathcal{N}_1$ be the symmetric $\mathbb{T}$-pattern matrix of even order n defined by

$$(\mathcal{N}_1)_{11} = -,\ (\mathcal{N}_1)_{22} = +_0,\ (\mathcal{N}_1)_{n-1,n-1} = +_0,\ (\mathcal{N}_1)_{nn} = -;$$

$$(\mathcal{N}_1)_{ij} = 0, \qquad \text{if } |i-j| > 1;$$

$$(\mathcal{N}_1)_{ii} = 0, \qquad \text{if } i \neq 1, 2, n-1, n;$$

$$(\mathcal{N}_1)_{i,i+1} = \begin{cases} +, & \text{if } i \text{ is odd;} \\ -, & \text{if } i \text{ is even.} \end{cases}$$

Therefore,

$$\mathcal{N}_1 = \begin{bmatrix} - & + & 0 & \cdots & \cdots & \cdots & 0 \\ + & +_0 & - & \ddots & & & \vdots \\ 0 & - & 0 & + & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & + & 0 & - & 0 \\ \vdots & & & \ddots & - & +_0 & + \\ 0 & \cdots & \cdots & \cdots & 0 & + & - \end{bmatrix}.$$

Lemma 3.36. *$\mathcal{N}_1$ requires algebraic positivity.*

Proof. Let $A \in Q(\mathcal{N}_1)$ and $j < k$. Then

$$\det(A(\{j\},\{k\})) = \det(A[\alpha]) \cdot \det(A[\beta]) \cdot \prod_{m=j+1}^{k} a_{m,m-1},$$

where $\alpha = \{1, \ldots, j-1\}$ and $\beta = \{k+1, \ldots, n\}$.

Observe the following details. Here n is even.

$$\operatorname{sgn}\det(A[\alpha]) = \begin{cases} (-1)^{\frac{j-1}{2}}, & \text{if } j \text{ is odd;} \\ (-1)^{\frac{j}{2}}, & \text{if } j \text{ is even.} \end{cases}$$

$$\operatorname{sgn}\det(A[\beta]) = \begin{cases} (-1)^{\frac{n-k+1}{2}}, & \text{if } k \text{ is odd;} \\ (-1)^{\frac{n-k}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

$$\operatorname{sgn} \prod_{m=j+1}^{k} a_{m,m-1} = \begin{cases} (-1)^{\frac{k-j}{2}}, & \text{if both of } j,k \text{ are odd or even;} \\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{k-j+1}{2}}, & \text{if } j \text{ is even and } k \text{ is odd.} \end{cases}$$

Therefore

$$\operatorname{sgn}\det(A(\{j\},\{k\})) = \begin{cases} (-1)^{\frac{n-2}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{n+2}{2}}, & \text{if } j \text{ is even and } k \text{ is odd;} \\ (-1)^{\frac{n}{2}}, & \text{if both of } j,k \text{ are odd or even.} \end{cases}$$

Since A is sign-symmetric, $\operatorname{adj}(A)$ is also sign-symmetric. Thus for any two distinct $j,k \in \{1,2,\ldots,n\}$ with $j<k$, the digraph of $\operatorname{adj}(A)$ have both arcs $(j,k),(k,j)$. Therefore the digraph of $\operatorname{adj}(A)$ is strongly connected, and hence by Theorem 1.1, $\operatorname{adj}(A)$ is irreducible. Again all nonzero off-diagonal entries of $\operatorname{adj}(A)$ are of the same sign. So by Theorem 2.3.3, $\operatorname{adj}(A)$ is algebraically positive. Therefore by Theorem 2.8, A is algebraically positive. Hence the result follows. □

Let $\mathcal{N}_2$ be the symmetric $\mathbb{T}$-pattern matrix of even order n defined by

$$(\mathcal{N}_2)_{11} = -,\ (\mathcal{N}_2)_{22} = +_0,\ (\mathcal{N}_2)_{n-1,n-1} = \#;$$

$$(\mathcal{N}_2)_{ij} = 0, \qquad \text{if } |i-j| > 1;$$

$$(\mathcal{N}_2)_{ii} = \begin{cases} -_0, & \text{if } i \neq 1, n-1 \text{ and } i \text{ is odd;} \\ 0, & \text{if } i \neq 2 \text{ and } i \text{ is even;} \end{cases}$$

$$(\mathcal{N}_2)_{i,i+1} = \begin{cases} +, & \text{if } i \text{ is odd;} \\ -, & \text{if } i \text{ is even.} \end{cases}$$

Therefore,

$$\mathcal{N}_2 = \begin{bmatrix} - & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ + & +_0 & - & \ddots & & & & & \vdots \\ 0 & - & -_0 & + & \ddots & & & & \vdots \\ \vdots & \ddots & + & 0 & - & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & - & -_0 & + & \ddots & \vdots \\ \vdots & & & & \ddots & + & 0 & - & 0 \\ \vdots & & & & & \ddots & - & \# & + \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + & 0 \end{bmatrix}.$$

Lemma 3.37. *$\mathcal{N}_2$ requires algebraic positivity.*

Proof. Let $A \in Q(\mathcal{N}_2)$ and $j < k$. Then

$$\det(A(\{j\},\{k\})) = \det(A[\alpha]) \cdot \det(A[\beta]) \cdot \prod_{m=j+1}^{k} a_{m,m-1},$$

where $\alpha = \{1, \ldots, j-1\}$ and $\beta = \{k+1, \ldots, n\}$.

Observe the following details. Here n is even.

$$\operatorname{sgn}\det(A[\alpha]) = \begin{cases} (-1)^{\frac{j-1}{2}}, & \text{if } j \text{ is odd;} \\ (-1)^{\frac{j}{2}}, & \text{if } j \text{ is even.} \end{cases}$$

$$\operatorname{sgn}\det(A[\beta]) = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ (-1)^{\frac{n-k}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

$$\operatorname{sgn} \prod_{m=j+1}^{k} a_{m,m-1} = \begin{cases} (-1)^{\frac{k-j}{2}}, & \text{if both of } j,k \text{ are odd or even;} \\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{k-j+1}{2}}, & \text{if } j \text{ is even and } k \text{ is odd.} \end{cases}$$

Therefore

$$\operatorname{sgn} \det(A(\{j\},\{k\})) = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ (-1)^{\frac{n-2}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{n}{2}}, & \text{if both } j,k \text{ are even.} \end{cases}$$

Since A is sign-symmetric, $\operatorname{adj}(A)$ is also sign-symmetric. Thus for each

$$j \in \{1,2,3,\ldots,n-1\} \text{ and } k \in \{2,4,6,\ldots,n\} \text{ with } j < k,$$

the digraph of $\operatorname{adj}(A)$ have both arcs $(j,k),(k,j)$. So for any two distinct $j,k \in \{1,2,\ldots,n\}$ with $j < k$, there is a directed path

1. $j \to k$, if k is even;

2. $j \to n \to k$, if k is odd

from j to k in the digraph of $\operatorname{adj}(A)$. Since $\operatorname{adj}(A)$ is sign-symmetric, there is also a directed path from k to j in the digraph of $\operatorname{adj}(A)$. Therefore the digraph of $\operatorname{adj}(A)$ is strongly connected, and hence by Theorem 1.1, $\operatorname{adj}(A)$ is irreducible. Again all nonzero off-diagonal entries of $\operatorname{adj}(A)$ are of the same sign. So by Theorem 2.3.3, $\operatorname{adj}(A)$ is algebraically positive. Therefore by Theorem 2.8, A is algebraically positive. Hence the result follows. □

Let $\mathcal{N}_3$ be the symmetric $\mathbb{T}$-pattern matrix of even order n defined by

$$(\mathcal{N}_3)_{22} = \#, \; (\mathcal{N}_3)_{n-1,n-1} = \#;$$
$$(\mathcal{N}_3)_{ij} = 0, \quad \text{if } |i-j| > 1;$$
$$(\mathcal{N}_3)_{ii} = 0, \quad \text{if } i \neq 2, n-1;$$
$$(\mathcal{N}_3)_{i,i+1} = \begin{cases} +, & \text{if } i \text{ is odd;} \\ -, & \text{if } i \text{ is even.} \end{cases}$$

Therefore,

$$\mathcal{N}_3 = \begin{bmatrix} 0 & + & 0 & \cdots & \cdots & \cdots & 0 \\ + & \# & - & \ddots & & & \vdots \\ 0 & - & 0 & + & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & + & 0 & - & 0 \\ \vdots & & & \ddots & - & \# & + \\ 0 & \cdots & \cdots & \cdots & 0 & + & 0 \end{bmatrix}.$$

Lemma 3.38. *$\mathcal{N}_3$ requires algebraic positivity.*

Proof. Let $A \in Q(\mathcal{N}_3)$ and $j < k$. Then

$$\det(A(\{j\},\{k\})) = \det(A[\alpha]) \cdot \det(A[\beta]) \cdot \prod_{m=j+1}^{k} a_{m,m-1},$$

where $\alpha = \{1, \ldots, j-1\}$ and $\beta = \{k+1, \ldots, n\}$.

Observe the following details. Here n is even.

$$\operatorname{sgn}\det(A[\alpha]) = \begin{cases} (-1)^{\frac{j-1}{2}}, & \text{if } j \text{ is odd;} \\ 0, & \text{if } j \text{ is even.} \end{cases}$$

$$\operatorname{sgn}\det(A[\beta]) = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ (-1)^{\frac{n-k}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

$$\operatorname{sgn} \prod_{m=j+1}^{k} a_{m,m-1} = \begin{cases} (-1)^{\frac{k-j}{2}}, & \text{if both of } j,k \text{ are odd or even;} \\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;} \\ (-1)^{\frac{k-j+1}{2}}, & \text{if } j \text{ is even and } k \text{ is odd.} \end{cases}$$

Therefore

$$\operatorname{sgn}\det(A(\{j\},\{k\})) = \begin{cases} 0, & \text{if } j \text{ is even or } k \text{ is odd;} \\ (-1)^{\frac{n-2}{2}}, & \text{if } j \text{ is odd and } k \text{ is even.} \end{cases}$$

Since A is sign-symmetric, $\mathrm{adj}(A)$ is also sign-symmetric. Thus for each

$$j \in \{1, 3, 5, \ldots, n-1\} \text{ and } k \in \{2, 4, 6, \ldots, n\} \text{ with } j < k,$$

the digraph of $\mathrm{adj}(A)$ have both arcs $(j,k), (k,j)$. So for any two distinct $j, k \in \{1, 2, \ldots, n\}$ with $j < k$, there is a directed path

1. $j \to n \to k$, if j is odd and k is odd;
2. $j \to k$, if j is odd and k is even;
3. $j \to 1 \to n \to k$, if j is even and k is odd;
4. $j \to 1 \to k$, if j is even and k is even

from j to k in the digraph of $\mathrm{adj}(A)$. Since $\mathrm{adj}(A)$ is sign-symmetric, there is also a directed path from k to j in the digraph of $\mathrm{adj}(A)$. Therefore the digraph of $\mathrm{adj}(A)$ is strongly connected, and hence by Theorem 1.1, $\mathrm{adj}(A)$ is irreducible. Again all nonzero off-diagonal entries of $\mathrm{adj}(A)$ are of the same sign. So by Theorem 2.3.3, $\mathrm{adj}(A)$ is algebraically positive. Therefore by Theorem 2.8, A is algebraically positive. Hence the result follows. □

For each odd $p \in \{3, 4, \ldots, n-2\}$, let $\mathcal{N}_{4,p}$ be the symmetric $\mathbb{T}$-pattern matrix of even order n defined by

$$(\mathcal{N}_{4,p})_{22} = \#, \ (\mathcal{N}_{4,p})_{pp} = -, \ (\mathcal{N}_{4,p})_{n-1,n-1} = \#;$$

$$(\mathcal{N}_{4,p})_{ij} = 0, \qquad \text{if } |i-j| > 1;$$

$$(\mathcal{N}_{4,p})_{ii} = \begin{cases} 0, & \text{if } i \in \{1, 3, \ldots, p-2\} \cup \{p+1, p+3, \ldots, n\}; \\ -_0, & \text{if } i \in \{4, 6, \ldots, p-1\} \cup \{p+2, p+4, \ldots, n-3\}. \end{cases}$$

$$(\mathcal{N}_{4,p})_{i,i+1} = \begin{cases} +, & \text{if } i \text{ is odd}; \\ -, & \text{if } i \text{ is even}. \end{cases}$$

Therefore,

$$\mathcal{N}_{4,p} = \begin{bmatrix}
0 & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
+ & \# & - & \ddots & & & & & & & & & & \vdots \\
0 & - & 0 & + & \ddots & & & & & & & & & \vdots \\
\vdots & \ddots & + & -_0 & - & \ddots & & & & & & & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & \vdots \\
\vdots & & & \ddots & - & 0 & + & \ddots & & & & & & \vdots \\
\vdots & & & & \ddots & + & -_0 & - & \ddots & & & & & \vdots \\
\vdots & & & & & \ddots & - & - & + & \ddots & & & & \vdots \\
\vdots & & & & & & \ddots & + & 0 & - & \ddots & & & \vdots \\
\vdots & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & & & & & & \ddots & - & -_0 & + & \ddots & \vdots \\
\vdots & & & & & & & & & \ddots & + & 0 & - & 0 \\
\vdots & & & & & & & & & & \ddots & - & \# & + \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + & 0
\end{bmatrix},$$

where the symbol '$-$' on the diagonal is the (p,p)-th entry.

Lemma 3.39. *For each odd* $p \in \{3,4,\ldots,n-2\}$, $\mathcal{N}_{4,p}$ *requires algebraic positivity.*

Proof. Let $A \in Q(\mathcal{N}_{4,p})$ for some odd $p \in \{3,4,\ldots,n-2\}$ and $j<k$. Then

$$\det(A(\{j\},\{k\})) = \det(A[\alpha]) \cdot \det(A[\beta]) \cdot \prod_{m=j+1}^{k} a_{m,m-1},$$

where $\alpha = \{1,\ldots,j-1\}$ and $\beta = \{k+1,\ldots,n\}$.

Observe the following details. Here n is even.

$$\operatorname{sgn}\det(A[\alpha]) = \begin{cases} (-1)^{\frac{j-1}{2}}, & \text{if } j \text{ is odd;} \\ 0, & \text{if } j \text{ is even and } j<p; \\ (-1)^{\frac{j}{2}}, & \text{if } j \text{ is even and } j>p. \end{cases}$$

$$\operatorname{sgn}\det(A[\beta]) = \begin{cases} 0, & \text{if } k \text{ is odd and } k \geq p; \\ 0 \text{ or } (-1)^{\frac{n-k+1}{2}}, & \text{if } k \text{ is odd and } k<p; \\ (-1)^{\frac{n-k}{2}}, & \text{if } k \text{ is even.} \end{cases}$$

$$\operatorname{sgn}\prod_{m=j+1}^{k} a_{m,m-1} = \begin{cases} (-1)^{\frac{k-j}{2}}, & \text{if both of } j,k \text{ are odd or even;}\\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;}\\ (-1)^{\frac{k-j+1}{2}}, & \text{if } j \text{ is even and } k \text{ is odd.}\end{cases}$$

Therefore

$$\operatorname{sgn}\det(A(\{j\},\{k\})) = \begin{cases} (-1)^{\frac{n-2}{2}}, & \text{if } j \text{ is odd and } k \text{ is even;}\\ 0, & \text{if } j \text{ is even and } k \text{ is odd;}\\ 0 \text{ or } (-1)^{\frac{n}{2}}, & \text{if both } j,k \text{ are odd and } k<p;\\ (-1)^{\frac{n}{2}}, & \text{if both } j,k \text{ are even and } j>p;\\ 0, & \text{otherwise.}\end{cases}$$

Since A is sign-symmetric, $\operatorname{adj}(A)$ is also sign-symmetric. Thus for each

$$j \in \{1,3,5,\dots,n-1\} \text{ and } k \in \{2,4,6,\dots,n\} \text{ with } j<k,$$

the digraph of $\operatorname{adj}(A)$ have both arcs $(j,k),(k,j)$. So for any two distinct $j,k \in \{1,2,\dots,n\}$ with $j<k$, there is a directed path

1. $j \to n \to k$, if j is odd and k is odd;
2. $j \to k$, if j is odd and k is even;
3. $j \to 1 \to n \to k$, if j is even and k is odd;
4. $j \to 1 \to k$, if j is even and k is even

from j to k in the digraph of $\operatorname{adj}(A)$. Since $\operatorname{adj}(A)$ is sign-symmetric, there is also a directed path from k to j in the digraph of $\operatorname{adj}(A)$. Therefore the digraph of $\operatorname{adj}(A)$ is strongly connected, and hence by Theorem 1.1, $\operatorname{adj}(A)$ is irreducible. Again all nonzero off-diagonal entries of $\operatorname{adj}(A)$ are of the same sign. So by Theorem 2.3.3, $\operatorname{adj}(A)$ is algebraically positive. Therefore by Theorem 2.8, A is algebraically positive. Hence the result follows. □

For each odd $p \in \{3, 4, \ldots, n-2\}$, let $\mathcal{N}_{5,p}$ be the symmetric $\mathbb{T}$-pattern matrix of odd order n defined by

$$
\begin{aligned}
&(\mathcal{N}_{5,p})_{22} = \#,\ (\mathcal{N}_{5,p})_{pp} = +,\ (\mathcal{N}_{5,p})_{n-1,n-1} = \#;\\
&(\mathcal{N}_{5,p})_{ij} = 0, \qquad \text{if } |i-j| > 1;\\
&(\mathcal{N}_{5,p})_{ii} = \begin{cases} 0, & \text{if } i \neq p \text{ and } i \text{ is odd};\\ -_0, & \text{if } i \neq 2, n-1 \text{ and } i \text{ is even}.\end{cases}\\
&(\mathcal{N}_{5,p})_{i,i+1} = \begin{cases} +, & \text{if } i \in \{1, 3, \ldots, p-2\} \cup \{p+1, p+3, \ldots, n-1\};\\ -, & \text{if } i \in \{2, 4, \ldots, p-1\} \cup \{p, p+2, \ldots, n-2\}.\end{cases}
\end{aligned}
$$

Therefore,

$$
\mathcal{N}_{5,p} = \begin{bmatrix}
0 & + & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
+ & \# & - & \ddots & & & & & & & & & & & \vdots\\
0 & - & 0 & + & \ddots & & & & & & & & & & \vdots\\
\vdots & \ddots & + & -_0 & - & \ddots & & & & & & & & & \vdots\\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & & \vdots\\
\vdots & & & \ddots & - & 0 & + & \ddots & & & & & & & \vdots\\
\vdots & & & & \ddots & + & -_0 & - & \ddots & & & & & & \vdots\\
\vdots & & & & & \ddots & - & + & - & \ddots & & & & & \vdots\\
\vdots & & & & & & \ddots & - & -_0 & + & \ddots & & & & \vdots\\
\vdots & & & & & & & \ddots & + & 0 & - & \ddots & & & \vdots\\
\vdots & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots\\
\vdots & & & & & & & & & \ddots & - & -_0 & + & \ddots & \vdots\\
\vdots & & & & & & & & & & \ddots & + & 0 & - & 0\\
\vdots & & & & & & & & & & & \ddots & - & \# & +\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & + & 0
\end{bmatrix},
$$

where the symbol '+' on the diagonal is the (p,p)-th entry.

Lemma 3.40. *For each odd $p \in \{3, 4, \ldots, n-2\}$, $\mathcal{N}_{5,p}$ requires algebraic positivity.*

Proof. Let $A \in Q(\mathcal{N}_{5,p})$ for some odd $p \in \{3, 4, \ldots, n-2\}$ and $j < k$. Then

$$\det(A(\{j\},\{k\})) = \det(A[\alpha]) \cdot \det(A[\beta]) \cdot \prod_{m=j+1}^{k} a_{m,m-1},$$

where $\alpha = \{1, \ldots, j-1\}$ and $\beta = \{k+1, \ldots, n\}$.

Observe the following details. Here n is odd.

$$\operatorname{sgn}\det(A[\alpha]) = \begin{cases} (-1)^{\frac{j-1}{2}}, & \text{if } j \text{ is odd;} \\ (-1)^{\frac{j-2}{2}}, & \text{if } j \text{ is even and } j > p; \\ 0, & \text{otherwise.} \end{cases}$$

$$\operatorname{sgn}\det(A[\beta]) = \begin{cases} (-1)^{\frac{n-k}{2}}, & \text{if } k \text{ is odd;} \\ (-1)^{\frac{n-k-1}{2}}, & \text{if } k \text{ is even and } k < p; \\ 0, & \text{otherwise.} \end{cases}$$

$$\operatorname{sgn}\prod_{m=j+1}^{k} a_{m,m-1} = \begin{cases} (-1)^{\frac{k-j}{2}}, & \text{if both of } j,k \text{ are odd;} \\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is odd}, k \text{ is even and } k < p; \\ (-1)^{\frac{k-j-1}{2}}, & \text{if } j \text{ is even}, k \text{ is odd and } j > p. \end{cases}$$

Therefore

$$\operatorname{sgn}\det(A(\{j\},\{k\})) = \begin{cases} (-1)^{\frac{n-1}{2}}, & \text{if both of } j,k \text{ are odd;} \\ 0, & \text{if both of } j,k \text{ are even;} \\ 0, & \text{if } j \text{ is odd}, k \text{ is even and } k > p; \\ (-1)^{\frac{n-3}{2}}, & \text{if } j \text{ is odd}, k \text{ is even and } k < p; \\ 0, & \text{if } j \text{ is even}, k \text{ is odd and } j < p; \\ (-1)^{\frac{n-3}{2}}, & \text{if } j \text{ is even}, k \text{ is odd and } j > p. \end{cases}$$

Since A is sign-symmetric, $\operatorname{adj}(A)$ is also sign-symmetric. Thus for each

$$j, k \in \{1, 3, 5, \ldots, n\} \text{ with } j < k,$$

the digraph of adj(A) have both arcs $(j,k),(k,j)$. Again for each $k \in \{2,4,\dots,p-1\}$, the digraph of adj(A) have both arcs $(1,k),(k,1)$, and for each $j \in \{p+1,p+3,\dots,n-1\}$, the digraph of adj$(A)$ have both arcs $(j,n),(n,j)$. So for any two distinct $j,k \in \{1,2,\dots,n\}$ with $j<k$, there is a directed path

1. $j \to k$, if j is odd and k is odd;
2. $j \to 1 \to k$, if j is odd and k is even with $k<p$;
3. $j \to n \to k$, if j is odd and k is even with $k>p$;
4. $j \to 1 \to k$, if j is even and k is odd with $j<p$;
5. $j \to n \to k$, if j is even and k is odd with $j>p$;
6. $j \to 1 \to k$, if j is even and k is even with $j,k<p$;
7. $j \to 1 \to n \to k$, if j is even and k is even with $j<p<k$;
8. $j \to n \to k$, if j is even and k is even with $j,k>p$

from j to k in the digraph of adj(A). Since adj(A) is sign-symmetric, there is also a directed path from k to j in the digraph of adj(A). Therefore the digraph of adj(A) is strongly connected, and hence by Theorem 1.1, adj(A) is irreducible. Again all nonzero off-diagonal entries of adj(A) are of the same sign. So by Theorem 2.3.3, adj(A) is algebraically positive. Therefore by Theorem 2.8, A is algebraically positive. Hence the result follows. □

Lemma 3.41. *Suppose A is a sign pattern matrix of order n as in (3.16) with $b_1=+$, $b_{n-1}=+$ and $b_k=-$ for some k. If A requires algebraic positivity, then A is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where*

$$\mathcal{S} \in \{\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3\} \cup \{\mathcal{N}_{4,p},\mathcal{N}_{5,p} : \ p \text{ is odd and } 3 \le p \le n-2\}.$$

Proof. Suppose A requires algebraic positivity. Then a_1,a_n cannot be $+$ by Lemma 3.13 and Lemma 3.17.

`Case I:` Suppose at least one of a_1, a_n is nonzero. Then the possibilities of the ordered pair (a_1, a_n) are $(-,-), (-,0), (0,-)$. Now by Corollary 3.29, n is even, $b_k = +$ for all odd k and $b_k = -$ for all even k. Therefore off-diagonal entries of both P^TAP and A have the same sign pattern, where P is the anti-diagonal permutation sign pattern matrix of order n. So the possibility $(0,-)$ is equivalent to the possibility $(-,0)$, by Lemma 3.13. So by Lemma 3.23 and Lemma 3.33, A is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where $\mathcal{S} \in \{\mathcal{N}_1, \mathcal{N}_2\}$.

`Case II:` Suppose $a_1, a_n = 0$ and n is even. Then by Lemma 3.27 and Lemma 3.28, $b_k = +$ for all odd k and $b_k = -$ for all even k. So by Lemma 3.31, a_k cannot be $+$ for any k except possibly for $k = 2, n-1$.

If $a_p = -$ for some p other than $1, 2, n-1, n$, we can choose p to be the smallest index such that p is odd and $a_p = -$ or to be the largest index such that p is even and $a_p = -$. Then by Lemma 3.13 and Lemma 3.34, A is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{N}_{4,p})$ for some odd p with $3 \leq p \leq n-2$.

If there is no p such that $a_p = -$, then $a_k = 0$ for all k except possibly $k = 2, n-1$. Therefore $A \in \mathcal{F}(\mathcal{N}_3)$.

`Case III:` Suppose $a_1, a_n = 0$ and n is odd. Therefore by Lemma 3.27 and Lemma 3.35, $A \in \mathcal{F}(\mathcal{N}_{5,p})$ for some odd p with $3 \leq p \leq n-2$. □

The following result characterizes all path sign pattern matrices of order at least 4 requiring algebraic positivity.

Theorem 3.42. *Let A be a path sign pattern matrix of order at least 4. Then A requires algebraic positivity if and only if one of the following conditions hold:*

1. *all nonzero off-diagonal entries of A are $+$;*

2. *all nonzero off-diagonal entries of A are $-$;*

3. *A or $-A$ is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where*

$$\mathcal{S} \in \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\} \cup \{\mathcal{N}_{4,p}, \mathcal{N}_{5,p} : \ p \text{ is odd and } 3 \leq p \leq n-2\}.$$

Proof. Let A be such that it requires algebraic positivity. Then by Theorem 3.2, A is a symmetric sign pattern matrix. Suppose that the first two conditions do not hold. Then A or $-A$ is permutationally similar to a sign pattern matrix of the form (3.16) with $b_1 = +$ and $b_k = -$ for some k.

If $b_1 = +$ and $b_{n-1} = -$, then by Lemma 3.26, A or $-A$ is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where $\mathcal{S} \in \{\mathcal{M}_1, \mathcal{M}_2\}$.

If $b_1, b_{n-1} = +$, then by Lemma 3.41, A or $-A$ is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where

$$\mathcal{S} \in \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\} \cup \{\mathcal{N}_{4,p}, \mathcal{N}_{5,p} : \; p \text{ is odd and } 3 \leq p \leq n-2\}.$$

The converse statement follows from Lemma 3.8, Lemma 3.26, Lemma 3.36, Lemma 3.37, Lemma 3.38, Lemma 3.39 and Lemma 3.40. □

3.3.3 5-by-5 Tree Sign Pattern Matrices Requiring Algebraic Positivity

Since any tree of order at most four is either a star or a path, all tree sign pattern matrices of order less than or equal to 4 requiring algebraic positivity are identified in the previous sections. There is only one tree, with five vertices, which is neither a star nor a path. In this section, we give a complete description of all tree sign pattern matrices of order 5 requiring algebraic positivity, which are neither a star nor a path. As earlier, we can assume that the sign pattern matrix is symmetric and both the symbols $+$ and $-$ appear as off-diagonal entries. Let us consider the tree sign pattern matrix

$$R = \begin{bmatrix} a_1 & b_1 & b_2 & b_3 & 0 \\ b_1 & a_2 & 0 & 0 & 0 \\ b_2 & 0 & a_3 & 0 & 0 \\ b_3 & 0 & 0 & a_4 & b_4 \\ 0 & 0 & 0 & b_4 & a_5 \end{bmatrix}, \qquad (3.43)$$

where $a_1, a_2, a_3, a_4, a_5 \in \{+, -, 0\}$, $b_1, b_2, b_3 \in \{+, -\}$, $b_4 = -$ and at least one of b_1, b_2, b_3 is $+$.

Lemma 3.44. *If R requires algebraic positivity, then $b_3 = +$ and at least one of b_1, b_2 is $-$.*

Proof. Suppose $b_3 = -$. Without loss of generality we can assume that $b_1 = +$. Consider the matrix

$$A = \begin{bmatrix} v(a_1) & 1 & v(b_2) & -1 & 0 \\ 1 & v(a_2) & 0 & 0 & 0 \\ v(b_2) & 0 & v(a_3) & 0 & 0 \\ -1 & 0 & 0 & v(a_4) & -5 \\ 0 & 0 & 0 & -5 & v(a_5) \end{bmatrix} \in Q(R).$$

Suppose λ be an eigenvalue of A corresponding to a positive right eigenvector. Using Lemma 3.14 we can conclude that $-1 < \lambda < 1$. Then $R_4(A-\lambda I)+R_5(A-\lambda I)$ is a nonzero, nonpositive vector. So $\mathcal{N}(A - \lambda I)$ does not contain any positive vector, which is a contradiction. Hence $b_3 = +$.

Suppose $b_1, b_2 = +$. Consider the matrix

$$A = \begin{bmatrix} v(a_1) & 5 & 1 & 1 & 0 \\ 5 & v(a_2) & 0 & 0 & 0 \\ 1 & 0 & v(a_3) & 0 & 0 \\ 1 & 0 & 0 & v(a_4) & -1 \\ 0 & 0 & 0 & -1 & v(a_5) \end{bmatrix} \in Q(R).$$

If λ be an eigenvalue of A corresponding to a positive right eigenvector, then by Lemma 3.14 we have $-1 < \lambda < 1$. So $R_1(A - \lambda I) + R_2(A - \lambda I)$ is a nonzero, nonnegative vector. Therefore $\mathcal{N}(A - \lambda I)$ does not contain any positive vector, which is a contradiction. Hence at least one of b_1, b_2 is $-$. □

So the possible sign pattern matrices of the form R requiring algebraic positivity are

$$R_1 = \begin{bmatrix} a_1 & + & - & + & 0 \\ + & a_2 & 0 & 0 & 0 \\ - & 0 & a_3 & 0 & 0 \\ + & 0 & 0 & a_4 & - \\ 0 & 0 & 0 & - & a_5 \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} a_1 & - & - & + & 0 \\ - & a_2 & 0 & 0 & 0 \\ - & 0 & a_3 & 0 & 0 \\ + & 0 & 0 & a_4 & - \\ 0 & 0 & 0 & - & a_5 \end{bmatrix},$$

where $a_1, a_2, a_3, a_4, a_5 \in \{+, -, 0\}$

Lemma 3.45. *Suppose R_1 requires algebraic positivity. Then the following conditions hold.*

1. $a_3 = 0, a_2 = -$ *and* a_5 *cannot be* $-$.
2. *If* $a_5 = +$, *then* a_4 *cannot be* $+$.

Proof.

1. By Lemma 3.15, a_2 cannot be $+$ and a_3, a_5 cannot be $-$.

   Suppose $a_3 = +$. Consider the matrix

$$A = \begin{bmatrix} v(a_1) & 5 & -1 & 1 & 0 \\ 5 & v(a_2) & 0 & 0 & 0 \\ -1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & v(a_4) & -1 \\ 0 & 0 & 0 & -1 & v(a_5) \end{bmatrix} \in Q(R_1).$$

   If λ be an eigenvalue of A corresponding to a positive right eigenvector, then by Lemma 3.14, $-1 < \lambda < 1$. So $R_1(A-\lambda I) + R_2(A-\lambda I) + R_3(A-\lambda I)$ is a nonzero, nonnegative vector. Thus $\mathcal{N}(A-\lambda I)$ does not contain any positive vector, which is a contradiction. Hence $a_3 = 0$. Therefore by Lemma 3.15, $a_2 = -$.

2. Suppose $a_4, a_5 = +$. Since $a_3 = 0, a_2 = -$, consider the matrix

$$A = \begin{bmatrix} v(a_1) & 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & -1 & 5 \end{bmatrix} \in Q(R_1).$$

   If λ be an eigenvalue of A corresponding to a positive right eigenvector, then by Lemma 3.14, $-1 < \lambda < 0$. So $R_4(A-\lambda I) + R_5(A-\lambda I)$ is a nonzero, nonnegative vector. Thus $\mathcal{N}(A-\lambda I)$ does not contain any positive vector, which is a contradiction. Thus if $a_5 = +$, then a_4 cannot be $+$. □

Lemma 3.46. *If R_2 requires algebraic positivity, then the following conditions hold.*

1. *a_2, a_3, a_5 cannot be $-$.*
2. *If $a_5 = +$, then a_4 cannot be $+$.*
3. *If $a_2, a_3 = +$, then a_1 cannot be $+$.*

Proof.

1. Suppose $a_3 = -$ and consider the matrix

$$A = \begin{bmatrix} v(a_1) & -1 & -1 & 1 & 0 \\ -1 & v(a_2) & 0 & 0 & 0 \\ -1 & 0 & -5 & 0 & 0 \\ 1 & 0 & 0 & v(a_4) & -1 \\ 0 & 0 & 0 & -1 & v(a_5) \end{bmatrix} \in Q(R_2).$$

If λ be an eigenvalue of A corresponding to a positive right eigenvector, then by Lemma 3.14, $\lambda < -5$. So $R_4(A - \lambda I) + R_5(A - \lambda I)$ is a nonzero, nonnegative vector. Thus $\mathcal{N}(A - \lambda I)$ does not contain any positive vector, which is a contradiction. So a_3 cannot be $-$. Similarly a_2 cannot be $-$.

Suppose $a_5 = -$ and consider the matrix

$$A = \begin{bmatrix} v(a_1) & -1 & -1 & 1 & 0 \\ -1 & v(a_2) & 0 & 0 & 0 \\ -1 & 0 & v(a_3) & 0 & 0 \\ 1 & 0 & 0 & v(a_4) & -1 \\ 0 & 0 & 0 & -1 & -5 \end{bmatrix} \in Q(R_2).$$

If λ be an eigenvalue of A corresponding to a positive right eigenvector, then by Lemma 3.14, $\lambda < -5$. So $R_1(A - \lambda I) + R_2(A - \lambda I) + R_3(A - \lambda I)$ is a nonzero, nonnegative vector. Thus $\mathcal{N}(A - \lambda I)$ does not contain any positive vector, which is a contradiction. So a_5 cannot be $-$.

2. It can be proved using similar arguments as in Lemma 3.45.2.

3. Suppose $a_1 = +$. Consider the matrix

$$A = \begin{bmatrix} 5 & -1 & -1 & 1 & 0 \\ -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & v(a_4) & -1 \\ 0 & 0 & 0 & -1 & v(a_5) \end{bmatrix} \in Q(R_2).$$

If λ be an eigenvalue of A corresponding to a positive right eigenvector, then $\lambda < 1$, by Lemma 3.14. So $R_1(A - \lambda I) + R_2(A - \lambda I) + R_3(A - \lambda I)$ is a nonzero, nonnegative vector. Thus $\mathcal{N}(A - \lambda I)$ does not contain any positive vector, which is a contradiction. So a_1 cannot be $+$. □

So any tree sign pattern matrix requiring algebraic positivity of the form R in (3.43) must be in $\mathcal{F}(\mathcal{S})$, where

$$\mathcal{S} \in \left\{ \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} -_0 & - & - & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} -_0 & - & - & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} \# & + & - & + & 0 \\ + & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} \# & + & - & + & 0 \\ + & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix} \right\}.$$

Lemma 3.47. *The following $\mathbb{T}$-pattern matrices require algebraic positivity.*

$$\begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

Proof. Any matrix in the qualitative class of these patterns is similar (through a diagonal matrix with positive diagonal entries) to a matrix of the form

$$A = \begin{bmatrix} c & -c_1 & -c_2 & c_3 & 0 \\ -c_1 & 0 & 0 & 0 & 0 \\ -c_2 & 0 & 0 & 0 & 0 \\ c_3 & 0 & 0 & d_4 & -c_4 \\ 0 & 0 & 0 & -c_4 & d_5 \end{bmatrix},$$

where $c_1, c_2, c_3, c_4 > 0, d_4 d_5 \leq 0, d_5 \geq 0$ and c is of arbitrary sign. So it is enough to show that A is algebraically positive. The characteristic polynomial of A is

$$f(x) = x^2(x-c)\{(x-d_4)(x-d_5)-c_4^2\} - (c_1^2+c_2^2)x\{(x-d_4)(x-d_5)-c_4^2\} - c_3^2x^2(x-d_5).$$

Suppose $x^2-(d_4+d_5)x+d_4d_5-c_4^2 = (x-\alpha)(x-\beta)$, where $\alpha, \beta \in \mathbb{R}$. Then $\alpha\beta = d_4d_5 - c_4^2 < 0$. Without loss of generality assume $\alpha < 0$. Eigenvalues of $A(\{1\})$ are $\alpha, 0, 0, \beta$. Now we have

$$\left.\frac{f(x)}{x}\right|_{x=0} = (c_1^2+c_2^2)(c_4^2-d_4d_5) \neq 0 \text{ and } f(\alpha) = -c_3^2\alpha^2(\alpha-d_5) \neq 0.$$

So using Cauchy's interlacing theorem, we can conclude that A has exactly two negative eigenvalues and they are simple. We will show that A has a positive right eigenvector corresponding to the eigenvalue λ satisfying $\alpha < \lambda < 0$. Suppose $(x_1, x_2, x_3, x_4, x_5)^T$ be a right eigenvector of A corresponding to λ. Then

$$-c_1x_1 = \lambda x_2, \quad -c_2x_1 = \lambda x_3, \text{ and } x_1 \begin{bmatrix} c_3 \\ 0 \end{bmatrix} + \begin{bmatrix} d_4 & -c_4 \\ -c_4 & d_5 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \lambda \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}.$$

From the third matrix equation we have

$$\begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} \lambda - d_4 & c_4 \\ c_4 & \lambda - d_5 \end{bmatrix}^{-1} \begin{bmatrix} c_3 \\ 0 \end{bmatrix} = \frac{c_3x_1}{(\lambda-\alpha)(\lambda-\beta)} \begin{bmatrix} \lambda - d_5 \\ -c_4 \end{bmatrix}.$$

If $x_1 = 0$, then $x_2 = x_3 = x_4 = x_5 = 0$. So assume $x_1 = 1$. Then $x_2, x_3, x_4, x_5 > 0$. Since for a symmetric matrix left and right eigenvectors corresponding to an eigenvalue are same, A is algebraically positive. Therefore the given $\mathbb{T}$-pattern matrices require algebraic positivity. □

Theorem 3.48. *Let A be a 5×5 tree sign pattern matrix such that its graph is neither a star nor a path. Then A requires algebraic positivity if and only if one of the following conditions hold:*

1. *all nonzero off-diagonal entries of A are $+$;*
2. *all nonzero off-diagonal entries of A are $-$;*
3. *A or $-A$ is permutationally similar to some sign pattern matrix from $\mathcal{F}(\mathcal{S})$, where*

$$\mathcal{S} \in \left\{ \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} -_0 & - & - & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} -_0 & - & - & + & 0 \\ - & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} \# & - & - & + & 0 \\ - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} \# & + & - & + & 0 \\ + & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & -_0 & - \\ 0 & 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} \# & + & - & + & 0 \\ + & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & \# & - \\ 0 & 0 & 0 & - & 0 \end{bmatrix} \right\}.$$

Proof. If A requires algebraic positivity and all nonzero off-diagonal entries of A are not of the same sign, then by Lemma 3.44, A or $-A$ is permutationally similar to some sign pattern matrix from R_1 and R_2. By Lemma 3.45, any sign pattern matrix of the form R_1 requiring algebraic positivity must be in $\mathcal{F}(\mathcal{S})$, where $\mathcal{S}$ is either the seventh or the eighth in the above list. By Lemma 3.46, any sign pattern matrix of the form R_2 requiring algebraic positivity must be in $\mathcal{F}(\mathcal{S})$, where $\mathcal{S}$ is one of the first six in the above list.

For the converse part, if all nonzero off-diagonal entries of A have the same sign, then the result follows from Lemma 3.8. Sign patterns of the off-diagonal entries of the adjugate of each of the matrices in the qualitative classes of the first, second, third, fourth, seventh and eighth from the above list are respectively

$$\begin{bmatrix} \cdot & - & 0 & 0 & 0 \\ - & \cdot & - & 0 & - \\ 0 & - & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & - \\ 0 & - & 0 & - & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & - & - & 0 & - \\ - & \cdot & - & 0 & - \\ - & - & \cdot & 0 & - \\ 0 & 0 & 0 & \cdot & - \\ - & - & - & - & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & - & 0 & 0 & 0 \\ - & \cdot & - & - & - \\ 0 & - & \cdot & 0 & 0 \\ 0 & - & 0 & \cdot & - \\ 0 & - & 0 & - & \cdot \end{bmatrix},$$

$$\begin{bmatrix} \cdot & - & - & - & - \\ - & \cdot & - & - & - \\ - & - & \cdot & - & - \\ - & - & - & \cdot & - \\ - & - & - & - & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 0 & + & 0 & 0 \\ 0 & \cdot & + & 0 & 0 \\ + & + & \cdot & + & + \\ 0 & 0 & + & \cdot & + \\ 0 & 0 & + & + & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 0 & + & 0 & 0 \\ 0 & \cdot & + & 0 & 0 \\ + & + & \cdot & 0 & + \\ 0 & 0 & 0 & \cdot & + \\ 0 & 0 & + & + & \cdot \end{bmatrix}.$$

Each of these patterns is irreducible and all the nonzero off-diagonal entries have the same sign. So by Lemma 3.8, these six patterns require algebraic positivity. Hence by Theorem 2.8, first, second, third, fourth, seventh and eighth from the above list require algebraically positivity. The remaining two sign pattern matrices require algebraic positivity by Lemma 3.47. Hence the result follows. □

Theorem 3.12, Theorem 3.42 and Theorem 3.48 taken together give the set of all tree sign pattern matrices of order less than 6 requiring algebraic positivity.

CHAPTER 4

Sign Patterns that Allow or Require Diagonalizability

In this chapter, we discuss about sign pattern matrices that allow or require diagonalizability. Sign pattern matrices allowing diagonalizability have been studied in [16, 19, 20, 48]. In Section 4.1, we define essential index for a tree sign pattern matrix, and in Section 4.2, we use this concept to give necessary and/or sufficient conditions for some class of sign pattern matrices that allow diagonalizability. Finally, in Section 4.3, we characterize all star sign pattern matrices requiring diagonalizability.

Throughout this chapter, $c(A)$ denotes the maximum cycle length in A and $[u, v]$ denotes the edge joining two vertices u, v of an undirected graph. We have the following result from [26, Section 42.6].

Lemma 4.1 ([26]). $\mathrm{MR}(A)$ *is the maximum number of nonzero entries of* A *no two of which are on the same row or the same column.*

4.1 Essential Index

Let A be a tree sign pattern matrix of order n such that $V(G(A))$ denotes the set of all vertices in $G(A)$. If $S \subseteq V(G(A))$, then $G(A) - S$ is the graph obtained from $G(A)$ by deleting all the vertices in S and the edges incident to them. Let

$$L_A = \{S \subseteq V(G(A)) : G(A) - S \text{ has a perfect matching}\}$$

and

$$I_A = \{S \in L_A : T \subseteq S \text{ for some } T \in L_A \text{ implies } T = S\}.$$

Clearly, $G(A)$ has a perfect matching if and only if $I_A = \{\emptyset\}$.

Let us recall that the standard determinant expansion of a square matrix $A = [a_{ij}]$ is

$$\det A = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1i_1} a_{2i_2} \cdots a_{ni_n} \tag{4.2}$$

where the summation extends over all permutations $\sigma = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. We have the following characterization of sign pattern matrices requiring singularity.

Lemma 4.3 ([9, 26]). *If A is a sign pattern matrix of order n, then the following statements are equivalent.*

1. *A requires singularity.*
2. *Every term in the standard determinant expansion of A is zero.*
3. *A has no composite n-cycle.*

Theorem 4.4. *If A is a tree sign pattern matrix, then A requires singularity if and only if $I_A \neq \{\emptyset\}$ and $\prod_{i \in S} a_{ii} = 0$ for all $S \in I_A$.*

Proof. Suppose A requires singularity. So by Lemma 4.3, A has no composite n-cycle and thus $G(A)$ has no perfect matching. Therefore $I_A \neq \{\emptyset\}$. For the second part, let $S \in I_A$ and M be the perfect matching for $G(A) - S$. Then $\det A$ has a term $\pm \prod_{i \in S} a_{ii} \cdot \prod_{[i,j] \in M} a_{ij} a_{ji}$ in its expansion as (4.2). Since $a_{ij} a_{ji} \neq 0$ for all $[i, j] \in M$, by Lemma 4.3, $\prod_{i \in S} a_{ii} = 0$.

Conversely, suppose $I_A \neq \{\emptyset\}$ and $\prod_{i \in S} a_{ii} = 0$ for all $S \in I_A$. Since A is a tree sign pattern matrix, every term in the standard determinant expansion of A is of the form $\pm \prod_{i \in T} a_{ii} \cdot \prod_{[i,j] \in M} a_{ij} a_{ji}$ for some $T \subseteq V(G(A))$, where M is the perfect matching for $G(A) - T$. Since $I_A \neq \{\emptyset\}$, $G(A)$ does not have a perfect matching and thus $\emptyset \notin I_A$. Further T is a super set of an $S \in I_A$ and hence the above term is zero. Therefore by Lemma 4.3, A requires singularity. □

The above theorem asserts that for a tree sign pattern matrix A, singularity depends on some terms of the form a_{ii}. We call the index of such a term as an 'essential index'.

Definition 4.5. *Let A be a tree sign pattern matrix of order n. An index i is said to be an essential index of A if there exists $S \in I_A$ such that $i \in S$ and the sign pattern matrix $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. All other indices in $\{1, 2, \dots, n\}$ are said to be non-essential indices of A.*

Theorem 4.6. *Let A be a tree sign pattern matrix. If i is an essential index of A, then $a_{ii} = 0$.*

Proof. Suppose $a_{ii} \neq 0$ for some index i and $S \in I_A$ containing i. If M is the perfect matching for $G(A) - S$, then $\det A(S \setminus \{i\})$ has a nonzero term $\pm a_{ii} \cdot \prod_{[i,j] \in M} a_{ij}a_{ji}$ in its standard expansion, and thus by Lemma 4.3, $A(S \setminus \{i\})$ allows nonsingularity. Therefore i is a non-essential index of A. Hence if i is an essential index of A, then $a_{ii} = 0$. □

The following example illustrates the essential and non-essential indices of a tree sign pattern matrix and establishes that the converse of Theorem 4.6 is not true.

Example 4.7. *Let us consider a tree sign pattern matrix A with its graph $G(A)$ as follows.*

$$A = \begin{bmatrix} 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \\ + & - & + & + & 0 & 0 \\ 0 & 0 & - & - & + & 0 \\ 0 & 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & + & 0 \end{bmatrix}$$

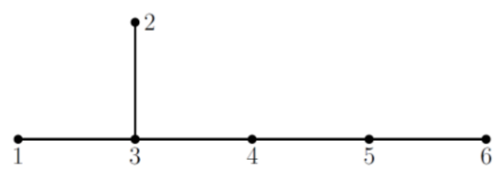


Figure 4.8: A tree sign pattern matrix A with its graph $G(A)$.

From the above figure it is clear that $I_A = \{\{1,2\}, \{1,4\}, \{2,4\}, \{1,6\}, \{2,6\}\}$. Further,

$$\{i : i \in S \text{ for some } S \in I_A \text{ and } a_{ii} = 0\} = \{1, 2, 6\}.$$

If $S = \{1,4\} \in I_A$, then all $T \subseteq S$ containing 1 are $\{1\}$ and $\{1,4\}$. Since both A and $A(\{4\})$ require singularity, $A(T \setminus \{1\})$ requires singularity for all $T \subseteq S$ containing 1. So 1 is an essential index of A. Similarly, 2 is an essential index of A.

All the sets in I_A *containing* 6 *are* $\{1,6\}$ *and* $\{2,6\}$. *If* $S=\{1,6\}$, *then for* $T=S$, $A(T\setminus\{6\})=A(\{1\})$ *allows nonsingularity. If* $S=\{2,6\}$, *then for* $T=S$, $A(T\setminus\{6\})=A(\{2\})$ *allows nonsingularity. So* 6 *is a non-essential index of* A.

Therefore the essential indices of A *are* $1,2$, *and the non-essential indices of* A *are* $3,4,5,6$.

Theorem 4.9. *If a tree sign pattern matrix* A *has an essential index, then* A *requires singularity.*

Proof. Suppose i is an essential index of A. Then there exists $S\in I_A$ such that $i\in S$ and the sign pattern matrix $A(T\setminus\{i\})$ requires singularity for all $T\subseteq S$ containing i. Choosing $T=\{i\}$, we have A requires singularity. □

The converse is also true, which will be proved using the following lemmas.

Lemma 4.10. *Let* A *be a tree sign pattern matrix and* $G(A)$ *has vertices* $1,2$ *such that* 2 *is adjacent to* 1 *and* $\deg(1)=1$. *Then* $2\notin S$ *for any* $S\in I_A$ *and thus* 2 *is a non-essential index of* A.

Proof. Suppose there exists $S\in I_A$ such that $2\in S$. Since $G(A)-S$ has a perfect matching, S must contain the vertex 1. Then $G(A)-\tilde{S}$ also has a perfect matching, where $\tilde{S}=S\setminus\{1,2\}$. This is a contradiction to the fact that $S\in I_A$. □

Lemma 4.11. *Let* A *be a tree sign pattern matrix. If* $a_{ii}\neq 0$ *for all pendant vertices* i *in* $G(A)$, *then* A *allows nonsingularity.*

Proof. Let A be of order n, and r be the number of vertices of degree ≥ 3 in $G(A)$. We prove the result by induction on r. For $r=0$, A is a path sign pattern matrix having pendant vertices i with $a_{ii}\neq 0$. So A has a composite n-cycle and thus by Lemma 4.3, A allows nonsingularity. So the result is true for $r=0$. Suppose the result is true for $r=k-1$. Let $r=k$ and A be a tree sign pattern matrix having k vertices of degree ≥ 3. There exists a vertex in $G(A)$ of degree $d\geq 3$, say 1, and $d-1$ pendant vertices such that the paths joining 1 and each of the $d-1$ pendant vertices contains no vertex of degree ≥ 3 except 1. Since each of the principal submatrices of A corresponding to the indices from those paths excluding 1 is a path sign pattern matrix, they allow nonsingularity. Let α be the set of vertices corresponding to the indices from $d-2$ of those paths excluding 1. So $A[\alpha]$ is direct sum of $d-2$ sign pattern matrices allowing nonsingularity and thus $A[\alpha]$ allows nonsingularity. Further, if we delete all the vertices in α from $G(A)$, then the new graph will have $k-1$ vertices of degree ≥ 3 (since 1 now has degree 2) and $a_{ii} \neq 0$ for all pendant vertices i in the new graph. So by the induction hypothesis, the principal submatrix of A corresponding to the new graph, i.e., $A(\alpha)$ allows nonsingularity.

Since both $A[\alpha]$ and $A(\alpha)$ allow nonsingularity, by Lemma 4.3, $A[\alpha]$ has a composite $|\alpha|$-cycle and $A(\alpha)$ has a composite $(n-|\alpha|)$-cycle. Since the index sets of those two cycles are disjoint, the product of those two cycles is a composite n-cycle of A. Therefore by Lemma 4.3, A does not require singularity that is, A allows nonsingularity. □

Therefore if a tree sign pattern matrix A requires singularity, then $a_{ii} = 0$ for at least one pendant vertex i in $G(A)$.

Lemma 4.12. *Suppose A is a tree sign pattern matrix of order n with $a_{nn} = 0$, and $G(A)$ has vertices $n-2, n-1, n$ such that $n-1$ is adjacent to both $n-2, n$ and $\deg(n) = 1, \deg(n-1) = 2$. Let $i \in \{1, 2, \ldots, n-2\}$. If i is an essential index of A, then there exists $S \in I_A$ containing i such that $n \notin S$ and $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i.*

Proof. According to the given conditions we can represent $G(A)$ as follows.

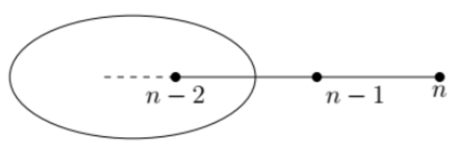


Figure 4.13: A representation of $G(A)$.

Since i is an essential index of A, there exists $S \in I_A$ containing i such that $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. If $n \in S$, then $n-2 \notin S$, and thus $i < n-2$. Now $S \in I_A$ implies $R = (S \setminus \{n\}) \cup \{n-2\} \in I_A$. Let $T' \subseteq R$ contains i. If $n-2 \notin T'$, then

$T' \subseteq S$. So $A(T' \setminus \{i\})$ requires singularity. If $n-2 \in T'$, let $T = (T' \setminus \{n-2\}) \cup \{n\}$. Then $T \subseteq S$ and thus $A(T \setminus \{i\})$ requires singularity. Since $T \setminus \{n\} = T' \setminus \{n-2\} \subseteq \{1, 2, \ldots, n-3\}$, $n-2 \in T'$, $n \in T$ and $i < n-2$, assuming $T \setminus \{n\} = T' \setminus \{n-2\} = Z$ we have

$$A(T' \setminus \{i\}) = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & A_1(Z \setminus \{i\}) & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & 0 \end{array}\right]$$

and

$$A(T \setminus \{i\}) = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & A_1(Z \setminus \{i\}) & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & c & 0 \\ \hline 0 & \cdots & 0 & d & a_{n-2,n-2} & a_{n-2,n-1} \\ 0 & \cdots & 0 & 0 & a_{n-1,n-2} & a_{n-1,n-1} \end{array}\right],$$

where $A_1 = A(\{n-2, n-1, n\})$, and either both c, d are zero or both are nonzero.

So $-a_{n-2,n-1}a_{n-1,n-2} \cdot \det A_1(Z \setminus \{i\})$ is a sum of some terms in the standard expansion of $\det A(T \setminus \{i\})$. Since $A(T \setminus \{i\})$ requires singularity, by Lemma 4.3, each term in the standard expansion of $\det A(T \setminus \{i\})$ is zero. Since $a_{n-2,n-1}, a_{n-1,n-2} \neq 0$, each term in the standard expansion of $\det A_1(Z \setminus \{i\})$ is zero.

Now each term in the standard expansion of $\det A(T' \setminus \{i\})$ is a product of $-a_{n-1,n}a_{n,n-1}$ and a term in the standard expansion of $\det A_1(Z \setminus \{i\})$. Therefore each term in the standard expansion of $\det A(T' \setminus \{i\})$ is zero and thus by Lemma 4.3, $A(T' \setminus \{i\})$ requires singularity.

Hence $R \in I_A$ containing i is such that $n \notin R$ and $A(T' \setminus \{i\})$ requires singularity for all $T' \subseteq R$ containing i. □

Lemma 4.14. *Suppose A is a tree sign pattern matrix of order n, and $G(A)$ has vertices $n-2, n-1, n$ such that $n-1$ is adjacent to both $n-2, n$ and $\deg(n) = 1, \deg(n-1) = 2$. Let $a_{nn} = 0$, $\tilde{A} = A(\{n-1, n\})$ and $G(\tilde{A}) = G(A) - \{n-1, n\}$. Then we have the following.*

1. *n is an essential index of A if and only if $n-2$ is an essential index of A.*

2. *For each $i \in \{1, 2, \ldots, n-2\}$, i is an essential index of $\tilde{A}$ if and only if i is an essential index of A.*

Proof. According to the given conditions we can represent $G(A)$ as follows.

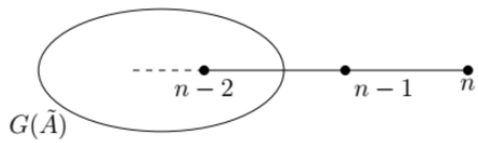


Figure 4.15: A representation of $G(A)$.

1. Let $n-2$ be an essential index of A. Then there exists $S \in I_A$ such that $n-2 \in S$ and $A(T \setminus \{n-2\})$ requires singularity for all $T \subseteq S$ containing $n-2$. Let $R = (S \setminus \{n-2\}) \cup \{n\}$. If M is the perfect matching for $G(A) - S$, then $[n-1, n] \in M$ and thus $(M \setminus \{[n-1, n]\}) \cup \{[n-2, n-1]\}$ is the perfect matching for $G(A) - R$. Therefore $S \in I_A$ implies $R \in I_A$. Now $S \setminus \{n-2\} = R \setminus \{n\}$. If $T' \subseteq R$, then $T' \setminus \{n\} = T \setminus \{n-2\}$ for some $T \subseteq S$. Since $A(T \setminus \{n-2\})$ requires singularity for all $T \subseteq S$ containing $n-2$, $A(T' \setminus \{n\})$ requires singularity for all $T' \subseteq R$ containing n. Therefore n is an essential index of A.

Similarly, if n is an essential index of A, then $n-2$ is an essential index of A.

2. Here $a_{nn} = 0$. Thus if $S \in I_A$ such that $n-1, n \notin S$, then for each $i \in \{1, 2, \ldots, n-2\}$ and $T \subseteq S$, we can express $A(T \setminus \{i\})$ as

$$A(T \setminus \{i\}) = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & \tilde{A}(T \setminus \{i\}) & & & 0 & 0 \\ & & & & c & 0 \\ \hline 0 & \cdots & 0 & d & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & 0 & a_{n,n-1} & 0 \end{array}\right], \tag{4.16}$$

where $a_{n,n-1}, a_{n-1,n} \neq 0$, and either both c, d are zero or both are nonzero. Thus we have

$$\det A(T \setminus \{i\}) = -a_{n-1,n} a_{n,n-1} \cdot \det \tilde{A}(T \setminus \{i\}). \tag{4.17}$$

Suppose i is an essential index of $\tilde{A}$ for some $i < n-1$. Then there exists $S \in I_{\tilde{A}}$ containing i such that $\tilde{A}(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. If M is the perfect matching for $G(\tilde{A}) - S$, then $M \cup \{[n-1, n]\}$ is the perfect matching for $G(A) - S$. Therefore $S \in I_{\tilde{A}}$ implies $S \in I_A$. Since $\tilde{A}(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i, by (4.17), $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. Therefore i is an essential index of A.

Conversely, suppose i is an essential index of A for some $i < n-1$. Then there exists $S \in I_A$ containing i such that $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. By Lemma 4.12, we may assume without loss of generality that $n \notin S$. Then $[n-1, n]$ is in the perfect matching of $G(A) - S$. If M is the perfect matching for $G(A) - S$, then $M \setminus \{[n-1, n]\}$ is the perfect matching for $G(\tilde{A}) - S$, and $S \in I_A$ implies $S \in I_{\tilde{A}}$. Since $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i and $a_{n-1,n}, a_{n,n-1} \neq 0$, from (4.17) we can conclude that $\tilde{A}(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. Therefore i is an essential index of $\tilde{A}$. □

Lemma 4.18. *Suppose A is a tree sign pattern matrix of order n with $a_{nn} = 0$, and $G(A)$ has vertices $n-1, n$ such that $n-1$ is adjacent to n and $\deg(n) = 1, \deg(n-1) \geq 3$. Let the principal submatrices of A corresponding to the components of $G(A) - \{n-1, n\}$ be $A_1, A_2, \ldots, A_r$.*

1. *If for some $t \in \{1, 2, \ldots, r\}$, the vertices of $G(A_t)$ be $1, 2, \ldots, k$ such that k is adjacent to $n-1$, then we have the following.*
   - *(a) For each $i < k$, i is an essential index of A_t if and only if i is an essential index of A.*
   - *(b) If k is an essential index of A_t, then both k, n are essential indices of A.*
   - *(c) If k is an essential index of A, then k is an essential index of A_t.*
2. *If n is an essential index of A, then there exists $l \in \{1, 2, \ldots, r\}$ such that p is an essential index of A_l, where p is the vertex in $G(A_l)$ adjacent to $n-1$.*

Proof. According to the given conditions we can represent $G(A)$ as follows.

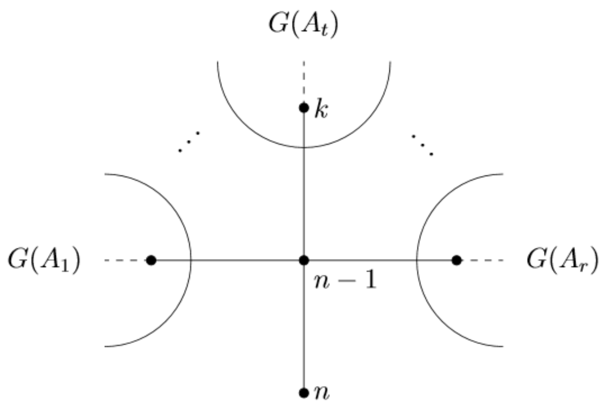


Figure 4.19: A representation of $G(A)$.

Let the vertices of $G(A_t)$ be $1, 2, \ldots, k$ such that k is adjacent to $n-1$. Let $S \in I_A$ be such that $n-1, n \notin S$, and $S_t \in I_{A_t}$ such that $S_t = \{s \in S : s \leq k\}$. If $\tilde{S} = \{s \in S : s \geq k+1\}$ and $i \in \{1, 2, \ldots, k\}$, then for each $T \subseteq S$, we can write $A(T \setminus \{i\})$ as

$$A(T \setminus \{i\}) = \left[\begin{array}{cccc|cccc|cc} & & & & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ & \multicolumn{3}{c|}{A_t(T \cap S_t \setminus \{i\})} & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & 0 & 0 & \cdots & 0 & c & 0 \\ \hline 0 & \cdots & 0 & 0 & & & & & & \\ \vdots & & \vdots & \vdots & & \multicolumn{2}{c}{\tilde{A}(T \cap \tilde{S})} & & \mathbf{u} & \mathbf{0} \\ 0 & \cdots & 0 & 0 & & & & & & \\ 0 & \cdots & 0 & 0 & & & & & & \\ \hline 0 & \cdots & 0 & d & & \multicolumn{2}{c}{\mathbf{v}^T} & & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & 0 & & \multicolumn{2}{c}{\mathbf{0}^T} & & a_{n,n-1} & 0 \end{array}\right], \tag{4.20}$$

where $\tilde{A} = A[\{k+1, \ldots, n-2\}]$, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, and either both c, d are zero or both are nonzero. Therefore

$$\det A(T \setminus \{i\}) = -a_{n-1,n}a_{n,n-1} \cdot \det A_t(T \cap S_t \setminus \{i\}) \cdot \det \tilde{A}(T \cap \tilde{S}). \tag{4.21}$$

If $\tilde{S} \subseteq T$, then

$$\det A(T \setminus \{i\}) = -a_{n-1,n}a_{n,n-1} \cdot \det A_t(T \cap S_t \setminus \{i\}) \cdot \det \tilde{A}(\tilde{S}). \tag{4.22}$$

`1(a)`. Suppose i is an essential index of A_t for some $i < k$. Then there exists $S_t \in I_{A_t}$ containing i such that $A_t(T \setminus \{i\})$ requires singularity for all $T \subseteq S_t$ containing i. If $S = S_1 \cup \cdots \cup S_r$, where $S_j \in I_{A_j}$ for $j = 1, \ldots, r$, then $S \in I_A$. Since $a_{nn} = 0$ and $A_t(T \setminus \{i\})$ requires singularity for all $T \subseteq S_t$ containing i, by (4.21), $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. Therefore i is an essential index of A.

Conversely, suppose i is an essential index of A for some $i < k$. Then there exists $S \in I_A$ containing i such that $A(T \setminus \{i\})$ requires singularity for all $T \subseteq S$ containing i. Proceeding as in the proof of Lemma 4.12, we can assume without loss of generality that $n \notin S$. Let $S_t = \{i \in S : i \leq k\}$. Now $G(A) - S$ has a perfect matching, say M. Let $M_t = \{[p, q] : [p, q] \in M \text{ and } p, q \leq k\}$. Then M_t is a perfect matching for $G(A_t) - S_t$, which together with the fact $S \in I_A$ implies that $S_t \in I_{A_t}$. If $S_j \in I_{A_j}$ for all $j \neq t$, then $S = S_1 \cup S_2 \cup \cdots \cup S_r$.

Let $T_1 \subseteq S_t$ contains i, and $T_2 = T_1 \cup (S_1 \cup \cdots \cup S_{t-1}) \cup (S_{t+1} \cup \cdots \cup S_r)$. Since i is an essential index of A, $A(T_2 \setminus \{i\})$ requires singularity. Therefore by Lemma 4.3, each term in the standard expansion of $\det A(T_2 \setminus \{i\})$ is zero. If $E = \{[i, j] \in M : i, j > k\}$, then by (4.22), $\det A_t(T_2 \cap S_t \setminus \{i\}) \cdot \prod_{[i,j] \in E}(-a_{ij}a_{ji})$ gives the sum of some terms in the standard expansion of $\det A(T_2 \setminus \{i\})$. So each term in the standard expansion of $\det A_t(T_2 \cap S_t \setminus \{i\})$ is zero. Therefore by Lemma 4.3, $A_t(T_1 \setminus \{i\}) = A_t(T_2 \cap S_t \setminus \{i\})$ requires singularity. Hence i is an essential index of A_t.

`1(b)`. Suppose k is an essential index of A_t. Then there exists $S_t \in I_{A_t}$ such that $k \in S_t$ and $A_t(T \setminus \{k\})$ requires singularity for all $T \subseteq S_t$ containing k. Let $S = S_1 \cup \cdots \cup S_r$, where $S_j \in I_{A_j}$ for $j = 1, \ldots, r$. Then $S \in I_A$ and $G(A) - S$ has a perfect matching, say M. Let $R = (S \setminus \{k\}) \cup \{n\}$. Then $(M \setminus \{[n-1, n]\}) \cup \{[k, n-1]\}$ is a perfect matching for $G(A) - R$. Therefore $S \in I_A$ implies $R \in I_A$. Since $a_{nn} = 0$ and $A_t(T \setminus \{k\})$ requires singularity for all $T \subseteq S_t$ containing k, by (4.21), $A(T \setminus \{k\})$ requires singularity for all $T \subseteq S$ containing k. Now $S \setminus \{k\} = R \setminus \{n\}$. If $T' \subseteq R$, then $T' \setminus \{n\} = T \setminus \{k\}$ for some $T \subseteq S$. Since $A(T \setminus \{k\})$

requires singularity for all $T \subseteq S$ containing k, $A(T' \setminus \{n\})$ requires singularity for all $T' \subseteq R$ containing n. Therefore both k, n are essential indices of A.

1(c). Suppose k is an essential index of A. Then by the similar arguments as in the converse part of 1(a), k is an essential index of A_t.

2. Suppose n is an essential index of A. Then there exists $S \in I_A$ such that $n \in S$ and $A(T \setminus \{n\})$ requires singularity for all $T \subseteq S$ containing n. Suppose there exist no $t \in \{1, 2, \ldots, r\}$ such that k_t is an essential index of A_t, where k_t is a vertex of $G(A_t)$ adjacent to $n-1$. Since $S \in I_A$ and $n-1 \notin S$, there exists an $l \in \{1, 2, \ldots, r\}$, where $G(A_l)$ (after a relabelling of vertices, if required) has vertices $1, 2, \ldots, p$ with p being adjacent to $n-1$ such that $[p, n-1]$ is in the perfect matching of $G(A) - S$. If $S_l = \{p\} \cup \{i \in S : i < p\}$, then $S \in I_A$ implies $S_l \in I_{A_l}$. Since p is a non-essential index of A_l, $A_l(T \setminus \{p\})$ allows nonsingularity for some $T \subseteq S_l$ containing p. Therefore by (4.21), $A(T_1 \setminus \{p\})$ allows nonsingularity, where $T_1 = T \cup \{i : i \in S \text{ and } p < i < n\}$. If $T_2 = (T_1 \setminus \{p\}) \cup \{n\}$, then $T_2 \subseteq S$ and $A(T_2 \setminus \{n\}) = A(T_1 \setminus \{p\})$. So $A(T_2 \setminus \{n\})$ allows nonsingularity, which is a contradiction. Therefore there exists $l \in \{1, 2, \ldots, r\}$ such that p is an essential index of A_l, where p is the vertex in $G(A_l)$ adjacent to $n-1$. □

From Theorem 4.9 and 1(a), 1(c) of Lemma 4.18 we have the following corollary.

Corollary 4.23. *Suppose A is a tree sign pattern matrix of order n with $a_{nn} = 0$, and $G(A)$ has vertices $n-1, n$ such that $n-1$ is adjacent to n and $\deg(n) = 1, \deg(n-1) \geq 3$. Let the principal submatrices corresponding to the components of $G(A) - \{n-1, n\}$ be $A_1, A_2, \ldots, A_r$, and for some $t \in \{1, 2, \ldots, r\}$, let the vertices of $G(A_t)$ be $1, 2, \ldots, k$ such that k is adjacent to $n-1$. If A_t allows nonsingularity, then each index in $\{1, 2, \ldots, k\}$ is a non-essential index of A.*

The following result characterizes tree sign pattern matrices requiring singularity in terms of essential indices.

Theorem 4.24. *A tree sign pattern matrix A of order n requires singularity if and only if i is an essential index of A for some $i \in \{1, 2, \ldots, n\}$.*

Proof. Suppose A requires singularity. We prove by induction on n that i is an essential index of A for some $i \in \{1, 2, \ldots, n\}$. If $n = 3$, then A is given by

$$A = \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix},$$

where $a_{12}, a_{21}, a_{23}, a_{32} \in \{+, -\}$. Clearly, $I_A = \{\{1\}, \{3\}\}$. If $S = \{1\}$, then $A(S \setminus \{1\}) = A$ requires singularity. Therefore 1 is an essential index of A. Thus the result is true for $n = 3$. Suppose the result is true for any $k < n$. Let A be a tree sign pattern matrix of order n requiring singularity. Then by Lemma 4.11, we can assume without loss of generality that $a_{nn} = 0$ and $\deg(n) = 1$ in $G(A)$. Let n be adjacent to $n - 1$ in $G(A)$.

`Case I:` $\deg(n-1) = 2$. Let $\tilde{A} = A(\{n-1, n\})$ and $G(\tilde{A}) = G(A) - \{n-1, n\}$. Then $\tilde{A}$ is a tree sign pattern matrix of order $n - 2$ requiring singularity. By induction hypothesis i is an essential index of $\tilde{A}$ for some $i \in \{1, 2, \ldots, n-2\}$. Then using Lemma 4.14, we can conclude that i is an essential index of A.

`Case II:` $\deg(n-1) \geq 3$. Let the principal submatrices of A corresponding to the components of $G(A) - \{n-1, n\}$ be $A_1, A_2, \ldots, A_r$. Then there exists $t \in \{1, 2, \ldots, r\}$ such that A_t requires singularity. Let the vertices of $G(A_t)$ be $1, 2, \ldots, l$ such that l is adjacent to $n-1$ in $G(A)$. By induction hypothesis i is an essential index of A_t for some $i \in \{1, 2, \ldots, l\}$. Then by Lemma 4.18 we can conclude that i is an essential index of A.

The converse part follows from Theorem 4.9. □

Throughout the following discussions, $\mathbf{e}_i$ denotes the column vector whose i-th component is 1 and all other components are zero, $\mathbf{0}$ denotes the zero column vector, O denotes the zero matrix, and their orders will be clear from the context.

Lemma 4.25. *Suppose A is a tree sign pattern matrix of order n, and $G(A)$ has vertices $n-2$, $n-1, n$ such that $n-1$ is adjacent to both $n-2, n$ and $\deg(n) = 1, \deg(n-1) = 2$. Let $a_{nn} = 0$ and $\tilde{A} = A(\{n-1, n\})$. If $\mathbf{e}_i \in \mathcal{C}(\tilde{B})$ for some $\tilde{B} \in Q(\tilde{A})$, then $\mathbf{e}_i \in \mathcal{C}(B)$ for all $B \in Q(A)$ with $B(\{n-1, n\}) = \tilde{B}$. Further, $\mathbf{e}_{n-2} \in \mathcal{C}(\tilde{B})$ implies $\mathbf{e}_n \in \mathcal{C}(B)$.*

Proof. We can rewrite A as

$$A = \left[\begin{array}{c|cc} \tilde{A} & a_{n-2,n-1}\,\mathbf{e}_{n-2} & \mathbf{0} \\ \hline a_{n-1,n-2}\,\mathbf{e}_{n-2}^T & a_{n-1,n-1} & a_{n-1,n} \\ \mathbf{0}^T & a_{n,n-1} & 0 \end{array}\right],$$

where $a_{n-1,n-2}, a_{n-2,n-1}, a_{n,n-1}, a_{n-1,n} \neq 0$.

Let $\mathbf{e}_i \in \mathcal{C}(\tilde{B})$ for some $\tilde{B} \in Q(\tilde{A})$. Then there exists $\mathbf{p}_i \in \mathbb{R}^{n-2}$ such that $\tilde{B}\mathbf{p}_i = \mathbf{e}_i$.

Let $B \in Q(A)$ such that $B(\{n-1, n\}) = \tilde{B}$. Then there exist $b_{n-2,n-1}$, $b_{n-1,n-2}$, $b_{n-1,n-1}$, $b_{n-1,n}$, $b_{n,n-1} \in \mathbb{R}$ such that $b_{n-1,n}, b_{n,n-1} \neq 0$ and

$$B = \left[\begin{array}{c|cc} \tilde{B} & b_{n-2,n-1}\,\mathbf{e}_{n-2} & \mathbf{0} \\ \hline b_{n-1,n-2}\,\mathbf{e}_{n-2}^T & b_{n-1,n-1} & b_{n-1,n} \\ \mathbf{0}^T & b_{n,n-1} & 0 \end{array}\right].$$

Therefore

$$B\mathbf{m}_i = \left[\begin{array}{c} \mathbf{e}_i \\ \hline 0 \\ 0 \end{array}\right], \quad \text{where} \quad \mathbf{m}_i = \left[\begin{array}{c} \mathbf{p}_i \\ \hline 0 \\ -\frac{b_{n-1,n-2}}{b_{n-1,n}}\,\mathbf{e}_{n-2}^T\,\mathbf{p}_i \end{array}\right].$$

So $\mathbf{e}_i \in \mathcal{C}(B)$.

Suppose $\mathbf{e}_{n-2} \in \mathcal{C}(\tilde{B})$. Then there exists $\mathbf{p}_{n-2} \in \mathbb{R}^{n-2}$ such that $\tilde{B}\mathbf{p}_{n-2} = \mathbf{e}_{n-2}$. Therefore

$$B\mathbf{m}_{n-2} = \left[\begin{array}{c} \mathbf{0} \\ \hline 0 \\ b_{n,n-1} \end{array}\right], \text{ where } \mathbf{m}_{n-2} = \left[\begin{array}{c} -b_{n-2,n-1}\mathbf{p}_{n-2} \\ \hline 1 \\ \frac{1}{b_{n-1,n}}(b_{n-2,n-1}b_{n-1,n-2}\,\mathbf{e}_{n-2}^T\,\mathbf{p}_{n-2} - b_{n-1,n-1}) \end{array}\right].$$

So $\mathbf{e}_n \in \mathcal{C}(B)$. Hence the result follows. □

Lemma 4.26. *Suppose A is a tree sign pattern matrix of order n with $a_{nn} = 0$, and $G(A)$ has vertices $n-1, n$ such that $n-1$ is adjacent to n and $\deg(n) = 1, \deg(n-1) \geq 3$. Let the principal submatrix corresponding to one component of $G(A) - \{n-1, n\}$ be $\tilde{A}$ such that $G(\tilde{A})$ has the vertices $1, 2, \ldots, k$ with k being adjacent to $n-1$. If $\mathbf{e}_i \in \mathcal{C}(\tilde{B})$ for some $\tilde{B} \in Q(\tilde{A})$, then for each $B \in Q(A)$ with $B[\{1, \ldots, k\}] = \tilde{B}$, $\mathbf{e}_i \in \mathcal{C}(B)$.*

Proof. According to the given conditions we can express A as

$$A = \begin{bmatrix} \tilde{A} & O & a_{k,n-1}\,\mathbf{e}_k & \mathbf{0} \\ O & A[\alpha] & \mathbf{u} & \mathbf{0} \\ a_{n-1,k}\,\mathbf{e}_k^T & \mathbf{v}^T & a_{n-1,n-1} & a_{n-1,n} \\ \mathbf{0}^T & \mathbf{0}^T & a_{n,n-1} & 0 \end{bmatrix},$$

where $\alpha = \{k+1, k+2, \ldots, n-2\}$, $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ and $a_{n-1,k}, a_{k,n-1}, a_{n-1,n}, a_{n,n-1} \neq 0$.

Suppose there exists $\tilde{B} \in Q(\tilde{A})$ such that $\mathbf{e}_i \in \mathcal{C}(\tilde{B})$. Then there exists a vector $\mathbf{p}_i \in \mathbb{R}^k$ such that $\tilde{B}\mathbf{p}_i = \mathbf{e}_i$.

Let $\widehat{B} \in Q(A[\alpha]), \mathbf{x} \in Q(\mathbf{u}), \mathbf{y} \in Q(\mathbf{v})$, $b_{k,n-1}$, $b_{n-1,k}$, $b_{n-1,n-1}$, $b_{n-1,n}$, $b_{n,n-1} \in \mathbb{R}$ such that $b_{n-1,n} \neq 0$ and

$$B = \begin{bmatrix} \tilde{B} & O & b_{k,n-1}\,\mathbf{e}_k & \mathbf{0} \\ O & \widehat{B} & \mathbf{x} & \mathbf{0} \\ b_{n-1,k}\,\mathbf{e}_k^T & \mathbf{y}^T & b_{n-1,n-1} & b_{n-1,n} \\ \mathbf{0}^T & \mathbf{0}^T & b_{n,n-1} & 0 \end{bmatrix} \in Q(A), \quad \text{where} \quad B[\{1,\ldots,k\}] = \tilde{B}.$$

Then

$$B\mathbf{m}_i = \begin{bmatrix} \mathbf{e}_i \\ \mathbf{0} \\ 0 \\ 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{m}_i = \begin{bmatrix} \mathbf{p}_i \\ \mathbf{0} \\ 0 \\ -\frac{b_{n-1,k}}{b_{n-1,n}}\,\mathbf{e}_k^T\,\mathbf{p}_i \end{bmatrix}.$$

Therefore for each $B \in Q(A)$ with $B[\{1,\ldots,k\}] = \tilde{B}$, $\mathbf{e}_i \in \mathcal{C}(B)$. □

In the following theorem, we give some results related to the column spaces of tree sign pattern matrices requiring singularity.

Theorem 4.27. *If a tree sign pattern matrix A of order n requires singularity, then we have the following.*

1. *For each $M \in Q(A)$, $\mathbf{e}_k \notin \mathcal{C}(M)$ if k is an essential index of A.*

2. *For each $M \in Q(A)$ with $\operatorname{rank}(M) = \operatorname{MR}(A)$, $\mathbf{e}_k \in \mathcal{C}(M)$ if k is a non-essential index of A.*

Proof. We prove the result by induction on n.

`Induction base case.` For $n = 1$, the result is obvious. If $n = 2$, then A allows nonsingularity. For $n = 3$, since A requires singularity, A is given by

$$A = \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix}.$$

Here $I_A = \{\{1\},\{3\}\}$. Further, the set of all essential indices of A is $\{1,3\}$. If $M \in Q(A)$, then

$$M = \begin{bmatrix} 0 & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & 0 \end{bmatrix},$$

where $m_{12}, m_{21}, m_{23}, m_{32} \neq 0$. Clearly $\mathbf{e}_1, \mathbf{e}_3 \notin \mathcal{C}(M)$ and $\mathbf{e}_2 \in \mathcal{C}(M)$ for all $M \in Q(A)$. Further $\operatorname{rank}(M) = 2 = \operatorname{MR}(A)$. Thus the result is also true for $n = 3$.

`Induction hypothesis.` Suppose the result is true for any tree sign pattern matrix of order less than n.

`Induction final step.` Let A be a tree sign pattern matrix of order n such that A requires singularity. Then by Lemma 4.11, there exist vertices in $G(A)$, say $n-1, n$, such that n is adjacent to $n-1$, $a_{nn} = 0$ and $\deg(n) = 1$.

`Case I:` $\deg(n-1) = 2$ Let $n-2$ be adjacent to $n-1$. Then we can rewrite A as

$$A = \begin{bmatrix} \tilde{A} & a_{n-2,n-1}\,\mathbf{e}_{n-2} & \mathbf{0} \\ a_{n-1,n-2}\,\mathbf{e}_{n-2}^T & a_{n-1,n-1} & a_{n-1,n} \\ \mathbf{0}^T & a_{n,n-1} & 0 \end{bmatrix}, \tag{4.28}$$

where $\tilde{A} = A(\{n-1, n\})$ and $a_{n-1,n}, a_{n,n-1}, a_{n-1,n-2}, a_{n-2,n-1} \neq 0$. By Lemma 4.1, $\operatorname{MR}(A) = \operatorname{MR}(\tilde{A}) + 2$.

`First Part:` Let n be an essential index of A. Then by Lemma 4.14, $n-2$ is an essential index of $\tilde{A}$ and hence by induction hypothesis $\mathbf{e}_{n-2} \notin \mathcal{C}(\tilde{M})$ for all $\tilde{M} \in Q(\tilde{A})$. If $\mathbf{e}_n \in \mathcal{C}(M)$

for some $M \in Q(A)$, then there exist $c_{n-1}, c_n \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^{n-2}$ such that

$$\begin{bmatrix} \mathbf{0} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{M} \\ m_{n-1,n-2}\,\mathbf{e}_{n-2}^T \\ \mathbf{0}^T \end{bmatrix} \mathbf{p} + c_{n-1} \begin{bmatrix} m_{n-2,n-1}\,\mathbf{e}_{n-2} \\ m_{n-1,n-1} \\ m_{n,n-1} \end{bmatrix} + c_n \begin{bmatrix} \mathbf{0} \\ m_{n-1,n} \\ 0 \end{bmatrix},$$

where $\tilde{M} = M(\{n-1, n\}) \in Q(\tilde{A})$.

Since $c_{n-1}, m_{n-2,n-1} \neq 0$ and $\tilde{M}\mathbf{p} = -c_{n-1} m_{n-2,n-1}\,\mathbf{e}_{n-2}$, $\mathbf{e}_{n-2} \in \mathcal{C}(\tilde{M})$, which is a contradiction.

Let $i(\neq n)$ be an essential index of A. Then by Lemma 4.10, $i \neq n-1$, and by Lemma 4.14, i is an essential index of $\tilde{A}$. If $\mathbf{e}_i \in \mathcal{C}(M)$ for some $M \in Q(A)$, then from (4.28) we can conclude that $\mathbf{e}_i$ is a linear combination of all but the $(n-1)$-th column of M. Then for some $a_i \in \mathbb{R}$, $\mathbf{e}_i + a_i\,\mathbf{e}_{n-1}$ is a linear combination of the first $n-2$ columns of M and thus $\mathbf{e}_i \in \mathcal{C}(\tilde{M})$, where $\tilde{M} = M(\{n-1, n\}) \in Q(\tilde{A})$. This is a contradiction to the induction hypothesis.

`Second Part:` Let E be the set of all non-essential indices of $\tilde{A}$. Then from Lemma 4.14 we can conclude that the set of all non-essential indices of A is given by either $E \cup \{n-1, n\}$ or $E \cup \{n-1\}$ accordingly as $n-2 \in E$ or $n-2 \notin E$.

Let $M \in Q(A)$ such that $\text{rank}(M) = \text{MR}(A)$ and $\tilde{M} = M(\{n-1, n\}) \in Q(\tilde{A})$. Then there exist $m_{n-2,n-1}$, $m_{n-1,n-2}$, $m_{n-1,n-1}$, $m_{n-1,n}$, $m_{n,n-1} \in \mathbb{R}$ such that

$$M = \left[\begin{array}{c|cc} \tilde{M} & m_{n-2,n-1}\,\mathbf{e}_{n-2} & \mathbf{0} \\ \hline m_{n-1,n-2}\,\mathbf{e}_{n-2}^T & m_{n-1,n-1} & m_{n-1,n} \\ \mathbf{0}^T & m_{n,n-1} & 0 \end{array}\right],$$

where $m_{n-2,n-1}$, $m_{n-1,n-2}$, $m_{n-1,n}$, $m_{n,n-1} \neq 0$. It can be easily verified that $\text{rank}(\tilde{M}) = \text{rank}(M) - 2 = \text{MR}(A) - 2 = \text{MR}(\tilde{A})$. Therefore by induction hypothesis, $\mathbf{e}_i \in \mathcal{C}(\tilde{M})$ for all $i \in E$. Hence by Lemma 4.25, $\mathbf{e}_i \in \mathcal{C}(M)$ for all $i \in E$. Since $M\,\mathbf{e}_n = m_{n-1,n}\,\mathbf{e}_{n-1}$ with $m_{n-1,n} \neq 0$, $\mathbf{e}_{n-1} \in \mathcal{C}(M)$.

Suppose $n-2 \in E$. Then $\mathbf{e}_{n-2} \in \mathcal{C}(\tilde{M})$. Therefore by Lemma 4.25, $\mathbf{e}_n \in \mathcal{C}(M)$.

Therefore for each $M \in Q(A)$ with $\text{rank}(M) = \text{MR}(A)$, $\mathbf{e}_i \in \mathcal{C}(M)$ for all non-essential indices i of A.

`Case II:` $\deg(n-1) \geq 3$ Suppose $G(A)-\{n-1,n\}$ has r components. Then $A(\{n-1,n\})$ is a direct sum of r irreducible components, say $A_1, A_2, \ldots, A_r$.

`First Part:` Let n be an essential index of A. Then by Lemma 4.18, there exists $t \in \{1,2,\ldots,r\}$ such that k is an essential index of A_t, where k is the vertex of $G(A_t)$ adjacent to $n-1$. By induction hypothesis $\mathbf{e}_k \notin \mathcal{C}(M_t)$ for all $M_t \in Q(A_t)$. Without loss of generality, let us assume $G(A_t)$ has vertices $1, 2, \ldots, k$. So we can write A as

$$A = \begin{bmatrix} A_t & O & a_{k,n-1}\,\mathbf{e}_k & \mathbf{0} \\ O & A[\alpha] & \mathbf{u} & \mathbf{0} \\ a_{n-1,k}\,\mathbf{e}_k^T & \mathbf{v}^T & a_{n-1,n-1} & a_{n-1,n} \\ \mathbf{0}^T & \mathbf{0}^T & a_{n,n-1} & 0 \end{bmatrix}, \tag{4.29}$$

where $\alpha = \{k+1,\ldots,n-2\}$, $a_{n-1,k}, a_{k,n-1}, a_{n-1,n}, a_{n,n-1} \neq 0$ and $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$. If $\mathbf{e}_n \in \mathcal{C}(M)$ for some $M \in Q(A)$, then by similar arguments as provided in `Case I`, $\mathbf{e}_k \in \mathcal{C}(M_t)$, where $M_t = M[\{1,2,\ldots,k\}] \in Q(A_t)$. This is a contradiction to the induction hypothesis.

Let i be an essential index of A for some $i \in \{1,2,\ldots,k\}$. Then by Lemma 4.18, i is an essential index of A_t. If $\mathbf{e}_i \in \mathcal{C}(M)$ for some $M \in Q(A)$, then from (4.29) we can conclude that $\mathbf{e}_i$ is a linear combination of the first k columns and the n-th column of M. Then for some $a_i \in \mathbb{R}$, $\mathbf{e}_i + a_i\,\mathbf{e}_{n-1}$ is a linear combination of the first k columns of M and thus $\mathbf{e}_i \in \mathcal{C}(M_t)$, where $M_t = M[\{1,2,\ldots,k\}] \in Q(A_t)$. This is a contradiction to the induction hypothesis.

`Second Part:` Let the vertices of $G(A_1), \ldots, G(A_r)$ adjacent to $n-1$ be $k_1, \ldots, k_r$, respectively. Then A can be expressed as

$$A = \left[\begin{array}{cccc|cc} A_1 & O & \cdots & O & & \\ O & A_2 & \ddots & \vdots & \mathbf{u} & \mathbf{0} \\ \vdots & \ddots & \ddots & O & & \\ O & \cdots & O & A_r & & \\ \hline & \mathbf{v}^T & & & a_{n-1,n-1} & a_{n-1,n} \\ & \mathbf{0}^T & & & a_{n,n-1} & 0 \end{array}\right], \tag{4.30}$$

where $\mathbf{u} = a_{k_1,n-1}\,\mathbf{e}_{k_1} + \cdots + a_{k_r,n-1}\,\mathbf{e}_{k_r}$, $\mathbf{v} = a_{n-1,k_1}\,\mathbf{e}_{k_1} + \cdots + a_{n-1,k_r}\,\mathbf{e}_{k_r}$ for some $a_{k_1,n-1}$, $a_{n-1,k_1}, \ldots, a_{k_r,n-1}, a_{n-1,k_r} \neq 0$, and $a_{n-1,n}, a_{n,n-1} \neq 0$. Therefore by Lemma 4.1, $\mathrm{MR}(A) = \mathrm{MR}(A_1) + \cdots + \mathrm{MR}(A_r) + 2$.

Let $M \in Q(A)$ be such that $\text{rank}(M) = \text{MR}(A)$. Then there exist $\mathbf{x} \in Q(\mathbf{u})$, $\mathbf{y} \in Q(\mathbf{v})$ and $m_{n-1,n-1}, m_{n-1,n}, m_{n,n-1} \in \mathbb{R}$ with $m_{n-1,n}, m_{n,n-1} \neq 0$ such that

$$M = \left[\begin{array}{cccc|cc} M_1 & O & \cdots & O & & \\ O & M_2 & \ddots & \vdots & \mathbf{x} & \mathbf{0} \\ \vdots & \ddots & \ddots & O & & \\ O & \cdots & O & M_r & & \\ \hline & \mathbf{y}^T & & & m_{n-1,n-1} & m_{n-1,n} \\ & \mathbf{0}^T & & & m_{n,n-1} & 0 \end{array}\right] = \left[\begin{array}{c|cc} \tilde{M} & \mathbf{x} & \mathbf{0} \\ \hline \mathbf{y}^T & m_{n-1,n-1} & m_{n-1,n} \\ \mathbf{0}^T & m_{n,n-1} & 0 \end{array}\right],$$

where $\tilde{M} = M(\{n-1, n\})$ and $M_t \in Q(A_t)$ for $t = 1, 2, \ldots, r$. It can be easily verified that

$$\text{rank}(\tilde{M}) = \text{rank}(M) - 2 = \text{MR}(A) - 2 = \text{MR}(A_1) + \cdots + \text{MR}(A_r)$$

and thus $\text{rank}(M_t) = \text{MR}(A_t)$ for $t = 1, 2, \ldots, r$.

If $E = \{i : i \text{ is a non-essential index of } A_t \text{ for some } t \in \{1, 2, \ldots, r\}\}$, then the set of all non-essential indices of A is either $E \cup \{n-1\}$ or $E \cup \{n-1, n\}$.

If A_t allows nonsingularity, then M_t is invertible, and thus $\mathbf{e}_i \in \mathcal{C}(M_t)$ for all indices i of A_t. Further, by Theorem 4.24, each index of A_t is non-essential. If A_t requires singularity, then by induction hypothesis $\mathbf{e}_i \in \mathcal{C}(M_t)$ for all non-essential indices i of A_t. Now by Lemma 4.18, all non-essential indices of A_t are also non-essential for A. Therefore using Lemma 4.26, we can conclude that $\mathbf{e}_i \in \mathcal{C}(M)$ for all non-essential indices $i (< n-1)$ of A. Further, $M\,\mathbf{e}_n = m_{n-1,n}\,\mathbf{e}_{n-1}$ with $m_{n-1,n} \neq 0$ implies $\mathbf{e}_{n-1} \in \mathcal{C}(M)$.

If n is a non-essential index of A, then by Lemma 4.18.1(b), $k_1, \ldots, k_r$ are non-essential indices of $A_1, \ldots, A_r$ respectively. So by Lemma 4.26, $\mathbf{e}_i \in \mathcal{C}(\tilde{M})$ for $i = k_1, k_2, \ldots, k_r$. Since $\mathbf{x} \in Q(\mathbf{u})$, there exist nonzero $m_{k_1,n-1}, \ldots, m_{k_r,n-1}$ such that $\mathbf{x} = m_{k_1,n-1}\,\mathbf{e}_{k_1} + \cdots + m_{k_r,n-1}\,\mathbf{e}_{k_r}$. So $\mathbf{x} \in \mathcal{C}(\tilde{M})$. Therefore there exists $\mathbf{p} \in \mathbb{R}^{n-2}$ such that $\tilde{M}\mathbf{p} = \mathbf{x}$. Therefore

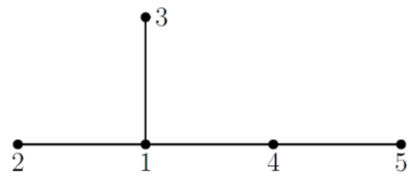
$$M\mathbf{m} = \left[\begin{array}{c} \mathbf{0} \\ \hline 0 \\ m_{n,n-1} \end{array}\right], \quad \text{where} \quad \mathbf{m} = \left[\begin{array}{c} -\mathbf{p} \\ \hline 1 \\ \frac{\mathbf{y}^T\mathbf{p} - m_{n-1,n-1}}{m_{n-1,n}} \end{array}\right].$$

So $\mathbf{e}_n \in \mathcal{C}(M)$. Hence for each $M \in Q(A)$ with $\text{rank}(M) = \text{MR}(A)$, $\mathbf{e}_i \in \mathcal{C}(M)$ for all non-essential indices i of A. □

The following example illustrates the necessity of maximum rank in the above theorem.

Example 4.31. *Let us consider a tree sign pattern matrix A with its graph $G(A)$ as follows.*

$$A = \begin{bmatrix} 0 & + & + & + & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & - & - \\ 0 & 0 & 0 & - & - \end{bmatrix}$$



Clearly, $I_A = \{\{2\}, \{3\}\}$. Since A requires singularity, both $2, 3$ are essential indices of A. The non-essential indices of A are $1, 4, 5$, as they do not belong to S for all $S \in I_A$. Let

$$M = \begin{bmatrix} 0 & 2 & 1 & 3 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{bmatrix} \in Q(A).$$

Then $\text{rank}(M) = 3 < 4 = \text{MR}(A)$. *Further,* $\mathbf{e}_4, \mathbf{e}_5 \notin \mathcal{C}(M)$.

Since A is combinatorially symmetric, we have the following corollary.

Corollary 4.32. *If a tree sign pattern matrix A of order n requires singularity, then we have the following.*

1. *For each $M \in Q(A)$, $\mathbf{e}_k \notin \mathcal{R}(M)$ if k is an essential index of A.*

2. *For each $M \in Q(A)$ with $\text{rank}(M) = \text{MR}(A)$, $\mathbf{e}_k \in \mathcal{R}(M)$ if k is a non-essential index of A.*

4.2 Sign Patterns that Allow diagonalizability

The problem of characterizing sign patterns allowing diagonalizability first came up in the study of sign patterns requiring repeated eigenvalues by Eschenbach and Johnson [16]. Shao and Gao [48] give the following sufficient condition for a sign pattern matrix to allow diagonalizability.

Theorem 4.33 ([48])**.** *If a sign pattern matrix A is combinatorially symmetric, then A allows diagonalizability.*

In this section, we consider the sign pattern matrices whose graphs are trees, but not necessarily combinatorially symmetric. We give some combinatorial structures based on their graphs which are necessary and sufficient for some specific graphs, for example the star and path to allow diagonalizability. We also give a sufficient condition for a more general class of sign pattern matrices A for which $G(A)$ is a tree to allow diagonalizability.

Let us recall the following definitions from [3, p. 39]. Let A be a square matrix of order n. For $1 \leq i, j \leq n$, we say that i has an access to j if in $D(A)$ there is a path from vertex i to vertex j, and that i, j communicate if i has an access to j and j has an access to i. This communication relation between the vertices of $D(A)$ is an equivalence relation, which partitions $\{1, 2, \ldots, n\}$ into equivalence classes, and each equivalence class will be called a class. We say a class α has an access to a class β if i has an access to j for all $i \in \alpha$ and for all $j \in \beta$. If $G(A)$ is a tree, then each principal submatrix of A corresponding to a class is a tree sign pattern matrix.

We find a sufficient condition for sign pattern matrices to allow diagonalizability whose graphs are trees but not necessarily combinatorially symmetric.

Let A be a real square matrix of order n. Throughout this chapter, $z(A)$ denotes the algebraic multiplicity of the eigenvalue 0 of A, $\sigma(A)$ denotes the set of all eigenvalues of A, $\sigma^*(A)$ denotes the set of all nonzero eigenvalues of A, and $P_A(x)$ denotes the characteristic polynomial of A in x. Further, I denotes the identity matrix, and $|S|$ denotes the cardinality of a finite set S.

Lemma 4.34. *Let R, S be two square matrices such that $\sigma^*(R) \cap \sigma^*(S) = \emptyset$. Then for any X of appropriate order, the matrix*

$$M = \begin{bmatrix} R & X \\ O & S \end{bmatrix}$$

is diagonalizable if and only if $\operatorname{rank}(M) = \operatorname{rank}(R) + \operatorname{rank}(S)$ *and both R, S are diagonalizable.*

Proof. Clearly $P_M(x) = P_R(x)P_S(x)$. Since $\sigma^*(R) \cap \sigma^*(S) = \emptyset$, for all $\lambda \in \sigma^*(R)$ and $\mu \in \sigma^*(S)$, $\text{nullity}(M - \lambda I) = \text{nullity}(R - \lambda I)$ and $\text{nullity}(M - \mu I) = \text{nullity}(S - \mu I)$. Further, $\text{rank}(M) \geq \text{rank}(R) + \text{rank}(S)$ and $z(M) = z(R) + z(S)$. Hence M is diagonalizable if and only if $\text{rank}(M) = \text{rank}(R) + \text{rank}(S)$ and both R, S are diagonalizable. □

From the details of the proof of [48, Theorem 2.6], we have the following result.

Lemma 4.35. *If a sign pattern matrix A is combinatorially symmetric, then $c(A) = \text{MR}(A)$.*

The following result was conjectured in [18] and proved in [11].

Theorem 4.36 ([11])**.** *Let A be an $n \times n$ invertible matrix. Then there exists an $n \times n$ invertible diagonal D such that AD has n distinct eigenvalues.*

By a minor modification to the proof of Theorem 1.1 in [11], the following result was obtained.

Theorem 4.37 ([19])**.** *Let A be an $n \times n$ invertible matrix. Then there exists an $n \times n$ invertible diagonal D with positive diagonal entries such that AD has n distinct eigenvalues.*

Since the eigenvalues of a matrix depend continuously on its entries, we have the following lemma.

Lemma 4.38. *If $A \in M_n$ has an invertible principal submatrix of order k such that all principal submatrices of higher orders are singular, then there are invertible diagonal matrices $D_1, D_2 \in M_n$ with positive diagonal entries such that both D_1A and AD_2 have exactly k distinct nonzero eigenvalues.*

Proof. Without loss of generality, we may assume that the leading principal minor of A of order k is invertible. Then we can write A as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1 is the leading principal minor of order k. So by Lemma 4.37, there exists a diagonal matrix D with positive diagonal entries such that DA_1 has distinct nonzero eigenvalues. Let

$D_\epsilon = D \oplus \epsilon I_{n-k}$. Since all principal submatrices of A of order $> k$ are singular, all principal submatrices of $D_\epsilon A$ of order $> k$ are singular for all $\epsilon > 0$. Again since the eigenvalues of a matrix depend continuously on its entries, we can choose $\epsilon > 0$ such that $D_\epsilon A$ has exactly k distinct nonzero eigenvalues. Thus there is an invertible diagonal matrix $D_1 \in M_n$ with positive diagonal entries such that $D_1 A$ has exactly k distinct nonzero eigenvalues.

Similarly, there exists an invertible diagonal matrix $D_2 \in M_n$ with positive diagonal entries such that AD_2 has exactly k distinct nonzero eigenvalues. □

Lemma 4.39. *If a sign pattern matrix A is combinatorially symmetric, then there exists a diagonalizable matrix $B \in Q(A)$ with* $\operatorname{rank}(B) = \operatorname{MR}(A)$.

Proof. We can choose $C \in Q(A)$ such that C has a nonsingular principal submatrix of order $c(A)$. So $c(A) \leq \operatorname{rank}(C) \leq \operatorname{MR}(A)$. Now by Lemma 4.35, $c(A) = \operatorname{MR}(A)$. Thus $\operatorname{rank}(C) = \operatorname{MR}(A)$.

Since $C \in Q(A)$, all principal submatrices of C of order higher than $c(A)$ are singular. Therefore by Lemma 4.38, there exists an invertible diagonal matrix $D \in M_n$ with positive diagonal entries such that DC has exactly $c(A)$ distinct nonzero eigenvalues.

Since $\operatorname{rank}(DC) = \operatorname{rank}(C) = c(A)$ and DC has exactly $c(A)$ distinct nonzero eigenvalues, DC is diagonalizable. Again $DC \in Q(A)$. Hence the result follows. □

We know that if a sign pattern matrix A allows nonsingularity or A is combinatorially symmetric, then A allows diagonalizability. The following theorem gives a sufficient condition for a class of sign pattern matrices (not necessarily combinatorially symmetric) that require singularity to allow diagonalizability.

Theorem 4.40. *Let A be a sign pattern matrix which requires singularity such that $G(A)$ is a tree. If there are no directed paths in $D(A)$ between the essential indices of the principal submatrices corresponding to two distinct classes, or if all possible directed paths in $D(A)$ between the essential indices of the principal submatrices corresponding to two distinct classes in $D(A)$ contain non-essential indices from those classes, then there exists a diagonalizable matrix $B \in Q(A)$ with* $\operatorname{rank}(B) = \operatorname{MR}(A)$.

Proof. We prove this by induction on r, the number of distinct classes in $D(A)$. If $r = 1$, then A is combinatorially symmetric and thus by Lemma 4.39, there exists a diagonalizable matrix $B \in Q(A)$ with $\text{rank}(B) = \text{MR}(A)$. Suppose the theorem is true for any matrix A with the number of distinct classes in $D(A)$ less than r.

Let A be a sign pattern matrix satisfying the given conditions such that the number of distinct classes in $D(A)$ is r. Let 1 be a pendant vertex of $G(A)$ and A_1 be the principal submatrix of A corresponding to the class in $D(A)$ containing 1. Let A_1 be of order k with vertices $1, 2, \ldots, k$ such that k is adjacent to a vertex say $k+1$, from a different class.

`Case I:` k `is a non-essential index of` A_1. Then A can be written as

$$A = \begin{bmatrix} A_1 & A_2 \\ O & A_3 \end{bmatrix},$$

where all entries of A_2 are zero except the $(k,1)$-th entry. Since k is non-essential for A_1, by Theorem 4.27, $\mathbf{e}_k \in \mathcal{C}(B_1)$ for all $B_1 \in Q(A_1)$ with $\text{rank}(B_1) = \text{MR}(A_1)$, which implies $\text{MR}(A) = \text{MR}(A_1) + \text{MR}(A_3)$. Using induction hypothesis, we can choose $B_3 \in Q(A_3)$ such that B_3 is diagonalizable and $\text{rank}(B_3) = \text{MR}(A_3)$. By Lemma 4.39, we can choose a diagonalizable matrix $B_1 \in Q(A_1)$ with $\text{rank}(B_1) = \text{MR}(A_1)$ such that $\sigma^*(B_1) \cap \sigma^*(B_3) = \emptyset$. Since $\mathbf{e}_k \in \mathcal{C}(B_1)$,

$$B = \begin{bmatrix} B_1 & B_2 \\ O & B_3 \end{bmatrix}$$

implies $\text{rank}(B) = \text{rank}(B_1) + \text{rank}(B_3)$. Therefore $\text{rank}(B) = \text{MR}(A)$, and by Lemma 4.34, B is diagonalizable.

`Case II:` k `is an essential index of` A_1. Let

$$A = \left[\begin{array}{c|cc} A_1 & O & O \\ \hline A_2 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{A}$}} \\ O & & \end{array}\right] \quad \text{with} \quad \tilde{A} = \begin{bmatrix} A_3 & A_4 \\ O & A_5 \end{bmatrix},$$

where A_3 is the principal submatrix of A corresponding to the class in $D(A)$ containing $k+1$ and all entries of A_2 are zero except the $(1,k)$-th entry.

Since all possible directed paths in $D(A)$ between the essential indices of the principal submatrices corresponding to two distinct classes in $D(A)$ contain non-essential indices from those classes, without loss of generality we may assume that A is of the above form so that the vertex $k+1$ in $G(A)$, a representation of the index 1 for A_3, is non-essential for A_3.

From the induction hypothesis, there exists a diagonalizable matrix $\tilde{B} \in Q(\tilde{A})$ with $\text{rank}(\tilde{B}) = \text{MR}(\tilde{A})$. Let

$$\tilde{B} = \begin{bmatrix} B_3 & B_4 \\ O & B_5 \end{bmatrix},$$

where $B_3 \in Q(A_3)$ and $B_5 \in Q(A_5)$. Since $\tilde{B}$ is diagonalizable,

$$z(\tilde{B}) = \text{nullity}(\tilde{B}) \leq \text{nullity}(B_3) + \text{nullity}(B_5) \leq z(B_3) + z(B_5) = z(\tilde{B}).$$

Therefore $\text{nullity}(\tilde{B}) = \text{nullity}(B_3) + \text{nullity}(B_5)$ and thus $\text{rank}(\tilde{B}) = \text{rank}(B_3) + \text{rank}(B_5)$. Since $\text{rank}(\tilde{B}) = \text{MR}(\tilde{A}) \geq \text{MR}(A_3) + \text{MR}(A_5) \geq \text{rank}(B_3) + \text{rank}(B_5)$, $\text{rank}(B_3) = \text{MR}(A_3)$. Therefore by Theorem 4.27, $\mathbf{e}_1 \in \mathcal{C}(B_3)$ and thus $\mathbf{e}_1 \in \mathcal{C}(\tilde{B})$. Hence $\text{MR}(A) = \text{MR}(A_1) + \text{MR}(\tilde{A})$.

By Lemma 4.39, we can choose a diagonalizable matrix $B_1 \in Q(A_1)$ with $\text{rank}(B_1) = \text{MR}(A_1)$ such that $\sigma^*(B_1) \cap \sigma^*(\tilde{B}) = \emptyset$. Let

$$B = \left[\begin{array}{c|cc} B_1 & O & O \\ \hline B_2 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{B}$}} \\ O & & \end{array}\right] \in Q(A).$$

Since $\mathbf{e}_1 \in \mathcal{C}(\tilde{B})$, $\text{rank}(B) = \text{rank}(B_1) + \text{rank}(\tilde{B}) = \text{MR}(A)$, and therefore by Lemma 4.34, B is diagonalizable. $\square$

The following example establishes that the converse of Theorem 4.40 is not true.

Example 4.41. *Let us consider the sign pattern*

$$A = \left[\begin{array}{cc|c|c} 0 & + & 0 & + \\ + & 0 & 0 & 0 \\ \hline + & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_1}} & 0 & + \\ & & 0 & 0 \\ \hline + & 0 & A_2 & 0 \\ \hline 0 & 0 & 0 & A_3 \end{array}\right].$$

Here $G(A)$ is a tree, and there are three distinct classes in $D(A)$, namely $\{1,2\},\{3\},\{4\}$.

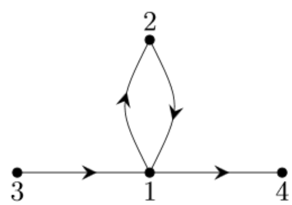


Figure 4.42: A diagram of $D(A)$.

So there is a directed path in $D(A)$ between essential indices of A_2 and A_3, namely $3 \to 1 \to 4$, containing no other vertex from $D(A_2), D(A_3)$. So A does not satisfy the conditions of Theorem 4.40.

Let

$$B = \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \in Q(A).$$

Then $P_B(x) = x^2(x^2-1)$. So $\sigma(B) = \{0,1,-1\}$, where the algebraic multiplicity of 0 is 2. Further, $\text{nullity}(B) = 2$. Therefore B is diagonalizable.

If we restrict $G(A)$ in Theorem 4.40 to be a path, then the converse is also true.

A matrix of order n, whose graph is a path, is permutationally similar to a matrix of the form

$$\begin{bmatrix} a_1 & b_2 & 0 & \cdots & 0 \\ c_2 & a_2 & b_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & a_{n-1} & b_n \\ 0 & \cdots & 0 & c_n & a_n \end{bmatrix}, \tag{4.43}$$

where $|b_i| + |c_i| \neq 0$.

A path sign pattern matrix requiring singularity is of the form

$$\begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & & \vdots \\ 0 & c_2 & a_3 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & c_{2n-1} & a_{2n} & b_{2n} \\ 0 & \cdots & \cdots & 0 & c_{2n} & a_{2n+1} \end{bmatrix}, \tag{4.44}$$

where $a_1 = a_3 = \cdots = a_{2n+1} = 0$ and $b_i c_i \neq 0$ for $i = 1, 2, \ldots, 2n$. In this case, the essential indices are $1, 3, 5, \ldots, 2n+1$.

Theorem 4.45. *Let A be a sign pattern matrix such that $G(A)$ is a path. Then A allows diagonalizability if and only if A allows nonsingularity, or A requires singularity and there are no directed paths in $D(A)$ between the essential indices of the principal submatrices corresponding to any two distinct classes in $D(A)$.*

Proof. The 'if' part is established by Theorem 4.37 and Theorem 4.40. For the 'only if' part, suppose A requires singularity, and there is a directed path in $D(A)$ between essential indices of two distinct irreducible components in $D(A)$. Suppose A has the form (4.43).

Let $M \in Q(A)$ and

$$M = \begin{bmatrix} X & P & O & O & O \\ Q & R & O & O & O \\ O & B & S & O & O \\ O & O & C & T & D \\ O & O & O & E & Y \end{bmatrix},$$

where X, Y may be vacuous, R, T are matrices of the form (4.44), S is a tridiagonal matrix such that $s_{i+1,i} \neq 0$ for all i, exactly one of P, Q is a zero matrix, exactly one of D, E is a zero matrix, and $b_{1n_1}, c_{1n_2} \neq 0$ and $b_{ij}, c_{ij} = 0$ for all other i, j (assuming that R, S, T have the orders n_1, n_2, n_3 respectively). Characteristic polynomial of M is $P_M(x) = P_X(x)P_R(x)P_S(x)P_T(x)P_Y(x)$. Since both R, T are singular, $z(M) \geq z(X) + z(Y) + 2$. Further, $\text{rank}(M) \geq \text{rank}(X) + \text{rank}(Y) + n_1 + n_2 + n_3 - 1$. Therefore $\text{nullity}(M) \leq$

nullity(X) + nullity$(Y) + 1 < z(X) + z(Y) + 2 \leq z(M)$. So M is not diagonalizable. Thus A does not allow diagonalizability. □

If $G(A)$ is a star, then the converse of Theorem 4.40 is not valid, as it is shown in Example 4.41. However, the following result characterizes all sign pattern matrices whose graph is a star to allow diagonalizability. From [8, p. 294], we note that the Hadamard product of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, denoted by $A \circ B$, is defined by $A \circ B = [a_{ij}b_{ij}]$.

A matrix of order n, whose graph is a star, is permutationally similar to a matrix of the form

$$\begin{bmatrix} a_1 & b_2 & b_3 & \cdots & b_n \\ c_2 & a_2 & 0 & \cdots & 0 \\ c_3 & 0 & a_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_n & 0 & \cdots & 0 & a_n \end{bmatrix}, \tag{4.46}$$

where $|b_i| + |c_i| \neq 0$.

Theorem 4.47. *Let A be a sign pattern matrix of the form (4.46) with $n \geq 2$ and $P = \{2, 3, \ldots, n\}$. Let $S = \{i \in P : a_i = 0\}$ and $T = \{i \in P : b_ic_i \neq 0\}$. Then A allows diagonalizability if and only if one of the following conditions hold.*

1. *$S = \emptyset$.*
2. *$S \neq \emptyset, T = \emptyset, a_1 \neq 0$ and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$.*
3. *$S, T \neq \emptyset$, $S \cap T = \emptyset$ and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$.*
4. *$S \cap T \neq \emptyset$.*

Proof. We first show that if one of the conditions 1,2,3,4 holds, then A allows diagonalizability.

1. If $S = \emptyset$, then A allows nonsingularity and thus by Theorem 4.37, A allows diagonalizability.

2. Suppose $S \neq \emptyset, T = \emptyset, a_1 \neq 0$ and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$. We can choose $M \in Q(A)$ such that all nonzero diagonal entries of M are distinct. Since $T = \emptyset$, the eigenvalues of M are its diagonal entries. Since $a_1 \neq 0$, $z(M) = |S| =$ nullity(M). Therefore M is diagonalizable, and thus A allows diagonalizability.

3. Suppose $S, T \neq \emptyset$, $S \cap T = \emptyset$ and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$. Then A or A^T is permutationally similar to

$$\begin{bmatrix} A(S) & B \\ O & O \end{bmatrix},$$

which is of the form (4.46). Since $T \neq \emptyset$, $A(S)$ allows nonsingularity. So we can choose $M \in Q(A)$ such that all eigenvalues of $M(S)$ are nonzero and distinct. Further, $z(M) = |S| = \text{nullity}(M)$. Therefore M is diagonalizable, and thus A allows diagonalizability.

4. Suppose $S \cap T \neq \emptyset$. Then A is permutationally similar to

$$\begin{bmatrix} A[T] & B \\ C & D \end{bmatrix},$$

which is of the form (4.46) such that $B \circ C^T = O$ and D is a diagonal sign pattern matrix. So we can choose $M \in Q(A)$ such that all nonzero diagonal entries of $M(T)$ are distinct, and all nonzero eigenvalues of $M[T]$ are distinct (by Lemma 4.38) such that $\sigma^*(M[T]) \cap \sigma^*(M(T)) = \emptyset$. Since $B \circ C^T = O$, $P_M(x) = P_{M[T]}(x) \cdot P_{M(T)}(x)$ and thus $z(M) = z(M[T]) + z(M(T))$. Since $S \cap T \neq \emptyset$, for some $i > 1$, the i-th column of each one of $M[T], M[T]^T$ is a nonzero scalar multiple of $[1\ 0\ \cdots\ 0]^T$. Therefore $\text{rank}(M) = \text{rank}(M[T]) + \text{rank}(M(T))$. So $z(M) = \text{nullity}(M)$. Therefore M is diagonalizable, and thus A allows diagonalizability.

Now we prove the converse part by contradiction. Suppose none of the conditions 1,2,3,4 are satisfied. Then either

`Case I:` A is of the form (4.46) such that $S \neq \emptyset$, $a_1 = 0$ and $b_i c_i = 0$ for all $i \in P$; or

`Case II`: A is permutationally similar to

$$\begin{bmatrix} A(S) & B \\ C & O \end{bmatrix},$$

which is of the form (4.46) such that $B \circ C^T = O$ and both B, C are nonzero.

In `Case I`, for any $M \in Q(A)$, $z(M) = |S| + 1$ and $\text{rank}(M) = n - |S|$. So $\text{nullity}(M) = |S| < z(M)$. Therefore M is not diagonalizable and hence A does not allow diagonalizability.

In `Case II`, for all $M \in Q(A)$, $P_M(x) = x^{|S|} P_{M(S)}(x)$ and thus $z(M) = |S| + z(M(S))$. Since $\text{rank}(M) \geq \text{rank}(M(S)) + 1$, $\text{nullity}(M) < z(M)$ and thus M is not diagonalizable for all $M \in Q(A)$. Hence A does not allow diagonalizability. $\square$

The following result is a special case of [29, Theorem 2.4.6.1].

Lemma 4.48 ([29]). *If T_1, T_2 are square matrices such that $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, then for any X of appropriate order,*

$$\begin{bmatrix} T_1 & X \\ O & T_2 \end{bmatrix} \quad \textit{is similar to} \quad \begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix}.$$

From Lemma 4.38, we can conclude that if a sign pattern matrix A does not allow diagonalizability, then $z(M) \geq 2$ for all $M \in Q(A)$. The following results give the number of Jordan blocks corresponding to the eigenvalue zero of M, and the index of M corresponding to the eigenvalue zero for all $M \in Q(A)$.

Theorem 4.49. *Let A be a sign pattern matrix of the form (4.46) with $n \geq 2$ that does not allow diagonalizability. Let $P = \{2, 3, \ldots, n\}$, $S = \{i \in P : a_i = 0\}$, $T = \{i \in P : b_i c_i \neq 0\}$. If $M \in Q(A)$ and $J_0(M)$ be the number of Jordan blocks corresponding to the eigenvalue zero, then we have the following.*

1. *If $S \neq \emptyset, T = \emptyset$, $a_1 = 0$, and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$, then $J_0(M) = |S|$.*

2. *If $S \neq \emptyset, S \cap T = \emptyset$, $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$, then $J_0(M) = |S| - 1$.*

Proof.

1. $S \neq \emptyset, T = \emptyset$. Suppose $a_1 = 0$, and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$.

   Then $\text{rank}(M) = n - |S|$. Therefore $J_0(M) = n - \text{rank}(M) = |S|$.

2. $S \neq \emptyset, S \cap T = \emptyset$. Suppose $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$.

   Then $\text{rank}(M) = n - |S| + 1$. Therefore $J_0(M) = n - \text{rank}(M) = |S| - 1$. □

Theorem 4.50. *Let A be a sign pattern matrix of the form (4.46) with $n \geq 2$ that does not allow diagonalizability. Let $P = \{2, 3, \ldots, n\}$, $S = \{i \in P : a_i = 0\}$, $T = \{i \in P : b_i c_i \neq 0\}$. If $M \in Q(A)$ and $I_M(0)$ be the index of A corresponding to the eigenvalue zero, then we have the following.*

1. *If $S \neq \emptyset, T = \emptyset$, $a_1 = 0$, and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$, then $I_M(0) = 2$.*

2. *If $S \neq \emptyset, T = \emptyset$, $a_1 \neq 0$, $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$, then $I_M(0) = 2$.*

3. *If $S \neq \emptyset, T = \emptyset$, $a_1 = 0$, $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$, then $I_M(0) = 3$.*

4. *If $S, T \neq \emptyset, S \cap T = \emptyset$, $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$, then $I_M(0) = z(M(S)) + 2$.*

Proof.

1. $S \neq \emptyset, T = \emptyset$. Suppose $a_1 = 0$, and either $b_i = 0$ for all $i \in S$ or $c_i = 0$ for all $i \in S$.

   Then $\text{rank}(M) = n - |S|$. Since $a_1 = 0$, $z(M) = |S| + 1$ and thus $z(M) - \text{nullity}(M) = 1$. Therefore $I_M(0) = 2$.

2. $S \neq \emptyset, T = \emptyset$. Suppose $a_1 \neq 0$, $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$.

   Then $\text{rank}(M) = n - |S| + 1$. Since $a_1 \neq 0$, $z(M) = |S|$ and thus $z(M) - \text{nullity}(M) = 1$. Therefore $I_M(0) = 2$.

3. $S \neq \emptyset, T = \emptyset$. Suppose $a_1 = 0$, $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$.

   Let $\tilde{S} = S \cup \{1\}$. Then M is permutationally similar to a matrix of the form

$$\begin{bmatrix} M[\tilde{S}] & X & O \\ O & T_1 & O \\ Y & O & T_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} M[\tilde{S}] & X \\ O & T_1 \end{bmatrix},$$

   where T_1, T_2 are diagonal matrices with nonzero diagonal entries. Since $M[\tilde{S}]$ is nilpotent, $\sigma(M[\tilde{S}]) \cap [\sigma(T_1) \cup \sigma(T_2)] = \emptyset$. Therefore by Lemma 4.48, first matrix is similar to

$$\begin{bmatrix} M[\tilde{S}] & O & O \\ O & T_1 & O \\ Y_1 & T_{21} & T_2 \end{bmatrix}, \text{ which again by Lemma 4.48, is similar to } \begin{bmatrix} M[\tilde{S}] & O & O \\ O & T_1 & O \\ O & T_{21} & T_2 \end{bmatrix}.$$

   Further, by Lemma 4.48, second matrix is similar to

$$\begin{bmatrix} M[\tilde{S}] & O \\ O & T_1 \end{bmatrix}.$$

   Now $\text{rank}(M) = n - |S| + 1$. Since $a_1 = 0$, $z(M) = |S| + 1$ and thus $z(M) - \text{nullity}(M) = 2$. Therefore, $z(M[\tilde{S}]) - \text{nullity}(M[\tilde{S}]) = 2$. If $b_r \neq 0, c_s \neq 0$ for some $r, s \in S$, then $[(M[\tilde{S}])^2]_{sr} = m_{s1}m_{1r} \neq 0$. Therefore $I_M(0) = 3$.

4. $S, T \neq \emptyset, S \cap T = \emptyset$. Suppose $b_i \neq 0$ for some $i \in S$ and $c_i \neq 0$ for some $i \in S$.

   Then M is similar to

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$M(S)$}} & 1 & 0 \\ & & \mathbf{0} & \mathbf{0} \\ \hline 0 & \mathbf{0}^T & 0 & 0 \\ 1 & \mathbf{0}^T & 0 & 0 \end{array}\right] \oplus O.$$

   Now $\text{rank}(M) = n - |S| + 1$. Therefore $I_M(0) = z(M(S)) + 2$. □

4.3 Sign Patterns that Require Diagonalizability

If a sign pattern matrix requires all distinct eigenvalues, then it requires diagonalizability. Some necessary and/or sufficient conditions for sign patterns to require distinct eigenvalues can be found in [17, 34, 40]. Lists of sign patterns upto order 4 requiring distinct eigenvalues are given in [34, 40].

Lemma 4.51 ([48]). *If A is a sign pattern matrix, then $c(A) \leq \mathrm{MR}(A)$.*

In the following result, we give a necessary condition in terms of the maximum cycle length and the maximum rank for a sign pattern matrix to require diagonalizability.

Theorem 4.52. *If a sign pattern matrix A requires diagonalizability, then $c(A) = \mathrm{MR}(A)$.*

Proof. Let A be a sign pattern matrix of order n. By Lemma 4.51, $c(A) \leq \mathrm{MR}(A)$. Let $c(A) < \mathrm{MR}(A)$. Let $B \in Q(A)$ be such that $\mathrm{rank}(B) = \mathrm{MR}(A)$. If $c(A) = k$, then $E_i(B) = 0$ for all $i \geq k+1$, where $E_i(B)$ is the sum of all $i \times i$ principal minors of B for $i = 1, 2, \ldots, n$. Then x^{n-k} is a factor of $P_B(x)$. So $\mathrm{nullity}(B) = n - \mathrm{MR}(A) < n - k \leq z(B)$. Therefore B is not diagonalizable. Hence $c(A) = \mathrm{MR}(A)$. □

The following example shows that the converse of Theorem 4.52 is not true.

Example 4.53. *Let us consider the sign pattern matrix*

$$A = \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

Here $c(A) = \mathrm{MR}(A) = 2$. But A does not require diagonalizability, since

$$B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \in Q(A)$$

and B is not diagonalizable.

In the above example, all terms in the standard determinant expansion of A have different signs. When $c(A) = \mathrm{MR}(A)$, then even the condition "all terms in the standard determinant expansion of A have the same sign" does not necessarily imply that A requires diagonalizability. The following example illustrates this for both reducible and irreducible sign pattern matrices.

Example 4.54. *Let us consider the sign pattern matrices*

$$A_1 = \begin{bmatrix} + & + \\ 0 & + \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} + & + \\ - & + \end{bmatrix}.$$

Here $c(A_1) = \mathrm{MR}(A_1) = 2$ *and* $c(A_2) = \mathrm{MR}(A_2) = 2$. *But neither* A_1 *nor* A_2 *require diagonalizability, since*

$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in Q(A_1) \quad \text{and} \quad B_2 = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \in Q(A_2)$$

and both B_1, B_2 *are not diagonalizable.*

The next few results enable us to reduce the problem of determining star sign pattern matrices, which require diagonalizability for any arbitrary order n to the same problem for the matrices of orders less than or equal to 4.

Any star sign pattern matrix of order n is permutation and signature similar to a matrix of the form

$$S = \begin{bmatrix} a_1 & + & + & \cdots & + \\ b_2 & a_2 & 0 & \cdots & 0 \\ b_3 & 0 & a_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_n & 0 & \cdots & 0 & a_n \end{bmatrix}, \tag{4.55}$$

where $a_1, \ldots, a_n \in \{+, -, 0\}$ and $b_2, \ldots, b_n \in \{+, -\}$.

Lemma 4.56. *If a star sign pattern matrix of the form (4.55) requires diagonalizability, then there is no* i, j *with* $i \neq j$ *and* $i, j \geq 2$ *such that* $b_i \neq b_j$ *and* $a_i = a_j$.

Proof. Let a star sign pattern matrix S of the form (4.55) be such that $b_i \neq b_j$ and $a_i = a_j$ for some $i \neq j$ with $i, j \geq 2$. Suppose the symbol a_i appears multiple times for some $i \geq 2$, and let $\{t \geq 2 : a_t = a_i\} = \{i_1, i_2, \ldots, i_k\}$ such that all the symbols $b_{i_1}, \ldots, b_{i_k}$ are not the same. Therefore we can choose

$$M = \begin{bmatrix} \alpha_1 & 1 & 1 & \cdots & 1 \\ \beta_2 & \alpha_2 & 0 & \cdots & 0 \\ \beta_3 & 0 & \alpha_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \beta_n & 0 & \cdots & 0 & \alpha_n \end{bmatrix} \in Q(S)$$

such that $\beta_{i_1} + \cdots + \beta_{i_k} = 0$ and $\alpha_{i_1} = \cdots = \alpha_{i_k} = \alpha_i$, say.

The characteristic polynomial of M is

$$f(x) = \prod_{p=1}^{n}(x - \alpha_p) - \sum_{p=2}^{n} \beta_p \prod_{\substack{q=2 \\ q \neq p}}^{n} (x - \alpha_q). \tag{4.57}$$

Since $\beta_{i_1} + \cdots + \beta_{i_k} = 0$, $(x - \alpha_i)^k$ is a factor of $f(x)$. So α_i is an eigenvalue of M with algebraic multiplicity $\geq k$. Further, $\text{rank}(M - \alpha_i I) = n - k + 1$ and thus $\text{nullity}(M - \alpha_i I) = k - 1$. So M is not diagonalizable. Hence the result follows. □

Lemma 4.58. *Let M_1, M_2 be the matrices given by*

$$M_1 = \begin{bmatrix} \alpha_1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \beta_2 & \alpha_2 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & & \vdots \\ \beta_k & \vdots & \ddots & \alpha_k & \ddots & & \vdots \\ \beta_{k+1} & \vdots & & \ddots & a & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & 0 \\ \beta_n & 0 & \cdots & \cdots & \cdots & 0 & a \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} \alpha_1 & 1 & \cdots & \cdots & 1 \\ \beta_2 & \alpha_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \beta_k & \vdots & \ddots & \alpha_k & 0 \\ \sum_{i=k+1}^{n} \beta_i & 0 & \cdots & 0 & a \end{bmatrix},$$

where $a \neq \alpha_i$ for all $i \in \{2, 3, \ldots, k\}$ and $\sum_{i=k+1}^{n} \beta_i \neq 0$. Then M_1 is diagonalizable if and only if M_2 is diagonalizable.

Proof. Note that $P_{M_1}(x) = (x - a)^{n-k-1} P_{M_2}(x)$. If $\lambda \neq a$, then

$$\text{rank}(M_1 - \lambda I) = \text{rank}(M_2 - \lambda I) + n - k - 1. \tag{4.59}$$

Further, $\text{rank}(M_1 - aI) = \text{rank}(M_2 - aI) = k + 1$.

Suppose M_1 is diagonalizable. If $\lambda \in \sigma(M_2)$, then $\lambda \neq a$ and thus by (4.59), we have $\text{nullity}(M_1 - \lambda I) = \text{nullity}(M_2 - \lambda I)$. Further, the algebraic multiplicity of λ for M_1 is same as the algebraic multiplicity of λ for M_2. Hence M_2 is diagonalizable.

Suppose M_2 is diagonalizable. Let $\lambda \in \sigma(M_1)$. Note that $\text{nullity}(M_1 - aI) = z(M_1 - aI) = n - k - 1$. If $\lambda \neq a$, then by (4.59), $\text{nullity}(M_1 - \lambda I) = \text{nullity}(M_2 - \lambda I)$. Further, the algebraic multiplicity of λ for M_1 is same as the algebraic multiplicity of λ for M_2. Hence M_1 is diagonalizable. □

Remark 4.60. *We use Lemma 4.58 to identify star sign pattern matrices of any order n which does not require diagonalizability by identifying the same for $n = 2, 3, 4$.*

We know that if a matrix A belongs to the qualitative class of a symmetric tree sign pattern matrix, then A is similar to a symmetric matrix. If A belongs to the qualitative class of a skew-symmetric tree sign pattern matrix with zero diagonal entries, then A is similar to a skew-symmetric matrix. To identify the star sign pattern matrices that do not require diagonalizability, we need to consider skew-symmetric star sign patterns with some nonzero diagonal entries and the star sign pattern matrices those are neither symmetric nor skew-symmetric.

Lemma 4.61. *Any 2×2 skew-symmetric star sign pattern matrix with some nonzero diagonal entries does not require diagonalizability.*

Proof. Non-equivalent skew-symmetric star sign pattern matrices of order 2 with some nonzero diagonal entries are

$$\begin{bmatrix} 0 & + \\ - & - \end{bmatrix}, \begin{bmatrix} - & + \\ - & - \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix}.$$

None of the above sign pattern matrices require diagonalizability, and examples of non-diagonalizable matrices in their qualitative classes are respectively

$$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

These are not diagonalizable because their characteristic polynomials are respectively

$$(x+1)^2, \quad (x+2)^2, \quad x^2,$$

whereas all the eigenvalues have geometric multiplicity 1. □

Lemma 4.62. *Any* 3×3 *skew-symmetric star sign pattern matrix with some nonzero diagonal entries does not require diagonalizability.*

Proof. Non-equivalent skew-symmetric star sign pattern matrices of order 3 with some nonzero diagonal entries are

$$\begin{bmatrix} 0 & + & + \\ - & - & 0 \\ - & 0 & - \end{bmatrix}, \begin{bmatrix} - & + & + \\ - & - & 0 \\ - & 0 & - \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & - & 0 \\ - & 0 & - \end{bmatrix}, \begin{bmatrix} - & + & + \\ - & 0 & 0 \\ - & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & + \\ - & - & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} - & + & + \\ - & - & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & - & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} - & + & + \\ - & - & 0 \\ - & 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ - & - & 0 \\ - & 0 & + \end{bmatrix}.$$

None of the first four sign pattern matrices from the above list require diagonalizability because we can find non-diagonalizable matrices in their qualitative classes using Lemma 4.58 and Lemma 4.61. None of the next five sign pattern matrices from the above list require diagonalizability, and examples of non-diagonalizable matrices in their qualitative classes are respectively

$$\begin{bmatrix} 0 & 6 & 3 \\ -4 & -9 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 1 \\ -2 & -4 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 9 & 1 \\ -3 & -8 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 \\ -2 & -3 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$$

These are not diagonalizable because their characteristic polynomials are respectively

$$(x+3)^3, \quad (x+2)^3, \quad (x+2)^3, \quad (x+1)^3, \quad x^3,$$

whereas all the eigenvalues have geometric multiplicity 1. □

Lemma 4.63. *Any* 3×3 *star sign pattern matrix, which is neither symmetric nor skew-symmetric, does not require diagonalizability.*

Proof. Each matrix in the qualitative class of such a sign pattern matrix is equivalent to a matrix of the form

$$\begin{bmatrix} \alpha_1 & 1 & 1 \\ -\beta_2 & \alpha_2 & 0 \\ \beta_3 & 0 & \alpha_3 \end{bmatrix}, \tag{4.64}$$

where $\beta_2, \beta_3 > 0$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. If the qualitative class of a sign pattern matrix contains a matrix of this form such that $\operatorname{sgn}(\alpha_2) = \operatorname{sgn}(\alpha_3)$, then by Lemma 4.56, that sign pattern matrix does not require diagonalizability. Now the remaining non-equivalent sign pattern matrices of the form (4.64) are

$$\begin{bmatrix} 0 & + & + \\ - & - & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & - & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} - & + & + \\ - & - & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ - & 0 & 0 \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} - & + & + \\ - & 0 & 0 \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & 0 & 0 \\ + & 0 & + \end{bmatrix},$$

$$\begin{bmatrix} + & + & + \\ - & - & 0 \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} - & + & + \\ - & - & 0 \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ - & - & 0 \\ + & 0 & + \end{bmatrix}.$$

None of the above sign pattern matrices require diagonalizability, and examples of non-diagonalizable matrices in their qualitative classes are respectively

$$\begin{bmatrix} 0 & 3 & 5 \\ -8 & -1 & 0 \\ 9 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 4 \\ -3 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 7 & 8 \\ -9 & 0 & 0 \\ 12 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 9 & 8 \\ -5 & 0 & 0 \\ 8 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 1 & 1 \\ -5 & 0 & 0 \\ 1 & 0 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 & 1 \\ -5 & -3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 4 & 4 \\ -3 & -1 & 0 \\ 8 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 5 & 6 \\ -4 & -12 & 0 \\ 6 & 0 & 4 \end{bmatrix}.$$

These are not diagonalizable because their characteristic polynomials are respectively

$$(x+3)^2(x-5), \quad (x+2)^2(x-1), \quad (x+2)^2(x-2), \quad (x+3)^2(x-7), \quad (x+3)^2(x-5),$$

$$(x-2)^2(x-5), \quad x(x+1)^2, \quad (x+3)^2(x-5), \quad (x+8)^2(x-8),$$

whereas all the eigenvalues have geometric multiplicity 1. □

Lemma 4.65. *Any* 4×4 *skew-symmetric star sign pattern matrix with some nonzero diagonal entries does not require diagonalizability.*

Proof. Each matrix in the qualitative class of such a sign pattern matrix is equivalent to a matrix of the form

$$\begin{bmatrix} \alpha_1 & 1 & 1 & 1 \\ -\beta_2 & \alpha_2 & 0 & 0 \\ -\beta_3 & 0 & \alpha_3 & 0 \\ -\beta_4 & 0 & 0 & \alpha_4 \end{bmatrix} \tag{4.66}$$

where $\beta_2, \beta_3, \beta_4 > 0$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$. If the qualitative class of a sign pattern matrix contains a matrix of this form such that any two of $\text{sgn}(\alpha_2), \text{sgn}(\alpha_3), \text{sgn}(\alpha_4)$ are equal, then by Lemma 4.58, Lemma 4.61 and Lemma 4.62, that sign pattern matrix does not require diagonalizability. Now the remaining non-equivalent sign pattern matrices of the form (4.66) are

$$\begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ - & 0 & 0 & 0 \\ - & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ - & 0 & 0 & 0 \\ - & 0 & 0 & + \end{bmatrix}.$$

None of the above sign pattern matrices require diagonalizability, and examples of non-diagonalizable matrices in their qualitative classes are respectively

$$\begin{bmatrix} 5 & 5 & 5 & 5 \\ -4 & -5\sqrt{5} & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -4 & 0 & 0 & 5\sqrt{5} \end{bmatrix}, \begin{bmatrix} 0 & 10 & 4 & 21 \\ -25 & -32 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ -18 & 0 & 0 & 32 \end{bmatrix}.$$

These are not diagonalizable because their characteristic polynomials are respectively

$$(x-5)^3(x+10), \quad (x-8)^3(x+24),$$

whereas all the eigenvalues have geometric multiplicity 1. □

Lemma 4.67. *Any* 4×4 *star sign pattern matrix, which is neither symmetric nor skew-symmetric, does not require diagonalizability.*

Proof. Each matrix in the qualitative class of such a sign pattern matrix is equivalent to a matrix of the form

$$\begin{bmatrix} \alpha_1 & 1 & 1 & 1 \\ -\beta_2 & \alpha_2 & 0 & 0 \\ \beta_3 & 0 & \alpha_3 & 0 \\ \beta_4 & 0 & 0 & \alpha_4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha_1 & 1 & 1 & 1 \\ -\beta_2 & \alpha_2 & 0 & 0 \\ -\beta_3 & 0 & \alpha_3 & 0 \\ \beta_4 & 0 & 0 & \alpha_4 \end{bmatrix},$$

where $\beta_2, \beta_3, \beta_4 > 0$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$. If the qualitative class of a sign pattern matrix contains a matrix of the first form, then we have the following.

1. If $\text{sgn}(\alpha_2)$ is equal to one of $\text{sgn}(\alpha_3), \text{sgn}(\alpha_4)$, then by Lemma 4.56, that sign pattern matrix does not require diagonalizability.

2. If $\text{sgn}(\alpha_3) = \text{sgn}(\alpha_4)$, then by Lemma 4.58 and Lemma 4.63, that sign pattern matrix does not require diagonalizability.

Similarly, if the qualitative class of a sign pattern matrix contains a matrix of the second form, then we have the following.

1. If $\text{sgn}(\alpha_4)$ is equal to one of $\text{sgn}(\alpha_2), \text{sgn}(\alpha_3)$, then by Lemma 4.56, that sign pattern matrix does not require diagonalizability.

2. If $\text{sgn}(\alpha_2) = \text{sgn}(\alpha_3)$, then by Lemma 4.58 and Lemma 4.63, that sign pattern matrix does not require diagonalizability.

So the remaining non-equivalent sign pattern matrices of the above forms are

$$\begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & + & + \\ - & 0 & 0 & 0 \\ + & 0 & - & 0 \\ + & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + & + & + \\ - & 0 & 0 & 0 \\ + & 0 & - & 0 \\ + & 0 & 0 & + \end{bmatrix},$$

$$\begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ - & 0 & 0 & 0 \\ + & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ - & 0 & 0 & 0 \\ + & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ - & 0 & 0 & 0 \\ + & 0 & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ - & 0 & + & 0 \\ + & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ - & 0 & + & 0 \\ + & 0 & 0 & 0 \end{bmatrix}.$$

None of the above sign pattern matrices require diagonalizability, and examples of non-diagonalizable matrices in their qualitative classes are respectively

$$\begin{bmatrix} 105 & 84 & 21 & 21 \\ -81 & -84 & 0 & 0 \\ 14 & 0 & 0 & 0 \\ 16 & 0 & 0 & 63 \end{bmatrix}, \begin{bmatrix} 0 & 10 & 24 & 21 \\ -9 & -18 & 0 & 0 \\ 32 & 0 & 0 & 0 \\ 14 & 0 & 0 & 18 \end{bmatrix}, \begin{bmatrix} -36 & 21 & 72 & 21 \\ -30 & -54 & 0 & 0 \\ 64 & 0 & 0 & 0 \\ 42 & 0 & 0 & 54 \end{bmatrix},$$

$$\begin{bmatrix} 12 & 8 & 9 & 2 \\ -16 & 0 & 0 & 0 \\ 10 & 0 & -2 & 0 \\ 9 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 16 & 18 & 2 \\ -12 & 0 & 0 & 0 \\ 15 & 0 & -2 & 0 \\ 7 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 5 & 5 & 5 \\ -42 & -25 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 15 & 0 & 0 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 10 & 10 & 9 \\ -9 & -20 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 10 & 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} -6 & 6 & 12 & 9 \\ -5 & -18 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 18 & 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 1 & 1 \\ -2 & -\sqrt{3} & 0 & 0 \\ -2 & 0 & \sqrt{3} & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 2 & 2 \\ -9 & -8 & 0 & 0 \\ -9 & 0 & 8 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

These are not diagonalizable because their characteristic polynomials are respectively

$$(x+21)^2(x-42)(x-84), \quad (x+24)^2(x-12)(x-36), \quad (x+72)^2(x-72)(x-36),$$

$$(x-4)^3(x+12), \quad (x-3)^2(x+1)^2, \quad (x+10)^2(x+5)(x-10),$$

$$(x+10)^2(x+5)(x-10), \quad (x+12)^2(x+6)(x-12), \quad (x-4)^2(x-8)(x+4), \quad (x-4)^2(x+4)^2$$

and all the eigenvalues have geometric multiplicity 1. □

Theorem 4.68. *A star sign pattern matrix S requires diagonalizability if and only if S is a symmetric sign pattern matrix or a skew-symmetric sign pattern matrix with all diagonal entries zero.*

Proof. If S is a symmetric star sign pattern matrix or a skew-symmetric star sign pattern matrix with all diagonal entries zero, then every matrix in $Q(S)$ is similar to a symmetric or a skew-symmetric matrix. So S requires diagonalizability.

For the converse part, suppose S is neither a symmetric sign pattern matrix nor a skew-symmetric sign pattern matrix with all diagonal entries zero. Suppose S is of the form (4.55). If there are i, j with $i \neq j$ and $i, j \geq 2$ such that $b_i \neq b_j$ and $a_i = a_j$, then by Lemma 4.56, S does not require diagonalizability. Otherwise, that is if S satisfies the condition that $a_i = a_j$ implies $b_i = b_j$ for all $i, j \geq 2$, then we can find a non-diagonalizable matrix in $Q(S)$ using Lemma 4.58 and one of Lemma 4.61, Lemma 4.62, Lemma 4.63, Lemma 4.65 and Lemma 4.67. So S does not require diagonalizability. □

The following example shows that Theorem 4.68 cannot be extended to the path sign pattern matrices.

Example 4.69. *Let us consider the path sign pattern matrix*

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

which is neither symmetric nor skew-symmetric. Any matrix in its qualitative class is similar to a matrix of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 0 \end{bmatrix},$$

where $a, b, c > 0$. The characteristic polynomial of A is $x^4 + (a - b - c)x^2 - ac$ i.e.,

$$\left(x^2 - \frac{b + c - a + \sqrt{(b + c - a)^2 + 4ac}}{2}\right)\left(x^2 - \frac{b + c - a - \sqrt{(b + c - a)^2 + 4ac}}{2}\right).$$

So A has one positive, one negative and two purely imaginary eigenvalues and thus A is diagonalizable. Hence the above sign pattern matrix requires diagonalizability.

CHAPTER 5

Potentially Stable Tree Sign Patterns with Negative Edges

In this chapter, we consider only tree sign pattern matrices with negative edges. A square matrix A is stable, if all eigenvalues of A have negative real part. A sign pattern matrix A is potentially stable, if there exists a stable matrix $B \in Q(A)$. A sign pattern matrix A allows a properly signed nest if there exist $B \in Q(A)$ and a permutation $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $\{1, 2, \ldots, n\}$ such that

$$\operatorname{sgn} \det B[\{\alpha_1, \alpha_2, \ldots, \alpha_k\}] = (-1)^k \text{ for all } k \in \{1, 2, \ldots, n\}.$$

Johnson et al. [33] proved that if a sign pattern matrix allows a properly signed nest, then it is potentially stable. However, the converse is not true, even for tree sign pattern matrices. For example, the tree sign pattern matrix

$$A = \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & + \end{bmatrix}$$

is potentially stable, since the matrix

$$\begin{bmatrix} -5 & 1 & 0 & 0 \\ 25 & 0 & 1 & 0 \\ 0 & -700 & 0 & 1 \\ 0 & 0 & 150 & 1 \end{bmatrix}$$

has eigenvalues $-3.8699, -0.1244$, and $-0.0029 \pm 22.7925i$, and thus is stable. If A allows a properly signed nest, then there exists $B \in Q(A)$ and a permutation $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of $\{1, 2, 3, 4\}$ such that $\operatorname{sgn}\det B[\{\alpha_1, \alpha_2, \ldots, \alpha_k\}] = (-1)^k$ for all $k \in \{1, 2, 3, 4\}$. Then $\alpha_1 = 1$, and thus $\det B[\{\alpha_1, \alpha_2\}] \leq 0$. This is a contradiction, and therefore A does not allow a properly signed nest. We try to see if the converse of the above result is true for tree sign pattern matrices with negative edges. In Section 5.1, we give sufficient conditions for a sign pattern matrix to allow a properly signed nest. We also characterize potentially stable star and path sign pattern matrices with negative edges. Finally in Section 5.2, we identify all 5-by-5 spectrally arbitrary tree sign pattern matrices with negative edges.

5.1 Potentially Stable Tree Sign Pattern Matrices with all Edges Negative

Characterization of potentially stable sign pattern matrices has been a long standing open problem. Gao and Li [21] gave a necessary and sufficient condition for potentially stable star sign pattern matrices. Bone [5] and Grundy et al. [25] gave some methods to construct higher order potentially stable sign pattern matrices from lower order potentially stable sign pattern matrices. Johnson et al. [33] introduced the concept of properly signed nest and proved that a sign pattern matrix allowing a properly signed nest is potentially stable. Olesky et al. [44] characterized sign pattern matrices allowing a properly signed nest in terms of another allow problem.

The following result was given by Johnson et al. [33] in an attempt to obtain a sufficient condition for a tree sign pattern matrix with negative edges to allow a properly signed nest.

[33, Corollary 3.7]. *If A is a tree sign pattern matrix in which at least one diagonal entry is negative and every edge is negative (except possibly those with both end vertices negative), and A allows a nonzero determinant, then A allows a properly signed nest.*

However, the following example shows that the above result is not true in general.

Example 5.1. *Let us consider the tree sign pattern matrix*

$$A = \begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & - & + \end{bmatrix}.$$

Let $B \in Q(A)$. Then without loss of generality we may assume that

$$B = \begin{bmatrix} a & 1 & 0 \\ -d & -b & 1 \\ 0 & -e & c \end{bmatrix},$$

where $a, b, c, d, e > 0$. If A allows a properly signed nest, then we must have

$$-b < 0, \quad e - bc > 0 \text{ or } d - ab > 0 \quad \text{and} \quad ae + cd - abc < 0$$

for some $a, b, c, d, e > 0$, which is not possible. So A does not allow a properly signed nest.

We give some sufficient conditions for a sign pattern matrix with all edges negative to allow a properly signed nest. The following result by Jeffries and Johnson [30] is very useful in identifying some tree sign pattern matrices which are not potentially stable. Let us recall from Chapter 1 that for a tree sign pattern matrix A, the symmetric factorization of A is $A = S_2A_2$, where S_2 is a signature sign pattern matrix with $(1,1)$ entry $+$ and A_2 is a symmetric tree sign pattern matrix. Let $i_+(S)$ denotes the number of $+$ of a signature sign pattern matrix S.

Lemma 5.2 ([30]). *Let $A = S_2A_2$ be the symmetric factorization of an $n \times n$ tree sign pattern matrix A. If A is potentially stable, then there is a symmetric matrix $B_2 \in Q(A_2)$ such that*

$$i_+(B_2) = n - i_+(S_2).$$

Lemma 5.3. *Let T be a tree with a vertex v such that $T - \{v\}$ has a perfect matching. Then for each vertex u adjacent to v, there is a path $(v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}, v_{i_k})$ such that $\deg(v_{i_k}) = 1$ and $\deg(v_{i_{k-1}}) = 2$, where $v_{i_1} = v$ and $v_{i_2} = u$.*

Proof. Let u be a vertex adjacent to v. Since $T-\{v\}$ has a perfect matching, u is not pendant. We consider a longest path $(v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}, v_{i_k})$ in T, where $v_{i_1}=v$ and $v_{i_2}=u$. Then v_{i_k} must be a pendant vertex. If $\deg(v_{i_{k-1}})>2$, then $v_{i_{k-1}}$ must be adjacent to another pendant vertex, say w. So we have two pendant vertices w, v_{i_k} distinct from v such that $v_{i_{k-1}}$ is adjacent to both w, v_{i_k}. Then $T-\{v\}$ cannot have a perfect matching, a contradiction. So $\deg(v_{i_{k-1}})=2$. □

If G is an undirected graph with the vertex set $V(G)$ and the edge set $E(G)$, then the subgraph of G induced by S is the graph having the vertex set S and the edge set $\{[i,j]\in E(G) : i,j\in S\}$, where $S\subseteq V(G)$. We denote this induced subgraph by $G(S)$. Throughout this chapter, $\langle n\rangle$ denotes the set $\{1,2,\ldots,n\}$.

Lemma 5.4. *Let T be a tree with $2n+1$ vertices and let v be a vertex of T. If $T-\{v\}$ has a perfect matching, then we can label the vertices with the numbers from $\langle 2n+1\rangle$ such that the vertex v is labelled as 1 and for each $s\in\langle 2n+1\rangle$, $G(\langle s\rangle)$ has a perfect matching when s is even, and $G(\langle s\rangle)-\{1\}$ has a perfect matching when s is odd.*

Proof. We prove this by induction on n. For $n=1$, T is a tree with three vertices. Since $T-\{v\}$ has a perfect matching, v must be pendant. If we label v as 1, the vertex adjacent to v as 2 and the other vertex as 3, then we get the desired result.

Suppose the required result is true for any tree with $2n-1$ vertices. Let T be a tree with $2n+1$ vertices, and T has a vertex v such that $T-\{v\}$ has a perfect matching. Then by Lemma 5.3, there is a path $(v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}, v_{i_k})$ such that $\deg(v_{i_k})=1$ and $\deg(v_{i_{k-1}})=2$, where $v_{i_1}=v$ and $v_{i_{k-1}}, v_{i_k}\neq v$. So $\tilde{T}=T-\{v_{i_{k-1}}, v_{i_k}\}$ is a tree such that $\tilde{T}-\{v\}$ has a perfect matching. Further, $\tilde{T}$ is a tree with $2n-1$ vertices. So by induction hypothesis, we can label $\tilde{T}$ such that the vertex v is labelled as 1; and for each $s\in\langle 2n-1\rangle$, $G(\langle s\rangle)$ has a perfect matching when s is even, and $G(\langle s\rangle)-\{1\}$ has a perfect matching when s is odd. If $v_{i_{k-2}}$ be labelled as m in the above labelling, then label $v_{i_{k-1}}$ as $m+1$, v_{i_k} as $m+2$, and relabel all vertices $j(>m)$ of $\tilde{T}$ as $j+2$.

If m is even, then by induction hypothesis, both $G(\langle m\rangle)$ and $G(\langle m-1\rangle)-\{1\}$ have perfect matchings in $\tilde{T}$. So $G(\langle m+1\rangle)-\{1\}$ has a perfect matching, namely the perfect

Theorem 5.14. *If A is an irreducible tridiagonal sign pattern matrix of order n with all edges negative, then the following statements are equivalent.*

1. *A is potentially stable.*
2. *A allows a properly signed nest.*
3. *Exactly one of the following statements is true.*
   (a) *n is even and A has a negative diagonal entry.*
   (b) *n is odd and the i-th diagonal entry of A is negative for some odd i.*

Proof. Theorem 5.6 and Theorem 5.10 imply $3 \Rightarrow 2$. By Lemma 5.12, $2 \Rightarrow 1$. Now we show $1 \Rightarrow 3$.

A is potentially stable implies that A has a negative diagonal entry. So $1 \Rightarrow 3$ when n is even. If n is odd, then suppose $a_{ii} \neq -$ for any odd i. Consider a path sign pattern matrix B of order n such that A is a subpattern of B and $(2i-1)$-th diagonal entry of B is positive for all i with $1 \leq i \leq \frac{n+1}{2}$. Let $B = S_2B_2$ be the symmetric factorization of B. If B is potentially stable, then by Lemma 5.2, there exists a symmetric matrix $C_2 \in Q(B_2)$ such that $i_+(C_2) = n - i_+(S_2) = \frac{n-1}{2}$. But B_2 has a principal submatrix, which is a direct sum of $\frac{n+1}{2}$ $[+]$s. So by Cauchy's interlacing theorem for real symmetric matrices, $i_+(M_2) \geq \frac{n+1}{2}$ for all $M_2 \in Q(B_2)$. Thus B is not potentially stable, and therefore by Lemma 5.13, A is not potentially stable, which is a contradiction. So A is a potentially stable sign pattern matrix of order n implies that the i-th diagonal entry of A is negative for some odd i. Therefore $1 \Rightarrow 3$. □

The following result characterizes all potentially stable star sign pattern matrices with all edges negative.

Theorem 5.15. *If A is a star sign pattern matrix of order $n \geq 2$ with all edges negative, then the following statements are equivalent.*

1. *A is potentially stable.*
2. *A allows a properly signed nest.*
3. *Diagonal entry corresponding to at most one pendant vertex of $G(A)$ is nonnegative.*

Proof. Corollary 5.9 implies that $3 \Rightarrow 2$. Again by Lemma 5.12, $2 \Rightarrow 1$. Now we show $1 \Rightarrow 3$.

For $n = 2$, the potentially stable star sign pattern matrices (upto equivalence) with all edges negative are one of the following sign pattern matrices.

$$\begin{bmatrix} - & + \\ - & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} - & + \\ - & - \end{bmatrix}.$$

Clearly, each of these matrices satisfies condition 3. For $n \geq 3$, if the condition 3 is not true, then at least two diagonal entries corresponding to pendant vertices of $G(A)$ are nonnegative. Let B be a star sign pattern matrix of order n such that A is a subpattern of B and two diagonal entries of B corresponding to pendant vertices of $G(A)$ are $+$. Without loss of generality we may assume that one of these pendant vertices is labelled as 1. Let $B = S_2B_2$ be the symmetric factorization of B. Since the number of vertices at even distance from the vertex 1 including itself is $n-1$, $i_+(S_2) = n-1$.

If B is potentially stable, then by Lemma 5.2, there exists a symmetric matrix $C_2 \in Q(B_2)$ such that $i_+(C_2) = n - i_+(S_2) = 1$. But B_2 has a principal submatrix $[+] \oplus [+]$. So by Cauchy's interlacing theorem for real symmetric matrices, $i_+(M_2) \geq 2$ for all $M_2 \in Q(B_2)$. Thus B is not potentially stable. So by Lemma 5.13, A is not potentially stable, which is a contradiction. Therefore $1 \Rightarrow 3$. □

We get from Lemma 5.12 that if a sign pattern matrix allows a properly signed nest then it is potentially stable. Since from Theorem 5.14 and Theorem 5.15, we get that for path and star sign pattern matrices with all edges negative, potential stability implies the existence of a properly signed nest, we believe that the same is true for all tree sign pattern matrices with all edges negative.

Conjecture 5.16. *A tree sign pattern matrix A with all edges negative is potentially stable if and only if it allows a properly signed nest.*

We do not have a complete answer to the above conjecture as yet, however the following theorem shows that the above conjecture is true for tree sign pattern matrices upto order 6.

$$A_{23} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 \end{bmatrix}, \quad A_{24} = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ - & + & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & - & + & 0 \\ 0 & 0 & 0 & - & + & + \\ 0 & 0 & 0 & 0 & - & - \end{bmatrix}.$$

Now by Corollary 5.9, A_{21}, A_{22}, A_{23} allow properly signed nests. Further, by Example 5.7, $A_{24}(\{6\})$ allows a properly signed nest, and thus by Lemma 5.5 and Lemma 5.8, A_{24} allows a properly signed nest. Therefore by Lemma 5.5, A is potentially stable implies A allows a properly signed nest. Hence the result holds true by Lemma 5.12.

3. If $G(A)$ is equivalent to G_3, then we can consider

$$A = \begin{bmatrix} s_1 & + & + & + & + & 0 \\ - & s_2 & 0 & 0 & 0 & 0 \\ - & 0 & s_3 & 0 & 0 & 0 \\ - & 0 & 0 & s_4 & 0 & 0 \\ - & 0 & 0 & 0 & s_5 & + \\ 0 & 0 & 0 & 0 & - & s_6 \end{bmatrix}.$$

The symmetric factorization of A is

$$A = \begin{bmatrix} + & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} s_1 & + & + & + & + & 0 \\ + & -s_2 & 0 & 0 & 0 & 0 \\ + & 0 & -s_3 & 0 & 0 & 0 \\ + & 0 & 0 & -s_4 & 0 & 0 \\ + & 0 & 0 & 0 & -s_5 & - \\ 0 & 0 & 0 & 0 & - & s_6 \end{bmatrix} = S_2 A_2.$$

If A is potentially stable, then by Lemma 5.2, there exists a symmetric matrix $M_2 \in Q(A_2)$ such that $i_+(M_2) = 6 - i_+(S_2) = 4$ and thus $i_-(M_2) \leq 2$. Therefore by Cauchy's interlacing theorem and Lemma 5.13, we can conclude the following.

(a) If any three of s_2, s_3, s_4, s_5 is $+$, then A and its subpatterns are not potentially stable.

(b) If any two of s_2, s_3, s_4 is $+$ and $s_6 = -$, then A and its subpatterns are not potentially stable.

Further, any two of s_2, s_3, s_4 are 0 implies A requires singularity, and thus A is not potentially stable. Therefore A is potentially stable implies that A is permutation similar to some super-pattern of one among

$$A_{31} = \begin{bmatrix} 0 & + & + & + & + & 0 \\ - & - & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} 0 & + & + & + & + & 0 \\ - & + & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & - & + \\ 0 & 0 & 0 & 0 & - & + \end{bmatrix}.$$

Now by Corollary 5.9, A_{31} allows a properly signed nest. Further, by Example 5.7, $A_{32}(\{3\})$ allows a properly signed nest, and thus by Lemma 5.5 and Lemma 5.8, A_{32} allows a properly signed nest. Therefore by Lemma 5.5, A is potentially stable implies A allows a properly signed nest. Hence the result holds true by Lemma 5.12.

4. If $G(A)$ is equivalent to G_4, then we can consider

$$A = \begin{bmatrix} s_1 & + & + & + & 0 & 0 \\ - & s_2 & 0 & 0 & 0 & 0 \\ - & 0 & s_3 & 0 & 0 & 0 \\ - & 0 & 0 & s_4 & + & + \\ 0 & 0 & 0 & - & s_5 & 0 \\ 0 & 0 & 0 & - & 0 & s_6 \end{bmatrix}.$$

The symmetric factorization of A is

$$A = \begin{bmatrix} + & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} s_1 & + & + & + & 0 & 0 \\ + & -s_2 & 0 & 0 & 0 & 0 \\ + & 0 & -s_3 & 0 & 0 & 0 \\ + & 0 & 0 & -s_4 & - & - \\ 0 & 0 & 0 & - & s_5 & 0 \\ 0 & 0 & 0 & - & 0 & s_6 \end{bmatrix} = S_2 A_2.$$