Indian Institute of Technology Guwahati
Department of Physics

Asymptotic symmetry and its role in black hole thermodynamics

A thesis submitted by:
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Thesis supervisor: Dr. Debaprasad Maity,
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Declaration

I hereby declare that the papers included in this thesis entitled "Asymptotic symmetry and its role in black hole thermodynamics" are the result of the work carried out by me, under the supervision of Dr. Debaprasad Maity and Dr. Bibhas Ranjan Majhi, Department of Physics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for the award of any degree.

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Disclaimer

The bibliography included in this thesis is, by no means complete but contains the ones which I have consulted thoroughly. I apologize for unintentionally missing out some of the research papers, review articles and other scientific documents pertaining to the focus of this thesis which should also have been cited.

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Certificate

It is certified that the work contained in this thesis entitled "Asymptotic symmetry and its role in black hole thermodynamics" by Mousumi Maitra, a student in Department of Physics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under our supervisions and this work has not been submitted elsewhere for a degree.

December, 2021

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To
My Parents

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"Thinking is the capital, Enterprise is the way, Hard work is the solution."

-Dr. A.P.J. Abdul Kalam

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Abstract

The astonishing connection between the asymptotic symmetries and the thermodynamics of the black hole was realized by the scientific communities about forty years ago. One of the unsolved tasks in the study of the black hole thermodynamics is to uncover and understand the microscopic degrees of freedom which are responsible for the entropy of the black hole. In this context, the Noether charges and current related to the asymptotic symmetries play a pivotal role in resolving this issue. In 1962, surprisingly, an infinite-dimensional group of symmetries near null infinity of the asymptotically flat spacetime was discovered as the asymptotic symmetry group (BMS), which has the flat space Poincaé group as the subgroup. Initially, it was revealed that the asymptotic symmetry analysis near null boundaries could shed light on the gravitational scattering phenomena, but later the symmetry analysis was well extended to the near-horizon region of the black hole. In diffeomorphism invariant gravity theory, the commutator algebra between the charges corresponding to the near horizon symmetries were shown to be the Virasoro algebra having a non-trivial central extension which produce correctly the well-known Bekenstein-Hawking result, through the Cardy formula. It is expected that probably the microscopic information of the black hole entropy can be extracted from the central charges and the zero mode Q_0. In these endeavors, the study of the Noether charges corresponding to the asymptotic symmetries has gained immense interest in the recent past. The near horizon symmetry analysis is further explored near a generic null hypersurface which may not represent the solutions of the Einstein equations. Therefore, understanding the symmetries, specifically under the spacetime-dependent transformation, has been a central part of gravity for a long time.

In the present thesis, we investigate the various aspects of the asymptotic symmetries in gravitational theories. At first, we try to analyze the asymptotic symmetries and the conserved charges near a generic null hypersurface having electromagnetic charge, in higher-order theory of gravity with the presence of the non-linear gauge field. We hope that this result will illuminate the physical importance of the charges in a more general context. However, by these lines of work, consequently, it is found out that the supertranslation and superrotation parameters modify the macroscopic parameters (i.e., mass, angular momentum) of the black hole. We argue that this can be treated as the breaking of the symmetry of the arbitrariness of the solutions, by the black hole backgrounds. Consequently, we promote the supertranslation parameter as the Goldstone boson modes. In Rindler

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and Schwarzschild black hole backgrounds, we study the classical dynamics of these modes by proposing the appropriate Lagrangian. Interestingly the Fourier modes of the supertranslation parameter come out to be the unstable ones. In the semi-classical regime, we found that this instability can lead to the thermalization of the horizon. Therefore, we hope that these Goldstone modes can shed light on the microscopic origin of the thermal nature of the horizon. Next, the same analysis is further performed in the rotating black hole background. Finally, We investigate the asymptotic symmetries near a timelike hypersurface following the standard procedure. For our discussion, the timelike surface is assumed to be positioned at a finite distance outside the horizon of the black hole.

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Publication list

This thesis is based on the following works:

- **BMS Goldstone modes near the horizon of a Kerr black hole are thermal**

  Mousumi Maitra, Debaprasad Maity and Bibhas Ranjan Majhi

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- **Symmetries near a generic timelike surface in black hole spacetime.**

  Mousumi Maitra, Debaprasad Maity and Bibhas Ranjan Majhi

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- **Near horizon symmetries, emergence of Goldstone modes and thermality.**

  Mousumi Maitra, Debaprasad Maity and Bibhas Ranjan Majhi

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- **Symmetries near a generic charged null surface and associated algebra: An off-shell analysis.**

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Introduction 1

1.1 General background

One of the most remarkable scientific developments of the 20th century is Einstein's General Theory of Relativity (GTR), which culminated in 1915. GTR is a field theoretical framework that successfully describes the gravitational dynamics in terms of the geometric components of spacetime metric. The theory can explain those physical phenomena that Newtonian gravity cannot. In this framework, many intriguing physical phenomena such as perihelion precession time dilation, gravitational lensing, the gravitational redshift of light [1–9] to name a few can be explained by simply analyzing the geodesic dynamic in curved spacetime background. When it comes to an understanding the spacetime dynamics itself, the theory elegantly provides us a set of non-linear field equations, known as Einstein's equations [2]. The recent detection of gravitational waves, which is one of the many profound predictions of those Einstein field equations, from a binary black hole merger [10] by *LIGO Scientific Collaboration and Virgo Collaboration*, paved the new era of gravitational-wave astronomy, which the scientists were longing for over half a century.

Since the advent of general relativity and despite its complicated non-linear field equation, attempts have been made to realize the closed-form solution of the Einstein equation. After one year of Einstein's discovery, in 1916, Schwarzschild [11] first presented a non-trivial exact solution of Einstein's vacuum field equations for a point mass, known as Schwarzschild solution. Droste [12], around the same time as Schwarzschild, arrived independently at the same answer. In fact, he

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published the Schwarzschild solution in the form which is used in the present day. In the same year, Hans Reissner [13] generalized Schwarzschild solution and found the solutions of the Einstein-Maxwell equations having electrically charged objects. Gunnar Nordström [14] later independently arrived at the same solution, which is known as the Reissner-Nordström metric. In 1963, Roy Kerr [15] found the static axially symmetric vacuum solution of the Einstein equation. Since its theoretical inception to the present day detection [16–19] of over a hundred years of overwhelming journey, those solutions still fascinate physicists/non-physicists because of their unparalleled exoticness, which are yet to be unveiled. The existence of essential singularities is one such exotic property that was at first difficult to comprehend, but gradually physicists accepted this fact and believed that GTR is necessarily not the complete description of gravity. The very existence of such singularity comes with another exotic property: a hypothetical surface called horizon, and nothing can come out once it enters the horizon. John Archibald Wheeler named these peculiar objects as 'black hole.' After the gravitational collapse, the black hole is formed, covering a region of spacetime from where nothing can escape to infinity. In 1958 David Finkelstein described that the black holes are separated from the rest of the universe by a unidirectional membrane known as the event horizon. Recently in 2019, EHT collaboration [16] produced the first-ever image of black holes, thus ending the long-standing debate over the existence of black holes in reality.

Starting from 1970, in the past forty years, a new direction has been opened up in the study of black hole theory, which shows the deep connection of black hole physics with the laws of thermodynamics, known to describe complicated equilibrium statistical systems with very few physical variables. When matter collapses, the endpoint of such a physical process generically settles down to a stationary black hole which is known to be uniquely described by very few macroscopic parameters- mass, charge and angular momentum. Hence all the microscopic information of the collapsing matters seems to be hidden and this is the famous 'no- hair' theorem [20–22] of the black hole. The first hint of the thermodynamical properties of black hole get surfaced when Hawking proposed his area theorem [23] stating that the area of the black hole event horizon can never decrease in time if the null energy condition is satisfied. Then the similarities between the area theorem with the second law of thermodynamics and also the idea that the information about any object is lost when it falls inside the black hole, inspired Bekenstein to propose black hole entropy as the multiple of the area of its

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event horizon [24, 25]. Later Bardeen, Carter, and Hawking, in their seminal paper [26] introduced a detailed explanation of the four laws of black hole mechanics, which have direct resemblance with the corresponding laws of thermodynamics. The main three laws are as follows,

- The zeroth law of the black hole mechanics says that the surface gravity κ is constant over the horizon of a stationary black hole. This law resembles the zeroth law of thermodynamics which ensures uniform temperature throughout a system in thermal equilibrium. An important consequence of the zeroth law is that the stationary black holes can be two types: extremal black holes (for which, $\kappa = 0$) and a non-extremal black hole with bifurcate horizons.
- The first law of black hole mechanics provides the relation among the variations in mass M, area of the horizon A, angular momentum J, and charge Q of a stationary perturbed black hole;

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta Q \tag{1.1}$$

  Here Ω and Φ are the angular velocity and electrostatic potentials respectively. Thus the first law gives the differential change between masses of two nearby stationary black hole solutions where the surface gravity κ is analogous to the temperature of the system. Although original derivation [26] was done for stationary perturbations but later the derivation was generalized for non-stationary perturbations also [27, 28].
- Now, the second law of the black hole mechanics says that the area of the horizon of the black hole never decreases with time.

$$\delta A \geq 0. \tag{1.2}$$

Bekenstein argued [24, 25] the entropy of the black hole to be proportional to the the surface area A of its event horizon with a multiplicative factor β as given by,

$$S_{BH} = \beta A. \tag{1.3}$$

The exact value of β was not determined till then and from the concept of information theory and using Shannon's formula , Bekenstein conjectured the value of β to be $\frac{ln2}{8\pi}$. Then the temperature T, being proportional to surface gravity, was given by $T = \lambda\kappa$. Now the first law of black hole mechanics (1.1) shows that λ and β are related by,

$$8\pi\beta\lambda = 1. \tag{1.4}$$

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Then λ was found to be $1/(ln2)$. Later by incorporating quantum field theoretical treatment in curved spacetime background Hawking [29, 30] showed that the black holes could emit thermal radiation at a temperature T_{BH} known as Hawking temperature as given by (in Planck units),

$$T_{BH} = \frac{\kappa}{2\pi}. \tag{1.5}$$

Then one can say $\lambda = 1/(2\pi)$ and hence from (1.4) the factor β was determined to be $1/4$. Thus, Hawking's derivation of thermal radiation of black hole establishes Bekenstein's result.

Unlike standard thermodynamics, the microscopic nature of black hole thermodynamics is a poorly understood subject till now. Over the years, numerous efforts have been made to give a proper geometric description of the black hole entropy from the microscopic point of view. Earlier it was thought that a valid quantum theory of gravity was necessary to uncover and solve this issue. The first attempt in this direction was found in the context of Euclidean quantum gravity. After one year of Hawking's semi-classical derivation of the thermal spectrum of black hole, Gibbons and Hawking [31] derived the entropy for Kerr-Newman solutions and also for de-Sitter space, by evaluating the canonical partition functions in path-integral approach. Although their derivation confirms Bekenstein's classical results, it does not provide much information on the microscopic description of the black hole entropy. A fruitful computation of the entropy of the black hole with the help of the microcanonical ensemble is presented in [32]. However another important approach to determine the black hole entropy is to derive quantum field correlations of the inside region with the outside of the black hole. Thus from these correlations, one can define entanglement entropy [33–35], whose leading term is shown to be the same as 'Bekenstein-Hawking' entropy. However, in this process, the black hole is treated as classical and the field is quantized in that background. It does not provide much insight on the underlying degrees of freedom which are responsible for the entropy. There have been other attempts [36] where the entropy of the black hole is derived by analyzing the quantum structure near the horizon. Here it is assumed that the black hole obeys the rules of quantum mechanics like ordinary matter systems in the thermal atmosphere and also contains a large number of particles and have energy levels. So the density of states of the system having energy E is calculated using statistical description and finally, entropy density comes out to be proportional to T^3, where T is the temperature of the system. However, in this process total entropy of the thermal system becomes divergent near the horizon. This divergence is then removed by

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imposing an appropriate cut-off on the frequency of the modes corresponding to the thermal system. This approach has similarities with entanglement entropy formulations (more can be found in [37]).

String theory [38–40] and loop quantum gravity (LQG) [41–47] are two powerful theories which give us extremely useful perspectives and direction in approaching the complete understanding of the black hole. In the framework of LQG, a plausible explanation of the nature of the entropy has been proposed by directly quantizing the geometric quantities (such as the area) which are assumed to be the fundamental field variables. In this approach, at first one has to quantize the associated phase space of the isolated horizons and then have to specify certain quantum states which describe the geometry of the horizon. In the classical regime, the fields in the bulk regions help to determine the fields on the surface and as a result, no additional surface degrees of freedom arise. However in quantum treatment, independent degrees of freedom appear on the horizon surface due to the counting of the surface states, and these degrees of freedom account for the entropy of the black hole, being proportional to the area of the horizon. The exact value of this proportionality constant is determined by a free parameter (known as Immirzi parameter [48, 49]) which is the result of ambiguities of the loop quantum procedure. However, the suitable choice of this parameter can lead to correct Bekenstein-Hawking entropy of the black hole. On the other hand, another fruitful derivation of black hole entropy in higher dimension is achieved in string theory. It is anticipated that the massive degenerate states found in string theory may be the potential reason for the origin of the black hole entropy [39, 50, 51]. Precisely in weak coupling limit of string theory [52, 53] the states are formed perturbatively and black hole solutions can be identified from these states at low energy having definite charges. Using the usual formalism of the statistical mechanics, the entropy can be derived from these microscopic states and surprisingly, the expression of entropy matches with the Bekeinstein-Hawking result.

Along with the afore-described development, another approach which gained significant interest in understanding black holes is through the idea of the asymptotic symmetries. Diffeomorphism plays a fundamental role in the theory of gravity. The intertwining connection between the thermodynamics of black hole and Noether charges corresponding to symmetries of the Killing horizon (i.e., a null hypersurface having Killing vector as null generators) in the diffeomorphism invariant gravity theory, was explored first in the seminal work of Wald [28, 54, 55].

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Most importantly, the symmetries near the horizon are identified with part of diffeomorphism symmetry, details of which will be discussed later. It was shown in [28, 54] by covariant phase space formalism that the Noether charge associated with the symmetries near the horizon of a black hole is directly connected with the Bekenstein-Hawking entropy. In addition, the laws of black hole mechanics were derived correctly from the Noether charge of the Killing horizon by Wald's prescription for both stationary and non-stationary perturbations [27, 28]. Moreover, it was found that for a general class of gravitational theory, Bekenstein-Hawking entropy does not hold, whereas Wald's formalism gives correct entropy for a wider class of gravity, for example, Lanczos-Lovelock theory of gravity.

Before Wald's prescription, an astonishing result was reported in 1985 by Brown and Henneaux [56] relating gravity and conformal field theories [57] living in one less dimension. The original proposal has been proposed in $(2+1)$ dimensional gravity theory with asymptotically anti-de Sitter spacetime. In this context, the conserved charges corresponding to the asymptotic symmetries at spatial infinity, were shown to satisfy the conformal algebra associated with the field theory on the boundary of AdS space. The Fourier mode $\mathcal{L}_m$ of the asymptotic symmetry generators in AdS_3 spacetime obey Virasoro algebra having a central extension,

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{1.6}$$

Here c is known as the central charge expressed as,

$$c = \frac{3}{2G\sqrt{-\Lambda}} \tag{1.7}$$

where G is the gravitational constant and Λ is cosmological constant. It is well known that the central charge plays a vital role in determining the degrees of freedom entertained by the system which is conformally invariant (for more detail, see [57]). Also, the central charge specifies the density of states, and thus the free energy and entropy of the system can be computed from the partition functions using the laws of the statistical mechanics. Meanwhile, in $(2+1)$ dimensions, the black hole horizon is two-dimensional and therefore, the $2d$ conformal field theory technique can be applied for $3d$ black hole. Later inspired by the work of Brown and Henneaux, in [58] Strominger has explored more in this direction. He showed that the central charge (1.7) could be incorporated in the statistical formula of Cardy, which gives the entropy from the density of states as follows,[59–61]

$$S \approx ln\rho \approx 2\pi\sqrt{\frac{c\Delta}{6}}. \tag{1.8}$$

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Here Δ is the zero-mode eigenvalue of the symmetry generators L_0 and can be calculated from the expression of the mass and the angular momentum of the BTZ black hole. Thus (1.8) produce the exact expression of Bekenstein-Hawking entropy in three-dimensional gravity theory microscopically. Later, Carlip [62, 63] generalized the above formulation for Killing horizon in an arbitrary spacetime dimension, there the diffeomorphism vectors are determined by imposing suitably defined boundary conditions near the horizon. Therefore following the previous arguments, one can construct the Virasoro subalgebra with a term which is the central charge. Then from this central charge, the correct Bekenstein-Hawkings entropy can be easily calculated using the Cardy formula. These results indicate that probably the microscopic description of the entropy can be obtained from these central charges and zero mode $\mathcal{L}_0$. In these endeavours, exploration of asymptotic symmetries near asymptotic null infinity as well as near the black hole horizon has gained profound importance among scientific communities over the years.

1.2 Review on asymptotic symmetries and BMS group

In this section, we introduce a detailed description of the asymptotic symmetry group widely studied in gravitational theories. The idea of asymptotic symmetries was first proposed near null infinities (both past and future) for asymptotically flat-four dimensional spacetime, back in 1962 by Bondi, van der Berg, Metzner and Sachs [64–67]. The primary interest was to resolve the problems of gravitational waves and radiation. More specifically, they tried to know the detailed mechanism behind the mass loss phenomenon when the gravitational waves propagate at a large distance from the source. At the time of analyzing the gravitational radiation, surprisingly, it was found that the inhomogeneous Lorentz group (viz. Poincaré group) does not represent the correct group of symmetries at the asymptotically flat region where the effect of gravitational fields are very weak. Rather an infinite-dimensional group emerges as the asymptotic symmetry group which was found to be the semi-direct product of usual Poincare symmetry and another infinite-dimensional symmetry transformation. The group of symmetries at asymptotically flat spacetime was at first named "the Generalized Bondi-Metzner" group. After the work of Sachs [67], the group was renamed as "Bondi-Metzner-Sachs" group or BMS group. With the discovery of a non-trivial infinite-dimensional group

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at asymptotic infinity, it was established that at large distances and for weak gravitational forces, general relativity does not simply reduce to special relativity.

As described in [64] [65], in the neighbourhood of the future null infinity of asymptotically flat spacetime, the line element ds^2 in the special kind of coordinate system (u, r, θ, ϕ) (detail in [65]) takes the following form in four spacetime dimension,

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - e^{2\beta}dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du) \tag{1.9}$$

where, u represents the retarded time coordinate and u = constant hypersurfaces are null. r is the radial coordinate and also acts as the affine parameter for the null geodesics. x^A=(θ, ϕ) and $h_{AB} = e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2$, such that $Det[h_{AB}] = \sin^2\theta$. These particular coordinate system was later known as the *Bondi-Sachs coordinates*. Here V, β, U^A and h_{AB} are arbitrary six functions of the coordinates whose asymptotic forms are determined by solving linearized Einstein field equations [64] [65]. At first the four components of Bianchi identites $\nabla^b G_{bc} = 0$ are thoroughly analyzed and the results directly yield that the vacuum Einstein equations $R_{ab} = 0$ can be split into three parts.

- The one part consists of six main equations given as:

$$R_{rr} = 0; \quad R_{rA} = 0; \quad R_{AB} = 0. \tag{1.10}$$

- $R_{ur} = 0$ is trivially satisfied due to the symmetry property of the metric.

- The remaining components of R_{ab} will provide the non-trivial constraints,

$$R_{uu} = 0; \quad R_{uA} = 0. \tag{1.11}$$

In linearized gravity theory the asymptotic solutions of the field equations were derived in [65, 66] by analyzing the behaviour of the gravitational waves far away from the source. However it is well known that the gravitational radiation must follow Sommerfeld's boundary condition at null infinities so that the inward travelling wave is eliminated and the principle of causality is maintained (see [68, 69] and the references therein). In our present context Sommerfeld's outgoing radiation condition says that $[\partial(r\gamma)/\partial r]|_{u=const} \to 0$ as $r \to \infty$. To satisfy this conditon, the asymptotic behaviour of γ must be ,

$$\gamma = \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{c_3}{r^3} + \ldots . \tag{1.12}$$

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Here all c_i are functions of (u, x^A).

Now to obtain the detailed asymptotic behaviour of the other metric components, the main equations (1.10) are solved algebrically for the metric ansatz (1.9). For this reason it is useful to introduce the tetrad system of four null vectors m^a, k^a, t^a and $\bar{t}^a$ for the null hypersurfaces $u = constant$ (more detail on the null tetrad can be found in [65, 70, 71]), such that the following conditions are satisfied,

$$m^a k_a = t^a \bar{t}_a = -1; \quad m^a m_a = k^a k_a = t^a t_a = \bar{t}^a \bar{t}_a = 0. \tag{1.13}$$

The components of these vectors can be obtained such that the conditions (1.13) are satisfied. Now the main equations (1.10) can be split again in terms of the three hypersurface and one standard equation as follows [64],

$$R_{ab}k^a k^b = \partial_r \beta - \frac{1}{2} r (\partial_r \gamma)^2 = 0; \tag{1.14}$$

$$R_{ab}k^a t^b = \partial_r \left(r^4 + e^{2(\gamma-\beta)} \partial_r U \right) - 2r^2 [(\partial_r \partial_\theta \beta)(\partial_r \partial_\theta \gamma) + 2\partial_r \gamma \partial_\theta \gamma - 2\frac{\partial_r \beta}{r}$$
$$-2\cot\theta \partial_r \gamma] = 0. \tag{1.15}$$

$$R_{ab}t^a \bar{t}^b = 2\partial_r V + \frac{1}{2} r^4 e^{2(\gamma-\beta)} (\partial_r U)^2 - r\partial_\theta (r\partial_r U + 4U) - r\cot\theta (r\partial_r U + 4U)$$
$$+2e^{-2(\gamma-\beta)}[-1 - \cot\theta \partial_\theta (2\gamma - \beta) - \partial_\theta^2 (\gamma - \beta) + (\partial_\theta \beta)^2 + 2\partial_\theta \gamma \partial_\theta (\gamma - \beta)] = 0 \tag{1.16}$$

$$R_{ab}t^a t^b = 2r\partial_u \partial_r (r\gamma) + (1 - r\partial_r \gamma)(\partial_r V - r\partial_\theta U) - (r\partial_r^2 \gamma + \partial_r \gamma) V - r^2 (\cot\theta$$
$$-\partial_\theta \gamma)\partial_r U + r\partial_\theta (2r\partial_r \gamma - 2\gamma) U + r\cot\theta U (r\partial_r \gamma - 3) + e^{-2(\gamma-\beta)}[-1$$
$$- \cot\theta \partial_\theta (3\gamma - 2\beta) - \partial_\theta^2 \gamma + 2\partial_\theta \gamma \partial_\theta (\gamma - \beta)] = 0. \tag{1.17}$$

These equations are solved for an axially symmetric isolated system having $U^\theta = U$ and $U^\phi = 0$. Now having the asymptotic behaviour of γ given in (1.12), one can solve (1.14) to get the form of β. Then equations (1.15) and (1.16) can determine the behaviour of U and V respectively. However Eqn. (1.17) gives the derivative of γ w.r.t the coordinate u and from the solution of this equation, the specific form of γ can be obtained. Then with this form of γ, one can again solve the above equations (1.14-1.16) to get the forms of the other metric coefficients. Finally, for an asymptotically flat spacetime, the asymptotic solutions of the metric components yield as follows,

$$h_{AB} = \bar{\gamma}_{AB} + \frac{c_{AB}}{r} + \mathcal{O}(1/r^2); \quad \frac{V}{r} = -1 + \frac{2M(u,\theta,\phi)}{r} + \mathcal{O}(1/r^2); \tag{1.18}$$
$$\beta = -\frac{c_1(u,\theta,\phi)c_1^*}{4r^2} + \mathcal{O}(1/r^4); \quad U^A = -\frac{\nabla_B c^{BA}}{r^2} + \frac{N^A(u,\theta,\phi)}{r^3} + \mathcal{O}(1/r^4).$$

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Here $\bar{\gamma}_{AB}$ is the metric on unit two-sphere. $M(u,\theta,\phi)$ and $N^A(u,\theta,\phi)$ are two arbitrary unknown integrating constants. Now for the solutions (1.18), the trivial equation is automatically satisfied and the three supplementary equations (1.11) generate the constraint relations between the form of the arbitrary functions $M(u,\theta,\phi)$ and $N^A(u,\theta,\phi)$ as follows;

$$\partial_u N_A = \partial_A M + \frac{1}{16}\partial_A(c_{BC}\partial_u c^{BC}) - \frac{1}{4}\partial_u c_{BC}\nabla_A c^{BC} - \frac{1}{4}\nabla_B(\nabla^B\nabla^C c_{AC}$$
$$\nabla_A\nabla_C c^{BC} + c^{BC}\partial_u c_{AC} - c_{AC}\partial_u c^{CB}),$$

and

$$\partial_u M = \frac{1}{4}\nabla_A\nabla_B(\partial_u c^{AB}) - \frac{1}{8}\partial_u c^{AB}c_{AB}. \tag{1.19}$$

The function M is known as *Bondi Mass aspect* whereas N^A is known as *angular momentum aspect* (Detail can be found in [64] [65]) . Therefore for (1.18), the metric (1.9) reduces to the flat Minkowki metric at large value of r.

In the seminal papers [64, 65] Bondi and Metzner analyzed and found out a set of coordinate transformations from (u,θ,ϕ) to (u',θ',ϕ'), such that the asymptotic behaviour of the solutions of the field equations (given in (1.18)) are preserved under those transformations. In other words , one can say that after those coordinate transformations, the transformed metric expressed in new coordinate system should also be Minkowski at the limit $r \to \infty$ and spacetime must remain Ricci flat at null infinities. The general form of the aforementioned transformations of coordinates are given by,

$$\theta' = \theta + H(\theta,\phi);\;\; \phi' = \phi + K(\theta,\phi);\;\; u' = L^{-1}[u + \alpha(\theta,\phi)] \tag{1.20}$$

Here $H(\theta,\phi)$ and $K(\theta,\phi)$ are two arbitrary functions which are responsible for the conformal transformation of the metric of the unit two-sphere i.e. $ds^2_{2D} = L^2(\theta',\phi')ds'^2_{2D}$. Here L is the conformal factor. Following the reference [67] the transformations of coordinates found in (1.20) are called Generalized Bondi-Metzner transformation, and these transformations form the "Generalized Bondi-Metzner" (GBM) group which has the flat Poincaré group as the subgroup. This GBM group consists of the semi-direct product of the conformal transformation on the unit two-sphere with the time translation given by α, which is an arbitrary function of angular coordinates and also twice differentiable. For the specific case, when there is no transformation of the angular coordinates (i.e., $\theta = \theta'$ and $\phi = \phi'$), this special type of time translation was termed as the *supertranslation*.

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It was proved that supertranslation is a normal subgroup of the asymptotic symmetry group. So having the spherical harmonics expansion of the parameters, the supertranslation subgroup was found to be infinite-dimensional. Moreover supertranslation reduces to usual four-dimensional Poincaré translation for a particular form of α, being the linear superposition of $l = 0$ and $l = 1$ spherical harmonics. This result immediately implies that the supertranslations form the more generalized version of the translation group.

In contrary to the aforesaid finite transformation, Bondi in his pioneering paper [65] analyzed and explored the infinitesimal asymptotic symmetry transformations near the null infinity. This attempt was inspired by the idea of the isometries of spacetime which has Killing symmetries. So the infinitesimal transformations of spacetime coordinates (also known as *diffeomorphism*) are given by $x'^a(\beta, x^c) = x'^a(0, x^c) + (\partial x'^a/\partial\beta)|_{\beta=0} = x^a + \zeta^a$, where x^a denotes four coordinates (viz. u, r, θ, ϕ) and ζ^a is any contravariant vector. Then the infinitesimal change of the metric tensor can be defined as the Lie derivative of that metric along the direction ζ^a,

$$\delta g_{ab} \equiv \pounds_\zeta g_{ab} = \zeta^a \partial_a g_{ab} + g_{ac}\partial_b \zeta^c + g_{bc}\partial_a \zeta^c. \tag{1.21}$$

After the infinitesimal variation is obtained, a set of conditions has been imposed on these variations so that the asymptotic form of the solutions (1.18) remain unaltered near the null infinity. The proposed conditions are,

$$\begin{aligned} &\delta g_{rr} = 0; \;\; \delta g_{rA} = 0; \;\; \delta g_{AB}\, g^{AB} = 0. \\ &\delta g_{uu} \approx \mathcal{O}(1/r); \;\; \delta g_{uA} \approx \mathcal{O}(1); \;\; \delta g_{ur} \approx \mathcal{O}(1/r^2); \delta g_{AB} \approx \mathcal{O}(r). \end{aligned} \tag{1.22}$$

Now we try to find out the physical arguments of writing these conditions given in (1.22). The first two conditions on g_{rr} and g_{rA} are imposed as the original metric (1.9) which is written in Bondi gauges, does not have these components. So to preserve the asymptotic structure of all the metric coefficients near the null infinity, after the infinitesimal transformations, the components g'_{rr} and g'_{rA} of the transformed metric should also vanish, i.e., variation must be considered to be zero. On the other hand, the determinant of the metric must be invariant even after the given transformations. So we can write that $\delta g/g = \delta g_{AB}\, g^{AB}$ is zero. These three conditions in the first line of (1.22) are the *gauge choices*. Next, we concentrate on the second line of (1.22). Following (1.18), the asymptotic behaviour of g_{uu} says

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that

$$g_{uu} = e^{2\beta}\frac{V}{r} + r^2 h_{AB}U^A U^B$$
$$\approx (1 + \frac{c_1(u,\theta,\phi)c_1^*}{2r^2})[-1 + \frac{2M(u,\theta,\phi)}{r}] + \mathcal{O}(\frac{1}{r^2}) \approx -1 + \mathcal{O}(\frac{1}{r}). \quad (1.23)$$

In the above expansion, the leading order term corresponds to the flat Minkowski metric and the subleading term yields the fall-off behaviour of this component as the radial coordinate, $r \to \infty$. Now the infinitesimally transformed metric must follow the same asymptotic condition as the original one, to satisfy the symmetry arguments. This yields the first condition given in the second line of (1.22). The other three conditions in the second line of (1.22) can be easily derived in a similar way such that the asymptotic forms of the metric solution are kept unchanged. These conditions in the second line of (1.22) are known as the *fall-off conditions* as they are determined by the fall-off behaviour of the Bondi metric. Therefore all these infinitesimal transformations given in (1.22), constitute the infinitesimal GBM group, later known as *BMS group*. Now one can easily solve the gauge choices in (1.22) to get the components of ζ^a as follows,

$$\zeta^u = T(u,x^A); \;\; \zeta^A = Y^A(u,x^B) - \partial_B T \int e^{2\beta} g^{AB} dr;$$
$$\zeta^r = -\frac{r}{2}[\nabla_A Y^A - U^C \partial_C T]. \quad (1.24)$$

Here T and R^A are arbitrary constants of integration. However the fall-off conditions in (1.22) imposes extra restriction on these two functions and thus obtain,

$$T(u,x^A) = F(x^A) + \frac{u}{2}\nabla_C Y^C; \;\; \partial_u Y^A = 0;$$
$$\pounds_Y(\bar{\gamma}_{AB}) = \nabla_C Y^C \bar{\gamma}_{AB}. \quad (1.25)$$

The last one denotes the conformal transformation of the two-sphere metric $\bar{\gamma}_{AB}$. Therefore we can say that the equations (1.24) and (1.25) generate the infinitesimal coordinate transformation which are analogous to the finite transformation (1.20). Thus these transformations construct the BMS group. Like the finite transformations, in the present context, F is the parameter for the *supertranslation* which is a function of angles and generates infinitesimal symmetry transformation along the time direction. However, another parameter Y^A did not get any such nomenclature in the earlier work [65]. Then by having the expansion of the parameters in terms of spherical harmonic modes, the standard Lie algebra between the symmetry vectors can be constructed. The resultant asymptotic symmetry algebra comes out

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to be the semi-direct sum between globally defined conformal algebra on the unit two-sphere and the supertranslation which is infinite-dimensional. This global conformal algebra is also isomorphic with the homogeneous Lorentz algebra.

Recently Barnich and his collaborators [72–74] have relooked at the well-known BMS group found near null infinity and found a new version of it, by incorporating the ideas of the infinitesimal *local conformal transformation in two dimensions*. In this scenario, those infinitesimal transformations were considered which keep, not only the structure of the metric invariant near asymptotic region but also conformally rescale the metric component g_{AB}. Then the associated symmetry parameter F and Y^A, expressed in complex stereographic coordinates $(z, \bar{z})$, are shown to be the meromorphic functions, which may not be regular at all points on the Riemann sphere. However, those singularities can be avoided by considering local regions of spacetime for the symmetry analysis. So by having the Laurent expansion of the parameter Y^z as $Y^z_n = z^{n+1}$ for any integer n, the infinitesimal local conformal group was generated. The parameters Y^z and $Y^{\bar{z}}$ are named as *superrotations* [74] which are indeed infinite-dimensional. Also, the algebra of the aforesaid BMS group consists of the semi-direct sum of the local conformal transformations with the infinite-dimensional supertranslations denoted by F. So we can expand the parameters in the Laurent series as, $F_{mn} = z^m \bar{z}^n$ and also $Y^{\bar{z}}_n = \bar{z}^{n+1}$. Now the standard definition of Lie algebra between two vectors ζ_1 and ζ_2 is given by,

$$[\zeta_1, \zeta_2]^x = \zeta_1^a \partial_a \zeta_2^x - \zeta_2^a \partial_a \zeta_1^x \,. \tag{1.26}$$

With the help of this definition and for the Laurent mode expansion given above, one can compute the BMS algebra between symmetry parameters as,

$$\begin{aligned} &[Y^z_m, Y^z_n] = (m-n)Y^z_{m+n}. \;\; [Y^{\bar{z}}_m, Y^{\bar{z}}_n] = (m-n)Y^{\bar{z}}_{m+n}; \;\; [Y^z_m, Y^{\bar{z}}_n] = 0; \\ &[Y^z_m, F_{nk}] = (\frac{m+1}{2} - n)F_{m+n\;k}; \;\; [Y^{\bar{z}}_m, F_{nk}] = (\frac{m+1}{2} - n)F_{m\;n+k} \end{aligned} \tag{1.27}$$

The commutation relations in the first line are the Witt algebra (two copies) which is actually the Virasoro algebra without the central extension. The supertranslation and superrotation generators come out to be non-commutative. Also, the Poincarë algebra which is spanned by the ten number of generators in four dimensions, can be shown to be the subalgebra of this BMS algebra. Contrary to the original work by Bondi et al., this local version of BMS algebra has wider relevance with other works in this direction, for instance, with the work of Brown and Henneaux [56] which we have discussed in the last section. Specifically it can be proved that BMS_3 charge algebra [74] has non-zero central extension term (as shown in (1.7) and (1.8)) which is exactly same as obtained in [56].

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Noether's theorem says that, for every continuous symmetry, there must be associated conserved quantities. Various mechanisms can be found in the standard literature to obtain the conserved charges and current associated with asymptotic symmetries at the null boundaries. However, the Noether charges associated with BMS transformations at the null infinities are not conserved in general due to the outgoing gravitational radiation flux. Iyer-Wald formalism [55] first presented a general prescription for calculating those "conserved quantities" for the asymptotic boundaries of spacetime following the covariant phase space method for the diffeomorphism invariant Lagrangian. There are other important approaches to derive conserved charges near the spacetime boundaries, such as Lagrangian Noether methods [75–77], and also quasi-local techniques [78] and conformal methods (see [79, 80] and the references therein). Because of the universal properties of the surface charges in linearized theory, Barnich in [81, 82] developed the covariant approach to calculate surface charges for asymptotically flat spacetime using the cohomological technique originally used in [83]. This approach has a particular physical interest in the case of the black hole whose geometry is asymptotically AdS_3. In this scenario, the central charges appear in the algebra of surface charges, and this central extension plays a pivotal role in determining the macroscopic black hole entropy. Most recently it has been found out [84–88] that two non-trivial sets of the infinite number of conserved charges corresponding to BMS symmetries appear in the context of classical gravitational scattering between past null infinity $\mathcal{I}^-$ and future null infinity $\mathcal{I}^+$ of asymptotically flat spacetime (see the diagram (1.1)). One set of those charges is known as the conserved *supertranslation charges* and another one is the conserved *superrotation charges*. Thus an asymptotically flat black hole can carry these infinite number of conserved charges identified as *soft hair* [84, 85, 89], which come in addition to the other well-known conserved quantities of the black hole i.e energy, linear momentum, intrinsic angular momentum and boost charges. It is hoped that these newly found supertranslation and superrotation hair may shed light on the information loss paradox of the black hole.

Most interestingly, the Killing horizon of the static or stationary black hole act as another null boundary of spacetime and asymptotic symmetry analysis has been investigated there (see [62, 63, 90–96] and other references thereafter). In particular, Carlip in [62, 63] constructed the Virasoro algebra between the asymptotic symmetry generators at the horizon boundary in connection with the microscopic states responsible for the Bekenstein-Hawking entropy. This part has been discussed in

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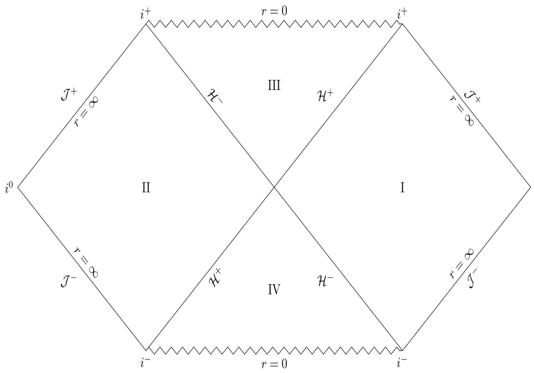

FIGURE 1.1: Penrose diagram of the Schwarzschild black hole. In the diagram the past null infinity is denoted by $\mathcal{I}^-$ and future nul infinity is $\mathcal{I}^+$. spacelike infinity i^0 and timelike infinities are i^- and i^+. Similarly past and future horizon are $\mathcal{H}^-$ and $\mathcal{H}^+$ respectively. The black hole region is denoted by III and $r = 0$ is the physical singularity.

the previous section. The near horizon symmetry analysis was further impoverished in [92, 97] where the zero mode of the conserved charges and central charge were derived in a more concrete way to obtain the Bekenstein-Hawking result. Recently in [91] both the infinite-dimensional supertranslation and superrotation symmetries were shown to appear near the horizon. The symmetry arguments require that under the subset of diffeomorphism, the structure of the metric (expressed in Gaussian null coordinates) must be preserved near the horizon. It has been found out that the form of the near horizon BMS_4 algebra [91] is different from the standard BMS_4 algebra derived at null infinity [74]. Also, the zero mode of the conserved quantities corresponding to the near horizon symmetries have been shown to exactly reproduce the entropy and the angular momentum of the black hole. We will discuss in detail the charge calculation as we go along. Nevertheless, it was also proved that a slightly different set of boundary conditions which are consistent with the near horizon symmetries, lead to Heisenberg algebra [98–103]. The associated conserved charges have been shown to provide the 'soft hair' to the black hole, but only the zero mode of those charges determines the horizon entropy. Later this approach of near horizon symmetry analysis was further extended for a generic null surface [104–114] which may not be the solution of field equations. In any spacetime, the null hypersurface is perceived as the one-way membrane (i.e., information cannot pass through it) by a specific family of observers (for instance, Rindler observers) who may be situated at a fixed spatial distance from this surface (see e.g., chapter 8 of [115]). The entropy can be associated with this null surface locally with the help of the Virasoro algebra

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and central charge formalism by choosing the appropriate diffeomorphism vector which keeps intact the near null surface geometries. The results obtained in this case are more general in all aspects. Now one important physical interpretation of the BMS conserved quantities was established by the relation between asymptotic symmetries and membrane paradigm [116]. By virtue of this paradigm, the null boundaries of spacetime (i.e., both null infinity and black hole horizon) behave as the lower dimensional viscous fluids [117–121]. Therefore the infinite set of conserved charges corresponding to fluid dynamics of the membrane can be shown to be equivalent to BMS conserved quantities.

It should be mentioned that the asymptotic symmetry analysis has been further explored and extended near null infinities of asymptotically AdS and dS spacetime [55, 122–126] and for gauge theories including Maxwell, Yang-Mills and Chern-Simons theories [127–134] and also near other two distinct regimes of spacetime-timelike and spacelike infinity, based on Hamiltonian (ADM) approach [135–139]. However, one of the most astonishing and important observations in this context is the deep connection of the asymptotic symmetries with other seemingly unrelated areas of physics [85] viz. infrared phenomena found in QFT and gravitational memory effects and also holographic principles. Infrared physics is based on the soft theorems, which reveal the interesting features of Feynman diagrams and scattering amplitudes when the massless external particles carry zero energy (i.e., *soft*). The soft theorems were introduced in 1937 by Bloch and Nordsieck [140], then explored further in [141–143]. However, Weinberg [144] first presented soft photon and graviton theorems in a concrete way and thus generalized the theorem for gravitational theory. Soft theorems state that the infinite number of soft particles (maybe photons, gravitons, or pions) having energy and momentum zero, is produced during any quantum scattering process and thus resolve the infrared divergence problem found in the quantum theory of electrodynamics and gravity. Recently it was found out that because of the BMS invariance of the S-matrix of the quantum gravity, the soft graviton theorem can directly reproduce the Ward identities associated with supertranslatons [86–88, 145–150]. Similarly, equivalence has been established between Weinberg's soft photon theorem and large gauge symmetries of QED [145, 151]. On the other hand, the BMS transformations near infinity as well as near the horizon are shown to be closely connected with the gravitational memory effect [147, 152, 153]. Also, BMS symmetries can be realised as one important example of the bulk-boundary correspondence known as the holographic duality [154–159]. Precisely holographic dictionary relates gauge

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symmetries of the bulk theory with the global symmetries on the boundary theory, and thus asymptotic symmetries of the bulk solutions act as the global symmetries. This correspondence has not been firmly established yet, but there are some recent successful attempts in this direction [160–163].

So far, we have discussed the various features of asymptotic symmetries and their connection with the different realms of classical and quantum theories of gravity and also with field theories. It is quite evident that BMS symmetries can be explored further in some specific directions. Motivated by this, in the present thesis, we will investigate more elaborately some of those topics. At first, our primary focus will be on the asymptotic symmetries near a generic null hypersurface. It is well known that the thermodynamics properties of the null surface encourage to think the gravitational theories to be 'an emergent phenomena' (we will discuss this regard in chapter 2 and also more elaborate discussion can be found in [108, 111]). Although the symmetries of a generic null surface have been analyzed extensively in the standard literature for not only Einstein gravity but also for the higher derivative gravity model, however, it will be illuminating to consider the generic null surface which may have electromagnetic charges. Therefore, in this case, symmetry analysis will be more interesting for the unified gauge-gravity theory and in this thesis, we have tried to explore in this direction. On the other hand, generally, the explicit forms of the near horizon symmetry parameters are determined by having Fourier modes decomposition depending on the periodicity of the time coordinate, and corresponding zero modes are shown to contribute to the horizon entropy. Nevertheless, in this thesis, we will try to obtain the other possible solution of these parameters by exploring another important physical phenomenon, i.e., the spontaneous symmetry breaking by the background solutions. Although the appearance of the Goldstone modes near the black hole horizon has already been discussed by the quantum analysis in [164, 165], its dynamics have not been studied in a concrete way and this is one of the main goal in the present thesis. Other than the null surface, also we have focussed on the symmetries near a timelike hypersurface which is assumed to act as the boundary situated in the bulk spacetime. In the upcoming section, we have briefly presented the chapter-wise overview of the thesis.

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1.3 Chapter-wise description of the thesis

The results discussed in this thesis are based on the works in [166–169]. In this section, we have summarized the research guidelines of the thesis as follows,

1.3.1 Chapter 2:

To understand the underlying degrees of freedom, near horizon symmetry analysis of a black hole has gained significant interest in the recent past. In this chapter, we generalize that analysis first by taking into account a generic null surface carrying $U(1)$ electromagnetic charge. With the appropriate boundary conditions near the surface under study, we identify the symmetry algebra among the subset of diffeomorphism and gauge generators which preserve the metric of the null surface and the form of the gauge field configuration. With the knowledge of those symmetries, we further derive the algebra among the associated charges considering general Lanczos-Lovelock gravity theory and also non-linear $U(1)$ gauge theory. Importantly while computing the charges, not only we consider general theory, but also used off-shell formalism which is believed to play crucial role in understanding quantum gravity. Both the nonextremal and extremal cases are addressed here.

1.3.2 Chapter 3:

For a long time, it is believed that black hole horizon are thermal and quantum mechanical in nature. The microscopic origin of this thermality is the main question behind our present investigation in this chapter, which reveals the possible importance of near horizon symmetry. It is this symmetry which is assumed to be spontaneously broken by the background spacetime, generates the associated Goldstone modes. In this chapter, we construct a suitable classical action for those Goldstone modes and show that all the momentum modes experience nearly the same inverted harmonic potential, leading to instability. Thanks to the recent conjectures on the chaos and thermal quantum system, particularly in the context of an inverted harmonic oscillator system. Going into the quantum regime, the system of a large number of Goldstone modes with the aforementioned instability is shown to be inherently thermal. Interestingly the temperature of the system also turns out to be proportional to that of the well-known horizon temperature. Therefore, we hope that our present study in this chapter can illuminate an intimate

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connection between the horizon symmetries and the associated Goldstone modes as a possible mechanism of the microscopic origin of the horizon thermality.

1.3.3 Chapter 4:

In the previous chapter, we have discussed how the near horizon Bondi-Metzner-Sachs (BMS) like symmetry is spontaneously broken by the black hole background itself and hence gives rise to Goldstone mode. The associated Goldstone mode for the near horizon BMS-like symmetry of a Schwarzschild black hole was found to behave like inverted harmonic oscillators, which has been further shown to lead to thermodynamic temperature in the semi-classical regime. Here we investigate the generalization of these previous findings for the Kerr black hole. The analysis is being performed for two different situations. Firstly, we analyze Goldstone mode dynamics considering slowly rotating Kerr. In another case, the problem is solved in the frame of zero angular momentum observer (ZAMO) with an arbitrary value of rotation. In both analyses, the effective semi-classical temperature of Goldstone modes turns out to be proportional to that of Hawking temperature. Due to such similarity and generality, we feel that these Goldstone modes may play an important role in understanding the underlying microscopic description of horizon thermalization.

1.3.4 Chapter 5:

Recently symmetries of gravity and gauge fields in the asymptotic regions of spacetime have been shown to play a vital role in their low energy scattering phenomena. Further, for the black hole spacetime, near horizon symmetry has been observed to play a possible role in understanding the underlying degrees of freedom for thermodynamic behaviour of horizon. Following the similar idea, in this chapter, we analyzed the symmetry and associated algebra near a time-like surface that is situated at any arbitrary radial position and is embedded in black hole spacetime. Here we consider both Schwarzschild and Kerr black hole spacetimes. The families of hypersurfaces with constant radial coordinate (outside the horizon) in these spacetimes are timelike in nature and divide the space into two distinct regions. The symmetry algebra turned out to be reminiscent of Bondi-Metzner-Sach (BMS) symmetries found in the asymptotic null boundaries.

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1.3.5 Chapter 6:

Here we have presented the conclusion of the thesis and also discussed some interesting scope for future work.

From the next chapter onwards, we have the detail analysis of the thesis. Each chapter of the thesis contains several appendices that are added at the end of the respective chapter. Here we will consider the signature of the Lorentzian metric to be $(-,+,+,+)$.

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Symmetries near a generic charged null surface and associated algebra: An off-shell analysis 2

2.1 Introduction

In the introduction chapter of the present thesis, we have widely discussed that for a generic diffeomorphism invariant gravity theory, the Noether current and charge are very important in understanding the thermodynamic properties of black holes [28, 54, 55, 61–63, 97, 104, 105]. In this regard, one of the significant results is the commutator algebra among the charges associated with the asymptotic symmetries of spacetime under study. Those generally lead to Virasoro algebra with a central charge [56]. This central charge is found to be intimately connected with the entropy of black holes through well known Cardy formula [58].

The asymptotic symmetries near the null infinities of asymptotically flat spacetime and the horizon lead to an infinite-dimensional BMS group, which is a semidirect product of usual Poincare symmetry and the infinite-dimensional supertranslation symmetry transformation. Later this idea has been extended to different situations; among them, one crucial extension includes gauge fields in the exploration of boundary symmetries [66, 129–132]. For instance, the elaborate symmetry structure of three-dimensional Einstein-Maxwell systems with non-trivial asymptotics at null infinity has been explored in [131] which leads to Virasoro-Kac-Moody algebra, which is an extension of BMS_3 algebra of pure

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gravitational case. In [132] the analysis for gravity in four spacetime dimensions at null infinity has been extended to include Yang-Mills fields. Most recently, in [170] the unified treatment of asymptotic symmetries for the Einstein-Maxwell system has been discussed for Kerr-Newman (A)dS black hole horizon. Therefore asymptotic symmetry analysis for Einstein-Maxwell theory was extended and explored near a static or stationary horizon (see [91, 171, 172] for different cases), which are solutions of Einstein's equations of motion. The general strategy is to choose a subset of diffeomorphism such that under those transformations, the solutions of the field equation must remain invariant near the null boundaries.

So far, the boundary diffeomorphism symmetries have been explored at first near null infinity for asymptotically flat spacetime, later at the black hole horizon, which acts as another null boundary of spacetime. However, one can have a generic null surface that will serve as a horizon for a class of observers in any spacetime. It has been observed that not only black hole horizon has a thermodynamics interpretation, but also any generic null surface in gravity theory has this property [108]. The idea stems from the equivalence principle - locally, an accelerated frame known as the Rindler frame can mimic gravity. Hence, it can be a good candidate for exploring various properties of gravity. Therefore, an accelerated observer in flat spacetime background is equivalent to a static observer in curved spacetime. This stimulates us to think the gravity as an "emergent phenomenon" [111]. Nevertheless, from the above discussions, one would tend to believe that understanding the behavior of a generic null surface not only can provide the desired results of on-shell properties of the theory under study but also can shed light on the off-shell behavior which naturally appears in quantum theory.

2.2 Objective of the chapter

In this chapter, our aim is twofold. Firstly we study asymptotic symmetries and the associated algebra, which keep a generic charged null surface invariant. In this analysis, we look at generic non-extremal and extremal charged null surfaces separately. To keep our study even more general while calculating various charges and their algebra, we also consider the Lanczos-Lovelock (LL) theory of gravity [173] in the presence of an arbitrary $U(1)$ gauge field. This will help us understand the properties of charges while incorporating the interactions among the fields. Second, we will do our whole analysis off-shell. By this, we mean that neither

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Einstein's equations of motion nor the gauge field equation of motion will be used in our final results. Let us point out that this has not been looked at in earlier work.

We choose a generic null surface in the presence of gauge fields for our discussion. Imposing the relevant fall-off conditions for the metric coefficients and gauge fields, which asymptotically preserves the null structure, we find the associated diffeomorphism and gauge symmetry transformations. Then the algebra of the corresponding Fourier modes is obtained. The computed charges for the diffeomorphism and gauge symmetries are off-shell. We will also calculate the associated symmetry algebra for transformation parameters, an arbitrary function of null surface coordinates. It is evident that our analysis will be completely off-shell not only by choice of metric but also by the derivation of charges as nowhere the information of equations of motion is being used. Also, our results will be valid for any order LL gravity in the presence of $U(1)$ gauge field. Here we will consider both non-extremal and extremal situations. Hence, we demand that our present analysis be more general and reflect the properties of a wide class of theories.

2.3 Null surface with $U(1)$ charge : a brief overview

In this section, we shall briefly discuss about the relevant properties of a generic charged null-hypersurface in an arbitrary spacetime dimension. For the description of the null surface we consider well known Gaussian null coordinate as (u, r, x), with $x = \{x^A\}$, $A = 2, 3, \ldots, d$ where A corresponds to the different angular coordinates and the number of spacetime dimension is $d+1$. The line element in this coordinate system is expressed as [174, 175],

$$ds^2 = \quad M(u,r,x)du^2 + 2dudr + 2h_A(u,r,x)dx^A du + \mu_{AB}dx^A dx^B \,, \tag{2.1}$$

where, we assume the null surface to be located at $r = 0$. The behavior of the metric components near the null surface are assumed as,

$$\begin{aligned} M(u,r,x) &= \quad -2\alpha(u,x)r + \mathcal{O}(r^2)\,; \\ h_A(u,r,x) &= \quad -r\beta_A(u,x) + \mathcal{O}(r^2)\,; \\ \mu_{AB}(u,r,x) &= \quad \mu^{(0)}_{AB}(u,x) + 2\mu^{(1)}_{AB}(u,x)r + \mathcal{O}(r^2)\,. \end{aligned} \tag{2.2}$$

Here metric on the two surface (where $u =$ constant and $r = 0$) is represented by $\mu^{(0)}_{AB}$. At this point, let us point out that the leading order behavior for g_{uu} has been chosen for the non-extremal null surface. We will discuss extremal cases separately

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at the end. As has been emphasized in the introduction of this chapter, we will consider the most generic null surface with all the metric components M, h_A, and μ_{AB} being functions of all the spacetime coordinates (u, r, x). Therefore by choosing this particular coordinate system, all redundant gauge degrees of freedom are eliminated and left only $d(d-1)/2$ free functions in the metric tensor as given by M, h_A, and μ_{AB}. Moreover, we assume that the null surface is charged under the $U(1)$ gauge field A_a. Therefore, in general, all the metric components will also be a function of electric and magnetic charges. For a generic null surface we can define a null vector and its complimentary null vector $k^a = (1,0,0)$ and $l^a = (0,-1,0)$ respectively such that $g_{ab}l^a k^b = -1$ holds. For convenience we also mention here the covariant components of those two vectors as $k_a = (-2r\alpha, 1, -r\beta_A)$, and $l_a = (-1,0,0)$. We also can see that the $r = 0$ surface is a null $(d-2)$ dimensional sphere with an elementary surface area $d\Sigma_{ab} = -d^{(d-2)}x\sqrt{\mu}(k_a l_b - k_b l_a)$, where μ is the determinant of the induced metric on null surface [176].

For generic charged null surface, we also consider the following fall off conditions for the gauge field near the surface,

$$A_u = C^{(0)} + \mathcal{O}(r); \quad A_r = 0; \quad A_B = C_B^{(0)}(u, x) + \mathcal{O}(r)\,, \tag{2.3}$$

where the time component of the gauge field assumes a constant value $C^{(0)}$ to the leading order in r. The time component of the gauge field is generally identified as a scalar potential. Therefore, it must be a constant on a particular surface. We also choose the following gauge condition, $A_r = 0$. The choice of A_B needs further explanation. However, for Kerr-Newmann black hole the condition appears to be true. As has been observed in [177, 178], a static observer sitting outside the horizon will express the energy density of the electromagnetic field as $U = T_{uu}(1/2r\alpha)$, where T_{uu} is the outgoing null-null component of energy-momentum tensor. This component must be divergent as one approach towards the null horizon $r = 0$. Therefore it is sufficient to consider T_{uu} to be finite. Considering the explicit expression for the electromagnetic energy-momentum tensor one can immediately show that to the leading order in r, $A_B \approx C_B^{(0)}(u, x)$. A detailed discussion on this issue is given in Appendix 2.A.

As emphasized before, we study the near horizon symmetries of a generic charged null surface. Therefore, it generalizes a special class of null metric obtained from the near horizon expansion of the Kerr-Newmann black hole, which is stationary (the procedure can be followed from [179]). Here we are considering the case with all the metric coefficients to be depending on all spacetime coordinates. Moreover, the null metric Eq.(2.1) may not be the solution to Einstein's equations

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of motion. Therefore our analysis will be much more general and covers a wide class of spacetimes. In that respect, the present one differs from the analysis given in [91, 172].

2.4 Symmetries near the null surface

To understand the symmetry properties near a surface, the general approach is to define the appropriate fall-off conditions for the metric coefficients and the gauge field components. The suitable fall-off conditions are such that it keeps all the gauge choices intact and the remaining components of the metric and the gauge field assume the same form near the null surface $r = 0$ after the symmetry transformations. For the present purpose, we will simultaneously consider the symmetries under diffeomorphism and $U(1)$ gauge transformations. After the transformation, we solve the aforementioned boundary conditions and identify the appropriate generators and their algebraic properties.

Let us first concentrate on the boundary condition of metric coefficients. These boundary conditions can be divided into two categories [171, 172]. One category is related to the gauge fixing conditions which we call "strong" ones,

$$\pounds_\zeta g_{rr} = 0; \quad \pounds_\zeta g_{ur} = 0; \quad \pounds_\zeta g_{Ar} = 0\,, \tag{2.4}$$

while remaining conditions are the "weak" ones, and those are

$$\pounds_\zeta g_{uu} \approx \mathcal{O}(r); \quad \pounds_\zeta g_{uA} \approx \mathcal{O}(r); \quad \pounds_\zeta g_{AB} \approx \mathcal{O}(1)\,. \tag{2.5}$$

In the above $\pounds_\zeta$ denotes the Lie derivative along the vector ζ^a corresponding to the diffeomorphism $x^a \to x^a + \zeta^a$. These strong conditions say that the metric components, which are zero or constant, must remain unaltered after the diffeomorphism and the weak conditions come from the leading order behavior of the metric coefficients. Now as emphasized earlier, we also need to consider the behavior of the gauge field. The combined symmetry transformation of gauge and diffeomorphism will lead to the following transformation for the gauge field A_r and satisfy,

$$\delta A_r = \pounds_\zeta A_r + \partial_r \epsilon = 0\,, \tag{2.6}$$

while the other components must satisfy,

$$\delta A_u = \pounds_\zeta A_u + \partial_u \epsilon \approx \mathcal{O}(1); \quad \delta A_B = \pounds_\zeta A_B + \partial_B \epsilon \approx \mathcal{O}(1)\,. \tag{2.7}$$

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In the above ϵ is the $U(1)$ gauge transformation parameter. Our aim now is to find out the diffeomorphism vector ζ^a and the gauge parameter ϵ, which satisfy the above-imposed conditions. This can be done in the following way.

First we solve the strong conditions (2.4) to find different components of ζ^a and then we impose the weak conditions (2.5) on aforementioned solutions. From Eq.(2.4) we find

$$\begin{aligned}
\zeta^u &= F(u,x)\,; \\
\zeta^r &= T(u,x) - r\partial_u F - \partial_B F \int r\beta^B dr\,; \\
\zeta^A &= -\partial_B F \int \mu^{AB} dr + R^A(u,x)\,,
\end{aligned} \tag{2.8}$$

where F, T and R^A are the integration constants which are unknown at this moment (for a schematic derivation, please see Appendix 2.B). The weak conditions for the metric components g_{uu}, g_{vA} in Eq.(2.5) give us the following constraint on the diffeomorphism parameters derived in Eq. (2.8):

$$\begin{aligned}
&\partial_u T - \alpha(u,x)T = 0\,; \\
&\partial_A T - T(u,x)\beta_A(u,x) + \mu^{(0)}_{AB}\partial_u R^B = 0\,.
\end{aligned} \tag{2.9}$$

The fall-of condition on g_{AB} does not give any new condition (see Appendix 2.B for detail derivation). Our goal is to understand the symmetry properties of the null surface located at $r = 0$. Therefore, any transformation that changes the null surface's position should vanish as we approach the null surface. This implies the vanishing $T(u,x)$ from Eq.(2.8). Therefore, with the condition $T = 0$, we just mentioned all the constraints Eqs. (2.10) will automatically be satisfied provided the parameter R^A is independent of u. In a similar manner, the Eq.(2.6) solves for gauge parameter ϵ. To find ϵ, one has to use the components of ζ^a, which have been found out by the conditions on g_{ab}. Finally one finds the relevant diffeomorphism parameters as,

$$\begin{aligned}
&\zeta^v = F(u,x)\,; \\
&\zeta^r = -r\partial_u F - \partial_B F \int r\beta^B dr\,; \\
&\zeta^A = -\partial_B F \int \mu^{AB} dr + R^A(x)\,; \\
&\epsilon = E(u,x) + \int_r dr[A_B(\partial_C F)\mu^{BC}]\,.
\end{aligned} \tag{2.10}$$

Here E is an another integration constant. The above expressions are the components of the diffeomorphism symmetry vector which keep the null surface

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structure invariant near $r = 0$. Here the diffeomorphism parameter F is known as the super-translation, R^A is the super-rotation and ϵ is the supergauge parameter.

2.5 Algebra of the symmetry parameters

We are now interested to explore the algebra of the Fourier modes of the symmetry parameters very near to the null surface. In this case the non vanishing parameters of our importance are,

$$\zeta^u = F(u,x); \quad \zeta^A\partial_A = R^A(x)\partial_A; \quad \epsilon = E(u,x). \tag{2.11}$$

For generality we assume the differmorphism parameter R^A to be function of all the transverse coordinates namely $\zeta^\theta = R^\theta(x), \zeta^\phi = R^\phi(x)$. The Fourier modes of the components F, R^A, E are expressed in the following manner,

$$\begin{aligned}
\zeta^u &= F = \sum_{m,n} \frac{B_{mn}}{\alpha} e^{i(mau+\sum_A n\, x^A)} = \sum_{m,n} B_{mn}F_{mn}; \\
\zeta^A\partial_A &= R^A(x)\partial_A = \sum_k \bar{B}_k^A e^{i\sum_A k_A\, x^A}\partial_A = \sum_k \bar{B}_k^A R_A^k; \\
E &= \sum_{j,l} \bar{E}_{jl} e^{i(jau+\sum_A l_A\, x^A)} = \sum_{j,l} \bar{E}_{jl}E_{jl}.
\end{aligned} \tag{2.12}$$

where m, n, k, j and l are both positive and negative integers. B_{mn}, $\bar{B}_k^A$ and $\bar{E}_{jl}$ are the series expansion coefficient. α is a constant having dimension of inverse length. Hence the periodicity of the coordinate u is taken to be $2\pi/\alpha$. The associated symmetry algebra of the aforementioned Fourier modes will be coming from the Lie algebra satisfied by the various components of diffeomorphism vector ζ^a and the gauge parameter ϵ. However in some literatures [73, 180], the modified version of the Lie algebra has been used to compute the required bracket between the diffeomorphism vectors. So the definition of the modified Lie bracket among two vectors is given by (Detail derivation of this bracket can be found in Appendix 2.C),

$$[\zeta_1, \zeta_2]_M = [\zeta_1, \zeta_2] - \delta^g_{\zeta_1}\zeta_2 - \delta^g_{\zeta_2}\zeta_1\,, \tag{2.13}$$

where,

$$[\zeta_1, \zeta_2]^x = \zeta_1^a\partial_a\zeta_2^x - \zeta_2^a\partial_a\zeta_1^x\,. \tag{2.14}$$

Nevertheless, as discussed in Appendix 2.C, in the present context, we have used only the Lie bracket shown in Eq.(2.14), instead of the modified version of it. So

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the required symmetry algebra are found out as follows,

$$\begin{aligned} i[F_{mn}, F_{pq}] &= (m-p)\, F_{m+p\; n+q}; \quad i[R^m_A, R^n_{A'}] = m_{A'} R^{m+n}_A - n_A R^{m+n}_{A'}. \\ i[R^k_A, F_{mn}] &= -n_A\, F_{m\; n+k}; \quad i[F_{mn}, E_{jl}] = -j\, E_{m+j\; n+l} \\ i[R^m_A, E_{jl}] &= -l_A\, E_{j\; m+l}; \quad [E_{jl}, E_{mn}] = 0\,. \end{aligned} \tag{2.15}$$

Detailed computation of bracket is presented in Appendix 2.D.1. Here it is clear that superrotation vectors (R^n_A) are noncommutative for different angular variables. This is similar to the usual rotation algebra. Moreover, supertranslation vector $F_{m,n}$ is noncommutative with itself. This happens because of supertranslation generator F is a function of both space and time coordinates. In this sense, our analysis is the generalization of studies reported in [91]. In the subsequent section, we will calculate various charges and their associated algebra corresponding to the diffeomorphism and $U(1)$ gauge symmetry transformations which keep our generic charged null surface invariant.

2.6 Charge and its algebra: an off-shell analysis

In this section our aim is to find the algebra among the Fourier modes of the charges corresponding to the aforementioned diffeomorphism vector and gauge parameter. For both the cases we shall first identify the most general expression for the Noether charges corresponding to general two derivative LL gravity in presence of matter and general $U(1)$ gauge invariant theories. The general action for gravity and minimally coupled $U(1)$ gauge invariant theory is taken to be,

$$\mathcal{L} = \int d^d x \sqrt{-g}\left(\frac{L(g_{ab}, R^a{}_{bcd})}{16\pi G} + f(\mathcal{F}_{ab})\right)\,, \tag{2.16}$$

where $L(g_{ab}, R^a{}_{bcd})$ corresponds to a general LL gravity theory. $f(\mathcal{F}_{ab})$ is a generic scalar function in terms of $U(1)$ field strength tensor $\mathcal{F}_{ab} = \nabla_a A_b - \nabla_b A_a$. For instance, in case of $U(1)$ Yang-Mills theory, it is given by $f = (1/16\pi)\mathcal{F}_{ab}\mathcal{F}^{ab}$.

In this case, the charge due to both the diffeomorphism and gauge symmetries is given by $Q_{tot} = Q[\xi] + Q[\epsilon]$, where the first term is the contribution originating from the gravity while the other one is the matter part. Let us first concentrate on the gravity part. Hence associated with the diffeomorphism symmetry $x^a \to x^a + \zeta^a$, the conserved charge can naturally be,

$$Q_t = \int_{\mathcal{V}} d\Sigma_a J^a; \tag{2.17}$$

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where $d\Sigma_a$ is the volume element on a t = constant surface. Now J^a can be expressed as $J^a = \nabla_b J^{ab}$ where J^{ab} is anti-symmetric. Using Stokes' theorem charge Q_t is transformed into following closed surface integral,

$$Q_t = \oint_{\partial\mathcal{V}} d\Sigma_{ab} J^{ab}; \tag{2.18}$$

where $d\Sigma_{ab}$ is a surface element of the closed boundary $\partial\mathcal{V}$ which encloses the volume $\mathcal{V}$. Depending upon the system under the study boundary may contain multiple disconnected closed surfaces. For instance, black hole spacetime has two natural sets of boundaries at the two-dimensional cross-section of the null infinity and the horizon. If the boundary surfaces are disconnected, one can compute the charge for every individual surface and understand their properties. However, one should keep in mind that individual boundary charges may not be conserved in general unless one considers all the surface contributions.

So for LL gravity, one obtains the part of the charge on the null surface as follows [115]

$$Q[\zeta] = \frac{1}{2}\int_{\mathcal{H}} d\Sigma_{ab} J^{ab}\,, \tag{2.19}$$

where

$$J^{ab} = \frac{1}{8\pi G} P^{abcd}\nabla_c \zeta_d\,, \tag{2.20}$$

with $P_{abcd} = \partial L/\partial R^{abcd}$. For the sake of generality we considered the above charge to be off-shell in the sense that one does not need to use the Einstein's equation of motion to derive this. Detail derivation of the off-shell nature of the charge is shown in Appendix 2.F. This is very important for our purpose as we emphasized earlier that the generic null metric Eq.(2.1) under our present study does not need to be a solution of Einstein's equations of motion. Hence the algebra will of off-shell in nature.

Since our charged null surface is located at $r = 0$, the surface integral will survive only for $d\Sigma^{ur}$ component. Therefore, the expression of diffeomorphism charges comes out to be,

$$Q[\zeta] = \frac{1}{8\pi G}\int_{\mathcal{H}} d\Sigma_{ur} P^{urcd}\nabla_c \zeta_d\,. \tag{2.21}$$

Using the symmetric properties of P^{abcd} the above expression can be expanded as

$$Q[\zeta] = \frac{1}{8\pi G}\int_{\mathcal{H}} d\Sigma_{ur}[P^{urur}(\partial_u\zeta_r - \partial_r\zeta_u) + P^{uruA}(\partial_u\zeta_A - \partial_A\zeta_u)$$
$$+P^{urrA}(\partial_r\zeta_A - \partial_A\zeta_r) + \frac{1}{2}P^{urAB}(\partial_A\zeta_B - \partial_B\zeta_A)]\,. \tag{2.22}$$

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In the next step, we lowered the last two indices of P^{abcd} so that it gives a nonzero finite value near the null surface. Finally using the explicit expression for the symmetry transformation parameter Eq.(2.11) and then taking the limit $r=0$ we found,

$$Q[F,\mathcal{R}] = -\frac{1}{8\pi G}\int_{\mathcal{H}} d^{(d-2)}x\sqrt{\mu}\Big[P^{ur}{}_{ru}(2\alpha F+2\partial_u F+\beta_A R^A)-P^{ur}{}_{uB}\mu^{AB}\partial_A F$$
$$+P^{ur}{}_{rB}\mu^{BC}(\partial_u\mu_{AC})R^A+P^{ur}{}_{EF}\mu^{EC}\mu^{FD}\Big(\partial_C(\mu_{DA}R^A)-\partial_D(\mu_{CA}R^A)\Big)\Big]\,. \tag{2.23}$$

In terms of Fourier modes of the symmetry parameters (ζ^a,ϵ) as shown in Eqs.(2.12), from (2.23) the supertranslation charges separately can be written as ,

$$\begin{aligned} Q[F] &= -\sum_{mn}\frac{C_{mn}}{8\pi G}\int_{\mathcal{H}} d^{(d-2)}x\sqrt{\mu}[P^{ur}{}_{ru}(2\alpha F_{mn}+2\partial_u F_{mn})-P^{ur}{}_{uB}\mu^{AB}\partial_A F_{mn}] \\ &= \sum_{mn} C_{mn}Q[F_{mn}] \end{aligned} \tag{2.24}$$

where for individual mode we have defined,

$$Q[F_{mn}] = -\frac{1}{8\pi G}\int_{\mathcal{H}} d^{(d-2)}x\sqrt{\mu}[P^{ur}{}_{ru}(2\alpha F_{mn}+2\partial_u F_{mn})-P^{ur}{}_{uB}\mu^{AB}\partial_A F_{mn}]\,. \tag{2.25}$$

The superrotation charges corresponding to all the angular variables can also be expressed in this way as

$$Q_A[\mathcal{R}] = \sum_k \bar{C}_k Q_A[R^k]; \tag{2.26}$$

where,

$$Q_A[R^k] = -\frac{1}{8\pi G}\int_{\mathcal{H}} d^{(d-2)}x\sqrt{\mu}\Big[P^{ur}{}_{ru}\beta_A R^k+P^{ur}{}_{rB}\mu^{BC}(\partial_u\mu_{AC})R^k$$
$$+P^{ur}{}_{EF}\mu^{EC}\mu^{FD}[\partial_C(\mu_{DA}R^k)-\partial_D(\mu_{CA}R^k)]\Big]\,. \tag{2.27}$$

In a similar manner we will calculate the charge associated with the $U(1)$ gauge transformation. As has already been pointed out, our goal is to compute the charge for a general nonlinear $U(1)$ invariant Lagrangian. One such well known theory is called Born-Infeld electrodynamics [181–183] with $f(\mathcal{F}_{ab}) = \lambda^2(-1+\sqrt{1+\mathcal{F}_{ab}\mathcal{F}^{ab}/(8\pi\lambda^2)})$. Where λ is the Born-Infeld parameter. Clearly for large λ limit one gets back the usual $U(1)$ electromagnetic theory. Goal is to

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keep our discussions as general as possible, therefore, we will not consider any specific form of $f(\mathcal{F}_{ab})$. The Noether current due to gauge symmetry is given by

$$J^a = \nabla_b(f^{ab}\epsilon)\,, \tag{2.28}$$

where ϵ is the gauge parameter. An off-shell derivation of this current is presented in Appendix 2.G. Using Stoke's theorem and considering only the null boundary located at $r = 0$, one obtains the associated charge on the null surface as

$$Q[\epsilon] = \int_{\mathcal{H}} d\Sigma_{ab} f^{ab}\epsilon \tag{2.29}$$

where, $f^{ab} = \partial f(\mathcal{F})/\partial\mathcal{F}_{ab}$. This charge is also defined off-shell as no condition of the equation of motion has been imposed in the derivation.

However for a constant parameter ϵ, the on-shell Noether current J^a in (2.28) vanishes (by satisfying the equation of motion $\nabla_a f^{ab} = 0$). But still, the quantity $f^{ab}\epsilon$ may not be zero when calculated on the part of the closed boundary, and thus, one can get a non-zero charge from (2.29). In the present context ϵ is space-time dependent function given in (2.11). Hence the aforementioned difficulty may not appear here.

The Fourier modes of the $U(1)$ gauge charge turns out to be

$$Q[E] = \sum_{pq} \tilde{E}_{pq} Q[E_{pq}] \tag{2.30}$$

$$Q[E_{pq}] = \int_{\mathcal{H}} 2d^{(d-2)}x\sqrt{\mu} f^{ur} E_{pq}\,. \tag{2.31}$$

We have all three different types of charges for a generic charged null surface. Out of those, $(Q[F_{m,n}], Q_A[R^k])$ are identified as the super-translation and super-rotation charges respectively. Similarly, we call $Q[E_{pq}]$ as the super-gauge charge.

Hence, the parameters associated with a null surface such as M, h_A, and μ_{AB} get transformed under the aforementioned symmetry transformation. Furthermore, In the solution space of the complete Einstein equation, such as black hole spacetime, those conserved charges indicate the existence of soft hair near the horizon of the black hole. This can potentially solve the so-called information loss paradox of black holes (for recent discussion, see [89, 184]).

We use the fundamental Lie bracket among the charges given by (detail in Appendix 2.D.2),

$$[Q[\zeta_m], Q[\zeta_n]] = \pounds_{\zeta_m} Q[\zeta_n], \tag{2.32}$$

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then the Lie bracket algebra among the various charges can be expressed as,

$$\begin{aligned}
& i\big[Q[F_{mn}], Q[F_{pq}]\big] = (m-p)Q[F_{m+p\ n+q}];\\
& i\big[Q_A[R^m], Q_{A'}[R^n]\big] = m_{A'}Q_A[R^{m+n}] - n_A Q_{A'}[R^{m+n}];\\
& i\big[Q_A[R^k], Q[F_{mn}]\big] = -n_A\, Q[F_{m\ n+k}]; \quad i\big[Q[F_{mn}], Q[E_{jl}]\big] = -jQ[E_{m+j\ n+l}];\\
& i\big[Q_A[R^m], Q[E_{jl}]\big] = -l_A Q[E_{j\ m+l}]; \quad \big[Q[E_{jl}], Q[E_{mn}]\big] = 0\,.
\end{aligned} \tag{2.33}$$

It is clear from the above equation (2.33) that the symmetry bracket among the charges is isomorphic to that among diffeomorphism vectors. Here, the gauge symmetry and the diffeomorphism symmetry together form a closed algebra that sharply contrasts with the usual transformation. The implication of this could be interesting to explore further.

At this point, let us again emphasize the fact that our analysis does not depend upon the equation of motion of the fields under consideration. We started with a general charged null surface which is not a solution to Einstein's equation of motion. After this, we follow the usual asymptotic symmetry analysis with a physically motivated fall of conditions of all the fields under study near the surface. Associated with those symmetries, we constructed conserved charges without taking into account the equation motion. Therefore, our off-shell approach not only helps us to understand the symmetry properties of a generic null surface but also applies to the on-shell solution. Therefore, it is much more general than the earlier analysis [73, 91, 131, 132, 185].

Before we complete our analysis, we show that the same algebra can also be obtained from the Noether charge corresponding to the surface term of the gravitational action. For simplicity, we will only consider the usual Einstein-Hilbert action and its associated boundary term called Gibbons-Hawking-York (GHY) surface term. The idea is the following. It is well known that the GHY term itself, calculated on the horizon, leads to horizon entropy. Moreover, its Noether charge plays the same role (see Sec. 2 of [186] for a detail discussion). The possible reason behind this is that both the terms (i.e., Noether charges for GHY term and GHY term itself) will coincide on the null surface, corresponding to a timelike Killing vector. Since we did not find such discussion in the literature, in Appendix 2.H, we show this similarity explicitly for a static spacetime.

The conserved Noether current for GHY term is given by [105],

$$J^a[\zeta] = \nabla_b\, J^{ab}[\zeta] = \frac{1}{8\pi G}\nabla_b\left(K\zeta^a N^b - K\zeta^b N^a\right), \tag{2.34}$$

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where N^a is the unit normal to the boundary ∂V of a region of spacetime V. $K = -\nabla_a N^a$ is the trace of the extrinsic curvature of this boundary surface and J_{ab} is the Noether potential associated with diffeomorphism symmetry of the theory. Now since both the Noether charges of the Einstein-Hilbert action and GHY term lead to entropy when calculated on the horizon, we expect that the GHY Noether charge also leads to the same algebra (2.33) for the parameters (2.11) obtained here. For the given null surface (2.1) only relevant surface element is $d\Sigma_{ur} = d^{(d-2)}x$. The unit spacelike normal vector N^a on $r = constant$ surface comes out as $N_a = (0, (r^2\beta_A\beta^A + 2r\alpha)^{-1/2}, 0)$ and so the contravariant components are as follows:

$$N^a = \left((r^2\beta_A\beta^A + 2r\alpha)^{-1/2}, \sqrt{r^2\beta_A\beta^A + 2r\alpha}, (r\beta^A/\sqrt{r^2\beta_A\beta^A + 2r\alpha})\right). \quad (2.35)$$

Therefore, the trace of the extrinsic curvature is calculated to be;

$$\begin{aligned}
K &= -(1/\sqrt{\mu})\,[\partial_a(\sqrt{\mu})N^a + (\sqrt{\mu})\,\partial_a N^a] \\
&= [\frac{1}{2}\mu^{AB}[(\partial_u\mu_{AB})(r^2\beta_A\beta^A + 2r\alpha)^{-1/2} + (\partial_r\mu_{AB})\sqrt{r^2\beta_A\beta^A + 2r\alpha} \\
&+(\partial_C\mu_{AB})(r\beta^C/\sqrt{r^2\beta_A\beta^A + 2r\alpha})] - \partial_u(r^2\beta_A\beta^A + 2r\alpha)^{-1/2} \\
&-\partial_r[\sqrt{r^2\beta_A\beta^A + 2r\alpha}] - \partial_A[r\beta^A/\sqrt{r^2\beta_A\beta^A + 2r\alpha}]]\,.
\end{aligned} \quad (2.36)$$

Substituting all the relevant quantities in the charge expression for the parameters (2.11), one obtains

$$Q[F, \mathcal{R}] = \frac{1}{16\pi G}\int_H d^{(d-2)}x\sqrt{\mu}[2\alpha F + \partial_u F + \beta_A R^A] \quad (2.37)$$

which is the similar expression obtained earlier from the usual Noether charge [see Eqn. (2.23)] with the value $P^{ur}_{ru} = (1/2), P^{ur}_{rB} = P^{ur}_{AB} = P^{ur}_{rA} = 0$ for GR). Therefore it is obvious that this will also lead to the algebra (2.33). This further indicates that the surface term of gravitational action carries the information of the bulk theory (for more to this direction, see [186] and the references therein).

2.7 Null surface: Extremal case

In recent years the study of the extremal black hole has got a profound interest in the search for the microscopic degrees of freedom of black hole [187–189]. The extremal black hole is defined as having surface gravity $\kappa = 0$, which follows that the Hawking radiation vanishes in the extremal limit. The near-horizon geometry of the extremal Killing horizon has been studied extensively in Gaussian null

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coordinate in [179, 190]. In the present context, we look at the diffeomorphism symmetries for a generic charged extremal null surface which can be defined as a zero-temperature limit of a non-extremal null surface considered so far. In this case, also we will perform the off-shell symmetry analysis for a generic gravity and $U(1)$ gauge-invariant theory. The neighborhood of an extremal null surface is parametrized by Gaussian null coordinate as [179]:

$$ds^2 = -r^2\bar{\alpha}(u,x)du^2 + 2dudr - 2r\bar{\beta}_A(u,x)dudx^A + \bar{\mu}_{AB}(u,x)dx^A dx^B \,. \tag{2.38}$$

The extremality condition, which is equivalent to zero temperature limit, is manifested into the fall-off condition of $g_{uu} \sim \mathcal{O}(r^2)$ as one approach toward the surface at $r = 0$.

As one can observe the behavior of the remaining metric components $\bar{\beta}_A$ and $\bar{\mu}_{AB}$ are same as non-extremal surface defined in Eq.(2.3). The extremality condition on the metric does not have any effect on the gauge field A_μ configuration near the surface Eq.(2.3). Therefore, given the metric and the gauge field configurations in the extremal null background, we will carry out the same analysis as before with the following modified fall off conditions,

$$\pounds_\zeta g_{uu} \approx \mathcal{O}(r^2); \;\; \pounds_\zeta g_{uA} \approx \mathcal{O}(r); \;\; \pounds_\zeta g_{AB} \approx \mathcal{O}(1) \,. \tag{2.39}$$

As emphasized, all the remaining conditions remain the same. Therefore, the diffeomorphism parameters derived in Eq.(2.8) have to satisfy the modified constraint relations as follows,

$$\begin{aligned} &\partial_u T = 0\,; \\ &T(u,x)\bar{\alpha} + \partial_u^2 F + \bar{\beta}_A \partial_u R^A = 0\,; \\ &\partial_A T - T(u,x)\bar{\beta}_A(u,x) + \bar{\mu}^{(0)}_{AB}\partial_u R^B = 0\,. \end{aligned} \tag{2.40}$$

As stated earlier, radial component of the diffeomorphism vector T must be zero near the surface. Therefore, all the constraint Eqs.(2.40) will automatically be satisfied if we consider,

$$\partial_u R^A(u,x) = 0 \;\; ; \;\; \partial_u^2 F(u,x) = 0\,;. \tag{2.41}$$

Hence, R^A will be independent of u coordinate. The general solution of $F(u,x)$ can be written as,

$$F(u,x) = M(x)u + N(x)\,;. \tag{2.42}$$

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We have two independent arbitrary functions $(M(x), N(x))$, in the time diffeomorphism symmetry. To have finite values near the horizon where $u \to \infty$, M must vanish. Otherwise, close to the horizon F will diverge. So we set $M = 0$. Therefore, F becomes independent of u, which is in sharp contrast with the non-extremal case described earlier. Therefore, for extremal null surface, the asymptotic symmetry generators are-

$$\zeta^u = F(x)\ ; \quad \zeta^r = -\partial_B F \int r\bar{\beta}^B dr\ ;$$
$$\zeta^A = -\partial_B F \int \bar{\mu}^{AB} dr + R^A(x)\ ; \quad \epsilon = E(u,x) + \int_r dr[A_B(\partial_C F)\bar{\mu}^{BC}]. \tag{2.43}$$

Next we have Fourier mode decomposition like non-extremal case given in (2.12). Following the same procedure as has been discussed for non-nextremal null surface, the associated symmetry algebra will take the following form (detail in Appendix 2.E.1),

$$i[F_m, F_n] = 0; \quad i[R^k_A, F_n] = -n_A\ F_{n+k}; \quad i[F_n, E_{jl}] = -j\ E_{(j\ n+l)}. \tag{2.44}$$

Henceforth we observe that for the extremal null surface, the supertranslation vector field F commutes with itself, which was noncommutative for non-extremal case (2.15). Similarly algebra between charges will change only for those associated with supertranslation charges $Q[F_n]$ as,

$$[Q[F_m], Q[F_n]] = 0; \quad i[Q[R^k_A], Q[F_n]] = -n_A\ Q[F_{n+k}];$$
$$i[Q[F_n], Q[E_{jl}]] = -jQ[E_{j\ n+l}]. \tag{2.45}$$

Here also brackets among charges are isomorphic to that among the vector fields. Other results from (2.12) remain exactly the same as before. It would be fascinating to understand the physical interpretation of the difference in symmetry algebra for two different null surfaces. How the zero-temperature limit plays a role in determining symmetry could be an interesting point to study. This topic is yet to explore in the future.

2.8 Conclusions

One of the main goals of the analysis presented in this chapter was to understand the symmetry properties of a generic null surface defined in gravity theory minimally coupled with the electromagnetic gauge theory. As we have emphasized throughout our analysis, we have made two important generalizations of the

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existing research. In one direction, we have considered the most general $U(1)$ invariant electromagnetic theory minimally coupled with a gravity theory at any arbitrary order in LL gravity. On the other hand, in our derivation of symmetry algebra among the charges, we use the off-shell formalism, where we have not considered any equation of motions. Therefore, our study can automatically give the near horizon symmetry of any black hole of the theory under consideration. As pointed out in the recent papers [89, 184] that those near horizon symmetries are spontaneously broken in the black hole background. Therefore, in quantum theory, those symmetry breaking will lead to the associated Goldstone modes, which will behave as soft hairs. This may play an essential role in solving the black hole information loss paradox.

Nonetheless, we found the near horizon symmetries, categorized as supertranslation and superrotation acting on the null surface under study. As discussed, those transformations asymptotically preserve the structure of the null structure in the presence of a gauge charge. Finally, the algebra of the corresponding charges on the null surface has been computed. Our algebra for the parameters and the charges are different from the earlier ones. This difference is because, in general, the super translation parameter can be a function of a null coordinate (u), and we have considered this situation. At the same time, the earlier literature (see [91]) did not take into account this. Therefore, our analysis takes care of the most general situation in all respect. We Hope the present study will illuminate this paradigm more. At the end of this chapter, we have also tried to explore the symmetry properties of a generic charged extremal null surface. In this context, one of the symmetry parameters F became independent of the null coordinate. As a result, the symmetry algebras of the generators were quite different compared to the algebras constructed for a generic non-extremal charged null surface. However, the physical reason for this difference is not apparent. We will try to understand and explore this topic in the future.

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Appendix

2.A Why is the angular component of gauge field $A_B \approx O(1)$ near the null surface?

Here we are considering the static observer very near to the null surface $r = 0$. It has been pointed out in the main text of this chapter that for such an observer the energy density $U = T_{ab}u^a u^b$ diverges, where u^a is the four velocity. Now for the given metric, u^a is given by $u^a = (1/2r\alpha, 0, 0)$, as $u^a u_a = -1$. Therefore energy density turns out to be

$$U = \frac{T_{uu}}{2r\alpha}\,. \tag{2.A.1}$$

For U to be divergent, T_{uu} should be finite as we approach toward $r = 0$. Hence T_{uu} must be independent of r.

Now near the null surface the $U(1)$ electromagnetic gauge field energy-momentum tensor will take,

$$\begin{aligned}\lim_{r\to 0} T_{uu} &= \lim_{r\to 0} \mathcal{F}_{ua}\,\mathcal{F}^a{}_u - \frac{1}{4}g_{uu}\mathcal{F}_{ab}\mathcal{F}^{ab}\\ &= \lim_{r\to 0}\Big(r(\partial_r A_u)^2(r\beta_B\beta^B + 2\alpha) + \mu^{CB}(\partial_u A_C)(\partial_u A_B)\\ &-r\beta^B\partial_r A_u(\partial_u A_B - \partial_B A_u) - \frac{r\alpha}{2}(\mathcal{F}_{ab}\mathcal{F}^{ab}).\Big)\end{aligned} \tag{2.A.2}$$

Considering the components A_u and A_r from (2.3) and the upper components of the metric (2.1) one can easily show that $\mathcal{F}_{ab}\mathcal{F}^{ab}$ will be $\mathcal{O}(r)$. Therefore the result will be,

$$\lim_{r\to 0} T_{uu} == \mu^{AB}(\partial_u A_A)(\partial_u A_B)\,. \tag{2.A.3}$$

Hence, in order for T_{uu} to be finite, $A_B \approx O(1)$.

2.B Derivation of diffeomorphism and gauge parameters

2.B.1 Diffeomorphism vectors (2.8)

The first equation of (2.4) implies that

$$\pounds_\zeta g_{rr} = \zeta^c\partial_c g_{rr} + 2g_{cr}\partial_r\zeta^c = 2g_{ur}\,\partial_r\zeta^u = 0\,, \tag{2.B.1}$$

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which immediately implies the form of ζ^u given in (2.8). Using this in the last condition of (2.4) one finds

$$\mu_{AB}\,\partial_r\zeta^B + \partial_A F = 0\,. \tag{2.B.2}$$

Solution of which leads to ζ^A. Finally, use of these components in the second condition of (2.4) yields

$$\partial_r\zeta^r + r\beta^A\partial_A F + \partial_u F = 0\,, \tag{2.B.3}$$

whose solution is the radial component of ζ^a.

2.B.2 Equations in (2.10)

Putting the components of ζ^a from (2.8) in the first condition of (2.5), near null surface we get,

$$\pounds_\zeta g_{uu} = \partial_u T(u,x) - \alpha(u,x)T(u,x). \tag{2.B.4}$$

In the above expression, we have written down the leading order term. Now given the fall-off condition $\pounds_\zeta g_{uu} = \mathcal{O}(r)$ as shown in Eq.(2.5), the right-hand side of the above equation must vanish as it is $\mathcal{O}(1)$ in r. Similarly near $r = 0$, variation of the other metric components read,

$$\pounds_\zeta g_{uA} = \partial_A T(u,x) - T(u,x)\beta_A(u,x) + \mu^{(0)}_{AB}\partial_u R^B(u,x)\,, \tag{2.B.5}$$

which must vanish according to $\pounds_\zeta g_{uA} \approx O(r)$. This yields the other equation in (2.10). With these one can verify that the remaining conditions of Eq.(2.5) are automatically satisfied as one approaches toward the null surface. The variation of g_{AB} is given by,

$$\pounds_\zeta g_{AB} = F\partial_u\mu^{(0)}_{AB} + T\mu^{(1)}_{AB} + R^E(u,x)]\partial_E\mu^{(0)}_{AB} + \mu^{(0)}_{AD}\,\partial_B R^D(u,x)\,. \tag{2.B.6}$$

This variation does not give us any new constraints as it is already matching with the assumed fall off condition $\pounds_\zeta g_{AB} = \mathcal{O}(1)$.

2.B.3 Derivation of gauge parameter ϵ

Using the derived forms of ζ^a, the condition (2.6) leads to,

$$A_B[-\partial_C F\,\mu^{BC}] + \partial_r\epsilon = 0 \tag{2.B.7}$$

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whose solution yields the expression of gauge parameter ϵ given in Eq. (2.10).

The other conditions do not give any new constraints as they are now automatically satisfied. For instance one obtains near the null surface,

$$\pounds_\zeta A_u + \partial_u \epsilon = C_0\, \partial_u F + A_B \partial_u R^A + \partial_u E(u,x)\,, \tag{2.B.8}$$

and

$$\pounds_\zeta A_B + \partial_u \epsilon = F\partial_u A_B + R^C \partial_C A_B + (\partial_B F)C_0 + A_C \partial_B R^C + \partial_B E(u,x)\,, \tag{2.B.9}$$

which are $\mathcal{O}(1)$.

2.C Modified Lie bracket

Following [73, 180] here, we will try to present the detail about the modified version of the Lie bracket (2.13), which should be considered in the computation of algebras among vectors. As described in the main text, by solving gauge choices and satisfying the fall-off conditions, one can get the components of ζ_1^a and ζ_2^a. Now to understand the origin of the modified version of the Lie bracket, we have to consider the variation of metric g_{ab} under the variation of the vector field ζ_1^a, followed by the variation of another vector ζ_2^a. So the variation of the metric components under the variation of ζ_1^a is given by,

$$g_{ab} \to g_{ab} + h_{ab} \tag{2.C.1}$$

Here $h_{ab} = \pounds_{\zeta_1} g_{ab}$ is the first order perturbation part of the original metric g_{ab}. Next we consider that the vector field ζ_2^a act on the perturbed metric $g_{ab} + h_{ab}$. Then there will be additional perturbation as,

$$\zeta_2^a \to \zeta_2^a + \mu_2^a; \quad g_{ab} \to g_{ab} + h_{ab} + K_{ab}. \tag{2.C.2}$$

where μ_2^a is the first order perturbation to the vector field ζ_2^a and K_{ab} is the second order variation of the metric g_{ab}. Then K_{ab} is given by,

$$K_{ab} = \pounds_{\zeta_2'}(g_{ab} + h_{ab}); \tag{2.C.3}$$

here $\zeta_2'^a = \zeta_2^a + \mu_2^a$. Now we can find the form of μ_2^a by demanding that the second order variation of the metric given in (2.C.3) satisfies all the boundary conditions as (2.4) and (2.5). In (2.13) the expression $\delta^g_{\zeta_1}\zeta_2^a$ denotes μ_2^a. We have to repeat the same process where first ζ_2^a and then ζ_1^a will act on the metric. Then we will get the

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first order variation of ζ_1^a which is represented as $\delta^g_{\zeta_2}\zeta_1^a$ in (2.13). Therefore these two terms must be subtracted to the original form of Lie bracket in order to take into account the variation of the vector fields due to the higher order variation of the metric. However in our present analysis we will neglect all these higher order variations of the metric components. Moreover here the diffeomorphism vector (2.11) is not dependent on the metric coefficients near the null surface $r = 0$. Hence we have considered the original definition of Lie bracket given in (2.14).

2.D Non-extremal null surface

2.D.1 Derivation of the symmetry algebras among the components of the vector and gauge parameter given in (2.15)

In this Appendix we have discussed about the detail computation of the algebras among the components of the symmetry vector ζ^a and gauge transformation parameter ϵ. For this reason we have considered each component as a separately vector quantity. Hence to comute the algebras let us construct the three symmetry vectors separately as given by, $\chi^a\partial_a = F\partial_u$, $\eta^a\partial_a = R^A\partial_A$ and $\lambda^a\partial_a = E\partial_{\epsilon_0}$. Here each vector consists of $d+1$ no of components where the first d components correspond to the time, radial and $(d-2)$ number of angular components and ϵ_0 is the $(d+1)^{th}$ component corresponding to the $U(1)$ gauge parameter ϵ. However χ^a has only time component non-zero, whereas all the $(d-2)$ no of the angular components of η^a and the gauge component of the vector λ^a are respectively non-vanishing. Now the algebras are computed as follows.

- $[\mathcal{F}, \mathcal{F}]$ commutator:

  In terms of Fourier modes given in (2.12) we can write the commutator between χ_1 and χ_2 as ,

$$\Big[\chi_1, \chi_2\Big]^u \partial_u = \frac{1}{\alpha^2} \sum_{m,n,p,l} B_{mn}\tilde{B}_{pl}\Big[F_{(mn)}, F_{(pl)}\Big]^u \partial_u. \tag{2.D.1}$$

  Hence in coordinate basis the temporal component of the aforementioned Lie bracket $[\chi_1, \chi_2]$ is non-zero only. It can be checked easily that the other components of the bracket are zero automatically. With the help of the Lie

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algebra defined in (2.14), (2.D.1) becomes,

$$\begin{aligned}
&\frac{1}{\alpha^2}\sum_{m,n,p,l} B_{mn}\tilde{B}_{pl}\left[F_{(mn)},F_{(pl)}\right]^u\partial_u \\
&=\frac{1}{\alpha^2}\sum_{m,n,p,l} B_{mn}\tilde{B}_{pl}[(F_{mn})^a\partial_a F_{pl}-(F_{pl})^a\partial_a F_{mn}]\partial_u \\
&=\frac{1}{\alpha^2}\sum_{m,n,p,l} i(p-m)B_{mn}\tilde{B}_{pl}e^{i((m+p)u+\sum_A(n+l)_A x^A)}\partial_u. \\
&=\frac{1}{\alpha^2}\sum_{m,n,p,l} i(p-m)B_{mn}\tilde{B}_{pl}F_{(m+p\ n+l)}\partial_u. \qquad (2.D.2)
\end{aligned}$$

Comparing both the equations (2.D.1) and (2.D.2) we get the bracket algebra for $[\mathcal{F},\mathcal{F}]$ commutator given in (2.15).

- $[\mathcal{R},\mathcal{R}]$ commutator:
  To derive the aforesaid commutator we can write Lie bracket between η_1 and η_2 as,

$$[\eta_1,\eta_2]^{A''}\partial_{A''}=\sum_{mn}B^A_m\bar{B}^{A'}_n[R^m_A,R^n_{A'}]^{A''}\partial_{A''}. \qquad (2.D.3)$$

with the help of (2.14), (2.D.3) can be written,

$$\begin{aligned}
&\sum_{mn}B^A_m\bar{B}^{A'}_n[R^m_A,R^n_{A'}]^{A''}\partial_{A''} \\
&=\sum_{mn}B^A_m\bar{B}^{A'}_n[(R^m_A)^a\partial_a(R^n_{A'})^{A''}-(R^n_{A'})^a\partial_a(R^m_A)^{A''}]\partial_{A''} \\
&=\sum_{mn}B^A_m\bar{B}^{A'}_n\Big[e^{i\sum_A m_A x^A}\partial_A(e^{i\sum_{A'}n_{A'}x^{A'}}\delta^{A''}_{A'}) \\
&-e^{i\sum_{A'}n_{A'}x^{A'}}\partial_{A'}(e^{(i\sum_A m_A x^A)}\delta^{A''}_A)\Big]\partial_{A''}. \\
&=\sum_{mn}B^A_m\bar{B}^{A'}_n\Big[in_A e^{i\sum_{A'}(m+n)_{A'}x^{A'}}\delta^{A''}_{A'})-im_{A'}e^{i\sum_A(m+n)_A x^A}\delta^{A''}_A)\Big]\partial_{A''}. \\
&=\sum_{mn}iB^A_m\bar{B}^{A'}_n[n_A(R^{m+n}_{A'})^{A''}-m_{A'}(R^{m+n}_A)^{A''}]\partial_{A''}. \qquad (2.D.4)
\end{aligned}$$

Now comparing the equations (2.D.3) and (2.D.4) we get the algebra for $[\mathcal{R},\mathcal{R}]$ commutator given in (2.15).

- $[\mathcal{R},\mathcal{F}]$ commutator:
  Following the previous manner, we can write the non-zero component of the Lie bracket $[\eta_1,\chi_2]$ as,

$$[\eta_1,\chi_2]^u\partial_u=\frac{1}{\alpha}\sum_{mnk}B_{mn}\bar{B}^A_k[R^k_A,F_{mn}]^u\partial_u. \qquad (2.D.5)$$

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Using the definition of Lie algebra, (2.D.5) is given by,

$$\frac{1}{\alpha}\sum_{mnk} B_{mn}\bar{B}_k^A[R_A^k, F_{mn}]^u\partial_u = \frac{1}{\alpha}\sum_{mnk} B_{mn}\bar{B}_k^A(R_A^k)^a\partial_a F_{mn}\partial_u$$
$$= \frac{1}{\alpha}\sum_{mnk} B_{mn} i l_A \bar{B}_k^A e^{(iku+\sum_A i(m+l)_A x^A)}\partial_u = \frac{1}{\alpha}\sum_{mnk} B_{mn} i l_A \bar{B}_k^A F_{(m\ k+l)}\partial_u. \tag{2.D.6}$$

Comparing (2.D.5) with (2.D.6), the required $[\mathcal{R}, \mathcal{F}]$ commutator algebra in (2.15) can be obtained.

- $[\mathcal{F}, E]$ commutator:
The commutator between χ_1 and λ_2 is calculated in the following way,

$$[\chi_1, \lambda_2]^{\epsilon_0}\partial_{\epsilon_0} = \frac{1}{\alpha}\sum_{mnkl} B_{mn}\bar{E}_{kl}[F_{mn}, E_{kl}]^{\epsilon_0}\partial_{\epsilon_0}. \tag{2.D.7}$$

Here the Lie bracket $[\chi_1, \lambda_2]$ have one non-zero component (which is ϵ_0 component) corresponding to the $U(1)$ gauge parameter ϵ. Hence from (2.D.7) one can write,

$$\frac{1}{\alpha}\sum_{mnkl} B_{mn}\bar{E}_{kl}[F_{mn}, E_{kl}]^{\epsilon_0}\partial_{\epsilon_0} = \frac{1}{\alpha}\sum_{mnkl} B_{mn}\bar{E}_{kl}F_{mn}(\partial_u E_{kl})\partial_{\epsilon_0}$$
$$= \frac{1}{\alpha}\sum_{mnkl} ikB_{mn}\bar{E}_{kl}e^{[i(m+k)u+i\sum_A(n+l)_A x^A]}\partial_{\epsilon_0}$$
$$= \frac{1}{\alpha}\sum_{mnkl} ikB_{mn}\bar{E}_{kl}E_{m+k\ n+l}\partial_{\epsilon_0}. \tag{2.D.8}$$

From (2.D.7) and (2.D.8) we can get the $[\mathcal{F}, E]$ bracket as given in (2.15).

- $[\mathcal{R}, E]$ commutator: Now we have to compute bracket among η_1 and λ_2 as,

$$[\eta_1, \lambda_2]^{\epsilon_0}\partial_{\epsilon_0} = \sum_{mnk} B_{mn}\bar{B}_k^A[R_A^k, E_{mn}]^{\epsilon_0}\partial_{\epsilon_0}. \tag{2.D.9}$$

Using the definition of Lie algebra, (2.D.9) is given by,

$$\sum_{mnk} B_{mn}\bar{B}_k^A[R_A^k, E_{mn}]^{\epsilon_0}\partial_{\epsilon_0} = \sum_{mnk} B_{mn}\bar{B}_k^A(R_A^k)^a\partial_a E_{mn}\partial_{\epsilon_0}$$
$$= \sum_{mnk} B_{mn} i l_A \bar{B}_k^A e^{(iku+\sum_A i(m+l)_A x^A)}\partial_{\epsilon_0} = \sum_{mnk} B_{mn} i l_A \bar{B}_k^A E_{(m\ k+l)}\partial_{\epsilon_0}. \tag{2.D.10}$$

Comparing (2.D.9) with (2.D.10), the required $[\mathcal{R}, E]$ commutator algebra in (2.15) can be obtained.

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- $[E, E]$ commutator:

$$[\lambda_1, \lambda_2]^{\epsilon_0}\partial_{\epsilon_0} = \sum_{mnkl} E'_{mn}\bar{E}_{kl}[E_{mn}, E_{kl}]^{\epsilon_0}\partial_{\epsilon_0}. \tag{2.D.11}$$

Like before, here also the Lie bracket $[\lambda_1, \lambda_2]$ have one non-zero component corresponding to the $U(1)$ gauge parameter ϵ. Hence from (2.D.11) one can write,

$$\begin{aligned}&\sum_{mnkl} E'_{mn}\bar{E}_{kl}[E_{mn}, E_{kl}]^{\epsilon_0}\partial_{\epsilon_0}\\ &= \sum_{mnkl} E'_{mn}\bar{E}_{kl}\Big(E_{mn}(\partial_{\epsilon_0}E_{kl} - E_{kl}(\partial_{\epsilon_0}E_{mn})\Big)\partial_{\epsilon_0} = 0.\end{aligned} \tag{2.D.12}$$

In (2.D.12), the derivative with respect to ϵ_0 will be zero, as parameters are functions of time and space coordinates only. Thus $[E, E]$ bracket vanishes in (2.15).

2.D.2 Algebra among symmetry charges as found in (2.33)

Using the definition of bracket given in (2.32), we calculate commutator of the supertranslation charges with itself as follows,

$$\begin{aligned}&[Q[F_1], Q[F_2]] = \pounds_{F_1}Q[F_2]\\ &= -\frac{1}{8\pi G}\int_{\mathcal{H}} d^{(d-2)}x\sqrt{\mu}\Big(P^{ur}{}_{ru}(2\alpha + 2\partial_u) - P^{ur}{}_{uB}\mu^{AB}\partial_A\Big)[F_1, F_2]\\ &= -\sum_{mnpq}\frac{C_{mn}\bar{C}_{pq}}{8\pi G}\int_{\mathcal{H}} d^{(d-2)}x\sqrt{\mu}\Big(P^{ur}{}_{ru}(2\alpha + 2\partial_u) - P^{ur}{}_{uB}\mu^{AB}\partial_A\Big)[F_{mn}, F_{pq}].\end{aligned} \tag{2.D.13}$$

From (2.24) it follows that,

$$[Q[F_1], Q[F_2]] = \sum_{mnpq} C_{mn}\bar{C}_{pq}[Q[F_{mn}], Q[F_{pq}]] \tag{2.D.14}$$

Now with the help of the result derived in (2.15), we can write from (2.D.13) that,

$$\begin{aligned}[Q[F_1], Q[F_2]] &= -\sum_{mnpq}\frac{i(p-m)C_{mn}\bar{C}_{pq}}{8\pi G}\int_{\mathcal{H}} d^{(d-2)}x\sqrt{\mu}\Big(P^{ur}{}_{ru}(2\alpha + 2\partial_u)\\ &\quad -P^{ur}{}_{uB}\mu^{AB}\partial_A\Big)F_{(m+p\ n+q)}\\ &= \sum_{mnpq} i(p-m)C_{mn}\bar{C}_{pq}Q[F_{(m+p\ n+q)}].\end{aligned} \tag{2.D.15}$$

Comparing the result in (2.D.14) with (2.D.15) one can get the commutator of the supertransaltion charges given in (2.33). Similarly, we can derive other brackets given in (2.33,).

2.E Extremal null surface

2.E.1 Bracket algebra given in (2.44)

Like non-extremal surface, here also we compute bracket algebra among the components of symmetry vector ζ^A and gauge parameter ϵ. For this reason we have considered each component of the vector and also gauge parameter ϵ as a separately vector quantity. Hence like before the three symmetry vectors are given by,$\chi^a\partial_a = F\partial_u$, $\eta^a\partial_a = R^A\partial_A$ and $\lambda^a\partial_a = E\partial_{\epsilon_0}$. However the form of the algebra is simplified compared to non-extremal case. Using the definition of Lie bracket (2.13) we compute $[\mathcal{F},\mathcal{F}]$ commutator as,

$$\begin{aligned}\left[\chi_1,\chi_2\right]^u\partial_u &= \frac{1}{\alpha^2}\sum_{m,p} B_m\tilde{B}_p\left[F_m,F_p\right]^u\partial_u \\ &= \frac{1}{\alpha^2}\sum_{m,p} B_m\tilde{B}_p[(F_m)^a\partial_a F_p - (F_p)^a\partial_a F_m]\partial_u = 0.\end{aligned} \tag{2.E.1}$$

As supertranslation parameter F is independent of u, this gives,$[F_m,F_p]=0$.

Next, following the same procedure as we did for the non-extremal case in (2.D.4), we have computed the $[\mathcal{R},\mathcal{R}]$ commutator and get the second bracket in (2.44).

Then we move to calculate the bracket among η_1 and χ_2. Here also we have followed the same procedure as non-extremal surfaces. One can write the non-zero component of the Lie bracket $[\eta_1,\chi_2]$ as,

$$[\eta_1,\chi_2]^u\partial_u = \frac{1}{\alpha}\sum_{mk} B_m\bar{B}^A_k[R^k_A,F_m]^u\partial_u. \tag{2.E.2}$$

Using the definition of Lie algebra, (2.E.2) is given by,

$$\begin{aligned}\frac{1}{\alpha}\sum_{mk} B_m\bar{B}^A_k[R^k_A,F_m]^u\partial_u &= \frac{1}{\alpha}\sum_{mk} B_m\bar{B}^A_k(R^k_A)^a\partial_a F_m\partial_u \\ = \frac{1}{\alpha}\sum_{mk} B_m i l_A\bar{B}^A_k e^{(\sum_A i(m+k)_A x^A)}\partial_u &= \frac{1}{\alpha}\sum_{mk} B_m i l_A\bar{B}^A_k F_{(m+k)}\partial_u.\end{aligned} \tag{2.E.3}$$

Comparing (2.E.2) with (2.E.3), the required $[\mathcal{R},\mathcal{F}]$ commutator algebra in (2.44) can be obtained.

Now we compute $[\mathcal{F},E]$ commutator as follows.

$$[\chi_1,\lambda_2]^{\epsilon_0}\partial_{\epsilon_0} = \frac{1}{\alpha}\sum_{mkl} B_m\bar{E}_{kl}[F_m,E_{kl}]^{\epsilon_0}\partial_{\epsilon_0}. \tag{2.E.4}$$

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Here the Lie bracket $[\chi_1, \lambda_2]$ have one non-zero component ϵ_0 which corresponds to the $U(1)$ gauge parameter ϵ. Hence from (2.E.4) one can write,

$$\frac{1}{\alpha}\sum_{mkl} B_m \bar{E}_{kl}[F_m, E_{kl}]^{\epsilon_0}\partial_{\epsilon_0} = \frac{1}{\alpha}\sum_{mkl} B_m \bar{E}_{kl} F_m(\partial_u E_{kl})\partial_{\epsilon_0}$$
$$= \frac{1}{\alpha}\sum_{mkl} ikB_m \bar{E}_{kl} e^{[iku + i\sum_A (m+l)_A x^A]}\partial_{\epsilon_0} = \frac{1}{\alpha}\sum_{mkl} ikB_m \bar{E}_{kl} E_{(k\ m+l)}\partial_{\epsilon_0}. \tag{2.E.5}$$

From (2.E.4) and (2.E.5), we can get the $[\mathcal{F}, E]$ bracket given in (2.44).

Now following the precisely same procedure as we did for the non-extremal case in (2.D.1), we have computed the $[\mathcal{R}, E]$ and the $[E, E]$ commutator and get the corresponding brackets in (2.44).

2.F An off-shell derivation of Noether current for the general theory of gravity.

following [115], here we have presented a off-shell derivation of Noether current from the Lagrangian in gravity. Let us consider a general action of the form,

$$\mathcal{S} = \int d^d x \sqrt{-g}\mathcal{L}[g^{ab}, R^a{}_{bcd}]. \tag{2.F.1}$$

We have ignored higher order derivative of $R^a{}_{bcd}$. Now we consider the variation of $\mathcal{L}\sqrt{-g}$ where $\mathcal{L}$ is a covariant scalar made of g^{ab} and $R^a{}_{bcd}$. Its variation will be,

$$\delta(\mathcal{L}\sqrt{-g}) = (\frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{ab}})\delta g^{ab} + (\frac{\partial \mathcal{L}\sqrt{-g}}{\partial R^a{}_{bcd}})\delta R^a{}_{bcd}$$
$$= (\frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{ab}})\delta g^{ab} + \sqrt{-g}P_a{}^{bcd}\delta R^a{}_{bcd} \tag{2.F.2}$$

The first term of (2.F.2) is calculated as,

$$\frac{\partial \mathcal{L}\sqrt{-g}}{\partial g^{ab}} = \frac{\partial \mathcal{L}}{\partial g^{ab}}\sqrt{-g} + \frac{\partial \sqrt{-g}}{\partial g^{ab}}\mathcal{L} = \sqrt{-g}\frac{\partial \mathcal{L}}{\partial R^{kl}_{ij}}\frac{\partial R^{kl}_{ij}}{\partial g^{ab}} - \frac{1}{2}\sqrt{-g}g_{ab}\mathcal{L}.$$
$$= \sqrt{-g}[P^{ij}_{kl}\frac{\partial(g^{ml}R^k{}_{mij})}{\partial g^{ab}} - \frac{1}{2}g_{ab}\mathcal{L}] = \sqrt{-g}[P^{ij}_{kl}\delta^m_{(a}\delta^l_{b)}R^k{}_{mij} - \frac{1}{2}g_{ab}\mathcal{L}]$$
$$= \sqrt{-g}[P^{ij}_{k(b}R^k{}_{a)ij} - \frac{1}{2}g_{ab}\mathcal{L}] = \sqrt{-g}[P^{kij}_{(b}R_{a)kij} - \frac{1}{2}g_{ab}\mathcal{L}]. \tag{2.F.3}$$

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Here $\delta^m_{(a}\delta^l_{b)} = \frac{1}{2}(\delta^m_a\delta^l_b + \delta^m_b\delta^l_a)$. Next we concentrate on the second term of (2.F.2). The variation of $R^a{}_{bcd}$ is given by,

$$\delta R^a{}_{bcd} = \nabla_c(\delta\Gamma^a_{db}) - \nabla_d(\delta\Gamma^a_{cb})$$
$$= \frac{1}{2}\nabla_c[g^{ai}(-\nabla_i\delta g_{db} + \nabla_d\delta g_{bi} + \nabla_b\delta g_{di})] - (c \leftrightarrow d). \tag{2.F.4}$$

Now by multiplying the above expression by P^{ibcd} and using the antisymmetry properties of P^{ibcd} in index i and b, we can write from (2.F.4) that ,

$$P_a{}^{bcd}\delta R^a{}_{bcd} = 2P^{ibcd}\nabla_c\nabla_b\delta g_{di}. \tag{2.F.5}$$

Using the properties of covariant derivatives, (2.F.5) can be written as,

$$P_a{}^{bcd}\delta R^a{}_{bcd} = 2\nabla_c[P^{ibcd}\nabla_b\delta g_{di}] - 2\nabla_b[\delta g_{di}\nabla_c P^{ibcd}] + 2\delta g_{di}\nabla_b\nabla_c P^{ibcd}. \tag{2.F.6}$$

Now combining the result (2.F.3) with (2.F.6) and putting in (2.F.2), we get the variation of action as,

$$\delta S = \delta\int d^dx\sqrt{-g}\mathcal{L}[g^{ab}, R^a{}_{bcd}] = \int d^dx\sqrt{-g}[E_{ab}\delta g^{ab} + \nabla_j\delta v^j]; \tag{2.F.7}$$

where ,

$$E_{ab} = \sqrt{-g}[P_{(b}{}^{kij}R_{a)kij} - \frac{1}{2}g_{ab}\mathcal{L} - 2\nabla^m\nabla^n P_{amnb}]; \tag{2.F.8}$$

and

$$\nabla_j\delta v^j = 2\nabla_c[P^{ibcd}\nabla_b\delta g_{di}] - 2\nabla_b[\delta g_{di}\nabla_c P^{ibcd}]. \tag{2.F.9}$$

Let us consider the variation of action due to the diffeomorphism $x^a \to x^a + \chi^a$. Under diffeomorphism, variation of Lagrangian density will be,

$$\pounds_\chi(\sqrt{-g}\mathcal{L}) = \sqrt{-g}[E_{ab}(\pounds_\chi g^{ab}) + \nabla_j(\pounds_\chi v^j)]. \tag{2.F.10}$$

The second term in the right hand side of (2.F.10) is boundary term. Lie variation of $(\sqrt{-g}\mathcal{L})$ will be,

$$\pounds_\chi(\sqrt{-g}\mathcal{L}) = (\pounds_\chi\mathcal{L})\sqrt{-g} + \mathcal{L}(\pounds_\chi\sqrt{-g})$$
$$= \chi^a\nabla_a\mathcal{L}(\sqrt{-g}) + \mathcal{L}(-\frac{1}{2}g_{ab}\,\pounds_\chi g^{ab})\sqrt{-g} = \sqrt{-g}\chi^a\nabla_a\mathcal{L} + \sqrt{-g}\mathcal{L}\nabla_a\chi^a$$
$$= \sqrt{-g}\nabla_a(\mathcal{L}\chi^a) \tag{2.F.11}$$

Now $E_{ab}(\pounds_\chi g^{ab})$ is calculated as,

$$E_{ab}(\pounds_\chi g^{ab}) = -E_{ab}(\nabla^a\chi^b + \nabla^b\chi^a) = -2E_{ab}\nabla^a\chi^b = -2\nabla_a(E^a_b\chi^b). \tag{2.F.12}$$

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Here we have used the Bianchi identity that $\nabla_a E^{ab} = 0$. Hence combining (2.F.12) and (2.F.11) we can write,

$$\begin{aligned}\sqrt{-g}\nabla_a(\mathcal{L}\chi^a) &= \sqrt{-g}[-2\nabla_a(E^a_b\chi^b) + \nabla_j(\pounds_\chi v^j)] \\ or \sqrt{-g}\nabla_a[\mathcal{L}\chi^a &+ 2E^a_b\chi^b - \pounds_\chi v^a] = 0.\end{aligned} \tag{2.F.13}$$

So under diffeomorphism, variation of action must vanish and hence we identify conserved current as,

$$J^a = \mathcal{L}\chi^a + 2E^a_b\chi^b - \pounds_\chi v^a. \tag{2.F.14}$$

Here we have obtained the expression of conserved current J^a without using the field equations. So the charges will be off-shell in nature.

Now we elaborately compute all the term appeared in (2.F.14) separately. First we concentrate on the term $\pounds_\chi v^a$. It can be expressed as,

$$\begin{aligned}\pounds_\chi v^c &= 2P^{ibcd}\nabla_b(\pounds_\chi g_{id}) - 2(\pounds_\chi g_{id})\nabla_b P^{icbd} \\ &= 2P^{ibcd}\nabla_b(\nabla_i\chi_d + \nabla_d\chi_i) - 2(\nabla_i\chi_d + \nabla_d\chi_i)\nabla_b P^{ibcd}.\end{aligned} \tag{2.F.15}$$

Hence substituting the expression of E_{ab} from (2.F.8) and also putting the result (2.F.15) in (2.F.14), finally J^a will be,

$$\begin{aligned}J^a &= 2P^{akij}\, R_{bkij}\chi^b - 4\chi_b\nabla_m\nabla_n P^{amnb} - 2P^{ibcd}\nabla_b(\nabla_i\chi_d + \nabla_d\chi_i) \\ &+2(\nabla_i\chi_d + \nabla_d\chi_i)\nabla_b P^{ibcd}.\end{aligned} \tag{2.F.16}$$

Now we use the algebraic properties of curvature tensor to simplify the expression (2.F.16). Note that P^{abcd} has same algebraic symmetries of curvature tensor R^{abcd}. The properties are as follows,

$$R_{abcd} = -R_{bacd}; \quad R_{abcd} = -R_{abdc}; \quad R_{abcd} = R_{cdab}; \quad R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{2.F.17}$$

The covariant derivative satisfy the following relation as given by,

$$[\nabla_a, \nabla_b]\chi^i = R^i{}_{cab}\chi^c. \tag{2.F.18}$$

Using the properties (2.F.17) and (2.F.18) in (2.F.16) repeatedly, after rearranging the terms, (2.F.16) becomes,

$$\begin{aligned}J^a &= -4\chi_d\nabla_b\nabla_c P^{abcd} - 2\nabla_b(P^{acbd} + P^{adbc})(\nabla^c\chi_d) \\ &+2P^{acbd}\nabla_b\nabla_c\chi_d.\end{aligned} \tag{2.F.19}$$

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The above can be re-expressed as follows,

$$\begin{aligned}
J^a &= -2\nabla_b(P^{abcd}+P^{adbc})(\nabla_c\chi_d)+2\nabla_b(P^{abcd}\nabla_c\chi_d)-2\nabla_b P^{abcd}(\nabla_c\chi_d) \\
&\quad -4\nabla_b(\chi_d\nabla_c P^{abcd})+4(\nabla_b\chi_d)\nabla_c P^{abcd} \\
&= \nabla_b[2P^{abcd}\nabla_c\chi_d-4\chi_d\nabla_d P^{abcd}]+2\nabla_b[P^{acbd}-P^{adbc}-P^{abcd}](\nabla_c\chi_d) \\
&= \nabla_b[2P^{abcd}\nabla_c\chi_d-4\chi_d\nabla_d P^{abcd}]+2\nabla_b[P^{acbd}+P^{adcb}+P^{abdc}](\nabla_c\chi_d)
\end{aligned} \tag{2.F.20}$$

In the above expression second term will be zero by the properties given in (2.F.17). Then the first term of (2.F.20) gives the expression of J^{ab} as presented in (2.20) provided $\nabla_d P^{abcd}=0$.

2.G An off-shell derivation of gauge current

The variation of the matter action (2.16) for an arbitrary change in field $A_a \to A_a+\delta A_a$ is given by

$$\delta\mathcal{L}_{em} = \int d^dx\sqrt{-g}\Big[\frac{\partial f}{\partial A_a}\delta A_a+\frac{\partial f}{\partial(\nabla_a A_b)}\delta(\nabla_a A_b)\Big]\,. \tag{2.G.1}$$

Now since f is function of F_{ab} only, the first term will vanish. Denoting $\partial f/\partial F_{ab}=f^{ab}$, we find

$$\begin{aligned}
\delta\mathcal{L}_{em} &= \int d^dx\sqrt{-g}f^{mn}\delta(\nabla_m A_n-\nabla_n A_m) \\
&= 2\int d^dx\sqrt{-g}f^{mn}\nabla_m\delta A_n\,.
\end{aligned} \tag{2.G.2}$$

Now if this variation is due to the gauge transformation $A_a\to A_a+\nabla_a\epsilon$, then the above equation reduces to

$$\delta\mathcal{L}_{em}=2\int d^dx\sqrt{-g}f^{mn}\nabla_m\nabla_n\epsilon\,. \tag{2.G.3}$$

Since f^{mn} is an antisymmetric tensor, the above equation can be expressed as a total derivative without using the equation of motion. The steps are as follows:

$$\begin{aligned}
\delta\mathcal{L}_{em} &= 2\int d^dx\sqrt{-g}\Big[\nabla_m(f^{mn}\nabla_n\epsilon)-(\nabla_m f^{mn})(\nabla_n\epsilon)\Big] \\
&= 2\int d^dx\sqrt{-g}\Big[\nabla_m\Big\{\nabla_n(f^{mn}\epsilon)-\epsilon\nabla_n f^{mn}\Big\}-(\nabla_m f^{mn})(\nabla_n\epsilon)\Big] \\
&= 2\int d^dx\sqrt{-g}\Big[\nabla_m\nabla_n(f^{mn}\epsilon)-(\nabla_m\epsilon)(\nabla_n f^{mn})-\epsilon\nabla_m\nabla_n f^{mn} \\
&\quad -(\nabla_m f^{mn})(\nabla_n\epsilon)\Big].
\end{aligned} \tag{2.G.4}$$

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The third term vanishes as f^{mn} is antisymmetric tensor while the second and last terms cancel each other. Therefore we are left with

$$\delta\mathcal{L}_{em} = 2\int d^dx\sqrt{-g}\nabla_m\nabla_n(f^{mn}\epsilon) = 2\int d^dx\sqrt{-g}\nabla_m J^m \tag{2.G.5}$$

Since the action has this gauge symmetry, its variation must vanish, and hence we identify the conserved current as given in eq. (2.28).

2.H Surface terms and corresponding Noether charge are same on horizon

The entropy of the horizon is identified as

$$\begin{aligned} S &= \frac{2\pi}{\kappa}\frac{1}{2}\int d\Sigma_{ab}J^{ab} \\ &= \frac{1}{16\pi G}\int dtd^{d-2}x\sqrt{\mu}(N_aT_b - N_bT_a)J^{ab}\,, \end{aligned} \tag{2.H.1}$$

where the periodicity of the Euclidean time has been adopted in the last step. For horizon N^a is the spacelike unit normal; i.e. $N^2 = +1$ while T^a is the unit timelike normal: $T^2 = -1$. μ is the determinant of the horizon induced metric. Next substituting the value of J^{ab} from (2.34) and using the fact that ζ^a is the timelike Killing vector, we find

$$S = \frac{1}{8\pi G}\int d^{d-1}x\sqrt{\mu}K(-T_a\zeta^a) = \frac{1}{8\pi G}\int d^{d-1}x\sqrt{h}K\,, \tag{2.H.2}$$

where h is the determinant of the induced metric on the radial coordinate constant surface, which is timelike. This clearly shows that the two quantities, one is the GHY term and the other one is the corresponding Noether charge multiplied by the periodicity of the Euclidean time, are the same on the horizon. This is why both of them give the same quantity – entropy of the horizon.

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Near horizon symmetries, emergence of Goldstone modes and thermality 3

3.1 Introduction

In the last chapter, we have elaborately studied the diffeomorphism symmetries near arbitrary charged null hypersurfaces for extremal and non-extremal cases. The analysis was completely off-shell, and also, we have shown that our results are valid for any order Lanczos-Lovelock theory of gravity. In the present chapter, we will follow a slightly different direction compared to the earlier one. In an earlier chapter, we have assumed that the symmetry parameters will have Fourier mode expansion in v (in the form e^{imv}) because of the periodicity of v. This assumption matches with the analysis found in the existing literature (see [97, 104] and the other references therein). However, in the present chapter, we will have no prior assumption like that one. Rather we will try to find the possible mode solution of those near horizon symmetry parameters by incorporating some interesting physical phenomena which is the spontaneous symmetry breaking. Here we will show in detail that under the specific set of diffeomorphism corresponding to null boundary symmetries, the macroscopic quantities associated with black hole get transformed. In analogy with $U(1)$ symmetry breaking, this phenomenon can be described as the spontaneous symmetry breaking caused by the background spacetime itself. Therefore we will identify the associated symmetry parameters as the Goldstone modes. Then by studying the dynamics of those modes, we will determine a possible set of mode solutions for the parameters.

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We know that the symmetries in nature are broadly classified into two categories. The symmetry that acts globally on the physical fields is called global symmetry. Most importantly, for each continuous global symmetry, there is an associated conserved charge that encodes essential properties of the system under consideration. Another class of symmetry that acts locally on the fields, generally known as gauge symmetry, makes the description of the system redundant. Unlike global continuous symmetries, gauge symmetry does not have associated nontrivial conserved charge [191][192]. However, the unique property of a continuous global symmetry lies in its spontaneous breaking phenomena, which plays a significant role in understanding the low energy behaviour of the system under consideration. If a continuous global symmetry of a system breaks spontaneously, the associated Goldstone boson mode emerges, whose dynamics will characterise the underlying states and their properties of the system [193]. On the other hand, breaking of gauge symmetry is inherently inconsistent with the theory under consideration.

However, the generic underlying symmetry of a gravitational theory is spacetime diffeomorphism which is a set of local general coordinate transformations. Therefore, diffeomorphism can be thought of as a gauge symmetry of the gravitational theory. But it is well known from the idea of the bulk-boundary correspondence that a gauge symmetry in bulk acts as a nontrivial global symmetry at the boundary. One of the popular and important examples of such a correspondence is the well-known global Bondi-Metzner-Sachs (BMS) group [64, 65, 67] of transformation. Therefore, even if the gravitational theory can be formulated as a gauge theory, the theory of Goldstone modes can still be applicable, and information about the microscopic gravitational states may be extracted from the global boundary symmetry. Hence the Goldstone modes phenomenon in the context of BMS symmetry has been elaborately studied and explored in some recent literatures [164, 165, 184, 194, 195] and also in [171, 196]. Specifically, in [164, 184] it has been discussed that the spontaneous breaking of symmetries by the black hole geometries gives rise to the Goldstone modes, which can act as the promising candidate to understand the thermodynamics properties of the black hole, especially to determine the microscopic degrees of freedom of black hole entropy. Although the appearance of Goldstone modes in the context of BMS symmetry exists, its dynamical behavior has not been studied concretely. It is believed that the dynamics of those modes should also play a crucial role in uncovering the microscopic nature of the black holes. Having set this motivation,

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in the present chapter, we will study the dynamics of those Goldstone modes following the standard procedure.

In order to clarify and better understand the methodology followed in this chapter, let us consider the emergence of Goldstone boson mode for a well known U(1) invariant complex scalar field theory with the following Lagrangian,

$$\mathcal{L} = 1/2(\partial_\mu \phi \partial^\mu \phi^\dagger) - V(\phi\phi^\dagger). \tag{3.1.1}$$

The background solution such as $\phi_0 = c$ naturally breaks U(1) symmetry which transforms the vacuum as

$$\phi_0' \to \mathrm{e}^{i\pi(x)}\phi_0 = c + ic\pi(x). \tag{3.1.2}$$

Now we can identify the $\pi(x)$ as Goldstone boson field, and calculate the Lagrangian as follows

$$\begin{aligned}\mathcal{L}_\pi &= \frac{1}{2}(\partial_\mu \phi_0' \partial^\mu \phi_0'^\dagger) - V(\phi_0'\phi_0'^\dagger)\\ &= \partial_\mu (c + ic\pi(x))\partial^\mu (c - ic\pi(x)) - V(c + ic\pi(x))\\ &= \frac{c^2}{2}\partial_\mu \pi(x))\partial^\mu \pi(x) + \cdots .\end{aligned} \tag{3.1.3}$$

The last expression should be the leading order Goldstone boson Lagrangian associated with the broken U(1) symmetry (more detail can be found in [197]). Throughout our following discussions in this chapter, we will use this analogy to understand the dynamics of the Goldstone mode in the gravity sector.

3.2 Objective of the chapter

In the present chapter, we will be trying to understand the dynamics of the Goldstone boson modes associated with a special class of global symmetry arising at the boundary of spacetime with a nontrivial gravitational background. In the first half of the present chapter, our focus will be on the Killing horizon of Rindler spacetime with a flat spatial section. In the latter half, we will consider the asymptotically flat Schwarzchild black hole. Those horizons behave like another null boundary where bulk diffeomorphism acts non-trivially in terms of BMS-like global symmetry (see [92, 172, 185, 198] and the other references therein). Once we have a gravitational background, we will first identify the global symmetry associated with the null boundary surface imposing the appropriate boundary conditions. Boundary conditions will be such that the near horizon

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form of the metric must remain unchanged after the symmetry transformation. However, macroscopic quantities such as mass, charge, and angular momentum characterizing the physical states of a black hole under consideration will change under those symmetry transformations. Such transformation of the black hole parameters can be understood as the spontaneous symmetry breaking phenomena near the horizon. We, therefore, expect the associated dynamical Goldstone boson modes. As mentioned earlier in this chapter, we will study the dynamics of those Goldstone boson modes, which may shed some light on the possible microscopic states of the black holes.

3.3 Rindler Background

In this section, we will consider the simplest background and try to understand the symmetry breaking phenomena described in the introduction. The Rindler metric, in the Gaussian null coordinate is expressed as

$$ds^2 = -2r\alpha dv^2 + 2dvdr + \delta_{AB}dx^A dx^B \, . \tag{3.3.1}$$

The Rindler horizon is located at $r = 0$. x^A stands for the flat Rindler coordinate y and z. α is the acceleration parameter which characterizes the macroscopic state of the background spacetime. Symmetry properties of the horizon, following the last chapter, can be extracted from the following fall off and gauge conditions,

$$\pounds_\zeta g_{rr} = 0, \quad \pounds_\zeta g_{vr} = 0, \quad \pounds_\zeta g_{Ar} = 0\,; \tag{3.3.2}$$

$$\pounds_\zeta g_{vv} \approx \mathcal{O}(r); \quad \pounds_\zeta g_{vA} \approx \mathcal{O}(r); \quad \pounds_\zeta g_{AB} \approx \mathcal{O}(r)\, . \tag{3.3.3}$$

Here, $\pounds_\zeta$ corresponds to the Lie variation for the diffeomorphism $x^a \to x^a + \zeta^a$. The above conditions are satisfied for the following form of the diffeomorphism vector,

$$\zeta^a \partial_a = \quad F(v,y,z)\partial_v - r\partial_v F(v,y,z)\partial_r - r\partial^A F(v,y,z)\partial_A\, . \tag{3.3.4}$$

This is derived following the prescription of chapter 2. Note that in this case, we have only one diffeomorphism parameter F, which characterizes the symmetry of the Rindler horizon. Since for constant F, it essentially gives the time translation, the general form of this time diffemorphism, which acts non-trivially on the $r = 0$ hypersurface, is called supertranslation. For details of this analysis, we refer to [91, 166, 172].

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Next, let us first obtain the modified metric which is consistent with the aforementioned gauge (3.3.2) and fall-off (3.3.3) conditions. The reason for constructing this transformed form of the Rindler metric will be clear later. Here important point to remember is that the Lie variation of the metric component in our analysis is defined up to the linear order in ζ^a and hence we express the form of ζ^a (3.3.4) valid up to linear order in F. Under this diffeomorphism vector (3.3.4), the modified metric takes the following form (details are given in Appendix 3.A.1):

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b = \Big[g^{(0)}_{ab} + h_{ab}\Big]dx^a dx^b \\ &= -2r\alpha dv^2 + 2dvdr + \delta_{AB}dx^A dx^B \\ &+ \Big[-2r\Big(\alpha\partial_v F + \partial_v^2 F\Big)\Big]dv^2 + \Big[-4r\Big(\alpha\,\partial_A F + \partial_A\partial_v F\Big)\Big]dvdx^A \\ &+ \Big[-4r\partial_A\partial_B F\Big]dx^A dx^B\,. \end{aligned} \tag{3.3.5}$$

In the above, $g^{(0)}_{ab}$ is the original unperturbed metric (3.3.1), whereas all linear in F terms are incorporated in $h_{ab} = \pounds_\zeta g^{(0)}_{ab}$. Under the following supertranslation symmetry transformation,

$$\begin{aligned} &v' = v + F(v,x^A)\,,\ x'^A = x^A - r\partial^A F(v,x^A), \\ &r' = r - r\partial_v F(v,x^A). \end{aligned} \tag{3.3.6}$$

we can clearly see the macroscopic state parameter α of the original Rindler background transforms into

$$\alpha \to \alpha + \Big(\alpha\partial_v F + \partial_v^2 F\Big). \tag{3.3.7}$$

We have shown that under the diffeomorphism (3.3.4), some of the metric coefficients have transformed (3.3.5). This can be thought of as similar to the transformation (3.1.2), which breaks the $U(1)$ symmetry. The corrections in the metric coefficients are determined by the supertranslation parameter F. Consequently, the macroscopic parameters of the original metric will be modified. For instance, the parameter α gets shifted to α' keeping the near horizon boundary conditions intact. Strictly speaking, the above phenomena does not show the breaking of the near horizon symmetries. However, this indicates that among all possible metric solutions, characterised by the arbitrariness of F, Rindler is only one choice. Hence the background Rindler spacetime breaks the symmetry of considering all possible values of the parameter F. This nature is identical to spontaneous symmetry breaking by the vacuum expectation value (VEV) of fields in QFT if we think the choice as Rindler is similar to VEV in QFT. Therefore the parameter F

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which characterizes this symmetry breaking, is promoted as the Goldstone boson modes [184, 195].

Since α appears as a Lagrange multiplier in the Hamiltonian formulation, one usually chooses the gauge where variation of α is zero everywhere [171, 196]. However strictly speaking this is not a generic choice. It is sufficient to set the variation of α to be zero only at the boundary for consistency,

$$\delta\alpha(-\infty, x^A) = \lim_{v\to-\infty}\left(\alpha\partial_v F + \partial_v^2 F\right) = 0, \tag{3.3.8}$$

where horizon is located at $v \to -\infty$. One of the obvious choices to satisfy the above condition is to set the total variation $\delta\alpha$ to be zero everywhere. This naturally sets the boundary condition at the horizon and furthermore makes the field F non-dynamical. Therefore, we believe this restrictive condition does not capture the full potential of the Goldstone modes. This analysis aims to go beyond and understand the dynamics of these Goldstone modes, which could be the potential candidate for the underlying degrees of freedom of the black hole. Therefore, we first construct an appropriate Lagrangian of this mode, and finally, at the solution level, we set the boundary condition such that Eq. (3.3.8) is automatically satisfied at the horizon. Important to note that if one allows the fluctuation of α even at the boundary, one needs to take care of the appropriate boundary terms (e.g., see [199, 200]).

3.3.1 Dynamical equation for F

As we have already pointed out, in order to study the dynamics for F we propose the Lagrangian $\mathcal{L}_F$ associated with the newly perturbed metric (3.3.5) near the $r = 0$ surface:

$$\mathcal{L}_F = \sqrt{-g}R\,. \tag{3.3.9}$$

Here R is the Ricci scalar calculated for the newly constructed metric g_{ab} (3.3.5) and g is the corresponding determinant. Before proceeding further, we want to mention an important point of our proposed form of the Goldstone boson Lagrangian. Since the modified metric (3.3.5) has been constructed by taking into account a particular type of diffeomorphism, one always concludes that the Lagrangian must be invariant up to a total derivative. The contribution of the total derivative term vanishes over the closed boundaries, which encloses the bulk of the manifold. For instance, in the case of $\sqrt{-g}R$, the variation of it under

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diffeomorphism $x^a \to x^a + \xi^a$ leads to $\sqrt{-g}\nabla_a(R\xi^a) = \partial_a(\sqrt{-g}R\xi^a)$, which is a total derivative term. In this analysis, we are interested in building a theory on the horizon (i.e., $r = 0$), and the horizon is a part of the closed boundary of the bulk manifold. Therefore it is expected that the total derivative term will give a non-vanishing contribution on the part of the closed surfaces such as the horizon. In the case of $\sqrt{-g}R$, the boundary term on $r =$ constant surface in the variation of action is given by $\int d^3x\hat{n}_a\xi^a\sqrt{-g}R$, where $\hat{n}_a$ is the normal to the surface with components $(0, 1, 0, 0)$. Therefore, on this surface, our proposal for the Lagrangian density (loosely call it as Lagrangian) is $\sqrt{-g}R$. This is precisely considered here. The Lagrangian (3.3.9) is not the one that is defined for the whole spacetime, rather it is calculated on the $r =$ constant surface, and hence coming out to be nontrivial.

To study the dynamics of the Goldstone mode associated with the horizon symmetry, first, we will compute the above Lagrangian Eq. (3.3.9) at an arbitrary r value in the bulk spacetime. Then we take the limit $r \to 0$. This procedure is similar to the stretched horizon approached in black hole thermodynamics (for example, see the discussion in section 4 of [62]). In this approach, if we are interested in finding any quantity on a particular surface (say $r = 0$), one usually first calculates the same just away from this surface (say $r = \epsilon$, with ϵ, is very small). After that, the obtained value is derived by taking the limit $\epsilon \to 0$.

Now we are in a position to expand our Lagrangian (3.3.9) in terms of the transformed metric (3.3.5). If the background metric components are $g_{ab} = g^{(0)}_{ab} + h_{ab}$, with h_{ab} as a small fluctuation, in general the Taylor series expansion of the Lagrangian around background metric $g_{ab} = g^{(0)}_{ab}$ can be written as

$$\mathcal{L}_{\mathcal{F}} = \mathcal{L}_{\mathcal{F}}(g^{(0)}_{ab}) + h_{ab}\Big(\frac{\delta\mathcal{L}_{\mathcal{F}}}{\delta g_{ab}}\Big)^{(0)} + h_{ab}h_{cd}\Big(\frac{\delta^2\mathcal{L}_{\mathcal{F}}}{\delta g_{ab}\delta g_{cd}}\Big)^{(0)} + \dots \tag{3.3.10}$$

The first term of the above equation obviously does not contribute to the dynamics. Given the background metric to be a solution of the equation of motion, the second term vanishes as it is essentially proportional to Einstein's equation of motion. The third term introduces the quadratic form for the Goldstone field F. For our purpose in the present analysis, we will restrict only to Lagrangian for the Goldstone mode, which is at the quadratic order. It should be mentioned that any higher-order contribution to g_{ab} in (3.3.5), will not affect the second-order term given in (3.3.10). All the higher order in F-terms we left for our future discussions.

The final form of the Lagrangian (3.3.9) after taking the near Horizon limit

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comes out as (detail procedure has been discussed in Appendix 3.A.2),

$$\begin{aligned}\mathcal{L}_F &= \lim_{r\to 0}\left(\sqrt{-g}R\right)\\ &= \Big[-6\alpha^2\partial_y F\partial_y F - 6\alpha^2\partial_z F\partial_z F + 4\alpha\partial_v F\partial_y^2 F - 12\alpha\partial_z F\partial_v\partial_z F - 6(\partial_v\partial_z F)^2\\ &- 12\alpha\partial_y F\partial_v\partial_y F - 6(\partial_v\partial_y F)^2 + 4\partial_y^2 F\partial_v^2 F + 4\partial_z^2 F(\alpha\partial_v F + \partial_v^2 F)\Big]\,.\end{aligned} \tag{3.3.11}$$

Since $\mathcal{L}_F$ is calculated on a $r =$ constant surface, the action can be defined as the integration of the above Lagrangian on v, y and z.

The induced horizon geometry has a flat spatial section. We, therefore, consider the following generic form of F:

$$F_{mn} = f_{mn}(v)\frac{1}{\alpha}\exp\Big[i(my+nz)\Big]\,. \tag{3.3.12}$$

Hence the general solution for Goldstone mode would be,

$$F(v,y,z) = \sum_{m,n} C_{mn}F_{mn}\,. \tag{3.3.13}$$

Here we need to find the form of $f_{mn}(v)$ from the solution of the equation of motion obtained from (3.3.11). It is quite apparent that substitution of the above ansatz in (3.3.11) and then integrating over transverse coordinates, one can get a one-dimensional action which determines the evaluation of $f_{mn}(v)$ with respect to v. Since the total derivative terms in action keep the dynamics unchanged, it may be verified (given in Appendix 3.A.2) that under the integration of transverse coordinates, third, fourth, sixth and ninth terms in (3.3.11) are total derivative terms with respect to v. So, those terms can be neglected. Ignoring total derivative terms the final form of the Lagrangian (3.3.11) is given by (detail in Appendix 3.A.3);

$$\begin{aligned}\mathcal{L}_F &= \Big[-6\alpha^2\partial_y F\partial_y F - 6\alpha^2\partial_z F\partial_z F - 6(\partial_v\partial_y F)^2\\ &- 6(\partial_v\partial_z F)^2 + 4\partial_y^2 F\partial_v^2 F + 4\partial_z^2 F\partial_v^2 F\Big]\,.\end{aligned} \tag{3.3.14}$$

Next we concentrate on Gibbons-Hawking-York (GHY) boundary term

$$\mathcal{S}_2 = -\frac{1}{8\pi G}\int d^3x\sqrt{h}K\,, \tag{3.3.15}$$

which is usually added to the action in order to define a proper variation of the action. The trace of the extrinsic curvature of the boundary surface ($r \to$

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0) is given by $K = -\nabla_a N^a$, where N^a is considered as the unit normal to the $r = constant$ hyper-surface. For metric (3.3.5), its lower component is given by $N_a = (0, 1/\sqrt{2r(\alpha + \alpha\partial_v F + \partial_v^2 F)}, 0, 0)$. Therefore in the near horizon limit ($r \to 0$), one gets the following form of the action coming from the GHY term :

$$\mathcal{S}_2 = -\frac{1}{8\pi G}\int d^3x\Big[\alpha + \Big(\alpha\partial_v F + \frac{1}{2}\partial_v^2 F + \frac{1}{2\alpha}\partial_v^3 F\Big) + \frac{1}{2\alpha^2}\Big(\alpha^2\partial_v F\partial_v^2 F + \alpha(\partial_v^2 F)^2 + \alpha\partial_v F\partial_v^3 F + \partial_v^2 F\partial_v^3 F\Big)\Big]\,. \tag{3.3.16}$$

However, we observed that this term does not contribute to our required equation of motion as it produces a constant term which is independent of F plus total derivative terms on the horizon boundary at $r = 0$ (detail in Appendix 3.A.4). In fact, the above boundary term in action is turned out to be related to the horizon entropy, which is discussed in Appendix 3.D.

In this scenario, one important point remains left to be discussed here. As discussed before, the Lagrangian is calculated using a stretched horizon approach by considering a timelike surface very near the horizon. So one may consider the GHY term as the correct boundary term to have a proper variational principle of the action. However, it has been shown in [201, 202] that if one wants to perform any physical analysis exactly on the null surface, then from the first principle, one can prescribe a well defined counter term on this surface. Therefore it will be interesting to study the present analysis considering the null boundary counter term.

Note that the aforesaid Lagrangian (3.3.11) contains higher derivative terms of F. Therefore, the theory of Goldstone boson modes emerging on the boundary of a gravitational theory turns out to be higher derivative in nature. However, if we want to trace back the origin of this higher derivative action, it is from the diffeomorphically transformed metric components that contain the derivative term. However, we will see those higher derivative terms will be crucial for our subsequent discussions on the horizon properties. This connection could be an interesting topic to investigate further. The generalized Euler-Lagrangian equation, defined for higher derivative theory, is (Derivation is shown in Appendix 3.C):

$$\frac{\partial L}{\partial F} - \partial_\mu\Big(\frac{\partial L}{\partial(\partial_\mu F)}\Big) + \partial_\mu\partial_\nu\Big(\frac{\partial L}{\partial(\partial_\mu\partial_\nu F)}\Big) = 0. \tag{3.3.17}$$

With this the equation of motion for the Lagrangian corresponding to (3.3.14) is found to be (detail in Appendix 3.A.5),

$$3\alpha^2\partial_y^2 F + 3\alpha^2\partial_z^2 F - 4\partial_y^2\partial_v^2 F - 4\partial_z^2\partial_v^2 F = 0\,. \tag{3.3.18}$$

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Important to note again, the contribution on the equation of motion comes only from from (3.3.14). GHY (3.3.16) term does not contribute.

Substitution of (3.3.12) in (3.3.18) yields

$$(m^2+n^2)[\partial_v^2 f_{mn}(v) - \frac{3\alpha^2}{4} f_{mn}(v)] = 0\,. \tag{3.3.19}$$

Important point to note that every individual mode (m,n), will follow the same equation of motion in an inverted harmonic potential. The solution will be,

$$f_{mn}(v) \quad = A\exp\left[(\sqrt{3/4})\alpha v\right] + B\exp\left[-(\sqrt{3/4})\alpha v\right] + f_1(y,z)\delta_{m,0}\delta_{n,0}\,, \tag{3.3.20}$$

for all m,n. In the above, A and B and $f_1(y,z)$ are arbitrary constants to be determined. The last term in (3.3.20) has been appeared as the equation of motion given in (3.3.19) is trivially satisfied when $m = n = 0$. In this situation $f_{00}(v)$ becomes completely arbitrary. So we have to add that term to the solution of f_{mn}.

So far, we have talked about the classical dynamics of the Goldstone mode. It is apparent that at least at the tree level Lagrangian, the system is unstable because of the inverted harmonic potential. This is also apparent from the solution (3.3.20). As we are interested in the near-horizon region where $v \to -\infty$, the above solution grows rapidly and makes the mode unstable. Therefore, the appropriate boundary condition one can set is $B = 0$, leading to

$$F_{mn}(v,y,z) \quad = [A\exp\left[(\sqrt{3}/2)\alpha v\right] + f_1(y,z)\delta_{m,0}\delta_{n,0}]\,\frac{1}{\alpha}\exp[i(my+nz)]\,. \tag{3.3.21}$$

Interestingly this is precisely the boundary condition that satisfies the requirement of vanishing fluctuation of surface gravity $\delta\alpha = 0$ at the horizon defined by the Eq. (3.3.8).

We already know that the horizon is a special place in the entire spacetime region, as any two hypothetical observers spatially separated by the horizon can never communicate with each other. Therefore, it would have been unusual had there been just a simple, stable free field like Lagrangian for the Goldstone modes. The connection between the special nature of the horizon and the emergence of instability has been the subject of study for a long time. Our goal of this analysis would be to shed some light on this issue. *Does the emergence of the inverted harmonic potential has anything to do with the thermal nature of the black hole horizon?* Of course, in order to understand this, we need to go beyond the classical regime. In the next section, we will try to make this connection considering a recent proposal [203, 204].

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3.3.2 Thermal behaviour of the mode solution

In this section, we consider the quantum mechanical treatment of the Goldstone boson model discussed so far. It has recently been conjectured that Lyapunov exponent λ of a thermal quantum system, in the presence of quantum chaos, is bounded by the temperature T of the system as $\lambda \leq 2\pi T/\hbar$ [205]. Based on this result, further conjecture has been made in the reference [204, 206] which says a chaotic system with a definite Lyapunov exponent could be fundamentally thermal by reversing the above inequality. To justify the argument, one of the interesting examples the author has studied is the semi-classical dynamics of a particle in an inverted harmonic potential*, and showed that the quantum correction induces an energy emission by the particle under study obeying thermal probability distribution. Therefore, the connection between the semi-classical chaotic system and the thermal nature has emerged. Interestingly, for our present system, each Goldstone boson mode behaves like an inverted harmonic oscillator. Hence, the aforesaid connection between the thermal emission and the semi-classical chaotic dynamics could be a potential reason for the thermal nature of the black hole horizon. Interestingly, every individual Goldstone boson model parameterized by (m, n) sees the same inverted potential, which may also indicate the universality of the thermal nature of the horizon. *Our present claim is ambitious and exciting, which needs detailed future exploration and generality*. In the next chapter, we will explore this possibility for Kerr black hole.

Before we resort to our discussion of the thermal nature of the black hole, following from the reference [203][204], let us briefly describe the connection between the thermality and the inverted harmonic oscillator. These are connected with the finite quantum mechanical transition probability through a potential barrier. The equation of motion of the particle moving in a inverted harmonic potential is given by,

$$\mu\ddot{x} - \bar{\omega}x = 0 \tag{3.3.22}$$

Here potential $V = -\frac{\bar{\omega}x^2}{2}$ and $\bar{\omega} = \mu\omega^2$. The angular frequency is given by ω, and μ is the particle's mass. $\mathcal{E}$ gives the energy of the particle. An important case would be, if one considers the energy of the particle $\mathcal{E} < 0$, For which the potential energy of the particle is greater than its kinetic energy. With this energy, if the

*The choice of the inverted harmonic oscillator stems from the fact that the particle motion is unstable under this potential and hence, at the classical level, any small perturbation can lead induction of chaos in the motion (for example, see [207–210]).

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particle travels toward the potential from the left ($x < 0$), classically, it cannot pass through the potential towards the right ($x > 0$). However, quantum mechanically, the particle will have finite tunneling probability to go across the potential barrier. Therefore, the particle will have a finite probability of transmission through the barrier even for $\mathcal{E} < 0$. Similarly, for $\mathcal{E} > 0$, the particle will have the finite quantum mechanical probability of reflection off the barrier, which otherwise was not possible classically.

Therefore, to describe the above quantum mechanical phenomena, the appropriate Hamiltonian for the wave function $\Phi(x)$ associated with the particle is expressed as

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} - \frac{\bar{\omega}x^2}{2} \tag{3.3.23}$$

with the Schrödinger equation,

$$-\frac{\hbar^2}{2\mu}\frac{d^2\Phi}{dx^2} - \frac{\bar{\omega}x^2}{2}\Phi = \mathcal{E}\Phi\,. \tag{3.3.24}$$

The well known expression for the probability of transmission (P_T) and the reflection (P_R) using WKB approximation [211] are (also the derivation is shown in Appendix 3.E as well) ,

$$P_{T/R} = \frac{1}{e^{\frac{2\pi}{\hbar}\sqrt{\frac{\mu}{\bar{\omega}}}|\mathcal{E}|} + 1} = \frac{1}{e^{\beta|\mathcal{E}|} + 1}\,. \tag{3.3.25}$$

An exciting interpretation of this expression is that for a large absolute value of the energy $\mathcal{E}$, probability amplitude from the classical path to quantum transmission or reflection will be $\exp[-\beta|\mathcal{E}|]$. Therefore, the quantum harmonic oscillator system can be mapped to a two-level system with temperature T, whose ground state is represented as the classical trajectories and excited state is a quantum one. And the temperature of the system can be easily identified as

$$T = \frac{\hbar}{2\pi}\sqrt{\frac{\bar{\omega}}{\mu}}\,. \tag{3.3.26}$$

For further detail of this interesting interpretation the reader can look into the reference [203, 204]). In this context, it is worth mentioning that recently the inverted harmonic oscillator has been shown to give rise to the temperature at the quantum level in an independent and completely different way [212].

In our present analysis, we have obtained the dynamical equation of motion for individual mode as given in (3.3.19). Comparing this with Eq. (3.3.22) one can

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easily conclude that for $m \neq 0$ and/or $n \neq 0$ or vice-versa, the dynamics of the mode along v is governed by inverted harmonic oscillator potential. To clarify our analogy, each mode $f_{mn}(v)$ can be thought of as the position $x(t)$ of a particle of mass unity with v playing the role of the time coordinate as t. Therefore, we have the following equivalence table:

$$f_{mn} \equiv x; \quad v \equiv t; \tag{3.3.27}$$

accompanied by the identifications

$$\mu \equiv 1; \quad \omega \equiv \frac{3\alpha^2}{4}. \tag{3.3.28}$$

Hence by the earlier argument, we can conclude that each mode, at the quantum level, is thermal. The temperature is evaluated as (3.3.26) with the following substitutions: $\mu = 1$ and $\omega = (3\alpha^2/4)$. Therefore in our case it is given by

$$T = \frac{\hbar}{2\pi}\frac{\sqrt{3}\alpha}{2}. \tag{3.3.29}$$

Even more interestingly, what is emerged from our present calculation is that all the modes with the quantum number (l, m) are degenerate with respect to $\mathcal{E}$. *This observation seems to suggest that the horizon understudy can carry entropy because of those degenerate quantum states*. However, in order to have finite entropy, we need to have an upper limit on the value of (p, q), which must be proportional to the only scale available in theory, namely the Planck scale. Our naive analysis based on [203] shows that semi-classical Goldstone boson dynamics can capture the well-known thermal behavior of the horizon. *Moreover, the temperature turned out to be proportional to the acceleration of the Rindler frame*. This is an important observation as an accelerated observer associates thermality on the Rindler horizon. In this case also the temperature is proportional to α, known as Unruh temperature [213]. However, the proportionality constant appeared to be different. *Another essential outcome of our analysis is the emergence of an infinite number of degenerate states, which can be associated with the entropy on this horizon*. We will take up this issue in the future. The microscopic origin of horizon thermodynamics has been a subject of intensive research for a long time. Our present analysis hints towards an important fact that the BMS-like symmetry near the horizon could play an important role in understanding the thermal nature and possible origin of the underlying microscopic states of a black hole. Motivated by the present analysis, in the subsequent section, we will try to investigate the generality of these arguments for a black hole viz, the Schwarzschild black hole.

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3.4 Schwarzschild black hole

So far, we have discussed about the dynamics of Goldstone boson mode in the Rindler background. To this end, we perform a similar analysis considering Schwarzschild black hole background. The near-horizon geometry of the Schwarzchild black is again Rindler, however, with the two-dimensional sphere at each point. Therefore, we expect similar behavior of the Goldstone mode for this case as well. As we go along, we also notice the main differences with the flat Rindler case.

The Schwarzschild metric in Eddington-Finkelstein coordinate (v, r, θ, ϕ) is expressed as,

$$ds^2 = -(1 - 2M/r)dv^2 + 2dvdr + r^2\gamma_{AB}dx^A dx^B \,. \tag{3.4.1}$$

The event horizon is located at $r = 2M$. M is the mass of the black hole which characterizes the macroscopic state of the background spacetime. Asymptotic symmetry properties of the horizon can be extracted from similar fall off and gauge conditions for the metric components,

$$\pounds_\zeta g_{rr} = 0, \quad \pounds_\zeta g_{vr} = 0, \quad \pounds_\zeta g_{Ar} = 0\,; \tag{3.4.2}$$

$$\pounds_\zeta g_{vv} \approx \mathcal{O}(r - 2M); \quad \pounds_\zeta g_{vA} \approx \mathcal{O}(r - 2M); \quad \pounds_\zeta g_{AB} \approx \mathcal{O}(r - 2M) \tag{3.4.3}$$

Here, $\pounds_\zeta$ corresponds to the Lie variation for the diffeomorphism $x^a \to x^a + \zeta^a$. The primary motivation to consider the aforementioned conditions is essentially to preserve the form of the metric under the diffeomorphism. As has already been observed in our previous case, those differmorphsim in turn renormalizes the state of the black hole parameter such as mass M of the Schwarzschild black hole. Similar to our previous analysis after solving the above gauge fixing conditions with the imposed fall-off conditions, the diffeomorphism vectors turned out to be ,

$$\zeta^a\partial_a = F(v, x^A)\partial_v - (r - 2M)\partial_v F\partial_r + (1/r - 1/2M)\gamma^{AB}\partial_B F\,\partial_A. \tag{3.4.4}$$

Again we have one unknown function F which is identified as supertranslation generator. Under this transformation the background metric takes of following

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form (details are shown in Appendix 3.B.1) [164],

$$
\begin{aligned}
ds^2 &= \left[g^{(0)}_{ab} + \pounds_{\zeta} g^{(0)}_{ab}\right] dx^a dx^b \\
&= -(1-2M/r)dv^2 + 2dvdr + r^2\gamma_{AB}dx^A dx^B \\
&+ \left[2M/r(1-2M/r)\partial_v F - 2(1-2M/r)\partial_v F - 2(r-2M)\partial_v^2 F\right] dv^2 \\
&+ \Big[-2(1-2M/r)\partial_A F - 2(r-2M)\partial_A\partial_v F \\
&\quad +2r^2\partial_A\partial_v F(1/r - 1/2M)\Big] dvdx^A \\
&+ \Big[-2(2M-r)r\gamma_{AB}\partial_v F - (1/r - 1/2M)(\partial_E F\gamma^{DE}\partial_D\gamma_{AB} \\
&+ \gamma_{AD}\,\partial_B(\partial_E F\gamma^{DE}) + \gamma_{BD}\,\partial_A(\partial_E F\gamma^{DE}))\Big] dx^A dx^B\,. \qquad (3.4.5)
\end{aligned}
$$

As has already been discussed for the Rindler metric with flat spatial section, for the present case the modification h_{ab} due to following super-translation,

$$
\begin{aligned}
& v' = v + F\,;\; x'^A = x^A + (1/r - 1/2M)\gamma^{AB}\partial_B F, \\
& r' = r - (r-2M)\partial_v F. \qquad (3.4.6)
\end{aligned}
$$

The macroscopic black hole parameter M changes to,

$$
\frac{1}{M} \to \frac{1}{M} + \frac{1}{M}\Big(\partial_v F + 4M\partial_v^2 F\Big). \qquad (3.4.7)
$$

Therefore, like the Rindler case, this change of macroscopic parameter by the symmetry transformation can be understood as the spontaneous symmetry breaking phenomenon that happened by the black hole background itself (for more detail see the discussion after equation (3.3.7)). Here F is treated as the Goldstone boson modes.

Now it is clear that the solution of the Einstein equations will be modified, having a transformed form of the black hole parameters where the transformation is given by the first-order derivative of F (see (3.4.5)). Einstein's equation of motion is derived from the variation of g_{ab}. It may be noted that in the case of modified metric, g_{ab} is now not simply F, instead, it is a nontrivial function of F. Therefore the dynamical equation of motion for F cannot be derived from the Einstein equations as it is not picking the exact flavor of variation of F. Hence to find an equation of F, it is always necessary to find an action for this parameter, which we have done in the subsequent analysis.

Following the same procedure as for the Rindler case, the Lagrangian $\mathcal{L}_F$ of the Goldstone mode on the horizon surface takes the following form (detail procedure

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has been discussed in Appendix 3.B.2),

$$\begin{aligned}
\mathcal{L}_{\mathcal{F}} &= \Big[\frac{-3}{2(2M)^2}\csc\theta\,\partial_\phi F\partial_\phi F - \frac{3}{2(2M)^2}\sin\theta\,\partial_\theta F\partial_\theta F + 4\sin\theta\,\partial_v F\partial_v F \\
&- \frac{3}{M}\csc\theta\,\partial_\phi F\partial_v\partial_\phi F + \frac{1}{M}\cos\theta\,\partial_\theta F\partial_v F - \frac{3}{M}\sin\theta\,\partial_\theta F\partial_\theta\partial_v F \\
&+ 4\cos\theta\,\partial_\theta F\partial_v^2 F + \frac{1}{M}\csc\theta\,\partial_\phi^2 F\partial_v F + 4\csc\theta\,\partial_\phi^2 F\partial_v^2 F \\
&+ \frac{1}{M}\sin\theta\,\partial_\theta^2 F\partial_v F + 4\sin\theta\,\partial_\theta^2 F\partial_v^2 F - 6\csc\theta\,(\partial_v\partial_\phi F)^2 \\
&-6\sin\theta\,(\partial_v\partial_\theta F)^2 + 8\sin\theta\,\partial_v F\partial_v^2 F\Big]\,. \qquad (3.4.8)
\end{aligned}$$

Since the action has the rotational symmetry, we can take the following solution ansatz for Goldstone boson mode in terms of spherical harmonics,

$$F(v,\theta,\phi) = \frac{1}{k}\sum_{lm} c_{lm} f_{lm}(v) Y_{lm}(\theta,\phi), \qquad (3.4.9)$$

with c_{lm} are constant coefficients and f_{lm} are the time dependent mode function. This is consistent with the spherically symmetric Schwarzschild geometry. The factor $1/k = 4M$ is introduced for dimensional reason.

Substituting the form of the solution ansatz (3.4.9) of F in (3.4.8) and neglecting the total derivative terms (details are shown in Appendix 3.B.3), we can write final Lagrangian as,

$$\begin{aligned}
\mathcal{L}_{\mathcal{F}} &= \Big[\frac{-3}{2(2M)^2}\csc\theta\,\partial_\phi F\partial_\phi F - \frac{3}{2(2M)^2}\sin\theta\,\partial_\theta F\partial_\theta F + 4\sin\theta\,\partial_v F\partial_v F \\
&+ 4\cos\theta\,\partial_\theta F\partial_v^2 F + 4\csc\theta\,\partial_\phi^2 F\partial_v^2 F + 4\sin\theta\,\partial_\theta^2 F\partial_v^2 F \\
&- 6\csc\theta\,(\partial_v\partial_\phi F)^2 - 6\sin\theta\,(\partial_v\partial_\theta F)^2\Big]\,. \qquad (3.4.10)
\end{aligned}$$

Here also we concentrate on GHY term which is added to the EH action to have proper variation. The non-vanishing lower components of N^a is given by

$$N_r = \frac{1}{\sqrt{f(r) - (2M/r)f(r)\partial_v F + 2f(r)\partial_v F + 2rf(r)\partial_v^2 F}}\,, \qquad (3.4.11)$$

where $f(r) = 1 - 2M/r$. Hence for GHY boundary term the action can be expressed as,

$$\begin{aligned}
\mathcal{S}_2 &= -\frac{M}{8\pi G}\int d^3x \sin\theta\Big[1 + (\partial_v F + 2M\partial_v^2 F) + (2M\partial_v F\partial_v^2 F \\
&+ 8M^2(\partial_v^2 F)^2 + 8M^2\partial_v F\partial_v^3 F + 32M^3\partial_v^2 F\partial_v^3 F)\Big]\,, \qquad (3.4.12)
\end{aligned}$$

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which have a constant plus total derivative terms and hence does not contribute to the equation of motion as was the case for Rindler space (detail has been shown in Appendix 3.B.4). The dynamics of the Goldstone mode will be governed by the action corresponding to $\mathcal{L}_{\mathcal{F}}$, and the equation of motion is given by (detail derivation has been presented in Appendix 3.B.5),

$$\begin{aligned} &-8\sin\theta\partial_v^2 F + \frac{3}{(2M)^2}\cos\theta\partial_\theta F + \frac{3}{(2M)^2}\sin\theta\partial_\theta^2 F + \frac{3}{(2M)^2}\csc\theta\partial_\phi^2 F \\ -\quad &16\sin\theta\partial_v^2\partial_\theta^2 F - 16\cos\theta\partial_v^2\partial_\theta F - 16\csc\theta\partial_v^2\partial_\phi^2 F = 0\,. \end{aligned} \tag{3.4.13}$$

In this analysis, the full metric has been considered. Since we are interested in the near horizon symmetries, the near horizon metric could be enough to obtain the same result. For completeness, we explicitly demonstrated this in Appendix 3.B.7.

Now substituting the form of F (3.4.9) in (3.4.13), we get following equation of motion for $f_{lm}(v)$ (derivation is shown in Appendix 3.B.6),

$$[2l(l+1)-1]\partial_v^2 f_{lm} - \frac{3}{32M^2}l(l+1)f_{lm} = 0. \tag{3.4.14}$$

Since the near horizon geometry of the Schwarzschild black hole is Rindler with the sphere as spatial section, one notices some significant differences in the mode dynamics governed by Eq. (3.4.14) and that of the previous case in Eq.(3.3.19). Most importantly, for spatial spherical geometry, the effective potential perceived by every individual mode parametrized by (l, m) is no longer universal but dependent upon the angular momentum l. Before we discuss the implications of this dependence, let us look at the behaviour of individual modes.

- For $l = 0$ mode, the equation reduces to,

$$\partial_v^2 f_{00}(v) = 0\,. \tag{3.4.15}$$

  The solution of the above equation is $f_{00} = c_1(x^A)v + c_2(x^A)$. As f_{00} needs to be finite near horizon at $v \to -\infty$, we choose $c_1 = 0$. Then the final solution will be $f_{00}(v) = c_2(x^A)$.

- For all remaining modes $l \geq 1$, we get the inverted harmonic oscillator potential similar to our previous case. One important difference is the angular momentum dependence of the inverted harmonic potential. Therefore, the universality of all the modes with respect to their time dynamics is lost as opposed to our previous study in the Rindler metric with spatial section. However, it can be checked that numerically the inverted potential depends

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very weakly on the value of l, which we will discuss in terms of temperature in the next subsection. Nonetheless, the mode equation looks likes,

$$\partial_v^2 f_{lm} - k^2\Omega^2 f_{lm}(v) = 0\,, \tag{3.4.16}$$

where,

$$\Omega = \sqrt{\frac{3l(l+1)}{2(2l(l+1)-1)}}\,. \tag{3.4.17}$$

We get the inverted harmonic oscillator potential similar to our previous case. The complete solution for all modes can therefore be,

- for $l = 0$;

$$F(x^A) = \sum_{lm} \frac{1}{k} c_2(x^A) Y_{lm}(x^A); \tag{3.4.18}$$

- for $l \geq 1$,

$$F(v, x^A) = \sum_{lm} \frac{A}{k} e^{\Omega(l)kv} Y_{lm}(x^A)\,. \tag{3.4.19}$$

Hereafter we can proceed along the same line as discussed before. Important difference would be the mode dependent inverted harmonic potential

$$V_{harmonic} = -\frac{1}{2}\Omega(l)^2 k^2 f_{lm}^2\,. \tag{3.4.20}$$

Therefore, strictly speaking, for the present case, degenerate states will be only for m within $(-l, l)$. However, let us point out that if we consider numerical values consideration, the value of Ω is confined within a very narrow region.

$$\sqrt{\frac{3}{4}} \leq \Omega(l) \leq 1\,. \tag{3.4.21}$$

Hence, one can approximately consider all the quantum states of the Goldstone boson parametrized by (l, m) with $l \geq 1$ are quasi-degenerate. Unlike the previous case for the Rindler spacetime with a flat spatial section, the emission probability for the present case would be identified with Boltzmann distribution with temperature,

$$T_l = \frac{\hbar}{8\pi M}\Omega(l), \tag{3.4.22}$$

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which will weakly depend upon the value of angular momentum quantum number l. Interestingly for $l = 1$ mode, the above expression came out exactly the same as usual black hole temperature T_{BH}, given by the Hawking expression [29]. However, considering other modes we can define an average temperature.

$$T_{avg} = \frac{\hbar}{8\pi M}\left(\frac{\sum_l \Omega(l)}{\sum_l 1}\right) = \frac{\hbar}{8\pi M}\left(\sqrt{\frac{3}{4}}\right) = \frac{\sqrt{3}}{2}T_{BH}\,, \tag{3.4.23}$$

Here again, we observed that the Goldstone modes are inherently thermal in nature. The obtained temperature is proportional to the Hawking expression for that of the Schwarzschild horizon.

From the analysis so far, we can infer that since the origin of the Goldstone modes is associated with the breaking of symmetries of the horizon, those modes can be a potential candidate for the microscopic states of a black hole. Quantum mechanically, all those states turned out to be thermal with a specific temperature. However, the origin of different expressions for the temperature compared with that of the usual Hawking temperature needs to be explored in detail. Furthermore, the nature of degeneracy of those Goldstone states appears to be dependent upon the spacetime background. Such as for Rindler spacetime with the plane-symmetric horizon, all the modes emerged as degenerate and, therefore, each mode feels the same temperature. On the other hand, this is not the case for Schwarzschild black hole as the degeneracy of states has been lifted by the less symmetric spherical horizon. Nevertheless, we hope that this thermal nature of the Goldstone modes at the quantum level can be inferred for all horizon types. Of course, the present treatment will be complete once the field theoretical description is done, in which the definition of relevant vacuum state will be cleared. We keep this for our future projects.

3.5 Summary and conclusions

The microscopic origin of the thermodynamic nature of the black hole is one of the fundamental questions in the theory of gravity. But within the framework of classical treatment of Einsteinian gravity, this question may not be answered. For that, a proper quantum prescription will be required. However, the recent understanding of infrared behavior of gravity opens up a new avenue towards understanding this question. In the gravitational theory, one of the interesting infrared properties is the emergence of infinite-dimensional symmetry at null

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infinity, which leads to the soft graviton theorem. Over the years, it has been observed that analogous symmetry exists near the null horizon, which can play an important role in explaining the microscopic origin of horizon thermodynamics. Here we mainly concentrated on the BMS-like symmetry in the near-horizon region. It is observed that in this process, the mass (or the surface gravity) of the black hole gets modified. This change in macroscopic parameters has been argued to be the phenomena of the spontaneous symmetry breaking of the metric solutions because of the arbitrariness of the diffeomorphism parameter, and the corresponding parameter has been viewed as the Goldstone mode.

In the present analysis, our main effort was to explore the dynamics of these Goldstone modes. For our current study, we consider two simple gravitational backgrounds. One is a simple Rindler spacetime with a flat Killing horizon, and the other is Schwarzschild black hole. Our preliminary investigation at the tree level reveals that the horizon is a special place where an inverse harmonic potential governs Goldstone mode's dynamics in momentum space. As mentioned earlier, in the framework of classical Einsteinian gravity, it is difficult to understand this situation as those modes are simply unstable. Interestingly, at the quantum level, this instability [204] can have a nice interpretation of inherent thermality in connection with its chaotic behavior, which may provide us a first glimpse of a microscopic view of the horizon thermodynamics. Interestingly, for Rindler and the gravitational backgrounds, as expected, the temperature turned out to be proportional to the surface gravity, which is similar to the expression (except a numerical factor) given by Unruh [213], and Hawking [29]. This led us to think that these Goldstone modes might be candidates for the microscopic description of the horizon thermality. Even more, interestingly, we found many degenerate states for Rindler and quasi-degenerate states for Schwarzschild black holes that may be responsible for the horizon entropy. We will take up these issues in more detail in the future.

So far, we have considered the black hole spacetime, which is static and generating only one Goldstone field. However, for a gravitational background having intrinsic rotation such as Kerr spacetime, we expect that the corresponding analysis of the Goldstone mode dynamics will be more general. This is because, in this case, there will be another symmetry generator which is superrotation. We have investigated this topic in the next chapter.

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Appendix

3.A Appendix I: Rindler Background

3.A.1 Detail derivation of the modified Rindler metric given in (3.3.5)

Following the gauge choices given in Eq.(3.3.2), the associated diffeomorphism vectors can be expressed as,

$$\begin{aligned} \zeta^v &= F(v,y,z)\,; \\ \zeta^r &= T(v,y,z) - r\partial_v F; \\ \zeta^A &= -\partial_B F \int \delta^{AB} dr + R^A(v,y,z)\,. \end{aligned} \tag{3.A.1}$$

The detail procedure of deriving these vectors from the three gauge choices has been already presented in (2.B).

Now using the components of ζ^a from (3.A.1), one can find the variation of the metric component $g^{(0)}_{vv}$ upto $\mathcal{O}(r)$ as,

$$\begin{aligned} £_\zeta g^{(0)}_{vv} &= \zeta^r \partial_r g^{(0)}_{vv} + 2g^{(0)}_{vv}\partial_v\zeta^v + 2g^{(0)}_{vr}\partial_v\zeta^r \\ &= [T(v,y,z) - r\partial_v F]\partial_r(-2r\alpha) - 2r\alpha\partial_v F + 2\partial_v[T(v,y,z) - r\partial_v F] \\ &= [2\partial_v T - 2\alpha T] + r[-2\alpha\partial_v F - 2\partial_v^2 F] \end{aligned} \tag{3.A.2}$$

Now given the fall-off condition $£_\zeta g^{(0)}_{vv} \approx \mathcal{O}(r)$ as shown in (3.3.3), the r independent term of the right hand side of (3.A.2) must vanish which give rise the constraint relation as,

$$\partial_v T - \alpha T = 0 \tag{3.A.3}$$

Similarly the variation of $g^{(0)}_{vA}$ and $g^{(0)}_{AB}$ upto $\mathcal{O}(r)$ will be,

$$\begin{aligned} £_\zeta g^{(0)}_{vA} &= g^{(0)}_{vv}\partial_A\zeta^v + g^{(0)}_{vr}\partial_A\zeta^r + g^{(0)}_{AB}\partial_v\zeta^B \\ &= -2r\alpha\partial_A F + \partial_A[T(v,y,z) - r\partial_v F] + \delta_{AB}\partial_v[-\partial_C F\int\delta^{CB}dr + R^B(v,y,z)] \\ &= [\partial_A T + \delta_{AB}\partial_v R^B] + r[-2r\alpha\partial_A F - 2\partial_A\partial_v F] \end{aligned} \tag{3.A.4}$$

$$\begin{aligned} £_\zeta g^{(0)}_{AB} &= g^{(0)}_{DA}\partial_B\zeta^D + g^{(0)}_{DB}\partial_A\zeta^D \\ &= \delta_{DA}\partial_B[-\partial_C F\int\delta^{CD}dr + R^D(v,y,z)] + \delta_{DB}\partial_A[-\partial_C F\int\delta^{CD}dr + R^D(v,y,z)] \\ &= [\delta_{AD}\partial_B R^D + \delta_{BD}\partial_A R^D] + r[-2\partial_A\partial_B F]. \end{aligned} \tag{3.A.5}$$

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Now following the weak fall off conditions given in (3.3.3) for the metric components g_{vA} and g_{AB}, we can say that the r independent terms of (3.A.4) and (3.A.5) must be zero. Thus we get the two other constraints relations for the diffeomorphism vectors Eq. (3.A.1) as follows,

$$\partial_A T + \delta_{AB}\partial_v R^B = 0\,; \tag{3.A.6}$$

$$\delta_{AD}\partial_B R^D + \delta_{BD}\partial_A R^D = 0\,. \tag{3.A.7}$$

Now we demand that the position of the null surface at $r = 0$ must remain unchanged after the symmetry transformations. Therefore there will be no radial transformation which suggest $T(v, y, z) = 0$ from Eq.(3.A.1). Since Rindler background geometry has flat spatial section with no rotation, we can assume the angular component of the diffeomorphism vector ζ^A to be zero at the horizon and thus we got that $R^A = 0$. Then the three constraints relations given in (4.3.13), (4.3.14) and (4.3.15) are automatically satisfied. Finally the components of the diffeomporphism vectors are given by (3.3.4).

Now from the varitation of metric components as given in (3.A.2), (3.A.4) and in (3.A.5), we collect all the $\mathcal{O}(r)$ terms which generate the modified metric in (3.3.5).

3.A.2 Construction of the Lagrangian (3.3.11)

The Lagrangian is computed from the modified metric with the help of the Mathematica 10.00 packages. From the modified metric (3.4.5) we calculate $\sqrt{-g}$ as,

$$\sqrt{-g} = \sqrt{1 - 2r(\partial_\phi^2 F + \partial_\theta^2 F) - 4r^2(\partial_\theta\partial_\phi F)^2 + 4r^2\partial_\phi^2 F\partial_\theta^2 F}. \tag{3.A.8}$$

Then we have calculated the components of the Christoffel symbols as,

$$\Gamma^a_{bc} = \frac{1}{2}g^{ia}(\partial_b g_{ic} + \partial_c g_{ib} - \partial_i g_{bc}). \tag{3.A.9}$$

and then we have computed Riemannian tensor as follows,

$$R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^s_{bd}\Gamma^a_{cs} - \Gamma^s_{bc}\Gamma^a_{ds}. \tag{3.A.10}$$

Then we got the components of Ricci tensor using the definition,

$$R_{ij} = R^i_{kij}, \tag{3.A.11}$$

and also Ricci scalar as,

$$R = g^{ij}R_{ij}. \tag{3.A.12}$$

Now by computing the product of $\sqrt{-g}$ and R and taking limit $r \to 0$ we finally get the result in (3.3.11).

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3.A.3 Derivation of the final Lagrangian in (3.3.14) from the form given in (3.3.11)

After substituting the solution ansatz for F as given in (3.3.12) in the form of the Lagrangian density (3.3.11), we compute the form of the three dimensional action on the $r =$ constant surface as follows,

$$S = \int d^3x \mathcal{L}_{\mathcal{F}} = \frac{1}{\alpha^2}\sum_{mn}\sum_{kl} C_{mn}\bar{C}_{kl}\int dv \int_{y=0}^{L}\int_{z=0}^{L} dydz e^{i((m-k)y+(n-l)z)}[6\alpha^2(mk + nl)f_{mn}f_{kl} - 4\alpha(k^2+l^2)\partial_v f_{mn} f_{kl} + 12\alpha(mk+nl)f_{mn}\partial_v f_{kl} + 6(mk+nl)\partial_v f_{mn}\partial_v f_{kl} - 4(m^2+n^2)f_{mn}\partial_v^2 f_{kl}]. \tag{3.A.13}$$

The coordinates y and z run from $-\infty$ to $+\infty$ and so $y - z(i.e.x = 0)$ surface (the horizon here) is a infinitely extended plane. However here we can perform the integration over y and z coordinates assuming them periodic between the ranges 0 and $+L$ and thus obtain,

$$\sum_{mn}\sum_{kl}\frac{4\pi^2 C_{mn}\bar{C}_{kl}}{\alpha^2}\delta_{m,k}\delta_{n,l}\int dv\Big[6\alpha^2(mk+nl)f_{mn}f_{kl} - 4\alpha(k^2+l^2)f_{mn}\partial_v f_{kl} + 12\alpha(mk+nl)\partial_v f_{mn} f_{kl} + 6(mk+nl)\partial_v f_{mn}\partial_v f_{kl} - 4(m^2+n^2)f_{mn}\partial_v^2 f_{kl}\Big]. \tag{3.A.14}$$

$$= \sum_{mn}\frac{4\pi^2 C_{mn}\bar{C}_{mn}}{\alpha^2}\int dv\Big[6\alpha^2(m^2+n^2)f_{mn}f_{mn} + 8\alpha(m^2+n^2)f_{mn}\partial_v f_{mn} + 6(m^2+n^2)\partial_v f_{mn}\partial_v f_{mn} - 4(m^2+n^2)f_{mn}\partial_v^2 f_{mn}\Big]. \tag{3.A.15}$$

Hence after integrating, the last line of (3.A.15) represents the one dimensional action denoting the evaluation of $f_{mn}(v)$ with respect to v. So in the above expression (3.A.15), the second term $f_{mn}\partial_v f_{mn}$ can be expressed as $\frac{1}{2}\partial_v(f_{mn}^2)$ which is a total derivative of v and can be neglected in construction of dynamics of F. Correspondingly we neglect the third, fourth, sixth and ninth terms of (3.3.11) and write down the Lagrangian as given in (3.3.14).

3.A.4 Why GHY terms does not contribute in the dynamics?

In the expression of GHY term given in (3.3.16) the first term is a constant, while the terms in the first bracket in the first line are the total derivative in v. Now the remaining terms in the second line can also be transformed into total derivative as follows,

$$\partial_v F \partial_v^2 F = \frac{1}{2}\partial_v[(\partial_v F)^2]; \quad \partial_v^2 F \partial_v^3 F = \frac{1}{2}\partial_v[(\partial_v^2 F)^2]$$
$$(\partial_v^2 F)^2 + \partial_v F \partial_v^3 F = \partial_v[\partial_v F \partial_v^2 F]. \tag{3.A.16}$$

Hence all these total derivative terms should not contribute to the dynamics of F.

3.A.5 Derivation of the equation of motion (3.3.18) from the Lagrangian (3.3.14).

Using the Generalized Euler-Lagrangian equation given in (3.3.17) we can easily construct the equation of motion of F from the Lagrangian (3.3.14) as follows,

$$\begin{aligned}
&\partial_y(\frac{\partial \mathcal{L}}{\partial_y F}) - \partial_z(\frac{\partial \mathcal{L}}{\partial_z F}) + 2[\partial_v\partial_y(\frac{\partial \mathcal{L}}{\partial(\partial_v\partial_y F)}) + \partial_v\partial_z(\frac{\partial \mathcal{L}}{\partial(\partial_v\partial_z F)})] + \partial_v^2(\frac{\partial \mathcal{L}}{\partial(\partial_v^2 F)}) \\
&+\partial_y^2(\frac{\partial \mathcal{L}}{\partial(\partial_y^2 F)}) + \partial_z^2(\frac{\partial \mathcal{L}}{\partial(\partial_z^2 F)}) = 0; \\
&\Rightarrow 6\alpha^2[\partial_y(2\partial_y F) + \partial_z(2\partial_z F)] - 24[\partial_v\partial_y(\partial_v\partial_y F) + \partial_v\partial_z(\partial_v\partial_z F)] \\
&+4[\partial_y^2(\partial_v^2 F) + \partial_z^2(\partial_v^2 F)] + 4[\partial_v^2(\partial_y^2 F) + \partial_v^2(\partial_z^2 F)] = 0. \qquad (3.A.17) \\
&\Rightarrow 12\alpha^2[\partial_y^2 F + \partial_z^2 F] - 24[\partial_v^2\partial_y^2 F + \partial_v^2\partial_z^2 F] + 8[\partial_y^2\partial_v^2 F + \partial_z^2\partial_v^2 F] = 0.
\end{aligned}$$

From the above expression we can easily get the equation of motion of F as given in (3.3.18).

3.B Appendix II: Schwarzschild background

3.B.1 Derivation of diffeomorphism vectors and formation of the modified metric given in (3.4.5)

We have solved the equations of gauge choices given in (3.4.2) and found the components of the diffeomorphism vector as,

$$\begin{aligned}
&\pounds_\zeta g_{rr}^{(0)} = 2g_{vr}^{(0)}\partial_r\zeta^v = 0; \rightarrow \zeta^v = F(v,\theta,\phi); \\
&\pounds_\zeta g_{vr}^{(0)} = \partial_r\zeta^r + \partial_v F = 0; \rightarrow \zeta^r = -\int_{2M}^r \partial_v F dr + T(v,\theta,\phi) \\
&= -(r-2M)\partial_v F + T(v,\theta,\phi). \\
&\pounds_\zeta g_{AB}^{(0)} = g_{AB}^{(0)}\partial_r\zeta^B + \partial_A F = 0; \rightarrow \zeta^B = -\gamma^{AB}\partial_A F\int_{2M}^r \frac{dr}{r^2} + R^B(v,\theta,\phi) \\
&= (\frac{1}{r} - \frac{1}{2M})\gamma^{AB}\partial_A F + R^B(v,\theta,\phi).
\end{aligned} \tag{3.B.1}$$

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Now the symmetry requires that the position of the horizon surface must remains unaltered after the diffeomorphism. Therefore there will be no radial transformation which suggests that $T(v,\theta,\phi) = 0$. Now background geometry have no explicit rotation which implies $R^A(v,\theta,\phi) = 0$. Then the results found in (3.B.1) reduces to that as given in (3.4.4).

Now with the help of the components of the diffeomorphism vector found in (3.4.4), one can find Lie variation of the metric components $g^{(0)}_{vv}$, $g^{(0)}_{vA}$ and $g^{(0)}_{AB}$ as follows,

$$\begin{aligned}\pounds_\zeta g^{(0)}_{vv} &= \zeta^r \partial_r g^{(0)}_{vv} + 2g^{(0)}_{vv}\partial_v\zeta^v + 2g^{(0)}_{vr}\partial_v\zeta^r \\ &= (r-2M)\partial_v F(\frac{2M}{r^2}) - 2(1-\frac{2M}{r})\partial_v F - 2(r-2M)\partial_v^2 F.\end{aligned} \tag{3.B.2}$$

$$\begin{aligned}\pounds_\zeta g^{(0)}_{vA} &= g^{(0)}_{vv}\partial_A\zeta^v + g^{(0)}_{vr}\partial_A\zeta^r + g^{(0)}_{AB}\partial_v\zeta^B \\ &= -(1-\frac{2M}{r})\partial_A F - (r-2M)\partial_v\partial_A F + r^2\partial_v\partial_A F(\frac{1}{r}-\frac{1}{2M}).\end{aligned} \tag{3.B.3}$$

$$\begin{aligned}\pounds_\zeta g^{(0)}_{AB} &= \zeta^r\partial_r g^{(0)}_{AB} + \zeta^D\partial_D g^{(0)}_{AB} + g^{(0)}_{DA}\partial_B\zeta^D + g^{(0)}_{DB}\partial_A\zeta^D \\ &= \Big(-(r-2M)\partial_v F\Big)(2r\gamma_{AB}) + \Big((\frac{1}{r}-\frac{1}{2M})\gamma^{DE}\partial_E F\Big)r^2\partial_D\gamma_{AB} \\ &+ r^2\gamma_{AD}\partial_B\Big((\frac{1}{r}-\frac{1}{2M})\gamma^{DE}\partial_E F\Big) + r^2\gamma_{BD}\partial_A\Big((\frac{1}{r}-\frac{1}{2M})\gamma^{DE}\partial_E F\Big).\end{aligned} \tag{3.B.4}$$

Therefore adding the variation of the components as found in (3.B.2), (3.B.3) and (3.B.4) with the components of the background metric $g^{(0)}_{ab}$, we will get the the form of the modified metric as given in (3.4.5).

3.B.2 Construction of the Lagrangian (3.4.8)

Like before as we did in Rindler case, here also the Lagrangian is computed from the modified metric with the help of the Mathematica packages. We have calculated determinant $\sqrt{-g}$ from the modified metric (3.4.5). Where $(-g)$ is given by the following expression,

$$\begin{aligned}(-g) = \frac{r^2}{4M^2}\Big(&-4(r-2M)^2(\cot\theta\partial_\phi F - \partial_\theta\partial_\phi F)^2 + (2rM - (r-2M)(\partial_\theta^2 F) \\ &+2M\partial_v F)(2rM\sin^2\theta - 2(r-2M)(\partial_\phi^2 F + \sin\theta\cos\theta\partial_\theta F + 2M\sin\theta\partial_v F))\Big).\end{aligned} \tag{3.B.5}$$

Then we have calculated the components of the Christoffel symbols Γ^a_{bc} and then Reimannian tensor $R^a{}_{bcd}$ as given by (3.A.9) and (3.A.10) respectively. After that

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we got the components of Ricci tensor R_{ij} using the definition (3.A.11) and also Ricci scalar by (3.A.12). By computing the product of $\sqrt{-g}$ and R, then taking near horizon limit we finally get the result in (3.4.8).

3.B.3 Derivation of final Lagrangian in (3.4.10) from the form given in (3.4.8)

After substituting the solution ansatz for F given in (3.4.9) in the form of the Lagrangian density given in (3.4.8), we compute the form of the three dimensional action on the $r =$ constant surface as given by,

$$\begin{aligned}
S &= \int d^3x \mathcal{L}_{\mathcal{F}} \\
&= \frac{1}{\kappa^2}\sum_{lm}\sum_{l'm'} c_{lm}\bar{c}_{l'm'} \int dv \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \Big[\frac{-3mm'}{2(2M)^2}(\csc\theta\, Y_l^m Y_{l'}^{m'}) f_{lm} f_{l'm'} \\
&-\frac{3}{2(2M)^2}(\sin\theta\, \partial_\theta Y_l^m \partial_\theta Y_{l'}^{m'}) f_{lm} f_{l'm'} + 4(\sin\theta\, Y_l^m Y_{l'}^{m'})\partial_v f_{lm}\partial_v f_{l'm'} \\
&+\frac{3}{M}(\sin\theta\, \partial_\theta Y_l^m \partial_\theta Y_{l'}^{m'}) f_{lm}\partial_v f_{l'm'} + \frac{3mm'}{M}(\csc\theta\, Y_l^m Y_{l'}^{m'}) f_{lm}\partial_v f_{l'm'} \\
&+\frac{1}{M}(\cos\theta\, \partial_\theta Y_l^m Y_{l'}^{m'}) f_{lm}\partial_v f_{l'm'} + 4(\cos\theta\, \partial_\theta Y_l^m Y_{l'}^{m'}) f_{lm}\partial_v^2 f_{l'm'} \\
&-\frac{m^2}{M}(\csc\theta\, Y_l^m Y_{l'}^{m'}) f_{lm}\partial_v f_{l'm'} + \frac{1}{M}(\sin\theta\, \partial_\theta^2 Y_l^m Y_{l'}^{m'}) f_{lm}\partial_v f_{l'm'} \\
&-4m^2(\csc\theta\, Y_l^m Y_{l'}^{m'}) f_{lm}\partial_v^2 f_{l'm'} + 4(\sin\theta\, \partial_\theta^2 Y_l^m Y_{l'}^{m'}) f_{lm}\partial_v^2 f_{l'm'} \\
&+6mm'(\csc\theta\, Y_l^m Y_{l'}^{m'})\partial_v f_{lm}\partial_v f_{l'm'} - 6(\sin\theta \partial_\theta Y_l^m \partial_\theta Y_{l'}^{m'})\partial_v f_{lm}\partial_v f_{l'm'} \\
&+8(\sin\theta\, Y_l^m Y_{l'}^{m'})\partial_v f_{lm}\partial_v^2 f_{l'm'}\Big].
\end{aligned} \tag{3.B.6}$$

Now from the above expression the fourth term can be written separately as,

$$\begin{aligned}
&\frac{3}{M\kappa^2}\sum_{lm}\sum_{l'm'} c_{lm}\bar{c}_{l'm'} \int dv \int_0^{\pi} d\theta \int_0^{2\pi} d\phi (\sin\theta\, \partial_\theta Y_m^l \partial_\theta Y_{m'}^{l'}) f_{lm}\partial_v f_{l'm'} \\
&= (\frac{1}{M\kappa^2} + \frac{2}{M\kappa^2})\sum_{lm}\sum_{l'm'} c_{lm}\bar{c}_{l'm'} \int dv \int_0^{\pi} d\theta \int_0^{2\pi} d\phi (\sin\theta\, \partial_\theta Y_l^m \partial_\theta Y_{l'}^{m'}) f_{lm}\partial_v f_{l'm'}.
\end{aligned} \tag{3.B.7}$$

Now the first term of (3.B.7) is added with sixth and ninth terms of (3.B.6) and we get the result as,

$$\frac{3N^2}{M\kappa^2}\sum_{lm}\sum_{l'm'} c_{lm}\bar{c}_{l'm'} \int_{\phi=0}^{2\pi} e^{[i(m-m')\phi]} d\phi \int_{\theta=0}^{\pi} d\theta \partial_\theta \Big(\sin\theta\, \partial_\theta P_l^m P_{l'}^{m'}\Big) \int dv f_{lm}\partial_v f_{l'm'} = 0.$$

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Here we have decomposed Y_l^m as, $Y_l^m(\theta,\phi) = NP_l^m(\theta)e^{im\phi}$ and N is the normalization constant of the spherical harmonics. So after integrating over the coordinate θ from the ranges 0 to π, we found out that the result is zero. Now we concentrate on the fifth term of (3.B.6) which can be expressed as,

$$\begin{aligned} & \frac{3mm'N^2}{M\kappa^2}\sum_{lm}\sum_{l'm'} c_{lm}\bar{c}_{l'm'}\delta_{mm'}\int dv\int_0^\pi d\theta(\csc\theta\, P_l^m P_{l'}^{m'})f_{lm}\partial_v f_{l'm'}; \\ = & \frac{3m^2N^2}{2M\kappa^2}\sum_{lm}\sum_{l'} c_{lm}\bar{c}_{l'm}\int dv\int_0^\pi d\theta(\csc\theta\, P_m^l P_m^{l'})(f_{lm}\partial_v f_{l'm} + f_{l'm}\partial_v f_{lm}); \\ = & \frac{3mm'N^2}{2M\kappa^2}\sum_{lm}\sum_{l'} c_{lm}\bar{c}_{l'm}\int dv\int_0^\pi d\theta(\csc\theta\, P_l^m P_{l'}^m)\partial_v(f_{lm}f_{l'm}). \end{aligned} \tag{3.B.8}$$

In the first line of (3.B.8) we have integrated over the coordinate ϕ and the result implies that $m = m'$. Here l, m, l' and m' all are dummy indices. Hence we have interchanged between l and l' indices provided $c_{lm} = \bar{c}_{lm}$. A factor $(1/2)$ is multiplied to avoid double counting. Hence the result (3.B.8) shows that the fifth term of (3.B.6) can be expressed as total derivative of v and can be neglected.

Now one can easily show that the eighth term of (3.B.6) and second term of (3.B.7) can be expressed as total derivative of coordinate v following the same procedure that we did in (3.B.8). Then we concentrate on the last term of (3.B.6). Using the orthonormality properties of Y_l^m we can perform the integration over transverse coordinates and obtain,

$$8\sum_{lm}\sum_{l'm'} c_{lm}\bar{c}_{l'm'}\delta_{ll'}\delta_{mm'}\int dv f_{lm}\partial_v f_{l'm'} = \sum_{lm} c_{lm}\bar{c}_{lm}\int dv\frac{1}{2}\partial_v((\partial_v f_{lm})^2);$$

which is again a total derivative of v and can be neglected.

Therefore the other remaining terms left in the expression (3.B.6) have significant contribution in the dynamics of F and with those terms we have formed the Lagrangian given in (3.4.10).

3.B.4 Why GHY boundary terms have no contribution in dynamics?

In the expression of GHY term given in (3.4.12) the first term is a constant, while the terms in the first bracket in that same line are the total derivative in v. After some algebraic calculation other terms in the second line of (3.4.12) can also be transformed into total derivative as follows,

$$\begin{aligned} & 2M\partial_v F\partial_v^2 F = M\partial_v[(\partial_v F)^2]; \quad 32M^3\partial_v^2 F\partial_v^3 F = 16M^3\partial_v[(\partial_v^2 F)^2] \\ & 8M^2((\partial_v^2 F)^2 + \partial_v F\partial_v^3 F) = 8M^2\partial_v[\partial_v F\partial_v^2 F]. \end{aligned} \tag{3.B.9}$$

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Hence all these total derivative terms behave as the boundary term and should not contribute to the dynamics of F.

3.B.5 Derivation of the equation of motion from the Lagrangian (3.4.10)

Using the Generalized Euler-Lagrangian equation of motion as given in (3.3.17), we derive the equation of motion from the Lagrangian (3.4.10) as follows,

$$\begin{aligned}&\frac{3}{(2M)^2}\partial_\phi[\csc\theta\partial_\phi F]+\frac{3}{(2M)^2}\partial_\theta[\sin\theta\partial_\theta F]-4\partial_v[\sin\theta\partial_v F]-4\partial_\theta[\cos\theta\partial_v^2 F]\\&+4\partial_v^2[\cos\theta\partial_\theta F]+4\partial_\phi^2[\csc\theta\partial_v^2 F]+4\partial_v^2[\csc\theta\partial_\phi^2 F]+4\partial_v^2[\sin\theta\partial_\theta^2 F]+4\partial_\theta^2[\sin\theta\partial_v^2 F]\\&-24\partial_v\partial_\theta[\sin\theta\partial_v\partial_\theta F]-24\partial_v\partial_\phi[\csc\theta\partial_v\partial_\phi F]=0;\end{aligned} \tag{3.B.10}$$

Where,

$$\begin{aligned}&\partial_\theta[\sin\theta\partial_\theta F]=\sin\theta\partial_\theta^2 F+\cos\theta\partial_\theta F;\qquad \partial_\theta[\cos\theta\partial_v^2 F]=\cos\theta\partial_\theta\partial_v^2 F-\sin\theta\partial_v^2 F;\\&\partial_\theta^2[\sin\theta\partial_v^2 F]=-\sin\theta\partial_v^2 F+2\cos\theta\partial_\theta\partial_v^2 F+\sin\theta\partial_\theta^2\partial_v^2 F;\\&\partial_v\partial_\theta[\sin\theta\partial_v\partial_\theta F]=\sin\theta\partial_v^2\partial_\theta^2 F+\cos\theta\partial_v^2\partial_\theta F.\end{aligned} \tag{3.B.11}$$

Hence substituting the result (3.B.11) in (3.B.10) we get the equation of F given in (3.4.13).

3.B.6 Derivation of (3.3.19) from (3.4.13)

Substituting (3.4.9) in (3.4.13) and dividing the equation (3.4.13) with $(\sin\theta)$ gives,

$$\begin{aligned}&-8\partial_v^2 f_{lm}Y_l^m+\frac{3}{4M^2}f_{lm}[\cot\theta\partial_\theta+\partial_\theta^2+\csc\theta\partial_\phi]Y_l^m\\&-4\partial_v^2 f_{lm}[\cot\theta\partial_\theta+\partial_\theta^2+\csc\theta\partial_\phi]Y_l^m=0.\\&-8\partial_v^2 f_{lm}Y_l^m-\frac{3}{4M^2}f_{lm}(\mathcal{J}^2Y_l^m)+4\partial_v^2 f_{lm}(\mathcal{J}^2Y_l^m)=0\\&-8\partial_v^2 f_{lm}Y_l^m-\frac{3}{4M^2}f_{lm}(l(l+1)Y_l^m)+4\partial_v^2 f_{lm}(l(l+1)Y_l^m)=0.\end{aligned} \tag{3.B.12}$$

Where we have, $\mathcal{J}^2Y_l^m=-(\cot\theta\partial_\theta+\partial_\theta^2+\csc\theta\partial_\phi)Y_l^m=l(l+1)Y_l^m$. Now from (3.B.12) we get directly the form given in (3.3.19).

3.B.7 Near horizon analysis of Schwarzschild black hole

As mentioned in the main text of this chapter, in this section we will argue that same dynamical equations and solution for the Goldstone modes can be obtained

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starting from the near horizon metric of the Schwarzschild black hole. The near horizon form of the Schwarzschild metric can be obtained by Taylor expansion of all the metric components around $(r-2M)=\tilde{r}=0$. Therefore, the form of the metric is given by (in Gaussian null coordinates);

$$ds^2 = -2\tilde{r}\kappa\, dv^2 + 2\, dvd\tilde{r} + (4M^2 + 4\tilde{r}M)\, d\Omega^2\,, \tag{3.B.13}$$

where the parameter $\kappa = 1/4M$ is essentially the surface gravity. The diffeomorphism vector components takes following form near the near horizon

$$\zeta^a\partial_a = F(v,y,z)\partial_v - \tilde{r}\partial_v F(v,y,z) - \tilde{r}\partial_A F(v,y,z)\partial_A\,; \tag{3.B.14}$$

F is the only unknown function of coordinates, called as super-translation parameter. To construct the modified metric we follow the same procedure as described in our main text, and it takes the following form:

$$\begin{aligned}
g'_{ab} &= g^{(0)}_{ab} + h_{ab} \\
&= -2\tilde{r}k\, dv^2 + 2\, dvd\tilde{r} + (4M^2 + 4\tilde{r}M)\gamma_{AB}dx^A dx^B \\
&+ \Big[-2\tilde{r}(k\partial_v F + \partial_v^2 F)\Big]dv^2 + \Big[-4\tilde{r}(k\,\partial_A F \\
&+ \partial_A\partial_v F)\Big]dvdx^A + \Big[-4\tilde{r}M\gamma_{AB}\partial_v F \\
&- 2\tilde{r}(\partial_E F\gamma^{DE}\partial_D\gamma_{AB} + \gamma_{AD}\,\partial_B(\partial_E F\gamma^{DE}))\Big]dx^A dx^B\,.
\end{aligned}$$

Following the same prescription described in the main text we obtain the action for the Goldstone mode as,

$$\mathcal{L} = \frac{1}{8\pi G}M\int d^3x\Big[-2\sin\theta\; + 8\csc\theta\partial_v\partial_\phi^2 F + \mathcal{L}_{F^2}\Big] \tag{3.B.15}$$

In the above expression $\mathcal{L}_{F^2}$ contains all the second order terms of F which are exactly same with the terms given in (3.4.8). Now we can easily check that the action constructed from the near horizon metric will contains three types of terms. The ones which are independent and linear in F, can be traced back from their origin which can be transmitted to the fact that the near horizon geometry of the Schwarzschild black hole is Rindler times a sphere, and it does not satisfy the background Einstein's equation. We, therefore, ignore those terms as they can also be made total derivative. Non-trivial dynamics of the Goldstone modes are attributed to second order terms in F arising in the action as given in (3.4.8) up to a total derivative. As a result with a proper prescription, full spacetime geometry as well as near horizon geometry of the Schwarzschild background are giving rise to the same Goldstone mode dynamics.

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3.C Derivation of the Generalized Euler-Lagrangian equation for higher derivative theory

In this Appendix we have briefly reviewed the generalization of the Euler-Lagrangian equations for higher derivative theories[214–216]. At first we consider a classical system describing by scalar field ϕ in D dimensional spacetime. Further we assume that the Lagrangian density $\mathcal{L}$ depends on field ϕ and its N-th order spacetime derivatives. So we can write $\mathcal{L}$ as,

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, \partial_{\mu_1}\partial_{\mu_2}\phi, \ldots \partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_N}\phi). \tag{3.C.1}$$

And action is given by,

$$\mathcal{S} = \int_{\mathcal{V}} d^D x \mathcal{L}. \tag{3.C.2}$$

Now to determine the dynamics of the system, the variation of action must be zero under the arbitrary variation of ϕ and its all order derivatives which also vanish on the boundary $\partial\mathcal{V}$ enclosing a closed volume $\mathcal{V}$. This is expressed in the following way,

$$\delta\mathcal{S} = \int_{\mathcal{V}} d^D x \sum_{n=0}^{N} \frac{\mathcal{L}}{\partial(\partial_{\mu_{(n)}}\phi)} \delta(\partial_{\mu_{(n)}}\phi) = 0. \tag{3.C.3}$$

where we have used the notation that,

$$\partial_{\mu_{(n)}}\phi = \partial_{\mu_1}\partial_{\mu_2}\ldots\mu_n\phi; \qquad \partial_{\mu_{(0)}}\phi \equiv \phi. \tag{3.C.4}$$

Now variation of ϕ and its n-th order derivatives must vanish on the boundary $\delta\mathcal{V}$. This imples,

$$\delta(\partial_{\mu_{(n)}}\phi)|_{\delta\mathcal{V}} = 0; \qquad n = 0, \ldots, N-1. \tag{3.C.5}$$

Now the right hand side of (3.C.3) can be written elaborately as,

$$\sum_{n=0}^{N} \frac{\mathcal{L}}{\partial(\partial_{\mu_{(n)}}\phi)} \delta(\partial_{\mu_{(n)}}\phi) = \frac{\partial\mathcal{L}}{\partial\phi} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\phi)}\delta(\partial_{\mu_1}\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\delta(\partial_{\mu_1}\partial_{\mu_2}\phi) + \ldots$$
$$+ \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_N}\phi)}\delta(\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_N}\phi). \tag{3.C.6}$$

Now each term of (3.C.6) can be written using Leibniz product derivative rule as follows,

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\phi)}\delta(\partial_{\mu_1}\phi) = \partial_{\mu_1}(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\phi)}\delta\phi) - \partial_{\mu_1}(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\phi)})\delta\phi.$$
$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\delta(\partial_{\mu_1}\partial_{\mu_2}\phi) = \partial_{\mu_1}(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\delta(\partial_{\mu_2}\phi)) - \partial_{\mu_1}(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)})\delta(\partial_{\mu_2}\phi). \tag{3.C.7}$$

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It is known from (3.C.5) that $\delta\phi = 0$ and $\delta(\partial_{\mu_2}\phi) = 0$ on the closed boundary $\delta\mathcal{V}$. Then we have from (3.C.7),

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\phi)}\delta(\partial_{\mu_1}\phi) = -\partial_{\mu_1}(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\phi)})\delta\phi$$
$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\delta(\partial_{\mu_1}\partial_{\mu_2}\phi) = -\partial_{\mu_1}(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)})\delta(\partial_{\mu_2}\phi) = -\partial_{\mu_2}\Big(\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\delta\phi\Big)$$
$$+\partial_{\mu_2}\Big(\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\Big)\delta\phi = \partial_{\mu_2}\partial_{\mu_1}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi)}\delta\phi. \tag{3.C.8}$$

here also $\delta\phi = 0$ by the boundary condition (3.C.5). In the similar way after neglecting boundary term, we get from the last term of (3.C.6) that,

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_N}\phi)}\delta(\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_N}\phi)$$
$$= (-1)^n\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_N}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_N}\phi)}\Big)\delta\phi. \tag{3.C.9}$$

Hence using the results found in (3.C.8) and (3.C.9), one can write from (3.C.3) that,

$$\int_{\mathcal{V}} d^Dx\sum_{n=0}^{n}(-1)^N[\partial_{\mu_{(n)}}\frac{\mathcal{L}}{\partial(\partial_{\mu_{(n)}}\phi)}]\delta\phi = 0. \tag{3.C.10}$$

Since the variation in the field $\delta\phi$ is arbitrary, this leads immediately to the generalized Euler-Lagrange field equation as given by,

$$\sum_{n=0}^{N}(-1)^n\partial_{\mu_{(n)}}\frac{\mathcal{L}}{\partial(\partial_{\mu_{(n)}}\phi)} = 0. \tag{3.C.11}$$

In particular, upto second order derivative of the fields, i.e. for $N = 2$, the equation (3.C.11) boils down to the form given in (3.3.17).

3.D Surface Hamiltonian and heat content

In the main text we have constructed the GHY boundary term in the action formulation which did not contribute in the dynamics of the Goldstone mode. However an important analysis is left to be discussed there. It is well known that the boundary term of the Einstein-Hilbert action in gravitational theroy leads to surface Hamiltonian which is directly related to the heat content of the Horizon (detail discussion is given in [217] [218]). The expression of the surface Hamiltonian comes out to be the product of temperature and entropy of the

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horizon. Keeping this in mind we can write surface Hamiltonian corresponding to the GHY boundary term (3.3.16):

$$H_{sur} = -\frac{\partial S_2}{\partial v} \,. \tag{3.D.1}$$

Hence to show the aforementioned connection with the heat content in the present analysis, we have to calculate the form of the Hamiltonian for the mode solution of F given in (3.3.21). Substituting this form of the solution in the expression (3.3.16) and integrating the boundary term , the Hamiltonian (3.D.1) comes out:

$$H_{sur} = \frac{\bar{A}}{8\pi G}\left[\alpha + f_3(\alpha)e^{(f_2(\alpha)v)}\right] , \tag{3.D.2}$$

$\bar{A}$ denotes the transverse area of the Rindler horizon. In the evaluation of the Hamiltonian H_{sur}, we found out that all the first and higher derivative terms of F will generate some complicated expressions (functions of α) which are exponentials in v. However the explicit forms of these terms are not needed in our present discussion as all these terms are exponentially suppressed towards the horizon in the limit $v \to -\infty$. Therefore to show the nature of those terms in the Hamiltonian, we have kept the form $f_3(\alpha, x^2)e^{f_2(\alpha,x^2)v}$ in (3.D.2), where the functional form of $f_3(\alpha, x^2)$ and $f_2(\alpha, x^2)$ are not explicitly given but they are finite near the horizon. Hence the second term in (3.D.2) vanishes near the boundary. Hence the result comes out as:

$$H_{sur} = \frac{1}{8\pi G}\bar{A}\alpha, \tag{3.D.3}$$

Now the Horizon entropy and temperature are given by $S = \bar{A}/4G$ and $T = \alpha/2\pi$ are respectively. Hence with the help of these two expressions, we can write the surface Hamiltonian (3.D.3) as,

$$H_{sur} = TS. \tag{3.D.4}$$

The result clearly indicates that irrespective of Goldstone modes $F_{m,n}$, the GHY boundary term in the action is related to the heat content of the horizon. Similar conclusion can be drawn for Schwarzschild black hole horizon also.

3.E The derivation of (P_T) an (P_R)

In this appendix we have presented a brief review about the quantum mechanics in inverted harmonic potential to derive the probability of transmission (P_T) and

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the reflection (P_R) following [203, 204, 211]. At first we consider Hamiltonian for the wave function $\Phi(x)$ associated with the particle as,

$$\bar{H} = -\frac{\hbar^2}{2\mu}\partial_x^2 - \frac{\bar{\omega}}{2}x^2. \tag{3.E.1}$$

with Schrödinger equation,

$$\bar{H}\Phi(x) = \mathcal{E}\Phi(x); \rightarrow \Phi'' + (\frac{\mu\bar{\omega}x^2}{\hbar^2} + \frac{2\mu\mathcal{E}}{\hbar^2})\Phi. \tag{3.E.2}$$

Now the Schrödinger equation can be expressed as ,

$$\partial_\alpha^2\Phi + (\frac{1}{4}\alpha^2 - a)\Phi = 0; \tag{3.E.3}$$

where we have defined,

$$\alpha = (\frac{4\mu\bar{\omega}}{\hbar^2})^{(1/4)}x; \qquad a = -\sqrt{\frac{\mu}{\bar{\omega}}}\frac{\mathcal{E}}{\hbar} \tag{3.E.4}$$

The two linearly independent solutions of (3.E.3) are given by parabolic cyllinder functions $\mathcal{Z}(a,\alpha)$ and its complex conjugate $\mathcal{Z}^*(a,\alpha)$ [219]. For $\alpha \gg \sqrt{|a|}$, the function $\mathcal{Z}(a,\alpha)$ behaves asymptotically as,

$$\mathcal{Z}(a,\alpha) = \bar{\mathcal{Z}}e^{i\chi}; \qquad \chi \approx \frac{1}{4}\alpha^2 - a\log\alpha + \ldots.$$
$$\bar{\mathcal{Z}} \approx \sqrt{\frac{2}{\alpha^2}}(1 + \mathcal{O}(\alpha^{-2})) \tag{3.E.5}$$

Now to find probabilities we quote a standard result given in 19.18.3 of [219]. So energy eigenfunction corresponding to the particle incident from left is expressed as,

$$\sqrt{1+e^{2\pi a}}\mathcal{Z}(a,\alpha) = e^{\pi a}\mathcal{Z}^*(a,\alpha) + i\mathcal{Z}^*(a,-\alpha);$$
$$\rightarrow \mathcal{Z}(a,\alpha) = i\sqrt{1+e^{2\pi a}}\mathcal{Z}^*(a,-\alpha) - ie^{\pi a}\mathcal{Z}(a,\alpha). \tag{3.E.6}$$

At large positive or negative value of α, the asymptotic expressions are,

$$\mathcal{Z}(a,\alpha\rightarrow\infty) \approx \sqrt{\frac{2}{|\alpha|}}\exp\Big[i(\frac{\alpha^2}{4} + \ldots)\Big]. \tag{3.E.7}$$

$$\mathcal{Z}(a,\alpha\rightarrow-\infty) \approx i\sqrt{1+e^{2\pi a}}\sqrt{\frac{2}{|\alpha|}}\exp\Big[-i(\frac{\alpha^2}{4} + \ldots)\Big]$$
$$-e^{\pi a}\sqrt{\frac{2}{|\alpha|}}\exp\Big[i(\frac{\alpha^2}{4} + \ldots)\Big]. \tag{3.E.8}$$

Now the physically relevent propagation direction should be determined by the probability flux satisfying continuity equation, as given by,

$$j = -i(\Phi^* \Phi' - \Phi^{*'} \Phi). \tag{3.E.9}$$

It can be easily checked that the flux due to (3.E.7) and due to the first part of (3.E.8) are positive. Therefore these two terms represent transmitted and incident waves respectively, whereas the second one in (3.E.8) have negative flux. Hence this part corresponds to the reflected wave.

Now the reflectance and transmittance are defined as the ratio of the intensity of the reflected wave to the incident wave and that of the transmitted wave to the incident wave respectively. So we normalize the coefficient of incident wave to 1 to calculate the probabilies and obtain,

$$\begin{aligned} \mathcal{Z}^*(a, -\alpha) &= \frac{1}{\sqrt{1+e^{-2\pi a}}} \mathcal{Z}(a, -\alpha) - i \frac{1}{\sqrt{1+e^{2\pi a}}} \mathcal{Z}(a, \alpha); \\ &= \frac{1}{\sqrt{1+e^{\frac{2\pi}{\hbar}\sqrt{\frac{\mu}{\omega}}\mathcal{E}}}} \mathcal{Z}(a, -\alpha) - i \frac{1}{\sqrt{1+e^{-\frac{2\pi}{\hbar}\sqrt{\frac{\mu}{\omega}}\mathcal{E}}}} \mathcal{Z}(a, \alpha). \end{aligned} \tag{3.E.10}$$

Then from the coefficients of the reflected wave $\mathcal{Z}(a, -\alpha)$ and that of the transmitted wave $\mathcal{Z}(a, \alpha)$, one can easily get the intensities which generate (P_T) and (P_R) as given in (3.3.25).

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BMS Goldstone modes near the horizon of a Kerr black hole are thermal 4

4.1 Introduction

Recently, it has been argued that the near horizon symmetries are spontaneously broken by the background spacetime (see [164, 165, 171, 184, 196]). Therefore, there exists associated Goldston modes that are conjectured to be connected with the soft hair of the black hole spacetime. Furthermore, interesting one-to-one correspondence between the black hole microstates and the aforementioned soft hairs has been conjectured in (see [89, 100, 164, 165, 171, 184, 194, 196] and other references therein). Following these lines of arguments, in the previous chapter, we proposed a Lagrangian dynamics of those Goldston modes emerging due to spontaneous breaking of the aforementioned near horizon BMS symmetries by the vacuum Schwarzschild black hole. The dynamics of the Goldstone modes have been shown to behave like inverse harmonic oscillators. Quantization of those inverse harmonic oscillators was found to have an interesting connection with the thermal properties of the black hole. Following conjecture proposed in [203, 204] and also in [205–209, 211, 212], we were able to define a thermodynamic temperature which turned out to be proportional to the Hawking temperature of the Schwarzschild black hole (see also [212, 220–224] for connecting inverse harmonic oscillator with thermalization). The whole idea has been successfully extended to black hole solution in three-dimensional massive gravity [225].

In the present chapter, we generalize our previous construction for the Kerr

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black hole. Our current study not only sheds light on the effect of the black hole spin but also can be thought of as a first non-trivial check of our earlier proposal [167] of physics of Goldstone modes in the gravitational sector.

4.2 Objective of the chapter

In this analysis, we will observe that the dynamics of Goldstone modes are associated with the supertranslation parameter, which is being governed by an inverse harmonic oscillator equation with spin-dependent inverted potential. This inverted harmonic oscillator dynamics will be further conjectured to lead to the semi-classical temperature, which turned out to be proportional to the Hawking temperature of the rotating black hole. We will perform our present analysis for two different cases. One is for a slowly rotating black hole with the arbitrary observer. There we will have a perturbative analysis of the static Schwarzschild background. For the second case, we will compute the temperature for a special class of observers, namely, the Zero Angular Momentum Observers (ZAMO) for an arbitrary spin parameter. Our present analysis will reconfirm the fact that the BMS-like symmetries may play an important role in understanding the microscopic origin of black hole thermodynamics.

4.3 Near horizon symmetries and modified metric

In order to investigate the BMS-like symmetry near the horizon, the suitable form of the metric is usually converted in Gaussian null coordinates (GNC) near the horizon. However, the derivation of the Kerr metric in GNC has been explicitly presented in [179] and also near horizon symmetry vector has been derived in details in the literatures (see [91, 172] and other references therein). Nevertheless, below we present a brief description of the derivations as those results are important and also useful for the subsequent part of the present analysis.

4.3.1 Construction of the metric in GNC

In Eddington-Finkelstein coordinate Kerr metric is written as [179],

$$ds^2 = -(1 - \frac{\Delta - \chi}{\Sigma})dv^2 + 2dvdr - \frac{2a(\chi - \Delta)\sin^2\theta}{\Sigma}dvd\phi - 2a\sin^2\theta drd\phi$$
$$+\Sigma d\theta^2 + \frac{\sin^2\theta[\chi^2 - a^2\sin^2\theta\Delta]}{\Sigma}d\phi^2\,, \tag{4.3.1}$$

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where $\Delta = r^2 + a^2 - 2Mr$; $\chi = r^2 + a^2$; $\Sigma = r^2 + a^2\cos^2\theta$. It is well known that within ergosphere of the Kerr black hole observers cannot be static, rather they must feel frame-dragging effect and corotate with the black hole because of the presence of $g_{v\phi}$ term in the metric. So in order to have reasonable physical picture of the near horizon geometry in stationary rotating background, one has to introduce a coordinate transformation to the dragging frame as given by,

$$v \to v, \quad r \to r, \quad \theta \to \theta, \quad \phi \to \phi + (a/\chi_H)v. \tag{4.3.2}$$

By this transformation $g_{v\phi}$ vanishes on the horizon of the Kerr metric. In this coordinate system, null vector $\frac{\partial}{\partial v}$ will be orthogonal to the surfaces of $v =$constant. The transformed metric becomes,

$$\begin{aligned} ds^2 &= -\Big(\frac{\Sigma_H^2\Delta - a^2\sin^2\theta(r-r_H)^2}{\Sigma\chi_H^2}\Big)dv^2 + 2\frac{\Sigma_H}{\chi_H}dvdr \\ &+2a\sin^2\theta\frac{\Delta\Sigma_H + \chi(r^2-r_H^2)}{\Sigma(r_H^2+a^2)}dvd\phi - 2a\sin^2\theta drd\phi \\ &+\Sigma d\theta^2 + \frac{\sin^2\theta[\chi^2 - a^2\sin^2\theta\Delta]}{\Sigma}d\phi^2 \,. \end{aligned} \tag{4.3.3}$$

Now we will follow the standard procedure (presented in [179]) to construct near horizon metric in Gaussian null coordinate system. For this reason we have to define the suitable pair of null normals on the horizon as,

$$l^a = (1,0,0,0); \quad n^a = \Big(\frac{a^2\sin^2\theta}{2\Sigma_H}, \frac{\chi_H}{\Sigma_H}, 0, \frac{a}{\chi_H}\Big). \tag{4.3.4}$$

These normals are defined such that at the horizon $n^a n_a = 0; l^a l_a = 0$; and $l^a n_a = 1$ (proof is shown in Appendix 4.A.1).

Next, we consider a set of incoming null geodesics which crosses the horizon having tangent vector n^a. So the geodesics can be parametrized with affine parameter ρ ($\rho \approx r - r_H$) such that the horizon surface is identified as $\rho = 0$, but increases towards the inside. Therefore the null geodesics curve $X^a(\rho)$ with $X^a = (v, r, \theta, \phi)$, can be expanded up to second order of ρ as follows:

$$X^a(\rho) \approx X^a\Big|_{\rho=0} + \rho\frac{dX^a}{d\rho}\Big|_{\rho=0} + \frac{\rho^2}{2}\frac{d^2X^a}{d\rho^2}\Big|_{\rho=0} + \dots \,. \tag{4.3.5}$$

In the right hand expression the components of first term is identified as $X^a|_{\rho=0} = (v, r_H, \theta, \phi)$. The second one can be expressed as the tangent vector to the curve, $\frac{dX^a}{d\rho}|_{\rho=0} = n^a|_{\rho=0}$. The third term can be written with the help of null geodesics equation at the horizon as,

$$\frac{d^2X^a}{d\rho^2}\Big|_{\rho=0} = -\Gamma^a_{bc}n^b n^c\Big|_{\rho=0}. \tag{4.3.6}$$

Using all these results in Eq. (4.3.5) and keeping upto first order in ρ, we can define a transformation of coordinates from (v,ρ,θ,ϕ) to (v',r,θ',ϕ') in the following way,

$$\begin{aligned} v' &= v+\rho(\frac{a^2\sin^2\theta}{2\Sigma_H}); \quad r = r_H+\rho(\frac{\chi_H}{\Sigma_H})\,; \\ \theta' &= \theta; \quad \phi' = \phi+\rho(\frac{a}{\chi_H})\,. \end{aligned} \tag{4.3.7}$$

So by having a tensor transformation rule of the components, we can obtain the metric in Gaussian null coordinate system (v,ρ,θ,ϕ) as follows,

$$ds^2 = -2\rho\kappa dv^2+2dvd\rho+2\rho\beta_A dvdx^A+(\mu_{AB}+\rho\lambda_{AB})dx^A dx^B\,. \tag{4.3.8}$$

Here the horizon is located at $\rho=0$. The expressions of the metric coefficients and detail derivation of these components are given in Appendix 4.A.2.

4.3.2 Diffeomorphism symmetries near the horizon

To obtain the asymptotic symmetries near the horizon, the appropriate fall-off conditions for the metric coefficients are assumed to be

$$\pounds_\zeta g_{\rho\rho}=0, \quad \pounds_\zeta g_{v\rho}=0, \quad \pounds_\zeta g_{A\rho}=0\,; \tag{4.3.9}$$

$$\pounds_\zeta g_{vv}\approx\mathcal{O}(\rho); \quad \pounds_\zeta g_{vA}\approx\mathcal{O}(\rho); \quad \pounds_\zeta g_{AB}\approx\mathcal{O}(\rho)\,, \tag{4.3.10}$$

such that the transformed metric assume the same form near the horizon $\rho=0$. Here, $\pounds_\zeta$ corresponds to the Lie variation associated with the diffeomorphism $x^a\to x^a+\zeta^a$ and is given by

$$\pounds_\zeta g_{ab} = \nabla_a\zeta_b+\nabla_b\zeta_a \tag{4.3.11}$$

Following the gauge choices given in Eq.(4.3.9), the associated diffeomorphism vectors can be expressed in the following form,

$$\begin{aligned} \zeta^v &= F(v,x^A)\,; \\ \zeta^\rho &= T(v,x^A)-\rho\partial_v F-\partial_B F\int\rho\beta^B d\rho\,; \\ \zeta^A &= -\partial_B F\int\mu^{AB}d\rho+R^A(v,x^B)\,. \end{aligned} \tag{4.3.12}$$

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Now we impose the weak fall off conditions given in (4.3.10), on the aforementioned solutions (4.3.12). From there the following constraints relations have appeared among the diffeomorphism parameters T and R^A,

$$\partial_v T + \kappa T = 0\,; \tag{4.3.13}$$
$$\partial_A T - T(v, x^A)\beta_A + \mu_{AB}\partial_v R^B = 0\,; \tag{4.3.14}$$
$$R^D\partial_D\mu_{AB} + \mu_{AD}\partial_B R^D + \mu_{BD}\partial_A R^D = 0\,. \tag{4.3.15}$$

The equation (4.3.15) boils down to the three component equations as,

$$\begin{aligned} &R^\theta\partial_\theta\mu_{\theta\theta} + 2\mu_{\theta\theta}\partial_\theta R^\theta = 0; \\ &\mu_{\theta\theta}\partial_\phi R^\theta + \mu_{\phi\phi}\partial_\theta R^\phi = 0; \\ &R^\theta\partial_\theta\mu_{\phi\phi} + 2\mu_{\phi\phi}\partial_\phi R^\phi = 0\,. \end{aligned} \tag{4.3.16}$$

Now, we solve the above equations with the condition that the position of the null surface at $\rho = 0$ remains unaltered, which leads to $T(v, x^A) = 0$ from Eq.(4.3.12). Then (4.3.14) yields that R^A is independent of v. Furthermore, the leading order term in β^A is independent of ρ, and the last term in ζ^ρ yields sub-leading $\mathcal{O}(\rho^2)$ contribution. With this, the diffeomorphism vector ζ^a corresponding to above conditions assumes the following form [91, 172]

$$\zeta^a\partial_a = \quad F(v, x^A)\partial_v - \rho\partial_v F(v, x^A)\partial_\rho + \Big(-\rho\partial^A F(v, x^A) + R^A(x^A)\Big)\partial_A\,. \tag{4.3.17}$$

Near the horizon $\rho = 0$ surface we have two diffeomorphism parameters F and R^A associated with supertranslation and superrotation symmetry parameters respectively [91, 166, 172].

4.3.3 The corrected metric

Under the asymptotic symmetry transformation discussed above, the corrected metric is expressed as $g_{ab} = g^{(0)}_{ab} + h_{ab}$, where $g^{(0)}_{ab}$ is given by Eq.(4.3.8) and the components of h_{ab} are derived by the Lie derivative of the metric as given by (4.3.11); i.e. $h_{ab} = \pounds_\zeta g_{ab}$. So these corrected components are expressed up to linear order by the symmetry transformations as follows, (detail construction can be

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found in Appendix 4.B),

$$
\begin{aligned}
& h_{ab}dx^a dx^b = -[2\rho(\kappa\partial_v F + \partial_v^2 F)]dv^2 + 2\rho[R^B\partial_B\beta_A - 2\kappa\partial_A F \\
- \;& \partial_A\partial_v F + \beta_B\partial_A R^B - \mu_{AB}\partial_v\partial_D F\mu^{BD}]dv dx^A \\
+ \;& \rho\Big[-\lambda_{AB}\partial_v F - \partial_E F\mu^{DE}\partial_D\mu_{AB} + \beta_A\partial_B F + \beta_B\partial_A F - \mu_{AD}\partial_B(\partial_E F\mu^{DE}) \\
- \;& \mu_{BD}\partial_A(\partial_E F\mu^{DE}) + R^D\partial_D\lambda_{AB} + \lambda_{AD}\partial_B R^D + \lambda_{BD}\partial_A R^D\Big]dx^A dx^B. \qquad (4.3.18)
\end{aligned}
$$

The corrected metric can be understood as the change of physical parameters of the background black hole with the following transformation,

$$
\begin{aligned}
& \kappa \to \kappa + \kappa\partial_v F + \partial_v^2 F; \\
& \beta_A \to \beta_A + R^B\partial_B\beta_A - 2\kappa\partial_A F - \partial_A\partial_v F + \beta_B\partial_A R^B - \mu_{AB}\partial_v\partial_D F\mu^{BD}; \\
& \lambda_{AB} \to \lambda_{AB} + \Big(-\lambda_{AB}\partial_v F - \partial_E F\mu^{DE}\partial_D\mu_{AB} + \beta_A\partial_B F + \beta_B\partial_A F \\
& -\mu_{AD}\partial_B(\partial_E F\mu^{DE}) - \mu_{BD}\partial_A(\partial_E F\mu^{DE}) + R^D\partial_D\lambda_{AB} + \lambda_{AD}\partial_B R^D + \lambda_{BD}\partial_A R^D\Big).
\end{aligned} \qquad (4.3.19)
$$

Analogous to the analysis presented in chapter 3, In this chapter, we have studied symmetry breaking phenomena near the horizon of Kerr spacetime. Hence we can say that the Goldstone Boson should appear due to the spontaneous breaking of the BMS-like boundary global symmetry. At first, we have identified those specific sets of diffeomorphism under which the asymptotic structure of the near horizon Kerr metric remains intact. Nevertheless, the black hole solution is transformed in this process, and mass M and angular momentum a of the Kerr black hole get shifted to M' and a' respectively, keeping the near horizon boundary conditions invariant. However, because of the arbitrariness of the symmetry parameters F and R^A, one may construct various solutions of the Einstein equations depending on the values of these parameters. Nevertheless, the background Kerr is only one choice among all these solutions. Thus one can say that the Background solution breaks the symmetry of taking into account all arbitrary values of the parameters. This phenomenon can be compared with the spontaneous symmetry breaking of the global $U(1)$ symmetry. Here the corresponding parameters F and R^A are taken as the Goldstone boson modes.

Furthermore, as discuded in the last chapter, in the Hamiltonian formulation we know κ is associated with the Lagrangian multipliers of the theory, which are usually kept fixed everywhere in the spacetime [171, 196]. Then appropriate boundary terms has to be added with action such that on-shell variation vanishes.

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With this condition on κ, the Eq.(4.3.19) is solved for F, which becomes non-dynamical. However, if we impose more general condition on $\delta\kappa$ being non-vanishing everywhere in the bulk except at the horizon boundary (denoted by $r = r_H$ or $v \to -\infty$), satisfying (3.3.8), then the super-translation parameter F becomes dynamical in nature. Therefore keeping this in mind, we first construct an appropriate Lagrangian of F and pick those solutions of F which will naturally satisfy Eq. (3.3.8).

It may be pointed out that R^A does not depend upon timelike coordinate v, and F depends on v. Consequently, F will be the physical dynamical Goldstone mode for Kerr background. The rotational parameter will be non-dynamical in nature, and that must be consistent with the three constraint conditions expressed in Eq.(4.3.16).

Therefore, consistent with the constraints as discussed above, the simplest possible choice for R^A is

$$R^\theta = 0; \qquad R^\phi = C = \text{constant}\,. \tag{4.3.20}$$

We will later see that such a choice makes $\delta\beta_A$ vanish automatically at the horizon. This is reminiscent of the condition $\delta\kappa = 0$ at the horizon boundary. The constraint relations in (4.3.16) impose over restriction on parameters R^θ and R^ϕ. The above solution Eq.(4.3.20) is the simplest one that will satisfy all the conditions. There may exist other solutions. However, for our present purpose, we restrict to this specific choice.

4.4 Lagrangian and the equation of motion

In this section, our aim is to construct dynamical equations of the two Goldstone modes with the aforementioned constraint conditions on the black hole parameters. Following the methodology formulated in chapter 3, appropriate dynamical Lagrangian is conjectured to have two terms. The important term is the Einstein-Hilbert Lagrangian corresponding to the corrected metric g_{ab} near the black hole horizon $\rho = 0$,

$$\mathcal{L}_{\mathcal{F},\mathcal{R}^\mathcal{A}} = \sqrt{-g'}\mathcal{R}'\,. \tag{4.4.1}$$

g' and $\mathcal{R}'$ are the determinant and Ricci scalar of the corrected metric g_{ab}. Here the modified metric g_{ab} has been constructed by taking into account a subset of diffeomorphism, which preserves the near horizon boundary conditions. It is well

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known that due to this diffeomorphism, the Einstein-Hilbert Lagrangian must be invariant up to total derivative terms, which also become boundary terms on the closed boundary enclosing a bulk region of the manifold. However, in this analysis, we have focussed near the horizon ($r = 0$), which is a part of this closed boundary. Therefore this total derivative term will have a finite contribution on this part of the boundary. The form of the Lagrangian presented in the main text comes from this non-zero contribution. (more detail can be found in the chapter 3, Section 3.3.1 after the Eq. (3.3.9)).

Now to calculate the above Lagrangian, we follow the procedure usually adopted in the context of the stretched horizon (detail discussion can be found in section 4 of [62]). We first calculate (4.4.1) on the radial coordinate $\rho = \epsilon$ very near the horizon, then $\epsilon \to 0$ limit will be taken to get the required expression of the Goldstone boson Lagrangian.

Now we can expand the above Lagrangian in Taylor series with respect to the background metric $g^{(0)}_{ab}$ (4.3.8), assuming h_{ab} as the small fluctuation. The expansion of the Lagrangian is given by,

$$\mathcal{L}_{\mathcal{F},\mathcal{R}^{\mathcal{A}}} = \mathcal{L}_{\mathcal{F},\mathcal{R}^{\mathcal{A}}}(g^{(0)}_{ab}) + h_{ab}\left(\frac{\delta \mathcal{L}_{\mathcal{F},\mathcal{R}^{\mathcal{A}}}}{\delta g_{ab}}\right)^{(0)} + h_{ab}h_{cd}\left(\frac{\delta^2 \mathcal{L}_{\mathcal{F},\mathcal{R}^{\mathcal{A}}}}{\delta g_{ab}\delta g_{cd}}\right)^{(0)} + \dots . \tag{4.4.2}$$

As has been explained in the previous chapter, the free Lagrangian for the Goldstone mode will be the third term which is quadratic in the parameters. For the present purpose, we will restrict ourselves only up to second order in Goldstone modes. Using the constraints on the superrotation parameter as given in (4.3.20), the Lagrangian for the super-translation modes becomes,

$$\begin{aligned} \mathcal{L}_{(F,R^{\theta}=0,,R^{\phi}=C)} &= c_1(\theta)(\partial_v F)^2 + c_2(\theta)(\partial_\theta F)^2 + c_3(\theta)(\partial_\phi F)^2 \\ &+ c_4(\theta)(\partial_v\partial_\theta F)^2 + c_5(\theta)(\partial_v\partial_\phi F)^2 + c_6(\theta)\partial_\theta F\partial_v^2 F \\ &+ c_7(\theta)\partial_\phi F\partial_v^2 F + c_8(\theta)\partial_\theta^2 F\partial_v^2 F + c_9(\theta)\partial_\phi^2 F\partial_v^2 F. \end{aligned} \tag{4.4.3}$$

See Appendix 4.C for a detail derivation of the Lagrangian (4.4.3). The explicit forms of c_is are,

$$\begin{aligned} &c_1 = \frac{B(\theta)}{2\Sigma_H^2\chi_H\sin\theta}; \;\; c_2 = -6\alpha_2^2\frac{\sin\theta}{\chi_H\Sigma_H}; \;\; c_3 = -6\frac{\alpha_2^2\Sigma_H(\csc\theta)}{\chi_H^3}; \\ &c_4 = -6\chi_H\frac{(\sin\theta)}{\Sigma_H}; \;\; c_5 = -6\Sigma_H\frac{(\csc\theta)}{\chi_H^2}; \;\; c_6 = 4\chi_H\frac{(\cos\theta)}{\Sigma_H}; \\ &c_8 = 4\chi_H\frac{(\sin\theta)}{\Sigma_H}; \;\; c_9 = 4\Sigma_H\frac{(\csc\theta)}{\chi_H}. \end{aligned} \tag{4.4.4}$$

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where,

$$B(\theta) = \alpha_2 \sin^2\theta \Big(a^4(M+7r_H) + 4a^2Mr_H^2 + +12a^2r_H^3 + 8r_H^5 + 4a^2\alpha_2 r_H^2 \cos 2\theta$$
$$+ a^4\alpha_2 \cos 4\theta + 8r_H\chi_H(r_H^2 + a^2\cos 2\theta)\Big); \tag{4.4.5}$$
$$\alpha_1 = (M+r_H);\ \ \alpha_2 = (r_H - M);\ \ \alpha_3 = (a^2 + Mr_H).$$

As has already been described in detail in the previous chapter, the Lagrangian is so constructed that the action from the modified metric g_{ab} will describe the Goldstone mode dynamics near the rotating black hole horizon. Another part of the proposed Lagrangian is the Gibbons-Hawking-York (GHY) boundary term which is required to have consistent variational principle and is given by,

$$\mathcal{S}_2 = -\frac{1}{8\pi G}\int d^3x\sqrt{h}K\,. \tag{4.4.6}$$

In the above expression the quantity K is given by $K = -\nabla_a M^a$, which is the trace of the extrinsic curvature of the boundary surface (at $\rho \to 0$). Here M^a is considered as the unit normal to the $\rho = constant$ hyper-surface. For the corrected metric (4.3.18), the lower components of M^a are given by,

$$M_a = (0, 1/\sqrt{2r(\kappa + \kappa\partial_v F + \partial_v^2 F)}, 0, 0). \tag{4.4.7}$$

Hence for the GHY term (4.4.6) we can write action in the following form by taking near horizon limit ($\rho \to 0$),

$$\mathcal{S}_2 = -\frac{\chi_H}{8\pi G}\int d^3x \sin\theta \Big[\kappa + \Big(\kappa\partial_v F + \frac{1}{2}\partial_v^2 F + \frac{1}{2\kappa}\partial_v^3 F\Big)$$
$$+ \frac{1}{2\kappa^2}\Big(\kappa^2\partial_v F\partial_v^2 F + \kappa(\partial_v^2 F)^2 + \kappa\partial_v F\partial_v^3 F + \partial_v^2 F\partial_v^3 F\Big)\Big]. \tag{4.4.8}$$

In Appendix 4.D we showed that (4.4.8) gives a total derivative term in v and therefore does not contribute to the dynamics.

Once we have the appropriate Lagrangian expressed in Eq.(4.4.3), the equation of motion for F yields as follows (detail is given in Appendix 4.F),

$$12\alpha_2^2\Sigma_H\Big(\Sigma_H\chi_H^2\partial_\theta^2 F + (r_H^2 + 2a^2 - a^2\cos^2\theta)\chi_H^2(\cot\theta)\partial_\theta F + \frac{\Sigma_H^3}{\sin^2\theta}\partial_\phi^2 F\Big)$$
$$-\chi_H^2\Big(16\Sigma_H^2\chi_H^2\partial_v^2\partial_\theta^2 F + 16(r_H^2 + 2a^2 - a^2\cos^2\theta)\Sigma_H\chi_H^2(\cot\theta)\partial_v^2\partial_\theta F$$
$$+ \frac{16\Sigma_H^4}{\sin^2\theta}\partial_v^2\partial_\phi^2 F - \alpha\partial_v^2 F\Big) = 0\,. \tag{4.4.9}$$

Here α is given by

$$\alpha = -8r_H^3\Big(a^2\alpha_1\alpha_3 - 4Mr_H^3\alpha_2\Big) + 8a^2\Big(8a^2\chi_H(7r_H^2+3a^2) + r_H^3\alpha_1\alpha_3 + 2a^2r_H^2\alpha_2^2\Big)\cos^2\theta + 8a^4\Big(-3\chi_H^2 + 4Mr_H\alpha_3 + \alpha_2^2(a^2-2r_H^2)\Big)\cos^4\theta - 8a^6\alpha_2^2\cos^6\theta. \tag{4.4.10}$$

$$\alpha_1 = (M+r_H);\;\; \alpha_2 = (r_H - M);\;\; \alpha_3 = (a^2+Mr_H).$$

For the general value of background angular momentum parameter a, it turned out to be very difficult to solve for the mode function f_{lm}. Hence, we will solve the above problem for the slowly rotating black holes. Later we will discuss the same for Zero Angular Momentum Observers (ZAMO).

4.4.1 Slowly rotating Kerr spacetime

We consider slowly rotating background where rotation parameter a is such that $a \ll M$. Considering $a/M = x$, we can expand (4.4.9) upto $\mathcal{O}(x^2)$ as follows (detail can be found in Appendix 4.G and also in Appendix 4.H.1) :

$$\Big[64x^2\sin^2\theta\partial_\theta^2\partial_v^2F + \Big(12\cos^2\theta + 84\Big)x^2\partial_v^2F + 128x^2\partial^2\partial_\phi^2F - 64(2-3x^2)\Big(\partial_\theta^2\partial_v^2F + \cot\theta\partial_\theta\partial_v^2F + \frac{1}{\sin^2\theta}\partial_\phi^2\partial_v^2F\Big)\Big] - \frac{1}{M^2}\Big[3x^2\sin^2\theta\partial_\theta^2F + 6x^2\partial_\phi^2F - 6(1-2x^2)\Big(\partial_\theta^2F + \cot\theta\partial_\theta F + \frac{1}{\sin^2\theta}\partial_\phi^2F\Big)\Big] = 0 \tag{4.4.11}$$

In the original stationery axially symmetric background, the angular part of the field must satisfy the equation for spheroidal harmonics [219, 226]. However, in the present analysis, because of the fluctuation around the background, the equation of F does not satisfy the equation of the spheroidal harmonic. But still, we have considered the angular part of the solution ansatz in the form of spherical harmonics. The validity of this choice is clear in this section, as we have focussed on slowly rotating spacetime. Here we will try to find the dynamics of F by the perturbative analysis where the zeroth-order solution is the result found on a spherically symmetric Schwarzschild background. Therefore in a slowly rotating background, the final solution of F has been found as the first-order correction to that zeroth-order solution. So we assume that the separability assumption in terms of spherical harmonics should work in the present analysis. We, therefore,

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expand field F in terms of the generic spherical harmonic basic,

$$F(v,\theta,\phi) = \sum_{l,m} f_{lm}(v) Y_{lm}(\theta,\phi). \tag{4.4.12}$$

Putting solution ansatz (4.4.12) in the equation (4.4.11), we get,

$$\begin{aligned} &4(\partial_v^2 f_{lm})\Big[16x^2\sin^2\theta(\partial_\theta^2 Y_{lm}) + \Big(32l(l+1) - 16 - x^2(48l(l+1) - 3\cos^2\theta \\ &+32m^2)\Big)Y_{lm}\Big] \\ &-\frac{3}{M^2} f_{lm}(v)\Big[x^2\sin^2\theta(\partial_\theta^2 Y_{lm}) + 2\Big(l(l+1) - x^2(2l(l+1)+m^2)\Big)Y_{lm}\Big] = 0. \end{aligned} \tag{4.4.13}$$

The mode $l=0$ and $m=0$, satisfies,

$$\partial_v^2 f_{00} = 0, \tag{4.4.14}$$

whose solution is

$$f_{00} = b_1(x^A) + b_2(x^A)v. \tag{4.4.15}$$

The boundary condition $\delta\kappa = 0$ at $v \to -\infty$ implies $b_2 = 0$. Therefore, the final solution will be

$$f_{00}(v) = b_1. \tag{4.4.16}$$

Now to get the equation of motion for generic mode f_{lm}, we use the relation between trigonometric functions and spherical harmonics and also we have to use the results which is found from the second order derivative of Y_{lm} with respect to θ, as given in Appendix 4.H.2. Hence by substituting (4.H.3) and (4.H.4) in (4.4.13) and considering constant and x dependent parts of the equation (4.4.13) separately, the equation can be expressed as,

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$$
\begin{aligned}
&\sum_{lm}\Big[32(l(l+1)-1)\partial_v^2 f_{lm}-\frac{3}{M^2}l(l+1)f_{lm}\Big]Y_{lm}\\
&+x^2\Big[\partial_v^2 f_{lm}\Big(-96l(l+1)Y_{lm}+8\pi(\frac{1}{2\sqrt{\pi}}Y_{00}+\frac{1}{\sqrt{5\pi}}Y_{20})Y_{lm}\\
&+\frac{128\pi}{3}\Big(m^2(\frac{1}{2\sqrt{\pi}}Y_{00}+\frac{1}{\sqrt{5\pi}}Y_{20})-m\Big)Y_{lm}\\
&+64\sqrt{\frac{2\pi}{15}}\sqrt{(l-m)(l+m+1)}(2m+1)\times Y_{2-1}Y_{l\ m+1}\\
&+128\sqrt{\frac{2\pi}{15}}\sqrt{(l-m)(l-m-1)(l+m+2)(l+m+1)}Y_{2-2}Y_{l\ m+2}\\
&-2(32m^2-37)Y_{lm}\Big)-\frac{1}{M^2}f_{lm}\Big(-6l(l+1)Y_{lm}\\
&+2\pi\Big(m^2(\frac{1}{2\sqrt{\pi}}Y_{00}+\frac{1}{\sqrt{5\pi}}Y_{20})-m\Big)Y_{lm}\\
&+3\sqrt{\frac{2\pi}{15}}\sqrt{(l-m)(l+m+1)}(2m+1)Y_{2-1}Y_{l\ m+1}+6\sqrt{\frac{2\pi}{15}}\\
&\times\sqrt{(l-m)(l-m-1)(l+m+2)(l+m+1)}Y_{2-2}Y_{l\ m+2}-3m^2Y_{lm}\Big)\Big]=0.
\end{aligned}
\tag{4.4.17}
$$

Now we use (4.H.5) to calculate $Y_{20}Y_{lm}$, $Y_{2-1}Y_{l\ m+1}$ and $Y_{2-2}Y_{l\ m+2}$ as follows,

$$
\begin{aligned}
Y_{20}Y_{lm} &= \sum_{m_3=-l_3}^{l_3}\sum_{l_3=|2-l|}^{|2+l|}\Lambda^{20}_{lml_3m_3}Y_{l_3m_3};\\
Y_{2-1}Y_{lm+1} &= \sum_{m_3=-l_3}^{l_3}\sum_{l_3=|2-l|}^{|2+l|}\Lambda^{2-1}_{lm+1l_3m_3}Y_{l_3m_3};\\
Y_{2-2}Y_{lm+2} &= \sum_{m_3=-l_3}^{l_3}\sum_{l_3=|2-l|}^{|2+l|}\Lambda^{2-2}_{lm+2l_3m_3}Y_{l_3m_3}.
\end{aligned}
\tag{4.4.18}
$$

Then using the above relations we express the equation (4.4.17) in the combined spherical harmonic basis as follows,

$$
\begin{aligned}
&\sum_{lm}\Big[(l(l+1)-1)\partial_v^2 f_{lm}-\frac{3}{32M^2}l(l+1)f_{lm}\Big]Y_{lm}\\
&+\sum_{lm}x^2\Big[\beta_1(l,m)\partial_v^2 f_{lm}+\beta_2(l,m)f_{lm}+\sum_{m_3=-l_3}^{l_3}\sum_{l_3=|l-2|}^{|l+2|}\Big(\beta_{lml_3m_3}(l,m)\partial_v^2 f_{lm}\\
&+\bar{\beta}_{lml_3m_3}(l,m)f_{lm}\Big)\Big]Y_{l_3m_3}=0.
\end{aligned}
\tag{4.4.19}
$$

where we have defined following quantities,

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$$\beta_1(l,m) = \Big(\frac{-72l(l+1)+57-40m^2-24m}{24}\Big);$$
$$\beta_2(l,m) = \Big(\frac{12l(l+1)+5m^2+3m}{64M^2}\Big);$$

$$\beta_{lml_3m_3}(l,m) = 2\sqrt{\pi}[\Lambda^{20}_{lml_3m_3}(\frac{2m^2+24}{3\sqrt{5}}) + (2m+1)\Big(\Lambda^{2-1}_{l(m+1)l_3m_3}$$
$$\times(\sqrt{\frac{2(l-m)(l+m+1)}{15}})$$
$$+2\Lambda^{2-2}_{l(m+2)l_3m_3}\sqrt{\frac{2(l^2-(m+1)^2)(l(l+2)-m(m+2))}{15}}\Big)];$$

$$\bar{\beta}_{lml_3m_3}(l,m) = 2\sqrt{\pi}[\Lambda^{20}_{lml_3m_3}\frac{m^2}{32\sqrt{5}M^2} + \Lambda^{2-1}_{l(m+1)l_3m_3}$$
$$\times\frac{3(2m+1)\sqrt{2(l-m)(l+m+1)}}{64M^2\sqrt{15}}$$
$$+\Lambda^{2-2}_{l(m+2)l_3m_3}\frac{3(2m+1)\sqrt{2(l^2-(m+1)^2)(l(l+2)-m(m+2))}}{32M^2\sqrt{15}}]. \tag{4.4.20}$$

Now in the second line of (4.4.19), at first we replace dummy indices such that $l \to l'$ and $m \to m'$ in the third and fourth terms, after that we again replace $l_3 \to l$ and $m_3 \to m$. Thus finally from (4.4.19) we get,

$$\Big(\sum_{lm}\Big[(l(l+1)-1)\partial_v^2 f_{lm} - \frac{3}{32M^2}l(l+1)f_{lm}\Big]$$
$$+\sum_{lm}x^2\Big[\beta_1(l,m)\partial_v^2 f_{lm} + \beta_2(l,m)f_{lm}\Big] + x^2\Big[\sum_{l'm'}\sum_{m=-l}^{l}\sum_{l=|l'-2|}^{|l'+2|}\Big(\beta_{l'm'lm}(l',m')\partial_v^2 f_{l'm'}$$
$$+\bar{\beta}_{l'm'lm}(l',m')f_{l'm'}\Big)\Big]\Big)Y_{lm} = 0. \tag{4.4.21}$$

Now using the selection rule, one can write,

$$\sum_{m'=-l'}^{l'}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l}\sum_{l=|l'-2|}^{|l'+2|} \equiv \sum_{m'=-l'}^{l'}\sum_{l'=|l-2|}^{|l+2|}\sum_{m=-l}^{l}\sum_{l=0}^{\infty}. \tag{4.4.22}$$

Also ,

$$\sum_{lm} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}.$$

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Now using (4.4.22), the equation (4.4.21) boils down to

$$\sum_{lm}\Big((2l(l+1)-1)\partial_v^2 f_{lm}-\frac{3l(l+1)}{32M^2}f_{lm}+x^2\Big[(\beta_1\partial_v^2 f_{lm}+\beta_2 f_{lm})$$
$$+\sum_{m'=-l'}^{l'}\sum_{l'=|l-2|}^{|l+2|}(\beta_{l'm'lm}\partial_v^2 f_{l'm'}+\bar{\beta}_{l'm'lm}f_{l'm'})\Big]\Big)Y_{lm}=0. \quad (4.4.23)$$

Since Y_{lm} are linearly independent, we now have generic coupled mode equation,

$$(2l(l+1)-1)\partial_v^2 f_{lm}-\frac{3l(l+1)}{32M^2}f_{lm}+x^2\Big[(\beta_1\partial_v^2 f_{lm}+\beta_2 f_{lm})$$
$$+\sum_{m'=-l'}^{l'}\sum_{l'=|2-l|}^{|2+l|}(\beta_{l'm'lm}\partial_v^2 f_{l'm'}+\bar{\beta}_{l'm'lm}f_{l'm'})\Big]=0; \quad (4.4.24)$$

We solve the aforesaid equation by using perturbative method in terms of rotation parameter x. The zeroth order part of the equation will correspond to that of the Schwarzschild background given in chapter 3. Therefore, the solution is taken as, (for $l\geq 1$)

$$f_{lm}=f_{lm}^{sc}+x^2 f_{lm}^{(1)}+\dots, \quad (4.4.25)$$

where, f_{lm}^{sc} is the form of the solution of $f_{lm}(v)$ in Schwarzschild background as given by (detail discussion can be found in chapter 3),

$$f_{lm}^{sc}=\frac{c_{sc}^{(0)}}{k_{sc}}\exp[\Omega(l)\kappa_{sc}v],\ \ \Omega(l)=\sqrt{\frac{3l(l+1)}{2(2l(l+1)-1)}}. \quad (4.4.26)$$

Here κ_{sc} is the surface gravity of Schwarzschild black hole and $c_{sc}^{(0)}$ is a dimensionless constant. Hence f_{lm}^{sc} have dimension of length. $f_{lm}^{(1)}$ is the leading order contribution due to slowly rotating Kerr black hole. Now substituting (4.4.25) in (4.4.24) till $\mathcal{O}(x^2)$, we find that

$$(2l(l+1)-1)\partial_v^2 f_{lm}^{sc}-\frac{3l(l+1)}{32M^2}f_{lm}^{sc}$$
$$+x^2\Big(2l(l+1)-1)\partial_v^2 f_{lm}^{(1)}-\frac{3l(l+1)}{32M^2}f_{lm}^{(1)}\Big)+x^2\Big[(\beta_1\partial_v^2 f_{lm}^{sc}+\beta_2 f_{lm}^{sc})$$
$$+\Big(\beta_{l+2mlm}\partial_v^2 f_{l+2m}^{sc}+\bar{\beta}_{l+2mlm}f_{l+2m}^{sc}+\beta_{lmlm}\partial_v^2 f_{lm}^{sc}$$
$$+\bar{\beta}_{lmlm}f_{lm}^{sc}+\beta_{|l-2|mlm}\partial_v^2 f_{|l-2|m}^{sc}+\bar{\beta}_{|l-2|mlm}f_{|l-2|m}^{sc}\Big)\Big]=0; \quad (4.4.27)$$

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Following the contraction rule as given in Appendix 4.H.2 and also considering (4.4.18), we can say that for $m' = m$ only, one can get non-zero values of Λ"s . It is clear that the zeroth order part of the above equation which matches with the equation of f^{sc}_{lm} (given in chapter 3) automatically vanishes. Then putting the solution (4.4.26), in (4.4.27), the coefficients of x^2 are given by,

- For $l = 1$,

$$\partial_v^2 f^{(1)}_{1m} - \Omega^2 \kappa_{sc}^2 f^{(1)}_{1m} = c^{(0)}_{sc} \kappa_{sc} \left[\bar{b} \exp[\bar{\Omega} \kappa_{sc} v] + \bar{b}_1 \exp[\bar{\Omega}_1 \kappa_{sc} v] \right], \tag{4.4.28}$$

  where $-1 \leq m \leq 1$ and the other quantities in the above equation are defined in (4.I.1).

- For $l \geq 2$,
  the coefficients of x^2 in the equation (4.4.27) reduces to the following form;

$$\partial_v^2 f^{(1)}_{lm} - \Omega^2 \kappa_{sc}^2 f^{(1)}_{lm} = c^{(0)}_{sc} \kappa_{sc} \Big(b(l,m) \exp[\Omega \kappa_{sc} v] + b_1(l,m) \exp[\Omega_1 \kappa_{sc} v] \\ + b_2(l,m) \exp[\Omega_2 \kappa_{sc} v] \Big). \tag{4.4.29}$$

This is the equation of the first order perturbation $f^{(1)}_{lm}$. The corresponding quantities in the above equations are expressed in Appendix 4.I. Now as discussed in the Appendix 4.J, we neglect the second term in the right hand side of (4.4.28) and also the second and the third source terms in the R.H.S of (4.4.29). Then the equation of motion for $f^{(1)}_{lm}$ boils down to ($l \geq 1$),

$$\partial_v^2 f^{(1)}_{lm} - \Omega^2 \kappa_{sc}^2 f^{(1)}_{lm} = c^{(0)}_{sc} \kappa_{sc} b(l,m) \exp[\Omega \kappa_{sc} v] = \kappa_{sc}^2 b(l,m) f^{sc}_{lm}. \tag{4.4.30}$$

Upto $\mathcal{O}(x^2)$, one can approximate,

$$x^2 f_{lm} = x^2 (f^{sc}_{lm} + x^2 f^{(0)}_{lm}) \approx x^2 f^{sc}_{lm}. \tag{4.4.31}$$

Then multiplying the equation (4.4.30) with x^2 , then substituting (4.4.31) in (4.4.30), at last adding zeroth order equation of f^{sc}_{lm} with it, the equation (4.4.30) boils down to,

$$\left[\partial_v^2 f^{sc}_{lm} - \Omega^2 \kappa_{sc}^2 f^{sc}_{lm} \right] + x^2 \left[\partial_v^2 f^{(1)}_{lm} - \Omega^2 \kappa_{sc}^2 f^{(1)}_{lm} - \kappa_{sc}^2 b(l,m) f_{lm} \right] = 0. \\ \partial_v^2 \left[f^{sc}_{lm} + x^2 f^{(1)}_{lm} \right] - \Omega^2 \kappa_{sc}^2 \left[f^{sc}_{lm} + x^2 f^{(1)}_{lm} \right] - x^2 \kappa_{sc}^2 b(l,m) f_{lm} = 0. \\ \partial_v^2 f_{lm} - \kappa_{sc}^2 (\Omega^2 + b x^2) f_{lm} = 0. \tag{4.4.32}$$

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This solution contains already the zeroth order solution corresponding to Schwarzschild background Eq.(4.4.26). Now the solution of f_{lm} will be,

$$f_{lm} = c^{(1)}_{kerr}\exp\Big\{[(\sqrt{\Omega^2+bx^2})\kappa_{sc}v]\Big\} + c^{(2)}_{kerr}\exp\Big\{[-(\sqrt{\Omega^2+bx^2})\kappa_{sc}v]\Big\}. \tag{4.4.33}$$

Here $c^{(1)}_{kerr}$ and $c^{(2)}_{kerr}$ are the two undetermined constants of integration having dimension of length. In the above solution second term is diverging near the horizon boundary at $v \to -\infty$, and hence we set $c^{(2)}_{kerr} = 0$. Therefore the required solution turns into,

$$f_{lm} = c^{(1)}_{kerr}\exp\Big\{[(\sqrt{\Omega^2+bx^2})\kappa_{sc}v]\Big\} \approx c^{(1)}_{kerr}\exp\Big\{[\Omega(1+\frac{b(l,m)x^2}{2\Omega^2})\kappa_{sc}v]\Big\}. \tag{4.4.34}$$

The leading term in expansion of the above i.e. $\mathcal{O}(x^0)$ yields f^{sc}_{lm}. Finally, the complete solution of F will be,

$$F(v,\theta,\phi) = \sum_{l,m} c^{(1)}_{kerr}\exp\left[(\Omega+\frac{b(l,m)x^2}{2\Omega})\kappa_{sc}v\right]Y_{lm}(\theta,\phi). \tag{4.4.35}$$

One may check that the above one automatically makes $\delta\beta_A$ in (4.3.19), vanishing at the horizon for the choice (4.3.20). Hence the equation (4.4.32) shows that the dynamics of each Goldstone mode f_{lm} is governed by inverted harmonics potential. Following the argument described in the references [203, 204] and also in chapter 3, near horizon modes living in the inverted harmonic potential can be related to the thermal nature of the horizon through its chaotic dynamics. Therefore, one can identify the temperature perceived by every individual mode near the horizon of the black hole as,

$$T(lm) = \frac{\hbar}{2\pi}\Omega(l)\kappa_{sc}\Big(1+\frac{b(l,m)x^2}{2\Omega^2(l)}\Big). \tag{4.4.36}$$

Detail discussions regarding the connection between the inverted harmonic potential and thermality of the horizon can be found in the section (3.3.2) of the last chapter. The average temperature perceived by individual l mode can be calculated from the expression (4.4.36) as follows,

$$T_{avg} = \frac{\hbar}{2\pi}\kappa_{sc}\Big(\frac{\sum_l\Omega(l)}{\sum_l 1} + \frac{x^2}{2\sum_l(2l+1)}\times\sum_l\sum_{m=-l}^{l}\frac{b(l,m)}{\Omega(l)}\Big) = \frac{\hbar}{2\pi}\kappa_{sc}(\sqrt{\frac{3}{4}} - \frac{7}{25}x^2) = \frac{\hbar}{2\pi}\kappa_{sc}\sqrt{\frac{3}{4}}(1-0.32x^2). \tag{4.4.37}$$

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To this end, it would be illuminating to compare the above expression for the mode temperature due to underlying Goldstone mode degrees of freedom, with that of Hawking temperature of Kerr black hole [29], $T_{BH} = \hbar\kappa/2\pi$. For slowly rotating ($x = a/M \ll 1$) Kerr black hole, using $r_H/M \approx (2 - \frac{x^2}{2})$, the surface gravity κ is expanded as,

$$\begin{aligned} \kappa &= \frac{\sqrt{M^2 - a^2}}{4Mr_H} \approx \frac{1 - \frac{x^2}{2}}{2M(1 - \frac{x^2}{4})} \\ &\approx \kappa_{sc}\Big(1 - \frac{1}{4}x^2 + \mathcal{O}(x^4)\Big). \end{aligned} \tag{4.4.38}$$

In this approximation, the expression of the Hawking temperature for Kerr black hole can be expressed as the correction to the Hawking temperature of Schwarzschild black hole horizon up to $\mathcal{O}(x^2)$ as follows,

$$T_{BH} = \frac{\hbar}{2\pi}\kappa_{sc}\Big(1 - \frac{1}{4}x^2\Big). \tag{4.4.39}$$

Therefore comparing (4.4.37) with (4.4.39), we can conclude that the obtained expression of the temperature of unstable Goldstone modes resembles the usual Hawking temperature for slowly rotating black hole with different numerical constant, which is matching with our previous analysis. Our present analysis again suggests the fact that the thermal nature of the horizon of a black hole spacetime is intimately tied with the symmetry breaking and is associated with the emergence of supertranslation Goldstone modes in the gravitational sector.

4.4.2 ZAMO observer trajectory

In this section, we study the dynamics of the Goldstone mode F with respect to the trajectory of zero angular momentum observers. In stationary rotating spacetime within the ergosphere region, observers will not be static due to the frame-dragging effect, rather will be co-rotating with the black hole. In this section, we will consider the family of those co-rotating observers with zero angular momentum (ZAMO) in their proper frame. The trajectory of a ZAMO observer is defined by $r =$ constant and $\theta =$ constant $= \theta_c$ (say) (more detail can be found in [115]). In this scenario, the aforementioned Goldstone mode F is a function of v and ϕ only.

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For those observers, the Lagrangian (4.4.3) reduces to,

$$\begin{aligned}\mathcal{L}_F\big|_{\theta=\theta_c} &= c_1(\theta_c)(\partial_v F(v,\phi))^2 + c_3(\theta_c)(\partial_\phi F(v,\phi))^2 \\ &+ c_5(\theta_c)(\partial_v\partial_\phi F(v,\phi))^2 + c_7(\theta_c)\partial_\phi F(v,\phi)\partial_v^2 F(v,\phi) + c_9(\theta_c)\partial_\phi^2 F(v,\phi)\,\partial_v^2 F(v,\phi).\end{aligned} \tag{4.4.40}$$

Hence with the expression of c_is from (4.4.4), the equation of motion will be,

$$\chi_H^2\Big(B(\theta_c)\partial_v^2 F(v,\phi) - 16\Sigma_H^3\partial_v^2\partial_\phi^2 F(v,\phi)\Big) + 12\alpha_2^2\Sigma_H^3\partial_\phi^2 F(v,\phi) = 0\,. \tag{4.4.41}$$

The expression of B is given in (4.4.5). Using the azimuthal symmetry the solution can be chosen as $F(v,\phi) = \sum_m f_m(v)\exp(im\phi)$, such that the equation of motion for F given in (4.4.41) corresponds to a particular ZAMO observer situated at fixed angle θ_c. Different value of θ_c yields different ZAMO observer. It is then reasonable to talk about an average equation over the all observers. It has been shown in Appendix 4.K that after averaging over the different directions of θ, the equation of motion for f_m yields,

$$\frac{\partial^2 f_m}{\partial v^2} - 3N(m)\kappa^2 f_m = 0. \tag{4.4.42}$$

where κ is the surface gravity of Kerr black hole. The expression of $N(m)$ is given by,

$$N = m^2\frac{(P_1\ \text{Elliptic}E[-a^2/r_H^2] + P_2\ \text{Elliptic}K[-a^2/r_H^2])}{(P_3\ \text{Elliptic}E[-a^2/r_H^2] + P_4\ \text{Elliptic}K[-a^2/r_H^2])} \tag{4.4.43}$$

where,

$$\begin{aligned}P_1 &= 2\Big(-96a^8m^2 - 2r_H^6\alpha_2(3M-17r_H) + 2a^2r_H^4(M^2 - 135Mr_H \\ &+2r_H^2(67-88m^2)) - 2a^4r_H^2(64M^2 + 5Mr_H + 3r_H^2(-23+88m^2)) \\ &-a^6(69M^2 - 110Mr_H + r_H^2(41+368m^2))\Big);\end{aligned}$$

$$\begin{aligned}P_2 &= \chi_H\Big(-96a^6m^2 + 4r_H^4(3M^2 - 20Mr_H + 17r_H^2) - 2a^2r_H^2(M^2 + 40Mr_H \\ &+r_H^2(-41+142m^2)) - a^4(69M^2 - 110Mr_H + r_H^2(41+284m^2))\Big);\end{aligned}$$

$$\begin{aligned}P_3 &= 8a^6(6a^2 + 23r_H^2) + 11a^2r_H^4(3a^2 + 2r_H^2); \\ P_4 &= a^6(24a^2 + 95r_H^2) + a^2r_H^4(142a^2 + 71r_H^2)\,.\end{aligned} \tag{4.4.44}$$

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In the expression of $N(m)$ (4.4.43), Elliptic$K[x]$ and Elliptic$E[x]$ are complete Elliptic integrals of first and second kind respectively. The corresonding expressions are given by,

$$\text{Elliptic}E[x] = \int_0^{\pi/2} (1 - x\sin^2\theta)^{(-1/2)} d\theta;$$
$$\text{Elliptic}K[x] = \int_0^{\pi/2} (1 - x\sin^2\theta)^{(1/2)} d\theta. \tag{4.4.45}$$

So the equation (4.4.42) shows that the dynamics of f_m is governed by inverse harmonic potential with solution,

$$f_m(v) = \bar{A}_1 \exp\left\{[\sqrt{3N(m)}\kappa v]\right\}$$
$$+\bar{A}_2 \exp\left\{[-\sqrt{3N(m)}\kappa v]\right\}; \tag{4.4.46}$$

Here $\bar{A}_1$ and $\bar{A}_2$ are two undetermined constant of integration having dimension of length. Since we are interested in the near-horizon region where $v \to -\infty$, we have to discard the second part of the solution (4.4.46), which grows rapidly and makes the mode unstable. Therefore suitable boundary condition can be set as $\bar{A}_2 = 0$. So the complete solution of F will be,

$$F(v,\phi) = \sum_m \bar{A}_1 \exp\left\{[\sqrt{3N(m)}\kappa v + im\phi]\right\}. \tag{4.4.47}$$

Following the discussion in section 3.3.2 about the connection between thermality and dynamics of chaotic system in the semi classical regime, the temperature in the present case is given by,

$$T_{ZAMO}(m) = \frac{\hbar}{2\pi}\sqrt{3N(m)}\kappa\,. \tag{4.4.48}$$

Therefore, in the present discussion, the thermal nature of the black hole horizon is captured through the quantum dynamics of the possible candidates of the BH microstates which are conjectured to be the Goldstone mode F associated with the breaking of supertranslation symmetry near the horizon.

4.5 Conclusion

Due to the spontaneous breakdown of global symmetry, the emergence of Goldstone modes and their dynamics play a fundamental role in many branches of

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physics. This same phenomenon has recently gained widespread interest in gravitational physics, stimulated by the discovery of an important connection between infinite-dimensional BMS symmetry at null infinity and the well-known soft graviton theorem [85–87, 144, 145, 227]. Soft graviton modes have been conjectured to play as Goldstone modes in the black hole background. In chapter 3 we used this very idea of spontaneous symmetry breaking near the Schwarzschild black hole horizon instead of asymptotic null infinity and investigated the dynamics of those modes. There we considered spherically symmetric black hole spacetime. It turned out that in the free field limit, Goldstone mode dynamics are governed by the inverted harmonic potential. At the quantum level, this instability [204] is interpreted as the deep-rooted cause of the thermodynamic nature of the underlying black holes. Through the analysis, we could define an average effective thermodynamic temperature T_{avg}, which turned out to be $T_{avg} = \sqrt{3/4}\, T^{sc}_{BH}$, with T^{sc}_{BH} as the Bekenstein-Hawking temperature of the Schwarzschild black hole.

In our present chapter, we extended the aforementioned analysis for rotating black hole. In the slow rotation limit, following the same methodology, we have arrived at perturbatively corrected temperature as,

$$T_{avg} = \sqrt{\frac{3}{4}} \left(1 - 0.32 \left(\frac{a}{M}\right)^2\right) T^{sc}_{BH}. \tag{4.5.1}$$

In the second part of this chapter, we have analyzed a special class of ZAMO observers, and the result leads to thermalization of the BMS modes whose temperature is again proportional to Hawking expression,

$$T_{ZAMO} = \frac{\hbar}{2\pi} \sqrt{3N(m)}\kappa \; = \sqrt{3N(m)} T^{kerr}_{BH}. \tag{4.5.2}$$

These are the two main results (Eqs. (4.5.1), (4.5.2)) of our present analysis. Although our obtained results of the horizon temperature are not exactly matching with Hawking's expression, it is an interesting hint to investigate further the symmetry breaking phenomena in the gravitational sector to understand the deeper underlying reasons for the thermodynamic nature of black holes. Our analysis seems to suggest the direction where all the (lm) modes of F could be the underlying microstates responsible for the horizon entropy, which requires in-depth investigation.

Another important comment can be mentioned here. The symmetry analysis has been performed for the near horizon Kerr metric constructed in GNC. It is evident that only those observers sitting in this coordinate system will identify

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these symmetries and correspondingly, this vector near the horizon. Thus the present symmetry analysis is observer-dependent. In this sense, among all possible diffeomorphism symmetries, only a subset has been chosen by our GNC observer, which incorporates a thermalization of the horizon at the semi-classical level and thereby provides the observer dependence of the thermal nature of black holes.

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Appendix

4.A Kerr metric in Gaussian null coordinate

4.A.1 The norm of the normal vectors

- Proof of $n^a n_a = 0$:

$$\begin{aligned} n^a n_a\Big|_H = g_{ab} n^a n^b\Big|_H &= \Big(g_{vv} n^v n^v + 2 g_{vr} n^v n^r + 2 g_{v\phi} n^v n^\phi + 2 g_{r\phi} n^r n^\phi \\ &+ g_{\phi\phi} n^\phi n^\phi \Big)\Big|_H \\ &= 2 g_{vr}|_H \, n^v n^r + 2 g_{r\phi}|_H \, n^r n^\phi + g_{\phi\phi}|_H \, n^\phi n^\phi \\ &= \frac{a^2 \sin^2\theta}{\Sigma_H} - 2\frac{a^2 \sin^2\theta}{\Sigma_H} + \frac{a^2 \sin^2\theta}{\Sigma_H} = 0 \end{aligned} \tag{4.A.1}$$

- Proof of $l^a l_a = 0$:

$$l^a l_a|_H = g_{ab} l^a l^b|_H = g_{vv} l^v l^v|_H = 0. \tag{4.A.2}$$

- Proof of $l^a n_a = 1$:

$$\begin{aligned} l^a n_a|_H = g_{ab} n^a n^b|_H &= (g_{vv} l^v n^v + g_{vr} l^v n^r + g_{v\phi} l^v n^\phi)|_H \\ &= (\frac{\Sigma_H}{\chi_H})(\frac{\chi_H}{\Sigma_H}) = 1. \end{aligned} \tag{4.A.3}$$

Here it is clear from (4.3.3) that $g_{vv}|_H = g_{v\phi}|_H = 0$.

4.A.2 Construction of the metric

By having tensor transformation rule of the components, we can obtain the components of the metric in (v, ρ, θ, ϕ) using the transformation of coordinates (4.3.8), as follows,

$$g_{vv} = \frac{\partial v' \partial v'}{\partial v \partial v} g_{v'v'} = -(\frac{\Sigma_H^2 \Delta - a^2 \sin^2\theta (r - r_H)^2}{\Sigma \chi_H^2})\Big|_{r \Rightarrow r_H + \rho(\frac{\chi_H}{\Sigma_H})}$$

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Now by Taylor expansion around $(r = r_H)$, we have,

$$\begin{aligned}
g_{vv} &= -\Big(\frac{\Sigma_H^2\Delta - a^2\sin^2\theta(r-r_H)^2}{\Sigma\chi_H^2}\Big)\Big|_{r=r_H} \\
&\quad -(r-r_H)\,\partial_r\Big(\frac{\Sigma_H^2\Delta - a^2\sin^2\theta(r-r_H)^2}{\Sigma\chi_H^2}\Big)\Big|_{r=r_H} \\
&= -\rho\Big(\frac{2\chi_H\Sigma(\Sigma_H^2(r-M) - a^2(r-r_H)\sin^2\theta)}{\Sigma^2\chi_H^2\Sigma_H} \\
&\quad -\frac{2r(\Sigma_H^2\Delta - a^2\sin^2\theta(r-r_H)^2)}{\Sigma^2\chi_H^2\Sigma_H}\Big)\Big|_{r=r_H} \\
&= -\rho\,\frac{2(r_H - M)}{\chi_H} = \rho\frac{\Delta'(r_H)}{\chi_H} = -\rho\,\kappa. \qquad (4.A.4)
\end{aligned}$$

Next we compute $g_{v\rho}$ compnent as,

$$\begin{aligned}
g_{v\rho} &= \frac{\partial v'}{\partial v}\frac{\partial r}{\partial\rho}g_{v'r} + \frac{\partial v'}{\partial v}\frac{\partial v'}{\partial\rho}g_{v'v'} + \frac{\partial v'}{\partial v}\frac{\partial\phi'}{\partial\rho}g_{v'\phi'} \\
&= (\frac{\chi_H}{\Sigma_H})\frac{\Sigma_H}{\chi_H} - \Big((\frac{a^2\sin^2\theta}{2\Sigma_H})(\frac{\Sigma_H^2\Delta - a^2\sin^2\theta(r-r_H)^2}{\Sigma\chi_H^2}) \\
&\quad -\frac{a\sin^2\theta[\chi^2 - a^2\sin^2\theta\Delta]}{\Sigma\chi_H}\Big)\Big|_{r\Rightarrow r_H+\rho(\frac{\chi_H}{\Sigma_H})} \approx 1 + \mathcal{O}(\rho). \qquad (4.A.5)
\end{aligned}$$

Now we calculate component $g_{\rho\rho}$:

$$\begin{aligned}
g_{\rho\rho} &= \frac{\partial v'}{\partial\rho}\frac{\partial v'}{\partial\rho}g_{v'v'} + 2\frac{\partial v'}{\partial\rho}\frac{\partial\phi'}{\partial\rho}g_{v'\phi'} + \frac{\partial\phi'}{\partial\rho}\frac{\partial\phi'}{\partial\rho}g_{\phi'\phi'} + 2\frac{\partial r}{\partial\rho}\frac{\partial\phi'}{\partial\rho}g_{r\phi'} + 2\frac{\partial v'}{\partial\rho}\frac{\partial r}{\partial\rho}g_{v'r} \\
&= \Big((\frac{a^2\sin^2\theta}{2\Sigma_H})^2 g_{v'v'} + 2(\frac{a^3\sin^2\theta}{2\Sigma_H\chi_H})g_{v'\phi} + \frac{a^2}{\chi_H^2}g_{\phi\phi} + 2(\frac{a}{\Sigma_H})g_{r\phi} \\
&\quad +2(\frac{a^2\chi_H\sin^2\theta}{2\Sigma_H^2})\Big)\Big|_{r\Rightarrow r_H+\rho(\frac{\chi_H}{\Sigma_H})}. \qquad (4.A.6)
\end{aligned}$$

After Taylor expansion about $(r = r_H)$, at the leading order we have,

$$\begin{aligned}
g_{\rho\rho} &= (\frac{a^2\sin^2\theta}{\Sigma_H}) - 2a\sin^2\theta(\frac{a}{\Sigma_H}) + \frac{a^2\sin^2\theta}{\Sigma_H} + \mathcal{O}(\rho) \\
&= 0 + \mathcal{O}(\rho). \qquad (4.A.7)
\end{aligned}$$

Similarly one can show that the component $g_{\rho\phi}$ vanishes at the leading order.

Next we transform $g_{v\phi}$ in new coordinate as,

$$g_{v\phi} = \frac{\partial v'}{\partial v}\frac{\partial \phi'}{\partial \phi} g_{v'\phi'} = a\sin^2\theta\Big(\frac{\Delta\Sigma_H + \chi(r^2 - r_H^2)}{\Sigma(r_H^2 + a^2)}\Big)\Big|_{r\Rightarrow r_H + \rho(\frac{\chi_H}{\Sigma_H})}$$
$$= a\sin^2\theta\Big(\frac{\Delta\Sigma_H + \chi(r^2 - r_H^2)}{\Sigma(r_H^2 + a^2)}\Big)\Big|_{r=r_H}$$
$$+ (r - r_H)\,\partial_r\Big(a\sin^2\theta\frac{\Delta\Sigma_H + \chi(r^2 - r_H^2)}{\Sigma(r_H^2 + a^2)}\Big)\Big|_{r=r_H}$$
$$= 2a\rho\sin^2\theta\Big(\frac{\chi_H}{\Sigma_H}\Big)\frac{\Sigma_H(r_H - M) + r\chi_H}{\Sigma_H\chi_H} = \frac{a\sin^2\theta}{\Sigma_H}\Delta'(r_H) + \frac{2ar_H\chi_H\sin^2\theta}{\Sigma_H^2}.$$

Then the transformation of $g_{\theta\theta}$ is derived as,

$$g_{\theta\theta} = \frac{\partial \theta'}{\partial \theta}\frac{\partial \theta'}{\partial \theta} g_{\theta'\theta'} = \Sigma\Big|_{r\Rightarrow r_H + \rho(\frac{\chi_H}{\Sigma_H})}$$
$$= \Sigma_H + 2r_H\rho\Big(\frac{\chi_H}{\Sigma_H}\Big). \tag{4.A.8}$$

Next is the transformation of $g_{\phi\phi}$:

$$g_{\phi\phi} = \frac{\partial \phi'}{\partial \phi}\frac{\partial \phi'}{\partial \phi} g_{\phi'\phi'} = \frac{\sin^2\theta[\chi^2 - a^2\sin^2\theta\Delta]}{\Sigma}\Big|_{r\Rightarrow r_H + \rho(\frac{\chi_H}{\Sigma_H})}. \tag{4.A.9}$$

Now Taylor expansion about $r = r_H$ gives,

$$g_{\phi\phi} = \frac{\sin^2\theta\chi_H^2}{\Sigma_H} + \rho\ \sin^2\theta\chi_H\Big(\frac{\Sigma_H(4r_H\chi_H - 2a^2\sin^2\theta(r_H - M)) - 2r_H\chi_H^2}{\Sigma_H^3}\Big)$$
$$= \frac{\sin^2\theta\chi_H^2}{\Sigma_H} + \rho\Big(-\frac{a^2\chi_H\sin^4\theta}{\Sigma_H^2}\Delta'(r_H) + \frac{2r_H\chi_H^2\sin^2\theta}{\Sigma_H^3}(2\Sigma_H - \chi_H)\Big)$$
$$= \frac{\sin^2\theta\chi_H^2}{\Sigma_H} + \rho\Big(-\frac{a^2\chi_H\sin^4\theta}{\Sigma_H^2}\Delta'(r_H) + \frac{2r_H\chi_H^2\sin^2\theta(\Sigma_H - a^2\sin^2\theta)}{\Sigma_H^3}\Big).$$

Finally with the metric components derived above, the kerr metric in Gaussian null coordinates (v, ρ, θ, ϕ), boils down to (4.3.8) with the following quantities as,

$$\kappa = \frac{\Delta'(r_H)}{2\chi_H};\quad \beta_\theta = \frac{2a^2\sin\theta\cos\theta}{\Sigma_H};\quad \beta_\phi = \frac{a\sin^2\theta}{\Sigma_H}\Delta'(r_H) + \frac{2ar_H\chi_H\sin^2\theta}{\Sigma_H^2}$$
$$\mu_{\theta\theta} = \Sigma_H;\quad \mu_{\phi\phi} = \frac{\chi_H^2\sin^2\theta}{\Sigma_H};\quad \lambda_{\theta\theta} = \frac{2r_H\chi_H}{\Sigma_H};\quad \lambda_{\theta\phi} = \frac{2a^3\chi_H\sin^3\theta\cos\theta}{\Sigma_H^2};$$
$$\lambda_{\phi\phi} = \Big(-\frac{a^2\chi_H\sin^4\theta}{\Sigma_H^2}\Delta'(r_H) + \frac{2r_H\chi_H^2\sin^2\theta(\Sigma_H - a^2\sin^2\theta)}{\Sigma_H^3}\Big). \tag{4.A.10}$$

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4.B Derivation of the modified metric (4.3.18)

Now putting components of ζ^a from (4.3.12), one can find the variation of the metric component $g^{(0)}_{vv}$ upto $\mathcal{O}(r)$ as,

$$\begin{aligned}
&\pounds_\zeta g^{(0)}_{vv} = \zeta^\rho \partial_\rho g^{(0)}_{vv} + 2g^{(0)}_{vv}\partial_v \zeta^v + 2g^{(0)}_{v\rho}\partial_v\zeta^\rho + 2g^{(0)}_{vA}\partial_v\zeta^A \\
&= [T(v,x^A) - \rho\partial_v F - \partial_B F \int \rho\beta^B d\rho]\,\partial_\rho(-2\rho\kappa) - 2\rho\kappa\partial_v F + 2\partial_v[T(v,\theta,\phi) \\
&-\rho\partial_v F - \partial_B F\int \rho\beta^B d\rho] + 2\rho\beta_A\partial_v[-\partial_B F\int \mu^{AB} d\rho + R^A(v,x^A)] \\
&= [2\partial_v T - 2\kappa T] + \rho[-2\kappa\partial_v F - 2\partial_v^2 F + 2\rho\beta_A\partial_v R^A].
\end{aligned} \tag{4.B.1}$$

Then upto $\mathcal{O}(r)$, variation of $g^{(0)}_{vA}$ is given by,

$$\begin{aligned}
&\pounds_\zeta g^{(0)}_{vA} = \zeta^c\partial_c g^{(0)}_{vA} + g^{(0)}_{vv}\partial_A\zeta^v + g^{(0)}_{v\rho}\partial_A\zeta^\rho + g^{(0)}_{vA}\partial_v\zeta^v + g^{(0)}_{Bv}\partial_A\zeta^B + g^{(0)}_{AB}\partial_v\zeta^B \\
&= [T - \rho\partial_v F - \partial_B F\int \rho\beta^B d\rho]\,\partial_\rho(\rho\beta_A) + [-\partial_D F\int \mu^{BD} d\rho + R^B]\partial_B(\rho\beta_A) \\
&-2\rho\kappa\partial_A F + \partial_A[T - \rho\partial_v F - \partial_B F\int \rho\beta^B d\rho] + \rho\beta_A\partial_v F \\
&+\rho\beta_B\partial_A[-\partial_C F\int \mu^{BC} d\rho + R^B] + (\mu_{AB} + \rho\lambda_{AB})\partial_v[-\partial_C F\int \mu^{BC} d\rho + R^B] \\
&= [\partial_A T + T\beta_A + \mu_{AB}\partial_v R^B] + \rho\,[R^B\partial_B\beta_A - 2\rho\kappa\partial_A F - \partial_A\partial_v F + \beta_B\partial_A R^B \\
&-\mu_{AB}\partial_v\partial_D F\mu^{BD} + \lambda_{AB}\partial_v R^B].
\end{aligned} \tag{4.B.2}$$

Similarly with (4.3.12), the variation of $g^{(0)}_{AB}$ upto $\mathcal{O}(r)$ will be,

$$\begin{aligned}
&\pounds_\zeta g^{(0)}_{AB} = \zeta^\rho\partial_\rho g^{(0)}_{AB} + \zeta^C\partial_C g^{(0)}_{AB} + g^{(0)}_{vA}\partial_B\zeta^v + g^{(0)}_{vB}\partial_A\zeta^v + g^{(0)}_{DA}\partial_B\zeta^D + g^{(0)}_{DB}\partial_A\zeta^D \\
&= [T - \rho\partial_v F - \partial_C F\int \rho\beta^C d\rho]\,\partial_\rho(\rho\lambda_{AB}) + [-\partial_D F\int \mu^{CD} d\rho + R^C]\partial_C(\mu_{AB} \\
&+\rho\lambda_{AB}) + \rho\beta_A\partial_B F + \rho\beta_B\partial_A F + (\mu_{DA} + \rho\lambda_{DA})\partial_B[-\partial_C F\int \mu^{DC} d\rho + R^D] \\
&+(\mu_{DB} + \rho\lambda_{DB})\partial_A[-\partial_C F\int \mu^{DC} d\rho + R^D] \\
&= [\lambda_{AB}\,T + R^C\partial_C\mu_{AB} + \mu_{DA}\partial_B R^D + \mu_{DB}\partial_A R^D] + \rho[-\lambda_{AB}\partial_v F \\
&-\partial_D F\mu^{DC}\partial_C\mu_{AB} + \rho\beta_A\partial_B F + \rho\beta_B\partial_A F - \mu_{AD}\partial_B(\partial_C\mu^{CD}) - \mu_{BD}\partial_A(\partial_C\mu^{CD})]
\end{aligned} \tag{4.B.3}$$

Now given the fall-off conditions in (4.3.10), the r independent term of the right hand side of (4.B.1) , (4.B.2) and also (4.B.3) must vanish which give rise the constraint relations given in the main text as (4.3.13) ,(4.3.14) and (4.3.15).

Now following all the $\mathcal{O}(\rho)$ terms found in the varitation of metric components as given in (4.B.1), (4.B.2) and in (4.B.3), also with the condition that R^A is function of angular coordinates only, we can easily get the modified metric in (4.3.18).

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4.C Lagrangian (4.4.3)

4.C.1 Construction of the Lagrangian

Using mathematica packages, for the modified metric (4.3.18), at first we get the determinant of the metric as $\sqrt{-g}$. Then we have calculated the components of the Christoffel symbols Γ^a_{bc} and then Reimannian tensor $R^a{}_{bcd}$ as given by (3.A.9) and (3.A.10) respectively. After that we got the components of Ricci tensor R_{ij} using the definition (3.A.11) and also Ricci scalar by (3.A.12). By computing the product of $\sqrt{-g}$ and R, then taking near horizon limit and For the choice (4.3.20), we finally get the form of the Einstein-Hilbert Lagrangian (4.4.1) as follows,

$$
\begin{aligned}
\mathcal{L}_{(F,R^\theta,R^\phi)} = {} & d_1(\theta) + \Big[d_2(\theta)R^\theta + \partial_\phi(d_3(\theta)R^\theta + d_4(\theta)R^\phi) + \partial^2_\phi(d_5(\theta)R^\theta \\
& + d_6(\theta)R^\phi) + d_7(\theta)\partial_\theta R^\theta + d_8(\theta)\partial_\theta R^\phi + d_9(\theta)\partial^2_\theta R^\theta + d_{10}(\theta)\partial^2_\theta R^\phi \\
& + \partial_\phi(d_{11}(\theta)F + d_{12}(\theta)\partial_v F) + d_{13}(\theta)\partial_\theta F + d_{14}(\theta)\partial_v\partial_\theta F\Big] + \\
& + \Big[c_1(\theta)(\partial_v F)^2 + c_2(\theta)(\partial_\theta F)^2 + c_3(\theta)(\partial_\phi F)^2 + c_4(\theta)(\partial_v\partial_\theta F)^2 + c_5(\theta)(\partial_v\partial_\phi F)^2 \\
& + c_6(\theta)\partial_\theta F\partial^2_v F + c_7(\theta)\partial_\phi F\partial^2_v F + c_8(\theta)\partial^2_\theta F\partial^2_v F + c_9(\theta)\partial^2_\phi F\partial^2_v F + c_{10}(\theta)\partial_v F\partial^2_v F \\
& + c_{11}(\theta)\partial_\theta F\partial_v\partial_\theta F + c_{12}(\theta)\partial_\phi F\partial_v\partial_\phi F + c_{13}(\theta)\partial^2_\theta F\partial_v F + c_{14}(\theta)\partial_\theta F\partial_v F \\
& + c_{15}(\theta)\partial^2_\phi F\partial_v F + c_{16}(\theta)\partial_\phi F\partial_v F + c_{17}(\theta)(R^\theta)^2 + R^\theta\Big(c_{18}(\theta)\partial_\phi R^\theta + c_{19}(\theta)\partial_\phi R^\phi \\
& + c_{20}(\theta)\partial_\theta R^\theta + c_{21}(\theta)\partial_\phi R^\phi + c_{22}(\theta)\partial_\phi F + c_{23}(\theta)\partial_\theta F + c_{24}(\theta)\partial_v F + c_{25}(\theta)\partial^2_v F \\
& + c_{26}(\theta)\partial_v\partial_\phi F + c_{27}(\theta)\partial_v\partial_\theta F\Big) + c_{28}(\theta)(\partial_\phi R^\theta)^2 + c_{29}(\theta)\partial_\phi R^\theta\partial_\phi R^\phi \\
& + c_{30}(\theta)(\partial_\phi R^\phi)^2 + c_{31}(\theta)(\partial_\theta R^\theta)^2 + c_{32}(\theta)\partial_\theta R^\theta\partial_\theta R^\phi + c_{33}(\theta)(\partial_\theta R^\phi)^2 \\
& + c_{34}(\theta)\partial_\phi R^\theta\partial\phi F + \partial_\phi F(c_{35}(\theta)\partial_\phi R^\phi + c_{36}(\theta)\partial_\phi R^\theta) + \partial_\theta F(c_{37}(\theta)\partial_\theta R^\theta \\
& + c_{38}(\theta)\partial_\theta R^\phi) + \partial_v\partial_\phi F(c_{39}(\theta)\partial_\phi R^\theta + c_{40}(\theta)\partial_\phi R^\phi) + \partial_v\partial_\theta F(c_{41}(\theta)\partial_\theta R^\theta \\
& + c_{42}(\theta)\partial_\theta R^\phi) + \partial^2_v F(c_{43}(\theta)\partial_\phi R^\theta + c_{44}(\theta)\partial_\phi R^\phi + c_{45}(\theta)\partial_\theta R^\theta + c_{46}(\theta)\partial_\theta R^\phi)]
\end{aligned}
\tag{4.C.1}
$$

In the above form, the coefficients $d_1....d_{14}$ and $c_1...c_{46}$ are functions of θ and have been generated from the original kerr metric coefficients $g^{(o)}_{ab}$. However for the choice (4.3.20), all the terms containing R^θ and R^ϕ and its derivatives in (4.C.1) will vanish. Hence the explicit expressions of the coefficients of those terms having R^θ and R^ϕ (viz the expressions of $d_2 \dots d_{10}$ and also $c_{17} \dots c_{46}$) are not required in the present context. Finally after removing those terms, the Lagrangian boils

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down to,

$$\begin{aligned}\mathcal{L}_{(F,R^\theta=0,R^\phi=C)} &= d_1(\theta)+\partial_\phi(d_{11}(\theta)F+d_{12}(\theta)\partial_v F)+d_{13}(\theta)\partial_\theta F+d_{14}(\theta)\partial_v\partial_\theta F\\ &+c_1(\theta)(\partial_v F)^2+c_2(\theta)(\partial_\theta F)^2+c_3(\theta)(\partial_\phi F)^2\\ &+c_4(\theta)(\partial_v\partial_\theta F)^2+c_5(\theta)(\partial_v\partial_\phi F)^2+c_6(\theta)\partial_\theta F\partial_v^2 F+c_7(\theta)\partial_\phi F\partial_v^2 F+c_8(\theta)\partial_\theta^2 F\partial_v^2 F\\ &+c_9(\theta)\partial_\phi^2 F\partial_v^2 F+c_{10}(\theta)\partial_v F\partial_v^2 F+c_{11}(\theta)\partial_\theta F\partial_v\partial_\theta F+c_{12}(\theta)\partial_\phi F\partial_v\partial_\phi F\\ &+c_{13}(\theta)\partial_\theta^2 F\partial_v F+c_{14}(\theta)\partial_\theta F\partial_v F+c_{15}(\theta)\partial_\phi^2 F\partial_v F+c_{16}(\theta)\partial_\phi F\partial_v F. \end{aligned} \tag{4.C.2}$$

$\mathcal{L}$ is calculated on the stretched horizon at ρ = constant surface, then the near horizon limit $\rho \to 0$ has be taken. Therefore action has been defined as the integration of the Lagrangian over the coordinates v, θ and ϕ. The reduced form of the Lagrangian (4.C.2) correspond to the action for the super-translation mode F.

4.C.2 Form of the final Lagrangian (4.4.3) from (4.C.2)

To construct the final form as given in (4.4.3), let us assume that for the general rotation, the solution ansatz of F can be expressed in terms of spheroidal harmonics [219] as follows,

$$F(v,\theta,\phi) = \sum_{l,m} f_{lm}(v) Z_{lm}(\theta,\phi). \tag{4.C.3}$$

Now by substituting (4.C.3) in the Lagrangian density (3.3.11), we can obtain the form of the three dimensional action on the r = constant surface as,

$$S = \int d^3x \mathcal{L}_{(F,R^\theta=0,R^\phi=C)}.$$

However here we will compute each term of this action separately. So at first let us concentrate on the two first order terms $d_{12}(\theta)\partial_v\partial_\phi F$ and $d_{14}(\theta)\partial_v\partial_\theta F$ of the Lagrangian density (4.C.2) which can be integrated as follows,

$$\begin{aligned}\int d_{12}(\theta)\partial_v\partial_\phi F dv d\theta d\phi &= \sum_{lm}\int d_{12}(\theta)\partial_\phi Z_{lm} d\theta d\phi\Big(\int dv\partial_v f_{lm}\Big);\\ \int d_{14}(\theta)\partial_v\partial_\theta F dv d\theta d\phi &= \sum_{lm}\int d_{14}(\theta)\partial_\theta Z_{lm} d\theta d\phi\Big(\int dv\partial_v f_{lm}\Big); \end{aligned} \tag{4.C.4}$$

In the right hand side, the integration over the function Z_{lm} will yield finite values. So the rest part of these terms are the total derivative of v and thus these two terms can be neglected. Now d_1 term will not contribute in the equation of motion for F. The other two terms $d_{11}(\theta)\partial_\phi F$ and $d_{13}(\theta)\partial_\theta F$ are the first order in F. In the formulation of the equation of motion of F, these will generate a θ-dependent

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function which will have no significant contribution in the dynamics . Hence in (4.C.2) we neglect all the first order terms and consider only those terms which are quadratic in F .

Next we concentrate on the last seven terms in (4.C.2) which contain first order derivative of F with respect to v. Putting the solution ansatz of F (4.C.3), after integrating over coordinates v, θ and ϕ, the term $c_{10}(\theta)\partial_v F\partial_v^2 F$ yields,

$$
\begin{aligned}
&\int (c_{10}(\theta)\partial_v F\partial_v^2 F)dvd\theta d\phi \\
&= \sum_{lm}\sum_{l'm'}\int d\phi\, d\theta (c_{10}(\theta)\, Z_{lm}Z_{l'm'})\partial_v^2 f_{lm}\partial_v f_{l'm'}; \\
&= \frac{1}{2}\sum_{lm}\sum_{l'm'}\int d\phi\, d\theta (c_{10}(\theta)\, Z_{lm}Z_{l'm'})\int dv(\partial_v^2 f_{lm}\partial_v f_{l'm'} + \partial_v^2 f_{l'm'}\partial_v f_{lm}); \\
&= \frac{1}{2}\sum_{lm}\sum_{l'm'}\int d\phi\, d\theta (c_{10}(\theta)\, Z_{lm}Z_{l'm'})\int dv\partial_v(\partial_v f_{lm}\partial_v f_{l'm'});
\end{aligned}
\tag{4.C.5}
$$

here c_{10} is given by,

$$
c_{10} = 4\sin\theta\left(\frac{r_H\chi_H^2 - a^2\alpha_2^2\Sigma_H\sin^2\theta + r_H\chi_H(r_H^2 + a^2\cos 2\theta)}{\Sigma_H^2}\right). \tag{4.C.6}
$$

Here l, m, l' and m' all are dummy indices. Hence we can easily interchange between l and l' indices. A factor $(1/2)$ is multiplied to avoid double counting. Hence in the last line of (4.C.5), integration over transverse coordinates will generate finite result. But the v part of this term has been expressed as total derivative and so this term can be neglected. In the similar way $c_{12}(\theta)\partial_\phi F\partial_v\partial_\phi F$ from (4.C.2), can be computed as,

$$
\begin{aligned}
&\int (c_{12}(\theta)\partial_\phi F\, \partial_v\partial_\phi F)dvd\theta d\phi \\
&= \sum_{lm}\sum_{l'm'}\Big(\int d\theta d\phi (c_{12}(\theta)\, \partial_\phi Z_{lm}\partial_\phi Z_{l'm'}) f_{lm}\partial_v f_{l'm'} \\
&= \frac{1}{2}\sum_{lm}\sum_{l'm'}\Big(\int d\theta d\phi (c_{12}(\theta)\, \partial_\phi Z_{lm}\partial_\phi Z_{l'm'})\int dv(f_{lm}\partial_v f_{l'm'} + f_{l'm'}\partial_v f_{lm}); \\
&= \frac{1}{2}\sum_{lm}\sum_{l'm'}\Big(\int d\theta d\phi (c_{12}(\theta)\, \partial_\phi Z_{lm}\partial_\phi Z_{l'm'})\int dv\partial_v(f_{lm}f_{l'm'});
\end{aligned}
\tag{4.C.7}
$$

where integration over coordinate v gives the total derivative and so this term is neglected in the contruction of the dynamics. Here $c_{12} = -\frac{12\alpha_2\Sigma_H}{\chi_H\sin\theta}$. Similarly other terms i.e. $c_{15}(\theta)\partial_\phi^2 F\partial_v F$ and $c_{16}(\theta)\partial_\phi F\partial_v F$ can be expressed as total derivative of v and thus neglected.

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Next we concentrate on $c_{11}(\theta)\partial_\theta F\partial_v\partial_\theta F$ and write this term as follows,

$$\begin{aligned} c_{11}(\theta)\partial_\theta F\partial_v\partial_\theta F &= -\frac{12\alpha_2\sin\theta}{\Sigma_H}\partial_\theta F\partial_v\partial_\theta F \\ &= \Big(\frac{4\alpha_2\sin\theta}{\Sigma_H} - \frac{16\alpha_2\sin\theta}{\Sigma_H}\Big)\partial_\theta F\partial_v\partial_\theta F. \end{aligned} \tag{4.C.8}$$

Then we concentrate on the two terms $c_{13}(\theta)\partial_\theta^2 F\partial_v F$ and $c_{14}(\theta)\partial_\theta F\partial_v F$. Here the expression of c_{13} and c_{14} are,

$$c_{13}(\theta) = \frac{4\alpha_2\sin\theta}{\Sigma_H};\;\; c_{14} = \frac{4\alpha_2\cos\theta}{\Sigma_H}. \tag{4.C.9}$$

Now adding the terms $c_{13}(\theta)\partial_\theta^2 F\partial_v F$ and $c_{14}(\theta)\partial_\theta F\partial_v F$ with the first one in (4.C.8) we get,

$$\begin{aligned} & c_{13}(\theta)\partial_\theta^2 F\partial_v F + c_{14}(\theta)\partial_\theta F\partial_v F + (\frac{4\alpha_2\sin\theta}{\Sigma_H})\partial_\theta F\partial_v\partial_\theta F \\ &= (\frac{4\alpha_2\sin\theta}{\Sigma_H})(\partial_\theta^2 F\partial_v F + \partial_\theta F\partial_v\partial_\theta F) + \frac{4\alpha_2\cos\theta}{\Sigma_H}\partial_\theta F\partial_v F. \\ &= \frac{4\alpha}{\Sigma_H}\partial_\theta\Big(\sin\theta\partial_\theta F\partial_v F\Big). \end{aligned} \tag{4.C.10}$$

Like before, putting (4.C.3) in (4.C.10), we integrate over coordinates v, θ and ϕ and the result yields,

$$\frac{4\alpha_2}{\Sigma_H}\sum_{lm}\sum_{l'm'}\int d\phi\int_0^\pi d\theta\partial_\theta\Big(\sin\theta\;\partial_\theta Z_{lm}Z_{l'm'}\Big)\int dv f_{lm}\partial_v f_{l'm'} = 0.$$

Here close integration over θ from ranges 0 to π gives zero. Now the second term in (4.C.8) can be shown to be the total derivative with respect to v as we did earlier in (4.C.7). Thus in the expression (4.C.2) all the terms containing first derivative of F with respect to v can be expressed as total derivative.

In this way we have shown that the all the first order F terms and also the last seven second order F terms of the Lagrangian (4.C.2) can be neglected easily in order to study the dynamics of the modes. Finally removing those terms, the form of the Lagrangian density (4.C.2) boils down into (4.4.3).

4.D GHY boundary term

The GHY term is given by (4.4.8). Here the first term is a constant, while the terms in the first bracket are the total derivative in v. Other terms can also be

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transformed into total derivative as follows,

$$\partial_v F \partial_v^2 F = \frac{1}{2}\partial_v[(\partial_v F)^2]; \quad \partial_v^2 F \partial_v^3 F = \frac{1}{2}\partial_v[(\partial_v^2 F)^2]$$
$$(\partial_v^2 F)^2 + \partial_v F \partial_v^3 F = \partial_v[\partial_v F \partial_v^2 F]. \tag{4.D.1}$$

Hence all these total derivative terms should not contribute to the dynamics of F. However the boundary term added to the action could be related to the horizon entropy which is discussed in Appendix 4.E.

4.E Contribution of boundary term in the heat content of the horizon

In the section 4.4 the GHY boundary term has been evaluated for our metric and added to the EH action which does not have effect on the mode dynamics. However this boundary term gains importance in the description of surface hamiltonian, defined as $H_{sur} = -\frac{\partial S_2}{\partial v}$, which is directly related to the heat content of the horizon [217, 218]. Hence to show the aforementioned connection with the heat content in the present analysis, we have to calculate the form of the Hamiltonian for the mode solution of F found in (4.4.35) for slowly rotating spacetime. So putting this form of F in the GHY action S_2 (4.4.8), after integrating we get the resultant expression of the Hamiltonian as the following,

$$H_{sur} = \frac{\bar{A}_{kerr}}{8\pi G}\left[\kappa + f_3(\kappa, x^2) e^{f_2(\kappa,x^2)v}\right], \tag{4.E.1}$$

$\bar{A}_{kerr}$ denotes the transverse area of the horizon of kerr spacetime. In the evaluation of the Hamiltonian H_{sur}, we found out that all the first and higher derivative terms of F will generate some complicated expressions (functions of κ and x^2) which are exponentials in v. However the explicit forms of these terms are not needed in our present discussion as all these terms are exponentially suppressed towards the horizon in the limit $v \to -\infty$. Therefore to show the nature of those terms in the Hamiltonian, we have kept the form $f_3(\kappa, x^2)e^{f_2(\kappa,x^2)v}$ in (4.E.1), where the functional form of $f_3(\kappa, x^2)$ and $f_2(\kappa, x^2)$ are not explicitly given but they are finite near the horizon. Hence the second term in (4.E.1) vanishes near the boundary. The above Hamiltonian then is simplified into,

$$H_{sur} = \frac{1}{8\pi G}\bar{A}_{kerr}\kappa\,. \tag{4.E.2}$$

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It is known that the horizon entropy and the temperature of the horizon are given by $S = \bar{A}_{kerr}/4G$ and $T = \kappa/2\pi$ respectively. Hence with the help of these two expressions, we can write the surface Hamiltonian (4.E.2) as,

$$H_{sur} = TS. \tag{4.E.3}$$

This result clearly shows that the GHY boundary term in the action is directly connected with the heat content of the horizon.

4.F Derivation of Eq.(4.4.9)

Using the generalized Euler-Lagrangian equation of motion given in (3.3.17), we can compute the equation of motion of F from the Lagrangian (4.4.3) as follows,

$$\begin{aligned}
&-2c_1\partial_v^2 F - 2c_2\partial_\theta^2 F - 2\partial_\theta c_2\,\partial_\theta F - 2c_3\partial_\phi^2 F + 4\partial_\theta c_4\,\partial_v^2\partial_\theta F + 4c_4\partial_v^2\partial_\theta^2 F + 4c_5\partial_v^2\partial_\phi^2 F\\
&+c_6\partial_v^2\partial_\theta F - c_6\partial_\theta\partial_v^2 F - \partial_\theta c_6\,\partial_v^2 F + c_7\partial_v^2\partial_\phi F - c_7\partial_\phi\partial_v^2 F + \partial_\theta^2 c_8\,\partial_v^2 F + 2\partial_\theta c_8\,\partial_v^2\partial_\theta F\\
&+2c_8\partial_v^2\partial_\theta^2 F + 2c_9\partial_v^2\partial_\phi^2 F = 0;\\
&\Rightarrow (-2c_1 - \partial_\theta c_6 + \partial_\theta^2 c_8)\partial_v^2 F - 2c_2\partial_\theta^2 F - 2\partial_\theta c_2\,\partial_\theta F - c_3\partial_\phi^2 F + (4\partial_\theta c_4\\
&+2\partial_\theta c_8)\partial_v^2\partial_\theta F + (4c_4 + 2c_8)\partial_v^2\partial_\theta^2 F + (4c_5 + 2c_9)\partial_v^2\partial_\phi^2 F = 0.
\end{aligned} \tag{4.F.1}$$

Now putting the expression of $c_i's$ from (4.4.4) in the above equation, and rearranging the terms, we get,

$$\begin{aligned}
&\Big(\frac{12\alpha_2^2\sin\theta}{\Sigma_H\chi_H}\Big)\partial_\theta^2 F - \Big(\frac{16\chi_H\sin\theta}{\Sigma_H}\Big)\partial_v^2\partial_\theta^2 F + \Big(\frac{12\alpha_2^2\cos\theta(\chi_H + a^2\sin^2\theta)}{\Sigma_H^2\chi_H}\Big)\partial_\theta F\\
&-\Big(\frac{16\chi_H(\chi_H + a^2\sin^2\theta)\cos\theta}{\Sigma_H^2}\Big)\partial_v^2\partial_\theta F + \Big(\frac{12\alpha_2^2\Sigma_H}{\chi_H^3\sin\theta}\Big)\partial_\phi^2 F - \Big(\frac{16\Sigma_H}{\chi_H\sin\theta}\Big)\partial_v^2\partial_\phi^2 F\\
&+\Big(-\frac{B(\theta)}{\Sigma_H^2\chi_H\sin\theta} - \frac{4\chi_H\sin\theta(-r_H^2 + a^2\cos^2\theta)}{\Sigma_H^2}\\
&-\frac{\chi_H\sin\theta(21a^4 + 8a^2r_H^2 - 8r_H^4 + 4a^2(5a^2 + 6r_H^2)\cos 2\theta - a^4\cos 4\theta)}{2\Sigma_H^3}\Big)\partial_v^2 F = 0.
\end{aligned} \tag{4.F.2}$$

where the expression of B is given in (4.4.5).

Now the coefficients of the term $\partial_v^2 F$ in (4.F.2), is simplified as,

$$-\frac{\alpha_2 \sin^2\theta}{\Sigma_H^2 \chi_H \sin\theta}\Big(a^4(M+7r_H)+4a^2Mr_H^2+12a^2r_H^3+8r_H^5+4a^2\alpha_2 r_H^2\cos 2\theta$$
$$+a^4\alpha_2\cos 4\theta+8r_H\chi_H(r_H^2+a^2\cos 2\theta)\Big)-\frac{4\chi_H\sin\theta(-r_H^2+a^2\cos^2\theta)}{\Sigma_H^2}$$
$$-\frac{\chi_H\sin\theta(21a^4+8a^2r_H^2-8r_H^4+4a^2(5a^2+6r_H^2)\cos 2\theta-a^4\cos 4\theta)}{2\Sigma_H^3}$$
$$=\frac{\sin\theta}{2\Sigma_H^3\chi_H}\Big[2\Sigma_H\alpha_2\Big(-a^4(M+7r_H)-4a^2Mr_H^2-12a^2r_H^3-8r_H^5$$
$$-4a^2\alpha_2r_H^2\cos 2\theta-a^4\alpha_2\cos 4\theta-8r_H\chi_H(r_H^2+a^2\cos 2\theta)\Big)$$
$$-8\Sigma_H\chi_H^2\Big(-r_H^2+a^2\cos^2\theta\Big)-\chi_H^2(21a^4+8a^2r_H^2-8r_H^4$$
$$+4a^2(5a^2+6r_H^2)\cos 2\theta-a^4\cos 4\theta)\Big]=\frac{\alpha\sin\theta}{\Sigma_H^3\chi_H}. \tag{4.F.3}$$

Where the expression of α is given in (4.4.10). Then multiplying (4.F.2) by $\Sigma_H^3\chi_H^3$ and dividing by $(\sin\theta)$, then rearranging, we get the equation as given in (4.4.9).

4.G Details of Eq. (4.4.9)

After rearranging the terms in the equation (4.4.9) and simplifying, we can write,

$$\Big[-16\chi_H^4(\chi_H^2-2a^2\chi_H\sin^2\theta+a^4\sin^4\theta)\partial_\theta^2\partial_v^2F-16\cot\theta\chi_H^4(\chi_H^2$$
$$-a^4\sin^4\theta)\partial_\theta\partial_v^2F-\frac{16\chi_H^2}{\sin^2\theta}(\chi_H^4-4a^2\chi_H^3\sin^2\theta+6a^4\chi_H^2\sin^4\theta-4a^6\chi_H\sin^6\theta$$
$$+a^8\sin^8\theta)\partial_\phi^2\partial_v^2F+\alpha\chi_H^2\partial_v^2F\Big]$$
$$+12\alpha_2^2\Big[\chi_H^2(\chi_H^2-2a^2\chi_H\sin^2\theta+a^4\sin^4\theta)\partial_\theta^2F+\cot\theta\chi_H^2(\chi_H^2$$
$$-a^4\sin^4\theta)\partial_\theta F+\frac{1}{\sin^2\theta}(\chi_H^4-4a^2\chi_H^3\sin^2\theta+6a^4\chi_H^2\sin^4\theta$$
$$-4a^6\chi_H\sin^6\theta+a^8\sin^8\theta)\partial_\phi^2F\Big]=0 \tag{4.G.1}$$

In the above equation we have used the expansion of the quantities as,

$$\Sigma_H^2=(\chi_H-a^2\sin^2\theta)^2=\chi_H^2-2a^2\chi_H\sin^2\theta+a^4\sin^4\theta.$$
$$\Sigma_H^4=(\chi_H^2-a^2\sin^2\theta)^4=\chi_H^4-4a^2\chi_H^3\sin^2\theta+6a^4\chi_H^2\sin^4\theta$$
$$-4a^6\chi_H\sin^6\theta+a^8\sin^8\theta. \tag{4.G.2}$$

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Now the equation (4.G.1) can be more simplified and written as,

$$\Big[-16\chi_H^6\Big(\partial_\theta^2\partial_v^2F+\cot\theta\partial_\theta\partial_v^2F+\frac{1}{\sin^2\theta}\partial_\phi^2\partial_v^2F\Big)-16\chi_H^4(-2a^2\chi_H\sin^2\theta$$
$$+a^4\sin^4\theta)\partial_\theta^2\partial_v^2F+(16a^4\chi_H^4\sin^4\theta\cot\theta)\partial_\theta\partial_v^2F-\chi_H^2(-64a^2\chi_H^3$$
$$+96a^4\chi_H^2\sin^2\theta-64a^6\chi_H\sin^4\theta+16a^8\sin^6\theta+\alpha)\partial_v^2\partial_\phi^2F\Big]$$
$$+12\alpha_2^2\Big[\chi_H^4\Big(\partial_\theta^2F+\cot\theta\partial_\theta F+\frac{1}{\sin^2\theta}\partial_\phi^2F\Big)+\chi_H^2(-2a^2\chi_H\sin^2\theta$$
$$+a^4\sin^4\theta)\partial_\theta^2F-(a^4\chi_H^2\sin^4\theta\cot\theta)\partial_\theta F-(4a^2\chi_H^3$$
$$-6a^4\chi_H^2\sin^2\theta+4a^6\chi_H\sin^4\theta-a^8\sin^6\theta)\partial_\phi^2F\Big]=0. \tag{4.G.3}$$

From the above equation one can derive dynamical equation of F in slow rotation approximation as shown in the Appendix 4.H.1.

4.H Results in slowly rotating background

4.H.1 Derivation of the Eq. (4.4.11) from Eq. (4.4.9)

In slowly rotating background for $a \ll M$, we have considered $a/M \approx x$, which imples that

$$\frac{r_H}{M}=1+\sqrt{1-\frac{a^2}{M^2}}=1+\sqrt{1-x^2}\approx 1+1-\frac{1}{2}x^2=2-\frac{1}{2}x^2. \tag{4.H.1}$$

Now using (4.H.1), from the equation (4.G.3) we expand the term χ_H^6 and also the coefficient of $\partial_\theta^2\partial_v^2F$ respectively in slowly rotation approximation as follows,

$$\chi_H^6=(a^2+r_H^2)^6=2^6M^6r_H^6=2^6M^{12}(\frac{r_H}{M})^6=2^6M^{12}(2-\frac{1}{2}x^2)^6$$
$$=2^{11}(2-3x^2)+\mathcal{O}(x^4).$$

Also,

$$2a^2(a^2+r_H^2)^5\sin^2\theta-a^4(a^2+r_H^2)^4\sin^4\theta=2^6a^2M^5r_H^5\sin^2\theta$$
$$-(2aMr_H\sin\theta)^4=2^6M^{12}(\frac{a}{M})^2(\frac{r_H}{M})^5\sin^2\theta-2^4M^{12}(\frac{a}{M})^4(\frac{r_H}{M})^4\sin^4\theta$$
$$=2^6M^{12}x^2(2-\frac{1}{2}x^2)^5\sin^2\theta-2^4M^{12}x^4(2-\frac{1}{2}x^2)^4\sin^4\theta$$
$$\approx 2^{11}M^{12}x^2\sin^2\theta+\mathcal{O}(x^4). \tag{4.H.2}$$

Similarly the other terms of (4.G.3) can be expanded upto $\mathcal{O}(x^2)$ keeping only M and x in the equation. Thus from (4.G.3) we get the equation (4.4.11).

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4.H.2 The details on some properties of spherical harmonics

Eq. (4.4.13) for all the modes $l \geq 1$, can be expressed in the form as given in (4.4.23) using the various well known properties of spherical harmonic functions. The the derivative of spherical harmonics are given by,

$$\begin{aligned}
&\sin^2\theta(\partial_\theta^2 Y_{lm}) \\
&= (m^2\cos^2\theta - m)Y_{lm} + \sqrt{(l-m)(l+m+1)}(2m+1)e^{-i\phi}\sin\theta\cos\theta\, Y_{l(m+1)} \\
&+\sqrt{(l-m)(l-m-1)(l+m+2)(l+m+1)}\sin^2\theta e^{-2i\phi}\, Y_{l(m+2)}.
\end{aligned} \tag{4.H.3}$$

Now we can express $\cos^2\theta$ and $\sin\theta\cos\theta$ in terms of spherical harmonics function as,

$$\begin{aligned}
&\sin^2\theta e^{-2i\phi} = \frac{4\sqrt{2\pi}}{\sqrt{15}}Y_{2-2}\ ;\sin\theta\cos\theta e^{-i\phi} = -\frac{2\sqrt{2\pi}}{\sqrt{15}}Y_{2-1}; \\
&\cos^2\theta = \frac{4\pi}{3}Y_{10}Y_{10} = \frac{4\pi}{3}(\frac{1}{2\sqrt{\pi}}Y_{00} + \frac{1}{\sqrt{5\pi}}Y_{20}).
\end{aligned} \tag{4.H.4}$$

We will use the following contraction rule of spherical harmonics,

$$Y_{l_1m_1}Y_{l_2m_2} = \sum_{m_3=-l_3}^{l_3}\sum_{l_3=|l_1-l_2|}^{|l_1+l_2|}\Lambda^{l_1m_1}_{l_2m_2l_3m_3}Y_{l_3m_3}; \tag{4.H.5}$$

where Λ is expressed by the Wigner 3-j symbols defined for the product of two spherical harmonics as,

$$\Lambda^{l_1m_1}_{l_2m_2l_3m_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.H.6}$$

The selection rules on Wigner 3-j symbols are given by

$$l_1 + l_2 \geq l_3;\ \ l_2 + l_3 \geq l_1;\ \ l_3 + l_1 \geq l_2;\ \ \ m_3 = m_1 + m_2. \tag{4.H.7}$$

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4.I The expression of the quantities given in (4.4.28) (4.4.29)

The corresponding quantities in the equations (4.4.28) and (4.4.29) are expressed as follows,

$$\Omega_1(l) = \Omega(l \to l+2); \quad \Omega_2(l) = \Omega(l \to \mid l-2 \mid),\ \bar{\Omega} = \Omega|_{l=1};\ \bar{\Omega}_1 = \Omega_1|_{l=1}.$$

$$b(l,m) = \frac{1}{2(2l(l+1)-1)c_{sc}^{(0)}\kappa_{sc}}\Big((\beta_1 + \beta_{lmlm})\partial_v^2 f_{lm}^{sc} + (\beta_2 + \bar{\beta}_{lmlm})f_{lm}^{sc}\Big)$$
$$= -\frac{1}{2l(l+1)-1}\Big(\frac{-72l(l+1)+57-40m^2-24m}{24}\Omega^2(l)$$
$$+\frac{12l(l+1)+5m^2+3m}{4}\Big) + \Lambda^{20}_{lmlm}\, b_0(l,m) + \Lambda^{2-1}_{l(m+1)lm}\, b'(l,m)$$
$$+\Lambda^{2-2}_{l(m+2)lm}\, b''(l,m);$$

where we have defined,

$$b_0(l,m) = -\frac{2\sqrt{\pi}}{\sqrt{5}(2l(l+1)-1)}\Big(\frac{\Omega^2}{16} - m^2\Big);$$
$$b'(l,m) = -\frac{2\sqrt{\pi}}{(2l(l+1)-1)}(2m+1)\sqrt{\frac{2(l-m)(l+m+1)}{15}}\Big(\Omega^2 - \frac{3}{4}\Big);$$
$$b''(l,m) = -\frac{2\sqrt{\pi}}{2l(l+1)-1}\sqrt{\frac{2(l^2-(m+1)^2)(l(l+2)-m(m+2))}{15}}$$
$$\times\Big(2\Omega^2 - \frac{3}{2}\Big);$$

Now with the expression of f_{lm}^{sc} as given in (4.4.26), the quantities in (4.4.29) are defined as,

$$b_1(l,m) = \frac{1}{2(2l(l+1)-1)c_{sc}^{(0)}\kappa_{sc}}\Big((\beta_1 + \beta_{l+2mlm})\partial_v^2 f_{l+2m}^{sc} + (\beta_2$$
$$+\bar{\beta}_{l+2mlm})f_{l+2m}^{sc}\Big)$$
$$= \Lambda^{20}_{(l+2)mlm}\, b_0(l \to l+2) + \Lambda^{2-1}_{(l+2)(m+1)lm}\, b'(l \to l+2)$$
$$+\Lambda^{2-2}_{(l+2)(m+2)lm}\, b''(l \to l+2);$$

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and,

$$
\begin{aligned}
b_2(l,m) &= \frac{1}{2(2l(l+1)-1)c_{sc}^{(0)}\kappa_{sc}}\Big((\beta_1+\beta_{l-2mlm})\partial_v^2 f^{sc}_{l-2m}+(\beta_2\\
&+\bar{\beta}_{l-2mlm})f^{sc}_{l-2m}\Big)\\
&= \Lambda^{20}_{(|l-2|)mlm}\, b_0(l\to |\, l-2\,|) + \Lambda^{2-1}_{(|l-2|)(m+1)lm}\, b'(l\to |\, l-2\,|)\\
&+\Lambda^{2-2}_{(|l-2|)(m+2)lm}\, b''(l\to |\, l-2\,|);\\
\bar{b} &= b|_{l=1},\ \ \bar{b}_1 = b_1|_{l=1}.
\end{aligned}
\tag{4.I.1}
$$

4.J Comparison of the numerical values of the source terms found in (4.4.28) and (4.4.29)

It is clear that all the source terms which are present in the right hand side of the equations (4.4.29) and (4.4.28) are exponentially decaying and vanishes near the horizon in the limit $v \to -\infty$. However if we closely look at the numerical values of b, b_1 and b_2, it can be shown that $|\, b_1 \,|<|\, b\,|$ and also $|\, b_2\,|<|\, b\,|$ for all (l,m). Therefore, the terms containing b_1 and b_2 are decaying faster than those terms containing b as one approaches near the horizon. We have tabulated the numerical values of these source terms for few sample (l,m) modes in the table 4.1 with a particular choice that $\kappa_{sc}v = 1$. It is observed that the numerical values of $b_1 \exp\{[\Omega_1]\}$ and $b_2 \exp\{[\Omega_2]\}$ are smaller compared to that of $b \exp\{[\Omega]\}$ in the near horizon limit. Therefore, near the horizon dominating contribution comes from the term $b \exp\{[\Omega]\}$ in (4.4.29) and also in (4.4.28). Hence for all $l \geq 0$, the equations (4.4.28) and (4.4.29) reduce to (4.4.30).

4.K ZAMO trajectory: Derivation of Eq. (4.4.42)

After the substitution of the ansatz for F in (4.4.41), for each mode m the equation can be written as,

$$
[B(\theta_c)+16m^2(r_H^2+a^2\cos^2\theta_c)]\frac{\partial^2 f_m}{\partial v^2} - \frac{12(M-r_H)^2 m^2}{(a^2+r_H^2)^2}(r_H^2+a^2\cos^2\theta_c)^3 f_m(v) = 0.
\tag{4.K.1}
$$

Hence the one dimensional equation of motion presented in (4.K.1) gives the dynamics of $f_m(v)$ for a specific ZAMO observer for a given angle θ_c. However there will be a large number of ZAMO observers situated at different angles in

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Table 4.1: Numerical values of the terms in the R.H.S of (4.4.29) and (4.4.28) and $\frac{b}{2\Omega}$

l	m	b $\exp\{[\Omega]\}$	$b_1 \exp\{[\Omega_1]\}$	$b_2 \exp\{[\Omega_2]\}$	$\frac{b}{2\Omega}$
1	0	-2.3	0.2	|	-0.43
1	1	-1.6	0.1	|	-0.5
1	-1	-2.8	0.2	|	-0.26
2	0	-1.5	0.2	-0.05	-0.3
2	1	-1.3	0.15	0	-0.32
2	-1	-1.5	0.18	0	-0.3
2	-2	-1.16	0.1	0	-0.35
2	2	-1.65	0.2	0	-0.4
⋮	⋮	⋮	⋮	⋮	⋮
10	0	-1.2	0.16	0.13	-0.32
10	10	-1.13	0.04	0	-0.3
10	-10	-1.24	0.04	0	-0.3
⋮	⋮	⋮	⋮	⋮	⋮
100	0	-1.2	0.15	0.15	-0.3
100	100	-1.2	0.005	0	-0.3
100	-100	-1.2	0.006	0	-0.3

the spacetime. Hence to study the mode dynamics of $f_m(v)$ corresponding to all those observers it is customary to take average over the angle θ. We compute the required average by multiplying the equation by $\sqrt{g_{\theta\theta}}\big|_{\theta=\theta_c} = \sqrt{\Sigma_H(\theta_c)}$ and integrating θ_c for all values ranging from 0 to π. So from (4.K.1) we get,

$$\frac{1}{\int_0^\pi \sqrt{\Sigma_H(\theta_c)}d\theta_c}\Big(\frac{\partial^2 f_m}{\partial v^2}\int_0^\pi [B(\theta_c)+16m^2(r_H^2+a^2\cos^2\theta_c)]\sqrt{\Sigma_H(\theta_c)}d\theta_c$$
$$-f_m(v)\int_0^\pi \frac{12(M-r_H)^2m^2}{(a^2+r_H^2)^2}(r_H^2+a^2\cos^2\theta_c)^3\sqrt{\Sigma_H(\theta_c)}d\theta_c\Big) = 0. \tag{4.K.2}$$

Then after averaging over all possible ZAMO observers finally the equation of motion comes out as given in (4.4.42).

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Diffeomorphism symmetries near a timelike surface in black hole spacetime 5

So far most of the studies on asymptotic symmetry has been performed either at asymptotic null infinity [64, 65, 67] or near the black hole horizon ([91, 92, 185] and the references therein). Being motivated by that analysis, in chapter 2, we have studied diffeomorphism symmetries for the Einstein-Maxwell system near a charged null hypersurface, which may not be the solution of field equations. On the other hand, inspired by $U(1)$ symmetry breaking phenomena, in the chapters 3 and 4, we have tried to find out the non-trivial mode solutions of the diffeomorphism symmetry parameters (known as *supertranslation* and *superrotation*) by incorporating the idea of the spontaneous symmetry breaking happened by Rindler and gravitational (Schwarzschild and Kerr) backgrounds themselves. Later we have found by quantum analysis that those mode solutions reveal the thermal nature of the black hole. However, a natural question arises, *what will be the physical implication of the diffeomorphism symmetries on an another type of physical boundary, like a timelike hypersurface located in the bulk spacetime?* To explore the full power of this asymptotic symmetry in understanding the nature of gravity, we believe it is timely to explore the asymptotic symmetries near the families of non-null hypersurface positioned at any arbitrary radial coordinate in the bulk spacetime manifold. The hypersurface can be thought of as a domain wall dividing the spacetime into two regions. In our present discussion, we will consider a timelike hypersurface situated at a finite radial distance outside the black hole horizon.

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To motivate further with regard to our choice of the timelike boundary in the bulk spacetime, it is important to mention the recent developments in the correspondence between gravity and fluid dynamics, where timelike surface away from the horizon plays an important part. The interesting connection has been established between Einstein field equations of gravity and Navier-Stokes (NS) equation of viscous fluid on a timelike surface [228–233]. Wherein the $(p+2)$-dimensional metric solution of Einstein's equations of motion describes a $(p+1)$-dimensional non-relativistic incompressible fluid on the timelike surface. This is obtained from the conservation Brown-York surface stress-energy tensor on the timelike hypersurface. An interesting connection has been uncovered between the fluid dynamics on the timelike membrane and infinite set of symmetries and associated conserved charges which are similar to the BMS symmetries and charges [116]. All these important results motivate us to perform the detail investigation of the diffeomorphism symmetries of the timelike surface under consideration.

5.1 Objective of the chapter

In this chapter we will consider a timelike hypersurface defined by the radial coordinate $r = r_c > r_H$, assumed to be situated outside the black hole horizon. Here r_H is the location of the horizon. We will consider both Schwarzschild and Kerr black hole spacetimes. The first step would be to express the metric near the surface in Gaussian normal coordinate. In order to do that we have followed the methodology outlined in [179]. Once the metric is derived, we will follow the usual procedure for the asymptotic symmetry analysis. By imposing appropriate boundary conditions near the surface, we will determine the diffeomorphism symmetry parameters. The boundary conditions will be such that the form of the metric near the timelike surface must remain invariant under arbitrary diffeomorphism. The conserved charges due to the diffeomorphism symmetries will be also calculated in a spacelike subspace of this timelike surface and the associated symmetry algebra will be constructed thereafter. At the end, we will show an interesting connection between those charges with the local heat content of the surface under consideration.

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5.2 The Gaussian normal coordinate system: a brief review

In this section we give a quick overview on the construction of the Gaussian normal coordinate (GNC) system which will be used in the later part of our analysis. These coordinate system is suitable to explain physical quantities on an $(p+1)$-dimensional hypersurface $\mathcal{S}$, embedded in the $(p+2)$-dimensional manifold $\mathcal{N}$. For the present discussion we consider sub-manifold $\mathcal{S}$ as a timelike hypersurface. One can choose a unique vector N^a in the tangent space of the manifold $\mathcal{N}$ in such a way that the vector will be orthogonal to all the vectors in the tangent space of the submanifold $\mathcal{S}$. So for $\mathcal{S}$, we can normalize N^a such that $N^a N_a = 1$; i.e. N^a is spacelike corresponding to signature $(-,+,+,+)$ of the metric. Thus the vector N^a is the unit normal to $\mathcal{S}$.

Now we can construct unique geodesic with tangent N^a emanating from an arbitrary point $\mathcal{P}$ on the aforesaid hypersurface $\mathcal{S}$. So the geodesic must pass through another arbitrary point $\mathcal{Q}$ situated in the small neighbourhood of the hypersurface $\mathcal{S}$. In this setup, one defines a coordinate system in the form (x^1, x^2, x^3, ρ), among them (x^1, x^2, x^3) denote the coordinates of the point $\mathcal{P}$ on the surface. On the other hand fourth coordinate ρ labels the point $\mathcal{Q}$ which lies on the geodesic, very near to $\mathcal{S}$. Here ρ is a spacelike coordinate. Also one of (x^1, x^2, x^3) is timelike and others are spacelike. So the GNC system (x^1, x^2, x^3, ρ) will be well defined in the small neighbourhood of the point $\mathcal{P}$ in such a way that the congruence of spacelike geodesics will be orthogonal to the hypersurface. The spacetime metric in the neighbourhood of the surface can be expressed in the following form (more details in this regard can be found in [175]):

$$ds^2 = g_{ab}dx^a dx^b = d\rho^2 + h_{\mu\nu}dx^\mu dx^\nu \, . \tag{5.2.1}$$

Here μ, ν indices are defined on the hypersurface $\mathcal{S}$ whose induced metric is given by $h_{\mu\nu}$. Whereas, a, b correspond to spacetime indices. Here we have metric coefficient $g_{\rho\rho} = 1$ due to the normalization of the vector field N^a. Furthermore $g_{\rho\mu} = 0$ since vector N^a is orthogonal to the surface. Hence in this newly formed coordinate system, $\rho =$ constant hypersurface describes timelike hypersurface $\mathcal{S}$ in GNC.

Next in the upcoming sections, we will express Schwarzschild and Kerr geometries in this coordinate system in order to study the symmetry properties of the timelike surface which is situated at any arbitrary radial position in the black

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hole spacetime backgrounds.

5.3 Schwarzschild black hole

5.3.1 Schwarzschild metric in Gaussian Normal coordinate system

The Schwarzschild metric in original Schwarzschild coordinates is given by,

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{(1 - 2M/r)} + r^2 d\Omega^2 \,, \tag{5.3.1}$$

where, $d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$. Since our main objective is to study the symmetry of this spacetime near a timelike surface, we shall express the above metric in Gaussian Normal Coordinates (GNC) near this hypersurface. We choose $\mathcal{S}$ as $r = r_c$ (constant) hypersurface lies outside the horizon $r = 2M$. Below we shall follow the procedure, described in the last section, to express metric (5.3.1) in GNC (more detail in this regard can be found in [179]).

The unit spacelike normal vector to $\mathcal{S}$ for Eq. (5.3.1) is given by,

$$N^a = (0, \sqrt{1 - \frac{2M}{r}}, 0, 0). \tag{5.3.2}$$

The spacelike geodesic congruence that crosses the surface orthogonally will have N^a as a tangential vector near the surface. Near the surface we can parametrize the geodesics with affine parameter $\rho = r - r_c$. The timelike surface is identified as $\rho = 0$. Therefore, the geodesics curve $X^a(\rho)$ with $X^a = (t, r, \theta, \phi)$ and orthogonal to S at the intersection, can be expressed through the following Taylor series expansion,

$$X^a(\rho) \approx X^a\Big|_{\rho=0} + \rho \frac{dX^a}{d\rho}\Big|_{\rho=0} + \frac{\rho^2}{2}\frac{d^2X^a}{d\rho^2}\Big|_{\rho=0} + \dots \,. \tag{5.3.3}$$

The first term in the right hand side is identified as $X^a|_{\rho=0} = (t', r_c, \theta', \phi')$. The second one is given by the tangent vector to the curve, $\frac{dX^a}{d\rho}|_{\rho=0} = N^a|_{\rho=0}$. The third term can be evaluated by considering the geodesic equation $N^a\nabla_a N^b = 0$ at $\rho = 0$ (shown in Appendix 5.A.1). This yields

$$\frac{d^2r}{d\rho^2}\Big|_{\rho=0} = -\Gamma^r_{bc}N^bN^c\Big|_{\rho=0} = \frac{M}{r_c^2} \,. \tag{5.3.4}$$

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Where all the other component equations identically vanish. Using these in Eq. (5.3.3) and keeping upto second order in ρ, we find the transformation of coordinates from (t', r, θ', ϕ') to (t, ρ, θ, ϕ) as (also shown in Appendix 5.A.1),

$$t' = t; \quad r = r_c + \rho\sqrt{1 - \frac{2M}{r_c}} + \rho^2 \frac{M}{2r_c^2}\,; \qquad (5.3.5)$$
$$\theta' = \theta; \quad \phi' = \phi\,.$$

Application of these transformations into Eq. (5.3.1) leads to the following form of the metric:

$$ds^2 = g_{\rho\rho}(\rho)d\rho^2 + g_{tt}(\rho)dt^2 + g_{AB}(\rho, x)dx^A dx^B\,, \qquad (5.3.6)$$

where the argument x corresponds to the angular coordinates (θ, ϕ). Now the metric coefficients can be obtained by simply using tensor transformation and keeping up to $\mathcal{O}(\rho)$ as follows (details are shown in Appendix 5.A.2),

$$g_{\rho\rho}(\rho) = \simeq 1 + \mathcal{O}(\rho^2)$$
$$g_{tt}(\rho) \simeq -\Big(1 - \frac{2M}{r_c}\Big) - \Big(\frac{2M\sqrt{1 - \frac{2M}{r_c}}}{r_c^2}\Big)\rho\,;$$
$$g_{AB}(\rho, x) \simeq \Big(r_c^2 + 2r_c\rho\sqrt{1 - \frac{2M}{r_c}}\Big)\gamma_{AB}; \qquad (5.3.7)$$

where γ_{AB} is the metric on the two-sphere given by, $d\Omega^2 = d\theta'^2 + \sin^2\theta d\phi'^2$.

Note that in the sufficiently small neighbourhood of the surface the metric Eq. (5.3.6) assumes the form Eq. (5.2.1) to the leading order in ρ. Near the timelike hypersurface $r = r_c$, the asymptotic form of the metric will be exactly the same as given in (5.3.6) which is expressed in Gaussian normal coordinates. Our primary goal would be to study the symmetries near $r = r_c$ surface, analogous to the symmetry analysis near the horizon or near null infinity. Hence we will impose appropriate asymptotic conditions on the variation of the metric coefficients (5.3.6).

5.3.2 Symmetries near the surface

The key ideas in identifying the asymptotic symmetries are intertwined with the appropriate set of boundary conditions on the variation of the metric coefficients. Due to diffeomorphism the structure of the metric (5.3.6) is assumed to remain same in our region of interest. Generically the asymptotic conditions on the metric

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coefficients are of two different categories. In one category, the gauge fixed conditions are strictly assumed to be invariant under diffeomorphism transformation $x^a \to x^a + \zeta^a$ and those are

$$\pounds_\zeta g_{\rho\rho} = 0,\ \pounds_\zeta g_{\rho t} = 0,\ \pounds_\zeta g_{\rho A} = 0\,. \tag{5.3.8}$$

In the above expression $\pounds_\zeta$ implies the Lie derivative along the vector field ζ^a. We call the above set as "strong" conditions. These four conditions Eq. (5.3.8) essentially provide residual diffeomorhism transformation generators. The functional form of those generators can possibly be restricted by the other category of transformations under which the metric coefficients are assumed to change with the appropriate fall-off conditions as

$$\pounds_\zeta g_{tt} \approx \mathcal{O}(1);\ \pounds_\zeta g_{tA} \approx \mathcal{O}(\rho^2), \pounds_\zeta g_{AB} \approx \mathcal{O}(1). \tag{5.3.9}$$

Analogous to the analysis near the null boundary, the asymptotic conditions on the metric variation are given by four gauge conditions Eq. (5.3.8) and remaining fall-off conditions Eq. (5.3.9) defined near the timelike boundary in the present discussion. The weak fall off condition for g_{tt} is postulated considering the leading order behaviour of the metric coefficient. The off-diagonal component g_{tA} is absent in the original Schwarzschild solution, hence we have assumed the fall-off rate of g_{tA} as $\mathcal{O}(\rho^2)$. At the same time, the fall-off condition of g_{AB} is imposed as $\mathcal{O}(1)$ depending on the leading order structure of the metric. In the next part of the analysis, we will show that for the obtained form of the solution of the vector field ζ^a, the Lie variations of g_{tA} and g_{AB} automatically vanish. We now first solve the gauge fixed condition Eq. (5.3.8) to identify the appropriate symmetry parameter and the form of the solutions are as follows (detail is given in Appendix 5.A.3),

$$\begin{aligned}
\zeta^\rho &= T(t,x)\,;\\
\zeta^t &= 1/(1-2M/r_c)\int \partial_t T(t,x) d\rho + F(t,x)\,;\\
\zeta^A &= -g^{AC}\int \partial_C T\, d\rho + R^A(t,x)\,;
\end{aligned} \tag{5.3.10}$$

where T, F and R^A are the integration constants. These are the residual diffeomorphism symmetry parameters under which the gauge choice remains the same. To proceed further, we choose $R^A = 0$, as static Schwarzschild spacetime does not have intrinsic rotation. Further, we also set $T(t,x) = 0$ which keeps the position of the hypersurface intact at $\rho = 0$. Therefore, we are left with one diffeomorphism parameter F which is similar to the 'supertranslation' discussed in the literature for

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the asymptotic null boundary and null horizon. Now we have shown in Appendix 5.A.4 that near the surface at $\rho = 0$, the Lie variation of the component g_{tA} will give only one non-zero term, $-(1 - \frac{2M}{r_c})\partial_A F$. Therefore the functional form of F needs to be further constrained by the following condition,

$$\partial_A F = 0 \implies F = F(t), \tag{5.3.11}$$

so that the structure of the metric near S does not change. Finally we can write symmetry parameters very near to the surface as;

$$\zeta^\rho = 0 = \zeta^A \, ; \;\; \zeta^t = F(t) \, . \tag{5.3.12}$$

Hence for the above condition (5.3.11) on parameter F and for the solution set (5.3.12) of ζ^a, the Lie variation of the metric component g_{tA} and g_{AB} will be zero automatically (details can be found in Appendix 5.A.4, see Eqs. (5.A.14) and (5.A.15)). So we have left with only one fall-off condition on the component g_{tt} as given in (5.3.9). Correspondingly there is diffeomorphism along time direction only. This feature is completely different from what was obtained for the case of a general null-surface in chapter 2 where parameter F was the function of angular coordinates as well. In the later part of the discussion, we will discuss that the derived form of ζ^a is such that all the metric coefficients preserve their asymptotic form near the surface, while the black hole parameter mass gets modified with a time-dependent term. This phenomenon is similar to the near horizon symmetries or asymptotic symmetries near infinity, where it has been observed that under infinitesimal diffeomorphism structure of the metric near the boundaries does not change, but the black hole parameters, like mass, can be modified. Analogous to the near horizon symmetry, we call $F(t)$ as the 'supertranslation' parameter. *But unlike near horizon symmetries [91] and asymptotic null infinity BMS symmetries [67], symmetry generator near the timelike hypersurface embedded in the Schwarzschild spacetime, the 'supertranslation' parameter is not arbitrary function of spatial coordinates (θ, ϕ), rather it turned out to be dependent only on time.*

5.3.3 Algebra of symmetry parameter

Now we want to construct the bracket algebra between the symmetry generators constructed in the last section. We have only one non-zero component of $\zeta^a = (F(t), 0, 0)$ which will generate the symmetry algebra of the residual symmetry transformation. Since like horizon at $r = 2M$, there is no periodicity of Euclidean

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time at $r = r_c$, therefore associated Fourier mode of F will be continuous in frequency space ,

$$F(t) = \int_{-\infty}^{\infty} d\omega\ \alpha(\omega) e^{-i\omega t} = \int_{-\infty}^{\infty} d\omega\ \alpha(\omega) \bar{F}(\omega, t)\ . \tag{5.3.13}$$

Here $\alpha(\omega)$ represents individual Fourier mode of $F(t)$. The kernel function $\bar{F}(\omega, t) = e^{-i\omega t}$ will form the complete orthogonal basis with continuous frequency ω.

Now to get the required algebra, let us consider two symmetry vectors $\zeta_1^a = (F_1, 0, 0)$ and $\zeta_2^a = (F_2, 0, 0)$. The standard Lie algebra between these two diffeomorphism vectors is given in (2.13). As we have already discussed in the first chapter that the second and third terms in the definition of Lie bracket (2.13) are generally considered in order to take into account the variation of the vector fields due to the higher-order variation of the metric. But we will ignore the higher-order effect on the variation of diffeomorphism vectors (detail is shown in Appendix 2.C). Therefore, Eq. (2.14) will be our main equation to construct the symmetry algebra. In terms of F, the Lie bracket takes the following form,

$$[\zeta_1, \zeta_2]^t = [F_1, F_2] = F_1 \partial_t F_2 - F_2 \partial_t F_1\ . \tag{5.3.14}$$

In terms of Fourier modes, the left hand side becomes,

$$[F_1, F_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \alpha(\omega_1, \omega_2) [\bar{F}(\omega_1, t), \bar{F}(\omega_2, t)]\ , \tag{5.3.15}$$

with $\alpha(\omega_1, \omega_2) = \alpha(\omega_1)\alpha(\omega_2)$, and the right hand side of the equation Eq. (5.3.14) becomes,

$$\begin{aligned} & F_1 \partial_t F_2 - F_2 \partial_t F_1 \\ = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \alpha(\omega_1) \alpha(\omega_2) [-i(\omega_2 - \omega_1)] \bar{F}(\omega_1 + \omega_2, t)\ . \end{aligned} \tag{5.3.16}$$

Comparing both the equations Eq. (5.3.15) and Eq. (5.3.16), the algebra among the basis functions becomes,

$$i[\bar{F}(\omega_1, t), \bar{F}(\omega_2, t)] = (\omega_2 - \omega_1) \bar{F}(\omega_1 + \omega_2, t). \tag{5.3.17}$$

Here the important point to mention is that although the diffeomorphism symmetry parameter defined on the timelike surface in Schwarzschild spacetime came out to be dependent only on time, but the aforementioned bracket algebra (5.3.17) is similar to the that obtained for a generic null surface (see $[F, F]$ commutator given in Eq. (2.15) of the chapter 2).

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5.3.4 Charges on the surface

For any diffeomorphism invariant gravity theory, conserved charges corresponding to symmetries play a crucial role in understanding the thermodynamics of the black hole. Innumerable studies have been done where some specific set of diffeomorphism symmetric charges have been shown to be related to the black hole entropy [28, 54, 55, 62, 63, 104–106, 234]. It is well known that the Noether current J^a associated with the diffeomorphism symmetry $x^a \to x^a + \zeta^a$ for Einstein-Hilbert action satisfies covariant conservation equation for arbitrary ζ^a (for instance, see Appendix 2.F in chapter 2 for an off-shell derivation of charges). Here ζ^a, in principle, is completely arbitrary. Hence the conserved charge can naturally be expressed as (2.18). Detailed discussion regarding the conservation of Nother charges on a closed boundary has been presented in section (2.6) of chapter 2.

In the present discussion, we have considered a $t = constant$ slice of $r = r_c$ timelike hypersurface which act as a two-dimensional boundary between horizon and infinity. Here Q_t can be written as,

$$Q_t = \Big[Q_t(r = r_H) + Q_t(r = r_c, -N^a)\Big] + \Big[Q_t(r = r_c, N^a) + Q_t(r \to \infty)\Big]; \tag{5.3.18}$$

where the expression in the first part corresponds to region within horizon and r_c while the second part is connected to region between r_c to ∞. It is expected that $Q_t(r = r_c, -N^a)$ and $Q_t(r = r_c, N^a)$ must cancel each other, because the normal on $r = r_c$ surface are same but in opposite direction with respect to two regions. However, the value of individual terms may be non-vanishing. This suggests that, in principle, one can define a unique quantity $Q \sim \int d\Sigma_{ab} J^{ab}$ on any $t = r = constant$ two dimensional cross-sections which can be thought of as part of the conserved charge Q_t if the observer is confined in either one side of the surface. In this sense Q alone should not be a conserved quantity. However, we use the usual nomenclature as "charge" here in a very weak sense as this part is not alone conserved. This charge can be interpreted as the induced charge on an arbitrary time like hypersurface associated with a special class of diffeomorphism under study.

Therefore in Einstein gravity part of the 'charges' due to diffeomorphism symmetry on a two-surface (either on the horizon or at infinity) is expressed as given in (2.19), where J_{ab} is the Noether potential. For Einstein theory the

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expression of J_{ab} is given by

$$J_{ab} = \frac{1}{16\pi G}(\nabla_a \zeta_b - \nabla_b \zeta_a) . \tag{5.3.19}$$

The integration is being done on the two surface.

Now a natural question arises what symmetry does this charge should correspond to? For any arbitrary diffeomorphism, as we stated earlier, the charge Eq. (2.19) can be defined in principal on any arbitrary surface. This is defined for any arbitrary diffeomorphism. Therefore the choice of ζ^a (the gauge choice) from a particular condition will construct a charge which is conserved in the sense of our previous discussion. Even if one finds this diffeomorphism from a local (gauge) symmetry of metric, the total corresponding charge which is the sum of contributions from different parts of the boundary, still conserved as this is by construction conserved for any arbitrary diffeomorphism. In fact, given a specific form of the diffeomorphism associated with the underlying symmetry, the formula yields the well-known charges near the horizon or null-infinity. This procedure has been shown to be yielded horizon entropy in the context of Virasoro algebra (e.g. see [62, 63, 104, 105]). Note that even if the conserved charge here is due to gauge symmetry, its value on the part of the boundary may be non-vanishing. However, in the present analysis, what would be the physical characteristics and how to compute the charge on the arbitrary time like surface are the questions we have asked and explored in detail. Hence the charges are calculated for a class of diffeomorphism symmetries which preserves the form of the metric near the timelike surface under study. Once Q is defined as above, on any two dimensional cross-section (like, hypersurface defined by both $t =$ and $r =$ are constants), one can compute the associated diffeomorphism symmetry transformation which keeps the metric structure near the surface under study unchanged, following the usual procedure. One usually computes ζ^a from physically motivated boundary conditions on metric coefficients g_{ab} as just mentioned above. Using this subset of diffeomorphisms, constraint by certain physical boundary conditions on g_{ab}, one usually calculates Q. This well known procedure has been used in the case of horizon as well as asymptotic infinity structure-preserving symmetries. In both cases, it has been observed that the structure of the metric remains unchanged, but the black hole parameters, such as mass, angular momentum can be modified.

In the present discussion we are calculating this Q on the cross-section of a timelike surface $r = r_c$, denoted by $t = constant$ slice following the same procedure. We also observed that under suitable redefinition, the black hole

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parameters changes, e.g., in the case of Schwarzschild, the mass has been changed to,

$$M \to M + (M - \frac{r_c}{2})(\partial_t F). \tag{5.3.20}$$

However, this change may be thought of as a diffeomorphic change of the black hole mass.

In this context, as a side remark, it may be mentioned that since our interest is on the $t = constant$ slice of timelike hypersurface and Q can be associated on any two dimensional surface, it is not mandatory to exists a horizon in the spacetime, as least as far as the definition of Q is concerned. The only restriction has to be that the metric should be solution of diffeomorphism invariant gravity theory, like General theory of relativity (GR), and in that sense, the same can be applied to a star spacetime solution.

In the present situation, as mentioned above, we shall calculate Q on the one side of $\rho = 0$ surface at a particular time slice. Therefore this subspace is two dimensional and can be indicated by two unit normals: one is timelike and another is spacelike. Hence this particular subspace is in general spacelike, but not the two dimensional spacelike slice of the horizon as $r_c \neq 2M$. The surface element on the two-surface is

$$d\Sigma^{ab} = d^2x\sqrt{\sigma}(M^a n^b - M^b n^a) , \tag{5.3.21}$$

where σ is the determinant of the induced metric on the hypersurface under consideration. M^a is the unit spacelike normal vector directed outward and defined on the timelike surface ($\rho = constant$), and is given by, $M^a = (0, 1, 0, 0)$. Similarly n^a is the unit timelike normal on the spacelike hypersurface ($t = constant$) taking the following form (shown in Appendix 5.A.5.1),

$$n^t = -\frac{1}{\sqrt{1 - 2M/r_c}} + \rho\Big(\frac{M}{(r_c^2 - 2Mr_c)}\Big) . \tag{5.3.22}$$

As the spacelike surface is situated at $\rho = 0$ the only non-vanishing component of the surface element in Eq. (5.3.21) is,

$$d\Sigma^{t\rho} = -d^2x\sqrt{\sigma}n^t M^\rho. \tag{5.3.23}$$

Using Eq. (5.3.12) and Eq. (5.3.23) one obtains Q to be (detail in Appendix 5.A.5.2),

$$Q[F] = \frac{MF(t)}{2G} . \tag{5.3.24}$$

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As expected from the spherically symmetric background, the charge Q comes out to be independent of the position of the timelike surface under study. For Kerr black hole background, we will see this does not hold true. Note that it depends on a particular $t = constant$ slice of our preferred hypersurface $r = r_c$. In this case Q can have leakage on this spacelike slice if one moves with time. But it may be noted that similar leakage in Q can be found on the same $t = constant$ slice of the actual physical boundaries (at $r \to \infty$ and horizon) and also in another side of r_c surface. It may happen that all these leakages are such that the collective quantity vanishes. This feature is not new as it happens for charges on the horizon for general diffeomorphism vector ζ^a (for example, see Eq. (13) of [105]; also see [103]). The reason is obvious. As mentioned earlier, the covariantly conserved Noether current J^a and consequently the anti-symmetric potential J^{ab} are defined for any arbitrary ζ^a, even it can be time-dependent. Therefore the computation of charge on a particular boundary $\partial\mathcal{V}$ can be in general time-dependent. However, if one includes all part of $\partial\mathcal{V}$, total charges must be conserved by construction.

Next in order to compute bracket algebra between charges we express Eq. (5.3.24) in terms of the Fourier transform of the 'supertranslation' parameter given in (5.3.13):

$$Q[F] = \frac{M}{2G}\int_{-\infty}^{\infty} d\omega\, \alpha(\omega)\bar{F}(\omega,t)\,. \tag{5.3.25}$$

The Fourier mode of the charge is defined as

$$Q[\bar{F}(\omega,t)] = \frac{M}{2G}\bar{F}(\omega,t)\,. \tag{5.3.26}$$

Then one can write,

$$Q[F] = \int_{-\infty}^{\infty} d\omega\, \alpha(\omega) Q[\bar{F}(\omega,t)]\,. \tag{5.3.27}$$

Therefore, from the following definition,

$$[Q[\zeta_1], Q[\zeta_2]] = \pounds_{\zeta_1} Q[\zeta_2]\,, \tag{5.3.28}$$

the algebra among the charges can be directly computed as (detail is given in Appendix 5.A.6),

$$i[Q[\bar{F}(\omega_1,t)], Q[\bar{F}(\omega_2,t)]] \;=\; (\omega_2-\omega_1)Q[\bar{F}(\omega_1+\omega_2,t)]. \tag{5.3.29}$$

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5.3.5 Physical interpretation of 'charges'

Before we end the discussion for Schwarzschild spacetime, let us try to investigate whether the computed 'supertranslation charge' has any thermodynamic interpretation. From Eq. (5.3.26), the choice of global symmetry generator, $F(t) = 1$, yields

$$Q = \frac{A_c}{4G}\frac{g(r_c)}{2\pi} . \tag{5.3.30}$$

where $A_c = 4\pi r_c^2$ and the local gravitational acceleration at the radial coordinate r_c is given by $g(r_c) = M/r_c^2$. Hence, the quantity Q may be interpreted as the local heat content on the hypersurface at a fixed time if one identifies the surface entropy $S_c \propto A_c$ and local temperature $T_c \propto g(r_c)$. The temperature is identified as the Unruh temperature which is directly connected with the local gravitational acceleration $g(r_c)$.

Next we will show that the same thermodynamic interpretation can be drawn in a more general setting, i.e. for static spherically symmetric background also. Let us start with the general static, spherically symmetric metric of the form

$$ds^2 = -f_1(r)dt^2 + f_2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{5.3.31}$$

In this case the metric near timelike hypersurface r_c in GNC is determined to be as (shown in Appendix 5.B.2)

$$ds^2 = -[f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho]\,dt^2 + d\rho^2 + [r_c^2 + 2r_c\sqrt{1/f_2(r_c)}\,\rho]\,d\Omega^2 . \tag{5.3.32}$$

The above can be obtained by using the following transformations (derivation in Appendix 5.B.1.1),

$$\begin{aligned} r &= r_c + \rho\sqrt{1/f_2(r_c)} - \frac{\rho^2}{4f_2^2}f_2'(r_c); \\ t &\to t; \quad \theta \to \theta; \quad \phi \to \phi . \end{aligned} \tag{5.3.33}$$

Now like earlier the charge for the symmetries near r_c, given by (5.3.12), for $F(t) = 1$ at a particular time turns out to be (detail is given in Appendix 5.B.3),

$$Q = \frac{A_c}{16\pi G}\frac{f_1'(r_c)}{\sqrt{f_1(r_c)f_2(r_c)}} , \tag{5.3.34}$$

where $A_{r_c} = 4\pi r_c^2$ is the area of our subspace of timelike hypersurface.

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Now for general spherically symmetric black hole Eq. (5.3.31), it can be proved that local acceleration takes the from,

$$g(r_c) = f'(r_c)/\sqrt{4f_1(r_c)f_2(r_c)}. \tag{5.3.35}$$

A discussion regarding this has been presented in Appendix 5.B.4. Hence, with the form (5.3.35), (5.3.34) can be expressed as (5.3.30). Like Schwarzschild case, (5.3.34) may be interpreted as the local heat content on the hypersurface at a fixed time with T_c is identified as Unruh temperature [213, 235–238]. Hence the local observer hovering near the $r = r_c$ hypersurface, will see the surface as thermal object with the aforementioned Temperature T_c.

Another physical interpretation of the aforementioned 'charges' can be presented through an interesting perspective considering the Tolman relation [239–242], which gives rise to the temperature gradient of a thermodynamic equilibrium system in the gravitational field. The relation entails,

$$T(r)\sqrt{-g_{ab}k^a k^b} = T(r)\sqrt{-g_{tt}} = T_0, \tag{5.3.36}$$

where $k^a = (1,0,0,0)$ is the timelike Killing vector of the background under consideration and $T(r)$ is called Tolman temperature and T_0 is arbitrary constant equilibrium temperature. Using the relation $f_1' = -2f_1(T'/T)$ and $F(t) = 1$ which precisely corresponds to the time-like Killing vector k^a, we obtain the following interesting relation (detail in Appendix 5.C),

$$Q = -\frac{1}{2\pi}\frac{A_c}{4G}\left(\frac{T_0}{T}\right)\frac{\hat{n}^a\nabla_a T}{T}\bigg|_{r_c} \tag{5.3.37}$$

where we have chosen the normal to r = constant surface in such a way that $\hat{n}_a = (0, \sqrt{f_2}, 0, 0)$ and consequently $\hat{n}^a = (0, 1/\sqrt{f_2}, 0, 0)$. Now it has been shown in [243] that a classical gas of radiation under Newtonian acceleration satisfies $\nabla_a T/T = -g_a$. Therefore the temperature gradient part of (5.3.37) leads to gravitational acceleration along the normal to hypersurface and hence we obtain (5.3.30). It would be interesting to explore this relation Eq. (5.3.37) further in the context thermodynamic origin of gravitational theory.

Hence for a globally defined diffeomorphism vector $F(t) = 1$, the charge induced on the timelike hypersurface can be expressed as Eq.(5.3.37), for which we are able to provide a distinct thermodynamic interpretation. However, for arbitrary time dependent globally defined function of $F(t)$, we are unable to give any physical interpretation.

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5.3.6 Comparison of the results in Minkowski background

So far, we have tried to explore the symmetry of the non-null hypersurface situated at $r_c > r_H$ in the Schwarzschild black hole background. In this section, we consider its flat space limit $M \to 0$. One expected outcome naturally follows from the Eq. (5.3.25) so that the expression of charge Q vanishes on the timelike hypersurface. This can also be straightforwardly calculated considering a specific set of the timelike surface at any arbitrary radial position in the Minkowski background. In spherical polar coordinates, the metric is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2)\ . \tag{5.3.38}$$

Following the same procedure as discussed in detail in Schwarzschild case, we can obtain the induced metric on a time like surface defined by $r = r_c$ in Gaussian normal coordinates as,

$$ds^2 = d\rho^2 - dt^2 + (r_c^2 + 2r_c\rho)(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.3.39}$$

The Lie variation of the gauge conditions and other metric components as fall-off conditions will remain the same as given in Eq. (5.3.8) and Eq. (5.3.9). Therefore, the solution of the gauge choices will be the same as found in Eq. (5.3.10) with the limit $M \to 0$. The constraints coming from the fall-off condition will be the same as before Eq. (5.3.11). Hence symmetry algebra will remain the same as Eq. (5.3.17) as for the Schwarzschild black hole background. However, as mentioned in the beginning, our explicit computation also shows the associated charge Q becomes zero (detail derivation is given in Appendix 5.D). Therefore one can conclude that, unlike the gravitational case, for flat spacetime, there is no thermodynamic quantity related to the global symmetry parameter given by $F = 1$.

5.4 Kerr background

We will now extend the earlier discussion for a timelike surface $r = r_c$ located outside the horizon of the Kerr black hole.

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5.4.1 Kerr in GNC

The Kerr metric in Boyer-Lindquist coordinate is expressed as

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta[(r^2+a^2)^2 - a^2\sin^2\theta\Delta]}{\Sigma}d\phi^2 \,, \tag{5.4.1}$$

where, $\Delta = r^2 + a^2 - 2Mr$ and $\Sigma = r^2 + a^2\cos^2\theta$. We will follow the same methodology as elaborately discussed for the Schwarzschild black hole. Our task is to express the metric in the neighbourhood of the time like surface $r = r_c > r_H$ in Gaussian normal coordinate in the form of Eq. (5.2.1). The detail procedure in this regard has been discussed in the last section in case of Schwarzschild spacetime.

In the Kerr background spacetime, the non-vanishing component of the unit space like normal vector N^a to the hypersurface under consideration is given by,

$$N^r = \sqrt{\frac{(r^2 + a^2 - 2Mr)}{(r^2 + a^2\cos^2\theta)}} \,. \tag{5.4.2}$$

Now following the same procedure discussed in the previous section, and Taylor expanding the geodesics as given in Eq. (5.3.3), we can define the transformation between spacetime coordinates (t, r, θ, ϕ) and GN coordinate $(t', \rho, \theta', \phi')$ upto $\mathcal{O}(\rho^2)$ as (detail on Appendix 5.E.1),

$$t = t' \,; \quad r = r_c + \rho\, s_1(\theta) - \rho^2 s_2(\theta) \,; \\ \theta = \theta' . \ \phi = \phi' \,. \tag{5.4.3}$$

where we define $s_1(\theta) = N^r|_{r=r_c}$, and

$$s_2(\theta) = \frac{d^2X^a}{d\rho^2}\Big|_{\rho=0} = \left(\frac{a^2 r_c - M r_c^2 + (M - r_c)a^2\cos^2\theta}{2(r_c^2 + a^2\cos^2\theta)^2}\right) . \tag{5.4.4}$$

The above transformations upto $\mathcal{O}(\rho^2)$ will be sufficient for our subsequent discussion. Using the above transformation of coordinates, the induced metric near the timelike hypersurface under consideration will assume the GNC form as Eq. (5.2.1) where metric components upto $\mathcal{O}(\rho)$ are given by (detail is given in Appendix

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5.E.2),

$$
\begin{aligned}
g_{tt} &= -\frac{\Delta_{r_c} - a^2\sin^2\theta}{\Sigma_{r_c}} - 2\rho\, s_1(\theta)\left(\frac{Mr_c^2 - a^2M\cos^2\theta}{\Sigma_{r_c}^2}\right);\\
g_{t\phi} &= \frac{-4aMr_c\sin^2\theta}{\Sigma_{r_c}} + 4aMs_1(\theta)\rho\,\sin^2\theta\left(\frac{r_c^2 - a^2\cos^2\theta}{\Sigma^2{}_{r_c}}\right);\\
g_{\rho\rho} &= 1\,;\quad g_{\theta\theta} = \Sigma_{r_c} + 2\rho\, r_c\, s_1(\theta)\,;\\
g_{\phi\phi} &= \sin^2\theta\left(\frac{(r_c^2+a^2)^2 - a^2\sin^2\theta\Delta_{r_c}}{\Sigma_{r_c}}\right)\\
&+2\rho s_1(\theta)\sin^2\theta\Big[\left(\frac{2a^2r_c + 2r_c^3 + (a^2M - a^2r_c)\sin^2\theta}{\Sigma_{r_c}}\right)\\
&-\left(\frac{r_c(a^2+r_c^2)^2 - a^2r_c(a^2 - 2Mr_c + r_c^2)\sin^2\theta}{\Sigma^2{}_{r_c}}\right)\Big].
\end{aligned}
\tag{5.4.5}
$$

Here Σ_{r_c} and Δ_{r_c} are the corresponding quantities defined at $r = r_c$. These are given by,

$$
\begin{aligned}
\Sigma_{r_c} &= r_c^2 + a^2\cos^2\theta\,,\\
\Delta_{r_c} &= r_c^2 + a^2 - 2Mr_c\,.
\end{aligned}
\tag{5.4.6}
$$

Therefore, the above form (5.4.5) in GNC properly describes the spacetime geometry very near to the timelike surface situated at $\rho = 0$ (where $r - r_c \approx \rho$). Hence the components of the metric take the asymptotic form as given in (5.4.5) near the aforementioned timelike boundary. Our subsequent analysis will be the same as that of the Schwarzschild black hole background discussed above.

5.4.2 Symmetries near the surface

Unlike Schwarzschild black holes, Kerr spacetime is endowed with angular momentum, which plays a characteristically different role in defining the properties of the associated symmetry algebra. Main objective of the whole symmetry analysis program is to keep the asymptotic structure of the metric under investigation invariant under diffeomorphism transformation. Here also the following constraints on the behaviour of some of the metric components which are associated with the gauge condition, need to be maintained as before, and those are,

$$
\pounds_\zeta g_{\rho\rho} = 0;\ \pounds_\zeta g_{\rho t} = 0;\ \pounds_\zeta g_{\rho A} = 0\,,
\tag{5.4.7}
$$

where A stands for the transverse coordinates θ and ϕ. Because of the inherent rotational angular momentum of the spacetime, the weak fall off conditions of the

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remaining metric component in terms of ρ coordinate are assumed to follow,

$$\pounds_{\zeta} g_{tt} \approx \mathcal{O}(1);\ \pounds_{\zeta} g_{tA} \approx \mathcal{O}(1);\ \pounds_{\zeta} g_{AB} \approx \mathcal{O}(1)\ . \tag{5.4.8}$$

The fall-off conditions directly follow from the leading order behaviour of the metric coefficients g_{tt}, g_{tA} and g_{AB}. The two categories of boundary conditions keep the asymptotic form of the metric intact near the timelike surface at $r = r_c$. So these are the asymptotic conditions imposed on the fluctuation of the metric coefficients (5.4.5). Solving the three strong form of gauge conditions expressed in Eq. (5.4.7) we determine the general solutions of the symmetry vector as follows (detail is given in Appendix 5.E.3),

$$\begin{aligned} &\zeta^{\rho} = T(t,x); \quad \zeta^{t} = -\int \frac{\partial_t T}{g_{tt}} d\rho + \int \frac{g^{tA}}{g_{tt}} [g_{tt}\partial_A T + g_{tA}\partial_t T)] d\rho + F(t,x)\ ; \\ &\zeta^{A} = [-g^{tA} \int \partial_t T d\rho - g^{AB} \int \partial_B T d\rho] + R^{A}(t,x)\ , \end{aligned} \tag{5.4.9}$$

where x corresponds to the angular coordinates (θ, ϕ). Here T, F, and R^A are the unknown integration constants. These are the diffeomorphism vectors under which the form of the metric near the time like surface under consideration will remain invariant. Now further restrict on the above vectors under which the position of the surface does not change will lead to the following additional constraint $T(t,x) = 0$. The fall-off conditions expressed in Eq. (5.4.8) have been observed to be automatically satisfied by the solution space Eq. (5.4.9) (details are shown is given in Appendix 5.E.4). Finally the non-vanishing components of the symmetry vectors $\zeta = \zeta^a \partial_a$ are given by,

$$\zeta^a \partial_a = F(t,x)\partial_t + R^A(t,x)\partial_A. \tag{5.4.10}$$

In the above expression R^A denotes the rotation parameter. For generality we assume all the differmorphism parameters to be function of all the coordinates namely $\zeta^{\theta} = R^{\theta}(t,x)$ and $\zeta^{\phi} = R^{\phi}(t,x)$. Following the conventional definition, we describe the parameters F as the generator of the 'supertranslation' and R^A as the 'superrotation'. Important to note that for Kerr black hole background, the supertranslation generator near the timelike surface turned out to be an arbitrary function of both time and angular coordinates as opposed to that of the Schwarzschild black hole background discussed before.

Similar to the Schwarzschild case, the form of the vector field ζ^a is such that near the surface $\mathcal{S}$, the metric coefficients in Kerr background keep their asymptotic

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form intact, while from the Lie variation of g_{tt}, one can show that the black hole mass M has been changed to,

$$M \to M - \frac{(\Sigma_{r_c} - 2r_c M)\Sigma_{r_c}\partial_t F - 2Ma^2 r_c R^\theta \sin 2\theta - 4aMr_c\Sigma_{r_c}\sin^2\theta(\partial_t R^\phi)}{2r_c\Sigma_{r_c}}; \tag{5.4.11}$$

and from the variation of the $g_{t\phi}$ component, a/Σ_{r_c} transform to,

$$\begin{aligned} &\frac{a}{\Sigma_{r_c}} \to \frac{a}{\Sigma_{r_c}}(1 + \partial_\phi R^\phi - \partial_t F) \\ &+ \frac{4aMr_c(\Sigma_{r_c} + a^2\cos\theta)R^\theta \sin 2\theta - (\Sigma_{r_c} - 2Mr_c)\Sigma_{r_c}^2\partial_\phi F}{4Mr_c\Sigma_{r_c}^2\sin^2\theta} \\ &- \frac{\left((r_c^2 + a^2)^2 - a^2\sin^2\theta\Delta_{r_c}\right)\partial_t R^\phi}{4Mr_c\Sigma_{r_c}^2}. \end{aligned} \tag{5.4.12}$$

Following the same procedure discussed for the Schwarzschild black hole, we now explore the bracket algebra among the symmetry generators $\chi^a\partial_t = F\partial_t$ and $\eta^a\partial_a = R^A\partial_A$. The generators are expressed in terms of following Fourier expansion,

$$\begin{aligned} F(t,x)\partial_t &= \sum_{k_A}\int_{-\infty}^{\infty} d\omega\, \alpha(\omega,k_A)e^{-i\omega t + i\sum_A k_A x^A}\partial_t \\ &= \sum_{k_A}\int_{-\infty}^{\infty} d\omega\, \alpha(\omega,k_A)\, \bar{F}_k(\omega,t,x); \end{aligned} \tag{5.4.13}$$

$$\begin{aligned} R^A(t,x)\partial_A &= \sum_{l_A}\int_{-\infty}^{\infty} d\omega\, \bar{\alpha}^A(\omega,l_A)\, e^{-i\omega t + i\sum_A l_A x^A}\partial_A \\ &= \sum_{l_A}\int_{-\infty}^{\infty} d\omega\, \bar{\alpha}^A(\omega,l_A)\bar{R}^l_A(\omega,t,x). \end{aligned} \tag{5.4.14}$$

Here $\alpha(\omega)$ and $\bar{\alpha}^A(\omega)$ represent the Fourier modes in frequency domain of the functions $F(t,x)$ and $R^A(t,x)$ respectively. The argument x corresponds to angular coordinates (θ,ϕ). $\bar{F}_k(\omega,t,x)$ and $\bar{R}^l_A(\omega,t,x)$ are identified as basis generators. So among the generators, the commutator algebra will be computed as follows (details are given in Appendix 5.E.5),

$$i[\bar{F}_k(\omega_1,t,x), \bar{F}_l(\omega_2,t,x)] = (\omega_2 - \omega_1)\, \bar{F}_{k+l}(\omega_1+\omega_2,t,x). \tag{5.4.15}$$

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The Lie bracket for the 'superrotation' diffeomorphism vector with itself will take the following form,

$$i[\bar{R}^k_A(\omega_1,t,x)],\bar{R}^l_B(\omega_2,t,x)] = l_A\ \bar{R}^{k+l}_B(\omega_1+\omega_2,t,x) - k_B\ \bar{R}^{k+l}_A(\omega_1+\omega_2,t,x)\ . \tag{5.4.16}$$

By using the similar manner we can calculate the commutator algebra between 'supertranslation' and 'superrotation' generators as,

$$i[\bar{F}_k(\omega_1,t,x),\bar{R}^l_A(\omega_2,t,x)] = \omega_2\bar{R}^{k+l}_A(\omega_1+\omega_2,t,x)\ -\ k_A\bar{F}_{k+l}(\omega_1+\omega_2,t,x)\ ; \tag{5.4.17}$$

Let us point out the important difference between the results obtained for the time like surface embedded in Kerr spacetime from that of the null surface [166]. For the null surface case, the notable difference is that the general 'superrotation generators is time-independent as opposed to the present case. Therefore, this fact leads to difference result for the $[\mathcal{R}(t,x),\mathcal{F}(t,x)]$ commutator. On the null surface, the aforementioned commutator generates supertranslation only, whereas on the time like surface, it generates both 'superrotation and 'supertranslation' depending upon the parameter values of the transformation.

5.4.3 Charges Q on the surface

Associated with the symmetries discussed in the previous section, we now explicitly compute the charges Q defined on the two dimensional $r =$ constant, $t =$ constant surface, which is essentially a two-sphere. The sphere at a particular instant of time is characterized by spacelike and timelike normal vectors defined as M^a and n^a, respectively. For the metric Eq. (5.4.5), the unit spacelike normal vector M^a assumes $M^a = (0,1,0,0)$ and the non-zero components of the unit timelike normal vector n^a are:

$$n^t = -\sqrt{\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}};\quad n^\phi = -\frac{2aM(r_c+\rho\, s_1-\rho^2 s_2)}{\sqrt{\Sigma(\rho,\theta)\Delta(\rho,\theta)\alpha(\rho,\theta)}}\ , \tag{5.4.18}$$

where we have defined the following quantities in GNC,

$$\Sigma(\rho,\theta) = (r_c+\rho\, s_1-\rho^2 s_2)^2 + a^2\cos^2\theta, \tag{5.4.19}$$
$$\Delta(\rho,\theta) = (r_c+\rho\, s_1-\rho^2 s_2)^2 + a^2 - 2M(r_c+\rho\, s_1-\rho^2 s_2), \tag{5.4.20}$$
$$\alpha(\rho,\theta) = ((r_c+\rho\, s_1-\rho^2 s_2)^2+a^2)^2 - a^2\sin^2\theta\Delta(\rho,\theta), \tag{5.4.21}$$

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Following the same procedure as discussed for the Schwarzschild case, the area element $d\Sigma^{ab}$ survives with two non vanishing components: $d\Sigma^{t\rho}$ and $d\Sigma^{\rho\phi}$. With all these ingredients, the diffeomorphism charges can be expressed near the surface $\rho = 0$, as follows (details are presented in Appendix 5.E.6.2),

$$Q[\zeta] = Q[\mathcal{F}] + Q[\mathcal{R}]\,, \tag{5.4.22}$$

The first term is 'supertranslation' charge with the following explicit form,

$$Q[\mathcal{F}] = -\frac{1}{16\pi G}\int d^2x\Big[2M(\sin\theta)\mathcal{F}(t,x)\frac{(a^2+r_c^2)(r_c^4-a^4\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\Big]\,, \tag{5.4.23}$$

the second term is attributed to the charge associated with the 'superrotation',

$$Q[\mathcal{R}] = \frac{1}{16\pi G}\int d^2x\Big[2aM(\sin^3\theta)\mathcal{R}(t,x)\frac{(3r_c^6+a^2r_c^4+4a^2r_c^4\cos^2\theta-a^4(a^2-r_c^2)\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\Big]. \tag{5.4.24}$$

Now we are in a position to inspect the physical interpretation of the 'supertranslation' and 'superrotation' charges. For simplest choice, $F(t,x) = 1$ and $R^A(t,x) = 1$,the results (5.4.23) and (5.4.24) yield the followings form of the charges (detail can be found in Appendix 5.E.6.3),

$$Q[F(t,x)=1] = \frac{M}{2G}, \quad Q[R^A(t,x)=1] = -\frac{Ma}{G}; \tag{5.4.25}$$

which can be identified as the Komar conserved quantities [244–246] defined on the timelike surface at a particular instant of time in the Kerr black hole spacetime. However, with the given diffeomorphism vectors, we have not found any obvious thermodynamic interpretation as has been found for the Schwarzchild case. However, if we consider a slightly more general diffeomorphism vectors $\zeta = \partial_t + \Omega_c\partial_\phi$, where

$$\Omega_c = -\frac{g_{t\phi}}{g_{\phi\phi}}\Big|_{r=r_c} = \frac{2Mar_c}{(r_c^2+a^2)(r_c^2+a^2\cos^2\theta)+2aMr_c\sin^2\theta}\,, \tag{5.4.26}$$

is the the angular velocity on our timelike surface. In that case we have, (detail computation is given in Appendix 5.E.6.4)

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$$
\begin{aligned}
Q_\zeta &= Q[F(t,x)=1] + Q[R^A(t,x)=\Omega_c] \\
&= \frac{a^2-2Mr_c+r_c^2}{2aG}\tan^{-1}(\frac{a}{r_c}) \\
&- \frac{(a^2+r_c^2)(a^2r_c+a^2M+r_c^3-3Mr_c^2)}{2aG\sqrt{r_c}\sqrt{(a^2+r_c^2-2Mr_c)(r_c^3+2Ma^2+a^2r_c)}} \\
&\tan^{-1}[\frac{a\sqrt{a^2-2Mr_c+r_c^2}}{\sqrt{r_c(r_c^3+2Ma^2+a^2r_c)}}].
\end{aligned}
\tag{5.4.27}
$$

To find similar relation obtained in Eq. (5.3.37), following the reference [247], we generalize the Tolman relation for stationary spacetime as

$$
T(r)\sqrt{-g_{ab}\zeta^a\zeta^b} = T_0; \tag{5.4.28}
$$

where ζ^a is the timelike vector, constructed out from the linear combination of two existing Killing vectors $(\partial_t)^a$ and $(\partial_\phi)^a$ of the spacetime. Now, in this case the form of ζ^a is taken to be $\zeta^a = (1,0,0,\Omega_c)$ and we define

$$
|K| = \sqrt{-(g_{tt}+2\Omega_c g_{t\phi}+\Omega_c^2 g_{\phi\phi})}\,. \tag{5.4.29}
$$

Then the Tolman temperature gradient leads to the gravitational acceleration along the normal to the surface as follows,

$$
g = -M^a\frac{\nabla_a T}{T} = -\frac{\partial_r|K|^2}{|K|^2}, \tag{5.4.30}
$$

where M^a is the normal to the $\rho=$ constant surface given by, $(0,1,0,0)$. Associated expression for the charge (5.4.27), in differential form, will then take the identical form as (detail is given in Appendix 5.E.7),

$$
\delta Q_\zeta = -\frac{1}{2\pi}\left(\frac{\delta A_c}{4G}\right)\left(\frac{T_0}{T}\right)\left(\frac{M^a\nabla_a T}{T}\right)\Big|_{\rho=0}\,. \tag{5.4.31}
$$

Area element δA_c is given by $\sqrt{\alpha_c}\sin\theta\, d\theta d\phi$. It is now easy to check that after integration over transverse coordinates the expression (5.4.31) reduces to (5.4.27).

The charges can now be expressed in terms of the mode function of the symmetry generators as,

$$
\begin{aligned}
Q[F(t,x)] &= \sum_k\int_{-\infty}^{\infty} d\omega\, \alpha_k(\omega)\, Q[\bar{F}_k(\omega,t,x)] \\
Q[\mathcal{R}(t,x)] &= \sum_l\int_{-\infty}^{\infty} d\omega\, \bar{\alpha}_l^A(\omega)\, Q_A[\bar{R}^l(\omega,t,x)]\,,
\end{aligned}
\tag{5.4.32}
$$

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where each mode can be written as follows,

$$Q[\bar{F}_k(\omega,t,x)] = -\frac{1}{16\pi G}\int d^2x\Big[2M(\sin\theta)\bar{F}_k(\omega,t,x)\frac{(a^2+r_c^2)(r_c^4-a^4\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\Big],$$

and

$$\begin{aligned}&Q_A[\bar{R}^l(\omega,t,x)]\\&=\frac{1}{16\pi G}\int d^2x\Big[2aM(\sin^3\theta)\bar{R}^l_A(\omega,t,x)\\&\frac{(3r_c^6+a^2r_c^4+4a^2r_c^4\cos^2\theta-a^4(a^2-r_c^2)\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\Big]\end{aligned}$$

. (5.4.33)

Then using Eq. (5.3.28) the algebra between modes of charges are computed as follows (detail is given in Appendix 5.E.8):

$$\begin{aligned}&i[Q[\bar{F}_k(\omega_1,t,x)],Q[\bar{F}_l(\omega_2,t,x)]] = (\omega_2-\omega_1)Q[\bar{F}_{k+l}(\omega_1+\omega_2,t,x)]\,;\\&i[Q[\bar{F}_k(\omega_1,t,x)],Q_A[\bar{R}^l(\omega_2,t,x)]] = -k_A\,Q[\bar{F}_{k+l}(\omega_1+\omega_2,t,x)]\\&+\omega_2\,Q_A[\bar{R}^{k+l}(\omega_1+\omega_2,t,x)];\\&i[Q_A[\bar{R}^k(\omega_1,t,x)],Q_B[\bar{R}^l(\omega_2,t,x)]] = l_A\,Q_B[\bar{R}^{k+l}(\omega_1+\omega_2,t,x)]\\&-k_B\,Q_A[\bar{R}^{k+l}(\omega_1+\omega_2,t,x)].\end{aligned}$$

(5.4.34)

It is clear from the above Eq. (5.4.34) that the symmetry bracket among the charges is isomorphic to that among diffeomorphism vectors. Hence we can conclude that the components of the diffeomorphism symmetry generators together form a closed algebra which is slightly different from the standard near horizon BMS algebra.

5.5 Conclusion

In this chapter, we have studied in detail the symmetries of a timelike hypersurface positioned at any arbitrary radial coordinate embedded in black hole spacetime. We have considered two different black hole spacetime, where symmetries have been identified by considering the class of diffeomorphisms which keep the form of the metric near the timelike surface invariant. In the present analysis, the

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obtained bracket algebra of the asymptotic symmetry group near the timelike surface has similarities with those of the black hole horizon and null infinity in the Schwarzschild background, but for Kerr spacetime, there is a significant difference between the algebra near the timelike surface and those near actual physical boundaries of spacetime. This is because, in Schwarzschild spacetime, the symmetry generator defined on the timelike surface turns out to be dependent only on time, but the bracket algebra of the 'supertranslation' generator is similar to that obtained for a generic null surface. Whereas, for Kerr case 'superrotation' parameter $\mathcal{R}(t,x)$ is time-dependent, therefore notable differences has generated for the $[\mathcal{R}(t,x), \mathcal{F}(t,x)]$ commutator compared to null boundaries. The associated charges have also been computed and shown to follow the same algebra as diffeomorphism transformation vectors. We have also established an interesting connection between the charges with the local heat content on the surface under the study.

The timelike surface divides the space into two complementary regions, which are causally disconnected at any instant of time. Therefore, the symmetries near the surface could play an interesting role in understanding the entanglement phenomena in quantum theory. Usually, all the symmetry analysis have been studied so far in the asymptotic infinity or near the horizon, keeping in mind the causal physical phenomena such as the scattering of massive or massless particles. However, the role of symmetries in entanglement phenomena has not been discussed in the literature. Our present analysis of symmetries near timelike surfaces could be helpful in understanding this phenomenon which will be taken up in future work.

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Appendix

5.A Schwarzschild background

5.A.1 Derivation of (5.3.4) and (5.3.6)

For the metric (5.3.1), we evaluate the r component of the geodesic equation as,

$$\frac{d^2r}{d\rho^2}\Big|_{r=r_c} = [-\Gamma^r{}_{bc}N^bN^c]\Big|_{r=r_c} = [-\Gamma^r{}_{rr}N^rN^r]\Big|_{r=r_c}$$
$$= [-\frac{1}{2}g^{rr}\partial_r g_{rr}N^rN^r]\Big|_{r=r_c} = \Big(\frac{M}{r_c^2}(1-\frac{2M}{r_c})^2\Big)/(1-\frac{2M}{r_c})^2 = \frac{M}{r_c^2}. \quad (5.A.1)$$

which is the result given in (5.3.4). Here N^a has only r component non-zero. Then with the components of Schwarzschild metric as given in (5.3.1), Γ^t_{rr}, Γ^θ_{rr} and Γ^ϕ_{rr} are computed as,

$$\Gamma^t_{rr} = \frac{1}{2}g^{it}(2\partial_r g_{ir} - \partial_i g_{rr}) = 0; \quad \Gamma^\theta_{rr} = \frac{1}{2}g^{i\theta}(2\partial_r g_{ir} - \partial_i g_{rr}) = 0.$$
$$\Gamma^\phi_{rr} = \frac{1}{2}g^{i\phi}(2\partial_r g_{ir} - \partial_i g_{br}) = 0. \quad (5.A.2)$$

Hence we can say that,

$$\frac{d^2t}{d\rho^2}\Big|_{r=r_c} = \frac{d^2\theta}{d\rho^2}\Big|_{r=r_c} = \frac{d^2\phi}{d\rho^2}\Big|_{r=r_c} = 0. \quad (5.A.3)$$

So collecting the results (5.A.1) and (5.A.3), considering the value of N^r given in (5.3.2), from (5.3.3) we get the transformation of coordinates as given in (5.3.6).

5.A.2 Construction of the metric in (5.3.7)

- The component $g_{t't'}$:

  From (5.3.6), we use tensor transformation rule to construct the metric from the original Schwarzschild one (5.3.1) as,

$$g_{t't'} = \frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'}g_{tt} = -(1-\frac{2M}{r})\Big|_{r\to r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}}$$
$$= -(1-\frac{2M}{r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}}) \quad (5.A.4)$$

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Now the last line of (5.A.4) is expanded in Taylor series around $\rho = 0$ upto $\mathcal{O}(\rho)$ as,

$$
\begin{aligned}
g_{t't'} &= -(1 - \frac{2M}{r_c + \rho\sqrt{1-\frac{2M}{r_c}} + \rho^2\frac{M}{2r_c^2}}) \\
&= -(1-\frac{2M}{r_c}) - \rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}}}}{(r_c+\rho\sqrt{1-\frac{2M}{r_c}})^2}\Big)\Big|_{\rho=0}. \qquad (5.A.5)
\end{aligned}
$$

Fom the above expression we get the component g_{tt} given in (5.3.7) upto $\mathcal{O}(\rho)$.

- The component $g_{\rho\rho}$:
From the transformation (5.3.6) we get ,

$$
\begin{aligned}
g_{\rho\rho} &= g_{rr}\frac{\partial r}{\partial\rho}\frac{\partial r}{\partial\rho} = \Big(\sqrt{1-\frac{2M}{r_c}} + \rho\frac{M}{r_c^2}\Big)^2 \big(\frac{1}{1-\frac{2M}{r}}\big)\Big|_{r\to r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}} \\
&= \Big(\sqrt{1-\frac{2M}{r_c}} + \rho\frac{M}{r_c^2}\Big)^2 \frac{1}{\Big(1-\frac{2M}{(r_c+\rho\sqrt{1-2M/r_c}+\rho^2\frac{M}{2r_c^2})}\Big)} \\
&= (1-\frac{2M}{r_c} + 2\rho\sqrt{1-\frac{2M}{r_c}}\frac{M}{r_c^2} + \rho^2\frac{M^2}{r_c^4}\Big)\Big[\frac{1}{1-\frac{2M}{r_c}} \\
&\quad -\rho\Big(\frac{2M(\sqrt{1-\frac{2M}{r_c}}+\frac{M\rho}{r_c^2})}{(r_c-2M+\rho\sqrt{1-\frac{2M}{r_c}}+\frac{M\rho^2}{2r_c^2})^2}\Big)\Big|_{\rho=0}\Big]. \\
&= 1+\rho\Big(\frac{2M}{r_c^2}\frac{\sqrt{1-\frac{2M}{r_c}}}{1-\frac{2M}{r_c}} - \frac{2M(1-\frac{2M}{r_c})\sqrt{1-\frac{2M}{r_c}}}{(r_c-2M)^2}\Big) + \mathcal{O}(\rho^2). \qquad (5.A.6)
\end{aligned}
$$

From the above expression one can easily get $g_{\rho\rho}$ upto $\mathcal{O}(\rho)$ as given in (5.3.7).

- The component $g_{\theta'\theta'}$ and $g_{\phi'\phi'}$:
With the help of the tensor transformation rule and using the aforementioned

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transformation of coordinates (5.3.6) we get,

$$\begin{aligned} g_{\theta'\theta'} &= g_{\theta\theta}\frac{\partial\theta}{\partial\theta'}\frac{\partial\theta}{\partial\theta'} = r^2\Big|_{r\to r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}} \\ &= r_c^2 + 2\rho\, r\Big|_{r\to r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}}. \end{aligned} \tag{5.A.7}$$

$$\begin{aligned} g_{\phi'\phi'} &= g_{\phi\phi}\frac{\partial\phi}{\partial\phi'}\frac{\partial\phi}{\partial\phi'} = r^2\sin^2\theta\Big|_{r\to r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}} \\ &= \sin^2\theta(r_c^2 + 2\rho\;\; r\Big|_{r\to r_c+\rho\sqrt{1-\frac{2M}{r_c}}+\rho^2\frac{M}{2r_c^2}}). \end{aligned} \tag{5.A.8}$$

Hence from (5.A.7) and (5.A.8), we can easily get the components $g_{\theta'\theta'}$ and $g_{\phi'\phi'}$ upto $\mathcal{O}(\rho)$ as given in (5.3.7).

5.A.3 Derivation of diffeomorphism parameters (5.3.10)

The first equation of (5.3.8) implies that

$$\pounds_\zeta g_{\rho\rho} = \zeta^c\partial_c g_{\rho\rho} + 2g_{c\rho}\partial_\rho\zeta^c = 2g_{\rho\rho}\,\partial_\rho\zeta^\rho = 0. \tag{5.A.9}$$

which immediately gives the form of ζ^ρ as given in (5.3.10). Using this in the second condition of (5.3.8), one finds

$$\pounds_\zeta g_{t\rho} = g_{\rho\rho}\partial_t\zeta^\rho + g_{tt}\partial_\rho\zeta^t = 0 \tag{5.A.10}$$

$$\Big[1-\frac{2M}{r_c}+\rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)\Big]\partial_\rho\zeta^t - \partial_t\zeta^\rho = 0$$

$$\Rightarrow \zeta^t = \int d\rho\Big[(1-\frac{2M}{r_c})+\rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)\Big]^{-1}\partial_t T(t,x) + F(t,x). \tag{5.A.11}$$

which upto $\mathcal{O}(\rho)$ leads to ζ^t. Finally, using these components in the last condition of (5.3.8), yields

$$\begin{aligned} &\pounds_\zeta g_{\rho A} = g_{\rho\rho}\partial_A\zeta^\rho + g_{AB}\partial_\rho\zeta^B = 0; \quad \partial_A T(t,x) + g_{AB}\partial_\rho\zeta^B = 0 \\ &or,\, g^{AC}g_{AB}\partial_\rho\zeta^B = -g^{AC}\partial_A T(t,x); \\ &\delta^C_B\zeta^B = -\int d\rho[g^{AC}\partial_A T(t,x)] + R^C(t,x). \end{aligned} \tag{5.A.12}$$

whose solution is the angular component of ζ^A.

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5.A.4 Details of the fall-off conditions (5.3.9)

Putting the components of ζ^a from (5.3.10) in the first condition of (5.3.9), near the surface we get,

$$\begin{aligned}
\pounds_\zeta g_{tt} &= \zeta^c \partial_c g_{tt} + 2 g_{ct} \partial_t \zeta^c = \zeta^\rho \partial_\rho g_{tt} + g_{tt} \partial_t \zeta^t \\
&= -T(t,x)\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big) + \Big[-(1-\frac{2M}{r_c}) \\
&-\rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)\Big]\partial_t[1/(1-2M/r_c)\int \partial_t T(t,x) d\rho + F(t,x)] \\
&= -(1-\frac{2M}{r_c})\partial_t F(t,x) + \mathcal{O}(\rho).
\end{aligned} \tag{5.A.13}$$

For arriving at the last line in (5.A.13), we have imposed the condition that $T = 0$. So the variation of g_{tt} does not give us any new constraints as it is already matching with the assumed fall off condition $\pounds_\zeta g_{tt} = \mathcal{O}(1)$.

Next we concentrate on the variation of g_{tA} as,

$$\begin{aligned}
\pounds_\zeta g_{tA} &= g_{tt}\partial_A \zeta^t + g_{AB}\partial_t \zeta^B \\
&= \Big(-(1-\frac{2M}{r_c}) - \rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)\Big)\partial_A[1/(1-2M/r_c)\int \partial_t T(t,x) d\rho \\
&+F(t,x)] + \Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)\gamma_{AB}\partial_t[-g^{BC}\int \partial_C T\, d\rho + R^B(t,x)] \\
&= -(1-\frac{2M}{r_c})\partial_A F(t,x) + \Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)\gamma_{AB}\partial_t R^B + \mathcal{O}(\rho).
\end{aligned} \tag{5.A.14}$$

The static Schwarzschild space-time has no intrinsic rotation, which implies that $R^A = 0$ and then from (5.A.14), we get directly the constraint condition on F as given in (5.3.11). With that constraint condition and for the form of ζ^a as given in (5.3.12), it is clear from (5.A.14) that Lie variation of g_{tA} automatically vanishes at all order of ρ.

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Next the variation of g_{AB} is given by,

$$
\begin{aligned}
&\pounds_\zeta g_{AB} = g_{AC}\partial_B\zeta^C + g_{BC}\partial_A\zeta^C \\
&= \Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)\gamma_{AC}\partial_B[-g^{BC}\int\partial_C T\,d\rho + R^B] \\
&+\Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)\gamma_{BC}\partial_A[-g^{BC}\int\partial_C T\,d\rho + R^B]. \\
&= \gamma_{AC}\partial_B(-\gamma^{BC}\int\partial_C T\,d\rho) + \gamma_{BC}\partial_A(-\gamma^{AC}\int\partial_C T\,d\rho) \\
&+\Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)(\gamma_{AC}\partial_B R^B + \gamma_{BC}\partial_A R^B) \\
&= r_c^2(\gamma_{AC}\partial_B R^B + \gamma_{BC}\partial_A R^B) + \mathcal{O}(\rho)
\end{aligned}
\tag{5.A.15}
$$

which matches with the fall-off of g_{AB} given in (5.3.9). Like before here also for $R^A = 0$, variation of g_{AB} automatically vanishes.

5.A.5 Derivation of the Charges (5.3.24)

5.A.5.1 Derivation of unit timelike normal vector (5.3.22)

The unit timelike normal to the t = constant hypersurface can be defined by, $n_t = c\partial_t t = c$. Now the norm of n_t gives,

$$
\begin{aligned}
&n^t n_t = g^{tt}n_t n_t = -1 \Rightarrow -c^2\frac{1}{\Big(1-\frac{2M}{r_c}\Big)+\rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)} = -1; \\
&c = \sqrt{\Big(1-\frac{2M}{r_c}\Big)+\rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)} \\
&\Rightarrow n_t = \sqrt{\Big(1-\frac{2M}{r_c}\Big)+\rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)}; \quad \Rightarrow n^t = g^{tt}n_t \\
&= -\frac{1}{\sqrt{\Big(1-\frac{2M}{r_c}\Big)+\rho\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)}}.
\end{aligned}
\tag{5.A.16}
$$

by Taylor expansion of (5.A.16), we get the normal as given in (5.3.22).

5.A.5.2 Derivation of charges (5.3.24)

With the expression of J_{ab} as given in (2.20) and also $d\Sigma^{t\rho}$ from (5.3.21), for the diffeomorphism vector (5.3.12) we calculate the diffeomorphism charges as follows,

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$$
\begin{aligned}
Q[F] &= \frac{1}{16\pi G}\int_{\rho\to 0} d\Sigma^{t\rho} J_{t\rho} = -\frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{\sigma} n^t M^\rho(\partial_t\zeta_\rho - \partial_\rho\zeta_t). \\
&= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)\sin\theta\Big(\frac{1}{\sqrt{1-2M/r_c}} \\
&-\frac{M\rho}{(r_c^2-2Mr_c)}\Big)(g_{\rho\rho}\partial_t\zeta^\rho - \partial_\rho g_{tt}\zeta^t) \\
&= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)\sin\theta\Big(\frac{1}{\sqrt{1-2M/r_c}} \\
&-\frac{M\rho}{(r_c^2-2Mr_c)}\Big)\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)F(t). \\
&= \frac{1}{16\pi G}\int d^2x\frac{r_c^2\sin\theta}{\sqrt{1-\frac{2M}{r_c}}}\Big(\frac{2M\sqrt{1-\frac{2M}{r_c}}}{r_c^2}\Big)F(t). \\
&= \frac{2MF(t)}{16\pi G}\int_{\theta=0}^{\pi}(\sin\theta)d\theta\int_{\phi=0}^{2\pi}d\phi = \frac{2MF(t)}{16\pi G}(4\pi)
\end{aligned}
\tag{5.A.17}
$$

Here σ is the determinant of the induced metric on the r = constant surface as given by,

$$
\sigma = g_{\theta\theta}g_{\phi\phi} = \Big(r_c^2 + 2r_c\rho\sqrt{1-\frac{2M}{r_c}}\Big)\sin^2\theta. \tag{5.A.18}
$$

From (5.A.17), we get the final expression of charge as given in (5.3.24).

5.A.6 Algebra among charges (5.3.29)

Using the definition of bracket given in (5.3.28), we calculate commutator of the supertranslation charges with itself as follows,

$$
\begin{aligned}
&[Q[F_1], Q[F_2]] = \pounds_{F_1} Q[F_2] \\
&= \frac{M}{2G}[F_1, F_2] = \frac{M}{2G}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \alpha(\omega_1,\omega_2)[\bar{F}(\omega_1,t), \bar{F}(\omega_2,t)].
\end{aligned}
\tag{5.A.19}
$$

Also with the help of (5.3.27), we can write that,

$$
[Q[F_1], Q[F_2]] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \alpha(\omega_1)\alpha(\omega_2)[Q[\bar{F}(\omega_1,t)], Q[\bar{F}(\omega_2,t)]]. \tag{5.A.20}
$$

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Using the result as given in (5.3.17), we can express (5.A.19) as follows,

$$[Q[F_1], Q[F_2]] = \frac{M}{2G}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \alpha(\omega_1,\omega_2)[-i(\omega_2-\omega_1)]\bar{F}(\omega_1+\omega_2,t)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_1 d\omega_2 \alpha(\omega_1,\omega_2)[-i(\omega_2-\omega_1)]Q[\bar{F}(\omega_1+\omega_2,t)]. \tag{5.A.21}$$

Now comparing the result in (5.A.20) with (5.A.21), one can compute the bracket among supertranslation charges given in (5.3.29).

5.B General spherically symmetric metric background

5.B.1 Derivation of (5.3.33) and (5.3.32)

5.B.1.1 Finding the transformation of coordinates (5.3.33)

For the metric (5.3.31) the normal vector to the $r =$ constant surface, N_a will be,

$$N_r = c\partial_r r = c; \tag{5.B.1}$$

which satisfies, $N^r N_r = 1$. Using this relation we got,

$$g^{rr}c^2 = 1; \Rightarrow c = \sqrt{f_2(r)}. \Rightarrow N_r = \sqrt{f_2(r)}.$$
$$N^r = g^{rr}N_r = \frac{1}{\sqrt{f_2(r)}}. \tag{5.B.2}$$

Here the other components of N^a are zero for the given metric (5.3.31).

Like before in Schwarzschild case, we evaluate the geodesic equation, for the general spherically symmetric metric (5.3.31) as follows,

$$\frac{d^2 r}{d\rho^2}\Big|_{r=r_c} = [-\Gamma^r{}_{bc}N^b N^c]\Big|_{r=r_c} = [-\Gamma^r{}_{rr}N^r N^r]\Big|_{r=r_c}$$
$$= [-\frac{1}{2}g^{rr}\partial_r g_{rr}N^r N^r]\Big|_{r=r_c} = -\frac{\partial_r f_2(r_c)}{2f_2^2(r_c)}. \tag{5.B.3}$$

Then for the components of the metric as given in (5.3.31), we found that Christoffel symbols Γ^t_{rr}, Γ^θ_{rr} and Γ^ϕ_{rr} are zero. Now following (5.3.3), with the results (5.B.2) and (5.B.3), we construct the transfomration as given in (5.3.33).

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5.B.2 Derivation of the metric in GNC (5.3.32)

- The component $g_{t't'}$:

Having the transformation given in (5.3.33), we use tensor transformation rule to construct the metric in GNC from (5.3.31) as,

$$g_{t't'} = \frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'}g_{tt} = -f_1(r)\Big|_{r=r_c+\rho\sqrt{1/f_2(r_c)}-\frac{\rho^2}{4f_2^2}f_2'(r_c)}$$

which after Taylor expansion around $\rho = 0$ gives,

$$g_{t't'} = -[f_1(r_c) + \rho(\partial_\rho f_1)\Big|_{\rho=0}] = -[f_1(r_c) + \rho\partial_r f_1(\frac{1}{\sqrt{f_2(r_c)}})]. \tag{5.B.4}$$

This is the component $g_{t't'}$ as given in (5.3.32).

- $g_{\rho\rho}$:

From the transformation (5.3.33) we get ,

$$\begin{aligned} g_{\rho\rho} &= g_{rr}\frac{\partial r}{\partial \rho}\frac{\partial r}{\partial \rho} \\ &= \Big(\sqrt{1/f_2(r_c)} - \frac{\rho}{4f_2^2(r_c)}f_2'(r_c)\Big)^2\Big(f_2(r)\Big|_{r=r_c+\rho\sqrt{1/f_2(r_c)}-\frac{\rho^2}{4f_2^2}f_2'(r_c)}\Big) \\ &= \Big(\frac{1}{f_2(r_c)} - \frac{\rho}{f_2^2(r_c)\sqrt{f_2(r_c)}}f_2'(r_c)\Big)\Big(f_2(r_c) + \rho f_2'(r_c)\frac{1}{\sqrt{f_2(r_c)}}\Big) + \mathcal{O}(\rho^2) \\ &= 1 + \rho\Big(\frac{f_2'(r_c)}{f_2(r_c)\sqrt{f_2(r_c)}} - \frac{f_2'(r_c)}{f_2(r_c)\sqrt{f_2(r_c)}}\Big) + \mathcal{O}(\rho^2). \end{aligned} \tag{5.B.5}$$

The above expression gives the component $g_{\rho\rho}$ in (5.3.32).

- The component $g_{\theta'\theta'}$ and $g_{\phi'\phi'}$:

With the help of the tensor transformation rule and using the aforementioned transformation of coordinates (5.3.33) we get,

$$\begin{aligned} g_{\theta'\theta'} &= g_{\theta\theta}\frac{\partial\theta}{\partial\theta'}\frac{\partial\theta}{\partial\theta'} = r^2\Big|_{r=r_c+\rho\sqrt{1/f_2(r_c)}-\frac{\rho^2}{4f_2^2}f_2'(r_c)} \\ &= r_c^2 + 2\rho\, r_c\frac{1}{\sqrt{f_2(r_c)}}. \end{aligned} \tag{5.B.6}$$

$$\begin{aligned} g_{\phi'\phi'} &= g_{\phi\phi}\frac{\partial\phi}{\partial\phi'}\frac{\partial\phi}{\partial\phi'} = r^2\sin^2\theta\Big|_{r=r_c+\rho\sqrt{1/f_2(r_c)}-\frac{\rho^2}{4f_2^2}f_2'(r_c)} \\ &= \sin^2\theta(r_c^2 + 2\rho\, r_c\frac{1}{\sqrt{f_2(r_c)}}) \end{aligned} \tag{5.B.7}$$

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Hence (5.B.6) and (5.B.7) give the components $g_{\theta'\theta'}$ and $g_{\phi'\phi'}$ upto $\mathcal{O}(\rho)$ as given in (5.3.31).

5.B.3 Derivation of charges (5.3.34)

Like before as we did in Schwarzschild case, The normal to the t = constant hypersurface can be defined by, $n_t = c\partial_t t = c$.
Now the norm of n_t gives,

$$
\begin{aligned}
n^t n_t = g^{tt} n_t n_t = -1 &\Rightarrow -c^2 \frac{1}{[f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho]} = -1; \\
c &= \sqrt{[f_1(r_c) + \rho\, f_1'(r_c)\sqrt{1/f_2(r_c)}]} \\
n_t &= \sqrt{[f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho]}. \Rightarrow n^t = g^{tt} n_t; \\
n^t &= \frac{1}{\sqrt{[f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho]}}.
\end{aligned}
\tag{5.B.8}
$$

Here other components of n^a are zero. Also unit spacelike normal to ρ =constant surface will be $M^\rho = (0,1,0,0)$. Then for the symmetry vector components (5.3.12) we calculate charges as,

$$
\begin{aligned}
Q[F] &= \frac{1}{16\pi G}\int_{\rho\to 0} d\Sigma^{t\rho} J_{t\rho} = -\frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{\sigma} n^t M^\rho (\partial_t \zeta_\rho - \partial_\rho \zeta_t). \\
&= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x [r_c^2 + 2r_c\sqrt{1/f_2(r_c)}\,\rho](\sin\theta)\left(\frac{-\partial_\rho g_{tt} F(t)}{\sqrt{[f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho]}}\right) \\
&= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x [r_c^2 + 2r_c\sqrt{1/f_2(r_c)}\,\rho](\sin\theta)\frac{f_1'(r_c) F(t)}{\sqrt{f_2[f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho]}} \\
&= \frac{1}{16\pi G}\int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi \frac{r_c^2 F(t)\, f_1'(r_c)}{\sqrt{f_1(r_c) f_2(r_c)}}. = \frac{4\pi r_c^2}{16\pi G} F(t) \frac{f_1'}{\sqrt{f_1 f_2}}.
\end{aligned}
\tag{5.B.9}
$$

From the above expression we get the charges given in (5.3.34) with $A_c = 4\pi r_c^2$.

5.B.4 Local acceleration of the observer at $r = r_c$

The effective metric, from (5.3.32), for an observer moving along ρ in the vicinity of $r = r_c$ surface is given by

$$
ds^2 = -[f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho]\, dt^2 + d\rho^2. \tag{5.B.10}
$$

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The above one is in Rindler form and more conveniently can be expressed by the transformation $f_1(r_c) + f_1'(r_c)\sqrt{1/f_2(r_c)}\,\rho = x$. Then (5.B.10) reduces to,

$$ds^2 = -x dt^2 + \frac{f_2(r_c)}{f_1'^2(r_c)} dx^2 \,. \tag{5.B.11}$$

The metric (5.B.11) can be expressed in Minkowski form $ds^2 = -dT^2 + dX^2$, by having following transformation of coordinates,

$$X = \frac{\sqrt{x}}{a}\cosh(at)\,, \quad T = \frac{\sqrt{x}}{a}\sinh(at)\,, \tag{5.B.12}$$

where

$$a(x) = \frac{f_1'(r_c)}{2\sqrt{f_2(r_c)x}}\,. \tag{5.B.13}$$

The coordinates (t, x), adapted to the uniformly accelerated motion, are known as Rindler coordinates. Below we will show that a is identified to be the local acceleration of an observer in the Rindler frame.

To show this let us first calculate the proper acceleration this observer. For that we consider any x = constant trajectory and then for this value of x the coordinate time t is identified as the proper time τ. Then the magnitude of the proper acceleration, defined as $a_{prop} = \sqrt{a^i_{prop} a_{i_{prop}}}$ with $a^i_{prop} = dX^a/d\tau$, from (5.B.12) is obtained as,

$$a_{prop} = a(x)\sqrt{x}|_{x=const}\,. \tag{5.B.14}$$

Therefore the local acceleration is given by $a(x) = a_{prop}/\sqrt{x}$. At $r = r_c$, from (5.B.13), this is given by

$$g(r_c) = a(x = x_c) = \frac{a_{prop}}{\sqrt{x_c}} = \frac{f_1'(r_c)}{2\sqrt{f_2(r_c)f_1(r_c)}}. \tag{5.B.15}$$

5.C Tolman temperature and charges: derivation of (5.3.37)

From the Tolman relation (5.3.36) one can get the following,

$$T^2(r)(f_1(r)) = T_0 \Rightarrow 2T\partial_r T f_1 + T^2\partial_r f_1 = 0. \rightarrow \frac{\partial_r T(r)}{T(r)} = -\frac{\partial_r f_1(r)}{2f_1(r)}. \tag{5.C.1}$$

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At $r = r_c$, (5.C.1) will be,

$$\frac{\partial_r T(r)}{T(r)}\Big|_{r=r_c} = -\frac{\partial_r f_1(r)}{2f_1(r)}\Big|_{r=r_c}; \;\Rightarrow f_1'(r_c) = -2f_1(r_c)\frac{\partial_r T(r_c)}{T(r_c)}. \tag{5.C.2}$$

Now as shown in (5.B.2), the unit normal to the $r =$ constant hypersurface is given by,

$$\hat{n}^r = \frac{1}{\sqrt{f_2}}; \quad \hat{n}^t = \hat{n}^\theta = \hat{n}^\phi = 0. \tag{5.C.3}$$

So using (5.C.2) and (5.C.3), also for $F(t) = 1$, (5.3.34) can be expressed as,

$$\begin{aligned} Q = \;& \frac{A_c}{16\pi G}\Big(-2f_1(r_c)\frac{\partial_r T(r_c)}{T(r_c)}\Big)\hat{n}^r\Big|_{r=r_c}\Big(\frac{1}{\sqrt{f_1(r_c)}}\Big) \\ &= \frac{-A_c}{8\pi G}\sqrt{f_1(r_c)}\,\Big(\frac{\hat{n}^r\partial_r T(r)}{T(r)}\Big)\Big|_{r=r_c} \\ &= \frac{-A_c}{8\pi G}\Big(\frac{T_0}{T}\;\frac{\hat{n}^a\partial_a T(r)}{T(r)}\Big)\Big|_{r=r_c}; \end{aligned} \tag{5.C.4}$$

which directly boils down to (5.3.37).

5.D Results in Minkowski background

Considering the metric (5.3.38) in Minkowski background, we get the following transformation of coordinates ,

$$t = t'; \quad r = r_c + \rho; \quad \theta = \theta'; \quad \phi = \phi'. \tag{5.D.1}$$

In flat background all Γ's are zero. Also unit normal to the $r =$ constant surface will be, $N^a = (0, 1, 0, 0)$. Now following the tensor transformation rule as presented in (5.3.7), the components of the metric are derived as,

$$\begin{aligned} g_{t't'} &= \frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'}g_{tt} = g_{tt} = -1. \\ g_{\rho\rho} &= g_{rr}\frac{\partial r}{\partial \rho}\frac{\partial r}{\partial \rho} = 1. \\ g_{\theta'\theta'} &= g_{\theta\theta}\frac{\partial \theta}{\partial \theta'}\frac{\partial \theta}{\partial \theta'} = r^2\Big|_{r=r_c+\rho} = (r_c^2 + 2r_c\rho). \\ g_{\phi'\phi'} &= g_{\phi\phi}\frac{\partial \phi}{\partial \phi'}\frac{\partial \phi}{\partial \phi'} = r^2\sin^2\theta\Big|_{r=r_c+\rho} = (r_c^2 + 2r_c\rho)\sin^2\theta. \end{aligned} \tag{5.D.2}$$

With all these components, we get the metric transformed in the new coordinates as given in (5.3.39).

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Now keeping the gauge conditions as before like static Schwarzschild case, from (5.A.9) we got $\zeta^\rho = T(t,x)$. Then from (5.A.10) we got,

$$\partial_t \zeta^\rho - \partial_\rho \zeta^t = 0 \Rightarrow \zeta^t = \int \partial_t T d\rho + F(t,x). \tag{5.D.3}$$

Similarly following (5.A.12) , the component ζ^A will be,

$$\zeta^B = -\int d\rho [g^{BC} \partial_A T(t,x)] + R^B(t,x). \tag{5.D.4}$$

with g_{AB} is the transverse component of the flat metric found in (5.3.39).

Here the fall-off conditions (5.3.9) also remain same as in Schwarzschild case. The variation of g_{tt} is,

$$\pounds_\zeta g_{tt} = -\partial_t [\int \partial_t T(t,x) d\rho + F(t,x)] = -\partial_t F(t,x) + \mathcal{O}(\rho). \tag{5.D.5}$$

Like before there will be no transformation along radial coordinate which says that $T = 0$. The variation of g_{tt} does not give us any new constraints as it is already matching with assumed fall off condition of this metric component.

Next we concentrate on the variation of g_{tA} as,

$$\begin{aligned}\pounds_\zeta g_{tA} &= -\partial_A [\int \partial_t T(t,x) d\rho + F(t,x)] + \left(r_c^2 + 2r_c\rho\right) \gamma_{AB} \partial_t [-g^{BC} \int \partial_C T \, d\rho \\ &+ R^B(t,x)] = -\partial_A F(t,x) + \left(r_c^2 + 2r_c\rho\right) \gamma_{AB} \partial_t R^B + \mathcal{O}(\rho).\end{aligned} \tag{5.D.6}$$

Where γ_{AB} is the metric on the two-sphere which is given by, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. For flat spacetime there is no rotation, so $R^A = 0$. Then from (5.D.6), we get directly the constraint condition on F as given in (5.3.11).

Next the variation of g_{AB} is given by,

$$\begin{aligned}\pounds_\zeta g_{AB} &= g_{AC} \partial_B \zeta^C + g_{BC} \partial_A \zeta^C \\ &= \left(r_c^2 + 2r_c\rho\right) \gamma_{AC} \partial_B [-g^{BC} \int \partial_C T \, d\rho + R^B] \\ &+ \left(r_c^2 + 2r_c\rho\right) \gamma_{BC} \partial_A [-g^{BC} \int \partial_C T \, d\rho + R^B]. \\ &= \gamma_{AC} \partial_B (-\gamma^{BC} \int \partial_C T \, d\rho) + \gamma_{BC} \partial_A (-\gamma^{AC} \int \partial_C T \, d\rho) \\ &+ \left(r_c^2 + 2r_c\rho\right) (\gamma_{AC} \partial_B R^B + \gamma_{BC} \partial_A R^B);\end{aligned} \tag{5.D.7}$$

Here like before, considering $R^A = 0$ and also $T = 0$, very near to the surface at $\rho = 0$, variation of g_{AB} automatically vanishes.

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At last we calculate diffeomorphism charges like before. The unit normal to $\rho =$ constant surface is $M^a = (0,1,0,0)$ and the normal on the $t =$ constant surface is given by $n^a = (-1,0,0,0)$. For the symmetry parameters (5.3.12), charges can be calculated as,

$$\begin{aligned} Q[F] &= \frac{1}{16\pi G}\int_{\rho\to 0} d\Sigma^{t\rho} J_{t\rho} = -\frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{\sigma}n^t M^\rho(\partial_t\zeta_\rho - \partial_\rho\zeta_t). \\ &= -\frac{1}{16\pi G}\int_{\rho\to 0} d^2x\left(r_c^2 + 2r_c\rho\right)\sin\theta(\partial_\rho g_{tt}\zeta^t) = 0. \end{aligned} \tag{5.D.8}$$

In the expression (5.D.8), derivative of g_{tt} is zero for the flat metric (5.3.39). Then we found that the charges vanishes in Minkowski spacetime.

5.E Kerr background

5.E.1 Derivation of (5.4.3) and (5.4.4)

For the metric (5.4.1), the unit normal to the $r =$ constant surface is given by

$$N_r = c\partial_r r = c. \tag{5.E.1}$$

Using $N^a N_a = 1$, we get,

$$\begin{aligned} g^{rr}c^2 &= 1 \Rightarrow c = \frac{1}{\sqrt{g^{rr}}}; \quad N_r = \frac{1}{\sqrt{g^{rr}}}. \\ N^r &= g^{rr}N_r = \sqrt{g^{rr}} = \sqrt{\frac{(r^2+a^2-2Mr)}{(r^2+a^2\cos^2\theta)}}\,. \end{aligned} \tag{5.E.2}$$

Other components of N^a are zero. Then we evaluate the geodesic equation to get the expression of s_2 as given in (5.4.4). For the metric (5.4.1), the r component of the geodesic equation will be,

$$\begin{aligned} \frac{d^2r}{d\rho^2}\Big|_{r=r_c} &= [-\Gamma^r{}_{bc}N^bN^c]\Big|_{r=r_c} = [-\Gamma^r{}_{rr}N^rN^r]\Big|_{r=r_c} \\ &= \Big[-\frac{1}{2}g^{rr}\partial_r g_{rr}N^rN^r\Big]\Big|_{r=r_c} = [-\frac{1}{2}(\frac{r^2+a^2-2Mr}{r^2+a^2\cos^2\theta})^2\,\partial_r(\frac{r^2+a^2\cos^2\theta}{r^2+a^2-2Mr})]\Big|_{r=r_c} \\ &= -\Big(\frac{r_c(r_c^2+a^2-2Mr_c)-(r_c-M)(r_c^2+a^2\cos^2\theta)}{(r_c^2+a^2\cos^2\theta)^2}\Big) = -s_2. \end{aligned} \tag{5.E.3}$$

Here s_2 is same as defined in (5.4.4). Now with the help of (5.A.2), the components Γ^t_{rr}, Γ^θ_{rr} and Γ^ϕ_{rr} are evaluated to be zero for the metric (5.4.1). Thus here we can say that,

$$\frac{d^2t}{d\rho^2}\Big|_{r=r_c} = \frac{d^2\theta}{d\rho^2}\Big|_{r=r_c} = \frac{d^2\phi}{d\rho^2}\Big|_{r=r_c} = 0. \tag{5.E.4}$$

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So collecting the results (5.E.3) and (5.E.4), for the value of N^r given in (5.E.2), from (5.3.3) we get the transformation as given in (5.4.3).

5.E.2 Construction of the Kerr metric in GNC (5.4.5)

- The component $g_{t't'}$:

  Having the transformation given in (5.4.3), we use the tensor transformation rule to construct the metric in GNC from the form give in Boyer-Lindquist coordinate in (5.4.1) as follows,

$$g_{t't'} = \frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'}g_{tt} = -\Big(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\Big)\Big|_{r\to r_c+\rho\, s_1(\theta)-\rho^2 s_2(\theta)}$$
$$= \frac{(r_c+\rho\, s_1-\rho^2 s_2)^2 + a^2 - 2M(r_c+\rho\, s_1-\rho^2 s_2) - a^2\cos^2\theta}{(r_c+\rho\, s_1-\rho^2 s_2)^2 + a^2\cos^2\theta}. \tag{5.E.5}$$

  Next we have the Taylor expansion of $g_{t't'}$ around $\rho = 0$ upto $\mathcal{O}(\rho)$ and thus we obtain,

$$g_{t't'} = -\Big(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\Big)\Big|_{\rho=0} - \rho\,\partial_\rho\Big(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\Big)\Big|_{\rho=0}$$
$$= -\Big(\frac{r_c^2 + a^2 - 2Mr_c - a^2\sin^2\theta}{r_c^2 + a^2\cos^2\theta}\Big) - \rho\,\Big(\frac{\partial r}{\partial\rho}\Big)\,\partial_r\Big(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\Big)\Big|_{\rho=0}$$
$$= -\Big(\frac{r_c^2 + a^2 - 2Mr_c - a^2\sin^2\theta}{r_c^2 + a^2\cos^2\theta}\Big) - \rho\, s_1\frac{2\Sigma(r-M) - 2r(\Delta - a^2\sin^2\theta)}{(r^2 + a^2\cos^2\theta)^2}\Big|_{\rho=0}$$
$$= -\Big(\frac{r_c^2 + a^2 - 2Mr_c - a^2\sin^2\theta}{r_c^2 + a^2\cos^2\theta}\Big) - 2\rho\, s_1\Big(\frac{Mr^2 - a^2M\cos^2\theta}{(r^2 + a^2\cos^2\theta)^2}\Big)\Big|_{\rho=0}. \tag{5.E.6}$$

  From (5.E.6) we get directly the transformed component as given in (5.4.5).

- The component $g_{\rho\rho}$:
  From the transformation (5.4.3) we get ,

$$g_{\rho\rho} = g_{rr}\frac{\partial r}{\partial\rho}\frac{\partial r}{\partial\rho} = (s_1 - 2\rho s_2)^2\Big(\frac{\Sigma}{\Delta}\Big)\Big|_{r\to r_c+\rho\, s_1(\theta)-\rho^2 s_2(\theta)}$$

  Here also we have the Taylor expansion of $g_{\rho\rho}$ around $\rho = 0$ upto $\mathcal{O}(\rho)$ and then we got,

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$$g_{\rho\rho} = \Big[\sqrt{\frac{(r_c^2+a^2-2Mr_c)}{(r_c^2+a^2\cos^2\theta)}} - \rho\Big(\frac{a^2r_c - Mr_c^2 + (M-r_c)a^2\cos^2\theta}{(r_c^2+a^2\cos^2\theta)^2}\Big)\Big]^2$$
$$\times\Big[\frac{r_c^2+a^2\cos^2\theta}{r_c^2+a^2-2Mr_c} + \rho\sqrt{\frac{(r_c^2+a^2-2Mr_c)}{(r_c^2+a^2\cos^2\theta)}}$$
$$\frac{2r(r^2+a^2-2Mr)-2(r-M)(r^2+a^2\cos^2\theta)}{(r^2+a^2-2Mr)^2}\Big|_{\rho=0}\Big] + \mathcal{O}(\rho^2)$$

$$= \Big[\frac{(r_c^2+a^2-2Mr_c)}{(r_c^2+a^2\cos^2\theta)} - 2\rho\sqrt{\frac{(r_c^2+a^2-2Mr_c)}{(r_c^2+a^2\cos^2\theta)}}$$
$$\times\Big(\frac{a^2r_c - Mr_c^2 + (M-r_c)a^2\cos^2\theta}{2(r_c^2+a^2\cos^2\theta)^2}\Big)\Big]\Big[\frac{r_c^2+a^2\cos^2\theta}{r_c^2+a^2-2Mr_c}$$
$$+2\rho\sqrt{\frac{(r_c^2+a^2-2Mr_c)}{(r_c^2+a^2\cos^2\theta)}}\Big(\frac{a^2r_c\sin^2\theta + Ma^2\cos^2\theta - Mr_c^2}{(r_c^2+a^2-2Mr_c)^2}\Big)\Big] + \mathcal{O}(\rho^2)$$

$$= 1 - 2\rho\Big[\frac{1}{\sqrt{r_c^2+a^2-2Mr_c}}\Big(\frac{a^2r_c - Mr_c^2 + (M-r_c)a^2\cos^2\theta}{(r_c^2+a^2\cos^2\theta)^{3/2}}\Big)$$
$$-\frac{1}{(r_c^2+a^2\cos^2\theta)^{3/2}}\Big(\frac{a^2r_c\sin^2\theta + Ma^2\cos^2\theta - Mr_c^2}{\sqrt{(r_c^2+a^2-2Mr_c)}}\Big)\Big] + \mathcal{O}(\rho^2). \tag{5.E.7}$$

From the above expression one can easily get $g_{\rho\rho}$ as given in (5.4.5).

- The component $g_{t'\phi'}$:

  Using the transformation (5.4.3), we calculate,

$$g_{t'\phi'} = g_{t\phi}\frac{\partial t}{\partial t'}\frac{\partial\phi}{\partial\phi'} = -\frac{4aMr\sin^2\theta}{\Sigma}\Big|_{r\to r_c+\rho\, s_1(\theta)-\rho^2 s_2(\theta)}.$$
$$= -\frac{4aM(r_c+\rho\, s_1-\rho^2 s_2)\sin^2\theta}{(r_c+\rho\, s_1-\rho^2 s_2)^2+a^2\cos^2\theta} \tag{5.E.8}$$

After the Taylor expansion of $g_{t'\phi'}$ around $\rho = 0$ upto $\mathcal{O}(\rho)$, we got,

$$g_{t'\phi'} = -\frac{4aMr_c\sin^2\theta}{(r_c^2+a^2\cos^2\theta)} - \rho\,\Big(\frac{\partial r}{\partial\rho}\Big)\Big[\partial_r\Big(\frac{4aMr\sin^2\theta}{(r^2+a^2\cos^2\theta)}\Big)\Big]\Big|_{\rho=0}.$$
$$= -\frac{4aMr_c\sin^2\theta}{(r_c^2+a^2\cos^2\theta)} - \rho\, s_1(\theta)$$
$$\times\Big(\frac{4aM\sin^2\theta(r_c^2+a^2\cos^2\theta) - 8aMr_c^2\sin^2\theta}{(r_c^2+a^2\cos^2\theta)^2}\Big). \tag{5.E.9}$$

The above result directly boils down to the expression of $g_{t'\phi'}$ as given in (5.4.5).

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- The component $g_{\theta'\theta'}$ and $g_{\phi'\phi'}$:
  With the help of the tensor transformation rule and using the aforementioned transformation of coordinates (5.4.3), we get,

$$\begin{aligned} g_{\theta'\theta'} &= g_{\theta\theta}\frac{\partial\theta}{\partial\theta'}\frac{\partial\theta}{\partial\theta'} = \Sigma(r,\theta)\Big|_{r\to r_c+\rho\, s_1(\theta)-\rho^2 s_2(\theta)} \\ &= (r_c+\rho\, s_1-\rho^2 s_2)^2 + a^2\cos^2\theta. \end{aligned} \tag{5.E.10}$$

Like before after Taylor expansion around $\rho = 0$, it gives,

$$g_{\theta'\theta'} = (r_c^2 + a^2\cos^2\theta) + 2\rho\, r_c s_1(\theta). \tag{5.E.11}$$

Next the component $g_{\phi'\phi'}$ transforms as,

$$\begin{aligned} g_{\phi'\phi'} &= g_{\phi\phi}\frac{\partial\phi}{\partial\phi'}\frac{\partial\phi}{\partial\phi'} = \frac{\sin^2\theta[(r^2+a^2)^2 - a^2\sin^2\theta\Delta]}{\Sigma}\Big|_{r\to r_c+\rho\, s_1(\theta)-\rho^2 s_2(\theta)}. \\ &= \frac{\sin^2\theta[((r_c+\rho\, s_1-\rho^2 s_2)^2+a^2)^2 - a^2\sin^2\theta\Delta(\rho,\theta)]}{(r_c+\rho\, s_1-\rho^2 s_2)^2 + a^2\cos^2\theta} \end{aligned} \tag{5.E.12}$$

where $\Delta(\rho,\theta)$ is given by (5.4.20). Then we have the Taylor expansion about $\rho = 0$ and found out ,

$$\begin{aligned} g_{\phi'\phi'} &= \frac{\sin^2\theta[(r_c^2+a^2)^2 - a^2\sin^2\theta\Delta_{r_c}]}{\Sigma_{r_c}} + \rho\, s_1(\theta)\sin^2\theta \\ &\quad \left(\frac{[4r(r^2+a^2) - 2a^2(r-M)\sin^2\theta]}{(r^2+a^2\cos^2\theta)} - \frac{2r[(r^2+a^2)^2 - a^2\sin^2\theta\Delta]}{(r^2+a^2\cos^2\theta)^2}\right)\Big|_{\rho=0}. \\ &= \frac{\sin^2\theta[(r_c^2+a^2)^2 - a^2\sin^2\theta\Delta_{r_c}]}{\Sigma_{r_c}} + 2\rho\, s_1(\theta)\sin^2\theta \\ &\quad \times\left(\frac{[2r_c(r_c^2+a^2) - a^2(r_c-M)\sin^2\theta]}{(r_c^2+a^2\cos^2\theta)} - \frac{r_c[(r_c^2+a^2)^2 - a^2\sin^2\theta\Delta_{r_c}]}{(r_c^2+a^2\cos^2\theta)^2}\right). \end{aligned} \tag{5.E.13}$$

Hence from (5.E.11) and (5.E.13), we easily get the components $g_{\theta'\theta'}$ and $g_{\phi'\phi'}$ upto $\mathcal{O}(\rho)$ as given in (5.4.5).

5.E.3 Derivation of diffeomorphism parameters (5.4.9)

With the components of the newly transformed metric given in (5.4.5), the first equation of (5.4.7) imples that

$$\pounds_\zeta g_{\rho\rho} = \zeta^c\partial_c g_{\rho\rho} + 2g_{c\rho}\partial_\rho\zeta^c = 2g_{\rho\rho}\,\partial_\rho\zeta^\rho = 0; \tag{5.E.14}$$

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which immediately implies the form of ζ^ρ given in (5.4.9). Using this in the second and the third condition of (5.4.7) respectively one finds following two relations,

$$\pounds_\zeta g_{t\rho} = g_{\rho\rho}\partial_t\zeta^\rho + g_{t\phi}\partial_\rho\zeta^\phi + g_{tt}\partial_\rho\zeta^t = 0 \tag{5.E.15}$$

$$\pounds_\zeta g_{\rho A} = g_{tA}\partial_\rho\zeta^t + g_{AB}\partial_\rho\zeta^B + \partial_A T = 0. \tag{5.E.16}$$

Now the equation (5.E.16) can be expressed in component form as follows,

$$g_{\theta\theta}\partial_\rho\zeta^\theta + \partial_\theta T = 0 \tag{5.E.17}$$
$$g_{t\phi}\partial_\rho\zeta^t + g_{\phi\phi}\partial_\rho\zeta^\phi + \partial_\phi T = 0. \tag{5.E.18}$$

Now (5.E.17) gives directly the form of ζ^θ as,

$$\zeta^\theta = \int \frac{\partial_\theta T}{g_{\theta\theta}} d\rho + R^\theta(t,\theta,\phi). \tag{5.E.19}$$

Then with the solution of ζ^ρ, the equation (5.E.15) can be written as,

$$\partial_\rho\zeta^t = -\frac{1}{g_{tt}}(\partial_t T + g_{t\phi}\partial_\rho\zeta^\phi). \tag{5.E.20}$$

Substituting (5.E.20) in (5.E.18) we get,

$$-\frac{(g_{t\phi})^2\partial_\rho\zeta^\phi}{g_{tt}} + g_{\phi\phi}\partial_\rho\zeta^\phi = -\partial_\phi T + \frac{g_{t\phi}}{g_{tt}}\partial_t T; \tag{5.E.21}$$
$$\Rightarrow \zeta^\phi = \frac{1}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}\Big(g_{t\phi}\int \partial_t T d\rho - g_{tt}\int \partial_\phi T d\rho\Big) + R^\phi(t,\theta,\phi). \tag{5.E.22}$$

Next putting the expression of $\partial_\rho\zeta^\phi$ from (5.E.21) in (5.E.20), we get the form of ζ^t as follows,

$$\zeta^t = -\int \frac{\partial_t T}{g_{tt}} d\rho - \int \frac{g_{t\phi}}{g_{tt}}\Big(\frac{g_{tt}\partial_\phi T + g_{t\phi}\partial_t T}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}\Big) d\rho + F(t,\theta,\phi). \tag{5.E.23}$$

Here we know that the upper components are,

$$g^{t\phi} = \frac{g_{t\phi}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}; \quad g^{\phi\phi} = \frac{g_{tt}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}. \tag{5.E.24}$$

Hence (5.E.23) , (5.E.19) and (5.E.22) can be written in the index notation as given in (5.4.9), whereas the upper components of the metric are given by (5.E.24).

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5.E.4 Details of the fall-off condition (5.4.8)

Putting the components of ζ^a from (5.4.9) in the first condition of (5.4.8), we get,

$$
\begin{aligned}
&\pounds_\zeta g_{tt} = \zeta^c \partial_c g_{tt} + 2 g_{ct}\partial_t \zeta^c = \zeta^A \partial_A g_{tt} + 2 g_{tt}\partial_t \zeta^t + g_{tA}\partial_t \zeta^A \\
&= [\int \frac{\partial_\theta T}{g_{\theta\theta}} d\rho + R^\theta(t,\theta,\phi)]\partial_\theta g_{tt} \\
&+ g_{tt}\partial_t\Big(-\int \frac{\partial_t T}{g_{tt}} d\rho - \int \frac{g^{t\phi}}{g_{tt}}\Big(g_{tt}\partial_\phi T + g_{t\phi}\partial_t T\Big) d\rho + F(t,\theta,\phi)\Big) \\
&- g_{t\phi}\partial_t\Big(g^{t\phi}\int \partial_t T d\rho - g^{AB}\int \partial_\phi T d\rho + R^\phi(t,\theta,\phi)\Big).
\end{aligned}
\tag{5.E.25}
$$

Here the metric components are functions of ρ and θ as given in (5.4.5). Hence all the upper components of the metric which are present in (5.E.25), can have series expansion in the order of ρ about $\rho = 0$. Then the leading order terms in the expression (5.E.25) will be $\mathcal{O}(1)$ followed by the terms of $\mathcal{O}(\rho)$. So the variation of g_{tt} does not give us any new constraints as it is already matching with the assumed fall off condition given in (5.4.8).

Similarly next we evaluate the variation of g_{tA} as,

$$
\begin{aligned}
&\pounds_\zeta g_{tA} = \zeta^\rho \partial_\rho g_{tA} + \zeta^B \partial_B g_{tA} + g_{tt}\partial_A \zeta^t + g_{At}\partial_t \zeta^t + g_{tB}\partial_A \zeta^B + g_{AB}\partial_t \zeta^B \\
&= T\,\partial_\rho g_{tA} + [-g^{tB}\int \partial_t T d\rho - g^{CB}\int \partial_C T d\rho + R^B(t,x)]\partial_B g_{tA} \\
&+ g_{tt}\partial_A[-\int \frac{\partial_t T}{g_{tt}} d\rho + \int \frac{g^{tA}}{g_{tt}}[g_{tt}\partial_A T + g_{tA}\partial_t T)] d\rho + F(t,x)] \\
&+ g_{At}[-\int \frac{\partial_t^2 T}{g_{tt}} d\rho + \int \frac{g^{tA}}{g_{tt}}[g_{tt}\,\partial_t \partial_A T + g_{tA}\partial_t^2 T)] d\rho + \partial_t F(t,x)] \\
&+ g_{tB}\partial_A[-g^{tB}\int \partial_t T d\rho - g^{CB}\int \partial_C T d\rho + R^B(t,x)] + g_{AB}[-g^{tB}\int \partial_t^2 T d\rho \\
&- g^{CB}\int \partial_t \partial_C T d\rho + \partial_t R^B(t,x)].
\end{aligned}
\tag{5.E.26}
$$

And the variation of g_{AB} will be,

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$$
\begin{aligned}
&\pounds_\zeta g_{AB} = \zeta^\rho\partial_\rho g_{AB} + \zeta^C\partial_C g_{AB} + g_{At}\partial_B\zeta^t + g_{Bt}\partial_A\zeta^t + g_{AC}\partial_B\zeta^C + g_{BC}\partial_A\zeta^C.\\
&= T\,\partial_\rho g_{AB} + [-g^{tC}\int\partial_t T d\rho - g^{CD}\int\partial_D T d\rho + R^C(t,x)]\partial_C g_{AB}\\
&+g_{At}\partial_B[-\int\frac{\partial_t T}{g_{tt}}d\rho + \int\frac{g^{tC}}{g_{tt}}[g_{tt}\partial_C T + g_{tC}\partial_t T]d\rho + F(t,x)]\\
&+g_{Bt}\partial_A[-\int\frac{\partial_t T}{g_{tt}}d\rho + \int\frac{g^{tC}}{g_{tt}}[g_{tt}\partial_C T + g_{tC}\partial_t T]d\rho + F(t,x)]\\
&+g_{AC}\partial_B[-g^{tC}\int\partial_t T d\rho - g^{DC}\int\partial_D T d\rho + R^C(t,x)] + g_{BC}\partial_A[-g^{tC}\int\partial_t T d\rho\\
&-g^{CD}\int\partial_D T d\rho + R^C(t,x)].
\end{aligned}
\tag{5.E.27}
$$

It is clear from (5.E.26) and (5.E.27) that the leading order terms in the variation of g_{tA} and g_{AB} are $\mathcal{O}(1)$ followed by the terms of $\mathcal{O}(\rho)$. So these variations does not give us any new constraints as it is already matching with assumed fall off conditions given in (5.4.8).

5.E.5 Details of the algebra among diffeomorphism vectors

As given in the main text the symmetry vectors are, $\chi^a\partial_a = F\partial_t$ and $\eta^a\partial_a = R^A\partial_A$. At first, we calculate the commutator of the supertranslation generator with itself. The algebras are computed as follows.

- $[\mathcal{F},\mathcal{F}]$ commutator:

  In terms of Fourier modes given in (5.4.13) we can write the commutator between χ_1 and χ_2 as ,

$$
\begin{aligned}
&\Big[\chi_1,\chi_2\Big]^t\partial_t\\
&= \sum_{k_A,l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_A)\alpha(\omega_2,l_A)[\bar{F}_k(\omega_1,t,x),\bar{F}_l(\omega_2,t,x)]^t\partial_t.
\end{aligned}
\tag{5.E.28}
$$

  Hence in coordinate basis the temporal component of the aforementioned Lie bracket $[\chi_1,\chi_2]$ is non-zero only. It can be checked easily that the other components of the bracket are zero automatically. With the help of the Lie

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algebra defined in (5.3.28), (5.E.28) becomes,

$$
\begin{aligned}
&\sum_{k_A,l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_A)\alpha(\omega_2,l_A)[\bar{F}_k(\omega_1,t,x),\bar{F}_l(\omega_2,t,x)]^t\partial_t \\
&=\sum_{k_A,l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_A)\alpha(\omega_2,l_A)[(\bar{F}_k)^a\partial_a(\bar{F}_l)-(\bar{F}_l)^a\partial_a(\bar{F}_k)]\partial_t \\
&=\sum_{k_A,l_A}\int_{-\infty}^{\infty}\Big(d\omega_1\, d\omega_2\, \alpha(\omega_1,k_A)\alpha(\omega_2,l_A)i(\omega_1-\omega_2) \\
&\times e^{i(-(\omega_1+\omega_2)t+\sum_A(k+l)_A x^A)}\partial_t\Big). \\
&=\sum_{k_A,l_A}\int_{-\infty}^{\infty}\Big(d\omega_1\, d\omega_2\, \alpha(\omega_1,k_A)\alpha(\omega_2,l_A) \\
&\times i(\omega_1-\omega_2)\bar{F}_{k+l}(\omega_1+\omega_2,t,x)\partial_t\Big).
\end{aligned} \tag{5.E.29}
$$

Comparing both the equations (5.E.28) and (5.E.29), we get the bracket algebra for $[\mathcal{F}\mathcal{F}]$ commutator given in (5.4.15).

- $[\mathcal{R},\mathcal{R}]$ commutator:

To derive the aforesaid commutator we can write Lie bracket between η_1 and η_2 as ,

$$
\begin{aligned}
[\eta_1,\eta_2]^{A''}\partial_{A''} &= \sum_{l_A,k_{A'}}\int_{-\infty}^{\infty}\Big(d\omega_1\, d\omega_2\, \bar{\alpha}^A(\omega_1,l_A)\, \bar{\alpha}^{A'}(\omega_2,k_{A'}) \\
&\times[\bar{R}^l_A(\omega_1,t,x),\bar{R}^k_{A'}(\omega_1,t,x)]^{A''}\partial_{A''}\Big).
\end{aligned} \tag{5.E.30}
$$

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With the help of (5.3.28), the right hand side of (5.E.30) can be written,

$$\sum_{l_A,k_{A'}}\int_{-\infty}^{\infty}\Big(d\omega_1\, d\omega_2\, \bar{\alpha}^A(\omega_1,l_A)\,\bar{\alpha}^{A'}(\omega_2,k_{A'})$$
$$\times[\bar{R}^l_A(\omega_1,t,x),\bar{R}^k_{A'}(\omega_2,t,x)]^{A''}\partial_{A''}\Big)$$

$$=\sum_{l_A,k_{A'}}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \bar{\alpha}^A(\omega_1,l_A)\,\bar{\alpha}^{A'}(\omega_2,k_{A'})\Big((\bar{R}^l_A)^a\partial_a(\bar{R}^k_{A'})^{A''}$$
$$-(\bar{R}^k_{A'})^a\partial_a(\bar{R}^l_A)^{A''}\Big)\partial_{A''}$$

$$=\sum_{l_A,k_{A'}}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \bar{\alpha}^A(\omega_1,l_A)\,\bar{\alpha}^{A'}(\omega_2,k_{A'})\Big(e^{-i\omega_1 t+i\sum_A l_A x^A}$$
$$\times\partial_A(e^{-i\omega_2 t+i\sum_{A'}k_{A'}x^{A'}}\delta^{A''}_{A'})-e^{-i\omega_2 t+i\sum_{A'}k_{A'}x^{A'}}\partial_{A'}(e^{-i\omega_1 t+i\sum_A l_A x^A}\delta^{A''}_A)\Big)\partial_{A''}$$

$$=\sum_{l_A,k_{A'}}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \bar{\alpha}^A(\omega_1,l_A)\,\bar{\alpha}^{A'}(\omega_2,k_{A'})$$
$$\times\Big(ik_A\, e^{-i(\omega_1+\omega_2)t+i\sum_{A'}(l+k)_{A'}x^{A'}}\delta^{A''}_{A'}$$
$$-il_{A'}\, e^{-i(\omega_1+\omega_2)t+i\sum_A(l+k)_A x^A}\delta^{A''}_A\Big)$$

$$=\sum_{l_A,k_{A'}}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \bar{\alpha}^A(\omega_1,l_A)\,\bar{\alpha}^{A'}(\omega_2,l_{A'})$$
$$\times\Big(ik_A(\bar{R}^{l+k}_{A'}(\omega_1+\omega_2,t,x))^{A''}-il_{A'}(\bar{R}^{l+k}_A(\omega_1+\omega_2,t,x))^{A''}\Big)\partial_{A''}. \tag{5.E.31}$$

Now comparing the equations (5.E.30) and (5.E.31) we get the algebra for $[\mathcal{R},\mathcal{R}]$ commutator given in (5.4.16).

- $[\mathcal{F},\mathcal{R}]$ commutator:

  Following the previous manner here we will get two non-zero components of the Lie bracket $[\chi_1,\eta_2]$ as,

$$[\chi_1,\eta_2]^t\partial_t+[\chi_1,\eta_2]^{A''}\partial_{A''}$$
$$=\sum_{k_{A'},l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\,\Big([\bar{F}_k(\omega_1,t,x),\bar{R}^l_A(\omega_2,t,x)]^t\partial_t$$
$$+[\bar{F}_k(\omega_1,t,x),\bar{R}^l_A(\omega_2,t,x)]^{A''}\partial_{A''}.\Big) \tag{5.E.32}$$

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Using the definition of Lie algebra, we calculate the first bracket given in the the right hand side of (5.E.32) as follows,

$$\begin{aligned}
&\sum_{k_{A'},l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\ [\bar{F}_k(\omega_1,t,x),\bar{R}^l_A(\omega_2,t,x)]^t\partial_t\\
&=-\sum_{k_{A'},l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\Big((\bar{R}^l_A)^a\partial_a(\bar{F}_k)\Big)\partial_t\\
&=-\sum_{k_{A'},l_A}\int_{-\infty}^{\infty}\Big(ik_A\, d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\\
&\times e^{-i(\omega_1+\omega_2)t+\sum_A i(l+k)_A x^A}\partial_t\Big)\\
&=-\sum_{k_{A'},l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)ik_A\,\bar{F}_{k+l}(\omega_1+\omega_2,t,x)\partial_t
\end{aligned} \tag{5.E.33}$$

Next the second bracket given in the the right hand side of (5.E.32) is calculated as follows,

$$\begin{aligned}
&\sum_{k_{A'},l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\ [\bar{F}_k(\omega_1,t,x),\bar{R}^l_A(\omega_2,t,x)]^{A''}\partial_{A''}\\
&=\sum_{k_{A'},l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)(\bar{F}_k)^a\partial_a(\bar{R}^l_A)^{A''}\partial_{A''}\\
&=-\sum_{k_{A'},l_A}\int_{-\infty}^{\infty}\Big(d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\\
&\times(i\omega_2\, e^{-i(\omega_1+\omega_2)t+\sum_A i(l+k)_A x^A}\delta^{A''}_A)\partial_{A''}\Big)\\
&=\sum_{k_{A'},l_A}\int_{-\infty}^{\infty}\Big(d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\\
&\times(-i\omega_2)\ (\bar{R}^{k+l}_A(\omega_1+\omega_2,t,x))^{A''}\partial_{A''}\Big).
\end{aligned} \tag{5.E.34}$$

Hence adding (5.E.33) with (5.E.34) we get,

$$\begin{aligned}
&=\sum_{k_{A'},l_A}\int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1,k_{A'})\bar{\alpha}^A(\omega_2,l_A)\\
&\times\Big(-i\omega_2(\bar{R}^{k+l}_A(\omega_1+\omega_2,t,x))^{A''}\partial_{A''}-ik_A\,\bar{F}_{k+l}(\omega_1+\omega_2,t,x)\partial_t\Big).
\end{aligned} \tag{5.E.35}$$

Comparing (5.E.35) with (5.E.32), the required $[\mathcal{F},\mathcal{R}]$ commutator algebra in (5.4.17) can be obtained.

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5.E.6 Computation of charges (5.4.23) and (5.4.24)

5.E.6.1 Derivation of the unit time like normal vector (5.4.18)

The unit timelike normal to the $t =$ constant hypersurface can be defined by, $n_t = c\partial_t t = c$. Now the norm of n_t gives,

$$n^t n_t = g^{tt} n_t n_t = -1 \Rightarrow -c^2 \frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{\phi\phi} g_{tt}} = c^2 \left(\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}\right) = -1;$$
$$\Rightarrow n_t = \frac{g_{t\phi}^2 - g_{\phi\phi} g_{tt}}{g_{\phi\phi}} = \sqrt{\frac{\Sigma(\rho,\theta)\Delta(\rho,\theta)}{\alpha(\rho,\theta)}}.$$
$$n^t = g^{tt} n_t = -\sqrt{\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{\phi\phi} g_{tt}}} = -\left(\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}\right)\sqrt{\frac{\Sigma(\rho,\theta)\Delta(\rho,\theta)}{\alpha(\rho,\theta)}}. \tag{5.E.36}$$

$$n^\phi = g^{\phi t} n_t = \frac{g^{t\phi}}{\sqrt{g^{tt}}} = \frac{g_{t\phi}}{\sqrt{g_{\phi\phi}(g_{t\phi}^2 - g_{\phi\phi} g_{tt})}}$$
$$= \left(\frac{-2aM(r_c + \rho\, s_1 - \rho^2 s_2)\sin^2\theta}{\Delta(\rho,\theta)\Sigma(\rho,\theta)\sin^2\theta}\right)\left(\sqrt{\frac{\Sigma(\rho,\theta)\Delta(\rho,\theta)}{\alpha(\rho,\theta)}}\right);$$
$$n^r = g^{rt} n_t = 0; \quad n^\theta = g^{\theta t} n_t = 0. \tag{5.E.37}$$

where Σ, Δ and α are given by (5.4.19), (5.4.20) and (5.4.21) respectively. For the kerr metric constructed in (5.E.2), we have used the following result that,

$$g_{t\phi}^2 - g_{\phi\phi} g_{tt} = \Delta(\rho,\theta)\sin^2\theta. \tag{5.E.38}$$

Hence from (5.E.36) we get the components of the unit timelike normal as given in (5.4.18).

5.E.6.2 Charges (5.4.23) and (5.4.24)

Here we will calculate diffeomorphism charges with the expression of J_{ab} as given in (2.20) and also with two surviving area element from (5.3.21) as,

$$d\Sigma^{t\rho} = -d^2x\sqrt{\sigma} n^t M^\rho; \quad d\Sigma^{\rho\phi} = d^2x\sqrt{\sigma} M^\rho n^\phi. \tag{5.E.39}$$

Here σ is the determinant of the induced metric on the $r =$ constant surface as given by,

$$\sigma = g_{\theta\theta} g_{\phi\phi} = \alpha(\rho,\theta)\sin^2\theta. \tag{5.E.40}$$

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We compute the charges as follows,

$$
\begin{aligned}
Q[\zeta] &= \frac{1}{16\pi G}\int_{\rho\to 0}\Big(d\Sigma^{t\rho}J_{t\rho} + d\Sigma^{\rho\phi}J_{\rho\phi}\Big)\\
&= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{\sigma}\Big(-n^t M^\rho(\partial_t\zeta_\rho - \partial_\rho\zeta_t) + M^\rho n^\phi(\partial_\rho\zeta_\phi - \partial_\phi\zeta_\rho)\Big).
\end{aligned}
\tag{5.E.41}
$$

For the components of the metric in (5.4.5) and given that the components of the symmetry vector in (5.4.10), we calculate ,

$$
\begin{aligned}
&\partial_t\zeta_\rho = \partial_t(g_{\rho\rho}\zeta^\rho) = 0;\\
&\partial_\rho\zeta_t = \partial_\rho(g_{tt}\zeta^t + g_{t\phi}\zeta^\phi) = \partial_\rho g_{tt}F(t,\theta,\phi) + \partial_\rho g_{t\phi}R^\phi(t,\theta,\phi);\\
&\partial_\rho\zeta_\phi = \partial_\rho(g_{\phi t}\zeta^t + g_{\phi\phi}\zeta^\phi) = \partial_\rho g_{\phi t}F(t,\theta,\phi) + \partial_\rho g_{\phi\phi}R^\phi(t,\theta,\phi);\\
&\partial_\phi\zeta_\rho = \partial_\phi(g_{\rho\rho}\zeta^\rho) = 0.
\end{aligned}
\tag{5.E.42}
$$

Substituting the results (5.E.42) in (5.E.41) and for the timelike normal n^t as given in (5.4.18) and for the spacelike normal M^ρ, we found the charges as,

$$
Q[\zeta] = \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{g_{\theta\theta}g_{\phi\phi}}\sin\theta\Big(-\sqrt{\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{\phi\phi}g_{tt}}}(\partial_\rho g_{tt}F + \partial_\rho g_{t\phi}R^\phi)
+\frac{g_{t\phi}}{\sqrt{g_{\phi\phi}(g_{t\phi}^2 - g_{\phi\phi}g_{tt})}}(\partial_\rho g_{\phi t}F + \partial_\rho g_{\phi\phi}R^\phi)\Big)
\tag{5.E.43}
$$

$$
= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x(\sqrt{\alpha(\rho,\theta)}\sin\theta)\Big(-\sqrt{\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}}(\partial_\rho g_{tt}F + \partial_\rho g_{t\phi}R^\phi)
-\frac{2aM(r_c + \rho\, s_1 - \rho^2 s_2)}{\sqrt{\Sigma(\rho,\theta)\Delta(\rho,\theta)\alpha(\rho,\theta)}}(\partial_\rho g_{\phi t}F + \partial_\rho g_{\phi\phi}R^\phi)\Big).
\tag{5.E.44}
$$

Now we separate the above expression for the supertranslation and superrotation charges as given by,

$$
\begin{aligned}
Q[F] &= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\Big(\sqrt{\alpha(\rho,\theta)}\sin\theta\Big)\Big[-\sqrt{\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}}(\partial_\rho g_{tt})F \\
&-\frac{2aM(r_c+\rho\, s_1-\rho^2 s_2)}{\sqrt{\Sigma(\rho,\theta)\Delta(\rho,\theta)\alpha(\rho,\theta)}}(\partial_\rho g_{\phi t})F\Big] \\
&= -\frac{1}{16\pi G}\int_{\rho\to 0} d^2x(\sin\theta)\sqrt{\frac{\alpha(\rho,\theta)\Delta_{r_c}}{\Sigma_{r_c}}}\Big[2M\sqrt{\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}}\Big(\frac{r_c^2-a^2\cos^2\theta}{\Sigma_{r_c}^2}\Big)F \\
&-\frac{4a^2M^2\sin^2\theta(r_c+\rho\, s_1-\rho^2 s_2)}{\sqrt{\Sigma(\rho,\theta)\Delta(\rho,\theta)\alpha(\rho,\theta)}}\Big(\frac{r_c^2-a^2\cos^2\theta}{\Sigma^2{}_{r_c}}\Big)F\Big] \\
&= -\frac{1}{16\pi G}\int d^2x(\sin\theta)\sqrt{\frac{\alpha_{r_c}\Delta_{r_c}}{\Sigma_{r_c}}}\Big[2M\sqrt{\frac{\alpha_{r_c}}{\Sigma_{r_c}\Delta_{r_c}}}\Big(\frac{r_c^2-a^2\cos^2\theta}{\Sigma_{r_c}^2}\Big)F \\
&-\frac{4r_c a^2M^2\sin^2\theta}{\sqrt{\Sigma_{r_c}\Delta_{r_c}\alpha_{r_c}}}\Big(\frac{r_c^2-a^2\cos^2\theta}{\Sigma^2{}_{r_c}}\Big)F\Big] \\
&= -\frac{1}{16\pi G}\int d^2x\frac{2M\,(\sin\theta)}{\Sigma_{r_c}^3}\,F\,(r_c^2-a^2\cos^2\theta)(\alpha_{r_c}-2a^2M\,r_c\sin^2\theta) \\
&= -\frac{1}{16\pi G}\int d^2x\frac{2M\,(\sin\theta)}{\Sigma_{r_c}^3}\,F\,(r_c^2-a^2\cos^2\theta)[(r_c^2+a^2)\Sigma_{r_c}+2a^2M\,r_c\sin^2\theta \\
&-2a^2M\,r_c\sin^2\theta] \\
&= -\frac{1}{16\pi G}\int d^2x\frac{2M\,(\sin\theta)}{\Sigma_{r_c}^3}\,F\,(r_c^2-a^2\cos^2\theta)(r_c^2+a^2)(r_c^2+a^2\cos^2\theta). \quad (5.E.45)
\end{aligned}
$$

Here we have expressed α_{r_c} as,

$$\alpha_{r_c} = ((r_c)^2+a^2)^2 - a^2\sin^2\theta\Delta_{r_c} = (r_c^2+a^2)\Sigma_{r_c}+2a^2M\,r_c\sin^2\theta. \quad (5.E.46)$$

Then from the expression derived in (5.E.45) we easily get the supertranslation charges given in (5.4.23).

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Similarly we calculate the superrotation charges as,

$$
\begin{aligned}
Q[R] &= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\left(\sqrt{\alpha(\rho,\theta)}\sin\theta\right)\Big[\sqrt{\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}}(\partial_\rho g_{t\phi})R^\phi \\
&-\frac{2aM(r_c+\rho\, s_1-\rho^2 s_2)}{\sqrt{\Sigma(\rho,\theta)\Delta(\rho,\theta)\alpha(\rho,\theta)}}(\partial_\rho g_{\phi\phi})R^\phi\Big] \\
&= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{\alpha(\rho,\theta)}(\sin\theta)s_1\; R^\phi \\
&\Big[2aM(\sin^2\theta)\sqrt{\frac{\alpha(\rho,\theta)}{\Sigma(\rho,\theta)\Delta(\rho,\theta)}}\Big(\frac{r_c^2-a^2\cos^2\theta}{\Sigma^2{}_{r_c}}\Big) \\
&-\frac{2aM(r_c+\rho\, s_1-\rho^2 s_2)}{\sqrt{\Sigma(\rho,\theta)\Delta(\rho,\theta)\alpha(\rho,\theta)}}\sin^2\theta\Big(\frac{2a^2r_c+2r_c^3+(a^2M-a^2r_c)\sin^2\theta}{\Sigma_{r_c}} \\
&-\frac{r_c(a^2+r_c^2)^2-a^2r_c(a^2-2Mr_c+r_c^2)\sin^2\theta}{\Sigma^2{}_{r_c}}\Big)\Big] \\
&= \frac{1}{16\pi G}\int d^2x\sqrt{\alpha_{r_c}}(\sin\theta)\; R^\phi\sqrt{\frac{\Delta_{r_c}}{\Sigma_{r_c}}}\Big[2aM(\sin^2\theta)\sqrt{\frac{\alpha_{r_c}}{\Sigma_{r_c}\Delta_{r_c}}}\Big(\frac{r_c^2-a^2\cos^2\theta}{\Sigma^2{}_{r_c}}\Big) \\
&-\frac{2aMr_c}{\sqrt{\Sigma_{r_c}\Delta_{r_c}\alpha_{r_c}}}\sin^2\theta\Big(\frac{2a^2r_c+2r_c^3+(a^2M-a^2r_c)\sin^2\theta}{\Sigma_{r_c}} \\
&-\frac{r_c(a^2+r_c^2)^2-a^2r_c(a^2-2Mr_c+r_c^2)\sin^2\theta}{\Sigma^2{}_{r_c}}\Big)\Big] \\
&= \frac{1}{16\pi G}\int d^2x\frac{(\sin^3\theta)}{\Sigma^3_{r_c}}\; 2aM\, R^\phi\Big[\alpha_{r_c}(r_c^2-a^2\cos^2\theta)-r_c\Sigma_{r_c}\Big(2a^2r_c+2r_c^3 \\
&+(a^2M-a^2r_c)\sin^2\theta\Big)+r_c^2(a^2+r_c^2)^2-a^2r_c^2(a^2-2Mr_c+r_c^2)\sin^2\theta\Big] \\
&= \frac{1}{16\pi G}\int d^2x\frac{(\sin^3\theta)}{\Sigma^3_{r_c}}\; 2aM\, R^\phi\Big[2r_c^2(r_c^2+a^2)^2 \\
&-2r_c^2a^2(a^2-2Mr_c+r_c^2)(1-\cos^2\theta)-a^2(r_c^2+a^2)^2\cos^2\theta \\
&+a^4(a^2-2Mr_c+r_c^2)(\cos^2\theta-\cos^4\theta)-r_c(r_c^2+a^2\cos^2\theta)(2a^2r_c+2r_c^3 \\
&+a^2(M-r_c)(1-\cos^2\theta))\Big].
\end{aligned}
\tag{5.E.47}
$$

From (5.E.47) we directly get the expression of the superrotation charges as given in (5.4.24).

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5.E.6.3 Derivation of (5.4.25)

After integrating over transverse coordinates, from the expression given in (5.4.23), for $F(t,x) = 1$ we get,

$$\begin{aligned} Q[F=1] &= -\frac{2M}{16\pi G}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left[(\sin\theta)\frac{(a^2+r_c^2)(r_c^4-a^4\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\right] \\ &= -\frac{M}{4G}\int_{\theta=0}^{\pi}\left[(\sin\theta)\frac{(a^2+r_c^2)(r_c^4-a^4\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\right] \\ &= \frac{M}{2G}; \end{aligned} \tag{5.E.48}$$

which is the first result as given in (5.4.25).

In the similar way we integrate (5.4.24) over the transverse coordinates, then for $R^A(t,x) = 1$, we get,

$$\begin{aligned} Q[\mathcal{R}^A=\infty] &= \frac{1}{16\pi G}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\Big[2aM(\sin^3\theta) \\ &\times\frac{(3r_c^6+a^2r_c^4+4a^2r_c^4\cos^2\theta-a^4(a^2-r_c^2)\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\Big] \\ &= \frac{2aM}{8G}\int_{\theta=0}^{\pi}\left[(\sin^3\theta)\frac{(3r_c^6+a^2r_c^4+4a^2r_c^4\cos^2\theta-a^4(a^2-r_c^2)\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\right] \\ &= -\frac{Ma}{G}. \end{aligned} \tag{5.E.49}$$

The above result matches with the second one given in (5.4.25).

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5.E.6.4 Derivation of the result given in (5.4.27)

Now we calculate diffeomorphism charges corresponding to a general diffeomorphism vector $\zeta^a\partial_a = \partial_t + \Omega_c\partial_\phi$. From (5.E.43) we can calculate,

$$
\begin{aligned}
&Q_\zeta = Q[F(t,x)=1] + Q[R^A(t,x)=\Omega_c] \\
&= \frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{g_{\theta\theta}g_{\phi\phi}}\sin\theta\Big(-\sqrt{\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{\phi\phi}g_{tt}}}(\partial_\rho g_{tt} + \Omega_c\,\partial_\rho g_{t\phi}) \\
&+\frac{g_{t\phi}}{\sqrt{g_{\phi\phi}(g_{t\phi}^2 - g_{\phi\phi}g_{tt})}}(\partial_\rho g_{\phi t} + \Omega_c\,\partial_\rho g_{\phi\phi})\Big) \\
&= -\frac{1}{16\pi G}\int_{\rho\to 0} d^2x\sqrt{\frac{\alpha(\rho,\theta)}{\Delta(\rho,\theta)}}\Big(\frac{g_{\phi\phi}\partial_\rho g_{tt} - g_{t\phi}\partial_\rho g_{t\phi}}{\sqrt{g_{\phi\phi}}} \\
&+\Omega_c\frac{g_{\phi\phi}\partial_\rho g_{t\phi} - g_{t\phi}\partial_\rho g_{\phi\phi}}{\sqrt{g_{\phi\phi}}}\Big) \\
&= -\frac{1}{16\pi G}\int d^2x\csc\theta\sqrt{\frac{\Sigma_{r_c}}{\Delta_{r_c}}}\Big(g_{\phi\phi}\partial_\rho g_{tt} - g_{t\phi}\partial_\rho g_{t\phi} + \Omega_c(g_{\phi\phi}\partial_\rho g_{t\phi} - g_{t\phi}\partial_\rho g_{\phi\phi})\Big)
\end{aligned}
\tag{5.E.50}
$$

Here we have used the relation (5.E.38) and also that $g_{\phi\phi} = \frac{\alpha\sin^2\theta}{\Sigma}$. Hence after integrating over transverse coordinates, (5.E.50) boils down to the expression given in (5.4.27).

5.E.7 Tolman relation and kerr charges

- The proof of (5.4.29) and (5.4.30):

  For the form of $\zeta^a = (1,0,0,\Omega_c)$, we can write,

$$
\begin{aligned}
|K|^2 &= -\zeta^a\zeta_a = -g_{ab}\zeta^a\zeta^b \\
&= -(g_{tt}K^tK^t + 2g_{t\phi}K^tK^\phi + g_{\phi\phi}K^\phi K^\phi).
\end{aligned}
\tag{5.E.51}
$$

  where Ω_c is given by (5.4.26). From the above expression we get (5.4.29). Now the temperature gradient part given in (5.4.30), can be derived from (5.4.28)

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as,

$$\begin{aligned}
T(r) &= \frac{T_0}{\sqrt{-(g_{tt}K^tK^t + 2g_{t\phi}K^tK^\phi + g_{\phi\phi}K^\phi K^\phi)}} \\
\Rightarrow M^r\frac{\nabla_r T}{T} \\
&= g^{rr}\frac{\partial_\rho(g_{tt} + 2\Omega_c g_{t\phi} + \Omega_c^2 g_{\phi\phi})}{2(-g_{tt} - 2\Omega_c g_{t\phi} - \Omega_c^2 g_{\phi\phi})^{3/2}}\sqrt{-(g_{tt} + 2\Omega_c g_{t\phi} + \Omega_c^2 g_{\phi\phi})}. \\
&= -\frac{1}{2}\frac{-\partial_\rho(g_{tt} + 2\Omega_c g_{t\phi} + \Omega_c^2 g_{\phi\phi})}{(-g_{tt} - 2\Omega_c g_{t\phi} - \Omega_c^2 g_{\phi\phi})}; \qquad (5.E.52)
\end{aligned}$$

which is the form given in (5.4.30).

- Derivation of (5.4.31)

  Now the unit normal to the ρ = constant surface is defined by, $M^a = (0,1,0,0)$.

  Then the product of the temperature gradient term (5.4.30) , area element $\delta A_c = \sqrt{g_{\theta\theta}g_{\phi\phi}}d\theta d\phi$ and ratio T_0/T from (5.4.28), can be expressed near the surface at $r = r_c$ as follows,

$$\begin{aligned}
&\frac{\delta A_c}{8\pi G}\frac{T_0}{T(r)}\frac{M^a\nabla_a T}{T}\Big|_{\rho=0} \\
&= \frac{1}{8\pi G}\Big[\sqrt{\alpha}\sin\theta d\theta d\phi\,\sqrt{-(g_{tt} + 2\Omega_c g_{t\phi} + \Omega_c^2 g_{\phi\phi})} \\
&\times\frac{\partial_\rho(g_{tt} + 2\Omega_c g_{t\phi} + \Omega_c^2 g_{\phi\phi})}{2(g_{tt} + 2\Omega_c g_{t\phi} + \Omega_c^2 g_{\phi\phi})}\Big]\Big|_{\rho=0}. \qquad (5.E.53)
\end{aligned}$$

Now we can express that,

$$\begin{aligned}
&\Big(g_{tt} + 2\Omega_c g_{t\phi} + \Omega_c^2 g_{\phi\phi}\Big)\Big|_{\rho=0} = \Big(g_{tt} - 2\frac{g_{t\phi}}{g_{\phi\phi}}g_{t\phi} + (\frac{g_{t\phi}}{g_{\phi\phi}})^2 g_{\phi\phi}\Big)\Big|_{r=r_c} \\
&= \Big(g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}}\Big)\Big|_{r=r_c} = \Big(\frac{g_{tt}g_{\phi\phi} - g_{t\phi}^2}{g_{\phi\phi}}\Big)\Big|_{r=r_c} = -\frac{\Delta_{r_c}\Sigma_{r_c}}{\alpha_{r_c}} \qquad (5.E.54)
\end{aligned}$$

Putting (5.E.54) in (5.E.53) we get,

$$
\begin{aligned}
&-\frac{\delta A_c}{8\pi G}\frac{T_0}{T(r)}\frac{M^a\nabla_a T}{T}\Big|_{\rho=0}\\
&=-\frac{1}{16\pi G}\Big[\sqrt{\alpha_{r_c}}\sin\theta d\theta d\phi\sqrt{\frac{\Delta_{r_c}\Sigma_{r_c}}{\alpha_{r_c}}}\alpha_{r_c}\\
&\times\Big(\frac{\partial_\rho g_{tt}+2\Omega_c\,\partial_\rho g_{t\phi}+\Omega_c^2\,\partial_\rho g_{\phi\phi}}{\Delta_{r_c}\Sigma_{r_c}}\Big)\Big|_{\rho=0}\Big]\\
&=-\frac{1}{16\pi G}\Big[\frac{\alpha_{r_c}}{\sqrt{\Delta_{r_c}\Sigma_{r_c}}}\sin\theta d\theta d\phi\Big(\partial_\rho g_{tt}-2\frac{g_{t\phi}}{g_{\phi\phi}}\,\partial_\rho g_{t\phi}+(\frac{g_{t\phi}}{g_{\phi\phi}})^2\partial_\rho g_{\phi\phi}\Big)\\
&=-\frac{1}{16\pi G}\Big[\frac{\alpha_{r_c}}{\sqrt{\Delta_{r_c}\Sigma_{r_c}}}\sin\theta d\theta d\phi\Big(\frac{g_{\phi\phi}\partial_\rho g_{tt}-g_{t\phi}\partial_\rho g_{t\phi}}{g_{\phi\phi}}\\
&-\frac{g_{t\phi}}{g_{\phi\phi}}(\frac{g_{\phi\phi}\partial_\rho g_{t\phi}-g_{t\phi}\partial_\rho g_{\phi\phi}}{g_{\phi\phi}})\Big)\Big|_{\rho=0}\Big]\\
&=-\frac{1}{16\pi G}(\csc\theta\, d\theta d\phi)\sqrt{\frac{\Sigma_{r_c}}{\Delta_{r_c}}}\Big(g_{\phi\phi}\partial_\rho g_{tt}-g_{t\phi}\partial_\rho g_{t\phi}+\Omega_c(g_{\phi\phi}\partial_\rho g_{t\phi}\\
&-g_{t\phi}\partial_\rho g_{\phi\phi})\Big)\Big].
\end{aligned}
\tag{5.E.55}
$$

Hence the result (5.E.55) matches with the result found in (5.E.50) and after integrating over tranverse coordinates the expression (5.E.55) reduces to (5.4.27).

5.E.8 Algebra among charges (5.4.34)

Using the definition of bracket given in (5.3.28), we calculate commutator of the supertranslation charges with itself as follows,

$$
\begin{aligned}
&[Q[F_1],Q[F_2]]=\pounds_{F_1}Q[F_2]\\
&=-\frac{1}{16\pi G}\int d^2x\Big[2M(\sin\theta)\frac{(a^2+r_c^2)(r_c^4-a^4\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\\
&\times[F_1(\omega_1,t,x),F_2(\omega_1,t,x)]\Big]\\
&=-\frac{1}{16\pi G}\sum_{k_A,l_A}\int d^2x\int_{-\infty}^{\infty}d\omega_1\,d\omega_2\,\Big[2M(\sin\theta)\frac{(a^2+r_c^2)(r_c^4-a^4\cos^4\theta)}{(r_c^2+a^2\cos^2\theta)^3}\\
&\alpha(\omega_1,k_A)\alpha(\omega_2,l_A)[\bar F_k(\omega_1,t,x),\bar F_l(\omega_2,t,x)]\Big].
\end{aligned}
\tag{5.E.56}
$$

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Now following (5.4.32), $[Q[F_1], Q[F_2]]$ can be expressed as,

$$[Q[F_1], Q[F_2]] = \sum_{k,l} \int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1, k_A)\alpha(\omega_2, l_A) \times [Q[\bar{F}_k(\omega_1, t, x)], Q[\bar{F}_l(\omega_2, t, x)]]. \tag{5.E.57}$$

Following the $[F, F]$ commutator as given in (5.4.15), from (5.E.56) we get,

$$\begin{aligned}[Q[F_1], Q[F_2]] &= -\frac{1}{16\pi G} \sum_{k_A, l_A} \int d^2x \int_{-\infty}^{\infty} d\omega_1\, d\omega_2 \\ &\times \Big[2M(\sin\theta) \frac{(a^2 + r_c^2)(r_c^4 - a^4\cos^4\theta)}{(r_c^2 + a^2\cos^2\theta)^3} \alpha(\omega_1, k_A)\alpha(\omega_2, l_A)(i(\omega_1 - \omega_2)) \\ &\times \bar{F}_{k+l}(\omega_1 + \omega_2, t, x)\Big] \\ &= \sum_{k_A, l_A} \int_{-\infty}^{\infty} d\omega_1\, d\omega_2\, \alpha(\omega_1, k_A)\alpha(\omega_2, l_A)(i(\omega_1 - \omega_2)) \\ &\quad Q[\bar{F}_{k+l}(\omega_1 + \omega_2, t, x)]. \end{aligned} \tag{5.E.58}$$

Now comparing the result in (5.E.58) with (5.E.57), one can obtain the bracket among supertranslation charges given in (5.4.34). Now similarly other brackets given in (5.4.34) can be easily derived.

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Conclusions and Outlook 6

6.1 Conclusions

Symmetries and the corresponding conserved charges play a vital role in understanding the full dynamics of a theory. In these endeavors, the asymptotic symmetry analysis for the diffeomorphism invariant gravity theory has been one of the most delicate research areas for a long time. Instead of the usual Poincaré group, an infinite-dimensional group of symmetries emerged as the *asymptotic symmetry group* in the early sixties near the boundary of the asymptotically Minkowski spacetime. The basic strategy behind the symmetry mechanism of this group was to preserve the asymptotic forms of the solutions near the spacetime boundary. Near the null infinities, this group paves the way for a fruitful interpretation of the scattering of gravitational waves and mass loss phenomena of gravitational radiation, etc. Nevertheless, the null infinity (past or future) of the asymptotically flat spacetime is regarded as one part of the closed boundary enclosing the bulk manifold. In contrast, the black hole horizon is considered as another part of this boundary. Hence it was expected that like null infinity, the BMS symmetries must be explored and studied extensively near the horizon of the black hole also. It is well researched that the horizon BMS symmetries play a crucial role in the microscopic description of the black hole thermodynamics. In this thesis, our primary target was to inspect the various facts and facets of asymptotic symmetries near the black hole horizon and analyze the connection of these symmetries with the black hole thermodynamics.

In the second chapter, we studied the BMS symmetries near a generic charged

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null surface in a more general setting than the earlier analysis found in the existing literature [90–92, 97, 110, 113]. The results that we found in this chapter can be summarized as follows;

- The complete Lie bracket algebra between three symmetry parameters, supertranslation, superrotation, and super-gauge, has been computed. In these results, interestingly, the supertranslation parameter is shown to be non-commutative with itself. This result is in sharp contrast with the existing near horizon BMS algebra.
- In this analysis, the conserved charges have been derived for the higher-order Lanczos-Lovelock gravity model accompanied by the non-linear electromagnetic fields (Born-Infeld type). In this sense, the present analysis is most general from the aspects of the gauge-gravity theory. Also it can be shown that the zero mode of the supertranslation charges can be expressed as the product of the entropy and the temperature attributed to a generic null surface.
- To compute charges from the variation of the action, nowhere the field equations have been used (i.e., an off-shell derivation). This one is crucial in the present context as the null surface under consideration may not be the solution of the Einstein equations.
- Also, we have shown that the Noether charge derived from the GHY boundary term of the Einstein-Hilbert action gives rise to the same expression obtained before from the bulk action itself. Thus bracket algebra between the charges is also the same for the two cases. This result indicates the holographic relation of the bulk theory with the boundary.
- The bracket algebra between the symmetry generators near the extremal null surface is found to be a bit different compared to that of the non-extremal case. The difference appears because the supertranslation parameter becomes independent of the null coordinate. This result shows that the zero-temperature limit of the non-extremal surface has physically different symmetry properties. However, the reason for this difference is not understood in the realm of the present investigation.

In the third chapter, we have obtained the mode solutions of the diffeomorphism parameters by analyzing the near horizon spontaneous symmetry breaking phenomena that happened by the background solution itself. The corresponding parameters which characterize this symmetry breaking are identified as the

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Goldstone modes. The dynamics of those modes have been determined by the Lagrangian method in linearized theory. This analysis has been concluded with the following results;

- For both Rindler and Spherically symmetric backgrounds, the one-dimensional dynamics of the Goldstone modes along the null direction v is governed by an inverse harmonic potential. Thus it is concluded that the modes are unstable under this potential.
- For the Rindler case, irrespective of the values (l, m), all the modes have been found out to feel the same inverted harmonic potential. However, for the Schwarzschild background, inverse harmonic potential has emerged to depend on l.
- To capture the physical interpretation of these unstable modes, quantum analysis has been performed following the recently found connection between semi-classical chaotic motion and the thermal nature of the system. Hence it was revealed that the instability has a nice explanation for the thermality of the black hole horizon.
- In Rindler background, the temperature perceived by the mode is the same for all values of (l, m) and it has been proportional to the well-known expression of Unruh with the proportionality factor $\sqrt{3}/2$.
- In Schwarzschild spacetime, each mode realizes different temperature depending on the values of l. Therefore, the average temperature of all modes came out to be proportional to the Hawking temperature of the black hole, multiplied by the factor $\sqrt{3}/2$. Only for $l = 1$, the obtained result matches exactly with Hawking's expression.

In the fourth chapter, we have extended the analysis performed in the last chapter for Kerr black hole. Here we have concentrated on two different conditions.

- Firstly, we analyzed Goldstone modes' theory and found its non-trivial dynamics for slowly rotating spacetime. By analyzing the instability of the modes quantum mechanically, we found the thermal nature like before. The expression of the average temperature felt by the modes has emerged to be similar to the known expression given by Hawking in a slowly rotating case. This result has been obtained by determining the perturbatively corrected terms upon the corresponding expression found in the Schwarzschild background.

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- Next for ZAMO observers, we found that the temperature perceived by the modes is proportional to the Hawking temperature of Kerr black hole with a multiplicative factor given by m.

There are some important observations that should be mentioned here. In the chapter 3 and 4 of the present thesis, we did a quantum mechanical treatment by considering the Schrodinger equation corresponding to the Goldstone mode F to explore the thermal behaviour. However the parameter F can be treated as a quantum field which we leave for our further study. But it is expected that the behaviour of each mode of the quantum field is similar to the quantum mechanical wave function as far as temperature is concerned. Therefore the predicted thermalization and temperature presented here are expected to be well defined within the framework of the present analysis.

In the fifth chapter, we have investigated the asymptotic symmetries near a timelike hypersurface which is fixed at a constant radial distance from the black hole horizon. We have assumed the surface to be a physical boundary dividing the space into two regions at an instant of time. Here briefly we have the following results;

- Like null boundaries, the diffeomorphism symmetry vector obtained near the timelike surface has two components. One component denotes translation along time direction, named as *supertranslation* and another one gives angular transformation, named as *superrotation*.

- Interestingly, unlike null boundaries, in Schwarzschild spacetime, the supertranslation parameter came out to be a function of time only. However, for the Kerr case, the symmetry parameters are dependent on both time and angular coordinates.

- The bracket algebras found between the symmetry generators in the present context resemble near horizon symmetry algebra in Schwarzschild spacetime. But for Kerr background, the algebras constructed near the aforementioned surface are different from those found near null boundaries.

- The computed supertranslation and superrotation 'charges' have an exciting thermodynamics interpretation. Locally the charges can be described as the heat content on the aforesaid hypersurface at a particular time. So surface entropy and local temperature can be identified in terms of the local gravitational acceleration. This result was proved to behold for not only

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Schwarzschild black hole, but for a generally spherically symmetric spacetime also.

- Also, the corresponding charges for static and stationary spacetime have been shown to have another astonishing explanation by considering the well-known Tolman relation of the thermal equilibrium system in the gravitational fields.

By all these results, we have tried to unfold the interesting connection of the asymptotic symmetries with the thermodynamics of the black hole at the classical or semi-classical level. Next, we have presented a brief description of some directions of future research work.

6.2 Scope for future work

6.2.1 More detail investigation about the symmetries of extremal null surface

In the second chapter, we have noticed that the bracket algebra between symmetry generators of the extremal null surfaces is significantly different compared to that of a non-extremal null surface. However, the physical reason behind this difference was not clear there. The extremal surfaces correspond to a specific configuration of the non-extremal one when the temperature goes to zero. So it will be interesting to investigate more about the intertwining connection between asymptotic symmetries and thermodynamics of extremal null surfaces.

6.2.2 Field theoretical analysis of the parameter F and R^A

In the second and third chapters of the present thesis, we have found the non-trivial classical mode solution of the supertranslation parameter F for both static and stationary background spacetime. We have proposed the Einstein-Hilbert Lagrangian for the modified metric under diffeomorphism and finally found out that the one-dimensional equation of motion of F (along null coordinate v) resembles the inverse Harmonic oscillator equation. Then by having a quantum mechanical analysis with the help of the Schrodinger equation, the system of a large number of unstable Goldstone modes is shown to be inherently thermal. However, in that analysis, degenerate and quasi-degenerate modes have emerged for Rindler and Schwarzschild backgrounds, respectively. We expect that these

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modes can be responsible for the possible *mechanism* of the microscopic origin of the horizon entropy. But to study the corresponding microscopic states in detail, full quantum field theoretic treatment is extremely important to elucidate the thermal behavior of the system and for that relevant vacuum state have to be considered.

6.2.3 Shock wave interpretation of modified metric

In the second and third chapters, we have analyzed that the background metric can be modified by the supertranslation, which is the generator of the near horizon symmetries. However, in the recent paper [84] by Strominger, Hawking and Perry, it is shown that the background Schwarzschild metric is corrected due to the supertranslation-like symmetries near null infinities. This phenomenon has been physically realized as the effect of the propagation of shock waves in this spacetime. The idea of the generation of the shock wave as the effect of the motion of the massless particle near the Schwarzschild black hole horizon was earlier introduced by Dray, and 't Hooft in [248]. Motivated by that analysis, we want to investigate if the physical interpretation of the fluctuation metric which arises due to horizon BMS symmetries can be obtained in terms of the propagation of shock waves in that spacetime.

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