EIGENVALUE BACKWARD ERRORS OF POLYNOMIAL EIGENVALUE PROBLEMS UNDER STRUCTURE PRESERVING PERTURBATIONS

Ph.D. Thesis

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DEPARTMENT OF MATHEMATICS

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DECLARATION

It is certified that the work contained in the thesis titled **"Eigenvalue backward errors of polynomial eigenvalue problems under structure preserving perturbations"** has been done by me, a student in the Department of Mathematics, Indian Institute of Technology Guwahati under the guidance of **Dr. Shreemayee Bora** for the award of Doctor of Philosophy and that this work has not been submitted elsewhere for a degree.

August, 2015

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CERTIFICATE

It is certified that the work contained in the thesis titled **"Eigenvalue backward errors of polynomial eigenvalue problems under structure preserving perturbations"** by **Punit Sharma**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

August, 2015

Dr. Shreemayee Bora
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Dedicated to my family

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Abstract

Structured matrix polynomials have occurred in many engineering applications and have been studied widely for the last two decades. Structured eigenvalue-eigenpair backward error analysis of structured matrix polynomials is important in order to know the backward stability of algorithms that compute them without losing the structure of the polynomial. Structured eigenpair backward errors are known in literature but structured eigenvalue backward errors are not known even for the matrix pencils. These backward errors have been studied with respect to three different norms $|||\cdot|||_{w,2}$, $|||\cdot|||_{w,F}$ and $|||\cdot|||_{w,\infty}$, where w is a weight vector.

Explicit formulas for structured eigenvalue backward errors of matrix pencils and polynomials with Hermitian and related structures are derived when structure preserving perturbations are measured with respect to the $|||\cdot|||_{w,2}$ norm. These include skew-Hermitian and $*$-alternating polynomials. If perturbations affect only a few coefficients of the matrix polynomial then structured eigenvalue backward errors are obtained with respect to a restricted perturbation set. Also a minimal perturbation that attains the backward error is constructed in each case. Similar results are also derived for T-palindromic and T- even polynomials of degree at most 2, T-odd pencils and T-antipalindromic pencils. For all these structures, the structured eigenvalue backward errors for higher degree polynomials are estimated by bounds. Numerical experiments suggest that these bounds are tight. In fact in all these cases the lower bound gives the exact structured backward error under certain assumptions that are seen to be satisfied in numerical experiments.

Computable bounds are obtained for structured eigenvalue backward errors of Hermitian polynomials, $*$-palindromic polynomials and T-palindromic polynomials of degree at most 2 with respect to $|||\cdot|||_{w,\infty}$ norm. Once again in each case, the lower bound is equal to the exact backward error under some conditions that are satisfied in numerical experiments. Structured eigenvalue backward errors of some matrix polynomials with respect to $|||\cdot|||_{w,F}$ norm are also estimated tightly in terms of backward errors with respect to $|||\cdot|||_{w,2}$ norm.

If a matrix pencil is real then real eigenvalue backward error of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and real eigenpair backward errors of $(\lambda, x) \in (\mathbb{C} \setminus \mathbb{R}) \times (\mathbb{C}^n \setminus \{0\})$ have been estimated. For a real Hermitian polynomial it is proved that under certain assumptions, the real Hermitian eigenvalue backward error is equal to its complex Hermitian eigenvalue backward error with respect to $|||\cdot|||_{w,2}$ and $|||\cdot|||_{w,\infty}$. It has been observed that these assumptions are satisfied in

generic situations. If a T-palindromic polynomial is real, then eigenvalue backward errors of real numbers have been derived with respect to perturbations that preserve real as well as T-palindromic structure of the polynomial. Similar results are also obtained for real T-alternating, real skew-symmetric and real T-antipalindromic polynomials. Eigenvalue and eigenpair backward errors are also obtained under certain norms for some special block structured pencils that arise in continuous and discrete time linear-quadratic optimal control problems.

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Chapter 1

Introduction

A matrix polynomial or λ-matrix of size $n \times n$ is a polynomial $P(z) = \Sigma_{j=0}^{m} z^j A_j$ with coefficients $A_j, j = 0, \ldots, m$ that are $n \times n$ matrices. The polynomial is said to be real or complex according as whether the coefficient matrices are real or complex. Also the degree of $P(z)$ is m if $A_m \neq 0$. When the degree is 1 the polynomial $P(z) = A_0 + zA_1$ is also called a matrix pencil. The polynomial $P(z)$ is said to be *regular* if there exists a scalar λ such that $\det(P(\lambda)) \neq 0$. Given a regular polynomial $P(z)$, the eigenvalue problem associated with $P(z)$ consists of finding scalars λ and nonzero vectors x and y such that $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$. A special case of this is the matrix eigenvalue problem where $m = 1$ and A_1 is the $n \times n$ identity matrix. We will consider *only* regular matrix polynomials in this thesis. Polynomial eigenvalue problems arise naturally in the solution of higher order linear differential equations and hence they occur widely in applications [43, 31]. In many applications, the coefficient matrices of $P(z)$ have additional properties due to which the $(m+1)$-tuple $(A_0, \ldots, A_m)$ belongs to a subset, say $\mathbb{S}$, of $(\mathbb{C}^{n\times n})^{m+1}$. For example, $P(z)$ is a real matrix polynomial if $\mathbb{S} = (\mathbb{R}^{n\times n})^{m+1}$ and a Hermitian matrix polynomial if $A_j^* = A_j, j = 0, \ldots, m$. In such a case, we say that $P(z)$ is a structured matrix polynomial. Structured matrix polynomials arise in a wide range of applications. For example, Hermitian matrix polynomials occur in the vibration analysis of machines, buildings and vehicles (see [43] and references therein). Closely related to the Hermitian matrix polynomials are the $*$-even and $*$-odd matrix polynomials with coefficient matrices that are alternatingly Hermitian and skew-Hermitian due to which they are also collectively referred to as $*$-alternating matrix polynomials [32]. They occur in the study of gyroscopic systems [28] and continuous time linear quadratic optimal control problems [29, 34]. Another important class of structured matrix polynomials that arise in optimal control [32] are $*$-palindromic matrix polynomials $P(z) = \Sigma_{j=0}^{m} z^j A_j$ where $A_j^* = A_{m-j}$. These polynomials have the property that replacing every coefficient matrix by its complex conjugate transpose and reversing the order of the coefficients gives back the original polynomial, that is, $\mathrm{rev}P(z) = (P(\bar{z}))^*$ where $\mathrm{rev}P(z) = z^m P(1/z)$ is called the reversal of $P(z)$. When $*$ is replaced by T, we have the T-palindromic matrix polynomials which arise in the mathematical modeling and numerical simulation of Surface Acoustic Wave (SAW) filters [45] and in the vibration analysis of railway tracks excited by high speed trains [20].

The main problem considered in this thesis is as follows. Given a *regular matrix polynomial* $P(z) = \Sigma_{j=0}^{m} z^j A_j$ with $(A_0, \ldots, A_m) \in \mathbb{S} \subseteq (\mathbb{C}^{n\times n})^{m+1}$, and $\lambda \in \mathbb{C}$, find a smallest matrix polynomial $\Delta(z) = \Sigma_{j=0}^{m} z^j \Delta_j$ with $(\Delta_0, \ldots, \Delta_m) \in \mathbb{S}$ (with respect to a specified norm) such that λ is an eigenvalue of $(P - \Delta)(z)$. The answer to this problem is called the structured eigenvalue backward error of λ as an approximate eigenvalue of $P(z)$ whenever $\mathbb{S} \neq (\mathbb{C}^{n\times n})^{m+1}$. If $\mathbb{S} = (\mathbb{C}^{n\times n})^{m+1}$, it is called the unstructured eigenvalue backward error of λ as an approximate eigenvalue of $P(z)$. Closely related to this is the problem where $P(z) = \Sigma_{j=0}^{m} z^j A_j$, with $(A_0, \ldots, A_m) \in \mathbb{S} \subseteq (\mathbb{C}^{n\times n})^{m+1}$, $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \setminus \{0\}$ are given and the aim is to find a smallest matrix polynomial $\Delta(z) = \Sigma_{j=0}^{m} z^j \Delta_j$ with $(\Delta_0, \ldots, \Delta_m) \in \mathbb{S}$ (in some specified norm) such that $(P + \Delta)(\lambda)x = 0$. The answer to this problem is called the structured (resp. unstructured) eigenpair backward error of the approximate eigenpair (λ, x) of $P(z)$ whenever $\mathbb{S} \neq (\mathbb{C}^{n\times n})^{m+1}$ (resp. $\mathbb{S} = (\mathbb{C}^{n\times n})^{m+1}$). Indeed, for a given $\mathbb{S} \subset (\mathbb{C}^{n\times n})^{m+1}$, the infimum of the eigenpair backward errors corresponding to $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$, taken over all possible $x \in \mathbb{C}^n \setminus \{0\}$ gives the eigenvalue backward error corresponding to λ.

Table 1.0.1 gives a list of structures for which we compute the structured eigenvalue backward errors. As noted in the table, the structures are associated with a symmetry in their eigenvalue distribution and a corresponding critical set. For example, when the matrix polynomial is real or Hermitian, the eigenvalues are symmetrically placed with respect to the real line as they occur in pairs $(\lambda, \bar{\lambda})$ whereas for $*$-alternating matrix polynomials, the pairing is $(\lambda, -\bar{\lambda})$ which is symmetric with respect to the imaginary axis. The pairing coalesces if there are real eigenvalues in the first case and purely imaginary eigenvalues in the second case. Due to this, the real line and the imaginary axis are referred to as critical sets for the corresponding structures. The presence of critical eigenvalues leads to computational challenges [13, 34, 39] and other undesirable phenomena like loss of stability and passivation [7, 18]. Therefore, there is significant interest in minimal structure preserving perturbations that place eigenvalues on the critical sets or remove them from

Table 1.0.1: Structured matrix polynomials: spectral symmetry and critical sets

Structure	Property	Eigenvalue symmetry	Critical set C
Complex $*$-even	$(P(-z))^* = P(\bar{z})$	$(\lambda, -\bar{\lambda})$	imaginary axis
Complex $*$-odd	$(P(-z))^* = -P(\bar{z})$	$(\lambda, -\bar{\lambda})$	imaginary axis
Complex T-even	$(P(-z))^T = P(z)$	$(\lambda, -\lambda)$	$\{0, \infty\}$
Complex T-odd	$(P(-z))^T = -P(z)$	$(\lambda, -\lambda)$	$\{0, \infty\}$
Real T-even	$(P(-z))^T = P(z)$	$(\lambda, -\bar{\lambda}, -\lambda, \bar{\lambda})$	imaginary axis
Real T-odd	$(P(-z))^T = -P(z)$	$(\lambda, -\bar{\lambda}, -\lambda, \bar{\lambda})$	imaginary axis
Real	$\overline{P(z)} = P(\bar{z})$	$(\lambda, \bar{\lambda})$	real line
Hermitian	$(P(z))^* = P(\bar{z})$	$(\lambda, \bar{\lambda})$	real line
Real symmetric	$(P(z))^T = P(z)$	$(\lambda, \bar{\lambda})$	real line
Complex $*$-palindromic	$\mathrm{rev}P(\bar{z}) = (P(z))^*$	$(\lambda, 1/\bar{\lambda})$	unit circle
Complex $*$-antipalindromic	$\mathrm{rev}P(\bar{z}) = -(P(z))^*$	$(\lambda, 1/\bar{\lambda})$	unit circle
Complex T-palindromic	$\mathrm{rev}P(z) = (P(z))^T$	$(\lambda, 1/\lambda)$	$\{1, -1\}$
Complex T-antipalindromic	$\mathrm{rev}P(z) = -(P(z))^T$	$(\lambda, 1/\lambda)$	$\{1, -1\}$
Real T-palindromic	$\mathrm{rev}P(z) = (P(z))^T$	$(\lambda, 1/\bar{\lambda}, \bar{\lambda}, 1/\lambda)$	unit circle
Real T-antipalindromic	$\mathrm{rev}P(z) = -(P(z))^T$	$(\lambda, 1/\bar{\lambda}, \bar{\lambda}, 1/\lambda)$	unit circle

the critical set. The points on the critical sets have the same structured and unstructured eigenvalue backward errors in all cases [1, 2]. However, in all our examples we have found that there is a very large difference in the structured and unstructured eigenvalue backward errors of points that do not belong to the critical set but are close to them.

Given a structured matrix polynomial $P(z)$, it is well known that algorithms that preserve the structure of the problem when computing the eigenvalues/eigenvectors are more efficient as they often require less time and memory and produce physically meaningful results because they preserve the symmetry of the problem even in the presence of rounding error [43]. As noted in [2], the backward perturbation and sensitivity analysis with respect to structure preserving perturbations is very important for determining the accuracy of solutions obtained from structure preserving algorithms. In particular, structured eigenvalue backward errors analyse the backward stability of structure preserving algorithms like the QZ algorithm for computing the eigenvalues of Hermitian pencils. For matrix polynomials of degree more than one, the strategy for solving the eigenvalue problem is to linearize the matrix polynomial which converts it into an equivalent problem of finding the eigenvalues and eigenvectors of a matrix pencil [15]. If the original matrix polynomial is structured, then a structure preserving linearization is used in this process [32]. Because of this, the structured eigenvalue backward errors for the matrix pencils is an important special case. Besides, it is well known that eigenvalues of structured matrix polynomials display very different behaviours with respect to structured preserving and arbitrary perturbations [35, 7]. Structured eigenvalue backward errors are also a stepping stone for solving distance problems. For example, if $P(z)$ is a Hermitian matrix polynomial with only real eigenvalues, then the distance to a nearest Hermitian matrix polynomial with a nonreal eigenvalue is the infimum over all eigenvalue backward errors of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with respect to Hermitian perturbations.

The unstructured eigenvalue and eigenpair backward errors are already well known (see, for example, [42] and [4, 5] where they are obtained for a variety of norms). Structured eigenpair backward errors are also well known for a variety of structures [1, 2, 3]. However, *structured eigenvalue backward errors* are not known for most of the important structures except for a few cases when they are equal to the unstructured backward errors [1, 2]. The expressions for the structured eigenpair backward errors obtained in [2, 3] demonstrate that following the strategy of finding the infimum of the structured eigenpair backward error of the pair (λ, x) over all $x \in \mathbb{C}^n \setminus \{0\}$, to compute the structured backward error of λ leads to a very challenging optimization problem.

Although there are formulas for structured eigenpair backward error for a variety of structures, the matrix polynomials are complex in all these cases [1, 3]. The case when $\mathbb{S} = (\mathbb{R}^{n\times n})^{m+1}$ which we refer to as the real eigenvalue/eigenpair backward error is a particularly challenging one which is not known even for eigenpairs. We find tight bounds for the real eigenpair backward errors with respect to three different norms. We also estimate the real eigenvalue backward error for a specified norm which gives a lower bound on the backward error as well as its exact value in certain cases. Given a real structured matrix polynomial where $\mathbb{S} \subset (\mathbb{R}^{n\times n})^{m+1}$, we obtain formulas for the structured eigenvalue backward error for certain choices of $\mathbb{S}$ and λ, and bounds on the structured eigenvalue backward error in some other cases.

The thesis is organized as follows. Chapter 1 introduces the notations, terminologies and preliminaries. It also introduces the main problem of computing the structured backward error with respect to three different norms and gives the general framework for solving the problem. Finally it states the main results that are frequently used in the thesis with proofs wherever appropriate. In Chapter 2, we find formulas for the structured eigenvalue backward error for Hermitian and related structures like skew-Hermitian and $*$-alternating with respect to the norm

$$|||(A_0, \ldots, A_m)|||_{w,2} := \sqrt{w_0^2\|A_0\|^2 + \cdots + w_m^2\|A_m\|^2}$$

where $\|\cdot\|$ is the 2-norm or spectral norm on $\mathbb{C}^{n\times n}$ and $w = (w_0, w_1, \ldots, w_m)$ is a weight vector with positive entries. In Chapter 3, we derive formulas for the structured eigenvalue backward error with respect to the norm $|||\cdot|||_{w,2}$ for T-palindromic matrix polynomials as well as their antipalindromic counterparts. It also finds these backward errors for $*$-alternating and T-alternating matrix polynomials. In Chapter 4, we consider the counterparts of the problems solved in Chapters 2 and 3 with respect to the following norms:

$$\begin{aligned} |||(A_0, \ldots, A_m)|||_{w,\infty} &:= \max\{w_0\|A_0\|, \ldots, w_m\|A_m\|\} \\ |||(A_0, \ldots, A_m)|||_{w,F} &:= \sqrt{w_0^2\|A_0\|_F^2 + \cdots + w_m^2\|A_m\|_F^2} \end{aligned}$$

where $\|\cdot\|_F$ denotes the Frobenius norm on $\mathbb{C}^{n\times n}$. Finally in Chapter 5, we consider structured matrix polynomials with structure $\mathbb{S} \subseteq (\mathbb{R}^{n\times n})^{m+1}$ and obtain results for the computation and estimation of corresponding structured eigenvalue and eigenpair backward errors. The last section of Chapter 5 deals with matrix pencils having special block structures that arise in optimal control problems. For such pencils, we compute eigenvalue and eigenpair backward errors for points lying in the corresponding critical set with respect to special structure preserving perturbations. The results in Chapter 1 and 2 have appeared in [10] while some of the results in Chapters 3 and 5 have appeared in [11].

1.1 Notations and terminology

The following notations have been used throughout the thesis.

- $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, respectively.
- $\mathbb{R}^n$ and $\mathbb{C}^n$ are the sets of real and complex vectors of length n, respectively.
- $\mathbb{C}^{m\times n}$ is the set of complex matrices of size $m \times n$.
- $e_j \in \mathbb{C}^n$ denotes the j-th standard unit vector of $\mathbb{C}^n$.
- A^* and A^T denote the complex conjugate transpose and complex transpose of the matrix A, respectively.

- A^{-1} denotes the inverse of A.
- $A^\dagger$ denotes the Moore-Penrose pseudoinverse of matrix A.
- I_n denotes the identity matrix of size n.
- $\det(A)$ denotes the determinant of the matrix A.
- $\mathrm{Ker}(A)$ denotes the kernel of the matrix A.
- $\|\cdot\|$, $\|\cdot\|_F$ and $\|\cdot\|_\infty$ denote the spectral norm, Frobenius norm and infinity norm respectively, of a vector or a matrix.
- A matrix A is called Hermitian, skew-Hermitian, symmetric and skew-symmetric if A satisfies $A^* = A$, $A^* = -A$, $A^T = A$ and $A^T = -A$, respectively.
- $v_\lambda := \sum_{j=0}^{m} \lambda^j v_j$ where $v_0, \ldots, v_m \in \mathbb{C}^n$.
- $\mathrm{Herm}(n)$ denotes the set of Hermitian matrices of size $n \times n$.
- $\mathrm{SHerm}(n)$ denotes the set of skew-Hermitian matrices of size $n \times n$.
- $\mathrm{SSym}(n)$ denotes the set of skew-symmetric matrices of size $n \times n$.
- $\mathrm{Sym}(n)$ denotes the set of symmetric matrices of size $n \times n$.
- $\lambda_{\max}(H)$ denotes the largest eigenvalue of the Hermitian matrix H.
- $\lambda_k(H)$ denotes the k^{th} largest eigenvalue of the Hermitian matrix H.
- $\sigma_{\min}(A)$ denotes the smallest singular value of the matrix A.
- $\sigma_2(A)$ denotes the second largest singular value of the matrix A.
- Given two matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$, $A \otimes B$ denotes the Kronecker product of A and B defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}, \quad \text{where} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}. \text{ [23]}$$

- $P(z)$ denotes the matrix polynomial $P(z) = \Sigma_{j=0}^{m} z^j A_j$ of size n, where $A_j \in \mathbb{C}^{n\times n}$ for $j = 0, 1, \ldots, m$.

- $\Delta(z)$ denotes the matrix polynomial $\Delta(z) = \Sigma_{j=0}^{m} z^j \Delta_j$ of size n, where $\Delta_j \in \mathbb{C}^{n\times n}$ for $j = 0, 1, \dots, m$.
- The degree of $P(z)$ is m if $A_m \neq 0$.
- A regular matrix polynomial $P(z)$ is one for which $\det P(\lambda) \neq 0$ for some $\lambda \in \mathbb{C}$.
- $\operatorname{rev} P(z)$ denotes the reversal of $P(z)$ defined by $\operatorname{rev} P(z) = z^m \Sigma_{j=0}^{m} A_j / z^j$.
- $P^*(z)$ denotes the adjoint of $P(z)$ defined by $P^*(z) = \Sigma_{j=0}^{m} z^j A_j^*$.
- $P^T(z)$ denotes the transpose of $P(z)$ defined by $P^T(z) = \Sigma_{j=0}^{m} z^j A_j^T$.

1.2 Preliminaries

Throughout the thesis we assume that $P(z) = \Sigma_{j=0}^{m} z^j A_j$ is a regular matrix polynomial of degree m. In order to measure perturbations of matrix polynomials in a flexible way, we introduce some norms on $(\mathbb{C}^{n\times n})^{m+1}$ associated with a weight vector $w \in \mathbb{R}^{m+1}$.

Definition 1.2.1. Let $\|\cdot\|$ be the spectral norm and let $w = (w_0, \dots, w_m) \in \mathbb{R}^{m+1}$, where $w_0, \dots, w_m > 0$.

1) w is called a *weight vector* and its entries w_j are called *weights*.
2) The *reciprocal weight vector* of w is defined as $w^{-1} := (w_0^{-1}, \dots, w_m^{-1})$.
3) A weight vector $w = (w_0, \dots, w_m)$ is said to be a *palindromic weight vector* if for each $j = 0, \dots, m$, $w_j = w_{m-j}$.
4) For a tuple of matrices $\Delta_0, \dots, \Delta_m \in \mathbb{C}^{n\times n}$, we define

$$|||(\Delta_0, \dots, \Delta_m)|||_{w,2} := \sqrt{w_0^2\|\Delta_0\|^2 + \cdots + w_m^2\|\Delta_m\|^2}, \qquad (1.2.1)$$

$$|||(\Delta_0, \dots, \Delta_m)|||_{w,\infty} := \max\{w_0\|\Delta_0\|, \cdots, w_m\|\Delta_m\|\}, \qquad (1.2.2)$$

$$|||(\Delta_0, \dots, \Delta_m)|||_{w,F} := \sqrt{w_0^2\|\Delta_0\|_F^2 + \cdots + w_m^2\|\Delta_m\|_F^2}, \qquad (1.2.3)$$

where 2, F and ∞ stand for spectral, Frobenius and infinity norm on the space of $n \times n$ complex matrices, respectively.

Definition 1.2.2. Let $\mathbb{S} \subseteq (\mathbb{C}^{n\times n})^{m+1}$, $P(z) = z^m A_m + \cdots + zA_1 + A_0$ be a matrix polynomial, where $(A_0, \ldots, A_m) \in \mathbb{S}$ and let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$. Furthermore, let $w = (w_0, \ldots, w_m) \in \mathbb{R}^{m+1}$ be a weight vector. For $p = 2$, $p = \infty$ and $p = F$, let

$$\eta^{\mathbb{S}}_{w,p}(P, \lambda, x) := \inf \left\{ \|(\Delta_0, \ldots, \Delta_m)\|_{w,p} \,\middle|\, \left(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j) \right) x = 0,\ (\Delta_0, \ldots, \Delta_m) \in \mathbb{S} \right\} \tag{1.2.4}$$

and

$$\eta^{\mathbb{S}}_{w,p}(P, \lambda) := \inf \left\{ \|(\Delta_0, \ldots, \Delta_m)\|_{w,p} \,\middle|\, \det \left(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j) \right) = 0,\ (\Delta_0, \ldots, \Delta_m) \in \mathbb{S} \right\} \tag{1.2.5}$$

be the structured eigenpair backward error of (λ, x) and structured eigenvalue backward error of λ with respect to $P(z)$, $\mathbb{S}$ and w, respectively.

We note that structured eigenvalue backward errors obtained in Chapter 2 and Chapter 3, can also be considered with respect to norm $\|\cdot\|_{w,1}$ defined by

$$\|(\Delta_0, \ldots, \Delta_m)\|_{w,1} := w_0 \|\Delta_0\| + \cdots + w_m \|\Delta_m\|,$$

where $\|\cdot\|$ denotes the spectral norm. Although results with respect to norm $\|\cdot\|_{w,1}$ could not be derived in the framework proposed in the thesis, it is easy to see that when the structured polynomial is a pencil then the structured eigenvalue backward errors with respect to norm $\|\cdot\|_{w,1}$ are tightly bound by those with respect to the norm $\|\cdot\|_{w,2}$, since

$$\|(\Delta_0, \Delta_1)\|_{w,2} \leq \|(\Delta_0, \Delta_1)\|_{w,1} \leq \sqrt{2}\, \|(\Delta_0, \Delta_1)\|_{w,2}.$$

1.2.1 Problem definition

Our aim will be to solve the following problem.

Problem 1.2.3. Let $\mathbb{S} \subseteq (\mathbb{C}^{n\times n})^{m+1}$ and $P(z) = \sum_{j=0}^{m} z^j A_j$ be a regular matrix polynomial of degree m where $(A_0, \ldots A_m) \in \mathbb{S}$ and $\lambda \in \mathbb{C}$. Find the smallest perturbation $(\Delta_0, \ldots, \Delta_m)$ from $\mathbb{S}$ that makes λ an eigenvalue of the perturbed polynomial. More precisely, calculate $\eta^{\mathbb{S}}(P, \lambda)$. Further, if the infimum in the definition of $\eta^{\mathbb{S}}(P, \lambda)$ is attained, then construct a minimal perturbation $\Delta(z) = \sum_{j=0}^{m} z^j \Delta_j$ that attains the infimum.

We consider the above problem with respect to the norms $\|\cdot\|_{w,2}$, $\|\cdot\|_{w,F}$ and $\|\cdot\|_{w,\infty}$. The corresponding quantities $\eta^{\mathbb{S}}(P, \lambda)$ are denoted by $\eta^{\mathbb{S}}_{w,2}(P, \lambda)$, $\eta^{\mathbb{S}}_{w,F}(P, \lambda)$ and $\eta^{\mathbb{S}}_{w,\infty}(P, \lambda)$, respectively.

Clearly, we have $\eta^{\mathbb{S}}(P,\lambda)=0$ if the matrix $P(\lambda)\in\mathbb{C}^{n\times n}$ is singular, i.e., if λ is already an eigenvalue of $P(z)$ (including the case that the matrix polynomial $P(z)$ is singular). So, in the following we always assume that $P(z)$ is regular and that $P(\lambda)$ is nonsingular.

The case of computing $\eta^{\mathbb{S}}(P,\infty)$ is equivalent to computing $\eta^{\mathbb{S}'}(\mathrm{revP},0)$ where $\mathbb{S}'=\mathbb{S}$ or closely related to $\mathbb{S}$. In fact, except for the $\bullet$-even and $\bullet$-odd polynomials of odd degree where $\bullet=*$ or $\bullet=T$, for all other structures $\mathbb{S}'=\mathbb{S}$. For these exceptional cases, $\mathbb{S}'$ is $\bullet$-even if $\mathbb{S}$ is $\bullet$-odd and vice-versa. Therefore without loss of generality, we assume that $\lambda\neq\infty$.

Remark 1.2.4. If $(A_0,\ldots,A_m)\in\mathbb{S}$ then for any norm on $(\mathbb{C}^{n\times n})^{m+1}$ we have

$$\eta^{\mathbb{S}}(P,\lambda)\leq |\!|\!|(A_0,\ldots,A_m)|\!|\!|<\infty,$$

because the perturbation with the tuple $(A_0,\ldots,A_m)$ results in the zero polynomial.

When $\mathbb{S}=(\mathbb{R}^{n\times n})^{m+1}$, the structured backward errors are referred to as real eigenvalue backward errors and denoted by $\eta^{\mathbb{R}}_{w,p}(P,\lambda)$.

If $\mathbb{S}=(\mathbb{C}^{n\times n})^{m+1}$, the backward errors are referred to as unstructured backward errors and denoted by $\eta_{w,p}(P,\lambda)$. The unstructured backward errors $\eta_{w,p}(P,\lambda)$ are well-known and are given in [5, Proposition 4.6]. For completeness we restate the result here and include a proof.

Theorem 1.2.5. *Let $P(z)=\sum_{j=0}^m z^jA_j$, where $A_0,\ldots,A_m\in\mathbb{C}^{n\times n}$ and let $\lambda\in\mathbb{C}$. Then*

$$\eta_{w,2}(P,\lambda)=\eta_{w,F}(P,\lambda)=\frac{\sigma_{\min}\big(P(\lambda)\big)}{|\!|\!|(1,\lambda,\ldots,\lambda^m)|\!|\!|_{w^{-1},2}}\quad \textit{and}\quad \eta_{w,\infty}(P,\lambda)=\frac{\sigma_{\min}\big(P(\lambda)\big)}{|\!|\!|(1,\lambda,\ldots,\lambda^m)|\!|\!|_1},$$

where $|\!|\!|(1,\lambda,\ldots,\lambda^m)|\!|\!|_1:=1+|\lambda|+\cdots+|\lambda|^m$ and $\sigma_{\min}(M)$ stands for the smallest singular value of a matrix M.

Proof. We first prove the result for $\eta_{w,2}(P,\lambda)$. Let $x\in\mathbb{C}^n\setminus\{0\}$. Then the backward error $\eta_{w,2}(P,\lambda,x)$ of the eigenpair (λ,x) is given by

$$\eta_{w,2}(P,\lambda,x):=\frac{\|(P(\lambda)x\|}{\|x\|\cdot|\!|\!|(1,\lambda,\ldots,\lambda^m)|\!|\!|_{w^{-1},2}}. \tag{1.2.6}$$

Indeed, if $\Delta_0,\ldots,\Delta_m\in\mathbb{C}^{n\times n}$ are perturbation matrices such that

$$\Delta P(\lambda)x:=\sum_{j=0}^m\lambda^j\Delta_jx=P(\lambda)x,$$

that is, (λ, x) is an eigenpair of $\sum_{j=0}^{m} z^j(A_j - \Delta_j)$, then

$$\begin{aligned}\|P(\lambda)x\| &= \|\Delta P(\lambda)x\| \leq \left\| \sum_{j=0}^{m} \lambda^j \Delta_j \right\| \cdot \|x\| = \left\| \sum_{j=0}^{m} \frac{\lambda^j}{w_j}(w_j \Delta_j) \right\| \cdot \|x\| \\ &\leq \sum_{j=0}^{m} \frac{|\lambda^j|}{w_j} w_j \|\Delta_j\| \cdot \|x\| \leq \|\!|(1, \lambda, \ldots, \lambda^m)|\!\|_{w^{-1},2} \|\!|(\Delta_0, \ldots, \Delta_m)|\!\|_{w,2} \|x\|\end{aligned}$$

using the Cauchy-Schwarz inequality. This implies the "$\geq$"-inequality in (1.2.6). On the other hand, setting

$$\Delta_j := \frac{\bar{\lambda}^j P(\lambda) x x^*}{w_j^2 \, x^* x \|\!|(1, \lambda, \ldots, \lambda^m)|\!\|_{w^{-1},2}^2} \tag{1.2.7}$$

we easily obtain $\Delta P(\lambda)x = \sum_{j=0}^{m} \lambda^j \Delta_j x = P(\lambda)x$ and equality in (1.2.6). Clearly we have $\eta_{w,2}(P, \lambda) = \min\{\eta_{w,2}(P, \lambda, x) \,|\, x \in \mathbb{C}^n \setminus \{0\}\}$, so the assertion immediately follows from (1.2.6).

Note that Δ_j is a rank one matrix for each $j = 0, \ldots, m$, as a consequence we get $\eta_{w,F}(P, \lambda) = \eta_{w,2}(P, \lambda)$.

The assertion for $\eta_{w,\infty}(P, \lambda)$ follows similarly by considering

$$\Delta_j := \frac{\bar{\lambda}^j P(\lambda) x x^*}{w_j \, |\lambda|^j \, x^* x \|\!|(1, \lambda, \ldots, \lambda^m)|\!\|_1} \tag{1.2.8}$$

for $j = 0, \ldots, m$. □

Observe that $\|\!|\cdot|\!\|_{w,2}$, $\|\!|\cdot|\!\|_{w,F}$ and $\|\!|\cdot|\!\|_{w,\infty}$ are norms on $(\mathbb{C}^{n\times n})^{m+1}$. The weights can be used to balance the importance of perturbations of individual coefficients. Note that by restricting all the entries of weight vector to be positive in the Definition 1.2.1, we are allowing all perturbations to $P(z)$ that may affect all its coefficient matrices. But there are situations where it is necessary to find the backward error $\eta^{\mathbb{S}}(P, \lambda)$ under the restriction that perturbations can affect only some of the coefficient matrices of $P(z)$. This is achieved by putting zero weights in the weight vector w. We also consider such situations and obtain the backward error $\eta^{\mathbb{S}}(P, \lambda)$ with a restricted perturbation set.

1.2.2 Idea and framework

The following lemma reformulates the determinant equation in the definition of $\eta^{\mathbb{S}}(P, \lambda)$ in terms of a collection of mapping problems.

Lemma 1.2.6. *Let $P(z) = z^m A_m + \cdots + zA_1 + A_0$ be a matrix polynomial, where $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$, let $\Delta_0, \ldots, \Delta_m \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ such that $M := P(\lambda)^{-1}$ exists. Then the following statements are equivalent.*

(a) $\det\Big(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\Big) = 0.$

(b) *There exist vectors* $v_0, \ldots, v_m \in \mathbb{C}^n$ *satisfying* $\sum_{j=0}^m \lambda^j v_j \neq 0$ *such that*

$$v_j = \Delta_j M(v_0 + \lambda v_1 + \cdots + \lambda^m v_m), \quad \text{for } j = 0, \ldots, m.$$

Proof. Denote $\widetilde{P}(\lambda) := \sum_{j=0}^m \lambda^j (A_j - \Delta_j)$.
(a) $\Rightarrow$ (b): If (a) holds then there exists $x \neq 0$ such that $\widetilde{P}(\lambda)x = 0$. Let $v_j := \Delta_j\, x$ for $j = 0, \ldots, m$. Then we have

$$P(\lambda)x = P(\lambda)x - \widetilde{P}(\lambda)x = \sum_{j=0}^{m} \lambda^j \Delta_j\, x = \sum_{j=0}^{m} \lambda^j v_j =: v_\lambda. \tag{1.2.9}$$

We have $v_\lambda \neq 0$ because $P(\lambda) = M^{-1}$ is nonsingular by assumption. On multiplying (1.2.9) from the left with $\Delta_j M$ we obtain the identities $v_j = \Delta_j M v_\lambda$ for $j = 0, \ldots, m$.

(b) $\Rightarrow$ (a): Suppose that (b) holds and set $v_\lambda := \sum_{j=0}^m \lambda^j v_j$. Then

$$\widetilde{P}(\lambda) M v_\lambda = \Big(P(\lambda) - \sum_{j=0}^{m} \lambda^j \Delta_j\Big) M v_\lambda = v_\lambda - \sum_{j=0}^{m} \lambda^j \Delta_j\, M v_\lambda = 0,$$

because $\Delta_j\, M v_\lambda = v_j$ for $j = 0, \ldots, m$. Since $M v_\lambda \neq 0$, this implies (a). $\square$

Note that when $P(z) = \sum_{j=0}^m z^j A_j$ is structured, i.e. $(A_0, \ldots, A_m) \in \mathbb{S}$, then due to Lemma 1.2.6 the determinant equation in the definition of $\eta^{\mathbb{S}}(P, \lambda)$ can be reformulated in terms of finding solutions of a collection of mapping problems involving structured matrices which depend on the structure of $P(z)$. We will refer to such problems as structured mapping problems.

Our principal strategy for computing structured eigenvalue backward errors is as follows. For Hermitian and related structures like skew-Hermitian and $*$-alternating as well as $*$-palindromic and $*$-antipalindromic structures, the reformulation in Lemma 1.2.6 allows us to convert the original problem into an equivalent optimization problem of maximizing the Rayleigh Quotient of a Hermitian matrix with respect to specified conditions involving Hermitian matrices. We then follow the strategy suggested by Karow in [26] and convert the problem of computing the structured backward error into an equivalent problem of minimizing the maximal eigenvalue of a parameter-depending Hermitian matrix. This results in a convex optimization problem. Therefore the resulting structured backward error can be computed by using convex optimization algorithms like CVX [17, 16], as is done in our numerical examples.

In case of other structures like T-palindromic, T-alternating, etc., we reformulate the original problem of finding structured eigenvalue backward error into an equivalent optimization problem of maximizing the Rayleigh Quotient of a Hermitian matrix with respect to specified conditions involving symmetric matrices. When there is only one constraint, this problem becomes equivalent to minimizing the second smallest eigenvalue of a Hermitian matrix due to the following result from [26]. The resulting optimization problems are not convex but it is shown in the thesis later that these problems attain a global minimum inside a closed region. Such structured backward errors are computed using function *fminsearch* in MATLAB for small size structured matrix polynomials of degree at most 4.

Theorem 1.2.7. *[26] Let* $H \in \mathrm{Herm}(n)$ *and* $S \in \mathrm{Sym(n)}$ *with rank(S)* ≥ 2*. Then*

$$\sup\left\{v^* H v : v \in \mathbb{C}^n, v^T S v = 0, \|v\| = 1\right\} = \min_{0 \leq t \leq t_1} \lambda_2\left(\begin{bmatrix} H & t\bar{S} \\ tS & \bar{H} \end{bmatrix}\right),$$

where $\lambda_2\Big(\begin{bmatrix} H & t\bar{S} \\ tS & \bar{H} \end{bmatrix}\Big)$ *denotes the second largest eigenvalue of matrix* $\begin{bmatrix} H & t\bar{S} \\ tS & \bar{H} \end{bmatrix}$ *and* $t_1 = \dfrac{2\|H\|}{\sigma_2(S)}$*, where* $\sigma_2(S)$ *denotes the second largest singular value of* S*.*

1.2.3 Minimal norm mappings

Given $x, y \in \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the problem of finding a matrix $\Delta \in \mathbb{K}^{n\times n}$ such that $\Delta x = y$ has been solved in [30, 44]. The following result characterizes all such matrices.

Theorem 1.2.8. *[44] Let* $x \in \mathbb{K}^n \setminus \{0\}$ *and* $b \in \mathbb{K}^n$*, where* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$*. Then there exist* $\Delta \in \mathbb{K}^{n\times n}$ *satisfying* $\Delta x = b$ *if and only if* Δ *is of the form*

$$\Delta = bx^\dagger + Z\left(I_n - xx^\dagger\right),$$

where $Z \in \mathbb{K}^{n\times n}$ *is arbitrary and* $x^\dagger$ *is the Moore-Penrose pseudoinverse of* x*. Further the minimal spectral norm and Frobenius norm of such* Δ *is* $\|b\|/\|x\|$ *and is attained by* $\Delta = bx^\dagger$*.*

Complete solutions of more general mapping problems when Δ is restricted to have some extra properties are presented in [33]. In this section we state a few results for some such mapping problems that are repeatedly used in the thesis with proofs wherever appropriate.

Consider the following Hermitian mapping problem.

Under which conditions on vectors $x, y \in \mathbb{C}^n$ does there exists a $H \in \mathrm{Herm}(n)$ satisfying $Hx = y$?

The answer to this problem is well known, see, e.g., [33] where solutions that are minimal with respect to the spectral or Frobenius norm are also characterized. We also refer to [27] and [41] for the more general problem of the existence of a Hermitian matrix $H \in \mathbb{C}^{n\times n}$ such that $HX = Y$ for two matrices $X, Y \in \mathbb{C}^{n\times m}$. The following theorem gives an answer to the Hermitian mapping problem in terms that allow a direct application in this thesis.

Theorem 1.2.9. *Let $x, y \in \mathbb{C}^n$, $x \neq 0$. Then there exists a matrix $H \in \mathrm{Herm}(n)$ such that $Hx = y$ if and only if $\mathrm{Im}\,(x^*y) = 0$. If the latter condition is satisfied, then*

$$\min\left\{ \|H\| \;\middle|\; H \in \mathrm{Herm}(n),\ Hx = y\right\} = \frac{\|y\|}{\|x\|}$$

and the minimum is attained for

$$H_0 := \frac{\|y\|}{\|x\|}\begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|}\end{bmatrix}\begin{bmatrix} \frac{y^*x}{\|x\|\,\|y\|} & 1 \\ 1 & \frac{x^*y}{\|x\|\,\|y\|}\end{bmatrix}^{-1}\begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|}\end{bmatrix}^*. \tag{1.2.10}$$

if x and y are linearly independent, and for $H_0 := \dfrac{yx^*}{x^*x}$ otherwise.

Proof. The identity $Hx = y$ immediately implies $\mathrm{Im}\,(x^*y) = \mathrm{Im}\,(x^*Hx) = 0$, because H is Hermitian, and

$$\|H\| \geq \|y\|/\|x\| =: c.$$

In particular, this proves the "only if"-part of the statement of the theorem.

Conversely, let $\mathrm{Im}\,(x^*y) = 0$. Suppose first that x and y are linearly independent. Then H_0 given as in (1.2.10) is well defined and Hermitian, and we immediately obtain

$$H_0\begin{bmatrix} \frac{x}{\|x\|} & \frac{y}{\|y\|}\end{bmatrix} = \frac{\|y\|}{\|x\|}\begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|}\end{bmatrix}$$

which implies $H_0x = y$ and $H_0y = c^2x$. Thus, $y \pm cx$ are eigenvectors of H_0 associated with the eigenvalues $\pm c$, respectively, which implies $\|H_0\| = c$.

On the other hand, if x and y are linearly dependent, then $y = \alpha\, x$ with $\alpha \in \mathbb{R}$. Also the matrix $H_0 = \alpha xx^*/\|x\|^2$ is Hermitian and satisfies $H_0x = y$ and since H_0 has rank 1, $\|H_0\| = c$. ∎

The following result gives a minimal spectral norm Hermitian map that is also of minimal rank and minimal Frobenius norm.

Theorem 1.2.10. *[33] Let $x, b \in \mathbb{K}^n \setminus \{0\}$ and $\mathcal{M} = \{A \in \mathrm{Herm}(n) |\, Ax = b\}$. Assume further that x, b are vectors such that $\mathcal{M}$ is nonempty. Consider the following mapping*

$$A := \frac{\|b\|}{\|x\|} U^* \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} U,$$

where $R := sgn(\mu)$ if $b = \mu x$ for some $\mu \in \mathbb{K}$, and $R := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ otherwise, and U can be taken as the product of at most two unitary Householder reflectors; when $\mathbb{K} = \mathbb{R}$, U is real orthogonal. Then A is the unique solution of the minimal rank problem $\min_{A \in \mathcal{M}} rank(A)$ and the minimal Frobenius norm problem $\min_{A \in \mathcal{M}} \|A\|_F$. Moreover,

(1) $\|A\| = \|b\|/\|x\|$

(2) $\|A\|_F = \|b\|/\|x\|$, *when A has rank one.*

(2) $\|A\|_F = \sqrt{2}\|b\|/\|x\|$, *when A has rank two.*

The following theorem from [25] contains solutions to the minimal Frobenius and minimal spectral norm mapping problems that are also real matrices.

Theorem 1.2.11. *[25] Let $(x, y) \in \mathbb{C}^l \setminus \{0\} \times \mathbb{C}^q$. Set*

$$\Delta_0 = \begin{cases} \begin{bmatrix} \mathrm{Re}\, y & \mathrm{Im}\, y \end{bmatrix} \begin{bmatrix} \mathrm{Re}\, x & \mathrm{Im}\, x \end{bmatrix}^\dagger & \text{if } \mathrm{Re}\, x \text{ and } \mathrm{Im}\, x \text{ are linearly independent.} \\ \|x\|^{-2} y x^* & \text{if } \mathrm{Re}\, x \text{ and } \mathrm{Im}\, x \text{ are linearly dependent.} \end{cases}$$

Then

(1) $\Delta_0 \in \mathbb{R}^{q \times l}$ *and* $\Delta_0 x = y$.

(2) $\inf\{\|\Delta\| \mid \Delta \in \mathbb{R}^{q \times l},\ \Delta x = y\} = \|\Delta_0\|$.

Now consider the following problem. Let $X \in \mathbb{C}^{m \times p}$, $Y \in \mathbb{C}^{n \times p}$, $Z \in \mathbb{C}^{n \times k}$ and let $S \in \mathbb{C}^{m \times k}$. Define

$$\mathcal{S} := \left\{ \Delta \in \mathbb{C}^{n \times m} \,|\, \Delta X = Y,\ \Delta^* Z = S \right\}.$$

Under which conditions on vectors X, Y, Z and S, is $\mathcal{S}$ nonempty? Further if $\mathcal{S} \neq \phi$, then find the minimal spectral norm and minimal Frobenius norm maps from $\mathcal{S}$.

We answer this problem and find matrices in the set $\mathcal{S}$ that are minimal with respect to the spectral norm and the Frobenius norm. In doing so, we follow the strategy suggested in [1] that invokes the following theorem of Davis, Kahan and Weinberger [12].

Theorem 1.2.12. *[12] Let A, B, C be given matrices. Then for any positive number μ satisfying*

$$\mu \geq \max\left(\left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|, \left\| \begin{bmatrix} A & C \end{bmatrix} \right\| \right) \tag{1.2.11}$$

there exists D such that $\left\| \begin{bmatrix} A & C \\ B & D \end{bmatrix} \right\| \leq \mu$. All such matrices D which have this property are exactly of the form

$$D = -KA^*L + \mu(I - KK^*)^{1/2} Z (I - LL^*)^{1/2}$$

where $K^* = (\mu^2 I - A^*A)^{-1/2} B^*, L = (\mu^2 I - AA^*)^{-1/2} C$ and Z is an arbitrary contraction, that is, $\|Z\| \leq 1$.

Theorem 1.2.13. *Let $X \in \mathbb{C}^{m\times p}$, $Y \in \mathbb{C}^{n\times p}$, $Z \in \mathbb{C}^{n\times k}$ and $S \in \mathbb{C}^{m\times k}$. Assume that* $\mathrm{rank}(X) = r_1$ *and* $\mathrm{rank}(Z) = r_2$. *Consider the SVD: $X = U\Sigma V^*$ and $Z = \hat{U}\hat{\Sigma}\hat{V}^*$, where $U = [U_1,\, U_2]$, $U_1 \in \mathbb{C}^{m\times r_1}$ and $\hat{U} = [\hat{U}_1,\, \hat{U}_2]$, $\hat{U}_1 \in \mathbb{C}^{n\times r_2}$. Then*

(1) $\mathcal{S} \neq \phi$ if and only if $X^*S = Y^*Z$, $YX^\dagger X = Y$ and $SZ^\dagger Z = S$.

(2) $A \in \mathcal{S}$ if and only if A is of the form

$$A = YX^\dagger + (SZ^\dagger)^* - (SZ^\dagger)^* XX^\dagger + (I - ZZ^\dagger) R (I - XX^\dagger)$$

for some $R \in \mathbb{C}^{n\times m}$.

(3) ***Frobenius norm :*** *The matrix*

$$G(X, Y, Z, S) := YX^\dagger + (SZ^\dagger)^* - (SZ^\dagger)^* XX^\dagger \tag{1.2.12}$$

is a unique map such that $G(X, Y, Z, S)X = Y$ and $G(X, Y, Z, S)^*Z = S$ and

$$\inf_{\Delta \in \mathcal{S}} \|\Delta\|_F^2 = \|G(X, Y, Z, S)\|_F^2 = \|YX^\dagger\|_F^2 + \|SZ^\dagger\|_F^2 - \mathrm{Trace}\big((SZ^\dagger)(SZ^\dagger)^* XX^\dagger\big).$$

(4) ***Spectral norm :*** *We have*

$$\inf_{\Delta \in \mathcal{S}} \|\Delta\| = \max\left\{\|YX^\dagger\|,\, \|SZ^\dagger\|\right\} =: \mu \tag{1.2.13}$$

Indeed, the infimum in (1.2.13) *is attained by the matrix*

$$F(X, Y, Z, S) := YX^\dagger + (SZ^\dagger)^* - (SZ^\dagger)^* XX^\dagger + (I - ZZ^\dagger)\hat{U}_2 R U_2^* (I - XX^\dagger), \tag{1.2.14}$$

where

$$\begin{aligned} R &= -K\big(U_1^*(YX^\dagger)\hat{U}_1\big)L + \mu(I-KK^*)^{\frac{1}{2}}P(I-L^*L)^{\frac{1}{2}}, \\ K^* &= \big(\mu^2 I - U_1^*(YX^\dagger)^* ZZ^\dagger (YX^\dagger)U_1\big)^{-\frac{1}{2}}(\hat{U}_2^* YX^\dagger U_1) \\ \text{and} \qquad L &= \big(\mu^2 I - \hat{U}_1^*(YX^\dagger)(YX^\dagger)^*\hat{U}_1\big)^{-\frac{1}{2}}\big(\hat{U}_1^*(SZ^\dagger)^* U_2\big), \end{aligned}$$

P being an arbitrary contraction.

Proof. Proof of (1): Suppose that $\mathcal{S} \neq \emptyset$. Then there exists $\Delta \in \mathbb{C}^{n\times m}$ such that $\Delta X = Y$ and $\Delta^* Z = S$. This implies that $X^*S = Y^*Z$. Also the ranges of Y^* and S^* are contained in those of X^* and Z^* respectively. Since $X^\dagger X$ and $Z^\dagger Z$ are orthogonal projections onto the ranges of X^* and Z^* respectively, it follows that $YX^\dagger X = Y$ and $SZ^\dagger Z = S$.

Now suppose that $X^*S = Y^*Z$, $YX^\dagger X = Y$ and $SZ^\dagger Z = S$. Then $\mathcal{S} \neq \emptyset$ because for any $R \in \mathbb{C}^{n\times m}$, the matrix

$$A = YX^\dagger + (SZ^\dagger)^* - (SZ^\dagger)^* XX^\dagger + (I - ZZ^\dagger)R(I - XX^\dagger) \in \mathcal{S}.$$

Proof of (2): Let $\Sigma_1 \in \mathbb{R}^{r_1\times r_1}$ be the principal component of Σ formed by it first r_1 rows and columns. Similarly let $\hat{\Sigma}_1 \in \mathbb{R}^{r_2\times r_2}$ be the principal component of $\hat{\Sigma}$ formed by its first r_2 rows and columns. Also let $V = [V_1, V_2]$ and $\hat{V} = [\hat{V}_1, \hat{V}_2]$ be the conformal partitions of V and $\hat{V}$, respectively, such that $X = U_1\Sigma_1 V_1^*$ and $Z = \hat{U}_1\hat{\Sigma}_1\hat{V}_1^*$ are condensed SVDs of X and Z respectively. Let $\Delta \in \mathcal{S}$. If $R(X)$ and $R(Y)$ denote the ranges of X and Y respectively, then Δ is a linear map $\Delta : R(X) \oplus R(X)^\perp \to R(Z) \oplus R(Z)^\perp$ and $\Delta = \hat{U}\hat{U}^*\Delta UU^*$. Set $\hat{\Delta} := \hat{U}^*\Delta U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11} \in \mathbb{C}^{r_1\times r_2}$, $A_{12} \in \mathbb{C}^{r_1\times(m-r_2)}$, $A_{21} \in \mathbb{C}^{(n-r_1)\times r_2}$ and $A_{22} \in \mathbb{C}^{(n-r_1)\times(m-r_2)}$. Then $\|\Delta\|_F = \|\hat{\Delta}\|_F$ and $\|\Delta\| = \|\hat{\Delta}\|$. Also

$$\begin{aligned} \Delta X = Y \Rightarrow \hat{U}\hat{\Delta}U^*X = Y \ &\Rightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} X = \begin{bmatrix} \hat{U}_1^* \\ \hat{U}_2^* \end{bmatrix} Y \\ &\Rightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \Sigma_1 V_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{U}_1^* Y \\ \hat{U}_2^* Y \end{bmatrix}. \end{aligned}$$

Thus

$$A_{11} = \hat{U}_1^* Y V_1 \Sigma_1^{-1} \ \text{ and } \ A_{21} = \hat{U}_2^* Y V_1 \Sigma_1^{-1}. \tag{1.2.15}$$

Also,

$$\Delta^* Z = S \Rightarrow U\hat{\Delta}^*\hat{U}^* Z = S \;\Rightarrow \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} \hat{U}_1^* \\ \hat{U}_2^* \end{bmatrix} Z = \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} S$$
$$\Rightarrow \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_1 \hat{V}_1^* \\ 0 \end{bmatrix} = \begin{bmatrix} U_1^* S \\ U_2^* S \end{bmatrix}.$$

This gives

$$A_{11} = (\hat{\Sigma}_1^{-1})^* \hat{V}_1^* S^* U_1 \text{ and } A_{12} = (\hat{\Sigma}_1^{-1})^* \hat{V}_1^* S^* U_2. \tag{1.2.16}$$

From (1.2.15) and (1.2.16), $\hat{U}_1^* Y V_1 \Sigma_1^\dagger = (\hat{\Sigma}_1^\dagger)^* \hat{V}_1^* S^* U_1$. Note that the later equality holds as $X^*S = Y^*Z$. Therefore, we can write

$$\hat{\Delta} = \begin{bmatrix} \hat{U}_1^* Y V_1 \Sigma_1^{-1} & (\hat{\Sigma}_1^{-1})^* \hat{V}_1^* S^* U_2 \\ \hat{U}_2^* Y V_1 \Sigma_1^{-1} & A_{22} \end{bmatrix} = \begin{bmatrix} \hat{U}_1^* Y X^\dagger U_1 & \hat{U}_1^* (SZ^\dagger)^* U_2 \\ \hat{U}_2^* Y X^\dagger U_1 & A_{22} \end{bmatrix}. \tag{1.2.17}$$

Thus

$$\begin{aligned} \Delta &= \hat{U}\hat{\Delta}U^* = \hat{U}_1\hat{U}_1^* Y X^\dagger U_1 U_1^* + \hat{U}_2\hat{U}_2^* Y X^\dagger U_1 U_1^* + \hat{U}_1\hat{U}_1^*(SZ^\dagger)^* U_2 U_2^* + \hat{U}_2 A_{22} U_2^* \\ &= \hat{U}_1\hat{U}_1^* Y X^\dagger U_1 U_1^* + (I - \hat{U}_1\hat{U}_1^*) Y X^\dagger U_1 U_1^* + \hat{U}_1\hat{U}_1^*(SZ^\dagger)^*(I - U_1U_1^*) + \hat{U}_2 A_{22} U_2^* \\ &= YX^\dagger + (SZ^\dagger)^* - (SZ^\dagger)^* X X^\dagger + (I - ZZ^\dagger)\hat{U}_{22} A_{22} U_2^* (I - XX^\dagger). \end{aligned} \tag{1.2.18}$$

Proof of (3): Note that in the view of (1.2.17),

$$\begin{aligned} \|\Delta\|_F^2 = \|\hat{\Delta}\|_F^2 &= \left\| \begin{bmatrix} \hat{U}_1^* Y X^\dagger U_1 \\ \hat{U}_2^* Y X^\dagger U_1 \end{bmatrix} \right\|_F^2 + \|\hat{U}_1^*(SZ^\dagger)^* U_2\|_F^2 + \|A_{22}\|_F^2 \\ &= \|YX^\dagger\|_F^2 + \|SZ^\dagger\|_F^2 - \text{Trace}\big((SZ^\dagger)(SZ^\dagger)^* XX^\dagger\big) + \|A_{22}\|_F^2. \end{aligned}$$

Thus by setting $A_{22} = 0$ in (1.2.18), we get a unique map from $\mathcal{S}$ which minimizes the Frobenius norm, i.e.,

$$\inf_{\Delta \in \mathcal{S}} \|\Delta\|_F^2 = \|YX^\dagger\|_F^2 + \|SZ^\dagger\|_F^2 - \text{Trace}\big((SZ^\dagger)(SZ^\dagger)^* XX^\dagger\big).$$

Proof of (4): Again consider the map $\hat{\Delta}$ given in (1.2.17) and set

$$\begin{aligned} \mu : &= \max\left\{ \left\| \begin{bmatrix} \hat{U}_1^* Y X^\dagger U_1 \\ \hat{U}_2^* Y X^\dagger U_1 \end{bmatrix} \right\|, \left\| \begin{bmatrix} \hat{U}_1^* Y X^\dagger U_1 & \hat{U}_1^*(SZ^\dagger)^* U_2 \end{bmatrix} \right\| \right\} \\ &= \max\left\{ \|YX^\dagger\|,\ \|SZ^\dagger\| \right\}. \end{aligned}$$

Then it follows that for any $\Delta \in \mathcal{S}$, $\|\Delta\| = \|\hat{\Delta}\| \geq \mu$. Now by equation (1.2.11), we have $\|\Delta\| = \mu$, i.e., $\inf_{\Delta\in\mathcal{S}} \|\Delta\| = \mu$ which is attained by

$$\begin{aligned} A_{22} &= -K\big(U_1^*(YX^\dagger)\hat{U}_1\big)L + \mu(I-KK^*)^{\frac{1}{2}}P(I-L^*L)^{\frac{1}{2}}, \\ \text{where } K^* &= \big(\mu^2 I - U_1^*(YX^\dagger)^* ZZ^\dagger (YX^\dagger)U_1\big)^{-\frac{1}{2}}(\hat{U}_2^* YX^\dagger U_1), \\ L &= \big(\mu^2 I - \hat{U}_1^*(YX^\dagger)(YX^\dagger)^*\hat{U}_1\big)^{-\frac{1}{2}}\big(\hat{U}_1^*(SZ^\dagger)^* U_2\big), \end{aligned}$$

and P is an arbitrary contraction. Hence the proof follows by setting $R = A_{22}$. □

A particular case of Theorem 1.2.13 when $p = k = 1$ is obtained in [24, Theorem 2]. We restate the result for this case and include explicit formulae which give minimal Frobenius norm and minimal spectral norm solutions that are also of minimal rank.

For $u \in \mathbb{C}^m$, $r \in \mathbb{C}^n \setminus \{0\}$, $w \in \mathbb{C}^n$ and $s \in \mathbb{C}^m \setminus \{0\}$, define

$$\hat{\mathcal{S}} := \left\{\Delta \in \mathbb{C}^{n\times m} \,|\, \mathrm{rank}(\Delta) \leq 2,\ \Delta u = r,\ \Delta^* w = s\right\}.$$

Theorem 1.2.14. *Let $u \in \mathbb{C}^m$, $r \in \mathbb{C}^n \setminus \{0\}$, $w \in \mathbb{C}^n$ and $s \in \mathbb{C}^m \setminus \{0\}$ be such that $\|u\| = 1$ and $\|w\| = 1$. Define $\delta := s^*u$, $x := r - \delta w$ and $y := s - \bar{\delta}u$. Then*

(1) $\hat{\mathcal{S}} \neq \phi$ if and only if $u^*s = r^*w$.

(2) $\hat{\mathcal{S}} = \{E_\beta \in \mathbb{C}^{n\times m} \,|\, E_\beta = xu^* + wy^* + \delta wu^* - \beta xy^*,\ \beta \in \mathbb{C}\}$.

(3) ***Frobenius norm:***

$$\min_{\Delta\in\hat{\mathcal{S}}} \|\Delta\|_F^2 = \|r\|^2 + \|s\|^2 - |s^*u|^2. \tag{1.2.19}$$

Moreover, (1.2.19) *has a unique solution from $\hat{\mathcal{S}}$ given by*

$$\widetilde{\Delta} := \begin{bmatrix} w & \frac{x}{\|x\|} \end{bmatrix} \begin{bmatrix} s^*u & \|y\| \\ \|x\| & 0 \end{bmatrix} \begin{bmatrix} u & \frac{y}{\|y\|} \end{bmatrix}^*$$

if $\|r\|^2 - |\delta|^2 \neq 0$ and $\|s\|^2 - |\delta|^2 \neq 0$. If $\|r\|^2 = |\delta|^2$ then (1.2.19) *is attained by $\hat{\Delta} := ws^*$ and by $\hat{\Delta} := ru^*$ if $\|s\|^2 = |\delta|^2$.*

(4) ***Spectral norm:***

$$\min_{\Delta\in\hat{\mathcal{S}}} \|\Delta\| = \max\{\|r\|, \|s\|\}. \tag{1.2.20}$$

Moreover, if $\|r\| \geq \|s\|$ then the minimum in (1.2.20) *is attained by*

$$\hat{\Delta} := \begin{bmatrix} w & \frac{x}{\|x\|} \end{bmatrix} \begin{bmatrix} s^*u & \|y\| \\ \|x\| & -\frac{\|y\|}{(s^*u)\frac{\|y\|}{\|x\|}} \end{bmatrix} \begin{bmatrix} u & \frac{y}{\|y\|} \end{bmatrix}^*$$

if $\|r\|^2 - |\delta|^2 \neq 0$ *and by* $\hat{\Delta} := ws^*$ *if* $\|r\|^2 = |\delta|^2$. *If* $\|r\| \leq \|s\|$ *then the minimum in* (1.2.20) *is attained by*

$$\hat{\Delta} := \left[\begin{array}{cc} w & \frac{x}{\|x\|} \end{array}\right] \left[\begin{array}{cc} s^*u & \|y\| \\ \|x\| & -(s^*u)\frac{\|x\|}{\|y\|} \end{array}\right] \left[\begin{array}{cc} u & \frac{y}{\|y\|} \end{array}\right]^*$$

if $\|s\|^2 - |\delta|^2 \neq 0$ *and by* $\hat{\Delta} := ru^*$ *if* $\|s\|^2 = |\delta|^2$.

Note that we will restate Theorem 1.2.14 in Chapter 3 (Theorem 3.1.3, page 44) and give a more simplified proof for the case when $w = u$. This has a direct application in finding backward errors of palindromic matrix polynomials.

Lemma 1.2.15. *[7] Let* $\Delta_1, \Delta_2 \in \mathbb{C}^{n\times p}$. *Then* $[\Delta_1\ \overline{\Delta}_1]\,[\Delta_2\ \overline{\Delta}_2]^\dagger$ *is a real matrix.*

In the view of above lemma, the following corollary of the Theorem 1.2.13 gives a real minimal Frobenius norm solution of the mapping problem considered in the theorem.

Corollary 1.2.16. *Let* $X \in \mathbb{C}^{m\times p}$, $Y \in \mathbb{C}^{n\times p}$, $Z \in \mathbb{C}^{n\times k}$ *and* $S \in \mathbb{C}^{m\times k}$ *be such that* $\mathrm{rank}([X\ \overline{X}]) = 2p$ *and* $\mathrm{rank}([Z\ \overline{Z}]) = 2k$, *then there exist a real matrix* $\Delta \in \mathbb{R}^{n\times m}$ *such that* $\Delta X = Y$ *and* $\Delta^* Z = S$ *if and only if* $X^*S = Y^*Z$ *and* $X^TS = Y^TZ$. *In such a case, the matrix* $G_{\mathbb{R}} := G\big([X\ \overline{X}][Y\ \overline{Y}][Z\ \overline{Z}][S\ \overline{S}]\big)$ *obtained by replacing* X, Y, Z *and* S *by* $[X\ \overline{X}], [Y\ \overline{Y}], [Z\ \overline{Z}]$ *and* $[S\ \overline{S}]$ *respectively in (1.2.12) is a real matrix with* $G_{\mathbb{R}}X = Y$ *and* $G_{\mathbb{R}}^*Z = S$. *Furthermore, among all real matrices* Δ *with* $\Delta X = Y$ *and* $\Delta^*Z = S$ *the matrix* $G_{\mathbb{R}}$ *has the smallest Frobenius norm.*

Proof. If Δ is any real matrix with $\Delta X = Y$ and $\Delta^*Z = S$, then $\Delta\overline{X} = \overline{Y}$ and $\Delta^*\overline{Z} = \overline{S}$. Hence $\Delta[X\ \overline{X}] = [Y\ \overline{Y}]$ and $\Delta^*[Z\ \overline{Z}] = [S\ \overline{S}]$. By Theorem 1.2.13 there exist such a Δ iff the conditions $[X\ \overline{X}]^*[S\ \overline{S}] = [Y\ \overline{Y}]^*[Z\ \overline{Z}]$, $[Y\ \overline{Y}][X\ \overline{X}]^\dagger[X\ \overline{X}] = [Y\ \overline{Y}]$ and $[Z\ \overline{Z}][S\ \overline{S}]^\dagger[S\ \overline{S}] = [Z\ \overline{Z}]$ are satisfied. The first condition is equivalent to $X^*S = Y^*Z$ and $X^TS = Y^TZ$ and since $\mathrm{rank}([X\ \overline{X}]) = \mathrm{rank}([Z\ \overline{Z}]) = 2k$, the last two conditions are always satisfied. Therefore there exists $\Delta \in \mathbb{R}^{n\times m}$ such that $\Delta X = Y$ and $\Delta^*Z = S$ if and only if $X^*S = Y^*Z$ and $X^TS = Y^TZ$.

Clearly, if the given conditions hold then the matrix $G_{\mathbb{R}}$ satisfies $G_{\mathbb{R}}[X\ \overline{X}] = [Y\ \overline{Y}]$ and $G_{\mathbb{R}}^*[Z\ \overline{Z}] = [S\ \overline{S}]$. Moreover by Theorem 1.2.13, among all matrices Δ such that $\Delta[X\ \overline{X}] = [Y\ \overline{Y}]$ and $\Delta^*[Z\ \overline{Z}] = [S\ \overline{S}]$, $G_{\mathbb{R}}$ has the smallest Frobenius norm. The fact that $G_{\mathbb{R}}$ is real follows from Lemma 1.2.15. $\square$

1.2.4 Minimizing the maximal eigenvalue of a Hermitian matrix function

The computation of the structured backward error of eigenvalues of structured matrix polynomials of degree m will lead to a minimization problem of a function of the form

$$L : \mathbb{R}^{m+1} \to \mathbb{R}, \quad (t_0, \dots, t_m) \mapsto \lambda_{\max}(G + t_0 H_0 + \cdots + t_m H_m)$$

for some Hermitian matrices $G, H_0, \dots, H_m \in \mathbb{C}^{n\times n}$. In order to analyze the extrema of L, we first need information on the partial differentiability of these kinds of functions. To this end, the following theorem from [8, page 149] or [36] provides useful information.

Theorem 1.2.17. *[8] Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and let the map $L : \mathbb{R} \to \mathbb{R}$ be given by $L(t) := \lambda_{\max}(G + tH)$. Let the columns of the isometric matrix $U \in \mathbb{C}^{n\times m}$ form an (orthonormal) basis of the eigenspace of the eigenvalue $\lambda_{\max}(G)$ of G. Then the left and right directional derivatives of L in $t = 0$ exists and we have*

$$\begin{aligned} \frac{d}{dt}L(0)_+ &:= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\lambda_{\max}(G+\varepsilon H) - \lambda_{\max}(G)}{\varepsilon} = \lambda_{\max}(U^*HU), \\ \frac{d}{dt}L(0)_- &:= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\lambda_{\max}(G-\varepsilon H) - \lambda_{\max}(G)}{-\varepsilon} = \lambda_{\min}(U^*HU). \end{aligned}$$

If, in particular, $m = 1$, then L is differentiable in $t = 0$, $u := U \in \mathbb{C}^n \setminus \{0\}$, and

$$\frac{d}{dt}L(0) = \lambda_{\max}(U^*HU) = u^*Hu.$$

With these preparations, we are able to state and prove one of the main results of the thesis. This result will be used repeatedly in the successive chapters. Note that in this and all other results, we define a Hermitian matrix to be indefinite if it has at least one positive and one negative eigenvalues.

Theorem 1.2.18. *Let $G, H_0, \dots, H_m \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Assume that any nonzero linear combination $\alpha_0 H_0 + \cdots + \alpha_m H_m$, $(\alpha_0, \dots, \alpha_m) \in \mathbb{R}^{m+1} \setminus \{0\}$ is indefinite. Then the following statements hold:*

(1) *The function $L : \mathbb{R}^{m+1} \to \mathbb{R}$, $(t_0, \dots, t_m) \mapsto \lambda_{\max}(G + t_0 H_0 + \cdots + t_m H_m)$ is convex and has a global minimum*

$$\lambda^\star_{\max} := \min_{t_0,\dots,t_m\in\mathbb{R}} L(t_0, \dots, t_m).$$

(2) *If the minimum* $\lambda_{\max}^\star$ *of* L *is attained at* $(t_0^\star, \ldots, t_m^\star) \in \mathbb{R}^{m+1}$ *and is a simple eigenvalue of* $H_\star := G + t_0^\star H_0 + \cdots + t_m^\star H_m$*, then there exists an eigenvector* $u \in \mathbb{C}^n \setminus \{0\}$ *of* $H_\star$ *associated with* $\lambda_{\max}^\star$ *satisfying*

$$u^* H_j u = 0 \quad \text{for} \quad j = 0, \ldots, m. \tag{1.2.21}$$

(3) *Under the assumptions of* (2), we have

$$\sup \left\{ \frac{u^* G u}{u^* u} \,\middle|\, u \in \mathbb{C}^n \setminus \{0\},\ u^* H_j u = 0,\ j = 0, \ldots, m \right\} = \lambda_{\max}^\star. \tag{1.2.22}$$

In particular, the supremum of the left hand side of (1.2.22) is a maximum and is attained for the eigenvector u in (2).

Proof. (1) The convexity of L is straightforward to check. Concerning the proof that L has a global minimum, we will show that there exists a constant $\varrho > 0$ such that for all $(t_0, \ldots, t_m)$ with $t_0^2 + \cdots + t_m^2 > \varrho^2$ we have $L(t_0, \ldots, t_m) \geq L(0, \ldots, 0)$. Since the closed ball

$$\mathcal{B}_\varrho := \{(t_0, \ldots, t_m) \in \mathbb{R}^{m+1} \,|\, t_0^2 + \cdots + t_m^2 \leq \varrho^2\}$$

with center in the origin and radius ϱ is compact and since L is continuous as eigenvalues depend continuously on the entries of a matrix, L has a global minimum $\lambda_{\max}^\star \leq L(0, \ldots, 0)$ on $\mathcal{B}_\varrho$. By construction we then have $\lambda_{\max}^\star \leq L(t_0, \ldots, t_m)$ for all $(t_0, \ldots, t_m) \in \mathbb{R}^{m+1}$, i.e., $\lambda_{\max}^\star$ is the global minimum of L. Thus, define

$$c := \inf \left\{ \lambda_{\max}(\alpha_0 H_0 + \cdots + \alpha_m H_m) \,\middle|\, (\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1},\ \alpha_0^2 + \cdots + \alpha_m^2 = 1 \right\}.$$

Then $c \geq 0$, because by hypothesis the matrix $\alpha_0 H_0 + \cdots + \alpha_m H_m$ is indefinite for all $(\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1}$ with $\alpha_0^2 + \cdots + \alpha_m^2 = 1$, i.e., it always has at least one positive eigenvalue. Since the function $f : (\alpha_0, \ldots, \alpha_m) \mapsto \lambda_{\max}(\alpha_0 H_0 + \cdots + \alpha_m H_m)$ is continuous (again using the well known fact that eigenvalues depend continuously on the entries of a matrix), the infimum c is attained, because of the compactness of the unit sphere in $\mathbb{R}^{m+1}$. This implies $c > 0$, because the function f only takes positive values on the unit sphere. Next, define

$$\varrho := \frac{\lambda_{\max}(G) - \lambda_{\min}(G)}{c} \geq 0.$$

Let $(t_0, \ldots, t_m) \in \mathbb{R}^{m+1}$ and $r \geq \varrho$ so that $t_0^2 + \cdots + t_m^2 = r^2 \geq \varrho^2$. Using the fact that for two Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$ we have $\lambda_{\max}(A + B) \geq \lambda_{\max}(A) + \lambda_{\min}(B)$, (see

[22]), we obtain

$$\begin{aligned} L(t_0,\dots,t_m) &= \lambda_{\max}(G+t_0H_0+\cdots+t_mH_m) \geq \lambda_{\max}\big(t_0H_0+\cdots+t_mH_m\big)+\lambda_{\min}(G) \\ &= r\cdot\lambda_{\max}\left(\frac{t_0}{r}H_0+\cdots+\frac{t_m}{r}H_m\right)+\lambda_{\min}(G) \\ &\geq \varrho\cdot c+\lambda_{\min}(G)=\lambda_{\max}(G)=L(0,\dots,0), \end{aligned}$$

This finishes the proof of (1).

(2) By step (1), the minimum $\lambda_{\max}^{\star}$ of L exists and by assumption it is attained at some point $(t_0^\star,\dots,t_m^\star)\in\mathbb{R}^{m+1}$ and is a simple eigenvalue of the corresponding matrix $G+t_0^\star H_0+\cdots+t_m^\star H_m$. Then, it follows from Theorem 1.2.17 that L is differentiable at $(t_0^\star,\dots,t_m^\star)$ with partial derivatives,

$$\frac{\partial L}{\partial t_j}(t_0^\star,\dots,t_m^\star)=u^*H_j\,u,\quad j=0,\dots,m,$$

where u is an eigenvector of $G+t_0^\star H_0+\cdots+t_m^\star H_m$ associated with $\lambda_{\max}^{\star}$ satisfying $\|u\|=1$. Since $\lambda_{\max}^{\star}$ is the global minimum of L, this implies $u^*H_ju=0$ for $j=0,\dots,m$.

(3) Let $s^\star$ denote the left hand side of (1.2.22). We show that $s^\star=\lambda_{\max}^{\star}$.

"$\geq$": By (2), there exists an eigenvector $u\in\mathbb{C}^n\setminus\{0\}$ of $G+t_0^\star H_0+\cdots+t_m^\star H_m$ associated with $\lambda_{\max}^{\star}$ satisfying $u^*H_ju=0$ for $j=0,\dots,m$. Thus, we obtain

$$\lambda_{\max}^{\star}=\frac{u^*(G+t_0^\star H_0+\cdots+t_m^\star H_m)u}{u^*u}=\frac{u^*Gu}{u^*u}$$

which implies that $s^\star\geq\lambda_{\max}^{\star}$.

"$\leq$": Let $u\in\mathbb{C}^n\setminus\{0\}$ be an arbitrary vector satisfying $u^*H_j\,u=0$ for $j=0,\dots,m$ (by "$\geq$" there do exists such vectors). Then, we obtain

$$\frac{u^*Gu}{u^*u}=\frac{u^*(G+t_0^\star H_0+\cdots+t_m^\star H_m)u}{u^*u}\leq\lambda_{\max}(G+t_0^\star H_0+\cdots+t_m^\star H_m)=\lambda_{\max}^{\star}.$$

Since u was arbitrary, this implies $s^\star\leq\lambda_{\max}^{\star}$. This completes the proof. □

Remark 1.2.19. We highlight that the applicability of Theorem 1.2.18 relies heavily on the fact that the eigenvalue $\lambda_{\max}^{\star}$ is a simple eigenvalue. This need not be the case always as the following example shows.

Example 1.2.20. Consider the Hermitian 2×2 matrices $G=0$,

$$H_0=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\quad\text{and}\quad H_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Then for $t_0, t_1 \in \mathbb{R}$, the matrix

$$H(t_0, t_1) = G + t_0 H_0 + t_1 H_1 = \begin{bmatrix} t_0 & t_1 \\ t_1 & -t_0 \end{bmatrix}$$

has the eigenvalues $\pm\sqrt{t_0^2 + t_1^2}$ which implies in particular that any nonzero linear combination $\alpha_0 H_0 + \alpha_1 H_1$ is indefinite. Moreover, the function $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(t_0, t_1) := \lambda_{\max}\big(H(t_0, t_1)\big)$ has its minimum at $(t_0^\star, t_1^\star) = (0, 0)$ with value $\lambda_{\max}^\star = 0$ which happens to be a double eigenvalue of the zero matrix $H(0, 0)$. Nevertheless, the vector $u = \begin{bmatrix} 1 & i \end{bmatrix}^\top$ is an eigenvector of $H(0, 0)$ associated with $\lambda_{\max}^\star = 0$ satisfying $u^* H_0 u = 0 = u^* H_1 u$.

Example 1.2.20 suggests that the statement (2) of Theorem 1.2.18 may still be true even without the hypothesis of $\lambda_{\max}^\star$ being a simple eigenvalue. The next theorem shows that in the case of the pencils when $m = 1$, this is indeed always the case.

Theorem 1.2.21. *Let $G, H_0, H_1 \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Assume that any linear combination $\alpha_0 H_0 + \alpha_1 H_1$, $(\alpha_0, \alpha_1) \in \mathbb{R}^2 \setminus \{0\}$ is indefinite. Then the following statements hold:*

(1) *The function $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(t_0, t_1) := \lambda_{\max}(G + t_0 H_0 + t_1 H_1)$ is convex and has a global minimum $\lambda_{\max}^\star$.*

(2) *If the minimum $\lambda_{\max}^\star$ of L is attained at $(t_0^\star, t_1^\star) \in \mathbb{R}^2$, then there exists an eigenvector $u \in \mathbb{C}^n \setminus \{0\}$ of $G + t_0^\star H_0 + t_1^\star H_1$ associated with $\lambda_{\max}^\star$ satisfying*

$$u^* H_0 u = 0 = u^* H_1 u. \tag{1.2.23}$$

(3) *We have*

$$\sup\left\{ \frac{u^* G u}{u^* u} \,\middle|\, u \in \mathbb{C}^n \setminus \{0\},\ u^* H_0 u = 0,\ u^* H_1 u = 0 \right\} = \min_{t_0, t_1 \in \mathbb{R}} L(t_0, t_1) = \lambda_{\max}^\star. \tag{1.2.24}$$

In particular, the supremum of the left hand side of (1.2.24) *is a maximum and is attained for the eigenvector u in* (2).

Proof. In view of Theorem 1.2.18, it remains to prove (2) for the case that $\lambda_{\max}^\star$ is a multiple eigenvalue of $G + t_0^\star H_0 + t_1^\star H_1$. Let the columns of $U \in \mathbb{C}^{n\times m}$ form an orthonormal basis of the eigenspace of $G + t_0^\star H_0 + t_1^\star H_1$ associated with $\lambda_{\max}^\star$. Moreover, let $\alpha_0, \alpha_1 \in \mathbb{R}$ such that $\alpha_0^2 + \alpha_1^2 = 1$. By Theorem 1.2.17, we obtain the existence of the limit of the one-sided derivatives at $t = 0$ of the function

$$t \mapsto L(t_0^\star + \alpha_0 t, t_1^\star + \alpha_1 t) = \lambda_{\max}\big((G + t_0^\star H_0 + t_1^\star H_1) + t(\alpha_0 H_0 + \alpha_1 H_1)\big)$$

and this limit must be nonnegative, because there is a global minimum at $(t_0^\star, t_1^\star)$. More precisely, we obtain from Theorem 1.2.17 that

$$\lambda_{\max}\big(U^*(\alpha_0 H_0 + \alpha_1 H_1)U\big) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{L(t_0^\star + \alpha_0\varepsilon, t_1^\star + \alpha_1\varepsilon) - L(t_0^\star, t_1^\star)}{\varepsilon} \geq 0$$

for all $\alpha_0, \alpha_1 \in \mathbb{R}$ with $\alpha_0^2 + \alpha_1^2 = 1$. Thus, for all such $\alpha = (\alpha_0, \alpha_1)$ there exists an eigenvector $x_\alpha \in \mathbb{C}^m$, $\|x_\alpha\| = 1$ associated with $\lambda_{\max}\big(U^*(\alpha_0 H_0 + \alpha_1 H_1)U\big)$ such that

$$x_\alpha^* U^*(\alpha_0 H_0 + \alpha_1 H_1)U x_\alpha = \lambda_{\max}\big(U^*(\alpha_0 H_0 + \alpha_1 H_1)U\big) \geq 0 \tag{1.2.25}$$

We now show the existence of a vector $x \in \mathbb{C}^m$ with $\|x\| = 1$ such that

$$x^* U^* H_0 U x = 0 = x^* U^* H_1 U x. \tag{1.2.26}$$

Then $u = Ux$ is the desired eigenvector of $G + t_0^\star H_0 + t_1^\star H_1$ satisfying (1.2.21).

Recall that the joint numerical range of two Hermitian matrices $F_1, F_2 \in \mathbb{C}^{n \times n}$ is the set

$$\mathcal{W}_0(F_1, F_2) := \{(x^* F_1 x, x^* F_2 x) \in \mathbb{R}^2 \mid x \in \mathbb{C}^n, \|x\| = 1\}.$$

Thus the existence of a vector x with $\|x\| = 1$ satisfying (1.2.26) is equivalent to the fact that zero is in the joint numerical range $\mathcal{W}_0 := \mathcal{W}_0(U^* H_0 U, U^* H_1 U)$ of the matrices $U^* H_0 U$ and $U^* H_1 U$. Thus, let us assume that zero is *not* in $\mathcal{W}_0$. Since $\mathcal{W}_0$ is a closed convex set [23], by [21, Theorem 4.11, page 51] this implies the existence of $\widetilde{\alpha} = [\widetilde{\alpha}_0, \widetilde{\alpha}_1]^\top \in \mathbb{R}^2 \backslash \{0\}$ (without loss of generality we may assume $\widetilde{\alpha}_0^2 + \widetilde{\alpha}_1^2 = 1$) with

$$0 > \left\langle \widetilde{\alpha}, \begin{bmatrix} x^* U^* H_0 U x \\ x^* U^* H_1 U x \end{bmatrix} \right\rangle = x^* U^* (\widetilde{\alpha}_0 H_0 + \widetilde{\alpha}_1 H_1) U x$$

for all $x \in \mathbb{C}^m$ with $\|x\| = 1$ contradicting (1.2.25). Hence, zero is in the joint numerical range of $U^* H_0 U$ and $U^* H_1 U$ which finishes the proof of (2) and thus of the theorem. $\square$

Remark 1.2.22. If $m > 1$ in the above result, then since 0 is in the joint numerical range of the $m \times m$ Hermitian matrices $U^* H_0 U$ and $U^* H_1 U$, the Hermitian pencil $zU^* H_0 U + U^* H_1 U$ is not a definite pencil (see, [40] for details). Therefore its eigenvalues do not satisfy the conditions that characterize definite pencils as specified in Theorem 3.2 of [6]. These facts may be used in the numerical computation of the eigenvector x corresponding to $\lambda_{\max}^\star$ such that $x^* U^* H_0 U x = x^* U^* H_1 U x = 0$ when $\lambda_{\max}^\star$ is a multiple eigenvalue of $G + t_0^\star H_0 + t_1^\star H_1$.

Remark 1.2.23. Unfortunately, the argument in the proof of Theorem 1.2.21 cannot be generalized to the case $m > 1$, because the joint numerical range of three or more Hermitian matrices need not be convex.

Example 1.2.24. Consider the Hermitian 3×3 matrices $G = \mathrm{diag}(\alpha, \alpha, \beta)$, where $\alpha > \beta \geq 0$, and

$$H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad H_2 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then for $t_0, t_1, t_2 \in \mathbb{R}$, the matrix

$$H(t_0, t_1, t_2) = G + t_0 H_0 + t_1 H_1 + t_2 H_2 = \begin{bmatrix} \alpha + t_0 & t_1 + it_2 & 0 \\ t_1 - it_2 & \alpha - t_0 & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

has the eigenvalues β and $\alpha \pm \sqrt{t_0^2 + t_1^2 + t_2^2}$. Again, any nonzero linear combination $\alpha_0 H_0 + \alpha_1 H_1 + \alpha_2 H_2$ is indefinite. Similar to Example 1.2.20, the function $L : \mathbb{R}^3 \to \mathbb{R}$ given by $L(t_0, t_1, t_2) = \lambda_{\max}\big(H(t_0, t_1, t_2)\big)$ has its minimum in $(0, 0, 0)$ with value $\lambda_{\max}^\star = \alpha$ which happens to be a double eigenvalue of the matrix $H(0, 0, 0) = G$. In this case, a matrix whose columns form an orthonormal bases of the eigenspace of $H(0, 0, 0)$ associated with α is the 3×2 matrix $U = [e_1 \ e_2]$, where e_1 and e_2 denote the first two standard basis vectors. One easily checks that zero is not in the joint numerical range of

$$U^* H_0 U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; U^* H_1 U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \text{and } U^* H_2 U = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad (1.2.27)$$

and hence no eigenvector u of $H(0, 0)$ associated with $\lambda_{\max}^\star = \alpha$ satisfies $u^* H_j u = 0$ for $j = 0, 1, 2$.

Note that in this example, scalar multiples of the third standard basis vector e_3 are the only vectors u satisfying $u^* H_j u = 0$ for $j = 0, 1, 2$ which shows that the left hand side of (1.2.22) in Theorem 1.2.18 equals β which is strictly less than $\alpha = \lambda_{\max}^\star$.

The three Hermitian matrices in (1.2.27) are a classical example for Hermitian matrices whose joint numerical range is not convex [9, 19].

Chapter 2

Structured eigenvalue backward errors under $|||\cdot|||_{w,2}$ norm: Hermitian and related structures

In this chapter, we derive a formula for the backward error of a complex number λ when considered as an approximate eigenvalue of a regular Hermitian matrix pencil or polynomial $P(z) = \sum_{j=0}^{m} z^j A_j$ under Hermitian perturbations measured with respect to the $|||\cdot|||_{w,2}$ norm as defined in (1.2.1). If $P(z)$ is a skew-Hermitian matrix polynomial then $Q(z) = iP(z)$ is a Hermitian matrix polynomial. Likewise, if $P(z)$ is a $*$-even ($*$-odd) matrix polynomial then $Q(z) = P(iz)$ ($Q(z) = iP(iz)$) is a Hermitian matrix polynomial. Due to these facts, structured eigenvalue backward errors of approximate eigenvalues of skew-Hermitian, $*$-even and $*$-odd matrix pencils and polynomials are also obtained in terms of the appropriate Hermitian eigenvalue backward error. Numerical experiments suggest that in many cases there is a significant difference between the backward errors with respect to perturbations that preserve structure and those with respect to arbitrary perturbations.

2.1 Framework for the Hermitian backward error

Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a Hermitian matrix polynomial, i.e, $(A_0, \ldots, A_m) \in (\mathrm{Herm(n)})^{m+1}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ be such that $M = (P(\lambda))^{-1}$ exists. Let $w = (w_0, \ldots, w_m)$ be a weight vector. In the view of Lemma 1.2.6 we have the following.

$$\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)=\inf\Big\{|||(\Delta_0,\ldots,\Delta_m)|||_{w,2}\,\Big|\,(\Delta_0,\ldots,\Delta_m)\in(\mathrm{Herm(n)})^{m+1},\ \exists\, v_0,\ldots,v_m\in\mathbb{C}^n,\ v_\lambda:=\textstyle\sum_{j=0}^m\lambda^j v_j\neq 0,\ v_j=\Delta_j M v_\lambda,\ j=0,\ldots,m\Big\}. \tag{2.1.1}$$

By Theorem 1.2.9, there exist $\Delta_j\in\mathrm{Herm(n)}$ such that $\Delta_j M v_\lambda=v_j$ for $j=0,\ldots,m$ if and only if $v_j^* M v_\lambda\in\mathbb{R}$ for each $j=0,\ldots,m$. Also the minimal 2-norm of all such Δ_js is $\frac{\|v_j\|}{\|Mv_\lambda\|}$.

As briefly mentioned in Chapter 1, the aim is to reformulate the original problem of computing $\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)$ into an equivalent optimization problem of maximizing Rayleigh Quotient of a Hermitian matrix under some constraints involving Hermitian matrices. We further convert this reformulated problem into an equivalent problem of minimizing the maximal eigenvalue of a parameter-depending Hermitian matrix using the strategy suggested by M. Karow in [26].

In order to illustrate the main ideas, let us first consider the pencil case $m=1$. Thus, for the moment, assume that $P(z)=A_0+zA_1$ and for simplicity let us consider the $|||.|||_{w,2}$ norm (1.2.1) with weight vector $w=(1,1)$. In view of (2.1.1), we need to find vectors $v_0,v_1\in\mathbb{C}^n$ with $v_\lambda:=v_0+\lambda v_1\neq 0$ and matrices $\Delta_0,\Delta_1\in\mathrm{Herm}(n)$ of minimal norm such that

$$v_0=\Delta_0 M v_\lambda \quad\text{and}\quad v_1=\Delta_1 M v_\lambda, \tag{2.1.2}$$

where $M:=P(\lambda)^{-1}$. By Theorem 1.2.9 the minimal norm $|||(\Delta_0,\Delta_1)|||_{w,2}$ for a fixed pair (v_0,v_1) is then given by

$$|||(\Delta_0,\Delta_1)|||_{w,2}^2=\|\Delta_0\|^2+\|\Delta_1\|^2=\frac{\|v_0\|^2}{\|Mv_\lambda\|^2}+\frac{\|v_1\|^2}{\|Mv_\lambda\|^2}=\frac{\|v_0\|^2+\|v_1\|^2}{\|M(\lambda v_1+v_0)\|^2}.$$

Setting

$$v:=\begin{bmatrix} v_0\\ v_1\end{bmatrix},\quad\text{and}\quad G:=\begin{bmatrix} M^*M & \lambda\, M^*M\\ \bar\lambda\, M^*M & |\lambda|^2\, M^*M\end{bmatrix},$$

and using $\|M(v_0+\lambda v_1)\|^2=(v_0^*+\bar\lambda v_1^*)M^*M(v_0+\lambda v_1)$ we obtain

$$|||(\Delta_0,\Delta_1)|||_{w,2}^2=\frac{\|v_0\|^2+\|v_1\|^2}{\|M(v_0+\lambda v_1)\|^2}=\frac{v^*v}{v^*\begin{bmatrix} M^*M & \lambda\, M^*M\\ \bar\lambda\, M^*M & |\lambda|^2\, M^*M\end{bmatrix}v}=\frac{v^*v}{v^*Gv} \tag{2.1.3}$$

which is just the reciprocal of the Rayleigh quotient of v with respect to the Hermitian matrix G. Since this quantity is minimal in norm for a fixed pair (v_0,v_1), we now have to minimize (2.1.3) over all admissible pairs (v_0, v_1), i.e., all pairs for which there exists $\Delta_j \in \mathrm{Herm}(n)$, $j = 0, 1$ such that (2.1.2) is satisfied. By Theorem 1.2.9 those are exactly the pairs (v_0, v_1) satisfying $\mathrm{Im}\left(v_0^* M(v_0 + \lambda v_1)\right) = 0 = \mathrm{Im}\left(v_1^* M(v_0 + \lambda v_1)\right)$ and $v_0 + \lambda v_1 \neq 0$. Setting

$$H_0 := i \begin{bmatrix} M - M^* & \lambda M \\ -\bar{\lambda} M^* & 0 \end{bmatrix} \quad \text{and} \quad H_1 := i \begin{bmatrix} 0 & -M^* \\ M & \lambda M - \bar{\lambda} M^* \end{bmatrix}$$

these identities can be reformulated as

$$\begin{aligned} 0 &= -2\,\mathrm{Im}\left(v_0^* M(v_0 + \lambda v_1)\right) = i\left(v_0^* M(v_0 + \lambda v_1) - (M(v_0 + \lambda v_1))^* v_0\right) \\ &= i\left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} M & \lambda M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} - \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} M^* & 0 \\ \bar{\lambda} M^* & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}\right) \\ &= v^* H_0 v, \end{aligned} \tag{2.1.4}$$

and

$$\begin{aligned} 0 &= -2\,\mathrm{Im}\left(v_1^* M(v_0 + \lambda v_1)\right) = i\left(v_1^* M(v_0 + \lambda v_1) - (M(v_0 + \lambda v_1))^* v_1\right) \\ &= i\left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} 0 & 0 \\ M & \lambda M \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} - \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}^* \begin{bmatrix} 0 & M^* \\ 0 & \bar{\lambda} M^* \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}\right) \\ &= v^* H_1 v. \end{aligned} \tag{2.1.5}$$

Observe that $v^* G v = \|M(v_0 + \lambda v_1)\|^2 \neq 0$ if and only if $v_0 + \lambda v_1 \neq 0$. Thus, from (2.1.1) that for $\mathbb{S} = (\mathrm{Herm}(n))^2$ we have

$$\begin{aligned} \left(\eta_{w,2}^{\mathbb{S}}(P, \lambda)\right)^2 &= \inf\left\{ |||(\Delta_0, \Delta_1)|||_{w,2}^2 \,\middle|\, \Delta_j \in \mathrm{Herm}(n),\ \exists\, v_0, v_1 \in \mathbb{C}^n : \lambda v_1 + v_0 \neq 0, \right. \\ &\qquad\qquad \left. v_j = \Delta_j M(\lambda v_1 + v_0), j = 0, 1 \right\} \\ &= \inf\left\{ \frac{v^* v}{v^* G v} \,\middle|\, v \in \mathbb{C}^{2n},\ v^* G v \neq 0,\ v^* H_0 v = 0,\ v^* H_1 v = 0 \right\} \\ &= \left(\sup\left\{ \frac{v^* G v}{v^* v} \,\middle|\, v \in \mathbb{C}^{2n} \setminus \{0\},\ v^* H_0 v = 0,\ v^* H_1 v = 0 \right\}\right)^{-1}. \end{aligned} \tag{2.1.6}$$

Note that in the latter identity the condition $v^* G v \neq 0$ could be dropped, because $\eta_{w,2}^{\mathbb{S}}(P, \lambda)$ is finite which implies that the supremum in (2.1.6) will be positive. Therefore including vectors v satisfying $v^* G v = 0$ will not change the supremum of the considered set.

From these observations, we see that the structured backward error $\eta_{w,2}^{\mathbb{S}}(P, \lambda)$ can be computed by maximizing a Rayleigh quotient under two constraints. Since for Hermitian matrices the maximum of the Rayleigh quotient is equal to the maximal eigenvalue, the idea is to introduce the function

$$L : \mathbb{R}^2 \to \mathbb{R}, \quad (t_0, t_1) \mapsto \lambda_{\max}(G + t_0 H_0 + t_1 H_1). \tag{2.1.7}$$

and invoke Theorem 1.2.18. This will establish that the supremum in (2.1.6) coincides with the global minimum of L and give a formula to compute $\eta^{\mathbb{S}}_{w,2}(P, \lambda)$. With this objective in mind, we will prove that the matrices G, H_0 and H_1 satisfy the conditions of Theorem 1.2.18.

We extend these ideas of the pencil case in the next section to obtain backward errors of approximate eigenvalues of Hermitian polynomials under structure preserving perturbations.

2.2 Structured eigenvalue backward errors of Hermitian polynomials

In this section, we consider eigenvalue backward error problems for Hermitian polynomials. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $P(z) = \sum_{j=0}^{m} z^j A_j$ be a Hermitian matrix polynomial i.e., $(A_0, \ldots, A_m) \in (\mathrm{Herm}(n))^{m+1}$. Note that if $\lambda = 0$ then $\eta^{\mathrm{Herm}}_{w,2}(P, \lambda) = w_0 \sigma_{\min}(A_0)$. The unstructured backward error $\eta_{w,2}(P, \lambda) := \eta^{\mathbb{S}}_{w,2}(P, \lambda)$ with $\mathbb{S} = (\mathbb{C}^{n\times n})^{m+1}$ is well-known and as given in Theorem 1.2.5 we have,

$$\eta_{w,2}(P, \lambda) = \frac{\sigma_{\min}\big(P(\lambda)\big)}{\|\!|(1, \lambda, \ldots, \lambda^m)\|\!|_{w^{-1},2}}.$$

In the following, we require our perturbation matrices to be Hermitian as well, that is, we want to compute the structured backward error $\eta^{\mathrm{Herm}}_{w,2}(P, \lambda) := \eta^{\mathbb{S}}_{w,2}(P, \lambda)$, where $\mathbb{S} := (\mathrm{Herm}(n))^{m+1}$. If $\lambda \in \mathbb{R}$, then there is no difference between the structured and the unstructured case. This fact was shown in [1, 3] for the weight vector $w = (1, \ldots, 1)$ and easily generalizes to arbitrary weight vectors.

Theorem 2.2.1. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$, where $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$ are Hermitian, and let $\lambda \in \mathbb{R}$. Then*

$$\eta^{\mathrm{Herm}}_{w,2}(P, \lambda) = \eta_{w,2}(P, \lambda) = \frac{\sigma_{\min}\big(P(\lambda)\big)}{\|\!|(1, \lambda, \ldots, \lambda^m)\|\!|_{w^{-1},2}}.$$

Proof. If λ is real, then the perturbation matrices Δ_j in (1.2.7) are Hermitian, which implies the desired result. □

Due to Theorem 2.2.1 in the following we compute $\eta^{\mathrm{Herm}}_{w,2}(P, \lambda)$ if $\mathrm{Im}\,\lambda \neq 0$.

Theorem 2.2.2. *Let $P(z) = \sum_{j=0}^m z^j A_j$, where $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$ are Hermitian, and $w = (w_0, \ldots, w_m)$ be a weight vector. Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Im} \lambda \neq 0$ and $\det P(\lambda) \neq 0$ so that $M := P(\lambda)^{-1}$ exists. Let $\Lambda_m := [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$ and set*

$$\widetilde{G} := (\Lambda_m^* \Lambda_m) \otimes (M^*M) = \begin{bmatrix} M^*M & \lambda M^*M & \ldots & \lambda^m M^*M \\ \bar{\lambda}M^*M & |\lambda|^2 M^*M & \ldots & \bar{\lambda}\lambda^m M^*M \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\lambda}^m M^*M & \lambda\bar{\lambda}^m M^*M & \ldots & |\lambda|^{2m} M^*M \end{bmatrix},$$

$$\widetilde{H}_j := i\big((e_{j+1}\Lambda_m) \otimes M - (\Lambda_m^* e_{j+1}^*) \otimes M^*\big) = i \begin{bmatrix} & & -\bar{\lambda}^0 M^* & & \\ & & \vdots & & \\ \lambda^0 M \cdots & & \lambda^j M - \bar{\lambda}^j M^* & & \cdots \lambda^m M \\ & & \vdots & & \\ & & -\bar{\lambda}^m M^* & & \end{bmatrix},$$

for $j = 0, \ldots, m$, where e_{j+1} denotes the $(j+1)$th standard basis vector of $\mathbb{R}^{m+1}$ and

$$W := \operatorname{diag}(w_0, \ldots, w_m) \otimes I_n, \quad G = W^{-1}\widetilde{G}W^{-1}, \quad H_j = W^{-1}\widetilde{H}_j W^{-1} \tag{2.2.1}$$

for $j = 0, \ldots, m$. Then

$$\lambda_{\max}^\star := \min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0 H_0 + \cdots + t_m H_m)$$

is attained for some $(t_0^\star, \ldots, t_m^\star) \in \mathbb{R}^{m+1}$. If $m = 1$ or $\lambda_{\max}^\star$ is a simple eigenvalue of $G + t_0^\star H_0 + \cdots + t_m^\star H_m$, then

$$\eta_{w,2}^{\mathrm{Herm}}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^\star}} = \left(\min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0 H_0 + \cdots + t_m H_m) \right)^{-1/2}.$$

Proof. Let $v_0, \ldots, v_m \in \mathbb{C}^n$ with $v_\lambda := \sum_{j=0}^m \lambda^j v_j \neq 0$ and set $v := [v_0^T, \ldots, v_m^T]^T$. By Theorem 1.2.9, there exist $\Delta_j \in \operatorname{Herm}(n)$ satisfying

$$v_j = \Delta_j M v_\lambda, \quad j = 0, \ldots, m \tag{2.2.2}$$

if and only if $v_j^* M v_\lambda \in \mathbb{R}$ for $j = 0, \ldots, m$. As in (2.1.4) and (2.1.5) these conditions can be reformulated as $m+1$ Hermitian constraints $v^*\widetilde{H}_j v = 0$. If these conditions are fulfilled then according to Theorem 1.2.9 the minimal norms of $\Delta_j \in \operatorname{Herm}(n)$ satisfying (2.2.2) are given by $\|\Delta_j\| = \|v_j\|/\|Mv_\lambda\|$, $j = 0, \ldots, m$. Setting $u := Wv$, by reasons identical to those used to establish (2.1.3), the minimal norm of a tuple $(\Delta_0, \ldots, \Delta_m) \in \operatorname{Herm}(n)^{m+1}$ satisfying (2.2.2) is given by

$$|||(\Delta_0, \ldots, \Delta_m)|||_{w,2}^2 = \frac{w_0^2\|v_0\|^2 + \cdots + w_m^2\|v_m\|^2}{\|Mv_\lambda\|^2} = \frac{v^*W^2v}{v^*\widetilde{G}v} = \frac{u^*u}{u^*Gu}.$$

Observe that for any vector $v = [v_0^T, \ldots, v_m^T]^T$ we have $0 \neq u^*Gu = \|Mv_\lambda\|^2$ if and only if $v_\lambda = \lambda^m v_m + \cdots + \lambda v_1 + v_0 \neq 0$, and that $v^* \widetilde{H}_j v = u^* H_j u$. Thus, we have

$$\begin{aligned}\left(\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)\right)^2 &= \inf\left\{ \frac{u^*u}{u^*Gu} \,\middle|\, u \in \mathbb{C}^{2n},\ u^*Gu \neq 0,\ u^*H_ju = 0,\ j = 0,\ldots,m \right\} \\ &= \sup\left\{ \frac{u^*Gu}{u^*u} \,\middle|\, u \in \mathbb{C}^{2n} \setminus \{0\},\ u^*H_ju = 0,\ j = 0,\ldots,m \right\}^{-1}. \end{aligned} \tag{2.2.3}$$

Note that since $\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)$ is finite and positive, the supremum in the latter equality of (2.2.3) will not be attained by vectors u satisfying $u^*Gu = 0$ and therefore, the condition $u^*Gu \neq 0$ is superfluous for it.

Since our aim is to apply Theorem 1.2.21 or Theorem 1.2.18 for the case of the pencils and polynomials of degree greater than one, respectively, we need to check whether each nontrivial linear combination of $H_0, \ldots, H_m$, or, equivalently, of $\widetilde{H}_0, \ldots, \widetilde{H}_m$, is indefinite. For checking this, assume that $\alpha := [\alpha_0, \ldots, \alpha_m]^T \in \mathbb{R}^{m+1}$ is such that $H := \sum_{j=0}^m \alpha_j \widetilde{H}_j$ is semidefinite. Then

$$H = i\sum_{j=0}^{m} \alpha_j \left((e_{j+1}\Lambda_m) \otimes M - (\Lambda_m^* e_{j+1}^*) \otimes M^* \right) = i\left((\alpha\Lambda_m) \otimes M - (\Lambda_m^* \alpha^T) \otimes M^* \right)$$

and we have to show that $\alpha = 0$. Setting

$$Q := \begin{bmatrix} 1 & -\lambda & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & -\lambda \\ 0 & \ldots & 0 & 1 \end{bmatrix} \quad \text{and} \quad a := \begin{bmatrix} a_0 \\ \vdots \\ a_m \end{bmatrix} := Q^*\alpha,$$

we obtain $\Lambda_m Q = e_1^T$ and hence

$$\begin{aligned} &(Q \otimes I_n)^* H (Q \otimes I_n) \\ = \ & i\left((ae_1^T) \otimes M - (e_1 a^*) \otimes M^* \right) = i \begin{bmatrix} a_0M - \bar{a}_0M^* & -\bar{a}_1M^* & \ldots & -\bar{a}_mM^* \\ a_1M & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_mM & 0 & \ldots & 0 \end{bmatrix}. \end{aligned}$$

Since H is semidefinite and M is invertible, it follows that $a_1 = \cdots = a_m = 0$, i.e., $a = a_0e_1$. In particular, $a_1 = 0$ implies that $\alpha_1 - \bar{\lambda}\alpha_0 = 0$. But this implies that $\alpha_0 = \alpha_1 = 0$, because α_0, α_1 are real and $\operatorname{Im}\lambda \neq 0$. Finally, the first entry of $Q^*\alpha$ yields the identity $a_0 = \alpha_0 = 0$ and thus $a = 0$ which implies $\alpha = 0$.

If $m = 1$ then the assertion follows immediately from (2.2.3) and Theorem 1.2.21. On the other hand, if $m > 1$ then with the additional assumption on $\lambda_{\max}^{\star}$, the assertion follows similarly from (2.2.3) and Theorem 1.2.18. $\square$

Since $\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)$ can be computed without any assumptions when $m = 1$, whereas for the case $m > 1$ we need the assumption that $\lambda_{\max}^{\star}$ is simple, we state the pencil case separately for the sake of future reference.

Theorem 2.2.3. *Let* $P(z) = A_0 + zA_1$, *where* $A_0, A_1 \in \mathrm{Herm}(n)$. *Let* $\lambda \in \mathbb{C}$ *such that* $\operatorname{Im}\lambda \neq 0$ *and let* $w = (w_0, w_1)$ *be a weight vector. Suppose* $\det P(\lambda) \neq 0$ *so that* $M := P(\lambda)^{-1}$ *exists. Then*

$$\eta_{w,2}^{\mathrm{Herm}}(P,\lambda) = \left(\min_{t_0,t_1\in\mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + t_1\, H_1)\right)^{-1/2},$$

where

$$G \;:=\; W^{-1}\begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2 M^*M \end{bmatrix} W^{-1}, \qquad H_0 \;:=\; i\,W^{-1}\begin{bmatrix} M - M^* & \lambda M \\ -\bar{\lambda}M^* & 0 \end{bmatrix} W^{-1},$$

$$H_1 \;:=\; i\,W^{-1}\begin{bmatrix} 0 & -M^* \\ M & \lambda M - \bar{\lambda}M^* \end{bmatrix} W^{-1}, \qquad W := \mathrm{diag}(w_0 I_n, w_1 I_n).$$

Remark 2.2.4. In view of Example 1.2.24 it is crucial that the eigenvalue $\lambda_{\max}^{\star}$ in Theorem 2.2.2 is a simple eigenvalue when $m > 1$. Numerical experiments suggest that generically this is indeed the case. When $m = 1$, and $\lambda_{\max}^{\star}$ is an eigenvalue of $G + t_0^{\star}H_0 + t_1^{\star}H_1$ of multiplicity say $r > 1$ then a corresponding eigenvector u satisfying $u^*H_0u = u^*H_1u = 0$ may be computed as follows. By Theorem 1.2.21, 0 belongs to the joint numerical range of U^*H_0U and U^*H_1U where the columns of $U \in \mathbb{C}^{n\times r}$ form an orthonormal basis of the eigenspace of $G + t_0^{\star}H_0 + t_1^{\star}H_1$ corresponding to the eigenvalue $\lambda_{\max}^{\star}$. Therefore the matrix pencil $U^*H_0U + z\,U^*H_1U$ is Hermitian but not definite [40]. This implies the existence of an eigenvalue of $U^*H_0U + z\,U^*H_1U$ that is either non-real or is of mixed sign characteristic [for details about sign characteristic, see [14]]. In either case there exists a corresponding eigenvector x satisfying $x^*U^*H_0Ux = x^*U^*H_1Ux = 0$. Thus u may be calculated by setting $u := Ux$.

Remark 2.2.5. Once $\lambda_{\max}^{\star}$ and the corresponding eigenvector $u \in \mathbb{C}^{(m+1)n}$ satisfying $u^*H_j\,u = 0$, $j = 0,\dots,m$ have been computed, the optimal perturbation matrices can be easily constructed by using Theorem 1.2.9. For this, we set $v := W^{-1}u = [v_0^T,\dots,v_m^T]^T$ with

$v_j \in \mathbb{C}^n$ and $v_\lambda := \lambda^m v_m + \cdots + \lambda v_1 + v_0$. Then the required coefficients Δ_j, $j = 0, \ldots, m$ of the minimal Hermitian perturbation are given by

$$\Delta_j := \frac{\|v_j\|}{\|Mv_\lambda\|} \begin{bmatrix} \frac{v_j}{\|v_j\|} & \frac{Mv_\lambda}{\|Mv_\lambda\|} \end{bmatrix} \begin{bmatrix} \frac{v_j^* M v_\lambda}{\|Mv_\lambda\|\,\|v_j\|} & 1 \\ 1 & \frac{(Mv_\lambda)^* v_j}{\|Mv_\lambda\|\,\|v_j\|} \end{bmatrix}^{-1} \begin{bmatrix} \frac{v_j}{\|v_j\|} & \frac{Mv_\lambda}{\|Mv_\lambda\|} \end{bmatrix}^*,$$

if v_j and Mv_λ are linearly independent and by

$$\Delta_j := \frac{v_j v_\lambda^* M^*}{v_\lambda^* M^* M v_\lambda},$$

otherwise.

Remark 2.2.6. We highlight that there are situations when the perturbed matrix pencils or matrix polynomials turn out to be singular. For example, this is the case if $A_0 = [\, a\,]$ and $A_1 = [\, b\,]$ are real 1×1 matrices. As non-real eigenvalues of Hermitian matrix pencils always occur in pairs $(\lambda, \overline{\lambda})$, the only Hermitian perturbation that makes $\lambda \in \mathbb{C} \setminus \mathbb{R}$ an eigenvalue of $A_0 + zA_1$ is $(\Delta_0, \Delta_1) = ([-a], [-b])$ resulting in the zero pencil which is singular. Similar examples can be constructed for larger dimensions. However, numerical examples suggest that these cases are actually exceptional.

2.3 Matrix polynomials with related structures

The problem of computing structured backward errors for eigenvalues of skew-Hermitian and $*$-alternating polynomials can be reduced to the case of Hermitian polynomials. This leads to a formula for the structured backward error of approximate eigenvalues for such polynomials also.

Theorem 2.3.1. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be a skew-Hermitian matrix polynomial with $A_0, \ldots, A_m \in \mathrm{SHerm(n)}$, let $\mathbb{S} := (\mathrm{SHerm(n)})^{m+1}$, and let $w \in \mathbb{R}^{m+1}$ be a weight vector. Then*

$$\eta_{w,2}^{\mathrm{SHerm}}(P, \lambda) = \eta_{w,2}^{\mathrm{Herm}}(iP, \lambda).$$

Proof. This follows immediately from the fact that $P(z)$ is skew-Hermitian if and only if $iP(z)$ is Hermitian. □

For the next result let

$$\begin{aligned} \mathbb{S}_e &:= \left\{(\Delta_0, \ldots, \Delta_m) \middle| \Delta_{2j} \in \mathrm{Herm}(n),\ \Delta_{2j+1} \in \mathrm{SHerm(n)},\ j = 0, \ldots, \lfloor \tfrac{m}{2} \rfloor \right\} \\ \text{and} \quad \mathbb{S}_o &:= \left\{(\Delta_0, \ldots, \Delta_m) \middle| \Delta_{2j+1} \in \mathrm{Herm}(n),\ \Delta_{2j} \in \mathrm{SHerm(n)},\ j = 0, \ldots, \lfloor \tfrac{m}{2} \rfloor \right\}. \end{aligned}$$

Theorem 2.3.2. *Let $P(z) = A_0 + zA_1 + \cdots + z^m A_m$ be a $*$-alternating matrix polynomial with $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$, $Q(z) := P(iz) = \sum_{j=0}^{m} z^j(i^j A_j)$, and $w \in \mathbb{R}^{m+1}$ be a weight vector. Then*

$$\eta_{w,2}^{\mathbb{S}_e}(P,\lambda) = \eta_{w,2}^{\mathrm{Herm}}(Q,\lambda/i),$$

if $P(z)$ is $*$-even and

$$\eta_{w,2}^{\mathbb{S}_o}(P,\lambda) = \eta_{w,2}^{\mathrm{Herm}}(iQ,\lambda/i),$$

if $P(z)$ is $*$-odd.

Proof. This follows immediately from the fact that $Q(z/i) = P(z)$ and that $Q(z)$ is Hermitian if $P(z)$ is $*$-even and skew-Hermitian if $P(z)$ is $*$-odd. ∎

As mentioned in the introduction section of Chapter 1, the equations

$$\eta_{w,2}^{\mathbb{S}_e}(P,\lambda) = \eta_{w,2}(P,\lambda) \quad\text{and}\quad \eta_{w,2}^{\mathbb{S}_o}(P,\lambda) = \eta_{w,2}(P,\lambda)$$

are proved in [1, 2] whenever $\operatorname{Re}\lambda = 0$ for the case $w = (1,\ldots,1)$. These properties also hold for other weight vectors. Theorem 2.3.2 allows us to find formulas for $\eta_{w,2}^{\mathbb{S}_e}(P,\lambda)$ and $\eta_{w,2}^{\mathbb{S}_o}(P,\lambda)$ in terms of the corresponding Hermitian eigenvalue backward errors whenever $\operatorname{Re}\lambda \neq 0$.

2.4 Further restriction of perturbation sets

In some cases it may be of interest to further restrict the perturbation set $\mathbb{S} = (\mathrm{Herm}(n))^{m+1}$. In particular, it may be useful to perturb only some of the coefficients of the matrix polynomial. For example, a Hermitian pencil $P(z) = A_0 + zA_1$ can be canonically identified with the A_1-selfadjoint matrix $\mathcal{H} := A_1^{-1}A_0$ if A_1 is invertible. (A matrix $\mathcal{H}$ is called A_1-selfadjoint if $\mathcal{H}^*A_1 = A_1\mathcal{H}$, see, e.g., [15].) In this case, A_1 can be interpreted as a matrix that induces a (possibly indefinite) scalar product on $\mathbb{C}^n$. If perturbations of the pencil $P(z)$ that allow changes only to A_0 are considered, then this results in the effect that the matrix defining the scalar product remains constant. Therefore, we briefly explain in this section how our main results can be applied to such cases also.

To be more precise, let $I := \{i_0,\ldots,i_k\} \subseteq \{0,\ldots,m\}$ with $i_0 < \cdots < i_k$ be an index set and define

$$\mathbb{S} := \mathbb{S}(I) := \mathcal{S}_0 \times \cdots \times \mathcal{S}_m \subseteq (\mathrm{Herm}(n))^{m+1}, \tag{2.4.1}$$

where $\mathcal{S}_j = \mathrm{Herm}(n)$ if $j \in I$ and $\mathcal{S}_j = \{0\}$ if $j \notin I$. For example, if $m = 3$ and $I = \{1,2\}$, then $(\Delta_0,\ldots,\Delta_3) \in \mathbb{S}(I)$ if and only if $\Delta_0 = \Delta_3 = 0$ and $\Delta_1,\Delta_2 \in \mathrm{Herm}(n)$, i.e., perturbations from $\mathbb{S}$ will change only the coefficients A_1 and A_2 of a matrix polynomial $\sum_{j=0}^{3} z^j A_j$. Thus, each $(\Delta_0, \ldots, \Delta_m) \in \mathbb{S}$ can be canonically identified with a tuple $(\Delta_{i_0}, \ldots, \Delta_{i_k}) \in (\mathrm{Herm}(n))^{k+1}$. We consider

$$|||(\Delta_{i_0}, \ldots, \Delta_{i_k})|||_{\widehat{w},2} = |||(\Delta_0, \ldots, \Delta_m)|||_{w,2} = \sqrt{w_{i_0}^2 \|\Delta_{i_0}\|^2 + \cdots + w_{i_k}^2 \|\Delta_{i_k}\|^2},$$

which is a norm on $(\mathrm{Herm}(n))^{k+1}$ and the corresponding backward error

$$\eta^{\mathbb{S}}_{\widehat{w},2}(P,\lambda) := \inf \left\{ |||(\Delta_0, \ldots, \Delta_m)|||_{w,2} \,\middle|\, \det\left(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\right) = 0,\, (\Delta_0, \ldots, \Delta_m) \in \mathbb{S} \right\}.$$

Thus, the new weight vector $\widehat{w} := [w_{i_0}, \ldots, w_{i_k}]^T \in \mathbb{R}^{k+1}$ is obtained from the old weight vector $w \in \mathbb{R}^{m+1}$ by deleting the entries w_j with $j \notin I$.

Note that the computation of $\eta^{\mathbb{S}}_{\widehat{w},2}(P,\lambda)$ when $\mathrm{Im}\,\lambda = 0$ has already been considered in [1, 3]. Therefore, we only consider $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda \neq 0$ and obtain the following analogue of Theorem 2.2.2.

Theorem 2.4.1. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$, where $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$ are Hermitian and $\lambda \in \mathbb{C}$ be such that $\mathrm{Im}\,\lambda \neq 0$ and $M := P(\lambda)^{-1}$ exists. Let $I := \{i_0, \ldots, i_k\} \subseteq \{0, \ldots, m\}$, $\mathbb{S}$ be given by (2.4.1) and $\widehat{w} := (w_{i_0}, w_{i_1}, \ldots, w_{i_k}) \in \mathbb{R}^{k+1}$ be a weight vector with respect to I. Let $\Lambda_k := [\lambda^{i_0}, \ldots, \lambda^{i_k}]$ and set*

$$\widehat{G} := (\Lambda_k^* \Lambda_k) \otimes (M^* M) \quad \text{and} \quad \widehat{H}_j := i\big((e_{j+1}\Lambda_k) \otimes M - (\Lambda_k^* e_{j+1}^*) \otimes M^*\big)$$

for $j = 0, \ldots, k$, where e_{j+1} denotes the $(j+1)$th standard basis vector of $\mathbb{R}^{k+1}$. Also let

$$W := \mathrm{diag}(w_{i_0}, \ldots, w_{i_k}) \otimes I_n, \quad G := W^{-1}\widehat{G}W^{-1}, \quad H_j := W^{-1}\widehat{H}_j W^{-1}$$

for $j = 0, \ldots, k$. Then

$$\lambda^\star_{\max} := \min_{t_0,\ldots,t_k \in \mathbb{R}} \lambda_{\max}(G + t_0 H_0 + \cdots + t_k H_k)$$

is attained for some $(t_0^\star, \ldots, t_k^\star) \in \mathbb{R}^{k+1}$. Suppose that $\eta^{\mathbb{S}}_{\widehat{w},2}(P,\lambda)$ is finite. If either $k \leq 1$ or $\lambda^\star_{\max}$ is a simple eigenvalue of $G + t_0^\star H_0 + \cdots + t_k^\star H_k$, then

$$\eta^{\mathbb{S}}_{\widehat{w},2}(P,\lambda) = \frac{1}{\sqrt{\lambda^\star_{\max}}} = \left(\min_{t_0,\ldots,t_k \in \mathbb{R}} \lambda_{\max}(G + t_0 H_0 + \cdots + t_k H_k)\right)^{-1/2}.$$

Observe that $\widehat{G}$ and $\widehat{H}_j$ are obtained from the corresponding matrices $\widetilde{G}$ and $\widetilde{H}_{i_j}$ in Theorem 2.2.2 by deleting the block rows and columns with indices not in I.

Remark 2.4.2. The proof of Theorem 2.4.1 proceeds in exactly the same way as the proof of Theorem 2.2.2. It is based on a modified version of Lemma 1.2.6 with setting $\Delta_j = 0$ for $j \notin I$ and requiring $v_j = 0$ for $j \notin I$ in (b). (In the case $k = 0$ [26, Theorem 4.5] is applied in place of Theorem 1.2.21.)

The condition $\eta^{\mathbb{S}}_{\widehat{w},2}(P,\lambda) < \infty$ is indeed necessary, as there are number of instances when this is not the case. For example, if $m = 1$ and $I = \{0\}$, then $\eta^{\mathbb{S}}_{\widehat{w},2}(P,\lambda) = \infty$ for any non-real λ if A_1 is either positive or negative definite.

Thus, we see that by restricting the perturbation set in such a way that only k of m coefficient matrices are perturbed, the corresponding structured backward error can be computed by solving a $(k+1)$-parameter optimization problem rather than an $(m+1)$-parameter problem.

2.5 Numerical experiments

In this section we present some numerical examples to illustrate the proposed method for computing the structured backward error $\eta^{\mathbb{S}}_{w,2}(P,\lambda)$ of some $\lambda \in \mathbb{C} \setminus \mathbb{R}$ for the case $\mathbb{S} := (\mathrm{Herm(n)})^{\mathrm{m+1}}$ and $w := (1, 1, \ldots, 1)$. In all cases we have used the software package CVX [17, 16] in MATLAB to solve the associated optimization problem of finding

$$\lambda^{\star}_{\max} := \min_{t_0,t_1,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0 H_0 + \cdots + t_m H_m)$$

and the points $t^{\star}_0, t^{\star}_1, \ldots, t^{\star}_m \in \mathbb{R}$ that attain it as described in Theorem 2.2.2.

Example 2.5.1. $L(z) := A_0 + zA_1$ is a randomly generated Hermitian pencil of size 4×4 with eigenvalues $0.57661 \pm 1.0199i$, -1.0966 and -0.10193. The Hermitian backward error for the point $\lambda = -1.0966 + 0.5i$ which is close to the eigenvalue -1.0966 is 1.3058, while the unstructured backward error 0.47045 is much smaller as expected.

Figure 2.5.1 illustrates the movement of the eigenvalues of the pencil $L(z)$ (marked with stars surrounded by circles) under the homotopic perturbation $L(z) + t\Delta L(z)$ as t varies from 0 to 1. Observe that the target point $-1.0966 + 0.5i$ (marked with a star surrounded by a diamond) as well as its complex conjugate, become eigenvalues of $(L + \Delta L)(z)$ and this is produced by the splitting of a real eigenvalue of multiplicity 2 of $(L + t_0\Delta L)(z)$, for some $0 < t_0 < 1$. Here $\Delta L(z) := \Delta_0 + z\Delta_1$ is the optimal Hermitian perturbation satisfying $|||(\Delta_0, \Delta_1)|||_{w,2} = 1.3058$ such that $-1.0966 + 0.5i$ is an eigenvalue of $(L + \Delta L)(z)$.

Figure 2.5.2 illustrates the same effect with respect to unstructured homotopic perturbations $L(z) + t\widehat{\Delta L}(z)$ as t varies from 0 to 1. In this case $\widehat{\Delta L}(z) := \widehat{\Delta}_0 + z\widehat{\Delta}_1$ is a minimal non-Hermitian perturbation such that $-1.0966 + 0.5i$ is an eigenvalue of $(L + \widehat{\Delta L})(z)$. Observe that in this case the complex conjugate of $-1.0966 + 0.5i$ is not an eigenvalue of $(L + \widehat{\Delta L})(z)$ as it is not a Hermitian pencil.

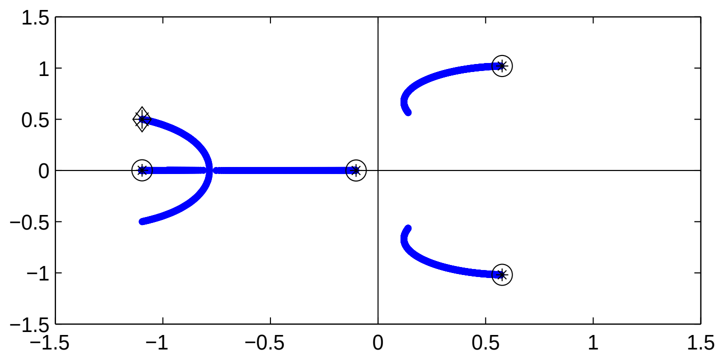

Table 2.5.1: Eigenvalue perturbation curves for the Hermitian pencil in Example 2.5.1 with respect to Hermitian perturbation.

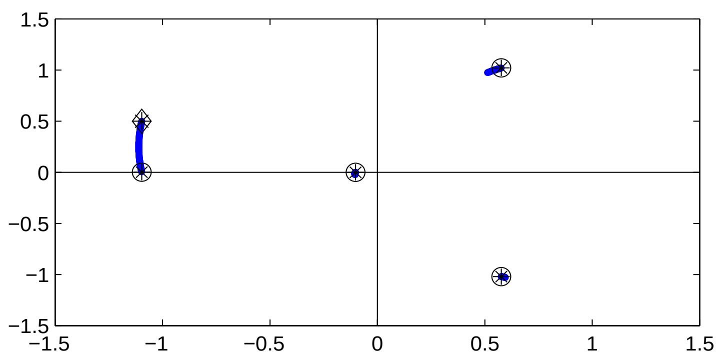

Table 2.5.2: Eigenvalue perturbation curves for the Hermitian pencil in Example 2.5.1 with respect to unstructured perturbation.

Example 2.5.2. $L(z) := A_0 + zA_1$ is a diagonal Hermitian pencil of size 3×3 with real eigenvalues $21.393, 4.2464$ and -3.5385. The Hermitian backward error of the point $-0.1241 + 1.4897i$ is 0.5608 while its unstructured backward error is 0.4246. This is an example for which $\lambda^\star_{\max}$ is a multiple eigenvalue of $G + t_0^\star H_0 + t_1^\star H_1$ where $t_0^\star = -0.3819$ and $t_1^\star = 0.6266$.

Figure 2.5.3 traces the movement of the eigenvalues of $L(z)$ with respect to perturbations $L(z) + t\Delta L(z)$ as t varies from 0 to 1, where $\Delta L(z) := z\Delta_1 + \Delta_0$ is the optimal Hermitian perturbation satisfying $|||(\Delta_0, \Delta_1)|||_{w,2} = 0.5608$ such that $-0.1241 + 1.4897i$ is an eigenvalue of $(L + \Delta L)(z)$.

The point $-0.1241 + 1.4897i$ (marked with a star surrounded by a diamond) and its complex conjugate, become eigenvalues of $(L + \Delta L)(z)$ after the splitting of a real eigenvalue of multiplicity 2 that arises from the meeting of eigenvalue curves that originated from the unperturbed eigenvalues 21.393 and -3.5385 of $L(z)$. It is interesting to note that the eigenvalue curve originating from 21.393 moves over ∞ before it meets the curve originating from -3.5385. Figure 2.5.4 illustrates the same effect with respect to unstructured homotopic perturbations $L(z) + t\widehat{\Delta L}(z)$ as t varies from 0 to 1. In this case $\widehat{\Delta L}(z) := \widehat{\Delta}_0 + z\widehat{\Delta}_1$ is a minimal non-Hermitian perturbation such that $-0.1241 + 1.4897i$ is an eigenvalue of $(L+\widehat{\Delta L})(z)$. The complex conjugate of $-0.1241+1.4897i$ is not an eigenvalue of $(L+\widehat{\Delta L})(z)$ as it is not a Hermitian pencil and therefore only a single eigenvalue curve originating from -3.5385 reaches this point for $t = 1$.

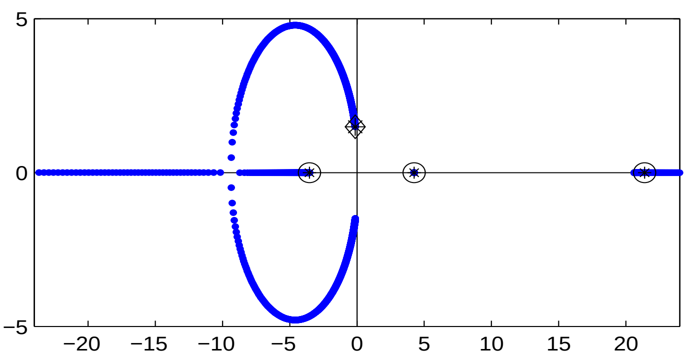

Table 2.5.3: Eigenvalue perturbation curves for the Hermitian pencil in Example 2.5.2 with respect to Hermitian perturbation.

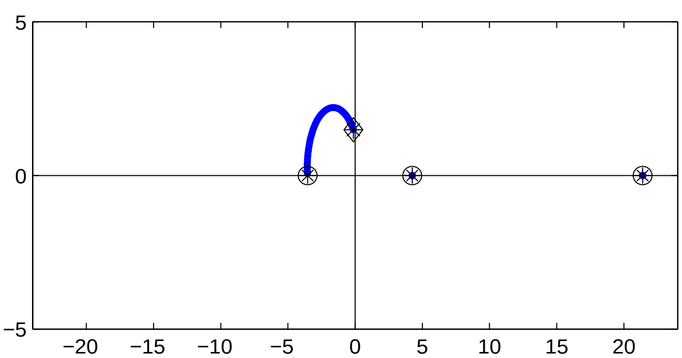

Table 2.5.4: Eigenvalue perturbation curves for the Hermitian pencil in Example 2.5.2 with respect to unstructured perturbation.

Example 2.5.3. $Q(z) := z^2A_2 + zA_1 + A_0$ is a Hermitian matrix polynomial of size 3×3 with eigenvalues $-0.8738 \pm 2.4984i, 0.3091 \pm 1.226i, 0.62802$ and 0.07796. The Hermitian backward error for the point $0.62802 + 0.5i$ which is close to the real eigenvalue 0.62802 is 1.9177 whereas the backward error with respect to arbitrary perturbations is 1.3279.

Figure 2.5.5 traces the movement of the eigenvalues of $Q(z)$ with respect to perturbations $Q(z) + t\Delta Q(z)$ as t moves from 0 to 1, $\Delta Q(z)$ being the minimal Hermitian perturbation that produces an eigenvalue at $0.62802 + 0.5i$. As expected, since $(Q + \Delta Q)(z)$ is Hermitian, it has a pair of eigenvalues at $0.62802 \pm 0.5i$ which are produced by the meeting (on the real line) and splitting of eigenvalue curves originating from the two real eigenvalues of $Q(z)$.

On the other hand, Figure 2.5.6 traces the movement of the eigenvalues of $Q(z)$ with respect to perturbations $Q(z)+t\widehat{\Delta Q}(z)$ as t moves from 0 to 1. Here $\widehat{\Delta Q}(z)$ is the minimal non Hermitian perturbation to $Q(z)$ such that $0.62802{+}0.5i$ is an eigenvalue of $(Q{+}\widehat{\Delta Q})(z)$.

In further numerical experiments we have observed that for diagonal Hermitian polynomials, $\lambda^\star_{\max}$ is a multiple eigenvalue of $G + t^\star_0 H_0 + \cdots t^\star_m H_m$. Despite this fact, it has been observed that in each of these cases it is possible to find an eigenvector x corresponding to $\lambda^\star_{\max}$ satisfying $x^\star H_j x = 0$ for $j = 0, 1, \ldots, m$. However, we have not yet encountered a case where $\lambda^\star_{\max}$ is multiple for Hermitian matrix polynomials whose coefficients are randomly generated.

We also computed the structured and unstructured backward errors of a non-real λ whose real part is a simple eigenvalue of the Hermitian matrix polynomial. We observed that as expected, the unstructured backward error approached zero as the imaginary part of λ was reduced. However, this did not decrease the structured backward error as significantly, leading to large differences between the two backward error values. These are recorded for the Hermitian pencil considered in Example 2.5.1 and the Hermitian quadratic polynomial considered in Example 2.5.3 in Table 2.5.7 and 2.5.8, respectively.

The situation is different if the selected complex valued λ are chosen in such a way that they converge to a non-real eigenvalue instead of a real one. In that case both the structured and unstructured backward errors tend to zero as expected. This is illustrated in Table 2.5.9 which records both structured and unstructured backward errors for non-real λ corresponding to the Hermitian pencil in Example 2.5.1 as they converge to the eigenvalue $0.57661 + 1.0199i$ of the pencil. Structured and unstructured backward errors for some non-real values of λ that are not necessarily close to eigenvalues of the same Hermitian pencil are also recorded. The latter values show that the difference between the structured

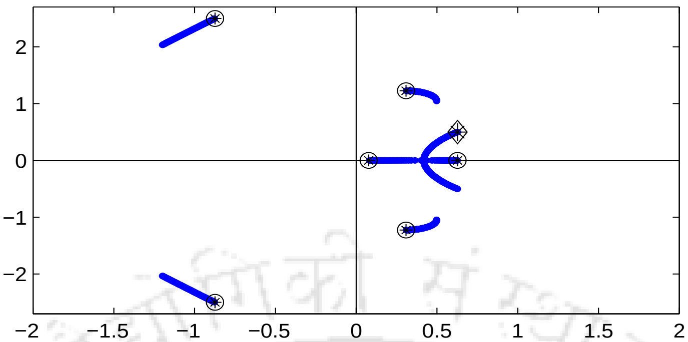

Table 2.5.5: Eigenvalue perturbation curves for the Hermitian polynomial $Q(z)$ of Example 2.5.3 with respect to Hermitian perturbation.

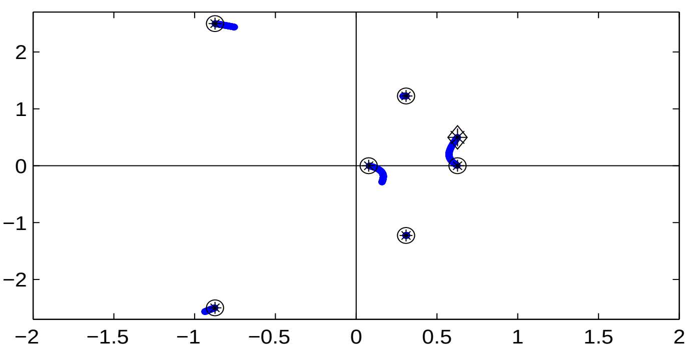

Table 2.5.6: Eigenvalue perturbation curves for the Hermitian polynomial $Q(z)$ of Example 2.5.3 with respect to non Hermitian perturbation.

Table 2.5.7: *Structured and unstructured eigenvalue backward errors for Hermitian pencils.*

λ	$t_0^\star$	$t_1^\star$	$\lambda_{\max}^\star$	$\eta_{w,2}(L,\lambda)$	$\eta_{w,2}^{\text{Herm}}(L,\lambda)$
-1.0966 + i	0.47	-0.73	0.5017	0.8450	1.4118
-1.0966 + 0.5i	1.12	-1.39	0.5865	0.4704	1.3058
-1.0966 + 0.1i	6.12	-6.75	0.6533	0.0978	1.2372
-1.0966 + 0.05i	12.27	-13.48	0.6561	0.0490	1.2345
-1.0966 + 0.01i	61.43	-67.37	0.6571	0.0098	1.2337
-1.0966 + 0.005i	122.87	-134.74	0.6571	0.0049	1.2337

and unstructured backward errors may be quite significant even if λ is not close to the real line.

Table 2.5.8: *Structured and unstructured eigenvalue backward errors for quadratic Hermitian polynomials.*

λ	$t_0^\star$	$t_1^\star$	$t_2^\star$	$\lambda_{\max}^\star$	$\eta_{w,2}(Q,\lambda)$	$\eta_{w,2}^{\text{Herm}}(Q,\lambda)$
0.62802 + i	0.06	0.09	0.03	0.81	1.0965	1.1099
0.62802 + 0.5i	-0.28	-0.54	-0.62	0.2719	1.3279	1.9177
0.62802 + 0.1i	-3.38	-5.33	-8.21	0.3852	0.2411	1.6113
0.62802 + 0.05i	-6.85	-10.88	-17.17	0.3882	0.1198	1.6051
0.62802 + 0.01i	-34.38	-54.73	-87.12	0.3891	0.0239	1.6032
0.62802 + 0.005i	-68.76	-109.48	-174.32	0.3891	0.0120	1.6031

Table 2.5.9: *Structured and unstructured eigenvalue backward errors for Hermitian pencils.*

λ	$\eta_{w,2}(L,\lambda)$	$\eta_{w,2}^{\text{Herm}}(L,\lambda)$
-0.3200 + 0.8000i	0.9073	1.0139
0.3000 + 0.8800i	0.4733	0.4851
0.4500 + 0.9000i	0.2717	0.2790
0.5200 + 0.9500i	0.1390	0.1426
0.5600 + 0.9800i	0.0663	0.0678
0.5760 + 1.0005i	0.0227	0.0231

λ	$\eta_{w,2}(L,\lambda)$	$\eta_{w}^{\text{Herm}}(L,\lambda)$
-0.1364 + 0.1139i	0.4247	0.8762
-1.1465 + 1.1909i	0.9531	1.4643
-1.2173 + 0.0412i	0.1213	1.3267
-1.4410 + 0.5711i	0.5717	1.4981
-1.6041 + 0.2573i	0.4800	1.5440
-2.1707 − 0.0592i	0.7687	1.7158

Chapter 3

Structured eigenvalue backward errors under $|||\cdot|||_{w,2}$ norm: palindromic and T-alternating structures

In this chapter, formulas for the backward errors of approximate eigenvalues of $*$-palindromic matrix polynomials with respect to $*$-palindromic perturbations are derived when the perturbations are measured by the $|||\cdot|||_{w,2}$ norm defined by (1.2.1). Such formulas are also obtained for T-palindromic polynomials of degree at most 2, T-antipalindromic pencils, T-even polynomials of degree at most 2 and T-odd pencils. In all cases, the corresponding minimal structure preserving perturbations are obtained as well. For higher degree T-palindromic and T-alternating polynomials, we estimate the backward error of an approximate eigenvalue by tight bounds. In all cases we only consider regular matrix polynomials. The results are illustrated by numerical experiments. These show that there is a significant difference between the backward errors with respect to structure preserving and arbitrary perturbations in many cases.

3.1 Palindromic polynomials

For the sake of brevity, whenever we make statements that are valid for both $*$-palindromic and T-palindromic structures, we use the term $\bullet$-palindromic where $\bullet = *$ or $\bullet = T$. Thus, denoting the $\bullet$-palindromic structure by $\mathrm{pal}_\bullet$, for $P(z) = \sum_{j=0}^{m} z^j A_j$, we have

$$\mathrm{pal}_\bullet = \begin{cases} \{(A_0, \ldots, A_m) \in (\mathbb{C}^{n\times n})^{m+1} : A_j^* = A_{m-j}\} & \text{if } \bullet = *, \\ \{(A_0, \ldots, A_m) \in (\mathbb{C}^{n\times n})^{m+1} : A_j^T = A_{m-j}\} & \text{if } \bullet = T. \end{cases}$$

The flexibility to perturb a polynomial with coefficients in $\mathrm{pal}_\bullet$ in a structure preserving way is restricted by the fact that equal weights must be given to coefficients in position j and position $m-j$. Therefore unless otherwise stated, we assume that the weight vector w is a palindromic weight vector as defined in Definition 1.2.1. In this section, our aim will be to solve the following problem where $\mathbb{S} = \mathrm{pal}_\bullet$.

Problem 3.1.1. Let $P(z) = \sum_{i=0}^{m} z^j A_j$ be $\bullet$-palindromic and $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $P(\lambda)$ is nonsingular. Find the smallest structured perturbation from $\mathrm{pal}_\bullet$ that makes λ an eigenvalue of the perturbed $\bullet$-palindromic polynomial. More precisely: calculate

$$\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda) := \inf\left\{ \|\!|(\Delta_0,\ldots,\Delta_m)\|\!|_{w,2} \,\middle|\, \det\Big(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\Big) = 0,\ (\Delta_0,\ldots,\Delta_m) \in \mathrm{pal}_\bullet \right\}.$$

Further, if the infimum is attained, then construct a minimum perturbation $\Delta P(z) = \sum_{j=0}^{m} \lambda^j \Delta_j$ that attains the infimum.

Note that the assumption $\lambda \in \mathbb{C} \setminus \{0\}$ is justified because

$$\eta_{w,2}^{\mathrm{pal}_\bullet}(P,0) = \sqrt{2} w_0 \sigma_{\min}(A_0).$$

Also note that by restricting all the entries of the weight vector to be positive in Definition 1.2.1, we are allowing perturbations to $P(z)$ that may affect all its coefficient matrices. We consider perturbations that leave certain coefficient matrices of $P(z)$ unchanged to be elements of some subset of $(\mathbb{C}^{n\times n})^p$ where $p < m$ is a positive integer determined by the number of coefficient matrices of $P(z)$ that are perturbed. The backward error $\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda)$ can be computed with respect to such perturbations on the lines of Section 2.5 of Chapter 2 where this is done for Hermitian matrix polynomials. Brief discussions on the corresponding strategies for the $*$-palindromic and T-palindromic matrix polynomials are provided in Section 3.1.4 .

We follow the strategy used in Chapter 2 to first reformulate the problem of computing $\eta_{w,2}^{\mathbb{S}}(P,\lambda)$ in terms of a structured mapping problem. A key result in this respect is Lemma 1.2.6. This yields the following alternative characterization of $\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda)$ in terms of mapping problems.

Lemma 3.1.2. *Let $P(z) = A_0 + zA_1 + \cdots + z^m A_m$ be $\bullet$-palindromic and $\lambda \in \mathbb{C} \setminus \{0\}$. Also let $k := \lfloor \frac{m-1}{2} \rfloor$ and $v_\lambda := \sum_{j=0}^{m} \lambda^j v_j$ where $v_0,\ldots,v_m \in \mathbb{C}^n$. Assume that $P(\lambda)$ is nonsingular and let $M = \big(P(\lambda)\big)^{-1}$. If m is odd,*

$$\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda) = \inf \left\{ \|\!|(\Delta_0,\ldots,\Delta_m)\|\!|_{w,2} \,\middle|\, \exists\, v_0,\ldots,v_m \in \mathbb{C}^n,\ v_\lambda \neq 0, (\Delta_0,\ldots,\Delta_m) \in \mathrm{pal}_\bullet, \right.$$
$$\left. \Delta_j M v_\lambda = v_j,\ \Delta_j^\bullet M v_\lambda = v_{m-j},\quad j = 0,\ldots,k \right\},$$

and if m is even,

$$\eta^{\mathrm{pal}_\bullet}_{w,2}(P,\lambda) = \inf \Big\{ \|\!|(\Delta_0,\ldots,\Delta_m)|\!\|_{w,2} \Big| \exists\, v_0,\ldots,v_m \in \mathbb{C}^n,\ v_\lambda \neq 0,\ (\Delta_0,\ldots,\Delta_m) \in \mathrm{pal}_\bullet,$$
$$\Delta_{\frac{m}{2}} M v_\lambda = v_{\frac{m}{2}},\ \Delta_j M v_\lambda = v_j,\ \Delta_j^\bullet M v_\lambda = v_{m-j},\quad j=0,\ldots,k \Big\}.$$

Necessary and sufficient conditions for the mapping problems

$$\Delta_j M v_\lambda = v_j,\ \Delta_j^\bullet M v_\lambda = v_{m-j}$$

in Lemma 3.1.2 to be solvable as well as minimal norm solutions to such problems have been obtained in Theorem 1.2.14. We restate the result with an alternative proof and include a formula for the desired minimal norm solution.

Theorem 3.1.3. *Let $x, y, z \in \mathbb{C}^n$ with $x \neq 0$. Then there exists a matrix $\Delta \in \mathbb{C}^{n\times n}$ such that $\Delta x = y$ and $\Delta^\bullet x = z$ if and only if $x^\bullet y = z^\bullet x$. If the latter condition is satisfied then*

$$\min \Big\{ \|\Delta\| \,\Big|\, \Delta \in \mathbb{C}^{n\times n},\ \Delta x = y, \Delta^\bullet x = z \Big\} = \max \Big\{ \frac{\|y\|}{\|x\|}, \frac{\|z\|}{\|x\|} \Big\}. \tag{3.1.1}$$

Furthermore, let $\hat{x} = x$ when $\bullet = *$ and $\hat{x} = \bar{x}$ when $\bullet = T$, and let y_1 and z_1 denote the orthogonal projections of y and z, respectively, onto the orthogonal complement of $\hat{x}$. If $\|z_1\| \leq \|y_1\|$, then the minimum in (3.1.1) is attained for

$$\widetilde{\Delta} = \frac{1}{\|x\|} \begin{bmatrix} \frac{\hat{x}}{\|x\|} & \frac{y_1}{\|y_1\|} \end{bmatrix} \begin{bmatrix} \frac{\hat{x}^* y}{\|x\|} & \|z_1\| \\ \|y_1\| & -\hat{x}^* y \frac{\|z_1\|}{\|x\|\,\|y_1\|} \end{bmatrix} \begin{bmatrix} \frac{\hat{x}}{\|x\|} & \frac{z_1}{\|z_1\|} \end{bmatrix}^\bullet$$

if $z_1 \neq 0$ and for $\widetilde{\Delta} = \frac{1}{\|x\|^2} y x^*$ if $z_1 = 0$. If $\|y_1\| \leq \|z_1\|$ then these formulas can be used to construct $\widetilde{\Delta}^\bullet$.

Proof. Clearly, the identities $\Delta x = y$ and $\Delta^\bullet x = z$ imply $x^\bullet y = z^\bullet x$ and

$$\|\Delta\| \geq \max \Big\{ \frac{\|y\|}{\|x\|}, \frac{\|z\|}{\|x\|} \Big\}.$$

Suppose now that $x^\bullet y = z^\bullet x$ holds true. Denote $x_0 = \hat{x}/\|x\|$. Then

$$y = (x_0^* y) x_0 + y_1, \quad x_0^* y_1 = 0, \quad \|y\|^2 = |x_0^* y|^2 + \|y_1\|^2,$$
$$z = (x_0^* z) x_0 + z_1, \quad x_0^* z_1 = 0, \quad \|z\|^2 = |x_0^* z|^2 + \|z_1\|^2.$$

Notice that $x_0^* y = z^* x_0$ and hence $\|y\|^2 - \|z\|^2 = \|y_1\|^2 - \|z_1\|^2$. For every $\alpha \in \mathbb{C}$, the matrix

$$\Delta_\alpha = \|x\|^{-1} \big((x_0^* y) x_0 x_0^\bullet + y_1 x_0^\bullet + x_0 z_1^\bullet + \alpha\, y_1 z_1^\bullet \big)$$

satisfies $\Delta_\alpha x = y$ and $\Delta_\alpha^\bullet x = z$. Hence,

$$\|\Delta_\alpha\| \geq \max\left\{\frac{\|y\|}{\|x\|}, \frac{\|z\|}{\|x\|}\right\}. \tag{3.1.2}$$

We show that equality holds in (3.1.2) for appropriate α. Without loss of generality we may assume $\|z\| \leq \|y\|$, or, equivalently, $\|z_1\| \leq \|y_1\|$. Otherwise, we may interchange the roles of z and y, and Δ and $\Delta^\bullet$, respectively. We consider two cases.

Case 1: $z_1 \neq 0$. Let $\alpha = -x_0^* y/\|y_1\|^2$. Then $\Delta_\alpha = \widetilde{\Delta}$ and $\|\Delta_\alpha\| = \|y\|/\|x\|$. In order to see that let $y_0 = y_1/\|y_1\|$ and $z_0 = z_1/\|z_1\|$. Then

$$\begin{aligned}
\Delta_\alpha &= \|x\|^{-1} \begin{bmatrix} x_0 & y_0 \end{bmatrix} \begin{bmatrix} x_0^* y & \|z_1\| \\ \|y_1\| & \alpha\|y_1\|\,\|z_1\| \end{bmatrix} \begin{bmatrix} x_0 & z_0 \end{bmatrix}^\bullet \\
&= \frac{\|y\|}{\|x\|} \begin{bmatrix} x_0 & y_0 \end{bmatrix} \underbrace{\frac{1}{\|y\|} \begin{bmatrix} x_0^* y & \|y_1\| \\ \|y_1\| & -x_0^* y \end{bmatrix}}_{=:C} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \|z_1\|/\|y_1\| \end{bmatrix}}_{=:D} \begin{bmatrix} x_0 & z_0 \end{bmatrix}^\bullet
\end{aligned}$$

Since $\begin{bmatrix} x_0 & y_0 \end{bmatrix}^* \begin{bmatrix} x_0 & y_0 \end{bmatrix} = \begin{bmatrix} x_0 & z_0 \end{bmatrix}^* \begin{bmatrix} x_0 & z_0 \end{bmatrix} = I$ we have

$$\|\begin{bmatrix} x_0 & y_0 \end{bmatrix}\| = \|\begin{bmatrix} x_0 & z_0 \end{bmatrix}^\bullet\| = 1.$$

The matrix C is easily seen to be unitary. Moreover, $\|D\| = \max\{1, \|z_1\|/\|y_1\|\} = 1$. Consequently, $\|\Delta_\alpha\| \leq \|y\|/\|x\|$. This inequality is actually an equality because of (3.1.2).

Case 2: $z_1 = 0$. Then for any α, $\Delta_\alpha = \|x\|^{-1} y x_0^*$, whence $\|\Delta_\alpha\| = \|y\|/\|x\|$. $\square$

3.1.1 Reformulation of the problem

Here, we further reformulate the already reformulated problem in Lemma 3.1.2 of finding the eigenvalue backward errors for $\bullet$-palindromic polynomials into an equivalent problem of maximizing the Rayleigh quotient of a Hermitian matrix with respect to specified constraints. These constraints involve Hermitian matrices when $\bullet = *$, and symmetric matrices when $\bullet = T$. Let $\lambda \in \mathbb{C} \setminus \{0\}$, let $P(z) = \sum_{j=0}^m z^j A_j$ be a $\bullet$-palindromic matrix polynomial such that $M = (P(\lambda))^{-1}$ exists and define $v_\lambda = \sum_{j=0}^m \lambda^j v_j$ where $v_0, \ldots, v_m \in \mathbb{C}^n$. Also let $k = \lfloor \frac{m-1}{2} \rfloor$.

Due to Theorem 3.1.3, for any $v_0, \ldots, v_m \in \mathbb{C}^n$ that satisfy $v_\lambda \neq 0$, there exists a $\boldsymbol{\Delta} = (\Delta_0, \ldots, \Delta_m) \in \mathrm{pal}_\bullet$ such that

$$\Delta_j M v_\lambda = v_j \text{ and } \Delta_j^\bullet M v_\lambda = v_{m-j},\ j = 0, \ldots, k \tag{3.1.3}$$

if and only if $(Mv_\lambda)^\bullet v_j = v_{m-j}^\bullet (Mv_\lambda)$. For any $\Delta_j \in \mathbb{C}^{n\times n}$ satisfying (3.1.3) which is minimal with respect to the 2-norm, we have $\|\Delta_j\| = \max\left\{\frac{\|v_j\|}{\|Mv_\lambda\|}, \frac{\|v_{m-j}\|}{\|Mv_\lambda\|}\right\}$.

If m is even, the matrix $\Delta_{\frac{m}{2}}$ of the tuple $\boldsymbol{\Delta} = (\Delta_0, \dots, \Delta_m)$ is Hermitian when $\bullet = *$ and symmetric when $\bullet = T$. In the case $\bullet = *$, the Hermitian matrix $\Delta_{\frac{m}{2}}$ may be chosen to satisfy $\Delta_{\frac{m}{2}} Mv_\lambda = v_{\frac{m}{2}}$, if and only if $(Mv_\lambda)^* v_{\frac{m}{2}} \in \mathbb{R}$, (by Theorem 1.2.9). On the other hand, when $\bullet = T$ the symmetric matrix $\Delta_{\frac{m}{2}}$ may be chosen to satisfy $\Delta_{\frac{m}{2}} Mv_\lambda = v_{\frac{m}{2}}$ without any restrictions on Mv_λ and $v_{\frac{m}{2}}$ and in either case, any minimal 2-norm solution of this mapping problem satisfies $\|\Delta_{\frac{m}{2}}\| = \frac{\|v_{m/2}\|}{\|Mv_\lambda\|}$, (see [33]). Therefore, if all the constraints are fulfilled, the minimal norm of $\boldsymbol{\Delta}$ is given by

$$\|\boldsymbol{\Delta}\|_{w,2}^2 = f(v_0, \dots, v_m),$$

where

$$f(v_0,\dots,v_m) := \begin{cases} \sum_{j=0}^{k} 2w_j^2 \max\left\{\frac{\|v_j\|^2}{\|Mv_\lambda\|^2}, \frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right\} & \text{if } m \text{ is odd,} \\ \sum_{j=0}^{k} 2w_j^2 \max\left\{\frac{\|v_j\|^2}{\|Mv_\lambda\|^2}, \frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right\} + w_{\frac{m}{2}}^2 \frac{\|v_{\frac{m}{2}}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even.} \end{cases}$$

Thus, Lemma 3.1.2 yields

$$\left(\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda)\right)^2 = \inf\left\{ f(v_0,\dots,v_m) \middle| (v_0,\dots,v_m) \in \mathcal{K}\right\}, \tag{3.1.4}$$

where $\mathcal{K} \subseteq (\mathbb{C}^n)^{m+1}$ is given by

$$\mathcal{K} := \left\{ (v_0,\dots,v_m) \,\middle|\, v_\lambda \neq 0,\ (Mv_\lambda)^\bullet v_j = v_{m-j}^\bullet M v_\lambda,\ j = 0\dots,k \right\} \tag{3.1.5}$$

if $\bullet = T$, or if m is odd and $\bullet = *$ and by

$$\mathcal{K} := \left\{ (v_0,\dots,v_m) \,\middle|\, v_\lambda \neq 0,\ (Mv_\lambda)^* v_{\frac{m}{2}} \in \mathbb{R}, (Mv_\lambda)^* v_j = v_{m-j}^* M v_\lambda,\ j = 0,\dots,k \right\} \tag{3.1.6}$$

otherwise (i.e., when $\bullet = *$ and m is even). Observe that $(Mv_\lambda)^\bullet v_j = v_{m-j}^\bullet (Mv_\lambda)$ for $j = 0,\dots,k$, if and only if

$$0 = \Big(M(v_0 + \dots + \lambda^m v_m)\Big)^\bullet v_j - v_{m-j}^\bullet \Big(M(v_0 + \dots + \lambda^m v_m)\Big) = v^\bullet \widetilde{C}_j v,$$

where $v := [v_0^T, \dots, v_m^T]^T$ and

$$\widetilde{C}_j := (\Lambda_m^\bullet e_{j+1}^\bullet) \otimes M^\bullet - (e_{m-j+1}\Lambda_m) \otimes M, \tag{3.1.7}$$

with $\Lambda_m := [1, \lambda, \dots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$. Similarly $(Mv_\lambda)^* v_{\frac{m}{2}} \in \mathbb{R}$ if and only if

$$0 = -2\,\mathrm{Im}\left((Mv_\lambda)^* v_{\frac{m}{2}}\right) = i\left(v_{\frac{m}{2}}^*(Mv_\lambda) - (Mv_\lambda)^* v_{\frac{m}{2}}\right) = v^* \widetilde{C}_{\frac{m}{2}} v,$$

where

$$\widetilde{C}_{\frac{m}{2}} := i\Big((\Lambda_m^* e_{\frac{m}{2}+1}^*) \otimes M^* - (e_{\frac{m}{2}+1}\Lambda_m) \otimes M\Big). \tag{3.1.8}$$

Note that $\widetilde{C}_{\frac{m}{2}}$ is a Hermitian matrix but the matrices $\widetilde{C}_j, j = 0, \ldots, k$ are not Hermitian. Thus, from (3.1.5) if $\bullet = T$, or if m is odd and $\bullet = *$

$$\mathcal{K} = \Big\{(v_0, \ldots, v_m) \,\Big|\, v_\lambda \neq 0,\, v^\bullet \widetilde{C}_j v = 0,\ j = 0, \ldots, k\Big\}, \tag{3.1.9}$$

and from (3.1.6)

$$\mathcal{K} = \Big\{(v_0, \ldots, v_m) \,\Big|\, v_\lambda \neq 0,\, v^* \widetilde{C}_{\frac{m}{2}} v = 0,\, v^* \widetilde{C}_j v = 0,\ j = 0, \ldots, k\Big\}, \tag{3.1.10}$$

otherwise.

As already stated in the beginning of this section, our aim is to reformulate the computation of the structured eigenvalue backward error as an equivalent problem of maximizing the Rayleigh quotient of a Hermitian matrix subject to specified constraints. The same strategy was applied in Chapter 2 to find the structured eigenvalue backward errors $\eta_{w,2}^{\mathbb{S}}(P, \lambda)$ for Hermitian and related structures. But the reformulation was aided by the fact that $\eta_{w,2}^{\mathbb{S}}(P, \lambda)$ satisfied

$$\eta_{w,2}^{\mathbb{S}}(P, \lambda) = \left(\sup\left\{\frac{\|Mv_\lambda\|^2}{\sum_{j=0}^m w_j^2 \|v_j\|^2} \,\Big|\, v_\lambda \neq 0, v_j^* M v_\lambda \in \mathbb{R}\right\}\right)^{-\frac{1}{2}} \tag{3.1.11}$$

for those structures, which made it possible to compute $\eta_{w,2}^{\mathbb{S}}(P, \lambda)$ by minimizing the Rayleigh quotient of a particular Hermitian matrix G as given in Theorem 2.2.2 subject to certain conditions involving Hermitian matrices. However, this is not the case for the structured eigenvalue backward error $\eta_{w,2}^{\mathrm{pal}_\bullet}(P, \lambda)$ for the $\bullet$-palindromic structures, because the function f in the right hand side of (3.1.4) involves taking a maximum instead of a sum of squares. The following lemma is a key step towards establishing a relationship similar to (3.1.11) for $\eta_{w,2}^{\mathrm{pal}_\bullet}(P, \lambda)$, because it shows that computing $\eta_{w,2}^{\mathrm{pal}_\bullet}(P, \lambda)$ is equivalent to minimizing a function g related to f that can be interpreted as a Rayleigh quotient of a certain Hermitian matrix.

Lemma 3.1.4. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be $\bullet$-palindromic and $\lambda \in \mathbb{C} \setminus \{0\}$. Assume further that $M = \big(P(\lambda)\big)^{-1}$ exists and $k = \lfloor \frac{m-1}{2} \rfloor$. Then*

$$\Big(\eta_{w,2}^{\mathrm{pal}_\bullet}(P, \lambda)\Big)^2 = \inf\Big\{g(v_0, \ldots, v_m) \,\Big|\, (v_0, \ldots, v_m) \in \mathcal{K}\Big\},$$

where

$$g(v_0,\ldots,v_m) := \begin{cases} \sum_{j=0}^{k} \frac{2\,w_j^2(\|v_j\|^2+|\lambda|^{m-2j}\|v_{m-j}\|^2)}{(1+|\lambda|^{m-2j})\,\|Mv_\lambda\|^2} & \text{if } m \text{ is odd,} \\ \sum_{j=0}^{k} \frac{2\,w_j^2(\|v_j\|^2+|\lambda|^{m-2j}\|v_{m-j}\|^2)}{(1+|\lambda|^{m-2j})\,\|Mv_\lambda\|^2} + \frac{w_{\frac{m}{2}}^2\|v_{\frac{m}{2}}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even,} \end{cases}$$

and $\mathcal{K}$ *is as defined in* (3.1.9) *and* (3.1.10), *respectively.*

Proof. Set $\nu := \inf\left\{g(v_0,\ldots,v_m)\,\middle|\,(v_0,\ldots,v_m)\in\mathcal{K}\right\}$. It is easily verified that

$$g(v_0,\ldots,v_m) \le f(v_0,\ldots,v_m) \ \text{ for all } \ (v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1} \ \text{ with } \ v_\lambda \neq 0.$$

This together with (3.1.4) implies $\nu \le \left(\eta_w^{\mathrm{pal}\bullet}(P,\lambda)\right)^2$. The opposite inequality is an immediate consequence of the following facts:

(a) *The infimum of g in the definition of ν is attained for some* $(\hat v_0,\ldots,\hat v_m)\in\mathcal{K}$.

(b) *For every minimizer* $(\hat v_0,\ldots,\hat v_m)\in\mathcal{K}$ *of g, we have*

$$g(\hat v_0,\ldots,\hat v_m) = f(\hat v_0,\ldots,\hat v_m).$$

Proof of (a): Since $\mathcal{K}$ is closed under scalar multiplication and since for all $t\in\mathbb{R}\backslash\{0\}$ and all $(v_0,\ldots,v_m)\in\mathcal{K}$ we have $g(v_0,\ldots,v_m)=g(t\,v_0,\ldots,t\,v_m)$, we obtain that $g(\mathcal{K})=g(\mathcal{K}\cap\mathcal{S})$, where $\mathcal{S}$ is defined as

$$\mathcal{S} = \left\{(v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1}\;\middle|\;\sum_{j=0}^{k}\|v_j\|^2 = 1\right\}.$$

Let $(v_0^{(\ell)},\ldots,v_m^{(\ell)})$, $\ell\in\mathbb{N}$, be a sequence in $\mathcal{K}\cap\mathcal{S}$ for which

$$\lim_{\ell\to\infty} g\big(v_0^{(\ell)},\ldots,v_m^{(\ell)}\big) = \nu.$$

Since $\mathcal{S}$ is compact, we may assume without loss of generality that the sequence $(v_0^{(\ell)},\ldots,v_m^{(\ell)})$ has a limit $(\hat v_0,\ldots,\hat v_m)\in\mathcal{S}$. Suppose that $(\hat v_0,\ldots,\hat v_m)$ does not belong to $\mathcal{K}$. Then we have $\hat v_\lambda := \sum_{j=0}^{m}\lambda^j\hat v_j = 0$, as $(\hat v_0,\ldots,\hat v_m)$ belongs to the closure of $\mathcal{K}$. This implies

$$\lim_{\ell\to\infty}\left\|M\left(v_0^{(\ell)}+\lambda v_1^{(\ell)}+\cdots+\lambda^m v_m^{(\ell)}\right)\right\|^{-1} = \infty$$

and hence

$$\lim_{\ell\to\infty} g(v_0^{(\ell)},\ldots,v_m^{(\ell)}) = \infty \neq \nu$$

which is a contradiction. Thus, $(\hat{v}_0,\ldots,\hat{v}_m)\in\mathcal{K}$ and $g(\hat{v}_0,\ldots,\hat{v}_m)=\nu$.

Proof of (b): Let $(\hat{v}_0,\ldots,\hat{v}_m)\in\mathcal{K}$ be such that $g(\hat{v}_0,\ldots,\hat{v}_m)=\nu$. Observe that, to show that $g(\hat{v}_0,\ldots,\hat{v}_m)=f(\hat{v}_0,\ldots,\hat{v}_m)$, it is sufficient to show that $\|\hat{v}_j\|=\|\hat{v}_{m-j}\|$ for all $j=0,\ldots,k$. Let

$$x_0=\begin{cases} M(\hat{v}_\lambda)/\|M(\hat{v}_\lambda)\| & \text{if } \bullet=*,\\ \overline{M(\hat{v}_\lambda)}/\|M(\hat{v}_\lambda)\| & \text{if } \bullet=T\end{cases}$$

and for each j such that $0\le j\le k$, let y_j, y_{m-j} be the projections of $\hat{v}_j$ and $\hat{v}_{m-j}$, respectively, onto the orthogonal complement of x_0, for $0\le j\le k$. Then

$$\hat{v}_j=y_j+c_jx_0 \quad\text{and}\quad \hat{v}_{m-j}=y_{m-j}+c_{m-j}x_0$$

for some $c_j,c_{m-j}\in\mathbb{C}$. Since $(\hat{v}_0,\ldots,\hat{v}_m)\in\mathcal{K}$ we have $\bar{c}_j=c_{m-j}$ when $\bullet=*$ and $c_j=c_{m-j}$ when $\bullet=T$. Hence

$$\|\hat{v}_j\|^2=\|y_j\|^2+|c_j|^2 \quad\text{and}\quad \|\hat{v}_{m-j}\|^2=\|y_{m-j}\|^2+|c_j|^2. \tag{3.1.12}$$

Let $y=\bar{\lambda}^{m-2j}\,y_j+|\lambda|^{m-2j}\,y_{m-j}$. Observe that

$$(\hat{v}_0,\ldots,\hat{v}_j+t\,\lambda^{m-2j}\,y,\ldots,\hat{v}_{m-j}-t\,y,\ldots,\hat{v}_m)\in\mathcal{K}$$

for all $t\in\mathbb{R}$. Thus as $(\hat{v}_0,\ldots,\hat{v}_m)$ is a minimizer of g over $\mathcal{K}$, we have

$$\begin{aligned}
0 &= \frac{d}{dt}g\big(\hat{v}_0,\ldots,\hat{v}_j+t\,\lambda^{m-2j}\,y,\ldots,\hat{v}_{m-j}-t\,y,\ldots,\hat{v}_m\big)\Big|_{t=0}\\
&= \frac{d}{dt}\left(\frac{2w_j^2(\|\hat{v}_j+t\,\lambda^{m-2j}\,y\|^2+|\lambda|^{m-2j}\|\hat{v}_{m-j}-t\,y\|^2)}{(1+|\lambda|^{m-2j})\,\|M\hat{v}_\lambda\|^2}\right)\Bigg|_{t=0}\\
&= \frac{2w_j^2\mathrm{Re}\big(\hat{v}_j^*(\lambda^{m-2j}\,y)-|\lambda|^{m-2j}\,\hat{v}_{m-j}^*y\big)}{(1+|\lambda|^{m-2j})\,\|M\hat{v}_\lambda\|^2} \qquad \left(\text{since, } \frac{d}{dt}\|v+ty\|^2\Big|_{t=0}=2\mathrm{Re}(v^*y)\right)\\
&= \frac{2w_j^2\mathrm{Re}\big(y_j^*(\lambda^{m-2j}\,y)-|\lambda|^{m-2j}\,y_{m-j}^*y\big)}{(1+|\lambda|^{m-2j})\,\|M\hat{v}_\lambda\|^2}\\
&= \frac{2w_j^2|\lambda|^{m-2j}\,\mathrm{Re}\big(|\lambda|^{m-2j}\,\|y_j\|^2+\lambda^{m-2j}\,y_j^*y_{m-j}-\bar{\lambda}^{m-2j}\,y_{m-j}^*y_j-|\lambda|^{m-2j}\,\|y_{m-j}\|^2\big)}{(1+|\lambda|^{m-2j})\,\|M\hat{v}_\lambda\|^2}\\
&= \frac{2w_j^2\,|\lambda|^{2(m-2j)}\big(\|y_j\|^2-\|y_{m-j}\|^2\big)}{(1+|\lambda|^{m-2j})\,\|M\hat{v}_\lambda\|^2},
\end{aligned}$$

which implies $\|y_j\|=\|y_{m-j}\|$. This together with (3.1.12) yields $\|\hat{v}_j\|=\|\hat{v}_{m-j}\|$. Hence $\|\hat{v}_j\|=\|\hat{v}_{m-j}\|$ for all j, and the latter implies $g(\hat{v}_0,\ldots,\hat{v}_m)=f(\hat{v}_0,\ldots,\hat{v}_m)$. This completes the proof. □

Recalling that $k = \lfloor \frac{m-1}{2} \rfloor$, define $\gamma_{j1} = w_j \sqrt{\frac{2}{1+|\lambda|^{m-2j}}}$, $\gamma_{j2} = w_j \sqrt{\frac{2|\lambda|^{m-2j}}{1+|\lambda|^{m-2j}}}$ for $j = 0, \ldots, k$, and

$$\Gamma := \begin{cases} \mathrm{diag}(\gamma_{01}, \ldots, \gamma_{k1}, \gamma_{k2}, \ldots, \gamma_{02}) \otimes I_n, & \text{if } m \text{ is odd,} \\ \mathrm{diag}(\gamma_{01}, \ldots, \gamma_{k1}, w_{\frac{m}{2}}, \gamma_{k2}, \ldots, \gamma_{02}) \otimes I_n & \text{if } m \text{ is even.} \end{cases} \tag{3.1.13}$$

Also recall that $\Lambda_m = [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$. Then we have

$$g(v_0, \ldots, v_m) = \frac{v^*\Gamma^2 v}{v^*\widetilde{G}v}, \quad \text{where } \widetilde{G} := (\Lambda_m^*\Lambda_m) \otimes (M^*M), \quad v = [v_0^T, \ldots, v_m^T]^T \tag{3.1.14}$$

and where $v^*\widetilde{G}v = \|Mv_\lambda\|^2 \neq 0$, or, equivalently, $v_\lambda \neq 0$. It follows that

$$\begin{aligned} \eta_{w,2}^{\mathrm{pal}_\bullet}(P, \lambda) &= \left(\inf\left\{ f(v_0, \ldots, v_m) \,\middle|\, (v_0, \ldots, v_m) \in \mathcal{K}\right\}\right)^{1/2} \\ &= \left(\inf\left\{ g(v_0, \ldots, v_m) \,\middle|\, (v_0, \ldots, v_m) \in \mathcal{K}\right\}\right)^{1/2} \qquad \text{(by Lemma 3.1.4)} \\ &= \left(\sup\left\{ g(v_0, \ldots, v_m)^{-1} \,\middle|\, (v_0, \ldots, v_m) \in \mathcal{K}\right\}\right)^{-1/2}. \end{aligned}$$

Set $u := \Gamma v$ and

$$G := \Gamma^{-1}\widetilde{G}\Gamma^{-1}, \quad C_j := \Gamma^{-1}\widetilde{C}_j\Gamma^{-1}, \quad C_{\frac{m}{2}} := \Gamma^{-1}\widetilde{C}_{\frac{m}{2}}\Gamma^{-1}, \tag{3.1.15}$$

where $\widetilde{G}, \widetilde{C}_j$ and $\widetilde{C}_{\frac{m}{2}}$ are as defined in (3.1.14), (3.1.7) and (3.1.8) respectively.

By Lemma 3.1.4 and (3.1.14), for $k = \lfloor \frac{m-1}{2} \rfloor$, we have

$$\begin{aligned} \left(\eta_{w,2}^{\mathrm{pal}_\bullet}(P, \lambda)\right)^{-2} &= \sup\left\{ \frac{v^*\widetilde{G}v}{v^*\Gamma^2 v} \,\middle|\, v \in \mathbb{C}^{n(m+1)} \setminus \{0\},\ v^\bullet\widetilde{C}_j v = 0,\ j = 0, \ldots, k\right\} \\ &= \sup\left\{ \frac{u^*Gu}{u^*u} \,\middle|\, u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\ u^\bullet C_j u = 0,\ j = 0, \ldots, k\right\}, \end{aligned}$$

if $\bullet = T$, or if m is odd and $\bullet = *$, and

$$\left(\eta_{w,2}^{\mathrm{pal}_*}(P, \lambda)\right)^{-2} = \sup\left\{ \frac{u^*Gu}{u^*u} \,\middle|\, u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\ u^*C_{\frac{m}{2}}u = 0,\ u^*C_j u = 0,\ j = 0, \ldots, k\right\}$$

otherwise. Note that the condition $v_\lambda \neq 0$ from the definition of $\mathcal{K}$ in (3.1.9) or (3.1.10), respectively or, equivalently, the conditions $v^*\widetilde{G}v \neq 0$ and $u^*Gu \neq 0$ can be dropped in the two expressions for $\left(\eta_{w,2}^{\mathrm{pal}_*}(P, \lambda)\right)^{-2}$, because $\widetilde{G}$ and G are semidefinite. This implies $\frac{u^*Gu}{u^*u} \geq 0$ and hence the supremum of this Rayleigh quotient over all nonzero vectors u satisfying some constraints will be the same with or without the additional condition $u^*Gu \neq 0$.

In order to state the main result of this section, for each $j = 0, \ldots, k$ we define

$$H_j := C_j + C_j^*,\ H_{m-j} \ := \ i(C_j - C_j^*),\ H_{\frac{m}{2}} := C_{\frac{m}{2}}, \tag{3.1.16}$$
$$S_j \ := \ C_j + C_j^T, \tag{3.1.17}$$

where C_j, for $j = 0, \ldots, k$, and $C_{\frac{m}{2}}$, are as in (3.1.15).

Observe that for $j = 0, \ldots, k$,

$$\begin{aligned} v^*\widetilde{C}_j v = 0 &\iff u^* H_j u = 0 \text{ and } u^* H_{m-j} u = 0,\\ v^T\widetilde{C}_j v = 0 &\iff u^T S_j u = 0,\\ v^*\widetilde{C}_{\frac{m}{2}} v = 0 &\iff u^* H_{\frac{m}{2}} u = 0. \end{aligned}$$

Therefore we have proved the following theorem which gives the desired reformulation.

Theorem 3.1.5. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be $\bullet$-palindromic and $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $P(\lambda)$ is nonsingular and $M = (P(\lambda))^{-1}$. Furthermore, let $k := \lfloor \frac{m-1}{2} \rfloor$, G be as in* (3.1.15), *H_j, for $j = 0, \ldots, m$, be defined by* (3.1.16) *and S_j, for $j = 0, \ldots, k$, be defined by* (3.1.17). *Then*

$$\eta_{w,2}^{\mathrm{pal}_T}(P,\lambda) \ = \ \left(\sup\left\{ \frac{u^*Gu}{u^*u} \ \middle|\ u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\ u^T S_j u = 0,\ j = 0, \ldots, k \right\}\right)^{-\frac{1}{2}} \tag{3.1.18}$$

and

$$\eta_{w,2}^{\mathrm{pal}_*}(P,\lambda) \ = \ \left(\sup\left\{ \frac{u^*Gu}{u^*u} \ \middle|\ u \in \mathbb{C}^{n(m+1)} \setminus \{0\},\ u^* H_j u = 0,\ j = 0, \ldots, m \right\}\right)^{-\frac{1}{2}}. \tag{3.1.19}$$

3.1.2 Eigenvalue backward errors of $*$-palindromic matrix polynomials

In this section, we obtain structured eigenvalue backward errors $\eta_{w,2}^{\mathrm{pal}_*}(\lambda, P)$ for matrix polynomials $P(z)$ with $*$-palindromic structure. If $\lambda \in \mathbb{C} \setminus \{0\}$ is such that $|\lambda| = 1$, then there is no difference between the eigenvalue backward errors with respect to structure preserving and arbitrary perturbations. This fact was shown in [1, 2] for the weight vector $w = (1, \ldots, 1)$ and easily generalizes to arbitrary choices of palindromic weight vectors. The situation is completely different if $|\lambda| \neq 1$. In this case, we obtain the structured backward error via minimization of the maximal eigenvalue of a parameter-dependent Hermitian matrix of the form $G + t_0 H_0 + \cdots + t_p H_p \in \mathbb{C}^{n(m+1) \times n(m+1)}$ with the property that any nonzero linear combination $\alpha_0 H_0 + \cdots + \alpha_p H_p$ is indefinite. Our aim is to use Theorem 1.2.18 or Theorem 1.2.21 when $m > 1$ and $m = 1$ respectively in the process.

We have the following result which gives a formula for $\eta_{w,2}^{\mathrm{pal}_*}(P, \lambda)$ when $|\lambda| \neq 1$.

Theorem 3.1.6. *Let* $P(z) = \sum_{j=0}^{m} z^j A_j$ *be* $*$*-palindromic and* $\lambda \in \mathbb{C} \setminus \{0\}$ *such that* $|\lambda| \neq 1$. *Suppose that* $P(\lambda)$ *is nonsingular and* $M = (P(\lambda))^{-1}$. *Then for* G *as defined in (3.1.15) and* H_j, *for* $j = 0, \ldots, m$, *as defined in* (3.1.16), *we have that*

$$\lambda_{\max}^{\star} := \min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + \cdots + t_m\, H_m)$$

is attained for some $(t_0^{\star}, \ldots, t_m^{\star}) \in \mathbb{R}^{m+1}$. *If* $m = 1$ *or* $\lambda_{\max}^{\star}$ *is a simple eigenvalue of* $G + t_0^{\star} H_0 + \cdots + t_m^{\star} H_m$, *then*

$$\eta_{w,2}^{\mathrm{pal}_*}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^{\star}}} = \left(\min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + \cdots + t_m\, H_m) \right)^{-1/2}.$$

Proof. Setting $k = \lfloor \frac{m-1}{2} \rfloor$, let $\widetilde{H}_j = \widetilde{C}_j + \widetilde{C}_j^*$, $\widetilde{H}_{m-j} = i\big(\widetilde{C}_j - \widetilde{C}_j^*\big)$ and $\widetilde{H}_{\frac{m}{2}} = \widetilde{C}_{\frac{m}{2}}$, where $\widetilde{C}_j$ for $j = 0, \ldots, k$ are as defined in (3.1.7) and (3.1.8). In view of Theorem 3.1.5, we aim to apply Theorem 1.2.18. Thus we check whether each nontrivial linear combination of $H_0, \ldots, H_m$, or, equivalently, of $\widetilde{H}_0, \ldots, \widetilde{H}_m$ is indefinite. Let $H := \sum_{j=0}^{m} \alpha_j \widetilde{H}_j$. Recalling that $\Lambda_m := [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1 \times (m+1)}$, easy calculations show that

$$H = (\Lambda_m^* \alpha^*) \otimes M^* + (\alpha \Lambda_m) \otimes M,$$

where $\alpha := [\alpha_0 - i\alpha_m, \ldots, \alpha_k - i\alpha_{m-k},\ -(\alpha_k + i\alpha_{m-k}), \ldots, -(\alpha_0 + i\alpha_m)]^T$ if m is odd and

$$\alpha := \big[\alpha_0 - i\alpha_m, \ldots, \alpha_k - i\alpha_{m-k},\ -i\alpha_{\frac{m}{2}},\ -(\alpha_k + i\alpha_{m-k}), \ldots, -(\alpha_0 + i\alpha_m)\big]^T$$

if m is even. To complete the proof, we show that if H is semidefinite then $\alpha = 0$ and hence $\alpha_0 = \cdots = \alpha_m = 0$. Let

$$Q := \begin{bmatrix} 1 & -\lambda & 0 & \ldots & 0 \\ 0 & 1 & -\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(m+1)\times(m+1)} \tag{3.1.20}$$

and $a = [a_0\ \ldots,\ a_m]^T := Q^* \alpha$. Since $\Lambda_m Q = e_1^*$ we have

$$\begin{aligned} (Q \otimes I_n)^* H (Q \otimes I_n) &= (Q^* \Lambda_m^* \alpha^* Q) \otimes M^* + (Q^* \alpha \Lambda_m Q) \otimes M \\ &= (e_1 a^*) \otimes M^* + (a e_1^*) \otimes M \\ &= \begin{bmatrix} a_0 M + \bar{a}_0 M^* & \bar{a}_1 M^* & \cdots & \bar{a}_m M^* \\ a_1 M & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_m M & 0 & \ldots & 0 \end{bmatrix} \end{aligned}$$

If H is semidefinite, then $a_1 = \cdots = a_m = 0$ and hence $Q^*\alpha = a = a_0 e_1$. When $m \geq 3$, observing that

$$\begin{aligned} a_1 = 0 \;&\Rightarrow\; \alpha_1 - i\alpha_{m-1} = \bar{\lambda}(\alpha_0 - i\alpha_m), \\ \bar{a}_m = 0 \;&\Rightarrow\; \alpha_0 - i\alpha_m = \lambda(\alpha_1 - i\alpha_{m-1}), \end{aligned}$$

we have $a_0 = \alpha_0 - i\alpha_m = \lambda(\alpha_1 - i\alpha_{m-1}) = \lambda\bar{\lambda}(\alpha_0 - i\alpha_m) = \lambda\bar{\lambda}a_0$.

Similarly, when $m = 1$,

$$a_1 = 0 \Rightarrow \alpha_0 + i\alpha_1 = -\bar{\lambda}(\alpha_0 - i\alpha_1) \text{ and } \alpha_0 - i\alpha_1 = -\lambda(\alpha_0 + i\alpha_1)$$

so that $a_0 = \alpha_0 - i\alpha_1 = -\lambda(\alpha_0 + i\alpha_1) = \lambda\bar{\lambda}(\alpha_0 - i\alpha_1) = \lambda\bar{\lambda}a_0$.

Finally when $m = 2$,

$$\begin{aligned} a_1 = 0 \;&\Rightarrow\; i\alpha_1 = -\bar{\lambda}(\alpha_0 - i\alpha_2) \\ \bar{a}_2 = 0 \;&\Rightarrow\; \alpha_0 - i\alpha_2 = -i\lambda\alpha_1 \end{aligned}$$

so that $a_0 = \alpha_0 - i\alpha_2 = -i\lambda\alpha_1 = \lambda\bar{\lambda}(\alpha_0 - i\alpha_2) = \lambda\bar{\lambda}a_0$.

In all cases we have $a_0 = \lambda\bar{\lambda}a_0$ and hence $a_0 = 0$ as $\lambda\bar{\lambda} \neq 1$. Therefore, $\alpha = (Q^*)^{-1}a = 0$ which implies that $\alpha_0 = \cdots = \alpha_m = 0$. ◻

Remark 3.1.7. Although it cannot be established that $\lambda^\star_{\max}$ is always simple for the particular matrices $G, H_0, \ldots, H_m$ when $m > 1$ in Theorem 3.1.6, numerical experiments suggests that this holds generically.

To highlight the fact that the assumption of simplicity of $\lambda^\star_{\max}$ is not required when $m = 1$, we state the result for the case of the $*$-palindromic pencils separately.

Theorem 3.1.8. *Let $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\lambda| \neq 1$. Suppose that the pencil $P(\lambda) = A + \lambda A^*$ is nonsingular and let $M := (A + \lambda A^*)^{-1}$ exists. Furthermore, define $\gamma_1 := \sqrt{\frac{2}{1+|\lambda|}}$, $\gamma_2 := \sqrt{\frac{2|\lambda|}{1+|\lambda|}}$,*

$$\widetilde{G} := \begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2 M^*M \end{bmatrix}, \; C := \begin{bmatrix} M^* & 0 \\ \bar{\lambda}M^* - M & -\lambda M \end{bmatrix}, \; \Gamma := \begin{bmatrix} w_0\gamma_1 I_n & 0 \\ 0 & w_0\gamma_2 I_n \end{bmatrix},$$

$$G := \Gamma^{-1}\widetilde{G}\Gamma^{-1}, \quad H_0 = \Gamma^{-1}(C + C^*)\Gamma^{-1}, \quad \textit{and} \;\; H_1 := i\,\Gamma^{-1}(C - C^*)\Gamma^{-1}.$$

Then

$$\eta^{\mathrm{pal}_*}_{w,2}(P,\lambda) = \left(\min_{t_0,t_1\in\mathbb{R}} \lambda_{\max}(G + t_0\,H_0 + t_1\,H_1)\right)^{-1/2}.$$

Remark 3.1.9. Let $P(z)$ be a $*$-palindromic polynomial of degree $m > 1$. To obtain an optimal $*$-palindromic perturbation to $P(z)$ with norm equal to the structured backward error $\eta_{w,2}^{\mathrm{pal}_*}(P, \lambda)$ such that the perturbed polynomial has an eigenvalue at λ, we first compute the eigenvector u corresponding to the eigenvalue $\lambda_{\max}^\star$ of $G + t_0^\star H_0 + \cdots + t_m^\star H_m$ that satisfies the constraints $u^* H_j u = 0$. Setting, $v := \Gamma^{-1} u$, the coefficient matrices Δ_j of the $*$-palindromic perturbation may be obtained from Theorem 3.1.3 and Theorem 1.2.9, the second result being necessary only to construct $\Delta_{\frac{m}{2}}$ when m is even. When $m = 1$ and $\lambda_{\max}^\star$ is not a simple eigenvalue of $G + t_0^\star H_0 + t_1^\star H_1$, the optimal $*$-palindromic perturbation may be calculated by the process described in Remark 2.2.4.

3.1.3 Eigenvalue backward errors of T-palindromic pencils and quadratics

In this section, we obtain the structured eigenvalue backward error $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T}}}(P, \lambda)$ for a matrix pencil or quadratic matrix polynomial $P(z)$ with T-palindromic structure.

Due to [1, Theorem 5.3.1], if $\lambda = \pm 1$ then there is no difference between the eigenvalue backward errors with respect to arbitrary perturbations and with respect to complex T-palindromic perturbations (when $P(z)$ is complex) and real T-palindromic perturbations (when $P(z)$ is real). This is proved for the weight vector $w = (1, \ldots, 1)$, but it may be easily generalized to arbitrary choices of palindromic weight vectors.

However, the situation is different if $\lambda \neq \pm 1$. Due to Theorem 3.1.5, the original Problem 3.1.1 of finding the structured backward error $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T}}}(P, \lambda)$ for T-palindromic polynomials is equivalent to an optimization problem which requires maximizing the Rayleigh quotient of a Hermitian matrix subject to a number of constraints involving symmetric matrices. In these cases, the structured backward error may be obtained by using Theorem 1.2.7. To state the results that follow from it, we recall that $\lambda_2(B)$ denotes the second largest eigenvalue of a Hermitian matrix B and $\sigma_2(S)$ denotes the second largest singular value of a matrix S.

The following theorem gives a formula for the structured eigenvalue backward error $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T}}}(P, \lambda)$ when $P(z)$ is a T-palindromic pencil and $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$.

Theorem 3.1.10. *Let $A \in \mathbb{C}^{n \times n}$ and suppose that $P(\lambda) := A + \lambda A^T$ is nonsingular for $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$. Let $M := (A + \lambda A^T)^{-1}$ and define $\gamma_1 := \sqrt{\frac{2}{1+|\lambda|}}$, $\gamma_2 := \sqrt{\frac{2|\lambda|}{1+|\lambda|}}$,*

$$\widetilde{G} := \begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda} M^*M & |\lambda|^2 M^*M \end{bmatrix}, \quad C := \begin{bmatrix} M^T & 0 \\ \lambda M^T - M & -\lambda M \end{bmatrix}, \quad \Gamma := \begin{bmatrix} w_0\gamma_1 I_n & 0 \\ 0 & w_0\gamma_2 I_n \end{bmatrix},$$

$$G := \Gamma^{-1}\widetilde{G}\Gamma^{-1} \quad and \quad S := \Gamma^{-1}(C + C^T)\Gamma^{-1}.$$

Then

$$\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda) = \left(\min_{0\le t\le t_1} \lambda_2\Big(\begin{bmatrix} G & t\bar{S} \\ tS & \bar{G} \end{bmatrix} \Big) \right)^{-1/2},$$

where $t_1 = \dfrac{2\|G\|}{\sigma_2(S)}$.

Proof. Since $P(z) = A + zA^T$, (3.1.18) implies that

$$\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda) \quad = \quad \left(\sup \left\{ \frac{u^*Gu}{u^*u} \;\middle|\; u \in \mathbb{C}^{2n}\setminus\{0\},\, u^TS_0u = 0, \right\} \right)^{-\frac{1}{2}}$$

where $S_0 = S$. The proof then follows by applying Theorem 1.2.7. ◻

Similarly, the following theorem gives $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$ when $P(z)$ is a T-palindromic quadratic matrix polynomial and $\lambda \in \mathbb{C}\setminus\{0,1,-1\}$.

Theorem 3.1.11. *Let* $P(z) = A_0 + zA_1 + z^2A_0^T$ *be a T-palindromic quadratic polynomial and* $\lambda \in \mathbb{C}\setminus\{0,1,-1\}$. *Suppose that* $\det(P(\lambda)) \neq 0$, *and let* $M := (P(\lambda))^{-1}$. *Furthermore, let* $\gamma_1 := \sqrt{\frac{2}{1+|\lambda|^2}}$ *and* $\gamma_2 := \sqrt{\frac{2|\lambda|^2}{1+|\lambda|^2}}$, *and define*

$$\widetilde{G} := \begin{bmatrix} 1 & \lambda & \lambda^2 \\ \bar{\lambda} & |\lambda|^2 & \lambda|\lambda|^2 \\ \bar{\lambda}^2 & \bar{\lambda}|\lambda|^2 & |\lambda|^4 \end{bmatrix} \otimes M^*M, \quad C := \begin{bmatrix} M^T & 0 & 0 \\ \lambda M^T & 0 & 0 \\ \lambda^2M^T - M & -\lambda M & -\lambda^2 M \end{bmatrix},$$

$$\Gamma := \mathrm{diag}(w_0\gamma_1, w_1, w_0\gamma_2) \otimes I_n, \quad G := \Gamma^{-1}\widetilde{G}\Gamma^{-1} \quad and \quad S = \Gamma^{-1}(C + C^T)\Gamma^{-1}.$$

Then

$$\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda) = \left(\min_{0\le t\le t_1} \lambda_2\Big(\begin{bmatrix} G & t\bar{S} \\ tS & \bar{G} \end{bmatrix} \Big) \right)^{-1/2},$$

where $t_1 = \dfrac{2\|G\|}{\sigma_2(S)}$.

Proof. Since $P(z) = A_0 + zA_1 + z^2A_0^T$, from (3.1.18) we have,

$$\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda) \quad = \quad \left(\sup \left\{ \frac{u^*Gu}{u^*u} \;\middle|\; w \in \mathbb{C}^{3n}\setminus\{0\},\, u^TS_0u = 0 \right\} \right)^{-\frac{1}{2}}$$

where $S_0 = S$. The proof then follows by applying Theorem 1.2.7. ◻

Remark 3.1.12. Due to (3.1.18), computing $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$ for polynomials of degree greater than 2 involves maximizing the Rayleigh quotient $\frac{v^*Gv}{v^*v}$ with respect to nonzero vectors v that satisfies more than one constraint each involving a symmetric matrix. Generalizing our approach to compute $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$ in these cases may involve obtaining appropriate extensions of Theorem 1.2.7. This does not seem to be straightforward and will be the subject of Section 3.2.

Remark 3.1.13. A strategy identical to the one suggested in Remark 3.1.9 gives an optimal T-palindromic perturbation to $P(z)$ corresponding to $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$ in Theorems 3.1.10 and 3.1.11.

3.1.4 Further restriction of perturbation sets

As mentioned in Section 2.4, there may be situations when it would be necessary to find the backward error $\eta^{\mathrm{pal}_\bullet}_{w,2}(P,\lambda)$ under the restriction that $\bullet$-palindromic perturbations can affect only some of the coefficient matrices. This is equivalent to setting some of the entries in the palindromic weight vector w to zero. Let

$$I := \{i_0, i_1, \ldots, i_\ell\} \subset \begin{cases} \{0,1,\ldots,\lfloor m-1\rfloor/2\}, & \text{if } m \text{ is odd,} \\ \{0,1,\ldots,m/2\} & \text{if } m \text{ is even.} \end{cases}, \tag{3.1.21}$$

assume that $i_0 < i_1 < \cdots < i_\ell$. Suppose that I is the set of indices such that only the coefficients A_j and A_{m-j}, $j \in I$ of $P(z)$ are affected by perturbations.

Let $\widehat{w}$ be a palindromic weight vector extracted from w by retaining only its nonzero entries. Then $\widehat{w}$ belongs to $\mathbb{R}^{2\ell+1}$ if m is even and $i_\ell = \frac{m}{2}$ and to $\mathbb{R}^{2\ell+2}$ otherwise. We call "if m is even and $i_\ell = \frac{m}{2}$" as case-1 and " otherwise" as case-2. Let $\mathrm{pal}_\bullet(I)$ be a subset of $\mathrm{pal}_\bullet$ such that $(\Delta_0,\ldots,\Delta_m) \in \mathrm{pal}_\bullet(I)$ implies $\Delta_j = \Delta_{m-j} = 0$ if $j \notin I$. Therefore for any $(\Delta_0,\ldots,\Delta_m) \in \mathrm{pal}_*(I)$, we then have

$$\begin{aligned}
& |\!|\!|(\Delta_0,\ldots,\Delta_m)|\!|\!|^2_{w,2} \\
= \; & |\!|\!|(\Delta_{i_0},\Delta_{i_1},\ldots,\Delta_{m-i_1},\Delta_{m-i_0})|\!|\!|^2_{\widehat{w},2} \\
= \; & \begin{cases} \sum\limits_{j=0}^{\ell-1}\left(w_{i_j}^2\|\Delta_{i_j}\|^2 + w_{m-i_j}^2\|\Delta_{m-i_j}\|^2\right) + w_{i_\ell}^2\|\Delta_{\frac{m}{2}}\|^2 & \text{if case-1 holds ,} \\ \sum\limits_{j=0}^{\ell}\left(w_{i_j}^2\|\Delta_{i_j}\|^2 + w_{m-i_j}^2\|\Delta_{m-i_j}\|^2\right) & \text{if case-2 holds.} \end{cases}
\end{aligned}$$

Then $|\!|\!|\cdot|\!|\!|_{\widehat{w},2}$ defines a norm on $(\mathbb{C}^{n\times n})^{2\ell+1}$ in the first case and on $(\mathbb{C}^{n\times n})^{2\ell+2}$ in the second case. By using this, and the weight vector $\widehat{w}$, we consider

$$\eta^{\mathrm{pal}_\bullet}_{\widehat{w},2}(P,\lambda) := \inf\left\{ \|(\Delta_0,\ldots,\Delta_m)\|_{w,2} \,\middle|\, \det\left(\sum_{j=0}^{m}\lambda^j(A_j - \Delta_j)\right) = 0,\ (\Delta_0,\ldots,\Delta_m) \in \mathrm{pal}_\bullet(I)\right\}.$$

The strategy of reformulation proposed in Section 2.4 may be used to compute the structured backward error $\eta^{\mathrm{pal}_\bullet}_{\widehat{w},2}(P,\lambda)$ with fewer constraints and smaller Hermitian matrices involved in each constraint.

To give an analogue of Theorem 3.1.5 to reformulate the problem of finding $\eta^{\mathrm{pal}_\bullet}_{\widehat{w},2}(P,\lambda)$ corresponding to the restricted perturbation set $\mathrm{pal}_\bullet(I)$, we define

$$\widehat{\Lambda}_\ell := \begin{cases} [\lambda^{i_0},\ldots,\lambda^{i_{\ell-1}},\lambda^{i_\ell},\lambda^{m-i_{\ell-1}},\ldots,\lambda^{m-i_0}], & \text{if case-1 holds,} \\ [\lambda^{i_0},\ldots,\lambda^{i_l},\lambda^{m-i_\ell},\ldots,\lambda^{m-i_0}] & \text{if case-2 holds.} \end{cases}$$

$$\widehat{C}_j := \begin{cases} (\widehat{\Lambda}^\bullet_\ell e^\bullet_{j+1}) \otimes M^\bullet - (e_{2\ell+2-j}\widehat{\Lambda}_\ell) \otimes M, & \text{j=0,}\ldots\text{,}\ell\text{-1, if case-1 holds,} \\ (\widehat{\Lambda}^*_\ell e^*_{i_\ell+1}) \otimes M^* - (e_{i_\ell+1}\widehat{\Lambda}_\ell) \otimes M, & \text{j=}\ell\text{, if case-1 holds and } \bullet = *, \\ (\widehat{\Lambda}^\bullet_\ell e^\bullet_{j+1}) \otimes M^\bullet - (e_{2l+3-j}\widehat{\Lambda}_\ell) \otimes M, & \text{for j=0,}\ldots,\ell\text{, if case-2 holds.} \end{cases}$$

Also, define $\widehat{\gamma}_{j1} = w_{i_j}\sqrt{\frac{2}{1+|\lambda|^{m-2i_j}}}$, $\widehat{\gamma}_{j2} = w_{i_j}\sqrt{\frac{2|\lambda|^{m-2i_j}}{1+|\lambda|^{m-2i_j}}}$,

$$\widehat{\Gamma} := \begin{cases} \mathrm{diag}(\widehat{\gamma}_{01},\ldots,\widehat{\gamma}_{\ell-1\,1},\widehat{w}_\ell,\widehat{\gamma}_{\ell-1\,2},\ldots,\widehat{\gamma}_{02}) \otimes I_n, & \text{if case-1 holds,} \\ \mathrm{diag}(\widehat{\gamma}_{01},\ldots,\widehat{\gamma}_{\ell 1},\widehat{\gamma}_{\ell 2},\ldots,\widehat{\gamma}_{02}) \otimes I_n & \text{if case-2 holds.} \end{cases}$$

and

$$\widehat{G} = \widehat{\Gamma}^{-1}(\widehat{\Lambda}^*_\ell\widehat{\Lambda}_\ell \otimes M^*M)\widehat{\Gamma}^{-1} \tag{3.1.22}$$

$$\widehat{S}_j = \widehat{\Gamma}^{-1}(\widehat{C}_j + \widehat{C}^T_j)\widehat{\Gamma}^{-1},\ j = 0,\ldots,p, \tag{3.1.23}$$

where $p = l-1$ if case-1 holds and $p = l$ otherwise. Further, when case-1 holds and $\bullet = *$, for $j = 0,\ldots,2l+1$ let

$$\widehat{H}_j = \widehat{\Gamma}^{-1}(\widehat{C}_j + \widehat{C}^*_j)\widehat{\Gamma}^{-1}, \quad \widehat{H}_{2l+1-j} = \widehat{\Gamma}^{-1}(i(\widehat{C}_j - \widehat{C}^*_j))\widehat{\Gamma}^{-1} \tag{3.1.24}$$

$$\text{and} \quad \widehat{H}_\ell = \widehat{\Gamma}^{-1}(\widehat{C}_\ell)\widehat{\Gamma}^{-1} \tag{3.1.25}$$

and when case-2 holds and $\bullet = *$, for $j = 0,\ldots,2l+2$, let

$$\widehat{H}_j = \widehat{\Gamma}^{-1}(\widehat{C}_j + \widehat{C}^*_j)\widehat{\Gamma}^{-1} \quad \text{and} \quad \widehat{H}_{2l+2-j} = \widehat{\Gamma}^{-1}(i(\widehat{C}_j - \widehat{C}^*_j))\widehat{\Gamma}^{-1}. \tag{3.1.26}$$

Theorem 3.1.14. *Let $P(z) = \sum_{i=0}^{m} z^i A_i$ be a $\bullet$-palindromic polynomial and $\lambda \in \mathbb{C} \setminus \{0\}$ be such that $M := (P(\lambda))^{-1}$ exists. Let $I = \{i_0, \ldots, i_\ell\}$ be as defined in (3.1.21) and $\widehat{w}$ be a weight vector corresponding to I. Let $\widehat{G}$, $\widehat{S}_j$ and $\widehat{H}_j$ be as defined in (3.1.22)-(3.1.26) with respect to I.*

(a) If $\bullet = T$, then

$$\eta^{\mathrm{pal_T}}_{\widehat{w},2}(P,\lambda) = \left(\sup\left\{ \frac{u^*\widehat{G}u}{u^*u} \;\middle|\; u \in \mathbb{C}^{n(2\ell+1)}\setminus\{0\},\, u^T\widehat{S}_j u = 0,\, j = 0,\ldots,\ell-1 \right\}\right)^{-\frac{1}{2}},$$

if case-1 holds, and

$$\eta^{\mathrm{pal_T}}_{\widehat{w},2}(P,\lambda) = \left(\sup\left\{ \frac{u^*\widehat{G}u}{u^*u} \;\middle|\; u \in \mathbb{C}^{n(2\ell+2)}\setminus\{0\},\, u^T\widehat{S}_j u = 0,\, j = 0,\ldots,\ell \right\}\right)^{-\frac{1}{2}},$$

otherwise.

(b) If $\bullet = *$, then

$$\eta^{\mathrm{pal_*}}_{\widehat{w},2}(P,\lambda) = \left(\sup\left\{ \frac{u^*\widehat{G}u}{u^*u} \;\middle|\; u \in \mathbb{C}^{n(2\ell+1)}\setminus\{0\},\, u^*\widehat{H}_j u = 0,\, j = 0,\ldots,2\ell+1 \right\}\right)^{-\frac{1}{2}},$$

if case-1 holds and

$$\eta^{\mathrm{pal_*}}_{\widehat{w},2}(P,\lambda) = \left(\sup\left\{ \frac{u^*\widehat{G}u}{u^*u} \;\middle|\; u \in \mathbb{C}^{n(2\ell+2)}\setminus\{0\},\, u^*\widehat{H}_j u = 0,\, j = 0,\ldots,2\ell+2 \right\}\right)^{-\frac{1}{2}},$$

otherwise.

Proof. The proof proceeds in the same way as of Theorem 3.1.5 with the observation that $\widehat{G}$, $\widehat{H}_j$, $\widehat{H}_{2\ell+1-j}$ or $\widehat{H}_{2\ell+2-j}$, and $\widehat{S}_j$ are obtained from the corresponding matrices G, H_{i_j}, H_{m-i_j} and S_{i_j} in Theorem 3.1.5 by deleting the block rows and block columns j and $m-j$ for each j not in I. □

The following result is an analogue of Theorem 3.1.6 and obtains $\eta^{\mathrm{pal_*}}_{\widehat{w},2}(P,\lambda)$ for $*$-palindromic polynomials when $|\lambda| \neq 1$.

Theorem 3.1.15. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a $\bullet$-palindromic polynomial. Let $\lambda \in \mathbb{C} \setminus \{0\}$ be such that $|\lambda| \neq 1$ and $M := (P(\lambda))^{-1}$ exists. Let $I = \{i_0, \ldots, i_\ell\}$ be as defined in (3.1.21) and $\widehat{w}$ be a weight vector according to I. Let $\widehat{G}$ and $\widehat{H}_j$ be as defined in (3.1.22) and (3.1.24)- (3.1.26) respectively, with respect to I. Then*

$$\lambda^\star_{\max} := \min_{t_0,\ldots,t_p\in\mathbb{R}} \lambda_{\max}(\widehat{G} + t_0\,\widehat{H}_0 + \cdots + t_p\,\widehat{H}_p)$$

is attained for some $(t_0^\star, \ldots, t_p^\star) \in \mathbb{R}^{p+1}$, *where* $p = 2\ell + 1$ *if case-1 holds and* $p = 2\ell + 2$ *if case-2 holds. Suppose that* $\eta_{\widehat{w},2}^{\mathrm{pal}_*}(P, \lambda)$ *is finite. If either* $\ell = 0$ *or* $\lambda_{\max}^\star$ *is a simple eigenvalue of the matrix* $\widehat{G} + t_0^\star \widehat{H}_0 + \cdots + t_p^\star \widehat{H}_p$, *then*

$$\eta_{\widehat{w},2}^{\mathrm{pal}_*}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^\star}} = \left(\min_{t_0, \ldots, t_p \in \mathbb{R}} \lambda_{\max}(\widehat{G} + t_0 \widehat{H}_0 + \cdots + t_p \widehat{H}_p) \right)^{-1/2}.$$

Proof. The proof follows immediately in the view of Theorem 3.1.14 and Theorem 1.2.18 by noting that matrices $\widehat{G}$ and $\widehat{H}_j$ for $j = 0, \ldots, p$ satisfy the assumption of Theorem 1.2.18. $\square$

Remark 3.1.16. Note that the assumption of $\eta_{\widehat{w},2}^{\mathrm{pal}_*}(P, \lambda)$ being finite in the above theorem is necessary because if $P(z) = A_0 + zA_1 + z^2 A_0^*$ where A_0 is a positive definite matrix and A_1 is a Hermitian matrix then $\eta_{\widehat{w},2}^{\mathrm{pal}_*}(P, \lambda) = \infty$ for any λ such that $\operatorname{Im} \lambda \neq 0$ and $|\lambda| \neq 1$ where $w = (0, 1, 0)$.

We can use Theorem 1.2.7 in Theorem 3.1.14 to obtain $\eta_{\widehat{w},2}^{\mathrm{pal}_\mathrm{T}}(P, \lambda)$ for T-palindromic polynomials if the number of coefficients that are perturbed are such that the computations involve a single constraint involving a symmetric matrix. For instance, for a T-palindromic polynomial of odd degree, $\eta_{\widehat{w},2}^{\mathrm{pal}_\mathrm{T}}(P, \lambda)$ may be computed provided I has only one element whereas for a T-palindromic polynomial of even degree, the same is possible provided the $I \subseteq \{j, \frac{m}{2}\}$, $j = 0, \ldots, \frac{m}{2} - 1$.

It may be noted that if $P(z)$ is a T-palindromic polynomial of even degree and only the coefficient $A_{\frac{m}{2}}$ is affected by perturbation, then there are no constraints in the computation of $\eta_{\widehat{w},2}^{\mathrm{pal}_\mathrm{T}}(P, \lambda)$ and therefore it is equal to the backward error $\eta_{\widehat{w},2}(P, \lambda)$ with respect to arbitrary perturbations.

Theorem 3.1.17. *Let* $P(z) = \sum_{j=0}^m z^j A_j$ *be a T-palindromic polynomial. Suppose that* $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$ *such that* $M = (P(\lambda))^{-1}$ *exists. Let* $I = \{i_0\}$ *or* $I = \{i_0, \frac{m}{2}\}$ *when* m *is even, where* $i_0 \in \{0, 1, \ldots, \lfloor \frac{m-1}{2} \rfloor\}$ *and* $\widehat{w}$ *be a weight vector corresponding to* I. *Let* $\widehat{G}$ *and* $\widehat{S}_0$ *be as defined in (3.1.22) and (3.1.23) respectively, with respect to* I. *If* $\eta_{\widehat{w},2}^{\mathrm{pal}_\mathrm{T}}(P, \lambda)$ *is finite, then*

$$\eta_{\widehat{w},2}^{\mathrm{pal}_\mathrm{T}}(P, \lambda) = \left(\min_{0 \le t \le t_1} \lambda_2 \left(\begin{bmatrix} \widehat{G} & t\bar{\widehat{S}}_0 \\ t\widehat{S}_0 & \bar{\widehat{G}} \end{bmatrix} \right) \right)^{-1/2},$$

where $t_1 = \dfrac{2\|\widehat{G}\|}{\sigma_2(\widehat{S}_0)}$.

Proof. If $I = \{i_0\}$, we can perturb coefficient matrices with indices i_0 and $m - i_0$. If $I = \{i_0, \frac{m}{2}\}$, we can perturb coefficient matrices with indices i_0, $m - i_0$ and $\frac{m}{2}$. Thus as a consequence of Theorem 3.1.14, we get

$$\eta^{\mathrm{pal_T}}_{\widehat{w},2}(P,\lambda) = \left(\sup\left\{ \frac{u^*\widehat{G}u}{u^*u} \,\middle|\, u \in \mathbb{C}^{2n}\backslash\{0\},\, u^T\widehat{S}_0u = 0\right\}\right)^{-\frac{1}{2}},$$

in the first case and

$$\eta^{\mathrm{pal_T}}_{\widehat{w},2}(P,\lambda) = \left(\sup\left\{ \frac{u^*\widehat{G}u}{u^*u} \,\middle|\, u \in \mathbb{C}^{3n}\backslash\{0\},\, u^T\widehat{S}_0u = 0\right\}\right)^{-\frac{1}{2}},$$

in the second case. Now the proof follows by Theorem 1.2.7. □

3.1.5 Numerical experiments

In this section we present some numerical examples to illustrate the proposed method for computing the structured backward error $\eta^{\mathbb{S}}_{w,2}(P,\lambda)$ of some $\lambda \in \mathbb{C} \setminus \{0\}$ for the structure $\mathbb{S} = \mathrm{pal}_*$ and for $w = (1, 1, \ldots, 1)$. In all cases we have used the software package CVX [16, 17] in MATLAB to solve the associated optimization problems. Numerical examples for the structure $\mathbb{S} = \mathrm{pal_T}$ are illustrated in Subsection 5.2.2.

Example 3.1.18. $L(z) = A + zA^*$ is a $*$-palindromic pencil of size 4 with eigenvalues $0.4624-0.8867i$, $-0.5697+1.7298i$, $-0.1718+0.5215i$, $-0.9765+0.2155i$. For $\lambda = 0.4853-0.5955i$, the backward error with respect to arbitrary perturbations is $\eta_{w,2}(L,\lambda) = 0.0912$ while the structured backward error satisfies $\eta^{\mathrm{pal}_*}_{w,2}(L,\lambda) = 0.3320$. The plot on the left of Figure 3.1.1 illustrates the movement of the eigenvalues of the pencil $L(z)$ under the homotopic perturbations $L(z) + t\Delta L(z)$ as t varies from 0 to 1 and $\Delta L(z)$ is an optimal $*$-palindromic perturbation corresponding to $\eta^{\mathrm{pal}_*}_{w,2}(L,\lambda)$ that induces eigenvalues at $(\lambda, 1/\bar{\lambda})$. The eigenvalue curves starting from $0.4624 - 0.8867i$ and $-0.9765 + 0.2155i$ (each marked by a star surrounded by a circle) come together on the unit circle and split out to form the pair of eigenvalues $(\lambda, 1/\bar{\lambda})$ (where λ is marked by a star surrounded by a diamond) of the pencil $L(z) + \Delta L(z)$.

On the other hand, the plot on the right hand side of Figure 3.1.1 gives the movement of the eigenvalues under the homotopic perturbations $L(z) + t\widetilde{\Delta L}(z)$ when t moves from 0 to 1 and $\widetilde{\Delta L}(z)$ is an optimal perturbation corresponding to $\eta_{w,2}(L,\lambda)$ that induces an eigenvalue at λ without preserving $*$-palindromic structure. In this case the perturbations move the nearest eigenvalue of the pencil to λ.

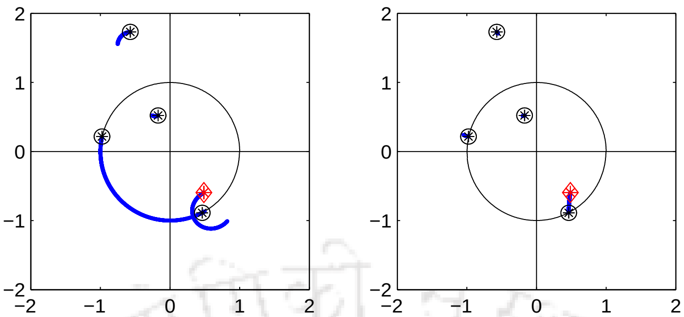

Figure 3.1.1: Eigenvalue perturbation curves for the $*$-palindromic pencil $L(z)$ of Example 3.1.18 with respect to $*$-palindromic perturbations (left) and arbitrary perturbations (right).

Example 3.1.19. $Q(z) = A + zB^* + z^2A^*$ is a $*$-palindromic polynomial of size 3 with eigenvalues $-3.2746-0.4165i$, $-0.3597+1.82221i$, $0.9896-0.1437i$, $-0.0961-0.9954i$, $-0.3005-0.0382i$, $-0.1043+0.5282i$. For $\lambda = 0.88+0.15i$, the eigenvalue backward error with respect to arbitrary perturbations is 0.609 while with respect to $*$-palindromic perturbations this is 1.7059. The plot on the left of Figure 3.1.2 illustrates the effect of perturbations $Q(z) + t\Delta Q(z)$ on the eigenvalues of $Q(z)$ as t varies from 0 to 1, $\Delta Q(z)$ being an optimal $*$-palindromic perturbation to $Q(z)$ corresponding to $\eta_{w,2}^{\mathrm{pal}_*}(Q,\lambda)$ that induces eigenvalues at $(\lambda, 1/\bar{\lambda})$. It shows eigenvalue curves starting from the eigenvalues $-0.1043+0.528i$ and $-0.3597+1.82221i$ (each marked by a star surrounded by circle) of $Q(z)$ coalescing on the unit circle and moving along the circle till they next coalesce with the eigenvalue curve starting from the eigenvalue $0.9896-0.1437i$ on the unit circle. After the second coalescence, the eigenvalues curves split out of the unit circle to form the pair of eigenvalues $(\lambda, 1/\bar{\lambda})$ (where λ is marked by a star surrounded by a diamond) of $Q(z) + \Delta Q(z)$.

The plot on the right of Figure 3.1.2 shows the movement of the eigenvalues of $Q(z)$ under perturbations $Q(z) + t\widetilde{\Delta Q}(z)$, where $\widetilde{\Delta Q}(z)$ is an optimal perturbation to $Q(z)$ corresponding to $\eta_{w,2}(Q,\lambda)$ that induces an eigenvalue at λ and is not $*$-palindromic. In this case the nearest eigenvalue of $Q(z)$ on the unit circle moves to form the eigenvalue λ of $Q(z) + \widetilde{\Delta Q}(z)$.

We also compare $\eta_{w,2}(P,\lambda)$ with $\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda)$ for the cases that the values of λ converge to an eigenvalue of $P(z)$ as well as for arbitrary values of λ.

Table 3.1.1 illustrates these comparisons for the $*$-palindromic pencil $L(z)$ of Example 3.1.18 as λ values converge to the eigenvalue $0.4624-0.8867i$ on the unit circle. Observe that while $\eta_{w,2}(L,\lambda)$ decreases to 0, this is not the case for $\eta_{w,2}^{\mathrm{pal}_*}(L,\lambda)$ leading to large dif-

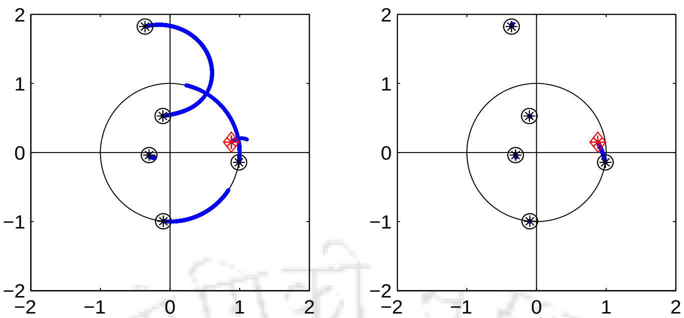

Figure 3.1.2: Eigenvalue perturbation curves for the $*$-palindromic polynomial $Q(z)$ of Example 3.1.19 with respect to $*$-palindromic perturbations (left) and arbitrary perturbations (right).

ferences in the values of the two backward errors.

Table 3.1.2 does the same comparison for values of λ that converge to the eigenvalue $-0.5697 + 1.7298i$ not on the unit circle as well as for arbitrary values of λ. In the first case, both $\eta_{w,2}(L, \lambda)$ and $\eta_{w,2}^{\mathrm{pal}_*}(L, \lambda)$ decrease to 0. However in the second case, there is significant difference between the two backward errors even when the values of λ are away from the unit circle.

Table 3.1.1: *Values of $\eta_{w,2}(L, \lambda)$ and $\eta_{w,2}^{\mathrm{pal}_*}(L, \lambda)$ for the $*-$palindromic pencil $L(z)$ in Example 3.1.18 where $\lambda \to 0.4624 - 0.8867i$.*

λ	$\eta_{w,2}(L, \lambda)$	$\eta_{w,2}^{\mathrm{pal}_*}(L, \lambda)$
0.600 - 1.200i	0.0710	0.2755
0.550 - 1.100i	0.0510	0.2772
0.520 - 1.000i	0.0300	0.2824
0.500 - 0.980i	0.0240	0.2807
0.480 - 0.950i	0.0159	0.2799
0.475 - 0.930i	0.0111	0.2811
0.470 - 0.900i	0.0038	0.2835
0.465 - 0.895i	0.0022	0.2830

Table 3.1.2: *Values of $\eta_{w,2}(L,\lambda)$ and $\eta_{w,2}^{\mathrm{pal}_*}(L,\lambda)$ for the $*-$palindromic pencil $L(z)$ in Example 3.1.18 where $\lambda \to -0.5697 + 1.7298i$ (left) and for arbitrary λ (right).*

λ	$\eta_{w,2}(L,\lambda)$	$\eta_{w,2}^{\mathrm{pal}_*}(L,\lambda)$
$-2.50 + 0.50i$	0.1134	0.1605
$-2.00 + 1.00i$	0.1006	0.1344
$-1.50 + 1.40i$	0.0882	0.1012
$-0.90 + 1.50i$	0.0569	0.0593
$-0.60 + 1.62i$	0.0196	0.0203
$-0.58 + 1.70i$	0.0053	0.0055

λ	$\eta_{w,2}(L,\lambda)$	$\eta_{w,2}^{\mathrm{pal}_*}(L,\lambda)$
$1.1890 + 0.0376i$	0.4726	0.5937
$0.2940 - 1.3362i$	0.0850	0.2190
$1.1910 - 1.2025i$	0.1571	0.3415
$0.9410 - 0.9921i$	0.1173	0.3494
$0.4850 - 0.5955i$	0.0912	0.3320
$0.6680 - 0.0783i$	0.3688	0.5149

3.2 Estimates for the structured eigenvalue backward errors of higher degree T-palindromic polynomials

The techniques used to establish Theorem 3.1.10 and its analogues for T-even and T-odd polynomials (in Section 3.4) do not extend to higher degree polynomials. We obtain estimates of the structured eigenvalue backward error for such polynomials by extending Theorem 1.2.7.

Given a T-palindromic polynomial $P(z)$ and $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$, we establish lower and upper bounds on the structured backward error $\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(P,\lambda)$ which estimate it very closely. In fact, we show that the lower bound gives the exact value of $\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(P,\lambda)$ under certain assumptions. Numerical experiments indicate that these assumptions are always satisfied in practice. The same kind of strategy can be used to estimate the eigenvalue backward errors of higher degree T-antipalindromic and T-alternating polynomials.

We first recall a few things from previous section. Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a regular T-palindromic polynomial and $\lambda \in \mathbb{C} \setminus \{0, \pm 1\}$ be such that $M = (P(\lambda))^{-1}$ exists. If $k = \lfloor \frac{m-1}{2} \rfloor$ then from (3.1.4) we have

$$\left(\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(P,\lambda)\right)^2 = \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{K} \right\}, \tag{3.2.1}$$

where $f(v_0,\ldots,v_m)$ is given by

$$f(v_0,\ldots,v_m)=\begin{cases}\sum\limits_{j=0}^{k} 2w_j^2\max\left\{\frac{\|v_j\|^2}{\|Mv_\lambda\|^2},\frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right\} & \text{if } m \text{ is odd,}\\ \sum\limits_{j=0}^{k} 2w_j^2\max\left\{\frac{\|v_j\|^2}{\|Mv_\lambda\|^2},\frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right\}+w_{\frac{m}{2}}^2\frac{\|v_{\frac{m}{2}}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even}\end{cases} \tag{3.2.2}$$

and $\mathcal{K}\subseteq(\mathbb{C}^n)^{m+1}$ is given by

$$\mathcal{K}=\left\{(v_0,\ldots,v_m)\,\Big|\, v_\lambda\neq 0,\,(Mv_\lambda)^Tv_j=v_{m-j}^TMv_\lambda,\ j=0\ldots,k\right\}. \tag{3.2.3}$$

Moreover, in view of Theorem 3.1.5 computing structured eigenvalue backward error of an approximate eigenvalue for a T-palindromic polynomial is equivalent to solving a minimization problem of maximizing the Rayleigh Quotient of a Hermitian matrix subject to some constraints involving symmetric matrices. Indeed,

$$\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)\;=\;\left(\sup\left\{\frac{u^*Gu}{u^*u}\,\Big|\,u\in\mathbb{C}^{n(m+1)}\setminus\{0\},\,u^TS_ju=0,\,j=0,\ldots,k\right\}\right)^{-\frac{1}{2}}, \tag{3.2.4}$$

where G and S_j, for $j=0,\ldots,k$, are defined by (3.1.15) and (3.1.17), respectively.

3.2.1 Framework for the lower bound of $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$

Let $P(z)=\sum_{j=0}^m z^jA_j$ be a T-palindromic polynomial. In this section, we build the framework for deriving a lower bound of the structured backward error $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$ for $\lambda\in\mathbb{C}\setminus\{0,1,-1\}$.

Given $H\in\mathrm{Herm}(n)$, $p\in\mathbb{N}$ and $S_j\in\mathrm{Sym(n)}$ for $j=1,\ldots,p$, we define

$$F_1:\mathbb{R}\mapsto\mathbb{C}^{2n\times 2n}\qquad\text{by}\qquad F_1(t_1):=\begin{bmatrix}H & t_1\bar{S}_1\\ t_1S_1 & \bar{H}\end{bmatrix}$$

and

$$F_{p+1}:\mathbb{R}^{p+1}\mapsto\mathbb{C}^{2^{p+1}n\times 2^{p+1}n}\qquad\text{by,}$$

$$F_{p+1}(t_1,\ldots,t_{p+1}):=\begin{bmatrix}F_p(t_1,\ldots,t_p) & S\\ S & F_p(t_1,\ldots,t_p)\end{bmatrix}, \tag{3.2.5}$$

where S is given by,

$$S:=\begin{bmatrix}0 & \ldots & 0 & t_{p+1}\bar{S}_{p+1}\\ \vdots & \iddots & t_{p+1}S_{p+1} & 0\\ \vdots\ \iddots & \iddots & \iddots & \vdots\\ 0 & t_{p+1}\bar{S}_{p+1} & \iddots & \vdots\\ t_{p+1}S_{p+1} & 0 & \ldots & 0\end{bmatrix}_{2^pn\times 2^pn}.$$

Also let

$$m_{hs_1\dots s_p}(H,S_1,\dots,S_p) := \sup\left\{v^*Hv \,|\, v\in\mathbb{C}^n,\ \|v\|=1,\ v^TS_jv=0 \text{ for } j=1,\dots,p\right\}. \tag{3.2.6}$$

For a unit vector $v\in\mathbb{C}^n$, if

$$\Upsilon_v^1 := \left\{\begin{bmatrix} z_1\, v^T & \bar{z}_2\bar{v}^T \end{bmatrix}^T \,|\, z_1,z_2\in\mathbb{C}^n\right\}$$

then Υ_v^1 is a 2−dimensional subspace of $\mathbb{C}^{2n}$, and

$$\begin{bmatrix} z_1v \\ \bar{z}_2\bar{v}\end{bmatrix}^* F_1(t_1)\begin{bmatrix} z_1v \\ \bar{z}_2\bar{v}\end{bmatrix} = (|z_1|^2+|z_2|^2)v^*Hv + 2t_1\mathrm{Re}(z_1z_2v^TS_1v), \quad z_1,z_2\in\mathbb{C}.$$

Now for $p\in\mathbb{N}$, we define Υ_v^{p+1} recursively as follows,

$$\Upsilon_v^{p+1} := \left\{\begin{bmatrix} y_1^T & y_2^T\end{bmatrix}^T \,|\, y_1,\, y_2\,\in \Upsilon_v^p\right\}.$$

Again observe that Υ_v^{p+1} is a 2^{p+1} dimensional subspace of $\mathbb{C}^{2^{p+1}n}$. The quantity m_{hs_1} is obtained explicitly in Theorem 1.2.7. This result together with Theorem 3.1.5 gives the structured eigenvalue backward error for T-palindromic polynomials of degree at most 2 in the Subsection 3.1.3. Following the strategy used in [26], we will first obtain a upper bound for the quantity $m_{hs_1,\dots,s_p}$.

Lemma 3.2.1. *Let $H\in\mathrm{Herm}(n)$ and $v\in\mathbb{C}^n$ be a unit vector. Let $p\in\mathbb{N}$, $S_i\in\mathrm{Sym(n)}$ and $t_i\in\mathbb{R}$ for $i=1,\dots,p+1$. Set*

$$u_1 = \begin{bmatrix} y_1v^T & \bar{y}_2\bar{v}^T & \dots & y_{2^p-1}v^T & \bar{y}_{2^p}\bar{v}^T\end{bmatrix}^T, u_2 = \begin{bmatrix} z_1v^T & \bar{z}_2\bar{v}^T & \dots & z_{2^p-1}v^T & \bar{z}_{2^p}\bar{v}^T\end{bmatrix}^T,$$

where $y_j,z_j\in\mathbb{C}$ for all j. Then

$$\begin{bmatrix} u_1\\ u_2\end{bmatrix}^* F_{p+1}(t_1,\dots,t_{p+1})\begin{bmatrix} u_1\\ u_2\end{bmatrix} = \left(|y_1|^2+\dots+|y_{2^p}|^2+|z_1|^2+\dots+|z_{2^p}|^2\right)v^*Hv + \Xi_{p+1},$$

where Ξ_{p+1} is some sum of the terms of the form $2t\mathrm{Re}(\alpha_1\alpha_2v^TS_iv)$, $i\in\{1,\dots,p+1\}$, $\alpha_1,\alpha_2\in\{y_1,\dots,y_{2^p},z_1,\dots,z_{2^p}\}$ and $t\in\{t_1,\dots,t_{p+1}\}$.

Proof. We prove the lemma by applying induction on p. For $p=1$ result follows, since

$$\begin{bmatrix} y_1\,v\\ \bar{y}_2\,\bar{v}\\ z_1\,v\\ \bar{z}_2\,\bar{v}\end{bmatrix}^* F_2(t_1,t_2)\begin{bmatrix} y_1\,v\\ \bar{y}_2\,\bar{v}\\ z_1\,v\\ \bar{z}_2\,\bar{v}\end{bmatrix} = \begin{bmatrix} y_1\,v\\ \bar{y}_2\,\bar{v}\\ z_1\,v\\ \bar{z}_2\,\bar{v}\end{bmatrix}^* \begin{bmatrix} H & t_1\bar{S}_1 & 0 & t_2\bar{S}_2\\ t_1S_1 & \bar{H} & t_2S_2 & 0\\ 0 & t_2\bar{S}_2 & H & t_1\bar{S}_1\\ t_2S_2 & 0 & t_1S_1 & \bar{H}\end{bmatrix}\begin{bmatrix} y_1\,v\\ \bar{y}_2\,\bar{v}\\ z_1\,v\\ \bar{z}_2\,\bar{v}\end{bmatrix}$$

$$= (|y_1|^2+|y_2|^2+|z_1|^2+|z_2|^2)v^*Hv + 2t_1\mathrm{Re}(y_1y_2v^TS_1v) + 2t_2\mathrm{Re}(y_1z_2v^TS_2v)$$
$$+2t_1\mathrm{Re}(z_1z_2v^TS_1v) + 2t_2\mathrm{Re}(y_2z_1v^TS_2v).$$

Next, we assume that result holds for $p = k-1$, $k > 2$. Now let $p = k$ and let

$$u_1 = \begin{bmatrix} y_1 v \\ \bar{y}_2 \bar{v} \\ \vdots \\ y_{2^k-1} v \\ \bar{y}_{2^k} \bar{v} \end{bmatrix}, \; u_2 = \begin{bmatrix} z_1 v \\ \bar{z}_2 \bar{v} \\ \vdots \\ z_{2^k-1} v \\ \bar{z}_{2^k} \bar{v} \end{bmatrix}, \; S = \begin{bmatrix} 0 & \dots & 0 & t_{k+1}\bar{S}_{k+1} \\ \vdots & \iddots & t_{k+1}S_{k+1} & 0 \\ \vdots \;\; \iddots & \iddots & \iddots & \vdots \\ 0 & t_{k+1}\bar{S}_{k+1} & \iddots & \vdots \\ t_{k+1}S_{k+1} & 0 & \dots & 0 \end{bmatrix}_{2^k n \times 2^k n}.$$

Then

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^* F_{k+1}(t_1,\dots,t_{k+1}) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^* \begin{bmatrix} F_k(t_1,\dots,t_k) & S \\ S & F_k(t_1,\dots,t_k) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= u_1^* F_k(t_1,\dots,t_k) u_1 + u_2^* F_k(t_1,\dots,t_k) u_2 + u_2^* S u_1 + u_1^* S u_2. \tag{3.2.7}$$

By induction hypothesis,

$$u_1^* F_k(t_1,\dots,t_k) u_1 = (|y_1|^2 + |y_2|^2 + \dots + |y_{2^k-1}|^2 + |y_{2^k}|^2) v^* H v + \Xi_k^1, \tag{3.2.8}$$

where Ξ_k^1 is some sum of terms of the form $2t\mathrm{Re}(\alpha_1\alpha_2 v^T S_i v)$, $i \in \{1,\dots,k\}$, $t \in \{t_1,\dots,t_k\}$ and $\alpha_1, \alpha_2 \in \{y_1,\dots,y_{2^k}\}$. Similarly,

$$u_2^* F_k(t_1,\dots,t_k) u_2 = (|z_1|^2 + |z_2|^2 + \dots + |z_{2^k-1}|^2 + |z_{2^k}|^2) v^* H v + \Xi_k^2, \tag{3.2.9}$$

where Ξ_k^2 is some sum of terms of form $2t\mathrm{Re}(\alpha_1\alpha_2 v^T S_i v)$, $i \in \{1,\dots,k\}$, $t \in \{t_1,\dots,t_k\}$, and $\alpha_1, \alpha_2 \in \{z_1,\dots,z_{2^k}\}$, and a straight forward calculation shows that

$$u_2^* S u_1 + u_1^* S u_2 = 2t_{k+1}\mathrm{Re}(y_1 z_{2^k} v^T S_{k+1} v + \dots + y_{2^k} z_1 v^T S_{k+1} v). \tag{3.2.10}$$

In view of (3.2.7), (3.2.8), (3.2.9) and (3.2.10),

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^* F_{k+1}(t_1,\dots,t_{k+1}) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \big(|y_1|^2 + \dots + |y_{2^k}|^2 + |z_1|^2 + \dots + |z_{2^k}|^2\big) v^* H v + \Xi_{k+1},$$

where Ξ_{k+1} is some sum of terms of the form $2t\mathrm{Re}(\alpha_1\alpha_2 v^T S_i v)$ where $i \in \{1,\dots,k+1\}$, $\alpha_1, \alpha_2 \in \{y_1,\dots,y_{2^k}, z_1,\dots,z_{2^k}\}$ and $t \in \{t_1,\dots,t_{k+1}\}$. Hence the statement holds for $p = k$ and the proof follows by induction. □

Theorem 3.2.2. *Let $H \in \mathrm{Herm}(n)$, $p \in \mathbb{N}$ and $S_i \in \mathrm{Sym(n)}$ for $i = 1,\dots,p$. Then*

$$m_{hs_1\dots s_p}(H, S_1,\dots,S_p) \le \inf_{t_1,\dots,t_p \in \mathbb{R}} \lambda_{2^p}\big(F_p(t_1,\dots,t_p)\big),$$

where $\lambda_{2^p}\big(F_p(t_1,\dots,t_p)\big)$ denotes the $(2^p)^{th}$ largest eigenvalue of $F_p(t_1,\dots,t_p)$.

Proof. Let $v \in \mathbb{C}^n$ be a unit vector satisfying $v^T S_i v = 0$ for all $i = 1, \ldots, p$. Then by Courant-Fischer max-min-principle [23] and Lemma 3.2.1,

$$\lambda_{2^p}\big(F_p(t_1, \ldots, t_p)\big) \geq \min_{x \in \Upsilon_v^p, \|x\|=1} x^* F_p(t_1, \ldots, t_p) x = v^* H v \quad \forall\, t_1, \ldots, t_p \in \mathbb{R}.$$

Hence, $\lambda_{2^p}\big(F_p(t_1, \ldots, t_p)\big) \geq m_{hs_1 \ldots s_p}(H, S_1, \ldots, S_p)$. $\square$

The function $\lambda_{2^p}\big(F_p(t_1, \ldots, t_p)\big)$ has a global minimum over $\mathbb{R}^p$. The lemma below proves this for 2 parameters. The general case follows by noticing that the same proof can be extended for any number of parameters. For $H \in \mathrm{Herm}(n)$, S_1, $S_2 \in \mathrm{Sym(n)}$ and $t_1, t_2 \in \mathbb{R}$ we can write $F_2(t_1, t_2)$ as

$$F_2(t_1, t_2) = \underbrace{\begin{bmatrix} H & 0 & 0 & 0 \\ 0 & \bar{H} & 0 & 0 \\ 0 & 0 & H & 0 \\ 0 & 0 & 0 & \bar{H} \end{bmatrix}}_{=:H_0} + t_1 \underbrace{\begin{bmatrix} 0 & \bar{S}_1 & 0 & 0 \\ S_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_1 \\ 0 & 0 & S_1 & 0 \end{bmatrix}}_{=:H_1} + t_2 \underbrace{\begin{bmatrix} 0 & 0 & 0 & \bar{S}_2 \\ 0 & 0 & S_2 & 0 \\ 0 & \bar{S}_2 & 0 & 0 \\ S_2 & 0 & 0 & 0 \end{bmatrix}}_{=:H_2}. \tag{3.2.11}$$

Lemma 3.2.3. *Let $H \in \mathrm{Herm}(n)$, S_1, $S_2 \in \mathrm{Sym(n)}$ with $S_1 \neq S_2$ and let $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be given by $\phi(t_1, t_2) = \lambda_4(F_2(t_1, t_2))$. Then ϕ satisfies*

$$\phi(t_1, t_2) = \phi(-t_1, t_2) = \phi(t_1, -t_2) = \phi(-t_1, -t_2) \quad \text{for all } t_1,\, t_2 \in \mathbb{R}.$$

If rank$(S_1) \geq 2$ and rank$(S_2) \geq 2$ then ϕ has a global minimum in the region $t_1^2 + t_2^2 \leq R^2$ for $t_1 \geq 0, t_2 \geq 0$, where $R = \frac{(\lambda_{\max}(H) - \lambda_{\min}(H))}{c}$ and

$$c = \min\left\{\lambda_4(t_1 H_1 + t_2 H_2) \,|\, t_1, t_2 \in \mathbb{R}\,;\, t_1^2 + t_2^2 = 1\right\},$$

H_1 and H_2 being defined by (3.2.11).

Proof. Set

$$T_1 = \mathrm{diag}\,(-1, 1, -1, 1) \otimes I_n,\ T_2 = \mathrm{diag}\,(-1, 1, 1, -1) \otimes I_n,\ T_3 = \mathrm{diag}\,(1, 1, -1, -1) \otimes I_n.$$

Then for $t_1, t_2 \in \mathbb{R}$,

$$-F_2(t_1, t_2) = T_1\big(F_2(t_1, t_2)\big)T_1^{-1},\ F_2(-t_1, t_2) = T_2\big(F_2(t_1, t_2)\big)T_2^{-1},\ F_2(t_1, -t_2) = T_3\big(F_2(t_1, t_2)\big)T_3^{-1} \tag{3.2.12}$$

which implies

$$\phi(t_1, t_2) = \phi(-t_1, t_2) = \phi(t_1, -t_2) = \phi(-t_1, -t_2) \text{ for all } t_1, t_2 \in \mathbb{R}.$$

Now define

$$G(t_1,t_2) := t_1H_1 + t_2H_2 = \begin{bmatrix} 0 & t_1\bar{S}_1 & 0 & t_2\bar{S}_2 \\ t_1S_1 & 0 & t_2S_2 & 0 \\ 0 & t_2\bar{S}_2 & 0 & t_1\bar{S}_1 \\ t_2S_2 & 0 & t_1S_1 & 0 \end{bmatrix} \text{ for all } \mathrm{t}_1, \mathrm{t}_2 \in \mathbb{R}.$$

Note that (3.2.12) also hold if $F(t_1,t_2)$ is replaced by $G(t_1,t_2)$. In particular, we have $-G(t_1,t_2) = T_1\big(G(t_1,t_2)\big)T_1^{-1}$. This implies λ is an eigenvalue of $G(t_1,t_2)$ if and only if $-\lambda$ is an eigenvalue of $G(t_1,t_2)$. To prove that ϕ has a global minimum we will show that there exist a constant $R>0$ such that for all (t_1,t_2) with $t_1^2+t_2^2 > R^2$ we have $\phi(t_1,t_2) \geq \phi(0,0)$. Since the closed ball

$$\mathcal{B}_R := \{(t_1,t_2) \in \mathbb{R}^2 \,|\, t_1^2+t_2^2 \leq R\}$$

is compact and ϕ is continuous due to the fact that eigenvalues depend continuously on the entries of a matrix, ϕ has a global minimum $\lambda_4^\star \leq \phi(t_1,t_2)$ on $\mathcal{B}_R$. By construction we then have $\lambda_4^\star \leq \phi(t_1,t_2)$ for all $(t_1,t_2) \in \mathbb{R}^2$, i.e., $\lambda_4^\star$ is the global minimum of ϕ. Thus, consider

$$c = \min\left\{\lambda_4\big(G(t_1,t_2)\big) \,|\, t_1,t_2 \in \mathbb{R}\,;\, t_1^2+t_2^2 = 1\right\}.$$

Now as $\mathrm{rank}(S_1) \geq 2$ and $\mathrm{rank}(S_2) \geq 2$, we have $\mathrm{rank}(G(t_1,t_2)) \geq 8$ for all $(t_1,t_2) \neq (0,0)$ and $\lambda_4(G(t_1,t_2)) > 0$. As the function $g : (t_1,t_2) \mapsto \lambda_4(G(t_1,t_2))$ is continuous, the infimum c is attained for some $(t_1,t_2) \in \mathbb{R}^2$ with $t_1^2+t_2^2 = 1$. Clearly $c > 0$ as the function g takes only positive values. Next, define

$$R = \frac{\big(\lambda_{\max}(H) - \lambda_{\min}(H)\big)}{c}.$$

Then $\phi(t_1,t_2) \geq \phi(0,0)$ whenever $t_1^2+t_2^2 \geq R^2$. Indeed, let $(t_1,t_2) \in \mathbb{R}^2$ be such that $t_1^2+t_2^2 \geq R^2$. This implies there exist $r \geq R$ and $\theta \in [0,2\pi[$ such that $t_1 = r\cos\theta$ and $t_2 = r\sin\theta$. Thus, we obtain

$$\begin{aligned} \phi(t_1,t_2) &= \lambda_4\big(F_2(t_1,t_2)\big) \\ &= \lambda_4(H_0 + t_1H_1 + t_2H_2) \\ &\geq \lambda_4(t_1H_1+t_2H_2) + \lambda_{\min}(H_0) \\ &= r\,\lambda_4\big(\cos(\theta)H_1 + \sin(\theta)H_2\big) + \lambda_{\min}(H_0) \\ &\geq R.c + \lambda_{\min}(H_0) \\ &= \phi(0,0), \end{aligned}$$

where the last equality holds because $\phi(0,0) = \lambda_4(H_0) = \lambda_{\max}(H) = R.c + \lambda_{\min}(H)$. This completes the proof. $\square$

In the next section, our aim will be to obtain a lower bound of $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$ that is also an upper bound for the unstructured eigenvalue backward error. We prove two lemmas which will be used for this purpose.

Lemma 3.2.4. *Let $M \in \mathbb{C}^{n\times n}$ be non singular, $m \in \mathbb{N}$, $k = \lfloor \frac{m-1}{2} \rfloor$, and $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $\mathcal{S} \subseteq (\mathbb{C}^n)^{m+1}$ is given by $\mathcal{S} = \{(v_0,\ldots,v_m) \,|\, v_0 + \lambda v_1 + \ldots + \lambda^m v_m \neq 0\}$. Let $v_\lambda = v_0 + \lambda v_1 + \ldots + \lambda^m v_m$ and $g : \mathcal{S} \mapsto \mathbb{R}$ be defined by*

$$g(v_0,\ldots,v_m) = \begin{cases} \sum_{j=0}^{k} \alpha_j \max\left\{\frac{\|v_j\|^2}{\|Mv_\lambda\|^2}, \frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right\} & \text{if } m \text{ is odd} \\ \sum_{j=0}^{k} \alpha_j \max\left\{\frac{\|v_j\|^2}{\|Mv_\lambda\|^2}, \frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right\} + \alpha_{m/2}\frac{\|v_{m/2}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even,} \end{cases} \tag{3.2.13}$$

where α_j, $j = 0,\ldots,k$ are non-negative real numbers. Further assume that g has its infimum over $\mathcal{S}$. Then every minimizer $(\hat{v}_0,\ldots,\hat{v}_m)$ of g with $\alpha_j \neq 0$ satisfies $\|\hat{v}_j\| = \|\hat{v}_{m-j}\|$, $j \in \{0,\ldots,k\}$.

Proof. Let the minimum of g be attained at $(\hat{v}_0,\ldots,\hat{v}_m) \in \mathcal{S}$. Also, for some $j \in \{0,\ldots,k\}$ let $\alpha_j \neq 0$ and $\|\hat{v}_j\| \neq \|\hat{v}_{m-j}\|$. We only consider the case that

$$\|\hat{v}_j\| > \|\hat{v}_{m-j}\|. \tag{3.2.14}$$

(The case $\|\hat{v}_j\| < \|\hat{v}_{m-j}\|$ leads to an analogous argument). We prove that every vector $y \in \mathbb{C}^n$ is perpendicular to $\hat{v}_j$ and hence in particular $\hat{v}_j \perp \hat{v}_j$ which implies $\hat{v}_j = 0$. This will complete the proof by contradiction as $\hat{v}_j \neq 0$.

Now let $y \in \mathbb{C}^n$, and consider $\widetilde{v}_j = \hat{v}_j + \beta y$ and $\widetilde{v}_{m-j} = \hat{v}_{m-j} - \frac{\beta}{\lambda^{m-2j}} y$. Also set $\widetilde{v}_i = \hat{v}_i$ for $i = 0,\ldots,m$ except $i = j$ and $i = m-j$. Then $\sum_{j=0}^{m} \lambda^j \widetilde{v}_j = \sum_{j=0}^{m} \lambda^j \hat{v}_j \neq 0$ implies that $(\widetilde{v}_0,\ldots,\widetilde{v}_m) \in \mathcal{S}$ and $\|M(\widetilde{v}_0 + \lambda\widetilde{v}_1 + \cdots + \lambda^m \widetilde{v}_m)\| = \|M(\hat{v}_0 + \lambda \hat{v}_1 + \cdots + \lambda^m \hat{v}_m)\|$. Note that $\|\hat{v}_j + \beta y\|^2 = \|\hat{v}_j\|^2 + 2\mathrm{Re}(\beta \hat{v}_j^* y) + |\beta|^2\|y\|^2$. Thus, unless $\hat{v}_j^* y = 0$, there exists $\beta_0 \neq 0$ sufficiently small such that

$$\|\hat{v}_j\| > \|\hat{v}_j + \beta_0 y\| > \left\|\hat{v}_{m-j} - \frac{\beta_0}{\lambda^{m-2j}} y\right\|.$$

Since $g(\widetilde{v}_0,\ldots,\widetilde{v}_m) < g(\hat{v}_0,\ldots,\hat{v}_m)$, this contradicts the fact that the minimum is attained at $(\hat{v}_0,\ldots,\hat{v}_m) \in \mathcal{S}$. Thus $\forall y \in \mathbb{C}^n$ we have $y \perp \hat{v}_j$. $\square$

Lemma 3.2.5. *Let $M \in \mathbb{C}^{n\times n}$ be non singular, $m \in \mathbb{N}$, $k = \lfloor \frac{m-1}{2} \rfloor$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $\mathcal{S} \subseteq (\mathbb{C}^n)^{m+1}$ is given by $\mathcal{S} = \{(v_0,\ldots,v_m) \,|\, v_0 + \lambda v_1 + \ldots + \lambda^m v_m \neq 0\}$. Let*

$v_\lambda := v_0 + \lambda v_1 + \ldots + \lambda^m v_m$ and $h : \mathcal{S} \mapsto \mathbb{R}$ be defined by

$$h(v_0,\ldots,v_m) = \begin{cases} \sum_{j=0}^{k} \alpha_j \left(\frac{\|v_j\|^2}{\|Mv_\lambda\|^2} + \frac{|\lambda|^{m-2j}\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right) & \text{if } m \text{ is odd} \\ \sum_{j=0}^{k} \alpha_j \left(\frac{\|v_j\|^2}{\|Mv_\lambda\|^2} + \frac{|\lambda|^{m-2j}\|v_{m-j}\|^2}{\|Mv_\lambda\|^2}\right) + \alpha_{\frac{m}{2}} \frac{\|v_{m/2}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even,} \end{cases}$$

where α_j, $j = 0,\ldots,k$ are non-negative real numbers. Further assume that h has its infimum over $\mathcal{S}$. Then every minimizer $(\hat{v}_0,\ldots,\hat{v}_m)$ of h over $\mathcal{S}$ with $\alpha_j \neq 0$ satisfies $\|\hat{v}_j\| = \|\hat{v}_{m-j}\|$, $j \in \{0,\ldots,k\}$.

Proof. We prove the lemma for the case that m is odd. (The case m is even, leads to an analogous argument). Let $(\hat{v}_0,\ldots,\hat{v}_m) \in \mathcal{S}$ be a minimizer of h and $\alpha_j \neq 0$ for some $j \in \{0,\ldots,k\}$. Set $x := M(\hat{v}_0 + \lambda\hat{v}_1 + \cdots + \lambda^m\hat{v}_m)$ and let y_j, $y_{m-j} \in \{\overline{x}\}^\perp$ such that $\hat{v}_j = y_j + c_j\overline{x}$ and $\hat{v}_{m-j} = y_{m-j} + c_{m-j}\overline{x}$ for some $c_j, c_{m-j} \in \mathbb{C}$. Set $z_j := c_j\overline{x}$ and $z_{m-j} := c_{m-j}\overline{x}$. Now, since y_j, $y_{m-j} \in \{\overline{x}\}^\perp$, we have

$$\|\hat{v}_j\|^2 = \|y_j\|^2 + \|z_j\|^2, \quad \|\hat{v}_{m-j}\|^2 = \|y_{m-j}\|^2 + \|z_{m-j}\|^2.$$

We show that $\|\hat{v}_j\| = \|\hat{v}_{m-j}\|$, by proving that $\|y_j\| = \|y_{m-j}\|$ and $\|z_j\| = \|z_{m-j}\|$. First we show that $\|y_j\| = \|y_{m-j}\|$. Set $y = \bar{\lambda}^{m-2j}y_j + |\lambda|^{m-2j}y_{m-j}$. Then for all $\epsilon \in \mathbb{R}$ we have $(\hat{v}_0,\ldots,\hat{v}_j + \epsilon\,\lambda^{m-2j}\,y,\ldots,\hat{v}_{m-j} - \epsilon\,y,\ldots,\hat{v}_m) \in \mathcal{S}$ and

$$x = M\left(\hat{v}_0 + \lambda\hat{v}_1 + \cdots + \lambda^j(\hat{v}_j + \epsilon\lambda^{m-2j}y) + \cdots + \lambda^{m-j}(\hat{v}_{m-j} - \epsilon y) + \cdots + \lambda^m\hat{v}_m\right).$$

Thus,

$$\begin{aligned}
0 &= \frac{d}{d\epsilon} h\big((\hat{v}_0,\ldots,\hat{v}_j + \epsilon\,\lambda^{m-2j}\,y,\ldots,\hat{v}_{m-j} - \epsilon\,y,\ldots,\hat{v}_m)\big)\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\Big(\sum_{i=0,i\neq j}^{k} \alpha_i\Big(\frac{\|\hat{v}_i\|^2}{\|x\|^2} + \frac{|\lambda|^{m-2i}\|\hat{v}_{m-i}\|^2}{\|x\|^2}\Big)\Big) + \alpha_j\frac{d}{d\epsilon}\Big(\frac{\|\hat{v}_j + \epsilon\,\lambda^{m-2j}y\|^2}{\|x\|^2} + \frac{\|\hat{v}_{m-j} - \epsilon y\|^2}{\|x\|^2}\Big)\Big|_{\epsilon=0} \\
&= \frac{2\alpha_j \mathrm{Re}\big(\hat{v}_j^*(\lambda^{m-2j}\,y) - |\lambda|^{m-2j}\,\hat{v}_{m-j}^* y\big)}{\|x\|^2} \qquad \Big(\because \frac{d}{d\epsilon}\|v + \epsilon y\|^2\Big|_{\epsilon=0} = 2\mathrm{Re}(v^*y)\Big) \\
&= \frac{2\alpha_j \mathrm{Re}\big(\hat{y}_j^*(\lambda^{m-2j}\,y) - |\lambda|^{m-2j}\,\hat{y}_{m-j}^* y\big)}{\|x\|^2} \\
&= \frac{2\alpha_j\,|\lambda|^{m-2j}\mathrm{Re}\big(|\lambda|^{m-2j}\,\|y_j\|^2 + \lambda^{m-2j}\,y_j^* y_{m-j} - \bar{\lambda}^{m-2j}\,y_{m-j}^* y_j - |\lambda|^{m-2j}\,\|y_{m-j}\|\big)}{\|x\|^2} \\
&= \frac{2\alpha_j\,|\lambda|^{2(m-2j)}\big(\|y_j\|^2 - \|y_{m-j}\|^2\big)}{\|x\|^2},
\end{aligned}$$

which implies $\|y_j\| = \|y_{m-j}\|$, since $\alpha_j \neq 0$. Similarly, one can show that $\|z_j\| = \|z_{m-j}\|$ by choosing $y = \bar{\lambda}^{m-2j}z_j + |\lambda|^{m-2j}z_{m-j}$ in the above. Hence result follows. ∎

3.2.2 A lower bound of $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$

Due to Theorem 3.1.5 and Theorem 3.2.2, for any $\lambda \in \mathbb{C}\setminus\{0,1,-1\}$, regular T-palindromic polynomial $P(z)=\sum_{j=0}^{m} z^j A_j$ and $k=\lfloor \frac{m-1}{2} \rfloor$, we have

$$\left(\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)\right)^2 = \left(m_{hs_0\ldots s_k}(G,S_0,\ldots,S_k)\right)^{-1} \geq \left(\inf_{t_0,\ldots,t_k\in\mathbb{R}} \lambda_{2^{k+1}}\big(F_{k+1}(t_0,\ldots,t_k)\big)\right)^{-1} \tag{3.2.15}$$

where $G \in \mathrm{Herm(n(m+1))}$ and $S_0,\ldots,S_k \in \mathrm{Sym(n(m+1))}$ are given by (3.1.15) and (3.1.17) respectively, and $F_{k+1}: \mathbb{R}^{k+1} \to \mathbb{C}^{2^{k+1}N\times 2^{k+1}N}$ is defined by (3.2.5) with respect to G and $S_0,\ldots,S_k$ for $N=n(m+1)$.

Theorem 3.2.6. *Let $P(z)=\sum_{j=0}^{m} z^j A_j$ be a T-palindromic polynomial and suppose that $\lambda\in\mathbb{C}\setminus\{0,\pm1\}$. Suppose that $P(\lambda)$ is non singular and $M=(P(\lambda))^{-1}$. Furthermore let $w=(w_0,\ldots,w_m)$ be a palindromic weight vector and $k=\lfloor\frac{m-1}{2}\rfloor$. Then*

$$\left(\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)\right)^2 = \left(m_{hs_0\ldots s_k}(G,S_0,\ldots,S_k)\right)^{-1} \geq \left(\inf_{t_0,\ldots,t_k\in\mathbb{R}} \lambda_{2^{k+1}}\big(F_{k+1}(t_0,\ldots,t_k)\big)\right)^{-1} \geq \left(\eta_{w,2}(P,\lambda)\right)^2, \tag{3.2.16}$$

where G and S_j for $j=0,\ldots,k$ are defined by (3.1.15) *and* (3.1.17), *respectively, $\eta_{w,2}(P,\lambda)$ denotes the unstructured eigenvalue backward error and $\lambda_{2^{k+1}}\big(F_{k+1}(t_0,\ldots,t_k)\big)$ denotes the $(2^{k+1})^{th}$ largest eigenvalue of $F_{k+1}(t_0,\ldots,t_k)$. There exist $t_0^\star,\ldots,t_k^\star\in\mathbb{R}$ such that*

$$\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big) := \inf_{t_0,\ldots,t_k\in\mathbb{R}} \lambda_{2^{k+1}}\big(F_{k+1}(t_0,\ldots,t_k)\big).$$

If there exist a unit vector $v\in\mathbb{C}^{n(m+1)}$ satisfying

$$v^*Gv = \lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big) \quad \text{and} \quad v^T S_j v = 0 \ \ \forall\ j=0,\ldots,k, \tag{3.2.17}$$

then

$$\left(\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)\right)^2 = \left(m_{hs_0\ldots s_k}(G,S_0,\ldots,S_k)\right)^{-1} = \left(\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big)\right)^{-1}.$$

Proof. In view of (3.2.15) to show that (3.2.16) holds, we only need to show that

$$\left(\inf_{t_0,\ldots,t_k\in\mathbb{R}} \lambda_{2^{k+1}}\big(F_{k+1}(t_0,\ldots,t_k)\big)\right)^{-1} \geq \left(\eta_{w,2}(P,\lambda)\right)^2.$$

We have,

$$\inf_{t_0,\ldots,t_k\in\mathbb{R}} \lambda_{2^{k+1}}\big(F_{k+1}(t_0,\ldots,t_k)\big) \leq \lambda_{2^{k+1}}\big(F_{k+1}(0,\ldots,0)\big) = \lambda_{\max}(G), \tag{3.2.18}$$

where $\lambda_{\max}(G)$ is the largest eigenvalue of G. So to establish (3.2.16), we show that $(\lambda_{\max}(G))^{-1} \geq (\eta_{w,2}(P,\lambda))^2$. By Lemma 1.2.6 we first write $\eta_{w,2}(P,\lambda)$ as follows.

$$(\eta_{w,2}(P,\lambda))^2 = \inf\left\{ \sum_{j=0}^{m} w_j^2 \|\Delta_j\|^2 \,\middle|\, \exists v_0,\ldots,v_m \in \mathbb{C}^n,\ v_\lambda \neq 0,\ (\Delta_0,\ldots,\Delta_m) \in (\mathbb{C}^{n\times n})^{m+1} \right.$$
$$\left. v_j = \Delta_j M v_\lambda,\ j=0,\ldots,m \right\}.$$

By using Theorem 1.2.8, we get

$$(\eta_{w,2}(P,\lambda))^2 = \inf\left\{ \sum_{j=0}^{m} w_j^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{S}_1 \right\}, \tag{3.2.19}$$

where $\mathcal{S}_1 := \left\{(v_0,\ldots,v_m) \in (\mathbb{C}^n)^{m+1} \,\middle|\, v_\lambda := v_0 + \ldots + \lambda^m v_m \neq 0\right\}$. Now from (3.2.1), we have

$$\begin{aligned} \infty > \left(\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)\right)^2 &= \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{K}\right\} \\ &\geq \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{S}_1\right\} \quad (\because\ \mathcal{K} \subseteq \mathcal{S}_1) \\ &= \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{S}_2\right\}, \end{aligned} \tag{3.2.20}$$

where $\mathcal{S}_2 := \left\{(v_0,\ldots,v_m) \in \mathcal{S}_1 \,\middle|\, \|v_j\| = \|v_{m-j}\|,\ j=0,\ldots,k\right\}$. The last equality in (3.2.20) holds because by arguments similar to those used in Lemma 3.1.4, it can be shown that f has its infimum over $\mathcal{S}_1$ and by Lemma 3.2.4 this is attained at a point of $\mathcal{S}_2$. Now since $(v_0,\ldots,v_m) \in \mathcal{S}_2 \Rightarrow \|v_j\| = \|v_{m-j}\|$ for $j=0,\ldots,k$, we can write f over $\mathcal{S}_2$ as

$$f(v_0,\ldots,v_m) = \begin{cases} \sum_{j=0}^{k} 2w_j^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is odd} \\ \sum_{j=0}^{k} 2w_j^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} + w_{\frac{m}{2}}^2 \frac{\|v_{m/2}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even.} \end{cases} \tag{3.2.21}$$

Again from (3.2.20) and (3.2.19)

$$\begin{aligned} \infty > \left(\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)\right)^2 &\geq \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{S}_2\right\} \\ &\geq \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{S}_1\right\} \quad (\because\ \mathcal{S}_2 \subseteq \mathcal{S}_1) \\ &= \inf\left\{ \sum_{i=0}^{m} w_i^2 \frac{\|v_i\|^2}{\|Mv_\lambda\|^2} \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{S}_1\right\} \\ &= \eta_{w,2}(L,\lambda)^2 \quad \big(\text{by } (3.2.19)\big). \end{aligned} \tag{3.2.22}$$

We now show that,

$$\mu^2 := \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{S}_2\right\} = \frac{1}{\lambda_{\max}(G)}.$$

For this, consider $\gamma_{j1} = \sqrt{\frac{2}{1+|\lambda|^{m-2j}}}$, $\gamma_{j2} = \sqrt{\frac{2|\lambda|^{m-2j}}{1+|\lambda|^{m-2j}}}$ for $j = 0, \ldots, k$. Now using the fact that $\gamma_{j1}^2 + \gamma_{j2}^2 = 2$ for all j, from (3.2.21) f can be written over $\mathcal{S}_2$ as

$$f(v_0, \ldots, v_m) = \begin{cases} \sum_{j=0}^{k} w_j^2 \gamma_{j1}^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} + w_j^2 \gamma_{j2}^2 \frac{\|v_{m-j}\|^2}{|Mv_\lambda\|^2} & \text{if } m \text{ is odd} \\ \sum_{j=0}^{k} w_j^2 \gamma_{j1}^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} + w_j^2 \gamma_{j2}^2 \frac{\|v_{m-j}\|^2}{|Mv_\lambda\|^2} + w_{\frac{m}{2}}^2 \frac{\|v_{m/2}\|^2}{\|Mv_\lambda\|^2} & \text{if } m \text{ is even.} \end{cases} \tag{3.2.23}$$

By using the values of γ_{j1} and γ_{j2}, we have $f(v_0, \ldots, v_m) = h(v_0, \ldots, v_m)$ over $\mathcal{S}_2$ with $\alpha_j = \frac{2w_j^2}{(1+|\lambda|^{m-2j})}$ and $\alpha_{\frac{m}{2}} = w_{\frac{m}{2}}^2$ for h as defined in Lemma 3.2.5. Now since $\mathcal{S}_2 \subseteq \mathcal{S}_1$, we have

$$\mu^2 \geq \inf \left\{ h(v_0, \ldots, v_m) \,\middle|\, (v_0, \ldots, v_m) \in \mathcal{S}_1 \right\}. \tag{3.2.24}$$

In fact, equality holds in (3.2.24) because infimum of h over $\mathcal{S}_1$ is finite (as μ^2 is finite by (3.2.20)) and arguments similar to those used in Lemma 3.1.4 show that h has its infimum over $\mathcal{S}_1$ which is attained at a point of $\mathcal{S}_2$ by Lemma 3.2.5. Therefore

$$\mu^2 = \inf \left\{ h(v_0, \ldots, v_m) \,\middle|\, (v_0, \ldots, v_m) \in \mathcal{S}_1 \right\}. \tag{3.2.25}$$

Set $u = [u_0^T, \ldots, u_m^T]^T$, where $u_j = w_j \gamma_{j1} v_j$, $u_{m-j} = w_j \gamma_{j2} v_{m-j}$ for $j = 0, \ldots, k$, and $u_{m/2} = w_{m/2} v_{m/2}$ when m is even. This implies $h(v_0, \ldots, v_m) = \dfrac{u^* u}{u^* G u}$. Also, note that $0 \neq u^* G u = \|Mv_\lambda\|^2 \iff v_\lambda \neq 0$. Thus, from (3.2.25), we have

$$\begin{aligned} \mu^2 &= \inf \left\{ \frac{u^* u}{u^* G u} \,\middle|\, u \in (\mathbb{C}^n)^{m+1},\ u^* G u \neq 0 \right\} \\ &= \left(\sup \left\{ \frac{u^* G u}{u^* u} \,\middle|\, u \in (\mathbb{C}^n)^{m+1} \setminus \{0\} \right\}^{-1} = (\lambda_{\max}(G)) \right)^{-1}. \end{aligned} \tag{3.2.26}$$

Note that since μ^2 is finite and positive, the supremum in the latter equality of (3.2.26) will not be attained by vectors u satisfying $u^* G u = 0$ and therefore, the condition $u^* G u \neq 0$ is superfluous for it. Now (3.2.26), (3.2.22) and (3.2.18) yield

$$\left(\inf_{t_0, \ldots, t_k \in \mathbb{R}} \lambda_{2^{k+1}} \big(F_{k+1}(t_0, \ldots, t_k) \big) \right)^{-1} \geq (\lambda_{\max}(G))^{-1} \geq (\eta_{w,2}(P, \lambda))^2$$

and this proves (3.2.16). By Lemma 3.2.3 there exist $t_0^\star, \ldots, t_k^\star \in \mathbb{R}$ such that

$$\lambda_{2^{k+1}} \big(F_{k+1}(t_0^\star, \ldots, t_k^\star) \big) = \inf_{t_0, \ldots, t_k \in \mathbb{R}} \lambda_{2^{k+1}} \big(F_{k+1}(t_0, \ldots, t_k) \big).$$

The next part of the theorem follows immediately by noting that if a unit vector $v \in \mathbb{C}^n$ satisfies

$$v^* G v = \lambda_{2^{k+1}} \big(F_{k+1}(t_0^\star, \ldots, t_k^\star) \big) \quad \text{and} \quad v^T S_j v = 0 \ \ \forall\, j = 0, \ldots, k, \tag{3.2.27}$$

then

$$m_{hs_1\dots s_m}(G, S_1, \dots, S_m) \geq \inf_{t_1,\dots,t_m \in \mathbb{R}} \lambda_{2^m}\big(F_m(t_1, \dots, t_m)\big).$$

□

In Chapter 5 we will prove that, when $P(z)$ is a real T-palindromic polynomial and $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$ then

$$\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star, \dots, t_k^\star)\big) = \lambda_{\max}(G + t_0^\star S_0 + \dots + t_k^\star S_k),$$

and $\Big(\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star, \dots, t_k^\star)\big)\Big)^{-1}$ gives the exact backward error of λ with respect to real T-palindromic perturbations whenever $\lambda_{\max}(G + t_0^\star S_0 + \dots + t_k^\star S_k)$ is a simple eigenvalue of $G + t_0^\star S_0 + \dots + t_k^\star S_k$.

Remark 3.2.7. In general, numerical experiments suggest that the lower bound in the above theorem is actually equal to the structured backward error when $\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star, \dots, t_k^\star)\big)$ is a simple eigenvalue of F_{k+1} at the optimal point $(t_0^\star, \dots, t_k^\star)$.

On the basis of numerical experiments we have the following conjecture.

Conjecture 3.2.8. *Let $P(z) = \sum_{i=0}^m z^i A_i$ be a T-palindromic polynomial. Suppose that $\lambda \in \mathbb{C} \setminus \{0, \pm 1\}$ such that $\det(P(\lambda)) \neq 0$. For the matrices G and S_j, $j = 0, \dots, k$ of Theorem 3.2.6, if*

$$\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star, \dots, t_k^\star)\big) := \inf_{t_0,\dots,t_k \in \mathbb{R}} \lambda_{2^{k+1}}\big(F_{k+1}(t_0, \dots, t_k)\big)$$

is a simple eigenvalue of $F_{k+1}(t_0^\star, \dots, t_k^\star)$, then there exist a unit vector $v \in \mathbb{C}^{n(m+1)}$ satisfying

$$v^* G v = \lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star, \dots, t_k^\star)\big) \quad \text{and} \quad v^T S_j v = 0 \ \ \forall\, j = 0, \dots, k.$$

3.2.3 An upper bound of $\eta_{w,2}^{\mathrm{pal_T}}(P, \lambda)$

We recall that for a matrix polynomial $P(z) = \sum_{j=0}^m z^j A_j$ with $(A_0, \dots, A_m) \in \mathbb{S}$ and pair $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$, the structured eigenpair backward error $\eta_{w,2}^{\mathbb{S}}(P, \lambda, x)$ with respect to $|||\cdot|||_{w,2}$ norm is defined by

$$\eta_{w,2}^{\mathbb{S}}(P, \lambda, x) := \inf\left\{ |||(\Delta_0, \dots, \Delta_m)|||_{w,2} \,\middle|\, \Big(\sum_{j=0}^m \lambda^j (A_j - \Delta_j)\Big) x = 0,\ (\Delta_0, \dots, \Delta_m) \in \mathbb{S} \right\}.$$

An exact formula for $\eta^{\mathbb{S}}_{w,2}(P,\lambda,x)$ has been derived in [1] for various classes of structures. As noted in Theorem 3.2.6, if there exist a vector $v = [v_0^T,\ldots,v_m^T]^T \in \mathbb{C}^{n(m+1)}$ such that

$$\frac{v^*Gv}{v^*v} = \lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big) \ \text{ and } \ v^TS_jv = 0, \text{ for } j = 0,\ldots,k,$$

then $\left(\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)\right)^{-2} = \lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big)$ and we get the exact value of $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$. If this is the case then by Remark 3.1.13 for $u = M\left(\sum_{j=0}^m \lambda^j u_j\right)$ where $[u_0^T,\ldots,u_m^T] = \Gamma^{-1}v$, Γ being defined by (3.1.13), we have $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda) = \eta^{\mathrm{pal_T}}_{w,2}(P,\lambda,u)$. Therefore we obtain an upper bound on $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$ as follows.

Let $v = [v_1,\ldots,v_{2^{k+1}}]^T$ be an eigenvector corresponding to eigenvalue $\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big)$ of $F_{k+1}(t_0^\star,\ldots,t_k^\star)$, where $v_i = [v_{i0}^T,\ldots,v_{im}^T]^T \in \mathbb{C}^{n(m+1)}$ for $i = 1,\ldots,2^{k+1}$. For each $i = 1,\ldots,2^{k+1}$, define

$$u_i := \Gamma^{-1}v_i = [u_{i0}^T,\ldots,u_{im}^T]^T \quad \text{and} \quad w_i := M\left(u_{i0} + \lambda u_{i1} + \cdots + \lambda^m u_{im}\right),$$

where Γ is defined by (3.1.13). Then

$$\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda) \le \min_{i\in\{1,\ldots,2^{k+1}\}} \eta^{\mathrm{pal_T}}_{w,2}(P,\lambda,w_i). \tag{3.2.28}$$

Set $R_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes I_{2^kN}$ and $R_2 := \mathrm{diag}(D,\ldots,D)\otimes I_N$, where $D := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $N := n(m+1)$. Then

$$R_1F_{k+1}(t_0^\star,\ldots,t_k^\star)R_1 = F_{k+1}(t_0^\star,\ldots,t_k^\star) \ \text{ and } \ R_2F_{k+1}(t_0^\star,\ldots,t_k^\star)R_2 = \overline{F_{k+1}(t_0^\star,\ldots,t_k^\star)}.$$

Now if $\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big)$ is simple then there exist a eigenvector $v = [\hat{x}^T, \pm\hat{x}^T]^T$ corresponding to eigenvalue $\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big)$, where $\hat{x} = [x^T\ \bar{x}^T\ \cdots\ x^T\ \bar{x}^T]^T \in \mathbb{C}^{2^kN}$ with $x \in \mathbb{C}^N$ such that

$$\frac{x^*Gx}{x^*x} = \lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big) \ \text{ and } \ \mathrm{Re}\,(x^TS_jx) = 0, \text{ for } j = 0,\ldots,k.$$

Numerical experiments suggest that in such cases $\mathrm{Im}\,(x^TS_jx)$ is also 0 for matrices G and S_j , for $j = 0,\ldots,k$ of Theorem 3.2.6. Thus this choice of vectors w_i in (3.2.28) gives a good upper bound whenever $\lambda_{2^{k+1}}\big(F_{k+1}(t_0^\star,\ldots,t_k^\star)\big)$ is a simple eigenvalue of $F_{k+1}(t_0^\star,\ldots,t_k^\star)$.

3.2.4 Numerical experiments

In this section, we present some numerical experiments to illustrate the bounds for $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$ with $w = (1,\ldots,1)$ obtained in Theorem 3.2.6. As mentioned in Remark 3.2.7, Conjecture

3.2.8 holds in all our numerical experiments. Therefore the lower bound in Theorem 3.2.6 indeed gives the value of $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$.

Example 3.2.9. $P(z) = \sum_{j=0}^{3} z^j A_j$ is a random T-palindromic polynomial of size 3×3 with eigenvalues $0.8430-3.6718i$, $1.2781+0.8043i$, $0.4717-1.3658i$, $-1.4242-0.2421i$, -1, $-0.6824+0.1160i$, $0.2259+0.6541i, 0.0594+0.2587i, 0.5605-0.3527i$. We recorded the lower and upper bounds for $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$ for different values of λ. If λ converges to an eigenvalue of $P(z)$ then both the unstructured and structured backward errors tend to zero as expected. The difference between unstructured and structured backward error may be quite significant for arbitrary λ. These are recorded in Table 3.2.1 for values of λ converging to the eigenvalue $1.2781+0.8043i$ as well as for arbitrary λ (not necessarily close to an eigenvalue). Here lbound stands for the lower bound obtained in Theorem 3.2.6 and ubound stands for the upper bound obtained in Section 3.2.3. In all cases lbound $= \eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$.

Table 3.2.1: *Estimates of $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$ of a $T-$palindromic cubic polynomial $P(z)$ where $\lambda \to 1.2781+0.8043i$ (left) and for arbitrary λ (right).*

λ	$\eta_{w,2}(P,\lambda)$	lbound $(=\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda))$	ubound	λ	$\eta_{w,2}(P,\lambda)$	lbound $(=\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda))$	ubound
$2.15+0.25i$	0.9047	1.1175	1.2505	$-0.2073+2.4453i$	1.5482	2.0843	2.1712
$2.00+0.35i$	0.8519	1.0320	1.1697	$0.4376+0.2017i$	0.9773	1.1949	1.3439
$1.85+0.45i$	0.7629	0.9050	1.0386	$-1.7307+1.1932i$	1.1819	1.5057	1.6726
$1.70+0.55i$	0.6288	0.7300	0.8473	$0.6861+1.8292i$	1.3233	1.7124	1.8774
$1.55+0.65i$	0.4433	0.5039	0.5903	$-0.1779-0.0049i$	0.9432	1.3399	1.3441
$1.40+0.75i$	0.2064	0.2301	0.2712	$-2.0310-1.5386i$	0.8108	1.1010	1.1615
$1.33+0.78i$	0.0920	0.1016	0.1200	$-1.3070-1.4864i$	0.9349	1.1860	1.3112

Table 3.2.2: *Estimates of $\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda)$ of a $T-$palindromic polynomial $P(z)$ of degree 4 where $\lambda \to -1.3434+0.5141i$ (left) and for arbitrary λ (right).*

λ	$\eta_{w,2}(P,\lambda)$	lbound $(=\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda))$	ubound	λ	$\eta_{w,2}(P,\lambda)$	lbound $(=\eta_{w,2}^{\mathrm{pal_T}}(P,\lambda))$	ubound
$-2.45+0.05i$	1.3495	1.8070	1.8566	$-2.9080+0.8252i$	1.4138	1.9494	1.9909
$-2.25+0.15i$	1.2121	1.5956	1.6525	$-2.4245+0.9594i$	1.1971	1.6266	1.6792
$-2.05+0.25i$	1.0173	1.3110	1.3711	$-0.2820+0.0335i$	1.3678	1.9066	1.9246
$-1.85+0.30i$	0.7987	1.0004	1.0580	$-0.0983+0.0414i$	1.1595	1.6393	1.6396
$-1.65+0.35i$	0.5341	0.6467	0.6921	$-0.0229-0.2620i$	1.1180	1.5679	1.5793
$-1.45+0.40i$	0.2448	0.2861	0.3096	$-0.0000+0.0549i$	1.1226	1.5876	1.5876
$-1.38+0.45i$	0.1142	0.1329	0.1447	$0.0226-0.0479i$	1.0877	1.5382	1.5382

Example 3.2.10. Table 3.2.2 has a record of similar experiments done for a random T-palindromic polynomial $P(z) = A_0 + zA_1 + z^2A_2 + z^3A_1^T + z^4A_0^T$ of degree 4 and size 4×4 having an eigenvalue at $-1.3434 + 0.5141i$.

3.3 Antipalindromic polynomials

In this section, we obtain structured eigenvalue backward errors of $\bullet$-antipalindromic polynomials where $\bullet \in \{*, T\}$ by techniques analogous to those for $\bullet$-palindromic polynomials. Thus denoting the $\bullet$-antipalindromic structure by antipal$_\bullet$, where

$$\text{antipal}_\bullet = \begin{cases} \{(A_0, \dots, A_m) \in (\mathbb{C}^{n\times n})^{m+1} : A_j^* = -A_{m-j}\} & \text{if } \bullet = * \\ \{(A_0, \dots, A_m) \in (\mathbb{C}^{n\times n})^{m+1} : A_j^T = -A_{m-j}\} & \text{if } \bullet = T \end{cases}.$$

Again, to preserve the antipalindromic structure of the perturbed polynomial equal weights must be given to coefficients in position j and $m-j$. Thus let $w = (w_0, \dots, w_m)$ be such that $w_j = w_{m-j}$, i.e., let w be a palindromic weight vector.

Let $P(z) = \sum_{j=0}^m z^j A_j$ be a $\bullet$-antipalindromic polynomial, i.e. $(A_0, \dots, A_m) \in \text{antipal}_\bullet$ and $\lambda \in \mathbb{C}$. Then we compute,

$$\eta_{w,2}^{\text{antipal}_\bullet}(P, \lambda) = \inf\left\{ |||(A_0, \dots, A_m)|||_{w,2} \middle| \det\Big(\sum_{j=0}^m \lambda^j (A_j - \Delta_j)\Big) = 0,\ (A_0, \dots, A_m) \in \text{antipal}_\bullet \right\}.$$

Note that $\eta_{w,2}^{\text{antipal}_\bullet}(P, 0) = \sqrt{2}w_0\sigma_{\min}(A_0)$ where $\sigma_{\min}(A_0)$ is the minimum singular value of A_0. It is shown in [1] with weight vector $w = (1, \dots, 1)$ that $\eta_{w,2}^{\text{antipal}_*}(P, \lambda) = \eta_{w,2}(P, \lambda)$ when $|\lambda| = 1$ and $\eta_{w,2}^{\text{antipal}_\text{T}}(P, \lambda) = \eta_{w,2}(P, \lambda)$ when $\lambda = \pm 1$. This fact can be generalized for arbitrary palindromic weight vectors w.

In order to reformulate the $\eta_{w,2}^{\text{antipal}_\bullet}(P, \lambda)$ into an equivalent optimization problem of the form (3.1.18) when $\bullet = T$ and (3.1.19) when $\bullet = *$, we define
$k := \lfloor \frac{m-1}{2} \rfloor$, $\gamma_{j1} := w_j \sqrt{\frac{2}{1+|\lambda|^{m-2j}}}$, $\gamma_{j2} := w_j \sqrt{\frac{2|\lambda|^{m-2j}}{1+|\lambda|^{m-2j}}}$ $j = 0, \dots, k$,

$$\Gamma := \begin{cases} \text{diag}(\gamma_{01}, \dots, \gamma_{k1}, \gamma_{k2} \dots, \gamma_{02}) \otimes I_n, & \text{if } m \text{ is odd} \\ \text{diag}(\gamma_{01}, \dots, \gamma_{k1}, w_{\frac{m}{2}}, \gamma_{k2} \dots, \gamma_{02}) \otimes I_n & \text{if } m \text{ is even} \end{cases} \tag{3.3.1}$$

and $\Lambda_m := [1, \lambda, \dots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$. Further

$$C_j := (\Lambda_m^\bullet e_{j+1}^\bullet) \otimes M^\bullet + (e_{m-j+1}\Lambda_m) \otimes M,\ j = 0, \dots, k, \tag{3.3.2}$$

$$C_{\frac{m}{2}} := (\Lambda_m^\bullet e_{\frac{m}{2}+1}^\bullet) \otimes M^\bullet + (e_{\frac{m}{2}+1}\Lambda_m) \otimes M, \tag{3.3.3}$$

$$G := \Gamma^{-1}\left((\Lambda_m^*\Lambda_m) \otimes (M^*M)\right)\Gamma^{-1}, \tag{3.3.4}$$

where e^j is the j-th standard unit vector of $\mathbb{C}^{m+1}$. Also, when $\bullet = *$ for $j = 0, \ldots, k$ define

$$H_j := \Gamma^{-1}\left(C_j + C_j^*\right)\Gamma^{-1}, \;\; H_{m-j} := i\,\Gamma^{-1}\left(C_j - C_j^*\right)\Gamma^{-1} \quad \text{and} \quad H_{\frac{m}{2}} := \Gamma^{-1}C_{\frac{m}{2}}\Gamma^{-1}, \tag{3.3.5}$$

and when $\bullet = T$ define

$$S_j := \Gamma^{-1}\left(C_j + C_j^T\right)\Gamma^{-1}, \quad \text{and} \quad S_{\frac{m}{2}} := \Gamma^{-1}C_{\frac{m}{2}}\Gamma^{-1}. \tag{3.3.6}$$

Thus by arguing as in Section 3.1.1, we have the following analogue of Theorem 3.1.5 for $\bullet$-antipalindromic polynomials.

Theorem 3.3.1. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be $\bullet$-antipalindromic and $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $P(\lambda)$ is nonsingular and $M = (P(\lambda))^{-1}$. Furthermore, let $k := \lfloor \frac{m-1}{2} \rfloor$, G be defined by* (3.3.4)*, H_j for each j be defined by* (3.3.5) *and S_j for each j be defined by* (3.3.6)*. Then*

$$\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T}}}(P,\lambda) \;=\; \left(\sup\left\{\frac{v^*Gv}{v^*v} \;\middle|\; v \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, v^T S_j v = 0,\, j = 0, \ldots, k\right\}\right)^{-1/2} \tag{3.3.7}$$

when m is odd and

$$\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T}}}(P,\lambda) \;=\; \left(\sup\left\{\frac{v^*Gv}{v^*v} \;\middle|\; v \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, v^T S_{\frac{m}{2}} v = 0,\, v^T S_j v = 0,\, j = 0, \ldots, k\right\}\right)^{-1/2} \tag{3.3.8}$$

when m is even, and

$$\eta_{w,2}^{antipal_*}(P,\lambda) \;=\; \left(\sup\left\{\frac{v^*Gv}{v^*v} \;\middle|\; v \in \mathbb{C}^{n(m+1)} \setminus \{0\},\, v^* H_j v = 0,\, j = 0, \ldots, m\right\}\right)^{-1/2}.$$

Proof. The proof is similar to that of Theorem 3.1.5. $\square$

In the view of Theorem 3.3.1 and Theorem 1.2.18, the following result gives a formula for $\eta_{w,2}^{\mathrm{antipal}_*}(P,\lambda)$ when $|\lambda| \neq 1$.

Theorem 3.3.2. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be $*$-antipalindromic and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $|\lambda| \neq 1$. Suppose that $P(\lambda)$ is nonsingular and $M = (P(\lambda))^{-1}$. Then for G as defined in (3.3.4) and H_j, $j = 0, \ldots, m$, as defined in* (3.3.5)*, we have that*

$$\lambda_{\max}^\star := \min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + \cdots + t_m\, H_m)$$

is attained for some $(t_0^\star, \ldots, t_m^\star) \in \mathbb{R}^{m+1}$. If $m = 1$ or $\lambda_{\max}^\star$ is a simple eigenvalue of $G + t_0^\star H_0 + \cdots + t_m^\star H_m$, then

$$\eta_{w,2}^{\mathrm{antipal}_*}(P,\lambda) = \frac{1}{\sqrt{\lambda_{\max}^\star}} = \left(\min_{t_0,\ldots,t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + \cdots + t_m\, H_m)\right)^{-1/2}.$$

Proof. The proof follows by Theorem 1.2.18 and Theorem 3.3.1 with arguments similar to that in Theorem 3.1.6. □

Note that Remark 3.1.7 holds for $*$-antipalindromic polynomials also and thus making the case of the $\bullet$-antipalindromic pencils a special one.

Theorem 3.3.3. *Let $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\lambda| \neq 1$. Suppose that the pencil $P(\lambda) = A - \lambda A^*$ is nonsingular and let $M := (A - \lambda A^*)^{-1}$ exists. Furthermore define $\gamma_1 := \sqrt{\frac{2}{1+|\lambda|}}$, $\gamma_2 := \sqrt{\frac{2|\lambda|}{1+|\lambda|}}$,*

$$\widetilde{G} := \begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2M^*M \end{bmatrix},\ C := \begin{bmatrix} M^* & 0 \\ \bar{\lambda}M^* + M & \lambda M \end{bmatrix},\ \Gamma := \begin{bmatrix} w_0\gamma_1 I_n & 0 \\ 0 & w_0\gamma_2 I_n \end{bmatrix},$$

$$G := \Gamma^{-1}\widetilde{G}\Gamma^{-1}, \quad H_0 = \Gamma^{-1}(C + C^*)\Gamma^{-1} \ \ \text{and} \ \ H_1 := i\,\Gamma^{-1}(C - C^*)\Gamma^{-1}.$$

Then

$$\eta_{w,2}^{\mathrm{antipal}_*}(P,\lambda) = \left(\min_{t_0,t_1\in\mathbb{R}} \lambda_{\max}(G + t_0\,H_0 + t_1\,H_1)\right)^{-1/2}.$$

An optimal perturbation $(\Delta_0,\ldots,\Delta_m) \in$ antipal$_*$ that attains the backward error $\eta_{w,2}^{\mathrm{antipal}_*}(P,\lambda)$ can be constructed on the lines of Remark 3.1.9.

Now let $P(z) = A - zA^T$ where $A \in \mathbb{C}^{n\times n}$ be a T-antipalindromic pencil and $\lambda \in \mathbb{C}\setminus\{0\}$ be such that $M = (P(\lambda))^{-1}$ exists. Then by Theorem 3.3.1, we have

$$\eta_{w,2}^{\mathrm{antipal}_T}(P,\lambda) \;=\; \left(\sup\left\{ \frac{v^*Gv}{v^*v} \;\middle|\; v \in \mathbb{C}^{2n}\setminus\{0\},\ v^TSv = 0\right\}\right)^{-1/2}$$

where G is Hermitian and S is symmetric. Therefore as an easy consequence of Theorem 1.2.7 we obtain $\eta_{w,2}^{\mathrm{antipal}_T}(P,\lambda)$. More precisely, we have the following result for T-antipalindromic pencils.

Theorem 3.3.4. *Let $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ with $\lambda \neq \pm 1$. Suppose that the pencil $P(\lambda) = A - \lambda A^T$ is nonsingular and let $M := (A - \lambda A^T)^{-1}$ exists. Furthermore define $\gamma_1 := \sqrt{\frac{2}{1+|\lambda|}}$, $\gamma_2 := \sqrt{\frac{2|\lambda|}{1+|\lambda|}}$,*

$$\widetilde{G} := \begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2M^*M \end{bmatrix},\ C := \begin{bmatrix} M^T & 0 \\ \lambda M^T + M & \lambda M \end{bmatrix},\ \Gamma := \begin{bmatrix} w_0\gamma_1 I_n & 0 \\ 0 & w_0\gamma_2 I_n \end{bmatrix},$$

$$G := \Gamma^{-1}\widetilde{G}\Gamma^{-1} \quad \text{and} \quad S := \Gamma^{-1}(C + C^T)\Gamma^{-1}.$$

Then

$$\eta_{w,2}^{\mathrm{antipal}_T}(P,\lambda) = \left(\min_{0\le t\le t_1} \lambda_2\Big(\begin{bmatrix} G & t\bar{S} \\ tS & \bar{G} \end{bmatrix} \Big)\right)^{-1/2},$$

where $\lambda_2\Big(\begin{bmatrix} G & t\bar{S} \\ tS & \bar{G} \end{bmatrix}\Big)$ *denotes the second largest eigenvalue of matrix* $\begin{bmatrix} G & t\bar{S} \\ tS & \bar{G} \end{bmatrix}$ *and* $t_1 = \dfrac{2\|G\|}{\sigma_2(S)}$, *where* $\sigma_2(S)$ *denotes the second largest singular value of* S.

Remark 3.3.5. By following the techniques of Section 3.1.4, we can allow zero weights in the palindromic weight vector w. Thus $\eta^{\mathrm{antipal}_\bullet}_{\widehat{w},2}(P,\lambda)$ with a restricted perturbation set can be obtained on the lines of Section 3.1.4 by appropriate changes in matrices $\widehat{C}_j$ and hence $\widehat{H}_j$ and $\widehat{S}_j$. Note that if m is even then we have a symmetric matrix corresponding to $(\frac{m}{2})^{th}$-index. Hence when $P(z)$ is T-antipalindromic and we allow perturbation to at most two coefficient matrices of $P(z)$, then $\eta^{\mathrm{antipal}_\bullet}_{\widehat{w},2}(P,\lambda)$ may be computed by using Theorem 3.3.4.

For higher degree T-antipalindromic polynomial $P(z)$, we can estimate $\eta^{\mathrm{antipal_T}}_{w,2}(P,\lambda)$ by following the techniques used to estimate $\eta^{\mathrm{pal_T}}_{w,2}(P,\lambda)$ in Section 3.2. In the following, we have a analogue of Theorem 3.2.6 for T-antipalindromic polynomials that estimates $\eta^{\mathrm{antipal_T}}_{w,2}(P,\lambda)$ fairly tightly.

Theorem 3.3.6. *Let* $P(z) = \sum_{j=0}^m z^j A_j$ *be a T-antipalindromic polynomial and suppose that* $\lambda \in \mathbb{C}\setminus\{0,\pm 1\}$. *Suppose that* $P(\lambda)$ *is non singular and* $M = (P(\lambda))^{-1}$. *Furthermore, let* $w = (w_0,\ldots,w_m)$ *be a palindromic weight vector,* $k = \lfloor \frac{m-1}{2} \rfloor$, *and* $p = k$ *when* m *is odd and* $p = k+1$ *when* m *is even. Then*

$$\left(\eta^{\mathrm{antipal_T}}_{w,2}(P,\lambda)\right)^2 = \left(m_{hs_0\ldots s_p}(G,S_0,\ldots,S_p)\right)^{-1} \geq \left(\inf_{t_0,\ldots,t_p\in\mathbb{R}} \lambda_{2^{p+1}}\big(F_{p+1}(t_0,\ldots,t_p)\big)\right)^{-1} \geq \left(\eta_{w,2}(P,\lambda)\right)^2, \tag{3.3.9}$$

where $F_{p+1}(t_0,\ldots,t_p)$ *is defined by (3.2.5),* $m_{hs_0\ldots s_p}$ *is defined by (3.2.6),* S_j *for* $j = 0,\ldots,p$ *and* G *are defined by* (3.3.6) *and* (3.3.4) *respectively, and* $\lambda_{2^{p+1}}\big(F_{p+1}(t_0,\ldots,t_p)\big)$ *denotes the* $(2^{p+1})^{th}$ *largest eigenvalue of* $F_{p+1}(t_0,\ldots,t_p)$. *Let,* $t_0^\star,\ldots,t_p^\star \in \mathbb{R}$ *such that*

$$\lambda_{2^{p+1}}\big(F_{p+1}(t_0^\star,\ldots,t_p^\star)\big) := \inf_{t_0,\ldots,t_p\in\mathbb{R}} \lambda_{2^{p+1}}\big(F_{p+1}(t_0,\ldots,t_p)\big)$$

If there exist a unit vector $v \in \mathbb{C}^{n(m+1)}$ *satisfying*

$$v^* G v = \lambda_{2^{p+1}}\big(F_{p+1}(t_0^\star,\ldots,t_p^\star)\big) \quad \text{and} \quad v^T S_j v = 0 \;\; \forall\, j = 0,\ldots,p, \tag{3.3.10}$$

then

$$\left(\eta^{\mathrm{antipal_T}}_{w,2}(P,\lambda)\right)^2 = \left(m_{hs_0\ldots s_p}(G,S_0,\ldots,S_p)\right)^{-1} = \left(\lambda_{2^{p+1}}\big(F_{p+1}(t_0^\star,\ldots,t_p^\star)\big)\right)^{-1}.$$

Proof. The proof is similar to that of Theorem 3.2.6. ◻

Note that, in general numerical experiments suggest that the lower bound in the above theorem is equal to the exact backward error, thus the counterpart of Conjecture 3.2.8 holds for T-antipalindromic polynomials. Moreover, we can obtain a tight upper bound of $\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T}}}(P,\lambda)$ by arguments similar to that of Section 3.2.3.

Table 3.3.1 records the experiments done for randomly generated T-antipalindromic quadratic polynomial $P(z)$ of size 4, randomly chosen $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $w = (1,1,1)$. Here lbound is the lower bound obtained in Theorem 3.3.6 and ubound is an upper bound obtained by following Section 3.2.3. This shows that the unstructured backward error and T-antipalindromic backward error are significantly different. Also in each case the sufficient condition for the lower bound to be equal to $\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T}}}(P,\lambda)$ is satisfied and hence gives the exact value of $\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T}}}(P,\lambda)$.

Table 3.3.1: *Estimates of $\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T}}}(P,\lambda)$ of a T-antipalindromic quadratic polynomial $P(z)$ for arbitrary λ.*

λ	$\eta_{w,2}(P,\lambda)$	lbound $(=\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T}}}(P,\lambda))$	ubound
$0.0983 + 0.0414i$	1.1039	1.5476	1.5637
$0.0000 - 0.0549i$	1.1005	1.5501	1.5550
$0.2458 + 0.0700i$	1.0650	1.4625	1.5378
$1.7013 - 0.5097i$	0.8223	1.1268	1.2773
$2.0243 - 2.3595i$	1.0490	1.4116	1.4953
$-0.2132 - 0.1345i$	0.9635	1.2716	1.3507

3.4 T-alternating polynomials

The problem of computing structured backward error for eigenvalues of T-alternating polynomials can also be transformed to a problem of maximizing the Rayleigh Quotient of a Hermitian matrix with respect to some conditions involving symmetric matrices. The structured eigenvalue backward errors for eigenvalues of T-even polynomials of degree at most 2 and T-odd pencils can then be found by applying Theorem 1.2.7. We denote the T-alternating structure of matrix polynomials by $\mathrm{alt}_{\mathrm{T}} \subseteq (\mathbb{C}^{n\times n})^{m+1}$, where alt = even if

$P(z)$ is T-even and alt = odd if P(z) is T-odd, and define it as follows.

$$\mathrm{alt_T} = \begin{cases} \{(A_0, \ldots, A_m) : A_{2j} \in \mathrm{Sym(n)},\ A_{2j+1} \in \mathrm{SSym(n)}\} & \text{if alt = even} \\ \{(A_0, \ldots, A_m) : A_{2j+1} \in \mathrm{Sym(n)},\ A_{2j} \in \mathrm{SSym(n)}\} & \text{if alt = odd.} \end{cases}$$

Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a T-alternating polynomial, i,e., $(A_0, \ldots, A_m) \in \mathrm{alt_T}$. From [1, 2], $\eta_{w,2}^{\mathrm{alt_T}}(P, 0) = \eta_{w,2}(P, 0)$. Therefore let $\lambda \in \mathbb{C} \setminus \{0\}$ such that $P(\lambda)$ is nonsingular. We calculate,

$$\eta_{w,2}^{\mathrm{alt_T}}(P, \lambda) := \inf \left\{ \|\!|(\Delta_0, \ldots, \Delta_m)\|\!|_{w,2} \,\middle|\, \det\Big(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\Big) = 0,\ (\Delta_0, \ldots, \Delta_m) \in \mathrm{alt_T} \right\}$$

and construct the corresponding perturbation $\Delta(z) = \sum_{j=0}^{m} \lambda^j \Delta_j$ that attains the infimum.

Lemma 1.2.6 yields the following alternative characterization of $\eta_{w,2}^{\mathrm{alt_T}}(P, \lambda)$ in terms of mapping problems.

$$\eta_{w,2}^{\mathrm{alt_T}}(P, \lambda) = \inf \Big\{ \|\!|(\Delta_0, \ldots, \Delta_m)\|\!|_{w,2} \,\Big|\, \exists\, v_0, \ldots, v_m \in \mathbb{C}^n \quad , \quad v_\lambda \neq 0, (\Delta_0, \ldots, \Delta_m) \in \mathrm{alt_T}, \\ \Delta_j M v_\lambda = v_j,\ j = 0, \ldots, m \Big\}.$$

The matrices Δ_j for $j = 0, \ldots, m$ of the tuple $(\Delta_0, \ldots, \Delta_m) \in \mathrm{alt_T}$ are either symmetric or skew symmetric. For any $(v_0, \ldots, v_m)$ such that $v_\lambda \neq 0$, a symmetric matrix Δ may be chosen to satisfy $\Delta M v_\lambda = v_j$ without any restriction on $M v_\lambda$ and v_j. On the other hand, a skew-symmetric matrix Δ may be chosen to satisfy $\Delta M v_\lambda = v_j$ if and only if $(M v_\lambda)^T v_j = 0 \iff v^T \hat{S}_j v = 0$, where $v = [v_0^T, \ldots, v_m^T]$ and $\hat{S}_j = \Lambda^T e_{j+1}^T \otimes M^T$. Due to these facts, the following theorem gives the desired reformulation of the problem of computing $\eta_{w,2}^{\mathrm{alt_T}}(P, \lambda)$.

Theorem 3.4.1. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be T-alternating matrix polynomial. Suppose that $\lambda \in \mathbb{C} \setminus \{0\}$ such that $M = (P(\lambda))^{-1}$ exists. Let $\Lambda_m := [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$ and set*

$$\widetilde{G} := (\Lambda_m^* \Lambda_m) \otimes (M^* M), \quad \Gamma := \mathrm{diag}\big(w_0, w_1, \ldots, w_m\big) \otimes I_n, \quad G = \Gamma^{-1} \widetilde{G} \Gamma^{-1},$$

as well as

$$\widetilde{S}_{ej} := ((\Lambda_m^T e_{2j+2}^T) \otimes M^T),\ S_{ej} = \Gamma^{-1}(\widetilde{S}_{ej} + \widetilde{S}_{ej}^T)\Gamma^{-1} \text{ for } j = 0, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \tag{3.4.1}$$

$$\widetilde{S}_{oj} := ((\Lambda_m^T e_{2j+1}^T) \otimes M^T),\ S_{oj} = \Gamma^{-1}(\widetilde{S}_{oj} + \widetilde{S}_{oj}^T)\Gamma^{-1} \text{ for } j = 0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, \tag{3.4.2}$$

where e_j denotes the j-th standard basis vector.

(1) If $\mathrm{alt} = \mathrm{even}$*, then*

$$\left(\eta_{w,2}^{\mathrm{even}_{\mathrm{T}}}(P,\lambda)\right)^{-2} = \sup\left\{ \frac{u^*Gu}{u^*u} \,\middle|\, u \in (\mathbb{C}^n)^{m+1} \setminus \{0\},\ u^T S_{ej} u = 0,\ j = 0, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \right\}. \tag{3.4.3}$$

(2) If $\mathrm{alt} = \mathrm{odd}$*, then*

$$\left(\eta_{w,2}^{\mathrm{odd}_{\mathrm{T}}}(P,\lambda)\right)^{-2} = \sup\left\{ \frac{u^*Gu}{u^*u} \,\middle|\, u \in (\mathbb{C}^n)^{m+1} \setminus \{0\},\ u^T S_{oj} u = 0,\ j = 0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \right\}. \tag{3.4.4}$$

Proof. The proof is similar to that of Theorem 3.1.5 for the case when $\bullet = T$. □

Clearly, when $P(z)$ is either a T-even polynomial of degree at most 2 or a T-odd pencil, then (3.4.3) and (3.4.4) respectively imply that calculating $\eta_{w,2}^{\mathrm{alt}_{\mathrm{T}}}(P,\lambda)$ involves a single condition associated with a symmetric matrix. Thus, a straightforward application of Theorem 1.2.7 gives the structured eigenvalue backward errors in these cases.

Theorem 3.4.2. *Let* $P(z) = A_0 + zA_1$ *be a T-alternating pencil and* $\lambda \in \mathbb{C} \setminus \{0\}$*. Suppose that* $\det(P(\lambda)) \neq 0$ *so that* $M = (P(\lambda))^{-1}$ *exists. Set*

$$G := \begin{bmatrix} \frac{1}{w_0^2} & \frac{\lambda}{w_0 w_1} \\ \frac{\bar{\lambda}}{w_0 w_1} & \frac{|\lambda|^2}{w_1^2} \end{bmatrix} \otimes M^*M, \quad S_e := \begin{bmatrix} 0 & \frac{M^T}{w_0 w_1} \\ \frac{M}{w_0 w_1} & \frac{\lambda}{w_1^2}(M^T + M) \end{bmatrix}$$

and

$$S_o := \begin{bmatrix} \frac{1}{w_0^2}(M^T + M) & \frac{\lambda}{w_0 w_1} M \\ \frac{\lambda}{w_0 w_1} M^T & 0 \end{bmatrix}.$$

Then

$$\eta_{w,2}^{\mathrm{even}_{\mathrm{T}}}(P,\lambda) = \left(\min_{0 \le t \le t_1} \lambda_2\left(\begin{bmatrix} G & t\bar{S}_e \\ tS_e & \bar{G} \end{bmatrix} \right) \right)^{-1/2}$$

where $t_1 = \dfrac{2\|G\|}{\sigma_2(S_e)}$*, and*

$$\eta_{w,2}^{\mathrm{odd}_{\mathrm{T}}}(P,\lambda) = \left(\min_{0 \le t \le t_1} \lambda_2\left(\begin{bmatrix} G & t\bar{S}_o \\ tS_o & \bar{G} \end{bmatrix} \right) \right)^{-1/2}$$

where $t_1 = \dfrac{2\|G\|}{\sigma_2(S_o)}$.

Proof. Since $P(z) = A_0 + zA_1$, Theorem 3.4.1 implies that

$$\eta_{w,2}^{\mathrm{alt_T}}(P,\lambda)^{-2} = \sup\left\{ \frac{u^*Gu}{u^*u} \,\middle|\, u \in \mathbb{C}^{2n} \setminus \{0\},\ u^T Su = 0 \right\}, \tag{3.4.5}$$

where $S = S_e$ if $P(z)$ is T-even and $S = S_o$ if $P(z)$ is T-odd pencil. Thus the proof follows by applying Theorem 1.2.7. □

Similarly, the following result gives $\eta_{w,2}^{\mathrm{even_T}}(P,\lambda)$ when $P(z)$ is T-even quadratic matrix polynomial and $\lambda \in \mathbb{C} \setminus \{0\}$.

Theorem 3.4.3. *Let $P(z) = A_0 + zA_1 + z^2A_2$ be a T-even quadratic polynomial and $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $\det(P(\lambda)) \neq 0$ so that $M = (P(\lambda))^{-1}$ exists. Set*

$$G := \begin{bmatrix} \frac{1}{w_0^2} & \frac{\lambda}{w_0w_1} & \frac{\lambda^2}{w_0w_2} \\ \frac{\bar\lambda}{w_0w_1} & \frac{|\lambda|^2}{w_1^2} & \frac{|\lambda|^2\lambda}{w_1w_2} \\ \frac{\bar\lambda^2}{w_0w_2} & \frac{|\lambda|^2\bar\lambda}{w_1w_2} & \frac{|\lambda|^4}{w_2^2} \end{bmatrix} \otimes M^*M \quad \text{and} \quad S_e := \begin{bmatrix} 0 & \frac{M^T}{w_0w_1} & 0 \\ \frac{M}{w_0w_1} & \frac{\lambda}{w_1^2}(M^T+M) & \frac{\lambda^2}{w_1w_2}M \\ 0 & \frac{\lambda^2}{w_1w_2}M^T & 0 \end{bmatrix}.$$

Then

$$\eta_{w,2}^{\mathrm{even_T}}(P,\lambda) = \left(\min_{0 \le t \le t_1} \lambda_2\left(\begin{bmatrix} G & t\bar S_e \\ tS_e & \bar G \end{bmatrix} \right) \right)^{-1/2},$$

where $t_1 = \dfrac{2\|G\|}{\sigma_2(S_e)}$.

Proof. Since $P(z) = A_0 + zA_1 + z^2A_2$ is T-even, Theorem 3.4.1 implies that

$$\eta_{w,2}^{\mathrm{alt_T}}(P,\lambda)^{-2} = \sup\left\{ \frac{u^*Gu}{u^*u} \,\middle|\, u \in \mathbb{C}^{3n} \setminus \{0\},\ u^T Su = 0 \right\}, \tag{3.4.6}$$

where $S = S_e$. Thus proof follows by applying Theorem 1.2.7. □

We can follow the strategy used in Section 2.4 to obtain analogous results for $\eta_{\widehat{w},2}^{\mathrm{alt_T}}(P,\lambda)$ with a restricted perturbation set by allowing zero weights in the weight vector w. Let $P(z) = \sum_{j=0}^m z^jA_j$ be a T-alternating polynomial i.e., $(A_0,\ldots,A_m) \in \mathrm{alt_T}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $M = (P(\lambda))^{-1}$ exists. Let $I := \{i_0,\ldots,i_\ell\} \subseteq \{0,\ldots,m\}$ and $\widehat{w} = (w_{i_0},\ldots,w_{i_\ell})$ be a weight vector obtained by retaining only the nonzero entries of $w = (w_0,\ldots,w_m)$. Define

$$\Lambda_\ell := [\lambda^{i_0},\ldots,\lambda^{i_\ell}] \in \mathbb{C}^{1\times \ell+1}, \quad \widehat{W} := \mathrm{diag}\,(w_{i_0},\ldots,w_{i_\ell}) \otimes I_n,$$

$$\widehat{G} := \widehat{W}^{-1}\left((\Lambda_\ell^*\Lambda_\ell) \otimes M^*M\right)\widehat{W}^{-1}. \tag{3.4.7}$$

(a) If alt = even, let

$$R := \{i_{r_0}, \ldots, i_{r_p}\} \subseteq I, \quad i_{r_j} \text{ is odd for each } \; j \tag{3.4.8}$$

be exactly the set of all odd numbers from I. If $R \neq \emptyset$ then define

$$\widehat{S}_{ej} := \widehat{W}^{-1}\left((\Lambda_\ell^T e_{r_j+1}^T) \otimes M^T + (e_{r_j+1}\Lambda_\ell) \otimes M\right)\widehat{W}^{-1} \tag{3.4.9}$$

for $j = 0, \ldots, p$, where e_j is the j-th standard basis vector of $\mathbb{C}^{\ell+1}$.

(b) If alt = odd, let

$$R' := \{i_{r'_0}, \ldots, i_{r'_q}\} \subseteq I, \quad i_{r'_j} \text{ is even for each } \; j \tag{3.4.10}$$

be exactly the set of all even numbers from I. If $R' \neq \emptyset$ then define

$$\widehat{S}_{oj} := \widehat{W}^{-1}\left((\Lambda_\ell^T e_{r'_j+1}^T) \otimes M^T + (e_{r'_j+1}\Lambda_\ell) \otimes M\right)\widehat{W}^{-1} \tag{3.4.11}$$

for $j = 0, \ldots, q$, where e_j is the j-th standard basis vector of $\mathbb{C}^{\ell+1}$.

Now we are in a position to give an analogue of Theorem 3.4.1 with a restricted perturbation set.

Theorem 3.4.4. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be T-alternating i.e., $(A_0, \ldots, A_m) \in \mathrm{alt_T}$. Let $\lambda \in \mathbb{C} \setminus \{0\}$ such that $M = (P(\lambda))^{-1}$ exists. Let $I := \{i_0, \ldots, i_\ell\} \subseteq \{0, \ldots, m\}$ and $\widehat{w} = (w_{i_0}, \ldots, w_{i_\ell}) \in \mathbb{C}^{\ell+1}$ be a weight vector. Let $\widehat{G}$ be as defined in (3.4.7).*
(1) If alt = even*, let $R \subseteq I$ be as defined in (3.4.8).*
If $R = \emptyset$ then

$$\eta_{\widehat{w},2}^{\mathrm{even_T}}(P, \lambda) = \left(\lambda_{max}(\widehat{G})\right)^{-1/2} = \eta_{\widehat{w},2}(P, \lambda).$$

If $R \neq \emptyset$ then

$$\eta_{\widehat{w},2}^{\mathrm{even_T}}(P, \lambda) = \left(\sup\left\{\frac{u^*\widehat{G}u}{u^*u}\Big| u \in \mathbb{C}^{n(\ell+1)} \setminus \{0\}, \; u^*\widehat{S}_{ej}u = 0, \; j = 0, \ldots, p\right\}\right)^{-1/2} \tag{3.4.12}$$

where $\widehat{S}_{ej}$ for $j = 0, \ldots, p$ are defined by (3.4.9).
(2) If alt = odd*, let $R' \subseteq I$ be as defined in (3.4.10).*
If $R' = \emptyset$ then

$$\eta_{\widehat{w},2}^{\mathrm{odd_T}}(P, \lambda) = \left(\lambda_{\max}(\widehat{G})\right)^{-1/2} = \eta_{\widehat{w},2}(P, \lambda).$$

If $R' \neq \emptyset$ then

$$\eta^{\mathrm{odd_T}}_{\widehat{w},2}(P,\lambda) = \left(\sup\left\{\frac{u^*\widehat{G}u}{u^*u}\Big| u \in \mathbb{C}^{n(\ell+1)} \setminus \{0\},\ u^*\widehat{S}_{oj}u = 0,\ j = 0,\ldots,q\right\}\right)^{-1/2} \tag{3.4.13}$$

where $\widehat{S}_{oj}$ for $j = 0,\ldots,q$ are defined by (3.4.11).

Proof. The proof proceeds in the same way as that of Theorem 3.4.1 by noting that $\widehat{G}$, $\widehat{S}_{ej}$ and $\widehat{S}_{oj}$ are obtained by deleting the block rows and columns corresponding to indices that are not in I of G, $S_{e\tilde{p}}$ and $S_{o\tilde{q}}$ (as given in Theorem 3.4.1) respectively for $\tilde{p} = \lfloor\frac{i_{r_j}-1}{2}\rfloor$ and $\tilde{q} = \lfloor\frac{i_{r'_j}}{2}\rfloor$. $\square$

Again, we can apply Theorem 1.2.7 in Theorem 3.4.4 to obtain explicit formula of $\eta^{\mathrm{alt_T}}_{\widehat{w},2}(P,\lambda)$ for T-alternating polynomial $P(z)$, if the number of coefficients that are perturbed are such that the computation involves a single constraint involving symmetric matrix. This is the case when we restrict perturbations to perturb only a single skew-symmetric coefficient matrix of $P(z)$. Note that perturbation in a symmetric coefficient matrix of $P(z)$ does not result in any extra constraint in the computation. This allows us to perturb any number of symmetric coefficient matrices. More precisely, we have the following result.

Theorem 3.4.5. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be T-alternating i.e., $(A_0,\ldots,A_m) \in \mathrm{alt_T}$. Let $\lambda \in \mathbb{C}\setminus\{0\}$ such that $M = (P(\lambda))^{-1}$ exists. Let $I := \{i_0,\ldots,i_\ell\} \subseteq \{0,\ldots,m\}$ such that $R = \{i_{r_0}\}$ in (3.4.8) when $\mathrm{alt} = \mathrm{even}$ and $R' = \{i_{r'_0}\}$ in (3.4.10) when $\mathrm{alt} = \mathrm{odd}$. Let $\widehat{w}$ be a weight vector corresponding to I.*

- *If $\mathrm{alt} = \mathrm{even}$ then*

$$\eta^{\mathrm{even_T}}_{\widehat{w},2}(P,\lambda) = \left(\min_{0\le t\le t_1}\lambda_2\Big(\begin{bmatrix}\widehat{G} & t\overline{\widehat{S}_{e0}}\\ t\widehat{S}_{e0} & \overline{\widehat{G}}\end{bmatrix}\Big)\right)^{-1/2}$$

where $t_1 = \dfrac{2\|\widehat{G}\|}{\sigma_2(\widehat{S}_{e0})}$, $\widehat{G} \in \mathbb{C}^{n(\ell+1)\times n(\ell+1)}$ is defined by (3.4.7) and $\widehat{S}_{e0} \in \mathbb{C}^{n(\ell+1)\times n(\ell+1)}$ is defined by (3.4.9).

- *If $\mathrm{alt} = \mathrm{odd}$ then*

$$\eta^{\mathrm{odd_T}}_{\widehat{w},2}(P,\lambda) = \left(\min_{0\le t\le t_1}\lambda_2\Big(\begin{bmatrix}\widehat{G} & t\overline{\widehat{S}_{o0}}\\ t\widehat{S}_{o0} & \overline{\widehat{G}}\end{bmatrix}\Big)\right)^{-1/2}$$

where $t_1 = \dfrac{2\|\widehat{G}\|}{\sigma_2(\widehat{S}_{o0})}$, $\widehat{G} \in \mathbb{C}^{n(\ell+1)\times n(\ell+1)}$ *is defined by (3.4.7) and* $\widehat{S}_{o0} \in \mathbb{C}^{n(\ell+1)\times n(\ell+1)}$ *is defined by (3.4.11).*

Remark 3.4.6. We can extend the ideas from Section 3.2 to estimate the backward error $\eta_{w,2}^{\mathrm{alt}_\mathrm{T}}(P,\lambda)$ for higher degree T-alternating polynomials. In particular, we get a lower bound of $\eta_{w,2}^{\mathrm{alt}_\mathrm{T}}(P,\lambda)$ from Theorem 3.2.2 by using Hermitian matrix G and symmetric matrices S_{ej} and S_{oj} as defined in Theorem 3.4.1. Indeed, we have

$$\eta_{w,2}(P,\lambda) \leq \left(\inf_{t_0,\ldots,t_p \in \mathbb{R}} \lambda_{2^{p+1}}\big(F_{p+1}(t_0,\ldots,t_p)\big)\right)^{-1/2} \leq \eta_{w,2}^{\mathrm{alt}_\mathrm{T}}(P,\lambda), \qquad (3.4.14)$$

where $p = \lfloor \frac{m-1}{2} \rfloor$ if alt = even and $p = \lfloor \frac{m}{2} \rfloor$ if alt = odd, and $F_{p+1} : \mathbb{R}^{p+1} \to \mathbb{C}^{2^{p+1}N \times 2^{p+1}N}$ is defined by (3.2.5) with $h = G$, $S_j = S_{ej}$ for $j = 0,\ldots,p$ (as given in (3.4.1)) if alt = even and $S_j = S_{oj}$ for $j = 0,\ldots,p$ (as given in (3.4.2)) if alt = odd. We can also obtain an upper bound to $\eta_{w,2}^{\mathrm{alt}_\mathrm{T}}(P,\lambda)$ on the lines of the upper bound for $\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(P,\lambda)$ obtained in Section 3.2.3.

Numerical experiments suggest that the lower bound in (3.4.14) is equal to the exact backward error when $\lambda_{2^{p+1}}\big(F_{p+1}(t_0^\star,\ldots,t_p^\star)\big)$ is a simple eigenvalue of F_{p+1} at an optimal point $(t_0^\star,\ldots,t_p^\star)$.

Tables 3.4.1 and 3.4.2 record the experiments done for a randomly generated T-even cubic polynomial $P(z)$ of size 4 and a T-odd quadratic polynomial $Q(z)$ of size 3, and randomly chosen scalars $\lambda \in \mathbb{C}\setminus\mathbb{R}$. The terms lbound and ubound denote the lower and upper bounds of $\eta_{w,2}^{\mathrm{even}_\mathrm{T}}(P,\lambda)$ and $\eta_{w,2}^{\mathrm{odd}_\mathrm{T}}(Q,\lambda)$ in Table 3.4.1 and Table 3.4.2, respectively. These tables show that the two backward errors are significantly different. Also in each case the sufficient condition for the lower bound to be equal to $\eta_{w,2}^{\mathrm{alt}_\mathrm{T}}(P,\lambda)$ is satisfied and hence it gives the exact value of $\eta_{w,2}^{\mathrm{alt}_\mathrm{T}}(P,\lambda)$.

Table 3.4.1: *Estimates of* $\eta_{w,2}^{\mathrm{even_T}}(P,\lambda)$ *of a* $T-$*even cubic polynomial* $P(z)$ *for arbitrary* λ.

λ	$\eta_{w,2}(P,\lambda)$	lbound $\big(=\eta_{w,2}^{\mathrm{even_T}}(P,\lambda)\big)$	ubound
$1.7463-0.0740i$	1.4898	1.9249	1.9253
$0.0477-1.1651i$	1.2138	1.4026	1.4029
$-1.7898+0.4113i$	1.3535	1.7605	1.7701
$-2.2942+0.6491i$	1.6015	2.0043	2.0091
$1.9598-0.1706i$	1.5723	2.0565	2.0579
$-1.8076+0.2152i$	1.4599	1.9322	1.9355

Table 3.4.2: *Estimates of* $\eta_{w,2}^{\mathrm{odd_T}}(Q,\lambda)$ *of a* $T-$*odd polynomial* $Q(z)$ *of degree 4 for arbitrary* λ.

λ	$\eta_{w,2}(Q,\lambda)$	lbound $\big(=\eta_{w,2}^{\mathrm{odd_T}}(Q,\lambda)\big)$	ubound
$-0.0766-0.5027i$	1.0542	1.4937	1.4978
$0.2571+0.2434i$	0.4903	1.0415	1.0722
$0.0371-0.4582i$	0.9539	1.4503	1.4513
$-1.9129-1.5769i$	1.2804	1.7627	1.7828
$-2.6045+0.9416i$	1.2048	1.7354	1.7412
$-0.0135+0.3394i$	0.6761	1.4075	1.4077

Chapter 4

Structured eigenvalue backward errors with respect to norms $|||\cdot|||_{w,\infty}$ and $|||\cdot|||_{w,F}$

In this chapter, we extend the work of finding structured eigenvalue backward error of a matrix polynomial $P(z) = \sum_{j=0}^{m} z^j A_m$ with respect to structure preserving perturbations done in Chapter 1 and 2 to the case where the norm on $(A_0, \ldots, A_m)$ is $|||\cdot|||_{w,\infty}$ or $|||\cdot|||_{w,F}$ which are as defined in (1.2.2) and (1.2.3), respectively. In particular, we derive tight bounds for structured eigenvalue backward errors of matrix pencils and polynomials with Hermitian structure. When the norm is $|||\cdot|||_{w,\infty}$, the lower bound is equal to the exact backward error under some assumptions that are seen to be satisfied in our extensive numerical experiments. Similar results are also obtained for $*$-palindromic and T-palindromic polynomials. These ideas can be used to estimate the backward errors for other structures like $*$-alternating, T-alternating, skew-Hermitian, $*$-antipalindromic and T-antipalindromic.

4.1 Structured eigenvalue backward error with respect to $|||\cdot|||_{w,\infty}$ norm

We follow the notation and terminology from Chapter 2 and Chapter 3. In this section, we consider the problem of computing the structured eigenvalue backward error of approximate eigenvalues with respect to $|||\cdot|||_{w,\infty}$ norm defined over the set $(\mathbb{C}^{n\times n})^{m+1}$ by

$$|||(\Delta_0, \ldots, \Delta_m)|||_{w,\infty} = \max\{w_0\|\Delta_0\|, \ldots, w_m\|\Delta_m\|\},$$

where $\|.\|$ denotes the 2-norm of a matrix or a vector. We first recall a few things from Chapter 1.

Let $\mathbb{S} \subseteq (\mathbb{C}^{n\times n})^{m+1}$. Let $P(z) = \sum_{j=0}^{m} z^j A_j$, where $(A_0, \ldots, A_m) \in \mathbb{S}$ and let $\lambda \in \mathbb{C}$. For the sake of notational simplicity, we assume that all the weights are equal to 1 in our results. The generalization to the case of other nonzero weight vectors is straightforward. Since the weight vector is $w = (1, \ldots, 1)$, we denote $|\!|\!|\cdot|\!|\!|_{w,\infty}$ by $|\!|\!|\cdot|\!|\!|_{\infty}$ and the structured eigenvalue backward error of λ as an approximate eigenvalue of $P(z)$ under structure preserving perturbations measured with respect to the norm $|\!|\!|\cdot|\!|\!|_{w,\infty}$ by $\eta_{\infty}^{\mathbb{S}}(P, \lambda)$. That is,

$$\eta_{\infty}^{\mathbb{S}}(P,\lambda) = \inf \left\{ \max\{\|\Delta_0\|, \ldots, \|\Delta_m\|\} \,\middle|\, (\Delta_0, \ldots, \Delta_m) \in \mathbb{S}, \;\; \Delta P(z) = \sum_{j=0}^{m} z^j \Delta_j \,, \right. \\ \left. \det\big(P(\lambda) - \Delta P(\lambda)\big) = 0 \right\}.$$

When $\mathbb{S} = (\mathbb{C}^{n\times n})^{m+1}$, $\eta_{\infty}^{\mathbb{S}}(P, \lambda)$ is called the unstructured eigenvalue backward error and we denote it by $\eta_{\infty}(P, \lambda)$ instead of $\eta_{\infty}^{\mathbb{S}}(P, \lambda)$. Note that $\eta_{\infty}^{\mathbb{S}}(P, 0) = \sigma_{\min}(A_0)$ where $\sigma_{\min}(A_0)$ is the smallest singular value of A_0.

Though the unstructured eigenvalue backward error $\eta_{\infty}(P, \lambda)$ is well known due to Theorem 1.2.5, the following result gives a new characterization for the unstructured backward error which will be used later.

Lemma 4.1.1. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a matrix polynomial, where $A_0, \ldots, A_m \in \mathbb{C}^{n\times n}$ and let $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $M = (P(\lambda))^{-1}$ exists and define $v_\lambda := \sum_{j=0}^{m} \lambda^j v_j$ for $(v_0, \ldots, v_m) \in (\mathbb{C}^n)^{m+1}$. Furthermore, define*

$$\widetilde{\gamma}_i := \sqrt{\frac{|\lambda|^i}{\sum_{j=0}^{m} |\lambda|^j}}, \text{ for } i = 0, \ldots, m, \quad \widetilde{\Gamma} := \operatorname{diag}\left(\widetilde{\gamma}_0, \ldots, \widetilde{\gamma}_m\right),$$

$$\Lambda_m := [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)} \quad \text{and} \quad G(\widetilde{\Gamma}) := \widetilde{\Gamma}^{-1}\big(\Lambda_m^* \Lambda_m\big)\widetilde{\Gamma}^{-1} \otimes M^* M.$$

Then

$$(\eta_{\infty}(P,\lambda))^2 = \inf\left\{ g_{\widetilde{\gamma}_0,\ldots,\widetilde{\gamma}_m}(v_0, \ldots, v_m) \,\middle|\, (v_0, \ldots, v_m) \in (\mathbb{C}^n)^{m+1}, \; v_\lambda \neq 0 \right\} = \frac{1}{\lambda_{\max}(G(\widetilde{\Gamma}))}, \tag{4.1.1}$$

where $g_{\widetilde{\gamma}_0,\ldots,\widetilde{\gamma}_m}(v_0, \ldots, v_m)$ is defined by

$$g_{\widetilde{\gamma}_0,\ldots,\widetilde{\gamma}_m}(v_0, \ldots, v_m) = \frac{\widetilde{\gamma}_0^2\|v_0\|^2 + \cdots + \widetilde{\gamma}_m^2\|v_m\|^2}{\|Mv_\lambda\|^2} \quad \text{for all } (v_0, \ldots, v_m) \in (\mathbb{C}^n)^{m+1} \text{ with } v_\lambda \neq 0.$$

Proof. It is well established by Theorem 1.2.5 that

$$\eta_\infty(P,\lambda) = \frac{\sigma_{\min}(P(\lambda))}{1+|\lambda|+\cdots+|\lambda|^m},$$

where $\sigma_{\min}\big(P(\lambda)\big)$ is the minimum singular value of $P(\lambda)$. Now the proof is an easy consequence of the following two observations.

(1)

$$\begin{aligned}
\lambda_{\max}(G(\widetilde{\Gamma})) &= \sup_{u\in\mathbb{C}^{n(m+1)}\setminus\{0\}} \frac{u^*G(\widetilde{\Gamma})u}{u^*u} \\
&= \sup\left\{ \frac{u^*G(\widetilde{\Gamma})u}{u^*u} \,\middle|\, u\in\mathbb{C}^{n(m+1)}\setminus\{0\},\ u^*G(\widetilde{\Gamma})u\neq 0\right\} \qquad (4.1.2)\\
&= \Big(\inf\Big\{ g_{\widetilde{\gamma}_0,\ldots,\widetilde{\gamma}_m}(v_0,\ldots,v_m) \,\Big|\, (v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1},\ v_\lambda\neq 0\Big\}\Big)^{-1} \qquad (4.1.3)
\end{aligned}$$

Since $\lambda_{\max}(G(\widetilde{\Gamma}))$ is finite and positive, the supremum in the equality of (4.1.2) will not be attained by vectors u satisfying $u^*G(\widetilde{\Gamma})u=0$. Therefore, the condition $u^*G(\widetilde{\Gamma})u=0$ is superfluous for it. Also, the later equality of (4.1.3) holds as

$$\frac{u^*u}{u^*G(\widetilde{\Gamma})u} = \frac{\|\widetilde{\gamma}_0v_0\|^2+\cdots+\|\widetilde{\gamma}_mv_m\|^2}{\|Mv_\lambda\|^2} \quad \text{for some} \quad (v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1}$$

and

$$u^*G(\widetilde{\Gamma})u\neq 0 \iff Mv_\lambda\neq 0 \iff v_\lambda\neq 0.$$

(2) If $\rho(\neq 0)$ is an eigenvalue of $G(\widetilde{\Gamma})$, then $\rho=\frac{(1+|\lambda|+\cdots+|\lambda|^m)^2}{\sigma^2}$ for some singular value σ of $P(\lambda)$. This follows from the fact that eigenvalues of $G(\widetilde{\Gamma})=\widetilde{\Gamma}^{-1}\big(\Lambda_m^*\Lambda_m\big)\widetilde{\Gamma}^{-1}\otimes M^*M$ are the product of eigenvalues of $\widetilde{\Gamma}^{-1}\big(\Lambda_m^*\Lambda_m\big)\widetilde{\Gamma}^{-1}$ and M^*M, and $\widetilde{\Gamma}^{-1}\big(\Lambda_m^*\Lambda_m\big)\widetilde{\Gamma}^{-1}$ is a rank one matrix with nonzero eigenvalue $(1+|\lambda|+\cdots+|\lambda|^m)^2$. □

Let $P(z)=\sum_{j=0}^m z^jA_j$ be a matrix polynomial such that $(A_0,\ldots,A_m)\in\mathbb{S}$ and let $\lambda\in\mathbb{C}\setminus\{0\}$. Our strategy for computing $\eta_\infty^{\mathbb{S}}(P,\lambda)$ is as follows. We first establish that

$$\eta_\infty^{\mathbb{S}}(P,\lambda) = \inf\left\{\max\left\{\frac{\|v_0\|}{\|Mv_\lambda\|},\ldots,\frac{\|\Delta_0\|}{\|Mv_\lambda\|}\right\} \,\middle|\, (v_0,\ldots,v_m)\in\mathcal{K}\right\}, \qquad (4.1.4)$$

where

$$\mathcal{K} = \Big\{(v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1} \,\Big|\, v_\lambda := \sum_{j=0}^m \lambda^jv_j\neq 0,\quad (\Delta_0,\ldots,\Delta_m)\in\mathbb{S},$$
$$\Delta_jMv_\lambda = v_j,\ j=0,\ldots,m\Big\}.$$

Then, our attempt is to find a certain weight vector $\gamma = (\gamma_0, \ldots, \gamma_m) \in \mathbb{R}^{m+1}$ with $\gamma_0^2 + \cdots + \gamma_m^2 = 1$, so that

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} \max\{\|\Delta_0\|, \ldots, \|\Delta_m\|\} = \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} \sqrt{\sum_{j=0}^{m} \gamma_j^2 \|\Delta_j\|^2},$$

and to use the ideas from Chapter 2 and Chapter 3 to reformulate the problem into an equivalent constrained optimization problem involving Hermitian or symmetric matrices. In the following subsections we give the details of this approach for Hermitian and related structures, and $*$-palindromic polynomials. Similar results will also hold for some other structures.

4.1.1 Hermitian polynomials

Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a Hermitian polynomial i.e., $A_j \in \mathrm{Herm(n)}$ for $j = 0, \ldots, m$ and $\lambda \in \mathbb{C} \setminus \{0\}$. We follow the notation from Chapter 2 and denote the Hermitian backward error with respect to $|\!|\!|\cdot|\!|\!|_\infty$ norm by $\eta_\infty^{\mathrm{Herm}}(P, \lambda)$. Note that for a Hermitian polynomial $P(z)$, if $\lambda \in \mathbb{R}$ then there is no difference between structured and unstructured backward error, i.e. $\eta_\infty^{\mathrm{Herm}}(P, \lambda) = \eta_\infty(P, \lambda)$. This fact is shown in [42]. The situation is completely different for $\lambda \notin \mathbb{R}$. Thus in the following our attempt is to calculate $\eta_\infty^{\mathrm{Herm}}(P, \lambda)$ when $\lambda \notin \mathbb{R}$. In the view of Lemma 1.2.6, we have

$$\eta_\infty^{\mathrm{Herm}}(P, \lambda) = \inf\Big\{ |\!|\!|(\Delta_0, \ldots, \Delta_m)|\!|\!|_\infty \,\Big|\, (\Delta_0, \ldots, \Delta_m) \in (\mathrm{Herm(n)})^{m+1},\ \exists v_0, \ldots, v_m \in \mathbb{C}^n, \\ v_\lambda := \textstyle\sum_{j=0}^{m} \lambda^j v_j \neq 0,\ v_j = \Delta_j M v_\lambda,\ j = 0, \ldots, m \Big\}. \tag{4.1.5}$$

The following result provides an expression for structured eigenvalue backward error $\eta_\infty^{\mathrm{Herm}}(P, \lambda)$ of a Hermitian polynomial $P(z)$.

Theorem 4.1.2. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a Hermitian polynomial i.e., $A_j \in \mathrm{Herm(n)}$ for $j = 0, \ldots, m$. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be such that $\det(P(\lambda)) \neq 0$. Set $M = (P(\lambda))^{-1}$ and define $v_\lambda := \sum_{j=0}^{m} \lambda^j v_j$ for $(v_0, \ldots, v_m) \in (\mathbb{C}^n)^{m+1}$. Then*

$$\left(\eta_\infty^{\mathrm{Herm}}(P, \lambda)\right)^2 = \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} \ \sup_{\gamma_0^2+\cdots+\gamma_m^2=1} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m), \tag{4.1.6}$$

where

$$\mathcal{K} = \left\{ (v_0, \ldots, v_m) \in (\mathbb{C}^n)^{m+1} \,\middle|\, v_\lambda \neq 0,\ \mathrm{Im}\, v_j^*(M v_\lambda) = 0,\ j = 0, \ldots, m \right\} \tag{4.1.7}$$

and for $\gamma_0, \ldots, \gamma_m \in \mathbb{R}$, $g_{\gamma_0,\ldots,\gamma_m} : \mathcal{K} \to \mathbb{R}$ is defined by

$$g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) = \frac{\gamma_0^2 \|v_0\|^2 + \cdots + \gamma_m^2 \|v_m\|^2}{\|Mv_\lambda\|^2}. \tag{4.1.8}$$

Proof. By Theorem 1.2.9, for $x(\neq 0), y \in \mathbb{C}^n$ there exist $\Delta \in \mathrm{Herm(n)}$ such that $\Delta x = y$ if and only if $\operatorname{Im} x^* y = 0$. Furthermore minimal 2-norm of such a Δ is $\|\Delta\| = \frac{\|y\|}{\|x\|}$. Therefore in the view of (4.1.5)

$$\left(\eta_\infty^{\mathrm{Herm}}(P, \lambda)\right)^2 = \inf \left\{ f(v_0, \ldots, v_m) \;\middle|\; (v_0, \ldots, v_m) \in \mathcal{K} \right\}, \tag{4.1.9}$$

where $f(v_0, \ldots, v_m)$ is defined for all $(v_0, \ldots, v_m) \in (\mathbb{C}^n)^{m+1}$ with $v_\lambda \neq 0$ by

$$f(v_0, \ldots, v_m) = \max \left\{ \frac{\|v_0\|^2}{\|Mv_\lambda\|^2}, \ldots, \frac{\|v_m\|^2}{\|Mv_\lambda\|^2} \right\}. \tag{4.1.10}$$

Being the supremum of a continuous function on a compact set, $\sup\limits_{\gamma_0^2+\cdots+\gamma_m^2=1} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m)$ is a continuous function of $(v_0, \ldots, v_m)$. Also, for a fixed $(v_0, \ldots, v_m) \in \mathcal{K}$ and for any $\gamma_0, \ldots, \gamma_m \in \mathbb{R}$ with $\gamma_0^2 + \cdots + \gamma_m^2 = 1$, we have $g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) \leq f(v_0, \ldots, v_m)$. This implies that

$$\sup_{\gamma_0^2+\cdots+\gamma_m^2=1} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) \leq f(v_0, \ldots, v_m) \tag{4.1.11}$$

and

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} \sup_{\gamma_0^2+\cdots+\gamma_m^2=1} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) \leq \left(\eta_\infty^{\mathrm{Herm}}(P, \lambda)\right)^2 < \infty.$$

Therefore, by arguments similar to those in Lemma 3.1.4, the infimum is attained at some element of $\mathcal{K}$. Now if $\max\left\{\frac{\|v_0\|^2}{\|Mv_\lambda\|^2}, \ldots, \frac{\|v_m\|^2}{\|Mv_\lambda\|^2}\right\} = \frac{\|v_{j_0}\|^2}{\|Mv_\lambda\|^2}$ for some $j_0 \in \{0, \ldots, m\}$, then

$$\max\left\{\frac{\|v_0\|^2}{\|Mv_\lambda\|^2}, \ldots, \frac{\|v_m\|^2}{\|Mv_\lambda\|^2}\right\} = \sum_{j=0}^{m} \hat{\gamma}_j^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} = g_{\hat{\gamma}_0,\ldots,\hat{\gamma}_m}(v_0, \ldots, v_m), \tag{4.1.12}$$

where $\hat{\gamma}_j = 1$ if $j = j_0$ and $\hat{\gamma}_j = 0$, otherwise. Therefore from (4.1.11) and (4.1.12), we get

$$\sup_{\gamma_0^2+\cdots+\gamma_m^2=1} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) = f(v_0, \ldots, v_m) \text{ for each } (v_0, \ldots, v_m) \in \mathcal{K}.$$

This implies

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} \sup_{\gamma_0^2+\cdots+\gamma_m^2=1} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) = \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} f(v_0, \ldots, v_m) = \left(\eta_\infty^{\mathrm{Herm}}(P, \lambda)\right)^2. \qquad \square$$

Note that the backward error obtained in Theorem 4.1.2 is not easy to compute. Therefore the main aim of this section is to derive computable tight bounds for $\eta_\infty^{\mathrm{Herm}}(P,\lambda)$ and also give sufficient conditions for the bounds to be equal to the exact backward error. We first state a lemma which is used to prove the main result.

Lemma 4.1.3. *Let $M \in \mathbb{C}^{n\times n}$ be a nonsingular matrix and $\lambda \in \mathbb{C}$ such that $\operatorname{Im}\lambda \neq 0$. Let $\Lambda_m := [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$ and set*

$$G := (\Lambda_m^*\Lambda_m) \otimes (M^*M), \quad H_j := i((e_{j+1}\Lambda_m) \otimes M - (\Lambda_m^* e_{j+1}^*) \otimes M^*)$$

for $j = 0, \ldots, m$, where e_j denotes the j-th standard basis vector of $\mathbb{R}^{m+1}$. For nonzero real numbers $\gamma_0, \ldots, \gamma_m$ let

$$\Gamma := \operatorname{diag}\big(\gamma_0, \ldots, \gamma_m\big), \quad G(\Gamma) := \Gamma^{-1}G\Gamma^{-1} \quad \textit{and} \quad H_j(\Gamma) := \Gamma^{-1}H_j\Gamma^{-1}$$

for $j = 0, \ldots, m$. Then the following statements hold.

(1) *For each $\gamma_0, \ldots, \gamma_m \in \mathbb{R}\setminus\{0\}$, the function*

$$L_{\gamma_0,\ldots,\gamma_m}(t_0, \ldots, t_m) := \lambda_{\max}\big(G(\Gamma) + t_0\, H_0(\Gamma) + \cdots + t_m\, H_m(\Gamma)\big) \tag{4.1.13}$$

attains a global minimum at some $t_0, \ldots, t_m \in \mathbb{R}$.

(2) *Let $L_{\gamma_0,\ldots,\gamma_m}(t_0^\star, \ldots, t_m^\star) := \min_{t_0,\ldots,t_m\in\mathbb{R}} L_{\gamma_0,\ldots,\gamma_m}(t_0, \ldots, t_m)$ for some $t_0^\star, \ldots, t_m^\star \in \mathbb{R}$. If $L_{\gamma_0,\ldots,\gamma_m}(t_0^\star, \ldots, t_m^\star)$ is a simple eigenvalue of $G(\Gamma) + t_0^\star\, H_0(\Gamma) + \cdots + t_m^\star\, H_m(\Gamma)$ or $m = 1$, then*

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) = \Big(L_{\gamma_0,\ldots,\gamma_m}(t_0^\star, \ldots, t_m^\star)\Big)^{-1}, \tag{4.1.14}$$

where $g_{\gamma_0,\ldots,\gamma_m}$ and $\mathcal{K}$ are defined by (4.1.8) and (4.1.7), respectively.

Proof. The proof of (1) follows immediately from Theorem 1.2.18 if we can show that any linear combination $\alpha_0 H_0(\Gamma) + \cdots + \alpha_m H_m(\Gamma)$ for $(\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1}\setminus\{0\}$ is indefinite. Since H_j and $H_j(\Gamma)$, $j = 0, \ldots, m$ are congruent and the congruence transformation does not depend on j, by Sylvester's Law of Inertia, it is sufficient to show that $\alpha_0 H_0 + \cdots + \alpha_m H_m$ is indefinite for any $(\alpha_0, \ldots, \alpha_m) \in \mathbb{R}^{m+1}\setminus\{0\}$. But this can be shown on the lines of Theorem 2.2.2.

Arguments similar to those used in Lemma 3.1.4 imply that for each $(\gamma_0, \ldots, \gamma_m) \in \mathbb{R}^n\setminus\{0\}$,

$g_{\gamma_0,\ldots,\gamma_m}$ attains its infimum over $\mathcal{K}$. Also following the arguments on pages 2 and 3 in Chapter 2, we can write infimum of $g_{\gamma_0,\ldots,\gamma_m}$ over $\mathcal{K}$ as

$$\begin{aligned}&\inf\left\{\, g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m) \,|\, (v_0,\ldots,v_m)\in\mathcal{K}\right\}\\ &\qquad = \left(\sup\left\{ \frac{u^*(\Gamma\, G\,\Gamma)u}{u^*u} \,\middle|\, u\in(\mathbb{C}^n)^{m+1}\setminus\{0\},\ u^*(\Gamma H_j \Gamma)u = 0,\ j=0,\ldots,m \right\}\right)^{-1}.\end{aligned}$$

Now the proof of (2) follows immediately by Theorem 1.2.18. □

Theorem 4.1.4. *Let $P(z)=\sum_{j=0}^m z^j A_j$ be a Hermitian polynomial i.e., $A_j\in\mathrm{Herm(n)}$ for $j=0,\ldots,m$. Let $\lambda\in\mathbb{C}\setminus\mathbb{R}$ be such that $M=(P(\lambda))^{-1}$ exists. For $\gamma_0,\ldots,\gamma_m\in\mathbb{R}$, let $g_{\gamma_0,\ldots,\gamma_m}$ and $\mathcal{K}$ be as defined by (4.1.8) and (4.1.7) respectively. Then the following hold.*

(a)

$$\begin{aligned}(\eta_\infty(P,\lambda))^2 &\le \sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m) \le \left(\eta_\infty^{\mathrm{Herm}}(P,\lambda)\right)^2\\ &\le (m+1)\sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m). \qquad (4.1.15)\end{aligned}$$

(b) For $\gamma_0,\ldots,\gamma_m\in\mathbb{R}\setminus\{0\}$ and $\Gamma=\mathrm{diag}(\gamma_0,\ldots,\gamma_m)$, let $L_{\gamma_0,\ldots,\gamma_m}(t_0,\ldots,t_m)$ be defined by (4.1.13) for Hermitian matrices $G(\Gamma)$ and $H_j(\Gamma)$, $j=0,\ldots,m$ of Lemma 4.1.3. Suppose there exist $\gamma_0^\star,\ldots,\gamma_m^\star$ where $\gamma_i^\star$ are all nonzero such that ${\gamma_0^\star}^2+\cdots+{\gamma_m^\star}^2=1$ and

$$\inf_{v_0,\ldots,v_m\in\mathcal{K}} g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0,\ldots,v_m) = \sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{v_0,\ldots,v_m\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m).$$

Then

$$\lambda_{\max}^\star := \inf_{t_0,\ldots,t_m\in\mathbb{R}} L_{\gamma_0^\star,\ldots,\gamma_m^\star}(t_0,\ldots,t_m) \qquad (4.1.16)$$

is attained for some $t_0^\star,\ldots,t_m^\star\in\mathbb{R}$. If $m=1$ or $\lambda_{\max}^\star$ is a simple eigenvalue of $G(\Gamma^\star)+t_0H_0(\Gamma^\star)+\cdots+t_mH_m(\Gamma^\star)$ where $\Gamma^\star=\mathrm{diag}\left(\gamma_0^\star,\ldots,\gamma_m^\star\right)$, then

$$\begin{aligned}\eta_\infty(P,\lambda)\le\frac{1}{\sqrt{\lambda_{\max}^\star}}\le\eta_\infty^{\mathrm{Herm}}(P,\lambda) &\le \left(\inf_{t_0,\ldots,t_m\in\mathbb{R}} L_{1,\ldots,1}(t_0,\ldots,t_m)\right)^{-\frac{1}{2}}\\ &\le \sqrt{(m+1)}\frac{1}{\sqrt{\lambda_{\max}^\star}}. \qquad (4.1.17)\end{aligned}$$

(c) We have,

$$\eta_\infty^{\mathrm{Herm}}(P,\lambda) = \left(\inf_{v_0,\ldots,v_m\in\mathcal{K}} g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0,\ldots,v_m)\right)^{1/2} = \frac{1}{\sqrt{\lambda_{\max}^\star}}, \qquad (4.1.18)$$

if $m=1$ or $\lambda_{\max}^\star$ is a simple eigenvalue of $G(\Gamma^\star)+t_0H_0(\Gamma^\star)+\cdots+t_mH_m(\Gamma^\star)$, and either of the following statements hold.

(i) There exists a $(\hat v_0,\ldots,\hat v_m)\in\mathcal{K}$ with $\|\hat v_j\|=\|\hat v_m\|$ for all $j\in 0,\ldots,m-1$ such that

$$g_{\gamma_0^\star,\ldots,\gamma_m^\star}(\hat v_0,\ldots,\hat v_m)=\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0,\ldots,v_m).$$

(ii)
$$\sup_{\gamma_0^2+\ldots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)=\inf_{(v_0,\ldots,v_m)\in\mathcal{K}}\ \sup_{\gamma_0^2+\cdots+\gamma_m^2=1} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m).$$

Proof. By (4.1.9)

$$\left(\eta_\infty^{\mathrm{Herm}}(P,\lambda)\right)^2=\inf\left\{f(v_0,\ldots,v_m)\;\middle|\;(v_0,\ldots,v_m)\in\mathcal{K}\right\},\tag{4.1.19}$$

where $f(v_0,\ldots,v_m)$ is defined by (4.1.10) and $\mathcal{K}$ is defined by (4.1.7). Now by Lemma 4.1.1, for $\widetilde\gamma_j=\sqrt{\frac{|\lambda|^j}{1+|\lambda|+\cdots+|\lambda^m|}}$, $j\in\{0,\ldots,m\}$ we have

$$\begin{aligned}
(\eta_\infty(P,\lambda))^2 &= \inf\left\{g_{\widetilde\gamma_0,\ldots,\widetilde\gamma_m}(v_0,\ldots,v_m)\,|\,(v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1},\ v_\lambda\neq 0\right\}\\
&\le \inf\left\{g_{\widetilde\gamma_0,\ldots,\widetilde\gamma_m}(v_0,\ldots,v_m)\,|\,(v_0,\ldots,v_m)\in\mathcal{K}\right\}\\
&\le \sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m).
\end{aligned}\tag{4.1.20}$$

For all $(\gamma_0,\ldots,\gamma_m)\in\mathbb{R}^{m+1}$ with $\gamma_0^2+\cdots+\gamma_m^2=1$ and $(v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1}$ with $v_\lambda\neq 0$, we have

$$g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)\le f(v_0,\ldots,v_m)\le g_{1,\ldots,1}(v_0,\ldots,v_m).$$

Therefore

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)\le\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} f(v_0,\ldots,v_m)\le\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{1,\ldots,1}(v_0,\ldots,v_m)$$

for all $(\gamma_0,\ldots\gamma_m)\in\mathbb{R}^{m+1}$ with $\gamma_0^2+\cdots+\gamma_m^2=1$. Thus

$$\sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)\le\left(\eta_\infty^{\mathrm{Herm}}(P,\lambda)\right)^2\le\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{1,\ldots,1}(v_0,\ldots,v_m).\tag{4.1.21}$$

Also observe that

$$(m+1)\left(\sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)\right) = \sup_{\gamma_0^2+\cdots+\gamma_m^2=m+1}\ \inf_{(v_0,\ldots,v_1)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m),$$

which implies

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{1,\ldots,1}(v_0,\ldots,v_m) \le (m+1)\left(\sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)\right) \tag{4.1.22}$$

Thus (4.1.15) follows immediately from (4.1.20), (4.1.21) and (4.1.22). Note that $\inf\limits_{(v_0,\ldots,v_m\in\mathcal{K})} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)$ is a continuous function of γ_j for $j=0,\ldots,m$. Therefore its supremum over the compact set $\{(\gamma_0,\ldots,\gamma_m)\in\mathbb{R}^{m+1}\,|\,\gamma_0^2+\cdots+\gamma_m^2=1\}$ is attained by some $(\gamma_0^\star,\ldots,\gamma_m^\star)$. Thus with the assumption that $\gamma_j^\star$ are nonzero for all $j=0,\ldots,m$, inequality (4.1.17) follows immediately by using Lemma 4.1.3 in (4.1.21) and (4.1.22).

Again let $(\hat v_0,\ldots,\hat v_m)$ be a minimizer of $g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0,\ldots,v_m)$ over $\mathcal{K}$ such that for each $j\in\{0,\ldots,m\}$, $\|\hat v_j\|=\|\hat v_m\|$. Let $\mathcal{K}' := \{(v_0,\ldots,v_m)\in\mathcal{K}\,|\,\|\hat v_j\|=\|\hat v_m\|,\ j=0,\ldots,m-1\}$. Then

$$\begin{aligned}
\sup_{\gamma_0^2+\cdots+\gamma_m^2=1}\ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m) &= g_{\gamma_0^\star,\ldots,\gamma_m^\star}(\hat v_0,\ldots,\hat v_m) = \frac{\|\hat v_0\|^2}{\|M\hat v_\lambda\|^2}\\
&\ge \inf_{(v_0,\ldots,v_m)\in\mathcal{K}'} \max\left\{\frac{\|v_0\|^2}{\|Mv_\lambda\|^2},\ldots,\frac{\|v_m\|^2}{\|Mv_\lambda\|^2}\right\}\\
&\ge \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} \max\left\{\frac{\|v_0\|^2}{\|Mv_\lambda\|^2},\ldots,\frac{\|v_m\|^2}{\|Mv_\lambda\|^2}\right\}\\
&= \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} f(v_0,\ldots,v_m).
\end{aligned} \tag{4.1.23}$$

Therefore from (4.1.19), (4.1.21) and (4.1.23), we have

$$\left(\eta_\infty^{\mathrm{Herm}}(P,\lambda)\right)^2 = \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0,\ldots,v_m).$$

Hence (4.1.18) holds due to Lemma 4.1.3. ∎

Remark 4.1.5. The function $\inf\limits_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0,\ldots,v_m)$ is a continuous concave function of $(\gamma_0,\ldots,\gamma_m)\in\mathbb{R}^{m+1}$ and hence its supremum over the compact set

$$\{(\gamma_0,\ldots,\gamma_m)\in\mathbb{R}^{m+1}\,|\,\gamma_0^2+\cdots+\gamma_m^2=1\}$$

is attained for some $(\gamma_0^\star,\ldots,\gamma_m^\star)$. The assumption on $(\gamma_0^\star,\ldots,\gamma_m^\star)$ in the above theorem that $\gamma_j^\star$ are nonzero for all $j\in\{0,\ldots,m\}$, is not a very strong assumption. This is because if $\gamma_j^\star=0$ for some $j\in\{0,\ldots,m\}$, then for any given $\epsilon>0$ we can choose a point $(\hat{\gamma}_0, \ldots, \hat{\gamma}_m) \in \mathbb{R}^{m+1}$ close to $(\gamma_0^\star, \ldots, \gamma_m^\star)$ such that $\hat{\gamma}_j \neq 0$ for all $j \in \{0, \ldots, m\}$, $\hat{\gamma}_0^2 + \cdots + \hat{\gamma}_m^2 = 1$ and

$$\left| \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0, \ldots, v_m) - \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\hat{\gamma}_0,\ldots,\hat{\gamma}_m}(v_0, \ldots, v_m) \right| < \epsilon.$$

This is possible due to the continuous dependence of the function $\inf\limits_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m)$ on $(\gamma_0, \ldots, \gamma_m)$. As a result tight lower and upper bounds are always possible to compute in terms of $\inf\limits_{t_0,\ldots,t_m} \big(L_{\hat{\gamma}_0,\ldots,\hat{\gamma}_m}(t_0, \ldots, t_m)\big)$.

Remark 4.1.6. Although, in the above theorem we cannot claim that

$$\mu^\star := \sup_{\gamma_0^2+\cdots+\gamma_m^2=1} \ \inf_{v_0,\ldots,v_m\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m)$$

is attained for some $(\gamma_0^\star, \ldots, \gamma_m^\star) \in \mathbb{R}^{m+1}$ where $\gamma_j^\star$ are all nonzero, and at least one of the sufficient conditions for (4.1.18) to hold is always satisfied, numerical experiments suggest that this hold generically. We have observed in all numerical experiments that $\mu^\star$ is attained for some $(\gamma_0^\star, \ldots, \gamma_m^\star) \in \mathbb{R}^{m+1}$ where $\gamma_j^\star$ is very different from zero for each j. Also every minimizer $(\hat{v}_0, \ldots, \hat{v}_m)$ of $g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0, \ldots, v_m)$ over $\mathcal{K}$ satisfies $\|\hat{v}_j\| = \|\hat{v}_m\|$, for $j \in 0, \ldots, m-1$. Thus equality (4.1.18) is always satisfied in numerical experiments.

On the basis of numerical experiments we have the following conjecture.

Conjecture 4.1.7. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be a Hermitian polynomial i.e., $A_j \in \mathrm{Herm(n)}$ for $j = 0, \ldots, m$. Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Im} \lambda \neq 0$ and $\det(P(\lambda)) \neq 0$. There exist $\gamma_0^\star, \ldots, \gamma_m^\star \in \mathbb{R} \setminus \{0\}$ and $(\hat{v}_0, \ldots, \hat{v}_m) \in \mathcal{K}$ with $\|\hat{v}_j\| = \|\hat{v}_k\|$ for all $j, k \in \{0, \ldots, m\}$ such that*

$$g_{\gamma_0^\star,\ldots,\gamma_m^\star}(\hat{v}_0, \ldots, \hat{v}_m) = \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0^\star,\ldots,\gamma_m^\star}(v_0, \ldots, v_m) = \sup_{\gamma_0^2+\cdots+\gamma_m^2=1} \ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m),$$

where $g_{\gamma_0,\ldots,\gamma_m}$ and $\mathcal{K}$ are as defined in (4.1.8) *and* (4.1.7)*, respectively.*

Remark 4.1.8. To obtain an optimal Hermitian perturbation (provided Conjecture 4.1.7 is true) with norm equals to $\eta_\infty^{\mathrm{Herm}}(P, \lambda)$, we first compute an unit eigenvector u corresponding to $L_{\gamma_0^\star,\ldots,\gamma_m^\star}(t_0^\star, \ldots, t_m^\star) = \lambda_{\max}\big(G(\Gamma^\star) + t_0^\star H_0(\Gamma^\star) + \cdots + t_m^\star H_m(\Gamma^\star)\big)$ satisfying

$$u^* G(\Gamma^\star) u = L_{\gamma_0^\star,\ldots,\gamma_m^\star}(t_0^\star, \ldots, t_m^\star) \quad \text{and} \quad u^* H_i(\Gamma^\star) u = 0 \text{ for } i = 0, \ldots, m.$$

The vectors $(v_0, \ldots, v_m) \in \mathcal{K}$ with $\|v_j\| = \|v_k\|$, $j, k \in \{0, \ldots, m\}$ are obtained by using the relation $[v_0^T, \ \ldots \ , v_m^T]^T = \Gamma^{\star -1} u$, where $\Gamma^\star = \operatorname{diag}\{\gamma_0^\star, \ldots, \gamma_{\mathrm{m}}^\star\} \otimes I_n$. The coefficients

Δ_j satisfying $\Delta_j(M(v_0 + \lambda v_1 + \cdots + \lambda^m v_m)) = v_j$ for $j = 0, \ldots, m$, and

$$\left(\eta_\infty^{\mathrm{Herm}}(P, \lambda)\right)^2 = \max\{\|\Delta_0\|^2, \ldots, \|\Delta_m\|^2\} = \sum_{j=0}^{m} \gamma_j^{\star 2} \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} = \left(L_{\gamma_0^\star, \ldots, \gamma_m^\star}(t_0^\star, \ldots, t_m^\star)\right)^{-1}$$

may be constructed from Theorem 1.2.9.

Remark 4.1.9. In the view of the relations in Section 2.3, we can calculate the structured backward error $\eta_\infty^{\mathbb{S}}(P, \lambda)$ for $*$-alternating and skew-Hermitian polynomials by converting the polynomial into an equivalent Hermitian polynomial.

Remark 4.1.10. Note that zero weights in the weight vector w can be allowed to restrict the perturbations set. Thus counterparts of Theorem 4.1.4 and Conjecture 4.1.7, for $\eta_{\widehat{w},\infty}^{\mathrm{Herm}}(P, \lambda)$ with a restricted perturbation set can be obtained by following the strategy used in Section 2.4.

4.1.2 Numerical Experiments

In this section we present some numerical examples to illustrate the tightness of bounds obtained in Theorem 4.1.4. As mentioned in Remark 4.1.6, Conjecture 4.1.7 is satisfied in all our numerical experiments. This allows us to compute $\eta_\infty^{\mathrm{Herm}}(P, \lambda)$.

Example 4.1.11. Let $Q(z) = A + zB + z^2C$ be a random Hermitian quadratic polynomial of size 3×3 with eigenvalues 4.2442, $0.7905 \pm 0.6383i$, $-1.3319 \pm 0.6955i$, -0.4273. We recorded the structured as well as unstructured backward errors of $\lambda \in \mathbb{C} \setminus \mathbb{R}$ satisfying $\mathrm{Re}\,\lambda = -0.4273$ which is a simple eigenvalue of $Q(z)$. As expected the unstructured backward error tends to zero as imaginary part of λ tends to zero while structured backward error does not. This leads to large differences between the structured and the unstructured backward errors. This fact is depicted in Table- 4.1.1 for the Hermitian polynomial $Q(z)$.

The situation is different if the points $\lambda \in \mathbb{C} \setminus \mathbb{R}$ are chosen in such a way that they converge to a non-real eigenvalue instead of a real one. In that case both the structured and unstructured backward errors tend to zero as expected. The values are recorded in Table 4.1.2 for the same Hermitian quadratic polynomial $Q(z)$ and values λ converging to the eigenvalue $0.7905 - 0.6383i$. The Hermitian backward errors for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$ values that are not necessarily close to eigenvalues are recorded in Table 4.1.3. The latter values show that the difference between the structured and unstructured backward errors may be quite significant even if λ is not close to the real line. As illustrated in tables, note that

Table 4.1.1: *Structured and unstructured eigenvalue backward errors for Hermitian quadratic polynomial Q(z).*

λ	$\gamma_0^\star$	$\gamma_1^\star$	$\gamma_2^\star$	$\|\hat{v}_0\|$	$\|\hat{v}_1\|$	$\|\hat{v}_2\|$	$\eta_\infty(Q,\lambda)$	$\eta_\infty^{\mathrm{Herm}}(Q,\lambda)$
-0.4273+i	0.5502	0.6047	0.5759	1.298	1.298	1.298	1.1907	1.2979
-0.4273+.5i	0.6436	0.6048	0.4691	1.397	1.397	1.397	1.1308	1.3975
-0.4273+.1i	0.6785	0.5717	0.4614	1.355	1.355	1.355	0.2989	1.3555
-0.4273+.05i	0.6803	0.5697	0.4611	1.353	1.353	1.353	0.1510	1.3536
-0.4273+.01i	0.6810	0.5691	0.4609	1.353	1.353	1.353	0.0303	1.3530
-0.4273+.005i	0.6810	0.5691	0.4609	1.353	1.353	1.353	0.0151	1.3530

Conjecture 4.1.7 holds in every case so that $\eta_\infty^{\mathrm{Herm}}(Q,\lambda)$ can be computed due to Theorem 4.1.4.

Table 4.1.2: *Structured and unstructured eigenvalue backward errors for Hermitian quadratic polynomial $Q(z)$.*

λ	$\gamma_0^\star$	$\gamma_1^\star$	$\gamma_2^\star$	$\|\hat{v}_0\|$	$\|\hat{v}_1\|$	$\|\hat{v}_2\|$	$\eta_\infty(Q,\lambda)$	$\eta_\infty^{\mathrm{Herm}}(Q,\lambda)$
1.50-1.30i	0.4235	0.5517	0.7185	1.064	1.064	1.064	0.8805	1.0639
1.00-1.10i	0.4852	0.5722	0.6612	0.713	0.713	0.713	0.6523	0.7130
0.95-0.90i	0.5200	0.5728	0.6337	0.485	0.485	0.485	0.4491	0.4849
0.90-0.80i	0.5428	0.5733	0.6137	0.330	0.330	0.330	0.3067	0.3304
0.85-0.70i	0.5662	0.5708	0.5946	0.155	0.155	0.155	0.1438	0.1558
0.80-0.65i	0.5856	0.5718	0.5745	0.028	0.028	0.028	0.0263	0.0284

Table 4.1.3: *Structured and unstructured eigenvalue backward errors for Hermitian quadratic polynomial $Q(z)$.*

λ	$\gamma_0^\star$	$\gamma_1^\star$	$\gamma_2^\star$	$\|\hat{v}_0\|$	$\|\hat{v}_1\|$	$\|\hat{v}_2\|$	$\eta_\infty(Q,\lambda)$	$\eta_\infty^{\mathrm{Herm}}(Q,\lambda)$
0.1253+0.2877i	0.8142	0.5114	0.2750	1.764	1.764	1.764	1.6279	1.7647
0.3273+0.1746i	0.8109	0.4949	0.3124	1.672	1.672	1.672	1.5858	1.6722
0.2311+1.1909i	0.4994	0.6182	0.6070	1.109	1.109	1.109	0.9928	1.1097
0.4860-0.0376i	0.7843	0.5162	0.3441	1.556	1.556	1.556	1.4246	1.5559
0.3273+0.1746i	0.8109	0.4949	0.3124	1.672	1.672	1.672	1.5858	1.6722
-0.5883+2.1832i	0.3387	0.5511	0.7626	1.410	1.410	1.410	1.2876	1.4101

Example 4.1.12. Let $L(z) = A_0 + zA_1$ be the Hermitian pencil of Example 2.5.1 with eigenvalue $0.5766 \pm 1.0199i$, -1.0966 and -0.1019. The Hermitian eigenvalue backward error $\eta_\infty^{\mathrm{Herm}}(L,\lambda)$ of the point $-1.0966 + 0.5i$ is 0.9238 while its unstructured backward

error is 0.3341. Also the optimal $\gamma_0^\star = 0.6946$ and $\gamma_1^\star = 0.7194$ and vectors $(\hat{v}_0, \hat{v}_1) \in \mathcal{K}$ satisfy $\|\hat{v}_0\| = \|\hat{v}_1\| = 0.923$

Figure 4.1.1 shows the movement of eigenvalue curves originating from eigenvalues (marked with stars surrounded by circles) of $L(z)$ under homotopic perturbation $L(z) + t\Delta L(z)$ as t varies from 0 to 1. The figure on the left illustrates that point $-1.0966 + 0.5i$ (marked with a star surrounded by a diamond) and its complex conjugate become eigenvalues of $L(z) + \Delta L(z)$ by the splitting of a real eigenvalue of multiplicity 2 of $L(z) + t_0\Delta L(z)$ for some $0 < t_0 < 1$. Here $\Delta L(z) := \Delta_0 + z\Delta_1$ is the optimal Hermitian perturbation satisfying $\max\{\|\Delta_0\|, \|\Delta_1\|\} = 0.9238$ such that $-1.0966+0.5i$ is an eigenvalue of $L(z)+\Delta L(z)$.

The figure on the right illustrates the same effect with respect to unstructured perturbations. In this case the eigenvalue symmetry fails as the perturbed pencil is no longer Hermitian and the nearest eigenvalue moves towards the point $-1.0966 + 0.5i$.

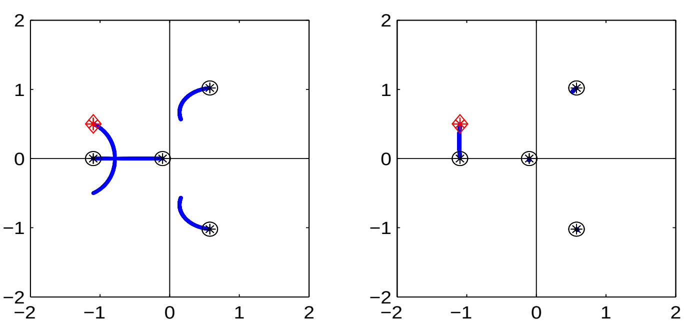

Figure 4.1.1: Eigenvalue perturbation curves for the Hermitian pencil $L(z)$ of Example 4.1.12 with respect to Hermitian perturbations (left) and arbitrary perturbations (right)..

Example 4.1.13. Let $Q(z) = A_0 + zA_1 + z^2A_2$ be the Hermitian quadratic polynomial of Example 2.5.3 with eigenvalues $-0.8738 \pm 2.4984i$, $0.3091 \pm 1.2260i$, 0.6280 and 0.0780. The Hermitian backward error $\eta_\infty^{\mathrm{Herm}}(Q, \lambda)$ of the point $0.6280 + 0.5i$ is 1.1622 while its unstructured backward error is 0.7788. Also the optimal $\gamma_0^\star = 0.6823$, $\gamma_1^\star = 0.5249$ and $\gamma_2^\star = 0.5090$ and vectors $(\hat{v}_0, \hat{v}_1, \hat{v}_2) \in \mathcal{K}$ satisfy $\|\hat{v}_0\| = \|\hat{v}_1\| = \|\hat{v}_2\| = 1.162$.

Figure 4.1.2 illustrates the movement of eigenvalues (marked with stars surrounded by circles) of $Q(z)$ under homotopic perturbations $Q(z) + t\Delta Q(z)$ as t varies from 0 to 1. In the figure on the left, eigenvalue curves originating from eigenvalues 0.6280 and 0.0780 meet on the real line and split into two curves to make the point $0.6280 + 0.5i$ (marked with a star surrounded by a diamond) and its complex conjugate as eigenvalues of $Q(z) + \Delta Q(z)$. Here, $\Delta Q(z) = \Delta_0 + z\Delta_1 + z^2\Delta_2$ is the optimal Hermitian perturbation satisfying $\max\{\|\Delta_0\|, \|\Delta_2\|, \|\Delta_2\|\} = 1.1622$ such that $0.6280 + 0.5i$ is an eigenvalue of $Q(z) + \Delta Q(z)$.

The right hand side figure traces the same effect with respect to unstructured perturbations. Again, in this case the eigenvalue symmetry degenerates and the nearest eigenvalue moves towards the point $0.6280 + 0.5i$.

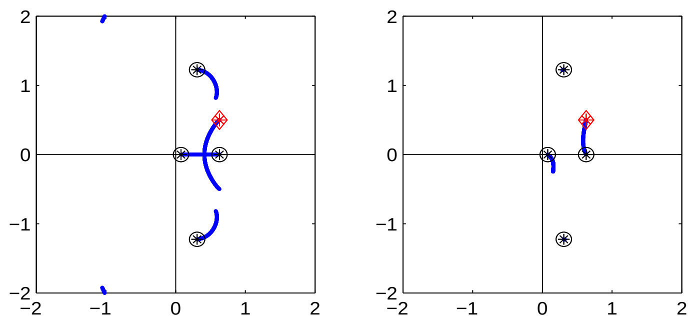

Figure 4.1.2: Eigenvalue perturbation curves for the Hermitian quadratic polynomial $Q(z)$ of Example 4.1.13 with respect to Hermitian perturbations (left) and arbitrary perturbations (right).

4.1.3 Palindromic polynomials

In this section, we extend the ideas of the previous section to derive tight bounds for the structured eigenvalue backward error of $*$-palindromic polynomials with respect to $|||\cdot|||_\infty$ norm. Similar results are also obtained for T-palindromic polynomials of degree at most 2.

Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a $\bullet$-palindromic polynomial and $\lambda \in \mathbb{C} \setminus \{0\}$. For this case $\mathbb{S} = \mathrm{pal}_\bullet$, $\bullet \in \{*, T\}$, where $\mathrm{pal}_\bullet$ is defined by

$$\mathrm{pal}_\bullet := \left\{(\Delta_0, \ldots, \Delta_m) \in (\mathbb{C}^{n\times n})^{m+1} \;\middle|\; \Delta_{m-j} = \Delta_j^\bullet, \text{ for } j = 0, \ldots, m\right\}.$$

Set $k := \lfloor \frac{m-1}{2} \rfloor$. Following the notations from Chapter 3 we denote the palindromic backward error with respect to norm $|||\cdot|||_\infty$ by $\eta_\infty^{\mathrm{pal}_\bullet}(P, \lambda)$, i.e.,

$$\eta_\infty^{\mathrm{pal}_\bullet}(P, \lambda) = \inf\left\{ \max\{\|\Delta_0\|, \ldots, \|\Delta_m\|\} \;\middle|\; (\Delta_0, \ldots, \Delta_m) \in \mathrm{pal}_\bullet, \Delta P(z) = \sum_{j=0}^{m} z^j \Delta_j\,, \right.$$
$$\left. \det\big(P(\lambda) - \Delta P(\lambda)\big) = 0 \right\}.$$

Note that there is no difference between structured and unstructured eigenvalue backward errors of a $*$-palindromic polynomial if $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$. The same is true for T-palindromic polynomials when $\lambda = \pm 1$. This fact is shown in [42]. The situation is completely different in other cases.

In the view of Lemma 1.2.6, Theorem 3.1.3 and Theorem 1.2.9,

$$\left(\eta_\infty^{\mathrm{pal}_\bullet}(P,\lambda)\right)^2 = \inf\left\{ f(v_0,\ldots,v_m) \,\middle|\, (v_0,\ldots,v_m) \in \mathcal{M} \right\}, \tag{4.1.24}$$

where $f(v_0,\ldots,v_m)$ for all $(v_0,\ldots,v_m)$ with $v_\lambda := \sum_{j=0}^m \lambda^j v_j \neq 0$ is defined by

$$f(v_0,\ldots,v_m) = \max\left\{ \frac{\|v_0\|^2}{\|Mv_\lambda\|^2}, \ldots, \frac{\|v_m\|^2}{\|Mv_\lambda\|^2} \right\} \tag{4.1.25}$$

and $\mathcal{M} \subseteq (\mathbb{C}^n)^{m+1}$ is given by

$$\mathcal{M} = \left\{ (v_0,\ldots,v_m) \,\middle|\, v_\lambda \neq 0,\ (Mv_\lambda)^\bullet v_j = v_{m-j}^\bullet (Mv_\lambda) \text{ for } j = 0,\ldots,k \right\} \tag{4.1.26}$$

if either $\bullet = T$ or m is odd and $\bullet = *$ and by

$$\mathcal{M} = \left\{ (v_0,\ldots,v_m) \,\middle|\, v_\lambda \neq 0,\ (Mv_\lambda)^* v_{\frac{m}{2}} \in \mathbb{R},\ (Mv_\lambda)^* v_j = v_{m-j}^* (Mv_\lambda) \text{ for } j = 0,\ldots,k \right\} \tag{4.1.27}$$

otherwise (i.e., if m is even and $\bullet = *$).

Note that for palindromic structures, the pencil case $m = 1$ differs from the polynomial case $m > 1$. It is very clear by the definition of norms that the structured eigenvalue backward error for $\bullet$-palindromic pencil $L(z) = A + zA^\bullet$, $\bullet \in \{*, T\}$ with respect to norm $\max\{\|A\|, \|A^\bullet\|\}$ is $\frac{1}{\sqrt{2}}$ times of the structured eigenvalue backward error with respect to norm $\sqrt{\|A\|^2 + \|A^\bullet\|^2}$. More precisely, we have the following result for structured eigenvalue backward errors of palindromic pencils.

Theorem 4.1.14. *Let $P(z) = A + zA^\bullet$ be $\bullet$-palindromic pencil, where $\bullet \in \{*, T\}$ and $A \in \mathbb{C}^{n\times n}$. Let $\lambda \in \mathbb{C}\setminus\{0\}$ be such that $\det(P(\lambda)) \neq 0$. Then*

$$\eta_\infty^{\mathrm{pal}_\bullet}(P,\lambda) = \frac{1}{\sqrt{2}}\, \eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda),$$

where $\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda)$ is the structured eigenvalue backward error of λ with respect to weight vector $w = (1,1)$ and the $|||\cdot|||_2$ norm.

Therefore for $\bullet$-palindromic pencils, $\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda)$ is obtained from Theorem 3.1.8 when $\bullet = *$ and from Theorem 3.1.10 when $\bullet = T$.

In the following we obtain an analogue to Theorem 4.1.2 that provides an expression for $\eta_\infty^{\mathrm{pal}_\bullet}(P,\lambda)$.

Theorem 4.1.15. *Let $P(z)=\sum_{j=0}^{m} z^j A_j$ be a $\bullet$-palindromic polynomial where $\bullet \in \{*, T\}$ and $(A_0,\ldots,A_m) \in \mathrm{pal}_\bullet$. Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ be such that $M=(P(\lambda))^{-1}$ exists. Then*

$$\left(\eta_\infty^{\mathrm{pal}_\bullet}(P,\lambda)\right)^2 = \inf_{(v_0,\ldots,v_m)\in\mathcal{M}} \sup_{\epsilon_0^2+\cdots+\epsilon_m^2=1} g_{\epsilon_0,\ldots,\epsilon_m}(v_0,\ldots,v_m), \tag{4.1.28}$$

where $\mathcal{M}$ is as defined in (4.1.26)-(4.1.27) and $g_{\epsilon_0,\ldots,\epsilon_m}$ is defined by

$$g_{\epsilon_0,\ldots,\epsilon_m}(v_0,\ldots,v_m) := \sum_{j=0}^{m} \epsilon_j^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} \tag{4.1.29}$$

for all $(v_0,\ldots,v_m)\in(\mathbb{C}^n)^{m+1}$ such that $v_\lambda := \sum_{j=0}^{m}\lambda^j v_j \neq 0$.

Proof. The proof follows on the lines of Theorem 4.1.2 by replacing $\mathcal{K}$ with $\mathcal{M}$. $\square$

To estimate $\eta_\infty^{\mathrm{pal}_*}(P,\lambda)$ for higher degree $*$-palindromic polynomials, we first recall a few things from Chapter 3 and redefine them here. Set $\Lambda_m := [1,\lambda,\ldots,\lambda^m] \in \mathbb{C}^{1\times(m+1)}$ and $k := \lfloor \frac{m-1}{2} \rfloor$. For $j=0,\ldots,k$ define

$$\begin{aligned}
\widetilde{C}_j &= (\Lambda_m^* e_{j+1}^*)\otimes M^* - (e_{m-j+1}\Lambda_m)\otimes M,\\
\widetilde{C}_{\frac{m}{2}} &= i\big((\Lambda_m^* e_{\frac{m}{2}+1}^*)\otimes M^* - (e_{\frac{m}{2}+1}\Lambda_m)\otimes M\big),
\end{aligned}$$

where e_j be the j-th standard basis vector of $\mathbb{R}^{m+1}$. Let $\epsilon_0,\ldots,\epsilon_m \in \mathbb{R}\setminus\{0\}$, define

$$\widetilde{G} := (\Lambda_m^*\Lambda_m)\otimes(M^*M) \quad \text{and} \quad \Upsilon := \mathrm{diag}(\epsilon_0,\ldots,\epsilon_m).$$

Also for each $j=0,\ldots,k$, define

$$G(\Upsilon) := \Upsilon^{-1}\widetilde{G}\Upsilon^{-1}, \tag{4.1.30}$$

$$H_j(\Upsilon) := \Upsilon^{-1}(\widetilde{C}_j+\widetilde{C}_j^*)\Upsilon^{-1}, \tag{4.1.31}$$

$$H_{m-j}(\Upsilon) := i\Upsilon^{-1}(\widetilde{C}_j-\widetilde{C}_j^*)\Upsilon^{-1}, \tag{4.1.32}$$

$$H_{\frac{m}{2}}(\Upsilon) := \Upsilon^{-1}\widetilde{C}_{\frac{m}{2}}\Upsilon^{-1}, \tag{4.1.33}$$

The following is a analogous to Lemma 4.1.3 for $*$-palindromic polynomials.

Lemma 4.1.16. *Let $M\in\mathbb{C}^{n\times n}$ be a nonsingular matrix and $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Then, the following statements hold.*

(1) *For each $\epsilon_0,\ldots,\epsilon_m\in\mathbb{R}\setminus\{0\}$, the function*

$$L_{\epsilon_0,\ldots,\epsilon_m}(t_0,\ldots,t_m) := \lambda_{\max}\big(G(\Upsilon)+t_0 H_0(\Upsilon)+\cdots+t_m H_m(\Upsilon)\big) \tag{4.1.34}$$

attains a global minimum at some $t_0,\ldots,t_m\in\mathbb{R}$, where $\Upsilon=\mathrm{diag}(\epsilon_0,\ldots,\epsilon_m)$, and $G(\Upsilon)$ and $H_j(\Upsilon)$ for $j=0,\ldots,m$ are as defined in (4.1.30)- (4.1.33).

(2) *For* $\epsilon_0, \ldots, \epsilon_m \in \mathbb{R} \setminus \{0\}$, *let* $L_{\epsilon_0,\ldots,\epsilon_m}(t_0^\star, \ldots, t_m^\star) := \min_{t_0,\ldots,t_m \in \mathbb{R}} L_{\epsilon_0,\ldots,\epsilon_m}(t_0, \ldots, t_m)$. *If* $L_{\epsilon_0,\ldots,\epsilon_m}(t_0^\star, \ldots, t_m^\star)$ *is a simple eigenvalue of* $G(\Upsilon) + t_0^\star H_0(\Upsilon) + \cdots + t_m^\star H_m(\Upsilon)$ *or* $m = 1$, *then*

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{M}} g_{\epsilon_0,\ldots,\epsilon_m}(v_0, \ldots, v_m) = \Big(L_{\epsilon_0,\ldots,\epsilon_m}(t_0^\star, \ldots, t_m^\star)\Big)^{-1}, \tag{4.1.35}$$

where $\mathcal{M}$ *is defined by (4.1.26)-(4.1.27) and* $g_{\epsilon_0,\ldots,\epsilon_m}$ *is defined by* (4.1.29).

Proof. The proof follows by applying Theorem 3.1.6 with arguments similar to those in Lemma 4.1.3. ◻

Theorem 4.1.17. *Let* $P(z) = \sum_{j=0}^m z^j A_j$ *be* $*$*-palindromic i.e.,* $(A_0, \ldots, A_m) \in \mathrm{pal}_*$. *Let* $\lambda \in \mathbb{C} \setminus \{0\}$ *be such that* $|\lambda| \neq 1$ *and* $M = (P(\lambda))^{-1}$ *exists. Then*

$$\begin{aligned} (\eta_\infty(P,\lambda))^2 &\leq \sup_{\epsilon_0^2+\cdots+\epsilon_m^2=1} \inf_{(v_0,\ldots,v_m)\in\mathcal{M}} g_{\epsilon_0,\ldots,\epsilon_m}(v_0, \ldots, v_m) \leq \Big(\eta_\infty^{\mathrm{pal}_*}(P,\lambda)\Big)^2 \\ &\leq (m+1) \sup_{\epsilon_0^2+\cdots+\epsilon_m^2=1} \inf_{(v_0,\ldots,v_m)\in\mathcal{M}} g_{\epsilon_0,\ldots,\epsilon_m}(v_0, \ldots, v_m), \end{aligned} \tag{4.1.36}$$

where $\mathcal{M}$ *is defined by (4.1.26)-(4.1.27) and* $g_{\epsilon_0,\ldots,\epsilon_m}$ *is defined by* (4.1.29). *Furthermore the following hold.*

(1) *Let* $L_{\epsilon_0,\ldots,\epsilon_m}(t_0, \ldots, t_m)$ *be as defined in (4.1.13) for* $\Upsilon = \mathrm{diag}\big(\epsilon_0, \ldots, \epsilon_m\big)$, *and Hermitian matrices* $G(\Upsilon)$ *and* $H_j(\Upsilon)$, $j = 0, \ldots, m$ *of Lemma 4.1.16. Suppose* $\exists\ \epsilon_0^\star, \ldots, \epsilon_m^\star$ *where* $\epsilon_i^\star$ *are all nonzero such that* ${\epsilon_0^\star}^2 + \cdots + {\epsilon_m^\star}^2 = 1$ *and*

$$\inf_{v_0,\ldots,v_m\in\mathcal{M}} g_{\epsilon_0^\star,\ldots,\epsilon_m^\star}(v_0, \ldots, v_m) = \sup_{\epsilon_0^2+\cdots+\epsilon_m^2=1} \inf_{v_0,\ldots,v_m\in\mathcal{M}} g_{\epsilon_0,\ldots,\epsilon_m}(v_0, \ldots, v_m).$$

Then

$$\lambda_{\max}^\star := \inf_{t_0,\ldots,t_m\in\mathbb{R}} L_{\epsilon_0^\star,\ldots,\epsilon_m^\star}(t_0, \ldots, t_m) \tag{4.1.37}$$

is attained for some $t_0^\star, \ldots, t_m^\star \in \mathbb{R}$. *If* $m = 1$ *or* $\lambda_{\max}^\star$ *is a simple eigenvalue of* $G(\Upsilon^\star) + t_0 H_0(\Upsilon^\star) + \cdots + t_m H_m(\Upsilon^\star)$, *then*

$$\begin{aligned} \eta_\infty(P,\lambda) \leq \frac{1}{\sqrt{\lambda_{\max}^\star}} \leq \eta_\infty^{\mathrm{pal}_*}(P,\lambda) &\leq \left(\inf_{t_0,\ldots,t_m\in\mathbb{R}} L_{1,\ldots,1}(t_0, \ldots, t_m)\right)^{-\frac{1}{2}} \\ &\leq \sqrt{(m+1)}\frac{1}{\sqrt{\lambda_{\max}^\star}}. \end{aligned} \tag{4.1.38}$$

(2) We have

$$\eta_\infty^{\mathrm{pal}_*}(P,\lambda) = \left(\inf_{v_0,\ldots,v_m \in \mathcal{M}} g_{\epsilon_0^\star,\ldots,\epsilon_m^\star}(v_0,\ldots,v_m) \right)^{1/2} = \frac{1}{\sqrt{\lambda_{\max}^\star}}, \qquad (4.1.39)$$

if $m=1$ or $\lambda_{\max}^\star$ is a simple eigenvalue of $G(\Upsilon^\star)+t_0H_0(\Upsilon^\star)+\cdots+t_mH_m(\Upsilon^\star)$, and either of the following statements hold.

(a) There exists a $(\hat{v}_0,\ldots,\hat{v}_m)\in\mathcal{M}$ with $\|\hat{v}_j\|=\|\hat{v}_m\|$ for all $j\in 0,\ldots,m-1$ such that

$$g_{\epsilon_0^\star,\ldots,\epsilon_m^\star}(\hat{v}_0,\ldots,\hat{v}_m) = \inf_{(v_0,\ldots,v_m)\in\mathcal{M}} g_{\epsilon_0^\star,\ldots,\epsilon_m^\star}(v_0,\ldots,v_m).$$

(b) $$\sup_{\epsilon_0^2+\ldots+\epsilon_m^2=1} \inf_{(v_0,\ldots,v_m)\in\mathcal{M}} g_{\epsilon_0,\ldots,\epsilon_m}(v_0,\ldots,v_m) = \inf_{(v_0,\ldots,v_m)\in\mathcal{M}} \sup_{\epsilon_0^2+\cdots+\epsilon_m^2=1} g_{\epsilon_0,\ldots,\epsilon_m}(v_0,\ldots,v_m).$$

Proof. The proof is similar to that of Theorem 4.1.4. □

Remark 4.1.18. The analogue of Remarks 4.1.5, 4.1.6, 4.1.8, 4.1.10 and Conjecture 4.1.7 hold for $*$-palindromic polynomials.

Note that sufficient condition (a) of Theorem 4.1.17 for (4.1.39) to hold is often satisfied. Thus we have the value of $\eta_\infty^{\mathrm{pal}_*}(P,\lambda)$ in our numerical experiments. Table 4.1.4 records the values of $\eta_\infty(P,\lambda)$ and $\eta_\infty^{\mathrm{pal}_*}(P,\lambda)$ for a random $*$-palindromic polynomial $P(z) = A_0+zA_1+z^2A_2$ of size 3×3 and random $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Note that in each case, the optimal $\epsilon_j^\star$ for each $j=0,1,2$ is nonzero and the optimal $\hat{v}_j$ for $j=0,1,2$ satisfy $\|\hat{v}_0\|=\|\hat{v}_1\|=\|\hat{v}_2\|$. This allows us to compute $\eta_\infty^{\mathrm{pal}_*}(P,\lambda)$ due to Theorem 4.1.17. Large differences are observed between the unstructured and the structured backward errors.

Table 4.1.4: *Structured and unstructured eigenvalue backward errors for $*$-palindromic quadratic polynomial $P(z)$.*

λ	$\epsilon_0^\star$	$\epsilon_1^\star$	$\epsilon_2^\star$	$\|\hat{v}_0\|$	$\|\hat{v}_1\|$	$\|\hat{v}_2\|$	$\eta_\infty(P,\lambda)$	$\eta_\infty^{\mathrm{pal}_*}(P,\lambda)$
0.5913−0.6436i	0.6323	0.5429	0.5527	0.786	0.786	0.786	0.1716	0.7865
0.2944−1.3362i	0.4957	0.5424	0.6783	1.079	1.079	1.079	0.7903	1.0791
-0.6918+0.8580i	0.5503	0.5738	0.6066	0.587	0.587	0.587	.2034	0.5870
0.6500−0.7500i	0.5959	0.5434	0.5913	0.813	0.813	0.813	0.1014	0.8130
-0.7118+0.8080i	0.5571	0.5741	0.6000	0.607	0.607	0.607	0.2119	0.6077
0.2244−1.4062i	0.4831	0.5417	0.6878	1.090	1.090	1.090	0.8529	1.0909

In the view of Theorem 1.2.7, the following result estimates the backward error $\eta_\infty^{\mathrm{pal}_T}(P,\lambda)$ for T-palindromic quadratic polynomials.

Theorem 4.1.19. *Let $P(z) = A_0 + zA_1 + z^2 A_0^T$, $A_0, A_1 \in \mathbb{C}^{n\times n}$ be a T-palindromic polynomial and $\lambda \in C \setminus \{0\}$. Suppose that $M = (P(\lambda))^{-1}$ exists. Then*

$$\begin{aligned}(\eta_\infty(P,\lambda))^2 &\leq \sup_{\epsilon_0^2+\epsilon_1^2+\epsilon_2^2=1} \inf_{(v_0,v_1,v_2)\in\mathcal{M}} g_{\epsilon_0,\epsilon_1,\epsilon_2}(v_0,v_1,v_2) \leq \left(\eta_\infty^{\mathrm{pal_T}}(P,\lambda)\right)^2 \\ &\leq 3 \sup_{\epsilon_0^2+\epsilon_1^2+\epsilon_2^2=1} \inf_{(v_0,v_1,v_2)\in\mathcal{M}} g_{\epsilon_0,\epsilon_1,\epsilon_2}(v_0,v_1,v_2),\end{aligned}$$

where $\mathcal{M}$ is defined by (4.1.26) with $\bullet = T$ and $g_{\epsilon_0,\epsilon_1,\epsilon_2}$ is defined by (4.1.29) *with $m = 2$. Furthermore, the following hold.*

(1) Suppose that there exist $\epsilon_0^\star, \epsilon_1^\star, \epsilon_2^\star \in \mathbb{R}\setminus\{0\}$ such that ${\epsilon_0^\star}^2 + {\epsilon_1^\star}^2 + {\epsilon_2^\star}^2 = 1$ and

$$\inf_{(v_0,v_1,v_2)\in\mathcal{M}} g_{\epsilon_0^\star,\epsilon_1^\star,\epsilon_2^\star}(v_0,v_1,v_2) = \sup_{\epsilon_0^2+\epsilon_1^2+\epsilon_2^2=1} \inf_{(v_0,v_1,v_2)\in\mathcal{M}} g_{\epsilon_0,\epsilon_1,\epsilon_2}(v_0,v_1,v_2).$$

Let $\Upsilon^\star := \mathrm{diag}\,(\epsilon_0^\star, \epsilon_1^\star, \epsilon_2^\star)$, $\Lambda_2 := [1\ \lambda\ \lambda^2]$, $G(\Upsilon^\star) := \Upsilon^{\star-1}\big((\Lambda_2^*\Lambda_2)\otimes M^*M\big)\Upsilon^{\star-1}$ and

$$S_0(\Upsilon^\star) := \Upsilon^{\star-1}\big(\Lambda_2^T(e_1^T - e_3^T)\otimes M^T + (e_1-e_3)\Lambda_2\otimes M\big)\Upsilon^{\star-1}.$$

Then

$$\begin{aligned}\eta_\infty(P,\lambda) &\leq \left(\min_{0<t\leq t_1} \lambda_2\left(\begin{bmatrix} G(\Upsilon^\star) & tS_0(\Upsilon^\star) \\ t\overline{S_0}(\Upsilon^\star) & \overline{G}(\Upsilon^\star)\end{bmatrix}\right)\right)^{-\frac{1}{2}} \leq \eta_\infty^{\mathrm{pal_T}}(P,\lambda) \\ &\leq \sqrt{3}\left(\min_{0<t\leq t_1} \lambda_2\left(\begin{bmatrix} G(\Upsilon^\star) & tS_0(\Upsilon^\star) \\ t\overline{S_0}(\Upsilon^\star) & \overline{G}(\Upsilon^\star)\end{bmatrix}\right)\right)^{-\frac{1}{2}},\end{aligned}$$

where $t_1 = \dfrac{2\|G(\Upsilon^\star)\|}{\sigma_2(S_0(\Upsilon^\star))}$.

(2) We have

$$\eta_\infty^{\mathrm{pal_T}}(P,\lambda) = \left(\min_{0<t\leq t_1} \lambda_2\left(\begin{bmatrix} G(\Upsilon^\star) & tS_0(\Upsilon^\star) \\ t\overline{S_0}(\Upsilon^\star) & \overline{G}(\Upsilon^\star)\end{bmatrix}\right)\right)^{-\frac{1}{2}}, \qquad (4.1.40)$$

if either of the following statements hold.

(a) There exist a $(\hat{v}_0,\ \hat{v}_1,\ \hat{v}_2) \in \mathcal{M}$ with $\|\hat{v}_0\| = \|\hat{v}_1\| = \|\hat{v}_2\|$ such that

$$g_{\epsilon_0^\star,\epsilon_1^\star,\epsilon_2^\star}(\hat{v}_0,\hat{v}_1,\hat{v}_2) = \inf_{(v_0,v_1,v_2)} g_{\epsilon_0^\star,\epsilon_1^\star,\epsilon_2^\star}(v_0,v_1,v_2).$$

(b)

$$\sup_{\epsilon_0^2+\epsilon_1^2+\epsilon_2^2} \inf_{(v_0,v_1,v_2)} g_{\epsilon_0^\star,\epsilon_1^\star,\epsilon_2^\star}(\hat{v}_0,v_1,v_2) = \inf_{(v_0,v_1,v_2)} \sup_{\epsilon_0^2+\epsilon_1^2+\epsilon_2^2=1} g_{\epsilon_0^\star,\epsilon_1^\star,\epsilon_2^\star}(\hat{v}_0,v_1,v_2).$$

Proof. The proof follows by applying Theorem 1.2.7 with arguments similar to those in the proof of Theorem 4.1.4. $\square$

Just like the case of the $*$-palindromic polynomials, we have observed that the assumptions in Theorem 4.1.19 for (4.1.40) to hold are fulfilled in all numerical experiments. We have also observed that $\sup_{\gamma_0^2+\gamma_1^2+\gamma_2^2=1} \inf_{v_0,v_1,v_2\in\mathcal{M}} g_{\gamma_0,\gamma_1,\gamma_2}(v_0,v_1,v_2)$ is attained for some $(\gamma_0^\star,\gamma_1^\star,\gamma_2^\star)\in\mathbb{R}^3$ where $\gamma_j^\star$ is very different from zero for each $j=0,1,2$. Also every minimizer $(\hat{v}_0,\hat{v}_1,\hat{v}_2)$ of $g_{\gamma_0^\star,\gamma_1^\star,\gamma_2^\star}(v_0,v_1,v_2)$ over $\mathcal{M}$ satisfies $\|\hat{v}_0\| = \|\hat{v}_1\| = \|\hat{v}_2\|$. Therefore on the basis of numerical experiments we have the following conjecture.

Conjecture 4.1.20. *Let $P(z) = A_0 + zA_1 + z^2A_0^T$, $A_0, A_1 \in \mathbb{C}^{n\times n}$ be a T-palindromic polynomial. Let $\lambda \in C\setminus\{0\}$ and $M := (P(\lambda))^{-1}$. There exist $\gamma_0^\star,\gamma_1^\star,\gamma_2^\star \in \mathbb{R}\setminus\{0\}$ and $(\hat{v}_0,\hat{v}_1,\hat{v}_2)\in\mathcal{M}$ with $\|\hat{v}_0\| = \|\hat{v}_1\| = \|\hat{v}_2\|$ such that*

$$g_{\gamma_0^\star,\gamma_1^\star,\gamma_2^\star}(\hat{v}_0,\hat{v}_1,\hat{v}_2) = \inf_{(v_0,v_1,v_2)\in\mathcal{M}} g_{\gamma_0^\star,\gamma_1^\star,\gamma_2^\star}(v_0,v_1,v_2) = \sup_{\gamma_0^2+\gamma_1^2+\gamma_2^2=1} \inf_{(v_0,v_1,v_2)\in\mathcal{M}} g_{\gamma_0,\gamma_1,\gamma_2}(v_0,v_1,v_2)$$

where $\mathcal{M}$ is defined by (4.1.26) with $\bullet = T$ and $g_{\epsilon_0,\epsilon_1,\epsilon_2}$ is defined by (4.1.29) *with $m=2$.*

The above conjecture is supported by Table 4.1.5 which records the values of $\eta_\infty(P,\lambda)$ and $\eta_\infty^{\mathrm{pal_T}}(P,\lambda)$ for a random T-palindromic polynomial $P(z) = A_0 + zA_1 + z^2A_0^T$ of size 3×3 and $\lambda\in\mathbb{C}\setminus\mathbb{R}$. This shows that the unstructured and structured backward errors are significantly different.

Table 4.1.5: *Structured and unstructured eigenvalue backward errors for T-palindromic quadratic polynomial $P(z)$.*

λ	$\epsilon_0^\star$	$\epsilon_1^\star$	$\epsilon_2^\star$	$\|\hat{v}_0\|$	$\|\hat{v}_1\|$	$\|\hat{v}_2\|$	$\eta_\infty(P,\lambda)$	$\eta_\infty^{\mathrm{pal_T}}(P,\lambda)$
0.3252-0.7549i	0.6023	0.6263	0.4951	0.928	0.928	0.928	0.7752	0.9285
0.4252+0.8549i	0.5638	0.6264	0.5383	1.090	1.090	1.090	0.9093	1.0900
-0.2256+1.1174i	0.5368	0.5810	0.6118	1.541	1.541	1.541	1.4458	1.5416
0.5525+1.1006i	0.4922	0.6248	0.6061	0.862	0.862	0.862	0.7262	0.8626
-0.6156+0.7481i	0.5818	0.5863	0.5637	1.893	1.893	1.893	1.7555	1.8934
-0.1924+0.8886i	0.5989	0.5872	0.5445	1.618	1.618	1.618	1.4833	1.6182

Remark 4.1.21. The counterparts of Theorem 4.1.19 and Conjecture 4.1.20 also hold for T-antipalindromic pencils, T-odd pencils and T-even polynomials of degree at most 2. These allow us to estimate their corresponding structured eigenvalue backward errors with respect to $|\!|\!|\cdot|\!|\!|_\infty$.

4.2 Structured eigenvalue backward error with respect to $|||\cdot|||_{w,F}$ norm

In this section, we consider the problem of computing the structured eigenvalue backward error for matrix polynomials with respect to $|||\cdot|||_{w,F}$. Let $(\Delta_0, \ldots, \Delta_m) \in (\mathbb{C}^{n\times n})^{m+1}$. Recall that $|||(\Delta_0, \ldots, \Delta_m)|||_{w,F} := \left(w_0^2\|\Delta_0\|_F^2 + \cdots + w_m^2\|\Delta_m\|_F^2\right)^{1/2}$ defines a norm on the set $(\mathbb{C}^{n\times n})^{m+1}$. Here $\|.\|_F$ stands for the Frobenius norm on $\mathbb{C}^{n\times n}$ and $w = (w_0, \ldots, w_m)$ is a weight vector.

Let $\mathbb{S} \subseteq (\mathbb{C}^{n\times n})^{m+1}$, $P(z) = \sum_{j=0}^m z^j A_j$ be a matrix polynomial, where $(A_0, \ldots, A_m) \in \mathbb{S}$ and $\lambda \in \mathbb{C}\setminus\{0\}$. We denote the backward error of λ as an approximate eigenvalue of $P(z)$ under structure preserving perturbations with respect to norm $|||\cdot|||_F$ by $\eta^{\mathbb{S}}_{w,F}(P, \lambda)$, i.e.,

$$\eta^{\mathbb{S}}_{w,F}(P, \lambda) = \inf\left\{|||(\Delta_0, \ldots, \Delta_m)|||_{w,F} \,\middle|\, (\Delta_0, \ldots, \Delta_m) \in \mathbb{S},\ \det\Big(\sum_{j=0}^m \lambda^j (A_j - \Delta_j)\Big) = 0\right\}.$$

The strategy used to compute structured eigenvalue backward errors $\eta^{\mathbb{S}}_{w,2}(P, \lambda)$ of structured matrix polynomials with respect to $|||\cdot|||_{w,2}$ in Chapter 2 and Chapter 3 allows us to estimate $\eta^{\mathbb{S}}_{w,F}(P, \lambda)$ fairly tightly with respect to the $|||\cdot|||_{w,F}$ norm also.

The following result is a corollary of [33, Theorem 5.6] and is used to estimate eigenvalue backward errors of Hermitian polynomial eigenvalue problem.

Theorem 4.2.1. *[33] Let $x(\neq 0), y \in \mathbb{C}^n$ such that $x^*y \in \mathbb{R}$. Then*

$$\inf\left\{\|\Delta\|_F^2 \,|\, \Delta \in \mathrm{Herm}(n),\ \Delta x = y\right\} = 2\frac{\|y\|_2^2}{\|x\|_2^2} - \frac{|y^*x|^2}{\|x\|_2^4}.$$

Moreover, this infimum is attained by

$$\Delta_0 = \frac{yx^*}{x^*x} + \frac{xy^*}{x^*x}\left(I - \frac{xx^*}{x^*x}\right).$$

Theorem 4.2.2. *Let $\mathbb{S} = (\mathrm{Herm(n)})^{\mathrm{m+1}}$ or $\mathbb{S} = \mathrm{pal}_\bullet$, where $\bullet \in \{*, \mathrm{T}\}$. Suppose that $w \in \mathbb{R}^{m+1}$ is a weight vector which is palindromic when $\mathbb{S} = \mathrm{pal}_\bullet$. Let $P(z) = \sum_{j=0}^m z^j A_j$ with $(A_0, \ldots, A_m) \in \mathbb{S}$ and let $\lambda \in \mathbb{C}\setminus\{0\}$ such that $\det(P(\lambda)) \neq 0$. Then*

$$\eta^{\mathbb{S}}_{w,2}(P, \lambda) \leq \eta^{\mathbb{S}}_{w,F}(P, \lambda) \leq \sqrt{2}\,\eta^{\mathbb{S}}_{w,2}(P, \lambda). \tag{4.2.1}$$

Proof. We first prove the case when $\mathbb{S} = (\mathrm{Herm(n)})^{\mathrm{m+1}}$. Let $P(z) = \sum_{j=0}^m z^j A_j$ be a Hermitian polynomial and $\lambda \in \mathbb{C}\setminus\mathbb{R}$ such that $M = (P(\lambda))^{-1}$ exists. Let $v_\lambda := \sum_{j=0}^m \lambda^j v_j$, where $v_j \in \mathbb{C}^n$, $j = 0, \ldots, m$. By Lemma 1.2.6 and Theorem 4.2.1, we have

$$\left(\eta^{\mathrm{Herm}}_{w,F}(P, \lambda)\right)^2 = \inf\{f(v_0, \ldots, v_m) \,|\, (v_0, \ldots, v_m) \in \mathcal{K}\}, \tag{4.2.2}$$

where

$$\mathcal{K} := \left\{(v_0,\ldots,v_m) \in (\mathbb{C}^n)^{m+1} \,|\, v_\lambda \neq 0,\, (Mv_\lambda)^* v_j \in \mathbb{R},\, j = 0,\ldots,m\right\}$$

and $f : \mathcal{K} \to \mathbb{R}$ is given by

$$f(v_0,\ldots,v_m) := \sum_{j=0}^{m} w_j^2 \left(2\frac{\|v_j\|_2^2}{\|Mv_\lambda\|_2^2} - \frac{|v_j^*(Mv_\lambda)|^2}{\|Mv_\lambda\|_2^4}\right).$$

By the Cauchy Schwarz inequality

$$\frac{\|v_j\|_2^2}{\|Mv_\lambda\|_2^2} \leq 2\frac{\|v_j\|_2^2}{\|Mv_\lambda\|_2^2} - \frac{|v_j^*(Mv_\lambda)|^2}{\|Mv_\lambda\|_2^4}$$

for each $j = 0,\ldots,m$. This implies that,

$$\inf_{(v_0,\ldots,v_m)\in\mathcal{K}} \sum_{j=0}^{m} w_j^2 \frac{\|v_j\|_2^2}{\|Mv_\lambda\|_2^2} \leq \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} f(v_0,\ldots,v_m) \leq \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} 2\sum_{j=0}^{m} w_j^2 \frac{\|v_j\|_2^2}{\|Mv_\lambda\|_2^2}.$$

Thus, we have

$$\eta_{w,2}^{\mathrm{Herm}}(P,\lambda) \leq \eta_{w,F}^{\mathrm{Herm}}(P,\lambda) \leq \sqrt{2}\,\eta_{w,2}^{\mathrm{Herm}}(P,\lambda),$$

where the last inequality holds from Lemma 1.2.6 and Theorem 1.2.9.

Now assume that $\mathbb{S} = \mathrm{pal}_\bullet$ and $P(z) = \sum_{j=0}^{m} z^j A_j$ is $\bullet$-palindromic. Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ such that $M = (P(\lambda))^{-1}$ exists. We assume that $|\lambda| \neq 1$ when $\bullet = *$ and $\lambda \neq \pm 1$ when $\bullet = T$. Assume that m is odd (the proof when m is even follows similarly). Again by Lemma 1.2.6, we can write

$$\left(\eta_{w,F}^{\mathrm{pal}_\bullet}(P,\lambda)\right)^2 = \inf\left\{\sum_{j=0}^{m} w_j^2\|\Delta_j\|_F^2 \,\middle|\, (\Delta_0,\ldots,\Delta_m) \in \mathrm{pal}_\bullet, v_0,\ldots,v_m \in \mathbb{C}^n,\, v_\lambda \neq 0, \right.$$
$$\left. \Delta_j M v_\lambda = v_j,\, \Delta_j^\bullet M v_\lambda = v_{m-j}, \text{ for } j = 0,\ldots,k\right\} \qquad (4.2.3)$$

Now from Theorem 1.2.14, for each $j = 0,\ldots,k$ there exist Δ satisfying $\Delta M v_\lambda = v_j$ and $\Delta^\bullet M v_\lambda = v_{m-j}$ if and only if $(Mv_\lambda)^\bullet v_j = v_{m-j}^\bullet(Mv_\lambda)$. Also minimal Frobenius norm of such a $\widetilde{\Delta}$ is

$$\|\widetilde{\Delta}\|_F^2 = \frac{\|v_j\|^2}{\|Mv_\lambda\|^2} + \frac{\|v_{m-j}\|^2}{\|Mv_\lambda\|^2} - \frac{|v_{m-j}^\bullet(Mv_\lambda)|^2}{\|Mv_\lambda\|^2}.$$

Clearly by Theorem 1.2.14, we have

$$\begin{aligned} \|\widetilde{\Delta}\|_F &\leq \sqrt{2}\max\left\{\frac{\|v_j\|}{\|Mv_\lambda\|}, \frac{\|v_{m-j}\|}{\|Mv_\lambda\|}\right\} \\ &= \sqrt{2}\left\{\|\Delta\| \,\middle|\, \Delta \in \mathbb{C}^{n\times n},\ \Delta M v_\lambda = v_j,\, \Delta^\bullet M v_\lambda = v_{m-j}\right\} \qquad (4.2.4) \end{aligned}$$

Therefore from (4.2.3) and (4.2.4), we get

$$\left(\eta_{w,F}^{\mathrm{pal}_\bullet}(P,\lambda)\right)^2 \le \inf\left\{ \sum_{j=0}^{m} 2\,\|\Delta_j\|^2 \;\middle|\; \begin{array}{l} (\Delta_0,\ldots,\Delta_m)\in \mathrm{pal}_\bullet, v_0,\ldots,v_m\in\mathbb{C}^n,\, v_\lambda\neq 0, \\ \Delta_j M v_\lambda = v_j,\, \Delta_j^\bullet M v_\lambda = v_{m-j},\, \text{for}\, j=0,\ldots,k \end{array}\right\}.$$

Thus from Lemma 3.1.2, we have $\eta_{w,F}^{\mathrm{pal}_\bullet}(P,\lambda) \le \sqrt{2}\,\eta_{w,2}^{\mathrm{pal}_\bullet}(P,\lambda)$. Also the other inequality in (4.2.1) follows immediately by noticing that for any $\Delta\in\mathbb{C}^{n\times n}$, $\|\Delta\|\le\|\Delta\|_F$. $\square$

Chapter 5

Structured backward errors: real and other special structures

In this chapter, we study the eigenpair and eigenvalue backward errors of real regular matrix pencils and polynomials with respect to real perturbations. If such a pencil or polynomial has additional structure like symmetric, skew-symmetric, T-alternating, T-palindromic or T-antipalindromic, then we consider eigenvalue backward errors with respect to real perturbations that also preserve the additional structure. Finally we consider matrix pencils with special block structures that arise in optimal control and compute eigenvalue and eigenpair backward errors associated with points on the corresponding critical set with respect to structure preserving perturbations.

5.1 Backward errors with preserving real structures of matrix pencils

If the matrix pencil or polynomial is real, then algorithms that solve the associated eigenvalue problem while staying in real arithmetic are faster, require less memory and produce results that are meaningful for applications even in finite precision. Therefore the best numerical methods to solve real problems are the ones which are backward stable and preserve the real structure of the problem. However, very little is known about the backward errors of complex eigenvalues and eigenvectors of such problems with respect to perturbations that are real.

In this section, we undertake the problem of computing eigenvalue and eigenpair backward errors of real matrix pencils with respect to real perturbations. We first recall the definition of these backward errors from Chapter 1.

Definition 5.1.1. Let $P(z) = A_0 + zA_1 + \cdots + z^m A_m$ be a regular real matrix polynomial, i.e, $A_0, \ldots, A_m \in \mathbb{R}^{n\times n}$ and let $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n \setminus \{0\}$. Then

$$\eta^{\mathbb{R}}(P,\lambda,x) := \inf\left\{ |||(\Delta_0,\ldots,\Delta_m)||| \,\Big|\, \Big(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\Big)x = 0,\ (\Delta_0,\ldots,\Delta_m) \in (\mathbb{R}^{n\times n})^{m+1} \right\} \tag{5.1.1}$$

and

$$\eta^{\mathbb{R}}(P,\lambda) := \inf\left\{ |||(\Delta_0,\ldots,\Delta_m)||| \,\Big|\, \det\Big(\sum_{j=0}^{m} \lambda^j (A_j - \Delta_j)\Big) = 0,\ (\Delta_0,\ldots,\Delta_m) \in (\mathbb{R}^{n\times n})^{m+1} \right\}, \tag{5.1.2}$$

the real eigenpair backward error of (λ, x) and real eigenvalue backward error of λ, respectively with respect to some norm $|||\cdot|||$ on $(\mathbb{R}^{n\times n})^{m+1}$.

Note that the real eigenvalue and eigenpair backward errors are not known for matrix polynomials even for the pencil case. We obtain tight bounds for real eigenvalue and eigenpair backward errors of real matrix pencils with respect to certain norms.

5.1.1 Real eigenpair backward errors

In this section, we obtain tight bounds for real eigenpair backward errors of matrix pencils with respect to the norms $|||\cdot|||_{w,2}$, $|||\cdot|||_{w,F}$ and $|||\cdot|||_{\infty}$ defined in (1.2.1), (1.2.3) and (1.2.2), respectively. The weight vector is $w = (1, 1)$ in all cases, although the results may also be generalized to other weight vectors. The following lemma relates various norms on $\mathbb{C}^{n\times n}$.

Lemma 5.1.2. *Let $\Delta_0, \Delta_1, A_0, A_1 \in \mathbb{C}^{n\times n}$, $\mathbf{\Delta} := [\Delta_0\ \Delta_1]$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $x \in \mathbb{C}^n \setminus \{0\}$, then*

(1) $(A_0 - \Delta_0)x + \lambda(A_1 - \Delta_1)x = 0 \Longleftrightarrow \begin{bmatrix} \Delta_0 & \Delta_1 \end{bmatrix} \begin{bmatrix} x^T & \lambda x^T \end{bmatrix}^T = (A_0 + \lambda A_1)x,$

(2) $\|\mathbf{\Delta}\|_F = \sqrt{\|\Delta_0\|_F^2 + \|\Delta_1\|_F^2},$

(3) $\max\{\|\Delta_0\|, \|\Delta_1\|\} \le \|\mathbf{\Delta}\| \le \sqrt{\|\Delta_0\|^2 + \|\Delta_1\|^2},$

where $\|.\|$ and $\|.\|_F$ denote the matrix 2-norm and Frobenius norm, respectively.

Let $P(z) = A_0 + zA_1$ where $A_0, A_1 \in \mathbb{R}^{n\times n}$. If $\lambda \in \mathbb{R}$, then it is easy to see that $\eta^{\mathbb{R}}(P,\lambda,x) = \eta(P,\lambda,x)$ for each of the norms $|||\cdot|||_{w,2}$, $|||\cdot|||_{w,F}$ and $|||\cdot|||_{\infty}$. Therefore in the next result we consider $\eta^{\mathbb{R}}(P,\lambda,x)$ for $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$.

Theorem 5.1.3. *Let $P(z) = A_0 + zA_1,\ A_0, A_1 \in \mathbb{R}^{n\times n}$ be a real matrix pencil and let $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ with $\operatorname{Im}\lambda \neq 0$. Set $\hat{x} := \begin{bmatrix} x^T & \lambda x^T \end{bmatrix}^T$, $X := \begin{bmatrix} \operatorname{Re}\hat{x} & \operatorname{Im}\hat{x} \end{bmatrix}$, $R := \begin{bmatrix} \operatorname{Re} r & \operatorname{Im} r \end{bmatrix}$, where $r := P(\lambda)x$. Then*

(1) $\eta^{\mathbb{R}}_{w,F}(P, \lambda, x) = \|RX^\dagger\|_F$

(2) $\|RX^\dagger\|_2 \leq \eta^{\mathbb{R}}_{w,2}(P, \lambda, x) \leq \sqrt{2}\|RX^\dagger\|_2$

(3) $\frac{1}{\sqrt{2}}\|RX^\dagger\|_2 \leq \eta^{\mathbb{R}}_{\infty}(P, \lambda, x) \leq \|RX^\dagger\|_2,$

where $X^\dagger$ is the Moore-Penrose pseudoinverse of X, and $\eta^{\mathbb{R}}_{w,2}(P, \lambda, x)$, $\eta^{\mathbb{R}}_{w,F}(P, \lambda, x)$ and $\eta^{\mathbb{R}}_{\infty}(P, \lambda, x)$ are the real eigenpair backward errors with respect to norms $|\!|\!|\cdot|\!|\!|_{w,2}$, $|\!|\!|\cdot|\!|\!|_{w,F}$ (where $w = (1, 1)$) and $|\!|\!|\cdot|\!|\!|_{\infty}$, respectively.

Proof. Observe that

$$\left\{\begin{bmatrix} \Delta_0 & \Delta_1 \end{bmatrix} \,\middle|\, \Delta_0, \Delta_1 \in \mathbb{C}^{n\times n},\ \begin{bmatrix} \Delta_0 & \Delta_1 \end{bmatrix}\hat{x} = r\right\} = \left\{\Delta \,\middle|\, \Delta \in \mathbb{C}^{n\times 2n},\ \Delta\hat{x} = r\right\}. \quad (5.1.3)$$

Since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\operatorname{Re}\hat{x}$ and $\operatorname{Im}\hat{x}$ are linearly independent. Therefore $\operatorname{rank}(X) = 2$.
Proof of (1): In the view of part (1) of Lemma 5.1.2, we have

$$\begin{aligned}
\left(\eta^{\mathbb{R}}_{w,F}(P, \lambda, x)\right)^2 &= \inf\left\{\|\Delta_0\|_F^2 + \|\Delta_1\|_F^2 \,\middle|\, \big((A_0 - \Delta_0) + \lambda\,(A_1 - \Delta_1)\big)x = 0,\ \Delta_0, \Delta_1 \in \mathbb{R}^{n\times n}\right\} \\
&= \inf\left\{\left\|\begin{bmatrix} \Delta_0 & \Delta_1 \end{bmatrix}\right\|_F^2 \,\middle|\, \Delta_0,\ \Delta_1 \in \mathbb{R}^{n\times n},\ \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix}\hat{x} = r\right\} \\
&= \inf\left\{\|\Delta\|_F^2 \,\middle|\, \Delta \in \mathbb{R}^{n\times 2n},\ \Delta\hat{x} = r\right\} \qquad \text{(by 5.1.3)} \\
&= \|RX^\dagger\|_F^2
\end{aligned}$$

where the last equality follows by Theorem 1.2.11 as rank(X) = 2.
Proof of (2): Let $s := \inf\left\{\|\Delta\| \,\middle|\, \Delta \in \mathbb{R}^{n\times 2n},\ \Delta\hat{x} = r\right\}$. Now by definition

$$\left(\eta^{\mathbb{R}}_{w,2}(P, \lambda, x)\right)^2 = \inf\left\{\|\Delta_0\|^2 + \|\Delta_1\|^2 \,\middle|\, \big((A_0 - \Delta_0) + \lambda\,(A_1 - \Delta_1)\big)x = 0,\ \Delta_0, \Delta_1 \in \mathbb{R}^{n\times n}\right\}$$

By (5.1.3) and part (3) of Lemma 5.1.2, we have

$$s \leq \eta^{\mathbb{R}}_{w,2}(P, \lambda, x) \leq \sqrt{2}\, s$$

and since $\operatorname{rank}(X) = 2$ by Theorem 1.2.11

$$\|RX^\dagger\| \leq \eta^{\mathbb{R}}_{w,2}(P, \lambda, x) \leq \sqrt{2}\|RX^\dagger\|.$$

Proof of (3): Proof is similar to that of (2). ∎

The bounds on the real eigenpair backward errors in Theorem 5.1.3 may be calculated by using the following result.

Theorem 5.1.4. *Let* $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{R}^{n\times 2}$ *with* $\mathrm{rank(X)} = 2$, $B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \in \mathbb{R}^{n\times 2}$ *and let* $L = (B^TB)(X^TX)^{-1}$. *Then*

$$\|BX^\dagger\|_F^2 = \mathrm{trace(L)} \tag{5.1.4}$$

and

$$\|BX^\dagger\| = \frac{\sqrt{\mathrm{trace(L)} + 2\sqrt{\det(\mathrm{L})}} + \sqrt{\mathrm{trace(L)} - 2\sqrt{\det(\mathrm{L})}}}{2}, \tag{5.1.5}$$

where

$$\mathrm{trace(L)} = \frac{\|b_1\|^2\|x_2\|^2 + \|b_2\|^2\|x_1\|^2 - 2(b_1^Tb_2)(x_1^Tx_2)}{\|x_1\|^2\|x_2\|^2 - (x_1^Tx_2)^2}$$

and

$$\det(\mathrm{L}) = \frac{\|b_1\|^2\|b_2\|^2 - (b_1^Tb_2)^2}{\|x_1\|^2\|x_2\|^2 - (x_1^Tx_2)^2}.$$

Proof. Set $M = BX^\dagger$ then $\mathrm{rank(M)} \leq 2$. If $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the singular values of M, then $\sigma_k = 0$ for every $k > 2$. The squares of the singular values of M are the eigenvalues of M^TM. We can write

$$M^TM = (BX^\dagger)^T(BX^\dagger) = (X^\dagger)^TB^TBX^\dagger = CD,$$

where $C = (X^\dagger)^T$ and $D = B^TBX^\dagger$. As CD and

$$DC = B^TBX^\dagger(X^\dagger)^T = B^TB(X^TX)^\dagger = B^TB(X^TX)^{-1} = L$$

have the same nonzero eigenvalues, we have

$$\mathrm{trace(L)} = \sigma_1^2 + \sigma_2^2 \quad \text{and} \quad \det(\mathrm{L}) = \sigma_1^2\sigma_2^2. \tag{5.1.6}$$

Now (5.1.4) and (5.1.5) follow from the facts that $\|BX^\dagger\|_F^2 = \sigma_1^2 + \sigma_2^2$ and $\|BX^\dagger\| = \sigma_1$, respectively. $\square$

5.1.2 Real eigenvalue backward errors

In this section, we consider the real eigenvalue backward errors for real matrix pencils with respect to norm $|||\cdot|||_\infty$, i.e., for a real pencil $P(z) = A_0 + zA_1$, $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}$, calculate

$$\eta_\infty^{\mathbb{R}}(P,\lambda) := \inf\left\{\max\{\|\Delta_0\|, \|\Delta_1\|\} \,\middle|\, (\Delta_0,\Delta_1) \in (\mathbb{R}^{n\times n})^2,\ \det((A_0 - \Delta_0) + (A_1 - \Delta_1)) = 0\right\}.$$

Recall that the unstructured backward error is denoted by $\eta_\infty(P,\lambda)$ and defined as

$$\eta_\infty(P,\lambda) := \inf\left\{\max\{\|\Delta_0\|,\|\Delta_1\|\} \;\middle|\; (\Delta_0,\Delta_1)\in(\mathbb{C}^{n\times n})^2,\ \det((A_0-\Delta_0)+(A_1-\Delta_1))=0\right\}.$$

The following results from [38] will be required to estimate $\eta_\infty^{\mathbb{R}}(P,\lambda)$.

Theorem 5.1.5. *[38] Let $M\in\mathbb{C}^{n\times n}$. Then*

$$\begin{aligned}
\mu_{\mathbb{C}}(M) &:= \left(\inf\left\{\|\Delta\| \;\middle|\; \Delta\in\mathbb{C}^{n\times n},\ \det(I-\Delta M)=0\right\}\right)^{-1} = \sigma_{\max}(M) \quad \text{and}\\
\mu_{\mathbb{R}}(M) &:= \left(\inf\left\{\|\Delta\| \;\middle|\; \Delta\in\mathbb{R}^{n\times n},\ \det(I-\Delta M)=0\right\}\right)^{-1} = \inf_{\gamma\in(0,1]}\sigma_2(R(\gamma)),
\end{aligned}$$

where $R(\gamma) := \begin{bmatrix} \operatorname{Re} M & -\gamma\operatorname{Im} M\\ \gamma^{-1}\operatorname{Im} M & \operatorname{Re} M\end{bmatrix}$ and $\sigma_2(R(\gamma))$ is the second largest singular value of $R(\gamma)$.

Theorem 5.1.6. *[38] Let $M=\begin{bmatrix} M_1 & M_2\\ M_3 & M_4\end{bmatrix}\in\mathbb{C}^{2n\times 2n}$. Then*

$$\mu_D(M) := \left(\inf\left\{\|\Delta\| \;\middle|\; \Delta\in\operatorname{diag}_{\mathbb{C}}(2n),\ \det(I-\Delta M)=0\right\}\right)^{-1} = \inf_{\delta>0}\sigma_{\max}(M_\delta),$$

where $\operatorname{diag}_{\mathbb{C}}(2n) := \left\{\operatorname{diag}(\Delta_1,\Delta_2) \;\middle|\; \Delta_1,\ \Delta_2\in\mathbb{C}^{n\times n}\right\}$, $M_\delta := \begin{bmatrix} M_1 & \delta M_2\\ \delta^{-1}M_3 & M_4\end{bmatrix}$ and $\sigma_{\max}(M_\delta)$ is the largest singular value of M_δ.

We refer to $\mu_{\mathbb{C}}(M)$, $\mu_{\mathbb{R}}(M)$ and $\mu_D(M)$ as μ-values of M. Let

$$\operatorname{diag}_{\mathbb{R}}(2n) := \left\{\operatorname{diag}(\Delta_1,\Delta_2) \;\middle|\; \Delta_1,\ \Delta_2\in\mathbb{R}^{n\times n}\right\}$$

and

$$\mu_D^{\mathbb{R}}(M) := \left(\inf\left\{\|\Delta\| \;\middle|\; \Delta\in\operatorname{diag}_{\mathbb{R}}(2n),\ \det(I-\Delta M)=0\right\}\right)^{-1}.$$

In view of Lemma 1.2.6, the following corollary gives an equivalent characterization of $\eta_\infty^{\mathbb{R}}(P,\lambda)$ in terms of the μ-value $\mu_D^{\mathbb{R}}(\hat{M})$ of some matrix $\hat{M}$ associated with $P(z)$ and λ.

Corollary 5.1.7. *Let $P(z)=A_0+zA_1$ be a real matrix pencil where $A_0,A_1\in R^{n\times n}$ and let $\lambda\in\mathbb{C}$. Suppose that $\det(P(\lambda))\neq 0$, and let $M=(P(\lambda))^{-1}$. Then*

$$\eta_\infty^{\mathbb{R}}(P,\lambda) = \left(\mu_D^{\mathbb{R}}(\hat{M})\right)^{-1},$$

where $\hat{M} := \begin{bmatrix} M & \lambda M\\ M & \lambda M\end{bmatrix}$.

Proof. Let $\Delta_0,\, \Delta_1 \in \mathbb{R}^{n\times n}$ such that $\det\left(\Delta_0 + \lambda\Delta_1 - P(\lambda)\right) = 0$. From the proof of Lemma 1.2.6 there exist $v_0, v_1 \in \mathbb{C}^n$ with $v_0 + \lambda v_1 \neq 0$ such that $\Delta_0 M(v_0 + \lambda v_1) = v_0$ and $\Delta_1 M(v_0+\lambda v_1) = v_1$, which can be combined as $\begin{bmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{bmatrix} \begin{bmatrix} M & \lambda M \\ M & \lambda M \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$. This is equivalent to the fact that $\det(I - \hat{\Delta}\hat{M}) = 0$, where $\hat{\Delta} = \begin{bmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{bmatrix}$. Hence proof follows by definition of $\eta_\infty^{\mathbb{R}}(P,\lambda)$. ◻

The following result obtains real eigenvalue backward errors of a real matrix pencil $P(z) = A_0 + zA_1$ in terms of the real μ-value of some matrix $\hat{M}$ when only one of the coefficient matrices of the pencil $P(z)$ is perturbed.

Theorem 5.1.8. *Let $P(z) = A_0 + zA_1$ be a real matrix pencil ,i.e., $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\{0\}$. Suppose that $\det(P(\lambda)) \neq 0$ and $M = P(\lambda)^{-1}$. Then*

(a) $\eta^{\mathbb{R}}_{\infty,A_0}(P,\lambda) := \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathbb{R}^{n\times n},\ \det((A_0 - \Delta) + \lambda A_1) = 0\right\} = \left(\mu_{\mathbb{R}}(M)\right)^{-1} < \infty$,

(b) $\eta^{\mathbb{R}}_{\infty,A_1}(P,\lambda) := \inf\left\{\|\Delta\| \middle| \Delta \in \mathbb{R}^{n\times n},\ \det(A_0 + \lambda(A_1 - \Delta)) = 0\right\} = \left(\mu_{\mathbb{R}}(\lambda M)\right)^{-1} < \infty$,

where $\mu_{\mathbb{R}}(M)$ is as defined in Theorem 5.1.5.

Proof. The proof of (a) and (b) follow from Theorem 5.1.5 in the view of following observations. For any $\Delta \in \mathbb{R}^{n\times n}$,

$$\det((A_0 - \Delta) + \lambda A_1) = 0 \Longleftrightarrow \det(I - \Delta\, M) = 0,$$

and

$$\det(A_0 + \lambda(A_1 - \Delta)) = 0 \Longleftrightarrow \det(I - \Delta\, \lambda M) = 0.$$

◻

Given a real $n \times n$ pencil $P(z)$, computing the real eigenvalue backward error $\eta_\infty^{\mathbb{R}}(P,\lambda)$ of $\lambda \in \mathbb{C}$ is a difficult task when $\operatorname{Im}\lambda \neq 0$. We derive a lower bound of $\eta_\infty^{\mathbb{R}}(P,\lambda)$ which is equal to the backward error $\eta_\infty^{\mathbb{R}}(P,\lambda)$ under certain conditions. The following lemma will be used to derive this.

Lemma 5.1.9. *[38] Let $U \in \mathbb{R}^{p\times k}$ and $V \in \mathbb{R}^{m\times k}$. If $U^TU = V^TV \neq 0$ then*

$$\sigma_{\max}(VU^\dagger) = 1 \quad \text{and} \quad VU^\dagger U = V.$$

Theorem 5.1.10. *Let $P(z) = A_0 + zA_1$, $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\{0\}$. Suppose $M = (P(\lambda))^{-1}$ exists. For $\gamma > 0$ and $\delta \in \mathbb{C}\setminus\{0\}$, define*

$$R(\gamma,\delta) := \begin{bmatrix} \operatorname{Re} M_\delta & -\gamma \operatorname{Im} M_\delta \\ \gamma^{-1}\operatorname{Im} M_\delta & \operatorname{Re} M_\delta \end{bmatrix} \quad \text{where} \quad M_\delta := \begin{bmatrix} M & \delta\lambda M \\ \delta^{-1}M & \lambda M \end{bmatrix}. \quad \textit{Then}$$

$$\eta_\infty(P,\lambda) \le \underbrace{\left(\inf_{\gamma\in(0,1]}\ \inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_2(R(\gamma,\delta))\right)^{-1}}_{=:\,\text{lbound}} \le \eta^{\mathbb{R}}_\infty(P,\lambda) \le \min\left\{\eta^{\mathbb{R}}_{\infty,A_0},\, \eta^{\mathbb{R}}_{\infty,A_1}\right\}, \tag{5.1.7}$$

where $\eta^{\mathbb{R}}_{\infty,A_0}$ and $\eta^{\mathbb{R}}_{\infty,A_1}$ are the real eigenvalue backward errors with respect to perturbations to only A_0 and A_1, respectively. Further,

(a) *If $\operatorname{rank}(\operatorname{Im} M) > 1$ then infimum of $\sigma_2(R(\gamma,\delta))$ is attained by a pair $(\gamma^\star,\delta^\star)$ where $(\gamma^\star,\delta^\star) \in (0,1]\times\mathbb{C}\setminus\{0\}$.*

(b) *Let $\sigma^\star := \inf_{\gamma\in(0,1]}\ \inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_2(R(\gamma,\delta))$ and let $(\gamma^\star,\delta^\star) \in (0,1]\times\mathbb{C}\setminus\{0\}$ be such that $\sigma^\star = \sigma_2(R(\gamma^\star,\delta^\star))$. If*

$$u = \begin{bmatrix} u_{11}^T & u_{12}^T & u_{21}^T & u_{22}^T \end{bmatrix}^T \in \mathbb{R}^{4n} \quad \text{and} \quad v = \begin{bmatrix} v_{11}^T & v_{12}^T & v_{21}^T & v_{22}^T \end{bmatrix}^T \in \mathbb{R}^{4n}$$

are respectively the left and right singular vectors of $R(\gamma^\star,\delta^\star)$ corresponding to singular value $\sigma_2(R(\gamma^\star,\delta^\star))$ satisfying

$$\begin{bmatrix} u_{11} & u_{21} \end{bmatrix}^T \begin{bmatrix} u_{11} & u_{21} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \end{bmatrix}^T \begin{bmatrix} v_{11} & v_{21} \end{bmatrix} \tag{5.1.8}$$

and

$$\begin{bmatrix} u_{12} & u_{22} \end{bmatrix}^T \begin{bmatrix} u_{12} & u_{22} \end{bmatrix} = \begin{bmatrix} v_{12} & v_{22} \end{bmatrix}^T \begin{bmatrix} v_{12} & v_{22} \end{bmatrix}, \tag{5.1.9}$$

then

$$\eta^{\mathbb{R}}_\infty(P,\lambda) = \frac{1}{\sigma^\star}.$$

Proof. We note that by the definition of $\eta_\infty(P,\lambda)$, Corollary 5.1.7 and Theorem 5.1.6, a different characterization for $\eta_\infty(P,\lambda)$ can be given as

$$(\eta_\infty(P,\lambda))^{-1} = \mu_D(\hat{M}) = \inf_{\delta>0} \sigma_{\max}(M_\delta), \tag{5.1.10}$$

where $\hat{M} = \begin{bmatrix} M & \lambda M \\ M & \lambda M \end{bmatrix}$, $M_\delta = \begin{bmatrix} M & \delta\lambda M \\ \delta^{-1}M & \lambda M \end{bmatrix}$ and $\mu_D(\hat{M})$ is as defined in Theorem 5.1.6. Now by Corollary 5.1.7

$$\begin{aligned} \eta^{\mathbb{R}}_\infty(P,\lambda) &= \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \operatorname{diag}_{\mathbb{R}}(2n),\ \det(I-\Delta\hat{M}) = 0\right\} \\ &\ge \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathbb{R}^{2n\times 2n},\ \det(I-\Delta\hat{M}) = 0\right\}. \end{aligned}$$

Note that for $\delta \in \mathbb{C} \setminus \{0\}$, $\det(I - \Delta \hat{M}) = 0 \Longleftrightarrow \det(I - \Delta M_\delta) = 0$. Therefore

$$\begin{aligned}
\eta_\infty^{\mathbb{R}}(P,\lambda) &\geq \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathbb{R}^{2n\times 2n},\ \det(I-\Delta M_\delta)=0\right\} && (5.1.11)\\
&\geq \inf\left\{\|\Delta\| \;\middle|\; \Delta \in \mathbb{C}^{2n\times 2n},\ \det(I-\Delta M_\delta)=0\right\}\\
&= (\sigma_{\max}(M_\delta))^{-1} \quad \big(\text{by Theorem 5.1.5}\big) && (5.1.12)
\end{aligned}$$

Thus by (5.1.11) and Theorem 5.1.6, for any $\delta \in \mathbb{C} \setminus \{0\}$ we have

$$\eta_\infty^{\mathbb{R}}(P,\lambda) \geq \left(\inf_{\gamma\in(0,1]} \sigma_2(R(\gamma,\delta))\right)^{-1} \geq (\sigma_{\max}(M_\delta))^{-1}$$

which implies

$$\eta_\infty^{\mathbb{R}}(P,\lambda) \geq \left(\inf_{\gamma\in(0,1]} \inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_2(R(\gamma,\delta))\right)^{-1} \geq \left(\inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_{\max}(M_\delta)\right)^{-1}.$$

Hence by (5.1.10)

$$\eta_\infty^{\mathbb{R}}(P,\lambda) \geq \left(\inf_{\gamma\in(0,1]} \inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_2(R(\gamma,\delta))\right)^{-1} \geq \left(\inf_{\delta>0} \sigma_{\max}(M_\delta)\right)^{-1} = \eta_\infty(P,\lambda).$$

Therefore (5.1.7) follows immediately from Theorem 5.1.8.
Proof of (a): Let $\sigma^\star = \inf_{\gamma\in(0,1]} \inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_2(R(\gamma,\delta))$ and $\operatorname{rank}(\operatorname{Im} M) > 1$. Then by (5.1.7) $\sigma^\star$ is finite. Also note that $\sigma_2(R(\gamma,\delta))$ is a continuous function of γ and δ. We show that there exist $(\gamma^\star,\delta^\star) \in (0,1] \times \mathbb{C} \setminus \{0\}$ such that $\sigma^\star = \sigma_2(R(\gamma^\star,\delta^\star))$ by establishing the following

(i) $\lim_{(\gamma,\delta)\to(0,0)} \sigma_2(R(\gamma,\delta)) = \infty$. (ii) For each fixed $\gamma \in (0,1]$, $\lim_{\delta\to\infty} \sigma_2(R(\gamma,\delta)) = \infty$.

For this first recall that

$$R(\gamma,\delta) = \begin{bmatrix} \operatorname{Re} M_\delta & -\gamma \operatorname{Im} M_\delta \\ \gamma^{-1} \operatorname{Im} M_\delta & \operatorname{Re} M_\delta \end{bmatrix} \text{ where } M_\delta = \begin{bmatrix} M & \delta\lambda M \\ \delta^{-1} M & \lambda M \end{bmatrix}.$$

Now for $\gamma \in (0,1]$ and $\delta = |\delta| e^{i\theta} \in \mathbb{C} \setminus \{0\}$, by repeated use of ([22], p. 189, Theorem 4.3.15) and ([22], p. 181, Theorem 4.3.1), we obtain

$$\begin{aligned}
\sigma_2(R(\gamma,\delta)) &= (\lambda_2((R(\gamma,\delta))^*(R(\gamma,\delta))))^{1/2} \\
&\geq (\lambda_2((\operatorname{Re} M_\delta)^*(\operatorname{Re} M_\delta)+\gamma^{-2}(\operatorname{Im} M_\delta)^*(\operatorname{Im} M_\delta)))^{1/2} \\
&\geq (\lambda_2(\gamma^{-2}(\operatorname{Im} M_\delta)^*(\operatorname{Im} M_\delta))))^{1/2} \\
&= \gamma^{-1}(\lambda_2((\operatorname{Im} M_\delta)^*(\operatorname{Im} M_\delta))))^{1/2} \qquad (5.1.13)\\
&\geq \gamma^{-1}(\lambda_2((\operatorname{Im} M)^*(\operatorname{Im} M)+\frac{1}{|\delta|^2}(\operatorname{Im}(e^{-i\theta}M))^*(\operatorname{Im}(e^{-i\theta}M))))^{1/2} \\
&\geq \frac{1}{\gamma}\frac{1}{|\delta|}(\lambda_2((\operatorname{Im}(e^{-i\theta}M))^*(\operatorname{Im}(e^{-i\theta}M))))^{1/2} \\
&= \frac{1}{\gamma}\frac{1}{|\delta|}\sigma_2(\operatorname{Im}(e^{-i\theta}M)). \qquad (5.1.14)
\end{aligned}$$

Again from (5.1.13)

$$\begin{aligned}
\sigma_2(R(\gamma,\delta)) &\geq \gamma^{-1}(\lambda_2((\operatorname{Im} M_\delta)^*(\operatorname{Im} M_\delta))))^{1/2} \\
&\geq \gamma^{-1}(\lambda_2((\operatorname{Im}(\lambda M))^*(\operatorname{Im}(\lambda M))+|\delta|^2(\operatorname{Im}(e^{i\theta}\lambda M))^*(\operatorname{Im}(e^{i\theta}\lambda M))))^{1/2} \\
&\geq \frac{1}{\gamma}|\delta|(\lambda_2((\operatorname{Im}(e^{i\theta}\lambda M))^*(\operatorname{Im}(e^{i\theta}\lambda M))))^{1/2} \\
&= \frac{1}{\gamma}|\delta|\sigma_2(\operatorname{Im}(e^{i\theta}\lambda M)). \qquad (5.1.15)
\end{aligned}$$

Thus for any $\gamma \in (0,1]$ and $\delta = |\delta|e^{i\theta} \in \mathbb{C}\setminus\{0\}$ from (5.1.14) and (5.1.15), we get

$$\sigma_2\big(R(\gamma,\delta)\big) \geq \max\left\{\frac{1}{\gamma\,|\delta|}\sigma_2\big(\operatorname{Im}(e^{-i\theta}M)\big), \frac{|\delta|}{\gamma}\sigma_2\big(\operatorname{Im}(e^{i\theta}\lambda M)\big)\right\}. \qquad (5.1.16)$$

Since rank(Im M) > 1 and $\lambda \neq 0$, we have

$$\operatorname{rank}(\operatorname{Im}(e^{-i\theta}M)) = \operatorname{rank}(\operatorname{Im}(e^{i\theta}\lambda M)) = \operatorname{rank}(\operatorname{Im} M) > 1.$$

Also $\sigma^\star$ satisfies $0 < \sigma^\star < \infty$. Therefore (i) and (ii) follow from (5.1.16). Hence the proof of (a).

Proof of (b): By (5.1.7) we have

$$\eta_\infty^{\mathbb{R}}(P,\lambda) \geq \frac{1}{\sigma^\star}. \qquad (5.1.17)$$

We show that equality holds in (5.1.17) by constructing a $\Delta \in \operatorname{diag}_{\mathbb{R}}(2n)$ with the assumptions (5.1.8) and (5.1.9) on u and v such that

$$\det(I_{2n} - \Delta M_{\delta^\star}) = 0 \quad \text{and} \quad \|\Delta\| = \frac{1}{\sigma^\star}.$$

For this, set

$$v_1 := \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix},\ v_2 := \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix},\ u_1 := \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix},\ u_2 := \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix},\ \Delta := \mathrm{diag}(\Delta_1, \Delta_2),$$

where

$$\Delta_1 := \sigma^{\star -1} \begin{bmatrix} v_{11} & v_{21} \end{bmatrix} \begin{bmatrix} u_{11} & u_{21} \end{bmatrix}^{\dagger} \quad \text{and} \quad \Delta_2 := \sigma^{\star -1} \begin{bmatrix} v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} u_{12} & u_{22} \end{bmatrix}^{\dagger}. \tag{5.1.18}$$

Noting that $v \neq 0$ implies $v_1 + i\gamma^\star v_2 \neq 0$, we have

$$\begin{aligned}
(I_{2n} - \Delta M_{\delta^\star})(v_1 + i\gamma^\star v_2) &= (I_{2n} - \Delta(\mathrm{Re}\, M_{\delta^\star} + i\, \mathrm{Im}\, M_{\delta^\star}))(v_1 + i\gamma^\star v_2) \\
&= v_1 + i\gamma^\star v_2 - \Delta((\mathrm{Re}\, M_{\delta^\star} v_1 - \gamma^\star\, \mathrm{Im}\, M_{\delta^\star} v_2) + i(\mathrm{Im}\, M_{\delta^\star} v_1 + \gamma^\star\, \mathrm{Re}\, M_{\delta^\star} v_2)) \\
&= v_1 + i\gamma^\star v_2 - \Delta(\sigma^\star u_1 + i\gamma^\star \sigma^\star u_2) \qquad (\because R(\gamma^\star, \delta^\star) v = \sigma^\star u) \\
&= v_1 + i\gamma^\star v_2 - \sigma^\star \begin{bmatrix} \Delta_1(u_{11} + i\gamma^\star u_{21}) \\ \Delta_2(u_{12} + i\gamma^\star u_{22}) \end{bmatrix} \\
&= v_1 + i\gamma^\star v_2 - \begin{bmatrix} v_{11} + i\gamma^\star v_{21} \\ v_{12} + i\gamma^\star v_{22} \end{bmatrix} \qquad (\text{ By (5.1.18) and Lemma 5.1.9}) \\
&= 0.
\end{aligned}$$

Due to assumptions (5.1.8) and (5.1.9), and Lemma 5.1.9, we have $\|\Delta_1\| = \|\Delta_2\| = \frac{1}{\sigma^\star}$, which implies

$$\|\Delta\| = \max\{\|\Delta_1\|, \|\Delta_2\|\} = \frac{1}{\sigma^\star}.$$

□

Remark 5.1.11. The assumption that $\mathrm{rank}(\mathrm{Im}\, M) > 1$ in (a) of the above theorem is satisfied generically. This is because if either A_1 is invertible or more generally $\mathrm{rank}(A_1) \geq 3$ whenever $n \geq 4$, then $\mathrm{rank}(\mathrm{Im}\, M) > 1$. In such cases the assumption $\mathrm{rank}(\mathrm{Im}\, M) > 1$ is not required.

5.1.3 Numerical experiments

We have made several numerical experiments to illustrate the result obtained in Theorem 5.1.10. We will prove in Theorem 5.3.1 that given a real Hermitian polynomial $P(z)$ and $\lambda \in \mathbb{C}$ under certain assumptions, the real Hermitian backward error $\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P, \lambda)$ preserving the real as well as the Hermitian structure of $P(z)$ is equal to its Hermitian

Table 5.1.1: Lower bound of $\eta_\infty^{\mathbb{R}}(P,\lambda)$ for a Hermitian pencil with an eigenvalue 2.7922

λ	$\eta_\infty(P,\lambda)$	lbound	$\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P,\lambda)$
2.7922+i	0.3145	1.4794	2.0078
2.7922+0.5i	0.1635	1.4265	1.9914
2.7922+0.1i	0.0331	1.4069	1.9855
2.7922+0.05i	0.0166	1.4063	1.9853
2.7922+0.01i	0.0033	1.4061	1.9853

Table 5.1.2: Lower bound of $\eta_\infty^{\mathbb{R}}(P,\lambda)$ for arbitrary Hermitian pencils where $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda \neq 0$

size	$\eta_\infty(P,\lambda)$	lbound	$\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P,\lambda)$
2×2	0.4929	0.6056	0.6059
3×3	0.2909	0.3394	0.3741
4×4	0.5205	0.8181	0.8646
5×5	0.3135	0.3475	0.3611
6×6	0.5156	0.5753	0.6046

Table 5.1.3: $\eta_\infty^{\mathbb{R}}(P,\lambda)$ for arbitrary pencils and $\lambda \in \mathbb{C}\setminus\mathbb{R}$

size	$\eta_\infty(P,\lambda)$	$\eta_\infty^{\mathbb{R}}(P,\lambda)$
2×2	0.8801	0.9243
3×3	0.4498	0.4638
4×4	0.4913	0.5320
5×5	0.6085	0.6132
6×6	0.4986	0.8227

Table 5.1.4: $\eta_\infty^{\mathbb{R}}(P,\lambda)$ for arbitrary Hermitian pencils and $\lambda \in \mathbb{C}\setminus\mathbb{R}$

size	$\eta_\infty(P,\lambda)$	$\eta_\infty^{\mathbb{R}}(P,\lambda)$	$\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P,\lambda)$
2×2	0.4093	0.5301	0.5301
3×3	1.1851	1.2479	1.2479
4×4	0.4637	0.6708	0.6708
5×5	0.4425	0.5152	0.5152
6×6	1.0419	1.1466	1.1466

backward error, i.e., $\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P,\lambda) = \eta_\infty^{\mathrm{Herm}}(P,\lambda)$. We use this fact to show the tightness of lower bound for $\eta_\infty^{\mathbb{R}}(P,\lambda)$ obtained in Theorem 5.1.10.

Firstly, let $P(z)$ be a random real Hermitian pencil of size 3×3 with eigenvalues 2.7922 and $0.4155 \pm 0.0099i$. In Table 5.1.1, we compute the lower bound (lbound) of $\eta_\infty^{\mathbb{R}}(P,\lambda)$ and the unstructured backward error $\eta_\infty(P,\lambda)$ of a complex λ. We also compute $\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P,\lambda)$ by using Theorem 5.3.1. Note that $\eta_\infty^{\mathbb{R}}(P,\lambda)$ which lies between lbound and $\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P,\lambda)$ is estimated tightly by these values. We observe that as expected $\eta_\infty(P,\lambda)$ goes to zero as λ converges to the real eigenvalue 2.7922. However this is not true for $\eta_\infty^{\mathbb{R}}(P,\lambda)$ as lbound does not go to zero when $\lambda \to 2.7922$. This lead to large differences between $\eta_\infty(P,\lambda)$ and $\eta_\infty^{\mathbb{R}}(P,\lambda)$.

In Table 5.1.2, we record the same bounds on $\eta_\infty^{\mathbb{R}}(P,\lambda)$ for randomly chosen real Hermitian pencils $P(z)$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Here also the values of lbound and $\eta_\infty^{\mathrm{Herm}_\mathbb{R}}(P,\lambda)$ are seen to bound $\eta_\infty^{\mathbb{R}}(P,\lambda)$ tightly.

In our numerical examples we found several instances of randomly generated real pencils and values of $\lambda \in \mathbb{C}\setminus\mathbb{R}$ which satisfied the sufficient condition in Theorem 5.1.10 for $\eta_\infty^{\mathbb{R}}(P,\lambda)$ to be given by $\left(\inf_{\gamma\in(0,1]} \inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_2(R(\gamma,\delta))\right)^{-1}$. The value of $\eta_\infty^{\mathbb{R}}(P,\lambda)$ for a few such cases is reported in Table 5.1.3.

Finally, in Table 5.1.4 we record $\eta_\infty^{\mathbb{R}}(P,\lambda)$ for arbitrary chosen real Hermitian pencils $P(z)$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$ for which $\eta_\infty^{\mathbb{R}}(P,\lambda) = \left(\inf_{\gamma\in(0,1]} \inf_{\delta\in\mathbb{C}\setminus\{0\}} \sigma_2(R(\gamma,\delta))\right)^{-1}$. Interestingly, we observe that for such pencils, $\eta_\infty^{\mathbb{R}}(P,\lambda) = \eta_\infty^{\mathrm{Herm}_{\mathbb{R}}}(P,\lambda)$.

5.2 Real structured eigenvalue backward errors with respect to $|||\cdot|||_{w,2}$ norm

Let $P(z) = A_0 + zA_1 + \cdots + z^m A_m$ be a real structured matrix polynomial with coefficients $(A_0,\ldots,A_m) \in \mathbb{S}_{\mathbb{R}} \subseteq \mathbb{R}^{n\times n}$. In this section, we obtain results for eigenvalue backward errors of $P(z)$ when perturbations belong to the set $\mathbb{S}_{\mathbb{R}}$ and λ belongs to some specific subset of $\mathbb{C}$.

5.2.1 Real Hermitian polynomials

Let $\mathrm{Herm}_{\mathbb{R}}(\mathrm{n})$ denote the set of all real symmetric/Hermitian matrices of size $n\times n$. Let $P(z) = \sum_{j=0}^m z^j A_j$ be a real Hermitian polynomial, i.e., $(A_0,\ldots,A_m) \in (\mathrm{Herm}_{\mathbb{R}}(\mathrm{n}))^{m+1}$, $\lambda \in \mathbb{C}$ and $w = (w_0,\ldots,w_m) \in \mathbb{R}^{m+1}$ be a weight vector. Then we compute the real Hermitian eigenvalue backward error $\eta_{w,2}^{\mathrm{Herm}_{\mathbb{R}}}(P,\lambda)$, i.e.,

$$\eta_{w,2}^{\mathrm{Herm}_{\mathbb{R}}}(P,\lambda) := \inf\left\{ |||(\Delta_0,\ldots,\Delta_m)|||_{w,2} \,\middle|\, \det\left(\sum_{j=0}^m \lambda^j (A_j - \Delta_j)\right) = 0, \; \Delta_0,\ldots,\Delta_m \in (\mathrm{Herm}_{\mathbb{R}}(\mathrm{n}))^{m+1} \right\}.$$

Note that if $\lambda \in \mathbb{R}$, then there is no difference between $\eta_{w,2}^{\mathrm{Herm}_{\mathbb{R}}}(P,\lambda)$ and unstructured backward error $\eta_{w,2}(P,\lambda)$. This fact is shown in [1, 3] for weight vector $w = (1,\ldots,1)$ and can easily be generalized to arbitrary weight vector w. If $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then we have observed that in many cases the real Hermitian backward error $\eta_{w,2}^{\mathrm{Herm}_{\mathbb{R}}}(P,\lambda)$ preserving the real as well as the Hermitian structure of $P(z)$ is equal to its Hermitian backward error $\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)$ obtained in Theorem 2.2.2. The following examples illustrate this observation.

Example 5.2.1. Let

$$P(z) = \begin{bmatrix} -3.8902 & -1.6716 & -0.7747 & 0.4359 \\ -1.6716 & 1.6739 & -0.7427 & -2.4721 \\ -0.7747 & -0.7427 & 0.5578 & 1.7556 \\ 0.4359 & -2.4721 & 1.7556 & 1.6230 \end{bmatrix} + z \begin{bmatrix} 1.2727 & 1.1608 & -0.2640 & -0.2936 \\ 1.1608 & -4.8980 & -0.1814 & -0.2134 \\ -0.2640 & -0.1814 & -2.1613 & -0.5476 \\ -0.2936 & -0.2134 & -0.5476 & -0.0720 \end{bmatrix}$$

be a real Hermitian pencil, $\lambda = -0.1748 - 0.9573i$ and $w = (1, 1)$. For this unstructured and structured backward errors are $\eta_{w,2}(P, \lambda) = 0.6549$ and $\eta^{\text{Herm}}_{w,2}(P, \lambda) = 0.8894$ respectively. In view of Remark 2.2.5, an optimal Hermitian perturbation $\Delta_0 + z\Delta_1$ satisfying $\det(P(\lambda) - \Delta_0 - \lambda\Delta_1) = 0$ and $|||(\Delta_0, \Delta_1)|||_{w,2} = 0.8894$ can be constructed. In fact, we found that this optimal perturbation is real and given by

$$\Delta_0 = \begin{bmatrix} .0047 & -.0398 & .0283 & .0295 \\ -.0398 & .1395 & .0359 & -.3959 \\ .0283 & .0359 & -.2157 & .3847 \\ .0295 & -.3959 & .3847 & .0715 \end{bmatrix} \text{ and } \Delta_1 = \begin{bmatrix} -.0004 & -.0323 & .0479 & -.0300 \\ -.0323 & .3469 & -.2998 & -.1431 \\ .0479 & -.2998 & .1421 & .3769 \\ -.0300 & -.1431 & .3769 & -.4885 \end{bmatrix}.$$

This implies that $\eta^{\text{Herm}_{\mathbb{R}}}_{w,2}(P, \lambda)$ is also equal to 0.8894.

Similarly let $Q(z) = A_0 + zA_1 + z^2A_2$ be a real Hermitian polynomial where A_0, A_1 and A_2 be given by

$$A_0 = \begin{bmatrix} .9997 & 1.5389 & 1.5885 \\ 1.5389 & -.0264 & -.8379 \\ 1.5885 & -.8379 & -2.8191 \end{bmatrix}, A_1 = \begin{bmatrix} 3.5402 & .9459 & -.9839 \\ .9459 & 2.5396 & -1.0351 \\ -.9839 & -1.0351 & -2.3268 \end{bmatrix}, A_2 = \begin{bmatrix} 2.3674 & -.7319 & .6430 \\ -.7319 & -1.3111 & 2.1626 \\ .6430 & 2.1626 & -.5502 \end{bmatrix}$$

$\lambda = 2.2126 + 1.5085i$ and $w = (1, 1, 1)$. For this unstructured and structured backward errors are $\eta_{w,2}(Q, \lambda) = 0.6371$ and $\eta^{\text{Herm}}_{w,2}(Q, \lambda) = 0.7518$ respectively. An optimal Hermitian perturbation $\Delta_0 + z\Delta_1 + z^2\Delta_2$ satisfying $\det(Q(\lambda) - (\Delta_0 + \lambda\Delta_1 + \lambda^2\Delta_2)) = 0$ and $|||(\Delta_0, \Delta_1, \Delta_2)|||_{w,2} = 0.7518$ can be constructed by Remark 2.2.5. Again we found that this optimal perturbation is real and given by

$$\Delta_0 = \begin{bmatrix} .0028 & -.0044 & .0166 \\ -.0044 & -.1004 & -.0253 \\ .0166 & -.0253 & .0975 \end{bmatrix}, \Delta_1 = \begin{bmatrix} .0057 & .0189 & .0333 \\ .0189 & -.2003 & .1122 \\ .0333 & .1122 & .1946 \end{bmatrix}, \Delta_2 = \begin{bmatrix} .0050 & .1155 & .0284 \\ .1155 & -.1668 & .6784 \\ .0284 & .6784 & .1618 \end{bmatrix}.$$

This implies that $\eta^{\text{Herm}_{\mathbb{R}}}_{w,2}(Q, \lambda) = 0.7518 = \eta^{\text{Herm}}_{w,2}(Q, \lambda)$.

The reason for the equality of $\eta^{\text{Herm}_{\mathbb{R}}}_{w,2}(P, \lambda)$ and $\eta^{\text{Herm}}_{w,2}(P, \lambda)$ in the above example is given in the proof of the next theorem.

Theorem 5.2.2. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a real Hermitian polynomial, that is, let $(A_0, \ldots, A_m) \in (\text{Herm}_{\mathbb{R}}(\text{n}))^{m+1}$ and $w = (w_0, \ldots, w_m)$ be a weight vector. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be such that $\det(\text{P}(\lambda)) \neq 0$ and the Hermitian matrices $G, H_0, \ldots, H_m$ and $(t_0^\star, \ldots, t_m^\star) \in \mathbb{R}^{m+1}$ be as defined in Theorem 2.2.2. If $\lambda^\star_{\max} := \lambda_{\max}(G + t_0^\star H_0 + \cdots + t_m^\star H_m)$ is a simple eigenvalue of $G + t_0^\star H_0 + \cdots + t_m^\star H_m$, then*

$$\eta^{\text{Herm}_{\mathbb{R}}}_{w,2}(P, \lambda) = \eta^{\text{Herm}}_{w,2}(P, \lambda) = \frac{1}{\sqrt{\lambda^\star_{\max}}}.$$

Proof. Recall that from Theorem 2.2.2, we have

$$\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)^{-2} = \min_{t_0,\ldots,t_m\in\mathbb{R}} \lambda_{\max}(G + t_0\, H_0 + \cdots + t_m\, H_m) = \lambda_{\max}^{\star}.$$

Let the perturbation $\boldsymbol{\Delta} = (\Delta_0,\ldots,\Delta_m) \in (\mathrm{Herm(n)})^{m+1}$ be such that

$$\det\Big(\sum_{\mathrm{j=0}}^{\mathrm{m}} \lambda^{\mathrm{j}}(\mathrm{A_j} - \Delta_{\mathrm{j}})\Big) = 0 \quad \text{and} \quad \eta_{w,2}^{\mathrm{Herm}}(P,\lambda)^2 = \sum_{j=0}^{m} w_j^2 \|\Delta_j\|^2. \tag{5.2.1}$$

The procedure for the construction of the matrices $\Delta_j, j = 0,\ldots,m$ as given by Remark 2.2.5 imply that they satisfy

$$\Delta_j M\left(\frac{v_0}{w_0} + \lambda\frac{v_1}{w_1} + \cdots + \lambda^m \frac{v_m}{w_m}\right) = \frac{v_j}{w_j} \quad \text{for} \quad j = 0,\ldots,m, \tag{5.2.2}$$

where $v = [v_0^T,\ldots,v_m^T]^T \in \mathbb{C}^{n(m+1)}$ is an eigenvector of $G + t_0^{\star}\, H_0 + \cdots + t_m^{\star}\, H_m$ corresponding to $\lambda_{\max}^{\star}$. Moreover, in view of Theorem 1.2.10, each $\Delta_j, j = 0,\ldots,m$, is a unique Hermitian map of minimal rank and minimal Frobenius norm that satisfies (5.2.2). Now clearly

$$\eta_{w,2}^{\mathrm{Herm}_{\mathbb{R}}}(P,\lambda) \geq \eta_{w,2}^{\mathrm{Herm}}(P,\lambda) \quad \text{as} \quad (\mathrm{Herm}_{\mathbb{R}}(\mathrm{n}))^{m+1} \subseteq (\mathrm{Herm(n)})^{m+1}. \tag{5.2.3}$$

To establish equality in (5.2.3) we show $\boldsymbol{\Delta} \in (\mathrm{Herm}_{\mathbb{R}}(\mathrm{n}))^{m+1}$. Note that λ is an eigenvalue of $P(z)$ if and only if $\overline{\lambda}$ is an eigenvalue of $P(z)$. This implies $\exists\, x \in \mathbb{C}^n \setminus \{0\}$ such that

$$\Big(P(\overline{\lambda}) - \sum_{j=0}^{m} \overline{\lambda}^j \Delta_j\Big)x = 0.$$

Let $u_j := w_j \Delta_j x$ and $\hat{u}_j := \frac{u_j}{w_j}$ for $j = 0,\ldots,m$. Then for $\hat{u}_{\overline{\lambda}} := \sum_{j=0}^{m} \overline{\lambda}^j \hat{u}_j$,

$$P(\overline{\lambda})x = \hat{u}_{\overline{\lambda}} \implies x = \overline{M}\hat{u}_{\overline{\lambda}}.$$

Thus

$$\Delta_j \overline{M}\hat{u}_{\overline{\lambda}} = \hat{u}_j \quad \text{for} \quad j = 0,\ldots,m. \tag{5.2.4}$$

Set $u := [u_0^T,\ldots,u_m^T]^T$. Since each $\Delta_j,\ j = 0,\ldots,m$ is Hermitian, therefore, $\hat{u}_j^* \overline{M}\hat{u}_{\overline{\lambda}} \in \mathbb{R}$. Again from (5.2.1) and (5.2.4)

$$\left(\eta_{w,2}^{\mathrm{Herm}}(P,\lambda)\right)^2 = \sum_{j=0}^{m} w_j^2 \|\Delta_j\|^2 \geq \sum_{j=0}^{m} \frac{w_j^2 \|\hat{u}_j\|^2}{\|\overline{M}\hat{u}_{\overline{\lambda}}\|^2} = \sum_{j=0}^{m} \frac{\|u_j\|^2}{\|\overline{M}\hat{u}_{\overline{\lambda}}\|^2} = \left(\frac{u^*\overline{G}u}{u^*u}\right)^{-1}, \tag{5.2.5}$$

where G is given by (2.2.1). Since $\hat{u}_j^* \overline{M}\hat{u}_{\overline{\lambda}} \in \mathbb{R}$, we have

$$u^* \overline{H}_j u = 0 \quad \text{for} \quad j = 0,\ldots,m \tag{5.2.6}$$

where H_j, $j = 0, \ldots, m$ are as given in (2.2.1). By Theorem 1.2.9, there exist Hermitian matrices $\tilde{\Delta}_j, j = 0, \ldots, m$ such that

$$\tilde{\Delta}_j \overline{M} \hat{u}_{\overline{\lambda}} = \hat{u}_j, \text{ and } \|\tilde{\Delta}_j\| = \frac{\|\hat{u}_j\|}{\|\overline{M}\hat{u}_{\overline{\lambda}}\|} \; j = 0, \ldots, m.$$

By Lemma 1.2.6, $\overline{\lambda}$ is an eigenvalue of $P(\lambda) - \Sigma_{j=0}^{m} \lambda^j \tilde{\Delta}_j$. Hence

$$\left(\eta_{w,2}^{\mathrm{Herm}}(P, \overline{\lambda})\right)^2 \leq \sum_{j=0}^{k} \frac{w_j^2 \|\hat{u}_j\|^2}{\|\overline{M}\hat{u}_{\overline{\lambda}}\|^2} = \left(\frac{u^*\overline{G}u}{u^*u}\right)^{-1}. \tag{5.2.7}$$

However since $\eta_{w,2}^{\mathrm{Herm}}(P, \lambda) = \eta_{w,2}^{\mathrm{Herm}}(P, \overline{\lambda})$, from (5.2.5) and (5.2.7) we have,

$$\left(\eta_{w,2}^{\mathrm{Herm}}(P, \lambda)\right)^2 = \left(\frac{u^*\overline{G}u}{u^*u}\right)^{-1}. \tag{5.2.8}$$

Since

$$\lambda_{\max}(G + t_0^\star H_0 + \cdots + t_m^\star H_m) = \lambda_{\max}(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)$$

and $\lambda_{\max}(G + t_0^\star H_0 + \cdots + t_m^\star H_m)$ is a simple eigenvalue of $G + t_0^\star H_0 + \cdots + t_m^\star H_m$, with v as a corresponding eigenvector, $\lambda_{\max}(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)$ is also a simple eigenvalue of $\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m$ with $\overline{v}$ as a corresponding eigenvector. Therefore,

$$\begin{aligned}
\left(\frac{\overline{v}^*(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)\overline{v}}{\overline{v}^*\overline{v}}\right)^{-1} &= \left(\lambda_{\max}(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)\right)^{-1} \\
&= \left(\eta_{w,2}^{\mathrm{Herm}}(P, \lambda)\right)^2 \\
&= \left(\frac{u^*(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)u}{u^*u}\right)^{-1},
\end{aligned}$$

where the last equality follows from (5.2.6) and (5.2.8). Setting $\widetilde{v} := \frac{\overline{v}}{\|v\|}$ and $\widetilde{u} := \frac{u}{\|u\|}$

$$\widetilde{v}^*(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)\widetilde{v} = \lambda_{\max}^\star = \widetilde{u}^*(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)\widetilde{u}. \tag{5.2.9}$$

Let $\lambda_2, \ldots, \lambda_{n(m+1)}$ be the other eigenvalues of $\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m$ with corresponding eigenvectors $\widetilde{v}_2, \ldots, \widetilde{v}_{n(m+1)}$ such that $\{\hat{v}, \widetilde{v}_2 \ldots, \widetilde{v}_{n(m+1)}\}$ is an orthonormal basis of $\mathbb{C}^{n(m+1)}$. Thus there exist $\alpha_1, \ldots, \alpha_{n(m+1)} \in \mathbb{C}$, $\sum_{j=0}^{n(m+1)} |\alpha_j|^2 = 1$ such that

$$\widetilde{u} = \alpha_1 \widetilde{v} + \alpha_2 \widetilde{v}_2 + \cdots + \alpha_{n(m+1)} \widetilde{v}_{n(m+1)}.$$

Thus

$$\begin{aligned}
\lambda_{\max}^\star &= \widetilde{u}^*(\overline{G} + t_0^\star \overline{H}_0 + \cdots + t_m^\star \overline{H}_m)\widetilde{u} \\
&= |\alpha_1|^2 \lambda_{\max}^\star + |\alpha_2|^2 \lambda_2 + \cdots + |\alpha_{n(m+1)}|^2 \lambda_{n(m+1)} \\
&\leq \lambda_{\max}^\star.
\end{aligned}$$

Now since $\lambda_{\max}^{\star}$ is a simple eigenvalue of $\overline{G} + t_0^{\star}\overline{H}_0 + \cdots + t_m^{\star}\overline{H}_m$, if $\alpha_j \neq 0$ for any $j \in \{2, \ldots, n(m+1)\}$, the last inequality must be strict which is impossible. Thus $\widetilde{u} = \alpha_1 \widetilde{v}$ where $|\alpha_1| = 1$. This implies

$$u = \alpha_1 \frac{\|u\|}{\|v\|}\overline{v} \quad \Longrightarrow \quad u_j = \alpha_1 \frac{\|u\|}{\|v\|}\overline{v}_j \quad \text{for } j = 0, \ldots, m. \tag{5.2.10}$$

Using (5.2.10) in (5.2.4), we get

$$\overline{\Delta_j M \left(\frac{v_0}{w_0} + \lambda \frac{v_1}{w_1} + \cdots + \lambda^m \frac{v_m}{w_m} \right)} = \overline{\frac{v_j}{w_j}} \quad \text{for} \quad j = 0, \ldots, m. \tag{5.2.11}$$

However by (5.2.2) we also have

$$\overline{\overline{\Delta}_j M \left(\frac{v_0}{w_0} + \lambda \frac{v_1}{w_1} + \cdots + \lambda^m \frac{v_m}{w_m} \right)} = \overline{\frac{v_j}{w_j}} \quad \text{for} \quad j = 0, \ldots, m. \tag{5.2.12}$$

By the uniqueness of each Δ_j, $j = 0, \ldots, m$, we have

$$\Delta_j = \overline{\Delta_j}, j = 0, \ldots, m.$$

Hence $\mathbf{\Delta} \in (\mathrm{Herm}_{\mathbb{R}}(\mathrm{n}))^{m+1}$ and equality holds in (5.2.3). $\square$

Remark 5.2.3. The assumption that $\lambda_{\max}^{\star}$ is a simple eigenvalue of $G + t_0^{\star}H_0 + \cdots + t_m^{\star}H_m$ even when $m = 1$ in Theorem 5.2.2 is indeed necessary because the optimal Hermitian perturbations in Example 2.5.2 which attain the value of $\eta_{w,2}^{\mathrm{Herm}}(P, \lambda)$ are not real. Note that in this case $\lambda_{\max}^{\star}$ is not a simple eigenvalue of $G + t_0^{\star}H_0 + \cdots + t_m^{\star}H_m$.

Remark 5.2.4. We note that Theorem 5.2.2 also holds for the case that some of the entries in weight vector $w = (w_0, \ldots, w_m)$ are zero. This is due to Theorem 2.4.1.

5.2.2 Real T-palindromic polynomials

In this section, we obtain the structured backward error for approximate real eigenvalues of real T-palindromic polynomials without restrictions on the degree of such polynomials.

Note that in Theorem 3.1.5 if $\lambda \in \mathbb{R} \setminus \{0, 1, -1\}$ and $P(z)$ is a real T-palindromic polynomial, then G and S_i are real Hermitian matrices. Therefore, denoting the real $\bullet$-palindromic structure by $\mathrm{pal}_{\bullet,\mathbb{R}}$, that is,

$$\mathrm{pal}_{\bullet,\mathbb{R}} := \left\{ (\Delta_0, \ldots, \Delta_m) \in (\mathbb{R}^{n\times n})^{m+1} \middle| \Delta_j^{\bullet} = \Delta_{m-j}, j = 0, \ldots, m \right\},$$

we have,

$$\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P, \lambda) = \eta_{w,2}^{\mathrm{pal}_{*,\mathbb{R}}}(P, \lambda). \tag{5.2.13}$$

The main result of this section gives the structured backward error $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ when $\lambda \in \mathbb{R} \setminus \{0, 1, -1\}$ and $P(z)$ is a real T-palindromic polynomial of any degree. It may be noted that despite the equality (5.2.13), this result is not a corollary of Theorem 3.1.6. However the proof is based on similar arguments that make use of the real version of a combination of Theorem 1.2.18 and Theorem 1.2.21 with $\mathbb{C}$ replaced by $\mathbb{R}$ as given below. We recall that a Hermitian matrix is indefinite if it has at least one positive and at least negative eigenvalues.

Theorem 5.2.5. *Let $G, H_0, \ldots, H_p \in \mathbb{R}^{n\times n}$ be Hermitian matrices. Assume that any nonzero linear combination $\alpha_0 H_0 + \cdots + \alpha_p H_p$, $(\alpha_0, \ldots, \alpha_p) \in \mathbb{R}^{p+1} \setminus \{0\}$ is indefinite. Then the following statements hold:*

(1) *The function $L : \mathbb{R}^{p+1} \to \mathbb{R}$, $(t_0, \ldots, t_p) \mapsto \lambda_{\max}(G + t_0 H_0 + \cdots + t_p H_p)$ is convex and has a global minimum*

$$\lambda_{\max}^{\star} = \min_{t_0,\ldots,t_p \in \mathbb{R}} L(t_0, \ldots, t_p).$$

(2) *If $p = 0$ or the minimum $\lambda_{\max}^{\star}$ of L is attained at $(\hat{t}_0, \ldots, \hat{t}_p) \in \mathbb{R}^{p+1}$ and is a simple eigenvalue of $H_\star := G + \hat{t}_0 H_0 + \cdots + \hat{t}_p H_p$, then there exists an eigenvector $u \in \mathbb{R}^n \setminus \{0\}$ of $H_\star$ associated with $\lambda_{\max}^{\star}$ satisfying*

$$u^T H_j u = 0 \quad \text{for } j = 0, \ldots, p. \tag{5.2.14}$$

(3) *Under the assumptions of (2) we have*

$$\sup \left\{ \frac{u^T G u}{u^T u} \,\middle|\, u \in \mathbb{R}^n \setminus \{0\},\ u^T H_j u = 0,\ j = 0, \ldots, p \right\} = \lambda_{\max}^{\star}. \tag{5.2.15}$$

In particular, the supremum of the left hand side of (5.2.15) is a maximum and attained for the eigenvector u from (5.2.14).

Proof. The proofs of (1) and (3) follow from the proofs of the corresponding parts of Theorem 1.2.18.

If $p = 0$, then the proof of part (2) follows by arguing as in the proof of Theorem 1.2.21 due to the fact that the set

$$\left\{ x^T H_0 x \,\middle|\, x \in \mathbb{R}^n, x^T x = 1 \right\}$$

is convex. If $p > 0$ and $\lambda_{\max}^{\star}$ is a simple eigenvalue of $H_\star$ then, part (2) of Theorem 1.2.18 implies that there exists a non zero (possibly complex) eigenvector u corresponding to

$\lambda^\star_{\max}$ of $H_\star$ that satisfies (5.2.14). However as $H_\star$ is real and $\lambda^\star_{\max}$ is real and simple, the eigenvector u can be chosen to be real. This proves part (2) and completes the proof of the theorem. $\square$

It is important to note that the assumption of simplicity of $\lambda^\star_{\max}$ made in the hypothesis of Theorem 5.2.5 is necessary even when $p = 1$. This is evident from the following example which is a slight modification of Example 1.2.24.

Example 5.2.6. Let $G = \text{diag}(\alpha, \alpha, \beta)$ where $\alpha > \beta \geq 0$. Also let

$$H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then any non zero real linear combination of H_0 and H_1 is indefinite and for $t_0, t_1 \in \mathbb{R}$, the matrix

$$H(t_0, t_1) = G + t_0 H_0 + t_1 H_1 = \begin{bmatrix} \alpha + t_0 & t_1 & 0 \\ t_1 & \alpha - t_0 & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

has eigenvalues $\alpha \pm \sqrt{t_0^2 + t_1^2}$ and β. Clearly $L : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$L(t_0, t_1) = \lambda_{\max}(H(t_0, t_1)) = \alpha + \sqrt{t_0^2 + t_1^2}$$

has its minimum $\lambda^\star_{\max}$ at $(t_0, t_1) = (0, 0)$, i.e., when $H(0, 0) = G$. But the maximal eigenvalue α of G is a double eigenvalue with corresponding eigenvectors e_1 and e_2 which are the first two basis vectors of $\mathbb{R}^3$. Therefore, the matrix whose columns form an orthonormal basis of the eigenspace of G corresponding to α is $U = [\ e_1 \quad e_2\ ] \in \mathbb{R}^{3\times 2}$. There exists a real non zero vector x in the eigenspace of G corresponding to α satisfying $x^* H_0 x = x^* H_1 x = 0$ if and only if the real joint numerical range of the matrices

$$U_1 := U^T H_0 U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } U_2 := U^T H_1 U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

defined by

$$\{(x^T U_1 x, x^T U_2 x) \,\Big|\, x \in \mathbb{R}^2, \|x\|_2 = 1\}$$

contains 0. Clearly this is not true in this case as this set represents the unit circle.

Note that this example does not contradict Theorem 1.2.18 as the eigenvector of G with respect to α that satisfies $u^* H_0 u = u^* H_1 u = 0$ is the complex vector $u = [1 \quad i]^T$.

The following theorem gives a formula for the structured eigenvalue backward error $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ when $P(z)$ is a real T-palindromic polynomial and $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$.

Theorem 5.2.7. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a real T-palindromic polynomial. Choose $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. Suppose that $\det(P(\lambda)) \neq 0$ so that $M = (P(\lambda))^{-1}$ exists and set $k = \lfloor \frac{m-1}{2} \rfloor$. Then*

$$\lambda_{\max}^{\star} := \min_{t_0,\ldots,t_k \in \mathbb{R}} \lambda_{\max}(G + t_0\, S_0 + \cdots + t_k\, S_k)$$

is attained for some $(\hat{t}_0, \ldots, \hat{t}_k) \in \mathbb{R}^{k+1}$ where G is defined by (3.1.15) and $S_j, j = 0, \ldots, k$ are defined by (3.1.17), respectively. Whenever $m \leq 2$ or $\lambda_{\max}^{\star}$ is a simple eigenvalue of $G + \hat{t}_0 S_0 + \cdots + \hat{t}_k S_k$, then

$$\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P,\lambda) = \frac{1}{\sqrt{\lambda_{\max}^{\star}}} = \left(\min_{t_0,\ldots,t_k \in \mathbb{R}} \lambda_{\max}(G + t_0\, S_0 + \cdots + t_k\, S_k) \right)^{-1/2}.$$

Proof. Consider the Hermitian matrices $\widetilde{S}_j := \widetilde{C}_j + \widetilde{C}_j^T$, $j = 0, \ldots, k$, where $\widetilde{C}_j$ are as defined in (3.1.7). Observe that $k = 0$ whenever $m = 1$ or $m = 2$. Therefore, the proof follows from Theorem 5.2.5 if it is established that each nontrivial linear combination of $S_0, \ldots, S_k$ given by (3.1.17), or equivalently, of $\widetilde{S}_0, \ldots, \widetilde{S}_k$ is indefinite. Suppose there exists $\begin{bmatrix} \alpha_0, \ldots, \alpha_k \end{bmatrix}^T \in \mathbb{R}^{k+1}$ such that $S := \sum_{j=0}^{k} \alpha_j \widetilde{S}_j$ is semidefinite. Then recalling that $\Lambda_m = [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$, we have

$$\begin{aligned} S &= \sum_{j=0}^{k} \alpha_j \Big(\big(\Lambda_m^T (e_{j+1}^T - e_{m+1-j}^T) \big) \otimes M^T + \big((e_{j+1} - e_{m+1-j}) \Lambda_m \big) \otimes M \Big) \\ &= (\Lambda_m^T \alpha^T) \otimes M^T + (\alpha \Lambda_m) \otimes M \end{aligned}$$

where the vector α is given by $\alpha := [\alpha_0, \ldots, \alpha_k,\ -\alpha_k, \ldots, -\alpha_0]^T$ when m is odd and by $\alpha := [\alpha_0, \ldots, \alpha_k, 0, -\alpha_k, \ldots, -\alpha_0]^T$ when m is even.

Setting

$$Q := \begin{bmatrix} 1 & -\lambda & 0 & \ldots & 0 \\ 0 & 1 & -\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(m+1)\times(m+1)} \tag{5.2.16}$$

and $a = [a_0, \ldots, a_m]^T := Q^T\alpha$, since $\Lambda_m Q = e_1^T$, we have

$$\begin{aligned}(Q\otimes I_n)^T S(Q\otimes I_n) &= (Q^T\Lambda_m^T\alpha^T Q)\otimes M^T + (Q^T\alpha\Lambda_m Q_m)\otimes M\\ &= (e_1a^T)\otimes M^T + (ae_1^T)\otimes M\\ &= \begin{bmatrix} a_0(M+M^T) & a_1M^T & \cdots & a_mM^T\\ a_1M & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ a_mM & 0 & \cdots & 0\end{bmatrix}.\end{aligned}$$

If S is semidefinite, then $a_1 = \cdots = a_m = 0$ which implies that $Q^T\alpha = a = a_0e_1$. Therefore, $a_0 = \alpha_0$ and

$$-\lambda\alpha_{j-1} + \alpha_j = 0,\ j = 1, \ldots, k. \tag{5.2.17}$$

Also,

$$\begin{aligned}(\lambda+1)\alpha_k &= 0 \text{ when } m \text{ is odd and} &(5.2.18)\\ \lambda\alpha_k &= 0 \text{ when } m \text{ is even.} &(5.2.19)\end{aligned}$$

The identities (5.2.17) imply that

$$\alpha_k = \lambda^k\alpha_0. \tag{5.2.20}$$

When m is odd, we have $\alpha_k = 0$ from (5.2.18) since $\lambda \neq -1$. Similarly when m is even, (5.2.19) gives $\alpha_k = 0$ as $\lambda \neq 0$. In either case, (5.2.20) implies that $a_0 = \alpha_0 = 0$ as $\lambda \neq 0$ and this completes the proof. $\square$

Remark 5.2.8. An optimal real T-palindromic perturbation to $P(z)$ corresponding to $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ in Theorem 5.2.7 can be constructed by following the procedure mentioned in Remark 3.1.9.

An analogue of Theorem 5.2.7 that obtains $\eta_{\widehat{w},2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ for a real T-palindromic polynomial $P(z)$ with a restricted perturbation set $\mathrm{pal}_{\mathrm{T},\mathbb{R}}(I)$ defined by

$$\mathrm{pal}_{\mathrm{T},\mathbb{R}}(I) := \left\{(\Delta_0, \ldots, \Delta_m) \in \mathrm{pal}_{\mathrm{T},\mathbb{R}} \;\middle|\; \Delta_j = 0 \text{ for } j \notin I\right\}$$

may also be derived.

Example 5.2.9. $L(z) = A + zA^T$ is a real T-palindromic pencil of size 3 with eigenvalues -1 and $-0.5954 \pm 0.8034i$ all on the unit circle. For $\lambda = -1.6656$ and $w = (1,1)$, the eigenvalue backward error is 0.6563 with respect to real T-palindromic perturbations (obtained in Theorem 5.2.7) and 0.5614 with respect to complex T-palindromic perturbations (obtained in Theorem 3.1.10). With respect to arbitrary perturbations, the eigenvalue backward error is 0.3177. Figure 5.2.1 illustrates the effect of real T-palindromic, complex T-palindromic and arbitrary perturbations on the eigenvalues of $L(z)$ so that they move to form an eigenvalue at λ for the respective perturbed pencils.

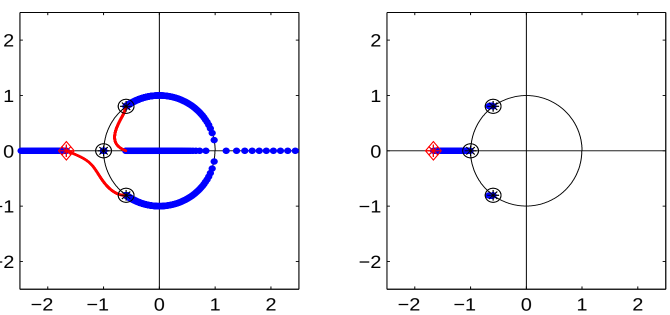

Figure 5.2.1: Eigenvalue perturbation curves for the real T-palindromic pencil $L(z)$ of Example 2.5.3 with respect to real and complex T-palindromic perturbations (left) and arbitrary perturbations (right).

The plot on the left of Figure 5.2.1 shows the effect of perturbations $L(z)+t\Delta L(z)$ on the eigenvalues of $L(z)$ (in thick curves) as t moves from 0 to 1, $\Delta L(z)$ being the minimal real T-palindromic perturbation to $L(z)$ corresponding to $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(L,\lambda)$ that induces eigenvalues at $(\lambda, 1/\lambda)$. In this case eigenvalue curves starting from the eigenvalues $-0.5954 \pm 0.8034i$ (each marked by a star surrounded by a circle) on the unit circle coalesce on the unit circle and split out with one of the branches moving over ∞ to form the eigenvalue λ (marked by a star surrounded by a diamond) and the other moving out to form the eigenvalue $1/\lambda$ (inside the unit circle) as t moves from 0 to 1.

The movement of the perturbed eigenvalues under the given real T-palindromic perturbations may partly be attributed to two facts. The first is that -1 is always an eigenvalue of a T-palindromic polynomial of odd degree and odd size (since $P(-1)$ is then a skew symmetric matrix of odd size and thus singular) of which the given pencil $L(z)$ is a particular case. The second fact is that eigenvalues of real T-palindromic polynomials occur in quadruples $(\mu, \bar{\mu}, 1/\mu, 1/\bar{\mu})$. This symmetry breaks down only on the unit circle and on the real line where it reduces to the pairing $(\mu, 1/\mu)$. Since the only eigenvalues of $L(z)$ other than -1 are also on the unit circle, in order to maintain the eigenvalue symmetry, they have to pass through the intersection of the unit circle and the real line to form the eigenvalues at λ and $1/\lambda$.

The plot on the left of Figure 5.2.1 also shows the effect of perturbations $L(z)+t\widehat{\Delta L}(z)$ on the eigenvalues of $L(z)$ (in thin curves) as t moves from 0 to 1, $\widehat{\Delta L}(z)$ being the minimal complex T-palindromic perturbation to $L(z)$ corresponding to $\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(L,\lambda)$ that induces eigenvalues at $(\lambda, 1/\lambda)$. Also in this case -1 cannot be moved to λ with respect to T-palindromic perturbations. Instead, an eigenvalue curve starting from $-0.5954+0.8034i$ moves to λ while another starting from $-0.5954-0.8034i$ moves to $1/\lambda$ (inside the unit circle) as t moves from 0 to 1.

Finally, the plot on the right of Figure 5.2.1 shows the effect of perturbations $L(z)+t\widetilde{\Delta L}(z)$ on the eigenvalues of $L(z)$ as t moves from 0 to 1, $\widetilde{\Delta L}(z)$ being an optimal perturbation corresponding to $\eta_{w,2}(L,\lambda)$ that induces an eigenvalue at λ and is not T-palindromic. In this case, the nearest eigenvalue -1 of $L(z)$ moves to form the eigenvalue λ of $L(z)+\widetilde{\Delta L}(z)$.

Table 5.2.1 compares the backward errors $\eta_{w,2}(L,\lambda)$, $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(L,\lambda)$ and $\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(L,\lambda)$ for the T-palindromic pencil $L(z)$ in Example 5.2.9 as λ converges to -1 along the real line. Observe that while both $\eta_{w,2}(L,\lambda)$, and $\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(L,\lambda)$ decrease, $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(L,\lambda)$ increases as λ approaches -1. This leads to large differences between $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(L,\lambda)$ and the other backward errors at values of λ close to -1.

Table 5.2.1: *Values of $\eta_{w,2}(L,\lambda)$, $\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(L,\lambda)$ and $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(L,\lambda)$ for the T-palindromic pencil $L(z)$ in Example 5.2.9 as $\lambda \to -1$.*

λ	$\eta_{w,2}(P,\lambda)$	$\eta_{w,2}^{\mathrm{pal}_\mathrm{T}}(P,\lambda)$	$\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$
-1.6656	0.1692	0.3177	0.5614
-1.5500	0.1501	0.3076	0.5623
-1.4500	0.1308	0.2992	0.5631
-1.3500	0.1086	0.2912	0.5638
-1.2500	0.0827	0.2842	0.5644
-1.1500	0.0528	0.2788	0.5649

5.2.3 Real T-antipalindromic polynomials

Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a T-antipalindromic polynomial i.e., ${A_j}^T = -A_{m-j}$ for all j. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $k = \lfloor \frac{m-1}{2} \rfloor$. A formula for $\eta_{w,2}^{\mathrm{antipal_T}}(P,\lambda)$ was derived for the case that $m = 1$ in Section 3.3 by applying Theorem 1.2.7. But the same approach does not work if $m > 1$. However, in this section we obtained real T-antipalindromic backward error of a real λ for any real T-antipalindromic polynomial. Therefore, denoting the real $\bullet$-antipalindromic structure by $\mathrm{antipal}_{\bullet,\mathbb{R}}$ where $\bullet \in \{*, T\}$, that is

$$\mathrm{antipal}_{\bullet,\mathbb{R}} := \left\{ (\Delta_0, \ldots, \Delta_m) \in (\mathbb{R}^{n\times n})^{m+1} \middle|\ \Delta_j^\bullet = -\Delta_{m-j},\ j = 0, \ldots, m \right\}$$

we have

$$\eta_{w,2}^{\mathrm{antipal_{T,\mathbb{R}}}}(P,\lambda) = \eta_{w,2}^{\mathrm{antipal_{*,\mathbb{R}}}}(P,\lambda)$$

in such cases.

Note that in Theorem 3.3.1 if $\lambda \in \mathbb{R} \setminus \{0, 1, -1\}$ and $P(z)$ is a real T-antipalindromic polynomial, then G and S_j are *real* Hermitian matrices where G is defined by (3.3.4) and S_j for each $j = 0, \ldots, p$ with $p = k$ when m is odd and $p = k+1$ when m is even, are defined by (3.3.6). Indeed (3.3.7) and (3.3.8) become

$$\eta_{w,2}^{\mathrm{antipal_{T,\mathbb{R}}}}(P,\lambda) = \left(\sup\left\{ \frac{v^T G v}{v^T v} \middle|\ v \in \mathbb{C}^{n(m+1)} \setminus \{0\},\ v^T S_j v = 0,\ j = 0, \ldots, p \right\} \right)^{-1}. \tag{5.2.21}$$

Thus in the view of (5.2.21), we can apply Theorem 5.2.5 to obtain an analogue of Theorem 5.2.7 for real T-antipalindromic polynomials when $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$.

Theorem 5.2.10. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a real T-antipalindromic polynomial and let $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. Suppose that $\det(P(\lambda)) \neq 0$ so that $M = (P(\lambda))^{-1}$ exists and set $k = \lfloor \frac{m-1}{2} \rfloor$. Then*

$$\lambda_{\max}^{\star} := \min_{t_0,\ldots,t_p \in \mathbb{R}} \lambda_{\max}(G + t_0\, S_0 + \cdots + t_p\, S_p)$$

is attained for some $(\hat{t}_0, \ldots, \hat{t}_p) \in \mathbb{R}^{p+1}$ where $p = k$ when m is odd and $p = k+1$ when m is even, G is defined by (3.1.15) and $S_j, j = 0, \ldots, p$ are defined by (3.1.17), respectively. If $m = 1$ or if $\lambda_{\max}^{\star}$ is a simple eigenvalue of $G + \hat{t}_0 S_0 + \cdots + \hat{t}_p S_p$, then

$$\eta_{w,2}^{\mathrm{antipal_{T,\mathbb{R}}}}(P,\lambda) = \frac{1}{\sqrt{\lambda_{\max}^{\star}}} = \left(\min_{t_0,\ldots,t_p \in \mathbb{R}} \lambda_{\max}(G + t_0\, S_0 + \cdots + t_p\, S_p) \right)^{-1/2}.$$

Proof. The proof is similar to that of Theorem 5.2.7. $\square$

Remark 5.2.11. In the view of Remark 3.3.5 an analogue of Theorem 5.2.10 that obtains $\eta^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}_{\widehat{w},2}(P,\lambda)$ for a real T-antipalindromic polynomial $P(z)$ with a restricted perturbation set $\mathrm{antipal}_{\mathrm{T},\mathbb{R}}(I)$ defined by

$$\mathrm{antipal}_{\mathrm{T},\mathbb{R}}(I) := \left\{(\Delta_0,\ldots,\Delta_m) \in \mathrm{antipal}_{\bullet,\mathbb{R}} \middle|\ \Delta_j = 0 \text{ for } j \notin I\right\}$$

may also be derived. Also an optimal T-antipalindromic perturbation with norm equal to $\eta^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}_{\widehat{w},2}(P,\lambda)$ may be constructed by following the procedure in Remark 3.1.9.

To compare the backward errors $\eta_{w,2}(P,\lambda)$, $\eta^{\mathrm{antipal}_{\mathrm{T}}}_{w,2}(P,\lambda)$ and $\eta^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}_{w,2}(P,\lambda)$, we take a random real T-antipalindromic pencil $P(z) = A - zA^T$ of size 4×4. Let $w = (1,1)$. In Table 5.2.2 we observed that as expected both $\eta_{w,2}(P,\lambda)$ and $\eta^{\mathrm{antipal}_{\mathrm{T}}}_{w,2}(P,\lambda)$ are approaching the same number 0.2518 ($\eta_{w,2}(P,1) = 0.2518$) when λ approaches 1. But $\eta^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}_{w,2}(P,\lambda)$ remains away from 0.2518 as λ approaches 1. This leads to large differences between $\eta^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}_{w,2}(L,\lambda)$ and the other backward errors at values of λ close to 1.

Table 5.2.2: *Values of $\eta_{w,2}(P,\lambda)$, $\eta^{\mathrm{antipal}_{\mathrm{T}}}_{w,2}(P,\lambda)$ and $\eta^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}_{w,2}(P,\lambda)$ for the T-antipalindromic pencil $P(z)$ as $\lambda \to 1$.*

λ	$\eta_{w,2}(P,\lambda)$	$\eta^{\mathrm{antipal}_{\mathrm{T}}}_{w,2}(P,\lambda)$	$\eta^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}_{w,2}(P,\lambda)$
1.3000	0.2769	0.3209	0.8062
1.2500	0.2661	0.3035	0.8059
1.2000	0.2569	0.2875	0.8057
1.1500	0.2498	0.2734	0.8055
1.1000	0.2459	0.2620	0.8053
1.0500	0.2462	0.2545	0.8052
1.0200	0.2488	0.2522	0.8052

Given a real pencil $P(z) = A_0 + zA_1$ with T-palindromic or T-antipalindromic structure, the next result bounds the real structured backward error in terms of the unstructured backward error $\eta_{w,2}(L,\lambda)$ whenever λ comes from the unit circle. To keep the notation simple we prove the result for the weight vector $w = (1,1)$.

Theorem 5.2.12. *(1) Let $P(z) = A + zA^T$, where $A \in \mathbb{R}^{n\times n}$ be a real T-palindromic pencil. Then*

$$\eta_{w,2}(P,e^{i\theta}) \leq \eta^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}_{w,2}(P,e^{i\theta}) \leq \sqrt{1+\tan^2\left(\frac{\theta}{2}\right)}\,\eta_{w,2}(P,e^{i\theta}), \quad \theta \in [0,2\pi) \tag{5.2.22}$$

where $\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P, e^{i\theta})$ denotes the eigenvalue backward error of $e^{i\theta}$ under real T-palindromic perturbations with weight vector $w = (1, 1)$.

(2) Let $P(z) = A - zA^T$, where $A \in \mathbb{R}^{n\times n}$ be a real T-antipalindromic pencil. Then

$$\eta_{w,2}(P, e^{i\theta}) \le \eta_{w,2}^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}(P, e^{i\theta}) \le \sqrt{1 + \cot^2\left(\frac{\theta}{2}\right)}\, \eta_{w,2}(P, e^{i\theta}), \quad \theta \in [0, 2\pi) \qquad (5.2.23)$$

where $\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}(P, e^{i\theta})$ denotes the eigenvalue backward error of $e^{i\theta}$ with respect to real T-antipalindromic perturbations with weight vector $w = (1, 1)$.

Proof. Proof of (1): Let $\lambda = e^{i\theta}$ $\theta \in [0, 2\pi)$. Note that

$$\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P, \lambda) = \eta_{w,2}(P, \lambda) \quad \text{if } \lambda = \pm 1.$$

Thus assume that $\lambda \neq \pm 1$. Set $\mu = \frac{1-\lambda}{1+\lambda}$. Then $\mu = -i \tan\left(\frac{\theta}{2}\right)$. Now

$$P(\lambda) = A + \lambda A^T = \frac{1}{1+\mu}(\hat{A} + \mu\hat{B}), \quad \text{where} \quad \hat{A} = A + A^T \quad \text{and} \quad B = A - A^T.$$

$$\text{Consider} \quad \Delta = \sigma_{\min}(\hat{A} + \mu\hat{B})[u\ \overline{u}]\,[u\ \overline{u}]^\dagger,$$

where u is a unit right singular vector of $\hat{A} + \mu\hat{B}$ corresponding to the minimum singular value $\sigma_{\min}(\hat{A} + \mu\hat{B})$. By Lemma 1.2.15, $\Delta \in \mathbb{R}^{n\times n}$ and since $(\hat{A} + \mu\hat{B})^* = (\hat{A} + \mu\hat{B})$, we have

$$(\hat{A} + \mu\hat{B} - \Delta)u = (\hat{A} + \mu\hat{B})u - \Delta u = \sigma_{\min}(\hat{A} + \mu\hat{B})u - \sigma_{\min}(\hat{A} + \mu\hat{B})u = 0,$$

Again

$$\begin{aligned}
\det\big((\hat{A} + \mu\hat{B}) - \Delta\big) = 0 \iff & \det\Big((\hat{A} - \Delta) + \frac{1-\lambda}{1+\lambda}\hat{B}\Big) = 0 \\
\iff & \det\Big(\big(A - \frac{\Delta}{2}\big) + \lambda\big(A^T - \frac{\Delta}{2}\big)\Big) = 0 \\
\iff & \det\Big(\big(A + \lambda A^T\big) - \big(\frac{\Delta}{2} + \lambda\frac{\Delta}{2}\big)\Big) = 0
\end{aligned}$$

This implies

$$\begin{aligned}
\Big(\eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P, \lambda)\Big)^2 &\le \big\|\Delta/2\big\|^2 + \big\|\Delta/2\big\|^2 \\
&= \frac{\|\Delta\|^2}{2} = \frac{(\sigma_{\min}(\hat{A} + \mu\hat{B}))^2}{2} = |1 + \mu|^2 \frac{(\sigma_{\min}(A + \lambda A^T))^2}{2} \\
&= \left(1 + \tan^2\left(\frac{\theta}{2}\right)\right) \frac{(\sigma_{\min}(A + \lambda A^T))^2}{2}. \qquad (5.2.24)
\end{aligned}$$

Also from Theorem 1.2.5, we have

$$\eta_{w,2}(P,\lambda) = \frac{\sigma_{\min}(A+\lambda A^T)}{\sqrt{2}} \quad \text{as} \quad |\lambda| = 1. \tag{5.2.25}$$

Hence the proof of (5.2.22) follows from (5.2.24) and (5.2.25).

Proof of (2): Proof is similar to that of (1). Let $\lambda = e^{i\theta},\ \theta \in [0, 2\pi)$. Again note that $\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}(L,\lambda) = \eta_{w,2}(L,\lambda)$ if $\lambda = \pm 1$. Thus assume that $\lambda \neq \pm 1$ and set $\mu = \frac{1-\lambda}{1+\lambda}$. Then $\mu = -i\tan\left(\frac{\theta}{2}\right)$. This implies

$$P(\lambda) = \frac{\mu}{1+\mu}\big(\hat{A} + \rho\hat{B}\big), \quad \text{where} \quad \hat{A} = A + A^T, \quad \hat{B} = A - A^T \quad \text{and} \quad \rho = \frac{1}{\mu}.$$

Now consider

$$\Delta = \sigma_{\min}\big(\hat{A} + \rho\hat{B}\big)\,[u\ \overline{u}]\,[u\ \overline{u}]^{\dagger},$$

where u is a unit right singular vector of $\hat{A} + \rho\hat{B}$ corresponding to the minimum singular value $\sigma_{\min}(\hat{A} + \rho\hat{B})$. By Lemma 1.2.15, $\Delta \in \mathbb{R}^{n\times n}$ and since $(\hat{A} + \rho\hat{B})^* = (\hat{A} + \rho\hat{B})$, we have

$$(\hat{A} + \rho\hat{B} - \Delta)u = (\hat{A} + \rho\hat{B})u - \Delta u = \sigma_{\min}(\hat{A} + \rho\hat{B})u - \sigma_{\min}(\hat{A} + \rho\hat{B})u = 0,$$

Again

$$\begin{aligned}
\det\big((\hat{A} + \rho\hat{B}) - \Delta\big) = 0 \quad &\Longleftrightarrow \quad \det\Big((\hat{A} - \Delta) + \frac{1+\lambda}{1-\lambda}\hat{B}\Big) = 0 \\
&\Longleftrightarrow \quad \det\Big(\big(A - \frac{\Delta}{2}\big) - \lambda\big(A^T - \frac{\Delta}{2}\big)\Big) = 0 \\
&\Longleftrightarrow \quad \det\Big((A - \lambda A^T) - \big(\frac{\Delta}{2} - \lambda\frac{\Delta}{2}\big)\Big) = 0
\end{aligned}$$

This implies

$$\begin{aligned}
\Big(\eta_{w,2}^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}(P,\lambda)\Big)^2 &\leq \big\|\Delta/2\big\|^2 + \big\|\Delta/2\big\|^2 \\
&= \frac{\|\Delta\|^2}{2} = \frac{(\sigma_{\min}(\hat{A} + \rho\hat{B}))^2}{2} = \big|1 + \mu^{-1}\big|^2\frac{(\sigma_{\min}(A - \lambda A^T))^2}{2} \\
&= \left(1 + \cot^2\left(\frac{\theta}{2}\right)\right)\frac{(\sigma_{\min}(A - \lambda A^T))^2}{2}.
\end{aligned} \tag{5.2.26}$$

Also from Theorem 1.2.5, we have

$$\eta_{w,2}(P,\lambda) = \frac{\sigma_{\min}(A-\lambda A^T)}{\sqrt{2}} \quad \text{as} \quad |\lambda| = 1. \tag{5.2.27}$$

Thus (5.2.23) follows from (5.2.26) and (5.2.27). $\square$

Remark 5.2.13. By the above theorem, note that

$$\eta_{w,2}(P,e^{i\theta}) \leq \eta_{w,2}^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P,e^{i\theta}) \leq \sqrt{2}\,\eta_{w,2}(P,e^{i\theta}), \quad \text{if} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

This shows that the real T-palindromic backward error and unstructured backward error are close to each other when λ is on the right half of the unit circle. The same also holds for real T-antipalindromic pencils when λ is on the left half of the unit circle, i.e. $\lambda = e^{i\theta}$ with $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.

5.2.4 Real T-alternating polynomials

Note that in Theorem 3.4.1 if $\lambda \in \mathbb{R} \setminus \{0\}$ and $P(z)$ is a real T-alternating matrix polynomial, then G, S_{ej} and S_{oj} are real Hermitian matrices. Therefore, denoting the real $\bullet$-alternating structure by $\mathrm{alt}_{\bullet,\mathbb{R}}$, where $\bullet = T$ or $\bullet = *$, we have

$$\mathrm{alt}_{\bullet,\mathbb{R}} := \left\{(\Delta_0,\ldots,\Delta_m) \in \mathrm{alt}_\bullet \middle| \Delta_j \in \mathbb{R}^{n\times n} \ \text{ for } j = 0,\ldots,m\right\} \tag{5.2.28}$$

In such cases, we have

$$\eta_{w,2}^{\mathrm{alt}_{\mathrm{T},\mathbb{R}}}(P,\lambda) = \eta_{w,2}^{\mathrm{alt}_{*,\mathbb{R}}}(P,\lambda). \tag{5.2.29}$$

Note that the formula for $\eta_{w,2}^{\mathrm{alt}_{\mathrm{T}}}(P,\lambda)$ was derived for complex T-alternating pencils and T-even quadratic polynomials in Theorem 3.4.2 and Theorem 3.4.3, respectively. But in view of Theorem 5.2.5, the real structured backward error $\eta_{w,2}^{\mathrm{alt}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ can be derived for any real T-alternating matrix polynomial whenever $\lambda \in \mathbb{R} \setminus \{0\}$.

Theorem 5.2.14. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a real T-even matrix polynomial and let $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that $\det(P(\lambda)) \neq 0$ so that $M = (P(\lambda))^{-1}$ exists and $k = \lfloor\frac{m-1}{2}\rfloor$. Then*

$$\lambda_{\max}^{\star} := \min_{t_0,\ldots,t_k \in \mathbb{R}} \lambda_{\max}(G + t_0\,S_{e0} + \cdots + t_k\,S_{ek})$$

is attained for some $(t_0^\star,\ldots,t_k^\star) \in \mathbb{R}^{k+1}$ where G and S_{ej} are as defined in Theorem 3.4.1. If $m \leq 2$ or $\lambda_{\max}^\star$ is a simple eigenvalue of $G + t_0^\star\,S_{e0} + \cdots + t_k^\star\,S_{ek}$ then

$$\eta_{w,2}^{\mathrm{even}_{\mathrm{T},\mathbb{R}}}(P,\lambda) = \frac{1}{\sqrt{\lambda_{\max}^\star}} = \left(\min_{t_0,\ldots,t_k \in \mathbb{R}} \lambda_{\max}(G + t_0\,S_{e0} + \cdots + t_k\,S_{ek})\right)^{-1/2}.$$

Proof. Let $\hat{S}_{ej} = \widetilde{S}_{ej} + \widetilde{S}_{ej}^T$, where $\widetilde{S}_{ej}$ for $j = 0 : k$ are as defined in (**??**). Since we aim to apply Theorem 1.2.18 with real Hermitian matrices, we have to check whether each nontrivial linear combination of $S_{e0},\ldots,S_{ek}$, or, equivalently, of $\hat{S}_{e0},\ldots,\hat{S}_{ek}$ is indefinite. Thus, assume that $\alpha := \begin{bmatrix} \alpha_0, & \ldots, & \alpha_k \end{bmatrix}^T \in \mathbb{R}^{k+1}\setminus\{0\}$ is such that

$$S := \sum_{j=0}^{k} \alpha_j \hat{S}_{ej} = \sum_{j=0}^{k} \alpha_j \Big(\Lambda_m^T e_{2j+2}^T \otimes M^T + e_{2j+2}\Lambda_m \otimes M\Big)$$

is semidefinite. Then we have to show that $\alpha = 0$. By setting

$$Q := \begin{bmatrix} 1 & -\lambda & 0 & \ldots & 0 \\ 0 & 1 & -\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix}, \quad \text{we obtain}$$

$$(Q\otimes I_n)^T S (Q\otimes I_n) = \begin{bmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,m+1} \\ A_{1,2}^T & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ A_{1,m+1}^T & 0 & \ldots & 0 \end{bmatrix}, \tag{5.2.30}$$

where $A_{1,l}$ for $l = 2,\ldots,m+1$ are given by:

$$\begin{aligned} A_{1,2j} &= \alpha_{j-1}M^T, \text{ for } j = 1,\ldots,k \\ A_{1,2j+1} &= -\alpha_{j-1}\,\lambda M^T, \text{ for } j = 1,\ldots,k \\ A_{1,m+1} &= \alpha_k M^T, \end{aligned}$$

when m is odd and by

$$\begin{aligned} A_{1,2j} &= \alpha_{j-1}M^T, \text{ for } j = 1,\ldots,k+1 \\ A_{1,2j+1} &= -\alpha_{j-1}\,\lambda M^T, \text{ for } j = 1,\ldots,k+1, \end{aligned}$$

when m is even. Since S is semidefinite, it follows that $A_{1,j} = 0$ for all $j = 2 : m+1$, But then $\alpha = 0$ as M is invertible and $\lambda \neq 0$. Hence the proof. $\square$

Similarly, the following theorem gives a formula for $\eta_{w,2}^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ when $\lambda \in \mathbb{R}$ and $P(z)$ is a real T-odd matrix polynomial of any degree.

Theorem 5.2.15. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a real T-odd matrix polynomial and λ be a nonzero real number. Suppose that $\det(P(\lambda)) \neq 0$ so that $M = (P(\lambda))^{-1}$ exists and let $k = \lfloor \frac{m}{2} \rfloor$. Then*

$$\lambda_{\max}^{\star} := \min_{t_0,\ldots,t_k\in\mathbb{R}} \lambda_{\max}(G + t_0\, S_{o0} + \cdots + t_k\, S_{ok})$$

is attained for some $(t_0^\star,\ldots,t_k^\star)\in\mathbb{R}^{k+1}$ *where* G *and* S_{oj} *are as defined in Theorem 3.4.1. If* $m=1$ *or* $\lambda_{\max}^\star$ *is a simple eigenvalue of* $G+t_0^\star S_{o0}+\cdots+t_k^\star S_{ok}$ *then*

$$\eta_{w,2}^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P,\lambda)=\frac{1}{\sqrt{\lambda_{\max}^\star}}=\left(\min_{t_0,\ldots,t_k\in\mathbb{R}}\lambda_{\max}(G+t_0\,S_{o0}+\cdots+t_k\,S_{ok})\right)^{-1/2}.$$

Proof. Proof follows by replacing S_{ej} matrices from S_{oj} matrices in the proof of Theorem 5.2.14. In this case $A_{1,l}$ coefficients for $l=2,\ldots m+1$ of (5.2.30) are given by

$$\begin{aligned}A_{1,2j}&=-\alpha_{j-1}\,\lambda M^T,\ \text{for}\,j=1,\ldots,k\\A_{1,2j+1}&=\alpha_j M^T,\ \text{for}\,j=1,\ldots,k\\A_{1,m+1}&=-\alpha_k\lambda M^T,\end{aligned}$$

when m is odd and by

$$\begin{aligned}A_{1,2j}&=-\alpha_{j-1}\,\lambda M^T,\ \text{for}\,j=1,\ldots,k\\A_{1,2j+1}&=\alpha_j M^T,\ \text{for}\,j=1,\ldots,k,\end{aligned}$$

when m is even. Here too S is semidefinite, we have $A_{1,k}=0$ for $k=1,\ldots,m+1$ which implies that $\alpha=0$. □

Remark 5.2.16. Optimal real T-even and real T-odd perturbations with norms equal to $\eta_{w,2}^{\mathrm{even}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ and $\eta_{w,2}^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ respectively can be constructed by following the ideas in Remark 3.1.9. If zero weights are allowed in the weight vector w then $\eta_{\widehat{w},2}^{\mathrm{alt}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ may be obtained with a restricted perturbation set. This can be proved via arguments similar to those in the proof of Theorem 3.4.4.

Table 5.2.3 compares the backward errors $\eta_{w,2}(P,\lambda)$, $\eta_{w,2}^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ and $\eta_{w,2}^{\mathrm{odd}_{\mathrm{T}}}(P,\lambda)$ with $w=(1,1)$ for a random T-odd pencil $P(z)$ of size 4×4. we observed that as expected both $\eta_{w,2}(P,\lambda)$ and $\eta_{w,2}^{\mathrm{odd}_{\mathrm{T}}}(P,\lambda)$ are approaching the same number 0.3560 ($\eta_{w,2}(P,0)=0.3560$) when λ approaches 0. But $\eta_{w,2}^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ remains away from 0.3560 as λ approaches 0. This leads to large differences between $\eta_{w,2}^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(L,\lambda)$ and the other backward errors at values of λ close to 0.

Remark 5.2.17. Note that $\mathrm{rev}P(z)$ of the T-odd pencil $P(z)=A+zB$ in Table 5.2.3 is T-even. Therefore backward errors $\eta_{w,2}(P,\lambda)$, $\eta_{w,2}^{\mathrm{even}_{\mathrm{T}}}(P,\lambda)$ and $\eta_{w,2}^{\mathrm{even}_{\mathrm{T},\mathbb{R}}}(P,\lambda)$ exhibit the same behaviour for λ values close to ∞.

The following theorem gives an analogue of Theorem 5.2.12 for T-alternating pencils.

Table 5.2.3: *Values of* $\eta_{w,2}(P,\lambda)$, $\eta_{w,2}^{\mathrm{odd_T}}(P,\lambda)$ *and* $\eta_{w,2}^{\mathrm{odd_{T,\mathbb{R}}}}(P,\lambda)$ *for the T-odd pencil* $P(z)$ *as* $\lambda \to 0$.

λ	$\eta_{w,2}(P,\lambda)$	$\eta_{w,2}^{\mathrm{odd_T}}(P,\lambda)$	$\eta_{w,2}^{\mathrm{odd_{T,\mathbb{R}}}}(P,\lambda)$
0.1364	0.3968	0.4576	1.1325
0.1136	0.3783	0.4296	1.1344
0.0909	0.3633	0.4050	1.1360
0.0682	0.3528	0.3845	1.1372
0.0455	0.3475	0.3690	1.1380
0.0227	0.3484	0.3593	1.1385
0.0091	0.3522	0.3566	1.1387

Theorem 5.2.18. *Let* $P(z) = A + zB$, *where* $A,\ B \in \mathbb{R}^{n\times n}$ *and* $w = (1,1)$.
(1) Suppose $A^T = A$ *and* $B^T = -B$. *Then*

$$\eta_{w,2}(P,\lambda) \leq \eta_{w,2}^{\mathrm{even_{T,\mathbb{R}}}}(P,\lambda) \leq \sqrt{1+|\lambda|^2}\,\eta_{w,2}(P,\lambda), \quad \text{if } \lambda \in i\mathbb{R}\setminus\{0\}, \tag{5.2.31}$$

where $\eta_{w,2}^{\mathrm{even_{T,\mathbb{R}}}}(P,\lambda)$ *denotes the eigenvalue backward error with respect to real* T*-even perturbations.*
(2) Suppose $A^T = -A$ *and* $B^T = B$. *Then*

$$\eta_{w,2}(P,\lambda) \leq \eta_{w,2}^{\mathrm{odd_{T,\mathbb{R}}}}(P,\lambda) \leq \sqrt{1+\frac{1}{|\lambda|^2}}\,\eta_{w,2}(P,\lambda), \quad \text{if } \lambda \in i\mathbb{R}\setminus\{0\}, \tag{5.2.32}$$

where $\eta_{w,2}^{\mathrm{odd_{T,\mathbb{R}}}}(P,\lambda)$ *denotes the eigenvalue backward error with respect to real* T*-odd perturbations.*

Proof. Let $\lambda = i\gamma,\ \gamma \in \mathbb{R}\setminus\{0\}$.

Proof of (1): Let $\Delta := \sigma_{\min}(A+i\gamma B)\,[u\ \overline{u}]\,[u\ \overline{u}]^{\dagger}$, where u is a unit right singular vector of $(A+i\gamma B)$ corresponding to minimum singular value $\sigma_{\min}(A+i\gamma B)$. Then by Lemma 1.2.15, $\Delta \in \mathbb{R}^{n\times n}$ and since $A+i\gamma B$ is Hermitian, we have

$$\big((A-\Delta)+i\gamma B\big)u = (A+i\gamma B)u - \Delta u = \sigma_{\min}(A+i\gamma B)u - \sigma_{\min}(A+i\gamma B)u = 0.$$

This implies

$$\eta_{w,2}^{\mathrm{even_{T,\mathbb{R}}}}(P,i\gamma) \leq \|\Delta\| = \sigma_{\min}(A+i\gamma B). \tag{5.2.33}$$

Also from Theorem 1.2.5
$$\eta_{w,2}(P,\lambda) = \frac{\sigma_{\min}(A+i\gamma B)}{\sqrt{1+\gamma^2}}. \tag{5.2.34}$$
Therefore (5.2.31) follows from (5.2.33) and (5.2.34).

Proof of (2): Note that $P(\lambda) = \lambda(B+\mu A)$, where $\mu = \frac{1}{\lambda}$. Since pencil $(B+\mu A)$ is T-even, consider
$$\Delta = \sigma_{\min}(B+\mu A)\,[u\ \overline{u}]\,[u\ \overline{u}]^{\dagger},$$
where u is a unit right singular vector of $B+\mu A$ corresponding to minimum singular value $\sigma_{\min}(B+\mu A)$. As $B+\mu A$ is Hermitian by Lemma 1.2.15, $\Delta \in \mathbb{R}^{n\times n}$ and we have
$$\big((B-\Delta)+\mu A\big)u = (B+\mu A)u - \Delta u = \sigma_{\min}(B+\mu A)u - \sigma_{\min}(B+\mu A)u = 0.$$
This implies
$$\eta_{w,2}^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P,i\gamma) \le \|\Delta\| = \sigma_{\min}(B+\mu A) = \frac{\sigma_{\min}(A+i\gamma B)}{|\gamma|}. \tag{5.2.35}$$
Also from Theorem 1.2.5
$$\eta_{w,2}(P,\lambda) = \frac{\sigma_{\min}(A+i\gamma B)}{\sqrt{1+\gamma^2}}. \tag{5.2.36}$$
Therefore (5.2.32) follows from (5.2.35) and (5.2.36). □

Remark 5.2.19. *Theorem 5.2.18 implies that*
$$\eta_{w,2}(P,i\gamma) \le \eta_{w,2}^{\mathrm{even}_{\mathrm{T},\mathbb{R}}}(P,i\gamma) \le \sqrt{2}\,\eta_{w,2}(P,i\gamma), \quad \textit{if}\ \ \gamma \in [-1,1].$$
This shows the real T-even backward error is close to the unstructured backward error when γ belongs to the interval $[-1,1]$. The same also holds for real T-odd pencils when $\lambda = i\gamma$ and γ is outside the interval $[-1,1]$.

5.2.5 Real skew-symmetric polynomials

Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a skew-symmetric polynomial, i.e., $(A_0,\ldots,A_m) \in (\mathrm{SSym(n)})^{m+1}$ and $\lambda \in \mathbb{C}\setminus\{0\}$. Then as a consequence of Lemma 1.2.6 we can write skew-symmetric backward error $\eta_{w,2}^{\mathrm{ssym}}(P,\lambda)$ in terms of mapping problems as
$$\eta_{w,2}^{\mathrm{ssym}}(P,\lambda) = \inf\Big\{ |||(\Delta_0,\ldots,\Delta_m)|||_{w,2} \,\Big|\, \exists\, v_0,\ldots,v_m \in \mathbb{C}^n, \quad v_\lambda \neq 0, \Delta_j \in \mathrm{SSym(n)}, \quad \Delta_j M v_\lambda = v_j, \ \ \text{for } j = 0,\ldots,m \Big\}.$$

Since Δ_j is skew symmetric for each $j = 0, \ldots, m$, therefore for any $(v_0, \ldots, v_m)$ such that $v_\lambda \neq 0$ a skew-symmetric matrix Δ may be chosen to satisfy $\Delta M v_\lambda = v_j$ if and only if $(Mv_\lambda)^T v_j = 0 \iff v^T \widetilde{S}_j v = 0$, where $v = [v_0^T, \ldots, v_m^T]$ and $\widetilde{S}_j = \Lambda^T e_{j+1}^T \otimes M^T$. The following theorem gives the analogue of Theorem 3.4.1 for skew-symmetric matrix polynomials.

Theorem 5.2.20. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be a skew-symmetric matrix polynomial. Suppose that $\lambda \in \mathbb{C} \setminus \{0\}$ such that $M = (P(\lambda))^{-1}$ exists. Let $\Lambda_m := [1, \lambda, \ldots, \lambda^m] \in \mathbb{C}^{1\times(m+1)}$ and set*

$$\widetilde{G} := (\Lambda_m^* \Lambda_m) \otimes M^* M \quad \text{and} \quad \widetilde{S}_j := \left((\Lambda_m^T e_{j+1}^T) \otimes M^T\right) \tag{5.2.37}$$

for $j = 0, \ldots, m$, where e_j denotes the j-th standard basis vector, as well as

$$\Gamma := \operatorname{diag}\left(w_0, \ldots, w_m\right) \otimes I_n, \quad G = \Gamma^{-1} \widetilde{G} \Gamma^{-1}$$

$$S_j = \Gamma^{-1}(\widetilde{S}_j + \widetilde{S}_j^T)\Gamma^{-1}, \quad \text{for} \;\; \mathrm{j} = 0, \ldots, \mathrm{m}.$$

Then

$$\eta_{w,2}^{\text{ssym}}(P, \lambda)^2 = \left(\sup \left\{ \frac{u^* G u}{u^* u} \;\middle|\; u \in (\mathbb{C}^n)^{m+1} \setminus \{0\},\, u^T S_j u = 0,\, j = 0 : m \right\} \right)^{-1}. \tag{5.2.38}$$

If $P(z)$ is a complex skew-symmetric matrix polynomial and $\lambda \in \mathbb{C} \setminus \{0\}$, then (5.2.38) shows that computing $\eta_{w,2}^{\text{ssym}}(P, \lambda)$ involves finding the supremum of $\frac{u^* G u}{u^* u}$ subject to the conditions $u^T S_0 u = 0$ and $u^T S_1 u = 0$ for the case $m = 1$. Therefore Theorem 1.2.7 cannot be applied to find $\eta_{w,2}^{\text{ssym}}(P, \lambda)$ even for the case that $P(z)$ is a pencil.

However, if $\lambda \in \mathbb{R} \setminus \{0\}$ and $P(z)$ is a real skew-symmetric matrix polynomial, then G and S_j are real Hermitian matrices. Thus denoting the real skew-symmetric backward error by $\eta_{w,2}^{\text{ssym}_\mathbb{R}}(P, \lambda)$, (5.2.38) becomes

$$\eta_{w,2}^{\text{ssym}_\mathbb{R}}(P, \lambda)^2 = \left(\sup \left\{ \frac{u^T G u}{u^T u} \;\middle|\; u \in (\mathbb{R}^n)^{m+1} \setminus \{0\},\, u^T S_j u = 0,\, j = 0 : m \right\} \right)^{-1}. \tag{5.2.39}$$

Therefore in the following, we obtain $\eta_{w,2}^{\text{ssym}_\mathbb{R}}(P, \lambda)$ by using Theorem 5.2.5.

Theorem 5.2.21. *Let $P(z) = \sum_{j=0}^m z^j A_j$ be a real skew-symmetric matrix polynomial and $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that $\det(P(\lambda)) \neq 0$ so that $M = (P(\lambda))^{-1}$ exists. Then*

$$\lambda_{\max}^\star := \min_{t_0, \ldots, t_m \in \mathbb{R}} \lambda_{\max}(G + t_0 S_0 + \cdots + t_m S_m)$$

is attained for some $(t_0^\star, \ldots, t_m^\star) \in \mathbb{R}^{m+1}$ *where* G *and* S_j *are as defined in Theorem 5.2.20. If* $\lambda_{\max}^\star$ *is a simple eigenvalue of* $G + t_0^\star S_0 + \cdots + t_k^\star S_k$*, then*

$$\eta_{w,2}^{\mathrm{ssym}_{\mathbb{R}}}(P, \lambda) = \frac{1}{\sqrt{\lambda_{\max}^\star}} = \left(\min_{t_0, \ldots, t_m \in \mathbb{R}} \lambda_{\max}(G + t_0\, S_0 + \cdots + t_m\, S_m) \right)^{-1/2}.$$

Proof. Set $\hat{S}_j = \widetilde{S}_j + \widetilde{S}_j^T$ where $\widetilde{S}_j$ for $j = 0 : m$ are as defined in (5.2.37). Since our aim is to apply Theorem 5.2.5 we check whether each nontrivial linear combination of $S_0, \ldots, S_m$, or equivalently, of $\hat{S}_0, \ldots, \hat{S}_m$ is indefinite. Suppose $\alpha := \begin{bmatrix} \alpha_0, & \ldots, & \alpha_m \end{bmatrix}^T \in \mathbb{R}^{m+1} \setminus \{0\}$ is such that

$$S := \sum_{j=0}^{m} \alpha_j \hat{S}_j = \sum_{j=0}^{m} \alpha_j \Big(\Lambda_m^T e_{j+1}^T \otimes M^T + e_{j+1} \Lambda_m \otimes M \Big)$$

is semidefinite. To complete the proof we show that $\alpha = 0$. By setting

$$Q := \begin{bmatrix} 1 & -\lambda & 0 & \ldots & 0 \\ 0 & 1 & -\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\lambda \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix}, \text{ we obtain}$$

$$(Q \otimes I_n)^T S (Q \otimes I_n) = \begin{bmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,m+1} \\ A_{1,2}^T & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ A_{1,m+1}^T & 0 & \ldots & 0 \end{bmatrix},$$

where $A_{1,l}$ for $l = 2, \ldots, m+1$ are given by:

$$A_{1,j} = \big(-\alpha_{j-2}\, \lambda + \alpha_{j-1} \big) M^T, \text{ for } j = 2, \ldots, m+1,$$

when m is odd and by

$$\begin{aligned} A_{1,j} &= \big(-\alpha_{j-2}\, \lambda + \alpha_{j-1} \big) M^T, \text{ for } j = 2, \ldots, m \\ A_{1,m+1} &= \alpha_m M^T, \end{aligned}$$

when m is even. Since S is semidefinite, it follows that $A_{1,j} = 0$ for all $j = 2 : m+1$, but then $\alpha = 0$ as M is invertible and $\lambda \neq 0$. Hence the proof. □

Remark 5.2.22. In the above theorem, an optimal real skew-symmetric matrix polynomial $\Delta(z) = \sum_{j=0}^{m} z^j \Delta_j$ satisfying $\det\big((P-\Delta)\lambda\big) = 0$ and with $|\!|\!|(\Delta_0, \ldots, \Delta_m)|\!|\!|_{w,2} = \eta_{w,2}^{\mathrm{ssym}_{\mathbb{R}}}(P, \lambda)$ may be constructed by following the procedure in Remark 2.2.5. Also an analogue result for $\eta_{\widehat{w},2}^{\mathrm{ssym}_{\mathbb{R}}}(P, \lambda)$ can be derived for the case that some of the entries in weight vector w are zero.

5.3 Real structured eigenvalue backward errors with respect to $|\!|\!|\cdot|\!|\!|_\infty$ norm

5.3.1 Real Hermitian polynomials

Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a real Hermitian polynomial, i.e. $(A_0, \ldots, A_m) \in (\mathrm{Herm}_{\mathbb{R}}(\mathrm{n}))^{m+1}$ and let $\lambda \in \mathbb{C}$. In Section 5.2.1, we have shown that the real Hermitian backward error $\eta_{w,2}^{\mathrm{Herm}_{\mathbb{R}}}(P, \lambda)$ of λ as an approximate eigenvalue of $P(z)$ is its Hermitian backward error $\eta_{w,2}^{\mathrm{Herm}}(P, \lambda)$ under certain conditions that are seen to be satisfied in most cases. We observed that whenever the assumptions in Theorem 4.1.4 hold, the same is true with respect to $|\!|\!|\cdot|\!|\!|_\infty$ also. In fact the optimal perturbations corresponding to $\eta_{\infty}^{\mathrm{Herm}}(P, \lambda)$ in Example 5.2.1 were found to be real. This is explained in the proof of the following theorem.

Theorem 5.3.1. *Let $P(z) = \sum_{j=0}^{m} z^j A_j$ be a real Hermitian polynomial. Suppose that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $M = (P(\lambda))^{-1}$ exists. Let $\mathcal{K}$ be defined by (4.1.7). For $\gamma_0, \ldots, \gamma_m \in \mathbb{R}$ with $\gamma_0^2 + \cdots + \gamma_m^2 = 1$, define*

$$g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m) = \sum_{j=0}^{m} \gamma_j^2 \frac{\|v_j\|^2}{\|Mv_\lambda\|^2}$$

for $(v_0, \ldots, v_m) \in \mathcal{K}$. Suppose that there exists $\gamma_0^\star, \ldots, \gamma_m^\star \in \mathbb{R} \setminus \{0\}$ with ${\gamma_0^\star}^2 + \cdots + {\gamma_m^\star}^2 = 1$ and $(\hat{v}_0, \ldots, \hat{v}_m) \in \mathcal{K}$ with $\|\hat{v}_j\| = \|\hat{v}_k\|\ \forall\, j, k \in \{0, \ldots, m\}$ such that

$$g_{\gamma_0^\star,\ldots,\gamma_m^\star}(\hat{v}_0, \ldots, \hat{v}_m) = \sup_{\gamma_0^2+\cdots+\gamma_m^2=1} \ \inf_{(v_0,\ldots,v_m)\in\mathcal{K}} g_{\gamma_0,\ldots,\gamma_m}(v_0, \ldots, v_m). \tag{5.3.1}$$

For $\Gamma^\star = \mathrm{diag}\,(\gamma_0^\star, \ldots, \gamma_m^\star)$, let $G(\Gamma^\star)$ and $H_j(\Gamma^\star)$, $j = 0, \ldots, m$ be as defined in Lemma 4.1.3. Let $t_0^\star, \ldots, t_m^\star \in \mathbb{R}$ such that

$$\begin{aligned} \lambda_{\max}^\star : &= \lambda_{\max}\left(G(\Gamma^\star) + t_0^\star H_0(\Gamma^\star) + \cdots + t_m^\star H_m(\Gamma^\star)\right) \\ &= \inf_{t_0,\ldots,t_m\in\mathbb{R}} \lambda_{\max}\left(G(\Gamma^\star) + t_0 H_0(\Gamma^\star) + \cdots + t_m H_m(\Gamma^\star)\right). \end{aligned}$$

If $\lambda^\star_{\max}$ is a simple eigenvalue of $G(\Gamma^\star) + t^\star_0 H_0(\Gamma^\star) + \cdots + t^\star_m H_m(\Gamma^\star)$ then

$$\eta^{\mathrm{Herm}_{\mathbb{R}}}_\infty(P,\lambda) = (\lambda^\star_{\max})^{-1/2} = \eta^{\mathrm{Herm}}_\infty(P,\lambda).$$

Proof. Note that since $(\mathrm{Herm(n)})^{\mathrm{m+1}} \supseteq (\mathrm{Herm}_{\mathbb{R}}(\mathrm{n}))^{\mathrm{m+1}}$, $\eta^{\mathrm{Herm}}_\infty(P,\lambda) \leq \eta^{\mathrm{Herm}_{\mathbb{R}}}_\infty(P,\lambda)$. We show that $\eta^{\mathrm{Herm}_{\mathbb{R}}}_\infty(P,\lambda) \leq \eta^{\mathrm{Herm}}_\infty(P,\lambda)$. Due to (5.3.1), Theorem 4.1.4 implies that

$$\eta^{\mathrm{Herm}}_\infty(P,\lambda) = \gamma_0^{\star 2}\frac{\|\hat{v}_0\|^2}{\|M\hat{v}_\lambda\|^2} + \cdots + \gamma_0^{\star 2}\frac{\|\hat{v}_m\|^2}{\|M\hat{v}_\lambda\|^2} = \eta^{\mathrm{Herm}}_{\gamma^\star,2}(P,\lambda)$$

where $\gamma^\star = (\gamma^\star_0,\ldots,\gamma^\star_m)$. Since $\gamma^\star_j \neq 0$ for all $j = 0,\ldots,m$ by Theorem 5.2.2 we have $\eta^{\mathrm{Herm}}_{\gamma^\star,2}(P,\lambda) = \eta^{\mathrm{Herm}_{\mathbb{R}}}_{\gamma^\star,2}(P,\lambda)$. Therefore there exist $\hat{\Delta}_j \in \mathrm{Herm}_{\mathbb{R}}(\mathrm{n})$, $j = 0,\ldots,m$ such that

$$\hat{\Delta}_j M\hat{v}_\lambda = \hat{v}_j \quad \text{and} \quad \|\hat{\Delta}_j\| = \frac{\|\hat{v}_j\|}{\|M\hat{v}_\lambda\|}.$$

Also due to (5.3.1), by arguing as in the proof of Theorem 4.1.4

$$\big(\eta^{\mathrm{Herm}}_\infty(P,\lambda)\big)^2 = \gamma_0^{\star 2}\frac{\|\hat{v}_0\|^2}{\|M\hat{v}_\lambda\|^2} + \cdots + \gamma_0^{\star 2}\frac{\|\hat{v}_m\|^2}{\|M\hat{v}_\lambda\|^2} = \frac{\|\hat{v}_j\|^2}{\|M\hat{v}_\lambda\|^2}$$

for all $j = 0,\ldots,m$. Thus

$$\big(\eta^{\mathrm{Herm}}_\infty(P,\lambda)\big)^2 = \|\hat{\Delta}_j\|^2 = \max\{\|\hat{\Delta}_0\|^2,\ldots,\|\hat{\Delta}_m\|^2\} \geq \eta^{\mathrm{Herm}_{\mathbb{R}}}_\infty(P,\lambda)^2$$

where the last inequality holds as $\hat{\Delta}_j \in \mathrm{Herm}_{\mathbb{R}}(\mathrm{n})$ for all j with $\det\big(P(\lambda) - \sum_{j=0}^m \lambda^j \hat{\Delta}_j\big) = 0$. Therefore

$$\eta^{\mathrm{Herm}_{\mathbb{R}}}_\infty(P,\lambda) = \eta^{\mathrm{Herm}}_\infty(P,\lambda).$$

Now the proof follows from the fact that since (5.3.1) holds, by Theorem 4.1.4 we have

$$\eta^{\mathrm{Herm}}_\infty(P,\lambda) = (\lambda^\star_{\max})^{-1/2}.$$

□

5.3.2 Real palindromic and alternating polynomials

Given a real pencil $P(z) = A + zB$ which is either T-alternating, T-palindromic or T-antipalindromic, the following result which is a counterpart of Theorems 5.2.12 and 5.2.18, bounds $\eta^{S_{\mathbb{R}}}_\infty(P,\lambda)$ in terms of $\eta_\infty(P,\lambda)$ for specific values of λ.

Theorem 5.3.2. *(1) Let $P(z) = A + zB$, where $A, B \in \mathbb{R}^{n\times n}$ with $A^T = A$ and $B^T = -B$, and $w = (1, 1)$ be a weight vector. Then*

$$\eta_\infty(P, \lambda) \le \eta_\infty^{\mathrm{even}_{\mathrm{T},\mathbb{R}}}(P, \lambda) \le (1 + |\lambda|)\, \eta_\infty(P, \lambda), \quad \text{if } \lambda \in i\mathbb{R},$$

where $\eta_\infty^{\mathrm{even}_{\mathbb{R}}}(P, \lambda)$ denotes the eigenvalue backward error with respect to real T-even perturbations.

(2) Let $P(z) = A + zB$, where $A, B \in \mathbb{R}^{n\times n}$ with $A^T = -A$ and $B^T = B$. Then

$$\eta_\infty(P, \lambda) \le \eta_\infty^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P, \lambda) \le \left(1 + \frac{1}{|\lambda|}\right) \eta_\infty(P, \lambda), \quad \text{if } \lambda \in i\mathbb{R},$$

where $\eta_\infty^{\mathrm{odd}_{\mathrm{T},\mathbb{R}}}(P, \lambda)$ denotes the eigenvalue backward error with respect to real T-odd perturbations.

(3) Let $P(z) = A + zA^T$, where $A \in \mathbb{R}^{n\times n}$. Then

$$\eta_\infty(P, \lambda) \le \eta_\infty^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P, \lambda) \le \sqrt{1 + \tan^2\left(\frac{\theta}{2}\right)}\, \eta_\infty(P, \lambda), \quad \text{if } \lambda = e^{i\theta},$$

where $\eta_\infty^{\mathrm{pal}_{\mathrm{T},\mathbb{R}}}(P, \lambda)$ denotes the eigenvalue backward error with respect to real T-palindromic perturbations.

(4) Let $P(z) = A - zA^T$, where $A \in \mathbb{R}^{n\times n}$. Then

$$\eta_\infty(P, \lambda) \le \eta_\infty^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}(P, \lambda) \le \sqrt{1 + \cot^2\left(\frac{\theta}{2}\right)}\, \eta_\infty(P, \lambda), \quad \text{if } \lambda = e^{i\theta},$$

where $\eta_\infty^{\mathrm{antipal}_{\mathrm{T},\mathbb{R}}}(P, \lambda)$ denotes the eigenvalue backward error under real T-antipalindromic perturbations.

Proof. The proof follows by replacing $|\!|\!|\cdot|\!|\!|_{w,2}$ by $|\!|\!|\cdot|\!|\!|_\infty$ and arguing as in the proof of Theorem 5.2.12 for T-palindromic/T-antipalindromic structures and of Theorem 5.2.18 for T-alternating pencils. □

5.4 Pencils arising in linear quadratic optimal control problems

In this section, we consider the pencil $L_c(z) := H_c - zN_c$ where

$$H_c := \begin{bmatrix} 0 & A & B \\ A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \quad \text{and} \quad N_c := \begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.4.1}$$

with $A, E, Q \in \mathbb{C}^{n\times n}$, $S, B \in \mathbb{C}^{n\times m}$ and $R \in \mathbb{C}^{m\times m}$. Such pencils frequently arise in linear quadratic optimal control problems in the continuous time case where Q and R are positive definite and $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$ is positive semidefinite [34, 37]. Eigenvalues of $L_c(z)$ display Hamiltonian symmetry, i.e., eigenvalues occur in pairs $(\lambda, -\bar{\lambda})$ or in quadruples $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$ if pencil is real. Therefore the eigenvalue symmetry degenerates on the imaginary axis. This makes important to find eigenvalue and eigenpair backward errors when λ lies on the imaginary axis. We use Theorem 1.2.13 and Theorem 1.2.14 to find structured eigenvalue and eigenpair backward errors of $L_c(z)$ associated with points on the imaginary axis. For this, define $\mathcal{A} := \begin{bmatrix} A & B \end{bmatrix}$, $\mathcal{E} := \begin{bmatrix} E & 0 \end{bmatrix}$ and $C := \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$.

5.4.1 Perturbation only to matrices A and B

Let $\Delta H_c = \begin{bmatrix} 0 & \Delta A & \Delta B \\ (\Delta A)^* & 0 & 0 \\ (\Delta B)^* & 0 & 0 \end{bmatrix}$, $\Delta N_c = 0$ and $\mathbb{S}$ be the collection of all pencils $\Delta L_c(z)$ of the form $\Delta L_c(z) = \Delta H_c - z\Delta N_c$. For $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^{2n+m}\setminus\{0\}$, we define the following eigenpair and eigenvalue backward errors with respect to perturbations only to blocks A and B.

$$\eta_p^{\mathbb{S}}(H_c, N_c, \lambda, x) = \inf\Big\{ \|[\Delta A\ \Delta B]\|_p \Big| \ \Delta A \in \mathbb{C}^{n\times n},\ \Delta B \in \mathbb{C}^{n\times m},\ \Delta H_c - z\Delta N_c \in \mathbb{S}, \\ \big((H_c - \Delta H_c) - \lambda(N_c - \Delta N_c)\big)x = 0\Big\}$$

and

$$\eta_p^{\mathbb{S}}(H_c, N_c, \lambda) = \inf\Big\{ \|[\Delta A\ \Delta B]\|_p \Big| \ \Delta A \in \mathbb{C}^{n\times n},\ \Delta B \in \mathbb{C}^{n\times m},\ \Delta H_c - z\Delta N_c \in \mathbb{S}, \\ \det\big((H_c - \Delta H_c) - \lambda(N_c - \Delta N_c)\big) = 0\Big\},$$

with respect to some norm $\|.\|_p$. We consider the case that $\|.\|_p$ denotes either the spectral norm or the Frobenius norm, i.e., $p = 2$ or $p = F$. If the perturbations are restricted to be real then the above backward errors are denoted by $\eta_p^{\mathbb{S}_\mathbb{R}}(H_c, N_c, \lambda, x)$ and $\eta_p^{\mathbb{S}_\mathbb{R}}(H_c, N_c, \lambda)$, respectively.

Let $\lambda \in \mathbb{C}$ and $x = [x_1^T\ x_2^T]^T \in \mathbb{C}^{2n+m} \setminus \{0\}$ where $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^{n+m}$. Then for any

$$\Delta H_c = \begin{bmatrix} 0 & \Delta\mathcal{A} \\ (\Delta\mathcal{A})^* & 0 \end{bmatrix} \quad \text{and} \quad \Delta N_c = 0,$$

where $\Delta\mathcal{A} = \begin{bmatrix} \Delta A & \Delta B \end{bmatrix}$, $\Delta A \in \mathbb{C}^{n\times n}$, and $\Delta B \in \mathbb{C}^{n\times m}$, we have $(L_c - \Delta L_c)(z)x = 0$, or equivalently

$$\begin{bmatrix} 0 & \Delta\mathcal{A} \\ (\Delta\mathcal{A})^* & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{A}-\lambda\mathcal{E} \\ \mathcal{A}^*+\lambda\mathcal{E}^* & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This implies

$$\Delta\mathcal{A}x_2 = (\mathcal{A}-\lambda\mathcal{E})x_2 \quad \text{and} \quad (\Delta\mathcal{A})^* x_1 = (\mathcal{A}^*+\lambda\mathcal{E}^*)x_1 + Cx_2. \tag{5.4.2}$$

By Theorem 1.2.14, there exist such a $\Delta\mathcal{A} \in \mathbb{C}^{n\times(n+m)}$ if and only if

$$\begin{aligned} x_2^*\big((\mathcal{A}^*+\lambda\mathcal{E}^*)x_1 + Cx_2\big) &= \big((\mathcal{A}-\lambda\mathcal{E})x_2\big)^* x_1 \\ \iff x_2^* C x_2 &= -2\operatorname{Re}(\lambda) x_2^* \mathcal{E}^* x_1 \end{aligned} \tag{5.4.3}$$

Thus provided $x = [x_1^T \ x_2^T]^T$ satisfies (5.4.3), we can find minimal Frobenius norm and minimal spectral norm maps satisfying (5.4.2) from Theorem 1.2.14. In general, if $x = [x_1^T \ x_2^T]^T$ satisfies (5.4.3), then

$$\big(\eta_F^{\mathbb{S}}(H_c, N_c, \lambda, x)\big)^2 = \frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2} + \frac{\|(\mathcal{A}^*+\lambda\mathcal{E}^*)x_1 + Cx_2\|^2}{\|x_1\|^2} - \frac{|x_1^*(\mathcal{A}-\lambda\mathcal{E})x_2|^2}{\|x_1\|^2\|x_2\|^2} \tag{5.4.4}$$

and

$$\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x) = \max\left\{\frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|}{\|x_2\|}, \frac{\|(\mathcal{A}^*+\lambda\mathcal{E}^*)x_1 + Cx_2\|}{\|x_1\|}\right\}. \tag{5.4.5}$$

We can further simplify the expressions in (5.4.4) and (5.4.5) if λ is purely imaginary and C is either definite or semidefinite.

Theorem 5.4.1. *Let $L_c(z)$ be a pencil as defined in (5.4.1), $\gamma \in \mathbb{R}$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $x = [x_1^T \ x_2^T]^T$ with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^{n+m}$. Then the following hold.*

(1) If C is definite, then $\eta_{2,F}^{\mathbb{S}}(H_c, N_c, i\gamma, x)$ is finite if and only if $x_2 = 0$. In such a case,

$$\eta_F^{\mathbb{S}}(H_c, N_c, i\gamma, x) = \frac{\|(\mathcal{A}-i\gamma\mathcal{E})^* x_1\|}{\|x_1\|} = \eta_2^{\mathbb{S}}(H_c, N_c, i\gamma, x) \tag{5.4.6}$$

and

$$\eta_F^{\mathbb{S}}(H_c, N_c, i\gamma) = \sigma_{\min}\big((\mathcal{A}-i\gamma\mathcal{E})^*\big) = \eta_2^{\mathbb{S}}(H_c, N_c, i\gamma). \tag{5.4.7}$$

(2) If C is semidefinite then $\eta^{\mathbb{S}}_{2,F}(H_c, N_c, i\gamma, x)$ is finite if and only if $Cx_2 = 0$. In such a case, let $r := (\mathcal{A} - i\gamma\mathcal{E})x_2$ and $s := (\mathcal{A} - i\gamma\mathcal{E})^* x_1$. Then

$$\left(\eta^{\mathbb{S}}_F(H_c, N_c, i\gamma, x)\right)^2 = \begin{cases} \frac{\|r\|^2}{\|x_2\|^2} + \frac{\|s\|^2}{\|x_1\|^2} - \frac{|x_1^* r|^2}{\|x_1\|^2\|x_2\|^2} & \text{if } x_1 \neq 0,\ x_2 \neq 0, \\ \frac{\|r\|^2}{\|x_2\|^2} & \text{if } x_1 = 0, \\ \frac{\|s\|^2}{\|x_1\|^2} & \text{if } x_2 = 0, \end{cases} \tag{5.4.8}$$

$$\eta^{\mathbb{S}}_2(H_c, N_c, i\gamma, x) = \begin{cases} \max\left\{\frac{\|r\|}{\|x_2\|}, \frac{\|s\|}{\|x_1\|}\right\} & \text{if } x_1 \neq 0,\ x_2 \neq 0, \\ \frac{\|r\|}{\|x_2\|} & \text{if } x_1 = 0, \\ \frac{\|s\|}{\|x_1\|} & \text{if } x_2 = 0, \end{cases} \tag{5.4.9}$$

and

$$\eta^{\mathbb{S}}_2(H_c, N_c, i\gamma) = \sigma_{\min}\left((\mathcal{A} - i\gamma\mathcal{E})^*\right) = \eta^{\mathbb{S}}_F(H_c, N_c, i\gamma). \tag{5.4.10}$$

Proof. Set $\lambda = i\gamma$, then in the view of (5.4.3) there exist $\Delta\mathcal{A}$ satisfying (5.4.2) if and only if $x_2^* C x_2 = 0$.
(1) If C is definite then $x_2^* C x_2 = 0$ implies $x_2 = 0$. This implies $x = [x_1^T\ 0]^T$, $x_1 \in \mathbb{C}^n \setminus \{0\}$ and from (5.4.4) and (5.4.5), we have

$$\eta^{\mathbb{S}}_F(H_c, N_c, \lambda, x) = \frac{\|(\mathcal{A} - \lambda\mathcal{E})^* x_1\|}{\|x_1\|} = \eta^{\mathbb{S}}_2(H_c, N_c, \lambda, x). \tag{5.4.11}$$

This proves (5.4.6).

To obtain the eigenvalue backward error, we can minimize (5.4.11) over all possible x of the form $x = [x_1^T\ 0]^T$. Therefore

$$\begin{aligned} \eta^{\mathbb{S}}_F(H_c, N_c, \lambda) &= \inf\left\{\eta^{\mathbb{S}}_F(H_c, N_c, \lambda, x) \,\middle|\, x = [x_1^T\ 0]^T \in \mathbb{C}^{2n+m},\ x_1 \in \mathbb{C}^n \setminus \{0\}\right\} \\ &= \inf_{x_1 \in \mathbb{C}^n \setminus \{0\}} \frac{\|(\mathcal{A} - \lambda\mathcal{E})^* x_1\|}{\|x_1\|} \\ &= \inf\left\{\eta^{\mathbb{S}}_2(H_c, N_c, \lambda, x) \,\middle|\, x = [x_1^T\ 0]^T \in \mathbb{C}^{2n+m},\ x_1 \in \mathbb{C}^n \setminus \{0\}\right\} \\ &= \eta^{\mathbb{S}}_2(H_c, N_c, \lambda). \end{aligned}$$

This implies

$$\eta^{\mathbb{S}}_F(H_c, N_c, \lambda) = \sigma_{\min}((\mathcal{A} - \lambda\mathcal{E})^*) = \eta^{\mathbb{S}}_2(H_c, N_c, \lambda).$$

Indeed, let u and v be unit left and right singular vectors of $(\mathcal{A}-\lambda\mathcal{E})$ corresponding to singular value $\sigma^\star := \sigma_{\min}((\mathcal{A}-\lambda\mathcal{E}))$ and consider $\Delta\hat{\mathcal{A}} = \sigma^\star uv^*$, then $\|\Delta\hat{\mathcal{A}}\|_{F,2} = \sigma^\star$ and

$$\begin{aligned}\begin{bmatrix} 0 & \Delta\hat{\mathcal{A}} \\ (\Delta\hat{\mathcal{A}})^* & 0 \end{bmatrix}\begin{bmatrix} u \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ (\Delta\hat{\mathcal{A}})^* u \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma^\star v \end{bmatrix} = \begin{bmatrix} 0 \\ (\mathcal{A}-\lambda\mathcal{E})^* u \end{bmatrix} \\ &= \begin{bmatrix} 0 & \mathcal{A}-\lambda\mathcal{E} \\ (\mathcal{A}-\lambda\mathcal{E})^* & C \end{bmatrix}\begin{bmatrix} u \\ 0 \end{bmatrix} = L_c(\lambda)\begin{bmatrix} u \\ 0 \end{bmatrix}.\end{aligned}$$

(2) If C is semidefinite. Then $x_2^* C x_2 = 0$ implies $Cx_2 = 0$.

(subcase-1): If $x = [x_1^T \ x_2^T]^T$ be such that $x_1 \in \mathbb{C}^n \setminus \{0\}$, $x_2 \in \mathbb{C}^{n+m} \setminus \{0\}$ and $Cx_2 = 0$, then from (5.4.4) and (5.4.5) we have

$$\big(\eta_F^{\mathbb{S}}(H_c, N_c, \lambda, x)\big)^2 = \frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2} + \frac{\|(\mathcal{A}^*+\lambda\mathcal{E}^*)x_1\|^2}{\|x_1\|^2} - \frac{|x_1^*(\mathcal{A}-\lambda\mathcal{E})x_2|^2}{\|x_1\|^2\|x_2\|^2} \quad (5.4.12)$$

and

$$\big(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x)\big)^2 = \max\left\{\frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2}, \frac{\|(\mathcal{A}-\lambda\mathcal{E})^* x_1\|^2}{\|x_1\|^2}\right\}. \quad (5.4.13)$$

To obtain the eigenvalue backward error $\eta_2^{\mathbb{S}}(H_c, N_c, \lambda)$ we further minimize (5.4.13) over all $x \in \mathcal{S}$ where $\mathcal{S} := \{(x_1, x_2) \,|\, x_1 \in \mathbb{C}^n \setminus \{0\},\, x_2 \in \mathbb{C}^{n+m} \setminus \{0\},\, Cx_2 = 0\}$.

$$\begin{aligned}\mu_1 : &= \inf\left\{(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x))^2, \,\middle|\, x = [x_1^T \ x_2^T]^T,\, (x_1, x_2) \in \mathcal{S}\right\} \\ &= \inf_{(x_1,x_2)\in\mathcal{S}} \max\left\{\frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2}, \frac{\|(\mathcal{A}-\lambda\mathcal{E})^* x_1\|^2}{\|x_1\|^2}\right\} \\ &= \max\left\{\inf_{x_2\in\mathbb{C}^{n+m}\setminus\{0\},(x_1,x_2)\in\mathcal{S}} \frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2}, \inf_{x_1\in\mathbb{C}^{n}\setminus\{0\},(x_1,x_2)\in\mathcal{S}} \frac{\|(\mathcal{A}-\lambda\mathcal{E})^* x_1\|^2}{\|x_1\|^2}\right\} \\ &= \max\left\{\inf_{x_2\in\mathbb{C}^{n+m}\setminus\{0\},Cx_2=0} \frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2}, \inf_{x_1\in\mathbb{C}^{n}\setminus\{0\}} \frac{\|(\mathcal{A}-\lambda\mathcal{E})^* x_1\|^2}{\|x_1\|^2}\right\} \\ &= \max\left\{\inf_{x_2\in\mathbb{C}^{n+m}\setminus\{0\},Cx_2=0} \frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2}, \sigma_{\min}^2(\mathcal{A}-\lambda\mathcal{E})\right\}. \quad (5.4.14)\end{aligned}$$

Let $\dim\big(\text{null(C)}\big) = r$. Consider $U = [u_1, \ldots, u_r]$ where $\{u_1, \ldots, u_r\}$ is an orthonormal basis of null(C). Then

$$\begin{aligned}\inf_{x_2\in\mathbb{C}^{n+m}\setminus\{0\},Cx_2=0} \frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2} &= \inf_{x_2\in\text{null(C)}\setminus\{0\}} \frac{\|(\mathcal{A}-\lambda\mathcal{E})x_2\|^2}{\|x_2\|^2} \\ &= \inf_{\alpha\in\mathbb{C}^r\setminus\{0\}} \frac{\|(\mathcal{A}-\lambda\mathcal{E})U\alpha\|^2}{\|U\alpha\|^2} \\ &= \inf_{\alpha\in\mathbb{C}^r\setminus\{0\}} \frac{\|\big((\mathcal{A}-\lambda\mathcal{E})U\big)\alpha\|^2}{\|\alpha\|^2} = \sigma_{\min}^2\big((\mathcal{A}-\lambda\mathcal{E})U\big).\end{aligned}$$

Thus from (5.4.14), we have

$$\mu_1 = \max\left\{\sigma^2_{\min}\big((\mathcal{A}-\lambda\mathcal{E})U\big), \sigma^2_{\min}(\mathcal{A}-\lambda\mathcal{E})\right\} = \sigma^2_{\min}\big((\mathcal{A}-\lambda\mathcal{E})U\big). \tag{5.4.15}$$

(subcase-2): If $x = [x_1^T \; x_2^T]^T \in \mathbb{C}^{2n+m} \setminus \{0\}$ be such that $x_2 = 0$, then from (5.4.4) and (5.4.5) we have

$$\left(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x)\right)^2 = \frac{\|(\mathcal{A}-\lambda\mathcal{E})^* x_1\|^2}{\|x_1\|^2} = \left(\eta_F^{\mathbb{S}}(H_c, N_c, \lambda, x)\right)^2. \tag{5.4.16}$$

Therefore,

$$\begin{aligned} \mu_2 : &= \inf\left\{(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x))^2, \,\middle|\, x = [x_1^T \; 0]^T \in \mathbb{C}^{2n+m}, \; x_1 \in \mathbb{C}^n \setminus \{0\}\right\} \\ &= \inf_{x_1 \in \mathbb{C}^n \setminus \{0\}} \frac{\|(\mathcal{A}-\lambda\mathcal{E})^* x_1\|^2}{\|x_1\|^2} \\ &= \sigma^2_{\min}\big((A-\lambda\mathcal{E})^*\big). \end{aligned} \tag{5.4.17}$$

(subcase-3): If $x = [x_1^T \; x_2^T]^T \in \mathbb{C}^{2n+m}$ be such that $x_1 = 0$, $x_2 \in \mathbb{C}^{n+m} \setminus \{0\}$ and $Cx_2 = 0$, then from (5.4.4) and (5.4.5) we have

$$\left(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x)\right)^2 = \frac{\|(\mathcal{A}-\lambda\mathcal{E}) x_2\|^2}{\|x_2\|^2} = \left(\eta_F^{\mathbb{S}}(H_c, N_c, \lambda, x)\right)^2. \tag{5.4.18}$$

Also

$$\begin{aligned} \mu_3 : &= \inf\left\{(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x))^2, \,\middle|\, x = [0 \; x_2^T]^T \in \mathbb{C}^{2n+m}, \, x_2 \in \mathbb{C}^{n+m} \setminus \{0\}, \, Cx_2 = 0\right\} \\ &= \inf_{x_2 \in \mathrm{null(C)} \setminus \{0\}} \frac{\|(\mathcal{A}-\lambda\mathcal{E}) x_2\|^2}{\|x_2\|^2} = \mu_1. \end{aligned} \tag{5.4.19}$$

Thus (5.4.8) and (5.4.9) follow from (5.4.12), (5.4.13), (5.4.16) and (5.4.18). Now

$$\begin{aligned} \left(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda)\right)^2 = \inf\Big\{(\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x))^2 \,\Big|\, & \; x = [x_1^T x_2^T]^T \in \mathbb{C}^{2n+m} \setminus \{0\}, \\ & x_1 \in \mathbb{C}^n, \, x_2 \in \mathbb{C}^{n+m}, \, Cx_2 = 0\Big\} \end{aligned}$$

$$\begin{aligned} &= \min\{\mu_1, \, \mu_2\} \qquad \big(\text{ From } \; (5.4.15), \; (5.4.17) \text{ and } \; (5.4.19)\big) \\ &= \min\left\{\sigma^2_{\min}\big((\mathcal{A}-\lambda\mathcal{E})U\big), \, \sigma^2_{\min}\big((A-\lambda\mathcal{E})^*\big)\right\} \\ &= \sigma^2_{\min}\big((\mathcal{A}-\lambda\mathcal{E})^*\big). \end{aligned} \tag{5.4.20}$$

Indeed, let u and v be unit left and right singular vectors of $\mathcal{A} - \lambda\mathcal{E}$ corresponding to singular value $\sigma^* := \sigma_{\min}((\mathcal{A} - \lambda\mathcal{E}))$ and consider $\Delta\hat{\mathcal{A}} = \sigma^* uv^*$, then $\|\Delta\hat{\mathcal{A}}\|_2 = \sigma^*$ and

$$\begin{bmatrix} 0 & \Delta\hat{\mathcal{A}} \\ (\Delta\hat{\mathcal{A}})^* & 0 \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ (\Delta\hat{\mathcal{A}})^* u \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma^* v \end{bmatrix} = \begin{bmatrix} 0 \\ (\mathcal{A} - \lambda\mathcal{E})^* u \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \mathcal{A} - \lambda\mathcal{E} \\ (\mathcal{A} - \lambda\mathcal{E})^* & C \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = L_c(\lambda) \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

Note that in general $\eta_2^{\mathbb{S}}(H_c, N_c, \lambda) \leq \eta_F^{\mathbb{S}}(H_c, N_c, \lambda)$. But as $\Delta\hat{\mathcal{A}}$ is a rank one matrix such that $\eta_2^{\mathbb{S}}(H_c, N_c, \lambda) = \|\Delta\hat{\mathcal{A}}\|$, therefore $\|\hat{\Delta}\mathcal{A}\|_F = \sigma^\star$ and hence

$$\eta_F^{\mathbb{S}}(H_c, N_c, \lambda) = \sigma^* = \eta_2^{\mathbb{S}}(H_c, N_c, \lambda).$$

$\square$

Remark 5.4.2. Following the same strategy as above and by using Corollary 1.2.16, we can obtain real backward error $\eta_F^{\mathbb{S}_{\mathbb{R}}}(H_c, N_c, \lambda, x)$ of an approximate eigenpair (λ, x) of $L_c(z)$ with perturbation only to matrices A and B. Indeed, let $\lambda \in \mathbb{C} \setminus \{0\}$ and $x = [x_1^T \ x_2^T]^T \in \mathbb{C}^{2n+m} \setminus \{0\}$ such that $\operatorname{rank}([x_1 \ \overline{x}_1]) = 2$ and $\operatorname{rank}([x_2 \ \overline{x}_2]) = 2$. If x_1, x_2 satisfy $x_2^* C x_2 = -2\operatorname{Re}(\lambda) x_2^* \mathcal{E}^* x_1$ and $x_2^T C x_2 = -2\lambda x_2^T \mathcal{E}^T x_1$ then by Corollary 1.2.16

$$\eta_F^{\mathbb{S}_{\mathbb{R}}}(H_c, N_c, \lambda, x)^2 := \inf \left\{ \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\|_F^2 \,\middle|\, \Delta A \in \mathbb{R}^{n \times k},\ \Delta B \in \mathbb{R}^{n \times m},\ \Delta = \begin{bmatrix} \Delta A & \Delta B \end{bmatrix}, \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} x = L_c(\lambda) x \right\}$$

$$= \|[y\,\overline{y}]\,[x_2\,\overline{x}_2]^\dagger\|_F^2 + \|[s\,\overline{s}]\,[x_1\,\overline{x}_1]^\dagger\|_F^2 - \operatorname{Trace}\left(([s\,\overline{s}]\,[x_1\,\overline{x}_1]^\dagger)([s\,\overline{s}]\,[x_1\,\overline{x}_1]^\dagger)^*([x_2\,\overline{x}_2]\,[x_2\,\overline{x}_2]^\dagger)\right),$$

where $y := (\mathcal{A} - \lambda\mathcal{E})x_2$ and $s := (\mathcal{A} - \lambda\mathcal{E})^* x_1 + Cx_2$.

5.4.2 Perturbation only to matrices A, B and E

Let $L_c(z) = H_c - zN_c$ be a pencil defined by (5.5.2). Let $\mathbb{S}'$ be the collection of all pencils of the form $\Delta H_c - z\Delta N_c$, where

$$\Delta H_c = \begin{bmatrix} 0 & \Delta A & \Delta B \\ (\Delta A)^* & 0 & 0 \\ (\Delta B)^* & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta N_c = \begin{bmatrix} 0 & \Delta E & 0 \\ -(\Delta E)^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$, we define the following eigenpair and eigenvalue backward errors with respect to perturbations only to blocks A, B and E.

$$\eta_F^{\mathbb{S}'}(H_c, N_c, \lambda, x) = \inf \Big\{ \|[\Delta A \; \Delta B \; \Delta E]\|_F \quad \Big| \Delta A, \Delta E \in \mathbb{C}^{n\times n}, \, \Delta B \in \mathbb{C}^{n\times m},$$
$$\Delta H_c - z\Delta N_c \in \mathbb{S}', \; \big((H_c - \Delta H_c) - \lambda(N_c - \Delta N_c)\big)x = 0 \Big\},$$

and

$$\eta_F^{\mathbb{S}'}(H_c, N_c, \lambda) = \inf \Big\{ \|[\Delta A \; \Delta B \; \Delta E]\|_F \quad \Big| \Delta A, \Delta E \in \mathbb{C}^{n\times n}, \, \Delta B \in \mathbb{C}^{n\times m},$$
$$\Delta H_c - z\Delta N_c \in \mathbb{S}', \; \big((H_c - \Delta H_c) - \lambda(N_c - \Delta N_c)\big)x = 0 \Big\},$$

where $\|.\|_F$ denotes the Frobenius norm of a matrix.

Let $\lambda \in \mathbb{C}$ and $x = [x_1^T \;\; x_2^T \;\; x_3^T] \in \mathbb{C}^{2n+m} \setminus \{0\}$ with $x_1, \, x_2 \in \mathbb{C}^n$ and $x_3 \in \mathbb{C}^m$. Then for any $\Delta L_c(z) = \Delta H_c - z\Delta N_c$, $\Delta L_c(\lambda)x - L_c(\lambda)x = 0$ if and only if

$$\begin{bmatrix} 0 & \Delta A - \lambda \Delta E & \Delta B \\ (\Delta A)^* + \lambda(\Delta E)^* & 0 & 0 \\ (\Delta B)^* & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & A - \lambda E & B \\ A^* + \lambda E^* & Q & S \\ B^* & S^* & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

or equivalently, if and only if

$$(\Delta A - \lambda \Delta E)x_2 + \Delta B x_3 = (A - \lambda E)x_2 + Bx_3, \tag{5.4.21}$$
$$((\Delta A)^* + \lambda(\Delta E)^*)\, x_1 = (A^* + \lambda E^*)x_1 + Qx_2 + Sx_3, \tag{5.4.22}$$
$$(\Delta B)^* x_1 = B^* x_1 + S^* x_2 + Rx_3. \tag{5.4.23}$$

The following theorem obtains backward errors for the case that λ lies on the imaginary axis.

Theorem 5.4.3. *Let $\gamma \in \mathbb{R}$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $x = \begin{bmatrix} x_1^T & x_2^T & x_3^T \end{bmatrix}^T$ where $x_1, \; x_2 \in \mathbb{C}^n, \; x_3 \in \mathbb{C}^m$. Define*

$$u := \begin{bmatrix} x_2^T & x_3^T \end{bmatrix}^T, \; w := x_1, \; r := \begin{bmatrix} A - i\gamma E & B \end{bmatrix} u, \; s := \begin{bmatrix} A - i\gamma E & B \end{bmatrix}^* x_1.$$

(1) If C is semidefinite, then $\eta_F^{\mathbb{S}'}(H_c, N_C, i\gamma, x)$ is finite if and only if $Cu = 0$. In such a case

$$\eta_F^{\mathbb{S}'}(H_c, N_c, i\gamma, x) = \left(\|\hat{\Delta}_1\|_F^2/(1+\gamma^2) + \|\hat{\Delta}_2\|_F^2 \right)^{1/2},$$

where $\hat{\Delta}_1 \in \mathbb{C}^{n\times n}, \hat{\Delta}_2 \in \mathbb{C}^{n\times m}$ are such that

$$\begin{bmatrix} \hat{\Delta}_1 & \hat{\Delta}_2 \end{bmatrix} = \begin{cases} \frac{ru^*}{\|u\|^2} & \text{if } x_1 = 0, \\ \frac{ws^*}{\|w\|^2} & \text{if } x_2 = 0, x_3 = 0 \\ \frac{ru^*}{\|u\|^2} + \frac{sw^*}{\|w\|^2}\left(I_{n+m} - \frac{uu^*}{\|u\|^2}\right) & \text{otherwise.} \end{cases} \tag{5.4.24}$$

(2) If C is definite, then $\eta_F^{\mathbb{S}'}(H_c, N_c, i\gamma, x)$ is finite if and only if $u = 0$. In such a case

$$\eta_F^{\mathbb{S}'}(H_c, N_c, i\gamma, x) = \left(\|\widetilde{\Delta}_1\|_F^2/(1+\gamma^2) + \|\widetilde{\Delta}_2\|_F^2\right)^{1/2},$$

where $\left[\begin{array}{cc} \widetilde{\Delta}_1 & \widetilde{\Delta}_2 \end{array}\right] = \frac{1}{\|w\|^2} s w^*$.

Proof. Let $\lambda = i\gamma$, $\gamma \in \mathbb{R} \setminus \{0\}$. Then (5.4.21) - (5.4.23) may also be written as follows

$$\left[\begin{array}{cc} \Delta A - i\gamma \Delta E & \Delta B \end{array}\right] \underbrace{\left[\begin{array}{c} x_2 \\ x_3 \end{array}\right]}_{=u} = \underbrace{(A - i\gamma E)x_2 + Bx_3}_{=r} \quad \text{and} \tag{5.4.25}$$

$$\left[\begin{array}{cc} \Delta A - i\gamma \Delta E & \Delta B \end{array}\right]^* \underbrace{x_1}_{=w} = \underbrace{\left[\begin{array}{ccc} A^* + i\gamma E^* & Q & S \\ B^* & S^* & R \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]}_{=:\hat{s}} \tag{5.4.26}$$

Note that there exist ΔA, ΔB, and ΔE satisfying (5.4.25) and (5.4.26) if and only if there exist $\Delta_1 \in \mathbb{C}^{n\times n}$ and $\Delta_2 \in \mathbb{C}^{n\times m}$ such that

$$\left[\begin{array}{cc} \Delta_1 & \Delta_2 \end{array}\right] u = r \quad \text{and} \quad \left[\begin{array}{cc} \Delta_1 & \Delta_2 \end{array}\right]^* w = \hat{s}. \tag{5.4.27}$$

From Theorem 1.2.14 this is equivalent to $u^*\hat{s} = r^*w$. Now

$$u^*\hat{s} = r^*w \iff u^*Cu = 0\,, \quad \text{where} \quad C = \left[\begin{array}{cc} Q & S \\ S^* & R \end{array}\right]. \tag{5.4.28}$$

(1) If C is semidefinite then $u^*Cu = 0$ implies $Cu = 0$. Thus (5.4.27) implies

$$\left[\begin{array}{cc} \Delta_1 & \Delta_2 \end{array}\right] u = r \quad \text{and} \quad \left[\begin{array}{cc} \Delta_1 & \Delta_2 \end{array}\right]^* w = s. \tag{5.4.29}$$

Also note that

$$\begin{aligned} &\inf\left\{ \|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta E\|_F^2 \;\middle|\; \Delta A,\, \Delta B,\, \Delta E \text{ satisfying } \ (5.4.25) \text{ and } (5.4.26)\right\} \\ &\qquad = \inf\left\{ \tfrac{\|\Delta_1\|_F^2}{1+\gamma^2} + \|\Delta_2\|_F^2 \;\middle|\; \Delta_1,\ \Delta_2 \text{ satisfy } (5.4.29)\right\}. \end{aligned} \tag{5.4.30}$$

To see this, let Δ_1 and Δ_2 be such that they satisfy (5.4.29). Now set

$$\Delta A := \frac{\Delta_1}{1+\gamma^2}, \quad \Delta E := \frac{i\gamma \Delta_1}{1+\gamma^2} \quad \text{and} \quad \Delta B := \Delta_2.$$

Then $\Delta A - i\gamma\Delta E = \Delta_1$ and $\|\Delta A\|_F^2 + \|\Delta E\|_F^2 = \frac{\|\Delta_1\|_F^2}{1+\gamma^2}$. This shows ($\leq$) in (5.4.30). Again, let ΔA, ΔB and ΔE be such that they satisfy (5.4.25) and (5.4.26) and assume that $\Delta_1 = \Delta A - i\gamma\Delta E$ and $\Delta_2 = \Delta B$. Then Δ_1 and Δ_2 satisfy (5.4.29). Also

$$\begin{aligned}
&\|\Delta_1\|_F \leq \|\Delta A\|_F + |\gamma|\|\Delta E\|_F \\
\Longrightarrow\ &\|\Delta_1\|_F^2 \leq \big(\|\Delta A\|_F^2 + \|\Delta B\|_F^2\big)\big(1+\gamma^2\big) \ \ \text{(by Holder's inequality)} \\
\Longrightarrow\ &\frac{\|\Delta_1\|_F^2}{1+\gamma^2} + \|\Delta_2\|_F^2 \leq \|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta E\|_F^2.
\end{aligned}$$

This implies ($\geq$) in (5.4.30). Thus we have,

$$\begin{aligned}
&\big(\eta_F^{\mathcal{S}}(H_c, N_c, i\gamma, x)\big)^2 \\
&\quad = \inf\left\{ \left\| \begin{bmatrix} \Delta A & \Delta B & \Delta E \end{bmatrix} \right\|_F^2 \,\middle|\, \Delta A,\, \Delta B,\, \Delta E \text{ satisfying } (5.4.25) \text{ and } (5.4.26) \right\} \\
&\quad = \inf\left\{ \frac{\|\Delta_1\|_F^2}{1+\gamma^2} + \|\Delta_2\|_F^2 \,\middle|\, \Delta_1,\ \Delta_2 \text{ satisfying } (5.4.29) \right\} \\
&\quad = \frac{\|\hat{\Delta}_1\|_F^2}{1+\gamma^2} + \|\hat{\Delta}_2\|_F^2,
\end{aligned}$$

where $\hat{\Delta} = [\hat{\Delta}_1\ \hat{\Delta}_2]$ is the unique minimal Frobenius norm solution of (5.4.29). Hence (5.4.24) follows from Theorem 1.2.14.

(2) If C is definite then from (5.4.28) $u^*Cu = 0$ implies $u = 0$. Thus x must be of the form $x = [x_1^T\ 0\ 0]^T \in \mathbb{C}^{2n+m}$, $x_1 \in \mathbb{C}^n \setminus \{0\}$. In this case from (5.4.26), ΔA, ΔB and ΔE satisfy

$$\begin{bmatrix} \Delta A - i\gamma\Delta E & \Delta B \end{bmatrix}^* w = \begin{bmatrix} A - i\gamma E & B \end{bmatrix}^* w. \tag{5.4.31}$$

Thus following the proof of (1), we have

$$\begin{aligned}
\big(\eta_F^{\mathcal{S}}(H_c, N_c, i\gamma, x)\big)^2 &= \inf\left\{ \left\| \begin{bmatrix} \Delta A & \Delta B & \Delta E \end{bmatrix} \right\|_F^2 \,\middle|\, \Delta A,\, \Delta B,\, \Delta E \text{ satisfying } (5.4.31) \right\} \\
&= \inf\left\{ \frac{\|\Delta_1\|_F^2}{1+\gamma^2} + \|\Delta_2\|_F^2 \,\middle|\, \Delta_1 \in \mathbb{C}^{n\times n},\ \Delta_2 \in \mathbb{C}^{n\times m},\ [\Delta_1\ \Delta_2]^* w = s \right\} \\
&= \frac{\|\widetilde{\Delta}_1\|_F^2}{1+\gamma^2} + \|\widetilde{\Delta}_2\|_F^2,
\end{aligned}$$

where $\begin{bmatrix} \widetilde{\Delta}_1 & \widetilde{\Delta}_2 \end{bmatrix} = \frac{1}{\|w\|^2} s w^*$ is the minimal Frobenius norm map satisfying $[\widetilde{\Delta}_1\ \widetilde{\Delta}_2] w = s$.
$\square$

5.5 Pencils arising in discrete-time linear quadratic optimal control problems

A discrete-time analogue to the linear quadratic optimal control problem leads to a slightly different matrix pencil of the form $L_d(z) = H_d - zN_d$ where

$$H_d := \begin{bmatrix} 0 & A & B \\ -E^* & Q & S \\ 0 & S^* & R \end{bmatrix} \quad \text{and} \quad N_d := \begin{bmatrix} 0 & E & 0 \\ -A^* & 0 & 0 \\ -B^* & 0 & 0 \end{bmatrix}, \tag{5.5.1}$$

with $A, E, Q \in \mathbb{C}^{n\times n}$, $S, B \in \mathbb{C}^{n\times m}$ and $R \in \mathbb{C}^{m\times m}$ [34, 37]. The eigenvalues of $L_d(z)$ occur in pairs $(\lambda, 1/\bar{\lambda})$ when the pencil is complex and in quadruples $(\lambda, 1/\lambda, \bar{\lambda}, 1/\bar{\lambda})$ when the pencil is real. Therefore eigenvalues follow symplectic symmetry and the unit circle is the critical set for $L_d(z)$. In the following, results corresponding to Theorems 5.4.1 and 5.4.3 are obtained for these pencils when $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. For this, define

$$\mathcal{A} := \begin{bmatrix} A & B \end{bmatrix}, \; \mathcal{E} := \begin{bmatrix} E & 0 \end{bmatrix} \text{ and } C := \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}.$$

5.5.1 Perturbation only to matrices A and B

Let

$$\Delta H_d = \begin{bmatrix} 0 & \Delta A & \Delta B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \Delta N_d = \begin{bmatrix} 0 & 0 & 0 \\ -(\Delta A)^* & 0 & 0 \\ -(\Delta B)^* & 0 & 0 \end{bmatrix}$$

and $\mathbb{S}$ be the collection of all pencils of the form $\Delta L_d(z) = \Delta H_d - z\Delta N_d$. For $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$, define the following eigenpair and eigenvalue backward errors with respect to perturbation that affect only blocks A and B as follows.

$$\eta_p^{\mathbb{S}}(H_d, N_d, \lambda, x) = \inf \Big\{ \|[\Delta A \; \Delta B]\|_p \Big| \; \Delta A \in \mathbb{C}^{n\times n}, \, \Delta B \in \mathbb{C}^{n\times m}, \, \Delta H_d - z\Delta N_d \in \mathbb{S}, \\ \big((H_d - \Delta H_d) - \lambda(N_d - \Delta N_d)\big)x = 0 \Big\}$$

and

$$\eta_p^{\mathbb{S}}(H_d, N_d, \lambda) = \inf \Big\{ \|[\Delta A \; \Delta B]\|_p \Big| \; \Delta A \in \mathbb{C}^{n\times n}, \, \Delta B \in \mathbb{C}^{n\times m}, \, \Delta H_d - z\Delta N_d \in \mathbb{S}, \\ \det\big((H_d - \Delta H_d) - \lambda(N_d - \Delta N_d)\big) = 0 \Big\},$$

where $\|.\|_p$ denotes either spectral norm $\|.\|_2$ or Frobenius norm $\|.\|_F$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $x = [x_1^T \;\; x_2^T]^T \in \mathbb{C}^{2n+m} \setminus \{0\}$ where $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^{n+m}$. Then for any

$$\Delta H_d = \begin{bmatrix} 0 & \Delta\mathcal{A} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta N_d = \begin{bmatrix} 0 & 0 \\ -(\Delta\mathcal{A})^* & 0 \end{bmatrix},$$

where $\Delta\mathcal{A} = \begin{bmatrix} \Delta A & \Delta B \end{bmatrix}$, $\Delta A \in \mathbb{C}^{n\times n}$, and $\Delta B \in \mathbb{C}^{n\times m}$, we have $(L_d - \Delta L_d)(\lambda)x = 0$, or equivalently

$$\begin{bmatrix} 0 & \Delta\mathcal{A} \\ \lambda(\Delta\mathcal{A})^* & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{A} - \lambda\mathcal{E} \\ \lambda\mathcal{A}^* - \mathcal{E}^* & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This implies

$$\Delta\mathcal{A}x_2 = (\mathcal{A} - \lambda\mathcal{E})x_2 \quad \text{and} \quad (\Delta\mathcal{A})^* x_1 = \frac{1}{\lambda}\left((\lambda\mathcal{A}^* - \mathcal{E}^*)x_1 + Cx_2\right). \tag{5.5.2}$$

By Theorem 1.2.14, there exist such a $\Delta\mathcal{A} \in \mathbb{C}^{n\times(n+m)}$ if and only if

$$\begin{aligned} x_2^*\left(\frac{1}{\lambda}\left((\lambda\mathcal{A}^* - \mathcal{E}^*)x_1 + Cx_2\right)\right) &= \left((\mathcal{A} - \lambda\mathcal{E})x_2\right)^* x_1 \quad \text{if and only if} \\ x_2^* C x_2 &= (1 - |\lambda|^2) x_2^* \mathcal{E}^* x_1. \end{aligned} \tag{5.5.3}$$

Thus provided $x = [x_1^T \;\; x_2^T]^T$ satisfies (5.5.3), we can find minimal Frobenius norm and minimal spectral norm maps satisfying (5.5.2) from Theorem 1.2.14. In general, if $x = [x_1^T \;\; x_2^T]^T$ satisfies (5.5.3), then

$$\left(\eta_F^{\mathbb{S}}(H_c, N_c, \lambda, x)\right)^2 = \frac{\|(\mathcal{A} - \lambda\mathcal{E})x_2\|^2}{\|x_2\|^2} + \frac{\|(-\mathcal{E}^* + \lambda\mathcal{A}^*)x_1 + Cx_2\|^2}{|\lambda|^2\|x_1\|^2} - \frac{|x_1^*(\mathcal{A} - \lambda\mathcal{E})x_2|^2}{\|x_1\|^2\|x_2\|^2} \tag{5.5.4}$$

and

$$\eta_2^{\mathbb{S}}(H_c, N_c, \lambda, x) = \max\left\{\frac{\|(\mathcal{A} - \lambda\mathcal{E})x_2\|}{\|x_2\|}, \frac{\|(-\mathcal{E}^* + \lambda\mathcal{A}^*)x_1 + Cx_2\|}{|\lambda|\|x_1\|}\right\}. \tag{5.5.5}$$

We can further simplify the expressions in (5.5.4) and (5.5.5) if $|\lambda| = 1$ and C is either definite or semidefinite. Therefore in view of (5.5.3), (5.5.4) and (5.5.5), the arguments similar to those in the previous section yield the following analogue of Theorem 5.4.1 when $|\lambda| = 1$.

Theorem 5.5.1. *Let $L_d(z)$ be a pencil as defined in (5.5.1), $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $x = [x_1^T \; x_2^T]^T$ with $x_1 \in \mathbb{C}^n$ and $x_2 \in \mathbb{C}^{n+m}$. Then the following hold.*

(1) If C is definite, then $\eta^{\mathbb{S}}_{2,F}(H_d, N_d, \lambda, x)$ is finite if and only if $x_2 = 0$. In such a case,

$$\eta^{\mathbb{S}}_{F}(H_d, N_d, \lambda, x) = \frac{\|(\mathcal{A} - \lambda\mathcal{E})^* x_1\|}{\|x_1\|} = \eta^{\mathbb{S}}_{2}(H_d, N_d, \lambda, x)$$

and

$$\eta^{\mathbb{S}}_{F}(H_d, N_d, \lambda) = \sigma_{\min}\big((\mathcal{A} - \lambda\mathcal{E})^*\big) = \eta^{\mathbb{S}}_{2}(H_d, N_d, \lambda).$$

(2) If C is semidefinite then $\eta^{\mathbb{S}}_{2,F}(H_d, N_d, \lambda, x)$ is finite if and only if $Cx_2 = 0$. In such a case, let $r := (\mathcal{A} - \lambda\mathcal{E})x_2$ and $s := (\mathcal{A} - \lambda\mathcal{E})^* x_1$. Then

$$\big(\eta^{\mathbb{S}}_{F}(H_d, N_d, \lambda, x)\big)^2 = \begin{cases} \frac{\|r\|^2}{\|x_2\|^2} + \frac{\|s\|^2}{\|x_1\|^2} - \frac{|x_1^* r|^2}{\|x_1\|^2\|x_2\|^2} & \text{if } x_1 \neq 0,\ x_2 \neq 0, \\ \frac{\|r\|^2}{\|x_2\|^2} & \text{if } x_1 = 0, \\ \frac{\|s\|^2}{\|x_1\|^2} & \text{if } x_2 = 0, \end{cases}$$

$$\eta^{\mathbb{S}}_{2}(H_d, N_d, \lambda, x) = \begin{cases} \max\left\{\frac{\|r\|}{\|x_2\|}, \frac{\|s\|}{\|x_1\|}\right\} & \text{if } x_1 \neq 0,\ x_2 \neq 0, \\ \frac{\|r\|}{\|x_2\|} & \text{if } x_1 = 0, \\ \frac{\|s\|}{\|x_1\|} & \text{if } x_2 = 0, \end{cases}$$

and

$$\eta^{\mathbb{S}}_{2}(H_d, N_d, \lambda) = \sigma_{\min}\big((\mathcal{A} - \lambda\mathcal{E})^*\big) = \eta^{\mathbb{S}}_{F}(H_d, N_d, \lambda).$$

5.5.2 Perturbation only to matrices A, B and E

Let $L_d(z) = H_d - zN_d$ be a pencil defined by (5.5.1). Also let $\mathbb{S}'$ be the collection of all pencils of the form $\Delta H_d - z\Delta N_d$, where ΔH_d and ΔN_d are given by

$$\Delta H_d = \begin{bmatrix} 0 & \Delta A & \Delta B \\ -(\Delta E)^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta N_d = \begin{bmatrix} 0 & \Delta E & 0 \\ -(\Delta A)^* & 0 & 0 \\ -(\Delta B)^* & 0 & 0 \end{bmatrix}.$$

For $\lambda \in \mathbb{C} \setminus \{0\}$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$, we define the following eigenpair and eigenvalue backward errors with respect to perturbations that affect only blocks A, B and E.

$$\eta^{\mathbb{S}'}_{F}(H_d, N_d, \lambda, x) = \inf\Big\{\|[\Delta A\ \Delta B\ \Delta E]\|_F \ \Big|\ \Delta A, \Delta E \in \mathbb{C}^{n\times n},\ \Delta B \in \mathbb{C}^{n\times m},$$
$$\Delta H_d - z\Delta N_d \in \mathbb{S}',\ \big((H_d - \Delta H_d) - \lambda(N_d - \Delta N_d)\big)x = 0\Big\},$$

and

$$\eta^{\mathbb{S}'}_{F}(H_d, N_d, \lambda) = \inf\Big\{\|[\Delta A\ \Delta B\ \Delta E]\|_F \ \Big|\ \Delta A, \Delta E \in \mathbb{C}^{n\times n},\ \Delta B \in \mathbb{C}^{n\times m},$$
$$\Delta H_d - z\Delta N_d \in \mathbb{S}',\ \big((H_d - \Delta H_d) - \lambda(N_d - \Delta N_d)\big)x = 0\Big\},$$

where $\|.\|_F$ denotes the Frobenius norm of a matrix.

The following theorem is an analogue of Theorem 5.4.3 for $L_d(z)$ and obtains $\eta_F^{\mathbb{S}'}(H_d, N_d, \lambda)$ and $\eta_F^{\mathbb{S}'}(H_d, N_d, \lambda, x)$ when $|\lambda| = 1$. The proof is similar to that of Theorem 5.4.3.

Theorem 5.5.2. *Let $L_d(z)$ be a pencil as defined in (5.5.1), $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$ and $x \in \mathbb{C}^{2n+m} \setminus \{0\}$. Partition $x = [x_1^T \ \ x_2^T \ \ x_3^T]^T$ where $x_1, \ x_2 \in \mathbb{C}^n, \ x_3 \in \mathbb{C}^m$. Define*

$$u := \left[\begin{array}{cc} x_2^T & x_3^T \end{array}\right]^T, \ w := x_1, \ r := \left[\begin{array}{cc} A - \lambda E & B \end{array}\right] u, \ s := \left[\begin{array}{cc} A - \lambda E & B \end{array}\right]^* x_1.$$

(1) *If C is semidefinite, then $\eta_F^{\mathbb{S}'}(H_c, N_C, i\gamma, x)$ is finite if and only if $Cu = 0$. In such a case the following holds.*

$$\eta_F^{\mathbb{S}'}(H_d, N_d, \lambda, x) = \left(\|\hat{\Delta}_1\|_F^2/(1+\gamma^2) + \|\hat{\Delta}_2\|_F^2\right)^{1/2},$$

where $\hat{\Delta}_1 \in \mathbb{C}^{n \times n}, \hat{\Delta}_2 \in \mathbb{C}^{n \times m}$ are such that

$$\left[\begin{array}{cc} \hat{\Delta}_1 & \hat{\Delta}_2 \end{array}\right] = \begin{cases} \frac{ru^*}{\|u\|^2} & \text{if } x_1 = 0, \\ \frac{ws^*}{\|w\|^2} & \text{if } x_2 = 0, x_3 = 0 \\ \frac{ru^*}{\|u\|^2} + \frac{sw^*}{\|w\|^2}\left(I_{n+m} - \frac{uu^*}{\|u\|^2}\right) & \text{otherwise.} \end{cases} \tag{5.5.6}$$

(2) *If C is definite, then $\eta_F^{\mathbb{S}'}(H_d, N_d, \lambda, x)$ is finite if and only if $u = 0$. In such a case the following holds.*

$$\eta_F^{\mathbb{S}'}(H_d, N_d, \lambda, x) = \left(\|\widetilde{\Delta}_1\|_F^2/(1+\gamma^2) + \|\widetilde{\Delta}_2\|_F^2\right)^{1/2},$$

where $\left[\begin{array}{cc} \widetilde{\Delta}_1 & \widetilde{\Delta}_2 \end{array}\right] = \frac{1}{\|w\|^2} sw^*.$

Conclusion

We have developed a new framework to obtain computable formulas for structured eigenvalue backward errors of matrix polynomials with various structures under some prespecified norms. In particular, we have undertaken a detailed analysis of structured eigenvalue backward errors of structured matrix polynomials when the perturbations are measured with respect to $|||.|||_{w,2}$ norm. We have obtained explicit formulas for structured eigenvalue backward errors of matrix polynomials with Hermitian, skew-Hermitian, $*$-alternating, $*$-palindromic and $*$-antipalindromic structures in terms of the maximal eigenvalue of a parameter depending Hermitian matrix. We have also derived structured eigenvalue backward errors of T-even and T-palindromic polynomials of degree at most 2, T-odd and T-antipalindromic pencils in terms of second largest eigenvalue of a parameter depending Hermitian matrix. For higher degree T-palindromic and T-alternating polynomials we have estimated the structured eigenvalue backward error by tight bounds.

Under the same framework, we have generalized these ideas to obtain computable bounds for structured eigenvalue backward errors of structured matrix polynomials with respect to $|||.|||_\infty$ and $|||.|||_{w,F}$ norms. In most cases, the lower bound gives the exact value of the backward error when certain assumptions are met. We have shown by numerical experiments that these assumptions are satisfied in practice, thus giving the exact eigenvalue backward error.

Finally we have estimated real eigenpair and eigenvalue backward errors of real matrix polynomials under real perturbations. If the real matrix polynomial has additional structure like Hermitian, $*$-alternating, T-palindromic etc., then in many cases eigenvalue backward errors are computed with respect to perturbations that preserve these additional structures also. We have also computed structured eigenvalue and eigenpair backward errors of some special block structured pencils that arise in linear-quadratic optimal control problems with respect to special structure preserving perturbations. In most cases, we have observed that there is a significant difference between the backward errors with respect to perturbations that preserve structure and those with respect to arbitrary perturbations.

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