INTEGRAL MIXED CAYLEY GRAPHS

by

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INTEGRAL MIXED CAYLEY GRAPHS

A thesis submitted

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DOCTOR OF PHILOSOPHY

by

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DECLARATION

I do hereby declare that the work contained in this thesis entitled **"INTEGRAL MIXED CAYLEY GRAPHS"** has been done by me under the supervision of **Dr. Bikash Bhattacharjya,** Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy and that this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is certified that the work contained in this thesis entitled **"INTEGRAL MIXED CAYLEY GRAPHS"** by **Monu Kadyan,** a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

February, 2023

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ABSTRACT

In spectral graph theory, mixed graphs are crucial because they give a structure in which directed and undirected edges may coexist. If every edge of a mixed graph is undirected, then such a graph is known as an undirected graph.

An important problem in spectral graph theory is to understand when all eigenvalues of the $(0,1)$-adjacency matrix of an undirected graph are integers. An undirected graph with this property is called integral. The notion of integral undirected graph was first introduced by Harary and Schwenk in 1974, and they raised the problem of determining undirected integral graphs. Many researchers attempted to solve this problem in the last few decades, yet it remains unsolved completely. In general, the problem of characterizing integral undirected graphs seems challenging to answer. Many researchers investigated some special classes of graphs such as trees, graphs with restricted degrees, regular graphs and undirected Cayley graphs for their integrality.

A *mixed graph* G is a pair $(V(G), E(G))$ of sets, where $V(G) \neq \emptyset$ and

$$E(G) \subseteq (V(G) \times V(G)) \setminus \{(u,u) \colon \ u \in V(G)\}.$$

In spectral graph theory, various types of adjacency matrices of graphs are defined and studied. In the thesis, we consider the following three adjacency matrices of a mixed graph G.

(i) The (0,1)-*adjacency matrix* of G, denoted $\mathcal{A}(G)$, is the matrix $[a_{uv}]$, where a_{uv} is given by

$$a_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The *Hermitian-adjacency matrix* of G, denoted $\mathcal{H}(G)$, is the matrix $[h_{uv}]$, where h_{uv} is given by

$$h_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \text{ and } (v,u) \in E(G) \\ \mathbf{i} & \text{if } (u,v) \in E(G) \text{ and } (v,u) \notin E(G) \\ -\mathbf{i} & \text{if } (u,v) \notin E(G) \text{ and } (v,u) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

(iii) The *Hermitian-adjacency matrix of second kind* of G, denoted $\mathcal{K}(G)$, is the matrix $[k_{uv}]$, where k_{uv} is given by

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$$k_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \text{ and } (v,u) \in E(G) \\ \frac{1+\mathbf{i}\sqrt{3}}{2} & \text{if } (u,v) \in E(G) \text{ and } (v,u) \notin E(G) \\ \frac{1-\mathbf{i}\sqrt{3}}{2} & \text{if } (u,v) \notin E(G) \text{ and } (v,u) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

In this thesis, we discuss the following four types of integrality of mixed graphs.

(i) H-integral mixed graph: A mixed graph is said to be H-*integral* if the eigenvalues of its Hermitian-adjacency matrix are integers.

(ii) HS-integral mixed graph: A mixed graph is said to be HS-*integral* if the eigenvalues of its Hermitian-adjacency matrix of second kind are integers.

(iii) Gaussian integral mixed graph: A mixed graph is said to be *Gaussian integral* if the eigenvalues of its $(0,1)$-adjacency matrix are Gaussian integers.

(iv) Eisenstein integral mixed graph: A mixed graph is said to be *Eisenstein integral* if the eigenvalues of its $(0,1)$-adjacency matrix are Eisenstein integers.

We first characterize H-integral and HS-integral mixed Cayley graphs over abelian groups. Thereafter, we generalize these characterizations to normal mixed Cayley graphs. We also show that a normal mixed Cayley graph is Gaussian integral (respectively Eisenstein integral) if and only if it is H-integral (respectively HS-integral). Further, we introduce two sums that are equal to an integer multiple of the Ramanujan sum. Indeed, the eigenvalues of the Hermitian-adjacency matrix (respectively the Hermitian-adjacency matrix of second kind) of a mixed circulant graph can be expressed in terms of these sums. We also express these sums in terms of the generalized Möbius function.

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List of Symbols

$\mathbb{N}$: set of natural numbers

$\mathbb{Z}$: set of integers

$\mathbb{Q}$: set of rational numbers

$\mathbb{C}$: set of complex numbers

$\mathbb{C}^*$: set of non-zero complex numbers

$\mathbb{Z}_n$: group of integers modulo n

$a \mid b$: a divides b

$a \nmid b$: a does not divide b

φ : Euler phi function

μ : classical Möbius function

μ_P : generalized Möbius function

δ : indicator function

$\mathbf{i}$: $\sqrt{-1}$

$\overline{z}$: conjugate of the complex number z

$\Re(\alpha)$: real part of α

$\Im(\alpha)$: imaginary part of α

ω_n : $\exp\left(\frac{2\pi\mathbf{i}}{n}\right)$

$\mathcal{A}(G)$: (0,1)-adjacency matrix of G

$\mathcal{H}(G)$: Hermitian-adjacency matrix of G

$\mathcal{K}(G)$: Hermitian-adjacency matrix of second kind of G

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$A \times B$	: cartesian product of A and B
$\Gamma_1 \otimes \Gamma_2$	: direct product of Γ_1 and Γ_2
S^c	: compliment of S
$\lvert S \rvert$	: number of elements of S
$\overline{S}$	: $\{u \in S \colon u^{-1} \notin S\}$
S^{-1}	: $\{u^{-1} \colon u \in S\}$
$-S$	: $\{-s \colon s \in S\}$
aS	: $\{as \colon s \in S\}$
$a + S$	: $\{a + s \colon s \in S\}$
$\mathbf{1}$	: identity element of a group
$\mathrm{ord}(x)$	: order of x
$\langle x \rangle$	: cyclic group generated by x
$\mathrm{Cl}(x)$	: set of all conjugates of x
$C_\Gamma(x)$	: set of all elements of Γ that commute with x
$\exp(\Gamma)$	: least common multiple of the orders of all elements of Γ
$\mathbb{F}\Gamma$	: group algebra of Γ over $\mathbb{F}$
$\mathbb{Z}(\mathbf{i})$	: $\{a + \mathbf{i}b \colon a, b \in \mathbb{Z}\}$
$\mathbb{Z}(\omega_3)$	: $\{a + \omega_3 b \colon a, b \in \mathbb{Z}\}$
$\mathbb{F}(a)$	: smallest field containing $\mathbb{F}$ and a
$\mathbb{F}(a, b)$	: smallest field containing $\mathbb{F}$, a and b
$\mathbb{F}[x]$	: polynomial ring in the variable x with coefficients in $\mathbb{F}$
$\mathbb{Z}[x]$	: polynomial ring in the variable x with coefficients in $\mathbb{Z}$
$\mathbb{Z}(\mathbf{i})[x]$	: polynomial ring in the variable x with coefficients in $\mathbb{Z}(\mathbf{i})$
$\mathbb{Z}(\omega_3)[x]$	: polynomial ring in the variable x with coefficients in $\mathbb{Z}(\omega_3)$
$\mathrm{Cay}(\Gamma, S)$	: mixed Cayley graph
$\mathrm{Circ}(\mathbb{Z}_n, S)$	: mixed circulant graph
$\mathrm{GL}_n(\mathbb{C})$	: set of all $n \times n$ invertible matrices with complex entries
I_n	: identity matrix of size $n \times n$
$\mathbb{Q}^n$	: set of $n \times 1$ matrices with rational entries
$\mathbb{C}^n$	: set of $n \times 1$ matrices with complex entries
$\mathbf{0}$	: zero element of $\mathbb{C}^n$

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E^*	: conjugate transpose of the matrix E
$[\mathbb{K} : \mathbb{F}]$	: dimension of $\mathbb{K}$ over $\mathbb{F}$
$\mathrm{Gal}(\mathbb{K}/\mathbb{F})$	: Galois group of $\mathbb{K}$ over $\mathbb{F}$
$\mathrm{IRR}(\Gamma)$	: complete set of non-equivalent irreducible representations of Γ
$\mathrm{Irr}(\Gamma)$	: complete set of non-equivalent irreducible characters of Γ
$M_n(d)$	: $\{dk \colon 1 \leq dk \leq n-1\}$
$G_n(d)$	: $\{dk \colon 1 \leq dk \leq n-1, \gcd(dk, n) = d\}$
$M_n^r(d)$	: $\{dk \colon 0 \leq dk \leq n-1, k \equiv r \pmod 4\}$
$G_n^r(d)$	: $\{dk \colon 1 \leq dk \leq n-1, \gcd(dk, n) = d, k \equiv r \pmod 4\}$
$M_{n,3}^r(d)$	: $\{dk \colon 0 \leq dk \leq n-1, k \equiv r \pmod 3\}$
$G_{n,3}^r(d)$	: $\{dk \colon 1 \leq dk \leq n-1, \gcd(dk, n) = d, k \equiv r \pmod 3\}$
D_g	: $\{k \colon k$ is an odd divisor of $g\}$
D_g^r	: $\{k \colon k$ divides $g, k \equiv r \pmod 4\}$
$D_{g,3}$	: $\{k \colon k$ divides $g, k \not\equiv 0 \pmod 3\}$
$D_{g,3}^r$	: $\{k \colon k$ divides $g, k \equiv r \pmod 3\}$
$\Phi_n(x)$	: $\prod\limits_{a \in G_n(1)} (x - \omega_n^a)$
$\Phi_n^1(x)$	: $\prod\limits_{a \in G_n^1(1)} (x - \omega_n^a)$
$\Phi_n^3(x)$	: $\prod\limits_{a \in G_n^3(1)} (x - \omega_n^a)$
$\Phi_{n,3}^1(x)$	: $\prod\limits_{a \in G_{n,3}^1(1)} (x - \omega_n^a)$
$\Phi_{n,3}^2(x)$	: $\prod\limits_{a \in G_{n,3}^2(1)} (x - \omega_n^a)$
$\Gamma(4)$	: $\{x \in \Gamma \colon \mathrm{ord}(x) \equiv 0 \pmod 4\}$
$\Gamma(3)$	: $\{x \in \Gamma \colon \mathrm{ord}(x) \equiv 0 \pmod 3\}$
$M_r(x)$	: $\{x^k \colon 1 \leq k \leq \mathrm{ord}(x), k \equiv r \pmod 4\}$
$M_{r,3}(x)$	: $\{x^k \colon 1 \leq k \leq \mathrm{ord}(x), k \equiv r \pmod 3\}$
$\sim$	: $x \sim y$ iff $y = x^k$ for some $k \in G_m(1)$, where $m = \mathrm{ord}(x)$ and $x, y \in \Gamma$
$\approx$	: $x \approx y$ iff $y = x^k$ for some $k \in G_m^1(1)$, where $m = \mathrm{ord}(x)$ and $x, y \in \Gamma(4)$
$\simeq$	: $x \simeq y$ iff $y = x^k$ for some $k \in G_{m,3}^1(1)$, where $m = \mathrm{ord}(x)$ and $x, y \in \Gamma(3)$

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$[x]$: equivalence class of x with respect to the relation $\sim$

$[\![x]\!]$: equivalence class of x with respect to the relation $\approx$

$\langle\!\langle x\rangle\!\rangle$: equivalence class of x with respect to the relation $\simeq$

$\mathbb{B}(\Gamma)$: $\{[x_1] \cup \cdots \cup [x_k] \colon x_1, \ldots, x_k \in \Gamma, k \in \mathbb{N}\}$

$$\mathbb{D}(\Gamma) : \begin{cases} \{[\![x_1]\!] \cup \cdots \cup [\![x_k]\!] \colon x_1, \ldots, x_k \in \Gamma(4), k \in \mathbb{N}, \\ \qquad \text{and } x_i^{-1} \notin [\![x_j]\!] \text{ for all } 1 \leq i, j \leq k\} & \text{if } \Gamma(4) \neq \emptyset \\ \{\emptyset\} & \text{if } \Gamma(4) = \emptyset \end{cases}$$

$$\mathbb{E}(\Gamma) : \begin{cases} \{\langle\!\langle x_1\rangle\!\rangle \cup \cdots \cup \langle\!\langle x_k\rangle\!\rangle \colon x_1, \ldots, x_k \in \Gamma(3), k \in \mathbb{N}, \\ \qquad \text{and } x_i^{-1} \notin \langle\!\langle x_j\rangle\!\rangle \text{ for all } 1 \leq i, j \leq k\} & \text{if } \Gamma(3) \neq \emptyset \\ \{\emptyset\} & \text{if } \Gamma(3) = \emptyset \end{cases}$$

S_x^1 : smallest symmetric subset of Γ containing x that is closed under both conjugacy and the equivalence relation $\sim$

S_y^4 : smallest skew-symmetric subset of $\Gamma(4)$ containing y that is closed under both conjugacy and the equivalence relation $\approx$

S_y^2 : smallest skew-symmetric subset of $\Gamma(3)$ containing y that is closed under both conjugacy and the equivalence relation $\simeq$

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Introduction

The study of eigenvalues of matrices associated to a graph is an essential component of algebraic graph theory. It is used in numerous fields, such as chemistry, social science, electrical engineering, architecture, computer science, and many others. Graph theorists use different types of matrices to represent a graph, depending on the problem and their preferences. The $(0,1)$-adjacency matrix and the Laplacian matrix are two of the most widely used ones. Specifically, eigenvalues of these matrices associated to undirected Cayley graphs gained popularity due to their significance in algebraic graph theory and applications in expanders, chemical graph theory, and quantum computing. A large number of findings on eigenvalues of matrices associated to Cayley graphs was made in the last few decades, see [33] for a survey.

An important problem in spectral graph theory is to understand when all eigenvalues of a matrix associated to a graph are integers. An undirected graph with this property is called integral. In this chapter, we introduce some basic concepts and results that are used in the subsequent chapters. Also, we give an overview of the research on integral undirected Cayley graphs that are available in the literature. An overview of the thesis is given in the last section.

1.1 Mixed graphs

A *mixed graph* G is a pair $(V(G), E(G))$ of sets, where $V(G) \neq \emptyset$ and

$$E(G) \subseteq \big(V(G) \times V(G)\big) \setminus \{(u,u)\colon\ u \in V(G)\}.$$

We call $V(G)$ (respectively $E(G)$) the *vertex* set (respectively *edge* set) of the mixed graph G. The elements of $V(G)$ (respectively $E(G)$) are called vertices (respectively edges) of G.

In the absence of any specification, the vertex set $V(G)$ is assumed to be $\{1, \ldots, n\}$. If $(u,v), (v,u) \in E(G)$, then we say that there is an undirected edge from the vertex u to the vertex v. Similarly, if $(u,v) \in E(G)$ but $(v,u) \notin E(G)$, then we say that there is a directed edge from the vertex u to the vertex v. A mixed graph can have both undirected and directed edges. For an example of a mixed graph, see Figure 1.1. A mixed graph G is said to be an

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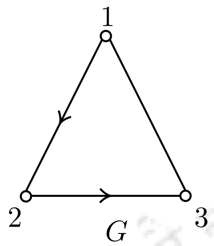
undirected graph if all the edges of G are undirected. A mixed graph G is said to be a *directed* graph if all the edges of G are directed. Throughout the thesis, we consider $\mathbf{i} = \sqrt{-1}$ and $\omega_n := \exp\left(\frac{2\pi\mathbf{i}}{n}\right)$. For $z \in \mathbb{C}$, let $\overline{z}$ denote the complex conjugate of z.



$V(G) = \{1, 2, 3\}$

$E(G) = \{(1, 2), (2, 3), (1, 3), (3, 1)\}$

Figure 1.1: A mixed graph

Assume that G is a mixed graph with n vertices. In spectral graph theory, various types of adjacency matrices are defined and studied. In the thesis, we consider the following three adjacency matrices of a mixed graph G.

(i) The (0,1)-*adjacency matrix* of G, denoted $\mathcal{A}(G)$, is the matrix $[a_{uv}]$, where a_{uv} is given by

$$a_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $\mathcal{A}(G)$ are called the *eigenvalues* of G. The *spectrum* of G is the multi-set of the eigenvalues of G. If G has at least one directed edge, then $\mathcal{A}(G)$ is not symmetric, and so the eigenvalues of G need not be real numbers.

(ii) The *Hermitian-adjacency matrix* of G, denoted $\mathcal{H}(G)$, is the matrix $[h_{uv}]$, where h_{uv} is given by

$$h_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E(G) \text{ and } (v, u) \in E(G) \\ \mathbf{i} & \text{if } (u, v) \in E(G) \text{ and } (v, u) \notin E(G) \\ -\mathbf{i} & \text{if } (u, v) \notin E(G) \text{ and } (v, u) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

In 2012, Bapat et al. [8] introduced the notion of a 3-colored digraph and its adjacency matrix. The Hermitian adjacency matrix is a special case of the adjacency matrix of a 3-colored digraph; see [8] for details. Later on, Liu et al. [32] and Guo et al. [21] discussed it independently. The eigenvalues of $\mathcal{H}(G)$ are called the *H-eigenvalues* of G. The *H-spectrum* of G is the multi-set of the H-eigenvalues of G.

(iii) The *Hermitian-adjacency matrix of second kind* of G, denoted $\mathcal{K}(G)$, is the matrix $[k_{uv}]$, where k_{uv} is given by

$$k_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E(G) \text{ and } (v,u) \in E(G) \\ \frac{1+\mathbf{i}\sqrt{3}}{2} & \text{if } (u,v) \in E(G) \text{ and } (v,u) \notin E(G) \\ \frac{1-\mathbf{i}\sqrt{3}}{2} & \text{if } (u,v) \notin E(G) \text{ and } (v,u) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

The Hermitian-adjacency matrix of second kind was introduced by Mohar [35]. The eigenvalues of $\mathcal{K}(G)$ are called the *HS-eigenvalues* of G. The *HS-spectrum* of G is the multi-set of the HS-eigenvalues of G.

Let us consider the mixed graph G shown in Figure 1.1. The (0,1)-adjacency matrix, the Hermitian-adjacency matrix and the Hermitian-adjacency matrix of second kind of G are given below.

$$\mathcal{A}(G) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathcal{H}(G) = \begin{bmatrix} 0 & \mathbf{i} & 1 \\ -\mathbf{i} & 0 & \mathbf{i} \\ 1 & -\mathbf{i} & 0 \end{bmatrix} \qquad \mathcal{K}(G) = \begin{bmatrix} 0 & \frac{1+\mathbf{i}\sqrt{3}}{2} & 1 \\ \frac{1-\mathbf{i}\sqrt{3}}{2} & 0 & \frac{1+\mathbf{i}\sqrt{3}}{2} \\ 1 & \frac{1-\mathbf{i}\sqrt{3}}{2} & 0 \end{bmatrix}$$

A major drawback of the (0,1)-adjacency matrix of a mixed graph is that it need not be symmetric, and hence it need not have real eigenvalues. An alternative of the (0,1)-adjacency matrix is the $(0,1,-1)$-adjacency matrix. The (u,v)-th entry of this matrix is 1 if there is a directed edge from u to v or an undirected edge from u to v, -1 if there is a directed edge from v to u, and 0 otherwise. This alternative is surely natural. However, $(0,1,-1)$-adjacency matrix is skew-symmetric only for directed graphs. To overcome this disadvantage for mixed graphs, the Hermitian-adjacency matrix is defined. The Hermitian-adjacency matrix incorporates both the (0,1)-adjacency matrix of an undirected graph and the $(0,1,-1)$-adjacency matrix of a directed graph. However, it is not clear whether the Hermitian-adjacency matrix of a mixed graph is the best choice in respect of investigating relationship between its eigenvalues and combinatorial properties. In this context, Mohar [35] introduced the Hermitian-adjacency matrix of second kind by replacing $\mathbf{i}$ with ω_6 and $-\mathbf{i}$ with ω_6^5 in the Hermitian-adjacency matrix. He observed that the sixth root of unity is more natural in relation to combinatorial properties since $\omega_6.\omega_6^5 = 1$ and $\omega_6 + \omega_6^5 = 1$. Since $\omega_6 + \omega_6^5 = 1$, two oppositely directed edges between u and v contribute 1 to the corresponding entries of the adjacency matrix.

Note that the numbers in the set $\{a + \mathbf{i}b : a, b \in \mathbb{Z}\}$ are called Gaussian integers. Similarly, the numbers in the set $\{a + \omega_3 b : a, b \in \mathbb{Z}\}$ are called Eisenstein integers. Assume that G is a mixed graph.

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(i) The graph G is said to be H-*integral*, if all the H-eigenvalues of G are integers.

(ii) The graph G is said to be HS-*integral*, if all the HS-eigenvalues of G are integers.

(iii) The graph G is said to be *Gaussian integral*, if all the eigenvalues of the $(0, 1)$-adjacency matrix of G are Gaussian integers.

(iv) The graph G is said to be *Eisenstein integral*, if all the eigenvalues of the $(0, 1)$-adjacency matrix of G are Eisenstein integers.

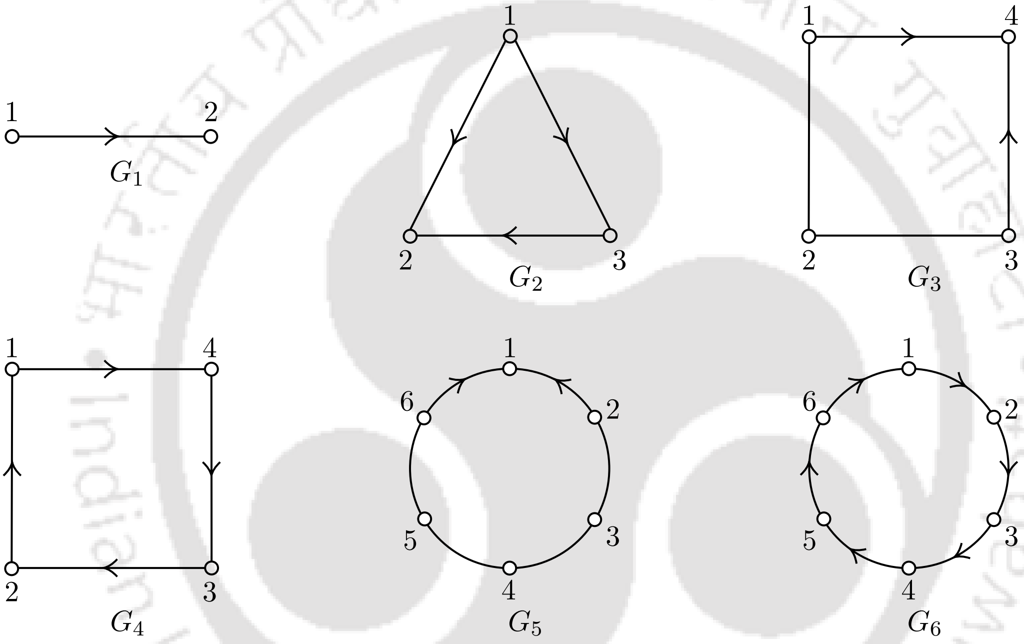

Figure 1.2: The mixed graphs G_1, G_2, G_3, G_4, G_5, and G_6

We remark here that if G is an undirected graph, then $\mathcal{A}(G) = \mathcal{H}(G) = \mathcal{K}(G)$, and therefore the terms H-integral, HS-integral, Gaussian integral and Eisenstein integral have the same meaning. However, if G has at least one directed edge, then $\mathcal{A}(G) \neq \mathcal{H}(G) \neq \mathcal{K}(G)$. Consider the mixed graphs G_1, G_2, G_3, G_4, G_5, and G_6 shown in Figure 1.2. In Table 1.1, we mention the integralities of these mixed graphs.

An H-integral (HS-integral, Gaussian integral or Eisenstein integral) undirected graph is called an *integral* undirected graph. The concept of integral undirected graphs was first introduced by Harary and Schwenk [22] in 1974, and they raised the question of determining integral undirected graphs. In the last few decades, many researchers attempted to solve this problem, which is still not solved completely. In general, the problem of determining all integral undirected graphs seems challenging to answer. Many researchers explored some special classes of undirected graphs such as trees, graphs with restricted degrees, and regular graphs for their

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Mixed graph	H-integral ?	HS-integral ?	Gaussian integral ?	Eisenstein integral ?
G_1	Yes	Yes	Yes	Yes
G_2	No	No	Yes	Yes
G_3	Yes	Yes	No	No
G_4	Yes	No	Yes	No
G_5	Yes	Yes	No	No
G_6	No	Yes	No	Yes

Table 1.1: Integralities of the mixed graphs G_1, G_2, G_3, G_4, G_5, and G_6

integrality. We summarize some significant achievements on integral undirected graphs in the following.

- In 1976, Bussemaker and Cvetković [12] proved that there are exactly 13 connected cubic integral undirected graphs. A few months later, Schwenk [38] reported the similar result independently using some other techniques.

- In 1979, Watanabe [44] showed that an integral undirected tree other than K_2 does not have a complete matching. In the same year, all integral undirected trees with at most one vertex of degree more than 2 were determined by Watanabe and Schwenk [45].

- In 1998, Cvetković et al. [16] found 1888 possible 4-regular bipartite integral undirected graphs.

- In 2000, Balińska et al. [5, 6, 7] proved that there are exactly 263 connected integral undirected graphs with at most 11 vertices.

- In 2003, Stevanović [41] determined all 24 connected 4-regular integral undirected graphs avoiding ± 3 in the spectrum.

- In 2005, Lepović et al. [29] proved that there are 93 non-regular, bipartite integral undirected graphs with maximum degree four.

- In 2009, Ahmadi et al. [2] showed that the total number of adjacency matrices of integral undirected graphs with n vertices is at most $2^{\binom{n}{2}-\frac{n}{400}}$ for a sufficiently large n.

- In 2010, Csikvari [15] constructed integral undirected trees with arbitrarily large diameters. Further research on integral undirected trees can be found in [10, 11, 42, 43].

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1.2 Mixed Cayley graphs

Throughout the thesis, Γ is considered to be a finite group with identity element $\mathbf{1}$. For $x \in \Gamma$, let $\mathrm{ord}(x)$ denote the order of x. If g and h are elements of the group Γ, then we call h a *conjugate* of g if $g = x^{-1}hx$ for some $x \in \Gamma$. The *conjugacy class* of g, denoted $\mathrm{Cl}(g)$, is the set of all conjugates of g in Γ. Define $C_\Gamma(g)$ to be the set of all elements of Γ that commute with g. The *exponent* of a group Γ, denoted $\exp(\Gamma)$, is the least common multiple of the orders of all elements of Γ. We denote the *group algebra* of Γ over a field $\mathbb{F}$ by $\mathbb{F}\Gamma$. That is, $\mathbb{F}\Gamma$ is the set of all formal sums $\sum\limits_{g\in\Gamma} a_g g$, where $a_g \in \mathbb{F}$, and we assume $1.g = g$ to have $\Gamma \subseteq \mathbb{F}\Gamma$.

Let S be a subset of Γ that does not contain the identity element $\mathbf{1}$. If S is closed under inverse, that is, $a^{-1} \in S$ for all $a \in S$, then it is said to be a *symmetric* set. Similarly, if $a^{-1} \notin S$ for all $a \in S$, then it is said to be a *skew-symmetric* set. Define $\overline{S} := \{u \in S \colon u^{-1} \notin S\}$ and $S^{-1} := \{u^{-1} \colon u \in S\}$. Clearly, S is symmetric if and only if $S = S^{-1}$. Similarly, S is symmetric if and only if $\overline{S} = \emptyset$. Further, $S \setminus \overline{S}$ is symmetric, while $\overline{S}$ is skew-symmetric. The *mixed Cayley graph* $\mathrm{Cay}(\Gamma, S)$ is a mixed graph, where $V(\mathrm{Cay}(\Gamma, S)) = \Gamma$ and

$$E(\mathrm{Cay}(\Gamma, S)) = \{(a, b) \colon a, b \in \Gamma, ba^{-1} \in S\}.$$

If S is symmetric (respectively skew-symmetric), then we call $\mathrm{Cay}(\Gamma, S)$ an *undirected Cayley graph* (respectively a *directed Cayley graph*). A mixed Cayley graph $\mathrm{Cay}(\Gamma, S)$ is called *normal* if S is the union of some conjugacy classes of Γ. The mixed Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ is known as a mixed circulant graph, and it is denoted by $\mathrm{Circ}(\mathbb{Z}_n, S)$.

An H-integral (HS-integral, Gaussian integral or Eisenstein integral) undirected Cayley graph is referred to be an *integral undirected Cayley graph*. We summarize some major achievements on integral undirected Cayley graphs in the following.

- In 1982, Bridge and Mena [9] gave a characterization of integral undirected Cayley graphs over abelian groups. Later on, the same characterization was rediscovered by Wasin So [39] for cyclic groups in 2006.
- In 2009, Abdollahi and Vatandoost [1] proved that there are exactly seven connected cubic integral undirected Cayley graphs.
- In 2010, Klotz and Sander [25] proved that if an undirected Cayley graph $\mathrm{Cay}(\Gamma, S)$ over an abelian group Γ is integral, then the set S belongs to the Boolean algebra $\mathbb{B}(\Gamma)$ generated by the subgroups of Γ. Moreover, they conjectured that the converse is also true, which was proved by Alperin and Peterson [3].
- In 2014, Ku et al. [28] proved that normal undirected Cayley graphs over symmetric groups are integral.

- In 2017, Lu et al. [34] gave necessary and sufficient condition for the integrality of undirected Cayley graphs over the dihedral group $\mathcal{D}_n := \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle$. In particular, they completely determined all integral undirected Cayley graphs over the dihedral group $\mathcal{D}_p$ for a prime p.

- In 2019, Cheng et al.[13] obtained several sufficient conditions for the integrality of undirected Cayley graphs over the group $T_{4n} := \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$. In particular, they also completely determined all integral undirected Cayley graphs over the group T_{4p} for a prime p.

In 2011, Xu and Meng [46] gave a sufficient condition for Gaussian integrality of mixed circulant graphs. Additionally, they conjectured that the condition is also necessary. Later, Li [30] confirmed this conjecture in 2013.

1.3 Some basics of representation theory

In this section, we discuss some basic definitions and results of representation theory that are used throughout the thesis. A *representation* of a finite group Γ is a homomorphism $\rho \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ is the set of all $n \times n$ invertible matrices with complex entries. Here, the number n is called the *degree* of ρ. Two representations ρ_1 and ρ_2 of Γ of degree n are *equivalent* if there is a $T \in \mathrm{GL}_n(\mathbb{C})$ such that $T\rho_1(x) = \rho_2(x)T$ for each $x \in \Gamma$.

Let $\rho \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ be a representation of Γ. The *character* $\chi_\rho \colon \Gamma \to \mathbb{C}$ of ρ is defined by setting $\chi_\rho(x) := \mathrm{Tr}(\rho(x))$ for $x \in \Gamma$, where $\mathrm{Tr}(\rho(x))$ is the trace of $\rho(x)$. By degree of χ_ρ, we mean the degree of ρ, which is simply $\chi_\rho(\mathbf{1})$. If W is a $\rho(x)$-invariant subspace of $\mathbb{C}^n$ for each $x \in \Gamma$, then we say that W is a $\rho(\Gamma)$-invariant subspace of $\mathbb{C}^n$. If $\{\mathbf{0}\}$ and $\mathbb{C}^n$ are the only $\rho(\Gamma)$-invariant subspaces of $\mathbb{C}^n$, then we say ρ an *irreducible representation* of Γ, and the corresponding character χ_ρ an *irreducible character* of Γ.

For a group Γ, we denote by $\mathrm{IRR}(\Gamma)$ and $\mathrm{Irr}(\Gamma)$ the complete set of non-equivalent irreducible representations of Γ and the complete set of non-equivalent irreducible characters of Γ, respectively.

Lemma 1.3.1 ([40]). *If $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ is the cyclic group under addition modulo n, then $\mathrm{Irr}(\mathbb{Z}_n) = \{\phi_k \colon 0 \leq k \leq n-1\}$, where $\phi_k(j) = \omega_n^{jk}$ for each $j, k \in \{0, 1, \ldots, n-1\}$.*

Lemma 1.3.2 ([40]). *Let Γ_1 and Γ_2 be two abelian groups of order m and n, respectively. If $\mathrm{Irr}(\Gamma_1) = \{\phi_1, \ldots, \phi_m\}$ and $\mathrm{Irr}(\Gamma_2) = \{\rho_1, \ldots, \rho_n\}$, then*

$$\mathrm{Irr}(\Gamma_1 \otimes \Gamma_2) = \{\psi_{kl} \colon 1 \leq k \leq m, 1 \leq l \leq n\},$$

where $\psi_{kl} \colon \Gamma_1 \otimes \Gamma_2 \to \mathbb{C}^*$ and $\psi_{kl}(x, y) = \phi_k(x)\rho_l(y)$ for $x \in \Gamma_1, y \in \Gamma_2$.

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Theorem 1.3.3 ([40]). *Let Γ be a finite group and ρ be a representation of Γ of degree k with corresponding character χ. If $x \in \Gamma$ and $\operatorname{ord}(x) = m$, then the following assertions hold.*

(i) $\rho(x)$ *is similar to a diagonal matrix with diagonal entries* $\epsilon_1, \ldots, \epsilon_k$*, where* $\epsilon_i^m = 1$ *for each* $i \in \{1, \ldots, k\}$.

(ii) $\chi(x) = \sum\limits_{i=1}^{k} \epsilon_i$*, where* $\epsilon_i^m = 1$ *for each* $i \in \{1, \ldots, k\}$.

(iii) $\chi(x^{-1}) = \overline{\chi(x)}$.

Proof. Note that $\rho(x)^m$ is an identity matrix. Therefore, $\rho(x)$ is diagonalizable, and that its eigenvalues are m-th roots of unity. Thus the proofs of Part (i) and Part (ii) follow.

Again, $xx^{-1} = \mathbf{1}$ gives that $\rho(x^{-1}) = \rho(x)^{-1}$. Therefore if $\chi(x) = \sum_{i=1}^{k} \epsilon_i$, we have that $\chi(x^{-1}) = \sum_{i=1}^{k} \epsilon_i^{-1} = \sum_{i=1}^{k} \overline{\epsilon}_i = \overline{\chi(x)}$. □

For a representation $\rho \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ of Γ, define $\overline{\rho} \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ by $\overline{\rho}(x) := \overline{\rho(x)}$, where $\overline{\rho(x)}$ is the matrix whose entries are the complex conjugates of the corresponding entries of $\rho(x)$. Note that if ρ is irreducible, then $\overline{\rho}$ is also irreducible. Hence we have the following lemma. See Proposition 9.1.1 and Corollary 9.1.2 in [40] for details.

Lemma 1.3.4 ([40]). *Let Γ be a finite group and $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. If $j \in \{1, \ldots, h\}$, then there exists $k \in \{1, \ldots, h\}$ satisfying $\overline{\chi}_k = \chi_j$, where $\overline{\chi}_k \colon \Gamma \to \mathbb{C}$ such that $\overline{\chi}_k(x) = \overline{\chi_k(x)}$ for each $x \in \Gamma$.*

Theorem 1.3.5 ([40]). *Let Γ be a finite group and $x, y \in \Gamma$. If $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then*

(i)

$$\sum_{x \in \Gamma} \chi_j(x)\overline{\chi_k(x)} = \begin{cases} |\Gamma| & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

(ii)

$$\sum_{j=1}^{h} \chi_j(x)\overline{\chi_j(y)} = \begin{cases} |C_\Gamma(x)| & \text{if } x \text{ and } y \text{ are conjugates to each other} \\ 0 & \text{otherwise.} \end{cases}$$

For a function $f \colon \Gamma \to \mathbb{C}$, let $[f(yx^{-1})]_{x,y \in \Gamma}$ be the matrix whose rows and columns are indexed by the elements of Γ, and for $x, y \in \Gamma$, the (x, y)-th entry of the matrix is $f(yx^{-1})$.

Theorem 1.3.6 ([18]). *Let Γ be a finite group and $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. If $f \colon \Gamma \to \mathbb{C}$ is a class function, then the spectrum of the matrix $[f(yx^{-1})]_{x,y \in \Gamma}$ is $\{[\gamma_1]^{d_1^2}, \ldots, [\gamma_h]^{d_h^2}\}$, where*

$$\gamma_j = \frac{1}{\chi_j(\mathbf{1})} \sum_{x \in \Gamma} f(x)\chi_j(x) \quad \text{and} \quad d_j = \chi_j(\mathbf{1})$$

for each $j \in \{1, \ldots, h\}$.

1.4 Irreducible factors of cyclotomic polynomial over $\mathbb{Q}(\mathbf{i})$

Let $\mathbb{K}$ be a field containing a subfield $\mathbb{F}$. Then the field $\mathbb{K}$ is said to be an *extension* of $\mathbb{F}$. The dimension of the vector space $\mathbb{K}$ over $\mathbb{F}$ is denoted by $[\mathbb{K} : \mathbb{F}]$. If $a \in \mathbb{K}$ then the smallest subfield of $\mathbb{K}$, containing both $\mathbb{F}$ and the element a, is denoted by $\mathbb{F}(a)$. Similarly, if $a, b \in \mathbb{K}$ then the smallest subfield of $\mathbb{K}$, containing both $\mathbb{F}$ and the elements a and b, is denoted by $\mathbb{F}(a, b)$.

In this section, we factorize the n-th degree cyclotomic polynomial into two irreducible factors over $\mathbb{Q}(\mathbf{i})$ for $n \equiv 0 \pmod 4$. This factorization is useful in characterizing H-integral mixed Cayley graphs over abelian groups in Chapter 2. Let n be a fixed positive integer such that $n \geq 2$. We review some basic definitions and notations from [39]. For a divisor d of n, define

$$M_n(d) := \{dk \colon 1 \leq dk \leq n-1\} \text{ and}$$

$$G_n(d) := \{dk \colon 1 \leq dk \leq n-1, \gcd(dk, n) = d\}.$$

It is clear that $M_n(n) = G_n(n) = \emptyset$, $M_n(d) = dM_{\frac{n}{d}}(1)$ and $G_n(d) = dG_{\frac{n}{d}}(1)$. For $h, g \in \mathbb{N}$, we write $h \mid g$ to mean that h divides g.

Lemma 1.4.1 ([39]). *If $n = dg$ for some $d, g \in \mathbb{N}$, then $M_n(d) = \bigcup\limits_{h \mid g} G_n(hd)$.*

Let $n \equiv 0 \pmod 4$. For a divisor d of $\frac{n}{4}$ and $r \in \{0, 1, 2, 3\}$, define

$$M_n^r(d) := \{dk \colon 0 \leq dk \leq n-1, k \equiv r \pmod 4\}.$$

For a divisor d of $\frac{n}{4}, r \in \{1, 3\}$ and $g \in \mathbb{N}$, define the sets

$$G_n^r(d) := \{dk \colon 1 \leq dk \leq n-1, \gcd(dk, n) = d, k \equiv r \pmod 4\},$$

$$D_g := \{k \colon k \text{ is an odd divisor of } g\} \text{ and}$$

$$D_g^r := \{k \colon k \text{ divides } g, k \equiv r \pmod 4\}.$$

It is clear that $D_g = D_g^1 \cup D_g^3$. Also, $G_n^1(d) = dG_{\frac{n}{d}}^1(1)$ and $G_n^3(d) = dG_{\frac{n}{d}}^3(1)$. In the next two lemmas, we partition the set $M_n^r(d)$ in terms of $G_n^r(d)$. These partitions are used to find some suitable factorizations of the polynomials $x^{\frac{n}{4}} - \mathbf{i}$ and $x^{\frac{n}{4}} + \mathbf{i}$ in Lemma 1.4.6. Thereafter, Lemma 1.4.6 is used to prove the irreducibility over $\mathbb{Q}(\mathbf{i})$ of two specific factors of the cyclotomic polynomial in Theorem 1.4.11.

Lemma 1.4.2. *If $n = 4dg$ for some $d, g \in \mathbb{N}$, then the following assertions hold.*

(i) $M_n^1(d) \cup M_n^3(d) = \bigcup\limits_{h \in D_g} G_n(hd)$.

(ii) $M_n^2(d) = \bigcup_{h\in D_g} G_n(2hd)$.

(iii) $M_n^0(d) = M_n(4d) \cup \{0\}$.

Proof. (i) Let $dk \in M_n^1(d) \cup M_n^3(d)$ and $h = \gcd(k, g)$. Since $k \equiv 1$ or $3 \pmod 4$, it follows that k is an odd integer, and so $h \in D_g$. Hence $\gcd(dk, n) = \gcd(dk, 4dg) = hd$, and it implies that $M_n^1(d) \cup M_n^3(d) \subseteq \bigcup_{h\in D_g} G_n(hd)$. On the other hand, if $x \in \bigcup_{h\in D_g} G_n(hd)$ then there exists an $h \in D_g$ such that $\gcd(x, n) = hd$. So there exists $x_0 \in \mathbb{Z}$ such that $x = hdx_0$ and $\gcd(x_0, \frac{n}{hd}) = 1$. Since $\frac{n}{hd}$ is an even integer and $\gcd(x_0, \frac{n}{hd}) = 1$, x is an odd multiple of d. Note that $M_n^1(d) \cup M_n^3(d)$ is the set of all odd multiples of d between 0 and n. Thus $x \in M_n^1(d) \cup M_n^3(d)$, and so $\bigcup_{h\in D_g} G_n(hd) \subseteq M_n^1(d) \cup M_n^3(d)$. Hence the desired equality follows.

(ii) Let $dk \in M_n^2(d)$ and $h = \gcd(\frac{k}{2}, g)$. Since $k \equiv 2 \pmod 4$, we see that $\gcd(k, 4g) = 2h$. This gives $\gcd(dk, n) = \gcd(dk, 4dg) = 2hd$, and so $M_n^2(d) \subseteq \bigcup_{h\in D_g} G_n(2hd)$. On the other hand, if $x \in \bigcup_{h\in D_g} G_n(2hd)$, then there exists $h \in D_g$ such that $\gcd(x, n) = 2hd$. So there exists $x_0 \in \mathbb{Z}$ such that $x = 2hdx_0$ and $\gcd(x_0, \frac{n}{2hd}) = 1$. Since $\frac{n}{2hd}$ is an even integer and $\gcd(x_0, \frac{n}{2hd}) = 1$, x_0 is an odd integer. Now h and x_0 are odd integers, and so letting $\alpha = 2hx_0$, we get $x = \alpha d$, where $\alpha \equiv 2 \pmod 4$. Hence $x \in M_n^2(d)$, and so $\bigcup_{h\in D_g} G_n(2hd) \subseteq M_n^2(d)$. Thus the desired equality follows.

(iii) By definition, we have

$$\begin{aligned} M_n^0(d) &= \{dk \colon 0 \le dk \le n-1, k = 4\alpha \text{ for some } \alpha \in \mathbb{Z}\} \\ &= \{4\alpha d \colon 1 \le 4\alpha d \le n-1 \text{ for some } \alpha \in \mathbb{Z}\} \cup \{0\} \\ &= M_n(4d) \cup \{0\}. \end{aligned}$$

□

Lemma 1.4.3. *If $n = 4dg$ for some $d, g \in \mathbb{N}$, then the following assertions hold.*

(i) $G_n^1(d) \cap G_n^3(d) = \emptyset$.

(ii) $G_n(d) = G_n^1(d) \cup G_n^3(d)$.

(iii) $M_n^1(d) = \left(\bigcup_{h\in D_g^1} G_n^1(hd)\right) \cup \left(\bigcup_{h\in D_g^3} G_n^3(hd)\right)$.

(iv) $M_n^3(d) = \left(\bigcup_{h\in D_g^1} G_n^3(hd)\right) \cup \left(\bigcup_{h\in D_g^3} G_n^1(hd)\right)$.

Proof. (i) It is clear from the definitions of $G_n^1(d)$ and $G_n^3(d)$ that $G_n^1(d) \cap G_n^3(d) = \emptyset$.

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(ii) Since $G_n^1(d) \subseteq G_n(d)$ and $G_n^3(d) \subseteq G_n(d)$, we have $G_n^1(d) \cup G_n^3(d) \subseteq G_n(d)$. On the other hand, if $x \in G_n(d)$ then $x = d\alpha$ for some α. Note that $\gcd(\alpha, \frac{n}{d}) = 1$, and so α is an odd integer. Thus $\alpha \equiv 1 \pmod 4$ or $\alpha \equiv 3 \pmod 4$. Hence $G_n(d) \subseteq G_n^1(d) \cup G_n^3(d)$. Thus the desired equality follows.

(iii) Let $dk \in M_n^1(d)$ so that $k \equiv 1 \pmod 4$. Lemma 1.4.2 gives an $h \in D_g$ satisfying $dk \in G_n(hd)$. Thus by Part (ii), we have $dk \in G_n^1(hd)$ or $dk \in G_n^3(hd)$.
Case 1. Assume that $h \equiv 1 \pmod 4$. Suppose, on the contrary, that $dk \in G_n^3(hd)$, that is, $dk = \alpha hd$ with $\alpha \equiv 3 \pmod 4$ and $\gcd(\alpha, \frac{n}{hd}) = 1$. Thus we have $k = \alpha h \equiv 3 \pmod 4$, a contradiction. Hence $dk \in G_n^1(hd)$.
Case 2. Assume that $h \equiv 3 \pmod 4$. Suppose, on the contrary, that $dk \in G_n^1(hd)$. As in Case 1, we get a contradiction in this case also. Hence $dk \in G_n^3(hd)$. Thus

$$M_n^1(d) \subseteq \left(\bigcup_{h\in D_g^1} G_n^1(hd)\right) \cup \left(\bigcup_{h\in D_g^3} G_n^3(hd)\right).$$

On the other hand, if $\alpha hd \in G_n^1(hd)$ with $h \in D_g^1$ and $\alpha \equiv 1 \pmod 4$, then we get $\alpha h \equiv 1 \pmod 4$, that is, $\alpha hd \in M_n^1(d)$. Similarly, if $\beta hd \in G_n^3(hd)$ for $h \in D_g^3$ and $\beta \equiv 3 \pmod 4$, then also $\beta hd \in M_n^1(d)$. Therefore

$$\left(\bigcup_{h\in D_g^1} G_n^1(hd)\right) \cup \left(\bigcup_{h\in D_g^3} G_n^3(hd)\right) \subseteq M_n^1(d).$$

Thus the desired equality follows.

(iv) The proof of this part is similar to the proof of Part (iii). For the sake of completeness, we provide the proof. Let $dk \in M_n^3(d)$ so that $k \equiv 3 \pmod 4$. Lemma 1.4.2 gives an $h \in D_g$ satisfying $dk \in G_n(hd)$. Thus by Part (ii), we have $dk \in G_n^1(hd)$ or $dk \in G_n^3(hd)$.
Case 1. Assume that $h \equiv 1 \pmod 4$. Suppose, on the contrary, that $dk \in G_n^1(hd)$, that is, $dk = \alpha hd$ with $\alpha \equiv 1 \pmod 4$ and $\gcd(\alpha, \frac{n}{hd}) = 1$. Thus we have $k = \alpha h \equiv 1 \pmod 4$, a contradiction. Hence $dk \in G_n^3(hd)$.
Case 2. Assume that $h \equiv 3 \pmod 4$. Suppose, on the contrary, that $dk \in G_n^3(hd)$. As in Case 1, we get a contradiction in this case also. Hence $dk \in G_n^1(hd)$. Thus

$$M_n^3(d) \subseteq \left(\bigcup_{h\in D_g^1} G_n^3(hd)\right) \cup \left(\bigcup_{h\in D_g^3} G_n^1(hd)\right).$$

On the other hand, if $\alpha hd \in G_n^3(hd)$ with $h \in D_g^1$ and $\alpha \equiv 3 \pmod 4$, then we get $\alpha h \equiv 3 \pmod 4$, that is, $\alpha hd \in M_n^3(d)$. Similarly, if $\beta hd \in G_n^1(hd)$ for $h \in D_g^3$ and $\beta \equiv 1 \pmod 4$, then also $\beta hd \in M_n^3(d)$. Therefore

$$\left(\bigcup_{h\in D_g^1} G_n^3(hd)\right) \cup \left(\bigcup_{h\in D_g^3} G_n^1(hd)\right) \subseteq M_n^3(d).$$

Thus the desired equality follows. □

Let n be a fixed positive integer, where $n \geq 2$. The *cyclotomic polynomial* $\Phi_n(x)$ is the monic polynomial whose zeros are the primitive n^{th} roots of unity. That is,

$$\Phi_n(x) = \prod_{a \in G_n(1)} (x - \omega_n^a).$$

Clearly, the degree of $\Phi_n(x)$ is $\varphi(n)$, where φ is the Euler phi function, that is, $\phi(n) = |G_n(1)|$. See [23] for more details about cyclotomic polynomials.

Lemma 1.4.4 ([23])**.** $x^n - 1 = \prod_{d|n} \Phi_d(x)$ *and* $\Phi_n(x) \in \mathbb{Z}[x]$.

Theorem 1.4.5 ([23])**.** *The cyclotomic polynomial* $\Phi_n(x)$ *is irreducible in* $\mathbb{Z}[x]$.

Note that $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\alpha)$ if and only if $[\mathbb{Q}(\alpha, \omega_n) : \mathbb{Q}(\alpha)] = \varphi(n)$, where $\alpha \in \mathbb{C}$. See Section 13.2 in [17] for more details. In particular, $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\mathbf{i})$ if and only if $[\mathbb{Q}(\mathbf{i}, \omega_n) : \mathbb{Q}(\mathbf{i})] = \varphi(n)$. Also, $\mathbb{Q}(\omega_n)$ does not contain the number $\mathbf{i}$ if and only if $n \not\equiv 0 \pmod 4$. Thus, if $n \not\equiv 0 \pmod 4$ then $[\mathbb{Q}(\mathbf{i}, \omega_n) : \mathbb{Q}(\omega_n)] = 2 = [\mathbb{Q}(\mathbf{i}), \mathbb{Q}]$, and therefore

$$[\mathbb{Q}(\mathbf{i}, \omega_n) : \mathbb{Q}(\mathbf{i})] = \frac{[\mathbb{Q}(\mathbf{i}, \omega_n) : \mathbb{Q}(\omega_n)] \times [\mathbb{Q}(\omega_n) : \mathbb{Q}]}{[\mathbb{Q}(\mathbf{i}) : \mathbb{Q}]} = [\mathbb{Q}(\omega_n) : \mathbb{Q}] = \varphi(n).$$

Hence for $n \not\equiv 0 \pmod 4$, the polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\mathbf{i})$.

If $n \equiv 0 \pmod 4$ then $\mathbb{Q}(\mathbf{i}, \omega_n) = \mathbb{Q}(\omega_n)$, and so

$$[\mathbb{Q}(\mathbf{i}, \omega_n) : \mathbb{Q}(\mathbf{i})] = \frac{[\mathbb{Q}(\mathbf{i}, \omega_n) : \mathbb{Q}]}{[\mathbb{Q}(\mathbf{i}) : \mathbb{Q}]} = \frac{\varphi(n)}{2}.$$

Hence the polynomial $\Phi_n(x)$ is not irreducible over $\mathbb{Q}(\mathbf{i})$ for $n \equiv 0 \pmod 4$.

For $n \equiv 0 \pmod 4$, we factorize $\Phi_n(x)$ into two irreducible monic polynomials over $\mathbb{Q}(\mathbf{i})$. From Lemma 1.4.3, we know that $G_n(1)$ is a disjoint union of $G_n^1(1)$ and $G_n^3(1)$. Define

$$\Phi_n^1(x) := \prod_{a \in G_n^1(1)} (x - \omega_n^a) \text{ and } \Phi_n^3(x) := \prod_{a \in G_n^3(1)} (x - \omega_n^a).$$

It is clear from the definition that $\Phi_n(x) = \Phi_n^1(x)\Phi_n^3(x)$.

Lemma 1.4.6. *If* $n \equiv 0 \pmod 4$*, then the following assertions hold.*

(i) $x^{\frac{n}{4}} - \mathbf{i} = \prod_{h \in D_n^1} \Phi^1_{\frac{n}{h}}(x) \prod_{h \in D_n^3} \Phi^3_{\frac{n}{h}}(x)$.

(ii) $x^{\frac{n}{4}} + \mathbf{i} = \prod_{h \in D_n^1} \Phi^3_{\frac{n}{h}}(x) \prod_{h \in D_n^3} \Phi^1_{\frac{n}{h}}(x)$.

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Proof. (i) Note that $|M_n^1(1)| = \frac{n}{4}$ and ω_n^a is a root of $x^{\frac{n}{4}} - \mathbf{i}$ for each $a \in M_n^1(1)$. Therefore

$$\begin{aligned}
x^{\frac{n}{4}} - \mathbf{i} &= \prod_{a\in M_n^1(1)} (x-\omega_n^a) \\
&= \prod_{h\in D^1_{\frac{n}{4}}} \prod_{a\in G_n^1(h)} (x-\omega_n^a) \prod_{h\in D^3_{\frac{n}{4}}} \prod_{a\in G_n^3(h)} (x-\omega_n^a), \qquad \text{using Lemma 1.4.3} \\
&= \prod_{h\in D^1_n} \prod_{a\in hG^1_{\frac{n}{h}}(1)} (x-\omega_n^a) \prod_{h\in D^3_n} \prod_{a\in hG^3_{\frac{n}{h}}(1)} (x-\omega_n^a), \quad \text{since } D_n^1 = D^1_{\frac{n}{4}}, D_n^3 = D^3_{\frac{n}{4}} \\
&= \prod_{h\in D^1_n} \prod_{a\in G^1_{\frac{n}{h}}(1)} (x-(\omega_n^h)^a) \prod_{h\in D^3_n} \prod_{a\in G^3_{\frac{n}{h}}(1)} (x-(\omega_n^h)^a) \\
&= \prod_{h\in D^1_n} \Phi^1_{\frac{n}{h}}(x) \prod_{h\in D^3_n} \Phi^3_{\frac{n}{h}}(x).
\end{aligned}$$

In the last line of the preceding equalities, we use the fact that ω_n^h is a primitive $\frac{n}{h}$-th root of unity.

(ii) Note that $|M_n^3(1)| = \frac{n}{4}$ and ω_n^a is a root of $x^{\frac{n}{4}} + \mathbf{i}$ for each $a \in M_n^3(1)$. Therefore

$$\begin{aligned}
x^{\frac{n}{4}} + \mathbf{i} &= \prod_{a\in M_n^3(1)} (x-\omega_n^a) \\
&= \prod_{h\in D^1_{\frac{n}{4}}} \prod_{a\in G_n^3(h)} (x-\omega_n^a) \prod_{h\in D^3_{\frac{n}{4}}} \prod_{a\in G_n^1(h)} (x-\omega_n^a), \qquad \text{using Lemma 1.4.3} \\
&= \prod_{h\in D^1_n} \prod_{a\in hG^3_{\frac{n}{h}}(1)} (x-\omega_n^a) \prod_{h\in D^3_n} \prod_{a\in hG^1_{\frac{n}{h}}(1)} (x-\omega_n^a) \\
&= \prod_{h\in D^1_n} \prod_{a\in G^3_{\frac{n}{h}}(1)} (x-(\omega_n^h)^a) \prod_{h\in D^3_n} \prod_{a\in G^1_{\frac{n}{h}}(1)} (x-(\omega_n^h)^a) \\
&= \prod_{h\in D^1_n} \Phi^3_{\frac{n}{h}}(x) \prod_{h\in D^3_n} \Phi^1_{\frac{n}{h}}(x).
\end{aligned}$$

□

Corollary 1.4.7. *If $n \equiv 0 \pmod 4$, then the factors $\Phi_n^1(x)$ and $\Phi_n^3(x)$ of $\Phi_n(x)$ are monic polynomials in $\mathbb{Z}(\mathbf{i})[x]$ of degree $\frac{\varphi(n)}{2}$.*

Proof. By definition, $\Phi_n^1(x)$ and $\Phi_n^3(x)$ are monic. Also, $G_n(1) = G_n^1(1) \cup G_n^3(1)$, a disjoint union and that $|G_n^1(1)| = |G_n^3(1)|$. Therefore, the degree of each of $\Phi_n^1(x)$ and $\Phi_n^3(x)$ is $\varphi(n)/2$. Now we use induction on n to show that $\Phi_n^1(x), \Phi_n^3(x) \in \mathbb{Z}(\mathbf{i})[x]$. For $n = 4$, the polynomials $\Phi_4^1(x) := x - \mathbf{i}$ and $\Phi_4^3(x) := x + \mathbf{i}$ are clearly in $\mathbb{Z}(\mathbf{i})[x]$. Assume that $\Phi_k^1(x)$ and $\Phi_k^3(x)$ are in $\mathbb{Z}(\mathbf{i})[x]$ for $k < n$ with $k \equiv 0 \pmod 4$. By Lemma 1.4.6, $\Phi_n^1(x) = \frac{x^{\frac{n}{4}} - \mathbf{i}}{f(x)}$ and $\Phi_n^3(x) = \frac{x^{\frac{n}{4}} + \mathbf{i}}{g(x)}$, where

$$f(x) = \prod_{\substack{h\in D^1_n \\ h\neq 1}} \Phi^1_{\frac{n}{h}}(x) \prod_{h\in D^3_n} \Phi^3_{\frac{n}{h}}(x) \text{ and } g(x) = \prod_{\substack{h\in D^1_n \\ h\neq 1}} \Phi^3_{\frac{n}{h}}(x) \prod_{h\in D^3_n} \Phi^1_{\frac{n}{h}}(x).$$

Clearly, $f(x)$ and $g(x)$ are monic polynomials. By induction hypothesis $f(x), g(x) \in \mathbb{Z}(\mathbf{i})[x]$. Now it follows by "long division" that $\Phi_n^1(x) \in \mathbb{Z}(\mathbf{i})[x]$ and $\Phi_n^3(x) \in \mathbb{Z}(\mathbf{i})[x]$. Hence the proof is complete by induction $\square$

Corollary 1.4.8. *If* $n \equiv 0 \pmod 4$, *then* $\frac{n}{2} = \sum\limits_{d \in D_n} \varphi\left(\frac{n}{d}\right)$.

Proof. Using Lemma 1.4.6, we get $x^{\frac{n}{2}} + 1 = \prod\limits_{d \in D_n} \Phi_{\frac{n}{d}}(x)$. Now the proof follows by comparing the degree of both side of this equation. $\square$

Corollary 1.4.9. *If n is a positive even integer, then* $\frac{n}{2} = \sum\limits_{d \in D_n} \varphi\left(\frac{n}{d}\right)$.

Proof. Since $2n \equiv 0 \pmod 4$ and $\frac{2n}{d} \equiv 0 \pmod 4$ for all $d \in D_n$, we find that $\varphi(\frac{2n}{d}) = 2\varphi(\frac{n}{d})$ for $d \in D_n = D_{2n}$. Thus by Corollary 1.4.8, $\frac{2n}{2} = \sum\limits_{d \in D_{2n}} \varphi(\frac{2n}{d}) = \sum\limits_{d \in D_n} 2\varphi(\frac{n}{d})$. $\square$

From Lemma 1.4.6, the next corollary follows easily.

Corollary 1.4.10. *If* $n = 2^k$ *with* $k \geq 2$, *then* $\Phi_n^1(x) = x^{\frac{n}{4}} + \mathbf{i}$ and $\Phi_n^3(x) = x^{\frac{n}{4}} - \mathbf{i}$.

Note that if $[\mathbb{F}(\alpha) : \mathbb{F}] = k$, then there is a unique irreducible monic polynomial $p(x) \in \mathbb{F}[x]$ of degree k having α as a root. See Section 13.2 in [17] for more details.

Theorem 1.4.11. *If* $n \equiv 0 \pmod 4$, *then the factors* $\Phi_n^1(x)$ *and* $\Phi_n^3(x)$ *of* $\Phi_n(x)$ *are irreducible monic polynomials in* $\mathbb{Q}(\mathbf{i})[x]$ *of degree* $\frac{\varphi(n)}{2}$.

Proof. Let $n \equiv 0 \pmod 4$. In view of Corollary 1.4.7, we only need to show that $\Phi_n^1(x)$ and $\Phi_n^3(x)$ are irreducible over $\mathbb{Q}(\mathbf{i})$. We have $[\mathbb{Q}(\mathbf{i}, \omega_n) : \mathbb{Q}(\mathbf{i})] = \frac{\varphi(n)}{2}$. Then there is a unique irreducible monic polynomial $p(x) \in \mathbb{Q}(\mathbf{i})[x]$ of degree $\frac{\varphi(n)}{2}$ having ω_n as a root. Since ω_n is also a root of $\Phi_n^1(x)$, we have $\Phi_n^1(x) = p(x)f(x)$ for some $f(x) \in \mathbb{Q}(\mathbf{i})[x]$. As $\Phi_n^1(x)$ is a monic polynomial of degree $\varphi(n)/2$, we have $f(x) = 1$. Hence $\Phi_n^1(x) = p(x)$, that is, $\Phi_n^1(x)$ is irreducible. Similarly, for an $a \in G_n^3(1)$, we have $[\mathbb{Q}(\mathbf{i}, \omega_n^a) : \mathbb{Q}(\mathbf{i})] = \frac{\varphi(n)}{2}$. Now, proceeding as in the case of irreducibility of $\Phi_n^1(x)$, we find that $\Phi_n^3(x)$ is also irreducible. $\square$

1.5 Irreducible factors of cyclotomic polynomial over $\mathbb{Q}(\omega_3)$

In this section, we factorize the n-th degree cyclotomic polynomial into two irreducible factors over $\mathbb{Q}(\omega_3)$ where $n \equiv 0 \pmod 3$. This factorization is useful in characterizing HS-integral mixed Cayley graphs over abelian groups in Chapter 3. Let $n \equiv 0 \pmod 3$. For a divisor d of $\frac{n}{3}, r \in \{0, 1, 2\}$ and $g \in \mathbb{N}$, define the sets

$$M_{n,3}^r(d) := \{dk \colon 0 \leq dk \leq n-1, k \equiv r \pmod 3\},$$

$$G_{n,3}^r(d) := \{dk \colon 1 \leq dk \leq n-1, \gcd(dk, n) = d, k \equiv r \pmod 3\},$$

$$D_{g,3} := \{k \colon k \text{ divides } g, k \not\equiv 0 \pmod 3\} \text{ and}$$

$$D_{g,3}^r := \{k \colon k \text{ divides } g, k \equiv r \pmod 3\}.$$

It is clear that $D_{g,3} = D^1_{g,3} \cup D^2_{g,3}$. In the next two lemmas, we partition the sets $M^1_{n,3}(d)$ and $M^2_{n,3}(d)$ in terms of $G^r_{n,3}(d)$. These partitions are used to find some suitable factorizations of the polynomials $x^{\frac{n}{3}} + \omega_3$ and $x^{\frac{n}{4}} + \omega_3^2$ in Lemma 1.5.3. Thereafter, Lemma 1.5.3 is used to prove the irreducibility over $\mathbb{Q}(\omega_3)$ of two specific factors of the cyclotomic polynomial in Theorem 1.5.7.

Lemma 1.5.1. *If $n \equiv 0 \pmod 3$, d divides $\frac{n}{3}$, and $g = \frac{n}{3d}$, then*

$$M^1_{n,3}(d) \cup M^2_{n,3}(d) = \bigcup_{h \in D_{g,3}} G_n(hd).$$

Proof. Let $dk \in M^1_{n,3}(d) \cup M^2_{n,3}(d)$. Lemma 1.4.1 gives that

$$M^1_{n,3}(d) \cup M^2_{n,3}(d) \subseteq M_n(d) = \bigcup_{h|3g} G_n(hd).$$

Thus there exists a divisor h of $3g$ such that $dk = \alpha hd$, for some $\alpha \in \mathbb{Z}$ with $\gcd(\alpha, \frac{3g}{h}) = 1$. Now we have $h = \frac{k}{\alpha}$ and k is not a multiple of 3, which implies that $h \mid g$ and h is not a multiple of 3. Thus $h \in D_{g,3}$, and so $dk \in \bigcup_{h \in D_{g,3}} G_n(hd)$. Conversely, let $x \in \bigcup_{h \in D_{g,3}} G_n(hd)$. Then there exists $h \in D_{g,3}$ such that $x = \alpha hd$, where $\alpha \in \mathbb{Z}$ and $\gcd(\alpha, \frac{3g}{h}) = 1$. Note that α and h are not multiples of 3. Thus $\alpha h \equiv 1$ or $2 \pmod 3$, and so $x \in M^1_{n,3}(d) \cup M^2_{n,3}(d)$. Thus the desired equality follows. □

We now prove that $G_n(d)$ is a disjoint union of $G^1_{n,3}(d)$ and $G^2_{n,3}(d)$.

Lemma 1.5.2. *If $n \equiv 0 \pmod 3$, d divides $\frac{n}{3}$ and $g = \frac{n}{3d}$, then the following assertions hold.*

(i) $G^1_{n,3}(d) \cap G^2_{n,3}(d) = \emptyset$.

(ii) $G_n(d) = G^1_{n,3}(d) \cup G^2_{n,3}(d)$.

(iii) $M^1_{n,3}(d) = \left(\bigcup_{h \in D^1_{g,3}} G^1_{n,3}(hd) \right) \cup \left(\bigcup_{h \in D^2_{g,3}} G^2_{n,3}(hd) \right)$.

(iv) $M^2_{n,3}(d) = \left(\bigcup_{h \in D^1_{g,3}} G^2_{n,3}(hd) \right) \cup \left(\bigcup_{h \in D^2_{g,3}} G^1_{n,3}(hd) \right)$.

Proof. (i) It is clear from the definitions of $G^1_{n,3}(d)$ and $G^2_{n,3}(d)$ that $G^1_{n,3}(d) \cap G^2_{n,3}(d) = \emptyset$.

(ii) Since $G^1_{n,3}(d) \subseteq G_n(d)$ and $G^2_{n,3}(d) \subseteq G_n(d)$, we have $G^1_{n,3}(d) \cup G^2_{n,3}(d) \subseteq G_n(d)$. Conversely, let $x \in G_n(d)$. Then $x = d\alpha$ for some α satisfying $\gcd(\alpha, 3g) = 1$, and therefore $\alpha \equiv 1$ or $2 \pmod 3$. Hence $G_n(d) \subseteq G^1_{n,3}(d) \cup G^2_{n,3}(d)$. Thus the desired equality follows.

(iii) Let $dk \in M^1_{n,3}(d)$ so that $k \equiv 1 \pmod 3$. By Lemma 1.5.1, there exists $h \in D_{g,3}$ satisfying $dk \in G_n(hd)$. Thus $dk \in G^1_{n,3}(hd)$ or $dk \in G^2_{n,3}(hd)$.

Case 1. Assume that $h \equiv 1 \pmod 3$. Suppose, on the contrary, that $dk \in G^2_{n,3}(hd)$, that is, $dk = \alpha hd$ for some $\alpha \equiv 2 \pmod 3$ satisfying $\gcd(\alpha, \frac{3g}{h}) = 1$. Then we have $k = \alpha h \equiv 2 \pmod 3$, a contradiction. Hence $dk \in G^1_{n,3}(hd)$.

Case 2. Assume that $h \equiv 2 \pmod 3$. Proceeding as in Case 1, we get $dk \in G^2_{n,3}(hd)$.

Case 1 and Case 2 altogether implies $M^1_{n,3}(d) \subseteq \left(\bigcup_{h\in D^1_{g,3}} G^1_{n,3}(hd)\right) \cup \left(\bigcup_{h\in D^2_{g,3}} G^2_{n,3}(hd)\right)$. Conversely, if $\alpha hd \in G^1_{n,3}(hd)$ with $h \in D^1_{g,3}$ and $\alpha \equiv 1 \pmod 3$, we get $\alpha h \equiv 1 \pmod 3$, that is, $\alpha hd \in M^1_{n,3}(d)$. Similarly, $\beta hd \in G^2_{n,3}(hd)$ with $h \in D^2_{g,3}$ and $\beta \equiv 2 \pmod 3$, implies that $\beta hd \in M^1_{n,3}(d)$. Therefore $\left(\bigcup_{h\in D^1_{g,3}} G^1_{n,3}(hd)\right) \cup \left(\bigcup_{h\in D^2_{g,3}} G^2_{n,3}(hd)\right) \subseteq M^1_{n,3}(d)$. Thus the desired equality follows.

(iv) The proof of this part is similar to the proof of Part (iii). For the sake of completeness, we provide the proof. Let $dk \in M^2_{n,3}(d)$ so that $k \equiv 2 \pmod 3$. By Lemma 1.5.1, there exists $h \in D_{g,3}$ satisfying $dk \in G_n(hd)$. Thus $dk \in G^1_{n,3}(hd)$ or $dk \in G^2_{n,3}(hd)$.

Case 1. Assume that $h \equiv 1 \pmod 3$. Suppose, on the contrary, that $dk \in G^1_{n,3}(hd)$, that is, $dk = \alpha hd$ for some $\alpha \equiv 1 \pmod 3$ satisfying $\gcd(\alpha, \frac{3g}{h}) = 1$. Then we have $k = \alpha h \equiv 1 \pmod 3$, a contradiction. Hence $dk \in G^2_{n,3}(hd)$.

Case 2. Assume that $h \equiv 2 \pmod 3$. Proceeding as in Case 1, we get $dk \in G^1_{n,3}(hd)$.

Case 1 and Case 2 altogether implies $M^2_{n,3}(d) \subseteq \left(\bigcup_{h\in D^1_{g,3}} G^2_{n,3}(hd)\right) \cup \left(\bigcup_{h\in D^2_{g,3}} G^1_{n,3}(hd)\right)$. Conversely, if $\alpha hd \in G^2_{n,3}(hd)$ with $h \in D^1_{g,3}$ and $\alpha \equiv 2 \pmod 3$, we get $\alpha h \equiv 2 \pmod 3$, that is, $\alpha hd \in M^2_{n,3}(d)$. Similarly, $\beta hd \in G^1_{n,3}(hd)$ with $h \in D^2_{g,3}$ and $\beta \equiv 1 \pmod 3$, implies that $\beta hd \in M^2_{n,3}(d)$. Therefore $\left(\bigcup_{h\in D^1_{g,3}} G^2_{n,3}(hd)\right) \cup \left(\bigcup_{h\in D^2_{g,3}} G^1_{n,3}(hd)\right) \subseteq M^2_{n,3}(d)$. Thus the desired equality follows. □

Note that $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\omega_3)$ if and only if $[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \varphi(n)$. See Section 13.2 in [17] for more details. Also, $\mathbb{Q}(\omega_n)$ does not contain the number ω_3 if and only if $n \not\equiv 0 \pmod 3$. Thus, if $n \not\equiv 0 \pmod 3$ then $[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_n)] = 2 = [\mathbb{Q}(\omega_3), \mathbb{Q}]$, and therefore

$$[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \frac{[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_n)] \times [\mathbb{Q}(\omega_n) : \mathbb{Q}]}{[\mathbb{Q}(\omega_3) : \mathbb{Q}]} = [\mathbb{Q}(\omega_n) : \mathbb{Q}] = \varphi(n).$$

Further, if $n \equiv 0 \pmod 3$ then $\mathbb{Q}(\omega_3, \omega_n) = \mathbb{Q}(\omega_n)$, and so

$$[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \frac{[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}]}{[\mathbb{Q}(\omega_3) : \mathbb{Q}]} = \frac{\varphi(n)}{2}.$$

Note that $\mathbb{Q}(\omega_3) = \mathbb{Q}(\omega_6) = \mathbb{Q}(\mathbf{i}\sqrt{3})$. Therefore, $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\omega_3), \mathbb{Q}(\omega_6)$ or $\mathbb{Q}(\mathbf{i}\sqrt{3})$ if and only if $n \not\equiv 0 \pmod 3$.

Let $n \equiv 0 \pmod 3$. From Lemma 1.5.2, we know that $G_n(1)$ is a disjoint union of $G^1_{n,3}(1)$ and $G^2_{n,3}(1)$. Define

$$\Phi^1_{n,3}(x) := \prod_{a \in G^1_{n,3}(1)} (x - \omega_n^a) \quad \text{and} \quad \Phi^2_{n,3}(x) := \prod_{a \in G^2_{n,3}(1)} (x - \omega_n^a).$$

It is clear from the definition that $\Phi_n(x) = \Phi^1_{n,3}(x)\Phi^2_{n,3}(x)$.

Lemma 1.5.3. *If* $n \equiv 0 \pmod 3$, *then the following assertions hold.*

(i) $x^{\frac{n}{3}} - \omega_3 = \prod_{h \in D^1_{n,3}} \Phi^1_{\frac{n}{h},3}(x) \prod_{h \in D^2_{n,3}} \Phi^2_{\frac{n}{h},3}(x).$

(ii) $x^{\frac{n}{3}} - \omega_3^2 = \prod_{h \in D^1_{n,3}} \Phi^2_{\frac{n}{h},3}(x) \prod_{h \in D^2_{n,3}} \Phi^1_{\frac{n}{h},3}(x).$

Proof. (i) We have $|M^1_{n,3}(1)| = \frac{n}{3}$ and ω_n^a is a root of $x^{\frac{n}{3}} - \omega_3$ for each $a \in M^1_{n,3}(1)$. Also note that $D^1_{n,3} = D^1_{\frac{n}{3},3}$ and $D^2_{n,3} = D^2_{\frac{n}{3},3}$. Therefore

$$\begin{aligned}
x^{\frac{n}{3}} - \omega_3 &= \prod_{a \in M^1_{n,3}(1)} (x - \omega_n^a) \\
&= \prod_{h \in D^1_{\frac{n}{3},3}} \prod_{a \in G^1_{n,3}(h)} (x - \omega_n^a) \prod_{h \in D^2_{\frac{n}{3},3}} \prod_{a \in G^2_{n,3}(h)} (x - \omega_n^a), \qquad \text{using Lemma 1.5.2} \\
&= \prod_{h \in D^1_{n,3}} \prod_{a \in hG^1_{\frac{n}{h},3}(1)} (x - \omega_n^a) \prod_{h \in D^2_{n,3}} \prod_{a \in hG^2_{\frac{n}{h},3}(1)} (x - \omega_n^a) \\
&= \prod_{h \in D^1_{n,3}} \prod_{a \in G^1_{\frac{n}{h},3}(1)} (x - (\omega_n^h)^a) \prod_{h \in D^2_{n,3}} \prod_{a \in G^2_{\frac{n}{h},3}(1)} (x - (\omega_n^h)^a) \\
&= \prod_{h \in D^1_{n,3}} \Phi^1_{\frac{n}{h},3}(x) \prod_{h \in D^2_{n,3}} \Phi^2_{\frac{n}{h},3}(x).
\end{aligned}$$

In the last line of the preceding equalities, we use the fact that ω_n^h is a primitive $\frac{n}{h}$-th root of unity.

(ii) We have $|M^2_{n,3}(1)| = \frac{n}{3}$ and ω_n^a is a root of $x^{\frac{n}{3}} - \omega_3^2$ for each $a \in M^2_{n,3}(1)$. Therefore

$$\begin{aligned}
x^{\frac{n}{3}} - \omega_3^2 &= \prod_{a\in M^2_{n,3}(1)} (x-\omega_n^a) \\
&= \prod_{h\in D^1_{\frac{n}{3},3}} \prod_{a\in G^2_{n,3}(h)} (x-\omega_n^a) \prod_{h\in D^2_{\frac{n}{3},3}} \prod_{a\in G^1_{n,3}(h)} (x-\omega_n^a), \qquad \text{using Lemma 1.5.2} \\
&= \prod_{h\in D^1_{n,3}} \prod_{a\in hG^2_{\frac{n}{h},3}(1)} (x-\omega_n^a) \prod_{h\in D^2_{n,3}} \prod_{a\in hG^1_{\frac{n}{h},3}(1)} (x-\omega_n^a) \\
&= \prod_{h\in D^1_{n,3}} \prod_{a\in G^2_{\frac{n}{h},3}(1)} (x-(\omega_n^h)^a) \prod_{h\in D^2_{n,3}} \prod_{a\in G^1_{\frac{n}{h},3}(1)} (x-(\omega_n^h)^a) \\
&= \prod_{h\in D^1_{n,3}} \Phi^2_{\frac{n}{h},3}(x) \prod_{h\in D^2_{n,3}} \Phi^1_{\frac{n}{h},3}(x).
\end{aligned}$$

□

Corollary 1.5.4. *If $n \equiv 0 \pmod 3$, then $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ are monic polynomials in $\mathbb{Z}(\omega_3)[x]$ of degree $\frac{\varphi(n)}{2}$.*

Proof. By definition, $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ are monic polynomials. Since the sizes of $G^1_{n,3}(1)$ and $G^2_{n,3}(1)$ are equal to $\frac{\varphi(n)}{2}$, the polynomials $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ are of degree $\frac{\varphi(n)}{2}$. Now apply induction on n to show that $\Phi^1_{n,3}(x), \Phi^2_{n,3}(x) \in \mathbb{Z}(\omega_3)[x]$. For $n = 3$, the polynomials $\Phi^1_{3,3}(x) := x - \omega_3$ and $\Phi^2_{3,3}(x) := x - \omega_3^2$ are clearly in $\mathbb{Z}(\omega_3)[x]$. Assume that $\Phi^1_{k,3}(x)$ and $\Phi^2_{k,3}(x)$ are in $\mathbb{Z}(\omega_3)[x]$ for $k < n$ with $k \equiv 0 \pmod 3$. By Lemma 1.5.3, $\Phi^1_{n,3}(x) = \frac{x^{\frac{n}{3}} - \omega_3}{f(x)}$ and $\Phi^2_{n,3}(x) = \frac{x^{\frac{n}{3}} - \omega_3^2}{g(x)}$, where

$$f(x) = \prod_{\substack{h\in D^1_{n,3}\\ h\neq 1}} \Phi^1_{\frac{n}{h},3}(x) \prod_{h\in D^2_{n,3}} \Phi^2_{\frac{n}{h},3}(x) \text{ and } g(x) = \prod_{\substack{h\in D^1_{n,3}\\ h\neq 1}} \Phi^2_{\frac{n}{h},3}(x) \prod_{h\in D^2_{n,3}} \Phi^1_{\frac{n}{h},3}(x).$$

Clearly, $f(x)$ and $g(x)$ are monic polynomials. By induction hypothesis, $f(x), g(x) \in \mathbb{Z}(\omega_3)[x]$. Now it follows by "long division" that $\Phi^1_{n,3}(x) \in \mathbb{Z}(\omega_3)[x]$ and $\Phi^2_{n,3}(x) \in \mathbb{Z}(\omega_3)[x]$. Hence the proof is complete by induction. □

Corollary 1.5.5. *If $n \equiv 0 \pmod 3$, then*

$$\frac{2n}{3} = \sum_{d\in D_{n,3}} \varphi\left(\frac{n}{d}\right).$$

Proof. Using Lemma 1.5.3, we get $x^{\frac{2n}{3}} + x^{\frac{n}{3}} + 1 = \prod_{d\in D_{n,3}} \Phi_{\frac{n}{d}}(x)$. Now the proof follows by comparing the degree of both side of this equation. □

From Lemma 1.5.3, the next corollary follows easily.

Corollary 1.5.6. *If $n = 3^k$ with $k \geq 1$, then $\Phi^1_{n,3}(x) = x^{\frac{n}{3}} - \omega_3$ and $\Phi^2_{n,3}(x) = x^{\frac{n}{3}} - \omega_3^2$.*

Theorem 1.5.7. *If $n \equiv 0 \pmod 3$, then $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ are irreducible monic polynomials in $\mathbb{Q}(\omega_3)[x]$ of degree $\frac{\varphi(n)}{2}$.*

Proof. Let $n \equiv 0 \pmod 3$. In view of Corollary 1.5.4, we only need to show the irreducibility of $\Phi^1_{n,3}(x)$ and $\Phi^2_{n,3}(x)$ over $\mathbb{Q}(\omega_3)$. We have $[\mathbb{Q}(\omega_3, \omega_n) : \mathbb{Q}(\omega_3)] = \frac{\varphi(n)}{2}$. Then there is a unique irreducible monic polynomial $p(x) \in \mathbb{Q}(\omega_3)[x]$ of degree $\frac{\varphi(n)}{2}$ having ω_n as a root. Since ω_n is also a root of $\Phi^1_{n,3}(x)$, we have that $\Phi^1_{n,3}(x) = p(x)f(x)$ for some $f(x) \in \mathbb{Q}(\omega_3)[x]$. Also $\Phi^1_{n,3}(x)$ is a monic polynomial of degree $\varphi(n)/2$, and so $f(x) = 1$. Hence $\Phi^1_{n,3}(x) = p(x)$, that is, $\Phi^1_{n,3}(x)$ is irreducible. Similarly, $[\mathbb{Q}(\omega_3, \omega_n^a) : \mathbb{Q}(\omega_3)] = \frac{\varphi(n)}{2}$ for $a \in G^2_{n,3}(1)$. Now, proceeding as in the case of irreducibility of $\Phi^1_{n,3}(x)$, we find that $\Phi^2_{n,3}(x)$ is also irreducible. □

1.6 Integral undirected Cayley graphs

In this section, we discuss some known characterizations of integral undirected Cayley graphs.

Suppose Γ is a finite group. Let $\mathbb{B}(\Gamma)$ be the boolean algebra generated by the subgroups of Γ. That is, $\mathbb{B}(\Gamma)$ is the set whose elements are obtained by intersections, unions and complements of subgroups of Γ. For example, $\mathbb{B}(\mathbb{Z}_6)$ is generated by the subgroups $\{0\}$, $\{0, 3\}$, $\{0, 2, 4\}$, and $\mathbb{Z}_6$. Define an equivalence relation $\sim$ on Γ such that $x \sim y$ if and only if $y = x^k$ for some $k \in G_m(1)$, where $m = \text{ord}(x)$. See [3] for details. For $x \in \Gamma$, let $[x]$ denote the equivalence class of x with respect to the relation $\sim$. Note that minimal non-empty sets in a boolean algebra are called its *atoms*. For example, the atoms of $\mathbb{B}(\mathbb{Z}_6)$ are the sets $\{0\}$, $\{3\}$, $\{1, 5\}$, and $\{2, 4\}$.

We notice that $x \sim y$ if and only if for all subgroup S of Γ, either $x, y \in S$ or $x, y \notin S$. That is, for all subgroup S of Γ either $[x] \subseteq S$ or $[x] \subseteq S^c$ for each $x \in \Gamma$. Further, we see that

$$[x] = \langle x \rangle \cap \left(\bigcup_{r \notin G_m(1)} \langle x^r \rangle \right)^c,$$

where $m = \text{ord}(x)$. Thus $[x] \in \mathbb{B}(\Gamma)$. The following theorem shows that the atoms of $\mathbb{B}(\Gamma)$ are precisely the equivalence classes $[x]$.

Theorem 1.6.1 ([3])**.** *The atoms of the boolean algebra $\mathbb{B}(\Gamma)$ are the sets $[x]$ for each $x \in \Gamma$.*

Proof. Assume that A is an atom of $\mathbb{B}(\Gamma)$ and $x, y \in A$. Since A is an atom, $A \subseteq \langle x \rangle$ and $A \subseteq \langle y \rangle$, and hence $A \subseteq \langle x \rangle \cap \langle y \rangle$. This gives $y = x^r$, where $1 \leq r \leq m$ and $m = \text{ord}(x)$. Now $x \in A \subseteq \langle x \rangle \cap \langle x^r \rangle$, and so $x \in \langle x^r \rangle$, which gives $r \in G_m(1)$. Thus $x \sim y$, and hence $A = [x]$. Conversely, assume that $x \in \Gamma$. Using the fact that either $[x] \subseteq S$ or $[x] \subseteq S^c$ for all subgroup S of Γ, we find that either $[x] \subseteq S$ or $[x] \subseteq S^c$ for all $S \in \mathbb{B}(\Gamma)$. Hence, $[x]$ is an atom. □

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By Theorem 1.6.1, we observe that each element of $\mathbb{B}(\Gamma)$ can be expressed as a disjoint union of the equivalence classes of the relation $\sim$ on Γ. Thus

$$\mathbb{B}(\Gamma) = \{[x_1] \cup \cdots \cup [x_k] \colon x_1, \ldots, x_k \in \Gamma, k \in \mathbb{N}\}.$$

In 2006, Wasin So [39] characterized integral circulant graphs.

Theorem 1.6.2 ([39])**.** *An undirected circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is integral if and only if*

$$S = \bigcup_{d \in \mathscr{D}} G_n(d),$$

where $\mathscr{D} \subseteq \{d \colon d \mid n\}$*.*

The following result by Klotz and Sander [25] provides a sufficient condition for the integrality of undirected Cayley graphs over abelian groups.

Theorem 1.6.3 ([25])**.** *If* Γ *is an abelian group and* $S \in \mathbb{B}(\Gamma)$*, then the undirected Cayley graph* $\text{Cay}(\Gamma, S)$ *is integral.*

For the particular case when Γ is a cyclic group, Klotz and Sander used Theorem 1.6.2 to show that the sufficient condition in Theorem 1.6.3 is also necessary.

Theorem 1.6.4 ([25])**.** *An undirected circulant graph* $\text{Circ}(\mathbb{Z}_n, S)$ *is integral if and only if* $S \in \mathbb{B}(\mathbb{Z}_n)$*.*

However, Theorem 1.6.2 and Theorem 1.6.4 are equivalent to each other due to the fact that $G_n(d)$ is an atom of $\mathbb{B}(\mathbb{Z}_n)$ if $d < n$ and $d \mid n$. In the same paper [25], Klotz and Sander further conjectured that Theorem 1.6.4 would be true for any finite abelian group. One year later, they proved their own conjecture for a particular class of abelian groups.

Theorem 1.6.5 ([26])**.** *Let* $\Gamma = \mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k}$ *be an abelian group. If* $\gcd(n_i, n_j) \leq 2$ *for each* $i \neq j$*, then an undirected Cayley graph* $\text{Cay}(\Gamma, S)$ *is integral if and only if* $S \in \mathbb{B}(\Gamma)$*.*

The conjecture of Klotz and Sander [25] was finally confirmed by Alperin and Peterson in [3]. However, the same result was proved by Bridges and Mena [9] in 1982 using some terminologies that are different from [3].

Theorem 1.6.6 ([3],[9])**.** *If* Γ *is an abelian group, then an undirected Cayley graph* $\text{Cay}(\Gamma, S)$ *is integral if and only if* $S \in \mathbb{B}(\Gamma)$*.*

Godsil and Spiga [19] improved Theorem 1.6.6 further, as stated in the next theorem. Additionally, the "if" portion of this result was also proved in [20] and [27].

Theorem 1.6.7 ([19])**.** *Let* Γ *be a finite group and* $\text{Cay}(\Gamma, S)$ *be a normal undirected Cayley graph. Then* $\text{Cay}(\Gamma, S)$ *is integral if and only if* $S \in \mathbb{B}(\Gamma)$*.*

1.7 Two important equivalence relations

In this section, we present two equivalence relations on specific subsets of a finite group. The equivalence classes of these two relations play major roles in the characterization of H-integral and HS-integral mixed Cayley graphs that are discussed in the subsequent chapters.

Let Γ a finite group. Let $\Gamma(4)$ be the set of all $x \in \Gamma$ satisfying $\text{ord}(x) \equiv 0 \pmod 4$. That is, $\Gamma(4) := \{x \in \Gamma \colon \text{ord}(x) \equiv 0 \pmod 4\}$. Define an equivalence relation $\approx$ on $\Gamma(4)$ such that $x \approx y$ if and only if $y = x^k$ for some $k \in G_m^1(1)$, where $m = \text{ord}(x)$. Observe that if $x, y \in \Gamma(4)$ and $x \approx y$ then $x \sim y$, but the converse need not be true. For example, consider $x = 5 \pmod{12}$, $y = 11 \pmod{12}$ in $\mathbb{Z}_{12}$. Here $x, y \in \mathbb{Z}_{12}(4)$ and $x \sim y$, but $x \not\approx y$. For $x \in \Gamma(4)$, we denote the equivalence class of x with respect to the relation $\approx$ by $[\![x]\!]$. For $\Gamma(4) \neq \emptyset$, define $\mathbb{D}(\Gamma)$ to be the class of all skew-symmetric subsets S, where $S = [\![x_1]\!] \cup \cdots \cup [\![x_k]\!]$ for some $x_1, \ldots, x_k \in \Gamma(4)$. For $\Gamma(4) = \emptyset$, define $\mathbb{D}(\Gamma) := \{\emptyset\}$. Thus

$$\mathbb{D}(\Gamma) = \begin{cases} \{[\![x_1]\!] \cup \cdots \cup [\![x_k]\!] \colon x_1, \ldots, x_k \in \Gamma(4), k \in \mathbb{N}, \\ \qquad\qquad \text{and } x_i^{-1} \not\in [\![x_j]\!] \text{ for all } 1 \leq i, j \leq k\} & \text{if } \Gamma(4) \neq \emptyset \\ \{\emptyset\} & \text{if } \Gamma(4) = \emptyset. \end{cases}$$

Similarly, let $\Gamma(3) := \{x \in \Gamma \colon \text{ord}(x) \equiv 0 \pmod 3\}$. Define an equivalence relation $\simeq$ on $\Gamma(3)$ such that $x \simeq y$ if and only if $y = x^k$ for some $k \in G_{m,3}^1(1)$, where $m = \text{ord}(x)$. Observe that if $x, y \in \Gamma(3)$ and $x \simeq y$ then $x \sim y$, but the converse need not be true. For example, consider $x = 5 \pmod{12}$, $y = 7 \pmod{12}$ in $\mathbb{Z}_{12}$. Here $x, y \in \mathbb{Z}_{12}(3)$ and $x \sim y$, but $x \not\simeq y$. For $x \in \Gamma(3)$, we denote the equivalence class of x with respect to the relation $\simeq$ by $\langle\!\langle x \rangle\!\rangle$. For $\Gamma(3) \neq \emptyset$, define $\mathbb{E}(\Gamma)$ to be the set of all skew-symmetric subsets S, where $S = \langle\!\langle x_1 \rangle\!\rangle \cup \cdots \cup \langle\!\langle x_k \rangle\!\rangle$ for some $x_1, \ldots, x_k \in \Gamma(3)$. For $\Gamma(3) = \emptyset$, define $\mathbb{E}(\Gamma) := \{\emptyset\}$. Thus

$$\mathbb{E}(\Gamma) = \begin{cases} \{\langle\!\langle x_1 \rangle\!\rangle \cup \cdots \cup \langle\!\langle x_k \rangle\!\rangle \colon x_1, \ldots, x_k \in \Gamma(3), k \in \mathbb{N}, \\ \qquad\qquad \text{and } x_i^{-1} \not\in \langle\!\langle x_j \rangle\!\rangle \text{ for all } 1 \leq i, j \leq k\} & \text{if } \Gamma(3) \neq \emptyset \\ \{\emptyset\} & \text{if } \Gamma(3) = \emptyset. \end{cases}$$

1.8 Overview of the thesis

After presenting the basic fundamentals in Chapter 1, the main work of the thesis is divided into the following six chapters:

Chapter 2: H-integral mixed Cayley graphs over abelian groups. The results of this chapter are published in *The Electronic Journal of Combinatorics*, 28(4):P4.46, 2021.

Chapter 3: HS-integral mixed Cayley graphs over abelian groups. The results of this chapter are published in *Linear Algebra and its Applications*, 645:68–90, 2022.

Chapter 4: H-integral and Gaussian integral normal mixed Cayley graphs. The results of this chapter are available in arXiv:2110.03268.

Chapter 5: HS-integral and Eisenstein integral normal mixed Cayley graphs. The results of this chapter are available in *Linear Algebra and its Applications*, 669:1–23, 2023.

Chapter 6: Ramanujan type sum. The results of this chapter are published in *Discrete Mathematics*, 346(1):113142, 2022 and *Theory and Applications of Graphs*, 10(1):3, 2023.

Chapter 7: Future work.

In Chapters 2 and 3, we give characterizations of H-integrality and HS-integrality of mixed Cayley graphs over abelian groups, respectively. We extend the main results of Chapters 2 and 3 to normal mixed Cayley graphs over finite groups in Chapters 4 and 5, respectively. In addition, we characterize Gaussian integral normal mixed Cayley graphs in Chapter 4. Similarly, we characterize Eisenstein integral normal mixed Cayley graphs in Chapter 5. In Chapter 6, we introduce two sums that are equal to an integer multiple of the Ramanujan sum. These sums play an important role in the expression of the eigenvalues of H-integral and HS-integral mixed circulant graphs. We express these sums in terms of the generalized Möbius function. In the last chapter, we outline some possible future direction of research.

We refer the reader to [17, 40] for all terminologies and results that are used in the thesis but not defined or presented explicitly.

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2

H-integral mixed Cayley graphs over abelian groups

In 1982, Bridges and Mena [9] characterized integral undirected Cayley graphs over abelian groups (Theorem 1.6.6). In 2012, the same result was proved by Alperin and Peterson [3] using some terminologies that are different from [9]. However, Bridges and Mena proved the result in a broader sense. For $s \in \Gamma$, let P_s be the permutation matrix whose rows and columns are indexed by elements of Γ such that the (x, xs)-th entry of P_s is 1 for each $x \in \Gamma$. For $S \subseteq \Gamma$, let $P_S = \sum_{s \in S} P_s$. Bridges and Mena showed that for an abelian group Γ, a complex linear combination of the matrices in $\{P_s \colon s \in \Gamma\}$ is a rational matrix with rational eigenvalues if and only if it is a rational linear combination of the matrices in $\{P_Q \colon Q \text{ is an atom of } \mathbb{B}(\Gamma)\}$.

In this chapter, we use Theorem 1.6.6 to characterize H-integral mixed Cayley graphs over abelian groups. In the first section, we present some notations for an abelian group and express the H-eigenvalues of a mixed Cayley graph over an abelian group in terms of the irreducible characters of the group. We further show that a mixed Cayley graph over an abelian group is H-integral if and only if both of its undirected and directed portions are H-integral. In the second section, we find a sufficient condition for which a mixed Cayley graph over an abelian group is H-integral. In the last section, we show that the sufficient condition obtained in Section 2 is also necessary.

2.1 Preliminaries

Let Γ be an abelian group of order n. Then Γ is isomorphic to the direct product of cyclic groups of prime power order, that is

$$\Gamma \cong \mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k},$$

where $n = n_1 \cdots n_k$, and n_j is a power of a prime number for each $j \in \{1, \ldots, k\}$. If Γ is an abelian group, then throughout the thesis, we use the following terminologies:

- We consider Γ as $\mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k}$ of order n, where $n = n_1 \ldots n_k$.

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- We consider the elements $x \in \Gamma$ as elements of the cartesian product $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, that is
$$x = (x_1, \ldots, x_k), \text{ where } x_j \in \mathbb{Z}_{n_j} \text{ for each } j \in \{1, \ldots, k\}.$$
- Addition in Γ is done coordinate-wise modulo n_j for each j.
- For a positive integer k and $a \in \Gamma$, we denote by ka or a^k the k-fold sum of a to itself, $(-k)a = k(-a)$, $0a = 0$, and inverse of a by $-a$.
- Since Γ is an abelian group, every irreducible representation of Γ is 1-dimensional, and thus it can be identified with its characters. Hence $\mathrm{IRR}(\Gamma) = \mathrm{Irr}(\Gamma)$.
- By Lemma 1.3.1 and Lemma 1.3.2, we have $\mathrm{Irr}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$, where
$$\psi_\alpha(x) = \prod_{j=1}^{k} \omega_{n_j}^{\alpha_j x_j} \text{ for each } \alpha = (\alpha_1, \ldots, \alpha_k), x = (x_1, \ldots, x_k) \in \Gamma. \tag{2.1}$$

The following lemma can be easily proved.

Lemma 2.1.1. *If Γ is an abelian group, then the following assertions hold.*

(i) $\psi_\alpha(x) = \psi_x(\alpha)$ *for all* $x, \alpha \in \Gamma$.

(ii) $(\psi_\alpha(x))^{\mathrm{ord}(x)} = (\psi_\alpha(x))^{\mathrm{ord}(\alpha)} = 1$ *for all* $x, \alpha \in \Gamma$.

Lemma 2.1.2 ([4, 18])**.** *If Γ is an abelian group, then the H-spectrum of the mixed Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *is* $\{\gamma_\alpha : \alpha \in \Gamma\}$*, where* $\gamma_\alpha = \lambda_\alpha + \mu_\alpha$*,*
$$\lambda_\alpha = \sum_{s \in S \setminus \overline{S}} \psi_\alpha(s) \text{ and } \mu_\alpha = \mathbf{i} \sum_{s \in \overline{S}} (\psi_\alpha(s) - \psi_\alpha(-s)) \text{ for each } \alpha \in \Gamma.$$

Proof. Define $f : \Gamma \to \{0, 1, \mathbf{i}, -\mathbf{i}\}$ such that
$$f(s) = \begin{cases} 1 & \text{if } s \in S \setminus \overline{S} \\ \mathbf{i} & \text{if } s \in \overline{S} \\ -\mathbf{i} & \text{if } s \in (\overline{S})^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Since Γ is an abelian group, the conjugacy classes of Γ are singleton sets. Therefore, f is a class function. The Hermitian adjacency matrix of the mixed Cayley graph $\mathrm{Cay}(\Gamma, S)$ is equal to $[f(y - x)]_{x,y \in \Gamma}$. Thus the result follows from Theorem 1.3.6. □

Next two corollaries are special cases of Lemma 2.1.2.

Corollary 2.1.3 ([25]). *If Γ is an abelian group, then the H-spectrum (or spectrum) of the undirected Cayley graph* $\text{Cay}(\Gamma, S)$ *is* $\{\lambda_\alpha : \alpha \in \Gamma\}$, *where* $\lambda_\alpha = \lambda_{-\alpha}$ *and*

$$\lambda_\alpha = \sum_{s \in S} \psi_\alpha(s) \text{ for each } \alpha \in \Gamma.$$

Proof. Note that $\overline{S} = \emptyset$, and so $s \in S$ if and only if $-s \in S$. Using Lemma 2.1.2, we have

$$\lambda_\alpha = \sum_{s \in S} \psi_\alpha(s) = \sum_{s \in S} \psi_{-\alpha}(-s) = \sum_{s \in S} \psi_{-\alpha}(s) = \lambda_{-\alpha}.$$ □

Corollary 2.1.4. *If Γ is an abelian group, then the H-spectrum of the directed Cayley graph* $\text{Cay}(\Gamma, S)$ *is* $\{\mu_\alpha : \alpha \in \Gamma\}$, *where* $\mu_\alpha = -\mu_{-\alpha}$ *and*

$$\mu_\alpha = \mathbf{i} \sum_{s \in S} (\psi_\alpha(s) - \psi_\alpha(-s)) \text{ for each } \alpha \in \Gamma.$$

Proof. Note that $S \setminus \overline{S} = \emptyset$. Using Lemma 2.1.2, we have

$$\mu_\alpha = \mathbf{i} \sum_{s \in S} (\psi_\alpha(s) - \psi_\alpha(-s)) = \mathbf{i} \sum_{s \in S} (\psi_{-\alpha}(-s) - \psi_{-\alpha}(s)) = -\mu_{-\alpha}.$$ □

By Part (ii) of Lemma 2.1.1, we have that $\psi_\alpha(x)$ is an algebraic integer for each $\alpha, x \in \Gamma$. Further, it is well known that the sum, difference, and product of algebraic integers are also algebraic integers. Hence γ_α, λ_α and μ_α are algebraic integers for each $\alpha \in \Gamma$. It is also well known that if a rational number is an algebraic integer, then it is an integer.

Theorem 2.1.5. *If Γ is an abelian group, then the mixed Cayley graph* $\text{Cay}(\Gamma, S)$ *is H-integral if and only if each of the undirected Cayley graph* $\text{Cay}(\Gamma, S \setminus \overline{S})$ *and the directed Cayley graph* $\text{Cay}(\Gamma, \overline{S})$ *are H-integral.*

Proof. Assume that the mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral. Let γ_α be an H-eigenvalue of the mixed Cayley graph $\text{Cay}(\Gamma, S)$. By Lemma 2.1.2, Corollary 2.1.3 and Corollary 2.1.4, we have $\gamma_\alpha = \lambda_\alpha + \mu_\alpha$ and $\gamma_{-\alpha} = \lambda_\alpha - \mu_\alpha$ for each $\alpha \in \Gamma$, where λ_α is an H-eigenvalue of the undirected Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$ and μ_α is an H-eigenvalue of the directed Cayley graph $\text{Cay}(\Gamma, \overline{S})$. Thus $\lambda_\alpha = \frac{\gamma_\alpha + \gamma_{-\alpha}}{2} \in \mathbb{Q}$ and $\mu_\alpha = \frac{\gamma_\alpha - \gamma_{-\alpha}}{2} \in \mathbb{Q}$. As λ_α and μ_α are algebraic integers, so $\lambda_\alpha, \mu_\alpha \in \mathbb{Q}$ implies that λ_α and μ_α are integers. Thus the undirected Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$ and the directed Cayley graph $\text{Cay}(\Gamma, \overline{S})$ are H-integral.

Conversely, suppose each of $\text{Cay}(\Gamma, S \setminus \overline{S})$ and $\text{Cay}(\Gamma, \overline{S})$ are H-integral. Then Lemma 2.1.2 implies that $\text{Cay}(\Gamma, S)$ is H-integral. □

2.2 A sufficient condition for H-integrality of mixed Cayley graphs over abelian groups

Due to Theorem 2.1.5, to find a characterization of the H-integral mixed Cayley graph $\text{Cay}(\Gamma, S)$, it is enough to find individual characterizations of the H-integral undirected Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$ and the H-integral directed Cayley graph $\text{Cay}(\Gamma, \overline{S})$. The H-integral undirected Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$ is characterized in Theorem 1.6.6. So our attempt is to characterize the H-integral directed Cayley graph $\text{Cay}(\Gamma, \overline{S})$.

Recall that $\Gamma(4)$ is the set of all $x \in \Gamma$ satisfying $\text{ord}(x) \equiv 0 \pmod 4$. It is clear that $\exp(\Gamma) \equiv 0 \pmod 4$ if and only if $\Gamma(4) \neq \emptyset$. For $x \in \Gamma(4)$ and $r \in \{0, 1, 2, 3\}$, define

$$M_r(x) := \{x^k \colon 1 \leq k \leq \text{ord}(x), k \equiv r \pmod 4\}.$$

For $a \in \Gamma$ and $S \subseteq \Gamma$, define $a + S := \{a + s \colon s \in S\}$ and $-S := \{-s \colon s \in S\}$. Also, $\langle a \rangle$ denotes the cyclic subgroup of Γ generated by a. Note that $-s$ denotes the inverse of s, that is $-s = s^{m-1}$, where $m = \text{ord}(s)$.

Lemma 2.2.1. *If Γ is an abelian group and $x \in \Gamma(4)$, then the following assertions hold.*

(i) $\bigcup\limits_{r=0}^{3} M_r(x) = \langle x \rangle$.

(ii) The sets $M_1(x)$ and $M_3(x)$ are skew-symmetric subsets of Γ.

(iii) $-M_1(x) = M_3(x)$ *and* $-M_3(x) = M_1(x)$.

(iv) $a + M_1(x) = M_3(x)$ *and* $a + M_3(x) = M_1(x)$ *for all* $a \in M_2(x)$.

(v) $a + M_1(x) = M_1(x)$ *and* $a + M_3(x) = M_3(x)$ *for all* $a \in M_0(x)$.

Proof. (i) The proof follows from the definitions of $M_r(x)$ and $\langle x \rangle$.

(ii) If $x^k \in M_1(x)$ then $-x^k = x^{n-k} \notin M_1(x)$, as $k \equiv 1 \pmod 4$ implies $n - k \equiv 3 \pmod 4$. Thus $M_1(x)$ is a skew-symmetric subset of Γ. Similarly, $M_3(x)$ is also a skew-symmetric subset of Γ.

(iii) Note that $k \equiv 1 \pmod 4$ if and only if $n - k \equiv 3 \pmod 4$, and $-x^k = x^{n-k}$. Using these facts, we get $-M_1(x) = M_3(x)$ and $-M_3(x) = M_1(x)$.

(iv) Let $a \in M_2(x)$ and $y \in a + M_1(x)$. Then $a = x^{k_1}$ and $y = x^{k_1} + x^{k_2} = x^{k_1 + k_2}$, where $k_1 \equiv 2 \pmod 4$ and $k_2 \equiv 1 \pmod 4$. Since $k_1 + k_2 \equiv 3 \pmod 4$, we have $y \in M_3(x)$ implying that $a + M_1(x) \subseteq M_3(x)$. Since sizes of the sets $M_1(x)$ and $M_3(x)$ are equal, we have $a + M_1(x) = M_3(x)$. Similarly, $a + M_3(x) = M_1(x)$ for all $a \in M_2(x)$.

(v) The proof is similar to the proof of Part (iv). □

Lemma 2.2.2. *If Γ is an abelian group and $x \in \Gamma(4)$, then*

$$\mathbf{i}\Big(\sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s)\Big) \in \mathbb{Z}$$

for each $\alpha \in \Gamma$.

Proof. Let $x \in \Gamma(4)$, $\alpha \in \Gamma$ and

$$\mu_\alpha = \mathbf{i}\Big(\sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s) \Big).$$

Case 1. Assume that there exists $a \in M_2(x)$ such that $\psi_\alpha(a) \neq -1$. We have

$$\begin{aligned}
\mu_\alpha &= -\mathbf{i}\Big(\sum_{s \in M_3(x)} \psi_\alpha(s) - \sum_{s \in M_1(x)} \psi_\alpha(s) \Big) \\
&= -\mathbf{i}\Big(\sum_{s \in a+M_1(x)} \psi_\alpha(s) - \sum_{s \in a+M_3(x)} \psi_\alpha(s) \Big) \\
&= -\mathbf{i}\Big(\sum_{s \in M_1(x)} \psi_\alpha(a+s) - \sum_{s \in M_3(x)} \psi_\alpha(a+s) \Big) \\
&= -\mathbf{i}\psi_\alpha(a)\Big(\sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s) \Big) \\
&= -\psi_\alpha(a)\mu_\alpha.
\end{aligned}$$

Thus $(1 + \psi_\alpha(a))\mu_\alpha = 0$. Since $\psi_\alpha(a) \neq -1$, we have $\mu_\alpha = 0 \in \mathbb{Z}$.

Case 2. Assume that there exists $a \in M_0(x)$ such that $\psi_\alpha(a) \neq 1$. Proceeding as in Case 1, we get $\mu_\alpha = 0 \in \mathbb{Z}$.

Case 3. Assume that $\psi_\alpha(a) = -1$ for all $a \in M_2(x)$ and $\psi_\alpha(a) = 1$ for all $a \in M_0(x)$. Let $s \in M_3(x)$, so that $s = x^{4r+3}$ for some $r \in \mathbb{Z}$. We have $x^{4r} \in M_0(x)$ and $x^2 \in M_2(x)$, and so $\psi_\alpha(x^{4r}) = 1$ and $\psi_\alpha(x^2) = -1$. Therefore $\psi_\alpha(s) = \psi_\alpha(x^{4r})\psi_\alpha(x^2)\psi_\alpha(x) = -\psi_\alpha(x)$. Similarly, if $s \in M_1(x)$ then $\psi_\alpha(s) = \psi_\alpha(x)$. Therefore

$$\mu_\alpha = \mathbf{i}\Big(\sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s) \Big) = 2\mathbf{i}\psi_\alpha(x)|M_1(x)|.$$

Note that $x^4 \in M_0(x)$ and μ_α is an eigenvalue of a Hermitian matrix. Hence $\psi_\alpha(x)^4 = 1$ and μ_α is real, and so $\psi_\alpha(x) = \pm\mathbf{i}$. Thus $\mu_\alpha = \pm 2|M_1(x)| \in \mathbb{Z}$. □

For $m \equiv 0 \pmod 4$ and $r \in \{1, 3\}$, recall that

$$G_m^r(1) = \{k \colon k \equiv r \pmod 4, \gcd(k, m) = 1\}.$$

Also, recall the equivalence relation $\approx$ on $\Gamma(4)$, for which $x \approx y$ if and only if $y = x^k$ for some $k \in G_m^1(1)$, where $m = \operatorname{ord}(x)$. For $x \in \Gamma(4)$, recall that $[\![x]\!]$ is the equivalence class of x with respect to the relation $\approx$.

Lemma 2.2.3. *If Γ is an abelian group, $x \in \Gamma(4)$ and $m = \mathrm{ord}(x)$, then the following assertions hold.*

(i) $[\![x]\!] = \{x^k : k \in G_m^1(1)\}$.

(ii) $[\![-x]\!] = \{x^k : k \in G_m^3(1)\}$.

(iii) $[\![x]\!] \cap [\![-x]\!] = \emptyset$.

(iv) $[x] = [\![x]\!] \cup [\![-x]\!]$.

Proof. (i) Let $y \in [\![x]\!]$. Then $y \approx x$, and so $\mathrm{ord}(y) = \mathrm{ord}(x) = m$ and there exists $k \in G_m^1(x)$ such that $y = x^k$. Thus $[\![x]\!] \subseteq \{x^k : k \in G_m^1(1)\}$. On the other hand, let $z = x^k$ for some $k \in G_m^1(1)$. Then $\mathrm{ord}(z) = \mathrm{ord}(x)$, and so $x \approx z$. Thus $\{x^k : k \in G_m^1(1)\} \subseteq [\![x]\!]$. Hence the desired equality follows.

(ii) Note that $-x = x^{m-1}$ and $m - 1 \equiv 3 \pmod 4$. By Part (i), we have

$$\begin{aligned}[\![-x]\!] = \{(-x)^k : k \in G_m^1(1)\} &= \{x^{(m-1)k} : k \in G_m^1(1)\} \\ &= \{x^{-k} : k \in G_m^1(1)\} \\ &= \{x^k : k \in G_m^3(1)\}.\end{aligned}$$

(iii) Since $G_m^1(1) \cap G_m^3(1) = \emptyset$, we have by Part (i) and Part (ii) that $[\![x]\!] \cap [\![-x]\!] = \emptyset$.

(iv) Since $[x] = \{x^k : k \in G_m(1)\}$ and $G_m(1)$ is a disjoint union of $G_m^1(1)$ and $G_m^3(1)$, we have by Part (i) and Part (ii) that $[x] = [\![x]\!] \cup [\![-x]\!]$. □

Note that Lemma 2.2.3 also holds for non-abelian group. For $g \in \mathbb{Z}$ and $r \in \{1, 3\}$, recall the sets

$$\begin{aligned}D_g &:= \{k : k \text{ is an odd divisor of } g\} \quad \text{and} \\ D_g^r &:= \{k : k \text{ divides } g, k \equiv r \pmod 4\}.\end{aligned}$$

Lemma 2.2.4. *If Γ is an abelian group, $x \in \Gamma(4)$, $m = \mathrm{ord}(x)$ and $g = \frac{m}{4}$, then the following assertions hold.*

(i) $M_1(x) \cup M_3(x) = \bigcup_{h \in D_g} [x^h]$.

(ii) $M_1(x) = \left(\bigcup_{h \in D_g^1} [\![x^h]\!] \right) \cup \left(\bigcup_{h \in D_g^3} [\![-x^h]\!] \right)$.

(iii) $M_3(x) = \left(\bigcup_{h \in D_g^1} [\![-x^h]\!] \right) \cup \left(\bigcup_{h \in D_g^3} [\![x^h]\!] \right)$.

Proof. (i) Let $x^k \in M_1(x) \cup M_3(x)$, where $k \equiv 1$ or $3 \pmod 4$. To show $x^k \in \bigcup\limits_{h\in D_g} [x^h]$, it is enough to show that $x^k \sim x^h$ for some $h \in D_g$. Let $h = \gcd(k,g) \in D_g$. Note that

$$\mathrm{ord}(x^k) = \frac{m}{\gcd(m,k)} = \frac{m}{\gcd(g,k)} = \frac{m}{h} = \mathrm{ord}(x^h).$$

Also, as $h = \gcd(k,m)$, we have $\langle x^k\rangle = \langle x^h\rangle$. Therefore $x^k = x^{hj}$ for some $j \in G_q(1)$, where $q = \mathrm{ord}(x^h) = \frac{m}{h}$. Thus $x^k \sim x^h$, where $h = \gcd(k,g) \in D_g$. Therefore, we get $x^k \in \bigcup\limits_{h\in D_g} [x^h]$. Conversely, let $z \in \bigcup\limits_{h\in D_g} [x^h]$. Then there exists $h \in D_g$ such that $z = x^{hj}$ where $j \in G_q(1)$ and $q = \frac{m}{\gcd(m,h)}$. Now $h \in D_g$ and $q \equiv 0 \pmod 4$ altogether implies that both h and j are odd integers. Thus $hj \equiv 1$ or $3 \pmod 4$, and so $z \in M_1(x)\cup M_3(x)$. Hence $M_1(x) \cup M_3(x) = \bigcup\limits_{h\in D_g} [x^h]$.

(ii) Let $x^k \in M_1(x)$, where $k \equiv 1 \pmod 4$. By Part (i), there exists $h \in D_g$ and $j \in G_q(1)$ such that $x^k = x^{hj}$, where $q = \frac{m}{\gcd(m,h)}$. Note that $k \equiv jh \pmod m$. If $h \equiv 1 \pmod 4$ then $j \in G_q^1(1)$, else $j \in G_q^3(1)$. Thus using Part (i) and Part (ii) of Lemma 2.2.3, if $h \equiv 1 \pmod 4$ then $x^k \approx x^h$, else $x^k \approx -x^h$. Hence

$$M_1(x) \subseteq \Big(\bigcup_{h\in D_g^1} [\![x^h]\!]\Big) \cup \Big(\bigcup_{h\in D_g^3} [\![-x^h]\!]\Big).$$

Conversely, assume that $z \in \Big(\bigcup\limits_{h\in D_g^1} [\![x^h]\!]\Big) \cup \Big(\bigcup\limits_{h\in D_g^3} [\![-x^h]\!]\Big)$. This gives $z \in [\![x^h]\!]$ for an $h \in D_g^1$ or $z \in [\![-x^h]\!]$ for an $h \in D_g^3$. In the first case, by Part (i) of Lemma 2.2.3, there exists $j \in G_q^1(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. Similarly, for the second case, by Part (ii) of Lemma 2.2.3, there exists $j \in G_q^3(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. In both the cases, $hj \equiv 1 \pmod 4$. Thus $z \in M_1(x)$, and so

$$\Big(\bigcup_{h\in D_g^1} [\![x^h]\!]\Big) \cup \Big(\bigcup_{h\in D_g^3} [\![-x^h]\!]\Big) \subseteq M_1(x).$$

Hence the desired equality follows.

(iii) The proof of this part is similar to the proof of Part (ii). For the sake of completeness, we provide the proof. Let $x^k \in M_3(x)$, where $k \equiv 3 \pmod 4$. By Part (i), there exists $h \in D_g$ and $j \in G_q(1)$ such that $x^k = x^{hj}$, where $q = \frac{m}{\gcd(m,h)}$. Note that $k \equiv jh \pmod m$. If $h \equiv 1 \pmod 4$ then $j \in G_q^3(1)$, else $j \in G_q^1(1)$. Thus using Part (i) and Part (ii) of Lemma 2.2.3, if $h \equiv 1 \pmod 4$ then $x^k \approx -x^h$, else $x^k \approx x^h$. Hence

$$M_3(x) \subseteq \Big(\bigcup_{h\in D_g^1} [\![-x^h]\!]\Big) \cup \Big(\bigcup_{h\in D_g^3} [\![x^h]\!]\Big).$$

Conversely, assume that $z \in \Big(\bigcup\limits_{h\in D_g^1} [\![-x^h]\!]\Big) \cup \Big(\bigcup\limits_{h\in D_g^3} [\![x^h]\!]\Big)$. This gives $z \in [\![-x^h]\!]$ for an $h \in D_g^1$ or $z \in [\![x^h]\!]$ for an $h \in D_g^3$. In the first case, by Part (ii) of Lemma 2.2.3, there exists $j \in G_q^3(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. Similarly, for the second case, by Part (i) of Lemma 2.2.3, there exists $j \in G_q^1(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. In both the cases, $hj \equiv 3 \pmod 4$. Thus $z \in M_3(x)$, and so

$$\Big(\bigcup_{h\in D_g^1} [\![-x^h]\!]\Big) \cup \Big(\bigcup_{h\in D_g^3} [\![x^h]\!]\Big) \subseteq M_3(x).$$

Hence the desired equality follows. □

Lemma 2.2.5. *If Γ is an abelian group and $x \in \Gamma(4)$, then $\mathbf{i}\Big(\sum\limits_{s\in[\![x]\!]} \psi_\alpha(s) - \sum\limits_{s\in[\![-x]\!]} \psi_\alpha(s)\Big) \in \mathbb{Z}$ for all $\alpha \in \Gamma$.*

Proof. Note that there exists $x \in \Gamma(4)$ with $\mathrm{ord}(x) = 4$. Apply induction on $\mathrm{ord}(x)$. If $\mathrm{ord}(x) = 4$, then $M_1(x) = [\![x]\!]$ and $M_3(x) = [\![-x]\!]$. Hence by Lemma 2.2.2,

$$\mathbf{i}\Big(\sum_{s\in[\![x]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x]\!]} \psi_\alpha(s)\Big) \in \mathbb{Z} \quad \text{for all } \alpha \in \Gamma.$$

Assume that the statement holds for all $x \in \Gamma(4)$ with $\mathrm{ord}(x) \in \{4, \ldots, 4(g-1)\}$. We prove the statement for $\mathrm{ord}(x) = 4g$. Lemma 2.2.4 gives that

$$M_1(x) = \Big(\bigcup_{h\in D_g^1} [\![x^h]\!]\Big) \cup \Big(\bigcup_{h\in D_g^3} [\![-x^h]\!]\Big) \quad \text{and} \quad M_3(x) = \Big(\bigcup_{h\in D_g^1} [\![-x^h]\!]\Big) \cup \Big(\bigcup_{h\in D_g^3} [\![x^h]\!]\Big).$$

If $\mathrm{ord}(x) = 4g$, $h \in D_g$ and $h > 1$, then $\mathrm{ord}(x^h) \in \{4, \ldots, 4(g-1)\}$. By induction hypothesis

$$\mathbf{i}\Big(\sum_{s\in[\![x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x^h]\!]} \psi_\alpha(s)\Big) \in \mathbb{Z} \text{ for all } \alpha \in \Gamma.$$

Now we have

$$\begin{aligned}\mathbf{i}\Big(\sum_{s\in M_1(x)} \psi_\alpha(s) - \sum_{s\in M_3(x)} \psi_\alpha(s)\Big) =& \mathbf{i}\Big(\sum_{s\in[\![x]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x]\!]} \psi_\alpha(s)\Big)\\ &+ \sum_{h\in D_g^1, h>1} \mathbf{i}\Big(\sum_{s\in[\![x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x^h]\!]} \psi_\alpha(s)\Big)\\ &+ \sum_{h\in D_g^3, h>1} \mathbf{i}\Big(\sum_{s\in[\![-x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![x^h]\!]} \psi_\alpha(s)\Big).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{i}\Big(\sum_{s\in[\![x]\!]}\psi_\alpha(s)-\sum_{s\in[\![-x]\!]}\psi_\alpha(s)\Big)=&\mathbf{i}\Big(\sum_{s\in M_1(x)}\psi_\alpha(s)-\sum_{s\in M_3(x)}\psi_\alpha(s)\Big)\\&-\sum_{h\in D_g^1,h>1}\mathbf{i}\Big(\sum_{s\in[\![x^h]\!]}\psi_\alpha(s)-\sum_{s\in[\![-x^h]\!]}\psi_\alpha(s)\Big)\\&+\sum_{h\in D_g^3,h>1}\mathbf{i}\Big(\sum_{s\in[\![x^h]\!]}\psi_\alpha(s)-\sum_{s\in[\![-x^h]\!]}\psi_\alpha(s)\Big).\end{aligned}$$

Hence by Lemma 2.2.2 and induction hypothesis, $\mathbf{i}\Big(\sum_{s\in[\![x]\!]}\psi_\alpha(s)-\sum_{s\in[\![-x]\!]}\psi_\alpha(s)\Big)$ is an integer for each $\alpha\in\Gamma$. Thus the proof follows by induction. □

For $\Gamma(4)\neq\emptyset$, recall that $\mathbb{D}(\Gamma)$ is the set of all skew-symmetric subsets of Γ of the form $[\![x_1]\!]\cup\cdots\cup[\![x_k]\!]$ for some $x_1,\ldots,x_k\in\Gamma(4)$. For $\Gamma(4)=\emptyset$, recall that $\mathbb{D}(\Gamma)=\{\emptyset\}$.

Theorem 2.2.6. *Let Γ be an abelian group. If $S\in\mathbb{D}(\Gamma)$, then the directed Cayley graph* $\text{Cay}(\Gamma,S)$ *is H-integral.*

Proof. Assume that $S\in\mathbb{D}(\Gamma)$. We get $S=[\![x_1]\!]\cup\cdots\cup[\![x_k]\!]$ for some $x_1,\ldots,x_k\in\Gamma(4)$. Let μ_α be an H-eigenvalue of the directed Cayley graph $\text{Cay}(\Gamma,S)$. By Lemma 2.1.4, we have

$$\mu_\alpha=\mathbf{i}\sum_{s\in S}\Big(\psi_\alpha(s)-\psi_\alpha(-s)\Big)=\sum_{j=1}^{k}\sum_{s\in[\![x_j]\!]}\mathbf{i}\Big(\psi_\alpha(s)-\psi_\alpha(-s)\Big). \tag{2.2}$$

Applying Lemma 2.2.5 in Equation (2.2), we find that μ_α is an integer for each $\alpha\in\Gamma$. Hence, the directed Cayley graph $\text{Cay}(\Gamma,S)$ is H-integral. □

Theorem 2.2.7. *Let Γ be an abelian group. If $S\setminus\overline{S}\in\mathbb{B}(\Gamma)$ and $\overline{S}\in\mathbb{D}(\Gamma)$, then the mixed Cayley graph* $\text{Cay}(\Gamma,S)$ *is H-integral.*

Proof. By Theorem 2.1.5, $\text{Cay}(\Gamma,S)$ is H-integral if and only if each of $\text{Cay}(\Gamma,S\setminus\overline{S})$ and $\text{Cay}(\Gamma,\overline{S})$ are H-integral. Thus the result follows from Theorem 1.6.6 and Theorem 2.2.6. □

2.3 Characterization of H-integral mixed Cayley graphs over abelian groups

In this section, first we prove that there does not exist an H-integral directed Cayley graph $\text{Cay}(\Gamma,S)$ for $\exp(\Gamma)\not\equiv 0 \pmod 4$ and $S\neq\emptyset$. Thereafter, we prove that the sufficient condition on S for the H-integrality of the mixed Cayley graph $\text{Cay}(\Gamma,S)$, as stated in Theorem 2.2.7, is also necessary.

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Theorem 2.3.1. *If* Γ *is an abelian group and* $\exp(\Gamma) \not\equiv 0 \pmod 4$, *then the directed Cayley graph* $\text{Cay}(\Gamma, S)$ *is H-integral if and only if* $S = \emptyset$.

Proof. Let the H-spectrum of the directed Cayley graph $\text{Cay}(\Gamma, S)$ be $\{\mu_\alpha \colon \alpha \in \Gamma\}$ and $\ell = \exp(\Gamma)$. Assume that $\ell \not\equiv 0 \pmod 4$ and $\text{Cay}(\Gamma, S)$ is H-integral. By Corollary 2.1.4, we have $\mu_\alpha = -\mu_{-\alpha} \in \mathbb{Q}$ and

$$\mu_\alpha = \mathbf{i} \sum_{s \in S} (\psi_\alpha(s) - \psi_\alpha(-s)) \text{ for each } \alpha \in \Gamma.$$

Note that, $\psi_\alpha(s)$ and $\psi_\alpha(-s)$ are ℓ^{th} roots of unity for $\alpha \in \Gamma, s \in S$. Fix a primitive ℓ^{th} root ω of unity and express $\psi_\alpha(s)$ in the form ω^j for some $j \in \{0, 1, \ldots, \ell - 1\}$. Thus

$$\mu_\alpha = \mathbf{i} \sum_{s \in S} (\psi_\alpha(s) - \psi_\alpha(-s)) = \sum_{j=0}^{\ell-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(\mathbf{i})$. Since $\mu_\alpha \in \mathbb{Z}$, we have that ω is a root of the polynomial $p(x)$, where $p(x) := \sum_{j=0}^{\ell-1} a_j x^j - \mu_\alpha \in \mathbb{Q}(\mathbf{i})[x]$. Since $\ell \not\equiv 0(\mod 4)$, $\Phi_\ell(x)$ is irreducible in $\mathbb{Q}(\mathbf{i})[x]$. Thus $p(\omega) = 0$ and $\Phi_\ell(x)$ is a monic irreducible polynomial over $\mathbb{Q}(\mathbf{i})$ having ω as a root. Therefore $\Phi_\ell(x)$ divides $p(x)$, and so ω^{-1} is also a root of $p(x)$. Note that, if $\psi_\alpha(s) = \omega^j$ for some $j \in \{0, 1, \ldots, \ell - 1\}$ then $\psi_{-\alpha}(s) = \omega^{-j}$. We have

$$0 = p(\omega^{-1}) = \sum_{j=0}^{\ell-1} a_j \omega^{-j} - \mu_\alpha = \mu_{-\alpha} - \mu_\alpha.$$

Thus $\mu_{-\alpha} = \mu_\alpha$, and hence $\mu_\alpha = 0$ for all $\alpha \in \Gamma$. Hence $S = \emptyset$.

Conversely, if $S = \emptyset$ then all the H-eigenvalues of $\text{Cay}(\Gamma, S)$ are zero. Thus $\text{Cay}(\Gamma, S)$ is H-integral. □

Lemma 2.2.3 gives that corresponding to each equivalence class of the relation $\sim$, we get two equivalence classes of the relation $\approx$. Define F to be the matrix $[F_{xy}]$ of size $n \times n$, whose rows and columns are indexed by the elements of Γ such that $F_{xy} := \mathbf{i}\psi_x(y)$ for each $x, y \in \Gamma$. Note that each row of F corresponds to a character of Γ and $FF^* = nI_n$, where F^* is the conjugate transpose of F. Let the coordinates of $\mathbf{v} \in \mathbb{Q}^n$ be indexed by the elements of Γ, and let $\mathbf{v}_x$ denote the coordinate of $\mathbf{v}$ indexed by x. We note here that there exists $\mathbf{v} \in \mathbb{Q}^n$ such that $F\mathbf{v} \in \mathbb{Q}^n$. For example, let $a \in \Gamma(4)$ and $\mathbf{u}$ be the vector in $\mathbb{Q}^n$ whose coordinates are indexed by the elements of Γ, where

$$\mathbf{u}_x = \begin{cases} 1 & \text{if } x \in [\![a]\!] \\ -1 & \text{if } x \in [\![a^{-1}]\!] \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.2.5, we have $F\mathbf{u} \in \mathbb{Q}^n$.

Lemma 2.3.2. *Let Γ be an abelian group, $\mathbf{v} \in \mathbb{Q}^n$, and let the coordinates of $\mathbf{v}$ be indexed by the elements of Γ. If $F\mathbf{v} \in \mathbb{Q}^n$, then*

(i) $\mathbf{v}_x = -\mathbf{v}_{-x}$ *for each* $x \in \Gamma$;

(ii) $\mathbf{v}_x = \mathbf{v}_y$ *for all* $x, y \in \Gamma(4)$ *satisfying* $x \approx y$;

(iii) $\mathbf{v}_x = 0$ *for all* $x \in \Gamma \setminus \Gamma(4)$.

Proof. Let F_x and F_y denote the columns of F indexed by x and y, respectively, and assume that $\mathbf{u} = F\mathbf{v} \in \mathbb{Q}^n$.

(i) We use the fact that $\overline{\psi_x(y)} = \psi_{-x}(y) = \psi_x(-y)$ for each $x, y \in \Gamma$. Again

$$\mathbf{u} = F\mathbf{v} \quad \text{or,} \quad F^*\mathbf{u} = F^*F\mathbf{v} = (nI_n)\mathbf{v} \quad \text{or,} \quad \frac{1}{n}F^*\mathbf{u} = \mathbf{v} \in \mathbb{Q}^n.$$

Thus

$$\begin{aligned}\mathbf{v}_x = \frac{1}{n}(F^*\mathbf{u})_x = \frac{1}{n}\sum_{a\in\Gamma} F^*_{xa}\mathbf{u}_a = \frac{1}{n}\sum_{a\in\Gamma}\overline{\mathbf{i}\psi_a(x)}\mathbf{u}_a &= -\frac{1}{n}\sum_{a\in\Gamma}\mathbf{i}\psi_a(-x)\mathbf{u}_a \\ &= -\overline{\frac{1}{n}\sum_{a\in\Gamma}\overline{\mathbf{i}\psi_a(-x)}\mathbf{u}_a} \\ &= -\overline{\frac{1}{n}\sum_{a\in\Gamma}F^*_{-xa}\mathbf{u}_a} \\ &= -\frac{1}{n}\overline{(F^*\mathbf{u})_{-x}} \\ &= -\overline{\mathbf{v}}_{-x} = -\mathbf{v}_{-x}.\end{aligned}$$

(ii) If $\Gamma(4) = \emptyset$ then there is nothing to prove. Now assume that $\Gamma(4) \neq \emptyset$. Let $x, y \in \Gamma(4)$. If $x \approx y$, then there exists $k \in G^1_m(1)$ such that $y = x^k$, where $m = \text{ord}(x) = \text{ord}(y)$. Assume that $x \neq y$, so that $k \geq 2$. Using Lemma 2.1.1, we see that the entries of F_x and F_y are $\mathbf{i}$ times an m^{th} root of unity. Fix a primitive m^{th} root ω of unity, and express each entry of F_x and F_y in the form $\mathbf{i}\omega^j$ for some $j \in \{0, 1, \ldots, m-1\}$. Thus

$$n\mathbf{v}_x = (F^*\mathbf{u})_x = \sum_{j=0}^{m-1} a_j\omega^j,$$

where $a_j \in \mathbb{Q}(\mathbf{i})$ for all j. Thus ω is a root of $p(x)$, where $p(x) := \sum_{j=0}^{m-1} a_jx^j - n\mathbf{v}_x \in \mathbb{Q}(\mathbf{i})[x]$. Therefore $p(x)$ is a multiple of the irreducible polynomial $\Phi^1_m(x)$. This, along with $k \in G^1_m(1)$, gives that ω^k is also a root of $p(x)$. As $y = x^k$ implies that $\psi_a(y) = \psi_a(x)^k$ for all $a \in \Gamma$, we have $(F^*\mathbf{u})_y = \sum_{j=0}^{m-1} a_j\omega^{kj}$. Hence

$$0 = p(\omega^k) = \sum_{j=0}^{m-1} a_j\omega^{kj} - n\mathbf{v}_x = (F^*\mathbf{u})_y - n\mathbf{v}_x = n\mathbf{v}_y - n\mathbf{v}_x.$$

This gives $\mathbf{v}_x = \mathbf{v}_y$.

(iii) Let $x \in \Gamma \setminus \Gamma(4)$ and $r = \text{ord}(x) \not\equiv 0 \pmod 4$. Fix a primitive r^{th} root ω of unity, and express each entry of F_x in the form $\mathbf{i}\omega^j$ for some $j \in \{0, 1, \dots, r-1\}$. We have

$$n\mathbf{v}_x = (F^*\mathbf{u})_x = \sum_{j=0}^{r-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(\mathbf{i})$ for all j. Thus ω is a root of $p(x)$, where $p(x) := \sum_{j=0}^{r-1} a_j x^j - n\mathbf{v}_x \in \mathbb{Q}(\mathbf{i})[x]$. Therefore $p(x)$ is a multiple of the irreducible polynomial $\Phi_r(x)$, and so ω^{-1} is also a root of $p(x)$. Since $\psi_a(-x) = \psi_a(x)^{-1}$ for all $a \in \Gamma$, we have $(F^*\mathbf{u})_{-x} = \sum_{j=0}^{r-1} a_j \omega^{-j}$. Hence

$$0 = p(\omega^{-1}) = \sum_{j=0}^{r-1} a_j \omega^{-j} - n\mathbf{v}_x = (F^*\mathbf{u})_{-x} - n\mathbf{v}_x = n\mathbf{v}_{-x} - n\mathbf{v}_x.$$

Thus $\mathbf{v}_x = \mathbf{v}_{-x}$. This, together with Part (i), implies that $\mathbf{v}_x = 0$ for all $x \in \Gamma \setminus \Gamma(4)$. □

Theorem 2.3.3. *If Γ is an abelian group, then the directed Cayley graph* $\text{Cay}(\Gamma, S)$ *is H-integral if and only if $S \in \mathbb{D}(\Gamma)$.*

Proof. Assume that the directed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral. If $\Gamma(4) = \emptyset$, then by Theorem 2.3.1 we have $S = \emptyset$, and so $S \in \mathbb{D}(\Gamma)$. Now assume that $\Gamma(4) \neq \emptyset$. Let $\mathbf{v}$ be the vector in $\mathbb{Q}^n$ whose coordinates are indexed by the elements of Γ, where

$$\mathbf{v}_x = \begin{cases} 1 & \text{if } x \in S \\ -1 & \text{if } x \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(F\mathbf{v})_a = \sum_{x \in \Gamma} F_{ax}\mathbf{v}_x = \sum_{x \in S} F_{ax} - \sum_{x \in S^{-1}} F_{ax} = \mathbf{i} \sum_{x \in S} (\psi_a(x) - \psi_a(-x)).$$

Thus $(F\mathbf{v})_a$ is an H-eigenvalue of the H-integral directed Cayley graph $\text{Cay}(\Gamma, S)$. Therefore $F\mathbf{v} \in \mathbb{Q}^n$, and hence all the three conditions of Lemma 2.3.2 are satisfied for $\mathbf{v}$.

By the third condition of Lemma 2.3.2, $\mathbf{v}_x = 0$ for all $x \in \Gamma \setminus \Gamma(4)$, and so we must have $S \cup S^{-1} \subseteq \Gamma(4)$. Again, let $x \in S$, $y \in \Gamma(4)$ and $x \approx y$. The second condition of Lemma 2.3.2 gives $\mathbf{v}_x = \mathbf{v}_y$, which implies that $y \in S$. Thus $x \in S$ implies $[\![x]\!] \subseteq S$. Hence $S \in \mathbb{D}(\Gamma)$. The converse part follows from Theorem 2.2.6. □

The following example illustrates Theorem 2.3.3.

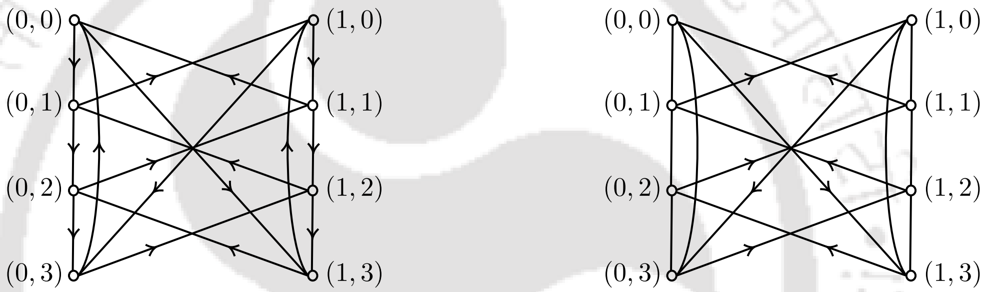
Example 2.3.1. Consider $\Gamma = \mathbb{Z}_2 \otimes \mathbb{Z}_4$ and $S = \{(0,1),(1,3)\}$. The directed Cayley graph $\text{Cay}(\mathbb{Z}_2 \otimes \mathbb{Z}_4, S)$ is shown in Figure 2.1a. We see that $[\![(0,1)]\!] = \{(0,1)\}$ and $[\![(1,3)]\!] = \{(1,3)\}$. Therefore $S \in \mathbb{D}(\Gamma)$. Further, using Corollary 2.1.4 and Equation (2.1), the H-eigenvalues of $\text{Cay}(\mathbb{Z}_2 \otimes \mathbb{Z}_4, S)$ are obtained as

$$\mu_\alpha = \mathbf{i}[\psi_\alpha(0,1) - \psi_\alpha(0,3)] + \mathbf{i}[\psi_\alpha(1,3) - \psi_\alpha(1,1)] \text{ for each } \alpha \in \mathbb{Z}_2 \otimes \mathbb{Z}_4,$$

where

$$\psi_\alpha(x) = (-1)^{\alpha_1 x_1} \mathbf{i}^{\alpha_2 x_2} \text{ for each } \alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_2 \otimes \mathbb{Z}_4.$$

It can be seen that $\mu_{(0,0)} = \mu_{(0,1)} = \mu_{(0,2)} = \mu_{(0,3)} = \mu_{(1,0)} = \mu_{(1,2)} = 0$, $\mu_{(1,1)} = -4$ and $\mu_{(1,3)} = 4$. Thus $\text{Cay}(\mathbb{Z}_2 \otimes \mathbb{Z}_4, S)$ is H-integral.



(a) $S = \{(0,1),(1,3)\}$ (b) $S = \{(0,1),(0,3),(1,3)\}$

Figure 2.1: The mixed graph $\text{Cay}(\mathbb{Z}_2 \otimes \mathbb{Z}_4, S)$

Theorem 2.3.4. *If Γ is an abelian group, then the mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral if and only if $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$ and $\overline{S} \in \mathbb{D}(\Gamma)$.*

Proof. By Theorem 2.1.5, the mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral if and only if each of $\text{Cay}(\Gamma, S \setminus \overline{S})$ and $\text{Cay}(\Gamma, \overline{S})$ are H-integral. Note that $S \setminus \overline{S}$ is a symmetric set and $\overline{S}$ is a skew-symmetric set. Thus by Theorem 1.6.6, $\text{Cay}(\Gamma, S \setminus \overline{S})$ is H-integral if and only if $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$. By Theorem 2.3.3, $\text{Cay}(\Gamma, \overline{S})$ is H-integral if and only if $\overline{S} \in \mathbb{D}(\Gamma)$. Hence the result follows. □

The following example illustrates Theorem 2.3.4.

Example 2.3.2. Consider $\Gamma = \mathbb{Z}_2 \otimes \mathbb{Z}_4$ and $S = \{(0,1),(0,3),(1,3)\}$. The mixed Cayley graph $\text{Cay}(\mathbb{Z}_2 \otimes \mathbb{Z}_4, S)$ is shown in Figure 2.1b. Observe that $\overline{S} = \{(1,3)\} = [\![(1,3)]\!] \in \mathbb{D}(\Gamma)$ and $S \setminus \overline{S} = \{(0,1),(0,3)\} = [(0,1)] \in \mathbb{B}(\Gamma)$. Further, using Lemma 2.1.2 and Equation (2.1), the H-eigenvalues of $\text{Cay}(\mathbb{Z}_2 \otimes \mathbb{Z}_4, S)$ are obtained as

$$\mu_\alpha = [\psi_\alpha(0,1) + \psi_\alpha(0,3)] + \mathbf{i}[\psi_\alpha(1,3) - \psi_\alpha(1,1)] \text{ for each } \alpha \in \mathbb{Z}_2 \otimes \mathbb{Z}_4,$$

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where

$$\psi_\alpha(x) = (-1)^{\alpha_1 x_1}\mathbf{i}^{\alpha_2 x_2} \text{ for each } \alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_2 \otimes \mathbb{Z}_4.$$

We find that $\mu_{(0,0)} = \mu_{(0,1)} = \mu_{(1,0)} = \mu_{(1,3)} = 2$ and $\mu_{(0,2)} = \mu_{(0,3)} = \mu_{(1,1)} = \mu_{(1,2)} = -2$. Thus $\text{Cay}(\mathbb{Z}_2 \otimes \mathbb{Z}_4, S)$ is H-integral.

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3

HS-integral mixed Cayley graphs over abelian groups

In this chapter, we use Theorem 1.6.6 to characterize HS-integral mixed Cayley graphs over abelian groups. In Section 1, we express the HS-eigenvalues of mixed Cayley graphs over an abelian group in terms of the irreducible characters of the group, and present some preliminary results. In the second section, we find a sufficient condition for which a directed Cayley graph over an abelian group is HS-integral. In the last section, we characterize HS-integral mixed Cayley graphs over abelian groups. The definitions, results, and the flow of arguments in this chapter have similarities with those in Chapter 2.

3.1 Preliminaries

In this chapter, Γ is considered to be an abelian group. Recall all the notations associated to an abelian group from Section 2.1.

Lemma 3.1.1 ([4, 18]). *If Γ is an abelian group, then the HS-spectrum of the mixed Cayley graph* $\text{Cay}(\Gamma, S)$ *is* $\{\gamma_\alpha \colon \alpha \in \Gamma\}$, *where* $\gamma_\alpha = \lambda_\alpha + \mu_\alpha$,

$$\lambda_\alpha = \sum_{s \in S \setminus \overline{S}} \psi_\alpha(s) \text{ and } \mu_\alpha = \sum_{s \in \overline{S}} \left(\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)\right) \text{ for each } \alpha \in \Gamma.$$

Proof. Define $f \colon \Gamma \to \{0, 1, \omega_6, \omega_6^5\}$ such that

$$f(s) = \begin{cases} 1 & \text{if } s \in S \setminus \overline{S} \\ \omega_6 & \text{if } s \in \overline{S} \\ \omega_6^5 & \text{if } s \in (\overline{S})^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Since Γ is an abelian group, f is a class function. The Hermitian adjacency matrix of second kind of $\text{Cay}(\Gamma, S)$ is equal to $[f(y-x)]_{x,y \in \Gamma}$. Thus the result follows from Theorem 1.3.6. □

The next corollary is a special case of Lemma 3.1.1.

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Corollary 3.1.2. *If* Γ *is an abelian group, then the HS-spectrum of the directed Cayley graph* $\text{Cay}(\Gamma, S)$ *is* $\{\mu_\alpha \colon \alpha \in \Gamma\}$, *where*

$$\mu_\alpha = \sum_{s \in S} \big(\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)\big) \text{ for each } \alpha \in \Gamma.$$

Recall that $\Gamma(3)$ is the set of all $x \in \Gamma$ satisfying $\text{ord}(x) \equiv 0 \pmod 3$. For $x \in \Gamma(3)$ and $r \in \{0, 1, 2\}$, define

$$M_{r,3}(x) := \{x^k \colon 1 \leq k \leq \text{ord}(x), k \equiv r \pmod 3\}.$$

For $a \in \Gamma$ and $S \subseteq \Gamma$, recall that $a + S := \{a + s \colon s \in S\}$ and $-S := \{-s \colon s \in S\}$. Also, $\langle a \rangle$ denotes the cyclic subgroup of Γ generated by a. Note that $-s$ denotes the inverse of s, that is $-s = s^{m-1}$, where $m = \text{ord}(s)$.

Lemma 3.1.3. *If* Γ *is an abelian group and* $x \in \Gamma(3)$, *then the following assertions hold.*

(i) $\bigcup\limits_{r=0}^{2} M_{r,3}(x) = \langle x \rangle$.

(ii) The sets $M_{1,3}(x)$ *and* $M_{2,3}(x)$ *are skew-symmetric subsets of* Γ.

(iii) $-M_{1,3}(x) = M_{2,3}(x)$ *and* $-M_{2,3}(x) = M_{1,3}(x)$.

(iv) $a + M_{1,3}(x) = M_{1,3}(x)$ *and* $a + M_{2,3}(x) = M_{2,3}(x)$ *for all* $a \in M_{0,3}(x)$.

Proof. (i) The proof follows from the definitions of $M_{r,3}(x)$ and $\langle x \rangle$.

(ii) Let $\text{ord}(x) = m$. If $x^k \in M_{1,3}(x)$ then $-x^k = x^{m-k} \notin M_{1,3}(x)$, as $k \equiv 1 \pmod 3$ gives $m - k \equiv 2 \pmod 3$. Thus $M_{1,3}(x)$ is a skew-symmetric subset of Γ. Similarly, $M_{2,3}(x)$ is also a skew-symmetric subset of Γ.

(iii) Let $\text{ord}(x) = m$. As $k \equiv 1 \pmod 3$ if and only if $m - k \equiv 2 \pmod 3$, and $-x^k = x^{m-k}$, we get $-M_{1,3}(x) = M_{2,3}(x)$. Similarly, $-M_{2,3}(x) = M_{1,3}(x)$.

(iv) Let $a \in M_{0,3}(x)$ and $y \in a + M_{1,3}(x)$. Then $a = x^{k_1}$ and $y = x^{k_1} + x^{k_2} = x^{k_1+k_2}$, where $k_1 \equiv 0 \pmod 3$ and $k_2 \equiv 1 \pmod 3$. Since $k_1 + k_2 \equiv 1 \pmod 3$, we have $y \in M_{1,3}(x)$. This gives $a + M_{1,3}(x) \subseteq M_{1,3}(x)$, and hence $a + M_{1,3}(x) = M_{1,3}(x)$. Similarly, $a + M_{2,3}(x) = M_{2,3}(x)$ for all $a \in M_{0,3}(x)$. □

Let $m \equiv 0 \pmod 3$, $r \in \{1, 2\}$ and $g \in \mathbb{Z}$. Recall the following definitions from Chapter 1:

$$G_{m,3}^r(1) = \{k \colon 1 \leq k \leq m-1, \gcd(k, m) = 1, k \equiv r \pmod 3\},$$

$$D_{g,3} = \{k \colon k \text{ divides } g, k \not\equiv 0 \pmod 3\} \text{ and}$$

$$D_{g,3}^r = \{k \colon k \text{ divides } g, k \equiv r \pmod 3\}.$$

Recall the equivalence relation $\simeq$ on $\Gamma(3)$, for which $x \simeq y$ if and only if $y = x^k$ for some $k \in G^1_{m,3}(1)$, where $m = \text{ord}(x)$. For $x \in \Gamma(3)$, recall that $\langle\langle x \rangle\rangle$ is the equivalence class of x with respect to the relation $\simeq$.

Lemma 3.1.4. *If Γ is an abelian group, $x \in \Gamma(3)$ and $m = \text{ord}(x)$, then the following assertions hold.*

(i) $\langle\langle x \rangle\rangle = \{x^k : k \in G^1_{m,3}(1)\}$.

(ii) $\langle\langle -x \rangle\rangle = \{x^k : k \in G^2_{m,3}(1)\}$.

(iii) $\langle\langle x \rangle\rangle \cap \langle\langle -x \rangle\rangle = \emptyset$.

(iv) $[x] = \langle\langle x \rangle\rangle \cup \langle\langle -x \rangle\rangle$.

Proof. (i) Let $y \in \langle\langle x \rangle\rangle$. Then $y \simeq x$, and so $\text{ord}(y) = \text{ord}(x) = m$ and there exists $k \in G^1_{m,3}(x)$ such that $y = x^k$. Thus $\langle\langle x \rangle\rangle \subseteq \{x^k : k \in G^1_{m,3}(1)\}$. On the other hand, let $z = x^k$ for some $k \in G^1_{m,3}(1)$. Then $\text{ord}(z) = \text{ord}(x)$, and so $x \simeq z$. Thus we have $\{x^k : k \in G^1_{m,3}(1)\} \subseteq \langle\langle x \rangle\rangle$. Hence the desired equality follows.

(ii) Note that $-x = x^{m-1}$ and $m - 1 \equiv 2 \pmod 3$. By Part (i), we have

$$\begin{aligned}\langle\langle -x \rangle\rangle = \{(-x)^k : k \in G^1_{m,3}(1)\} = \{x^{(m-1)k} : k \in G^1_{m,3}(1)\} &= \{x^{-k} : k \in G^1_{m,3}(1)\} \\ &= \{x^k : k \in G^2_{m,3}(1)\}.\end{aligned}$$

(iii) Since $G^1_{m,3}(1) \cap G^2_{m,3}(1) = \emptyset$, we have by Part (i) and Part (ii) that $\langle\langle x \rangle\rangle \cap \langle\langle -x \rangle\rangle = \emptyset$.

(iv) Since $[x] = \{x^k : k \in G_m(1)\}$, and $G_m(1)$ is a disjoint union of $G^1_{m,3}(1)$ and $G^3_{m,3}(1)$, we have by Part (i) and Part (ii) that $[x] = \langle\langle x \rangle\rangle \cup \langle\langle -x \rangle\rangle$. □

Note that Lemma 3.1.4 also holds for non-abelian group.

Lemma 3.1.5. *If Γ be an abelian group, $x \in \Gamma(3)$, $m = \text{ord}(x)$ and $g = \frac{m}{3}$, then the following assertions hold.*

(i) $M_{1,3}(x) \cup M_{2,3}(x) = \bigcup\limits_{h \in D_{g,3}} [x^h]$.

(ii) $M_{1,3}(x) = \left(\bigcup\limits_{h \in D^1_{g,3}} \langle\langle x^h \rangle\rangle \right) \cup \left(\bigcup\limits_{h \in D^2_{g,3}} \langle\langle -x^h \rangle\rangle \right)$.

(iii) $M_{2,3}(x) = \left(\bigcup\limits_{h \in D^1_{g,3}} \langle\langle -x^h \rangle\rangle \right) \cup \left(\bigcup\limits_{h \in D^2_{g,3}} \langle\langle x^h \rangle\rangle \right)$.

Proof. (i) Let $x^k \in M_{1,3}(x) \cup M_{2,3}(x)$, where $k \equiv 1$ or $2 \pmod 3$. To show that $x^k \in \bigcup\limits_{h\in D_{g,3}} [x^h]$, it is enough to show $x^k \sim x^h$ for some $h \in D_{g,3}$. Let $h = \gcd(k,g) \in D_{g,3}$. Note that

$$\text{ord}(x^k) = \frac{m}{\gcd(m,k)} = \frac{m}{\gcd(g,k)} = \frac{m}{h} = \text{ord}(x^h).$$

Also, as $h = \gcd(k,m)$, we have $\langle x^k\rangle = \langle x^h\rangle$, and so $x^k = x^{hj}$ for some $j \in G_q(1)$, where $q = \text{ord}(x^h) = \frac{m}{h}$. Thus $x^k \sim x^h$, where $h = \gcd(k,g) \in D_{g,3}$. Therefore $x^k \in \bigcup\limits_{h\in D_{g,3}} [x^h]$. Conversely, let $z \in \bigcup\limits_{h\in D_{g,3}} [x^h]$. Then there exists $h \in D_{g,3}$ such that $z = x^{hj}$, where $j \in G_q(1)$ and $q = \frac{m}{\gcd(m,h)}$. Now $h \in D_{g,3}$ and $q \equiv 0 \pmod 3$ altogether implies that $hj \equiv 1$ or $2 \pmod 3$, and so $z \in M_{1,3}(x) \cup M_{2,3}(x)$. Hence $M_{1,3}(x) \cup M_{2,3}(x) = \bigcup\limits_{h\in D_{g,3}} [x^h]$.

(ii) Let $x^k \in M_{1,3}(x)$, where $k \equiv 1 \pmod 3$. By Part (i), there exists $h \in D_{g,3}$ and $j \in G_q(1)$ such that $x^k = x^{hj}$, where $q = \frac{m}{\gcd(m,h)}$. Note that $k \equiv jh \pmod m$. If $h \equiv 1 \pmod 3$ then $j \in G^1_{q,3}(1)$, else $j \in G^2_{q,3}(1)$. Thus using Part (i) and Part (ii) of Lemma 3.1.4, if $h \equiv 1 \pmod 3$ then $x^k \simeq x^h$, else $x^k \simeq -x^h$. Hence

$$M_{1,3}(x) \subseteq \left(\bigcup_{h\in D^1_{g,3}} \langle\!\langle x^h\rangle\!\rangle\right) \cup \left(\bigcup_{h\in D^2_{g,3}} \langle\!\langle -x^h\rangle\!\rangle\right).$$

Conversely, assume that $z \in \left(\bigcup\limits_{h\in D^1_{g,3}} \langle\!\langle x^h\rangle\!\rangle\right) \cup \left(\bigcup\limits_{h\in D^2_{g,3}} \langle\!\langle -x^h\rangle\!\rangle\right)$. This gives $z \in \langle\!\langle x^h\rangle\!\rangle$ for an $h \in D^1_{g,3}$ or $z \in \langle\!\langle -x^h\rangle\!\rangle$ for an $h \in D^2_{g,3}$. In the first case, by Part (i) of Lemma 3.1.4, there exists $j \in G^1_{q,3}(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. Similarly, for the second case, by Part (ii) of Lemma 3.1.4, there exists $j \in G^2_{q,3}(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. In both the cases, $hj \equiv 1 \pmod 3$. Thus $z \in M_{1,3}(x)$, and so

$$\left(\bigcup_{h\in D^1_{g,3}} \langle\!\langle x^h\rangle\!\rangle\right) \cup \left(\bigcup_{h\in D^2_{g,3}} \langle\!\langle -x^h\rangle\!\rangle\right) \subseteq M_{1,3}(x).$$

Hence the desired equality follows.

(iii) The proof of this part is similar to the proof of Part (ii). For the sake of completeness, we provide the proof. Let $x^k \in M_{2,3}(x)$, where $k \equiv 2 \pmod 3$. By Part (i), there exists $h \in D_{g,3}$ and $j \in G_q(1)$ such that $x^k = x^{hj}$, where $q = \frac{m}{\gcd(m,h)}$. Note that $k \equiv jh \pmod m$. If $h \equiv 1 \pmod 3$ then $j \in G^2_{q,3}(1)$, else $j \in G^1_{q,3}(1)$. Thus using Part (i) and Part (ii) of Lemma 3.1.4, if $h \equiv 1 \pmod 3$ then $x^k \simeq -x^h$, else $x^k \simeq x^h$. Hence

$$M_{2,3}(x) \subseteq \left(\bigcup_{h\in D^1_{g,3}} \langle\!\langle -x^h\rangle\!\rangle\right) \cup \left(\bigcup_{h\in D^2_{g,3}} \langle\!\langle x^h\rangle\!\rangle\right).$$

Conversely, assume that $z \in \left(\bigcup\limits_{h\in D^1_{g,3}} \langle\!\langle -x^h\rangle\!\rangle\right) \cup \left(\bigcup\limits_{h\in D^2_{g,3}} \langle\!\langle x^h\rangle\!\rangle\right)$. This gives $z \in \langle\!\langle -x^h\rangle\!\rangle$ for an $h \in D^1_{g,3}$ or $z \in \langle\!\langle x^h\rangle\!\rangle$ for an $h \in D^2_{g,3}$. In the first case, by Part (ii) of Lemma 3.1.4, there exists $j \in G^2_{q,3}(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. Similarly, for the second case, by Part (i) of Lemma 3.1.4, there exists $j \in G^1_{q,3}(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. In both the cases, $hj \equiv 2 \pmod 3$. Thus $z \in M_{2,3}(x)$, and so

$$\left(\bigcup_{h \in D^1_{g,3}} \langle\langle -x^h \rangle\rangle \right) \cup \left(\bigcup_{h \in D^2_{g,3}} \langle\langle x^h \rangle\rangle \right) \subseteq M_{2,3}(x).$$

Hence the desired equality follows. □

3.2 A sufficient condition for HS-integrality of directed Cayley graphs over abelian groups

In this section, first we prove that $S = \emptyset$ is the only connection set for an HS-integral directed Cayley graph $\text{Cay}(\Gamma, S)$ whenever $\Gamma(3) = \emptyset$. After that we obtain a sufficient condition on the set S for which the directed Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral.

Lemma 3.2.1. *Let S be a skew-symmetric subset of an abelian group Γ. If*

$$\sum_{s \in S} \mathbf{i}\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) = 0$$

for all $\alpha \in \Gamma$, then $S = \emptyset$.

Proof. Define $f \colon \Gamma \to \{0, \mathbf{i}\sqrt{3}, -\mathbf{i}\sqrt{3}\}$ such that

$$f(s) = \begin{cases} \mathbf{i}\sqrt{3} & \text{if } s \in S \\ -\mathbf{i}\sqrt{3} & \text{if } s \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 1.3.6, $\lambda_\alpha := \sum_{s \in S} \mathbf{i}\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))$ is an eigenvalue of the matrix $[f(y-x)]_{x,y \in \Gamma}$ for each $\alpha \in \Gamma$. If $\lambda_\alpha = 0$ for all $\alpha \in \Gamma$, then all the entries of this matrix must be zero. Hence $S = \emptyset$. □

Theorem 3.2.2. *If Γ is an abelian group and $\Gamma(3) = \emptyset$, then the directed Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral if and only if $S = \emptyset$.*

Proof. Let the HS-spectrum of the directed Cayley graph $\text{Cay}(\Gamma, S)$ be $\{\mu_\alpha \colon \alpha \in \Gamma\}$ and $\ell = \exp(\Gamma)$. Assume that $\text{Cay}(\Gamma, S)$ is HS-integral and $\Gamma(3) = \emptyset$, so that $\ell \not\equiv 0 \pmod 3$. By Corollary 3.1.2,

$$\mu_\alpha = \sum_{s \in S} (\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s)) \in \mathbb{Z} \text{ for all } \alpha \in \Gamma.$$

Note that $\psi_\alpha(s)$ and $\psi_\alpha(-s)$ are ℓ^{th} roots of unity for all $\alpha \in \Gamma, s \in S$. Fix a primitive ℓ^{th} root ω of unity and express $\psi_\alpha(s)$ in the form ω^j for some $j \in \{0, 1, \ldots, \ell-1\}$. Thus

$$\mu_\alpha = \sum_{s\in S}(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(-s)) = \sum_{j=0}^{\ell-1} a_j\omega^j,$$

where $a_j \in \mathbb{Q}(\omega_3)$. Since $\mu_\alpha \in \mathbb{Z}$, we have that ω is a root of the polynomial $p(x)$, where $p(x) := \sum\limits_{j=0}^{\ell-1} a_j x^j - \mu_\alpha \in \mathbb{Q}(\omega_3)[x]$. Since $\ell \not\equiv 0 \pmod 3$, the polynomial $\Phi_\ell(x)$ is irreducible in $\mathbb{Q}(\omega_3)[x]$. Thus $p(\omega) = 0$ and $\Phi_\ell(x)$ is a monic irreducible polynomial over $\mathbb{Q}(\omega_3)$ having ω as a root. Therefore $\Phi_\ell(x)$ divides $p(x)$, and so ω^{-1} is also a root of $p(x)$. Note that, if $\psi_\alpha(s) = \omega^j$ for some $j \in \{0, 1, \ldots, \ell-1\}$ then $\psi_{-\alpha}(s) = \omega^{-j}$. We have

$$\begin{aligned}\sum_{s\in S}\mathbf{i}\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s)) &= \sum_{s\in S}[(\omega_6 - \omega_6^5)\psi_\alpha(s) + (\omega_6^5 - \omega_6)\psi_\alpha(-s)]\\ &= \mu_\alpha - \mu_{-\alpha} = \mu_\alpha - \sum_{j=0}^{\ell-1} a_j\omega^{-j} = -p(\omega^{-1}) = 0.\end{aligned}$$

By Lemma 3.2.1, $S = \emptyset$. Conversely, if $S = \emptyset$ then all the HS-eigenvalues of $\text{Cay}(\Gamma, S)$ are zero. Thus $\text{Cay}(\Gamma, S)$ is HS-integral. □

Lemma 3.2.3. *If Γ is an abelian group and $x \in \Gamma(3)$, then $\sum\limits_{s\in M_{1,3}(x)} \big(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(-s)\big)$ is an integer for each $\alpha \in \Gamma$.*

Proof. Let $x \in \Gamma(3)$, $\alpha \in \Gamma$ and $\mu_\alpha = \sum\limits_{s\in M_{1,3}(x)} \big(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(-s)\big)$.

Case 1. Assume that there exists $a \in M_{0,3}(x)$ such that $\psi_\alpha(a) \neq 1$. We have

$$\begin{aligned}\mu_\alpha = \sum_{s\in M_{1,3}(x)} \big(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(-s)\big) &= \sum_{s\in M_{1,3}(x)} \omega_6\psi_\alpha(s) + \sum_{s\in M_{2,3}(x)} \omega_6^5\psi_\alpha(s)\\ &= \sum_{s\in a+M_{1,3}(x)} \omega_6\psi_\alpha(s) + \sum_{s\in a+M_{2,3}(x)} \omega_6^5\psi_\alpha(s)\\ &= \psi_\alpha(a)\sum_{s\in M_{1,3}(x)} \omega_6\psi_\alpha(s) + \psi_\alpha(a)\sum_{s\in M_{2,3}(x)} \omega_6^5\psi_\alpha(s)\\ &= \psi_\alpha(a)\mu_\alpha.\end{aligned}$$

Thus we have $(1 - \psi_\alpha(a))\mu_\alpha = 0$. Since $\psi_\alpha(a) \neq 1$, we get $\mu_\alpha = 0 \in \mathbb{Z}$.

Case 2. Assume that $\psi_\alpha(a) = 1$ for all $a \in M_{0,3}(x)$. Let $s \in M_{1,3}(x)$, so that $s = x^{3r+1}$ for some $r \in \mathbb{Z}$. Note that we have $x^{3r} \in M_{0,3}(x)$, and so $\psi_\alpha(x^{3r}) = 1$. Therefore, we get

$\psi_\alpha(s) = \psi_\alpha(x^{3r})\psi_\alpha(x) = \psi_\alpha(x)$. Similarly, $\psi_\alpha(s) = \psi_\alpha(x^2)$ for all $s \in M_{2,3}(x)$. Thus

$$\begin{aligned}\mu_\alpha = \sum_{s\in M_{1,3}(x)} \left(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(-s)\right) &= \sum_{s\in M_{1,3}(x)} \omega_6\psi_\alpha(s) + \sum_{s\in M_{2,3}(x)} \omega_6^5\psi_\alpha(s)\\ &= |M_{1,3}(x)|(\omega_6\psi_\alpha(x) + \omega_6^5\psi_\alpha(x^2))\\ &= -|M_{1,3}(x)|(\omega_3^2\psi_\alpha(x) + \omega_3\psi_\alpha(x^2)).\end{aligned}$$

Note that $x^3 \in M_{0,3}(x)$, and so $\psi_\alpha(x^3) = 1$. Therefore $\psi_\alpha(x) = \omega_3$ or ω_3^2. If $\psi_\alpha(x) = \omega_3$, then $\mu_\alpha = -2|M_{1,3}(x)|$. If $\psi_\alpha(x) = \omega_3^2$, then $\mu_\alpha = |M_{1,3}(x)|$. In both the cases, μ_α is an integer for each $\alpha \in \Gamma$. □

For $x \in \Gamma(3)$ and $\alpha \in \Gamma$, define

$$Z_x(\alpha) := \sum_{s\in\langle\langle x\rangle\rangle} \left(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(-s)\right).$$

Lemma 3.2.4. *If Γ is an abelian group and $x \in \Gamma(3)$, then $Z_x(\alpha)$ is an integer for each $\alpha \in \Gamma$.*

Proof. Note that there exists $x \in \Gamma(3)$ with $\mathrm{ord}(x) = 3$. Apply induction on $\mathrm{ord}(x)$. If $\mathrm{ord}(x) = 3$, then $M_{1,3}(x) = \langle\langle x\rangle\rangle$. Hence by Lemma 3.2.3, $Z_x(\alpha)$ is an integer for each $\alpha \in \Gamma$. Assume that the statement holds for each $x \in \Gamma(3)$ with $\mathrm{ord}(x) \in \{3, \ldots, 3(g-1)\}$. We prove the statement for $\mathrm{ord}(x) = 3g$. Lemma 3.1.5 gives that

$$M_{1,3}(x) = \left(\bigcup_{h\in D^1_{g,3}} \langle\langle x^h\rangle\rangle\right) \cup \left(\bigcup_{h\in D^2_{g,3}} \langle\langle -x^h\rangle\rangle\right).$$

If $\mathrm{ord}(x) = 3g, h \in D^1_{g,3} \cup D^2_{g,3}$ and $h > 1$, then $\mathrm{ord}(x^h), \mathrm{ord}(-x^h) \in \{3, \ldots, 3(g-1)\}$. By induction hypothesis, $Z_{x^h}(\alpha)$ and $Z_{-x^h}(\alpha)$ are integers for all $\alpha \in \Gamma$. Now we have

$$\sum_{s\in M_{1,3}(x)} \left(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(s)\right) = Z_x(\alpha) + \sum_{h\in D^1_{g,3}, h>1} Z_{x^h}(\alpha) + \sum_{h\in D^2_{g,3}, h>1} Z_{-x^h}(\alpha).$$

This gives

$$Z_x(\alpha) = \sum_{s\in M_{1,3}(x)} \left(\omega_6\psi_\alpha(s) + \omega_6^5\psi_\alpha(s)\right) - \sum_{h\in D^1_{g,3}, h>1} Z_{x^h}(\alpha) - \sum_{h\in D^2_{g,3}, h>1} Z_{-x^h}(\alpha).$$

Hence by Lemma 3.2.3 and induction hypothesis, we find that $Z_x(\alpha)$ is an integer for each $\alpha \in \Gamma$. Thus the proof is complete by induction. □

For $\Gamma(3) \neq \emptyset$, recall that $\mathbb{E}(\Gamma)$ is the set of all skew-symmetric subsets of Γ of the form $\langle\langle x_1\rangle\rangle \cup \ldots \cup \langle\langle x_k\rangle\rangle$ for some $x_1, \ldots, x_k \in \Gamma(3)$. For $\Gamma(3) = \emptyset$, recall that $\mathbb{E}(\Gamma) = \{\emptyset\}$.

Theorem 3.2.5. *Let Γ be an abelian group. If $S \in \mathbb{E}(\Gamma)$, then the directed Cayley graph $\mathrm{Cay}(\Gamma, S)$ is HS-integral.*

Proof. If $S \in \mathbb{E}(\Gamma)$, then $S = \langle\!\langle x_1 \rangle\!\rangle \cup \cdots \cup \langle\!\langle x_k \rangle\!\rangle$ for some $x_1, \ldots, x_k \in \Gamma(3)$. We have

$$\mu_\alpha = \sum_{s \in S} \left(\omega_6 \psi_\alpha(s) + \omega_6^5 \psi_\alpha(-s) \right) = \sum_{j=1}^{k} Z_{x_j}(\alpha).$$

Now by Lemma 3.2.4, μ_α is an integer for each $\alpha \in \Gamma$. Hence the directed Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral. □

3.3 Characterization of HS-integral mixed Cayley graphs over abelian groups

Let Γ be an abelian group of order n. Define E to be the matrix $[E_{xy}]$ of size $n \times n$, whose rows and columns are indexed by the elements of Γ such that $E_{xy} := \psi_x(y)$ for each $x, y \in \Gamma$. Note that each row of E corresponds to a character of Γ and $EE^* = nI_n$. We note here that there exists $\mathbf{v} \in \mathbb{Q}(\omega_3)^n$ such that $E\mathbf{v} \in \mathbb{Q}^n$. For example, let $a \in \Gamma(3)$ and $\mathbf{u}$ be the vector in $\mathbb{Q}(\omega_3)^n$ whose coordinates are indexed by the elements of Γ, where

$$\mathbf{u}_x = \begin{cases} \omega_6 & \text{if } x \in \langle\!\langle a \rangle\!\rangle \\ \omega_6^5 & \text{if } x \in \langle\!\langle -a \rangle\!\rangle \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3.2.4, we have $E\mathbf{u} \in \mathbb{Q}^n$. For $z \in \mathbb{C}$, let $\Re(z)$ and $\Im(z)$ denote the real part and imaginary part of z, respectively.

Lemma 3.3.1. *Let Γ be an abelian group, $\mathbf{v} \in \mathbb{Q}(\omega_3)^n$, and let the coordinates of $\mathbf{v}$ be indexed by the elements of Γ. If $E\mathbf{v} \in \mathbb{Q}^n$, then*

(i) $\overline{\mathbf{v}}_x = \mathbf{v}_{-x}$ *for all* $x \in \Gamma$;

(ii) $\mathbf{v}_x = \mathbf{v}_y$ *for all* $x, y \in \Gamma(3)$ *satisfying* $x \simeq y$;

(iii) $\Re(\mathbf{v}_x) = \Re(\mathbf{v}_{-x})$ *and* $\Im(\mathbf{v}_x) = \Im(\mathbf{v}_{-x}) = 0$ *for all* $x \in \Gamma \setminus \Gamma(3)$.

Proof. Let E_x and E_y denote the columns of E indexed by x and y, respectively, and assume that $\mathbf{u} = E\mathbf{v} \in \mathbb{Q}^n$.

(i) We use the fact that $\overline{\psi_x(y)} = \psi_{-x}(y) = \psi_x(-y)$ for all $x, y \in \Gamma$. Again

$$\mathbf{u} = E\mathbf{v} \quad \text{or,} \quad E^*\mathbf{u} = E^*E\mathbf{v} = (nI_n)\mathbf{v} \quad \text{or,} \quad \frac{1}{n}E^*\mathbf{u} = \mathbf{v} \in \mathbb{Q}(\omega_3)^n.$$

Thus

$$\mathbf{v}_x = \frac{1}{n}(E^*\mathbf{u})_x = \frac{1}{n}\sum_{a\in\Gamma} E^*_{xa}\mathbf{u}_a = \frac{1}{n}\sum_{a\in\Gamma} \overline{\psi_a(x)}\mathbf{u}_a = \frac{1}{n}\sum_{a\in\Gamma} \psi_a(-x)\mathbf{u}_a$$
$$= \overline{\frac{1}{n}\sum_{a\in\Gamma} \overline{\psi_a(-x)}\mathbf{u}_a}$$
$$= \overline{\frac{1}{n}\sum_{a\in\Gamma} E^*_{-xa}\mathbf{u}_a} = \frac{1}{n}\overline{(E^*\mathbf{u})_{-x}} = \overline{\mathbf{v}}_{-x}.$$

(ii) If $\Gamma(3) = \emptyset$, then there is nothing to prove. Now assume that $\Gamma(3) \neq \emptyset$. Let $x, y \in \Gamma(3)$. If $x \simeq y$, then there exists $k \in G^1_{m,3}(1)$ such that $y = x^k$, where $m = \text{ord}(x)$. Assume $x \neq y$, so that $k \geq 2$. Using Lemma 2.1.1, we find that the entries of E_x and E_y are m^{th} roots of unity. Fix a primitive m^{th} root ω of unity, and express each entry of E_x and E_y in the form ω^j for some $j \in \{0, 1, \ldots, m-1\}$. Thus

$$n\mathbf{v}_x = (E^*\mathbf{u})_x = \sum_{j=0}^{m-1} a_j\omega^j,$$

where $a_j \in \mathbb{Q}$ for all j. Thus ω is a root of $p(x)$, where $p(x) := \sum_{j=0}^{m-1} a_jx^j - n\mathbf{v}_x \in \mathbb{Q}(\omega_3)[x]$. Therefore $p(x)$ is a multiple of the irreducible polynomial $\Phi^1_{m,3}(x)$. This, along with $k \in G^1_{m,3}(1)$, implies that ω^k is also a root of $p(x)$. As $y = x^k$ implies that $\psi_a(y) = \psi_a(x)^k$ for each $a \in \Gamma$, we have $(E^*\mathbf{u})_y = \sum_{j=0}^{m-1} a_j\omega^{kj}$. Hence

$$0 = p(\omega^k) = \sum_{j=0}^{m-1} a_j\omega^{kj} - n\mathbf{v}_x = (E^*\mathbf{u})_y - n\mathbf{v}_x = n\mathbf{v}_y - n\mathbf{v}_x.$$

This gives $\mathbf{v}_x = \mathbf{v}_y$.

(iii) Let $x \in \Gamma \setminus \Gamma(3)$ and $r = \text{ord}(x) \not\equiv 0 \pmod 3$. Fix a primitive r^{th} root ω of unity, and express each entry of E_x in the form ω^j for some $j \in \{0, 1, \ldots, r-1\}$. Thus

$$n\mathbf{v}_x = (E^*\mathbf{u})_x = \sum_{j=0}^{r-1} a_j\omega^j,$$

where $a_j \in \mathbb{Q}$ for all j. Thus ω is a root of $p(x)$, where $p(x) := \sum_{j=0}^{r-1} a_jx^j - n\mathbf{v}_x \in \mathbb{Q}(\omega_3)[x]$. Therefore $p(x)$ is a multiple of the irreducible polynomial $\Phi_r(x)$, and so ω^{-1} is also a root of $p(x)$. Since $\psi_a(-x) = \psi_a(x)^{-1}$ for each $a \in \Gamma$, we have $(E^*\mathbf{u})_{-x} = \sum_{j=0}^{r-1} a_j\omega^{-j}$. Hence

$$0 = p(\omega^{-1}) = \sum_{j=0}^{r-1} a_j\omega^{-j} - n\mathbf{v}_x = (E^*\mathbf{u})_{-x} - n\mathbf{v}_x = n\mathbf{v}_{-x} - n\mathbf{v}_x.$$

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Therefore $\mathbf{v}_x = \mathbf{v}_{-x}$. This, together with Part (i), implies that $\Re(\mathbf{v}_x) = \Re(\mathbf{v}_{-x})$, and that $\Im(\mathbf{v}_x) = \Im(\mathbf{v}_{-x}) = 0$ for all $x \in \Gamma \setminus \Gamma(3)$. □

Theorem 3.3.2. *If* Γ *is an abelian group, then the directed Cayley graph* $\text{Cay}(\Gamma, S)$ *is HS-integral if and only if* $S \in \mathbb{E}(\Gamma)$.

Proof. Assume that the directed Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral. If $\Gamma(3) = \emptyset$, then by Theorem 3.2.2 we have $S = \emptyset$, and so $S \in \mathbb{E}(\Gamma)$. Now assume that $\Gamma(3) \neq \emptyset$. Let $\mathbf{v}$ be the vector in $\mathbb{Q}(\omega_3)^n$ whose coordinates are indexed by the elements of Γ, where

$$\mathbf{v}_x = \begin{cases} \omega_6 & \text{if } x \in S \\ \omega_6^5 & \text{if } x \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(E\mathbf{v})_a = \sum_{x\in\Gamma} E_{ax}\mathbf{v}_x = \sum_{x\in S} \omega_6 E_{ax} + \sum_{x\in S^{-1}} \omega_6^5 E_{ax} = \sum_{x\in S} \left(\omega_6\psi_a(x) + \omega_6^5\psi_a(-x)\right).$$

Thus $(E\mathbf{v})_a$ is an HS-eigenvalue of the HS-integral directed Cayley graph $\text{Cay}(\Gamma, S)$ for each $a \in \Gamma$. Therefore $E\mathbf{v} \in \mathbb{Q}^n$, and hence all the three conditions of Lemma 3.3.1 are satisfied for $\mathbf{v}$.

By the third condition of Lemma 3.3.1, $\mathbf{v}_x = 0$ for all $x \in \Gamma \setminus \Gamma(3)$, and so we must have $S \cup S^{-1} \subseteq \Gamma(3)$. Again, let $x \in S$, $y \in \Gamma(3)$ and $x \simeq y$. The second condition of Lemma 3.3.1 gives $\mathbf{v}_x = \mathbf{v}_y$, which implies that $y \in S$. Thus $x \in S$ implies $\langle\langle x \rangle\rangle \subseteq S$. Hence $S \in \mathbb{E}(\Gamma)$. The converse part follows from Theorem 3.2.5. □

The following example illustrates Theorem 3.3.2.

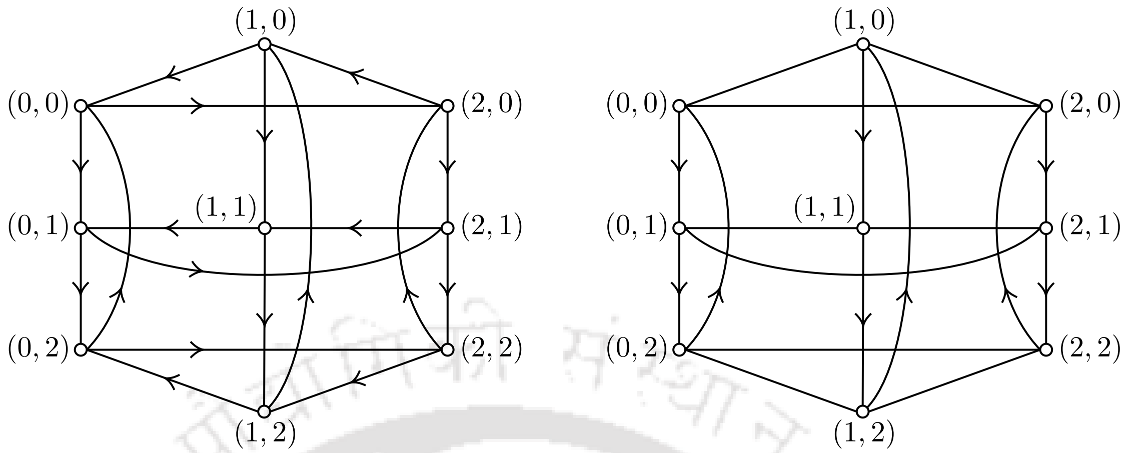
Example 3.3.1. Consider the abelian group $\mathbb{Z}_3 \otimes \mathbb{Z}_3$ and $S = \{(0,1),(2,0)\}$. The directed Cayley graph $\text{Cay}(\mathbb{Z}_3 \otimes \mathbb{Z}_3, S)$ is shown in Figure 3.1a. We see that $\langle\langle(0,1)\rangle\rangle = \{(0,1)\}$ and $\langle\langle(2,0)\rangle\rangle = \{(2,0)\}$. Therefore $S \in \mathbb{E}(\Gamma)$. Further, using Corollary 3.1.2 and Equation (2.1), the HS-eigenvalues of $\text{Cay}(\mathbb{Z}_3 \otimes \mathbb{Z}_3, S)$ are obtained as

$$\mu_\alpha = [\omega_6\psi_\alpha(0,1) + \omega_6^5\psi_\alpha(0,2)] + [\omega_6\psi_\alpha(2,0) + \omega_6^5\psi_\alpha(1,0)] \text{ for each } \alpha \in \mathbb{Z}_3 \otimes \mathbb{Z}_3,$$

where

$$\psi_\alpha(x) = \omega_3^{\alpha_1 x_1}\omega_3^{\alpha_2 x_2} \text{ for each } \alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_3 \otimes \mathbb{Z}_3.$$

It can be seen that $\mu_{(0,0)} = 2, \mu_{(0,1)} = -1, \mu_{(0,2)} = 2, \mu_{(1,0)} = 2, \mu_{(1,1)} = -1, \mu_{(1,2)} = 2$, $\mu_{(2,0)} = -1$, $\mu_{(2,1)} = -4$ and $\mu_{(2,2)} = -1$. Thus $\text{Cay}(\mathbb{Z}_3 \otimes \mathbb{Z}_3, S)$ is HS-integral.



(a) $S = \{(0,1),(2,0)\}$ (b) $S = \{(0,1),(1,0),(2,0)\}$

Figure 3.1: The mixed graph $\text{Cay}(\mathbb{Z}_3 \otimes \mathbb{Z}_3, S)$

Lemma 3.3.3. *Let S be a skew-symmetric subset of an abelian group Γ and $t(\neq 0) \in \mathbb{Q}$. If*

$$\sum_{s\in S} \mathbf{i}t\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))$$

is an integer for each $\alpha \in \Gamma$, then $S \in \mathbb{E}(\Gamma)$.

Proof. Let the vector $\mathbf{v}$, whose coordinates are indexed by the elements of Γ, be defined by

$$\mathbf{v}_x = \begin{cases} \mathbf{i}t\sqrt{3} & \text{if } x \in S \\ -\mathbf{i}t\sqrt{3} & \text{if } x \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{v} \in \mathbb{Q}(\omega_3)^n$ and α-th coordinate of $E\mathbf{v}$ is $\sum_{s\in S} \mathbf{i}t\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))$, we get $E\mathbf{v} \in \mathbb{Q}^n$. Therefore, all the three condition of Lemma 3.3.1 are satisfied for $\mathbf{v}$. By the third condition of Lemma 3.3.1, $\Im(\mathbf{v}_x) = 0$, and so $\mathbf{v}_x = 0$ for all $x \in \Gamma \setminus \Gamma(3)$. Thus we must have $S \cup S^{-1} \subseteq \Gamma(3)$. Again, let $x \in S$, $y \in \Gamma(3)$ and $x \simeq y$. The second condition of Lemma 3.3.1 gives $\mathbf{v}_x = \mathbf{v}_y$, which implies that $y \in S$. Thus $x \in S$ implies $\langle\langle x \rangle\rangle \subseteq S$. Hence $S \in \mathbb{E}(\Gamma)$. □

Lemma 3.3.4. *Let S be a skew-symmetric subset of an abelian group Γ and $t(\neq 0) \in \mathbb{Q}$. If*

$$\sum_{s\in S} \mathbf{i}t\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))$$

is an integer for each $\alpha \in \Gamma$, then $\sum_{s \in S \cup S^{-1}} \psi_\alpha(s)$ is an integer for each $\alpha \in \Gamma$.

Proof. Assume that $\sum_{s\in S} \mathbf{i}t\sqrt{3}(\psi_\alpha(s) - \psi_\alpha(-s))$ is an integer for each $\alpha \in \Gamma$. By Lemma 3.3.3, we have $S \in \mathbb{E}(\Gamma)$, and so $S = \langle\langle x_1 \rangle\rangle \cup \cdots \cup \langle\langle x_k \rangle\rangle$ for some $x_1, \ldots, x_k \in \Gamma(3)$. Therefore using

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Lemma 3.1.4, we get $S\cup S^{-1}=[x_1]\cup\cdots\cup[x_k]\in\mathbb{B}(\Gamma)$. Thus by Theorem 1.6.6, $\text{Cay}(\Gamma,S\cup S^{-1})$ is integral, that is, $\sum\limits_{s\in S\cup S^{-1}}\psi_\alpha(s)$ is an integer for each $\alpha\in\Gamma$. □

Lemma 3.3.5. *If* Γ *is an abelian group, then the mixed Cayley graph* $\text{Cay}(\Gamma,S)$ *is HS-integral if and only if* $\text{Cay}(\Gamma,S\setminus\overline{S})$ *is integral and* $\text{Cay}(\Gamma,\overline{S})$ *is HS-integral.*

Proof. Assume that the mixed Cayley graph $\text{Cay}(\Gamma,S)$ is HS-integral. Let the HS-spectrum of $\text{Cay}(\Gamma,S)$ be $\{\gamma_\alpha\colon\alpha\in\Gamma\}$, where $\gamma_\alpha=\lambda_\alpha+\mu_\alpha$,

$$\lambda_\alpha=\sum_{s\in S\setminus\overline{S}}\psi_\alpha(s)\text{ and }\mu_\alpha=\sum_{s\in\overline{S}}(\omega_6\psi_\alpha(s)+\omega_6^5\psi_\alpha(-s))\text{ for each }\alpha\in\Gamma.$$

Note that $\{\lambda_\alpha\colon\alpha\in\Gamma\}$ is the spectrum of $\text{Cay}(\Gamma,S\setminus\overline{S})$ and $\{\mu_\alpha\colon\alpha\in\Gamma\}$ is the HS-spectrum of $\text{Cay}(\Gamma,\overline{S})$. By assumption $\gamma_\alpha\in\mathbb{Z}$, and so $\sum\limits_{s\in\overline{S}}\mathbf{i}\sqrt{3}(\psi_\alpha(s)-\psi_\alpha(-s))=\gamma_\alpha-\gamma_{-\alpha}\in\mathbb{Z}$ for all $\alpha\in\Gamma$. By Lemma 3.3.4, we get $\sum\limits_{s\in\overline{S}\cup\overline{S}^{-1}}\psi_\alpha(s)\in\mathbb{Z}$ for all $\alpha\in\Gamma$. Note that μ_α, being a sum of products of algebraic integers, is an algebraic integer. Also,

$$\mu_\alpha=\frac{1}{2}\sum_{s\in\overline{S}\cup\overline{S}^{-1}}\psi_\alpha(s)+\frac{1}{2}\sum_{s\in\overline{S}}\mathbf{i}\sqrt{3}(\psi_\alpha(s)-\psi_\alpha(-s))\in\mathbb{Q}.$$

Hence μ_α is an integer for each $\alpha\in\Gamma$. Thus $\text{Cay}(\Gamma,\overline{S})$ is HS-integral. Now we have $\gamma_\alpha,\mu_\alpha\in\mathbb{Z}$, and so $\lambda_\alpha=\gamma_\alpha-\mu_\alpha\in\mathbb{Z}$ for each $\alpha\in\Gamma$. Hence $\text{Cay}(\Gamma,S\setminus\overline{S})$ is also integral.

Conversely, assume that $\text{Cay}(\Gamma,S\setminus\overline{S})$ is integral and $\text{Cay}(\Gamma,\overline{S})$ is HS-integral. Then Lemma 3.1.1 implies that $\text{Cay}(\Gamma,S)$ is HS-integral. □

Theorem 3.3.6. *If* Γ *is an abelian group, then the mixed Cayley graph* $\text{Cay}(\Gamma,S)$ *is HS-integral if and only if* $S\setminus\overline{S}\in\mathbb{B}(\Gamma)$ *and* $\overline{S}\in\mathbb{E}(\Gamma)$.

Proof. By Lemma 3.3.5, the mixed Cayley graph $\text{Cay}(\Gamma,S)$ is HS-integral if and only if $\text{Cay}(\Gamma,S\setminus\overline{S})$ is integral and $\text{Cay}(\Gamma,\overline{S})$ is HS-integral. Note that $S\setminus\overline{S}$ is a symmetric set and $\overline{S}$ is a skew-symmetric set. Thus by Theorem 1.6.6, $\text{Cay}(\Gamma,S\setminus\overline{S})$ is integral if and only if $S\setminus\overline{S}\in\mathbb{B}(\Gamma)$. By Theorem 3.3.2, $\text{Cay}(\Gamma,\overline{S})$ is HS-integral if and only if $\overline{S}\in\mathbb{E}(\Gamma)$. Hence the result follows. □

The following example illustrates Theorem 3.3.6.

Example 3.3.2. Consider $\Gamma=\mathbb{Z}_3\otimes\mathbb{Z}_3$ and $S=\{(0,1),(1,0),(2,0)\}$. The mixed Cayley graph $\text{Cay}(\mathbb{Z}_3\otimes\mathbb{Z}_3,S)$ is shown in Figure 3.1b. Here $\overline{S}=\{(0,1)\}=\langle\!\langle(0,1)\rangle\!\rangle\in\mathbb{E}(\Gamma)$ and $S\setminus\overline{S}=\{(1,0),(2,0)\}=[(1,0)]\in\mathbb{B}(\Gamma)$. Further, using Lemma 3.1.1 and Equation (2.1), the HS-eigenvalues of $\text{Cay}(\mathbb{Z}_3\otimes\mathbb{Z}_3,S)$ are obtained as

$$\gamma_\alpha=[\psi_\alpha(1,0)+\psi_\alpha(2,0)]+[\omega_6\psi_\alpha(0,1)+\omega_6^5\psi_\alpha(0,2)]\ \text{ for each }\alpha\in\mathbb{Z}_3\otimes\mathbb{Z}_3.$$

We find $\gamma_{(0,1)}=\gamma_{(1,0)}=\gamma_{(1,2)}=\gamma_{(2,0)}=\gamma_{(2,2)}=0$, $\gamma_{(0,0)}=\gamma_{(0,2)}=3$ and $\gamma_{(1,1)}=\gamma_{(2,1)}=-3$. Thus $\text{Cay}(\mathbb{Z}_3\otimes\mathbb{Z}_3,S)$ is HS-integral.

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4

H-integral and Gaussian integral normal mixed Cayley graphs

Recall that a mixed Cayley graph $\text{Cay}(\Gamma, S)$ is called normal if S is a union of some conjugacy classes of Γ. The class of mixed Cayley graphs over an abelian group is a subclass of the class of normal mixed Cayley graphs. In this chapter, we extend Theorem 2.3.4 to normal mixed Cayley graphs. Consequently, some definitions and results of this chapter have similarities with those in Chapter 2. Xu et al. [46] and Li [30] presented a characterization of Gaussian integral mixed circulant graphs. We find a similar characterization of Gaussian integral normal mixed Cayley graphs.

4.1 Preliminaries

In this section, we determine the H-eigenvalues of a normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ in terms of the irreducible characters of Γ. Finally, we show that the normal mixed Cayley graph is H-integral if and only if each of its directed and undirected portions are H-integral.

Lemma 4.1.1. *Let Γ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then the H-spectrum of the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\gamma_1]^{d_1^2}, \ldots, [\gamma_h]^{d_h^2}\}$, where $\gamma_j = \lambda_j + \mu_j$,*

$$\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \overline{S}} \chi_j(s), \;\; \mu_j = \frac{\mathbf{i}}{\chi_j(1)} \sum_{s \in \overline{S}} (\chi_j(s) - \chi_j(s^{-1})),$$

and $d_j = \chi_j(\mathbf{1})$ for each $j \in \{1, \ldots, h\}$.

Proof. Let $f \colon \Gamma \to \{0, 1, \mathbf{i}, -\mathbf{i}\}$ be the function such that

$$f(s) = \begin{cases} 1 & \text{if } s \in S \setminus \overline{S} \\ \mathbf{i} & \text{if } s \in \overline{S} \\ -\mathbf{i} & \text{if } s \in (\overline{S})^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

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Since S is a union of some conjugacy classes of Γ, f is a class function. The Hermitian adjacency matrix of $\text{Cay}(\Gamma, S)$ is equal to $[f(yx^{-1})]_{x,y\in\Gamma}$. By Theorem 1.3.6,

$$\gamma_j = \frac{1}{\chi_j(1)}\Bigg(\sum_{s\in S\setminus\overline{S}} \chi_j(s) + \sum_{s\in\overline{S}} \mathbf{i}\chi_j(s) + \sum_{s\in\overline{S}^{-1}} (-\mathbf{i})\chi_j(s)\Bigg),$$

and the result follows. □

As special cases of Lemma 4.1.1, we have the following two corollaries.

Corollary 4.1.2. *Let Γ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \dots, \chi_h\}$, then the H-spectrum (or spectrum) of the normal undirected Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\lambda_1]^{d_1^2}, \dots, [\lambda_h]^{d_h^2}\}$, where*

$$\lambda_j = \frac{1}{\chi_j(1)}\sum_{s\in S} \chi_j(s) \text{ and } d_j = \chi_j(\mathbf{1}) \quad \text{for each } j \in \{1, \dots, h\}.$$

Corollary 4.1.3. *Let Γ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \dots, \chi_h\}$, then the H-spectrum of the normal directed Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\mu_1]^{d_1^2}, \dots, [\mu_h]^{d_h^2}\}$, where*

$$\mu_j = \frac{\mathbf{i}}{\chi_j(1)}\sum_{s\in S}(\chi_j(s) - \chi_j(s^{-1})) \text{ and } d_j = \chi_j(\mathbf{1}) \quad \text{for each } j \in \{1, \dots, h\}.$$

Lemma 4.1.4. *If Γ is a finite group, then the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral if and only if $\text{Cay}(\Gamma, S \setminus \overline{S})$ is integral (or H-integral) and $\text{Cay}(\Gamma, \overline{S})$ is H-integral.*

Proof. Let $\text{Irr}(\Gamma) = \{\chi_1, \dots, \chi_h\}$ and γ_j be an H-eigenvalue of the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$. By Lemma 4.1.1, we have $\gamma_j = \lambda_j + \mu_j$, where

$$\lambda_j = \frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S\setminus\overline{S}} \chi_j(s) \text{ and } \mu_j = \frac{\mathbf{i}}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}(\chi_j(s) - \chi_j(s^{-1}))$$

for each $j \in \{1, \dots, h\}$. Assume that $\text{Cay}(\Gamma, S)$ is H-integral and $j \in \{1, \dots, h\}$. By Lemma 1.3.4, there exists $k \in \{1, \dots, h\}$ such that $\overline{\chi_k} = \chi_j$. If $k = j$, then we find that $\mu_j = 0$ and $\lambda_j = \gamma_j$. Thus λ_j and μ_j are integers. For $k \neq j$, we have

$$\gamma_k = \overline{\gamma}_k = \overline{\lambda}_k + \overline{\mu}_k = \lambda_j - \mu_j.$$

By assumption, γ_j and γ_k are integers. As $\gamma_j = \lambda_j + \mu_j$ and $\gamma_k = \lambda_j - \mu_j$, we get $\lambda_j = \frac{\gamma_j+\gamma_k}{2}$ and $\mu_j = \frac{\gamma_j-\gamma_k}{2}$. Thus λ_j and μ_j are rational algebraic integers, and so they are integers. Hence by Corollaries 4.1.2 and 4.1.3, $\text{Cay}(\Gamma, S \setminus \overline{S})$ is integral and $\text{Cay}(\Gamma, \overline{S})$ is H-integral.

Conversely, assume that $\text{Cay}(\Gamma, S \setminus \overline{S})$ is integral and $\text{Cay}(\Gamma, \overline{S})$ is H-integral. Using Lemma 4.1.1, $\text{Cay}(\Gamma, S)$ is H-integral. □

4.2 H-integral normal mixed Cayley graphs

Let the order of the group Γ be n and $\chi \in \text{Irr}(\Gamma)$ be of degree d. Let $x \in \mathbb{Q}(\mathbf{i})\Gamma$ be such that $x = \sum\limits_{g\in\Gamma} \mathbf{i}c_g g$, where $c_g \in \mathbb{Z}$ for all $g \in \Gamma$. Define $\chi(x) := \sum\limits_{g\in\Gamma} \mathbf{i}c_g\chi(g)$. Note that $\chi(g)^n = \chi(g^n) = \chi(\mathbf{1}) = d$, and so $\chi(g)$ is an algebraic integer for each $g \in \Gamma$. Therefore $\mathbf{i}c_g\chi(g)$ is an algebraic integer for each $g \in \Gamma$, and hence $\chi(x)$ is an algebraic integer.

Let $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Let E be the matrix $[E_{jg}]$ of size $h \times n$, whose rows are indexed by $1, \ldots, h$ and columns are indexed by the elements of Γ such that $E_{jg} = \chi_j(g)$. Note that $EE^* = nI_h$ and the rank of E is h, where E^* is the conjugate transpose of E and I_h is the $h \times h$ identity matrix.

Let $\text{Gal}(\mathbb{K}/\mathbb{F})$ denote the Galois group of an extension $\mathbb{K}$ over the field $\mathbb{F}$. It is well known that $\text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) = \{\sigma_r : r \in G_m(1), \sigma_r(\omega_m) = \omega_m^r\}$. For example, see Section 14.5 in [17]. If $m \equiv 0 \pmod 4$, then $\mathbb{Q}(\mathbf{i}, \omega_m) = \mathbb{Q}(\omega_m)$. Therefore, $\text{Gal}(\mathbb{Q}(\mathbf{i}, \omega_m)/\mathbb{Q}(\mathbf{i}))$ is a subgroup of $\text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q})$. Thus $\text{Gal}(\mathbb{Q}(\mathbf{i}, \omega_m)/\mathbb{Q}(\mathbf{i}))$ contains those automorphisms in $\text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q})$ that fix $\mathbf{i}$. Recall that $G_m(1) = G_m^1(1) \cup G_m^3(1)$ and $G_m^1(1) \cap G_m^3(1) = \emptyset$. If $r \in G_m^1(1)$ then $\sigma_r(\mathbf{i}) = \mathbf{i}$, and if $r \in G_m^3(1)$ then $\sigma_r(\mathbf{i}) = -\mathbf{i}$. Thus

$$\text{Gal}(\mathbb{Q}(\mathbf{i}, \omega_m)/\mathbb{Q}(\mathbf{i})) = \text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}(\mathbf{i})) = \{\sigma_r : r \in G_m^1(1), \sigma_r(\omega_m) = \omega_m^r\}.$$

If $m \not\equiv 0 \pmod 4$, then $[\mathbb{Q}(\mathbf{i}, \omega_m) : \mathbb{Q}(\mathbf{i})] = \varphi(m)$. Thus the field $\mathbb{Q}(\mathbf{i}, \omega_m)$ is a Galois extension of $\mathbb{Q}(\mathbf{i})$ of degree $\varphi(m)$. Any automorphism of the field $\mathbb{Q}(\mathbf{i}, \omega_m)$ is uniquely determined by its action on ω_m. Hence $\text{Gal}(\mathbb{Q}(\mathbf{i}, \omega_m)/\mathbb{Q}(\mathbf{i})) = \{\tau_r : r \in G_m(1), \tau_r(\omega_m) = \omega_m^r \text{ and } \tau_r(\mathbf{i}) = \mathbf{i}\}$.

Let $g \in \Gamma$, $m = \text{ord}(g)$, and χ be a character of Γ. By Theorem 1.3.3, $\chi(g) = \sum_{i=1}^k \epsilon_i$, where $\epsilon_1, \ldots, \epsilon_k$ are some m-th roots of unity. If $m \equiv 0 \pmod 4$ and $\sigma_r \in \text{Gal}(\mathbb{Q}(\mathbf{i}, \omega_m)/\mathbb{Q}(\mathbf{i}))$, then

$$\sigma_r(\chi(g)) = \sigma_r\left(\sum_{i=1}^k \epsilon_i\right) = \sum_{i=1}^k \sigma_r(\epsilon_i) = \sum_{i=1}^k \epsilon_i^r = \chi(g^r).$$

Similarly, if $m \not\equiv 0 \pmod 4$ and $\tau_r \in \text{Gal}(\mathbb{Q}(\mathbf{i}, \omega_m)/\mathbb{Q}(\mathbf{i}))$, then also $\tau_r(\chi(g)) = \chi(g^r)$.

Theorem 4.2.1. *Let Γ be a finite group and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. If $x = \sum\limits_{g\in\Gamma} \mathbf{i}c_g g$, where $c_g \in \mathbb{Z}$ for all $g \in \Gamma$, then $\chi_j(x)$ is an integer for each $j \in \{1, \ldots, h\}$ if and only if the following conditions hold:*

(i) $\sum\limits_{s\in \text{Cl}(g_1)} c_s = \sum\limits_{s\in \text{Cl}(g_2)} c_s$ *for each $g_1, g_2 \in \Gamma(4)$ and $g_1 \approx g_2$;*

(ii) $\sum\limits_{s\in \text{Cl}(g)} c_s = -\sum\limits_{s\in \text{Cl}(g^{-1})} c_s$ *for each $g \in \Gamma$;*

(iii) $\sum\limits_{s\in \text{Cl}(g)} c_s = 0$ *for all $g \in \Gamma \setminus \Gamma(4)$.*

Proof. Let L be a set of representatives of the conjugacy classes in Γ. Since characters are class functions, we have

$$\chi_j(x) = \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)} \mathbf{i}c_s\Big)\chi_j(g) \text{ for each } j\in\{1,\dots,h\}. \tag{4.1}$$

Assume that $\chi_j(x)$ is an integer for each $j\in\{1,\dots,h\}$. Let $g_1,g_2\in\Gamma(4)$, $g_1\approx g_2$ and $m=\mathrm{ord}(g_1)$. Therefore, there is $r\in G_m^1(1)$ and $\sigma_r\in \mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}(\mathbf{i}))$ such that $g_2=g_1^r$ and $\sigma_r(\omega_m)=\omega_m^r$. Note that $\sigma_r(\chi_j(g_1))=\chi_j(g_1^r)$ for each $j\in\{1,\dots,h\}$. For $t\in\Gamma$, let $\theta_t=\sum\limits_{j=1}^{h}\chi_j(t)\overline{\chi}_j$, where $\overline{\chi}_j(g)=\overline{\chi_j(g)}$ for each $g\in\Gamma$. By Theorem 1.3.5, we have

$$\theta_t(u)=\begin{cases} |C_\Gamma(t)| & \text{if } u \text{ and } t \text{ are conjugates to each other}\\ 0 & \text{otherwise.}\end{cases}$$

So $\theta_t(x)=|C_\Gamma(t)|\sum\limits_{s\in \mathrm{Cl}(t)}\mathbf{i}c_s\in\mathbb{Q}(\mathbf{i})$, and it gives that $\sigma_r(\theta_t(x))=\theta_t(x)$. Since $\chi_j(x)$ is assumed to be an integer, we have $\sigma_r(\chi_j(x))=\chi_j(x)$ for each $j\in\{1,\dots,h\}$. Thus

$$\begin{aligned}|C_\Gamma(g_1)|\sum_{s\in \mathrm{Cl}(g_1)}\mathbf{i}c_s=\theta_{g_1}(x)=\sigma_r(\theta_{g_1}(x)) &= \sum_{j=1}^{h}\sigma_r(\chi_j(g_1))\sigma_r(\overline{\chi}_j(x))\\ &= \sum_{j=1}^{h}\chi_j(g_1^r)\overline{\chi}_j(x)\\ &= \theta_{g_1^r}(x)=\theta_{g_2}(x)=|C_\Gamma(g_2)|\sum_{s\in \mathrm{Cl}(g_2)}\mathbf{i}c_s. \end{aligned}\tag{4.2}$$

Since $g_1\approx g_2$, we have $C_\Gamma(g_1)=C_\Gamma(g_2)$. So Equation (4.2) implies that $\sum\limits_{s\in \mathrm{Cl}(g_1)}c_s=\sum\limits_{s\in \mathrm{Cl}(g_2)}c_s$. Hence condition (i) holds. Again

$$\begin{aligned}0=\chi_j(x)-\overline{\chi_j(x)} &= \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)}\mathbf{i}c_s\Big)\chi_j(g)-\sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)}-\mathbf{i}c_s\Big)\overline{\chi_j(g)}\\ &= \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)}\mathbf{i}c_s\Big)\chi_j(g)+\sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)}\mathbf{i}c_s\Big)\chi_j(g^{-1})\\ &= \sum_{g\in L}\mathbf{i}\Big(\sum_{s\in \mathrm{Cl}(g)}c_s+\sum_{s\in \mathrm{Cl}(g^{-1})}c_s\Big)\chi_j(g),\end{aligned}$$

and so

$$\sum_{g\in L}\mathbf{i}\Big(\sum_{s\in \mathrm{Cl}(g)}c_s+\sum_{s\in \mathrm{Cl}(g^{-1})}c_s\Big)\begin{bmatrix}\chi_1(g)\\ \vdots\\ \chi_h(g)\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}. \tag{4.3}$$

Note that the number of irreducible characters of Γ is equal to the number of conjugacy classes of Γ, that is, $|L| = h$. Since characters are class functions and the rank of E is h, the columns of E corresponding to the elements of L are linearly independent. Thus by Equation (4.3),

$$\sum_{s\in \mathrm{Cl}(g)} c_s + \sum_{s\in \mathrm{Cl}(g^{-1})} c_s = 0$$

for all $g \in L$, and so condition (ii) holds.

Let $g \in \Gamma \setminus \Gamma(4)$ and $m = \mathrm{ord}(g)$. Then there exists $\tau_{m-1} \in \mathrm{Gal}(\mathbb{Q}(\mathbf{i}, \omega_m)/\mathbb{Q}(\mathbf{i}))$ such that $\tau_{m-1}(\omega_m) = \omega_m^{m-1}$. Note that $\tau_{m-1}(\chi_j(g)) = \chi_j(g^{m-1})$ for each $j \in \{1, \ldots, h\}$. Now

$$\begin{aligned} |C_\Gamma(g)| \sum_{s\in \mathrm{Cl}(g)} \mathbf{i}c_s = \theta_g(x) &= \tau_{m-1}(\theta_g(x)) \\ &= \sum_{j=1}^{h} \tau_{m-1}(\chi_j(g))\tau_{m-1}(\overline{\chi}_j(x)) \\ &= \sum_{j=1}^{h} \chi_j(g^{m-1})\overline{\chi}_j(x) \\ &= \theta_{g^{m-1}}(x) = \theta_{g^{-1}}(x) = |C_\Gamma(g^{-1})| \sum_{s\in \mathrm{Cl}(g^{-1})} \mathbf{i}c_s. \end{aligned} \tag{4.4}$$

Since $C_\Gamma(g) = C_\Gamma(g^{-1})$, Equation (4.4) implies that $\sum\limits_{s\in \mathrm{Cl}(g)} c_s = \sum\limits_{s\in \mathrm{Cl}(g^{-1})} c_s$. This, together with condition (ii), gives $\sum\limits_{s\in \mathrm{Cl}(g)} c_s = 0$ for all $g \in \Gamma \setminus \Gamma(4)$. Hence condition (iii) also holds.

Conversely, assume that all the three conditions of the theorem hold. Let n be the order of Γ. If $n \not\equiv 0 \pmod 4$ then $\Gamma(4) = \emptyset$. Therefore by condition (iii) and Equation (4.1), we have $\chi_j(x) = 0$. Thus, $\chi_j(x)$ is an integer for each $j \in \{1, \ldots, h\}$. Now assume that $n \equiv 0 \pmod 4$. Let $L(4)$ be a set of representatives of the conjugacy classes of $\Gamma(4)$. Since characters are class functions, using condition (iii) we have

$$\chi_j(x) = \sum_{g\in L(4)} \left(\sum_{s\in \mathrm{Cl}(g)} \mathbf{i}c_s \right) \chi_j(g) \text{ for each } j \in \{1, \ldots, h\}. \tag{4.5}$$

Let $\sigma_k \in \mathrm{Gal}(\mathbb{Q}(\mathbf{i}, \omega_n)/\mathbb{Q}(\mathbf{i}))$. Therefore $\sigma_k(\omega_n) = \omega_n^k$ and $k \in G_n^1(1)$. Thus

$$\begin{aligned} \sigma_k(\chi_j(x)) &= \sum_{g\in L(4)} \left(\sum_{s\in \mathrm{Cl}(g)} \mathbf{i}c_s \right) \sigma_k(\chi_j(g)) \\ &= \sum_{g\in L(4)} \left(\sum_{s\in \mathrm{Cl}(g)} \mathbf{i}c_s \right) \chi_j(g^k). \end{aligned} \tag{4.6}$$

Since $g \approx g^k$, by condition (i) we have $\sum\limits_{s\in \mathrm{Cl}(g)} c_s = \sum\limits_{s\in \mathrm{Cl}(g^k)} c_s$. From Equation (4.6), we get

$$\sigma_k(\chi_j(x)) = \sum_{g\in L(4)} \left(\sum_{s\in \mathrm{Cl}(g^k)} \mathbf{i}c_s \right) \chi_j(g^k) = \chi_j(x). \tag{4.7}$$

The second equality in Equation (4.7) holds, because $\{g^k \colon g \in L(4)\}$ is also a set of representatives of conjugacy classes of $\Gamma(4)$. Now $\sigma_k(\chi_j(x)) = \chi_j(x)$ for each $k \in G_n^1(1)$, and so $\chi_j(x) \in \mathbb{Q}(\mathbf{i})$. Taking complex conjugates in Equation (4.5), we have

$$\begin{aligned}\overline{\chi_j(x)} = \sum_{g\in L(4)} \Bigg(\sum_{s\in \mathrm{Cl}(g)} -\mathbf{i}c_s \Bigg) \overline{\chi_j(g)} &= \sum_{g\in L(4)} \Bigg(\sum_{s\in \mathrm{Cl}(g)} -\mathbf{i}c_s \Bigg) \chi_j(g^{-1}) \\ &= \sum_{g\in L(4)} \Bigg(\sum_{s\in \mathrm{Cl}(g^{-1})} \mathbf{i}c_s \Bigg) \chi_j(g^{-1}) \\ &= \chi_j(x). \end{aligned} \tag{4.8}$$

Thus Equation (4.8) implies that $\chi_j(x) \in \mathbb{Q}$. As $\chi_j(x)$ is a rational algebraic integer, it must be an integer for each $j \in \{1, \ldots, h\}$. □

For $\Gamma(4) \neq \emptyset$, recall that $\mathbb{D}(\Gamma)$ is the class of all skew-symmetric subsets S of Γ, where $S = [\![x_1]\!] \cup \cdots \cup [\![x_k]\!]$ for some $x_1, \ldots, x_k \in \Gamma(4)$. For $\Gamma(4) = \emptyset$, recall that $\mathbb{D}(\Gamma) = \{\emptyset\}$. Indeed, we can replace condition (i) of Theorem 4.2.1 by $\sum\limits_{s\in \mathrm{Cl}(x)} c_s = \sum\limits_{s\in \mathrm{Cl}(y)} c_s$ for all $x, y \in [\![g]\!]$ and $g \in \Gamma(4)$.

Theorem 4.2.2. *Let Γ be a finite group and $\mathrm{Cay}(\Gamma, S)$ be a normal directed Cayley graph. Then $\mathrm{Cay}(\Gamma, S)$ is H-integral if and only if $S \in \mathbb{D}(\Gamma)$.*

Proof. Let $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$ and $x = \sum\limits_{g\in\Gamma} \mathbf{i}c_g g$, where

$$c_g = \begin{cases} 1 & \text{if } g \in S \\ -1 & \text{if } g \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\chi_j(x) = \sum\limits_{s\in S} \mathbf{i}(\chi_j(s) - \chi_j(s^{-1}))$, and so $\frac{\chi_j(x)}{\chi_j(\mathbf{1})}$ is an H-eigenvalue of $\mathrm{Cay}(\Gamma, S)$. Assume that the normal directed Cayley graph $\mathrm{Cay}(\Gamma, S)$ is H-integral. Thus $\chi_j(x)$ is an integer for each $j \in \{1, \ldots, h\}$, and therefore all the three conditions of Theorem 4.2.1 are satisfied for x. By the third condition of Theorem 4.2.1, we get $\sum\limits_{s\in \mathrm{Cl}(g)} c_s = 0$ for all $g \in \Gamma \backslash \Gamma(4)$. Note that S is a union of some conjugacy classes of Γ. Therefore, if $g \in S$ then $\mathrm{Cl}(g) \subseteq S$, and so by the definition of c_g we get $\sum\limits_{s\in \mathrm{Cl}(g)} c_s = |\mathrm{Cl}(g)| \neq 0$. Thus $S \cap (\Gamma \setminus \Gamma(4)) = \emptyset$, that is, $S \subseteq \Gamma(4)$. Again, let $g_1 \in S$, $g_2 \in \Gamma(4)$ and $g_1 \approx g_2$. By the first condition of Theorem 4.2.1, we get $0 < \sum\limits_{s\in \mathrm{Cl}(g_1)} c_s = \sum\limits_{s\in \mathrm{Cl}(g_2)} c_s$, which implies that $g_2 \in S$. Thus $g_1 \in S$ gives $[\![g_1]\!] \subseteq S$. Hence $S \in \mathbb{D}(\Gamma)$.

Conversely, assume that $S \in \mathbb{D}(\Gamma)$. Let $\mathrm{Cay}(\Gamma, S)$ be a normal directed Cayley graph, so that S is a union of some conjugacy classes of Γ. Let

$$S = [\![x_1]\!] \cup \cdots \cup [\![x_r]\!] = \mathrm{Cl}(y_1) \cup \cdots \cup \mathrm{Cl}(y_k) \subseteq \Gamma(4)$$

for some $x_1, \dots, x_r, y_1, \dots, y_k \in \Gamma(4)$. Therefore

$$S^{-1} = [\![x_1^{-1}]\!] \cup \cdots \cup [\![x_r^{-1}]\!] = \mathrm{Cl}(y_1^{-1}) \cup \cdots \cup \mathrm{Cl}(y_k^{-1}) \subseteq \Gamma(4).$$

Now for $g_1, g_2 \in \Gamma(4)$, if $g_1 \approx g_2$ then $\mathrm{Cl}(g_1), \mathrm{Cl}(g_2) \subseteq S$ or $\mathrm{Cl}(g_1), \mathrm{Cl}(g_2) \subseteq S^{-1}$ or $\mathrm{Cl}(g_1), \mathrm{Cl}(g_2) \subseteq (S \cup S^{-1})^c$. Here $(S \cup S^{-1})^c$ is the complement of $S \cup S^{-1}$ in Γ. Note that $|\mathrm{Cl}(g_1)| = |\mathrm{Cl}(g_2)|$. For all the cases, using the definition of c_g, we have

$$\sum_{s \in \mathrm{Cl}(g_1)} c_s = \sum_{s \in \mathrm{Cl}(g_2)} c_s.$$

Again, we see that $\mathrm{Cl}(g) \subseteq S$ if and only if $\mathrm{Cl}(g^{-1}) \subseteq S^{-1}$. Therefore

$$\sum_{s \in \mathrm{Cl}(g)} c_s = -\sum_{s \in \mathrm{Cl}(g^{-1})} c_s = 0 \quad \text{or,} \quad \sum_{s \in \mathrm{Cl}(g)} c_s = \pm|\mathrm{Cl}(g)| = -\sum_{s \in \mathrm{Cl}(g^{-1})} c_s.$$

Further, if $g \notin \Gamma(4)$ then $\mathrm{Cl}(g) \cap (S \cup S^{-1}) = \emptyset$, and so $\sum\limits_{s \in \mathrm{Cl}(g)} c_s = 0$. Thus the three conditions of Theorem 4.2.1 are satisfied, and therefore $\chi_j(x)$ is an integer for each $j \in \{1, \dots, h\}$. Consequently, the H-eigenvalue $\mu_j := \frac{\chi_j(x)}{\chi_j(\mathbf{1})}$ of $\mathrm{Cay}(\Gamma, S)$ is a rational algebraic integer, and hence it an integer for each $j \in \{1, \dots, h\}$. □

We give the following example to illustrate Theorem 4.2.2.

Example 4.2.1. Consider the group $M_{16} := \langle a, x \mid a^8 = x^2 = \mathbf{1}, xax^{-1} = a^5 \rangle$, and let $S = \{a, a^5, a^3x, a^7x\}$. The conjugacy classes of M_{16} are $\{\mathbf{1}\}$,$\{a^4\}$,$\{a^2\}$,$\{a^6\}$,$\{a, a^5\}$, $\{a^3, a^7\}$, $\{ax, a^5x\}$, $\{a^3x, a^7x\}$, $\{x, a^4x\}$ and $\{a^2x, a^6x\}$. The normal directed Cayley graph $\mathrm{Cay}(M_{16}, S)$ is shown in Figure 4.1a. We see that $S = [\![a]\!] \cup [\![a^3x]\!] = \mathrm{Cl}(a) \cup \mathrm{Cl}(a^3x)$. Thus $S \in \mathbb{D}(M_{16})$, and hence $\mathrm{Cay}(M_{16}, S)$ is H-integral. We can also confirm this by finding its H-eigenvalues. Using the GAP software, the character table of M_{16} is obtained and given in Table 4.1, where $\mathrm{Irr}(M_{16}) = \{\chi_1, \dots, \chi_{10}\}$. Further, using Corollary 4.1.3, the H-spectrum of $\mathrm{Cay}(M_{16}, S)$ is obtained as $\{[\mu_j]^1 \colon 1 \leq j \leq 8\} \cup \{[\mu_9]^4, [\mu_{10}]^4\}$, where $\mu_j = 0$ for $j \notin \{5, 6\}$, $\mu_5 = -8$ and $\mu_6 = 8$. Thus all the H-eigenvalues of $\mathrm{Cay}(M_{16}, S)$ are integers.

Theorem 4.2.3. *Let Γ be a finite group and $\mathrm{Cay}(\Gamma, S)$ be a normal mixed Cayley graph. Then $\mathrm{Cay}(\Gamma, S)$ is H-integral if and only if $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$ and $\overline{S} \in \mathbb{D}(\Gamma)$.*

Proof. By Lemma 4.1.4, $\mathrm{Cay}(\Gamma, S)$ is H-integral if and only if $\mathrm{Cay}(\Gamma, S \setminus \overline{S})$ is integral and $\mathrm{Cay}(\Gamma, \overline{S})$ is H-integral. Now the proof follows from Theorem 1.6.7 and Theorem 4.2.2. □

The following example uses Theorem 4.2.3 to check H-integrality of a normal mixed Cayley graph.

Example 4.2.2. Consider the group M_{16} of Example 4.2.1 and $S = \{a, a^3, a^5, a^7, a^3x, a^7x\}$. The normal mixed Cayley graph $\text{Cay}(M_{16}, S)$ is shown in Figure 4.1b. We have

$$S = [a] \cup [\![a^3x]\!] = \text{Cl}(a) \cup \text{Cl}(a^3) \cup \text{Cl}(a^3x).$$

Therefore $S \setminus \overline{S} \in \mathbb{B}(M_{16})$ and $\overline{S} \in \mathbb{D}(M_{16})$. Further, using Lemma 4.1.1, the H-spectrum of $\text{Cay}(M_{16}, S)$ is obtained as $\{[\gamma_j]^1 \colon 1 \leq j \leq 8\} \cup \{[\gamma_9]^4, [\gamma_{10}]^4\}$, where $\gamma_1 = \gamma_3 = \gamma_6 = \gamma_7 = 4$, $\gamma_2 = \gamma_4 = \gamma_5 = \gamma_8 = -4$ and $\gamma_9 = \gamma_{10} = 0$. Thus $\text{Cay}(M_{16}, S)$ is H-integral.

	$\{\mathbf{1}\}$	$\{a^4\}$	$\{a^2\}$	$\{a^6\}$	$\{a, a^5\}$	$\{a^3, a^7\}$	$\{ax, a^5x\}$	$\{a^3x, a^7x\}$	$\{x, a^4x\}$	$\{a^2x, a^6x\}$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	1	1
χ_3	1	1	1	1	1	1	-1	-1	-1	-1
χ_4	1	1	1	1	-1	-1	1	1	-1	-1
χ_5	1	1	-1	-1	$\mathbf{i}$	$-\mathbf{i}$	$-\mathbf{i}$	$\mathbf{i}$	-1	1
χ_6	1	1	-1	-1	$-\mathbf{i}$	$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{i}$	-1	1
χ_7	1	1	-1	-1	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{i}$	$-\mathbf{i}$	1	-1
χ_8	1	1	-1	-1	$-\mathbf{i}$	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{i}$	1	-1
χ_9	2	-2	$2\mathbf{i}$	$-2\mathbf{i}$	0	0	0	0	0	0
χ_{10}	2	-2	$-2\mathbf{i}$	$2\mathbf{i}$	0	0	0	0	0	0

Table 4.1: Character table of M_{16}

4.3 Gaussian integral normal mixed Cayley graphs

In Chapter 2, we proved that if Γ is an abelian group, then $[\![x]\!] \cup [\![x^{-1}]\!] = [x]$ for each $x \in \Gamma(4)$. Note that this result and its proof also hold for non-abelian group. In the subsequent discussion, we use this fact for non-abelian groups.

Let S be a union of some conjugacy classes of a finite group Γ that does not contain $\mathbf{1}$, and let $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Consider the function $f \colon \Gamma \to \{0, 1\}$ defined by

$$f(s) = \begin{cases} 1 & \text{if } s \in S \\ 0 & \text{otherwise} \end{cases}$$

in Theorem 1.3.6. We see that $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s \in S} \chi_j(s)$ is an eigenvalue of the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ for each $j \in \{1, \ldots, h\}$. Indeed, all the eigenvalues of $\text{Cay}(\Gamma, S)$ are of this form. For each $j \in \{1, \ldots, h\}$, define

$$f_j(S) := \frac{1}{\chi_j(\mathbf{1})} \sum_{s \in S \setminus \overline{S}} \chi_j(s) + \frac{1}{2\chi_j(\mathbf{1})} \sum_{s \in \overline{S} \cup (\overline{S})^{-1}} \chi_j(s)$$

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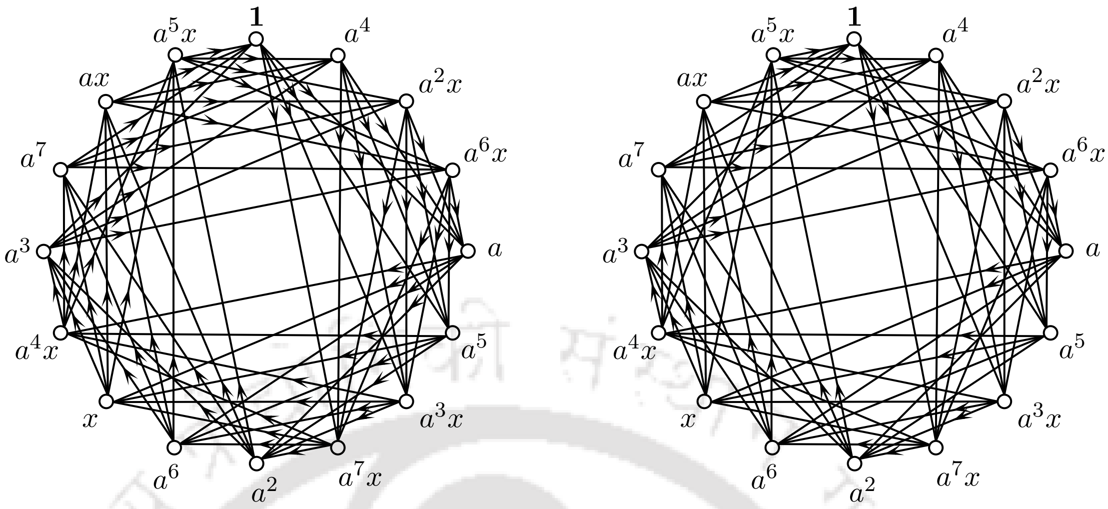

(a) $S = \{a, a^5, a^3x, a^7x\}$ (b) $S = \{a, a^3, a^5, a^7, a^3x, a^7x\}$

Figure 4.1: The mixed graph $\text{Cay}(M_{16}, S)$

and

$$g_j(S) := \frac{\mathbf{i}}{2\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} (\chi_j(s) - \chi_j(s^{-1})).$$

We have

$$\frac{1}{\chi_j(\mathbf{1})} \sum_{s\in S} \chi_j(s) = f_j(S) - \mathbf{i}g_j(S) \text{ for each } j \in \{1,\ldots,h\}. \tag{4.9}$$

Therefore, the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Gaussian integral if and only if $f_j(S)$ and $g_j(S)$ are integers for each $j \in \{1,\ldots,h\}$.

Theorem 4.3.1. *Let Γ be a finite group. If the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Gaussian integral, then it is H-integral.*

Proof. Let the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ be Gaussian integral. Therefore, $f_j(S)$ and $g_j(S)$ are integers for each $j \in \{1,\ldots,h\}$. Note that $2g_j(S)$ is an integer H-eigenvalue of the normal directed Cayley graph $\text{Cay}(\Gamma, \overline{S})$ for each $j \in \{1,\ldots,h\}$. By Theorem 4.2.2, we get $\overline{S} \in \mathbb{D}(\Gamma)$. Let $\overline{S} = \bigcup_{j=1}^{k} [\![x_j]\!] = \bigcup_{j=1}^{s} \text{Cl}(y_j)$ for some $x_1,\ldots,x_k, y_1,\ldots,y_s \in \Gamma(4)$. Thus

$$\overline{S} \cup (\overline{S})^{-1} = \bigcup_{j=1}^{k} \left([\![x_j]\!] \cup [\![x_j^{-1}]\!]\right) = \bigcup_{j=1}^{k} [x_j] = \bigcup_{j=1}^{s} \left(\text{Cl}(y_j) \cup \text{Cl}(y_j^{-1})\right) \in \mathbb{B}(\Gamma).$$

Note that $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s\in\overline{S}\cup(\overline{S})^{-1}} \chi_j(s)$ is an eigenvalue of the normal undirected Cayley graph $\text{Cay}(\Gamma, \overline{S}\cup(\overline{S})^{-1})$ for each $j \in \{1,\ldots,h\}$. By Theorem 1.6.7, $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s\in\overline{S}\cup(\overline{S})^{-1}} \chi_j(s)$ is an integer

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for each $j \in \{1, \dots, h\}$. Therefore

$$\frac{1}{\chi_j(\mathbf{1})} \sum_{s \in S \setminus \overline{S}} \chi_j(s) = f_j(S) - \frac{1}{2\chi_j(\mathbf{1})} \sum_{s \in \overline{S} \cup (\overline{S})^{-1}} \chi_j(s),$$

and hence $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s \in S \setminus \overline{S}} \chi_j(s)$ is a rational number. Thus, the eigenvalue $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s \in S \setminus \overline{S}} \chi_j(s)$ of the normal undirected Cayley graph $\mathrm{Cay}(\Gamma, S \setminus \overline{S})$ is a rational algebraic integer, and hence it is an integer for each $j \in \{1, \dots, h\}$. Hence by Theorem 1.6.7, we get $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$. Now the result follows from Theorem 4.2.3. $\square$

Lemma 4.3.2. *Let $x \in \Gamma$ and $\mathrm{ord}(x) = 2^t m$. If $t \geq 2$ and m is odd, then the following assertions hold.*

(i) $[\![x]\!] = \begin{cases} x^{3m}[x^2] & \text{if } m \equiv 1 \pmod 4 \\ x^m[x^2] & \text{if } m \equiv 3 \pmod 4. \end{cases}$

(ii) $[\![x^{-1}]\!] = \begin{cases} x^{m}[x^2] & \text{if } m \equiv 1 \pmod 4 \\ x^{3m}[x^2] & \text{if } m \equiv 3 \pmod 4. \end{cases}$

(iii) $[x] = x^m[x^2] \cup x^{3m}[x^2]$.

Proof. (i) Assume that $t \geq 2$ and $k = 2^t m$. Let $m \equiv 1 \pmod 4$ and $x^{3m+2r} \in x^{3m}[x^2]$ for some $r \in G_{\frac{k}{2}}(1)$. Now $\gcd(r, \frac{k}{2}) = 1$, and it implies that $\gcd(3m + 2r, k) = 1$ and $3m + 2r \equiv 1 \pmod 4$. We have $x^{3m+2r} \in [\![x]\!]$. Thus $x^{3m}[x^2] \subseteq [\![x]\!]$. Now since the sizes of $[\![x]\!]$ and $x^{3m}[x^2]$ are equal, we have $[\![x]\!] = x^{3m}[x^2]$. Similarly, if $m \equiv 3 \pmod 4$ then $[\![x]\!] = x^m[x^2]$.

(ii) The proof of this part is similar to the proof of Part (i). For the sake of completeness, we provide the proof. Assume that $t \geq 2$ and $k = 2^t m$. Let $m \equiv 1 \pmod 4$ and $x^{m+2r} \in x^m[x^2]$ for some $r \in G_{\frac{k}{2}}(1)$. Now $\gcd(r, \frac{k}{2}) = 1$ implies that $\gcd(m + 2r, k) = 1$ and $m + 2r \equiv 3 \pmod 4$. We have $x^{m+2r} \in [\![x^{-1}]\!]$. Thus $x^m[x^2] \subseteq [\![x^{-1}]\!]$. Now since the sizes of $[\![x^{-1}]\!]$ and $x^m[x^2]$ are equal, we have $[\![x^{-1}]\!] = x^m[x^2]$. Similarly, if $m \equiv 3 \pmod 4$ then $[\![x^{-1}]\!] = x^{3m}[x^2]$.

(iii) Using Part (i), Part (ii) and $[x] = [\![x]\!] \cup [\![x^{-1}]\!]$, we get the result in desired form. $\square$

For $x \in \Gamma$, define $S_x^1 := \bigcup\limits_{s \in \mathrm{Cl}(x)} [s]$. We see that if $m = \mathrm{ord}(x)$, then

$$S_x^1 = \{g^{-1}x^r g \colon g \in \Gamma, r \in G_m(1)\} = \bigcup_{s \in [x]} \mathrm{Cl}(s).$$

The set S_x^1 is also known as the rational conjugacy class of x. See [18] for details. For each $y \in S_x^1$, it is clear that $\mathrm{Cl}(y), [y] \subseteq S_x^1$. Now let A be a symmetric subset of Γ such that $x \in A$, and $\mathrm{Cl}(a), [a] \subseteq A$ for each $a \in A$. Let $g^{-1}x^r g \in S_x^1$, where $g \in \Gamma$, $r \in G_m(1)$ and $m = \mathrm{ord}(x)$. As $[x] \subseteq A$, we have $x^r \in A$. Now $\mathrm{Cl}(x^r) \subseteq A$, and so $g^{-1}x^r g \in A$. Thus $S_x^1 \subseteq A$, and therefore S_x^1 is the smallest symmetric subset of Γ containing x that is closed under both conjugacy and the equivalence relation $\sim$. Considering each of the repeated equivalence classes, if any, only once in $\bigcup_{s \in \mathrm{Cl}(x)} [s]$, we can write $S_x^1 = \bigcup_{i=1}^{\ell} [x_i]$, where the equivalence classes $[x_1], \ldots, [x_\ell]$ are distinct. We state this fact in the next lemma.

Lemma 4.3.3. *If $x \in \Gamma$, then there exist distinct equivalence classes $[x_1], \ldots, [x_\ell]$ such that $S_x^1 = \bigcup_{i=1}^{\ell} [x_i]$, where $x_1, \ldots, x_\ell \in \mathrm{Cl}(x)$.*

Lemma 4.3.4. *If $y \in S_x^1$, then $S_y^1 = S_x^1$.*

Proof. Let $y \in S_x^1$, so that $y = g^{-1}x^r g$ for some $g \in \Gamma$ and $r \in G_m(1)$, where $m = \mathrm{ord}(x)$. We see that $\mathrm{ord}(y) = \mathrm{ord}(x) = m$. Now let $z \in S_y^1$. Then $z = h^{-1}y^t h$ for some $h \in \Gamma$ and $t \in G_m(1)$. This gives $z = h^{-1}y^t h = h^{-1}g^{-1}x^{rt}gh \in S_x^1$. Conversely, let $w \in S_x^1$ so that $w = h^{-1}x^t h$ for some $h \in \Gamma$ and $t \in G_m(1)$. Therefore

$$w = h^{-1}x^t h = (h^{-1}g)g^{-1}(x^r)^{r^{-1}t} g(g^{-1}h) = (h^{-1}g)y^{r^{-1}t}(g^{-1}h) \in S_y^1.$$

Here r^{-1} is the multiplicative inverse of r in the group $G_m(1)$. Hence we conclude that $S_y^1 = S_x^1$. □

Due to Lemma 4.3.4, the sets S_x^1 and S_y^1 are either disjoint or equal. Hence the class of distinct subsets of Γ of the form S_x^1 is a partition of Γ.

Lemma 4.3.5. *Let $x \in \Gamma(4)$. If $S_x^1 = [x_1] \cup \cdots \cup [x_\ell]$ for some $x_1, \ldots, x_\ell \in \mathrm{Cl}(x)$, then $S_{x^2}^1 = [x_1^2] \cup \cdots \cup [x_\ell^2]$.*

Proof. Let $m = \mathrm{ord}(x)$ and $S_x^1 = [x_1] \cup \cdots \cup [x_\ell]$ for some $x_1 \ldots, x_\ell \in \mathrm{Cl}(x)$. Assume that the sets $[x_1], \ldots, [x_\ell]$ are all distinct. We see that

$$\begin{aligned}
S_{x^2}^1 &= \left\{g^{-1}x^{2r}g \colon g \in \Gamma, r \in G_{\frac{m}{2}}(1)\right\} \\
&= \left\{g^{-1}x^{2r}g \colon g \in \Gamma, r \in G_{\frac{m}{2}}(1)\right\} \cup \left\{g^{-1}x^{2(\frac{m}{2}+r)}g \colon g \in \Gamma, r \in G_{\frac{m}{2}}(1)\right\} \\
&= \left\{g^{-1}x^{2r}g \colon g \in \Gamma, r \in G_m(1), r < \frac{m}{2}\right\} \cup \left\{g^{-1}x^{2t}g \colon g \in \Gamma, t \in G_m(1), t > \frac{m}{2}\right\} \\
&= \{g^{-1}x^{2r}g \colon g \in \Gamma, r \in G_m(1)\} \\
&= \{y^2 \colon y \in S_x^1\}.
\end{aligned}$$

Now noting that $\{s^2 \colon s \in [x]\} = [x^2]$ and $S_x^1 = [x_1] \cup \cdots \cup [x_\ell]$, we have $S_{x^2}^1 = [x_1^2] \cup \cdots \cup [x_\ell^2]$. □

Let $x \in \Gamma(4)$ be an element of order m. The element x is said to be *admissible* if $x^r \notin \mathrm{Cl}(x)$ for all $r \in G_m^3(1)$. The following lemma characterizes admissible elements in terms of skew-symmetric sets.

Lemma 4.3.6. *If $x \in \Gamma(4)$, then x is admissible if and only if the set $\bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$ is skew-symmetric.*

Proof. We see that if $m = \mathrm{ord}(x)$, then

$$\bigcup_{s\in \mathrm{Cl}(x)} [\![s]\!] = \{g^{-1}x^r g \colon g \in \Gamma, r \in G_m^1(1)\} = \bigcup_{s\in [\![x]\!]} \mathrm{Cl}(s).$$

Assume that x is not admissible, so that $x^r \in \mathrm{Cl}(x)$ for some $r \in G_m^3(1)$. As $m-r \in G_m^1(1)$ and $\mathrm{Cl}(x) \subseteq \bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$, we find that $x^r, x^{m-r} \in \bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$. Hence $\bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$ is not skew-symmetric.

Now assume that $\bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$ is not skew-symmetric. Then there is an $y = g^{-1}x^r g \in \bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$ for some $r \in G_m^1(1)$ such that $y^{-1} \in \bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$. Therefore $g^{-1}x^{m-r}g = y^{-1} = h^{-1}x^k h$ for some $h \in \Gamma, k \in G_m^1(1)$. Let $t \in G_m(1)$ be the multlipicative inverse of $m-r$. We have $g^{-1}x^{(m-r)t}g = h^{-1}x^{kt}h$, and it gives $x^{kt} = hg^{-1}xgh^{-1} \in \mathrm{Cl}(x)$. Since $(m-r)t \equiv 1 \pmod 4$ and $m-r \in G_m^3(1)$, we have that $t \in G_m^3(1)$. Thus $kt \in G_m^3(1)$ with $x^{kt} \in \mathrm{Cl}(x)$, giving that x is not admissible. $\square$

Let $x \in \Gamma(4)$ be admissible, and define $S_x^4 := \bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$. The structure and properties of the set S_x^4 are similar to those of S_x^1. If Γ is abelian, then $S_x^4 = [\![x]\!]$ for each $x \in \Gamma(4)$.

For each $y \in S_x^4$, it is clear that $\mathrm{Cl}(y), [\![y]\!] \subseteq S_x^4$. Now let A be a skew-symmetric subset of Γ containing an admissible element x, and $\mathrm{Cl}(a), [\![a]\!] \subseteq A$ for each $a \in A$. It is easy to see that $S_x^4 \subseteq A$. Thus, S_x^4 is the smallest skew-symmetric subset of Γ containing x that is closed under both conjugacy and the equivalence relation $\approx$. Considering each of the repeated equivalence classes, if any, only once in $\bigcup\limits_{s\in \mathrm{Cl}(x)} [\![s]\!]$, we can write $S_x^4 = \bigcup\limits_{i=1}^{r} [\![y_i]\!]$, where the equivalence classes $[\![y_1]\!], \ldots, [\![y_r]\!]$ are distinct. We state this fact in the next lemma.

Lemma 4.3.7. *If x is an admissible element in $\Gamma(4)$, then there are distinct equivalence classes $[\![y_1]\!], \ldots, [\![y_r]\!]$ such that $S_x^4 = \bigcup\limits_{i=1}^{r} [\![y_i]\!]$, where $y_1, \ldots, y_r \in \mathrm{Cl}(x)$.*

Lemma 4.3.8. *If $y \in S_x^4$, then $S_y^4 = S_x^4$.*

Proof. Let $y \in S_x^4$, so that $y = g^{-1}x^r g$ for some $g \in \Gamma$ and $r \in G_m^1(1)$, where $m = \mathrm{ord}(x)$. We see that $\mathrm{ord}(y) = \mathrm{ord}(x) = m$. Now let $z \in S_y^4$. Then $z = h^{-1}y^t h$ for some $h \in \Gamma$ and $t \in G_m^1(1)$. This gives $z = h^{-1}y^t h = h^{-1}g^{-1}x^{rt}gh \in S_x^4$. Conversely, let $w \in S_x^4$ so that $w = h^{-1}x^t h$ for some $h \in \Gamma$ and $t \in G_m^1(1)$. Therefore

$$w = h^{-1}x^t h = (h^{-1}g)g^{-1}(x^r)^{r^{-1}t}g(g^{-1}h) = (h^{-1}g)y^{r^{-1}t}(g^{-1}h) \in S_y^4.$$

Here r^{-1} is the multiplicative inverse of r in the subgroup $G_m^1(1)$. Thus we conclude that $S_y^4 = S_x^4$. $\square$

Due to Lemma 4.3.8, the sets S_x^4 and S_y^4 are either disjoint or equal.

Lemma 4.3.9. *If $x \in \Gamma(4)$ is admissible, then $S_x^4 \cup S_{x^{-1}}^4 = S_x^1$.*

Proof. Let $m = \mathrm{ord}(x)$. We have

$$\begin{aligned} S_x^4 \cup S_{x^{-1}}^4 &= \{g^{-1}x^r g \colon g \in \Gamma, r \in G_m^1(1)\} \cup \{g^{-1}x^{-r} g \colon g \in \Gamma, r \in G_m^1(1)\} \\ &= \{g^{-1}x^r g \colon g \in \Gamma, r \in G_m^1(1)\} \cup \{g^{-1}x^r g \colon g \in \Gamma, r \in G_m^3(1)\} \\ &= \{g^{-1}x^r g \colon g \in \Gamma, r \in G_m(1)\} \\ &= S_x^1. \end{aligned}$$

□

Lemma 4.3.10. *Let $x \in \Gamma(4)$ be an admissible element. If $S_x^4 = [\![x_1]\!] \cup \cdots \cup [\![x_r]\!]$ for some $x_1, \dots, x_r \in \mathrm{Cl}(x)$, then $S_{x^2}^1 = [x_1^2] \cup \cdots \cup [x_r^2]$.*

Proof. Let $S_x^4 = [\![x_1]\!] \cup \cdots \cup [\![x_r]\!]$, where $x_1, \dots, x_r \in \mathrm{Cl}(x)$. Then $S_{x^{-1}}^4 = [\![x_1^{-1}]\!] \cup \cdots \cup [\![x_r^{-1}]\!]$. Therefore

$$S_x^1 = S_x^4 \cup S_{x^{-1}}^4 = \left([\![x_1]\!] \cup [\![x_1^{-1}]\!]\right) \cup \cdots \cup \left([\![x_r]\!] \cup [\![x_r^{-1}]\!]\right) = [x_1] \cup \cdots \cup [x_r].$$

Now the result follows from Lemma 4.3.5. □

For $x \in \Gamma$ and $j \in \{1, \dots, h\}$, define

$$C_x(j) := \frac{1}{\chi_j(\mathbf{1})} \sum_{s \in S_x^1} \chi_j(s).$$

Note that $S_x^1 \in \mathbb{B}(\Gamma)$ and $C_x(j)$ is an eigenvalue of the normal undirected Cayley graph $\mathrm{Cay}(\Gamma, S_x^1)$. As a consequence of Theorem 1.6.7, $C_x(j)$ is an integer for each $x \in \Gamma$ and $j \in \{1, \dots, h\}$. For a complex number α, recall that $\Re(\alpha)$ and $\Im(\alpha)$ denote the real and imaginary parts of α, respectively.

Lemma 4.3.11. *Let $x \in \Gamma$ and $\mathrm{ord}(x) = 2^t m$. If m is odd and $t \geq 2$, then*

$$C_x(j) = (\chi_j(x^m) + \chi_j(x^{3m}))C_{x^2}(j).$$

Moreover, $C_x(j)$ is an even integer for each $j \in \{1, \dots, h\}$.

Proof. Let $S_x^1 = [x_1] \cup \cdots \cup [x_k]$ for some $x_1, \dots, x_k \in \mathrm{Cl}(x)$. For $j \in \{1, \dots, h\}$, we have

$$\begin{aligned} C_x(j) &= \frac{1}{\chi_j(\mathbf{1})} \sum_{r=1}^{k} \sum_{s \in [x_r]} \chi_j(s) \\ &= \frac{1}{\chi_j(\mathbf{1})} \sum_{r=1}^{k} \left(\sum_{s \in [x_r^2]} \chi_j(x_r^m)\chi_j(s) + \sum_{s \in [x_r^2]} \chi_j(x_r^{3m})\chi_j(s) \right) \\ &= (\chi_j(x^m) + \chi_j(x^{3m})) \frac{1}{\chi_j(\mathbf{1})} \sum_{r=1}^{k} \sum_{s \in [x_r^2]} \chi_j(s) \\ &= (\chi_j(x^m) + \chi_j(x^{3m}))C_{x^2}(j). \end{aligned} \tag{4.10}$$

The second equality in the preceding equations follows from Part (iii) of Lemma 4.3.2 and the fourth equality follows from Lemma 4.3.5.

We apply induction on t to prove that $C_x(j)$ is an even integer. Let ρ_j be a representation corresponding to χ_j. If $t = 2$ then $\rho_j(x^m)^4$ is the identity matrix, and so each eigenvalue of $\rho_j(x^m)$ is a 4-th root of unity. Thus, $\chi_j(x^m)$ is the trace of a matrix whose eigenvalues are 4-th roots of unity. Therefore $\chi_j(x^m) + \chi_j(x^{3m}) = 2\Re(\chi_j(x^m))$, an even integer. Hence $C_x(j)$ is an even integer. Assume that the statement holds for each $z \in \Gamma$ with $\text{ord}(z) = 2^{t-1}m$, where m is odd and $t-1 \geq 2$. Let $x \in \Gamma$ with $\text{ord}(x) = 2^t m$, where m is odd and $t \geq 3$. Since the order of x^2 is $2^{t-1}m$, by induction hypothesis $C_{x^2}(j)$ is an even integer. If $C_{x^2}(j) = 0$, then clearly $C_x(j) = 0$, an even integer. By Equation (4.10), $\chi_j(x^m) + \chi_j(x^{3m})$ is a rational algebraic integer whenever $C_{x^2}(j) \neq 0$. Thus if $C_{x^2}(j) \neq 0$, then $\chi_j(x^m) + \chi_j(x^{3m})$ is an integer. Hence by Equation (4.10) and induction hypothesis, $C_x(j)$ is an even integer for each $j \in \{1, \ldots, h\}$. Thus the proof is complete by induction. □

Let $x \in \Gamma(4)$ be admissible. For $j \in \{1, \ldots, h\}$, define

$$S_x(j) := \frac{\mathbf{i}}{\chi_j(\mathbf{1})} \sum_{s \in S_x^4} (\chi_j(s) - \chi_j(s^{-1})).$$

Note that $S_x(j)$ is an H-eigenvalue of the normal directed Cayley graph $\text{Cay}(\Gamma, S_x^4)$ for each $j \in \{1, \ldots, h\}$. Since $S_x^4 \in \mathbb{D}(\Gamma)$, by Theorem 4.2.2 $S_x(j)$ is an integer for each $j \in \{1, \ldots, h\}$.

Lemma 4.3.12. *Let $x \in \Gamma(4)$ be admissible and* $\text{ord}(x) = 2^t m$*. If m is odd and $t \geq 2$, then*

$$S_x(j) = \begin{cases} -2\Im(\chi_j(x^{3m}))C_{x^2}(j) & \text{if } m \equiv 1 \pmod 4 \\ -2\Im(\chi_j(x^m))C_{x^2}(j) & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Moreover, $S_x(j)$ is an even integer for each $j \in \{1, \ldots, h\}$.

Proof. Let $S_x^4 = [\![x_1]\!] \cup \cdots \cup [\![x_k]\!]$ for some $x_1, \ldots, x_k \in \text{Cl}(x)$. For $j \in \{1, \ldots, h\}$, we have

$$
\begin{aligned}
S_x(j) &= \frac{\mathbf{i}}{\chi_j(\mathbf{1})} \sum_{r=1}^{k} \sum_{s \in [\![x_r]\!]} (\chi_j(s) - \chi_j(s^{-1})) \\
&= \begin{cases} \frac{\mathbf{i}}{\chi_j(\mathbf{1})} \sum\limits_{r=1}^{k} \sum\limits_{s \in [\![x_r]\!]} (\chi_j(s) - \chi_j(s^{-1})) & \text{if } m \equiv 1 \pmod 4 \\ \frac{\mathbf{i}}{\chi_j(\mathbf{1})} \sum\limits_{r=1}^{k} \sum\limits_{s \in [\![x_r]\!]} (\chi_j(s) - \chi_j(s^{-1})) & \text{if } m \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} \frac{\mathbf{i}}{\chi_j(\mathbf{1})} \sum\limits_{r=1}^{k} \sum\limits_{s \in [x_r^2]} (\chi_j(x_r^{3m})\chi_j(s) - \chi_j(x_r^{-3m})\chi_j(s^{-1})) & \text{if } m \equiv 1 \pmod 4 \\ \frac{\mathbf{i}}{\chi_j(\mathbf{1})} \sum\limits_{r=1}^{k} \sum\limits_{s \in [x_r^2]} (\chi_j(x_r^{m})\chi_j(s) - \chi_j(x_r^{-m})\chi_j(s^{-1})) & \text{if } m \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} -2\Im(\chi_j(x^{3m})) \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{r=1}^{k} \sum\limits_{s \in [x_r^2]} \chi_j(s) & \text{if } m \equiv 1 \pmod 4 \\ -2\Im(\chi_j(x^{m})) \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{r=1}^{k} \sum\limits_{s \in [x_r^2]} \chi_j(s) & \text{if } m \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} -2\Im(\chi_j(x^{3m}))C_{x^2}(j) & \text{if } m \equiv 1 \pmod 4 \\ -2\Im(\chi_j(x^{m}))C_{x^2}(j) & \text{if } m \equiv 3 \pmod 4). \end{cases}
\end{aligned}
$$

The third equality in the preceding equations follows from Part (i) of Lemma 4.3.2 and the fifth equality follows from Lemma 4.3.10. If $t = 2$, then $\chi_j(x^{3m})$ and $\chi_j(x^m)$ are traces of matrices whose eigenvalues are 4-th roots of unity. Therefore, $\Im(\chi_j(x^{3m}))$ and $\Im(\chi_j(x^m))$ are integers. Thus $S_x(j)$ is an even integer. Now assume that $t \geq 3$. If $C_{x^2}(j) = 0$, then clearly $S_x(j) = 0$, an even integer. Note that $2\Im(\chi_j(x^{3m}))$ and $2\Im(\chi_j(x^m))$ are rational algebraic integers whenever $C_{x^2}(j) \neq 0$. Thus if $C_{x^2}(j) \neq 0$, then $2\Im(\chi_j(x^{3m}))$ and $2\Im(\chi_j(x^m))$ are integers. Since the order of x^2 is $2^{t-1}m$, by Lemma 4.3.11 $S_x(j)$ is an even integer. □

Let S be a nonempty set in $\mathbb{D}(\Gamma)$ and that S be expressible as a union of some conjugacy classes of Γ. Then S is a skew-symmetric subset of Γ that is closed under both conjugacy and the equivalence relation $\approx$. Let $S = \mathrm{Cl}(x_1) \cup \cdots \cup \mathrm{Cl}(x_k) = [\![y_1]\!] \cup \cdots \cup [\![y_r]\!]$ for some $x_1, \ldots, x_k, y_1, \ldots, y_r \in \Gamma(4)$. We see that

$$
S = \mathrm{Cl}(x_1) \cup \cdots \cup \mathrm{Cl}(x_k) = \left(\bigcup_{s \in \mathrm{Cl}(x_1)} [\![s]\!] \right) \cup \cdots \cup \left(\bigcup_{s \in \mathrm{Cl}(x_k)} [\![s]\!] \right) = S_{x_1}^4 \cup \cdots \cup S_{x_k}^4.
$$

Due to Lemma 4.3.8, we can assume that the sets $S_{x_1}^4, \ldots, S_{x_k}^4$ are all distinct. In the next result, we prove the converse of Theorem 4.3.1.

Theorem 4.3.13. *If Γ is a finite group, then the normal mixed Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *is Gaussian integral if and only if it is H-integral.*

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Proof. Assume that the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is H-integral. It is enough to show that $f_j(S)$ and $g_j(S)$ are integers for $j \in \{1, \ldots, h\}$. By Theorem 4.2.3, we get $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$ and $\overline{S} \in \mathbb{D}(\Gamma)$. Therefore $\overline{S} = S^4_{x_1} \cup \cdots \cup S^4_{x_r}$ for some $x_1, \ldots, x_r \in \Gamma(4)$, where the sets $S^4_{x_1}, \ldots, S^4_{x_r}$ are all distinct. By Lemma 4.3.12, $S_{x_1}(j) + \cdots + S_{x_r}(j)$ is an even integer. As $g_j(S) = \frac{1}{2}(S_{x_1}(j) + \cdots + S_{x_r}(j))$, we find that $g_j(S)$ is an integer. Observe that $S^4_{x_i} \cup S^4_{x_i^{-1}} = S^1_{x_i}$, and so $\overline{S} \cup (\overline{S})^{-1} = S^1_{x_1} \cup \cdots \cup S^1_{x_r}$. Therefore $f_j(S) = \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s \in S \setminus \overline{S}} \chi_j(s) + \frac{1}{2}(C_{x_1}(j) + \ldots + C_{x_r}(j))$. By Theorem 1.6.7, $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s \in S \setminus \overline{S}} \chi_j(s)$ is an integer. Also, by Lemma 4.3.11, $C_{x_i}(j)$ is an even integer for each $i \in \{1, \ldots, r\}$. Hence we find that $f_j(S)$ is an integer. The other part of the theorem is already proved in Theorem 4.3.1. □

We give the following example to illustrate Theorem 4.3.13.

Example 4.3.1. Consider the normal mixed Cayley graph $\text{Cay}(M_{16}, S)$ of Example 4.2.2. We have already seen that it is H-integral, and hence it must be Gaussian integral. Indeed, using Theorem 1.3.6, the spectrum of $\text{Cay}(M_{16}, S)$ is obtained as

$$\{[\gamma_j]^1 \colon 1 \leq j \leq 8\} \cup \{[\gamma_9]^4, [\gamma_{10}]^4\}, \text{ where}$$

$$\gamma_j = \frac{1}{\chi_j(1)} \left[\chi_j(a) + \chi_j(a^3) + \chi_j(a^5) + \chi_j(a^7) + \chi_j(a^3 x) + \chi_j(a^7 x)\right] \text{ for each } j \in \{1, \ldots, 10\}.$$

We find that $\gamma_1 = 6$, $\gamma_2 = -6$, $\gamma_3 = 2$, $\gamma_4 = -2$, $\gamma_6 = \gamma_7 = 2\mathbf{i}$, $\gamma_5 = \gamma_8 = -2\mathbf{i}$, and $\gamma_9 = \gamma_{10} = 0$. Thus $\text{Cay}(M_{16}, S)$ is Gaussian integral.

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5

HS-integral and Eisenstein integral normal mixed Cayley graphs

This chapter extends Theorem 3.3.6 to normal mixed Cayley graphs. Additionally, we find a characterization of Eisenstein integral normal mixed Cayley graphs. We first express the HS-eigenvalues of a normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ in terms of the irreducible characters of Γ. In the second section, we find a characterization of HS-integral normal directed Cayley graphs. In the third section, we extend the characterization obtained in Section 2 to normal mixed Cayley graphs. In the last section, we show that a normal mixed Cayley graph is HS-integral if and only if it is Eisenstein integral. Some definitions and results of this chapter have similarities with those in the previous chapters.

5.1 Preliminaries

In this section, we express the HS-eigenvalues of a normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ in terms of the irreducible characters of Γ.

Lemma 5.1.1. *Let Γ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \dots, \chi_h\}$, then the HS-spectrum of the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\gamma_1]^{d_1^2}, \dots, [\gamma_h]^{d_h^2}\}$, where $\gamma_j = \lambda_j + \mu_j$,*

$$\lambda_j = \frac{1}{\chi_j(\mathbf{1})} \sum_{s \in S \setminus \overline{S}} \chi_j(s), \quad \mu_j = \frac{1}{\chi_j(\mathbf{1})} \sum_{s \in \overline{S}} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})),$$

and $d_j = \chi_j(\mathbf{1})$ for each $j \in \{1, \dots, h\}$.

Proof. Let $f \colon \Gamma \to \{0, 1, \omega_6, \omega_6^5\}$ be defined by

$$f(s) = \begin{cases} 1 & \text{if } s \in S \setminus \overline{S} \\ \omega_6 & \text{if } s \in \overline{S} \\ \omega_6^5 & \text{if } s \in (\overline{S})^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

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Since S is a union of some conjugacy classes of Γ, f is a class function. The Hermitian adjacency matrix of second kind of $\mathrm{Cay}(\Gamma, S)$ is given by $[f(yx^{-1})]_{x,y\in\Gamma}$. By Theorem 1.3.6,

$$\gamma_j = \frac{1}{\chi_j(\mathbf{1})}\Big(\sum_{s\in S\setminus\overline{S}} \chi_j(s) + \sum_{s\in\overline{S}} \omega_6\chi_j(s) + \sum_{s\in(\overline{S})^{-1}} \omega_6^5\chi_j(s)\Big),$$

and the result follows. □

As special cases of Lemma 5.1.1, we have the following two corollaries.

Corollary 5.1.2. *Let Γ be a finite group. If $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then the HS-spectrum (or spectrum) of the normal undirected Cayley graph $\mathrm{Cay}(\Gamma, S)$ is $\{[\lambda_1]^{d_1^2}, \ldots, [\lambda_h]^{d_h^2}\}$, where*

$$\lambda_j = \frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S}\chi_j(s) \text{ and } d_j = \chi_j(\mathbf{1}) \quad \text{for each } j \in \{1, \ldots, h\}.$$

Corollary 5.1.3. *Let Γ be a finite group. If $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then the HS-spectrum of the normal directed Cayley graph $\mathrm{Cay}(\Gamma, S)$ is $\{[\mu_1]^{d_1^2}, \ldots, [\mu_h]^{d_h^2}\}$, where*

$$\mu_j = \frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S}(\omega_6\chi_j(s) + \omega_6^5\chi_j(s^{-1})) \text{ and } d_j = \chi_j(\mathbf{1}) \quad \text{for each } j \in \{1, \ldots, h\}.$$

5.2 HS-integral normal directed Cayley graphs

Let $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Let E be the matrix $[E_{jg}]$ of size $h \times n$, whose rows are indexed by $1, \ldots, h$, and columns are indexed by the elements of Γ such that $E_{jg} = \chi_j(g)$. Note that $EE^* = nI_h$ and the rank of E is h, where E^* is the conjugate transpose of E. This matrix was also considered in Section 4.2.

Recall that $\mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) = \{\sigma_r \colon r \in G_m(1), \sigma_r(\omega_m) = \omega_m^r\}$. If $m \equiv 0 \pmod 3$, then $\mathbb{Q}(\omega_3, \omega_m) = \mathbb{Q}(\omega_m)$. Therefore, the Galois group $\mathrm{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$ is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q})$. Thus $\mathrm{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$ contains those automorphisms in $\mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q})$ that fix ω_3. Note that $G_m(1) = G_{m,3}^1(1) \cup G_{m,3}^2(1)$, a disjoint union. Using $\sigma_r(\omega_3) = \omega_3$ for all $r \in G_{m,3}^1(1)$ and $\sigma_r(\omega_3) = \omega_3^2$ for all $r \in G_{m,3}^2(1)$, we get

$$\mathrm{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3)) = \mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}(\omega_3)) = \{\sigma_r \colon r \in G_{m,3}^1(1), \sigma_r(\omega_m) = \omega_m^r\}.$$

If $m \not\equiv 0 \pmod 3$, then $[\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_3)] = \varphi(m)$. Thus the field $\mathbb{Q}(\omega_3, \omega_m)$ is a Galois extension of $\mathbb{Q}(\omega_3)$ of degree $\varphi(m)$. Any automorphism of the field $\mathbb{Q}(\omega_3, \omega_m)$ is uniquely determined by its action on ω_m. Hence

$$\mathrm{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3)) = \{\tau_r \colon r \in G_m(1), \tau_r(\omega_m) = \omega_m^r \text{ and } \tau_r(\omega_3) = \omega_3\}.$$

Let $g \in \Gamma$, $m = \mathrm{ord}(g)$ and χ be a character of Γ. By Theorem 1.3.3, $\chi(g) = \sum_{i=1}^k \epsilon_i$, where $\epsilon_1, \ldots, \epsilon_k$ are some m-th roots of unity. If $m \equiv 0 \pmod 3$ and $\sigma_r \in \mathrm{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$, then

$$\sigma_r(\chi(g)) = \sigma_r\left(\sum_{i=1}^{k} \epsilon_i\right) = \sum_{i=1}^{k} \sigma_r(\epsilon_i) = \sum_{i=1}^{k} \epsilon_i^r = \chi(g^r).$$

Similarly, if $m \not\equiv 0 \pmod 3$ and $\tau_r \in \mathrm{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$, then also $\tau_r(\chi(g)) = \chi(g^r)$.

Theorem 5.2.1. *Let Γ be a finite group and $\mathrm{Irr}(\Gamma) = \{\chi_1, \dots, \chi_h\}$. If $x = \sum_{g \in \Gamma} c_g g \in \mathbb{Q}(\omega_3)\Gamma$, then $\chi_j(x)$ is rational for each $j \in \{1, \dots, h\}$ if and only if the following conditions hold:*

(i) $\sum_{s \in \mathrm{Cl}(g_1)} c_s = \sum_{s \in \mathrm{Cl}(g_2)} c_s$ *for each $g_1, g_2 \in \Gamma(3)$ and $g_1 \simeq g_2$;*

(ii) $\sum_{s \in \mathrm{Cl}(g_1)} c_s = \sum_{s \in \mathrm{Cl}(g_2)} c_s$ *for each $g_1, g_2 \in \Gamma \setminus \Gamma(3)$ and $g_1 \sim g_2$;*

(iii) $\sum_{s \in \mathrm{Cl}(g)} c_s = \sum_{s \in \mathrm{Cl}(g^{-1})} \bar{c}_s$ *for each $g \in \Gamma$.*

Proof. Let L be a set of representatives of the conjugacy classes in Γ. Since characters are class functions, we have

$$\chi_j(x) = \sum_{g \in L} \left(\sum_{s \in \mathrm{Cl}(g)} c_s \right) \chi_j(g) \text{ for each } j \in \{1, \dots, h\}. \tag{5.1}$$

Assume that $\chi_j(x) \in \mathbb{Q}$ for each $j \in \{1, \dots, h\}$. Let $g_1, g_2 \in \Gamma(3)$, $g_1 \simeq g_2$ and $m = \mathrm{ord}(g_1)$. Therefore, there exist $r \in G_{m,3}^1(1)$ and $\sigma_r \in \mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}(\omega_3))$ such that $g_2 = g_1^r$ and $\sigma_r(\omega_m) = \omega_m^r$. Note that $\sigma_r(\chi_j(g_1)) = \chi_j(g_1^r)$ for each $j \in \{1, \dots, h\}$. For $t \in \Gamma$, let $\theta_t = \sum_{j=1}^{h} \chi_j(t)\overline{\chi}_j$, where $\overline{\chi}_j(g) = \overline{\chi_j(g)}$ for each $g \in \Gamma$. By Theorem 1.3.5, we have

$$\theta_t(u) = \begin{cases} |C_\Gamma(t)| & \text{if } u \text{ and } t \text{ are conjugates to each other} \\ 0 & \text{otherwise.} \end{cases}$$

So $\theta_t(x) = |C_\Gamma(t)| \sum_{s \in \mathrm{Cl}(t)} c_s \in \mathbb{Q}(\omega_3)$, and it gives that $\sigma_r(\theta_t(x)) = \theta_t(x)$. Since $\chi_j(x)$ is assumed to be a rational number, we have $\sigma_r(\chi_j(x)) = \chi_j(x)$ for each $j \in \{1, \dots, h\}$. Thus

$$\begin{aligned} |C_\Gamma(g_1)| \sum_{s \in \mathrm{Cl}(g_1)} c_s = \theta_{g_1}(x) = \sigma_r(\theta_{g_1}(x)) &= \sum_{j=1}^{h} \sigma_r(\chi_j(g_1))\sigma_r(\overline{\chi}_j(x)) \\ &= \sum_{j=1}^{h} \chi_j(g_1^r)\overline{\chi}_j(x) \\ &= \theta_{g_1^r}(x) = \theta_{g_2}(x) = |C_\Gamma(g_2)| \sum_{s \in \mathrm{Cl}(g_2)} c_s. \end{aligned} \tag{5.2}$$

Since $g_1 \simeq g_2$, we have $C_\Gamma(g_1) = C_\Gamma(g_2)$. So Equation (5.2) implies that $\sum_{s \in \mathrm{Cl}(g_1)} c_s = \sum_{s \in \mathrm{Cl}(g_2)} c_s$. Hence condition (i) holds.

Now let $g_1, g_2 \in \Gamma \setminus \Gamma(3)$, $g_1 \sim g_2$, and $m = \text{ord}(g_1)$. Then there is $r \in G_m(1)$ and $\tau_r \in \text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$ such that $g_2 = g_1^r$, $\tau_r(\omega_m) = \omega_m^r$ and $\tau_r(\omega_3) = \omega_3$. Now proceeding as in the proof of condition (i), we have $\sum\limits_{s \in \text{Cl}(g_1)} c_s = \sum\limits_{s \in \text{Cl}(g_2)} c_s$. Thus condition (ii) also holds.

Again

$$\begin{aligned} 0 = \chi_j(x) - \overline{\chi_j(x)} &= \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} c_s \Bigg) \chi_j(g) - \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} \overline{c}_s \Bigg) \overline{\chi_j(g)} \\ &= \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} c_s \Bigg) \chi_j(g) - \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} \overline{c}_s \Bigg) \chi_j(g^{-1}) \\ &= \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} c_s - \sum_{s \in \text{Cl}(g^{-1})} \overline{c}_s \Bigg) \chi_j(g), \end{aligned}$$

and so

$$\sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} c_s - \sum_{s \in \text{Cl}(g^{-1})} \overline{c}_s \Bigg) \begin{bmatrix} \chi_1(g) \\ \vdots \\ \chi_h(g) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.3}$$

Note that the number of irreducible characters of Γ is equal to the number of conjugacy classes of Γ, that is, $|L| = h$. Since characters are class functions and rank of E is h, the columns of E corresponding to the elements of L are linearly independent. Thus by Equation (5.3), $\sum\limits_{s \in \text{Cl}(g)} c_s - \sum\limits_{s \in \text{Cl}(g^{-1})} \overline{c}_s = 0$ for all $g \in L$, and so condition (iii) holds.

Conversely, assume that the three conditions of the theorem hold. Let n be the number of elements of Γ. We have the following two cases.

Case 1. Assume that $n \equiv 0 \pmod 3$. Let $\sigma_k \in \text{Gal}(\mathbb{Q}(\omega_3, \omega_n)/\mathbb{Q}(\omega_3))$. Then $\sigma_k(\omega_n) = \omega_n^k$ and $k \in G_{n,3}^1(1)$, and so $\sigma_k(\chi_j(g)) = \chi_j(g^k)$ for each $j \in \{1, \ldots, h\}$. Thus

$$\begin{aligned} \sigma_k(\chi_j(x)) &= \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} c_s \Bigg) \sigma_k(\chi_j(g)) \\ &= \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g)} c_s \Bigg) \chi_j(g^k). \end{aligned} \tag{5.4}$$

In the sum of Equation (5.4) we have two possible casses, namely, $g \in \Gamma(3)$ or $g \in \Gamma \setminus \Gamma(3)$. If $g \in \Gamma(3)$, then using the fact $g \simeq g^k$ and condition (i), we get $\sum\limits_{s \in \text{Cl}(g)} c_s = \sum\limits_{s \in \text{Cl}(g^k)} c_s$. Similarly, if $g \in \Gamma \setminus \Gamma(3)$, then using the fact $g \sim g^k$ and condition (ii), we get $\sum\limits_{s \in \text{Cl}(g)} c_s = \sum\limits_{s \in \text{Cl}(g^k)} c_s$. Therefore, we have $\sum\limits_{s \in \text{Cl}(g)} c_s = \sum\limits_{s \in \text{Cl}(g^k)} c_s$ for each $g \in \Gamma$. Now from Equation (5.4), we get

$$\sigma_k(\chi_j(x)) = \sum_{g \in L} \Bigg(\sum_{s \in \text{Cl}(g^k)} c_s \Bigg) \chi_j(g^k) = \chi_j(x). \tag{5.5}$$

The second equality in Equation (5.5) holds, because $\{g^k : g \in L\}$ is also a set of representatives of conjugacy classes of Γ. Now since $\sigma_k(\chi_j(x)) = \chi_j(x)$ for each $k \in G_{n,3}^1(1)$, we have that $\chi_j(x) \in \mathbb{Q}(\omega_3)$.

Case 2. Assume that $n \not\equiv 0 \pmod 3$. Let $\tau_r \in \mathrm{Gal}(\mathbb{Q}(\omega_3, \omega_n)/\mathbb{Q}(\omega_3))$. Then we have $\tau_r(\chi_j(g)) = \chi_j(g^r)$ for each $j \in \{1, \ldots, h\}$. Note that $g \sim g^r$. Therefore using Equation (5.1) and condition (ii), we have

$$\begin{aligned}\tau_r(\chi_j(x)) &= \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)} c_s\Big)\tau_r(\chi_j(g))\\ &= \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)} c_s\Big)\chi_j(g^r)\\ &= \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g^r)} c_s\Big)\chi_j(g^r)\\ &= \chi_j(x).\end{aligned}$$

This gives that $\chi_j(x) \in \mathbb{Q}(\omega_3)$. Thus in both the cases, we get $\chi_j(x) \in \mathbb{Q}(\omega_3)$. Taking complex conjugates in Equation (5.1), we get

$$\begin{aligned}\overline{\chi_j(x)} = \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)} \overline{c}_s\Big)\overline{\chi_j(g)} &= \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g)} \overline{c}_s\Big)\chi_j(g^{-1})\\ &= \sum_{g\in L}\Big(\sum_{s\in \mathrm{Cl}(g^{-1})} c_s\Big)\chi_j(g^{-1})\\ &= \chi_j(x). \end{aligned}\tag{5.6}$$

Equation (5.6) implies that $\chi_j(x) \in \mathbb{Q}$ for all $j \in \{1, \ldots, h\}$. □

For $\Gamma(3) = \emptyset$, recall that $\mathbb{E}(\Gamma) = \{\emptyset\}$. For $\Gamma(3) \neq \emptyset$, recall that $\mathbb{E}(\Gamma)$ is the set of all skew-symmetric subsets S, where $S = \langle\langle x_1 \rangle\rangle \cup \cdots \cup \langle\langle x_k \rangle\rangle$ for some $x_1, \ldots, x_k \in \Gamma(3)$. Indeed, we can replace condition (i) of Theorem 5.2.1 by $\sum\limits_{s\in \mathrm{Cl}(x)} c_s = \sum\limits_{s\in \mathrm{Cl}(y)} c_s$ for all $x, y \in \langle\langle g \rangle\rangle$ and $g \in \Gamma(3)$.

Theorem 5.2.2. *Let Γ be a finite group and $\mathrm{Cay}(\Gamma, S)$ be a normal directed Cayley graph. Then $\mathrm{Cay}(\Gamma, S)$ is HS-integral if and only if $S \in \mathbb{E}(\Gamma)$.*

Proof. Let $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$ and $x = \sum\limits_{g\in\Gamma} c_g g$, where

$$c_g = \begin{cases} -\omega_3^2 & \text{if } g \in S\\ -\omega_3 & \text{if } g \in S^{-1}\\ 0 & \text{otherwise.}\end{cases}$$

Note that $-\omega_3^2 = \omega_6$ and $-\omega_3 = \omega_6^5$. Thus $\chi_j(x) = \sum\limits_{s\in S}(-\omega_3^2\chi_j(s) - \omega_3\chi_j(s^{-1}))$, and so $\frac{\chi_j(x)}{\chi_j(\mathbf{1})}$ is an HS-eigenvalue of $\text{Cay}(\Gamma, S)$. Assume that the normal directed Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral. Thus $\chi_j(x)$ is an integer for each $j \in \{1,\ldots,h\}$, and therefore the three conditions of Theorem 5.2.1 are satisfied for x. Using the fact that $g \sim g^{-1}$, and conditions (ii) and (iii) of Theorem 5.2.1, we get $\Im\left(\sum\limits_{s\in \text{Cl}(g)} c_s\right) = 0$ for all $g \in \Gamma \setminus \Gamma(3)$. Note that S is a union of some conjugacy classes of Γ. Therefore, if $g \in S$ then $\text{Cl}(g) \subseteq S$, and so by the definition of c_g, we get $\Im\left(\sum\limits_{s\in \text{Cl}(g)} c_s\right) = \frac{\sqrt{3}|\,\text{Cl}(g)|}{2} \neq 0$. Thus $S \cap (\Gamma \setminus \Gamma(3)) = \emptyset$, that is, $S \subseteq \Gamma(3)$. Again, let $g_1 \in S$, $g_2 \in \Gamma(3)$ and $g_1 \simeq g_2$. By the first condition of Theorem 5.2.1, we get $0 \neq \sum\limits_{s\in \text{Cl}(g_1)} c_s = \sum\limits_{s\in \text{Cl}(g_2)} c_s$, which implies that $g_2 \in S$. Thus $g_1 \in S$ gives $\langle\langle g_1\rangle\rangle \subseteq S$. Hence $S \in \mathbb{E}(\Gamma)$.

Conversely, assume that $S \in \mathbb{E}(\Gamma)$. Let $\text{Cay}(\Gamma, S)$ be a normal directed Cayley graph, so that S is a union of some conjugacy classes of Γ. Let

$$S = \langle\langle x_1\rangle\rangle \cup \cdots \cup \langle\langle x_r\rangle\rangle = \text{Cl}(y_1) \cup \cdots \cup \text{Cl}(y_k) \subseteq \Gamma(3)$$

for some $x_1,\ldots,x_r,y_1,\ldots,y_k \in \Gamma(3)$. We have

$$S^{-1} = \langle\langle x_1^{-1}\rangle\rangle \cup \cdots \cup \langle\langle x_r^{-1}\rangle\rangle = \text{Cl}(y_1^{-1}) \cup \cdots \cup \text{Cl}(y_k^{-1}) \subseteq \Gamma(3).$$

Now for $g_1, g_2 \in \Gamma(3)$, if $g_1 \simeq g_2$ then $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq S$ or $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq S^{-1}$ or $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq (S \cup S^{-1})^c$. Note that $|\,\text{Cl}(g_1)| = |\,\text{Cl}(g_2)|$. For all the cases, using the definition of c_g, we find

$$\sum_{s\in \text{Cl}(g_1)} c_s = \sum_{s\in \text{Cl}(g_2)} c_s.$$

Thus condition (i) of Theorem 5.2.1 holds. If $g_1, g_2 \in \Gamma \setminus \Gamma(3)$ and $g_1 \sim g_2$, then clearly $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq \Gamma \setminus \Gamma(3)$. Therefore $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq (S \cup S^{-1})^c$. Accordingly,

$$\sum_{s\in \text{Cl}(g_1)} c_s = 0 = \sum_{s\in \text{Cl}(g_2)} c_s.$$

Hence condition (ii) of Theorem 5.2.1 also holds.

Again for $g \in \Gamma$, we have $\text{Cl}(g) \subseteq S$ if and only if $\text{Cl}(g^{-1}) \subseteq S^{-1}$. Therefore we have $\sum\limits_{s\in \text{Cl}(g)} c_s = \sum\limits_{s\in \text{Cl}(g^{-1})} \overline{c}_s$, and so condition (iii) of Theorem 5.2.1 also holds. Thus by Theorem 5.2.1, $\chi_j(x)$ is a rational number for each $j \in \{1,\ldots,h\}$. Consequently, the HS-eigenvalue $\mu_j := \frac{\chi_j(x)}{\chi_j(\mathbf{1})}$ of $\text{Cay}(\Gamma, S)$ is a rational algebraic integer, and hence an integer for each $j \in \{1,\ldots,h\}$. □

In the following example, we illustrate an use of Theorem 5.2.2.

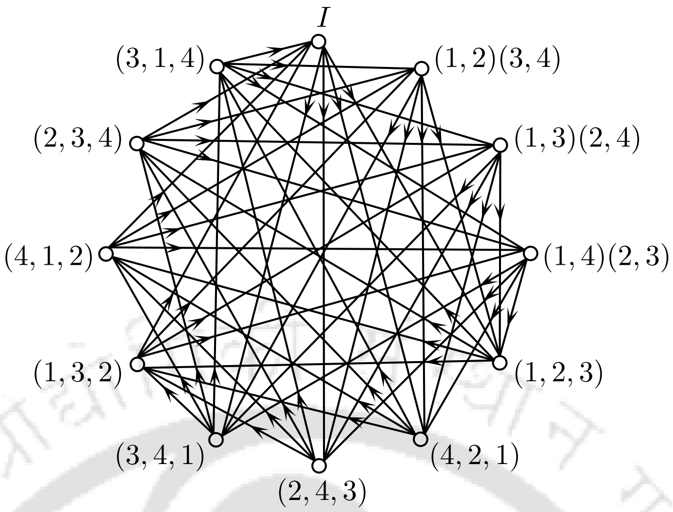

Figure 5.1: The directed graph $\text{Cay}(A_4, \{(1,2,3),(4,2,1),(2,4,3),(3,4,1)\})$

Example 5.2.1. Consider $S = \{(1,2,3),(4,2,1),(2,4,3),(3,4,1)\}$ in the alternating group A_4. The conjugacy classes of A_4 are $\{I\}, \text{Cl}((1,2)(3,4)), \text{Cl}((1,2,3))$ and $\text{Cl}((1,3,2))$, where

$$I = (1)(2)(3)(4),$$

$$\text{Cl}((1,2)(3,4)) = \{(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\},$$

$$\text{Cl}((1,2,3)) = \{(1,2,3),(4,2,1),(2,4,3),(3,4,1)\} \text{ and}$$

$$\text{Cl}((1,3,2)) = \{(1,3,2),(4,1,2),(2,3,4),(3,1,4)\}.$$

The normal directed Cayley graph $\text{Cay}(A_4, S)$ is shown in Figure 5.1. We see that $S = \langle\langle(1,2,3)\rangle\rangle \cup \langle\langle(4,2,1)\rangle\rangle \cup \langle\langle(2,4,3)\rangle\rangle \cup \langle\langle(3,4,1)\rangle\rangle = \text{Cl}((1,2,3))$. Therefore $S \in \mathbb{E}(\Gamma)$, and hence $\text{Cay}(A_4, S)$ is HS-integral by Theorem 5.2.2. The character table of the group A_4 is given in Table 5.1 [24], where $\text{Irr}(A_4) = \{\chi_1, \chi_2, \chi_3, \chi_4\}$. Further, using Corollary 5.1.3, the HS-spectrum of $\text{Cay}(A_4, S)$ is obtained as $\{[\mu_1]^1, [\mu_2]^1, [\mu_3]^1, [\mu_4]^9\}$, where $\mu_1 = 4(\omega_6 + \omega_6^5) = 4$, $\mu_2 = 4(\omega_6\omega_3 + \omega_6^5\omega_3^2) = -8$, $\mu_3 = 4(\omega_6\omega_3^2 + \omega_6^5\omega_3) = 4$ and $\mu_4 = 0$.

	I	$\text{Cl}((1,2)(3,4))$	$\text{Cl}((1,2,3))$	$\text{Cl}((1,3,2))$
χ_1	1	1	1	1
χ_2	1	1	ω_3	ω_3^2
χ_3	1	1	ω_3^2	ω_3
χ_4	3	-1	0	0

Table 5.1: Character table of A_4

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5.3 HS-integral normal mixed Cayley graphs

In this section, we extend Theorem 5.2.2 to normal mixed Cayley graphs.

Lemma 5.3.1. *Let S be a skew-symmetric subset of a finite group Γ and $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Let S be expressible as a union of some conjugacy classes of Γ and $t(\neq 0) \in \mathbb{Q}$. If*

$$\frac{1}{\chi_j(\mathbf{1})} \sum_{s\in S} \mathbf{i}t\sqrt{3}\left(\chi_j(s) - \chi_j(s^{-1})\right)$$

is an integer for each $j \in \{1, \ldots, h\}$, then $S \in \mathbb{E}(\Gamma)$.

Proof. Let $x = \sum\limits_{g\in\Gamma} c_g g \in \mathbb{Q}(\omega_3)\Gamma$, where

$$c_g = \begin{cases} \mathbf{i}t\sqrt{3} & \text{if } g \in S \\ -\mathbf{i}t\sqrt{3} & \text{if } g \in S^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\frac{\chi_j(x)}{\chi_j(\mathbf{1})} = \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s\in S} \mathbf{i}t\sqrt{3}\left(\chi_j(s) - \chi_j(s^{-1})\right)$. Assume that $\frac{\chi_j(x)}{\chi_j(\mathbf{1})}$ is an integer for each $j \in \{1, \ldots, h\}$. Therefore, all the three conditions of Theorem 5.2.1 are satisfied for x. Using the fact that $g \sim g^{-1}$, and conditions (ii) and (iii) of Theorem 5.2.1, we get $\Im\left(\sum\limits_{s\in \mathrm{Cl}(g)} c_s\right) = 0$ for all $g \in \Gamma \setminus \Gamma(3)$, and so we must have $S \cup S^{-1} \subseteq \Gamma(3)$. Again, let $g_1 \in S$, $g_2 \in \Gamma(3)$ and $g_1 \simeq g_2$. The first condition of Theorem 5.2.1 gives

$$\sum_{s\in \mathrm{Cl}(g_1)} c_s = \sum_{s\in \mathrm{Cl}(g_2)} c_s.$$

Note that $\sum\limits_{s\in \mathrm{Cl}(g_1)} c_s = \mathbf{i}t\sqrt{3}|\operatorname{Cl}(g_1)|$. Therefore $\sum\limits_{s\in \mathrm{Cl}(g_2)} c_s = \mathbf{i}t\sqrt{3}|\operatorname{Cl}(g_1)|$, and so $g_2 \in S$. Thus $g_1 \in S$ implies $\langle\!\langle g_1 \rangle\!\rangle \subseteq S$. Hence $S \in \mathbb{E}(\Gamma)$. □

In Chapter 3, we proved that if Γ is an abelian group, then $\langle\!\langle x \rangle\!\rangle \cup \langle\!\langle x^{-1} \rangle\!\rangle = [x]$ for each $x \in \Gamma(3)$. Note that this result and its proof also hold good for non-abelian group. In the subsequent discussion, we use this fact for non-abelian group.

Lemma 5.3.2. *Let S be a skew-symmetric subset of a finite group Γ and $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Let S be expressible as a union of some conjugacy classes of Γ and $t(\neq 0) \in \mathbb{Q}$. If*

$$\frac{1}{\chi_j(\mathbf{1})} \sum_{s\in S} \mathbf{i}t\sqrt{3}\left(\chi_j(s) - \chi_j(s^{-1})\right)$$

is an integer for each $j \in \{1, \ldots, h\}$, then $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s\in S\cup S^{-1}} \chi_j(s)$ is also an integer for each $j \in \{1, \ldots, h\}$.

Proof. Assume that $\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S}\mathbf{i}t\sqrt{3}\left(\chi_j(s)-\chi_j(s^{-1})\right)$ is an integer for each $j\in\{1,\dots,h\}$. By Lemma 5.3.1 we have $S\in\mathbb{E}(\Gamma)$, and so $S=\langle\langle x_1\rangle\rangle\cup\cdots\cup\langle\langle x_k\rangle\rangle$ for some $x_1,\dots,x_k\in\Gamma(3)$. Therefore, we get

$$S\cup S^{-1}=(\langle\langle x_1\rangle\rangle\cup\cdots\cup\langle\langle x_k\rangle\rangle)\cup\left(\langle\langle x_1^{-1}\rangle\rangle\cup\cdots\cup\langle\langle x_k^{-1}\rangle\rangle\right)=[x_1]\cup\cdots\cup[x_k]\in\mathbb{B}(\Gamma).$$

Thus by Theorem 1.6.7, $\mathrm{Cay}(\Gamma,S\cup S^{-1})$ is integral, that is, $\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S\cup S^{-1}}\chi_j(s)$ is an integer for each $j\in\{1,\dots,h\}$. □

In the next result, we use the fact that the HS-eigenvalues of a mixed Cayley graph are algebraic integers. See Theorem 2.6 of [31] for details.

Lemma 5.3.3. *If Γ is a finite group, then the normal mixed Cayley graph $\mathrm{Cay}(\Gamma,S)$ is HS-integral if and only if $\mathrm{Cay}(\Gamma,S\setminus\overline{S})$ is integral (or HS-integral) and $\mathrm{Cay}(\Gamma,\overline{S})$ is HS-integral.*

Proof. Let $\mathrm{Irr}(\Gamma)=\{\chi_1,\cdots,\chi_h\}$. By Lemma 5.1.1, the HS-spectrum of the normal mixed Cayley graph $\mathrm{Cay}(\Gamma,S)$ is $\{[\gamma_1]^{d_1^2},\dots,[\gamma_h]^{d_h^2}\}$, where $\gamma_j=\lambda_j+\mu_j$,

$$\lambda_j=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S\setminus\overline{S}}\chi_j(s),\ \ \mu_j=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}(\omega_6\chi_j(s)+\omega_6^5\chi_j(s^{-1})),$$

and $d_j=\chi_j(\mathbf{1})$ for each $j\in\{1,\dots,h\}$. Note that $\{[\lambda_1]^{d_1^2},\dots,[\lambda_h]^{d_h^2}\}$ is the spectrum of $\mathrm{Cay}(\Gamma,S\setminus\overline{S})$ and $\{[\mu_1]^{d_1^2},\dots,[\mu_h]^{d_h^2}\}$ is the HS-spectrum of $\mathrm{Cay}(\Gamma,\overline{S})$.

Assume that the mixed Cayley graph $\mathrm{Cay}(\Gamma,S)$ is HS-integral. Let $j\in\{1,\dots,h\}$. By Lemma 1.3.4, there exists $k\in\{1,\dots,h\}$ such that $\chi_k=\overline{\chi_j}$. Therefore, $\chi_j(\mathbf{1})=\chi_k(\mathbf{1})$ and

$$\lambda_j=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S\setminus\overline{S}}\chi_j(s^{-1})=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S\setminus\overline{S}}\overline{\chi_j(s)}=\frac{1}{\chi_k(\mathbf{1})}\sum_{s\in S\setminus\overline{S}}\chi_k(s)=\lambda_k.$$

Now we have

$$\begin{aligned}\gamma_j-\gamma_k&=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\left(\omega_6\chi_j(s)+\omega_6^5\chi_j(s^{-1})\right)-\frac{1}{\chi_k(\mathbf{1})}\sum_{s\in\overline{S}}\left(\omega_6\chi_k(s)+\omega_6^5\chi_k(s^{-1})\right)\\&=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\left(\omega_6\chi_j(s)+\omega_6^5\chi_j(s^{-1})\right)-\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\left(\omega_6\overline{\chi_j(s)}+\omega_6^5\overline{\chi_j(s^{-1})}\right)\\&=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\left(\omega_6\chi_j(s)+\omega_6^5\chi_j(s^{-1})\right)-\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\left(\omega_6\chi_j(s^{-1})+\omega_6^5\chi_j(s)\right)\\&=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\left((\omega_6-\omega_6^5)\chi_j(s)+(\omega_6^5-\omega_6)\chi_j(s^{-1})\right)\\&=\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\mathbf{i}\sqrt{3}\left(\chi_j(s)-\chi_j(s^{-1})\right).\end{aligned}$$

By assumption $\gamma_j, \gamma_k \in \mathbb{Z}$, and so $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s\in\overline{S}} \mathbf{i}\sqrt{3}\left(\chi_j(s) - \chi_j(s^{-1})\right) \in \mathbb{Z}$ for each $j \in \{1,\dots,h\}$. Therefore by Lemma 5.3.2, we get $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s\in\overline{S}\cup(\overline{S})^{-1}} \chi_j(s) \in \mathbb{Z}$ for each $j \in \{1,\dots,h\}$. Since

$$\mu_j = \frac{1}{2\chi_j(\mathbf{1})} \sum_{s\in\overline{S}\cup(\overline{S})^{-1}} \chi_j(s) + \frac{1}{2\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} \mathbf{i}\sqrt{3}\left(\chi_j(s) - \chi_j(s^{-1})\right),$$

μ_j is a rational algebraic integer, and hence it is an integer for each $j \in \{1,\dots,h\}$. Thus $\text{Cay}(\Gamma, \overline{S})$ is HS-integral. Now we have $\gamma_j, \mu_j \in \mathbb{Z}$, and so $\lambda_j = \gamma_j - \mu_j \in \mathbb{Z}$ for each $j \in \{1,\dots,h\}$. Hence $\text{Cay}(\Gamma, S\setminus\overline{S})$ is also integral.

Conversely, assume that $\text{Cay}(\Gamma, S\setminus\overline{S})$ is integral and $\text{Cay}(\Gamma, \overline{S})$ is HS-integral. Then Lemma 5.1.1 implies that $\text{Cay}(\Gamma, S)$ is HS-integral. □

Theorem 5.3.4. *Let Γ be a finite group and $\text{Cay}(\Gamma, S)$ be a normal mixed Cayley graph. Then $\text{Cay}(\Gamma, S)$ is HS-integral if and only if $S\setminus\overline{S} \in \mathbb{B}(\Gamma)$ and $\overline{S} \in \mathbb{E}(\Gamma)$.*

Proof. By Lemma 5.3.3, $\text{Cay}(\Gamma, S)$ is HS-integral if and only if $\text{Cay}(\Gamma, S\setminus\overline{S})$ is integral and $\text{Cay}(\Gamma, \overline{S})$ is HS-integral. Now the proof follows from Theorem 1.6.7 and Theorem 5.2.2. □

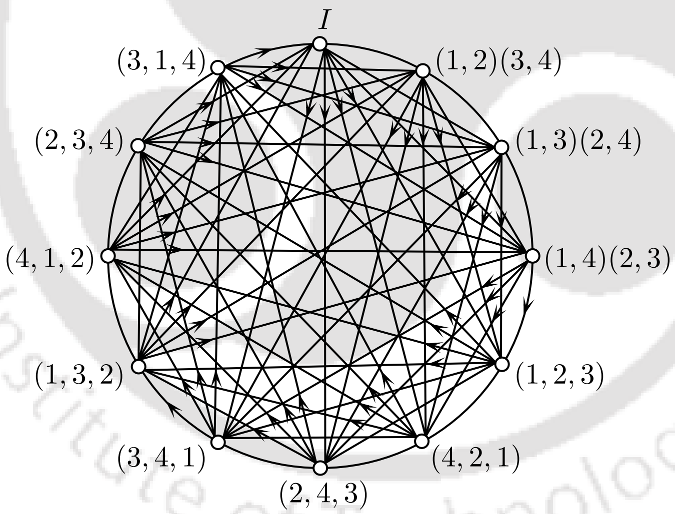

Figure 5.2: The mixed graph $\text{Cay}(A_4, S)$

We give the following example to illustrate Theorem 5.3.4.

Example 5.3.1. Consider

$$S = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (4,2,1), (2,4,3), (3,4,1)\}$$

in the alternating group A_4. The normal mixed Cayley graph $\text{Cay}(A_4, S)$ is shown in Figure 5.2. We find that

$$\overline{S} = \langle\langle(1,2,3)\rangle\rangle \cup \langle\langle(4,2,1)\rangle\rangle \cup \langle\langle(2,4,3)\rangle\rangle \cup \langle\langle(3,4,1)\rangle\rangle = \text{Cl}((1,2,3)) \in \mathbb{E}(\Gamma)$$

and

$$S \setminus \overline{S} = [(1,2)(3,4)] \cup [(1,3)(2,4)] \cup [(1,4)(2,3)] = \mathrm{Cl}((1,2)(3,4)) \in \mathbb{B}(\Gamma).$$

Using Theorem 5.3.4, $\mathrm{Cay}(A_4, S)$ is HS-integral. The character table of A_4 is given in Table 5.1. Further, using Lemma 5.1.1, the HS-spectrum of $\mathrm{Cay}(A_4, S)$ is obtained as $\{[\gamma_1]^1, [\gamma_2]^1, [\gamma_3]^1, [\gamma_4]^9\}$, where $\gamma_1 = 3 + 4(\omega_6 + \omega_6^5) = 7$, $\gamma_2 = 3 + 4(\omega_6\omega_3 + \omega_6^5\omega_3^2) = -5$, $\gamma_3 = 3 + 4(\omega_6\omega_3^2 + \omega_6^5\omega_3) = 7$ and $\gamma_4 = -1$.

5.4 Eisenstein integral normal mixed Cayley graphs

Assume that S is a union of some conjugacy classes of a finite group Γ, $\mathbf{1} \notin S$ and $\mathrm{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Using the function $f\colon \Gamma \to \{0,1\}$ defined by

$$f(s) = \begin{cases} 1 & \text{if } s \in S \\ 0 & \text{otherwise} \end{cases}$$

in Theorem 1.3.6, we find that $\frac{1}{\chi_j(\mathbf{1})} \sum\limits_{s\in S} \chi_j(s)$ is an eigenvalue of the normal mixed Cayley graph $\mathrm{Cay}(\Gamma, S)$ for each $j \in \{1, \ldots, h\}$. Indeed, all the eigenvalues of $\mathrm{Cay}(\Gamma, S)$ are of this form.

For each $j \in \{1, \ldots, h\}$, define

$$f_j(S) := \frac{1}{\chi_j(\mathbf{1})} \sum_{s \in S\setminus\overline{S}} \chi_j(s) \quad \text{and} \quad g_j(S) := \frac{1}{\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} (\omega\chi_j(s) + \overline{\omega}\chi_j(s^{-1})),$$

where $\omega = \frac{1}{2} - \frac{\mathbf{i}\sqrt{3}}{6}$. Let $j \in \{1, \ldots, h\}$. By Lemma 1.3.4, there exists $k \in \{1, \ldots, h\}$ such that $\chi_k = \overline{\chi}_j$. Note that

$$\begin{aligned} g_j(S) + \omega_3(g_j(S) - g_k(S)) =& (1+\omega_3)g_j(S) - \omega_3 g_k(S) \\ =& \frac{1+\mathbf{i}\sqrt{3}}{2\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} \left[\left(\frac{1}{2} - \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_j(s) + \left(\frac{1}{2} + \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_j(s^{-1}) \right] \\ &+ \frac{1-\mathbf{i}\sqrt{3}}{2\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} \left[\left(\frac{1}{2} - \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_k(s) + \left(\frac{1}{2} + \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_k(s^{-1}) \right] \\ =& \frac{1+\mathbf{i}\sqrt{3}}{2\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} \left[\left(\frac{1}{2} - \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_j(s) + \left(\frac{1}{2} + \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_j(s^{-1}) \right] \\ &+ \frac{1-\mathbf{i}\sqrt{3}}{2\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} \left[\left(\frac{1}{2} - \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_j(s^{-1}) + \left(\frac{1}{2} + \frac{\mathbf{i}\sqrt{3}}{6}\right)\chi_j(s) \right] \\ =& \frac{1}{\chi_j(\mathbf{1})} \sum_{s\in\overline{S}} \chi_j(s). \end{aligned}$$

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Therefore

$$\frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S}\chi_j(s) = f_j(S) + g_j(S) + \omega_3(g_j(S) - g_k(S)). \tag{5.7}$$

Note that if $\chi_k = \overline{\chi}_j$, then $f_j(S) = f_k(S)$ and $g_j(S) - g_k(S) = [f_j(S) + g_j(S)] - [f_k(S) + g_k(S)]$. Therefore if $f_j(S) + g_j(S)$ is an integer for each $j \in \{1, \ldots, h\}$, then $g_j(S) - g_k(S)$ is also an integer for each $j \in \{1, \ldots, h\}$. Hence the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral if and only if $f_j(S) + g_j(S)$ is an integer for each $j \in \{1, \ldots, h\}$.

Lemma 5.4.1. *If Γ is a finite group, then the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral if and only if $2f_j(S)$ and $2g_j(S)$ are integers of the same parity for each $j \in \{1, \ldots, h\}$.*

Proof. Assume that the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral. Then $f_j(S) + g_j(S)$ and $g_j(S) - g_k(S)$ are integers for each $j \in \{1, \ldots, h\}$, where $\chi_k = \overline{\chi}_j$. Note that

$$g_j(S) - g_k(S) = \frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\frac{-\mathbf{i}\sqrt{3}}{3}(\chi_j(s) - \chi_j(s^{-1})).$$

Therefore by Lemma 5.3.2, $\frac{1}{\chi_j(\mathbf{1})}\sum\limits_{s\in\overline{S}\cup(\overline{S})^{-1}}\chi_j(s) \in \mathbb{Z}$. Using

$$2g_j(S) = \frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}\cup(\overline{S})^{-1}}\chi_j(s) - \frac{1}{\chi_j(\mathbf{1})}\sum_{s\in\overline{S}}\frac{\mathbf{i}\sqrt{3}}{3}(\chi_j(s) - \chi_j(s^{-1})),$$

we find that $2g_j(S)$ is an integer. Since $2f_j(S) = 2(f_j(S) + g_j(S)) - 2g_j(S)$, we see that $2f_j(S)$ is also an integer of the same parity with $2g_j(S)$.

Conversely, assume that $2f_j(S)$ and $2g_j(S)$ are integers of the same parity for each $j \in \{1, \ldots, h\}$. Then $f_j(S) + g_j(S)$ is an integer for each $j \in \{1, \ldots, h\}$. Hence the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral. □

Lemma 5.4.2. *The normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral if and only if $f_j(S)$ and $g_j(S)$ are integers for each $j \in \{1, \ldots, h\}$.*

Proof. Let $j \in \{1, \ldots, h\}$. Due to Lemma 5.4.1, it is enough to prove that $2f_j(S)$ and $2g_j(S)$ are integers of the same parity if and only if $f_j(S)$ and $g_j(S)$ are integers. If $f_j(S)$ and $g_j(S)$ are integers, then clearly $2f_j(S)$ and $2g_j(S)$ are even integers. Conversely, assume that $2f_j(S)$ and $2g_j(S)$ are integers of the same parity. Since $f_j(S)$ is an algebraic integer, the integrality of $2f_j(S)$ implies that $f_j(S)$ is an integer. Thus $2f_j(S)$ is an even integer, and so by assumption $2g_j(S)$ is also an even integer. Hence $g_j(S)$ is an integer. □

Theorem 5.4.3. *Let Γ be a finite group. If the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral, then $\text{Cay}(\Gamma, S)$ is HS-integral.*

Proof. Assume that $\text{Cay}(\Gamma, S)$ is Eisenstein integral. By Lemma 5.4.2, we find that $f_j(S)$ and $g_j(S)$ are integers for each $j \in \{1, \ldots, h\}$. Note that $f_j(S)$ is an eigenvalue of the normal undirected Cayley graph $\text{Cay}(\Gamma, S \setminus \overline{S})$. By Theorem 1.6.7, $f_j(S)$ is an integer for each $j \in \{1, \ldots, h\}$ if and only if $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$. Further,

$$\frac{1}{\chi_j(\mathbf{1})} \sum_{s \in \overline{S}} \frac{-\mathbf{i}\sqrt{3}}{3} (\chi_j(s) - \chi_j(s^{-1})) = g_j(S) - g_k(S),$$

and that $g_j(S) - g_k(S)$ is an integer for each $j \in \{1, \ldots, h\}$, where $\chi_k = \overline{\chi_j}$. Using Lemma 5.3.1, we see that $\overline{S} \in \mathbb{E}(\Gamma)$. Thus by Theorem 5.3.4, $\text{Cay}(\Gamma, S)$ is HS-integral. □

Lemma 5.4.4. *Let $x \in \Gamma$ and $\text{ord}(x) = 3^t m$. If $m \not\equiv 0 \pmod 3$, then the following assertions hold.*

(i) If $t = 1$, then $[x] = x^m[x^3] \cup x^{2m}[x^3]$.

(ii) If $t = 1$, then

$$\langle\langle x \rangle\rangle = \begin{cases} x^m[x^3] & \text{if } m \equiv 1 \pmod 3 \\ x^{2m}[x^3] & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

(iii) If $t \geq 2$, then

$$[x] = \begin{cases} x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}\langle\langle x^{-3} \rangle\rangle \cup x^{5m}\langle\langle x^{-3} \rangle\rangle & \text{if } m \equiv 1 \pmod 3 \\ x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}\langle\langle x^{3} \rangle\rangle \cup x^{5m}\langle\langle x^{3} \rangle\rangle & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

(iv) If $t \geq 2$, then

$$[x] = \begin{cases} x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}\langle\langle x^{3} \rangle\rangle \cup x^{5m}\langle\langle x^{3} \rangle\rangle & \text{if } m \equiv 1 \pmod 3 \\ x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}\langle\langle x^{-3} \rangle\rangle \cup x^{5m}\langle\langle x^{-3} \rangle\rangle & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

(v) If $t \geq 2$, then $[x] = x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}[x^3] \cup x^{5m}[x^3] \cup x^{7m}[x^3] \cup x^{8m}[x^3]$.

(vi) If $t \geq 2$, then

$$\langle\langle x \rangle\rangle = \begin{cases} x^m[x^3] \cup x^{4m}\langle\langle x^{-3} \rangle\rangle & \text{if } m \equiv 1 \pmod 3 \\ x^{2m}[x^3] \cup x^{5m}\langle\langle x^{3} \rangle\rangle & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

(vii) If $t \geq 2$, then

$$\langle\langle x \rangle\rangle = \begin{cases} x^{7m}[x^3] \cup x^{4m}\langle\langle x^{3} \rangle\rangle & \text{if } m \equiv 1 \pmod 3 \\ x^{8m}[x^3] \cup x^{5m}\langle\langle x^{-3} \rangle\rangle & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

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(viii) If $t \geq 2$, then

$$\langle\langle x\rangle\rangle = \begin{cases} x^m[x^3] \cup x^{4m}[x^3] \cup x^{7m}[x^3] & \text{if } m \equiv 1 \pmod 3 \\ x^{2m}[x^3] \cup x^{5m}[x^3] \cup x^{8m}[x^3] & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Proof. (i) Assume that $\text{ord}(x) = 3m$ and $m \not\equiv 0 \pmod 3$. Let us take $x^{m+3r} \in x^m[x^3]$ for some $r \in G_m(1)$. Then $\gcd(r, m) = 1$, and so $\gcd(m + 3r, 3m) = 1$. Therefore $x^m[x^3] \subseteq [x]$. Similarly, we have $x^{2m}[x^3] \subseteq [x]$. Therefore $x^m[x^3] \cup x^{2m}[x^3] \subseteq [x]$. Note that $|[x]| = \varphi(3m) = 2\varphi(m)$, $|x^m[x^3]| = \varphi(m) = |x^{2m}[x^3]|$, and that $x^m[x^3] \cup x^{2m}[x^3]$ is a disjoint union. Thus, the sizes of $[x]$ and $x^m[x^3] \cup x^{2m}[x^3]$ are equal, and therefore $[x] = x^m[x^3] \cup x^{2m}[x^3]$.

(ii) Assume that $\text{ord}(x) = 3m$ and $m \not\equiv 0 \pmod 3$. Let $m \equiv 1 \pmod 3$. We see that $\gcd(r, m) = 1$ if and only if $\gcd(m + 3r, 3m) = 1$. Also $m + 3r \equiv 1 \pmod 3$. Therefore

$$x^m[x^3] = \{x^{m+3r} : r \in G_m(1)\} \subseteq \{x^k : k \in G^1_{3m,3}(1)\} = \langle\langle x\rangle\rangle.$$

Since the sets $x^m[x^3]$ and $\langle\langle x\rangle\rangle$ are of equal size, we get $x^m[x^3] = \langle\langle x\rangle\rangle$. Similarly, if $m \equiv 2 \pmod 3$, we have $x^{2m}[x^3] = \langle\langle x\rangle\rangle$.

(iii) Assume that $p = 3^t m$, $t \geq 2$ and $m \equiv 1 \pmod 3$. Let $x^{m+3r} \in x^m[x^3]$ for some $r \in G_{\frac{p}{3}}(1)$. Then $\gcd(r, \frac{p}{3}) = 1$, and so $\gcd(m+3r, p) = 1$. Thus $x^m[x^3] \subseteq [x]$. Similarly, $x^{2m}[x^3] \subseteq [x]$. Now let $x^{4m+3r} \in x^{4m}\langle\langle x^{-3}\rangle\rangle$ for some $r \in G^2_{\frac{p}{3},3}(1)$. Again, $\gcd(r, \frac{p}{3}) = 1$ implies that $\gcd(4m+3r, p) = 1$. Therefore $x^{4m}\langle\langle x^{-3}\rangle\rangle \subseteq [x]$. Similarly, $x^{5m}\langle\langle x^{-3}\rangle\rangle \subseteq [x]$. Thus $x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}\langle\langle x^{-3}\rangle\rangle \cup x^{5m}\langle\langle x^{-3}\rangle\rangle \subseteq [x]$. Note that $|[x]| = 2 \times 3^{t-1}\varphi(m)$. Also, $|x^m[x^3]| = 2 \times 3^{t-2}\varphi(m) = |x^{2m}[x^3]|$, $|x^{4m}\langle\langle x^{-3}\rangle\rangle| = 3^{t-2}\varphi(m) = |x^{5m}\langle\langle x^{-3}\rangle\rangle|$, and that $x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}\langle\langle x^{-3}\rangle\rangle \cup x^{5m}\langle\langle x^{-3}\rangle\rangle$ is a disjoint union. Thus, the sizes of $[x]$ and $x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}\langle\langle x^{-3}\rangle\rangle \cup x^{5m}\langle\langle x^{-3}\rangle\rangle$ are equal, and hence these two sets are equal. For $m \equiv 2 \pmod 3$, the proof follows the similar steps as in the case of $m \equiv 1 \pmod 3$.

(iv) The proof is similar to the proof Part (iii). For the sake of completeness, we provide the proof for the case $m \equiv 1 \pmod 3$. Assume that $p = 3^t m$, $t \geq 2$ and $m \equiv 1 \pmod 3$. Let $x^{7m+3r} \in x^{7m}[x^3]$ for some $r \in G_{\frac{p}{3}}(1)$. Then $\gcd(r, \frac{p}{3}) = 1$, and so $\gcd(7m + 3r, p) = 1$. Thus $x^{7m}[x^3] \subseteq [x]$. Similarly, $x^{8m}[x^3] \subseteq [x]$. Now let $x^{4m+3r} \in x^{4m}\langle\langle x^3\rangle\rangle$ for some $r \in G^1_{\frac{p}{3},3}(1)$. Again, $\gcd(r, \frac{p}{3}) = 1$ gives $\gcd(4m + 3r, p) = 1$. Thus, $x^{4m}\langle\langle x^3\rangle\rangle \subseteq [x]$. Similarly, $x^{5m}\langle\langle x^3\rangle\rangle \subseteq [x]$. Thus $x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}\langle\langle x^3\rangle\rangle \cup x^{5m}\langle\langle x^3\rangle\rangle \subseteq [x]$. Note that $x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}\langle\langle x^3\rangle\rangle \cup x^{5m}\langle\langle x^3\rangle\rangle$ is a disjoint union, and so its size is equal to $2 \times 3^{t-2}\varphi(m) + 2 \times 3^{t-2}\varphi(m) + 3^{t-2}\varphi(m) + 3^{t-2}\varphi(m)$, which is equal to the size $2 \times 3^{t-1}\varphi(m)$ of $[x]$. Hence we have the desired equality.

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(v) Combine Part (iii) and Part (iv), and use $[x^3] = \langle\langle x^3\rangle\rangle \cup \langle\langle x^{-3}\rangle\rangle$ to get the proof of this part.

(vi) Assume that $p = 3^t m$, $t \geq 2$ and $m \equiv 1 \pmod 3$. We see that if $r \in G_{\frac{p}{3}}(1)$, then $m + 3r \in G^1_{p,3}(1)$. Similarly, if $r \in G^2_{\frac{p}{3},3}(1)$, then $4m + 3r \in G^1_{p,3}(1)$. Thus we have $x^m[x^3] \cup x^{4m}\langle\langle x^{-3}\rangle\rangle \subseteq \langle\langle x\rangle\rangle$. Since the sizes of $x^m[x^3] \cup x^{4m}\langle\langle x^{-3}\rangle\rangle$ and $\langle\langle x\rangle\rangle$ are equal, we find that $x^m[x^3] \cup x^{4m}\langle\langle x^{-3}\rangle\rangle = \langle\langle x\rangle\rangle$. Similarly, we have $x^{2m}[x^3] \cup x^{5m}\langle\langle x^3\rangle\rangle = \langle\langle x\rangle\rangle$ for $m \equiv 2 \pmod 3$.

(vii) The proof of this part follows similar steps as in Part (vi). For the sake of completeness, we provide the proof for the case $m \equiv 2 \pmod 3$. Assume that $p = 3^t m$, $t \geq 2$ and $m \equiv 2 \pmod 3$. We see that if $r \in G_{\frac{p}{3}}(1)$, then $8m + 3r \in G^1_{p,3}(1)$. Also, if $r \in G^2_{\frac{p}{3},3}(1)$, then $5m + 3r \in G^1_{p,3}(1)$. Thus $x^{8m}[x^3] \cup x^{5m}\langle\langle x^{-3}\rangle\rangle \subseteq \langle\langle x\rangle\rangle$. Since the sizes of $x^{8m}[x^3] \cup x^{5m}\langle\langle x^{-3}\rangle\rangle$ and $\langle\langle x\rangle\rangle$ are equal, we find that $x^{8m}[x^3] \cup x^{5m}\langle\langle x^{-3}\rangle\rangle = \langle\langle x\rangle\rangle$.

(viii) Combine Part (vi) and Part (vii), and use $[x^3] = \langle\langle x^3\rangle\rangle \cup \langle\langle x^{-3}\rangle\rangle$ to get the proof of this part. □

Let $x \in \Gamma(3)$ be an element of order m. The element x is said to be *tolerable* if $x^r \notin \mathrm{Cl}(x)$ for all $r \in G^2_{m,3}(1)$. The following lemma characterizes tolerable elements in terms of skew-symmetric sets. This lemma is similar to Lemma 4.3.6.

Lemma 5.4.5. *If $x \in \Gamma(3)$, then x is tolerable if and only if the set $\bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$ is skew-symmetric.*

Proof. We see that if $m = \mathrm{ord}(x)$, then

$$\bigcup_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle = \{g^{-1}x^r g \colon g \in \Gamma, r \in G^1_{m,3}(1)\} = \bigcup_{s\in \langle\langle x\rangle\rangle} \mathrm{Cl}(s).$$

Assume that x is not tolerable, so that $x^r \in \mathrm{Cl}(x)$ for some $r \in G^2_{m,3}(1)$. As $m - r \in G^1_{m,3}(1)$ and $\mathrm{Cl}(x) \subseteq \bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$, we find that $x^r, x^{m-r} \in \bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$. Hence $\bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$ is not skew-symmetric.

On the other hand, assume that $\bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$ is not a skew-symmetric set. Then there is an $y = g^{-1}x^r g \in \bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$ for some $r \in G^1_{m,3}(1)$ such that $y^{-1} \in \bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$. Therefore we have $g^{-1}x^{m-r}g = y^{-1} = h^{-1}x^k h$ for some $h \in \Gamma, k \in G^1_{m,3}(1)$. Let $t \in G_m(1)$ be the multiplipicative inverse of $m - r$. We have $g^{-1}x^{(m-r)t}g = h^{-1}x^{kt}h$, and it gives $x^{kt} = hg^{-1}xgh^{-1} \in \mathrm{Cl}(x)$. Since $(m-r)t \equiv 1 \pmod 3$ and $m - r \in G^2_{m,3}(1)$, we have that $t \in G^2_{m,3}(1)$. Thus $kt \in G^2_{m,3}(1)$ with $x^{kt} \in \mathrm{Cl}(x)$, giving that x is not tolerable. □

Let $x \in \Gamma(3)$ be tolerable, and define $S^3_x := \bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$. The structure and properties of the set S^3_x are similar to those of S^1_x and S^4_x. If Γ is abelian, then $S^3_x = \langle\langle x\rangle\rangle$ for each $x \in \Gamma(3)$.

For each $y \in S_x^3$, it is clear that $\mathrm{Cl}(y), \langle\langle y\rangle\rangle \subseteq S_x^3$. Now let A be a skew-symmetric subset of Γ containing a tolerable element x, and $\mathrm{Cl}(a), \langle\langle a\rangle\rangle \subseteq A$ for each $a \in A$. It is easy to see that $S_x^3 \subseteq A$. Thus, S_x^3 is the smallest skew-symmetric subset of Γ containing x that is closed under both conjugacy and the equivalence relation $\simeq$. Considering each of the repeated equivalence classes, if any, only once in $\bigcup\limits_{s\in \mathrm{Cl}(x)} \langle\langle s\rangle\rangle$, we can write $S_x^3 = \bigcup\limits_{i=1}^{r} \langle\langle y_i\rangle\rangle$, where the equivalence classes $\langle\langle y_1\rangle\rangle, \ldots, \langle\langle y_r\rangle\rangle$ are distinct. We state this fact in the next lemma.

Lemma 5.4.6. *If x is a tolerable element in $\Gamma(3)$, then there are distinct equivalence classes $\langle\langle x_1\rangle\rangle, \ldots, \langle\langle x_r\rangle\rangle$ such that $S_x^3 = \bigcup\limits_{i=1}^{r} \langle\langle x_i\rangle\rangle$, where $x_1, \ldots, x_r \in \mathrm{Cl}(x)$.*

Lemma 5.4.7. *If $y \in S_x^3$, then $S_y^3 = S_x^3$.*

Proof. Let $y \in S_x^3$, so that $y = g^{-1}x^r g$ for some $g \in \Gamma$ and $r \in G_{m,3}^1(1)$, where $m = \mathrm{ord}(x)$. We see that $\mathrm{ord}(y) = \mathrm{ord}(x) = m$. Now let $z \in S_y^3$. Then $z = h^{-1}y^t h$ for some $h \in \Gamma$ and $t \in G_{m,3}^1(1)$. This gives $z = h^{-1}y^t h = h^{-1}g^{-1}x^{rt}gh \in S_x^3$. Conversely, let $w \in S_x^3$ so that $w = h^{-1}x^t h$ for some $h \in \Gamma$ and $t \in G_{m,3}^1(1)$. Therefore

$$w = h^{-1}x^t h = (h^{-1}g)g^{-1}(x^r)^{r^{-1}t}g(g^{-1}h) = (h^{-1}g)y^{r^{-1}t}(g^{-1}h) \in S_y^3.$$

Here r^{-1} is the multiplicative inverse of r in the subgroup $G_{m,3}^1(1)$. Thus we conclude that $S_y^3 = S_x^3$. □

Due to Lemma 5.4.7, the sets S_x^3 and S_y^3 are either disjoint or equal.

Lemma 5.4.8. *Let $x \in \Gamma(3)$. If $S_x^1 = [x_1] \cup \cdots \cup [x_k]$ for some $x_1, \ldots, x_k \in \mathrm{Cl}(x)$, then $S_{x^3}^1 = [x_1^3] \cup \cdots \cup [x_k^3]$.*

Proof. Let $m = \mathrm{ord}(x)$ and $S_x^1 = [x_1] \cup \cdots \cup [x_k]$ for some $x_1 \ldots, x_k \in \mathrm{Cl}(x)$. Assume that the sets $[x_1], \ldots, [x_k]$ are all distinct. We see that

$$\begin{aligned}
S_{x^3}^1 &= \left\{ g^{-1}x^{3r}g \colon g \in \Gamma, r \in G_{\frac{m}{3}}(1) \right\} \\
&= \left\{ g^{-1}x^{3r}g \colon g \in \Gamma, r \in G_{\frac{m}{3}}(1) \right\} \cup \left\{ g^{-1}x^{3(\frac{m}{3}+r)}g \colon g \in \Gamma, r \in G_{\frac{m}{3}}(1) \right\} \\
&\quad \cup \left\{ g^{-1}x^{3(\frac{2m}{3}+r)}g \colon g \in \Gamma, r \in G_{\frac{m}{3}}(1) \right\} \\
&= \left\{ g^{-1}x^{3r}g \colon g \in \Gamma, r \in G_m(1), r < \frac{m}{3} \right\} \cup \left\{ g^{-1}x^{3t}g \colon g \in \Gamma, t \in G_m(1), \frac{m}{3} < t < \frac{2m}{3} \right\} \\
&\quad \cup \left\{ g^{-1}x^{3t}g \colon g \in \Gamma, t \in G_m(1), \frac{2m}{3} < t \right\} \\
&= \left\{ g^{-1}x^{3r}g \colon g \in \Gamma, r \in G_m(1) \right\} \\
&= \left\{ y^3 \colon y \in S_x^1 \right\}.
\end{aligned}$$

Now noting that $\{s^3 \colon s \in [x]\} = [x^3]$ and $S_x^1 = [x_1] \cup \cdots \cup [x_k]$, we have $S_{x^3}^1 = [x_1^3] \cup \cdots \cup [x_k^3]$. □

Lemma 5.4.9. *If $x \in \Gamma(3)$ is tolerable, then $S_x^3 \cup S_{x^{-1}}^3 = S_x^1$.*

Proof. Let $m = \mathrm{ord}(x)$. We have

$$\begin{aligned} S_x^3 \cup S_{x^{-1}}^3 &= \{g^{-1}x^r g\colon g \in \Gamma, r \in G_{m,3}^1(1)\} \cup \{g^{-1}x^{-r}g\colon g \in \Gamma, r \in G_{m,3}^1(1)\} \\ &= \{g^{-1}x^r g\colon g \in \Gamma, r \in G_{m,3}^1(1)\} \cup \{g^{-1}x^r g\colon g \in \Gamma, r \in G_{m,3}^2(1)\} \\ &= \{g^{-1}x^r g\colon g \in \Gamma, r \in G_m(1)\} \\ &= S_x^1. \end{aligned}$$

□

Lemma 5.4.10. *Let $x \in \Gamma(3)$ be a tolerable element. If $S_x^3 = \langle\langle x_1 \rangle\rangle \cup \cdots \cup \langle\langle x_k \rangle\rangle$ for some $x_1, \ldots, x_k \in \mathrm{Cl}(x)$, then $S_{x^3}^1 = [x_1^3] \cup \cdots \cup [x_k^3]$.*

Proof. Assume that $S_x^3 = \langle\langle x_1 \rangle\rangle \cup \cdots \cup \langle\langle x_k \rangle\rangle$ for some $x_1, \ldots, x_k \in \mathrm{Cl}(x)$. Then we have $S_{x^{-1}}^3 = \langle\langle x_1^{-1} \rangle\rangle \cup \cdots \cup \langle\langle x_k^{-1} \rangle\rangle$. Therefore

$$S_x^1 = S_x^3 \cup S_{x^{-1}}^3 = (\langle\langle x_1 \rangle\rangle \cup \langle\langle x_1^{-1} \rangle\rangle) \cup \cdots \cup (\langle\langle x_1 \rangle\rangle \cup \langle\langle x_k^{-1} \rangle\rangle) = [x_1] \cup \cdots \cup [x_k].$$

Now the result follows from Lemma 5.4.8. □

For $x \in \Gamma$ and $j \in \{1, \ldots, h\}$, recall that

$$C_x(j) = \frac{1}{\chi_j(\mathbf{1})} \sum_{s \in S_x^1} \chi_j(s).$$

Note that $S_x^1 \in \mathbb{B}(\Gamma)$ and $C_x(j)$ is an eigenvalue of the normal undirected Cayley graph $\mathrm{Cay}(\Gamma, S_x^1)$. As a consequence of Theorem 1.6.7, $C_x(j)$ is an integer for each $x \in \Gamma$ and $j \in \{1, \ldots, h\}$.

Lemma 5.4.11. *Let $x \in \Gamma$ and $\mathrm{ord}(x) = 3^t m$. If $m \not\equiv 0 \pmod 3$ and $t \geq 2$, then*

$$2C_x(j) = \left(\sum_{s \in G_9(1)} \chi_j(x^{sm}) \right) C_{x^3}(j).$$

Moreover, $\frac{C_x(j)}{3}$ is an integer for each $j \in \{1, \ldots, h\}$.

Proof. Let $S_x^1 = [x_1] \cup \cdots \cup [x_k]$ for some $x_1, \ldots, x_k \in \mathrm{Cl}(x)$ and $j \in \{1, \ldots, h\}$. We use the fact that each $[x_i]$ can be written as disjoint unions in two different ways using Part (iii) and Part (iv) of Lemma 5.4.4. For $m \equiv 1 \pmod 3$, using Part (iii) and Part (iv) of Lemma 5.4.4, we have

$$\begin{aligned}2\sum_{s\in[x_i]}\chi_j(s) &= \sum_{s\in[x_i]}\chi_j(s)+\sum_{s\in[x_i]}\chi_j(s)\\ &= \sum_{s\in x_i^m[x_i^3]}\chi_j(s)+\sum_{s\in x_i^{2m}[x_i^3]}\chi_j(s)+\sum_{s\in x_i^{4m}\langle\langle x_i^{-3}\rangle\rangle}\chi_j(s)+\sum_{s\in x_i^{5m}\langle\langle x_i^{-3}\rangle\rangle}\chi_j(s)\\ &\quad+\sum_{s\in x_i^{7m}[x_i^3]}\chi_j(s)+\sum_{s\in x_i^{8m}[x_i^3]}\chi_j(s)+\sum_{s\in x_i^{4m}\langle\langle x_i^{3}\rangle\rangle}\chi_j(s)+\sum_{s\in x_i^{5m}\langle\langle x_i^{3}\rangle\rangle}\chi_j(s)\\ &= \sum_{s\in[x_i^3]}\chi_j(x_i^m)\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{2m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{4m})\chi_j(s)\\ &\quad+\sum_{s\in[x_i^3]}\chi_j(x_i^{5m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{7m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{8m})\chi_j(s) \qquad (5.8)\end{aligned}$$

for each $i\in\{1,\dots,k\}$. Similarly, for $m\equiv 2 \pmod 3$, using Part (iii) and Part (iv) of Lemma 5.4.4, we have

$$\begin{aligned}2\sum_{s\in[x_i]}\chi_j(s) &= \sum_{s\in[x_i^3]}\chi_j(x_i^m)\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{2m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{4m})\chi_j(s)\\ &\quad+\sum_{s\in[x_i^3]}\chi_j(x_i^{5m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{7m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{8m})\chi_j(s) \qquad (5.9)\end{aligned}$$

for each $i\in\{1,\dots,k\}$. Thus using Equations (5.8) and (5.9), we get

$$\begin{aligned}2C_x(j) =& \frac{1}{\chi_j(\mathbf{1})}\sum_{i=1}^{k}2\sum_{s\in[x_i]}\chi_j(s)\\ =& \frac{1}{\chi_j(\mathbf{1})}\sum_{i=1}^{k}\Bigg(\sum_{s\in[x_i^3]}\chi_j(x_i^m)\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{2m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{4m})\chi_j(s)\\ &+\sum_{s\in[x_i^3]}\chi_j(x_i^{5m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{7m})\chi_j(s)+\sum_{s\in[x_i^3]}\chi_j(x_i^{8m})\chi_j(s)\Bigg)\\ =& \Big(\chi_j(x^m)+\chi_j(x^{2m})+\chi_j(x^{4m})+\chi_j(x^{5m})+\chi_j(x^{7m})\\ &+\chi_j(x^{8m})\Big)\frac{1}{\chi_j(\mathbf{1})}\sum_{i=1}^{k}\sum_{s\in[x_i^3]}\chi_j(s)\\ =& \Bigg(\sum_{r\in G_9(1)}\chi_j(x^{rm})\Bigg)C_{x^3}(j). \qquad (5.10)\end{aligned}$$

Here the third equality in Equation (5.10) follows from the fact that $x_1,\dots,x_k\in \mathrm{Cl}(x)$, and the fourth equality in Equation (5.10) follows from Lemma 5.4.8.

Let $d_j=\chi_j(\mathbf{1})$. We apply induction on t to prove that $\frac{C_x(j)}{3}$ is an integer. Let $t=2$, so that $\mathrm{ord}(x)=9m$ with $m\not\equiv 0 \pmod 3$. By Theorem 1.3.3, we have $\chi_j(x^m)=\sum_{\ell=1}^{d_j}\epsilon_{j\ell}$, where

$\epsilon_{j1},\dots,\epsilon_{jd_j}$ are some 9-th roots of unity. We have

$$\sum_{r\in G_9(1)} \chi_j(x^{rm}) = \sum_{r\in G_9(1)} \sum_{\ell=1}^{d_j} \epsilon_{j\ell}^r = \sum_{\ell=1}^{d_j} \sum_{r\in G_9(1)} \epsilon_{j\ell}^r. \tag{5.11}$$

Note that $\sum\limits_{r\in G_9(1)} \epsilon_{j\ell}^r = (\epsilon_{j\ell}+\epsilon_{j\ell}^2)(1+\epsilon_{j\ell}^3+\epsilon_{j\ell}^6)$. Since $\epsilon_{j\ell}\in\{1,\omega_9,\omega_9^2,\dots,\omega_9^8\}$, we have

$$\sum_{r\in G_9(1)} \epsilon_{j\ell}^r = \begin{cases} 6 & \text{if } \epsilon_{j\ell}=1 \\ -3 & \text{if } \epsilon_{j\ell}\in\{\omega_9^3,\omega_9^6\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\sum\limits_{r\in G_9(1)} \epsilon_{j\ell}^r$ is an integer multiple of 3 for each $\ell\in\{1,\dots,d_j\}$. Therefore by Equation (5.11), $\sum\limits_{r\in G_9(1)} \chi_j(x^{rm})$ is an integer multiple of 3. Now Equation (5.10) gives that $\frac{2C_x(j)}{3}$ is an integer. Since $C_x(j)$ is an integer, integrality of $\frac{2C_x(j)}{3}$ gives that $\frac{C_x(j)}{3}$ is also an integer.

Assume that $\frac{C_y(j)}{3}$ is an integer for each $j\in\{1,\dots,h\}$ whenever $\mathrm{ord}(y)=3^{t-1}m$ with $m\not\equiv 0 \pmod 3$ and $t\geq 3$. Let $\mathrm{ord}(x)=3^t m$ with $m\not\equiv 0 \pmod 3$ and $t\geq 3$. Note that $\mathrm{ord}(x^3)=3^{t-1}m$. Therefore by induction hypothesis, $\frac{C_{x^3}(j)}{3}$ is an integer. By Equation (5.10), $\sum\limits_{s\in G_9(1)} \chi_j(x^{sm})$ is a rational algebraic integer whenever $C_{x^3}(j)\neq 0$. Thus, if $C_{x^3}(j)\neq 0$ then $\sum\limits_{s\in G_9(1)} \chi_j(x^{sm})$ is an integer. Therefore by Equation (5.10), $\frac{2C_x(j)}{3}$ is an integer, and accordingly $\frac{C_x(j)}{3}$ is an integer. Hence the proof is complete by induction. □

Let $x\in\Gamma(3)$ be tolerable. For each $j\in\{1,\dots,h\}$, define

$$T_x(j) := \frac{1}{\chi_j(\mathbf{1})} \sum_{s\in S_x^3} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})).$$

Let $j\in\{1,\dots,h\}$. Using $S_x^1 = S_x^3\cup S_{x^{-1}}^3$, we see that

$$\begin{aligned}\frac{C_x(j)+T_x(j)}{2} &= \frac{1}{2\chi_j(\mathbf{1})}\left[\sum_{s\in S_x^1}\chi_j(s) + \sum_{s\in S_x^3}\mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1}))\right] \\ &= \frac{1}{2\chi_j(\mathbf{1})}\left[\sum_{s\in S_x^3}\chi_j(s) + \sum_{s\in S_{x^{-1}}^3}\chi_j(s) + \sum_{s\in S_x^3}\mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1}))\right] \\ &= \frac{1}{\chi_j(\mathbf{1})}\left[\sum_{s\in S_x^3}\left(\omega_6\chi_j(s)+\omega_6^5\chi_j(s^{-1})\right)\right].\end{aligned}$$

Thus $\frac{C_x(j)+T_x(j)}{2}$ is an HS-eigenvalue of the normal directed Cayley graph $\mathrm{Cay}(\Gamma, S_x^3)$. Therefore by Theorem 5.2.2, $\frac{C_x(j)+T_x(j)}{2}$ is an integer. Since $C_x(j)$ is an integer (by Theorem 1.6.7), $T_x(j)$ is also an integer for each $j\in\{1,\dots,h\}$.

Lemma 5.4.12. *Let $x \in \Gamma(3)$ be tolerable and* $\mathrm{ord}(x) = 3m$. *If $m \not\equiv 0 \pmod 3$, then*

$$T_x(j) = \begin{cases} -2\sqrt{3}\Im(\chi_j(x^m))C_{x^3}(j) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Im(\chi_j(x^{2m}))C_{x^3}(j) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Moreover, $\frac{T_x(j)}{3}$ is an integer for each $j \in \{1, \dots, h\}$.

Proof. Let $S_x^3 = \langle\langle x_1 \rangle\rangle \cup \cdots \cup \langle\langle x_k \rangle\rangle$ for some $x_1, \dots, x_k \in \mathrm{Cl}(x)$ and $j \in \{1, \dots, h\}$. We get

$$\begin{aligned} T_x(j) &= \frac{1}{\chi_j(\mathbf{1})} \sum_{i=1}^{k} \sum_{s \in \langle\langle x_i \rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) \\ &= \begin{cases} \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \sum\limits_{s \in [x_i^3]} \mathbf{i}\sqrt{3}\left[\chi_j(x_i^m)\chi_j(s) - \chi_j(x_i^{-m})\chi_j(s^{-1})\right] & \text{if } m \equiv 1 \pmod 3 \\ \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \sum\limits_{s \in [x_i^3]} \mathbf{i}\sqrt{3}\left[\chi_j(x_i^{2m})\chi_j(s) - \chi_j(x_i^{-2m})\chi_j(s^{-1})\right] & \text{if } m \equiv 2 \pmod 3 \end{cases} \\ &= \begin{cases} \frac{\mathbf{i}\sqrt{3}}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \left[\chi_j(x_i^m) \sum\limits_{s \in [x_i^3]} \chi_j(s) - \overline{\chi_j(x_i^m)} \sum\limits_{s \in [x_i^3]} \chi_j(s^{-1})\right] & \text{if } m \equiv 1 \pmod 3 \\ \frac{\mathbf{i}\sqrt{3}}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \left[\chi_j(x_i^{2m}) \sum\limits_{s \in [x_i^3]} \chi_j(s) - \overline{\chi_j(x_i^{2m})} \sum\limits_{s \in [x_i^3]} \chi_j(s^{-1})\right] & \text{if } m \equiv 2 \pmod 3 \end{cases} \\ &= \begin{cases} \frac{\mathbf{i}\sqrt{3}}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \left[\chi_j(x_i^m) \sum\limits_{s \in [x_i^3]} \chi_j(s) - \overline{\chi_j(x_i^m)} \sum\limits_{s \in [x_i^3]} \chi_j(s)\right] & \text{if } m \equiv 1 \pmod 3 \\ \frac{\mathbf{i}\sqrt{3}}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \left[\chi_j(x_i^{2m}) \sum\limits_{s \in [x_i^3]} \chi_j(s) - \overline{\chi_j(x_i^{2m})} \sum\limits_{s \in [x_i^3]} \chi_j(s)\right] & \text{if } m \equiv 2 \pmod 3 \end{cases} \\ &= \begin{cases} -2\sqrt{3}\Im(\chi_j(x^m)) \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \sum\limits_{s \in [x_i^3]} \chi_j(s) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Im(\chi_j(x^{2m})) \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \sum\limits_{s \in [x_i^3]} \chi_j(s) & \text{if } m \equiv 2 \pmod 3 \end{cases} \\ &= \begin{cases} -2\sqrt{3}\Im(\chi_j(x^m))C_{x^3}(j) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Im(\chi_j(x^{2m}))C_{x^3}(j) & \text{if } m \equiv 2 \pmod 3. \end{cases} \end{aligned}$$

Here the second equality follows from Part (ii) of Lemma 5.4.4, and the fourth equality follows from Lemma 5.4.10. Let $d_j = \chi_j(\mathbf{1})$. By Theorem 1.3.3, we have $\chi_j(x^m) = \sum\limits_{\ell=1}^{d_j} \epsilon_{j\ell}$, where $\epsilon_{j1}, \dots, \epsilon_{jd_j}$ are cube roots of unity. Therefore, $2\sqrt{3}\Im(\chi_j(x^m))$ is an integer multiple of 3. Similarly, $2\sqrt{3}\Im(\chi_j(x^{2m}))$ is also an integer multiple of 3. Hence $\frac{T_x(j)}{3}$ is an integer for each $j \in \{1, \dots, h\}$. □

Lemma 5.4.13. *Let $x \in \Gamma$ be tolerable and* $\mathrm{ord}(x) = 3^t m$*. If* $m \not\equiv 0 \pmod 3$ *and* $t \geq 2$*, then*

$$2T_x(j) = \begin{cases} -2\sqrt{3}\Big(\sum\limits_{s\in G^1_{9,3}(1)} \Im(\chi_j(x^{sm}))\Big) C_{x^3}(j) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Big(\sum\limits_{s\in G^2_{9,3}(1)} \Im(\chi_j(x^{sm}))\Big) C_{x^3}(j) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Moreover, $\frac{T_x(j)}{3}$ *is an integer for each* $j \in \{1,\dots,h\}$*.*

Proof. Let $S_x^3 = \langle\langle x_1\rangle\rangle \cup \cdots \cup \langle\langle x_k\rangle\rangle$ for some $x_1,\dots,x_k \in \mathrm{Cl}(x)$ and $j \in \{1,\dots,h\}$. We use the fact that each $\langle\langle x_i\rangle\rangle$ can be written as disjoint unions in two different ways using Part (vi) and Part (vii) of Lemma 5.4.4. For $m \equiv 1 \pmod 3$, using Part (vi) and Part (vii) of Lemma 5.4.4, we have

$$\begin{aligned}
& 2\sum_{s\in\langle\langle x_i\rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) \\
={}& \sum_{s\in\langle\langle x_i\rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) + \sum_{s\in\langle\langle x_i\rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) \\
={}& \sum_{s\in x_i^m[x_i^3]} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) + \sum_{s\in x_i^{4m}\langle\langle x_i^{-3}\rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) \\
& + \sum_{s\in x_i^{7m}[x_i^3]} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) + \sum_{s\in x_i^{4m}\langle\langle x_i^{3}\rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) \\
={}& \sum_{s\in x_i^m[x_i^3]} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) + \sum_{s\in x_i^{4m}[x_i^3]} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) \\
& + \sum_{s\in x_i^{7m}[x_i^3]} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) \\
={}& -2\sqrt{3}\Im(\chi_j(x_i^m)) \sum_{s\in[x_i^3]} \chi_j(s) - 2\sqrt{3}\Im(\chi_j(x_i^{4m})) \sum_{s\in[x_i^3]} \chi_j(s) \\
& - 2\sqrt{3}\Im(\chi_j(x_i^{7m})) \sum_{s\in[x_i^3]} \chi_j(s) \\
={}& -2\sqrt{3}\Big(\sum_{r\in G^1_{9,3}(1)} \Im(\chi_j(x_i^{rm}))\Big) \sum_{s\in[x_i^3]} \chi_j(s)
\end{aligned} \tag{5.12}$$

for each $i \in \{1,\dots,k\}$. Similarly, for $m \equiv 2 \pmod 3$ we have

$$2\sum_{s\in\langle\langle x_i\rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s)-\chi_j(s^{-1})) = -2\sqrt{3}\Big(\sum_{r\in G^2_{9,3}(1)} \Im(\chi_j(x_i^{rm}))\Big) \sum_{s\in[x_i^3]} \chi_j(s) \tag{5.13}$$

for each $i \in \{1, \dots, k\}$. Using Equation (5.12) and Equation (5.13), we get

$$\begin{aligned}
2T_x(j) &= \frac{1}{\chi_j(\mathbf{1})} \sum_{i=1}^{k} 2 \sum_{s \in \langle\langle x_i \rangle\rangle} \mathbf{i}\sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) \\
&= \begin{cases} -2\sqrt{3}\Big(\sum\limits_{r \in G_{9,3}^1(1)} \Im(\chi_j(x^{rm})) \Big) \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \sum\limits_{s \in [x_i^3]} \chi_j(s) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Big(\sum\limits_{r \in G_{9,3}^2(1)} \Im(\chi_j(x^{rm})) \Big) \frac{1}{\chi_j(\mathbf{1})} \sum\limits_{i=1}^{k} \sum\limits_{s \in [x_i^3]} \chi_j(s) & \text{if } m \equiv 2 \pmod 3 \end{cases} \\
&= \begin{cases} -2\sqrt{3}\Big(\sum\limits_{r \in G_{9,3}^1(1)} \Im(\chi_j(x^{rm})) \Big) C_{x^3}(j) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Big(\sum\limits_{r \in G_{9,3}^2(1)} \Im(\chi_j(x^{rm})) \Big) C_{x^3}(j) & \text{if } m \equiv 2 \pmod 3. \end{cases}
\end{aligned} \tag{5.14}$$

The last equality in the preceding equations follows from Lemma 5.4.10.

Let $d_j = \chi_j(\mathbf{1})$. Assume that $t = 2$. By Theorem 1.3.3, we have $\chi_j(x^m) = \sum\limits_{\ell=1}^{d_j} \epsilon_{j\ell}$, where $\epsilon_{j1}, \dots, \epsilon_{jd_j}$ are some 9-th roots of unity. We have

$$\begin{aligned}
-2\sqrt{3} \sum_{r \in G_{9,3}^1(1)} \Im(\chi_j(x^{rm})) &= \mathbf{i}\sqrt{3} \sum_{r \in G_{9,3}^1(1)} (\chi_j(x^{rm}) - \chi_j(x^{-rm})) \\
&= \mathbf{i}\sqrt{3} \sum_{r \in G_{9,3}^1(1)} \left(\sum_{\ell=1}^{d_j} \epsilon_{j\ell}^{r} - \sum_{\ell=1}^{d_j} \epsilon_{j\ell}^{-r} \right) \\
&= \sum_{\ell=1}^{d_j} \sum_{r \in G_{9,3}^1(1)} \mathbf{i}\sqrt{3}(\epsilon_{j\ell}^{r} - \epsilon_{j\ell}^{-r}).
\end{aligned} \tag{5.15}$$

Note that $\sum\limits_{r \in G_{9,3}^1(1)} \mathbf{i}\sqrt{3}(\epsilon_{j\ell}^{r} - \epsilon_{j\ell}^{-r}) = \mathbf{i}\sqrt{3}(\epsilon_{j\ell} - \epsilon_{j\ell}^2)(1 + \epsilon_{j\ell}^3 + \epsilon_{j\ell}^6)$. Since $\epsilon_{j\ell} \in \{1, \omega_9, \omega_9^2, \dots, \omega_9^8\}$, we see that

$$\sum_{r \in G_{9,3}^1(1)} \mathbf{i}\sqrt{3}(\epsilon_{j\ell}^{r} - \epsilon_{j\ell}^{-r}) = \begin{cases} \pm 9 & \text{if } \epsilon_{j\ell} \in \{\omega_9^3, \omega_9^6\} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\sum\limits_{r \in G_{9,3}^1(1)} \mathbf{i}\sqrt{3}(\epsilon_{j\ell}^{r} - \epsilon_{j\ell}^{-r})$ is an integer multiple of 3. Therefore by Equation (5.15), we find that $-2\sqrt{3} \sum\limits_{r \in G_{9,3}^1(1)} \Im(\chi_j(x^{rm}))$ is an integer multiple of 3. Similarly, $-2\sqrt{3} \sum\limits_{r \in G_{9,3}^2(1)} \Im(\chi_j(x^{rm}))$ is also an integer multiple of 3. Using Equation (5.14), we find that $\frac{2T_x(j)}{3}$ is an integer. Since $T_x(j)$ is an integer, integrality of $\frac{2T_x(j)}{3}$ gives that $\frac{T_x(j)}{3}$ is also an integer for each $j \in \{1, \dots, h\}$.

Now assume that $t \geq 3$ and $j \in \{1, \dots h\}$. Let

$$A_x(j) := \begin{cases} -2\sqrt{3}\Big(\sum\limits_{r \in G_{9,3}^1(1)} \Im(\chi_j(x^{rm})) \Big) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Big(\sum\limits_{r \in G_{9,3}^2(1)} \Im(\chi_j(x^{rm})) \Big) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

By Equation (5.14), we find that $2T_x(j) = A_x(j)C_{x^3}(j)$. Therefore $A_x(j)$ is a rational algebraic integer whenever $C_{x^3}(j) \neq 0$. Thus, if $C_{x^3}(j) \neq 0$ then $A_x(j)$ is an integer. Now by Lemma 5.4.11 and Equation (5.14), $\frac{2T_x(j)}{3}$ is an integer, and hence $\frac{T_x(j)}{3}$ is also an integer. □

Let S be a nonempty set in $\mathbb{E}(\Gamma)$ and S be expressible as a union of some conjugacy classes of Γ. Then S is a skew-symmetric subset of Γ that is closed under both conjugacy and the equivalence relation $\simeq$. Let $S = \mathrm{Cl}(x_1) \cup \cdots \cup \mathrm{Cl}(x_k) = \langle\!\langle y_1 \rangle\!\rangle \cup \cdots \cup \langle\!\langle y_r \rangle\!\rangle$ for some $x_1, \ldots, x_k, y_1, \ldots, y_r \in \Gamma(3)$. We see that

$$S = \mathrm{Cl}(x_1) \cup \cdots \cup \mathrm{Cl}(x_k) = \left(\bigcup_{s \in \mathrm{Cl}(x_1)} \langle\!\langle s \rangle\!\rangle \right) \cup \cdots \cup \left(\bigcup_{s \in \mathrm{Cl}(x_k)} \langle\!\langle s \rangle\!\rangle \right) = S^3_{x_1} \cup \cdots \cup S^3_{x_k}.$$

Due to Lemma 5.4.7, we can assume that the sets $S^3_{x_1}, \ldots, S^3_{x_k}$ are all distinct. In the following result, we also prove the converse of Theorem 5.4.3.

Theorem 5.4.14. *If Γ is a finite group, then the normal mixed Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *is Eisenstein integral if and only if it is HS-integral.*

Proof. Assume that $\mathrm{Cay}(\Gamma, S)$ is HS-integral and $j \in \{1, \ldots, h\}$. Then $\mathrm{Cay}(\Gamma, S \setminus \overline{S})$ is integral, and so $f_j(S)$ is an integer. By Theorem 5.3.4, $\overline{S} \in \mathbb{E}(\Gamma)$, which implies that $\overline{S} = S^3_{x_1} \cup \cdots \cup S^3_{x_k}$ for some $x_1, \ldots, x_k \in \Gamma(3)$, where the sets $S^3_{x_1}, \ldots, S^3_{x_k}$ are all distinct. Using the fact that $S^3_{x_i} \cup S^3_{x_i^{-1}} = S^1_{x_i}$, we have $\overline{S} \cup (\overline{S})^{-1} = S^1_{x_1} \cup \cdots \cup S^1_{x_k}$. Therefore

$$\begin{aligned}
g_j(S) &= \frac{1}{2\chi_j(\mathbf{1})} \sum_{s \in \overline{S} \cup (\overline{S})^{-1}} \chi_j(s) - \frac{1}{6\chi_j(\mathbf{1})} \sum_{s \in \overline{S}} \mathbf{i}\sqrt{3}\left(\chi_j(s) - \chi_j(s^{-1})\right) \\
&= \frac{1}{2\chi_j(\mathbf{1})} \sum_{\ell=1}^{k} \sum_{s \in S^1_{x_\ell}} \chi_j(s) - \frac{1}{6\chi_j(\mathbf{1})} \sum_{\ell=1}^{k} \sum_{s \in S^3_{x_\ell}} \mathbf{i}\sqrt{3}\left(\chi_j(s) - \chi_j(s^{-1})\right) \\
&= \frac{1}{2} \sum_{\ell=1}^{k} C_{x_\ell}(j) - \frac{1}{6} \sum_{\ell=1}^{k} T_{x_\ell}(j) \\
&= \frac{1}{2} \sum_{\ell=1}^{k} \left(C_{x_\ell}(j) - \frac{1}{3} T_{x_\ell}(j) \right).
\end{aligned} \tag{5.16}$$

Let $1 \leq \ell \leq k$. Since $\frac{C_{x_\ell}(j) + T_{x_\ell}(j)}{2}$ is an HS-eigenvalue of the normal directed Cayley graph $\mathrm{Cay}(\Gamma, S^3_{x_\ell})$, the numbers $C_{x_\ell}(j)$ and $T_{x_\ell}(j)$ are integers of the same parity. By Lemma 5.4.12 and Lemma 5.4.13, $\frac{T_{x_\ell}(j)}{3}$ is an integer. Therefore, $C_{x_\ell}(j)$ and $\frac{T_{x_\ell}(j)}{3}$ are integers of the same parity. Thus $C_{x_\ell}(j) - \frac{1}{3}T_{x_\ell}(j)$ is an even integer, and so $g_j(S)$ is an integer by Equation (5.16). Hence by Lemma 5.4.2, $\mathrm{Cay}(\Gamma, S)$ is Eisenstein integral. The other part of the theorem is proved in Theorem 5.4.3. □

The following example illustrates an use of Theorem 5.4.14.

Example 5.4.1. Consider the mixed graph $\text{Cay}(A_4, S)$ of Example 5.3.1. We have already seen that it is HS-integral, and hence it must be Eisenstein integral. We find that the spectrum of $\text{Cay}(A_4, S)$ is $\{[\gamma_1]^1, [\gamma_2]^1, [\gamma_3]^1, [\gamma_4]^9\}$, where $\gamma_1 = 7, \gamma_2 = 3 + 4\omega_3, \gamma_3 = -1 - 4\omega_3$, and $\gamma_4 = -1$. It is clear that the eigenvalues of $\text{Cay}(A_4, S)$ are Eisenstein integers.

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6

Ramanujan type sums

In 1918, Ramanujan [37] published a seminal paper in which he introduced a sum (now called Ramanujan sum) for each $n \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{0\}$, defined by

$$C_n(q) := \sum_{a \in G_n(1)} \cos\left(\frac{2\pi aq}{n}\right). \tag{6.1}$$

Indeed, $C_n(q)$ is an integer eigenvalue of $\text{Circ}(\mathbb{Z}_n, G_n(1))$. It is known that $C_n(q)$ can be expressed in terms of certain arithmetic functions. Motivating by this, we introduce two sums that are equal to an integer multiple of the Ramanujan sum. Indeed, the H-eigenvalues and HS-eigenvalues of a mixed circulant graph can be expressed in terms of these sums. In Sections 6.2 and 6.3, we express these sums in terms of generalized Möbius function.

6.1 Ramanujan sum and arithmetic functions

For $n \geq 3$, the Ramanujan sum can also be written as

$$C_n(q) = \sum_{a \in G_n(1)} \omega_n^{aq} = \sum_{\substack{a \in G_n(1) \\ a < \frac{n}{2}}} 2\cos\left(\frac{2\pi aq}{n}\right).$$

Note that $C_n(q)$ is a special case of $C_x(j)$ defined in Chapter 4. For a divisor d of n, we have

$$C_{\frac{n}{d}}(q) = \sum_{a \in G_{\frac{n}{d}}(1)} \omega_{\frac{n}{d}}^{aq} = \sum_{a \in G_{\frac{n}{d}}(1)} \omega_n^{daq} = \sum_{a \in G_n(d)} \omega_n^{aq}.$$

The classical Möbius function $\mu(n)$ is defined by

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes}, \\ 0 & \text{otherwise}. \end{cases}$$

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Lemma 6.1.1 ([36]). *If $n \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{0\}$, then*

$$C_n(t) = \varphi(n)\frac{\mu(\frac{n}{\delta_t})}{\varphi(\frac{n}{\delta_t})},$$

where $\delta_t = \gcd(n, t)$.

Lemma 6.1.2. *Let p be a prime, $n \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{0\}$. If p^2 divides n, then $C_n(t)$ is a multiple of p.*

Proof. Let $n = p^k m$, where $k \geq 2$ and p does not divide m. By Lemma 6.1.1, we have $C_n(t) = \varphi(n)\frac{\mu(\frac{n}{\delta_t})}{\varphi(\frac{n}{\delta_t})}$, where $\delta_t = \gcd(n, t)$. If $\frac{n}{\delta_t}$ has a square factor, then clearly $\mu(\frac{n}{\delta_t}) = 0$, and hence $C_n(t) = 0$. Else $\frac{n}{\delta_t} = p^r m_1$, where $0 \leq r \leq 1$ and $m_1 \mid m$. We have

$$C_n(t) = \varphi(n)\frac{\mu(\frac{n}{\delta_t})}{\varphi(\frac{n}{\delta_t})} = \varphi(p^k m)\frac{\mu(p^r m_1)}{\varphi(p^r m_1)} = \frac{\varphi(p^k)\varphi(m)}{\varphi(p^r)\varphi(m_1)}\mu(p^r m_1).$$

As $\varphi(m_1) \mid \varphi(m)$ and $\frac{\varphi(p^k)}{\varphi(p^r)}$ is a multiple of p, we find that $C_n(t)$ is a multiple of p. □

Let δ be the indicator function defined by

$$\delta(n) := \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1. \end{cases}$$

Lemma 6.1.3 ([36]). *If $n \in \mathbb{N}$, then $\sum_{d|n} \mu(d) = \delta(n)$.*

Lemma 6.1.4 ([36]). *If $n \in \mathbb{N}$, then $\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$.*

Lemma 6.1.5. *If $a, b \in \mathbb{N}$, then $\frac{\varphi(ab)}{\varphi(b)} = a \sum_{\substack{e|a \\ \gcd(e,b)=1}} \frac{\mu(e)}{e}$.*

Proof. Let the prime factorizations of a and b be given by $a = p_1^{\alpha_1} \dots p_k^{\alpha_k} p_{k+1}^{\alpha_{k+1}} \dots p_r^{\alpha_r}$ and $b = p_{k+1}^{\beta_{k+1}} \dots p_r^{\beta_r} p_{r+1}^{\beta_{r+1}} \dots p_s^{\beta_s}$. We have

$$\begin{aligned}
\frac{\varphi(ab)}{\varphi(b)} &= \frac{ab\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_k}\right)\left(1-\frac{1}{p_{k+1}}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{p_{r+1}}\right)\cdots\left(1-\frac{1}{p_s}\right)}{b\left(1-\frac{1}{p_{k+1}}\right)\cdots\left(1-\frac{1}{p_r}\right)\left(1-\frac{1}{p_{r+1}}\right)\cdots\left(1-\frac{1}{p_s}\right)} \\
&= a\left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_k}\right) \\
&= a\frac{(p_1-1)\cdots(p_k-1)}{p_1\dots p_k} \\
&= a\frac{\varphi(p_1\dots p_k)}{p_1\dots p_k} \\
&= a\sum_{d|p_1\dots p_k}\frac{\mu(d)}{d} \\
&= a\sum_{\substack{e|a \\ \gcd(e,b)=1}}\frac{\mu(e)}{e}.
\end{aligned}$$

Here the fifth equality follows from Lemma 6.1.4. □

E. Cohen [14] introduced a generalized Möbius inversion formula of arbitrary direct factor sets. Let P and Q be two nonempty subsets of $\mathbb{N}$ such that if $n_1, n_2 \in \mathbb{N}$ with $\gcd(n_1, n_2) = 1$, then $n_1 n_2 \in P$ (respectively $n_1 n_2 \in Q$) if and only if $n_1, n_2 \in P$ (respectively $n_1, n_2 \in Q$). If each integer $n \in \mathbb{N}$ possesses a unique factorization of the form $n = ab$ with $a \in P, b \in Q$, then the sets P and Q are called direct factor sets of $\mathbb{N}$. In what follows, P denote such a direct factor set with (conjugate) factor set Q. The Möbius function can be generalized to an arbitrary direct factor set P by setting

$$\mu_P(n) := \sum_{d|n, d \in P} \mu\left(\frac{n}{d}\right),$$

where μ is the classical Möbius function. The function μ_P is called the generalized Möbius function with respect to the direct factor set P. For example, $\mu(n) = \mu_{\{1\}}(n)$ and $\mu_{\mathbb{N}}(n) = \delta(n)$.

Theorem 6.1.6 ([14]). $\sum\limits_{d|n, d \in Q} \mu_P\left(\frac{n}{d}\right) = \delta(n).$

The next result generalizes the Möbius inversion formula to arbitrary direct factor sets.

Theorem 6.1.7 ([14]). *If $f(n)$ and $g(n)$ are arithmetic functions, then*

$$f(n) = \sum_{d|n, d \in Q} g\left(\frac{n}{d}\right) \text{ if and only if } g(n) = \sum_{d|n} f(d) \mu_P\left(\frac{n}{d}\right).$$

Note that the sets $\{2^k : k \geq 0\}$ and $\{n : n \in \mathbb{N} \text{ and } 2 \nmid n\}$ are direct factor sets. Similarly, the sets $\{3^k : k \geq 0\}$ and $\{n : n \in \mathbb{N} \text{ and } 3 \nmid n\}$ are also direct factor sets.

Lemma 6.1.8. *If $P = \{2^k : k \geq 0\}$, then*

$$\mu_P(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \mu(n) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Assume that n is an even integer. Then there exists a positive integer t such that $n = 2^t m$, where m is an odd integer. Using $\mu(m) = -\mu(2m)$, we get

$$\mu_P(n) = \sum_{d|n, d \in P} \mu\left(\frac{n}{d}\right) = \sum_{i=0}^{t} \mu(2^i m) = \mu(m) + \mu(2m) = 0.$$

If n is an odd integer, then 1 is the only divisor of n in P, and so $\mu_P(n) = \mu(n)$. □

Lemma 6.1.9. *If $P = \{3^k : k \geq 0\}$, then*

$$\mu_P(n) = \begin{cases} 0 & \text{if } 3 \mid n \\ \mu(n) & \text{if } 3 \nmid n. \end{cases}$$

Proof. Assume that $3 \mid n$. Then there exists a positive integer t such that $n = 3^t m$, where $3 \nmid m$. Using $\mu(m) = -\mu(3m)$, we get

$$\mu_P(n) = \sum_{d|n, d\in P} \mu\left(\frac{n}{d}\right) = \sum_{i=0}^{t} \mu(3^i m) = \mu(m) + \mu(3m) = 0.$$

If $3 \nmid n$, then 1 is the only divisor of n in P, and so $\mu_P(n) = \mu(n)$. □

6.2 H-eigenvalues of mixed circulant graphs

The next result follows from Lemma 2.1.2.

Lemma 6.2.1. *The H-spectrum of the mixed graph* $\mathrm{Circ}(\mathbb{Z}_n, S)$ *is* $\{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$*, where* $\gamma_j = \lambda_j + \mu_j$*,*

$$\lambda_j = \sum_{k\in S\setminus\overline{S}} \omega_n^{jk} \text{ and } \mu_j = \mathbf{i}\sum_{k\in\overline{S}} (\omega_n^{jk} - \omega_n^{-jk}) \text{ for each } j \in \{0, 1, \ldots, n-1\}.$$

The next result follows from Theorem 2.3.4.

Theorem 6.2.2. *The mixed graph* $\mathrm{Circ}(\mathbb{Z}_n, S)$ *is H-integral if and only if* $S \setminus \overline{S} = \bigcup_{d\in\mathscr{D}_1} G_n(d)$ *and*

$$\overline{S} = \begin{cases} \emptyset & \text{if } n \not\equiv 0 \pmod 4 \\ \left(\bigcup_{d\in\mathscr{D}_2} G_n^1(d)\right) \cup \left(\bigcup_{d\in\mathscr{D}_3} G_n^3(d)\right) & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where $\mathscr{D}_1 \subseteq \{d \colon d \mid n\}$, $\mathscr{D}_2 \cup \mathscr{D}_3 \subseteq \{d \colon d \mid \frac{n}{4}\}$, $\mathscr{D}_1 \cap (\mathscr{D}_2 \cup \mathscr{D}_3) = \emptyset$ *and* $\mathscr{D}_2 \cap \mathscr{D}_3 = \emptyset$.

For $n \equiv 0 \pmod 4$ and $q \in \{0, 1, \ldots, n-1\}$, define

$$S_n(q) := \sum_{a\in G_n^1(1)} \frac{\omega_n^{aq} - \omega_n^{-aq}}{\mathbf{i}}.$$

Observe that

$$S_n(q) = \sum_{a\in G_n^1(1)} 2\sin\left(\frac{2\pi aq}{n}\right) \quad \text{and} \quad -S_n(q) = \sum_{a\in G_n^3(1)} \frac{\omega_n^{aq} - \omega_n^{-aq}}{\mathbf{i}}.$$

Note that $S_n(q)$ is a special case of $-S_x(j)$, defined in Chapter 4. By Lemma 6.2.1, $-S_n(q)$ is an H-eigenvalue of the mixed graph $\mathrm{Circ}(\mathbb{Z}_n, G_n^1(1))$ for each $q \in \{0, 1, \ldots, n-1\}$. Thus by Theorem 6.2.2, $S_n(q)$ is an integer for each $q \in \{0, 1, \ldots, n-1\}$.

For $n \equiv 0 \pmod 4$ and a divisor d of $\frac{n}{4}$, we have

$$S_{\frac{n}{d}}(q) = \sum_{a\in G_{\frac{n}{d}}^1(1)} \frac{(\omega_n^d)^{aq} - (\omega_n^d)^{-aq}}{\mathbf{i}} = \sum_{a\in G_n^1(d)} \frac{\omega_n^{aq} - \omega_n^{-aq}}{\mathbf{i}}.$$

Theorem 6.2.3 ([39]). *If $d \mid n$ and $d \neq n$, then the spectrum of the undirected graph* $\mathrm{Circ}(\mathbb{Z}_n, G_n(d))$ *is* $\{\lambda_0, \ldots, \lambda_{n-1}\}$, *where*

$$\lambda_j = C_{\frac{n}{d}}(j) \text{ for } j \in \{0, 1, \ldots, n-1\}.$$

Further, the spectrum is equal to d copies of $\{C_{\frac{n}{d}}(0), C_{\frac{n}{d}}(1), \ldots, C_{\frac{n}{d}}(\frac{n}{d}-1)\}$.

Theorem 6.2.4. *Let $\mathscr{D}_1 \subseteq \{d \colon d \mid n\}$ and $\mathscr{D}_2 \cup \mathscr{D}_3 \subseteq \{d \colon d \mid \frac{n}{4}\}$. If $\mathscr{D}_1 \cap (\mathscr{D}_2 \cup \mathscr{D}_3) = \emptyset$ and $\mathscr{D}_2 \cap \mathscr{D}_3 = \emptyset$, then*

$$\sum_{d \in \mathscr{D}_1} C_{\frac{n}{d}}(j) - \sum_{d \in \mathscr{D}_2} S_{\frac{n}{d}}(j) + \sum_{d \in \mathscr{D}_3} S_{\frac{n}{d}}(j)$$

is an H-eigenvalue of an H-integral mixed circulant graph for each $j \in \{0, 1, \ldots n-1\}$.

Proof. Assume that $\mathscr{D}_1 \subseteq \{d \colon d \mid n\}$, $\mathscr{D}_2 \cup \mathscr{D}_3 \subseteq \{d \colon d \mid \frac{n}{4}\}$, $\mathscr{D}_1 \cap (\mathscr{D}_2 \cup \mathscr{D}_3) = \emptyset$ and $\mathscr{D}_2 \cap \mathscr{D}_3 = \emptyset$. Define the set S by setting $S \setminus \overline{S} = \bigcup\limits_{d \in \mathscr{D}_1} G_n(d)$ and

$$\overline{S} = \begin{cases} \emptyset & \text{if } \quad n \not\equiv 0 \pmod 4 \\ \left(\bigcup\limits_{d \in \mathscr{D}_2} G_n^1(d) \right) \cup \left(\bigcup\limits_{d \in \mathscr{D}_3} G_n^3(d) \right) & \text{if } \quad n \equiv 0 \pmod 4. \end{cases}$$

By Theorem 6.2.2, $\mathrm{Circ}(\mathbb{Z}_n, S)$ is an H-integral mixed graph. By Lemma 6.2.1, the H-eigenvalues of $\mathrm{Circ}(\mathbb{Z}_n, S)$ are given by

$$\begin{aligned} \gamma_j &= \sum_{d \in \mathscr{D}_1} \sum_{a \in G_n(d)} \omega_n^{aj} - \sum_{d \in \mathscr{D}_2} \sum_{a \in G_n^1(d)} \frac{\omega_n^{aj} - \omega_n^{-aj}}{\mathbf{i}} - \sum_{d \in \mathscr{D}_3} \sum_{a \in G_n^3(d)} \frac{\omega_n^{aj} - \omega_n^{-aj}}{\mathbf{i}} \\ &= \sum_{d \in \mathscr{D}_1} C_{\frac{n}{d}}(j) - \sum_{d \in \mathscr{D}_2} S_{\frac{n}{d}}(j) + \sum_{d \in \mathscr{D}_3} S_{\frac{n}{d}}(j), \end{aligned}$$

where $j \in \{0, 1, \ldots, n-1\}$. □

From Theorem 6.2.4, it is clear that $S_n(q)$ plays an important role in the expression of the H-eigenvalues of H-integral mixed circulant graphs. In the next result, we see some basic properties of $S_n(q)$. The proofs easily follow from the definition of $S_n(q)$.

Theorem 6.2.5. *If $n \equiv 0 \pmod 4$, then the following assertions hold.*

(i) *If $t_1 \equiv t_2 \pmod n$, then $S_n(t_1) = S_n(t_2)$.*

(ii) $S_n(n-t) = -S_n(t)$.

(iii) $S_n(0) = S_n(\frac{n}{2}) = 0$.

(iv) $S_n(\frac{n}{4}) = \varphi(n)$.

(v) $S_n(\frac{3n}{4}) = -\varphi(n)$.

Example 6.2.1. *If $n = 2^k$ and $k \geq 2$, then*

$$S_n(t) = \begin{cases} 2^{k-1} & \text{if } t \equiv 2^{k-2} \pmod{n} \\ -2^{k-1} & \text{if } t \equiv 3 \times 2^{k-2} \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$S_n(t) = \sum_{a \in G_n^1(1)} \frac{\omega_n^{at} - \omega_n^{-at}}{\mathbf{i}} = \frac{1}{\mathbf{i}}(\omega_n^t - \omega_n^{3t} + \omega_n^{5t} - \ldots - \omega_n^{(n-1)t}).$$

Observe that $S_n(t)$ is a sum of a geometric progression of 2^{k-1} terms with common ratio $-\omega_n^{2t}$. If $t \equiv 2^{k-2} \pmod{n}$, then $\omega_n^t = \mathbf{i}$, and so $S_n(t) = 2^{k-1}$. Similarly, $S_n(t) = -2^{k-1}$ for $t \equiv 3 \times 2^{k-2} \pmod{n}$. Now assume that $t \not\equiv 2^{k-2} \pmod{n}$ and $t \not\equiv 3 \times 2^{k-2} \pmod{n}$, so that $\omega_n^{2t} \neq -1$. Using $(-\omega_n^{2t})^{2^{k-1}} = 1$, we have

$$S_n(t) = \frac{1}{\mathbf{i}}\omega_n^t \frac{(-\omega_n^{2t})^{2^{k-1}} - 1}{-\omega_n^{2t} - 1} = 0.$$

Thus the result follows. □

Lemma 6.2.6. *Let $n = 2^t m$, $t \geq 2$, and m an odd integer. Then the following assertions hold.*

(i) $G_n^1(1) = \begin{cases} 3m + 2G_{\frac{n}{2}}(1) & \text{if } m \equiv 1 \pmod{4} \\ m + 2G_{\frac{n}{2}}(1) & \text{if } m \equiv 3 \pmod{4}. \end{cases}$

(ii) $G_n^3(1) = \begin{cases} m + 2G_{\frac{n}{2}}(1) & \text{if } m \equiv 1 \pmod{4} \\ 3m + 2G_{\frac{n}{2}}(1) & \text{if } m \equiv 3 \pmod{4}. \end{cases}$

(iii) $G_n(1) = \big(m + 2G_{\frac{n}{2}}(1)\big) \cup \big(3m + 2G_{\frac{n}{2}}(1)\big)$.

Proof. Considering Γ to be the cyclic group $\mathbb{Z}_n$ and $x = 1 \pmod{n}$, the proof follows from Lemma 4.3.2. □

The next lemma is a special case of Lemma 4.3.12. For the sake of completeness, we provide the proof.

Lemma 6.2.7. *Let $n = 2^t m$. If m is odd and $t \geq 2$, then*

$$S_n(q) = \begin{cases} 2\sin\left(\frac{3\pi q}{2^{t-1}}\right) C_{\frac{n}{2}}(q) & \text{if } m \equiv 1 \pmod{4} \\ 2\sin\left(\frac{\pi q}{2^{t-1}}\right) C_{\frac{n}{2}}(q) & \text{if } m \equiv 3 \pmod{4}. \end{cases}$$

Proof. Noting that $\sum_{a\in G_{\frac{n}{2}}(1)} \omega_n^{2aq} = \sum_{a\in G_{\frac{n}{2}}(1)} \omega_n^{-2aq}$, we have

$$
\begin{aligned}
S_n(q) &= \sum_{a\in G_n^1(1)} \frac{\omega_n^{aq}-\omega_n^{-aq}}{\mathbf{i}} \\
&= \begin{cases} (-\mathbf{i}) \sum_{a\in G^1_{\frac{n}{2}}(1)} (\omega_n^{(3m+2a)q} - \omega_n^{-(3m+2a)q}) & \text{if } m \equiv 1 \pmod 4 \\ (-\mathbf{i}) \sum_{a\in G^1_{\frac{n}{2}}(1)} (\omega_n^{(m+2a)q} - \omega_n^{-(m+2a)q}) & \text{if } m \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} 2\Im(\omega_n^{3mq}) \sum_{a\in G^1_{\frac{n}{2}}(1)} \omega_n^{2aq} & \text{if } m \equiv 1 \pmod 4 \\ 2\Im(\omega_n^{mq}) \sum_{a\in G^1_{\frac{n}{2}}(1)} \omega_n^{2aq} & \text{if } m \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} 2\Im(\omega_n^{3mq}) C_{\frac{n}{2}}(q) & \text{if } m \equiv 1 \pmod 4 \\ 2\Im(\omega_n^{mq}) C_{\frac{n}{2}}(q) & \text{if } m \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} 2\Im(\omega_{2^t}^{3q}) C_{\frac{n}{2}}(q) & \text{if } m \equiv 1 \pmod 4 \\ 2\Im(\omega_{2^t}^{q}) C_{\frac{n}{2}}(q) & \text{if } m \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} 2\sin\left(\frac{3\pi q}{2^{t-1}}\right) C_{\frac{n}{2}}(q) & \text{if } m \equiv 1 \pmod 4 \\ 2\sin\left(\frac{\pi q}{2^{t-1}}\right) C_{\frac{n}{2}}(q) & \text{if } m \equiv 3 \pmod 4. \end{cases}
\end{aligned}
$$

Here the second equality follows from Part (i) of Lemma 6.2.6. □

Note that $2\sin\left(\frac{3\pi q}{2^{t-1}}\right)$ and $2\sin\left(\frac{\pi q}{2^{t-1}}\right)$ are algebraic integers. Since $S_n(q)$ and $C_{\frac{n}{2}}(q)$ are integers, Lemma 6.2.7 gives that $2\sin\left(\frac{3\pi q}{2^{t-1}}\right)$ and $2\sin\left(\frac{\pi q}{2^{t-1}}\right)$ are rational algebraic integers whenever $C_{\frac{n}{2}}(q) \neq 0$, and hence they are integers. Thus $S_n(q)$ is an integer multiple of the Ramanujan sum.

Theorem 6.2.8. *Let $n \equiv 0 \pmod 4$ and $P = \{2^k : k \geq 0\}$. If $D_n^3 = \emptyset$, then*

$$S_n(t) = 2\delta_t \mu_P(n_t) \sum_{\substack{e|\delta_t \\ \frac{te}{\delta_t} \text{ is odd} \\ \gcd(n_t, e)=1}} (-1)^{\frac{te-\delta_t}{2\delta_t}} \frac{\mu_P(e)}{e},$$

where $\delta_t = \gcd(\frac{n}{4}, t)$ and $n_t = \frac{n}{4\delta_t}$.

Proof. Let $f_n(t) := \sum_{a\in M_n^1(1)} \frac{\omega_n^{at}-\omega_n^{-at}}{\mathbf{i}}$. We observe that $f_n(t) = \frac{1}{\mathbf{i}}(\omega_n^t - \omega_n^{3t} + \omega_n^{5t} - \ldots - \omega_n^{(n-1)t})$. Proceeding as in the proof of Lemma 6.2.1, we find

$$f_n(t) = \begin{cases} \frac{n}{2} & \text{if } t \equiv \frac{n}{4} \pmod n \\ -\frac{n}{2} & \text{if } t \equiv \frac{3n}{4} \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

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Noting that $D_n^1 = D_{\frac{n}{4}}^1$, we use Lemma 1.4.3 to have

$$f_n(t) = \sum_{a\in M_n^1(1)} \frac{\omega_n^{at} - \omega_n^{-at}}{\mathbf{i}} = \sum_{d\in D_n^1} \sum_{a\in G_n^1(d)} \frac{\omega_n^{at} - \omega_n^{-at}}{\mathbf{i}} = \sum_{d\in D_n^1} \sum_{a\in dG_{n/d}^1(1)} \frac{\omega_n^{at} - \omega_n^{-at}}{\mathbf{i}}$$
$$= \sum_{d\in D_n^1} \sum_{a\in G_{n/d}^1(1)} \frac{(\omega_n^d)^{at} - (\omega_n^d)^{-at}}{\mathbf{i}}$$
$$= \sum_{d\in D_n^1} S_{\frac{n}{d}}(t).$$

In the last equality, we use the fact that ω_n^d is an $\frac{n}{d}$-th root of unity. Consider $P = \{2^k : k \geq 0\}$ and $Q = \{n : n \in \mathbb{N} \text{ and } 2 \nmid n\}$. We see that $Q \cap \{d : d \mid n\} = D_n^1 \cup D_n^3 = D_n^1 \cup \emptyset = D_n^1$. Thus

$$f_n(t) = \sum_{d\in Q, d|n} S_{\frac{n}{d}}(t).$$

Therefore Theorem 6.1.7 gives

$$S_n(t) = \sum_{\substack{d|n \\ t\equiv \frac{d}{4} \pmod d}} f_d(t)\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ t\equiv \frac{3d}{4} \pmod d}} f_d(t)\mu_P\left(\frac{n}{d}\right)$$
$$= \sum_{\substack{d|n \\ t\equiv \frac{d}{4} \pmod d}} \frac{d}{2}\mu_P\left(\frac{n}{d}\right) - \sum_{\substack{d|n \\ t\equiv \frac{3d}{4} \pmod d}} \frac{d}{2}\mu_P\left(\frac{n}{d}\right)$$
$$= \sum_{\substack{4d|n \\ t\equiv d \pmod{4d}}} \frac{4d}{2}\mu_P\left(\frac{n}{4d}\right) - \sum_{\substack{4d|n \\ t\equiv 3d \pmod{4d}}} \frac{4d}{2}\mu_P\left(\frac{n}{4d}\right)$$
$$= \sum_{\substack{d|\frac{n}{4} \\ \frac{t}{d}\equiv 1 \pmod 4}} 2d\mu_P\left(\frac{n}{4d}\right) - \sum_{\substack{d|\frac{n}{4} \\ \frac{t}{d}\equiv 3 \pmod 4}} 2d\mu_P\left(\frac{n}{4d}\right)$$
$$= \sum_{\substack{d|\gcd(\frac{n}{4},t) \\ \frac{t}{d} \text{ is odd}}} (-1)^{\frac{t-d}{2d}} 2d\mu_P\left(\frac{n}{4d}\right).$$

As μ_P is a multiplicative function [14], we have $\mu_P(rs) = \mu_P(r)\mu_P(s)$ whenever $\gcd(r,s) = 1$. Suppose that $\gcd(r,s) > 1$. In this case we use Lemma 6.1.8. We see that either rs is even for which $\mu_P(rs) = \mu(rs) = 0$ or rs is odd with a square factor for which $\mu_P(rs) = \mu(rs) = 0$.

Now let $\delta_t = \gcd(\frac{n}{4}, t)$ and $\frac{n}{4} = \delta_t n_t$. We have

$$S_n(t) = \sum_{\substack{d|\delta_t \\ \frac{t}{d} \text{ is odd}}} (-1)^{\frac{t-d}{2d}} 2d\mu_P\left(\frac{n}{4d}\right) = \sum_{\substack{de=\delta_t \\ \frac{t}{d} \text{ is odd}}} (-1)^{\frac{t-d}{2d}} 2d\mu_P(n_t e)$$

$$= 2\mu_P(n_t) \sum_{\substack{de=\delta_t \\ \frac{t}{d} \text{ is odd} \\ \gcd(n_t,e)=1}} (-1)^{\frac{t-d}{2d}} d\mu_P(e)$$

$$= 2\delta_t\mu_P(n_t) \sum_{\substack{e|\delta_t \\ \frac{te}{\delta_t} \text{ is odd} \\ \gcd(n_t,e)=1}} (-1)^{\frac{te-\delta_t}{2\delta_t}} \frac{\mu_P(e)}{e}.$$

Thus the proof is complete. □

Corollary 6.2.9. *Let $n \equiv 0 \pmod 4$. If $D_n^3 = \emptyset$ and $\gcd(\frac{n}{4}, t) = 1$, then*

$$S_n(t) = \begin{cases} 2(-1)^{\frac{t-1}{2}}\mu\left(\frac{n}{4}\right) & \text{if } t \text{ and } \frac{n}{4} \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P = \{2^k : k \geq 0\}$. Put $\delta_t = \gcd(\frac{n}{4}, t) = 1$ in Theorem 6.2.8. We see that $e = 1$ and $n_t = \frac{n}{4\delta_t} = \frac{n}{4}$. Therefore

$$S_n(t) = 2\mu_P\left(\frac{n}{4}\right) \sum_{t \text{ odd}} (-1)^{\frac{t-1}{2}}\mu_P(1) = 2\mu_P\left(\frac{n}{4}\right) \sum_{t \text{ odd}} (-1)^{\frac{t-1}{2}}$$

$$= \begin{cases} 2(-1)^{\frac{t-1}{2}}\mu_P\left(\frac{n}{4}\right) & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

By Lemma 6.1.8, we have $\mu_P\left(\frac{n}{4}\right) = 0$ if $\frac{n}{4}$ is even and $\mu_P\left(\frac{n}{4}\right) = \mu\left(\frac{n}{4}\right)$ if $\frac{n}{4}$ is odd. Therefore

$$S_n(t) = \begin{cases} 2(-1)^{\frac{t-1}{2}}\mu\left(\frac{n}{4}\right) & \text{if } t \text{ and } \frac{n}{4} \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the proof is complete. □

Corollary 6.2.10. *Let $n = 4m$ with m an odd integer. If $D_n^3 = \emptyset$, then*

$$S_n(t) = \begin{cases} 2(-1)^{\frac{t-1}{2}}C_m(t) & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

Proof. Let $P = \{2^k : k \geq 0\}$. We use the notations of Theorem 6.2.8. Since $\frac{n}{4}$ is given to be odd, we see that δ_t is odd. Therefore, $\frac{\delta_t}{e}$ is odd for each divisor e of δ_t. Thus if t is an even integer, then $\frac{te}{\delta_t}$ is not odd whenever $e \mid \delta_t$. Therefore by Theorem 6.2.8, we get $S_n(t) = 0$.

Now assume that t is odd. As δ_t is odd, each divisor e of δ_t is also odd. Further, $D_n^3 = \emptyset$ and m is odd. Therefore $\delta_t \equiv 1 \pmod 4$. Accordingly, $\frac{te}{\delta_t} \equiv 1 \pmod 4$ (respectively $3 \pmod 4$) if and only if $t \equiv 1 \pmod 4$ (respectively $3 \pmod 4$), whenever $e \mid \delta_t$. Therefore $(-1)^{\frac{t-1}{2}} = (-1)^{\frac{te-\delta_t}{2\delta_t}}$. Now for $0 \leq t \leq n-1$, using Lemma 6.1.8 and Theorem 6.2.8, we have

$$\begin{aligned} S_n(t) = 2\delta_t \mu_P(n_t) \sum_{\substack{e|\delta_t \\ \frac{te}{\delta_t} \text{ is odd} \\ \gcd(n_t,e)=1}} (-1)^{\frac{te-\delta_t}{2\delta_t}} \frac{\mu_P(e)}{e} &= 2\delta_t \mu(n_t) \sum_{\substack{e|\delta_t \\ \gcd(n_t,e)=1}} (-1)^{\frac{t-1}{2}} \frac{\mu(e)}{e} \\ &= 2(-1)^{\frac{t-1}{2}} \mu(n_t)\delta_t \sum_{\substack{e|\delta_t \\ \gcd(n_t,e)=1}} \frac{\mu(e)}{e} \\ &= 2(-1)^{\frac{t-1}{2}} \mu(n_t) \frac{\varphi(\delta_t n_t)}{\varphi(n_t)} \\ &= 2(-1)^{\frac{t-1}{2}} \frac{\mu(m/\delta_t)\varphi(m)}{\varphi(m/\delta_t)} \\ &= 2(-1)^{\frac{t-1}{2}} C_m(t). \end{aligned}$$

Here in the fourth step, we use Lemma 6.1.5 with $a = \delta_t$ and $b = n_t$. After that, we consider $n_t = \frac{m}{\delta_t}$, and use Lemma 6.1.1 in the last step. □

6.3 HS-eigenvalues of mixed circulant graphs

The next result follows from Lemma 3.1.1.

Lemma 6.3.1. *The HS-spectrum of the mixed graph* $\mathrm{Circ}(\mathbb{Z}_n, S)$ *is* $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$, *where* $\gamma_j = \lambda_j + \mu_j$,

$$\lambda_j = \sum_{k \in S \setminus \overline{S}} \omega_n^{jk} \quad \text{and} \quad \mu_j = \sum_{k \in \overline{S}} (\omega_6 \omega_n^{jk} + \omega_6^5 \omega_n^{-jk}) \text{ for each } j \in \{0, 1, \dots, n-1\}.$$

The next result follows from Theorem 3.3.6.

Theorem 6.3.2. *The mixed graph* $\mathrm{Circ}(\mathbb{Z}_n, S)$ *is HS-integral if and only if* $S \setminus \overline{S} = \bigcup_{d \in \mathscr{D}_1} G_n(d)$ *and*

$$\overline{S} = \begin{cases} \emptyset & \text{if } n \not\equiv 0 \pmod 3 \\ \left(\bigcup_{d \in \mathscr{D}_2} G^1_{n,3}(d) \right) \cup \left(\bigcup_{d \in \mathscr{D}_3} G^2_{n,3}(d) \right) & \text{if } n \equiv 0 \pmod 3, \end{cases}$$

where $\mathscr{D}_1 \subseteq \{d \colon d \mid n\}$, $\mathscr{D}_2 \cup \mathscr{D}_3 \subseteq \{d \colon d \mid \frac{n}{3}\}$, $\mathscr{D}_1 \cap (\mathscr{D}_2 \cup \mathscr{D}_3) = \emptyset$ *and* $\mathscr{D}_2 \cap \mathscr{D}_3 = \emptyset$.

For $n \equiv 0 \pmod 3$ and $q \in \{0, 1, \dots, n-1\}$, define

$$T_n(q) := \sum_{a \in G^1_{n,3}(1)} \mathbf{i}\sqrt{3}(\omega_n^{aq} - \omega_n^{-aq}).$$

Recall the definition of $T_x(j)$ from Chapter 5. Note that $T_n(q)$ is a special case of $T_x(j)$ for cyclic group, and that $T_n(q) = \sum\limits_{a\in G^1_{n,3}(1)} -2\sqrt{3}\sin\left(\frac{2\pi aq}{n}\right)$.

We have

$$\frac{C_n(q)}{2} + \frac{T_n(q)}{2} = \sum_{a\in G^1_{n,3}(1)} \left(\omega_6\omega_n^{aq} + \omega_6^5\omega_n^{-aq}\right)$$

and

$$\frac{C_n(q)}{2} - \frac{T_n(q)}{2} = \sum_{a\in G^2_{n,3}(1)} \left(\omega_6\omega_n^{aq} + \omega_6^5\omega_n^{-aq}\right).$$

By Lemma 6.3.1, $\frac{C_n(q)}{2} + \frac{T_n(q)}{2}$ (respectively $\frac{C_n(q)}{2} - \frac{T_n(q)}{2}$) is an HS-eigenvalue of the mixed graph $\mathrm{Circ}(\mathbb{Z}_n, G^1_{n,3}(1))$ (respectively $\mathrm{Circ}(\mathbb{Z}_n, G^2_{n,3}(1))$) for each $q \in \{0, 1, \ldots, n-1\}$.

For $n \equiv 0 \pmod 3$ and a divisor d of $\frac{n}{3}$, we have

$$\frac{C_{\frac{n}{d}}(q)}{2} + \frac{T_{\frac{n}{d}}(q)}{2} = \sum_{a\in G^1_{n,3}(d)} \left(\omega_6\omega_n^{aq} + \omega_6^5\omega_n^{-aq}\right)$$

and

$$\frac{C_{\frac{n}{d}}(q)}{2} - \frac{T_{\frac{n}{d}}(q)}{2} = \sum_{a\in G^2_{n,3}(d)} \left(\omega_6\omega_n^{aq} + \omega_6^5\omega_n^{-aq}\right).$$

Theorem 6.3.3. *Let $\mathscr{D}_1 \subseteq \{d\colon d \mid n\}$ and $\mathscr{D}_2 \cup \mathscr{D}_3 \subseteq \{d\colon d \mid \frac{n}{3}\}$. If $\mathscr{D}_1 \cap (\mathscr{D}_2 \cup \mathscr{D}_3) = \emptyset$ and $\mathscr{D}_2 \cap \mathscr{D}_3 = \emptyset$, then*

$$\sum_{d\in\mathscr{D}_1} C_{\frac{n}{d}}(j) + \sum_{d\in\mathscr{D}_2}\left(\frac{C_{\frac{n}{d}}(j)}{2} + \frac{T_{\frac{n}{d}}(j)}{2}\right) + \sum_{d\in\mathscr{D}_3}\left(\frac{C_{\frac{n}{d}}(j)}{2} - \frac{T_{\frac{n}{d}}(j)}{2}\right)$$

is an HS-eigenvalue of a mixed circulant graph for each $j \in \{0, 1, \ldots n-1\}$.

Proof. Let $\mathscr{D}_1 \subseteq \{d\colon d \mid n\}$, $\mathscr{D}_2 \cup \mathscr{D}_3 \subseteq \{d\colon d \mid \frac{n}{3}\}$, $\mathscr{D}_1 \cap (\mathscr{D}_2 \cup \mathscr{D}_3) = \emptyset$ and $\mathscr{D}_2 \cap \mathscr{D}_3 = \emptyset$. Define the set S by setting $S \setminus \overline{S} = \bigcup\limits_{d\in\mathscr{D}_1} G_n(d)$ and

$$\overline{S} = \begin{cases} \emptyset & \text{if } \ n \not\equiv 0 \pmod 3 \\ \left(\bigcup\limits_{d\in\mathscr{D}_2} G^1_{n,3}(d)\right) \cup \left(\bigcup\limits_{d\in\mathscr{D}_3} G^2_{n,3}(d)\right) & \text{if } \ n \equiv 0 \pmod 3. \end{cases}$$

Using Lemma 6.3.1, the HS-eigenvalues of $\mathrm{Circ}(\mathbb{Z}_n, S)$ are given by

$$\begin{aligned}\gamma_j &= \sum_{d\in\mathscr{D}_1}\sum_{a\in G_n(d)} \omega_n^{aj} + \sum_{d\in\mathscr{D}_2}\sum_{a\in G^1_{n,3}(d)} \left(\omega_6\omega_n^{aj} + \omega_6^5\omega_n^{-aj}\right) + \sum_{d\in\mathscr{D}_3}\sum_{a\in G^2_{n,3}(d)} \left(\omega_6\omega_n^{aj} + \omega_6^5\omega_n^{-aj}\right)\\ &= \sum_{d\in\mathscr{D}_1} C_{\frac{n}{d}}(j) + \sum_{d\in\mathscr{D}_2}\left(\frac{C_{\frac{n}{d}}(j)}{2} + \frac{T_{\frac{n}{d}}(j)}{2}\right) + \sum_{d\in\mathscr{D}_3}\left(\frac{C_{\frac{n}{d}}(j)}{2} - \frac{T_{\frac{n}{d}}(j)}{2}\right),\end{aligned}$$

where $j \in \{0, 1, \ldots, n-1\}$. □

Lemma 6.3.4. *Let $n = 3^t m$ and $m \not\equiv 0 \pmod 3$. Then the following assertions hold.*

(i) If $t = 1$ then $G_n(1) = \big(m + 3G_{\frac{n}{3}}(1)\big) \cup \big(2m + 3G_{\frac{n}{3}}(1)\big)$.

(ii) If $t = 1$ then

$$G^1_{n,3}(1) = \begin{cases} m + 3G_{\frac{n}{3}}(1) & \text{if } m \equiv 1 \pmod 3 \\ 2m + 3G_{\frac{n}{3}}(1) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

(iii) If $t \geq 2$ then

$$G^1_{n,3}(1) = \begin{cases} \big(m + 3G_{\frac{n}{3}}(1)\big) \cup \big(4m + 3G^2_{\frac{n}{3},3}(1)\big) & \text{if } m \equiv 1 \pmod 3 \\ \big(2m + 3G_{\frac{n}{3}}(1)\big) \cup \big(5m + 3G^1_{\frac{n}{3},3}(1)\big) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

(iv) If $t \geq 2$ then

$$G^1_{n,3}(1) = \begin{cases} \big(7m + 3G_{\frac{n}{3}}(1)\big) \cup \big(4m + 3G^1_{\frac{n}{3},3}(1)\big) & \text{if } m \equiv 1 \pmod 3 \\ \big(8m + 3G_{\frac{n}{3}}(1)\big) \cup \big(5m + 3G^2_{\frac{n}{3},3}(1)\big) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

(v) If $t \geq 2$ then

$$G^1_{n,3}(1) = \begin{cases} \big(m + 3G_{\frac{n}{3}}(1)\big) \cup \big(4m + 3G_{\frac{n}{3}}(1)\big) \cup \big(7m + 3G_{\frac{n}{3}}(1)\big) & \text{if } m \equiv 1 \pmod 3 \\ \big(2m + 3G_{\frac{n}{3}}(1)\big) \cup \big(5m + 3G_{\frac{n}{3}}(1)\big) \cup \big(8m + 3G_{\frac{n}{3}}(1)\big) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Proof. Considering Γ to be the cyclic group $\mathbb{Z}_n$ and $x = 1 \pmod n$, the proof follows from Lemma 5.4.4. □

The next two lemmas are special cases of Lemma 5.4.12 and Lemma 5.4.13, respectively. For the sake of completeness, we provide the proofs.

Lemma 6.3.5. *Let $n = 3m$. If $m \not\equiv 0 \pmod 3$, then*

$$T_n(q) = \begin{cases} T_3(q)C_{\frac{n}{3}}(q) & \text{if } m \equiv 1 \pmod 3 \\ -T_3(q)C_{\frac{n}{3}}(q) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Moreover, $\frac{T_n(q)}{3}$ is an integer for each $q \in \mathbb{Z}$.

Proof. Note that

$$T_3(q) = \sum_{a \in G^1_{3,3}(1)} \mathbf{i}\sqrt{3}(\omega_3^{aq} - \omega_3^{-aq}) = \mathbf{i}\sqrt{3}(\omega_3^{q} - \omega_3^{-q}) = -2\sqrt{3}\Im(\omega_3^{q}) = 2\sqrt{3}\Im(\omega_3^{2q}).$$

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Now we have

$$
\begin{aligned}
T_n(q) &= \sum_{a\in G^1_{n,3}(1)} \mathbf{i}\sqrt{3}(\omega_n^{aq}-\omega_n^{-aq})\\
&= \begin{cases} \sum_{a\in G_{\frac{n}{3}}(1)} \mathbf{i}\sqrt{3}(\omega_n^{mq}\omega_n^{3aq}-\omega_n^{-mq}\omega_n^{-3aq}) & \text{if } m\equiv 1 \pmod 3\\ \sum_{a\in G_{\frac{n}{3}}(1)} \mathbf{i}\sqrt{3}(\omega_n^{2mq}\omega_n^{3aq}-\omega_n^{-2mq}\omega_n^{-3aq}) & \text{if } m\equiv 2 \pmod 3\end{cases}\\
&= \begin{cases} -2\sqrt{3}\Im(\omega_n^{mq}) \sum_{a\in G_{\frac{n}{3}}(1)} \omega_{\frac{n}{3}}^{aq} & \text{if } m\equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_n^{2mq}) \sum_{a\in G_{\frac{n}{3}}(1)} \omega_{\frac{n}{3}}^{aq} & \text{if } m\equiv 2 \pmod 3\end{cases}\\
&= \begin{cases} T_3(q)C_{\frac{n}{3}}(q) & \text{if } m\equiv 1 \pmod 3\\ -T_3(q)C_{\frac{n}{3}}(q) & \text{if } m\equiv 2 \pmod 3.\end{cases}
\end{aligned}
$$

Here the second equality follows from Part (ii) of Lemma 6.3.4. Since $T_3(q)\in\{0,\pm 3\}$, we see that $\frac{T_n(q)}{3}$ is an integer for each $q\in\mathbb{Z}$. □

Lemma 6.3.6. *Let $n=3^t m$ and $m\not\equiv 0 \pmod 3$. If $t\geq 2$, then*

$$
2T_n(q) = \begin{cases} -2\sqrt{3}\Im(\omega_n^{mq}+\omega_n^{4mq}+\omega_n^{7mq})C_{\frac{n}{3}}(q) & \text{if } m\equiv 1 \pmod 3\\ -2\sqrt{3}\Im(\omega_n^{2mq}+\omega_n^{5mq}+\omega_n^{8mq})C_{\frac{n}{3}}(q) & \text{if } m\equiv 2 \pmod 3.\end{cases}
$$

Moreover, $\frac{T_n(q)}{3}$ is an integer for each $q\in\mathbb{Z}$.

Proof. We use the fact that $G^1_{n,3}(1)$ can be written as disjoint unions in two different ways using Part (iii) and Part (iv) of Lemma 6.3.4. We have

$$2T_n(q) = \sum_{a\in G^1_{n,3}(1)} \mathbf{i}\sqrt{3}(\omega_n^{aq} - \omega_n^{-aq}) + \sum_{a\in G^1_{n,3}(1)} \mathbf{i}\sqrt{3}(\omega_n^{aq} - \omega_n^{-aq})$$

$$= \begin{cases} \sum\limits_{a\in G_{\frac{n}{3}}(1)} \mathbf{i}\sqrt{3}\big[(\omega_n^{mq} + \omega_n^{4mq} + \omega_n^{7mq})\omega_n^{3aq} \\ \qquad -(\omega_n^{-mq} + \omega_n^{-4mq} + \omega_n^{-7mq})\omega_n^{-3aq}\big] & \text{if } m \equiv 1 \pmod 3 \\ \sum\limits_{a\in G_{\frac{n}{3}}(1)} \mathbf{i}\sqrt{3}\big[(\omega_n^{2mq} + \omega_n^{5mq} + \omega_n^{8mq})\omega_n^{3aq} \\ \qquad -(\omega_n^{-2mq} + \omega_n^{-5mq} + \omega_n^{-8mq})\omega_n^{-3aq}\big] & \text{if } m \equiv 2 \pmod 3 \end{cases}$$

$$= \begin{cases} -2\sqrt{3}\Im(\omega_n^{mq} + \omega_n^{4mq} + \omega_n^{7mq}) \sum\limits_{a\in G_{\frac{n}{3}}(1)} \omega_{\frac{n}{3}}^{aq} & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Im(\omega_n^{2mq} + \omega_n^{5mq} + \omega_n^{8mq}) \sum\limits_{a\in G_{\frac{n}{3}}(1)} \omega_{\frac{n}{3}}^{aq} & \text{if } m \equiv 2 \pmod 3 \end{cases}$$

$$= \begin{cases} -2\sqrt{3}\Im(\omega_n^{mq} + \omega_n^{4mq} + \omega_n^{7mq})C_{\frac{n}{3}}(q) & \text{if } m \equiv 1 \pmod 3 \\ -2\sqrt{3}\Im(\omega_n^{2mq} + \omega_n^{5mq} + \omega_n^{8mq})C_{\frac{n}{3}}(q) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

For $t = 2$, that is, for $n = 9m$, we see that

$$-2\sqrt{3}\Im(\omega_n^{mq} + \omega_n^{4mq} + \omega_n^{7mq}) = -2\sqrt{3}\Im(\omega_9^{q} + \omega_9^{4q} + \omega_9^{7q}) = \sum_{a\in G^1_{9,3}(1)} \mathbf{i}\sqrt{3}(\omega_9^{aq} - \omega_9^{-aq}) = T_9(q).$$

Similarly, $-2\sqrt{3}\Im(\omega_n^{2mq} + \omega_n^{5mq} + \omega_n^{8mq}) = -T_9(q)$. Thus

$$2T_n(q) = \begin{cases} T_9(q)C_{\frac{n}{3}}(q) & \text{if } m \equiv 1 \pmod 3 \\ -T_9(q)C_{\frac{n}{3}}(q) & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

We see that

$$T_9(q) = \begin{cases} -9 & \text{if } q \equiv 3 \pmod 9 \\ 9 & \text{if } q \equiv 6 \pmod 9 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $T_9(q)$ is a multiple of 3, giving that $\frac{2T_n(q)}{3}$ is an integer. Hence $\frac{T_n(q)}{3}$ is an integer for each $q \in \mathbb{Z}$. Now assume that $t \geq 3$. If $C_{\frac{n}{3}}(q) \neq 0$, then $-2\sqrt{3}\Im(\omega_n^{mq} + \omega_n^{4mq} + \omega_n^{7mq})$ and $-2\sqrt{3}\Im(\omega_n^{2mq} + \omega_n^{5mq} + \omega_n^{8mq})$ are rational algebraic integers, and hence both are integers for each $q \in \mathbb{Z}$. By Lemma 6.1.2, $C_{\frac{n}{3}}(q)$ is an integer multiple of 3. Hence $\frac{T_n(q)}{3}$ is an integer for each $q \in \mathbb{Z}$. □

Note that $T_3(q) \in \{0, \pm 3\}$. Therefore, if $n = 3m$ with $m \not\equiv 0 \pmod 3$, then by Lemma 6.3.5 $T_n(q)$ is an integer multiple of the Ramanujan sum. Further, by Lemma 6.3.6, $T_n(q)$ is an integer multiple of the Ramanujan sum whenever $n = 3^t m$ with $t \geq 2$ and $m \not\equiv 0 \pmod 3$.

Theorem 6.3.7. *Let $n \equiv 0 \pmod 3$ and $P = \{3^k \colon k \geq 0\}$. If $D^2_{n,3} = \emptyset$, then*

$$T_n(t) = \sum_{\substack{d \mid \frac{n}{3} \\ \frac{t}{d} \equiv 1 \pmod 3}} -3d\mu_P\left(\frac{n}{3d}\right) + \sum_{\substack{d \mid \frac{n}{3} \\ \frac{t}{d} \equiv 2 \pmod 3}} 3d\mu_P\left(\frac{n}{3d}\right).$$

Proof. Assume that $D^2_{n,3} = \emptyset$. Let $f_n(t) = \sum_{a \in M^1_{n,3}(1)} \mathbf{i}\sqrt{3}(\omega_n^{at} - \omega_n^{-at})$. We have

$$\begin{aligned} f_n(t) &= -2\sqrt{3} \sum_{a \in M^1_{n,3}(1)} \Im(\omega_n^{at}) \\ &= -2\sqrt{3}\Im(\omega_n^t + \omega_n^{4t} + \ldots + \omega_n^{(n-2)t}). \end{aligned}$$

Note that

$$\omega_n^t + \omega_n^{4t} + \ldots + \omega_n^{(n-2)t} = \begin{cases} \frac{n}{3}\omega_3 & \text{if } t \equiv \frac{n}{3} \pmod n \\ \frac{n}{3}\omega_3^2 & \text{if } t \equiv \frac{2n}{3} \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$f_n(t) = \begin{cases} -n & \text{if } t \equiv \frac{n}{3} \pmod n \\ n & \text{if } t \equiv \frac{2n}{3} \pmod n \\ 0 & \text{otherwise.} \end{cases}$$

Since $D^2_{\frac{n}{3},3} = D^2_{n,3} = \emptyset$ and $D^1_{\frac{n}{3},3} = D^1_{n,3}$, using Lemma 1.5.2 we have

$$\begin{aligned} f_n(t) = \sum_{a \in M^1_{n,3}(1)} \mathbf{i}\sqrt{3}(\omega_n^{at} - \omega_n^{-at}) &= \sum_{d \in D^1_{n,3}} \sum_{a \in G^1_{n,3}(d)} \mathbf{i}\sqrt{3}(\omega_n^{at} - \omega_n^{-at}) \\ &= \sum_{d \in D^1_{n,3}} \sum_{a \in dG^1_{\frac{n}{d},3}(1)} \mathbf{i}\sqrt{3}(\omega_n^{at} - \omega_n^{-at}) \\ &= \sum_{d \in D^1_{n,3}} \sum_{a \in G^1_{\frac{n}{d},3}(1)} \mathbf{i}\sqrt{3}[(\omega_n^d)^{at} - (\omega_n^d)^{-at}] \\ &= \sum_{d \in D^1_{n,3}} T_{\frac{n}{d}}(t). \end{aligned}$$

In the last equality, we use the fact that ω_n^d is a primitive $\frac{n}{d}$-th root of unity. Now consider $P = \{3^k \colon k \geq 0\}$ and $Q = \{n \colon n \in \mathbb{N} \text{ and } 3 \nmid n\}$. We see that

$$Q \cap \{d \colon d \mid n\} = D^1_{n,3} \cup D^2_{n,3} = D^1_{n,3} \cup \emptyset = D^1_{n,3}.$$

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Thus

$$f_n(t) = \sum_{d\in Q, d|n} T_{\frac{n}{d}}(t).$$

Therefore, Theorem 6.1.7 implies that

$$\begin{aligned}
T_n(t) &= \sum_{\substack{d|n \\ t\equiv \frac{d}{3} \pmod d}} f_d(t)\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ t\equiv \frac{2d}{3} \pmod d}} f_d(t)\mu_P\left(\frac{n}{d}\right) \\
&= \sum_{\substack{d|n \\ t\equiv \frac{d}{3} \pmod d}} -d\mu_P\left(\frac{n}{d}\right) + \sum_{\substack{d|n \\ t\equiv \frac{2d}{3} \pmod d}} d\mu_P\left(\frac{n}{d}\right) \\
&= \sum_{\substack{3d|n \\ t\equiv d \pmod{3d}}} -3d\mu_P\left(\frac{n}{3d}\right) + \sum_{\substack{3d|n \\ t\equiv 2d \pmod{3d}}} 3d\mu_P\left(\frac{n}{3d}\right) \\
&= \sum_{\substack{d|\frac{n}{3} \\ \frac{t}{d}\equiv 1 \pmod 3}} -3d\mu_P\left(\frac{n}{3d}\right) + \sum_{\substack{d|\frac{n}{3} \\ \frac{t}{d}\equiv 2 \pmod 3}} 3d\mu_P\left(\frac{n}{3d}\right).
\end{aligned}$$

□

Corollary 6.3.8. *Let $n = 3m$ with $3 \nmid m$. If $D^2_{n,3} = \emptyset$, then*

$$T_n(t) = \begin{cases} -3C_m(t) & \text{if } t \equiv 1 \pmod 3 \\ 3C_m(t) & \text{if } t \equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P = \{3^k : k \geq 0\}$. Since $3 \nmid \frac{n}{3}$ and $D^2_{n,3} = \emptyset$, we see that each divisor of $\frac{n}{3}$ is congruent to 1 modulo 3. Thus if $t \equiv 0 \pmod 3$, then $\frac{t}{d} \equiv 0 \pmod 3$ whenever $d \mid \frac{n}{3}$ and $d \mid t$. Therefore by Theorem 6.3.7, we get $T_n(t) = 0$. Now assume that $t \equiv 1 \pmod 3$. Then $\frac{t}{d} \equiv 1 \pmod 3$ whenever $d \mid \frac{n}{3}$ and $d \mid t$. Therefore by Theorem 6.3.7, we get

$$T_n(t) = \sum_{d|\frac{n}{3}, d|t} -3d\mu_P\left(\frac{n}{3d}\right).$$

Similarly, if $t \equiv 2 \pmod 3$, then $\frac{t}{d} \equiv 2 \pmod 3$ whenever $d \mid \frac{n}{3}$ and $d \mid t$. Again by Theorem 6.3.7, we get

$$T_n(t) = \sum_{d|\frac{n}{3}, d|t} 3d\mu_P\left(\frac{n}{3d}\right).$$

Thus

$$T_n(t) = \begin{cases} -\sum_{d|\gcd(\frac{n}{3},t)} 3d\mu_P\left(\frac{n}{3d}\right) & \text{if } t \equiv 1 \pmod 3 \\ \sum_{d|\gcd(\frac{n}{3},t)} 3d\mu_P\left(\frac{n}{3d}\right) & \text{if } t \equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\delta_t = \gcd(\frac{n}{3}, t)$ and $\frac{n}{3} = \delta_t n_t$. Using Lemma 6.1.9, we have

$$\begin{aligned}
\sum_{d|\gcd(\frac{n}{3},t)} 3d\mu_P\left(\frac{n}{3d}\right) = \sum_{d|\delta_t} 3d\mu\left(\frac{n}{3d}\right) &= \sum_{de=\delta_t} 3d\mu\left(\frac{\delta_t n_t}{d}\right) \\
&= \sum_{de=\delta_t} 3d\mu(n_t e) \\
&= 3\mu(n_t) \sum_{\substack{de=\delta_t \\ \gcd(n_t,e)=1}} d\mu(e) \\
&= 3\mu(n_t)\delta_t \sum_{\substack{e|\delta_t \\ \gcd(n_t,e)=1}} \frac{\mu(e)}{e} \\
&= 3\mu(n_t)\frac{\varphi(\delta_t n_t)}{\varphi(n_t)} \\
&= 3\frac{\mu(m/\delta_t)\varphi(m)}{\varphi(m/\delta_t)} \\
&= 3C_m(t).
\end{aligned}$$

Here in the sixth step, we use Lemma 6.1.5 with $a = \delta_t$ and $b = n_t$. After that, we consider $n_t = \frac{m}{\delta_t}$, and use Lemma 6.1.1 in the last step. Thus the proof is complete. □

Note that Corollary 6.3.8 is a special case of Lemma 6.3.5.

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7

Future work

In this chapter, we present few problems that came to our attention while working for this thesis.

In the thesis, we discuss characterizations of H-integral, HS-integral, Gaussian integral and Eisenstein integral normal mixed Cayley graphs. So, the search for these four types of integralities of mixed Cayley graphs are open for investigation for the non-normal case. One can start this search with the well-known non-abelian groups, namely the dihedral and dicyclic groups.

Problem 1. *Characterize H-integral, HS-integral, Gaussian integral and Eisenstein integral non-normal mixed Cayley graphs over the dihedral and dicyclic groups.*

Problem 2. *Characterize H-integral, HS-integral, Gaussian integral and Eisenstein integral non-normal mixed Cayley graphs over a finite group.*

Suppose Γ is a finite group and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. For $x \in \Gamma$ and $j \in \{1, \ldots, h\}$, recall that

$$C_x(j) = \frac{1}{\chi_j(\mathbf{1})} \sum_{s \in S_x^1} \chi_j(s).$$

If $\Gamma = \mathbb{Z}_n$, $\text{Irr}(\mathbb{Z}_n) = \{\chi_1, \ldots, \chi_n\}$ with $\chi_j(k) = \omega_n^{jk}$ for each $k \in \mathbb{Z}_n$ and $x = 1 \pmod n$, then $C_x(j) = \sum_{a \in G_n(1)} \omega_n^{aj} = C_n(j)$. Note that $C_n(j)$ is the Ramanujan sum. Using Lemma 6.1.1, the Ramanujan sum $C_n(j)$ can be calculated by the formula

$$\varphi(n)\frac{\mu(\frac{n}{\gcd(n,j)})}{\varphi(\frac{n}{\gcd(n,j)})}.$$

It is natural to ask whether a similar formula can be obtained for $C_x(j)$ in which the group Γ is non-cyclic.

Problem 3. *Determine a formula for $C_x(j)$, involving some arithmetic functions, where x is an element of a non-cyclic group Γ.*

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