ON THE CONVERGENCE OF H^1-GALERKIN MIXED FINITE ELEMENT METHOD FOR PARABOLIC PROBLEMS

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by

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Dedicated

To my Parents

Certificate

It is certified that the work contained in this thesis entitled **"On the Convergence of H^1-Galerkin Mixed Finite Element Method for Parabolic Problems"** by **Madhusmita Tripathy,** a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

December, 2008

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With Regards,

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Abstract

The purpose of the present work is to study the convergence of H^1-Galerkin mixed finite element method for the linear parabolic partial differential equations. The emphasis is on the theoretical aspects of such methods.

An attempt has been made in this thesis to study the error analysis for the semidiscrete and fully discrete schemes with lesser regularity assumptions on the initial data. More precisely, for homogeneous parabolic problem an energy technique is used to obtain error estimates of order $\mathcal{O}(h^2t^{-1/2})$ with positive time in the L^2-norm for both the solution and the flux when the given initial data is in $H^2(\Omega)\cap H_0^1(\Omega)$. Further, a parabolic duality argument is used to obtain optimal order error estimates of order $\mathcal{O}(h^2t^{-1})$ for both the solution and its flux when the given initial function is only in $H_0^1(\Omega)$. Analogous results are shown to hold for two dimensional parabolic problems. Since the smoothing property of the exact solution plays a significant role in the study of error analysis in the semidiscrete case we, therefore, first investigate the smoothing property of the exact solution of this problem using energy arguments. Based on backward Euler method, a fully discrete scheme is analyzed for one-dimensional homogeneous parabolic problems and almost optimal order error bounds are established.

Optimal order error estimate is the best that one can get between the exact solution and its numerical approximation when measured globally on the computational domain. But, there are places (points or lines) in the computational domain where the approximate solution is more closer to the exact solution than what is predicted by the global error estimates. It would be advantageous to make use of those points or lines in the modelling process. Therefore, we study superconvergence phenomenon for the semidiscrete H^1-Galerkin mixed finite element method for parabolic problems. A new approximate solution for the flux with superconvergence of order $\mathcal{O}(h^{k+3})$ is realized via a postprocessing technique, where $k \geq 1$ is the order of the approximating polynomials employed in the Raviart-Thomas element.

A priori error estimates can give asymptotic rates of convergence as the mesh parameter goes to zero, but often can not provide much practical information about the actual errors encountered on a given mesh. The question of quantifying the error brings attention to a posteriori estimates. A posteriori error estimators are computable quantities which bound the errors or approximate the errors by computed numerical solution and input data of the problem. Further, to guarantee a good convergence behavior of the discrete solution, one needs to apply a refinement algorithm based on a posteriori error estimates. Therefore, we study a posteriori error analysis for the semidiscrete and fully discrete H^1-Galerkin mixed finite element method for parabolic problems. The estimators are derived based on a residual approach.

Contents

Chapter 1

Introduction

The aim of this thesis is to present some theoretical results on the convergence of H^1-Galerkin mixed finite element method for parabolic problems.

1.1 Model Problem

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 1, 2)$ with smooth boundary $\partial\Omega$ and let $J = (0, T]$ with $T < \infty$. We shall consider the linear parabolic problems of the form

$$p_t - \nabla \cdot (a(\mathbf{x})\nabla p) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times J \tag{1.1.1}$$

subject to the homogeneous Dirichlet boundary condition

$$p(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times J, \tag{1.1.2}$$

and the initial condition

$$p(\mathbf{x}, 0) = p_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.1.3}$$

where $p_t = \frac{\partial p}{\partial t}$. The coefficient $a = a(\mathbf{x})$ is assumed to be smooth, symmetric and uniformly positive definite in Ω. Further, the source function f is assumed to be smooth for our purpose.

Equations of the type (1.1.1)-(1.1.3) and nonlinear variants thereof arise in many applications such as heat diffusion and fluid flow in porous media and stock option pricing. These problems are also known as evolution problems and they describe physical and mathematical systems with a time variable, and which behave essentially like heat diffusing through a medium like metal plate. The heat equation is often used in financial mathematics in the modelling of options. The famous Black-Scholes option pricing models differential equation can be transformed into the heat equation allowing relatively easy solutions.

1.2 Notations and Preliminaries

In this section, we shall introduce some standard notations and basic results to be used throughout of this work.

All functions considered here are real valued. Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 1, 2)$, d-dimensional Euclidean space and $\partial\Omega$ denote the boundary of Ω. Let $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \Omega$, and let $d\mathbf{x} = dx_1, \ldots dx_d$. Further, let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a d-tuple with nonnegative integer components and denote order of α as $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$. Then, by $D^\alpha\phi$, we shall mean the αth derivative of ϕ defined by

$$D^\alpha\phi = \frac{\partial^{|\alpha|}\phi}{\partial {x_1}^{\alpha_1}, \ldots, \partial {x_d}^{\alpha_d}}.$$

We shall make frequent reference to the following well-known function spaces. For $1 \leq p < \infty$, $L^p(\Omega)$ denotes the linear space of equivalence classes of measurable functions ϕ in Ω such that $\int_\Omega |\phi(x)|^p dx$ exists and is finite. The norm on $L^p(\Omega)$ is given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |\phi(x)|^p dx \right)^{\frac{1}{p}}, \;\; 1 \leq p < \infty.$$

When $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx.$$

By support of a function ϕ, supp ϕ, we mean the closure of all points x with $\phi(x) \neq 0$, i.e.,

$$\text{supp } \phi = \overline{\{x : \phi(x) \neq 0\}}.$$

For any nonnegative integer m, $\mathcal{C}^m(\overline{\Omega})$ denotes the space of functions with continuous derivatives up to and including order m in $\overline{\Omega}$. $\mathcal{C}_0^m(\Omega)$ is the space all $\mathcal{C}^m(\Omega)$ functions with compact support in Ω. Also $\mathcal{C}_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω.

We now introduce the notion of Sobolev spaces. Let m be the nonnegative integer and let p be such that $1 \leq p < \infty$. The Sobolev space of order (m, p) on Ω, denoted by $W^{m,p}(\Omega)$, is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives up to order m are also in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) = \{\phi : \; D^\alpha\phi \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\begin{aligned}\|\phi\|_{m,p} = \|\phi\|_{m,p,\Omega} &= \left(\int_{\Omega} \sum_{0\le|\alpha|\le m} |D^{\alpha}\phi(x)|^p dx\right)^{\frac{1}{p}} \\ &= \left(\sum_{0\le|\alpha|\le m} \|D^{\alpha}\phi\|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.\end{aligned}$$

For $p = 2$, these spaces will be denoted by $H^m(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space with natural inner product defined by

$$(\phi, \psi) = \sum_{0\le|\alpha|\le m} \int_{\Omega} D^{\alpha}\phi D^{\alpha}\psi dx, \quad \phi, \psi \in H^m(\Omega).$$

The Sobolev spaces $H^m(\Omega)$ (respectively, $H_0^m(\Omega)$) is also defined as the closure of $\mathcal{C}^m(\Omega)$ (respectively, $\mathcal{C}_0^{\infty}(\Omega)$) with respect to the norm $\|\phi\|_m = \|\phi\|_{m,2}$. This result is true under some smoothness assumption on the boundary $\partial\Omega$. Clearly, $L^2(\Omega) = H^0(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$. For $m = 1$, the space $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \{\phi : \phi, D\phi \in L^2(\Omega)\}$$

and

$$H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}.$$

We shall need the following space:

$$\mathbf{V} = H(\text{div}, \Omega) = \{\mathbf{v} \in (\mathrm{L}^2(\Omega))^2 : \nabla \cdot \mathbf{v} \in \mathrm{L}^2(\Omega)\}$$

equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{V}} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{1/2}.$$

For a more complete discussion on Sobolev spaces, see Adams [1, 40].

We shall also use the following time-space function spaces in our error analysis. For a Hilbert space X with norm $\|\cdot\|_X$, let $L^p(0,T;X)$ be the space of strongly measurable and pth integrable X-valued function $\phi : [0,T] \longrightarrow X$. The norm $\|\cdot\|_{L^p(0,T;X)}$ is defined as

$$\|\phi\|_{L^p(0,T;X)} = \left(\int_0^T \|\phi(s)\|_X^p ds\right)^{1/p}, \quad 1 \le p < \infty.$$

From time to time we shall make use of the following inequalities for our error analysis (see, Hardy *et al.* [42]):

(i) *Cauchy-Schwarz inequality:* Let ϕ and ψ are both real valued and $\phi,\ \psi \in L^2(\Omega)$, then

$$|\int_{\Omega} \phi\psi| \le \|\phi\|\ \|\psi\|$$

(*ii*) *Young's Inequality:* Let $a, b \geq 0$, $\epsilon > 0$, p, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{\epsilon a^p}{p} + \frac{\epsilon^{1-q} b^q}{q}.$$

(*iii*) *Hölder's inequality:* For $\phi \in L^p$, $\psi \in L^q$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_\Omega \phi\psi \leq \|\phi\|_{L^p} \|\psi\|_{L^q}.$$

For $p = q = 2$, the above inequality is known as *Schwarz's inequality.*
(*iv*) *Discrete Schwarz's inequality:* Let $\phi_j, \psi_j, j = 1, 2, \ldots, n$ be positive real numbers. Then

$$\sum_{j=1}^{n} \phi_j \psi_j \leq \left(\sum_{j=1}^{n} \phi_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \psi_j^2 \right)^{\frac{1}{2}}.$$

(*v*) *Poincaré's Inequality:* If Ω is a bounded domain in $\mathbb{R}^d$ $(d = 1, 2)$, then there exists a constant $C = C(\Omega)$ such that

$$\|w\| \leq C \|\nabla w\|, \ \forall w \in H_0^1(\Omega).$$

Below, we shall state without proof, the following continuous version of Gronwall's lemma. For a proof, see [69].

Lemma 1.2.1 (Gronwall's Lemma) *Let $G(t)$ be a continuous function and $H(t)$ a nonnegative continuous function on the interval $t_0 \leq t \leq t_0 + a$. If a continuous function $F(t)$ has the property*

$$F(t) \leq G(t) + \int_{t_0}^{t} F(s)H(s)ds \quad \textit{for } \ t \in [t_0, t_0 + a],$$

then

$$F(t) \leq G(t) + \int_{t_0}^{t} G(s)H(s) \exp \left[\int_{s}^{t} H(\tau)d\tau \right] ds \quad \textit{for } t \in [t_0, t_0 + a].$$

In particular, when $G(t) = C$ a nonnegative constant, we have

$$F(t) \leq C \exp \left[\int_{t_0}^{t} H(s)ds \right] \quad \textit{for } t \in [t_0, t_0 + a].$$

We note that for nondecreasing nonnegative G, we obtain the above result with C replaced by $G(t)$. We shall also use the following discrete version of Gronwall's lemma, a proof of which can be found in Pani *et al.* [67].

Lemma 1.2.2 (Discrete Gronwall's Lemma) *Let $\{\xi_n\}$ be a sequence of nonnegative numbers satisfying*

$$\xi_n \leq \alpha_n + \sum_{j=0}^{n-1} \beta_j \xi_j, \ for\, n \geq 0,$$

where $\{\alpha_n\}$ is a nondecreasing sequence and β_j are nonnegative. Then

$$\xi_n \leq \alpha_n \exp(\sum_{j=0}^{n-1} \beta_j), \ for\, n \geq 0.$$

Unless otherwise stated throughout this work C denotes a generic positive constant not necessarily the same at each occurrence.

1.3 Background and Motivation

In this section, we shall discuss a brief survey of the relevant literature concerning the classical mixed finite element method and the H^1-Galerkin mixed finite element method for parabolic problems. This section also elucidates the motivation for the present study.

The study of mixed finite element method comes from two fundamental mathematical models of physics and engineering: The Stokes problem for slow (creeping) incompressible viscous fluid flow and the Poisson problem arising from invisid fluid flow, fluid flow in Porous Media, stationary heat conduction and electrostatics. The Stokes problem can not be discretized by a straightforward Galerkin method as this method turns out to be unstable by giving non-physical oscillation in the pressure [75]. Further, when our primary concern is to compute fluxes or velocities the standard Galerkin methods lead to a loss of accuracy as they are computed from the approximate solutions via a postprocessing. However, the mixed finite element method provides direct approximation of the physical quantities such as fluxes or velocities and leads to schemes that are locally conservative. The main advantage of using mixed formulation lies on the possibility of introducing further unknowns with a physical interest, such as fluxes or velocities, so that they can be approximated directly and thus avoiding any numerical postprocessing yielding an additional source of error. The second variable is usually related with some derivatives of the primary variable. For example, in the elasticity equations, the stress can be introduced to be approximated at the same time with the displacement and in case of the Stoke problem, the two variables are pressure and velocity. The basic principle of the mixed finite element method is to reduce the given second order partial differential equation into a systems of first order equations. The word mixed refers to two independent approximations for each variable. Therefore, finite element methods in which two spaces are used to approximate two different variables receive the general denomination of mixed methods. The purpose of this thesis is to present some convergence results on H^1-Galerkin mixed finite element method for the problems (1.1.1)-(1.1.3). In order to put our results into proper perspective, we shall relate our work to the existing literature. For this purpose, we first recall the classical mixed finite element approximation [44] to the problem (1.1.1)-(1.1.3).

Classical Mixed Finite Element Method: Introducing $\mathbf{u} = a\nabla p$, we split the system (1.1.1) into a first order system as:

$$p_t - \nabla \cdot \mathbf{u} = f, \ (\mathbf{x}, t) \in \Omega \times J, \tag{1.3.1}$$

$$\alpha \mathbf{u} - \nabla p = 0, \ (\mathbf{x}, t) \in \Omega \times J, \tag{1.3.2}$$

where $\alpha = 1/a$. Let $\mathbf{V} = \{\mathbf{v} \in (L^2(\Omega))^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$. Now, multiplying (1.3.1) and (1.3.2) by test functions $\phi \in L^2(\Omega)$ and $\Psi \in \mathbf{V}$, respectively, and then integrating over Ω we obtain the weak mixed formulation of problem (1.3.1)-(1.3.2) as follows:

$$(p_t, \phi) - (\nabla \cdot \mathbf{u}, \phi) = (f, \phi), \ \forall \phi \in L^2(\Omega), \tag{1.3.3}$$

$$(\alpha \mathbf{u}, \Psi) + (p, \nabla \cdot \Psi) = 0, \ \forall \Psi \in \mathbf{V} \tag{1.3.4}$$

with $p(0) = p_0$. In (1.3.4), we have used integration by parts formula and the fact that $p = 0$ on $\partial\Omega$. Here, $(\cdot, \cdot)$ denotes the standard L^2-inner product on Ω. In order to define finite element approximations to the solutions $\{\mathbf{u}, p\}$ of (1.3.3)-(1.3.4) we need to introduce the finite dimensional subspaces of $\mathbf{V}$ and $L^2(\Omega)$ made of piecewise polynomial functions. Let $\mathcal{T}_h$ be a triangulation of Ω into a finite number of elements called simplexes, i.e., $\Omega = \bigcup_{K \in \mathcal{T}_h} K$. Let $\mathbf{V}_h$ and Q_h be finite dimensional subspaces of $\mathbf{V}$ and $L^2(\Omega)$, respectively. The mixed finite element approximation is then read as: Find $\{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times Q_h$ such that

$$(p_{h,t}, \phi_h) - (\nabla \cdot \mathbf{u}_h, \phi_h) = (f, \phi_h), \ \forall \phi_h \in Q_h, \tag{1.3.5}$$

$$(\alpha \mathbf{u}_h, \Psi_h) + (p_h, \nabla \cdot \Psi_h) = 0, \ \forall \Psi_h \in \mathbf{V}_h \tag{1.3.6}$$

with given $p_h(0) \in Q_h$. We note that $\mathbf{u}_h(0)$ is determined by $p_h(0)$ through the equation (1.3.6).

It is now well-known that the finite element spaces approximating $\mathbf{u}$ and p cannot be chosen independently. The so-called inf-sup or Ladyzhenskaya-Babuška-Brezzi (LBB) condition [4, 11, 14] is essential if one wants to construct unconditionally stable schemes with optimal convergence rates. To ensure the existence and convergence of the solution of the above formulation, it is assumed that

$$\nabla \cdot \mathbf{V}_h \subset Q_h,$$

and there exists a linear operator $\Pi_h : (H^1(\Omega))^2 \rightarrow \mathbf{V}_h$ such that

$$\nabla \cdot \Pi_h = L_h \nabla. \tag{1.3.7}$$

Here, the operator $L_h : L^2(\Omega) \rightarrow Q_h$ is the L^2-projection, i.e,

$$(\phi - L_h\phi, \phi_h) = 0, \ \ \forall \phi \in L^2(\Omega), \ \phi_h \in Q_h.$$

The identity (1.3.7) guarantees that the classical inf-sup condition is satisfied. Example of spaces of piecewise polynomials that satisfy the conditions stated above are the triangular and rectangular Raviart-Thomas elements from Raviart and Thomas [70] and BDM elements from Brezzi, Douglas and Marini [13]. For other examples, see Brezzi and Fortin [14].

Earlier, the error analysis of mixed finite element methods are developed and analyzed in [12, 13, 27, 48, 54, 64, 65] for elliptic equations. Subsequently, mixed finite element methods applied to parabolic problems are discussed in [19, 22, 33, 44, 45, 73]. There is now sizeable literature on mixed methods applied to oil reservoir engineering problems, see [25, 28, 37], and [71] for references therein. As mentioned before, the standard mixed finite element procedure has to satisfy the inf-sup or LBB condition on the approximating subspaces. This restricts the choice of finite element spaces. Therefore, the mixed formulation that circumvent this stringent condition has been the subject of intensive research in the past decade [60, 62, 66]. The proposed method which relaxes the LBB condition is referred to as H^1-Galerkin mixed finite element method. A notable advantage of this approach is that the approximating finite element spaces are allowed to be of different polynomial degree.

H^1-Galerkin Mixed Finite Element Method (H^1-Galerkin MFEM): To define the H^1-Galerkin mixed formulation for the problem (1.3.1)-(1.3.2) we proceed as follows. Multiplying (1.3.1) and (1.3.2), respectively, by $\nabla \cdot \Psi$ and $\nabla\phi$ and then integrating over Ω to have

$$(\alpha \mathbf{u}_t, \Psi) + (\nabla \cdot \mathbf{u}, \nabla \cdot \Psi) = -(f, \nabla \cdot \Psi), \ \forall \Psi \in \mathbf{V}, \tag{1.3.8}$$
$$(\nabla p, \nabla \phi) = (\alpha \mathbf{u}, \nabla \phi), \ \forall \phi \in H_0^1(\Omega), \tag{1.3.9}$$

where for the first term of (1.3.8), we have used integration by parts formula and the fact $p_t(\mathbf{x}, t) = 0$ on $\partial\Omega$. Then, the H^1-Galerkin mixed formulation is to find $\{\mathbf{u}, p\} : [0, T] \longrightarrow \mathbf{V} \times H_0^1(\Omega)$ such that

$$(\alpha \mathbf{u}_t, \Psi) + A(\mathbf{u}, \Psi) = -(f, \nabla \cdot \Psi) + \lambda(\mathbf{u}, \Psi), \ \forall \Psi \in \mathbf{V}, \tag{1.3.10}$$
$$(\nabla p, \nabla \phi) = (\alpha \mathbf{u}, \nabla \phi), \ \forall \phi \in H_0^1(\Omega) \tag{1.3.11}$$

with $\mathbf{u}(0) = a\nabla p_0$ and $\lambda > 0$. The bilinear form $A(\cdot,\cdot)$ is given by

$$A(\mathbf{u},\Psi) = (\nabla\cdot\mathbf{u},\nabla\cdot\Psi) + \lambda(\mathbf{u},\Psi).$$

Note that λ is chosen appropriately so that $A(\cdot,\cdot)$ is $\mathbf{V}$-coercive, i.e.,

$$A(\mathbf{v},\mathbf{v}) \geq c_1\|\mathbf{v}\|^2_{\mathbf{V}}, \quad \mathbf{v}\in\mathbf{V},$$

for some $c_1 > 0$. Further, there exists a positive constant C such that

$$|A(\mathbf{u},\mathbf{v})| \leq C\|\mathbf{u}\|_{\mathbf{V}}\|\mathbf{v}\|_{\mathbf{V}}.$$

Let $\mathcal{T}_h$ be a triangulation of Ω into finite number of elements called simplexes, i.e., $\Omega = \bigcup_{K\in\mathcal{T}_h} K$ with $h = \max\{\mathrm{diam(K)} : \mathrm{K}\in\mathcal{T}_\mathrm{h}\}$. Let $\mathbf{V}_h$ and W_h be finite dimensional subspaces of $\mathbf{V}$ and $H^1_0(\Omega)$, respectively. Standard examples of such finite element spaces are as follows:

$$\mathbf{V}_h = \left\{\mathbf{v}\in\mathbf{V} \ : \ \mathbf{v}|_K = p_K(\mathbf{x}) + \mathbf{x}\, q_K(\mathbf{x}), \ p_K\in(\mathcal{P}_k)^2, \ q_K\in\mathcal{P}_k, \ \forall\, K\in\mathcal{T}_h\right\},$$

and

$$W_h = \{w\in\mathcal{C}(\Omega) \ : \ w|_K\in\mathcal{P}_k, \ \forall\, K\in\mathcal{T}_h, \ w = 0 \text{ on } \partial\Omega\},$$

where $\mathcal{P}_s(K)$ is the space of polynomials of degree $\leq s$ on K. Other examples of approximating spaces can be found in Raviart and Thomas [70].

Now the semidiscrete H^1-Galerkin mixed finite element approximation is to find $\{\mathbf{u}_h, p_h\} : [0,T]\to\mathbf{V}_h\times W_h$ such that

$$(\alpha\mathbf{u}_{h,t},\Psi_h) + A(\mathbf{u}_h,\Psi_h) = -(f,\nabla\cdot\Psi_h) + \lambda(\mathbf{u}_h,\Psi_h), \ \forall\Psi_h\in\mathbf{V}_h, \qquad (1.3.12)$$

$$(\nabla p_h,\nabla\phi_h) = (\alpha\mathbf{u}_h,\nabla\phi_h), \ \forall\phi_h\in W_h \qquad (1.3.13)$$

with appropriately chosen $\{\mathbf{u}_h(0), p_h(0)\}$.

Compared to classical mixed finite element method, the H^1-Galerkin mixed method is not subject to the LBB-consistency condition. The approximating finite element spaces $\mathbf{V}_h$ and W_h are allowed to be of different polynomial degrees. In contrast to the standard H^1-Galerkin method [23, 61, 78, 79], $\mathcal{C}^1$-continuity on the approximating finite element spaces can be relaxed and gives us freedom to work with computationally attractive piecewise linear elements.

In an attempt to extend least-squares mixed method to parabolic problems, the author of [60] has introduced the H^1-Galerkin mixed finite element procedure. When $\Omega = (0,1)$, i.e, for one dimensional parabolic problem it is shown in [60] (with $k = 1$ and $r = 1$) that

$$\|(p-p_h)(t)\| + \|(u-u_h)(t)\| \leq Ch^2\left(\|p\|_{L^\infty(H^2)} + \|u\|_{L^\infty(H^2)} + \|u_t\|_{L^2(H^2)}\right), \qquad (1.3.14)$$

where p_h and u_h are H^1-Galerkin mixed finite element approximations to the solution p and its flux $u = ap_x$, respectively. The *a priori* error estimates in (1.3.14) demand $u_t \in H^2(\Omega)$ which in turn require $p_0 \in H^3(\Omega) \cap H_0^1(\Omega)$. Therefore, it is natural to enquire whether one would expect $\mathcal{O}(h^2)$ order of convergence as in (1.3.14) when p_0 has lesser regularity. An attempt has been made in this thesis to study the convergence analysis with lesser regularity assumption on the initial function p_0. More precisely, for homogeneous parabolic problem, error estimates of order $\mathcal{O}(h^2 t^{-1/2})$ for the solution and its flux are established for positive time with initial function $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ (see, Theorem 2.2.1). Further, error estimates of order $\mathcal{O}(h^2 t^{-1})$ for both the solution and its flux are derived for positive time with p_0 just in $H_0^1(\Omega)$ (see, Theorem 2.2.2). Since the smoothing property of the solution plays a significant role in the error analysis we derive some *a priori* bounds in terms of initial data in various Sobolev norms. The crucial technical tools used in our error analysis are the nonstandard energy formulation and parabolic duality argument. We extend the analysis of one dimensional situation to two space variables and analogous results are derived (see, Theorem 2.3.1 and Theorem 2.3.2).

We now turn to the time discretization of the homogeneous one dimensional parabolic problem with $\Omega = (0,1)$. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of the time interval $[0,T]$ with step length $\Delta t = T/N$, for some positive integer N. For a smooth function ϕ on $[0,T]$, define $\phi^n = \phi(t_n)$ and $\partial_t \phi^n = (\phi^n - \phi^{n-1})/\Delta t$. Let U^n and P^n, respectively, be the approximations of u and p at $t = t_n$ which we shall define through the following scheme: For $n \geq 1$, find $\{U^n, P^n\} \in V_h \times W_h$ satisfying

$$(\alpha \partial_t U^n, \psi_h) + A(U^n, \psi_h) = \lambda(U^n, \psi_h), \ \forall \psi_h \in V_h, \tag{1.3.15}$$

$$(P_x^n, \phi_{hx}) = (\alpha U^n, \phi_{hx}), \ \forall \phi_h \in W_h \tag{1.3.16}$$

with given $\{U^0, P^0\} \in V_h \times W_h$. Here, V_h is a finite dimensional subspace of $H^1(\Omega)$. Based on the backward Euler method, a fully discrete scheme is analyzed for the homogeneous parabolic problem and almost optimal order error estimates for the solution and its flux are established in the L^2-norm for smooth and nonsmooth initial data, (see, Theorem 3.2.1 and Theorem 3.3.1), respectively.

Our next objective is to investigate superconvergence phenomena for the H^1-Galerkin mixed finite element solution of the parabolic problems (1.1.1)-(1.1.3). Superconvergence results are important from an application point of view because they provide higher order accuracy under reasonable assumption on the grid and with additional smoothness of the solution [41]. Optimal order error estimate is the best which one can get between the exact solution and its numerical approximation when measured globally on the computational domain. But, there are places (points or lines) where the approximate solution is more closer to the exact solution [36, 80] than what is predicted by global optimal order error estimate. It would be advantageous to make use of those points or lines in the modelling process.

For the analysis of superconvergence, we have considered here a special case of the finite element partition $\widehat{\mathcal{T}}_h$ of the domain Ω consisting of rectangular elements only. We discretize Ω into rectangular sub-domains with boundaries parallel to either the x-axis or y-axis, respectively. The Raviart-Thomas finite element space $\mathbf{V}_h$ of $\mathbf{V}$ and the standard finite element space W_h of $H_0^1(\Omega)$ are defined as follows (cf. [14], [70]):

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v} \in \mathbf{V} \ : \ \mathbf{v}|_e \in Q_{k+1,k} \times Q_{k,k+1}, \ e \in \widehat{\mathcal{T}}_h\}, \\ W_h &:= \{w \in \mathcal{C}(\Omega) \ : \ w|_e \in Q_{k,k}, \ e \in \widehat{\mathcal{T}}_h, \ w = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

where $Q_{r,s}$ be the space of polynomials of degree no more than r in the x-direction and no more than s in the y-direction and $k \geq 1$. The central feature of our analysis of superconvergence results lie in the treatment of the following two linear forms

$$\mathcal{F}(\mathbf{v}) = (\mathbf{u} - \pi_h \mathbf{u}, \mathbf{v}),$$

and

$$\widetilde{\mathcal{F}}(\mathbf{v}) = ((\mathbf{u} - \pi_h \mathbf{u})_t, \mathbf{v}),$$

where $\pi_h \mathbf{u}$ is an appropriately defined local projection [14, 24] and $\mathbf{v}$ be any finite element function. These linear forms are estimated by expanding the interpolation errors $\mathbf{u} - \pi_h \mathbf{u}$ and $(\mathbf{u} - \pi_h \mathbf{u})_t$ as Taylor series involving only finite number of terms. Each of the terms in the Taylor expansion is a polynomial. The orthogonality of $\mathbf{u} - \pi_h \mathbf{u}$ and $(\mathbf{u} - \pi_h \mathbf{u})_t$ with certain class of polynomials and the cancellation of line integrals play a crucial role in deriving superconvergence results.

Earlier, superconvergence results for classical mixed finite element method are developed and analyzed by Squeff in [74] using quasi-projections of Douglas, Dupont and Wheeler [24] and the averaging method of Bramble and Schatz [9] for parabolic problems. Subsequently, one may refer to [19, 34, 38] and references therein for superconvergence in mixed finite element methods. Compared to [19, 74], the proposed method can offer the following added advantages:

- The present method is not subject to inf-sup or LBB-consistency condition.
- The approximating finite element spaces are allowed to be of different polynomial degrees.

- Assuming higher regularity on the solution, superconvergence estimate of order $\mathcal{O}(h^{k+3})$ is established for the flux.

The literature concerning superconvergence phenomena for classical mixed finite element methods can be found in [19, 29, 32, 35, 74] for elliptic and parabolic problems and for non-fickian flows in porous media, one may refer to [34, 38].

The present analysis yields better convergence result for the flux with higher regularity assumption. In particular, use of piecewise linear polynomial spaces yields optimal order error estimate of order $\mathcal{O}(h^2)$ in the L^2-norm for the flux $\mathbf{u}$ (see, [60]) whereas our estimate gives of order $\mathcal{O}(h^4)$ which indicates superconvergence.

Next, we turn our attention for a posteriori error analysis of H^1-Galerkin mixed finite element method for parabolic problems (1.1.1)-(1.1.3). *A priori* error estimates can give asymptotic rates of convergence as the mesh parameter h goes to zero, but often can not provide much practical information about the actual errors encountered on a given mesh. The question of quantifying the error brings attention to a posteriori estimates. A posteriori estimators are quantities which bound the errors or approximate the errors and can be computed from the knowledge of numerical solution and input data of the problem. In order to guarantee a good convergence behavior of the discrete solutions, one usually needs to apply a refinement algorithm based on a posteriori error estimates. The estimators are utilized in two important ways connected with adaptivity. The first is the indication for which element to be refined and the second is the stopping criteria.

In the thesis we concern ourselves on a posteriori error estimate for semidiscrete H^1-Galerkin mixed finite element method for parabolic problems (1.1.1)-(1.1.3). The semidiscrete a posteriori error analysis is based on the residual approach with mesh refinement technique. We now define the residuals as follows: For $t \in J$

$$\begin{aligned} < R_1(\mathbf{u}_h), \Psi > &= -(f, \nabla \cdot \Psi) - (\nabla \cdot \mathbf{u}_h, \nabla \cdot \Psi) - (\alpha \mathbf{u}_{h,t}, \Psi), \\ (R_2(\mathbf{u}_h, p_h), \nabla \phi) &= (\alpha \mathbf{u}_h, \nabla \phi) - (\nabla p_h, \nabla \phi), \end{aligned}$$

for all $\Psi \in \mathbf{V}$ and $\phi \in H_0^1(\Omega)$. Here, $< \cdot, \cdot >$ denotes the dual inner product and $(\cdot, \cdot)$ is the standard L^2-inner product. The global error estimator is then defined as

$$\eta_R := \sum_{K \in \mathcal{T}_h} \left(\|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))} + \|R_2(\mathbf{u}_h, p_h)\|^2_{L^2(K)} \right)^{1/2}.$$

For the purpose of deriving upper bounds for $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$ in the L^2-norm in terms of the estimator η_R, we make the following *saturation assumption*: There is a number $\beta < 1$ such that

$$\|\mathbf{u}(t) - \mathbf{u}_{h/2}(t)\| \leq \beta \|\mathbf{u}(t) - \mathbf{u}_h(t)\|, \; t \in J, \tag{1.3.17}$$

and

$$\|\nabla(p - p_{h/2})(t)\| \leq \beta\|\nabla(p - p_h)(t)\|, \; t \in J. \tag{1.3.18}$$

The assumptions (1.3.17)-(1.3.18) imply that on the refined mesh $\mathcal{T}_{h/2}$ the refined finite element approximation $(\mathbf{u}_{h/2}, p_{h/2})$ is a better approximation to the exact solution $(\mathbf{u}, p)$ than $(\mathbf{u}_h, p_h)$. Here $\mathcal{T}_{h/2}$ is a refinement of $\mathcal{T}_h$ by dividing each triangle into four congruent ones. These saturation assumptions are motivated by the well-known *a priori* error estimates for $\mathbf{u}(t) - \mathbf{u}_h(t)$ and $\nabla(p - p_h)(t)$ (see, e.g., [60], Theorem 3.1 with $k = r = 2$ and $k + 1 = r = 2$, respectively):

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| \leq C(p_0, \mathbf{u}, p)h^2, \tag{1.3.19}$$

and

$$\|\nabla(p - p_h)(t)\| \leq C(p_0, \mathbf{u}, p)h^2. \tag{1.3.20}$$

The above *a priori* estimates assures us that, in general, a refinement of the mesh and a reduction of h will lead to an improvement of the finite element solution. However, (1.3.19) and (1.3.20) provides no information on the improvement for individual cases. Therefore, we exclude the exceptional cases in which the improvement is very small (cf. [6]). Generally, one expects that by refining the mesh will significantly reduce the error, so this is a natural assumption. Based on the assumptions (1.3.17) and (1.3.18), we derive the upper bounds for the errors $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$ in the L^2-norm (cf. Theorem 5.3.1).

While there is a sizable literature available for a posteriori error analysis based on residual approach for classical mixed finite element methods for elliptic problems [3, 8, 16, 39, 49, 50, 51, 88], the literature seems to be limited for parabolic problems. The previous work on a posteriori error estimates by means of the standard Galerkin method for parabolic problem can be found in [5, 7, 31, 68, 84], and for nonlinear parabolic problems in [59, 82, 85]. Recently, a posteriori error analysis of H^1-Galerkin MFEM for second order elliptic equations has been studied in [81]. The analysis in [81] is performed under a saturation assumption and a Helmholtz decomposition for vector fields. There is no results available on a posteriori error analysis for parabolic problems in H^1-Galerkin mixed method.

Our next objective is to derive a posteriori error estimates for the H^1-Galerkin mixed finite element method for linear parabolic problems (1.1.1)-(1.1.3) by space-time discretization. The error analysis is based on Raviart-Thomas finite elements in space and backward Euler scheme with variable time steps.

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of the time interval $[0, T]$, for some $N \geq 1$. Set $\tau_n = t_n - t_{n-1}, \; 1 \leq n \leq N$. Now, for any smooth function ϕ on $[0, T]$, define

$\phi^n = \phi(t_n)$. With each time step t_n, we associate an affinely equivalent, admissible, and shape regular partition $\mathcal{T}_{h,n}$ of Ω and the corresponding conforming finite element spaces $\mathbf{V}_h^n$ and W_h^n of $\mathbf{V}$ and $H_0^1(\Omega)$, respectively. The discrete problem based on backward Euler method is defined as: Find $\{\mathbf{u}_h^n, p_h^n\} \in \mathbf{V}_h^n \times W_h^n$ such that for $n = 1, \ldots, N$,

$$(\alpha(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n}), \Psi_h) + A(\mathbf{u}_h^n, \Psi_h) \;=\; -(f^n, \nabla \cdot \Psi_h) + \lambda(\mathbf{u}_h^n, \Psi_h), \tag{1.3.21}$$

and

$$(\nabla p_h^n, \nabla \phi_h) \;=\; (\alpha \mathbf{u}_h^n, \nabla \phi_h), \tag{1.3.22}$$

for all $\Psi_h \in \mathbf{V}_h^n$ and $\phi_h \in W_h^n$ with given $\mathbf{u}_h(0)$ to be defined later. With the sequence of solutions $\{\mathbf{u}_h^n, p_h^n\}$, we associate the functions $\{\mathbf{u}_{h\tau}, p_{h\tau}\}$, which are piecewise affine on the time intervals $[t_{n-1}, t_n], 1 \le n \le N$, and equals $\{\mathbf{u}_h^n, p_h^n\}$ at time $t = t_n,\ 0 \le n \le N$. Similarly, we denote $f_{h\tau}$, the function which is piecewise constant on time intervals and on each interval $(t_{n-1}, t_n]$ is equal to the L^2-projection of f^n onto the finite element space $\mathbf{V}_h^n$. With the functions $\{\mathbf{u}_{h\tau}, p_{h\tau}\}$ as the solution of the discrete problems (1.3.21)-(1.3.22), define the residuals $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ and $R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})$, respectively, by

$$< R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi > = \; -(f, \nabla \cdot \Psi) - (\nabla \cdot \mathbf{u}_{h\tau}, \nabla \cdot \Psi) - (\alpha(\mathbf{u}_{h\tau})_t, \Psi), \tag{1.3.23}$$

and

$$(R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau}), \nabla \phi) = (\alpha \mathbf{u}_{h\tau}, \nabla \phi) - (\nabla p_{h\tau}, \nabla \phi), \tag{1.3.24}$$

for all $\Psi \in \mathbf{V}$ and $\phi \in H_0^1(\Omega)$. Here, $< \cdot, \cdot >$ denotes the dual inner product and $(\cdot)_t$ denotes the differentiation with respect to time t. We rewrite the residual $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ as

$$< R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi >= (f_{h\tau} - f, \nabla \cdot \Psi) + < R_{11,h}(\mathbf{u}_{h\tau}), \Psi > + (R_{12,\tau}(\mathbf{u}_{h\tau}), \Psi), \tag{1.3.25}$$

where $R_{11,h}(\mathbf{u}_{h\tau})$ is the spatial residual defined by

$$< R_{11,h}(\mathbf{u}_{h\tau}), \Psi > = \; -(f_{h\tau}, \nabla \cdot \Psi) - (\nabla \cdot \mathbf{u}_h^n, \nabla \cdot \Psi) - (\alpha(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n}), \Psi) \tag{1.3.26}$$

on $(t_{n-1}, t_n]$ and $R_{12,\tau}(\mathbf{u}_{h\tau})$ is the temporal residual defined by

$$(R_{12,\tau}(\mathbf{u}_{h\tau}), \Psi) = (\nabla \cdot \mathbf{u}_h^n - \nabla \cdot \mathbf{u}_{h\tau}, \nabla \cdot \Psi) \text{ on } (t_{n-1}, t_n], \tag{1.3.27}$$

for all $\Psi \in \mathbf{V}(\Omega)$ and $1 \le n \le N$. The error indicator corresponding to the residual $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ is defined as:

$$\begin{aligned} \eta_{1,h,\tau}^n \;:=\; & \{ \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \tau_n \| f_{h\tau} + \nabla \cdot \mathbf{u}_h^n \|_{L^2(K)}^2 + \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \frac{h_K^2}{\tau_n} \| \alpha(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) \|_{L^2(K)}^2 \\ & + \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \tau_n \| \nabla \cdot (\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) \|_{L^2(K)}^2 \}^{1/2}, \end{aligned} \tag{1.3.28}$$

The error indicators corresponding to the residuals $R_{11,h}(\mathbf{u}_{h\tau})$ and $R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})$ are given by

$$\eta_{11,h}^n := \left\{ \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|f_{h\tau} + \nabla\cdot\mathbf{u}_h^n\|_{L^2(K)}^2 + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \frac{h_K^2}{\tau_n^2}\|\alpha(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|_{L^2(K)}^2 \right\}^{1/2} \tag{1.3.29}$$

and

$$\eta_{2,h}^n := \left\{ \sum_{K\in\mathcal{T}_{h,n}} \|\alpha\mathbf{u}_h^n - \nabla p_h^n\|_{L^2(K)}^2 \right\}^{1/2}, \tag{1.3.30}$$

respectively. Here $\eta_{11,h}^n$ and $\eta_{2,h}^n$ denote the measure for the error of the space discretization and can be used to adapt the mesh size in space. The second term in $\eta_{1,h,\tau}^n$ can be interpreted as a measure for the error of the time discretization.

Our a posteriori error analysis consists of the following two steps. First, we bound the errors in terms of the residuals through the standard energy argument. In the second step, the residuals are bounded by the estimators $\eta_{1,h,\tau}^n$ and $\eta_{2,h}^n$. These estimators are computable quantities in terms of the problem data $\mathbf{u}_0, f_{h\tau}, \Omega, T$, computed solutions $\mathbf{u}_h^n$ and p_h^n, mesh size h_K and the time step τ_n. We derive the upper bounds for the errors $\mathbf{u}-\mathbf{u}_{h\tau}$ and $p - p_{h\tau}$ (see, Theorem 6.3.1). Such estimators provide bound for the errors and are useful for modifying meshes and time steps adaptively.

The previous work on space-time discretization a posteriori error analysis for classical mixed method has been carried out in [18] for parabolic problem. A duality argument is used in a crucial way to derive a posteriori error estimates in [18]. Compared to [18], our analysis do not involve edge residuals in the computation of error indicators and the method is not subject to LBB-consistency condition on finite element spaces.

1.4 Organization of the Thesis

The organization of the rest of the thesis is as follows: Chapter 2 deals with some new error estimates for H^1-Galerkin MFEM for homogeneous parabolic problems with smooth and nonsmooth initial data in one-space dimension. For the spatially discrete scheme, optimal order error estimates in L^2-norm are derived for the solution and its flux assuming $p_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $p_0 \in H_0^1(\Omega)$. Further, an extension to two space variables is considered and optimal error bounds are shown to hold for both the solution and its flux.

Chapter 3 deals with the time discretization of the H^1-Galerkin MFEM based on backward Euler scheme for one dimensional homogeneous parabolic problems. Non-standard energy formulation and a duality argument is used to derive almost optimal order error estimates in L^2-norm for the solution and the flux when the initial function $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $p_0 \in H_0^1(\Omega)$.

Chapter 4 is devoted to superconvergence phenomenon of H^1-Galerkin MFEM for parabolic problem (1.1.1)-(1.1.3).

Chapters 5 is devoted to the semidiscrete a posteriori error analysis for the H^1-Galerkin MFEM for parabolic problems. We derive upper bounds for the errors under a *saturation assumption*.

In Chapter 6, we study discrete-in-time a posteriori error analysis for the H^1-Galerkin MFEM for parabolic problems. The time discretization is based on backward Euler method with variable time step and the spatial discretization consists of Raviart-Thomas finite elements over the spatial mesh. The estimators yield upper bounds on the errors which are global in space and time.

Finally, Chapter 7 discusses the results highlighting the contributions made by the thesis and the scope for future investigations.

For clarity of presentation we have repeatedly given equation (1.1.1)-(1.1.3) at the beginning of every chapter.

Chapter 2

Semidiscrete H^1-Galerkin MFEM for Parabolic Problems

In this chapter, we derive some new error estimates for the semidiscrete H^1-Galerkin MFEM for homogeneous parabolic problems under lesser regularity assumption on the initial function p_0. Both one and two space variable problems are discussed. Optimal order error estimates for positive time are shown to hold for the solution and its flux in L^2-norm assuming $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $p_0 \in H_0^1(\Omega)$. We use simple energy techniques and duality argument.

2.1 Introduction

When our primary concern is to estimate both displacement p and flux $\mathbf{u} = a\nabla p$, we split the problem (1.1.1) into a system of equations and then apply classical mixed methods, see [44] and [70]. However, this procedure need to satisfy LBB-consistency condition on the approximating subspaces which restricts the choice of finite element spaces. For example, Raviart-Thomas spaces of index $k \geq 0$ are usually used for standard mixed methods. In order to avoid LBB-consistency condition, the H^1-Galerkin mixed finite element method for parabolic problems was introduced in [60]. Comparing with the standard H^1-Galerkin method, $\mathcal{C}^1$-continuity of the approximating spaces can be relaxed and gives us freedom to work with computationally attractive piecewise linear elements. The author of [60] has studied the H^1-Galerkin MFEM for parabolic equations and derived optimal order error estimates in the L^2 and H^1-norms by assuming initial data $p_0 \in H^3(\Omega) \cap H_0^1(\Omega)$. An attempt has been made in this chapter to study convergence analysis of the proposed method under lesser regularity assumption on the initial function p_0. More precisely, we establish error estimates of order $\mathcal{O}(h^2t^{-1/2})$ and $\mathcal{O}(h^2t^{-1})$ for the solution and its flux for positive time when $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $p_0 \in H_0^1(\Omega)$, respectively. The main crucial technical tools used in our analysis are energy techniques and parabolic duality argument.

The plan of this chapter is as follows. Section 2.2 deals with semidiscrete error analysis for one dimensional parabolic problem. Section 2.2 contains three subsections. Some *a priori* estimates for the solution u and stability estimates of its semidiscrete approximation u_h are derived in Subsection 2.2.1. While Subsection 2.2.2 is devoted to error estimates for H^1-Galerkin mixed finite element method with smooth initial data, the error estimates for nonsmooth initial data are carried out in Subsection 2.2.3. An extension to two space variable problem is considered in Section 2.3. Analogous results for smooth and nonsmooth data are established in Subsections 2.3.2 and 2.3.3, respectively.

Throughout this chapter C denotes a positive generic constant which may depend on time T but independent of the mesh parameters h, and may not be the same at each occurrence.

2.2 Error Estimates for One Dimensional Parabolic Problems

We consider the following one-dimensional homogeneous parabolic problem:

$$p_t - (ap_x)_x = 0, \quad (x,t) \in \Omega \times J, \tag{2.2.1}$$

$$p(0,t) = p(1,t) = 0, \quad t \in J, \tag{2.2.2}$$

$$p(x,0) = p_0(x), \quad x \in \Omega, \tag{2.2.3}$$

where $p_t = \frac{\partial p}{\partial t}$, $\Omega = (0,1)$ and J denotes the time interval $(0,T]$ with $T < \infty$. The coefficient $a = a(x)$ is assumed to be smooth. Further, a is bounded below and above by positive constants b_0 and b_1, respectively i.e.,

$$b_0 \leq a(x) \leq b_1, \ x \in \Omega. \tag{2.2.4}$$

Introducing $u = ap_x$, we split the problem (2.2.1) into two first order equations as:

$$p_t - u_x = 0, \quad p_x = \alpha u, \tag{2.2.5}$$

where $\alpha = 1/a$. For the error analysis, we shall need the following spaces: $H_0^1(\Omega) = \{w \in H^1(\Omega) : w(0) = w(1) = 0\}$ and $V = H^1(\Omega)$. The H^1-Galerkin mixed formulation is stated as: Find $\{u, p\} \in V \times H_0^1(\Omega)$ such that

$$(\alpha u_t, \psi) + A(u, \psi) = \lambda(u, \psi), \ \forall \, \psi \in V, \tag{2.2.6}$$

$$(p_x, \phi_x) = (\alpha u, \phi_x), \ \forall \, \phi \in H_0^1(\Omega) \tag{2.2.7}$$

with $u_0 = u(x, 0) = a\frac{dp_0}{dx}$. For the first term in (2.2.6), we have used integration by parts with respect to x and the Dirichlet boundary conditions $p_t(0, t) = p_t(1, t) = 0$. The bilinear form $A(\cdot, \cdot)$ is given by

$$A(u, v) = (u_x, v_x) + \lambda(u, v).$$

Note that λ is chosen appropriately so that $A(\cdot, \cdot)$ is V-coercive, i.e.,

$$A(v, v) \geq c_0\|v\|_1^2, \ v \in V,$$

for some $c_0 > 0$. Moreover, $A(\cdot, \cdot)$ is bounded. That is, there is a positive constant C such that $|A(u, v)| \leq C\|u\|_1\|v\|_1$.

Let V_h and W_h be finite dimensional subspaces of V and $H_0^1(\Omega)$, respectively. The finite dimensional spaces V_h and W_h satisfy the following approximation properties:

$$\inf_{v_h \in V_h} \{\|v - v_h\| + h\|v - v_h\|_1\} \leq Ch^2\|v\|_{H^2(\Omega)}, \ \forall \, v \in H^2(\Omega),$$

and

$$\inf_{w_h \in W_h} \{\|w - w_h\| + h\|w - w_h\|_1\} \leq Ch^2\|w\|_{H^2(\Omega)}, \ \forall \, w \in H^2(\Omega) \cap H_0^1(\Omega).$$

The semidiscrete H^1-Galerkin mixed finite element approximation is to find a pair $\{u_h, p_h\} \in V_h \times W_h$ such that

$$(\alpha u_{h,t}, \psi_h) + A(u_h, \psi_h) = \lambda(u_h, \psi_h), \ \forall \, \psi_h \in V_h, \tag{2.2.8}$$

$$(p_{hx}, \phi_{hx}) = (\alpha u_h, \phi_{hx}), \ \forall \, \phi_h \in W_h \tag{2.2.9}$$

with initial data $u_h(0) = L_h u_0$, where $L_h : L^2(\Omega) \longrightarrow V_h$ is the standard L^2 projection. Since the stiffness matrix associated with $(p_{h,x}, \phi_{h,x})$ is positive definite, the system is uniquely solvable for a consistent initial condition [15].

2.2.1 A Priori Estimates

In this subsection, we shall derive some *a priori* estimates for the solution u satisfying (2.2.5). In addition, some stability estimates for its semidiscrete solution u_h satisfying (2.2.8) and (2.2.9) are also obtained. These estimates will be useful in our subsequent error analysis.

Lemma 2.2.1 *Assume that $u_0 \in L^2(\Omega)$. Let u be the solution of (2.2.5). Then, there is a positive generic constant C such that*

$$\int_0^t s^i \|u(s)\|_1^2 ds \leq C\|u_0\|^2, \quad i = \{0, 1, 2\}.$$

Proof. Choose $\psi = u$ in (2.2.6) and use coercivity of $A(\cdot,\cdot)$ to have

$$\frac{1}{2}\frac{d}{dt}\{\|\alpha^{1/2}u(t)\|^2\} + c_0\|u(t)\|_1^2 \leq \lambda\|u(t)\|^2.$$

Integration with respect to time from 0 to t yields

$$\|\alpha^{1/2}u(t)\|^2 + 2c_0\int_0^t \|u(s)\|_1^2 ds \leq C\left(\|u_0\|^2 + \int_0^t \|u(s)\|^2 ds\right).$$

In view of (2.2.4) and an application of Gronwall's lemma yields

$$\|u(t)\|^2 + \int_0^t \|u(s)\|_1^2 ds \leq C\|u_0\|^2, \tag{2.2.10}$$

which proves the inequality for $i = 0$. For $i = 1$, we choose $\psi = tu$ in (2.2.6) and use coercivity of $A(\cdot,\cdot)$ to have

$$\frac{1}{2}\frac{d}{dt}\{t\|\alpha^{1/2}u(t)\|^2\} + c_0 t\|u(t)\|_1^2 \leq \lambda t\|u(t)\|^2 + \frac{1}{2}\|\alpha^{1/2}u(t)\|^2.$$

Integrating from 0 to t and using (2.2.4), we obtain

$$t\|u(t)\|^2 + \int_0^t s\|u(s)\|_1^2 ds \leq C\int_0^t \|u(s)\|^2 ds + C\int_0^t s\|u(s)\|^2 ds.$$

An application of Gronwall's lemma leads to

$$t\|u(t)\|^2 + \int_0^t s\|u(s)\|_1^2 ds \leq C\int_0^t \|u(s)\|^2 ds.$$

Using (2.2.10), we obtain

$$t\|u(t)\|^2 + \int_0^t s\|u(s)\|_1^2 ds \leq C\|u_0\|^2, \tag{2.2.11}$$

and this proves the desired inequality. Finally, for $i = 2$, set $\psi = t^2 u$ in (2.2.6) and use coercivity of $A(\cdot,\cdot)$ to have

$$\frac{1}{2}\frac{d}{dt}\{t^2\|\alpha^{1/2}u(t)\|^2\} + c_0 t^2\|u(t)\|_1^2 \leq \lambda t^2\|u(t)\|^2 + t\|\alpha^{1/2}u(t)\|^2.$$

Integrating the above equation from 0 to t and using (2.2.4), it now follows that

$$t^2\|u(t)\|^2 + \int_0^t s^2\|u(s)\|_1^2 ds \leq C\int_0^t s\|u(s)\|^2 ds + C\int_0^t s^2\|u(s)\|^2 ds.$$

An application of Gronwall's lemma leads to

$$t^2\|u(t)\|^2 + \int_0^t s^2\|u(s)\|_1^2 ds \le C\int_0^t s\|u(s)\|^2 ds.$$

An use of (2.2.11) now yields

$$t^2\|u(t)\|^2 + \int_0^t s^2\|u(s)\|_1^2 ds \le C\|u_0\|^2, \tag{2.2.12}$$

and this completes the rest of the proof. ■

Lemma 2.2.2 *Let u be the solution of (2.2.5). Then the following estimates hold true:*

(*a*) *If $u_0 \in V$, then* $\displaystyle\int_0^t \|u_s(s)\|^2 ds + \|u(t)\|_1^2 \le C\|u_0\|_1^2$.

If $u_0 \in L^2(\Omega)$, then

(*b*) $\displaystyle\int_0^t s\|u_s(s)\|^2 ds + t\|u(t)\|_1^2 \le C\|u_0\|^2$,

(*c*) $\displaystyle\int_0^t s^2\|u_s(s)\|^2 ds \le C\|u_0\|^2$.

Proof. Setting $\psi = u_t$ in (2.2.6), we have

$$\|\alpha^{1/2}u_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\{A(u(t),u(t))\} = \frac{\lambda}{2}\frac{d}{dt}\|u(t)\|^2.$$

Integrating the above equation from 0 to t, using (2.2.4) and coercivity of $A(\cdot,\cdot)$, we obtain

$$\int_0^t \|u_s(s)\|^2 ds + c_0\|u(t)\|_1^2 \le C\left(\|u_0\|_1^2 + \|u(t)\|^2\right).$$

Apply (2.2.10) to complete the proof of (a). To estimate (b), choose $\psi = tu_t$ in (2.2.6) to have

$$t\|\alpha^{1/2}u_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\{tA(u(t),u(t))\} = \frac{\lambda}{2}\frac{d}{dt}\{t\|u(t)\|^2\} - \frac{\lambda}{2}\|u(t)\|^2 + \frac{1}{2}A(u(t),u(t)).$$

Integrate the above equation from 0 to t. Then use coercivity and boundedness of $A(\cdot,\cdot)$ to obtain

$$\int_0^t s\|u_s(s)\|^2 ds + c_0 t\|u(t)\|_1^2 \le C\left(t\|u(t)\|^2 + \int_0^t \|u(s)\|_1^2 ds\right).$$

The estimate (b) now follows from (2.2.10) and (2.2.11). Finally, to prove (c), choose $\psi = t^2u_t$ in (2.2.6) to have

$$t^2\|\alpha^{1/2}u_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\{t^2A(u(t),u(t))\} = \frac{\lambda}{2}\frac{d}{dt}\{t^2\|u(t)\|^2\} - \frac{\lambda}{2}t\|u(t)\|^2 + \frac{1}{2}tA(u(t),u(t)).$$

Integrate from 0 to t. Then use of coercivity and boundedness of $A(\cdot,\cdot)$ yields

$$\int_0^t s^2\|u_s(s)\|^2 ds + c_0t^2\|u(t)\|_1^2 \le C\left(t^2\|u(t)\|^2 + \int_0^t s\|u(s)\|_1^2 ds + \int_0^t s\|u(s)\|^2 ds\right).$$

Now, the estimate (c) follows from (2.2.10), (2.2.11) and (2.2.12). This completes the proof of Lemma 2.2.2. ■

Lemma 2.2.3 *Let u be the solution of (2.2.5). Then the following estimates hold true:*

(a) *If* $u_0 \in V$, *then* $\displaystyle\int_0^t s\|u_s(s)\|_1^2 ds \le C\|u_0\|_1^2.$

(b) *If* $u_0 \in L^2(\Omega)$, *then* $\displaystyle\int_0^t s^2\|u_s(s)\|_1^2 ds \le C\|u_0\|^2.$

(c) *If* $u_0 \in V$, *then* $\displaystyle\int_0^t s^2\|u_{ss}(s)\|^2 ds \le C\|u_0\|_1^2.$

Proof. Differentiating (2.2.6) with respect to time t, we have

$$(\alpha u_{tt}, \psi) + A(u_t, \psi) = \lambda(u_t, \psi). \tag{2.2.13}$$

Set $\psi = tu_t$ in (2.2.13) and then use coercivity of $A(\cdot,\cdot)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\{t\|\alpha^{1/2}u_t(t)\|^2\} + c_0 t\|u_t(t)\|_1^2 \le \lambda t\|u_t(t)\|^2 + \frac{1}{2}\|\alpha^{1/2}u_t(t)\|^2.$$

Integrating the above equation from 0 to t and use (2.2.4) to have

$$t\|u_t(t)\|^2 + c_0\int_0^t s\|u_s(s)\|_1^2 ds \le C\int_0^t s\|u_s(s)\|^2 ds + C\int_0^t \|u_s(s)\|^2 ds. \tag{2.2.14}$$

Apply Lemma 2.2.2 for the terms appearing on the right hand side of (2.2.14) to obtain (a). Next, to prove (b), choose $\psi = t^2 u_t$ in (2.2.13). Then, use coercivity of $A(\cdot,\cdot)$ to have

$$\frac{1}{2}\frac{d}{dt}\{t^2\|\alpha^{1/2}u_t(t)\|^2\} + c_0 t^2\|u_t(t)\|_1^2 \le \lambda t^2\|u_t(t)\|^2 + t\|u_t(t)\|^2.$$

Integration from 0 to t now leads to

$$t^2\|u_t(t)\|^2 + \int_0^t s^2\|u_s(s)\|_1^2 ds \le C\left(\int_0^t s^2\|u_s(s)\|^2 ds + \int_0^t s\|u_s(s)\|^2 ds\right). \tag{2.2.15}$$

The desired estimate (b) now follows from Lemma 2.2.2 and Gronwall's lemma. Finally, to estimate (c), choose $\psi = t^2 u_{tt}$ in (2.2.13). Then, integrate the resulting equation from 0 to t and use (2.2.4) to have

$$\int_0^t s^2\|u_{ss}\|^2 ds + t^2\|u_t(t)\|_1^2 \le C\left(\int_0^t s^2\|u_s(s)\|^2 ds + \int_0^t s\|u_s(s)\|^2 ds + \int_0^t s\|u_s(s)\|_1^2 ds\right).$$

Now, apply (2.2.14) for the last term on the right hand side of the above equation and then use Lemma 2.2.2 to complete the proof of (c). ■

Lemma 2.2.4 *Let u be the solution of (2.2.5). Then, for $u_0 \in V$, the following estimates hold true:*

(a) $t\|u(t)\|_2^2 \le C\|u_0\|_1^2,$

(b) $\displaystyle\int_0^t \|u(s)\|_2^2 ds \leq C\|u_0\|_1^2,$

(c) $\displaystyle\int_0^t s^2\|u_s(s)\|_2^2 ds \leq C\|u_0\|_1^2.$

Further, if $u_0 \in L^2(\Omega)$, then

(d) $\displaystyle\int_0^t s^i\|u(s)\|_2^2 ds \leq C\|u_0\|^2, i = 1, 2.$

Proof. Note that

$$\begin{aligned}\|u\|_2^2 &= \|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 \\ &= \|u\|_1^2 + \|u_{xx}\|^2 = \|u\|_1^2 + \|p_{tx}\|^2, \end{aligned} \tag{2.2.16}$$

where we have used the fact that $u_x = p_t$. Differentiating (2.2.7) with respect to t, we have

$$(p_{xt}, \phi_x) = (\alpha u_t, \phi_x), \tag{2.2.17}$$

which yields $\|p_{tx}\| \leq C\|u_t(t)\|$. Thus, multiplying (2.2.16) by t and using (2.2.14) and Lemma 2.2.2, we obtain

$$t\|u\|_2^2 \leq t\|u\|_1^2 + Ct\|u_t\|^2 \leq C\|u_0\|_1^2,$$

which proves (a). To prove (b), integrate (2.2.16) from 0 to t and use the fact that $\|p_{tx}(t)\| \leq C\|u_t(t)\|$ to have

$$\int_0^t \|u(t)\|_2^2 ds \leq C\left(\int_0^t \|u(s)\|_1^2 ds + \int_0^t \|u_s(s)\|^2 ds\right) \leq C\|u_0\|_1^2,$$

where we have used Lemma 2.2.1 and Lemma 2.2.2. Next, to estimate (c), we first note that

$$\|u_t\|_2^2 = \|u_t\|^2 + \|u_{tx}\|^2 + \|u_{txx}\|^2 = \|u_t\|_1^2 + \|p_{ttx}\|^2, \tag{2.2.18}$$

where, we have used $p_{ttx} = u_{txx}$. Differentiating (2.2.7) twice with respect to t to have

$$(\alpha u_{tt}, \phi_x) = (p_{ttx}, \phi_x), \tag{2.2.19}$$

which yields $\|p_{ttx}\| \leq C\|u_{tt}\|$. Now, using $\|p_{ttx}\| \leq C\|u_{tt}\|$ and multiplying (2.2.18) by t^2 and integrating from 0 to t, we get

$$\int_0^t s^2\|u_s(s)\|_2^2 ds \leq \int_0^t s^2\|u_s(s)\|_1^2 ds + C\int_0^t s^2\|u_{ss}\|^2 ds.$$

The desired estimate now follows from Lemma 2.2.2 and Lemma 2.2.3. Finally, to estimate (d) for $i = 1$, multiply (2.2.16) by t and integrate from 0 to t and then use the fact that $\|p_{tx}(t)\| \leq C\|u_t(t)\|$ to have

$$\int_0^t t\|u(t)\|_2^2 ds \leq C\left(\int_0^t s\|u(s)\|_1^2 ds + \int_0^t s\|u_s(s)\|^2 ds\right) \leq C\|u_0\|^2,$$

where we have used Lemma 2.2.1 and Lemma 2.2.2. Similarly, for $i = 2$, multiply (2.2.16) by t^2 and integrate from 0 to t and use the fact that $\|p_{tx}(t)\| \leq C\|u_t(t)\|$ to have

$$\begin{aligned}\int_0^t t^2\|u(t)\|_2^2 ds &\leq C\left(\int_0^t s^2\|u(s)\|_1^2 ds + \int_0^t s^2\|u_s(s)\|^2 ds\right)\\ &\leq C\left(\int_0^t s^2\|u(s)\|_1^2 ds + t\int_0^t s\|u_s(s)\|^2 ds\right).\end{aligned}$$

Apply Lemma 2.2.1 and Lemma 2.2.2 to complete the proof of Lemma 2.2.4. ■

Below, we prove stability estimates of u_h satisfying (2.2.8) and (2.2.9).

Lemma 2.2.5 *Assume that $u_0 \in V$ with $u_h(0) = L_h u_0$. Then there is a positive generic constant C such that*

(a) $\int_0^t \|u_{h,s}(s)\|^2 ds + \|u_h(t)\|_1^2 \leq C\|u_0\|_1^2,$

(b) $\int_0^t s\|u_{h,s}(s)\|^2 ds + t\|u_h(t)\|_1^2 \leq C\|u_0\|^2,$

(c) $\int_0^t s\|u_{h,s}(s)\|_1^2 ds \leq C\|u_0\|_1^2,$

and

(d) $\int_0^t s^2\|u_{h,ss}(s)\|^2 ds \leq C\|u_0\|_1^2$

hold true.

Proof. Choose $\psi_h = u_h$ in (2.2.8) and use coercivity of $A(\cdot,\cdot)$ to have

$$\frac{1}{2}\frac{d}{dt}(\|\alpha^{1/2}u_h(t)\|^2) + c_0\|u_h(t)\|_1^2 \leq \lambda\|u_h(t)\|^2.$$

Integration with respect to time from 0 to t yields

$$\|\alpha^{1/2}u_h(t)\|^2 + 2c_0\int_0^t \|u_h(s)\|_1^2 ds \leq C\left(\|u_h(0)\|^2 + \int_0^t \|u_h(s)\|^2 ds\right).$$

In view of (2.2.4) and an application of Gronwall's lemma yields

$$\|u_h(t)\|^2 + \int_0^t \|u_h(s)\|_1^2 ds \leq C\|u_h(0)\|^2 \leq C\|u_0\|^2. \tag{2.2.20}$$

Now, choose $\psi_h = u_{h,t}$ in (2.2.8) to have

$$\|\alpha^{1/2}u_{h,t}(t)\|^2 + \frac{1}{2}\frac{d}{dt}A(u_h(t), u_h(t)) = \frac{\lambda}{2}\frac{d}{dt}\|u_h(t)\|^2.$$

Integrate the above equation from 0 to t. Then use (2.2.4) and coercivity of $A(\cdot,\cdot)$ to obtain

$$\int_0^t \|u_{h,s}(s)\|^2 ds + \|u_h(t)\|_1^2 \leq C\left(\|u_h(0)\|_1^2 + \|u_h(t)\|^2\right).$$

Using the estimate of (2.2.20) together with $\|L_h u_0\|_1 \leq C\|u_0\|_1$ completes the proof of (a). Following the arguments in Lemma 2.2.1-2.2.3, the rest of the stability estimates can be easily derived. We, therefore, omit the proof. ■

2.2.2 Error Estimates with Smooth Initial Data

This subsection is concerned with pointwise-in-time error estimates for the solution p and the flux u in the L^2-norm when initial data $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Using (2.2.6), (2.2.7), (2.2.8) and (2.2.9), we obtain the following error equations:

$$(\alpha(u-u_h)_t, \psi_h) + A(u-u_h, \psi_h) = \lambda(u-u_h, \psi_h), \ \forall\, \psi_h \in V_h, \tag{2.2.21}$$

$$((p-p_h)_x, \phi_{hx}) = (\alpha(u-u_h), \phi_{hx}), \ \forall\, \phi_h \in W_h. \tag{2.2.22}$$

Define the elliptic projections $\widetilde{u}_h : [0,T] \longrightarrow V_h$ and $\widetilde{p}_h : [0,T] \longrightarrow W_h$, by

$$A(u-\widetilde{u}_h, \psi_h) = 0, \ \forall\, \psi_h \in V_h, \tag{2.2.23}$$

$$((p-\widetilde{p}_h)_x, \phi_{hx}) = 0, \ \forall\, \phi_h \in W_h, \tag{2.2.24}$$

respectively. Since the direct comparisons between u, u_h and p, p_h does not yield optimal order of convergence, we, therefore, split the errors $u-u_h$ and $p-p_h$ as:

$$\begin{aligned} u-u_h &= (u-\widetilde{u}_h) + (\widetilde{u}_h - u_h) \\ &:= \eta + \theta_h, \end{aligned}$$

and

$$\begin{aligned} p-p_h &= (p-\widetilde{p}_h) + (\widetilde{p}_h - p_h) \\ &:= \rho + \rho_h. \end{aligned}$$

It is well-known [60] that η and ρ satisfy the following estimates:

$$\|\eta\| \le Ch^2\|u\|_2 \ \text{ and } \ \|\eta_t\| \le Ch^2\|u_t\|_2, \tag{2.2.25}$$

and

$$\|\rho\| \le Ch^2\|p\|_2 \ \text{ and } \ \|\rho\| \le Ch^2\|p_t\|_2. \tag{2.2.26}$$

Using (2.2.21)-(2.2.22) and auxiliary projections (2.2.23)-(2.2.24), we obtain the following error equations in θ_h and ρ_h as follows:

$$(\alpha\theta_{h,t}, \psi_h) + A(\theta_h, \psi_h) = \lambda(\eta, \psi_h) + \lambda(\theta_h, \psi_h) - (\alpha\eta_t, \psi_h), \ \psi_h \in V_h, \tag{2.2.27}$$

and

$$(\rho_{hx}, \phi_{hx}) = (\alpha(\eta+\theta_h), \phi_{hx}), \ \phi_h \in W_h. \tag{2.2.28}$$

Due to presence of the term η_t in (2.2.27), the standard energy formulation demands higher regularity on the flux u and which in turn requires higher regularity on p. We, therefore, use the following nonstandard energy formulation which is described below. Define $\widehat{\theta}(t)$ as

$$\widehat{\theta}(t) = \int_0^t \theta(\tau) d\tau, \quad t \in \bar{J}.$$

Note that $\widehat{\theta}(0) = 0$ and $\widehat{\theta}_t(t) = \theta(t)$. Integrating (2.2.28) from 0 to t, we have

$$(\widehat{\rho}_{hx}, \phi_{hx}) = (\alpha\widehat{\eta}, \phi_{hx}) + (\alpha\widehat{\theta}_h, \phi_{hx}). \tag{2.2.29}$$

Further, integrate (2.2.6) and (2.2.8) from 0 to t. Then, using $u_h(0) = L_h u_0$, where $L_h : L^2(\Omega) \longrightarrow V_h$ be the standard L^2 projection, we arrive at

$$(\alpha\theta_h, \psi_h) + A(\widehat{\theta}_h, \psi_h) = \lambda(\widehat{\eta}, \psi_h) + \lambda(\widehat{\theta}_h, \psi_h) - (\alpha\eta, \psi_h). \tag{2.2.30}$$

The main result concerning smooth data error estimates are given in the following theorem.

Theorem 2.2.1 *Let $\{u, p\}$ and $\{u_h, p_h\}$ be the solutions of (2.2.6)-(2.2.7) and (2.2.8)-(2.2.9), respectively. Further, let $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there is a positive constant C independent of h such that*

$$\|u(t) - u_h(t)\| \leq Ch^2 t^{-1/2} \|u_0\|_1 \leq Ch^2 t^{-1/2} \|p_0\|_2, \quad t > 0 \tag{2.2.31}$$

and

$$\|p(t) - p_h(t)\| \leq Ch^2 t^{-1/2} \|p_0\|_2, \quad t > 0. \tag{2.2.32}$$

The proof of the above theorem requires some preparations. Below, we shall prove a sequence of auxiliary results which altogether will lead to the desired estimate.

Lemma 2.2.6 *Let ρ_h satisfies (2.2.28). Then there is a positive constant C independent of h such that*

$$\|\rho_h\| \leq C \left(\|\eta\| + \|\theta_h\| \right).$$

Proof. Taking $\phi_h = \rho_h$ in (2.2.28) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\rho_{hx}\|^2 = (\rho_{hx}, \rho_{hx}) &= (\alpha\eta, \rho_{hx}) + (\alpha\theta_h, \rho_{hx}) \\ &\leq C(\|\eta\|^2 + \|\theta_h\|^2) + \frac{1}{2}\|\rho_{hx}\|^2 \\ &\leq C(\|\eta\|^2 + \|\theta_h\|^2). \end{aligned}$$

As $\rho_h \in H_0^1(\Omega)$, an application of Poincaré's inequality completes the proof. ■

Lemma 2.2.7 *Let θ_h satisfies (2.2.27). Then there exists a positive constant C such that*

$$t\|\theta_h\|^2 + \int_0^t s\|\theta_h\|_1^2 ds \le C\left[\int_0^t (s^2\|\eta_s\|^2 + s\|\eta\|^2)ds + \int_0^t \|\theta_h\|^2 ds\right].$$

Proof. Setting $\psi_h = t\theta_h$ in (2.2.27) to have

$$(\alpha\theta_{h,t}, t\theta_h) + A(\theta_h, t\theta_h) = \lambda(\eta, t\theta_h) + \lambda(\theta_h, t\theta_h) - (\alpha\eta_t, t\theta_h).$$

Using coercivity of $A(\cdot,\cdot)$ and Cauchy-Schwarz inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\{t\|\alpha^{1/2}\theta_h\|^2\} + c_0 t\|\theta_h\|_1^2 \le C(t\|\eta\|^2 + t\|\theta_h\|^2 + t^2\|\eta_t\|^2 + \|\theta_h\|^2).$$

Integration from 0 to t leads to

$$t\|\theta_h\|^2 + \int_0^t s\|\theta_h\|_1^2 ds \le C\left[\int_0^t (s^2\|\eta_s\|^2 + s\|\eta\|^2)ds + \int_0^t \|\theta_h\|^2 ds\right] + C\int_0^t s\|\theta_h\|^2 ds.$$

An application of Gronwall's lemma completes the rest of the proof. ■

Lemma 2.2.8 *Let θ_h satisfies (2.2.30). Then there is a positive constant C such that*

$$\int_0^t \|\theta_h\|^2 ds + \|\widehat{\theta}_h\|_1^2 \le C\int_0^t \left(\|\widehat{\eta}\|^2 + \|\eta\|^2\right) ds.$$

Proof. Choose $\psi_h = \theta_h$ in (2.2.30) to have

$$(\alpha\theta_h, \theta_h) + A(\widehat{\theta}_h, \theta_h) = \lambda(\widehat{\eta}, \theta_h) + \lambda(\widehat{\theta}_h, \theta_h) - (\alpha\eta, \theta_h).$$

Use coercivity of $A(\cdot,\cdot)$ and Cauchy-Schwarz inequality and Young's inequality for the terms on the right hand side of the above equation to obtain

$$\|\alpha^{1/2}\theta_h\|^2 + \frac{c_0}{2}\frac{d}{dt}\|\widehat{\theta}_h\|_1^2 \le C(\epsilon)\left(\|\eta\|^2 + \|\widehat{\eta}\|^2 + \|\widehat{\theta}_h\|^2\right) + \epsilon C\|\theta_h\|^2.$$

Integrating from 0 to t, it now follows that

$$(1-\epsilon C)\int_0^t \|\theta_h\|^2 ds + \|\widehat{\theta}_h\|_1^2 \;\le\; C(\epsilon)\int_0^t \left(\|\widehat{\eta}\|^2 + \|\eta\|^2\right) ds + C(\epsilon)\int_0^t \|\widehat{\theta}_h\|_1^2 ds,$$

where we have used the fact $\widehat{\theta}_h(0) = 0$. Choose ϵ appropriately such that $(1-\epsilon C) > 0$. Thus, we obtain

$$\int_0^t \|\theta_h\|^2 ds + \|\widehat{\theta}_h\|_1^2 \;\le\; C\int_0^t \left(\|\widehat{\eta}\|^2 + \|\eta\|^2\right) ds + C\int_0^t \|\widehat{\theta}_h\|_1^2 ds.$$

An application of Gronwall's lemma completes the rest of the proof. ■

Proof of Theorem 2.2.1. By triangle inequality, we have

$$\|u(t) - u_h(t)\| \leq \|\eta(t)\| + \|\theta_h(t)\|. \tag{2.2.33}$$

Using Lemma 2.2.8 in Lemma 2.2.7, we get

$$t\|\theta_h\|^2 \leq C\left[\int_0^t s^2\|\eta_s\|^2 ds + \int_0^t s\|\eta\|^2 ds + \int_0^t \|\eta\|^2 ds + \int_0^t \|\widehat{\eta}\|^2 ds\right].$$

Since

$$\int_0^t \|\widehat{\eta}\|^2 ds \leq C\int_0^t \|\eta\|^2 ds,$$

and from the approximation property (2.2.25), we obtain

$$t\|\theta_h\|^2 \leq Ch^4\left[\int_0^t s^2\|u_s\|_2^2 ds + \int_0^t s\|u\|_2^2 ds + \int_0^t \|u\|_2^2 ds\right]. \tag{2.2.34}$$

Further, using *a priori* estimates of Lemma 2.2.4, we obtain

$$\|\theta_h\| \leq Ch^2 t^{-1/2}\|u_0\|_1, \tag{2.2.35}$$

and

$$\|\eta\| \leq Ch^2\|u\|_2 \leq Ch^2 t^{-1/2}\|u_0\|_1. \tag{2.2.36}$$

Combining (2.2.33), (2.2.35) and (2.2.36) the first inequality (2.2.31) is easily obtained. To estimate (2.2.32), we again use triangle inequality and Lemma 2.2.6 to obtain

$$\begin{aligned} \|p(t) - p_h(t)\| &\leq \|\rho(t)\| + \|\rho_h(t)\| \\ &\leq \|\rho(t)\| + C(\|\eta(t)\| + \|\theta_h(t)\|). \end{aligned} \tag{2.2.37}$$

From the approximation property (2.2.26), we have

$$\|\rho\| \leq Ch^2\|p\|_2. \tag{2.2.38}$$

Note that

$$\begin{aligned} \|p\|_2^2 &= \|p\|^2 + \|p_x\|^2 + \|p_{xx}\|^2 \\ &\leq C(\|p_x\|^2 + \|p_{xx}\|^2), \end{aligned}$$

where in the second step, we have used Poincaré's inequality $\|p\| \leq C\|p_x\|$. Since $u = ap_x$, we have

$$\|p\|_2 \leq C(\|u\| + \|u_x\|) = C\|u\|_1. \tag{2.2.39}$$

Using (2.2.39) in (2.2.38) and *a priori* estimate of Lemma 2.2.2, we obtain

$$\|\rho\| \leq Ch^2\|u_0\|_1. \tag{2.2.40}$$

Now, putting (2.2.35), (2.2.36) and (2.2.40) in (2.2.37), we arrive at

$$\begin{aligned} \|p(t) - p_h(t)\| &\leq Ch^2 t^{-1/2}\|u_0\|_1 \\ &\leq Ch^2 t^{-1/2}\|p_0\|_2. \end{aligned}$$

and this completes the proof of Theorem 2.2.1. ■

2.2.3 Error Estimates with Nonsmooth Initial Data

This subsection is devoted to pointwise-in-time error estimates of $u-u_h$ and $p-p_h$ when initial data $p_0 \in H_0^1(\Omega)$. Let V^* be the dual space of V equipped with the norm

$$\|v\|_{V^*} = \sup_{\psi \in V} \frac{(v,\psi)}{\|\psi\|_V}.$$

The main tool used in our error analysis is the following parabolic duality argument [19, 63]: For fixed $t > 0$ and $g \in V$, let $\{v(s), q(s)\} : [0,t) \longrightarrow V \times H_0^1(\Omega)$ be the solution of the following mixed problem

$$(\alpha v_s, \psi) - A(v,\psi) = -\lambda(v,\psi),\ \forall\, \psi \in V,\ s < t, \tag{2.2.41}$$
$$(q_x, \phi_x) = (\alpha v, \phi_x),\ \forall\, \phi \in H_0^1(\Omega) \tag{2.2.42}$$

with $v(t) = g$ and $v(s) = aq_x(s)$. The corresponding semidiscrete H^1-Galerkin mixed finite element approximation read as: Find $\{v_h(s), q_h(s)\} : [0,t) \longrightarrow V_h \times W_h$ such that

$$(\alpha v_{h,s}, \psi_h) - A(v_h,\psi_h) = -\lambda(v_h,\psi_h),\ \forall\, \psi_h \in V_h,\ s < t, \tag{2.2.43}$$
$$(q_{hx}, \phi_{hx}) = (\alpha v_h, \phi_{hx}),\ \forall\, \phi_h \in W_h \tag{2.2.44}$$

with $v_h(t) = L_h g$.

The main result for the rough data error estimate is given in the following theorem.

Theorem 2.2.2 *Let $\{u,p\}$ and $\{u_h,p_h\}$ be the solutions of (2.2.6)-(2.2.7) and (2.2.8)-(2.2.9), respectively. Let $p_0 \in H_0^1(\Omega)$ and $u_h(0) = L_h u_0$. Then there is a positive constant C independent of h such that*

$$\|u(t) - u_h(t)\| \leq Ch^2 t^{-1} \|p_0\|_1,\ t > 0 \tag{2.2.45}$$

and

$$\|p(t) - p_h(t)\| \leq Ch^2 t^{-1} \|p_0\|_1,\ t > 0. \tag{2.2.46}$$

The proof of the above theorem requires some preparations. Using (2.2.41)-(2.2.44) and (2.2.6)-(2.2.9), we note that

$$\frac{d}{ds}\{(\alpha u(s), v(s)) - (\alpha u_h(s), v_h(s))\} = 0. \tag{2.2.47}$$

Now integrating (2.2.47) from 0 to t, we obtain

$$(\alpha u(t), v(t)) - (\alpha u_h(t), v_h(t)) = (\alpha u(0), v(0)) - (\alpha u_h(0), v_h(0)). \tag{2.2.48}$$

With $u_h(0) = L_h u_0$ and $v_h(t) = L_h v(t) = L_h g$, we obtain

$$(\alpha e_2(t), g) = (\alpha u_0, \widetilde{e}_2(0)), \tag{2.2.49}$$

where $e_2(t) = u(t) - u_h(t)$ and $\widetilde{e}_2(s) = v(s) - v_h(s)$ the errors associated with the forward problem (2.2.6)-(2.2.9) and backward problem (2.2.41)-(2.2.44), respectively. Here, for the term on the right hand side of (2.2.48), we have used the fact that

$$(\alpha L_h u_0, v_h(0)) = (L_h u_0, \alpha v_h(0)) = (u_0, \alpha v_h(0)) = (\alpha u_0, v_h(0)).$$

The following lemma proves to be convenient for nonsmooth data error estimate.

Lemma 2.2.9 *If $p_0 \in H_0^1(\Omega)$, then*

$$\|e_2(t)\|_{V^*} \leq Ch^2 t^{-1/2}\|u_0\| \leq Ch^2 t^{-1/2}\|p_0\|_1. \tag{2.2.50}$$

Proof. From Theorem 2.2.1, we have

$$\|e_2(t)\| \leq Ch^2 t^{-1/2}\|u_0\|_1. \tag{2.2.51}$$

From (2.2.49), we notice that

$$\begin{aligned} |(\alpha e_2(t), g)| &= |(\alpha u_0, \widetilde{e}_2(0))| \\ &\leq C\|u_0\| \, \|\widetilde{e}_2(0)\|. \end{aligned} \tag{2.2.52}$$

Applying the estimate (2.2.51) to the backward problem (2.2.41)-(2.2.44), we obtain

$$\|\widetilde{e}_2(s)\| \leq Ch^2(t-s)^{-1/2}\|g\|_1, \; s < t.$$

As an immediate consequence of the above estimate, we get

$$\|\widetilde{e}_2(0)\| \leq Ch^2 t^{-1/2}\|g\|_1. \tag{2.2.53}$$

The estimates (2.2.52) and (2.2.53) now yields

$$\begin{aligned} |(e_2(t), \alpha g)| &\leq Ch^2 t^{-1/2}\|g\|_1 \, \|u_0\| \\ &\leq Ch^2 t^{-1/2}\|\alpha g\|_1 \, \|u_0\|, \end{aligned}$$

where we have used $\|g\|_1 = \|\alpha(\frac{1}{\alpha})g\|_1 \leq C\|\alpha g\|_1$ and this completes proof of Lemma 2.2.9. ∎

Proof of Theorem 2.2.2. Integrating (2.2.47) from $t/2$ to t, we obtain

$$\begin{aligned} (\alpha u(t), v(t)) - (\alpha u_h(t), v_h(t)) &= (\alpha u(t/2), v(t/2)) - (\alpha u_h(t/2), v_h(t/2)) \\ &= (\alpha e_2(t/2), v(t/2)) + (\alpha u_h(t/2), \widetilde{e}_2(t/2)). \end{aligned}$$

Since $v(t) = g$ and $v_h(t) = L_h g$, we obtain

$$\begin{aligned} |(\alpha e_2(t), g)| &= |(\alpha(u(t) - u_h(t)), g)| \\ &\leq C\|e_2(t/2)\|_{V^*}\|v(t/2)\|_1 + C\|\widetilde{e}_2(t/2)\|_{V^*}\|u_h(t/2)\|_1. \quad (2.2.54) \end{aligned}$$

The estimate (2.2.50) applied to $e_2(t/2)$ and $\widetilde{e}_2(t/2)$ yield,

$$\|e_2(t/2)\|_{V^*} \leq Ch^2 t^{-1/2}\|u_0\|,$$

and

$$\|\widetilde{e}_2(t/2)\|_{V^*} \leq Ch^2 t^{-1/2}\|g\|.$$

With appropriate modifications in the proof of estimate (b) in Lemma 2.2.2, it is easy to derive the following *a priori* estimate for the backward solution v

$$\|v(s)\|_1 \leq C(t-s)^{-1/2}\|g\|, \quad s < t,$$

and hence,

$$\|v(t/2)\|_1 \leq Ct^{-1/2}\|g\|.$$

Further, the estimate (b) of Lemma 2.2.5 gives

$$\|u_h(t/2)\|_1 \leq Ct^{-1/2}\|u_0\|.$$

Using the above estimates in (2.2.54), we obtain

$$\begin{aligned} |(e_2(t), \alpha g)| &\leq Ch^2 t^{-1}\|u_0\|\ \|g\| \\ &\leq Ch^2 t^{-1}\|u_0\|\ \|\alpha g\|. \end{aligned}$$

With $\overline{\psi} = \alpha g$, we get

$$\|e_2(t)\| = \sup_{\overline{\psi} \in L^2(\Omega)} \frac{(e_2(t), \overline{\psi})}{\|\overline{\psi}\|} \leq Ch^2 t^{-1}\|u_0\| \leq Ch^2 t^{-1}\|p_0\|_1,$$

and this proves the first estimate (2.2.45). Next, to prove (2.2.46), using (2.2.22) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} (e_{1,x}, \phi_{h,x}) &= (\alpha e_2, \phi_{h,x}) \\ &\leq C\|e_2\|\ \|\phi_{h,x}\|, \end{aligned}$$

where $e_1(t) = (p - p_h)(t)$ and $e_2(t) = (u - u_h)(t)$. Thus,

$$\|e_{1,x}\| \leq C\|e_2\|.$$

As $e_1 \in H_0^1(\Omega)$, an application of Poincaré's inequality and (2.2.45) completes the rest of the proof. ■

Remark. In this section, we discuss semidiscrete error analysis for H^1-Galerkin mixed finite element method for one-dimensional parabolic problems with both smooth and nonsmooth initial data. More precisely, optimal order error estimates of order $\mathcal{O}(h^2 t^{-1/2})$ in the L^2-norm are established for the solution p and its flux u for positive time with initial function $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Further, error estimates of order $\mathcal{O}(h^2 t^{-1})$ with positive time are derived in the L^2-norm for both the solution p and its flux u with initial data $p_0 \in H_0^1(\Omega)$. The proposed method is not subject to LBB-consistency condition and we don't require quasiuniformity condition on the finite element mesh. Compared to [60], our results retain the same order of convergence but with lesser regularity assumption on the initial data p_0.

2.3 Error Estimates for Two Dimensional Parabolic Problems

In this section, we extend the analysis presented in the previous section to two space variable parabolic problems. Analogous results are shown to hold for both the solution and its flux. We recall the following parabolic problem:

$$p_t - \nabla \cdot (a \nabla p) = 0, \quad (\mathbf{x}, t) \in \Omega \times J, \tag{2.3.1}$$

$$p(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times J, \tag{2.3.2}$$

$$p(\mathbf{x}, 0) = p_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.3.3}$$

where Ω is an open bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $J = (0, T]$ with $T < \infty$ and $p_t = \frac{\partial p}{\partial t}$. The coefficient matrix $a = a(\mathbf{x})$ is assumed to be smooth, symmetric and uniformly positive definite in Ω.

With the flux variable $\mathbf{u} = a\nabla p$, where $\alpha = 1/a$, (2.3.1) reduces to the following system:

$$p_t - \nabla \cdot \mathbf{u} = 0, \quad (\mathbf{x}, t) \in \Omega \times J, \tag{2.3.4}$$

$$\alpha \mathbf{u} - \nabla p = 0, \quad (\mathbf{x}, t) \in \Omega \times J. \tag{2.3.5}$$

For the error analysis, we shall need the spaces $\mathbf{V} = H(\mathrm{div}, \Omega)$ and $H_0^1(\Omega)$. The H^1-Galerkin mixed formulation is stated as: Find $\{\mathbf{u}, p\} : [0, T] \to \mathbf{V} \times H_0^1(\Omega)$ such that

$$(\alpha \mathbf{u}_t, \Psi) + A(\mathbf{u}, \Psi) = \lambda(\mathbf{u}, \Psi), \ \forall \, \Psi \in \mathbf{V}, \tag{2.3.6}$$

$$(\nabla p, \nabla \phi) = (\alpha \mathbf{u}, \nabla \phi), \ \forall \, \phi \in H_0^1(\Omega) \tag{2.3.7}$$

with $\mathbf{u}_0 = \mathbf{u}(\mathbf{x}, 0) = a\nabla p_0$. For the first term in (2.3.6), we have used integration by parts and the Dirichlet boundary condition $p_t(\mathbf{x}, t) = 0$ on $\partial\Omega$. The bilinear form $A(\cdot, \cdot)$ is given by

$$A(\mathbf{u}, \Psi) = (\nabla \cdot \mathbf{u}, \nabla \cdot \Psi) + \lambda(\mathbf{u}, \Psi).$$

Note that λ is chosen appropriately so that $A(\cdot, \cdot)$ is $\mathbf{V}$-coercive, i.e.,

$$A(\mathbf{v}, \mathbf{v}) \geq c_1 \|\mathbf{v}\|_{\mathbf{V}}^2, \quad \mathbf{v} \in \mathbf{V},$$

for some $c_1 > 0$. Moreover, $A(\cdot, \cdot)$ is bounded. That is, there is a positive constant C such that $|A(\mathbf{u}, \mathbf{v})| \leq C\|\mathbf{u}\|_{\mathbf{V}}\|\mathbf{v}\|_{\mathbf{V}}$.

Let $\mathbf{V}_h$ and W_h be finite dimensional subspaces of $\mathbf{V}$ and $H_0^1(\Omega)$, respectively, satisfying the following approximation properties [60]:

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \{\|\mathbf{v} - \mathbf{v}_h\| + h\|\mathbf{v} - \mathbf{v}_h\|_1\} \leq Ch^2\|\mathbf{v}\|_2, \ \forall \ \mathbf{v} \in (H^2(\Omega))^2.$$

and

$$\inf_{w_h \in W_h} \{\|w - w_h\| + h\|w - w_h\|_1\} \leq Ch^2\|w\|_2, \ \forall \ w \in H^2(\Omega) \cap H_0^1(\Omega).$$

Standard examples of approximating spaces can be found in [70]. The semidiscrete H^1-Galerkin mixed finite element approximation is to determine a pair $\{\mathbf{u}_h, p_h\} : [0, T] \to \mathbf{V}_h \times W_h$ satisfying

$$(\alpha \mathbf{u}_{h,t}, \Psi_h) + A(\mathbf{u}_h, \Psi_h) = \lambda(\mathbf{u}_h, \Psi_h), \ \forall \ \Psi_h \in \mathbf{V}_h, \tag{2.3.8}$$

$$(\nabla p_h, \nabla \phi_h) = (\alpha \mathbf{u}_h, \nabla \phi_h), \ \forall \ \phi_h \in W_h \tag{2.3.9}$$

with $\mathbf{u}_h(0) = L_h \mathbf{u}_0$, where $L_h : L^2(\Omega) \longrightarrow \mathbf{V}_h$ is the standard L^2 projection.

2.3.1 A Priori Estimates

In this subsection, we shall state some *a priori* estimates for the solution $\mathbf{u}$ satisfying (2.3.4)-(2.3.5). In addition, some stability estimates for its semidiscrete solution $\mathbf{u}_h$ satisfying (2.3.8) and (2.3.9) are also presented. Following the lines of arguments of Lemmas 2.2.1-2.2.5 in Subsection 2.2.1, it is easy to derive the following *a priori* estimates. The proof is, therefore, omitted.

Lemma 2.3.1 *Assume $\mathbf{u}_0 \in L^2(\Omega)$. Let $\mathbf{u}$ be the solution of (2.3.4)-(2.3.5). Then there is a positive generic constant C such that*

$$\int_0^t s^i \|\mathbf{u}(s)\|_{\mathbf{V}}^2 ds \leq C\|\mathbf{u}_0\|^2, \quad i = \{0, 1, 2\}.$$

Lemma 2.3.2 *Let* $\mathbf{u}$ *be the solution of (2.3.4)-(2.3.5). Then the following estimates hold true:*

(a) *If* $\mathbf{u}_0 \in \mathbf{V}$, *then* $\displaystyle\int_0^t \|\mathbf{u}_s(s)\|^2 ds + \|\mathbf{u}(t)\|_{\mathbf{V}}^2 \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2.$

If $\mathbf{u}_0 \in L^2(\Omega)$, *then*

(b) $\displaystyle\int_0^t s\|\mathbf{u}_s(s)\|^2 ds + t\|\mathbf{u}(t)\|_{\mathbf{V}}^2 \le C\|\mathbf{u}_0\|^2,$

(c) $\displaystyle\int_0^t s^2\|\mathbf{u}_s(s)\|^2 ds \le C\|\mathbf{u}_0\|^2.$

Lemma 2.3.3 *Let* $\mathbf{u}$ *be the solution of (2.3.4)-(2.3.5). Then the following estimates hold true:*

(a) *If* $\mathbf{u}_0 \in \mathbf{V}$, *then* $\displaystyle\int_0^t s\|\mathbf{u}_s(s)\|_{\mathbf{V}}^2 ds \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2,$

(b) *If* $\mathbf{u}_0 \in L^2(\Omega)$, *then* $\displaystyle\int_0^t s^2\|\mathbf{u}_s(s)\|_{\mathbf{V}}^2 ds \le C\|\mathbf{u}_0\|^2,$

(c) *If* $\mathbf{u}_0 \in \mathbf{V}$, *then* $\displaystyle\int_0^t s^2\|\mathbf{u}_{ss}(s)\|^2 ds \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2.$

Lemma 2.3.4 *Let* $\mathbf{u}$ *be the solution of (2.3.4)-(2.3.5). Then, for* $\mathbf{u}_0 \in \mathbf{V}$, *the following estimates hold true:*

(a) $t\|\mathbf{u}(t)\|_2^2 \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2,$

(b) $\displaystyle\int_0^t \|\mathbf{u}(s)\|_2^2 ds \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2,$

(c) $\displaystyle\int_0^t s^2\|\mathbf{u}_s(s)\|_2^2 ds \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2.$

Further, if $\mathbf{u}_0 \in L^2(\Omega)$, *then*

(d) $\displaystyle\int_0^t s^i\|\mathbf{u}(s)\|_2^2 ds \le C\|\mathbf{u}_0\|^2, i = 1, 2.$

Lemma 2.3.5 *Assume that* $\mathbf{u}_0 \in \mathbf{V}$ *with* $\mathbf{u}_h(0) = L_h\mathbf{u}_0$. *Then there is a positive generic constant* C *such that*

(a) $\displaystyle\int_0^t \|\mathbf{u}_{h,s}(s)\|^2 ds + \|\mathbf{u}_h(t)\|_{\mathbf{V}}^2 \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2,$

(b) $\displaystyle\int_0^t s\|\mathbf{u}_{h,s}(s)\|^2 ds + t\|\mathbf{u}_h(t)\|_{\mathbf{V}}^2 \le C\|\mathbf{u}_0\|^2,$

(c) $\displaystyle\int_0^t s\|\mathbf{u}_{h,s}(s)\|_{\mathbf{V}}^2 ds \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2,$

and

(d) $\displaystyle\int_0^t s^2\|\mathbf{u}_{h,ss}(s)\|^2 ds \le C\|\mathbf{u}_0\|_{\mathbf{V}}^2$

hold true.

2.3.2 Error Estimates with Smooth Initial Data

This subsection is devoted to the L^2-error estimates for the solution p and the flux $\mathbf{u}$ when $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Using the equations (2.3.6)-(2.3.9), we obtain the following error equations:

$$
\begin{aligned}
(\alpha(\mathbf{u}-\mathbf{u}_h)_t, \Psi_h) + A(\mathbf{u}-\mathbf{u}_h, \Psi_h) &= \lambda(\mathbf{u}-\mathbf{u}_h, \Psi_h), \ \forall \Psi_h \in \mathbf{V}_h, && (2.3.10)\\
(\nabla(p-p_h), \nabla\phi_h) &= (\alpha(\mathbf{u}-\mathbf{u}_h), \nabla\phi_h), \ \forall \phi_h \in W_h. && (2.3.11)
\end{aligned}
$$

We now define the elliptic projections $\widetilde{\mathbf{u}}_h : [0,T] \longrightarrow \mathbf{V}_h$ and $\widetilde{p}_h : [0,T] \longrightarrow W_h$, by

$$
A(\mathbf{u}-\widetilde{\mathbf{u}}_h, \Psi_h) = 0, \ \forall \ \Psi_h \in \mathbf{V}_h, \tag{2.3.12}
$$

and

$$
(\nabla(p-\widetilde{p}_h), \nabla\phi_h) = 0, \ \forall \ \phi_h \in W_h, \tag{2.3.13}
$$

respectively. We split the errors $\mathbf{u}-\mathbf{u}_h$ and $p-p_h$ as follows:

$$
\begin{aligned}
\mathbf{u}-\mathbf{u}_h &= (\mathbf{u}-\widetilde{\mathbf{u}}_h) + (\widetilde{\mathbf{u}}_h - \mathbf{u}_h)\\
&:= \boldsymbol{\eta} + \boldsymbol{\theta}_h,
\end{aligned}
$$

and

$$
\begin{aligned}
p-p_h &= (p-\widetilde{p}_h) + (\widetilde{p}_h - p_h)\\
&:= \rho + \rho_h.
\end{aligned}
$$

It is well-known [55, 56, 57] that $\boldsymbol{\eta}$ and ρ satisfy the following estimates:

$$
\|\boldsymbol{\eta}\| \le Ch^2\|\mathbf{u}\|_2 \quad \text{and} \quad \|\boldsymbol{\eta}_t\| \le Ch^2\|\mathbf{u}_t\|_2, \tag{2.3.14}
$$

and

$$
\|\rho\| \le Ch^2\|p\|_2 \quad \text{and} \quad \|\rho_t\| \le Ch^2\|p_t\|_2. \tag{2.3.15}
$$

Using (2.3.6)-(2.3.9) and auxiliary projections (2.3.12)-(2.3.13), we obtain the following error equations in $\boldsymbol{\theta}_h$ and ρ_h as follows:

$$
(\alpha\boldsymbol{\theta}_{h,t}, \Psi_h) + A(\boldsymbol{\theta}_h, \Psi_h) = \lambda(\boldsymbol{\eta}, \Psi_h) + \lambda(\boldsymbol{\theta}_h, \Psi_h) - (\alpha\boldsymbol{\eta}_t, \Psi_h), \ \Psi_h \in \mathbf{V}_h, \tag{2.3.16}
$$

and

$$
(\nabla\rho_h, \nabla\phi_h) = (\alpha(\boldsymbol{\eta}+\boldsymbol{\theta}_h), \nabla\phi_h), \ \phi_h \in W_h. \tag{2.3.17}
$$

We shall also use the following nonstandard energy formulation. Define $\widehat{\theta}(t)$ as

$$\widehat{\theta}(t) = \int_0^t \theta(\tau) d\tau, \quad t \in \bar{J}.$$

Note that $\widehat{\theta}(0) = 0$ and $\widehat{\theta}_t(t) = \theta(t)$. Integrating (2.3.17) from 0 to t, we have

$$(\nabla \widehat{\rho}_h, \nabla \phi_h) = (\alpha \widehat{\boldsymbol{\eta}}, \nabla \phi_h) + (\alpha \widehat{\boldsymbol{\theta}}_h, \nabla \phi_h). \tag{2.3.18}$$

Further, integrating (2.3.6) and (2.3.8) from 0 to t and using $\mathbf{u}_h(0) = L_h \mathbf{u}_0$, where $L_h : L^2(\Omega) \longrightarrow \mathbf{V}_h$ be the standard L^2 projection, we arrive at

$$(\alpha \boldsymbol{\theta}_h, \Psi_h) + A(\widehat{\boldsymbol{\theta}}_h, \Psi_h) = \lambda(\widehat{\boldsymbol{\eta}}, \Psi_h) + \lambda(\widehat{\boldsymbol{\theta}}_h, \Psi_h) - (\alpha \boldsymbol{\eta}, \Psi_h). \tag{2.3.19}$$

The main result concerning smooth data error estimates are given in the following theorem.

Theorem 2.3.1 *Let $\{\mathbf{u}, p\}$ and $\{\mathbf{u}_h, p_h\}$ be the solutions of (2.3.6)-(2.3.7) and (2.3.8)-(2.3.9), respectively. Further, let $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there is a positive constant C independent of h such that*

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\| \leq Ch^2 t^{-1/2} \|\mathbf{u}_0\|_{\mathbf{V}} \leq Ch^2 t^{-1/2} \|p_0\|_2, \quad t > 0 \tag{2.3.20}$$

and

$$\|p(t) - p_h(t)\| \leq Ch^2 t^{-1/2} \|p_0\|_2, \quad t > 0. \tag{2.3.21}$$

The proof of Theorem 2.3.1 requires some preparations. Below, we shall prove a sequence of auxiliary results which altogether will lead to the desired estimate.

Lemma 2.3.6 *Let ρ_h satisfies (2.3.17). Then there exists a positive constant C independent of h such that*

$$\|\rho_h\| \leq C(\|\boldsymbol{\eta}\| + \|\boldsymbol{\theta}_h\|).$$

Proof. Taking $\phi_h = \rho_h$ in (2.3.17) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\nabla \rho_h\|^2 = (\nabla \rho_h, \nabla \rho_h) &= (\alpha \boldsymbol{\eta}, \nabla \rho_h) + (\alpha \boldsymbol{\theta}_h, \nabla \rho_h) \\ &\leq C(\|\boldsymbol{\eta}\|^2 + \|\boldsymbol{\theta}_h\|^2) + \frac{1}{2}\|\nabla \rho_h\|^2 \\ &\leq C(\|\boldsymbol{\eta}\|^2 + \|\boldsymbol{\theta}_h\|^2). \end{aligned}$$

As $\rho_h \in H_0^1(\Omega)$, an application of Poincaré's inequality completes the proof. ∎

Lemma 2.3.7 *Let $\boldsymbol{\theta}_h$ satisfy (2.3.16). Then there exists a positive constant C such that*

$$t\|\boldsymbol{\theta}_h\|^2 + \int_0^t s\|\boldsymbol{\theta}_h\|_{\mathbf{V}}^2 ds \leq C\int_0^t s^2\|\boldsymbol{\eta}_s\|^2 ds + C\int_0^t s\|\boldsymbol{\eta}\|^2 ds + C\int_0^t \|\boldsymbol{\theta}_h\|^2 ds.$$

Proof. Set $\Psi_h = t\boldsymbol{\theta}_h$ in (2.3.16). Then, using coercivity of $A(\cdot,\cdot)$ and Cauchy-Schwarz inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\{t\|\alpha^{1/2}\boldsymbol{\theta}_h\|^2\} + c_1 t\|\boldsymbol{\theta}_h\|_{\mathbf{V}}^2 \leq C(t\|\boldsymbol{\eta}\|^2 + t\|\boldsymbol{\theta}_h\|^2 + t^2\|\boldsymbol{\eta}_t\|^2 + \|\boldsymbol{\theta}_h\|^2).$$

Integration from 0 to t leads to

$$t\|\boldsymbol{\theta}_h\|^2 + \int_0^t s\|\boldsymbol{\theta}_h\|_{\mathbf{V}}^2 ds \leq C\left[\int_0^t (s^2\|\boldsymbol{\eta}_s\|^2 + s\|\boldsymbol{\eta}\|^2)ds + \int_0^t \|\boldsymbol{\theta}_h\|^2 ds\right] + C\int_0^t s\|\boldsymbol{\theta}_h\|^2 ds.$$

An application of Gronwall's lemma completes the proof. ■

Lemma 2.3.8 *Let $\boldsymbol{\theta}_h$ satisfy (2.3.19). Then there exists a positive constant C such that*

$$\int_0^t \|\boldsymbol{\theta}_h\|^2 ds + \|\widehat{\boldsymbol{\theta}}_h\|_{\mathbf{V}}^2 \leq C\int_0^t \left(\|\boldsymbol{\eta}\|^2 + \|\widehat{\boldsymbol{\eta}}\|^2\right) ds.$$

Proof. Choose $\Psi_h = \boldsymbol{\theta}_h$ in (2.3.19). Then, using coercivity of $A(\cdot,\cdot)$ and Cauchy-Schwarz inequality, we obtain

$$\|\alpha^{1/2}\boldsymbol{\theta}_h\|^2 + \frac{c_1}{2}\frac{d}{dt}\|\widehat{\boldsymbol{\theta}}_h\|_{\mathbf{V}}^2 \leq C\left(\|\widehat{\boldsymbol{\eta}}\|\|\boldsymbol{\theta}_h\| + \|\widehat{\boldsymbol{\theta}}_h\|\|\boldsymbol{\theta}_h\| + \|\boldsymbol{\eta}\|\|\boldsymbol{\theta}_h\|\right).$$

Integrating from 0 to t and using Cauchy-Schwarz inequality and Young's inequality, we have

$$\int_0^t \|\boldsymbol{\theta}_h\|^2 ds + \|\widehat{\boldsymbol{\theta}}_h\|_{\mathbf{V}}^2 \quad\leq\quad C(\epsilon)\int_0^t \left(\|\widehat{\boldsymbol{\eta}}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\widehat{\boldsymbol{\theta}}_h\|^2\right) ds + \epsilon C\int_0^t \|\boldsymbol{\theta}_h\|^2 ds.$$

Choose ϵ appropriately such that $(1-\epsilon C) > 0$. Thus, we obtain

$$\int_0^t \|\boldsymbol{\theta}_h\|^2 ds + \|\widehat{\boldsymbol{\theta}}_h\|_{\mathbf{V}}^2 \quad\leq\quad C\int_0^t (\|\widehat{\boldsymbol{\eta}}\|^2 + \|\boldsymbol{\eta}\|^2)\, ds + C\int_0^t \|\widehat{\boldsymbol{\theta}}_h\|_{\mathbf{V}}^2 ds.$$

Finally, an application of Gronwall's lemma completes the rest of the proof. ■

Proof of Theorem 2.3.1. By triangle inequality, we have

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\| \leq \|\boldsymbol{\eta}(t)\| + \|\boldsymbol{\theta}_h(t)\|. \tag{2.3.22}$$

Using Lemma 2.3.8 in Lemma 2.3.7, we get

$$t\|\boldsymbol{\theta}_h\|^2 \leq C\left[\int_0^t s^2\|\boldsymbol{\eta}_s\|^2 ds + \int_0^t s\|\boldsymbol{\eta}\|^2 ds + \int_0^t \|\boldsymbol{\eta}\|^2 ds + \int_0^t \|\widehat{\boldsymbol{\eta}}\|^2 ds\right].$$

Since

$$\int_0^t \|\widehat{\boldsymbol{\eta}}\|^2 ds \le C \int_0^t \|\boldsymbol{\eta}\|^2 ds,$$

and using approximation property (2.3.14), we obtain

$$t\|\boldsymbol{\theta}_h\|^2 \le Ch^4 \left[\int_0^t s^2 \|\mathbf{u}_s(s)\|_2^2 ds + \int_0^t s\|\mathbf{u}(s)\|_2^2 ds + \int_0^t \|\mathbf{u}(s)\|_2^2 ds \right]. \tag{2.3.23}$$

Using *a priori* estimates of Lemma 2.3.4, we obtain

$$\|\boldsymbol{\theta}_h\| \le Ch^2 t^{-1/2} \|\mathbf{u}_0\|_{\mathbf{V}}, \tag{2.3.24}$$

and

$$\|\boldsymbol{\eta}\| \le Ch^2 \|\mathbf{u}\|_2 \le Ch^2 t^{-1/2} \|\mathbf{u}_0\|_{\mathbf{V}}. \tag{2.3.25}$$

Combine (2.3.22), (2.3.24) and (2.3.25) to prove the first inequality (2.3.20). To estimate (2.3.21), we again use triangle inequality and Lemma 2.3.6 to obtain

$$\begin{aligned} \|p(t) - p_h(t)\| &\le \|\rho(t)\| + \|\rho_h(t)\| \\ &\le \|\rho(t)\| + C(\|\boldsymbol{\eta}(t)\| + \|\boldsymbol{\theta}_h(t)\|). \end{aligned} \tag{2.3.26}$$

From the approximation property (2.3.15), we have

$$\|\rho\| \le Ch^2 \|p\|_2. \tag{2.3.27}$$

Arguing as in (2.2.39), it is easy to obtain

$$\|p\|_2 \le C\|\mathbf{u}\|_{\mathbf{V}}. \tag{2.3.28}$$

Use (2.3.28) in (2.3.27) and then apply Lemma 2.3.2 to obtain

$$\|\rho\| \le Ch^2 \|\mathbf{u}_0\|_{\mathbf{V}}. \tag{2.3.29}$$

Combing (2.3.24), (2.3.25), (2.3.26) and (2.3.29), we conclude

$$\begin{aligned} \|p(t) - p_h(t)\| &\le Ch^2 t^{-1/2} \|\mathbf{u}_0\|_{\mathbf{V}} \\ &\le Ch^2 t^{-1/2} \|p_0\|_2, \end{aligned}$$

and this completes the proof of Theorem 2.3.1. ■

2.3.3 Error Estimates with Nonsmooth Initial Data

This subsection is concerned about pointwise-in-time error estimates of $\mathbf{u}-\mathbf{u}_h$ and $p-p_h$ when initial data $p_0 \in H_0^1(\Omega)$. Let $\mathbf{V}^*$ be the dual space of $\mathbf{V}$ equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{V}^*} = \sup_{\Psi\in\mathbf{V}} \frac{(\mathbf{v},\Psi)}{\|\Psi\|_{\mathbf{V}}}.$$

The main tool used in our error analysis is the following parabolic duality argument: For fixed $t > 0$ and $\mathbf{g} \in \mathbf{V}$, let $\{\mathbf{v}(s), q(s)\} : [0,t) \longrightarrow \mathbf{V} \times H_0^1(\Omega)$ be the solution of the following mixed problem

$$(\alpha\mathbf{v}_s, \Psi) - A(\mathbf{v}, \Psi) \;=\; -\lambda(\mathbf{v}, \Psi),\ \forall\Psi \in \mathbf{V},\ s < t \tag{2.3.30}$$

$$(\nabla q, \nabla\phi) \;=\; (\alpha\mathbf{v}, \nabla\phi),\ \forall\phi \in H_0^1(\Omega) \tag{2.3.31}$$

with $\mathbf{v}(t) = \mathbf{g}$ and $\mathbf{v}(s) = a\nabla q(s)$. The corresponding semidiscrete H^1-Galerkin mixed finite element approximation is defined as: Find $\{\mathbf{v}_h(s), q_h(s)\} : [0,t) \longrightarrow \mathbf{V}_h \times W_h$ such that

$$(\alpha\mathbf{v}_{h,s}, \Psi_h) - A(\mathbf{v}_h, \Psi_h) \;=\; -\lambda(\mathbf{v}_h, \Psi_h),\ \forall\Psi_h \in \mathbf{V}_h,\ s < t \tag{2.3.32}$$

$$(\nabla q_h, \nabla\phi_h) \;=\; (\alpha\mathbf{v}_h, \nabla\phi_h),\ \forall\phi_h \in W_h \tag{2.3.33}$$

with $\mathbf{v}_h(t) = L_h\mathbf{g}$. Using (2.3.30)-(2.3.33) and (2.3.6)-(2.3.9), we observe that

$$\frac{d}{ds}\{(\alpha\mathbf{u}, \mathbf{v}) - (\alpha\mathbf{u}_h, \mathbf{v}_h)\} = 0. \tag{2.3.34}$$

Now integrating (2.3.34) from 0 to t, we obtain

$$(\alpha\mathbf{u}(t), \mathbf{v}(t)) - (\alpha\mathbf{u}_h(t), \mathbf{v}_h(t)) = (\alpha\mathbf{u}(0), \mathbf{v}(0)) - (\alpha\mathbf{u}_h(0), \mathbf{v}_h(0)). \tag{2.3.35}$$

With $\mathbf{u}_h(0) = L_h\mathbf{u}_0$ and $\mathbf{v}_h(t) = L_h\mathbf{v}(t) = L_h\mathbf{g}$, we obtain

$$(\alpha\mathbf{e}_2(t), \mathbf{g}) = (\alpha\mathbf{u}_0, \widetilde{\mathbf{e}}_2(0)), \tag{2.3.36}$$

where $\widetilde{\mathbf{e}}_2(s) = \mathbf{v}(s) - \mathbf{v}_h(s)$ denotes the error associated with the backward problem (2.3.30)-(2.3.33) for $s \leq t$ and $\mathbf{e}_2(t) = \mathbf{u}(t) - \mathbf{u}_h(t)$. To obtain (2.3.36), we have used the fact

$$(\alpha L_h\mathbf{u}_0, \mathbf{v}_h(0)) = (L_h\mathbf{u}_0, \alpha\mathbf{v}_h(0)) = (\mathbf{u}_0, \alpha\mathbf{v}_h(0)) = (\alpha\mathbf{u}_0, \mathbf{v}_h(0)).$$

The following lemma proves to be convenient for our future use.

Lemma 2.3.9 *Assume that* $p_0 \in H_0^1(\Omega)$. *Then*

$$\|\mathbf{e}_2(t)\|_{\mathbf{V}^*} \leq Ch^2t^{-1/2}\|\mathbf{u}_0\| \leq Ch^2t^{-1/2}\|p_0\|_1. \tag{2.3.37}$$

Proof. We have from Theorem 2.3.1

$$\|\mathbf{e}_2(t)\| \leq Ch^2 t^{-1/2} \|\mathbf{u}_0\|_{\mathbf{V}}. \tag{2.3.38}$$

From (2.3.36), we note that

$$\begin{aligned} |(\alpha \mathbf{e}_2(t), \mathbf{g})| &= |(\alpha \mathbf{u}_0, \widetilde{\mathbf{e}}_2(0))| \\ &\leq C\|\mathbf{u}_0\| \, \|\widetilde{\mathbf{e}}_2(0)\|. \end{aligned} \tag{2.3.39}$$

Apply (2.3.38) to the backward problem (2.3.30)-(2.3.33) to obtain

$$\|\widetilde{\mathbf{e}}_2(s)\| \leq Ch^2 (t-s)^{-1/2} \|\mathbf{g}\|_{\mathbf{V}}, \; s < t.$$

As a consequence, we have

$$\|\widetilde{\mathbf{e}}_2(0)\| \leq Ch^2 t^{-1/2} \|\mathbf{g}\|_{\mathbf{V}}. \tag{2.3.40}$$

Now (2.3.39) and (2.3.40) yields

$$\begin{aligned} |(\mathbf{e}_2(t), \alpha \mathbf{g})| &\leq Ch^2 t^{-1/2} \|\mathbf{g}\|_{\mathbf{V}} \, \|\mathbf{u}_0\| \\ &\leq Ch^2 t^{-1/2} \|\alpha \mathbf{g}\|_{\mathbf{V}} \, \|\mathbf{u}_0\|, \end{aligned}$$

and this completes the proof of Lemma 2.3.9. ■

The main result on nonsmooth data error estimate is given in the following theorem.

Theorem 2.3.2 *Let* $\{\mathbf{u}, p\}$ *and* $\{\mathbf{u}_h, p_h\}$ *be the solutions of (2.3.6)-(2.3.7) and (2.3.8)-(2.3.9), respectively. Let* $p_0 \in H_0^1(\Omega)$ *with* $\mathbf{u}_h(0) = L_h \mathbf{u}_0$. *Then there is a positive constant* C *independent of* h *such that*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^2 t^{-1} \|p_0\|_1, \; t > 0, \tag{2.3.41}$$

and

$$\|p - p_h\| \leq Ch^2 t^{-1} \|p_0\|_1, \; t > 0. \tag{2.3.42}$$

Proof. Integrating (2.3.34) from $t/2$ to t, we have

$$\begin{aligned} (\alpha \mathbf{u}(t), \mathbf{v}(t)) - (\alpha \mathbf{u}_h(t), \mathbf{v}_h(t)) &= (\alpha \mathbf{u}(t/2), \mathbf{v}(t/2)) - (\alpha \mathbf{u}_h(t/2), \mathbf{v}_h(t/2)) \\ &= (\alpha \mathbf{e}_2(t/2), \mathbf{v}(t/2)) + (\alpha \mathbf{u}_h(t/2), \widetilde{\mathbf{e}}_2(t/2)). \end{aligned}$$

With $\mathbf{v}(t) = \mathbf{g}$ and $\mathbf{v}_h(t) = L_h \mathbf{g}$, we obtain

$$\begin{aligned} |(\alpha \mathbf{e}_2(t), \mathbf{g})| &= |(\alpha(\mathbf{u}(t) - \mathbf{u}_h(t)), \mathbf{g})| \\ &\leq C\|\mathbf{e}_2(t/2)\|_{\mathbf{V}^*} \|\mathbf{v}(t/2)\|_{\mathbf{V}} + C\|\widetilde{\mathbf{e}}_2(t/2)\|_{\mathbf{V}^*} \|\mathbf{u}_h(t/2)\|_{\mathbf{V}}. \end{aligned} \tag{2.3.43}$$

The estimate (2.3.37) applied to $\mathbf{e}_2(t/2)$ and $\widetilde{\mathbf{e}}_2(t/2)$ yield,

$$\|\mathbf{e}_2(t/2)\|_{\mathbf{V}^*} \leq Ch^2t^{-1/2}\|\mathbf{u}_0\|,$$

and

$$\|\widetilde{\mathbf{e}}_2(t/2)\|_{\mathbf{V}^*} \leq Ch^2t^{-1/2}\|\mathbf{g}\|.$$

Now, applying *a priori* estimate (b) of Lemma 2.3.2 to $\mathbf{v}$, we obtain

$$\|\mathbf{v}(t/2)\|_{\mathbf{V}} \leq Ct^{-1/2}\|\mathbf{g}\|.$$

Further, the estimate (b) of Lemma 2.3.5 gives

$$\|\mathbf{u}_h(t/2)\|_{\mathbf{V}} \leq Ct^{-1/2}\|\mathbf{u}_0\|.$$

Using the above estimates in (2.3.43), we obtain

$$\begin{aligned} |(\mathbf{e}_2(t), \alpha\mathbf{g})| &\leq Ch^2t^{-1}\|\mathbf{u}_0\| \, \|\mathbf{g}\| \\ &\leq Ch^2t^{-1}\|\mathbf{u}_0\| \, \|\alpha\mathbf{g}\|. \end{aligned}$$

With $\Psi = \alpha\mathbf{g}$,

$$\|\mathbf{e}_2(t)\| = \sup_{\Psi \in L^2(\Omega)} \frac{(\mathbf{e}_2(t), \Psi)}{\|\Psi\|} \leq Ch^2t^{-1}\|\mathbf{u}_0\| \leq Ch^2t^{-1}\|p_0\|_1,$$

which yields (2.3.41). Next, to prove (2.3.42), we have from (2.3.11)

$$\begin{aligned} (\nabla e_1, \nabla\phi_h) &= (\alpha\mathbf{e}_2, \nabla\phi_h) \\ &\leq C\|\mathbf{e}_2\| \, \|\nabla\phi_h\|, \end{aligned}$$

where $e_1(t) = p(t) - p_h(t)$. Thus,

$$\|\nabla e_1(t)\| \leq C\|\mathbf{e}_2(t)\|.$$

As $e_1 \in H_0^1(\Omega)$, an application of Poincaré's inequality and (2.3.41) complete the rest of the proof. ∎

Remark. We have extended the semidiscrete error analysis of one dimensional parabolic problem to two space dimensions for the proposed H^1-Galerkin mixed finite element method with both smooth and nonsmooth initial data. Analogous convergence results for the solution p and the flux $\mathbf{u}$ are derived. Compared to [60], we require lesser regularity assumption on the initial data p_0. The method is not subject to LBB-consistency condition and we do not require quasiuniformity condition on the finite element mesh.

Chapter 3

Fully Discrete H^1-Galerkin MFEM for Parabolic Problems

In this chapter, the discrete-in-time backward Euler scheme is analyzed for one-dimensional homogeneous parabolic problems. Almost optimal order error estimates in the L^2-norm for the solution and the flux are derived when the initial function $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $p_0 \in H_0^1(\Omega)$. We use simple energy technique and duality argument.

3.1 Introduction

Considering the following one-dimensional homogeneous parabolic problem:

$$p_t - (ap_x)_x = 0, \ (x,t) \in \Omega \times J, \tag{3.1.1}$$

$$p(0,t) = p(1,t) = 0, \ t \in J, \tag{3.1.2}$$

$$p(x,0) = p_0(x), \ x \in \Omega, \tag{3.1.3}$$

where $p_t = \frac{\partial p}{\partial t}$, $\Omega = (0,1)$ and J denotes the time interval $(0,T]$ with $T < \infty$. The coefficient $a = a(x)$ is assumed to be smooth. Further, a is bounded above and below by positive constants b_0 and b_1, i.e.,

$$b_0 \leq a(x) \leq b_1, \ x \in \Omega. \tag{3.1.4}$$

Introducing $u = ap_x$, we split the problem (3.1.1) into a first order system as follows:

$$p_t - u_x = 0, \quad p_x = \alpha u, \tag{3.1.5}$$

where $\alpha = 1/a$. For the error analysis, we need the following spaces: $H_0^1(\Omega) = \{w \in H^1(\Omega) : w(0) = w(1) = 0\}$ and $V = H^1(\Omega)$.

Let V_h and W_h be the finite dimensional subspaces of V and $H_0^1(\Omega)$, respectively. Further, let $0 = t_0 < t_1 < \ldots < t_M = T$ be a partition of the time interval $[0, T]$ with step length $\Delta t = T/M$, for some positive integer M. For a smooth function ϕ on $[0, T]$, define $\phi^n = \phi(t_n)$ and $\partial_t \phi^n = (\phi^n - \phi^{n-1})/\Delta t$. Let U^n and P^n be the approximations of u and p at $t = t_n$, respectively. Then, the discrete problem based on the backward Euler method is stated as follows: For $n \geq 1$, find $\{U^n, P^n\} \in V_h \times W_h$ satisfying

$$(\alpha \partial_t U^n, \psi_h) + A(U^n, \psi_h) = \lambda(U^n, \psi_h), \ \forall \psi_h \in V_h, \tag{3.1.6}$$

$$(P_x^n, \phi_{hx}) = (\alpha U^n, \phi_{hx}), \ \forall \phi_h \in W_h \tag{3.1.7}$$

with given $\{U^0, P^0\} \in V_h \times W_h$.

The layout of this chapter is as follows. Section 3.2 deals with error estimates for H^1-Galerkin MFEM with smooth initial data. The error estimates for nonsmooth initial data are carried out in Section 3.3.

Throughout this chapter C denotes a positive generic constant independent of h, Δt and may not be same at each occurrence.

3.2 Error Estimates with Smooth Initial Data

The aim of this section is to derive L^2-error estimates for $u(t_n) - U^n$ and $p(t_n) - P^n$ when the initial function $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. At $t = t_n$, the semidiscrete H^1-Galerkin mixed finite element approximation (2.2.8)-(2.2.9) read as:

$$(\alpha u_{h,t}(t_n), \psi_h) + A(u_h(t_n), \psi_h) = \lambda(u_h(t_n), \psi_h), \ \forall \ \psi_h \in V_h, \tag{3.2.1}$$

$$(p_{hx}(t_n), \phi_{hx}) = (\alpha u_h(t_n), \phi_{hx}), \ \forall \ \phi_h \in W_h. \tag{3.2.2}$$

Now adding the term $\alpha \partial_t u_h(t_n)$ to both sides of (3.2.1), we obtain

$$(\alpha \partial_t u_h(t_n), \psi_h) + A(u_h(t_n), \psi_h) = \lambda(u_h(t_n), \psi_h) + (\alpha \partial_t u_h(t_n) - \alpha u_{h,t}(t_n), \psi_h). \tag{3.2.3}$$

Using (3.1.6)-(3.1.7) and (3.2.2)-(3.2.3), we obtain the following error equations:

$$(\alpha \partial_t (u_h(t_n) - U^n), \psi_h) + A(u_h(t_n) - U^n, \psi_h) = \lambda(u_h(t_n) - U^n, \psi_h) + (\sigma(t_n), \psi_h), \tag{3.2.4}$$

and

$$(p_{hx}(t_n) - P_x^n, \phi_{hx}) = (\alpha(u_h(t_n) - U^n), \phi_{hx}), \tag{3.2.5}$$

for all $\psi_h \in V_h$ and $\phi_h \in W_h$. Here $\sigma(t_n) = \alpha(\partial_t u_h(t_n) - u_{h,t}(t_n))$. Setting $u_h(t_n) - U^n = \zeta^n$ and $p_h(t_n) - P^n = \xi^n$, the error equations (3.2.4)-(3.2.5) reduce to

$$(\alpha \partial_t \zeta^n, \psi_h) + A(\zeta^n, \psi_h) = \lambda(\zeta^n, \psi_h) + (\sigma^n, \psi_h), \tag{3.2.6}$$

and

$$(\xi_x^n, \phi_{hx}) \quad = \quad (\alpha\zeta^n, \phi_{hx}). \tag{3.2.7}$$

Along with the standard energy formulation, we also use nonstandard energy formulation which is described as follows:

Nonstandard Formulation: Multiplying (3.1.6) by Δt and taking summation over n from $n = 1$ to $n = m$ with $1 \le n \le m \le N$, we obtain

$$\Delta t \sum_{n=1}^{m} (\alpha\partial_t U^n, \psi_h) + \Delta t \sum_{n=1}^{m} A(U^n, \psi_h) = \lambda \Delta t \sum_{n=1}^{m} (U^n, \psi_h),$$

and hence

$$(\alpha U^m, \psi_h) + \Delta t \sum_{n=1}^{m} A(U^n, \psi_h) = (\alpha U^0, \psi_h) + \lambda \Delta t \sum_{n=1}^{m} (U^n, \psi_h). \tag{3.2.8}$$

Integrating (2.2.8) from 0 to t, we have

$$(\alpha u_h(t), \psi_h) + \int_0^t A(u_h(s), \psi_h) ds = (\alpha u_h(0), \psi_h) + \lambda \int_0^t (u_h(s), \psi_h) ds.$$

At $t = t_m$, we obtain

$$(\alpha u_h(t_m), \psi_h) + \int_0^{t_m} A(u_h(s), \psi_h) ds = (\alpha u_h(0), \psi_h) + \lambda \int_0^{t_m} (u_h(s), \psi_h) ds. \tag{3.2.9}$$

Using (3.2.8) and (3.2.9) with $U^0 = u_h(0)$, we obtain

$$\begin{aligned}(\alpha\zeta^m, \psi_h) + \Delta t \sum_{n=1}^{m} A(\zeta^n, \psi_h) \quad &= \quad \lambda \Delta t \sum_{n=1}^{m} (\zeta^n, \psi_h) + Q_1^m(u_h)(\psi_h) \\ &\quad + Q_2^m(u_h)(\psi_h),\end{aligned} \tag{3.2.10}$$

where

$$\begin{aligned} Q_1^m(u_h)(\psi_h) \quad &= \quad \Delta t \sum_{n=1}^{m} A(u_h(t_n), \psi_h) - \int_0^{t_m} A(u_h(s), \psi_h) ds, \\ Q_2^m(u_h)(\psi_h) \quad &= \quad \int_0^{t_m} \lambda(u_h(s), \psi_h) ds - \lambda \Delta t \sum_{n=1}^{m} (u_h(t_n), \psi_h). \end{aligned}$$

Define $\widetilde{\zeta}^n = \Delta t \sum_{j=0}^{n} \zeta^j$. Clearly, $\partial_t \widetilde{\zeta}^n = \zeta^n$ and $\widetilde{\zeta}^0 = \zeta^0 = 0$. Now using the fact that $\Delta t \sum_{n=1}^{m} A(\zeta^m, \psi_h) = A(\widetilde{\zeta}^m, \psi_h)$, (3.2.10) reduces to

$$(\alpha\zeta^m, \psi_h) + A(\widetilde{\zeta}^m, \psi_h) = \lambda(\widetilde{\zeta}^m, \psi_h) + Q_1^m(u_h)(\psi_h) + Q_2^m(u_h)(\psi_h). \tag{3.2.11}$$

The main result concerning smooth data error estimate is given in the following theorem.

Theorem 3.2.1 *Let $\{u, p\}$ be the exact solution of (3.1.5) and $\{U^n, P^n\}$ be the backward Euler approximation defined by (3.1.6)-(3.1.7). Then, for $n \geq 1$ and $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\begin{aligned} \|u(t_n) - U^n\| &\leq C\left(h^2 + \Delta t\left(1 + \log\frac{1}{\Delta t}\right)^{1/2}\right) t_n^{-1/2}\|u_0\|_1 \\ &\leq C\left(h^2 + \Delta t\left(1 + \log\frac{1}{\Delta t}\right)^{1/2}\right) t_n^{-1/2}\|p_0\|_2,\ t_n > 0 \quad (3.2.12) \end{aligned}$$

and

$$\|p(t_n) - P^n\| \leq C\left(h^2 + \Delta t\left(1 + \log\frac{1}{\Delta t}\right)^{1/2}\right) t_n^{-1/2}\|p_0\|_2,\ t_n > 0. \quad (3.2.13)$$

The proof of the above theorem requires some preparatory results which are proved below in a sequence of lemmas.

Lemma 3.2.1 *Let ζ^n satisfies (3.2.6). Then there exists a positive constant C such that*

$$t_m\|\zeta^m\|^2 + \Delta t\sum_{n=1}^{m} t_n\|\zeta^n\|_1^2 \leq C\left(\Delta t\sum_{n=1}^{m} t_n^2\|\sigma^n\|^2 + \Delta t\sum_{n=1}^{m}\|\zeta^n\|^2\right).$$

Proof. Taking $\psi_h = t_n\zeta^n$ in (3.2.6) and using the identity

$$(\alpha\partial_t\zeta^n, t_n\zeta^n) = \frac{1}{2}\partial_t\{t_n\|\alpha^{1/2}\zeta^n\|^2\} + \frac{\Delta t}{2}t_n\|\alpha^{1/2}\partial_t\zeta^n\|^2 - \frac{1}{2}\|\alpha^{1/2}\zeta^{n-1}\|^2,$$

we obtain

$$\begin{aligned} \frac{1}{2}\partial_t\{t_n\|\alpha^{1/2}\zeta^n\|^2\} \quad &+ \quad A(\zeta^n, t_n\zeta^n) + \frac{\Delta t}{2}t_n\|\alpha^{1/2}\partial_t\zeta^n\|^2 = \frac{1}{2}\|\alpha^{1/2}\zeta^{n-1}\|^2 \\ &+(\sigma^n, t_n\zeta^n) + \lambda(\zeta^n, t_n\zeta^n). \end{aligned}$$

Using coercivity of $A(\cdot,\cdot)$ and applying Cauchy-Schwarz inequality to the right hand side of the above equation, we obtain

$$\begin{aligned} \frac{1}{2}\partial_t\{t_n\|\alpha^{1/2}\zeta^n\|^2\} \quad &+ \quad \frac{\Delta t}{2}t_n\|\alpha^{1/2}\partial_t\zeta^n\|^2 + c_0 t_n\|\zeta^n\|_1^2 \\ &\leq \frac{1}{2}\|\alpha^{1/2}\zeta^{n-1}\|^2 + Ct_n^2\|\sigma^n\|^2 + C\|\zeta^n\|^2 + Ct_n\|\zeta^n\|^2. \end{aligned}$$

Multiplying by $2\Delta t$ and taking summation $n = 1, \ldots, m$, we obtain

$$\begin{aligned} Ct_m\|\zeta^m\|^2 \quad &+ \quad c_0\Delta t\sum_{n=1}^{m} t_n\|\zeta^n\|_1^2 + (\Delta t)^2\sum_{n=1}^{m} t_n\|\alpha^{1/2}\partial_t\zeta^n\|^2 \leq \Delta t\sum_{n=1}^{m}\|\alpha^{1/2}\zeta^{n-1}\|^2 \\ &+C\left(\Delta t\sum_{n=1}^{m} t_n^2\|\sigma^n\|^2 + \Delta t\sum_{n=1}^{m}\|\zeta^n\|^2 + \Delta t\sum_{n=1}^{m} t_n\|\zeta^n\|^2\right), \end{aligned}$$

which may be rewritten as

$$
\begin{aligned}
(C - C\Delta t)t_m\|\zeta^m\|^2 + (\Delta t)^2 \sum_{n=1}^{m} t_n \|\alpha^{1/2}\partial_t \zeta^n\|^2 + c_0 \Delta t \sum_{n=1}^{m} t_n \|\zeta^n\|_1^2 \\
\leq C\left(\Delta t \sum_{n=1}^{m} t_n^2 \|\sigma^n\|^2 + \Delta t \sum_{n=1}^{m} \|\zeta^n\|^2 \right) + C\Delta t \sum_{n=1}^{m-1} t_n \|\zeta^n\|^2.
\end{aligned}
$$

Choose Δt sufficiently small so that $(C - C\Delta t) > 0$. Then, apply discrete Gronwall's lemma to complete the rest of the proof. ■

Lemma 3.2.2 *Let ζ^m satisfies (3.2.11). Then there exists a positive constant C such that*

$$
\Delta t \sum_{m=1}^{l} \|\zeta^m\|^2 + \|\widetilde{\zeta}^l\|_1^2 \leq |\Delta t \sum_{m=1}^{l} Q_2^m(u_h)(\zeta^m)| + |\Delta t \sum_{m=1}^{l} Q_2^m(u_h)(\zeta^m)|.
$$

Proof. Taking $\psi_h = \zeta^m$ in (3.2.11) and using the identity

$$
A(\widetilde{\zeta}^m, \partial_t \widetilde{\zeta}^m) = \frac{1}{2}\partial_t\{A(\widetilde{\zeta}^m, \widetilde{\zeta}^m)\} + \frac{\Delta t}{2} A(\partial_t \widetilde{\zeta}^m, \partial_t \widetilde{\zeta}^m),
$$

we obtain

$$
\begin{aligned}
\|\alpha^{1/2}\zeta^m\|^2 + \frac{1}{2}\partial_t\{A(\widetilde{\zeta}^m, \widetilde{\zeta}^m)\} + \frac{\Delta t}{2} A(\partial_t \widetilde{\zeta}^m, \partial_t \widetilde{\zeta}^m) = Q_1^m(u_h)(\zeta^m) \\
+ Q_2^m(u_h)(\zeta^m) + \lambda(\widetilde{\zeta}^m, \zeta^m),
\end{aligned}
$$

where we have used $\partial_t \widetilde{\zeta}^n = \zeta^n$. Multiplying by Δt and summing over $m = 1$ to $m = l$, we obtain

$$
\begin{aligned}
\Delta t \sum_{m=1}^{l} \|\zeta^m\|^2 + \frac{1}{2} A(\widetilde{\zeta}^l, \widetilde{\zeta}^l) + \frac{(\Delta t)^2}{2} \sum_{m=1}^{l} A(\zeta^m, \zeta^m) = \Delta t \sum_{m=1}^{l} Q_1^m(u_h)(\zeta^m) \\
+ \Delta t \sum_{m=1}^{l} Q_2^m(u_h)(\zeta^m) + \Delta t \sum_{m=1}^{l} \lambda(\widetilde{\zeta}^m, \zeta^m),
\end{aligned}
$$

where we have used $\widetilde{\zeta}^0 = 0$. Now, using coercivity of $A(\cdot, \cdot)$ and applying Cauchy-Schwarz inequality and Young's inequality to the right hand side of the above equation, we obtain

$$
\begin{aligned}
(1 - \epsilon C)\Delta t \sum_{m=1}^{l} \|\zeta^m\|^2 + \frac{c_0}{2}\|\widetilde{\zeta}^l\|_1^2 \;\leq\; & |\Delta t \sum_{m=1}^{l} Q_1^m(u_h)(\zeta^m)| + |\Delta t \sum_{m=1}^{l} Q_2^m(u_h)(\zeta^m)| \\
& + C(\epsilon)\Delta t \sum_{m=1}^{l} \|\widetilde{\zeta}^m\|_1^2.
\end{aligned}
$$

Choose ϵ appropriately so that $(1-\epsilon C)>0$ and hence, we obtain

$$\begin{aligned}\Delta t\sum_{m=1}^{l}\|\zeta^m\|^2+(\frac{c_0}{2}-C\Delta t)\|\widetilde{\zeta}^l\|_1^2 &\le |\Delta t\sum_{m=1}^{l}Q_1^m(u_h)(\zeta^m)|+|\Delta t\sum_{m=1}^{l}Q_2^m(u_h)(\zeta^m)|\\ &\quad +C\Delta t\sum_{m=1}^{l-1}\|\widetilde{\zeta}^m\|_1^2.\end{aligned}$$

For sufficiently small Δt, $(\frac{c_0}{2}-C\Delta t)>0$. An application of discrete Gronwall's lemma completes the rest of the proof. ■

Lemma 3.2.3 *There is a positive generic constant C such that*

$$|\Delta t\sum_{m=1}^{l}Q_1^m(u_h)(\zeta^m)|\le C(\Delta t)^2\left(1+\log\frac{1}{\Delta t}\right)\|u_0\|_1^2+\frac{1}{2}\|\widetilde{\zeta}^l\|_1^2.$$

Proof. Since $\widetilde{\zeta}^0=0$, we first note that

$$\begin{aligned}|\Delta t\sum_{m=1}^{l}Q_1^m(u_h)(\zeta^m)| &= |\Delta t\sum_{m=1}^{l}Q_1^m(u_h)(\partial_t\widetilde{\zeta}^m)|\\ &= |\Delta t\sum_{m=1}^{l}\partial_t\{Q_1^m(u_h)(\widetilde{\zeta}^m)\}|=|Q_1^l(u_h)(\widetilde{\zeta}^l)|.\end{aligned}$$

An application of the rectangular rule yields

$$\begin{aligned}|Q_1^l(u_h)(\widetilde{\zeta}^l)| &= |\sum_{j=1}^{l}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\frac{\partial}{\partial s}\{A(u_h(s),\widetilde{\zeta}^l)\}ds|\\ &\le \sum_{j=1}^{l}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\|u_{h,s}(s)\|_1\|\widetilde{\zeta}^l\|_1ds\\ &= \|\widetilde{\zeta}^l\|_1\int_0^{t_1}s\|u_{h,s}\|_1ds+\|\widetilde{\zeta}^l\|_1\sum_{j=2}^{l}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\|u_{h,s}(s)\|_1ds\\ &\le \|\widetilde{\zeta}^l\|_1\left(\int_0^{t_1}sds\right)^{1/2}\left(\int_0^{t_1}s\|u_{h,s}\|_1^2ds\right)^{1/2}+\Delta t\|\widetilde{\zeta}^l\|_1\sum_{j=2}^{l}\int_{t_{j-1}}^{t_j}\|u_{h,s}\|_1ds\\ &\le C\Delta t\|\widetilde{\zeta}^l\|_1\left(\int_0^{t_1}s\|u_{h,s}\|_1^2ds\right)^{1/2}\\ &\quad +\Delta t\|\widetilde{\zeta}^l\|_1\left(\sum_{j=2}^{l}\int_{t_{j-1}}^{t_j}\frac{1}{s}ds\right)^{1/2}\left(\sum_{j=2}^{l}\int_{t_{j-1}}^{t_j}s\|u_{h,s}\|_1^2ds\right)^{1/2}\\ &\le C\Delta t\|\widetilde{\zeta}^l\|_1\|u_0\|_1+C\Delta t\|\widetilde{\zeta}^l\|_1\left(\log\frac{1}{\Delta t}\right)^{1/2}\|u_0\|_1\\ &\le C(\Delta t)^2\|u_0\|_1^2+C(\Delta t)^2\left(\log\frac{1}{\Delta t}\right)\|u_0\|_1^2+\frac{1}{2}\|\widetilde{\zeta}^l\|_1^2,\end{aligned}$$

where in the sixth step we have used Lemma 2.2.5 and this completes the proof. ■

Lemma 3.2.4 *There is a positive generic constant C such that*

$$|\Delta t \sum_{m=1}^{l} Q_2^m(u_h)(\zeta^m)| \leq C(\Delta t)^2\|u_0\|_1^2 + \frac{\Delta t}{2}\sum_{m=1}^{l}\|\zeta^m\|^2.$$

Proof. An application of the rectangular rule leads to

$$\begin{aligned}
|Q_2^m(u_h)(\zeta^m)| &= |\lambda \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}(t_{j-1}-s)\partial/\partial s(u_h(s),\zeta^m)ds| \\
&\leq \lambda \sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\|u_{h,s}(s)\|\|\zeta^m\|ds \\
&\leq C\Delta t\|\zeta^m\|\sum_{j=1}^{m}\int_{t_{j-1}}^{t_j}\|u_{h,s}(s)\|ds \\
&\leq C\Delta t\|\zeta^m\|\sum_{j=1}^{m}(\Delta t)^{1/2}\left(\int_{t_{j-1}}^{t_j}\|u_{h,s}\|^2ds\right)^{1/2} \\
&\leq C\Delta t\|\zeta^m\|\left(\int_0^{t_m}\|u_{h,s}\|^2ds\right)^{1/2} \\
&\leq C\Delta t\|\zeta^m\|\;\|u_0\|_1 \\
&\leq C(\Delta t)^2\|u_0\|_1^2 + \frac{1}{2}\|\zeta^m\|^2, \qquad (3.2.14)
\end{aligned}$$

where in the sixth step we have used Lemma 2.2.5. Now, multiplying by Δt and taking sum $m = 1, \ldots, l$ yields the desired estimate. ■

Lemma 3.2.5 *Let ζ^m satisfies (3.2.11). Then there exists a positive generic constant C such that*

$$\Delta t \sum_{m=1}^{l}\|\zeta^m\|^2 + \|\widetilde{\zeta}^l\|_1^2 \leq C(\Delta t)^2\left(1+\log\frac{1}{\Delta t}\right)\|u_0\|_1^2.$$

Proof. The desired result follows from Lemma 3.2.2, Lemma 3.2.3 and Lemma 3.2.4, and this completes the proof. ■

Lemma 3.2.6 *Let $u_0 \in V$. Then there exists a positive generic constant C such that*

$$\sum_{n=1}^{m} t_n^2\|\sigma^n\|^2 \leq C\Delta t\|u_0\|_1^2.$$

Proof. We write σ^n as

$$\sigma^n = \frac{\alpha}{\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})u_{h,ss}(s)ds.$$

Note that

$$\begin{aligned}\|\sigma^n\| &\le \frac{C}{\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})\|u_{h,ss}(s)\|ds\\ &\le \frac{C}{(\Delta t)^{1/2}}\left(\int_{t_{n-1}}^{t_n}(s-t_{n-1})^2\|u_{h,ss}(s)\|^2ds\right)^{1/2}.\end{aligned}$$

Hence,

$$\|\sigma^n\|^2 \le \frac{C}{\Delta t}\int_{t_{n-1}}^{t_n}(s-t_{n-1})^2\|u_{h,ss}(s)\|^2ds.$$

Multiplying by t_n^2 and summing $n=1,\ldots,m$, we have

$$\begin{aligned}\sum_{n=1}^{m} t_n^2\|\sigma^n\|^2 &\le \frac{C}{\Delta t}\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} t_n^2(s-t_{n-1})^2\|u_{h,ss}(s)\|^2ds\\ &\le C\Delta t\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} s^2\|u_{h,ss}(s)\|^2ds\\ &\le C\Delta t\int_0^{t_m} s^2\|u_{h,ss}\|^2ds \le C\Delta t\|u_0\|_1^2,\end{aligned}$$

where in the second step we have used $t_n^2(s-t_{n-1})^2 \le s^2(\Delta t)^2$ and in the third step Lemma 2.2.5. This completes the proof. ■

Lemma 3.2.7 *Let ζ^m satisfies (3.2.6). Then there exists a positive generic constant C such that*

$$\|\zeta^m\| \le C\Delta t\left(1+\log\frac{1}{\Delta t}\right)^{1/2} t_m^{-1/2}\|u_0\|_1,\ t_m>0.$$

Proof. Combining Lemma 3.2.1 with Lemma 3.2.5 and Lemma 3.2.6, we arrive at the desired estimate. ■

Proof of Theorem 3.2.1. We rewrite $u(t_n)-U^n$ as

$$u(t_n)-U^n = (u(t_n)-u_h(t_n))+\zeta^n,$$

By triangle inequality, we have

$$\|u(t_n)-U^n\| \le \|u(t_n)-u_h(t_n)\|+\|\zeta^n\|. \tag{3.2.15}$$

Using the estimate (2.2.31) at $t=t_n$ and Lemma 3.2.7 in (3.2.15) leads to (3.2.12). To estimate (3.2.13), we can rewrite $p(t_n)-P^n$ as

$$p(t_n)-P^n = (p(t_n)-p_h(t_n))+\xi^n.$$

Again, using triangle inequality, we have

$$\|p(t_n) - P^n\| \leq \|p(t_n) - p_h(t_n)\| + \|\xi^n\|. \tag{3.2.16}$$

From (3.2.7), we have

$$\begin{aligned} (\xi^n_x, \phi_{hx}) &= (\alpha\zeta^n, \phi_{hx}) \\ &\leq C\|\zeta^n\| \, \|\phi_{hx}\|, \end{aligned}$$

and hence,

$$\|\xi^n_x\| \leq C\|\zeta^n\|.$$

An application of Lemma 3.2.7 now leads to

$$\|\xi^n_x\| \leq C\|\zeta^n\| \leq C\Delta t(1 + \log \frac{1}{\Delta t})^{1/2} t_n^{-1/2} \|p_0\|_2.$$

As $\xi^n \in H^1_0(\Omega)$, use Poincaré's inequality to have

$$\|\xi^n\| \leq C\|\zeta^n\| \leq C\Delta t(1 + \log \frac{1}{\Delta t})^{1/2} t_n^{-1/2} \|p_0\|_2. \tag{3.2.17}$$

Now, using the estimate (2.2.32) at $t = t_n$ and (3.2.17) in (3.2.16) proves (3.2.13). This completes the proof of Theorem 3.2.1 . ■

3.3 Error Estimates with Nonsmooth Initial Data

This section is devoted to the L^2-error estimates of $u(t_n) - U^n$ and $p(t_n) - P^n$ when the initial function $p_0 \in H^1_0(\Omega)$. The main tool used in our error analysis is the following discrete parabolic duality argument: For any fixed time $t_n > 0$ and any function $G \in V_h$, define $\{v_h(s), q_h(s)\} \in V_h \times W_h$ and $\{(V^m)^n_{m=0}, (Q^m)^n_{m=0}\} \in V_h \times W_h$ to be the continuous and discrete solutions to the following mixed problems:
Continuous case: Find $\{v_h, q_h\} : [0, T] \longrightarrow V_h \times W_h$ such that

$$(\alpha v_{h,s}, \psi_h) - A(v_h, \psi_h) = -\lambda(v_h, \psi_h), \ \forall \psi_h \in V_h, \ s \leq t_n, \tag{3.3.1}$$

$$(q_{hx}, \phi_{hx}) = (\alpha v_h, \phi_{hx}), \ \forall \phi_h \in W_h \tag{3.3.2}$$

with $v_h(t_n) = G$.
Discrete case: Find $\{V^m, Q^m\} \in V_h \times W_h$ such that

$$(\alpha \partial_t V^m, \psi_h) - A(V^{m-1}, \psi_h) = -\lambda(V^{m-1}, \psi_h), \ \forall \psi_h \in V_h, \ m = n, \ldots, 1 \tag{3.3.3}$$

$$(Q^m_x, \phi_{hx}) = (\alpha V^m, \phi_{hx}), \ \forall \phi_h \in W_h \tag{3.3.4}$$

with $V^n = G$.
The main result for nonsmooth data error estimate is given in the following theorem.

Theorem 3.3.1 *Let $\{u, p\}$ be the exact solution of (3.1.5) and $\{U^n, P^n\}$ be the backward Euler approximation defined by (3.1.6)-(3.1.7). Then, for $n \geq 1$ and $p_0 \in H_0^1(\Omega)$, with $u_h(0) = L_h u_0$, we have*

$$\|u(t_n) - U^n\| \leq C \left(h^2 + \Delta t \left(1 + \log \frac{1}{\Delta t} \right)^{1/2} \right) t_n^{-1} \|p_0\|_1, \ t_n > 0, \tag{3.3.5}$$

and

$$\|p(t_n) - P^n\| \leq C \left(h^2 + \Delta t \left(1 + \log \frac{1}{\Delta t} \right)^{1/2} \right) t_n^{-1} \|p_0\|_1, \ t_n > 0. \tag{3.3.6}$$

The proof of the theorem requires some preparations. Using (2.2.8) and (3.3.1), we note that

$$\frac{d}{ds}\{\alpha u_h, v_h\} = (\alpha u_{h,s}, v_h) + (\alpha u_h, v_{h,s}) = 0. \tag{3.3.7}$$

Further, using (3.1.6) and (3.3.3), we have

$$\begin{aligned} \partial_t(\alpha U, V)^m &= \frac{(\alpha U, V)^m - (\alpha U, V)^{m-1}}{\Delta t} \\ &= \frac{(\alpha U^m, V^m) - (\alpha U^{m-1}, V^{m-1})}{\Delta t} \\ &= \frac{(\alpha U^m, V^m) - (\alpha U^m, V^{m-1}) + (\alpha U^m, V^{m-1}) - (\alpha U^{m-1}, V^{m-1})}{\Delta t} \\ &= (U^m, \alpha \partial_t V^m) + (\alpha \partial_t U^m, V^{m-1}) \\ &= (V_x^{m-1}, U_x^m) - (U_x^m, V_x^{m-1}) = 0. \end{aligned} \tag{3.3.8}$$

The discrete analogue of (2.2.47) reads:

$$\partial_t \{(\alpha u_h, v_h) - (\alpha U, V)\}^m = 0. \tag{3.3.9}$$

Summing over $m = 1, \ldots, n$, (3.3.9) leads to

$$\{(\alpha u_h, v_h) - (\alpha U, V)\}^n - \{(\alpha u_h, v_h) - (\alpha U, V)\}^0 = 0,$$

With $v_h(t_n) = L_h V(t_n) = L_h G$ and $u_h(0) = L_h u_0$, we obtain

$$(\alpha u_h^n, L_h V(t_n)) - (\alpha U^n, V^n) = (\alpha u_h(0), v_h(0)) - (\alpha U^0, V^0),$$

and hence,

$$(\alpha(u_h^n - U^n), G) = (\alpha u_h(0), v_h(0) - V^0). \tag{3.3.10}$$

With $\bar{\zeta}^n = v_h(t_n) - V^n$, $n \leq m$, and the error associated with discrete backward problem we have

$$|(\alpha \zeta^n, G)| \leq c\|u_h(0)\| \, \|\bar{\zeta}^0\|, \tag{3.3.11}$$

where $\bar{\zeta}^0 = v_h(0) - V^0$. Now, apply Lemma 3.2.7 to $\bar{\zeta}^n = v_h(t_n) - V^n$ with time reversed to obtain

$$\begin{aligned} \|\bar{\zeta}^p\| &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} (t_n - t_p)^{-1/2} \|v(t_n)\|_1, \; t_p \leq t_n, \\ &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} (t_n - t_p)^{-1/2} \|G\|_1. \end{aligned}$$

At $t_p = t_0$, we have

$$\|\bar{\zeta}^0\| \leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2} \|G\|_1. \tag{3.3.12}$$

The following lemmas prove to be convenient for nonsmooth data error estimate.

Lemma 3.3.1 *With $\zeta^n = u_h(t_n) - U^n$, we have*

$$\|\zeta^n\|_{V^*} \leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2} \|p_0\|_1, \; n \geq 1.$$

Proof. From (3.3.11) and (3.3.12), it follows that

$$\begin{aligned} |(\zeta^n, \alpha G)| &\leq C\|u_h(0)\| \, \|\bar{\zeta}^0\| \\ &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2} \|G\|_1 \, \|u_h(0)\| \\ &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2} \|\alpha G\|_1 \, \|u_0\|, \end{aligned}$$

where we have used $u_h(0) = L_h u_0$ and the fact $\|u_h(0)\| \leq C\|u_0\|$ to obtain the desired estimate. This completes the proof. ∎

Lemma 3.3.2 *With $\zeta^n = u_h(t_n) - U^n$, we have*

$$\|\zeta^n\| \leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1} \|p_0\|_1.$$

Proof. By summing (3.3.9) for $m = r+1, \ldots, n$, where $r = [\frac{n}{2}]$, we have

$$(\alpha u_h^n, v_h^n) - (\alpha U^n, V^n) = (\alpha u_h^r, v_h^r) - (\alpha U^r, V^r).$$

With $v_h^n = L_h G$, it follows that

$$(\alpha \zeta^n, G) = (\alpha \zeta^r, v_h^r) + (\alpha U^r, \bar{\zeta}^r). \tag{3.3.13}$$

Thus,

$$|(\alpha \zeta^n, G)| \leq C\|\zeta^r\|_{V^*} \, \|v_h^r\|_1 + C\|\bar{\zeta}^r\|_{V^*} \, \|U^r\|_1. \tag{3.3.14}$$

Lemma 3.3.1 applied to the backward error $\bar{\zeta}^r$ to have

$$\|\bar{\zeta}^r\|_{V^*} \leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2}\|G\|. \tag{3.3.15}$$

Lemma 3.3.1 with $t = t_r$ yields

$$\|\zeta^r\|_{V^*} \leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2}\|u_0\|. \tag{3.3.16}$$

Using *a priori* estimates

$$\|U^r\|_1 \leq Ct_n^{-1/2}\|u_0\|,$$

and

$$\|v_h^r\|_1 \leq Ct_n^{-1/2}\|G\|,$$

with (3.3.15) and (3.3.16), we obtain

$$\begin{aligned} |(\zeta^n, \alpha G)| &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2}\|u_0\| \, \|G\| \\ &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2}\|u_0\| \, \|\alpha G\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|\zeta^n\| &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2}\|u_0\| \\ &\leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1/2}\|p_0\|_1, \end{aligned}$$

and this completes the proof of Lemma 3.3.2. ■

Proof of Theorem 3.3.1. As before, rewriting $u(t_n) - U^n$ as

$$u(t_n) - U^n = (u(t_n) - u_h(t_n)) + \zeta^n.$$

By triangle inequality, we have

$$\|u(t_n) - U^n\| \leq \|u(t_n) - u_h(t_n)\| + \|\zeta^n\|. \tag{3.3.17}$$

Using the estimate (2.2.45) at $t = t_n$ and Lemma 3.3.2 in (3.3.17) yields (3.3.5). To estimate (3.3.6), we can rewrite $p(t_n) - P^n$ as

$$p(t_n) - P^n = (p(t_n) - p_h(t_n)) + \xi^n.$$

Again, using triangle inequality, we have

$$\|p(t_n) - P^n\| \leq \|p(t_n) - p_h(t_n)\| + \|\xi^n\|. \tag{3.3.18}$$

From (3.2.7), we have

$$\begin{aligned}(\xi_x^n, \phi_{hx}) &= (\alpha\zeta^n, \phi_{hx}) \\ &\leq C\|\zeta^n\| \, \|\phi_{hx}\|,\end{aligned}$$

and hence,

$$\|\xi_x^n\| \leq C\|\zeta^n\|.$$

An application of Lemma 3.3.2 leads to

$$\|\xi_x^n\| \leq C\|\zeta^n\| \leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1}\|p_0\|_1.$$

As $\xi^n \in H_0^1(\Omega)$, apply Poincaré's inequality to have

$$\|\xi^n\| \leq C\|\xi_x^n\| \leq C\Delta t \left(1 + \log \frac{1}{\Delta t}\right)^{1/2} t_n^{-1}\|p_0\|_1. \tag{3.3.19}$$

Combing (3.3.18), (3.3.19) and (2.2.46) we obtain (3.3.6). This completes the proof of Theorem 3.3.1. ■

Remark. In this chapter, we discuss a fully discrete scheme based on backward Euler method for H^1-Galerkin mixed finite element method for one-dimensional homogeneous parabolic problems with both smooth and nonsmooth initial data. More precisely, almost optimal order error estimates of order $\mathcal{O}((h^2 + \Delta t(1 + \log \frac{1}{\Delta t})^{1/2})t_n^{-1/2})$ in the L^2-norm are established for the solution p and the flux u with initial function $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Further, error estimates of order $\mathcal{O}((h^2 + \Delta t(1 + \log \frac{1}{\Delta t})^{1/2})t_n^{-1})$ with positive time are derived in the L^2-norm for both the solution p and its flux u with initial data $p_0 \in H_0^1(\Omega)$. The method is not subject to LBB-consistency condition and we do not require quasiuniformity condition on the finite element mesh.

Chapter 4

Superconvergence of H^1-Galerkin MFEM for Parabolic Problems

In this chapter, we study superconvergence phenomenon for the semidiscrete H^1-Galerkin MFEM for parabolic problems (1.1.1)-(1.1.3). The well-known optimal order error estimate in the L^2-norm for the flux is of order $\mathcal{O}(h^{k+1})$, where $k \geq 1$ is the order of the approximating polynomials employed in the Raviart-Thomas element [60]. We derive superconvergence estimate of order $\mathcal{O}(h^{k+3})$ between the H^1-Galerkin mixed finite element approximation and an appropriately defined local projection of the flux variable when $k \geq 1$. A new approximate solution for the flux with superconvergence of order $\mathcal{O}(h^{k+3})$ is realized via a postprocessing technique using local projection.

4.1 Introduction

Recalling the parabolic problem:

$$p_t - \nabla \cdot (a \nabla p) = f(\mathbf{x}, t), \ (\mathbf{x}, t) \in \Omega \times J, \tag{4.1.1}$$

$$p(\mathbf{x}, t) = 0, \ (\mathbf{x}, t) \in \partial\Omega \times J, \tag{4.1.2}$$

$$p(\mathbf{x}, 0) = p_0(\mathbf{x}), \ \mathbf{x} \in \Omega, \tag{4.1.3}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $J = (0, T]$ with $T < \infty$ and $p_t = \frac{\partial p}{\partial t}$. The coefficient matrix $a = a(\mathbf{x})$ is assumed to be smooth, symmetric and uniformly positive definite in Ω. The initial function $p_0(\mathbf{x})$ and the source function $f(\mathbf{x}, t)$ are assumed to be smooth for our purpose. Our main objective is to investigate superconvergence phenomena for the H^1-Galerkin mixed finite element method for the problem (4.1.1)-(4.1.3).

Optimal order error estimate is the best that one can get between the exact solution and its numerical approximation when measured globally on the computational domain. But, there are places (points or lines) where the approximate solution is more closer to the exact solution than what is predicted by the global optimal order error estimate [32, 36]. It would be advantageous to make use of those points or lines in the modelling process. Superconvergence results are important from an application point of view because they provide higher order accuracy under reasonable assumption on the grid and with additional smoothness of the solution.

Our analysis for the superconvergence results is based on the treatment of the linear functionals of the forms

$$\mathcal{F}(\mathbf{v}) = (\mathbf{u} - \pi_h \mathbf{u}, \mathbf{v}), \tag{4.1.4}$$

and

$$\widetilde{\mathcal{F}}(\mathbf{v}) = ((\mathbf{u} - \pi_h \mathbf{u})_t, \mathbf{v}), \tag{4.1.5}$$

where $\pi_h \mathbf{u}$ is an appropriately defined local projection of $\mathbf{u}$ [14, 24] and $\mathbf{v}$ be any finite element function. These linear forms are estimated by expanding the interpolation errors $\mathbf{u} - \pi_h \mathbf{u}$ and $(\mathbf{u} - \pi_h \mathbf{u})_t$ in Taylor series involving only finite number of terms. Each of the terms in the Taylor expansion is a polynomial. The orthogonality property of $\mathbf{u} - \pi_h \mathbf{u}$ and $(\mathbf{u} - \pi_h \mathbf{u})_t$ with certain class of polynomials play crucial role for deriving superconvergence result.

This chapter is organized as follows. Section 4.2 deals with some introductory material for the H^1-Galerkin MFEM. In Section 4.3, we derive some auxiliary estimates useful for analyzing the linear forms. Superconvergence results for the H^1-Galerkin MFEM are obtained in Section 4.4.

Throughout this chapter, C denotes a positive generic constant which is independent of the mesh parameter h and may not be the same at each occurrence.

4.2 Mixed Finite Element Discretization

With the flux variable $\mathbf{u} = a\nabla p$, problem (4.1.1) can be rewritten in the form of a first-order system as

$$p_t - \nabla \cdot \mathbf{u} = f, \ (\mathbf{x}, t) \in \Omega \times J, \tag{4.2.1}$$

$$\alpha \mathbf{u} - \nabla p = 0, \ (\mathbf{x}, t) \in \Omega \times J, \tag{4.2.2}$$

where $\alpha = 1/a$. Here, for our analysis we shall need the following spaces:

$$\mathbf{V} = \{\mathbf{v} \in (L^2(\Omega))^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{V}} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{1/2},$$

and $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$ with usual Sobolev norm. The H^1-Galerkin mixed formulation is stated as: Find $\{\mathbf{u}, p\} : [0,T] \longrightarrow \mathbf{V} \times H_0^1(\Omega)$ such that

$$(\alpha \mathbf{u}_t, \Psi) + A(\mathbf{u}, \Psi) = -(f, \nabla \cdot \Psi) + \lambda(\mathbf{u}, \Psi), \ \forall \Psi \in \mathbf{V}, \qquad (4.2.3)$$

$$(\nabla p, \nabla \phi) = (\alpha \mathbf{u}, \nabla \phi), \ \forall \phi \in H_0^1(\Omega) \qquad (4.2.4)$$

with $\mathbf{u}(0) = a\nabla p_0$. For the first term in (4.2.3), we have used integration by parts and the Dirichlet boundary condition $p_t(\mathbf{x}, t) = 0$ on $\partial\Omega$. The bilinear form $A(\cdot,\cdot)$ is given by

$$A(\mathbf{u}, \Psi) = (\nabla \cdot \mathbf{u}, \nabla \cdot \Psi) + \lambda(\mathbf{u}, \Psi).$$

Note that λ is chosen appropriately so that $A(\cdot,\cdot)$ is $\mathbf{V}$-coercive, i.e.,

$$A(\mathbf{v}, \mathbf{v}) \geq c_1 \|\mathbf{v}\|_{\mathbf{V}}^2, \quad \mathbf{v} \in \mathbf{V},$$

for some $c_1 > 0$. Moreover, $A(\cdot,\cdot)$ is bounded. That is, there is a positive constant C such that $|A(\mathbf{u}, \mathbf{v})| \leq C\|\mathbf{u}\|_{\mathbf{V}}\|\mathbf{v}\|_{\mathbf{V}}$.

In this work, we have considered a special case of the finite element partition $\widehat{\mathcal{T}}_h$ of Ω consisting of rectangular elements only. This means that the domain Ω have to be made of rectangular subdomains with boundaries parallel to either the x-axis or y-axis. Let $\mathbf{V}_h$ and W_h, respectively, be finite dimensional subspaces of $\mathbf{V}$ and $H_0^1(\Omega)$ associated with a rectangular finite element partition $\widehat{\mathcal{T}}_h$ for the domain Ω. The Raviart-Thomas space $\mathbf{V}_h$ and the standard finite dimensional space W_h are defined as follows (cf. [14, 70]):

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{V} \ : \ \mathbf{v}|_e \in Q_{k+1,k} \times Q_{k,k+1}, \ e \in \widehat{\mathcal{T}}_h\},$$

$$W_h := \{w \in \mathcal{C}(\Omega) \ : \ w|_e \in Q_{k,k}, \ e \in \widehat{\mathcal{T}}_h, \ w = 0 \text{ on } \partial\Omega\},$$

where $Q_{r,s}$ be the space of polynomials with degree no more than r in the x direction and no more than s in the y direction and $k \geq 1$. The semidiscrete H^1-Galerkin mixed finite element approximation is thus defined as follows: Find $\{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times W_h$ such that

$$(\alpha \mathbf{u}_{h,t}, \Psi_h) + A(\mathbf{u}_h, \Psi_h) = -(f, \nabla \cdot \Psi_h) + \lambda(\mathbf{u}_h, \Psi_h), \ \forall \Psi_h \in \mathbf{V}_{0,h}, \qquad (4.2.5)$$

$$(\nabla p_h, \nabla \phi_h) = (\alpha \mathbf{u}_h, \nabla \phi_h), \ \forall \phi_h \in W_h \qquad (4.2.6)$$

with a suitable choice of $\mathbf{u}_h(0)$ to be defined later. Here, the subspace $\mathbf{V}_{0,h}$ consists of all finite element functions which have a varnishing component on the boundary $\partial\Omega$ in the normal direction. For our analysis, we now define the following auxiliary projection $\pi_h : [0, T] \longrightarrow \mathbf{V}_{0,h}$ satisfying

$$A(\mathbf{v} - \pi_h \mathbf{v}, \Psi_h) = 0, \ \forall \ \Psi_h \in \mathbf{V}_{0,h}. \tag{4.2.7}$$

To construct such a projection operator π_h, let $e \in \widehat{\mathcal{T}}_h$ be any rectangular element. For any sufficiently smooth vector-valued function $\mathbf{v} \in \mathbf{V}$, define its local projection $\pi_h^e \mathbf{v} \in Q_{k+1,k} \times Q_{k,k+1}$ over each element by using the following system of linear equations [14]:

$$\int_{l_i} (\mathbf{v} - \pi_h^e \mathbf{v}) \cdot \mathbf{n}_i \phi ds = 0, \ \phi \in \mathcal{P}_k(l_i), \ i = 1, 2, 3, 4,$$
$$\int_e (\mathbf{v} - \pi_h^e \mathbf{v}) \cdot \Psi dxdy = 0, \ \Psi \in Q_{k-1,k}(e) \times Q_{k,k-1}(e),$$

where $\{l_i, i = 1, 2, 3, 4\}$ stands for the edges of the element e. Using the local operator π_h^e, we can define the global projection operator $\pi_h : [0, T] \longrightarrow \mathbf{V}_{0,h}$ by setting

$$(\pi_h \mathbf{v})(\mathbf{x}, t) = \pi_h^e \mathbf{v}(\mathbf{x}, t), \ \forall \ \mathbf{x} = (x, y) \in e, \ e \in \widehat{\mathcal{T}}_h, \tag{4.2.8}$$

which also satisfies the relation (4.2.7). The operator π_h can be split into two components as follows:

$$\pi_h \mathbf{v} = (\pi_1 v_1, \pi_2 v_2),$$

where π_1 and π_2 are defined independently to each other.

In this chapter, we shall make use of the following notations. Denote by

$$|\mathbf{u}|_{m,q,\Omega} = \sum_{|\alpha|=m} \|D^\alpha \mathbf{u}\|_{L^q(\Omega)}$$

the seminorm in the Sobolev space $W^{m,q}(\Omega)$ with integer $m \geq 0$ and real number $q \geq 1$. In addition, we denote

$$|\phi|_{m,q,h} = \sum_{|\alpha|=m} \left(\sum_{e \in \widehat{\mathcal{T}}_h} \|D^\alpha \phi\|^q_{L^q(e)} \right)^{1/q}$$

as discrete seminorm for any piecewise polynomials $\phi = \phi(x, y)$. For the purpose of our error analysis, we define the dual norm by

$$\|\mathbf{v}\|_{\mathbf{V}^*} = \sup_{\Psi_h \in \mathbf{V}_{0,h}} \frac{(\mathbf{v}, \Psi_h)}{\|\Psi_h\|_{\mathbf{V}}},$$

where $\mathbf{V}^*$ is the dual space of $\mathbf{V}_{0,h}$.

4.3 A Framework for Superconvergence

In this section, we present a framework for superconvergence in H^1-Galerkin MFEM. There are two major steps involved in our analysis. In the first step, we compare the H^1-Galerkin mixed finite element approximation with an appropriately chosen interpolation of the exact solution in the finite element space. This difference is often smaller than the global optimal error estimate. In the second step, we investigate the relation between the exact solution and its interpolation. The interpolant is usually locally defined so that the second step is easily carried out. Using (4.2.3), (4.2.5) and the auxiliary projection (4.2.7), we have the following error equation:

$$\begin{aligned}(\alpha(\pi_h\mathbf{u}-\mathbf{u}_h)_t,\Psi_h)+A(\pi_h\mathbf{u}-\mathbf{u}_h,\Psi_h) &= -(\alpha(\mathbf{u}-\pi_h\mathbf{u})_t,\Psi_h)\\ &+\lambda(\mathbf{u}-\pi_h\mathbf{u},\Psi_h)+\lambda(\pi_h\mathbf{u}-\mathbf{u}_h,\Psi_h) \qquad (4.3.1)\end{aligned}$$

for all $\Psi_h \in \mathbf{V}_{0,h}$. Now, we shall prove the following lemma.

Lemma 4.3.1 *Let $\{\mathbf{u},p\}$ and $\{\mathbf{u}_h,p_h\}$ be the solutions of the mixed problems (4.2.3)-(4.2.4) and (4.2.5)-(4.2.6), respectively. Further, let $\mathbf{u}_h(0)=\pi_h\mathbf{u}_0$. Then there is a positive constant C independent of h such that*

$$\begin{aligned}\|(\pi_h\mathbf{u}-\mathbf{u}_h)(t)\|^2 &+ \int_0^t \|\pi_h\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{V}}ds\\ &\leq C\left\{\int_0^t \|\mathbf{u}-\pi_h\mathbf{u}\|^2_{\mathbf{V}^*}ds+\int_0^t \|(\mathbf{u}-\pi_h\mathbf{u})_t\|^2_{\mathbf{V}^*}ds\right\}. \quad (4.3.2)\end{aligned}$$

Proof. Choose $\Psi_h=\pi_h\mathbf{u}-\mathbf{u}_h$ in (4.3.1) and use the coercivity of $A(\cdot,\cdot)$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|\alpha^{1/2}(\pi_h\mathbf{u}-\mathbf{u}_h)\|^2\} &+ c_1\|\pi_h\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{V}} \leq C\|(\mathbf{u}-\pi_h\mathbf{u})_t\|_{\mathbf{V}^*}\|\pi_h\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}}\\ &+\lambda\|\mathbf{u}-\pi_h\mathbf{u}\|_{\mathbf{V}^*}\|\pi_h\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}}+\lambda\|\pi_h\mathbf{u}-\mathbf{u}_h\|^2\\ &\leq C(\epsilon)\left(\|\mathbf{u}-\pi_h\mathbf{u}\|^2_{\mathbf{V}^*}+\|(\mathbf{u}-\pi_h\mathbf{u})_t\|^2_{\mathbf{V}^*}\right)+\epsilon C\|\pi_h\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{V}}.\end{aligned}$$

Rewriting the above inequality as

$$\frac{1}{2}\frac{d}{dt}\{\|\alpha^{1/2}(\pi_h\mathbf{u}-\mathbf{u}_h)\|^2\}+(c_1-\epsilon C)\|\pi_h\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{V}} \leq C(\epsilon)\left(\|\mathbf{u}-\pi_h\mathbf{u}\|^2_{\mathbf{V}^*}+\|(\mathbf{u}-\pi_h\mathbf{u})_t\|^2_{\mathbf{V}^*}\right).$$

Choose ϵ appropriately so that $(c_1-\epsilon C)>0$. Integrating from 0 to t and using $\mathbf{u}_h(0)=\pi_h\mathbf{u}_0$, we obtain

$$\|(\pi_h\mathbf{u}-\mathbf{u}_h)(t)\|^2+\int_0^t \|\pi_h\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{V}}ds \leq C\int_0^t \{\|(\mathbf{u}-\pi_h\mathbf{u})\|^2_{\mathbf{V}^*}+\|(\mathbf{u}-\pi_h\mathbf{u})_t\|^2_{\mathbf{V}^*}\}\,ds.$$

This completes the proof. ■

Next, our objective is to estimate the terms $\|\mathbf{u}-\pi_h\mathbf{u}\|_{\mathbf{V}^*}$ and $\|(\mathbf{u}-\pi_h\mathbf{u})_t\|_{\mathbf{V}^*}$ separately. For the vector valued function $\Psi_h \in \mathbf{V}_{0,h}$ with $\Psi_h = (\psi_{1h}, \psi_{2h})$, we note that

$$\begin{aligned}
\mathcal{F}(\Psi_h) &= (\mathbf{u}-\pi_h\mathbf{u}, \Psi_h) \\
&= (u_1 - \pi_1 u_1, \psi_{1h}) + (u_2 - \pi_2 u_2, \psi_{2h}) \\
&= \sum_{e\in\widehat{\mathcal{T}}_h} (u_1 - \pi_1 u_1, \psi_{1h})_e + \sum_{e\in\widehat{\mathcal{T}}_h} (u_2 - \pi_2 u_2, \psi_{2h})_e \\
&:= \sum_{e\in\widehat{\mathcal{T}}_h} (\rho_1(t), \psi_{1h})_e + \sum_{e\in\widehat{\mathcal{T}}_h} (\rho_2(t), \psi_{2h})_e,
\end{aligned} \tag{4.3.3}$$

where $\rho_1(t) = (u_1 - \pi_1 u_1)(t)$ and $\rho_2(t) = (u_2 - \pi_2 u_2)(t)$.
Similarly,

$$\begin{aligned}
\widetilde{\mathcal{F}}(\Psi_h) &= ((\mathbf{u}-\pi_h\mathbf{u})_t, \Psi_h) \\
&= ((u_1 - \pi_1 u_1)_t, \psi_{1h}) + ((u_2 - \pi_2 u_2)_t, \psi_{2h}) \\
&= \sum_{e\in\widehat{\mathcal{T}}_h} ((u_1 - \pi_1 u_1)_t, \psi_{1h})_e + \sum_{e\in\widehat{\mathcal{T}}_h} ((u_2 - \pi_2 u_2)_t, \psi_{2h})_e \\
&:= \sum_{e\in\widehat{\mathcal{T}}_h} (\rho_{1,t}(t), \psi_{1h})_e + \sum_{e\in\widehat{\mathcal{T}}_h} (\rho_{2,t}(t), \psi_{2h})_e.
\end{aligned} \tag{4.3.4}$$

where $\rho_{1,t}(t) = (u_1 - \pi_1 u_1)_t(t)$ and $\rho_{2,t}(t) = (u_2 - \pi_2 u_2)_t(t)$.

We now recall from [36] the following Lemmas. These lemmas will be useful to estimate the terms appearing on the right hand side of (4.3.3) and (4.3.4).

Lemma 4.3.2 *Let $e \in \widehat{\mathcal{T}}_h$ be a rectangular element with $e = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Let ψ be a sufficiently smooth function defined on e and satisfy*

$$\int_{l_i} \psi dy = 0, \ i = 1, 3, \tag{4.3.5}$$

$$\int_e x^j \psi dx dy = 0, \ 0 \leq j \leq \widetilde{k} - 1. \tag{4.3.6}$$

Then, for any integer $m \leq \widetilde{k} + 1$, we have

$$\int_e (x - x_e)^m \psi dx dy = \frac{(-1)^m m!}{(2m+2)!} \int_e E^{m+1}(x) \partial_x^{m+2} \psi dx dy, \tag{4.3.7}$$

where l_1 and l_3 are two vertical edges of the element e, $2h_e = x_{i+1} - x_i$ is the length of l_2, (x_e, y_e) is the center of e, and $E(x) = (x - x_e)^2 - h_e^2$.

Lemma 4.3.3 *Let $e \in \widehat{\mathcal{T}}_h$ be a rectangular element with $e = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Let $\tilde{\psi}$ be a sufficiently smooth function defined on e and satisfy*

$$\int_{l_i} \tilde{\psi} dx = 0, \ i = 2, 4, \tag{4.3.8}$$

$$\int_e y^j \tilde{\psi} dx dy = 0, \ 0 \leq j \leq \widetilde{k} - 1. \tag{4.3.9}$$

Then, for any integer $m \leq \widetilde{k} + 1$, we have

$$\int_e (y - y_e)^m \tilde{\psi} dx dy = \frac{(-1)^m m!}{(2m+2)!} \int_e E^{m+1}(y) \partial_y^{m+2} \tilde{\psi} dx dy, \tag{4.3.10}$$

where l_2 and l_4 are two horizontal edges of the element e, $2\tau_e = y_{j+1} - y_j$ is the length of l_1, (x_e, y_e) is the center of e, and $E(y) = (y - y_e)^2 - \tau_e^2$.

To estimate the linear forms (4.3.3) and (4.3.4), we borrow the proof technique from [36]. An estimate for the first term on the right hand side of (4.3.3) is given in the following lemma.

Lemma 4.3.4 *Let $k \geq 1$ be an integer and $u_1(t)$ be a sufficiently smooth function. Let ψ_{1h} be a polynomial of degree no more than $k+1$ in x and k in y, respectively. Let ψ_{2h} be any sufficiently smooth function on e with $\Psi_h = (\psi_{1h}, \psi_{2h})$. Then*

$$(\rho_1(t), \psi_{1h})_e = J_{1,e} + \frac{(-1)^k}{(2k+2)!} (\int_{l_4} - \int_{l_2}) E^{k+1}(x) \partial_x^{k+2} u_1 \partial_x^{k-1} \psi_{2h} dx, \tag{4.3.11}$$

where the term $J_{1,e}$ represents the area integrals over the element e with the following estimate:

$$\begin{aligned} |J_{1,e}| \leq \; & \frac{h_e^{2k+2}}{(2k+2)!} |u_1|_{k+2,p,e} |\nabla \cdot \Psi_h|_{k-1,q,e} \\ & + \frac{h_e^{2k+2}}{(2k+2)!} |u_1|_{k+3,p,e} |\psi_{2h}|_{k-1,q,e} \\ & + (\frac{1}{(2k+2)!(2k+4)} + \frac{1}{(2k+4)!}) h_e^{2k+4} |u_1|_{k+3,p,e} |\psi_{1h}|_{k+1,q,e}, \end{aligned} \tag{4.3.12}$$

where $2h_e = x_{i+1} - x_i$ is the width of the element and q is the conjugate of $p \geq 1$ satisfying $1/p + 1/q = 1$.

Proof. Expanding the polynomial ψ_{1h} in x as

$$\psi_{1h}(x, y) = \sum_{i=0}^{k+1} \frac{1}{i!} (x - x_e)^i \partial_x^i \psi_{1h}(x_e, y),$$

where each of $\partial_x^i \psi_{1h}(x_e, y)$ is a polynomial of degree no more than k in y. The definition of π_1 implies that $\rho_1(t)$ is orthogonal to the polynomial space $Q_{k-1,k}$ (polynomials of degree no more than $k-1$ in x and k in y). Thus,

$$\begin{aligned}(\rho_1(t), \psi_{1h})_e &= \frac{1}{k!}\int_e (x-x_e)^k \rho_1(t)\partial_x^k \psi_{1h}(x_e,y)dxdy \\ &\quad + \frac{1}{(k+1)!}\int_e (x-x_e)^{k+1}\rho_1(t)\partial_x^{k+1}\psi_{1h}(x_e,y)dxdy \\ &:= I_1 + I_2, \end{aligned} \tag{4.3.13}$$

where $I_j, j = 1,2$, are defined accordingly. Notice that $\partial_x^k \psi_{1h}(x_e, y) \in Q_{1,k}$. Setting $\psi = \rho_1(t)\partial_x^k \psi_{1h}(x_e, y)$, we note that the conditions of Lemma 4.3.2 are satisfied with $\widetilde{k} = k-1$ and $m = k$. Thus, from (4.3.7) it follows that

$$I_1 = \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_1 \partial_x^k \psi_{1h}(x_e,y)dxdy. \tag{4.3.14}$$

Since $\partial_x^k \psi_{1h}(x, y)$ is linear in x, we have

$$\partial_x^k \psi_{1h}(x_e,y) = \partial_x^k \psi_{1h}(x,y) + (x_e - x)\partial_x^{k+1}\psi_{1h}(x,y).$$

Substituting the above identity into (4.3.14) gives

$$\begin{aligned} I_1 &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_1 \partial_x^k \psi_{1h}(x,y)dxdy \\ &\quad - \frac{(-1)^k}{(2k+2)!}\int_e (x-x_e)E^{k+1}(x)\partial_x^{k+2}u_1 \partial_x^{k+1}\psi_{1h}(x,y)dxdy. \end{aligned} \tag{4.3.15}$$

For the second term of (4.3.15), we use

$$(x-x_e)E^{k+1}(x) = \frac{1}{2k+4}\partial_x E^{k+2}(x)$$

to obtain

$$\begin{aligned} &\int_e (x-x_e)E^{k+1}(x)\partial_x^{k+2}u_1 \partial_x^{k+1}\psi_{1h}(x,y)dxdy \\ &= \frac{1}{2k+4}\int_e \partial_x E^{k+2}(x)\partial_x^{k+2}u_1 \partial_x^{k+1}\psi_{1h}(x,y)dxdy \\ &= \frac{-1}{2k+4}\int_e E^{k+2}(x)\partial_x^{k+3}u_1 \partial_x^{k+1}\psi_{1h}(x,y)dxdy. \end{aligned} \tag{4.3.16}$$

Substituting the above estimate in (4.3.15) yields

$$\begin{aligned} I_1 &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_1 \partial_x^k \psi_{1h}(x,y)dxdy \\ &\quad + \frac{(-1)^k}{(2k+2)!(2k+4)}\int_e E^{k+2}(x)\partial_x^{k+3}u_1 \partial_x^{k+1}\psi_{1h}(x,y)dxdy \\ &:= I_{11} + I_{12}. \end{aligned} \tag{4.3.17}$$

For any sufficiently smooth function $\psi_{2h}(x,y)$, we rewrite the first term I_{11} of I_1 as follows:

$$\begin{aligned} I_{11} &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_1\partial_x^{k-1}(\partial_x\psi_{1h}+\partial_y\psi_{2h})dxdy \\ &\quad -\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_1\partial_x^{k-1}\partial_y\psi_{2h}dxdy. \end{aligned} \tag{4.3.18}$$

The second term on the right of (4.3.18) can be further simplified by using the integration by parts in y, yielding

$$\begin{aligned} I_{11} &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_1\partial_x^{k-1}(\nabla\cdot\Psi_h)dxdy \\ &\quad +\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_y\partial_x^{k+2}u_1\partial_x^{k-1}\psi_{2h}dxdy \\ &\quad +\frac{(-1)^k}{(2k+2)!}(\int_{l_4}-\int_{l_2})E^{k+1}(x)\partial_x^{k+2}u_1\partial_x^{k-1}\psi_{2h}dx. \end{aligned} \tag{4.3.19}$$

Now consider the term I_2. Noting that $\partial_x^{k+1}\psi_{1h}(x_e,y)\in Q_{0,k}$. Let $\psi=\rho_1(t)\partial_x^{k+1}\psi_{1h}(x_e,y)$. Then the conditions of Lemma 4.3.2 are satisfied with $\widetilde{k}=k$ and $m=k+1$. Thus, it follows from (4.3.7) that

$$I_2=\frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(x)\partial_x^{k+3}u_1\partial_x^{k+1}\psi_{1h}(x_e,y)dxdy. \tag{4.3.20}$$

Since $\partial_x^{k+1}\psi_{1h}(x_e,y)\in Q_{0,k}$ is constant in x, we have

$$\partial_x^{k+1}\psi_{1h}(x_e,y)=\partial_x^{k+1}\psi_{1h}(x,y).$$

Substituting the above identity in (4.3.20), we arrive at

$$I_2=\frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(x)\partial_x^{k+3}u_1\partial_x^{k+1}\psi_{1h}(x,y)dxdy. \tag{4.3.21}$$

Combining (4.3.17), (4.3.19), (4.3.21) with (4.3.13), we obtain

$$(\rho_1(t),\psi_{1h})_e=I_1+I_2=I_{11}+I_{12}+I_2,$$

where I_{11} is given in (4.3.19), I_{12} can be seen in (4.3.17) and I_2 is the corresponding integral in (4.3.21). Setting

$$\begin{aligned} J_{1,e} &= I_{12}+I_2+\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_1\partial_x^{k-1}(\nabla\cdot\Psi_h)dxdy \\ &\quad +\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_y\partial_x^{k+2}u_1\partial_x^{k-1}\psi_{2h}dxdy. \end{aligned} \tag{4.3.22}$$

we obtain

$$(\rho_1(t), \psi_{1h})_e = J_{1,e} + \frac{(-1)^k}{(2k+2)!}\Big(\int_{l_4} - \int_{l_2}\Big)E^{k+1}(x)\partial_x^{k+2}u_1\partial_x^{k-1}\psi_{2h}dx.$$

Here, $J_{1,e}$ represent the combined area integrals. Now we estimate the total area integrals $J_{1,e}$. Since $|E(x)| \leq h_e^2$, using Hölder's inequality, we obtain the desired estimate given in (4.3.23). ■

Below, we estimate the second term on the right hand side of (4.3.3).

Lemma 4.3.5 *Let $k \geq 1$ be an integer and $u_2(t)$ be a sufficiently smooth function on the rectangle e. Let ψ_{2h} be a polynomial of degree no more than k in x and $k+1$ in y, respectively. Let ψ_{1h} be any sufficiently smooth function on e with $\Psi_h = (\psi_{1h}, \psi_{2h})$. Then*

$$(\rho_2(t), \psi_{2h})_e = J_{2,e} + \frac{(-1)^k}{(2k+2)!}\Big(\int_{l_3} - \int_{l_1}\Big)E^{k+1}(x)\partial_y^{k+2}u_2\partial_y^{k-1}\psi_{1h}dy, \qquad (4.3.23)$$

where the term $J_{2,e}$ represent the area integral over the element e with the following estimate:

$$\begin{aligned} |J_{2,e}| \;\leq\; & \frac{\tau_e^{2k+2}}{(2k+2)!}|u_2|_{k+2,p,e}|\nabla\cdot\Psi_h|_{k-1,q,e} \\ & + \frac{\tau_e^{2k+2}}{(2k+2)!}|u_2|_{k+3,p,e}|\psi_{1h}|_{k-1,q,e} \\ & + \left(\frac{1}{(2k+2)!(2k+4)} + \frac{1}{(2k+4)!}\right)\tau_e^{2k+4}|u_2|_{k+3,p,e}|\psi_{2h}|_{k+1,q,e}, \end{aligned} \qquad (4.3.24)$$

where $2\tau_e = y_{j+1} - y_j$ is the width of the element and q is the conjugate of $p \geq 1$ satisfying $1/p + 1/q = 1$.

Proof. Expanding the polynomial ψ_{2h} in y as,

$$\psi_{2h}(x,y) = \sum_{j=0}^{k+1}\frac{1}{j!}(y-y_e)^j\partial_y^j\psi_{2h}(x,y_e),$$

where each of $\partial_y^j\psi_{2h}(x,y_e)$ is a polynomial of degree no more than k in x. The definition of π_2 implies that $\rho_2(t)$ is orthogonal to the polynomial space $Q_{k,k-1}$ (polynomials of degree no more than k in x and $k-1$ in y). Thus,

$$\begin{aligned} (\rho_2(t), \psi_{2h})_e \;=\; & \frac{1}{k!}\int_e (y-y_e)^k\rho_2(t)\partial_y^k\psi_{2h}(x,y_e)dxdy \\ & + \frac{1}{(k+1)!}\int_e (y-y_e)^{k+1}\rho_2(t)\partial_y^{k+1}\psi_{2h}(x,y_e)dxdy, \\ :=\; & I_3 + I_4, \end{aligned} \qquad (4.3.25)$$

where $I_j, j = 3, 4$, are defined accordingly. Notice that $\partial_y^k \psi_{2h}(x, y_e) \in Q_{k,1}$. Setting $\tilde{\psi} = \rho_2(t)\partial_y^k \psi_{2h}(x, y_e)$, we note that the conditions of Lemma 4.3.3 are satisfied with $\widetilde{k} = k - 1$ and $m = k$. Thus, it follows from (4.3.10) that

$$I_3 = \frac{(-1)^k}{(2k+2)!} \int_e E^{k+1}(y)\partial_y^{k+2} u_2 \partial_y^k \psi_{2h}(x, y_e) dxdy. \tag{4.3.26}$$

Since $\partial_y^k \psi_{2h}(x, y)$ is linear in y, we have

$$\partial_y^k \psi_{2h}(x, y_e) = \partial_y^k \psi_{2h}(x, y) + (y_e - y)\partial_y^{k+1} \psi_{2h}(x, y).$$

Substituting the above identity into (4.3.26) gives

$$\begin{aligned} I_3 &= \frac{(-1)^k}{(2k+2)!} \int_e E^{k+1}(y)\partial_y^{k+2} u_2 \partial_y^k \psi_{2h}(x, y) dxdy \\ &\quad - \frac{(-1)^k}{(2k+2)!} \int_e (y - y_e) E^{k+1}(y)\partial_y^{k+2} u_2 \partial_y^{k+1} \psi_{2h}(x, y) dxdy. \end{aligned} \tag{4.3.27}$$

To deal with the second term of (4.3.27), we use

$$(y - y_e)E^{k+1}(y) = \frac{1}{2k+4}\partial_y E^{k+2}(y)$$

to obtain

$$\begin{aligned} &\int_e (y - y_e)E^{k+1}(y)\partial_y^{k+2} u_2 \partial_y^{k+1} \psi_{2h}(x, y) dxdy \\ &\quad = \frac{1}{2k+4} \int_e \partial_y E^{k+2}(y)\partial_y^{k+2} u_2 \partial_y^{k+1} \psi_{2h}(x, y) dxdy \\ &\quad = \frac{-1}{2k+4} \int_e E^{k+2}(y)\partial_y^{k+3} u_2 \partial_y^{k+1} \psi_{2h}(x, y) dxdy. \end{aligned} \tag{4.3.28}$$

Substituting (4.3.28) into (4.3.27) yields

$$\begin{aligned} I_3 &= \frac{(-1)^k}{(2k+2)!} \int_e E^{k+1}(y)\partial_y^{k+2} u_2 \partial_y^k \psi_{2h}(x, y) dxdy \\ &\quad + \frac{(-1)^k}{(2k+2)!(2k+4)} \int_e E^{k+2}(y)\partial_y^{k+3} u_2 \partial_y^{k+1} \psi_{2h}(x, y) dxdy \\ &:= I_{31} + I_{32}. \end{aligned} \tag{4.3.29}$$

For any given smooth function ψ_{1h}, we rewrite the first term I_{31} of I_3 as follows:

$$\begin{aligned} I_{31} &= \frac{(-1)^k}{(2k+2)!} \int_e E^{k+1}(y)\partial_y^{k+2} u_2 \partial_y^{k-1} (\partial_y \psi_{2h} + \partial_x \psi_{1h}) dxdy \\ &\quad - \frac{(-1)^k}{(2k+2)!} \int_e E^{k+1}(y)\partial_y^{k+2} u_2 \partial_y^{k-1} \partial_x \psi_{1h} dxdy. \end{aligned} \tag{4.3.30}$$

The second term above can be further simplified by using the integration by parts in x, yielding

$$\begin{aligned} I_{31} &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_2\partial_y^{k-1}(\nabla\cdot\Psi_h)dxdy \\ &\quad+\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_x\partial_y^{k+2}u_2\partial_y^{k-1}\psi_{1h}dxdy \\ &\quad+\frac{(-1)^k}{(2k+2)!}(\int_{l_3}-\int_{l_1})E^{k+1}(y)\partial_y^{k+2}u_2\partial_y^{k-1}\psi_{1h}dy. \end{aligned} \tag{4.3.31}$$

Let us consider the term I_4. Note that $\partial_y^{k+1}\psi_{2h}(x,y_e)\in Q_{k,0}$. Set $\tilde{\psi}=\rho_2(t)\partial_y^{k+1}\psi_{2h}(x,y_e)$. We note that conditions of Lemma 4.3.3 are satisfied with $\widetilde{k}=k$ and $m=k+1$. Thus, it follows from (4.3.10) that

$$I_4=\frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(y)\partial_y^{k+3}u_2\partial_y^{k+1}\psi_{2h}(x,y_e)dxdy. \tag{4.3.32}$$

Since $\partial_y^{k+1}\psi_{2h}(x,y_e)\in Q_{k,0}$ is constant in y, we get

$$\partial_y^{k+1}\psi_{2h}(x,y_e)=\partial_y^{k+1}\psi_{2h}(x,y).$$

Use of the above identity in (4.3.32) gives

$$I_4=\frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(y)\partial_y^{k+3}u_2\partial_y^{k+1}\psi_{2h}(x,y)dxdy. \tag{4.3.33}$$

Combining (4.3.29), (4.3.31), (4.3.33) with (4.3.25), we obtain

$$(\rho_2(t),\psi_{2h})_e=I_3+I_4=I_{31}+I_{32}+I_4,$$

where I_{31} is given in (4.3.31), I_{32} can be seen in (4.3.29) and I_4 is the corresponding integral in (4.3.33). Setting

$$\begin{aligned} J_{2,e} &= I_{32}+I_4+\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_2\partial_y^{k-1}(\nabla\cdot\Psi_h)dxdy \\ &\quad+\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_x\partial_y^{k+2}u_2\partial_y^{k-1}\psi_{1h}dxdy, \end{aligned} \tag{4.3.34}$$

we arrive at

$$(\rho_2(t),\psi_{2h})_e=J_{2,e}+\frac{(-1)^k}{(2k+2)!}(\int_{l_3}-\int_{l_1})E^{k+1}(y)\partial_y^{k+2}u_2\partial_y^{k-1}\psi_{1h}dy,$$

where $J_{2,e}$ represent the combined area integrals. Now we will estimate the total area integrals $J_{2,e}$. Since $|E(y)|\le\tau_e^2$. Using Hölder's inequality, we obtain the desired estimate. ■

The following two lemmas are concerned with the terms on the right hand side of (4.3.4). The proof techniques are similar as in Lemma 4.3.4 and Lemma 4.3.5, respectively. However, for the sake of clarity, we present the proof.

Lemma 4.3.6 *Let $k \geq 1$ be an integer and $u_{1,t}(t)$ be a sufficiently smooth function on the rectangle e. Let ψ_{1h} be a polynomial of degree no more than $k+1$ in x and k in y, respectively. Let ψ_{2h} be any sufficiently smooth function on e with $\Psi_h = (\psi_{1h}, \psi_{2h})$. Then*

$$(\rho_{1,t}(t), \psi_{1h})_e = J_{3,e} + \frac{(-1)^k}{(2k+2)!}\Big(\int_{l_4} - \int_{l_2}\Big)E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}\psi_{2h}dx, \tag{4.3.35}$$

where the term $J_{3,e}$ represents the area integral over the element e with the following estimate:

$$\begin{aligned} |J_{3,e}| \;\leq\; & \frac{h_e^{2k+2}}{(2k+2)!}|u_{1,t}|_{k+2,p,e}|\nabla\cdot\Psi_h|_{k-1,q,e} \\ & +\frac{h_e^{2k+2}}{(2k+2)!}|u_{1,t}|_{k+3,p,e}|\psi_{2h}|_{k-1,q,e} \\ & +\left(\frac{1}{(2k+2)!(2k+4)} + \frac{1}{(2k+4)!}\right)h_e^{2k+4}|u_{1,t}|_{k+3,p,e}|\psi_{1h}|_{k+1,q,e}, \end{aligned} \tag{4.3.36}$$

where $2h_e = x_{i+1} - x_i$ is the width of the element and q is the conjugate of $p \geq 1$ satisfying $1/p + 1/q = 1$.

Proof. Expanding the polynomial ψ_{1h} in x as

$$\psi_{1h}(x,y) = \sum_{i=0}^{k+1}\frac{1}{i!}(x-x_e)^i\partial_x^i\psi_{1h}(x_e,y),$$

where each of $\partial_x^i\psi_{1h}(x_e,y)$ is a polynomial of degree no more than k in y. The definition of π_1 implies that $\rho_{1,t}(t)$ is orthogonal to the polynomial space $Q_{k-1,k}$ (polynomials of degree no more than $k-1$ in x and k in y). Thus,

$$\begin{aligned} (\rho_{1,t}(t), \psi_{1h})_e &= \frac{1}{k!}\int_e (x-x_e)^k\rho_{1,t}(t)\,\partial_x^k\psi_{1h}(x_e,y)dxdy \\ &\quad + \frac{1}{(k+1)!}\int_e (x-x_e)^{k+1}\rho_{1,t}(t)\,\partial_x^{k+1}\psi_{1h}(x_e,y)dxdy \\ &= I_5 + I_6, \end{aligned} \tag{4.3.37}$$

where $I_j, j = 5,6$, are defined accordingly. Notice that $\partial_x^k\psi_{1h}(x_e,y) \in Q_{1,k}$. Let $\psi = \rho_{1,t}(t)\,\partial_x^k\psi_{1h}(x_e,y)$. We note that conditions of Lemma 4.3.2 are satisfied with $\widetilde{k} = k-1$ and $m = k$. Thus, it follows from (4.3.7) that

$$I_5 = \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^k\psi_{1h}(x_e,y)dxdy. \tag{4.3.38}$$

Since $\partial_x^k\psi_{1h}(x,y)$ is linear in x, we have

$$\partial_x^k\psi_{1h}(x_e,y) = \partial_x^k\psi_{1h}(x,y) + (x_e - x)\partial_x^{k+1}\psi_{1h}(x,y).$$

Substituting the above identity into (4.3.38) gives

$$\begin{aligned} I_5 &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^k\psi_{1h}(x,y)dxdy \\ &-\frac{(-1)^k}{(2k+2)!}\int_e (x-x_e)E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k+1}\psi_{1h}(x,y)dxdy. \end{aligned} \tag{4.3.39}$$

To deal with the second term of (4.3.39), we use

$$(x-x_e)E^{k+1}(x) = \frac{1}{2k+4}\partial_x E^{k+2}(x)$$

to obtain

$$\begin{aligned} &\int_e (x-x_e)E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k+1}\psi_{1h}(x,y)dxdy \\ &= \frac{1}{2k+4}\int_e \partial_x E^{k+2}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k+1}\psi_{1h}(x,y)dxdy \\ &= \frac{-1}{2k+4}\int_e E^{k+2}(x)\partial_x^{k+3}u_{1,t}\,\partial_x^{k+1}\psi_{1h}(x,y)dxdy. \end{aligned} \tag{4.3.40}$$

Substituting the above estimate in (4.3.39) yields

$$\begin{aligned} I_5 &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^k\psi_{1h}(x,y)dxdy \\ &+\frac{(-1)^k}{(2k+2)!(2k+4)}\int_e E^{k+2}(x)\partial_x^{k+3}u_{1,t}\,\partial_x^{k+1}\psi_{1h}(x,y)dxdy \\ := &\; I_{51}+I_{52}. \end{aligned} \tag{4.3.41}$$

For any sufficiently smooth function $\psi_{2h}(x,y)$, we rewrite the first term I_{51} of I_5 as follows:

$$\begin{aligned} I_{51} &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}(\partial_x\psi_{1h}+\partial_y\psi_{2h})dxdy \\ &-\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}\partial_y\psi_{2h}dxdy. \end{aligned} \tag{4.3.42}$$

The second term on the right of (4.3.42) can be further simplified by using the integration by parts in y, yielding

$$\begin{aligned} I_{51} &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}(\nabla\cdot\Psi_h)dxdy \\ &+\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_y\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}\psi_{2h}dxdy \\ &+\frac{(-1)^k}{(2k+2)!}\Big(\int_{l_4}-\int_{l_2}\Big)E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}\psi_{2h}dx. \end{aligned} \tag{4.3.43}$$

Now consider the term I_6. Noting that $\partial_x^{k+1}\psi_{1h}(x_e, y) \in Q_{0,k}$. Choose $\psi = \rho_{1,t}(t)\,\partial_x^{k+1}\psi_{1h}(x_e, y)$. We note that conditions of Lemma 4.3.2 are satisfied with $\widetilde{k} = k$ and $m = k+1$. Thus, it follows from (4.3.7) that

$$I_6 = \frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(x)\partial_x^{k+3}u_{1,t}\,\partial_x^{k+1}\psi_{1h}(x_e, y)dxdy. \qquad (4.3.44)$$

Since $\partial_x^{k+1}\psi_{1h}(x_e, y) \in Q_{0,k}$ is constant in x, we have

$$\partial_x^{k+1}\psi_{1h}(x_e, y) = \partial_x^{k+1}\psi_{1h}(x, y).$$

Substituting the above identity in (4.3.44), we arrive at

$$I_6 = \frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(x)\partial_x^{k+3}u_{1,t}\,\partial_x^{k+1}\psi_{1h}(x, y)dxdy. \qquad (4.3.45)$$

Combining (4.3.41), (4.3.43), (4.3.45) with (4.3.37), we obtain

$$(\rho_{1,t}(t), \psi_{1h})_e = I_5 + I_6 = I_{51} + I_{52} + I_6,$$

where I_{51} is given in (4.3.43), I_{52} can be seen in (4.3.41) and I_6 is the corresponding integral in (4.3.45). Setting

$$\begin{aligned} J_{3,e} &= I_{52} + I_6 + \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}(\nabla\cdot\Psi_h)dxdy \\ &\quad + \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(x)\partial_y\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}\psi_{2h}dxdy. \end{aligned} \qquad (4.3.46)$$

we obtain

$$(\rho_{1,t}(t), \psi_{1h})_e = J_{3,e} + \frac{(-1)^k}{(2k+2)!}\Big(\int_{l_4} - \int_{l_2}\Big)E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}\psi_{2h}dx.$$

Here, $J_{3,e}$ represent the combined area integrals. Now we estimate the total area integral $J_{3,e}$. Since $|E(x)| \le h_e^2$, using Hölder's inequality, we obtain the desired estimate given in (4.3.36). ■

Next, we have the following result for the second term on the right of (4.3.4).

Lemma 4.3.7 *Let $k \ge 1$ be an integer and $u_{2,t}(t)$ be a sufficiently smooth function on the rectangle e. Let ψ_{2h} be a polynomial of degree no more than k in x and $k+1$ in y, respectively. Let ψ_{1h} be any sufficiently smooth function on e with $\Psi_h = (\psi_{1h}, \psi_{2h})$. Then*

$$(\rho_{2,t}(t), \psi_{2h})_e = J_{4,e} + \frac{(-1)^k}{(2k+2)!}\Big(\int_{l_3} - \int_{l_1}\Big)E^{k+1}(x)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}\psi_{1h}dy, \qquad (4.3.47)$$

where the term $J_{4,e}$ represent the area integral over the element e with the following estimate:

$$
\begin{aligned}
|J_{4,e}| \le\ & \frac{\tau_e^{2k+2}}{(2k+2)!}|u_{2,t}|_{k+2,p,e}|\nabla\cdot\Psi_h|_{k-1,q,e} \\
&+\frac{\tau_e^{2k+2}}{(2k+2)!}|u_{2,t}|_{k+3,p,e}|\psi_{1h}|_{k-1,q,e} \\
&+\left(\frac{1}{(2k+2)!(2k+4)}+\frac{1}{(2k+4)!}\right)\tau_e^{2k+4}|u_{2,t}|_{k+3,p,e}|\psi_{2h}|_{k+1,q,e},
\end{aligned}
\tag{4.3.48}
$$

where $2\tau_e = y_{j+1}-y_j$ is the width of the element and q is the conjugate of $p\ge 1$ satisfying $1/p+1/q=1$.

Proof. Expanding the polynomial ψ_{2h} in y as,

$$\psi_{2h}(x,y)=\sum_{j=0}^{k+1}\frac{1}{j!}(y-y_e)^j\partial_y^j\psi_{2h}(x,y_e),$$

where each of $\partial_y^j\psi_{2h}(x,y_e)$ is a polynomial of degree no more than k in x. The definition of π_2 implies that $\rho_{2,t}(t)$ is orthogonal to the polynomial space $Q_{k,k-1}$ (polynomials of degree no more than k in x and $k-1$ in y). Thus,

$$
\begin{aligned}
(\rho_{2,t}(t),\psi_{2h})_e &= \frac{1}{k!}\int_e(y-y_e)^k\rho_{2,t}(t)\,\partial_y^k\psi_{2h}(x,y_e)dxdy \\
&\quad+\frac{1}{(k+1)!}\int_e(y-y_e)^{k+1}\rho_{2,t}(t)\,\partial_y^{k+1}\psi_{2h}(x,y_e)dxdy, \\
&:= I_7+I_8,
\end{aligned}
\tag{4.3.49}
$$

where $I_j, j=7,8$, are defined accordingly. Notice that $\partial_y^k\psi_{2h}(x,y_e)\in Q_{k,1}$. Setting $\tilde{\psi}=\rho_{2,t}(t)\,\partial_y^k\psi_{2h}(x,y_e)$, we note that conditions of Lemma 4.3.3 are satisfied with $\widetilde{k}=k-1$ and $m=k$. Thus, it follows from (4.3.10) that

$$I_7=\frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^k\psi_{2h}(x,y_e)dxdy. \tag{4.3.50}$$

Since $\partial_y^k\psi_{2h}(x,y)$ is linear in y, we have

$$\partial_y^k\psi_{2h}(x,y_e)=\partial_y^k\psi_{2h}(x,y)+(y_e-y)\partial_y^{k+1}\psi_{2h}(x,y).$$

Substituting the above into (4.3.50) gives

$$
\begin{aligned}
I_7 =\ & \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^k\psi_{2h}(x,y)dxdy \\
&-\frac{(-1)^k}{(2k+2)!}\int_e(y-y_e)E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k+1}\psi_{2h}(x,y)dxdy.
\end{aligned}
\tag{4.3.51}
$$

To deal with the second term of (4.3.51), we use

$$(y-y_e)E^{k+1}(y) = \frac{1}{2k+4}\partial_y E^{k+2}(y)$$

to obtain

$$\begin{aligned}
&\int_e (y-y_e)E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k+1}\psi_{2h}(x,y)dxdy \\
&\quad = \frac{1}{2k+4}\int_e \partial_y E^{k+2}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k+1}\psi_{2h}(x,y)dxdy \\
&\quad = \frac{-1}{2k+4}\int_e E^{k+2}(y)\partial_y^{k+3}u_{2,t}\,\partial_y^{k+1}\psi_{2h}(x,y)dxdy. \qquad (4.3.52)
\end{aligned}$$

Substituting (4.3.52) into (4.3.51) yields

$$\begin{aligned}
I_7 &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k}\psi_{2h}(x,y)dxdy \\
&\quad + \frac{(-1)^k}{(2k+2)!(2k+4)}\int_e E^{k+2}(y)\partial_y^{k+3}u_{2,t}\,\partial_y^{k+1}\psi_{2h}(x,y)dxdy \\
&:= I_{71}+I_{72}. \qquad (4.3.53)
\end{aligned}$$

For any given smooth function ψ_{1h}, we rewrite the first term I_{71} of I_7 as follows:

$$\begin{aligned}
I_{71} &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}(\partial_y\psi_{2h}+\partial_x\psi_{1h})dxdy \\
&\quad - \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}\partial_x\psi_{1h}dxdy. \qquad (4.3.54)
\end{aligned}$$

The second term above can be further simplified by using the integration by parts in x, yielding

$$\begin{aligned}
I_{71} &= \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}(\nabla\cdot\Psi_h)dxdy \\
&\quad + \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_x\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}\psi_{1h}dxdy \\
&\quad + \frac{(-1)^k}{(2k+2)!}\Big(\int_{l_3}-\int_{l_1}\Big)E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}\psi_{1h}dy. \qquad (4.3.55)
\end{aligned}$$

Let us consider the term I_8. Note that $\partial_y^{k+1}\psi_{2h}(x,y_e)\in Q_{k,0}$. Set $\tilde{\psi}=\rho_{2,t}(t)\,\partial_y^{k+1}\psi_{2h}$. Then the conditions of Lemma 4.3.3 are satisfied with $\widetilde{k}=k$ and $m=k+1$. Thus it follows from (4.3.10) that

$$I_8 = \frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(y)\partial_y^{k+3}u_{2,t}\,\partial_y^{k+1}\psi_{2h}(x,y_e)dxdy. \qquad (4.3.56)$$

Since $\partial_y^{k+1}\psi_{2h}(x,y_e)\in Q_{k,0}$ is constant in y, we get

$$\partial_y^{k+1}\psi_{2h}(x,y_e) = \partial_y^{k+1}\psi_{2h}(x,y).$$

Use of the above identity in (4.3.56) gives

$$I_8 = \frac{(-1)^{k+1}}{(2k+4)!}\int_e E^{k+2}(y)\partial_y^{k+3}u_{2,t}\,\partial_y^{k+1}\psi_{2h}(x,y)dxdy. \tag{4.3.57}$$

Combining (4.3.55), (4.3.53), (4.3.57) with (4.3.49), we obtain

$$(\rho_{2,t}(t),\psi_{2h})_e = I_7 + I_8 = I_{71} + I_{72} + I_8,$$

where I_{71} is given in (4.3.55), I_{72} can be seen in (4.3.53) and I_8 is the corresponding integral in (4.3.57). Setting

$$\begin{aligned} J_{4,e} &= I_{72} + I_8 + \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}(\nabla\cdot\Psi_h)dxdy \\ &\quad + \frac{(-1)^k}{(2k+2)!}\int_e E^{k+1}(y)\partial_x\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}\psi_{1h}dxdy. \end{aligned} \tag{4.3.58}$$

we arrive at

$$(\rho_{2,t}(t),\psi_{2h})_e = J_{4,e} + \frac{(-1)^k}{(2k+2)!}\Big(\int_{l_3} - \int_{l_1}\Big)E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}\psi_{1h}dy,$$

where $J_{4,e}$ represent the combined area integrals. Now we will estimate the total area integrals $J_{4,e}$. Since $|E(y)| \le \tau_e^2$. Using Hölder's inequality, we obtain the desired estimate (4.3.47). ■

4.4 Superconvergence Result for the Flux

Recall that the linear form $\mathcal{F}(\Psi_h)$ in (4.3.3) is given by:

$$\begin{aligned} \mathcal{F}(\Psi_h) &:= \sum_{e\in\widehat{\mathcal{T}}_h}(\rho_1(t),\psi_{1h})_e + \sum_{e\in\widehat{\mathcal{T}}_h}(\rho_2(t),\psi_{2h})_e \\ &= \mathcal{F}_1(\psi_{1,h}) + \mathcal{F}_2(\psi_{2,h}). \end{aligned} \tag{4.4.1}$$

Applying Lemma 4.3.4 on each element e, we obtain

$$\mathcal{F}_1(\psi_{1,h}) = \sum_{e\in\widehat{\mathcal{T}}_h} J_{1,e} + \frac{(-1)^k}{(2k+2)!}\sum_{e\in\widehat{\mathcal{T}}_h}\Big(\int_{l_4} - \int_{l_2}\Big)E^{k+1}(x)\partial_x^{k+2}u_1\partial_x^{k-1}\psi_{2h}dx, \tag{4.4.2}$$

where ψ_{2h} is chosen to be the second component of the vector valued function Ψ_h. Observe that the line integral over l_4 is given as an integral along the bottom edge and the integral over l_2 is given as the one on the top edge of e. If l_4 is not a boundary edge, then there will be another element $\widetilde{e}$, for which l_4 is the top edge and the corresponding integral on l_4 has an opposite sign from the contribution of e. Thus all the line integrals over the interior edges must cancel each other in the last summation. The line integrals over the boundary edges varnishes due to the fact that $\psi_{2h} = 0$ along horizontal boundary edges. Thus, we obtain

$$\mathcal{F}_1(\psi_{1,h}) = \sum_{e\in\widehat{\mathcal{T}}_h} J_{1,e}. \tag{4.4.3}$$

Using estimate (4.3.12) with $p = q = 2$, we obtain

$$\begin{aligned}
|\mathcal{F}_1(\psi_{1,h})| &\leq \sum_{e\in\widehat{\mathcal{T}}_h} |J_{1,e}| \\
&\leq \sum_{e\in\widehat{\mathcal{T}}_h} \frac{h_e^{2k+2}}{(2k+2)!}|u_1|_{k+2,2,e}|\nabla\cdot\Psi_h|_{k-1,2,e} \\
&\quad + \frac{h_e^{2k+2}}{(2k+2)!}|u_1|_{k+3,2,e}|\psi_{2h}|_{k-1,2,e} \\
&\quad + \left(\frac{1}{(2k+2)!(2k+4)} + \frac{1}{(2k+4)!}\right)h_e^{2k+4}|u_1|_{k+3,2,e}|\psi_{1h}|_{k+1,2,e} \\
&\leq \frac{\widetilde{h}^{2k+2}}{(2k+2)!}|u_1|_{k+2,2,h}|\nabla\cdot\Psi_h|_{k-1,2,h} \\
&\quad + \frac{\widetilde{h}^{2k+2}}{(2k+2)!}|u_1|_{k+3,2,h}|\psi_{2h}|_{k-1,2,h} \\
&\quad + \left(\frac{1}{(2k+2)!(2k+4)} + \frac{1}{(2k+4)!}\right)\widetilde{h}^{2k+4}|u_1|_{k+3,2,h}|\psi_{1h}|_{k+1,2,h},
\end{aligned} \tag{4.4.4}$$

where $\widetilde{h} = \max_{e\in\widehat{\mathcal{T}}_h} h_e$. By applying the standard inverse inequality to various norms of Ψ_h, we arrive at

$$\begin{aligned}
|\mathcal{F}_1(\psi_{1,h})| &\leq C\widetilde{h}^{k+3}\|u_1\|_{k+3,2,\Omega}(\|\Psi_h\| + \|\nabla\cdot\Psi_h\|) \\
&\leq C\widetilde{h}^{k+3}\|u_1\|_{k+3,,2,\Omega}\|\Psi_h\|_{\mathbf{V}}.
\end{aligned} \tag{4.4.5}$$

Similarly, applying Lemma 4.3.5 on each element e, we will have

$$\mathcal{F}_2(\psi_{2,h}) = \sum_{e\in\widehat{\mathcal{T}}_h} J_{2,e} + \frac{(-1)^k}{(2k+2)!}\sum_{e\in\widehat{\mathcal{T}}_h}\left(\int_{l_3} - \int_{l_1}\right)E^{k+1}(y)\partial_y^{k+2}u_2\partial_y^{k-1}\psi_{1h}dy. \tag{4.4.6}$$

By using the above arguments to the linear form $\mathcal{F}_2(\psi_{2,h})$ given in (4.4.6) with $\widetilde{\tau} = \max_{e\in\widehat{\mathcal{T}}_h} \tau_e$, the corresponding estimate is given by

$$|\mathcal{F}_2(\psi_{2,h})| \leq C\widetilde{\tau}^{k+3}\|u_2\|_{k+3,2,\Omega}\|\Psi_h\|_{\mathbf{V}}. \tag{4.4.7}$$

Substituting (4.4.5) and (4.4.7) into (4.3.3), we obtain

$$|\mathcal{F}(\Psi_h)| \leq C(\widetilde{h}^{k+3} + \widetilde{\tau}^{k+3})\|\mathbf{u}\|_{k+3,2,\Omega}\|\Psi_h\|_{\mathbf{V}}. \tag{4.4.8}$$

Taking $h = \max(\widetilde{h}, \widetilde{\tau})$, we obtain

$$|\mathcal{F}(\Psi_h)| \leq Ch^{k+3}\|\mathbf{u}\|_{k+3,2,\Omega}\|\Psi_h\|_{\mathbf{V}}. \tag{4.4.9}$$

Thus, we have

$$\|(\mathbf{u}-\pi_h\mathbf{u})(t)\|_{\mathbf{V}^*} = \sup_{\Psi_h\in\mathbf{V}_{0,h}} \frac{((\mathbf{u}-\pi_h\mathbf{u})(t),\Psi_h)}{\|\Psi_h\|_{\mathbf{V}}} \le Ch^{k+3}\|\mathbf{u}\|_{k+3}. \tag{4.4.10}$$

Considering the second linear form (4.3.4) consisting of two components:

$$\begin{aligned}\widetilde{\mathcal{F}}(\Psi_h) &:= \sum_{e\in\widehat{\mathcal{T}}_h}(\rho_{1,t}(t),\psi_{1h})_e + \sum_{e\in\widehat{\mathcal{T}}_h}(\rho_{2,t}(t),\psi_{2h})_e \\ &:= \widetilde{\mathcal{F}}_1(\psi_{1,h}) + \widetilde{\mathcal{F}}_2(\psi_{2,h}).\end{aligned} \tag{4.4.11}$$

As before, we need to estimate $\widetilde{\mathcal{F}}_1(\psi_{1,h})$ and $\widetilde{\mathcal{F}}_2(\psi_{2,h})$. Application of Lemma 4.3.6 on each element e now leads to

$$\widetilde{\mathcal{F}}_1(\psi_{1,h}) = \sum_{e\in\widehat{\mathcal{T}}_h} J_{3,e} + \frac{(-1)^k}{(2k+2)!}\sum_{e\in\widehat{\mathcal{T}}_h}(\int_{l_4} - \int_{l_2})E^{k+1}(x)\partial_x^{k+2}u_{1,t}\,\partial_x^{k-1}\psi_{2h}dx, \tag{4.4.12}$$

where ψ_{2h} is chosen to be the second component of the vector valued function Ψ_h. Using the arguments as in deriving (4.4.3), we obtain

$$\widetilde{\mathcal{F}}_1(\psi_{1,h}) = \sum_{e\in\widehat{\mathcal{T}}_h} J_{3,e}. \tag{4.4.13}$$

Now using the estimate (4.3.36) with $p = q = 2$ to obtain

$$\begin{aligned}|\widetilde{\mathcal{F}}_1(\psi_{1,h})| &\le \sum_{e\in\widehat{\mathcal{T}}_h}|J_{3,e}| \\ &\le \sum_{e\in\widehat{\mathcal{T}}_h}\frac{h_e^{2k+2}}{(2k+2)!}|u_{1,t}|_{k+2,2,e}|\nabla\cdot\Psi_h|_{k-1,2,e} \\ &\quad+\frac{h_e^{2k+2}}{(2k+2)!}|u_{1,t}|_{k+3,2,e}|\psi_{2h}|_{k-1,2,e} \\ &\quad+(\frac{1}{(2k+2)!(2k+4)}+\frac{1}{(2k+4)!})h_e^{2k+4}|u_{1,t}|_{k+3,2,e}|\psi_{1h}|_{k+1,2,e} \\ &\le \frac{\widetilde{h}^{2k+2}}{(2k+2)!}|u_{1,t}|_{k+2,2,h}|\nabla\cdot\Psi_h|_{k-1,2,h} \\ &\quad+\frac{\widetilde{h}^{2k+2}}{(2k+2)!}|u_{1,t}|_{k+3,2,h}|\psi_{2h}|_{k-1,2,h} \\ &\quad+(\frac{1}{(2k+2)!(2k+4)}+\frac{1}{(2k+4)!})\widetilde{h}^{2k+4}|u_{1,t}|_{k+3,2,h}|\psi_{1h}|_{k+1,2,h},\end{aligned} \tag{4.4.14}$$

where $\widetilde{h} = \max_{e\in\widehat{\mathcal{T}}_h} h_e$. By applying the standard inverse inequality to various norms of Ψ_h, we arrive at

$$\begin{aligned}|\widetilde{\mathcal{F}}_1(\psi_{1,h})| &\le C\widetilde{h}^{k+3}\|u_{1,t}\|_{k+3,2,\Omega}(\|\Psi_h\| + \|\nabla\cdot\Psi_h\|) \\ &\le C\widetilde{h}^{k+3}\|u_{1,t}\|_{k+3,2,\Omega}\|\Psi_h\|_{\mathbf{V}}.\end{aligned} \tag{4.4.15}$$

Similarly, applying Lemma 4.3.7 on each element e, we have

$$\widetilde{\mathcal{F}}_2(\psi_{2,h}) = \sum_{e\in\widehat{\mathcal{T}}_h} J_{4,e} + \frac{(-1)^k}{(2k+2)!}\sum_{e\in\widehat{\mathcal{T}}_h}\Big(\int_{l_3} - \int_{l_1}\Big)E^{k+1}(y)\partial_y^{k+2}u_{2,t}\,\partial_y^{k-1}\psi_{1h}dy. \tag{4.4.16}$$

By using the above arguments to the linear form $\widetilde{\mathcal{F}}_2(\psi_{2,h})$ given in (4.4.16) with $\widetilde{\tau} = \max_{e\in\widehat{\mathcal{T}}_h}\tau_e$, the corresponding estimate is given by

$$|\widetilde{\mathcal{F}}_2(\psi_{2,h})| \leq C\widetilde{\tau}^{k+3}\|u_{2,t}\|_{k+3,2,\Omega}\|\Psi_h\|_{\mathbf{V}}. \tag{4.4.17}$$

Substituting (4.4.15) and (4.4.17) into (4.3.4), we obtain

$$|\widetilde{\mathcal{F}}(\Psi_h)| \leq C(\widetilde{h}^{k+3} + \widetilde{\tau}^{k+3})\|\mathbf{u}_t\|_{k+3,2,\Omega}\|\Psi_h\|_{\mathbf{V}}.$$

Taking $h = \max(\widetilde{h}, \widetilde{\tau})$,

$$|\widetilde{\mathcal{F}}(\Psi_h)| \ \leq \ Ch^{k+3}\|\mathbf{u}_t\|_{k+3,2,\Omega}\|\Psi_h\|_{\mathbf{V}}. \tag{4.4.18}$$

Thus,

$$\|(\mathbf{u}-\pi_h\mathbf{u})_t\|_{\mathbf{V}^*} = \sup_{\Psi_h\in\mathbf{V}_{0,h}}\frac{((\mathbf{u}-\pi_h\mathbf{u})_t, \Psi_h)}{\|\Psi_h\|_{\mathbf{V}}} \leq Ch^{k+3}\|\mathbf{u}_t\|_{k+3}. \tag{4.4.19}$$

Now, using (4.4.10) and (4.4.19) in Lemma 4.3.1 we obtain the following result.

Theorem 4.4.1 *Let $\{\mathbf{u}, p\}$ and $\{\mathbf{u}_h, p_h\}$ satisfy the mixed problems (4.2.3)-(4.2.4) and (4.2.5)-(4.2.6), respectively. Further, let $\mathbf{u}_h(0) = \pi_h\mathbf{u}_0$. Then there is a positive generic constant C independent of the mesh size h such that*

$$\|\pi_h\mathbf{u}(t) - \mathbf{u}_h(t)\| \leq Ch^{k+3}\left(\int_0^t \left\{\|\mathbf{u}\|_{k+3}^2 + \|\mathbf{u}_s\|_{k+3}^2\right\} ds\right)^{1/2} \tag{4.4.20}$$

Next, we provide a new approximation for $\mathbf{u}_h$ via a postprocessing of $\mathbf{u}_h$. Let $\mathbf{u}_h$ be the finite element approximation to $\mathbf{u}$ with the estimate (4.4.20), where $\pi_h\mathbf{u}$ is a projection of $\mathbf{u}$ defined by (4.2.7). The interpolation error between $\mathbf{u}$ and $\pi_h\mathbf{u}$ is assumed to be worse than $\mathcal{O}(h^{k+3})$. However, because of the estimate (4.4.20) and the locality of $\pi_h\mathbf{u}$, it is possible to construct a new approximate solution based on $\mathbf{u}_h$ which approximates $\mathbf{u}$ with the superconvergence order of $O(h^{k+3})$. This new approximate solution is obtained through an operator P_h from the finite element space to a new finite element space consisting of high order (e.g., of order r' on each element, where $r' = k + k'$, for some integer $k' > 0$) polynomials with the property:

$$P_h\pi_h\mathbf{u} = P_h\mathbf{u}. \tag{4.4.21}$$

For the Raviart-Thomas element of order r', we have (cf. [60])

$$\|\mathbf{u} - P_h\mathbf{u}\| \leq Ch^{r'+1}\|\mathbf{u}\|_{r'+1}. \tag{4.4.22}$$

Further, we have assumed the L^2-boundedness of the operator P_h. The construction of such an operator P_h is described in [36].

Theorem 4.4.2 *Let $\{\mathbf{u}, p\}$ and $\{\mathbf{u}_h, p_h\}$ satisfy the mixed problems (4.2.3)-(4.2.4) and (4.2.5)-(4.2.6), respectively. Further, let $\mathbf{u} \in [H^{k+3}]^2$, then there is a positive generic constant C independent of the mesh size h such that*

$$\|\mathbf{u}(t) - P_h\mathbf{u}_h(t)\| \leq Ch^{k+3}\left\{\|\mathbf{u}\|_{k+3} + \left(\int_0^t (\|\mathbf{u}\|^2_{k+3} + \|\mathbf{u}_s\|^2_{k+3})\, ds\right)^{1/2}\right\}. \tag{4.4.23}$$

Proof. By triangle inequality, we have

$$\begin{aligned}
\|\mathbf{u}(t) - P_h\mathbf{u}_h(t)\| &\leq \|\mathbf{u} - P_h\mathbf{u}\| + \|P_h\mathbf{u} - P_h\mathbf{u}_h\| \\
&\leq \|\mathbf{u} - P_h\mathbf{u}\| + \|P_h(\pi_h\mathbf{u} - \mathbf{u}_h)\| \\
&\leq \|\mathbf{u} - P_h\mathbf{u}\| + \|\pi_h\mathbf{u} - \mathbf{u}_h\| \\
&\leq Ch^{k+k'+1}\|\mathbf{u}\|_{k+k'+1} + Ch^{k+3}\left(\int_0^t (\|\mathbf{u}\|^2_{k+3} + \|\mathbf{u}_s\|^2_{k+3})\, ds\right)^{1/2} \\
&\leq Ch^{k+3}\left\{\|\mathbf{u}\|_{k+3} + \left(\int_0^t (\|\mathbf{u}\|^2_{k+3} + \|\mathbf{u}_s\|^2_{k+3})\, ds\right)^{1/2}\right\}
\end{aligned}$$

with $k' = 2$. Here, in the third step we have used the L^2-boundedness of the operator P_h and in the fourth step the estimate (4.4.20). This completes the proof. ■

Remark. (i) Once $\mathbf{u}_h$ is computed, find $p_h \in W_h$ such that

$$(\nabla p_h, \nabla\phi_h) = (\alpha P_h\mathbf{u}_h, \nabla\phi_h), \;\forall\phi_h \in W_h. \tag{4.4.24}$$

From (4.2.4) and (4.4.24), we have the following error equation

$$(\nabla(p - p_h), \nabla\phi_h) = (\alpha(\mathbf{u} - P_h\mathbf{u}_h), \nabla\phi_h), \tag{4.4.25}$$

and hence,

$$\|\nabla(p - p_h)\| \leq C\|\mathbf{u} - P_h\mathbf{u}_h\|. \tag{4.4.26}$$

An application of Poincaré's inequality leads to

$$\begin{aligned}
\|p - p_h\| &\leq C\|\mathbf{u} - P_h\mathbf{u}_h\| \\
&\leq Ch^{k+3}\left\{\|\mathbf{u}\|_{k+3} + \left(\int_0^t (\|\mathbf{u}\|^2_{k+3} + \|\mathbf{u}_s\|^2_{k+3})\, ds\right)^{1/2}\right\},
\end{aligned} \tag{4.4.27}$$

where in the second step, we have used Theorem 4.4.2.

(ii) In this chapter, a new approximate solution for the flux with superconvergence of order $\mathcal{O}(h^{k+3})$ is established via a postprocessing technique using a local projection. Compared to [19], the present analysis yields better convergence result for the flux with a higher regularity assumption on the exact solution. Use of piecewise linear polynomial spaces yields optimal order error estimate of order $\mathcal{O}(h^2)$ in the L^2-norm for the flux $\mathbf{u}$ (see, [60]) whereas Theorem 4.4.2 gives order $\mathcal{O}(h^4)$ for the flux which indicates superconvergence. The proposed method is not subject to LBB-consistency condition.

Chapter 5

Semidiscrete A Posteriori Error Analysis for H^1-Galerkin MFEM for Parabolic Problems

In this chapter, we study semidiscrete a posteriori error analysis for H^1-Galerkin mixed finite element method for parabolic problems. The upper bounds for the errors are derived under a saturation assumption. Our analysis is based on residual approach.

5.1 Introduction

Recalling the parabolic problem:

$$p_t - \nabla \cdot (a(\mathbf{x})\nabla p) = f(\mathbf{x},t), \ (\mathbf{x},t) \in \Omega \times J, \tag{5.1.1}$$

$$p(\mathbf{x},t) = 0, \ (\mathbf{x},t) \in \partial\Omega \times J, \tag{5.1.2}$$

$$p(\mathbf{x},0) = p_0(\mathbf{x}), \ \mathbf{x} \in \Omega, \tag{5.1.3}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $p_t = \frac{\partial p}{\partial t}$ and J denotes the time interval $(0,T]$ with $T < \infty$. The coefficient matrix $a = a(\mathbf{x})$ is symmetric and uniformly positive definite in Ω. The source function $f(\mathbf{x},t)$ and the initial data $p_0(\mathbf{x})$ are assumed to be smooth functions.
With the flux variable $\mathbf{u} = a\nabla p$, the problem (5.1.1) can be rewritten in the form of a first-order system as

$$p_t - \nabla \cdot \mathbf{u} = f, \ (\mathbf{x},t) \in \Omega \times J, \tag{5.1.4}$$

$$\alpha\mathbf{u} - \nabla p = 0, \ (\mathbf{x},t) \in \Omega \times J, \tag{5.1.5}$$

where $\alpha = 1/a$. For the purpose of error analysis, we shall need the following spaces:

$$\mathbf{V} = \left\{ \mathbf{v} \in (L^2(\Omega))^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}$$

with norm

$$\|\mathbf{v}\|_{\mathbf{V}} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{1/2},$$

and $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$ with the usual Sobolev norm, see [1, 40]. Let $\mathbf{V}^*$ be the dual space of $\mathbf{V}$. The H^1-Galerkin mixed formulation for the problem (5.1.4)-(5.1.5) is to determine a pair $\{\mathbf{u}, p\} : [0, T] \longrightarrow \mathbf{V} \times H_0^1(\Omega)$ such that

$$\begin{aligned} (\alpha \mathbf{u}_t, \Psi) + A(\mathbf{u}, \Psi) &= -(f, \nabla \cdot \Psi) + \lambda(\mathbf{u}, \Psi), \ \forall \Psi \in \mathbf{V}, && (5.1.6) \\ (\nabla p, \nabla \phi) &= (\alpha \mathbf{u}, \nabla \phi), \ \forall \phi \in H_0^1(\Omega) && (5.1.7) \end{aligned}$$

with $\mathbf{u}_0 = \mathbf{u}(\mathbf{x}, 0) = a\nabla p_0$. The bilinear form $A(\cdot, \cdot)$ is given by

$$A(\mathbf{u}, \mathbf{v}) = (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \lambda(\mathbf{u}, \mathbf{v}), \ \lambda > 0.$$

Note that λ is chosen appropriately such that $A(\cdot, \cdot)$ is $\mathbf{V}$-coercive, i.e.,

$$A(\mathbf{v}, \mathbf{v}) \geq c_1 \|\mathbf{v}\|_{\mathbf{V}}^2, \ \mathbf{v} \in \mathbf{V}, \tag{5.1.8}$$

for some $c_1 > 0$. For the purpose of H^1-Galerkin mixed finite element procedure, let $\mathbf{V}_h$ be a finite dimensional subspace of $\mathbf{V}$ consisting of Raviart-Thomas finite elements and W_h be the standard finite dimensional subspace of $H_0^1(\Omega)$. The semidiscrete H^1-Galerkin mixed finite element approximation is to determine a pair $\{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} (\alpha \mathbf{u}_{h,t}, \Psi_h) + A(\mathbf{u}_h, \Psi_h) &= -(f, \nabla \cdot \Psi_h) + \lambda(\mathbf{u}_h, \Psi_h), \ \forall \Psi_h \in \mathbf{V}_h, && (5.1.9) \\ (\nabla p_h, \nabla \phi_h) &= (\alpha \mathbf{u}_h, \nabla \phi_h), \ \forall \phi_h \in W_h && (5.1.10) \end{aligned}$$

with given $\{\mathbf{u}_h(0), p_h(0)\}$.

The literature concerning a posteriori error analysis for standard Galerkin method can be found in [5, 7, 20, 31, 68, 53] for linear parabolic problems. The previous work on a posteriori error analysis by means of the classical mixed method for elliptic problems are described in [3, 8, 16, 39, 47, 49, 50, 51, 86]. To the best of our knowledge there is no result available for H^1-Galerkin mixed finite element method for parabolic problems. In this chapter, we study the semidiscrete a posteriori error analysis for the problem (5.1.1)-(5.1.3) by H^1-Galerkin mixed finite element method.

This chapter is organized as follows. In Section 5.2, we define error indictors and derive some auxiliary estimates useful for our a posteriori error analysis. The upper bounds for the errors in the L^2-norms are established in Section 5.3.

Throughout this chapter, C denotes a positive generic constant which is independent of the mesh parameter h and may not be the same at each occurrence.

5.2 A Framework for A Posteriori Error Analysis

Our semidiscrete a posteriori error analysis is based on residual approach with mesh refinement technique. Let $\mathcal{T}_{h/2}$ be a refinement of $\mathcal{T}_h$ by dividing each triangle K into four congruent ones. We now define the residuals corresponding to the equations (5.1.9) and (5.1.10), respectively, as follows: For $t \in J$

$$< R_1(\mathbf{u}_h), \Psi > \; = \; -(f, \nabla \cdot \Psi) - (\nabla \cdot \mathbf{u}_h, \nabla \cdot \Psi) - (\alpha \mathbf{u}_{h,t}, \Psi), \qquad (5.2.1)$$

$$(R_2(\mathbf{u}_h, p_h), \nabla \phi) \; = \; (\alpha \mathbf{u}_h, \nabla \phi) - (\nabla p_h, \nabla \phi), \qquad (5.2.2)$$

for all $\Psi \in \mathbf{V}$ and $\phi \in H_0^1(\Omega)$. Here, $< \cdot, \cdot >$ denotes the dual inner product and $(\cdot, \cdot)$ is the standard L^2-inner product. The global error estimator is then defined as

$$\eta_R := \sum_{K \in \mathcal{T}_h} \left(\|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))} + \|R_2(\mathbf{u}_h, p_h)\|^2_{L^2(K)} \right)^{1/2}. \qquad (5.2.3)$$

Our objective is to establish upper bounds for the errors $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$ in L^2-norm in terms of global estimator η_R. We now make the following *saturation assumption*: there is a number $\beta < 1$ such that

$$\|\mathbf{u}(t) - \mathbf{u}_{h/2}(t)\| \;\leq\; \beta \|\mathbf{u}(t) - \mathbf{u}_h(t)\|, \; t \in J, \qquad (5.2.4)$$

$$\|\nabla(p - p_{h/2})(t)\| \;\leq\; \beta \|\nabla(p - p_h)(t)\|, \; t \in J. \qquad (5.2.5)$$

The assumptions (5.2.4)-(5.2.5) imply that on the refined mesh $\mathcal{T}_{h/2}$ the refined finite element approximation $(\mathbf{u}_{h/2}, p_{h/2})$ is a better approximation to the exact solution $(\mathbf{u}, p)$ than $(\mathbf{u}_h, p_h)$. These saturation assumptions are motivated by the well-known *a priori* error estimates for $\mathbf{u}(t) - \mathbf{u}_h(t)$ and $\nabla(p - p_h)(t)$ (see, e.g., [60], Theorem 3.1 with $k = r = 2$ and $k + 1 = r = 2$, respectively):

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| \leq C(p_0, \mathbf{u}, p) h^2, \qquad (5.2.6)$$

and

$$\|\nabla(p - p_h)(t)\| \leq C(p_0, \mathbf{u}, p) h^2. \qquad (5.2.7)$$

The above *a priori* estimates assures us that, in general, a refinement of the mesh and a reduction of h will lead to an improvement of the finite element solution. However, (5.2.6) and (5.2.7) provides no information on the improvement for individual cases. Therefore, we exclude the exceptional cases in which the improvement is very small. Generally, one expects that by refining the mesh will significantly reduce the error, so this is a natural assumption. In view of the saturation assumption (5.2.4)-(5.2.5), it will suffice to establish an upper bound for

$$\|\mathbf{u}_{h/2} - \mathbf{u}_h\| \quad \text{and} \quad \|p_{h/2} - p_h\|.$$

Lemma 5.2.1 *Let $R_1(\mathbf{u}_h)$ be the first residual given by (5.2.1). Further, let $\mathbf{u}_h(0) = \mathbf{u}_{h/2}(0)$. Then there is a positive constant C such that*

$$\|(\mathbf{u}_{h/2} - \mathbf{u}_h)(t)\| \leq C \left(\sum_{K \in \mathcal{T}_h} \|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))} \right)^{1/2}.$$

Proof. Let $\mathbf{u}_{h/2}$ be the solution to (5.1.9) on the refined triangulation $\mathcal{T}_{h/2}$ satisfying the following weak formulation

$$(\alpha \mathbf{u}_{h/2,t}, \Psi_{h/2}) + A(\mathbf{u}_{h/2}, \Psi_{h/2}) = -(f, \nabla \cdot \Psi_{h/2}) + \lambda(\mathbf{u}_{h/2}, \Psi_{h/2}), \ \forall \Psi_{h/2} \in \mathbf{V}_{h/2}, \quad (5.2.8)$$

where $\mathbf{V}_{h/2}$ is the finite dimensional subspace of $\mathbf{V}$ over the refined triangulation $\mathcal{T}_{h/2}$. Rewriting (5.2.8), we have

$$\begin{aligned}(\alpha(\mathbf{u}_{h/2} - \mathbf{u}_h)_t, \Psi_{h/2}) + A(\mathbf{u}_{h/2} - \mathbf{u}_h, \Psi_{h/2}) &= -(f, \nabla \cdot \Psi_{h/2}) - (\nabla \cdot \mathbf{u}_h, \nabla \cdot \Psi_{h/2}) \\ &\quad -(\alpha \mathbf{u}_{h,t}, \Psi_{h/2}) + \lambda(\mathbf{u}_{h/2} - \mathbf{u}_h, \Psi_{h/2}) \\ &= < R_1(\mathbf{u}_h), \Psi_{h/2} > + \lambda(\mathbf{u}_{h/2} - \mathbf{u}_h, \Psi_{h/2}).\end{aligned}$$

Choosing $\Psi_{h/2} = \mathbf{u}_{h/2} - \mathbf{u}_h$ in the above equation and using coercivity of $A(\cdot,\cdot)$, Cauchy-Schwarz inequality and the inequality $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$, $a, b \geq 0$, $\epsilon > 0$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|\alpha^{1/2}(\mathbf{u}_{h/2} - \mathbf{u}_h)\|^2\} + c_1\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2_{\mathbf{V}} &\leq \|R_1(\mathbf{u}_h)\|_{\mathbf{V}^*} \|\mathbf{u}_{h/2} - \mathbf{u}_h\|_{\mathbf{V}} + \lambda\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2 \\ &\leq C(\epsilon)\|R_1(\mathbf{u}_h)\|^2_{\mathbf{V}^*} + \epsilon C\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2_{\mathbf{V}} \\ &\quad + \lambda\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2.\end{aligned}$$

This can be further rewritten as

$$\frac{1}{2}\frac{d}{dt}\{\|\alpha^{1/2}(\mathbf{u}_{h/2} - \mathbf{u}_h)\|^2\} + (c_1 - \epsilon C)\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2_{\mathbf{V}} \leq C(\epsilon)\|R_1(\mathbf{u}_h)\|^2_{\mathbf{V}^*} + \lambda\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2.$$

Choose $\epsilon > 0$ so that $(c_1 - \epsilon C) > 0$. Integration from 0 to t now leads to

$$\begin{aligned}\|(\mathbf{u}_{h/2} - \mathbf{u}_h)(t)\|^2 + C\int_0^t \|(\mathbf{u}_{h/2} - \mathbf{u}_h)(s)\|^2_{\mathbf{V}} ds &\leq C\int_0^t \|R_1(\mathbf{u}_h)\|^2_{\mathbf{V}^*} ds \\ &\quad + C\int_0^t \|(\mathbf{u}_{h/2} - \mathbf{u}_h)(s)\|^2 ds.\end{aligned}$$

An application of the Grownwall's lemma yields,

$$\begin{aligned}\|(\mathbf{u}_{h/2} - \mathbf{u}_h)(t)\|^2 \;+\; &\int_0^t \|(\mathbf{u}_{h/2} - \mathbf{u}_h)(s)\|^2_{\mathbf{V}} ds \leq C\int_0^t \|R_1(u_h)\|^2_{\mathbf{V}^*} ds \\ &\leq C\int_0^t \sum_{K \in \mathcal{T}_h} \|R_1(\mathbf{u}_h)\|^2_{\mathbf{V}^*(K)} \\ &\leq C\sum_{K \in \mathcal{T}_h} \|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;V^*(K))}\end{aligned}$$

which proves the desired estimate. ■

Lemma 5.2.2 *Let $R_1(\mathbf{u}_h)$ and $R_2(\mathbf{u}_h, p_h)$ be the first and second residuals given by (5.2.1) and (5.2.2), respectively. Then there is a positive constant C such that*

$$\|(p_{h/2} - p_h)(t)\| \le C \left(\sum_{K \in \mathcal{T}_h} \left(\|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))} + \|R_2(\mathbf{u}_h, p_h)\|^2_{L^2(K)} \right) \right)^{1/2}.$$

Proof. Let $(\mathbf{u}_{h/2}, p_{h/2})$ be the solution to (5.1.10) on the refined triangulation $\mathcal{T}_{h/2}$ of Ω satisfying the weak formulation

$$(\nabla p_{h/2}, \nabla \phi_{h/2}) = (\alpha \mathbf{u}_{h/2}, \nabla \phi_{h/2}), \ \forall \ \phi_{h/2} \in W_{h/2}, \tag{5.2.9}$$

where $W_{h/2}$ is finite dimensional subspace of W over the refined triangulation $\mathcal{T}_{h/2}$. Rewriting (5.2.9), we have

$$(\nabla(p_{h/2} - p_h), \nabla \phi_{h/2}) = (\alpha(\mathbf{u}_{h/2} - \mathbf{u}_h), \nabla \phi_{h/2}) + (R_2(\mathbf{u}_h, p_h), \nabla \phi_{h/2}). \tag{5.2.10}$$

Choosing $\phi_{h/2} = p_{h/2} - p_h$ in (5.2.10) and using Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\begin{aligned} \|\nabla(p_{h/2} - p_h)\|^2 &\le C\|\mathbf{u}_{h/2} - \mathbf{u}_h\| \, \|\nabla(p_{h/2} - p_h)\| + \|R_2(\mathbf{u}_h, p_h)\| \, \|\nabla(p_{h/2} - p_h)\| \\ &\le \epsilon C \|\nabla(p_{h/2} - p_h)\|^2 + C(\epsilon) \left(\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2 + \|R_2(\mathbf{u}_h, p_h)\|^2 \right). \end{aligned}$$

Choose ϵ appropriately so that $(1 - \epsilon C) > 0$. Thus, we obtain

$$\|\nabla(p_{h/2} - p_h)\|^2 \le C(\|\mathbf{u}_{h/2} - \mathbf{u}_h\|^2 + \|R_2(\mathbf{u}_h, p_h)\|^2). \tag{5.2.11}$$

Next, we use Lemma 5.2.1 in (5.2.11) to obtain

$$\begin{aligned} \|\nabla(p_{h/2} - p_h)\|^2 &\le C \left(\sum_{K \in \mathcal{T}_h} \|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))} + \sum_{K \in \mathcal{T}_h} \|R_2(\mathbf{u}_h, p_h)\|^2_{L^2(K)} \right) \\ &\le C \sum_{K \in \mathcal{T}_h} \left(\|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))} + \|R_2(\mathbf{u}_h, p_h)\|^2_{L^2(K)} \right). \end{aligned} \tag{5.2.12}$$

As $(p_{h/2} - p_h) \in H_0^1(\Omega)$, an application of Poincaré's Inequality lead to the desired estimate. ■

5.3 Proof of the Upper Bound

In this section, we derive upper bounds for the errors $\mathbf{u}(t) - \mathbf{u}_h(t)$ and $p(t) - p_h(t)$ in terms of the error indicator η_R. The main result of this section is given in the following theorem.

Theorem 5.3.1 *Let η_R be the global error estimator given by (5.2.3). Then there is a positive constant C such that*

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\| \leq \frac{C}{1-\beta}\eta_R, \tag{5.3.1}$$

and

$$\|p(t) - p_h(t)\| \leq \frac{C}{1-\beta}\eta_R. \tag{5.3.2}$$

Proof. By triangle inequality, we have

$$\|\mathbf{u} - \mathbf{u}_h\| \leq \|\mathbf{u} - \mathbf{u}_{h/2}\| + \|\mathbf{u}_{h/2} - \mathbf{u}_h\|. \tag{5.3.3}$$

Using the saturation assumption (5.2.4) in (5.3.3), we obtain

$$\|\mathbf{u} - \mathbf{u}_h\| \leq \frac{1}{1-\beta}\|\mathbf{u}_{h/2} - \mathbf{u}_h\|. \tag{5.3.4}$$

Applying the estimate of Lemma 5.2.1 to (5.3.4), we obtain

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\| &\leq \frac{C}{1-\beta}\left(\sum_{K\in\mathcal{T}_h}\|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))}\right)^{1/2} \\
&\leq \frac{C}{1-\beta}\eta_R,
\end{aligned}$$

which proves the first inequality (5.3.1). Next, to prove (5.3.2), we have from the triangle inequality

$$\|\nabla(p - p_h)\| \leq \|\nabla(p - p_{h/2})\| + \|\nabla(p_{h/2} - p_h)\|.$$

Using (5.2.5) and (5.2.12) in the above inequality, we obtain

$$\begin{aligned}
\|\nabla(p - p_h)\| &\leq \frac{1}{1-\beta}\|\nabla(p_{h/2} - p_h)\| \\
&\leq \frac{C}{1-\beta}\left(\sum_{K\in\mathcal{T}_h}\left(\|R_1(\mathbf{u}_h)\|^2_{L^2(0,t;\mathbf{V}^*(K))} + \|R_2(\mathbf{u}_h, p_h)\|^2_{L^2(K)}\right)\right)^{1/2} \\
&\leq \frac{C}{1-\beta}\eta_R,
\end{aligned}$$

An application of Poincaré's Inequality now leads to the desired estimate. This completes the proof. ■

Remark. In this chapter, we discuss the semidiscrete a posteriori error analysis for H^1-Galerkin mixed finite element method for the problem (5.1.1)-(5.1.3). Our analysis is based on residual approach. The upper bounds for the errors $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$ are derived via a saturation assumption. Compared to [8], the present analysis is not subject to LBB-consistency condition and the error estimators are free from edge residuals.

Chapter 6

Space-Time Discretization A Posteriori Error Analysis for H^1-Galerkin MFEM for Parabolic Problems

In this chapter, we study discrete-in-time a posteriori error analysis for H^1-Galerkin mixed finite element method for parabolic problems. The time discretization is based on backward Euler scheme with variable time step and the spatial discretization consists of Raviart-Thomas finite elements. The main result of this chapter consists of building error indicators with respect to both time and space approximations. Further, these indicators yield upper bound for the errors which are global in space and time. Moreover, there is no restriction on the relation between the step sizes in space and time.

6.1 Introduction

Recalling the parabolic problem:

$$
\begin{aligned}
p_t - \nabla \cdot (a \nabla p) &= f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times J, && (6.1.1)\\
p(\mathbf{x}, t) &= 0, \quad (\mathbf{x}, t) \in \partial\Omega \times J, && (6.1.2)\\
p(\mathbf{x}, 0) &= p_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, && (6.1.3)
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $p_t = \frac{\partial p}{\partial t}$ and J denotes the time interval $(0, T]$ with $T < \infty$. The coefficient matrix $a = a(\mathbf{x})$ is assumed to be smooth, symmetric and uniformly positive definite in Ω. The source function $f(\mathbf{x}, t)$ and the initial function $p_0(\mathbf{x})$ are assumed to be smooth functions. With the flux variable

$\mathbf{u} = a\nabla p$, problem (6.1.1) reduces to a first-order system as:

$$p_t - \nabla \cdot \mathbf{u} = f, \quad (\mathbf{x}, t) \in \Omega \times J, \tag{6.1.4}$$

$$\alpha \mathbf{u} - \nabla p = 0, \quad (\mathbf{x}, t) \in \Omega \times J, \tag{6.1.5}$$

where $\alpha = 1/a$. For the purpose of error analysis, we shall need the following spaces:

$$\mathbf{V} = \{\mathbf{v} \in (L^2(\Omega))^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{V}} = (\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2)^{1/2},$$

and

$$H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$$

with the usual Sobolev norm [1, 40]. Let $\mathbf{V}^*$ is the dual space of $\mathbf{V}$. The H^1-Galerkin mixed formulation for the problem (6.1.4)-(6.1.5) is to seek a pair $\{\mathbf{u}, p\} : [0, T] \longrightarrow \mathbf{V} \times H_0^1(\Omega)$ such that

$$(\alpha \mathbf{u}_t, \Psi) + A(\mathbf{u}, \Psi) = -(f, \nabla \cdot \Psi) + \lambda(\mathbf{u}, \Psi), \ \forall\, \Psi \in \mathbf{V}, \tag{6.1.6}$$

$$(\nabla p, \nabla \phi) = (\alpha \mathbf{u}, \nabla \phi), \ \forall\, \phi \in H_0^1(\Omega) \tag{6.1.7}$$

with $\mathbf{u}(0) = \mathbf{u}(\mathbf{x}, 0) = a\nabla p_0$. Here, $(\cdot, \cdot)$ denotes the standard L^2 inner product and the bilinear form $A(\cdot, \cdot)$ is given by

$$A(\mathbf{u}, \mathbf{v}) = (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \lambda(\mathbf{u}, \mathbf{v}), \ \lambda > 0,$$

which is $\mathbf{V}$-coercive with suitable choice of λ.

To start with the time discretization based on backward Euler approximation, let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of the time interval $[0, T]$ for some $N \geq 1$. Set $\tau_n = t_n - t_{n-1}$, $1 \leq n \leq N$. Now, for a smooth function ϕ on $[0, T]$, define $\phi^n = \phi(t_n)$. With each time step $(t_{n-1}, t_n]$, we associate an affinely equivalent, admissible, and shape regular partition $\mathcal{T}_{h,n}$ of Ω and the corresponding conforming finite element spaces $\mathbf{V}_h^n$ and W_h^n of $\mathbf{V}$ and $H_0^1(\Omega)$, respectively. The discrete problem based on backward Euler method is stated as follows: Find $\{\mathbf{u}_h^n, p_h^n\} \in \mathbf{V}_h^n \times W_h^n$ such that for $n = 1, \ldots, N$,

$$(\alpha(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n}), \Psi_h) + A(\mathbf{u}_h^n, \Psi_h) = -(f^n, \nabla \cdot \Psi_h) + \lambda(\mathbf{u}_h^n, \Psi_h), \tag{6.1.8}$$

and

$$(\nabla p_h^n, \nabla \phi_h) = (\alpha \mathbf{u}_h^n, \nabla \phi_h), \tag{6.1.9}$$

for all $\Psi_h \in \mathbf{V}_h^n$ and $\phi_h \in W_h^n$ with given $\mathbf{u}_h(0)$ to be defined later. With the sequence of solutions $\{\mathbf{u}_h^n, p_h^n\}$, we associate the functions $\{\mathbf{u}_{h\tau}, p_{h\tau}\}$ which are piecewise affine on the time intervals $[t_{n-1}, t_n], 1 \leq n \leq N$, and equals $\{\mathbf{u}_h^n, p_h^n\}$ at time $t = t_n,\ 1 \leq n \leq N$. Similarly, we denote $f_{h\tau}$, the function which is piecewise constant on time intervals and on each interval $(t_{n-1}, t_n]$ is equal to the L^2-projection of f^n onto the finite element space $\mathbf{V}_h^n$. With each time step $(t_{n-1}, t_n]$, the partition $\mathcal{T}_{h,n}$ of Ω and the corresponding finite element spaces $\mathbf{V}_h^n$ and W_h^n satisfy the following conditions (cf. [18, 84]):

1. *Affine equivalence:* Every element $K \in \mathcal{T}_{h,n}$ can be mapped by an invertible affine mapping onto the standard reference element in $\mathbb{R}^2$.

2. *Admissibility:* Any two elements are either disjoint or share a vertex, or a complete edge.

3. *Shape regularity:* For any element K the ratio of its diameter h_K to the diameter ρ_K of the largest inscribed ball is bounded uniformly with respect to all partitions $\mathcal{T}_{h,n}$ and to N.

4. *Transition condition:* For $1 \leq n \leq N$, there is an affinely equivalent, admissible, and shape regular partition $\widetilde{\mathcal{T}}_{h,n}$ such that it is a refinement of both $\mathcal{T}_{h,n}$ and $\mathcal{T}_{h,n-1}$ and such that
$$\sup_{1\leq n\leq N}\ \sup_{K\in\widetilde{\mathcal{T}}_{h,n}}\ \sup_{K'\in\mathcal{T}_{h,n};K\subset K'} \frac{h_{K'}}{h_K} < \infty.$$

5. Each W_h^n is a subset of $H_0^1(\Omega)$ and consists of continuous functions which are piecewise polynomials. The degrees of the polynomials are assumed to be bounded uniformly with respect to all partitions $\mathcal{T}_{h,n}$ and to N.

6. $\mathbf{V}_h^n \subset \mathbf{V}$ is the Raviart-Thomas finite element space corresponding to $\mathcal{T}_{h,n}$.

Condition 1 restricts quadrilateral elements to parallelograms. One can also consider combination of both triangular and quadrilateral elements. Condition 2 excludes the hanging nodes. Condition 3 is a standard one and allows for locally refined meshes. However, it excludes anisotropic elements with large aspect ratios. Condition 4 is due to the simultaneous presence of finite element functions defined on different grids. In practice the partition $\mathcal{T}_{h,n}$ is usually obtained by a combination of refinement and coarsening. For $1 \leq n \leq N$, $\widetilde{\mathcal{T}}_{h,n}$ is a refinement of both $\mathcal{T}_{h,n}$ and $\mathcal{T}_{h,n-1}$. Also, there is a uniform bound with respect to n on the ratio of the diameters of the elements K' in $\mathcal{T}_{h,n}$ and of elements K in $\widetilde{\mathcal{T}}_{h,n}$ contained in K'. This condition is needed to handle the functions $\mathbf{u}_h^n$ and $\mathbf{u}_h^{n-1}$ simultaneously which are defined on different grids. In this case, condition 4 restricts only the coarsening.

The literature concerning space-time discretization a posteriori error analysis for standard Galerkin method can be found in [2, 43, 46, 76, 84] for linear parabolic problems and [59, 82, 85] for nonlinear parabolic problems. The previous work on fully discrete a posteriori error analysis by means of the classical mixed method for parabolic problems is contained in [86]. So far, there is no result available for H^1-Galerkin mixed finite element method for parabolic problems. In this chapter, we study a fully discrete a posteriori error analysis based on backward Euler method for the parabolic problem (5.1.1)-(5.1.3) by H^1-Galerkin mixed finite element method.

This chapter is organized as follows. Section 6.2 deals with the equivalence of the errors with the residuals. In Section 6.3, upper bounds for the errors are established.

Throughout this chapter, C denotes a positive generic constant which may not be the same at each occurrence.

6.2 Equivalence of Errors and Residuals

In this section, we show the equivalence of the errors $\mathbf{u} - \mathbf{u}_{h\tau}$ and $\nabla(p - p_{h\tau})$ with the residuals $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ and $R_{2,h}(u_{h\tau}, p_{h\tau})$, respectively. We now define the residuals $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ and $R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})$, respectively, by

$$< R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi >= -(f, \nabla \cdot \Psi) - (\nabla \cdot \mathbf{u}_{h\tau}, \nabla \cdot \Psi) - (\alpha(\mathbf{u}_{h\tau})_t, \Psi), \tag{6.2.1}$$

and

$$(R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau}), \nabla\phi) = (\alpha\mathbf{u}_{h\tau}, \nabla\phi) - (\nabla p_{h\tau}, \nabla\phi) \tag{6.2.2}$$

for all $\Psi \in \mathbf{V}$ and $\phi \in H_0^1(\Omega)$. Here, $< \cdot, \cdot >$ denotes the dual inner product and $(\cdot, \cdot)$ is the standard L^2-inner product. The symbol $(\cdot)_t$ denotes the differentiation with respect to time. To begin with, we shall first show the equivalence of the error $\mathbf{u} - \mathbf{u}_{h\tau}$ with the residual $R_{1,h,\tau}(\mathbf{u}_{h\tau})$. The proof is based on standard energy technique.

Lemma 6.2.1 *Assume that $\mathbf{u}_h(0) = L_h\mathbf{u}_0$, where L_h is the standard L^2-projection onto $\mathbf{V}_h^n$. Let $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ be defined by (6.2.1). Then, the following lower bound on the error $\mathbf{u} - \mathbf{u}_{h\tau}$*

$$\|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|_{L^2(0,t;\mathbf{V}^*(\Omega))} \leq C \left(\|\mathbf{u}_0 - L_h\mathbf{u}_0\| + \|\mathbf{u} - \mathbf{u}_{h\tau}\| + \|\mathbf{u} - \mathbf{u}_{h\tau}\|_{L^2(0,t;\mathbf{V}(\Omega))} \right) \tag{6.2.3}$$

holds. Conversely, for all $n \in [1, N]$ and $0 \leq t \leq t_n$, the error $\mathbf{u} - \mathbf{u}_{h\tau}$ can be bounded above by

$$\|\mathbf{u} - \mathbf{u}_{h\tau}\|^2 + \|\mathbf{u} - \mathbf{u}_{h\tau}\|^2_{L^2(0,t_n;\mathbf{V}(\Omega))} \leq C \left(\|\mathbf{u}_0 - L_h\mathbf{u}_0\|^2 + \|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|^2_{L^2(0,t_n;\mathbf{V}^*(\Omega))} \right). \tag{6.2.4}$$

Proof. The residual $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ can be rewritten as

$$\begin{aligned} < R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi > &= -(f, \nabla\cdot\Psi) - (\nabla\cdot\mathbf{u}_{h\tau}, \nabla\cdot\Psi) - (\alpha(\mathbf{u}_{h\tau})_t, \Psi) \\ &= (\alpha\mathbf{u}_t - \alpha(\mathbf{u}_{h\tau})_t, \Psi) + (\nabla\cdot(\mathbf{u}-\mathbf{u}_{h\tau}), \nabla\cdot\Psi), \end{aligned}$$

where we have used (6.1.6). Integrating the above equation from 0 to t with $0 \le t \le t_n$, and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \int_0^t < R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi > ds &= (\alpha\int_0^t (\mathbf{u}-\mathbf{u}_{h\tau})_t ds, \Psi) + (\int_0^t \nabla\cdot(\mathbf{u}-\mathbf{u}_{h\tau}) ds, \nabla\cdot\Psi) \\ &= (\alpha((\mathbf{u}-\mathbf{u}_{h\tau})(t) - (\mathbf{u}-\mathbf{u}_{h\tau})(0)), \Psi) \\ &\quad + \left(\int_0^t \nabla\cdot(\mathbf{u}-\mathbf{u}_{h\tau}) ds, \nabla\cdot\Psi\right) \\ &\le C\left\{\|\mathbf{u}-\mathbf{u}_{h\tau}\| + \|\mathbf{u}_0 - L_h\mathbf{u}_0\|\right\}\|\Psi\| \\ &\quad + \|\nabla\cdot(\mathbf{u}-\mathbf{u}_{h\tau})\|_{L^2(0,t;L^2(\Omega))}\|\nabla\cdot\Psi\| \\ &\le C\{\|\mathbf{u}_0 - L_h\mathbf{u}_0\| + \|\mathbf{u}-\mathbf{u}_{h\tau}\| \\ &\quad + \|\mathbf{u}-\mathbf{u}_{h\tau}\|_{L^2(0,t;\mathbf{V}(\Omega))}\}\|\Psi\|_{\mathbf{V}}, \end{aligned}$$

where in the third step we have used the fact $\mathbf{u}_h(0) = \mathbf{u}_{h\tau}(0) = L_h\mathbf{u}_0$ and this yields the estimate (6.2.3). Next, to prove (6.2.4), choose an integer $n \in [1, N]$ and time $0 \le t \le t_n$. From (6.1.6) and (6.2.1), we have

$$\begin{aligned} (\alpha(\mathbf{u}-\mathbf{u}_{h\tau})_t, \Psi) + A(\mathbf{u}-\mathbf{u}_{h\tau}, \Psi) &= -(f, \nabla\cdot\Psi) - (\nabla\cdot\mathbf{u}_{h\tau}, \nabla\cdot\Psi) \\ &\quad - (\alpha(\mathbf{u}_{h\tau})_t, \Psi) + \lambda(\mathbf{u}-\mathbf{u}_{h\tau}, \Psi) \\ &= < R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi > + \lambda(\mathbf{u}-\mathbf{u}_{h\tau}, \Psi) \end{aligned}$$

for all $\Psi \in \mathbf{V}(\Omega)$. Set $\Psi = (\mathbf{u}-\mathbf{u}_{h\tau})(\cdot,t)$ in the above equation. Then, applying coercivity of $A(\cdot,\cdot)$, Cauchy-Schwarz inequality and Young's inequality, the resulting equation becomes

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{\|\alpha^{1/2}(\mathbf{u}-\mathbf{u}_{h\tau})(\cdot,t)\|^2\right\} \; + \; & c_1\|(\mathbf{u}-\mathbf{u}_{h\tau})(\cdot,t)\|_{\mathbf{V}}^2 \\ &\le \|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*}\|\mathbf{u}-\mathbf{u}_{h\tau}\|_{\mathbf{V}} + \lambda\|\mathbf{u}-\mathbf{u}_{h\tau}\|^2 \\ &\le C(\epsilon)\|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*}^2 + \epsilon C\|\mathbf{u}-\mathbf{u}_{h\tau}\|_{\mathbf{V}}^2 \\ &\quad + \lambda\|\mathbf{u}-\mathbf{u}_{h\tau}\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{\|\alpha^{1/2}(\mathbf{u}-\mathbf{u}_{h\tau})(\cdot,t)\|^2\right\} \; + \; & (c_1 - \epsilon C)\|(\mathbf{u}-\mathbf{u}_{h\tau})(\cdot,t)\|_{\mathbf{V}}^2 \\ &\le C(\epsilon)\|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*}^2 + \lambda\|\mathbf{u}-\mathbf{u}_{h\tau}\|^2. \end{aligned}$$

Choose $\epsilon > 0$ such that $(c_1 - \epsilon C) > 0$. Now, integrating from 0 to t and using Gronwall's lemma, we obtain

$$\begin{aligned}\|\alpha^{1/2}(\mathbf{u}-\mathbf{u}_{h\tau})(\cdot,t)\|^2 &+ \int_0^t \|(\mathbf{u}-\mathbf{u}_{h\tau})(\cdot,s)\|^2_{\mathbf{V}} ds \\ &\leq C\left(\|\mathbf{u}_0 - L_h\mathbf{u}_0\|^2 + \int_0^t \|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|^2_{\mathbf{V}^*} ds\right)\end{aligned}$$

which leads to the desired estimate (6.2.4). Here, we have used the fact that $\mathbf{u}_{h\tau}(0) = \mathbf{u}_h(0) = L_h\mathbf{u}_0$. This completes the proof. ■

Our next result shows the equivalence between $\nabla(p - p_{h\tau})$ and $R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})$.

Lemma 6.2.2 *Let $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ and $R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})$ are defined by (6.2.1) and (6.2.2), respectively. Then, the following lower bound on the error $\nabla(p - p_{h\tau})$*

$$\|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\| \leq C\left\{\|\mathbf{u}-\mathbf{u}_{h\tau}\| + \|\nabla(p-p_{h\tau})\|\right\} \tag{6.2.5}$$

holds. Conversely, the error $\nabla(p - p_{h\tau})$ is bounded above by

$$\|\nabla(p-p_{h\tau})\| \leq C\left\{\|\mathbf{u}_0 - L_h\mathbf{u}_0\| + \|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|_{L^2(0,t_n;\mathbf{V}^*(\Omega))} + \|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\|\right\}. \tag{6.2.6}$$

Proof. Using (6.1.7) and (6.2.2), we note that

$$\begin{aligned}(R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau}), \nabla\phi) &= (\alpha\mathbf{u}_{h\tau}, \nabla\phi) - (\nabla p_{h\tau}, \nabla\phi) \\ &= (\alpha(\mathbf{u}_{h\tau} - \mathbf{u}), \nabla\phi) - (\nabla(p_{h\tau} - p), \nabla\phi) \\ &\leq C\left(\|\mathbf{u}-\mathbf{u}_{h\tau}\| + \|\nabla(p-p_{h\tau})\|\right)\|\nabla\phi\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\| &= \sup_{\nabla\phi\in L^2(\Omega)} \frac{(R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau}), \nabla\phi)}{\|\nabla\phi\|} \\ &\leq C\left(\|\mathbf{u}-\mathbf{u}_{h\tau}\| + \|\nabla(p-p_{h\tau})\|\right),\end{aligned}$$

and this proves the estimate (6.2.5). Conversely, for all $\phi \in H^1_0(\Omega)$, (6.1.7) and (6.2.2) leads to

$$(\nabla(p-p_{h\tau}), \nabla\phi) = (\alpha(\mathbf{u}-\mathbf{u}_{h\tau}), \nabla\phi) + (R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau}), \nabla\phi). \tag{6.2.7}$$

Choosing $\phi = p - p_{h\tau}$ in (6.2.7) and using Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\begin{aligned}\|\nabla(p-p_{h\tau})\|^2 &\leq C\|\mathbf{u}-\mathbf{u}_{h\tau}\|\,\|\nabla(p-p_{h\tau})\| + \|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\|\,\|\nabla(p-p_{h\tau})\| \\ &\leq C(\epsilon)\left(\|\mathbf{u}-\mathbf{u}_{h\tau}\|^2 + \|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\|^2\right) + \epsilon C\|\nabla(p-p_{h\tau})\|^2.\end{aligned}$$

Choose $\epsilon > 0$ such that $(1 - \epsilon C) > 0$ and hence, we obtain

$$\|\nabla(p - p_{h\tau})\|^2 \le C\left\{\|\mathbf{u} - \mathbf{u}_{h\tau}\|^2 + \|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\|^2\right\}. \tag{6.2.8}$$

Use (6.2.4) in (6.2.8) to have

$$\begin{aligned}\|\nabla(p - p_{h\tau})\|^2 \le\ & C\{\|\mathbf{u}_0 - L_h\mathbf{u}_0\|^2 + \|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|^2_{L^2(0,t_n;\mathbf{V}^*(\Omega))} \\ & + \|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\|^2\},\end{aligned} \tag{6.2.9}$$

and this completes the proof. ■

Remark. As a consequence of Lemma 6.2.2, we have the following estimate

$$\|p - p_{h\tau}\| \le C\{\|\mathbf{u}_0 - L_h\mathbf{u}_0\| + \|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|_{L^2(0,t_n;V^*(\Omega))} + \|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\|\},$$

where we have used Poincaré's inequality $\|p - p_{h\tau}\| \le C\|\nabla(p - p_{h\tau})\|$.

Next, we shall bound the residuals in terms of the error indicators. For this purpose, we decompose the residual $R_{1,h,\tau}(\mathbf{u}_{h\tau})$ as:

$$< R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi >= (f_{h\tau} - f, \nabla\cdot\Psi) + < R_{11,h}(\mathbf{u}_{h\tau}), \Psi > + (R_{12,\tau}(\mathbf{u}_{h\tau}), \Psi), \tag{6.2.10}$$

where the spatial residual $R_{11,h}(\mathbf{u}_{h\tau})$ and the temporal residual $R_{12,\tau}(\mathbf{u}_{h\tau})$ are defined by

$$< R_{11,h}(\mathbf{u}_{h\tau}), \Psi > = -(f_{h\tau}, \nabla\cdot\Psi) - (\nabla\cdot\mathbf{u}_h^n, \nabla\cdot\Psi) - (\alpha(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n}), \Psi), \tag{6.2.11}$$

and

$$(R_{12,\tau}(\mathbf{u}_{h\tau}), \Psi) = (\nabla\cdot\mathbf{u}_h^n - \nabla\cdot\mathbf{u}_{h\tau}, \nabla\cdot\Psi), \tag{6.2.12}$$

respectively, on $(t_{n-1}, t_n]$ for all $\Psi \in \mathbf{V}(\Omega)$ and $1 \le n \le N$. Since $(u_{h\tau})_t$ is piecewise constant, it equals $\frac{u_h^n - u_h^{n-1}}{\tau_n}$ on $(t_{n-1}, t_n]$. We now define the error indicator corresponding to the first residual $R_{1,h,\tau}(u_{h\tau})$ as

$$\begin{aligned}\eta_{1,h,\tau}^n :=\ & \{\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \tau_n\|f_{h\tau} + \nabla\cdot\mathbf{u}_h^n\|^2_{L^2(K)} + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \frac{h_K^2}{\tau_n}\|\alpha(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2_{L^2(K)} \\ & + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \tau_n\|\nabla\cdot(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2_{L^2(K)}\}^{1/2}.\end{aligned} \tag{6.2.13}$$

The spatial error indicators $\eta_{11,h}^n$ and $\eta_{2,h}^n$ corresponding to the residuals $R_{11,h}(\mathbf{u}_{h\tau})$ and $R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})$ are defined by

$$\eta_{11,h}^n := \left\{\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|f_{h\tau} + \nabla\cdot\mathbf{u}_h^n\|^2_{L^2(K)} + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \frac{h_K^2}{\tau_n^2}\|\alpha(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2_{L^2(K)}\right\}^{1/2}, \tag{6.2.14}$$

and

$$\eta_{2,h}^n := \left\{ \sum_{K \in \mathcal{T}_{h,n}} \|\alpha \mathbf{u}_h^n - \nabla p_h^n\|_{L^2(K)}^2 \right\}^{1/2}, \tag{6.2.15}$$

respectively. Here $\eta_{11,h}^n$ and $\eta_{2,h}^n$ measure the errors of the space discretization and can be used to adapt the mesh size in space. The third term in $\eta_{1,h,\tau}^n$ can be interpreted as a measure for the error of the time discretization. It can be used for controlling the step size in time.

6.3 Upper bounds for the Errors

In this section, we derive upper bounds for the errors $\mathbf{u} - \mathbf{u}_{h\tau}$ and $p - p_{h\tau}$ in terms of the error indicators and input data of the problem. First, we shall show the residuals are bounded by the error indicators. To begin with, we shall first show the bound for $R_{11,h}(\mathbf{u}_{h\tau})$ in terms of $\eta_{11,h}^n$.

Lemma 6.3.1 *Let $\eta_{11,h}^n$ be defined by (6.2.14). Then, the following estimate holds on each interval $(t_{n-1}, t_n]$, $1 \leq n \leq N$,*

$$\|R_{11,h}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*} \leq C^\dagger \eta_{11,h}^n, \tag{6.3.1}$$

where the constant $C^\dagger$ depends on the maximal ratio of the diameter of any element K to its largest inscribed ball. The constant $C^\dagger$ in addition depends on the maximal ratio of the diameter of any element K' in $\mathcal{T}_{h,n}$ to the diameter of any element K in $\widetilde{\mathcal{T}}_{h,n}$ contained in K'.

Proof. Choose an integer n between 1 and N and keep it fixed. The definition of $f_{h\tau}$ and equations (6.1.8) and (6.2.11) imply the following Galerkin orthogonality

$$< R_{11,h}(\mathbf{u}_{h\tau}), \Psi_h >= 0, \;\; \Psi_h \in \mathbf{V}_h. \tag{6.3.2}$$

Define $I_h : \mathbf{V}(\Omega) \longrightarrow \mathbf{V}_h^n$ the quasi interpolation operator [72, 83], having values in the space of continuous, piecewise linear finite element functions corresponding to the partition $\mathcal{T}_{h,n}$. Consider an arbitrary element K of $\widetilde{\mathcal{T}}_{h,n}$. Denote K' an element of $\mathcal{T}_{h,n}$ such that $K \subset K'$. The interpolation error yields the following estimates for all $\Psi \in \mathbf{V}(\Omega)$:

$$\begin{aligned} \|\nabla \cdot (\Psi - I_h\Psi)\|_{L^2(K)} &\leq \|\nabla \cdot (\Psi - I_h\Psi)\|_{L^2(K')} \\ &\leq C_0 \|\nabla \cdot \Psi\|_{L^2(\widetilde{\omega}_K)}, \end{aligned}$$

$$\|\Psi - I_h\Psi\|_{L^2(K)} \le \|\Psi - I_h\Psi\|_{L^2(K')} \le C_1 h_{K'} \|\nabla\cdot\Psi\|_{L^2(\widetilde{w}_K)} \le \widetilde{C}_1 h_K \|\nabla\cdot\Psi\|_{L^2(\widetilde{w}_K)}.$$

Here, the subset $\widetilde{w}_K$ consists of all elements that share at least a vertex with K'. The constants C_0 and C_1 depend on the maximal ratio of the diameter of any element to the diameter of its largest inscribed ball. The constant $\widetilde{C}_1$ depends on the ratio $\frac{h_{K'}}{h_K}$.

Since $\widetilde{\mathcal{T}}_{h,n}$ is a refinement of both $\mathcal{T}_{h,n}$ and $\mathcal{T}_{h,n-1}$, integrating over the elements of $\widetilde{\mathcal{T}}_{h,n}$ gives the following representation of the spatial residual

$$< R_{11,h}(\mathbf{u}_{h\tau}), \Psi >= \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \{(-f_{h\tau}, \nabla\cdot\Psi)_K - (\nabla\cdot\mathbf{u}_h^n, \nabla\cdot\Psi)_K - (\alpha(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n}), \Psi)_K\}$$

for any $\Psi \in \mathbf{V}(\Omega)$. Using Cauchy-Schwarz inequality for integrals and sums and the Galerkin orthogonality (6.3.2), we arrive at the following inequality,

$$\begin{aligned}
< R_{11,h}(\mathbf{u}_{h\tau}), \Psi > &= < R_{11,h}(\mathbf{u}_{h\tau}), \Psi - I_h\Psi > + < R_{11,h}(\mathbf{u}_{h\tau}), I_h\Psi > \\
&= \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} (-f_{h\tau} - \nabla\cdot\mathbf{u}_h^n, \nabla\cdot(\Psi - I_h\Psi))_K \\
&\quad + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} (-\alpha(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n}), \Psi - I_h\Psi)_K \\
&\le \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|f_{h\tau} + \nabla\cdot\mathbf{u}_h^n\|_{L^2(K)} \|\nabla\cdot(\Psi - I_h\Psi)\|_{L^2(K)} \\
&\quad + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|\alpha(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n})\|_{L^2(K)} \|\Psi - I_h\Psi\|_{L^2(K)} \\
&\le C_0 \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|f_{h\tau} + \nabla\cdot\mathbf{u}_h^n\|_{L^2(K)} \|\nabla\cdot\Psi\|_{L^2(\widetilde{w}_K)} \\
&\quad + \widetilde{C}_1 \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \frac{h_K}{\tau_n} \|\alpha(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|_{L^2(K)} \|\nabla\cdot\Psi\|_{L^2(\widetilde{w}_K)} \\
&\le \max\{C_0, \widetilde{C}_1\}[\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|f_{h\tau} + \nabla\cdot\mathbf{u}_h^n\|^2_{L^2(K)} \\
&\quad + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \frac{h_K^2}{\tau_n^2} \|\alpha(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2_{L^2(K)}]^{1/2} (\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|\nabla\cdot\Psi\|^2_{L^2(\widetilde{w}_K)})^{1/2} \\
&\le C_2 \eta^n_{11,h} \|\Psi\|_{\mathbf{V}}.
\end{aligned}$$

Here, in the last step we have used the fact that the domain $\widetilde{w}_K$ only consists of a finite number of elements and this number is bounded by the maximal ratio of the diameter of any element to the diameter of its largest inscribed ball and depend on the ratios $\frac{h_K}{h_{K'}}$. Since $\Psi \in \mathbf{V}$ is arbitrary, we obtain

$$\|R_{11,h}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*} = \sup_{\Psi\in\mathbf{V}(\Omega)} \frac{< R_{11,h}(\mathbf{u}_{h\tau}), \Psi >}{\|\Psi\|_{\mathbf{V}}} \leq C^{\dagger}\eta^n_{11,h}. \tag{6.3.3}$$

This completes the proof of the Lemma. ■

Our next objective is to show the bound for the second residual $R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})$ in terms of $\eta^n_{2,h}$ which is given in the following lemma.

Lemma 6.3.2 *Let $\eta^n_{2,h}$ be defined by (6.2.15). Then, the following estimate holds on each interval $(t_{n-1}, t_n]$, $1 \leq n \leq N$,*

$$\|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\| \leq \eta^n_{2,h}. \tag{6.3.4}$$

Proof. With $\phi \in H^1_0(\Omega)$ and applying Cauchy-Schwarz inequality to (6.2.2), we obtain

$$\begin{aligned}
(R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau}), \nabla\phi) &= \sum_{K\in\mathcal{T}_{h,n}} (\alpha\mathbf{u}^n_h - \nabla p^n_h, \nabla\phi)_{L^2(K)} \\
&\leq \sum_{K\in\mathcal{T}_{h,n}} \|\alpha\mathbf{u}^n_h - \nabla p^n_h\|_{L^2(K)}\, \|\nabla\phi\|_{L^2(K)} \\
&\leq \Big(\sum_{K\in\mathcal{T}_{h,n}} \|\alpha\mathbf{u}^n_h - \nabla p^n_h\|^2_{L^2(K)}\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_{h,n}} \|\nabla\phi\|^2_{L^2(K)}\Big)^{1/2} \\
&\leq \eta^n_{2,h}\, \|\nabla\phi\|.
\end{aligned}$$

Thus,

$$\|R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau})\| = \sup_{\nabla\phi\in L^2(\Omega)} \frac{(R_{2,h}(\mathbf{u}_{h\tau}, p_{h\tau}), \nabla\phi)}{\|\nabla\phi\|} \leq \eta^n_{2,h}. \tag{6.3.5}$$

which completes the proof. ■

The main result of this chapter is stated in the following theorem.

Theorem 6.3.1 *Let $(\mathbf{u}, p)$ be the solution of the problems (6.1.6)-(6.1.7) and let $(\mathbf{u}_{h\tau}, p_{h\tau})$ be the solution of the discrete problem (6.1.8)-(6.1.9) with $\mathbf{u}_h(0) = L_h\mathbf{u}_0$, where L_h is the standard L^2-projection. Then, for all $1 \leq n \leq N$, the following upper bounds on the errors hold:*

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_{h\tau}\|^2 &+ \|\mathbf{u} - \mathbf{u}_{h\tau}\|^2_{L^2(0,t_n;\mathbf{V}^*(\Omega))} \leq C\{\|\mathbf{u}_0 - L_h\mathbf{u}_0\|^2 \\
&+\|f_{h\tau} - f\|^2_{L^2(0,t_n;\mathbf{V}^*(\Omega))} + \sum_{m=1}^{n}(\eta^m_{1,h,\tau})^2\},
\end{aligned} \tag{6.3.6}$$

and

$$\begin{aligned}
\|p - p_{h\tau}\|^2 &\leq C\{\|\mathbf{u}_0 - L_h\mathbf{u}_0\|^2 + \|f_{h\tau} - f\|^2_{L^2(0,t_n;\mathbf{V}^*(\Omega))} \\
&+ \sum_{m=1}^{n}(\eta^m_{1,h,\tau})^2 + (\eta^n_{2,h})^2\}.
\end{aligned} \tag{6.3.7}$$

Proof. From (6.2.10)-(6.2.12), we have

$$\begin{aligned} < R_{1,h,\tau}(\mathbf{u}_{h\tau}), \Psi > &= (f_{h\tau} - f, \nabla\cdot\Psi) + < R_{11,h}(\mathbf{u}_{h\tau}), \Psi > + (\nabla\cdot(\mathbf{u}_h^n - \mathbf{u}_{h\tau}), \nabla\cdot\Psi) \\ &\leq \|f_{h\tau} - f\|\,\|\nabla\cdot\Psi\| + \|R_{11,h}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*}\,\|\Psi\|_{\mathbf{V}} \\ &\quad + \|\nabla\cdot(\mathbf{u}_h^n - \mathbf{u}_{h\tau})\|\,\|\nabla\cdot\Psi\| \\ &\leq C\{\|f_{h\tau} - f\| + \|R_{11,h}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*} + \|\nabla\cdot(\mathbf{u}_h^n - \mathbf{u}_{h\tau})\|\}\|\Psi\|_{\mathbf{V}} \end{aligned}$$

which further leads to

$$\|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*} \leq C\{\|f_{h\tau} - f\| + \|R_{11,h}(\mathbf{u}_{h\tau})\|_{\mathbf{V}^*} + \|\nabla\cdot(\mathbf{u}_h^n - \mathbf{u}_{h\tau})\|\}.$$

Fix $n \in [1, N]$ and $0 \leq t \leq t_n$. Integrating the above equation from 0 to t, we obtain

$$\begin{aligned} \|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|^2_{L^2(0,t;\mathbf{V}^*(\Omega))} &\leq C\{\|f_{h\tau} - f\|^2_{L^2(0,t;L^2(\Omega))} + \|R_{11,h}(\mathbf{u}_{h\tau})\|^2_{L^2(0,t;\mathbf{V}^*(\Omega))}\} \\ &\quad + C\|\nabla\cdot(\mathbf{u}_h^n - \mathbf{u}_{h\tau})\|^2_{L^2(0,t;L^2(\Omega))}. \end{aligned} \tag{6.3.8}$$

At $t = t_n$, we have

$$\begin{aligned} \|R_{11,h}(\mathbf{u}_{h\tau})\|_{L^2(0,t_n;\mathbf{V}^*(\Omega))} &= \left(\int_0^{t_n} \|R_{11,h}(\mathbf{u}_{h\tau})\|^2_{\mathbf{V}^*(\Omega)} ds\right)^{1/2} \\ &= \left(\sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \|R_{11,h}(\mathbf{u}_{h\tau})\|^2_{\mathbf{V}^*(\Omega)} ds\right)^{1/2} \\ &\leq C\left(\sum_{m=1}^{n} \tau_m (\eta_{11,h}^m)^2\right)^{1/2}, \end{aligned} \tag{6.3.9}$$

where we have used Lemma 6.3.1. As $\mathbf{u}_{h\tau}$ is piecewise affine on each time interval $(t_{m-1}, t_m]$, we have

$$\mathbf{u}_h^m - \mathbf{u}_{h\tau} = \left(1 - \frac{t - t_{m-1}}{\tau_m}\right)\left(\mathbf{u}_h^m - \mathbf{u}_h^{m-1}\right). \tag{6.3.10}$$

Using the identity (6.3.10), we obtain

$$\begin{aligned} \|\nabla\cdot(\mathbf{u}_h^n - \mathbf{u}_{h,\tau})\|_{L^2(0,t_n;L^2(\Omega))} &= \left(\sum_{m=1}^{n} \|\nabla\cdot(\mathbf{u}_h^m - \mathbf{u}_h^{m-1})\|^2 \int_{t_{m-1}}^{t_m} \left(1 - \frac{s - t_{m-1}}{\tau_m}\right)^2 ds\right)^{1/2} \\ &= \left(\sum_{m=1}^{n} \|\nabla\cdot(\mathbf{u}_h^m - \mathbf{u}_h^{m-1})\|^2 \int_{t_{m-1}}^{t_m} \left(\frac{t_m - s}{\tau_m}\right)^2 ds\right)^{1/2} \\ &= \left(\sum_{m=1}^{n} \frac{\tau_m}{3}\|\nabla\cdot(\mathbf{u}_h^m - \mathbf{u}_h^{m-1})\|^2\right)^{1/2}. \end{aligned} \tag{6.3.11}$$

Using (6.3.9) and (6.3.11) in (6.3.8) with $t = t_n$, we obtain

$$\begin{aligned}\|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|^2_{L^2(0,t_n;\mathbf{V}^*(\Omega))} \ \leq \ & C\{\|f_{h\tau} - f\|^2_{L^2(0,t_n;L^2(\Omega))} + \sum_{m=1}^{n} \tau_m (\eta^m_{11,h})^2 \\ & + \sum_{m=1}^{n} \tau_m \, \|\nabla \cdot (\mathbf{u}^m_h - \mathbf{u}^{m-1}_h)\|^2\}.\end{aligned}$$

In view of (6.2.13), we have

$$\|R_{1,h,\tau}(\mathbf{u}_{h\tau})\|^2_{L^2(0,t_n;\mathbf{V}^*(\Omega))} \leq C \left\{ \|f_{h\tau} - f\|^2_{L^2(0,t_n;L^2(\Omega))} + \sum_{m=1}^{n} (\eta^m_{1,h,\tau})^2 \right\}. \qquad (6.3.12)$$

Now using (6.3.12) in (6.2.4) leads to the desired estimate (6.3.6). Next, using (6.3.12) and (6.3.4) in (6.2.9), we obtain

$$\|\nabla(p - p_{h\tau})\|^2 \leq C \left\{ \|\mathbf{u}_0 - L_h \mathbf{u}_0\|^2 + \|f_{h\tau} - f\|^2_{L^2(0,t_n;L^2(\Omega))} + \sum_{m=1}^{n} (\eta^m_{1,h,\tau})^2 + (\eta^n_{2,h})^2 \right\}.$$

An application of Poincaré's Inequality leads to the estimate (6.3.7). This completes the proof of Theorem 6.3.1. ■

Remark. In this chapter, we discuss space-time discretization a posteriori error analysis for H^1-Galerkin mixed finite element method based on backward Euler method with variable time step for the problem (6.1.1)-(6.1.3). Our analysis is based on residual approach. The upper bounds for the errors are derived in terms of the error indicators and input data of the problem. The upper bounds are global in space and time. Compared to [18, 86], the present analysis is not subject to LBB-consistency condition and the error indicators are free from edge residuals.

Chapter 7

Conclusion and Extension

This chapter is devoted to critical assessment of the results highlighting the contributions made by this thesis and techniques used in deriving these. It also provides information for the scope of possible extensions and future investigations.

7.1 Critical Review of the Results

In this thesis, we have studied the H^1-Galerkin MFEM for parabolic problems. The main advantage of using mixed formulation is that it provides direct approximations to physical quantities such as fluxes and velocities. Compared to the classical mixed finite element method, the proposed method circumvents the stringent LBB-consistency condition. In the standard H^1-Galerkin method we require $\mathcal{C}^1$-continuity of the approximating spaces [23, 61, 79]. However, in H^1-Galerkin MFEM, $\mathcal{C}^1$-continuity can be relaxed and gives us freedom to work with computationally attractive piecewise linear elements. Also, the proposed method also gives us flexibility to work with two different finite element spaces for approximating the solution and the flux.

In Chapter 2, *a priori* error estimates are presented for the semidiscrete H^1-Galerkin MFEM. Both one and two space variable problems are discussed. The author of [60] has derived optimal order error estimates in L^2 and H^1-norms by assuming initial data $p_0 \in H^3(\Omega) \cap H_0^1(\Omega)$. Compared to [60], we study convergence analysis of the proposed method under lesser regularity assumption on the initial function p_0. More precisely, we establish error estimates of order $\mathcal{O}(h^2 t^{-1/2})$ and $\mathcal{O}(h^2 t^{-1})$ for the solution and the flux in L^2-norm for positive time when $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $p_0 \in H_0^1(\Omega)$, respectively (see, Theorem 2.2.1 and Theorem 2.2.2). The main technical tools used in our error analysis are standard energy argument, non-standard energy formulation and parabolic duality argument.

Then, we study the fully discrete H^1-Galerkin MFEM for one dimensional homogeneous parabolic problem (3.1.1)-(3.1.3) based on backward Euler method in Chapter 3. We have shown that the fully discrete solution converges to the true solution at almost optimal rate in the L^2-norm for both smooth and nonsmooth initial data. More precisely, error estimates of order $\mathcal{O}((h^2 + \Delta t(1 + \log \frac{1}{\Delta t})^{1/2})t_n^{-1/2})$ in the L^2-norm are established for the solution p and its flux u for positive time with smooth initial function, i.e., $p_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ (see, Theorem 3.2.1). Further, for $p_0 \in H_0^1(\Omega)$, error estimates of order $\mathcal{O}((h^2 + \Delta t(1 + \log \frac{1}{\Delta t})^{1/2})t_n^{-1})$ with positive time are derived in L^2-norm for the solution p and its flux u (see, Theorem 3.3.1). We have used the standard energy technique, non-standard energy formulation and parabolic duality argument in deriving these results.

In Chapter 4, we discuss superconvergence phenomenon for the semidiscrete H^1-Galerkin MFEM for parabolic problems (1.1.1)-(1.1.3). The results obtained in this chapter show an improved accuracy of order $\mathcal{O}(h^{k+3})$ between the H^1-Galerkin mixed finite element approximation $\mathbf{u}_h$ and an appropriately defined local projection $\pi_h \mathbf{u}$ of the flux variable, where $k \geq 1$ is the order of the approximating polynomials employed in the Raviart-Thomas element and h is the mesh size of the corresponding finite element partition. In particular, assuming higher regularity on the solution, better convergence results for the solution and its flux are established. Superconvergence results are important from an application point of view because they provide higher order accuracy under reasonable assumption on the grid and with additional smoothness of the solution. The analysis involves two linear forms (4.1.4) and (4.1.5) in deriving superconvergence estimates. The linear forms are estimated by expanding the interpolation errors $\mathbf{u} - \pi_h \mathbf{u}$ and $(\mathbf{u} - \pi_h \mathbf{u})_t$ in Taylor series involving only finite number of terms. The orthogonality property of $\mathbf{u} - \pi_h \mathbf{u}$ and $(\mathbf{u} - \pi_h \mathbf{u})_t$ with certain class of polynomials play a crucial role for deriving superconvergence result. Superconvergence estimates of order $\mathcal{O}(h^{k+3})$ are established in the L^2-norm for the solution and the flux without using LBB-consistency condition (cf. [19]) on the finite element mesh.

Chapters 5 and 6 deal with a posteriori error estimates for H^1-Galerkin MFEM for parabolic problems (1.1.1)-(1.1.3). An adaptive finite element method for numerical solution of partial differential equations is of great practical importance and has recently been an active research area [17, 21, 30, 31, 43, 59]. In general, the nature of the exact solution is not known before hand then it is not clear how to locally refine the finite element mesh. Recently, methods for automatic mesh refinement the so-called adaptive methods have been developed which don't require the user to supply information on the smoothness of the exact solution. In these methods this information is instead obtained through a sequence of computed solutions on successively refined meshes.

The adaptive algorithms are based on sharp a posteriori error estimators. Therefore, an attempt has been made to obtain a posteriori error estimates for the H^1-Galerkin MFEM for parabolic problems (1.1.1)-(1.1.3). In Chapter 5, we study the semidiscrete a posteriori error analysis based on residual approach via a saturation assumption (5.2.4)-(5.2.5). The assumptions says that on the refined mesh $\mathcal{T}_{h/2}$, the refined finite element approximation $(\mathbf{u}_{h/2}, p_{h/2})$ is a better approximation to the exact solution $(\mathbf{u}, p)$ than $(\mathbf{u}_h, p_h)$. The upper bounds for the solution and its flux are derived (cf. Theorem 5.3.1). Our analysis is not subject to the LBB-consistency condition and the estimators are free from edge residuals.

Finally, in Chapter 6, we study a space-time discretization a posteriori error analysis with variable time step for H^1-Galerkin MFEM. The upper bounds are established for the solution and its flux in terms of the estimators (cf. Theorem 6.3.1). The error analysis is carried out in two steps. First, we bound the errors by the residuals through the standard energy argument. In the second step, the residuals are bounded by the estimators $\eta^n_{1,h,\tau}$ and $\eta^n_{2,h}$. These estimators are computable quantities in terms of the problem data $\mathbf{u}_0, f_{h\tau}, \Omega, T$, computed solutions $\mathbf{u}^n_h$ and p^n_h, mesh size h_K and the time step τ_n. Such estimators provide bounds for the errors and are very useful for modifying meshes and time steps adaptively.

7.2 Extensions and Remarks

In this section, we make some informal observations pertaining to the possible extension of our results to different problems. We shall briefly outline some interesting problems to be taken up in future.

Non-smooth Error Analysis. In Chapter 2, we study *a priori* error estimates for the semidiscrete H^1-Galerkin MFEM for both one and two dimensional homogeneous parabolic problems, when the initial function $p_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $p_0 \in H^1_0(\Omega)$. In future, we would like to study the convergence analysis of the proposed method under much lesser regularity assumption on the initial function, i.e., when the initial function $p_0 \in L^2(\Omega)$. Further, an extension of the proposed method to more general elliptic and parabolic problems will be considered. Also, a fully discrete scheme based on Crank-Nicolson method for the non-homogeneous parabolic problem with both smooth and nonsmooth initial data will be taken up in future.

A Posteriori Error Analysis for Miscible Displacement Problems in Porous Media. The miscible displacement of one incompressible fluid by another in a reservoir $\Omega \subset \mathbb{R}^2$ of unit thickness and local elevation $z(x)$, $x \in \Omega$ is governed by the equation of the form

$$\phi(\mathbf{x})\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(D\nabla c) = g,\ \mathbf{x}\in\Omega,\ t\in J. \tag{7.2.1}$$

subject to the initial condition

$$c(\mathbf{x},0) = c_0(\mathbf{x}), \mathbf{x}\in\Omega, \tag{7.2.2}$$

and boundary conditions

$$-\left(\frac{\kappa(\mathbf{x})}{\mu(c)}(\nabla p - \nu_0(c)\nabla z)\right)\cdot\nu = 0,\ \mathbf{x}\in\partial\Omega,\ t\in J, \tag{7.2.3}$$

where p is the pressure, κ the permeability of the medium, μ the concentration dependent viscosity, and ν_0, the density of the fluid. Incompressibility implies

$$\nabla\cdot\mathbf{u} = q, \tag{7.2.4}$$

where $q = q(\mathbf{x}, t)$ is the imposed external flow, positive for injection and negative for production. Note that the pressure does not appear explicitly in the equation (7.2.1) for the concentration. It appears implicitly in the concentration only through its velocity field given by

$$\nabla\cdot\mathbf{u} = -\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left[a_i(\mathbf{x},c)\left(\frac{\partial p}{\partial x_i} - \nu_i(\mathbf{x},c)\right)\right] = q. \tag{7.2.5}$$

We would like to study a posteriori error analysis for the above problem by H^1-Galerkin mixed method. The error estimators for the pressure and the velocity field to be derived which will be useful for adaptive algorithm. We wish to take up this problem in future.

Interior Error Estimates for H^1-Galerkin Mixed Methods. Interior error estimates for finite element discretizations were first introduced by Nitsche and Schatz in [10, 58] for elliptic problems and Thomée in [77] for parabolic problems. The authors of [58, 77] have shown that the error in the interior of the domain $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ can be estimated with the best order of accuracy that is possible locally for the finite dimensional subspace used plus the error in the weaker norm over a slightly larger domain which measures the effects from outside of the domain Ω. The interior estimates implicitly take into consideration possible pollution from effects outside of Ω_0. These may be due to the following: (i) the smoothness of the boundary; (ii) the way in which a given method treats the boundary conditions; (iii) the smoothness of the solution outside of Ω_1. Interior error estimates have also been used successfully to study a posteriori estimators. The author of [30, 31] introduced two a posteriori error estimators based on local difference quotients of the numerical solutions. Their analysis is based on interior convergence theory in [58]. The interior convergence theory is well understood for standard finite element method (cf. [87]). But, there are only few results in this area for classical mixed finite element methods [26]. In future, we would like study interior error estimates for H^1-Galerkin mixed methods for both elliptic and parabolic problems.

The omission within the realm of this thesis is computational experiments. As our main objective is to study the theoretical aspects of H^1-Galerkin MFEM and their convergence, we have not discussed computational issues. The computational experiments will be an interesting future work.

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