PSEUDOSPECTRA OF MATRIX PENCILS AND THEIR APPLICATIONS IN PERTURBATION ANALYSIS OF EIGENVALUES AND EIGENDECOMPOSITIONS

by

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CERTIFICATE

It is certified that the work contained in the thesis titled **“Pseudospectra of Matrix Pencils and their Applications in Perturbation Analysis of Eigenvalues and Eigendecompositions”** by **Sk. Safique Ahmad**, a student in the department of Mathematics, Indian Institute of Technology, Guwahati for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

December, 2007

Prof. Rafikul Alam
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Dedicated to my Parents.

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Abstract

The main theme of the thesis revolves around pseudospectra of matrix pencils and matrix polynomials and their applications in perturbation theory.

More specifically, first, we develop a general framework for defining and analyzing pseudospectra of matrix pencils and matrix polynomials. The framework so developed unifies various definitions of pseudospectra of matrix pencils proposed in the literature. We dispel the perception that there are many nonequivalent ways of defining pseudospectra of matrix pencils/polynomials and put the analysis of pseudospectra of matrix pencils/polynomials on the same footing as that of matrices.

Second, we analyze various properties of backward error functions associated with matrix pencils/polynomials. We introduce a notion of critical points of backward error functions and show that a critical point is a multiple eigenvalue of an appropriately perturbed pencil/polynomial. We now show that common boundary points of the components of pseudospectra of a matrix pencil/polynomial are critical points. We show that a minimal critical point can be read off from the pseudospectra of matrix pencils/polynomials. Hence a solution of Wilkinson's problem for matrix pencils/polynomials can be read off from the pseudospectra of matrix pencils/polynomials. Given a diagonal pencil with distinct eigenvalues, we provide a simple procedure for the construction of nearest defective pencils.

Third, we provide various pseudospectra inclusions for matrix pencils and show that pseudospectra inclusions can be gainfully used for analyzing stability of eigendecompositions. We introduce analogues of various notions of separation of matrices to the case of matrix pencils and show their utility in analyzing stability of eigendecompositions. We show that the separations such as $\mathrm{sep}, \mathrm{sep}_\lambda$ and gsep can be defined and analyzed on the same lines as that of matrices.

Fourth, we present a general framework for the sensitivity analysis of eigenvalues of matrix pencils and matrix polynomials. We lay bare the big picture that lies behind the notion of sensitivity of eigenvalues of matrix pencils/polynomials. We show that

our treatment unifies various measures of sensitivity of simple eigenvalues of matrix pencils/polynomials proposed in the literature.

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Chapter 1

Introduction

1.1 Introduction

Eigenvalue problems for matrix pencils and matrix polynomials, respectively, are commonly referred to as generalized and polynomial eigenvalue problems. The generalized and polynomial eigenvalue problems occur in many scientific applications (see, for example, [32, 35, 28, 29, 36, 45, 14] and the references therein). The sensitivity and perturbation analysis of generalized and polynomial eigenvalue problems play an important role in the accuracy assessment of approximate solutions of these problems and their implications in the numerical simulation of the underlying scientific problems. Pseudospectra provide a powerful framework for numerical analysis of scientific problems that crucially depend on solution of eigenvalue problems (see, [47] and the references therein).

Sensitivity and perturbation analysis of generalized as well as polynomial eigenvalue problems have been studied extensively over the years [40, 43, 16, 42, 39, 17, 26, 46, 18, 14]. The pseudospectra of matrices have been studied extensively covering both the theory as well as the applications (see, for example, [47, 2, 3, 21]). Similarly, the pseudospectra of matrix pencils and matrix polynomials have been studied covering theoretical as well as practical aspects (see, for example, [38, 22, 19, 30, 45, 25, 20, 10, 11, 31, 47, 12]). However, the theoretical framework for defining and analyzing pseudospectra of matrix pencils/polynomials is not as developed as their matrix counterpart. For example, several pseudospectra, that is, several nonequivalent definitions of pseudospectra of matrix pencils have been proposed in the literature over the years [38, 22, 19, 30, 45, 25, 20, 10, 11, 47]. In fact, there is a perception that, unlike the pseudospectra of matrices, there are several ways one could define pseudospectra of matrix pencils and matrix polynomials. So, one may ask: How do these pseudospectra differ from one another? Is it possible to develop a theoretical framework for pseudospectra of matrix pencils and matrix polynomials on the same lines as that of matrices?

The main theme of this thesis revolves around pseudospectra of matrix pencils and matrix polynomials. First, we develop a general theoretical framework for defining and analyzing pseudospectra of matrix pencils/polynomials. We show that our framework not only demystifies the definition of pseudospectra of matrix pencils/polynomials but also unifies various pseudospectra proposed in the literature. Our theoretical framework for pseudospectra of matrix pencils/polynomials parallels that of the matrices. Thus we show that the pseudospectra of matrix pencils/polynomials are defined as uniquely as their matrix counterpart.The crux of the matter is that we equip the space of pencils/polynomials with a norm and define pseudospectra in the abstract setting of Banach spaces of pencils/polynomials and show that various pseudospectra proposed in the literature correspond to appropriate choices of the norm on the space of pencils/polynomials.

Next, with a view to analyzing properties of pseudospectra, we analyze various properties of backward error functions associated with matrix pencils/polynomials. We analyze the level curves of backward error functions and boundaries of pseudospectra. Specially, we analyze the evolution of pseudospectra components and study their points of coalescence. To that end, we introduce a notion of generic and nongeneric critical points of the backward error functions. We show that a generic/nongeneric critical point is a multiple eigenvalue of an appropriately perturbed pencil/polynomial. We now show that common boundary points of the components of pseudospectra of a matrix pencil/polynomial are critical points. We show that points of coalesce of pseudospectra components are in fact critical points of the associated backward error function. Hence we show that a minimal critical point, that is, the critical point at which the associated backward error function has the smallest value among all the critical points, can be read off from the pseudospectra of matrix pencils/polynomials. The consequence of having a minimal critical point is that it provides a solution to Wilkinson's problem for matrix pencils/polynomials. Given a regular matrix pencil/polynimial $\mathbf{L}$ having distinct eigenvalues, let

$$\mathrm{d}(\mathbf{L}) := \inf\{|||\Delta\mathbf{L}||| : \mathbf{L} + \Delta\mathbf{L} \text{ has a multiple eigenvalue}\},$$

where $|||\cdot|||$ is a norm on the space of pencils/polynomials. Then determining $\mathrm{d}(\mathbf{L})$ and a pencil/polynomial $\Delta\mathbf{L}$ such that that $\mathbf{L}+\boldsymbol{\Delta}\mathbf{L}$ has a multiple eigenvalue and $\mathrm{d}(\mathbf{L}) = |||\Delta\mathbf{L}|||$ is widely known as Wilkinson's problem. We provide a general scheme for constructing $\Delta\mathbf{L}$ so that $\mathbf{L} + \Delta\mathbf{L}$ has a specified eigenvalue. Then we show that a minimal critical point is a multiple eigenvalue of the pencil/polynomial $\mathbf{L}+\Delta\mathbf{L}$. This is how we construct a solution to Wilkinson's problem.

Pseudospectra inclusions provide a powerful geometric framework for perturbation analysis of eigenvalues and eigendecompositions of matrices [3]. With a view to analyzing perturbation of eigenvalues and eigendecompositions of matrix pencils, we develop a general framework for obtaining pseudospectra inclusions for matrix pencils. Indeed, we obtain various pseudospectra inclusions for matrix pencils and show that pseudospectra inclusions can be gainfully used for analyzing stability of eigendecompositions. Next, various notions of separation of matrices such as sep, sep_λ and gsep have been introduced in the literature for analyzing stability of eigendecompositions of matrices [40, 43, 48, 15, 3]. We develop a theoretical framework for defining various notions of separation of matrix pencils. We define analogues of various notions of separation of matrices to the case of matrix pencils. We show that the framework so developed for defining and analyzing separation of matrix pencils parallels that of the matrices. We analyze stability of eigendecompositions of matrix pencils. We characterize stability of eigendecompositions of matrix pencils and show that separations of matrix pencils play an important role in obtaining sufficient condition for stability of eigendecompositions.

Finally, we consider the issue of sensitivity of eigenvalues of matrix pencils and matrix polynomials. Like the pseudospectra, several measures of sensitivity of eigenvalues of matrix pencils and matrix polynomials have been proposed in the literature [40, 43, 42, 39, 17, 26, 46, 36, 18]. We show that the theoretical framework developed for analyzing pseudospectra of matrix pencils/polynomials is also a natural framework for analyzing sensitivity of eigenvalues of matrix pencils/polynomials. We define a general measure of sensitivity of eigenvalues which unify various measures of sensitivity of eigenvalues proposed in the literature. Most importantly, we analyze the geometry of ill-conditioning of eigenvalues of matrix pencils/polynomials. We show that for each simple eigenvalue of a matrix pencils/polynomials there is a dual pencil/polynomial that determines the sensitivity of the eigenvalue. Indeed, we provide a whole new perspective to understanding sensitivity analysis of eigenvalues of matrix pencils/polynomials.

For simplicity of presentation, first we analyze matrix pencils and then matrix polynomials. Our theoretical framework for pseudospectra, critical points, multiple eigenvalues and Wilkinson's problem for matrix pencils are presented in Chapter-2. The analogues of these results for matrix polynomials are presented in Chapter-3. Pseudospectra inclusions, stability analysis of eigendecompositions and analysis of various notions of separation of matrix pencils are presented in Chapter-4. Finally, sensitivity analysis of eigenvalues of matrix pencils and matrix polynomials are presented in Chapter-5.

1.2 Preliminaries

We use standard notation such as $\mathbb{C}^n$ and $\mathbf{C}^{m\times n}$ to denote the vector space of n-tuples $(x_1, \ldots, x_n), x_i \in \mathbb{C}$, and the vector space of m-by-n matrices with real or complex entries. Given a matrix $A \in \mathbb{C}^{n\times n}$, we denote the spectrum of A by $\Lambda(A)$ which is given by

$$\Lambda(A) := \{\lambda \in \mathbb{C} : \text{rank}(A - \lambda I) < n\}.$$

Let $\lambda \in \Lambda(A)$. Then there are nonzero vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ such that $Ax = \lambda x$ and $y^*A = \lambda y^*$. We say that x is a right eigenvector and y is a left eigenvector of A corresponding to λ. For convenience, we say that (λ, y, x) is an eigentriple of A if $\lambda \in \Lambda(A)$ and y and x, respectively, are left and right eigenvectors of A corresponding to λ.

Let $N(A)$ denote the null space of A, that is, $N(A) := \{x \in \mathbb{C}^n : Ax = 0\}$. We denote the dimension of $N(A)$ by $\dim N(A)$. Let $\lambda \in \Lambda(A)$. Then $g(\lambda) := \dim N(A - \lambda I)$ is called the geometric multiplicity of λ and $m(\lambda) := \dim N((A - \lambda I)^n)$ is called the algebraic multiplicity of λ. If $m(\lambda) = 1$ then λ is said to be a simple eigenvalue of A. If $g(\lambda) = m(\lambda)$ then λ is said to be semi-simple. If $m(\lambda) > g(\lambda)$ then λ is said to be defective.

The singular value decomposition (SVD) of a matrix $A \in \mathbb{C}^{m\times n}$ is given by $A = U\Sigma V^*$, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{C}^{m\times n}$ is a diagonal matrix with nonnegative diagonal entries (appear in descending order of magnitude). We denote the smallest singular value of a matrix $A \in \mathbb{C}^{n\times n}$ by $\sigma_{\min}(A)$.

Let $p := (p_1, \ldots, p_N)^T \in \mathbb{R}^N$ and $A(p) \in \mathbb{C}^{m\times n}$. Suppose that $\sigma(p)$ is a singular value of $A(p)$. Then there exists unit right and left singular vectors $v(p) \in \mathbb{C}^n$ and $u(p) \in \mathbb{C}^m$, respectively, such that $A(p)v(p) = \sigma(p)u(p), (A(p))^*u(p) = \sigma(p)v(p)$.

Theorem 1.2.1. *[44] Let $p \in \mathbb{R}^N$ and $A(p) \in \mathbb{C}^{m\times n}$. Suppose that Re$[A(p)]$ and Im$[A(p)]$ are real analytic matrix valued functions of p in some neighborhood $\mathcal{B}(0)$ of the origin. Let σ be a simple nonzero singular value of $A(0)$ with associated unit right and left singular vectors $v \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$, respectively. Then there exists a simple nonzero singular value $\sigma(p)$ of $A(p)$ which is a real analytic function of p in a neighborhood $\mathcal{N}(0)$ of the origin such that $\sigma(0) = \sigma$ and*

$$\frac{\partial \sigma(p)}{\partial p_j} = Re\left[u^* \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v\right].$$

We now briefly consider vector and matrix norms to be used in the subsequent development.

Definition 1.2.2. *A function $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}$ is said to be norm on $\mathbb{C}^n$ (or a vector norm) if $\|\cdot\|$ satisfies the following conditions:*

- $\|x\| = 0 \Leftrightarrow x = 0.$
- $\|\alpha x\| = |\alpha|\|x\|$ *for* $\alpha \in \mathbb{C}$ *and* $x \in \mathbb{C}^n$.
- $\|x + y\| \le \|x\| + \|y\|$ *for* $x, y \in \mathbb{C}^n$.

Let $\|\cdot\|$ be a norm $\mathbb{C}^n$. Define $\|\cdot\|_* : \mathbb{C}^n \to \mathbb{R}$ by

$$\|y\|_* := \sup\{|y^*x| : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

Then it is easy to see that $\|\cdot\|_*$ is a norm and is called the dual norm of the norm $\|\cdot\|$. It follows that for $x, y \in \mathbb{C}^n$, we have $|y^*x| \leq \|x\| \, \|y\|_*$.

For $1 \leq p \leq \infty$, the Hölder's p-norm on $\mathbb{C}^n$ is given by

$$\|x\|_p := \begin{cases} (\sum_{j=1}^n |x_j|^p)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \max_{1\leq j \leq n} |x_j|, & \text{for } p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_q$ is dual norm of $\|\cdot\|_p$, where $p^{-1} + q^{-1} = 1$.

For $w := (w_1, \ldots, w_n) \in \mathbb{R}^n$ and $x := (x_1, \ldots, x_n) \in \mathbb{C}^n$, we define

$$\|x\|_{w,p} := \|(w_1 x_1, \ldots, w_n x_n)\|_p.$$

Then $\|\cdot\|_{w,p}$ is a seminorm on $\mathbb{C}^n$. It follows that $\|\cdot\|_{w,p}$ defines a norm if and only if each components of w is nonzero.

Now we consider matrix norm, that is, norm on $\mathbb{C}^{n\times n}$. Let $\|\cdot\|$ be norm on $\mathbb{C}^n$. Define $\|\cdot\| : \mathbb{C}^{n\times n} \to \mathbb{R}$ by

$$\|A\| := \sup\{\|Ax\| : x \in \mathbb{C}^n, \, \|x\| = 1\}.$$

Then $\|\cdot\|$ is a norm on $\mathbb{C}^{n\times n}$ and is referred to as induced operator norm or the subordinate norm. The subordinate norm induced by the 2-norm $\|\cdot\|_2$ on $\mathbb{C}^n$ is referred to as the spectral norm or the 2-norm on $\mathbb{C}^{n\times n}$. We denote the spectral norm on $\mathbb{C}^{n\times n}$ by $\|\cdot\|_2$. Thus

$$\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|_2.$$

The Frobenius norm on $\mathbb{C}^{n\times n}$ is denoted by $\|\cdot\|_F$ and is given by $\|A\|_F := (\text{trace} A^*A)^{1/2}$. The spectral and the Frobenius norms have the following useful properties

- $\|Ux\|_2 = \|x\|_2$ if $U^*U = I_n$,
- $\|UAV^*\|_{2,F} = \|A\|_{2,F}$ if $U^*U = I = V^*V$
- $\|AB\|_{2,F} \leq \|A\|_{2,F}\|B\|_{2,F}$,

where U^* denotes the conjugate transpose of U.

1.2.1 Generalized eigenvalue problem

Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$. Then $\mathbf{L}(z) := A - zB$ is commonly referred to as matrix pencil and the eigenvalue problem $Ax = \lambda Bx$, that is, $\mathbf{L}(\lambda)x = 0$, is called a generalized eigenvalue problem. We consider both homogeneous and nonhomogeneous n-by-n pencils of the form $\mathbf{L}(c, s) := cA - sB$ and $\mathbf{L}(z) := A - zB$. So, a pencil $\mathbf{L}$ is a linear (resp., affine) map from $\mathbb{C}^2$ (resp., $\mathbb{C}$) to $\mathbb{C}^{n\times n}$. We denote the set of n-by-n matrix pencils (either homogeneous or nonhomogeneous) by $\mathbb{L}(\mathbb{C}^{n\times n})$. A pencil $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ is said to be regular if $\det(\mathbf{L}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular nonhomogeneous pencil $\mathbf{L}$, denoted by $\Lambda(\mathbf{L})$, is given by

$$\Lambda(\mathbf{L}) := \{\lambda \in \mathbb{C} : \det(\mathbf{L}(\lambda)) = 0\}.$$

Let $\lambda \in \Lambda(\mathbf{L})$. Then there exists nonzero $u, v \in \mathbb{C}^n$ such that $\mathbf{L}(\lambda)v = 0$ and $u^*\mathbf{L}(\lambda) = 0$, where u^* denotes the conjugate transpose of u. The vectors u and v are called left and right eigenvectors, respectively, of $\mathbf{L}$ corresponding to λ. We refer to (λ, u, v) as an eigentriple of $\mathbf{L}$. Note that if $\mathbf{L}(\lambda_1)v_1 = 0$ and $\mathbf{L}(\lambda_2)v_2 = 0$ with $\lambda_1 \neq \lambda_2$ then v_1 and v_2 are linearly independent. It is possible for $\mathbf{L}$ to have an infinite eigenvalue which is not included in $\Lambda(\mathbf{L})$. However, the case of an infinite eigenvalue can be resolved by considering $\Lambda(\mathbf{L})$ as a subset of $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, the one-point compactification of $\mathbb{C}$, and putting ∞ in $\Lambda(\mathbf{L})$ whenever $\det(B) = 0$.

A more convenient setup to deal with an infinite eigenvalue is to consider the homogeneous pencils $\mathbf{L}$. Thus, when $\mathbf{L}$ is homogenous, that is $\mathbf{L}(c, s) := cA - sB$, the spectrum $\Lambda(\mathbf{L})$ is given by

$$\Lambda(\mathbf{L}) := \{(c, s) \in \mathbb{C}^2 : (c, s) \neq 0 \text{ and } \det(\mathbf{L}(c, s)) = 0\}.$$

An infinite eigenvalue of $\mathbf{L}$, if any, is then represented by $(0, 1)$. Normalizing $(c, s) \in \Lambda(\mathbf{L})$ as $|c|^2 + |s|^2 = 1$, we often identify $\Lambda(\mathbf{L})$ as a subset of the unit sphere $\mathbb{S}^1 := \{(c, s) \in \mathbb{C}^2 : |c|^2 + |s|^2 = 1\}$. However, for computational purposes, further restricting c to be real, s to be complex and with the normalization $c^2 + |s|^2 = 1$, the spectrum $\Lambda(\mathbf{L})$ can be identified with a subset of the Riemann sphere $\mathbb{S} := \{(x, y, z) \in \mathbb{R}^3 : |x|^2 + |y|^2 + |z|^2 = 1\}$. In such a case, an infinite eigenvalue of $\mathbf{L}$ is represented by the north pole $(0, 0, 1)$ of $\mathbb{S}$.

Definition 1.2.3. *Consider a pencil* $\mathbf{L}(z) := A - zB$. *For nonsingular matrices* X *and* Y, *let* $\hat{A} = Y^*AX$ *and* $\hat{B} = Y^*BX$. *Then the pencil* $\hat{A} - z\hat{B}$ *is called equivalent to* $A - zB$ *and,* X *and* Y *are called equivalent transformations. Further, if* (λ, y, x) *is an eigentriple of* $A - zB$ *then* $(\lambda, Y^{-1}y, X^{-1}x)$ *is an eigentriple of* $\hat{A} - \lambda\hat{B}$.

A regular pencil $\mathbf{L}(z) := A - zB$ is said to be diagonalizable if there exists nonsingular matrices X and Y such that $Y^*AX = \mathrm{diag}(\alpha_i)$ and $\mathbf{Y}^*BX = \mathrm{diag}(\beta_i)$, where $\mathrm{diag}(\alpha_i)$ is the diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$ (may or may not be distinct). In such a case $(\alpha/\beta_i, Ye_i, Xe_i)$ is an eigentriple of $\mathbf{L}$, where e_i is the i-th column of the identity matrix $I \in \mathbb{C}^{n \times n}$. By convention, $\alpha_i/\beta_i = \infty$ when $\alpha_i \neq 0$ and $\beta_i = 0$. Hence $\beta_i = 0$, if any, corresponds to infinite eigenvalues of $\mathbf{L}$. Now, by considering the homogeneous pencil $\mathbf{L}(c, s) = cA - sB$, we see that (β_i, α_i) is an eigenvalue of $\mathbf{L}$ and hence $((\beta_i, \alpha_i), Ye_i, X_i)$ is a homogeneous eigentriple of $\mathbf{L}$, that is, eigentriple of the homogeneous pencil $\mathbf{L}(c, s) = cA - sB$. If a pencil $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n \times n})$ has n distinct eigenvalues then it has n-independent eigenvectors and hence is diagonalizable [8].

Definition 1.2.4. *A regular pencil* $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ *is said to be simple if it has* n *distinct eigenvalues.*

Let $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ be a regular pencil given by $\mathbf{L}(z) = A - zB$. Suppose that B is nonsingular and that $\mathbf{L}$ has n linearly independent eigenvectors. Let v_i be an eigenvector of $\mathbf{L}$ corresponding to an eigenvalue λ_i, that is, $\mathbf{L}(\lambda_i)v_i = 0$. Set $V := [v_1, v_2, \ldots, v_n]$ and $\operatorname{diag}(\lambda_i) := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then we have $AV = [\lambda_1 Bv_1, \lambda_2 Bv_2, \ldots, \lambda_n Bv_n] = BV\operatorname{diag}(\lambda_i)$. This shows that $(BV)^{-1}AV = \operatorname{diag}(\lambda_i)$. Setting $U = ((BV)^{-1})^*$, we have $U^*BV = I$ and $U^*AV = \operatorname{diag}(\lambda_i)$. Let $u_i := Ue_i$. Then it follows that (λ_i, u_i, v_i) is an eigentriple of $\mathbf{L}$ such that

$$u_i^* A v_j = \delta_{ij}\lambda_i \quad \text{and} \quad u_i^* B v_j = \delta_{ij}$$

where $\delta_{ij} = 0$ if $i \neq j$. It follows that

$$\mathbf{L}(z)^{-1} = \sum_{i=1}^{n} \frac{v_i u_i^*}{z - \lambda_i}. \tag{1.1}$$

If the matrices A and B are real symmetric then the left and right eigenvectors are the same.

We say that a regular pencil $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ is defective if it has fewer than n linearly independent eigenvectors. Obviously, a defective pencil is not diagonalizable. However, there are invertible matrices X and Y such that we have the Kronecker Cannonical Form (KCF)

$$Y^{-1}\mathbf{L}(\lambda)X = \operatorname{diag}(J - \lambda I, I - \lambda N),$$

where $J - \lambda I = \operatorname{diag}(J_i - \lambda I_i)$, $I - \lambda N = \operatorname{diag}(I_i - \lambda N_i)$,

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix} \text{ and } N_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

The block N_i corresponds to an infinite eigenvalue of multiplicity k_i, where k_i is the size of N_i.

A pair of subspaces $\mathcal{X}$ and $\mathcal{Y}$ is called deflating subspaces of $\mathbf{L}$ if $\mathcal{Y} = A\mathcal{X} - B\mathcal{X}$ and $\operatorname{rank}(\mathcal{Y}) = \operatorname{rank}(\mathcal{X})$. They are generalization of invariant subspace for the standard eigenvalue problem $A - \lambda I$. Note that $\mathcal{Y}$ is an invariant subspace of A if $\mathcal{Y} = A\mathcal{Y} - \mathcal{Y}$ that is $A\mathcal{Y} \subset \mathcal{Y}$.

The generalized Schur decomposition of matrix pencils are computationally more useful than (KCF). Given a regular matrix pencil $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$, there exists unitary matrices U and V and upper triangular matrices S and T such that the Schur decomposition

$$\mathbf{L}(z) = U^*(S - zT)V$$

holds. Computation of Schur decomposition of a matrix pencil is a well-posed problem and can be computed stably. An important fact is that we can always choose a Schur decomposition of $\mathbf{L}$ satisfying

$$U^*\mathbf{L}V = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix} \text{ and } \Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset,$$

where the pencils $\mathbf{L}_1$ and $\mathbf{L}_2$ are in Schur upper triangular form. A decomposition of $\mathbf{L}$ of the form

$$Y^{-1}\mathbf{L}X = \begin{bmatrix} \mathbf{L}_1 & 0 \\ 0 & \mathbf{L}_2 \end{bmatrix} \text{ and } \Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset \tag{1.2}$$

is called an eigendecomposition of $\mathbf{L}$. Equivalently, an eigendecomposition of $\mathbf{L}$ can be specified by a disjoint partition of the spectrum $\Lambda(\mathbf{L})$.

1.2.2 Polynomial eigenvalue problem

Consider a matrix polynomial of degree m given by $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$, where $A_i \in \mathbb{C}^{n\times n}$. Then the eigenvalue problem $\mathbf{L}(\lambda)x = 0$ is referred to as a polynomial eigenvalue problem. We consider both homogeneous and nonhomogeneous polynomials of the form $\mathbf{L}(c,s) := \sum_{i=0}^{m} c^{m-i}s^i A_i$ and $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$, where $A_i \in \mathbb{C}^{n\times n}$ for $i = 0, 1, \ldots, m$. We denote the set of n-by-p matrix polynomials of degree $\leq m$ (either homogeneous or nonhomogeneous) by $\mathbb{L}_m(\mathbb{C}^{n\times p})$.

A matrix polynomial $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ is said to be regular if $\det(\mathbf{L}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular nonhomogeneous polynomial $\mathbf{L}$, denoted by $\Lambda(\mathbf{L})$, is given by

$$\Lambda(\mathbf{L}) := \{\lambda \in \mathbb{C} : \det(\mathbf{L}(\lambda)) = 0\}.$$

It is possible for $\mathbf{L}$ to have an infinite eigenvalue. However, the case of an infinite eigenvalue can be resolved by considering $\Lambda(\mathbf{L})$ as a subset of $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, the one-point compactification of $\mathbb{C}$, and adding ∞ to $\Lambda(\mathbf{L})$ whenever $\det(A_m) = 0$, where $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$.

A more convenient setup to deal with an infinite eigenvalue is to consider the homogeneous form of the polynomial $\mathbf{L}$. Thus, when $\mathbf{L}$ is homogenous, the spectrum $\Lambda(\mathbf{L})$ is given by

$$\Lambda(\mathbf{L}) := \{(c,s) \in \mathbb{C}^2 : (c,s) \neq 0 \text{ and } \det(\mathbf{L}(c,s)) = 0\}.$$

An infinite eigenvalue of $\mathbf{L}$, if any, is then represented by $(0,1)$. As in the case of matrix pencils, normalizing $(c,s) \in \Lambda(\mathbf{L})$ as $|c|^2 + |s|^2 = 1$, we often identify $\Lambda(\mathbf{L})$ as a subset of the unit sphere $\mathbb{S}^1 := \{(c,s) \in \mathbb{C}^2 : |c|^2 + |s|^2 = 1\}$. Further, for computational purposes, restricting c to be real, s to be complex and with the normalization $c^2 + |s|^2 = 1$, the spectrum $\Lambda(\mathbf{L})$ can identified with a subset of the Riemann sphere $\mathbb{S} := \{(x,y,z) \in \mathbb{R}^3 :$

$|x|^2 + |y|^2 + |z|^2 = 1\}$. In such a case, an infinite eigenvalue of $\mathbf{L}$ is represented by the north pole $(0, 0, 1)$ of $\mathbb{S}$.

Definition 1.2.5. *Let* $\mathbf{L}_1, \mathbf{L}_2 \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ *be regular matrix polynomials. Then* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *are said to be equivalent if there exists unimodular matrix polynomials* $E, F \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ *(that is,* $\det(E(z))$ *and* $\det(F(z))$ *are nonzero constants for all* $z \in \mathbb{C}$*) such that* $\mathbf{L}_1(z) = E(z)\mathbf{L}_2(z)F(z)$ *for all* $z \in \mathbb{C}$*.*

The standard way of solving a polynomial eigenvalue problem is to linearize the polynomial to a pencil and solve the corresponding generalized eigenvalue problem. For a given polynomial $\mathbf{L}$, there are infinitely many linearizations and they can have widely varying eigenvalue condition numbers.

Definition 1.2.6. *Let* $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ *a matrix polynomial of degree m. Then an mn-by-mn pencil* $\mathbf{X} - z\mathbf{Y}$ *is said to be a linearization of* $\mathbf{L}$ *if there exists unimodular matrix polynomial* $E, F \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ *such that* $\begin{bmatrix} \mathbf{L}(z) & 0 \\ 0 & I_{n(m-1)} \end{bmatrix} = E(z)(\mathbf{X} - z\mathbf{Y})F(z)$ *for all* $z \in \mathbb{C}$*.*

We have the following result which will be useful for our purposes.

Theorem 1.2.7. *[23] Consider the linearization* $\mathbf{C}_{\mathbf{L}}(z) := \mathbf{X} - z\mathbf{Y}$ *of the regular polynomial* $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$*, where*

$$\mathbf{X} = \begin{bmatrix} 0 & -I & 0 & \dots & 0 \\ 0 & 0 & -I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & \dots & -I \\ A_0 & A_1 & & \dots & A_{m-1} \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} -I & 0 & \dots & & 0 \\ 0 & -I & \dots & & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -I & 0 \\ 0 & 0 & \dots & 0 & -A_m \end{bmatrix}.$$

Then we have

$$\mathbf{L}(z)^{-1} = \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} \mathbf{C}_{\mathbf{L}}(z)^{-1} \begin{bmatrix} 0 & 0 & 0 & \dots & I \end{bmatrix}^T,$$

where T *represents the transpose of the block matrix* $\begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{C}^{n\times mn}$, $I \in \mathbb{C}^{n\times n}$ *is the identity matrix and* $0 \in \mathbb{C}^{n\times n}$ *is the zero matrix.*

Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$. Then the Kronecker product of A and B denoted by $A \otimes B$ is given by the mp-by-nq block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Let A, B, C and D are matrices of appropriate sizes such that one can form the matrix products AC and BD. Then we have $(A \otimes B)(C \otimes D) = AC \otimes BD$.

Theorem 1.2.8. *Let* $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ *be regular matrix polynomial given by* $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$*. Let* (λ, u, v) *be an eigentriple of* $\mathbf{L}$*. Set* $\Lambda := \begin{bmatrix} 1 & \lambda & \dots & \lambda^{m-1}\end{bmatrix}^T$*. Consider the linearization* $\mathbf{C}_{\mathbf{L}}(z) := \mathbf{X} - z\mathbf{Y}$ *of* $\mathbf{L}(z)$*, where*

$$\mathbf{X} = \begin{bmatrix} 0 & -I & 0 & \dots & 0 \\ 0 & 0 & -I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & \dots & -I \\ A_0 & A_1 & & \dots & A_{m-1} \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} -I & 0 & \dots & & 0 \\ 0 & -I & \dots & & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -I & 0 \\ 0 & 0 & \dots & 0 & -A_m \end{bmatrix}.$$

Set $w := \begin{bmatrix} \left(A_1 + \lambda A_2 + \cdots + \lambda^{m-1}A_m\right)^* u \\ \left(A_2 + \lambda A_3 + \cdots + \lambda^{m-2}A_m\right)^* u \\ \vdots \\ (A_{m-1} + \lambda A_m)^* u \\ u \end{bmatrix}$*. Then* $(\lambda, w, \Lambda \otimes v)$ *is an eigentriple of* $\mathbf{C}_{\mathbf{L}}$ *and*

$$w^* \mathbf{Y}(\Lambda \otimes v) = -u^* \partial_z \mathbf{L}(\lambda) v,$$

where $\partial_z \mathbf{L}(\lambda)$ *is the derivative of* $\mathbf{L}$ *evaluated at* λ*.*

Proof: We have

$$\mathbf{C}_{\mathbf{L}}(\lambda)(\Lambda \otimes v) = \begin{bmatrix} \lambda I & -I & 0 & \dots & 0 \\ 0 & \lambda I & -I & \dots & 0 \\ 0 & 0 & \lambda I & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -I \\ A_0 & A_1 & A_2 & \dots & A_{m-1} + \lambda A_m \end{bmatrix} \begin{bmatrix} 1 \\ \lambda v \\ \lambda^2 v \\ \vdots \\ \lambda^{m-1} v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now,

$$w^* \mathbf{C}_{\mathbf{L}}(\lambda) = \begin{bmatrix} \left(A_1 + \lambda A_2 + \cdots + \lambda^{m-1}A_m\right)^* u \\ \left(A_2 + \lambda A_3 + \cdots + \lambda^{m-2}A_m\right)^* u \\ \vdots \\ (A_{m-1} + \lambda A_m)^* u \\ u \end{bmatrix}^* \begin{bmatrix} \lambda I & -I & 0 & \dots & 0 \\ 0 & \lambda I & -I & \dots & 0 \\ 0 & 0 & \lambda I & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -I \\ A_0 & A_1 & A_2 & \dots & A_{m-1} + \lambda A_m \end{bmatrix}$$

$$= \begin{bmatrix} u^*(\sum_{i=0}^{m} \lambda^i A_i) \\ u^*(-A_1 - \lambda A_2 - \cdots - \lambda^{m-1}A_m + \lambda A_2 + \lambda^2 A_3 + \cdots + \lambda^{m-1}A_m + A_1) \\ u^*(-A_2 - \lambda A_3 - \cdots - \lambda^{m-2}A_m + A_2 + \lambda A_3 + \cdots + \lambda^{m-2}A_m + A_2) \\ \vdots \\ u^*(-A_{m-1} - \lambda A_m + A_{m-1} + \lambda A_m) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $(\lambda, w, \Lambda \otimes v)$ is an eigentriple of $\mathbf{C}_{\mathbf{L}}$. Let e_m be last column of the identity matrix $I \in \mathbb{C}^{m\times m}$. Then we have

$$\mathbf{C}_{\mathbf{L}}(z)(\Lambda \otimes I) = e_m \otimes \mathbf{L}(z). \tag{1.3}$$

Now, differentiating (1.3) w.r.t z and evaluating at λ, we have

$$\partial_z \mathbf{C_L}(\lambda)(\Lambda \otimes I) + \mathbf{C_L}(\lambda)\partial_z[(\Lambda \otimes I)] = e_m \otimes \partial_z(\mathbf{L}(\lambda)). \tag{1.4}$$

Now, multiplying on the left by w^* and on the right by $v = 1 \otimes v$ in (1.4), we have

$$-w^*\mathbf{Y}(\Lambda \otimes I)(1 \otimes v) + \underbrace{w^*\mathbf{C_L}(\lambda)}[\partial_z(\Lambda \otimes I)](1 \otimes v) = w^*(e_m \otimes \partial_z \mathbf{L}(\lambda))(1 \otimes v).$$

Since $w^*\mathbf{C_L}(\lambda) = 0$, we have

$$-w^*Y(\Lambda \otimes I)(1 \otimes v) = w^*(e_m \otimes \partial_z \mathbf{L}(\lambda))(1 \otimes v) \Rightarrow -w^*\mathbf{Y}(\Lambda \otimes v) \;=\; w^*(e_m \otimes \partial_z \mathbf{L}(\lambda)v).$$

Note that $w^*(e_m \otimes \partial_z \mathbf{L}(\lambda)v) = u^*\partial_z \mathbf{L}(\lambda)v$. This shows that $-w^*\mathbf{Y}(\Lambda \otimes v) = u^*\partial_z \mathbf{L}(\lambda)v$. Hence the result follows. ■

1.2.3 Subharmonic functions

Let $z \in \mathbb{C}$ and $r > 0$. Unless otherwise stated, $B(z,r)$ and $B[z,r]$ will always denote the open and closed discs of radius r centered at z, respectively. Thus

$$B(z,r) := \{w \in \mathbb{C} : |w - z| < r\} \text{ and } B[z,r] := \{w \in \mathbb{C} : |w - z| \leq r\}.$$

Set $\mathbb{R}^* := \mathbb{R} \cup \{-\infty\}$. Let U be open subset of $\mathbb{C}$ and $\mathbb{D}$ be a domain, that is, a connected open subset of $\mathbb{C}$. A function $f : U \to \mathbb{R}^*$ is said to be subharmonic on U if it is upper semicontinuous on U and satisfies the mean inequality

$$f(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})$$

for all closed discs $B[z_0, r] \subset U$.

Theorem 1.2.9. (E. Vesentini, [5], pp. 52-53) *Let $\mathcal{A}$ be complex Banach algebra. Let $f : \mathbb{D} \to \mathcal{A}$ be analytic. Then $z \mapsto \|f(z)\|$ and $z \mapsto \log \|f(z)\|$ are subharmonic on $\mathbb{D}$.*

Theorem 1.2.10. ([5], page-174) *Let $U \subset \mathbb{C}$ be open. Then we have the following.*

(a) If ϕ_1 and ϕ_2 are subharmonic on U then $\phi_1 + \phi_2$ is subharmonic.

(b) If ϕ is subharmonic on U and α is positive number then $\alpha.\phi$ is subharmonic on U.

(c) If ϕ_1 and ϕ_2 are subharmonic on U then $\max(\phi_1, \phi_2)$ is subharmonic on U.

(d) If ϕ is subharmonic on U and if f is real, convex and increasing function on $\mathbb{R}$ then $f \circ \phi$ is subharmonic on U.

Theorem 1.2.11. **([5], page-175)** *Let* ϕ_1 *and* ϕ_2 *be two positive functions such that* $\log \phi_1$ *and* $\log \phi_2$ *are subharmonic on an open set* U*. Then* $\log(\phi_1 + \phi_2)$ *is subharmonic on* U*.*

Subharmonic functions satisfy the maximum principle.

Theorem 1.2.12. *[37] (a)* **(Maximum principle)** *Let* f *be a subharmonic function on* D*. If there exists* $a \in D$ *such that* $f(z) \leq f(a)$ *for all* $z \in D$*. Then* $f(a) = f(z)$ *for all* $z \in D$*.*

(b) If the map $f : \mathbb{C} \to \mathbb{R}^*$ *is subharmonic and* $\limsup_{z\to\infty} \frac{f(z)}{\log|z|} \leq 0$*, then* f *is constant on* $\mathbb{C}$*. In particular, if* f *is subharmonic on* $\mathbb{C}$ *and is bounded above then* f *must be constant.*

Chapter 2

On pseudospectra, critical points and multiple eigenvalues of matrix pencils

We develop a general framework for defining and analyzing pseudospectra of matrix pencils. The framework so developed unifies various definitions of pseudospectra of matrix pencils proposed in the literature. We introduce and analyze critical points of backward errors of approximate eigenvalues of matrix pencils and show that each critical point is a multiple eigenvalue of an appropriately perturbed pencil. Finally, we show that common boundary points of the components of pseudospectra of a matrix pencil are critical points. We show that a minimal critical point can be read off from the pseudospectra of matrix pencils. Hence a solution of Wilkinson's problem for matrix pencils can be read off from the pseudospectra of matrix pencils.

2.1 Introduction

This chapter deals with three main issues, namely, pseudospectra of matrix pencils, critical points of backward errors and multiple eigenvalues of matrix pencils. Given a regular matrix pencil $\mathbf{L}(z) = A - zB$, one may ask, how to define the ϵ-pseudospectrum of $\mathbf{L}$? Pseudospectra of matrices are well known and have been studied extensively over the years (see, [47] and the references therein). By contrast, there is a perception that, unlike the pseudospectra of matrices, there are several (nonequivalent) ways to define pseudospectra of matrix pencils. Indeed, various definitions of pseudospectra of matrix pencils have been proposed in the literature over the years (see, for example, [38, 22, 19, 30, 20, 25, 45]). This raises some natural questions: In what ways all these pseudospectra of matrix pencils differ from one another? Is it possible to put the analysis of pseudospectra of matrix pencils on the same footing as that of matrices?

To address these questions, first, we demystify the definition of pseudospectra of matrix pencils. We develop a general framework for defining and analyzing pseudospectra of matrix pencils. The crux of the matter is that, like the pseudospectra of matrices, the pseudospectra of pencils are determined by the geometry of the space of pencils, that is, by the choice of a norm on the space of pencils. Thus we endow the space of pencils with a seminorm/norm $|||\cdot|||$ and define the ϵ-pseudospectrum of $\mathbf{L}$ by

$$\Lambda_\epsilon(\mathbf{L}) := \bigcup_{|||\Delta\mathbf{L}|||\leq\epsilon} \Lambda(\mathbf{L}+\Delta\mathbf{L}),$$

where $\Lambda(\mathbf{L})$ is the spectrum of $\mathbf{L}$. We show that various pseudospectra of matrix pencils proposed in the literature correspond to various choices of norms on the space of pencils. Thus we show that our framework unifies various definitions of pseudospectra of matrix pencils considered in the literature. Consequently, we show that the pseudospectra of matrix pencils are defined as uniquely as the pseudospectra of matrices.

Next, we consider the issue of critical points of backward errors of approximate eigenvalues of matrix pencils. For illustration, consider a matrix $A \in \mathbb{C}^{n\times n}$ and, for $\lambda \in \mathbb{C}$, define $\eta(\lambda) := \sigma_{\min}(A-\lambda I)$, where $\sigma_{\min}(A-\lambda I)$ is the smallest singular value of $A-\lambda I$. Then $\eta(\lambda)$ is the backward error of λ when λ is treated as an approximate eigenvalue of A. If $\eta(\lambda)$ is simple then η is real analytic in a neighbourhood of λ and $\nabla\eta(\lambda) = -v^*u$, where u and v, respectively, are normalized left and right singular vectors of $A-\lambda I$ corresponding to $\eta(\lambda)$ [44]. Thus if λ is a point of extremum of η then $-v^*u = \nabla\eta(\lambda) = 0$. Such a λ is said to be a **generic critical point** of η. Consequently, defining $\Delta A := -\eta(\lambda)uv^*$, it follows that u and v, respectively, are left and right eigenvectors of $A+\Delta A$ corresponding to λ and that $\|\Delta A\|_2 = \eta(\lambda)$. Since $u^*v = 0$, by a result of Wilkinson [50], λ is a multiple eigenvalue of $A+\Delta A$. As $\mathrm{rank}(A+\Delta A-\lambda I) = n-1$, λ is in fact a defective eigenvalue of $A+\Delta A$. This shows that a generic critical point λ of η is, in fact, a defective eigenvalue of a matrix which lies on the boundary of the $\eta(\lambda)$-neighbourhood of A. On the other hand, when λ is such that $\eta(\lambda)$ is multiple then λ is said to be a **nongeneric critical point** of η. In such a case, it is easy to construct ΔA such that $\|\Delta A\|_2 = \eta(\lambda)$ and λ is a multiple eigenvalue of $A+\Delta A$ of geometric multiplicity at least 2 (see, [2]). Thus a critical point (either generic or nongeneric) of η is a multiple eigenvalue of a matrix whose distance from A is equal to $\eta(\lambda)$. A **minimal critical point** of η, that is, a critical point at which η takes the smallest value among all the critical points, is of special interest. When A is simple and λ is a minimal critical point of η, then $\eta(\lambda)$ is the Wilkinson's distance [49, 50, 2], that is, $\eta(\lambda)$ is the distance from A to the nearest matrix having a multiple eigenvalue. When A is simple, it is shown in [2] that a minimal critical point of η can be read off from the pseudospectra of A.

Our next goal is to generalize these results to the case of matrix pencils. We endow the space of pencils with weighted Hölder's p-norm $|||\cdot|||_{w,p}$ (defined in section 3). Given a regular matrix pencil $\mathbf{L}(z) = A - zB$ and $\lambda \in \mathbb{C}$, we denote by $\eta_{w,p}(\lambda, \mathbf{L})$ the backward error of λ as an approximate eigenvalue of $\mathbf{L}$, that is,

$$\eta_{w,p}(\lambda, \mathbf{L}) := \inf\{|||\Delta\mathbf{L}|||_{w,p} : \det(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = 0\}.$$

Then we show that if λ is a critical point of $\eta_{w,p}$ then there exists a pencil $\Delta\mathbf{L}$ such that $|||\Delta\mathbf{L}|||_{w,p} = \eta_{w,p}(\lambda, \mathbf{L})$ and that λ is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. Indeed, when λ is a generic critical point of $\eta_{w,p}$, defining

$$\Delta A := -\eta_{w,p}(\lambda, \mathbf{L})\, \overline{(\nabla_1 H)(1, \lambda)}\, uv^* \text{ and } \Delta B := \eta_{w,p}(\lambda, \mathbf{L})\, \overline{(\nabla_2 H)(1, \lambda)}\, uv^*,$$

we obtain the desired pencil $\Delta\mathbf{L}(z) := \Delta A - z\Delta B$, where u and v, respectively, are normalized left and right singular vectors of $\mathbf{L}(\lambda)$ corresponding to the smallest singular value $\sigma_{\min}(\mathbf{L}(\lambda))$, $H(z_1, z_2) := \|(z_1, z_2)\|_{w^{-1},q}$, $(\nabla_i H)(1, \lambda)$ is the partial gradient of H evaluated at $(1, \lambda)$ and $p^{-1} + q^{-1} = 1$. When λ is a nongeneric critical point of $\eta_{w,p}$, we show that a similar construction for $\Delta\mathbf{L}$ holds. Thus we show that if λ is a critical point (either generic or nongeneric) of $\eta_{w,p}$ then λ is a multiple eigenvalue of a pencil which lies on the boundary of the $\eta_{w,p}(\lambda, \mathbf{L})$-neighbourhood of $\mathbf{L}$. We show that certain critical points of $\eta_{w,p}$ can be read off from the pseudospectra of $\mathbf{L}$. More specifically, we show that common boundary points of the components of pseudospectra of $\mathbf{L}$ are critical points of $\eta_{w,p}$. In particular, when $\mathbf{L}$ is simple (that is, has distinct eigenvalues) we show that a minimal critical point of $\eta_{w,p}$ can be read off from the pseudospectra of $\mathbf{L}$. Hence we show that the distance from $\mathbf{L}$ to the nearest pencil having a multiple eigenvalue can be read off from the pseudospectra of $\mathbf{L}$.

2.2 Preliminaries

We consider nonhomogeneous (resp., homogeneous) pencils of the form $\mathbf{L}(z) = A - zB$ (resp., $\mathbf{L}(c, s) := cA - sB$), where $A, B \in \mathbb{C}^{n\times n}$. A pencil $\mathbf{L}$ is said to be regular if $\det(\mathbf{L}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular nonhomogeneous pencil $\mathbf{L}$ is given by

$$\Lambda(\mathbf{L}) := \{\lambda \in \mathbb{C} : \det(\mathbf{L}(\lambda)) = 0\}.$$

It is possible for $\mathbf{L}$ to have an infinite eigenvalue which is not included in $\Lambda(\mathbf{L})$. However, the case of an infinite eigenvalue can be resolved by considering $\Lambda(\mathbf{L})$ as a subset of $\mathbb{C}_\infty$, the one-point compactification of $\mathbb{C}$, and putting ∞ in $\Lambda(\mathbf{L})$ whenever $\det(B) = 0$.

A more convenient setup to deal with an infinite eigenvalue is to consider a homogeneous form of the pencil $\mathbf{L}$. Thus, when $\mathbf{L}$ is homogenous, $\Lambda(\mathbf{L})$ is given by

$$\Lambda(\mathbf{L}) := \{(c, s) \in \mathbb{C}^2 : (c, s) \neq 0 \text{ and } \det(\mathbf{L}(c, s)) = 0\}.$$

An infinite eigenvalue of $\mathbf{L}$, if any, is then represented by $(0,1)$. Normalizing $(c,s) \in \Lambda(\mathbf{L})$ as $|c|^2+|s|^2 = 1$, we often identify $\Lambda(\mathbf{L})$ as a subset of the unit sphere $\mathbb{S}^1 := \{(c,s) \in \mathbb{C}^2 : |c|^2+|s|^2 = 1\}$. However, for computational purposes, further restricting c to be real, s to be complex and with the normalization $c^2+|s|^2 = 1$, the spectrum $\Lambda(\mathbf{L})$ can be identified with a subset of the Riemann sphere $\mathbb{S} := \{(x,y,z) \in \mathbb{R}^3 : |x|^2+|y|^2+|z|^2 = 1\}$. In such a case, an infinite eigenvalue of $\mathbf{L}$ is represented by the north pole $(0,0,1)$ of $\mathbb{S}$.

Pseudospectra of matrices and matrix pencils play an important role in analyzing and understanding systems governed by matrices and matrix pencils [47]. We now briefly review various definitions of pseudospectra of matrix pencils proposed in the literature. Let $\mathbf{L}(z) := A - zB$ be an n-by-n regular pencil. Then, unlike the ϵ-pseudospectrum of a matrix, the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ of $\mathbf{L}$ has been defined in several (nonequivalent) ways in the literature (see, for example, [38, 22, 19, 30, 25]). A brief survey of various definitions of $\Lambda_\epsilon(\mathbf{L})$ is as follows.

- **Riedel [38]:** Suppose that B is positive definite and that $B = F^*F$. Then Riedel defined $\Lambda_\epsilon(\mathbf{L})$ by

$$\Lambda_\epsilon(\mathbf{L}) := \{\lambda \in \mathbb{C} : \sigma_{\min}((F^{-1})^*AF^{-1} - \lambda I) \leq \epsilon\} = \Lambda_\epsilon((F^*)^{-1}AF^{-1}).$$

  Here $\Lambda_\epsilon((F^*)^{-1}AF^{-1})$ is the 2-norm ϵ-pseudospectrum of the matrix $(F^*)^{-1}AF^{-1}$.

- **Dorsselaer [19, 20]:** For a regular pencil $\mathbf{L}(\lambda) = A - \lambda B$, Dorsselaer defined $\Lambda_\epsilon(\mathbf{L})$ by

$$\begin{aligned}\Lambda_\epsilon(\mathbf{L}) &:= \{\lambda \in \mathbb{C} : \sigma_{\min}(\mathbf{L}(\lambda)) \leq \epsilon\} \\ &= \{\lambda \in \mathbb{C} : \det(A + \Delta A - \lambda B) = 0 \text{ and } \|\Delta A\|_2 \leq \epsilon\}.\end{aligned}$$

- **Frayssé et al. [22]:** For an induced operator norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, Frayssé et al. defined $\Lambda_\epsilon(\mathbf{L})$ by

$$\begin{aligned}\Lambda_\epsilon(\mathbf{L}) &:= \{\lambda \in \mathbb{C} : \det(A + \Delta A - \lambda(B + \Delta B)) = 0 \text{ and } \|\Delta A\| \leq \alpha\epsilon, \|\Delta B\| \leq \beta\epsilon\} \\ &= \{\lambda \in \mathbb{C} : \|\mathbf{L}(\lambda)^{-1}\|(\alpha + |z|\beta) \geq \epsilon^{-1}\},\end{aligned}$$

  where α and β are nonnegative numbers and $\|\mathbf{L}(\lambda)^{-1}\| = \infty$ whenever λ is an eigenvalue of $\mathbf{L}$.

- **Lavallée et al. [30]:** Considering homogeneous form of $\mathbf{L}$, that is, $\mathbf{L}(c,s) = cA - sB$, Lavallée et al. defined $\Lambda_\epsilon(\mathbf{L})$ by

$$\begin{aligned}\Lambda_\epsilon(\mathbf{L}) &:= \Big\{(c,s) \in \mathbb{C}^2 \setminus \{0\} : \det(c(A + \Delta A) - s(B + \Delta B)) = 0 \text{ and} \\ &\qquad \|\Delta A\|_2^2 + \|\Delta B\|_2^2 \leq \epsilon^2\Big\} \\ &= \left\{(c,s) \in \mathbb{C}^2 \setminus \{0\} : \sigma_{\min}(cA - sB) \leq \epsilon\sqrt{|c|^2 + |s|^2}\right\}.\end{aligned}$$

- **Higham et al. [25, 45]:** For an induced operator norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, Higham et al. defined $\Lambda_\epsilon(\mathbf{L})$ by

$$\begin{aligned}\Lambda_\epsilon(\mathbf{L}) &:= \big\{(c,s)\in\mathbb{C}^2\setminus\{0\} : \det(c(A+\Delta A)-s(B+\Delta B))=0 \text{ and} \\ &\quad \|\Delta A\|\leq \alpha\epsilon,\ \|\Delta B\|\leq\beta\epsilon\} \\ &= \big\{(c,s)\in\mathbb{C}^2\setminus\{0\} : \|\mathbf{L}(c,s)^{-1}\|(\alpha|c|+\beta|s|)\geq\epsilon^{-1}\big\},\end{aligned}$$

  where α and β are nonnegative numbers.

So, one may ask: Which definition of the pseudospectrum of $\mathbf{L}$ is the most natural one and hence to be chosen? As we shall see in the next section, each definition of $\Lambda_\epsilon(\mathbf{L})$ mentioned above is as natural as it possibly could be. As for which definition of $\Lambda_\epsilon(\mathbf{L})$ is to be chosen, the answer would mainly be determined by the context in which the pseudospectrum is employed. For illustration, consider the problem: Given a full rank pencil $\mathbf{L}(z) := A - zB$, where $A, B\in\mathbb{C}^{m\times n}$, solve the minimization problem

$$\min_{A_0,B_0,\lambda}\{\|A-A_0\|_F^2+\|B-B_0\|_F^2 : \text{rank}(A_0-\lambda B_0)<n\}.$$

This problem was formulated and solved by Boutry et al. in [13] where it was shown that the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ as defined by Lavallée et al. [30] provides an appropriate framework for analyzing this minimization problem.

2.3 A framework for pseudospectra of matrix pencils

There is a perception that, unlike the pseudospectra of matrices, there are several nonequivalent ways in which one could define pseudospectra of matrix pencils. Indeed, various definitions briefly reviewed in the previous section lend credence to such a perception. The crux of the matter is that the definitions of pseudospectra of matrix pencils proposed so far in the literature are ad hoc in nature. We now develop a general framework for a systematic analysis of pseudospectra of matrix pencils. We show that the pseudospectra of matrix pencils are defined as uniquely as those of matrices.

We begin with the definition of the ϵ-pseudospectrum of a matrix $A\in\mathbb{C}^{n\times n}$. Given a norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, the ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is given by [47]

$$\Lambda_\epsilon(A) := \bigcup\{\Lambda(A+\Delta A) : \Delta A\in\mathbb{C}^{n\times n} \text{ and } \|\Delta A\|\leq\epsilon\}. \tag{2.1}$$

Needless to mention that the pseudospectrum of a matrix is determined by the geometry of the space $\mathbb{C}^{n\times n}$. Therefore the first step towards defining pseudospectrum is to choose a norm on $\mathbb{C}^{n\times n}$. A priori the same should be true for matrix pencils as well. So, the first step towards defining pseudospectra of matrix pencils systematically is to choose a norm or a seminorm on the space of pencils.

We consider both homogeneous and nonhomogeneous n-by-n pencils of the form $\mathbf{L}(c, s) := cA - sB$ and $\mathbf{L}(\lambda) := A - \lambda B$. So, a pencil $\mathbf{L}$ is a linear (resp., affine) map from $\mathbb{C}^2$ (resp., $\mathbb{C}$) to $\mathbb{C}^{n\times n}$. We denote the set of n-by-n matrix pencils (either homogeneous or nonhomogeneous) by $\mathbb{L}(\mathbb{C}^{n\times n})$. Obviously $\mathbb{L}(\mathbb{C}^{n\times n})$ is a vector space.

Definition 2.3.1. *Let $|||\cdot|||$ be a norm on $\mathbb{L}(\mathbb{C}^{n\times n})$. Then for a regular pencil $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$, we define the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ of $\mathbf{L}$ by*

$$\Lambda_\epsilon(\mathbf{L}) = \bigcup \{\Lambda(\mathbf{L} + \Delta\mathbf{L}) : \Delta\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n}) \text{ and } |||\Delta\mathbf{L}||| \leq \epsilon\}.$$

We mention that Definition 2.3.1 is a natural generalization of (2.1) to the case of matrix pencils. Consequently, the pseudospectrum of a matrix pencil is defined as uniquely as that of a matrix. We show that various definitions of pseudospectra considered in section 2 correspond to different choices of the norm/seminorm $|||\cdot|||$ on the space of pencils $\mathbb{L}(\mathbb{C}^{n\times n})$.

Note that the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ of a pencil $\mathbf{L}$ as defined above is a natural generalization of the pseudospectrum $\Lambda_\epsilon(A)$ of matrix A.

More generally, we now define weighted pseudospectrum. First, we define the action of $\mathbb{R}^2$ on $\mathbb{L}(\mathbb{C}^{n\times n})$. Define $\mathbb{R}^2 \times \mathbb{L}(\mathbb{C}^{n\times n}) \to \mathbb{L}(\mathbb{C}^{n\times n}), (w, \mathbf{L}) \mapsto w \odot \mathbf{L}$ by $(w \odot \mathbf{L})(z) = w_1 A - w_2 z B$, where $w := (w_1, w_2)$ and $\mathbf{L}(z) := A - zB$. Obviously the map $(w, \mathbf{L}) \mapsto w \odot \mathbf{L}$ is bilinear and $v \odot (w \odot \mathbf{L}) = w \odot (v \odot \mathbf{L})$. Note that w acts as a linear map $\mathbf{L} \mapsto w \odot \mathbf{L}$ on $\mathbb{L}(\mathbb{C}^{n\times n})$. We say that $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ is a w-null pencil if $w \odot \mathbf{L} = 0$. The action of w is said to be injective if $w \odot \mathbf{L} = 0 \Rightarrow \mathbf{L} = 0$ for all $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$. Obviously the action of w is injective if and only if all the components of w are nonzero. We define $w^{-1} := (w_1^{-1}, w_2^{-1})$ with the convention that $w_j^{-1} = 0$ if $w_j = 0$. Note that if one of the components of w are equal to 0 then $w^{-1} \odot (w \odot \mathbf{L}) = \mathbf{L}$ holds only for certain pencils. Finally, if all the entries of w are nonnegative then we say that w is a **weight vector**.

Definition 2.3.2. *Let $|||\cdot|||$ be a norm on $\mathbb{L}(\mathbb{C}^{n\times n})$ and $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ be regular. Then for a weight vector $w \in \mathbb{R}^2$, we define the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ of $\mathbf{L}$ by*

$$\Lambda_\epsilon(\mathbf{L}) := \bigcup \{\Lambda(\mathbf{L} + \Delta\mathbf{L}) : \Delta\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n}), w^{-1} \odot (w \odot \Delta\mathbf{L}) = \Delta\mathbf{L} \text{ and } |||w \odot \Delta\mathbf{L}||| \leq \epsilon\}.$$

We mention that $w^{-1} \odot (w \odot \Delta\mathbf{L}) = \Delta\mathbf{L}$ is an admissibility condition on the perturbation pencil $\Delta\mathbf{L}$. For example, if $\Delta\mathbf{L}(z) = \Delta A - z\Delta B$ then the admissibility condition implies that $\Delta A = 0$ whenever the component w_1 of w is 0.

Definition 2.3.3. *Let $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ and $w \in \mathbb{R}^2$ be a weight vector. Then $\mathbf{L}$ is said to be w-admissible if $w^{-1} \odot (w \odot \mathbf{L}) = \mathbf{L}$.*

Assumption: Given a weight vector w, unless stated otherwise,
all perturbations of a pencil $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ will be assumed
to be w-admissible.

The importance of w-admissible perturbation can be seen as follows. Suppose that $\Delta\mathbf{L}$ is w-admissible and $\mathbf{L}(z) = A - zB$. Then if the component w_1 of the weight vector w is 0 then A remains unperturbed when $\mathbf{L}$ is perturbed to $\mathbf{L} + \Delta\mathbf{L}$.

We show that the pseudospectra considered in section 2 follow from Definition 2.3.1 and correspond to appropriate choices of weight w and the norm $|||\cdot|||$ on the space of pencils $\mathbb{L}(\mathbb{C}^{n\times n})$. To that end, first we define the Hölder's p-norm on $\mathbb{L}(\mathbb{C}^{n\times n})$ which we refer to as the pencil p-norm. Let $\|\cdot\|$ be a norm on $\mathbb{C}^{n\times n}$. For $1 \leq p \leq \infty$, we define $|||\cdot|||_p : \mathbb{L}(\mathbb{C}^{n\times n}) \to \mathbb{R}$ by

$$|||\mathbf{L}|||_p := \|(\|A\|, \|B\|)\|_p,$$

where $\mathbf{L}(z) := A - zB$ or $\mathbf{L}(c, s) := cA - sB$ and $\|\cdot\|_p$ is the Hölder's p norm on $\mathbb{C}^2$. Then it is easily seen that $|||\cdot|||_p$ is a norm. We denote the space $\mathbb{L}(\mathbb{C}^{n\times n})$ when equipped with the norm $|||\cdot|||_p$ by $\mathbb{L}^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. This notation emphasizes the fact that $(\mathbb{C}^{n\times n}, \|\cdot\|)$ is the base space from which the norm $|||\cdot|||_p$ is built up.

More generally, let $w \in \mathbb{R}^2$ be a weight vector. Then we define the weighted Hölder's p-norm/seminorm $|||\cdot|||_{w,p} : \mathbb{L}(\mathbb{C}^{n\times n}) \to \mathbb{R}$ by

$$|||\mathbf{L}|||_{w,p} := |||w \odot \mathbf{L}|||_p.$$

Obviously, $|||\cdot|||_{w,p}$ defines a seminorm on $\mathbb{L}(\mathbb{C}^{n\times n})$. Note that $|||\cdot|||_{w,p}$ defines a norm if and only if all the entries of w are nonzero. We denote the space $\mathbb{L}^p(\mathbb{C}^{n\times n})$ when equipped with the norm/seminorm $|||\cdot|||_{w,p}$ by $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$. If $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ is a w-admissible pencil then it follows that

$$\begin{aligned} \|\mathbf{L}(\lambda)\| &\leq |||\mathbf{L}|||_{w,p} \|(1,\, \lambda)\|_{w^{-1},q}, \\ \|\mathbf{L}(c, s)\| &\leq |||\mathbf{L}|||_{w,p} \|(c,\, s)\|_{w^{-1},q}, \end{aligned} \tag{2.2}$$

where $p^{-1} + q^{-1} = 1$. When $1 < p < \infty$, another choice of p-norm/seminorm on $\mathbb{L}(\mathbb{C}^{n\times n})$ is given by $|||\mathbf{L}|||_{w,p} := (w_1\|A\|^p + w_2\|B\|^p)^{1/p}$. We, however, will not have occasion to use this norm/seminorm.

Now consider the special case $\mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_F)$. For $\mathbf{L}_1, \mathbf{L}_2 \in \mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_F)$, define

$$\langle \mathbf{L}_1,\, \mathbf{L}_2\rangle_w := w_1^2 \ \mathrm{trace}(A_2^* A_1) + w_2^2 \ \mathrm{trace}(B_2^* B_1),$$

where $\mathbf{L}_1(\lambda) := A_1 - \lambda B_1$ and $\mathbf{L}_2(\lambda) := A_2 - \lambda B_2$. Then $\langle \cdot,\, \cdot\rangle_{\mathbb{L}}$ defines an inner product whenever the action of w is injective. Thus $\mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_F)$ is a Hilbert space whenever w acts injectively. For the rest of the chapter, we consider only the space $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ for appropriate choices of w, p and the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$.

Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular pencil. Then ϵ-pseudospectrum is a set valued map from $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ to $\mathbb{C}$ or $\mathbb{C}^2$. The crux of the matter is that the ϵ-pseudospectra of $\mathbf{L}$ are completely determined by the geometry of the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. Consequently, a regular pencil $\mathbf{L}$ have different ϵ-pseudospectrum for different choices of w, p and $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. Said differently, from the viewpoint of pseudospectra, the same pencil $\mathbf{L}$ becomes a different object for different choices of w, p and $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. There is nothing new in this observation and the same conclusion holds for ϵ-pseudospectrum of a matrix. The reason behind emphasizing this obvious fact is to set the priorities in proper order: that (a) the geometry of the space of pencils determines the pseudospectra and therefore, in order to define pseudospectra, first it is necessary to define a norm/seminorm on the space of pencils and that (b) the pseudospectra of a pencil get automatically specified once the space to which the pencil belongs is specified. This lays the mathematical foundation for defining and analyzing pseudospectra of matrix pencils. Various pseudospectra of matrix pencils proposed in the literature (reviewed in section 2) are ad hoc in nature because of the lack of clearly defined norms/seminorms on the space of pencils. The introduction of the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ not only brings out the big picture associated with pseudospectra but also it helps in simplifying notation for the pseudospectra of matrix pencils. Since the pseudospectrum of a pencil is determined by the geometry of the space to which the pencil belongs, for a regular pencil $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$, we can afford to denote the ϵ-pseudospectrum of $\mathbf{L}$ simply by $\Lambda_\epsilon(\mathbf{L})$ without having to specify w, p and $\|\cdot\|$ or showing the dependence of $\Lambda_\epsilon(\mathbf{L})$ on w, p and $\|\cdot\|$.

To proceed further, consider a regular pencil $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. Then by Definition 2.3.2, the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ of $\mathbf{L}$ is given by

$$\Lambda_\epsilon(\mathbf{L}) := \bigcup\{\Lambda(\mathbf{L} + \Delta\mathbf{L}) : \Delta\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|) \text{ and } |\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} \leq \epsilon\}. \tag{2.3}$$

Now, we provide a characterization of $\Lambda_\epsilon(\mathbf{L})$. To that end, for $z, c, s \in \mathbb{C}$, we define the backward errors (recall that all perturbations are w-admissible)

$$\begin{aligned} \eta_{w,p}(z, \mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} : z \in \Lambda(\mathbf{L} + \Delta\mathbf{L})\} \\ \eta_{w,p}(c, s, \mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} : (c, s) \in \Lambda(\mathbf{L} + \Delta\mathbf{L})\}. \end{aligned}$$

Then obviously the following result holds.

Corollary 2.3.4. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular pencil. Then we have*

$$\begin{aligned} \Lambda_\epsilon(\mathbf{L}) &= \{z \in \mathbb{C} : \eta_{w,p}(z, \mathbf{L}) \leq \epsilon\}, \text{ when } \mathbf{L} \text{ is nonhomogeneous} \\ \Lambda_\epsilon(\mathbf{L}) &= \{(c, s) \in \mathbb{C}^2 \setminus \{0\} : \eta_{w,p}(c, s, \mathbf{L}) \leq \epsilon\}, \text{ when } \mathbf{L} \text{ is homogeneous.} \end{aligned}$$

The following result determines the backward error function $\eta_{w,p}$ when $\mathbb{C}^{n\times n}$ is equipped with a subordinate matrix norm.

Proposition 2.3.5. *Consider the space* $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *corresponding to a subordinate matrix* $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be a regular pencil. Then we have*

$$\begin{aligned}\eta_{w,p}(\lambda, \mathbf{L}) &= \min_{\|x\|=1}\left\{\frac{\|\mathbf{L}(\lambda)x\|}{\|(1,\lambda)\|_{w^{-1},q}} : x \in \mathbb{C}^n\right\} \leq \|\!|\mathbf{L}|\!\|_{w,p},\\ \eta_{w,p}(c, s, \mathbf{L}) &= \min_{\|x\|=1}\left\{\frac{\|\mathbf{L}(c,s)x\|}{\|(c,s)\|_{w^{-1},q}} : x \in \mathbb{C}^n\right\} \leq \|\!|\mathbf{L}|\!\|_{w,p},\end{aligned}$$

where $p^{-1} + q^{-1} = 1$.

Proof: First note that if $\lambda \in \Lambda(\mathbf{L} + \Delta\mathbf{L})$ then there exists x such that $\|x\| = 1$ and $\mathbf{L}(\lambda)x + \Delta\mathbf{L}(\lambda)x = 0$. Since $\Delta\mathbf{L}$ is w-admissible, we have $\|\mathbf{L}(\lambda)x\| = \|\Delta\mathbf{L}(\lambda)x\| \leq \|\Delta\mathbf{L}(\lambda)\| \leq \|\!|\Delta\mathbf{L}|\!\|_{w,p}\|(1,\lambda)\|_{w^{-1},q}$, where $p^{-1} + q^{-1} = 1$. This shows that

$$\eta_{w,p}(\lambda, \mathbf{L}) \geq \min_{\|x\|=1} \frac{\|\mathbf{L}(\lambda)x\|}{\|(1,\lambda)\|_{w^{-1},q}}.$$

For the reverse inequality, choose $x \in \mathbb{C}^n$ such that $\|x\| = 1$. Then there is a (functional) $y \in \mathbb{C}^n$ such that $\|y\|_D = 1$ and $y^*x = 1$, where $\|\cdot\|_D$ is the dual norm of the norm $\|\cdot\|$ on $\mathbb{C}^n$. Now set $r := \mathbf{L}(\lambda)x$ and define $E : \mathbb{C}^n \to \mathbb{C}^n$ by $Ez := (y^*z)r$. Then $E \in \mathbb{C}^{n\times n}$, $Ex = r$ and $\|E\| = \|y\|_D\|r\| = \|r\|$. Set $\alpha := \|(1,\lambda)\|_{w^{-1},q}$, where $p^{-1} + q^{-1} = 1$. Now we define pencil $\Delta\mathbf{L}(z) := \Delta A - z\Delta B$ as follows. If $p \neq 1$ then define

$$\Delta A := -\frac{w_1^{-q}\,E}{\alpha^q} \text{ and } \Delta B := \frac{w_2^{-q}\,\mathrm{sign}(\lambda)\,|\lambda|^{q-1}\,E}{\alpha^q},$$

where $\mathrm{sign}(z) = \overline{z}/|z|$ for $z \neq 0$ and $\mathrm{sign}(0) = 0$. When $p = 1$, define

$$\Delta A := \begin{cases} -\dfrac{w_1^{-1}\,E}{\alpha}, & \text{if } \alpha = w_1^{-1}\\ 0, & \text{if } \alpha = w_2^{-1}|\lambda| \end{cases} \text{ and } \Delta B := \begin{cases} 0, & \text{if } \alpha = w_1^{-1}\\ \dfrac{w_2^{-1}\,\mathrm{sign}(\lambda)\,E}{\alpha}, & \text{if } \alpha = w_2^{-1}|\lambda|. \end{cases}$$

Then it is easy to see that $\mathbf{L}(\lambda)x + \Delta\mathbf{L}(\lambda)x = 0$ and

$$\|\!|\Delta\mathbf{L}|\!\|_{w,p} = \|r\|/\alpha = \frac{\|\mathbf{L}(\lambda)x\|}{\|(1,\lambda)\|_{w^{-1},q}}.$$

Now the desired result follows by taking minimum over all x such that $\|x\| = 1$. The proof is similar for homogeneous pencil. ■

When $\mathbb{C}^{n\times n}$ is equipped with either the spectral norm $\|\cdot\|_2$ or the Frobenius norm $\|\cdot\|_F$, it turns out that the backward error $\eta_{w,p}$ and the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ remain the same for both the norms. Indeed, we have the following result.

Proposition 2.3.6. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be regular. Set* $\eta_2(\lambda, \mathbf{L}) := \eta_{w,p}(\lambda, \mathbf{L})$ *and* $\Lambda^2_\epsilon(\mathbf{L}) := \Lambda_\epsilon(\mathbf{L})$. *Now considering* $\mathbf{L}$ *as an element of* $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|_F)$, *set* $\eta_F(\lambda, \mathbf{L}) := \eta_{w,p}(\lambda, \mathbf{L})$ *and* $\Lambda^F_\epsilon(\mathbf{L}) := \Lambda_\epsilon(\mathbf{L})$. *Then we have*

$$\eta_2(\lambda, \mathbf{L}) = \eta_F(\lambda, \mathbf{L}) = \frac{\sigma_{\min}(\mathbf{L}(\lambda))}{\|(1,\lambda)\|_{w^{-1},q}} \textit{ and } \Lambda^2_\epsilon(\mathbf{L}) = \Lambda^F_\epsilon(\mathbf{L}),$$

where $p^{-1}+q^{-1}=1$. *Similar results hold when* $\mathbf{L}$ *is considered as a homogeneous pencil.*

Proof: Let u and v be unit left and right singular vectors of $\mathbf{L}(\lambda)$ corresponding to the smallest singular value $\sigma_{\min}(\mathbf{L}(\lambda))$ of $\mathbf{L}(\lambda)$. Now set $\alpha := \|(1,\lambda)\|_{w^{-1},q}$ and define

$$\Delta A := -\frac{w_1^{-q}\,\sigma_{\min}(\mathbf{L}(\lambda))\,uv^*}{\alpha^q} \text{ and } \Delta B := \frac{w_2^{-q}\,\sigma_{\min}(\mathbf{L}(\lambda))\,\mathrm{sign}(\lambda)\,|\lambda|^{q-1}\,uv^*}{\alpha^q}.$$

Here it is assumed that $p \neq 1$. For $p=1$, ΔA and ΔB can be constructed as in the proof of Proposition 3.3.5. Now consider the pencil $\Delta\mathbf{L}(z) := \Delta A - z\Delta B$. By construction we have $\mathbf{L}(\lambda)v + \Delta\mathbf{L}(\lambda)v = 0$. Since ΔA is a rank one matrix, the spectral and the Frobenius norms of ΔA are the same. The same is the case for ΔB. Consequently, $|||\Delta\mathbf{L}|||_{w,p}$ is the same for the spectral and the Frobenius norms on $\mathbb{C}^{n\times n}$. By construction, we have $|||\Delta\mathbf{L}|||_{w,p} = \dfrac{\sigma_{\min}(\mathbf{L}(\lambda))}{\alpha}$. Hence the result follows. The proof for the homogeneous pencil is similar. ■

We are now ready to show that various pseudospectra defined in section 2 follow from (2.3) for appropriate choices of w, p and the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. To that end, consider a regular pencil $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$.

- **Riedel [38]:** Suppose that $\mathbf{L}(\lambda) := A - \lambda B$ and that B is positive definite. Consider $B = F^*F$. Recall from section 2 that $\Lambda_\epsilon(\mathbf{L}) := \{z \in \mathbb{C} : \sigma_{\min}((F^*)^{-1}AF^{-1}) \leq \epsilon\}$. Equivalently,

  $$\begin{aligned} \Lambda_\epsilon(\mathbf{L}) &= \{z \in \mathbb{C} : \det(A + F^*\Delta AF - zB) = 0 \text{ and } \|\Delta A\|_2 \leq \epsilon\} \\ &= \{z \in \mathbb{C} : \det(A + F^*\Delta AF - zB) = 0,\ \|(F^*)^{-1}(F^*\Delta AF)F^{-1}\|_2 \leq \epsilon\} \\ &= \{z \in \mathbb{C} : \det(A + G - zB) = 0 \text{ and } \|(F^*)^{-1}GF^{-1}\|_2 \leq \epsilon\}. \end{aligned}$$

  Now for $X \in \mathbb{C}^{n\times n}$, define $\|X\|_B := \|(F^*)^{-1}XF^{-1}\|_2$. Then $\|\cdot\|_B$ defines a norm on $\mathbb{C}^{n\times n}$. Consequently, considering the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_B)$ for the choice $w=(1,0)$ and $p=2$ (any p will do the job), we obtain from (2.3) the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ defined by Riedel.

- **Dorsselaer [19]:** Again, considering the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ for the choice $w=(1,0)$ and $p=2$ (or any p for that matter), we obtain from (2.3) the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ defined by Dorsselaer.

- **Frayssé et al. [22]:** Next, considering the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ for the choice $w=(\alpha^{-1},\beta^{-1})$, $p=\infty$ and a subordinate matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, we obtain from (2.3) the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ defined by Frayssé et al..

- **Lavallée et al. [30]:** Considering homogeneous form of $\mathbf{L}$, that is, $\mathbf{L}(c,s) = cA - sB$, and the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ for the choice $w=(1,1)$ and $p=2$, we obtain from (2.3) the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ considered by Lavallée et al..

- **Higham et al. [25, 45]:** Considering the homogeneous pencil $\mathbf{L}(c, s) = cA - sB$ and the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ for the choice $w = (\alpha^{-1}, \beta^{-1})$, $p = \infty$ and a subordinate matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, we obtain from (2.3) the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ defined by Higham et al..

We mention that homogeneous form of pencils and their pseudospectra provide a natural setting for dealing with a pencil having an infinite eigenvalue. An alternative approach to resolving an infinite eigenvalue of a pencil $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ given by $\mathbf{L}(\lambda) = A - \lambda B$ is to consider the reverse pencil $\mathrm{rev}\mathbf{L}(\lambda) = B - \lambda A$. Note that ∞ is an eigenvalue of $\mathbf{L}$ if and only if 0 is an eigenvalue of $\mathrm{rev}\mathbf{L}$. More precisely, in the space $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, the following spectral mapping holds:

$$\Lambda(\mathrm{rev}\mathbf{L}) = (\Lambda(\mathbf{L}))^{-1} := \{1/\lambda \in \mathbb{C}_\infty : \lambda \in \Lambda(\mathbf{L})\}.$$

Now, notice that $\eta_{w,p}(\lambda, \mathbf{L}) = \eta_{w,p}(1/\lambda, \mathrm{rev}\mathbf{L})$. Consequently, the following pseudospectral mapping holds:

$$\Lambda_\epsilon(\mathrm{rev}\mathbf{L}) = (\Lambda_\epsilon(\mathbf{L}))^{-1} := \{1/\lambda \in \mathbb{C}_\infty : \lambda \in \Lambda_\epsilon(\mathbf{L})\}.$$

This shows that the image of the component of $\Lambda_\epsilon(\mathrm{rev}\mathbf{L})$ containing the eigenvalue 0 of $\mathrm{rev}\mathbf{L}$ under the map $z \mapsto z^{-1}$ is precisely the component of $\Lambda_\epsilon(\mathbf{L})$ containing an infinite eigenvalue of $\mathbf{L}$. Thus for a pencil $\mathbf{L}$ having an infinite eigenvalue, adding ∞ to $\Lambda_\epsilon(\mathbf{L})$ whenever $\eta_{w,p}(0, \mathrm{rev}\mathbf{L}) \leq \epsilon$, we obtain the ϵ-pseudospectrum of $\mathbf{L}$ in $\mathbb{C}_\infty$.

Next, let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular pencil, where $\|\cdot\|$ is a subordinate matrix norm on $\mathbb{C}^{n\times n}$. Then it can be checked that $\Lambda_\epsilon(\mathbf{L})$ has the following properties:

- The pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ consists of at most n components and each component contains at least one eigenvalue of $\mathbf{L}$.
- Suppose that $\mathbf{L}$ is block diagonal, that is, $\mathbf{L} = \mathrm{diag}(\mathbf{L}_1, \mathbf{L}_2)$ and that the subordinate matrix norm $\|\cdot\|$ satisfies $\|\mathrm{diag}(A_1, A_2)\| = \max(\|A_1\|, \|A_2\|)$ for $\mathrm{diag}(A_1, A_2) \in \mathbb{C}^{n\times n}$. Then we have $\Lambda_\epsilon(\mathbf{L}) = \Lambda_\epsilon(\mathbf{L}_1) \cup \Lambda_\epsilon(\mathbf{L}_2)$.

For the rest of the chapter, we consider only the spectral norm $\|\cdot\|_2$ on $\mathbb{C}^{n\times n}$, that is, we consider only the space $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$. In this case, recall that we have

$$\eta_{w,p}(\lambda, \mathbf{L}) = \frac{\sigma_{\min}(\mathbf{L}(\lambda))}{\|(1, \lambda)\|_{w^{-1},q}} \quad \text{and} \quad \eta_{w,p}(c, s, \mathbf{L}) = \frac{\sigma_{\min}(\mathbf{L}(c, s))}{\|(c, s)\|_{w^{-1},q}}. \tag{2.4}$$

We often identify $\mathbb{C}$ with $\mathbb{R}^2$ and treat the map $\lambda \mapsto \eta_{w,p}(\lambda, \mathbf{L})$ as a function from $\mathbb{R}^2$ to $\mathbb{R}$. As a consequence, we often identify $\Lambda_\epsilon(\mathbf{L})$ as a subset of $\mathbb{R}^2$. Then it follows that

$$\partial\Lambda_\epsilon(\mathbf{L}) \subset \{(x, y) \in \mathbb{R}^2 : \eta_{w,p}(x + iy, \mathbf{L}) = \epsilon\},$$

where $\partial\Lambda_\epsilon(\mathbf{L})$ is the boundary of $\Lambda_\epsilon(\mathbf{L})$. The next result shows that $\partial\Lambda_\epsilon(\mathbf{L})$ is a real algebraic curve.

Proposition 2.3.7. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular nonhomogeneous pencil. Then the boundary* $\partial\Lambda_\epsilon(\mathbf{L})$ *of* $\Lambda_\epsilon(\mathbf{L})$ *is embedded in a real algebraic curve when* $p/(p-1)$ *is either an integer or* $p = \infty$.

Proof: First, suppose that $p = 1$. Then $\epsilon\|(1,\, x+iy)\|_{w^{-1},\infty}$ is a singular value of $\mathbf{L}(x+iy)$ if and only if $Z_\infty(x,y) := (\mathbf{L}(x+iy)^*\mathbf{L}(x+iy)) - (\epsilon\|(1,\, x+iy)\|_{w^{-1},\infty})^2 I$ is singular. Hence $\partial\Lambda_\epsilon(\mathbf{L}) \subset \Gamma_\infty := \{(x,y) \in \mathbb{R}^2 : \det(Z_\infty(x,y)) = 0\}$. Now it is easy to check that Γ_∞ is an algebraic curve.

Next, suppose that $p \neq 1$. Then we have $\partial\Lambda_\epsilon(\mathbf{L}) \subset \{(x,y) \in \mathbb{R}^2 : \sigma_{\min}(x+iy,\, \mathbf{L}) = \epsilon\|(1, x+iy)\|_{w^{-1},q}\}$. Note that $\epsilon\|(1,\, x+iy)\|_{w^{-1},q}$ is a singular value of $\mathbf{L}(x+iy)$ if and only if $Z_q(x,y) := (\mathbf{L}(x+iy)^*\mathbf{L}(x+iy))^q - (\epsilon\|(1,\, x+iy)\|_{w^{-1},q})^{2q} I$ is singular. Hence $\partial\Lambda_\epsilon(\mathbf{L}) \subset \Gamma_q := \{(x,y) \in \mathbb{R}^2 : \det(Z_q(x,y)) = 0\}$. Since $q = p/(p-1)$ is an integer, it is easy to see that Γ_q is a real algebraic curve. Indeed, if $q = p/(p-1)$ is even then it follows that $\det(Z_q(x,y))$ is a polynomial. On the other hand, if q is odd then it follows that $\det(Z_q(x,y)) = (\sqrt{x^2+y^2})t(x,y) + r(x,y)$ for some nonzero polynomials $t(x,y)$ and $r(x,y)$. Hence in either case Γ_q is an algebraic curve. This completes the proof. ■

It would be interesting to know if $\partial\Lambda_\epsilon(\mathbf{L})$ is embedded in an algebraic for the case when $p/(p-1)$ is not an integer. In such a case, $\partial\Lambda_\epsilon(\mathbf{L})$ is at least a real analytic curve whenever $0 \notin \partial\Lambda_\epsilon(\mathbf{L})$. We mention, however, that for all most all practical purposes, the values of p that matter are $p = 1, 2, \infty$, and for these cases $\partial\Lambda_\epsilon(\mathbf{L})$ is an algebraic curve.

For a homogeneous pencil $\mathbf{L}(c,s)$, the boundary $\partial\Lambda_\epsilon(\mathbf{L})$ is embedded in a projective curve. Indeed, for the special case when $\mathbf{L} \in \mathbb{L}_w^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$, we have

$$\begin{aligned}\partial\Lambda_\epsilon(\mathbf{L}) \;&\subset\; \{(x,y,z) \in \mathbb{R}^3 \setminus \{0\} : \sigma_{\min}(\mathbf{L}(x, y+iz))^2 = \epsilon^2(w_1^{-2}x^2 + w_2^{-2}(y^2+z^2))^2\} \\ &\subset\; \{(x,y,z) \in \mathbb{R}^3 : Q(x,y,z) = 0\} =: V(Q),\end{aligned}$$

where $Q(x,y,z) := \det(\mathbf{L}^*(z, x+iy)\mathbf{L}(z, x+iy) - I\epsilon^2(w_1^{-2}x^2 + w_2^{-2}(y^2+z^2)))$. On the other hand, treating $\Lambda_\epsilon(\mathbf{L})$ as a subset of the Riemann sphere $\mathbb{S} \subset \mathbb{R}^3$, we have

$$\partial\Lambda_\epsilon(\mathbf{L}) \subset V(Q) \cap V(x^2+y^2+z^2-1),$$

where $V(f)$ denotes the algebraic variety of the polynomial f.

2.4 Critical points and multiple eigenvalues

For a regular matrix pencil $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$, we now investigate the functions $\mathbb{C} \to \mathbb{R}, z \mapsto \eta_{w,p}(z, \mathbf{L})$ and $\mathbb{C}^2 \to \mathbb{R}, (c,s) \mapsto \eta_{w,p}(c,s,\mathbf{L})$. To that end, the gradients of the maps $\mathbb{C}^2 \to \mathbb{R}, (c,s) \mapsto \|(c,s)\|_{w,p}$ and $\mathbb{C} \to \mathbb{R}, z \mapsto \|(1,z)\|_{w,p}$ will play an important role. We identify $\mathbb{C}$ with $\mathbb{R}^2$ and express the gradient $\nabla(\|(1,z)\|_{w,p})$ as a complex number. Then we have the following result.

Proposition 2.4.1. *Let* $h_{w,p}(\lambda) := \|(1, \lambda)\|_{w,p}$ *for* $\lambda \in \mathbb{C}$ *and* $1 \le p \le \infty$.

(a) *If* $1 < p < \infty$ *then* $h_{w,p}$ *is differentiable on* $\mathbb{C}$ *and*

$$\nabla h_{w,p}(\lambda) = \frac{w_2^p\, \lambda\, |\lambda|^{p-2}}{h_{w,p}(\lambda)^{p-1}}.$$

(b) *If* $p = 1$ *then* $h_{w,p}$ *is differentiable on* $\mathbb{C} \setminus \{0\}$ *and* $\nabla h_{w,p}(\lambda) = \dfrac{w_2\, \lambda}{|\lambda|}$ *for* $\lambda \neq 0$.

(c) *If* $p = \infty$ *then* $h_{w,p}$ *is differentiable on* $\mathbb{C} \setminus \{z \in \mathbb{C} : |z| = w_1/w_2\}$ *and*

$$\nabla h_{w,p}(\lambda) = \begin{cases} 0, & \text{when } |\lambda| < w_1/w_2, \\ \dfrac{w_2\, \lambda}{|\lambda|}, & \text{when } |\lambda| > w_1/w_2. \end{cases}$$

Proof: The proof follows from the fact that $\nabla(|\lambda|^p) = p\lambda\, |\lambda|^{p-2}$. ■

Next, we analyze the gradient of the map $\mathbb{C}^2 \to \mathbb{R}, (c, s) \mapsto \|(c, s)\|_{w,p}$. First, we define the **partial gradients** $\nabla_c(\|(c, s)\|_{w,p})$ and $\nabla_s(\|(c, s)\|_{w,p})$. We define the partial gradient $\nabla_c(\|(c, s)\|_{w,p})$ to be the gradient of the map $\mathbb{C} \to \mathbb{R}, c \mapsto \|(c, s)\|_{w,p}$ treating s as constant. The partial gradient $\nabla_s(\|(c, s)\|_{w,p})$ is defined similarly. Then the gradient of the map $(c, s) \mapsto \|(c, s)\|_{w,p}$ when exists is given by

$$\nabla(\|(c, s)\|_{w,p}) = (\nabla_c(\|(c, s)\|_{w,p}),\ \nabla_s(\|(c, s)\|_{w,p})) \subset \mathbb{C}^2.$$

The gradient and the partial gradients of the map $(c, s) \mapsto \|(c, s)\|_{w,p}$ are given in the following result.

Proposition 2.4.2. *Let* $H_{w,p}(c, s) := \|(c, s)\|_{w,p}$ *for* $(c, s) \in \mathbb{C}^2$ *and* $1 \le p \le \infty$. *Then we have the following.*

(a) *If* $1 < p < \infty$ *then* $H_{w,p}$ *is differentiable on* $\mathbb{C}^2$ *and*

$$\nabla_c\, H_{w,p}(c, s) = \frac{w_1^p c\, |c|^{p-2}}{H_{w,p}(c, s)^{p-1}} \text{ and } \nabla_s\, H_{w,p}(c, s) = \frac{w_2^p s\, |s|^{p-2}}{H_{w,p}(c, s)^{p-1}}.$$

(b) *If* $p = 1$ *then* $\nabla_c\, H_{w,p}(c, s)$ *exists for* $c \neq 0$, $\nabla_s\, H_{w,p}(c, s)$ *exists for* $s \neq 0$ *and are given by*

$$\nabla_c\, H_{w,p}(c, s) = \frac{w_1 c}{|c|} \text{ and } \nabla_s\, H_{w,p}(c, s) = \frac{w_2 s}{|s|}.$$

In particular, $H_{w,p}(c, s)$ *is differentiable at* (c, s) *if* $cs \neq 0$.

(c) *If* $p = \infty$ *then* $H_{w,p}$ *is differentiable on* $\mathbb{C}^2 \setminus \{(c, s) : w_1|c| = w_2|s|\}$ *and we have*

$$\nabla H_{w,p}(c, s) = \begin{cases} \left(0,\ \dfrac{w_2 s}{|s|}\right), & \text{when } w_1|c| < w_2|s|, \\ \left(\dfrac{w_1 c}{|c|},\ 0\right), & \text{when } w_1|c| > w_2|s|. \end{cases}$$

Proof: The proof is easy and follows from the fact that $\nabla(|z|^p) = pz\,|z|^{p-2}$. ■

Note that the gradient $\nabla H_{w,p}(c,s)$ is an element of the dual space of $(\mathbb{C}^2, \|\cdot\|_{w,p})$ and the dual space of $(\mathbb{C}^2, \|\cdot\|_{w,p})$ is the space $(\mathbb{C}^2, \|\cdot\|_{w^{-1},q})$, where $p^{-1}+q^{-1}=1$. The next result shows that $\nabla H_{w,p}(c,s)$ is a unit vector in $(\mathbb{C}^2, \|\cdot\|_{w^{-1},q})$.

Lemma 2.4.3. *Let $H_{w,p}(c,s) := \|(c,s)\|_{w,p}$ for $(c,s) \in \mathbb{C}^2$. Then we have*

$$\|(\nabla_c\, H_{w,p}(c,s), \nabla_s\, H_{w,p}(c,s))\|_{w^{-1},q} = 1$$

and

$$c\,(\overline{\nabla_c H_{w,p}(c,s)}) + s\,(\overline{\nabla_s H_{w,p}(c,s)}) = H_{w,p}(c,s),$$

where $p^{-1}+q^{-1}=1$.

Proof: First suppose that $1 < p < \infty$. Then by Proposition 2.4.2, we have

$$\begin{aligned}
\|\nabla H_{w,p}(c,s)\|_{w^{-1},q} &= \|(w_1^{-1}|\nabla_c\, H_{w,p}(c,s)|, w_2^{-1}|\nabla_s\, H_{w,p}(c,s)|)\|_q \\
&= \frac{1}{H_{w,p}(c,s)^{p-1}}\left(w_1^{pq-q}|c|^{pq-q} + w_2^{pq-q}|s|^{pq-q}\right)^{1/q} \\
&= \frac{1}{H_{w,p}(c,s)^{p-1}}(w_1^p|c|^p + w_2^p|s|^p)^{1/q} = 1.
\end{aligned}$$

The proof is similar for the case $p=1$ and $p=\infty$. The rest of the proof follows from the Proposition 2.4.2. ■

Next, we analyze differentiability of $\eta_{w,p}(\lambda, \mathbf{L})$ and $\eta_{w,p}(c,s,\mathbf{L})$. Again, we treat $\eta_{w,p}(\lambda,\mathbf{L})$ as a function from $\mathbb{R}^2$ to $\mathbb{R}$ and express its gradient as a complex number.

Theorem 2.4.4. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular pencil and $1 \leq p \leq \infty$.*

(a) *Suppose that $\mathbf{L}$ is given by $\mathbf{L}(\lambda) = A - \lambda B$. Suppose also that $\sigma_{\min}(\mathbf{L}(\lambda))$ is simple. If the gradient $\nabla(\|(1,\lambda)\|_{w^{-1},q})$ exists then the gradient $\nabla\eta_{w,p}(\lambda,\mathbf{L})$ exists and is given by*

$$\nabla\eta_{w,p}(\lambda,\mathbf{L}) = -\frac{\overline{u^*Bv} + \eta_{w,p}(\lambda,\mathbf{L})\nabla(\|(1,\lambda)\|_{w^{-1},q})}{\|(1,\lambda)\|_{w^{-1},q}},$$

where u and v are unit left and right singular vectors of $\mathbf{L}(\lambda)$ corresponding to $\sigma_{\min}(\mathbf{L}(\lambda))$ and $p^{-1}+q^{-1}=1$.

(b) *Next, suppose that $\mathbf{L}$ is given by $\mathbf{L}(c,s) = cA - sB$, where $(c,s) \in \mathbb{C}^2\setminus\{0\}$. Suppose that $\sigma_{\min}(\mathbf{L}(c,s))$ is simple and that $\nabla_c(\|(c,s)\|_{w^{-1},q})$ and $\nabla_s(\|(c,s)\|_{w^{-1},q})$ exist. Then $\nabla_c\,\eta_{w,p}(c,s,\mathbf{L})$ and $\nabla_s\,\eta_{w,p}(c,s,\mathbf{L})$ exist and are given by*

$$\begin{aligned}
\nabla_c\,\eta_{w,p}(c,s,\mathbf{L}) &= \frac{\overline{u^*Av} - \eta_{w,p}(c,s,\mathbf{L})\,\nabla_c\,(\|(c,s)\|_{w^{-1},q})}{\|(c,s)\|_{w^{-1},q}}, \\
\nabla_s\,\eta_{w,p}(c,s,\mathbf{L}) &= -\frac{\overline{u^*Bv} + \eta_{w,p}(c,s,\mathbf{L})\,\nabla_s\,(\|(c,s)\|_{w^{-1},q})}{\|(c,s)\|_{w^{-1},q}},
\end{aligned}$$

where u *and* v *are unit left and right singular vectors of* $\mathbf{L}(c,s)$ *corresponding to* $\sigma_{\min}(\mathbf{L}(c,s))$ *and* $p^{-1}+q^{-1}=1$.

Proof: Let $\lambda = x+iy$ and $g(\lambda) := \sigma_{\min}(\mathbf{L}(\lambda))$. Since $g(\lambda)$ is simple, it is shown by Sun [44] that g is real analytic at (x,y) and $\nabla g(\lambda) = (\mathrm{Re}(u^*\partial_x \mathbf{L}(\lambda)v), \mathrm{Re}(u^*\partial_y \mathbf{L}(\lambda)v))$. Writing the gradient as a complex number, we have $\nabla g(\lambda) = -\overline{(u^*Bv)}$. Now, setting $f(\lambda) := \|(1,\lambda)\|_{w^{-1},q}$ and using the fact that $\eta_{w,p}(\lambda,\mathbf{L}) = g(\lambda)/f(\lambda)$ and $\nabla(g/f) = \dfrac{f\nabla g - g\nabla f}{f^2}$, the desired result follows. The proof is similar for $\eta_{w,p}(c,s)$. ■

Now we construct a matrix pencil with a specified eigenvalue. We denote the gradient of the map $\mathbb{C} \to \mathbb{R}, z \mapsto \|(1,z)\|_{w,p}$ evaluated at λ by $\nabla(\|(1,\lambda)\|_{w,p})$.

Theorem 2.4.5. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular pencil given by* $\mathbf{L}(z) := A - zB$. *Let* $\lambda \in \mathbb{C}$. *Consider the SVD* $\mathbf{L}(\lambda) = U\Sigma V^*$ *and set* $u := U(:, n-m+1:n)$ *and* $v := V(:, n-m+1:n)$, *where* m *is the multiplicity of* $\sigma_{\min}(\mathbf{L}(\lambda))$. *For* $(c,s) \in \mathbb{C}^2$, *define* $H_{w,p}(c,s) := \|(c,s)\|_{w,p}$. *With the convention that a partial gradient of* $H_{w,p}$ *at* $(1,\lambda)$ *is* 0 *if it does not exist, we define*

$$\Delta A := -\eta_{w,p}(\lambda,\mathbf{L})\, \overline{(\nabla_c H_{w^{-1},q})(1,\lambda)}\, uv^* \text{ and } \Delta B := \eta_{w,p}(\lambda,\mathbf{L})\, \overline{(\nabla_s H_{w^{-1},q})(1,\lambda)}\, uv^*$$

and consider the pencil $\Delta\mathbf{L}(z) = \Delta A - z\Delta B$, *where* $p^{-1}+q^{-1}=1$. *Then we have*

(*a*) $\|\Delta\mathbf{L}\|_{w,p} = \eta_{w,p}(\lambda,\mathbf{L})$.

(*b*) $(\mathbf{L}(\lambda)+\Delta\mathbf{L}(\lambda))v = 0$ *and* $u^*(\mathbf{L}(\lambda)+\Delta\mathbf{L}(\lambda)) = 0$.

(*c*) $u^*(B+\Delta B)v = u^*Bv + \eta_{w,p}(\lambda,\mathbf{L})\, \overline{\nabla(\|(1,\lambda)\|_{w^{-1},q})}\, I_m$, *where* $I_m \in \mathbb{C}^{m\times m}$ *is the identity matrix.*

In particular, if $\sigma_{\min}(\mathbf{L}(\lambda))$ *is simple and* $\nabla(\|(1,\lambda)\|_{w^{-1},q})$ *exists then we have*

$$u^*(B+\Delta B)v = u^*Bv + \eta_{w,p}(\lambda,\mathbf{L})\overline{\nabla(\|(1,\lambda)\|_{w^{-1},q})} = -\|(1,\lambda)\|_{w^{-1},q}\, \overline{\nabla\eta_{w,p}(\lambda,\mathbf{L})}.$$

Proof: By Lemma 2.4.3, we have $\|((\nabla_c H_{w^{-1},q})(1,\lambda),\, (\nabla_s H_{w^{-1},q})(1,\lambda))\|_{w,p} = 1$. Consequently, we have

$$\begin{aligned}
\|\Delta\mathbf{L}\|_{w,p} &= \|(\|\Delta A\|_2,\, \|\Delta B\|_2)\|_{w,p} \\
&= \eta_{w,p}(\lambda,\mathbf{L})\|(|(\nabla_c H_{w^{-1},q})(1,\lambda))|,\, (|(\nabla_s H_{w^{-1},q})(1,\lambda))|)\|_{w,p} \\
&= \eta_{w,p}(\lambda,\mathbf{L}).
\end{aligned}$$

Next, we have

$$(\mathbf{L}(\lambda)+\Delta\mathbf{L}(\lambda))v = \sigma_{\min}(\mathbf{L}(\lambda))u - \eta_{w,p}(\lambda,\mathbf{L})(\overline{(\nabla_c H_{w^{-1},q})(1,\lambda)} + \lambda\, \overline{(\nabla_s H_{w^{-1},q})(1,\lambda)})u.$$

Again, by Lemma 2.4.3, $\overline{(\nabla_c H_{w^{-1},q})(1,\lambda)} + \lambda\,\overline{(\nabla_s H_{w^{-1},q})(1,\lambda)} = H_{w^{-1},q}(1,\lambda)$. Thus

$$(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda))v = \sigma_{\min}(\mathbf{L}(\lambda))u - \frac{\sigma_{\min}(\mathbf{L}(\lambda))}{H_{w^{-1},q}(1,\lambda)} H_{w^{-1},q}(1,\lambda)\, u = 0.$$

The proof of (c) follows from the fact that $(\nabla_s\, H_{w^{-1},q})(1,\lambda) = \nabla(\|(1,\lambda)\|_{w^{-1},q})$. Finally, when $\sigma_{\min}(\mathbf{L}(\lambda))$ is simple and the gradient $\nabla(\|(1,\lambda)\|_{w^{-1},q})$ exists, the desired result follows from Theorem 2.4.4. ■

A few comments about the results in Theorem 2.4.5, especially the case $p = 1$ and $p = \infty$, are in order.

1. Recall that for a weight vector $w = (w_1, w_2)$, $w^{-1} = (w_1^{-1}, w_2^{-1})$ with the convention that $w_j^{-1} = 0$ if $w_j = 0$. So for example, if $w_1 = 0$ then $(\nabla_c\, H_{w^{-1},q})(1,\lambda) = 0$. Hence by Theorem 2.4.5 we have $\Delta A = 0$ which conforms with the fact that $w_1 = 0$ means A remains unperturbed. Similarly, when $w_2 = 0$, we have $(\nabla_s\, H_{w^{-1},q})(1,\lambda) = 0$ and hence $\Delta B = 0$. Thus B remains unperturbed.

2. Suppose that $w_1 w_2 \neq 0$ and $p = \infty$. Then $q = 1$ and we have the partial gradients $(\nabla_c\, H_{w^{-1},1})(1,\lambda) = w_1^{-1}$ and $(\nabla_s\, H_{w^{-1},1})(1,\lambda) = w_2^{-1}\lambda/|\lambda|$. This shows that the partial gradient $(\nabla_s\, H_{w^{-1},1})(1,\lambda)$ does not exist for $\lambda = 0$. Hence by convention $(\nabla_s\, H_{w^{-1},1})(1,\lambda) = 0$ for $\lambda = 0$. Consequently, when $\lambda = 0$ by Theorem 2.4.5, we have $\Delta A = -w_1^{-1}\eta_{w,\infty}(\lambda, \mathbf{L})\, uv^*$ and $\Delta B = 0$. Hence B remains unperturbed.

3. Next, suppose that $w_1 w_2 \neq 0$ and $p = 1$. Then $q = \infty$ and by Theorem 2.4.2, we have

   $$((\nabla_c\, H_{w^{-1},\infty})(1,\lambda),\, (\nabla_s\, H_{w^{-1},\infty})(1,\lambda)) = \begin{cases} \left(0,\ \dfrac{w_2^{-1}\lambda}{|\lambda|}\right), & \text{when } |\lambda| > w_2/w_1, \\[2ex] \left(w_1^{-1},\ 0\right), & \text{when } |\lambda| < w_2/w_1. \end{cases}$$

   This shows that when $|\lambda| > w_2/w_1$, by Theorem 2.4.5, we have $\Delta A = 0$ and $\Delta B = w_2^{-1}\eta_{w,1}(\lambda, \mathbf{L})\,\mathrm{sign}(\lambda)\, uv^*$. This shows that A remains unperturbed. On the other hand, when $|\lambda| < w_2/w_1$, by Theorem 2.4.5, $\Delta A := -w_1^{-1}\eta_{w,1}(\lambda, \mathbf{L})\, uv^*$ and $\Delta B := 0$. So, in this case, B remains unperturbed.

   Finally, when $|\lambda| = w_2/w_1$, the partial gradients of $H_{w^{-1},\infty}$ do not exist at $(1,\lambda)$. Consequently, by Theorem 2.4.5, we have $\Delta A = 0$ and $\Delta B = 0$. This shows that when $|\lambda| = w_2/w_1$, Theorem 2.4.5 does not provide a perturbed pencil for which λ is an eigenvalue. This problem can be fixed as follows. Instead of setting the partial gradients of $H_{w^{-1},\infty}$ at $(1,\lambda)$ to 0 (as they do not exist), we redefine $(\nabla_c\, H_{w^{-1},\infty})(1,\lambda)$ and $(\nabla_s\, H_{w^{-1},\infty})(1,\lambda)$, respectively, as limits of $(\nabla_c\, H_{w^{-1},\infty})(1,z)$ and $(\nabla_s\, H_{w^{-1},\infty})(1,z)$ as $z \to \lambda$. We can approach λ either from inside the disk $|z| < w_2/w_1$ or from outside the disk $|z| > w_2/w_1$. For the first case, we have $((\nabla_c H_{w^{-1},\infty})(1,\lambda), (\nabla_s H_{w^{-1},\infty})(1,\lambda)) = (w_1^{-1}, 0)$. Hence by Theorem 2.4.5, we have $\Delta A := -w_1^{-1}\eta_{w,1}(\lambda, \mathbf{L})\, uv^*$ and $\Delta B := 0$. For the second case, we have $((\nabla_c H_{w^{-1},\infty})(1,\lambda), (\nabla_s H_{w^{-1},\infty})(1,\lambda)) = (0, w_2^{-1}\lambda/|\lambda|)$. Hence by Theorem 2.4.5, we have $\Delta A := 0$ and $\Delta B := w_2^{-1}\eta_{w,1}(\lambda, \mathbf{L})\,\mathrm{sign}(\lambda)\, uv^*$. To sum up: when $|\lambda| = w_2/w_1$, we can either perturb A or B and construct the pencil $\Delta\mathbf{L}(z) := \Delta A - z\Delta B$ as outlined above.

The following result generalizes Theorem 2.4.5 to homogeneous pencils.

Theorem 2.4.6. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular pencil given by* $\mathbf{L}(c,s) = cA - sB$. *Let* $(\lambda,\mu) \in \mathbb{C}^2\setminus\{0\}$. *Consider the SVD* $\mathbf{L}(\lambda,\mu) = U\Sigma V^*$ *and set* $u := U(:, n-m+1 : n)$ *and* $v := V(:, n-m+1 : n)$, *where* m *is the multiplicity of* $\sigma_{\min}(\mathbf{L}(\lambda,\mu))$. *For* $(c,s) \in \mathbb{C}^2$, *define* $H_{w,p}(c,s) := \|(c,s)\|_{w,p}$. *With the convention that a partial gradient of* $H_{w,p}$ *at* (λ,μ) *is* 0 *if it does not exist, we define*

$$\Delta A := -\eta_{w,p}(\lambda,\mu,\mathbf{L})\,\overline{(\nabla_c H_{w^{-1},q})(\lambda,\mu)}\; uv^*,\;\; \Delta B := \eta_{w,p}(\lambda,\mu,\mathbf{L})\,\overline{(\nabla_s H_{w^{-1},q})(\lambda,\mu)}\, uv^*$$

and consider the pencil $\Delta\mathbf{L}(c,s) = c\Delta A - s\Delta B$, *where* $p^{-1} + q^{-1} = 1$. *Then we have*

- *(a)* $\|\Delta\mathbf{L}\|_{w,p} = \eta_{w,p}(\lambda,\mu,\mathbf{L})$.
- *(b)* $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))v = 0$ *and* $u^*(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu)) = 0$.
- *(c)* $u^*(A+\Delta A)v = u^*Av - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c H_{w^{-1},q})(\lambda,\mu)}\, I_m$.
- *(d)* $u^*(B+\Delta B)v = u^*Bv + \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s H_{w^{-1},q})(\lambda,\mu)}\, I_m$,

where $I_m \in \mathbb{C}^{m\times m}$ *is the identity matrix.*

If $\sigma_{\min}(\mathbf{L}(\lambda,\mu))$ *is simple and* $(\nabla_c H_{w^{-1},q})(\lambda,\mu)$ *and* $(\nabla_s H_{w^{-1},q})(\lambda,\mu)$ *exist then*

$$\begin{aligned}
u^*(A+\Delta A)v &= u^*Av - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c H_{w^{-1},q})(\lambda,\mu)}\\
&= H_{w^{-1},q}(\lambda,\mu)\,\overline{\nabla_c\,\eta_{w,p}(\lambda,\mu,\mathbf{L})},\\
u^*(B+\Delta B)v &= u^*Bv + \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s H_{w^{-1},q})(\lambda,\mu)}\\
&= -H_{w^{-1},q}(\lambda,\mu)\,\overline{\nabla_s\,\eta_{w,p}(\lambda,\mu,\mathbf{L})}.
\end{aligned}$$

Remark 2.4.7. *We mention that when* $p = 1$, *the partial gradients of* $H_{w^{-1},\infty}$ *do not exist at* (λ,μ) *if* $w_2|\lambda| = w_1|\mu|$. *In such a case, instead of setting the partial gradients of* $H_{w^{-1},\infty}$ *in Theorem 2.4.6 to* 0, *they should be redefined as limits of* $(\nabla_c H_{w^{-1},\infty}(x,y)$, $\nabla_s H_{w^{-1},\infty}(x,y))$ *as* $(x,y) \to (\lambda,\mu)$. *As we have seen in the case of Theorem 2.4.5, this will provide us with two choices of the perturbed pencil* $\Delta\mathbf{L}$ *by letting* (x,y) *approach* (λ,μ) *from inside and from outside the region* $\{(x,y) \in \mathbb{C}^2 : w_2|x| < w_1|y|\}$.

Now we define the notion of generic and nongeneric critical points of the map $z \mapsto \eta_{w,p}(z, \mathbf{L})$ and $(c, s) \mapsto \eta_{w,p}(c, s, \mathbf{L})$.

Definition 2.4.8. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular pencil. Then* $\lambda \in \mathbb{C}$ *is said to be a generic critical point of* $\eta_{w,p}(z, \mathbf{L})$ *if* $\sigma_{\min}(\mathbf{L}(\lambda))$ *is simple and* $\nabla\eta_{w,p}(\lambda, \mathbf{L}) = 0$. *If* $\sigma_{\min}(\mathbf{L}(\lambda))$ *is multiple then* λ *is said to be a nongeneric critical point of* $\eta_{w,p}(z, \mathbf{L})$. *If the multiplicity of* $\sigma_{\min}(\mathbf{L}(\lambda))$ *is* m *then* λ *is said to be a nongeneric critical point of multiplicity* m. *A complex number is said to be a critical point of* $\eta_{w,p}(z, \mathbf{L})$ *if it is either a generic or a nongeneric critical point of* $\eta_{w,p}(z, \mathbf{L})$. *Critical points of* $\eta_{w,p}(c, s, \mathbf{L})$ *are defined similarly.*

We now show that critical points of $\eta_{w,p}(z, \mathbf{L})$ are multiple eigenvalues of nearby pencils. To that end, we state a result due to Wilkinson [50] that provides a necessary and sufficient condition for an eigenvalue of a matrix to be multiple.

Theorem 2.4.9 (Wilkinson, [50]). *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\lambda \in \Lambda(A)$. *Then* λ *is a multiple eigenvalue if and only if there exists a pair of left and right eigenvectors of* A *corresponding to* λ *which are orthogonal, that is, there exists nonzero vectors* y *and* x *such that* $y^*A = \lambda y^*, Ax = \lambda x$ *and* $y^*x = 0$.

Similar result holds for matrix pencils as well (see, for example, [12]). More specifically, $\lambda \in \mathbb{C}$ is a multiple eigenvalue of a regular pencil $\mathbf{L}(z) = A - zB$ if and only if there exists left and right eigenvectors y and x of $\mathbf{L}$ corresponding to λ such that $y^*Bx = 0$. We provide an elementary proof of this result.

Suppose that y and x are normalized left and right eigenvectors of $\mathbf{L}$ corresponding to the eigenvalue λ, that is, $y^*\mathbf{L}(\lambda) = 0$ and $\mathbf{L}(\lambda)x = 0$. Note that if $y^*Bx = 0$ then $y^*Ax = 0$. In contrast, $y^*Ax = 0$ does not necessarily imply that $y^*Bx = 0$. However, if 0 is a multiple eigenvalue of A (and hence of $\mathbf{L}$) then it turns out that there exists normalized vectors y and x such that $y^*Ax = 0 \Rightarrow y^*Bx = 0$.

Theorem 2.4.10. *Let* $\lambda \in \mathbb{C}$ *be an eigenvalue of a regular pencil* $\mathbf{L}(z) = A - zB$. *Then* λ *is a multiple eigenvalue of* $\mathbf{L}$ *if and only if there exists normalized left and right eigenvectors* y *and* x *of* $\mathbf{L}$ *corresponding to* λ *such that* $y^*Bx = 0$.

Further, ∞ *is a multiple eigenvalue of* $\mathbf{L}$ *if and only if there exists normalized left and right eigenvectors* y *and* x *of* $\mathbf{L}$ *corresponding to* ∞ *such that* $y^*Ax = 0$.

Proof: Suppose that $\lambda \in \mathbb{C}$ is a multiple eigenvalue of $\mathbf{L}$. Then the result follows from Schur decomposition of $\mathbf{L}$.

Conversely, suppose that $y^*Bx = 0$. Let $z \notin \Lambda(\mathbf{L})$. Observe that λ is a multiple eigenvalue of $\mathbf{L}$ if and only if λ is a multiple of $\mathbf{L}(z)^{-1}\mathbf{L}(\lambda)$. Since $\mathbf{L}(z)^{-1}\mathbf{L}(\lambda) = I -$

$(\lambda - z)\mathbf{L}(z)^{-1}B$, it follows that λ is a multiple eigenvalue of $\mathbf{L}$ if and only if $(\lambda - z)^{-1}$ is a multiple eigenvalue of $\mathbf{L}(z)^{-1}B$.

Now $y^*Bx = 0 \Rightarrow y^*Ax = 0$ and hence $y^*\mathbf{L}(z)x = 0$. Set $w := \mathbf{L}(z)^*\, y$. Then we have $w^*\,\mathbf{L}(z)^{-1}B = (\lambda - z)^{-1}w^*$, $\mathbf{L}(z)^{-1}Bx = (\lambda - z)^{-1}x$ and $w^*x = y^*\,\mathbf{L}(z)x = 0$. Hence by Theorem 2.4.9, $(\lambda - z)^{-1}$ is a multiple eigenvalue of $\mathbf{L}(z)^{-1}B$. Consequently, λ is a multiple eigenvalue of $\mathbf{L}$.

Finally, ∞ is a multiple eigenvalue of $\mathbf{L}(z) = A - zB$ if and only if 0 is a multiple eigenvalue of $B - zA$. Hence the result follows. ■

For a homogeneous pencil $\mathbf{L}(c, s) = cA - sB$, the above result states that $(\lambda, \mu) \neq 0$ is a multiple eigenvalue of $\mathbf{L}$ if and only if there exists left and right eigenvectors y and x of $\mathbf{L}$ corresponding to (λ, μ) such that $(y^*Ax,\, y^*Bx) = 0$.

The following result shows that critical points of $\eta_{w,p}(z, \mathbf{L})$ are multiple eigenvalues of nearby pencils.

Theorem 2.4.11. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular pencil. Let* $\mathbf{L}$ *be given by* $\mathbf{L}(z) := A - zB$. *For* $\lambda \in \mathbb{C}$, *construct the pencil* $\Delta\mathbf{L}$ *as in Theorem 2.4.5. Then we have* $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \eta_{w,p}(\lambda, \mathbf{L})$. *If* λ *is a generic critical point of* $\eta_{w,p}(z, \mathbf{L})$ *then* λ *is a defective eigenvalue of* $\mathbf{L} + \Delta\mathbf{L}$. *On the other hand, if* λ *is a nongeneric critical point of* $\eta_{w,p}(z, \mathbf{L})$ *of multiplicity* m *then* λ *is a multiple eigenvalue of* $\mathbf{L} + \Delta\mathbf{L}$ *of geometric multiplicity* m.

Proof: By Theorem 2.4.5, we have $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \eta_{w,p}(\lambda, \mathbf{L})$, $\mathbf{L}(\lambda)v + \Delta\mathbf{L}(\lambda)v = 0$ and $u^*(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = 0$. If λ is a generic critical point then we have $\nabla\eta_{w,p}(\lambda, \mathbf{L}) = 0$. Hence by Theorem 2.4.5, we have $u^*(B + \Delta B)v = -\|(1, \lambda)\|_{w^{-1},q}\overline{\nabla\eta_{w,p}(\lambda, \mathbf{L})} = 0$. Consequently, by Theorem 2.4.10, λ is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. Since by construction $\mathrm{rank}(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = n - 1$, it follows that λ is a defective eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$.

When λ is a nongeneric critical point of multiplicity m, by Theorem 2.4.5, we have $(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda))v(:, j) = 0, \quad u(:, j)^*(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = 0$ for $j = 1 : m$. Hence the result follows. ■

Obviously similar result holds for a homogeneous pencil. Indeed, we have the following result for a homogeneous pencil whose proof follows from Theorem 2.4.6.

Theorem 2.4.12. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular pencil given by* $\mathbf{L}(c, s) := cA - sB$. *For nonzero* $(\lambda, \mu) \in \mathbb{C}^2$, *construct the pencil* $\Delta\mathbf{L}$ *as in Theorem 2.4.6. Then we have* $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \eta_{w,p}(\lambda, \mu, \mathbf{L})$. *If* (λ, μ) *is a generic critical point of* $\eta_{w,p}(c, s, \mathbf{L})$ *then* (λ, μ) *is a defective eigenvalue of* $\mathbf{L} + \Delta\mathbf{L}$. *On the other hand, if* (λ, μ) *is a nongeneric critical point of* $\eta_{w,p}(c, s, \mathbf{L})$ *of multiplicity* m *then* (λ, μ) *is a multiple eigenvalue of* $\mathbf{L} + \Delta\mathbf{L}$ *of geometric multiplicity* m.

Thus we see that critical points of the function $z \mapsto \eta_{w,p}(z, \mathbf{L})$ are multiple eigenvalues of appropriately perturbed pencils. Although determining all critical points of $\eta_{w,p}(z, \mathbf{L})$ is a nontrivial task, it turns out that some critical points of $\eta_{w,p}(z, \mathbf{L})$ can be read off from the pseudospectra of $\mathbf{L}$. More precisely, we now show that common boundary points of components of pseudospectra of $\mathbf{L}$ are in fact critical points of $\eta_{w,p}(z, \mathbf{L})$.

Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular pencil. Then recall that for a sufficiently small ϵ, the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ consists of at most n components. As ϵ grows gradually, the components of $\Lambda_\epsilon(\mathbf{L})$ became large in size and some of them coalesce with other components. The following result shows that the points of coalesce of components of $\Lambda_\epsilon(\mathbf{L})$ are actually critical points of $\eta_{w,p}$.

Theorem 2.4.13. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular pencil. Suppose that two components of $\Lambda_\epsilon(\mathbf{L})$ coalesce at μ as $\epsilon \to \delta$. Then $\eta_{w,p}(\mu, \mathbf{L}) = \delta$. Further, μ is either a nongeneric critical point of $\eta_{w,p}(z, \mathbf{L})$ or the following holds:*

(a) If $1 < p < \infty$ and $p/(p-1)$ an integer then μ is a generic critical point of $\eta_{w,p}$.

(b) If $p = \infty$ then μ is a generic critical point of $\eta_{w,p}$ provided that $\mu \neq 0$.

(c) If $p = 1$ then μ is a generic critical point of $\eta_{w,p}$ provided that $|\mu| \neq w_2/w_1$.

Proof: Recall that $\partial\Lambda_\delta(\mathbf{L}) \subset \Gamma := \{z \in \mathbb{C} : \eta_{w,p}(z, \mathbf{L}) = \delta\}$. Consequently, we have $\eta_{w,p}(\mu, \mathbf{L}) = \delta$. Now, if $\sigma_{\min}(\mathbf{L}(\mu))$ is multiple then μ is a nongeneric critical point of $\eta_{w,p}$. On the other hand, if $\sigma_{\min}(\mathbf{L}(\mu))$ is simple then by Proposition 2.4.1 and Theorem 2.4.4, $\eta_{w,p}(z, \mathbf{L})$ is differentiable in a neighbourhood $\mathcal{N}_\mu$ of μ for $1 < p < \infty$. Since μ lies on the common boundary of two components, in view Proposition 2.3.7 the curve $\Gamma \cap \mathcal{N}_\mu$ consists of two arcs intersecting at μ. Hence by Implicit Function Theorem $\nabla\eta_{w,p}(\mu, \mathbf{L}) = 0$. When $p = \infty$, $\eta_{w,p}(z, \mathbf{L})$ is differentiable for $\mu \neq 0$ and when $p = 1$, $\eta_{w,p}(z, \mathbf{L})$ is differentiable for $|\mu| \neq w_2/w_1$. Hence the result follows. ■

We mention that when $p/(p-1)$ is not an integer, the boundary $\partial\Lambda_\epsilon(\mathbf{L})$ is an analytic curve whenever $0 \notin \partial\Lambda_\epsilon(\mathbf{L})$ and the common boundary points of the components are singular points of the analytic curve. Hence, as long as these singular points are isolated, the conclusion in Theorem 2.4.13(a) holds even when $p/(p-1)$ is not an integer. Our guess is that these singular points (arising out of coalescence of components of $\Lambda_\epsilon(\mathbf{L})$) are always isolated irrespective of $p/(p-1)$ being an integer or not. For all most all practical purposes, however, the values of p that one really cares about are $p = 1, 2$ and ∞. For these values of p, Theorem 2.4.13 holds.

The above result shows that common boundary points of components of pseudospectra of $\mathbf{L}$ are critical points of $\eta_{w,p}(\lambda, \mathbf{L})$. Consequently, by Theorem 2.4.11 we conclude that common boundary points of components of pseudospectra $\Lambda_\epsilon(\mathbf{L})$ are in fact multiple eigenvalues of appropriately perturbed pencils whose distance from $\mathbf{L}$ is equal to ϵ.

Note that if $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ is a regular pencil and has n distinct eigenvalues then Theorem 2.4.11 provides a pencil having a multiple eigenvalue whenever a critical point of $\eta_{w,p}(z, \mathbf{L})$ is available. A **minimal critical point** of $\eta_{w,p}$, that is, the critical point at which $\eta_{w,p}$ takes the smallest value among all the critical points, is of special interest and Theorem 2.4.13 tells us where to look for it. This brings us to Wilkinson's problem for matrix pencils.

2.5 Wilkinson's problem for matrix pencils

Given a matrix $A \in \mathbb{C}^{n\times n}$ having n distinct eigenvalues, the problem of determining ΔA having smallest norm such that $A + \Delta A$ has a multiple eigenvalue is widely known as the Wilkinson's problem. It is shown in [1] that a solution of Wilkinson's problem can be constructed from the pseudospectra of A. Wilkinson's problem for matrix pencils can be stated as follows.

Wilkinson's problem: Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular pencil having n distinct eigenvalues. Define

$$\mathrm{d}(\mathbf{L}) := \inf\{\|\Delta\mathbf{L}\|_{w,p} : \Delta\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2) \text{ and } \mathbf{L} + \Delta\mathbf{L} \text{ has a multiple eigenvalue}\}.$$

Then determine $\Delta\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ such that $\mathbf{L} + \Delta\mathbf{L}$ has a multiple eigenvalue and that $\|\Delta\mathbf{L}\|_{w,p} = \mathrm{d}(\mathbf{L})$.

We now show that a solution of Wilkinson's problem can be constructed from the pseudospectra of matrix pencils. For simplicity we consider the space of pencils $\mathbb{L}_w^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$. So, let $\mathbf{L} \in \mathbb{L}_w^2(\mathbb{C}^{n}, \|\cdot\|_2)$ be a regular pencil having n distinct eigenvalues. Recall that $\Lambda_\epsilon(\mathbf{L})$ consists of at most n components and each component contains at least one eigenvalue of $\mathbf{L}$. Since $\mathbf{L}$ has n distinct eigenvalues, for sufficiently small ϵ, $\Lambda_\epsilon(\mathbf{L})$ consists of n components. Let $\#(\Lambda_\epsilon(\mathbf{L}))$ denote the number of components of $\Lambda_\epsilon(\mathbf{L})$. Then if ϵ is such that $\#(\Lambda_\epsilon(\mathbf{L})) = n$ then obviously we have $\mathrm{d}(\mathbf{L}) > \epsilon$. Thus, in view of Theorem 2.4.13, it is now clear that $\mathrm{d}(\mathbf{L})$ can be read off from the pseudospectra of $\mathbf{L}$. Indeed, let δ be such that $\#(\Lambda_\epsilon(\mathbf{L})) = n$ when $\epsilon < \delta$ and $\#(\Lambda_\epsilon(\mathbf{L})) \leq n - 1$ when $\epsilon \geq \delta$. Then we have $\mathrm{d}(\mathbf{L}) = \delta$. Note that at least two components of $\Lambda_\epsilon(\mathbf{L})$ coalesce as $\epsilon \to \delta$.

Theorem 2.5.1. *Let $\mathbf{L} \in \mathbb{L}_w^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular pencil having n distinct eigenvalues. Let $\#(\Lambda_\epsilon(\mathbf{L}))$ denote the number of components of $\Lambda_\epsilon(\mathbf{L})$. Let $\delta > 0$ be such that $\#(\Lambda_\epsilon(\mathbf{L})) = n$ if $\epsilon < \delta$ and $\#(\Lambda_\epsilon(\mathbf{L})) \leq n - 1$ if $\epsilon \geq \delta$. Then we have $\mathrm{d}(\mathbf{L}) = \delta$.*

Let μ be a common boundary point of two components of $\Lambda_\delta(\mathbf{L})$. Consider the singular value decomposition $\mathbf{L}(\mu) = U\Sigma V^*$ and set $\sigma_n := \Sigma(n, n)$. Then we have

$$\mathrm{d}(\mathbf{L}) = \eta_{w,2}(\mu, \mathbf{L}) = \frac{\sigma_n}{\sqrt{w_1^{-2} + w_2^{-2}|\mu|^2}} = \delta.$$

If σ_n is multiple then set $u := U(:, n-1:n)$, $v := V(:, n-1:n)$ else set $u := U(:, n)$, $v := V(:, n)$. Define

$$\Delta A := -\frac{w_1^{-2}\sigma_n\, uv^*}{(w_1^{-2} + w_2^{-2}|\mu|^2)},\ \ \Delta B := \frac{\overline{\mu}\, w_2^{-2}\sigma_n\, uv^*}{(w_1^{-2} + w_2^{-2}|\mu|^2)}$$

and consider the pencil $\Delta \mathbf{L}(\lambda) = \Delta A - \lambda \Delta B$. Then $\mathrm{d}(\mathbf{L}) = |||\Delta \mathbf{L}|||_{w,2} = \eta_{w,2}(\mu, \mathbf{L})$. Further μ is a multiple eigenvalue value of $\mathbf{L} + \Delta \mathbf{L}$ of geometric multiplicity 2 when σ_n is multiple and μ is a defective eigenvalue of $\mathbf{L} + \Delta \mathbf{L}$ when σ_n is simple.

Proof: Since at least two components of $\Lambda_\epsilon(\mathbf{L})$ coalesce at μ as $\epsilon \to \delta$, by Theorem 2.4.13, μ is a critical point of $\eta_{w,2}(z, \mathbf{L})$. Hence $\mathrm{d}(\mathbf{L}) = \delta$. Now the desired results follow from Theorem 2.4.11. ■

Recall that an infinite eigenvalue of $\mathbf{L}$, if any, can be treated at par with finite eigenvalues by considering homogenous form of $\mathbf{L}$. We now present an analogue of Theorem 2.5.1 for homogeneous pencils. For this purpose, we normalize eigenvalue $(c, s) \in \Lambda(\mathbf{L})$ so that $|s|^2 + |c|^2 = 1$. Hence we consider $\Lambda_\epsilon(\mathbf{L})$ as a subset of $\mathbb{S}^1$. Then we have the following.

Theorem 2.5.2. *Let $\mathbf{L} \in \mathbb{L}_w^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular homogeneous pencil having n distinct eigenvalues. Let $\#(\Lambda_\epsilon(\mathbf{L}))$ denote the number of components of $\Lambda_\epsilon(\mathbf{L})$. Let $\delta > 0$ be such that $\#(\Lambda_\epsilon(\mathbf{L})) = n$ if $\epsilon < \delta$ and $\#(\Lambda_\epsilon(\mathbf{L})) \leq n-1$ if $\epsilon \geq \delta$. Then we have $\mathrm{d}(\mathbf{L}) = \delta$.*

Let $(\lambda, \mu) \in \mathbb{S}^1$ be a common boundary point of two components of $\Lambda_\delta(\mathbf{L})$. Consider the singular value decomposition $\mathbf{L}(\lambda, \mu) = U\Sigma V^*$ and set $\sigma_n := \Sigma(n, n)$. Then we have

$$\mathrm{d}(\mathbf{L}) = \eta_{w,2}(\lambda, \mu, \mathbf{L}) = \frac{\sigma_n}{\sqrt{w_1^{-2}|\lambda|^2 + w_2^{-2}|\mu|^2}} = \delta.$$

If σ_n is multiple then set $u := U(:, n-1:n)$, $v := V(:, n-1:n)$ else set $u := U(:, n)$, $v := V(:, n)$. Define

$$\Delta A := -\frac{\overline{\lambda}\, w_1^{-2}\sigma_n\, uv^*}{(w_1^{-2}|\lambda|^2 + w_2^{-2}|\mu|^2)},\ \ \Delta B := \frac{\overline{\mu}\, w_2^{-2}\sigma_n\, uv^*}{(w_1^{-2}|\lambda|^2 + w_2^{-2}|\mu|^2)}$$

and consider the pencil $\Delta \mathbf{L}(c, s) = c\Delta A - s\Delta B$. Then $\mathrm{d}(\mathbf{L}) = |||\Delta \mathbf{L}|||_{w,2} = \eta_{w,2}(\mu, \mathbf{L})$. Further (λ, μ) is a multiple eigenvalue value of $\mathbf{L} + \Delta \mathbf{L}$ of geometric multiplicity 2 when σ_n is multiple and (λ, μ) is a defective eigenvalue of $\mathbf{L} + \Delta \mathbf{L}$ when σ_n is simple.

We mention that a solution of Wilkinson's problem has been provided in [12] for the case when $\mathbf{L} \in \mathbb{L}_w^\infty(\mathbb{C}^{n\times n}, \|\cdot\|_2)$. We now show that the solution provided in [12] follows from Theorem 2.4.11 and Theorem 2.4.13 for $p = \infty$. Indeed, let δ be the smallest value for which at least two components of $\Lambda_\epsilon(\mathbf{L})$ coalesce at μ. Then $\eta_{w,\infty}(\mu, \mathbf{L}) =$

$\sigma_{\min}(\mathbf{L}(\mu))/(w_1^{-1} + |\mu| w_2^{-1}) = \delta$. By Theorem 2.4.13, μ is a critical point of $\eta_{w,\infty}(z, \mathbf{L})$. Now, for $p = \infty$, the pencil $\Delta\mathbf{L}(z) := \Delta A - z\Delta B$ in Theorem 2.4.11 is given by

$$\Delta A := -\frac{w_1^{-1}\eta_{w,\infty}(\mu, \mathbf{L})uv^*}{(w_1^{-1} + w_2^{-1}|\mu|)}, \quad \Delta B := \frac{\operatorname{sign}(\mu)\, w_2^{-1}\eta_{w,\infty}(\mu, \mathbf{L})\, uv^*}{(w_1^{-1} + w_2^{-1}|\mu|)},$$

satisfies $|||\Delta\mathbf{L}|||_{w,\infty} = \eta_{w,\infty}(\mu, \mathbf{L})$ and μ is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$.

We illustrate our result by considering a simple example. Consider the pencil

$$\mathbf{L}(z) := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - z \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\Lambda(\mathbf{L}) = \{0.3333, 1.0000, 0.2500\}$. For $w := (1, 1)$, figure 2.1 shows that the components of $\Lambda_\epsilon(\mathbf{L})$ containing the eigenvalues $\lambda_1 := 0.2500$ and $\lambda_2 := 0.3333$ coalesce as $\epsilon \to 3.8071 \times 10^{-4}$. Hence by Theorem 2.5.1, we have $\mathrm{d}(\mathbf{L}) = 3.8071 \times 10^{-4}$.

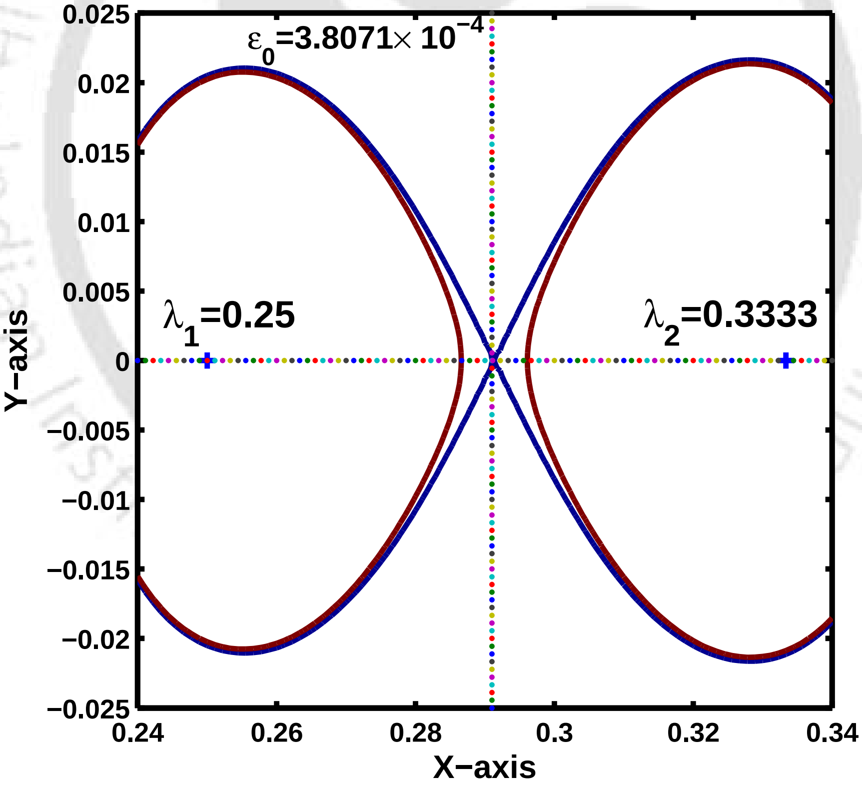


Figure 2.1: *Contour plot of* $\Lambda_\epsilon(\mathbf{L})$ *showing that the components containing* λ_1 *and* λ_2 *coalesce for* $\epsilon := 3.8071 \times 10^{-4}$.

2.6 Diagonal pencils

Now we consider the special case when $\mathbf{L}$ is a diagonal pencil or $\mathbf{L}$ is unitarily equivalent to a diagonal pencil and explore the possibility of determining solution of Wilkinson's problem for $\mathbf{L}$ without resorting to the pseudospectra of $\mathbf{L}$.

Wilkinson's problem for a normal matrix has an elegant solution that involves only eigenvalues and eigenvector of the matrix. More precisely, for a normal matrix the following result holds.

Theorem 2.6.1. *[1] Let $A \in \mathbb{C}^{n\times n}$ be normal having distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\mathrm{d}(A) := \min_{i\neq j} \frac{|\lambda_i-\lambda_j|}{2}$. Suppose that λ_i and λ_j are such that $|\lambda_i - \lambda_j|/2 = \mathrm{d}(A)$. Now define*

$$A' := A - \frac{(\lambda_i - \lambda_j)}{4}(x_i - x_j)(x_i + x_j)^* \text{ and } A'' := A - \frac{(\lambda_i - \lambda_j)}{4}(x_i + x_j)(x_i - x_j)^*,$$

where x_j is a normalized eigenvector of A corresponding to λ_j. Then A' and A'' are defective and $\mathrm{d}(A) = \|A - A'\|_2 = \|A - A''\|_2 = |\lambda_i - \lambda_j|/2$.

So, it is natural to ask: Is there an analogue of Theorem 2.6.1 for matrix pencils which are unitarily equivalent to diagonal pencils? We provide an affirmative answer to this question. We mention, however, that unlike Theorem 2.6.1, the solution of Wilkinson's problem even for a diagonal pencil turns out to be quite nontrivial. We proceed as follows.

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$, that is, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Given $x, y \in \mathbb{C}^n$, define $q : \mathbb{T} \to \mathbb{R}$ by $q(w) := \|x - wy\|_2$. Then the following result is easy to check.

Proposition 2.6.2. *For $w \in \mathbb{T}$, we have*

$$\sqrt{\|x\|_2^2 + \|y\|_2^2 - 2\,|\langle x,\, y\rangle|} \leq q(w) \leq \sqrt{\|x\|_2^2 + \|y\|_2^2 + 2\,|\langle x,\, y\rangle|},$$

where $\langle x,\, y\rangle := y^*x$.

For a complex number $z \in \mathbb{C}$, recall that $\mathrm{sign}(z) := \dfrac{\overline{z}}{|z|}$ if $z \neq 0$ and $\mathrm{sign}(0) := 1$. Then by definition $z\,\mathrm{sign}(z) = |z|$. The following result plays an important role in determining $\mathrm{d}(\mathbf{L})$.

Proposition 2.6.3. *Let x, y and $q(w)$ be as above. Define $w_{\max} := -\mathrm{sign}(\langle y,\, x\rangle)$ and $w_{\min} := \mathrm{sign}(\langle y,\, x\rangle)$. Then $w_{\max}, w_{\min} \in \mathbb{T}$ and we have*

$$\begin{aligned} \max_{w\in\mathbb{T}} q(w) &= q(w_{\max}) = \sqrt{\|x\|_2^2 + \|y\|_2^2 + 2\,|\langle x,\, y\rangle|} \\ \min_{w\in\mathbb{T}} q(w) &= q(w_{\min}) = \sqrt{\|x\|_2^2 + \|y\|_2^2 - 2\,|\langle x,\, y\rangle|} \end{aligned}$$

In particular, if $\|x\|_2 = 1 = \|y\|_2$ *and* $\cos(\theta) := |\langle x,\, y\rangle|$ *for* $0 \leq \theta \leq \pi/2$, *then we have*

$$q(w_{\max}) = 2\,\cos(\theta/2) \text{ and } q(w_{\min}) = 2\sin(\theta/2).$$

Proof: The desired results follow from the fact that

$$q(w) = \|x - wy\|_2 = \sqrt{\|x\|_2^2 + \|y\|_2^2 - 2\mathrm{Re}(\langle x,\, y\rangle \overline{w})}. \;\blacksquare$$

Next, we define the notion of **departure**, denoted by $\mathrm{Dep}(x, y)$, between points x and y in $\mathbb{C}^2$. We define $\mathrm{Dep} : \mathbb{C}^2 \to \mathbb{R}$ by

$$\mathrm{Dep}(x, y) := \frac{|y^T Jx|}{\sqrt{\|x\|_2^2 + \|y\|_2^2 + 2\,|\langle x,\, y\rangle|}}, \tag{2.5}$$

where $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and y^T is the transpose of y. It follows that

$$\mathrm{Dep}(x, y) \leq \frac{|y^T Jx|}{\|x - wy\|_2} \tag{2.6}$$

for $w \in \mathbb{T}$. If x and y are normalized unit vectors in $\mathbb{C}^2$ then we have

$$\mathrm{Dep}(x, y) = \frac{1}{2}\,|y^T Jx|\,\sec(\theta/2),$$

where $0 \leq \theta \leq \pi/2$ and is given by $\cos(\theta) := |\langle x,\, y\rangle|$. For $x := (\alpha_1, \beta_1) \in \mathbb{C}^2$ and $y := (\alpha_2, \beta_2) \in \mathbb{C}^2$, we also write $\mathrm{Dep}(x, y)$ as $\mathrm{Dep}(\alpha_1, \beta_1; \alpha_2, \beta_2)$. In such a case, we have

$$\mathrm{Dep}(\alpha_1, \beta_1; \alpha_2, \beta_2) = \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{\sqrt{\|(\alpha_1, \beta_1)\|_2^2 + \|(\alpha_2, \beta_2)\|^2 + 2\,|\langle(\alpha_1, \beta_1),\, (\alpha_2, \beta_2)\rangle|}}$$

and for normalized (α_1, β_1) and (α_2, β_2),

$$\mathrm{Dep}(\alpha_1, \beta_1; \alpha_2, \beta_2) = \frac{1}{2}\,|\alpha_1\beta_2 - \alpha_2\beta_1|\,\sec(\theta/2).$$

For ready reference, we list some essential properties of $\mathrm{Dep}(x, y)$.

- Obviously, $\mathrm{Dep}(x, y)$ is symmetric, that is, $\mathrm{Dep}(x, y) = \mathrm{Dep}(y, x)$.
- We have $\mathrm{Dep}(x, y) = 0 \iff x = y$ in $\mathbb{CP}^1$, the complex project line, that is, $x = ty$ for some $t \in \mathbb{C}$. Equivalently, $\mathrm{Dep}(\alpha_1, \beta_1; , \alpha_2, \beta_2) = 0 \iff \alpha_1/\beta_1 = \alpha_2/\beta_2$ in $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$.
- We endow $\mathbb{C}_\infty$ with chordal metric as follows: For $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{C}^2$,

$$\mathrm{chord}(\beta_1/\alpha_1, \beta_2/\alpha_2) := \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{\|(\alpha_1, \beta_1)\|_2 \|(\alpha_2, \beta_2)\|_2}.$$

Then we have

$$\mathrm{Dep}(\alpha_1, \beta_1; \alpha_2, \beta_2) \leq \frac{1}{\sqrt{2}}\sqrt{\|(\alpha_1, \beta_1)\|_2 \|(\alpha_2, \beta_2)\|_2}\,\mathrm{chord}(\beta_1/\alpha_1, \beta_2/\alpha_2).$$

In particular, for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{S}^1 := \{x \in \mathbb{C}^2 : \|x\|_2 = 1\}$, we have

$$\mathrm{Dep}(\alpha_1, \beta_1; \alpha_2, \beta_2) = \frac{1}{2}\mathrm{chord}(\beta_1/\alpha_1, \beta_2/\alpha_2)\sec(\theta/2) \leq \frac{1}{\sqrt{2}}\mathrm{chord}(\beta_1/\alpha_1, \beta_2/\alpha_2),$$

where $\theta \in [0,\ \pi/2]$ is given by $\cos(\theta) := |\alpha_1\,\overline{\alpha}_2 + \beta_1\,\overline{\beta}_2|$.

- We have $\mathrm{Dep}(tx, ty) = |t|\,\mathrm{Dep}(x, y)$ for $t \in \mathbb{C}$.

We have already demonstrated that weighted norm on the space of pencils present no extra difficulty in obtaining results. Therefore, for simplicity, we consider $w := (1, 1)$ and denote the $\mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ by $\mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$. Also, we denote the corresponding norm $|\!|\!|\cdot|\!|\!|_{w,2}$ on $\mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be $|\!|\!|\cdot|\!|\!|_2$. Note that the perturbation analysis of pencils in $\mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ then corresponds to absolute perturbations of matrix pencils.

Let $\mathbf{L} \in \mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular pencil having n distinct eigenvalues. We refer such a pencil as simple pencil. Since $\mathrm{d}(\mathbf{L})$ is invariant under unitary equivalence transformation of $\mathbf{L}$, without loss of generality, we assume that $\mathbf{L}$ is diagonal and is given by $\mathbf{L}(z) := \mathrm{diag}(\alpha_i) - \lambda\mathrm{diag}(\beta_i)$ or $\mathbf{L}(c, s) := c\,\mathrm{diag}(\alpha_i) - s\,\mathrm{diag}(\beta_i)$.

Proposition 2.6.4. *Let* $\mathbf{L} \in \mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular pencil given by* $\mathbf{L}(\lambda) := \mathrm{diag}(\alpha_i) - \lambda\,\mathrm{diag}(\beta_i)$. *Set* $x_j := (\beta_j, \alpha_j)$ *for* $j = 1 : n$. *Then* $\Lambda_\epsilon(\mathbf{L}) = \cup_{j=1}^n \Lambda_\epsilon(x_j)$, *where* $\Lambda_\epsilon(x_j) := \{z \in \mathbb{C} : |\alpha_j - z\beta_j| \leq \epsilon\,\|(1, z)\|_2\}$. *Further,* $\Lambda_\epsilon(x_i) \cap \Lambda_\epsilon(x_j) = \emptyset$ *if and only if* $\epsilon < \mathrm{Dep}(x_i, x_j)$. *If* $\epsilon = \mathrm{Dep}(x_i, x_j)$ *then* $z_w := \dfrac{\alpha_i + w\,\alpha_j}{\beta_i + w\,\beta_j}$ *is a common boundary point of* $\Lambda_\epsilon(x_i)$ *and* $\Lambda_\epsilon(x_j)$, *where* $w := \mathrm{sign}(\langle x_j,\, x_i\rangle)$.

Next, consider the homogeneous form of $\mathbf{L}$, *that is,* $\mathbf{L}(c, s) = c\,\mathrm{diag}(\alpha_i) - s\,\mathrm{diag}(\beta_i)$. *Then we have* $\Lambda_\epsilon(\mathbf{L}) = \cup_{j=1}^n \Lambda_\epsilon(x_j)$, *where* $\Lambda_\epsilon(x_j) := \{(c, s) \in \mathbb{S}^1 : |c\,\alpha_j - s\,\beta_j| \leq \epsilon\}$. *Further,* $\Lambda_\epsilon(x_i) \cap \Lambda_\epsilon(x_j) = \emptyset$ *if and only if* $\epsilon < \mathrm{Dep}(x_i, x_j)$. *If* $\epsilon = \mathrm{Dep}(x_i, x_j)$ *then* (c_w, s_w) *is a common boundary point of* $\Lambda_\epsilon(x_i)$ *and* $\Lambda_\epsilon(x_j)$, *where*

$$c_w := \frac{\beta_i + w\beta_j}{\|x_i + wx_j\|_2} \ \textit{and}\ s_w := \frac{\alpha_i + w\alpha_j}{\|x_i + wx_j\|_2}.$$

Proof: The fact that $\Lambda_\epsilon(\mathbf{L}) = \cup_{j=1}^n \Lambda_\epsilon(x_j)$ is easy to check.

Next, note that if $\Lambda_\epsilon(x_i) \cap \Lambda_\epsilon(x_j) \neq \emptyset$ then there is a complex number $z \in \mathbb{C}$ such that $|\alpha_i - z\beta_i| = |\alpha_j - z\beta_j|$. Hence $\alpha_i - z\beta_i = t(\alpha_j - z\beta_j)$ for some $t \in \mathbb{T}$. Consequently, we have $z = \dfrac{\alpha_i - t\alpha_j}{\beta_i - t\beta_j}$. Then by (2.6) we have

$$\epsilon = \frac{|\alpha_i - z\beta_i|}{\|(1, z)\|_2} = \frac{|\alpha_i\beta_j - \alpha_j\beta_i|}{\|x_i - tx_j\|_2} \geq \mathrm{Dep}(x_i, x_j).$$

On the other hand, we have $|\alpha_i - z_w\beta_i| = |\alpha_j - z_w\beta_j|$ and $\dfrac{|\alpha_i - z_w\beta_i|}{\|(1, z_w)\|_2} = \mathrm{Dep}(x_i, x_j)$. Hence the results follow. The proof is similar for homogeneous pencils. ■

This shows that if $\mathbf{L}(z) := \mathrm{diag}(\alpha_i) - z\,\mathrm{diag}(\beta_j)$ is a simple pencil then

$$\mathrm{d}(\mathbf{L}) \geq \min_{i \neq j} \mathrm{Dep}(x_i, x_j).$$

The following result shows that the equality holds.

Theorem 2.6.5. *Let* $\mathbf{L} \in \mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a simple pencil given by* $\mathbf{L}(z) := \mathrm{diag}(\alpha_i) - z\,\mathrm{diag}(\beta_i)$. *Set* $x_j := (\beta_j, \alpha_j)$ *for* $j = 1 : n$. *Then* $\mathrm{d}(\mathbf{L}) = \min_{i\neq j} \mathrm{Dep}(x_i, x_j)$. *Suppose that* $\mathrm{Dep}(x_i, x_j) = \min_{k\neq l} \mathrm{Dep}(x_k, x_l)$. *Let* $w := \mathrm{sign}(\langle x_j,\, x_i\rangle)$ *and* $z_w := \dfrac{\alpha_i + w\,\alpha_j}{\beta_i + w\,\beta_j}$.

Set $U := [e_i, e_j]$ *and* $V := [\mathrm{sign}(\alpha_i - z_w\beta_i)e_i,\, \mathrm{sign}(\alpha_j - z_w\beta_j)e_j]$, *where* e_j *is the* j*-th column of the identity matrix of size* n. *Define*

$$\Delta A := -\frac{\mathrm{Dep}(x_i, x_j)}{\|(1, z_w)\|_2}\, UV^* \text{ and } \Delta B := \frac{\overline{z_w}\,\mathrm{Dep}(x_i, x_j)}{\|(1, z_w)\|_2}\, UV^*$$

and consider the pencil $\Delta\mathbf{L}(z) := \Delta A - z\,\Delta B$. *Then we have* $|||\Delta\mathbf{L}|||_2 = \mathrm{d}(\mathbf{L})$ *and* z_w *is a multiple eigenvalue of the pencil* $\mathbf{L} + \Delta\mathbf{L}$ *of geometric multiplicity* 2.

Proof: Since $|\alpha_i - z_w\beta_i| = |\alpha_j - z_w\beta_j| = \mathrm{Dep}(x_i, x_j)\,\|(1, z_w)\|_2$, it is easy to check that $(\mathbf{L}(z_w) + \Delta\mathbf{L}(z_w))V = 0$ and $U^*(\mathbf{L}(z_w) + \Delta\mathbf{L}(z_w)) = 0$. Hence z_w is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. Consequently, we have $|||\Delta\mathbf{L}|||_2 = \mathrm{Dep}(x_i, x_j) \geq \mathrm{d}(\mathbf{L})$. On the other hand, by Proposition 2.6.4 we have $\mathrm{d}(\mathbf{L}) \geq \mathrm{Dep}(x_i, x_j)$. Hence the desired result follows. ■

The above construction just provides a nearest pencil having a multiple eigenvalue. We now show how to construct a pencil $\Delta\mathbf{L}$ such that $\mathbf{L} + \Delta\mathbf{L}$ is defective and that $|||\Delta\mathbf{L}|||_2 = \mathrm{d}(\mathbf{L})$. We proceed as follows. Recall from Theorem 2.6.5 that $\mathbf{L}(z) := \mathrm{diag}(\alpha_i) - z\,\mathrm{diag}(\beta_i)$ is a simple pencil, $x_i := (\beta_i, \alpha_i), x_j := (\beta_j, \alpha_j), w := \mathrm{sign}(\langle x_j,\, x_i\rangle)$, $z_w := \dfrac{\alpha_i + w\alpha_j}{\beta_i + w\beta_j}$ and $\mathrm{d}(\mathbf{L}) = \mathrm{Dep}(x_i, x_j)$. Then we have

$$|\alpha_i - z_w\beta_i| = |\alpha_j - z_w\beta_j| = \mathrm{Dep}(x_i, x_j)\|(1, z_w)\|_2$$

and

$$\|(1, z_w)\|_2 = \frac{\sqrt{\|x_i\|_2^2 + \|x_j\|_2^2 + 2|\langle x_i,\, x_j\rangle|}}{|\beta_i + w\beta_j|} = \frac{\|x_i + wx_j\|_2}{|\beta_i + w\beta_j|}.$$

Note that $\alpha_i - z_w\beta_i = -w(\alpha_j - z_w\beta_j)$. Hence $\mathrm{sign}(\alpha_i - z_w\beta_i) = -\overline{w}\,\mathrm{sign}(\alpha_j - z_w\beta_j)$. Note also that e_k and $e_k\mathrm{sign}(\alpha_k - z_w\beta_k)$ are left and right singular vectors of $\mathbf{L}(z_w)$ corresponding to $|\alpha_k - z_w\beta_k|$, for $k = i, j$. For $t \in (0, 1)$, we define

$$u := te_i + \sqrt{1 - t^2}\, e_j \text{ and } v := (te_i - \sqrt{1 - t^2}\, we_j)\mathrm{sign}(\alpha_i - z_w\beta_i).$$

Then it is easily checked that u and v are unit left and right singular vectors of $\mathbf{L}(z_w)$ corresponding to $|\alpha_i - z_w\beta_i|$. Now we determine t such that u and v satisfies

$$u^*Bv + \frac{\overline{z_w}\,|\alpha_i - z_w\beta_i|}{\|(1, z_w)\|_2^2} = 0.$$

This gives $t^2\beta_i - (1-t^2)w\beta_j + \dfrac{\overline{z_w}(\alpha_i - z_w\beta_i)}{\|(1, z_w)\|_2^2} = 0$. Consequently, we have

$$t^2(\beta_i + w\beta_j) = \frac{\|(1, z_w)\|_2^2(\beta_i + w\beta_j) - (\beta_i + \overline{z_w}\alpha_i)}{\|(1, z_w)\|_2^2} = (\beta_i + w\beta_j) - \frac{\|x_i\|_2^2 + |\langle x_i,\, x_j\rangle|}{(\beta_i + w\beta_j)\|(1, z_w)\|_2^2}$$

which gives

$$t^2 := \frac{\|x_j\|_2^2 + |\langle x_i,\, x_j\rangle|}{\|x_i + wx_j\|_2^2} \text{ and } 1 - t^2 := \frac{\|x_i\|_2^2 + |\langle x_i,\, x_j\rangle|}{\|x_i + wx_j\|_2^2}.$$

Thus, we have the following result.

Theorem 2.6.6. *Let* $\mathbf{L} \in \mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a simple pencil given by* $\mathbf{L}(z) = \operatorname{diag}(\alpha_i) - z\operatorname{diag}(\beta_i)$. *Let* x_i, x_j, w *and* z_w *be as in Theorem 2.6.5. Set*

$$t := \frac{\sqrt{\|x_j\|_2^2 + |\langle x_i,\, x_j\rangle|}}{\|x_i + wx_j\|_2}$$

and define $u := te_i + \sqrt{1-t^2}\; e_j$ *and* $v := (te_i - \sqrt{1-t^2}\; we_j)\operatorname{sign}(\alpha_i - z_w\beta_i)$. *Further, define*

$$\Delta A := -\frac{\operatorname{Dep}(x_i, x_j)}{\|(1, z_w)\|_2}\, uv^* \text{ and } \Delta B := \frac{\overline{z_w}\operatorname{Dep}(x_i, x_j)}{\|(1, z_w)\|_2}\, uv^*$$

and consider the pencil $\Delta\mathbf{L}(z) := \Delta A - z\,\Delta B$. *Then* $|\!|\!|\Delta\mathbf{L}|\!|\!|_2 = \mathrm{d}(\mathbf{L}) = \operatorname{Dep}(x_i, x_j)$ *and* z_w *is a defective eigenvalue of the pencil* $\mathbf{L} + \Delta\mathbf{L}$.

Proof: Note that by construction u and v are unit left and right singular vectors of $\mathbf{L}(z_w)$ corresponding to $\sigma := |\alpha_i - z_w\beta_i|$. Also, by construction u and v are unit left and right eigenvectors of $\mathbf{L} + \Delta\mathbf{L}$ corresponding to the eigenvalue z_w. Since $u^*Bv + \dfrac{\overline{z_w}\sigma}{\|(1, z_w)\|_2^2} = 0$, by Theorem 2.4.11, z_w is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. Note that $\operatorname{rank}(\mathbf{L}(z_w) + \Delta\mathbf{L}(z_w)) = n - 1$. Consequently, z_w is a defective eigenvalue $\mathbf{L} + \Delta\mathbf{L}$. By construction, we have $|\!|\!|\Delta\mathbf{L}|\!|\!|_2 = \operatorname{Dep}(x_i, x_j)$. Hence the proof. ■

For completeness, we now derive the homogeneous version of Theorem 2.6.6.

Theorem 2.6.7. *Let* $\mathbf{L} \in \mathbb{L}^2(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a simple pencil given by* $\mathbf{L}(c, s) = c\operatorname{diag}(\alpha_i) - s\operatorname{diag}(\beta_i)$. *Set* $x_j := (\beta_j, \alpha_j)$ *for* $j := 1 : n$. *Then we have*

$$\mathrm{d}(\mathbf{L}) = \min_{i\neq j}\operatorname{Dep}(x_i, x_j).$$

Let x_i *and* x_j *be such that* $\operatorname{Dep}(x_i, x_j) = \mathrm{d}(\mathbf{L})$. *Set* $w := \operatorname{sign}(\langle x_j,\, x_i\rangle)$ *and define*

$$c_w := \frac{\beta_i + w\beta_j}{\|x_i + wx_j\|_2} \text{ and } s_w := \frac{\alpha_i + w\alpha_j}{\|x_i + wx_j\|_2}.$$

Then $(c_w, s_w) \in \mathbb{S}^1$ *and* $|c_w\alpha_i - s_w\beta_i| = |c_w\alpha_j - s_w\beta_j| = \mathrm{Dep}(x_i, x_j)$. *Next, set*

$$t := \frac{\sqrt{\|x_j\|_2^2 + |\langle x_i,\, x_j\rangle|}}{\|x_i + wx_j\|_2}$$

and define $u := te_i + \sqrt{1-t^2}\; e_j$ *and* $v := (te_i - \sqrt{1-t^2}\; we_j)\mathrm{sign}(c_w\alpha_i - s_w\beta_i)$. *Further, define*

$$\Delta A := -\overline{c_w}\,\mathrm{Dep}(x_i, x_j)\, uv^* \;\; \textit{and} \;\; \Delta B := \overline{s_w}\,\mathrm{Dep}(x_i, x_j)\, uv^*$$

and consider the pencil $\Delta\mathbf{L}(c, s) := c\Delta A - s\,\Delta B$. *Then* $\|\Delta\mathbf{L}\|_2 = \mathrm{d}(\mathbf{L})$ *and* (c_w, s_w) *is a defective eigenvalue of the pencil* $\mathbf{L} + \Delta\mathbf{L}$.

Proof: By construction, we have

$$|c_w|^2 + |s_w|^2 = \frac{|\beta_i + w\beta_j|^2 + |\alpha_i + w\alpha_j|^2}{\|x_i + wx_j\|_2^2} = 1.$$

This shows that $(c_w, s_w) \in \mathbb{S}^1$. Now

$$c_w\alpha_i - s_w\beta_i = \frac{\beta_i + w\beta_j}{\|(x_i + wx_j)\|_2}\alpha_i - \frac{\alpha_i + w\alpha_j}{\|(x_i + wx_j)\|_2}\beta_i = \frac{w(\alpha_i\beta_j - \alpha_j\beta_i)}{\|x_i + wx_j\|_2}.$$

Consequently, we have

$$|c_w\alpha_i - s_w\beta_i| = \frac{|\alpha_j\beta_i - \alpha_i\beta_j|}{\|(x_i + wx_j)\|_2} = \frac{|\alpha_j\beta_i - \alpha_i\beta_j|}{\sqrt{(\|x_i\|^2 + \|x_j\|^2 + 2|\,\langle x_i, x_j\rangle\,|)}} = \mathrm{Dep}(x_i, x_j).$$

Similarly, we have that $|c_w\alpha_j - s_w\beta_j| = \mathrm{Dep}(x_i, x_j)$. Hence

$$|c_w\alpha_i - s_w\beta_i| = |c_w\alpha_j - s_w\beta_j| = \mathrm{Dep}(x_i, x_j).$$

By construction, u and v are unit left and right singular vectors of $\mathbf{L}(c_w, s_w)$ corresponding to the smallest singular value $\mathrm{Dep}(x_i, x_j)$. Also, by construction, we have $(\mathbf{L}(c_w, s_w) + \Delta\mathbf{L}(c_w, s_w))v = 0$ and $u^*(\mathbf{L}(c_w, s_w) + \Delta\mathbf{L}(c_w, s_w)) = 0$. We now show that

$$u^*Av - \overline{c_w}\mathrm{Dep}(x_i, x_j) = 0 \text{ and } u^*Bv + \overline{s_w}\mathrm{Dep}(x_i, x_j) = 0.$$

Now, we have

$$\begin{aligned} u^*Av - \overline{c_w}\mathrm{Dep}(x_i, x_j) &= \Big[t^2(\alpha_i + w\alpha_j) - w\alpha_j\Big]\mathrm{sign}(c_w\alpha_i - s_w\beta_i) - \overline{c_w}|c_w\alpha_i - s_w\beta_i| \\ &= \mathrm{sign}(c_w\alpha_i - s_w\beta_i)\Big[[t^2(\alpha_i + w\alpha_j) - w\alpha_j] - \overline{c_w}(c_w\alpha_i - s_w\beta_i)\Big]. \end{aligned}$$

Since $|c_w|^2 + |s_w|^2 = 1$, we have $[t^2(\alpha_i + w\alpha_j) - w\alpha_j] - \overline{c_w}(c_w\alpha_i - s_w\beta_i) = t^2(\alpha_i + w\alpha_j) - (\alpha_i + w\alpha_j) + s_w(\overline{s_w}\alpha_i + \overline{c_w}\beta_i) = (\alpha_i + w\alpha_j)\Big[t^2 - 1 + \dfrac{\|x_i\|_2^2 + \overline{w}\overline{\langle x_j, x_i\rangle}}{\|x_i + wx_j\|_2^2}\Big] = 0.$

This shows that $u^*Av - \overline{c_w}\mathrm{Dep}(x_i, x_j) = 0$. Similarly, we have $u^*Bv + \overline{s_w}\mathrm{Dep}(x_i, x_j) = 0$. Hence by Theorem 2.4.11, (c_w, s_w) is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. Note that $\mathrm{rank}(\mathbf{L}(c_w, s_w) + \Delta\mathbf{L}(c_w, s_w)) = n - 1$. Hence (c_w, s_w) is a defective eigenvalue. This completes the proof. ■

We mention that it is easy to derive analogues of the above results when $\mathbf{L} \in \mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$. Indeed, define the weighted scalar product on $\mathbb{C}^2$ by

$$\langle x,\, y\rangle_w := w_1^2 x_1\overline{y}_1 + w_2^2 x_2 \overline{y}_2 \quad \text{for } x, y \in \mathbb{C}^2.$$

Then $|\langle x,\, y\rangle_w| \leq \|x\|_{w,2}\, \|y\|_{w,2}$. Now for $x, y \in \mathbb{C}^2$, define

$$\mathrm{Dep}_w(x, y) := \frac{|x_2 y_1 - x_1 y_2|}{(\|x\|^2_{w^{-1},2} + \|y\|^2_{w^{-1},2} + 2|\langle x,\, y\rangle_{w^{-1}}|)^{1/2}}.$$

Then we have the following result.

Theorem 2.6.8. *Let* $\mathbf{L} \in \mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a simple pencil given by* $\mathbf{L}(c, s) = c\,\mathrm{diag}(\alpha_i) - s\,\mathrm{diag}(\beta_i)$. *Set* $x_j := (\beta_j, \alpha_j)$ *for* $j := 1 : n$. *Then we have*

$$\mathrm{d}(\mathbf{L}) = \min_{i\neq j} \mathrm{Dep}_w(x_i, x_j).$$

Let x_i *and* x_j *be such that* $\mathrm{Dep}_w(x_i, x_j) = \mathrm{d}(\mathbf{L})$. *Set* $e := \mathrm{sign}(\langle x_j,\, x_i\rangle_{w^{-1}})$ *and define*

$$c_e := \frac{\beta_i + e\beta_j}{\|x_i + e x_j\|_{w^{-1},2}} \text{ and } s_e := \frac{\alpha_i + e\alpha_j}{\|x_i + e x_j\|_{w^{-1},2}}.$$

Then $(c_e, s_e) \in \mathbb{S}^1$ *and* $|c_e\alpha_i - s_e\beta_i| = |c_e\alpha_j - s_e\beta_j| = \mathrm{Dep}_w(x_i, x_j)$. *Next, set*

$$t := \frac{\sqrt{\|x_j\|^2_{w^{-1},2} + |\langle x_i,\, x_j\rangle_{w^{-1}}|}}{\|x_i + e x_j\|_{w^{-1},2}}$$

and define $u := te_i + \sqrt{1-t^2}\; e_j$ *and* $v := (te_i - \sqrt{1-t^2}\; ee_j)\mathrm{sign}(c_e\alpha_i - s_e\beta_i)$. *Further, define*

$$\Delta A := -\frac{\overline{c_e}\, w_1^{-2}\, \mathrm{Dep}(x_i, x_j)}{\|(c_e, s_e)\|_{w^{-1},2}}\, uv^* \text{ and } \Delta B := \frac{\overline{s_e}\, w_2^{-2}\, \mathrm{Dep}(x_i, x_j)}{\|(c_e, s_e)\|_{w^{-1},2}}\, uv^*$$

and consider the pencil $\Delta\mathbf{L}(c, s) := c\Delta A - s\,\Delta B$. *Then* $\|\Delta\mathbf{L}\|_{w,2} = \mathrm{d}(\mathbf{L})$ *and* (c_e, s_e) *is a defective eigenvalue of the pencil* $\mathbf{L} + \Delta\mathbf{L}$.

We now illustrate Wilkinson's problem by considering a few simple examples. For simplicity, we assume that $w := (1, 1)$ and denote $\eta_{w,2}(z, \mathbf{L})$ by $\eta(z, \mathbf{L})$.

Example 2.6.9. Let $\mathbf{L}(z) := A - zB$ be a diagonal pencil given by

$$(a)\ A := \begin{bmatrix} -1+i & 0 \\ 0 & 1+2i \end{bmatrix},\ B := \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(b)\ A := \begin{bmatrix} 1+2i & 0 \\ 0 & 2+i \end{bmatrix},\ B := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then for (a) we have $\mathrm{d}(\mathbf{L}) := 0.3878$, $z_w := 0.1628 + 2.7905i$ and for (b) we have $\mathrm{d}(\mathbf{L}) := 0.5628$, $z_w := 1.1564 + 0.9877i$. Figure 2.2 shows the coalescence of components of $\Lambda_\epsilon(\mathbf{L})$. ■

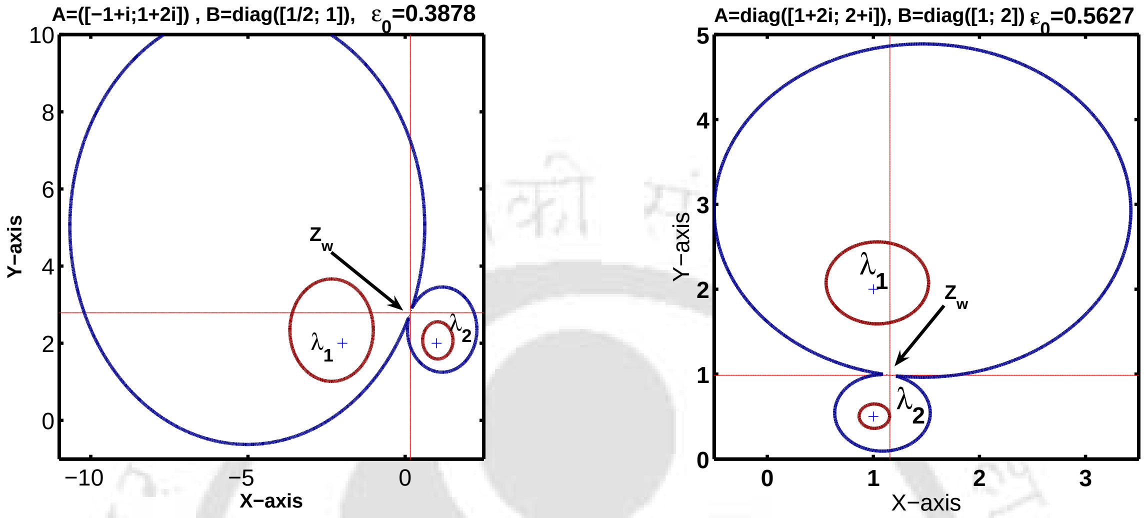


Figure 2.2: The left and right figures show coalescence of components of $\Lambda_\epsilon(\mathbf{L})$ corresponding to the pencil given in (a) and (b), respectively.

The pseudospectra plot of a matrix pencil $\mathbf{L}$ in $\mathbb{C}$ could be quite difficult to analyze when $\mathbf{L}$ has an infinite eigenvalue or when all the eigenvalues of $\mathbf{L}$ are finite but "infinity" enter into the pseudospectrum before the coalescence of two components. This is illustrated by the following example.

Example 2.6.10. Consider $\mathbf{L}(z) := A - zB$, where $A := \begin{bmatrix} -0.2+0.3i & 0 \\ 0 & 0.3-0.2i \end{bmatrix}$ and $B := \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$. The eigenvalues of $\mathbf{L}$ are given by $\lambda_1 := -2+3i$ and $\lambda_2 := 3/2 - i$. By Theorem 2.6.6, we have $\mathrm{d}(\mathbf{L}) := 0.145278$ and $z_w := 5.9567 - 1.0175i$. Figure 2.3 shows the contour plots of $\Lambda_\epsilon(\mathbf{L})$ and the point of coalescence of components. The evolution of the components of $\Lambda_\epsilon(\mathbf{L})$ is quite interesting in this case. Note that for sufficiently small ϵ, the components of $\Lambda_\epsilon(\mathbf{L})$, are bounded regions in $\mathbb{C}$ containing the eigenvalues λ_1 and λ_2 in their interiors. As ϵ grows gradually to 0.145278, the component containing λ_2 remains trapped in a bounded region of the complex plane. By contrast, as $\epsilon \to 0.145278$, the component containing λ_1 expands very fast engulfing the entire complex plane except for a small bounded region before coalescing with the component containing λ_2 at z_w. Indeed, for $\epsilon := 0.145278$, the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ is multiply connected and consists of entire complex plane except for the shaded region. Note that ∞ enters into the component of $\Lambda_\epsilon(\mathbf{L})$ containing λ_1 before it coalesces with the component containing λ_2 at z_w for $\epsilon := 0.145278$. The dynamics of the evolution of the components of $\Lambda_\epsilon(\mathbf{L})$ is better captured by the surface plot of the map $(x, y) \mapsto \eta(x + iy, \mathbf{L})$. Figure 2.4 clearly shows the evolution of the components of $\Lambda_\epsilon(\mathbf{L})$. ■

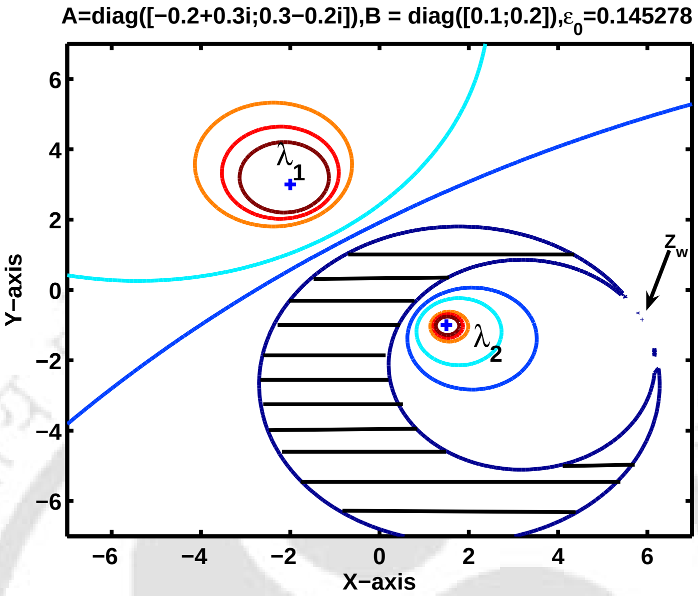


Figure 2.3: Contour plots of $\Lambda_\epsilon(\mathbf{L})$. The components coalesce at z_w for $\epsilon := 0.145278$. For $\epsilon := 0.145278$, $\Lambda_\epsilon(\mathbf{L})$ is multiply connected and consists of entire complex plane except for the shaded region.

For cases such as this, contour plots of $\Lambda_\epsilon(\mathbf{L})$ in the complex plane is inefficient and can be potentially confusing if not interpreted properly. The best way to overcome such problem as well as the case when $\mathbf{L}$ has an infinite eigenvalue is to consider homogeneous form of $\mathbf{L}$ and plot $\Lambda_\epsilon(\mathbf{L})$ on the Riemann sphere $\mathbb{S} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Indeed, by considering the homogeneous pencil $\mathbf{L}(c, s) = cA - sB$, we can restrict c to be real and s to be complex. Then the north pole $(0, 0, 1)$ of $\mathbb{S}$ represents ∞. Note that each point $(x, y, z) \in \mathbb{S}$, except the north pole $(0, 0, 1)$, can be associated with a complex number $w := \dfrac{x - iy}{1 - z}$. It is easily seen that this correspondence is one-to-one and $x = \dfrac{\overline{w} + w}{1 + |w|^2}$, $y = \dfrac{\overline{w} - w}{i(1 + |w|^2)}$. Further, it follows that the eigenvalue 0 of $\mathbf{L}$ is represented by the south pole $(0, 0, -1)$ of $\mathbb{S}$.

Now consider once again the pencil $\mathbf{L}$ given in Example 2.6.10. Considering homogeneous form of $\mathbf{L}$, Figure 2.5 shows the plot of $\Lambda_\epsilon(\mathbf{L})$ on the Riemann sphere $\mathbb{S}$. Note that as ϵ grows gradually, the component of $\Lambda_\epsilon(\mathbf{L})$ containing the (unnormalized) eigenvalue $(0.1, -0.2 + 0.3i)$ expands and contains $(0, 0, 1)$ before coalescing with the component containing the (unnormalized) eigenvalue $(0.2, 0.3 - 0.2i)$ at (c_w, s_w) for $\epsilon := 0.145278$, where the point (c_w, s_w) is given by Theorem 2.6.7.

Finally, we consider an example of a pencil for which ∞ is an eigenvalue.

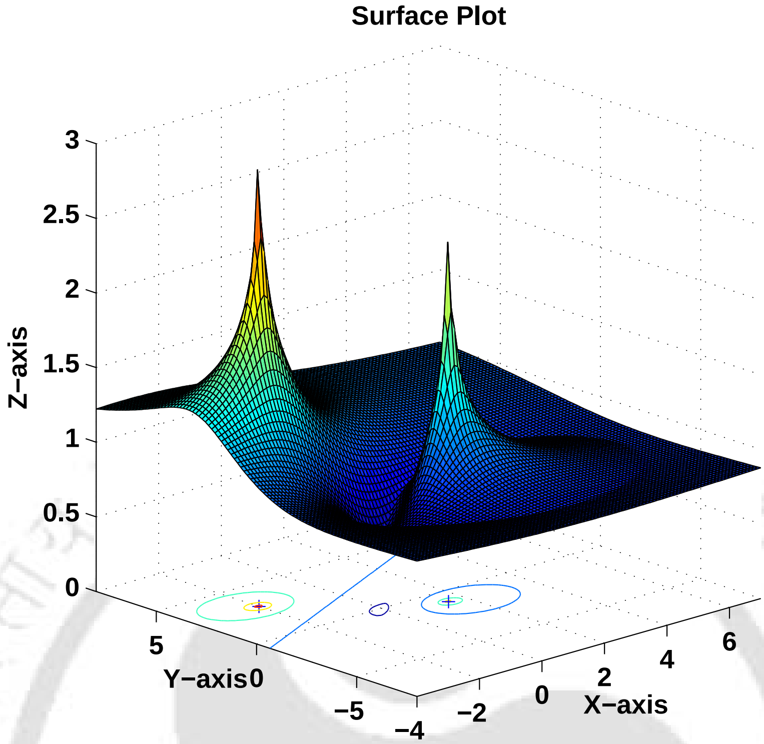


Figure 2.4: Surface plot of the map $(x, y) \mapsto \eta(x + iy, \mathbf{L})$ showing the evolution of components of $\Lambda_\epsilon(\mathbf{L})$.

Example 2.6.11. Consider the pencil $\mathbf{L}(z) = A - zB$, where $A := \mathrm{diag}(3, 1 + i)$ and $B := \mathrm{diag}(0, 2)$. Figure 2.6 shows the plot of $\Lambda_\epsilon(\mathbf{L})$ on the Riemann sphere $\mathbb{S}$ and the coalescence of the components of $\Lambda_\epsilon(\mathbf{L})$ for $\epsilon := 1.2381$. By Theorem 2.6.7, we have $\mathrm{d}(\mathbf{L}) := 1.2381$. ■

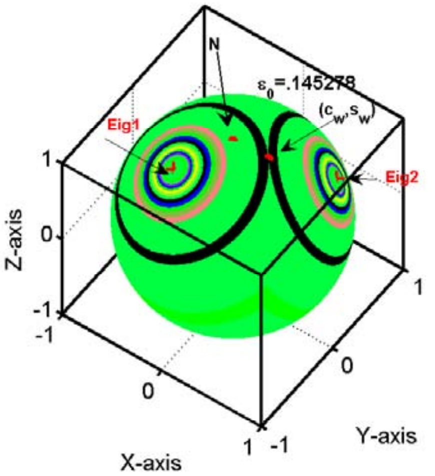


Figure 2.5: Plot of $\Lambda_\epsilon(\mathbf{L})$ on the Riemann sphere $\mathbb{S}$ for the pencil $\mathbf{L}$ given in Example 2.6.10 showing the coalescence of components.

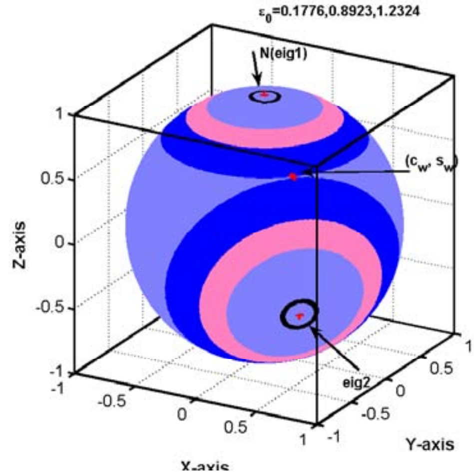


Figure 2.6: Plot of $\Lambda_\epsilon(\mathbf{L})$ on $\mathbb{S}$ showing coalescence of components.

Chapter 3

On pseudospectra, critical points and multiple eigenvalues of matrix polynomials

We develop a general framework for defining and analyzing pseudospectra of matrix polynomials. The framework so developed parallels the well developed framework for pseudospectra of matrices. We show that the pseudospectra of matrix polynomials well known in the literature follow as special cases from our framework. We also analyze critical points of backward errors of approximate eigenvalues of matrix polynomials and show that each critical point is a multiple eigenvalue of an appropriately perturbed polynomial. Finally, we show that common boundary points of the components of pseudospectra of a matrix polynomial are critical points. In particular, we show that a minimal critical point can be read off from the pseudospectra of matrix polynomials. Hence a solution of Wilkinson's problem for matrix polynomials can be read off from the pseudospectra of matrix polynomials.

3.1 Introduction

We further developed the framework introduced for defining and analyzing pseudospectra of matrix pencils to the case of matrix polynomials. We define the pseudospectra of regular matrix polynomials in the general framework of normed/seminormed linear space of matrix polynomials. We show that our framework for analyzing pseudospectra of matrix polynomials parallels the well developed framework that exists for analyzing pseudospectra of matrices. The crux of the matter is that, like the pseudospectra of matrices, the pseudospectra of matrix polynomials are determined by the geometry of the normed/seminormed space of polynomials, that is, by the choice of the norm/seminorm on the space of polynomials. We show that the pseudospectra of matrix polynomials well known in the literature correspond to appropriate choices of norm/seminorm on the space of polynomials.

Next, we consider critical points of backward errors of approximate eigenvalues of matrix polynomials and analyze their significance. We equip the space of polynomials with a weighted Hölder's p-norm/seminorm $|||\cdot|||_{w,p}$ (defined in section 3.3). Given a regular matrix polynomial $\mathbf{L}(z) = \sum_{i=0}^{m} z^i A_i$ and $\lambda \in \mathbb{C}$, we denote by $\eta_{w,p}(\lambda, \mathbf{L})$ the backward error of λ as an approximate eigenvalue of $\mathbf{L}$, that is,

$$\eta_{w,p}(\lambda, \mathbf{L}) := \inf\{|||\Delta\mathbf{L}|||_{w,p} : \det(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = 0\}.$$

Then we show that if λ is a critical point of $\eta_{w,p}$ then there exists a polynomial $\Delta\mathbf{L}$ such that $|||\Delta\mathbf{L}|||_{w,p} = \eta_{w,p}(\lambda, \mathbf{L})$ and that λ is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. Indeed, when λ is a generic critical point of $\eta_{w,p}$, setting

$$\Delta A_i := -\eta_{w,p}(\lambda, \mathbf{L})\, \overline{(\nabla_i N)(1, \lambda, \ldots, \lambda^m)}\, uv^*$$

and defining $\Delta\mathbf{L}(z) := \sum_{i=0}^{m} z^i \Delta A_i$ we obtain the desired polynomial, where u and v, respectively, are normalized left and right singular vectors of $\mathbf{L}(\lambda)$ corresponding to the smallest singular value $\sigma_{\min}(\mathbf{L}(\lambda))$, $N(z_1, z_2, \ldots, z_m) := \|(z_1, z_2, \ldots, z_m)\|_{w^{-1},q}$, $(\nabla_i N)(1, \lambda, \ldots, \lambda^m)$ is the partial gradient of N evaluated at $(1, \lambda, \ldots, \lambda^m)$ and $p^{-1} + q^{-1} = 1$. When λ is a nongeneric critical point of $\eta_{w,p}$, we show that a similar construction for $\Delta\mathbf{L}$ holds. Thus we show that if λ is a critical point (either generic or nongeneric) of $\eta_{w,p}$ then λ is a multiple eigenvalue of a polynomial which lies on the boundary of the $\eta_{w,p}(\lambda, \mathbf{L})$-neighbourhood of $\mathbf{L}$. We show that certain critical points of $\eta_{w,p}$ can be read off from the pseudospectra of $\mathbf{L}$. More specifically, we show that common boundary points of the components of pseudospectra of $\mathbf{L}$ are critical points of $\eta_{w,p}$. In particular, when $\mathbf{L}$ is simple (that is, it has distinct eigenvalues) we show that a minimal critical point of $\eta_{w,p}$ can be read off from the pseudospectra of $\mathbf{L}$. Hence we show that the distance from $\mathbf{L}$ to the nearest polynomial having a multiple eigenvalue can be read off from the pseudospectra of $\mathbf{L}$. Thus we show that a solution of Wilkinson's problem for matrix polynomials can be read off from the pseudospectra of the polynomials.

3.2 Preliminaries

We consider nonhomogeneous (resp., homogeneous) polynomials of the form $\mathbf{L}(z) = \displaystyle\sum_{i=0}^{m} z^i A_i$ (resp., $\mathbf{L}(c, s) := \displaystyle\sum_{i=0}^{m} c^{m-i} s^i A_i$), where $A_i \in \mathbb{C}^{n\times n}$. A polynomial $\mathbf{L}$ is said to be regular if $\det(\mathbf{L}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular nonhomogeneous pencil $\mathbf{L}$ is given by

$$\Lambda(\mathbf{L}) := \{\lambda \in \mathbb{C} : \operatorname{rank}(\mathbf{L}(\lambda)) < n\}.$$

It is possible for $\mathbf{L}$ to have an infinite eigenvalue. However, the case of an infinite eigenvalue can be resolved by considering $\Lambda(\mathbf{L})$ as a subset of $\mathbb{C}_\infty$, the one-point compactification of $\mathbb{C}$, and adding ∞ to $\Lambda(\mathbf{L})$ whenever $\operatorname{rank}(A_m) < n$.

A more convenient setup to deal with an infinite eigenvalue is to consider the homogeneous form of the polynomial $\mathbf{L}$. Thus, when $\mathbf{L}$ is homogenous, $\Lambda(\mathbf{L})$ is given by

$$\Lambda(\mathbf{L}) := \{(c, s) \in \mathbb{C}^2 : (c, s) \neq 0 \text{ and } \operatorname{rank}(\mathbf{L}(c, s)) < n\}.$$

An infinite eigenvalue of $\mathbf{L}$, if any, is then represented by $(0, 1)$. Normalizing $(c, s) \in \Lambda(\mathbf{L})$ as $|c|^2 + |s|^2 = 1$, we often identify $\Lambda(\mathbf{L})$ as a subset of the unit sphere $\mathbb{S}^1 := \{(c, s) \in \mathbb{C}^2 : |c|^2 + |s|^2 = 1\}$. Further, for computational purposes, restricting c to be real, s to be complex and with the normalization $c^2 + |s|^2 = 1$, the spectrum $\Lambda(\mathbf{L})$ can be identified with a subset of the Riemann sphere $\mathbb{S} := \{(x, y, z) \in \mathbb{R}^3 : |x|^2 + |y|^2 + |z|^2 = 1\}$. In such a case, an infinite eigenvalue of $\mathbf{L}$ is represented by the north pole $(0, 0, 1)$ of $\mathbb{S}$.

The pseudospectra of matrix polynomials have been developed and studied systematically by Higham and Tisseur [25, 45]. Further analysis of pseudospectra of matrix polynomials has been undertaken by Lancaster et al. [31, 12]. Let $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$ be an n-by-n regular polynomial. Then for a subordinate matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ as defined in [45] is given by

$$\begin{aligned}\Lambda_\epsilon(\mathbf{L}) &= \{\lambda \in \mathbb{C} : \det(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = 0 \text{ and } \|\Delta A_k\| \leq \epsilon\alpha_k, k = 0, 1, \ldots, m\} \\ &= \left\{\lambda \in \mathbb{C} : \|\mathbf{L}(\lambda)^{-1}\| \left(\sum_{i=0}^{m} \alpha_i |\lambda|^i\right) \geq \epsilon^{-1}\right\},\end{aligned}$$

where α_k are non-negative scalars. On the other hand, considering homogeneous form of $\mathbf{L}$, the ϵ-pseudospectrum as defined in [25] is given by

$$\begin{aligned}\Lambda_\epsilon(\mathbf{L}) &:= \{(c, s) \in \mathbb{C}^2 \setminus \{0\} : \det(\mathbf{L}(c, s) + \boldsymbol{\Delta}\mathbf{L}(c, s)) = 0 \text{ and } \|\Delta A_k\| \leq \alpha_k\epsilon\} \\ &= \left\{(c, s) \in \mathbb{C}^2 \setminus \{0\} : \|\mathbf{L}(c, s)^{-1}\| \left(\sum_{k=0}^{m} \alpha_k |c|^{m-k} |s|^k\right) \geq \epsilon^{-1}\right\}.\end{aligned}$$

3.3 A framework for pseudospectra of matrix polynomials

We now develop a general framework for defining and analyzing pseudospectra of matrix polynomials. We show that the pseudospectra of matrix polynomials can be treated at par with pseudospectra of matrices and matrix pencils.

We consider both homogeneous and nonhomogeneous polynomials of the form $\mathbf{L}(c, s) := \sum_{i=0}^{m} c^{m-i} s^i A_i$ and $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$, where $A_i \in \mathbb{C}^{n\times n}$ for $i = 0, 1, \ldots, m$. We denote the set of n-by-n matrix polynomials of degree $\leq m$ (either homogeneous or nonhomogeneous) by $\mathbb{L}_m(\mathbb{C}^{n\times n})$. Obviously $\mathbb{L}_m(\mathbb{C}^{n\times n})$ is a vector space.

Definition 3.3.1. *Let* $|||\cdot|||$ *be a norm on* $\mathbb{L}_m(\mathbb{C}^{n\times n})$*. Then for a regular polynomial* $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$*, we define the* ϵ*-pseudospectrum* $\Lambda_\epsilon(\mathbf{L})$ *of* $\mathbf{L}$ *by*

$$\Lambda_\epsilon(\mathbf{L}) := \bigcup \{\Lambda(\mathbf{L} + \Delta\mathbf{L}) : \Delta\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n}) \text{ and } |||\Delta\mathbf{L}||| \leq \epsilon\}.$$

Note that the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ of a polynomial $\mathbf{L}$ as defined above is a natural generalization of the pseudospectrum $\Lambda_\epsilon(A)$ of matrix A.

More generally, we now define weighted pseudospectrum. First, we define the action of $\mathbb{R}^{m+1}$ on $\mathbb{L}_m(\mathbb{C}^{n\times n})$. Define $\mathbb{R}^{m+1} \times \mathbb{L}_m(\mathbb{C}^{n\times n}) \to \mathbb{L}_m(\mathbb{C}^{n\times n}), (w, \mathbf{L}) \mapsto w \odot \mathbf{L}$ by $(w \odot \mathbf{L})(z) = \sum_{j=0}^{m} w_j z^j A_j$, where $w := (w_0, w_1, \ldots, w_m)$ and $\mathbf{L}(z) := \sum_{j=0}^{m} z^j A_j$. Obviously the map $(w, \mathbf{L}) \mapsto w \odot \mathbf{L}$ is bilinear and $v \odot (w \odot \mathbf{L}) = w \odot (v \odot \mathbf{L})$. Note that w acts a linear map $\mathbf{L} \mapsto w \odot \mathbf{L}$ on $\mathbb{L}_m(\mathbb{C}^{n\times n})$. We say that $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ is a w-null polynomial if $w \odot \mathbf{L} = 0$. The action of w is said to be injective if $w \odot \mathbf{L} = 0 \Rightarrow \mathbf{L} = 0$ for all $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$. Obviously the action of w is injective if and only if all the components of w are nonzero. We define $w^{-1} := (w_0^{-1}, w_1^{-1}, \ldots, w_m^{-1})$ with the convention that $w_j^{-1} = 0$ if $w_j = 0$. Note that if some components of w are equal to 0 then $w^{-1} \odot (w \odot \mathbf{L}) = \mathbf{L}$ holds only for certain polynomials. Finally, if all the entries of w are nonnegative then we say that w is a **weight vector**.

Definition 3.3.2. *Let* $|||\cdot|||$ *be a norm on* $\mathbb{L}_m(\mathbb{C}^{n\times n})$ *and* $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ *be regular. Then for a weight vector* $w \in \mathbb{R}^{m+1}$*, we define the* ϵ*-pseudospectrum* $\Lambda_\epsilon(\mathbf{L})$ *of* $\mathbf{L}$ *by*

$$\Lambda_\epsilon(\mathbf{L}) := \bigcup \{\Lambda(\mathbf{L} + \Delta\mathbf{L}) : \Delta\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n}), w^{-1} \odot (w \odot \Delta\mathbf{L}) = \Delta\mathbf{L} \text{ and } |||w \odot \Delta\mathbf{L}||| \leq \epsilon\}.$$

We mention that $w^{-1} \odot (w \odot \Delta\mathbf{L}) = \Delta\mathbf{L}$ is an admissibility condition on the perturbation polynomial $\Delta\mathbf{L}$. For example, if $\Delta\mathbf{L}(z) = \sum_{i=0}^{m} z^i \Delta A_i$ then the admissibility condition implies that $\Delta A_j = 0$ whenever the j-th component of w is 0.

Definition 3.3.3. *Let* $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ *and* $w \in \mathbb{R}^{m+1}$ *be a weight vector. Then* $\mathbf{L}$ *is said to be* w*-admissible if* $w^{-1} \odot (w \odot \mathbf{L}) = \mathbf{L}$.

Assumption: Given a weight vector w, unless stated otherwise,
all perturbations of a polynomial $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ will be assumed
to be w-admissible.

The importance of w-admissible perturbation can be seen as follows. Suppose that $\Delta\mathbf{L}$ is w-admissible and $\mathbf{L}(z) = \sum_{i=0}^{m} z^i A_i$. Then if the j-th component of the weight vector w is 0 then A_j remains unperturbed when $\mathbf{L}$ is perturbed to $\mathbf{L} + \Delta\mathbf{L}$.

We show that the pseudospectra considered in section 2 follow from Definition 3.3.2 and correspond to appropriate choices of weight w and the norm $|||\cdot|||$ on the space of polynomials $\mathbb{L}_m(\mathbb{C}^{n\times n})$. To that end, first we define the Hölder's p-norm on $\mathbb{L}_m(\mathbb{C}^{n\times n})$

which we refer to as the polynomial p-norm. Let $\|\cdot\|$ be a norm on $\mathbb{C}^{n\times n}$. For $1 \leq p \leq \infty$, we define $|\!|\!|\cdot|\!|\!|_p : \mathbb{L}_m(\mathbb{C}^{n\times n}) \to \mathbb{R}$ by

$$|\!|\!|\mathbf{L}|\!|\!|_p := \|(\|A_0\|, \cdots, \|A_m\|)\|_p,$$

where $\mathbf{L}(z) := \sum_{j=0}^{m} z^j A_j$ or $\mathbf{L}(c, s) := \sum_{i=0}^{m} c^{m-i} s^i A_i$ and $\|\cdot\|_p$ is the Hölder's p norm on $\mathbb{C}^{m+1}$. Then it is easily seen that $|\!|\!|\cdot|\!|\!|_p$ is a norm. We denote the space $\mathbb{L}_m(\mathbb{C}^{n\times n})$ when equipped with the norm $|\!|\!|\cdot|\!|\!|_p$ by $\mathbb{L}_m^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. This notation emphasizes the fact that $(\mathbb{C}^{n\times n}, \|\cdot\|)$ is the base space from which the norm $|\!|\!|\cdot|\!|\!|_p$ is built up.

More generally, let $w \in \mathbb{R}^{m+1}$ be a weight vector. Then we define the weighted Hölder's p-norm/seminorm $|\!|\!|\cdot|\!|\!|_{w,p} : \mathbb{L}_m(\mathbb{C}^{n\times n}) \to \mathbb{R}$ by

$$|\!|\!|\mathbf{L}|\!|\!|_{w,p} := |\!|\!|w \odot \mathbf{L}|\!|\!|_p.$$

Obviously, $|\!|\!|\cdot|\!|\!|_{w,p}$ defines a seminorm on $\mathbb{L}_m(\mathbb{C}^{n\times n})$. Note that $|\!|\!|\cdot|\!|\!|_{w,p}$ defines a norm if and only if all the entries of w are nonzero. We denote the space $\mathbb{L}_m^p(\mathbb{C}^{n\times n})$ when equipped with the norm/seminorm $|\!|\!|\cdot|\!|\!|_{w,p}$ by $\mathbb{L}_{m,w}^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. If $\mathbf{L} \in \mathbb{L}_{m,w}^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ is a w-admissible polynomial then it follows that

$$\begin{aligned} \|\mathbf{L}(\lambda)\| &\leq |\!|\!|\mathbf{L}|\!|\!|_{w,p} \|(1,\, \lambda,\, \ldots,\, \lambda^m)\|_{w^{-1},q}, \\ \|\mathbf{L}(c, s)\| &\leq |\!|\!|\mathbf{L}|\!|\!|_{w,p} \|(c^m,\, c^{m-1}s,\, \ldots,\, s^m)\|_{w^{-1},q}, \end{aligned} \tag{3.1}$$

where $p^{-1} + q^{-1} = 1$. When $1 < p < \infty$, another choice of p-norm/seminorm on $\mathbb{L}_m(\mathbb{C}^{n\times n})$ is given by $|\!|\!|\mathbf{L}|\!|\!|_{w,p} := (w_0\|A_0\|^p + w_1\|A_1\|^p + \ldots + w_m\|A_m\|^p)^{1/p}$. We, however, will not have occasion to use this norm/seminorm.

Now consider the special case $\mathbb{L}_{m,w}^2(\mathbb{C}^{n\times n}, \|\cdot\|_F)$. For $\mathbf{L}_1, \mathbf{L}_2 \in \mathbb{L}_{m,w}^2(\mathbb{C}^{n\times n}, \|\cdot\|_F)$, define

$$\langle \mathbf{L}_1,\, \mathbf{L}_2 \rangle_{\mathbb{L}} := \sum_{i=0}^{m} w_i^2 \operatorname{trace}(B_i^* A_i),$$

where $\mathbf{L}_1(\lambda) := \sum_{i=0}^{m} \lambda^i A_i$ and $\mathbf{L}_2(\lambda) := \sum_{i=0}^{m} \lambda^i B_i$. Then $\langle \cdot,\, \cdot \rangle_{\mathbb{L}}$ defines an inner product whenever the action of w is injective. Thus $\mathbb{L}_{m,w}^2(\mathbb{C}^{n\times n}, \|\cdot\|_F)$ is a Hilbert space whenever w acts injectively. For the rest of the chapter, we consider only the space $\mathbb{L}_{m,w}^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ for appropriate choices of w, p and the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$.

We emphasize that in order to define pseudospectra of a matrix polynomial $\mathbf{L} \in \mathbb{L}_m(\mathbb{C}^{n\times n})$, first it is necessary to define a norm/seminorm on $\mathbb{L}_m(\mathbb{C}^{n\times n})$ so as to measure the magnitude of perturbations of $\mathbf{L}$. There is nothing new in this observation and the same conclusion holds for ϵ-pseudospectrum of a matrix. However, the importance of this fact has not been emphasized enough in the literature while dealing with pseudospectra of matrix polynomials. As a consequence, the pseudospectra of matrix polynomials as defined in the literature are ad hoc in nature.

The ϵ-pseudospectra of regular polynomials in $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ can be thought of as a set valued map from $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ to $\mathbb{C}$ or $\mathbb{C}^2$. The crux of the matter is that the ϵ-pseudospectrum of $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ is determined by the geometry of the space $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$. Consequently, the same polynomial $\mathbf{L}$ will have different ϵ-pseudospectrum for different choices of w, p and $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. Said differently, from the viewpoint of pseudospectra, the same polynomial $\mathbf{L}$ becomes a different object for different choices of w, p and $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. Thus the pseudospectra of a polynomial get automatically specified once the space to which the polynomial belongs is specified. This helps in simplifying notation for pseudospectra of matrix polynomials. For regular polynomial $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$, we can afford to denote the ϵ-pseudospectrum of $\mathbf{L}$ simply by $\Lambda_\epsilon(\mathbf{L})$ without having to specify w, p and $\|\cdot\|$ or showing the dependence of $\Lambda_\epsilon(\mathbf{L})$ on w, p and $\|\cdot\|$.

Now, consider a regular polynomial $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$. Then by Definition 3.3.2, the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ of $\mathbf{L}$ is given by

$$\Lambda_\epsilon(\mathbf{L}) := \bigcup\{\Lambda(\mathbf{L}+\Delta\mathbf{L}) : \Delta\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|) \text{ and } |\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} \leq \epsilon\}. \tag{3.2}$$

We now provide a characterization of $\Lambda_\epsilon(\mathbf{L})$. To that end, for $z, c, s \in \mathbb{C}$, we define the backward errors (recall that all perturbations are w-admissible)

$$\begin{aligned} \eta_{w,p}(z, \mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} : z \in \Lambda(\mathbf{L}+\Delta\mathbf{L})\} \text{ and} \\ \eta_{w,p}(c, s, \mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} : (c,s) \in \Lambda(\mathbf{L}+\Delta\mathbf{L})\}. \end{aligned}$$

Then obviously the following result holds.

Corollary 3.3.4. *Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular polynomial. Then we have*

$$\begin{aligned} \Lambda_\epsilon(\mathbf{L}) &= \{z \in \mathbb{C} : \eta_{w,p}(z, \mathbf{L}) \leq \epsilon\}, \text{ if } \mathbf{L} \text{ is nonhomogeneous} \\ \Lambda_\epsilon(\mathbf{L}) &= \{(c,s) \in \mathbb{C}^2 \setminus \{0\} : \eta_{w,p}(c, s, \mathbf{L}) \leq \epsilon\}, \text{ if } \mathbf{L} \text{ is homogeneous.} \end{aligned}$$

The following result determines the backward error function $\eta_{w,p}$ when $\mathbb{C}^{n\times n}$ is equipped with a subordinate matrix norm.

Proposition 3.3.5. *Consider the space $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ corresponding to a subordinate matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular polynomial. Then we have*

$$\begin{aligned} \eta_{w,p}(\lambda, \mathbf{L}) &= \min_{\|x\|=1}\left\{\frac{\|\mathbf{L}(\lambda)x\|}{\|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}} : x \in \mathbb{C}^n\right\} \leq |\!|\!|\mathbf{L}|\!|\!|_{w,p}, \\ \eta_{w,p}(c, s, \mathbf{L}) &= \min_{\|x\|=1}\left\{\frac{\|\mathbf{L}(c,s)x\|}{\|(c^m, c^{m-1}s, \ldots, s^m)\|_{w^{-1},q}} : x \in \mathbb{C}^n\right\} \leq |\!|\!|\mathbf{L}|\!|\!|_{w,p}, \end{aligned}$$

where $p^{-1} + q^{-1} = 1$.

Proof: First note that if $\lambda \in \Lambda(\mathbf{L} + \Delta\mathbf{L})$ then there exists x such that $\|x\| = 1$ and $\mathbf{L}(\lambda)x + \Delta\mathbf{L}(\lambda)x = 0$. Since $\Delta\mathbf{L}$ is w-admissible, we have $\|\mathbf{L}(\lambda)x\| = \|\Delta\mathbf{L}(\lambda)x\| \leq \|\Delta\mathbf{L}(\lambda)\| \leq |\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p}\|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}$, where $p^{-1} + q^{-1} = 1$. This shows that

$$\eta_{w,p}(\lambda, \mathbf{L}) \geq \min_{\|x\|=1} \frac{\|\mathbf{L}(\lambda)x\|}{\|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}}.$$

For the reverse inequality, choose $x \in \mathbb{C}^n$ such that $\|x\| = 1$. Then there is a (functional) $y \in \mathbb{C}^n$ such that $\|y\|_D = 1$ and $y^*x = 1$, where $\|\cdot\|_D$ is the dual norm of the norm $\|\cdot\|$ on $\mathbb{C}^n$. Now set $r := \mathbf{L}(\lambda)x$ and define $E : \mathbb{C}^n \to \mathbb{C}^n$ by $Ez := (y^*z)r$. Then $E \in \mathbb{C}^{n\times n}$, $Ex = r$ and $\|E\| = \|y\|_D\|r\| = \|r\|$. Set $\alpha := \|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}$, where $p^{-1} + q^{-1} = 1$. For $1 < p \leq \infty$, define

$$\Delta A_i := -\frac{w_i^{-q}\,\mathrm{sign}(\lambda^i)\,|\lambda|^{i(q-1)}\,E}{\alpha^q} \quad \text{for } i = 0 : m,$$

where $\mathrm{sign}(z) = \overline{z}/|z|$, if $z \neq 0$ and $\mathrm{sign}(0) = 0$. When $p = 1$, define $\Delta A_j := -\dfrac{w_j^{-1}\,\mathrm{sign}(\lambda^j)\,E}{\alpha}$ and $\Delta A_i = 0$ for $i \neq j$ if $\alpha = w_j^{-1}|\lambda^j|$. Consider the polynomial $\Delta\mathbf{L}(z) := \sum_{i=0}^m z^i \Delta A_i$. Then we have $\mathbf{L}(\lambda)x + \Delta\mathbf{L}(\lambda)x = \mathbf{L}(\lambda)x - Ex = 0$ and

$$|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \|r\|/\alpha = \frac{\|\mathbf{L}(\lambda)x\|}{\|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}}.$$

Now the desired result follows by taking minimum over all x such that $\|x\| = 1$. Hence the proof. ■

We mention that Proposition 3.3.5 has been considered in [45] for the special case when $p = \infty$.

When $\mathbb{C}^{n\times n}$ is equipped with either the spectral norm $\|\cdot\|_2$ or the Frobenius norm $\|\cdot\|_F$, it turns out that the backward error $\eta_{w,p}$ and the ϵ-pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ remain same for both the norms. Indeed, we have the following result.

Proposition 3.3.6. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be regular. Set* $\eta_2(\lambda, \mathbf{L}) := \eta_{w,p}(\lambda, \mathbf{L})$ *and* $\Lambda^2_\epsilon(\mathbf{L}) := \Lambda_\epsilon(\mathbf{L})$. *Now considering* $\mathbf{L}$ *as an element of* $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_F)$, *set* $\eta_F(\lambda, \mathbf{L}) := \eta_{w,p}(\lambda, \mathbf{L})$ *and* $\Lambda^F_\epsilon(\mathbf{L}) := \Lambda_\epsilon(\mathbf{L})$. *Then we have*

$$\eta_2(\lambda, \mathbf{L}) = \eta_F(\lambda, \mathbf{L}) = \frac{\sigma_{\min}(\mathbf{L}(\lambda))}{\|(1,\ \lambda,\ \ldots,\ \lambda^m)\|_{w^{-1},q}} \quad \textit{and } \Lambda^2_\epsilon(\mathbf{L}) = \Lambda^F_\epsilon(\mathbf{L}),$$

where $p^{-1} + q^{-1} = 1$. *Similar results hold when* $\mathbf{L}$ *is considered as a homogeneous polynomial.*

Proof: Let u and v be unit left and right singular vectors of $\mathbf{L}(\lambda)$ corresponding to the smallest singular value $\sigma_{\min}(\mathbf{L}(\lambda))$. Now set $\alpha := \|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}$ and for $1 < p \leq \infty$, define

$$\Delta A_i := -\frac{w_i^{-q}\,\sigma_{\min}(\mathbf{L}(\lambda))\,\mathrm{sign}(\lambda^i)\,|\lambda|^{i(q-1)}\,uv^*}{\alpha^q} \quad \text{for } i = 0 : m,$$

where $\operatorname{sign}(z) := \overline{z}/|z|$, if $z \neq 0$ and $\operatorname{sign}(0) = 0$. Now consider the polynomial $\Delta\mathbf{L}(z) := \sum_{i=0}^{m} z^i \Delta A_i$. Then by construction, we have $\mathbf{L}(\lambda)v + \Delta\mathbf{L}(\lambda)v = 0$. Since ΔA_i are of rank one matrix, the spectral and the Frobenius norms of ΔA_i are same. Consequently, $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p}$ is the same for the spectral and the Frobenius norms on $\mathbb{C}^{n\times n}$. Now, we have $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = (\sum_{i=0}^{m} w_i^p \, \|\Delta A_i\|^p)^{1/p} = \dfrac{\sigma_{\min}(\mathbf{L}(\lambda))}{\alpha}$.

When $p = 1$, the result follows by defining $\Delta A_j := -\dfrac{w_j^{-1}\,\sigma_{\min}(\mathbf{L}(\lambda))\,\operatorname{sign}(\lambda^j)\,uv^*}{\alpha}$ and $\Delta A_i = 0$ for $i \neq j$, if $\alpha = w_j^{-1}|\lambda^j|$. For homogeneous polynomial, the result follows by setting $\alpha := \|(\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)\|_{w^{-1},q}$ and defining

$$\Delta A_i := -\frac{w_i^{-q}\,\sigma_{\min}(\mathbf{L}(\lambda,\mu))\,\operatorname{sign}(\lambda^{m-i}\mu^i)\,|\lambda|^{(m-i)(q-1)}|\mu|^{i(q-1)}\,uv^*}{\alpha^q}, \quad i = 0:m$$

when $1 < p \leq \infty$, and $\Delta A_j := -\dfrac{w_j^{-1}\sigma_{\min}(\mathbf{L}(\lambda,\mu))\operatorname{sign}(\lambda^{m-j}\mu^j)uv^*}{\alpha}$ and $\Delta A_i := 0$ for $i \neq j$ if $\alpha = w_j^{-1}|\lambda^{m-j}\mu^j|$ when $p = 1$. Hence the proof. ■

We are now ready to show that the pseudospectra defined in section 2 follow from (3.2) for appropriate choices of w, p and the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$.

- **Tisseur and Higham [45]:** Considering a polynomial $\mathbf{L}(z) = \sum_{i=0}^{m} z^i A_i$ and the space $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$, for the choice $w := (\alpha_0^{-1}, \alpha_1^{-1}, \ldots, \alpha_m^{-1})$, $p = \infty$ and a subordinate matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, we obtain from (3.2), the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ defined by Tisseur and Higham.

- **Higham and Tisseur [25]:** Considering the homogeneous polynomial $\mathbf{L}(c, s) = \displaystyle\sum_{i=0}^{m} c^{m-i}s^i A_i$ and the space $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$, for the choice $w := (\alpha_0^{-1}, \alpha_1^{-1}, \ldots, \alpha_m^{-1})$, $p = \infty$ and a subordinate matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, we obtain from (3.2), the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ defined by Higham and Tisseur.

We mention that homogeneous polynomials and their pseudospectra provide a natural setting for dealing with a polynomial having an infinite eigenvalue. An alternative approach to resolving an infinite eigenvalue of a polynomial $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ given by $\mathbf{L}(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ is to consider the reverse polynomial $\operatorname{rev}\mathbf{L}(\lambda) = \lambda^m \mathbf{L}(1/\lambda)$. Note that ∞ is an eigenvalue of $\mathbf{L}$ if and only if 0 is an eigenvalue of $\operatorname{rev}\mathbf{L}$. More precisely, in the space $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, the following spectral mapping holds:

$$\Lambda(\operatorname{rev}\mathbf{L}) = (\Lambda(\mathbf{L}))^{-1} := \{1/\lambda \in \mathbb{C}_\infty : \lambda \in \Lambda(\mathbf{L})\}.$$

Now, notice that $\eta_{w,p}(\lambda, \mathbf{L}) = \eta_{w,p}(1/\lambda, \operatorname{rev}\mathbf{L})$. Consequently, the following pseudospectral mapping holds:

$$\Lambda_\epsilon(\operatorname{rev}\mathbf{L}) = (\Lambda_\epsilon(\mathbf{L}))^{-1} := \{1/\lambda \in \mathbb{C}_\infty : \lambda \in \Lambda_\epsilon(\mathbf{L})\}.$$

This shows that the image of the component of $\Lambda_\epsilon(\text{rev}\mathbf{L})$ containing the eigenvalue 0 of $\text{rev}\mathbf{L}$ under the map $z \mapsto z^{-1}$ is precisely the component of $\Lambda_\epsilon(\mathbf{L})$ containing an infinite eigenvalue of $\mathbf{L}$. Thus for a polynomial $\mathbf{L}$ having an infinite eigenvalue, adding ∞ to $\Lambda_\epsilon(\mathbf{L})$ whenever $\eta_{w,p}(0, \text{rev}\mathbf{L}) \leq \epsilon$, we obtain the ϵ-pseudospectrum of $\mathbf{L}$ in $\mathbb{C}_\infty$.

3.4 Properties of pseudospectra of polynomial

Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular polynomial. Then it is easy to check that the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ has the following properties.

- The map $\epsilon \mapsto \Lambda_\epsilon(\mathbf{L})$ is monotone increasing, that is, if $\epsilon < \delta$ then $\Lambda_\epsilon(\mathbf{L}) \subset \Lambda_\delta(\mathbf{L})$.
- The pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ consists of at most mn components and each component contains an eigenvalue of $\mathbf{L}$. Further, we have $\Lambda_\epsilon(\mathbf{L}^*) = \Lambda_\epsilon(\mathbf{L})^*$.
- If $\mathbf{L}$ is block diagonal, that is, $\mathbf{L} = \text{diag}(\mathbf{L}_1, \mathbf{L}_2)$ and the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ satisfies $\|\text{diag}(A_1, A_2)\| = \max(\|A_1\|, \|A_2\|)$ then $\Lambda_\epsilon(\mathbf{L}) = \Lambda_\epsilon(\mathbf{L}_1) \cup \Lambda_\epsilon(\mathbf{L}_2)$.

The next result shows that $\Lambda_\epsilon(\mathbf{L})$ does not have isolated points.

Proposition 3.4.1. *Suppose that $\Lambda_\epsilon(\mathbf{L})$ is bounded subset of $\mathbb{C}$. Then $\Lambda_\epsilon(\mathbf{L})$ is compact and does not have isolated points. Thus $\Lambda_\epsilon(\mathbf{L})$ consists of nontrivial components.*

Proof: It is obvious that $\Lambda_\epsilon(\mathbf{L})$ is compact. So, suppose that λ is isolated point of $\Lambda_\epsilon(\mathbf{L})$. There is an open set $U \subset \mathbb{C}$ and a matrix polynomial $\Delta\mathbf{L}(\lambda) = \sum_{i=0}^m \lambda^i \Delta A_i$ with $\|\Delta\mathbf{L}\|_{w,p} = \eta_{w,p}(\lambda, \mathbf{L}) \leq \epsilon$ such that $\Lambda_\epsilon(\mathbf{L}) \cap U = \{\lambda\}$ and $\lambda \in \Lambda(\mathbf{L} + \Delta\mathbf{L})$. Let $\Delta\mathbf{L}_m$ be such that $\|\Delta\mathbf{L}_m - \Delta\mathbf{L}\|_{w,p} \to 0$ and $\|\Delta\mathbf{L}_m\|_{w,p} < \|\Delta\mathbf{L}\|_{w,p}$. Then for all large m, we have $\lambda \notin \Lambda(\mathbf{L} + \Delta\mathbf{L}_m)$ and $\Lambda(\mathbf{L} + \Delta\mathbf{L}_m) \cap U \neq \emptyset$ which contradicts that λ is an isolated of $\Lambda_\epsilon(\mathbf{L})$. ■

Theorem 3.4.2. *Let $\mathbf{L}(z) = A - zB$ be a regular pencil. Then the function $z \mapsto \mathbf{L}(z)^{-1}$ is analytic on $\mathbb{C}\backslash\Lambda(\mathbf{L})$. In particular, if $z_0 \in \mathbb{C}\backslash\Lambda(\mathbf{L})$ and $|z - z_0| \leq \|\mathbf{L}(z_0)^{-1}B\|^{-1}$. Then $\mathbf{L}(z)^{-1} = \left[\sum_{i=0}^{\infty}(z - z_0)^k(\mathbf{L}(z_0)^{-1}B)^k\right]\mathbf{L}(z_0)^{-1}$ and $\|\mathbf{L}(z)^{-1}\| \leq \dfrac{\|\mathbf{L}(z_0)^{-1}\|}{1 - |z - z_0|\|\mathbf{L}(z_0)^{-1}B\|}$.*

Proof: For $z_0 \in \mathbb{C} \setminus \Lambda(\mathbf{L})$ we have

$$\begin{aligned} (A - zB) &= (A - z_0B)[I - (z - z_0)(A - z_0B)^{-1}B] \\ \Rightarrow (A - zB)^{-1} &= [I - (z - z_0)(A - z_0B)^{-1}B]^{-1}(A - z_0B)^{-1} \\ \Rightarrow \mathbf{L}(z)^{-1} &= [I - (z - z_0)\mathbf{L}(z_0)^{-1}B]^{-1}\mathbf{L}(z_0)^{-1} \\ &= \left[\sum_{i=0}^{\infty}(z - z_0)^k(\mathbf{L}(z_0)^{-1}B)^k\right]\mathbf{L}(z_0)^{-1}. \end{aligned}$$

This shows that $\mathbf{L}(z)^{-1}$ is analytic on $\{z \in \mathbb{C} : \ |z - z_0| < \|\mathbf{L}(z_0)^{-1}B\|^{-1}\}$ and

$$\|\mathbf{L}(z)^{-1}\| \leq \frac{\|\mathbf{L}(z_0)^{-1}\|}{1 - |z - z_0|\|\mathbf{L}(z_0)^{-1}B\|}. \ \blacksquare$$

Now we prove the following proposition.

Proposition 3.4.3. *Let* $\mathbf{L}(z) := A - zB$ *be a regular pencil. Then the functions* $z \mapsto \|\mathbf{L}(z)^{-1}\|$ *and* $z \mapsto \log \|\mathbf{L}(z)^{-1}\|$ *are subharmonic on* $\mathbb{C} \setminus \Lambda(\mathbf{L})$.

Proof: By Theorem 3.4.2, the map $z \mapsto \mathbf{L}(z)^{-1}$ is analytic. Hence by Theorem 1.2.9, $z \mapsto \|\mathbf{L}(z)^{-1}\|$ and $z \mapsto \log \|\mathbf{L}(z)^{-1}\|$ are subharmonic. $\blacksquare$

For a regular matrix polynomial we have the following.

Theorem 3.4.4. *Let* $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$ *be a regular polynomial. Then the functions* $z \mapsto \|\mathbf{L}(z)^{-1}\|$ *and* $z \mapsto \log \|\mathbf{L}(z)^{-1}\|$ *are subharmonic on* $\mathbb{C} \setminus \Lambda(\mathbf{L})$.

Proof: By Theorem 3.4.2 and Theorem 1.2.7, the function $z \mapsto \mathbf{L}(z)^{-1}$ is analytic on $\{z \in \mathbb{C} : \ |z - z_0| < \|(X - z_0Y)^{-1}Y\|^{-1}\}$. Hence by Theorem 1.2.9, the functions $z \mapsto \|\mathbf{L}(z)^{-1}\|$ and $z \mapsto \log \|\mathbf{L}(z)^{-1}\|$ are subharmonic. $\blacksquare$

Proposition 3.4.5. *For* $1 \leq p \leq \infty$ *and a weight vector* $w \in \mathbb{R}^{m+1}$, *the functions* $z \mapsto \|(1, z, \ldots, z^m)\|_{w,p}$ *and* $z \mapsto \log \|(1, z, \ldots, z^m)\|_{w,p}$ *are subharmonic on* $\mathbb{C}$.

Proof: First, we show that $z \mapsto \log \|(1, z, \ldots, z^m)\|_{w,p}$ is subharmonic. Without loss of generality assume that $w_j \neq 0$ for $j = 0 : m$.

First, assume that $1 \leq p < \infty$. Then the map $z \mapsto w_i^p z^{ip}$ is analytic for $i = 0, 1, \ldots, m$. Hence by Theorem 1.2.9, $z \mapsto w_i^p |z|^{ip}$ and $z \mapsto \log(w_i^p |z|^{ip})$ are subharmonic for $i = 0, 1, \ldots, m$. Consequently, by Theorem 1.2.3, we have $z \mapsto \log\left(\sum_{i=0}^{m}(w_i^p|z|^{ip})\right)$ is subharmonic which implies that $z \mapsto \log(\|(1, z, \ldots, z^m)\|_{w,p}^p)$ is subharmonic. This shows that $z \mapsto \log(\|(1, z, \ldots, z^m)\|_{w,p})$ is subharmonic.

Next, assume that $p = \infty$. Since $z \mapsto w_i z^i$ is analytic, by Theorem 1.2.9, $z \mapsto w_i|z|^i$ and $z \mapsto \log(w_i|z|^i)$ are subharmonic on $\mathbb{C}$ for $i = 0, 1, \ldots, m$. By Theorem 1.2.10(c)

$$z \mapsto \max(\log(w_0), \log(w_1|z|), \ldots, \log(w_m|z|^m))$$

is subharmonic. Further, we have

$$\max(\log(w_0), \log(w_1|z|), \ldots, \log(w_m|z|^m)) = \log(\max(w_0, w_1|z|, \ldots, w_m|z|^m)).$$

Consequently, $z \mapsto \log(\|(1, z, \ldots, z^m)\|_{w,\infty})$ is subharmonic.

Now for $1 \leq p \leq \infty$, let $h(z) := \|(1, z, \ldots, z^m)\|_{w,p}$. Then $h(z) = e^{\log(h(z))}$. Since e^x is positive, convex and increasing and $\log(h(z))$ is subharmonic, by Theorem 1.2.10(d), $h(z)$ is subharmonic. $\blacksquare$

The next result shows that $1/\eta_{w,p}(z, \mathbf{L})$ is subharmonic.

Theorem 3.4.6. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular matrix polynomial. Then the functions* $z \mapsto 1/\eta_{w,p}(z, \mathbf{L})$ *and* $z \mapsto -\log \eta_{w,p}(z, \mathbf{L})$ *are subharmonic on* $\mathbb{C} \setminus \Lambda(\mathbf{L})$ *for* $1 \leq p \leq \infty$.

Proof: By Theorem 3.4.4 and Proposition 3.4.5, the functions $z \mapsto \log \|\mathbf{L}(z)^{-1}\|$ and $z \mapsto \log \|(1, z, \ldots, z^m)\|_{w^{-1},q}$, where $p^{-1} + q^{-1} = 1$, are subharmonic. Now by Theorem 1.2.10(a), $z \mapsto \log \|\mathbf{L}(z)^{-1}\| + \log(\|(1, z, \ldots, z^m)\|_{w^{-1},q})$ is subharmonic. Hence $z \mapsto \log(1/\eta_{w,p}(z, \mathbf{L}))$ is subharmonic. Next, we show that $z \mapsto 1/\eta_{w,p}(z, \mathbf{L})$ is subharmonic. So let $h(z) = 1/\eta_{w,p}(z, \mathbf{L})$. Then $h(z) = e^{\log(h(z))}$. Since e^x is positive, convex and increasing and $\log(h(z))$ is subharmonic, by Theorem 1.2.10(d), $h(z) = 1/\eta_{w,p}(z, \mathbf{L})$ is subharmonic. This completes the proof. ■

For the rest of the chapter, we consider only the spectral norm $\|\cdot\|_2$ on $\mathbb{C}^{n\times n}$, that is, we consider only the space $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$. In this case, recall that we have

$$\eta_{w,p}(\lambda, \mathbf{L}) = \frac{\sigma_{\min}(\mathbf{L}(\lambda))}{\|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}} \text{ and } \eta_{w,p}(c, s, \mathbf{L}) = \frac{\sigma_{\min}(\mathbf{L}(c, s))}{\|(c^m, c^{m-1}s, \ldots, s^m)\|_{w^{-1},q}}. \quad (3.3)$$

We often identify $\mathbb{C}$ with $\mathbb{R}^2$ and treat the map $\lambda \mapsto \eta_{w,p}(\lambda, \mathbf{L})$ as a function from $\mathbb{R}^2$ to $\mathbb{R}$. As a consequence, we often identify $\Lambda_\epsilon(\mathbf{L})$ as a subset of $\mathbb{R}^2$. Then it follows that

$$\partial\Lambda_\epsilon(\mathbf{L}) \subset \{(x, y) \in \mathbb{R}^2 : \eta_{w,p}(x + iy, \mathbf{L}) = \epsilon\},$$

where $\partial\Lambda_\epsilon(\mathbf{L})$ is the boundary of $\Lambda_\epsilon(\mathbf{L})$. It is shown in [12] that $\partial\Lambda_\epsilon(\mathbf{L})$ is an algebraic curve when $p = \infty$. The next result shows that $\partial\Lambda_\epsilon(\mathbf{L})$ is a real algebraic curve when $p/(p-1)$ is an integer or $p = \infty$.

Proposition 3.4.7. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular nonhomogeneous polynomial. Then the boundary* $\partial\Lambda_\epsilon(\mathbf{L})$ *of* $\Lambda_\epsilon(\mathbf{L})$ *is embedded in a real algebraic curve when* $p/(p-1)$ *is an integer or* $p = \infty$.

Proof: We have $\partial\Lambda_\epsilon(\mathbf{L}) \subset \{(x, y) \in \mathbb{R}^2 : \sigma_{\min}(x + iy, \mathbf{L}) = \epsilon\|(1, x + iy, \ldots, (x + iy)^m)\|_{w^{-1},q}\}$. First, suppose that $p = 1$. Then note that $\epsilon\|(1,\, x+iy, \ldots, (x+iy)^m)\|_{w^{-1},\infty}$ is a singular value of $\mathbf{L}(x + iy)$ if and only if $Z_\infty(x, y) := (\mathbf{L}(x + iy)^*\mathbf{L}(x + iy)) - (\epsilon\|(1,\, x + iy, \ldots, (x + iy)^m)\|_{w^{-1},\infty})^2 I$ is singular. Hence $\partial\Lambda_\epsilon(\mathbf{L}) \subset \Gamma_\infty := \{(x, y) \in \mathbb{R}^2 : \det(Z_\infty(x, y)) = 0\}$. Now it is easy to check that Γ_∞ is an algebraic curve.

Next, suppose that $p \neq 1$. Then $\epsilon\|(1,\, x + iy, \ldots, (x + iy)^m)\|_{w^{-1},q}$ is a singular value of $\mathbf{L}(x + iy)$ if and only if $Z_q(x, y) := (\mathbf{L}(x + iy)^*\mathbf{L}(x + iy))^q - (\epsilon\|(1,\, x + iy, \ldots, (x + iy)^m)\|_{w^{-1},q})^{2q} I$ is singular. Hence $\partial\Lambda_\epsilon(\mathbf{L}) \subset \Gamma_q := \{(x, y) \in \mathbb{R}^2 : \det(Z_q(x, y)) = 0\}$. Since $q = p/(p-1)$ is an integer, it is easy to see that Γ_q is a real algebraic curve. Indeed, if $q = p/(p-1)$ is even then it follows that $\det(Z_q(x, y))$ is a polynomial. On the other hand, if q is odd then it follows that $\det(Z_q(x, y)) = (\sqrt{x^2 + y^2})t(x, y) + r(x, y)$ for some nonzero polynomials $t(x, y)$ and $r(x, y)$. Hence in either case Γ_q is an algebraic curve. This completes the proof. ■

If $\partial\Lambda_\epsilon(\mathbf{L})$ does not pass through the origin $0 \in \mathbb{C}$ then it can be shown that $\partial\Lambda_\epsilon(\mathbf{L})$ is a real analytic curve for all $1 \leq p \leq \infty$.

For a homogeneous polynomial $\mathbf{L}(c, s)$, the boundary $\partial\Lambda_\epsilon(\mathbf{L})$ is embedded in a projective curve. Indeed, for the special case when $\mathbf{L} \in \mathbb{L}^2_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$, we have for non-zero $(x, y, z) \in \mathbb{R}^3$

$$\begin{aligned} \partial\Lambda_\epsilon(\mathbf{L}) &\subset \left\{(x, y, z) \in \mathbb{R}^3 : \sigma_{\min}(\mathbf{L}(x, y+iz))^2 = \epsilon^2 \left(\sum_{i=0}^{m} w_i^{-2} x^{2(m-i)} (y^2+z^2)^i\right)\right\} \\ &\subset \{(x, y, z) \in \mathbb{R}^3 : Q(x, y, z) = 0\} =: V(Q), \end{aligned}$$

where $Q(x, y, z) := \det\left(\mathbf{L}^*(z, x+iy)\mathbf{L}(z, x+iy) - \epsilon^2 \left(\sum_{i=0}^{m} w_i^{-2} x^{2(m-i)} (y^2+z^2)^i\right)\right)$. Now, treating $\Lambda_\epsilon(\mathbf{L})$ as a subset of the Riemann sphere $\mathbb{S} \subset \mathbb{R}^3$, we have

$$\partial\Lambda_\epsilon(\mathbf{L}) \subset V(Q) \cap V(x^2+y^2+z^2-1),$$

where $V(f)$ denotes the algebraic variety of the polynomial f.

Theorem 3.4.8. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be regular and* $U \subset \mathbb{C}$ *be open. Then either* $\eta_{w,p}(z, \mathbf{L})$ *is non-constant on* U *or if* $\eta_{w,p}(z, \mathbf{L}) = \delta$ *for all* $z \in U$ *then* $\Lambda_\delta(\mathbf{L}) = \mathbb{C}$.

Proof: The proof follows from Proposition 3.4.7 when $p/(p-1)$ is an integer or $p = \infty$. Indeed, if $\eta_{w,p}(z, \mathbf{L}) = \delta$ for all $z \in U$ then $U \subset \Gamma := \{z \in \mathbb{C} : \eta_{w,p}(z, \mathbf{L}) = \delta\}$. Since Γ is an algebraic curve, we must have $\Lambda_\delta(\mathbf{L}) = \mathbb{C}$.

For the general case, suppose that $\eta_{w,p}(z, \mathbf{L}) = \delta$ for $z \in U$. Let $\overline{U}$ be the closure of U. Then we have $\overline{U} \cap \Lambda(\mathbf{L}) = \emptyset$. Indeed, if $\overline{U}$ contains an eigenvalue $\lambda \in \Lambda(\mathbf{L})$ then $\delta = \eta_{w,p}(\lambda, \mathbf{L}) = 0$. Consequently $U \subset \Lambda(\mathbf{L})$ which is impossible. Now, if possible, suppose that $\Lambda_\delta(\mathbf{L}) \neq \mathbb{C}$. Then $\overline{U}$ is a component of $\Lambda_\delta(\mathbf{L})$ not containing an eigenvalue of $\mathbf{L}$ which is not possible. Hence the proof. ■

As a consequence, we have the following result which shows that the only local minimizers of $\eta_{w,p}$ are eigenvalues of $\mathbf{L}$.

Theorem 3.4.9. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular. Let* $\mu \in \mathbb{C}$ *be a local minimum of* $\eta_{w,p}(z, \mathbf{L})$. *Then either* $\mu \in \Lambda(\mathbf{L})$ *or* $\Lambda_\delta(\mathbf{L}) = \mathbb{C}$, *where* $\delta := \eta_{w,p}(\mu, \mathbf{L})$.

Proof: Since μ is a local minimizer, there is an open set U such that $\mu \in U$ and that $\eta_{w,p}(\mu, \mathbf{L}) \leq \eta_{w,p}(z, \mathbf{L})$ for all $z \in U$. Note that if μ is an eigenvalue then obviously μ is a local minimizer of $\eta_{w,p}(z, \mathbf{L})$.

Now suppose that μ is not an eigenvalue of $\mathbf{L}$. Then μ is a maximum of $1/\eta_{w,p}(z, \mathbf{L})$ on U. Since $1/\eta_{w,p}(z, \mathbf{L})$ is subharmonic on U, by maximum principle, $1/\eta_{w,p}(z, \mathbf{L})$ is constant on U. Consequently, by Theorem 3.4.8, we have $\Lambda_\delta(\mathbf{L}) = \mathbb{C}$. ■

3.5 Critical points and multiple eigenvalues

For a regular matrix polynomial $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$, we now investigate the functions $z \mapsto \eta_{w,p}(z, \mathbf{L})$ and $(c, s) \mapsto \eta_{w,p}(c, s, \mathbf{L})$. To that end, the gradients of the maps $\mathbb{C}^2 \to \mathbb{R}, (c, s) \mapsto \|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p}$ and $\mathbb{C} \to \mathbb{R}, z \mapsto \|(1, z, \ldots, z^m)\|_{w,p}$ will play an important role. We identify $\mathbb{C}$ with $\mathbb{R}^2$ and express the gradients as complex numbers. Then we have the following result.

Proposition 3.5.1. *Let $h_{w,p}(z) := \|(1, z, \ldots, z^m)\|_{w,p}$ for $z \in \mathbb{C}$ and $1 \leq p \leq \infty$.*

(1) *If $1 < p < \infty$ then $h_{w,p}$ is differentiable on $\mathbb{C}$ and $\nabla h_{w,p}(z) := \dfrac{\sum_{i=1}^m i w_i^p z|z|^{ip-2}}{h_{w,p}(z)^{p-1}}$.*

(2) *If $p = 1$ then $h_{w,p}$ is differentiable on $\mathbb{C} \setminus \{0\}$ and $\nabla h_{w,1}(z) = \sum_{i=1}^m i w_i z|z|^{i-2}$ for $z \neq 0$.*

(3) *Let $D := \{z \in \mathbb{C} : \; h_{w,\infty}(z) = w_j|z^j| = w_i|z^i|$ for some $i \neq j\}$. Then $h_{w,\infty}$ is differentiable on $\mathbb{C} \setminus D$ and is given by $\nabla h_{w,\infty}(z) = j w_j z|z|^{j-2}$, if j is such that $h_{w,\infty}(z) = w_j|z^j|$.*

Proof: The proof follows from the fact that $\nabla(|\lambda|^p) = p\lambda\,|\lambda|^{p-2}$ and the chain rule $\nabla(g \circ h)(\lambda) = \nabla g(h(\lambda))\nabla h(\lambda)$. ■

Next, we analyze the gradient of the map $\mathbb{C}^2 \to \mathbb{R}, (c, s) \mapsto \|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p}$. We define the **partial gradient** $\nabla_c(\|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p})$ to be the gradient of the map $\mathbb{C} \to \mathbb{R}, c \mapsto \|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p}$ treating s as constant. The partial gradient $\nabla_s(\|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p})$ is defined similarly. Then the gradient of the map $(c, s) \mapsto \|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p}$ when exists is given by

$$\nabla(\|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p}) = (\nabla_c(\|(c^m, \ldots, s^m)\|_{w,p}), \, \nabla_s(\|(c^m, \ldots, s^m)\|_{w,p})) \subset \mathbb{C}^2.$$

The gradient and the partial gradients of the map $(c, s) \mapsto \|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p}$ are given in the following result.

Proposition 3.5.2. *Define $H_{w,p} : \mathbb{C}^2 \to \mathbb{R}$ by $H_{w,p}(c, s) := \|(c^m, c^{m-1}s, \ldots, s^m)\|_{w,p}$, where $1 \leq p \leq \infty$.*
(1) *If $1 < p < \infty$ then the partial gradients $\nabla_c H_{w,p}$ and $\nabla_s H_{w,p}$ exist on $\mathbb{C}^2$ and are given by*

$$\nabla_c H_{w,p}(c, s) = \frac{\displaystyle\sum_{j=0}^{m-1}(m-j) w_j^p c|c|^{(m-j)p-2}|s|^{jp}}{H_{w,p}(c, s)^{p-1}},$$

$$\nabla_s H_{w,p}(c, s) = \frac{\displaystyle\sum_{j=1}^{m} j w_j^p s|s|^{jp-2}|c|^{(m-j)p}}{H_{w,p}(c, s)^{p-1}}.$$

(2) *The partial gradient* $\nabla_c H_{w,1}(c,s)$ *exists when* $c \neq 0$ *and the partial gradient* $\nabla_s H_{w,1}(c,s)$ *exists when* $s \neq 0$ *and are given by*

$$\nabla_c H_{w,1}(c,s) = \sum_{j=0}^{m-1}(m-j)w_j c|c|^{m-j-2}|s|^j,$$

$$\nabla_s H_{w,1}(c,s) = \sum_{j=1}^{m} j w_j s|s|^{j-2}|c|^{(m-j)}.$$

(3) *Let* $D := \{(c,s) \in \mathbb{C}^2 \setminus \{0\} : H_{w,\infty}(c,s) = w_j|c|^{m-j}|s|^j = w_i|c|^{m-i}|s|^i$ *for some* $i \neq j\}$. *Then* $\nabla_c H_{w,\infty}$ *and* $\nabla_s H_{w,\infty}$ *exist on* $(c,s) \in \mathbb{C}^2 \setminus D$ *and are given by*

$$\nabla_c H_{w,\infty}(c,s) = (m-j)w_j c|c|^{m-j-2}|s|^j, \text{ if } w_j|c|^{m-j}|s|^j = H_{w,\infty}(c,s),$$

$$\nabla_s H_{w,\infty}(c,s) = j w_j s|s|^{j-2}|c|^{m-j}, \text{ if } w_j|c|^{m-j}|s|^j = H_{w,\infty}(c,s).$$

Proof: Suppose that $1 \leq p < \infty$. Since $H_{w,p}(c,s) = (\sum_{j=0}^m w_j^p|c|^{(m-j)p}|s|^{jp})^{1/p}$ and $\nabla(|c|^p) = pc\,|c|^{p-2}$, by the chain rule of derivative we have

$$\begin{aligned}\nabla_c\, H_{w,p}(c,s) &= \frac{\displaystyle\sum_{j=0}^{m-1}(m-j)\, w_j^p\, c\,|c|^{(m-j)p-2}|s|^{jp}}{H_{w,p}(c,s)^{p-1}}\\ \nabla_s H_{w,p}(c,s) &= \frac{\displaystyle\sum_{j=1}^{m} j\, w_j^p\, s\,|s|^{jp-2}\,|c|^{(m-j)p}}{H_{w,p}(c,s)^{p-1}}.\end{aligned}$$

The proof is similar for the case $p = \infty$. ■

Next, we analyze the gradient of the map $\mathbb{C}^{m+1} \to \mathbb{R}, z \mapsto \|z\|_{w,p}$. First, we define the **partial gradients** $\nabla_i(\|z\|_{w,p})$. We define the partial gradient $\nabla_i(\|z\|_{w,p})$ to be the gradient of the map $\mathbb{C} \to \mathbb{R}, z_i \mapsto \|(z_0,\ldots,z_m)\|_{w,p}$ treating $z_0, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$ as constants. Then the gradient of the map $\mathbb{C}^{m+1} \to \mathbb{R}, z \mapsto \|z\|_{w,p}$, when exists, is given by

$$\nabla(\|z\|_{w,p}) = (\nabla_0\|z\|_{w,p},\ \nabla_1\|z\|_{w,p},\ \ldots,\ \nabla_m\|z\|_{w,p}) \subset \mathbb{C}^{m+1}.$$

Proposition 3.5.3. *Define* $N_{w,p} : \mathbb{C}^{m+1} \to \mathbb{R}$ *by* $N_{w,p}(z) := \|(z_0,\ldots,z_m)\|_{w,p}$, *where* $1 \leq p \leq \infty$.

(1) *If* $1 < p < \infty$ *then* $N_{w,p}$ *is differentiable on* $\mathbb{C}^{m+1}$ *and*

$$\nabla_i N_{w,p}(z) = \frac{w_i^p z_i|z_i|^{p-2}}{N_{w,p}(z)^{p-1}}, \text{ for } i = 0 : m.$$

(2) *If* $p = 1$ *then* $\nabla_i\, N_{w,p}(z)$ *exists for* $z_i \neq 0$ *and is given by*

$$\nabla_i N_{w,p}(z) = \frac{w_i z_i}{|z_i|}.$$

In particular, $N_{w,p}$ is differentiable at $(z_0, z_1, \ldots, z_m)$ if $z_j \neq 0$ for $j = 0 : m$.

(3) Let $D := \{z \in \mathbb{C}^{m+1} : \; N_{w,\infty}(z) = w_j|z_j| = w_i|z_i| \text{ for some } i \neq j\}$. Then $N_{w,\infty}$ is differentiable on $\mathbb{C}^{m+1} \setminus D$ and $\nabla N_{w,\infty}(z) = (0, \ldots, w_j \frac{z_j}{|z_j|}, \ldots, 0)$, if $w_j|z_j| = N_{w,\infty}(z)$. Also let $D_i := \{z \in \mathbb{C}^{m+1} : \; w_j|z_j| = w_i|z_i| \text{ for some } j \neq i\}$. Then $\nabla_i N_{w,\infty}(z)$ exists for $z \in \mathbb{C}^{m+1} \setminus D_i$ and

$$\nabla_i N_{w,\infty}(z) = \begin{cases} 0, & \text{if} \quad w_i|z_i| < N_{w,\infty}(z), \\ w_i \dfrac{z_i}{|z_i|}, & \text{if} \quad w_i|z_i| = N_{w,\infty}(z). \end{cases}$$

Proof: The proof is easy and follows from the fact that $\nabla(|z_i|^p) = pz_i\,|z_i|^{p-2}$. ■

Note that the gradient $\nabla N_{w,p}(z)$ is an element of the dual space of $(\mathbb{C}^{m+1}, \|\cdot\|_{w,p})$ and the dual space of $(\mathbb{C}^{m+1}, \|\cdot\|_{w,p})$ is the space $(\mathbb{C}^{m+1}, \|\cdot\|_{w^{-1},q})$, where $p^{-1} + q^{-1} = 1$. The next result shows that $\nabla N_{w,p}(z_0, z_1, \ldots, z_m)$ is a unit vector in $(\mathbb{C}^{m+1}, \|\cdot\|_{w^{-1},q})$.

Lemma 3.5.4. *Let $z := (z_0, z_1, \ldots, z_m) \in \mathbb{C}^{m+1}, N_{w,p}(z) := \|z\|_{w,p}$ and $p^{-1} + q^{-1} = 1$. Suppose that $\nabla_i N_{w,p}(z)$ exists for $i = 0, 1, \ldots, m$. Then*

- $\|(\nabla_0 N_{w,p}(z), \nabla_1 N_{w,p}(z), \ldots, \nabla_m N_{w,p}(z))\|_{w^{-1},q} = 1.$
- $(z_0\overline{\nabla_0 N_{w,p}(z)} + \cdots + z_i\overline{\nabla_i N_{w,p}(z)} + \cdots + z_m\overline{\nabla_m N_{w,p}(z)}) = N_{w,p}(z).$

Proof: For $1 \leq p < \infty$, by Proposition 3.5.3, we have

$$\begin{aligned} &\|(\nabla_0 N_{w,p}(z), \nabla_1 N_{w,p}(z), \ldots, \nabla_m N_{w,p}(z))\|_{w^{-1},q} \\ &= \frac{1}{N_{w,p}(z)^{p-1}} \left(w_0^{pq-q}|z_0|^{pq-q} + w_1^{pq-q}|z_1|^{pq-q} + \cdots + w_m^{pq-q}|z_m|^{pq-q}\right)^{1/q} \\ &= \frac{1}{N_{w,p}(z)^{p-1}} (w_0^p|z_0|^p + w_1^p|z_1|^p + \cdots + w_m^p|z_m|^p)^{1/q} = 1. \end{aligned}$$

The proof is similar for $p = \infty$. ■

The next result shows the relation between gradients of $N_{w,p}$, $H_{w,p}$ and $h_{w,p}$.

Lemma 3.5.5. *Let $N_{w,p}, H_{w,p}$ and $h_{w,p}$ be as above. Then we have the following.*

(a) $\displaystyle\sum_{j=1}^{m} j\;\overline{\lambda^{j-1}}\;(\nabla_j N_{w,p})(1, \lambda, \ldots, \lambda^m) = \nabla h_{w,p}(\lambda).$

(b) $\displaystyle\sum_{j=0}^{m-1} (m-j)\;\overline{c^{m-j-1}s^j}\;(\nabla_j N_{w,p})(c^m, c^{m-1}s, \ldots, s^m) = \nabla_c H_{w,p}(c, s).$

(c) $\displaystyle\sum_{j=1}^{m} j\;\overline{c^{m-j}s^{j-1}}\;(\nabla_j N_{w,p})(c^m, c^{m-1}s, \ldots, s^m) = \nabla_s H_{w,p}(c, s).$

Proof: Suppose that $1 \leq p < \infty$ and set $z_\lambda := (1, \lambda, \ldots, \lambda^m)$, $z_{cs} := (c^m, c^{m-1}s, \ldots, s^m)$. Then by Proposition 3.5.1 and Proposition 3.5.3, we have

$$\begin{aligned}\sum_{j=1}^{m} j\, \overline{\lambda^{j-1}}\, (\nabla_j N_{w,p})(z_\lambda) &= \frac{\sum_{j=1}^{m} j\, w_j^p\, \overline{\lambda^{j-1}}\, \lambda^{j-1}\, \lambda\, |\lambda^j|^{p-2}}{h_{w,p}(\lambda)^{p-1}} \\ &= \frac{\sum_{j=1}^{m} j\, w_j^p\, \lambda\, |\lambda|^{jp-2}}{h_{w,p}(\lambda)^{p-1}} = \nabla h_{w,p}(\lambda).\end{aligned}$$

Next, by Proposition 3.5.2 and Proposition 3.5.3 we have

$$\begin{aligned}\sum_{j=0}^{m-1} (m-j)\, \overline{c^{m-j-1}s^j}\, (\nabla_j N_{w,p})(z_{cs}) &= \frac{\sum_{j=0}^{m-1} (m-j)\, w_j^p\, c\, |c|^{(m-j)p-2}\, |s|^{jp}}{H_{w,p}(c,s)^{p-1}} \\ &= \nabla_c H_{w,p}(c,s).\end{aligned}$$

Finally, we have

$$\begin{aligned}\sum_{j=1}^{m} j\, \overline{c^{m-j}\, s^{j-1}}\, (\nabla_j N_{w,p})(z_{cs}) &= \frac{\displaystyle\sum_{j=1}^{m} j\, w_j^p\, \overline{c^{m-j}\, s^{j-1}}\, c^{m-j}\, s^j\, |c|^{(m-j)(p-2)}\, |s|^{j(p-2)}}{H_{w,p}(c,s)^{p-1}} \\ &= \frac{\displaystyle\sum_{j=1}^{m} j\, w_j^p\, s\, |s|^{jp-2}\, |c|^{(m-j)p}}{H_{w,p}(c,s)^{p-1}} = \nabla_s H_{w,p}(c,s).\end{aligned}$$

The proof is similar for $p = \infty$. ■

Let $p := (p_1, \ldots, p_N)^T \in \mathbb{R}^N$ and $A(p) \in \mathbb{C}^{m \times n}$. Suppose that σ is a singular value of A. Then there exists unit right and left singular vectors $v \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$, respectively, such that $Av = \sigma u$ and $A^* u = \sigma v$.

Theorem 3.5.6 (Sun, [44]). *Let $p \in \mathbb{R}^N$ and $A(p) \in \mathbb{C}^{m \times n}$. Suppose that $Re[A(p)]$ and $Im[A(p)]$ are real analytic matrix valued functions of p in some neighborhood $\mathcal{B}(0)$ of the origin. Let σ be a simple nonzero singular value of $A(0)$ and $v \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$ be the corresponding unit right and left singular vectors, respectively. Then there exists a simple nonzero singular value $\sigma(p)$ of $A(p)$ which is real analytic function of p in a neighborhood $\mathcal{N}(0)$ of the origin such that $\sigma(0) = \sigma$ and*

$$\frac{\partial \sigma(p)}{\partial p_j} = Re\left[u^* \left(\frac{\partial A(p)}{\partial p_j}\right)_{p=0} v\right].$$

Next, we analyze differentiability of $\eta_{w,p}(\lambda, \mathbf{L})$ and $\eta_{w,p}(c, s, \mathbf{L})$. Again, we treat $\eta_{w,p}(\lambda, \mathbf{L})$ as a function from $\mathbb{R}^2$ to $\mathbb{R}$ and express its gradient as a complex number.

Theorem 3.5.7. *Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n \times n}, \|\cdot\|_2)$ be a regular polynomial and $1 \leq p \leq \infty$.*

(a) *Suppose that* $\mathbf{L}$ *is given by* $\mathbf{L}(z) = \sum_{i=0}^{m} z^i A_i$. *Suppose also that* $\sigma_{\min}(\mathbf{L}(\lambda))$ *is simple. If the gradient* $\nabla h_{w^{-1},q}(z)$ *exists at* λ *then the gradient* $\nabla \eta_{w,p}(\lambda, \mathbf{L})$ *exists and is given by*

$$\nabla \eta_{w,p}(\lambda, \mathbf{L}) = \frac{\overline{u^* \partial_z \mathbf{L}(\lambda) v} - \eta_{w,p}(\lambda, \mathbf{L}) \nabla h_{w^{-1},q}(\lambda)}{h_{w^{-1},q}(\lambda)},$$

where u *and* v *are unit left and right singular vectors of* $\mathbf{L}(\lambda)$ *corresponding to* $\sigma_{\min}(\mathbf{L}(\lambda))$ *and* $p^{-1} + q^{-1} = 1$. *Here* $\partial_z \mathbf{L}(\lambda)$ *denotes derivative of* $\mathbf{L}(z)$ *evaluated at* λ.

(b) *Next, consider homogeneous form of* $\mathbf{L}$ *given by* $\mathbf{L}(c, s) = \sum_{i=0}^{m} c^{m-i} s^i A_i$, *where* $(c, s) \in \mathbb{C}^2 \setminus \{0\}$. *Suppose that* $\sigma_{\min}(\mathbf{L}(\lambda, \mu))$ *is simple and partial gradients of* $H_{w^{-1},q}$ *exist. Then the partial gradients of* $\eta_{w,p}$ *exist and are given by*

$$\begin{aligned} \nabla_c \, \eta_{w,p}(\lambda, \mu, \mathbf{L}) &= \frac{\overline{u^* \partial_c \mathbf{L}(\lambda, \mu) v} - \eta_{w,p}(\lambda, \mu, \mathbf{L})(\nabla_c \, H_{w^{-1},q})(\lambda, \mu)}{H_{w^{-1},q}(\lambda, \mu)}, \\ \nabla_s \, \eta_{w,p}(\lambda, \mu, \mathbf{L}) &= \frac{\overline{u^* \partial_s \mathbf{L}(\lambda, \mu) v} - \eta_{w,p}(\lambda, \mu, \mathbf{L})(\nabla_s \, H_{w^{-1},q})(\lambda, \mu)}{H_{w^{-1},q}(\lambda, \mu)}, \end{aligned}$$

where u *and* v *are left and right singular vectors of* $\mathbf{L}(c, s)$ *corresponding to* $\sigma_{\min}(\mathbf{L}(\lambda, \mu))$ *and* $p^{-1} + q^{-1} = 1$.

Proof: Let $\lambda = x + iy$ and $g(\lambda) := \sigma_{\min}(\mathbf{L}(\lambda))$. Since $g(\lambda)$ is simple, by Theorem 3.5.6, g is real analytic at (x, y) and $\nabla g(\lambda) = (\mathrm{Re}(u^* \partial_x \mathbf{L}(\lambda) v),\ \mathrm{Re}(u^* \partial_y \mathbf{L}(\lambda) v))$. Writing the gradient as a complex number, we have $\nabla g(\lambda) = \overline{u^* \partial_z \mathbf{L}(\lambda) v}$. Now, using the fact that $\nabla(g(\lambda)/h_{w^{-1},q}(\lambda)) = \dfrac{h_{w^{-1},q}(\lambda) \nabla g(\lambda) - g(\lambda) \nabla h_{w^{-1},q}(\lambda)}{h_{w^{-1},q}(\lambda)^2}$ and $\eta_{w,p}(\lambda, \mathbf{L}) = g(\lambda)/h_{w^{-1},q}(\lambda)$, the desired result follows.

Next, set $g(\lambda, \mu) := \sigma_{\min}(\mathbf{L}(\lambda, \mu))$. Since $g(\lambda, \mu)$ is simple, by Theorem 3.5.6, g is real analytic at (λ, μ) and $\nabla g(\lambda, \mu) = (\nabla_c g(\lambda, \mu), \nabla_s g(\lambda, \mu))$. Writing the gradient as a complex number, we have $\nabla_c \, g(\lambda, \mu) = \overline{u^* \partial_c \mathbf{L}(\lambda, \mu) v}$. Now, using the fact that $\nabla_c \, (g(\lambda, \mu)/H_{w^{-1},q}(\lambda, \mu)) = \dfrac{H_{w^{-1},q}(\lambda, \mu) \nabla_c \, g(\lambda, \mu) - g(\lambda, \mu)(\nabla_c \, H_{w^{-1},q})(\lambda, \mu)}{H_{w^{-1},q}(\lambda, \mu)^2}$ and $\eta_{w,p}(\lambda, \mu, \mathbf{L}) = g(\lambda, \mu)/H_{w^{-1},q}(\lambda, \mu)$, the desired result follows. ■

Now we construct a matrix polynomial with a specified eigenvalue.

Theorem 3.5.8. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n \times n}, \|\cdot\|_2)$ *be a regular polynomial given by* $\mathbf{L}(z) := \sum_{i=0}^{m} z^i A_i$. *Let* $\lambda \in \mathbb{C}$. *Consider the SVD* $\mathbf{L}(\lambda) = U \Sigma V^*$. *Suppose that* k *is the multiplicity of* $\sigma_{\min}(\mathbf{L}(\lambda))$. *Set* $u := U(:, n - k + 1 : n)$ *and* $v := V(:, n - k + 1 : n)$. *For* $z \in \mathbb{C}^{m+1}$, *define* $N_{w,p}(z) := \|z\|_{w,p}$. *With the convention that a partial gradient of* $N_{w,p}$ *at* $(1, \lambda, \ldots, \lambda^m)$ *is* 0 *if it does not exist, we define*

$$\Delta A_i := -\eta_{w,p}(\lambda, \mathbf{L}) \, \overline{(\nabla_i \, N_{w^{-1},q})(1, \lambda, \ldots, \lambda^m)} u v^*$$

and consider the polynomial $\mathbf{\Delta L}(z) := \sum_{i=0}^{m} z^i \Delta A_i$, *where* $p^{-1} + q^{-1} = 1$. *Then we have*

(*a*) $\Delta\mathbf{L}(\lambda) = -\eta_{w,p}(\lambda, \mathbf{L}) h_{w^{-1},q}(\lambda)\, uv^*$.

(*b*) $\partial_z \Delta\mathbf{L}(\lambda) = -\eta_{w,p}(\lambda, \mathbf{L})\, \overline{\nabla h_{w^{-1},q}(\lambda)}\, uv^*$.

(*c*) $|||\mathbf{\Delta L}|||_{w,p} = \eta_{w,p}(\lambda, \mathbf{L})$.

(*d*) $(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda))v = 0$ *and* $u^*\,(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = 0$.

(*e*)
$$\begin{aligned} u^*\,(\partial_z\mathbf{L}(\lambda) + \partial_z\Delta\mathbf{L}(\lambda))v &= u^*\,\partial_z\mathbf{L}(\lambda)v + u^*\partial_z\Delta\mathbf{L}(\lambda)v \\ &= u^*\,\partial_z\mathbf{L}(\lambda)v - \eta_{w,p}(\lambda, \mathbf{L})\overline{\nabla h_{w^{-1},q}(\lambda)}\; I_k, \end{aligned}$$

where $I_k \in \mathbb{C}^{k\times k}$ *is the identity matrix.*

If $\sigma_{\min}(\mathbf{L}(\lambda))$ *is simple and the gradient* $\nabla h_{w^{-1},q}(\lambda)$ *exists then we have*

$$\begin{aligned} u^*(\partial_z\mathbf{L}(\lambda) + \partial_z\Delta\mathbf{L}(\lambda))v &= u^*(\partial_z\mathbf{L}(\lambda))v - \eta_{w,p}(\lambda, \mathbf{L})\overline{\nabla h_{w^{-1},q}(\lambda)} \\ &= h_{w^{-1},q}(\lambda)\overline{\nabla\eta_{w,p}(\lambda, \mathbf{L})}. \end{aligned}$$

Proof: (*a*) Consider the polynomial $\Delta\mathbf{L}(z) := \sum_{i=0}^{m} \Delta A_i z^i$ and set $z_\lambda := (1, \lambda, \ldots, \lambda^m)$. Then we have $\mathbf{\Delta L}(\lambda) = -\eta_{w,p}(\lambda, \mathbf{L}) \sum_{i=0}^{m} \overline{(\nabla_i N_{w^{-1},q})\,(z_\lambda)}\; u\,v^*\,\lambda^i$. By Lemma 3.5.4, we have $\sum_{i=0}^{m} \lambda^i\; \overline{(\nabla_i N_{w^{-1},q})\,(z_\lambda)} = N_{w^{-1},q}(z_\lambda) = h_{w^{-1},q}(\lambda)$. Hence

$$\mathbf{\Delta L}(\lambda) = -\eta_{w,p}(\lambda, \mathbf{L}) h_{w^{-1},q}(\lambda) uv^*.$$

(*b*) Now, $\partial_z \Delta\mathbf{L}(\lambda) = \sum_{i=1}^{m} i\;\Delta A_i\,\lambda^{i-1} = -\eta_{w,p}(\lambda, \mathbf{L}) \sum_{i=1}^{m} i\;\overline{(\nabla_i N_{w^{-1},q})(z_\lambda)}\,\lambda^{i-1}$. By Lemma 3.5.5, we have $\partial_z\mathbf{\Delta L}(\lambda) = -\eta_{w,p}(\lambda, \mathbf{L})\overline{\nabla h_{w^{-1},q}(\lambda)}uv^*$.

(*c*) We have $|||\mathbf{\Delta L}|||_{w,p} = \big\|(\|\Delta A_0\|,\, \|\Delta A_1\|,\, \ldots,\, \|\Delta A_m\|)\big\|_{w,p} = \eta_{w,p}(\lambda, \mathbf{L})\big\|(\nabla_0 N_{w^{-1},q}(z_\lambda),\, \nabla_1 N_{w^{-1},q}(z_\lambda),\, \ldots,\, \nabla_m N_{w^{-1},q}(z_\lambda))\big\|_{w,p}$. By Lemma 3.5.4, we have $\big\|(\nabla_0 N_{w^{-1},q}(z_\lambda), \nabla_1 N_{w^{-1},q}(z_\lambda), \ldots, \nabla_m N_{w^{-1},q}(z_\lambda))\big\|_{w,p} = 1$. Consequently, we have $|||\mathbf{\Delta L}|||_{w,p} = \eta_{w,p}(\lambda, \mathbf{L})$.

(*d*) Next, by (*a*) we have $(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda))v = \mathbf{L}(\lambda)v + \Delta\mathbf{L}(\lambda)v = \sigma_{\min}(\mathbf{L}(\lambda))u - \eta_{w,p}(\lambda, \mathbf{L})h_{w^{-1},q}(\lambda)u = \sigma_{\min}(\mathbf{L}(\lambda))u - \sigma_{\min}(\mathbf{L}(\lambda))u = 0$. Also $u^*\,(\mathbf{L}(\lambda) + \mathbf{\Delta L}(\lambda)) = u^*\mathbf{L}(\lambda) + u^*\mathbf{\Delta L}(\lambda) = \sigma_{\min}(\mathbf{L}(\lambda))v^* - \eta_{w,p}(\lambda, \mathbf{L})h_{w^{-1},q}(\lambda)v^* = 0$. Hence λ is an eigenvalue of $\mathbf{L} + \mathbf{\Delta L}$.

(*e*) By (*b*) we have $u^*(\partial_z\mathbf{L}(\lambda) + \partial_z\Delta\mathbf{L}(\lambda))v = u^*\partial_z\mathbf{L}(\lambda)v - \eta_{w,p}(\lambda, \mathbf{L})\overline{\nabla h_{w^{-1},q}(\lambda)}$. Finally, when $\sigma_{\min}(\mathbf{L}(\lambda))$ is simple, the desired result follows from Theorem 3.5.7 (a). ■

A few comments about the results in Theorem 3.5.8, especially the case for $p = 1$ and $p = \infty$, are in order. Set $z_\lambda := (1, \lambda, \dots, \lambda^m)$.

1. Recall that for a weight vector $w := (w_0, w_1, \dots, w_m)$, $w^{-1} = (w_0^{-1}, w_1^{-1}, \dots, w_m^{-1})$ with the convention that $w_j^{-1} = 0$ if $w_j = 0$. So for example, if $w_i = 0$ then $\nabla_i N_{w^{-1},q}(z_\lambda) = 0$. Hence by Theorem 3.5.8 we have $\Delta A_i = 0$ which conforms with the fact that $w_i = 0$ means A_i remains unperturbed.

2. Now suppose that $p = \infty$. Then $q = 1$ and $\nabla_0 N_{w^{-1},1}(z_\lambda) = w_0^{-1}$ and $\nabla_i N_{w^{-1},1}(z_\lambda) = w_i^{-1}\dfrac{\lambda^i}{|\lambda^i|}$. Recall that, for $i = 1 : m$, the partial gradient $\nabla_i N_{w^{-1},1}(z_\lambda)$ does not exist when $\lambda = 0$. Hence by convention $\nabla_i N_{w^{-1},1}(z_\lambda) = 0$ when $\lambda = 0$. Consequently, when $\lambda = 0$ by Theorem 3.5.8, we have $\Delta A_0 = -w_0^{-1}\eta_{w,\infty}(\lambda, \mathbf{L})\, uv^*$ and $\Delta A_i = 0$ for $i = 1 : m$. Thus A_j remains unperturbed for $j = 1 : m$.

3. Finally, suppose that $p = 1$. Recall that $D_i := \{z \in \mathbb{C}^{m+1} : N_{w^{-1},\infty}(z) = w_j^{-1}|z_j| = w_i^{-1}|z_i| \text{ for some } j \neq i\}$. Then the partial gradient $\nabla_i N_{w^{-1},\infty}(z_\lambda)$ does not exist if $z_\lambda \in D_i$. Hence by Theorem 3.5.8, we have $\Delta A_i = 0$. Thus Theorem 3.5.8 does not provide us with a nonzero perturbed polynomial $\Delta\mathbf{L}$ such that $\lambda \in \Lambda(\mathbf{L} + \Delta\mathbf{L})$. This problem can easily be fixed by redefining ΔA_i as $\Delta A_i := -\eta_{w,1}(\lambda, \mathbf{L})\, \mathrm{sign}(\lambda^i)\, uv^*$ and $\Delta A_j = 0$ for $j \neq i$. This choice of ΔA_i corresponds to defining $\nabla_j N_{w^{-1},\infty}(z_\lambda) := \lim_{\mu\to\lambda} \nabla_j N_{w^{-1},\infty}(z_\mu)$ for $j = 0 : m$, where $z_\mu := (1, \mu, \dots, \mu^m)$ is such that $z_\mu \notin D_i$ and $N_{w^{-1},\infty}(z_\mu) = w_i^{-1}|\mu|^i$. This shows that if $N_{w^{-1},\infty}(z_\lambda) = w_{i_1}^{-1}|\lambda|^{i_1} = \cdots = w_{i_k}^{-1}|\lambda|^{i_k}$ then anyone of the matrices A_j, $j \in \{i_1, i_2, \dots, i_k\}$, can be perturbed by $\Delta A_j := -\eta_{w,1}(\lambda, \mathbf{L})\, \mathrm{sign}(\lambda^j) uv^*$ and each such choice will provide us with a perturbation $\Delta\mathbf{L}$ such that $\lambda \in \Lambda(\mathbf{L} + \Delta\mathbf{L})$.

The following result generalizes Theorem 3.5.8 to the case of homogeneous matrix polynomials.

Theorem 3.5.9. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular polynomial given by* $\mathbf{L}(c, s) = \sum_{i=0}^m c^{m-i}s^iA_i$. *Let* $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\}$. *Consider the SVD* $\mathbf{L}(\lambda, \mu) = U\Sigma V^*$ *and set* $u := U(:, n-k+1 : n)$ *and* $v := V(:, n-k+1 : n)$, *where* k *is the multiplicity of* $\sigma_{\min}(\mathbf{L}(\lambda, \mu))$. *For* $z \in \mathbb{C}^{m+1}$, *define* $N_{w,p}(z) := \|z\|_{w,p}$ *and* $H_{w,p}(c, s) := N_{w,p}(c^m, c^{m-1}s, \dots, s^m)$. *With the convention that a partial gradient of* $N_{w,p}$ *at* z *is* 0 *if it does not exist, we define*

$$\Delta A_i := -\eta_{w,p}(\lambda, \mu, \mathbf{L})\, \overline{(\nabla_i N_{w^{-1},q})(\lambda^m, \lambda^{m-1}\mu, \dots, \mu^m)}\, uv^*,$$

and consider the polynomial $\Delta\mathbf{L}(c, s) = \sum_{i=0}^m c^{m-i}s^i\Delta A_i$, *where* $p^{-1} + q^{-1} = 1$. *Then we have*

(*a*) $\mathbf{\Delta L}(\lambda, \mu) = -\eta_{w,p}(\lambda, \mu, \mathbf{L})H_{w^{-1},q}(\lambda, \mu)uv^*$.

(b) $\partial_c \mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c\, H_{w^{-1},q})(\lambda,\mu)}uv^*$.

(c) $\partial_s \mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s\, H_{w^{-1},q})(\lambda,\mu)}uv^*$.

(d) $|||\mathbf{\Delta L}|||_{w,p} = \eta_{w,p}(\lambda,\mu,\mathbf{L})$.

(e) $(\mathbf{L}(\lambda,\mu)+\mathbf{\Delta L}(\lambda,\mu))v = 0$ *and* $u^*(\mathbf{L}(\lambda,\mu)+\mathbf{\Delta L}(\lambda,\mu)) = 0$.

(f) $u^*\left(\partial_c\mathbf{L}(\lambda,\mu)+\partial_c\mathbf{\Delta L}(\lambda,\mu)\right)v = u^*\partial_c\mathbf{L}(\lambda,\mu)v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c\, H_{w^{-1},q})(\lambda,\mu)}\; I_k$.

(g) $u^*\left(\partial_s\mathbf{L}(\lambda,\mu)+\partial_s\mathbf{\Delta L}(\lambda,\mu)\right)v = u^*\partial_s\mathbf{L}(\lambda,\mu)v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s\, H_{w^{-1},q})(\lambda,\mu)}\; I_k$,

where $I_k \in \mathbb{C}^{k\times k}$ *is the identity matrix.*

If $\sigma_{\min}(\mathbf{L}(c,s))$ *is simple,* $(\nabla_c\, H_{w^{-1},q})(\lambda,\mu)$ *and* $(\nabla_s H_{w^{-1},q})(\lambda,\mu)$ *exist then we have*

$$\begin{aligned}
u^*\left(\partial_s\mathbf{L}(\lambda,\mu)+\partial_s\mathbf{\Delta L}(\lambda,\mu)\right)v &= u^*\partial_s\mathbf{L}(\lambda,\mu)v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s\, H_{w^{-1},q})(\lambda,\mu)}\\
&= H_{w^{-1},q}(\lambda,\mu)\overline{\nabla_s\eta_{w,p}(\lambda,\mu,\mathbf{L})},\\
u^*\left(\partial_c\mathbf{L}(\lambda,\mu)+\partial_c\mathbf{\Delta L}(\lambda,\mu)\right)v &= u^*\partial_c\mathbf{L}(\lambda,\mu)v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c\, H_{w^{-1},q})(\lambda,\mu)}\\
&= H_{w^{-1},q}(\lambda,\mu)\overline{\nabla_c\eta_{w,p}(\lambda,\mu,\mathbf{L})}.
\end{aligned}$$

Proof: (a) Set $z_{\lambda\mu} := (\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)$ and consider the polynomial $\mathbf{\Delta L}(c,s) = \sum_{i=0}^{m} c^{m-i}s^i\Delta A_i$. Then $\mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\sum_{i=0}^{m}\lambda^{m-i}\mu^m\;\overline{\nabla_i\, N_{w^{-1},q}(z_{\lambda\mu})}\;uv^*$. By Lemma 3.5.4, we have $\sum_{i=0}^{m}\;\lambda^{m-i}\mu^m\;\overline{\nabla_i N_{w^{-1},q}(z_{\lambda\mu})} = H_{w^{-1},q}(\lambda,\mu)$. Hence

$$\mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})H_{w^{-1},q}(\lambda,\mu)uv^*.$$

(b) Now, $\partial_c\mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\sum_{i=0}^{m-1}(m-i)\lambda^{m-i-1}\mu^i\;\overline{\nabla_i N_{w^{-1},q}(z_{\lambda\mu})}uv^*$. By Lemma 3.5.5, we have

$$\sum_{i=0}^{m-1}(m-i)\lambda^{m-i-1}\mu^i\;\overline{\nabla_i N_{w^{-1},q}(z_{\lambda\mu})} = \overline{\nabla_c H_{w^{-1},q}(\lambda,\mu)}.$$

Hence $\partial_c\mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\;\overline{(\nabla_c H_{w^{-1},q})(\lambda,\mu)}\;uv^*$.

(c) Similarly, we have $\partial_s\mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\sum_{i=1}^{m} i\;\lambda^{m-i}\,\mu^{i-1}\;\overline{\nabla_i N_{w^{-1},q}\,(z_{\lambda\mu})}\,uv^*$. By Lemma 3.5.5, we have $\sum_{i=1}^{m-1} i\,\lambda^{m-i}\mu^{i-1}\,\overline{\nabla_i N_{w^{-1},q}(z_{\lambda\mu})} = \overline{(\nabla_s H_{w^{-1},q})(\lambda,\mu)}$. Hence

$$\partial_s\mathbf{\Delta L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s\, H_{w^{-1},q})(\lambda,\mu)}\;uv^*.$$

(d) Next, we have

$$\begin{aligned}\|\mathbf{\Delta L}\|_{w,p} &= \big\|(\|\Delta A_0\|, \|\Delta A_1\|, \ldots, \|\Delta A_m\|)\big\|_{w,p} \\ &= \eta_{w,p}(\lambda,\mu,\mathbf{L})\big\|(\nabla_0 N_{w^{-1},q}(z_{\lambda\mu}), \nabla_1 N_{w^{-1},q}(z_{\lambda\mu}), \ldots, \nabla_m N_{w^{-1},q}(z_{\lambda\mu}))\big\|_{w,p}.\end{aligned}$$

By Lemma 3.5.4, we have $\big\|(\nabla_0 N_{w^{-1},q}(z_{\lambda\mu}), \ldots, \nabla_m N_{w^{-1},q}(z_{\lambda\mu}))\big\|_{w,p} = 1$. Consequently, we have $\|\mathbf{\Delta L}\|_{w,p} = \eta_{w,p}(\lambda,\mu,\mathbf{L})$.

(e) By (a) we have $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))v = \mathbf{L}(\lambda,\mu)v + \Delta\mathbf{L}(\lambda,\mu)v = \sigma_{\min}(\mathbf{L}(\lambda,\mu))u - \eta_{w,p}(\lambda,\mu,\mathbf{L})H_{w^{-1},q}(\lambda,\mu)u = \sigma_{\min}(\mathbf{L}(\lambda,\mu))u - \sigma_{\min}(\mathbf{L}(\lambda,\mu))u = 0$. Similarly, $u^*(\mathbf{L}(\lambda,\mu) + \mathbf{\Delta L}(\lambda,\mu)) = \sigma_{\min}(\mathbf{L}(\lambda,\mu))v^* - \eta_{w,p}(\lambda,\mu,\mathbf{L})H_{w^{-1},q}(\lambda,\mu)v^* = 0$. Hence λ is an eigenvalue of $\mathbf{L} + \mathbf{\Delta L}$.

(f) Now, $u^*(\partial_c\mathbf{L}(\lambda,\mu) + \partial_c\mathbf{\Delta L}(\lambda,\mu))v = u^*\partial_c\mathbf{L}(\lambda,\mu)v + u^*\partial_c\mathbf{\Delta L}(\lambda,\mu)v$. By ($b$) we have $\partial_c\mathbf{L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c H_{w^{-1},q})(\lambda,\mu)}\; uv^*$. Hence $u^*(\partial_c\mathbf{L}(\lambda,\mu) + \partial_c\mathbf{\Delta L}(\lambda,\mu))v = u^*(\partial_c\mathbf{L}(\lambda,\mu))v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c H_{w^{-1},q})(\lambda,\mu)}\; I_k$.

(g) Again $u^*(\partial_s\mathbf{L}(\lambda,\mu) + \partial_s\mathbf{\Delta L}(\lambda,\mu))v = u^*\partial_s\mathbf{L}(\lambda,\mu)v + u^*\partial_s\mathbf{\Delta L}(\lambda,\mu)v$. By (c) we have $\partial_s\mathbf{L}(\lambda,\mu) = -\eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s H_{w^{-1},q})(\lambda,\mu)}\; uv^*$. Hence $u^*(\partial_s\mathbf{L}(\lambda,\mu) + \partial_s\mathbf{\Delta L}(\lambda,\mu))v$ $= u^*(\partial_s\mathbf{L}(\lambda,\mu))v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s H_{w^{-1},q})(\lambda,\mu)}\; I_k$.

Finally if $\sigma_{\min}(\mathbf{L}(\lambda,\mu))$ is simple, $\nabla_c H_{w^{-1},q}(\lambda,\mu)$ and $\nabla_s H_{w^{-1},q}(\lambda,\mu)$ exist then from Theorem 3.5.7, we have

$$\begin{aligned}u^*\left(\partial_s\mathbf{L}(\lambda,\mu) + \partial_s\mathbf{\Delta L}(\lambda,\mu)\right)v &= u^*\partial_s\mathbf{L}(\lambda,\mu)v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_s H_{w^{-1},q})(\lambda,\mu)} \\ &= H_{w^{-1},q}(\lambda,\mu)\overline{\nabla_s\,\eta_{w,p}(\lambda,\mu,\mathbf{L})}, \\ u^*\left(\partial_c\mathbf{L}(\lambda,\mu) + \partial_c\mathbf{\Delta L}(\lambda,\mu)\right)v &= u^*\partial_c\mathbf{L}(\lambda,\mu)v - \eta_{w,p}(\lambda,\mu,\mathbf{L})\overline{(\nabla_c H_{w^{-1},q})(\lambda,\mu)} \\ &= H_{w^{-1},q}(\lambda,\mu)\overline{\nabla_c\,\eta_{w,p}(\lambda,\mu,\mathbf{L})}. \blacksquare\end{aligned}$$

Remark 3.5.10. *Recall that* $z_{\lambda\mu} := (\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)$. *Now suppose that* $p = 1$ *and that* $N_{w^{-1},\infty}(z_{\lambda\mu}) = w_{i_1}^{-1}|\lambda^{m-i_1}\mu^{i_1}| = \ldots = w_{i_k}^{-1}|\lambda^{m-i_k}\mu^{i_k}|$. *Then the partial gradient* $\nabla_j N_{w^{-1},\infty}(z_{\lambda\mu})$ *does not exist for* $j \in \{i_1, \ldots, i_k\}$. *Consequently, Theorem 3.5.9 does not provide us with a nonzero polynomial* $\mathbf{\Delta L}$ *such that* $(\lambda,\mu) \in \Lambda(\mathbf{L} + \mathbf{\Delta L})$. *However, as in the case of nonhomogeneous polynomials, perturbing anyone of the matrices* $A_j, j \in \{i_1, \ldots, i_k\}$, *by* $\Delta A_j := -\eta_{w,1}(\lambda,\mu,\mathbf{L})\,\mathrm{sign}(\lambda^{m-j}\mu^j)uv^*$ *and leaving the other matrices unperturbed we obtain a polynomial* $\mathbf{\Delta L}$ *such that* $(\lambda,\mu) \in \Lambda(\mathbf{L} + \mathbf{\Delta L})$. *For* $i \in \{i_1, \ldots, i_k\}$, *the perturbed matrix* ΔA_i *as defined above corresponds to defining* $\nabla_j N_{w^{-1},\infty}(z_{\lambda\mu}) := \lim_{(c,s)\to(\lambda,\mu)} \nabla_j N_{w^{-1},\infty}(z_{cs})$, *where* $z_{cs} := (c^m, c^{m-1}s, \ldots, s^m)$ *is such that* $N_{w^{-1},\infty}(z_{cs}) = w_i^{-1}|c^{m-i}s^i|$ *and* $w_j^{-1}|c^{m-j}s^j| < N_{w^{-1},\infty}(z_{cs})$ *for* $j \neq i$.

Now we define the notion of generic and nongeneric critical points of the map $z \mapsto \eta_{w,p}(z, \mathbf{L})$ and $(c, s) \mapsto \eta_{w,p}(c, s, \mathbf{L})$.

Definition 3.5.11. *Let $\lambda \in \mathbb{C}$. Then λ is said to be a generic critical point of $\eta_{w,p}$ if $\sigma_{\min}(\mathbf{L}(\lambda))$ is simple and $\nabla \eta_{w,p}(\lambda, \mathbf{L}) = 0$. If $\sigma_{\min}(\mathbf{L}(\lambda))$ is multiple then λ is said to be a nongeneric critical point of $\eta_{w,p}$. If the multiplicity of $\sigma_{\min}(\mathbf{L}(\lambda))$ is k then μ is said to be a nongeneric critical point of multiplicity k. A complex number is said to be a critical point of $\eta_{w,p}$ if it is either a generic or a nongeneric critical point of $\eta_{w,p}$. Critical points of $\eta_{w,p}(c, s, \mathbf{L})$ are defined similarly.*

We now show that critical points of $\eta_{w,p}$ are multiple eigenvalues of nearby polynomials. To that end, we state a result that provides a necessary and sufficient condition for an eigenvalue of a matrix polynomial to be multiple. Let λ be a simple eigenvalue of a regular polynomial $\mathbf{L}$. Then λ is a multiple eigenvalue of $\mathbf{L}$ if and only if there exists left and right eigenvectors u and v of $\mathbf{L}$ corresponding to λ such that $u^*\partial_z\mathbf{L}(\lambda)v = 0$. A proof of this result can be found, for example, in [12]. We provide a simple proof of this fact.

Given a regular polynomial $\mathbf{L}(z) = \sum_{i=0}^{m} z^i A_i$, consider the pencils $\mathbf{C_L}(z) := \mathbf{X} - z\mathbf{Y}$ and $\mathbf{C_L}(c, s) := c\mathbf{X} - s\mathbf{Y}$, where

$$\mathbf{X} := \begin{bmatrix} 0 & -I & 0 & \dots & 0 \\ 0 & 0 & -I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & \dots & -I \\ A_0 & A_1 & & \dots & A_{m-1} \end{bmatrix} \text{ and } \mathbf{Y} := \begin{bmatrix} -I & 0 & \dots & & 0 \\ 0 & -I & \dots & & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -I & 0 \\ 0 & 0 & \dots & 0 & -A_m \end{bmatrix}.$$

Note that $\mathbf{C_L}$ is a linearization of $\mathbf{L}$. Then we have the following.

Theorem 3.5.12. *Let $\mathbf{L}(z) = \sum_{i=0}^{m} z^i A_i$ be regular and $\lambda \in \Lambda(\mathbf{L})$. Then λ is a multiple eigenvalue if and only if there exist left and right eigenvectors u and v of $\mathbf{L}$ corresponding to λ such that $u^*\partial_z\mathbf{L}(\lambda)v = 0$.*

Proof: Let $Z := \begin{bmatrix} v \\ \lambda v \\ \vdots \\ \lambda^{m-1} v \end{bmatrix}$ and $W := \begin{bmatrix} \left(A_1 + \lambda A_2 + \cdots + \lambda^{m-1} A_m\right)^* u \\ \left(A_2 + \lambda A_3 + \cdots + \lambda^{m-2} A_m\right)^* u \\ \vdots \\ (A_{m-1} + \lambda A_m)^* u \\ u \end{bmatrix}$. Then it is easy to see that $W^*\mathbf{C_L}(\lambda) = 0$ and $\mathbf{C_L}(\lambda)Z = 0$. Hence λ is an eigenvalue of $\mathbf{C_L}(z)$ and W and Z are left and right eigenvectors corresponding to λ. Then by Theorem 2.4.10, λ is a multiple eigenvalue of $\mathbf{C_L}(z)$ if and only if $W^*YZ = 0$. By Theorem 1.2.8, we have $W^*YZ = -u^*\partial_z\mathbf{L}(\lambda)v$. Hence λ is multiple eigenvalue of $\mathbf{L}(z)$ if and only if $u^*\partial_z\mathbf{L}(\lambda)v = 0$. ■

The homogeneous version of Theorem 3.5.12 is as follows.

Theorem 3.5.13. *Let* $\mathbf{L}(c,s) = \sum_{i=0}^{m} c^{m-i}s^i A_i$ *regular and* $(c_0, s_0) \in \Lambda(\mathbf{L})$*. Then* (c_0, s_0) *is a multiple eigenvalue if and only if there exist left and right eigenvectors* u *and* v *of* $\mathbf{L}$ *corresponding to* (c_0, s_0) *such that* $(u^*\,\partial_c\mathbf{L}(c_0,s_0)\,v,\ u^*\partial_s\mathbf{L}(c_0,s_0)v) = 0$.

Proof: For $(c,s) \in \mathbb{C}^2 \setminus \{0\}$, set $\lambda := s/c$ when $c \neq 0$ and $\mu := c/s$ when $s \neq 0$. Then $\mathbf{L}(c,s) = c^m\,\mathbf{L}(\lambda)$ when $c \neq 0$ and $\mathbf{L}(c,s) = s^m\,\mathrm{rev}\mathbf{L}(\mu)$ when $s \neq 0$, where $\mathrm{rev}\mathbf{L}$ is the reversal of $\mathbf{L}$, that is, $\mathrm{rev}\mathbf{L}(z) = z^m\mathbf{L}(1/z)$. Let u and v be left and right eigenvectors of $\mathbf{L}$ corresponding to (c_0, s_0). Then taking partial derivatives of $\mathbf{L}(c,s)$, we have

$$u^*\partial_c\mathbf{L}(c_0,s_0)v = -c_0^{m-2}s_0u^*\partial_\lambda\mathbf{L}(\lambda_0)v,\ \ u^*\partial_s\mathbf{L}(c_0,s_0)v = c_0^{m-1}u^*\partial_\lambda\mathbf{L}(\lambda_0)v$$

when $c_0 \neq 0$, and

$$u^*\partial_c\mathbf{L}(c_0,s_0)v = s_0^{m-1}u^*\partial_\mu\mathrm{rev}\mathbf{L}(\mu_0)v,\ \ u^*\partial_s\mathrm{rev}\mathbf{L}(c_0,s_0)v = -s_0^{m-2}c_0u^*\partial_\mu\mathrm{rev}\mathbf{L}(\mu_0)v$$

when $s_0 \neq 0$. Note that ∞ is a multiple eigenvalue of $\mathbf{L}(z)$ if and only if 0 is a multiple eigenvalue of $\mathrm{rev}\mathbf{L}(z)$. Hence the desired result follows from Theorem 3.5.12.■

The following result shows that critical points of $\eta_{w,p}(z,\mathbf{L})$ are multiple eigenvalues of appropriately perturbed polynomials.

Theorem 3.5.14. *Let* $\mathbf{L} \in \mathbb{L}_{m,w}^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular polynomial given by* $\mathbf{L}(z) := \sum_{i=0}^m z^iA_i$*. For* $\lambda \in \mathbb{C}$*, construct the polynomial* $\Delta\mathbf{L}$ *as in Theorem 3.5.8. Then we have* $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \eta_{w,p}(\lambda,\mathbf{L})$*. If* λ *is a generic critical point of* $\eta_{w,p}(z,\mathbf{L})$ *then* λ *is a defective eigenvalue of* $\mathbf{L}+\Delta\mathbf{L}$*. On the other hand, if* λ *is a nongeneric critical point of* $\eta_{w,p}(z,\mathbf{L})$ *of multiplicity* k *then* λ *is a multiple eigenvalue of* $\mathbf{L}+\Delta\mathbf{L}$ *of geometric multiplicity* k*.*

Proof: By Theorem 3.5.8, we have $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \eta_{w,p}(\lambda,\mathbf{L})$, $\mathbf{L}(\lambda)v + \Delta\mathbf{L}(\lambda)v = 0$ and $u^*(\mathbf{L}(\lambda) + \Delta\mathbf{L}(\lambda)) = 0$. If λ is a generic critical point then we have $\nabla\eta_{w,p}(\lambda,\mathbf{L}) = 0$. Hence by Theorem 3.5.8, we have $u^*\partial_z(\mathbf{L}(\lambda) + \mathbf{\Delta L}(\lambda))v = h_{w^{-1},q}(\lambda)\overline{\nabla\,\eta_{w,p}(\lambda,\mathbf{L})} = 0$. Consequently, by Theorem 3.5.12, λ is a multiple eigenvalue of $\mathbf{L}+\Delta\mathbf{L}$. Since by construction $\mathrm{rank}(\mathbf{L}(\lambda)+\Delta\mathbf{L}(\lambda)) = n-1$, it follows that λ is a defective eigenvalue of $\mathbf{L}+\Delta\mathbf{L}$.

When λ is a nongeneric critical point of multiplicity k, by Theorem 3.5.8, we have $(\mathbf{L}(\lambda)+\Delta\mathbf{L}(\lambda))v(:,j) = 0,\ \ u(:,j)^*(\mathbf{L}(\lambda)+\Delta\mathbf{L}(\lambda)) = 0$ for $j = 1:k$. Hence the result follows. ■

Obviously similar result holds for a homogeneous polynomial. Indeed, we have the following result for a homogeneous polynomial whose proof follows from Theorem 3.5.9.

Theorem 3.5.15. *Let* $\mathbf{L} \in \mathbb{L}_{m,w}^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular polynomial given by* $\mathbf{L}(c,s) := \sum_{i=0}^m c^{m-i}s^iA_i$*. For nonzero* $(\lambda,\mu) \in \mathbb{C}^2$*, construct the polynomial* $\Delta\mathbf{L}$ *as in Theorem 3.5.9. Then we have* $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \eta_{w,p}(\lambda,\mu,\mathbf{L})$*. If* (λ,μ) *is a generic critical point of*

$\eta_{w,p}(c, s, \mathbf{L})$ then (λ, μ) is a defective eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$. On the other hand, if (λ, μ) is a nongeneric critical point of $\eta_{w,p}(c, s, \mathbf{L})$ of multiplicity k then (λ, μ) is a multiple eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$ of geometric multiplicity k.

Thus we see that critical points of the function $z \mapsto \eta_{w,p}(z, \mathbf{L})$ are multiple eigenvalues of appropriately perturbed polynomials. Although determining all critical points of $\eta_{w,p}(z, \mathbf{L})$ is a nontrivial task, it turns out that some critical points of $\eta_{w,p}(z, \mathbf{L})$ can be read off from the pseudospectra of $\mathbf{L}$. More precisely, we now show that common boundary points of components of pseudospectra of $\mathbf{L}$ are in fact critical points of $\eta_{w,p}(z, \mathbf{L})$.

Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular polynomial of degree m. As ϵ grows gradually, the components of $\Lambda_\epsilon(\mathbf{L})$ became large in size and some of them coalesce with other components. The following result shows that the points of coalescence of components of $\Lambda_\epsilon(\mathbf{L})$ are actually critical points of $\eta_{w,p}$.

Theorem 3.5.16. *Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular polynomial. Suppose that two components of $\Lambda_\epsilon(\mathbf{L})$ coalesce at μ as $\epsilon \to \delta$. Then $\eta_{w,p}(\mu, \mathbf{L}) = \delta$. Further, μ is either a nongeneric critical point of $\eta_{w,p}(z, \mathbf{L})$ for $1 \leq p \leq \infty$ or the following holds:*

(a) *If $1 < p < \infty$ and $p/(p-1)$ is an integer then μ is a generic critical point of $\eta_{w,p}$.*

(b) *If $p = \infty$ then μ is a generic critical point of $\eta_{w,p}$ provided that $\mu \neq 0$.*

(c) *If $p = 1$ then μ is a generic critical point of $\eta_{w,p}$ provided that $(1, \mu, \ldots, \mu^m) \in \mathbb{C}^{m+1} \setminus D$, where $D := \{z \in \mathbb{C}^{m+1} : w_i^{-1}|z_i| = w_j^{-1}|z_j| \text{ for some } i \neq j\}$.*

Proof: Recall that $\partial\Lambda_\delta(\mathbf{L}) \subset \Gamma := \{z \in \mathbb{C} : \eta_{w,p}(z, \mathbf{L}) = \delta\}$. Consequently, we have $\eta_{w,p}(\mu, \mathbf{L}) = \delta$. Now, if $\sigma_{\min}(\mathbf{L}(\mu))$ is multiple then μ is a nongeneric critical point of $\eta_{w,p}$. On the other hand, if $\sigma_{\min}(\mathbf{L}(\mu))$ is simple then by Proposition 3.5.3 and Theorem 3.5.7, $\eta_{w,p}(z, \mathbf{L})$ is differentiable in a neighbourhood $\mathcal{N}_\mu$ of μ for $1 < p < \infty$. Since μ lies on the common boundary of two components, in view Proposition 3.4.7, the curve $\Gamma \cap \mathcal{N}_\mu$ consists of two arcs intersecting at μ. Hence by Implicit Function Theorem $\nabla\eta_{w,p}(\mu, \mathbf{L}) = 0$. When $p = \infty$, $\eta_{w,p}(z, \mathbf{L})$ is differentiable for $\mu \neq 0$ and when $p = 1$, $\eta_{w,p}(z, \mathbf{L})$ is differentiable at μ if $(1, \mu, \ldots, \mu^m) \in \mathbb{C}^{m+1} \setminus D$. Hence the result follows. ■

This shows that common boundary points of components of pseudospectra of $\mathbf{L}$ are critical points of $\eta_{w,p}(\lambda, \mathbf{L})$. Consequently, by Theorem 3.5.14 we conclude that common boundary points of components of the pseudospectrum $\Lambda_\epsilon(\mathbf{L})$ are in fact multiple eigenvalues of appropriately perturbed polynomials whose distance from $\mathbf{L}$ is equal to ϵ.

Note that if $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ is a regular polynomial and has mn distinct eigenvalues then Theorem 3.5.14 provides a polynomial having a multiple eigenvalue whenever a critical point of $\eta_{w,p}(z, \mathbf{L})$ is available. A **minimal critical point** of $\eta_{w,p}$, that is, the critical point at which $\eta_{w,p}$ takes the smallest value among all the critical points, is of special interest and Theorem 3.5.16 tells us where to look for it. This brings us to Wilkinson's problem for matrix polynomials.

3.6 Wilkinson's problem for matrix polynomials

Given a matrix $A \in \mathbb{C}^{n\times n}$ having n distinct eigenvalues, the problem of determining ΔA having smallest norm such that $A + \Delta A$ has a multiple eigenvalue is widely known as Wilkinson's problem. It is shown in [1] that a solution of Wilkinson's problem can be constructed from the pseudospectra of A. Wilkinson's problem for matrix polynomials can be stated as follows.

Wilkinson's problem: Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular polynomial having n distinct eigenvalues. Define

$$\mathrm{d}(\mathbf{L}) := \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} : \Delta\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2) \text{ and } \mathbf{L}+\Delta\mathbf{L} \text{ has a multiple eigenvalue}\}.$$

Then determine $\Delta\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ such that $\mathbf{L}+\Delta\mathbf{L}$ has a multiple eigenvalue and that $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} = \mathrm{d}(\mathbf{L})$.

We now show that a solution of Wilkinson's problem can be constructed from the pseudospectra of matrix polynomials. We consider the space of polynomials $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ and assume that $p/(p-1)$ is an integer or $p=\infty$. So, let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n}, \|\cdot\|_2)$ be a regular polynomials having mn distinct eigenvalues. Recall that $\Lambda_\epsilon(\mathbf{L})$ consists of at most mn components and each component contains at least one eigenvalue of $\mathbf{L}$. Since $\mathbf{L}$ has mn distinct eigenvalues, for sufficiently small ϵ, $\Lambda_\epsilon(\mathbf{L})$ consists of mn components. Let $\#(\Lambda_\epsilon(\mathbf{L}))$ denote the number of components of $\Lambda_\epsilon(\mathbf{L})$. Then if ϵ is such that $\#(\Lambda_\epsilon(\mathbf{L})) = mn$ then obviously we have $\mathrm{d}(\mathbf{L}) > \epsilon$. Thus, in view of Theorem 3.5.16, it is now clear that $\mathrm{d}(\mathbf{L})$ can be read off from the pseudospectra of $\mathbf{L}$. Indeed, let δ be such that $\#(\Lambda_\epsilon(\mathbf{L})) = mn$ when $\epsilon < \delta$ and $\#(\Lambda_\epsilon(\mathbf{L})) \leq mn-1$ when $\epsilon \geq \delta$. Then we have $\mathrm{d}(\mathbf{L}) = \delta$. Note that at least two components of $\Lambda_\epsilon(\mathbf{L})$ coalesce as $\epsilon \to \delta$.

Theorem 3.6.1. *Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular polynomial having mn distinct eigenvalues. Let $\#(\Lambda_\epsilon(\mathbf{L}))$ denote the number of components of $\Lambda_\epsilon(\mathbf{L})$. Let $\delta > 0$ be such that $\#(\Lambda_\epsilon(\mathbf{L})) = mn$ if $\epsilon < \delta$ and $\#(\Lambda_\epsilon(\mathbf{L})) \leq mn-1$ if $\epsilon \geq \delta$. Then we have $\mathrm{d}(\mathbf{L}) = \delta$. Let μ be a common boundary point of two components of $\Lambda_\delta(\mathbf{L})$. Consider the SVD $\mathbf{L}(\mu) = U\Sigma V^*$ and set $\sigma := \Sigma(n,n)$. Then we have*

$$\mathrm{d}(\mathbf{L}) = \eta_{w,p}(\mu, \mathbf{L}) = \frac{\sigma}{\|(1,\mu,\ldots,\mu^m)\|_{w^{-1},q}} = \delta.$$

If σ is multiple then set $u := U(:, n-1:n)$ and $v := V(:, n-1:n)$ else set $u := U(:, n)$

and $v := V(:, n)$. *Define*

$$\Delta A_i := -\frac{w_i^{-q}\,\sigma\,\operatorname{sign}(\mu^i)\,|\mu|^{i(q-1)}\;u\,v^*}{\|(1, \mu, \ldots, \mu^m)\|_{w^{-1},q}^q} \text{ for } i = 0 : m,$$

where $\operatorname{sign}(z) := \overline{z}/|z|$ *if* $z \neq 0$ *and* $\operatorname{sign}(0) = 0$. *Now consider the polynomial* $\Delta\mathbf{L}(z) = \sum_{i=0}^m z^i \Delta A_i$. *Then* $\mathrm{d}(\mathbf{L}) = |||\Delta\mathbf{L}|||_{w,p} = \eta_{w,p}(\mu, \mathbf{L})$. *Further* μ *is a multiple eigenvalue value of* $\mathbf{L} + \Delta\mathbf{L}$ *of geometric multiplicity* 2 *when* σ *is multiple and* μ *is a defective eigenvalue of* $\mathbf{L} + \Delta\mathbf{L}$ *when* σ *is simple.*

Proof: Since at least two components of $\Lambda_\epsilon(\mathbf{L})$ coalesce at μ as $\epsilon \to \delta$, by Theorem 3.5.16, μ is a critical point of $\eta_{w,p}(z, \mathbf{L})$. Hence $\mathrm{d}(\mathbf{L}) = \delta$. Now the desired results follow from Theorem 3.5.14. ■

We mention that Boulton et al. [12] provided a solution of Wilkinson's problem for matrix polynomial. Their analysis corresponds to the case $p = \infty$ and their solution of Wilkinson's problem follows from Theorem 3.6.1 by setting $p = \infty$.

Recall that an infinite eigenvalue of $\mathbf{L}$, if any, can be treated at par with finite eigenvalues by considering homogenous form of $\mathbf{L}$. We now present analogue of Theorem 3.6.1 for homogeneous polynomials. For this purpose, we normalize eigenvalue $(c, s) \in \Lambda(\mathbf{L})$ so that $|c|^2 + |s|^2 = 1$. Hence we consider $\Lambda_\epsilon(\mathbf{L})$ as a subset of $\mathbb{S}^1$. Then we have the following.

Theorem 3.6.2. *Let* $\mathbf{L} \in \mathbb{L}_{m,w}^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular homogeneous polynomial having* mn *distinct eigenvalues. Let* $\#(\Lambda_\epsilon(\mathbf{L}))$ *denote the number of components of* $\Lambda_\epsilon(\mathbf{L})$. *Let* $\delta > 0$ *be such that* $\#(\Lambda_\epsilon(\mathbf{L})) = mn$ *if* $\epsilon < \delta$ *and* $\#(\Lambda_\epsilon(\mathbf{L})) \leq mn - 1$ *if* $\epsilon \geq \delta$. *Then we have* $\mathrm{d}(\mathbf{L}) = \delta$. *Let* $(\lambda, \mu) \in \mathbb{S}^1$ *be a common boundary point of two components of* $\Lambda_\delta(\mathbf{L})$. *Consider the singular decomposition* $\mathbf{L}(\lambda, \mu) = U\Sigma V^*$ *and set* $\sigma := \Sigma(n, n)$. *Then we have*

$$\mathrm{d}(\mathbf{L}) = \eta_{w,p}(\lambda, \mu, \mathbf{L}) = \frac{\sigma}{\|(\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)\|_{w^{-1},q}} = \delta.$$

If σ *is multiple then set* $u := U(:, n-1 : n)$ *and* $v := V(:, n-1 : n)$ *else set* $u := U(:, n)$ *and* $v := V(:, n)$. *Define*

$$\Delta A_i := -\frac{w_i^{-q}\,\sigma\,\operatorname{sign}(\lambda^{m-i}\mu^i)\,|\lambda|^{(m-i)(q-1)}\,|\mu|^{i(q-1)}\,u\,v^*}{\|(\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)\|_{w^{-1},q}^q} \text{ for } i = 0 : m$$

and consider the polynomial $\Delta\mathbf{L}(c, s) = \sum_{i=0}^m c^{m-i} s^i A_i$. *Then* $\mathrm{d}(\mathbf{L}) = |||\Delta\mathbf{L}|||_{w,p} = \eta_{w,p}(\lambda, \mu, \mathbf{L})$. *Further* (λ, μ) *is a multiple eigenvalue value of* $\mathbf{L} + \Delta\mathbf{L}$ *of geometric multiplicity* 2 *when* σ *is multiple and* (λ, μ) *is a defective eigenvalue of* $\mathbf{L} + \Delta\mathbf{L}$ *when* σ *is simple.*

Chapter 4

Pseudospectra inclusions, stability of eigendecompositions and separation of matrix pencils

We investigate pseudospectra inclusions for matrix pencils and show that pseudospectra inclusions can be gainfully used for analyzing stability of eigendecompositions. We introduce and analyze various notions of separation of matrix pencils and show their usefulness in analyzing stability of eigendecompositions.

4.1 Introduction

We investigate pseudospectra inclusions, continuous evolution of eigendecompositions and various notions of separation of matrix pencils. Given a regular matrix pencil $\mathbf{L}$ such that $Y^{-1}\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1, \mathbf{L}_2)$ for some nonsingular matrices X and Y, our first aim is to derive pseudospectra inclusions of the form

$$\Lambda_\epsilon(\mathbf{L}) \subset \Lambda_{\phi(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{\phi(\epsilon)}(\mathbf{L}_2)$$

for some strictly increasing function ϕ. We provide various choices of ϕ.

Various notions of separation of matrices such as $\mathrm{sep}, \mathrm{sep}_\lambda$ and gsep play important roles [3, 43, 15, 48] in perturbation analysis of eigenvalues and eigendecompositions. We provide a general framework for defining various notions of separation of matrix pencils such as $\mathrm{sep}, \mathrm{sep}_\lambda$ and gsep. We show that these separations possess similar properties as those possess by their matrix counterparts. Further, we show that the conceptual foundations of these separations are the same as those of their matrix counterparts. Hence these separations of matrix pencils can be analyzed and handled with the same ease as their matrix counterparts.

Next, given a regular pencil $\mathbf{L}$, we analyze continuous evolution of its eigendecompositions. By an eigendecomposition of $\mathbf{L}$ we mean a decomposition of the form

$\mathbf{L} = Y \operatorname{diag}(\mathbf{L}_1, \ldots, \mathbf{L}_m) X^{-1}$, where $\Lambda(\mathbf{L}_i) \cap \Lambda(\mathbf{L}_j) = \emptyset$ for $i \neq j$ and X and Y are nonsingular matrices. Thus X^{-1}, Y and $\mathbf{L}_j$ can be thought as functions of $\mathbf{L}$. An eigendecomposition of $\mathbf{L}$ is said to evolve continuously on an open set U containing $\mathbf{L}$ if X^{-1}, Y and $\mathbf{L}_j$ are continuous on U. Continuous evolution of eigendecompositions of matrix pencils has been investigated by Demmel et al. in [16]. We investigate conditions for continuous evolution of eigendecompositions of $\mathbf{L}$ on a given open ball centred at $\mathbf{L}$. We show that a sufficient condition for continuous evolution of eigendecompositions of $\mathbf{L}$ can be read off from the pseudospectra of $\mathbf{L}$. Further, we show that for an appropriate norm a necessary and sufficient condition for continuous evolution of eigendecompositions of $\mathbf{L}$ can be read off from the pseudospectra of $\mathbf{L}$.

4.2 Localization of pseudospectra of matrix pencils

For simplicity, throughout this chapter, we consider the space of pencils $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ and assume that each components of w is nonzero. This makes $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ a normed space. We further assume that the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ a subordinate norm and has the property that $\|\operatorname{diag}(A_1, A_2)\| = \max(\|A_1\|, \|A_2\|)$.

First, we consider pseudospectra of scalar pencils which will be used in the subsequent development.

Proposition 4.2.1. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}, |\cdot|)$ be a scalar pencil given by $\mathbf{L}(z) := \alpha - z\beta$. Then the pseudospectrum of $\mathbf{L}$ is given by $\Lambda_\epsilon(\mathbf{L}) := \{(z \in \mathbb{C} : \dfrac{|\alpha - z\beta|}{\|(1, z)\|_{w^{-1},q}} \leq \epsilon\}$. Now considering the homogeneous form $\mathbf{L}(c, s) := c\alpha - s\beta$, the pseudospectrum is given by $\Lambda_\epsilon(\mathbf{L}) := \left\{(c, s) \in \mathbb{C}^2 \setminus \{0\} : \dfrac{|c\alpha - s\beta|}{\|(c, s)\|_{w^{-1},q}} \leq \epsilon\right\}$.*

We now consider pseudospectra of diagonal and block diagonal pencils which will be crucial in the later development. We have the following result whose proof is immediate.

Theorem 4.2.2. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular pencil. Suppose that $\mathbf{L}$ is block diagonal and is given by $\mathbf{L} = \operatorname{diag}(\mathbf{L}_1, \mathbf{L}_2)$. Then we have $\Lambda_\epsilon(\mathbf{L}) = \Lambda_\epsilon(\mathbf{L}_1) \cup \Lambda_\epsilon(\mathbf{L}_2)$.*

In particular, suppose that $\mathbf{L}$ is a diagonal pencil and is given by $\mathbf{L} := \operatorname{diag}(\mathbf{L}_i)$, where each $\mathbf{L}_i$ is a scalar pencil. Then we have $\Lambda_\epsilon(\mathbf{L}) = \cup_{i=1}^n \Lambda_\epsilon(\mathbf{L}_i)$.

Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ given by $\mathbf{L}(z) = A - zB$ and $X \in \mathbb{C}^{n\times n}$. Then $X\mathbf{L}$ and $\mathbf{L}X$ denote the pencils given by $X\mathbf{L}(z) = XA - zXB$ and $\mathbf{L}X(z) = AX - zBX$. Then for the backward error function $\eta_{w,p}(z, \mathbf{L})$ we have the following.

Theorem 4.2.3. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ be regular. Then for $\mathbf{\Delta L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ we have*

$$\eta_{w,p}(\lambda, \mathbf{L}) - \|\mathbf{\Delta L}\|_{w,p} \leq \eta_{w,p}(\lambda, \mathbf{L} + \mathbf{\Delta L}) \leq \eta_{w,p}(\lambda, \mathbf{L}) + \|\mathbf{\Delta L}\|_{w,p}.$$

Let X and Y be invertible matrices in $\mathbb{C}^{n\times n}$ and $\mathrm{cond}(X,Y) := \|X^{-1}\|\,\|Y\|$. *Then*

$$\frac{\eta_{w,p}(\lambda,\mathbf{L})}{\mathrm{cond}(X,Y)} \leq \eta_{w,p}(\lambda, Y^{-1}\mathbf{L}X) \leq \mathrm{cond}(Y,X)\eta_{w,p}(\lambda,\mathbf{L}).$$

Similar result hold for homogeneous matrix pencils.

Proof: We have

$$\begin{aligned}\eta_{w,p}(\lambda,\mathbf{L}+\mathbf{\Delta L}) &:= \inf\{|\!|\!|G|\!|\!|_{w,p} : \lambda\in\Lambda(\mathbf{L}+\mathbf{\Delta L}+G)\}\\ &= \inf\{|\!|\!|(\mathbf{\Delta L}+G)-\mathbf{\Delta L}|\!|\!|_{w,p} : \lambda\in\Lambda(\mathbf{L}+\mathbf{\Delta L}+G)\}.\end{aligned}$$

This gives $\eta_{w,p}(\lambda,\mathbf{L}+\mathbf{\Delta L}) \leq \eta_{w,p}(\lambda,\mathbf{L}) + |\!|\!|\mathbf{\Delta L}|\!|\!|_{w,p}$ and $\eta_{w,p}(\lambda,\mathbf{L}+\mathbf{\Delta L}) \geq \eta_{w,p}(\lambda,\mathbf{L}) - |\!|\!|\mathbf{\Delta L}|\!|\!|_{w,p}$. Hence the result follows.

Next, we have

$$\begin{aligned}\eta_{w,p}(\lambda,Y^{-1}\mathbf{L}X) &= \inf\{|\!|\!|\mathbf{\Delta L}|\!|\!|_{w,p} : \lambda\in\Lambda(Y^{-1}\mathbf{L}X+\mathbf{\Delta L})\}\\ &= \inf\{|\!|\!|\mathbf{\Delta L}|\!|\!|_{w,p} : \lambda\in\Lambda(\mathbf{L}+Y\mathbf{\Delta L}X^{-1})\}\\ &= \inf\{|\!|\!|Y^{-1}Y\mathbf{\Delta L}X^{-1}X|\!|\!|_{w,p} : \lambda\in\Lambda(\mathbf{L}+Y\mathbf{\Delta L}X^{-1})\}\\ &\leq \mathrm{cond}(Y,X)\eta_{w,p}(\lambda,\mathbf{L}) \qquad (4.1)\end{aligned}$$

and

$$\begin{aligned}\eta_{w,p}(\lambda,\mathbf{L}) &= \eta_{w,p}(\lambda, YY^{-1}\mathbf{L}XX^{-1})\\ &\leq \|Y\|\|X^{-1}\|\eta_{w,p}(\lambda,Y^{-1}\mathbf{L}X) = \mathrm{cond}(X,Y)\eta_{w,p}(\lambda,Y^{-1}\mathbf{L}X). \qquad (4.2)\end{aligned}$$

Hence the desired result follows. ■

Now we have the following Bauer-Fike type inclusion theorems for pseudospectra.

Theorem 4.2.4. *Let* $\mathbf{L}\in\mathbb{L}_w^p(\mathbb{C}^{n\times n},\|\cdot\|)$ *be regular. Then we have the following.*

(*a*) *Let* $X,Y\in\mathbb{C}^{n\times n}$ *be non-singular. Set* $\kappa_1 := \|X^{-1}\|\|Y\|$ *and* $\kappa_2 := \|Y^{-1}\|\|X\|$. *Then*

$$\Lambda_\epsilon(\mathbf{L}) \subset \Lambda_{\kappa_2\epsilon}(Y^{-1}\mathbf{L}X) \text{ and } \Lambda_\epsilon(Y^{-1}\mathbf{L}X) \subset \Lambda_{\kappa_1\epsilon}(\mathbf{L}).$$

(*b*) *Let* $\kappa := \inf\{\|Y^{-1}\|\,\|X\| : Y^{-1}\,\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1,\,\mathbf{L}_2)\}$. *Then we have*

$$\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\kappa\epsilon}(\mathbf{L}_1)\cup\Lambda_{\kappa\epsilon}(\mathbf{L}_2).$$

(*c*) *Let X and Y be non-singular such that* $Y^{-1}\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1,\mathbf{L}_2)$. *Let* $X := \begin{bmatrix}X_1 & X_2\end{bmatrix}$ *and* $(Y^{-1})^* := \begin{bmatrix}Y_1 & Y_2\end{bmatrix}$ *be conformal partitions. Set* $\kappa := \|X_1\|\|Y_1^*\| + \|X_2\|\|Y_2^*\|$. *Then we have*

$$\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\kappa\epsilon}(\mathbf{L}_1)\cup\Lambda_{\kappa\epsilon}(\mathbf{L}_2).$$

(*d*) *Let X and Y be non-singular such that* $Y^{-1}\mathbf{L}X = \operatorname{diag}(\mathbf{L}_1, \ldots, \mathbf{L}_k)$. *Let* $X := \begin{bmatrix} X_1 & \cdots & X_k \end{bmatrix}$ *and* $(Y^{-1})^* := \begin{bmatrix} Y_1 & \cdots & Y_k \end{bmatrix}$ *be conformal partitions. Set* $\phi_j(\epsilon) := k\|X_j\|\,\|Y_j^*\|\epsilon$. *Then we have*

$$\Lambda_\epsilon(\mathbf{L}) \subseteq \cup_{j=1}^{k} \Lambda_{\phi_j(\epsilon)}(\mathbf{L}_j).$$

Proof: Without loss of generality we prove the results for non-homogeneous pencils. The proof of (a) follows from the fact that $\eta_{w,p}(\lambda, Y^{-1}\mathbf{L}X) \leq \|Y^{-1}\|\,\|X\|\,\eta_{w,p}(\lambda, \mathbf{L})$ and $\eta_{w,p}(\lambda, \mathbf{L}) \leq \|Y\|\,\|X^{-1}\|\,\eta_{w,p}(\lambda, Y^{-1}\mathbf{L}X)$.
By (*a*), we have that $\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\kappa\epsilon}(Y^{-1}\mathbf{L}X)$, where $\kappa = \|Y^{-1}\|\,\|X\|$. Now from Theorem 4.2.2, we have $\Lambda_{\kappa\epsilon}(Y^{-1}\mathbf{L}X) = \Lambda_{\kappa\epsilon}(\mathbf{L}_1) \cup \Lambda_{\kappa\epsilon}(\mathbf{L}_2)$. Thus $\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\kappa\epsilon}(\mathbf{L}_1) \cup \Lambda_{\kappa\epsilon}(\mathbf{L}_2)$.
Next, we have $\mathbf{L}(z) = Y\operatorname{diag}(\mathbf{L}_1(z),\, \mathbf{L}_2(z))X^{-1} \Rightarrow \mathbf{L}(z)^{-1} = X\operatorname{diag}(\mathbf{L}_1(z)^{-1},\, \mathbf{L}_2(z)^{-1})Y^{-1}$ $= X_1\mathbf{L}_1(z)^{-1}Y_1^* + X_2\mathbf{L}_2(z)^{-1}Y_2^* \Rightarrow \|\mathbf{L}(z)^{-1}\| \leq \max(\|\mathbf{L}_1(z)^{-1}\|, \|\mathbf{L}_2(z)^{-1}\|)(\|X_1\|\|Y_1^*\| + \|X_2\|\|Y_2^*\|) = \max(\|\mathbf{L}_1(z)^{-1}\|, \|\mathbf{L}_2(z)^{-1}\|)\kappa$. Hence the proof follows .
Finally, $\mathbf{L}(z)^{-1} = \sum_{j=1}^{k} X_j\mathbf{L}_j(z)^{-1}Y_j^* \Rightarrow \|\mathbf{L}(z)^{-1}\| \leq k \max\limits_{1\leq j\leq k} \|X_j\|\|Y_j^*\|\|\mathbf{L}_j(z)^{-1}\|$. This implies that $\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\phi_j(\epsilon)}(\mathbf{L})$, where $\phi_j(\epsilon) = k\|X_j\|\,\|Y_j^*\|$. ■

For the special case when $\mathbf{L}$ is simple, we have the following result.

Corollary 4.2.5. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be given by* $\mathbf{L}(z) = A - zB$ *or* $\mathbf{L}(c, s) = cA - sB$. *Suppose that* $Y^*AX = \operatorname{diag}(\alpha_i)$ *and* $Y^*BX = \operatorname{diag}(\beta_i)$. *Set* $y_i := Ye_i$ *and* $x_i := Xe_i$. *Then* $(\alpha_i/\beta_i, y_i, x_i)$ *is an eigentriple of* $\mathbf{L}(z) = A - zB$ *and* $((\beta_i, \alpha_i), y_i, x_i)$ *is an eigentriple of* $\mathbf{L}(c, s) = cA - sB$. *Further, we have*

$$\Lambda_\epsilon(\mathbf{L}) \subseteq \cup_{i=1}^{n} \Lambda_{\kappa\epsilon}(\mathbf{L}_i) \text{ and } \Lambda_\epsilon(\mathbf{L}) \subseteq \cup_{i=1}^{n} \Lambda_{n\kappa_i\epsilon}(\mathbf{L}_i),$$

where $\mathbf{L}_i(z) = \alpha_i - z\beta_i$ *or* $\mathbf{L}(c, s) = c\alpha_i - s\beta_i$ *and* $\kappa := \|Y^*\|\|X\|$, $\kappa_i := \|x_i\|\|y_i\|$.

If $\mathbf{L}(z) = A - zB$ is diagonalizable and B is non-singular then we can choose X and Y such that $Y^*BX = I$ and $Y^*AX = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Consequently, we have the following special case.

Corollary 4.2.6. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be given by* $\mathbf{L}(z) = A - zB$. *Suppose that* B *is non-singular. Let* Y *and* X *be such that* $Y^*BX = I$ *and* $Y^*AX = \operatorname{diag}(\lambda_i)$. *Let* $y_i := Ye_i$ *and* $x_i := Xe_i$. *Then* (λ_i, x_i, y_i) *is an eigentriple of* $\mathbf{L}$. *Further, we have*

$$\Lambda_\epsilon(\mathbf{L}) \subseteq \cup_{i=1}^{n} \Lambda_{\kappa\epsilon}(\mathbf{L}_i) \text{ and } \Lambda_\epsilon(\mathbf{L}) \subseteq \cup_{i=1}^{n} \Lambda_{n\kappa_i\epsilon}(\mathbf{L}_i),$$

where $\mathbf{L}_i(z) = z - \lambda_i$, $\kappa := \|Y^*\|\,\|X\|$ *and* $\kappa_i := \|x_i\|\,\|y_i\|$.

To obtain further inclusions for pseudospectra, we need the following result.

Lemma 4.2.7. *[28, 24] Let* $\begin{bmatrix} I_1 & M \\ 0 & I_2 \end{bmatrix} \in \mathbb{C}^{n\times n}$*, where* I_1 *and* I_2 *are the identity matrices of size* k *and* $n-k$*, respectively, and* $M \in \mathbb{C}^{k\times(n-k)}$*. Then*

$$\left\| \begin{bmatrix} I_1 & M \\ 0 & I_2 \end{bmatrix} \right\|_2 = \sqrt{1 + \frac{1}{2}(\|M\|_2^2 + \sqrt{\|M\|_2^4 + 4\|M\|_2^2})}.$$

We now generalize this result and obtain the following.

Lemma 4.2.8. *Let* $Z = \begin{bmatrix} \alpha I_1 & M \\ 0 & \beta I_2 \end{bmatrix} \in \mathbb{C}^{n\times n}$*, where* I_1 *and* I_2 *are the identity matrices of size* k *and* $n-k$*, respectively,* $M \in \mathbb{C}^{k\times(n-k)}$ *and* α, β *are nonzero positive real numbers. Then*

$$\sigma_{\max}(Z) = \sqrt{\frac{(\alpha^2+\beta^2+\|M\|_2^2) + \sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2 - 4\alpha^2\beta^2}}{2}},$$

$$\sigma_{\min}(Z) = \sqrt{\frac{(\alpha^2+\beta^2+\|M\|_2^2) - \sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2 - 4\alpha^2\beta^2}}{2}},$$

$$\mathrm{cond}(Z) = \frac{\sigma_{\max}(Z)}{\sigma_{\min}(Z)} = \frac{(\alpha^2+\beta^2+\|M\|_2^2) + \sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2 - 4\alpha^2\beta^2}}{2\alpha\beta} \quad \textit{and}$$

$\sigma_{\max}(Z)\sigma_{\min}(Z) = \alpha\beta$*, where* $\sigma_{\min}(Z)$ *and* $\sigma_{\max}(Z)$ *are the smallest and the largest singular values of* Z*, respectively.*

Proof: We have $\|Z\|_2^2 = \rho(Z^*Z)$, where $\rho(Z^*Z) = \sup\limits_{\|(x,y)\|_2=1} \left\{ \begin{bmatrix} x^* & y^* \end{bmatrix} Z^*Z \begin{bmatrix} x \\ y \end{bmatrix} ; \; x \in \mathbb{R}^k, \; y \in \mathbb{R}^{n-k} \right\}$. Now,

$$\begin{aligned}
\begin{bmatrix} x^* & y^* \end{bmatrix} Z^*Z \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x^* & y^* \end{bmatrix} \begin{bmatrix} \alpha^2 & \alpha M \\ \alpha M & \beta^2 + M^*M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} x^* & y^* \end{bmatrix} \begin{bmatrix} \alpha^2 x + \alpha M y \\ \alpha M^* x + (\beta^2 + M^* M) y \end{bmatrix} \\
&= \alpha^2 \|x\|_2^2 + \beta^2 \|y\|_2^2 + \alpha x^* M y + \alpha y^* M^* y + \alpha y^* M^* x + y^* M^* M y \\
&= \alpha^2 \|x\|_2^2 + \beta^2 \|y\|_2^2 + 2\,Re(\alpha\, x^*\, M\, y) + \|y\|_2^2 \|M\|_2^2.
\end{aligned}$$

Note that there exists vectors x and y such that $\|x\|_2 = 1 = \|y\|_2$ and $x^*My = \|My\|_2 = \|M\|_2$. Let $(x_1, x_2) := \left[\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right]$. Then $\|(x_1,x_2)\|_2 = 1$ and $\begin{bmatrix} x_1^* & x_2^* \end{bmatrix} Z^*Z \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\{\alpha^2\,\|x_1\|_2^2+\beta^2\,\|x_2\|_2^2+2\alpha\|x_1\|_2\|M\|_2\,\|x_2\|_2+\|x_2\|_2^2\|M\|_2^2\}$. Consequently, we have

$$\begin{aligned}
\rho(Z^*Z) &= \sup_{\|(x,y)\|_2=1}\{\alpha^2\,\|x\|_2^2+\beta^2\,\|y\|_2^2+2\alpha\|x\|_2\|M\|_2\,\|y\|_2+\|y\|_2^2\|M\|_2^2\}\\
&= \rho\begin{bmatrix}\alpha^2 & \alpha\,\|M\|_2\\ \alpha\,\|M\|_2 & \|M\|_2^2+\beta^2\end{bmatrix}\\
&= \frac{(\alpha^2+\beta^2+\|M\|_2^2)+\sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2-4\alpha^2\beta^2}}{2}\\
\Rightarrow \rho(Z^*Z) &= \frac{(\alpha^2+\beta^2+\|M\|_2^2)+\sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2-4\alpha^2\beta^2}}{2}\\
\Rightarrow \sigma_{\max}(Z) &= \|Z\|_2=\sqrt{\frac{(\alpha^2+\beta^2+\|M\|_2^2)+\sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2-4\alpha^2\beta^2}}{2}}. \qquad (4.3)
\end{aligned}$$

Now, we have $\sigma_{\min}(Z)=\dfrac{1}{\sigma_{\max}(Z^{-1})}$, where $Z^{-1}=\begin{bmatrix}\alpha^{-1}I_1 & -\alpha^{-1}M\beta^{-1}\\ 0 & \beta^{-1}I_2\end{bmatrix}$. Thus

$$\sigma_{\max}(Z^{-1})=\frac{\sqrt{2}}{\sqrt{(\alpha^2+\beta^2+\|M\|_2^2)-\sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2-4\alpha^2\beta^2}}}.$$

Therefore, we have

$$\sigma_{\min}(Z)=\sqrt{\frac{(\alpha^2+\beta^2+\|M\|_2^2)-\sqrt{(\alpha^2+\beta^2+\|M\|_2^2)^2-4\alpha^2\beta^2}}{2}}. \qquad (4.4)$$

From (4.3) and (4.4) we have $\sigma_{\min}(Z)\sigma_{\max}(Z)=\alpha\beta$. Hence the proof. ■

Now, we have the following result which will be useful in deriving pseudospectra inclusions.

Theorem 4.2.9. *Set* $P:=\sqrt{1+\|Z_1\|_2^2}$, $Q:=\sqrt{1+\|Z_2\|_2^2}$ *and let*

$$X:=\begin{bmatrix}I_1 & Z_2\\ 0 & I_2\end{bmatrix}\begin{bmatrix}Q^{1/2} & 0\\ 0 & P^{-1/2}\end{bmatrix}, Y:=\begin{bmatrix}I_1 & Z_1\\ 0 & I_2\end{bmatrix}\begin{bmatrix}Q^{1/2} & 0\\ 0 & P^{-1/2}\end{bmatrix}. \textit{ Then}$$

$\|X\|_2=\sqrt{Q/P}\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}}$, $\|X^{-1}\|_2=\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}}$,

$\|Y\|_2=\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}}$, $\|Y^{-1}\|_2=\sqrt{P/Q}\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}}$ *and*

$\mathrm{cond}(X)=\sqrt{Q/P}\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}$, $\mathrm{cond}(Y)=\sqrt{P/Q}\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}$.

Further, we have $\mathrm{cond}(Y,X)=\mathrm{cond}(X,Y)=\sqrt{\mathrm{cond}(X)\mathrm{cond}(Y)}\leq P+Q$.

Proof: We have $X=\begin{bmatrix}Q^{1/2}I_1 & P^{-1/2}Z_2\\ 0 & P^{-1/2}I_2\end{bmatrix}$. Then for $\alpha=Q^{1/2},\beta=P^{-1/2},M=P^{-1/2}Z_2$, by Lemma 4.2.8 we obtain $\mathrm{cond}(X)=\sqrt{Q/P}\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}$, $\|X\|_2=\sqrt{Q/P}\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}}$ and $\|X^{-1}\|_2=\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}}$.

Similarly, for $Y = \begin{bmatrix} Q^{1/2}I_1 & P^{-1/2}Z_1 \\ 0 & P^{-1/2}I_2 \end{bmatrix}$, by Lemma 4.2.8 we obtain $\mathrm{cond}(Y) = \sqrt{P/Q}\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}$, $\|Y\|_2 = \sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}}$ and

$$\|Y^{-1}\|_2 = \sqrt{P/Q}\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}}.$$

Again by Lemma 4.2.8, we have $\sigma_{\min}(X)\sigma_{\max}(X) = \sigma_{\min}(Y)\sigma_{\max}(Y) = \alpha\beta = \sqrt{\frac{Q}{P}} \Rightarrow \|X\|_2\ \|Y^{-1}\|_2 = \|Y\|_2\ \|X^{-1}\|_2 \Rightarrow \mathrm{cond}(Y,X) = \mathrm{cond}(X,Y)$. Now

$$\begin{aligned} \mathrm{cond}(X,Y) &= \sqrt{\mathrm{cond}(X,Y)\mathrm{cond}(Y,X)} \\ &= \sqrt{\|X^{-1}\|_2\ \|Y\|_2\ \|Y^{-1}\|_2\ \|X\|_2} \\ &= \sqrt{\mathrm{cond}(X)\mathrm{cond}(Y)}. \end{aligned}$$

Now we show that $\mathrm{cond}(X,Y) \leq P+Q$. We have

$$\begin{aligned} \mathrm{cond}(X,Y) &= \sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}}\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}} \\ &\leq \frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{4} + \frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{4} \\ &\leq \frac{(P+Q)}{2} + \frac{P+Q}{4} + \frac{P+Q}{4} = P+Q. \end{aligned}$$

Hence the proof. ■

Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be a regular pencil given by $\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix}$, where $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_{12}$ are pencils of size $m, n-m$ and $m \times n-m$, respectively, and $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. Now, consider the Sylvester operator

$$S : \mathbb{C}^{m\times n-m} \times \mathbb{C}^{m\times n-m} \to \mathbb{L}_w^p(\mathbb{C}^{m\times n-m}, \|\cdot\|)$$

defined by $S(Z_1, Z_2) := \mathbf{L}_1 Z_1 - Z_2 \mathbf{L}_2$. Then as we shall see, $S(Z_1, Z_2) = -\mathbf{L}_{12}$ has a unique solution. So let Z_1 and Z_2 be unique solution of $S(Z_1, Z_2) = -\mathbf{L}_{12}$. Set $P := \sqrt{1+\|Z_1\|^2}$ and $Q := \sqrt{1+\|Z_2\|^2}$. Define

$$X := \begin{bmatrix} Q^{1/2}I_1 & P^{-1/2}Z_2 \\ 0 & P^{-1/2}I_2 \end{bmatrix} \text{ and } Y := \begin{bmatrix} Q^{1/2}I_1 & P^{-1/2}Z_1 \\ 0 & P^{-1/2}I_2 \end{bmatrix}. \tag{4.5}$$

Then we have $Y^{-1}\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1, \mathbf{L}_2)$.

Theorem 4.2.10. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be a regular pencil given by* $\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix}$, *where* $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. *Let* Z_1 *and* Z_2 *be solution of the generalized Sylvester equation* $\mathbf{L}_1 Z_1 - Z_2 \mathbf{L}_2 = -\mathbf{L}_{12}$. *Set* $P := \sqrt{1+\|Z_1\|^2}$ *and* $Q := \sqrt{1+\|Z_2\|^2}$. *Define*

$$\kappa_1 := \sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4Q/P}}{2}}\sqrt{\frac{(P+Q)+\sqrt{(P+Q)^2-4P/Q}}{2}}$$

and $\kappa_2 := P + Q$. *Then we have*

$$\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\kappa_1\epsilon}(\mathbf{L}_1) \cup \Lambda_{\kappa_1\epsilon}(\mathbf{L}_2) \subseteq \Lambda_{\kappa_2\epsilon}(\mathbf{L}_1) \cup \Lambda_{\kappa_2\epsilon}(\mathbf{L}_2).$$

Proof: Defining X and Y as in (4.5), we have $Y^{-1}\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1, \mathbf{L}_2)$. By Theorem 4.2.9, we have $\mathrm{cond}(X, Y) = \kappa_1 \leq \kappa_2$. Hence the result follows from Theorem 4.2.4. ■

Now, consider the matrix $A := \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}$, where $A_1 \in \mathbb{C}^{m\times m}, A_2 \in \mathbb{C}^{(n-m)\times(n-m)}$. Set $R(z) := \|(A - zI)^{-1}\|_2,\ R_1(z) := \|(A_1 - zI)^{-1}\|_2$ and $R_2(z) := \|(A_2 - zI)^{-1}\|_2$. Then the following holds.

Proposition 4.2.11. *[24] Let* $a_{12} := \|A_{12}\|_2,\ r_m(z) := \min\left(R_1(z),\ R_2(z)\right)$ *and* $r_M(z) := \max\left(R_1(z),\ R_2(z)\right)$. *Then we have*

$$\|R(z)\|_2 \leq \sqrt{1 + \frac{a_{12} r_m(z)}{2}\left((a_{12}^2 r_m(z)^2 + 4)^{1/2} + a_{12} r_m(z)\right)}.$$

Then we have following pseudospectra inclusion. Recall that $h_{w,p}(z) = \|(1, z)\|_{w,p}$.

Theorem 4.2.12. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be given by* $\mathbf{L} := \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix}$, *where* $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. *Set* $g_1(\epsilon) := \epsilon\sqrt{1 + \dfrac{\kappa}{\epsilon}}$ *and* $\kappa := \|\mathbf{L}_{12}\|_{w,p}$. *Then for all* $\epsilon > 0$ *we have*

$$\Lambda_\epsilon(\mathbf{L}) \subset \Lambda_{g_1(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{g_1(\epsilon)}(\mathbf{L}_2).$$

Proof: Without loss of generality, we consider $\mathbf{L}$ to be nonhomogeneous. So, let $r_M(\lambda) = \max\left(\|\mathbf{L}_1(\lambda)^{-1}\|, \|\mathbf{L}_2(\lambda)^{-1}\|\right), r_m(\lambda) = \min\left(\|\mathbf{L}_1(\lambda)^{-1}\|, \|\mathbf{L}_2(\lambda)^{-1}\|\right)$. Then by Proposition 4.2.11,

$$\|\mathbf{L}(\lambda)^{-1}\| \leq r_M(\lambda)\sqrt{1 + \frac{\|\mathbf{L}_{12}(\lambda)\| r_m(\lambda)}{2}\left((\|\mathbf{L}_{12}(\lambda)\|^2 r_m(\lambda)^2 + 4)^{1/2} + \|\mathbf{L}_{12}(\lambda)\| r_m(\lambda)\right)}.$$

Now, $(\|\mathbf{L}(\lambda)^{-1}\|) \geq \dfrac{1}{\epsilon h_{w^{-1},q}(\lambda)}$ gives

$$\begin{aligned} &\frac{1}{\epsilon^2 h_{w^{-1},q}(\lambda)^2} \leq r_M(\lambda)^2\left(1 + \frac{\|\mathbf{L}_{12}(\lambda)\| r_m(\lambda)}{2}\left((\|\mathbf{L}_{12}(\lambda)\|^2 r_m(\lambda)^2 + 4)^{1/2} + \|\mathbf{L}_{12}(\lambda)\| r_m(\lambda)\right)\right) \\ &\leq r_M(\lambda)^2\left(1 + \frac{\|\mathbf{L}_{12}(\lambda)\| r_M(\lambda)}{2}\left((\|\mathbf{L}_{12}(\lambda)\|^2 r_M(\lambda)^2 + 4)^{1/2} + \|\mathbf{L}_{12}(\lambda)\| r_M(\lambda)\right)\right) \leq r_M(\lambda)^2 + \\ &\frac{\kappa h_{w^{-1},q}(\lambda) r_M(\lambda)^3}{2}\left(\kappa^2 h_{w^{-1},q}(\lambda)^2 r_M(\lambda)^2 + 4\right)^{1/2} + \frac{\kappa^2 h_{w^{-1},q}(\lambda)^2 r_M(\lambda)^4}{2}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{1}{\epsilon^2 h_{w^{-1},q}(\lambda)^2} - r_M(\lambda)^2\left(1 + \frac{\kappa^2 h_{w^{-1},q}(\lambda)^2 r_M(\lambda)^2}{2}\right) \\ &\leq \frac{\kappa h_{w^{-1},q}(\lambda) r_M(\lambda)^3}{2}\left(\kappa^2 h_{w^{-1},q}(\lambda)^2 r_M(\lambda)^2 + 4\right)^{1/2}. \end{aligned}$$

For simplicity of notations, set $\mathrm{d} := h_{w^{-1},q}(\lambda), r_M := r_M(\lambda)$ and recall that $\kappa = \|\mathbf{\Delta L}_{12}\|_{w,p}$. Then we have

$$\frac{1}{\epsilon^2\mathrm{d}^2} - r_M^2\left(1+\frac{\kappa^2\mathrm{d}^2r_M^2}{2}\right) \leq \frac{\kappa \mathrm{d} r_M^3}{2}\left(\sqrt{\kappa^2\mathrm{d}^2r_M^2+4}\right). \tag{4.6}$$

Now two choices arise, namely.
(a) $\frac{1}{\epsilon^2\mathrm{d}^2} - r_M^2\left(1+\frac{\kappa^2\mathrm{d}^2r_M^2}{2}\right) \geq 0$ and **(b)** $\frac{1}{\epsilon^2\mathrm{d}^2} - r_M^2\left(1+\frac{\kappa^2\mathrm{d}^2r_M^2}{2}\right) \leq 0.$

Now under the case (a) squaring both sides of (4.6) we get $\frac{1}{\epsilon^4\mathrm{d}^4} + r_M^4\left(1+\frac{\kappa^2\mathrm{d}^2r_M^2}{2}\right)^2 - \frac{2}{\epsilon^2}\mathrm{d}^2r_M^2\left(1+\frac{\kappa^2\mathrm{d}^2r_M^2}{2}\right) \leq \frac{\kappa^2\mathrm{d}^2r_M^6}{4}(\kappa^2\mathrm{d}^2r_M^2+4) \Rightarrow \frac{1}{\epsilon^4\mathrm{d}^4} + r_M^4 - \frac{2}{\epsilon^2\mathrm{d}^2}r_M^2 - \frac{\kappa^2}{\epsilon^2}r_M^4 \leq 0 \Rightarrow \left(r_M^2 - \frac{1}{\epsilon^2\mathrm{d}^2}\right)^2 \leq r_M^4\frac{\kappa^2}{\epsilon^2} \Rightarrow \left[\left(r_M^2 - \frac{1}{\epsilon^2\mathrm{d}^2}\right) - r_M^2\frac{\kappa}{\epsilon}\right]\left[\left(r_M^2 - \frac{1}{\epsilon^2\mathrm{d}^2}\right) + r_M^2\frac{\kappa}{\epsilon}\right] \leq 0$. Hence we have

$$\left[r_M^2\left(1-\frac{\kappa}{\epsilon}\right) - \frac{1}{\epsilon^2\mathrm{d}^2}\right]\left[r_M^2\left(1+\frac{\kappa}{\epsilon}\right) - \frac{1}{\epsilon^2\mathrm{d}^2}\right] \leq 0. \tag{4.7}$$

Case-I: Let $\kappa \leq \epsilon$. Then either

$$r_M^2\left(1-\frac{\kappa}{\epsilon}\right) \leq \frac{1}{\epsilon^2\mathrm{d}^2} \text{ and } r_M^2\left(1+\frac{\kappa}{\epsilon}\right) \geq \frac{1}{\epsilon^2\mathrm{d}^2}$$

or

$$r_M^2\left(1-\frac{\kappa}{\epsilon}\right) \geq \frac{1}{\epsilon^2\mathrm{d}^2} \text{ and } r_M^2\left(1+\frac{\kappa}{\epsilon}\right) \leq \frac{1}{\epsilon^2\mathrm{d}^2}.$$

In the first case we have $\frac{1}{\epsilon\sqrt{1+\kappa/\epsilon}} \leq r_M\mathrm{d} \leq \frac{1}{\epsilon\sqrt{1-\kappa/\epsilon}}$ and in the second case, we have $\frac{1}{\epsilon\sqrt{1-\kappa/\epsilon}} \leq r_M\mathrm{d} \leq \frac{1}{\epsilon\sqrt{1+\kappa/\epsilon}}$, which is not possible.

Case-II: Let $\kappa \geq \epsilon$. Then $\left[r_M^2\left(1-\frac{\kappa}{\epsilon}\right) - \frac{1}{\epsilon^2\mathrm{d}}\right] \leq 0$. Hence from (4.7) we have

$$\left[r_M^2\left(1+\frac{\kappa}{\epsilon}\right) - \frac{1}{\epsilon^2\mathrm{d}}\right] \geq 0 \Rightarrow r_M\mathrm{d} \geq \frac{1}{\epsilon\sqrt{1+\kappa/\epsilon}}.$$

This shows that $r_M\mathrm{d} \geq \frac{1}{\epsilon\sqrt{1+\kappa/\epsilon}}$ for all ϵ.

Next consider the case **(b)**. In this case, we have $\frac{1}{\epsilon^2\mathrm{d}^2} - r_M^2\left(1+\frac{\kappa^2\mathrm{d}^2r_M^2}{2}\right) \leq 0$ which gives $r_M(z)^2 \geq \frac{-1+\sqrt{1+2\frac{\kappa^2}{\epsilon^2}}}{\kappa^2\mathrm{d}^2}$. We now show that for all $\epsilon \geq 0$, we have

$$\frac{-1+\sqrt{1+2\frac{\kappa^2}{\epsilon^2}}}{\kappa^2\mathrm{d}^2} \geq \frac{1}{\epsilon^2\mathrm{d}^2(1+\kappa/\epsilon)}.$$

Indeed, $-1+\sqrt{1+2\frac{\kappa^2}{\epsilon^2}} \geq \frac{\kappa^2}{\epsilon^2(1+\kappa/\epsilon)}$ implies that $1 \geq \left[1+\frac{\kappa^2}{\epsilon^2(1+\frac{\kappa}{\epsilon})}\right]^2 - 2\kappa^2/\epsilon^2 \Rightarrow \frac{-2\kappa^3}{\epsilon^2(\kappa+\epsilon)} + \frac{\kappa^4}{\epsilon^2(\epsilon+\kappa)^2} \leq 0 \Rightarrow \kappa + 2\epsilon \geq 0$ which is obviously true. Hence we have

$$r_M(z)^2 \geq \frac{-1+\sqrt{1+2\frac{\kappa^2}{\epsilon^2}}}{\kappa^2 \mathrm{d}^2} \geq \frac{1}{\epsilon^2 \mathrm{d}^2(1+\kappa/\epsilon)} \text{ for all } \epsilon \geq 0.$$

This shows that $r_M \mathrm{d} \geq \frac{1}{\epsilon\sqrt{1+\kappa/\epsilon}} = \frac{1}{g_1(\epsilon)}$. Consequently, we have

$$\Lambda_\epsilon(\mathbf{L}) \subset \Lambda_{g_1(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{g_1(\epsilon)}(\mathbf{L}_2) \text{ for all } \epsilon. \blacksquare$$

For $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$, we have the following pseudospectra inclusion.

Theorem 4.2.13. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *be given by* $\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix}$, *where* $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. *Let* (Z_1, Z_2) *be the solution of the generalized Sylvester equation* $\mathbf{L}_1 Z_1 - Z_2 \mathbf{L}_2 = -\mathbf{L}_{12}$. *Set* $\phi(\epsilon) := \epsilon(1+\|\mathrm{Z}_1\|+\|\mathrm{Z}_2\|)$ *and* $f(\epsilon) := \frac{\epsilon+\sqrt{\epsilon^2+4\epsilon\,\kappa}}{2}$, *where* $\kappa := \|\mathbf{L}_{12}\|_{w,p}$. *Then we have*

$$(a)\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\phi(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{\phi(\epsilon)}(\mathbf{L}_2), \;\; (b)\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{f(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{f(\epsilon)}(\mathbf{L}_2).$$

Proof: Without loss of generality, assume that $\mathbf{L}$ is nonhomogeneous. For $\lambda \in \mathbb{C}$, set $\mathrm{d}(\lambda) := \max\left(\|\mathbf{L}_1(\lambda)^{-1}\|,\ \|\mathbf{L}_2(\lambda)^{-1}\|\right)$. Then

$$\mathbf{L}^{-1}(\lambda) = \begin{bmatrix} \mathbf{L}_1(\lambda)^{-1} & -\mathbf{L}_1(\lambda)^{-1}\mathbf{L}_{12}(\lambda)\mathbf{L}_2(\lambda)^{-1} \\ 0 & \mathbf{L}_2(\lambda)^{-1} \end{bmatrix}.$$

Consequently, we have

$$\|\mathbf{L}^{-1}(\lambda)\| \leq \max\left(\|\mathbf{L}_1(\lambda)^{-1}\|, \|\mathbf{L}_2(\lambda)^{-1}\|\right) + \left\|\mathbf{L}_1(\lambda)^{-1}\,\mathbf{L}_{12}(\lambda)\,\mathbf{L}_2(\lambda)^{-1}\right\|.$$

Since $\mathbf{L}_1(\lambda)Z_1 - Z_2\mathbf{L}_2(\lambda) = -\mathbf{L}_{12}(\lambda)$, we have

$$\begin{aligned} \left\|\mathbf{L}_1(\lambda)^{-1}\,\mathbf{L}_{12}(\lambda)\,\mathbf{L}_2(\lambda)^{-1}\right\| &= \left\|\mathbf{L}_1(\lambda)^{-1}\,(\mathrm{Z}_2\mathbf{L}_2(\lambda) - \mathbf{L}_1(\lambda)\mathrm{Z}_1)\,\mathbf{L}_2(\lambda)^{-1}\right\| \\ &\leq \mathrm{d}(\lambda)\,(\|\mathrm{Z}_1\| + \|\mathrm{Z}_2\|) \end{aligned}$$

and $\|\mathbf{L}(\lambda)^{-1}\| \leq \mathrm{d}(\lambda) + \mathrm{d}(\lambda)\,(\|\mathrm{Z}_1\|+\|\mathrm{Z}_2\|) = \mathrm{d}(\lambda)\,(1+\|\mathrm{Z}_1\|+\|\mathrm{Z}_2\|)$.

Now, $\|\mathbf{L}(\lambda)^{-1}\| \geq (h_{w^{-1},q}(\lambda)\epsilon)^{-1}$ gives $(h_{w^{-1},q}(\lambda)\epsilon)^{-1} \leq \|\mathbf{L}(\lambda)^{-1}\| \leq \mathrm{d}(\lambda)(1+\|\mathrm{Z}_1\|+\|\mathrm{Z}_2\|) = \mathrm{d}(\lambda)\phi(\epsilon) \Rightarrow \mathrm{d}(\lambda)h_{w^{-1},q}(\lambda) \geq \frac{1}{\epsilon(1+\|\mathrm{Z}_1\|+\|\mathrm{Z}_2\|)} = \frac{1}{\phi(\epsilon)}$. Hence $\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\phi(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{\phi(\epsilon)}(\mathbf{L}_2)$.

Next, we have $\|\mathbf{L}(\lambda)^{-1}\| \leq \mathrm{d}(\lambda) + \mathrm{d}(\lambda)^2\,\kappa\,h_{w^{-1},q}(\lambda) \Rightarrow \epsilon^{-1} \leq \mathrm{d}(\lambda)\,h_{w^{-1},q}(\lambda) + (\mathrm{d}(\lambda)\,h_{w^{-1},q}(\lambda))^2\kappa$. Now, set $\mathrm{d}'(\lambda) = \mathrm{d}(\lambda)h_{w^{-1},q}(\lambda)$. Then $\epsilon\,\mathrm{d}'(\lambda) + \epsilon\,\kappa\,\mathrm{d}'(\lambda)^2 - 1 \geq 0$ which implies that $\mathrm{d}'(\lambda) \geq \frac{-\epsilon+\sqrt{\epsilon^2+4\,\epsilon\,\kappa}}{2\epsilon\,\kappa} = \frac{2}{\epsilon+\sqrt{\epsilon^2+4\epsilon\,\kappa}} \Rightarrow \mathrm{d}(\lambda)h_{w^{-1},q}(\lambda) \geq \frac{1}{f(\epsilon)}$. Thus $\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{f(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{f(\epsilon)}(\mathbf{L}_2)$. $\blacksquare$

4.3 Separation of pencils

Various notions of separation of matrices such as $\mathrm{sep}, \mathrm{sep}_\lambda$ and gsep have been introduced in the literature [43, 48, 15, 3] for perturbation analysis of eigenvalues and invariant subspaces of matrices. To deal with perturbations of matrix pencils, the notions of separation such as dif and dif_λ have been introduced in the literature [40, 43, 16]). We mention, however, that dif and dif_λ are defined when the norm involved is either the spectral norm or the Frobenius norm.

We now introduce and systematically develop various notions of separations such as $\mathrm{sep}, \mathrm{sep}_\lambda$ and gsep for matrix pencils. Our separations are defined for any norm and are natural generalizations of separations of matrices. We show that dif is a special case of sep and relate dif_λ with sep_λ. For a special choice of norm, we show that $\mathrm{dif}_\lambda = \sqrt{2}\,\mathrm{sep}_\lambda$.

First, we define the separation sep between two pencils. To that end, we briefly review the definition of the separation sep between matrices. Given $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$, consider the Sylvester operator $\mathbf{T} : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}, X \mapsto AX - XB$. Then the separation of A and B, denoted by $\mathrm{sep}(A, B)$, is given by [43]

$$\mathrm{sep}(A, B) := \min_{\|X\|=1} \|\mathbf{T}(X)\|.$$

It is well known that $\mathrm{sep}(A, B) = 0$ if and only if $\Lambda(A) \cap \Lambda(B) \neq \emptyset$. Consequently, $\mathbf{T}$ is invertible if and only if $\mathrm{sep}(A, B) \neq 0$. We now generalize the notion of separation to the case of matrix pencils. We proceed as follows.

We define the left action of $\mathbb{C}^{m\times n}$ on $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$. The left action of $\mathbb{C}^{m\times n}$ on $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ is the map

$$\mathbb{C}^{m\times n} \times \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|) \longrightarrow \mathbb{L}^p_w(\mathbb{C}^{m\times n}, \|\cdot\|), (X, \mathbf{L}) \mapsto X\mathbf{L}$$

given by $X\mathbf{L}(z) = XA - zXB$, where $\mathbf{L}(z) := A - zB$. Then it is easy to check that $|||X\mathbf{L}|||_{w,p} \leq \|X\|\, |||\mathbf{L}|||_{w,p}$. Similarly, the right action of $\mathbb{C}^{n\times k}$ on $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ is the map

$$\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|) \times \mathbb{C}^{n\times k} \longrightarrow \mathbb{L}^p_w(\mathbb{C}^{n\times k}, \|\cdot\|), (\mathbf{L}, Y) \mapsto \mathbf{L}Y$$

given by $\mathbf{L}Y(z) := AY - zBY$, where $\mathbf{L}(z) := A - zB$. Then again it is easily seen that $|||\mathbf{L}Y|||_{w,p} \leq |||\mathbf{L}|||_{w,p}\, \|Y\|$.

Now, we define generalized Sylvester operator. Let $\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ and $\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$. We define the generalized Sylvester operator as

$$\mathcal{S} : \mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n} \longrightarrow \mathbb{L}^p_w(\mathbb{C}^{m\times n}, \ \|\cdot\|), \ (Z_1, Z_2) \longmapsto \mathbf{L}_1 Z_1 - Z_2 \mathbf{L}_2.$$

Then it follows that $\mathcal{S}$ is linear. Further, when $\mathbf{L}_1$ and $\mathbf{L}_2$ are regular pencils, it is easy to see that $\mathcal{S}$ is injective if and only if $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) \neq \emptyset$.

It is well known that Sylvester operator plays an important role in block diagonalizing a matrix. Similarly, generalized Sylvester operator plays an important role in block diagonalizing a matrix pencil. Indeed, $\mathbf{L}$ be a regular pencil given $\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix}$. Let $X := \begin{bmatrix} I & Z_1 \\ 0 & I \end{bmatrix} \in \mathbb{C}^{n\times n}$ and $Y := \begin{bmatrix} I & Z_2 \\ 0 & I \end{bmatrix} \in \mathbb{C}^{n\times n}$ be nonsingular matrices such that $Y^{-1}\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1, \mathbf{L}_2)$. Then we have

$$\begin{bmatrix} I & -Z_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix} \begin{bmatrix} I & Z_1 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 & 0 \\ 0 & \mathbf{L}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_1 Z_1 - Z_2\mathbf{L}_2 + \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 & 0 \\ 0 & \mathbf{L}_2 \end{bmatrix}.$$

This shows that $\mathbf{L}_1 Z_1 - Z_2\mathbf{L}_2 = -\mathbf{L}_{12}$, that is, $\mathcal{S}(Z_1, Z_2) = -\mathbf{L}_{12}$. Thus, when the spectra of $\mathbf{L}_1$ and $\mathbf{L}_2$ are disjoint, the generalized Sylvester equation $\mathcal{S}(Z_1, Z_2) = -\mathbf{L}_{12}$ has a unique solution and hence $\mathbf{L}$ is block diagonalized by X and Y.

Now consider $\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ and $\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$. In order to define separation of $\mathbf{L}_1$ and $\mathbf{L}_2$, we need a norm on the product space $\mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n}$. So, let $|\!|\!|\cdot|\!|\!|_{\mathrm{prod}}$ be a norm on the product space.

Definition 4.3.1. *Let* $\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular. Let* $|\!|\!|\cdot|\!|\!|_{\mathrm{prod}}$ *be a norm on the product space* $\mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n}$*. Consider the operator*

$$\mathcal{S} : \mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n} \longrightarrow \mathbb{L}^p_w(\mathbb{C}^{m\times n}, \|\cdot\|), \ (Z_1, Z_2) \longmapsto \mathbf{L}_1 Z_1 - Z_2\mathbf{L}_2.$$

Then the separation of $\mathbf{L}_1$ *and* $\mathbf{L}_2$*, denoted by* $\mathrm{sep}(\mathbf{L}_1, \mathbf{L}_2)$*, is defined by*

$$\mathrm{sep}(\mathbf{L}_1, \mathbf{L}_2) := \min\{|\!|\!|\mathcal{S}(Z_1, Z_2)|\!|\!|_{w,p} : \ |\!|\!|(Z_1, Z_2)|\!|\!|_{\mathrm{prod}} = 1\}.$$

Note that the $\mathrm{sep}(\mathbf{L}_1, \mathbf{L}_2)$ as defined above is a natural generalization of the separation $\mathrm{sep}(A, B)$ of matrices. Further, as in the case of separation of matrices, we have $\mathrm{sep}(\mathbf{L}_1, \mathbf{L}_2) = 0$ if and only if $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$, that is, $\mathcal{S}$ is invertible if and only if $\mathrm{sep}(\mathbf{L}_1, \mathbf{L}_2) \neq 0$.

For our purpose, we consider $|\!|\!|\cdot|\!|\!|_{\mathrm{prod}}$ to be the norm $|\!|\!|(Z_1, Z_2)|\!|\!|_r := \|(\|Z_1\|, \|Z_2\|)\|_r$ where $\|\cdot\|$ is the norm on $\mathbb{C}^{m\times n}$ (same as the norm on $\mathbb{C}^{n\times n}$) and $\|\cdot\|_r$ is the Hölder's norm on $\mathbb{C}^2$. More precisely, for all practical purposes, the special cases when $r = 1, 2, \infty$ are all that that one really cares about. We denote $\mathrm{sep}(\mathbf{L}_1, \mathbf{L}_2)$ by $\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2)$ when the product norm is $|\!|\!|\cdot|\!|\!|_r$. Now observe that

$$|\!|\!|\mathcal{S}(Z_1, Z_2)|\!|\!|_{w,p} \leq |\!|\!|\mathbf{L}_1|\!|\!|_{w,p} \|Z_1\| + |\!|\!|\mathbf{L}_2|\!|\!|_{w,p} \|Z_2\| \leq |\!|\!|(Z_1, Z_2)|\!|\!|_r \, \|(|\!|\!|\mathbf{L}_1|\!|\!|_{w,p}, |\!|\!|\mathbf{L}_2|\!|\!|_{w,p})\|_s,$$

where $r^{-1} + s^{-1} = 1$. This shows that for $r^{-1} + s^{-1} = 1$, we have

$$\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) \leq \|(|\!|\!|\mathbf{L}_1|\!|\!|_{w,p}, |\!|\!|\mathbf{L}_2|\!|\!|_{w,p})\|_s.$$

The following result shows that $\mathrm{sep}^{(r)}$ is a Lipschitz continuous function.

Proposition 4.3.2. *Let* $\mathbf{\Delta L}_1,\ \mathbf{L}_1 \in \mathbb{L}_w^p(\mathbb{C}^{m\times m},\ \|\cdot\|)$ *and* $\mathbf{\Delta L}_2,\ \mathbf{L}_2 \in \mathbb{L}_w^p(\mathbb{C}^{n\times n},\ \|\cdot\|)$. *Suppose that* $\mathbf{L}_i$ *and* $\mathbf{L}_i + \Delta\mathbf{L}_i, i = 1, 2,$ *are regular. Then we have*

$$\begin{aligned}\mathrm{sep}^{(r)}(\mathbf{L}_1 + \mathbf{\Delta L}_1,\ \mathbf{L}_2 + \mathbf{\Delta L}_2) &\leq \mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) + \|(\|\!|\mathbf{\Delta L}_1|\!\|_{w,p}, \|\!|\mathbf{\Delta L}_2|\!\|_{w,p})\|_s,\\ \mathrm{sep}^{(r)}(\mathbf{L}_1 + \mathbf{\Delta L}_1,\ \mathbf{L}_2 + \mathbf{\Delta L}_2) &\geq \mathrm{sep}^{(r)}(\mathbf{L}_1,\ \mathbf{L}_2) - \|(\|\!|\mathbf{\Delta L}_1|\!\|_{w,p},\ \|\!|\mathbf{\Delta L}_2|\!\|_{w,p})\|_s,\end{aligned}$$

where $r^{-1} + s^{-1} = 1$.

Proof: Consider the generalized Sylvester operators $\mathcal{S}(Z_1,\ Z_2) := \mathbf{L}_1 Z_1 - Z_2\mathbf{L}_2$ and $\Delta\mathcal{S}(Z_1, Z_2) := \mathbf{\Delta L}_1 Z_1 - Z_2\mathbf{\Delta L}_2$. Then $(\mathcal{S}+\Delta\mathcal{S})(Z_1, Z_2) = (\mathbf{L}_1+\mathbf{\Delta L}_1)Z_1 - Z_2(\mathbf{L}_2+\mathbf{\Delta L}_2)$. Now, $\|\!|(\mathcal{S} + \Delta\mathcal{S})(Z_1, Z_2)|\!\|_{w,p} \geq \|\!|\mathcal{S}(Z_1, Z_2)|\!\|_{w,p} - \|\!|\Delta\mathcal{S}(Z_1, Z_2)|\!\|_{w,p} \geq \|\!|\mathcal{S}(Z_1, Z_2)|\!\|_{w,p} - \|(Z_1, Z_2)\|_r\|(\|\!|\mathbf{\Delta L}_1|\!\|_{w,p}, \|\!|\mathbf{\Delta L}_2|\!\|_{w,p})\|_s$. Hence taking minimum, we have

$$\mathrm{sep}^{(r)}(\mathbf{L}_1 + \mathbf{\Delta L}_1,\ \mathbf{L}_2 + \mathbf{\Delta L}_2) \geq \mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) - \|(\|\!|\mathbf{\Delta L}_1|\!\|_{w,p},\ \|\!|\mathbf{\Delta L}_2|\!\|_{w,p})\|_s.$$

Next, we have

$$\begin{aligned}\|\!|(\mathcal{S} + \Delta\mathcal{S})(Z_1, Z_2)|\!\|_{w,p} &\leq \|\!|\mathcal{S}(Z_1, Z_2)|\!\|_{w,p} + \|\!|\Delta\mathcal{S}(Z_1, Z_2)|\!\|_{w,p}\\ &\leq \|\!|\mathcal{S}(Z_1, Z_2)|\!\|_{w,p} + \|(Z_1, Z_2)\|_r\ \|(\|\!|\mathbf{\Delta L}_1|\!\|_{w,p}, \|\!|\mathbf{\Delta L}_2|\!\|_{w,p})\|_s.\end{aligned}$$

Again taking minimum, we have

$$\mathrm{sep}^{(r)}(\mathbf{L}_1 + \mathbf{\Delta L}_1,\ \mathbf{L}_2 + \mathbf{\Delta L}_2) \leq \mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2)\ +\ \|(\|\!|\mathbf{\Delta L}_1|\!\|_{w,p}, \|\!|\mathbf{\Delta L}_2|\!\|_{w,p})\|_s.$$

This completes the proof. ■

The next result shows the influence of equivalence of transformations of $\mathbf{L}_1$ and $\mathbf{L}_2$ on $\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2)$. Recall that if $X, Y \in \mathbb{C}^{n\times n}$ are nonsingular matrices then $\mathrm{cond}(X, Y) := \|X^{-1}\|\,\|Y\|$ and $\mathrm{cond}(X) := \|X\|\,\|X^{-1}\|$.

Proposition 4.3.3. *Let* $\mathbf{L}_1 \in \mathbb{L}_w^p(\mathbb{C}^{m\times m},\ \|\cdot\|)$ *and* $\mathbf{L}_2 \in \mathbb{L}_w^p(\mathbb{C}^{n\times n},\ \|\cdot\|)$ *be regular pencils. Also, let* $Y_1, X_1 \in \mathbb{C}^{m\times m}$ *and* $Y_2, X_2 \in \mathbb{C}^{n\times n}$ *be nonsingular matrices. Set*

$$\begin{aligned}\alpha_1 &:= \mathrm{cond}(Y_1, X_1)\mathrm{cond}(X_2),\ \ \beta_1 := \mathrm{cond}(Y_2, X_2)\mathrm{cond}(Y_1),\\ \alpha_2 &:= \mathrm{cond}(X_1, Y_1)\mathrm{cond}(X_2),\ \ \beta_2 := \mathrm{cond}(X_2, Y_2)\mathrm{cond}(Y_1).\end{aligned}$$

Then we have

$$\frac{\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2)}{\max(\alpha_2, \beta_2)} \leq \mathrm{sep}^{(r)}(Y_1^{-1}\mathbf{L}_1 X_1,\ Y_2^{-1}\mathbf{L}_2 X_2) \leq \max(\alpha_1, \beta_1)\,\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2).$$

Proof: We have $\mathrm{sep}^{(r)}(Y_1^{-1}\mathbf{L}_1X_1,\ Y_2^{-1}\mathbf{L}_2X_2) = \min\{\|\!|(Y_1^{-1}\mathbf{L}_1X_1)Z_1 - Z_2(Y_2^{-1}\mathbf{L}_2X_2)|\!\|_{w,p} : \|\!|(Z_1, Z_2)|\!\|_r = 1\}$. Now consider $\mathcal{S}(Z_1, Z_2) := \mathbf{L}_1Z_1 - Z_2\mathbf{L}_2$. Then we have

$$(Y_1^{-1}\mathbf{L}_1X_1)Z_1 - Z_2(Y_2^{-1}\mathbf{L}_2X_2) = Y_1^{-1}\mathcal{S}(X_1Z_1X_2^{-1},\ Y_1Z_2Y_2^{-1})X_2.$$

Now using the fact that

$$|||(X_1Z_1X_2^{-1},\ Y_1Z_2Y_2^{-1})|||_r \leq \max(\|X_1\|\,\|X_2^{-1}\|,\ \|Y_1\|\,\|Y_2^{-1}\|)|||(Z_1,Z_2)|||_r$$

and that

$$\frac{|||\mathcal{S}(X_1Z_1X_2^{-1},\ Y_1Z_2Y_2^{-1})|||_{w,p}}{|||(Z_1,Z_2)|||_r} = \frac{|||\mathcal{S}(X_1Z_1X_2^{-1},\ Y_1Z_2Y_2^{-1})|||_{w,p}}{|||(X_1Z_1X_2^{-1},\ Y_1Z_2Y_2^{-1})|||_r}\,\frac{|||(X_1Z_1X_2^{-1},\ Y_1Z_2Y_2^{-1})|||_r}{|||(Z_1,\ Z_2)|||_r},$$

taking minimum over (Z_1, Z_2), we have

$$\mathrm{sep}^{(r)}(Y_1^{-1}\mathbf{L}_1X_1,\ Y_2^{-1}\mathbf{L}_2X_2) \leq \max(\alpha_1, \beta_1)\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2).$$

Finally, using the fact that $\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) = \mathrm{sep}^{(r)}(Y_1Y_1^{-1}\mathbf{L}_1X_1X_1^{-1},\ Y_2Y_2^{-1}\mathbf{L}_2X_2X_2^{-1})$, the other inequality follows. This completes the proof. ■

Now we relate $\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2)$ with the notion of separation of pencils proposed by Stewart [40, 43] and has been used extensively by Demmel et al. [16]. Given $\mathbf{L}_i(z) = A_i - zB_i, i = 1, 2$, the notion of separation of $\mathbf{L}_1$ and $\mathbf{L}_2$ denoted by $\mathrm{dif}(A_1, A_2; B_1, B_2)$ was first introduced by Stewart [40]. More precisely, $\mathrm{dif}(A_1, A_2; B_1, B_2)$ is given by

$$\mathrm{dif}(A_1, A_2; B_1, B_2) := \min\{|||(A_1Z_1 - Z_2A_2, B_1Z_1 - Z_2B_2)|||_\infty : |||(Z_1, Z_2)|||_\infty = 1\},$$

where $|||(X,Y)|||_\infty := \max(\|X\|_F, \|Y\|_F)$. This shows that for the special case when $w := (1,1)$, $\mathbf{L}_1 \in \mathbb{L}_w^\infty(\mathbb{C}^{m\times m}, \|\cdot\|_F)$ and $\mathbf{L}_2 \in \mathbb{L}_w^\infty(\mathbb{C}^{n\times n}, \|\cdot\|_F)$, we have

$$\mathrm{sep}^{(\infty)}(\mathbf{L}_1, \mathbf{L}_2) = \mathrm{dif}(A_1, A_2; B_1, B_2).$$

On the other hand, dif is defined in [43] by

$$\mathrm{dif}(A_1, A_2; B_1, B_2) := \min\{|||(A_1Z_1 - Z_2A_2, B_1Z_1 - Z_2B_2)|||_2 : |||(Z_1, Z_2)|||_2 = 1\},$$

where $|||(X,Y)|||_2 := (\|X\|_F^2 + \|Y\|_F^2)^{1/2}$. Again this shows that for the special case when $w := (1,1)$, $\mathbf{L}_1 \in \mathbb{L}_w^2(\mathbb{C}^{m\times m}, \|\cdot\|_F)$ and $\mathbf{L}_2 \in \mathbb{L}_w^2(\mathbb{C}^{n\times n}, \|\cdot\|_F)$, we have

$$\mathrm{sep}^{(2)}(\mathbf{L}_1, \mathbf{L}_2) = \mathrm{dif}(A_1, A_2; B_1, B_2).$$

This shows that our definition of $\mathrm{sep}(\mathbf{L}_1, \mathbf{L}_2)$ not only generalizes the notion of separation $\mathrm{sep}(A, B)$ of matrices to the case of pencils but also it unifies the notion of separation dif that exists for matrix pencils. The main advantage of our approach is that the separation sep of matrices and the separation $\mathrm{sep}^{(r)}$ of matrix pencils can be handled at equal ease. The crux of the matter is that conceptually there is no difference between the separation sep of matrices and the separation $\mathrm{sep}^{(r)}$ of matrix pencils. Our abstract approach to defining $\mathrm{sep}^{(r)}$ makes this fact clear.

Next, we introduce the notion of sep_λ for matrix pencils. The notion of sep_λ for matrices is well known and has been studied extensively in ([3], also see [15, 48]). The notion of sep_λ was introduced by Varah [48] and was subsequently modified by other researchers [3, 15]. Briefly, for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times n}$, the sep_λ is given by [3]

$$\mathrm{sep}_\lambda(A, B) := \min\{\epsilon : \Lambda_\epsilon(A) \cap \Lambda_\epsilon(B) \neq \emptyset\}.$$

Then $\mathrm{sep}_\lambda(A, B)$ is the smallest value of $\|\Delta A\|$ and $\|\Delta B\|$ such that $A+\Delta A$ and $B+\Delta B$ have a common eigenvalue. We now generalize the notion of sep_λ to the case of matrix pencils.

Definition 4.3.4. *Let* $\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$*. Then define*

$$\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2) := \min\{\epsilon : \Lambda_\epsilon(\mathbf{L}_1) \cap \Lambda_\epsilon(\mathbf{L}_2) \neq \emptyset\}.$$

Evidently, if $\epsilon < \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$ then the ϵ-pseudospectra of $\mathbf{L}_1$ and $\mathbf{L}_2$ are disjoint. Hence $\Lambda(\mathbf{L}_1+\boldsymbol{\Delta}\mathbf{L}_1)\cap\Lambda(\mathbf{L}_2+\boldsymbol{\Delta}\mathbf{L}_2) = \emptyset$ whenever $\max(|||\boldsymbol{\Delta}\mathbf{L}_1|||_{w,p}, |||\boldsymbol{\Delta}\mathbf{L}_2|||_{w,p}) < \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$. The next result shows that $\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$ is indeed the smallest value of $|||\boldsymbol{\Delta}\mathbf{L}_1|||_{w,p}$ and $|||\boldsymbol{\Delta}\mathbf{L}_2|||_{w,p}$ for which $\mathbf{L}_1 + \boldsymbol{\Delta}\mathbf{L}_1$ and $\mathbf{L}_2 + \boldsymbol{\Delta}\mathbf{L}_2$ have a common eigenvalue.

Theorem 4.3.5. *Let* $\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular. Then there exists* $\boldsymbol{\Delta}\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\boldsymbol{\Delta}\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *such that* $|||\boldsymbol{\Delta}\mathbf{L}_1|||_{w,p} = |||\boldsymbol{\Delta}\mathbf{L}_2|||_{w,p} = \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$ *and* $\Lambda(\mathbf{L}_1 + \boldsymbol{\Delta}\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2 + \boldsymbol{\Delta}\mathbf{L}_2) \neq \emptyset$*.*

Proof: Without loss of generality, we outline the proof for nonhomogeneous pencils. Let $\epsilon := \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$. Then by the definition there exists $z_0 \in \partial\Lambda_\epsilon(\mathbf{L}_1) \cap \partial\Lambda_\epsilon(\mathbf{L}_2)$. Since $z_0 \in \partial\Lambda_\epsilon(\mathbf{L}_1)$, there exists $\boldsymbol{\Delta}\mathbf{L}_1$ such that $z_0 \in \Lambda(\mathbf{L}+\boldsymbol{\Delta}\mathbf{L}_1)$ and $|||\boldsymbol{\Delta}\mathbf{L}_1|||_{w,p} = \eta_{w,p}(z_0, \mathbf{L}_1)$. Again since $z_0 \in \partial\Lambda_\epsilon(\mathbf{L}_2)$, there exists $|||\boldsymbol{\Delta}\mathbf{L}_2|||_{w,p} = \eta_{w,p}(z_0, \mathbf{L}_2) = \epsilon$. This implies there exists $\boldsymbol{\Delta}\mathbf{L}_1$ and $\boldsymbol{\Delta}\mathbf{L}_2$ such that $z_0 \in \Lambda(\mathbf{L}_1 + \boldsymbol{\Delta}\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2 + \boldsymbol{\Delta}\mathbf{L}_2)$ and $|||\boldsymbol{\Delta}\mathbf{L}_1|||_{w,p} = |||\boldsymbol{\Delta}\mathbf{L}_2|||_{w,p} = \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$. ■

We provide a characterization of $\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$.

Proposition 4.3.6. *Let* $\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$*. Then we have*

$$\mathrm{sep}_\lambda(\mathbf{L}_1,\, \mathbf{L}_2) := \inf_z\{\max(\eta_{w,p}(z, \mathbf{L}_1),\, \eta_{w,p}(z, \mathbf{L}_2))\}.$$

Similar result holds for homogeneous pencils.

Proof: Let $\epsilon_0 := \min\{\epsilon : \Lambda_\epsilon(\mathbf{L}_1) \cap \Lambda_\epsilon(\mathbf{L}_2) \neq \emptyset\}$. If $\epsilon < \epsilon_0$ then $\Lambda_\epsilon(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. Let $z \in \mathbb{C}$. Then either $z \in \Lambda_\epsilon(\mathbf{L}_1)$ or $z \in \Lambda_\epsilon(\mathbf{L}_2)$ or $z \notin \Lambda_\epsilon(\mathbf{L}_1) \cup \Lambda_\epsilon(\mathbf{L}_2)$. This shows that $\max(\eta_{w,p}(z, \mathbf{L}), \eta_{w,p}(z, \mathbf{L})) > \epsilon \Rightarrow \inf_z\{\max(\eta_{w,p}(z, \mathbf{L}_1), \eta_{w,p}(z, \mathbf{L}_2))\} > \epsilon$. Since this is true for any $\epsilon < \epsilon_0$. It follows that $\inf_z\{\max(\eta_{w,p}(z, \mathbf{L}_1),\, \eta_{w,p}(z, \mathbf{L}_2))\} \geq \epsilon_0$.

Conversely, let $\epsilon \geq \epsilon_0$. Then $\Lambda_\epsilon(\mathbf{L}_1) \cap \Lambda_\epsilon(\mathbf{L}_2) \neq \emptyset$. Let $z \in \Lambda_\epsilon(\mathbf{L}_1) \cap \Lambda_\epsilon(\mathbf{L}_2)$. Then $\max(\eta_{w,p}(z,\, \mathbf{L}_1),\, \eta_{w,p}(z,\, \mathbf{L}_2)) \leq \epsilon \Rightarrow \inf_z\{\max(\eta_{w,p}(z,\, \mathbf{L}_1),\, \eta_{w,p}(z,\, \mathbf{L}_2))\} \leq \epsilon$. Since $\epsilon \geq \epsilon_0$ is arbitrary, we have $\inf_z\{\max(\eta_{w,p}(z, \mathbf{L}_1),\, \eta_{w,p}(z, \mathbf{L}_2))\} \leq \epsilon_0$. Hence the proof. A verbatim proof holds for homogeneous pencils. ■

The next result shows the effect of equivalence transformations on pencils on sep_λ. Recall that for nonsingular matrices $X, Y \in \mathbb{C}^{n\times n}$, $\mathrm{cond}(X, Y) := \|X^{-1}\|\, \|Y\|$. Then we have the following.

Proposition 4.3.7. *Let* $\mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$. *Also, let* $X_1, Y_1 \in \mathbb{C}^{m\times m}$ *and* $X_2, Y_2 \in \mathbb{C}^{n\times n}$. *Set* $\kappa_1 := \mathrm{cond}(Y_1, X_1)$, $\kappa_2 := \mathrm{cond}(Y_2, X_2)$, $\kappa_3 := \mathrm{cond}(X_1, Y_1)$ *and* $\kappa_4 := \mathrm{cond}(X_2, Y_2)$. *Then we have*

$$\frac{\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)}{\max(\kappa_3,\ \kappa_4)} \leq \mathrm{sep}_\lambda(Y_1^{-1}\mathbf{L}_1 X_1, Y_2^{-1}\mathbf{L}_2 X_2) \leq \max(\kappa_1,\ \kappa_2)\, \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2).$$

Proof: Recall that for nonsingular matrices $X, Y \in \mathbb{C}^{n\times n}$ and $\alpha_1 := \mathrm{cond}(X, Y)$ and $\alpha_2 := \mathrm{cond}(Y, X)$, by Theorem 4.2.4(a), we have

$$\Lambda_\epsilon(\mathbf{L}) \subset \Lambda_{\alpha_2\epsilon}(Y^{-1}\mathbf{L}X)\ \text{and}\ \Lambda_\epsilon(Y^{-1}\mathbf{L}X) \subset \Lambda_{\alpha_1\epsilon}(\mathbf{L}).$$

Now using this fact to the pencils $\mathbf{L}_1$ and $\mathbf{L}_2$, the desired result follows. Alternatively, the proof follows from Proposition 4.3.6. ■

The following result shows that a small perturbations in $\mathbf{L}_1$ and $\mathbf{L}_2$ induces a small change in $\mathrm{sep}_\lambda(\mathbf{L}_1,\, \mathbf{L}_2)$.

Proposition 4.3.8. *Let* $\mathbf{\Delta L}_1, \mathbf{L}_1 \in \mathbb{L}^p_w(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{\Delta L}_2, \mathbf{L}_2 \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$. *Then for* $1 \leq s \leq \infty$, *we have*

$$\begin{aligned} \mathrm{sep}_\lambda(\mathbf{L}_1 + \mathbf{\Delta L}_1,\ \mathbf{L}_2 + \mathbf{\Delta L}_2) &\leq \mathrm{sep}_\lambda(\mathbf{L}_1,\ \mathbf{L}_2) + \|(\|\mathbf{\Delta L}_1\|_{w,p},\ \|\mathbf{\Delta L}_2\|_{w,p})\|_s, \\ \mathrm{sep}_\lambda(\mathbf{L}_1 + \mathbf{\Delta L}_1,\ \mathbf{L}_2 + \mathbf{\Delta L}_2) &\geq \mathrm{sep}_\lambda(\mathbf{L}_1,\ \mathbf{L}_2) - \|(\|\mathbf{\Delta L}_1\|_{w,p},\ \|\mathbf{\Delta L}_2\|_{w,p})\|_s. \end{aligned}$$

Proof: Set $t := \|(\|\mathbf{\Delta L}_1\|_{w,p},\ \|\mathbf{\Delta L}_2\|_{w,p})\|_s$. Then the proof follows from the fact that for all ϵ, we have the inclusion $\Lambda_{\epsilon-t}(\mathbf{L}_i) \subset \Lambda_\epsilon(\mathbf{L}_i + \Delta\mathbf{L}_i) \subset \Lambda_{\epsilon+t}(\mathbf{L}_i), i = 1, 2$.

Alternatively, by Proposition 4.2.3 and Proposition 4.3.6, we have

$$\begin{aligned} \mathrm{sep}_\lambda(\mathbf{L}_1 + \mathbf{\Delta L}_1,\, \mathbf{L}_2 + \mathbf{\Delta L}_2) &\leq \inf_z \left\{ \max\left(\eta_{w,p}(z, \mathbf{L}_1),\, \eta_{w,p}(z,\, \mathbf{L}_2)\right)\right\} + t \\ &\leq \mathrm{sep}_\lambda(\mathbf{L}_1,\, \mathbf{L}_2) + t. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathrm{sep}_\lambda(\mathbf{L}_1 + \mathbf{\Delta L}_1,\, \mathbf{L}_2 + \mathbf{\Delta L}_2) &\geq \inf_z \left\{ \max\left(\eta_{w,p}(z, \mathbf{L}_1),\, \eta_{w,p}(z,\, \mathbf{L}_2)\right)\right\} - t \\ &\geq \mathrm{sep}_\lambda(\mathbf{L}_1,\, \mathbf{L}_2) - t. \end{aligned}$$

The proof is similar for homogeneous pencils. ■

For diagonal pencils we have the following result.

Proposition 4.3.9. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{G} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be block diagonal pencils given by* $\mathbf{L} := \mathrm{diag}(\mathbf{L}_1, \ldots, \mathbf{L}_k)$ *and* $\mathbf{G} := \mathrm{diag}(\mathbf{G}_1, \ldots, \mathbf{G}_l)$. *Then we have*

$$\mathrm{sep}_\lambda(\mathbf{L}, \mathbf{G}) = \min_{i,j} \mathrm{sep}_\lambda(\mathbf{L}_i, \mathbf{G}_j).$$

Proof: We have $\mathrm{sep}_\lambda(\mathbf{L}, \mathbf{G}) := \min\{\epsilon : \Lambda_\epsilon(\mathbf{L}) \cap \Lambda_\epsilon(\mathbf{G}) \neq \emptyset\}$. Since $\mathbf{L}$ and $\mathbf{G}$ are block diagonal, by Theorem 4.2.2, we have $\Lambda_\epsilon(\mathbf{L}) = \cup_{i=1}^k \Lambda_\epsilon(\mathbf{L}_i)$ and $\Lambda_\epsilon(\mathbf{G}) = \cup_{j=1}^l \Lambda_\epsilon(\mathbf{G}_j)$. Now $\Lambda_\epsilon(\mathbf{L}) \cap \Lambda_\epsilon(\mathbf{G}) = (\cup_{i=1}^k \Lambda_\epsilon(\mathbf{L}_i)) \cap (\cup_{j=1}^l \Lambda_\epsilon(\mathbf{G}_j))$. This shows that $\Lambda_\epsilon(\mathbf{L}) \cap \Lambda_\epsilon(\mathbf{G}) \neq \emptyset \Rightarrow \Lambda_\epsilon(\mathbf{L}_i) \cap \Lambda_\epsilon(\mathbf{G}_j) \neq \emptyset$ for some i and j. Hence the result follows. ■

Now we establish relationship between sep and sep_λ. To that end, recall that $\mathcal{S}(Z_1, Z_2) := \mathbf{L}_1 Z_1 - Z_2 \mathbf{L}_2$ and the product norm $|||(Z_1, Z_2)|||_r := \|(\|Z_1\|, \|Z_2\|)\|_r$, for $1 \leq r \leq \infty$.

Theorem 4.3.10. *Let* $\mathbf{L}_1 \in \mathbb{L}_w^p(\mathbb{C}^{m\times m}, \|\cdot\|)$ *and* $\mathbf{L}_2 \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. *Then for* $r^{-1} + s^{-1} = 1$, *we have*

$$\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) \leq 2^{1/s}\, \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2).$$

In particular, we have following special cases

$$\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) \leq \begin{cases} \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2), & \textit{when } r = 1, \\ 2\,\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2), & \textit{when } r = \infty, \\ \sqrt{2}\ \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2), & \textit{when } r = 2. \end{cases}$$

Proof: By the definition of $\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$ there exist $\mathbf{\Delta L}_1$ and $\mathbf{\Delta L}_2$ such that $\Lambda(\mathbf{L}_1 + \mathbf{\Delta L}_1) \cap \Lambda(\mathbf{L}_2 + \mathbf{\Delta L}_2) \neq \emptyset$ and $|||\mathbf{\Delta L}_1|||_{w,p} = |||\mathbf{\Delta L}_2|||_{w,p} = \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$. Since $\Lambda(\mathbf{L}_1 + \mathbf{\Delta L}_1) \cap \Lambda(\mathbf{L}_2 + \mathbf{\Delta L}_2) \neq \emptyset$. Hence there exists $(Z_1, Z_2) \neq 0$ such that $(\mathcal{S} + \Delta\mathcal{S})(Z_1, Z_2) := (\mathbf{L}_1 + \mathbf{\Delta L}_1) Z_1 - Z_2(\mathbf{L}_2 + \mathbf{\Delta L}_2)$ is singular. This implies $(\mathcal{S} + \Delta\mathcal{S})(Z_1, Z_2) = 0$. Then we have $|||\mathcal{S}(Z_1, Z_2)|||_{w,p} = |||\Delta\mathcal{S}(Z_1, Z_2)|||_{w,p} = |||\mathbf{\Delta L}_1 Z_1 - Z_2 \mathbf{\Delta L}_2|||_{w,p} \leq \|Z_1\|\, |||\mathbf{\Delta L}_1|||_{w,p} + \|Z_2\|\, |||\mathbf{\Delta L}_2|||_{w,p} \leq |||(Z_1, Z_2)|||_r \|(|||\mathbf{\Delta L}_1|||_{w,p}, |||\mathbf{\Delta L}_2|||_{w,p})\|_s$, where $r^{-1} + s^{-1} = 1$. Thus,

$$\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) = \min_{|||(Z_1, Z_2)|||_r} \frac{|||\mathcal{S}(Z_1, Z_2)|||_{w,p}}{|||(Z_1, Z_2)|||_r} \leq \|(|||\mathbf{\Delta L}_1|||_{w,p}, |||\mathbf{\Delta L}_2|||_{w,p})\|_s.$$

Since $|||\mathbf{\Delta L}_1|||_{w,p} = |||\mathbf{\Delta L}_2|||_{w,p} = \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$, we have $\mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2) \leq 2^{1/s}\, \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$. Hence the results follow. ■

We mention that for matrices A and B, it is well known that $\mathrm{sep}(A, B) \leq 2\,\mathrm{sep}_\lambda(A, B)$. In Theorem 4.3.10, we have generalized this relationship to the case of matrix pencils.

Finally, we mention that with a view to analyzing stability of eigendecomposition of matrix pencils, Demmel et al. [16] introduced a notion of separation which is denoted by Dif_λ. More precisely, Dif_λ is defined by

$$\mathrm{Dif}_\lambda(\mathbf{L}_1, \mathbf{L}_2) := \inf_{|c|^2 + |s|^2 = 1} \{\left(\sigma_{\min}(\mathbf{L}_1(c, s))^2 + \sigma_{\min}(\mathbf{L}_2(c, s))^2\right)^{1/2}\}.$$

We now show that $\mathrm{Dif}_\lambda(\mathbf{L}_1, \mathbf{L}_2) = \sqrt{2}\,\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$ for the special case when $w := (1,1)$ and $\mathbf{L}_1 \in \mathbb{L}_w^2(\mathbb{C}^{m\times m}, \|\cdot\|_F)$ and $\mathbf{L}_2 \in \mathbb{L}_w^2(\mathbb{C}^{n\times n}, \|\cdot\|_F)$. First, note that $\eta_{w,2}(c, s, \mathbf{L}_1) = \frac{\sigma_{\min}(\mathbf{L}_1(c,s))}{\|(c,s)\|_2}$. Hence we have

$$\begin{aligned} \mathrm{Dif}_\lambda(\mathbf{L}_1, \mathbf{L}_2) &:= \inf_{|c|^2+|s|^2=1} \left\{ \left(\sigma_{\min}(\mathbf{L}_1(c,s))^2 + \sigma_{\min}(\mathbf{L}_2(c,s))^2\right)^{1/2} \right\} \\ &\leq \sqrt{2} \min_{|c|^2+|s|^2=1} \max(\sigma_{\min}(\mathbf{L}_1(c,s)),\, \sigma_{\min}(\mathbf{L}_2(c,s))) \\ &= \sqrt{2}\,\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2). \end{aligned}$$

On the other hand, it is immediate that $\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2) \leq \mathrm{Dif}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$. Now it follows that $\mathrm{Dif}_\lambda(\mathbf{L}_1, \mathbf{L}_2) = \min_{|c|^2+|s|^2=1}\{\sqrt{|\!|\!|\boldsymbol{\Delta}\mathbf{L}_1|\!|\!|_{w,2}^2 + |\!|\!|\boldsymbol{\Delta}\mathbf{L}_2|\!|\!|_{w,2}^2} : (c,s) \in \Lambda(\mathbf{L}_1 + \boldsymbol{\Delta}\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2 + \boldsymbol{\Delta}\mathbf{L}_2)\}$. By Theorem 4.3.5, there exists $\boldsymbol{\Delta}\mathbf{L}_1$ and $\boldsymbol{\Delta}\mathbf{L}_2$ such that $|\!|\!|\boldsymbol{\Delta}\mathbf{L}_1|\!|\!|_{w,2} = |\!|\!|\boldsymbol{\Delta}\mathbf{L}_2|\!|\!|_{w,2} = \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$ and that $\mathbf{L}_1 + \boldsymbol{\Delta}\mathbf{L}_1$ and $\mathbf{L}_2 + \boldsymbol{\Delta}\mathbf{L}_2$ have a common eigenvalue. This shows that $\mathrm{Dif}_\lambda(\mathbf{L}_1, \mathbf{L}_2) = \sqrt{2}\,\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$.

4.4 Geometric separation and stable eigendecompositions

We now define the geometric separation of eigenvalues of matrix pencils. The notion of geometric separation of eigenvalues of matrices was introduced in [2, 3].

Definition 4.4.1. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular and* Λ *be a nonempty subset of* $\Lambda(\mathbf{L})$. *Then the geometric separation of* Λ *from the rest of* $\Lambda(\mathbf{L})$, *denoted by* $\mathrm{gsep}(\Lambda)$, *is the smallest value of* ϵ *for which a component of* $\Lambda_\epsilon(\mathbf{L})$ *containing an eigenvalue from* Λ *coalesces with a component containing an eigenvalue from* $\Lambda(\mathbf{L}) \setminus \Lambda$.

Equivalently, let δ be such that if $\epsilon < \delta$ then each component of $\Lambda_\epsilon(\mathbf{L})$ either intersects Λ or $\Lambda(\mathbf{L}) \setminus \Lambda$ but not both and if $\epsilon \geq \delta$ then there is a component of $\Lambda_\epsilon(\mathbf{L})$ which intersects Λ as well as $\Lambda(\mathbf{L}) \setminus \Lambda$. Then we have $\mathrm{gsep}(\Lambda) = \delta$. In other words, $\mathrm{gsep}(\Lambda) = \min\{\epsilon : \Lambda_\epsilon(\mathbf{L})$ has a component Δ s.t. $\Delta \cap \Lambda \neq \emptyset$ and $\Delta \cap (\Lambda(\mathbf{L}) \setminus \Lambda) \neq \emptyset\}$.

It is evident that if $\epsilon < \mathrm{gsep}(\Lambda)$ then the (right and left) spectral projections of $\mathbf{L}$ associated with Λ vary continuously when $\mathbf{L}$ is perturbed to $\mathbf{L} + \boldsymbol{\Delta}\mathbf{L}$ and $|\!|\!|\boldsymbol{\Delta}\mathbf{L}|\!|\!|_{w,p} \leq \epsilon$.

The notion of successor introduced by Wilkinson [50, 51] is a convenient tool for describing the evolution of eigenvalues of matrices. We now describe the notion of successors of eigenvalues of matrix pencils. Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. We say that $z \in \mathbb{C}$ is an ϵ-successor of $\lambda \in \Lambda(\mathbf{L})$ if there is a matrix pencil $\boldsymbol{\Delta}\mathbf{L}$ with $|\!|\!|\boldsymbol{\Delta}\mathbf{L}|\!|\!|_{w,p} \leq \epsilon$ such that $z \in \Lambda(\mathbf{L} + \boldsymbol{\Delta}\mathbf{L})$ and $\lambda \to z$ as $\mathbf{L} \to \mathbf{L} + \boldsymbol{\Delta}\mathbf{L}$. If Δ is a component of $\Lambda_\epsilon(\mathbf{L})$ which isolates an eigenvalue λ from the rest of $\Lambda(\mathbf{L})$ then each $z \in \Delta$ is an ϵ-successor of λ. The successors of eigenvalues of homogeneous pencils are defined similarly. Note that if Λ is a nonempty subset of $\Lambda(\mathbf{L})$ and $\epsilon < \mathrm{gsep}(\Lambda)$ then the set of successors of eigenvalues belonging to Λ remains disjoint from the set of successors of the eigenvalues belonging to $\Lambda(\mathbf{L}) \setminus \Lambda$ as $\mathbf{L}$ is perturbed to $\mathbf{L} + \mathbf{\Delta L}$ and $\|\!|\mathbf{\Delta L}|\!\|_{w,p} \leq \epsilon$.

Definition 4.4.2. *We say that* z *(resp,* (c, s)*) is a common successor of two eigenvalues* λ_1 *and* λ_2 *(resp.,* (c_1, s_1) *and* (c_2, s_2)*) of* $\mathbf{L}$ *if* z *(resp.,* (c, s)*) is a successor of both* λ_1 *and* λ_2 *(resp.,* (c_1, s_1) *and* (c_2, s_2)*).*

The following result shows the effect of equivalence transformation of pencils on the gsep.

Corollary 4.4.3. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular and* Λ *be a non-empty subset of* $\Lambda_\epsilon(\mathbf{L})$*. For nonsingular matrices* $X, Y \in \mathbb{C}^{n\times n}$*, set* $\mathrm{cond}(X, Y) := \|X^{-1}\|\,\|Y\|$*. Then*

$$\frac{\mathrm{gsep}(\Lambda, \mathbf{L})}{\mathrm{cond}(X, Y)} \leq \mathrm{gsep}(\Lambda, Y^{-1}\mathbf{L}X) \leq \mathrm{cond}(Y, X)\mathrm{gsep}(\Lambda, \mathbf{L}).$$

Proof: The proof follows from Theorem 4.2.4(a). ■

The following result shows that a small change in $\mathbf{L}$ results a small change in gsep.

Proposition 4.4.4. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular and* Λ *be a nonempty subset of* $\Lambda(\mathbf{L})$*. Let* $\mathbf{\Delta L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be such that* $\|\!|\mathbf{\Delta L}|\!\|_{w,p} < \mathrm{gsep}(\Lambda, \mathbf{L})$*. Then there is a subset* Λ' *of* $\Lambda(\mathbf{L} + \mathbf{\Delta L})$ *such that the total algebraic multiplicity of* Λ' *is the same as that of* Λ *and that*

$$\mathrm{gsep}(\Lambda, \mathbf{L}) - \|\!|\mathbf{\Delta L}|\!\|_{w,p} \leq \mathrm{gsep}(\Lambda', \mathbf{L} + \mathbf{\Delta L}) \leq \mathrm{gsep}(\Lambda, \mathbf{L}) + \|\!|\mathbf{\Delta L}|\!\|_{w,p}.$$

Proof: Since $\|\!|\mathbf{\Delta L}|\!\|_{w,p} < \mathrm{gsep}(\lambda, \mathbf{L})$, the $\|\!|\mathbf{\Delta L}|\!\|_{w,p}$-successors of Λ remain disjoint from the $\|\!|\mathbf{\Delta L}|\!\|_{w,p}$-successors of $\Lambda(\mathbf{L}) \setminus \Lambda$. Hence the total algebraic multiplicity of Λ' is the same as that of Λ. Now for $t := \|\!|\mathbf{\Delta L}|\!\|_{w,p}$, we have $\Lambda_{\epsilon-t}(\mathbf{L}) \subset \Lambda_\epsilon(\mathbf{L} + \mathbf{\Delta L}) \subset \Lambda_{\epsilon+t}(\mathbf{L})$. Hence the result follows. ■

The following result relates gsep with $\eta_{w,p}(z, \mathbf{L})$.

Proposition 4.4.5. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *and* Λ *be a nonempty subset of* $\Lambda(\mathbf{L})$*. Let* G *denote the set of closed curves in* $\mathbb{C} \setminus \Lambda(\mathbf{L})$ *with the following property: If* $\Gamma \in G$ *then* Γ *isolates* Λ *from the rest of* $\Lambda(\mathbf{L})$ *and either* Γ *is a closed curve or finite union of disjoint closed curves. Then we have*

$$\mathrm{gsep}(\Lambda) = \sup_{\Gamma\in G} \min_{\lambda\in\Gamma} \eta_{w,p}(\lambda, \mathbf{L}).$$

Proof: Suppose that $\epsilon < \sup_{\Gamma\in G} \min_{\lambda\in\Gamma} \eta_{w,p}(\lambda, \mathbf{L})$. Then there is a curve $\Gamma \in G$ such that $\eta_{w,p}(\lambda, \mathbf{L}) > \epsilon$ for all $\lambda \in \Gamma$. This shows that each component of $\Lambda_\epsilon(\mathbf{L})$ either intersects Λ or $\Lambda(\mathbf{L}) \setminus \Lambda$, that is, $\epsilon < \mathrm{gsep}(\Lambda)$. Hence we have

$$\sup_{\Gamma\in G} \min_{\lambda\in\Gamma} \eta_{w,p}(\lambda, \mathbf{L}) \leq \mathrm{gsep}(\Lambda).$$

Next, if possible, suppose that the above inequality is strict. Then $\min_{\lambda \in \Gamma} \eta_{w,p}(\lambda, \mathbf{L}) < \mathrm{gsep}(\Lambda)$ for all $\Gamma \in G$. Set $\epsilon := \mathrm{gsep}(\Lambda)$. Consider $\Omega := \{z \in \mathbb{C} : \eta_{w,p}(z, \mathbf{L}) < \epsilon\}$. Then the closure of Ω is equal to $\Lambda_\epsilon(\mathbf{L})$. Note that each component of Ω either intersects Λ or $\Lambda(\mathbf{L}) \setminus \Lambda$ but not the both. Let U be the union of components of Ω which intersects with Λ. Then $\partial U \in G$ and $\sup_{z \in \partial U} \eta_{w,p}(z, \mathbf{L}) = \epsilon = \mathrm{gsep}(\Lambda)$, a contradiction. Hence the proof. ■

We now consider a simple example to illustrate gsep and coalescence of components.

Example 4.4.6. Let $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{-i}{\sqrt{2}} \end{bmatrix}$ and $B = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} \end{bmatrix}$. Consider the pencil $\mathbf{L}(z) = A - zB$.

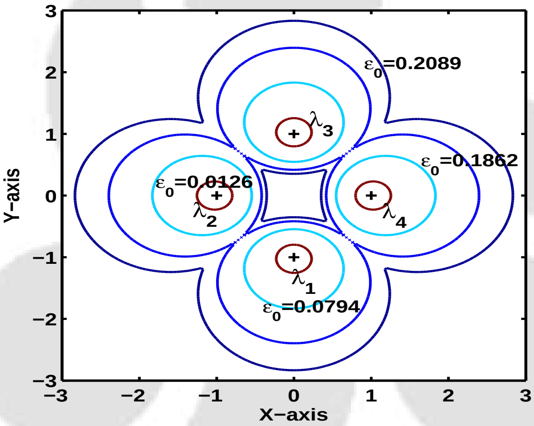


Figure 4.1: *Contour plot of $\Lambda_\epsilon(\mathbf{L})$ showing coalescence of components.*

We have $\Lambda(\mathbf{L}) = \{-i, -1, i, 1\}$. Figure 4.1, shows that contour plots of $\Lambda_\epsilon(\mathbf{L})$ for various values of ϵ. For $\epsilon_0 = 0.0126$, $\Lambda_\epsilon(\mathbf{L})$ consists of 4 components as shown by the inner most contours in Figure 4.1. As ϵ grows gradually, the components enlarge and coalesce for $\epsilon_0 = 0.1862$. Thus we have gsep $= 0.1862$. For $\epsilon = 0.2089$ the components of $\Lambda_\epsilon(\mathbf{L})$ overlap and form a single multiply connected component. ■

Now we analyze stability of eigendecompositions of matrix pencils.

Definition 4.4.7. *Let* $\mathbf{L}$ *be a regular pencil. Then by an eigendecomposition of* $\mathbf{L}$ *we mean a decomposition of the form*

$$Y^{-1}\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1, \ldots, \mathbf{L}_m), \text{ where } \Lambda(\mathbf{L}_i) \cap \Lambda(\mathbf{L}_j) = \emptyset \text{ for } i \neq j$$

and X, Y *are invertible.*

Equivalently, an eigendecomposition of $\mathbf{L}$ can also be specified by a partition of the spectrum $\Lambda(\mathbf{L})$ of the form

$$\Lambda(\mathbf{L}) = \cup_{j=1}^{m} \Lambda_j, \text{ where } \Lambda_i \cap \Lambda_j = \emptyset \text{ for } i \neq j. \tag{4.8}$$

We analyze continuous evolution of eigendecompositions of matrix pencils. More precisely, given an eigendecomposition $\mathbf{L} = Y \operatorname{diag}(\mathbf{L}_1, \ldots, \mathbf{L}_m) X^{-1}$, we investigate the largest open ball in $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ on which X^{-1}, Y and $\mathbf{L}_j$ vary as continuous functions of $\mathbf{L}$. To that end, for $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$, we define

$$\mathbb{B}(\mathbf{L}, \epsilon) := \{\mathbf{G} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|) : |\!|\!|\mathbf{L} - \mathbf{G}|\!|\!|_{w,p} \leq \epsilon\}.$$

Let Λ be a nonempty subset of $\mathbf{L}$ and Γ be a rectifiable closed curve which isolates Λ from the rest of $\Lambda(\mathbf{L})$. Then the left and the right spectral projections associated with $\mathbf{L}$ and Λ are given by

$$\mathbb{P}_L := -\frac{1}{2\pi i}\int_\Gamma B\mathbf{L}(z)^{-1}dz \text{ and } \mathbb{P}_R := -\frac{1}{2\pi i}\int_\Gamma \mathbf{L}(z)^{-1}Bdz,$$

where $\mathbf{L}(z) := A - zB$.

Definition 4.4.8. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular. We say that an eigendecomposition of* $\mathbf{L}$ *of the form (4.8) is* ϵ*-stable if the spectral projections associated with* $\mathbf{L}$ *and* Λ_j *vary continuously when* $\mathbf{L}$ *varies in* $\mathbb{B}(\mathbf{L}, \epsilon)$ *for all* $j = 1 : m$.

Obviously, (4.8) is ϵ-stable if and only if an eigenvalue from Λ_i and an eigenvalue from Λ_j do not move simultaneously and coalesce when $\mathbf{L}$ varies in $\mathbb{B}(\mathbf{L}, \epsilon)$ for all $i \neq j$. In particular, if the set of ϵ-successors of Λ_j remain disjoint form the set of ϵ-successors of Λ_i for all $i \neq j$ then (4.8) is ϵ-stable. Consequently, if $\epsilon < \min_j \operatorname{gsep}(\Lambda_j)$, then (4.8) is ϵ-stable. Thus a sufficient condition for ϵ-stability of an eigendecomposition of $\mathbf{L}$ can be read off from the pseudospectra of $\mathbf{L}$.

Note that in order to analyze ϵ-stability of eigendecomposition of $\mathbf{L}$ of the form (4.8), it is enough to consider an eigendecomposition of the $\Lambda(\mathbf{L}) = \Lambda_1 \cup \Lambda_2$, where Λ_1 and Λ_2 are disjoint.

Proposition 4.4.9. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be of the form* $\mathbf{L} := \operatorname{diag}(\mathbf{L}_1, \mathbf{L}_2)$ *with* $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. *Then the eigendecomposition* $\Lambda(\mathbf{L}) = \Lambda(\mathbf{L}_1) \cup \Lambda(\mathbf{L}_2)$ *is* ϵ*-stable if and only if* $\epsilon < \operatorname{gsep}(\Lambda(\mathbf{L}_1))$.

Proof: We have $\Lambda_\epsilon(\mathbf{L}) = \Lambda_\epsilon(\mathbf{L}_1) \cup \Lambda_\epsilon(\mathbf{L}_2)$. Hence $\operatorname{gsep}(\Lambda(\mathbf{L}_1)) = \min\{\epsilon : \partial\Lambda_\epsilon(\mathbf{L}_1) \cap \partial\Lambda_\epsilon(\mathbf{L}_2) \neq \emptyset\}$. For $\epsilon := \operatorname{gsep}(\Lambda(\mathbf{L}_1))$, let $z_0 \in \partial\Lambda_\epsilon(\mathbf{L}_1) \cap \partial\Lambda_\epsilon(\mathbf{L}_2)$ be a common boundary point. Then there are $\mathbf{\Delta L}_1$ and $\mathbf{\Delta L}_2$ with $|\!|\!|\mathbf{\Delta L}_1|\!|\!|_{w,p} = \eta_{w,p}(z_0, \mathbf{L}_1) = \epsilon$ and $|\!|\!|\mathbf{\Delta L}_2|\!|\!|_{w,p} = \eta_{w,p}(z_0, \mathbf{L}_2) = \epsilon$ such that $z_0 \in \Lambda(\mathbf{L}_1 + \mathbf{\Delta L}_1) \cap \Lambda(\mathbf{L}_2 + \mathbf{\Delta L}_2)$. Taking

$\mathbf{\Delta L} := \mathrm{diag}(\mathbf{\Delta L}_1,\ \mathbf{\Delta L}_2)$ we see that two eigenvalues from $\Lambda(\mathbf{L}_1)$ and $\Lambda(\mathbf{L}_2)$ coalesce at z_0 as $\mathbf{L} \to \mathbf{L} + \mathbf{\Delta L}$. Hence the result follows. The proof is similar for homogeneous pencils. ■

Let Δ_1 be a component of $\Lambda_\epsilon(\mathbf{L})$ containing an eigenvalue $\lambda_1 \in \Lambda(\mathbf{L})$. Then each point in Δ_1 is an ϵ-successor of λ_1. As the size of the ϵ increases the component increases in size and coalesce with another component Δ_2 containing another eigenvalue λ_2 of $\Lambda_\epsilon(\mathbf{L})$ and form a bigger component Δ_{12}. If the value of ϵ is such that the component Δ_1 and Δ_2 just coalesce then the set of points in Δ_{12} which forms the common boundary of Δ_1 and Δ_2 is the set of common successors of λ_1 and λ_2. We now show that common successors are the points where eigenvalues of $\mathbf{L}$ coalesce.

Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ and recall that $h_{w,p}(z) := \|(1,z)\|_{w,p}$ and $H_{w,p}(c,s) := \|(c,s)\|_{w,p}$. Further, $\nabla_1 H_{w,p}(\lambda,\mu)$ and $\nabla_2 H_{w,p}(\lambda,\mu)$ denote the partial gradients of $H_{w,p}$ evaluated at (λ,μ).

Theorem 4.4.10. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be regular and $p^{-1} + q^{-1} = 1$. Let Δ be a component of $\Lambda_\epsilon(\mathbf{L})$. Then each $\lambda \in \Delta \cup \partial\Delta$ is an ϵ-successor of some eigenvalue in $\Lambda := \Lambda(\mathbf{L}) \cap \Delta$. Set $\tau := \sigma_{\min}(\mathbf{L}(\lambda))$. Then $\eta_{w,p}(\lambda,\mathbf{L}) = \dfrac{\tau}{h_{w^{-1},q}(\lambda)} \le \epsilon$ is the smallest value for which λ is an $\eta_{w,p}(\lambda,\mathbf{L})$-successor of an eigenvalue in Λ. Let the multiplicity of τ be m. Let U and V be $n\times m$ matrices whose columns are orthonormal left and right singular vectors of $\mathbf{L}(\lambda)$ corresponding to τ, respectively. Define*

$$\Delta A := -\eta_{w,p}(\lambda,\mathbf{L})\,\overline{(\nabla_1 H_{w^{-1},q})(1,\lambda)}\,UV^* \ \text{ and } \ \Delta B := \eta_{w,p}(\lambda,\mathbf{L})\,\overline{(\nabla_2 H_{w^{-1},q})(1,\lambda)}\,UV^*$$

and consider the pencil $\mathbf{\Delta L}(z) := \Delta A - z\Delta B$. Then an eigenvalue in Λ moves to λ as $\mathbf{L} \to \mathbf{L} + \mathbf{\Delta L}$. Similar result holds for homogeneous pencils.

Proof: By our construction $\eta_{w,p}(\lambda,\mathbf{L})$ is the smallest value of $|||\mathbf{\Delta L}|||_{w,p}$ such that λ is an eigenvalue of $\mathbf{L} + \mathbf{\Delta L}$. Suppose that λ is not a common boundary of the components of $\Lambda_\epsilon(\mathbf{L})$. Then evidently an eigenvalue from $\Lambda_1 \subset \Lambda(\mathbf{L})$ moves to λ as $\mathbf{L} \to \mathbf{L} + \mathbf{\Delta L}$.

Suppose that λ is a common boundary point of two components of $\Lambda_\epsilon(\mathbf{L})$. Let $\{\lambda_n\} \in \Delta$ be $\eta_{w,p}(\lambda,\mathbf{L})$-successors of an eigenvalue in Λ such that $\lambda_n \to \lambda$. Let $\tau_n := \sigma_{\min}(\mathbf{L}(\lambda_n))$. Then $\eta_{w,p}(\lambda_n,\mathbf{L}) = \dfrac{\tau_n}{H_{w^{-1},q}(1,\lambda_n)} \le \epsilon$. Since $\|\mathbf{L}(\lambda) - \mathbf{L}(\lambda_n)\| \to 0$ as $n \to \infty$, for sufficiently large n, $\mathbf{L}(\lambda_n)$ has exactly m singular values, $\tau_{n,1},\ldots,\tau_{n,m}$, counting multiplicity, of which τ_n is a member such that each $\tau_{n,j} \to \tau$ as $n \to \infty$. Set $\mathbb{D}_n := \mathrm{diag}(\tau_{n,1},\ldots,\tau_{n,m})$. Let U_n and V_n be n-by-m matrices whose columns are orthonormal left and right singular vectors of $\mathbf{L}(\lambda_n)$ corresponding the singular values in $\mathbb{D}_n$. Then $\mathbf{L}(\lambda_n)V_n = U_n\mathbb{D}_n$ and $\mathbf{L}(\lambda_n)^*U_n = V_n\mathbb{D}_n$. Set $\Delta A_n := -\eta_{w,p}(\lambda_n,\mathbf{L})\,\overline{(\nabla_1 H_{w^{-1},q})(1,\lambda_n)}\,U_nV_n$ and $\Delta B_n := \eta_{w,p}(\lambda_n,\mathbf{L})\overline{(\nabla_2 H_{w^{-1},q})(1,\lambda_n)}\,U_nV_n$.

Then

$$\|\mathbf{\Delta L}_n\|_{w,p} = \frac{\tau_n}{H_{w^{-1},q}(1,\lambda_n)} = \eta_{w,p}(\lambda_n,\mathbf{L}) < \eta_{w,p}(\lambda,\mathbf{L}) = \epsilon.$$

Now we show that $\|\mathbf{\Delta L}_n - \mathbf{\Delta L}\|_{w,p} \to 0$ as $n \to \infty$. Set $T(\lambda) := \sqrt{\mathbf{L}(\lambda)\,\mathbf{L}^*(\lambda)}$. Since $\mathbf{L}(\lambda)V = \tau U$, $\mathbf{P} := UU^*$ is the projection associated with $T(\lambda)$ and τ. Similarly, $\mathbf{P}_n := U_nU_n^*$ is the spectral projection associated with $T(\lambda_n)$ and $\{\tau_{n,1},\ldots,\tau_{n,m}\}$. Note that $\|\mathbf{P}_n - \mathbf{P}\| \to 0$ as $n \to \infty$. Now we have

$$U_nV_n^* = U_n(\mathbf{L}(\lambda_n)^{-1}\,U_n\,\mathbb{D}_n)^* = U_n\mathbb{D}_n^*\,U_n^*\,(\mathbf{L}(\lambda_n)^{-1})^* = U_n\mathbb{D}_nU_n^*\,(\mathbf{L}(\lambda_n)^{-1})^*. \tag{4.9}$$

Hence $T(\lambda_n)^2\,U_nU_n^* = \mathbf{L}(\lambda_n)\mathbf{L}(\lambda_n)^*U_n\,U_n^* = \mathbf{L}(\lambda_n)V_n\mathbb{D}_nU_n^* = U_n\mathbb{D}_n^2\,U_n^* \Rightarrow T(\lambda_n)^2\,\mathbf{P}_n = U_n\mathbb{D}_n^2\,U_n^* \Rightarrow (T(\lambda_n)\,\mathbf{P}_n)^2 = U_n\mathbb{D}_n^2\,U_n^* \Rightarrow T(\lambda_n)\mathbf{P}_n = U_n\mathbb{D}_nU_n^*$. Since $V_n = \mathbf{L}(\lambda_n)^{-1}U_n\mathbb{D}_n$ and $T(\lambda_n)\mathbf{P}_n = \mathbf{P}_nT(\lambda_n)$, by (4.9), we have

$$\begin{aligned} U_nV_n^* &= U_n\mathbb{D}_nU_n^*(\mathbf{L}(\lambda_n)^{-1})^* \\ &= T(\lambda_n)\mathbf{P}_n(\mathbf{L}(\lambda_n)^{-1})^* \to T(\lambda)\mathbf{P}(\mathbf{L}(\lambda)^{-1})^* \\ &= T(\lambda)UU^*(\mathbf{L}(\lambda)^{-1})^* = \tau UU^*(\mathbf{L}(\lambda)^{-1})^*. \end{aligned}$$

Since $\mathbf{L}(\lambda)V = \tau U \Rightarrow V^* = \tau U^*(\mathbf{L}(\lambda)^{-1})^*$ and $T(\lambda)U = \tau U$, we see that $U_nV_n^* \to UV^*$ as $n \to \infty$. Consequently, we have $\|\mathbf{\Delta L}_n - \mathbf{\Delta L}\|_{w,p} \to 0$ as $n \to \infty$. ■

The following result characterizes ϵ-stability of eigendecompositions of a regular pencil $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$.

Theorem 4.4.11. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ be regular. Then an eigendecomposition of $\mathbf{L}$ of the form $\Lambda(\mathbf{L}) = \Lambda_1 \cup \Lambda_2$ is ϵ-stable if and only if $\epsilon < \mathrm{gsep}(\Lambda_1)$.*

Proof: Let $\epsilon := \mathrm{gsep}(\Lambda)$. Without loss of generality assume that $\mathbf{L}$ is nonhomogeneous. Then there are at least two components Δ_1 and Δ_2 of $\Omega := \{z \in \mathbb{C} : \eta_{w,p}(z,\mathbf{L}) < \epsilon\}$ containing eigenvalues from Λ and $\Lambda(\mathbf{L}) \setminus \Lambda$, respectively, such that $\partial\Delta_1 \cap \partial\Delta_2 \neq \emptyset$. Let $\lambda \in \partial\Delta_1 \cap \partial\Delta_2$. Set $\Delta A := -\eta_{w,p}(\lambda,\mathbf{L})\overline{(\nabla_1 H_{w^{-1},q})(1,\lambda)}UV^*$, $\Delta B := \eta_{w,p}(\lambda,\mathbf{L})\overline{(\nabla_2 H_{w^{-1},q})(1,\lambda)}UV^*$, where columns of U and V are orthonormal left and right singular vectors of $\sigma_{\min}(\mathbf{L}(\lambda))$. Then $\|\mathbf{\Delta L}\|_{w,p} = \eta_{w,p}(\lambda,\mathbf{L})$ and by Theorem 2.4.11, λ is a multiple eigenvalue of $\mathbf{L} + \mathbf{\Delta L}$. To prove the result we show that an eigenvalue from λ and an eigenvalue from $\Lambda(\mathbf{L}) \setminus \Lambda$ move simultaneously and coalesce at z as $\mathbf{L} \to \mathbf{L} + \mathbf{\Delta L}$.

Since λ is a boundary point of Δ_1, by Proposition 4.4.10, an eigenvalue in Λ moves to z as $\mathbf{L} \to \mathbf{L} + \mathbf{\Delta L}$. But z is also a boundary point of Δ_2 hence by Proposition 4.4.10, an eigenvalue in $\Lambda(\mathbf{L}) \setminus \Lambda$ moves to z as $\mathbf{L} \to \mathbf{L} + \mathbf{\Delta L}$. This shows that $\mathbf{L} \to \mathbf{L} + \mathbf{\Delta L}$, two eigenvalues from Λ and $\Lambda(\mathbf{L}) \setminus \Lambda$ moves simultaneously and coalesce at z. This completes the proof. ■

The above result shows that a lower bound of gsep provides a sufficient condition for ϵ-stability of eigendecompositions of matrix pencils. We now derive lower bounds of gsep along the way we establish relationship between gsep and sep_λ.

4.5 Lower bounds of geometric separation

Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ be regular. Then result provides a general framework for obtaining lower bounds of gsep.

Theorem 4.5.1. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular and* X *and* Y *be nonsingular matrices such that* $Y^{-1}\mathbf{L}X = \begin{bmatrix}\mathbf{L}_1 & 0 \\ 0 & \mathbf{L}_2\end{bmatrix}$, *where* $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) \neq \emptyset$. *Set* $\Lambda_1 := \Lambda(\mathbf{L}_1)$. *If there exists a strictly increasing function* ϕ *such that* $\Lambda_\epsilon(\mathbf{L}) \subseteq \Lambda_{\phi(\epsilon)}(\mathbf{L}_1) \cup \Lambda_{\phi(\epsilon)}(\mathbf{L}_2)$, *then* $\phi^{-1}\big(\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)\big) \leq \mathrm{gsep}(\Lambda_1)$ *and* $\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2) \leq \phi\big(\mathrm{gsep}(\Lambda_1)\big)$.

Proof: Note that $\epsilon < \mathrm{gsep}(\Lambda_1)$ if $\Lambda_{\phi(\epsilon)}(\mathbf{L}_1)$ and $\Lambda_{\phi(\epsilon)}(\mathbf{L}_2)$ are disjoint. But $\Lambda_{\phi(\epsilon)}(\mathbf{L}_1)$ and $\Lambda_{\phi(\epsilon)}(\mathbf{L}_2)$ remain disjoint if and only if $\phi(\epsilon) < \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$. This shows that as ϕ is strictly increasing, we have $\epsilon < \mathrm{gsep}(\Lambda_1)$ whenever $\epsilon < \phi^{-1}(\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2))$. Hence $\phi^{-1}(\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)) \leq \mathrm{gsep}(\Lambda_1)$ and $\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2) \leq \phi(\mathrm{gsep}(\Lambda_1))$. ■

Now we have the following bounds which gives the upper bound of ϵ for the stability of eigendecomposition of matrix pencil.

Theorem 4.5.2. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular and* X *and* Y *be nonsingular matrices such that* $Y^{-1}\mathbf{L}X = \mathrm{diag}(\mathbf{L}_1, \mathbf{L}_2)$, *where* $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) \neq \emptyset$. *Let* $X := [X_1, X_2]$ *and* $(Y^{-1})^* := [Y_1, Y_2]$ *be conformal partitions of* X *and* $(Y^{-1})^*$. *Set* $\kappa_1 := \|Y^{-1}\|\,\|X\|$ *and* $\kappa_2 := \|X_1\|\,\|Y_1^*\| + \|X_2\|\,\|Y_2^*\|$. *Set* $\Lambda := \Lambda(\mathbf{L}_1)$. *Then we have*

$$(a)\ \mathrm{gsep}(\Lambda) \geq \frac{\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)}{\kappa_1} \quad (b)\ \mathrm{gsep}(\Lambda) \geq \frac{\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)}{\kappa_2}.$$

Proof: By Theorem 4.2.4 (b), we have $\phi(\epsilon) = \kappa_1\epsilon$ for the first bound and by Theorem 4.2.4 (c), we have $\phi(\epsilon) = \kappa_2\epsilon$ for the second bound. Hence the results follow from Theorem 4.5.1. ■

For the rest of this section, we assume that $\mathbf{L}$ is of the form

$$\mathbf{L} := \begin{bmatrix}\mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2\end{bmatrix}, \tag{4.10}$$

where $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. Then we have the following bounds.

Theorem 4.5.3. *Let* $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|_2), \kappa_1$ *and* κ_2 *be as given in Theorem 4.2.10. Set* $\Lambda := \Lambda(\mathbf{L}_1)$. *Then we have*

$$(a)\ \mathrm{gsep}(\Lambda) \geq \frac{\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)}{\kappa_1}, \quad (b)\ \mathrm{gsep}(\Lambda_1) \geq \frac{\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)}{\kappa_2}.$$

Proof: By Theorem 4.2.10 we have $\phi(\epsilon) = \kappa_1\epsilon$ for the first bound and $\phi(\epsilon) = \kappa_2\epsilon$ for the second bound. Hence the results follow from Theorem 4.5.1. ■

Next, we have the following lower bound for gsep.

Theorem 4.5.4. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ *and* κ *be as in Theorem 4.2.12. Set* $\Lambda := \Lambda(\mathbf{L})$, $\mathrm{sep}_\lambda := \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)$ *and* $\mathrm{sep} := \mathrm{sep}^{(r)}(\mathbf{L}_1, \mathbf{L}_2)$. *Then for* $r^{-1} + s^{-1} = 1$, *we have*

$$\mathrm{gsep}(\Lambda_1) \geq \frac{2\mathrm{sep}_\lambda^2}{\kappa + \sqrt{\kappa^2 + 4\mathrm{sep}_\lambda^2}} \geq \frac{2^{1-2/s}\,\mathrm{sep}^2}{(\kappa + \sqrt{\kappa^2 + 4^{1-1/s}\,\mathrm{sep}^2})}.$$

Proof: By Theorem 4.2.12, we have $\phi(\epsilon) = \epsilon\sqrt{1 + \frac{\kappa}{\epsilon}}$. Since ϕ is strictly increasing on $[0\ \infty)$ and $\phi^{-1}(\epsilon) = \dfrac{2\epsilon^2}{\kappa + \sqrt{\kappa^2 + 4\epsilon^2}}$, the first inequality follows from Theorem 4.5.1. Since ϕ^{-1} is increasing, second inequality follows from Theorem 4.3.10. ■

Next, we have the following lower bounds.

Theorem 4.5.5. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be given by (4.10). Let* $\mathbf{L}_1 Z_1 - Z_2\mathbf{L}_2 = -\mathbf{L}_{12}$. *Set* $\kappa := |\!|\!|\mathbf{L}_{12}|\!|\!|_{w,p}$ *and* $\Lambda := \Lambda(\mathbf{L}_1)$. *Then we have*

$$(a)\,\mathrm{gsep}(\Lambda) \geq \frac{\mathrm{sep}_\lambda^2(\mathbf{L}_1, \mathbf{L}_2)}{\kappa + \mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)},\quad (b)\,\mathrm{gsep}(\Lambda) \geq \frac{\mathrm{sep}_\lambda(\mathbf{L}_1, \mathbf{L}_2)}{1 + \|Z_2\|_2 + \|Z_1\|_2}.$$

Proof: By Theorem 4.2.13(b), we have $\phi(\epsilon) = \dfrac{\epsilon + \sqrt{\epsilon^2 + 4\epsilon\kappa}}{2}$. Since ϕ is strictly increasing on $[0\ \infty)$ and $\phi^{-1}(\epsilon) = \dfrac{\epsilon^2}{(\epsilon + \kappa)}$, then the first bound follows from Theorem 4.5.1.

Now by Theorem 4.2.13(a) we have $\phi(\epsilon) = \epsilon(1 + \|Z_1\| + \|Z_2\|)$. Hence the second bound follows from Theorem 4.5.1. ■

4.6 Numerical examples

We now present some numerical examples to illustrate gsep, sep_λ and various bounds discussed above. We consider regular pencil $\mathbf{L} \in \mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$ of the form $\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_{12} \\ 0 & \mathbf{L}_2 \end{bmatrix}$, where $\mathbf{L}_1(z) := A_1 - zB_1, \mathbf{L}_2(z) := A_2 - zB_2, \mathbf{L}_{12}(z) := A_{12} - zB_{12}$ and $\Lambda(\mathbf{L}_1) \cap \Lambda(\mathbf{L}_2) = \emptyset$. Also, we consider only absolute perturbation, that is, $w := (1, 1)$ and the space $\mathbb{L}^2_w(\mathbb{C}^{n\times n}, \|\cdot\|_2)$. For the following examples, A_i and B_i remain the same and we only vary the matrices A_{12} and B_{12}. Therefore for all these examples, we have $\Lambda(\mathbf{L}_1) = \{0.05\}$ and $\Lambda(\mathbf{L}_2) = \{-0.05\}$. Table 4.1 gives the values of gsep and its various lower bounds for these pencils. For these examples, the table shows that the lower bounds provide good estimates of gsep.

Example 4.6.1. Consider $A_1 := \begin{bmatrix} 0.1 & 4 & 20 \\ 0 & 0.1 & 5 \\ 0 & 0 & 0.1 \end{bmatrix}, A_2 := \begin{bmatrix} -0.2 & 3 & 100 \\ 0 & -0.2 & 50 \\ 0 & 0 & -0.2 \end{bmatrix},$
$B_1 := \begin{bmatrix} 2 & 30 & 10 \\ 0 & 2 & 50 \\ 0 & 0 & 2 \end{bmatrix}, B_2 := \begin{bmatrix} 4 & 6 & 20 \\ 0 & 4 & 40 \\ 0 & 0 & 4 \end{bmatrix},$
$A_{12} = \begin{bmatrix} 2.2857 \times 10^{-2} & -2.3547 \times 10^{-2} & -6.8279 \times 10^{-2} \\ 9.3914 \times 10^{-2} & -9.6719 \times 10^{-2} & -2.8049 \times 10^{-1} \\ 2.8585 \times 10^{-1} & -2.9443 \times 10^{-1} & -8.5382 \times 10^{-1} \end{bmatrix}$ and
$B_{12} = \begin{bmatrix} 1.2606 \times 10^{-1} & -4.6007 \times 10^{-1} & 7.0963 \times 10^{-3} \\ 1.8156 \times 10^{-1} & -6.6259 \times 10^{-1} & 1.0235 \times 10^{-2} \\ 1.4481 \times 10^{-1} & -5.2845 \times 10^{-1} & 8.1625 \times 10^{-3} \end{bmatrix}.$

Figure 4.2(a) shows a portion of the contour plots of $\Lambda_\epsilon(\mathbf{L})$ for various values of ϵ. This shows that components coalesce for $\epsilon := 4.6626 \times 10^{-9}$. Hence gsep $= 4.6626 \times 10^{-9}$.

Next, we replace A_{12} and B_{12} by the following matrices.

Example 4.6.2. Consider $A_{12} = \begin{bmatrix} 9.5013 \times 10^{-1} & 4.8598 \times 10^{-1} & 4.5647 \times 10^{-1} \\ 2.3114 \times 10^{-1} & 8.9129 \times 10^{-1} & 1.8504 \times 10^{-2} \\ 6.0684 \times 10^{-1} & 7.6209 \times 10^{-1} & 8.2141 \times 10^{-1} \end{bmatrix},$

$B_{12} = \begin{bmatrix} 4.1027 \times 10^{-1} & 3.5287 \times 10^{-1} & 1.3889 \times 10^{-1} \\ 8.9365 \times 10^{-1} & 8.1317 \times 10^{-1} & 2.0277 \times 10^{-1} \\ 5.7891 \times 10^{-2} & 9.8613 \times 10^{-3} & 1.9872 \times 10^{-1} \end{bmatrix}.$

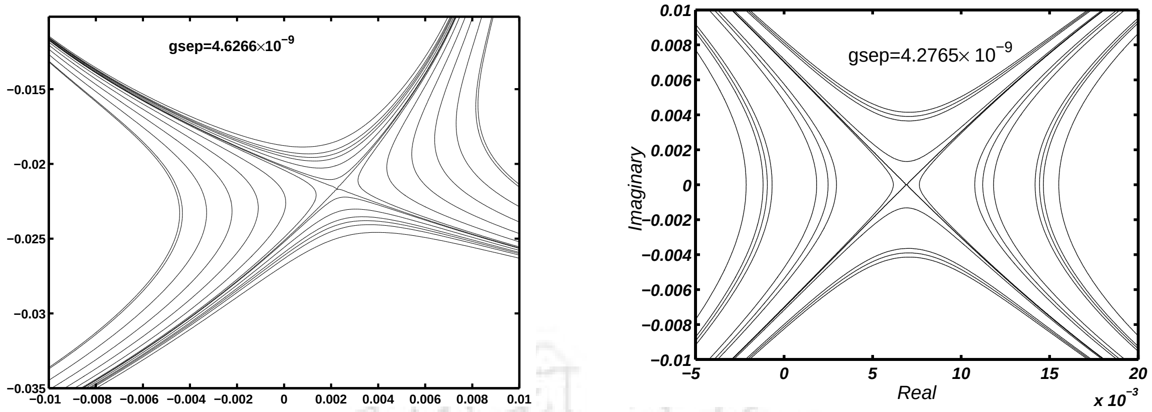
The figure 4.2(b) shows a portion of the contour plots of $\Lambda_\epsilon(\mathbf{L})$ for various values of ϵ. It shows that the components coalesce for $\epsilon := 4.2765 \times 10^{-9}$ giving gsep $= 4.2765 \times 10^{-9}$.

Now we choose A_{12} and B_{12} as follows.

Example 4.6.3. Consider

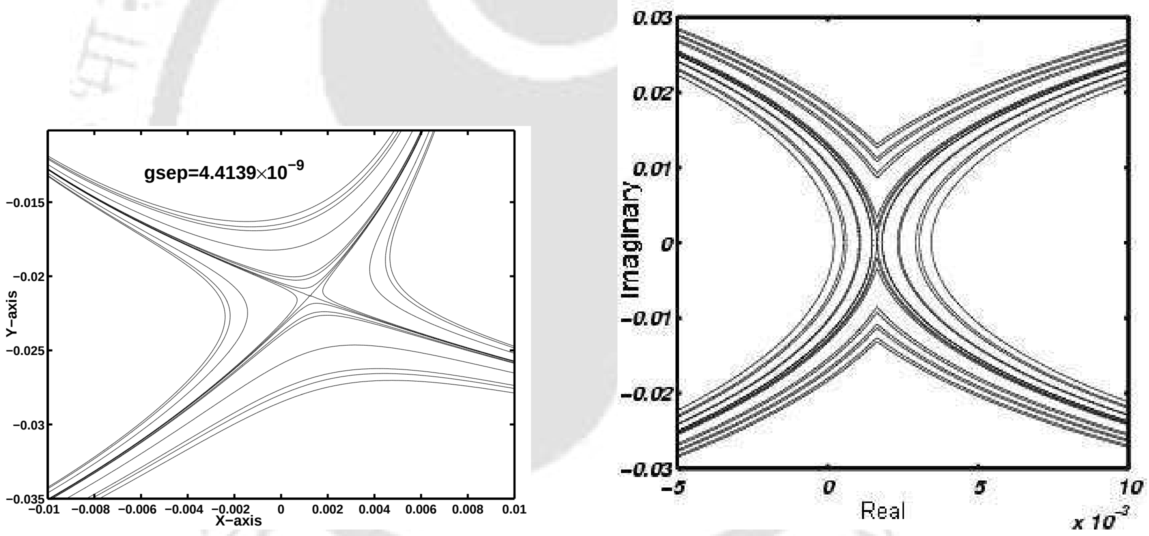
$$A_{12} = \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix} \text{ and } B_{12} = \begin{bmatrix} 1 & 5 \times 10^{-1} & 3.333 \times 10^{-1} \\ 5 \times 10^{-1} & 3.333 \times 10^{-1} & 2.5 \times 10^{-1} \\ 3.333 \times 10^{-1} & 2.5 \times 10^{-1} & 2 \times 10^{-1} \end{bmatrix}.$$

Then figure 4.3(a) shows a portion of the contour plots of $\Lambda_\epsilon(\mathbf{L})$ for various values of ϵ. It shows that the components coalesce for $\epsilon := 4.4139 \times 10^{-9}$ giving gsep $= 4.4139 \times 10^{-9}$. On the other hand, figure 4.3(b) shows a portion of the contour plot of $\Lambda_\epsilon(\text{diag}(\mathbf{L}_1, \mathbf{L}_2))$ for various values of ϵ. In this case, the components coalesce for $\epsilon := 5.1642 \times 10^{-5}$ giving $\text{sep}_\lambda = 5.1642 \times 10^{-5}$.



(a) (b)

Figure 4.2: Plot (a) shows that the components of $\Lambda_\epsilon(\mathbf{L})$ coalesce for $\epsilon = 4.6266 \times 10^{-9}$ giving gsep $= 4.6266 \times 10^{-9}$. Plot (b) shows that the components coalesce for $\epsilon = 4.2765 \times 10^{-9}$ giving gsep $= 4.2765 \times 10^{-9}$.



(a) (b)

Figure 4.3: Plot (a) shows that the components of $\Lambda_\epsilon(\mathbf{L})$ coalesce for $\epsilon = 4.4139 \times 10^{-9}$ giving gsep $= 4.4139 \times 10^{-9}$. Plot (b) shows that the components coalesce for $\epsilon = 5.1642 \times 10^{-5}$ giving $\mathrm{sep}_\lambda = 5.1642 \times 10^{-5}$.

Example	4.6.1	4.6.2	4.6.3
gsep	4.6266×10^{-9}	4.2765×10^{-9}	4.4139×10^{-9}
Theorem 4.5.4	3.2613×10^{-12}	1.18118×10^{-9}	3.2612×10^{-12}
Theorem 4.5.5(a)	3.2613×10^{-12}	1.18115×10^{-9}	3.2613×10^{-12}
Theorem 4.5.5(b)	3.15349×10^{-12}	4.3259×10^{-12}	2.8989×10^{-12}

Table 4.1: The table gives values of gsep and various lower bounds for the pencils given in Examples 4.6.1, 4.6.2 and 4.6.3.

Chapter 5

Sensitivity Analysis of Eigenvalues

We develop a general framework for the sensitivity analysis of eigenvalues of matrix pencils and matrix polynomials. We show that our treatment unifies various measures of the sensitivity of simple eigenvalues of matrix pencils and matrix polynomials proposed in the literature. We propose a general measure of the sensitivity of simple eigenvalues in an abstract framework of Banach spaces of matrix polynomials and show that various measures of the sensitivity of simple eigenvalues proposed in the literature follow from our general measure as special cases.

5.1 Introduction

Let $A \in \mathbb{C}^{n\times n}$. Let λ be a simple eigenvalue of A and, y and x, respectively, be left and right eigenvectors of A corresponding to λ, that is,

$$Ax = \lambda x \text{ and } y^*A = \lambda y^*.$$

We refer to (λ, y, x) as simple eigentriple of A. Having chosen a norm $\|\cdot\|$ on $\mathbb{C}^n$, in this setup, we treat y as a functional on $\mathbb{C}^n$ and hence $\|y\|$ would always mean the dual norm, that is, $\|y\| := \sup\{|y^*x| : x \in \mathbb{C}^n \text{ and } \|x\| = 1\}$. We, often, consider normalized eigentriple (λ, y, x) of A so that $\|x\| = 1$ and $\|y\| = 1$. It is well known [52] that the sensitivity of λ to small perturbations in A is measured by the condition number

$$\text{cond}(\lambda) := \frac{\|x\|\,\|y\|}{|y^*x|}.$$

Now, consider a matrix pencil $\mathbf{L}(z) := A - zB$, where $A, B \in \mathbb{C}^{n\times n}$. Then $\mathbf{L}$ is said to be regular if $\det(\mathbf{L}(z)) \neq 0$ for some $z \in \mathbb{C}$. The spectrum $\Lambda(\mathbf{L})$ of a regular pencil $\mathbf{L}$ is given by

$$\Lambda(\mathbf{L}) := \{z \in \mathbb{C} : \det(\mathbf{L}(z)) = 0\}.$$

It is possible for $\mathbf{L}$ to have an infinite eigenvalue. However, the case of an infinite eigenvalue can be resolved by considering $\Lambda(\mathbf{L})$ as a subset of $\mathbb{C}_\infty$, the one-point compactification of $\mathbb{C}$, and adding ∞ to $\Lambda(\mathbf{L})$ whenever $\det(B) = 0$. An alternative approach to resolving an infinite eigenvalue of $\mathbf{L}$ is to consider the reverse pencil rev$\mathbf{L}$ given by rev$\mathbf{L}(z) := B - zA$. Then ∞ is an eigenvalue of $\mathbf{L}$ if and only if 0 is an eigenvalue of rev$\mathbf{L}$. Yet another approach is to consider the homogenous form of $\mathbf{L}$, that is, $\mathbf{L}(c, s) := cA - sB$. Then defining the homogeneous spectrum $\Lambda(\mathbf{L})$ by

$$\Lambda(\mathbf{L}) := \{(c, s) \in \mathbb{C}^2 \setminus \{0\} : \det(\mathbf{L}(c, s)) = 0\}$$

it follows that an infinite eigenvalue of $\mathbf{L}(z) = A - zB$ is represented by $(0, 1)$. Note that, in this case, $\Lambda(\mathbf{L})$ is a subset of the projective space $\mathbb{CP}^1$. Normalizing the eigenvalues $(c, s) \in \Lambda(\mathbf{L})$ as $|c|^2 + |s|^2 = 1$, the spectrum $\Lambda(\mathbf{L})$ can be considered as a subset of the unit sphere $\mathbb{S}^1 := \{(c, s) \in \mathbb{C}^2 : |c|^2 + |s|^2 = 1\}$.

Let λ be a simple eigenvalue of a regular pencil $\mathbf{L}(z) = A - zB$ and y, x, respectively, be left and right eigenvectors of $\mathbf{L}$ corresponding to λ, that is, $\mathbf{L}(\lambda)x = 0$ and $y^*\mathbf{L}(\lambda) = 0$. As before, we refer to (λ, y, x) as a simple eigentriple of the pencil $\mathbf{L}$. Similarly, when $\mathbf{L}$ is homogeneous and $(c, s) \in \Lambda(\mathbf{L})$, we consider a homogeneous eigentriple $((c, s), y, x)$ of $\mathbf{L}$, that is, $\mathbf{L}(c, s)x = 0$ and $y^*\mathbf{L}(c, s) = 0$. How to measure the sensitivity of a simple eigenvalue λ (or (c, s)) to small perturbations in the pencil $\mathbf{L}$? In other words, how to define the condition number cond$(\lambda, \mathbf{L})$ of a simple eigenvalue λ of $\mathbf{L}$? Unlike the condition number of a simple eigenvalue a matrix, for a simple eigenvalue of a matrix pencil more than one condition number has been proposed in the literature. Let $((\lambda, \mu), y, x)$ be a simple (homogeneous) eigentriple of $\mathbf{L}$. Then Stewart [41, 42] defined cond$(\lambda, \mu, \mathbf{L})$ by

$$\mathrm{cond}(\lambda, \mu, \mathbf{L}) := \frac{\|x\|_2 \, \|y\|_2}{\sqrt{|y^*Ax|^2 + |y^*Bx|^2}}$$

and showed that cond$(\lambda, \mu, \mathbf{L})$ measures the sensitivity of (λ, μ) to small perturbations in $\mathbf{L}$ when the magnitude of a perturbation $\Delta\mathbf{L}(c, s) := c\Delta A - s\Delta B$ is measured by $\|[\Delta A, \ \Delta B]\|_2$. Dedieu [17] considered cond$(\lambda, \mu, \mathbf{L})$ as defined above but measured the magnitude of a perturbation $\Delta\mathbf{L}(c, s) := c\Delta A - s\Delta B$ by $\|[\Delta A, \ \Delta B]\|_F$. For a simple (nonhomogeneous) eigentriple (λ, y, x) of $\mathbf{L}$, Higham et al. [26] defined cond$(\lambda, \mathbf{L})$ by

$$\mathrm{cond}(\lambda, \mathbf{L}) := \frac{\|x\| \, \|y\| (\alpha + |\lambda| \, \beta)}{|y^*Bx|}$$

and showed that cond$(\lambda, \mathbf{L})$ measures the sensitivity of λ to small perturbations in $\mathbf{L}$ when the magnitude of a perturbation $\Delta\mathbf{L}(z) := \Delta A - z\Delta B$ is measured by $\max(\|\Delta A\|/\alpha, \ \|\Delta B\|/\beta)$ with the convention that $\|\Delta A\| = 0$ (resp., $\|\Delta B\| = 0$) when $\alpha = 0$ (resp., $\beta = 0$). Here we have considered only the absolute change in λ.

Finally, consider a matrix polynomial $\mathbf{L}(z) = \sum_{i=0}^{m} z^i A_i$, where $A_i \in \mathbb{C}^{n\times n}$. A polynomial $\mathbf{L}$ is said to be regular if $\det(\mathbf{L}(\lambda)) \neq 0$ for some $\lambda \in \mathbb{C}$. The spectrum of a regular polynomial $\mathbf{L}$ is given by

$$\Lambda(\mathbf{L}) := \{\lambda \in \mathbb{C} : \det(\mathbf{L}(\lambda)) = 0\}.$$

Again, it is possible for $\mathbf{L}$ to have an infinite eigenvalue. However, the case of an infinite eigenvalue can be resolved by considering the reverse polynomial $\mathrm{rev}\mathbf{L}(z) = z^m\mathbf{L}(1/z)$. Then ∞ is an eigenvalue of $\mathbf{L}$ if and only if 0 is an eigenvalue of $\mathrm{rev}\mathbf{L}$. An alternative approach is to consider the homogeneous form of $\mathbf{L}$, that is, $\mathbf{L}(c,s) := \sum_{i=0}^{m} c^{m-i}s^iA_i$. Then considering the homogeneous spectrum $\Lambda(\mathbf{L})$ given by

$$\Lambda(\mathbf{L}) := \{(c,s) \in \mathbb{C}^2 \setminus \{0\} : \det(\mathbf{L}(c,s)) = 0\} \subset \mathbb{CP}^1$$

it follows that an infinite eigenvalue of $\mathbf{L}(z) := \sum_{j=0}^{m} A_jz^j$ is represented by $(0,1)$. Let (λ, y, x) be a simple eigentriple of $\mathbf{L}$, that is, λ is simple, $\mathbf{L}(\lambda)x = 0$ and $y^*\mathbf{L}(\lambda) = 0$. Again one may ask: How to define the condition number $\mathrm{cond}(\lambda, \mathbf{L})$ of λ so as to measure the sensitivity of λ to small perturbations in $\mathbf{L}$? As in the case of matrix pencils, more than one condition number of λ has been proposed in the literature [46, 18]. For a simple eigentriple (λ, y, x) of $\mathbf{L}$, Tisseur [46] defined $\mathrm{cond}(\lambda, \mathbf{L})$ by

$$\mathrm{cond}(\lambda, \mathbf{L}) := \frac{\|x\|\,\|y\|\sum_{j=0}^{m}\alpha_j|\lambda|^j}{|y^*\partial_z\mathbf{L}(\lambda)x|}$$

and showed that $\mathrm{cond}(\lambda, \mathbf{L})$ measures the sensitivity of λ to small perturbations in $\mathbf{L}$ when the magnitude of a perturbation $\Delta\mathbf{L}(z) := \sum_{j=0}^{m}\Delta A_jz^j$ is measured by $\max(\|\Delta A_0\|/\alpha_0, \cdots, \|\Delta A_m\|/\alpha_m)$ with the convention that $\|\Delta A_j\| = 0$ whenever $\alpha_j = 0$. Here α_j's are nonnegative weights and $\partial_z\mathbf{L}(\lambda)$ is the derivative of $\mathbf{L}(z)$ evaluated at λ. On the other hand, considering a simple eigentriple $((\lambda, \mu), y, x)$ of the homogeneous form of $\mathbf{L}$, Dedieu et al. [18] defined $\mathrm{cond}(\lambda, \mu, \mathbf{L})$ by

$$\mathrm{cond}(\lambda, \mu, \mathbf{L}) := \frac{\|(\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)\|_2\,\|x\|_2\,\|y\|_2}{|y^*(\overline{\mu}\partial_c\mathbf{L}(\lambda,\mu) - \overline{\lambda}\partial_s\mathbf{L}(\lambda,\mu))x|}$$

and showed that $\mathrm{cond}(\lambda, \mu, \mathbf{L})$ measures the sensitivity of $(\lambda, \mu) \in \Lambda(\mathbf{L})$ to small perturbations in $\mathbf{L}$ when the magnitude of a perturbation $\Delta\mathbf{L}(c,s) := \sum_{j=0}^{m}\Delta A_jc^{m-j}s^j$ is measured by $\|[A_0, \cdots, A_m]\|_F$. Here $\partial_c\mathbf{L}(\lambda, \mu)$ and $\partial_s\mathbf{L}(\lambda, \mu)$ are the partial derivatives of $\mathbf{L}(c,s)$ evaluated at (λ, μ). For the special case of quadratic polynomials of the form $\mathbf{L}(z) := Az^2 + Bz + C$, Nichols et al. [36] defined $\mathrm{cond}(\lambda, \mathbf{L})$ by

$$\mathrm{cond}(\lambda, \mathbf{L}) := \frac{\|(1, \lambda, \lambda^2)\|_2\,\|y^*A\|_2\,\|x\|_2}{|y^*\partial_z\mathbf{L}(\lambda)x|}$$

and showed that $\mathrm{cond}(\lambda, \mathbf{L})$ measures the sensitivity of λ to small perturbations in $\mathbf{L}$ of the form $A\Delta\mathbf{L}(z) := A(\Delta Az^2 + \Delta Bz + \Delta C)$ when the magnitude of the perturbation is measured by $\|[\Delta A, \Delta B, \Delta C]\|_2$.

This raises a natural question: What is it that makes these various measures of sensitivity of eigenvalues differ from one another? What is the big picture? With a view to answering these questions, we develop a general framework for sensitivity analysis of eigenvalues. We provide a general measure of sensitivity of eigenvalues from which various measures of sensitivity considered above follow as special cases. Our analysis not only provides a unified treatment of the sensitivity analysis of eigenvalues but also makes the heart of the matter, so to speak, easy to understand and adds a new perspective to sensitivity analysis of polynomial eigenvalue problems.

5.2 Preliminaries

Let X and Y be Banach spaces and U be an open subset of X. A function $f : U \rightarrow Y$ is said to be (Fréchet) differentiable at $x \in U$ if there exists a bounded linear transformation $A_x : X \rightarrow Y$ such that

$$\lim_{\|h\|\rightarrow 0} \frac{\|f(x+h) - f(x) - A_x h\|}{\|h\|} = 0.$$

When f is differentiable at x, we write $\mathrm{D}f(x) = A_x$ and refer to $\mathrm{D}f(x)$ as the derivative of f at x. The derivative $\mathrm{D}f(x)$ is also referred to as the gradient of f at x and is denoted by $\nabla f(x)$. Note that $\mathrm{D}f(x) : X \rightarrow Y$ is a bounded linear transformation. If $\mathrm{D}f(x)$ exists for all $x \in U$ then f is said to be differentiable on U. Further, if f is differentiable and $\mathrm{D}f(x)$ is continuous in x for all $x \in U$ then f is said to be continuously differentiable on U or simply f is C^1 on U. Next, let $u \in X$ be a unit vector, that is, $\|u\| = 1$. For $x \in U$ and $t \in \mathbb{R}$, define

$$\mathrm{D}_u f(x) := \lim_{t\rightarrow 0} \frac{f(x+tu) - f(x)}{t}$$

if the limit exists. Then $\mathrm{D}_u f(x)$ is said to be the directional derivative of f at $x \in X$ in the direction u. If $\mathrm{D}f(x)$ exists then it follows that $\mathrm{D}_u f(x) = \mathrm{D}f(x)u$.

Let $(X^*, \|\cdot\|_*)$ be the dual space of a Banach space $(X, \|\cdot\|)$, that is, X^* is the set of bounded linear functionals on X and $\|x^*\|_* := \sup\{|x^*(x)| : \|x\| = 1\}$. Then $\|.\|_*$ is called the dual norm of $\|\cdot\|$. Consider the cartesian product $X^m := X \times \cdots \times X$. For $x \in X^m$, define $|\!|\!|x|\!|\!|_p := \|(\|x_1\|, \ldots, \|x_m\|)\|_p$, where $\|\cdot\|_p$ is the Hölder's p-norm on $\mathbb{C}^m$ and $1 \leq p \leq \infty$. Then $|\!|\!|\cdot|\!|\!|_p$ defines a norm on X^m. We denote the space $(X^m, |\!|\!|\cdot|\!|\!|_p)$ by $\ell_m^p(X, \|\cdot\|)$, where $\|\cdot\|$ is the norm on X. Then the dual space of $\ell_m^p(X, \|\cdot\|)$ is given by the following result.

Theorem 5.2.1 ([33]). *We have* $(\ell_m^p(X, \|\cdot\|))^* = \ell_m^q(X^*, \|\cdot\|_*)$, *where* $p^{-1}+q^{-1} = 1$. *In particular, if* $F \in (\ell_m^p(X, \|\cdot\|))^*$ *then there is a unique* $x^* := (x_1^*, \ldots, x_m^*) \in \ell_m^q(X^*, \|\cdot\|_*)$ *such that* $\|F\| = |\!|\!|x^*|\!|\!|_q$ *and for* $x := (x_1, \ldots, x_m) \in \ell_m^p(X, \|\cdot\|)$,

$$F(x) = x_1^*(x_1) + \cdots + x_m^*(x_m).$$

Now consider $\mathbb{C}^{n\times n}$ and define $\langle\cdot, \cdot\rangle : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \rightarrow \mathbb{C}$ by $\langle X, Y\rangle := \mathrm{trace}(Y^*X)$.

Then $\langle\cdot,\,\cdot\rangle$ defines an inner product on $\mathbb{C}^{n\times n}$ and $\|X\|_F := \sqrt{\langle X,\, X\rangle}$ is the Frobenius norm on $\mathbb{C}^{n\times n}$. For a fixed $Y \in \mathbb{C}^{n\times n}$, the map $X \mapsto \langle X,\, Y\rangle$ is a linear functional on $\mathbb{C}^{n\times n}$. On the other hand, if F is a linear functional on $\mathbb{C}^{n\times n}$ then (by the Riesz representation theorem) there exists a unique $Z \in \mathbb{C}^{n\times n}$ such that $F(X) = \langle X,\, Z\rangle$. Now, let $\|\cdot\|$ be a norm on $\mathbb{C}^{n\times n}$. Then $\|\cdot\|_* : \mathbb{C}^{n\times n} \to \mathbb{R}$ given by

$$\|Y\|_* := \sup\{|\langle X,\, Y\rangle| : X \in \mathbb{C}^{n\times n} \text{ and } \|X\| = 1\}$$

defines a norm and is referred to as the dual norm of $\|\cdot\|$. For $X, Y \in \mathbb{C}^{n\times n}$, we have $|\langle X,\, Y\rangle| \leq \|X\|\,\|Y\|_*$ and $(\mathbb{C}^{n\times n}, \|\cdot\|)^* = (\mathbb{C}^{n\times n}, \|\cdot\|_*)$. Thus, by Theorem 5.2.1, we have the following.

Theorem 5.2.2. *We have* $(\ell^p_m(\mathbb{C}^{n\times n}, \|\cdot\|))^* = \ell^q_m(\mathbb{C}^{n\times n}, \|\cdot\|_*)$, *where* $p^{-1} + q^{-1} = 1$ *and* $\|\cdot\|_*$ *is the dual norm of the norm* $\|\cdot\|$. *In particular, if* F *is a functional on* $\ell^p_m(\mathbb{C}^{n\times n}, \|\cdot\|)$ *then there is a unique* $Z := (Z_1, \ldots, Z_m) \in \ell^q_m(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ *such that* $\|F\| = |||Z|||_q$ *and for* $X := (X_1, \ldots, X_m) \in \ell^p_m(\mathbb{C}^{n\times n}, \|\cdot\|)$, *we have*

$$F(X) = \langle X_1,\, Z_1\rangle + \cdots + \langle X_m,\, Z_m\rangle.$$

Now we consider a few special norms on $\mathbb{C}^{n\times n}$. Note that $\mathbb{C}^{n\times n}$ is a Hilbert space when equipped with the Frobenius norm $\|\cdot\|_F$. Hence $(\mathbb{C}^{n\times n}, \|\cdot\|_F)^* = (\mathbb{C}^{n\times n}, \|\cdot\|_F)$. Next, we consider Schatten p-norm on $\mathbb{C}^{n\times n}$. For $X \in \mathbb{C}^{n\times n}$, let $\sigma_1(X) \geq \ldots, \geq \sigma_n(X)$ denote the singular values of X. Then, for $1 \leq p \leq \infty$, the Schatten p-norm of X is given by

$$\|X\|_{S^p} := \|(\sigma_1(X), \ldots, \sigma_n(X))\|_p, \,,$$

where $\|\cdot\|_p$ is the Hölder's p-norm on $\mathbb{C}^n$. Note that the Schatten 2-norm is the Frobenius norm on $\mathbb{C}^{n\times n}$, that is, $\|X\|_{S^2} = \|X\|_2$, and the Schatten ∞-norm is the spectral norm on $\mathbb{C}^{n\times n}$, that is, $\|X\|_{S^\infty} = \|X\|_2$. It is easy to see that the dual norm of the Schatten p-norm is the Schatten q-norm. More precisely, we have the following result whose proof is easy.

Theorem 5.2.3 ([6]). *We have* $\left(\mathbb{C}^{n\times n},\, \|.\|_{S^p}\right)^* = \left(\mathbb{C}^{n\times n},\, \|.\|_{S^q}\right)$, *where* $p^{-1} + q^{-1} = 1$.

Since the spectral norm on $\mathbb{C}^{n\times n}$ is the Schatten ∞-norm, we have the following result whose proof is immediate.

Corollary 5.2.4. *We have* $(\mathbb{C}^{n\times n}, \|\cdot\|_2)^* = (\mathbb{C}^{n\times n}, \|\cdot\|_{S^1})$. *Consequently,*

$$(\ell^p_m(\mathbb{C}^{n\times n}, \|\cdot\|_2))^* = (\ell^q_m(\mathbb{C}^{n\times n}, \|\cdot\|_{S^1})),$$

where $p^{-1} + q^{-1} = 1$ *and* $\|\cdot\|_2$ *is the spectral norm on* $\mathbb{C}^{n\times n}$.

5.3 Condition Number

Generically, a solution of a problem is a function of the given data. Thus for a given data a, the solution S of a problem is given by $S = f(a)$ for some function f. If the problem is well-posed at a then f is well defined and is continuous in a small neighbourhood of a. Thus, solving the problem for a given data a amounts to computing $f(a)$. For numerical computation of $f(a)$, it is of great importance to know how sensitive the solution S is to small perturbations in the data a. We therefore discuss the issue of sensitivity of solutions of a problem to small perturbations in the data in a formal abstract setting.

Let X and Y be Banach spaces. Let $BL(X, Y)$ denote the Banach space of all bounded linear transformations from X to Y. For an open set $U \subset X$ and $k \in \mathbf{N}$, let $C^k(U, Y)$ denote the space of k times continuously differentiable functions from U to Y. For $a \in X$ and $k \in \mathbf{N}$, let $C^k_a(X, Y)$ denote the space of terms of k times continuously differentiable Y-valued functions at a, that is, $C^k_a(X, Y)$ is the set of equivalence classes of Y-valued functions which are k times continuously differentiable in a neighbourhood of a. More specifically, for $a \in X$, let $\mathcal{F}_a$ denote the family of open subsets of X containing a. Then $C^k_a(X, Y)$ is the quotient space given by

$$C^k_a(X, Y) := \left(\bigcup_{U \in \mathcal{F}_a} C^k(U, Y) \right) / \sim,$$

where the relation $\sim$ is defined as follows: $f \sim g$ if $f_{|U} = g_{|U}$ for some $U \in \mathcal{F}_a$.

Now treating X as the space of data and Y as the space of solutions, a problem can often be stated by an implicit relation between data and solutions. Thus a general format of a problem is:

$$\begin{gathered}\text{Given } a \in X, \text{ find } S \in Y \text{ such that } G(a, S) = 0,\\ \text{where } G : X \times Y \to Z \text{ and } Z \text{ is a Banach space.}\end{gathered}$$

If a solution exists uniquely and depends smoothly on the data then $S = f(a)$ for some $f \in C^k_a(X, Y)$ and an appropriate $k \in \mathbf{N}$. We, therefore, define the **solution bundle** $\mathbb{M}(X, Y)$ by

$$\mathbb{M}(X, Y) := \bigcup_{a \in X} (C^1_a(X, Y) \times \{a\}) = \bigcup_{a \in X} \{(f, a) : f \in C^1_a(X, Y)\}. \tag{5.1}$$

We now define the condition operator and condition number as follows.

Definition 5.3.1. *The map* $\mathrm{D} : \mathbb{M}(X, Y) \longrightarrow BL(X, Y), (f, a) \mapsto \mathrm{D}f(a)$ *is called the* **condition operator,** *where* $\mathrm{D}f(a)$ *is the derivative of* f *at* a. *Suppose that* $S = f(a)$ *is a solution of the problem* $G(a, S) = 0$ *and that* $f \in C^1_a(X, Y)$. *Then* $\mathrm{cond}(f, a) := \|\mathrm{D}f(a)\|$ *is called the* **condition number** *of the solution* $S = f(a)$, *where* $\|\cdot\|$ *is the operator norm on* $BL(X, Y)$.

For the special case when $Y = \mathbb{C}$, we have $BL(X, \mathbb{C}) = X^*$, the dual space of X. Hence in such a case $\mathrm{D}f(a) \in X^*$ and hence $\mathrm{cond}(f, a) = \|\mathrm{D}f(a)\|_{X^*}$. If $f \in C_a^2(X, Y)$ then by Taylor's theorem, we have

$$f(a + h) = f(a) + \mathrm{D}f(a)h + \mathcal{O}(\|h\|_X^2), \tag{5.2}$$

where $\|\cdot\|_X$ denotes the norm on X. Hence for sufficiently small $\|h\|_X$, we have

$$f(a + h) \simeq f(a) + \mathrm{D}f(a)h \text{ and } \|f(a + h) - f(a)\|_Y \lesssim \mathrm{cond}(f, a)\|h\|_X,$$

where $\|\cdot\|_Y$ denotes the norm on Y. This shows that the condition operator $\mathrm{D}f(a)$ provides an important geometric insight into the sensitivity of $S = f(a)$ at a. On the other hand, $\mathrm{cond}(f, a) := \|\mathrm{D}f(a)\|$ quantifies the worst effect that a small change in a can have on the solution $S = f(a)$. When the changes in a is measured relative to $\|S\|_Y$, the condition number is then given by $\mathrm{cond}(f, a) := \dfrac{\|\mathrm{D}f(a)\|}{\|S\|_Y}$.

Alternatively, a structured approach to defining $\mathrm{cond}(f, a)$ is as follows. Suppose that $S = f(a)$ is a solution of $G(a, S) = 0$ and that $f \in C_a^1(X, Y)$. Let $u \in X$ be a unit vector and $t \in \mathbb{R}$. Then the directional derivative $\mathrm{D}_u f(a)$ of f at a in the direction of u is given by

$$\mathrm{D}_u f(a) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}.$$

Evidently, the sensitivity of the solution $S = f(a)$ to small perturbations in a in the direction of u, that is, the sensitivity of $S = f(a)$ relative to the perturbation $a + tu$ for small t is measured by $\|\mathrm{D}_u f(a)\|_Y$.

Definition 5.3.2. *Let $u \in X$ be a unit vector. Then the map $\mathrm{D}_u : \mathbb{M}(X, Y) \longrightarrow Y, (f, a) \mapsto \mathrm{D}_u f(a)$ is called the* **partial condition operator**, *where $\mathrm{D}_u f(a)$ is the directional derivative of f at a in the direction of u. Suppose that $S := f(a)$ is a solution of $G(a, S) = 0$ and that $f \in C_a^1(X, Y)$. Then $\mathrm{cond}_u(f, a) := \|\mathrm{D}_u f(a)\|_Y$ is called the* **partial condition number** *of the solution $S = f(a)$ relative to u.*

Now the sensitivity of $S = f(a)$ relative to small perturbation in a in the direction of u measured by $\mathrm{cond}_u(f, a)$. Hence the sensitivity of $S = f(a)$ relative to small arbitrary perturbations in a is measured by $\sup\limits_{\|u\|_X = 1} \mathrm{cond}_u(f, a)$.

Proposition 5.3.3. *Suppose that $S = f(a)$ is a solution of $G(a, S) = 0$ and that $f \in C_a^1(X, Y)$. Then we have $\sup_{\|u\|_X = 1} \mathrm{cond}_u(f, a) = \|\mathrm{D}f(a)\| = \mathrm{cond}(f, a)$.*

Proof: Since $f \in C_a^1(X, Y)$, we have $\mathrm{D}_u f(a) = \mathrm{D}f(a)u$. Consequently, we have $\sup_{\|u\|_X = 1} \mathrm{cond}_u(f, a) = \sup_{\|u\|_X = 1} \|\mathrm{D}_u f(a)\|_Y = \|\mathrm{D}f(a)\| = \mathrm{cond}(f, a)$. Hence the proof. ■

When $f \in C^2_a(X, Y)$ and $t \in \mathbb{R}$ is small, by Taylor's theorem, we have

$$f(a + tu) = f(a) + \mathrm{D}_u f(a)t + \mathcal{O}(|t|^2). \tag{5.3}$$

Hence $f(a + tu) \simeq f(a) + \mathrm{D}_u f(a)t$ and $\|f(a + tu) - f(a)\|_Y \lesssim \mathrm{cond}_u(f, a)|t|$ for small $t \in \mathbb{R}$.

5.4 Sensitivity of simple eigenvalues of a matrix

The sensitivity analysis of simple eigenvalues of matrices has been studied extensively over the years. We briefly review the sensitivity of a simple eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$. Let $\|\cdot\|$ be a norm on $\mathbb{C}^{n\times n}$. Then for $X := \mathbb{C}^{n\times n}$ and $Y := \mathbb{C}$, we have $\mathrm{D} : \mathbb{M}(\mathbb{C}^{n\times n}, \mathbb{C}) \to (\mathbb{C}^{n\times n}, \|\cdot\|)^*, (f, A) \mapsto \mathrm{D}f(A)$. Thus for $f \in C^1_A(\mathbb{C}^{n\times n}, \mathbb{C})$, we have $\mathrm{D}f(A) \in (\mathbb{C}^{n\times n}, \|\cdot\|)^*$. Since $(\mathbb{C}^{n\times n}, \|\cdot\|)^* = (\mathbb{C}^{n\times n}, \|\cdot\|_*)$, there is a unique matrix $K(f, A) \in (\mathbb{C}^{n\times n}, \|\cdot\|_*)$ such that $\mathrm{D}f(A)H = \langle H, K(f, A)\rangle$ for all $H \in (\mathbb{C}^{n\times n}, \|\cdot\|)$ and that $\|\mathrm{D}f(A)\| = \|K(f, A)\|_*$. Hence $\mathrm{cond}(f, A) = \|K(f, A)\|_*$. This shows that $K(f, A)$ is the matrix that represents the functional $\mathrm{D}f(A)$ and $K : \mathbb{M}(\mathbb{C}^{n\times n}, \mathbb{C}) \to (\mathbb{C}^{n\times n}, \|\cdot\|_*), (f, A) \mapsto K(f, A)$ is the matrix version of the condition operator $D : \mathbb{M}(\mathbb{C}^{n\times n}, \mathbb{C}) \to (\mathbb{C}^{n\times n}, \|\cdot\|)^*, (f, A) \mapsto \mathrm{D}f(A)$.

Definition 5.4.1. *Let $f \in C^1_A(\mathbb{C}^{n\times n}, \mathbb{C})$ and $\mathrm{D}f(A)$ be the derivative of f at A. If there is a matrix $K(f, A) \in \mathbb{C}^{n\times n}$ such that $\mathrm{D}f(A)Y = \langle Y, K(f, A)\rangle$ for all $Y \in \mathbb{C}^{n\times n}$ then $K(f, A)$ is said to be the condition matrix of (f, A).*

Let (λ, y, x) be a simple eigentriple, that is, $Ax = \lambda x, y^*A = \lambda y^*$. We often write λ as $\lambda(A)$ and treat λ as a function of A. Now defining $G : \mathbb{C}^{n\times n} \times \mathbb{C} \to \mathbb{C}$ by $G(A, s) := \det(A - sI)$, we see that λ is a solution of $G(A, s) = 0$. Since λ is simple, it is well known that there is an open set $U \subset \mathbb{C}^{n\times n}$ such that $A \in U$ and that $\lambda : U \to \mathbb{C}$, $A \mapsto \lambda(A)$ is a smooth function. For sufficiently small $t \in \mathbb{C}$ and $H \in \mathbb{C}^{n\times n}$, it is well known [27, 34] that $\lambda(A + tH)$ is holomorphic and that

$$\lambda(A + tH) = \lambda(A) + \frac{y^*Hx}{y^*x}t + \mathcal{O}(|t|^2). \tag{5.4}$$

Indeed, let $p(t, z) := \det(A + tH - zI)$ and consider the algebraic variety $V(A, H) := \{(t, z) \in \mathbb{C}^2 : p(t, z) = 0\}$. Let q be the reduced polynomial of p. Then $V(A, H) = V(q)$, where $V(q)$ is the algebraic variety of the polynomial q. Then following [4], we say that $(t_0, \lambda(t_0)) \in V(A, H)$ is a multiple point of $V(A, H)$ if $\lambda(t_0)$ is a multiple root of $q(t_0, z)$. If $q = q_1q_2 \ldots q_m$ is the prime factorization of q then $V(q) = \cup_{i=1}^m V(q_i)$. Consequently, $(t_0, \lambda(t_0))$ is a multiple point of $V(A, H)$ only when $(t_0, \lambda(t_0)) \in V(q_i) \cap V(q_j)$ for some $i \neq j$ or/and $(t_0, \lambda(t_0))$ is a multiple point of $V(q_j)$ for some j. Now by Bézout's theorem it can be shown [4] that the set of $t \in \mathbb{C}$ for which $\lambda(t)$ is a multiple eigenvalue of $A+tH$

is at most a finite set. Consequently, $\lambda(t)$ is holomorphic on $\mathbb{C} \setminus F$ for some finite set $F \subset \mathbb{C}$.

Recall that $\text{gsep}(\lambda)$ is the smallest positive number for which the component of $\Lambda_\epsilon(A)$ isolating λ from the rest of $\Lambda(A)$ coalesces with another component of $\Lambda_\epsilon(A)$ when $\epsilon \to \text{gsep}(\lambda)$. Now let $\epsilon < \text{gsep}(\lambda)$. Then $\Lambda_\epsilon(A)$ has a component Δ_λ which isolates λ from the rest of $\Lambda(A)$. Then evidently, for all $|t| \leq \epsilon/\|H\|$, we have $\lambda(t) \in \Delta_\lambda$. It is also evident that the spectral projection

$$P(t) := -\frac{1}{2\pi i}\int_{\partial\Delta_\lambda}(A + tH - zI)^{-1}dz$$

is holomorphic on $\{t \in \mathbb{C} : |t| < \epsilon/\|H\|\}$. Thus (5.4) follows from the power series expansion of $\lambda(A + tH) = \text{trace}((A + tH)P(t))$.

We now summarize these results in the following theorem. We mention that for a subordinate norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, we have $\|uv^*\| = \|u\|\,\|v\|_*$ and $\|uv^*\|_* = \|u\|_*\|v\|$, where $\|\cdot\|_*$ is the dual norm of the norm $\|\cdot\|$.

Theorem 5.4.2. *Let $A \in \mathbb{C}^{n\times n}$ and (λ, y, x) be a simple eigentriple of A. Let $H \in \mathbb{C}^{n\times n}$.*
(a) Then there is a finite set $F \subset \mathbb{C}$ and an eigenvalue $\lambda(A + tH)$ of $A + tH$ such that $\lambda(A + tH)$ is simple and $\lambda(A + tH)$ is holomorphic for all $t \in \mathbb{C} \setminus F$. Further, for small $|t|$, we have

$$\lambda(A + tH) = \lambda(A) + \frac{y^*Hx}{y^*x}t + \mathcal{O}(|t|^2).$$

Furthermore, we have $\lambda \in C^1_A(\mathbb{C}^{n\times n}, \mathbb{C})$ and for sufficiently small $\|H\|$, we have

$$\lambda(A + H) = \lambda(A) + \frac{y^*Hx}{y^*x} + \mathcal{O}(\|H\|^2).$$

(b) The condition matrix $K(\lambda, A) \in (\mathbb{C}^{n\times n}, \|\cdot\|_*)$ of (λ, A) is given by $K(\lambda, A) = \dfrac{yx^*}{x^*y}$. Further, for $H \in (\mathbb{C}^{n\times n}, \|\cdot\|)$, we have

$$\mathrm{D}\lambda(A)H = \langle H,\, K(\lambda, A)\rangle = \frac{y^*Hx}{y^*x} \text{ and } \text{cond}(\lambda, A) = \|K(\lambda, A)\|_* = \frac{\|yx^*\|_*}{|y^*x|}.$$

Hence $|\lambda(A + H) - \lambda(A)| \leq \text{cond}(\lambda, A)\|H\| + \mathcal{O}(\|H\|^2)$ for sufficiently small $\|H\|$.

For a subordinate norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, we have $\text{cond}(\lambda, A) = \dfrac{\|yx^*\|_*}{|y^*x|} = \dfrac{\|x\|\,\|y\|_*}{|y^*x|}$, where $\|y\|_*$ is the dual norm of the norm $\|x\|$ on $\mathbb{C}^n$.

(c) Suppose that $\|H\| = 1$. Then the partial condition operator $\mathrm{D}_H : \mathbb{M}(\mathbb{C}^{n\times n}, \mathbb{C}) \to \mathbb{C}, (\lambda, A) \mapsto \mathrm{D}_H\lambda(A)$ and the partial condition number $\text{cond}_H(\lambda, A)$ are given by

$$\mathrm{D}_H\lambda(A) = \frac{y^*Hx}{y^*x} = \langle H,\, K(\lambda, A)\rangle \text{ and } \text{cond}_H(\lambda, A) = \frac{|y^*Hx|}{|y^*x|} = |\langle H,\, K(\lambda, A)\rangle|.$$

Hence for small $|t|$, we have $|\lambda(A + tH) - \lambda(A)| \leq \text{cond}_H(\lambda, A)|t| + \mathcal{O}(|t|^2)$.

(d) For a subordinate norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, choose $u, v \in \mathbb{C}^n$ such that $\|u\| = 1$, $\|v\|_* = 1$, $y^*u = \|y\|_*$ and $v^*x = \|x\|$. Now define $H := uv^*$. Then $\|H\| = \|u\|\,\|v\|_* = 1$ and

$$\mathrm{cond}_H(\lambda, A) = |\langle H,\, K(\lambda, A)\rangle| = \|K(\lambda, A)\|_* = \mathrm{cond}(\lambda, A).$$

Proof: Defining $K(\lambda, A) := \dfrac{yx^*}{x^*y}$, it follows from (a) that $\langle H,\, K(\lambda, A)\rangle = \frac{y^*Hx}{y^*x} = \mathrm{D}\lambda(A)H$ for all $H \in (\mathbb{C}^{n\times n}, \|\cdot\|)$. Consequently, we have $\mathrm{cond}(\lambda, A) = \|K(\lambda, A)\|_* = \|yx^*\|_*/|y^*x|$. For a subordinate norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, we have $\|yx^*\|_* = \|x\|\,\|y\|_*$. Finally, when $\|H\| = 1$, we have

$$\mathrm{D}_H\lambda(A) = \mathrm{D}\lambda(A)H = \langle H,\, K(\lambda, A)\rangle = \frac{y^*Hx}{y^*x}.$$

Hence the desired results follow. ■

We now extend these results to the case of matrix pencils and matrix polynomials.

5.5 Sensitivity of simple eigenvalues of matrix pencils

Let $\mathbf{L} \in \mathbb{L}(\mathbb{C}^{n\times n})$ be regular. Then for $\mathbf{H} \in \mathbb{L}(\mathbb{C}^{n\times n})$, consider the algebraic variety $V(\mathbf{L}, \mathbf{H}) := \{(t, z) \in \mathbb{C}^2 : \det(\mathbf{L}(z) + t\mathbf{H}(z)) = 0\}$. Then for $t \in \mathbb{C}$, the eigenvalues of the perturbed pencil $\mathbf{L} + t\mathbf{H}$ evolve in $V(\mathbf{L}, \mathbf{H})$. Let $q(t, z)$ be the reduced polynomial of $\det(\mathbf{L}(z) + t\mathbf{H}(z))$. Then $V(\mathbf{L}, \mathbf{H}) = V(q)$. Let $(t_0, \lambda(t_0)) \in V(\mathbf{L}, \mathbf{H})$. Then $(t_0, \lambda(t_0))$ is said to be a multiple point of $V(\mathbf{L}, \mathbf{H})$ if $\lambda(t_0)$ is a multiple root of $q(t_0, z)$. Again, considering irreducible components of $V(\mathbf{L}, \mathbf{H})$ and invoking Bezout's theorem for the plane algebraic curve $V(q)$, it is easy to see that there are at most a finite number of multiple points of $V(\mathbf{L}, H)$. Consequently, finite eigenvalues of $\mathbf{L} + t\mathbf{H}$ evolve holomorphically on $\mathbb{C}$ except for a finitely many points. In particular, a simple eigenvalue of $\mathbf{L} + t\mathbf{H}, t \in \mathbb{C}$, evolves holomorphically for all but finitely many $t \in \mathbb{C}$.

Now consider the space of pencils $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ and assume that each component of w is nonzero. Then $|\!|\!|\cdot|\!|\!|_{w,p}$ is a norm and hence $\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ is a normed space. Now suppose that $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$. Let $(\lambda, y, x) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n$ be a simple eigentriple of $\mathbf{L}$, that is, λ is a simple eigenvalue of $\mathbf{L}$, y and x are left and right eigenvectors of $\mathbf{L}$ corresponding to λ. Thus we have $\mathbf{L}(\lambda)x = 0$ and $y^*\mathbf{L}(\lambda) = 0$. We often write λ as $\lambda(\mathbf{L})$ and treat λ as a function of $\mathbf{L}$. Then in view of the discussion above, there is an open set $U \subset \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ containing $\mathbf{L}$ such that the map $\lambda : U \to \mathbb{C}$, $\mathbf{L} \mapsto \lambda(\mathbf{L})$ defines a smooth function.

Theorem 5.5.1. *Let $\mathbf{L} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular pencil given by $\mathbf{L}(z) = A - zB$. Let (λ, y, x) be a simple eigentriple of $\mathbf{L}$. For $\mathbf{H} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$, consider the one parameter family of pencils $\mathbf{L} + t\mathbf{H}$, $t \in \mathbb{C}$. Then there is a finite set $F \subset \mathbb{C}$ and an* *eigenvalue* $\lambda(\mathbf{L}+t\mathbf{H})$ *of* $\mathbf{L}+t\mathbf{H}$ *such that* $\lambda(\mathbf{L}+t\mathbf{H})$ *is simple and holomorphic for all* $t \in \mathbb{C} \setminus F$*. Further, we have*

$$\lambda(\mathbf{L}+t\mathbf{H}) = \lambda(\mathbf{L}) + \frac{y^*\mathbf{H}(\lambda)x}{y^*Bx}t + \mathcal{O}(|t|^2)$$

for small $|t|$*. Furthermore, we have* $\lambda \in C^1_{\mathbf{L}}\left(\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|),\ \mathbb{C}\right)$ *and*

$$\lambda(\mathbf{L}+\mathbf{H}) = \lambda(\mathbf{L}) + \frac{y^*\mathbf{H}(\lambda)x}{y^*Bx} + \mathcal{O}((|||\mathbf{H}|||_{w,p})^2)$$

for sufficiently small $|||\mathbf{H}|||_{w,p}$*.*

Proof: We only need to show that $\lambda(\mathbf{L}+t\mathbf{H}) = \lambda(\mathbf{L}) + \dfrac{y^*H(\lambda)x}{y^*Bx}t + \mathcal{O}(|t|^2)$ for small $|t|$. Set $\lambda(t) := \lambda(\mathbf{L}+t\mathbf{H})$. Let $x(t)$ be a right eigenvectors of $\mathbf{L}+t\mathbf{H}$ corresponding to $\lambda(t)$. Then under appropriate normalization, we have $x(t) = x + x_1 t + \mathcal{O}(t^2)$. Also, we have $\lambda(t) = \lambda + \alpha t + \mathcal{O}(t^2)$. Now $y^*(\mathbf{L}+t\mathbf{H})(\lambda(t))x(t) = 0$ gives $\alpha = y^*\mathbf{H}(\lambda)x/y^*Bx + \mathcal{O}(t)$. Hence the result follows. ■

We mention that the first order perturbation expansion of $\lambda(\mathbf{L}+t\mathbf{H})$ in Theorem 5.5.1 is well known (see, for example, [43, 42]). We have stated this result in a form that will be useful for our purpose.

Recall that $\langle X,\, Y\rangle := \operatorname{trace}(Y^*X)$ defines an inner product on $\mathbb{C}^{n\times n}$. Also recall that $\mathbb{L}(\mathbb{C}^{n\times n})$ is the vector space of pencils (without a norm) of size n. We have already seen that $\langle,\rangle_{\mathbb{L}} : \mathbb{L}(\mathbb{C}^{n\times n}) \times \mathbb{L}(\mathbb{C}^{n\times n}) \longrightarrow \mathbb{C}$ defined by

$$\langle \mathbf{L}_1,\, \mathbf{L}_2\rangle_{\mathbb{L}} := \langle A_1,\, B_1\rangle + \langle A_2,\, B_2\rangle$$

is an inner product on $\mathbb{L}(\mathbb{C}^{n\times n})$, where $\mathbf{L}_i(z) = A_i - zB_i$ or $\mathbf{L}_i(c,s) = cA_i - sB_i$. Consequently, if $F : \mathbb{L}(\mathbb{C}^{n\times n}) \to \mathbb{C}$ is a linear functional then there is a unique $G \in \mathbb{L}(\mathbb{C}^{n\times n})$ such that $F(X) = \langle X,\, G\rangle_{\mathbb{L}}$ for all $X \in \mathbb{L}(\mathbb{C}^{n\times n})$. Finally, recall that the action of $\mathbb{R}^2$ on $\mathbb{L}(\mathbb{C}^{n\times n})$ is the map $\mathbb{R}^2 \times \mathbb{L}(\mathbb{C}^{n\times n}) \to \mathbb{L}(\mathbb{C}^{n\times n}), (w, \mathbf{L}) \mapsto w \odot \mathbf{L}$ given by $(w \odot \mathbf{L})(z) = w_1 A - z w_2 B$, where $\mathbf{L}(z) = A - zB$.

Theorem 5.5.2. *Consider the space* $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$*. Suppose that both the components of* w *are nonzero. Then* $(\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|))^* = (\mathbb{L}^q_{w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*))$*, where* $p^{-1} + q^{-1} = 1$ *and* $\|\cdot\|_*$ *is the dual norm of the norm* $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$*.*

Proof: Note that $\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ is a normed space, that is, $|||\cdot|||_{w,p}$ is a norm. Let $F : \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|) \to \mathbb{C}$ be linear. Then we have already seen that there exists a unique pencil Y such that $F(X) = \langle X,\, Y\rangle_{\mathbb{L}} = \langle X_1,\, Y_1\rangle + \langle X_2,\, Y_2\rangle$, where $X(z) = X_1 - zX_2$ and $Y(z) = Y_1 - zY_2$. Since $F(X) = \langle X,\, Y\rangle_{\mathbb{L}} = \langle w \odot X,\, w^{-1} \odot Y\rangle_{\mathbb{L}}$ and $|||X|||_{w,p} = |||w \odot X|||_p = \|(w_1\|X_1\|,\, w_2\|X_2\|)\|_p$, the desired result follows from Theorem 5.2.2. ■

Definition 5.5.3. *Let* $f \in C^1_{\mathbf{L}}(\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|), \mathbb{C})$ *and* $\mathrm{D}f(\mathbf{L})$ *be the derivative of* f *at* $\mathbf{L}$. *If there is a matrix pencil* $K(f,\mathbf{L}) \in \mathbb{L}^p_{w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ *such that* $\mathrm{D}f(\mathbf{L})Y = \langle Y,\, K(f,\mathbf{L})\rangle_{\mathbb{L}}$ *for all* $Y \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *then* $K(f,\mathbf{L})$ *is said to be the condition pencil of* $(f,\mathbf{L})$.

Let $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ be regular and (λ, y, x) be a simple eigentriple of $\mathbf{L}$. Then $\lambda \in C^1_{\mathbf{L}}(\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|), \mathbb{C})$ and there is a unique condition pencil $K(\lambda,\mathbf{L}) \in \mathbb{L}^q_{w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ such that $\mathrm{D}\lambda(\mathbf{L}) = \langle X,\, K(\lambda,\mathbf{L})\rangle_{\mathbb{L}}$ for all $X \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$. Hence $\mathrm{cond}(\lambda,\mathbf{L}) = \|\mathrm{D}\lambda(\mathbf{L})\| = |||K(\lambda,\mathbf{L})|||_{w^{-1},q}$. Further, for $\mathbf{H} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ with $|||\mathbf{H}|||_{w,p} = 1$, we have $\mathrm{D}_{\mathbf{H}}\lambda(\mathbf{L}) = \langle \mathbf{H},\, K(\lambda,\mathbf{L})\rangle_{\mathbb{L}}$. Now suppose that $\mathbf{H}(z) = H_1 - zH_2$. Then by Theorem 5.5.1, we have $\mathrm{D}_{\mathbf{H}}\lambda(\mathbf{L}) = \frac{y^*\mathbf{H}(\lambda)x}{y^*Bx} = \langle H_1,\, yx^*/\alpha\rangle + \langle H_2,\, \overline{\lambda}\, yx^*/\alpha\rangle = \langle \mathbf{H},\, K(\lambda,\mathbf{L})\rangle_{\mathbb{L}}$, where $\alpha := \overline{y^*Bx}$. This shows that the condition pencil $K(\lambda,\mathbf{L})$ is given by

$$K(\lambda,\mathbf{L})(z) := \frac{yx^* - z\overline{\lambda}\, yx^*}{\overline{y^*Bx}}.$$

Consequently, we have

$$\mathrm{cond}(\lambda,\mathbf{L}) = |||K(\lambda,\mathbf{L})|||_{w^{-1},q} = \frac{\|(1,\lambda)\|_{w^{-1},q}\, \|yx^*\|_*}{|y^*Bx|},$$

where $p^{-1} + q^{-1} = 1$. When the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ is a subordinate norm, we have

$$\mathrm{cond}(\lambda,\mathbf{L}) = \frac{\|(1,\lambda)\|_{w^{-1},q}\, \|yx^*\|_*}{|y^*Bx|} = \frac{\|(1,\lambda)\|_{w^{-1},q}\, \|y\|_*\|x\|}{|y^*Bx|}.$$

Now we summarize these results in the following theorem.

Theorem 5.5.4. *Let* $\mathbf{L} \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be a regular pencil given by* $\mathbf{L}(z) := A - zB$. *Suppose that both components of* w *are nonzero. Let* (λ, y, x) *be a simple eigentriple of* $\mathbf{L}$.

(a) The condition operator $\mathrm{D} : \mathbb{M}(\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|),\ \mathbb{C}) \to (\mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|))^*,\ (\lambda,\mathbf{L}) \mapsto \mathrm{D}\lambda(\mathbf{L})$ *is given by* $\mathrm{D}\lambda(\mathbf{L})X := \langle X,\, K(\lambda,\mathbf{L})\rangle_{\mathbb{L}}$ *for* $X \in \mathbb{L}^p_w(\mathbb{C}^{n\times n}, \|\cdot\|)$, *where* $K(\lambda,\mathbf{L}) \in \mathbb{L}^q_{w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$, $p^{-1} + q^{-1} = 1$, *is the condition pencil given by*

$$K(\lambda,\mathbf{L})(z) := \frac{yx^* - z\overline{\lambda}\, yx^*}{\overline{y^*Bx}}.$$

The condition number $\mathrm{cond}(\lambda,\mathbf{L})$ *is given by*

$$\mathrm{cond}(\lambda,\mathbf{L}) = |||K(\lambda,\mathbf{L})|||_{w^{-1},q} = \frac{\|(1,\lambda)\|_{w^{-1},q}\, \|yx^*\|_*}{|y^*Bx|}.$$

When the norm $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$ *is a subordinate norm, we have*

$$\mathrm{cond}(\lambda,\mathbf{L}) = \frac{\|(1,\lambda)\|_{w^{-1},q}\, \|yx^*\|_*}{|y^*Bx|} = \frac{\|(1,\lambda)\|_{w^{-1},q}\, \|y\|_*\|x\|}{|y^*Bx|}.$$

Hence we have $|\lambda(\mathbf{L}+\Delta\mathbf{L})-\lambda(\mathbf{L})| \leq \mathrm{cond}(\lambda, \mathbf{L})|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p} + \mathcal{O}((|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p})^2)$.

(b) Let $\mathbf{H} \in \mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be such that* $|\!|\!|\mathbf{H}|\!|\!|_{w,p} = 1$. *Then the partial condition operator* $\mathrm{D}_{\mathbf{H}} : \mathbb{M}(\mathbb{L}_w^p(\mathbb{C}^{n\times n}, \|\cdot\|), \mathbb{C}) \to \mathbb{C}, (\lambda, \mathbf{L}) \mapsto \mathrm{D}_{\mathbf{H}}\lambda(\mathbf{L})$ *and the partial condition number* $\mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L})$ *are given by*

$$\mathrm{D}_{\mathbf{H}}\lambda(\mathbf{L}) = \frac{y^*\mathbf{H}(\lambda)x}{y^*Bx} = \langle \mathbf{H},\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}} \text{ and } \mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L}) = \frac{|y^*\mathbf{H}(\lambda)x|}{|y^*Bx|} = |\langle \mathbf{H},\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}}|.$$

Hence for small $|t|$, *we have* $|\lambda(\mathbf{L}+t\mathbf{H})-\lambda(\mathbf{L})| \leq \mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L})|t| + \mathcal{O}(|t|^2)$.

(c) For a subordinate norm $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$, *choose* $u, v \in \mathbb{C}^n$ *such that* $\|u\| = 1$, $\|v\|_* = 1$, $y^*u = \|y\|_*$ *and* $v^*x = \|x\|$. *Now define*

$$H_1 := -\overline{\nabla_1(\|(1,\lambda)\|_{w^{-1},q})}\, uv^* \text{ and } H_2 := \overline{\nabla_2(\|(1,\lambda)\|_{w^{-1},q})}\, uv^*$$

and consider the pencil $\mathbf{H}(z) := H_1 - zH_2$, *where* $p^{-1}+q^{-1} = 1$. *Here* $\nabla_i(\|(1,\lambda)\|_{w^{-1},q})$ *is the partial gradient of the map* $\mathbb{C}^2 \to \mathbb{R}, (z_1, z_2) \mapsto \|(z_1, z_2)\|_{w^{-1},q}$ *evaluated at* $(1, \lambda)$. *Then we have* $|\!|\!|\mathbf{H}|\!|\!|_{w,p} = 1$ *and*

$$\mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L}) = |\langle \mathbf{H},\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}}| = \|K(\lambda, \mathbf{L})\|_{w^{-1},q} = \mathrm{cond}(\lambda, \mathbf{L}).$$

Proof: We only need to prove (c). First note that $\|H_1\| = |\nabla_1(\|(1,\lambda)\|_{w^{-1},q})|$ and $\|H_2\| = |\nabla_2(\|(1,\lambda)\|_{w^{-1},q})|$. Hence $|\!|\!|\mathbf{H}|\!|\!|_{w,p} = \|(\nabla_1(\|(1,\lambda)\|_{w^{-1},q}),\ \nabla_2(\|(1,\lambda)\|_{w^{-1},q}))\|_{w,p}$. By Lemma 2.4.3, we have $\|(\nabla_1(\|(1,\lambda)\|_{w^{-1},q}),\ \nabla_2(\|(1,\lambda)\|_{w^{-1},q}))\|_{w,p} = 1$. Hence $|\!|\!|\mathbf{H}|\!|\!|_{w,p} = 1$. Now $\mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L}) = |\langle \mathbf{H},\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}}| = |\dfrac{y^*\mathbf{H}(\lambda)x}{y^*Bx}|$. Again by Lemma 2.4.3, we have

$$|y^*\mathbf{H}(\lambda)x| = (|\overline{\nabla_1(\|(1,\lambda)\|_{w^{-1},q})} + \lambda\,\overline{\nabla_2(\|(1,\lambda)\|_{w^{-1},q})}|)\|y\|_*\|x\| = \|(1,\lambda)\|_{w^{-1},q}\|y\|_*\|x\|.$$

Hence the result follows. ■

Remark 5.5.5. *For the case when one of the components of* w *is* 0, *the results in Theorem 5.5.4 still hold. However, in such a case, these results are valid only for the* w*-admissible perturbation of* $\mathbf{L}$.

5.6 Sensitivity of simple eigenvalues of matrix polynomials

Following similar arguments as those employed for matrix pencils, it is easy to obtain analogues of Theorem 5.5.4 for matrix polynomial. Recall that $\mathbb{L}_m(\mathbb{C}^{n\times n})$ is the vector space (without a norm) of matrix polynomials of degree at most m. We have also seen that $\langle , \rangle_{\mathbb{L}} : \mathbb{L}_m(\mathbb{C}^{n\times n}) \times \mathbb{L}_m(\mathbb{C}^{n\times n}) \longrightarrow \mathbb{C}$ defined by

$$\langle \mathbf{L}_1,\, \mathbf{L}_2\rangle_{\mathbb{L}} := \langle A_0,\, B_0\rangle + \cdots + \langle A_m,\, B_m\rangle$$

is an inner product on $\mathbb{L}_m(\mathbb{C}^{n\times n})$, where $\mathbf{L}_1(z) = \sum_{i=0}^m z^i A_i$ and $\mathbf{L}_2(z) = \sum_{i=0}^m z^i B_i$. Consequently, if $F : \mathbb{L}_m(\mathbb{C}^{n\times n}) \to \mathbb{C}$ is a linear functional then there is a unique $G \in \mathbb{L}_m(\mathbb{C}^{n\times n})$ such that $F(X) = \langle X, G\rangle_{\mathbb{L}}$ for all $X \in \mathbb{L}(\mathbb{C}^{n\times n})$. Again recall that the action of $\mathbb{R}^{m+1}$ on $\mathbb{L}_m(\mathbb{C}^{n\times n})$ is the map $\mathbb{R}^{m+1} \times \mathbb{L}_m(\mathbb{C}^{n\times n}) \to \mathbb{L}_m(\mathbb{C}^{n\times n})$, $(w, \mathbf{L}) \mapsto w \odot \mathbf{L}$ given by $(w \odot \mathbf{L})(z) = \sum_{i=0}^m z^i w_i A_i$, where $\mathbf{L}(z) = \sum_{i=0}^m z^i A_i$. Now consider the normed space of polynomials $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$. Then we have the following.

Theorem 5.6.1. *Consider the space* $\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$. *Suppose that each component of* w *is nonzero. Then* $(\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|))^* = (\mathbb{L}^q_{m,w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*))$, *where* $p^{-1} + q^{-1} = 1$ *and* $\|\cdot\|_*$ *is the dual norm of the norm* $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$.

Proof: The proof is similar to that of Theorem 5.5.2. ■

We now define condition polynomial.

Definition 5.6.2. *Let* $f \in C^1_{\mathbf{L}}(\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|), \mathbb{C})$ *and* $\mathrm{D}f(\mathbf{L})$ *be the derivative of* f *at* $\mathbf{L}$. *If there is a matrix polynomial* $K(f, \mathbf{L}) \in \mathbb{L}^p_{m,w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ *such that* $\mathrm{D}f(\mathbf{L})Y = \langle Y, K(f, \mathbf{L})\rangle_{\mathbb{L}}$ *for all* $Y \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ *then* $K(f, \mathbf{L})$ *is said to be the condition polynomial of* $(f, \mathbf{L})$.

Let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ be regular. Let (λ, y, x) be a simple eigentriple of $\mathbf{L}$. As before we write λ as $\lambda(\mathbf{L})$ and treat λ as a function of $\mathbf{L}$. Now following similar arguments as those given for matrix pencils, it follows that there is an open set $U \subset \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ containing $\mathbf{L}$ such that the map $\lambda : U \to \mathbb{C}$, $\mathbf{L} \mapsto \lambda(\mathbf{L})$ defines a smooth function. Indeed, we have the following result.

Theorem 5.6.3. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular. Let* (λ, y, x) *be a simple eigentriple of* $\mathbf{L}$. *For* $\mathbf{H} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$, *consider the one parameter family of matrix polynomials* $\mathbf{L} + t\mathbf{H}$, $t \in \mathbb{C}$. *Then there is a finite set* $F \subset \mathbb{C}$ *and an eigenvalue* $\lambda(\mathbf{L} + t\mathbf{H})$ *of* $\mathbf{L} + t\mathbf{H}$ *such that* $\lambda(\mathbf{L} + t\mathbf{H})$ *is simple and holomorphic for all* $t \in \mathbb{C} \setminus F$. *Further, we have*

$$\lambda(\mathbf{L} + t\mathbf{H}) = \lambda(\mathbf{L}) - \frac{y^*\mathbf{H}(\lambda)x}{y^*\partial_\lambda \mathbf{L}(\lambda)x} t + \mathcal{O}(|t|^2)$$

for small $|t|$. *Furthermore, we have* $\lambda \in C^1_{\mathbf{L}}\left(\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|),\ \mathbb{C}\right)$ *and*

$$\lambda(\mathbf{L} + \mathbf{H}) = \lambda(\mathbf{L}) - \frac{y^*\mathbf{H}(\lambda)x}{y^*\partial_\lambda \mathbf{L}(\lambda)x} + \mathcal{O}((|\!|\!|\mathbf{H}|\!|\!|_{w,p})^2)$$

for sufficiently small $|\!|\!|\mathbf{H}|\!|\!|_{w,p}$. *Here* $\partial_\lambda \mathbf{L}(\lambda)$ *is the derivative of the polynomial* $\mathbf{L}(z)$ *evaluated at* λ.

Proof: The proof is similar to that of Theorem 5.5.1. Indeed, set $\lambda(t) := \lambda(\mathbf{L} + t\mathbf{H}) = \lambda + \alpha t + \mathcal{O}(t^2)$ and $x(t) = x + x_1 t + \mathcal{O}(t^2)$ be an associated right eigenvector. Now $\mathbf{L}(\lambda(t)) = \mathbf{L}(\lambda) + \partial_\lambda \mathbf{L}(\lambda)\alpha t + \mathcal{O}(t^2)$. Hence $y^*\mathbf{L}(\lambda(t)) = y^*\partial_\lambda \mathbf{L}(\lambda)\alpha t +$

$\mathcal{O}(t^2) \Rightarrow y^*\mathbf{L}(\lambda(t))x(t) = y^*\partial_\lambda\mathbf{L}(\lambda)x\alpha t + \mathcal{O}(t^2)$. Similarly, $\mathbf{H}(\lambda(t)) = \mathbf{H}(\lambda) + \mathcal{O}(t)$. Hence $y^*\mathbf{H}(\lambda(t))x(t) = y^*\mathbf{H}(\lambda)x + \mathcal{O}(t)$. Now $y^*(\mathbf{L}(\lambda(t))x(t) + t\mathbf{H}(\lambda(t))x(t) = 0 \Rightarrow \alpha t y^*\partial_\lambda\mathbf{L}(\lambda)x + ty^*H(\lambda)x + \mathcal{O}(t^2) = 0$. This shows that $\alpha = -\dfrac{y^*\mathbf{H}(\lambda)x}{y^*\partial_\lambda\mathbf{L}(\lambda)x} + \mathcal{O}(t)$. Consequently, we have $\lambda(t) = \lambda - \dfrac{y^*\mathbf{H}(\lambda)x}{y^*\partial_\lambda\mathbf{L}(\lambda)x} + \mathcal{O}(t^2)$. Hence the result follows. ■

We mention that the first order expansion of $\lambda(t)$ is well known [46, 18].

Now, let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ be a regular polynomial and (λ, y, x) be a simple eigentriple of $\mathbf{L}$. Then $\lambda \in C^1_{\mathbf{L}}(\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|), \mathbb{C})$ and there is a unique condition polynomial $K(\lambda, \mathbf{L}) \in \mathbb{L}^q_{m,w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ such that $\mathrm{D}\lambda(\mathbf{L}) = \langle X,\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}}$ for all $X \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$. Hence $\mathrm{cond}(\lambda, \mathbf{L}) = \|\mathrm{D}\lambda(\mathbf{L})\| = |\!|\!| K(\lambda, \mathbf{L}) |\!|\!|_{w^{-1},q}$. Define $K(\lambda, \mathbf{L}) \in \mathbb{L}^q_{m,w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ by

$$K(\lambda, \mathbf{L})(z) := \sum_{i=0}^{m} z^i K_i, \;\text{ where } K_i := -\frac{\overline{\lambda^i}\, yx^*}{\overline{y^*\partial_\lambda\mathbf{L}(\lambda)x}}.$$

Then by Theorem 5.6.3, we have

$$\mathrm{D}\lambda(\mathbf{L})\mathbf{H} = -\frac{y^*\mathbf{H}(\lambda)x}{y^*\partial_\lambda\mathbf{L}(\lambda)x} = \langle \mathbf{H},\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}}$$

for all $\mathbf{H} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$. Indeed, suppose that $\mathbf{H}(z) = \sum_{i=0}^m z^i H_i$. Then

$$-\frac{y^*\mathbf{H}(\lambda)x}{y^*\partial_\lambda\mathbf{L}(\lambda)x} = -(\langle H_0,\, yx^*/\alpha\rangle + \langle H_1,\, \overline{\lambda}\, yx^*/\alpha\rangle \cdots + \langle H_m,\, \overline{\lambda^m}\, yx^*/\alpha\rangle) = \langle \mathbf{H},\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}},$$

where $\alpha := \overline{y^*\partial_\lambda\mathbf{L}(\lambda)x}$. In particular, for $\mathbf{H} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ with $|\!|\!|\mathbf{H}|\!|\!|_{w,p} = 1$, we have $\mathrm{D}_{\mathbf{H}}\lambda(\mathbf{L}) = \langle \mathbf{H},\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}} = -\frac{y^*\mathbf{H}(\lambda)x}{y^*\partial_\lambda\mathbf{L}(\lambda)x}$.

Thus, we have

$$\mathrm{cond}(\lambda, \mathbf{L}) = |\!|\!| K(\lambda, \mathbf{L}) |\!|\!|_{w^{-1},q} = \frac{\|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}\, \|yx^*\|_*}{|y^*\partial_\lambda\mathbf{L}(\lambda)x|},$$

where $p^{-1} + q^{-1} = 1$. When the norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ is a subordinate norm, we have

$$\mathrm{cond}(\lambda, \mathbf{L}) = \frac{\|(1, \lambda, \ldots, \lambda^m)\|_{w^{-1},q}\, \|y\|_*\|x\|}{|y^*\partial_\lambda\mathbf{L}(\lambda)x|}.$$

Theorem 5.6.4. *Let* $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be regular and* $p^{-1} + q^{-1} = 1$. *Suppose that each component of* w *is nonzero. Let* (λ, y, x) *be a simple eigentriple of* $\mathbf{L}$.

(a) The condition operator $\mathrm{D} : \mathbb{M}\left(\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|),\ \mathbb{C}\right) \to (\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|))^*$, $(\lambda, \mathbf{L}) \mapsto \mathrm{D}\lambda(\mathbf{L})$ *is given by* $\mathrm{D}\lambda(\mathbf{L})X := \langle X,\, K(\lambda, \mathbf{L})\rangle_{\mathbb{L}}$ *for* $X \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$, *where* $K(\lambda, \mathbf{L}) \in \mathbb{L}^q_{m,w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ *is the condition polynomial given by*

$$K(\lambda, \mathbf{L})(z) := \sum_{i=0}^{m} z^i K_i \;\text{ and }\; K_i := -\frac{\overline{\lambda^i}\, yx^*}{\overline{y^*\partial_\lambda\mathbf{L}(\lambda)x}}.$$

The condition number $\mathrm{cond}(\lambda, \mathbf{L})$ *is given by*

$$\mathrm{cond}(\lambda, \mathbf{L}) = |\!|\!| K(\lambda, \mathbf{L}) |\!|\!|_{w^{-1},q} = \frac{\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q} \, \|yx^*\|_*}{|y^* \partial_\lambda \mathbf{L}(\lambda) x|}.$$

When the norm $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$ *is a subordinate norm,*

$$\mathrm{cond}(\lambda, \mathbf{L}) = \frac{\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q} \, \|y\|_* \|x\|}{|y^* \partial_\lambda \mathbf{L}(\lambda) x|}.$$

Hence we have $|\lambda(\mathbf{L} + \Delta\mathbf{L}) - \lambda(\mathbf{L})| \le \mathrm{cond}(\lambda, \mathbf{L}) |\!|\!| \Delta\mathbf{L} |\!|\!|_{w,p} + \mathcal{O}((|\!|\!| \Delta\mathbf{L} |\!|\!|_{w,p})^2)$.

(b) Let $\mathbf{H} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ *be such that* $|\!|\!| \mathbf{H} |\!|\!|_{w,p} = 1$. *Then the partial condition operator* $\mathrm{D}_{\mathbf{H}} : \mathbb{M}(\mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|), \mathbb{C}) \to \mathbb{C}, (\lambda, \mathbf{L}) \mapsto \mathrm{D}_{\mathbf{H}}\lambda(\mathbf{L})$ *and the partial condition number* $\mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L})$ *are given by*

$$\mathrm{D}_{\mathbf{H}}\lambda(\mathbf{L}) = -\frac{y^* \mathbf{H}(\lambda) x}{y^* \partial_\lambda \mathbf{L}(\lambda) x} = \langle \mathbf{H}, \, K(\lambda, \mathbf{L}) \rangle_{\mathbb{L}} \text{ and } \mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L}) = \frac{|y^* \mathbf{H}(\lambda) x|}{|y^* \partial_\lambda \mathbf{L}(\lambda) x|}.$$

Hence for small $|t|$, *we have* $|\lambda(\mathbf{L} + t\mathbf{H}) - \lambda(\mathbf{L})| \le \mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L})|t| + \mathcal{O}(|t|^2)$.

(c) For a subordinate norm $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$, *choose* $u, v \in \mathbb{C}^n$ *such that* $\|u\| = 1$, $\|v\|_* = 1$, $y^* u = \|y\|_*$ *and* $v^* x = \|x\|$. *Now define*

$$H_i := -\overline{\nabla_i(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q})} \, uv^*, \quad i = 0, 1, \dots, m,$$

and consider the polynomial $\mathbf{H}(z) := \sum_{i=0}^m z^i H_i$. *Here* $\nabla_i(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q})$ *is the partial gradient of the map* $\mathbb{C}^{m+1} \to \mathbb{R}, (z_0, z_1, \dots, z_m) \mapsto \|(z_0, z_1, \dots, z_m)\|_{w^{-1},q}$ *evaluated at* $(1, \lambda, \dots, \lambda^m)$. *Then we have* $|\!|\!| \mathbf{H} |\!|\!|_{w,p} = 1$ *and*

$$\mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L}) = |\langle \mathbf{H}, \, K(\lambda, \mathbf{L}) \rangle_{\mathbb{L}}| = \|K(\lambda, \mathbf{L})\|_{w^{-1},q} = \mathrm{cond}(\lambda, \mathbf{L}).$$

Proof: We only need to prove (c). Note that $\|H_i\| = |\nabla_i(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q})|$ and $|\!|\!| \mathbf{H} |\!|\!|_{w,p} = \|(\nabla_0(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q}), \dots, \nabla_m(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q}))\|_{w,p}$. By Lemma 3.5.4, we have $\|(\nabla_0(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q}), \dots, \nabla_m(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q}))\|_{w,p} = 1$. Hence $|\!|\!| \mathbf{H} |\!|\!|_{w,p} = 1$. Now $\mathrm{cond}_{\mathbf{H}}(\lambda, \mathbf{L}) = |\langle \mathbf{H}, \, K(\lambda, \mathbf{L}) \rangle_{\mathbb{L}}| = |\dfrac{y^* \mathbf{H}(\lambda) x}{y^* \partial_\lambda \mathbf{L}(\lambda) x}|$. By Lemma 3.5.4, we have

$$\begin{aligned} |y^* \mathbf{H}(\lambda) x| &= \left(|\sum_{i=0}^m \lambda^i \, \overline{\nabla_i(\|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q})}| \right) \|y\|_* \|x\| \\ &= \|(1, \lambda, \dots, \lambda^m)\|_{w^{-1},q} \|y\|_* \|x\|. \end{aligned}$$

Hence the result follows. ■

For most practical purposes one is only interested in knowing $\mathrm{cond}(\lambda, \mathbf{L})$. The crux of the matter is that when $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ and λ is a simple eigenvalue of $\mathbf{L}$, the sensitivity of λ is measured by

$$\mathrm{cond}(\lambda, \mathbf{L}) = \sup_{|\!|\!| \Delta\mathbf{L} |\!|\!|_{w,p} = 1} |\lim_{t \to 0} \frac{\lambda(\mathbf{L} + t\Delta\mathbf{L}) - \lambda(\mathbf{L})}{t}| = |\!|\!| K(\lambda, \mathbf{L}) |\!|\!|_{w^{-1},q}.$$

The sensitivity analysis of eigenvalues of matrix pencils for the case when some components of w are zero (which corresponds to some coefficients of the matrix polynomials remaining unperturbed) follows from Theorem 5.6.4 when the perturbations are restricted to be w-admissible.

We mention the similar results holds for homogeneous pencils and homogeneous matrix polynomials. Indeed, let $\mathbf{L} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$ be regular and $((\lambda,\mu),y,x)$ be a simple eigentriple of $\mathbf{L}$. Now for $\mathbf{H} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$, let $(\lambda(t),\mu(t)) := (\lambda(\mathbf{L}+t\mathbf{H}),\ \mu(\mathbf{L}+tH))$ be the eigenvalue of $\mathbf{L}+t\mathbf{H}$ such that $(\lambda(t),\mu(t))$ is the best approximation of (λ,μ), that is, $(\lambda(t)-\lambda, \mu(t)-\mu)$ is orthogonal to (λ,μ). Then it is easily seen that the maps $t \mapsto (\lambda(t),\mu(t))$ and $\mathbf{H} \mapsto (\lambda(\mathbf{L}+\mathbf{H}),\ \mu(\mathbf{L}+\mathbf{H}))$ are smooth functions. Now writing $\lambda(t) = \lambda + \alpha_1 t + \mathcal{O}(t^2)$, $\mu(t) = \mu + \alpha_2 t + \mathcal{O}(t^2)$ and $x(t) := x + x_1 t + \mathcal{O}(t^2)$, where $x(t)$ is a right eigenvector of $\mathbf{L}+t\mathbf{H}$ corresponding to $(\lambda(t),\mu(t))$, and arguing on the same lines as in Theorem 5.6.3, a little calculation shows that

$$\begin{aligned}
\alpha_1 &= -\frac{\overline{\mu} y^*\mathbf{H}(\lambda,\mu)x}{y^*(\overline{\mu}\partial_c\mathbf{L}(\lambda,\ \mu)) - \overline{\lambda}\partial_s\mathbf{L}(\lambda,\ \mu))x} + \mathcal{O}(t),\\
\alpha_2 &= -\frac{\overline{\lambda} y^*\mathbf{H}(\lambda,\ \mu)x}{y^*(\overline{\mu}\partial_c\mathbf{L}(\lambda,\ \mu)) - \overline{\lambda}\partial_s\mathbf{L}(\lambda,\ \mu))x} + \mathcal{O}(t).
\end{aligned}$$

Now define $K(\lambda,\mu,\mathbf{L}) \in \mathbb{L}^q_{m,w^{-1}}(\mathbb{C}^{n\times n}, \|\cdot\|_*)$ by

$$K(\lambda,\mu,\mathbf{L})(c,s) := \sum_{i=0}^{m} c^{m-i}s^i K_i, \text{ where } K_i := -\frac{\overline{\lambda^{m-i}\,\mu^i}\ yx^*}{(y^*(\overline{\mu}\partial_c\mathbf{L}(\lambda,\ \mu)) - \overline{\lambda}\partial_s\mathbf{L}(\lambda,\ \mu))x)^*}.$$

Then it follows that for $\mathbf{H} \in \mathbb{L}^p_{m,w}(\mathbb{C}^{n\times n}, \|\cdot\|)$, we have

$$\begin{aligned}
\mathrm{D}\lambda(\mathbf{L})\mathbf{H} &= -\frac{\overline{\mu} y^*\mathbf{H}(\lambda,\mu)x}{y^*(\overline{\mu}\partial_c\mathbf{L}(\lambda,\ \mu)) - \overline{\lambda}\partial_s\mathbf{L}(\lambda,\ \mu))x} = \langle \mathbf{H},\ \mu\, K(\lambda,\mu,\mathbf{L})\rangle_{\mathbb{L}},\\
\mathrm{D}\mu(\mathbf{L})\mathbf{H} &= -\frac{\overline{\lambda} y^*\mathbf{H}(\lambda,\ \mu)x}{y^*(\overline{\mu}\partial_c\mathbf{L}(\lambda,\ \mu)) - \overline{\lambda}\partial_s\mathbf{L}(\lambda,\ \mu))x} = \langle \mathbf{H},\ \lambda\, K(\lambda,\mu,\mathbf{L})\rangle_{\mathbb{L}}.
\end{aligned}$$

Consequently, the individual sensitivity of λ and μ are measured by $|\mu|\ |\!|\!|K(\lambda,\mu,\mathbf{L})|\!|\!|_{w^{-1},q}$ and $|\lambda|\ |\!|\!|K(\lambda,\mu,\mathbf{L})|\!|\!|_{w^{-1},q}$, respectively. This shows that

$$\begin{aligned}
\mathrm{cond}(\lambda,\mu,\mathbf{L}) &:= \sup_{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,p}=1} \lim_{t\to 0} \frac{\|(\lambda(\mathbf{L}+t\Delta\mathbf{L}),\mu(\mathbf{L}+t\mathbf{\Delta L})) - (\lambda(\mathbf{L}),\mu(\mathbf{L}))\|_2}{t\,\|(\lambda,\mu)\|_2}\\
&= |\!|\!|K(\lambda,\mu,\mathbf{L})|\!|\!|_{w^{-1},q} = \frac{\|(\lambda^m,\ \lambda^{m-1}\mu,\ \ldots,\ \mu^m)\|_{w^{-1},q}\|yx^*\|_*}{|y^*(\bar{\mu}\partial_c\mathbf{L}(\lambda,\mu) - \bar{\lambda}\partial_s\mathbf{L}(\lambda,\mu))x|}.
\end{aligned}$$

Finally, for a subordinate norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$, choose $u, v \in \mathbb{C}^n$ such that $\|u\| = 1$, $\|v\|_* = 1$, $y^*u = \|y\|_*$ and $v^*x = \|x\|$. Now define

$$H_i := -\overline{\nabla_i(\|(\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)\|_{w^{-1},q})}\, uv^*, \ \ i = 0, 1, \ldots, m$$

and consider $\mathbf{H}(c,s) := \sum_{i=0}^{m} c^{m-i}s^{i}H_i$, where $\nabla_i(\|(\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)\|_{w^{-1},q})$ is the partial gradient of the map $\mathbb{C}^{m+1} \to \mathbb{R}, (z_0, z_1, \ldots, z_m) \mapsto \|(z_0, z_1, \ldots, z_m)\|_{w^{-1},q}$ evaluated at $(\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m)$. Then we have $|\!|\!|\mathbf{H}|\!|\!|_{w,p} = 1$ and

$$\begin{aligned}
\operatorname{cond}_{\mathbf{H}}(\lambda, \mu, \mathbf{L}) &:= \lim_{t\to 0} \frac{\|(\lambda(\mathbf{L}+t\mathbf{H}), \mu(\mathbf{L}+t\mathbf{H})) - (\lambda(\mathbf{L}), \mu(\mathbf{L}))\|_2}{t\,\|(\lambda(\mathbf{L}), \mu(\mathbf{L}))\|_2} \\
&= \left|\frac{y^*\mathbf{H}(\lambda, \mu)x}{y^*(\overline{\mu}\partial_c\mathbf{L}(\lambda,\ \mu)) - \overline{\lambda}\partial_s\mathbf{L}(\lambda,\ \mu))x}\right| \\
&= |\langle \mathbf{H},\ K(\lambda, \mu, \mathbf{L})\rangle_{\mathbb{L}}| = |\!|\!|K(\lambda, \mu, \mathbf{L})|\!|\!|_{w^{-1},q} \\
&= \frac{\|(\lambda^m,\ \lambda^{m-1}\mu,\ \ldots,\ \mu^m)\|_{w^{-1},q}\|yx^*\|_*}{|y^*(\bar{\mu}\partial_c\mathbf{L}(\lambda, \mu) - \bar{\lambda}\partial_s\mathbf{L}(\lambda, \mu))x|} \\
&= \operatorname{cond}(\lambda, \mu, \mathbf{L}).
\end{aligned}$$

Note that various condition numbers discussed in the introduction of this chapter follow as special cases by choosing appropriate norms and weights.

Conclusion

We have developed a general framework for defining and analyzing pseudospectra of matrix pencils and matrix polynomial that unifies various definitions of pseudospectra of matrix pencils and polynomials proposed in the literature. We have shown that pseudospectra of matrix pencils/polynomials can be analyzed on the same lines as those of matrices.

We have undertaken a detailed analysis of backward error functions associated with matrix pencils and matrix polynomials. We have introduced a notion of critical points of backward errors of approximate eigenvalues of matrix pencils and shown that each critical point is a multiple eigenvalue of an appropriately perturbed pencil. We have shown that a minimal critical point can be read off from the pseudospectra of matrix pencils/polynomials. Further, we have shown that a solution of Wilkinson's problem for matrix pencils/polynomials can be read off from the pseudospectra of matrix pencils/polynomials. Further, for a diagonal pencil we have provided a simple procedure for the construction of nearest defective pencils.

We have derived various pseudospectra inclusions for matrix pencils and have shown that pseudospectra inclusions provides a powerful geometric framework for analyzing stability of eigendecompositions. We have introduced analogues of various separation of matrices to the case of matrix pencils and have shown their power and usefulness in analyzing stability of eigendecompositions.

Finally, we presented a general framework for the sensitivity analysis of eigenvalues of matrix pencils and matrix polynomials which lay bare the big picture that lies behind the notion of sensitivity of eigenvalues. We have shown that our treatment unifies various measures of the sensitivity of simple eigenvalues of matrix pencils and matrix polynomials proposed in the literature.

Bibliography

[1] R. Alam, *On the construction of nearest defective matrices to a normal matrix*, Linear Algebra Appl., 395 (2005), pp. 367–370.

[2] R. Alam and S. Bora, *On sensitivity of eigenvalues and eigendecompositions of matrices*, Linear Algebra Appl., 396 (2005), pp. 273–301.

[3] R. Alam and S. bora, *On stable eigendecompositions of matrices*, SIAM J. Matrix Anal. Appl., 26 (2005), pp. 830–848.

[4] R. Alam and S. Bora, *Effect of linear perturbation on spectra of matrices*, Linear Algebra Appl., 368 (2003), pp. 329–342.

[5] B. Aupetit, *A Primer on Spectral Theory*, Springer-Verlag, 1991.

[6] R. Bhatia, *Matrix Analysis*, Springer-Verlag, 1997.

[7] S. Bora, *A Geometric Analysis of Spectral Stability of Matrices and Operators*, Ph. D. Thesis, IIT Guwahati, 2001.

[8] Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H. van der Vorst, *Templets for the Solution of Algebraic Eigenvalue Problems: A Practical Guide*, SIAM, 2000.

[9] R. Byers, C. He, and V. Mehrmann, *Where is the nearest non-regular pencil?*, Linear Algebra Appl., 285 (1998), pp. 81–105.

[10] J. V. Burke, A. S. Lewis, and M. L. Overton, *A nonsmooth, nonconvex optimization approach to robust stabilization by static output feedback and low-order controllers*, Proceedings of Fourth IFAC Symposium on Robust Control Design, Milan, June 2003, pp. 175-181 (Elsevier, 2004).

[11] J. V. Burke, A. S. Lewis, and M. L. Overton, *Pseudospectral components and the distance to uncontrollability*, SIAM J. Matrix Anal. Appl., 26 (2004), pp. 350–361.

[12] L. Boulton, P. Lancaster and P. Psarrakos, *On pseudospectra of matrix polynomials and their boundaries*, Math. Comp., Electronically published, May 11, 2007.

[13] G. Boutry, M. Elad, G. H. Golub, and P. Milanfar, *The generalized eigenvalue problem for nonsquare pencils using a minimal perturbation approach*, SIAM J. Matrix Anal. Appl., 27 (2005), pp. 582–601.

[14] B. N. Datta, *Numerical Methods for Linear Control Systems*, Academic Press, 2003.

[15] J. W. Demmel, *Computing stable eigendecompositions of matrices*, Linear Algebra Appl., 79 (1986), pp. 163–193.

[16] J. W. Demmel and B. Kågström, *Computing stable eigendecompositions of matrix pencils*, Linear Algebra Appl., 88–89 (1987), pp. 139–186.

[17] J. P. Dedieu, *Condition operators, condition numbers, and condition number theorem for generalized eigenvalue problem*, Linear Algebra Appl., 263 (1997), pp. 1–24.

[18] J. P. Dedieu and F. Tisseur, *Perturbation theory for homogeneous polynomial eigenvalue problems*, Linear Algebra Appl., 358 (2003), pp. 71–94.

[19] J. L. M. V. Dorsselaer, *Pseudospectra of matrix pencils and stability of equlibria*, BIT, 37 (1997), pp. 833–845.

[20] J. L. M. V. Dorsselaer, *Several concepts to investigate strongly nonnormal eigenvalue problems*, SIAM J. Sci. Comput., 24 (2003), pp. 1031–1053.

[21] M. Embree and L. N. Trefethen, Pseudospectra Gateway. Available from: <http://web.comlab.ox.ac.uk/pseudospectra> .

[22] V. Frayssé, M. Gueury, F. Nicoud, and V. Toumazou, *Spectral portraits for matrix pencils*, TR/PA/96/19, Cerfacs, France, 1996.

[23] I. Gohberb, P. Lancaster, and L. Rodman, *Matrix Polynomials*, Academic Press, 1982.

[24] L. Grammont and A. Largillier, *On ϵ-spectra and stability radii*, J. Comp. Appl. Math., 147 (2002), pp. 453–469.

[25] N. J. Higham and F. Tisseur, *More on pseudospectra for polynomial eigenvalue problems and applications in control theory*, Linear Algebra Appl., 351-352 (2002), pp. 435–453.

[26] D. J. Higham and N. J. Higham, *Structured backward error and condition of generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 20 (1998), pp. 493–512.

[27] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1980.

[28] B. Kågström and P. Van Dooren, *A generalized state-space approach for the additive decomposition of a transfer matrix*, J. Numer. Linear Algebra Appl., 1 (1992), pp. 165–181.

[29] P. Kunkel and V. Mehrmann *A new look at pencils of matrix valued functions*, Linear Algebra Appl., 212–213 (1994), pp. 215–248.

[30] P. F. Lavallée and M. Sadkane, *Pseudospectra of linear matrix pencils by block diagonalization*, Computing, 60 (1998), pp. 133–156.

[31] P. Lancaster and P. Psarrakos, *On the pseudospectra of matrix polynomials*, SIAM J. Matrix Anal. Appl., 27 (2005), pp. 115–129.

[32] P. Lancaster, *Lambda Matrices and Vibrating Systems*, Dover Publication, 1966.

[33] B. V. Limaye, *Functional Analysis*, New Age Publishers, 1996.

[34] B. V. Limaye, *Spectral Perturbation and Approximation with Numerical Experiments*, Proceedings of the Centre for Math. Anal., Vol. 13, Australian National University, Canberra, 1987.

[35] V. Mehrmann and G. M. Krause, *Linear transformations which leave controllable multiinput descriptor systems controllable*, Linear Algebra Appl., 120 (1989), pp. 47–64.

[36] N. K. Nichols and J. Kautsky, *Robust eigenstructure assignment in quadratic matrix polynomials: nonsingular case*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 77–102.

[37] T. Ransford, *Potential theorey in the complex plane*, Cambridge University Press, 1995.

[38] K. S. Riedel, *Generalized epsilon-pseudospectra*, SIAM J. Numer. Anal., 31 (1994), pp. 1219–1225.

[39] M. Shub and S. Smale, *Complexity of Bezout's theorem IV: probabilty of success; extensions*, SIAM J. Numer. Anal., 33 (1996), pp. 128–148.

[40] G. W. Stewart, *On the sensitivity of the eigenvalue problem $Ax = \lambda Bx$*, SIAM J. Numer. Anal. Appl., 9 (1972), pp. 669–686.

[41] G. W. Stewart, *Gershgorin theory for generalized eigenvalue problem $Ax = \lambda Bx$*, Math. Comp., 29 (1975), pp. 600–606.

[42] G. W. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, 1990.

[43] G. W. Stewart, *Error and perturbation bounds for subspaces associated with certain eigenvalue problems*, SIAM Rev., 15 (1973), pp. 727–764.

[44] J. G. Sun, *A note on simple non-zero singular values*, J. Comput. Math., 6 (1988), pp. 258–266.

[45] F. Tisseur and N. J. Higham, *Structured pseudospectra for polynomial eigenvalue problems, with applications*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 187-208.

[46] F. Tisseur, *Backward error and condition of polynomial eigenvalue problems*, Linear Algebra Appl., 309 (2000), pp. 339–361.

[47] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra: The behavior of nonnormal matrices and operators*, Princeton University Press, 2005.

[48] J. M. Varah, *On the separation of two matrices*, SIAM J. Numer. Anal., 16 (1979), pp. 216–222.

[49] J. H. Wilkinson, *On neighbouring matrices with quadratic elementary divisors,* Numer. Math., 44 (1984), pp. 1–21.

[50] J. H. Wilkinson, *Sensitivity of eigenvalues*, Util. Math., 25 (1984), pp. 5–76.

[51] J. H. Wilkinson, *Sensitivity of eigenvalues II*, Util. Math., 30 (1986), pp. 243–286.

[52] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, Oxford, UK, 1996.